IDENTIFICATION VIA QUANTUM CHANNELS IN THE PRESENCE OF PRIOR CORRELATION AND FEEDBACK

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Continuing our earlier work (quant-ph/0401060), we give two alternative proofs of the result that a noiseless qubit channel has identification capacity 2: the first is direct by a “maximal code with random extension” argument, the second is by showing that 1 bit of entanglement (which can be generated by transmitting 1 qubit) and negligible (quantum) communication has identification capacity 2. This generalises a random hashing construction of Ahlswede and Dueck: that 1 shared random bit together with negligible communication has identification capacity 1. We then apply these results to prove capacity formulas for various quantum feedback channels: passive classical feedback for quantum–classical channels, a feedback model for classical–quantum channels, and “coherent feedback” for general channels.

1 Introduction

While the theory of identification via noisy channels has generated significant interest within the information theory community (the areas of, for instance, common randomness, channel resolvability, and watermarking were either developed in response or were discovered to have close connections to identification), the analogous theory where one uses a quantum channel has received comparably little attention: the only works extant at the time of writing are Löber’s starting of the theory, a strong converse for discrete memoryless classical-quantum channels by Ahlswede and Winter, and a recent paper by the present author.

This situation may have arisen from a perception that such a theory would not be very different from the classical identification theory, as indeed classical message transmission via quantum channels, at a fundamental mathematical level, does not deviate much from its classical counterpart coding theorem and converses are “just like” in Shannon’s classical channel coding theory, with Holevo information playing the role of Shannon’s mutual information. (Though we have to acknowledge that it took quite a while before this was understood, and that there are tantalising differences in detail, e.g. additivity problems.)

In our recent work, however, a quite startling discovery was made: it was shown that — contrary to the impression the earlier papers gave — the identification capacity of a (discrete memoryless, as always in this paper) quantum channel is in general not equal to its transmission capacity. Indeed, the identification capacity of a noiseless qubit was found to be 2. This means
that for quantum channels the rule that identification capacity equals common randomness capacity (see the discussion by Ahlswede and Kleinewächter) fails dramatically, even for the most ordinary of channels!

In the present paper we find some new results for identification via quantum systems: after a review of the necessary definitions and known results (section 2) and a collection of statements about what we called “random channels” in our earlier paper, we first give a direct proof that a qubit has identification capacity 2, in section 4. (Our earlier proof uses a reduction to quantum identification, which we avoid here.) Then, in section 5 we show the quantum analogue of Ahlswede and Dueck’s result that 1 bit of shared randomness plus negligible communication are sufficient to build an identification code of rate 1.5 namely, 1 bit of entanglement plus negligible (quantum) communication are sufficient to build an identification code of rate 2. In section 6 we briefly discuss the case of more general prior correlations between sender and receiver.

In section 7, we turn our attention to feedback channels: we first study quantum–classical channels with passive classical feedback, and prove a quantum generalisation of the capacity formula of Ahlswede and Dueck. Then, in section 8 we introduce a feedback model for general quantum channels which we call “coherent feedback”, and prove a capacity formula for these channels as well which can be understood as a quantum analogue of the feedback identification capacity of Ahlswede and Dueck. We also comment on a different feedback model for classical–quantum channels.

2 Review of definitions and known facts

For a broader review of identification (and, for comparison, transmission) via quantum channels we refer the reader to the introductory sections of our earlier paper, to Löber’s Ph.D. thesis and to the classical identification papers by Ahlswede and Dueck. Here we are content with repeating the bare definitions:

We are concerned with quantum systems, which are modelled as (finite) Hilbert spaces \( H \) (or rather the operator algebra \( B(H) \)). States on these systems we identify with density operators \( \rho \): positive semidefinite operators with trace 1.

A quantum channel is modelled in this context as a completely positive, trace preserving linear map \( T : B(H_1) \rightarrow B(H_2) \) between the operator algebras of Hilbert spaces \( H_1, H_2 \).

**Definition 1 (Löber, Ahlswede and Winter)** An identification code for the channel \( T \) with error probability \( \lambda_1 \) of first, and \( \lambda_2 \) of second kind is a set \( \{(\rho_i, D_i) : i = 1, \ldots, N\} \) of states \( \rho_i \) on \( H_1 \) and operators \( D_i \) on
$H_2$ with $0 \leq D_i \leq \mathbb{I}$, such that
\[
\forall i \quad \text{Tr}(T(\rho_i)D_i) \geq 1 - \lambda_1,
\]
\[
\forall i \neq j \quad \text{Tr}(T(\rho_i)D_j) \leq \lambda_2.
\]

For the identity channel $\text{id}_{\mathbb{C}^d}$ of the algebra $\mathcal{B}(\mathbb{C}^d)$ of a $d$-dimensional system we also speak of an identification code on $\mathbb{C}^d$.

For the special case of memoryless channels $T^\otimes n$ (where $T$ is implicitly fixed), we speak of an $(n, \lambda_1, \lambda_2)$–ID code, and denote the largest size $N$ of such a code $N(n, \lambda_1, \lambda_2)$.

An identification code as above is called simultaneous if all the $D_i$ are coexistent: this means that there exists a positive operator valued measure (POVM) $(E_k)_{k=1}^K$ and sets $D_i \subset \{1, \ldots, K\}$ such that $D_i = \sum_{k \in D_i} E_k$. The largest size of a simultaneous $(n, \lambda_1, \lambda_2)$–ID code is denoted $N_{\text{sim}}(n, \lambda_1, \lambda_2)$.

Most of the current knowledge about these concepts is summarised in the two following theorems.

**Theorem 2 (L"ober,21 Ahlswede and Winter,16)** Consider any channel $T$, with transmission capacity $C(T)$ (Holevo,17 Schumacher and Westmoreland,24). Then, the simultaneous identification capacity of $T$,
\[
C_{\text{sim–ID}}(T) := \inf_{\lambda_1, \lambda_2 > 0} \liminf_{n \to \infty} \frac{1}{n} \log \log N_{\text{sim}}(n, \lambda_1, \lambda_2) \geq C(T).
\]
(With log and exp in this paper understood to basis 2.)

For classical–quantum (cq) channels $T$ (see Holevo,16), even the strong converse for (non–simultaneous) identification holds:
\[
C_{\text{ID}}(T) = \lim_{n \to \infty} \frac{1}{n} \log \log N(n, \lambda_1, \lambda_2) = C(T),
\]
whenever $\lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 1$.

That the (non–simultaneous) identification capacity can be larger than the transmission capacity was shown only recently:

**Theorem 3 (Winter,29)** The identification capacity of the noiseless qubit channel, $\text{id}_{\mathbb{C}^2}$, is $C_{\text{ID}}(\text{id}_{\mathbb{C}^2}) = 2$, and the strong converse holds.

The main objective of the following three sections is to give two new proofs of the achievability of 2 in this theorem.

### 3 Random channels and auxiliary results

The main tool in the following results (as in our earlier paper,29) are random channels and in fact random states,22,15

**Definition 4** For positive integers $s, t, u$ with $s \leq tu$, the random channel $R^{t(u)}_s$ is a random variable taking values in quantum channels $\mathcal{B}(\mathbb{C}^s) \to \mathcal{B}(\mathbb{C}^t)$ with the following distribution:

There is a random isometry $V : \mathbb{C}^s \to \mathbb{C}^t \otimes \mathbb{C}^u$, by which we mean a random variable taking values in isometries whose distribution is left-/right-invariant under multiplication by unitaries on $\mathbb{C}^t \otimes \mathbb{C}^u$ on $\mathbb{C}^s$, respectively, such that

$$R_s^{(u)}(\rho) = \text{Tr}_{\mathbb{C}^u}(V \rho V^*) .$$

Note that the invariance demanded of the distribution of $V$ determines it uniquely — one way to generate the distribution is to pick an arbitrary fixed isometry $V_0 : \mathbb{C}^s \to \mathbb{C}^t \otimes \mathbb{C}^u$ and a random unitary $U$ on $\mathbb{C}^t \otimes \mathbb{C}^u$ according to the Haar measure, and let $V = UV_0$.

**Remark 5** Identifying $\mathbb{C}^{tu}$ with $\mathbb{C}^t \otimes \mathbb{C}^u$, we have $R_s^{(u)}(\rho) = \text{Tr}_{\mathbb{C}^u}(\text{R}_{tu}(1))$. Note that $R_s^{(1)}(1)$ is a random isometry from $\mathbb{C}^s$ into $\mathbb{C}^t$ in the sense of our definition, and that the distribution of $R_s^{(1)}$ is the Haar measure on the unitary group of $\mathbb{C}^s$.

**Remark 6** The one-dimensional Hilbert space $\mathbb{C}$ is a trivial system: it has only one state, $1$, and so the random channel $R_1^{(u)}$ is equivalently described by the image state it assigns to $1$, $R_1^{(u)}(1)$. For $s = 1$ we shall thus identify the random channel $R_s^{(u)}$ with the random state $R_s^{(u)}(1)$ on $\mathbb{C}^t$. A different way of describing this state is that there exists a random (Haar distributed) unitary $U$ and a pure state $\psi_0$ such that $R_1^{(u)} = \text{Tr}_{\mathbb{C}^u}(U \psi_0 U^*)$ — note that it has rank bounded by $u$. These are the objects we concentrate on in the following.

**Lemma 7** (see Bennett et al. 8 Winter 29) Let $\psi$ be a pure state, $P$ a projector of rank (at most) $r$ and let $U$ be a random unitary, distributed according to the Haar measure. Then for $\epsilon > 0$,

$$\Pr\{\text{Tr}(U\psi U^* P) \geq (1 + \epsilon)\frac{r}{d}\} \leq \exp\left(-r \frac{\epsilon^2}{6 \ln 2}\right).$$

For $0 < \epsilon \leq 1$, and rank $P = r$,

$$\Pr\{\text{Tr}(U\psi U^* P) \geq (1 + \epsilon)\frac{r}{d}\} \leq \exp\left(-r \frac{\epsilon^2}{6 \ln 2}\right),$$

$$\Pr\{\text{Tr}(U\psi U^* P) \leq (1 - \epsilon)\frac{r}{d}\} \leq \exp\left(-r \frac{\epsilon^2}{6 \ln 2}\right).$$

**Lemma 8** (Bennett et al. 8) For $\epsilon > 0$, there exists in the set of pure states on $\mathbb{C}^d$ an $\epsilon$–net $\mathcal{M}$ of cardinality $|\mathcal{M}| \leq (\frac{2}{\epsilon})^{2d}$; i.e.,

$$\forall \varphi \text{ pure } \exists \tilde{\varphi} \in \mathcal{M} \quad \|\varphi - \tilde{\varphi}\|_1 \leq \epsilon.$$

With these lemmas, we can prove an important auxiliary result:
Lemma 9 (see Harrow et al.\textsuperscript{[15]}) For $0 < \eta \leq 1$ and $t \leq u$, consider the random state $R_{1}^{t(u)}$ on $\mathbb{C}^{t}$. Then,

$$\Pr \left\{ R_{1}^{t(u)} \notin \left[ \frac{1 - \eta}{t} \mathbb{I}, \frac{1 + \eta}{t} \mathbb{I} \right] \right\} \leq 2 \left( \frac{10t}{\eta} \right)^{2t} \exp \left( -u \frac{\eta^{2}}{24 \ln 2} \right).$$

Proof. We begin with the observation that $R_{1}^{t(u)} \in [\alpha \mathbb{I}; \beta \mathbb{I}]$ if and only if for all pure states (rank one projectors) $\varphi$,

$$\text{Tr} \left( R_{1}^{t(u)} \varphi \right) = \text{Tr} \left( R_{1}^{t(u)} (\varphi \otimes \mathbb{I}_{u}) \right) \begin{cases} \geq \alpha, \\ \leq \beta. \end{cases}$$

Due to the triangle inequality, we have to ensure this only for $\varphi$ from an $\eta/2t$–net and with $\alpha = \left( 1 - \frac{\eta}{2} \right)/t$, $\beta = \left( 1 + \frac{\eta}{2} \right)/t$. Then the probability bound claimed above follows from lemmas 7 and 9 with the union bound. $\square$

4 ID capacity of a qubit

Here we give a new, direct proof of theorem\textsuperscript{[8]} — in fact, we prove the following proposition from which it follows directly.

Proposition 10 For every $0 < \lambda < 1$, there exists on the quantum system $\mathcal{B}(\mathbb{C}^{d})$ an ID code with

$$N = \left\lceil \frac{1}{2} \exp \left( \left( \frac{\lambda}{3000 \log d} \right)^{2} \right) \right\rceil$$

messages, with error probability of first kind equal to 0 and error probability of second kind bounded by $\lambda$.

Proof. We shall prove even a bit more: that such a code exists which is of the form $\{(\rho_{i}, D_{i}) : i = 1, \ldots, N\}$ with

$$D_{i} = \text{supp} \rho_{i}, \quad \text{rank} \rho_{i} = \delta := \alpha \frac{d}{\log d}, \quad \rho_{i} \leq \frac{1 + \eta}{\delta} D_{i}. \quad (1)$$

The constants $\alpha \leq \lambda/4$ and $\eta \leq 1/3$ will be fixed in the course of this proof. Let a maximal code $C$ of this form be given. We shall show that if $N$ is “not large”, a random codestate as follows will give a larger code, contradicting maximality.

Let $R = R_{1}^{\delta(d)}$ (the random state in dimension $d$ with $\delta$–dimensional ancillary system, see definition\textsuperscript{[4]}, and $D := \text{supp} R$. Then, according to the Schmidt decomposition and lemma\textsuperscript{[4]}

$$\Pr \left\{ R \notin \left[ \frac{1 - \eta}{\delta} D; \frac{1 + \eta}{\delta} D \right] \right\} = \Pr \left\{ R_{1}^{\delta(d)} \notin \left[ \frac{1 - \eta}{\delta} \mathbb{1}_{d}; \frac{1 + \eta}{\delta} \mathbb{1}_{d} \right] \right\} \leq 2 \left( \frac{10\delta}{\eta} \right)^{2\delta} \exp \left( -\frac{d \eta^{2}}{24 \ln 2} \right). \quad (2)$$
This is \( \leq 1/2 \) if
\[
d \geq \left( \frac{96 \ln 2}{\eta^2} \log \frac{10}{\eta} \right) \delta \log \delta,
\]
which we ensure by choosing \( \alpha \leq \lambda \left( \frac{96 \ln 2}{\eta^4} \log \frac{10}{\eta} \right)^{-1} \leq \lambda/4. \)

In the event that \( \frac{1-\eta}{\delta} D \leq R \leq \frac{1+\eta}{\delta} D \), we have on the one hand
\[
\text{Tr}(\rho_i D) \leq \text{Tr} \left( \frac{1+\eta}{\delta} D_i \frac{\delta}{1-\eta} R \right) \leq 2\text{Tr}(RD_i). \tag{3}
\]

On the other hand, because of \( R_i^{d(\delta)} = \text{Tr}_{C\otimes \bar{D}} R_i^{d(1)} \), we can rewrite \( \text{Tr}(RD_i) = \text{Tr}(R_i^{d(1)} (D_i \otimes \mathbb{I}_\delta)) \), hence by lemma 4
\[
\Pr\{\text{Tr}(RD_i) > \lambda/2\} \leq \exp(-\delta^2). \tag{4}
\]

So, by the union bound, eqs. (3) and (4) yield
\[
\Pr\{\mathcal{C} \cup \{(R, D)\} \text{ has error probability of second kind larger than } \lambda \text{ or violates eq. (1)}\} \leq \frac{1}{2} + N \exp(-\delta^2).
\]

If this is less than 1, there must exist a pair \((R, D)\) extending our code while preserving the error probabilities and the properties of eq. (1), which would contradict maximality. Hence,
\[
N \geq \frac{1}{2} \exp(\delta^2),
\]
and we are done, fixing \( \eta = 1/3 \) and \( \alpha = \lambda/3000. \)

The proof of theorem 3 is now obtained by applying the above proposition to \( d = 2^n \), the Hilbert space dimension of \( n \) qubits, and arbitrarily small \( \lambda \). That the capacity is not more than 2 is by a simple dimension counting argument, which we don’t repeat here.

5 **ID capacity of an ebit**

Ahlswede and Dueck\cite{ahlswede1989capacity} have shown that the identification capacity of any system, as soon as it allows — even negligible — communication, is at least as large as its common randomness capacity: the maximum rate at which shared randomness can be generated. (We may add, that except for pathological examples expressly constructed for that purpose, in practically all classical systems for which these two capacities exist, they turn out to be equal\cite{ahlswede1993information}, which we don’t repeat here.)
Proposition 11 (Ahlswede and Dueck) There exist, for \( \lambda > 0 \) and \( N \geq 4^{1/\lambda} \), functions \( f_i : \{1, \ldots, M\} \rightarrow \{1, \ldots, N\} \) \( (i = 1, \ldots, 2^M) \) such that the distributions \( P_i \) on \( \{1, \ldots, M\} \times \{1, \ldots, N\} \) defined by

\[
P_i(\mu, \nu) = \begin{cases} \frac{1}{M} & \text{if } \nu = f_i(\mu), \\ 0 & \text{otherwise.} \end{cases}
\]

and the sets \( D_i = \text{supp} P_i \) form an identification code with error probability of first kind 0 and error probability of second kind \( \lambda \).

In other words, prior shared randomness in the form of uniformly distributed \( \mu \in \{1, \ldots, M\} \) between sender and receiver, and transmission of \( \nu \in \{1, \ldots, N\} \) allow identification of \( 2^M \) messages. □

Thus, an alternative way to prove that a channel of capacity \( C \) allows identification at rate \( \geq C \), is given by the following scheme: use the channel \( n - O(1) \) times to generate \( Cn - o(n) \) shared random bits and the remaining \( O(1) \) times to transmit one out of \( N = 2^O(1) \) messages; then apply the above construction with \( M = 2^{Cn - o(n)} \). More generally, a rate \( R \) of common randomness and only negligible communication give identification codes of rate \( R \).

The quantum analogue of perfect correlation (i.e., shared randomness) being pure entanglement, substituting quantum state transmission wherever classical information was conveyed, and in the light of the result that a qubit has identification capacity 2, the following question appears rather natural (and we have indeed raised it, in remark 14 of our earlier paper): Does 1 bit of entanglement plus the ability to (even only negligibly) communicate result in an ID code of rate 2, asymptotically?

Proposition 12 For \( \lambda > 0 \), \( d \geq 2 \) and \( \Delta \geq \left( \frac{900 \log \frac{40d}{\lambda}}{\lambda} \right) \log d \), there exist quantum channels \( T_i : B(\mathbb{C}^d) \rightarrow B(\mathbb{C}^\Delta) \) \( (i = 1, \ldots, N' = \left\lceil \frac{1}{2} \exp(d^2) \right\rceil) \), such that the states \( \rho_i = (\text{id} \otimes T_i)\Phi_d \) (with state vector \( |\Phi_d\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d |j\rangle|j\rangle \)), and the operators \( D_i = \text{supp} \rho_i \) form an identification code on \( B(\mathbb{C}^d \otimes \mathbb{C}^\Delta) \) with error probability of first kind 0 and error probability of second kind \( \lambda \).

In other words, sender and receiver, initially sharing the maximally entangled state \( \Phi_d \), can use transmission of a \( \Delta \)-dimensional system to build an identification code with \( \left\lceil \frac{1}{2} \exp(d^2) \right\rceil \) messages.

Proof. Let a maximal code \( C \) as described in the proposition be given, such that additionally

\[
D_i = \text{supp} \rho_i, \quad \text{rank} \rho_i = d, \quad \rho_i \leq \frac{1 + \lambda}{d} D_i. \tag{5}
\]

Consider the random state \( R = R_1^{d\Delta(d)} \) on \( \mathbb{C}^d \otimes \mathbb{C}^\Delta \), and \( D := \text{supp} R \). Now, by Schmidt decomposition and with lemma (compare the proof of
proposition \(\text{(10)}\), for \(\eta := \lambda/3\)

\[
\Pr \left( R \notin \left[ \frac{1 - \eta}{d} D; \frac{1 + \eta}{d} D \right] \right) = \Pr \left( R^{d(\Delta d)} \notin \left[ \frac{1 - \eta}{d} \mathbb{1}_d; \frac{1 + \eta}{d} \mathbb{1}_d \right] \right) 
\leq 2 \left( \frac{10d}{\eta} \right)^{2d} \exp \left( -d \Delta \frac{\eta^2}{24 \ln 2} \right). 
\]

(6)

The very same estimate gives

\[
\Pr \left\{ \text{Tr} C \Delta R \notin \left[ \frac{1 - \eta}{d} \mathbb{1}_d; \frac{1 + \eta}{d} \mathbb{1}_d \right] \right\} = \Pr \left\{ R^{d(\Delta d)} \notin \left[ \frac{1 - \eta}{d} \mathbb{1}_d; \frac{1 + \eta}{d} \mathbb{1}_d \right] \right\} 
\leq 2 \left( \frac{10d}{\eta} \right)^{2d} \exp \left( -d \Delta \frac{\eta^2}{24 \ln 2} \right). 
\]

(7)

By choosing \(\Delta \geq \left( \frac{144 \ln 2}{\eta^2 \log \frac{10}{\eta}} \right) \log d\), as we indeed did, the sum of these two probabilities is at most \(1/2\).

In the event that \(1 - \eta d D \leq R \leq 1 + \eta d D\), we argue similar to the proof of proposition \(\text{(10)}\) (compare eq. (3)):

\[
\text{Tr}(\rho_i D) \leq \text{Tr} \left( 1 + \frac{\lambda d}{d \mathbb{1}_d; \frac{1 + \eta}{d} \mathbb{1}_d} \right) \leq 3 \text{Tr}(RD_i). 
\]

(8)

On the other hand (compare eq. \(\text{(4)}\)),

\[
\Pr \{ \text{Tr}(RD_i) > \lambda/3 \} \leq \exp(-d^2), 
\]

(9)

by lemma \(\text{(7)}\) and using \(\Delta^{-1} \leq \lambda/6\).

In the event that \(\frac{1 - \eta}{d} \mathbb{1}_d \leq \text{Tr}_C \Delta R \leq \frac{1 + \eta}{d} \mathbb{1}_d\), there exists an operator \(X\) on \(\mathbb{C}^d\) with \(\frac{1 + \eta}{d} \mathbb{1}_d \leq X \leq \frac{1 - \eta}{d} \mathbb{1}_d\), such that

\[
R_0 := \sqrt{R(X \otimes \mathbb{1})} \sqrt{R} \quad \text{(which has the same support \(D\) as \(R\))}
\]
satisfies \(\text{Tr}_C R_0 = \frac{1}{d} \mathbb{1}\). By the Jamiołkowski isomorphism \(\text{(18)}\) between quantum channels and states with maximally mixed reduction, this is equivalent to the existence of a quantum channel \(T_0\) such that \(R_0 = (\id \otimes T_0) \Phi_d\). Observe that \(R_0 \leq \frac{1 + \eta}{d} D\) and \(\text{Tr}(R_0 D_i) \leq \frac{2}{d} \text{Tr}(RD_i)\).

So, putting together the bounds of eqs. \(\text{(4)}, \text{(7)}, \text{(8)}\) and \(\text{(9)}\), we get, by the union bound,

\[
\Pr \{ \mathcal{C} \cup \{(R_0, D)\} \text{ has error probability of second kind larger than } \lambda \text{ or violates eq. } \text{(5)} \} \leq \frac{1}{2} + N \exp(-d^2).
\]

If this is less than 1, there will exist a state \(R_0 = (\id \otimes T_0) \Phi_d\) and an operator \(D\) enlarging the code and preserving the error probabilities as well as the properties in eq. \(\text{(14)}\), which contradicts maximality.

Hence, \(N \geq \frac{1}{7} \exp(d^2)\), and we are done. \(\square\)

This readily proves, answering the above question affirmatively:
Theorem 13 The identification capacity of a system in which entanglement (EPR pairs) between sender and receiver is available at rate $E$, and which allows (even only negligible) communication, is at least $2E$. This is tight for the case that the available resources are only the entanglement and negligible communication. □

Remark 14 Just as the Ahlswede–Dueck construction of proposition 11 can be understood as an application of random hashing, we are tempted to present our above construction as a kind of “quantum hashing”: indeed, the (small) quantum system transmitted contains, when held together with the other half of the prior shared entanglement, just enough of a signature of the functions/quantum channels used to distinguish them pairwise reliably.

6 General prior correlation

Proposition 11 quantifies the identification capacity of shared randomness, and proposition 12 does the same for shared (pure) entanglement. This of course raises the questions what the identification capacity of other, more general, correlations is: i.e., we are asking for code constructions and bounds if (negligible) quantum communication and $n$ copies of a bipartite state $\omega$ between sender and receiver are available.

For the special case that the correlation decomposes cleanly into entanglement and shared randomness,

$$\omega = \sum_{\mu} p_{\mu} \Psi^{AB}_{\mu} \otimes |\mu\rangle \langle \mu|^{A'} \otimes |\mu\rangle \langle \mu|^{B'},$$

with an arbitrary perfect classical correlation (between registers $A'$ and $B'$) distributed according to $p$ and arbitrary pure entangled states $\Psi_{\mu}$, we can easily give the answer (let the sender be in possession of $AA'$, the receiver of $BB'$):

$$C_{ID} = H(p) + 2 \sum_{\mu} p_{\mu} E(\Psi^{AB}_{\mu}); \quad (10)$$

here, $H(p)$ is the entropy of the classical perfect correlation $p$; $E(\Psi^{AB}) = S(\Psi^{A})$ is the entropy of entanglement [7] with the reduced state $\Psi^{A} = Tr_{B} \Psi^{AB}$. The achievability is seen as follows: by entanglement and randomness concentration [7], this state yields shared randomness and entanglement at rates $R = H(p)$ and $E = \sum_{\mu} p_{\mu} E(\Psi_{\mu})$, respectively (without the need of communication — note that both users learn which entangled state they have by looking at the primed registers). Proposition 12 yields an identification code of rate $2E$, while proposition 15 below shows how to increase this rate by $R$.

That the expression is an upper bound is then easy to see, along the lines of the arguments given in our earlier paper for the capacity of a “hybrid quantum memory” [20,21].
Proposition 15 (Winter29) Let \(\{ (\rho_i, D_i) : i = 1, \ldots, N \}\) be an identification code on the quantum system \(\mathcal{H}\) with error probabilities \(\lambda_1, \lambda_2\) of first and second kind, respectively, and let \(\mathcal{H}_C\) be a classical system of dimension \(M\) (by this we mean a Hilbert space only allowed to be in a state from a distinguished orthonormal basis \(\{|\mu\rangle\}_{\mu=1}^M\)). Then, for every \(\epsilon > 0\), there exists an identification code \(\{ (\sigma_f, D_f) : f = 1, \ldots, N' \}\) on \(\mathcal{H}_C \otimes \mathcal{H}\) with error probabilities \(\lambda_1, \lambda_2 + \epsilon\) of first and second kind, respectively, and \(N' \geq \left(\frac{1}{2}N^e\right)^M\). The \(f\) actually label functions (also denoted \(f\)) \(\{1, \ldots, M\} \rightarrow \{1, \ldots, N\}\), such that

\[
\sigma_f = \frac{1}{M} \sum_{\mu} |\mu\rangle \langle \mu | \otimes \rho_{f(k)}. \]

In other words, availability of shared randomness (\(\mu\) on the classical system \(\mathcal{H}_C\)) with an identification code allows us to construct a larger identification code.

The general case seems to be much more complex, and we cannot offer an approximation to the solution here. So, we restrict ourselves to highlighting two questions for further investigation:

1. What is the identification capacity of a bipartite state \(\omega\), together with negligible communication? For noisy correlations, this may not be the right question altogether, as a look at work by Ahlswede and Balakirsky2 shows: they have studied this problem for classical binary correlations with symmetric noise, and have found that — as in common randomness theory3 — one ought to include a limited rate of communication and study the relation between this additional rate and the obtained identification rate. Hence, we should ask: what is the identification capacity of \(\omega\) plus a rate of \(C\) bits of communication? An obvious thing to do in this scenario would be to use part of this rate to do entanglement distillation of which the communication cost is known in principle.11 This gives entanglement as well as shared randomness, so one can use the constructions above. It is not clear of course whether this is asymptotically optimal.

2. In the light of the code enlargement proposition, it would be most interesting to know if a stronger version of our proposition theorem holds: Does entanglement of rate \(E\) increase the rate of a given identification code by \(2E\)?

7 Identification in the presence of feedback:
quantum–classical channels

Feedback for quantum channels is a somewhat problematic issue, mainly because the output of the channel is a quantum state, of which there is in general
no physically consistent way of giving a copy to the sender. In addition, it should not even be a “copy” for the general case that the channel outputs a mixed state (which corresponds to the distribution of the output), but a copy of the exact symbol the receiver obtained; so the feedback should establish correlation between sender and receiver, and in the quantum case this appears to involve further choices, e.g. of basis. The approach taken in the small literature on the issue of feedback in quantum channels (see Fujiwara and Nagaoka, Bowen, and Bowen and Nagarajan) has largely been to look at active feedback, where the receiver decides what to give back to the sender, based on a partial evaluation of the received data.

We will begin our study by looking at a subclass of channels which do not lead into any of these conceptual problems: quantum–classical (qc) channels, i.e., destructive measurements, have a completely classical output anyway, so there is no problem in augmenting every use of the channel by instantaneous passive feedback.

Let a measurement POVM \((M_y)_{y \in \mathcal{Y}}\) be given; then its qc–channel is the map

\[
T : \rho \mapsto \sum_y \text{Tr}(\rho M_y) |y⟩⟨y|,
\]

with an orthogonal basis \((|y⟩)_{y \in \mathcal{Y}}\) of an appropriate Hilbert space \(\mathcal{F}\), say. We will denote this qc–channel as \(\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{Y}\).

For a qc–channels \(\mathcal{T}\), a (randomised) feedback strategy \(\Phi\) for block \(n\) is given by states \(\rho_t : y_{t-1} \in \mathcal{Y}_{t-1}\) on \(\mathcal{H}_1\) for each \(t = 1, \ldots, n\) and \(y_{t-1} \in \mathcal{Y}^{t-1}\): this is the state input to the channel in the \(t\)th timestep if the feedback from the previous rounds was \(y_{t-1} = y_1 \ldots y_{t-1}\). Clearly, this defines an output distribution \(Q\) on \(\mathcal{Y}^n\) by iteration of the feedback loop:

\[
Q(y^n) = \prod_{t=1}^n \text{Tr}(\rho_t : y_{t-1} M_{y_t}).
\] (11)

Remark 16 We could imagine a more general protocol for the sender: an initial state \(\sigma_0\) could be prepared on an ancillary system \(\mathcal{H}_A\), and the feedback strategy is a collection \(\Phi\) of completely positive, trace preserving maps

\[
\varphi_t : \mathcal{B}(\mathcal{F} \otimes (t-1) \otimes \mathcal{H}_A) \rightarrow \mathcal{B}((\mathcal{H}_A \otimes \mathcal{H})\),
\]

where \(\mathcal{F}\) is the quantum system representing the classical feedback by states from an orthogonal basis: this map creates the next channel input and a new state of the ancilla (potentially entangled) from the old ancilla state and the feedback.

This more general scheme allows for memory and even quantum correlations between successive uses of the channel, via the system \(\mathcal{H}_A\). However, the scheme has, for each “feedback history” \(y_{t-1}\) up to time \(t\), a certain state
\( \sigma_{t-1:y_{t-1}} \) on \( \mathcal{H}_A \) (starting with \( \sigma_0 \)), and consequently an input state \( \rho_{t:y_{t-1}} \) on \( \mathcal{H} \):

\[
\rho_{t:y_{t-1}} = \text{Tr}_{\mathcal{H}_A} \left( \varphi_t \left( |y_{t-1}\rangle \langle y_{t-1}| \otimes \sigma_{t-1:y_{t-1}} \right) \right),
\]

\[
\sigma_{t:y_{t}} = \frac{1}{\text{Tr} \left( \rho_{t:y_{t-1}} M_{y_{t}} \right)} \text{Tr}_{\mathcal{H}} \left( \varphi_t \left( |y_{t-1}\rangle \langle y_{t-1}| \otimes \sigma_{t-1:y_{t-1}} \right) \right).
\]

It is easy to check that the corresponding output distribution \( Q \) of this feedback strategy according to our definition (see eq. (11)) is the same as for the original, more general feedback scheme. So, we do not need to consider those to obtain ultimate generality.

An \((n, \lambda_1, \lambda_2)\)-feedback ID code for the qc–channel \( T \) with passive feedback is now a set \( \{(F_i, D_i) : i = 1, \ldots, N\} \) of feedback strategies \( F_i \) and of operators \( 0 \leq D_i \leq 1 \), such that the output states \( \omega_i = \sum_{y^n} Q_i(y^n)|y^n\rangle \langle y^n| \) with the operators \( D_i \) form an identification code with error probabilities \( \lambda_1 \) and \( \lambda_2 \) of first and second kind. Note that because the output is classical — i.e., the states are diagonal in the basis \( (|y^n\rangle) \) —, we may without loss of generality assume that all \( D_i = \sum_{y^n} D_i(y^n)|y^n\rangle \langle y^n| \), with certain \( 0 \leq D_i(y^n) \leq 1 \).

Finally, let \( N_F(n, \lambda_1, \lambda_2) \) be the maximal \( N \) such that there exists an \((n, \lambda_1, \lambda_2)\)-feedback ID code with \( N \) messages. Note that due to the classical nature of the channel output codes are automatically simultaneous.

To determine the capacity, we invoke the following result:

**Lemma 17 (Ahlswede and Dueck)** Consider a qc–channel \( T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{Y} \) and any randomised feedback strategy \( F \) for block \( n \). Then, for \( \epsilon > 0 \), there exists a set \( \mathcal{E} \subset \mathcal{Y}^n \) of probability \( Q(\mathcal{E}) \geq 1 - \epsilon \) and cardinality

\[
|\mathcal{E}| \leq \exp \left( n \max_{\rho} H(T(\rho)) + \alpha \sqrt{n} \right),
\]

where \( \alpha = |\mathcal{Y}| e^{-1/2} \).

The proof of Ahlswede and Dueck applies directly: a qc–channel with feedback is isomorphic to a classical feedback channel with an infinite input alphabet (the set of all states), but with finite output alphabet, which is the relevant fact.

This is the essential tool to prove the following generalisation of Ahlswede’s and Dueck’s capacity result:

**Theorem 18** For a qc–channel \( T \) and \( \lambda_1, \lambda_2 > 0 \), \( \lambda_1 + \lambda_2 < 1 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \log N_F(n, \lambda_1, \lambda_2) = C_{ID}^F(T) = \max_{\rho} H(T(\rho)),
\]

unless the transmission capacity of \( T \) is 0, in which case \( C_{ID}^F(T) = 0 \).

In other words, the capacity of a nontrivial qc–channel with feedback is its maximum output entropy and the strong converse holds.
Proof. Let’s first get the exceptional case out of the way: $C(T)$ can only be 0 for a constant channel (i.e., one mapping every input to the same output). Clearly such a channel allows not only no transmission but also no identification.

The achievability is explained in the paper of Ahlswede and Dueck\cite{Ahlswede87}: the sender uses $m = n - O(1)$ instances of the channel with the state $\rho$, which maximises the output entropy. Due to feedback they then share the outcomes of $m$ i.i.d. random experiments, which they can concentrate into $nH(T(\rho)) - o(n)$ uniformly distributed bits. (This is a bit simpler than in the original paper, they just cut up the space into type classes.) The remaining $O(1)$ uses of the channel (with an appropriate error correcting code) are then used to implement the identification code of proposition 11 based on the uniform shared randomness.

The strong converse is only a slight modification of the arguments of Ahlswede and Dueck\cite{Ahlswede87}, due to the fact that we allow probabilistic decoding procedures: first, for each message $i$ in a given code, lemma 17 gives us a set $E_i \subseteq Y^n$ of cardinality $\leq K = \exp \left(n \max_\rho H(T(\rho)) + 3|Y|\epsilon^{-1/2}/\sqrt{n}\right)$, with probability $1 - \epsilon/3$ under the feedback strategy $F_i$, where $\epsilon := 1 - \lambda_1 - \lambda_2 > 0$. Now let $c := \lceil \frac{3}{\epsilon} \rceil$, and define new decoding rules by letting

$$\tilde{D}_i(y^n) := \begin{cases} \frac{1}{c} \lfloor cD_i(y^n) \rfloor & \text{for } y^n \in E_i, \\ 0 & \text{for } y^n \not\in E_i. \end{cases}$$

(I.e., round the density $D_i(y^n)$ down to the nearest multiple of $1/c$ within $E_i$, and to 0 without.) It is straightforward to check that in this way we obtain an $(n, \lambda_1 + \frac{2}{3}\epsilon, \lambda_2)$-feedback ID code.

The argument is concluded by observing that the new decoding densities are (i) all distinct (otherwise $\lambda_1 + \frac{2}{3}\epsilon + \lambda_2 \geq 1$), and (ii) all have support $\leq K = \exp \left(n \max_\rho H(T(\rho)) + 3|Y|\epsilon^{-1/2}/\sqrt{n}\right)$. Hence

$$N \leq \left(\frac{|Y|^n}{K}\right)(c + 1)^K \leq \left[(c + 1)|Y|^n\right]^2n \max_\rho H(T(\rho)) + O(\sqrt{n}),$$

from which the claim follows. \qed

8 Identification in the presence of feedback: “coherent feedback channels”

Inspired by the work of Harrow\cite{Harrow11} we propose the following definition of “coherent feedback” as a substitute for full passive feedback: by Stinespring’s theorem we can view the channel $T$ as an isometry $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \otimes \mathcal{H}_3$, followed by the partial trace $\text{Tr}_3$ over $\mathcal{H}_3$: $T(\rho) = \text{Tr}_3(U\rho U^*)$. Coherent feedback is now defined as distributing, on input $\rho$, the bipartite state $\Theta(\rho) := U\rho U^*$ among sender and receiver, who get $\mathcal{H}_3$ and $\mathcal{H}_2$, respectively.
A coherent feedback strategy $\Phi$ for block $n$ consists of a system $\mathcal{H}_A$, initially in state $\sigma_0$, and quantum channels $\varphi_t : \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_3^{\otimes (t-1)}) \rightarrow \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_3^{\otimes (t-1)} \otimes \mathcal{H}_1)$, creating the $t$th round channel input from the memory in $\mathcal{H}_A$ and the previous coherent feedback $\mathcal{H}_3^{\otimes (t-1)}$. The output state on $\mathcal{H}_2^{\otimes n}$ after $n$ rounds of coherent feedback channel alternating with the $\varphi_t$, is

$$\omega = \text{Tr}_{\mathcal{H}_A \otimes \mathcal{H}_3^{\otimes n}} \left[ (\Theta \circ \varphi_n \circ \Theta \circ \varphi_{n-1} \circ \cdots \circ \Theta \circ \varphi_1) \sigma_0 \right],$$

where implicitly each $\Theta$ is patched up by an identity on all systems different from $\mathcal{H}_1$, and each $\varphi_t$ is patched up by an identity on $\mathcal{H}_2^{\otimes (t-1)}$.

Now, an $(n, \lambda_1, \lambda_2)$–coherent feedback ID code for the channel $T$ with coherent feedback consists of $N$ pairs $(\Phi_i, D_i)$ of coherent feedback strategies $\Phi_i$ (with output states $\omega_i$) and operators $0 \leq D_i \leq 1$ on $\mathcal{H}_2^{\otimes n}$, such that the $(\omega_i, D_i)$ form an $(n, \lambda_1, \lambda_2)$–ID code on $\mathcal{H}_2^{\otimes n}$.

As usual, we introduce the maximum size $N$ of an $(n, \lambda_1, \lambda_2)$–coherent feedback ID code, and denote it $N_{\mathcal{F}}(n, \lambda_1, \lambda_2)$. It is important to understand the difference to $N_F(n, \lambda_1, \lambda_2)$ at this point: for the qc–channel, the latter refers to codes making use of the classical feedback of the measurement result, but coherent feedback — even for qc–channels — creates entanglement between sender and receiver, which, as we have seen in section 5, allows for larger identification codes.

We begin by proving the analogue of lemma 17:

**Lemma 19** Consider a quantum channel $T : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ and any feedback strategy $\varphi$ on block $n$ with output state $\omega$ on $\mathcal{H}_2^{\otimes n}$. Then, for $\epsilon > 0$, there exists a projector $\Pi$ on $\mathcal{H}_2^{\otimes n}$ with probability $\text{Tr}(\omega \Pi) \geq 1 - \epsilon$ and rank

$$\text{rank} \Pi \leq \exp \left( n \max_\rho \mathcal{S}(\rho) + \alpha \sqrt{n} \right),$$

where $\alpha = (\dim \mathcal{H}_2) \epsilon^{-1/2}$.

**Proof.** The feedback strategy determines the output state $\omega$ on $\mathcal{H}_2^{\otimes n}$, and we choose complete von Neumann measurements on each of the $n$ tensor factors: namely, the measurement $M$ of an eigenbasis $\{|m_y\rangle\}_y$ of $\bar{\omega}$, the entropy-maximising output state of $T$ (which is unique, as easily follows from the strict concavity of $\mathcal{S}$).

Defining the qc–channel $\tilde{T} := M \circ T$ (i.e., the channel $T$ followed by the measurement $M$), we are in the situation of lemma 17 with $\mathcal{Y} = \{1, \ldots, \dim \mathcal{H}_2\}$. Indeed, we can transform the given quantum feedback strategy into one based solely on the classical feedback of the measurement results, as explained in remark 16. Note that the additional quantum information available now at the sender due to the coherent feedback does not impair the validity of the argument of that remark: the important thing is that the classical feedback of the measurement results collapses the sender’s state into one depending only on the message and the feedback.
By lemma 19 stated below, \( \max_\rho H(\tilde{T}(\rho)) = S(\tilde{\omega}) \), so lemma 17 gives us a set \( \mathcal{E} \) of probability \( Q(\mathcal{E}) \geq 1 - \epsilon \) and \( |\mathcal{E}| \leq \exp(nS(\tilde{\omega}) + \alpha \sqrt{n}) \). The operator

\[
\Pi := \sum_{y^n \in \mathcal{E}} |m_{y_1} \rangle \langle m_{y_1}| \otimes \cdots \otimes |m_{y_n} \rangle \langle m_{y_n}|
\]

then clearly satisfies \( \text{Tr}(\omega \Pi) = Q(\mathcal{E}) \geq 1 - \epsilon \), and \( \text{rank} \Pi = |\mathcal{E}| \) is bounded as in lemma 17. \( \square \)

**Lemma 20** Let \( T : \mathcal{B}(\mathbb{C}^{d_1}) \to \mathcal{B}(\mathbb{C}^{d_2}) \) be a quantum channel and let \( \tilde{\rho} \) maximise \( S(T(\rho)) \) among all input states \( \rho \). Denote \( \tilde{\omega} = T(\tilde{\rho}) \) (which is easily seen to be the unique entropy-maximising output state of \( T \)), and choose a diagonalisation \( \tilde{\omega} = \sum_j \lambda_j |e_j \rangle \langle e_j| \). Then, for the channel \( \tilde{T} \) defined by

\[
\tilde{T}(\rho) = \sum_j |e_j \rangle \langle e_j| T(\rho) |e_j \rangle \langle e_j|
\]

(i.e., \( T \) followed by dephasing of the eigenbasis of \( \tilde{\omega} \)),

\[
\max_\rho S(\tilde{T}(\rho)) = S(\tilde{\omega}) = \max_\rho S(T(\rho)).
\]

**Proof.** The inequality “\( \geq \)” is trivial because for input state \( \tilde{\rho} \), \( T \) and \( \tilde{T} \) have the same output state.

For the opposite inequality, let us first deal with the case that \( \tilde{\omega} \) is strictly positive (i.e., 0 is not an eigenvalue). The lemma is trivial if \( \tilde{\omega} = \frac{1}{d_2} \mathbb{1} \), so we assume \( \tilde{\omega} \neq \frac{1}{d_2} \mathbb{1} \) from now on. Observe that \( \mathcal{N} := \{ T(\rho) : \rho \text{ state on } \mathbb{C}^{d_1} \} \) is convex, as is the set \( \mathcal{S} := \{ \tau \text{ state on } \mathbb{C}^{d_2} : S(\tau) \geq S(\tilde{\omega}) \} \), and that \( \mathcal{N} \cap \mathcal{S} = \{ \tilde{\omega} \} \). Since we assume that \( \tilde{\omega} \) is not maximally mixed, \( \mathcal{S} \) is full-dimensional in the set of states, so the boundary \( \partial S = \{ \tau : S(\tau) = S(\tilde{\omega}) \} \) is a one-codimensional submanifold; from positivity of \( \tilde{\omega} \) (ensuring the existence of the derivative of \( S \)) it has a (unique) tangent plane \( H \) at this point:

\[
H = \{ \xi \text{ state on } \mathbb{C}^{d_2} : \text{Tr}[(\xi - \tilde{\omega})\nabla S(\tilde{\omega})] = 0 \}.
\]

Thus, \( H \) is the unique hyperplane separating \( \mathcal{S} \) from \( \mathcal{N} \):

\[
\mathcal{S} \subset H^+ = \{ \xi \text{ state on } \mathbb{C}^{d_2} : \text{Tr}[(\xi - \tilde{\omega})\nabla S(\tilde{\omega})] \geq 0 \},
\]

\[
\mathcal{N} \subset H^- = \{ \xi \text{ state on } \mathbb{C}^{d_2} : \text{Tr}[(\xi - \tilde{\omega})\nabla S(\tilde{\omega})] \leq 0 \}.
\]

Now consider, for phase angles \( \alpha = (\alpha_1, \ldots, \alpha_{d_2}) \), the unitary \( U_\alpha = \sum_j e^{i\alpha_j} |e_j \rangle \langle e_j| \), which clearly stabilises \( \mathcal{S} \) and leaves \( \tilde{\omega} \) invariant. Hence, also \( H \) and the two halfspaces \( H^+ \) and \( H^- \) are stabilised:

\[
U_\alpha HU_\alpha^* = H, \quad U_\alpha H^+ U_\alpha^* = H^+, \quad U_\alpha H^- U_\alpha^* = H^-.
\]
In particular, \( U_\alpha^* N U_\alpha \subset H^- \), implying the same for the convex hull of all these sets:
\[
\text{conv} \left\{ U_\alpha^* N U_\alpha \right\} \subset H^-.
\]
Since this convex hull includes (for \( \tau \in \mathcal{N} \)) the states
\[
\sum_j |e_j\rangle\langle e_j| \tau |e_j\rangle \langle e_j| = 1 \quad (2\pi)^d_2 \int d\alpha U_\alpha \tau U_\alpha^*,
\]
we conclude that for all \( \rho \), \( \tilde{T}(\rho) \in H^- \), forcing \( S(\tilde{T}(\rho)) \leq S(\tilde{\omega}) \).

We are left with the case of a degenerate \( \tilde{\omega} \): there we consider perturbations \( T_\epsilon = (1 - \epsilon)T + \epsilon \frac{1}{d_2} \mathbb{1} \) of the channel, whose output entropy is maximised by the same input states as \( T \), and the optimal output state is \( \tilde{\omega}_\epsilon = (1 - \epsilon)\tilde{\omega} + \epsilon \frac{1}{d_2} \mathbb{1} \). These are diagonal in any diagonalising basis for \( \tilde{\omega} \), so \( \tilde{T}_\epsilon = (1 - \epsilon)\tilde{T} + \epsilon \frac{1}{d_2} \mathbb{1} \).

Now our previous argument applies, and we get for all \( \rho \),
\[
S(\tilde{T}_\epsilon(\rho)) \leq S(\tilde{\omega}_\epsilon) \leq (1 - \epsilon)S(\tilde{\omega}) + \epsilon \log d_2 + H(\epsilon, 1 - \epsilon).
\]
On the other hand, by concavity,
\[
S(\tilde{T}_\epsilon(\rho)) \geq (1 - \epsilon)S(\tilde{T}(\rho)) + \epsilon \log d_2.
\]
Together, these yield for all \( \rho \),
\[
S(\tilde{T}(\rho)) \leq S(\tilde{\omega}) + \frac{1}{1 - \epsilon} H(\epsilon, 1 - \epsilon),
\]
and letting \( \epsilon \to 0 \) concludes the proof. \( \square \)

We are now in a position to prove

**Theorem 21** For a quantum channel \( T \) and \( \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 < 1 \),
\[
\lim_{n \to \infty} \frac{1}{n} \log \log N_{\mathbb{F}}(n, \lambda_1, \lambda_2) = C^{(F)}_{\text{ID}}(T) = 2 \max_{\rho} S(T(\rho)),
\]
unless the transmission capacity of \( T \) is 0, in which case \( C^{(F)}_{\text{ID}}(T) = 0 \).

In other words, the capacity of a nontrivial quantum channel with coherent feedback is twice its maximum output entropy and the strong converse holds.

**Proof.** The trivial channel is easiest, and the argument is just as in theorem [13]. Note just one thing: a nontrivial channel with maximal quantum feedback will always allow entanglement generation (either because of the feedback or because it is noiseless), so — by teleportation — it will always allow quantum state transmission.

For achievability, the sender uses \( m = n - O(\log n) \) instances of the channel to send one half of a purification \( \Psi_\rho \) of the output entropy maximising state \( \rho \) each. This creates \( m \) copies of a pure state which has reduced state
$T(\rho)$ at the receiver. After performing entanglement concentration, which yields $nS(T(\rho)) - o(n)$ EPR pairs, the remaining $O(\log n)$ instances of the channel are used (with an appropriate error correcting code and taking some of the entanglement for teleportation) to implement the construction of proposition 12 based on the maximal entanglement.

The converse is proved a bit differently than in theorem 18, where we counted the discretised decoders: now we have operators, and discretisation in Hilbert space is governed by slightly different rules. Instead, we do the following: given an identification code with feedback, form the uniform probabilistic mixture $\Phi$ of the feedback strategies $\Phi_i$ of messages $i$ — formally, $\Phi = \frac{1}{N} \sum_i \Phi_i$. Its output state $\omega$ clearly is the uniform mixture of the output states $\omega_i$ corresponding to message $i$: $\omega = \frac{1}{N} \sum_i \omega_i$. With $\epsilon = 1 - \lambda_1 - \lambda_2$, lemma 14 gives us a projector $\Pi$ of rank $K \leq \exp(n \max_{S(T(\rho))} + 48(\dim H)^2 \epsilon \sqrt{n})^2$. Thus, for half of the messages (which we may assume to be $i = 1, \ldots, \lfloor N/2 \rfloor$), $\text{Tr}(\omega_i \Pi) \geq 1 - \frac{1}{2} \left( \frac{\epsilon}{24} \right)^2$. Observe that the $\omega_i$ together with the decoding operators $D_i$ form an identification code on $B(H_2^n)$, with error probabilities of first and second kind $\lambda_1$ and $\lambda_2$, respectively. Now restrict all $\omega_i$ and $D_i$ ($i \leq N/2$) to the supporting subspace of $\Pi$ (which we identify with $C^K$):

$$\tilde{\omega}_i := \frac{1}{\text{Tr}(\omega_i \Pi)} \Pi \omega_i \Pi, \quad \tilde{D}_i := \Pi D_i \Pi.$$ 

This is now an identification code on $B(C^K)$, with error probabilities of first and second kind bounded by $\lambda_1 + \frac{1}{3} \epsilon$ and $\lambda_2 + \frac{1}{3} \epsilon$, respectively, as a consequence of the gentle measurement lemma, namely, $\frac{1}{2} ||\omega_i - \tilde{\omega}_i||_1 \leq \frac{1}{3} \epsilon$. So finally, we can invoke Proposition 11 of our earlier paper, which bounds the size of identification codes (this, by the way, is now the discretisation part of the argument):

$$\frac{N}{2} \leq \left( \frac{5}{1 - \lambda_1 - \epsilon/3 - \lambda_2 - \epsilon/3} \right)^{2K^2} = \left( \frac{15}{\epsilon} \right)^{2n \max_{S(T(\rho)) + O(\sqrt{n})}},$$

and we have the converse. \hfill \square

**Remark 22** For cq–channels $T : X \rightarrow B(H)$ (a map assigning a state $T(x) = \rho_x$ to every element $x$ from the finite set $X$), we can even study yet another kind of feedback (let us call it cq–feedback): fix purifications $\Psi_x$ of the $\rho_x$, on $H \otimes H$; then input of $x \in X$ to the channel leads to distribution of $\Psi_x$ between sender and receiver. In this way, the receiver still has the channel output state $\rho_x$, but is now entangled with the sender.

By the methods employed above we can easily see that in this model, the identification capacity is

$$C_{ID}^{FF}(T) \geq \max_{\rho_x} \left\{ S \left( \sum_x P(x)\rho_x \right) + \sum_x P(x)S(\rho_x) \right\}.$$
Achievability is seen as follows: for a given $P$ use a transmission code of rate $I(P; T) = S(\sum_x P(x)\rho_x) − \sum_x P(x)S(\rho_x)$ and with letter frequencies $P$ in the codewords. This is used to create shared randomness of the same rate, and the cq–feedback to obtain pure entangled states which are concentrated into EPR pairs at rate $\sum_x P(x)E(\Psi_x) = \sum_x P(x)S(\rho_x)$; then we use eq. (10).

The (strong) converse seems to be provable by combining the approximation of output statistics result of Ahlswede and Winter with a dimension counting argument as in our previous paper’s Proposition 11, but we won’t follow on this question here.

**Remark 23** Remarkably, the coherent feedback identification capacity $C_{\mathrm{ID}}(F|T)$ of a channel is at present the only one we actually “know” in the sense that we have a universally valid formula which can be evaluated (it is single–letter); this is in marked contrast to what we can say about the plain (non–simultaneous) identification capacity, whose determination remains the greatest challenge of the theory.

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