On Construction of Quantum Markov Chains on Cayley trees

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Abstract. The main aim of the present paper is to provide a new construction of quantum Markov chain (QMC) on arbitrary order Cayley tree. In that construction, a QMC is defined as a weak limit of finite volume states with boundary conditions, i.e. QMC depends on the boundary conditions. Note that this construction reminds statistical mechanics models with competing interactions on trees. If one considers one dimensional tree, then the provided construction reduces to well-known one, which was studied by the first author. Our construction will allow to investigate phase transition problem in a quantum setting.

1. Introduction
One of the basic open problems in quantum probability is the construction of a theory of quantum Markov fields, that is quantum process with multi-dimensional index set. This program concerns the generalization of the theory of Markov fields (see [17],[22]) to non-commutative setting, naturally arising in quantum statistical mechanics and quantum led theory.

The quantum analogues of Markov chains were first constructed in [1], where the notion of quantum Markov chain (QMC) on arbitrary order Cayley tree. In that construction, a QMC is defined as a weak limit of finite volume states with boundary conditions, i.e. QMC depends on the boundary conditions. Note that this construction reminds statistical mechanics models with competing interactions on trees. If one considers one dimensional tree, then the provided construction reduces to well-known one, which was studied by the first author. Our construction will allow to investigate phase transition problem in a quantum setting.

First attempts to construct a quantum analogue of classical Markov fields have been done in [3]-[6],[9, 20, 26]. In these papers the notion of quantum Markov state, introduced in [8] (see also [23, 20]), extended to fields as a sub-class of the quantum Markov chains (QMC) introduced in [1]. In [7] it has been proposed a definition of quantum Markov states and chains, which extend a proposed one in [31], and includes all the presently known examples. Note that in the mentioned papers quantum Markov fields were considered over multidimensional integer lattice $\mathbb{Z}$. This lattice has so-called amenability property. Moreover, there do not exist analytical solutions (for example, critical temperature) on such lattice. On the other hand, investigations of phase transitions of spin models on hierarchical lattices showed that there are exact calculations
of various physical quantities (see for example, [16]). Therefore, it is natural to investigate quantum Markov fields over hierarchical lattices. For example, a Cayley tree is the simplest hierarchical lattice with non-amenable graph structure [33]. First attempts to investigate QMC over such trees was done in [15], such studies were related to investigation of thermodynamic limit of valence-bond-solid models on a Cayley tree [18]. The mentioned considerations naturally suggest the study of the following problem: the extension to fields of the notion of generalized QMC. In [14] we have introduced a hierarchy of notions of Markovianity for states on discrete infinite tensor products of $C^*$-algebras and for each of these notions we constructed some explicit examples. We showed that the construction of [8] can be generalized to trees. It is worth to note that, in a different context and for quite different purposes, the special role of trees was already emphasized in [26]. In [18] the VBS-model was considered on the Cayley tree. It was established the existence of the phase transition for the model in term of finitely correlated states which describe ground states of the model. Note that more general structure of finitely correlated states was studied in [19]. We stress that finitely correlated states can be considered as quantum Markov chains. In [21, 27, 28, 29, 30] noncommutative extensions of classical Markov fields, associated with Ising and Potts models on a Cayley tree, were investigated. In the classical case, Markov fields on trees are also considered in [35, 36],[37]-[40].

In the present paper, for a given Hamiltonian, we provide a more general construction (than [10, 11]) of QMC associated with the Hamiltonian. Namely, in that construction, the Hamiltonian exhibits nearest-neighbor and next-nearest-neighbor interactions (in the previous papers [11]-[13] the Hamiltonian contained only nearest-neighbor interactions), and the corresponding QMC is defined as a weal limit of finite volume states (which depend on the Hamiltonian) with boundary conditions, i.e. QMC depends on the boundary conditions. We remark that in this construction, one can observe some similarities with Gibbs measures. We stress that all previous considered examples (in the literature) of QMC related to Hamiltonians with nearest-neighbor interactions. In classical setting, our construction, in particularly, leads to the Ising model with competing interactions, which has been rigorously investigated in many papers (see for example [29, 30, 34, 36].

2. Preliminaries
Let $\Gamma^k_+ = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root $x^0$ (i.e. each vertex of $\Gamma^k_+$ has exactly $k + 1$ edges, except for the root $x^0$, which has $k$ edges). Here $L$ is the set of vertices and $E$ is the set of edges. The vertices $x$ and $y$ are called nearest neighbors and they are denoted by $l = <x, y>$ if there exists an edge connecting them. A collection of the pairs $<x, x_1>, ..., <x_{d-1}, y>$ is called a path from the point $x$ to the point $y$. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$.

Recall a coordinate structure in $\Gamma^k_+$; every vertex $x$ (except for $x^0$) of $\Gamma^k_+$ has coordinates $(i_1, \ldots, i_n)$, here $i_m \in \{1, \ldots, k\}, 1 \leq m \leq n$ and for the vertex $x^0$ we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \ldots, i_n)$ form level $n$ (i.e. $d(x^0, x) = n$) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L : d(x, x_0) = n\}, \quad \Lambda_n = \bigcup_{k=0}^n W_k, \quad \Lambda_{[n,m]} = \bigcup_{k=n}^m W_k, \quad (n < m)$$

$$E_n = \{<x, y> : x, y \in \Lambda_n\}, \quad \Lambda_n^c = \bigcup_{k=n}^\infty W_k$$

For $x \in \Gamma^k_+, x = (i_1, \ldots, i_n)$ denote

$$S(x) = \{(x, i) : 1 \leq i \leq k\}.$$
Here \((x, i)\) means that \((i_1, \ldots, i_n, i)\). This set is called a set of direct successors of \(x\).

Two vertices \(x, y \in V\) is called one level next-nearest-neighbor vertices if there is a vertex \(z \in V\) such that \(x, y \in S(z)\), and they are denoted by \(> x, y <\). In this case the vertices \(x, z, y\) was called ternary and denoted by \(< x, z, y >\).

Let us define on \(\Gamma^k_+\) a binary operation \(\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+\) as follows: for any two elements \(x = (i_1, \ldots, i_n)\) and \(y = (j_1, \ldots, j_m)\) put
\[
x \circ y = (i_1, \ldots, i_n) \circ (j_1, \ldots, j_m) = (i_1, \ldots, i_n, j_1, \ldots, j_m)
\]
and
\[
x \circ x^0 = x^0 \circ x = (i_1, \ldots, i_n) \circ (0) = (i_1, \ldots, i_n).
\]

By means of the defined operation \(\Gamma^k_+\) becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations \(\tau_g : \Gamma^k_+ \to \Gamma^k_+\) by
\[
\tau_g(x) = g \circ x.
\]

It is clear that \(\tau(0) = \text{id}\).

The algebra of observables \(B_x\) for any single site \(x \in L\) will be taken as the algebra \(M_d\) of the complex \(d \times d\) matrices. The algebra of observables localized in the finite volume \(A \subset L\) is then given by \(B_A = \bigotimes_{x \in A} B_x\). As usual if \(A^1 \subset A^2 \subset L\), then \(B_{A^1}\) is identified as a subalgebra of \(B_{A^2}\) by tensoring with unit matrices on the sites \(x \in A^2 \setminus A^1\). Note that, in the sequel, by \(B_{A^+}\) we denote the positive part of \(B_A\). The full algebra \(B_L\) of the tree is obtained in the usual manner by an inductive limit
\[
B_L = \bigcup_{A_n} B_{A_n}.
\]

In what follows, by \(\mathcal{S}(B_A)\) we will denote the set of all states defined on the algebra \(B_A\).

Consider a triplet \(C \subset B \subset A\) of unital \(C^*\)-algebras. Recall [2] that a quasi-conditional expectation with respect to the given triplet is a completely positive (CP) linear map \(\mathcal{E} : A \to B\) such that \(\mathcal{E}(ca) = c\mathcal{E}(a)\), for all \(a \in A, c \in C\).

**Definition 2.1** ([14]). Let \(\varphi\) be a state on \(B_L\). Then \(\varphi\) is called

(i) a forward quantum d-Markov chain (QMC), associated to \(\{A_n\}\), if for each \(A_n\), there exist a quasi-conditional expectation \(\mathcal{E}_{A_n}\) with respect to the triplet
\[
B_{A_{n+1}} \subseteq B_{A_n} \subseteq B_{A_{n-1}}
\]
and a state $\hat{\varphi}_{\Lambda_n} \in \mathcal{S}(\mathcal{B}_{\Lambda_n})$ such that for any $n \in \mathbb{N}$ one has

$$\hat{\varphi}_{\Lambda_n}\mid_{\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n}} = \hat{\varphi}_{\Lambda_{n+1}} \circ \mathcal{E}_{\Lambda_{n+1}}\mid_{\mathcal{B}_{\Lambda_{n+1}\setminus\Lambda_n}}$$

(2.5)

and

$$\varphi = \lim_{n \to \infty} \hat{\varphi}_{\Lambda_n} \circ \mathcal{E}_{\Lambda_n} \circ \mathcal{E}_{\Lambda_{n-1}} \circ \cdots \circ \mathcal{E}_{\Lambda_1}$$

(2.6)

in the weak-* topology.

(ii) a backward quantum d-Markov chain, associated to $\{\Lambda_n\}$, if there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n}$ with respect to the triple $\mathcal{B}_{\Lambda_{n-1}} \subseteq \mathcal{B}_{\Lambda_n} \subseteq \mathcal{B}_{\Lambda_{n+1}}$ for each $n \in \mathbb{N}$ and an initial state $\rho_0 \in \mathcal{S}(\mathcal{B}_{\Lambda_0})$ such that

$$\varphi = \lim_{n \to \infty} \rho_0 \circ \mathcal{E}_{\Lambda_0} \circ \mathcal{E}_{\Lambda_1} \circ \cdots \circ \mathcal{E}_{\Lambda_n}$$

in the weak-* topology.

Note that (2.5) is an analogue of the DRL equation from classical statistical mechanics [17, 22], and QMC is thus the counterpart of the infinite-volume Gibbs measure.

3. Construction of Quantum Markov Chains on Cayley tree

In this section we are going to provide a general construction of quantum Markov chains. Note that in our construction generalizes our previous works [10]-[13].

Let us rewrite the elements of $W_n$ in the following order, i.e.

$$\overrightarrow{W_n} := \left( x^{(1)}_{W_n}, x^{(2)}_{W_n}, \cdots, x^{(|W_n|)}_{W_n} \right) , \quad \overleftarrow{W_n} := \left( x^{(|W_n|)}_{W_n}, x^{(|W_n|-1)}_{W_n}, \cdots, x^{(1)}_{W_n} \right).$$

Note that $|W_n| = k^n$. Vertices $x^{(1)}_{W_n}, x^{(2)}_{W_n}, \cdots, x^{(|W_n|)}_{W_n}$ of $W_n$ can be represented in terms of the coordinate system as follows:

$$x^{(1)}_{W_n} = (1, 1, \cdots, 1, 1), \quad x^{(2)}_{W_n} = (1, 1, \cdots, 1, 2), \quad \cdots \quad x^{(k)}_{W_n} = (1, 1, \cdots, 1, k),$$

(3.1)

$$x^{(k+1)}_{W_n} = (1, 1, \cdots, 2, 1), \quad x^{(2)}_{W_n} = (1, 1, \cdots, 2, 2), \quad \cdots \quad x^{(2k)}_{W_n} = (1, 1, \cdots, 2, k),$$

$$\vdots$$

$$x^{(|W_n|-k+1)}_{W_n} = (k, k, \cdots, k, 1), \quad x^{(|W_n|-k+2)}_{W_n} = (k, k, \cdots, k, 2), \quad \cdots \quad x^{(|W_n|)}_{W_n} = (k, k, \cdots, k, k).$$

Analogously, for a given vertex $x$, we shall use the following notation for the set of direct successors of $x$:

$$\overrightarrow{S(x)} := \{(x, 1), (x, 2), \cdots (x, k)\}, \quad \overleftarrow{S(x)} := \{(x, k), (x, k-1), \cdots (x, 1)\}.$$

In what follows, by $\prod$ we mean an ordered product, i.e.

$$\prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n,$$

where elements $\{a_k\} \subset \mathcal{B}_L$ are multiplied in the indicated order. This means that we are not allowed to change this order.
Assume that we are given a sequence of operators \( \{ K_{[n,n+1]} \}_{n \geq 0} \), where each operator \( K_{[n,n+1]} \) belongs to \( B_{\Lambda[0,\infty)} \) (\( n \geq 0 \)). Moreover, we suppose that boundary conditions \( w_0 \in B_{(0)}^+ \) and \( h = \{ h_n \in B_{W_n}^+ : n \in \mathbb{N} \} \) are also given.

For each \( n \in \mathbb{N} \) we denote

\[
K_n := w_0^{1/2} \prod_{m=0}^{n-1} K_{[m,m+1]} h_n^{1/2} \quad (3.2)
\]

\[
W_n^{(f)} := K_n K_n^*, \quad W_n^{(b)} := K_n^* K_n. \quad (3.3)
\]

One can see that \( W_n^{(f)} \) and \( W_n^{(b)} \) are positive.

In what follows, by \( \text{tr}_\Lambda : B_L \to B_{\Lambda} \) we mean normalized partial trace (i.e. \( \text{tr}_\Lambda (\mathbf{1}_L) = \mathbf{1}_\Lambda \), here \( \mathbf{1}_\Lambda = \bigotimes_{y \in \Lambda} \mathbf{1} \)), for any \( \Lambda \subseteq_{\text{fin}} L \). For the sake of shortness we put \( \text{tr}_n := \text{tr}_{\Lambda_n} \).

Let us define positive functionals \( \varphi_{w_0,h}^{(n,f)} \), \( \varphi_{w_0,h}^{(n,b)} \) on \( B_{\Lambda_n} \), respectively, by

\[
\varphi_{w_0,h}^{(n,f)} (a) = \text{tr} (W_n^{(f)} (a \otimes \mathbf{1}_{W_{n+1}})), \quad \varphi_{w_0,h}^{(n,b)} (a) = \text{tr} (W_n^{(b)} (a \otimes \mathbf{1}_{W_{n+1}})), \quad (3.4)
\]

for every \( a \in B_{\Lambda_n} \). Note that here, \( \text{tr} \) is a normalized trace on \( B_L \) (i.e. \( \text{tr}(\mathbf{1}_L) = 1 \)).

To get an infinite-volume state \( \varphi \) on \( B_L \) such that \( \varphi |_{B_{\Lambda_n}} = \varphi_{w_0,h}^{(n,e)} \) (\( e = f, b \)) we need to impose some constrains to the boundary conditions \( \{ w_0, h \} \) so that the functionals \( \{ \varphi_{w_0,h}^{(n,e)} \} \) satisfy the compatibility condition, i.e.

\[
\varphi_{w_0,h}^{(n+1,e)} |_{B_{\Lambda_n}} = \varphi_{w_0,h}^{(n,e)} \quad (3.5)
\]

In the following we need an auxiliary fact.

**Lemma 3.1.** Let \( \Lambda \subseteq \Lambda' \subseteq_{\text{fin}} L \), then for any \( A \in B_{\Lambda}, B \in B_{\Lambda'} \) one has \( \text{Tr}(AB) = \text{Tr}[A \text{Tr}_{\Lambda'}(B)] \).

**Theorem 3.2.** Let a sequence \( \{ K_{[n,n+1]} \}_{n \geq 0} \), \( K_{[n,n+1]} \in B_{\Lambda[0,\infty)} \) be given. Assume that boundary conditions \( w_0 \in B_{(0)}^+ \) and \( h = \{ h_x \in B_{x,1} : x \in L \} \) satisfy

\[
\text{Tr}(w_0 h_0) = 1, \quad (3.6)
\]

\[
\text{Tr}_n (K_{[n,n+1]} h_n K_{[n,n+1]}^*) = h_n, \quad \text{for every } n \in \mathbb{N}. \quad (3.7)
\]

Then the functionals \( \{ \varphi_{w_0,h}^{(n,f)} \} \) satisfy the compatibility condition \( (3.5) \). Moreover, there is a unique forward quantum Markov chain \( \varphi_{w_0,h}^{(f)} \) on \( B_L \) such that \( \varphi_{w_0,h}^{(f)} = w - \lim_{n \to \infty} \varphi_{w_0,h}^{(n,f)} \).

**Proof.** We first show that a sequence \( \{ W_n^{(f)} \} \) is projective with respect to \( \text{tr}_n \), i.e.

\[
\text{tr}_{n-1} (W_n^{(f)}) = W_n^{(f)} \quad \forall n \in \mathbb{N}. \quad (3.8)
\]

It is known [8] that the projectivity implies the compatibility condition.

Now let us check the equality \( (3.8) \). From \( (3.2) \) one has

\[
W_n^{(f)} = w_0^{1/2} \left( \prod_{m=0}^{n-1} K_{[m-1,m]} \right) K_{[n-1,n]} h_n K_{[n-1,n]}^* \left( \prod_{m=0}^{n-1} K_{[m-1,m]}^* \right)^* \quad w_0^{1/2}.
\]
Hence, from the last equality with (3.16) we get
\[
\text{tr}_{n-1}(W_{n|f}) = w_0^{1/2} \left( \prod_{m=0}^{n-1} K_{[m,n,1]} \right) \text{tr}_{n-1}(K_{[n-1,n]} h_n K_{[n-1,n]}^*) \left( \prod_{m=0}^{n-1} K_{[m,n,1]} \right)^* w_0^{1/2} = W_{n|f}^{(f)}.
\]

From the above argument and (3.15), one can show that \(W_{n|f}^{(f)}\) is density operator, i.e. \(\text{tr}(W_{n|f}^{(f)}) = 1\).

Let us show that the defined state \(\varphi_{w_0,h}^{(f)}\) is a forward QMC. Indeed, define quasi-conditional expectations \(\mathcal{E}_{\Lambda_n^c}\) as follows:

\[
\mathcal{E}_{\Lambda_1^c}(x|0) = \text{tr}_1(K_{[1,0,1]} w_0^{1/2} x_0 w_0^{1/2} K_{[1,0,1]}^*), \quad x_0 \in \mathcal{B}_0^c
\]

\[
\mathcal{E}_{\Lambda_1^c}(x|k) = \text{tr}_n(K_{[k-1,k]} x_{k-1} w_0^{1/2} K_{[k-1,k]}^*), \quad x_{k-1} \in \mathcal{B}_{k-1}^c, \quad k = 1, 2, \ldots, n + 1,
\]

here \(\text{tr}_n = \text{tr}_{\Lambda_n^c}\). Then for any monomial \(a_{\Lambda_1} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n} \otimes I_{W_{n+1}}\), where \(a_{\Lambda_1} \in B_{\Lambda_1}, a_{W_k} \in B_{W_k}, (k = 2, \ldots, n),\) we have

\[
\varphi_{w_0,h}^{(n,f)}(a_{\Lambda_1} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n}) = \text{tr}_n \left( h_{n+1} K_{[n,n+1]}^* \cdots K_{[1,2]}^* \mathcal{E}_{\Lambda_1^c}(a_{\Lambda_1}) a_{W_2} K_{[1,2]} \cdots a_{W_n} K_{[n,n+1]}^* \right) = \text{tr}_{n+1} \left( h_{n+1} \mathcal{E}_{\Lambda_n}^c \circ \mathcal{E}_{\Lambda_n}^c \circ \cdots \mathcal{E}_{\Lambda_2}^c \circ \mathcal{E}_{\Lambda_1^c}(a_{\Lambda_1} \otimes a_{W_2} \otimes \cdots \otimes a_{W_n}) \right).
\]

Hence, for any \(a \in \Lambda \subset \Lambda_{n+1}\) from (3.4) with (3.2), (3.9)-(3.11) one can see that

\[
\varphi_{w_0,h}^{(n,f)}(a) = \text{tr}_{n+1} \left( h_{n+1} \mathcal{E}_{\Lambda_{n+1}}^c \circ \mathcal{E}_{\Lambda_n^c} \circ \cdots \mathcal{E}_{\Lambda_2}^c \circ \mathcal{E}_{\Lambda_1^c}(a) \right).
\]

The projectivity of \(W_{n|f}^{(f)}\) yields the equality (2.5) for \(\varphi_{w_0,h}^{(n,f)}\), therefore, from (3.12) we conclude that \(\varphi_{w_0,h}^{(f)}\) is a forward QMC. \(\square\)

**Theorem 3.3.** Let a sequence \(\{K_{[n,n+1]}\}_{n \geq 0}\), \(K_{[n,n+1]} \in B_{\Lambda_{[n,n+1]}}\) be given. Assume that boundary conditions \(w_0 \in B_{\Lambda_{0}}\) and \(h = \{h_x \in B_{\Lambda_{x+1}}\}_{x \in L}\) satisfy (3.15),(3.16). Then the functionals \(\{\varphi_{w_0,h}^{(n,b)}\}\) satisfy the compatibility condition (3.5). Moreover, there is a unique backward quantum Markov chain \(\varphi_{w_0,h}^{(b)}\) on \(B_L\) such that \(\varphi_{w_0,h}^{(b)} = w - \lim_{n \to \infty} \varphi_{w_0,h}^{(n,b)}\).

**Proof.** Let us check that the states \(\{\varphi_{w_0,h}^{(n,b)}\}\) satisfy the compatibility condition. For every \(a \in B_{\Lambda_n}\)
we have
\[ \varphi_{w_0,b}^{(n+1,b)}(a \otimes I_{W_{n+1}}) = \text{tr}(K_n^* K_{n+1}^*(a \otimes I_{\Lambda_{n+1,n+2}})) \]
\[ = \text{tr} \left( K_{[n,n+1]}^* \cdots K_{[0,1]}^* W_0 K_{[0,1]}^* \cdots K_{[n+1,n+2]}^* h_{n+2}^{1/2} (a \otimes I_{\Lambda_{n+1,n+2}}) h_{n+2}^{1/2} K_{[n+1,n+2]}^* \right) \]
\[ = \text{tr} \left( K_{[n,n+1]}^* \cdots K_{[0,1]}^* W_0 K_{[0,1]}^* \cdots K_{[n+1,n+1]}(a \otimes I_{\Lambda_{n+1,n+2}}) K_{[n+1,n+2]}^* h_{n+2} K_{[n+1,n+2]}^* \right) \]
\[ = \text{tr} \left( K_{[n,n+1]}^* \cdots K_{[0,1]}^* W_0 K_{[0,1]}^* \cdots K_{[n,n+1]}(a \otimes I_{\Lambda_{n+1,n+2}}) K_{[n+1,n+2]}^* h_{n+2} K_{[n+1,n+2]}^* \right) \]
\[ = \text{tr} \left( K_{[n,n+1]}^* \cdots K_{[0,1]}^* W_0 K_{[0,1]}^* \cdots K_{[n,n+1]}(a \otimes I_{\Lambda_{n+1,n+2}}) K_{[n+1,n+2]}^* h_{n+1} \right) \]
\[ = \text{tr} \left( W_{n+1}(a \otimes I_{\Lambda_{n+1,n+2}}) \right) \]
\[ = \varphi_{w_0,b}^{(n+1,b)}(a) \]

Let us show that the defined state \( \varphi_{w_0,b}^{(b)} \) is a backward QMC. Indeed, define quasi-conditional expectations \( \mathcal{E}_n \) as follows:
\[ \mathcal{E}_n(x_{n+1}) = \text{tr}_n \left( K_{[n,n+1]} h_{n+1}^{1/2} x_{n+1} h_{n+1}^{1/2} K_{[n,n+1]}^* \right), \quad x_{n+1} \in \Lambda_{n+1} \]
\[ \mathcal{E}_k(x_{k+1}) = \text{tr}_k \left( K_{[k,k+1]} x_{k+1} K_{[k,k+1]}^* \right), \quad x_{k+1} \in \Lambda_{k+1}, \quad k = 0, 1, \ldots, n-1, \]

Then using the same argument as in the proof of Theorem 3.2 we have
\[ \varphi_{w_0,b}^{(n+1,b)}(a) = \rho_0 \left( \mathcal{E}_0 \circ \cdots \circ \mathcal{E}_{n-1} \circ \mathcal{E}_n \right)(a). \]

This means that the limit state \( \varphi_{w_0,b}^{(b)} \) is a backward QMC. This completes the proof. \( \square \)

A sequence \( \{K_{[n,n+1]}\}_{n \geq 0}, \ (K_{[n,n+1]} \in \Lambda_{[n+1,n+1]} \) is called localized if for each \( n \in \mathbb{N} \) one has
\[ K_{[n,n+1]} = \prod_{x \in W_n} A_{x,(x,1),\ldots,(x,k)} \] (3.13)

where \( A_{x,(x,1),\ldots,(x,k)} \in B_x \otimes B_{(x,1)} \otimes \cdots \otimes B_{(x,k)} \).

Now we assume that the boundary condition \( \mathfrak{h} \) is also localized, i.e. for every \( n \in \mathbb{N} \) one has
\[ h_n = \prod_{x \in W_n} h_x \] (3.14)

where \( h_x \in B_x^+, \ x \in W_n. \)

**Corollary 3.4.** Let \( \{K_{[n,n+1]}\}_{n \geq 0} \) be a localized sequence given by (3.13). Assume that boundary condition \( \mathfrak{h} \) is also localized by (3.14). If one has
\[ \text{Tr}(w_0 h_0) = 1, \] (3.15)
\[ \text{Tr}_2 \left( A_{x,(x,1),\ldots,(x,k)} \prod_{y \in S(x)} h_y A_{x,(x,1),\ldots,(x,k)}^* \right) = h_x, \] (3.16)

for all \( x \in L \setminus \{(0)\}. \) Then there is a unique backward quantum Markov chain \( \varphi_{w_0,b}^{(b)} \) on \( B_L. \)
**Theorem 3.5.** Let a sequence \( \{K_{[n,n+1]}\}_{n \geq 0} \) be given. Assume that two boundary conditions \((w_0, h)\) and \((\tilde{w}_0, \tilde{s})\) satisfy (3.15), (3.16), and \(\varphi^{(b)}_{w_0, h}\) and \(\varphi^{(b)}_{\tilde{w}_0, \tilde{s}}\) are the corresponding backward QMC. Then for every \( \lambda \in [0, 1] \) the state \( \lambda \varphi^{(b)}_{w_0, h} + (1 - \lambda) \varphi^{(b)}_{\tilde{w}_0, \tilde{s}} \) is also backward QMC.

**Proof.** Without loss of generality we may assume that for every boundary condition \( w_0 = 1 \). Otherwise, we re-denote \( w_0 h_0 \) by \( h'_0 \). In the latter case, one gets \( w'_0 = 1 \). Therefore, \( \varphi^{(b)}_{1, h} \) and \( \varphi^{(b)}_{1, s} \) are QMC corresponding to \((1, h)\) and \((1, s)\).

Now define a sequence \( g = \{g_n\} \) as follows:

\[
g_n = \lambda h_n + (1 - \lambda)s_n, \quad n \geq 0.
\]

It is clear that \( g_n \in B_W^+ \).

We first observe that the boundary condition \((1, g)\) satisfies (3.15), (3.16). Indeed, one can see that \( \text{tr}(g_0) = \text{tr}(\lambda h_0 + (1 - \lambda)s_0) = 1 \) and

\[
\text{tr}_{a'}(K_{[n,n+1]}K^*_{[n,n+1]}g_{n+1}) = \lambda \text{tr}_{a'}(K_{[n,n+1]}h_{n+1}K_{[n,n+1]}) + (1 - \lambda) \text{tr}_{a'}(K_{[n,n+1]}s_{n+1}K_{[n,n+1]}) = \lambda h_n + (1 - \lambda)s_n = g_n.
\]

Hence, due to Theorem 3.3 there is a backward QMC \( \varphi^{(b)}_{1, g} \) corresponding to the boundary condition \((1, g)\). Let us show that \( \varphi^{(b)}_{1, g} = \lambda \varphi^{(b)}_{w_0, h} + (1 - \lambda)\varphi^{(b)}_{\tilde{w}_0, \tilde{s}} \). Indeed, for every \( a \in B_{A_n} \), we have

\[
\varphi^{(b)}_{1, g}(a) = \text{tr}\left( g_{n+1}^{1/2}K^*_{[0,1]} \cdots K^*_{[n,n+1]} w_0 K_{[0,1]} \cdots K_{[n,n+1]} g_{n+1}^{1/2} (a \otimes 1_{W_{n+1}}) \right) = \text{tr}\left( K^*_{[n,n+1]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} \cdots K_{[n,n+1]}(\lambda h_{n+1} + (1 - \lambda)s_{n+1})(a \otimes 1_{W_{n+1}}) \right) = \lambda \text{tr}\left( K^*_{[n,n+1]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} \cdots K_{[n,n+1]} h_{n+1}(a \otimes 1_{W_{n+1}}) \right) + (1 - \lambda) \text{tr}\left( K^*_{[n,n+1]} \cdots K^*_{[0,1]} w_0 K_{[0,1]} \cdots K_{[n,n+1]} s_{n+1}(a \otimes 1_{W_{n+1}}) \right) = \lambda \varphi^{(b)}_{1, h}(a) + (1 - \lambda) \varphi^{(b)}_{1, s}(a).
\]

This completes the proof. \( \square \)

From the proved theorem we immediately infer the following result.

**Corollary 3.6.** If there are at least two backward QMC corresponding to the sequence \( \{K_{[n,n+1]}\}_{n \geq 0} \), \( K_{[n,n+1]} \in B_{\Lambda_{[n,n+1]}} \), then they are uncountable.

**References**

[1] Accardi L 1975 *Funct. Anal. Appl.*, 9 1–8
[2] Accardi L and Cecchini C 1982 *J. Funct. Anal.* 45 245–273
[3] Accardi L and Fidaleo F 2005 *Annali di Matematica Pura e Applicata*, 184 327–346
[4] Accardi L and Fidaleo F 2003 *Inf. Dim. Analysis, Quantum Probab. Related Topics* 6 123–138
[5] Accardi L and Fidaleo F 2003 *J. Funct. Anal.* 200 324–347
[6] Accardi L and Fidaleo F 2003 In book: Proceedings Burg Conference 15–20 March 2001, *W. Freudenberg* (ed.) (Singapore: World Scientific) 1–20
[7] Accardi L, Fidaleo F and Mukhamedov F 2007 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **10** 165–183
[8] Accardi L and Frigerio A 1983 *Proc. Royal Irish Acad.* **83A** 251-263
[9] Accardi L and Liebscher V 1999 *Inf. Dimens. Anal. Quantum Probab. Relat. Top.* **2** 645-661
[10] Accardi L, Mukhamedov F and Saburov M 2011 *Math. Notes* **90** 8–20
[11] Accardi L, Mukhamedov F and Saburov M 2011 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **14** 443–463
[12] Accardi L, Mukhamedov F and Saburov M 2011 *Ann. Henri Poincare* **12** 1109–1144
[13] Accardi L, Mukhamedov F and Saburov M 2014 *J. Stats. Phys.* **157** 303-329
[14] Accardi L, Ohno H and Mukhamedov F 2010 *Inf. Dim. Analysis, Quantum Probab. Related Topics* **13** 165–189
[15] Affleck L, Kennedy E, Lieb E H and Tasaki H 1988 *Commun. Math. Phys.* **115** 477–528
[16] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London-New York: Acad. Press).
[17] Dobrushin R L 1968 *Probab. Theory Appl.* **13** 201–229
[18] Fannes M, Nachtergaele B and Werner R F 1992 *J. Stat. Phys.* **66** 939–973
[19] Fannes M, Nachtergaele B and Werner R F 1992 *Commun. Math. Phys.* **144** 443–490
[20] Fidaleo F and Mukhamedov F 2004 *Probab. Math. Stat.* **24** 401–418
[21] Ganikhodzhaev N N. and Mukhamedov F M 2004 *Math. Notes* **76** 329–338
[22] Georgi H-O 1988 *Gibbs measures and phase transitions* (Berlin: Walter de Gruyter)
[23] Golodets V Y and Zholtkevich G N 1983 *Theor. Math. Phys.* **56** 686–690
[24] Ibinson B, Linden N and Winter A 2008 *Comm. Math. Phys.* **277** 289–304
[25] Kümmerer B 2006 *Lecture Notes in Math.*, **1866** 259–330
[26] Liebscher V 2003 In book: *Proceedings Burg Conference 15–20 March 2001, W. Freudenberg (ed.)* (Singapore: World Scientific) 151–159
[27] Mukhamedov F M 2004 *Rep. Math. Phys.* **53** 1–18
[28] Mukhamedov F M 2000 *Theor. Math. Phys.* **123** 489–493
[29] Mukhamedov F M and Rozikov U A 2004 *J. Stat. Phys.* **114**(2004),825–848.
[30] Mukhamedov F M and Rozikov U A 2005 *J. Stat. Phys.* **119** 427–446
[31] Ohno H. 2005 *Inf. Dim. Analysis, Quantum Probab. Related Topics*, **8** 141–152
[32] Ohyu M and Petz D 1993 *Quantum entropy and its use* (Berlin: Springer)
[33] Ostilli M 2012 *Physica A* **391** 3417–3423
[34] Ostilli M, Mukhamedov F and Mendes J F F 2008 *Physica A* **387** 2777–2792
[35] Preston C 1974, *Gibbs states on countable sets* (London: Cambridge University Press)
[36] Rozikov U A 2013 *Gibbs measures on Cayley trees*, (Singapore: World Scientific)
[37] Spatara A 1990 *Advances in Math* **81** 105–116
[38] Spitzer F 1975 *Ann. Prob.* **3** 387-398
[39] Zachary S 1983 *Ann. Prob.* **11** 894–903
[40] Zachary S 1985 *Stochastic Process. Appl.* **20** 247–256