We study in the present Letter the notion of weak measurement introduced by Aharonov and Albert [2], Bergmann, and Lebowitz in [1].

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1 Introduction

We study the weak values of a quantum observable from the point of view of the Wigner formalism. The main actor is here the cross-Wigner transform of two functions, which is in disguise the cross-ambiguity function familiar from radar theory and time-frequency analysis. It allows us to express weak values using a complex probability distribution. We suggest that our approach seems to confirm that the weak value of an observable is, as conjectured by several authors, due to the interference of two wavefunctions, one coming from the past, and the other from the future.
from the point of view of the Wigner phase space formalism. This will allow us to discuss the claim made by these authors that the weak value can be seen as the interference of two wavepackets, one going forward in time and the other backwards in time.

Let us briefly recall the difference between an ideal (also called strong, or von Neumann) measurement, and a weak measurement. Let $\hat{A}$ be a (quantum) observable, realized as an essentially self-adjoint operator; we assume for simplicity that $\hat{A}$ has a eigenvalues $a_1, a_2, \ldots$ with corresponding orthogonal eigenfunctions $\psi_1, \psi_2, \ldots$. In an ideal measurement the expectation value of $\hat{A}$ in a pre-selected state $\psi$ is

$$\langle \hat{A} \rangle^{\psi} = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle};$$

(1)

if the sequence of eigenvalues lies in some interval $[a_{\min}, a_{\max}]$ then we will have $a_{\min} \leq \langle \hat{A} \rangle^{\psi} \leq a_{\max}$. In fact, if one performs the ideal measurement the outcome will always be one of the eigenvalues $\lambda_j$, and the probability of this outcome is $|\lambda_j|^2/||\psi_j||^2$ where $\lambda_j$ is the coefficient of $\psi_j$ in the Fourier expansion $\psi = \sum_j \lambda_j \psi_j$. Moreover the system will be left in the state $\psi_j$ after the ideal measurement yielding the value $a_j$. The situation is very different for weak measurements. As is explained in Ritchie et al. [25] (also see Berry and Shukla [9], Steinberg [27]), in a weak measurement the eigenvalues are not fully resolved and the system is left in a superposition of the unresolved states. If an appropriate post-selection is made, this superposition can interfere to produce a measurement result which can be significantly outside the range of the eigenvalues of the observable $\hat{A}$. The post-selection can then be accomplished by making an ideal measurement of some other observable $\hat{B}$ and selecting one particular outcome. Thus, the post-selected state $|\phi\rangle$ is an eigenstate of $\hat{B}$ which can be expressed as a linear combination of the eigenstates of $\hat{A}$ (we note that, conversely, an ideal measurement can be expressed as a convex sum of weak values: see Hosoya and Shikano [21]). If $\langle \phi | \psi \rangle \neq 0$ (and if $\phi$, $\psi$ are square integrable) the weak value of $\hat{A}$ is then the complex number

$$\langle \hat{A} \rangle^{\phi,\psi}_{\text{weak}} = \frac{\langle \phi | \hat{A} | \psi \rangle}{\langle \phi | \psi \rangle}. \quad (2)$$

We will show that this weak value can be expressed in terms of the cross-Wigner transform

$$W(\phi, \psi)(x, p) = \left(\frac{1}{2\pi \hbar}\right)^N \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}py} \phi^*(x + \frac{1}{2}y) \psi(x - \frac{1}{2}y) dy \quad (3)$$
of the pair \((\phi, \psi)\) whose physical interpretation is that of an interference term in the Wigner distribution of the sum \(\phi + \psi\); we mention that the importance of these interference terms have been emphasized and studied by Zurek [29] in the context of the sub-Planckian structures in phase space. The cross-Wigner transform is a very important object being intensively studied in the harmonic analysis literature and in time-frequency analysis; see e.g. Cohen [11], Folland [14] Gröchenig [17], Hlawatsch and Flandrin [20]. Notice that \(W(\phi, \psi)(x, p)\) reduces to the familiar Wigner distribution (Hillery et al. [19], Littlejohn [22] when \(\phi = \psi\).

We will not address here the ontological debates arising around the problem of “Elements of Reality” (see Cohen and Hiley [12, 13]); these questions are difficult and have led to profound philosophical controversies.

**Notation.** We will work with systems having \(N\) degrees of freedom. Position (resp. momentum) variables are denoted \(x = (x_1, \ldots, x_N)\) (resp. \(p = (p_1, \ldots, p_N)\)); they are vectors in \(\mathbb{R}^N\). The corresponding phase space variable is \(z = (x, p)\); it is a vector in phase space \(\mathbb{R}^{2N}\). We will endow the phase space with the standard symplectic form \(\sigma(z, z') = px' - p'x\). When integrating we will use, where appropriate, the volume elements \(dx = dx_1 \cdots dx_N, dp = dp_1 \cdots dp_N, dz = dpdx\). The unitary \(\hbar\)-Fourier transform of a function \(\psi\) in \(L^2(\mathbb{R}^N)\) is defined by

\[
F\psi(p) = \left(\frac{1}{2\pi\hbar}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{i}{\hbar}px}\psi(y)dy.
\]

## 2 The Main Result

### 2.1 A complex probability distribution

The cross-Wigner transform \((3)\) satisfies the “marginal properties”

\[
\int_{\mathbb{R}^N} W(\phi, \psi)(z)dp = \phi(x)\ast \psi(x) \tag{4}
\]

and

\[
\int_{\mathbb{R}^N} W(\phi, \psi)(z)dx = F\phi(p)\ast F\psi(p). \tag{5}
\]

It follows from the equality \((4)\) that

\[
\int_{\mathbb{R}^N} W(\phi, \psi)(z)dz = \langle \phi|\psi \rangle. \tag{6}
\]
For $\langle \phi | \psi \rangle \neq 0$ we define

$$
\rho_{\phi,\psi}(z) = \frac{W(\phi,\psi)(z)}{\langle \phi | \psi \rangle}.
$$

(7)

Note the conjugation relation $\rho_{\phi,\psi}(z)^* = \rho_{\psi,\phi}(z)$; also $\rho_{\lambda\phi,\lambda\psi}(z) = \rho_{\phi,\psi}(z)$ for every complex $\lambda \neq 0$ hence the function $\rho_{\phi,\psi}$ only depends on the states $|\psi\rangle$ and $|\phi\rangle$. In view of Eqn. (3) we have

$$
\int_{\mathbb{R}^2N} \rho_{\phi,\psi}(z)dz = 1
$$

(8)

hence $\rho_{\phi,\psi}$ can be viewed as a complex probability distribution with respect to which the average of the classical observable $A$ is calculated; also, Eqn. (8) implies that

$$
\int_{\mathbb{R}^2N} \text{Re} \rho_{\phi,\psi}(z)dz = 1, \quad \int_{\mathbb{R}^2N} \text{Im} \rho_{\phi,\psi}(z)dz = 0
$$

(9)

so that $\text{Re} \rho_{\phi,\psi}$ can be viewed as a quasi-distribution, in the same way as the usual Wigner transform. When $\psi = \phi$ then $\text{Im} \rho_{\phi,\psi} = 0$ and $\text{Re} \rho_{\phi,\psi} = W\psi$.

Observe that it immediately follows from Eqns. (7) and (4), (5) that the marginals distributions of $\rho_{\phi,\psi}$ are given by

$$
\int_{\mathbb{R}^N} \rho_{\phi,\psi}(z)dp = \frac{\phi^*(x)\psi(x)}{\langle \phi | \psi \rangle}, \quad \int_{\mathbb{R}^N} \rho_{\phi,\psi}(z)dx = \frac{[F\phi(p)]^*F\psi(p)}{\langle \phi | \psi \rangle};
$$

(10)

note that anyone of these equalities allows by integrating in the conjugate variable to recover the normalization condition (5).

We point out that the consideration of complex probability densities has per se nothing unusual; such complex probabilities have been used in the context of stochastic processes (see Zak [28]), signal theory (multipath fading channels, see Chayawan [10]) and they also appear in the study of non-Hermitian quantum mechanics (see Barkay and Moiseyev [8]).

We claim that:

**Theorem 1** Let $A$ be a classical observable and $\hat{A}$ its Weyl quantization; we have

$$
\langle \hat{A} \rangle_{\text{weak}}^{\phi,\psi} = \int_{\mathbb{R}^2N} A(z)\rho_{\phi,\psi}(z)dz.
$$

(11)

The reader familiar with the Weyl–Wigner–Moyal formalism (de Gosson [15], Littlejohn [22]) will have noticed that when $\phi = \psi$ formula (11) reduces to the well-known relation

$$
\langle \psi | \hat{A} | \psi \rangle = \int_{\mathbb{R}^2N} A(z)W\psi(z)dz
$$
yielding the usual expectation value \( \langle \hat{A} \rangle^\psi = \langle \psi | \hat{A} | \psi \rangle / \langle \psi | \psi \rangle \). We will study the relative importance of these values when \( \phi \) and \( \psi \) are coherent states in Subsection 2.3.

### 2.2 Proof of Theorem 1

To prove formula (11) it is sufficient, in view of definition (7) of \( \rho_{\phi,\psi}(z) \), to show that

\[
\langle \phi | \hat{A} | \psi \rangle = \int_{\mathbb{R}^{2N}} W(\phi, \psi)(z) A(z) dz.
\]

To prove the latter we could perform a direct calculation staring from the right-hand side, inserting the expression (3) of \( W(\phi, \psi)(z) \) and making various changes of variables. We prefer to give a more elegant proof which has some conceptual advantages. The first step consists in observing that the cross-Wigner transform can be expressed in terms of the Grossmann–Royer \([18, 26]\) operator

\[
\hat{T}_{\text{GR}}(z_0) \phi(x) = e^{\frac{i}{\hbar} p_0(x-x_0)} \phi(2x_0 - x)
\]

(also see de Gosson \([16]\), Chapter 9). A simple calculation shows that we have

\[
W(\phi, \psi)(z) = \left( \frac{1}{\pi \hbar} \right)^{N} \langle \hat{T}_{\text{GR}}(z) \phi | \psi \rangle
\]

and that the Weyl quantization \( \hat{A} \) of the observable is given by

\[
\hat{A} \psi(x) = \left( \frac{1}{\pi \hbar} \right)^{N} \int_{\mathbb{R}^{2N}} A(z_0) \hat{T}_{\text{GR}}(z_0) \psi(x) dz_0.
\]

Using the latter we have

\[
\langle \phi | \hat{A} | \psi \rangle = \left( \frac{1}{\pi \hbar} \right)^{N} \int_{\mathbb{R}^{2N}} A(z_0) \langle \phi | \hat{T}_{\text{GR}}(z_0) \psi \rangle dz_0;
\]

we next observe that \( \hat{T}_{\text{GR}}(z_0) \) is both unitary and involutive (i.e. \( \hat{T}_{\text{GR}}(z_0) = \hat{T}_{\text{GR}}(z_0)^{-1} \)) and hence

\[
\langle \phi | \hat{T}_{\text{GR}}(z_0) \psi \rangle = \langle \hat{T}_{\text{GR}}(z_0)^{-1} \phi | \psi \rangle = \langle \hat{T}_{\text{GR}}(z_0) \phi | \psi \rangle
\]

so that (16) can be rewritten

\[
\langle \phi | \hat{A} | \psi \rangle = \left( \frac{1}{\pi \hbar} \right)^{N} \int_{\mathbb{R}^{2N}} A(z_0) \langle \hat{T}_{\text{GR}}(z_0) \phi | \psi \rangle dz_0
\]

\[
= \int_{\mathbb{R}^{2N}} A(z_0) W(\phi, \psi)(z) dz_0
\]

which was to be proven.
2.3 The case of coherent states

Suppose that both wavefunctions are normalized coherent states concentrated near $z_0 = (x_0, p_0)$ and $-z_0$ at time $t_m$, that is we choose $\theta$ and $\psi = \psi_{z_0}$ where

$$
\theta(x) = \left(\frac{1}{\pi \hbar}\right)^{N/4} \hat{T}(z_0) e^{-\frac{1}{\hbar}|x|^2}, \quad \psi(x) = \left(\frac{1}{\pi \hbar}\right)^{N/4} \hat{T}(-z_0) e^{-\frac{1}{\hbar}|x|^2};
$$

where $\hat{T}(z_0) = e^{-\frac{i}{\hbar}(x_0 \hat{p} - p_0 \hat{x})}$ is the Heisenberg–Weyl operator. These states are minimum uncertainty states (they saturate the Heisenberg inequalities $\Delta x_j \Delta p_j \geq \frac{\hbar}{2}$). A standard calculation of Gaussian integrals shows that the scalar product of these states is

$$
\langle \theta | \psi \rangle = e^{-\frac{1}{\hbar}|z_0|^2}. \quad (19)
$$

Let us calculate $W(\phi, \psi)$. Using the translation formula (see de Gosson [16])

$$
W(\hat{T}(\alpha)\phi, \hat{T}(\beta)\psi)(z) = e^{-\frac{i}{\hbar}\chi_{\alpha\beta}(z)}W(\phi, \psi)(z - \frac{1}{2}(\alpha + \beta)) \quad (20)
$$

where $\chi_{\alpha\beta}$ is the phase function defined by

$$
\chi_{\alpha\beta}(z) = \frac{1}{2}\sigma(z, \alpha - \beta) + \sigma(\alpha, \beta) \quad (21)
$$

($\sigma$ the standard symplectic form). We thus have

$$
W(\phi, \psi)(z) = e^{\frac{i}{\hbar}\sigma(z, z_0)}W(\xi_0, \xi_0)(z)
$$

where $\sigma(z, z_0) = px_0 - p_0 x$ and $\xi_0(x) = (\pi \hbar)^{-N/4} e^{-|x|^2/\hbar}$ is the standard fiducial coherent state (Littlejohn [22]). Now, $W(\xi_0, \xi_0) = W_{\xi_0}$, the Wigner distribution of $\xi_0$, which is given by

$$
W_{\xi_0}(z) = \left(\frac{1}{\pi \hbar}\right)^{N} e^{-\frac{1}{\hbar}|z|^2}, \quad |z|^2 = |x|^2 + |p|^2
$$

(de Gosson [15, 16], Littlejohn [22]). We thus conclude that

$$
W(\phi, \psi)(z) = \left(\frac{1}{\pi \hbar}\right)^{N} e^{\frac{i}{\hbar}\sigma(z, z_0)} e^{-\frac{1}{\hbar}|z|^2}. \quad (23)
$$

Using the scalar product formula (19) we see that the complex probability distribution $\rho_{\phi, \psi}$ is given by

$$
\rho_{\phi, \psi}(z) = \left(\frac{1}{\pi \hbar}\right)^{N} e^{\frac{i}{\hbar}\sigma(z, z_0)} e^{\frac{1}{\hbar}|z_0|^2} e^{-\frac{1}{\hbar}|z|^2}. \quad (24)
$$
This formula shows that \( \rho_{\alpha,\beta}(z) \) has an oscillatory behavior which is sharply peaked near the origin. We notice that since

\[
|\rho_{\phi,\psi}(z)| \leq (\frac{1}{\pi \hbar})^N e^{\frac{1}{\pi}|z_0|^2} e^{-\frac{1}{\pi}|z|^2}
\]

the weak value \( \langle \hat{A} \rangle_{\text{weak}}^{\phi,\psi} \) satisfies

\[
|\langle \hat{A} \rangle_{\text{weak}}^{\phi,\psi}| \leq \int_{\mathbb{R}^2N} |\rho_{\phi,\psi}(z)||A(z)|dz
= (\frac{1}{\pi \hbar})^N e^{\frac{1}{\pi}|z_0|^2} \int_{\mathbb{R}^2N} e^{-\frac{1}{\pi}|z|^2} |A(z)|dz
\leq (\frac{1}{\pi \hbar})^N e^{\frac{1}{\pi}|z_0|^2} \sup |A(z)| \int_{\mathbb{R}^2N} e^{-\frac{1}{\pi}|z|^2} dz.
\]

The integral in the third line is easy to evaluate; its value is \((\pi \hbar)^N\) hence we have the estimate

\[
|\langle \hat{A} \rangle_{\text{weak}}^{\phi,\psi}| \leq e^{\frac{1}{\pi}|z_0|^2} \sup |A(z)|.
\] (25)

This inequality shows that even if the observable \( A \) is small, the weak value can a priori take very large values provided that the phase space distance between both wavepackets \( \phi, \psi \) is large; this is in strong contrast with what happens for the individual states \( |\phi\rangle \) and \( |\psi\rangle \), for which lead to the estimates

\[
|\langle \hat{A} \rangle^{\phi}| \leq \sup |A(z)|, \quad |\langle \hat{A} \rangle^{\psi}| \leq \sup |A(z)|;
\]

the relative phase space localization of these states does not play any role in these inequalities. We will shortly discuss non-trivial extensions of the superposition considered above in the discussion below.

3 Discussion

Let us apply the phase space formalism to a discussion of the situation initially considered in [3, 4] where at a time \( t_{\text{in}} \) an observable \( \hat{A} \) is measured and a non-degenerate eigenvalue was found: \( |\psi(t_{\text{in}})\rangle = |\hat{A} = a\rangle \) (the pre-selected state); similarly at a later time \( t_{\text{fin}} \) a measurement of another observable \( \hat{B} \) yields \( |\phi(t_{\text{fin}})\rangle = |\hat{B} = b\rangle \) (the post-selected state). Let \( t \) be some intermediate time: \( t_{\text{in}} < t < t_{\text{fin}} \). Following the time-symmetric approach to quantum mechanics (see the review in [6]), at this intermediate time the system is described by the two wavefunctions

\[
\psi_t = U_{t,t_{\text{in}}}^H \psi(t_{\text{in}}) \quad \text{and} \quad \phi_t = U_{t,t_{\text{fin}}}^H \phi(t_{\text{fin}})
\] (26)
where $U^{H}_{t,t'} = e^{-i\hat{H}(t-t')/\hbar}$ is the Schrödinger unitary evolution operator ($\hat{H}$ the quantum Hamiltonian). Notice that $\phi_t$ travels backwards in time since $t < t_{\text{fin}}$. The situation is thus the following: at any time $t' < t$ the system under consideration is in the state $|\psi_t\rangle = U^{H}_{t,t_{\text{fin}}} |\psi(t_{\text{fin}})\rangle$ and has Wigner distribution $W_{\psi_t}$; at any time $t'' > t$ the system is in the state $|\phi_{t''}\rangle = U^{H}_{t'',t_{\text{fin}}} |\phi(t_{\text{fin}})\rangle$ and has Wigner distribution $W_{\phi_{t''}}$. But at time $t$ it is the superposition $|\psi_t\rangle + |\phi_t\rangle$ of both states, and the Wigner distribution of this cat-like state is

$$W(\phi_t + \psi_t) = W\phi_t + W\psi_t + 2 \text{Re} W(\phi_t, \psi_t). \tag{27}$$

This equality shows the abrupt emergence at time $t$ – and only at that time! – of the interference term $2 \text{Re} W(\phi_t, \psi_t)$, signalling a strong interaction between the states $|\psi_t\rangle$ and $|\phi_t\rangle$. Such an interaction is due to the wavelike nature of quantum mechanics, and is absent from classical mechanics. The appearance of interference terms described by the cross-Wigner transform is well-known and considered as an asset in time-frequency analysis (e.g. radar theory, see Cohen [11], Auslander and Tolimieri [7]). It seems therefore that our approach could well open new perspectives in the topic of weak measurements and values, by importing robust techniques from these Sciences (it is a fact, due mainly to historical and technical reasons, that the mathematical techniques related to the Wigner formalism have grown faster and are more sophisticated in signal theory and time-frequency analysis than they are in quantum mechanics, so a feedback seems to be more than welcome!).

How the weak values are related to sub-Planckian scales would also be interesting to investigate; the discussion in Zurek [29], and especially the results in Nicacio et al. [24] could certainly be useful in this context. These authors consider superpositions of an arbitrary number of Gaussian states, and study their motion under the action of arbitrary Hamiltonian flows. They show that the interference terms coming from the cross-Wigner transforms are always hyperbolic and survive the action of a thermal reservoir. While they mainly have in mind semiclassical dynamics, their approach could be implemented in the context of weak values. It is actually to a large extent sufficient to study the case of coherent states as in Subsection 2.3, because these states form an overcomplete set in the square-integrable functions. In fact, choosing an adequate lattice $\Lambda$ of points $z_0$ in phase space the functions $\hat{T}(z_0)\xi_0$ ($\xi_0(x) = (\pi\hbar)^{-N/4} e^{-|x|^2/\hbar}$) form a Gabor frame (Gröchenig [17]) allowing to write an arbitrary pure state as a linear superposition of the states $\hat{T}(z_0)\xi_0$. The net contribution of all cross-Wigner transforms of pairs $(\hat{T}(z_0)\xi_0, \hat{T}(z_1)\xi_0)$ with $z_0 \neq z_1$ is then the total interference leading
to weak values (in [29] Zurek considers a “compass state” consisting of four terms \( \hat{T}(z_0)\xi_0 \), of which he studies interference effects at the sub-Planckian scale; it would be interesting to interpret his results in terms of weak values).

There is another aspect of the theory of weak values we have not mentioned at all, if only because of lack of space and time. It is the possibility of reconstructing wave functions from weak values, as initiated in Lundeen et al. [23]. It turns out that the Wigner approach sketched in this Letter leads to useful formulas. For instance, one proves the following inversion formula (de Gosson [16], §9.4.2): Let \( \eta \) be an arbitrary square integrable function such that \( \langle \phi | \gamma \rangle \neq 0 \); then

\[
\psi(x) = \frac{2^N}{\langle \phi | \gamma \rangle} \int_{\mathbb{R}^{2N}} W(\phi, \psi) \langle \hat{T}_{GR}(z_0) \psi | \gamma \rangle dz_0.
\]

(28)

We can reconstruct \( \psi \) from the knowledge of the weak value provided that we know \( \langle \phi | \gamma \rangle \). This inversion formula together with the notion of mutually unbiased bases (MUB) could certainly play an important role in the reconstruction problem.

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