Global low regularity solutions for nonlinear elastic waves

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Abstract

We study the Cauchy problem for 3-D nonlinear elastic waves satisfying the null condition with low regularity initial data. In the radially symmetric case, we prove the global existence of a low regularity solution for every small data in $H^3 \times H^2$ with a low weight.

keywords: Elastic waves; global low regularity solutions; radial symmetry.

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1 Introduction

For elastic materials, the motion for the displacement is governed by the nonlinear elastic wave equation which is a second-order quasilinear hyperbolic system. For isotropic, homogeneous, hyperelastic materials, the motion for the displacement $u = u(t, x)$ satisfies

$$
\partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2) \nabla \cdot \nabla u = F(\nabla u, \nabla^2 u),
$$

where the nonlinear term $F(\nabla u, \nabla^2 u)$ is linear in $\nabla^2 u$ and will be described explicitly in later sections. Some physical backgrounds of nonlinear elastic waves can be found in Ciarlet [5] and Gurtin [8].

The short time existence of classical solutions for nonlinear elastic waves is standard now [17]. The research for long time existence of classical solutions for nonlinear elastic waves started from Fritz John’s pioneering work on elastodynamics (see Klainerman [25]). In the 3-D case, John showed that local classical solutions in general will develop singularities for radial and small initial data [20], and they almost globally exist for small data with compact support [21]. See also simplified proof of almost global existence in [28], [49] and a lower bound estimate in [29]. In order to ensure the global existence of classical solutions with small initial data, some structural condition on the nonlinearity which is called the null condition is necessary. We refer the reader to Agemi [1] and Sideris [40] (see also a previous result in Sideris [39]). The exterior domain analogues of John’s almost global existence result and Agemi and Sideris’s global existence result were obtained in [34] and [36], respectively. In the 2-D case, under the null condition global small classical solution with radial initial data

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was constructed in [47] and [46] for the Cauchy problem and the exterior domain problem, respectively. The non-radially symmetric case is still open.

All the above works are in the framework of classical solutions. Studies on local low regularity well-posedness for the Cauchy problem of nonlinear wave equations started from Klainerman-Machedon’s pioneering work [26] in the semilinear case. For the general quasilinear equations, we refer the reader to the sharpest result [42], [45] and the references therein. On the other hand, Lindblad constructed some counterexamples to show sharp results of ill-posedness [30], [31]. For the Einstein equation in wave coordinates, the sharpest result of well-posedness and ill-posedness was shown by Klainerman and Rodnianski [27] and Ettinger and Lindblad [6], respectively.

We note that a family of special solutions which are obtained via the Lorentz transformation has often played a key role in the proof of ill-posedness. It is therefore natural to expect that we can improve regularity requirements for initial data if we rule out such special solutions. It is indeed the case. It should be mentioned that improvement of regularity in the presence of radial symmetry was first observed in [26] for semilinear equations such as $\Box u = c_1(\partial_t u)^2 + c_2|\nabla u|^2$. (See also page 176 of [38], Section 5 of [33], and [14], [37], and [12] for related results.) We also mention that it was also observed for the wave equation with a power-type nonlinear term $\Box u = F(u)$, first by Lindbald and Sogge [32], and then by some authors (see [10], [18], [7]), and quasilinear wave equations of the form $\partial_t^2 u - a^2(u)\Delta u = c_1(\partial_t u)^2 + c_2|\nabla u|^2$, by [13] and [50]. Particularly, the almost global existence of low regularity radially symmetric solutions with small initial data was showed in [14] and [13] for the semilinear and the quasilinear case, respectively. A global regularity result can be found in [50], which is the first result for global existence of low regularity solutions to quasilinear wave equations.

In the present paper, we will study global low regularity solutions for the Cauchy problem of 3-D nonlinear elastic waves with the null condition, and focus on the radially symmetric case. For this purpose, we first note the fact that in the radially symmetric case, the nonlinear elastic wave equation can be reduced to a system of quasilinear wave equations with single wave speed. To analyze this system, we will use the ghost weight energy estimate of Alinhac to display the null condition and get enough decay in time on the region $r \geq \langle t \rangle /2$, and employ the Klainerman-Sideris type estimate and a refined Keel-Smith-Sogge (KSS, for short) type estimate to control the elastic waves with rather low regularity on the region $1/2 \leq r \leq \langle t \rangle /2$ and $r \leq 1/2$. Together with some weighted Sobolev inequalities for radially symmetric functions, we can get global existence for nonlinear elastic waves satisfying the null condition when radial data is small with respect to the $H^2$ norm with a low weight $\langle x \rangle$. Note that even the standard local existence theorem for nonlinear elastic waves (see [17]) need the regularity requirement $H^s \times H^{s-1}$ with $s > 7/2$. It is the first global low regularity existence result for nonlinear elastic waves. A almost global low regularity existence result has been got by the authors [15].

The outline of this paper is as follows. The remainder of this introduction will be devoted to the description of notation which will be used in the sequel, some reduction of the equations of motion and a statement of the low regularity global existence theorem. In Section 2, some necessary tools used to prove the global existence theorem are introduced, including some properties of the null condition, weighted Sobolev inequalities, the bounds
on the Klainerman-Sideris type energy and KSS type energy. Finally, the proof of global low regularity existence theorem will be given in Section 3.

1.1 Notation

Denote the space gradient and space-time gradient by $\nabla = (\partial_1, \partial_2, \partial_3)$ and $\partial = (\partial_t, \nabla)$, respectively. The scaling operator is the vector field

$$S = t\partial_t + r\partial_r, \quad (1.2)$$

where $r = |x|$, $\partial_r = \omega \cdot \nabla$, $\omega = (\omega_1, \omega_2, \omega_3)$, $\omega_i = x_i/r$. Denote the collection of vector fields $Z = (Z_1, Z_2, Z_3, Z_4) = (\nabla, S)$. For any multi-index $a = (a_1, a_2, a_3, a_4)$, we denote an ordered product of vector fields $Z_a = Z_{a_1} Z_{a_2} Z_{a_3} Z_{a_4} = \partial_{a_1} \partial_{a_2} \partial_{a_3} S_{a_4}$. Moreover $b \leq a$ means $b_i \leq a_i$ for each $i = 1, \ldots, 4$.

The energy associated to the linear wave operator is

$$E_1(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2) \, dx, \quad (1.3)$$

and the corresponding $k$-th order energy is given by

$$E_k(u(t)) = \sum_{|a| \leq k-1, a_4 \leq 1} E_1(Z^a u(t)). \quad (1.4)$$

We will use the so called “good derivatives”

$$T = (T_1, T_2, T_3) = \omega \partial_t + \nabla. \quad (1.5)$$

Let $\sigma = t - r$, $q(\sigma) = \arctan \sigma, q'(\sigma) = \frac{1}{1 + \sigma^2} = (t - r)^{-2}$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. Since $q$ is bounded, there exists a constant $c > 1$, such that

$$c^{-1} \leq e^{-q(\sigma)} \leq c. \quad (1.6)$$

Define the ghost weight energy (see [2, 3])

$$\mathcal{E}_1(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} e^{-q(\sigma)} \langle t - r \rangle^{-2} |T u|^2 \, dx, \quad (1.7)$$

and its higher-order version

$$\mathcal{E}_k(u(t)) = \sum_{|a| \leq k-1, a_4 \leq 1} \mathcal{E}_1(Z^a u(t)). \quad (1.8)$$

The KSS type energy (see [13, 22, 23, 35]) is defined through

$$\mathcal{M}_k(u(t)) = \int_0^t \mathcal{N}_k(u(\tau)) \, d\tau, \quad (1.9)$$

where

$$\mathcal{N}_k(u(t)) = \sum_{|a| \leq k-1, a_4 \leq 1} \| \langle r \rangle^{-1/4} r^{-1/4} \partial Z^a u \|_{L^2(\mathbb{R}^3)}^2 + \sum_{|a| \leq k-1, a_4 \leq 1} \| \langle r \rangle^{-1/4} r^{-5/4} Z^a u \|_{L^2(\mathbb{R}^3)}^2, \quad (1.10)$$
and its local version is
\[
L_k(u(t)) = \sum_{|a| \leq k-2} \left\| r^{-1/4} \partial Z^a u \right\|_{L^2(0,t;L^2(|x| \leq 1))}^2 + \sum_{|a| \leq k-1} \left\| r^{-5/4} Z^a u \right\|_{L^2(0,t;L^2(|x| \leq 1))}^2. \tag{1.11}
\]

We will also use the following Klainerman-Sideris type energy (see [28])
\[
X_k(u(t)) = \sum_{|a| \leq k-2} \left\| (t-r) \partial \nabla^a u(t) \right\|_{L^2(\mathbb{R}^3)}. \tag{1.12}
\]

Consider the space \( X^k(T) \), which is obtained by closing the set \( C^\infty([0,T); C^\infty_0(\mathbb{R}^3; \mathbb{R}^3)) \) in the norm \( \sup_{0 \leq t \leq T} E_k^{1/2}(u(t)) \).

The solution will be constructed in the space
\[
X^k_\text{rad}(T) = \{ u : u \text{ is radially symmetric}, u \in X^k(T) \}. \tag{1.13}
\]

Though we will consider radially symmetric solutions, the generators of spatial rotations and simultaneous rotations will be also important in our analysis. The angular momentum operators (generators of the spatial rotations) are the vector fields
\[
\Omega = (\Omega_1, \Omega_2, \Omega_3) = x \wedge \nabla, \tag{1.14}
\]
∧ being the usual vector cross product. The spatial derivatives can be conveniently decomposed into radial and angular components
\[
\nabla = \omega \partial_r - \frac{\omega}{r} \wedge \Omega. \tag{1.15}
\]

Noting the definitions (1.5) and (1.14), we also have the relationship between the angular momentum operators \( \Omega \) and the good derivatives \( T \)
\[
\frac{\Omega}{r} = \omega \wedge T, \tag{1.16}
\]
which implies
\[
\left| \frac{\Omega}{r} u \right| \leq |Tu|. \tag{1.17}
\]

Denote the generators of simultaneous rotations by \( \tilde{\Omega} = (\tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3) \), where
\[
\tilde{\Omega}_i = \Omega_i I + U_i, \tag{1.18}
\]
with \( U_1 = e_2 \otimes e_3 - e_3 \otimes e_2, U_2 = e_3 \otimes e_1 - e_1 \otimes e_3, U_3 = e_1 \otimes e_2 - e_2 \otimes e_1, \) \( \{ e_i \}_{i=1}^3 \) is the standard basis on \( \mathbb{R}^3 \) and \( \otimes \) stands for the tensor product on \( \mathbb{R}^3 \). It is easy to verify the following commutation relationship
\[
[\tilde{\Omega}, \nabla] = \nabla, \quad [\tilde{\Omega}, S] = 0. \tag{1.19}
\]

1.2 The equations of motion

Now we consider the equations of motion for 3-D homogeneous, isotropic and hyperelastic elastic waves. First we have the Lagrangian
\[
\mathcal{L}(u) = \iint \frac{1}{2} |u_t|^2 - W(\nabla u) \, dx dt, \tag{1.20}
\]
where \(u(t, x) = (u^1(t, x), u^2(t, x), u^3(t, x))\) denotes the displacement vector from the reference configuration and \(W\) is the stored energy function. Since we will consider small solutions, we can write

\[
W(\nabla u) = l_2(\nabla u) + l_3(\nabla u) + h.o.t., \tag{1.21}
\]

with \(l_2(\nabla u)\) and \(l_3(\nabla u)\) standing for the quadratic and cubic term in \(\nabla u\), respectively, and \(h.o.t.\) denoting higher order terms. Using the frame indifference and isotropic assumption, we can get

\[
l_2(\nabla u) = \frac{c_2^2}{2}|\nabla u|^2 + \frac{c_1^2 - c_2^2}{2}(\nabla \cdot u)^2, \tag{1.22}
\]

where the material constants \(c_1\) (pressure wave speed) and \(c_2\) (shear wave speed) satisfy \(0 < c_2 < c_1\), and

\[
l_3(\nabla u) = d_1(\nabla \cdot u)^3 + d_2(\nabla \cdot u)|\nabla \wedge u|^2 + d_3(\nabla \cdot u)Q_{ij}(u^i, u^j)
+ d_4(\partial_k u^j)Q_{ij}(u^i, u^k) + d_5(\partial_k u^j)Q_{ik}(u^i, u^j), \tag{1.23}
\]

where \(d_i\) \((i = 1, \ldots, 5)\) are also some constants which only depend on the stored energy function, and the null form \(Q_{ij}(f, g) = \partial_i f \partial_j g - \partial_j f \partial_i g\).

By the Hamilton’s principle we get the nonlinear elastic wave equation in 3-D

\[
\partial_t^2 u - c_2^2 \Delta u - (c_1^2 - c_2^2)\nabla \nabla \cdot u = F(\nabla u, \nabla^2 u), \tag{1.24}
\]

where

\[
F(\nabla u, \nabla^2 u) = 3d_1 \nabla(\nabla \cdot u)^2 + d_2(\nabla |\nabla \wedge u|^2 - 2\nabla \wedge (\nabla \cdot u \nabla \wedge u)) + Q(u, \nabla u), \tag{1.25}
\]

with

\[
Q(u, \nabla u)^i = (2d_3 + d_4)\left(Q_{ij}(\partial_k u^k, u^j) - Q_{jk}(\partial_i u^k, u^j)\right)
+ d_5(Q_{ij}(\partial_j u^k, u^k) + 2Q_{jk}(\partial_i u^k, u^k) - Q_{jk}(\partial_j u^k, u^i)). \tag{1.26}
\]

The derivation of the exact form of \(F(\nabla u, \nabla^2 u)\) can be also found in Agemi [1]. Following Agemi [1], we say that the nonlinear elastic wave equation (1.24) satisfies the null condition if

\[
d_1 = 0. \tag{1.27}
\]

Consider the Cauchy problem of (1.24)–(1.25) with initial data

\[
t = 0 : u = u_0, \; u_t = u_1. \tag{1.28}
\]

Noting the nonlinear elastic wave equation is invariant under simultaneous rotations (see page 860 of [40]), we can seek radially symmetric solution for the Cauchy problem

\footnote{We use the summation convention over repeated indices.}

\footnote{In 3-D case, the global existence of small solutions to quasilinear hyperbolic systems hinges on the specific form of the quadratic part of the nonlinearity in relation to the linear part. From the analytical point of view, therefore, it is enough to truncate at cubic order in \(u\), the higher order corrections having no influence on the existence of small solutions.}
(1.25)–(1.28) with radially symmetric initial data. In the radially symmetric case, we first derive some equivalent forms of the equation of motion and the null condition.

If \( u \in X^3_{\text{rad}}(T) \) is a radially symmetric solution to (1.24), then there exists a scalar function \( \psi \) such that

\[
    u(t, x) = x\psi(t, r).
\]

By (1.29), it is easy to see that

\[
    \nabla \times u = 0.
\]

It follows from the Hodge decomposition and (1.30) that

\[
    \Delta u = \nabla \cdot u - \nabla \times \nabla \times u = \nabla \nabla \cdot u.
\]

Thanks to (1.31), for the linear part of (1.24), we have

\[
    \partial^2_t u - c_1^2 \Delta u - (c_1^2 - c_2^2) \nabla \nabla \cdot u = \partial^2_t u - c_1^2 \Delta u.
\]

Next we consider the nonlinear part \( F(\nabla u, \nabla^2 u) \). Obviously (1.30) gives that the second term on the right-hand side of (1.25) vanishes. Furthermore, the null condition (1.27) implies that the first term on the right-hand side of (1.25) also vanishes. Without loss of generality we can take \( c_1 = 1 \). Hence \( u \) satisfies

\[
    \partial^2_t u - \Delta u = Q(u, \nabla u).
\]

Conversely, if \( u \in X^3_{\text{rad}}(T) \) is a radially symmetric solution to the equation (1.33), then \( u \) is also a radially symmetric solution to the equation (1.24) with the null condition (1.27).

For the convenience of the following analysis, we rewrite (1.33) as

\[
    \partial^2_t u - \Delta u = N(u, u),
\]

where

\[
    N(u, v)^i = \partial_i (g^{ijk}_{lmn} \partial_m u^j \partial_n v^k),
\]

and \( g = (g^{ijk}_{lmn}) \) is determined via \( N(u, u) = Q(u, \nabla u) \). We can verify that the coefficients are symmetric with respect to pairs of indices

\[
    g^{ijk}_{lmn} = g^{jik}_{mln} = g^{kji}_{nml},
\]

which implies that

\[
    N(u, v) = N(v, u),
\]

and the following null condition holds (see Lemma 3.1 of [1])

\[
    g^{ijk}_{lmn} \omega_l \omega_m \omega_n = 0, \quad \forall \ 1 \leq i, j, k \leq 3, \omega \in S^2.
\]

We note that (1.38) just coincides with the definition of standard null condition in the pioneering works Klainerman [24] and Christodoulou [4] for systems of quasilinear wave equations with single wave speed.

The above argument gives that in order to seek radially symmetric solutions to the Cauchy problem of nonlinear elastic waves (1.24)–(1.28) with the null condition (1.27), we only need to study the radially symmetric solutions to the Cauchy problem (1.34)–(1.28) with the null condition (1.38).

A vector function \( v \) is called radial if it has the form \( v(x) = x\phi(r) \) (\( r = |x| \)), where \( \phi \) is a scalar radial function. We refer the reader to Definition 4.4 and Lemma 4.5 of [19].
1.3 Global low regularity existence theorem

The main result in this paper is the following

**Theorem 1.1.** Consider the Cauchy problem for 3-D nonlinear elastic waves \((1.24)-(1.28)\). Assume that the null condition \((1.27)\) holds. Then there exists a constant \(\varepsilon_0 > 0\) such that for any given \(\varepsilon\) with \(0 < \varepsilon \leq \varepsilon_0\), if the initial data is radially symmetric and satisfies

\[
\sum_{|\alpha| \leq 2} \| \langle x \rangle \nabla^\alpha u_0 \|_{L^2(\mathbb{R}^3)} + \sum_{|\alpha| \leq 2} \| \langle x \rangle \nabla^\alpha u_1 \|_{L^2(\mathbb{R}^3)} \leq \varepsilon, \tag{1.39}
\]

then Cauchy problem \((1.25)-(1.28)\) admits a unique global solution \(u \in X^3_{\text{rad}}(T)\) for every \(T > 0\).

**Remark 1.1.** In the smallness condition \((1.39)\) of the initial data, we only need a low weight \(\langle x \rangle\). This is achieved by using the scaling operator \((1.2)\) only one time in our argument (note that in the definitions of higher-order energies \((1.4), (1.8), (1.10)\) and \((1.11)\), we restrict \(a_4 \leq 1\)).

2 Preliminaries

In this section, first we will give some properties on the null condition and some weighted Sobolev inequalities, which will play key roles in the decay estimate under rather low regularity. Then we will give some a priori estimates on the Klainerman-Sideris type energy \(X^3_3(u(t))\) and the KSS type energy \(M^3_3(u(t))\). It turns out that they can all be controlled by the general energy \(E^3_3(u(t))\), if it is sufficiently small.

2.1 Null condition

Denote the trilinear form

\[
\tilde{N}(u, v, w) = g^{ijk}_{lmn} \partial_r u^i \partial_m v^j \partial_n w^k. \tag{2.1}
\]

**Proposition 2.1.** If the null condition \((1.38)\) is satisfied, then for any \(u, v, w \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)\), we have

\[
|\langle u, N(v, w) \rangle| \leq \frac{C}{r} |u| \left( |\nabla \Omega| |\nabla w| + |\nabla \Omega| |\nabla v| + |\nabla \Omega| |\nabla^2 w| + |\nabla \Omega| |\nabla^2 v| \right), \tag{2.2}
\]

\[
|\tilde{N}(u, v, w)| \leq \frac{C}{r} \left( |\Omega u| |\nabla v| |\nabla w| + |\Omega u| |\nabla v| |\nabla w| + |\Omega u| |\nabla v| |\nabla w| \right). \tag{2.3}
\]

**Proof.** It follows from the radial-angular decomposition \((1.15)\) that we have the following pointwise equality

\[
\tilde{N}(u, v, w) = -g^{ijk}_{lmn} (\omega_r \wedge \Omega) u^i \partial_m v^j \partial_n w^k - g^{ijk}_{lmn} \omega_l \partial_r u^i (\omega_r \wedge \Omega) u^j \partial_n w^k \\
- g^{ijk}_{lmn} \omega_l \partial_r u^i \omega_m \partial_r v^j (\omega_r \wedge \Omega) u^k + g^{ijk}_{lmn} \omega_l \omega_m \omega_n \partial_r u^i \partial_r v^j \partial_r w^k. \tag{2.4}
\]

Thus \((2.3)\) follows from \((2.4)\) and the null condition \((1.38)\). \((2.2)\) can be proved similarly. \(\square\)
Lemma 2.1. Assume that the null condition (1.38) is satisfied and \( u \in X^3(T) \) is a solution to (1.34). Then for any multi-index \( a = (a_1, a_2, a_3, a_4), a_4 \leq 1 \), we have

\[
\square Z^a u = \sum_{b+c+d=a} N_d(Z^b u, Z^c u), \tag{2.5}
\]

where each \( N_d \) is a quadratic nonlinearity of the form (1.35) satisfying the null condition (1.38). Moreover, if \( b + c = a \), then \( N_d = N \).

Proof. See Lemma 6.6.5 of Hörmander [16] and Lemma 4.1 of Sideris and Tu [41].

2.2 Weighted Sobolev inequalities

Lemma 2.2. For \( u \in C^\infty_0(\mathbb{R}^3; \mathbb{R}^3) \), we have

\[
\|r u\|_{L^\infty} \leq C \sum_{|a| \leq 1} \|\nabla \tilde{\Omega}^a u\|_{L^2} + C \sum_{|a| \leq 2} \|\tilde{\Omega}^a u\|_{L^2}, \tag{2.6}
\]

\[
\|r^{1/2} u\|_{L^\infty} \leq C \sum_{|a| \leq 1} \|\nabla \tilde{\Omega}^a u\|_{L^2}, \tag{2.7}
\]

\[
\|r^{1/4} u\|_{L^\infty(r \leq 1)} \leq C \sum_{|a| \leq 1} \|(r)^{-1/4} r^{-1/4} \nabla \tilde{\Omega}^a u\|_{L^2} + C \sum_{|a| \leq 1} \|(r)^{-1/4} r^{-1/4} \tilde{\Omega}^a u\|_{L^2}
\]

\[
+ C \sum_{|a| \leq 1} \|(r)^{-1/4} r^{-5/4} \tilde{\Omega}^a u\|_{L^2}, \tag{2.8}
\]

\[
\|r (t-r) u\|_{L^\infty} \leq C \sum_{|a| \leq 1} \|(t-r) \nabla \tilde{\Omega}^a u\|_{L^2} + C \sum_{|a| \leq 2} \|(t-r) \tilde{\Omega}^a u\|_{L^2}, \tag{2.9}
\]

\[
\|r^{1/2} (t-r) u\|_{L^\infty} \leq C \sum_{|a| \leq 1} \|\tilde{\Omega}^a u\|_{L^2} + C \sum_{|a| \leq 1} \|(t-r) \nabla \tilde{\Omega}^a u\|_{L^2}, \tag{2.10}
\]

\[
\|(t-r) u\|_{L^\infty} \leq C \|\nabla u\|_{L^2} + C \|(t-r) \nabla^2 u\|_{L^2}. \tag{2.11}
\]

Proof. For (2.6), (2.7) and (2.9), see Lemma 3.3 of Sideris [40]. For (2.8), noting that

\[
\|r^{1/4} u\|_{L^\infty(r \leq 1)} \leq C \|r^{1/2} (r)^{-1/4} r^{-1/4} u\|_{L^\infty(\mathbb{R}^3)}, \tag{2.12}
\]

we can prove (2.8) by replacing \( u \) by \((r)^{-1/4} r^{-1/4} u \) in (2.7). (2.10) can be found in Lemma 4.1 of Hidano [9]. As for (2.11), see (2.13) of Hidano [11] and (37) of Zha [48].

Proposition 2.2. For radially symmetric \( u \in C^\infty_0(\mathbb{R}^3; \mathbb{R}^3) \), we have

\[
\|r \partial Z^a u\|_{L^\infty} \leq CE_3^{1/2}(u(t)), \ |a| \leq 1, \tag{2.13}
\]

\[
\|r^{1/2} Z^a u\|_{L^\infty} \leq CE_3^{1/2}(u(t)), \ |a| \leq 2, a_4 \leq 1, \tag{2.14}
\]

\[
\|r^{1/4} \nabla Z^a u\|_{L^\infty(r \leq 1)} \leq C \sum_{|b| \leq 1} \|(r)^{-1/4} r^{-1/4} \nabla^2 Z^b u\|_{L^2} + C \sum_{|b| \leq 1} \|(r)^{-1/4} r^{-1/4} \nabla Z^b u\|_{L^2}
\]

\[
+ C \sum_{|b| \leq 1} \|(r)^{-1/4} r^{-5/4} \nabla Z^b u\|_{L^2}, \tag{2.15}
\]

\[
\|r (t-r) \partial \nabla u\|_{L^\infty} \leq C \chi_3(u(t)), \tag{2.16}
\]

\[
\|r^{1/2}(t-r) \partial Z^a u\|_{L^\infty} \leq CE_2^{1/2}(u(t)) + C \chi_3(u(t)), \ |a| \leq 1, a_4 = 0, \tag{2.17}
\]

\[
\|(t-r) \nabla u\|_{L^\infty} \leq CE_2^{1/2}(u(t)) + C \chi_3(u(t)). \tag{2.18}
\]

Proof. (2.18) is a direct application of (2.11). Since \( u \) is radially symmetric, \( \tilde{\Omega} u = 0 \).

Noting the commutation relationship (1.19), we can derive (2.13)–(2.17) from (2.6)–(2.10), respectively.
2.3 Klainerman-Sideris bound

**Lemma 2.3.** Let radially symmetric \( u \in C^\infty([0, T); C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)) \). Then
\[
\mathcal{X}_3(u(t)) \leq CE_3^{1/2}(u(t)) + C t \sum_{|a| \leq 1} \| \Box \nabla^a u(t) \|_{L^2(\mathbb{R}^3)}.
\]
(2.19)

*Proof.* See Lemma 3.1 of Klainerman and Sideris [28]. □

**Proposition 2.3.** Let \( u \) be a radially symmetric and smooth solution to equation (1.34). Assume that
\[
\varepsilon_1 \equiv \sup_{0 \leq t \leq T} E_3^{1/2}(u(t))
\]
(2.20)
is sufficiently small. Then for \( 0 \leq t < T \), we have
\[
\mathcal{X}_3(u(t)) \leq CE_3^{1/2}(u(t)).
\]
(2.21)

*Proof.* By Lemma 2.3, we need to estimate the second term on the right-hand side of (2.19). For \( |a| \leq 1 \), we have
\[
\| \Box \nabla^a u(t) \|_{L^2} \leq C \sum_{b+c=a} \| N(\nabla^b u, \nabla^c u) \|_{L^2}.
\]
(2.22)
Noting (1.37), we only need to consider the case \( b = 0, c = a \). It is obvious that
\[
\| N(u, \nabla^a u) \|_{L^2} \leq C \| \nabla u \nabla^2 \nabla^a u \|_{L^2} + C \| \nabla^2 u \nabla \nabla^a u \|_{L^2}.
\]
(2.23)

We first consider on the region \( r \geq \langle t \rangle / 2 \). It follows from (2.13) that
\[
\| \nabla u \nabla^2 \nabla^a u \|_{L^2(r \geq \langle t \rangle / 2)} + \| \nabla^2 u \nabla \nabla^a u \|_{L^2(r \geq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| r \nabla u \nabla^2 \nabla^a u \|_{L^2(r \geq \langle t \rangle / 2)} + C\langle t \rangle^{-1} \| r \nabla^2 u \nabla \nabla^a u \|_{L^2(r \geq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| r \nabla u \|_{L^\infty} \| \nabla^2 \nabla^a u \|_{L^2} + C\langle t \rangle^{-1} \| r \nabla^2 u \|_{L^\infty} \| \nabla \nabla^a u \|_{L^2}.
\]
(2.24)

While on the region \( r \leq \langle t \rangle / 2 \), by (2.18) we have
\[
\| \nabla u \nabla^2 \nabla^a u \|_{L^2(r \leq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| (t - r) \nabla u \nabla^2 \nabla^a u \|_{L^2(r \leq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| (t - r) \nabla u \|_{L^\infty} \| \nabla^2 \nabla^a u \|_{L^2} \\
\leq C\langle t \rangle^{-1} (E_2^{1/2}(u(t)) + \mathcal{X}_3(u(t))) E_3^{1/2}(u(t)),
\]
(2.25)
and it follows from the Sobolev embedding \( H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3) \) and \( H^1(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \) that
\[
\| \nabla^2 u \nabla \nabla^a u \|_{L^2(r \leq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| (t - r) \nabla^2 u \nabla \nabla^a u \|_{L^2(r \leq \langle t \rangle / 2)} \\
\leq C\langle t \rangle^{-1} \| (t - r) \nabla^2 u \|_{L^6} \| \nabla \nabla^a u \|_{L^3} \\
\leq C\langle t \rangle^{-1} \| (t - r) \nabla^2 u \|_{L^1} \| \nabla \nabla^a u \|_{L^1} \\
\leq C\langle t \rangle^{-1} (E_2^{1/2}(u(t)) + \mathcal{X}_3(u(t))) E_3^{1/2}(u(t)).
\]
(2.26)

Hence it follows from (2.19), (2.20), (2.22)–(2.26) that
\[
\mathcal{X}_3(u(t)) \leq CE_3^{1/2}(u(t)) + CE_3(u(t)) + CE_3^{1/2}(u(t)) \mathcal{X}_3(u(t)) \\
\leq CE_3^{1/2}(u(t)) + C \varepsilon_1 E_3^{1/2}(u(t)) + C \varepsilon_1 \mathcal{X}_3(u(t)).
\]
(2.27)
If \( \varepsilon_1 \) is sufficiently small, we can get (2.21). □
2.4 KSS bound

Lemma 2.4. Consider the following perturbed linear wave operator

$$\square h u = \square u + Hu,$$  \hspace{1cm} (2.28)

where the perturbed term

$$(Hu)^i = \partial_t (h_{lm}^{ij}(t,x) \partial^j u), \quad i = 1, 2, 3.$$ \hspace{1cm} (2.29)

Assume that $h = (h_{lm}^{ij})$ satisfies the following symmetric condition

$$h_{lm}^{ij} = h_{ml}^{ji}$$ \hspace{1cm} (2.30)

and the smallness condition

$$|h| = \sum_{i,j,l,m=1}^{3} |h_{lm}^{ij}| \ll 1.$$ \hspace{1cm} (2.31)

Suppose that $u \in C^\infty([0,T); C^\infty_0(\mathbb{R}^3; \mathbb{R}^3))$. Then for any $0 < t < T$ we have

$$\left(\log(2 + t)\right)^{-1} \| \langle r \rangle^{-1/4} r^{-1/4} \partial u \|_{L^2_t L^2_x(S_t)}^2 + \left(\log(2 + t)\right)^{-1} \| \langle r \rangle^{-1/4} r^{-5/4} u \|_{L^2_t L^2_x(S_t)}^2$$

$$+ \| r^{-1/4} \partial u \|_{L^2(0,t; L^2(|x| \leq 1))}^2 + \| r^{-5/4} u \|_{L^2(0,t; L^2(|x| \leq 1))}^2$$

$$\leq C \| \partial u(0) \|_{L^2_t L^2_x(S_t)}^2 + C \| (|\nabla u| + \langle r \rangle^{-1/2} r^{-1/2} |u|) \partial h \|_{L^1_t L^1_x(S_t)}^1$$

$$+ C \| (|\partial h| + \langle r \rangle^{-1/2} r^{-1/2} |h|) |\nabla u| (|\nabla u| + r^{-1} |u|) \|_{L^1_t L^1_x(S_t)},$$  \hspace{1cm} (2.32)

where the strip $S_t = [0, t] \times \mathbb{R}^3$.

Proof. See Appendix A.

Proposition 2.4. Let $u$ be a radially symmetric and smooth solution to equation (1.34). Assume that

$$\varepsilon_1 = \sup_{0 \leq t < T} E_3^{1/2}(u(t))$$ \hspace{1cm} (2.33)

is sufficiently small. Then for $0 \leq t < T$, we have

$$\mathcal{M}_3^{1/2}(u(t)) \leq C \left(\log(2 + t)\right)^{1/2} (1 + t)^{\delta} \sup_{0 \leq \tau \leq t} E_3^{1/2}(u(\tau)),$$  \hspace{1cm} (2.34)

with $\delta = \varepsilon_1/2$.

Proof. By Lemma 2.1 and Lemma 2.4, we have that

$$\left(\log(2 + t)\right)^{-1} \mathcal{M}_3(u(t)) + e_3(u(t))$$

$$\leq C e_3(u(0)) + C \sum_{|a| \leq 2} \| (|\partial \nabla u| + \langle r \rangle^{-1/2} r^{-1/2} |\nabla u|) |\nabla Z^a u| + r^{-1} |Z^a u| \|_{L^1_t L^1_x(S_t)}$$

$$+ C \sum_{|a| \leq 2} \sum_{a_1 \leq 1} \sum_{b,c \neq a} \| (|\nabla Z^a u| + \langle r \rangle^{-1/2} r^{-1/2} |Z^a u|) N_d(Z^b u, Z^c u) \|_{L^1_t L^1_x(S_t)};$$  \hspace{1cm} (2.35)

$$\square h u = \square u + Hu,$$  \hspace{1cm} (2.28)

where the perturbed term

$$(Hu)^i = \partial_t (h_{lm}^{ij}(t,x) \partial^j u), \quad i = 1, 2, 3.$$ \hspace{1cm} (2.29)
We first estimate all the terms on the right-hand side of (2.35) on the region $r \geq 1/2$. Note that when $r \geq 1/2$, $(t) \leq Cr(t - r)$. For any $|a| \leq 2$, $a_4 \leq 1$, it follows from (2.16), (2.18), the Hardy inequality and Proposition 2.3 that

\[
\begin{align*}
&\left\| \left( (\partial \nabla u + (r)^{-1/2} \nabla u) | \nabla Z^a u | + r^{-1} | Z^a u | \right) \right\|_{L^1_1(r \geq 1/2)} \\
&\leq \left\| \left( (\partial \nabla u + r^{-1} | \nabla u |) | \nabla Z^a u | + r^{-1} | \nabla Z^a u | \right) \right\|_{L^1_1(r \geq 1/2)} \\
&\leq C(t)^{-1} \left\| (r(t - r)) | \partial \nabla u | + (t - r)^{-1} | \nabla u | (| \nabla Z^a u | + r^{-1} | \nabla Z^a u |) \right\|_{L^1_1(r \geq 1/2)} \\
&\leq C(t)^{-1} \left( (r(t - r)) | \partial \nabla u |_{L^\infty} + (t - r)^{-1} | \nabla u |_{L^\infty} \right) \| \nabla Z^a u \|_{L^2} (\| \nabla Z^a u \|_{L^2} + \| r^{-1} Z^a u \|_{L^2}) \\
&\leq C(t)^{-1} (\mathcal{A}_3(u(t))) + E_2^{1/2}(u(t)) \mathcal{E}_3(u(t)) \\
&\leq C(t)^{-1} E_3^{3/2}(u(t)). \tag{2.36}
\end{align*}
\]

For any $|a| \leq 2$, $a_4 \leq 1$, $b + c + d = a$, $b, c \neq a$, if $c_4 = 0$, by (2.13), the Hardy inequality and Proposition 2.3, we have that it holds that

\[
\begin{align*}
&\left\| \left( | \nabla Z^a u | + (r)^{-1/2} r^{-1/2} | Z^a u | \right) N_d(Z^b u, Z^c u) \right\|_{L^1_1(r \geq 1)} \\
&\leq C(t)^{-1} \left\| \left( | \nabla Z^a u | + r^{-1} | Z^a u | \right) r \nabla Z^b u (t - r)^{-1} | \nabla Z^c u | \right\|_{L^1_1(r \geq 1)} \\
&\leq C(t)^{-1} \left( \| \nabla Z^a u \|_{L^2} + \| r^{-1} Z^a u \|_{L^2} \right) \| r \nabla Z^b u \|_{L^\infty} \| (t - r)^{-1} \nabla Z^c u \|_{L^2} \\
&\leq C(t)^{-1} \mathcal{A}_3(u(t)) \mathcal{E}_3(u(t)) \\
&\leq C(t)^{-1} E_3^{3/2}(u(t)). \tag{2.37}
\end{align*}
\]

If $b_4 = 0$, we further need to split the region $r \geq 1/2$ into $1/2 \leq r \leq \langle t \rangle / 2$ and $r \geq \langle t \rangle / 2$. On the region $1/2 \leq r \leq \langle t \rangle / 2$, it follows from (2.17), the Hardy inequality and Proposition 2.3 that

\[
\begin{align*}
&\left\| \left( | \nabla Z^a u | + \langle r \rangle^{-1/2} r^{-1/2} | Z^a u | \right) N_d(Z^b u, Z^c u) \right\|_{L^1_1(1/2 \leq r \leq \langle t \rangle / 2)} \\
&\leq C(t)^{-1} \left\| \left( | \nabla Z^a u | + r^{-1} | Z^a u | \right) r^{1/2} (t - r) \nabla Z^b u \nabla^2 Z^c u \right\|_{L^1_1(1/2 \leq r \leq \langle t \rangle / 2)} \\
&\leq C(t)^{-1} \left( \| \nabla Z^a u \|_{L^2} + \| r^{-1} Z^a u \|_{L^2} \right) \| r^{1/2} (t - r) \nabla Z^b u \|_{L^\infty} \| \nabla^2 Z^c u \|_{L^2} \\
&\leq C(t)^{-1} \left( E_2^{1/2}(u(t)) + \mathcal{A}_3(u(t)) \right) \mathcal{E}_3(u(t)) \\
&\leq C(t)^{-1} E_3^{3/2}(u(t)). \tag{2.38}
\end{align*}
\]

And on the region $r \geq \langle t \rangle / 2$, it follows from (2.13), the Hardy inequality and Proposition 2.3 that

\[
\begin{align*}
&\left\| \left( | \nabla Z^a u | + \langle r \rangle^{-1/2} r^{-1/2} | Z^a u | \right) N_d(Z^b u, Z^c u) \right\|_{L^1_1(\langle t \rangle / 2 \leq r \leq \langle t \rangle / 2)} \\
&\leq C(t)^{-1} \left\| \left( | \nabla Z^a u | + r^{-1} | Z^a u | \right) r \nabla Z^b u \nabla^2 Z^c u \right\|_{L^1_1(\langle t \rangle / 2 \leq r \leq \langle t \rangle / 2)} \\
&\leq C(t)^{-1} \left( \| \nabla Z^a u \|_{L^2} + \| r^{-1} Z^a u \|_{L^2} \right) \| r \nabla Z^b u \|_{L^\infty} \| \nabla^2 Z^c u \|_{L^2} \\
&\leq C(t)^{-1} E_3^{3/2}(u(t)). \tag{2.39}
\end{align*}
\]

Now we will estimate the terms on the right-hand side of (2.35) on the region $r \leq 1/2$. \hfill \Box
(2.14) gives that for any $|a| \leq 2, a_4 \leq 1$,
\[
\left\|\left(\|\partial_t u\| + (r)^{-1/2} r^{-1/2} |\nabla u|\right) |\nabla Z^a u| \right\|_{L^1_t(x)} \leq \left\|\left(\|\partial_t u\| + (r)^{-1/2} r^{-1/2} |\nabla u|\right) |\nabla Z^a u| \right\|_{L^1_t(x)} \leq C\left\|\left(\|\partial_t u\| + (r)^{-1/2} r^{-1/2} |\nabla u|\right) |\nabla Z^a u| \right\|_{L^1_t(x)} \leq C\left\|\left(\|\partial_t u\| + (r)^{-1/2} r^{-1/2} |\nabla u|\right) |\nabla Z^a u| \right\|_{L^1_t(x)} \leq C\sup_{0 \leq \tau \leq t} E^{1/2}_3(u(\tau)) \mathcal{L}_3(u(t)).
\] (2.40)

Similarly, via (2.14) we also have that for any $|a| \leq 2, a_4 \leq 1, b + c = d = a, a, c \neq a$,
\[
\left\|\left(\|\nabla Z^a u| + (r)^{-1/2} r^{-1/2} |\nabla Z^a u| \right) N_\delta(Z^b u, Z^c u)\right\|_{L^1_t(x)} \leq \left\|\left(\|\nabla Z^a u| + (r)^{-1/2} r^{-1/2} |\nabla Z^a u| \right) N_\delta(Z^b u, Z^c u)\right\|_{L^1_t(x)} \leq \left\|\left(\|\nabla Z^a u| + (r)^{-1/2} r^{-1/2} |\nabla Z^a u| \right) N_\delta(Z^b u, Z^c u)\right\|_{L^1_t(x)} \leq \left\|\left(\|\nabla Z^a u| + (r)^{-1/2} r^{-1/2} |\nabla Z^a u| \right) N_\delta(Z^b u, Z^c u)\right\|_{L^1_t(x)} \leq C\sup_{0 \leq \tau \leq t} E^{1/2}_3(u(\tau)) \mathcal{L}_3(u(t)).
\] (2.41)

The above argument gives
\[
\left(\log(2 + t)\right)^{-1} \mathcal{M}_3(u(t)) + \mathcal{L}_3(u(t)) \leq CE_3(u(0)) + C \int_0^t (1 + \tau)^{-1} E^{3/2}_3(u(\tau)) d\tau + C \sup_{0 \leq \tau \leq t} E^{1/2}_3(u(\tau)) \mathcal{L}_3(u(t)) \leq CE_3(u(0)) + C \log(1 + t) \sup_{0 \leq \tau \leq t} E^{3/2}_3(u(\tau)) + C \sup_{0 \leq \tau \leq t} E^{1/2}_3(u(\tau)) \mathcal{L}_3(u(t)) \leq CE_3(u(0)) + C_{\varepsilon_1} \log(1 + t) \sup_{0 \leq \tau \leq t} E_3(u(\tau)) + C_{\varepsilon_1} \mathcal{L}_3(u(t)).
\] (2.42)

If $\varepsilon_1$ is sufficiently small, we have
\[
\left(\log(2 + t)\right)^{-1} \mathcal{M}_3(u(t)) + \mathcal{L}_3(u(t)) \leq CE_3(u(0)) + C_{\varepsilon_1} \log(1 + t) \sup_{0 \leq \tau \leq t} E_3(u(\tau)) \leq CE_3(u(0)) + C \log(1 + t)^{\varepsilon_1} \sup_{0 \leq \tau \leq t} E_3(u(\tau)) \leq CE_3(u(0)) + (1 + t)^{\varepsilon_1} \sup_{0 \leq \tau \leq t} E_3(u(\tau)) \leq C(1 + t)^{\varepsilon_1} \sup_{0 \leq \tau \leq t} E_3(u(\tau)),
\] (2.43)
which completes the proof of Proposition 2.4. \(\square\)

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, the key point is the following a priori estimate.

Proposition 3.1. There exist positive constants $\varepsilon_0$ and $A$ such that for any given $\varepsilon$ with $0 < \varepsilon \leq \varepsilon_0$, if the initial data is radially symmetric and satisfies (1.39) and $u$ is a smooth

\[ \text{...} \]
and radially symmetric solution to the Cauchy problem (1.24)–(1.28) with the null condition (1.27), then for any $T > 0$,
\[
\sup_{0 \leq t < T} E_3^{1/2}(u(t)) \leq A\varepsilon. \tag{3.1}
\]

Based on the method of proving Proposition 3.1, by some density argument and contraction-mapping argument, we can show Theorem 1.1. Because this procedure is routine and in order to keep the paper to a moderate length, we will omit it and refer the reader to [13] and [14].

Now we will prove Proposition 3.1. We only need to analyze the Cauchy problem (1.34)–(1.28) with the null condition (1.38). Assume that $u = u(t, x)$ is a smooth and radially symmetric solution of the Cauchy problem (1.34)–(1.28) on $[0, T)$. We will show that there exist positive constants $\varepsilon_0$ and $A$ such that for any $T > 0$, we have
\[
\sup_{0 \leq t < T} E_3^{1/2}(u(t)) \leq A\varepsilon \tag{3.1}
\]
under the assumption (1.39) and
\[
\sup_{0 \leq t < T} E_3^{1/2}(u(t)) \leq 2A\varepsilon, \tag{3.1}
\]
where $0 < \varepsilon \leq \varepsilon_0$.

3.1 General energy and ghost weight energy estimates

Following Alinhac [2, 3], we will use the ghost weight energy method. By Lemma 2.1, we have
\[
\sum_{|a| \leq 2, a_4 \leq 1} \int \langle e^{-q(\sigma)} \partial_t Z^a u, \Box Z^a u \rangle \, dx = \sum_{|a| \leq 2, b + c + d = a} \sum_{a_4 \leq 1} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_d(Z^b u, Z^c u) \rangle \, dx. \tag{3.2}
\]
As we are now treating a quasilinear system, for the right-hand side of (3.2), special attention should be paid on terms with $b = a$ or $c = a$ with $|a| = 2$. Noting (1.37), we can rewrite (3.2) as
\[
\sum_{|a| \leq 2, a_4 \leq 1} \int \langle e^{-q(\sigma)} \partial_t Z^a u, \Box Z^a u \rangle \, dx
\]
\[= 2 \sum_{|a| = 2, a_4 \leq 1} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N(u, Z^a u) \rangle \, dx + \sum_{|a| = 2, b + c + d = a} \sum_{a_4 \leq 1, b, c \neq a} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_d(Z^b u, Z^c u) \rangle \, dx
\]
\[+ \sum_{|a| \leq 1} \sum_{b + c + d = a} \int \langle e^{-q(\sigma)} \partial_t Z^a u, N_d(Z^b u, Z^c u) \rangle \, dx. \tag{3.3}
\]
For the left-hand side of (3.3), by integration by parts we have
\[
\int \langle e^{-q(\sigma)} \partial_t Z^a u, \Box Z^a u \rangle \, dx \]
\[= \frac{1}{2} \frac{d}{dt} \int e^{-q(\sigma)} |\partial_t Z^a u|^2 \, dx + \frac{1}{2} \int e^{-q(\sigma)} (t - r)^{-2} |TZ^a u|^2 \, dx. \tag{3.4}
\]
For the right-hand side of (3.3), it follows from the symmetric conditions (1.36) and the
integration by parts that
\[
\int (e^{-q(\sigma)} \partial_t Z^a u, N(u, Z^a u)) \, dx \\
= g^{ijk}_{lmn} \int e^{-q(\sigma)} \partial_t (Z^a u)^i \partial_j \partial_m u^j \partial_n (Z^a u)^k \, dx \\
= -g^{ijk}_{lmn} \int e^{-q(\sigma)} \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx - g^{ijk}_{lmn} \int e^{-q(\sigma)} q'(\sigma) \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx \\
= -\frac{1}{2} g^{ijk}_{lmn} \partial_t \int e^{-q(\sigma)} \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx + \frac{1}{2} g^{ijk}_{lmn} \int e^{-q(\sigma)} \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx \\
+ \frac{1}{2} g^{ijk}_{lmn} \int e^{-q(\sigma)} q'(\sigma) \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx - g^{ijk}_{lmn} \int e^{-q(\sigma)} q'(\sigma) T_l (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx.
\]

Define the perturbed energy
\[
\tilde{E}_3(u(t)) = \frac{1}{2} \sum_{|a| \leq 2} \int e^{-q(\sigma)} |\partial Z^a u|^2 \, dx + \sum_{|a| = 2} \sum_{a_4 \leq 1} g^{ijk}_{lmn} \int e^{-q(\sigma)} \partial_t (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx.
\]

Noting that \(|\partial u(t)||_{L^\infty} \leq CE_3^{1/2}(u(t))\), by (1.6), for small solutions we have
\[
(2c)^{-1} E_3(u(t)) \leq \tilde{E}_3(u(t)) \leq 2c E_3(u(t)).
\]

Returning to (3.3), by (3.4), (3.5) and (3.6), we have the following energy identity:
\[
\frac{d}{dt} \tilde{E}_3(u(t)) + E_3(u(t)) \\
= \sum_{|a| = 2} \int e^{-q(\sigma)} \tilde{N}(Z^a u, \partial_t u, Z^a u) \, dx + \sum_{|a| = 2} \sum_{a_4 \leq 1} g^{ijk}_{lmn} \int e^{-q(\sigma)} q'(\sigma) \tilde{N}(Z^a u, u, Z^a u) \, dx \\
- 2 \sum_{|a| = 2} \sum_{a_4 \leq 1} g^{ijk}_{lmn} \int e^{-q(\sigma)} q'(\sigma) T_l (Z^a u)^i \partial_m u^j \partial_n (Z^a u)^k \, dx \\
+ \sum_{|a| = 2} \sum_{a_4 \leq 1} \sum_{|b| + c + d = a} \sum_{|c, c| \leq 1} \int e^{-q(\sigma)} \langle \partial_t Z^a u, N_d(Z^b u, Z^c u) \rangle \, dx.
\]

Now we are ready to estimate all the terms on the right-hand side of (3.8). For this purpose, we will separate integrals over the regions \(r \geq \langle t \rangle / 2\) and \(r \leq \langle t \rangle / 2\), respectively. To get enough decay in time on the region \(r \geq \langle t \rangle / 2\), we need to exploit the null condition.

**On the region** \(r \geq \langle t \rangle / 2\). Now we consider all the terms on the right-hand side of (3.8) on the region \(r \geq \langle t \rangle / 2\). For the first term on the right-hand side of (3.8), for \(|a| = 2, a_4 \leq 1\), it follows from (2.3) that
\[
|\tilde{N}(Z^a u, \partial_t u, Z^a u)| \leq C \frac{\langle t \rangle}{r} (|\Omega Z^a u| |\nabla \partial_t u| |\nabla Z^a u| + |\nabla Z^a u| |\Omega \partial_t u| |\nabla Z^a u|).
\]

Noting (1.17), we have that
\[
\left\| e^{-q(\sigma)} \tilde{N}(Z^a u, \partial_t u, Z^a u) \right\|_{L^1(r \geq \langle t \rangle / 2)} \\
\leq C \| T Z^a u \nabla \partial_t u \nabla Z^a u \|_{L^1(r \geq \langle t \rangle / 2)} + C \langle t \rangle^{-\frac{1}{2}} \| \nabla Z^a u \partial_t u \nabla Z^a u \|_{L^1(r \geq \langle t \rangle / 2)}. 
\]
Via (2.16) and Proposition 2.3, we can get

\[\|TZ^a u \nabla \partial_t u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} \|(t - r)^{-1} T Z^a u \ r \langle t - r \rangle \nabla \partial_t u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} \|(t - r)^{-1} T Z^a u\|_{L^2} \|r \langle t - r \rangle \nabla u\|_{L^\infty} \|\nabla Z^a u\|_{L^2}\]
\[\leq C(t)^{-1} E_3^{1/2}(u(t)) \mathcal{X}_3(u(t)) E_3^{1/2}(u(t))\]

(3.11)

It follows from (2.13) that

\[\langle t \rangle^{-1} \|\nabla Z^a u \partial_t u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-2} \|\nabla Z^a u \ r \partial_t u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-2} \|\nabla Z^a u\|_{L^2} \|r \partial_t u\|_{L^\infty} \|\nabla Z^a u\|_{L^2}\]
\[\leq C(t)^{-2} E_3^{3/2}(u(t)).\]

(3.12)

So it follows from (3.10)–(3.12) that

\[\sum_{|a| = 2, \ a_1 \leq 1} \|e^{-q(\sigma)} \tilde{N}(Z^a u, \partial_t u, Z^a u)\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} E_3(u(t)) E_3^{1/2}(u(t)) + C(t)^{-2} E_3^{3/2}(u(t)).\]

(3.13)

Similarly to the first term, for the second term on the right-hand side of (3.8), for

\[|a| = 2, \ a_4 \leq 1\], we have

\[\|e^{-q(\sigma)} q'(\sigma) \tilde{N}(Z^a u, u, Z^a u)\|_{L^1(\tau \geq t)/2}\]
\[\leq C\|q'(\sigma) T Z^a u \nabla u \nabla Z^a u\|_{L^1(\tau \geq t)/2} + C(t)^{-1} \|q'(\sigma) \nabla Z^a u u \nabla Z^a u\|_{L^1(\tau \geq t)/2}.\]

(3.14)

For the first term on the right-hand side of (3.14), by (2.13) we have

\[\|q'(\sigma) T Z^a u \nabla u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} \|(t - r)^{-1} T Z^a u \ r \nabla u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} \|(t - r)^{-1} T Z^a u\|_{L^2} \|r \nabla u\|_{L^\infty} \|\nabla Z^a u\|_{L^2}\]
\[\leq C(t)^{-1} E_3(u(t)) E_3^{1/2}(u(t)).\]

(3.15)

For the second term on the right-hand side of (3.14), by (2.14) we have

\[\langle t \rangle^{-1} \|q'(\sigma) \nabla Z^a u u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-3/2} \|(t - r)^{-2} \nabla Z^a u \ r^{1/2} u \nabla Z^a u\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-3/2} \|\nabla Z^a u\|_{L^2} \|r^{1/2} u\|_{L^\infty} \|\nabla Z^a u\|_{L^2}\]
\[\leq C(t)^{-3/2} E_3^{3/2}(u(t)).\]

(3.16)

It follows from (3.14)–(3.16)

\[\sum_{|a| = 2, \ a_4 \leq 1} \|e^{-q(\sigma)} q'(\sigma) \tilde{N}(Z^a u, u, Z^a u)\|_{L^1(\tau \geq t)/2}\]
\[\leq C(t)^{-1} E_3(u(t)) E_3^{1/2}(u(t)) + C(t)^{-3/2} E_3^{3/2}(u(t)).\]

(3.17)
The third term on the right-hand side of (3.8) can be estimated by the same way as (3.15).

For the fourth term on the right-hand side of (3.8), for any \(|a| \leq 2, a_4 \leq 1, b + c + d = a, |b|, |c| \leq 1\), by Lemma 2.1, (2.2), (1.18), (1.19) and noting that \(\tilde{\Omega}u = 0\) we have

\[
\frac{1}{r} |\partial_t Z^a u| \left( |\nabla \Omega Z^b u||\nabla Z^c u| + |\nabla \Omega Z^c u||\nabla Z^b u| + |\Omega Z^b u||\nabla Z^c u| + |\Omega Z^c u||\nabla Z^b u| \right)
\leq C \frac{1}{r} |\partial_t Z^a u| \left( |\nabla Z^b u||\nabla Z^c u| + |Z^b u||\nabla Z^c u| + |Z^c u||\nabla Z^b u| \right).
\]

(3.18)

So it follows from (2.13) and (2.14) that

\[
\sum_{|a| \leq 2 \text{ and } a_4 \leq 1} \sum_{|b|, |c| \leq 1} \left\| e^{-q(r)} \langle \partial_t Z^a u, N_{\omega}(Z^b u, Z^c u) \rangle \right\|_{L^1(r \geq \langle t \rangle/2)}
\leq C \langle t \rangle^{-3/2} \sum_{|a| \leq 2 \text{ and } a_4 \leq 1} \sum_{|b|, |c| \leq 1} \frac{1}{r} |\partial_t Z^a u| \left( |\nabla Z^b u| \|\nabla Z^c u| \right. \\
\left. + |Z^b u| \|\nabla Z^c u| + |Z^c u| \|\nabla Z^b u| \right)
\leq C \langle t \rangle^{-3/2} E_3^{3/2} (u(t)).
\]

(3.19)

Hence by the above argument, we know that on the region \(r \geq \langle t \rangle/2\), the right-hand side of (3.8) admits an upper bound of the form

\[
\langle t \rangle^{-3/2} E_3^{3/2} (u(t)) + \langle t \rangle^{-1} E_3 (u(t)) E_3^{1/2} (u(t)).
\]

(3.20)

**On the region** \(1/2 \leq r \leq \langle t \rangle/2\). On the region \(1/2 \leq r \leq \langle t \rangle/2\), all terms on the right-hand side of (3.8) are bounded above by

\[
\sum_{|a| = 2 \text{ and } a_4 \leq 1} \| \langle t - r \rangle^{-2} \nabla Z^a u \nabla u \nabla Z^a u \|_{L^1(1/2 \leq r \leq \langle t \rangle/2)} + \sum_{|a| = 2 \text{ and } a_4 \leq 1} \| \nabla Z^a u \partial \nabla u \nabla Z^a u \|_{L^1(1/2 \leq r \leq \langle t \rangle/2)}
\]

\[
+ \sum_{|a| \leq 2 \text{ and } a_4 \leq 1} \sum_{|b|, |c| \leq 1} \| \partial_t Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(1/2 \leq r \leq \langle t \rangle/2)}.
\]

(3.21)

For the first term on the right-hand side of (3.21), for \(|a| = 2, a_4 \leq 1\), it follows from the Sobolev embedding \(H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)\) that

\[
\| \langle t - r \rangle^{-2} \nabla Z^a u \nabla u \nabla Z^a u \|_{L^1(1/2 \leq r \leq \langle t \rangle/2)}
\leq C \langle t \rangle^{-2} \| \nabla Z^a u \nabla Z^a u \|_{L^1(1/2 \leq r \leq \langle t \rangle/2)}
\leq C \langle t \rangle^{-2} \| \nabla Z^a u \|_{L^2}^2 \| \nabla u \|_{L^\infty}
\leq C \langle t \rangle^{-2} E_3^{3/2} (u(t)).
\]

(3.22)

For the second term on the right-hand side of (3.21), for \(|a| = 2, a_4 \leq 1\), by (2.17) and
Proposition 2.3 we have
\[ \| \nabla Z^a u \nabla \nabla Z^a u \|_{L^1(1/2 \leq r \leq (t)/2)} \leq C(t)^{-1} \| \nabla Z^a u \rightangle r^{1/2} (t - r) \nabla u (r) \nabla \nabla Z^a u \|_{L^1(1/2 \leq r \leq (t)/2)} \]
\[ \leq C(t)^{-1} \| \nabla Z^a u \|_{L^2} \| r^{1/2} (t - r) \partial \nabla u \|_{L^\infty} \| (t - r)^{-1/2} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E^2_3 (u(t)) \left( E^2_4 (u(t)) + \lambda_3 (u(t)) \right) \| (t - r)^{-1/2} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E^3_3 (u(t)) \| (t - r)^{-1/4} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E_3 (u(t)) \lambda_3^{1/2} (u(t)). \] (3.23)

As for the third term on the right-hand side of (3.21), for \( |a| \leq 2, a_4 \leq 1, b + c \leq a, \ |b|, \ |c| \leq 1 \), if \( b_4 = 0 \), by (2.17) and Proposition 2.3 we have
\[ \| \partial \nabla Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(1/2 \leq r \leq (t)/2)} \]
\[ \leq C(t)^{-1} \| \nabla Z^a u \rightangle r^{1/2} (t - r) \nabla Z^b u \rightangle r^{-1/2} \partial \nabla Z^a u \|_{L^1(1/2 \leq r \leq (t)/2)} \]
\[ \leq C(t)^{-1} \| \nabla Z^a u \|_{L^2} \| r^{1/2} (t - r) \nabla Z^b u \|_{L^\infty} \| (t - r)^{-1/2} \partial \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E^2_3 (u(t)) \left( E^2_4 (u(t)) + \lambda_3 (u(t)) \right) \| (t - r)^{-1/2} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E_3 (u(t)) \| (t - r)^{-1/4} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E_3 (u(t)) \lambda_3^{1/2} (u(t)). \] (3.24)

If \( c_4 = 0 \), it follows from (2.14) and Proposition 2.3 that
\[ \| \partial \nabla Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(1/2 \leq r \leq (t)/2)} \]
\[ \leq C(t)^{-1} \| (t - r) \nabla Z^a u \rightangle r^{1/2} \nabla Z^b u \rightangle r^{-1/2} \partial \nabla Z^a u \|_{L^1(1/2 \leq r \leq (t)/2)} \]
\[ \leq C(t)^{-1} \| (t - r) \nabla Z^a u \|_{L^2} \| r^{1/2} \nabla Z^b u \|_{L^\infty} \| (t - r)^{-1/2} \partial \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} \lambda_3 (u(t)) E^2_3 (u(t)) \| (t - r)^{-1/2} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E_3 (u(t)) \| (t - r)^{-1/4} \nabla Z^a u \|_{L^2} \]
\[ \leq C(t)^{-1} E_3 (u(t)) \lambda_3^{1/2} (u(t)). \] (3.25)

Hence by the above argument, we know that on the region \( 1/2 \leq r \leq (t)/2 \), the right-hand side of (3.8) admits an upper bound of the form
\[ (t)^{-2} E^3_3 (u(t)) + \| (t - r)^{1/2} \nabla Z^a u \|_{L^1(1/2 \leq r \leq (t)/2)}. \] (3.26)

**On the region** \( r \leq 1/2 \). Similarly to (3.21), on the region \( r \leq 1/2 \), all terms on the right-hand side of (3.8) are bounded above by
\[
\sum_{|a|=2 \atop a_4 \leq 1} |a| = 2 \sum_{a_4 \leq 1} \| \nabla Z^a u \nabla \nabla Z^a u \|_{L^1(1 \leq r \leq 1/2)} + \sum_{|a|=2 \atop a_4 \leq 1} \| \nabla Z^a u \partial \nabla u \nabla Z^a u \|_{L^1(1 \leq r \leq 1/2)}
\]
\[ + \sum_{|a|=2 \atop b + c \leq a \atop a_4 \leq 1 \ |b|, \ |c| \leq 1} \| \partial \nabla Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(1 \leq r \leq 1/2)}. \] (3.27)

The first term on the right-hand side of (3.27) can be estimated by the same way as
(2.22). For \( |a| = 2, a_4 \leq 1 \), we have
\[
\| \langle t - r \rangle^{-2} \nabla Z^a u \nabla u \nabla Z^a u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-2} \| \nabla Z^a u \nabla u \nabla Z^a u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-2} \| \nabla Z^a u \|^2_{L^2} \| \nabla u \|_{L^\infty} \\
\leq C \langle t \rangle^{-2} E_3^{3/2}(u(t)).
\]
(3.28)

For the second term on the right-hand side of (3.27), for \( |a| = 2, a_4 \leq 1 \), by (2.17) and Proposition 2.3 we have
\[
\| \nabla Z^a u \partial \nabla u \nabla Z^a u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \nabla Z^a u \ | t - r \rangle \partial \nabla u \ | r^{-1/4} \nabla Z^a u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \nabla Z^a u \|^2_{L^2(r \leq 1/2)} \| r^{1/2} \partial \nabla u \|_{L^\infty} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \nabla Z^a u \|^2_{L^2(r \leq 1/2)} (E_2^{1/2}(u(t)) + \kappa_3(u(t))) \\
\leq C \langle t \rangle^{-1} E_3^{1/2}(u(t)) \| \langle t - r \rangle^{-1/4} r^{-1/4} \nabla Z^a u \|^2_{L^2} \\
\leq C \langle t \rangle^{-1} E_3^{1/2}(u(t)) \kappa_3(u(t)).
\]
(3.29)

For the third term on the right-hand side of (3.27), for \( |a| = 2, a_4 \leq 1, b + c \leq a, |b|, |c| \leq 1 \), if \( b_4 = 0 \), by (2.17) and Proposition 2.3 we have
\[
\| \partial Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \partial Z^a u \ | t - r \rangle \nabla Z^b u \ | r^{-1/4} \nabla Z^c u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \partial Z^a u \|^2_{L^2(r \leq 1/2)} \| r^{1/2} \nabla Z^b u \|_{L^\infty} \| \langle t - r \rangle^{-1/4} \nabla Z^c u \|_{L^2} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \partial Z^a u \|^2_{L^2(r \leq 1/2)} (E_2^{1/2}(u(t)) + \kappa_3(u(t))) \| r^{-1/4} \nabla Z^b u \|_{L^2} \| \langle t - r \rangle^{-1/4} \nabla Z^c u \|_{L^2} \\
\leq C \langle t \rangle^{-1} E_3^{1/2}(u(t)) \kappa_3(u(t)).
\]
(3.30)

If \( c_4 = 0 \), by (2.15) and Proposition 2.3 we have
\[
\| \partial Z^a u \nabla Z^b u \nabla Z^c u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \partial Z^a u \ | t - r \rangle \nabla Z^b u \ | \langle t - r \rangle^{-1/4} \nabla Z^c u \|_{L^1(r \leq 1/2)} \\
\leq C \langle t \rangle^{-1} \| r^{-1/4} \partial Z^a u \|^2_{L^2(r \leq 1/2)} \| r^{1/2} \nabla Z^b u \|_{L^\infty} \| \langle t - r \rangle^{-1/4} \nabla Z^c u \|_{L^2} \\
\leq C \langle t \rangle^{-1} \kappa_3(u(t)) \| r^{-1/4} \partial Z^a u \|^2_{L^2(r \leq 1/2)} \| \langle t - r \rangle^{-1/4} \nabla Z^b u \|_{L^2} \\
\| \langle t \rangle^{-1/4} r^{-1/4} \nabla^2 Z^b u \|_{L^2} + \| \langle t \rangle^{-1/4} r^{-1/4} \nabla Z^b u \|_{L^2} + \| \langle t \rangle^{-1/4} r^{-3/4} \nabla Z^b u \|_{L^2} \\
\leq C \langle t \rangle^{-1} E_3^{1/2}(u(t)) \| \langle t \rangle^{-1/4} r^{-1/4} \partial Z^a u \|_{L^2} \\
\| \langle t \rangle^{-1/4} r^{-1/4} \nabla^2 Z^b u \|_{L^2} + \| \langle t \rangle^{-1/4} r^{-1/4} \nabla Z^b u \|_{L^2} + \| \langle t \rangle^{-1/4} r^{-3/4} \nabla Z^b u \|_{L^2} \\
\leq C \langle t \rangle^{-1} E_3^{1/2}(u(t)) \kappa_3(u(t)).
\]
(3.31)

Hence by the above argument, we know that on the region \( r \leq 1/2 \), the right-hand side of (3.8) admits an upper bound of the form
\[
\langle t \rangle^{-2} E_3^{3/2}(u(t)) + \langle t \rangle^{-1} E_3^{1/2}(u(t)) \kappa_3(u(t)).
\]
(3.32)
3.2 Conclusion of the proof

From the above three parts argument, we have that

\[
\frac{d}{dt} \tilde{E}_3(u(t)) + E_3(u(t)) \\
\leq C(t)^{-1}E_3(u(t))E_3^{1/2}(u(t)) + C(t)^{-3/2}E_3^{3/2}(u(t)) \\
+ C(t)^{-1}E_3(u(t))N_3^{1/2}(u(t)) + C(t)^{-1}E_3^{1/2}(u(t))N_3(u(t)) \\
\leq \frac{1}{4}E_3(u(t)) + C(t)^{-2}E_3^2(u(t)) + C(t)^{-3/2}E_3^{3/2}(u(t)) \\
+ C(t)^{-1}E_3(u(t))N_3^{1/2}(u(t)) + C(t)^{-1}E_3^{1/2}(u(t))N_3(u(t)).
\] (3.33)

The first term on the right-hand side of (3.33) can be absorbed to the left-hand side. So we have

\[
\frac{d}{dt} \tilde{E}_3(u(t)) \\
\leq C(t)^{-3/2}E_3^{3/2}(u(t)) + C(t)^{-1}E_3(u(t))N_3^{1/2}(u(t)) + C(t)^{-1}E_3^{1/2}(u(t))N_3(u(t)).
\] (3.34)

Integrating on time from 0 to \(T\) and noting (3.7), we have that

\[
\sup_{0 \leq t < T} E_3(u(t)) \\
\leq CE_3(u(0)) + C \int_0^T \langle t \rangle^{-3/2}E_3^{3/2}(u(t))dt \\
+ C \int_0^T \langle t \rangle^{-1}E_3(u(t))N_3^{1/2}(u(t))dt \\
+ C \int_0^T \langle t \rangle^{-1}E_3^{1/2}(u(t))N_3(u(t))dt.
\] (3.35)

It is obvious that

\[
\int_0^T \langle t \rangle^{-3/2}E_3^{3/2}(u(t))dt \leq \int_0^{+\infty} \langle t \rangle^{-3/2}dt \sup_{0 \leq t < T} E_3^{3/2}(u(t)) \leq C \sup_{0 \leq t < T} E_3^{3/2}(u(t)).
\] (3.36)

In order to treat the last two terms on the right-hand side of (3.35), we will employ the KSS type energy \(\mathcal{M}_3(u(t))\) and the dyadic decomposition technique in time (see page 363 of [43]).

Without loss of generality, we can assume \(T > 1\). It follows from the Hölder inequality and Proposition 2.4 that

\[
\int_0^1 \langle t \rangle^{-1}E_3(u(t))N_3^{1/2}(u(t))dt \\
\leq C \sup_{0 \leq t \leq 1} E_3(u(t)) \left( \int_0^1 N_3(u(t))dt \right)^{1/2} \\
\leq C \sup_{0 \leq t \leq 1} E_3(u(t)) \mathcal{M}_3^{1/2}(u(1)) \\
\leq C \sup_{0 \leq t < T} E_3^{3/2}(u(t)).
\] (3.37)

Take an integer \(N\) such that \(2^N < T \leq 2^{N+1}\). By the Hölder inequality and Proposition 2.4,
we have
\[
\int_1^T \langle t \rangle^{-1} E_3(u(t)) N_3^{1/2}(u(t)) dt \\
\leq \sum_{k=0}^{N} 2^{-k} \int_{2k}^{2k+1} N_3^{1/2}(u(t)) dt \sup_{0 \leq t < T} E_3(u(t)) \\
\leq \sum_{k=0}^{N} 2^{-k/2} \left( \int_{2k}^{2k+1} N_3(u(t)) dt \right)^{1/2} \sup_{0 \leq t < T} E_3(u(t)) \\
\leq C \sum_{k=0}^{N} 2^{-k/2} M_3^{1/2}(u(2^{k+1})) \sup_{0 \leq t < T} E_3(u(t)) \\
\leq C \sum_{k=0}^{N} 2^{-k/2} 2^{\delta k} (\log(2 + 2^{k+1}))^{1/2} \sup_{0 \leq t < T} E_3^{3/2}(u(t)). \tag{3.38}
\]

Noting that \( \delta \) is sufficiently small, we have
\[
\sum_{k=0}^{N} 2^{-k/2} 2^{\delta k} (\log(2 + 2^{k+1}))^{1/2} \leq \sum_{k=0}^{+\infty} 2^{-k/2} 2^{\delta k} (\log(2 + 2^{k+1}))^{1/2} \leq C. \tag{3.39}
\]
The combination of (3.37)–(3.39) gives
\[
\int_0^T \langle t \rangle^{-1} E_3(u(t)) N_3^{1/2}(u(t)) dt \leq C \sup_{0 \leq t < T} E_3^{3/2}(u(t)). \tag{3.40}
\]

By Proposition 2.4, we have
\[
\int_0^1 \langle t \rangle^{-1} E_3^{1/2}(u(t)) N_3(u(t)) dt \\
\leq C \sup_{0 \leq t \leq 1} E_3^{1/2}(u(t)) \int_0^1 N_3(u(t)) dt \\
\leq C \sup_{0 \leq t \leq 1} E_3^{1/2}(u(t)) M_3(u(1)) \\
\leq C \sup_{0 \leq t < T} E_3^{3/2}(u(t)). \tag{3.41}
\]

Similarly to (3.38), it follows from Proposition 2.4 that
\[
\int_1^T \langle t \rangle^{-1} E_3^{1/2}(u(t)) N_3(u(t)) dt dt \\
\leq \sum_{k=0}^{N} 2^{-k} \int_{2k}^{2k+1} N_3(u(t)) dt \sup_{0 \leq t < T} E_3^{1/2}(u(t)) \\
\leq C \sum_{k=0}^{N} 2^{-k} M_3(u(2^{k+1})) \sup_{0 \leq t < T} E_3^{1/2}(u(t)) \\
\leq C \sum_{k=0}^{N} 2^{-k} 2^{\delta k} (\log(2 + 2^{k+1})) \sup_{0 \leq t < T} E_3^{3/2}(u(t)). \tag{3.42}
\]

Noting that \( \delta \) is sufficiently small, we have
\[
\sum_{k=0}^{N} 2^{-k} 2^{\delta k} (\log(2 + 2^{k+1})) \leq \sum_{k=0}^{+\infty} 2^{-k} 2^{\delta k} (\log(2 + 2^{k+1})) \leq C. \tag{3.43}
\]
The combination of (3.41)–(3.43) gives
\[ \int_0^T (t)^{-1} E_3^{1/2}(u(t)) N_3(u(t)) dt \leq C \sup_{0 \leq t < T} E_3^{3/2}(u(t)). \] (3.44)

It follows from (3.35), (3.36), (3.40) and (3.44) that
\[ \sup_{0 \leq t < T} E_3(u(t)) \leq CE_3(u(0)) + C_2 \sup_{0 \leq t < T} E_3^{3/2}(u(t)) \leq C_1 \varepsilon^2 + 8C_2 A^3 \varepsilon^3. \] (3.45)

Take \( A^2 = 2C_1 \) and \( \varepsilon_0 \) so small that \( 16C_2 A \varepsilon_0 \leq 1. \) (3.46)

Then for any \( \varepsilon \) with \( 0 < \varepsilon \leq \varepsilon_0 \) we have
\[ \sup_{0 \leq t < T} E_k^{1/2}(u(t)) \leq A \varepsilon, \] (3.47)
which completes the proof of Proposition 3.1.

Appendix A Proof of Lemma 2.4

The proof of Lemma 2.4 repeats essentially the same arguments as in the preceding papers [44, 35, 13, 49, and 15]. It therefore suffices to give only the sketch of the proof. In the following, we use the notation \( M := f(r)\partial_r + (1/r)f(r) \) for a function \( f(r) \) to be given later. Proceeding as in (2.12)-(2.15) of [15], we can establish the differential identity for \( \mathbb{R}^3 \)-valued functions \( u = (u^1, u^2, u^3) \)
\[ \langle Mu, \Box u \rangle = \partial_t e + \nabla \cdot \hat{p} + \hat{q}, \] (A.1)
where, by \( \langle \cdot, \cdot \rangle \) we mean the inner product in \( \mathbb{R}^3 \), and
\[ e = f(r)(\partial_t u^i)(\partial_r u^i + \frac{1}{r} u^i), \] (A.2)
\[ \hat{p} = \frac{1}{2} f(r)(|\nabla u|^2 - |\partial_t u|^2) \omega - f(r)(\partial_r u^i + \frac{1}{r} u^i) \nabla u^i + \frac{rf'(r) - f(r)}{2r^2} |u|^2 \omega, \] (A.3)
\[ \hat{q} = \frac{f'(r)}{2} |\partial_t u|^2 + \frac{f'(r)}{2} |\partial_r u|^2 + \left( \frac{f(r)}{r} - \frac{f'(r)}{2} \right) |\nabla \omega u|^2 - \frac{1}{2} \left( \Delta \frac{f(r)}{r} \right) |u|^2, \] (A.4)
and \( \omega \in S^2 \). Recall that repeated indices are summed, regardless of their position up or down. Also, as in (2.21) of [15], we get
\[ \langle Mu, Hu \rangle = \nabla \cdot \hat{p} + \hat{q}, \] (A.5)
where
\[
\tilde{p}_l = f(r)\omega_k h^{ij}_{lm}(\partial_k u^i)(\partial_m u^j) - \frac{1}{2} f(r)\omega_k h^{ij}_{km}(\partial_k u^i)(\partial_m u^j) \\
+ \frac{1}{r} f(r)h^{ij}_{lm}u^l\partial_m u^j, \quad l = 1, 2, 3, \tag{A.6}
\]
\[
\tilde{q} = -\frac{r f'(r) - f(r)}{r}\omega_k h^{ij}_{lm}(\partial_k u^i)(\partial_m u^j) + \frac{1}{2} f'(r)h^{ij}_{lm}(\partial_l u^i)(\partial_m u^j) \\
+ \frac{1}{2} f(r)\omega_k (\partial_k h^{ij}_{lm})(\partial_l u^i)(\partial_m u^j) - \frac{1}{r} f(r)h^{ij}_{lm}(\partial_l u^i)(\partial_m u^j) \\
- \frac{r f'(r) - f(r)}{r^2} \omega_l h^{ij}_{lm}u^l\partial_m u^j. \tag{A.7}
\]

First, we take as in [13] and [15]
\[
f(r) = \left(\frac{r}{1 + r}\right)^{1/2}. \tag{A.8}
\]

Proceeding as in (2.37)-(2.42) of [13], we can obtain
\[
\|r^{-1/4}\partial u\|_{L^2(0, t; L^2(\{x| \leq 1\}))} + \|r^{-5/4} u\|_{L^2(0, t; L^2(\{x| \leq 1\}))} \\
\leq C \sup_{0 \leq \tau \leq t} \int_{\mathbb{R}^3} \|e\|dx + C \int_0^t \int_{\mathbb{R}^3} |\tilde{q}|dx d\tau + C \int_0^t \int_{\mathbb{R}^3} |\langle M u, \Box u\rangle|dx d\tau \\
\leq C\|\partial u(0, \cdot)\|_{L^2(\mathbb{R}^3)} + C \int_0^t \int_{\mathbb{R}^3} |\nabla h| |\nabla u|^2 dx d\tau \\
+ C \int_0^t \int_{\mathbb{R}^3} \left(\frac{|h|}{r^{1/2}(1 + r)^{1/2}} \left(\frac{|u|}{r} + |\nabla u|\right) |\nabla u|\right) dx d\tau \\
+ C \int_0^t \int_{\mathbb{R}^3} \left(|\nabla u| + \frac{|u|}{r^{1/2}(1 + r)^{1/2}}\right) |\Box u| dx d\tau. \tag{A.9}
\]

Here, we have also used the fact that thanks to the symmetry and smallness conditions (2.30)-(2.31), the standard energy inequality
\[
\sup_{0 \leq \tau \leq t} \|\partial u(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \\
\leq C\|\partial u(0, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + C \int_0^t \int_{\mathbb{R}^3} |\partial_h| |\nabla u|^2 dx d\tau + C \int_0^t \int_{\mathbb{R}^3} |\partial_l u||\Box u| dx d\tau \tag{A.10}
\]
holds. We note that owing to the Hardy inequality, (A.10) also yields
\[
\|r^{-1/4}\partial u\|_{L^2(0, t; L^2(\{x| \geq 1\}))} + \|r^{-3/2} u\|_{L^2(0, t; L^2(\{x| \geq 1\}))} \\
\leq C \sup_{0 \leq \tau \leq t} \|\partial u(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}^2 \\
\leq C\|\partial u(0, \cdot)\|_{L^2(\mathbb{R}^3)}^2 + C \int_0^t \int_{\mathbb{R}^3} |\partial_h| |\nabla u|^2 dx d\tau + C \int_0^t \int_{\mathbb{R}^3} |\partial_l u||\Box u| dx d\tau. \tag{A.11}
\]

It remains to consider the weighted $L^2$ estimate over $[0, t] \times \{x \in \mathbb{R}^3 : 1 < |x| < t\}$ with $t > 1$. We go back to (A.8), and take as in [44], [35], and [49]
\[
f(r) = \frac{r}{r + \rho}, \quad \rho \geq 1. \tag{A.12}
\]
Proceeding as in (2.52), (2.53), and (2.57) of [49], we can obtain

\[
\int_0^t \int_{\mathbb{R}^3} \left( \frac{\rho}{(r + \rho)^2} |\partial_r u|^2 + \frac{\rho}{(r + \rho)^2} |\partial_r u|^2 + \frac{2r + \rho}{r(r + \rho)^2} |\partial_t u|^2 + \frac{\rho}{r(r + \rho)^3} |u|^2 \right) dx \, dt \\
\leq C \| \partial u(0, \cdot) \|_{L^2(\mathbb{R}^3)}^2 \\
+ C \int_0^t \int_{\mathbb{R}^3} \left( |\partial h| + \frac{|h|}{r + 1} \right) |\nabla u| \left( |\nabla u| + \frac{|u|}{r + 1} \right) dx \, dt \\
+ C \int_0^t \int_{\mathbb{R}^3} \left( |\nabla u| + \frac{|u|}{r + 1} \right) |\Box u| dx \, dt.
\]  
(A.13)

Here all the constants \( C \) on the right-hand side above are independent of \( \rho \geq 1 \). It is now a routine practice (see, e.g, (2.70)-(2.71) of [49]) to obtain from (A.13)

\[
(\log(2 + t))^{-1} \left( \| (r)^{-1/2} \partial u \|_{L^2(t;L^2(1 < |x| < t)))}^2 + \| (r)^{-3/2} u \|_{L^2(t;L^2(1 < |x| < t)))}^2 \right) \\
\leq C \| \partial u(0, \cdot) \|_{L^2(\mathbb{R}^3)}^2 \\
+ C \int_0^t \int_{\mathbb{R}^3} \left( |\partial h| + \frac{|h|}{r + 1} \right) |\nabla u| \left( |\nabla u| + \frac{|u|}{r + 1} \right) dx \, dt \\
+ C \int_0^t \int_{\mathbb{R}^3} \left( |\nabla u| + \frac{|u|}{r + 1} \right) |\Box u| dx \, dt.
\]  
(A.14)

Combining (A.9), (A.11), and (A.14), we have finished the proof of Lemma 2.4.

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References

[1] R. Agemi, Global existence of nonlinear elastic waves, Invent. Math. 142 (2000) 225–250.

[2] S. Alinhac, The null condition for quasilinear wave equations in two space dimensions I, Invent. Math. 145 (2001) 597–618.

[3] S. Alinhac, Geometric analysis of hyperbolic differential equations: an introduction, London Mathematical Society Lecture Note Series, vol. 374, Cambridge University Press, Cambridge, 2010.

[4] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, Comm. Pure Appl. Math. 39 (1986) 267–282.

[5] P. G. Ciarlet, Mathematical elasticity. Vol. I: Three-dimensional elasticity, Studies in Mathematics and its Applications, vol. 20, North-Holland Publishing Co., Amsterdam, 1988.
[6] B. Ettinger, H. Lindblad, A sharp counterexample to local existence of low regularity solutions to Einstein equations in wave coordinates, Ann. of Math. (2) 185 (2017) 311–330.

[7] Z. Guo, Y. Wang, Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations, J. Anal. Math. 124 (2014) 1–38.

[8] M. E. Gurtin, Topics in finite elasticity, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 35, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa., 1981.

[9] K. Hidano, An elementary proof of global or almost global existence for quasi-linear wave equations, Tohoku Math. J. (2) 56 (2004) 271–287.

[10] K. Hidano, Small solutions to semi-linear wave equations with radial data of critical regularity, Rev. Mat. Iberoam. 25 (2009) 693–708.

[11] K. Hidano, Regularity and lifespan of small solutions to systems of quasi-linear wave equations with multiple speeds, I: almost global existence, arXiv:1610.04824v2. Accepted for publication in RIMS Kôkyûroku Bessatsu. (2016).

[12] K. Hidano, J. Jiang, S. Lee, C. Wang, Weighted fractional chain rule and nonlinear wave equations with minimal regularity, arXiv:1605.06748v2 (2017).

[13] K. Hidano, C. Wang, K. Yokoyama, On almost global existence and local well posedness for some 3-D quasi-linear wave equations, Adv. Differential Equations 17 (2012) 267–306.

[14] K. Hidano, K. Yokoyama, Space-time $L^2$-estimates and life span of the Klainerman-Machedon radial solutions to some semi-linear wave equations, Differential Integral Equations 19 (2006) 961–980.

[15] K. Hidano, D. Zha, Space-time $L^2$ estimates, regularity and almost global existence for elastic waves, arXiv:1710.05180 (2017).

[16] L. Hörmander, Lectures on nonlinear hyperbolic differential equations, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 26, Springer-Verlag, Berlin, 1997.

[17] T. J. R. Hughes, T. Kato, J. E. Marsden, Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity, Arch. Rational Mech. Anal. 63 (1976) 273–294.

[18] J.-C. Jiang, C. Wang, X. Yu, Generalized and weighted Strichartz estimates, Commun. Pure Appl. Anal. 11 (2012) 1723–1752.

[19] S. Jiang, R. Racke, Evolution equations in thermoelasticity, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 112, Chapman & Hall/CRC, Boca Raton, FL, 2000.
[20] F. John, *Formation of singularities in elastic waves*, in *Trends and applications of pure mathematics to mechanics (Palaiseau, 1983)*, Lecture Notes in Phys., vol. 195, Springer, Berlin, 1984, 194–210.

[21] F. John, *Almost global existence of elastic waves of finite amplitude arising from small initial disturbances*, Comm. Pure Appl. Math. 41 (1988) 615–666.

[22] M. Keel, H. F. Smith, C. D. Sogge, *Almost global existence for some semilinear wave equations*, J. Anal. Math. 87 (2002) 265–279.

[23] M. Keel, H. F. Smith, C. D. Sogge, *Almost global existence for quasilinear wave equations in three space dimensions*, J. Amer. Math. Soc. 17 (2004) 109–153 (electronic).

[24] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, in *Nonlinear systems of partial differential equations in applied mathematics, Part 1 (Santa Fe, N.M., 1984)*, Lectures in Appl. Math., vol. 23, Amer. Math. Soc., Providence, RI, 1986, 293–326.

[25] S. Klainerman, *On the work and legacy of Fritz John, 1934–1991*, Comm. Pure Appl. Math. 51 (1998) 991–1017.

[26] S. Klainerman, M. Machedon, *Space-time estimates for null forms and the local existence theorem*, Comm. Pure Appl. Math. 46 (1993) 1221–1268.

[27] S. Klainerman, I. Rodnianski, *Rough solutions of the Einstein-vacuum equations*, Ann. of Math. (2) 161 (2005) 1143–1193.

[28] S. Klainerman, T. C. Sideris, *On almost global existence for nonrelativistic wave equations in 3D*, Comm. Pure Appl. Math. 49 (1996) 307–321.

[29] H. Kubo, *Lower bounds for the lifespan of solutions to nonlinear wave equations in elasticity*, in *Evolution equations of hyperbolic and Schrödinger type*, Progr. Math., vol. 301, Birkhäuser/Springer Basel AG, Basel, 2012, 187–212.

[30] H. Lindblad, *Counterexamples to local existence for semi-linear wave equations*, Amer. J. Math. 118 (1996) 1–16.

[31] H. Lindblad, *Counterexamples to local existence for quasilinear wave equations*, Math. Res. Lett. 5 (1998) 605–622.

[32] H. Lindblad, C. D. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations*, J. Funct. Anal. 130 (1995) 357–426.

[33] S. Machihara, M. Nakamura, K. Nakanishi, T. Ozawa, *Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation*, J. Funct. Anal. 219 (2005) 1–20.

[34] J. Metcalfe, *Elastic waves in exterior domains. I. Almost global existence*, Int. Math. Res. Not. (2006) Art. ID 69826, 41.

[35] J. Metcalfe, C. D. Sogge, *Long-time existence of quasilinear wave equations exterior to star-shaped obstacles via energy methods*, SIAM J. Math. Anal. 38 (2006) 188–209.
[36] J. Metcalfe, B. Thomases, *Elastic waves in exterior domains. II. Global existence with a null structure*, Int. Math. Res. Not. IMRN (2007) Art. ID rnm034, 43.

[37] E. Y. Ovcharov, *Radial Strichartz estimates with application to the 2-D Dirac-Klein-Gordon system*, Comm. Partial Differential Equations 37 (2012) 1754–1788.

[38] G. Ponce, T. C. Sideris, *Local regularity of nonlinear wave equations in three space dimensions*, Comm. Partial Differential Equations 18 (1993) 169–177.

[39] T. C. Sideris, *The null condition and global existence of nonlinear elastic waves*, Invent. Math. 123 (1996) 323–342.

[40] T. C. Sideris, *Nonresonance and global existence of prestressed nonlinear elastic waves*, Ann. of Math. (2) 151 (2000) 849–874.

[41] T. C. Sideris, S.-Y. Tu, *Global existence for systems of nonlinear wave equations in 3D with multiple speeds*, SIAM J. Math. Anal. 33 (2001) 477–488 (electronic).

[42] H. F. Smith, D. Tataru, *Sharp local well-posedness results for the nonlinear wave equation*, Ann. of Math. (2) 162 (2005) 291–366.

[43] C. D. Sogge, *Global existence for nonlinear wave equations with multiple speeds*, in *Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001)*, Contemp. Math., vol. 320, Amer. Math. Soc., Providence, RI, 2003, 353–366.

[44] J. Sterbenz, *Angular regularity and Strichartz estimates for the wave equation, with an appendix by Igor Rodnianski*, Int. Math. Res. Not. (2005) 187–231.

[45] Q. Wang, *A geometric approach for sharp local well-posedness of quasilinear wave equations*, Ann. PDE 3 (2017) Art. 12, 108.

[46] D. Zha, *On nonlinear elastic waves in the radial symmetry in 2-D: exterior problem*, preprint (2016).

[47] D. Zha, *Remarks on nonlinear elastic waves in the radial symmetry in 2-D*, Discrete and Continuous Dynamical Systems–Series A 36 (2016) 4051–4062.

[48] D. Zha, *Some remarks on quasilinear wave equations with null condition in 3-D*, Math. Methods Appl. Sci. 39 (2016) 4484–4495.

[49] D. Zha, *Space–time $L^2$ estimates for elastic waves and applications*, J. Differential Equations 263 (2017) 1947–1965.

[50] Y. Zhou, Z. Lei, *Global low regularity solutions of quasi-linear wave equations*, Adv. Differential Equations 13 (2008) 55–104.