Productivity within the ETAS seismicity model

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SUMMARY
The productivity of a magnitude $m$ event can be characterized in term of triggered events of magnitude above $m - \Delta$: it is the number of direct ‘descendants’ $\nu_\Delta$ and the number of all ‘descendants’ $V_\Delta$. There is evidence in favor of the discrete exponential distribution for both $\nu_\Delta$ and $V_\Delta$ with a dominant magnitude $m$. We have modified the ETAS model to have any distribution of $\nu_\Delta$. It turned out that the branching structure of the model excludes the possibility of having exponential distributions for both productivity characteristics at once. We have analytically investigated the features of the $V_\Delta$ distribution within a wide class of ETAS models. We show the fundamental difference in tail behavior of the $V_\Delta$-distributions for general-type clusters and clusters with a dominant initial magnitude: the tail is heavy in the former case and light in the latter. The real data demonstrate the possibilities of this kind. This result can be considered as a weaker version of the $V_\Delta$-distribution law.

Keywords: Statistical seismology; ETAS model; productivity.

1 INTRODUCTION
Let $V_\Delta$ denote the number of seismic events of magnitude $m \geq m_0 - \Delta$ caused by an earthquake of magnitude $m_0 \geq m_c + \Delta$, where $m_c$ is the lower magnitude threshold and $\Delta$ a positive default value. $V_\Delta$ is further referred to as the total $\Delta$-productivity.
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Considering an earthquake cluster as the result of a branching process [Zaliapin et al., 2008], we can also deal with the directly triggered events of $m_0$, i.e. the first generation events of an $m_0$-event. The number $\nu_\Delta$ of such events is in the same magnitude range $m \geq m_0 - \Delta$ and is later named $\Delta$-productivity of $m_0$.

The distributions of quantities $V_\Delta$ and $\nu_\Delta$ have recently become the object of active analysis in the literature. According to Baranov and Shebalin [2019], Shebalin et al. [2020], Shebalin and Baranov [2021], the $\nu_\Delta$-distribution $f(n) = P(\nu_\Delta = n)$ is universal in shape, namely, it is geometric (or discrete exponential):

$$f_G(n) = p^n(1-p)$$

(1)

in which $p$ does not depend on the magnitude of triggering event. This result is relevant in statistical seismology because the popular Epidemic Type Aftershock sequence (ETAS) model is based on the Poisson distribution

$$f_P(n) = e^{-\lambda} \lambda^n / n!$$

(2)

an assumption that has never been tested.

Similarly, the geometric distribution (1) was adopted for statistics $V_\Delta$, provided that it refers to clusters with a dominant initial magnitude $m_0$, that is, to aftershocks. For the first time this hypothesis appeared in the work by Solovyev and Solovyeva [1962]; it was based on relatively scarce data (1954-1961) for the Pacific Belt ($m_o = 7; \Delta = 2$) and for the Kamchatka & Kuril Islands region ($m_o = 6; \Delta = 2$). Shebalin et al. [2018] expanded the analysis and confirmed the hypothesis using 850 aftershock sequences with main shocks $m_o = 6.5; \Delta = 2$ from the ANSS(1975-2018) catalog. In the same time, Kagan [2010], using aftershocks $m \geq 4.7$ from PDE (1977-2007) catalog for $m_o = 7.1 - 7.2$, found that the empirical distribution of $V_\Delta$ is bimodal with the dominant peak at $n = 0$.

Zaliapin and Ben-Zion (2016) gave a thorough analysis of earthquake clustering and showed its spatial dependence and connection with heat flow production. They showed that on the global scale the total $\Delta$-productivity distribution (for clusters with $\max m \geq 6$ and $\Delta = 2$) looks like a power-law for locations with high heat flow ($H > 0.2 W / m^2$) and as a log-normal distribution otherwise. These conclusions are qualitative because they are based on log-log plots. Nevertheless, the data for the regions of high heat flow differ significantly from the geometric hypothesis (1) for aftershocks. The same should be expected for any cluster with
max \( m \leq 5 \). In this case, according to Zaliapin and Ben-Zion (2016), the cluster size distribution in zones with both high and low heat flow is a power type distribution, \( f(n) \propto n^{-\kappa-1}, \kappa = 2.5 \).

The purpose of this note is twofold: first, we consider an extension of the ETAS model that allow any \( \nu_\lambda \) - distribution; and second, working within the framework of the extended model, we are going to prove analytically the fundamental difference between the \( V_\lambda \) - distributions for general-type clusters and clusters with a dominant initial magnitude.

2 ETAS MODELS

Let us consider the seismic events \( x = (t, g, m) \) occurred in the time-location-magnitude space \( S = (t \geq 0, g \in \mathbb{R}^2, m \geq 0) \). The lower magnitude threshold \( m_\lambda \) is taken here as the zero reference point without loss of generality. According to the branching structure of the ETAS model, an earthquake cluster can be represented as follows (see e.g. Saichev and Sornette [2017]). The initial event \( x_0 = (t_0, g_0, m_0) \) generates a random number \( \nu(m_0) \) of events \( \{x_i, i = 1, ..., \nu(m_0)\} \) where \( \nu(m_0) \) follows a special distribution \( F \) and the characteristic \( \lambda(m_0) = E\nu(m_0) \). The offspring events are distributed in \( S \) as independent variables with the probability density

\[
f(x \mid x_0) = f_1(m) f_2(t-t_0) f_3(g-g_0 \mid m_0).
\]

Each new event independently generates its own family of events following the generation law described above, and so on. If the mean of the direct triggered events is \( \Lambda_1 = E\lambda(m) < 1 \), then the resulting cluster is almost surely finite.

The components of the regular ETAS model are specified as follows:

\begin{align*}
\nu(m) & \text{ distribution} \quad \text{Poission} \\
\lambda(m) & = \lambda e^{-\alpha m}, m \geq 0 \quad \text{a copy of the Utsu law for aftershocks} \\
f_1(m) & = \beta \exp(-\beta m), m \geq 0 ; \beta > \alpha \quad \text{the Gutenberg-Richter law} \\
f_2(t) & = (p-1)(t/c+1)^{-\beta}/c \quad \text{the time scattering (a copy of the Omori law for aftershocks)} \\
f_3(g \mid m) & = (q-1)[(g / \psi_m)^2 + 1]^{-\gamma}/(\pi \psi_m^2) \quad \text{one of the laws of scattering in space,} \\
\psi_m & = de^{\gamma m/2} \quad \text{the scale parameter.}
\end{align*}

A theoretical analysis of various aspects of the regular ETAS model is contained in the works of Saichev and co-authors. In particular, they have proved the power law behavior of the
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cluster size distribution \( f_c(n) \propto n^{-1-\kappa} \) with \( \kappa \in [0.5,1] \) (Saichev et al, 2005; Baró, 2020). Note that, according to Zaliapin and Ben-Zion (2016), \( \kappa \in [1,2.5] \) for real data.

Hereinafter, the ETAS model with F distribution of \( \nu(m) \) will be designated ETAS(F). In particular, we set F=P for Poisson, and F=G for Geometric distribution. The characteristic \( \lambda(m) = E \nu(m) \) will be common to all models. The relation of \( \lambda(m) \) with the parameter \( p(m) \) of the geometric distribution is determined by the equality \( \lambda = p/(1 - p) \), i.e.

\[
p(m) = \frac{\lambda(m)}{1 + \lambda(m)}
\]

Usually the regular ETAS model is defined in terms of the conditional intensity of events

\[
\lambda'(x|\tau_i : t_i < t) = \mu(g) f_1(m) + \sum_i \lambda(m_i) f(x|x_i),
\]

where \( x = (t,g,m) \) and \( \mu(g) \) is the rate of background seismicity (Ogata, 1998). The above definition is broader because it allows for any \( \nu(m) \) distribution. The dual definition (5) for ETAS(F) is possible but looks differently.

**Statement 1.** Consider a model in which background events form a Poisson field of rate \( \mu(g) f_1(m) \). Each of these events independently generates an ETAS(F) type cluster with the components \( \lambda(m) \) and (3).

If \( q(z) = Ez^{\nu(m)} \) is a generating function of \( \nu(m) \), then the conditional rate of events is

\[
\lambda'(x|\tau_i : t_i < t) = \mu(g) f_1(m) + \sum_i R(m_i,t-t_i) \lambda(m_i) f(x|x_i)
\]

where

\[
\lambda(m) R(m,\tau) = - (d/dz) \ln q(z(\tau)), \quad z(\tau) = \int_t^\infty f_2(s)ds.
\]

In particular,

\[
R(m,\tau) = \begin{cases} 1 & F=P \\ \frac{1}{(1+\lambda(m) \int_0^\infty f_2(s)ds)^{-1}} & F=G \end{cases}.
\]

**Remark.** Let \( x_i \) be a strong event distant in time, \( t_i << t \) and \( \lambda(m_i) >> 1 \). The structure of the \( R(m,\tau) \) factor shows that the contribution of \( x_i \) to the occurrence of new event at the current time \( t \) is noticeably weaker in the ETAS(G) model than in the regular one, F=P.

The proof of the statement and all subsequent ones are contained in the Appendix.

To provide the G-distribution of \( \Delta \)-productivity for any \( \Delta \), it is natural to consider the Geometric distribution of \( \nu(m) \). It follows from the following statement.
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**Statement 2.** Let \( \nu(m_0) \) have the distribution \( F=G \) or \( P \). Then the \( \Delta \)-productivity of the magnitude \( m_0 \approx \Delta \) preserves the distributions type of \( \nu(m_0) \). In both cases the statistics \( \nu_\Delta(m_0) \) and \( \nu(m_0) \) differ only in average values:

\[
\lambda_\Delta(m_0) = \lambda(m_0) \bar{F}_i(m_0 - \Delta) / \bar{F}_i(\Delta)
\]

(6)

where \( \bar{F}_i(m) = \int_0^m f_i(u) du, \bar{F}_i(0) = 1 \).

Let \( a = f_i(0) > 0 \) and \( \lambda_\Delta(m_0) \) be independent of \( m_0 \) in the area \( 0 < \Delta \leq m_0 < M \). Then \( \lambda(m) = ce^{am} \) and \( \bar{F}_i(m) = e^{-am}, m \in (0, M) \).

**Remark.** The conditions for the independence of the \( \nu_\Delta(m_0) \)-distribution from \( m_0 \) are very strict, and therefore, in contrast with Shebalin et al. (2020), this property should be unstable. By combining \( \nu_\Delta(m_0), m_0 \in \delta M \) observations given \( \lambda_\Delta(m_0) \neq \text{const.} \), we get a mixture of \( \nu_\Delta(m_0) \) distributions with different values of the \( \lambda_\Delta(m_0) \) characteristic. The operation of averaging distributions, \( \int f(n|p)\sigma(dp) \), preserves the monotony of the geometric distribution (1) but not its type. This circumstance makes it difficult to prove the universality of the type of \( \nu_\Delta(m_0) \) distribution.

To get the geometric distribution of \( \nu_\Delta \) in framework of ETAS model, Shebalin et al. (2020) used randomization of the parameter \( \lambda(m_0) \). They used the fact that a Poisson distribution with an exponentially distributed random parameter is equivalent to a geometric distribution. The resulting ETAS* model is identical to ETAS(G) model in which the average productivity does not depend on the initial magnitude, namely \( \lambda(m_0) = \Lambda \). Because of (6), in this case \( \lambda_\Delta(m_0) \) becomes dependent on \( m_0 \) for any \( f_i(m) > 0 \).

The ETAS* cluster model may be interesting due to its connection with the classical branching Galton-Watson model (Harris, 1963). Both models are identical if we omit the space-time component in ETAS*. This gives us an exact formula for the distribution of the total productivity of any initial event (cluster size not including the initial event):

\[
P(V = n) = \frac{\Gamma(1/2 + n)}{\Gamma(2 + n) \times (4pq)^n} (q/\sqrt{\pi})^n, \quad q = 1 - p
\]

(7)

and the following asymptotic approximation:

\[
P(V = n) \asymp n^{-3/2} \exp(-C_1n), \quad n \gg 1,
\]

(8)

Where \( \Gamma(x) \) is the Gamma function, \( p \) is the parameter of the geometric distribution, \( C_1 = -\ln(4pq) \geq 0 \).
It's easy to see that the distribution (7) is a strictly decreasing function with a light tail. At the stage of empirical analysis, such distributions are easily perceived as exponential.

It is well known that the asymptotic (8) is typical for the Galton-Watson models (Harris, 1963) and, consequently, for any ETAS(F) model with \( \lambda(m) = \text{const} \).

The light tail of the \( V \) distribution in (8) belongs to a non-aftershock type cluster. This contrasts with the typical heavy tails of the \( V_\Delta \) distribution in the regular ETAS model. The explanation of this fact will follow from the condition \( \lambda(m) \leq \text{const} \).

### 3 THE MAIN RESULT

We will consider the ETAS(F) model in a subcritical regime: \( \Lambda_i = E\lambda(m) < 1 \); the productivity distribution \( F \) may be Poisson (\( F=P \)) or Geometric (\( F=G \)), for which \( Ev(m) = \lambda(m) \) is a non-decreasing, finite, positive function; the other components: \( f_i(m), f_2(t), f_3(g|m) \) are free from additional restriction.

As above \( m_0 > 0 \) is the magnitude of an initial cluster event and \( V_\Delta (m_0) \) is the total \( \Delta \)-productivity of \( m_0 \). Any initial event can be fixed or random, that is, have the \( f_i(m) \) distribution. A further distinction concerns the dominant initial events, i.e. those initial events having maximum magnitude in their cluster. The case of a random \( m_0 \) is important because identifying the initial cluster event is an unstable practical procedure.

**Statement 3.** Under the specified conditions, the following is true:

a) if \( m_0 \) is fixed or random, the number \( \mathcal{N}(V_\Delta) \) of finite moments of \( V_\Delta (m_0) \) coincides with the number \( \mathcal{N}(\lambda) \) of finite integrals \( \int \lambda^n (m) f_i(m) dm < \infty, n = 1,2, \ldots \) :

\[
\mathcal{N}(V_\Delta) = \mathcal{N}(\lambda). \tag{9}
\]

b) if \( m_0 \) is fixed and dominant, then \( \mathcal{N}(V_\Delta) = \infty \) regardless of the \( \mathcal{N}(\lambda) \)-value. This is also true for the random dominant \( m_0 \)-event if

\[
\lambda(m) \int_0^\Delta f_i(m-u) du \leq C. \tag{10}
\]

**Statement 4.** Let \( m_0 \) is a dominant initial magnitude in the ETAS(G) cluster and \( f_i(m) \) is strictly positive. Then the statistics \( V_\Delta (m_0) \) can have a geometric distribution for no more than one value \( \Delta \in (0,m_0) \).
Remarks:
Statement 3 allows us to judge on the tail attenuation of the $V_\lambda$ distribution because by the Chebyshev-Markov inequality, $P(\xi \geq u) \leq E\xi^n / u^n$ for any random variable $\xi \geq 0$. Distributions having all moments will be called distributions with light tails. Otherwise, we will talk about heavy tails.

Let’s consider clusters with fixed or random initial magnitude. The conditions of the Statement 3 are easily verified for the regular ETAS components: $\lambda(m) = \lambda e^{\alpha m}, \alpha > 0$ and $f_i(m) = \beta \exp(-\beta m), \beta > \alpha$. In this case we have $\mathcal{N}(\lambda) < \beta / \alpha$. This means that the total $\Delta-$productivity distribution has only a heavy tail. The data of Zalapin and Ben-Zion (2016) demonstrate the possibilities of this kind.

The case $\alpha = 0$ corresponds to the ETAS* model; here $\mathcal{N}(\lambda) = \infty$ in accordance with (7).

Given $\alpha \leq \beta$, the condition (10) is satisfied. This guarantees the light tails in the distribution of the total $\Delta-$productivity of main shocks. All the observations mentioned above support this fact.

The example of Kagan mentioned earlier shows that the distribution of $V_\lambda$ for aftershocks does not necessarily have to be exponential. Moreover, according to Statement 4 in the framework of the ETAS(F) model, the geometric laws for $\nu_\lambda$ and $V_\lambda$ in the case of aftershocks are incompatible. Visually, a monotone distribution with a light tail is perceived as exponential. Therefore, a more thorough statistical analysis is required here.

The difference in the tail behavior of $V_\lambda$ statistics can be partially explained as follows. The total $\Delta-$productivity of the main shock with a fixed magnitude $m_0$ is based on events in the finite range of magnitudes $(0,m_0)$. In this case, the frequency magnitude law $f_i(m)$ is replaced by their truncated counterpart in the interval $(0,m_0)$ where all moments of $\lambda(m)$ are finite. By virtue of (9), the tail of the $V_\lambda$ distribution becomes typical for the case $\mathcal{N}(\lambda) = \infty$ that is quasi exponential. The similar effect for a random dominant magnitude is not so obvious and is a non-trivial analytical fact.

Numerical examples
To support the theoretical conclusions, we simulated a synthetic earthquake catalog from the regular ETAS(P) model. To mimic the features of real seismicity, the set of ETAS parameters was mainly derived from the analysis of the earthquake catalog of Northeastern Italy (Benali et al., 2020). The simulation region also corresponds to that area, namely: Lon 11.5-14.0 and Lat 45.5-47.0.
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The model was based on the following parameters:

- the uniform background seismicity with the intensity $\mu = 0.04 \text{ (events/ (degree}^2 \times \text{day})$, which roughly corresponds to the rate of background events identified in the earthquake catalog of Northeastern Italy. The uniformity makes it easier the declustering process, as it allows to better identify the clustered component;

- magnitude range $m>2$; the Gutenberg -Richter parameter $\beta = 2.07$ which corresponds to the b-value 0.9;

- the Utsu Law parameters: $\lambda = 0.22 (\text{event/day}), \alpha = 1.54$. The estimated $\lambda = 0.67$ for Northeastern Italy was reduced to 0.22 to satisfy the stability conditions $\Lambda_1 = E\lambda(m) < 1$;

- the Omori law parameters: $c = 0.015 (\text{day}), p = 1.037$. The estimated $p$ parameter very close to 1 implies an extremely long tail of $f_2(t)$, so that offspring events may span several thousands years (Harte, 2013); this practically hinders the analysis of cluster size for synthetic catalogs. However, we recall that, from the theoretical viewpoint, the temporal component of the ETAS model does not affect cluster size. Therefore, in order to prevent this inconvenience, instead of increasing the $p$ parameter, we considered a truncated distribution $f_2(t)$ on the support $t < 0.7$ years, which corresponds to the 95% quantile of the cluster lifetime for North- eastern Italy;

- the space scattering law parameters: $d = 0.0085 (\text{degree}); q = 2.25, \gamma = 0.62$.

The synthetic earthquake catalog, consisting of 143568 events, spans about 500 years and includes information about the links between each event and its direct descendants. The synthetic time series were processed by the Nearest-Neighbor method (Zaliapin & Ben-Zion 2013) to identify clusters of events. This method allows partitioning earthquakes into background and clustered components, based on nearest-neighbor distances between earthquakes in the space–time–magnitude domain (Baiesi & Paczuski 2004; Zaliapin et al. 2008). In this study, the parameters necessary for the computation of nearest neighbor distances, namely the b-value and the fractal dimension of epicenters $d_f$, were set according to the values estimated from the analysis of Northeastern Italy catalog (Benali et al. 2020, and references therein): $b$–value $= 0.9$ and $d_f = 1.1$. The threshold of the nearest neighbor distance, which separates the clustered and background components, was automatically set ($\log_{10}(\eta_0) = -4.2$) following a criterion based on a one-dimensional Gaussian mixture model with two modes, where the threshold is the maximum likelihood boundary between the two modes (Zaliapin & Ben-Zion, 2013). The total $\Delta$-productivity distribution was examined considering the clusters extracted from the synthetic earthquake catalog.
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Fig. 1 represents the total $\Delta$–productivity distributions ($\Delta = 2$) a) for arbitrary initial events and b) for main shocks, as identified by nearest-neighbor method applied to the synthetic ETAS($P$) seismicity. In the first case, the distribution $V_\Delta$ demonstrates a heavy tail, which is also confirmed by the coefficient of variation $\sigma(V_\Delta)/EV_\Delta = 10.8$. On the contrary, the distribution $V_\Delta$ for main shocks (Fig1b) demonstrates the light tail behavior; for comparison with the previous case $\sigma(V_\Delta)/EV_\Delta = 0.7$.

Fig.2 shows similar results based on the clusters originally defined by the branching structure of the synthetic ETAS($P$) catalog. The coefficients of variation are also consistent with previous results: (a) $\sigma(V_\Delta)/EV_\Delta = 9.7$ for arbitrary initial events and (b) $\sigma(V_\Delta)/EV_\Delta = 0.8$ for main shocks.
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(a) The empirical data: \( N = 612 \) \( EV_\Delta = 39.8 \) \( \sigma(V_\Delta) = 431.3 \) \( \max V_\Delta = 10079 \)

![Graph showing N vs \log_{10}(N-Cluster) for the empirical data.]

(b) The empirical data: \( N = 1074 \) \( EV_\Delta = 9.2 \) \( \sigma(V_\Delta) = 6.5 \) \( \max V_\Delta = 52 \)

![Graph showing N vs N-Cluster for the empirical data.]

**Figure 1.** Clusters identified by nearest-neighbor method applied to the synthetic (ETAS(\( P \))) seismicity: \( V_\Delta, \Delta = 2 \) distributions for a) arbitrary initial events, \( m_0 \geq 4 \) and b) main shocks, \( m_0 \geq 4 \).
(a) The empirical data: $N = 481 \ E V_{\Delta} = 46.5 \ \sigma(V_{\Delta}) = 452.3 \ \text{max} V_{\Delta} = 9470$

(b) The empirical data: $N = 1074 \ E V_{\Delta} = 9.9 \ \sigma(V_{\Delta}) = 7.9 \ \text{max} V_{\Delta} = 56$

Figure 2. Clusters in the synthetic (ETAS($P$)) seismicity: $V_{\Delta}, \Delta = 2$ distributions for a) arbitrary initial events, $m_0 \geq 4$ and b) main shocks, $m_0 \geq 4$. 
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4 CONCLUSION

The concept of the $\Delta$-productivity $\nu_\Delta$ is associated with a priori ideas about the structure of a cluster of seismic events in the form of a random branching tree. Such a representation is not the only possible one. Therefore, the Geometric distribution of $\nu_\Delta$ can be considered as a useful alternative within the framework of the ETAS model. The other characteristic of the $\Delta$-productivity $V_\Delta$ for main shocks is more objective and stable.

For large class of the ETAS models, we answered the question when the total $\Delta$ productivity distribution has light tails for main shocks and heavy ones for arbitrary events. The real data demonstrate the possibilities of this kind.

Note the difficulties of substantiating the mentioned geometric laws for $\nu_\Delta$ and aftershock’s $V_\Delta$. They are associated both with the declusterization of seismicity and with the effect of averaging the distributions of $\nu_\Delta$ and $V_\Delta$ when grouping data.

The first difficulty can be judged by experiments of Zaliapin and Ben-Zion (2013) with the Nearest -Neighbor method. This method adapted to the ETAS structure has 40% error of incorrectly determining the parent of an event. The stability of the conclusions in this case may depend on the strict similarity in the real hierarchical structure of the seismic cluster.

The second difficulty is expressed in the fact that when averaging exponential distributions, their monotony is preserved, but not their type. This is unavoidable if the distribution parameter is dependent on location or the magnitude of the initial event (see (6)). Therefore, we can speak more confidently about the light tail of the $V_\Delta$- distribution for the main shocks.

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APPENDIX: proofs

Statement 1

Let \( q(z) = E z^{\nu(m)} \) be a generating function of \( \nu(m) \) variable in the ETAS(F) model and \( \Omega_t = \{ x_i : t_i < t \} \) is the past of the process up to the moment \( t \). The latter means that the first descendant, \( y_j \), of \( x_i \) is such that its time \( t(y_j) \geq t \). Let \( P_t(t) = \Pr(t(y_j) \geq t) \). By definition of the ETAS(F) model,

\[
P_t(t) := P(t(y_j) \geq t) = \sum_{m \geq 0} \Pr(\nu(m) = n) \int_{t-x_i}^{x} f_2(s) ds = q(F_z(t-t_j)).
\]

Suppose that the first event \( y \) after the moment \( t \) is contained in the \( \delta x = \delta \times \delta g \times \delta m \) neighborhood of the point \( x = (t, g, m) \). Then

\[
P(y \in \delta x|\Omega_t) = P(\cup_i \{ y_i \in \delta x \}|\Omega_t)
\]

\[
= \sum_i P\{ y_i \in \delta x|\Omega_t \} + o(\delta x) = \sum_i P\{ y_i \in \delta x \}|(y_i) \geq t \} + o(\delta x)
\]

where \( |\delta x| \) is the neighborhood volume. Here we used the independence of \( y_i \) from \( (x_j, y_j), j \neq i \), and \( P\{ y_i \in \delta x, y_j \in \delta x|\Omega_t \} = o(\delta x) \). Note that

\[
\Pr(y_i \in \delta x|t(y_i) \geq t) = \left[ - \delta P_t(t)/P(t) \right] f_1(m) \delta m f_1(g-g_i|m_i) \delta g
\]

As a result, the conditional rate to have a descendant at moment \( t \) given \( \Omega_t = \{ x_i : t_i < t \} \) is

\[
\lambda(x|\Omega_t) = -\sum d/dt \ln P_t(t) \times f_1(m) f_2(g-g_i|m_i) .
\]

Background and cluster events are independent. The contribution of the Poisson background events to the conditional rate is determined by the term \( \mu(g) f_1(m) \). Finally, the conditional rate will have the form

\[
\dot{\lambda}(x|\Omega(t)) = \mu(g) f_1(m) + \sum d/dz \ln q(z(t-t_i)) f_2(t-t_i) f_1(m) f_3(g-g_i|m_i)
\]

where \( z = \int_{t}^{s} f_2(s) ds \).

It remains to take into account that \( \ln q(z) = \lambda(m)(z-1) \) for the Poisson case and \( \ln q(z) = -\ln(1 + \lambda(m)(1-z)) \) for F=G.

Statement 2

Let \( E(z^{\nu(m)}|m_i) = q(z) \) be the generating function of \( \nu(m) \) with the characteristic \( \lambda(m) = E \nu(m) \). If \( \{ m_i, i = 1, ..., \nu(m_0) \} \) are magnitudes of the direct descendants of the \( m_0 \) event,
then \( \nu_{\lambda}(m_0) \) has the same distribution as \( \sum_{i}^{v(m_0)} \varepsilon(m_i) \), where \( \varepsilon(m) = [m > m_0 - \Delta] \) is the logical 1-0 function. Since \( \{m_i\} \) are independent

\[
\varphi_{\lambda}(z) = \mathbb{E}(z^{\varepsilon(m_0)}|m_0) = \sum_{m_0} (\mathbb{E}z^{\varepsilon(m)})^{m} P(v(m_0) = n) =\varphi(\pi_{m_0-\Delta} + \pi_{m_0-\Delta}z)
\]  

(A.1)

where \( \pi_{m_0-\Delta} = P(m \geq m_0 - \Delta| m_0 \geq \Delta) \) and \( \pi_{m} = 1 - \pi_{m} \).

Recall that \( \varphi(z) = (1 - p)/(1 - pz) = [1 - \lambda(z - 1)]^{-1} \) for \( F=G \) and \( \varphi(z) = \exp(\lambda(z - 1)) \) for \( F=P \). Substituting any of the functions \( \varphi(z) \) in the right part of (A.1), we get the same function with the replacement of \( \lambda(m_0) \) by

\[
\dot{\lambda}_{\lambda}(m_0) = \lambda(m_0) F_1(m_0 - \Delta)/F_1(\Delta).
\]  

(A.2)

Assume that the right part of (A.2) is independent of \( m_0 \) in the interval \( \Delta \leq m_0 < M \). Then \( \lambda(m) \) and \( F_1(m) \) are differentiable functions simultaneously. By (A.2)

\[
0 = (d/dm_0)\dot{\lambda}_{\lambda}(m_0)_{m_0=\Delta} = [\dot{\lambda}(\Delta) - \lambda(\Delta)f_1(0)]/F_1(\Delta).
\]

Here we used condition \( F_1(0) = 1 \). As a result, we get \( \lambda(m) = ce^{am} \) and \( F_1(m) = e^{-am}, m \in (0, M) \), where \( a = f_1(0) \).

**Statement 3**

**General remarks.**

We will need the following notation: \( N_M(m) \) is the total number of events of magnitude \( \geq M \) triggered by an event of magnitude \( m \). In particular, the total \( \Delta \)-productivity \( V_\lambda(m_0) \) of the initial event with magnitude \( m_0 \) is \( N_M(m_0), M(m) = (m - \Delta)_+, \) where \( a_+ = \max(a, 0) \). The notation \( E(\cdot|m) \) denotes the conditional mean given \( m \).

The spatial-temporal components of the ETAS model do not affect cluster size, since the law of generating new events depends only on the parent magnitude, and the problem of de-clustering does not arise in our theoretical analysis. Therefore, the spatial-temporal components in theoretical analysis can be any and the distribution of \( N_M(m) \) is the same for any \( x = (l, g, m) \).

By definition, if \( \{m_i, i = 1, \ldots, v(m_0)\} \) are offspring magnitudes of \( m_0 \) then \( \{N_M(m_i), i = 1, \ldots, v(m_0)\} \) are independent for a given \( m_0 \). Moreover, they are identically distributed together with \( N_M(m_0) \) since the magnitudes \( m_i, i \geq 0 \) have a common distribution \( f_i(m) \). Let us consider a new variable

\[
\tilde{N}_M(m_0) = \sum_{i}^{v(m_0)} \{N_M(m_i) + [m_i \geq M]\}, \quad (A.3)
\]
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where the symbol \([a > b]\) denotes the logical function 1-0. Obviously, (A.3) is nothing more than counting events with magnitude \(\geq M\) in the cluster through first-generation events and its triggered events of magnitude \(\geq M\). The component \([m_i \geq M]\) takes into account \(i\)-th event of the first generation if its value is greater than \(M\). Therefore \(N_M(m_0)\) and \(\tilde{N}_M(m_0)\) are identically distributed, that is \(N_M(m_0) = \tilde{N}_M(m_0)\).

The moments \(EV^\alpha\).

We will use the notation:

\[
\Lambda_n = \int_r \mathcal{R}(m) f_i(m) dm, \quad \Lambda_1 = 1 - \Lambda_1, \quad \lambda(m) = p(m)/(1 - p(m)) \text{ in the case } F=G.
\]

\[
b_M^{(\alpha)}(m) = \mathbb{E}[N_M^\alpha(m)|n], \quad b_M^{(\alpha)} = \mathbb{E}N_M^\alpha(m), \quad \pi_M = \mathbb{E}[m > M].
\]

Mean value: \(EV_\alpha\). Averaging (A.3) given \(m_0 = m\) we have

\[
b_M^{(1)}(m) = \lambda(m)(b^{(1)}_M + \pi_M).
\]

After averaging over \(m\), we obtain the equation for \(b_M^{(1)}:\)

\[
b_M^{(1)} = \Lambda_1(b^{(1)}_M + \pi_M).
\]

For finite \(b_M^{(1)}\), this equality can be true, if \(\Lambda_1 < 1\). Hence

\[
b_M^{(1)} = \pi_M \cdot \Lambda_1/\overline{\Lambda}_1, \quad b_M^{(1)}(m) = \lambda(m)\pi_M/\overline{\Lambda}_1.
\]  
(A.4)

Since \(V_\alpha = N_{M(m)}(m), \quad M(m) = (m - \Delta)_+\), we have

\[
\mathbb{E}(V_\alpha|m) = \lambda(m)\overline{F}_1(m - \Delta)/\overline{\Lambda}_1, \quad \mathbb{E}(V_\alpha) \leq \Lambda_1/\overline{\Lambda}_1,
\]

\[
\mathbb{E}(V_\alpha|m \geq \Delta) \leq (\Lambda_1/\overline{\Lambda}_1)/\overline{F}_1(\Delta),
\]

that is, \(V_\alpha\) has finite mean value in conditional and unconditional situations if \(\Lambda_1 < 1\).

The case \(n > 1\).

a) Fixed/random initial \(m_0\)

It is easy to see that for any \(\lambda(m) \geq 0\) there exists \(K \leq \infty\) such that \(\Lambda_k = E\lambda^k(m)\) is finite for \(k \leq K\) and is infinite for \(k > K\). Let us show that

\[
b_M^{(1)} \leq B_k, \quad b_M^{(k)}(m) < \infty \quad \text{iff} \quad \Lambda_k < \infty, \Lambda_1 < 1.
\]  
(A.5)

We have shown (see above) that (A.5) is true for \(k = 1\). We will prove (A.3) by the method of induction on \(k\). Assume that all values in (A.5) are finite for \(k \leq n - 1\).

Putting \(\xi_i = N_M(m_i) + [m_i \geq M]\) in (A.3), one has

\[
N_M^\alpha(m) = \sum_{i=1}^{\nu(m)} \xi_i^{(\alpha)} + b_M^{(\alpha)} + \sum_{i=1}^{\nu(m)} \xi_i^{(\alpha)}.
\]  
(A.6)
The number of terms in the first sum is \( \nu(m)(\nu(m) - 1) \ldots (\nu(m) - n + 1) \). For a random variable \( \nu(m) \) with a Poisson/Geometric distribution

\[
E[\nu(m)(\nu(m) - 1) \ldots (\nu(m) - n + 1)|m] = a_n(F) \lambda^m \left( m \right),
\]

(7.7)

where \( a_n(F = P) = 1, \ a_n(F = G) = n! \).

Therefore

\[
E(\sum_{i=1}^{\nu(m)} \xi_i \ldots \xi_i |m) = a_n(F) \lambda^m [b_M^{(i)} + \pi_M^m].
\]

Similarly

\[
E(R_m^{(i)} |m) = \sum_{\Omega} c_n(\omega_1) b_1(F) \lambda^k (m) \tilde{b}_M^{(i)} - \tilde{b}_M^{(i)},
\]

(7.8)

\[
\tilde{b}_M^{(i)} = E\{N_M(m_i) + [m_i \geq M]\} +
\]

where

\[
c_n(\omega) = n!/(k_1! \ldots k_r! m!), \ \Omega = \{\omega = (k, r_1, \ldots, r_k) : 1 < k < n, r_i \geq 1, r_i + \ldots + r_k = n\}.
\]

Since for any random variable \( \tilde{\xi} \geq 0 \):

\[
E_{\tilde{\xi}}^n E_{\tilde{\xi}}^m \leq E_{\tilde{\xi}}^{n+m},
\]

we get from (7.8):

\[
E(R_m^{(i)} |m) \leq C_n(1 + \lambda(m)) \max_{1 \leq k < n} (\tilde{b}_M^{(k)} b_M^{(i-k)}).
\]

(7.9)

The expectation of (7.6), given \( m \), is

\[
b_M^{(i)}(m) = a_n(F) \lambda^m (m) \tilde{b}_M^{(i)} + E(R_m^{(i)} |m) + \lambda(m) [b_M^{(i)} + \delta_M^{(i)}],
\]

(7.10)

where \( b_M^{(i)} + \delta_M^{(i)} = \tilde{b}_M^{(i)} \). By (7.8),

\[
\delta_M^{(i)} < E(N_M(m) + 1) - N_M^m(m) \]

\[
< nE(N_M(m) + 1)^n - n(2^{n-1} b_M^{(n-1)} + 1).
\]

(7.11)

Averaging (7.10) over \( m \), we get

\[
b_M^{(i)}(1 - \Lambda_i) = a_n(F) \Lambda_n(\pi_M \Lambda_i / \Lambda_i)^n + E R_m^{(i)} + \Lambda_i \delta_M^{(i)}.
\]

(7.12)

From (7.10-12) it can be seen that \( b_M^{(i)} \) and \( b_M^{(i)}(m) \) are finite only when \( \Lambda_n < \infty \) and \( \Lambda_i < 1 \), that is, (7.5) holds for \( k = n \).

By setting \( M = (m - \Delta)_+ \), we get:

\[
E(\nu_M^{(i)} |m) = b_M^{(i)}(m) < \infty \text{ only when } \Lambda_n < \infty.
\]

It remains to consider \( E(\nu_\Lambda^{(i)}) \). By (7.10, 7.12),

\[
b_M^{(i)}(1 - \Lambda_i) = a_n(F) \Lambda_n(\pi_M \Lambda_i / \Lambda_i)^n + E R_m^{(i)} + \Lambda_i \delta_M^{(i)}
\]

\[
E(\nu_\Lambda^{(i)}) = E b_M^{(i)}(m) \geq E \lambda(m) b_M^{(i)}(m) \geq E \lambda(m) a_n(F) \Lambda_n \pi_M^{(i)} \Lambda_i / \Lambda_i^{e+1}.
\]

Therefore \( E(\nu_\Lambda^{(i)}) = \infty \) if \( \Lambda_n \) is infinite.
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Let \( \Lambda_n < \infty \). Using (A.10) again, we have

\[
E \nu^\alpha_\Lambda \leq a_n(F)\Lambda_n(\Lambda_1/\bar{\Lambda}_i)^{\alpha} + ER_n^{\alpha}(m) + E\lambda(m)[\hat{b}_{\alpha}(\nu) + \delta_{i}(\nu)]
\]

Due to (A.9), we have \( ER_n^{\alpha}(m) \leq \tilde{\nu}_n \max(1, \Lambda_1, \ldots, \Lambda_{n-1}) \); by (A.9), \( E\lambda(m)\delta_{i}(\nu) \leq D_n\Lambda_1 \).

Finally, \( E\lambda(m)\hat{b}_{\alpha}(\nu) \leq B_n\Lambda_1 \), because we have proved (A.3) for index \( n \). The proof is complete.

\( \text{b) Fixed dominant } m_0 \)

Let us consider the ETAS cluster with a dominant initial magnitude \( m_0 \). Let \( \hat{N}_{0M}(m) \) be the number of events of magnitude \( > M \) in the cluster with the initial event \( m \leq m_0 \) in which all events have magnitude \( < m_0 \). In such cluster, direct descendants of any initial magnitude \( m \) are realized under the condition \( \Omega(m) = \{ m_i < m_0, i = 1, \ldots, \nu(m_0) \} \). The probability of \( \Omega(m) \) is

\[
P(\Omega(m)) = e^{-\lambda(m)} \hat{F}(m_0)
\]

and

\[
P(\Omega(m)) = \bar{p}(m)/\{d - \hat{F}(m_0)\} = (1 + \lambda(m) \hat{F}(m_0))^{-1} \text{ [G-model].}
\]

Here we used the relation: \( p(m) = \lambda(m)/(1 + \lambda(m)) \).

Therefore the conditional probability density of the direct descendants

\[
P(m_i = m_i, i = 1, \ldots, n, \nu(m_0) = n|\Omega(m))
\]

\[
= \frac{\hat{\lambda}(m) e^{-\hat{\lambda}(m)}}{n!} \times \hat{f}_{01}(m_i) \nu(m_0)
\]

\[
= \hat{\lambda}(m) F_i(m_0) [1 + \lambda(m) \hat{F}(m_0)]^{-1}, F = P
\]

\[
\hat{f}_{01}(m_i) = f_i(m)/(m_i), 0 \leq m \leq m_0, F_i(m) = \int_{m_0}^m f_i(u) du
\]

\[
\hat{\lambda}(m) = \frac{1}{(1 + \lambda(m) \hat{F}(m_0))^{-1}}, F = G
\]

Therefore, given the dominance of the initial magnitude \( m_0 \), we are again dealing with an ETAS model in which the magnitude \( m \) is transformed into \( \hat{m} \) with a distribution (A.16), and the productivity \( \nu(m) \) into \( \hat{\nu}(\hat{m}) \) with the same distribution (Poisson/Geometric), but with new parameters \( \hat{\lambda}(\hat{m}) \) as in (A.17). Therefore, as above, we will have the following relations

\[
\hat{N}_{0M}(\hat{m}) = \sum_{i=1}^{\hat{\nu}(\hat{m})} (\hat{N}_{0M}(\hat{m}_i) + [\hat{m}_i - M]),
\]

where \( \{ \hat{N}_{0M}(\hat{m}_i), i = 1, \ldots \} \) are independent and identical distributed given \( \{ \hat{m}_i, i \geq 0 \} \).
As a result
\[ \nu_{\Delta}(m_0) = \hat{N}_{0(m_0-\Delta)}(m_0), \quad m_0 \geq \Delta. \]  

(A.19)

Since \( m_0 \) is a parameter, in the random case of initial event, \( m_0 \) has distribution \( F_{\ast}(m)/\bar{F}_{\ast}(\Delta) \).

As we can see, equation (A.18) is a special case of (A.3), in which the magnitude range is finite. By (A.17), \( \hat{\lambda}_0(m) \leq \lambda(m_0) \) and therefore \( \hat{\lambda}_0(m) \) has all moments. Consequently, the total \( \Delta \)-productivity will have all the moments at any fixed \( m_0 \).

c) Random dominant \( m_0 \). We need additional notation:
\[ \hat{b}_{0M}^{(a)}(m) = E[\hat{N}_{0M}^{a}(\hat{m})][\hat{m}], \quad \hat{b}_{0M}^{(a)} = E\hat{N}_{0M}^{a}(\hat{m}) \],
\[ \tilde{b}_{0M}^{(a)}(m) = E(\hat{N}_{0M}(\hat{m}) + [\hat{m} > M])^a, \]
\[ \hat{x}_0(M) = E[\hat{m} > M] = [F_{\ast}(m_0) - F_{\ast}(M)]_{0}/F_{\ast}(m_0), \]
\[ \hat{\lambda}_0 = E\hat{\lambda}_0(m). \]

Using (A.17) and non-decreasing function \( \lambda(m) \), we have for the P-model
\[ \hat{\lambda}_0 = \int_{0}^{m_0} \lambda^{a-1}(m) F_{\ast}(m) f_\ast(x) dx \leq \hat{\lambda}_0^{a-1}(m_0) \lambda_1. \]  

(A.21)

The same is true for the G-model. Really, by (A.17) we have the following representation:
\[ \hat{\lambda}_0 = \int_{0}^{m_0} \lambda^{a-1}(m) \int_{0}^{m} [\psi^{a-1}(\lambda)/(1 + \lambda^a(m) \bar{F}(m_0))] \lambda(x) f_\ast(x) dx, \]

where
\[ \psi(m) = (\lambda^{a-1}(m) + \bar{F}(m_0)) / (\lambda^a(m) + \bar{F}(m_0)). \]

Since \( \lambda(m) \) is a non-decreasing function, \( \psi(m) \leq 1 \) on the interval \((0, m_0)\). Therefore, the integrand in square brackets is less than 1, which proves (A.21).

As above, the relation (A.18) gives,
\[ \hat{b}_{0M}^{(1)} = \hat{\lambda}_0 \hat{x}_0(M)/(1 - \hat{\lambda}_0) \leq \Lambda \hat{x}_0(M)/(1 - \Lambda_1), \]  

(A.22)
\[ E\nu_{\Delta} \leq \Lambda_1/(1 - \Lambda_1). \]

Setting \( \xi_t = \hat{\lambda}_0(m_t) + [m_t - M] \) in (A.18) and using (A.6), we get an analogue of (A.10):
\[ \hat{b}_{0M}^{a}(m) = \sum_{k=1}^{n} \sum_{r_k} c_k(r_1, ..., r_k) \hat{\lambda}_0^{(r_1)}(m) \hat{b}_{0M}^{(r_1)} ..., \hat{b}_{0M}^{(r_k)}), \]  

(A.23)

where the summation is over all integer vectors \( r^{(k)} = (r_1, ..., r_k) : r_1 + ... + r_k = n; r_i \geq 1 \). Averaging (A.23) over \( \hat{m} \) we will also have
\[ \hat{b}_{0M}^{a} = \sum_{k=1}^{n} \sum_{r_k} c_k(r_1, ..., r_k) \hat{\lambda}_0^{(r_1)} \hat{b}_{0M}^{(r_1)} ..., \hat{b}_{0M}^{(r_k)} + \hat{\lambda}_0 \hat{b}_{0M}^{a}. \]  

(A.24)
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Let's extract the term \( \hat{\Lambda}_0 \hat{\nu}^{(n)} \) from \( \hat{\Lambda}_0 \hat{\nu}^{(n)} \) and move it to the left part of (A.24); then apply (A.21) to \( \hat{\Lambda}_{0k} \) and take into account the relation \( \hat{\Lambda}_{01} / (1 - \hat{\Lambda}_{01}) \leq \Lambda_1 / (1 - \Lambda_1) \). Then multiplying (A.24) by \( \hat{\Lambda}_0(m_0) \), we get

\[
\hat{\lambda}_0(m_0) \hat{\nu}_{0M}^{(n)} \leq \sum_{\beta=\alpha} \left[ c_{\beta}(r_1, \ldots, r_k)(\hat{\lambda}_0(m_0) \hat{\nu}_{0M}^{(1)}) \cdots (\hat{\lambda}_0(m_0) \hat{\nu}_{0M}^{(n)}) + \hat{\lambda}_0(m_0)(\hat{b}_{0M}^{(n)} - \hat{\nu}_{0M}^{(n)}) \right] \Lambda_1 / (1 - \Lambda_1).
\]

(A.25)

Since \( \hat{b}_{0M}^{(n)} = E(\hat{N}_{0M}(\hat{m}_1) + [\hat{m}_1 > M]^n \), the elements of the right part of (A.25) can be obtained by summing and multiplying elements of the form: \( \hat{\lambda}_0(m_0) \hat{\nu}_0(M) \) and

\[
D_{k,\varepsilon} = \hat{\lambda}_0(m_0) E\hat{\nu}_{0M}^{(k)}(\hat{m} \geq M)^k, k = 1, \ldots, n - 1; \varepsilon = 0, 1,
\]

where \( D_{k,\varepsilon} \leq \hat{\lambda}_0(m_0) \hat{b}_{0M}^{(k)} \).

We have assumed that \( \lambda(m) \int_0^\Delta f_j(m - x) dx \leq C \). Hence

\[
\hat{\lambda}_0(m_0) \hat{\nu}_0(m_0 - \Delta) = \lambda(m_0) F_j(m_0)[F_j(m_0) - F_j(m_0 - \Delta)] / F_j(m_0) \leq C
\]

and by virtue of (A.22)

\[
\hat{\lambda}_0 \hat{\nu}_{0M}^{(1)}(m_0) \leq C_1, \quad M(m_0) = (m_0 - \Delta)_+.
\]

Using (A.25) and doing induction over \( n \), we can conclude that \( \hat{\lambda}_0 \hat{\nu}_{0M}^{(n)}(m_0) \leq C_n \) for any \( n \). But then, due to (A.19), \( E\nu^n_{\Delta} = E\hat{b}_{0M}^{(n)}(m_0) \leq K_n < \infty \) for any \( n \).

The proof is complete.

**Statement 4**

Let \( \xi \) be a random variable with a geometric distribution. Then

\[
E\xi(E\xi - 1) / (E\xi)^2 = 2.
\]

(A.27)

Consider \( \xi = V_\Lambda(m_0) \) where \( m_0 \) is the initial dominant magnitude in the ETAS(G) cluster. We will show that the relation (A.27) is possible for at most one parameter \( \Delta \in (0, m_0) \).

For simplicity of notation, we will temporarily proceed from equation (A.3). Recall that

\[
\Lambda_n = \int \lambda^\Delta(m) f_1(m) dm, \quad \pi_M = E[m > M],
\]

\[
b_M^{(n)}(m) = E[N_M^{n+1}(m) | m], \quad b_M^{(n)} = EN_M^{n}(m).
\]

Above we show that

\[
b_M^{(i)}(m) = \lambda(m) \pi_M / \Lambda_i, \quad b_M^{(i)} = \pi_M \cdot \Lambda_i / \Lambda_i, \quad b_M^{(i)} + \pi_M = \pi_M / \Lambda_i.
\]

(A.28)

According to (A.10) for the ETAS(G) model, we have
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\[ b_M^{(2)}(m) = 2(\lambda(m)\pi_M/\Lambda_1) + \lambda(m)(b_M^{(1)} + 2b_M^{(1)}\pi_M + \pi_M^2). \] (A.29)

By (A.28),

\[ 2b_M^{(1)}\pi_M + \pi_M^2 = (1 + \Lambda_1)\pi_M^2/\Lambda_1. \] (A.30)

Averaging (A.29) over \( m \), we can find \( b_M^{(2)} \); using (A.30), we will have

\[ b_M^{(2)} = 2\Lambda_2\pi_M^3/\Lambda_1 + \Lambda_1(1 + \Lambda_1)\pi_M^2/\Lambda_1. \] (A.31)

If we substitute (A.31) into (A.29), we get

\[ b_M^{(2)}(m) = 2(b_M^{(1)} + 2\lambda^2(m)/(2\Lambda_2 + 1 - \Lambda_1^2)/\Lambda_1. \]

As a result,

\[ (b_M^{(2)}(m) - b_M^{(1)})/\pi_M^2 = 2 + \lambda^{-1}(m)[(2\Lambda_2 + 1 - \Lambda_1)/\Lambda_1 - \Lambda_1/\pi_M]. \] (A.32)

The condition (A.27) is satisfied only if

\[ \pi_M = (1 - \Lambda_1^2)/(2\Lambda_2 + 1 - \Lambda_1^2), \] (A.33)

where \( \Lambda_2 - \Lambda_1^2 = \sigma^2(\lambda) > 0 \).

The equation (A.33) with respect to \( M \) has unique solution if \( \pi_M \) is strictly monotone.

Now we can return to the ETAS(G) cluster with dominant initial magnitude \( m_0 \). In this case the main equation is (A.18). It is identical to (A.10), but requires the following replacement of the characteristics used (see A.16, A.17):

\[ f_1(m) \Rightarrow f_1(m)/F_1(m_0), 0 \leq m \leq m_0, \]

\[ \lambda(m) \Rightarrow \lambda(m)F_1(m_0)/(1 + \lambda(m)F_1(m_0)), \]

\[ \pi_M \Rightarrow [F_1(m_0) - F_1(M)]_+ / F_1(m_0), \] (A.34)

\[ \Lambda_n = \int_0^{m_0} \lambda^{n-1}(m)F_1(m_0)(1 + \lambda(m)F_1(m_0))^{-n} f_1(m)dm. \] (A.35)

Since \( V_\Delta(m_0) = N_{m_0 - \Delta}(m_0) \), we have to consider equation (A.33) given the substitutions (A34, A35) and \( M = m_0 - \Delta \). Given \( f_1(m) > 0 \) the function (A.34) will be strictly monotonic with respect to \( \Delta \). Hence, the relation (A.27) is possible for at most one value of \( \Delta = \Delta(m_0) \).

The proof is complete.