HOCHSCHILD COHOMOLOGY OF ALGEBRAS IN MONOIDAL CATEGORIES AND SPLITTING MORPHISMS OF BIALGEBRAS

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Abstract. The main goal of this paper is to investigate the structure of Hopf algebras with the property that either its Jacobson radical is a Hopf ideal or its coradical is a subalgebra. In order to do that we define the Hochschild cohomology of an algebra in an abelian monoidal category. Then we characterize those algebras which have dimension less than or equal to 1 with respect to Hochschild cohomology. Now let us consider a Hopf algebra $A$ such that its Jacobson radical $J$ is a nilpotent Hopf ideal and $H := A/J$ is a semisimple algebra. By using our homological results, we prove that the canonical projection of $A$ on $H$ has a section which is an $H$–colinear algebra map. Furthermore, if $H$ is cosemisimple too, then we can choose this section to be an $(H, H)$–bicolinear algebra morphism. This fact allows us to describe $A$ as a ‘generalized bosonization’ of a certain algebra $R$ in the category of Yetter–Drinfeld modules over $H$. As an application we give a categorical proof of Radford’s result about Hopf algebras with projections. We also consider the dual situation. In this case, many results that we obtain hold true for a large enough class of $H$–module coalgebras, where $H$ is a cosemisimple Hopf algebra.

Introduction

Let $H$ be a Hopf algebra. The categories $H \mathcal{YD}$ and $H \mathcal{M}_H$, of Yetter–Drinfeld modules and respectively Hopf bimodules, appeared, in particular, as an attempt to construct new solutions to the Yang–Baxter equation. Nowadays we can recognize their most important properties into the definition of braided categories, a very general and abstract setting useful, not only for providing new solutions to the Yang–Baxter equation, but also in many other areas of mathematics, like the theory of quantum groups and low dimensional topology.

Partially motivated by these applications, the theory of Hopf algebras knew in 80’s an outstanding development. Besides many striking results obtained since then, we would like to recall, more or less chronologically, a few of them that will play a very important role in our paper.

- The description of the coradical filtration of a pointed coalgebra, result due to Taft and Wilson [TW], that is crucial in the classification of finite dimensional pointed Hopf algebras.
- The characterization of bialgebras with projection [Ra1] – later Majid [Maj1] showed that this result can be interpreted in terms of bialgebras in a braided category.
- The equivalence of braided categories $H \mathcal{YD} \simeq H \mathcal{M}_H$ [Sch1] and [AD].
- The classification of certain classes of pointed Hopf algebras of finite dimension. One of the used method is the ‘lifting’ method [AS1], [AS2], [AS3], [AS4]. Let $A$ be a Hopf algebra such that its coradical is a Hopf subalgebra $H$. Then the coradical filtration of $A$ is a filtration of Hopf algebras, so $\text{gr} A$ is a graded Hopf algebra. One of the main steps of the ‘lifting’ method is to describe $\text{gr} A$, by using the second mentioned result, as the ‘bosonization’ of a certain Hopf algebra $R$ in $H \mathcal{YD}$ by $H$. The next step is to find all Hopf algebras $A$ having a given graded Hopf algebra $\text{gr} A$.
- Let $A$ be a finite dimensional Hopf algebra over a field $k$ of characteristic zero whose coradical, say $H$, forms a Hopf subalgebra. Then the left $H$–module coalgebra $A$ is a cosmash in the sense that there exists a $H$–linear coalgebra map $\gamma : A \to H$ such that $\gamma |_H = \text{Id}_H$, see [SvO]. In [Mas] it

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is shown, with a different method, that the above result still holds true without any assumption on the dimension of $A$ and \text{char} $k$.

- For a Hopf algebra $A$ a conjectural formula for $A_1$, the first component of the coradical filtration of $A$, is proposed in [AS5]. This formula is proved in the same paper in the case when $A$ is a graded Hopf algebra such that its coradical is a Hopf subalgebra of $A$. In [CDMM] the conjecture is proved in the ungraded case.

One of the main aims of this paper is to strengthen some of the results that we mentioned above, by combining the formalism of homological algebra and monoidal categories.

Let $(\mathcal{M}, \otimes, \mathbf{1})$ be a monoidal category. We start by recalling some basic facts about algebras and $(A, A)$–bimodules in $\mathcal{M}$. Then, for an algebra $A$ and an $(A, A)$–bimodule $M$ in $\mathcal{M}$, we define the Hochschild cohomology $H^i(A, M)$ of $A$ with coefficients in $M$, and show that many classical results can be extended to this more general context. For example, we call an algebra $A$ separable if $A$ is $\mathcal{E}$–projective as an $(A, A)$–bimodule in $\mathcal{M}$. Such an $A$ is characterized by the fact that its Hochschild dimension is zero, that is $H^1(A, M) = 0$, for every bimodule $M$. Also, by defining in an appropriate way Hochschild extensions of $A$ with kernel $M$, we prove that the set of their equivalence classes is in one–to–one correspondence with $H^2(A, M)$. An algebra $A$ will be called (non–commutative) formally smooth if it has no non–trivial extensions. In the particular case when $\mathcal{M}$ is the category of $K$–vector spaces, we recognize the definition of quasi–free algebras introduced by J. Cuntz and D. Quillen in [CQ]. The first important results, Theorem 2.3.2 and Theorem 2.5.3, give different equivalent characterizations of formally smooth algebras in a monoidal category, and can be interpreted as generalizations of Wedderburn–Malcev Theorem. We obtain immediately that separable algebras are formally smooth. In view of these theorems, roughly speaking, formally smoothness is useful to prove that certain algebra morphisms have algebra morphism sections in the category that we work in.

In the third section of the paper we apply this technique to produce algebra sections in the case when $\mathcal{M}$ is either $\mathcal{M}^H$ (the category of right $H$–comodules) or $\mathcal{H}\mathcal{M}^H$ (the category of $(H, H)$–bicomodules), where $H$ is a semisimple Hopf algebra. Let $A$ be a Hopf algebra such that its Jacobson radical $J$ is a Hopf ideal. We denote the Hopf algebras quotient $A/J$ by $H$. If $J$ is nilpotent and $H$ is semisimple, in the main application of this section, Corollary 3.15 we prove that the canonical projection $\pi : A \to H$ has a section which is a morphism of algebras in $\mathcal{M}^H$. This is a generalization of one of the main results of [SyO]. The second part of the corollary establishes that $\pi$ has a section which is an algebra map in $\mathcal{H}\mathcal{M}^H$, if we assume in addition that $H$ is cosemisimple too (in fact we prove a more general result, that we shall not mention here to simplify the exposition). This corollary recall us [Ra1], where it is assumed that a Hopf algebra morphism $\pi : A \to H$ has a section $\sigma : H \to A$ which is a morphism of Hopf algebras. In [Ra1] it is shown that there is a bialgebra $R$ in $H\mathcal{YD}$ such that $A$ is the smash product algebra and the smash product coalgebra of $R$ by $H$.

Taking into account Corollary 3.15, it is natural to look for a similar description of a bialgebra $A$, supposing that $\pi : A \to H$ has a section $\sigma$ which is only a morphism of algebras in $\mathcal{H}\mathcal{M}^H$. This will be done in the fourth section of the paper.

The starting point is the simple observation that $A$ becomes in a natural way an object in $H\mathcal{M}^H$. Of course the left and right comodule structures are induced by $\pi$. Since $\sigma$ is a morphism of algebras, $A$ is a bimodule over $H$, and the fact that $\sigma$ is a morphism of bicomodules is enough to have the required compatibility relations. By using the equivalence $H\mathcal{M}^H \simeq H\mathcal{YD}$ we have $A \simeq R \otimes H$ (isomorphism in $H\mathcal{M}^H$), where $R = A^{(H)}$. Moreover, the multiplication of $A$ is a morphism in $H\mathcal{M}^H$ and the unit of $A$ is in $R$. Therefore $R$ becomes an algebra in $H\mathcal{YD}$ and $A$ can be identified as an algebra with the smash product $R\#H$. We can not repeat this argument for the coalgebra structure since $\Delta$ is only $(H, H)$–colinear. Thus, by identifying $A$ and $R\#H$ as algebras, the problem of describing all bialgebras $A$ as above is equivalent to find all coalgebras structures on $R\#H$ such that the comultiplication is a morphism of $(H, H)$–bicomodules. We prove that $\Delta_{R\#H}$ is uniquely determined by a pair of $K$–linear maps $\delta : R \to R \otimes R$ and $\omega : H \to R \otimes R$. Let $\varepsilon$ be the restriction of the counit of $A$ to $R$. The properties of $\delta$, $\omega$ and $\varepsilon$ necessary to get a
bialgebra structure on $R\# H$ are listed in Definition 4.41. The result that we obtain is stated in Theorem 4.41.

Let us remark that $(R, \delta, \varepsilon)$ is not a coalgebra since $\delta$ is not coassociative in general. In fact the coassociativity rule is broken by the map $\omega$, which is a normalized (non-commutative) 1-cocycle of $H$. If $\omega$ is the trivial cocycle then $R$ is a bialgebra in $H\# \mathcal{YD}$ and $A$ is isomorphic as an algebra and coalgebra with the smash product, i.e. with the ‘bosonization’ of $R$ by $H$. The main results of this section are Theorem 4.41 and Theorem 4.43.

In the last section of the paper we state the dual results and use them to prove some applications. We start by defining the Hochschild cohomology of a coalgebra in a monoidal category $\mathcal{M}$ and by giving homological characterizations of coseparable and formally smooth coalgebras. As a consequence, by taking the monoidal category $\mathcal{M}$ to be either $\mathfrak{M}_H$ or $H\mathfrak{M}_H$, we prove Theorem 5.19. According to this theorem, under some assumptions on $H$, for every coalgebra $C$ in $\mathcal{M}$ such that $C_0 = H$, there exists a morphism of coalgebras $\pi : C \to H$ in $\mathcal{M}$ so that $\pi|_H = \text{Id}_H$. The first assertion of Theorem 5.19 had already been proved by A. Masuoka in the case when $C$ is the underlying coalgebra structure of a Hopf algebra $A$ with with the property that its coradical is a subalgebra. Now we can describe the coradical filtration of such a coalgebra $C$ as in Theorem 5.17. Finally, we prove that a Hopf algebra $A$, having the coradical a semisimple and cosemisimple Hopf subalgebra, is as a Hopf algebra, not only as a coalgebra, a kind of smash product, see Theorem 5.24. We expect that this last result is strongly connected with the lifting method introduced by N. Andruskiewitsch and H.J. Schneider. Probably Theorem 5.24 can be used to get direct information about a Hopf algebra $A$ with the property that its coradical is a subalgebra, skipping the step when the associated graded Hopf algebra $gr A$ is investigated.

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Notation. In a category $\mathcal{M}$ the set of morphisms from $X$ to $Y$ will be denoted by $\mathcal{M}(X,Y)$. If $X$ is an object in $\mathcal{M}$ then the functor $\mathcal{M}(X,-)$ from $\mathcal{M}$ to $\text{Sets}$ associates to any morphism $u : U \to V$ in $\mathcal{M}$ the function that will be denoted by $\mathcal{M}(X,u)$.

1. Hochschild cohomology in monoidal categories

In this section we define and study the Hochschild cohomology of an algebra in a monoidal category. We start by recalling the definitions of monoidal categories and of algebras in such categories. In order to define Hochschild cohomology we will use relative homological algebra, for details on this matter see [HS Chapter IX].

1.1. A monoidal category means a category $\mathcal{M}$ that is endowed with a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, an object $1 \in \mathcal{M}$ and functorial isomorphisms: $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\), $l_X : 1 \otimes X \to X$ and $r_X : X \otimes 1 \to X$. The functorial morphism $a$ is called the associativity constraint and satisfies the Pentagon Axiom, that is the first diagram below is commutative, for every $U, V, W, X$ in $\mathcal{M}$. The morphisms $l$ and $r$ are called the unit constraints and they are assumed to satisfy the Triangle Axiom, i.e. the second diagram is commutative. The object $1$ is called the unit of $\mathcal{M}$.

\[
\begin{array}{ccc}
(U \otimes (V \otimes W)) \otimes X & \xrightarrow{a_{U,V,W} \otimes X} & (U \otimes V) \otimes (W \otimes X) \\
\downarrow a_{U,V,W,X} & & \downarrow a_{U \otimes V,W,X} \\
U \otimes ((V \otimes W) \otimes X) & \xrightarrow{U \otimes a_{V,W,X}} & U \otimes (V \otimes (W \otimes X))
\end{array}
\]

\[
\begin{array}{ccc}
(U \otimes V) \otimes (W \otimes X) & \xrightarrow{a_{U,V,W} \otimes X} & (U \otimes V) \otimes W \\
\downarrow a_{U,V,W} & \downarrow r_{V \otimes W} & \downarrow a_{V,W} \\
U \otimes (V \otimes W) & \xrightarrow{U \otimes l_W} & U \otimes l_W
\end{array}
\]

For details on monoidal categories we refer to [Ka Chapter XI]. A monoidal category is called strict if the associativity constraint and unit constraints are the corresponding identity morphisms.
1.2. As it is noticed in [Ma2, p. 420], the Pentagon Axiom solves the consistency problem that appears because there are two ways to go from \(((U \otimes V) \otimes W) \otimes X\) to \(U \otimes (V \otimes (W \otimes X))\). The coherence theorem, due to S. Mac Lane, solves the similar problem for the tensor product of an arbitrary number of objects in \(\mathcal{M}\). Accordingly with this theorem, we can always omit all brackets and simply write \(X_1 \otimes \cdots \otimes X_n\) for any object obtained from \(X_1, \ldots, X_n\) by using \(\otimes\) and brackets. Also as a consequence of the coherence theorem, the morphisms \(a, l, r\) take care of themselves, so they can be omitted in any computation involving morphisms in \(\mathcal{M}\).

1.3. A **monoidal functor** between two monoidal categories \((\mathcal{M}, \otimes, 1, a, l, r)\) and \((\mathcal{M}', \otimes, 1, a, l, r)\) is a triple \((F, \phi_0, \phi_2)\), where \(F : \mathcal{M} \to \mathcal{M}'\) is a functor, \(\phi_0 : 1 \to F(1)\) is an isomorphism such that

\[
\begin{align*}
1 \otimes F(U) & \xrightarrow{l_{F(U)}} F(U) & F(U) \otimes 1 & \xrightarrow{r_{F(U)}} F(U) \\
\phi_0 \otimes F(U) & \xrightarrow{} F(l_U) & F(U) \otimes \phi_0 & \xrightarrow{} F(r_U) \\
F(1) \otimes F(U) & \xrightarrow{\phi_2(1, U)} F(1 \otimes U) & F(U) \otimes F(1) & \xrightarrow{\phi_2(U, 1)} F(U \otimes 1)
\end{align*}
\]

are commutative, and \(\phi_2(U, V) : F(U \otimes V) \to F(U) \otimes F(V)\) is a family of functorial isomorphisms such that the following diagram is commutative.

\[
\begin{align*}
(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{a_{F(U), F(V), F(W)}} F(U) \otimes (F(V) \otimes F(W)) \\
\phi_2(U, V) \otimes F(W) & \xrightarrow{} F(U \otimes V) \otimes F(W) \\
F(U \otimes V) \otimes F(W) & \xrightarrow{\phi_2(U, V, W)} F((U \otimes V) \otimes W) \\
F(U \otimes V \otimes W) & \xrightarrow{F(a_{U, V, W})} F(U \otimes (V \otimes W))
\end{align*}
\]

**Examples 1.4.**

a) The category \(\mathfrak{M}_K\) of all modules over a commutative ring \(K\), is a monoidal category with the tensor product of \(K\)-modules, that will be denoted by \(\otimes_K\).

b) Suppose that \(H\) is a Hopf algebra over a commutative ring \(K\). The category \(\mathfrak{M}_H\) of right \(H\)-modules is a monoidal category with respect to the tensor product defined as follows. For any right \(H\)-modules \(M\) and \(N\), let \(M \otimes N = M \otimes_K N\), regarded as an right \(H\)-module with the structure:

\[
(m \otimes n)h := \sum m h_{(1)} \otimes nh_{(2)}, \quad \forall m \in M, \forall n \in N, \forall h \in H
\]

where \(\Delta_H(h) = \sum h_{(1)} \otimes h_{(2)}\) is the \(\Sigma\)-notation that we use for the comultiplication of \(H\). The unit object in \(\mathfrak{M}_H\) is \(K\), which is a right \(H\)-module via \(\varepsilon_H\), the counit of \(H\).

c) The category \(\mathfrak{M}_H^*\), of right \(H\)-comodules, is a monoidal category. The structures on the tensor product of two bicomodules are obtained by duality from the previous example.

d) Suppose that \(B\) is an arbitrary associative ring with unity. The category \(\mathfrak{M}_B\) of all \((B, B)\)-bimodules is a monoidal category with the tensor product \(\otimes_B\) and unit object \(1 := B\).

1.5. Following [Ma2, Definition 9.2.11], let us recall the definition of associative algebras in a monoidal category \((\mathcal{M}, \otimes, 1, a, l, r)\). Let \(A\) be an object in \(\mathcal{M}\). Suppose that \(m : A \otimes A \to A\) and \(u : 1 \to A\) are morphisms in \(\mathcal{M}\). If \(m\) and \(u\) obey the **associativity** and **unity** axioms:

\[
\begin{align*}
A \otimes (A \otimes A) & \xrightarrow{A \otimes m} A \otimes A \\
(A \otimes A) \otimes A & \xrightarrow{(A \otimes A) \otimes m} A \otimes (A \otimes A) \\
m \otimes A & \xrightarrow{m} A \\
A \otimes A & \xrightarrow{A \otimes 1} A \otimes 1 \\
\end{align*}
\]

we say that \((A,m,u)\) is an (associative) algebra with multiplication \(m\) and unit \(u\) in \(\mathcal{M}\). As we explained in [12], we can omit the maps \(a, l\) and \(r\), so we shall draw these diagrams in a more simple way as follows.

\[
\begin{align*}
A \otimes A \otimes A & \xrightarrow{A \otimes m} A \otimes A & 1 \otimes A & \xrightarrow{=} A \otimes 1 \\
m \otimes A & \xrightarrow{m} A & u \otimes A & \xrightarrow{=} A \otimes u \\
A \otimes A & \xrightarrow{m} A & A \otimes A & \xleftarrow{m} A \otimes A
\end{align*}
\]

**Examples 1.6.**

a) An algebra in \((\mathcal{M}_K, \otimes_K, K)\) is an unitary associative ring \(A\) together with a ring morphism \(i: K \to A\) such that the image of \(i\) is included in the center of \(A\). We recognize the usual definition of algebras over a commutative ring.

b) Let \(H\) be a Hopf algebra. An algebra in the category \(\mathcal{M}_H\) is an associative algebra \(A\), in the usual sense, which is a right \(H\)–module such that

\[
(xy)h = \sum xh_{(1)} \otimes yh_{(2)}, \quad \forall x, y \in A, \quad \forall h \in H,
\]

\[
1_A h = \varepsilon_H(h)1, \quad \forall h \in H.
\]

We recognize the definition of \(H\)–module algebras [13, Definition 4.1.1], sometimes called \(H\)–differential algebras (see for example [BDK]).

c) An algebra in \(\mathcal{M}_H\) is a right \(H\)–comodule algebra, see [13, Definition 4.1.2]. Recall that this means an algebra \(A\) which is a right \(H\)–comodule such that

\[
\rho(xy) = \sum x_{(0)}y_{(0)} \otimes x_{(1)}y_{(1)}, \quad \forall x, y \in A,
\]

\[
\rho(1) = 1 \otimes 1.
\]

d) A bimodule over \(B\), say \(A\), is an algebra in \((\mathcal{M}_B, \otimes_B, B)\) iff \(A\) is an associative ring with unity \(1_A\) such that \(1_A \in \{a \in A \mid ba = ab, \forall b \in B\}\). This set will be denoted \(A^B\) (more generally, if \(M \in \mathcal{M}_B\) then \(M^B\) will denote the set of all \(m \in M\) such that \(bm = mb, \forall b \in B\)). For example, any morphism of associative rings \(i: B \to A\) gives an algebra in \(\mathcal{M}_B\) where \(A\) is a \((B,B)\)–bimodule with the restriction of scalars via \(i\).

1.7. Now we are going to define the representations of algebras in monoidal categories. We shall proceed as in the case of algebras in \(\mathcal{M}_K\). Let us assume that \((A,m,u)\) is an algebra in the monoidal category \((\mathcal{M}, \otimes, 1)\). By a left \(A\)–module we mean an object \(M \in \mathcal{M}\) together with a morphism \(\mu: A \otimes M \to M\) such that

\[
\begin{align*}
A \otimes A \otimes M & \xrightarrow{A \otimes \mu} A \otimes M \\
m \otimes M & \xrightarrow{\mu} M \\
1 \otimes M & \xrightarrow{=} M
\end{align*}
\]

are commutative. If \((M, \mu)\) and \((N, \nu)\) are two left \(A\)–modules a morphism of modules from \(M\) to \(N\) is a morphism \(f: M \to N\) in \(\mathcal{M}\) such that \(\nu(A \otimes f) = f \mu\). The category of left \(A\)–modules will be denoted by \(A\mathcal{M}\). Let us remark that \(A\mathcal{M}\) is an abelian category if \(\mathcal{M}\) is so.

Similarly, we construct the category of right modules \(\mathcal{M}_A\). Combining left and right modules we get an \((A,A)\)–bimodule. More precisely, an \((A,A)\)–bimodule is an object in \(\mathcal{M}\) together with two maps, \(\mu_l: A \otimes M \to M\) and \(\mu_r: M \otimes A \to M\), such that \((M, \mu_l) \in A\mathcal{M}\) and \((M, \mu_r) \in A\mathcal{M}\) and the structures are compatible, that is the following diagram is commutative.

\[
\begin{align*}
A \otimes M \otimes A & \xrightarrow{A \otimes \mu_r} A \otimes M \\
\mu_l \otimes A & \xrightarrow{=} A \otimes A \\
M \otimes A & \xrightarrow{\mu_r} M
\end{align*}
\]
A morphism \( f : M \to N \) between two bimodules is a morphism in \( \mathfrak{M} \) which is both a morphism of left ant right modules. For the category of \((A, A)\)-bimodules we shall use the notation \( _{A}\mathcal{M}_A \).

Of course, if \( \mathcal{M} \) is abelian then \( _{A}\mathcal{M}_A \) is abelian too.

**Examples 1.8.** a) \( A \) always is an \((A, A)\)-bimodule, having both left and right module structures defined by the multiplication \( m \).

b) Suppose that \((A, m, u)\) is an algebra in \( (\mathcal{M}, \otimes, 1) \). Then \( A \otimes X \in _{A}\mathcal{M}_A \), for any \( X \in \mathcal{M} \), where the left structure is given by \( \mu := m \otimes X \). Thus we have a functor \( _A F : \mathcal{M} \to _{A}\mathcal{M}_A \), which is defined by \( _A F(X) = A \otimes X \) and \( _A F(f) = A \otimes f \).

Similarly \( X \otimes A \) is a right \( A \)-module, so we obtain a functor: \( F_A : \mathcal{M} \to \mathcal{M}_A \) given by \( F_A(X) = X \otimes A \) and \( F_A(f) = f \otimes A \).

c) Let \( M \in _A\mathcal{M}_A \). Then \( M \otimes A \) is a right \( A \)-module as in the previous example, and is a left \( A \)-module via \( \nu = \mu \otimes A \). These two structures are compatible, defining an \((A, A)\)-bimodule on \( M \otimes A \). Similarly, if \( M \in \mathcal{M}_A \) then \( A \otimes M \) is an \((A, A)\)-bimodule.

In particular \((A \otimes X) \otimes A \) is an \((A, A)\)-bimodule and \( _A F_A : \mathcal{M} \to _{A}\mathcal{M}_A \), \( _A F_A(X) = (A \otimes X) \otimes A \) and \( _A F_A(f) = (A \otimes f) \otimes A \) is a functor.

Analogously, \( A \otimes (X \otimes A) \) can be regarded as an \((A, A)\)-bimodule, and one can easily prove that \( _{A,A} \mu : (A \otimes X) \otimes A \to A \otimes (X \otimes A) \) is a functorial isomorphism of bimodules.

**Proposition 1.9.**

a) \( _A F \) is a left adjoint of \( _A U : \mathcal{M}_A \to \mathcal{M} \), the functor that “forgets” the module structure.

b) \( F_A \) is a left adjoint of \( U_A : \mathcal{M}_A \to \mathcal{M} \), the functor that “forgets” the module structure.

c) \( _A F_A \) is a left adjoint of \( _A U_A : _{A}\mathcal{M}_A \to \mathcal{M} \), the functor that “forgets” the bimodule structure.

**Proof.**
a) To prove that \( _A F \) is a left adjoint of \( _A U : \mathcal{M}_A \to \mathcal{M} \) we need functorial morphisms:

\[ _A \mathcal{M}(A \otimes X, M) \cong \mathcal{M}(X, M), \]

that are inverses each other. We define \( \phi_l(X, M)(f) := f(u \otimes X)|_{X}^{-1} \) and \( \psi_l(X, M)(g) := \mu(A \otimes g) \), where \( \mu \) is the module structure of \( M \). It is easy to prove that \( \psi_l(X, M)(g) \) is a morphism of left modules, and that \( \psi_l(X, M) \) is the inverse of \( \phi_l(X, M) \).

b) The isomorphisms

\[ \mathcal{M}_A(X \otimes A, M) \cong \mathcal{M}(X, M) \]

are now given by \( \phi_r(X, M)(f) := f(X \otimes u) \mu_X \) and \( \psi_l(X, M)(g) := \mu(g \otimes A) \), where \( \mu \) is the module structure of \( M \).

c) The isomorphisms \( _A \mathcal{M}_A((A \otimes X) \otimes A, M) \cong \mathcal{M}(X, M) \) are obtained by combining the isomorphisms constructed above: \( \phi(X, M) = \phi_l(X, M) \phi_r(A \otimes X, M) \), and similarly for \( \psi(X, M) \).

For future references, we explicitly write them down:

\[ \phi(X, M)(f) = f((A \otimes X) \otimes u) \mu_{A \otimes X}(u \otimes X)|_{X}^{-1}, \]
\[ \psi(X, M)(g) = \mu_r(\mu \otimes A) (\mu(A \otimes g) \otimes A), \]

where \( \mu_r \) and \( \mu_l \) give respectively the right and left \( A \)-module structures of \( M \).

**Corollary 1.10.** The functors \( _A F, F_A \) and \( _A F_A \) are right exact.

1.11. Since one of our main goals is to investigate the relative derived functors of \( _A \mathcal{M}_A(A, -) \), with respect to a certain projective class of epimorphisms in \( \mathcal{M} \), we need \( _A \mathcal{M}_A \) to be an abelian category. One can prove easily that \( _A \mathcal{M}_A \) is abelian, if we assume that \( \mathcal{M} \) is so. Thus, from now on, we shall assume that \( \mathcal{M} \) is an abelian category.

In order to produce projective resolutions we will apply the machinery of bar resolutions, see [We Chapter 6.6]. The pair of adjoint resolutions \( (F_A, U_A) \) defines a cotriple \( \langle \perp_A, \varepsilon_A, \delta_A \rangle \) on \( \mathcal{M}_A \). The functor \( \perp_A \) is defined by \( \perp_A := F_A U_A \). The functorial morphism \( \varepsilon_A \) is the counit of the adjunction, and \( \delta_A(M) := F_A(\eta_{U_A(M)}) \), where \( \eta \) is the unit of the adjunction. A quick computation shows us that, for any \((M, \mu) \in \mathcal{M}_A \), we have \( \varepsilon_A(M) = \mu \), and \( \delta_A(M) = (M \otimes u) \mu_{M}^{-1} \otimes A \).
By following the construction in [We 8.6.4], for any \((M, \mu) \in \mathcal{M}_A\) we obtain a simplicial object \((\beta_\bullet(A, M), \partial_\bullet, \sigma_\bullet)\), where \(\beta_n(A, M) = \perp_{A}^{\perp} M\). Its face and degeneracy operators are:
\[
\partial_i = \perp_{A}^{\perp}(\varepsilon_A(\perp_{A}^{\perp} M)) \quad \text{and} \quad \sigma_i = \perp_{A}^{\perp}(\delta_A(\perp_{A}^{\perp} M)).
\]
For \(i < n\) the module structure on \(\perp_{A}^{\perp} M\) is defined by \((\perp_{A}^{\perp} M \otimes m)a_{\perp_{A}^{\perp} M, A, A, M}\), so we have:
\[
\begin{align*}
\partial_i &= \begin{cases}
\perp_{A}^{\perp}(1) & \text{for } 0 < i < n; \\
\perp_{A}^{\perp}(\mu), & \text{for } i = n.
\end{cases} \\
\sigma_i &= \perp_{A}^{\perp}(1) \perp_{A}^{\perp} M \otimes u r_{-1}^{-1}(\perp_{A}^{\perp} M) \otimes A). \tag{3}
\end{align*}
\]
By [We Proposition 8.6.10], the augmented simplicial object \(U(\beta_\bullet(A, M)) \xrightarrow{U(\mu)} U(M)\) is aspherical, so the associated augmented chain complex is exact. Since \(U\) is faithfully exact, it results that \(\beta_\bullet(A, M) \xrightarrow{\beta} M\) is aspherical too. Its associated exact sequence \(\beta_\bullet(A, M)\) will be called, as in the classical case, the bar resolution of \(M\) in \(\mathcal{M}_A\). Our next aim is to give a new interpretation of \(\beta(A, A)\) in terms of \(\mathcal{E}\)–projective resolutions, where \(\mathcal{E}\) is an appropriate class of projective epimorphisms.

**Lemma 1.12.** If \(M\) is an \((A, A)\)–bimodule in \(\mathcal{M}\) then \(\beta_\bullet(A, M)\) is an exact complex in \(_\mathcal{A}\mathcal{M}_A\).

**Proof.** Since \(M \in _\mathcal{A}\mathcal{M}_A\) it follows that \(U(M)\) is a left module, so \(_\mathcal{U}_A^\perp U(M)\) is an \((A, A)\)–bimodule, with the structures as in example [LS]. By induction \(\perp^n M\) is an \((A, A)\)–bimodule for any \(n \geq 0\). If remains to show that the differential maps are morphisms of \((A, A)\)–bimodules. But \(\partial_i : \perp^{n+1} M \to \perp^n M\) is given by \(\partial_i = \perp_{A}^{\perp}(\varepsilon_A(\perp_{A}^{\perp} M))\) and in our case \(\varepsilon_A(\perp_{A}^{\perp} M)\) defines the right structure on \(\perp_{A}^{\perp} M\). Obviously for any bimodule \(N\) the maps \(\mu_r : N \otimes A \to N\) and \(\mu_l : A \otimes N \to N\) defining the module structures are morphisms of left, respectively right, \(A\)–modules. Moreover, if \(f : N \to P\) is a morphism of bimodules, then \(\perp^n_f(f)\) is a morphism of bimodules too. Then \(\partial_i\) is a morphism of bimodules, which ends the proof of the lemma because the differential maps of \(\beta_\bullet(A, M)\) are defined by \(d_n = \sum_{i=0}^{n} (-1)^i \partial_i\). \(\square\)

1.13. Let \(\mathcal{M}\) be an abelian category and let \(\mathcal{E}\) be a class of epimorphisms in \(\mathcal{M}\). We recall that an object \(P\) in \(\mathcal{M}\) is called projective rel \(\varepsilon\), where \(\varepsilon : X \to Y\) is an epimorphism in \(\mathcal{E}\), if \(\mathcal{M}(P, \varepsilon) : \mathcal{M}(P, X) \to \mathcal{M}(P, Y)\) is surjective. \(P\) is called \(\mathcal{E}\)–projective if it is projective rel \(\varepsilon\) for every \(\varepsilon \in \mathcal{E}\). The closure of \(\mathcal{E}\) is the class \(\mathcal{C}(\mathcal{E})\) containing all epimorphism \(\mathcal{E}\) in \(\mathcal{M}\) such that every \(\mathcal{E}\)–projective object is also projective rel \(\varepsilon\). The class \(\mathcal{E}\) is called projective if for any object \(M\) in \(\mathcal{M}\) there is an epimorphism \(\varepsilon : P \to M\) in \(\mathcal{E}\) such that \(P\) is \(\mathcal{E}\)–projective. Suppose now that \(\mathcal{E}\) is a closed class of epimorphisms in \(\mathcal{M}\). A morphism \(f : X \to Y\) in \(\mathcal{M}\) is called \(\mathcal{E}\)–admissible if in the canonical splitting \(f = ip\), \(i\) monic and \(p\) epic, we have \(p \in \mathcal{E}\). Finally an \(\mathcal{E}\)–projective resolution of \(M\) is an exact sequence:
\[
\cdots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{P_1} \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \xrightarrow{0}
\]
such that all maps are \(\mathcal{E}\)–admissible and \(P_n\) is \(\mathcal{E}\)–projective, for every \(n \geq 0\). As in the usual case (\(\mathcal{E}\) is the class of all epimorphisms) one can show that any object in \(\mathcal{M}\) has an \(\mathcal{E}\)–projective resolution, which is unique up to a homotopy. The theory of derived functors can be adapted to the relative context without difficulties. For details the reader is referred to [HS] Chapter XI.

1.14. We are now going to define the projective class of epimorphisms that we are interested in. Let \((\mathcal{M}, \otimes, 1)\) be a monoidal category and let \((A, m, u)\) be an algebra in \(\mathcal{M}\). In the abelian category \(_\mathcal{A}\mathcal{M}_A\) (recall that we always assume that \(\mathcal{M}\) is abelian) we consider the class \(\mathcal{E}\) of all epimorphisms that have a section in \(\mathcal{M}\). To prove that \(\mathcal{E}\) is projective we note that \(\mathcal{E} = \mathcal{U}_A^{-1}(\mathcal{E}_0)\), where \((\mathcal{A}F_A, \mathcal{U}_A)\) is the pair of adjoint functors from Proposition [EM] and \(\mathcal{E}_0\) is the class of all epimorphism in \(\mathcal{M}\) that have a section in \(\mathcal{M}\). Obviously \(\mathcal{E}_0\) is projective since any object in \(\mathcal{M}\) is \(\mathcal{E}_0\)–projective. Theorem IX.4.1 in [HS] reads in our situation as follows.

**Proposition 1.15.** The class \(\mathcal{E}\) of all epimorphisms in \(_\mathcal{A}\mathcal{M}_A\) that split in \(\mathcal{M}\) is projective. The objects \(_\mathcal{A}F_A\) where \(P \in \mathcal{M}\), are \(\mathcal{E}\)–projective and are sufficient for \(\mathcal{E}\)–presenting objects of \(_\mathcal{A}\mathcal{M}_A\), so that the \(\mathcal{E}\)–projectives are precisely the direct summands of objects \(_\mathcal{A}F_A\).
1.16. Following [HS], in view of foregoing proposition, we can now consider, for every \( M \in A\mathcal{M}_A \), the right \( \mathcal{E} \)-derived functors \( R^\bullet \mathcal{E} F_M \) of \( F_M := A\mathcal{M}_A(-, M) \). Then, for every \( M, N \in A\mathcal{M}_A \), we set:

\[
\text{Ext}^\bullet_\mathcal{E}(N, M) = R^\bullet \mathcal{E} F_M(N).
\]

The following well known result can be proved as in the non-relative case.

**Proposition 1.17.** Let \( (A, m, u) \) be an algebra in a monoidal category \( (\mathcal{M}, \otimes, 1) \) and let \( N \in A\mathcal{M}_A \).

The following assertions are equivalent:

1. \( N \) is \( \mathcal{E} \)-projective.
2. \( \text{Ext}^\bullet_\mathcal{E}(N, M) = 0 \), for all \( M \in A\mathcal{M}_A \).
3. \( \text{Ext}^\bullet_\mathcal{E}(N, M) = 0 \), for all \( M \in A\mathcal{M}_A \), for all \( n > 0 \).

1.18. We now want to prove that the bar resolution \( \beta_\bullet(A, A) \) is made of \( \mathcal{E} \)-projective modules. To this aim, let us prove that we have a canonical isomorphism of bimodules:

\[
\varphi_n : \bigwedge^{n+1} A \to (A \otimes \bigwedge^n A) \otimes A, \quad \text{for all } n \geq 1.
\]

Indeed, if \( n = 1 \) then \( a_{\cdot,A,A} : (A \otimes A) \otimes A \to A \otimes (A \otimes A) \) is an isomorphism of bimodules, as we noticed in example [LS](c). So we can take \( \varphi_1 := a_{\cdot,A,A} \). Let us assume that we have constructed \( \varphi_1, \ldots, \varphi_n \). Then:

\[
\bigwedge^n A = \bigwedge_A \bigwedge^n A = \bigwedge_A ((A \otimes \bigwedge^{n-1} A) \otimes A) \cong \bigwedge_A (A \otimes (\bigwedge^{n-1} A \otimes A)).
\]

As \( \bigwedge_A (A \otimes (\bigwedge^{n-1} A \otimes A)) = \bigwedge_A (A \otimes (\bigwedge^{n-1} A)) = (A \otimes (\bigwedge^{n-1} A)) \otimes A \), we can take by definition \( \varphi_{n+1} := a_{\bigwedge_A (\bigwedge^{n-1} A), A} \). Since \( \varphi_n \) and \( a_{\bigwedge_A (\bigwedge^{n-1} A), A} \) are morphisms of \( (A, A) \)-bimodules we deduce that \( \bigwedge_A (\bigwedge^{n} A) \) and \( \bigwedge_A (a_{\bigwedge_A (\bigwedge^{n-1} A), A}) \) are so. Hence \( \varphi_{n+1} \) is an isomorphism of bimodules. Now we can prove the following lemma.

**Lemma 1.19.** \( \beta_n(A, A) \) is \( \mathcal{E} \)-projective for every \( n \in \mathbb{N} \).

**Proof.** By [LS] we know that \( \beta_n(A, A) \cong (A \otimes \bigwedge^n A) \otimes A = \bigwedge_A (A \otimes (\bigwedge^n A \otimes A)) \). Since \( \mathcal{E}_0 \) is the class of all splitting epimorphisms, it follows that any object in \( \mathcal{M} \) is \( \mathcal{E}_0 \)-projective. By [HS] Theorem IX.4.1] we deduce that \( \bigwedge_A (\bigwedge^n A \otimes A) \) is \( \mathcal{E} \)-projective (see also the definition of \( \mathcal{E} \) for every \( X \in \mathcal{M} \)). In particular \( \beta_n(A, A) \cong \bigwedge_A (A \otimes (\bigwedge^n A \otimes A)) \) is \( \mathcal{E} \)-projective. \( \square \)

**Theorem 1.20.** \( \beta_\bullet(A, A) \) is an \( \mathcal{E} \)-projective resolution of \( A \) in \( A\mathcal{M}_A \).

**Proof.** We already know that \( \beta_\bullet(A, A) \) is an exact sequence in \( A\mathcal{M}_A \), see Lemma 1.12. By Lemma 1.19 it results that \( \beta_n(A, A) \) is \( \mathcal{E} \)-projective for every \( n \in \mathbb{N} \). It remains to show that the differential maps of \( \beta_\bullet(A, A) \) are \( \mathcal{E} \)-admissible. By [We] Proposition 8.6.10 it follows that the augmented simplicial object \( U_A(\beta_\bullet(A, A)) \) \( U^{(m)} \) \( U_A(A) \) constructed in [LS] is contractible in \( \mathcal{M} \). Here, \( m \) denotes the multiplication of \( A \). It follows that \( \beta_\bullet(A, A) \) is split exact in \( \mathcal{M} \), see [We] Exercise 8.4.6]. Let \( s_n : \beta_n(A, A) \to \beta_{n+1}(A, A) \) be a morphism such that \( d_n = d_n s_{n-1} d_n \), where \( d_n : \beta_n(A, A) \to \beta_{n-1}(A, A) \) are the differentials of \( \beta_\bullet(A, A) \). If \( d_n = i_n p_n \) is the canonical decomposition in \( \mathcal{M} \), with \( p_n \) an epimorphism and \( i_n \) a monomorphism, then \( p_n(s_{n-1} i_n) = \text{Id} \). Thus \( d_n \) splits in \( \mathcal{M} \), so \( d_n \) is admissible for any \( n \in \mathbb{N} \). \( \square \)

**Definition 1.21.** Let \( (\mathcal{M}, \otimes, 1) \) be a monoidal category. Suppose that \( A \) is an algebra in \( \mathcal{M} \) and that \( M \) is an \( (A, A) \)-bimodule. The **Hochschild cohomology** of \( A \) with coefficients in \( M \) is:

\[
\text{H}^\bullet(A, M) = \text{Ext}^\bullet_\mathcal{E}(A, M).
\]

1.22. In order to compute \( \text{H}^\bullet(A, M) \) we can apply the functor \( A\mathcal{M}_A(-, M) \) to \( \beta_\bullet(A, A) \), the bar resolution of \( A \) in \( A\mathcal{M}_A \). We obtain the **standard complex**:

\[
0 \to \mathcal{M}(1, M) \xrightarrow{b^0} \mathcal{M}(A, M) \xrightarrow{b^1} \mathcal{M}(A \otimes A, M) \xrightarrow{b^2} \mathcal{M}(A \otimes A \otimes A, M) \xrightarrow{b^3} \cdots
\]
where for \( n \in \{0, 1, 2\} \) the differentials \( b^n \) are given by:

\[
\begin{align*}
b^0(f) &= \mu_0(A \otimes f) r_A^{-1} - \mu_r(f \otimes A) l_A^{-1}; \\
b^1(f) &= \mu_1(A \otimes f) - f m + \mu_r(f \otimes A); \\
b^2(f) &= \mu_2(A \otimes f) a_{A, A, A} - f(m \otimes A) + f(A \otimes m) a_{A, A, A} - \mu_r(f \otimes A).
\end{align*}
\]

We will not write down the formula for \( b^n \) in general, because we shall need an explicit computation only in degree \( n \in \{0, 1, 2\} \). Note that if we omit the associativity constraint \( a_{A, A, A} \) then the formulas for \( b^0, b^1 \) and \( b^2 \) becomes the usual ones, well–known from the case \( \mathcal{M} = 2\mathcal{R}_K \). This observation holds true in general, for all \( b^n, n \in \mathbb{N} \).

2. Hochschild dimension of algebras in monoidal categories

In this section we define the Hochschild dimension of an algebra \( A \) in a monoidal category \((\mathcal{M}, \otimes, 1)\). Then we will give characterizations of algebras of Hochschild dimension less than or equal to 1. Our next goal will be to study separable algebras in monoidal categories and to prove a Wedderburn-Malcev type theorem. This last theorem will be used in the next sections to study the structure of Hopf algebras having a “nice” radical (Section 3) or coradical (Section 5).

Definition 2.1. An algebra \((A, m, u)\) in a monoidal category \((\mathcal{M}, \otimes, 1)\) is called separable if the multiplication \( m : A \otimes A \to A \) has a section in the category of \((A, A)\)–bimodules \(_A\mathcal{M}_A\).

Remark 2.2. The multiplication \( m \) always has a section in \(_A\mathcal{M}\) and in \(_A\mathcal{M}_A\), namely \( A \otimes u \) and respectively \( u \otimes A \).

Proposition 2.3. \((A, m, u)\) is separable iff \( A \) is \( \mathcal{E} \)–projective.

Proof. Recall that \( \mathcal{E} \) is the projective class of all epimorphism in \(_A\mathcal{M}_A\) that have a section in \( \mathcal{M} \). Therefore an \((A, A)\)–bimodule \( P \) is \( \mathcal{E} \)–projective if there is an object \( X \) in \( \mathcal{M} \) and an epimorphism \( \pi : A \otimes (X \otimes A) \to P \) in \(_A\mathcal{M}_A\) that splits in \(_A\mathcal{M}_A\). Thus \( A \) is \( \mathcal{E} \)–projective if \( m : A \otimes A \to A \) has a section in \(_A\mathcal{M}_A\), since \( A \otimes X \cong A \otimes (1 \otimes A) \). Conversely, if \( A \) is \( \mathcal{E} \)–projective, then \( m \) has a section in \(_A\mathcal{M}_A\) since \( m \) is an epimorphism in \( \mathcal{E} \).

Theorem 2.4. Let \((A, m, u)\) be an algebra in a monoidal category \((\mathcal{M}, \otimes, 1)\). The following assertions are equivalent:

(a) \( A \) is separable.
(b) \( \mathcal{H}^1(A, M) = 0 \), for all \( M \in _A\mathcal{M}_A \).
(c) \( \mathcal{H}^n(A, M) = 0 \), for all \( M \in _A\mathcal{M}_A \), for all \( n > 0 \).

Proof. Follows by Proposition 1.1.7 in the case when \( N = A \), and by Proposition 2.3. 

Definition 2.5. The Hochschild dimension of an algebra \( A \) in the monoidal category \( \mathcal{M} \) is the smallest \( n \in \mathbb{N} \) (if it exists) such that \( \mathcal{H}^{n+1}(A, M) = 0, \forall M \in _A\mathcal{M}_A \). If such an \( n \) does not exist, we will say that the Hochschild dimension of \( A \) is infinite. We shall denote the Hochschild dimension of \( A \) by \( \text{Hdim}(A) \).

Corollary 2.6. An algebra \((A, m, u)\) in \((\mathcal{M}, \otimes, 1)\) is separable iff \( \text{Hdim}(A) = 0 \).

For the characterization of algebras of Hochschild dimension 1 we need the interpretation of \( \mathcal{H}^2(A, -) \) in terms of algebra extensions. First some definitions.

Definition 2.7. Let \( A \) and \( B \) be two algebras in a monoidal category \((\mathcal{M}, \otimes, 1)\). A morphism \( \sigma : B \to A \) in \( \mathcal{M} \) is called unital if \( \sigma u_B = u_A \), where \( u_A \) and \( u_B \) are the units of \( A \) and \( B \), respectively. Moreover, if \( f : A \to B \) is a morphism of algebras in \( \mathcal{M} \) we shall say that \( \sigma \) is an unital section of \( f \) if \( f \sigma = \text{Id}_B \) and \( \sigma \) is an unital morphism.

Definitions 2.8. a) An ideal of an algebra \((A, m, u)\) in \((\mathcal{M}, \otimes, 1)\) is a pair \((I, i)\) such that \( I \) is an \((A, A)\)–bimodule and \( i : I \to A \) is a monomorphism of \((A, A)\)–bimodules.

b) If \((I, i)\) is an ideal in \( A \) and \( n \geq 2 \), we define \( I^n \) to be the image of \( m_ni \otimes n \) where \( m_n : A \otimes n \to A \) is the \( n^{th} \) iterated multiplication of \( A \) (\( m_2 := m \)).

c) An ideal \((I, i)\) is called nilpotent if there is \( n \geq 2 \) such that \( I^n = 0 \) (equivalently \( m_ni \otimes n = 0 \)).
REMARKS 2.9. a) If \((I, i)\) is a ideal then there is a unique algebra structure on \(A/I = \text{Coker } i\) such that the canonical morphism \(A \to A/I\) is an algebra map.

b) There is a canonical monomorphism \(i_n : I^n \to A\) in \(_A\mathcal{M}_A\). Thus \((i_n, i_n)\) is an ideal of \(A\). The map \(i_n\) factors through a morphism of bimodules \(i' : I^n \to I\) (i.e. \(i'_n = i_n\)). Moreover, for every \(m \geq n \geq 2\), there is a morphism of bimodules \(i_{n,m} : I^m \to I^n\) such that \(i_n i_{n,m} = i_m\).

**Lemma 2.10.** Let \((A, m, u)\) and \((E, m_E, u_E)\) be algebras. Let \(\pi : E \to A\) be a morphism of algebras in \((\mathcal{M}, \otimes, 1)\) that has a section \(\sigma\) in \(\mathcal{M}\). Assume that \((\text{Ker } \pi)^2 = 0\).

a) If \(\sigma : A \to E\) is a section of \(\pi\) then: 
\[m_E(\sigma u \otimes \sigma u)t_1^{-1} = 2\sigma u - u_E.\]

b) The morphism \(\pi\) admits a unital section.

c) \(\text{Ker } \pi\) has a natural structure of \((A, A)\)–bimodule given by \(\mu_l : A \otimes \text{Ker } \pi \to \text{Ker } \pi\) and \(\mu_r : \text{Ker } \pi \otimes A \to \text{Ker } \pi\) that are uniquely defined by:
\[(7) \quad i\mu_l = m_E(E \otimes i)(\sigma \otimes \text{Ker } \pi), \quad (8) \quad i\mu_r = m_E(i \otimes E)(\text{Ker } \pi \otimes \sigma),\]

where \(i : \text{Ker } \pi \to E\) is the canonical inclusion. The morphisms \(\mu_l\) and \(\mu_r\) do not depend on the choice of the section \(\sigma\).

**Proof.** a) The relation \(\pi(\sigma u - u_E) = 0\) tell us that there exists a unique morphism \(\lambda : 1 \to E\) such that \(\sigma u - u_E = i\lambda\). On the other hand, \((\text{Ker } \pi)^2 = 0\) so \(m_E(i \otimes i) = 0\). Thus:
\[m_E[(\sigma u - u_E) \otimes (\sigma u - u_E)]t_1^{-1} = 0.\]

By expanding this relation and using that \(u_E\) is the unit of \(E\) we obtain the formula from the first part of the lemma.

b) The unital section is given by \(\sigma' := 2\sigma - m_E(\sigma \otimes \sigma)(A \otimes u)r_A^{-1}\). The details are left to the reader.

c) Left also as an exercise. \(\square\)

**Definitions 2.11.** 1) Let \((A, m, u)\) be an algebra in \((\mathcal{M}, \otimes, 1)\) and let \(M\) be an \((A, A)\)–bimodule. An **Hochschild extension of \(A\)** with kernel \(M\), is an algebra homomorphism \(\pi : E \to A\) that satisfies the following conditions:

a) there is a section \(\sigma\) of \(\pi\);

b) there is a morphism \(i : M \to E\) such that \((M, i)\) is the kernel of \(\pi\) in \(\mathcal{M}\);

c) \(m(i \otimes i) = 0\) (i.e. \(M^2 = 0\));

d) the \((A, A)\)–bimodule structure of \(M\) coincides with the one induced by \(i\) (by the previous lemma \(M\) is an \((A, A)\)–bimodule with the module structure \(\otimes\) and \(\otimes\)).

2) Two Hochschild extensions \(\pi : E \to A\) and \(\pi' : E' \to A\) of \(A\) with kernel \(M\) are equivalent if there is a morphism of algebras \(f : E \to E'\) such that \(\pi'f = \pi\) and \(f' : \text{Ker } \pi \to \text{Ker } \pi'\), the restriction of \(f\), is an isomorphism in \(\mathcal{M}\).

**Remark 2.12.** The morphism \(f'\) is an isomorphism of \((A, A)\)–bimodules. By \(5\)–Lemma \(f\) is always an isomorphism of algebras.

**Lemma 2.13.** Let \((A, m, u)\) be an algebra and let \((M, \mu_l, \mu_r)\) an \((A, A)\)–bimodule. Suppose that \(\omega : A \otimes A \to M\) is a Hochschild 2–cocycle. If \(m_\omega : (A \otimes M) \otimes (A \otimes M) \to A \otimes M\) and \(u_\omega : 1 \to A \otimes M\) are defined by:
\[m_\omega : = i_A m(p_A \otimes p_A) + i_M [\mu_r(p_M \otimes p_A) + \mu_l(p_A \otimes p_M) - \omega(p_A \otimes p_A)], \quad u_\omega : = i_A u + i_M \omega(u \otimes u)t_1^{-1},\]

where \(i_A, i_M\) are the canonical injections in \(A \oplus M\) and \(p_A, p_M\) are the canonical projections. Then \((A \oplus M, m_\omega, u_\omega)\) is an algebra in \(\mathcal{M}\). Moreover \(p_A : A \oplus M \to A\) is a Hochschild extension of \(A\) with kernel \(\text{Ker } p_A = (M, i_M)\). This extension will be denoted by \(E_\omega\).

**Proof.** Very tedious computations, that will be skipped. We just remark that the \(m_\omega\) defines an associative multiplication because \(\omega\) is a cocycle. \(\square\)
DEFINITIONS 2.14. a) The Hochschild extension $p_A : E_\omega \to A$, introduced in the Lemma above, is called the Hochschild extension associated to $\omega$.

b) If $(A, m_A, u_A)$ and $(E, m_E, u_E)$ are algebras and $\sigma : A \to E$, is a unital morphism in $\mathcal{M}$ we define the curvature of $\sigma$ to be the morphism:

\[ \theta_\sigma : A \otimes A \to E, \quad \theta_\sigma := \sigma m_A - m_E(\sigma \otimes \sigma) \]

PROPOSITION 2.15. Let $\pi : E \to A$ be a Hochschild extension of $A$ with kernel $(M, i)$, let $\sigma : A \to E$ be a section of $\pi$ and let $\theta_\sigma$ be the morphism defined by formula (9). Then there exists a unique morphism $\omega : A \otimes A \to M$, such that $i \omega = \theta_\sigma$. Moreover $\omega$ is a 2-cocycle whose class $[\omega]$ in $H^2(A, M)$ does not depend on the choice of $\sigma$. If $p_A : E_\omega \to A$ is the Hochschild extension associated to $\omega$, the morphism $f_\omega := \sigma p_A + i p_M : E_\omega \to E$
defines an equivalence of Hochschild extensions.

Proof. The morphism $\pi$ is an algebra homomorphism, so that $\pi \theta_\sigma = 0$. Then there exists a unique morphism $\omega : A \otimes A \to M$ such that $i \omega = \theta_\sigma$. Let $\mu_l$ and $\mu_r$ be the morphisms that define the module structure of $M$ and let $m$ and $m_E$ be the multiplications of $A$ and $E$. We have:

\[ ib^2(\omega) = m_E(\sigma \otimes \theta_\sigma) - \theta_\sigma(m \otimes A) + \theta_\sigma(A \otimes m) - m_E(\theta_\sigma \otimes \sigma), \]

and, in view of the definition of $\theta_\sigma$, we obtain $ib^2(\omega) = 0$ so that $b^2(\omega) = 0$.

Let $\sigma' : A \to E$ be another section of $\pi$. Since $\pi(\sigma - \sigma') = 0$, there exists a unique morphism $\tau : A \to M$ such that $i \tau = \sigma - \sigma'$. If $\omega'$ is the associated to $\sigma'$, a straightforward computation shows us that:

\[ i \left( \omega' - b^1(\tau) \right) = i \omega, \]

so $\omega' = b^1(\tau) + \omega$. Thus $[\omega] = [\omega']$.

As $m(i \otimes i) = 0$ one can check that $f_\omega$ is an algebra homomorphism. Moreover $\pi f_\omega = p_A$ and $f_{\omega,i_M} = i$ (the restriction of $f_\omega$ to $M$ is the isomorphism $Id_M$). Thus $f_\omega$ is an equivalence of Hochschild extensions.

□

DEFINITION 2.16. With the notations of the previous Proposition, the class $[\omega]$ is called the cohomology class associated to the Hochschild extension $\pi : E \to A$.

LEMMA 2.17. Let $\omega : A \otimes A \to M$ be a 2-cocycle and let $p_A : E_\omega \to A$ be the Hochschild extension associated to $\omega$. Then the cohomology class associated to the Hochschild extension $p_A : E_\omega \to A$ is exactly $[\omega]$.

Proof. Since $i_A : A \to E_\omega$ is a section of $p_A$, we have:

\[ \theta_{i_A} = i_A m - m_\omega(i_A \otimes i_A) = i_A m - i_A m + i_M \omega = i_M \omega. \]

Thus, in view of Proposition 2.17 the cohomology class associated to this extension is $[\omega]$. □

2.18. Let $A$ be an algebra and let $M$ be an $(A, A)$–bimodule. If $\pi : E \to A$ is an Hochschild extension, we will denote by $[E]$ the class of the Hochschild extensions equivalent to it. We define:

\[ \text{Ext}(A, M) := \{ [E] \mid E \to A \text{ is a Hochschild extension of } A \text{ with kernel } M \}. \]

PROPOSITION 2.19. Let $A$ be an algebra and let $M$ be an $(A, A)$–bimodule. If $\omega, \omega' : A \otimes A \to M$ are 2–cocycle, then:

\[ [\omega] = [\omega'] \iff [E_\omega] = [E_{\omega'}]. \]

Proof. Suppose that $[E_\omega] = [E_{\omega'}]$. There exists an algebra homomorphism $g : E_\omega \to E_{\omega'}$ that is an equivalence of Hochschild extensions. As $g i_A$ is a section of $p'_{\omega} : E_{\omega'} \to A$, we have:

\[ \theta_{g i_A}' = i_M' \omega, \]

so that, by Lemma 2.17 $[\omega] = [\omega']$.

If $[\omega] = [\omega']$, there exists a morphism $\tau : A \to M$ such that $\omega = \omega' + b^1(\tau)$. The morphism $\sigma := i_A + i_M \tau : A \to E_\omega$ is a section of $p_A : E_\omega \to A$. Thus:

\[ \theta_\sigma = i_M' \omega'. \]
Applying Proposition 2.19 in the case when $E := E_\omega$, we get that there is an equivalence between $p'_A : E_\omega \to A$ and $p_A : E_\omega \to A$, so that $[E_\omega] = [E_\omega]$.

**Theorem 2.20.** Let $A$ be an algebra and let $M$ be an $(A,A)$–bimodule. The map:

$$\Phi : H^2(A,M) \to \text{Ext}(A,M),$$

where $\Phi ([\omega]) := [E_\omega]$, is well defined and is a bijection.

**Proof.** $\Phi$ is well defined and injective by Proposition 2.19 and it is surjective by Proposition 2.15. \qed

**Definition 2.21.** An extension $\pi : E \to A$ is a **trivial extension** whenever it admits a section that is an algebra homomorphism.

**Corollary 2.22.** Let $A$ be an algebra and let $M$ be an $(A,A)$–bimodule. Then a Hochschild extension of $A$ with kernel $M$ is trivial if and only if the associated cohomology class is zero.

**Proof.** Let $\pi : E \to A$ be a Hochschild extension of $A$ with kernel $M$, and let $i : M \to E$ the canonical injection. By the definition of trivial extensions, there exists a section $\sigma : A \to E$ of $\pi$ that is an algebra homomorphism. Thus $i\omega = \theta_\sigma = \sigma m - m_E (\sigma \circ \sigma) = 0$, so that $[\omega] = 0$.

If $[\omega] = 0$, where $\omega$ is the 2–cocycle associated to $\pi : E \to A$, then $[E] = [E_0]$ that is there exists an algebra homomorphism $f : E_0 \to E$ that is an equivalence of Hochschild extensions. Let $\sigma_0 : A \to E_0$ be a unital section of $p_0 : E_0 \to A$. Then $0 = i0 = \theta_{\sigma_0} = \sigma_0 m - m_E (\sigma_0 \circ \sigma_0) = 0$ so that $\sigma_0$ is an algebra homomorphism. It is easy to see that $f\sigma_0$ is a section of $\pi$ that is an algebra homomorphism. \qed

**Corollary 2.23.** Let $A$ be an algebra and let $M$ be an $(A,A)$–bimodule. Then $H^2(A,M) = 0$ if and only if any Hochschild extension of $A$ with kernel $M$ is trivial.

**Proof.** It follows by Theorem 2.20 and Corollary 2.22. \qed

**Definition 2.24.** Let $(A,m,u)$ be an algebra and let $f : X \to A$ be a morphism in $\mathcal{M}$. If $A_f := m (m \otimes A) (A \otimes f \otimes A)$ then the two–sided ideal of $A$ generated by $f$ is defined by:

$$(A(f)_A,i_f) := \text{Im}(A_f).$$

**Lemma 2.25.** Let $(A,m,u)$ be an algebra and let $f : X \to A$ be a morphism in $\mathcal{M}$. Then:

a) $(A(f)_A,i_f)$ is an ideal of $A$.

b) If $\xi : A \to Y$ is a morphism such that $\xi m = m(\xi \otimes \xi)$ and $\xi f = 0$, then $\xi i_f = 0$.

c) If $\xi : A \to Y$ is a morphism such that $\xi i_f = 0$, then $\xi f = 0$.

2.26. Let $(A,m_A,u_A)$ be an algebra in $(\mathcal{M}, \otimes, 1)$. Let us consider the tensor algebra $T(A) := \oplus_{n \in \mathbb{N}} T_n(A)$, where $T_0(A) := 1$ and $T_{n+1}(A) := T_n(A) \otimes A, \forall n > 0$. We set:

$$(I, \varsigma_1) := T(A) \langle u_{T(A)} - i_A u_A \rangle_{T(A)},$$

where $u_{T(A)} : 1 \to T(A)$ and $i_A : A \to T(A)$ are the canonical morphisms. Moreover we set:

$$(E_A, \rho'_A) := \text{Coker}(\varsigma_1).$$

Since $I$ an ideal of $T(A)$, $E_A$ is an algebra and $\rho'_A$ an algebra homomorphism: by the previous Lemma, $\rho'_A (u_{T(A)} - i_A u_A) = 0$. Let $\rho_A = \rho'_A i_A : A \to E_A$. Then we have:

$$\rho_A u_A = \rho'_A i_A u_A = \rho'_A u_{T(A)} = u_{E_A}.$$

So, by construction, $\rho_A$ is a unital morphism.

**Definition 2.27.** $(E_A, \rho_A)$ is called the **universal extension**. The following proposition justifies this name.

**Proposition 2.28.** Let $A,B$ be algebras in $(\mathcal{M}, \otimes, 1)$. Given a unital morphism $\rho : A \to B$, there exists a unique algebra homomorphism $v : E_A \to B$ such that $v \rho_A = \rho$. 

Proof. By the universal property of the tensor algebra $T(A)$, there exists a unique algebra homomorphism $\xi : T(A) \to B$ such that $\xi \iota_A = \rho$. Then:

$$\xi (u_{T(A)} - i_A u_A) = \xi u_{T(A)} - \rho u_A = u_B - u_B = 0,$$

where $u_B$ is the unit of $B$. By Lemma (2.25) b), $\xi \iota_l = 0$ so that there exists a unique morphism $v : E_A \to B$ such that $v\rho_A = \xi$ and hence $v\rho_A = v\rho_A i_A = \xi i_A = \rho$. Moreover such a morphism is an algebra map. The uniqueness is due to the universal property of $T(A)$. \qed

Corollary 2.29. There exists a unique algebra map $\pi_A : E_A \to A$ such that $\pi_A \rho_A = \text{Id}_A$.

2.30. Let $(e_A, i_A) = \text{Ker} \pi_A$. We have the exact sequence:

$$0 \to e_A \xrightarrow{i_A} E_A \xrightarrow{\pi_A} A \to 0.$$

From this sequence, we obtain an Hochschild extension of $A$, namely:

$$0 \to e_A \xrightarrow{e_A} E_A \xrightarrow{\pi_A} A \to 0,$$

where the section of $E_A/e_A^2 \to A$ is given by the composition of $E_A \to E_A/e_A^2$ and $\rho_A : A \to E_A$. The extension (10) is called the universal Hochschild extension of $A$.

Proposition 2.31. Let $A, B$ be algebras in $(\mathcal{M}, \otimes, 1)$, let $0 \to M \xrightarrow{i} E \xrightarrow{\pi} B \to 0$ be an Hochschild extension of $B$ with kernel $M$ and let $f : A \to B$ be an algebra homomorphism. Then, there exists an algebra homomorphism $\pi_f : E_A/e_A^2 \to E$ and an $(A, A)$-bimodule homomorphism $g : e_A/e_A^2 \to M$ such the following diagram commutes:

$$
\begin{array}{ccc}
0 & \to & e_A/e_A^2 \\
\downarrow & & \downarrow \\
E_A/e_A^2 & \xrightarrow{\pi_f} & A \\
\downarrow & & \downarrow \\
0 & \to & M \\
\end{array}
$$

Proof. Let $\rho : B \to E$ be a unital section of $\pi$. By Proposition 2.28 there exists a unique algebra homomorphism $\pi_f : E_A \to E$ such that $\rho f = \pi_f \rho_A$. Therefore we get:

$$f \pi_A \rho_A = f = \pi f \rho = \pi \pi_f \rho_A.$$ 

Thus, by Proposition 2.28 we get $f \pi_A = \pi \pi_f$. Since $M = \text{Ker} \pi$ the relation $\pi \pi_f i_A = f \pi i_A = 0$ implies the existence of a unique morphism $\gamma : e_A \to M$ such that $i_A = \pi_f i_A$. Then, from $M^2 = 0$ we deduce:

$$f \pi_A \rho_A = f = \pi f \rho = \pi \pi_f \rho_A.$$ 

so that there exists a unique morphism $\pi_f : E_A/e_A^2 \to E$ which, composed with the canonical projection $E_A \to E_A/e_A^2$, gives $\pi_f$. Since $e_A/e_A^2$ is the kernel of $\pi_A$ there is a unique $g$ such that the left square of the above diagram in commutative. \qed

Theorem 2.32. Let $A$ be an algebra in $(\mathcal{M}, \otimes, 1)$. Then the following conditions are equivalent:

(a) The universal Hochschild extension of $A$ is trivial.
(b) If $\pi : E \to B$ is an algebra homomorphism that splits in $\mathcal{M}$ and $(\ker \pi)^2 = 0$, then any algebra homomorphism $f : A \to B$ can be lifted to an algebra homomorphism $A \to E$.
(c) Given an algebra homomorphism $f : A \to B$ which admits a section that is an algebra homomorphism, then any Hochschild extension of $B$ is trivial.
(d) Any Hochschild extension of $A$ is trivial.
(e) $H^2(A, M) = 0, \forall M \in A\mathcal{M}_A$.
(f) Given an algebra homomorphism $f : A \to B$ which admits a section that is an algebra homomorphism, then the universal Hochschild extension of $B$ is trivial.
Proof. (a) ⇒ (b) If \( \pi : E \to B \) is an algebra homomorphism that splits in \( \mathcal{M} \) and \( (\text{Ker} \, \pi )^2 = 0 \), then \( \pi \) is an Hochschild extension of \( B \). If \( \sigma : A \to E_A/e_A^2 \) is an algebra morphism that is a section of the morphism \( E_A/e_A^2 \to A \), then \( \pi_f \sigma : A \to E \) is the algebra morphism that lifts \( f \).

(b) ⇒ (c) If \( \pi : E \to B \) is an Hochschild extension, then \( f : A \to B \) can be lifted to an algebra homomorphism \( g : A \to E \). If \( \sigma : B \to E \) is a section of \( f \) that is an algebra homomorphism, then \( g \sigma \) is a section of \( \pi \) that is an algebra homomorphism, so that \( \pi \) is trivial.

(c) ⇒ (d) The identity of \( A \) is an algebra homomorphism and is its own section, so that any Hochschild extension of \( A \) is trivial.

(d) ⇒ (a) It is obvious.

(d) ⇔ (e) It follows by Corollary 2.32 and obviously (f) is equivalent to the others. \( \square \)

**Lemma 2.33.** Let \( A \) be an algebra and let \( \theta_A : A \otimes A \to E_A \) be the curvature associated to the canonical morphism \( \rho_A : A \to E_A \). Then \( e_A \simeq A \langle \theta_A \rangle_{E_A} \).

**Proof.** Let us denote by \( (X, \phi) \) the cokernel of \( \Lambda_{\theta_A} \). By definition, \( E_A \langle \theta_A \rangle_{E_A} = \text{Im} \, \Lambda_{\theta_A} \). As \( \pi_A : E_A \to A \) is an algebra homomorphism, then \( \pi_A \theta_A = 0 \). So, by Lemma 2.26, \( \pi_A i_{\theta_A} = 0 \).

Let \( \beta : X \to A \) be such that \( \pi_A = \beta \phi \). Since \( X \simeq \text{Coker} \, i_{\theta_A} \) and \( \langle E_A \langle \theta_A \rangle_{E_A}, i_{\theta_A} \rangle \) is an ideal of \( E_A \) it follows that \( X \) has an algebra structure so that \( \phi \) is an algebra homomorphism. As, by definition, \( \phi i_{\theta_A} = 0 \) we have \( \phi \theta_A = 0 \). This relation, the fact that \( \phi \) is a morphism and \( \rho_A \) a unital morphism imply that \( \phi \rho_A : A \to X \) is an algebra homomorphism. We have:

\[
\phi \rho_A \pi_A = \phi \rho_A.
\]

By Proposition 2.32, we deduce that

\[
\phi \rho_A \pi_A = \phi.
\]

In particular, \( \phi \rho_A \) is an epimorphism. As \( \beta \phi \rho_A = \text{Id}_A \) then \( \phi \rho_A \) is a monomorphism too. Therefore we get \( (A, \pi_A) \simeq X \), so that \( E_A \langle \theta_A \rangle_{E_A} \simeq \text{Ker} \, \pi_A = e_A \). \( \square \)

**Theorem 2.34.** Let \( A \) be an algebra in \( \mathcal{M} \). Then the following assertions are equivalent:

(a) Ker \( m \) is \( \mathcal{E} \)-projective, where \( m \) is the multiplication of \( A \).

(b) \( H^2(\Lambda, M) = 0 \), \( \forall \Lambda \in A \mathcal{M}_A \).

(c) Let \( \pi : E \to B \) be an algebra epimorphism and let \( I \) denote the kernel of \( \pi \). Assume that there is \( n \in \mathbb{N} \) so that \( I^n = 0 \). If for any \( r = 1, \ldots, n - 1 \) the canonical projection \( p_r : E/I^{r+1} \to E/I^r \) splits in \( \mathcal{M} \) then any algebra homomorphism \( f : A \to B \) can be lifted to an algebra homomorphism \( g : A \to E \).

(d) Let \( \pi : E \to A \) be an algebra epimorphism and let \( I \) denote the kernel of \( \pi \). Assume that there is \( n \in \mathbb{N} \) so that \( I^n = 0 \). If for any \( r = 1, \ldots, n - 1 \) the canonical projection \( p_r : E/I^{r+1} \to E/I^r \) splits in \( \mathcal{M} \) then \( \pi \) has a section which is an algebra homomorphism.

**Proof.** (a) ⇔ (b) Let \( (L, j) := \text{Ker} \, m \) and let us consider the exact sequence:

\[
0 \to L \xrightarrow{j} A \otimes A \xrightarrow{\pi_1} A \to 0.
\]

We know that \( m \) has a section in \( \mathcal{M} \) so that \( m \in \mathcal{E} \) and the sequence above is \( \mathcal{E} \)-exact. Given any \( M \in A \mathcal{M}_A \), we apply the functor \( F := A \mathcal{M}_A(\cdot, M) \) to the sequence above and find:

\[
\text{Ext}^1_{\mathcal{E}}(A, M) \to \text{Ext}^1_{\mathcal{E}}(L, M) \to \text{Ext}^2_{\mathcal{E}}(A, M) \to \text{Ext}^2_{\mathcal{E}}(A \otimes A, M).
\]

Since \( A \otimes A \) is \( \mathcal{E} \)-projective, we get that \( \text{Ext}^1_{\mathcal{E}}(L, M) \simeq \text{Ext}^2_{\mathcal{E}}(A, M) = H^2(A, M) \). Then (a) and (b) are equivalent in view of Proposition 2.31(b). Hence, by the same theorem, \( H^2(A, M) = 0 \), for every \((A, \Lambda)\)–bimodule \( M \).

Now let us assume that the second Hochschild cohomology group of \( A \) with coefficients in \( M \) is trivial, for any \( M \in A \mathcal{M}_A \). By Theorem 2.32 we know that we have the required lifting property for all epimorphisms \( \pi \) splitting in \( \mathcal{M} \) and satisfying \( I^2 = 0 \). Let now \( \pi \) be an arbitrary epimorphisms as in (c). Since \( p_r : E/I^{r+1} \to E/I^r \) splits in \( \mathcal{M} \) and the square of its kernel is trivial, inductively we can construct a sequence of algebra morphisms \( f_1 := f, f_2, \ldots, f_n \) such that \( f_r : A \to E/I^r \) and \( p_r f_{r+1} = f_r \). We conclude this implication by remarking that \( E = E/I^n \), so the lifting of \( f \) can be chosen to be \( f_n \).
(b) ⇔ (d) Similarly to the proof of (b) ⇔ (c), by using the fact that in Theorem 2.32 the second and the fourth assertions are equivalent. □

Definition 2.35. Any algebra \((A, m, u)\) in \((\mathcal{M}, \otimes, 1)\), satisfying one of the conditions of Theorem 2.32 or of Theorem 2.34 is called formally smooth.

Corollary 2.36. Any separable algebra is formally smooth.

Proposition 2.37. If \(A\) is a formally smooth algebra and \(M\) is an \(\mathcal{E}\)-projective bimodule in \(A\mathcal{M}_A\), then the tensor algebra \(T_A(M)\) is also formally smooth.

Proof. Let \(\pi : E \to B\) be an algebra homomorphism that splits in \(\mathcal{M}\) and such that \((\text{Ker }\pi)^2 = 0\). Let \(f : T_A(M) \to B\) be an algebra homomorphism. Since \(A\) formally smooth, by the second condition from Theorem 2.32 there exists an algebra homomorphism \(g_0 : A \to E\) such that \(\pi g_0 = f i_A\), where \(i_A : A \to T_A(M)\) is the canonical inclusion. The objects \(E\) and \(B\) have a natural \((A, A)\)-bimodule induced by \(g_0\) and \(f i_A\), respectively. Thus \(\pi\) and \(f\) become homomorphisms of \((A, A)\)-bimodules. Let \(i_M : M \to T_A(M)\) be the canonical inclusion. Since \(M\) is \(\mathcal{E}\)-projective there exists a morphism of \((A, A)\)-bimodules \(g_1 : M \to E\) such that \(\pi g_1 = f i_M\). By the universal property of \(T_A(M)\), there exists a unique algebra homomorphism \(g : T_A(M) \to E\) such that \(gi_A = g_0\) and \(gi_M = g_1\) then \(\pi g_A = \pi g_0 = f i_A\) and \(\pi g_M = \pi g_1 = f i_M\). Finally \(\pi g = f\). □

Corollary 2.38. If \((A, m, u)\) is a formally smooth algebra, the tensor algebra \(T_A(\text{Ker }m)\) is also formally smooth.

3. Separable and formally smooth algebras in \(H\mathfrak{M}_H\)

In this section we shall apply the results of the previous section to study separability and formally smoothness of algebras in the monoidal category of all \((H, H)\)-bicomodules, where \(H\) is a given Hopf algebra.

3.1. Let \(H\) be a Hopf algebra. Let us consider the category \(\mathcal{M} := H\mathfrak{M}_H\) of \((H, H)\)-bicomodules with the tensor product \((-) \otimes (-)\) as in Example 1.4 c). Hence an algebra in \(\mathcal{M}\) is an algebra \(A\) which is an \((H, H)\)-bicomodule such that \(A\) is a left and a right \(H\)-comodule algebra. We shall say that \(A\) is an \(H\)-bimodule algebra.

Let \(A\) be an \(H\)-bicomodule algebra. The category of all \((A, A)\)-bimodules in \(\mathcal{M}\) will be denoted by \(H\mathfrak{M}_A^A\). An \((H, H)\)-bicomodule \(M\) is an object in \(H\mathfrak{M}_A^A\) if it is an \((A, A)\)-bimodule too such that \(\mu_l : A \otimes M \to M\) and \(\mu_r : M \otimes A \to M\) are morphisms of \((H, H)\)-bicomodules. Here \(\mu_l\) and \(\mu_r\) define the module structures on \(M\) and \(A \otimes M\) is an \((H, H)\)-bimodule with the diagonal coactions. For \(A = K\) with trivial \(H\)-comodule structures we get the category of \((H, H)\)-bicomodules. Also for the trivial Hopf algebra \(H = K\) we get that \(A\) is just a \(K\)-algebra, and \(H\mathfrak{M}_A^A = A\mathfrak{M}_A^A\).

\(H\mathfrak{M}_A^A\) is a monoidal category with the usual tensor product of two \((A, A)\)-bimodules \((-) \otimes (-)\). If \(V, W \in H\mathfrak{M}_A^A\) then the left structures on \(V \otimes_A W\) are given by:

\[
\rho_{V \otimes_A W} (v \otimes_A w) = \sum v(-1) w(-1) \otimes (v(0) \otimes_A w(0)).
\]

The right structures are defined similarly. The unit in \(H\mathfrak{M}_A^A\) is \(A\).

By definition, an algebra \(A\) in \(H\mathfrak{M}_H\) is separable if and only if the multiplication \(m : A \otimes A \to A\) has a section \(\sigma : A \to A \otimes A\) which is a morphism of \((A, A)\)-bimodules and \((H, H)\)-comodules. Obviously, then \(A\) is separable as an algebra in \(\mathfrak{M}_K\), but the converse does not hold in general. Nevertheless, if the forgetful functor \(F : H\mathfrak{M}_A^A \to A\mathfrak{M}_A^A\) is separable, then \(A\) is separable as an algebra in \(\mathcal{M}\). Before to prove this result, let us recall the definition and basic properties of separable functors.

3.2. A functor \(F : \mathfrak{C} \to \mathfrak{D}\) is called separable if, for all objects \(C_1, C_2 \in \mathfrak{C}\), there is a map \(\varphi : \text{Hom}_\mathfrak{D}(FC_1, FC_2) \to \text{Hom}_\mathfrak{C}(C_1, C_2)\) such that:

1) For all \(f \in \text{Hom}_\mathfrak{C}(C_1, C_2)\), \(\varphi(F(f)) = f\)
2) We have \( \varphi(k)f = g\varphi(h) \) for every commutative diagram in \( \mathcal{D} \) of type:

\[
\begin{array}{c}
F(C_1) \xrightarrow{h} F(C_2) \\
F(f) \downarrow \quad \downarrow F(g) \\
F(C_3) \xrightarrow{k} F(C_4)
\end{array}
\]

If \( F : \mathcal{C} \to \mathcal{D} \) has a right adjoint \( G : \mathcal{D} \to \mathcal{C} \), then \( F \) is separable (see [Ra1]) iff the unit \( \sigma : 1_\mathcal{C} \to GF \) splits, i.e. there is a functorial morphism \( \mu : GF \to 1_\mathcal{C} \) such that \( \mu \sigma = \text{Id}_{1_\mathcal{C}} \). If \( F \) is separable and \( F(f) \) has a section in \( \mathcal{D} \), then \( f \) has a section in \( \mathcal{C} \).

3.3. The forgetful functor \( F : \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A} \to \mathcal{A} \mathcal{M}_{\lambda} \) has a right adjoint \( G : \mathcal{A} \mathcal{M}_{\lambda} \to \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A} \), \( G(M) = H \otimes M \otimes H \), where \( G(M) \) is a bicomodule via \( \Delta_{H} \otimes M \otimes H \) and \( H \otimes M \otimes \Delta_{H} \), and \( G(M) \) is a bimodule with diagonal actions:

\[
a(h \otimes m \otimes k) = \sum a_{<1>}h \otimes a_{<0>}m \otimes a_{<1>}k \\
(h \otimes m \otimes k)a = \sum ha_{<1>} \otimes ma_{<0>} \otimes ka_{<1>}.
\]

Here we used the \( \Sigma \)-notation: \( (\rho_{\lambda}^{A} \otimes A)(\rho_{\lambda}^{A}) = \sum a_{<1>} \otimes a_{<0>} \otimes a_{<1>} \). For any \( M \in \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A} \) the unit of the adjunction is the map \( \sigma_{M} : M \to H \otimes M \otimes H, \sigma_{M} = (\rho_{\lambda}^{A} \otimes H)(\rho_{\lambda}^{M}) \).

**Lemma 3.4.** Let \( H \) be a semisimple and cosemisimple Hopf algebra over a field \( K \). Then there is a left and a right integral \( \lambda \) such that \( \lambda(1) = 1 \) and

\[
\lambda \left( \sum h_{(1)}xSh_{(2)} \right) = \lambda \left( \sum Sh_{(1)}xh_{(2)} \right) = \varepsilon(h) \lambda(x), \forall h, x \in H.
\]

**Proof.** First let us note that any semisimple Hopf algebra is finite dimensional. Hence, in view of [EG Corollary 3.2], we get that \( S^{2} = \text{Id}_{H} \). On the other hand, by [DN Corollary 3.2], \( H \) is unimodular and there is a (unique) right and left integral \( \lambda \in H^{*} \) such that \( \lambda(1) = 1 \). Hence equation 1(a) in [Ra2 Theorem 3] becomes in this particular case \( \lambda(hk) = \lambda(kh), \forall h, k \in H \), as \( S^{2} = \text{Id}_{H} \) and \( H \) is unimodular. Therefore:

\[
\lambda \left( \sum Sh_{(1)}xh_{(2)} \right) = \sum \lambda(xh_{(2)}Sh_{(1)}) = \varepsilon(h)\lambda(x),
\]

where for the last equality we used \( \sum h_{(2)}Sh_{(1)} = \varepsilon(h) \), relation that holds since \( S^{2} = \text{Id}_{H} \). The second equation of (11) can be proved similarly. \( \square \)

**Remark 3.5.** Suppose that \( H \) is a Hopf algebra. Then \( H \) is cosemisimple and has a non–zero left and right integral \( \lambda \) verifying (11) if and only if there is a (necessarily unique) left and right integral \( \lambda \) such that (11) holds true and \( \lambda(1) = 1 \). Indeed, two non–zero integrals are proportional, hence any non–zero integral verifies (11). On the other hand \( H \) is cosemisimple if and only if there is a (unique) integral \( \lambda \) such that \( \lambda(1) = 1 \).

**Definition 3.6.** A left and right integral \( \lambda \) verifying (11) and \( \lambda(1) = 1 \) will be called an \( ad \)-invariant integral.

**Theorem 3.7.** Let \( H \) be a Hopf algebra with an \( ad \)-invariant integral \( \lambda \). Then \( F : \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A} \to \mathcal{A} \mathcal{M}_{\lambda} \) is a separable functor.

**Proof.** We have to construct a functorial section of \( (\sigma_{M})_{M \in \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A}} \). Let \( \lambda \) be an \( ad \)-invariant integral. Let

\[
\mu_{M} : H \otimes M \otimes H \to M, \quad \mu_{M}(h \otimes m \otimes k) = \sum \lambda(Shm_{<1>})m_{<0>}\lambda(m_{<1>}Sk).
\]

Obviously \( (\mu_{M})_{M \in \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A}} \) is a functorial morphism. Let us check that \( \mu_{M} \) is a morphism in \( \text{H}_{\lambda}^{\mathcal{H}} \mathcal{A} \), i.e. \( \mu_{M} \) is a morphism of \( (A, A) \)-bicomodules and a morphism of \( (H, H) \)-bicomodules. Let \( x = \mu_{M}(a(h \otimes m \otimes k)) \). Then we have:

\[
x = \sum \lambda(ShSa_{<2>}a_{<1>}m_{<1>}m_{<0>}m_{<0>}\lambda(a_{<1>}m_{<1>}SkSa_{<2>}).
\]
Hence $\mu_M(a(h \otimes m \otimes k)) = a\mu_M(h \otimes m \otimes k)$. This relation proves that $\mu_M$ is left $A$-linear. Similarly, using the second equality of (12), one can show that $\mu_M$ is right $A$-linear. We have:

$$\sum h_{(1)} \otimes \mu_M(h_{(2)} \otimes m \otimes k) = \sum h_{(1)}\lambda \left(Sh_m <_{-1}> \right) \otimes m_{<0>}\lambda (m_{<1>Sk}) .$$

Let $y := \sum h_{(1)}\lambda \left(Sh_m <_{-1}> \right)$. Then, since $\lambda$ is a right integral, we have:

$$Sy = \sum \lambda \left(Sh_m <_{-1}> \right) Sm_{<1>}. $$

Thus $y = \sum \lambda \left(Sh_m <_{-1}> \right)m_{<1>}$; so:

$$\sum h_{(1)} \otimes \mu_M(h_{(2)} \otimes m \otimes k) = \sum \lambda \left(Sh_m <_{-2}> \right)m_{<1>} \otimes m_{<0>}\lambda (m_{<1>Sk}) .$$

As $\rho_M (h \otimes m \otimes k)$ equals the right hand side of the above equation, we have shown that $\mu_M$ is left–colinear. Analogously it can be proved that $\mu_M$ is right $H$–colinear. It remains to show that $\mu_M$ is a retraction of $\sigma_M$. But:

$$(\mu_M\sigma_M)(m) = \sum \lambda \left(Shm_{<2> m_{<1>}} \right)m_{<0>}\lambda (m_{<1>Sm_{<2>}}) = m,$$

so the theorem is proved. $\square$

**Theorem 3.8.** Let $H$ be a Hopf algebra over a field $K$ and assume that $H$ has an ad–invariant integral. An algebra $A$ in the category $^H\mathfrak{M}$ is separable iff $A$ is separable as an algebra in $(\mathfrak{M}_K, \otimes K, K)$, i.e. as an usual algebra.

**Proof.** It is enough to prove that if $A$ is separable as an algebra in $\mathfrak{M}_K$ then it is separable as an algebra in $^H\mathfrak{M}_A$. If $m : A \otimes A \to A$ is the multiplication of the algebra $A$ in the monoidal category $^H\mathfrak{M}_A$, then $m$ also defines the multiplication of $A$ as an algebra in $\mathfrak{M}_K$. Thus $F(m) = m$ has a section in $A\mathfrak{M}_A$. Since $F$ is separable, then $m$ has a section in $^H\mathfrak{M}_A$. Thus $A$ is separable in $^H\mathfrak{M}$. $\square$

**Corollary 3.9.** Let $H$ be a semisimple and cosemisimple Hopf algebra over a field $K$. If $A$ is an algebra in the category $^H\mathfrak{M}$ then $A$ is separable as an algebra in $^H\mathfrak{M}$ iff $A$ is separable as an algebra in $(\mathfrak{M}_K, \otimes K, K)$.

**Proof.** By Lemma 3.4 $H$ has a non–zero ad–invariant integral. $\square$

**Proposition 3.10.** Let $H$ be a semisimple and cosemisimple Hopf algebra. If $\pi : A \to B$ is a surjective morphism of algebras in $^H\mathfrak{M}$ such that $B$ is separable (as an algebra in $\mathfrak{M}_K$) and the kernel of $\pi$ is nilpotent then there is a section $\sigma : B \to A$ of $\pi$ which is a morphism of algebras in $^H\mathfrak{M}$.

**Proof.** By assumption $H$ is a semisimple and a cosemisimple Hopf algebra and hence $H$ is separable and coseparable by [DNC, Exercise 5.2.12]. Moreover by Corollary 3.6 $B$ is separable as an algebra in the category $^H\mathfrak{M}$. Let $n$ be a natural number such that $I^n = 0$, where $I = \text{Ker} \pi$. Since $H$ is coseparable, any epimorphism in the category $^H\mathfrak{M}$ splits in $^H\mathfrak{M}$, see [DQ]. In particular, for every $r = 1, \ldots, n - 1$ the canonical morphism $\pi_r : A/I^r \to A/I^{r+1}$ has a section in the category $^H\mathfrak{M}$. We can now conclude by applying Theorem 2.31 to the algebra homomorphism $\pi : A \to B$. $\square$

Let $H$ be a semisimple and cosemisimple Hopf algebra. In particular $H$ is separable and coseparable by [DNC, Exercise 5.2.12] (note that $H$ is necessarily finite dimensional). Hence, in the previous proposition, we can choose $B = H$.

Actually, in this case, we can relax the assumptions made on $H$. In order to do this we first prove the following lemma.

**Lemma 3.11.** Let $H$ be a Hopf algebra.

a) $H$ is separable as an algebra in $\mathfrak{M}$ if and only if $H$ is semisimple.

b) $H$ is separable as an algebra in $^H\mathfrak{M}$ if and only if there is an integral $t \in H$ such that $\varepsilon(t) = 1$ and $\sum t_{(1)}St_{(3)} \otimes t_{(2)} = 1 \otimes t$. 


Proof. We prove first (b). The category of \((H, H)\)–bimodules in \(\mathcal{M}^H\) is \(\mathcal{M}^H_H\). Hence we need a section of the multiplication of \(H\) in \(\mathcal{M}^H_H\). By the equivalence \(\mathcal{M}^H_H \simeq H^\mathcal{YD}\) (see [14]), it results that:

\[
\mathcal{M}^H_H(H, H \otimes H) \simeq H^\mathcal{YD}\left(H^{\operatorname{co}(H)}, (H \otimes H)^{\operatorname{co}(H)}\right),
\]

the isomorphism being given by the restriction to \(H^{\operatorname{co}(H)}\). Since the right comodule structure of \(H \otimes H\) is defined by \(H \otimes \Delta\), we have \((H \otimes H)^{\operatorname{co}(H)} = H\), where \(H\) is regarded as a left module via the multiplication of \(H\), and as a left comodule via the adjoint coaction. Hence:

\[
\mathcal{M}^H_H(H^{\operatorname{co}(H)}, (H \otimes H)^{\operatorname{co}(H)}) \simeq H^\mathcal{YD}(K, H),
\]

In conclusion, there is an one–to–one correspondence:

\[
\mathcal{M}^H_H(H, H \otimes H) \simeq \{t \in H | \sum t_{(1)} \mathcal{St}(3) \otimes t_{(2)} = 1 \otimes t \text{ and } ht = \varepsilon(h)t, \forall h \in H\}.
\]

Through this bijection a section of the multiplication corresponds to an element \(t\) such that \(\varepsilon(t) = 1\).

To prove (a) we first remark that the category of \((H, H)\)–bimodules in \(\mathcal{M}^H\) is \(\mathcal{M}^H_H\). Proceeding as in the proof of (b), but neglecting the left comodule structure, one can show that there is a bijection between \(\mathcal{M}^H_H(H, H \otimes H)\) and \(\{t \in H | ht = \varepsilon(h)t, \forall h \in H\}\). Moreover, the set of sections of the multiplication is bijectively equivalent with the set of all \(t\) as above such that \(\varepsilon(t) = 1\).

**Definition 3.12.** An integral \(t\) in a Hopf algebra \(H\) will be called \(ad\)–coinvariant if \(\varepsilon(t) = 1\) and \(\sum t_{(1)} \mathcal{St}(3) \otimes t_{(2)} = 1 \otimes t\).

**Remark 3.13.** Note that if \(H\) has an \(ad\)–coinvariant integral then \(H\) is semisimple, by Maschke Theorem.

**Theorem 3.14.** Let \(H\) be a semisimple Hopf algebra. Let \(\mathcal{M}\) be either the monoidal category \(\mathcal{M}^H\) or \(\mathcal{M}^H_H\). Suppose that \(\pi : A \to H\) is a morphisms of algebras in \(\mathcal{M}\) such that \(\operatorname{Ker} \pi\) is the Jacobson radical \(J\) of \(A\) and \(J\) is nilpotent.

a) Let \(\mathcal{M} = \mathcal{M}^H\). Then \(\pi : A \to A/J \simeq H\) has a section \(\sigma\) in \(\mathcal{M}^H\) which is an algebra map.

b) Let \(\mathcal{M} = H^\mathcal{YD}\). Assume that \(H\) has an \(ad\)–coinvariant integral and that every canonical map \(A/J^{n+1} \to A/J^n\) splits in \(H^\mathcal{YD}\). Then \(\pi : A \to H\) has a section \(\sigma\) in \(H^\mathcal{YD}\) which is an algebra map.

**Proof.** a) The Jacobson radical \(J\) of \(A\) is an \(H\)–subcomodule of \(A\) since \(\pi\) is a morphism of \(H\)–comodules. Hence, for every \(n > 0\), \(J^n\) is a subcomodule of \(A\) too such that the canonical map \(A/J^n+1 \to A/J^n\) is \(H\)–colinear. Furthermore, \(J^n/J^{n+1}\) has a natural module structure over \(A/J \simeq H\), and with respect to this structure \(J^n/J^{n+1}\) is an object in \(\mathcal{M}^H_H\). Hence \(J^n/J^{n+1}\) is a cofree right comodule (i.e. \(J^n/J^{n+1} \simeq V \otimes H\)). In particular \(J^n/J^{n+1}\) is an injective comodule, so the canonical map \(A/J^{n+1} \to A/J^n\) has a section in \(\mathcal{M}^H\). By the previous lemma we know that \(H\) is separable as an algebra in \(\mathcal{M}^H\), therefore we can apply Theorem 2.34.

b) We first remark that \(J^n\) is an \((H, H)\)–subbicomodule of \(A\) and that the canonical maps \(A/J^{n+1} \to A/J^n\) are morphisms of bicomodules. By the preceding lemma it results that \(H\) is separable in \(H^\mathcal{YD}\), so we conclude by applying Theorem 2.34.

**Corollary 3.15.** Let \(A\) be a Hopf algebra such that \(J\), the Jacobson radical of \(A\) is a nilpotent coideal in \(A\). Let \(H := A/J\), and let \(\pi : A \to H\) be the canonical projection.

a) If \(H\) is semisimple then there is an algebra morphism in \(\mathcal{M}^H\) that is a section of \(\pi\).

b) If \(H\) has an \(ad\)–coinvariant integral and every canonical map \(A/J^{n+1} \to A/J^n\) splits in \(H^\mathcal{YD}\) then there is an algebra morphism in \(H^\mathcal{YD}\) that is a section of \(\pi\).

c) If \(H\) has an \(ad\)–coinvariant integral and any object in \(H^\mathcal{YD}\) is injective as an \((H, H)\)–bicomodule (note that this holds whenever \(H\) is both semisimple and cosemisimple; e.g. when \(H\) is semisimple over a field of characteristic 0) then there is a section of \(\pi\) as in (b).

**Proof.** The first two assertions follows directly from the previous theorem, since we can regard \(A\) as an algebra in \(\mathcal{M}^H\) and as an algebra in \(H^\mathcal{YD}\), as \(\pi\) is a morphism of bialgebras.

Let us prove (c). In view of (b) it is enough to show that the canonical epimorphisms \(A/J^{n+1} \to A/J^n\) splits in \(H^\mathcal{YD}\). Since \(A/J^n\) is an object in \(H^\mathcal{YD}\) and the canonical epimorphism \(A/J^{n+1} \to A/J^n\) splits in \(H^\mathcal{YD}\), we conclude by applying Theorem 2.34.
$A/J^n$ is a morphism in $\mathcal{H}_M^H_R$ it follows that $J^n/J^{n+1} \in \mathcal{H}_M^H_R$, so it is an injective $(H, H)$–bicomodule. Therefore $A/J^{n+1} \rightarrow A/J^n$ has a section in $\mathcal{H}_M^H_R$.

If $H$ is semisimple and cosemisimple then $H$ is finite dimensional. By applying Lemma 3.4 to the dual Hopf algebra $H^*$ it results that $H$ has an $ad$–coinvariant integral. Furthermore, $H$ is coseparable, so every bicomodule is injective. In particular $J^n/J^{n+1}$ is injective. Thus $A/J^{n+1} \rightarrow A/J^n$ splits in $\mathcal{H}_M^H$.

### 4. Splitting morphisms of bialgebras

Let $H$ be a Hopf algebra and let $(A, m, u, \Delta, \varepsilon)$ be a bialgebra. Motivated by the result that we obtained in (3.13) we are going to investigate those bialgebras $A$ with the property that there is a pair of $K$–linear maps:

$$\pi : A \rightarrow H \quad \text{and} \quad \sigma : H \rightarrow A$$

such that $\pi$ is a morphism of bialgebras and $\sigma$ is an $(H, H)$–bicolinear algebra section of $\pi$ such that $\pi \sigma = \text{Id}_H$.

#### 4.1. Our approach is based on the observation that such a bialgebra can be viewed in a natural way as an object $A \in \mathcal{H}_M^H_R$ such that $A$ is an algebra in $(\mathcal{H}_M^H, \otimes_H)$. It is a coalgebra in $(\mathcal{H}_M^H, \square_H)$. Let us explain the exact meaning of this sentence.

Since $\pi$ is a morphism of coalgebras, $A$ is an $(H, H)$–bicomodule with the structures induced by $\pi$. Similarly $\sigma$ defines an $(H, H)$–bimodule structure on $A$.

Let us prove that these structures make $A$ a Hopf bimodule. We will check that $\rho^l : A \rightarrow H \otimes A$ and $\rho^r : A \rightarrow A \otimes H$ are bimodule morphisms. By definition $\rho^l(a) = \sum \pi(a_{(1)}) \otimes a_{(2)}$. Hence:

$$\rho^l(ha) = \rho^l(\sigma(h)a) = \sum \pi(\sigma(h_{(1)})a_{(1)}) \otimes \sigma(h_{(2)})a_{(2)} = \sum \pi(\sigma(h_{(1)})\pi(a_{(1)}) \otimes \sigma(h_{(2)})a_{(2)}),$$

where the last equality has been deduced by using the fact that $\pi$ is a morphism of algebras. Thus, by the definition of the left $H$–coaction on $A$ and the fact that $\sigma$ is left $H$–colinear, we get:

$$\rho^l(ha) = \sum \sigma(h_{(-1)})a_{(-1)} \otimes \sigma(h_{(0)})a_{(0)} = \sum h_1a_{(-1)} \otimes h_2a_{(0)}.$$

In a similar way one can prove that $\rho^r$ is right $H$–linear and that $\rho^r$ is a morphism of bimodules.

By assumption, $\sigma$ is a morphism of $(H, H)$–bicomodules. Moreover, since $\sigma$ is also an algebra morphism, we get that $\sigma$ is a morphism of $(H, H)$–bimodules and $m : A \otimes A \rightarrow A$ factorizes to a map $\overline{m} : A \otimes_H A \rightarrow A$. Furthermore, $m$ is left $H$–colinear. Indeed, we have:

$$(H \otimes m)(\rho^l_{A \otimes H} a \otimes b) = \sum a_{(-1)}b_{(-1)} \otimes a_{(0)}b_{(0)} = \sum \pi((ab)_{(1)}) \otimes (ab)_{(2)} = \rho^l(m(ab)).$$

In a similar way one proves that $m$ is also right $H$–colinear. Clearly $m$ is also $(H, H)$–bilinear. We have proved that $(A, \overline{m}, \sigma)$ is an algebra in $(\mathcal{H}_M^H, \otimes_H, H)$.

One can easily check that the image of $\Delta$ is contained in $A \square_H A$. We denote by $\overline{\Delta}$ the corestriction of $\Delta$ to $A \square_H A$. We have to prove that $\overline{\Delta}$ is a morphism in $\mathcal{H}_M^H$. Since the left $H$–coaction on $A \square_H A$ is given by the restriction of $\rho^l_{A} \otimes A$ to $A \square_H A$ it results:

$$\rho^l_{A \square_H A}(\overline{\Delta}(a)) = \sum \pi(a_{(1)}) \otimes a_{(2)} \otimes a_{(3)} = (H \otimes \overline{\Delta})\rho^l_{A}(a),$$

and similarly for the right coaction. Since $\pi$ is a coalgebra map, we have that $\varepsilon_H \pi = \varepsilon$. Moreover $\pi$ is $(H, H)$–bicolinear. Hence $(A, \overline{\Delta}, \overline{\pi})$ is a coalgebra in $(\mathcal{H}_M^H, \square_H, H)$.

These considerations lead us to the following definition (see also Definition 4.14).

#### Definition 4.2. Let $R$ be an $H$–bicomodule algebra. Let $A$ be an object in $\mathcal{H}_M^H_R$ which is an algebra in the category of vector spaces with multiplication $m : A \otimes A \rightarrow A$ and unit $u : K \rightarrow A$. We say that $(A, m, u)$ becomes an algebra in $(\mathcal{H}_M^H_R, \otimes, R)$ if $(A, m, u)$ is an $H$–bicomodule algebra and $m$ factorizes to a morphism $\overline{m} : A \otimes_R A \rightarrow A$ in $\mathcal{H}_M^H_R$.

#### 4.3. Note that $A$ becomes an algebra in $\mathcal{H}_M^H_R$ iff $A$ is an $H$–bicomodule algebra and $m$ is a morphism of $(R, R)$–bimodules which is $R$–balanced. Let us denote $m(a \otimes b) = \overline{m}(a \otimes_R b)$ by $ab$.

Then, for $a, b \in A$ and $r \in R$, we have

$$\begin{align*}
(ar)b &= a(rb) \\
ra &= (r1_A)a \\
ar &= a(1_Ar),
\end{align*}$$

where $a, b \in A$ and $r \in R$.
since by definition $m$ is an $R$–balanced morphism of $(R,R)$–bimodules. In particular the first relation gives us $r_1 A = 1_A r$, for all $r \in R$, so the unique left $R$–linear map $\overline{\varpi} : R \to A$, $\overline{\varpi}(r) = r 1_A$ is a morphism of $(R,R)$–bimodules. Since $A$ is an object in $\mathcal{H}_R \mathcal{M}_R$ and $u$ is $(H,H)$–bicolinear one can check easily that $\overline{\varpi}$ is a morphism of $(H,H)$–bimodules too, so $(A,\overline{\varpi},\overline{\varpi})$ is an algebra in the monoidal category $(\mathcal{H}_R \mathcal{M}_R, \otimes_R, R)$. This remark explains the terminology we introduced in the previous definition.

**Proposition 4.4.** Let $R$ be an $H$–bimodule algebra. Let $\phi : (A, m, u) \to (B, n, v)$ be an isomorphism of algebras in the category of vector spaces. If $A \in \mathcal{H}_R \mathcal{M}_R$ then $B$ can be endowed, via $\phi$, with obvious Hopf bimodule structures and $\phi : A \to B$ is an isomorphism in $\mathcal{H}_R \mathcal{M}_R$. Moreover if $A$ becomes an algebra in $(\mathcal{H}_R \mathcal{M}_R, \otimes_R, R)$, then $(B, n, v)$ also becomes an algebra in $(\mathcal{H}_R \mathcal{M}_R, \otimes_R, R)$ such that $\phi : (A,\overline{\varpi},\overline{\varpi}) \to (B,\overline{\varpi},\overline{\varpi})$ is an algebra isomorphism in the category $(\mathcal{H}_R \mathcal{M}_R, \otimes_R, R)$.

**Proof.** Obvious. □

4.5. It is well known that $\mathcal{H}_R \mathcal{M}_H$ is a monoidal category equivalent to the category of Yetter–Drinfeld modules $\mathcal{H}_R \mathcal{YD}$. This category consists of left $H$–modules $V$ which are left $H$–comodules such that
\[
\rho \left( h v \right) = \sum h_{(1)} v_{(-1)} S h_{(3)} \otimes h_{(2)} v_{(0)}, \forall h \in H, v \in V,
\]
where $h v$ is the notation that we shall use for the multiplication of $v \in V$ by $h \in H$ in an Yetter–Drinfeld module. The tensor product is $(-) \otimes_K (-)$, endowed with diagonal action and coaction. The equivalence between $\mathcal{H}_R \mathcal{M}_H$ and $\mathcal{H}_R \mathcal{YD}$ is given by $V \mapsto V^{\operatorname{co}(H)}$, where
\[
V^{\operatorname{co}(H)} = \left\{ v \in V \mid \rho^H_V(v) = v \otimes 1 \right\}.
\]
The structures making $V^{\operatorname{co}(H)}$ a left Yetter–Drinfeld module are the left adjoint action and the restriction of the left comodule structure of $V$:
\[
\begin{align*}
\rho(h^v) &= \sum h_{(1)} v S h_{(2)} \\
\rho(h v) &= \rho^H_V|_{V^{\operatorname{co}(H)}}.
\end{align*}
\]
Conversely if $W \in \mathcal{H}_R \mathcal{YD}$, then $W \otimes H$ becomes an object in $\mathcal{H}_R \mathcal{M}_H$ with the canonical right structures (coming from $H$) and with diagonal left action and coaction:
\[
\begin{align*}
\rho^H(w \otimes k) &= \sum w_{(-1)} h_{(1)} \otimes w_{(0)} \otimes h_{(2)}, \forall w \in W, \forall h, k \in H \\
\rho^H(w \otimes h) &= \sum h_{(1)} w \otimes h_{(2)} \otimes k, \forall w \in W, \forall h \in H
\end{align*}
\]
The functor that associates to $W \in \mathcal{H}_R \mathcal{YD}$ the Hopf bimodule $W \otimes H$ is an inverse of the monoidal functor defined above. Let us remark that, for every $V \in \mathcal{H}_R \mathcal{M}_H$, the isomorphism in $\mathcal{H}_R \mathcal{M}_H$ between $V^{\operatorname{co}(H)} \otimes H$ and $V$ is given by:
\[
\phi_V : V^{\operatorname{co}(H)} \otimes H \to V, \quad \phi_V(v \otimes h) = v h.
\]

**Example 4.6.** Let $R$ be a left $H$–module algebra. Recall that the smash product $R \# H$ of $R$ and $H$ is the associative algebra defined on $R \otimes H$ by setting:
\[
(r \# h)(s \# k) = \sum r \left( h^{(1)} s \right) \# h^{(2)} k.
\]
This algebra is unitary, with unit $1_R \# 1_H$. Here $r \# h := r \otimes h$. Moreover, if we assume in addition that $R$ is an algebra in $\mathcal{H}_R \mathcal{YD}$ then $R \# H$ becomes an algebra in $(\mathcal{H}_R \mathcal{M}_H, \otimes_H, H)$, with respect to the structures:
\[
\begin{align*}
\rho^H_{R \# H}(r \# h) &= \sum r_{(-1)} h_{(1)} \otimes (r \# h_{(2)}) \\
\rho^H_{R \# H}(r \# h) &= \sum \left( r \# h_{(1)} \right) \otimes h_{(2)} \\
\rho^H_{R \# H}(s \# k) &= s \# k h.
\end{align*}
\]
Our next aim is to prove that any algebra $A$ that becomes an algebra in $(\mathcal{H}_R \mathcal{M}_H, \otimes_H, H)$ is of this type, i.e. there is an algebra $R$ in $\mathcal{H}_R \mathcal{YD}$ such that $A \simeq R \# H$. 

\[
\begin{align*}
\rho^H_{R \# H}(r \# h) &= \sum r_{(-1)} h_{(1)} \otimes (r \# h_{(2)}) \\
\rho^H_{R \# H}(r \# h) &= \sum \left( r \# h_{(1)} \right) \otimes h_{(2)} \\
\rho^H_{R \# H}(s \# k) &= s \# k h.
\end{align*}
\]
Definition 4.7. Let $V$ be a Hopf bimodule. The space of right coinvariant elements of $V$ will be called the diagram of $V$ and it will be denoted by $R_V$, or shortly by $R$ if there is no danger of confusion.

Proposition 4.8. Let $(A, m, u)$ be an algebra. Suppose that $A$ is an object in $H_RM^H_H$ such that $A$ becomes an algebra in $H_RM_H^H$. If $R = A^{oH}$ is the diagram of $A$ then $R$ is an algebra in $H^{oD}$ and the canonical isomorphism $\phi_A : R^H_H \to A$ is a morphism of algebras in $(H_RM_H^H, \otimes H, H)$.

Proof. Since $A$ becomes an algebra in $(H_RM_H^H, \otimes H, H)$ the multiplication $m$ factors to a map $\overline{m} : A \otimes H A \to A$ which is a morphism of Hopf bimodules. As $(-)^{oH} : H_RM_H^H \to H^{oD}$ is a monoidal equivalence, we have

$$(A \otimes H A)^{oH} \simeq A^{oH} \otimes A^{oH}.$$ 

The morphisms of Yetter–Drinfeld modules that corresponds $\overline{m}$ through this equivalence is the restriction of $m$ to $R \otimes R$. Therefore we shall denoted by $m : R \otimes R \to R$ too. Obviously $m$ defines an associative multiplication on $R$.

Let $\overline{m} : H \to A$ be the unique left $H$–linear map such that $\overline{m}(1_H) = u(1_K) = 1_A$. By assumption $\overline{m}$ is a morphism in $H_RM_H^H$ and $(A, \overline{m}, \overline{m})$ is an algebra in $(H_RM_H^H, \otimes H, H)$. In particular $1_A$ is right coinvariant, so we can regard $u$ as a $K$–linear map from $K$ to $R$. As $u : K \to R$ is the morphism of Yetter–Drinfeld modules that corresponds to $\overline{m}$ through the equivalence $(-)^{oH}$ we deduce that $(R, m, u)$ is an algebra in $H^{oD}$.

It remains to prove that $\phi_A$ is an isomorphism of algebras in $(H_RM_H^H, \otimes H, H)$, see [Sch1] for the definition of $\phi_A$. By [Sch1] we know that $\phi_A$ is a bijective morphism in $H_RM_H^H$. Therefore it is enough to see that $\phi_A$ is a morphism of algebras. We have:

$$\phi_A((r \# h)(s \# k)) = \sum r(h^{(1, s)})[(h^{(2, s)})(s\# k) = \sum r[(h^{(1, s)})(h^{(2, s)})(s\# k)] = r[(h\# s)](s\# k) = (rh)(sk).$$

To deduce the first equality we used the definitions of the multiplication in $R^H_H$ and of $\phi_A$, and the fact that $\overline{m} : A \otimes H A \to A$ is right $H$–linear. The second equality comes from the definition of the left module structure on $R$, see [13]. To obtain the last equalities we applied the associativity relations $[12, 13]$ and the fact $m$ is $H$–balanced. We conclude the proof of the proposition by remarking that $\phi(r \# h)\phi(s \# k) = (rh)(sk)$.

\[\square\]

4.9. We can dualize all construction above. In particular, the category $H_RM^H_H$ of $H$–bimodules is a monoidal category with the tensor product $(-) \otimes_K (-)$. A coalgebra $D$ in this category is a coalgebra which is an $(H, H)$–bimodule such that $D$ is a left and a right $H$–module coalgebra, i.e. an $H$–bimodule coalgebra. For such a coalgebra we can consider the category $H_RM^D_H$ of all $(D, D)$–bicomodules which are also $(H, H)$–bimodules such that $\mu^H : H \otimes M \to M$ and $\mu^D : M \otimes H \to M$ are morphism of bicomodules ($\mu^H$ and $\mu^D$ define the module structures on $M$ and $H$). We get $\mu^H = H_RM_M^H$, and for $H = K$ we have $H_RM_M^D = D_RM_D$.

$H_RM^D_H$ is a monoidal category with respect to the tensor product given by $(-) \square_D (-)$, the cocommutator of two $(D, D)$–bicomodules. If $V, W \in H_RM^D_H$, then $V \square_D W$ is an $(H, H)$–bimodule with diagonal actions, and its comodule structures are defined by $\rho'_V \otimes W$ and $V \otimes \rho'_W$.

The monoidal category $(H_RM^D_H, \square_H, H)$ is also monoidal equivalent to $H^{oD}$. We are not going into details, the reader can find them in [Sch1].

Definition 4.10. Let $H$ be a Hopf algebra and let $D$ be an $(H, H)$–bimodule coalgebra. Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra and assume that $C \in H_RM^D_H$. We say that $(C, \Delta_C, \varepsilon_C)$ becomes a coalgebra in $(H_RM^D_H, \square_D, D)$ if $(C, \Delta_C, \varepsilon_C)$ is an $(H, H)$–bimodule coalgebra, the image of $\Delta_C$ is included in $C \square_D C$ and $\Delta_C : C \to C \square_D C$ (the corestriction of $\Delta_C$) is a morphism in $H_RM^D_H$.

4.11. Note that $(C, \Delta_C, \varepsilon_C)$ becomes a coalgebra in $H_RM^D_H$ if $(C, \Delta_C, \varepsilon_C)$ is an $(H, H)$–bimodule coalgebra and $\Delta_C$ is a morphism of $(D, D)$–bicomodules such that $\text{Im}(\Delta_C) \subseteq C \square_D C$.

Since $\text{Im}(\Delta_C) \subseteq C \square_D C$, for every $c \in C$, we have:

$$\sum (c(1))_0 \otimes (c(1))_1 \otimes (c(2)) = \sum (c(1))_1 \otimes (c(2))_1 \otimes (c(2))_1.$$
hence by applying $\varepsilon_C$ to the first and third factor we get:

$$\sum \varepsilon_C(c_{(0)})c_{(1)} = \sum \varepsilon_C(c_{(0)})c_{(-1)}.$$  \hfill (17)

In particular this relation means that the $K$–linear map $\tau_C : C \to D$, $\tau_C(c) = \sum \varepsilon_C(c_{(0)})c_{(-1)}$ is a morphism of $(D, D)$–comodules. In fact one can check easily that it is a morphism of $(H, H)$–bimodules too, since $C$ is a bimodule coalgebra over $H$. Thus $\tau_C$ is a morphism in $D \mathcal{M}_H$ such that

$$\varepsilon_C = \varepsilon_D \tau_C$$

In this case $(C, \Sigma_C, \tau_C)$ is a coalgebra in $(D \mathcal{M}_H, \boxtimes_D, D)$. Indeed, $\Sigma_C$ is obviously coassociative, and one can check easily that that the squares in the following diagram are commutative

$$\begin{array}{ccc}
C \boxtimes_D C & \xrightarrow{\Sigma_C} & C \\
\tau_C \boxtimes_D C & \xrightarrow{\lambda^l} & C \boxtimes_D C \\
\rho \boxtimes_D C & \xrightarrow{\lambda^r} & C \boxtimes_D D
\end{array}$$

where $\lambda^l : D \boxtimes_D C \to C \lambda^r : C \boxtimes_D D \to C$ are the canonical isomorphisms.

**Example 4.12.** Let $C$ be a left $H$–comodule coalgebra. Recall that the smash coproduct $C \# H$ of $C$ and $H$ is the coassociative and counitary coalgebra defined on $C \otimes H$ by setting:

$$\begin{align*}
\Delta(c \# h) &= \sum \varepsilon(1) \# (c_{(2)})_{(-1)} h_{(1)} \otimes (c_{(2)})_{(0)} \# h_{(2)} \\
\varepsilon(c \# h) &= \varepsilon_C(c) \varepsilon_H(h)
\end{align*}$$

Moreover, if we assume in addition that $C$ is a coalgebra in $H \mathcal{M}_H$ then $C \# H$ becomes a coalgebra in $(H \mathcal{M}_H, \boxtimes_H, H)$ with respect to the left diagonal action and diagonal coaction, and right canonical structures (coming from the corresponding structures of $H$).

The following Proposition is the dual statement of (13).

**Proposition 4.13.** Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. Suppose that $C$ is an object in $H \mathcal{M}_H$ such that $C$ becomes a coalgebra in $(H \mathcal{M}_H, \boxtimes_H, H)$. If $R = C^{\text{Co}(H)}$ is the diagram of $C$ then $R$ is a coalgebra in $H \mathcal{M}_H$ and the canonical isomorphism $\phi_C : R \# H \to C$ is a morphism of coalgebras in $(H \mathcal{M}_H, \boxtimes_H, H)$.

**Remark 4.14.** We keep the notation from the previous proposition. Let us denote the comultiplication of $R$ by $\delta : R \to R \otimes R$. Then $\delta = \psi_C \Delta \big|_R$, where $\psi_C$ is the isomorphism

$$(C \boxtimes_H C)^{\text{Co}(H)} \simeq C^{\text{Co}(H)} \otimes C^{\text{Co}(H)}$$

that comes from the equivalence $(H \mathcal{M}_H, \boxtimes_H, H) \simeq (H \mathcal{M}_H, \boxtimes_H, K)$. More generally, the isomorphism $(V \boxtimes_H W)^{\text{Co}(H)} \simeq V^{\text{Co}(H)} \otimes W^{\text{Co}(H)}$ maps an element $\sum_{i=1}^n v_i \otimes w_i$ to $\sum_{i=1}^n v_i S(w_{i(-1)}) \otimes w_{i(0)}$.

4.15. Suppose that $H$ is a Hopf algebra. Let $V \in \mathcal{M}_K$ and $W \in H \mathcal{M}$. It is well–known that we have a functorial isomorphism:

$$(V \otimes H) \boxtimes_H W \simeq V \otimes W$$

which is given by $\sum_{i=1}^n v_i \boxtimes_H w_i \mapsto \sum_{i=1}^n \varepsilon(h_i) v_i \otimes w_i$. The inverse of this map is $V \otimes \rho_V$.

Furthermore, the functor $F : \mathcal{M}_K \to \mathcal{M}_H$, $F(V) = V \otimes H$, has as a left adjoint the functor $G : \mathcal{M}_H \to \mathcal{M}_K$ that “forgets” the comodule structure. The maps that define this adjunction are:

$$\begin{align*}
\alpha_{V,W} : \mathcal{M}_H(V, W \otimes H) &\to \mathcal{M}_K(V, W), \quad \alpha_{V,W}(f) = (V \otimes \varepsilon) f \\
\beta_{V,W} : \mathcal{M}_K(V, W) &\to \mathcal{M}_H, \quad \beta_{V,W}(g) = (g \otimes H) \rho_V
\end{align*}$$

where $\rho_V$ defines the comodule structure on $V$. 

Lemma 4.16. Let $V$, $W$ be two vector spaces and let $Z$ be a left $H$–comodule. If we regard $Z \otimes H$ as a left $H$–comodule with diagonal coaction and a right comodule via $Z \otimes \Delta_H$ then there is an one–to–one correspondence between $\mathfrak{m}_K(V \otimes H, W \otimes Z)$ and $\mathfrak{m}^H(V \otimes H, (W \otimes H) \Box_H(Z \otimes H))$. If $\gamma \in \mathfrak{m}_K(V \otimes H, W \otimes Z)$ and $\Gamma \in \mathfrak{m}^H(V \otimes H, (W \otimes H) \Box_H(Z \otimes H))$ correspond to each other through this bijective map then they are related by the following relations:

\begin{equation}
\overline{\gamma} = (W \otimes \varepsilon_H \otimes Z \otimes \varepsilon_H) \Gamma, \tag{21}
\end{equation}

\begin{equation}
\Gamma(v \otimes h) = \sum \overline{\gamma}^1(v \otimes h_{(1)}) \otimes \overline{\gamma}^2(v \otimes h_{(1)})(-1)^{h_{(2)} \otimes \overline{\gamma}^2}(v \otimes h_{(1)})_{(0)} \otimes h_{(3)}, \tag{22}
\end{equation}

where $\overline{\gamma}(v \otimes h) = \sum \overline{\gamma}^1(v \otimes h) \otimes \overline{\gamma}^2(v \otimes h) \in W \otimes Z$ is a Sweedler–like notation for $\overline{\gamma}(v \otimes h)$.

Proof. By (20) we have:

$$(W \otimes H) \Box_H(Z \otimes H) \simeq W \otimes Z \otimes H.$$ 

Hence

$$\mathfrak{m}^H(V \otimes H, (W \otimes H) \Box_H(Z \otimes H)) \simeq \mathfrak{m}^H(V \otimes H, W \otimes Z \otimes H).$$

By composing this isomorphism with $\alpha_{V \otimes H, W \otimes Z}$ we obtain a bijective map:

$$\mathfrak{m}^H(V \otimes H, (W \otimes H) \Box_H(Z \otimes H)) \rightarrow \mathfrak{m}_K(V \otimes H, W \otimes Z).$$

Suppose now that $\overline{\gamma}$ and $\Gamma$ correspond each other through the above $K$–linear isomorphism. A straightforward but tedious computation shows us that $\overline{\gamma}$ and $\Gamma$ verifies (21) and (22). □

4.17. Let $R \in H^1$ and let $\Delta_{R\#H} : (R \# H) \rightarrow (R \# H) \Box_H(R \# H)$ be a right $H$–colinear map. By the previous lemma if

\begin{equation}
\delta = (R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H) \Delta_{R \# H}, \tag{23}
\end{equation}

and for $r \in R$, $h \in H$ we write $\delta(r \# h) = \sum \delta^1(r \# h) \otimes \delta^2(r \# h) \in R \otimes R$, then:

\begin{equation}
\Delta_{R \# H}(r \# h) = \sum \delta^1(r \# h_{(1)}) \# \delta^2(r \# h_{(1)})(-1)^{h_{(2)} \otimes \delta^2(r \# h_{(1)})_{(0)} \# h_3), \tag{24}
\end{equation}

Conversely if $\delta : R \otimes H \rightarrow R \otimes R$ is a linear map and $\Delta_{R \# H}$ is defined by (24), then $\Delta_{R \# H}$ is a right $H$–colinear map and $\text{Im}(\Delta_{R \# H}) \subseteq (R \# H) \Box_H(R \# H)$.

4.18. Let $A \in H^1 \mathfrak{m}_H^H$ be a Hopf bimodule. We assume that $A$ is a bialgebra with multiplication $m$, unit $u$, comultiplication $\Delta$ and counit $\varepsilon$. We are looking for necessary and sufficient conditions such that $A$ becomes an algebra in $(H^1 \mathfrak{m}_H^H, \otimes_H, H)$ and a coalgebra in $(H^1 \mathfrak{m}_H^H, \Box_H, H)$. Note that the latter monoidal category is equal to $(H^1 \mathfrak{m}_K^H, \Box_H, H)$, see 4.19. Thus it makes sense to talk about a bialgebra that becomes a coalgebra in the category of $(H, H)$–bicomodules, see 4.11.

By Proposition 4.3 the diagram $(R, m, u)$ of $A$ is an algebra in $(H^1 \mathfrak{m}_H^H, \otimes_H, H)$ and $R \# H$ is an algebra and the map $\phi_A : R \# H \rightarrow A$, $\phi_A(r \otimes h) = rh$, is an isomorphism of algebras in $(H^1 \mathfrak{m}_H^H, \otimes_H, H)$. Obviously, $R \# H$ is a bialgebra with comultiplication $\Delta_{R \# H}$ and counit $\varepsilon_{R \# H}$ given by:

$$\Delta_{R \# H} := (\phi_A^{-1} \otimes \phi_A^{-1}) \Delta \phi_A \quad \text{and} \quad \varepsilon_{R \# H} := \varepsilon \phi_A.$$

Of course, with respect to this bialgebra structure, $\phi_A$ becomes an isomorphism of bialgebras.

Furthermore, since $A$ becomes an algebra in $(H^1 \mathfrak{m}_H^H, \otimes_H, H)$ and a coalgebra in $H^1 \mathfrak{m}_H^H$, the smash $R \# H$ has the same properties. In particular the image of $\Delta_{R \# H}$ is included in $\Delta_{R \# H} \Box_H \Delta_{R \# H}$ and $\Delta_{R \# H}$ can be regarded as a morphism of right $H$–comodules $\Delta_{R \# H} : R \# H \rightarrow (R \# H) \Box_H(R \# H)$.

Hence, by Corollary 4.15 $\Delta_{R \# H}$ is uniquely determined by the $K$–linear map $\delta : R \# H \rightarrow R \otimes R$. In order to determine the counit $\varepsilon_{R \# H}$ we consider the restriction of $\varepsilon$ to $R$. For simplifying the notation we shall denote it by $\varepsilon$ too.

Lemma 4.19. We have $\varepsilon_{R \# H}(r \# h) = \varepsilon(r) \varepsilon_H(h)$, for all $r \in R$ and $h \in H$, if and only if $\varepsilon(1 \# h) = \varepsilon_H(h)$, for all $h \in H$ (equivalently, $\varepsilon$ is right $H$–linear).

Proof. Let us assume that $\varepsilon(1 \# h) = \varepsilon_H(h)$, for all $h \in H$. By definition and the middle relation in (12) we have:

$$\varepsilon_{R \# H}(r \# h) = \varepsilon(rh) = \varepsilon(r(1 \# h)) = \varepsilon(r) \varepsilon(1 \# h) = \varepsilon(r) \varepsilon_H(h).$$

The other implication is trivial since $\varepsilon_H(h) = \varepsilon_{R \# H}(1 \# h)$. □
Here we shall only mention that the morphisms are represented by arrows oriented downwards. In a braided category by diagrams. The reader is referred to for details to [Ka, Chapter XIV.1].

Stated) the following notation as in (4.18). Therefore, throughout the remaining part of this section we will keep (if not otherwise stated) the notation and the assumptions of this paragraph.

Figure 1. Definitions of $\rho_{R#H}$ and $\Delta_{R#H}$.

All considerations above still hold if we work with an arbitrary algebra $R$ in $H^H \mathcal{YD}$. To be more precise we reformulate our problem of characterizing algebras $A$ as above with the additional property that $\varepsilon$ is right $H$–linear) in the following way.

**Problem 4.20.** Let $R$ be an algebra in $H^H \mathcal{YD}$. Suppose that $\delta : R#H \to R \otimes R$, $\varepsilon : R \to K$ are $K$–linear maps. Let $\Delta_{R#H}$ be defined by (23) and let $\varepsilon_{R#H} := \varepsilon \otimes \varepsilon_H$. Find necessary and sufficient condition such that $(R#H, \Delta_{R#H}, \varepsilon_{R#H})$ is a bialgebra that becomes a coalgebra in $(H \mathfrak{M}^H, \Box_H, H)$.

Note that $R#H$ always becomes an algebra in $(H \mathfrak{M}^H, \otimes_H, H)$. Of course, by solving the above problem we also get an answer to our initial question (of finding all bialgebras $A$ that become an algebra in $H \mathfrak{M}^H$ and a coalgebra in $H \mathfrak{M}^H$). It is enough to take $R$ to be the diagram of $A$ and $\delta$, $\varepsilon$ as in (4.18). Therefore, throughout the remaining part of this section we will keep (if not otherwise stated) the following notation:

- $R$ is an algebra in $H^H \mathcal{YD}$;
- $\delta : R#H \to R \otimes R$ and $\varepsilon : R \to K$ are $K$–linear maps;
- $\Delta_{R#H}$ is defined by (23);
- $\varepsilon_{R#H} := \varepsilon \otimes \varepsilon_H$;
- $\rho_{R#H} : R#H \to H \otimes R#H$ denotes the map that defines the left coaction on $R#H$, coming from the monoidal structure of $H^H \mathcal{YD}$.

4.21. To simplify the computation sometimes we shall use the method of representing morphisms in a braided category by diagrams. The reader is referred to for details to [Ka, Chapter XIV.1]. Here we shall only mention that the morphisms are represented by arrows oriented downwards.

We shall apply this method in the category $H^H \mathcal{YD}$ of Yetter–Drinfeld modules. Recall that, for every $V, W \in H^H \mathcal{YD}$ the braiding is given by:

$$c_{V, W} : V \otimes W \to W \otimes V \quad c_{V, W}(v \otimes w) = \sum v_{(-1)} w \otimes v_{(0)}.$$  

Two examples of diagrams in this category can be found in Figure 1. Note that in both pictures the crossings represent $c_{R, H}$. Throughout the remaining part of this section we shall keep the notation and the assumptions of this paragraph.

**Lemma 4.22.** Let $H$ be a Hopf algebra. Then:

a) $(\varepsilon_H \otimes R) c_{R, H} = c_{R, H}(R \otimes \varepsilon_H)$.

b) $(\Delta_H \otimes R) c_{R, H} = (H \otimes c_{R, H})(c_{R, H} \otimes H)(R \otimes \Delta_H)$.

**Proof.** Trivial. □

**Remark 4.23.** The equations from the previous lemma admits the representations from Figure 2.
Figure 2. Properties of $\varepsilon_H$ and $\Delta_H$.

Proof. Note that (26) means that $\Delta_{R\#H}$ is left $H$–colinear, and that the equivalence that we have to prove can be represented as in Figure 3. We prove that (26) $\Rightarrow$ (27) in Figure 4. The first equality there was obtained by composing with $H \otimes R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H$ both sides of (26). The second equation holds because $\varepsilon_H$ and $\Delta_H$ can be pulled under the string in a crossing, see Remark 4.23. We conclude the proof of this implication by using that $\varepsilon_H$ is the counit of $H$.

The other implication is proved in Figure 5. By Remark 4.23 we can drag $\Delta_H$ under the braiding, so we get the first equality. Since the comultiplication in $H$ is coassociative we have the second and last relations. The third one follows since, by assumption, (27) holds.  

Lemma 4.25. Assume that $\Delta_{R\#H}$ is left $H$–colinear (i.e. satisfies (26)). Then the following two relations are equivalent:

\begin{align*}
(28) & \quad [\Delta_{R\#H} \otimes (R\#H)]\Delta_{R\#H} = [(R\#H) \otimes \Delta_{R\#H}]\Delta_{R\#H} \\
(29) & \quad (\bar{\delta} \otimes R)(\bar{R} \otimes c_{R,H})(\bar{\delta} \otimes R)(\bar{R} \otimes \Delta_H) = (\bar{R} \otimes \bar{\delta})(\bar{\delta} \otimes H)(\bar{R} \otimes \Delta_H)
\end{align*}

Proof. The diagrammatic representation of the equivalence is given in Figure 6. It is easy to see that (28) implies (29). Indeed it is enough to add $(R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H)$ on the bottom of the diagram representing (28), then to drag $\varepsilon_H$ under the crossings and to use that $\varepsilon_H$ is a counit. The other implication in proved in Figure 7.
4.26. Let \( R \) and \( S \) be two algebras in the braided category \( \mathcal{H} \). We can define a new algebra structure on \( R \otimes S \), by using the braiding \( (25) \), and not the usual flip morphism. The multiplication in this case is defined by the formula:

\[
(r \otimes s)(t \otimes v) = \sum h(r^{(1)}t) \otimes s_{(0)}v.
\]

Let us remark that for any algebra \( R \) in \( \mathcal{H} \) the smash product \( R \# H \) is a particular case of this construction. Just take \( S = H \) with the left adjoint coaction and usual left \( H \)–module structure.

Another example that we are interested in is \( R \otimes R \), where \( R \) is the diagram of a bialgebra \( A \) as in (4.18). For such an algebra \( R \) in \( \mathcal{H} \) we shall always use this algebra structure on \( R \otimes R \).

**Lemma 4.27.** Let \( \tilde{\delta} : R \otimes H \rightarrow R \otimes R \) be a \( K \)–linear map. Then the following two relations are equivalent:

\[
\Delta_{R \# H}((r \# h)(s \# k)) = \Delta_{R \# H}(r \# h)\Delta_{R \# H}(s \# k),
\]

\[
\tilde{\delta}((r \# h)(s \# k)) = \sum \tilde{\delta}(r \# h^{(1)}) h^{(2)} \delta(s \# k).
\]

where, for every \( h \in H \) and \( r, t \in R \) we have \( h(r \otimes t) = \sum h^{(1)} r \otimes h^{(2)} t \).
Proof. Let \( r \# h \) and \( s \# k \in R \# H \). Thus we have:

\[
\begin{align*}
\Delta(r \# h) &= \sum \delta^1(r \# h_{(1)}) \# \delta^2(r \# h_{(1)}) \langle -1 \rangle h_{(2)} \otimes \delta^2(r \# h_{(1)}) \langle 0 \rangle \# h_{(3)} \\
\Delta(s \# k) &= \sum \delta^1(s \# k_{(1)}) \# \delta^2(s \# k_{(1)}) \langle -1 \rangle k_{(2)} \otimes \delta^2(s \# k_{(1)}) \langle 0 \rangle \# k_{(3)},
\end{align*}
\]

\[
\Delta_{R \# H}((r \# h)(s \# k)) = \sum \delta^1(r h_{(1)} s \# h_{(2)} k_{(1)}) \# \delta^2(r h_{(1)} s \# h_{(2)} k_{(1)}) \langle -1 \rangle h_{(3)} k_{(2)} \otimes \delta^2(r h_{(1)} s \# h_{(2)} k_{(1)}) \langle 0 \rangle \# h_{(4)} k_{(3)}.
\]

By substituting in (33) the elements involving \( \Delta_{R \# H} \) with the right hand sides of the above three relations, and then by applying \( R \otimes \varepsilon_H \otimes R \otimes \varepsilon_H \) it results:

\[
\delta((r \# h)(s \# k)) = \sum \delta^1(r \# h_{(1)}) \# \delta^2(r \# h_{(1)}) \langle -1 \rangle h_{(2)} \delta^1(s \# k) \otimes \delta^2(r \# h_{(1)}) \langle 0 \rangle \# h_{(3)} \delta^2(s \# k)
\]

Since in \( R \otimes R \) the multiplication is defined by (30) it follows that the right hand sides of (32) and (33) are equal, so the equality (32) holds.

Conversely, if (32) holds true then we have (33). We can replace the left hand side of this relation by \( \sum \delta^1(r h_{(1)} s \# h_{(2)} k) \otimes \delta^2(r h_{(1)} s \# h_{(2)} k) \). A very long computation, using this equivalent form of (33), ends the proof of the proposition. \( \square \)
4.28. Let $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ be a $K$–linear map. For every $r \in R$ and $h \in H$ we introduce the notation:

$$\tilde{\delta}(r) = \tilde{\delta}(r \# 1) \quad \omega(h) = \tilde{\delta}(1 \# h).$$

Then $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$ are $K$–linear maps. Recall that $R \otimes R$ is an algebra in $H \mathcal{YD}$ with the multiplication defined in (4.26). For example we can compute the product $\delta(r)\omega(h)$ in $R \otimes R$. Now, using the notation above, we can give a new interpretation of (31).

**Lemma 4.29.** Let $\tilde{\delta} : R \otimes H \rightarrow R \otimes R$ be a $K$–linear map. Then $\Delta_{R\#H}$ is a morphism of algebras iff $\delta(1_R) = 1_R \otimes 1_R$, $\omega(1_H) = 1_R \otimes 1_R$ and $\tilde{\delta}$ and $\omega$ satisfy the following four relations:

\begin{align*}
(35) & \quad \tilde{\delta}(r \# h) = \delta(r)\omega(h) \\
(36) & \quad \tilde{\delta}(rs) = \delta(r)\delta(s) \\
(37) & \quad \omega(hk) = \sum \omega(h_{(1)}) h_{(2)} \omega(k), \\
(38) & \quad \sum \delta(h_{(1)} r) \omega(h_{(2)}) = \sum \omega(h_{(1)}) h_{(2)} \delta(r)
\end{align*}

**Proof.** By Lemma 2.27 the map $\Delta_{R\#H}$ is multiplicative if and only if (32) hold i.e.:

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \tilde{\delta}(r \# h_{(1)}) h_{(2)} \tilde{\delta}(s \# k).$$

Now assume that (32) holds. Then setting $h = 1_H = k$ we obtain (36), while for $r = 1_R = s$ we obtain (37). Also for $h = 1_H$ and $s = 1_H$ we get (35) and for $r = 1_R$ and $k = 1_H$ we get (38), by means of (35). Conversely assume that (35), (36), (37) and (38) hold true. Then by (35), (36) and (37) we have:

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \delta(r) h_{(1)} \omega(h_{(2)} k) = \sum \delta(r) \delta(h_{(1)} s) \omega(h_{(2)}) \omega(k).$$

So, by (35) and by the fact that $R \otimes R$ is an algebra in $H \mathcal{YD}$ (hence an $H$–module algebra), we get:

$$\tilde{\delta}((r \# h)(s \# k)) = \sum \delta(r) \omega(h_{(1)}) h_{(2)} \delta(s) h_{(3)} \omega(k) = \sum \delta(r) \omega(h_{(1)}) h_{(2)} [\delta(s) \omega(k)].$$

Now we can prove (32) by using (35) once again. Obviously $\Delta_{R\#H}$ is a morphism of unitary rings if and only if $\delta(1_R) = 1_R \otimes 1_R$ and $\omega(1_H) = 1_R \otimes 1_R$. \hfill $\square$

**Remark 4.30.** By (35) we can recover $\tilde{\delta}$ from $\delta : R \rightarrow R \otimes R$ and $\omega : H \rightarrow R \otimes R$. Equation (35) says that $\delta$ is multiplicative with the algebra structure on $R \otimes R$ introduced in 1.26. We have already noticed that $R \otimes R$ is a left $H$–module algebra. Now if $A$ is an arbitrary left $H$–module algebra, then Sweedler in [Sw] defined a noncommutative 1–cocycle with coefficient in $A$ to be a $K$–linear map $\theta : H \rightarrow A$ such that

$$\theta(hk) = \sum \theta(h_{(1)}) h_{(2)} \theta(k).$$

Hence (32) means that $\omega$ is a 1–cocycle with coefficients in $A$.

**Lemma 4.31.** Assume that $\Delta_{R\#H}$ is multiplicative. Then (27) holds iff $\delta$ and $\omega$ are left $H$–colinear (where $H$ is a left $H$–comodule with the left adjoint coaction).

**Proof.** Assume that (27) holds and let $r \in R$, $h \in H$. By evaluating (27) at $r \# 1$ we get:

$$\rho^1_{R \otimes R}(\delta(r)) = \sum r(-1) \otimes \delta(r(0)), $$

so $\delta$ is $H$–colinear. Similarly, for $1 \# h$ we have:

$$\sum h_{(1)} \otimes \omega^1(h_{(2)}) \otimes \omega^2(h_{(2)}) = \sum \omega^1(h_{(1)}) (-1) \omega^2(h_{(1)}) (-1) h_{(2)} \otimes \omega^1(h_{(1)}) (0) \otimes \omega^2(h_{(1)}) (0)$$

i.e. we get

$$\sum h_{(1)} \otimes \omega(h_{(2)}) = \sum \omega(h_{(1)}) (-1) h_{(2)} \otimes \omega(h_{(1)}) (0).$$

On the other hand:

$$\sum \omega(h)(-1) \otimes \omega(h)(0) = \sum \omega(h_{(1)}) (-1) h_{(2)} S(h_{(3)}) \otimes \omega(h_{(1)}) (0) = \sum h_{(1)} S(h_{(3)}) \otimes \omega(h_{(2)}),$$

where $S(H)$ introduced in (4.26).
where the last equality holds in view of (33). Hence \( \omega \) is left \( H \)-colinear.

Conversely assume that \( \delta \) and \( \omega \) are left \( H \)-colinear. The relation (35) that we have to prove is equivalent to \( A_r(s, t) = A_r(s, t) \), where:

\[
A_r(s, t) = \sum \delta (r \# h_1) (\delta 2 (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2)
\]

(40)

\[
A_r(s, t) = \sum \delta (r \# h_1) (\delta 2 (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2)
\]

(41)

Then, since \( \Delta_{R \# H} \) is multiplicative, by (35) we have:

\[
A_r(s, t) = \sum \delta (r \# h_1) (\delta 2 (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2)
\]

(42)

\[
A_r(s, t) = \sum \delta (r \# h_1) (\delta 2 (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2) \otimes \delta (r \# h_1) (l_1) h_2)
\]

(43)

Since \( \delta \) and \( \omega \) are left colinear it results:

\[
\sum \delta (r) (\omega (h_1)) (\omega (h_2)) \otimes \delta (r \# h_1) (\omega (h_1) \otimes \delta (r \# h_1) (\omega (h_1) \otimes \delta (r \# h_1)) \otimes \omega (h_2)) = A_r(s, t),
\]

so \( A_r(s, t) = A_r(s, t) \), thus the lemma has been proved.

4.32. To simplify the notation, for every \( r \in R \), let \( \delta (r) := r(1) \otimes r(2) \). This is a kind of \( \Sigma \)-notation that we shall use for \( \delta \).

**Lemma 4.33.** Assume that \( \Delta_{R \# H} \) is a morphism of algebras such that \( \delta \) is left \( H \)-colinear. Then (32) holds iff the following two relations hold true for any \( r \in R \) and \( h \in H \):

\[
\sum \delta (r) (\omega (h_1)) (\omega (h_2)) = \sum \delta (r) (\omega (h_1)) (\omega (h_2)) = \sum \delta (r) (\omega (h_1)) (\omega (h_2))
\]

(40)

\[
\sum \delta (r) (\omega (h_1)) (\omega (h_2)) = \sum \delta (r) (\omega (h_1)) (\omega (h_2)) = \sum \delta (r) (\omega (h_1)) (\omega (h_2))
\]

(41)

Proof. Since \( \Delta_{R \# H} \) is multiplicative it is straightforward to prove that (32) holds iff, for every \( r \in R \) and \( h \in H \), we have \( B_r(h) = B_r(h) \), where:

\[
B_r(h) = \sum r(1) \left( \omega (h_1) \right) \otimes \delta \left( r(2) \omega (h_2) \right) \omega (h_2).
\]

(44)

\[
B_r(h) = \sum r(1) \left( \omega (h_1) \right) \otimes \delta \left( r(2) \omega (h_2) \right) \omega (h_2).
\]

(45)

Since \( \Delta_{R \# H} \) is a morphism of algebras we have \( \delta (1_R) = 1_R \otimes 1_R \) and \( \omega (1_R) = 1_R \otimes 1_R \). Hence one can see easily that (12) and (13) are equivalent to \( B_r(r, 1) = B_r(r, 1) \) and \( B_1(1, h) = B_r(1, h) \), respectively. In particular, (29) implies (12) and (13). In order to prove the converse, let us denote by \( C_l(h) \) and \( C_r(h) \) the left and right hand sides of (13). Since \( \delta \) is left \( H \)-colinear, and by using (30), it results:

\[
B_r(h) = \sum \left( \delta (r(1)) \omega (r(2)) \right) C_l(h),
\]

where the product is performed in \( R \otimes R \otimes R \), which is an algebra with the multiplication:

\[
(r \otimes s \otimes t)(r' \otimes s' \otimes t') = \sum r^{s(t(-1))} s^{l(-1)} t^{l(1)}.
\]

Similarly, by (35) and (32), it follows:

\[
B_r(h) = \sum \left( \delta (r(1)) \omega (r(2)) \right) C_r(h).
\]

We deduce that \( B_r(h) = B_r(h) \) by multiplying (12) and (13) side by side in \( R \otimes R \otimes R \).
Now let us consider the following particular case. Take $M$ to be a left $H$–comodule and let $W := M \otimes H$. Then $W$ is an $(H, H)$–bimodule with respect to the diagonal left coaction and canonical right comodule structure. Let $\varepsilon_M := \varepsilon_M : M \to K$ be a $K$–linear map and define $\varepsilon : W \to K$ by \( \varepsilon(\varepsilon(m \otimes h)) = \varepsilon_M(m)\varepsilon_H(h). \) Since the right comodule structure on $W$ is induced by $\Delta_H$ we have $W^{Co(H)} = \{ m \otimes 1_H \mid m \in M \}$.

By the foregoing, if $\tau_{M \otimes H}$ is a morphism of $(H, H)$–bicomodules then:

\[ \sum \varepsilon_M(m(0))m(-1) = \varepsilon_M(1)H, \forall m \in M, \]

that is $\varepsilon_M$ is left $H$–colinear. Actually, in this particular case, we can prove that the converse also holds true. In conclusion, $\tau_{M \otimes H}$ is a morphism of $(H, H)$–bicomodules iff $\varepsilon_M$ is left $H$–colinear.

**Lemma 4.35.** Let $R$ be an algebra in $H^YD$ and let $\varepsilon : R \to K$ be a $K$–linear map. The map $\varepsilon_{\#R} : R^\#H \to R, \varepsilon_{\#R}(r \otimes h) := \varepsilon(r)\varepsilon_H(h)$, is an algebra map and $\tau_{R^\#H} : R^\#H \to H$ is a left $H$–colinear map if and only if $\varepsilon$ is an algebra map in $H^YD$.

**Proof.** “⇒” By (4.34) it follows that $\varepsilon : R \to K$ is left $H$–colinear. By the definition of the multiplication in $R^\#H$ and the definition of $\varepsilon_{\#R}$ we get:

\[ \sum \varepsilon(r^{(1)}s)\varepsilon_H(h^{(2)}v) = \varepsilon_{\#R}((r\#k)(s\#v)) = \varepsilon(r\#k)\varepsilon_{\#R}(s\#v) = \varepsilon(r)\varepsilon_H(h)\varepsilon(s)\varepsilon_H(v). \]

Thus $\varepsilon(b) = \varepsilon_H(h)\varepsilon(s)$ and $\varepsilon(rs) = \varepsilon(r)\varepsilon(s)$, i.e. $\varepsilon$ is an $H$–colinear algebra map.

“⇐” We know already by (4.34) that $\tau_{R^\#H} : R^\#H \to H$ is a left $H$–colinear map. Furthermore:

\[ \tau_{R^\#H}((r\#h)(s\#k)) = \sum \tau_{R^\#H}(r^{(1)}s\#h^{(2)}k) = \sum \varepsilon(r)\varepsilon_H(h^{(1)})\varepsilon(s)\varepsilon_H(h^{(2)})\varepsilon_H(k) \]

Hence $\tau_{R^\#H}((r\#h)(s\#k)) = \tau_{R^\#H}(r\#h)s\#k$, so $\varepsilon_{\#R}$ is an algebra map.

**Lemma 4.36.** Assume that $\varepsilon$ is an algebra map in $H^YD$. Then $\varepsilon_{\#R} : R^\#H \to K$ is a counit for $\Delta_{\#R}^H$ if and only if, for every $r \in R$ and $h \in H$, we have:

\[ \sum \varepsilon(\delta^1(r \otimes h))\delta^2(r \otimes h) = \varepsilon_H(h)r = \sum \delta^1(r \otimes h)\varepsilon \left( \delta^2(r \otimes h) \right). \]

**Proof.** Assume that $\varepsilon_{\#R}$ is a counit for $\Delta_{\#R}^H$. Then, by the definition of $\Delta_{\#R}^H$, see (24), it results:

\[ r \otimes h = \sum \delta^1(r \otimes h^{(1)}) \otimes \delta^2(r \otimes h^{(1)})(-1)h^{(2)} \varepsilon \left( \delta^2(r \otimes h^{(1)})0 \right) \varepsilon_H(h^{(3)}). \]

By applying $R \otimes \varepsilon_H$ to this relation we get the second equality of (47). The other one can be proved similarly.

Conversely assume that the equality (47) holds. Since $\varepsilon$ is left $H$–colinear, we have

\[ (R^\#H \otimes \varepsilon_{\#R})\Delta_{\#R}^H = \sum \delta^1(r \otimes h^{(1)}) \varepsilon \left( \delta^2(r \otimes h^{(1)})0 \right) \otimes h^{(2)} = \sum r\varepsilon_H(h^{(1)}) \otimes h^{(2)} = r \otimes h. \]

We can prove the second relation analogously.

**Lemma 4.37.** Assume that $\Delta_{\#R}^H$ is multiplicative and that $\varepsilon : R \to K$ is an algebra map in $H^YD$. Then (47) holds if and only if:

\begin{align*}
(\varepsilon \otimes R)\delta &= (R \otimes \varepsilon)\delta = \Id_R \\
(\varepsilon \otimes R)\omega &= (R \otimes \varepsilon)\omega = \varepsilon_H 1_R
\end{align*}

**Proof.** First let us observe that $\varepsilon \otimes R : R \otimes R \to R$ and $R \otimes \varepsilon : R \otimes R \to R$ are algebra morphisms (recall that $R \otimes R$ is an algebra with the multiplication $(m_R \otimes m_R \otimes R)(R \otimes c_R \otimes h)$, where $c$ is the braiding in $H^YD$). Clearly (47) holds if and only if:

\[ (\varepsilon \otimes R)\delta (r\#h) = \varepsilon_H(h)r = (R \otimes \varepsilon)\delta (r\#h), \forall r \in R, \forall h \in H. \]

Assume now that (48) and (49) holds. Then:

\[ (\varepsilon \otimes R)\delta (r\#h) = (\varepsilon \otimes R)\delta (r) \cdot (\varepsilon \otimes R)\omega (h) = \varepsilon_H(h)r. \]

Analogously we can deduce the second equality of (47). The other implication is trivial.
To state easier the main results of this part we collect together in the next definition all required properties of $\delta$, $\omega$ and $\varepsilon$.

**Definition 4.38.** Let $H$ be a Hopf algebra and let $R$ be an algebra in $H^H \mathcal{YD}$. Assume that $\varepsilon : R \to K$, $\delta : R \to R \otimes R$ and $\omega : H \to R \otimes R$ are $K$–linear maps. The quadruple $(R, \varepsilon, \delta, \omega)$ will be called a **Yetter–Drinfeld quadruple** if and only if, for all $r, s \in R$ and $h, k \in H$, the following relations are satisfied:

1. $\varepsilon(r s) = \varepsilon(r) \varepsilon(s)$ and $\varepsilon(1_R) = 1$;
2. $\rho_{R \otimes R} (\delta(r)) = \sum r_{(-1)} \otimes \delta(r_{(0)})$;
3. $\rho_{R \otimes R} (\omega(h)) = \sum h_{(1)} S(h_{(3)}) \otimes \omega(h_{(2)})$;
4. $\delta(r s) = \delta(r) \delta(s)$ and $\delta(1_R) = 1_R \otimes 1_R$;
5. $\omega(hk) = \sum \omega(h_{(1)}) \delta^1(h_{(2)}) \omega(k)$ and $\omega(1_H) = 1_R \otimes 1_R$;
6. $\sum \delta^1(h_{(1)} r) \omega(h_{(2)}) = \sum \omega^1(h_{(1)}) \omega(\omega^2(h_{(1)})(r_{(-1)}) \otimes r_{(0)});$
7. $\sum r_{(1)} \otimes \delta(r_{(2)}) = \sum \delta(r_{(1)}) \omega(r_{(2)});$
8. $\sum \omega^1(h_{(1)}) \otimes \delta(\omega^2(h_{(1)})) \omega(h_{(2)}) = \sum \delta(\omega^1(h_{(1)})) \omega(\omega^2(h_{(1)}) h_{(2)}) \otimes \omega^2(h_{(1)});$
9. $(\omega \otimes R) \delta = (R \otimes \omega) \delta = \text{Id}_R$;
10. $(\varepsilon \otimes R) \omega = (R \otimes \varepsilon) \omega = \varepsilon \Omega_H 1_R$.

**Remark 4.39.** Note that these relations can be interpreted as follows:

- $\varepsilon$ is a morphism in $H^H \mathcal{YD}$;
- $\delta$ is left $H$–colinear;
- $\omega$ is left $H$–colinear, where $H$ is a comodule with the adjoint coaction;
- $\delta$ is a morphism of algebras where on $R \otimes R$ we consider the algebra structure that uses the braiding $\varepsilon$;
- $\omega$ is a normalized cocycle;
- $\omega$ measures how far $\delta$ is to be a morphism of left $H$–modules (if $\omega$ is trivial, i.e. for every $h \in H$ we have $\omega(h) = \varepsilon(h) 1_R \otimes 1_R$, then $\delta$ is left $H$–linear); we shall say that $\delta$ is a twisted morphism of left $H$–modules;
- $\delta$ is a cocycle (when $\omega$ is trivial then $\delta$ is equivalent to the fact that $\delta$ is coassociative);
- $\omega$ is the only property that has not an equivalent in the theory of bialgebras; we shall just say that $\delta$ and $\omega$ are compatible;
- $\varepsilon$ is a counitary map with respect to $\omega$;
- $\omega$ is a counitary map with respect to $\varepsilon$;

Since $\varepsilon$ satisfies the last two relation we shall call it the counit of the Yetter–Drinfeld quadruple $R$. By analogy $\delta$ will be called the **comultiplication** of $R$. Finally, we shall say that $\omega$ is the **cocode** of $R$.

**Theorem 4.40.** To every Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$ we associate the $K$–linear maps: $\Delta_{R \# H} : R \# H \to (R \# H) \otimes (R \# H)$ and $\varepsilon_{R \# H} : R \# H \to K$, which are defined by:

1. $\Delta_{R \# H}(r \otimes h) = \sum \delta^1(r \otimes h_{(1)}) \otimes \delta^2(r \otimes h_{(1)})(-1) h_{(2)} \otimes \delta^3(r \otimes h_{(1)})(0) \otimes h_{(3)}$
2. $\varepsilon_{R \# H}(r \# h) = \varepsilon(r) \varepsilon_{H}(h)$

where $\delta(r \# h) := \delta(r) \omega(h)$ and $\delta(r \# h) = \sum \delta^1(r \otimes h) \otimes \delta^2(r \otimes h)$.

**Theorem 4.41.** Let $R$ be an algebra in $H^H \mathcal{YD}$. If $\varepsilon : R \to K$, $\delta : R \to R \otimes R$ and $\omega : H \to R \otimes R$ are linear maps, then the following assertions are equivalent:

- (a) $(R, \varepsilon, \delta, \omega)$ is a Yetter–Drinfeld quadruple.
(b) The smash product algebra $R \# H$ is a bialgebra with the comultiplication $\Delta_{R \# H}$ and the counit $\varepsilon_{R \# H}$ defined by \[ (62) \] and \[ (63) \] such that $R \# H$ becomes an algebra in $(H \# \mathcal{M}_H^H, \otimes_H, H)$ and a coalgebra in $(H \# \mathcal{M}_H^H, \square_H, H)$.

Proof. (a) $\Rightarrow$ (b) By Lemma 4.24 it results that $\Delta_{R \# H}$ is multiplicative, in view of \[ (51) \], \[ (53) \], \[ (55) \]. Note that we have \[ (53) \] by the definition of $\Delta_{R \# H}$. Since $\delta(1_R) = \omega(1_H) = 1_R \otimes 1_R$ we have $\Delta_{R \# H}(1_R \# 1_H) = (1_R \# 1_H) \otimes (1_R \# 1_H)$. In conclusion, $\Delta_{R \# H}$ is a morphism of unitary algebras.

Since $\Delta_{R \# H}$ is multiplicative we can apply Lemma 4.24 and Lemma 4.31 to deduce that $\Delta_{R \# H}$ is left $H$–colinear by using relations \[ (52) \] and \[ (53) \], i.e. that $\delta$ and $\omega$ are left $H$–colinear. On the other hand, by \[ (4.17) \], we get that $\Delta_{R \# H}$ is right colinear, so $\Delta_{R \# H}$ is a morphism of $(H, H)$–bicomodules. Also by \[ (4.17) \] it follows that the image of $\Delta_{R \# H}$ is included into $(R \# H) \square_H (R \# H)$.

Since $\Delta_{R \# H}$ is multiplicative and left $H$–colinear and since $\delta$ is also left $H$–colinear, by \[ (57) \] and \[ (58) \], it results that $\Delta_{R \# H}$ is coassociative (use Lemma 4.33 and Lemma 4.36).

To prove that $\varepsilon_{R \# H}$ is a morphism of algebras we use Lemma 4.36, \[ (51) \] and \[ (50) \]. Finally, in view of \[ (50) \] and \[ (51) \], Lemma 4.37 implies that $\varepsilon_{R \# H}$ is a counit for $\Delta_{R \# H}$. All these properties together mean that $R \# H$ is a bialgebra that becomes a coalgebra in $(H \# \mathcal{M}_H^H, \square_H, H)$. We conclude by remarking that $R \# H$ always becomes an algebra in $(H \# \mathcal{M}_H^H, \otimes_H, H)$, see Example 4.6.

In conclusion the object $R \# H$ is an algebra in $(H \# \mathcal{M}_H^H, \otimes_H, H)$. As $\varepsilon$ is an algebra map in $H \# \mathcal{M}_H^H$, by Lemma 1.35 $\varepsilon : R \# H \rightarrow H$ is a left $H$–colinear map so that it is a map in $H \# \mathcal{M}_H^H$ (it is always right $H$–colinear). Finally it is easy to check that the image of $\Delta_{R \# H}$ is included in $R \# H \square_H R \# H$.

(b) $\Rightarrow$ (a) Since $\Delta_{R \# H}$ is morphism of algebras, by Lemma 4.24 it follows that \[ (53) \], \[ (55) \] and \[ (58) \] hold true. As $\Delta_{R \# H}$ is left $H$–colinear and multiplicative, by Lemma 4.24 and Lemma 4.31 $\delta$ and $\omega$ are $H$–colinear, so that \[ (52) \] and \[ (53) \] hold.

Since we have already proved that $\delta$ is left $H$–colinear, we can apply Lemma 4.36 and Lemma 1.24 to deduce \[ (57) \] and \[ (58) \] from the fact that $\Delta_{R \# H}$ is coassociative.

Since $R \# H$ becomes a coalgebra in $(H \# \mathcal{M}_H^H, \square_H, H)$ it results that the canonical map $\pi_{R \# H}$ that is associated to $\varepsilon_{R \# H}$ is left $H$–colinear (see \[ (4.11) \]). This property and the fact that $\varepsilon_{R \# H}$ is an algebra map imply \[ (50) \] and \[ (51) \], in view of Lemma 1.36.

Since $\varepsilon_{R \# H}$ is a counit for $\Delta_{R \# H}$, and since $\Delta_{R \# H}$ is multiplicative, by Lemma 4.36 and Lemma 1.24 we conclude that \[ (50) \] and \[ (51) \] hold. Thus $(R, \varepsilon, \delta, \omega)$ is an Yetter–Drinfel’d quadruple. □

DEFINITION 4.42. Let $(R, \varepsilon, \delta, \omega)$ be a Yetter–Drinfel’d quadruple. The smash algebra $R \# H$ endowed with the bialgebra structure described in Theorem 4.41 will be called the bosonization of $(R, \varepsilon, \delta, \omega)$ and will be denoted by $R \#_b H$.

PROPOSITION 4.43. Let $H$ be a Hopf algebra and let $(R, \varepsilon, \delta, \omega)$ be a Yetter–Drinfel’d quadruple.

(a) The map $\pi : R \#_b H \rightarrow H$, $\pi(r \# h) := \varepsilon(r) h$, is a map in $(H \# \mathcal{M}_H^H)$ and a bialgebra morphism that induces the $(H, H)$–bicomodule structure of $R \# H$.

(b) The map $\sigma : H \rightarrow R \#_b H$, $\sigma(h) := 1 \# h$, is an $(H, H)$–bilinear section of $\pi$ and an algebra morphism that induces the $(H, H)$–bimodule structure of $R \#_b H$.

Proof. (a) Since $\Delta_{R \# H}$ is defined by \[ (51) \] we have

\[ (63) \]

and the second relation of \[ (51) \] we get that

\[ ((\pi \otimes \pi) \Delta_{R \# H})(r \# h) = \sum \varepsilon(\delta^1(r \# h_{(1)})) \delta^2(r \# h_{(1)})_{(2)} \otimes \varepsilon(\delta^2(r \# h_{(1)})(0)) h_{(3)}. \]

Hence by \[ (4.17) \] we get that

\[ ((\pi \otimes \pi) \Delta_{R \# H})(r \# h) = \sum \varepsilon(r) h_{(1)} \otimes h_{(2)}. \]

Clearly $\varepsilon_H \pi = \varepsilon_{R \# H}$, so $\pi$ is morphism of coalgebras. The first equality in \[ (50) \] implies easily that $\pi$ is a morphism of algebras. As the right $H$–module and the right $H$–comodule structures are induced from the corresponding ones of $H$ obviously $\pi$ is right $H$–linear and right $H$–colinear.

The fact that $\pi$ is a morphism of left modules follows by the first relation of \[ (50) \]. To prove that $\pi$ is a morphism of left comodules we use the second equality of \[ (50) \] again.

By a straightforward computation, similar to that on that we performed to prove \[ (4.17) \], we get:

\[ ((R \# H \otimes \pi) \Delta_{R \# H})(r \# h) = \sum \delta^1(r \# h_{(1)}) \varepsilon(\delta^2(r \# h_{(1)})) \# h_{(2)} \otimes h_{(3)} = \sum r \# h_{(1)} \otimes h_{(2)}. \]
This relation means that \( \pi \) induces the usual right \( H \)-comodule structure on \( R \# H \). Analogously one can prove that \( \pi \) induces the diagonal left coaction on \( R \# H \).

(b) Very easy, left to the reader. \( \square \)

The following Theorem was presented by the second author during her talk at the ”2003 Spring Eastern Sectional AMS Meeting” (Special Session on Hopf Algebras and Quantum Groups), New York, NY (U.S.A.), 12-13 April, 2003. There we were informed that the dual form of the equivalence (b) \( \iff \) (c) below, as stated in Theorem 5.23 has already been proved by P. Schauenburg (see 6.1 and Theorem 5.1 in [Sch2]). Nevertheless, for sake of completeness, we decided to keep our proof.

**Theorem 4.44.** Let \( A \) be a bialgebra and let \( H \) be a Hopf algebra. The following assertions are equivalent:

(a) \( A \) is an object in \( \mathcal{H}_H^H \), the map \( \varepsilon_A : A \to K \) is right \( H \)-linear and \( A \) becomes an algebra in \( (\mathcal{H}_H^H, \otimes_H, H) \) and a coalgebra in \( (\mathcal{H}_H^H, \square_H, H) \).

(b) There is an algebra \( R \) in \( \mathcal{H}_H^H \) and there are maps \( \varepsilon_R : R \to k, \delta : R \to R \otimes R, \omega : H \to R \otimes R \) such that \((R, \varepsilon_R, \delta, \omega)\) is a Yetter-Drienfeld quadruple and \( A \) is isomorphic as a bialgebra to the bosonization \( R \#_b H \) of this Yetter-Drienfeld quadruple.

(c) There is a bialgebra map \( \pi : A \to H \) and an \((H, H)\)-bilinear algebra map \( \sigma : H \to A \) such that \( \sigma \varepsilon = \text{Id}_H \).

In this case, we can choose \( R = A^{C\text{co}(H)} \) the diagram of \( A \).

**Proof.** (a) \( \Rightarrow \) (b) By [4.13] the canonical map \( \phi_A : R \# H \to A \) in \( \mathcal{H}_H^H \) is an isomorphism of bialgebras, where the coalgebra structure on \( R \# H \) is defined by \( \Delta_{R \# H} := (\phi_A^{-1} \otimes \phi_A^{-1}) \Delta_A \) and \( \varepsilon_{R \# H} := \varepsilon_A \phi_A \). Clearly \( R \# H \) becomes an algebra in \( (\mathcal{H}_H^H, \otimes_H, H) \) and a coalgebra in \( (\mathcal{H}_H^H, \square_H, H) \) because \( A \) does. Let \( \varepsilon \) be the restriction of \( \varepsilon_A \) to \( R \). Furthermore let us define the \( K \)-linear maps \( \delta, \delta \) and \( \omega \) as in [28] and [44]. By Theorem 4.31 \((R, \varepsilon, \delta, \omega)\) is an Yetter–Drinfeld quadruple. Of course the bosonization of this Yetter–Drinfeld quadruple is the bialgebra \( R \# H \) constructed above.

(b) \( \Rightarrow \) (c) Apply Proposition 4.13 to get a bialgebra map \( \pi' : R \# H \to H \) and an \((H, H)\)-bilinear algebra map \( \sigma' : H \to R \# H \) such that \( \pi' \sigma' = \text{Id}_H \) and \( H \) acts on \( R \# H \) via \( \sigma' \) and coacts on \( R \# H \) via \( \pi' \). Suppose that the isomorphism between \( R \# H \) and \( A \) is given by \( f : A \to R \# H \). Then \( \pi = \pi' f \) and \( \sigma = f^{-1} \sigma' \) are the required morphisms.

(c) \( \Rightarrow \) (a) Since \( \pi \) is a morphism of coalgebras, \( A \) is an \((H, H)\)-comodule with the structures induced by \( \pi \). Similarly \( \sigma \) defines an \((H, H)\)-bimodule structure on \( A \). One can check easily that these structures define a structure of Hopf bimodule on \( A \). Also with respect to these structures \( A \) becomes an algebra in \( (\mathcal{H}_H^H, \otimes_H, H) \) and a coalgebra in \( (\mathcal{H}_H^H, \square_H, H) \). The only thing that we have to prove is that \( \varepsilon_A \) is right \( H \)-linear.

Since \( \sigma \) is right \( H \)-colinear and the right coaction on \( A \) is induced by \( \pi \) we have:

\[
\sum \sigma(h_{(1)}) \otimes h_{(2)} = \sum \sigma(h_{(1)}) \otimes \pi \sigma(h_{(2)}).
\]

By applying \( \varepsilon_A \otimes H \) to this equation we obtain:

\[
\sum \varepsilon_A(\sigma(h_{(1)})) h_{(2)} = (\pi \sigma)(h) = h.
\]

By applying \( \varepsilon_A \) again we get \( \varepsilon_A(\sigma(h)) = \varepsilon_H(h) \). We conclude by remarking that \( ah = a \sigma(h) \), since the right action of \( H \) on \( A \) is induced by \( \sigma \). Thus \( \varepsilon_A(ah) = \varepsilon_H(h) \varepsilon_A(a) \), that is \( \varepsilon_A \) is right \( H \)-linear. \( \square \)

**Remark 4.45.** Let \((R, \varepsilon, \delta, \omega)\) be an Yetter–Drinfeld quadruple such that \( \omega \) is trivial. Recall that this means that:

\[
\omega(h) = \varepsilon_H(h) \text{Id}_R \otimes 1_R, \text{ for all } h \in H.
\]

Then it is easy to check that relations [50]–[53] are equivalent to the fact \((R, \delta, \varepsilon)\) is a bialgebra in \( (\mathcal{H}_H^H \otimes K) \). Conversely, starting with a bialgebra \((R, \delta, \varepsilon)\) in the monoidal category \( \mathcal{H}_H^H \otimes \mathcal{K} \), we can regard \( R \) as an Yetter–Drinfeld quadruple with respect to the trivial cocycle \( \omega \). Furthermore,
the bosonization of this Yetter–Drinfeld quadruple is the usual bosonization of the bialgebra $R$, i.e. as an algebra is the smash product $R \# H$ and as a coalgebra is the cosmash product i.e.

$$\Delta_{R \# H}(r \# h) = \sum r^{(1)} \otimes r^{(2)} (-1)^{\varepsilon(h)} (r^{(0)} \otimes h)$$

$$\varepsilon_{R \# H}(r \# h) = \varepsilon(r) \varepsilon(h),$$

where, by notation, $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$.

**Corollary 4.46.** (D. Radford) Let $H$ be a Hopf algebra and let $A$ be a bialgebra. Then the following statements are equivalent:

(a) $A$ is an object in $H \# \mathcal{H} H$; the counit of $A$ is right $H$–linear, $A$ becomes an algebra in $(H \# \mathcal{H} H, \otimes, H)$ and a coalgebra in $(H \# \mathcal{H} H, \Box, H)$.

(b) The diagram $R$ of $A$ is a bialgebra in $(H \# \mathcal{YD}, \otimes, K)$ such that $A$ is isomorphic as a bialgebra to the usual bosonization of $R$ by $H$.

(c) There are two bialgebra morphisms $\pi : A \to H$, $\sigma : H \to A$ such that $\pi \sigma = \text{Id}_H$.

**Proof.** (a) $\Rightarrow$ (b) By Theorem 4.44 and by the previous remark it is enough to show that $\omega$ is trivial. Let $\Delta_{R \# H} := (\phi_A^{-1} \phi_A^{-1}) \Delta_A \phi_A^{-1}$, where $\phi_A$ is the canonical isomorphism between $R \# H$ and $A$. Since $\Delta_A$ is a morphism in $H \# \mathcal{H} H$ from $A$ to $A \Box H$, we get immediately:

$$\Delta_{R \# H}(1_{R \# H}) = [(1_{R \# H}) \otimes (1_{R \# H})] h = \sum (1_{R \# H_1}) \otimes (1_{R \# H_2})$$

so that

$$\omega(h) = (R \otimes \epsilon_H \otimes R \otimes \epsilon_H) \Delta(1_{R \# H}) = \epsilon_H(h) 1_R \otimes 1_R.$$

(b) $\Rightarrow$ (c) In view of Theorem 4.44 we have only to prove that $\sigma$ is a morphism of coalgebras. For an arbitrary Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$ we showed that $\varepsilon_A(\sigma(h)) = \varepsilon_A(h)$, see the proof of Theorem 4.44. Assume that $\theta : A \to B$ is a bialgebra isomorphism where $(B, m, u, \Delta, \varepsilon)$ is the usual bosonization of a bialgebra $R$ in $(H \# \mathcal{YD}, \otimes)$. Also, by the proof of the above mentioned theorem we have $\omega = \phi_A \sigma'$, where $\sigma'(h) = 1 \# h$. By (64) it follows that $\sigma'$ is a morphism of coalgebras, hence $\sigma$ is so.

(c) $\Rightarrow$ (a) It is trivial, as $\sigma$ is a morphism of coalgebras. \qed

**Lemma 4.47.** Let $A$ be a bialgebra over a field $K$ and let $I$ be a nilpotent ideal and coideal of $A$. If the quotient bialgebra $A/I$ has an antipode, then $A$ is a Hopf algebra.

**Proof.** Let us point out that an element $x$ in a ring $R$ is invertible if it is invertible modulo a nil ideal $L$ of $R$. We apply this to the ring $R = \text{Hom}_K(A, I)$ endowed with the convolution product, to the nil ideal $L = \text{Hom}_K(A, I)$ and to $x = \text{Id}_A$. The quotient $R/L$ is isomorphic to the algebra $\text{Hom}_K(A, A/I)$ and through this identification the class of $\text{Id}_A$ corresponds to the canonical projection of $p : A \to A/I$. We conclude by remarking that the inverse of $p$ in $\text{Hom}_K(A, A/I)$ is $p \circ S$, where $S$ is the antipode of $A/I$. \qed

**Theorem 4.48.** Let $A$ be a bialgebra over a field $K$. If the Jacobson radical $J$ of $A$ is a nilpotent coideal such that $H := A/J$ is a Hopf algebra which has an ad–coinvariant integral and that every canonical map $A/J^{n+1} \to A/J^n$ splits in $H \# \mathcal{H} H$, then $A$ is isomorphic as a bialgebra with the bosonization $R \# H$ of a certain Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$. In fact $A$ and $R \# H$ are isomorphic Hopf algebras.

**Proof.** By Theorem 4.44 there is an $(H, H)$–bilinear algebra section $\sigma : H \to A$ of the canonical projection $\pi : A \to H$. We conclude by applying Theorem 4.44 and Lemma 4.47. \qed

**Theorem 4.49.** Let $A$ be a bialgebra over a field $K$. If the Jacobson radical $J$ of $A$ is a nilpotent coideal such that $H := A/J$ is a Hopf algebra which is both semisimple and cosemisimple (e.g. when $H$ is semisimple over a field of characteristic 0), then $A$ is isomorphic as a bialgebra with the bosonization of a certain Yetter–Drinfeld quadruple $(R, \varepsilon, \delta, \omega)$. In fact $A$ and $R \# H$ are isomorphic Hopf algebras.

**Proof.** Apply Corollary 4.15, Theorem 4.44 and Lemma 4.47. \qed
5. Dual results and applications

Of course all result of the previous sections can be dualized. Because this process is based only on some elaborate computation and does not require new ideas we shall just state the main results that we shall use in this part of the paper.

5.1. We start by defining the Hochschild cohomology of a coalgebra in a monoidal category \((\mathcal{M}, \otimes, 1)\). A triple \((C, \Delta, \varepsilon)\) such that \(C\) is an object in \(\mathcal{M}\) and \(\Delta : C \rightarrow C \otimes C\) and \(\varepsilon : C \rightarrow 1\) are morphisms in \(\mathcal{M}\) is a coalgebra in \(\mathcal{M}\) if it is an algebra in the dual monoidal category \(\mathcal{M}^\circ\) of \(\mathcal{M}\). Recall that \(\mathcal{M}^\circ\) and \(\mathcal{M}\) have same objects but \(\mathcal{M}^\circ(Y, X) = \mathcal{M}(Y, X)\). Similarly, for any coalgebra in \(\mathcal{M}\), we define a \((C, C)\)-bicomodule in \(\mathcal{M}\) to be an \((C, C)\)-bimodule in \(\mathcal{M}^\circ\). The category of all \((C, C)\)-bicomodules will be denoted by \(\mathcal{C} \mathcal{M}^C\). It is an abelian category (if \(\mathcal{M}\) is so). Furthermore, the class \(\mathcal{I}\) of all monomorphism in \(\mathcal{C} \mathcal{M}^C\) that have a retraction in \(\mathcal{M}\) is an injective class of monomorphisms. Note that if we regard \(C\) as an algebra in \(\mathcal{M}^\circ\) then \(\mathcal{I}\) is the projective class associated to this algebra, as in \([19]\).

We fix a coalgebra in a monoidal category \((\mathcal{C} \mathcal{M}^C, \otimes, 1)\). Now, for any \((C, C)\)-bicomodule \(M \in \mathcal{C} \mathcal{M}^C\), we define the Hochschild cohomology of \(C\) with coefficients in \(M\) by:

\[
\mathbf{H}^n(M, C) = \operatorname{Ext}^n_{\mathcal{C} \mathcal{M}^C}(M, C),
\]

where \(\operatorname{Ext}^n_{\mathcal{C} \mathcal{M}^C}(M, -)\) are the relative left derived functors of \(\mathcal{C} \mathcal{M}^C(M, -)\). Note that \(\mathbf{H}^*(M, C)\) is the Hochschild cohomology of the algebra \(C\) with the coefficients in \(M\) (regarded as objects in \(\mathcal{M}^\circ\)).

**Definition 5.2.** A coalgebra in \(\mathcal{M}\) is called *coseparable* if and only if the comultiplication \(\Delta : C \rightarrow C \otimes C\) has a retraction in \(\mathcal{C} \mathcal{M}^C\).

**Proposition 5.3.** Let \(C\) be a coalgebra in \(\mathcal{M}\). The following assertions are equivalent:

(a) \(C\) is coseparable.

(b) \(C\) is \(\mathcal{I}\)-injective in \(\mathcal{C} \mathcal{M}^C\).

(c) \(\mathbf{H}^1(M, C) = 0\), for all \(M \in \mathcal{C} \mathcal{M}^C\).

(d) \(\mathbf{H}^n(M, C) = 0\), for all \(M \in \mathcal{C} \mathcal{M}^C\), for all \(n > 0\).

**Proof.** Regard \(C\) and \(M\) as objects in \(\mathcal{M}^\circ\) and apply Proposition 2.8 and Theorem 2.10. \(\square\)

5.4. Also by working in \(\mathcal{M}^\circ\) we obtain the natural definition of a formally smooth coalgebra \(C\) in a monoidal category \((\mathcal{C} \mathcal{M}^C, \otimes, 1)\).

Let us consider a morphism of coalgebras \(i : D \rightarrow E\) which has a retraction in \(\mathcal{M}\). We define \(D \wedge D := \ker (\pi \otimes \pi)\Delta_E\), where \((M, \pi)\) is the cokernel of \(i\). Note that \(D \wedge D = E\) if and only if \((\ker i)^2 = 0\), where now \(i\) is regarded as a morphism of algebras in \(\mathcal{M}^\circ\) from \(E\) to \(D\). Hence by applying Theorem 2.32 we have proved the following theorem.

**Theorem 5.5.** Let \(C\) be a coalgebra in \((\mathcal{M}, \otimes, 1)\). Then the following conditions are equivalent:

(a) \(\mathbf{H}^2(M, C) = 0\), for all \(M \in \mathcal{C} \mathcal{M}^C\).

(b) If \(i : D \rightarrow E\) is a coalgebra homomorphism that has a retraction in \(\mathcal{M}\) and \(D \wedge D = E\), then any coalgebra homomorphism \(f : D \rightarrow C\) can be extended to a coalgebra homomorphism \(E \rightarrow C\).

**Definition 5.6.** A coalgebra \(C\) will be called *formally smooth* if it satisfies one of the above equivalent conditions.

**Corollary 5.7.** Any coseparable coalgebra in a monoidal category is formally smooth.

**Lemma 5.8.** Let \(H\) be Hopf algebra.

(a) \(H\) is coseparable as a coalgebra in \(\mathfrak{M}_H\) if and only if \(H\) is cosemisimple.

(b) \(H\) is coseparable as a coalgebra in \(\mathfrak{H} \mathfrak{M}_H\) if and only if there is an ad–invariant integral \(\lambda \in H^*\) (see Definition 3.7). In particular, if \(H\) is semisimple and cosemisimple, then \(H\) is coseparable in \(\mathfrak{H} \mathfrak{M}_H\).

**Proof.** Dual to Lemma 3.11. \(\square\)
Theorem 5.9. Let H be a Hopf algebra.

a) Let C be a coalgebra in $\mathcal{M}_H$. If the coradical $C_0$ of C is H then there is a coalgebra map $\pi_C : C \to \mathcal{M}_H$ which is a morphism in $\mathcal{M}_H$, such that $\pi_C|_{H} = \text{Id}_H$.

b) Let C be a coalgebra in $\mathcal{M}_H$. If $C_0 = H$, H has an ad–invariant integral and every C is a direct summand in $\mathcal{M}_H$ as an object in $\mathcal{M}_H$, then there is a coalgebra map $\pi_C : C \to \mathcal{M}_H$ which is a morphism in $\mathcal{M}_H$ such that $\pi_C|_{H} = \text{Id}_H$.

c) Let C be a coalgebra in $\mathcal{M}_H$. If $C_0 = H$ is semisimple then a morphism $\pi$ as in (b) exists.

Proof. a) Let us consider the coradical filtration $(C_n)_{n \in \mathbb{N}}$. Obviously, every $C_n$ is a coalgebra in $\mathcal{M}_H$ and $C_{n+1} = C_n \cap C_0$ (the wedge product is performed in the coalgebra $C_{n+1}$). Indeed, by the definition of the coradical filtration we have:

$$C_{n+1} = \{x \in C \mid \Delta(x) \in C \otimes C_n + C_0 \otimes C\},$$

for every $n \geq 0$. Let $f_0 := \text{Id}_H = \text{Id}_{C_0}$. Let us assume that we have constructed $f_0, \ldots, f_n$ morphisms of coalgebras in $\mathcal{M}_H$ such that $f_{i+1}|_{C_i} = f_i : C_i \to H$, for all $i \in \{0, \ldots, n-1\}$. On the other hand, by (55), $C_{n+1}/C_n$ becomes a right $H = C_0$–comodule with the structure induced by $\Delta$. Hence $C_{n+1}/C_n$ is an object in $\mathcal{M}_H$, so it is free as a right $H$–module (by the fundamental theorem for Hopf modules). In conclusion the inclusion $C_n \subseteq C_{n+1}$ has a retraction in $\mathcal{M}_H$. Since $H = C_0$ is cosemisimple, by Lemma 5.3(a), it is coseparable in $\mathcal{M}_H$ so that we can apply Theorem 5.1 to find a morphism of coalgebras $f_{n+1} : C_{n+1} \to H$ such that $f_{n+1}|_{C_n} = f_n$. Hence there is a unique morphism of coalgebras $\pi_C : C \to H$ in $\mathcal{M}_H$ such that $\pi_C|_{C_n} = f_n$ for all natural numbers $n$.

b) By the Lemma 5.3(b), $H$ is coseparable in $\mathcal{M}_H$ and moreover by assumption $C_n$ is a direct summand of $C_{n+1}$ as an object in $\mathcal{M}_H$. Hence we can apply Theorem 5.1 in the case when $\mathcal{M} = \mathcal{M}_H$, to find a morphism of coalgebras $f_{n+1} : C_{n+1} \to H$ in $\mathcal{M}_H$ such that $f_{n+1}|_{C_n} = f_n$.

c) Since $H$ is semisimple, it is separable so that the category $\mathcal{M}_H$ is semisimple. Hence any $C_n$ is a direct summand of $C_{n+1}$ as a $H$–module. Since $H = C_0$ is also cosemisimple, by Lemma 5.4 $H$ has an ad–invariant integral and so that conclude by b).

Corollary 5.10. Let A be a Hopf algebra such that $A_0$, the coradical of A, is a Hopf subalgebra.

a) There is a coalgebra map $\pi : A \to A_0$ which is a morphism in $\mathcal{M}_{A_0}$ such that $\pi|_{A_0} = \text{Id}_{A_0}$.

b) Suppose that $A_0$ has an ad–invariant integral and every $A_n$ is a direct summand of $A_{n+1}$ as an $(A_0, A_0)$–bimodule. Then there is a coalgebra map $\pi_A : A \to A_0$ which is a morphism in $\mathcal{M}_{A_0}$ such that $\pi_A|_{A_0} = \text{Id}_{A_0}$.

Remarks 5.11. a) A. Masuoka informed us that the first statement of Theorem 5.9 follows easily from [Mas] Theorem 4.1.

b) Statement (a) in Corollary 5.10 has already been proved by Masuoka, see [Mas] Theorem 3.1.

5.12. Let $H$ be a cosemisimple Hopf algebra. Suppose that $C$ is a coalgebra in $\mathcal{M}_H$ such that the coradical of $C$ is $H$. Then by the above theorem there is a coalgebra map $\pi_C : C \to H$ which is right $H$–linear and $\pi_C(h) = h$, for any $h \in H$. Since $\pi_C$ is a morphism of coalgebras it follows that $C$ is an $(H, H)$–bicomodule. In fact, as $\pi_C$ is a morphism of right $H$–modules, we can prove easily that $C$ is an object in $\mathcal{M}_H$, the comultiplication $\Delta_C$ is a morphism in $\mathcal{M}_H$, and $\text{Im} \Delta_C \subseteq \mathcal{M}_H$. Let $R$ be the subspace of right coinvariant elements in $C$. By the fundamental theorem of Hopf modules $\varphi_C : R \otimes H \to C$, $r \otimes h \mapsto rh$ is an isomorphism in $\mathcal{M}_H$. It is also left $H$–colinear, since $C$ belongs to $\mathcal{M}_H$. Thus the comultiplication $\Delta_C : C \to \mathcal{M}_H$ can be identified with a morphism $\Delta_{R\otimes H} : R \otimes H \to (R \otimes H) \mathcal{M}_H$ in $\mathcal{M}_H$. By Lemma 5.10 and (4.18) there is $\delta : R \otimes H \to R \otimes R$ such that relations (26) and (21) hold. Since $\Delta_{R \otimes H}$ is right $H$–linear we have:

$$\Delta_{R \otimes H}(r \otimes h) = \Delta_{R \otimes H}(r \otimes 1)h = \sum \left(\delta^1(r \otimes 1) \otimes \delta^2(r \otimes 1)(\cdot 1)_{h(1)}\right) \otimes \left(\delta^1(r \otimes 1)(\cdot 0) \otimes h_2\right).$$

If $\delta(r) = \bar{\delta}(r \otimes 1)$, and we use the notation $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$ then the above relation becomes:

$$\Delta(r \# h) = \sum \left(r^{(1)} \otimes r^{(2)}_{(-1)}h_{(1)}\right) \otimes \left(r^{(2)}_{(0)} \otimes h_{(2)}\right).$$
An easy computation shows that $\delta : R \to R \otimes R$ is coassociative and left $H$–linear since $\Delta_{R \otimes H}$ is so. Moreover, if $\varepsilon$ is the restriction of $\varepsilon_C$ to $R$, then $\varepsilon$ is a counit for $\delta$. In conclusion we have proved the following proposition.

**Proposition 5.13.** Let $H$ be a cosemisimple Hopf algebra. Suppose that $C$ is a coalgebra in $\mathfrak{M}_H$ such that the coradical $C_0$ of $C$ is $H$. Then $C$ is an object in $H\mathfrak{M}^H_H$ such that $R$, the space of right coinvariant elements of $C$, is an $H$–comodule coalgebra and $C$ is isomorphic as a coalgebra, via a morphism in $H\mathfrak{M}^H_H$, with the smash product coalgebra $R \# H$ of $R$ by $H$.

**Corollary 5.14.** Keep the notation and assumptions from the preceding proposition. Then there is a right $H$–linear coalgebra morphism $\pi_R : C \to R$ such that $\pi_R|_R = \text{Id}_R$, where $R$ is regarded as a right module with trivial action.

**Proof.** Obviously $\pi' : R \# H \to R$, given by $\pi'(r \# h) = \varepsilon_R(h) r$, is a morphism of coalgebras, it is right $H$–linear and $\pi'|_H = \text{Id}_R$. Hence $\pi_R := \pi' \varphi_C^{-1}$ has the same properties, as the canonical map $\varphi_C : R \# H \to C$ is an isomorphism of coalgebras in $H\mathfrak{M}^H_H$ and $\varphi_C(r \# 1) = r$, for every $r \in R$. □

**Lemma 5.15.** Let $C$ be a coalgebra. Suppose that there is a group–like element $c_0 \in C$ such that $C_0 = Kc_0$, i.e. $C$ is connected. Let $(C_n)_{n \in \mathbb{N}}$ be a coalgebra filtration in $C$ such that $C_0 = C_1$. Then, for every $c \in C_n$, we have:

$$\Delta(c) - c \otimes c_0 - c_0 \otimes c \in C_{n-1} \otimes C_{n-1}.\number{66}$$

In particular, if $c \in C_1$ then $\Delta(c) = c \otimes c_0 + c_0 \otimes c - \varepsilon(c) c_0 \otimes c_0$.\number{67}

**Proof.** Since $(C_n)_{n \in \mathbb{N}}$ is a coalgebra filtration we have $\Delta(C_n) \subseteq \sum_{i+j=n} C_i \otimes C_j$. Hence there are $c'$, $c''$ and $x \in C_{n-1} \otimes C_{n-1}$ such that:

$$\Delta(c) = c' \otimes c_0 + c_0 \otimes c'' + x.\number{68}$$

By applying $\varepsilon \otimes C$ and $C \otimes \varepsilon$ to this relation we deduce that:

$$c = \varepsilon(c') c_0 + c'' + x_1,$$
$$c = \varepsilon(c'') c_0 + c' + x_2,$$

where $x_1$, $x_2$ are in $C_{n-1}$ since $x \in C_{n-1} \otimes C_{n-1}$. We conclude the first part of the lemma by substituting $c'$ and $c''$ in (67). Now, if $c \in C_1$ then $\Delta(c) = c \otimes c_0 + c_0 \otimes c + \alpha c_0 \otimes c_0$, for a certain $\alpha$ in $K$. By applying $\varepsilon \otimes \varepsilon$ we deduce that $\alpha = -\varepsilon(c)$. □

5.16. Let $H$ be a cosemisimple Hopf algebra. We shall denote by $\hat{H}$ the set of isomorphism classes of simple left $H$–comodules. It is well–known that for every $\tau \in H$ there is a simple subcoalgebra $C(\tau)$ of $H$ such that $\rho_V(V) \subseteq C(\tau) \otimes V$, where $(V, \rho_V)$ is an arbitrary comodule in $\tau$. Moreover, we have $H = \bigoplus_{\tau \in \hat{H}} C(\tau)$.

**Theorem 5.17.** Keep the notation and assumptions from the statement and the proof of Proposition 5.13. Let $(C_n)_{n \in \mathbb{N}}$ be the coradical filtration of $C$.

a) For every natural number $n$ we have $C_n \simeq R_n \# H$ (isomorphism in $H\mathfrak{M}^H_H$). In particular $C_n$ is freely generated as an $H$–module by elements $r \in C$ satisfying the relation:

$$\Delta(r) = \sum r_{(-1)} \otimes r_{(0)} + r \otimes 1_H + C_{n-1} \otimes C_{n-1}.\number{69}$$

b) $C_1$ verifies the following equation:

$$C_1 = C_0 + \sum_{\tau \in \hat{H}} (C(\tau) \wedge K1_H) H,$$

**Proof.** a) By Proposition 5.13, $C_n$ is the smash product coalgebra $R_n \# H$. By the construction of $R_n$ we have $R_n = R \cap C_n$. Since $C_n$ is isomorphic in $H\mathfrak{M}^H_H$ with $R_n \# H$ it results that $C_n$ is free as a right $H$–module.

Note that $(R_n)_{n \in \mathbb{N}}$ is not a priori a coalgebra filtration in $R$, since $R$ is not a subcoalgebra of $C$ (its comultiplication is $\delta$, see 5.12 for its definition). Recall also that we use the notation $\delta(r) = \sum r^{(1)} \otimes r^{(2)}$. 

□
Let us prove that \((R'_n)_{n \in \mathbb{N}}\) is indeed a coalgebra filtration. Let \(\pi_R\) be the coalgebra morphism from Corollary 5.14. Then \(R'_n = \pi_R(C_n)\), so \((R'_n)_{n \in \mathbb{N}}\) is a coalgebra filtration of \(R\), as \(\pi_R\) is surjective. By [CDMM Corollary 5.3.5] the coradical of \(R\) is included in \(\pi_R(H) = K1_H\), hence \(R\) is connected and \(R'_0 = R_0\). By Lemma 5.14 applied to the filtration \((R'_n)_{n \in \mathbb{N}}\) we deduce that:
\[
\delta(r) \in r \otimes 1_H + 1_H \otimes r + R'_n \otimes R'_n,
\]
for any \(r \in R'_{n+1}\). By induction it results that \(R'_n \subseteq R_n\), for every \(n\). On the other hand, for \(r \in R_{n+1}\) we have:
\[
\delta(r) \in r \otimes 1_H + 1_H \otimes r + R_n \otimes R_n.
\]
Since \(C\) is isomorphic to the smash product coalgebra via the canonical map \(\phi_C\) we get:
\[
(70) \quad \Delta(r) \in \sum r^{(1)}_n r^{(2)}_n \otimes r^{(3)}_0 + R_n H \otimes R_n H = \sum r^{(1)}_n \otimes r^{(3)}_0 + r \otimes 1_H + R_n H \otimes R_n H.
\]
If we assume, by induction, that \(R_n = R'_n\) then \(\Delta(r) \in C \otimes C_n + H \otimes C\), that is \(r \in C_{n+1}\). Thus \(r \in C_{n+1} \cap R = R'_{n+1}\). In conclusion the filtrations \((R'_n)_{n \in \mathbb{N}}\) and \((R_n)_{n \in \mathbb{N}}\) are equal, and \(C_n \simeq R_n \# H\). Note also that, by (70), every element in \(R_n\) satisfies (69), so (a) is proved.

b) By the proof of the first part it follows that every \(R_n\) is a subobject in \(\tilde{H}^{gr1H}\) of \(R\). Let us decompose \(R_1\) as a direct sum of left \(H^\#\)-comodules:
\[
(71) \quad R_1 = K1_H \oplus R'_1 = K1_H \oplus (\oplus_{i=1}^n V_i),
\]
where each \(V_i\) is simple. Let \(\tau_i\) be the isomorphism class of \(V_i\). Take \(i \in \{1, \ldots, n\}\) and \(r \in V_i\). As in the proof of (70), by using the second equality in Lemma 5.14 one can show that:
\[
\Delta(r) = \sum r^{(1)}_n \otimes r^{(3)}_0 + r \otimes 1_H - \varepsilon(r) 1_H \otimes 1_H = \sum r^{(1)}_n \otimes r^{(3)}_0 + (r - \varepsilon(r)1_H) \otimes 1_H.
\]
Hence \(\Delta(r) \in C(\tau_i) \otimes C + C \otimes K1_H\) which proves that \(r \in C(\tau_i) \wedge K1_H\). Thus, in view of the decomposition (71), we have proved the inclusion " \(\subseteq\) " of (69), as \(C\) is generated as a right \(H\)-module by \(R\). The other inclusion is trivial since, for \(r \in H^\# R\) and \(c \in C(\tau) \wedge K1_H\), we have:
\[
\Delta(c) \in C(\tau) \otimes C + C \otimes K1_H \subseteq H \otimes C + C \otimes H.
\]
Thus \(c \in H \wedge H = C_1\), so we deduce \((C(\tau) \wedge K1_H)H \subseteq C_1\), as \(C_1\) is a right submodule of \(C\). \(\square\)

**Remark 5.18.** Let \(A\) be a Hopf algebra such that \(A_0\), the coradical of \(A\), is a subalgebra. In [AS3 Lemma 4.2] it is shown that equation (69) holds true for \(C := grA\). In [CDMM Remark 3.2] it is pointed out that the proof of (69), given in [AS3] for \(grA\), also works in the case \(C := A\), as \(A\) is a cosmash by [Mas Theorem 3.1].

**Definition 5.19.** Let \(H\) be a Hopf algebra and let \((R, \delta, \varepsilon)\) be a coalgebra in the category \((\mathcal{H}^{\mathcal{YD}}, \otimes, K)\). Assume that \(m : R \otimes R \rightarrow R\), and \(\xi : R \otimes R \rightarrow H\) are \(K\)-linear maps and fix one element \(1 \in R\). The quadruple \((R, 1, m, \xi)\) will be called dual Yetter–Drinfeld quadruple if and only if, for all \(r, s, t \in R\) and \(h \in H\), the following relations hold:
\[
\begin{align*}
(72) & \quad h1 = \varepsilon_H(h)1 \quad \text{and} \quad \rho_R(1) = 1_H \otimes 1; \\
(73) & \quad \delta(1) = 1 \otimes 1 \quad \text{and} \quad \varepsilon(1) = 1_K; \\
(74) & \quad h m(r \otimes s) = \sum m(h^{(1)} r \otimes h^{(2)} s); \\
(75) & \quad \delta m = (m \otimes m) \delta_{R \otimes R}\quad \text{and} \quad \varepsilon m = m_K(\varepsilon \otimes \varepsilon); \\
(76) & \quad \Delta_H \xi = (m_H \otimes H)(\xi \otimes H)(\xi \otimes \xi)(R \otimes R \otimes \rho_{R \otimes R})\delta_{R \otimes R}\quad \text{and} \quad \varepsilon H \xi = m_K(\varepsilon \otimes \varepsilon); \\
(77) & \quad c_{R,H}(m \otimes \xi) \delta_{R \otimes R} = (m_H \otimes m)(\xi \otimes H \otimes m)(R \otimes R \otimes \rho_{R \otimes R})\delta_{R \otimes R}; \\
(78) & \quad m_R m = m(m \otimes R)(R \otimes R \otimes m_R)(\xi \otimes \xi \otimes R)(\delta_{R \otimes R} \otimes R); \\
(79) & \quad m_H (\xi \otimes H)(R \otimes m \otimes \xi)(R \otimes \delta_{R \otimes R}) = m_H (\xi \otimes H)(R \otimes c_{H,R})(m \otimes \xi \otimes R)(\delta_{R \otimes R} \otimes R); \\
(80) & \quad m(r \otimes 1) = r = m(1 \otimes r); \\
(81) & \quad \xi(r \otimes 1) = \xi(1 \otimes r) = \varepsilon(r)1_H.
\end{align*}
\]
Remark 5.20. Note that these relations can be interpreted as follows:

- 1 is left \( H \)–invariant and left \( H \)–coinvariant;
- 1 is a group–like element;
- \( m \) is left \( H \)–linear;
- \( \xi \) is left \( H \)–linear, where \( H \) is a module with the adjoint action;
- \( m \) is a morphism of coalgebras, where on \( R \otimes R \) we consider the coalgebra structure that uses the braiding \( c \);
- \( \xi \) is a normalized cocycle; more generally, if \( C \) is a left \( H \)–comodule coalgebra then a map \( \psi : C \to H \) is called a non–commutative 1 cocycle if

\[
\Delta_H(\psi(c)) = \sum \psi(\epsilon_1) \otimes \psi(\epsilon_2) = \sum \psi(\epsilon_1) \otimes \psi(\epsilon_2)
\]

Remark 5.21. Let \( R \) be a coalgebra in \( \mathcal{H} \mathcal{Y} \mathcal{D} \). If \( 1 \in R, m : R \otimes R \to R \) and \( \xi : R \otimes R \to H \) are linear maps then the following assertions are equivalent:

(a) \((R,1,m,\xi)\) is a dual Yetter–Drinfeld quadruple.

(b) The smash product coalgebra \( R \# H \) of \( R \) by \( H \) is a bialgebra with unit \( 1\#1_H \) and multiplication:

\[
m_{R \# H}(r \# h \otimes s \# k) = \sum m(r_1 \otimes h_1 s_1) \otimes \xi(r_0 \otimes h_2 s_2) h_3 k,
\]

and \( R \# H \) becomes a coalgebra in \( (\mathcal{H} \mathcal{M} \mathcal{H}_H, \mathcal{D}_H, H) \) and an algebra in \( (\mathcal{H} \mathcal{M} \mathcal{H}_H, \otimes, H) \).

Definition 5.22. Let \((R,1,m,\xi)\) be a dual Yetter–Drinfeld quadruple. The smash product coalgebra \( R \# H \) endowed with the bialgebra structure described in the preceding Theorem will be called the bosonization of \((R,1,m,\xi)\) and will be denoted by \( R^\# H \).

As we already remarked, before Theorem 5.21 the equivalence \((b) \iff (c)\) below has already been proved by P. Schauenburg (see 6.1 and Theorem 5.1 in [Sch2]).

Theorem 5.23. Let \( A \) be a bialgebra and let \( H \) be a Hopf algebra. The following assertions are equivalent:

(a) \( A \) is in \( \mathcal{H} \mathcal{M} \mathcal{H}_H, 1 \) is right \( H \)–coinvariant and \( A \) becomes a coalgebra in \( (\mathcal{H} \mathcal{M} \mathcal{H}_H, \mathcal{D}_H, H) \) and an algebra in \( (\mathcal{H} \mathcal{M} \mathcal{H}_H, \otimes, H) \).

(b) There is a coalgebra \( R \) in \( \mathcal{H} \mathcal{Y} \mathcal{D} \), there is an element \( 1 \in R \) and linear maps \( m : R \otimes R \to R \), \( \xi : R \otimes R \to H \) such that \((R,1,m,\xi)\) is a dual Yetter–Drinfeld quadruple and \( A \) is isomorphic, as a bialgebra, with the bosonization \( R \# H \) of \((R,1,m,\xi)\).

(c) There are a bialgebra map \( \sigma : H \to A \) and an \( (H,H) \)–bilinear coalgebra map \( \pi : A \to H \) such that \( \sigma \pi = \text{Id}_H \).

In this case, we can choose \( R = A^{\text{Co}(H)} \) the diagram of \( A \).
Theorem 5.24. Let $A$ be bialgebra over a field $K$. Suppose that the coradical $H$ of $A$ is a semisimple sub–bialgebra of $A$ with antipode. Then $A$ is isomorphic as a bialgebra with the bosonization $R^{\#}bH$ of a certain dual Yetter–Drinfeld quadruple $(R,1,m,\xi)$. In fact $A$ and $R^{\#}bH$ are isomorphic Hopf algebras.

Proof. The first assertion is dual to Theorem 4.44. In view of a famous Takeuchi's result (see [Ma] Lemma 5.2.10) $A$ and $R^{\#}bH$ are Hopf algebras and hence they are also isomorphic as Hopf algebras.

Example 5.25. Let $p$ be an odd prime and let $K$ an infinite field containing a primitive $p$–th root of the unit $\lambda$. Let $C$ be a cyclic group of order $p^2$ with generator $c$. For every $a \in K$, $a \neq 0$, let $A := H(a)$ be the Hopf algebra constructed by Beatty, Dăscălescu and Grünfelder in [BDC]. $A$ has dimension $p^4$, with basis $\{c^i x_1^j x_2^k 0 \leq i \leq p^2, 0 \leq j, r \leq p - 1\}$ where $c, x_1, x_2$ are subject to:

$$c^{p^2} = 1, x_1^p = c^p - 1, x_2^p = c^p - 1,$$

$$x_1 c = \lambda^{-1} x_1 c, x_2 c = \lambda c x_2, x_2 x_1 = \lambda x_1 x_2 + a (c^2 - 1),$$

$$\Delta (c) = c \otimes c, \Delta (x_1) = c \otimes x_1 + x_1 \otimes 1, \Delta (x_2) = c \otimes x_2 + x_2 \otimes 1.$$ 

$A$ is a pointed Hopf algebra with coradical $H := KC$. Let $\sigma : H \to A$ be the canonical injection and let $\pi : A \to H$ be the obvious projection. It is straightforward to show that $A, H, \pi$ and $\sigma$ fulfill the requirements of Theorem 5.23(c). Let

$$R = A^{coH} = \left\{ b \in A \mid \sum b_{(1)} \otimes \pi (b_{(2)}) = b \otimes 1 \right\}.$$ 

We have that $R$ is the $K$–subspace of $A$ spanned by the products $x_1^j x_2^k$, where $0 \leq j, r \leq p - 1$. In view of Theorem 5.23, one gets a dual Yetter–Drinfeld quadruple $(R,1,m,\xi)$ such that $A$ is isomorphic as a bialgebra, with the bosonization $R^{\#}bH$ of $R$ by $H$. We point out that $\xi$ is not trivial. In fact we have:

$$\xi (x_2 \otimes x_1) = a (c^2 - 1).$$

Clearly, the dual Hopf algebra $A^*$ fulfills the requirements of Theorem 4.44 with respect to $H^*, \sigma^*$ and $\tau^*$. Let $\iota : R \to A$ be the canonical injection. Then we have that the restriction $\Lambda$ of $\iota^*$ to $(A^*)^{coH^*}$

$$\Lambda : (A^*)^{coH^*} \to R^*$$

is an isomorphism. Let $\omega : R^* \otimes R^* \to (R \otimes R)^*$ be the usual isomorphism. Then we have the following commutative diagram:

$$\begin{array}{ccc}
H^* & \xrightarrow{\omega} & (A^*)^{coH^*} \otimes (A^*)^{coH^*} \\
\xi^* \downarrow & & \downarrow \Lambda \otimes \Lambda \\
(R \otimes R)^* & \xrightarrow{\alpha} & R^* \otimes R^*
\end{array}$$

In fact we have:

$$[(\alpha (\Lambda \otimes \Lambda) \omega) (\chi)] (r \otimes s) = (\varepsilon R^\# \chi) m_{R^\#H} (r \# 1 \otimes s \# 1) = \sum \varepsilon_R \left[ m (r^{(1)} \otimes r^{(2)}_{(-1)} s^{(1)}) \chi \left[ \xi (r^{(2)}_{(0)} \otimes s^{(2)}) \right] \right]$$

$$= \sum \varepsilon_R (r^{(1)}) \varepsilon_R (r^{(2)}_{(-1)} s^{(1)}) \chi \left[ \xi (r^{(2)}_{(0)} \otimes s^{(2)}) \right]$$

$$= \sum \varepsilon_R (r^{(1)}) \varepsilon_H (r^{(2)}_{(-1)}) \varepsilon_R (s^{(1)}) \chi \left[ \xi (r^{(2)}_{(0)} \otimes s^{(2)}) \right] = \chi [\xi (r \otimes s) = [K^* (\chi)] (r \otimes s).$$

It follows that we can identify the Yetter–Drinfeld quadruple $((A^*)^{coH^*}, \varepsilon, \delta, \omega)$ with the Yetter–Drinfeld quadruple $(R^*, (u_R)^*, m^*, \xi^*)$, where $(u_R)^* : R^* \to K$ is the evaluation at $1 \in R$. In particular we observe that we get a nontrivial bosonization since $\omega$ is not trivial.
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