Time-dependent spin-wave theory

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We generalize the spin-wave expansion in powers of the inverse spin to time-dependent quantum spin models describing rotating magnets or magnets in time-dependent external fields. We show that in these cases, the spin operators should be projected onto properly defined rotating reference frames before the spin components are bosonized using the Holstein-Primakoff transformation. As a first application of our approach, we calculate the reorganization of the magnetic state due to Bose-Einstein condensation of magnons in the magnetic insulator yttrium-iron garnet; we predict a characteristic dip in the magnetization which should be measurable in experiments.

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I. INTRODUCTION

At low temperatures the static and dynamic properties of magnets are often determined by spin-wave excitations, which are bosonic quasiparticles in a magnetically ordered state. The theory of spin waves\(^1\) has been extremely successful to explain experimental data for a great variety of magnets. The basic assumption is that the thermal and quantum fluctuations are sufficiently small, so that one can expand in fluctuations around the classical ground-state configuration. The first step in the spin-wave expansion is therefore the determination of the spin configuration in the classical limit, where the spin operators are treated as classical vectors. Deviations from the classical limit can then be obtained by projecting the spin operators onto a basis which matches the direction defined by the classical spin configuration, and then bosonizing the spin components using the Holstein-Primakoff transformation\(^2\). Assuming that the spin quantum number \(S\) is large, one can then calculate fluctuation corrections perturbatively in powers of \(1/S\).

It is not obvious how to generalize this strategy to explicitly time-dependent spin Hamiltonians, because in this case energy is not conserved and the proper basis for setting up the spin-wave expansion may not be determined by minimizing the classical ground-state energy. At the first sight one can avoid this problem by simply projecting the spin operators onto a fixed (laboratory) coordinate system and then introducing Holstein-Primakoff bosons as usual. However, as will be demonstrated below, this strategy is not suitable to describe a possible dynamic reorganization of the magnetic state. Moreover, in the laboratory basis it is often very cumbersome (and in practice impossible) to take into account the dominant fluctuation effects. In this work, we shall develop the general framework to set up a proper \(1/S\) expansion out of equilibrium and then use our method to calculate the magnetization dynamics of a simplified spin model for the pumped magnon gas in the magnetic insulator yttrium-iron garnet (YIG),\(^3\) where parametric resonance and Bose-Einstein condensation (BEC) of magnons has recently been observed\(^4\).

II. SPIN WAVE APPROACH

A. Spin-wave expansion in equilibrium

To explain the basic principles of the time-dependent spin-wave expansion, we first consider a Heisenberg ferromagnet in a time-dependent magnetic field,

\[
\mathcal{H}(t) = -\frac{1}{2} \sum_{ij} J_{ij} \hat{S}_i \cdot \hat{S}_j - \sum_i \mathbf{h}_i(t) \cdot \hat{S}_i, \tag{1}
\]

where the sums are over the \(N\) sites of a cubic lattice, and \(\hat{S}_i\) are quantum mechanical spin operators localized at the lattice sites \(\mathbf{R}_i\). The spins interact via exchange couplings \(J_{ij}\) and are exposed to an external space- and time-dependent magnetic field \(\mathbf{h}_i(t)\) which we measure in units of energy. Assuming that \(\mathbf{h}_i(t)\) is sufficiently large, the nonequilibrium expectation values \(\langle \hat{S}_i(t) \rangle\) are finite so that the time-dependent unit vectors \(\hat{\mathbf{m}}_i(t) = \langle \hat{S}_i(t) \rangle / |\langle \hat{S}_i(t) \rangle|\) in the direction of the local magnetic moments are well defined. If the time-dependence of the external field is sufficiently slow, we may use the adiabatic approximation to determine \(\hat{\mathbf{m}}_i(t)\). In this case we may set up the spin-wave expansion as in equilibrium\(^2\) by projecting the spin operators onto a time-dependent basis \(\{\mathbf{e}_i^{(1)}(t), \mathbf{e}_i^{(2)}(t), \hat{\mathbf{m}}_i(t)\}\), where \(\mathbf{e}_i^{(1)}(t)\) and \(\mathbf{e}_i^{(2)}(t)\) are time-dependent unit vectors orthogonal to \(\hat{\mathbf{m}}_i(t)\). The directions \(\hat{\mathbf{m}}_i(t)\) are determined by a time-dependent extension of the static minimization condition of the classical ground-state energy\(^2\):

\[
\hat{\mathbf{m}}_i(t) \times [\mathbf{h}_i(t) + S \sum_j J_{ij} \hat{\mathbf{m}}_j(t)] = 0. \tag{2}
\]

We then expand the spin operators as \(\hat{S}_i = S_i^\parallel \hat{\mathbf{m}}_i + S_i^\perp \mathbf{e}_i^\perp, \) where \(\mathbf{e}_i^\perp = e_i^{(1)} \pm ie_i^{(2)}\). Finally, we express the spin components in terms of canonical boson operators \(a_i\) using the Holstein-Primakoff transformation\(^2\):

\[
S_i^\parallel = S - a_i^\dagger a_i, \quad S_i^\perp = (S_i^\perp)^\dagger = [2S - a_i^\dagger a_i]^{1/2} a_i.
\]

For large \(S\) the square roots can be expanded and the interactions between spin-waves can be taken into account by means of a systematic expansion in powers \(1/S\).
find that in adiabatic approximation the time-dependent thermal equilibrium at inverse temperature $\beta$ field. Assuming that at time $t$ the absolute value $E$ where the ground-state energy has been dropped. Note theory we obtain the Hamiltonian

$$H \approx J \sum_{ij} S_i \cdot S_j - h_z \sum_i S_i^z,$$

and its direction $\hat{m}(t) = \sin \theta_0 [\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}] + \cos \theta_0 \hat{z}$. Here $\theta_0$ is the angle between the magnetic field and the rotation axis, i.e.

$$\cos \theta_0 = \frac{h_z}{h}$$

as shown in Fig. 1.

C. Perturbation theory in the laboratory basis

To see that Eq. (5) is only valid for $|\omega| \ll h$, let us repeat the calculation of the magnetization in a perturbative approach. To set up the spin-wave expansion we write our Hamiltonian

$$\mathcal{H}(t) = \mathcal{H}_z + \mathcal{V}(t)$$

as a sum of the time independent part

$$\mathcal{H}_z = -\frac{1}{2} \sum_{ij} J_{ij} S_i \cdot S_j - h_z \sum_i S_i^z,$$

and the time dependent perturbation

$$\mathcal{V}(t) = -h_\perp \sum_i [\cos(\omega t)S_i^x - \sin(\omega t)S_i^y].$$

We now project the spin operators onto the fixed laboratory basis. This strategy is usually adopted to discuss parametric resonance of magnons and has recently been used in Ref. 10 to calculate the nonequilibrium dynamics of magnons in a related spin model. After expressing the Hamiltonian in terms of laboratory-frame Holstein-Primakoff bosons $b_i$ and transforming to momentum space, $b_i = N^{-1/2} \sum_k e^{i k \cdot R_i} b_k$, the Hamiltonian reads in linear spin-wave theory

$$\mathcal{H}_z \approx \sum_k E_k^{\text{lab}} b_k^\dagger b_k.$$

The dispersion $E_k^{\text{lab}} = \epsilon_k + h_z$ now contains the static part of the magnetic field. The time-dependent perturbation Eq. (10) can be written as

$$\mathcal{V}(t) = -\frac{h_\perp}{2} \sqrt{2S} \sqrt{N} \left[ e^{i \omega t} b_{k=0} + e^{-i \omega t} b_{k=0}^\dagger \right].$$

Since the boson Hamiltonian contains linear terms, the laboratory boson operators $b_k=0$ and $b_{k=0}$ have finite expectation values, thus condense. The dynamics of these expectation values, as well as the time dependence of the magnon distribution function $(b_k(t) b_{-k}(t))$ can be easily obtained within linear spin-wave theory by solving the Heisenberg equations of motion. With appropriate initial conditions we obtain for the time evolution of the magnetization,

$$M_{\text{lab}}(t) = \frac{h_\perp}{h_z - \omega} [\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}] + M_h \hat{z}.$$
Formally the perturbation has been carried out as an expansion in powers of $h_{\perp}/h_z$, but we will see later that it is essentially an expansion in powers of $h_{\perp}/(h_z - \omega)$. An important difference to the adiabatic result is the singularity for $\omega \rightarrow h_z$, which is of course unphysical and one would need a resummation to all orders in $1/S$ to resolve this. Using a similar approach, such a singularity has also been found in Ref. \[10\] for a slightly different treatment of the adiabatic approximation nor exhibits the pathologies of the perturbative approach in the laboratory frame breaks down, Eq. \[13\] indicates that both the adiabatic approximation and the perturbative approach in the laboratory frame have serious limitations: while the adiabatic basis is restricted to slowly varying external field and misses possible dynamic instabilities, in the laboratory basis one generates unphysical singularities in linear spin-wave theory, indicating that important fluctuation effects have been neglected.

III. SPIN WAVES IN THE PROPER ROTATING REFERENCE FRAME

We now develop a time-dependent generalization of the spin-wave expansion which neither suffers from the limitations of the adiabatic approximation nor exhibits the pathologies of the perturbative approach in the laboratory frame. Our theory is guided by the following two insights: (i) the spin operators should be bosonized in a proper rotating basis whose third axis $\tilde{m}_i(t)$ matches the direction of the true nonequilibrium expectation value $\langle \tilde{S}_i(t) \rangle$ and (ii) the proper rotating basis in general does not agree with the adiabatic basis defined in Eq. \[2\].

To construct the proper rotating basis, consider the unitary time-evolution operator $U(t)$ of some arbitrary time-dependent spin Hamiltonian $H(t)$, which satisfies the operator equation

$$i\partial_t U(t) = H(t)U(t). \quad (14)$$

Making the factorization ansatz

$$U(t) = U_0(t)\tilde{U}(t) \quad (15)$$

with some suitable $U_0(t)$, we find that $\tilde{U}(t)$ satisfies $i\partial_t \tilde{U}(t) = \tilde{H}(t)\tilde{U}(t)$, with the effective Hamiltonian

$$\tilde{H}(t) = \tilde{H}_A(t) + \tilde{H}_B(t), \quad (16)$$

where

$$\tilde{H}_A(t) = U_0^\dagger(t)H(t)U_0(t) \quad (17)$$

corresponds to the adiabatic approximation, while

$$\tilde{H}_B(t) = -iU_0^\dagger(t)\partial_t U_0(t) \quad (18)$$

contains all corrections to the adiabatic approximation, including possible Berry phases. We now choose $U_0(t)$ such that for each lattice site it rotates the $z$ axis of the laboratory to an axis in the direction $\tilde{m}_i(t)$ of the true local magnetization. This is achieved by setting

$$U_0(t) = e^{-i\sum_i \alpha_i(t)S_i}, \quad (19)$$

with suitable rotation vectors $\alpha_i(t) = \alpha_i(t)\hat{\alpha}_i(t)$, where $\alpha_i(t)$ is the rotation angle and $\hat{\alpha}_i(t)$ is a unit vector in the direction of the rotation axis. The rotated spin operators can then be written as

$$\tilde{S}_i(t) = e^{i\alpha_i(t)}S_i e^{-i\alpha_i(t)}S_i = e^{\alpha_i(t)}S_i. \quad (20)$$

To calculate the corresponding Berry-phase contribution $\tilde{H}_B(t)$ to the effective Hamiltonian in the rotating reference frame, we use Feynman’s representation

$$\frac{d}{dt}e^A = \int_0^1 d\lambda e^{\lambda A} \frac{dA}{dt} e^{(1-\lambda)A} \quad (21)$$

of the time derivative of the exponential of an operator $A$ which does not necessarily commute with its time derivative $dA/dt$. It is convenient to decompose a general rotation into three successive rotations parametrized by the usual Euler angles $\varphi$, $\theta$ and $\psi$ as follows,

$$e^{\alpha_i(t)} = e^{\varphi_i(t)}e^{\theta_i(t)}e^{\psi_i(t)}, \quad (22)$$

where the rotation vectors are $\varphi_i(t) = \varphi_i(t)\hat{z}$, $\theta_i(t) = \theta_i(t)\hat{\theta}_i(t)$, and $\psi_i(t) = \psi_i(t)\hat{m}_i(t)$. Explicitly, the direction of the rotation vector $\theta_i(t)$ is $\theta_i(t) = \hat{z} \times \hat{m}_i(t)$. To define the spin waves in the proper rotating basis, we expand the rotated spin operators $\tilde{S}_i$ defined in Eq. \[20\] in the time-dependent right-handed basis formed by the following three unit vectors:

$$\tilde{e}_i^{(1)}(t) = \cos \psi_i(t)\hat{\theta}_i(t) + \sin \psi_i(t)\hat{m}_i(t) \times \hat{\theta}_i(t), \quad (23a)$$

$$\tilde{e}_i^{(2)}(t) = -\sin \psi_i(t)\hat{\theta}_i(t) + \cos \psi_i(t)\hat{m}_i(t) \times \hat{\theta}_i(t), \quad (23b)$$

and $\hat{m}_i(t)$. The corresponding spin components are defined by

$$\tilde{S}_i(t) = \tilde{e}_i^{(1)}(t)\hat{e}_i^{(1)}(t) + \tilde{e}_i^{(2)}(t)\hat{e}_i^{(2)}(t) + \hat{e}_i^\parallel\hat{m}_i(t). \quad (24)$$

Evaluating the time derivative in Eq. \[18\] with the help of the formula \[21\] and inserting the expansion \[24\] for the rotated spin operators we can rewrite the Berry-phase contribution to the effective Hamiltonian as

$$\tilde{H}_B(t) = -\sum_i \left[\omega_i^{(1)}(t)\tilde{S}_i^{(1)}(t) + \omega_i^{(2)}(t)\tilde{S}_i^{(2)}(t) + \omega_i^\parallel\hat{e}_i^\parallel\hat{m}_i(t)\right], \quad (25)$$

where the three time-dependent energies $\omega_i^{(1)}(t)$, $\omega_i^{(2)}(t)$, and $\omega_i^\parallel(t)$ can be identified with the well known Euler angle parametrization of the components of the angular velocity vector in the rotating reference frame.

$$\omega_i^{(1)}(t) = \dot{\varphi}_i \sin \theta_i \sin \psi_i - \dot{\theta}_i \cos \psi_i, \quad (26a)$$

$$\omega_i^{(2)}(t) = \dot{\varphi}_i \sin \theta_i \cos \psi_i - \dot{\theta}_i \sin \psi_i, \quad (26b)$$

$$\omega_i^\parallel(t) = \dot{\varphi}_i \cos \theta_i + \dot{\psi}_i. \quad (26c)$$
In the models discussed in this work the proper rotation of the comoving basis is irrelevant, so that we may focus on the special case $\psi_i(t) = 0$. The Berry-phase Hamiltonian \cite{25} then reduces to

$$\tilde{H}_B(t) = -\sum_i \left[ \hat{\theta}_i \tilde{S}_i^{(1)} + \varphi_i \sin \hat{\theta}_i \tilde{S}_i^{(2)} + \varphi_i \cos \hat{\theta}_i \tilde{S}_i^{\|} \right].$$  \hspace{1cm} (27)$$

For the rotating ferromagnet shown in Fig. 1 symmetry suggests that the proper rotating coordinate system is characterized by a time-dependent precession angle $\varphi_i(t) = -\omega t$ and a constant rotation angle $\theta$. The Berry-phase contribution \cite{27} to the Hamiltonian in the rotating basis is then

$$\tilde{H}_B = \omega \sum_i \left[ \sin \theta \tilde{S}_i^{(2)} + \cos \theta \tilde{S}_i^{\|} \right],$$  \hspace{1cm} (28)

which is independent of time. Next we express the spin components in the rotating reference frame in terms of a third type of Holstein-Primakoff boson $c_i$, which should not be confused with the Holstein-Primakoff boson $a_i$ introduced in the adiabatic basis, and also not with the laboratory basis Holstein-Primakoff boson $b_i$. The true tilt angle $\theta$ is determined from the requirement that the effective Hamiltonian contains no terms linear in the bosons, which yields the frequency-dependent result

$$\cos \theta = \frac{h_z - \omega}{\tilde{h}_\omega}$$  \hspace{1cm} (29)

where

$$\tilde{h}_\omega = [h_\perp + (h_z - \omega)^2]^{1/2}.$$  \hspace{1cm} (30)

Note that for finite $\omega$ the true tilt angle $\theta$ is larger than the angle $\theta_0$ between rotation axis and magnetic field. In fact, our result for $\theta$ agrees with the result for a single isolated spin in a rotating magnetic field given in the book by Bohm et al.\cite{12} For the specific geometry shown in Fig. 1 the proper rotating reference frame has also been discussed previously in Ref. \cite{13}, but our Eq. \cite{27} is more general. In fact, our many-body approach allows us to set up a systematic $1/S$ expansion and calculate the thermodynamics and the correlation functions of any time-dependent spin model with finite local moments. Following the steps of the spin-wave expansion we obtain the Hamiltonian

$$\mathcal{H} \approx \sum_k E_k c_k^{\dagger} c_k$$  \hspace{1cm} (31)

to quadratic order in the bosonic operators $c_k^{\dagger}$ and $c_k$ describing bosons in the proper rotating reference frame. The dispersion $E_k = \omega_k + \tilde{h}_\omega$ is modified by the finite oscillation frequency; see inset in Fig. 2. Imposing suitable initial conditions for our model, we obtain for the time-dependent magnetization in linear spin-wave theory,

$$M(t) = \tilde{M}_0 \hat{z} + \tilde{M}_0 ^{\perp} \left[ \cos(\omega t) \hat{x} - \sin(\omega t) \hat{y} \right]$$

$$= \tilde{m}_\omega(t) \hat{M}_\omega,$$  \hspace{1cm} (32)

where $\tilde{M}_0 ^{\|} = \cos \theta \tilde{M}_\omega$ and $\tilde{M}_0 ^{\perp} = \sin \theta \tilde{M}_\omega$ with

$$\tilde{M}_\omega = S - \frac{1}{N} \sum_k \frac{1}{e^{\beta E_k} - 1},$$  \hspace{1cm} (33)

and $\tilde{m}_\omega(t) = \sin \theta \left[ \cos(\omega t) \hat{x} - \sin(\omega t) \hat{y} \right] + \cos \theta \hat{z}$. In the limit $\omega \to 0$, Eq. \cite{32} reduces to the result \cite{5} of the adiabatic approximation, which is only accurate as long as $|\omega| \ll \tilde{h}_\omega$. In fact, for the two special cases $\omega = 0$ and $\omega = 2h_z$ where the effective field is equal to the external field $h = \tilde{h}_\omega$, the adiabatic approximation Eq. \cite{32} matches the correct result of Eq. \cite{5}. While the result \cite{13} for the magnetization obtained from perturbation theory in the laboratory basis approaches the more accurate rotating reference frame result \cite{32} for $\omega \to 0$ and for large frequencies $\omega \gtrsim 2h_z$, perturbation theory in the laboratory basis gives unphysical results in the vicinity of the resonance $|h_z - \omega| \lesssim h_\perp$ and is thus meaningless, whereas Eq. \cite{32} predicts that the magnetization simply rotates in the $xy$-plane ($\theta \approx \pi/2$). Note that the magnetization shown in Fig. 2 does not approach $\tilde{M}_0 ^{\perp} = S$ because thermal fluctuations suppress the total magnetic moment.

IV. PARAMETRIC RESONANCE AND BEC OF MAGNONS IN YIG

Next, let us study another time-dependent spin model which gives us some insight into the relation between

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FIG. 2. (Color online) Comparison of the results for the perpendicular magnetization $M_\perp$ as a function of the rotation frequency $\omega$ in the three different approaches for our model system given in Eq. \cite{1} with the parameters $J = h_z$, $h_\perp = 0.4h_z$, $S = 1/2$ at the temperature $T = 0.5h_z$: blue (dashed) line presents the results of the adiabatic approximation, Eq. \cite{5}, green (dash-dotted) line is perturbation theory in the laboratory frame, Eq. \cite{13} and red (solid) is the proper rotating frame. The shaded area indicates the unphysical region. The inset shows a sketch of the spin-wave gap $E_{k=0}/h_z$ as a function of rotation frequency for the same parameters.

Next, let us study another time-dependent spin model which gives us some insight into the relation between
parametric resonance, BEC of magnons, and the reorga-
nization of the magnetic state. Previously, this problem
has been addressed in Refs. [16 and 17] using a Heisen-
berg ferromagnet with static single-ion anisotropy in a
time-dependent magnetic field. For our purpose it is
more convenient to consider a modified version of this
model, involving a static magnetic field in $z$ direction
and a rotating single-ion anisotropy of magnitude $A$,

$$
\mathcal{H}(t) = -\frac{J_{ij}}{2} \sum_{ij} S_i \cdot S_j - h \sum_i S_i^z
$$

$$
- \frac{A}{2} \sum_i \left\{ |S_i \cdot n(t)|^2 - |S_i \cdot (\hat{z} \times n(t))|^2 \right\},
$$

where the anisotropy axis $n(t) = \cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}$
rotates clockwise in the $xy$ plane. An illustration of the
model (34) is shown in Fig. 3. After bosonization of the
spins using the Holstein-Primakoff transformation in the
laboratory basis, we obtain in linear spin-wave theory,

$$
\mathcal{H}(t) \approx \sum_k \left[ (\epsilon_k + h) b_k^\dagger b_k + \frac{h}{2} \left( e^{2i\omega t} b_k^\dagger b_k + \text{H.c.} \right) \right],
$$

with $h_c = AS$. Time-dependent boson models of this
form have been used as model systems for parametric
resonance in magnon gases. [18] In fact, with appro-
priate replacements [18], the magnon Hamiltonian for YIG
in an external microwave field parallel to the external
field has the same form as Eq. (35). It is well known
that the Hamiltonian (35) predicts a parametric insta-
bility of the magnons with wave-vectors in the regime
$h_c > |\epsilon_k + h|$. If this condition is satisfied, then
the magnon occupation grows exponentially during some
intermediate time interval, until it saturates and the sys-
tem approaches a new equilibrium state, which in prin-
ciple can be calculated by taking the interactions be-
tween the magnons into account. Here we show that
the dynamics of the local magnetization $\langle S_i(t) \rangle$ as
well as the magnon spectrum can be obtained using our
time-dependent spin-wave formalism without considering
interactions between magnons. Because the Hamiltonian
(35) has the same spin symmetries as the rotating fer-
romagnet discussed above, the proper rotating reference


\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{(Color online) Graph of the time dependent spin model defined in Eq. (34). The model can be interpreted as

a Heisenberg magnet with a single-ion anisotropy axis $\hat{n}$ that is

fixed in the laboratory frame. The magnet is exposed to a

static external field perpendicular to the anisotropy axis and

rotates counter-clockwise around an axis parallel to the field.

\end{figure}

frame is again given by a time-dependent precession angle
$\varphi_i(t) = -\omega t$ and a constant rotation angle $\theta$. The
Berry-phase Hamiltonian $\mathcal{H}_B = \omega \sum_i [\sin \theta S_i^z + \cos \theta S_i^x]\]$ is therefore identical with the rotating ferromagnet dis-
cussed above, see Eq. (29). It is then easy to show that
for $|h - \omega| > h_c$ all spins point in the direction of the
field so that the tilt angle $\theta$ vanishes and the magnon spectrum is

$$
E_k = \sqrt{(\epsilon_k + h - \omega)^2 - h_c^2},
$$

where again $\epsilon_k = S(J_0 - J_k)$. On the other hand, for
$|h - \omega| < h_c$ the angle between magnetic field and mag-
netization does not vanish,

$$
\cos \theta = \frac{h - \omega}{h_c},
$$

and the magnon spectrum is

$$
E_k^2 = \sqrt{\left[ \epsilon_k + \frac{3h_c}{2} - \frac{(h - \omega)^2}{2h_c} \right]^2 - \left[ \frac{h_c}{2} + \frac{(h - \omega)^2}{2h_c} \right]^2}.
$$

A graph of the spin-wave gap $E_{k=0}$ is presented in Fig. 4.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Plot of the spin-wave gap $E_{k=0}$ of the time-dependent spin model defined in Eq. (34) as a function of $|h - \omega|/h_c$.}
\end{figure}

In the tilted phase, the time-dependent magnetization
is in linear spin-wave theory $\mathbf{M}_\omega(t) = \mathbf{M}_\omega \cdot \hat{n}(t)$, where
$\mathbf{m}_\omega(t) = \sin \theta [\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}] + \cos \theta \hat{z}$ and

$$
\mathbf{M}_\omega = S \left[ 1 + \frac{1}{2} \frac{3h_c}{2} - \frac{(h - \omega)^2}{2h_c} \right] \times \left[ \frac{1}{\epsilon_k - \frac{h_c}{2}} + \frac{1}{2} \right].
$$

Note that the gap $E_{k=0}$ of the magnon energy vanishes
at the critical fields $h_c^\pm = \pm h_c + \omega$, signaling a quantum
phase transition. Because the magnetic state in the
tilted phase spontaneously breaks the $Z_2$-symmetry
$\mathbf{S}_i \cdot \mathbf{n} \rightarrow -\mathbf{S}_i \cdot \mathbf{n}$ of the spin Hamiltonian [34], this phase
transition belongs to the Ising universality class. If we
bosonize the spin operators in the laboratory frame, then
at the critical point the corresponding bosons acquire
a macroscopic expectation value, which corresponds to
BEC of magnons. However, as pointed out by Kohn and Sherrington, such a transition is neither accompanied by magnon superfluidity nor by off-diagonal long-range order, which distinguishes the magnon condensate from the BEC of trapped atoms or molecules. In fact, the macroscopic occupation of magnon modes is an artifact from the BEC of trapped atoms or molecules. In fact, the range order, which distinguishes the magnon condensate from unity is enhanced by a factor of 500.

The Geometric Phase in Quantum Systems, (Springer, Berlin, 2003).

V. CONCLUSIONS

In summary, we have developed a general method to set up the spin-wave expansion for time-dependent spin models. Our method is very general and should also be useful to study nonequilibrium phenomena in all kinds of ordered magnets, including quantum antiferromagnets and frustrated magnets with finite local moments. We have used our method to study a simplified spin model for the magnon gas in YIG, and have shown that magnon BEC in this system can be interpreted as a magnetic quantum phase transition belonging to the Ising universality class. Our prediction of a dip in the magnetization close to the threshold for BEC can be tested experimentally.

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