FOURIER FRAMES FOR THE CANTOR-4 SET

GABRIEL PICIOROAGA AND ERIC S. WEBER

ABSTRACT. The measure supported on the Cantor-4 set constructed by Jorgensen-Pedersen is known to have a Fourier basis, i.e. that it possess a sequence of exponentials which form an orthonormal basis. We construct Fourier frames for this measure via a dilation theory type construction. We expand the Cantor-4 set to a 2 dimensional fractal which admits a representation of a Cuntz algebra. Using the action of this algebra, an orthonormal set is generated on the larger fractal, which is then projected onto the Cantor-4 set to produce a Fourier frame.

Jorgensen and Pedersen [10] demonstrated that there exist singular measures $\nu$ which are spectral--that is, they possess a sequence of exponential functions which form an orthonormal basis in $L^2(\nu)$. The canonical example of such a singular and spectral measure is the uniform measure on the Cantor 4-set defined as follows:

$$C_4 = \{ x \in [0, 1] : x = \sum_{k=1}^{\infty} \frac{a_k}{4^k}, \ a_k \in \{0, 2\} \}.$$ 

This is analogous to the standard middle third Cantor set where $4^k$ replaces $3^k$. The set $C_4$ can also be described as the attractor set of the following iterated function system on $\mathbb{R}$:

$$\tau_0(x) = \frac{x}{4}, \quad \tau_2(x) = \frac{x + 2}{4}.$$ 

The uniform measure on the set $C_4$ then is the unique probability measure $\mu_4$ which is invariant under this iterated function system:

$$\int f(x) d\mu_4(x) = \frac{1}{2} \left( \int f(\tau_0(x)) d\mu_4(x) + \int f(\tau_2(x)) d\mu_4(x) \right)$$

for all $f \in C(\mathbb{R})$, see [9] for details. The standard spectrum for $\mu_4$ is $\Gamma_4 = \{ \sum_{n=0}^{N} l_n 4^n : l_n \in \{0, 1\} \}$, though there are many spectra [4, 2].

Remarkably, Jorgensen and Pedersen prove that the uniform measure $\mu_3$ on the standard middle third Cantor set is not spectral. Indeed, there are no three mutually orthogonal exponentials in $L^2(\mu_3)$. Thus, there has been much attention on whether there exists a Fourier frame for $L^2(\mu_3)$--the problem is still unresolved, but see [5, 6] for progress in this regard. In this paper, we will construct Fourier frames for $L^2(\mu_4)$ using a dilation theory type argument. The motivation is whether the construction we demonstrate here for $\mu_4$ will be applicable to $\mu_3$. Fourier frames for $\mu_4$ were constructed in [6] using a duality type construction.
A frame for a Hilbert space $H$ is a sequence $\{x_n\}_{n \in I} \subset H$ such that there exists constants $A, B > 0$ such that for all $v \in H$,
\[ A\|v\|^2 \leq \sum_{n \in I} |\langle v, x_n \rangle|^2 \leq B\|v\|^2. \]

The largest $A$ and smallest $B$ which satisfy these inequalities are called the frame bounds. The frame is called a Parseval frame if both frame bounds are 1. The sequence $\{x_n\}_{n \in I}$ is a Bessel sequence if there exists a constant $B$ which satisfies the second inequality, whether or not the first inequality holds; $B$ is called the Bessel bound. A Fourier frame for $L^2(\mu_4)$ is a sequence of frequencies $\{\lambda_n\}_{n \in I} \subset \mathbb{R}$ together with a sequence of “weights” $\{d_n\}_{n \in I} \subset \mathbb{C}$ such that $x_n = d_ne^{2\pi i \lambda_n x}$ is a frame. Fourier frames (unweighted) for Lebesgue measure were introduced by Duffin and Schaffer [3], see also Ortega-Cerda and Seip [13].

It was proven in [8] that a frame for a Hilbert space can be dilated to a Riesz basis for a bigger space, that is to say, that any frame is the image under a projection of a Riesz basis. Moreover, a Parseval frame is the image of an orthonormal basis under a projection. This result is now known to be a consequence of the Naimark dilation theory. This will be our recipe for constructing a Fourier frame: constructing a basis in a bigger space and then projecting onto a subspace. We require the following result along these lines [4]:

**Lemma 1.** Let $H$ be a Hilbert space, $V, K$ closed subspaces, and let $P_V$ be the projection onto $V$. If $\{x_n\}_{n \in I}$ is a frame in $K$ with frame bounds $A, B$, then:

1. $\{P_Vx_n\}_{n \in I}$ is a Bessel sequence in $V$ with Bessel bound no greater than $B$;
2. if the projection $P_V : K \to V$ is onto, then $\{P_Vx_n\}_{n \in I}$ is a frame in $V$;
3. if $V \subset K$, then then $\{P_Vx_n\}_{n \in I}$ is a frame in $V$ with frame bounds between $A$ and $B$.

Note that if $V \subset K$ and $\{x_n\}_{n \in I}$ is a Parseval frame for $K$, then $\{P_Vx_n\}_{n \in I}$ is a Parseval frame for $V$. In the second item above, it is possible that the lower frame bound for $\{P_Vx_n\}$ is smaller than $A$, but the upper frame bound is still no greater than $B$.

The foundation of our construction is a dilation theory type argument. Our first step, described in Section 1, is to consider the fractal like set $C_4 \times [0, 1]$, which we will view in terms of an iterated function system. This IFS will give rise to a representation of the Cuntz algebra $\mathcal{O}_4$ on $L^2(\mu_4 \times \lambda)$ since $\mu_4 \times \lambda$ is the invariant measure under the IFS. Then in Section 2 we will generate via the action of $\mathcal{O}_4$ an orthonormal set in $L^2(\mu_4 \times \lambda)$ whose vectors have a particular structure. In Section 3 we consider a subspace $V$ of $L^2(\mu_4 \times \lambda)$ which can be naturally identified with $L^2(\mu_4)$, and then project the orthonormal set onto $V$ to, ultimately, obtain a frame. Of paramount importance will be whether the orthonormal set generated by $\mathcal{O}_4$ spans the subspace $V$ so that the projection yields a Parseval frame. Section 4 demonstrates concrete constructions in which this occurs, and identifies all possible Fourier frames that can be constructed using this method.

We note here that there may be Fourier frames for $L^2(\mu_4)$ which cannot be constructed in this manner, but we are unaware of such an example.

1. Dilation of the Cantor-4 Set

We wish to construct a Hilbert space $H$ which contains $L^2(\mu_4)$ as a subspace in a natural way. We will do this by making the fractal $C_4$ bigger as follows. We begin with an iterated
function system on $\mathbb{R}^2$ given by:

$\Upsilon_0(x, y) = \left(\frac{x}{4}, \frac{y}{2}\right)$, $\Upsilon_1(x, y) = \left(\frac{x + 2}{4}, \frac{y}{2}\right)$, $\Upsilon_2(x, y) = \left(\frac{x}{4}, \frac{y + 1}{2}\right)$, $\Upsilon_3(x, y) = \left(\frac{x + 2}{4}, \frac{y + 1}{2}\right)$.

As these are contractions on $\mathbb{R}^2$, there exists a compact attractor set, which is readily verified to be $C_4 \times [0, 1]$. Likewise, by Hutchinson [1], there exists an invariant probability measure supported on $C_4 \times [0, 1]$; it is readily verified that this invariant measure is $\mu_4 \times \lambda$, where $\lambda$ denotes the Lebesgue measure restricted to $[0, 1]$. Thus, for every continuous function $f : \mathbb{R}^2 \to \mathbb{C}$,

$$\int f(x, y) \, d(\mu_4 \times \lambda) = \frac{1}{4} \left( \int f\left(\frac{x}{4}, \frac{y}{2}\right) \, d(\mu_4 \times \lambda) + \int f\left(\frac{x + 2}{4}, \frac{y}{2}\right) \, d(\mu_4 \times \lambda) \right.$$ 

$$+ \int f\left(\frac{x}{4}, \frac{y + 1}{2}\right) \, d(\mu_4 \times \lambda) + \int f\left(\frac{x + 2}{4}, \frac{y + 1}{2}\right) \, d(\mu_4 \times \lambda) \right).$$

The iterated function system $\Upsilon_j$ has a left inverse on $C_4 \times [0, 1]$, given by

$$R : C_4 \times [0, 1] \to C_4 \times [0, 1] : (x, y) \mapsto (4x, 2y) \mod 1,$$

so that $R \circ \Upsilon_j(x, y) = (x, y)$ for $j = 0, 1, 2, 3$.

We will use the iterated function system to define an action of the Cuntz algebra $\mathcal{O}_4$ on $L^2(\mu_4 \times \lambda)$. To do so, we choose filters

$$m_0(x, y) = H_0(x, y)$$

$$m_1(x, y) = e^{2\pi ix} H_1(x, y)$$

$$m_2(x, y) = e^{4\pi ix} H_2(x, y)$$

$$m_3(x, y) = e^{6\pi ix} H_3(x, y)$$

where

$$H_j(x, y) = \sum_{k=0}^{3} a_{jk} \chi_{\Upsilon_k(C_4 \times [0, 1])}(x, y)$$

for some choice of scalar coefficients $a_{jk}$. In order to obtain a representation of $\mathcal{O}_4$ on $L^2(\mu_4 \times \lambda)$, we require that the above filters satisfy the matrix equation $\mathcal{M}^*(x, y) \mathcal{M}(x, y) = I$ for $\mu_4 \times \lambda$ almost every $(x, y)$, where

$$\mathcal{M}(x, y) = \begin{pmatrix}
m_0(\Upsilon_0(x, y)) & m_0(\Upsilon_1(x, y)) & m_0(\Upsilon_2(x, y)) & m_0(\Upsilon_3(x, y)) \\
m_1(\Upsilon_0(x, y)) & m_1(\Upsilon_1(x, y)) & m_1(\Upsilon_2(x, y)) & m_1(\Upsilon_3(x, y)) \\
m_2(\Upsilon_0(x, y)) & m_2(\Upsilon_1(x, y)) & m_2(\Upsilon_2(x, y)) & m_2(\Upsilon_3(x, y)) \\
m_3(\Upsilon_0(x, y)) & m_3(\Upsilon_1(x, y)) & m_3(\Upsilon_2(x, y)) & m_3(\Upsilon_3(x, y))
\end{pmatrix}$$

For our choice of filters, the matrix $\mathcal{M}$ becomes

$$\mathcal{M}(x, y) = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} \\
a_{10} & -e^{\pi ix/2} a_{11} & e^{\pi ix/2} a_{12} & -e^{\pi ix/2} a_{13} \\
a_{20} & e^{\pi ix/2} a_{21} & e^{\pi ix/2} a_{22} & e^{\pi ix/2} a_{23} \\
a_{30} & -e^{3\pi ix/2} a_{31} & e^{3\pi ix/2} a_{32} & -e^{3\pi ix/2} a_{33}
\end{pmatrix}.$$
which is unitary if and only if the matrix
\[
H = \begin{pmatrix}
  a_{00} & a_{01} & a_{02} & a_{03} \\
  a_{10} & -a_{11} & a_{12} & -a_{13} \\
  a_{20} & a_{21} & a_{22} & a_{23} \\
  a_{30} & -a_{31} & a_{32} & -a_{33}
\end{pmatrix}
\]
is unitary. For the remainder of this section, we assume that \( H \) is unitary.

**Lemma 2.** The operator \( S_j : L^2(\mu_4 \times \lambda) \rightarrow L^2(\mu_4 \times \lambda) \) given by
\[
[S_j f](x, y) = \sqrt{4} m_j(x, y) f(R(x, y))
\]
is an isometry.

**Proof.** We calculate:
\[
\|S_j f\|^2 = \int |\sqrt{4} m_j(x, y) f(R(x, y))|^2 \, d(\mu_4 \times \lambda)
\]
\[= \frac{1}{4} \sum_{k=0}^{3} \int |4 m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y)))|^2 \, d(\mu_4 \times \lambda)
\]
\[= \int \left( \sum_{k=0}^{3} |m_j(\Upsilon_k(x, y))|^2 \right) |f(x, y)|^2 \, d(\mu_4 \times \lambda).
\]
We used Equation (1) in the second line. The sum in the integral is the square of the Euclidean norm of the \( j \)-th row of the matrix \( M \), which is unitary. Hence, the sum is 1, so the integral is \( \|f\|^2 \), as required. \( \square \)

**Lemma 3.** The adjoint is given by
\[
[S_j^* f](x, y) = \frac{1}{2} \sum_{k=0}^{3} m_j(\Upsilon_k(x, y)) f(\Upsilon_j(x, y)).
\]

**Proof.** Let \( f, g \in L^2(\mu_4 \times \lambda) \). We calculate
\[
\langle S_j f, g \rangle = \int \sqrt{4} m_j(x, y) f(R(x, y)) g(x, y) \, d(\mu_4 \times \lambda)
\]
\[= \frac{1}{4} \sum_{k=0}^{3} \int \sqrt{4} m_j(\Upsilon_k(x, y)) f(R(\Upsilon_k(x, y))) g(\Upsilon_k(x, y)) \, d(\mu_4 \times \lambda)
\]
\[= \int f(x, y) \left( \frac{1}{2} \sum_{k=0}^{3} m_j(\Upsilon_k(x, y)) g(\Upsilon_k(x, y)) \right) \, d(\mu_4 \times \lambda)
\]
where we use Equation (1) and the fact that \( R \) is a left inverse of \( \Upsilon_k \). \( \square \)

**Lemma 4.** The isometries \( S_j \) satisfy the Cuntz relations:
\[
S_j^* S_k = \delta_{jk} I, \quad \sum_{k=0}^{3} S_k S_k^* = I.
\]
Proof. We consider the orthogonality relation first. Let \( f \in L^2(\mu_4 \times \lambda) \). We calculate:

\[
[S_j^* S_k f](x, y) = \frac{1}{2} \sum_{\ell=0}^{3} m_j(\Upsilon_\ell(x, y)) [S_k f](\Upsilon_\ell(x, y))
\]

\[
= \frac{1}{2} \sum_{\ell=0}^{3} m_j(\Upsilon_\ell(x, y)) \sqrt{4} m_k(\Upsilon_\ell(x, y)) f(R(\Upsilon_\ell(x, y)))
\]

\[
= \left( \sum_{\ell=0}^{3} m_j(\Upsilon_\ell(x, y)) m_k(\Upsilon_\ell(x, y)) \right) f(x, y).
\]

Note that the sum is the scalar product of the \( k \)-th row with the \( j \)-th row of the matrix \( M \), which is unitary. Hence, the sum is \( \delta_{jk} \) as required.

Now for the identity relation, let \( f, g \in L^2(\mu_4 \times \lambda) \). We calculate:

\[
\langle \sum_{k=0}^{3} S_k S_k^* f, g \rangle = \sum_{k=0}^{3} \langle S_k^* f, S_k^* g \rangle
\]

\[
= \sum_{k=0}^{3} \int \left( \frac{1}{2} \sum_{\ell=0}^{3} m_k(\Upsilon_\ell(x, y)) f(\Upsilon_\ell(x, y)) \right) \left( \frac{1}{2} \sum_{n=0}^{3} m_k(\Upsilon_n(x, y)) g(\Upsilon_n(x, y)) \right) d(\mu_4 \times \lambda)
\]

\[
= \sum_{\ell=0}^{3} \sum_{n=0}^{3} \frac{1}{4} \int \left( \sum_{k=0}^{3} m_k(\Upsilon_\ell(x, y)) m_k(\Upsilon_n(x, y)) \right) f(\Upsilon_\ell(x, y)) g(\Upsilon_n(x, y)) d(\mu_4 \times \lambda)
\]

\[
= \frac{1}{4} \sum_{n=0}^{3} \int f(\Upsilon_n(x, y)) g(\Upsilon_n(x, y)) d(\mu_4 \times \lambda)
\]

\[
= \langle f, g \rangle.
\]

Note that the sum over \( k \) in the third line is the scalar product of the \( \ell \)-th column with the \( n \)-th column of \( M \), so the sum collapses to \( \delta_{\ell n} \). The sum on \( n \) in the fourth line collapses by Equation (1). \( \square \)

2. Orthonormal Sets in \( L^2(\mu_4 \times \lambda) \)

Since the isometries \( S_j \) satisfy the Cuntz relations, we can use them to generate orthonormal sets in the space \( L^2(\mu_4 \times \lambda) \). We do so by having the isometries act on a generating vector. We consider words in the alphabet \( \{0, 1, 2, 3\} \); let \( W_4 \) denote the set of all such words. For a word \( \omega = j_K j_{K-1} \ldots j_1 \), we denote by \( |\omega| = K \) the length of the word, and define

\[
S_\omega f = S_{j_K} S_{j_{K-1}} \ldots S_{j_1} f.
\]

Definition 1. Let

\[
X_4 = \{ \omega \in W_4 : |\omega| = 1 \} \cup \{ \omega \in W_4 : |\omega| \geq 2, j_1 \neq 0 \}.
\]

For convenience, we allow the empty word \( \omega_\emptyset \) with length 0, and define \( S_{\omega_\emptyset} = I \), the identity.
Lemma 5. Suppose \( f \in L^2(\mu_4 \times \lambda) \) with \( ||f|| = 1 \), and that \( S_0 f = f \). Then,
\[
\{ S_\omega f : \omega \in X_4 \}
\]
is an orthonormal set.

Proof. Suppose \( \omega, \omega' \in X_4 \) with \( \omega \neq \omega' \). First consider \( |\omega| = |\omega'| \), with \( \omega = j_K \ldots j_1 \) and \( \omega' = i_K \ldots i_1 \). Suppose that \( \ell \) is the largest index such that \( j_\ell \neq i_\ell \). Then we have
\[
\langle S_\omega f, S_{\omega'} f \rangle = \langle S_{j_\ell} \ldots S_{j_1} f, S_{i_\ell} \ldots S_{i_1} f \rangle = \langle S_{i_\ell}^* S_{j_\ell} \ldots S_{j_1} f, S_{i_\ell} \ldots S_{i_1} f \rangle = 0
\]
by the orthogonality condition of the Cuntz relations.

Now, if \( K = |\omega| > |\omega'| = M \), with \( \omega' = i_M \ldots i_1 \), we define the word \( \rho = i_M \ldots i_1 0 \ldots 0 \) so that \( |\rho| = K \). Note that \( \rho \notin X_4 \) so \( \omega \neq \rho \). Note further that \( S_\omega f = S_\rho f \). Thus, by a similar argument to that above, we have
\[
\langle S_\omega f, S_{\omega'} f \rangle = 0.
\]
\( \square \)

Remark 1. The set \( \{ S_\omega f : \omega \in X_4 \} \) need not be complete. We will provide an example of this in Example 4 in Section 4.

Our goal is to project the set \( \{ S_\omega f : \omega \in X_4 \} \) onto some subspace \( V \) of \( L^2(\mu_4 \times \lambda) \) to obtain a frame. To that end, we need to know when the projection \( \{ P_V S_\omega f : \omega \in X_4 \} \) is a frame, which by Lemma 4 requires the projection \( P_V : K \to V \) to be onto, where \( K \) is the subspace spanned by \( \{ S_\omega f : \omega \in X_4 \} \). The tool we will use is the following result, which is a minor adaptation of a result from [7]; we will not use this result directly, but will use all of the critical components.

Theorem 1. Let \( \mathcal{H} \) be a Hilbert space, \( \mathcal{K} \subset \mathcal{H} \) a closed subspace, and \((S_i)_{i=0}^{N-1}\) be a representation of the Cuntz algebra \( \mathcal{O}_N \). Let \( \mathcal{E} \) be an orthonormal set in \( \mathcal{H} \) and \( f : X \to \mathcal{K} \) a norm continuous function on a topological space \( X \) with the following properties:

i) \( \mathcal{E} = \bigcup_{i=0}^{N-1} S_i \mathcal{E} \) where the union is disjoint.

ii) \( \text{span}\{ f(t) : t \in X \} = \mathcal{K} \) and \( ||f(t)|| = 1 \), for all \( t \in X \).

iii) There exist functions \( m_i : X \to \mathbb{C}, g_i : X \to X, i = 0, \ldots, N-1 \) such that
\begin{equation}
S_i^* f(t) = m_i(t) f(g_i(t)), \quad t \in X.
\end{equation}

iv) There exist \( c_0 \in X \) such that \( f(c_0) \in \text{span}\mathcal{E} \).

v) The only function \( h \in \mathcal{C}(X) \) with \( h \geq 0 \), \( h(c) = 1 \), \( \forall c \in \{ x \in X : f(x) \in \text{span}\mathcal{E} \} \), and
\begin{equation}
h(t) = \sum_{i=0}^{N-1} |m_i(t)|^2 h(g_i(t)), \quad t \in X
\end{equation}
are the constant functions.

Then \( \mathcal{K} \subset \text{span}\mathcal{E} \).

3. The Projection

Recall the definition of the filters \( m_j(x, y) = e^{2\pi i j x} H_j(x, y) \) from Section 4. We choose the filter coefficients \( a_{jk} \) so that the matrix \( H \) is unitary. We place the additional constraint that
\[
a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2},
\]
so that $S_0 \mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ the function in $L^2(\mu_4 \times \lambda)$ which is identically 1. As $S_0 \mathbb{1} = \mathbb{1}$, by Lemma 5 the set $\{S_\omega \mathbb{1} : \omega \in X_4\}$ is orthonormal. Moreover, we place the additional constraint that for every $j$, $a_{j0} + a_{j2} = a_{j1} + a_{j3}$, which will be required for our calculation of the projection.

**Definition 2.** We define the subspace $V = \{f \in L^2(\mu_4 \times \lambda) : f(x, y) = g(x) \chi_{[0,1]}(y), g \in L^2(\mu_4)\}$. Note that the subspace $V$ can be identified with $L^2(\mu_4)$ via the isometric isomorphism $g \mapsto g(x) \chi_{[0,1]}(y)$. We will suppress the $y$ variable in the future.

**Definition 3.** We define a function $c : X_4 \to \mathbb{N}_0$ as follows: for a word $\omega = j_K j_{K-1} \ldots j_1$,

$$c(\omega) = \sum_{k=1}^K j_k 4^{K-k}.$$

Here $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. It is readily verified that $c$ is a bijection.

**Lemma 6.** For a word $\omega = j_K j_{K-1} \ldots j_1$,

$$S_\omega \mathbb{1} = e^{2\pi i c(\omega)x} \left( \prod_{k=1}^K 2H_{j_k}(R^{K-k}(x, y)) \right).$$

**Proof.** We proceed by induction on the length of the word $\omega$. The equality is readily verified for $|\omega| = 1$. Let $\omega_0 = j_{K-1} j_{n-2} \ldots j_1$. We have

$$S_\omega \mathbb{1} = S_{j_K} S_{\omega_0} \mathbb{1}$$

$$= S_{j_K} \left[ e^{2\pi i c(\omega_0)x} \left( \prod_{k=1}^{K-1} 2H_{j_k}(R^{K-1-k}(x, y)) \right) \right]$$

$$= 2 e^{2\pi i \lambda_{j_K} x} H_{j_K}(x, y) e^{2\pi i c(\omega_0) 4x} \left( \prod_{k=1}^{K-1} H_{j_k}(R^{K-k}(x, y)) \right)$$

$$= 2 e^{2\pi i (\lambda_{j_K} + 4c(\omega_0)) x} H_{j_K}(R^{K-K}(x, y)) \left( \prod_{k=1}^{K-1} 2H_{j_k}(R^{K-k}(x, y)) \right)$$

$$= e^{2\pi i c(\omega)x} \left( \prod_{k=1}^K 2H_{j_k}(R^{K-k}(x, y)) \right).$$

The last line above is justified by the following calculation:

$$\lambda_{j_K} + 4c(\omega_0) = \lambda_{j_K} + 4 \left( \sum_{k=1}^{K-1} \lambda_{j_k} 4^{K-1-k} \right)$$

$$= \lambda_{j_K} 4^{K-K} + \sum_{k=1}^{K-1} \lambda_{j_k} 4^{K-k}$$

$$= \sum_{k=1}^K \lambda_{j_k} 4^{K-k}$$

$$= c(\omega).$$

□
We wish to project the vectors $S_\omega 1$ onto the subspace $V$. The following lemma calculates that projection, where $P_V$ denotes the projection onto the subspace $V$.

**Lemma 7.** If $f(x, y) = g(x) h(x, y)$ with $g \in L^2(\mu_4)$ and $h \in L^\infty(\mu_4 \times \lambda)$, then

$$[P_V f](x, y) = g(x) G(x)$$

where $G(x) = \int_{[0,1]} h(x, y) d\lambda(y)$.

**Proof.** We verify that for every $F(x) \in L^2(\mu_4)$, $f(x, y) - g(x) G(x)$ is orthogonal to $F(x)$. We calculate utilizing Fubini’s theorem:

$$\langle f - gG, F \rangle = \int \int g(x) h(x, y) \overline{F(x)} \, d(\mu_4 \times \lambda) - \int \int g(x) G(x) \overline{F(x)} \, d(\mu_4 \times \lambda)$$

$$= \int_{C_4} g(x) \overline{F(x)} \left( \int_{[0,1]} h(x, y) - G(x) \, d\lambda(y) \right) \, d\mu_4(x)$$

$$= \int_{C_4} g(x) \overline{F(x)} (G(x) - G(x)) \, d\mu_4(x)$$

$$= 0.$$

For the purposes of the following lemma, $\alpha x$ and $\beta y$ are understood to be modulo 1.

**Lemma 8.** For any word $\omega = j_k j_{k-1} \ldots j_1$,

$$\int \prod_{k=1}^K 2H_{j_k}(R^{k-1}(x, y)) \, d\lambda(y) = \prod_{k=1}^K 2 \int H_{j_k}(4^{k-1}x, y) \, d\lambda(y).$$

**Proof.** Let $F_m(x, y) = \prod_{k=m}^K 2H_{j_k}(4^{k-1}x, 2^{k-m}y)$. Note that

$$F_m(x, \frac{y}{2}) = 2H_{j_m}(4^{m-1}x, \frac{y}{2}) \left( \prod_{k=m+1}^K 2H_{j_k}(4^{k-1}x, 2^{k-(m+1)}y) \right) = 2H_{j_m}(4^{m-1}x, \frac{y}{2}) F_{m+1}(x, y).$$

Likewise for $F_m(x, \frac{y+1}{2})$.

Since $\lambda$ is the invariant measure for the iterated function system $y \mapsto \frac{y}{2}$, $y \mapsto \frac{y+1}{2}$, we calculate:

$$\int_0^1 F_m(x, y) \, d\lambda(y) = \frac{1}{2} \left[ \int_0^1 F_m(x, \frac{y}{2}) \, d\lambda(y) + \int_0^1 F_m(x, \frac{y+1}{2}) \, d\lambda(y) \right]$$

$$= \frac{1}{2} \left[ \int_0^1 2H_{j_m}(4^{m-1}x, \frac{y}{2}) F_{m+1}(x, y) + 2H_{j_m}(4^{m-1}x, \frac{y+1}{2}) F_{m+1}(x, y) \, d\lambda(y) \right]$$

$$= \frac{1}{2} \left[ \int_0^1 2a_{j_m, q} F_{m+1}(x, y) + 2a_{j_m, q+2} F_{m+1}(x, y) \, d\lambda(y) \right]$$

$$= \frac{1}{2} \left[ 2a_{j_m, q} + 2a_{j_m, q+2} \right] \cdot \left[ \int_0^1 F_{m+1}(x, y) \, d\lambda(y) \right]$$

$$= \left[ \int_0^1 2H_{j_m}(4^{m-1}x, y) \, d\lambda(y) \right] \cdot \left[ \int_0^1 F_{m+1}(x, y) \, d\lambda(y) \right]$$

where $q = 0$ if $0 \leq 4^{m-1}x < \frac{1}{2}$, and $q = 1$ if $\frac{1}{2} \leq 4^{m-1}x < 1$. 
The result now follows by a standard induction argument.

**Proposition 1.** Suppose the filters \( m_j(x, y) \) are chosen so that

i) the matrix \( H \) is unitary,

ii) \( a_{00} = a_{01} = a_{02} = a_{03} = \frac{1}{2} \), and

iii) for \( j = 0, 1, 2, 3 \), \( a_{j0} + a_{j2} = a_{j1} + a_{j3} \).

Then for any word \( \omega = j_K \ldots j_1 \),

\[
P_V S_\omega 1 = d_\omega e^{2\pi i c(\omega)x},
\]

where

\[
d_\omega = \prod_{k=1}^{K} (a_{j_k0} + a_{j_k2}).
\]

**Proof.** We apply the previous three Lemmas to obtain

\[
[P_V S_\omega 1](x, y) = e^{2\pi i c(\omega)x} \int \prod_{k=1}^{K} 2H_{j_k}(4^{k-1}x, y)d\lambda(y)
\]

\[
= e^{2\pi i c(\omega)x} \prod_{k=1}^{K} 2 \int H_{j_k}(4^{k-1}x, y)d\lambda(y)
\]

By assumption iii), the integral \( \int H_{j_k}(4^{k-1}x, y)d\lambda(y) \) is independent of \( x \), and the value of the integral is \( \frac{a_{j0}}{2} + \frac{a_{j2}}{2} \). Equation 4 now follows. \( \square \)

4. **Concrete Constructions**

We now turn to concrete constructions of Fourier frames for \( \mu_4 \). The hypotheses of Lemma 5 and Proposition 1 require \( H \) to be unitary and requires the matrix

\[
A = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} \\
a_{10} & a_{11} & a_{12} & a_{13} \\
a_{20} & a_{21} & a_{22} & a_{23} \\
a_{30} & a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

to have the first row be identically \( \frac{1}{2} \) and to have the vector \( (1 -1 1 -1)^T \) in the kernel.

We can use Hadamard matrices to construct examples of such a matrix \( A \). Every \( 4 \times 4 \) Hadamard matrix is a permutation of the following matrix:

\[
U_\rho = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & \rho & -\rho \\
1 & 1 & -1 & -1 \\
1 & -1 & -\rho & \rho
\end{pmatrix}
\]

where \( \rho \) is any complex number of modulus 1.

If we set \( H = U_\rho \), we obtain

\[
A = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & \rho & \rho \\
1 & 1 & -1 & -1 \\
1 & 1 & -\rho & -\rho
\end{pmatrix}
\]
which has the requisite properties to apply Lemma 4 and Proposition 1.

We define for \( k = 1, 2, 3 \), \( l_k : \mathbb{N}_0 \to \mathbb{N}_0 \) by \( l_k(n) \) is the number of digits equal to \( k \) in the base 4 expansion of \( n \). Note that \( l_k(0) = 0 \), and we follow the convention that \( 0^0 = 1 \).

**Theorem 2.** For the choice \( A \) as in Equation (5) with \( \rho \neq -1 \), the sequence

\[
\left\{ \left( \frac{1 + \rho}{2} \right)^{l_1(n)} \ 0^{l_2(n)} \left( \frac{1 - \rho}{2} \right)^{l_3(n)} e^{2\pi i n x} : n \in \mathbb{N}_0 \right\}
\]

is a Parseval frame in \( L^2(\mu_4) \).

**Proof.** By Lemma 4, we have that \( \{S_\omega 1 : \omega \in X_4\} \) is an orthonormal set. For a word \( \omega = jKjK-1 \ldots j_1 \), Proposition 1 yields that

\[
P_{V} S_\omega 1 = e^{2\pi i c(\omega)x} \prod_{k=1}^{K} (a_{j_k0} + a_{j_k2}).
\]

Then, setting \( n = c(\omega) \), we obtain

\[
P_{V} S_\omega 1 = e^{2\pi i n x} (a_{00} + a_{02})^{K-l_1(n)} \prod_{j=1}^{3} (a_{j0} + a_{j2})^{l_j(n)}.
\]

Since

\[
a_{00} + a_{02} = 1, \quad a_{10} + a_{12} = \frac{1 + \rho}{2}, \quad a_{20} + a_{22} = 0, \quad a_{30} + a_{32} = \frac{1 - \rho}{2},
\]

it follows that

\[
P_{V} S_\omega 1 = \left( \frac{1 + \rho}{2} \right)^{l_1(n)} \ 0^{l_2(n)} \left( \frac{1 - \rho}{2} \right)^{l_3(n)} e^{2\pi i n x}.
\]

Since \( c \) is a bijection, the set \( \{P_{V} S_\omega 1 : \omega \in X_4\} \) coincides with the set in (6).

In order to establish the bijection that the set (6) is a Parseval frame, we wish to apply Lemma 5 which requires that the subspace \( V \) is contained in the closed span of \( \{S_\omega 1 : \omega \in X_4\} \). Denote the closed span by \( \mathcal{K} \). We will proceed in a manner similar to Theorem 1. Define the function \( f : \mathbb{R} \to V \) by \( f(t) = e_t \) where \( e_t(x, y) = e^{2\pi i t x} \). Note that \( f(0) = 1 \in \mathcal{K} \). Likewise, define a function \( h_X : \mathbb{R} \to \mathbb{R} \) by

\[
h_X(t) = \sum_{\omega \in X_4} |\langle f(t), S_\omega 1 \rangle|^2 = \|P_{\mathcal{K}} f(t)\|^2.
\]

**Claim 1.** We have \( h_X \equiv 1 \).

Assuming for the moment that the claim holds, we deduce that \( f(t) \in \mathcal{K} \) for every \( t \in \mathbb{R} \). Since \( \{f(\gamma) : \gamma \in \Gamma_4\} \) is an orthonormal basis for \( V \), it follows that the closed span of \( \{f(t) : t \in \mathbb{R}\} \) is all of \( V \). We conclude that \( V \subset \mathcal{K} \), and so Lemma 5 implies that \( \{P_{V} S_\omega 1 : \omega \in X_4\} \) is a Parseval frame for \( V \), from which the Theorem follows.

Thus, we turn to the proof of Claim 1. First, we require \( \{S_\omega 1 : \omega \in X_4\} = \bigcup_{j=0}^{3} \{S_j S_\omega 1 : \omega \in X_4\} \), where the union is disjoint. Clearly, the RHS is a subset of the LHS, and the union is disjoint. Consider an element of the LHS: \( S_\omega 1 \). If \( |\omega| \geq 2 \), we write \( S_\omega 1 = S_j S_{\omega_0} 1 \) for some \( j \) and some \( \omega_0 \in X_4 \), whence \( S_\omega 1 \) is in the RHS. If \( |\omega| = 1 \), then we write \( S_\omega 1 = S_j 1 = S_j S_0 1 \), which is again an element of the RHS. Equality now follows.
As a consequence,

\[ h_X(t) = \sum_{\omega \in X_4} |\langle f(t), S_\omega \mathbb{1} \rangle|^2 \]

\[ = \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle f(t), S_j S_\omega \mathbb{1} \rangle|^2 \]

\[ = \sum_{j=0}^{3} \sum_{\omega \in X_4} |\langle S_j^* f(t), S_\omega \mathbb{1} \rangle|^2. \]

We calculate:

\[ [S_j^* f(t)](x, y) = \frac{1}{2} \sum_{k=0}^{3} m_j(\Upsilon_k(x, y)) e_t(\Upsilon_k(x, y)) \]

\[ = \frac{1}{2} \left[ \frac{1}{2} \left[ a_{j0} e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) + e^{-\pi ij} \frac{a_{j1}}{a_{j1}} e^{-2\pi ij x/4} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \right. \]

\[ + \left. a_{j2} e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) + e^{-\pi ij} \frac{a_{j3}}{a_{j3}} e^{-2\pi ij x/4} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \]

\[ = \frac{1}{2} \left[ \frac{1}{2} \left[ a_{j0} + e^{-\pi ij} a_{j1} e^{\pi it} + a_{j2} + e^{-\pi ij} a_{j3} e^{\pi it} \right] e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \right. \]

\[ + \left. a_{j2} e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) + e^{-\pi ij} a_{j3} e^{-2\pi ij x/4} e^{\pi it} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \]

\[ = \frac{1}{2} \left[ \frac{1}{2} \left[ a_{j0} + e^{-\pi ij} a_{j1} e^{\pi it} + a_{j2} + e^{-\pi ij} a_{j3} e^{\pi it} \right] e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \right. \]

\[ + \left. a_{j2} e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) + e^{-\pi ij} a_{j3} e^{-2\pi ij x/4} e^{\pi it} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \]

\[ = \frac{1}{2} \left[ \frac{1}{2} \left[ a_{j0} + e^{-\pi ij} a_{j1} e^{\pi it} + a_{j2} + e^{-\pi ij} a_{j3} e^{\pi it} \right] e^{-2\pi i j x/4} e_t\left(\frac{x}{4}, \frac{y}{2}\right) \right. \]

\[ + \left. a_{j2} e^{-2\pi ij x/4} e_t\left(\frac{x}{4}, \frac{y+1}{2}\right) + e^{-\pi ij} a_{j3} e^{-2\pi ij x/4} e^{\pi it} e_t\left(\frac{x+2}{4}, \frac{y+1}{2}\right) \right] \]

Thus, we define

\[ m_j(t) = \frac{1}{2} (a_{j0} + a_{j2}) + \frac{e^{-\pi ij}}{2} (a_{j1} + a_{j3}) e^{\pi it}, \]

and

\[ g_j(t) = \frac{t - j}{4}. \]
As a consequence, we obtain

\[ h_X(t) = \sum_{j=0}^{3} \sum_{\omega \in \mathcal{N}_4} |\langle S_j^* f(t), S_{\omega} \mathbf{1} \rangle|^2 \]

\[ = \sum_{j=0}^{3} \sum_{\omega \in \mathcal{N}_4} |\langle m_j(t) f(g_j(t)), S_{\omega} \mathbf{1} \rangle|^2 \]

\[ = \sum_{j=0}^{3} |m_j(t)|^2 h_X(g_j(t)). \quad (7) \]

Because of our choice of coefficients in the matrix \( A \), which has the vector \((1 \quad -1 \quad 1 \quad -1)^T\) in the kernel, we have for every \( j \): \( a_{j0} + a_{j2} = a_{j1} + a_{j3} \). Thus, if we let \( b_j = \frac{a_{j0} + a_{j2}}{2} \), the functions \( m_j \) simplify to

\[ m_j(t) = b_j e^{\pi i t} \cos(\pi t) \]

for \( j = 0, 2 \), and

\[ m_j(t) = -ib_j e^{\pi i t} \sin(\pi t) \]

for \( j = 1, 3 \). Substituting these into Equation (7),

\[ h_X(t) = \cos^2 \left( \frac{\pi t}{2} \right) h_X \left( \frac{t}{4} \right) + \sin^2 \left( \frac{\pi t}{2} \right) \left( \frac{1 + \rho}{2} \right) h_X \left( \frac{t - 1}{4} \right) + \sin^2 \left( \frac{\pi t}{2} \right) \left( \frac{1 - \rho}{2} \right) h_X \left( \frac{t - 3}{4} \right). \quad (8) \]

**Claim 2.** The function \( h_X \) can be extended to an entire function.

Assume for the moment that Claim 2 holds, we finish the proof of Claim 1. If \( h_X(t) = 1 \) for \( t \in [-1, 0] \), then \( h_X(z) = 1 \) for all \( z \in \mathbb{C} \), and Claim 1 holds.

Now, assume to the contrary that \( h_X(t) \) is not identically 1 on \([-1, 0]\). Since \( 0 \leq h_X(t) \leq 1 \) for \( t \) real, then \( \beta = \min \{ h_X(t) : t \in [-1, 0] \} < 1 \). Because constant functions satisfy (8), \( h_1 := h_X - \beta \) also satisfies Equation (8). There exists \( t_0 \) such that \( h_1(t_0) = 0 \) and \( t_0 \neq 0 \) as \( h_X(0) = 1 \). Since \( h_1 \geq 0 \) each of the terms in (8) must vanish:

\[ \cos^2 \left( \frac{\pi t_0}{2} \right) h_1 \left( \frac{t_0}{4} \right) = 0 \quad \text{(9)} \]

\[ \sin^2 \left( \frac{\pi t_0}{2} \right) \left( \frac{1 + \rho}{2} \right) h_1 \left( \frac{t_0 - 1}{4} \right) = 0 \quad \text{(10)} \]

\[ \sin^2 \left( \frac{\pi t_0}{2} \right) \left( \frac{1 - \rho}{2} \right) h_1 \left( \frac{t_0 - 3}{4} \right) = 0 \quad \text{(11)} \]

Our hypothesis is that \( \rho \neq -1 \), so in Equation (10), the coefficient \( \frac{1 + \rho}{2} \) \neq 0.

Case 1: If \( t_0 \neq -1 \) then Equation (9) implies \( h_1(t_0/4) = 0 = h_1(g_0(t_0)) \). Let \( t_1 := g_0(t_0) \in (-1, 0) \); iterating the previous argument implies that \( h_1(g_0(t_1)) = 0 \). Thus, we obtain an infinite sequence of zeroes of \( h_1 \).

Case 2: If \( t_0 = -1 \), then the previous argument does not hold. However, we can construct another zero of \( h_1 \), \( t_0' \in (-1, 0) \) to which the previous argument will hold. Indeed, if \( t_0 = -1 \),
Equation (10) implies \( h_1((t_0 - 1)/4) = h_1(-1/2) = 0 \). Let \( t_0' = -1/2 \) and continue as in Case 1.

In either case, \( h_1 \) vanishes on a (countable) set with an accumulation point, and since \( h_1 \) is analytic it follows that \( h_1 \equiv 0 \), a contradiction, and Claim 1 holds.

Now, to prove Claim 2, we follow the proof of Lemma 4.2 of [10]. For a fixed \( \omega \in X_4 \), define \( f_{\omega} : \mathbb{C} \to \mathbb{C} \) by

\[
f_{\omega}(z) = \langle e_z, S_\omega \mathbb{1} \rangle = \int e^{2\pi iz} \overline{[S_\omega \mathbb{1}]}(x, y) \, d(\mu_4 \times \lambda).
\]

Since the distribution \( \overline{[S_\omega \mathbb{1}]}(x, y) \, d(\mu_4 \times \lambda) \) is compactly supported, a standard convergence argument demonstrates that \( f_\omega \) is entire. Likewise, \( f^*_\omega(z) = f_\omega(\overline{z}) \) is entire, and for \( t \) real,

\[
f_\omega(t) f^*_\omega(t) = (\langle e_t, S_\omega \mathbb{1} \rangle) (\langle e_t, S_\omega \mathbb{1} \rangle) = |\langle e_t, S_\omega \mathbb{1} \rangle|^2.
\]

Thus,

\[
h_X(t) = \sum_{\omega \in X_4} f_\omega(t) f^*_\omega(t).
\]

For \( n \in \mathbb{N} \), let \( h_n(z) = \sum_{|\omega| \leq n} f_\omega(z) f^*_\omega(z) \), which is entire. By Hölder’s inequality,

\[
\sum_{\omega \in X_4} |f_\omega(z) f^*_\omega(z)| \leq \left( \sum_{\omega \in X_4} |\langle e_z, S_\omega \mathbb{1} \rangle|^2 \right)^{1/2} \left( \sum_{\omega \in X_4} |\langle e_\overline{z}, S_\omega \mathbb{1} \rangle|^2 \right)^{1/2}
\]

\[
\leq ||e_z|| ||e_\overline{z}||
\]

\[
\leq e^{K \text{Im}(z)}
\]

for some constant \( K \). Thus, the sequence \( h_n(z) \) converges pointwise to a function \( h(z) \), and are uniformly bounded on strips \( \text{Im}(z) \leq C \). By the theorems of Montel and Vitali, the limit function \( h \) is entire, which coincides with \( h_X \) for real \( t \), and Claim 2 is proved.

\[ \square \]

**Example 1.** As mentioned in Section 2 in general, \( \{S_\omega \mathbb{1}\} \) need not be complete, and the exceptional point \( \rho = -1 \) in Theorem 2 provides the example. In the case \( \rho = -1 \), the set \( \{ \} \) becomes

\[
\{ d_n e^{2\pi i n x} : n \in \mathbb{N}_0 \}
\]

where the coefficients \( d_n = 1 \) if \( n \in \Gamma_3 \) and 0 otherwise. Here,

\[
\Gamma_3 = \{ \sum_{n=0}^{N} l_n 4^n : l_n \in \{0, 3\} \}
\]

and it is known [11] that the sequence \( \{ e^{2\pi i n x} : n \in \Gamma_3 \} \) is incomplete in \( L^2(\mu_4) \). Thus, \( \{ P_{\mathbb{V}} S_\omega \mathbb{1} \} \) is incomplete in \( \mathbb{V} \), so \( \{ S_\omega \mathbb{1} \} \) is incomplete in \( L^2(\mu_4 \times \lambda) \).

We can generalize the construction of Theorem 2 as follows. We want to choose a matrix

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]
such that \((1 \ -1 \ 1 \ -1)^T\) is in the kernel of \(H\) and the matrix

\[
H = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -h_{11} & h_{12} & -h_{13} \\
\frac{1}{2} & h_{21} & -h_{22} & h_{23} \\
\frac{1}{2} & h_{31} & h_{32} & -h_{33}
\end{pmatrix}
\]

is unitary. We obtain a system of nonlinear equations in the 12 unknowns. To parametrize all solutions, we consider the following row vectors:

\[
\vec{v}_0 = \frac{1}{2} (1 \ 1 \ 1 \ 1) \quad \vec{w}_0 = \frac{1}{2} (1 \ -1 \ 1 \ -1)
\]

\[
\vec{v}_1 = \frac{1}{2} (1 \ -1 \ -1 \ 1) \quad \vec{w}_1 = \frac{1}{2} (1 \ 1 \ -1 \ -1)
\]

\[
\vec{v}_2 = \frac{1}{2} (1 \ 1 \ -1 \ -1) \quad \vec{w}_2 = \frac{1}{2} (1 \ -1 \ -1 \ 1)
\]

If we construct the matrix \(A\) so that the rows are linear combinations of \(\{\vec{v}_0, \vec{v}_1, \vec{v}_2\}\), then \(A\) will satisfy the desired condition on the kernel. Note that if the \(j\)-th row of \(A\) is \(\alpha_{j0} \vec{v}_0 + \alpha_{j1} \vec{v}_1 + \alpha_{j2} \vec{v}_2\) for \(j = 1, 3\), then the \(j\)-th row of \(H\) is \(\alpha_{j0} \vec{w}_0 + \alpha_{j1} \vec{w}_1 + \alpha_{j2} \vec{w}_2\), whereas if \(j = 0, 2\), then the \(j\)-th row of \(H\) is equal to the \(j\)-th row of \(A\).

Thus, we want to choose coefficients \(\alpha_{jk}, \ j = 0, 1, 2, 3, k = 1, 2, 3\) so that the matrix

\[
H = \begin{pmatrix}
\alpha_{00} \vec{v}_0 + \alpha_{01} \vec{v}_1 + \alpha_{02} \vec{v}_2 \\
\alpha_{10} \vec{w}_0 + \alpha_{11} \vec{w}_1 + \alpha_{12} \vec{w}_2 \\
\alpha_{20} \vec{v}_0 + \alpha_{21} \vec{v}_1 + \alpha_{22} \vec{v}_2 \\
\alpha_{30} \vec{w}_0 + \alpha_{31} \vec{w}_1 + \alpha_{32} \vec{w}_2
\end{pmatrix}
\]

is unitary. To satisfy the requirement on the first row, we choose \(\alpha_{00} = 1\) and \(\alpha_{01} = \alpha_{02} = 0\). Calculating the inner products of the rows of \(H\), we obtain the following necessary and sufficient conditions:

\[
|\alpha_{j0}|^2 + |\alpha_{j1}|^2 + |\alpha_{j2}|^2 = 1
\]

\[
\alpha_{00} \bar{\alpha}_{20} = 0
\]

\[
\alpha_{11} \bar{\alpha}_{22} + \alpha_{12} \bar{\alpha}_{21} = 0
\]

\[
\alpha_{10} \bar{\alpha}_{30} + \alpha_{11} \bar{\alpha}_{31} + \alpha_{12} \bar{\alpha}_{32} = 0
\]

\[
\alpha_{21} \bar{\alpha}_{32} + \alpha_{22} \bar{\alpha}_{31} = 0
\]

**Proposition 2.** Fix \(\alpha_{00} = 1\). There exists a solution to the Equations (16) - (20) if and only if \(\alpha_{10}, \alpha_{30} \in \mathbb{C}\) with

\[
|\alpha_{10}|^2 + |\alpha_{30}|^2 = 1.
\]

**Proof.** \((\Leftarrow)\) If \(|\alpha_{10}|^2 = 1\), then we choose \(\alpha_{21} = \alpha_{31} = 1\) and all other coefficients to be 0 to obtain a solution to Equations (16) - (20). Likewise, if \(|\alpha_{10}|^2 = 0\), then choose \(\alpha_{11} = \alpha_{21} = 1\) and all other coefficients to be 0.
Now suppose that $0 < |\alpha_{10}| < 1$, and we choose $\lambda = \frac{-\alpha_{10}\alpha_{30}}{1 - |\alpha_{10}|^2}$. Then choose $\alpha_{11}$ and $\alpha_{12}$ such that $|\alpha_{11}|^2 + |\alpha_{12}|^2 = 1 - |\alpha_{10}|^2$. Now let $\alpha_{31} = \lambda \alpha_{11}$ and $\alpha_{32} = \lambda \alpha_{12}$. We have

$$\alpha_{10}\alpha_{30} + \alpha_{11}\alpha_{31} + \alpha_{12}\alpha_{32} = \alpha_{10}\alpha_{30} + \lambda |\alpha_{11}|^2 + \lambda |\alpha_{12}|^2$$

$$= \alpha_{10}\alpha_{30} + \lambda (1 - |\alpha_{10}|^2)$$

$$= 0,$$

so Equation (19) is satisfied.

Equation (17) forces $\alpha_{20} = 0$; choose $\alpha_{21}$ and $\alpha_{22}$ such that $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$ and $\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} = 0$. Thus, Equations (18) and (20) are satisfied. Finally, regarding Equation (16), it is satisfied for $j = 0, 1, 2$ by construction. For $j = 3$, we calculate:

$$|\alpha_{30}|^2 + |\alpha_{31}|^2 + |\alpha_{32}|^2 = |\alpha_{30}|^2 + |\lambda|^2 (|\alpha_{11}|^2 + |\alpha_{12}|^2)$$

$$= |\alpha_{30}|^2 + \frac{|\alpha_{10}|^2 |\alpha_{30}|^2}{(1 - |\alpha_{10}|^2)^2} (1 - |\alpha_{10}|^2)$$

$$= |\alpha_{30}|^2 \left( 1 + \frac{|\alpha_{10}|^2}{1 - |\alpha_{10}|^2} \right)$$

$$= \frac{|\alpha_{30}|^2}{1 - |\alpha_{10}|^2}$$

(23)

as required.

$(\Rightarrow)$ Suppose that we have a solution to Equations (16) - (20). If $|\alpha_{10}| = 1$, then we must have $\alpha_{11} = \alpha_{12} = 0$, and thus Equation (19) requires $\alpha_{30} = 0$, so Equation (21) holds.

Now suppose $|\alpha_{10}| < 1$. Since $\alpha_{20} = 0$, we must have that $|\alpha_{21}|^2 + |\alpha_{22}|^2 = 1$. Combining this with Equations (18) and (20) imply that the matrix

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{pmatrix}$$

is singular. Thus, there exists a $\lambda$ such that $\alpha_{31} = \lambda \alpha_{11}$ and $\alpha_{32} = \lambda \alpha_{12}$. Using the same computation as in Equation (22), we conclude that $\lambda = \frac{-\alpha_{10}\alpha_{30}}{1 - |\alpha_{10}|^2}$, then Equation (23) implies (21).

The coefficient matrix we obtain from this construction is

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \alpha_{10} + \alpha_{11} + \alpha_{12} & \alpha_{10} - \alpha_{11} + \alpha_{12} & \alpha_{10} + \alpha_{11} - \alpha_{12} & \alpha_{10} + \alpha_{11} - \alpha_{12} \\ \alpha_{21} + \alpha_{22} & -\alpha_{21} + \alpha_{22} & -\alpha_{21} - \alpha_{22} & -\alpha_{21} - \alpha_{22} \\ \alpha_{30} + \lambda \alpha_{11} + \lambda \alpha_{12} & \alpha_{30} - \lambda \alpha_{11} + \lambda \alpha_{12} & \alpha_{30} - \lambda \alpha_{11} - \lambda \alpha_{12} & \alpha_{30} + \lambda \alpha_{11} - \lambda \alpha_{12} \end{pmatrix}$$

where we are allowed to choose $\alpha_{11}, \alpha_{12}, \alpha_{21}$ and $\alpha_{22}$ subject to the normalization condition in Equation (16). However, those choices do not affect the construction, since if we apply Proposition 1 and the calculation from Theorem 2 we obtain

$$P_V S_\omega \mathbf{1} = (\alpha_{10})^{\ell_1(n)} (0)^{\ell_2(n)} (\alpha_{30})^{\ell_3(n)} e^{2\pi i nx}.$$

This will in fact be a Parseval frame for $L^2(\mu_4)$, provided $V \subset K$, as in the proof of Theorem 2.
Theorem 3. Suppose $p, q \in \mathbb{C}$ with $|p|^2 + |q|^2 = 1$. Then
\[ \{ p^{\ell_1(n)} \cdot q^{\ell_2(n)} \cdot e^{2\pi i n x} : n \in \mathbb{N}_0 \} \]
is a Parseval frame for $L^2(\mu_4)$, provided $p \neq 0$.

Proof. Substitute $\alpha_{10} = p$ and $\alpha_{30} = q$ in Proposition \[2\] and Equation \[24\]. As noted, we only need to verify $V \subset K$. We proceed as in the proof of Theorem \[2\], indeed, define $f$, $h_X$, $m_j$ and $g_j$ as previously. We obtain $b_0 = 1$, $b_1 = \overline{p}$, $b_2 = 0$, and $b_3 = \overline{q}$, so Equation \[8\] becomes
\[ h_X(t) = \cos^2 \left( \frac{\pi t}{2} \right) h_X \left( \frac{t}{4} \right) + |\overline{p}|^2 \sin^2 \left( \frac{\pi t}{2} \right) h_X \left( \frac{t - 1}{4} \right) + |\overline{q}|^2 \sin^2 \left( \frac{\pi t}{2} \right) h_X \left( \frac{t - 3}{4} \right) \].

From here, the same argument shows that $h_X \equiv 1$, and $V \subset K$. \hfill \Box

5. Concluding Remarks

We remark here that the constructions given above for $\mu_4$ does not work for $\mu_3$. Indeed, we have the following no-go result. To obtain the measure $\mu_3 \times \lambda$, we consider the iterated function system:
\[ \begin{align*}
\Upsilon_0(x, y) &= \left( \frac{x}{3}, \frac{y}{2} \right), \\
\Upsilon_1(x, y) &= \left( \frac{x + 2}{3}, \frac{y}{2} \right), \\
\Upsilon_2(x, y) &= \left( \frac{x}{3}, \frac{y + 1}{2} \right), \\
\Upsilon_3(x, y) &= \left( \frac{x + 2}{3}, \frac{y + 1}{2} \right).
\end{align*} \]

Using the same choice of filters, the matrix $\mathcal{M}(x, y)$ reduces to
\[ H = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} \\
a_{10} & e^{4\pi i / 3}a_{11} & a_{12} & e^{4\pi i / 3}a_{13} \\
a_{20} & e^{2\pi i / 3}a_{21} & a_{22} & e^{2\pi i / 3}a_{23} \\
a_{30} & a_{31} & a_{32} & a_{33}
\end{pmatrix} \]

which we require to be unitary. Additionally, we require the same conditions as for $\mu_4$, namely, the first row of $H$ must have all entries $\frac{1}{2}$, and $a_{j0} + a_{j2} = a_{j1} + a_{j3}$. The inner product of the first two rows must be 0. Hence,
\[ \frac{1}{2} (a_{10} + e^{4\pi i / 3}a_{11} + a_{12} + e^{4\pi i / 3}a_{13}) = \frac{1}{2} (a_{10} + a_{12}) (1 + e^{4\pi i / 3}) = 0. \]

Consequently, $a_{10} + a_{12} = 0$. Likewise, $a_{20} + a_{22} = a_{30} + a_{32} = 0$. As a result,
\[ H \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{00} + a_{02} \\ a_{10} + a_{12} \\ a_{20} + a_{22} \\ a_{30} + a_{32} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \]

and so $H$ cannot be unitary.

It may be possible to extend the construction for $\mu_4$ to $\mu_3$ by considering a representation of $O_n$ for some sufficiently large $n$, or by considering $\mu_3 \times \rho$ for some other fractal measure $\rho$ rather than $\lambda$.

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FOURIER FRAMES FOR THE CANTOR-4 SET

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Department of Mathematical Sciences, 414 E. Clark St., University of South Dakota, Vermillion, SD 57069

E-mail address: gabriel.picioroaga@usd.edu

Department of Mathematics, Iowa State University, 396 Carver Hall, Ames, IA 50011

E-mail address: esweber@iastate.edu