Fischer decompositions for entire functions and the Dirichlet problem for parabolas

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Abstract
Let $P_{2k}$ be a homogeneous polynomial of degree $2k$ and assume that there exist $C > 0$, $D > 0$ and $\alpha \geq 0$ such that

$$\langle P_{2k} f_m, f_m \rangle_{L^2(S^{d-1})} \geq \frac{1}{C (m + D)^{\alpha}} \langle f_m, f_m \rangle_{S^{d-1}}$$

for all homogeneous polynomials $f_m$ of degree $m$. Assume that $P_j$ for $j = 0, \ldots, \beta < 2k$ are homogeneous polynomials of degree $j$. The main result of the paper states that for any entire function $f$ of order $\rho < (2k - \beta) / \alpha$ there exist entire functions $q$ and $h$ of order bounded by $\rho$ such that

$$f = (P_{2k} - P_\beta - \cdots - P_0) q + h$$

This result is used to establish the existence of entire harmonic solutions of the Dirichlet problem for parabola-shaped domains on the plane, with data given by entire functions of order smaller than $1/2$.

Keywords Fischer decomposition \cdot Fischer pair \cdot Entire harmonic function

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In memory of Harold Seymour Shapiro (1928-2021).

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1 Introduction

This paper is dedicated to the memory of Harold S. Shapiro (1928–2021), whose scientific work has had a very high and long-lasting impact on the mathematical community, easily recognizable in recent research. For a description of H. S. Shapiro’s main achievements, the interested reader is referred to [32].

Here we discuss a rather specific topic, studied and promoted by H. S. Shapiro in his celebrated paper [51] (see also [43, 44] written in collaboration with D.J. Newman). In [51] Shapiro introduced the notion of the Fischer decomposition of an entire function, which goes back to an old paper of Ernst Fischer 1 from 1917 discussing the polynomial case, see [20]. This method has many applications: in [14, 15] it was used to generalize the Cauchy–Kowaleskaya theorem to a much wider setting (see [17, 18] for further developments). In [34] this approach was used to explore analytical extension properties of solutions of the Dirichlet problem for entire data, with respect to an ellipsoid and to more general domains, see also [3, 5–7, 10, 13, 16, 30]. In [34] the Khavinson-Shapiro conjecture was stated, which initiated a wide range of research activity, see (in alphabetical order) [33, 37–39, 45, 47–49]. Further work on Fischer decompositions can be found in [19, 31, 41, 42]. In passing we note that Fischer decompositions also play an important role in Clifford Analysis, see e.g. [9] and the references therein.

Let us recall some notations and terminology in order to formulate Fischer’s theorem: we denote by $\mathcal{P}(\mathbb{R}^d)$ the set of all polynomials in the variable $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with complex coefficients, by $\mathbb{N}_0$ the set of natural numbers (emphasizing the fact that 0 is included), and by $\mathcal{P}_m(\mathbb{R}^d)$ the subspace of all homogeneous polynomials of degree $m$. Recall that a polynomial $P(x)$ is homogeneous of degree $\alpha$ if $P(tx) = t^\alpha P(x)$ for all $t > 0$ and for all $x$. Given a polynomial $P(x)$, we denote by $P^*(x)$ the polynomial obtained from $P(x)$ by conjugating its coefficients, and by $P(D)$ be the linear differential operator obtained by replacing the variable $x_j$ by the differential operator $\frac{\partial}{\partial x_j}$. It is well known that a polynomial $P(x)$ of degree $k$ can be written as a sum of homogeneous polynomials $P_j(x)$ of degree $j$ for $j = 0, \ldots, k$, so

$$P(x) = P_k(x) + \cdots + P_0(x),$$

and we call the homogeneous polynomial $P_k(x)$ the leading term. Fischer’s theorem states that given a homogeneous polynomial $P$, the following decomposition holds: for each polynomial $f \in \mathcal{P}(\mathbb{R}^d)$ there exist unique polynomials $q \in \mathcal{P}(\mathbb{R}^d)$ and $h \in \mathcal{P}(\mathbb{R}^d)$ such that

$$f = P \cdot q + h \text{ and } P^*(D) h = 0.$$

---

1 E. Fischer made important contributions to the development of abstract Hilbert spaces (e.g. the Riesz-Fischer theorem), so, not surprisingly, Hilbert space arguments play an important role in his paper. See [29], p. 217 and p. 228, for some biographical comments and remarks.
We recall the standard notation for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$: set $x^{\alpha} = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\alpha! = \alpha_1! \cdots \alpha_d!$, and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Let $P$ and $Q$ be given by

$$P(x) = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq N} c_{\alpha} x^{\alpha} \quad \text{and} \quad Q(x) = \sum_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq M} d_{\alpha} x^{\alpha},$$

where $c_{\alpha}, d_{\alpha} \in \mathbb{C}$.

An important ingredient in the proof of Fischer’s Theorem is the Fischer inner product $[\cdot, \cdot]_F$ on $P(\mathbb{R}^d)$, defined by

$$[P, Q]_F := (Q^*(D) P)(0) = \sum_{\alpha \in \mathbb{N}_0^d} \alpha! c_{\alpha} d_{\alpha}.$$  \hspace{1cm} (1)

The Fischer inner product has been used by many authors under different names, see [11, 50], and the references therein. In [43, 51] the corresponding Hilbert space norm $\sqrt{[P, P]_F}$ is called the Fischer norm, while in [8, 54] the term Bombieri norm is used.

One aim in Shapiro’s paper [51] is to provide Fischer decompositions in a wider setting, going beyond the case of polynomials to more general function spaces, in particular to the space $E(\mathbb{C}^d)$ of all entire functions $f: \mathbb{C}^d \to \mathbb{C}$. It is convenient to adopt a notion introduced in [51, p. 522]; suppose that $E$ is a vector space of infinitely differentiable functions $f : G \to \mathbb{C}$ (defined on an open subset $G$ in $\mathbb{R}^d$ or $\mathbb{C}^d$) that is a module over $P(\mathbb{R}^d)$: then we say that a polynomial $P$ and a differential operator $Q(D)$ form a Fischer pair for the space $E$, if for each $f \in E$ there exist unique elements $q \in E$ and $h \in E$ such that

$$f = P \cdot q + h \quad \text{and} \quad Q(D) h = 0.$$ \hspace{1cm} (2)

Shapiro proved in [51, Theorem 1] that Fischer’s theorem is also true when $P(\mathbb{R}^d)$ is replaced by $E(\mathbb{C}^d)$. His approach is based on homogeneous expansions of entire functions and estimates of the Fischer norms for homogeneous polynomials.

Shapiro also raised the question whether other types of Fischer pairs could be found. In [47] the first author identified new kinds of Fischer pairs, related to the polyharmonic operator $\Delta^k$, where $\Delta^k$ is the $k$-th iterate of the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}.$$  

It is shown in [47] that a polynomial $P(x)$ of degree $2k$ and the differential operator $Q(D) := \Delta^k$ form a Fischer pair for $P(\mathbb{R}^d)$ provided that the leading term $P_{2k}$ is non-zero and non-negative (i.e., $P_{2k} \geq 0$), thus guaranteeing Fischer decompositions for the pair $(P, \Delta^k)$ with respect to the vector space $P(\mathbb{R}^d)$. To pass from $P(\mathbb{R}^d)$ to the space of entire functions $E(\mathbb{C}^d)$ is a non-trivial task which involves careful
analysis: it is shown in [47] that \((P, \Delta^k)\) is Fischer pair for \(E(\mathbb{C}^d)\) if the leading polynomial \(P_{2k}(x)\) is elliptic, i.e., if there exists a constant \(C > 0\) such that
\[
P_{2k}(x) \geq C |x|^{2k} \quad \text{for all } x \in \mathbb{R}^d.
\]
Thus a Fischer decomposition holds for \textit{entire} functions when \(P_{2k}\) is elliptic.

In the present paper we want to discuss Fischer decompositions for entire functions when we relax the assumption of ellipticity. We remark that while the definition of “Fischer pair” entails the uniqueness of the decomposition, we shall use Fischer decomposition in a wider sense, that includes the possibility of not having uniqueness.

Let us denote the unit sphere by \(S^{d-1} = \{ \theta \in \mathbb{R}^d : |\theta| = 1 \}\), its surface area measure by \(d\theta\), and its area by \(\omega_{d-1} = |S^{d-1}|\). We define for \(f, g \in \mathcal{P}(\mathbb{R}^d)\) the inner product, together with its associated norm,
\[
(f, g)_{L^2(S^{d-1})} := \int_{S^{d-1}} f(\theta) g(\theta) d\theta \quad \text{and} \quad \|f\|_{L^2(S^{d-1})} = \sqrt{(f, f)_{L^2(S^{d-1})}}, \tag{3}
\]

Given a polynomial \(P\), to avoid excessive subindices we shall denote by both \(P\) and \(MP\) the multiplication operator associated to \(P\): for every function \(f\), we set \(MP(f) := Pf\).

With the convention that in the case \(\alpha = 0\), the expression \((2k - \beta)/\alpha\) is to be interpreted as \(\infty\), our first main result states the following:

\textbf{Theorem 1} Let \(P_{2k}\) be a homogeneous polynomial of degree \(2k > 0\) such that there exist \(C > 0, D > 0\) and \(\alpha \geq 0\) with
\[
(P_{2k}f_m, f_m)_{L^2(S^{d-1})} \geq \frac{1}{C(m + D)^\alpha} (f_m, f_m)_{S^{d-1}}
\]
for all homogeneous polynomials \(f_m\) of degree \(m\). Let \(0 \leq \beta < 2k\) and for \(j = 0, \ldots, \beta\), let the polynomials \(P_j\) be homogeneous of degree \(j\). Then for every entire function \(f\) of order \(\rho < (2k - \beta)/\alpha\), there exist entire functions \(q\) and \(r\) of order \(\leq \rho\) such that
\[
f = (P_{2k} - P_\beta - \cdots - P_0)q + r \quad \text{and} \quad \Delta^k r = 0.
\]

\textbf{Proof} By Proposition 11 with \(C_m := C^{-1}(m + D)^{-\alpha}\), we have that \((P_{2k}, \Delta^k)\) is a Fischer pair for \(\mathcal{P}(\mathbb{R}^d)\) and furthermore,
\[
\|Tf_m\|_{L^2(S^{d-1})} \leq C (m + D)^\alpha \|f_m\|_{L^2(S^{d-1})}
\]
for every homogeneous polynomial \(f_m\) of degree \(m\). Now the result follows from Theorem 9. \qed
First, let us see how Theorem 1 together with Theorem 10 imply the following result of D. Armitage, cf. [3, Theorem 1], which is a refinement of a result of D. Khavinson and H. Shapiro, cf. [34, Theorem 1]. Actually, the Theorem of Armitage is improved, since there we have that if \( 0 < \rho (h) = \rho (f) < \infty \) then \( \tau (h) \leq C_1 \tau (f) \), where \( C_1 \) depends only on the domain \( \Omega \), and in the case where \( \Omega \) is a ball, \( C_1 = 1 \). It follows from our results that \( C_1 \) can be taken to be 1 for arbitrary ellipsoids, and not just the ball.

**Theorem 2** Let \( \Omega = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_d^2}{a_d^2} < 1 \right\} \) be an ellipsoid, where the constants \( a_1, \ldots, a_d \) are assumed to be positive numbers. Then for every entire function \( f \) on \( \mathbb{C}^d \), the solution \( h \) of the Dirichlet problem for \( \Omega \) with data function \( f \) (restricted to the boundary) has a harmonic continuation to \( \mathbb{R}^d \) and hence a continuation to an entire function on \( \mathbb{C}^d \). Furthermore, denoting also by \( h \) the said extension, we have \( \rho (h) \leq \rho (f) \), and if \( 0 < \rho (h) = \rho (f) < \infty \), then \( \tau (h) \leq \tau (f) \).

**Proof** In order to apply Theorem 1 we take \( k = 1 \), \( P_2 (x) = \frac{x_2^2}{a_1^2} + \cdots + \frac{x_2^2}{a_d^2} \), \( P_0 = 1 \) and \( \beta = 0 \). Set \( C = \max \{ a_1^2, \ldots, a_d^2 \} \) and note that the restriction of \( P_2 \) to \( \mathbb{S}^{d-1} \) satisfies \( P_2 \geq 1/C \). Thus, the integral inequality from Theorem 1 with \( \alpha = 0 \) follows:

\[
\langle P_2 f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})} \geq \frac{1}{C} \langle f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})},
\]

for all homogeneous polynomials \( f_m \) of degree \( m \). Then \( (2k - \beta) / \alpha = \infty \) and Theorem 1 says that for any entire function \( f \) of order \( \rho < \infty \) there exist entire functions \( q \) and \( h \) of order \( \leq \rho \) such that \( f = (P_2 - 1) q + h \) and \( \Delta h = 0 \). Thus \( h \) is an entire harmonic function with \( f (\xi) = h (\xi) \) for all \( \xi \in \partial \Omega \). By Theorem 10 the function \( q \) has either order \( < \rho \), or order \( \rho \) and type \( \leq \tau (f) \). It follows that if \( h \) has order exactly \( \rho \), its type must satisfy \( \tau (h) \leq \tau (f) \).

The results in the present paper can also be used to deal with the Dirichlet problem for parabolas, including degenerate cases such as the strip, with boundary given by \( \{ x_2 + a \} (x_2 - a) = 0 \). For an arbitrary nondegenerate parabola, after a translation and a rotation we may assume that it is symmetric with respect to the \( x_2 \)-axis and has the origin as its vertex, so it is defined by the equation \( ax_1 = x_2^2 \).

In [21] it is shown that there exists a non-zero harmonic entire function of order 1/2 which vanishes on a parabola. In the sequel we will see that for entire functions of order < 1/2 one can provide a Fischer decomposition.

In order to apply Theorem 1 to these examples the assumed integral inequality needs to be proven. We do so for \( d = 2 \) and \( P_2 (x_1, x_2) = x_2^2 \) in the second main result. Higher dimensional generalizations will be pursued elsewhere.

**Theorem 3** Let \( d = 2 \). Then for all homogeneous polynomials \( f_m \) of degree \( m \) the following inequality holds:

\[
\left\langle \frac{x_2^2 f_m, f_m} {L^2(\mathbb{S}^1)} \right\rangle \geq \frac{\pi^2}{4 (m + 4)^2} \langle f_m, f_m \rangle_{L^2(\mathbb{S}^1)}.
\]
Theorem 4 Let $d = 2$, let $f(x_1, x_2)$ be an entire function of order $\rho(f) < \frac{1}{2}$, and let $ax_1 = x_2^2$ define the locus of a parabola. Then there exists an entire harmonic function $h$ of order $\leq \rho$ such that $h = f$ on $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : ax_1 = x_2^2\}$.

Proof Set $P_2(x_1, x_2) = x_2^2$ (so $k = 1$), $P_1(x_1, x_2) = ax_1$, and $P_0(x_1, x_2) = 0$. Since $\alpha = 2$ in Theorem 3, it follows that $(2k - \beta)/\alpha = \frac{1}{2}$. By Theorems 3 and 1 there exist entire functions $q$ and $h$ of order at most $\rho < 1/2$, such that $h$ is harmonic and $f(x_1, x_2) = (x_2^2 - ax_1)q(x_1, x_2) + h(x_1, x_2)$. Thus, $f = h$ on $\Omega$. □

Theorem 5 Let $d = 2$, let $f(x_1, x_2)$ be an entire function of order $\rho(f) < 1$, and consider the strip with locus defined by $(x_1 - a)(x_1 + a) = 0$, where $a > 0$. Then there exists an entire harmonic function $h$ of order $\leq \rho$ such that $h = f$ on $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 = a^2\}$.

Proof Set $P_2(x_1, x_2) = x_2^2$ (so $k = 1$), $P_1(x_1, x_2) = 0$, and $P_0(x_1, x_2) = a^2$. Then $(2k - \beta)/\alpha = 1$. By Theorems 3 and 1 there exist entire functions $q$ and $h$ of order at most $\rho$, such that $h$ is harmonic and $f(x_1, x_2) = (x_2^2 - a^2)q(x_1, x_2) + h(x_1, x_2)$. Thus, $f = h$ on $\Omega$. □

The Dirichlet problem on the strip in $\mathbb{R}^2$ was discussed in the classical paper [53], see also [12]. For the Dirichlet problem on the slab $S_{a,b} := (a, b) \times \mathbb{R}^{d-1}$ (which is a strip for $d = 2$) we mention the following result in [36]: each entire function $f$ has a decomposition

$$f(x_1, \ldots, x_d) = (x_1 - a)(x_1 - b)q(x_1, \ldots, x_d) + h(x_1, \ldots, x_d)$$

where $q$ is an entire function and $h$ is entire and harmonic. Thus the existence of a Fischer decomposition is proved, but uniqueness of the representation is lost. For example, the function $f(x_1, x_2) = \sin\left(\frac{\pi}{a}x_1\right)e^{\frac{\pi}{a}x_2}$ is harmonic on $\mathbb{R}^2$ and it vanishes on the boundary of the strip. Hence, the function $f$ has at least two decompositions, the first where $q = 0$ and $h = f$, the second where $h = 0$ and

$$q = \frac{\sin\left(\frac{\pi}{a}x_1\right)e^{\frac{\pi}{a}x_2}}{(x_1 - a)(x_1 + a)},$$

which is an entire function.

Another interesting example is the ellipsoidal cylinder

$$\Omega_{cyl} = \left\{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \frac{x_1^2}{a_1^2} + \cdots + \frac{x_{d-1}^2}{a_{d-1}^2} < 1 \right\}$$

for given positive numbers $a_1, \ldots, a_{d-1}$ and $d \geq 2$. One can use the results in this paper to conclude that for each entire function $f$ of order $\rho < 1$, there exist entire functions $q$ and $h$ of order bounded by $\rho$, such that

$$f = \left(\frac{x_1^2}{a_1^2} + \cdots + \frac{x_{d-1}^2}{a_{d-1}^2} - 1\right)q + h$$

and $\Delta h = 0$. 

\[
\begin{align*}
\text{Proof}\quad & \text{Set } P_2(x_1, x_2) = x_2^2\text{ (so } k = 1\text{), } P_1(x_1, x_2) = ax_1, \text{ and } P_0(x_1, x_2) = 0. \text{ Since } \alpha = 2 \text{ in Theorem 3, it follows that } (2k - \beta)/\alpha = \frac{1}{2}. \text{ By Theorems 3 and 1 there exist entire functions } q \text{ and } h \text{ of order at most } \rho < 1/2, \text{ such that } h \text{ is harmonic and } f(x_1, x_2) = (x_2^2 - ax_1)q(x_1, x_2) + h(x_1, x_2). \text{ Thus, } f = h \text{ on } \Omega. \quad \square
\end{align*}
\]
Thus $h$ is an entire harmonic function of order $\leq \rho$ solving the Dirichlet problem for the function $f$ and the cylinder—a result which was proven by a different method in [35]. It is an open question whether for any entire data function $f$ (restricted to the boundary of a cylinder) there exists a harmonic entire function $h$ that solves the Dirichlet problem for $f$ and the cylinder. Results in [27, 28] for extending harmonic functions vanishing on the boundary of the cylinder indicate that a positive answer is possible.

Finally let us mention that the Dirichlet problem for the halfspace is discussed in [2, 25], see also [40]. For the Dirichlet problem for general unbounded domains we refer to [26].

\section{2 Entire functions}

A point in $\mathbb{C}^d$ is denoted by $z = (z_1, \ldots, z_d)$, and by $|z| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$ its Euclidean norm. For a continuous function $f : \mathbb{C}^d \to \mathbb{C}$ we define

$$M_{\mathbb{C}^d}(f, r) := \sup \left\{ |f(z)| : z \in \mathbb{C}^d, |z| = r \right\},$$

and then the order $\rho_{\mathbb{C}^d}(f)$ of $f$ is

$$\rho_{\mathbb{C}^d}(f) = \lim_{r \to \infty} \sup \frac{\log \log M_{\mathbb{C}^d}(f, r)}{\log r} \in [0, \infty].$$

If $0 < \rho_{\mathbb{C}^d}(f) < \infty$ then the type of $f$ is given by

$$\tau_{\mathbb{C}^d}(f) = \lim_{r \to \infty} \sup \frac{\log M_{\mathbb{C}^d}(f, r)}{r^{\rho(f)}}.$$

There is a vast literature about entire analytic functions $f : \mathbb{C}^d \to \mathbb{C}$ of finite order. In this paper we will consider also the order of a harmonic function $f : \mathbb{R}^d \to \mathbb{R}$, which is defined in terms of real variables, see [22, 23], [24] or [3] for details, or the exposition below.

For our purposes it is necessary to introduce now some terminology and notations: let $B_R := \{ x \in \mathbb{R}^d : |x| < R \}$ be the open ball in $\mathbb{R}^d$ with center 0 and radius $0 < R \leq \infty$. Assume that $f$ is an infinitely differentiable function on $B_R$. Define a homogeneous polynomial of degree $m$ by

$$f_m(x) = \sum_{|\alpha| = m}^{\infty} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(0) \ x^\alpha \text{ for } m \in \mathbb{N}_0. \quad (5)$$

Let $A(B_R)$ be the set of all infinitely differentiable functions $f : B_R \to \mathbb{C}$ such that for every compact subset $K \subset B_R$, the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m(x)$ converges absolutely, and uniformly to $f$ on $K$. 


For \( x \in \mathbb{R}^d \) define \( r := |x| \) and \( \theta = x / |x| \), so \( \theta \in S^{d-1} \) and \( x = r\theta \). Then

\[
 f(x) = \sum_{m=0}^{\infty} f_m(x) = \sum_{m=0}^{\infty} f_m(r\theta) = \sum_{m=0}^{\infty} r^m f_m(\theta) = f(r\theta), \tag{6}
\]

which can be seen as a power series in the real variable \( r \) and the coefficients \( f_m(\theta) \), where \( \theta \in S^{d-1} \). Clearly we can replace \( r \) by a complex variable \( \zeta \), defining, for \( z = \zeta \vartheta := (\zeta \theta_1, \ldots, \zeta \theta_d) \) and \( \vartheta = (\theta_1, \ldots, \theta_d) \in S^{d-1} \), the function

\[
 f(\zeta \vartheta) = \sum_{m=0}^{\infty} \zeta^m f_m(\theta) ,
\]

which is entire in \( \zeta \in \mathbb{C} \) for any fixed \( \vartheta \in S^{d-1} \). One can also replace \( x = (x_1, \ldots, x_d) \) in formula (6) by the complex vector \( z = (z_1, \ldots, z_d) \in \mathbb{C}^d \), and the convergence of the sum \( f(z) \) follows from a result due to Siciak [52], stating that for each homogeneous polynomial \( f_m(z) \), the following estimate holds:

\[
 \sup_{z \in \mathbb{C}^d, |z| \leq 1} |f_m(z)| \leq \sqrt{2} \sup_{\zeta \in \mathbb{C}, |\zeta| \leq 1} \sup_{\vartheta \in S^{d-1}} |f_m(\zeta \vartheta)| = \sqrt{2} \sup_{\vartheta \in S^{d-1}} |f_m(\vartheta)| .
\]

Thus each function \( f \in A(B_R) \) with \( R = \infty \) has an extension to an entire function \( f : \mathbb{C}^d \to \mathbb{C} \). In general it is known that \( A(B_R) \) is isomorphic to the set of all holomorphic functions on the harmonicity hull of \( B_R \), see e.g. [47] for more details.

In analogy to (2), one defines

\[
 M_{\mathbb{R}^d}(f, r) := \sup \left\{ |f(\zeta \vartheta)| : \zeta \in \mathbb{C}, |\zeta| = r, \vartheta \in S^{d-1} \right\} ,
\]

and the corresponding order \( \rho_{\mathbb{R}^d}(f) \) of a non-constant function \( f \in A(B_\infty) \):

\[
 \rho_{\mathbb{R}^d}(f) = \lim_{r \to \infty} \sup \frac{\log \log M_{\mathbb{R}^d}(f, r)}{\log r} .
\]

For an entire function \( f \) one has the estimate

\[
 M_{\mathbb{R}^d}(f, r) \leq M_{\mathbb{C}^d}(f, r) \leq M_{\mathbb{R}^d}(f, \sqrt{2} r) ,
\]

which leads to the statements

\[
 \rho_{\mathbb{C}^d}(f) = \rho_{\mathbb{R}^d}(f) \quad \text{and} \quad \tau_{\mathbb{C}^d}(f) = \sqrt{2} \tau_{\mathbb{R}^d}(f) ,
\]

where \( \tau_{\mathbb{R}^d}(f) \) is the type with respect to \( M_{\mathbb{R}^d}(f, r) \), defined as

\[
 \tau_{\mathbb{R}^d}(f) = \lim_{r \to \infty} \sup \frac{\log M_{\mathbb{R}^d}(f, r)}{r^p} ,
\]
provided $0 < \rho := \rho_{\mathbb{R}^d} < \infty$. In this paper we shall work with $\rho_{\mathbb{R}^d}(f)$ and $\tau_{\mathbb{R}^d}(f)$, which are more natural in the setting of real euclidean spaces.

It is not difficult to prove that for $f \in A(B_{\infty})$ one has

$$
\rho_{\mathbb{R}^d}(f) = \lim_{m \to \infty} \sup \frac{\log m}{\log \left( \frac{1}{\max_{\theta \in S^{d-1}} |f_m(\theta)|} \right)}.
$$

Frequently we shall use the following formulation: let $f \in A(B_{\infty})$ and let $\rho \geq 0$. Then $\rho_{\mathbb{R}^d}(f) \leq \rho < \infty$ if and only if for every $\varepsilon > 0$ there exists an $m_0 \geq 0$ such that for every $m \geq m_0$ we have

$$
\max_{\theta \in S^{d-1}} |f_m(\theta)| \leq \frac{1}{m^{m/(\rho+\varepsilon)}}.
$$

Let $f$ be an entire function of order $0 < \rho < \infty$ and type $\tau$. According to a theorem of Lindelöf and Pringsheim,

$$
\lim_{m \to \infty} \sup \left( m \cdot \max_{\theta \in S^{d-1}} |f_m(\theta)|^{\frac{d}{d-1}} \right) = e^{\rho \tau}.
$$

Thus for every $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$
\max_{\theta \in S^{d-1}} |f_m(\theta)| \leq \frac{(e^{\rho \tau} + \varepsilon)^{m/\rho}}{m^{m/\rho}}.
$$

On the other hand, if we know that $f$ is entire and there exists an $m_0$ such that for all $m \geq m_0$ inequality (8) is satisfied (for some constants $\rho$ and $\tau$ and every $\varepsilon > 0$) standard arguments show that the order of $f$ is at most $\rho$, and if it is equal to $\rho$, then its type is at most $\tau$. An analogous remark can be made regarding inequality (7).

It can be proved that the sum (6) also converges if we replace $x \in \mathbb{R}^d$ by a complex vector $z \in \mathbb{C}^d$ in the polynomials $f_m(x)$.

The next result can be found in [47, Theorem 11] (save for a normalization error: $\sqrt{\omega_{d-1}}$ should appear in the denominator, as it does below).

**Theorem 6** For all homogeneous polynomials $f_m$ of degree $m \in \mathbb{N}_0$ we have

$$
\max_{\theta \in S^{d-1}} |f_m(\theta)| \leq \frac{\sqrt{2}}{\sqrt{\omega_{d-1}}} \frac{1 + m^{(d-1)/2}}{\|f_m\|_{L^2(S^{d-1})}}.
$$

## 3 Estimates for the Fischer decomposition

Assume that $(P, Q)$ is a Fischer pair for the vector space $P(\mathbb{R}^d)$. By definition, for each polynomial $f$ there exist unique polynomials $q$ and $r$ such that

$$
f = Pq + r \text{ with } Q(D)r = 0.
$$
Since the decomposition in (9) is unique we can define operators $T_P : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ and $R_P : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\mathbb{R}^d)$ by setting $T_P(f) := q$ and $R_P(f) := r$. So we write (as in [47] or [34])

$$f = P \cdot T_P(f) + R_P(f).$$

(10)

It is easy to see that $T_P$ and $R_P$ are linear operators. Now assume that $P$ is a polynomial of degree $2k$, and write

$$P = P_{2k} - P_{2k-1} - \cdots - P_0,$$

(11)

where the $P_j$ are homogeneous polynomials for $j = 0, \ldots, 2k$. The minus signs are chosen to have a simplified expression in the next Theorem, which describes the operator $T_P$ using just the multiplication operators $P_j$ and the operator $T := T_{P_{2k}}$ for the leading term $P_{2k}$.

**Theorem 7** Let $Q$ be a homogeneous polynomial of degree $2k > 0$, let $P$ be a polynomial of degree $2k$ of the form (11), and assume that $(P_{2k}, Q)$ is a Fischer pair for $\mathcal{P}(\mathbb{R}^d)$. Setting $T := T_{P_{2k}}$, we have

$$T_P(f_m) = \sum_{j=-1}^m \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} T(P_{s_j} T(\cdots P_{s_1} T(P_{s_0} T(f_m)) \cdots))$$

(12)

for all homogeneous polynomials $f_m$ of degree $m$, with the convention that the summand for $j = -1$ is $T f_m$.

**Proof** We will simplify the above expression by omitting parentheses. The proof follows by induction over the degree $m$. First we formulate the statement in a slightly different form:

(i) For each homogeneous polynomial $f_m$ of degree $m$ there exists a decomposition

$$f_m = P q_m + r_m,$$

where $q_m := T_P(f_m)$ and $r_m$ is a polynomial of degree $\leq m$ satisfying $Q(D) r_m = 0$.

For $m = 0$ the polynomial $f_0$ is constant and $Q(D) f_0 = 0$, since $Q$ is homogeneous of degree $2k > 0$. So we have the decomposition $f_0 = P \cdot q_0 + f_0$ with $q_0 = 0$. On the other hand, for $m = 0$ the right hand side has summands for $j = -1$ and $j = 0$, namely $T f_0$ and $\sum_{s_0=0}^{2k-1} T P_{s_0} T f_0$. Now use that $T f_0 = 0$.

Assume that the statement holds for all homogeneous polynomials of degree $\leq m$. Let $f_{m+1}$ be a homogeneous polynomial of degree $m + 1$. Since $(P_{2k}, Q)$ is a Fischer pair we can write

$$f_{m+1} = P_{2k} \cdot T f_{m+1} + r_{m+1}$$
where $Q(D) r_{m+1} = 0$. Then

$$f_{m+1} = P \cdot T f_{m+1} + \sum_{s_{m+1}=0}^{2k-1} P_{s_{m+1}} T f_{m+1} + r_{m+1}.$$ 

For $0 \leq s_{m+1} \leq k - 1$ define $g_{s_{m+1}} = P_{s_{m+1}} T f_{m+1}$, which is a homogeneous polynomial of degree $\leq m$. By the induction hypothesis we can write

$$g_{s_{m+1}} = P \cdot q_{s_{m+1}} + r_{s_{m+1}},$$

where $Q(D) r_{s_{m+1}} = 0$ and $q_{s_{m+1}}$ is given by

$$q_{s_{m+1}} = \sum_{j=-1}^{m} \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} T P_{s_j} \cdots T P_{s_0} T g_{s_{m+1}}.$$ 

It follows that

$$f_{m+1} = P \left( T f_{m+1} + \sum_{s_{m+1}=0}^{2k-1} q_{s_{m+1}} \right) + \sum_{s_{m+1}=0}^{2k-1} r_{s_{m+1}} + r_{m+1}.$$ 

This shows that we have a decomposition of $f_{m+1}$ of the desired form. Using the induction hypothesis we see that $q_{m+1} := T f_{m+1} + \sum_{s_{m+1}=0}^{2k-1} q_{s_{m+1}}$, so

$$q_{m+1} = T f_{m+1} + \sum_{s_{m+1}=0}^{2k-1} \sum_{j=-1}^{m} \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} T P_{s_j} \cdots T P_{s_0} T (P_{s_{m+1}} T f_{m+1})$$

$$= \sum_{j=-1}^{m+1} \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} T P_{s_j} \cdots T P_{s_0} T f_{m+1}.$$ 

\square

For the special case $P = P_{2k} - P_0$ we have contributions only when $s_0 = s_1 = \cdots = s_j = 0$, so

$$T_P (f_m) = \sum_{j=-1}^{m} (T \circ M_{P_0})^{j+1} \circ (T f_m).$$

Given a real number $a$, set $a^+ := \max\{a, 0\}$.

**Proposition 8** Let $P_{2k}$ and $Q$ be homogeneous polynomials in $d$ variables, of degree $2k > 0$, and assume that $(P_{2k}, Q)$ is a Fischer pair for $P \left( \mathbb{R}^d \right)$. Write $T := T_{P_{2k}}$. 
Suppose there exist $C > 0$, $D > 0$ and $\alpha \geq 0$ such that

\[ \| T f_m \|_{L^2(S^{d-1})} \leq C (m + D)^\alpha \| f_m \|_{L^2(S^{d-1})} \quad (13) \]

for every homogeneous polynomial $f_m(x)$ of degree $m$. Let the polynomials $P_j(x)$ be homogeneous of degree $j$ for $j = 0, \ldots, 2k$, and define

$D_s = \max_{\theta \in S^{d-1}} |P_s(\theta)|$ for $s = 0, \ldots, 2k - 1$.

If $\beta < 2k$ is a natural number such that $P_j = 0$ whenever $j = \beta + 1, \ldots, 2k - 1$, then the following estimate holds for $s_0, \ldots, s_j \in \{0, \ldots, 2k - 1\}$ and every homogeneous polynomial $f_m(x)$ of degree $m$:

\[ \| TP_{s_j} \cdots TP_{s_0} f_m \|_{L^2(S^{d-1})} \leq (m + D)^{\alpha(j+1)} C^{j+1} D_{s_j} \cdots D_{s_0} \| f_m \|_{L^2(S^{d-1})} \]

\[ \leq (m + D)^{\frac{\beta}{2k-\beta}} C^{j+1} D_{s_j} \cdots D_{s_0} \| f_m \|_{L^2(S^{d-1})}. \]

**Proof** Let us write $C_m = C (m + D)^\alpha$. Note that $P_{s_j} \cdots P_{s_0} T f_m$ is a homogeneous polynomial of degree

\[ d(s_0, \ldots, s_j) := \deg P_{s_j} \cdots P_{s_0} T f_m \]

\[ = [m + (s_0 - 2k) + \cdots + (s_j - 2k)]^+ < m. \quad (14) \]

If $d(s_0, \ldots, s_j) - 2k$ is negative, this means $P_{s_j} \cdots P_{s_0} T f_m$ has degree $< 2k$, hence we see that

\[ T \left[ P_{s_j} \cdots P_{s_0} T f_m \right] = 0. \]

If $d(s_0, \ldots, s_j) - 2k \geq 0$, using $P_{s_j} = 0$ when $s_j > \beta$, from (14) get the following estimate for $j$:

\[ j + 1 \leq j + 2 \leq \frac{m + s_0 + \cdots + s_j}{2k} \leq \frac{m + \beta (j + 1)}{2k}. \]

This is equivalent to

\[ j + 1 \leq \frac{m}{2k - \beta}. \]

Now use the estimate (13) to see that

\[ \| TP_{s_j} \cdots TP_{s_0} T f_m \|_{L^2(S^{d-1})} \leq C_{d_j} \| P_{s_j} T P_{s_{j-1}} \cdots TP_{s_0} T f_m \|_{L^2(S^{d-1})} \]

\[ \leq C_{d_j} D_{s_j} \| TP_{s_{j-1}} \cdots TP_{s_0} T f_m \|_{L^2(S^{d-1})}. \]
Note that $C_{d_j} \leq C_m$. We can iterate the process, obtaining
\[
\left\| T P_{s_j} \cdots T P_{s_0} T f_m \right\|_{L^2(\mathbb{S}^{d-1})} \leq C_m^{j+1} \cdot D_{s_j} \cdots D_{s_0} \left\| f_m \right\|_{L^2(\mathbb{S}^{d-1})} \\
\leq (m + D)^{\frac{ma}{2k-\beta}} C^{j+1} D_{s_j} \cdots D_{s_0} \left\| f_m \right\|_{L^2(\mathbb{S}^{d-1})}. \\
\]
\[\square\]

The main difference between the next result and Theorem 1 is that here we assume $(P, \Delta^k)$ is a Fischer pair for $P(\mathbb{R}^d)$.

**Theorem 9** Let $P_{2k}$ be a homogeneous polynomial in $d$ variables, of degree $2k > 0$, and let $P$ be a polynomial of degree $2k$, having the form (11). Assume that $(P_{2k}, \Delta^k)$ is a Fischer pair for $P(\mathbb{R}^d)$. Write $T := T P_{2k}$. Suppose there exist $C > 0$, $D > 0$ and $\alpha \geq 0$ such that
\[
\left\| T f_m \right\|_{L^2(\mathbb{S}^{d-1})} \leq C (m + D)^{\alpha} \left\| f_m \right\|_{L^2(\mathbb{S}^{d-1})}
\]
for every homogeneous polynomial $f_m$ of degree $m$, and let $\beta < 2k$ be a natural number such that $P_j = 0$ whenever $j = \beta + 1, \ldots, 2k - 1$. Then for every entire function $f$ of order $\rho < (2k - \beta)/\alpha$, there are entire functions $q$ and $h$ of order $\leq \rho$, satisfying
\[
f = (P_{2k} - P_{\beta} - \cdots - P_0) q + h \text{ and } \Delta^k h = 0. \quad (15)
\]

**Proof** Assume $f$ is an entire function $f$ of order $< (2k - \beta)/\alpha$ and write $f = \sum_{m=0}^{\infty} f_m$, where the homogeneous polynomials $f_m$ are given by (5). Then $T_P (f_m)$ is either the zero polynomial or a polynomial of degree $< m$ (not necessarily homogeneous). Our strategy is to show that
\[
q := \sum_{m=0}^{\infty} T_P (f_m) \quad (16)
\]
converges absolutely and uniformly on compacta, and hence it defines an entire function $q : \mathbb{R}^d \to \mathbb{R}$.

Next, we determine the order of $q$ by writing $q (x) = \sum_{M=0}^{\infty} G_M (x)$, where each $G_M$ is a homogeneous polynomial of degree $M$ (as a reminder, we mention that when it exists this representation is unique). Recall that
\[
T_P (f_m) = \sum_{j=-1}^{m} \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} T P_{s_j} \cdots T P_{s_0} T f_m, \quad (17)
\]
and clearly, $T P_{s_j} \cdots T P_{s_0} T f_m$ is a homogeneous polynomial of degree
\[
s_0 + \cdots + s_j + m - 2k (j + 2).
\]
If $s_0, \ldots, s_n \in \{0, \ldots, 2k - 1\}$ are given, where $n > m$, then $TP_{s_n} \cdots TP_{s_0} Tf_m$ is zero by inspection of its degree

$$m + s_0 + \cdots + s_n - 2k (n + 2) \leq m + (n + 1) (2k - 1) - 2k (n + 2) = m - 2k - n - 1 < 0,$$

so the $m$ in the first summatory of (17) can be replaced by $\infty$. By hypothesis, $P_s = 0$ for all $s \in \{\beta + 1, \ldots, 2k - 1\}$, so we can write

$$TP (f_m) = \sum_{j=-\infty}^{\infty} \sum_{s_0=0}^{\beta} \sum_{s_1=0}^{\beta} \cdots \sum_{s_j=0}^{\beta} TP_{s_j} \cdots TP_{s_0} Tf_m. \quad (18)$$

In order to show that the sum in (16) converges absolutely and uniformly on compacta it suffices to show that

$$G := \sum_{j=-\infty}^{\infty} \sum_{s_0=0}^{\beta} \sum_{s_1=0}^{\beta} \cdots \sum_{s_j=0}^{\beta} TP_{s_j} \cdots TP_{s_0} Tf_m$$

does so. Then the sum can be reordered and shown to be equal to $g$. Next we collect all summands having degree $M \geq 0$. The requirement

$$\deg TP_{s_j} \cdots TP_{s_0} Tf_m = s_0 + \cdots + s_j + m - 2k (j + 2) = M$$

means that $m = M + 2k (j + 2) - (s_0 + \cdots + s_j)$, and therefore we consider the sum

$$G_M := \sum_{j=-\infty}^{\infty} \sum_{s_0=0}^{\beta} \sum_{s_1=0}^{\beta} \cdots \sum_{s_j=0}^{\beta} TP_{s_j} \cdots TP_{s_0} Tf_{M+2k(j+2)-(s_0+\cdots+s_j)}.$$

We show next that $G_M$ converges absolutely everywhere. By Theorem 6, for every $\theta \in S^{d-1}$ we have

$$\left| TP_{s_j} \cdots TP_{s_0} Tf_{M+2k(j+2)-(s_0+\cdots+s_j)} (\theta) \right| \leq \frac{\sqrt{2}}{\sqrt{\omega_{d-1}}} (1 + M)^{(d-1)/2} \left\| TP_{s_j} \cdots TP_{s_0} Tf_{M+2k(j+2)-(s_0+\cdots+s_j)} \right\|_{L^2(S^{d-1})} . \quad (19)$$

Recall the following notation (used in Proposition 8):

$$D_s = \max_{\theta \in S^{d-1}} |P_s (\theta)| \text{ for } s = 0, \ldots, 2k - 1.$$
Since \( m \leq M + 2k (j + 2) \), by Proposition \( 8 \)

\[
\left\| TP_{s_j} \cdots TP_{s_0} T f_{M+2k(j+2)-(s_0+\cdots+s_j)} \right\|_{L^2(S^{d-1})} \leq (M + 2k (j + 2) + D)^{\alpha(j+1)} C^{j+1} D_{s_j} \cdots D_{s_0} \tag{21}
\]

\[
\times \left\| f_{M+2k(j+2)-(s_0+\cdots+s_j)} \right\|_{L^2(S^{d-1})}. \tag{22}
\]

By assumption \( \frac{2k-\beta}{\rho} > \alpha \). Since \( f \) has order \( \rho \), the bound (7) entails that for every \( \varepsilon > 0 \) there exists a constant \( A_\varepsilon \) such that

\[
\left\| f_m \right\|_{L^2(S^{d-1})} \leq \frac{A_\varepsilon}{m^{\rho+\varepsilon}} \tag{23}
\]

for all natural numbers \( m > 0 \). We may, without loss of generality, assume that

\[
\frac{2k-\beta}{\rho + \varepsilon} - \alpha > 0.
\]

Note that the right hand side of (23) is a decreasing function of \( m \) and that

\[
m \geq M + 2k (j + 2) - \beta (j + 1) = M + 2k + (2k - \beta) (j + 1).
\]

We infer that

\[
\left\| f_{M+2k(j+2)-(s_0+\cdots+s_j)} \right\|_{L^2(S^{d-1})} \leq \frac{A_\varepsilon}{(M + 2k + (2k - \beta) (j + 1))^{\frac{M+2k+(2k-\beta)(j+1)}{\rho+\varepsilon}}}. \tag{24}
\]

Now

\[
(M + 2k + (2k - \beta) (j + 1))^{\frac{M+2k+(2k-\beta)(j+1)}{\rho+\varepsilon}} \leq (M + 2k + (2k - \beta) (j + 1))^{\frac{M+2k}{\rho+\varepsilon}} (M + 2k + (2k - \beta) (j + 1))^{\frac{(2k-\beta)(j+1)}{\rho+\varepsilon}} \\
\geq (M + 2k)^{\frac{M+2k}{\rho+\varepsilon}} (M + 2k + (2k - \beta) (j + 1))^{\frac{(2k-\beta)(j+1)}{\rho+\varepsilon}}.
\]

Furthermore, we have

\[
\sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} D_{s_j} \cdots D_{s_0} = (D_0 + \cdots + D_{2k-1})^{j+1}.
\]
Define $\tilde{D} := D_0 + \cdots + D_{2k-1}$. Using (19)–(20), (21)–(22) and (24) we get

\[
G_M^{(j)} := \sum_{s_0=0}^{2k-1} \sum_{s_1=0}^{2k-1} \cdots \sum_{s_j=0}^{2k-1} \left| T P_{s_j} \cdots T P_{s_0} T f_{M+2k(j+2)}(s_0, \ldots, s_j) \right|
\]

\[
\leq \frac{\sqrt{2} \alpha_{\ell}}{\sqrt{\omega d^{-1}}} \frac{(M+1)^{d-1}}{(M+2k)^{\frac{M+2k}{\rho + \epsilon}}} \frac{(M+2k(j+2)+D)^{\alpha(j+1)} C^{j+1}}{(M+2k+(2k-\beta)(j+1))^{\frac{(2k-\beta)(j+1)}{\rho + \epsilon}}} \tilde{D}^{j+1}.
\]

Note that

\[
\frac{(M+2k(j+2)+D)^{\alpha}}{(M+2k+(2k-\beta)(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}} = \frac{(2k)^{\alpha}}{(2k-\beta)^{\frac{(2k-\beta)}{\rho + \epsilon}}} \frac{(M+2k+D)^{\alpha}}{(M+2k+\beta(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}}.
\]

For all $M, j \in \mathbb{N}_0$ the inequality

\[
\frac{M+2k+D}{2k} + j + 1 \leq (D+1) \left( \frac{M+2k}{2k-\beta} + j + 1 \right)
\]

holds since

\[
(D+1)(M+2k)2k - (M+2k+D)(2k-\beta) = D(M+2k)2k + \beta(M+2k + D) - 2Dk \geq 0.
\]

Thus,

\[
\frac{(M+2k(j+2)+D)^{\alpha}}{(M+2k+(2k-\beta)(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}} \leq \frac{(2k)^{\alpha}}{(2k-\beta)^{\frac{(2k-\beta)}{\rho + \epsilon}}} \frac{(M+2k+D)^{\alpha}}{(M+2k+\beta(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}},
\]

so

\[
G_M^{(j)} \leq \frac{\sqrt{2} \alpha_{\ell}}{\sqrt{\omega d^{-1}}} \frac{(M+1)^{d-1}}{(M+2k)^{\frac{M+2k}{\rho + \epsilon}}} \frac{C \tilde{D} (2k)^{\alpha}}{(2k-\beta)^{\frac{(2k-\beta)}{\rho + \epsilon}}} \left( \frac{(M+2k+D)^{\alpha}}{(M+2k+\beta(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}} - \alpha \right)^{j+1}.
\]

Recall that we have chosen $\epsilon > 0$ such that $\gamma := \frac{2k-\beta}{\rho + \epsilon} - \alpha > 0$. For any fixed $M \geq 0$, we can find a natural number $j_0$ such that for all $j \geq j_0$

\[
\frac{C \tilde{D} (2k)^{\alpha}}{(2k-\beta)^{\frac{(2k-\beta)}{\rho + \epsilon}}} \left( \frac{(M+2k+D)^{\alpha}}{(M+2k+\beta(j+1))^{\frac{(2k-\beta)}{\rho + \epsilon}}} - \alpha \right)^{j+1} \leq \frac{1}{2^j}.
\]
It follows that $|G_M| \leq \sum_{j=0}^{\infty} G_M^{(j)}$ and the latter series converges, so $G_M$ is well-defined for every $M \in \mathbb{N}_0$.

Next we want to estimate $G_M$ for $M \gg 1$. Take $M_0$ so large such that for all $M \geq M_0$

$$C \bar{D} (2k)^{\alpha} (D+1)^{\alpha} \frac{1}{(2k-\beta)^{(2k-\beta)(\rho+\epsilon)}(M+2k)^{(2k-\beta)(\rho+\epsilon)-\alpha}} \leq \frac{1}{2}.$$ 

Then for every $M \geq M_0$,

$$|G_M| \leq \sum_{j=0}^{\infty} G_M^{(j)} \leq \frac{\sqrt{2}A_\delta}{\omega_{d-1}} \frac{(M+1)^{\frac{d-1}{2}}}{(M+2k)^{\frac{d}{2}+\epsilon}} \sum_{j=0}^{\infty} \frac{1}{2^{j+1}}.$$ 

Using the criterion from (7) and the observation after (8), it follows that $G = \sum_{M=0}^{\infty} G_M$ is a well-defined entire function of order $\leq \rho$. Finally, since both $f$ and $(P_{2k} - P_\beta - \cdots - P_0) g$ have order $\leq \rho$, so does $h$. $\square$

If $\alpha = 0$ then $(2k-\beta)/\alpha = \infty$, so by Theorem 9, for every entire function $f$ of order $0 < \rho < \infty$ we can find entire functions $q$ and $h$ of order $\leq \rho$ such that (15) holds. Since we work only with upper estimates we cannot conclude that $q$ and $h$ have exact order $\rho$. However, if $q$ does have order $\rho$, then the type of $q$ is bounded by the type of $f$. When $\alpha > 0$, Theorem 9 provides a Fischer decomposition only for entire functions of order $\rho < (2k-\beta)/\alpha$, and it is unclear whether one can find Fischer decompositions for entire functions of higher order. However, when $\rho = (2k-\beta)/\alpha$, we can prove the existence of Fischer decompositions by assuming that the type of $f$ is sufficiently small for $\alpha > 0$ (no such restriction is needed if $\alpha = 0$). This is the content of the next theorem. Let us also recall the notation $D_s := \max_{\theta \in \mathbb{S}^{d-1}} |P_s(\theta)|$.

**Theorem 10** Let $P_{2k}$ be a homogeneous polynomial in $d$ variables, of degree $2k > 0$, and let $P$ be a polynomial of degree $2k$, having the form (11). Assume that $(P_{2k}, \Delta^k)$ is a Fischer pair for $\mathcal{P}(\mathbb{R}^d)$. Write $T := T_{P_{2k}}$. Suppose there exist $C > 0$, $D > 0$ and $\alpha \geq 0$ such that

$$\|Tf_m\|_{L^2(S^{d-1})} \leq C (m + D)^{\alpha} \|f_m\|_{L^2(S^{d-1})}$$

for every homogeneous polynomial $f_m$ of degree $m$, and let $\beta < 2k$ be a natural number such that $P_j = 0$ whenever $j = \beta + 1, \ldots, 2k - 1$. If $\alpha = 0$, then for every entire function $f$ of finite order $0 < \rho < \infty$ there exist entire functions $q$ and $h$ of order bounded by $\rho$, with

$$f = (P_{2k} - P_\beta - \cdots - P_0) q + h and \Delta^k h = 0. \quad (27)$$
Moreover, if \( q \) has order \( \rho \) then the type of \( q \) is smaller than or equal to the type of \( f \). If \( \alpha > 0 \), then for every entire function \( f \) of order \( \rho = \frac{2k-\beta}{\alpha} \) and type \( \tau \) satisfying

\[
\frac{(2k)^{\frac{2k-\beta}{\rho}}}{(2k-\beta)^{\frac{2k-\beta}{\rho}}} C \left( D_0 + \cdots + D_{\beta} \right) (e^{\rho \tau})^{\frac{2k-\beta}{\rho}} < 1,
\]

there exist entire functions \( q \) and \( h \) of order \( \leq \rho \) such that (27) holds. Moreover, if \( q \) has order \( \rho \) then the type of \( q \) is smaller than or equal to the type of \( f \).

**Proof** We proceed as in the last proof, but unlike the preceding result, for which the estimate from (7) sufficed, here we will need the sharper bound given in (8): from the Theorem of Lindelöf and Pringsheim we conclude that for every \( \varepsilon > 0 \) there exists an \( A_\varepsilon > 0 \) such that for every \( m \geq 1 \),

\[
\|f_m\|_{L^2(S^{d-1})} \leq \frac{A_\varepsilon}{m^{\rho/\rho}} (e^{\rho \tau + \varepsilon})^{m/\rho}.
\]  

(28)

The right hand side is clearly decreasing on \( m \) for \( m \geq e^{\rho \tau + \varepsilon} \). Arguing as in the preceding proof and using (28), for \( j \geq e^{\rho \tau + \varepsilon} \) or \( M \geq e^{\rho \tau + \varepsilon} \) we have

\[
\left\| f_M + 2k(j+2)-(s_0+\cdots+s_j) \right\|_{L^2(S^{d-1})} \leq \frac{A_\varepsilon (e^{\rho \tau + \varepsilon})^{M+2k+(2k-\beta)(j+1)}}{(M+2k+(2k-\beta)(j+1))^{\rho}}.
\]

Following the arguments and using the notation from the last proof, we obtain the estimate

\[
G_M^{(j)} \leq \frac{\sqrt{2}A_\varepsilon}{\sqrt{\omega_d-1}} \frac{(M+1)^{\frac{d-1}{2}} (M+2k (j+2) + D)^{\alpha(j+1)}}{(M+2k)^{\frac{M+2k}{\rho}}} \frac{C^{j+1} \tilde{D}^{j+1}}{(M+2k+(2k-\beta)(j+1))^{\frac{2k-\beta}{\rho}}} \times (e^{\rho \tau + \varepsilon})^\frac{M+2k+(2k-\beta)(j+1)}{\rho}.
\]

It follows that

\[
G_M^{(j)} \leq \frac{\sqrt{2}A_\varepsilon}{\sqrt{\omega_d-1}} \frac{(M+1)^{\frac{d-1}{2}} (M+2k (j+2) + D)^{\alpha(j+1)}}{(M+2k)^{\frac{M+2k}{\rho}}} \frac{C^{j+1} \tilde{D}^{j+1}}{(M+2k+(2k-\beta)(j+1))^{\frac{2k-\beta}{\rho}}} \times (e^{\rho \tau + \varepsilon})^\frac{M+2k+(2k-\beta)(j+1)}{\rho}.
\]

If \( \alpha = 0 \) then there exists a \( j_0 \in \mathbb{N} \) such that for all \( j \geq j_0 \) and for all \( M \geq 0 \),

\[
C \tilde{D} \left( \frac{e^{\rho \tau + \varepsilon}}{(M+2k+(2k-\beta)(j+1))^{\frac{2k-\beta}{\rho}}} \right) \leq \frac{1}{2}.
\]
Thus there exists a constant $B_{j_0}$ such that

$$|G_M| \leq \sum_{j=0}^{\infty} G_M^{(j)} \leq B_{j_0} \sqrt{2A_\epsilon} \frac{(M + 1)^{d-1}}{\sqrt{(\omega d - 1)} (M + 2k)} (e \rho \tau + \varepsilon)^{M + 2k}.$$

From this we see that

$$\lim_{M \to \infty} \sup (M \cdot \max_{\theta \in S} |G_M(\theta)|^\rho) \leq e \rho \tau + \varepsilon$$

for every $\varepsilon > 0$. Now let $\varepsilon \to 0$. Using the remark after (8), this implies that $G(x)$ is a well-defined entire function of order $\leq \rho$, and if $G$ has order $\rho$ it follows that the type of $G$ is smaller or equal to $\tau$.

Now assume that $\alpha = \frac{2k - \beta}{\rho} > 0$. Then

$$\frac{(M + 2k (j + 2) + D)^{\alpha(j+1)}}{(M + 2k + (2k - \beta) (j + 1))^{\frac{(2k - \beta)(j+1)}{\rho}}} = \frac{(2k)^{\frac{2k - \beta}{\rho}}}{(2k - \beta)^{\frac{2k - \beta}{\rho}}} \left(\frac{M + 2k + D}{2k} + j + 1\right)^{\frac{2k - \beta}{\rho}}.$$

We now choose $\delta > 0$ and $\varepsilon > 0$ such that

$$\frac{(2k)^{\frac{2k - \beta}{\rho}}}{(2k - \beta)^{\frac{2k - \beta}{\rho}}} C \tilde{D} (1 + \delta) (e \rho \tau + \varepsilon)^{\frac{2k - \beta}{\rho}} < 1.$$

Given fixed $M$ and $\delta > 0$ there exists $j_M$ such that for all $j \geq j_M$ we have

$$\frac{M + 2k + D}{2k} + j + 1 \leq 1 + \delta.$$

It follows that

$$G_M^{(j)} \leq \frac{\sqrt{2A_\epsilon}}{\sqrt{(\omega d - 1)} (M + 2k)} (e \rho \tau + \varepsilon)^{M + 2k} \times \left(\frac{2k - \beta}{\rho} (1 + \delta) C \tilde{D} (e \rho \tau + \varepsilon)^{\frac{2k - \beta}{\rho}}\right)^{j+1}.$$

Thus the series $\sum_{j=0}^{\infty} G_M^{(j)}$ converges. Next we want to estimate $G_M$ for large $M$. Take $M_0$ so large such that for all $M \geq M_0$ and for all $j \geq 0$

$$\frac{M + 2k + D}{2k} + j + 1 \leq 1 + \delta.$$
Then we have for all \( M \geq M_0 \) that
\[
|G_M| \leq \sum_{j=0}^{\infty} G_M^{(j)} \leq \frac{\sqrt{2}A_k}{\sqrt{\omega d-1}} \frac{(M + 1)^{\frac{d-1}{2}}}{(M + 2k)^{\frac{M+2k}{\rho}}} (e^{\rho \tau} + \epsilon)^{\frac{M+2k}{\rho}} \cdot \Gamma
\]
where
\[
\Gamma = \sum_{j=0}^{\infty} \left( \frac{(2k)^{\frac{2k-\beta}{\rho}}}{(2k-\beta)^{\frac{2k-\beta}{\rho}}} (1 + \delta) C \tilde{D} (e^{\rho \tau} + \epsilon)^{\frac{2k-\beta}{\rho}} \right)^{j+1} < \infty.
\]
Using the remark after (8), this implies that that \( G (x) \) is a well-defined entire function of order \( \leq \rho \).

\[\Box\]

4 Estimates for the norm of \( T (f) \)

The following result reduces the question about a norm estimate of \( T (f) \) to a question about an integral inequality for homogeneous polynomials:

**Proposition 11** Let \( P_{2k} \) be a homogeneous polynomial of degree \( 2k > 0 \). Suppose that for every \( m \in \mathbb{N}_0 \), there is a constant \( C_m > 0 \) such that
\[
\langle P_{2k} f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})} \geq C_m \langle f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})}
\]
whenever \( f_m \) is a homogeneous polynomial of degree \( m \). Then \( (P_{2k}, \Delta^k) \) is a Fischer pair for \( \mathcal{P} (\mathbb{R}^d) \) and
\[
\| T_{P_{2k}} (f_m) \|_{L^2(\mathbb{S}^{d-1})} \leq \frac{1}{C_{m-2k}} \| f_m \|_{L^2(\mathbb{S}^{d-1})}.
\]

**Proof** First we show that \( (P_{2k}, \Delta^k) \) is a Fischer pair for \( \mathcal{P} (\mathbb{R}^d) \). By [47, Theorem 37], it suffices to prove the injectivity of \( q \mapsto \Delta^k (P_{2k} q) \). If \( \Delta^k (P_{2k} q_m) = 0 \) for some homogeneous polynomial \( q_m \) of degree \( m \) then
\[
\langle P_{2k} q_m, f \rangle_{L^2(\mathbb{S}^{d-1})} = 0
\]
for all polynomials \( f \) with \( \deg f + 2k - 2 < m + 2k \), see Theorem 2 in [47]. Taking \( f = q_m \), we obtain
\[
0 = \langle P_{2k} q_m, q_m \rangle_{L^2(\mathbb{S}^{d-1})} \geq C_m \langle q_m, q_m \rangle_{L^2(\mathbb{S}^{d-1})} \geq 0.
\]
Thus \( q_m = 0 \) and the Fischer operator is injective.

Next, let \( f_m \) be a homogeneous polynomial of degree \( m \). We write
\[
f_m = P_{2k} \cdot T_{P_{2k}} (f_m) + h_m,
\]
where \( \Delta^k h_m = 0 \). If \( T_{P_{2k}} (f_m) = 0 \), there is nothing to prove, so assume otherwise. Then \( m - 2k \geq 0 \), and \( T_{P_{2k}} (f_m) \) is a homogeneous polynomial of degree \( m - 2k \), so \( h_m \) is either the zero polynomial or a homogeneous polynomial of degree \( m \). Using \( \Delta^k h_m = 0 \) and Theorem 2 of [47], we conclude that \( \langle h_m, T_{P_{2k}} (f_m) \rangle_{L^2(\mathbb{S}^{d-1})} = 0 \). Thus

\[
\langle f_m, T_{P_{2k}} (f_m) \rangle_{L^2(\mathbb{S}^{d-1})} = \langle P_{2k} T_{P_{2k}} (f_m), T_{P_{2k}} (f_m) \rangle_{L^2(\mathbb{S}^{d-1})} \\
\geq C_{m-2k} \langle T_{P_{2k}} (f_m), T_{P_{2k}} (f_m) \rangle_{L^2(\mathbb{S}^{d-1})}.
\]

By the Cauchy–Schwarz inequality,

\[
C_{m-2k} \| T_{P_{2k}} (f_m) \|^2_{L^2(\mathbb{S}^{d-1})} \leq \| f_m \|_{L^2(\mathbb{S}^{d-1})} \| T_{P_{2k}} (f_m) \|_{L^2(\mathbb{S}^{d-1})}.
\]

After dividing both sides of the inequality by \( \| T_{P_{2k}} (f_m) \|_{L^2(\mathbb{S}^{d-1})} \), we obtain the result.

Next we note that it is enough to assume \( \langle P_{2k} f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})} \geq C_m \langle f_m, f_m \rangle_{L^2(\mathbb{S}^{d-1})} \) for \( m \) even.

**Lemma 12** Let \( P_{2k} \) be a homogeneous polynomial of degree \( 2k > 0 \). Suppose that for each \( m \in \mathbb{N}_0 \) there exists a constant \( C_{2m} > 0 \) such that

\[
\langle P_{2k} f_{2m}, f_{2m} \rangle_{L^2(\mathbb{S}^{d-1})} \geq C_{2m} \langle f_{2m}, f_{2m} \rangle_{L^2(\mathbb{S}^{d-1})}
\]

for all homogeneous polynomials \( f_{2m} \) of degree \( 2m \), where \( m \geq 0 \). Then

\[
\langle P_{2k} f_{2m+1}, f_{2m+1} \rangle_{L^2(\mathbb{S}^{d-1})} \geq C_{2m+2} \langle f_{2m+1}, f_{2m+1} \rangle_{L^2(\mathbb{S}^{d-1})}
\]

for all homogeneous polynomials \( f_{2m+1} \) of degree \( 2m + 1 \).

**Proof** Let \( f_{2m+1} \) be a homogeneous polynomial of degree \( 2m + 1 \). Define \( F_j (x) := x_j f_{2m+1} (x) \) for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) and \( j = 1, \ldots, d \). Recall that \( |x|^2 = x_1^2 + \cdots + x_d^2 \) and \( |\theta|^2 = 1 \) for \( \theta \in \mathbb{S}^{d-1} \). Then

\[
\langle P_{2k} f_{2m+1}, f_{2m+1} \rangle_{L^2(\mathbb{S}^{d-1})} = \left\langle P_{2k} |x|^2 f_{2m+1}, f_{2m+1} \right\rangle_{L^2(\mathbb{S}^{d-1})} \\
= \sum_{j=1}^{d} \left\langle P_{2k} F_j, F_j \right\rangle_{L^2(\mathbb{S}^{d-1})}.
\]

Since \( F_j \) is a homogeneous polynomial of degree \( 2m + 2 \) our assumption implies that

\[
\langle P_{2k} f_{2m+1}, f_{2m+1} \rangle_{L^2(\mathbb{S}^{d-1})} \geq \sum_{j=1}^{d} C_{2m+2} \left\langle F_j, F_j \right\rangle_{L^2(\mathbb{S}^{d-1})} \\
= C_{2m+2} \langle f_{2m+1}, f_{2m+1} \rangle_{L^2(\mathbb{S}^{d-1})}.
\]
Next we use the orthogonality relations for spherical harmonics in order to derive bounds for homogeneous polynomials of even degree.

**Lemma 13** Let $P_{2k}$ be a homogeneous polynomial of degree $2k$ and let

$$H_{2m} := \left\{ \sum_{l=0}^{m} h_{2l} : h_{2l} \text{ is a harmonic homogeneous polynomial of degree } 2l \right\}.$$ 

Suppose that for each $m \in \mathbb{N}_0$ there exists a constant $C_{2m} > 0$ such that for all $h \in H_{2m}$,

$$\langle P_{2k} h, h \rangle_{L^2(S^{d-1})} \geq C_{2m} \langle h, h \rangle_{L^2(S^{d-1})}.$$

Then

$$\langle P_{2k} f_{2m}, f_{2m} \rangle_{L^2(S^{d-1})} \geq C_{2m} \langle f_{2m}, f_{2m} \rangle_{L^2(S^{d-1})}.$$ 

for all homogeneous polynomials $f_{2m}$ of degree $2m$.

**Proof** Let $f_{2m}$ be a homogeneous polynomial of degree $2m$. By the Gauss decomposition (see Theorem 5.5 in [4] or Theorem 5.7 in the 2001 edition) there exist homogeneous harmonic polynomials $h_{2l}$ of degree $2l$ for $l = 0, \ldots, m$, such that

$$f_{2m}(x) = \sum_{l=0}^{m} h_{2l}(x) |x|^{2m-2l}. \quad (29)$$

Define

$$h = \sum_{l=0}^{m} h_{2l}.$$ 

Since the harmonic polynomials $h_{2l}$ have different degrees for different values of $l$, the orthogonality relations for spherical harmonics yield

$$\langle f_{2m}, f_{2m} \rangle_{L^2(S^{d-1})} = \sum_{l=0}^{m} \langle h_{2l}, h_{2l} \rangle_{L^2(S^{d-1})} = \langle h, h \rangle_{L^2(S^{d-1})}.$$ 

Thus

$$\langle P_{2k} f_{2m}, f_{2m} \rangle_{S^{d-1}} = \sum_{l_1=0}^{m} \sum_{l_2=0}^{m} \langle P_{2k} h_{2l_1}, h_{2l_2} \rangle_{L^2(S^{d-1})} = \langle P_{2k} h, h \rangle_{L^2(S^{d-1})} \geq C_{2m} \langle h, h \rangle_{L^2(S^{d-1})} = C_{2m} \langle f_{2m}, f_{2m} \rangle_{L^2(S^{d-1})}. \quad \square$$
5 The integral inequality for dimension 2 and $P_2(x_1, x_2) = x_2^2$

First we include two auxiliary results.

**Lemma 14** For all $n \geq 2$ one has the estimate

$$\sin \frac{\pi}{n} \geq \frac{\pi}{n+2}.$$

**Proof** By taking $x = \pi/n$ it suffices to show that for all $0 < x \leq \pi/2$

$$\sin x \geq \frac{1}{\frac{1}{x} + \frac{2}{\pi}} = \frac{x}{1 + \frac{2}{\pi}x}, \text{ i.e. } \frac{x}{\sin x} \leq 1 + \frac{2}{\pi}x.$$

This inequality follows from the general inequality

$$\frac{x}{\sin x} \leq 1 + \frac{x}{\pi} \tan \frac{x}{2} \text{ for all } |x| < \pi$$

and the fact that $\tan \frac{x}{2} \leq \tan \frac{\pi}{4} = 1$ for $0 < x \leq \pi/2$. Inequality (30) follows from the representation (see [46, p. 159])

$$\frac{x}{\sin x} = 1 + x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n - 2}{(2n)!} B_{2n} x^{2n-1}$$

where $B_{2n}$ are the Bernoulli numbers. Using the trivial estimate $4^n - 2 \leq 4^n - 1$ and the positivity of $(-1)^{n-1} B_{2n}$ we obtain

$$\frac{x}{\sin x} \leq 1 + x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n - 1}{(2n)!} B_{2n} x^{2n-1} = 1 + \frac{x}{2} \tan \frac{x}{2}$$

since $\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n - 1}{(2n)!} 2^n B_{2n} x^{2n-1}$, see [46, p. 158]. □

The following result is well known; the proof is included for the reader’s convenience:

**Proposition 15** Let $P_n(\lambda) = \det (A_n - \lambda I)$ where $A_n$ is the $n \times n$-matrix

$$A_n = \begin{pmatrix} 0 & \sqrt{2} & & & \\ \sqrt{2} & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}$$

Then $P_n(\lambda) = 2T_n(-\frac{\lambda}{2})$ where $T_n$ is the Chebyshev polynomial of degree $n$. 
Theorem 16  Let $d = 2$. Then for all homogeneous polynomials $f_m$ of degree $m$ the following inequality holds:

$$
\left\langle \chi^2_{\lambda} f_m, f_m \right\rangle_{L^2(\mathbb{S}^1)} \geq \frac{\pi^2}{4(m+4)^2} \left\langle f_m, f_m \right\rangle_{L^2(\mathbb{S}^1)} .
$$

(31)

Proof  By Lemma 12 it suffices to show that

$$
\left\langle \chi^2_{\lambda} f_{2m}, f_{2m} \right\rangle_{L^2(\mathbb{S}^1)} \geq \frac{\pi^2}{4(2m+3)^2} \left\langle f_{2m}, f_{2m} \right\rangle_{L^2(\mathbb{S}^1)} .
$$

(32)

Clearly (32) implies (4) for even indices. Now (32) and Lemma 12 show that

$$
\left\langle \chi^2_{\lambda} f_{2m+1}, f_{2m+1} \right\rangle_{L^2(\mathbb{S}^1)} \geq \frac{\pi^2}{4(2m+2+3)^2} \left\langle f_{2m+1}, f_{2m+1} \right\rangle_{L^2(\mathbb{S}^1)}
$$

and (31) holds also for odd indices. We use now polar coordinates $x = r \cos t$ and $y = r \sin t$. An orthonormal basis of spherical harmonics (restrictions of harmonic homogeneous polynomials $h_\kappa (x, y)$ of degree $\kappa \geq 1$ to the unit circle) is given by

$$
Y_{\kappa,0} (t) := \frac{1}{\sqrt{\pi}} \cos \kappa t \text{ and } Y_{\kappa,1} (t) = \frac{1}{\sqrt{\pi}} \sin \kappa t
$$

for $\kappa \geq 1$, and for $\kappa = 0$ we define $Y_{0,0} (t) = 1/\sqrt{2\pi}$. It is convenient to set $Y_{0,1} (t) := 0$. For every integer $\kappa$ the following identities hold:

$$
-4 \sin^2 t \cdot \cos \kappa t = \cos (\kappa + 2) t - 2 \cos \kappa t + \cos (\kappa - 2) t . \quad (33)
$$

$$
-4 \sin^2 t \cdot \sin \kappa t = \sin (\kappa + 2) t - 2 \sin \kappa t + \sin (\kappa - 2) t . \quad (34)
$$
For $k \geq 1$, replace $\kappa$ with $2k \geq 1$ in the identities (33) and (34) and multiply by $1/\sqrt{\pi}$. If $k \geq 2$, so $2k - 2 \geq 1$, we have

$$-4 \sin^2 t \cdot Y_{2k,s} = Y_{2k+2,s} - 2Y_{2k,s} + Y_{2k-2,s}.$$ (35)

If $k = 1$ and $s = 1$ then (35) holds using the convention that $Y_{0,1} = 0$. For $k = 1$ and $s = 0$ we obtain

$$-4 \sin^2 t \cdot Y_{2,0} = Y_{4,0} - 2Y_{2,0} + \sqrt{2}Y_{0,0}.$$ (36)

For the case $k = 0$, formula (33) leads to

$$-4 \sin^2 t \cdot Y_{0,0} = \sqrt{2}Y_{2,0} - 2Y_{0,0}.$$ (37)

By Lemma 13 it is enough to prove the inequality for $h \in H_{2m}$. Let us write

$$h(\cos t, \sin t) = \sum_{s=0}^{m} \sum_{k=0}^{m} c_{k,s} Y_{2k,s}(t)$$ (38)

with complex coefficients $c_{k,s}$ (using the convention that $Y_{0,1} = 0$). With (38) we arrive at

$$-4 \left\langle \sin^2 t \cdot h, h \right\rangle_{L^2(S^1)} = \sum_{k=0}^{m} \sum_{s=0}^{m} c_{k,s} ar{c}_{k_1,s_1} \left\langle -4 \sin^2 t \cdot Y_{2k,s}, Y_{2k_1,s_1} \right\rangle_{L^2(S^1)}.$$ (39)

Since $\left\langle \sin^2 t \cdot Y_{2k,s}(t), Y_{2k_1,s_1}(t) \right\rangle_{L^2(S^1)} = 0$ for $s \neq s_1$ we can express

$$-4 \left\langle \sin^2 t \cdot h, h \right\rangle_{L^2(S^1)} = \Sigma_0 + \Sigma_1$$ (40)

where

$$\Sigma_s = \sum_{k=0}^{m} \sum_{k_1=0}^{m} c_{k,s} \bar{c}_{k_1,s} \left\langle -4 \sin^2 t \cdot Y_{2k,s}, Y_{2k_1,s} \right\rangle_{L^2(S^1)}$$

for $s = 0, 1$. Let us write $c_0 = (c_{0,0}, \ldots, c_{m,0})$. We see from (35), (36) and (37) that the matrix representing the operator “multiplication by $-4 \sin^2 t$” on the space generated by $\{Y_{0,0}, \ldots, Y_{2m,0}\}$ (with respect to that basis) is $-2I_{(m+1) \times (m+1)} + A_{m+1}$,
where $I_{(m+1)\times (m+1)}$ is the identity matrix and $A_{m+1}$ the $(m + 1) \times (m + 1)$-matrix

$$A_{m+1} = \begin{pmatrix} 0 & \sqrt{2} & \cdots & \sqrt{2} \\ \sqrt{2} & 0 & \cdots & 1 \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & \ddots \\ & & & & 1 & 0 \end{pmatrix}.$$  

It now follows from (39) that

$$\Sigma_0 = -2c_0^tA_{m+1}\overline{c_0}.$$  

Similarly, recalling that $Y_{0,1} = 0$, we see from (35), which is valid for $k \geq 1$, that

$$\Sigma_1 = -2c_1^t\overline{c_1} + c_1^tB_{m \times m}\overline{c_1}$$  

for $c_1 = (c_{1,1}, \ldots, c_{m,1})^t$, where $B_{m \times m}$ is the matrix

$$B_{m \times m} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots \\ 1 & 0 \end{pmatrix}.$$  

Let $\mu^*$ be the maximal eigenvalue of $A_{m+1}$. Then

$$c_0^tA_{m+1}\overline{c_0} \leq \mu^*c_0^t\overline{c_0}$$  

for all $c_0 \in \mathbb{R}^{m+1}$. Since $B_{m \times m}$ is a submatrix of $A_{m+1}$ we infer that

$$c_1^tB_{m \times m}\overline{c_1} \leq \mu^*c_1^t\overline{c_1}$$  

for all $c_1 \in \mathbb{R}^m$. Thus

$$-4\left(\sin^2 t \cdot h, h\right)_{S^1} = \Sigma_0 + \Sigma_1 \leq (\mu^* - 2) (c_0^t\overline{c_0} + c_1^t\overline{c_1}).$$  

(41)

By Proposition 15,

$$P_{m+1} := (\lambda) \det (A_{m+1} - \lambda I) = 2T_{m+1} \left( -\frac{\lambda}{2} \right),$$

where $T_n$ is the Chebyshev polynomial of degree $n$. The zeros of $T_{m+1}$ are given by $x_k := \cos \left( \frac{2k-1}{m+1} \pi \right)$ for $k = 1, \ldots, m + 1$. Since $T_{m+1}(x) = \frac{1}{2}P_{m+1}(-2x)$ clearly
−2x_k with k = 1, . . . , m are the roots of the polynomial P_{m+1}. The largest zero μ of P_{m+1} is then

$$\mu^* = -2 \cos \left( \frac{2m + 1 \pi}{m + 1} \right).$$

Dividing the inequality (41) by −4 we obtain

$$\left\langle \sin^2 t \cdot h, h \right\rangle_{L^1(S^1)} \geq \frac{1}{2} \left( 1 + \cos \left( \frac{m + \frac{1}{2}}{m + 1} \pi \right) \right) \left( c_0 \overline{c_0} + c_1 \overline{c_1} \right).$$

Since \( \cos (x \pi) = -\cos (x - 1) \pi \) we obtain for \( x = \frac{m + \frac{1}{2}}{m + 1} \pi \)

$$\cos \left( \frac{m + \frac{1}{2}}{m + 1} \pi \right) = -\cos \left( \frac{1}{m + 1} \pi \right).$$

Now the identity \( \frac{1}{2} (1 - \cos y) = \sin^2 \frac{y}{2} \), together with Lemma 14 and \( \left\langle h, h \right\rangle_{L^2(S^1)} = (c_0 \overline{c_0} + c_1 \overline{c_1}) \), yield

$$\left\langle \sin^2 t \cdot h, h \right\rangle_{L^2(S^1)} \geq \left( c_0 \overline{c_0} + c_1 \overline{c_1} \right) \left( \sin^2 \frac{\pi}{4(m + 4)} \right) \geq \left\langle h, h \right\rangle_{L^2(S^1)} \cdot \left( \frac{\pi}{4(m + 4 + 2)} \right)^2.$$

Thus we have

$$\left\langle \sin^2 t \cdot h, h \right\rangle_{L^1(S^1)} \geq \left\langle h, h \right\rangle_{L^2(S^1)} C_{2m} \text{ with } C_{2m} = \frac{\pi^2}{4} \frac{1}{(2m + 3)^2}. \quad \blacksquare$$

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**Conflict of interests** The authors declare that they have no competing interests.

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