LOW REGULARITY THEORY FOR THE INVERSE FRACTIONAL CONDUCTIVITY PROBLEM

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ABSTRACT. We characterize partial data uniqueness for the inverse fractional conductivity problem with $H^{s,n/s}$ regularity assumptions in all dimensions. This extends the earlier results for $H^{2s} \cap H^s$ conductivities by Covì and the authors. We construct counterexamples to uniqueness on domains bounded in one direction whenever measurements are performed in disjoint open sets having positive distance to the domain. In particular, we provide counterexamples in the special cases $s \in (n/4, 1)$, $n = 2, 3$, missing in the literature due to the earlier regularity conditions. We also give a new proof of the uniqueness result which is not based on the Runge approximation property. Our work can be seen as a fractional counterpart of Haberman’s uniqueness theorem for the classical Calderón problem with $W^{1,n}$ conductivities when $n = 3, 4$. One motivation of this work is Brown’s conjecture that uniqueness for the classical Calderón problem holds for $W^{1,n}$ conductivities also in dimensions $n \geq 5$.

1. Introduction

We study the global inverse fractional conductivity problem in a low regularity setting which extends the earlier theory in [Cov21, CRZ22, RZ22b]. The considered setting resembles the classical inverse conductivity problem with $W^{1,n}$ conductivities [Hab15]. The inverse conductivity problem is also known as the Calderón problem due to the seminal work of Calderón [Cal06], first published in 1980. The classical Calderón problem forms the mathematical model of electrical impedance tomography [Uhl14]. The mathematical studies of the inverse conductivity problem date at least back to the work of Langer [Lan33]. The proof of uniqueness for the classical Calderón problem is based on the reduction to the inverse problem for the Schrödinger equation (called Liouville reduction), on the construction of complex geometric optics (CGO) solutions [SU87], and on boundary determination results [KV84]. The proof of global uniqueness for the fractional Calderón problem is based on similar ideas [Cov21, CRZ22, RZ22b] but the construction of CGO solutions can be replaced by the UCP of the fractional Laplacians [GSU20, RS20] and boundary determination results are replaced by exterior determination
results, using sequences of special solutions whose energies can be concentrated in the limit to any point in the exterior \([CRZ22, RZ22b]\).

Let \(V \subset \mathbb{R}^n\) be a measurable set. We say that \(\gamma \in L^\infty(V)\) is uniformly elliptic if \(a \leq \gamma \leq b\) a.e. in \(V\) for some \(0 < a < b < \infty\). We assume that all conductivities are uniformly elliptic throughout the article. Suppose that the electric potential \(u_f \in H^1(\Omega)\) solves the Dirichlet problem for the classical conductivity equation

\[
\text{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega,
\]
\[
u u_f \quad \text{on } \partial \Omega
\]

for a given boundary voltage \(f \in H^{1/2}(\partial \Omega)\). In the Calderón problem, one aims to recover the conductivity \(\gamma\) from the knowledge of the voltage/current measurements given in the form of the Dirichlet-to-Neumann (DN) map \(\Lambda_\gamma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega)\) mapping \(f \mapsto \gamma \partial_\nu u_f|_{\partial \Omega}\).

Let now \(s \in (0, 1)\) and consider the Dirichlet problem for the fractional conductivity equation

\[
\text{div}_s(\Theta_\gamma \nabla^s u) = 0 \quad \text{in } \Omega,
\]
\[
u u_f \quad \text{in } \Omega_c,
\]

where \(\Omega_c := \mathbb{R}^n \setminus \overline{\Omega}\) is the exterior of the domain \(\Omega\), \(\Theta_\gamma\) is an appropriate matrix depending on the conductivity \(\gamma\), and \(\nabla^s\), \(\text{div}_s\) are the fractional gradient, divergence, respectively (see \([CRZ22, Cov21, RZ22b]\)). The inverse problem for the fractional conductivity equation asks to recover the conductivity \(\gamma\) from a nonlocal analogue of the DN data, which in the case of Lipschitz domains maps as \(\Lambda^s_\gamma : H^s(\Omega_c) \to H_{-s}^1(\mathbb{R}^n)\) (see Lemma 2.8 for the general definition). We say \(u \in H^s(\mathbb{R}^n)\) is a (weak) solution of (1) if there holds

\[
B_\gamma(u, \phi) := \frac{C_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{\gamma^{1/2}(x)\gamma^{1/2}(y)}{|x - y|^{n+2s}} (u(x) - u(y))(\phi(x) - \phi(y)) \, dx \, dy = 0
\]

for all \(\phi \in C_\infty(\mathbb{R}^n)\) and \(u - f \in \dot{H}^s(\Omega)\). See for instance the survey \([RO16]\) on more general nonlocal equations of similar type, and the nonlocal vector calculus developed in \([DGLZ12]\). As \(s \uparrow 1\), then the fractional conductivity operator converges in the sense of distributions to the classical conductivity operator when applied to sufficiently regular functions (cf. \([Cov21, Lemma 4.2]\)). In the rest of the article, we will simply write \(\Lambda_\gamma\) instead of \(\Lambda^s_\gamma\) also in the fractional case. We also define \(m_\gamma := \gamma^{1/2} - 1\) and call it the background deviation of \(\gamma\).

The fractional Calderón problems have been studied intensively in the past few years \([Sal17]\), starting from the work of Ghosh, Salo, and Uhlmann \([GSU20]\). The research in this area has mainly focused on the recovery of additive perturbations of a priori known nonlocal operators from the exterior DN maps \([Cov21, CMRU22, LL22, RZ22b]\). More recently, there has been growing interest towards inverse problems for nonlocal variable coefficient operators \([Cov20, CRZ22, GU21]\) and for nonlocal geometric problems \([FGKU21]\). Inverse problems for time-fractional, space-fractional and spacetime-fractional equations are considered recently for example in
More references of non-local inverse problems can be found from the aforesaid works.

1.1. Regularity theory for the inverse conductivity problem and Brown’s conjecture. In this subsection, we briefly recall some of the important contributions to low regularity theory for the inverse conductivity problem. More references can be found from the cited works.

One important step in the uniqueness of classical Calderón problem is the determination of conductivities on the boundary of a domain. Kohn and Vogelius proved that the boundary jet of a smooth function can be uniquely determined from the local DN map [KV84]. They applied this result to show uniqueness for piecewise analytic conductivities when $n \geq 2$ [KV85]. Sylvester and Uhlmann proved uniqueness for $C^2$ conductivities when $n \geq 3$ in their celebrated work [SU87]. This result was improved to cover $C^1$ conductivities by Haberman and Tataru [HT13] and extended to Lipschitz conductivities by Caro and Rogers [CR16].

A low regularity boundary determination result for conductivities, which are continuous a.e. on $\partial \Omega$, was obtained by Brown (see [Bro01] for the precise formulation). The results in [Bro01] improved the earlier boundary determination results for $C^\infty$ conductivities [KV84], Lipschitz by Alessandrini [Ale90], $W^{1,p}$ ($p > n$) by Nachman [Nac96], and $C^0$ by Sylvester and Uhlmann [SU88]. We have proved a low regularity exterior uniqueness result in Theorem 3.2 for the fractional Calderón problem. This is analogous to the boundary determination result in [Bro01]. Furthermore, it generalizes the result in [CRZ22, Theorem 1.2].

It has been conjectured by Brown that the classical Calderón problem is uniquely solvable for uniformly elliptic conductivities in $W^{1,n}$ whenever $n \geq 3$:

**Conjecture 1.1** (Brown [BT03, p. 565]). Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $\gamma_1, \gamma_2 \in W^{1,n}(\Omega)$ are uniformly elliptic. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$.

**Remark 1.2.** In [BT03], Brown actually conjectures a slightly weaker statement: uniqueness for the inverse conductivity problem holds for the uniformly elliptic conductivities in $W^{1,p}(\Omega)$ when $p > n$. We state Conjecture 1.1 as it is due to Theorem 1.4 of Haberman. We also mention here that Uhlmann had earlier conjectured uniqueness for the Lipschitz conductivities and this conjecture was eventually solved in the work of Caro and Rogers [CR16].

Brown’s conjecture is still open when $n \geq 5$, to the best of our knowledge. We have proved in Theorem 1.5 that a fractional counterpart of Brown’s conjecture holds in all dimensions $n \geq 1$. Finally, we recall two important achievements in this research area. The theorem of Astala and Päivärinta considers the low regularity inverse conductivity problem when $n = 2$. See also [ALP16, NRT20] for other advances in the low regularity theory of the classical Calderón problem when $n = 2$. The theorem of Haberman shows that Brown’s conjecture holds when $n = 3, 4$. 

Theorem 1.3 (Astala–Päivärinta [AP06]). Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain. Suppose that $\gamma_1, \gamma_2 \in L^\infty(\Omega)$ are uniformly elliptic. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$.

Theorem 1.4 (Haberman [Hab15]). Let $n = 3, 4$ and $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Suppose that $\gamma_1, \gamma_2 \in W^{1,n}(\Omega)$ are uniformly elliptic. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$.

1.2. Main results. We present and discuss our main theorems next. Theorem 1.5 generalizes the main result in [CRZ22, Theorem 1.3] from $H^{2,s,n/2} \cap H^s$ regularity to $H^{s,n/s}$ regularity. These results are not contained into each other and there is a tradeoff between differentiability and integrability. We have developed theory for the conductivities having $H^{s,n/s}$ regular background deviations for two reasons:

- This regularity permits to construct new counterexamples to partial data problems when $n = 2, 3$ for the cases missing in the earlier literature due to integrability issues.
- This regularity assumption is analogous to the classical Calderón problem for $W^{1,n}$ conductivities (see the previous discussion about Brown’s conjecture).

We believe that the $H^{s,n/s}$ regularity assumption cannot be substantially improved without coming up with a new proof strategy (see also Question 1.10 and later discussions). In general, one might be able to develop theory for all conductivities having their background deviations in the interpolation spaces between $H^{2,s,n/2}$ and $H^{s,n/s}$ but this is out of our scope here. The fractional Liouville reduction from the fractional conductivity equation to the fractional Schrödinger equation breaks down with weaker Bessel regularity assumptions. Our theory in this article achieves the best Bessel regularity assumptions permitted by the current method of reducing the problem into an inverse problem for the fractional Schrödinger equation.

Our main theorem is as follows:

Theorem 1.5 (Global uniqueness). Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic with $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n)$. Suppose that $W \subset \Omega_e$ is a nonempty open set such that $\gamma_1, \gamma_2$ are continuous a.e. in $W$. Then $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C^\infty_c(W)$ if and only if $\gamma_1 = \gamma_2$ in $\mathbb{R}^n$.

The assumption $H^{s,n/s}(\mathbb{R}^n)$ is needed for the Liouville reduction from the fractional conductivity equation to the fractional Schrödinger equation. We use Bessel potential spaces $H^{s,p}$ to model conductivities rather than the fractional Sobolev spaces $W^{s,p}$ since the fractional Laplacian has in these spaces strong mapping properties, i.e. $(-\Delta)^s : H^{s,p}(\mathbb{R}^n) \to H^{t-2s,p}(\mathbb{R}^n)$ is bounded. We assume that $\gamma, i = 1, 2$, have representatives, which are continuous a.e. in $W$ due to the exterior determination method (see Theorem 3.2).

We state the existence of counterexamples for general disjoint measurements in our next theorem. This extends [RZ22a, Theorem 1.3] to cover all cases of $s \in (0, 1)$ when $n \geq 2$. The earlier theory developed in [CRZ22,
THE INVERSE FRACTIONAL CONDUCTIVITY PROBLEM

RZ22b, RZ22a] had integrability issues in the construction of counterexamples when $s \in (n/4, 1)$, $n = 2, 3$, as the background deviations had to be $H^{2s/n} \subseteq \mathbb{R}$. 

**Theorem 1.6** (Counterexamples). Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction, $0 < s < \min(1, n/2)$. For any nonempty open disjoint sets $W_1, W_2 \subset \Omega$ with $\text{dist}(W_1 \cup W_2, \Omega) > 0$ there exist two different conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$ such that $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$, $m_1, m_2 \in H^{s,n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, and $\Lambda_{\gamma_1}f|_{W_2} = \Lambda_{\gamma_2}f|_{W_2}$ for all $f \in C_c(W_1)$.

The proofs of Theorems 1.5 and 1.6 are given in Sections 3 and 4, respectively.

1.3. **On the contributions of this work.** The proofs of Theorems 1.5 and 1.6 require that we analyze the two problems at hand individually, contrary to the earlier works [CRZ22, RZ22b, RZ22a] where the regularity assumptions were such that the uniqueness and nonuniqueness results had almost shared proofs. In the construction of counterexamples, our argument still requires one to work under the assumption that the background deviations are $H^s$ regular but in the proof of uniqueness, one of the merits in our present work is to get rid of this assumption.

We can remove the assumption that the background deviations belong to $H^s$ by a unique continuation property (UCP) for the fractional Laplacians in $H^{s,p}$ proved recently by Kar and the authors in [KRZ22]. The proof of Theorem 1.7 is based on $L^2$ Carleman estimates of R"uland with additional $L^p$ estimates and a localization argument for the Caffarelli–Silvestre extensions [CS14, CS07, KRZ22, R"ul15]. One also needs analytic regularity theory for elliptic equations and a certain iteration argument for the higher order cases [CMR21, GSU20, KRZ22]. The following holds:

**Theorem 1.7** (UCP in Bessel potential spaces [KRZ22, Theorem 2.2]). Let $V \subset \mathbb{R}^n$ be a nonempty open set, $s \in \mathbb{R}_+ \setminus \mathbb{N}$, $t \in \mathbb{R}$ and $1 \leq p < \infty$. If $u \in H^{t,p}(\mathbb{R}^n)$ and $u|_V = (-\Delta)^t u|_V = 0$, then $u \equiv 0$.

We remark that the UCP in the range $1 \leq p \leq 2$ directly follows from the UCP for functions $u$ belonging to $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, and the embedding theorems in Bessel potential spaces (see [CMR21, Corollary 3.5]).

We improve the assumption from $H^{2s/n} \subseteq \mathbb{R}$ to $H^{s,n/s}$ using sharp multiplication results in Bessel potential spaces, the Runst–Sickel lemmas, among a few other useful estimates in Bessel potential spaces. In this analysis, we are required to use Sobolev multipliers in the fractional Liouville reduction rather than $L^p$ functions as in the previous literature. However, the analysis of the Calderon problem for the fractional Schrodinger equations with Sobolev multiplier perturbations, the so called singular potentials, was already established by Riland and Salo in [RS20]. Most notably one does not have to use the complex geometric optics solutions. The fractional Liouville reduction in low regularity shares many similarities with Brown’s work on the classical Calderon problem [Bro96], e.g. both deal with Sobolev multipliers, multiplication estimates and global equations. There are some
differences too, only in the fractional case one has to deal with singular integrals related to nonlocal operators. One also has to make certain weak* approximations and use many estimates for the fractional order Bessel potential spaces. Nevertheless, the Sobolev extensions of conductivities are not encountered in our considerations like in the construction of Brown [Bro96].

We also give a new strategy of proof for the uniqueness result in Section 3.4 (see the second proof of Theorem 1.5). The proof is based on a special relation of solutions for two conductivity equations with the same exterior condition when the DN maps agree (see Lemma 3.14), the UCP, and the Alessandri identity. This argument uses the UCP twice like the earlier proof but does not rely on the Runge approximation property in any step. We believe that this method of proof and Lemma 3.14 are of independent interest and may be useful in other problems.

Finally, the sharper regularity assumptions in the exterior determination method are based on a more direct proof than the one given in [CRZ22]. In particular, we avoid using the fractional Liouville reduction completely in the proof of Theorem 3.2. The proof of exterior determination is otherwise similar to [CRZ22].

1.4. Further discussion and open problems. Our first questions concerns what happens when one takes the limit $s \uparrow 1$:

**Question 1.8.** Is it possible to obtain new insight about the classical Calderón problem using the analysis of the inverse fractional conductivity problem?

There is one mathematical necessity in such considerations: One should use or approximate exterior conditions, which do not vanish near $\partial \Omega$. If the analysis is only based on exterior conditions which vanish on $\partial \Omega$ (like often is the case in different kind of nonlocal Calderón problems), in the limit $s \uparrow 1$ such solutions cannot contain useful information of $\gamma$ in $\Omega$. This can be seen as in the limit one obtains, at least heuristically speaking, a solution to the associated conductivity equation, which is 0 in $\Omega$ as the conductivity equation is local and the exterior condition vanishes on $\partial \Omega$. However, these approximations may get more informative and interesting when one uses exterior conditions, which do not vanish identically on $\partial \Omega$. Therefore, Question 1.8 is not only interesting itself but may additionally give new light to the understanding of the classical Calderón problem.

It might be possible to improve our assumptions in exterior determination (Theorem 3.2) by making more precise $L^p$ estimates and constructing suitable special solutions. It remains an open problem if it suffices to only use some Lebesgue point conditions for the conductivities and the Lebesgue differentiation theorem (or similar concepts). We believe that the positive answer to the following question could be true:

**Question 1.9.** Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ are uniformly elliptic and $W \subset \Omega$ is an open set. Does $\Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W$ for all $f \in C_c^\infty(W)$ imply that $\gamma_1 = \gamma_2$ a.e. in $W$?

Our two main results motivate the following two problems, which we have not been able to answer despite of some efforts:
Question 1.10 (Fractional Astala–Päivärinta theorem). Does the partial/full data uniqueness hold for the uniformly elliptic conductivities that only satisfy $\gamma \in L^\infty(\mathbb{R}^n)$? Can one remove the assumption that conductivities converge to the trivial conductivity at infinity by some other regularity assumptions?

The positive answer to the first question would be analogous to Theorem 1.3 of Astala and Päivärinta for the classical Calderón problem in two dimensions. The other question of a sufficient/necessary decay at infinity is open as long as one does not assume that $\gamma$ is from a generic class like the class of real analytic conductivities. Such genericity assumptions "localize" the problem into the exterior determination problem which can be uniquely solved. We further note that the low regularity theory of the inverse fractional conductivity problem in all dimensions, as developed in [CRZ22, Cov21, RZ22b] and here, resembles more the higher dimensional than the two dimensional classical Calderón problem. Only little is known about uniqueness or nonuniqueness for the classical Calderón problem with uniformly elliptic $L^\infty$ conductivities when $n \geq 3$, see e.g. [San19] and the references therein for a more detailed discussion.

Question 1.11. Are there counterexamples to uniqueness in the partial data inverse problem for all nonempty open sets $W_1, W_2 \subset \mathbb{R}^n$ such that $W_1 \cap W_2 = \emptyset$ and $\text{dist}(W_1 \cup W_2, \Omega) = 0$?

These counterexamples would be new and interesting with any regularity assumption. We do not know the answer to Question 1.11 even when $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded open set or $n = 1$. The complete understanding of Question 1.11 would imply the full characterization of uniqueness for the inverse fractional conductivity problem with partial data (in the given regularity) in combination with Theorems 1.5 and 1.6.

![Figure 1.1](image-url)

**Figure 1.1.** An example illustrating a prototypical situation to be considered in Question 1.11. The major difficulty in such geometrical settings is that one does not have enough room to use the method of mollification (see [RZ22a]) to guarantee the required regularity conditions for the background deviations $m_i := \gamma_i^{1/2} - 1$ for $i = 1, 2$.

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2. Preliminaries

2.1. Basic notation. Throughout this article the space of Schwartz functions and the space of tempered distributions will be denoted by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, respectively. The Fourier transform of $u \in \mathcal{S}(\mathbb{R}^n)$ is defined as

$$\mathcal{F}u(\xi) := \hat{u}(\xi) := \int_{\mathbb{R}^n} u(x)e^{-ix\cdot \xi} \, dx.$$  

Moreover, the Fourier transform acts as an isomorphism on the spaces $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and we denote the inverse of the Fourier transform by $\mathcal{F}^{-1}$ in each case. The Bessel potential of order $s \in \mathbb{R}$ is the Fourier multiplier $\langle D \rangle^s : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, that is

$$\langle D \rangle^s u := \mathcal{F}^{-1}(\langle |\xi|^s \rangle \hat{u}),$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. For any $s \in \mathbb{R}$ and $1 \leq p < \infty$, the Bessel potential space $H^{s,p}(\mathbb{R}^n)$ is defined by

$$H^{s,p}(\mathbb{R}^n) := \{ u \in \mathcal{S}(\mathbb{R}^n) \mid \langle D \rangle^s u \in L^p(\mathbb{R}^n) \},$$

which we endow with the norm $\|u\|_{H^{s,p}(\mathbb{R}^n)} := \|\langle D \rangle^s u\|_{L^p(\mathbb{R}^n)}$. If $\Omega \subset \mathbb{R}^n$, $F \subset \mathbb{R}^n$ are given open and closed sets, then we define the following local Bessel potential spaces:

$$\tilde{H}^{s,p}(\Omega) := \text{closure of } C_c^\infty(\Omega) \text{ in } H^{s,p}(\mathbb{R}^n),$$

$$H^{s,p}_F(\mathbb{R}^n) := \{ u \in H^{s,p}(\mathbb{R}^n) \mid \text{supp}(u) \subset F \}.$$ 

We see that $\tilde{H}^{s,p}(\Omega), H^{s,p}_F(\mathbb{R}^n)$ are closed subspaces of $H^{s,p}(\mathbb{R}^n)$. As customary, we omit the index $p$ from the above notations in the case $p = 2$.

If $u \in \mathcal{S}(\mathbb{R}^n)$ is a tempered distribution and $s \geq 0$, the fractional Laplacian of order $s$ of $u$ is the Fourier multiplier

$$(\mathcal{-}\Delta)^s u := \mathcal{F}^{-1}(\langle |\xi|^{2s} \rangle \hat{u}),$$

whenever the right hand side is well-defined. If $p \geq 1$ and $t \in \mathbb{R}$, the fractional Laplacian is a bounded linear operator $(-\Delta)^s : H^{t,p}(\mathbb{R}^n) \to H^{t-2s,p}(\mathbb{R}^n)$. In the special case $u \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0, 1)$, we have the identities (see e.g. [DNPV12, Section 3])

$$(\mathcal{-}\Delta)^s u(x) = C_{n,s} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy,$$

where $C_{n,s} := \frac{4\Gamma(n/2+s)}{\pi^{n/2}\Gamma(s)}$. Moreover, in the study of the inverse problem related to the fractional conductivity equation one property of the fractional Laplacian turns out to be essential, namely the UCP (see Theorem 1.7).

Another important property of the fractional Laplacian which allows to study the fractional conductivity equation on domains bounded in one direction (cf. [RZ22b, Definition 2.1]) is the Poincaré inequality, which says that for any $\Omega \subset \mathbb{R}^n$ bounded in one direction there exists a constant $C > 0$ such that

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C \|(-\Delta)^{s/2} u\|_{L^p(\mathbb{R}^n)}$$
for all \( u \in C_c^\infty(\Omega) \), where \( s \geq 0, p \geq 2 \) or \( s \geq 1, 1 < p < 2 \) (cf. [RZ22b, Theorem 2.2]). The Poincaré inequality in the range \( 0 < s < 1, 1 < p < 2 \), for unbounded domains is not present in the existing literature to the best of our knowledge.

For the rest of this section we fix \( s \in (0, 1) \). The fractional gradient of order \( s \) is the bounded linear operator \( \nabla^s : H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}; \mathbb{R}^n) \) given by (see [Cov21, DGLZ12, RZ22b])

\[
\nabla^s u(x, y) := \sqrt{\frac{C_{n,s}}{2}} \frac{u(x) - u(y)}{|x-y|^{n/2+s+1}}(x-y),
\]

and satisfies

\[
\|\nabla^s u\|_{L^2(\mathbb{R}^n)} = \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{H^s(\mathbb{R}^n)}
\]

for all \( u \in H^s(\mathbb{R}^n) \). The adjoint of \( \nabla^s \) is called fractional divergence of order \( s \) and denoted by \( \text{div}_s \). More concretely, the fractional divergence of order \( s \) is the bounded linear operator \( \text{div}_s : L^2(\mathbb{R}^{2n}; \mathbb{R}^n) \to H^{-s}(\mathbb{R}^n) \) satisfying

\[
\langle \text{div}_s u, v \rangle_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} = \langle u, \nabla^s v \rangle_{L^2(\mathbb{R}^{2n})}
\]

for all \( u \in L^2(\mathbb{R}^{2n}; \mathbb{R}^n), v \in H^s(\mathbb{R}^n) \). One can show that (see [RZ22b, Section 8])

\[
\|\text{div}_s(u)\|_{H^{-s}(\mathbb{R}^n)} \leq \|u\|_{L^2(\mathbb{R}^{2n})}
\]

for all \( u \in L^2(\mathbb{R}^{2n}; \mathbb{R}^n) \), and also \( (-\Delta)^s u = \text{div}_s(\nabla^s u) \) weakly for all \( u \in H^s(\mathbb{R}^n) \) (see [Cov21, Lemma 2.1]).

2.2. Triebel– Lizorkin spaces and the Runst– Sickel lemma. Before recalling several important results from harmonic analysis which will be used throughout this article, we introduce the Triebel–Lizorkin spaces \( F^s_{p,q}(\mathbb{R}^n) \) following the exposition in [BM01] or [Tri83, Tri92]. In the sequel, we will write for brevity \( B_r \) instead of \( B_r(0) \). Fix any \( \psi_0 \in C_c^\infty(\mathbb{R}^n) \) satisfying

\[
0 \leq \psi_0 \leq 1, \quad \psi_0(\xi) = 1 \quad \text{for} \quad |\xi| \leq 1 \quad \text{and} \quad \psi_0(\xi) = 0 \quad \text{for} \quad |\xi| \geq 2.
\]

Let \( \psi_j \in C_c^\infty(B_{2^{j+1}}), j \geq 1 \), be given by

\[
\psi_j(\xi) = \psi_0(\xi/2^j) - \psi_0(\xi/2^{j-1}).
\]

In this section, we write \( u_j := u * \phi_j = F^{-1}(\psi_j \hat{u}) \in \mathcal{S}(\mathbb{R}^n) \) for any \( u \in \mathcal{S}'(\mathbb{R}^n) \), where \( \phi_j = F^{-1}(\psi_j) \) (\( j \geq 1 \)) and one has the Littlewood– Paley decomposition

\[
\text{u} = \sum_{j \geq 0} u_j \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^n).
\]

For all \( s \in \mathbb{R} \) and \( 0 < p, q \leq \infty \), we set ([Tri83, Section 2.3.1])

\[
F^s_{p,q} = \{ u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{F^s_{p,q}} = \|2^{js} u_j(x)\|_{L^p(\mathbb{R}^n)} < \infty \}.
\]
Remark 2.1. (i) For $0 < p < \infty$ different choices of $\psi_0$ yield equivalent quasi-norms (see [Tri83, Section 2.3.5]), but for $p = \infty$, $0 < q < \infty$ this is in general wrong as shown in [Tri92, Section 2.3.2], and for $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ the Triebel-Lizorkin spaces are quasi-Banach spaces and Banach spaces if $p, q \geq 1$ (see [Tri83, Section 2.3.3]).

(ii) By the embedding $\ell^p \hookrightarrow \ell^q$ for $0 < q_1 \leq q_2 \leq \infty$, we have $F^s_{p,q_1} \subset F^s_{p,q_2}$ when $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q_1 \leq q_2 \leq \infty$.

(iii) We have the following identifications with equivalent norms ([BM01, Tri83]):

(I) $F^0_{p,2} = L^p(\mathbb{R}^n)$ for $1 < p < \infty$,

(II) $F^s_{p,2} = H^{s,p}(\mathbb{R}^n)$ for $1 < p < \infty$, $s \in \mathbb{R}$,

(III) $F^s_{p,p} = W^{s,p}(\mathbb{R}^n)$ for $1 \leq p < \infty$, $s \in (0, \infty) \setminus \mathbb{N}$,

(IV) $L^\infty(\mathbb{R}^n) \hookrightarrow F^0_{p,\infty,\infty}$ with
\[
\|u\|_{F^0_{p,\infty,\infty}} = \sup_{j \in \mathbb{N}_0, x \in \mathbb{R}^n} |u_j(x)| \leq C\|u\|_{L^\infty(\mathbb{R}^n)}.
\]

Proposition 2.2 (Runst-Sickel lemma I, [BM01, Lemma 5] or [RS96, p. 345]). Let $0 < s < \infty$, $1 < q < \infty$, $1 < p_1, p_2, r_1, r_2 \leq \infty$ satisfy

\[
0 < \frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2} < 1,
\]
then for all $f \in F^s_{p_1,q} \cap L^{r_1}(\mathbb{R}^n)$, $g \in F^s_{p_2,q} \cap L^{r_2}(\mathbb{R}^n)$ there holds
\[
\|fg\|_{F^s_{p,q}} \leq C(\|Mf(x)\|_2 \|f\|_{L^p(\mathbb{R}^n)} + \|Mg(x)\|_2 \|g\|_{L^p(\mathbb{R}^n)})
\]
and
\[
\|fg\|_{F^s_{p,q}} \leq C(\|f\|_{F^s_{p_1,q}} \|g\|_{L^{r_2}(\mathbb{R}^n)} + \|g\|_{F^s_{p_2,q}} \|f\|_{L^{r_1}(\mathbb{R}^n)}).
\]

Remark 2.3. Note that the Kato–Ponce inequality for $1 < p < \infty$ (see [GO14, Theorem 1], [GK96, Theorem 1.4]) can be seen as a special case of the estimate (2) by choosing $q = 2$ and using the fact that $F^s_{p,2} = H^{s,p}$ (cf. Remark 2.1(II)).

Proposition 2.4 (Runst-Sickel lemma II, [BM01, Corollary 3]). Let $0 < s < \infty$, $1 < q < \infty$. Then the following assertions hold:

(i) If $1 < p_1, p_2, r_1, r_2 < \infty$ satisfy
\[
0 < \frac{1}{p} := \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2} < 1,
\]
then the multiplication map
\[
(F^s_{p_1,q} \cap L^{r_1}(\mathbb{R}^n)) \times (F^s_{p_2,q} \cap L^{r_2}(\mathbb{R}^n)) \ni (f, g) \mapsto fg \in F^s_{p,q}
\]
is continuous.

(ii) If $1 < p < \infty$ and there holds
\[
\begin{cases}
  f^k \to f \text{ in } F^s_{p_1,q}, & \|f^k\|_{L^\infty(\mathbb{R}^n)} \leq C, \\
  g^k \to g \text{ in } F^s_{p_2,q}, & \|g^k\|_{L^\infty(\mathbb{R}^n)} \leq C
\end{cases}
\]
for some $C > 0$, then $f^k g^k \to fg$ in $F^s_{p,q}$.
(iii) Let $1 < p_1, r, p < \infty$ be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{r}$$

and there holds

$$\begin{cases}
  f^k \to f & \text{in } F_{p_1,q}^s \\
  g^k \to g & \text{in } F_{p,q}^s \cap L^r
\end{cases}$$

for some $C > 0$, then $f^kg^k \to fg$ in $F_{p,q}^s$.

### 2.3. Sobolev multipliers

We recall now the definition of Sobolev multipliers. We mainly follow the exposition in [MS09]. Applications to inverse problems can be found in [RS20, CMRU22, RZ22b]. We let $\mathcal{D}'(\mathbb{R}^n)$ stand for the space of distributions and by $E^s$ the space of distributions. For any $s \in \mathbb{R}$, we denote by $M(H^s \to H^{-s})$ the space of Sobolev multipliers of order $s$, which consists of all distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{s,-s} := \sup \{|\langle f, uv \rangle| : u,v \in C_c^{\infty}(\mathbb{R}^n), \|u\|_{H^s(\mathbb{R}^n)} = \|v\|_{H^{-s}(\mathbb{R}^n)} = 1\}$$

is finite. Moreover, we let $M_0(H^s \to H^{-s})$ be the closure of $C_c^{\infty}(\mathbb{R}^n)$ in $M(H^s \to H^{-s})$. According to the authors knowledge it is still an open problem to fully characterize $M_0(H^s \to H^{-s})$ and to show that $M_0(H^s \to H^{-s}) \subsetneq M(H^s \to H^{-s})$. A partial answer to this problem is given in [RS20, Remark 2.5], which states that $C_c^{\infty}(\mathbb{R}^n)$ is dense in $M(H^{s-\delta} \to H^{-s+\delta}) \cap \mathcal{D}'(\mathbb{R}^n)$ if the later space is endowed with the norm inherited from $M(H^s \to H^{-s})$. If $f \in M(H^s \to H^{-s})$ and $u,v \in C_c^{\infty}(\mathbb{R}^n)$ are both nonvanishing, we have the multiplier inequality

$$|\langle f, uv \rangle| \leq \|f\|_{s,-s} \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^{-s}(\mathbb{R}^n)}.$$

By density of $C_c^{\infty}(\mathbb{R}^n) \times C_c^{\infty}(\mathbb{R}^n)$ in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, there is a unique continuous extension $(u,v) \mapsto \langle f, uv \rangle$ for $(u,v) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$. More precisely, each $f \in M(H^s \to H^{-s})$ gives rise to a linear multiplication map $m_f: H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ defined by

$$\langle m_f(u), v \rangle := \lim_{i \to \infty} \langle f, u_i v_i \rangle$$

for all $(u,v) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$, where $(u_i, v_i) \in C_c^{\infty}(\mathbb{R}^n) \times C_c^{\infty}(\mathbb{R}^n)$ is any sequence in $H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ converging to $(u,v)$. We will just write $fu$ instead of $m_f(u)$.

### 2.4. Direct problem

**Lemma 2.5** (Definition of bilinear forms and conductivity matrix (cf. [RZ22b, Lemma 8.3])). Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < s < 1$, $\gamma \in L^\infty(\mathbb{R}^n)$ and define the conductivity matrix associated to $\gamma$ by

$$\Theta_\gamma: \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}, \quad \Theta_\gamma(x,y) := \gamma^{1/2}(x)\gamma^{1/2}(y)1_{n \times n}$$

for $x,y \in \mathbb{R}^n$. Then the map defined by

$$B_\gamma: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}, \quad B_\gamma(u,v) := \int_{\mathbb{R}^{2n}} \Theta_\gamma \nabla^s u \cdot \nabla^s v \, dx dy$$

is continuous bilinear form.
Remark 2.6. If no confusion can arise, we will drop the subscript $\gamma$ in the definition for the conductivity matrix $\Theta$.  

Definition 2.7 (Weak solutions). Let $\Omega \subset \mathbb{R}^n$ be an open set, $0 < s < 1$ and $\gamma \in L^\infty(\mathbb{R}^n)$ with conductivity matrix $\Theta : \mathbb{R}^{2n} \to \mathbb{R}^{n \times n}$. If $f \in H^s(\mathbb{R}^n)$ and $F \in (\tilde{H}^s(\Omega))^*$, then we say that $u \in H^s(\mathbb{R}^n)$ is a weak solution to the fractional conductivity equation
\[
\text{div}_s(\Theta \nabla^s u) = F \quad \text{in} \quad \Omega,
\]
\[
u = f \quad \text{in} \quad \Omega_e
\]
if there holds
\[
B_\gamma(u, \phi) = F(\phi) \quad \text{and} \quad u - f \in \tilde{H}^s(\Omega)
\]
for all $\phi \in \tilde{H}^s(\Omega)$.  

Lemma 2.8 (Well-posedness and DN maps (cf. [RZ22b, Lemma 8.10])). Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < 1$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with conductivity matrix $\Theta$ satisfies $\gamma \geq \gamma_0 > 0$. Then the following assertions hold:

(i) For all $f \in X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ there is a unique weak solution $u_f \in H^s(\mathbb{R}^n)$ of the fractional conductivity equation
\[
\text{div}_s(\Theta \nabla^s u) = 0 \quad \text{in} \quad \Omega,
\]
\[
u = f \quad \text{in} \quad \Omega_e.
\]

(ii) The exterior DN map $\Lambda_\gamma : X \to X^*$ given by
\[
\langle \Lambda_\gamma f, g \rangle := B_\gamma(u_f, g),
\]
where $u_f \in H^s(\mathbb{R}^n)$ is the unique solution to the fractional conductivity equation with exterior value $f$, is a well-defined bounded linear map.  

Remark 2.9. We note that the additional condition $m \in H^{2s, \frac{n}{2}}(\mathbb{R}^n)$ in [RZ22b, Lemma 8.10] were only needed to establish the related assertions for the fractional Schrödinger equation.

3. Low regularity uniqueness for the inverse fractional conductivity problem

3.1. Low regularity exterior determination. We need to establish an exterior determination result for the fractional Calderón problem. We prove a generalization of [CRZ22, Theorem 1.2] without any Sobolev regularity for the conductivities. We will be very brief in the proof since the proof of [CRZ22, Theorem 1.2] holds almost identically. The only concern is to check that the required elliptic energy estimates (see [CRZ22, Corollary 5.4]) are still valid for the related exterior value problems. We also remark that one does not have to reduce this estimate to the similar property of Schrödinger equations (via the Liouville reduction) as done earlier in [CRZ22].

Lemma 3.1. Let $0 < s < 1$. Suppose that $\Omega \subset \mathbb{R}^n$ is an open set which is bounded in one direction, $W \subset \Omega_e$ an open set with finite measure and positive distance from $\Omega$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with conductivity matrix
\( \Theta \) satisfies \( \gamma(x) \geq \gamma_0 > 0 \). Then for any \( f \in C_c^\infty(W) \) the associated unique solution \( u_f \in H^s(\mathbb{R}^n) \) of

\[
\text{div}_x(\Theta \nabla u) = 0 \quad \text{in} \quad \Omega,
\]

\[
u = f \quad \text{in} \quad \Omega_e
\]

satisfies the estimate

\[
\|u_f - f\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)}
\]

for some \( C > 0 \) depending only on \( n, s, \gamma_0, \|\gamma\|_{L^\infty(\Omega \cup W)}, |W| \), the distance between \( W \) and \( \Omega \), and the Poincaré constant of \( \Omega \).

**Proof.** By [RZ22b, Lemma 8.10] the unique solution \( u_f \in H^s(\mathbb{R}^n) \) to (4) satisfies the estimate

\[
\|u_f - f\|_{H^s(\mathbb{R}^n)} \leq C\|B_\gamma(f, \cdot)\|_{(\tilde{H}^s(\Omega))'}.
\]

Therefore, by the definition of \( B_\gamma \), supp\( (f) \subset \Omega_e \) and the support argument carried out in detail in [CRZ22, Proof of Lemma 5.3] we have

\[
|B_\gamma(f, \phi)| = \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \gamma^{1/2}(x)\gamma^{1/2}(y) \frac{|f(x) - f(y)|}{|x - y|^{n+2s}} dxdy
\]

\[
= C_{n,s} \int_{W \times \Omega} \gamma^{1/2}(x)\gamma^{1/2}(y) \frac{f(y)\phi(x)}{|x - y|^{n+2s}} dxdy
\]

\[
= C_{n,s} \int_{W \times \Omega} \frac{\gamma^{1/2}(y)f(y)\gamma^{1/2}(x)\phi(x)}{|x - y|^{n+2s}} dxdy.
\]

for all \( \phi \in \tilde{H}^s(\Omega) \).

Following [CRZ22, Proof of Lemma 5.3], we obtain by the Cauchy–Schwartz and the Minkowski inequality the estimate

\[
|B_\gamma(f, \phi)| \leq C_{n,s} \sqrt{\frac{\omega_n}{(n + 4s)^{n+4s}}} |W|^{1/2} \gamma^{1/2}f\|_{L^2(\Omega)} \gamma^{1/2}\phi\|_{L^2(\Omega)}
\]

\[
\leq C_{n,s} \gamma\|\gamma\|_{L^\infty(\Omega)}^{1/2} \gamma\|\gamma\|_{L^\infty(\Omega)}^{1/2} \frac{\omega_n}{(n + 4s)^{n+4s}} |W|^{1/2} \|f\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)},
\]

where \( \omega_n \) denotes the Lebesgue measure of the unit ball. This shows

\[
\|u_f - f\|_{H^s(\mathbb{R}^n)} \leq C C_{n,s} \gamma\|\gamma\|_{L^\infty(\Omega)}^{1/2} \gamma\|\gamma\|_{L^\infty(\Omega)}^{1/2} \frac{\omega_n}{(n + 4s)^{n+4s}} |W|^{1/2} \|f\|_{L^2(\Omega)}.
\]

\( \square \)

**Theorem 3.2** (Exterior determination). Let \( \Omega \subset \mathbb{R}^n \) be an open set which is bounded in one direction and \( 0 < s < 1 \). Assume that \( \gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n) \) satisfy \( \gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0 \). Suppose that \( W \subset \Omega_e \) is a nonempty open set such that \( \gamma_1, \gamma_2 \) are continuous a.e. in \( W \). If \( \Lambda_{\gamma_1}f|_W = \Lambda_{\gamma_2}f|_W \) for all \( f \in C_c^\infty(W) \), then \( \gamma_1 = \gamma_2 \) a.e. in \( W \).

**Proof.** First choose a set \( N \subset W \) such that \( |N| = 0 \) and \( \gamma_1, \gamma_2 \) are continuous on \( V := W \setminus N \). Let \( x_0 \in V \). Next note that [CRZ22, Lemma 5.5] holds for any \( 0 < s < 1 \) and therefore by [CRZ22, Remark 5.6] there exists a sequence \( (\phi_N)_{N \in \mathbb{N}} \subset C_c^\infty(W) \) such that

(i) \( \|\phi_N\|_{L^2(\mathbb{R}^n)}^2 + \|(-\Delta)^{s/2}\phi_N\|_{L^2(\mathbb{R}^n)}^2 = 1 \) for all \( N \in \mathbb{N} \),
By the product rule for the fractional gradient

\[ \nabla^s(uv)(x, y) = v(y)\nabla^s u(x, y) + u(x)\nabla^s v(x, y) \]

for all \( u, v \in H^s(\mathbb{R}^n) \) and (ii) one obtains (cf. [CRZ22, eq. (18)])

\[
\limsup_{N \to \infty} \int_{\mathbb{R}^{2n}} \left( \gamma_i^{1/2}(y) - \gamma_i^{1/2}(x_0) \right) g(x) |\nabla^s \phi_N|^2 \, dx \, dy \\
\leq C \|g\|_{L_\infty(\mathbb{R}^n)} \|\gamma_i^{1/2} - \gamma_i^{1/2}(x_0)\|_{L_\infty(Q_{1/M}(x_0))}
\]

for any bounded function \( g \in L_\infty(\mathbb{R}^n) \), \( M \in \mathbb{N} \) and \( i = 1, 2 \). Here the cube \( Q_r(x) \) is given for all \( x \in \mathbb{R}^n, r > 0 \) by

\[ Q_r(x) := \{ y \in \mathbb{R}^n; |y_j - x_j| < r \text{ for all } j = 1, \ldots, n \}. \]

By the assumptions the conductivities \( \gamma_1, \gamma_2 \) are continuous at \( x_0 \) and therefore passing to the limit \( M \to \infty \) gives

\[
\lim_{N \to \infty} \int_{\mathbb{R}^{2n}} \left( \gamma_i^{1/2}(y) - \gamma_i^{1/2}(x_0) \right) g(x) |\nabla^s \phi_N|^2 \, dx \, dy = 0
\]

for any \( g \in L_\infty(\mathbb{R}^n) \) and \( i = 1, 2 \). This ensures

\[
\gamma_i(x_0) = \lim_{N \to \infty} \int_{\mathbb{R}^{2n}} \gamma_i^{1/2}(x) (\gamma_i^{1/2}(y) - \gamma_i^{1/2}(x_0)) |\nabla^s \phi_N|^2 \, dx \, dy \\
+ \gamma_i^{1/2}(x_0) \lim_{N \to \infty} \int_{\mathbb{R}^{2n}} (\gamma_i^{1/2}(x) - \gamma_i^{1/2}(x_0)) |\nabla^s \phi_N|^2 \, dx \, dy \\
+ \gamma_i(x_0) \lim_{N \to \infty} \int_{\mathbb{R}^{2n}} |\nabla^s \phi_N|^2 \, dx \, dy
\]

\[
= \lim_{N \to \infty} \int_{\mathbb{R}^{2n}} \gamma_i^{1/2}(x) \gamma_i^{1/2}(y) |\nabla^s \phi_N|^2 \, dx \, dy \\
= \lim_{N \to \infty} \langle \Theta_{\gamma_i} \nabla^s \phi_N, \nabla^s \phi_N \rangle_{L_2(\mathbb{R}^n)} \\
= \lim_{N \to \infty} E_{\gamma_i}(\phi_N)
\]

for \( i = 1, 2 \). In the first equality sign, we used the equation (5) for \( g = \gamma_i \) and \( g = 1 \) as well as \( \lim_{N \to \infty} |\nabla^s \phi_N|^2_{L_2(\mathbb{R}^n)} = 1 \), which follows from the properties (i) and (ii).

As in [CRZ22, eq. (20)] we have

\[ E_{\gamma_i}(u_N^i) = E_{\gamma_i}(u_N^i - \phi_N) + 2 \langle \Theta_{\gamma_i} \nabla^s \phi_N, \nabla^s (u_N^i - \phi_N) \rangle_{L_2(\mathbb{R}^{2n})} + E_{\gamma_i}(\phi_N) \]

for all \( N \in \mathbb{N}, i = 1, 2 \), where \( u_N^i \in H^s(\mathbb{R}^n) \) is the unique solution to the homogeneous fractional conductivity equation with conductivity \( \gamma_i \) and exterior value \( \phi_N \). By Lemma 3.1 and (ii) we deduce \( \|u_N - \phi_N\|_{H^s(\mathbb{R}^n)} \to 0 \) as \( N \to \infty \), which in turn implies

\[ \lim_{N \to \infty} E_{\gamma_i}(u_N^i) = \lim_{N \to \infty} E_{\gamma_i}(\phi_N). \]

Therefore, by (6) there holds

\[ \gamma_i(x_0) = \lim_{N \to \infty} E_{\gamma_i}(u_N^i) = \lim_{N \to \infty} B_{\gamma_i}(u_N^i, \phi_N) = \langle A_{\gamma_i} \phi_N, \phi_N \rangle \]
for all $x_0 \in V$, where in the second and third equality we used (ii) of Lemma 2.8. This immediately proves the assertion of Theorem 3.2. □

**Remark 3.3.** If the assumptions of Theorem 3.2 hold for $\gamma_1$ and $\gamma_2$, then the proof shows by following the same strategy as in [CRZ22] the reconstruction formula

$$\gamma_i(x_0) = \lim_{N \to \infty} \langle \Lambda_i, \phi_N, \phi_N \rangle$$

for a.e. $x_0 \in W$ and the stability estimate

$$\|\gamma_1 - \gamma_2\|_{L^\infty(W)} \lesssim \|\Lambda_1 - \Lambda_2\|_{X \to X^*}.$$  

We note that the stability estimate could be also formulated with partial data.

### 3.2. Low regularity Liouville reduction.

We first prove basic estimates for products and convolutions of Bessel potential functions. These are needed to assure that the potentials associated with the studied conductivities are in the right space of Sobolev multipliers (Lemma 3.7). This is then used in the solution of the inverse problem by making the Liouville reduction to the fractional Schrödinger equation.

**Lemma 3.4** (Multiplication estimates). Let $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with background deviation $m$ satisfies $\gamma(x) \geq \gamma_0 > 0$ and $m \in H^{s,n/s}(\mathbb{R}^n)$. If $u, v \in H^s(\mathbb{R}^n)$, then there holds $uv \in H^{s,n/s}(\mathbb{R}^n)$, $mu \in H^s(\mathbb{R}^n)$ with

$$\|uv\|_{H^{s,n/s}(\mathbb{R}^n)} \leq C \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}$$

and

$$\|mu\|_{H^s(\mathbb{R}^n)} \leq C \left( \|m\|_{L^\infty(\mathbb{R}^n)} + \|m\|_{H^{s,n/s}(\mathbb{R}^n)} \right) \|u\|_{H^s(\mathbb{R}^n)},$$

where $C = C(n, s) > 0$.

**Proof.** We first prove the estimate (7). Let $u, v \in H^s(\mathbb{R}^n)$ and observe that by the Sobolev embedding we have $u, v \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$. Next note that

$$\frac{1}{2} + \frac{n - 2s}{2n} = \frac{n - s}{n}$$

and hence the exponents $p = \frac{n}{n-2s}$, $p_1 = p_2 = 2$, $r_1 = r_2 = \frac{2n}{n-2s}$ satisfy the assumptions in Proposition 2.2. Now if we choose $q = 2$ in Proposition 2.2, use the identification (II) of Remark 2.1 and apply the Sobolev embedding we deduce that $uv \in H^{s,n/s}(\mathbb{R}^n)$ satisfying

$$\|uv\|_{H^{s,n/s}(\mathbb{R}^n)} \leq C \left( \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} + \|v\|_{H^s(\mathbb{R}^n)} \|u\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} \right)$$

$$\leq C \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)}.$$

Next observe that the background deviation $m$ satisfies $m \in H^{s,n/s}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and there holds

$$\frac{s}{n} + \frac{n - 2s}{2n} = \frac{1}{2}.$$  

Therefore the exponents $p = 2$, $p_1 = n/s$, $p_2 = 2$, $r_1 = \infty$, $r_2 = \frac{2n}{n-2s}$ fulfill the conditions in Proposition 2.2. Now again applying Proposition 2.2 with
Lemma 3.5 (Convergence of mollifications). Let $0 < s < n/2$ and assume that $(v_k)_{k \in \mathbb{N}} \subset H^s(\mathbb{R}^n)$ satisfies $v_k \to v$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$ for some $v \in H^s(\mathbb{R}^n)$. If $m \in H^{s,n/s}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then there holds $m_k v_k \to mv$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$, where $m_k := \rho_k * m$ for some decreasing sequence $\epsilon_k \to 0$ and a sequence of standard mollifiers $(\rho_k)_{\epsilon > 0} \subset C_c(\mathbb{R}^n)$.

Proof. By properties of mollification and $\int_{\mathbb{R}^n} \rho \, dx = 1$, $\rho_k \geq 0$ for all $\epsilon > 0$, we have

$$\|m_k\|_{L^\infty(\mathbb{R}^n)} \leq \|\rho_k\|_{L^1(\mathbb{R}^n)} \|m\|_{L^\infty(\mathbb{R}^n)} \leq \|m\|_{L^\infty(\mathbb{R}^n)} < \infty.$$  

By the Sobolev embedding there holds $v_k \to v$ in $L^{\frac{2n}{n-2}}(\mathbb{R}^n)$ as $k \to \infty$. Since (9) holds, all conditions in assertion (iii) of Proposition 2.4 with $q = 2$ are satisfied and by using statement (II) of Remark 2.1 we can conclude that $m_k v_k \to mv$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. □

Corollary 3.6 (Exterior conditions). Let $\Omega \subset \mathbb{R}^n$ be an open set and $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with background deviation $m$ satisfies $\gamma(x) \geq \gamma_0 > 0$ and $m \in H^{s,n/s}(\mathbb{R}^n)$. If $u \in \tilde{H}^s(\Omega)$, then there holds $\gamma^{1/2} u, \gamma^{-1/2} u \in \tilde{H}^s(\Omega)$.

Proof. Let $(\rho_k)_{\epsilon > 0} \subset C_c^\infty(\mathbb{R}^n)$ be a sequence of standard mollifiers and choose a sequence $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ in $H^s(\mathbb{R}^n)$ as $n \to \infty$. We have $m_n := \rho_k * m \in C_c^\infty(\mathbb{R}^n)$ and therefore by Lemma 3.5 the sequence $(m_n u_k)_{k \in \mathbb{N}} \subset C_c^\infty(\Omega)$ satisfies $m_n u_k \to mu$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. Hence, there holds $\gamma^{1/2} u = mu + u \in \tilde{H}^s(\Omega)$. Next note that we can write

$$\frac{1}{\gamma^{1/2}} = 1 - \frac{m}{m + 1}$$

and set $\Gamma_0 := \min(0, \gamma_0^{1/2} - 1)$. Let $\Gamma \in C_b^1(\mathbb{R})$ satisfy $\Gamma(t) = \frac{t}{t+1}$ for $t \geq \Gamma_0$. By [AF92, p. 156] and $m \in H^{s,n/s}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we deduce $\Gamma(m) \in H^{s,n/s}(\mathbb{R}^n)$. Since $m \geq \gamma_0^{1/2} - 1$ it follows that $\frac{m}{m+1} \in H^{s,n/s}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. We again obtain that $\frac{m}{m+1} u \in \tilde{H}^s(\Omega)$ and therefore $\gamma^{-1/2} u \in \tilde{H}^s(\Omega)$. □

Lemma 3.7 (Sobolev multiplier property). Let $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with background deviation $m$ satisfies $\gamma(x) \geq \gamma_0 > 0$ and $m \in H^{s,n/s}(\mathbb{R}^n)$. Then the distribution $q_\gamma = -\frac{(-\Delta)^{s/2} m}{\gamma^{1/2}}$, defined by

$$\langle q_\gamma, \phi \rangle := -\langle (-\Delta)^{s/2} m, (-\Delta)^{s/2} (\gamma^{-1/2} \phi) \rangle_{L^2(\mathbb{R}^n)}$$

for all $\phi \in C_c^\infty(\mathbb{R}^n)$, belongs to $M(H^s \to H^{-s})$. Moreover, for all $u, \phi \in H^s(\mathbb{R}^n)$, we have

$$\langle q_\gamma u, \phi \rangle = -\langle (-\Delta)^{s/2} m, (-\Delta)^{s/2} (\gamma^{-1/2} u \phi) \rangle_{L^2(\mathbb{R}^n)}$$
In the rest of this article, we set $q_\gamma := -\frac{(\Delta)^{s/2}}{\gamma}$ to belong to $M(H^s \to H^{-s})$ and refer to it as a potential. If no confusion can arise, we will drop the subscript $\gamma$.

**Proof.** Let $u, \phi \in H^s(\mathbb{R}^n)$ and note that by Corollary 3.6 we have $\gamma^{-1/2} \phi \in H^s(\mathbb{R}^n)$. Hence by Lemma 3.4, it follows that $uv/\gamma^{1/2} \in H^{\frac{m}{m+1}}(\mathbb{R}^n)$ and the mapping properties of the fractional Laplacian show $(-\Delta)^{s/2}(uv/\gamma^{1/2}) \in L^{\frac{m}{m+1}}(\mathbb{R}^n)$. Therefore, we obtain by Hölder’s inequality with

$$\frac{s}{n} + \frac{n-s}{n} = 1$$

the estimate

$$\left| \langle (-\Delta)^{s/2}m, (-\Delta)^{s/2}(\gamma^{-1/2}u\phi) \rangle \right|_{L^2(\mathbb{R}^n)}$$

$$\leq \left\| (-\Delta)^{s/2}m \right\|_{L^{n/n+s}(\mathbb{R}^n)} \left\| (-\Delta)^{s/2}(\gamma^{-1/2}u\phi) \right\|_{L^{n/n+s}(\mathbb{R}^n)}$$

$$\leq \left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)} \left\| (-\Delta)^{s/2}(\gamma^{-1/2}u\phi) \right\|_{L^{n/n+s}(\mathbb{R}^n)}$$

$$\leq \left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)}\left( \| \frac{m}{m+1} u \phi \|_{H^{s,n/s}(\mathbb{R}^n)} + \left\| \frac{m}{m+1} u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)} \right).$$

In the last inequality we used the decompositions $\gamma^{-1/2} = 1 - \frac{m}{m+1}$ with $\frac{m}{m+1} \in H^{s,n/s}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ (see the proof of Corollary 3.6).

Using (7) we can estimate the expression in brackets in the equation (11), (8) as

$$\left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)} + \left\| \frac{m}{m+1} u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)}$$

$$\leq C \left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)} + \left\| \frac{m}{m+1} u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)}$$

$$\leq C \left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)}$$

Next observe that by [AF92, p. 156] there holds

$$\left\| \frac{m}{m+1} \right\|_{H^{s,n/s}(\mathbb{R}^n)} \leq C \left\| m \right\|_{H^{s,n/s}(\mathbb{R}^n)}$$

for some $C > 0$ and $0 < s < 1$. This implies

$$\left| \langle (-\Delta)^{s/2}m, (-\Delta)^{s/2}(\gamma^{-1/2}u\phi) \rangle \right|_{L^2(\mathbb{R}^n)}$$

$$\leq C \left\| (-\Delta)^{s/2}m \right\|_{L^{\infty}(\mathbb{R}^n)} + \left\| m \right\|_{H^{s,n/s}(\mathbb{R}^n)} \left\| m \right\|_{H^{s,n/s}(\mathbb{R}^n)}$$

$$\cdot \left\| u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)} \left\| \frac{m}{m+1} u \phi \right\|_{H^{s,n/s}(\mathbb{R}^n)}.$$
Lemma 3.9 (Liouville reduction). Let \(0 < s < \min(1, n/2)\). Assume that \(\gamma \in L^\infty(\mathbb{R}^n)\) with conductivity matrix \(\Theta\) and background deviation \(m\) satisfies \(\gamma(x) \geq \gamma_0 > 0\) and \(m \in H^{s,n/s}(\mathbb{R}^n)\). Then there holds
\[
\langle (\Theta \nabla^s u, \nabla^s \phi)_{L^2(\mathbb{R}^n)} \rangle = \langle (\nabla^{s/2}(\gamma^{1/2} u), (-\Delta)^{s/2}(\gamma^{1/2} \phi))_{L^2(\mathbb{R}^n)} \rangle
\]
for all \(u, \phi \in H^s(\mathbb{R}^n)\).

**Proof.** We first prove identity (13) for Schwartz functions and then extend it to functions in \(H^s(\mathbb{R}^n)\) by an approximation argument.

**Step 1:** Let \(u, \phi \in \mathcal{S}(\mathbb{R}^n)\) and define \(\gamma^{1/2} := \gamma^{1/2} \ast \rho \in C_0^\infty(\mathbb{R}^n), m_\epsilon := m \ast \rho \in C_0^\infty(\mathbb{R}^n) \cap H^{s,n/s}(\mathbb{R}^n)\). Here \((\rho)_{\epsilon > 0} \subset C_0^\infty(\mathbb{R}^n)\) is a sequence of standard mollifiers and \(C_0^\infty(\mathbb{R}^n)\) denotes the space of smooth functions with bounded derivatives. We have \(m_\epsilon = \gamma^{1/2}_\epsilon - 1\). From [RZ22b, Proof of Theorem 8.6], we know
\[
\lim_{\epsilon \to 0} \langle (\Theta_\epsilon \nabla^s u, \nabla^s \phi)_{L^2(\mathbb{R}^n)} \rangle = \langle (\Theta \nabla^s u, \nabla^s \phi)_{L^2(\mathbb{R}^n)} \rangle
\]
for all \(u, \phi \in H^s(\mathbb{R}^n)\), where \(\Theta_\epsilon = \gamma^{1/2}_\epsilon(x)\gamma^{1/2}_\epsilon(y)1_{n \times n}\). Using [RZ22b, Remark 8.9], we deduce that \(m_\epsilon \in C^t(\mathbb{R}^n) \cap L_n(\mathbb{R}^n)\) for all \(t \in \mathbb{R}_+ \setminus \mathbb{N}\) and by [Sil05, Proposition 2.1.4] there holds
\[
(-\Delta)^s m_\epsilon(x) = -\frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{\delta m_\epsilon(x, y)}{|y|^{n+2s}} dy
\]
for all \(x \in \mathbb{R}^n\). Here \(C^t(\mathbb{R}^n), t \in \mathbb{R}_+ \setminus \mathbb{N}\), denotes the space of all Hölder continuous functions and \(L_n(\mathbb{R}^n)\) stands for the space of all functions \(f \in L^1_{loc}(\mathbb{R}^n)\) such that
\[
\int_{\mathbb{R}^n} \frac{|f(x)|}{1 + |x|^{n+2s}} dx < \infty.
\]
Thus, the proof of [RZ22b, Theorem 8.6] shows
\[
\langle (\Theta \nabla^s u, \nabla^s \phi)_{L^2(\mathbb{R}^n)} \rangle = \lim_{\epsilon \to 0} \langle (\nabla^{s/2}(\gamma^{1/2}_\epsilon u), (-\Delta)^{s/2}(\gamma^{1/2}_\epsilon \phi))_{L^2(\mathbb{R}^n)} \rangle
\]
for all \(u, \phi \in \mathcal{S}(\mathbb{R}^n)\).

Since \(u, \phi \in \mathcal{S}(\mathbb{R}^n)\) and \(\gamma^{1/2}_\epsilon \in C_0^\infty(\mathbb{R}^n)\), we have \(\gamma^{1/2} u \phi \in \mathcal{S}(\mathbb{R}^n)\). Therefore, by the fact that \(m_\epsilon \in H^{t,n/s}(\mathbb{R}^n)\) for all \(t \in \mathbb{R}\), we have by approximation
\[
\langle (-\Delta)^s m_\epsilon, (\gamma^{1/2}_\epsilon u \phi)_{L^2(\mathbb{R}^n)} \rangle = \langle (-\Delta)^s/2 m_\epsilon, (-\Delta)^{s/2}(\gamma^{1/2}_\epsilon u \phi)_{L^2(\mathbb{R}^n)} \rangle
\]
Using \(m_\epsilon = \gamma^{1/2}_\epsilon - 1\), we can decompose the last expression as
\[
\langle (-\Delta)^s m_\epsilon, (\gamma^{1/2}_\epsilon u \phi)_{L^2(\mathbb{R}^n)} \rangle = \langle (-\Delta)^{s/2} m_\epsilon, (-\Delta)^{s/2}(m_\epsilon u \phi)_{L^2(\mathbb{R}^n)} \rangle
\]
for all \(u, \phi \in \mathcal{S}(\mathbb{R}^n)\). By our regularity assumptions, we know \((-\Delta)^{s/2} m_\epsilon \to (-\Delta)^{s/2} m\) in \(L^{n/s}(\mathbb{R}^n)\). Thus, by the mapping properties of the fractional Laplacian and Hölder’s inequality, we have
\[
\langle (-\Delta)^{s/2} m_\epsilon, (-\Delta)^{s/2}(u \phi) \rangle_{L^2(\mathbb{R}^n)} \to \langle (-\Delta)^{s/2} m, (-\Delta)^{s/2}(u \phi) \rangle_{L^2(\mathbb{R}^n)}
\]
as $\epsilon \to 0$. By Lemma 3.4, we can estimate
\[
\|m_\epsilon u\phi - m\phi\|_{H^{s/2}(\mathbb{R}^n)} \leq C\|m_\epsilon u - mu\|_{H^{s}(\mathbb{R}^n)}\|\phi\|_{H^{s}(\mathbb{R}^n)}
\]
and since Lemma 3.5 implies $m_\epsilon u \to mu$ in $H^s(\mathbb{R}^n)$ as $\epsilon \to 0$ we deduce
\[
m_\epsilon u\phi \to m\phi \text{ in } H^{s/2}(\mathbb{R}^n).
\]
By the mapping properties of the fractional Laplacian, we have
\[
(-\Delta)^{s/2}(m_\epsilon u\phi) \to (-\Delta)^{s/2}(mu\phi) \quad \text{in } L^{\frac{n}{n-s}}(\mathbb{R}^n)
\]
as $\epsilon \to 0$ and therefore Hölder’s inequality shows
\[
\langle ((-\Delta)^{s/2} m_\epsilon, (-\Delta)^{s/2} (m_\epsilon u\phi))_{L^2(\mathbb{R}^n)} \to \langle ((-\Delta)^{s/2} m, (-\Delta)^{s/2} (mu\phi))_{L^2(\mathbb{R}^n)}
\]
as $\epsilon \to 0$. Thus, we have shown that there holds
\[
\lim_{\epsilon \to 0} \langle ((-\Delta)^{s/2} m_\epsilon, \gamma_\epsilon^{1/2} u\phi)\rangle_{L^2(\mathbb{R}^n)} = \langle ((-\Delta)^{s/2} m, (-\Delta)^{s/2} (mu\phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[+ \langle ((-\Delta)^{s/2} m_\epsilon, (-\Delta)^{s/2} (u\phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[= \langle ((-\Delta)^{s/2} m, (-\Delta)^{s/2} (\gamma_\epsilon^{1/2} u\phi))\rangle_{L^2(\mathbb{R}^n)}
\]
for all $u, \phi \in \mathcal{S}(\mathbb{R}^n)$.

By the usual splitting $\gamma_\epsilon^{1/2} = m_\epsilon + 1$, we see that the first term in (14) can be decomposed as
\[
\langle ((-\Delta)^{s/2} (\gamma_\epsilon^{1/2} u), (-\Delta)^{s/2} (\gamma_\epsilon^{1/2} \phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[= \langle ((-\Delta)^{s/2} (m_\epsilon u), (-\Delta)^{s/2} (m_\epsilon \phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[+ \langle ((-\Delta)^{s/2} (m_\epsilon u), (-\Delta)^{s/2} (\phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[+ \langle ((-\Delta)^{s/2} u, (-\Delta)^{s/2} (m_\epsilon \phi))\rangle_{L^2(\mathbb{R}^n)}
\]
\[+ \langle ((-\Delta)^{s/2} u, (-\Delta)^{s/2} (\phi))\rangle_{L^2(\mathbb{R}^n)}
\]
(15)

Now Lemma 3.5 and the continuity of the fractional Laplacian imply
\[
(-\Delta)^{s/2} (m_\epsilon u) \to (-\Delta)^{s/2} (mu) \quad \text{in } L^2(\mathbb{R}^n)
\]
as $\epsilon \to 0$ for all $v \in H^s(\mathbb{R}^n)$. This shows
\[
\langle ((-\Delta)^{s/2} \gamma_\epsilon^{1/2} u), (-\Delta)^{s/2} (\gamma_\epsilon^{1/2} \phi)\rangle_{L^2(\mathbb{R}^n)}
\]
\[\to \langle ((-\Delta)^{s/2} (\gamma^{1/2} u), (-\Delta)^{s/2} (\gamma^{1/2} \phi))\rangle_{L^2(\mathbb{R}^n)}
\]
for all $u, \phi \in \mathcal{S}(\mathbb{R}^n)$ and hence by the definition of $q$ the identity (13) holds for all $u, \phi \in \mathcal{S}(\mathbb{R}^n)$.

**Step 2:** Let $(u_k)_{k \in \mathbb{N}}, (\phi_k)_{k \in \mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)$ such that $u_k \to u$ and $\phi_k \to \phi$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$. By the first step, we have
\[
\langle \Theta \nabla^s u_k, \nabla^s \phi_k\rangle_{L^2(\mathbb{R}^{2n})} = \langle ((-\Delta)^{s/2} (\gamma^{1/2} u_k), (-\Delta)^{s/2} (\gamma^{1/2} \phi_k))\rangle_{L^2(\mathbb{R}^n)}
\]
\[- \langle q(\gamma^{1/2} u_k), (\gamma^{1/2} \phi_k)\rangle_{L^2(\mathbb{R}^n)}
\]
(16)

for all $k \in \mathbb{N}$. By continuity of the fractional gradient from $H^s(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$ and $\phi_k \to \phi$, $u_k \to u$ in $H^s(\mathbb{R}^n)$ as $k \to \infty$, we have
\[
\langle \Theta \nabla^s u_k, \nabla^s \phi_k\rangle_{L^2(\mathbb{R}^{2n})} \to \langle \Theta \nabla^s u, \nabla^s \phi\rangle_{L^2(\mathbb{R}^{2n})}
\]
as $k \to \infty$. Using again the splitting (15) and Lemma 3.5, we deduce
\[
\langle (-\Delta)^{s/2}(\gamma^{1/2}u_k), (-\Delta)^{s/2}(\gamma^{1/2}\phi_k) \rangle_{L^2(\mathbb{R}^n)}
\to \langle (-\Delta)^{s/2}(\gamma^{1/2}u), (-\Delta)^{s/2}(\gamma^{1/2}\phi) \rangle_{L^2(\mathbb{R}^n)}
\]
as $k \to \infty$. Finally, the convergence of the second term in (16) follows from the estimate (10) in Lemma 3.7 and Lemma 3.5. Hence, we can conclude the proof. 

\[\square\]

**Definition 3.10** (Bilinear form and weak solutions to the Schrödinger equation). Let $\Omega \subset \mathbb{R}^n$ be an open set with nonempty exterior, $s > 0$ and $q \in M(H^s \to H^{-s})$. Then we define the (continuous) bilinear form related to the fractional Schrödinger equation with potential $q$ by $B_q: H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \to \mathbb{R}$, where
\[
B_q(u, v) := \int_{\mathbb{R}^n} (-\Delta)^{s/2}u (-\Delta)^{s/2}v \, dx + \langle qu, v \rangle
\]
for all $u, v \in H^s(\mathbb{R}^n)$. Moreover, if $f \in H^s(\mathbb{R}^n)$ and $F \in (\tilde{H}^s(\Omega))^s$, then we say that $v \in H^s(\mathbb{R}^n)$ is a weak solution to the fractional Schrödinger equation
\[
((-\Delta)^s + q)v = 0 \quad \text{in} \quad \Omega,
\]
\[
v = f \quad \text{in} \quad \Omega_e
\]
if there holds $B_q(v, \phi) = F(\phi)$ for all $\phi \in \tilde{H}^s(\Omega)$ and $v - f \in \tilde{H}^s(\Omega)$.

**Lemma 3.11** (Well-posedness and DN maps for the Schrödinger equation). Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma \in L^\infty(\mathbb{R}^n)$ with conductivity matrix $\Theta$, background deviation $m$ and potential $q$ satisfies $\gamma(x) \geq \gamma_0 > 0$ and $m \in H^{s,n/s}(\mathbb{R}^n)$. Then the following assertions hold:

(i) If $u \in H^s(\mathbb{R}^n)$, $g \in X := H^s(\mathbb{R}^n)/\tilde{H}^s(\Omega)$ and $v := \gamma^{1/2}u$, $f := \gamma^{1/2}g$. Then $v \in H^s(\mathbb{R}^n)$, $f \in X$ and $u$ is a weak solution of the fractional conductivity equation
\[
\text{div}_s(\Theta \nabla u) = 0 \quad \text{in} \quad \Omega,
\]
\[
u = g \quad \text{in} \quad \Omega_e
\]
if and only if $v$ is a weak solution of the fractional Schrödinger equation
\[
((-\Delta)^s + q_v)v = 0 \quad \text{in} \quad \Omega,
\]
\[
v = f \quad \text{in} \quad \Omega_e.
\]

(ii) Conversely, if $v \in H^s(\mathbb{R}^n)$, $f \in X$ and $u := \gamma^{-1/2}v$, $g := \gamma^{-1/2}f$. Then $v$ is a weak solution of (18) if and only if $u$ is a weak solution of (17).

(iii) For all $f \in X$ there is a unique weak solutions $v_f \in H^s(\mathbb{R}^n)$ of the fractional Schrödinger equation
\[
((-\Delta)^s + q)v = 0 \quad \text{in} \quad \Omega,
\]
\[
v = f \quad \text{in} \quad \Omega_e.
(iv) The exterior DN map \( \Lambda_q : X \to X^* \) given by
\[
\langle \Lambda_q f, g \rangle := B_q(v_f, g),
\]
where \( v_f \in H^s(\mathbb{R}^n) \) is the unique solution to the Schrödinger equation with exterior value \( f \), is a well-defined bounded linear map.

**Proof.** Since the proofs for the assertion (i) and (ii) are very similar we only show (i).

(i) First of all note that by Lemma 3.7 the notion of weak solutions to the fractional Schrödinger equation is well-defined. Moreover, from Corollary 3.6 and the same argument as in [RZ22b, Theorem 8.6] it follows that \( v \in H^s(\mathbb{R}^n) \) and \( f \in X \). Now the rest of the assertion follows from the Liouville reduction (Lemma 3.9).

(iii) Let \( g := \gamma^{-1/2} f \in X \) then by Lemma 2.8 there is a unique solution \( u_g \in H^s(\mathbb{R}^n) \) to the fractional conductivity equation with exterior value \( g \). Using assertion (i) we deduce that \( v_f := \gamma^{1/2} u_g \in H^s(\mathbb{R}^n) \) solves (18) with exterior value \( f \). The obtained solution is unique since solutions to the fractional conductivity equation are.

(iv) This last assertion follows from the continuity of the fractional Laplacian, Lemma 3.7 and the fact that the solutions to the Schrödinger equation depend continuously on the data. \( \square \)

**Remark 3.12.** The equations (17) and (18) also have the Runge approximation property. This follows from the abstract theory in [RZ22b, Section 4], and in particular from [RZ22b, Remark 4.2 and Theorem 4.3]. The original argument is based on the work [GSU20]. We give two proofs of Theorem 1.5 in this article. The first one is based on the Runge approximation property and standard arguments. The second proof, based on a new strategy, does not rely on the Runge approximation property.

3.3. **First proof of Theorem 1.5.** We present the first proof of Theorem 1.5 in this section. It follows the standard structure applied earlier in different regularity settings in [CRZ22, Cov21, RZ22b].

First, we state a useful basic lemma as a preparation. The proof of Lemma 3.13 is virtually identical to that of [CRZ22, Lemma 4.1] and follows from Corollary 3.6, and Lemmas 3.9 and 3.11 considering the low regularity setting. Therefore, we do not repeat these details here.

**Lemma 3.13.** Let \( \Omega \subset \mathbb{R}^n \) be an open set which is bounded in one direction, \( W \subset \Omega \) an open set and \( 0 < s < \min(1, n/2) \). Assume that \( \gamma, \Gamma \in L^\infty(\mathbb{R}^n) \) with background deviations \( m_\gamma, m_\Gamma \) satisfy \( \gamma(x), \Gamma(x) \geq \gamma_0 > 0 \) and \( m_\gamma, m_\Gamma \in H^{s,n/s}(\mathbb{R}^n) \). If \( \gamma|_W = \Gamma|_W \), then
\[
\langle \Lambda_\gamma f, g \rangle = \langle \Lambda_\psi, (\Gamma^{1/2} f), (\Gamma^{1/2} g) \rangle
\]
holds for all \( f, g \in \tilde{H}^s(W) \).

We can now give the first proof of Theorem 1.5. **First proof of Theorem 1.5.** We have that \( \gamma_1|_W = \gamma_2|_W \) a.e. by exterior determination (Theorem 3.2). Throughout the proof we choose any function \( \Gamma \in L^\infty(\mathbb{R}^n) \) satisfying the conditions \( \Gamma \geq \gamma_0, m_\Gamma := \Gamma^{1/2} - 1 \in H^{s,n/s}(\mathbb{R}^n) \) and \( \Gamma = \gamma_1 = \gamma_2 \) in \( W \).
By the argument in [CRZ22, Proof of Theorem 1.1] we know that there holds
\[ \langle \Lambda_{\gamma_1} f, g \rangle = \langle \Lambda_{\gamma_2} f, g \rangle \]
for all \( f, g \in \widetilde{H}^s(W) \) if and only if we have
\[ \langle \Lambda_{\gamma_1} f, g \rangle = \langle \Lambda_{\gamma_2} f, g \rangle \]
for all \( f, g \in C_c^\infty(W) \). Similarly, one can show by Lemma 3.5 and 3.7 that there holds
\[ \langle \Lambda_{q_1} f, g \rangle = \langle \Lambda_{q_2} f, g \rangle \]
for all \( f, g \in C_c^\infty(W) \) if and only if
\[ \langle \Lambda_{q_1} f, g \rangle = \langle \Lambda_{q_2} f, g \rangle \]
for all \( f, g \in \widetilde{H}^s(W) \).

Now using Lemma 3.13, we deduce that the condition \( \Lambda_{\gamma_1} f|_W = \Lambda_{\gamma_2} f|_W \) for all \( f \in C_c^\infty(W) \) is equivalent to
\[ \langle \Lambda_{q_1} (\Gamma^{1/2} f), (\Gamma^{1/2} g) \rangle = \langle \Lambda_{q_2} (\Gamma^{1/2} f), (\Gamma^{1/2} g) \rangle \]
for all \( f, g \in \widetilde{H}^s(W) \). By Corollary 3.6, we can replace \( f, g \) in this identity by \( \Gamma^{-1/2} f, \Gamma^{-1/2} g \in \widetilde{H}^s(W) \) and obtain (19) is equivalent to
\[ \langle \Lambda_{q_1} f, g \rangle = \langle \Lambda_{q_2} f, g \rangle \]
for all \( f, g \in \widetilde{H}^s(W) \).

Using (20) and the uniqueness results for the fractional Schrödinger equations [RZ22b, Theorem 2.2, Corollary 2.7 and Remark 4.2], we obtain \( q_1 = q_2 \) in the weak sense in \( \widetilde{H}^s(W \cup \Omega) \). This implies that
\[ \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(fg) \rangle = 0 \]
for all \( f, g \in \widetilde{H}^s(W) \), where we have again used Corollary 3.6 to replaced \( g \in \widetilde{H}^s(W) \) by \( \Gamma^{1/2} g \) and \( \gamma_1 = \gamma_2 = \Gamma \) in \( W \). Now choosing \( f \in C_c^\infty(\omega) \) and a cutoff function \( g \in C_c^\infty(W) \) with \( g|_\omega = 1 \), where \( \omega \subseteq W \) is some nonempty open set, we see that \( (-\Delta)^s(m_1 - m_2) = 0 \) in \( \omega \). On the other hand, we have by the assumption \( \gamma_1 = \gamma_2 \) in \( \omega \) and hence the UCP (Theorem 1.7) implies \( m_1 = m_2 \). This shows \( \gamma_1 = \gamma_2 \), which concludes the proof. \( \square \)

### 3.4. Second proof of Theorem 1.5

In this subsection, we give an alternative proof of Theorem 1.5 which is not appearing in the earlier literature. This argument is based on using energies of the solutions in combination with the UCP. The merit of this argument is that it avoids using the Runge approximation property and a full scale reduction to the fractional Schrödinger equation. The approach resembles the proof of single measurement uniqueness for the Calderón problem of the fractional Schrödinger equation [GRSU20]. Heuristically speaking, this proof is based more directly on the properties of the fractional conductivity equation than the other argument, which reduces the problem to the data \( \Lambda_{q_1} = \Lambda_{q_2} \).

Let \( s \geq 0 \) and \( \gamma_0 > 0 \). We write that \( \gamma \in M_{\gamma_0}^s \) if the following holds:

(i) \( \gamma \in L^\infty(\mathbb{R}^n) \) with \( \gamma \geq \gamma_0 > 0 \) a.e. in \( \mathbb{R}^n \);

(ii) \( \gamma^{1/2} u \in \widetilde{H}^s(\Omega) \) and \( \gamma^{-1/2} u \in \widetilde{H}^s(\Omega) \) for any open set \( \Omega \subseteq \mathbb{R}^n \) and \( u \in \widetilde{H}^s(\Omega) \).
Lemma 3.14 (Relation of solutions). Let $\Omega \subset \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < 1$. Assume that $\gamma_1, \gamma_2 \in M^s_\emptyset$ satisfy $\gamma_1|_{W_1} = \gamma_2|_{W_2}$ and let $W_1, W_2 \subset \Omega_e$ be nonempty open sets. If 
$\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for some $f \in \tilde{H}^s(W_1)$ with $W_2 \setminus \text{supp}(f) \neq \emptyset$, then there holds
\begin{equation}
\gamma_1^{1/2} u_1^j = \gamma_2^{1/2} u_2^j \quad \text{a.e. in } \mathbb{R}^n.
\end{equation}
Proof. Choose first some $f, \phi \in H^s(\mathbb{R}^n)$ with disjoint supports. For any $\gamma \in M^s_\emptyset$ there holds $\gamma^{1/2} f, \gamma^{1/2} \phi \in H^s(\mathbb{R}^n)$, and hence we obtain
\begin{equation}
B_{\gamma}(f, \phi) = \langle (\Delta)^{s/2}(\gamma^{1/2} f), (\Delta)^{s/2}(\gamma^{1/2} \phi) \rangle_{L^2(\mathbb{R}^n)}
\end{equation}
by expanding both terms into their quadratic forms and applying the disjoint support condition (cf. [MN19, eq. (3) and Theorem 2]).

Let now $f \in \tilde{H}^s(W_1)$ be as in the statement and denote by $u_1^j, u_2^j$ the corresponding unique solutions to the homogeneous fractional conductivity equations with the conductivities $\gamma_1, \gamma_2$ and exterior value $f$. By the assumption $V := W_2 \setminus \text{supp}(f)$ is a nonempty open set. Then for all $\phi \in \tilde{H}^s(V)$ we have by the definition of the DN maps and $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ the identity
\[ B_{\gamma_1}(u_1^j, \phi) = \langle \Lambda_{\gamma_1} f, \phi \rangle = \langle \Lambda_{\gamma_2} f, \phi \rangle = B_{\gamma_2}(u_2^j, \phi). \]
Using (22) and $\text{supp}(f) \cap \text{supp}(\phi) = \emptyset$, we obtain
\[ \langle (\Delta)^{s/2}(\gamma_1^{1/2} u_1^j), (\Delta)^{s/2}(\gamma_1^{1/2} \phi) \rangle_{L^2(\mathbb{R}^n)} = \langle (\Delta)^{s/2}(\gamma_2^{1/2} u_2^j), (\Delta)^{s/2}(\gamma_2^{1/2} \phi) \rangle_{L^2(\mathbb{R}^n)}. \]
Since $\gamma_1^{1/2}|_{W_2} = \gamma_2^{1/2}|_{W_2}$, we have by approximation $\gamma_1^{1/2} \phi = \gamma_2^{1/2} \phi$ in $\mathbb{R}^n$ and obtain
\[ \langle (\Delta)^{s/2}(\gamma_1^{1/2} u_1^j - \gamma_2^{1/2} u_2^j), (\Delta)^{s/2}(\gamma_1^{1/2} \phi) \rangle_{L^2(\mathbb{R}^n)} = 0. \]
By choosing $\phi = \gamma_1^{-1/2} g \in \tilde{H}^s(V)$, we get
\[ \langle (\Delta)^{s/2}(\gamma_1^{1/2} u_1^j - \gamma_2^{1/2} u_2^j), (\Delta)^{s/2} g \rangle_{L^2(\mathbb{R}^n)} = 0. \]
for any $g \in \tilde{H}^s(V)$. Since $\gamma_1^{1/2} u_1^j - \gamma_2^{1/2} u_2^j \in H^s(\mathbb{R}^n)$ and $\gamma_1^{1/2} u_1^j - \gamma_2^{1/2} u_2^j = 0$ in $V$, the UCP of the fractional Laplacian gives $\gamma_1^{1/2} u_1^j = \gamma_2^{1/2} u_2^j$ in $\mathbb{R}^n$. \hfill \Box

Remark 3.15. Suppose that $\Omega, \gamma_1, \gamma_2$ satisfy the assumptions of Lemma 3.14 and $W_1, W_2 \subset \Omega_e$ are nonempty open sets. Suppose that the following holds: For all $V \Subset W_1$ it holds that $W_2 \setminus V \neq \emptyset$. Then $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$ implies that $\gamma_1^{1/2} u_1^j = \gamma_2^{1/2} u_2^j$ a.e. in $\mathbb{R}^n$ for any $g \in \tilde{H}^s(W_1)$. This follows by approximation and Lemma 3.14: We have by the linearity of the fractional conductivity operator and the Lax–Milgram theorem that the solutions depend continuously on the exterior conditions in $H^s(\mathbb{R}^n)$. Therefore, the general case $g \in \tilde{H}^s(W_1)$ follows by taking a sequence $(f_j)_{j \in \mathbb{N}} \subset C^\infty_c(W_1)$ such that $f_j \to g$ in $H^s(\mathbb{R}^n)$. Now there holds $\gamma_1^{1/2} u_1^j = \gamma_2^{1/2} u_2^j$ a.e. in $\mathbb{R}^n$ for all $j \in \mathbb{N}$ by Lemma 3.14. Hence, the identity $\gamma_1^{1/2} u_1^j = \gamma_2^{1/2} u_2^j$ also holds a.e. in $\mathbb{R}^n$ by taking the limit $j \to \infty$ of a suitable subsequence. One can for example take $W := W_1 = W_2$. 

\[ \]
Second proof of Theorem 1.5. We have that $\gamma_1|_W = \gamma_2|_W$ a.e. by exterior determination (Theorem 3.2). By the Alessandrini identity [RZ22b, Lemma 4.4] and the symmetry of the bilinear forms, there holds
\[(23) \quad 0 = \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f, g \rangle = (B_{\gamma_1} - B_{\gamma_2})(u^1_f, u^2_g)
\]
for any $f, g \in X$. Let now $f = g \in \tilde{H}^s(V)$ for some $V \Subset W$. By the Liouville reduction (Lemma 3.9), (23), and $\gamma_1|_W = \gamma_2|_W$, we obtain
\[
0 = \langle (\Lambda_{\gamma_1} - \Lambda_{\gamma_2}) f, g \rangle = (B_{\gamma_1} - B_{\gamma_2})(u^1_f, u^2_f)
\]
\[
= \langle (-\Delta)^{s/2}(\gamma_1)^{1/2} u^1_f \rangle, (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
\[
- \langle (-\Delta)^{s/2} m_1, (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
\[
- \langle (-\Delta)^{s/2} (\gamma_2)^{1/2} u^2_f \rangle, (-\Delta)^{s/2}(\gamma_2)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
\[
+ \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2}(\gamma_2)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
\[
= \langle (-\Delta)^{s/2}(\gamma_1)^{1/2} u^1_f - (\gamma_2)^{1/2} u^2_f \rangle, (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
\[
- \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
Now the conclusion (21) of Lemma 3.14 (with $W_1 = V, W_2 = W$) implies that $\gamma_1^{1/2} u^1_f - \gamma_2^{1/2} u^2_f = 0$, and hence
\[(24) \quad \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)} = 0
\]
for any $f \in \tilde{H}^s(V)$.

Let $f \in \tilde{H}^s(U)$ where $U \Subset V$ is a nonempty open set, $\phi|_U = 1$, $\phi \in C_c^\infty(V)$, and define $g := \phi - f \in \tilde{H}^s(V)$. Now $f^2 - g^2 = (f + g)(f - g) = \phi(2f - \phi) = 2f - \phi^2$. We can use the identity (24) to compute
\[
0 = \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(\gamma_1)^{1/2} (f^2 - g^2 + \phi^2) \rangle_{L^2(\mathbb{R}^n)}
\]
\[
= \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(\gamma_1)^{1/2} (2f) \rangle_{L^2(\mathbb{R}^n)}
\]
This ensures
\[
0 = \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2}(\gamma_1)^{1/2} f \rangle_{L^2(\mathbb{R}^n)}
\]
for all $f \in \tilde{H}^s(U)$. By taking $f = \gamma_1^{-1/2} h \in \tilde{H}^s(U)$, we obtain that
\[
0 = \langle (-\Delta)^{s/2}(m_1 - m_2), (-\Delta)^{s/2} h \rangle_{L^2(\mathbb{R}^n)}
\]
for all $h \in \tilde{H}^s(U)$. Now since $m_1|_U = m_2|_U$, the UCP implies that $m_1 = m_2$ and eventually $\gamma_1 = \gamma_2$ in $\mathbb{R}^n$. $\square$

4. Construction of counterexamples

We construct counterexamples to the inverse fractional conductivity equation in this section. These counterexamples are constructed following the strategy introduced in [RZ22a]. In bounded domains, these counterexamples are known to exist in the generality presented here (see [RZ22a, Theorem 1.2]). In the case of domains that are bounded in one direction, counterexamples were also constructed when $0 < s < \min(1, n/4)$ for $n \geq 2$ [RZ22a, Theorem 1.3]. The integrability conditions were the main obstruction in the earlier work to construct these counterexamples in general. In the present
work, we have changed the integrability requirements from $L^{n/2s}$ to $L^{n/s}$, which allow us to construct counterexamples also in the cases $n/4 < s < 1$ when $n = 2, 3$, missing in the earlier literature.

We will need Lemma 4.1 to verify that the constructed counterexamples have the desired properties. Lemma 4.1 is an alternative to [CRZ22, Theorem 1.1 (ii)], which states a similar result with the $H^{2s, 2s}$ regularity assumption in the place of $H^{s, n/s}$.

**Lemma 4.1** (Invariance of data). Let $\Omega \subseteq \mathbb{R}^n$ be an open set which is bounded in one direction and $0 < s < \min(1, n/2)$. Assume that $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations $m_1, m_2$ satisfy $\gamma_1(x), \gamma_2(x) \geq \gamma_0 > 0$ and $m_1, m_2 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$. Finally, assume that $W_1, W_2 \subseteq \Omega$ are nonempty disjoint open sets and that $\gamma_1|_{W_1 \cup W_2} = \gamma_2|_{W_1 \cup W_2}$ holds. Then there holds $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$ if and only if $m_0 := m_1 - m_2 \in H^s(\mathbb{R}^n)$ is the unique solution of

\[
(-\Delta)^s m + q_{\gamma_2} m = 0 \quad \text{in} \quad \Omega,
\]

\[
m = m_0 \quad \text{in} \quad \Omega_c.
\]

**Proof.** As in the first proof of Theorem 1.5, we may conclude that $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$ if and only if $\Lambda_{q_1} f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$.

Suppose first that $\Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2}$. It follows from the results in [RZ22b] that $q_1 = q_2$ in $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$. Next note that $q_1 = q_2$ in $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$ is equivalent to

\[
\langle (-\Delta)^{s/2} m_1, (-\Delta)^{s/2} (vw/\gamma_1^{1/2}) \rangle = \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (vw/\gamma_2^{1/2}) \rangle
\]

for all $v, w \in \tilde{H}^s(\Omega)$. This implies for $m = m_1 - m_2 \in H^{s, n/s}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ that

\[
\langle (-\Delta)^{s/2} m, (-\Delta)^{s/2} (vw/\gamma_1^{1/2}) \rangle
\]

\[
= \langle (-\Delta)^{s/2} m_1, (-\Delta)^{s/2} (vw/\gamma_1^{1/2}) \rangle - \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (vw/\gamma_1^{1/2}) \rangle
\]

\[
= \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (vw/\gamma_2^{1/2} - 1/\gamma_1^{1/2}) \rangle
\]

\[
= \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (vw \frac{\gamma_1^{1/2} - \gamma_2^{1/2}}{\gamma_1^{1/2} \gamma_2^{1/2}}) \rangle
\]

\[
= \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (vw \frac{m}{\gamma_1^{1/2} \gamma_2^{1/2}}) \rangle
\]

for all $v, w \in \tilde{H}^s(\Omega)$. Let $w = \gamma_1^{1/2} \phi$, then we obtain

\[
\langle (-\Delta)^{s/2} m, (-\Delta)^{s/2} (v\phi) \rangle - \langle (-\Delta)^{s/2} m_2, (-\Delta)^{s/2} (v\phi \frac{m}{\gamma_2^{1/2}}) \rangle = 0
\]

for all $v, \phi \in \tilde{H}^s(\Omega)$.

Next let $\phi \in C^\infty_c(\Omega)$ and choose $v \in C^\infty_c(\Omega)$ such that $v|_{\text{supp}(\phi)} = 1$. Then $m$ solves

\[
(-\Delta)^s m + q_{\gamma_2} m = 0
\]
in the sense of distributions on $\Omega$, $m \in H^s(\mathbb{R}^n)$ and $m = m_0$ in $\Omega_e$. Since (25) has a unique solution in $H^s(\mathbb{R}^n)$ by Lemma 3.11, this shows the first direction.

To see the converse, suppose that $m = m_1 - m_2$ is the unique solution of (25). Now

$$\langle (-\Delta)^{s/2}m, (-\Delta)^{s/2}\phi \rangle - \langle (-\Delta)^{s/2}m_2, (-\Delta)^{s/2}(\phi \frac{m}{\gamma_2^{1/2}}) \rangle = 0$$

for all $\phi \in C^\infty_c(\Omega)$. We may use the computation (26) with $\phi = vw$, $v, w \in C^\infty_c(\Omega)$ and suitable approximation, to obtain that

$$\langle (-\Delta)^{s/2}m_1, (-\Delta)^{s/2}(vw/\gamma_1^{1/2}) \rangle = \langle (-\Delta)^{s/2}m_2, (-\Delta)^{s/2}(vw/\gamma_2^{1/2}) \rangle.$$  

This clearly implies that $q_1 = q_2$ in $\tilde{H}^s(\Omega) \times \tilde{H}^s(\Omega)$ since $q_1, q_2 \in M(H^s \to H^{-s})$. Therefore, it holds that $u_j^f = u_j^g$ for any $f \in C^\infty_c(W_1)$ since $(-\Delta)^s + q_1 = (-\Delta)^s + q_2$ agree in the weak sense in $\Omega$ and $B_q(f, \phi) = B_{q_2}(f, \phi)$ for any $\phi \in C^\infty_c(\Omega)$. In particular,

$$\langle \Lambda_q f, g \rangle = B_q(u_j^f, g) = B_{q_2}(u_j^g, g) = \langle \Lambda_{q_2} f, g \rangle$$

for all $f \in C^\infty_c(W_1)$, $g \in C^\infty_c(W_2)$ by the support conditions. It follows that $\Lambda_q f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$, and hence $\Lambda f|_{W_2} = \Lambda_{q_2} f|_{W_2}$ for all $f \in C^\infty_c(W_1)$ as desired. \hfill $\Box$

We can now prove Theorem 1.6. The proof given here follows the one for [RZ22a, Theorem 1.3] with minor changes that are possible due to Lemma 4.1.

**Proof of Theorem 1.6.** For any $\delta > 0$, we denote by $A_\delta$ the open $\delta$-neighborhood of the set $A \subset \mathbb{R}^n$ (this should not be confused with the notation $\Omega_\epsilon$ for the exterior) and by $(\rho_\epsilon)_{\epsilon > 0} \subset C^\infty_c(\mathbb{R}^n)$ the standard mollifiers. First assume that the conductivities $\gamma_1, \gamma_2 \in L^\infty(\mathbb{R}^n)$ with background deviations $m_1, m_2 \in H^{s+n/2}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ satisfy the assumptions of Lemma 4.1 for some $\gamma_0 > 0$ and $m := m_1 - m_2 \in H^s(\mathbb{R}^n)$ solves (25) with exterior value $m_0 \in H^s(\mathbb{R}^n)$. Then by the computation in [RZ22a, Proof of Theorem 1.3] we know that $m_1 \in H^s(\mathbb{R}^n)$ solves

$$(-\Delta)^s m_1 - \frac{(-\Delta)^s m_2}{\gamma_2^{1/2}} \cdot m_1 = \frac{(-\Delta)^s m_2}{\gamma_2^{1/2}} \quad \text{in } \Omega, \ m_1 = m_2 + m_0 \quad \text{in } \Omega_e.$$

Now let $\gamma_2 \equiv 1$. Then $m_1 \in H^s(\mathbb{R}^n)$ is a $s$-harmonic function in $\Omega$; more precisely $m_1$ solves

$$(-\Delta)^s m_1 = 0 \quad \text{in } \Omega, \ m_1 = m_0 \quad \text{in } \Omega_e.$$

Now choose $\omega \subset \Omega_\epsilon \setminus \overline{W}_1 \cup \overline{W}_2$ and let $\epsilon > 0$ be such that the sets $\Omega_{2\epsilon}, \omega_{2\epsilon} \subset \mathbb{R}^n \setminus (W_1 \cup W_2)$ are disjoint. Let $\eta \in C^\infty_c(\omega_{2\epsilon})$ be a nonnegative cutoff function satisfying $\eta|_{\Omega_{2\epsilon}} = 1$. By the Lax–Milgram theorem and the fractional Poincaré inequality on domains bounded in one direction (cf. [RZ22b, Theorem 2.2]) there is a unique solution $\tilde{m}_1 \in H^s(\mathbb{R}^n)$ to

$$(-\Delta)^s \tilde{m}_1 = 0 \quad \text{in } \Omega_{2\epsilon}, \ \tilde{m}_1 = \eta \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}_{2\epsilon}.$$
Proceeding as in [RZ22b, Proof of Theorem 1.2 and 1.3] one can show that
\[ m_1 := C_\epsilon \rho_\epsilon * \tilde{m}_1 \in H^s(\mathbb{R}^n) \] with
\[ C_\epsilon := \left( \frac{\epsilon^{n/2}}{2|B_1|^{1/2}\|ho\|_{L^\infty(\mathbb{R}^n)}^{1/2}\|	ilde{m}_1\|_{L^2(\mathbb{R}^n)}} \right) \]
solves
\[ (-\Delta)^s m = 0 \text{ in } \Omega, \quad m = m_1 \text{ in } \Omega_\epsilon. \]
Moreover, \( m_1 \) has the following properties
(i) \( m_1 \in L^\infty(\mathbb{R}^n) \) with \( \|m_1\|_{L^\infty(\mathbb{R}^n)} \leq 1/2 \),
(ii) \( m_1 \in H^s(\mathbb{R}^n) \cap H^{s,n/s}(\mathbb{R}^n) \)
(iii) and \( \text{supp}(m_1) \subset \mathbb{R}^n \setminus (W_1 \cup W_2) \).
The statement (ii) follows from Young’s inequality since \( n/s > 2 \) implies
\[ p := \frac{2n}{n + 2s} \in (1, \frac{n}{2s}) \text{ and } 1/p + 1/2 = 1 + s/n. \]
Moreover, the support conditions imply that \( \gamma_1 = 1 \) in \( W_1 \cup W_2 \).
This ensures that the conductivity \( \gamma_1 \) defined by \( \gamma_1^{1/2} := m_1 + 1 \)
and the background deviation \( m_1 \) satisfy all required properties but \( \gamma_1 \neq \gamma_2 \).
Now since \( W_1, W_2 \subset \Omega_\epsilon \) are two disjoint open sets, we have found two
conductivities \( \gamma_1, \gamma_2 \) satisfying the properties of Lemma 4.1 and \( m := m_1 - m_2 \)
is the unique solution to (25), which in turn implies that the induced
DN maps fulfill \( \Lambda_{\gamma_1} f|_{W_2} = \Lambda_{\gamma_2} f|_{W_2} \) for all \( f \in C_0^\infty(W_1) \).

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