Nonlocal regularisation and two-dimensional induced actions

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Abstract

We present an invariant regularisation scheme to compute two dimensional induced gauge theory actions, that is local in Polyakov’s variables, but nonlocal in the original gauge potentials. Our method sheds light on the locality of this induced action, and leads to a straightforward proof that the $\varepsilon$-anomaly in $W_3$-gravity is completely given by the one loop term.

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In gauge theories, the regularisation scheme plays an essential (though often hidden) role in the study of anomalies, or induced gauge field actions that arise when integrating out matter fields. In particular, the classical symmetries respected by the regularisation survive the quantisation, whereas others are potentially anomalous. One particular scheme is the Pauli-Villars regularisation \[1\]. There, if one uses covariant derivatives, the potential gauge symmetry breaking is reflected in a non-invariant mass term. If one has a choice of mass terms, this may result \[2\] in a choice of which symmetries one keeps. Furthermore, these different choices can be related to each other by the addition of (finite) “counterterms”. An explicit formula for this counterterm and explicit examples are given in \[3, 4\].

In this letter we present a somewhat unusual application of this choice of regularisation scheme (mass term). We will introduce, in specific 2-dimensional gauge theories, a class of mass terms related by a continuous parameter \(0 \leq \alpha \leq 1\). The unusual feature of this mass term will be, that it is non-local in terms of the ordinary gauge potential, although it is local in terms of a different parametrisation (due to Polyakov \[5\]). For \(\alpha = 1\) the mass term will be invariant, leading straightforwardly to an invariant induced action. For \(\alpha = 0\), the mass term will be non-invariant, but local. The relation between the two is given by the counterterm refered to above. Thus this procedure can actually be turned into a way to (re)calculate the usual induced action. It gives additional insight into why this action comes out to be local in terms of the Polyakov variables. And finally, our method allows for extension to more involved theories, like \(W_n\)-gravities. We will use it to show explicitly that the \(\epsilon\)-anomaly in \(W_3\)-gravity is completely given by the one-loop term. Finally, we will give another illustration of our method, computing the twodimensional gauge action induced by fermions.

Let us, as a first example, consider \(W_2\)-gravity \[3\]. The quantity of interest is the action \(\Gamma[h]\) for the gauge field \(h\) induced by matter fields \(\varphi\) which are scalars:

\[
\exp -\Gamma[h] = \int D\varphi \exp -S[\varphi, h]
\]

\[
S[\varphi, h] = \frac{1}{2\pi} \int d^2x (\partial \varphi.\bar{\partial} \varphi - h \partial \varphi.\partial \varphi).
\]  

(1)

The action \(S\) is invariant under the combined transformation of \(h\) and \(\varphi\).

\[
\delta_e \varphi = \epsilon \partial \varphi
\]

\[
\delta_e h = \bar{\partial} \epsilon - h \partial \epsilon + \epsilon \partial h.
\]  

(2)

To decide whether \(\Gamma[h]\) is invariant, the formal argument on eq.(1) is insufficient, as is well known. We regularise the path integral in eq.(1) in the Pauli-Villars spirit, by adding a PV-field \(\Phi\) transforming in the same way as \(\varphi\) : \(\delta \Phi = \epsilon \partial \Phi\). There is an additional mass term proportional to \(M^2\), which is taken to infinity at the end.

\[
S_{PV}[\Phi, h] = S[\varphi \rightarrow \Phi, h] + \text{mass term}.
\]

Also, a \(\Phi\) loop is given an extra minus sign \[1\].

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1One may avoid this sleight-of-hand by introducing two fermions and one boson with equal kinetic operators. Since this point is irrelevant for the (mainly one loop) considerations here, we will continue to use the customary language of inserting a minus sign ‘by hand’, although in fact all our considerations are valid on a completely conventional Lagrangian level.
There are only one-loop diagrams, so the PV method simply regularises the theory. Although it may appear that one doesn’t have much freedom in choosing a mass-term, in fact one has. Any term of the form \( \frac{1}{2} m^2(x) \Phi^2(x) \) will do, as long as \( m^2(x) \neq 0 \). Let us now consider the interesting possibilities for \( W_2 \) gravity. We can parametrise, after Polyakov [5],

\[
h = \bar{\partial} f / \partial f,
\]

where we implicitly assume \( \partial f \neq 0 \). The reparameterisation invariance of eq.(2) corresponds to \( \delta f = \epsilon \partial f \), i.e. \( \partial f \) is a density. As a consequence, taking \( m^2(x) = M^2 \partial f \) actually makes the mass term, and therefore also the regularisation, invariant! The induced action, calculated with this regularisation, is then invariant for \( \delta f \) (or \( \delta h \) in eq.(2)), and is independent of \( f \) (or \( h \)) as a result.

Now we discuss the relation between different regularisations. Two things have to be specified: firstly the regularized Lagrangian (of PV type in our case), i.e. the mass term, and secondly one may wish to include a finite counterterm. In our case, two regularisation schemes then correspond to

\[
\exp - \Gamma^{(i)}[h] = \int D\varphi D\Phi \exp \{ -S[\varphi, h] - S[\Phi, h] - S_M^{(i)}[\Phi] \} \exp - M^{(i)}[h] \quad (3)
\]

where \( i \in \{0, 1\} \) labels the scheme, \( S_M^{(i)} \) denotes the different mass terms, and \( M^{(i)} \) the finite counterterm, that depends on the gauge field only. The condition that \( \Gamma^{(0)} = \Gamma^{(1)} \) imposes a relation between \( M^{(0)} \) and \( M^{(1)} \), that depends on the \( S_M^{(i)} \). If there exists an ‘interpolating’ regularisation labeled by \( \alpha \), with mass term \( S_M^{(\alpha)} = -\frac{1}{2} M^2 \Phi T(\alpha) \Phi \), then this relation is given by [3]

\[
M^{(1)}[h] - M^{(0)}[h] = \frac{1}{2} \int_0^1 d\alpha \, \text{Tr} \left\{ T^{-1}(\alpha) \frac{\partial T(\alpha)}{\partial \alpha} \exp \frac{R(\alpha)}{M^2} \right\} \quad (4)
\]

where \( R \) is a Fujikawa-style regulator, which is unambiguously fixed by the PV Lagrangian \( \mathcal{L}_{PV,[\Phi]} = \frac{1}{2} \Phi (T R) \Phi - \frac{1}{2} M^2 \Phi T \Phi \), and where the trace is over functional space. This formula is given in [3] in a somewhat different context, and is foreshadowed in [6]. In all of the above, a limit \( M^2 \to \infty \), possibly after a renormalisation, is understood.

Let us apply it to our case, taking \( T(\alpha) = (\partial f)^\alpha \). For \( \alpha = 1 \), as argued above, there is no anomaly, and therefore

\[
\Gamma[h] = M^{(1)}[h].
\]

For \( \alpha = 0 \), the regularisation is non-invariant (the anomaly). The result for \( \Gamma[h] \) will be the same in both regularisations if we satisfy eq.(3). Thus

\[
\Gamma[h] = M^{(0)}[h] + \frac{1}{2} \int_0^1 d\alpha \text{Tr} \left\{ T^{-1} \frac{\partial T}{\partial \alpha} \exp \frac{R}{M^2} \right\}. \quad (5)
\]

The regularisation with \( \alpha = 0 \), which is local in \( h \), corresponds closest to the usual prepossession that any ‘fundamental’ (i.e. non-induced) action is local. Also, \( M^{(0)} \) is then to be considered as a possible additional action for \( h \), but not part of an action induced by the matterfields \( \varphi \). Therefore, we set \( M^{(0)} = 0 \). Then eq.(4) gives an explicit closed

2
formula for the induced action, which we will evaluate forthwith. From the action for $\Phi$ one reads off

$$T(\alpha) = \frac{1}{2\pi}(\partial f)^\alpha$$

$$T R = -\frac{1}{\pi}(\partial \bar{\partial} - \partial h \bar{\partial})$$

$$R(\alpha) = -2(\partial f)^{-\alpha}(\partial \bar{\partial} - \partial h \bar{\partial})$$

$$T^{-1} \frac{\partial T}{\partial \alpha} = \log(\partial f).$$

The functional trace can be evaluated by a variety of methods, but the easiest is to look up the result in [7]. One can rewrite the differential operator $R$ as a covariant Laplacian with $g_{\mu\nu} = \begin{pmatrix} 2h & -1 \\ -1 & 0 \end{pmatrix} (\partial f)^{-\alpha}$. The result of the trace is then

$$\lim_{M \to \infty} Tr\{\log(\partial f) \exp \frac{R[g^{\mu\nu}]}{M^2}\} = \lim_{M \to \infty} \int d(vol) \frac{1}{4\pi} \log(\partial f) \{M^2 - \frac{1}{6} R[g^{\mu\nu}]\}$$

where $d(vol)$ and $R[g^{\mu\nu}]$ are the invariant volume element and the Riemann scalar corresponding to the metric $g^{\mu\nu}$. Observe that

$$\sqrt{g} R[g^{\mu\nu}] = R[\tilde{g}^{\mu\nu}] + \frac{1}{2} \partial_\mu \{\sqrt{g} g^{\mu\nu} \partial_\nu \log g\}$$

(6)

where $\tilde{g}^{\mu\nu} = \sqrt{\tilde{g}} g^{\mu\nu} = \begin{pmatrix} 2h & -1 \\ -1 & 0 \end{pmatrix}$, and that $g^{\mu\nu}$ has a curvature scalar equal to $2\partial^2 h$. The second term in eq.(6) equals $-2\alpha \partial^2 h$. The induced action of eq.(5) then becomes

$$\Gamma[h] = \lim_{M \to \infty} \frac{1}{2} \int_0^1 d\alpha \frac{1}{4\pi} \int d^2 x (\partial f)^\alpha [M^2 - \frac{1}{6} (\partial f)^{\alpha} 2(1 - \alpha) \partial^2 h].$$

The necessary renormalisation is performed by dropping the $M^2$ terms, (more precisely, by considering in the traditional way several PV fields with $\Sigma c_i = 1$ and $\Sigma c_i M_i^2 = 0$. [3]), or alternatively by including a normalisation factor $[\int D\Phi \exp -S^{(i)}_M[\Phi]]^{-1}$ in eq.(4) from the start. The final result for the induced action is then the well known

$$\Gamma[h] = -\frac{1}{24\pi} \int d^2 x (\partial^2 h) \log(\partial f) = -\frac{1}{24\pi} \int d^2 x (\partial^2 h) \frac{1}{\square}(\partial^2 h).$$

Let us recapitulate how we obtained it: we started from the observation that different regularisations will give the same result for $\Gamma$ if one adds counterterms satisfying eq.(4). Then we presented a scheme that is invariant, albeit non local in $h$. The induced part of $\Gamma$ vanishes in this scheme, so that $\Gamma$ equals the corresponding finite counterterm. It is then computed from eq.(4).

Although locality in $h$ is not present, the whole treatment is local in $f$. The fact that the final induced action is local in $f$ is quite mysterious from alternative ways to compute $\Gamma$ (like Feynman diagrams for instance).

See footnote 1, and choose the masses of the PV boson and fermions appropriately.
Our technique offers a way to understand why, in more involved 2-dimensional gravities \([8]\), the \(\epsilon\)-anomaly is not changed by the presence of the couplings with higher spin fields \([9]\). First, these extra couplings are \(\epsilon\)-invariant by themselves. Then, with the help of the covariant derivative (we mimic here the two-step procedure followed in non-abelian gauge theories in \([10]\)), one can construct higher derivative terms, that are local in \(h\), not just in \(f\). Being covariant explicitly, this does not introduce anomalies. We will check that, as is the case in nonabelian gauge theories, this regularises all diagrams except 1-loop. Then the Pauli-Villars method for 1 loop finishes off the divergences. This is where Polyakov’s \(f\) comes in. The anomaly then arises in the transition from the invariant non-local regularisation to the noninvariant local one, eq.(5), exactly as in \(W_2\). We now prove these assertions.

We give as an explicit example the \(W_3\) case. Consider the classical action:

\[
S[\phi, h, B] = \frac{1}{\pi} \int d^2x \left\{ -\frac{1}{2} \partial^i \phi \partial^i \phi + \frac{1}{2} \partial^i h \partial^i \phi + \frac{1}{3} B \partial^{ijk} \partial^i \phi \partial^j \phi \partial^k \phi \right\}.
\]

(7)

This action is invariant under the \(\epsilon\)-symmetry, eq.(2) and 

\[
\delta \epsilon B = \epsilon \partial B - 2B \partial \epsilon.
\]

Remark that the \(B\)-interaction term is invariant by itself. The regularising term to be added to eq.(7) is

\[
S_{\text{reg}} = -\frac{1}{\pi} \int d^2x \frac{2}{\Lambda^2} \partial^i \phi \nabla^3 \phi^i,
\]

(8)

\[
\nabla^3 = \partial^3 - h \partial.
\]

This term is clearly \(\epsilon\)-invariant. (For an explicit treatment of the \(\epsilon\)-covariance, see for example \([9]\).)

The addition of eq.(8) results in a modification (cut-off) of the propagator, and in extra vertices. The \(\phi\)-propagator becomes

\[
\frac{1}{kk} \to \frac{1}{kk} \left( \frac{\Lambda^2}{\Lambda^2 - k^2} \right)
\]

where we use a complex notation for the momentum vectors, \(\vec{k}, \vec{x} = (\vec{k}z + k\vec{z})/2\). The new vertices all have two \(\phi\)-lines and a number of \(h\)-lines, and momentum factors. With \(L\) the number of loops, \(i_\phi\) the number of internal \(\phi\)-lines, \(n_H\) the number of \(h\)-vertices, \(n_B\) the number of \(B\)-lines, and \(n_\phi\) the number of external \(\phi\)-lines we compute the superficial degree of divergence \(D\) and find the following relations:

\[
L = 1 + (n_B - n_\phi)/2,
\]

\[
2i_\phi = 3n_B + 2n_H - n_\phi,
\]

\[
D \leq 2 - 2n_B.
\]

Clearly, the divergent diagrams have \(n_B \leq 1\), and then the number of loops is one or zero. This proves that the addition of the covariant higher derivative term eq.(8) has eliminated
all divergences except the single loops. Now duplicate the action for the PV-fields \( \Phi^i \). This renders the one-loop diagrams finite too. As shown above, one can take an invariant mass term (nonlocal in \( h \)). The transition to the non-invariant local mass term is made in the same way as before, through eq. (4). This finishes the argument.

As a second example, we now apply our scheme to the calculation of the induced action for a gauge field coupled to \( n \) chiral fermions \([11]\). The action is

\[
S[\psi, A] = \frac{1}{2\pi} \int d^2 x \, \psi^t (\bar{\partial} - A) \psi
\]

with \( \psi \) a column of fermionic degrees of freedom, and \( A = A^a t_a \) with the \( t_a \) the generators of a Lie algebra in a representation \( R \). This representation must be such that \( R \wedge R \) contains the adjoint representation. The matter fields transform as

\[
\delta \psi = \eta \psi, \\
\delta \psi^t = -\psi^t \eta
\]

with \( \eta = \eta^a t_a \), and \( \eta^a \) the transformation parameters. If we take

\[
\delta A = \bar{\partial} \eta + [\eta, A]
\]

the action remains invariant.

The path-integral is regularised by introducing PV-fields as follows:

\[
S_{\text{Reg}} = S[\psi, A] + S[\xi_1, A] - \frac{1}{2\pi} \int d^2 x \, \xi_1^t \partial \xi_2 + S_{\text{mass}}[\xi_1, \xi_2]
\]

Notice that, in order to construct a mass term, we have introduced another PV-field of opposite chirality. This PV field does not transform under the gauge transformation \([11]\). The standard mass term, eventually leading to the anomaly, or the WZWN induced action, is non-invariant:

\[
S_{\text{mass}}^{(0)}[\xi_1, \xi_2] = \frac{-M}{4\pi} \int d^2 x \, (\xi_1^t \xi_2 - \xi_2^t \xi_1).
\]

In this case also, one can construct an invariant mass term. If one reparametrises \([11]\) \( A = \bar{\partial} g g^{-1} \), and \( \delta g = \eta g \), then

\[
S_{\text{mass}}^{(1)}[\xi_1, \xi_2] = \frac{-M}{4\pi} \int d^2 x \, (\xi_1^t g \xi_2 - \xi_2^t g^{-1} \xi_1)
\]

is invariant, thus leading to a zero induced action. The reasoning leading to eq. (5) again applies, mutatis mutandis. The interpolating regularisation is now given by

\[
S_{\text{mass}}^{(\alpha)}[\xi_1, \xi_2] = \frac{-M}{4\pi} \int d^2 x \, (\xi_1^t g^\alpha \xi_2 - \xi_2^t g^{-\alpha} \xi_1)
\]

\footnote{The remarks of footnote 1 are applicable also here}
where we have assumed $g$ to be connected to the identity, so that $g^\alpha$ makes sense. The matrices $T(\alpha)$ and the matrix differential operator $\mathcal{R}$ are again unambiguously fixed by the PV Lagrangian. In the present case (see [3, 4] for details), writing

$$S[\xi_1, A] + S[\xi_2] + S^{(s)}_{\text{mass}}[\xi_1, \xi_2] = \int d^2x \frac{1}{2}(\xi_1^\tau, \xi_2^\tau)\mathcal{O} \left( \begin{array}{l} \xi_1 \\ \xi_2 \end{array} \right) - \frac{1}{2} M(\xi_1^\tau, \xi_2^\tau)T \left( \begin{array}{l} \xi_1 \\ \xi_2 \end{array} \right)$$

we have that $\mathcal{R} = \mathcal{O}^2$ is the Fujikawa-style regulator. More in detail

$$T(\alpha) = \frac{1}{2\pi} \left( \begin{array}{cc} 0 & g^\alpha \\ -g^{-\alpha} & 0 \end{array} \right)$$

and putting $g = e^\Theta = e^{\theta a}$ we find that

$$T^{-1}(\alpha) \frac{\partial}{\partial \alpha} T(\alpha) = \left( \begin{array}{cc} -\Theta & 0 \\ 0 & \Theta \end{array} \right)$$

and

$$\mathcal{R} = \left( \begin{array}{cc} 4g^\alpha \partial g^{-\alpha}(\bar{\partial} - A) & 0 \\ 0 & 4g^{-\alpha}(\bar{\partial} - A) g^\alpha \partial \end{array} \right).$$

The induced action eq.(5), now becomes

$$\Gamma[h] = \lim_{M^2 \to \infty} -\frac{1}{2} \int_0^1 d\alpha Tr \left( \Theta e^{4\alpha \partial g^{-\alpha}(\bar{\partial} - A)/M^2} - \Theta e^{4\alpha \partial A} g^\alpha \partial /M^2 \right). \quad (10)$$

where $Tr$ stands for taking the trace over function space and over matrices. One can proceed by direct computation, but it is convenient at this point to use cyclicity of the trace to rewrite the second term of eq.(10) as

$$Tr \left( -\Theta e^{4\alpha \partial A} g^\alpha \partial /M^2 \right).$$

Using the covariant derivatives

$$\nabla = \partial + B \equiv \partial + g^\alpha (\partial g^{-\alpha})$$

$$\bar{\nabla} = \bar{\partial} + \bar{B} \equiv \bar{\partial} - A$$

one obtains

$$\Gamma[h] = \lim_{M^2 \to \infty} -\frac{1}{2} \int_0^1 d\alpha Tr \left( \Theta e^{4\nabla \bar{\nabla} /M^2} - \Theta e^{4\bar{\nabla} \nabla /M^2} \right).$$

Again, one can save oneself a (non-negligible) amount of work by realising that this type of functional trace can be treated with the help of [6]. Essentially one rewrites the regulator in such a way that the second derivatives appear covariantly and symmetrically, while the first derivatives are absorbed into the connection on the fibre. If, as in our case, the space metric is flat, the final result becomes very simple.

One rewrites $4 \nabla \bar{\nabla} = 2\nabla \nabla - E_G$ and uses Theorem 4.2, (2) in [6] to obtain

$$\Gamma[h] = -\frac{1}{2} \int_0^1 d\alpha \int d^2x Tr \left( \frac{\Theta}{4\pi} \left[ M^2 + E_G \right] - \frac{\Theta}{4\pi} \left[ M^2 - E_G \right] \right). \quad (11)$$
For our case, we obtain
\[ E_G = 2(\bar{\partial}B - \partial\bar{B} + [\bar{B}, B]) \]
which is the curvature of the nonabelian \( B, \bar{B} \) field. Finally:
\[
\Gamma[h] = -\frac{1}{2\pi} \int d^2x \int_0^1 d\alpha \text{tr} \left[ \bar{\partial}(g^\alpha g^{-\alpha}) + \partial(\bar{\partial}g \cdot g^{-1}) - \bar{\partial}g \cdot g^{-1} \cdot g^\alpha \bar{\partial}g^{-\alpha} + g^\alpha \partial g^{-\alpha} \cdot \bar{\partial}g^{-1} \right].
\]
This expression can be rewritten in a multitude of different ways. A short road to identify the known result \([11]\) as the WZWN action \([12]\) is as follows. First, by using the formula for the derivative of the exponential of an operator (see for example \([13]\)) one can write, using \( g = e^\Theta \),
\[
\text{tr}(\partial g^{-1}\bar{\partial}g) = -\text{tr} \left( \int_0^1 d\alpha \ g^{-1+\alpha} \cdot \partial \Theta \cdot g^{-\alpha} \cdot \bar{\partial}g \right) = -\text{tr} \int_0^1 d\alpha \ g^\alpha \partial \Theta g^{-\alpha} \bar{\partial}g g^{-1}
\]
\[= \text{tr} \int_0^1 d\alpha \left( -\partial(g^\alpha \Theta g^{-\alpha}) \bar{\partial}g g^{-1} + g^\alpha \cdot \Theta \cdot g^{-\alpha} \bar{\partial}g g^{-1} \right) \]
\[+ g^\alpha \Theta \partial g^{-\alpha} \cdot \bar{\partial}g \cdot g^{-1} \]
(12)
in which one recognises the last three terms in eq.(12) (freely using partial integration, dropping the boundary terms, in the 2-dimensional space). The second trick is to use \( \partial_\alpha g^\alpha \cdot g^{-\alpha} = \Theta \) in
\[
\text{tr}(\partial g^{-1} \cdot \bar{\partial}g) = \text{tr} \int_0^1 d\alpha \ \partial_\alpha(g^\alpha \bar{\partial}g^{-\alpha}) = \text{tr} \int_0^1 d\alpha \ (\partial(\Theta g^\alpha) \bar{\partial}g^{-\alpha} - \partial g^\alpha \bar{\partial}(\Theta g^{-\alpha})) \]
\[= \text{tr} \int_0^1 d\alpha \ (\partial g^\alpha \Theta g^{-\alpha} + \bar{\partial}(\partial g^\alpha \cdot g^{-\alpha})) \]
(13)
to rewrite the first term of eq.(12) as
\[
\text{tr} \int_0^1 d\alpha \left( -\partial_\alpha g^\alpha \cdot g^{-\alpha} \left[ \bar{\partial}g^\alpha \cdot g^{-\alpha}, \partial g^\alpha \cdot g^{-\alpha} \right] + \Theta \partial (g^\alpha \bar{\partial}g^{-\alpha}) \right). \]
By combining this with eq.(13), where one also finds back the first term of eq.(12), to eliminate the left-over \( \Theta \)-terms, one obtains, using also eq.(12)
\[
\Gamma[h] = -\frac{1}{4\pi} \text{tr} \left\{ \int_0^1 d\alpha \ \partial_\alpha g^\alpha [\partial g^\alpha \cdot g^{-\alpha}, \bar{\partial}g^\alpha \cdot g^{-\alpha}] + \partial g^{-1} \bar{\partial}g \right\}
\]
(14)
which is proportional to the celebrated WZWN action, the proportionality constant depending on the representation. Note how the parameter \( \alpha \), which was introduced as an interpolating parameter between different regularisations, has now turned into the ‘extra’ dimension which is conveniently introduced to write down the topological term. The 3-dimensional volume of which the 2-dimensional \( x \)-space is the boundary can be viewed as having \( \alpha \) as the radial coordinate. The ‘extension’ of \( g(x) \) to the 3-dimensional volume is here simply \( g(x, \alpha) = [g(x)]^\alpha \).
Again, as in the 2-dimensional induced gravity example, the result eq.(14) is well known. Our derivation sheds new light on one of the most surprising aspects of that
result, viz. its locality in $g$. This hinges crucially on three ingredients. The first is
the reparametrisation $A = \bar{\partial} g \cdot g^{-1}$ (cf. $h = \bar{\partial} f / \partial f$). The second is that, given this
parametrisation, it becomes possible to write down an invariant regularisation (mass term)
which is local in $g$ (resp. $f$). And finally, one makes use of the fact that the functional
trace calculations arising from eq.(5) always give local expressions \[1\]. Needless to say,
once the locality in $g$ is established, the induced action is almost determined.

Thus, we have shown, at least in two dimensions, the use of (mildly) non-local but
invariant regularisations can be turned into a tool to compute the anomalies associated
to strictly local regularisations. This can be exploited for more complicated theories like
$W_n$ gravities, to regularise invariantly in a two-step process. Namely, first we regularised
all $n \geq 2$-loop diagrams using higher derivatives, covariantly. The remaining 1-loop
divergence is regularised with the Pauli-Villars method. This second step is taken in a
way that is non-local in the original variables, so that a counterterm as in eq.(5) is needed
to produce the result of a local regularisation. This counterterm is then the only source of
anomalies. We showed explicitly on the example of $W_3$ how this proofs that the one-loop
$\epsilon$-anomaly is an all-loop result. In default of a covariant calculus for the $\lambda$-symmetries, it
is not clear whether the above ideas can be adopted to the full symmetry algebra of $W_n$.

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