Lower bounds for the blow-up time of the heat equation in convex domains with local nonlinear boundary conditions

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Abstract

This paper studies the lower bound for the blow-up time $T^*$ of the heat equation $u_t = \Delta u$ in a bounded convex domain $\Omega$ in $\mathbb{R}^N (N \geq 2)$ with positive initial data $u_0$ and a local nonlinear Neumann boundary condition: the normal derivative $\partial u / \partial n = u^q$ on partial boundary $\Gamma_1 \subseteq \partial \Omega$ for some $q > 1$, while $\partial u / \partial n = 0$ on the other part. For any $\alpha < \frac{1}{N-1}$, we obtain a lower bound for $T^*$ which is of order $\frac{|\Gamma_1|}{\alpha}$ as $|\Gamma_1| \to 0^+$, where $|\Gamma_1|$ represents the surface area of $\Gamma_1$. As $|\Gamma_1| \to 0^+$, this result significantly improves the previous lower bound $\ln \left( \frac{|\Gamma_1|}{2} \right)$ and is almost optimal in dimension $N = 2$, since the existing upper bound is of order $\frac{|\Gamma_1|}{1}$ as $|\Gamma_1| \to 0^+$. In addition, the optimal asymptotic order of the lower bound for $T^*$ on $q$ (as $q \to 1^+$) and on $M_0$ (as $M_0 \to 0^+$) are obtained, where $M_0$ denotes the maximum of $u_0$.

1 Introduction

1.1 Problem and Results

In this paper, $\Omega$ represents a bounded open subset in $\mathbb{R}^N (N \geq 2)$ with $C^2$ boundary $\partial \Omega$; $\Gamma_1$ and $\Gamma_2$ denote two disjoint relatively open subsets of $\partial \Omega$ which satisfy $\Gamma_1 \neq \emptyset$ and $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial \Omega$. Moreover, $\overline{\Gamma} \triangleq \overline{\Gamma_1} \cap \overline{\Gamma_2}$ is assumed to be $C^1$ when being regarded as $\partial \Gamma_1$ or $\partial \Gamma_2$. We study the following problem:

\begin{equation}
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) & \text{in } \Omega \times (0, T], \\
\frac{\partial u}{\partial n}(x, t) = u^q(x, t) & \text{on } \Gamma_1 \times (0, T], \\
\frac{\partial u}{\partial n}(x, t) = 0 & \text{on } \Gamma_2 \times (0, T], \\
u(x, 0) = u_0(x) & \text{in } \Omega,
\end{cases}
\end{equation}

where

\begin{equation}
q > 1, \ u_0 \in C^1(\overline{\Omega}), \ u_0(x) \geq 0, \ u_0(x) \neq 0.
\end{equation}

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The normal derivative on the boundary is understood in the classical way: for any \((x,t) \in \partial \Omega \times (0,T]\),

\[
\frac{\partial u}{\partial n}(x,t) \equiv \lim_{h \to 0^+} \frac{u(x,t) - u(x-h\mathbf{n}(x),t)}{h},
\]

(1.3)

where \(\mathbf{n}(x)\) denotes the exterior unit normal vector at \(x\). \(\partial \Omega\) being \(C^2\) ensures that \(x-h\mathbf{n}(x)\) belongs to \(\Omega\) when \(h\) is positive and sufficiently small.

Throughout this paper, we write

\[
M_0 = \max_{x \in \Omega} u_0(x)
\]

(1.4)

and denote \(M(t)\) to be the supremum of the solution \(u\) to (1.1) on \(\Omega \times [0,t]\):

\[
M(t) = \sup_{(x,\tau) \in \Omega \times [0,t]} u(x,\tau).
\]

(1.5)

\(|\Gamma_1|\) represents the surface area of \(\Gamma_1\), that is

\[
|\Gamma_1| = \int_{\Gamma_1} dS(x),
\]

where \(dS(x)\) means the surface integral with respect to the variable \(x\). \(\Phi\) refers to the fundamental solution to the heat equation:

\[
\Phi(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad \forall (x,t) \in \mathbb{R}^N \times (0, \infty).
\]

(1.6)

In addition, the constants \(C = C(a,b\ldots)\) and \(C_i = C_i(a,b\ldots)\) will always be positive and finite and depend only on the parameters \(a,b\ldots\). One should also note that \(C\) and \(C_i\) may stand for different constants in different places.

The recent paper [27] studied (1.1) systematically and the motivation was the Space Shuttle Columbia disaster in 2003, we refer the reader to that paper for the detailed discussion of the background. As a summary of its conclusions, [27] first established the local existence and uniqueness theory for (1.1) in the following sense: there exists \(T > 0\) such that there is a unique function \(u\) in \(C^{2,1}(\Omega \times (0,T]) \cap C(\overline{\Omega} \times [0,T])\) which satisfies (1.1) pointwisely and also satisfies

\[
\frac{\partial u}{\partial n}(x,t) = \frac{1}{2} u^q(x,t), \quad \forall (x,t) \in \Gamma \times (0,T].
\]

(1.7)

Moreover, it is shown that this unique solution \(u\) becomes strictly positive as soon as \(t > 0\). We want to remark here that the solution constructed in [27] through the heat potential technique automatically satisfies (1.7) due to a generalized jump relation. The purpose of imposing this additional restriction (1.7) to the local solution is to ensure the uniqueness through the Hopf’s lemma, it is not clear whether the uniqueness will still hold without this restriction. After the local existence and uniqueness theory was set up, it also studied the blow-up phenomenon of (1.1). If \(T^*\) denotes the maximal existence time of the local solution \(u\), then it is proved that \(0 < T^* < \infty\) and \(\lim_{t \uparrow T^*} M(t) = \infty\). In other words, the maximal existence time \(T^*\) is just the blow-up time of \(u\). Moreover, if \(\min_{x \in \Omega} u_0(x) > 0\), then an explicit formula for an upper bound of \(T^*\) is obtained as below.

\[
T^* \leq \frac{1}{(q-1)|\Gamma_1|} \int_{\Omega} u_0^{1-q}(x) dx.
\]

(1.8)
On the other hand, a lower bound of $T^*$ is also provided:

$$T^* \geq C^{-\frac{\alpha}{(q-1)M_0^{q-1}|\Gamma_1|^\alpha}} \left( \min \left\{ 1, \frac{1}{q M_0^{q-1} |\Gamma_1|^\alpha} \right\} \right)^{\frac{1+\alpha}{1-(N-1)\alpha}}$$

(1.9)

where $C = C(N, \Omega, q)$ is some positive constant.

In some realistic problems, small $|\Gamma_1|$ is of interest. For example in [27], the motivation for the study of (1.1) is the Columbia space shuttle disaster and $\Gamma_1$ stands for the broken part on the left wing of the shuttle during launching, so the surface area $|\Gamma_1|$ is expected to be small. As $|\Gamma_1| \to 0^+$, the upper bound (1.8) is of order $|\Gamma_1|^{-1}$ while the lower bound (1.9) is only of order $\left( \ln (|\Gamma_1|^{-1}) \right)^{2/(N+2)}$, so there is a big gap between them. The natural question to ask is whether $T^*$ grows like a positive power of $|\Gamma_1|^{-1}$ or just like the logarithm of $|\Gamma_1|^{-1}$ as $|\Gamma_1| \to 0^+$?

Under the convexity assumption of the domain $\Omega$, this paper provides a lower bound for $T^*$ which grows like $|\Gamma_1|^{-\alpha}$ for any $\alpha < \frac{1}{N-1}$. In the meantime, the asymptotic behaviour of $T^*$ with respect to $q$ (as $q \to 1^+$) and $M_0$ (as $M_0 \to 0^+$ or $M_0 \to +\infty$) are also studied. The following is the main result of this paper.

**Theorem 1.1.** Assume (1.2). Let $\Omega$ be convex. Then for any $\alpha \in \left[ 0, \frac{1}{N-1} \right)$, there exists $C = C(N, \Omega, \alpha)$ such that

$$T^* \geq \frac{C}{(q-1)M_0^{q-1} |\Gamma_1|^\alpha} \left( \min \left\{ 1, \frac{1}{q M_0^{q-1} |\Gamma_1|^\alpha} \right\} \right)^{\frac{1+\alpha}{1-(N-1)\alpha}},$$

(1.10)

where $T^*$ is the maximal existence time for (1.1) and $M_0$ is given by (1.4). In particular, if $\alpha$ is chosen to be 0 in (1.10), then

$$T^* \geq \frac{C_1}{(q-1)M_0^{q-1}} \min \left\{ 1, \frac{1}{q M_0^{q-1}} \right\},$$

(1.11)

for some $C_1 = C_1(N, \Omega)$.

**Remark 1.2.** Theorem 1 can be used to study the asymptotic behaviour of $T^*$ with respect to $|\Gamma_1|$, $q$ and $M_0$. See the discussions below.

(1) Relation between $T^*$ and $|\Gamma_1|$: If $|\Gamma_1| \to 0^+$ and other factors are fixed, then it follows from (1.10) that for any $\alpha \in \left[ 0, \frac{1}{N-1} \right)$,

$$T^* \geq \frac{C}{(q-1)M_0^{q-1} |\Gamma_1|^\alpha} \sim |\Gamma_1|^{-\alpha}.$$

This lower bound improves the previous one $\left( \ln (|\Gamma_1|^{-1}) \right)^{2/(N+2)}$ significantly as $|\Gamma_1| \to 0^+$. In particular, when the dimension $N = 2$, this result is almost optimal since $\alpha$ can be arbitrarily close to 1 and the upper bound in (1.8) is of order $|\Gamma_1|^{-1}$ as $|\Gamma_1| \to 0^+$.

(2) Relation between $T^*$ and $q$: If $q \to 1^+$ and other factors are fixed, then (1.11) implies

$$T^* \geq \frac{C_1}{q-1}.$$

On the other hand, the upper bound (1.8) implies

$$T^* \leq \frac{C_2}{q-1}.$$

Thus the order of $T^*$ as $q \to 1^+$ is exactly $(q - 1)^{-1}$. 

3
(3) Relation between $T^*$ and $M_0$:

- If $M_0 \to 0^+$ and other factors are fixed, then (1.11) implies
  \[ T^* \geq C_1 M_0^{-(q-1)}. \]
  This order is optimal since if the initial data $u_0$ is kept to be constant, then it follows from (1.8) that
  \[ T^* \leq C_2 M_0^{-(q-1)}. \]

- If $M_0 \to +\infty$ and other factors are fixed, then (1.11) implies
  \[ T^* \geq C_1 M_0^{-2(q-1)}. \]
  In this case, there is gap from the upper bound in (1.8).

1.2 Historical Works

1.2.1 Blow-up phenomenon for the heat equation with nonlinear Neumann conditions

Starting from the pioneering papers by Kaplan [10] and Fujita [7], the blow-up phenomenon of parabolic type has been extensive studied in the literature for the Cauchy problem as well as the boundary value problems. We refer the readers to the surveys [3, 13], the books [8, 23] and the references therein.

One of the core objects in the area is the heat equation with Neumann boundary conditions in a bounded domain $\Omega$:

\[
\begin{cases}
  u_t(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0,T], \\
  \frac{\partial u}{\partial n}(x,t) = F(u(x,t)) & \text{on } \partial \Omega \times (0,T], \\
  u(x,0) = \psi(x) & \text{in } \Omega.
\end{cases}
\]  

(1.12)

Here the initial data $\psi$ is not assumed to be nonnegative. It is well-known that there are two ways to construct the classical solution to (1.12) depending on the smoothness of $\partial \Omega$, $F$ and $u_0$ (see Theorem 1.1 and 1.3 in [17], also see the books [0, 11, 16]).

(a) The first way is by Schauder estimate. Assume $\partial \Omega$ is $C^{2+\alpha}$, $F \in C^{1+\alpha}(\mathbb{R})$, $\psi \in C^{2+\alpha}(\overline{\Omega})$ and the compatibility condition

\[ \frac{\partial \psi}{\partial n}(x) = F(\psi(x)), \quad \forall x \in \partial \Omega. \]

Then there exists $T > 0$ and a unique function $u$ in $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$ which satisfies (1.12) pointwisely.

(b) The second way is by the heat potential technique. The requirements on the data can be relaxed and in particular, the compatibility condition is no longer needed, but accordingly the conclusion is also weaker. Assume $\partial \Omega$ is $C^{1+\alpha}$, $F \in C^{1}(\mathbb{R})$, $\psi \in C^{1}(\overline{\Omega})$. Then there exists $T > 0$ and a unique function $u$ in $C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times [0,T])$ which satisfies (1.12) pointwisely.

In most papers, the assumptions will fall into either case (a) or case (b) for which the meaning of the finite-time blowup is clear. For other cases, the finite-time blowup means by assuming the local existence and uniqueness of a classical solution, then such a local solution can not be extended to a global one. In the following statements, we will ignore these distinctions and just refer them to be the local (classical) solutions.
It has been already known that if $F$ is bounded on $\mathbb{R}$, then the local solutions can be extended globally. But if $F$ is unbounded, then the finite-time blowup may occur.

The first result on the blow-up phenomenon for (1.12) is due to Levin and Payne [14]. They used a concavity argument to conclude that any classical solution blows up in finite time under the two assumptions as below.

- First,
  \[ F(z) = |z|^q h(z), \quad (1.13) \]
  for some constant $q > 1$ and some differentiable, increasing function $h(z)$.

- Secondly,
  \[ \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \left( \int_0^{\psi(x)} F(z) \, dz \right) dS(x) > \frac{1}{2} \int_{\Omega} |D\psi(x)|^2 \, dx. \quad (1.14) \]

**Remark 1.3.** As a corollary of the result in [14], if $h(z)$ is positive and $\psi$ is a positive constant function, then the solution blows up in finite time. Combining with the maximum principle, this also implies that for any positive $h(z)$ and $\psi$, the solution blows up in finite time.

Later, Walter [26] gave a more complete characterization for the blow-up phenomenon by introducing some comparison functions. More precisely, if $F(z)$ is positive, increasing and convex for $z \geq z_0$ with some constant $z_0$, there are exactly two possibilities.

- First, if $\int_{z_0}^{\infty} \frac{1}{F(z) F'(z)} \, dz = \infty$, then the solution exists globally for any initial data $\psi$.
- Secondly, if $\int_{z_0}^{\infty} \frac{1}{F(z) F'(z)} \, dz < \infty$, then the solution blows up in finite time for large initial data $\psi$.

The result was further generalized by Rial and Rossi [24] (also see [17]). In [24], by assuming $F$ to be $C^2$, increasing and positive in $\mathbb{R}^+$, and also assuming $1/F$ to be locally integrable near $\infty$ (that is $\int_{z_0}^\infty \frac{1}{F(z)} \, dz < \infty$), it is shown that for any positive initial data $\psi > 0$, the classical solution blows up in finite time. The success of their method was due to a clever choice of an energy function which made the proof short and elementary.

Applying these earlier results to the simpler model (that is (1.1) with $\Gamma_2 = \emptyset$)

\[
\begin{cases}
  u_t(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0,T], \\
  \frac{\partial u}{\partial n}(x,t) = u^q(x,t) & \text{on } \partial\Omega \times (0,T], \\
  u(x,0) = u_0(x) & \text{in } \Omega,
\end{cases}
\]

where $q > 1$ and the initial data $u_0 \geq 0$ and $u_0 \not\equiv 0$, it can be shown that any solution to (1.15) blows up in finite time. In fact, by the maximum principle, the solution $u$ becomes positive as soon as $t > 0$. Then either Remark 1.3 or the result in [24] (also see [9]) implies the finite time blowup of the solution. However, when the nonlinear radiation condition is only imposed on partial boundary (that is when $\Gamma_2 \not= \emptyset$ in (1.1)), additional difficulties appear due to the discontinuity of the normal derivative along the interface $\tilde{\Gamma}$ between $\Gamma_1$ and $\Gamma_2$. To our knowledge, [27] was the first paper that dealt with this problem and gave the bounds of the blow-up time as in (1.8) and (1.9).

### 1.2.2 Estimate for the lower bound of the blow-up time

When considering the bounds of the blow-up time, the upper bound is usually related to the nonexistence of the global solutions and various methods on this issue have been developed (see [12] for a list of six methods).
The lower bound was not studied as much in the past and not many methods have been explored. However, the lower bound can be argued to be more useful in practice, since it provides an estimate of the safe time. In the existing literature, they either used a comparison argument or applied differential inequality techniques.

The first work on the lower bound estimate of the blow-up time was due to Kaplan [10]. Later, Payne and Schaefer developed a very robust method on this issue. For example, they derived the lower bound of the blow-up time for the nonlinear heat equation with homogeneous Dirichlet or Neumann boundary conditions in [20,21]. Later this idea was also applied to the problem (1.12) (see [22]) and many other types of problems (see e.g, [1,2,4,5,15,18,19,25]). Recently, in order to obtain the lower bound of blow-up time for the problem (1.1), the authors of this paper developed another method in [27] by analyzing the representation formula of the solution. Nonetheless, this method still relied on the differential inequality argument. In the next section, we will compare this current paper with [22] and [27] in more detail.

1.3 Comparison

In this section, we will compare the methods and results in [22] and [27] with those in the current paper concerning the model problem (1.1).

1.3.1 Methods

In [22], due to technical reasons, it required $\Omega$ to be convex in $\mathbb{R}^3$, $\Gamma_1 = \partial \Omega$ and $q \geq 3/2$. By introducing the energy function

$$\varphi(t) = \int_{\Omega} u^4(q-1)(x,t) \, dx$$

and adopting a Sobolev-type inequality developed in [20], they derived a first order differential inequality for $\varphi(t)$ and then obtain a lower bound for $T^*$:

$$T^* \geq C \int_{\varphi(0)}^{\infty} \frac{d\eta}{\eta + \eta^q},$$

where $C$ is some positive constant depending only on $\Omega$ and $q$.

[27] studied the problem (1.1) without any further assumptions. The authors considered the function $M(t)$ defined in (1.5). By analysing the representation formula (A.1) for the solution, they derive a nonlinear Gronwall-type inequality for $M^{N+2}(t)$ and then obtain the lower bound (1.9).

For this paper, also due to the technical reasons, the convexity of $\Omega$ is imposed, but this is the only additional assumption. The novelty of this paper is that we develop a new method which does not introduce any functionals or differential inequalities. Instead, we will chop the maximal existence time interval $[0,T^*)$ into small subintervals in a delicate way such that the length of each subinterval is at least $t_*>0$ and the total number of the subintervals has a lower bound $L_0$. As a result, $L_0t_*$ is a lower bound for $T^*$ as stated in Theorem 1.1.

1.3.2 Results

Finally we want to compare Theorem 1.1 with the previous lower bound estimates (1.9) in [27] and (1.16) in [22]. For convenience of statement, the lower bounds in (1.9), (1.16), (1.10), and (1.11) will be written as $T_1, T_2, T_1^*$, and $T_2^*$ respectively.

- Comparing Theorem 1.1 with (1.9):
  - As $|\Gamma_1| \to 0^+$, this has been discussed before: $T_1^*$ improves $T_1$ significantly.
As \( q \to 1^+ \), \( T_{2*} \) is of order \( (q-1)^{-1} \); however, \( T_1 \) is only of a logarithm order \( (\ln[(q-1)^{-1}])^{2/(N+2)} \).

As \( M_0 \to 0^+ \), \( T_{2*} \) is of order \( M_0^{-(q-1)} \); on the other hand, \( T_1 \) is only of a logarithm order \( (\ln[M_0^{-1}])^{2/(N+2)} \).

As \( M_0 \to \infty \), \( T_{2*} \) is of order \( M_0^{-2(q-1)} \); but \( T_1 \) is not applicable since it is negative when \( M_0 \) is large.

• Comparing Theorem 1.1 with 1.15.

On the one hand, \( T_{2*} \) is valid for any \( q > 1 \) and also give the exact asymptotic rate of \( T^* \) as \( q \to 1^+ \). On the other hand, [22] requires \( q \geq 3/2 \) due to the technical restriction.

\( T_{2*} \) holds for any nonempty partial boundary \( \Gamma_1 \), but [22] only considers the whole boundary case \( \Gamma_1 = \partial \Omega \).

As \( M_0 \to 0^+ \), \( T_{2*} \) is of order \( M_0^{-(q-1)} \); if the initial data \( u_0 \) does not oscillate too much, that is assuming

\[
\varphi(0) = \int_{\Omega} u_0^{4(q-1)}(x) \, dx \sim M_0^{4(q-1)} |\Omega|,
\]

then

\[
T_2 \sim \ln \left( \frac{1}{\varphi(0)} \right) \sim 4(q-1) \ln(M_0^{-1}),
\]

which is only a logarithm order \( \ln(M_0^{-1}) \).

As \( M_0 \to \infty \), \( T_{2*} \) is of order \( M_0^{-2(q-1)} \); again, if the initial data \( u_0 \) again does not oscillate too much, then

\[
T_2 \sim [\varphi(0)]^{-2} \sim M_0^{-8(q-1)}.
\]

Since \( M_0 \) is large,

\[
M_0^{-2(q-1)} \gg M_0^{-8(q-1)}.
\]

1.4 Future Works

• In practice, the domain \( \Omega \) may not be convex everywhere. Only local convexity is reasonable. In the subsequent work [28], we follow the strategy in this paper and obtain similar results by only assuming local convexity near \( \Gamma_1 \). But this method fails when the region near \( \Gamma_1 \) is concave. So how to deal with this case is an open problem.

• Even in the convex domain case, there is still gap between the lower bound \( |\Gamma_1|^{-1/(N-1)} \) and the upper bound \( |\Gamma_1|^{-1} \) as \( |\Gamma_1| \to 0^+ \). It is an interesting question whether we can further narrow this gap.

1.5 Organization

The organization of this paper is as follows. In Section 2 we state several basic results which will be used later. Section 3 is devoted to demonstrate the main idea and a detailed proof of Theorem 1.1. The appendix justifies a representation formula for the solution which plays an essential role in the proof of the main theorem.

2 Preliminary results

The first result is about the continuity in time \( t \) (as \( t \to 0^+ \)) of the integral of the fundamental solution to the heat equation over the domain \( \Omega \).
Lemma 2.1. Let $\Omega$ be a bounded open subset in $\mathbb{R}^N (N \geq 2)$ with $C^2$ boundary. Define $F : \partial \Omega \times [0,1] \to \mathbb{R}$ by

$$F(x,t) = \begin{cases} \int_{\partial \Omega} \Phi(x-y,t) \, dy & \text{for } x \in \partial \Omega, \ t \in (0,1], \\ 1/2 & \text{for } x \in \partial \Omega, \ t = 0. \end{cases}$$

Then $F$ is continuous on $\partial \Omega \times [0,1]$. As a result,

$$b_1 \triangleq \min_{\partial \Omega \times [0,1]} F$$

is a positive constant depending only on $\Omega$ and the dimension $N$.

Proof. Since $\partial \Omega$ has been assumed to be $C^2$, the proof can be carried out by standard analysis. We can also prove it by applying (2.2) and noticing the uniform decay of the term

$$\int_0^t \int_{\partial \Omega} |D_y[\Phi(x-y,t-\tau)] \cdot \vec{n}(y)| \, dS(y) \, d\tau$$

in $x \in \partial \Omega$ as $t \to 0^+$. The details are omitted here. $\square$

We want to remark that although $\int_{\mathbb{R}^N} \Phi(x-y,t) \, dy = 1$, the main contribution of the integral, when $t$ is small, only comes from the integral over a small ball $B_r(x)$ around $x$ (the smaller $t$ is, the smaller $r$ can be chosen). Now if the integral domain is a bounded region $\Omega$ instead of $\mathbb{R}^N$ and if $x \in \partial \Omega$, then the intersection $B_r(x) \cap \Omega$ will be nearly half of $B_r(x)$ as $r \to 0^+$. As a result, when $t \to 0^+$, the limit of $\int_{\Omega} \Phi(x-y,t) \, dy$ becomes $1/2$ instead of 1.

The second result shows two identities concerning the fundamental solution to the heat equation.

Lemma 2.2. Let $\Omega$ be a bounded open subset in $\mathbb{R}^N (N \geq 2)$ with $C^2$ boundary. Define $\Phi$ as in (1.1). Then

$$\int_{\Omega} \Phi(x-y,t) \, dy - \int_0^t \int_{\partial \Omega} D_y[\Phi(x-y,t-\tau)] \cdot \vec{n}(y) \, dS(y) \, d\tau = \frac{1}{2}, \ \forall x \in \partial \Omega, \ t > 0. \quad (2.2)$$

In addition, if $\Omega$ is convex, then

$$\int_{\Omega} \Phi(x-y,t) \, dy + \int_0^t \int_{\partial \Omega} D_y[\Phi(x-y,t-\tau)] \cdot \vec{n}(y) \, dS(y) \, d\tau = \frac{1}{2}, \ \forall x \in \partial \Omega, \ t > 0. \quad (2.3)$$

Proof. Consider the problem

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial n}(x,t) = 0 & \text{on } \partial \Omega \times (0,\infty), \\ u(x,0) = 1 & \text{in } \Omega, \end{cases} \quad (2.4)$$

it obviously has the unique solution $u \equiv 1$ on $\overline{\Omega} \times [0,\infty)$. As a result, (2.2) follows by plugging $u \equiv 1$ into the representation formula (A.1) (taking $\Gamma_1 = \emptyset$). Now if $\Omega$ is convex, then $D_y[\Phi(x-y,t-\tau)] \cdot \vec{n}(y) \leq 0$ for any $x, y \in \partial \Omega$ and $0 \leq \tau < t$. Thus, (2.2) implies (2.3). $\square$

Finally, we present an elementary boundary-time integral estimate which is obtained by applying Holder’s inequality. For the convenience of notation, for any $\alpha \in \left[0, \frac{1}{N-1}\right)$, we denote

$$N_\alpha = \frac{1 - (N-1)\alpha}{2}. \quad (2.5)$$
It is readily seen that $0 < N_\alpha \leq \frac{1}{2}$.

**Lemma 2.3.** Let $\Omega$ be a bounded open subset in $\mathbb{R}^N$ ($N \geq 2$) with $C^2$ boundary. Then there exists $C = C(N, \Omega)$ such that for any $\Gamma_1 \subseteq \Omega$, $\alpha \in \left[0, \frac{1}{N-1}\right)$, $x \in \partial \Omega$ and $t > 0$

$$\int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau) dS(y) d\tau \leq C |\Gamma_1|^\alpha t^{N_\alpha}, \quad (2.6)$$

where $N_\alpha$ is defined as in (2.5). In particular, if $\alpha = 0$, then

$$\int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau) dS(y) d\tau \leq C \sqrt{t}. \quad (2.7)$$

**Proof.** Fix $\Gamma_1 \subseteq \Omega$, $\alpha \in \left[0, \frac{1}{N-1}\right)$, $x \in \partial \Omega$ and $t > 0$. We denote

$LHS = \int_0^t \int_{\Gamma_1} \Phi(x-y,t-\tau) dS(y) d\tau$.

By a change of variable in time,

$$LHS = \int_0^t \int_{\Gamma_1} \Phi(x-y,\tau) dS(y) d\tau,$$

where

$$= \frac{1}{(4\pi)^{N/2}} \int_0^t \int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) d\tau. \quad (2.7)$$

For any $m \geq 1$, applying Holder’s inequality,

$$\int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) \leq \left( \int_{\Gamma_1} e^{-m|x-y|^2/(4\tau)} dS(y) \right)^{1/m} |\Gamma_1|^{(m-1)/m}. \quad (2.8)$$

Denote

$$B_1 = \sup_{\tau > 0} \sup_{x \in \partial \Omega} \frac{\tau^{-\frac{N-1}{m}}}{\int_{\partial \Omega} e^{-|x-y|^2/(4\tau)} dS(y)}. \quad (2.9)$$

It is shown in Lemma 3.1 of [27] that $B_1$ is a finite positive constant depending only on $\Omega$ and $N$. As a result,

$$\int_{\Gamma_1} e^{-m|x-y|^2/(4\tau)} dS(y) = \frac{1}{\tau^{(N-1)/2}} B_1 \leq \frac{1}{m} \tau^{(N-1)/2} B_1.$$

Combining this inequality with (2.8),

$$\int_{\Gamma_1} e^{-|x-y|^2/(4\tau)} dS(y) \leq B_1^{1/m} \tau^{(N-1)/(2m)} |\Gamma_1|^{(m-1)/m} \leq (B_1 + 1) \tau^{(N-1)/(2m)} |\Gamma_1|^{(m-1)/m}. \quad (2.10)$$

Plugging (2.10) into (2.7),

$$LHS \leq \frac{B_1 + 1}{(4\pi)^{N/2}} |\Gamma_1|^{(m-1)/m} \int_0^t \tau^{-\frac{N}{m} + \frac{N-1}{2m}} d\tau. \quad (2.11)$$
Choose 

\[ m = \frac{1}{1 - \alpha}. \]

Then \( m \geq 1 \) and \( (m - 1)/m = \alpha \). Therefore, (2.11) becomes

\[
LHS \leq \frac{B_1 + 1}{(4\pi)^{N/2}} |\Gamma_1|^\alpha \int_0^t \tau^{\frac{1-(N-1)\alpha}{N}} d\tau \leq \frac{2(B_1 + 1)}{(4\pi)^{N/2}} |\Gamma_1|^\alpha t^{\frac{1-(N-1)\alpha}{N}}
\]

where the last equality takes advantage of the assumption that \( \alpha < \frac{1}{N-1} \).

3 Lower bounds in the convex domain case

3.1 Main idea

In this section, \( \Omega \) is assumed to be convex. In addition, \( M_0 \) and \( M(t) \) are still defined as in (1.4) and (1.5).

First of all, let us recall the method in [27] on the lower bound estimate of \( T_1^* \). By considering the maximum of \( u \) on \( \partial \Omega \) for each time \( t \), the authors introduce

\[
\tilde{M}(t) = \max_{x \in \partial \Omega} u(x, t).
\] (3.1)

Note \( \tilde{M}(t) \) is different from \( M(t) \). For any \( t > 0 \), there exists \( x_0 \in \partial \Omega \) such that \( u(x_0, t) = \tilde{M}(t) \), so it follows from the representation formula (A.1) that

\[
\tilde{M}(t) \leq 2M_0 \int_\Omega \Phi(x^0 - y, t) \, dy + 2 \int_0^t \tilde{M}(\tau) \int_\Omega \left| D_y [\Phi(x^0 - y, t - \tau)] \cdot \nu(y) \right| dS(y) \, d\tau + \int_\Gamma_1 \Phi(x^0 - y, t - \tau) \, dS(y) \, d\tau = I + II + III.
\] (3.2)

After estimating

\[
\int_{\partial \Omega} \left| D_y [\Phi(x^0 - y, t - \tau)] \cdot \nu(y) \right| dS(y) \quad \text{and} \quad \int_{\Gamma_1} \Phi(x^0 - y, t - \tau) \, dS(y),
\]

in the terms \( II \) and \( III \), they achieve the lower bound (3.9) by applying a Gronwall-type technique to (3.2). However, this lower bound is only logarithmic of \( |\Gamma_1|^{-1} \) as \( |\Gamma_1| \to 0^+ \). The obstruction that prevents this method obtaining a polynomial order of \( |\Gamma_1|^{-1} \) is explained through the remark below.

Remark 3.1. Consider the following two simple integral inequalities. First,

\[
\begin{align*}
\phi_1(t) &\leq A + \int_0^t \phi_1(\tau) \, d\tau + |\Gamma_1| \int_0^t \phi_1^2(\tau) \, d\tau, & t > 0, \\
\phi_1(0) &= A > 0.
\end{align*}
\] (3.3)

It is easy to see by the Gronwall’s inequality that the blow-up time \( T_1^* \) of (3.3) satisfies

\[
T_1^* \geq \frac{1}{q - 1} \ln \left( 1 + \frac{1}{Aq^{-1} |\Gamma_1|} \right),
\]
which is of order \( \ln(|\Gamma_1|^{-1}) \) as \( |\Gamma_1| \to 0^+ \). Secondly,

\[
\begin{align*}
\phi_2(t) &\leq A + |\Gamma_1| \int_0^t \phi_2^* (\tau) \, d\tau, \quad t > 0, \\
\phi_2(0) &= A > 0.
\end{align*}
\]

(3.4)

Again by applying Gronwall’s inequality, the blow-up time \( T_2^* \) of (3.4) satisfies

\[ T_2^* \geq \frac{1}{(q-1)Aq^{-1}|\Gamma_1|}, \]

which is of order \( |\Gamma_1|^{-1} \). Comparing these two differential equations, (3.3) contains a linear term \( \int_0^t \phi_2(\tau) \, d\tau \), however, (3.4) does not. So this term is the obstruction that prevents the lower bound being a polynomial order of \( |\Gamma_1|^{-1} \).

Thus, coming back to (3.2), if the linear term II can be eliminated, then the lower bound is expected to be a polynomial order of \( |\Gamma_1|^{-1} \). Under the convexity assumption of \( \Omega \), we will develop a new method which eliminates the linear term II in a certain sense (see (3.3) through (3.6)). In addition, this new method is much more accurate than the Gronwall’s type argument. The details will be shown in the proof of Theorem 1.1.

The initial idea of the proof is as follows. First, we chop the range of \( M(t) \) into small pieces \([M_{k-1}, M_k]\) (\( k \geq 1 \)) and denote \( t_k \) to be the time that \( M(t) \) increases from \( M_{k-1} \) to \( M_k \). Secondly, we will find a lower bound \( t_{k_*} \) for each \( t_k \). Finally, \( \sum_{k=0}^{\infty} t_{k_*} \) is a lower bound for \( T^* \). However, it is very difficult to find a precise relation between the sum \( \sum_{k=1}^{\infty} t_{k_*} \) and the order of \( |\Gamma_1|^{-1} \) since the expression of \( \sum_{k=1}^{\infty} t_{k_*} \) is very complicated. As a result, in order to obtain a lower bound with simple formula, what we actually do in the proof of Theorem 1.1 is to find a uniform lower bound \( t_* \) for finitely many \( t_k \), say \( 1 \leq k \leq L \), then \( Lt_* \) serves as a lower bound. This way greatly reduces the computations and yields a lower bound with nice expression. The analysis based on the representation formula (A.2) and the identity (2.3). The delicate part is how to choose \( \{M_k\}_{1 \leq k \leq L} \) and \( t_* \) such that \( Lt_* \) is maximized.

### 3.2 Proof of Theorem 1.1

Now we start to prove the main result of this paper.

**Proof of Theorem 1.1** Let \( M(t) \) be defined as in (1.5) and let \( t_* \in (0, 1] \) be a positive constant which will be determined later. We will first use induction to construct a finite strictly increasing sequence \( \{M_j\}_{0 \leq j \leq L} \) such that if \( T_k \) denotes the first time that \( M(t) \) reaches \( M_k \), then \( T_k \geq kt_* \) for \( 0 \leq k \leq L \) (the total steps \( L \) depends on \( t_* \)). Then \( t_* \) will be chosen so that \( Lt_* \) is maximized.

- **Step 0.** Define \( M_0 \) as in (1.4). Then \( T_0 = 0 \).

- **Step \( k \) (\( k \geq 1 \)).** Suppose \( \{M_j\}_{0 \leq j \leq k-1} \) have been defined such that \( T_{k-1} \geq (k - 1)t_* \). We intend to construct \( M_k \) in this step such that \( T_k \geq kt_* \). For some \( \lambda_k > 1 \) to be determined, we define

\[
M_k = \lambda_k M_{k-1}
\]

(3.5)

and denote

\[
t_k = T_k - T_{k-1}.
\]

(3.6)

Next we want to choose suitable \( \lambda_k \) such that \( t_k \geq t_* \), which implies \( T_k \geq kt_* \).
Since $M_k > M_0$ and $T_k$ is the first time that $M(t)$ reaches $M_k$, it follows from the maximum principle that there exists $x^k \in \partial \Omega$ such that $u(x^k, T_k) = M_k$. Applying the time-shifted representation formula (A.2) with $T = T_{k-1}$ and $(x, t) = (x^k, t_k)$,

$$u(x^k, T_k) = 2 \int_\Omega \Phi(x^k - y, t_k) u(y, T_{k-1}) \, dy - 2 \int_0^{t_k} \int_{\partial \Omega} D_y \left[ \Phi(x^k - y, t_k - \tau) \right] \cdot \overrightarrow{n}(y) u(y, T_{k-1} + \tau) \, dS(y) \, d\tau + 2 \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) u^\alpha(y, T_{k-1} + \tau) \, dS(y) \, d\tau. \quad (3.7)$$

As a result,

$$M_k \leq 2M_{k-1} \int_\Omega \Phi(x^k - y, t_k) \, dy + 2M_k \int_0^{t_k} \int_{\partial \Omega} |D_y \left[ \Phi(x^k - y, t_k - \tau) \right] \cdot \overrightarrow{n}(y)| \, dS(y) \, d\tau + 2M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \quad (3.8)$$

Since $\Omega$ is convex, it follows from (2.3) that

$$\int_0^{t_k} \int_{\partial \Omega} \left| D_y \left[ \Phi(x^k - y, t_k - \tau) \right] \cdot \overrightarrow{n}(y) \right| \, dS(y) \, d\tau = \frac{1}{2} - \int_\Omega \Phi(x^k - y, t_k) \, dy. \quad (3.9)$$

Plugging this identity into (3.8) and simplifying,

$$M_k \int_\Omega \Phi(x^k - y, t_k) \, dy \leq M_{k-1} \int_\Omega \Phi(x^k - y, t_k) \, dy + M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \quad (3.10)$$

Equivalently,

$$(M_k - M_{k-1}) \int_\Omega \Phi(x^k - y, t_k) \, dy \leq M_k^q \int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau. \quad (3.11)$$

If $t_k > 1$, we automatically have $t_k \geq t_\ast$ since $t_\ast \leq 1$. So next we assume $t_k \leq 1$. Then it follows from Lemma 2.1 that

$$\int_\Omega \Phi(x^k - y, t_k) \, dy \geq b_1. \quad (3.12)$$

In addition, Lemma 2.3 implies the existence of a constant $C = C(N, \Omega)$ such that

$$\int_0^{t_k} \int_{\Gamma_1} \Phi(x^k - y, t_k - \tau) \, dS(y) \, d\tau \leq C |\Gamma_1|^a t_k^{N_\alpha}/N_\alpha, \quad (3.13)$$

where $N_\alpha$ is defined as in (2.5). Plugging (3.11) and (3.12) into (3.10),

$$\frac{M_k - M_{k-1}}{M_k^q} \leq \frac{C |\Gamma_1|^a t_k^{N_\alpha}}{b_1 N_\alpha}. \quad (3.13)$$
Recalling that $M_k = \lambda_k M_{k-1}$, then

$$\frac{\lambda_k - 1}{\lambda_k^q M_{k-1}} \leq \frac{C |\Gamma_1|^b t_{k}^{N_{\alpha}}}{b_1 N_{\alpha}},$$  \hspace{1cm} (3.14)

Based on this observation, if there exists $\lambda_k > 1$ such that

$$\frac{\lambda_k - 1}{\lambda_k^q M_{k-1}} = \delta_1,$$

where

$$\delta_1 \equiv \frac{C |\Gamma_1|^b t_{k}^{N_{\alpha}}}{b_1 N_{\alpha}},$$  \hspace{1cm} (3.15)

then (3.14) through (3.16) together implies that $t_k \geq t_*$. As a conclusion, in Step $k$, as long as (3.15) has a solution $\lambda_k > 1$, we define $M_k$ as in (3.5). Then $t_k \geq t_*$ and therefore $T_k \geq kt_*$. According to Lemma 3.3, if $1 \leq k \leq L$ with

$$L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1} \delta_1} - 9q \right),$$

then there exists a solution $\lambda_k > 1$ to (3.15). So we can construct a finite sequence $\{M_j\}_{0 \leq j \leq L}$ such that $T_L \geq L t_*$. As a result,

$$T^* > T_L \geq L t_* > \frac{1}{10(q-1)} \left( \frac{1}{M_0^{q-1} \delta_1} - 9q \right) t_*.$$

Note that when $t_* \to 0^+$, $\delta_1$ also tends to $0^+$. So if we choose $t_*$ to be small, then the right hand side of (3.17) is positive.

The final question is how to choose $t_* \in (0, 1]$ to maximize the right hand side of (3.17). Plugging (3.16) into (3.17),

$$T^* \geq \frac{1}{10(q-1)} \left( \frac{b_1 N_{\alpha}}{M_0^{q-1} C |\Gamma_1|^b t_{k}^{N_{\alpha}}} - 9q \right) t_* = \frac{9q}{10(q-1)} \left( \frac{C_1 N_{\alpha}}{q M_0^{q-1} |\Gamma_1|^b t_{k}^{1-N_{\alpha}}} - t_* \right),$$  \hspace{1cm} (3.18)

where $C_1 \equiv b_1/(9C)$ is a constant only depending on $N$ and $\Omega$. In order to maximize the right hand side of (3.18), define

$$A = \frac{C_1 N_{\alpha}}{q M_0^{q-1} |\Gamma_1|^b} \quad \text{and} \quad \beta = 1 - N_{\alpha} \in [1/2, 1).$$

Regarding the right hand side of (3.18) to be a function of $t_*$ on $[0, 1]$, it follows from Lemma 3.4 that the maximum of this function is

$$(1 - \beta) A \left( \min \{1, \beta A\} \right)^{\beta/(1-\beta)}$$

and this maximum occurs at $t_* = \left( \min \{1, \beta A\} \right)^{1/(1-\beta)}$. As a result,

$$T^* \geq \frac{9q}{10(q-1)} (1 - \beta) A \left( \min \{1, \beta A\} \right)^{\beta/(1-\beta)}.$$
Noticing that $\beta \geq 1/2$, so

$$T^* \geq \frac{9q}{10(q-1)} (1-\beta) A \left( \min \left\{ 1, \frac{A}{2} \right\} \right)^{\beta/(1-\beta)}$$

$$= \frac{9C_1 N\alpha^2}{10(q-1) M_0^{q-1} |\Gamma_1|^{2\alpha}} \left( \min \left\{ 1, \frac{C_1 N\alpha}{2q M_0^{q-1} |\Gamma_1|^\alpha} \right\} \right)^{\alpha\alpha^{-1}}$$

$$\geq \frac{C_2}{(q-1) M_0^{q-1} |\Gamma_1|^{2\alpha}} \left( \min \left\{ 1, \frac{1}{q M_0^{q-1} |\Gamma_1|^\alpha} \right\} \right)^{\alpha\alpha^{-1}}$$

where

$$C_2 = \frac{9C_1 N\alpha^2}{10} \left( \min \left\{ 1, \frac{C_1 N\alpha}{2} \right\} \right)^{\alpha\alpha^{-1}}$$

is a constant depending on $N$, $\Omega$ and $\alpha$. In particular, if we choose $\alpha = 0$ in (3.19) and (3.20), then it follows from

$$N\alpha = 1 - (N-1)\alpha = \frac{1}{2}$$

that

$$T^* \geq C_3 \left( \frac{1}{q-1} \right)^{1/2} \min \left\{ 1, \frac{1}{q M_0^{q-1}} \right\},$$

where $C_3$ is a positive constant only depending on $N$ and $\Omega$.

Finally we will prove two technical results, Lemma 3.3 and Lemma 3.4, which were used in the above proof. In the following, for any $q > 1$, we write

$$E_q = (q-1)^{q-1}/q^q.$$  \hspace{1cm} (3.21)

By elementary calculus,

$$\frac{1}{3q} < E_q < \min \left\{ \frac{1}{q}, \frac{1}{(q-1)e} \right\} < 1.$$  \hspace{1cm} (3.22)

Before stating Lemma 3.3 we discuss a simple property as below.

**Lemma 3.2.** For any $q > 1$, write $E_q$ as in (3.21) and define $g : (1, \infty) \to \mathbb{R}$ by

$$g(\lambda) = \frac{\lambda - 1}{\lambda^q}.  \hspace{1cm} (3.23)$$

Then the following two claims hold.

(1) For any $y \in (0, E_q]$, there exists unique $\lambda \in \left( 1, \frac{q}{q-1} \right]$ such that $g(\lambda) = y$.

(2) For any $y > E_q$, there does not exist $\lambda > 1$ such that $g(\lambda) = y$.

**Proof.** Since $g$ is strictly increasing on the interval $\left( 1, \frac{q}{q-1} \right]$ and strictly decreasing on the interval $\left[ \frac{q}{q-1}, \infty \right)$, it reaches the maximum at $\lambda = q/(q-1)$. Noticing that

$$g \left( \frac{q}{q-1} \right) = \frac{(q-1)^{q-1}}{q^q} = E_q,$$

then the claims (1) and (2) follow directly. \hfill \Box

**Lemma 3.3.** Given $q > 1$, $M_0 > 0$ and $\delta_1 > 0$. Denote $E_q$ as in (3.21) and construct a (finite) sequence $\{M_k\}_{k \geq 0}$ inductively as follows (note $M_0$ has been given). Suppose $M_{k-1}$ has been constructed for some $k \geq 1$, then whether defining $M_k$ depends on how large $M_{k-1}$ is.
\( M_{q-1}^\delta_1 \leq E_q \), then Lemma 3.2 asserts there exists a unique \( \lambda_k \in (1, \frac{q}{q-1}) \) such that
\[
\frac{\lambda_k - 1}{\lambda_k} = M_{q-1}^\delta_1. \tag{3.24}
\]
Then we define \( M_k = \lambda_k M_{k-1} \) and continue to the next step.

\( M_{q-1}^\delta_1 > E_q \), then Lemma 3.2 implies the nonexistence of \( \lambda_k > 1 \) such that (3.24) holds. So we stop the construction.

We claim this construction stops in finite steps and if the last term is denoted as \( M_L \), then
\[
L > \frac{1}{10(q-1)} \left( \frac{1}{M_0^q - 1} \delta_1 - 9q \right). \tag{3.25}
\]

**Proof.** First, we will show the construction has to stop in finite steps. Plugging \( \lambda_k = M_k/M_{k-1} \) into (3.24) and rearranging, we obtain
\[
M_k = M_{k-1} + M_k^q \delta_1. \tag{3.26}
\]
Then it follows from (3.26) that the sequence \( \{M_k\} \) is strictly increasing. Therefore,
\[
M_k \geq M_{k-1} + M_{k-1} M_0^q \delta_1 = (1 + M_0^q \delta_1) M_{k-1}.
\]
Consequently,
\[
M_k \geq (1 + M_0^q \delta_1)^k M_0.
\]
Hence, \( M_k^q \delta_1 \) will exceed \( E_q \) when \( k \) is sufficiently large. So the construction will stop in finite steps.

Next suppose the constructed sequence is \( \{M_k\}_{0 \leq k \leq L} \). We will derive the lower bound (3.25) for \( L \) as below. If
\[
M_0^q \delta_1 > \frac{1}{9q},
\]
then (3.25) holds automatically since the right hand side of (3.25) is negative. So we will just assume
\[
M_0^q \delta_1 \leq \frac{1}{9q}.
\]
Taking advantage of (3.22), we know
\[
M_0^q \delta_1 \leq \min\{1/2, E_q\}.
\]
In addition, since the last term of the constructed sequence is \( M_L \), then \( M_L^q \delta_1 > E_q \). As a result, \( L \geq 1 \) and there exists an index \( L_0 \in [1, L] \) such that
\[
M_{L_0}^q \delta_1 \leq \min\{1/2, E_q\} \quad \text{and} \quad M_L^q \delta_1 > \min\{1/2, E_q\}. \tag{3.27}
\]
The reason of considering \( \min\{1/2, E_q\} \) here instead of \( E_q \) is because later we need the upper bound 1/2 to justify (3.30).

According to (3.20),
\[
M_{k-1} = M_k - M_k^q \delta_1 = M_k (1 - M_k^q \delta_1).
\]
Raising both sides to the power \( q - 1 \) and multiplying by \( \delta_1 \),

\[
M_k^{q-1} \delta_1 = M_k^{q-1} (1 - M_k^{q-1} \delta_1)^{q-1} \delta_1.
\]

Define \( x_k = M_k^{q-1} \delta_1 \). Then

\[
x_{k-1} = x_k (1 - x_k)^{q-1}, \quad \forall 1 \leq k \leq L_0.
\]

Moreover,

\[
x_0 = M_0^{q-1} \delta_1, \quad x_{L_0-1} \leq \min\{1/2, E_q\} \quad \text{and} \quad x_{L_0} > \min\{1/2, E_q\}.
\]

Noticing that \( M_{L_0} = \lambda_{L_0} M_{L_0-1} \leq \frac{q}{q-1} M_{L_0-1} \), so

\[
x_{L_0} = \left( \frac{M_{L_0}}{M_{L_0-1}} \right)^{q-1} x_{L_0-1} \leq \left( \frac{q}{q-1} \right)^{q-1} E_q = \frac{1}{q}.
\]

Now we claim the following inequality:

\[
\frac{1}{x_0} \leq \frac{1}{x_{L_0-1}} + 10(q-1)(L_0 - 1).
\]

In fact, if \( L_0 = 1 \), then (3.29) automatically holds. If \( L_0 \geq 2 \), then for any \( 1 \leq k \leq L_0 - 1 \),

\[
x_{k-1} = x_k (1 - x_k)^{q-1} \geq x_k (1 - 2(q-1)x_k)
\]

(3.30)

since \( 0 < x_k \leq x_{L_0-1} \leq 1/2 \). Recalling the fact \( x_k \leq x_{L_0-1} \leq E_q \) and the estimate \( E_q < \frac{1}{(q-1)e} \) in (3.22), we have

\[
1 - 2(q-1)x_k \geq 1 - 2(q-1)E_q \geq \frac{1}{5}.
\]

Hence, taking the reciprocal in (3.30) yields

\[
\frac{1}{x_{k-1}} \leq \frac{1}{x_k [1 - 2(q-1)x_k]}
\]

\[
= \frac{1}{x_k} + \frac{2(q-1)}{1 - 2(q-1)x_k}
\]

\[
\leq \frac{1}{x_k} + 10(q-1).
\]

(3.31)

Summing up (3.31) for \( k \) from 1 to \( L_0 - 1 \), we also obtain (3.29).

Finally, since

\[
\frac{1}{3q} < \min\{\frac{1}{2}, E_q\} \leq x_{L_0} \leq \frac{1}{q},
\]

then

\[
x_{L_0-1} = x_{L_0} (1 - x_{L_0})^{q-1}
\]

\[
> \frac{1}{3q} \left( 1 - \frac{1}{q} \right)^{q-1} = \frac{E_q}{3}.
\]

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Plugging the above inequality and \( x_0 = M_0^{q-1}\delta_1 \) into (3.29), it gives
\[
\frac{1}{M_0^{q-1}\delta_1} < \frac{3}{E_q} + 10(q - 1)(L_0 - 1) < 9q + 10(q - 1)(L_0 - 1).
\]
Rearranging this inequality yields
\[
L_0 > \frac{1}{10(q - 1)} \left( \frac{1}{M_0^{q-1}\delta_1} - 9q \right) + 1.
\]
Hence,
\[
L \geq L_0 > \frac{1}{10(q - 1)} \left( \frac{1}{M_0^{q-1}\delta_1} - 9q \right).
\]

**Lemma 3.4.** Fix any two constants \( A > 0 \) and \( \beta \in (0,1) \). Define \( f : [0,1] \rightarrow \mathbb{R} \) by \( f(t) = At^\beta - t \). Then \( f \) attains the maximum at \( t = \min\{1, \beta A\}^{1/(1-\beta)} \) and
\[
\max_{0 \leq t \leq 1} f(t) \geq (1 - \beta)A \left( \min\{1, \beta A\}\right)^{\beta/(1-\beta)}. \tag{3.32}
\]

**Proof.** For any \( t \in (0,1) \), \( f'(t) = \beta A t^{\beta-1} - 1 \), so \( f \) is increasing when \( 0 \leq t \leq \min\{1, (\beta A)^{1/(1-\beta)}\} \) and decreasing when \( \min\{1, (\beta A)^{1/(1-\beta)}\} \leq t \leq 1 \). Hence, \( f \) attains its maximum at \( t = \min\{1, (\beta A)^{1/(1-\beta)}\} \).

In addition,

- If \( \beta A \geq 1 \), then
  \[
  \max_{0 \leq t \leq 1} f(t) = f(1) = A - 1 \geq (1 - \beta)A.
  \]
- If \( 0 < \beta A < 1 \), then
  \[
  \max_{0 \leq t \leq 1} f(t) = f\left( (\beta A)^{1/(1-\beta)} \right) = A (\beta A)^{\beta/(1-\beta)} - (\beta A)^{1/(1-\beta)} = (1 - \beta)A (\beta A)^{\beta/(1-\beta)}.
  \]

Combining these two cases, (3.32) is justified.

## A Time-shifted representation formula

In Corollary 3.9 of [27], it derived the following representation formula (A.1) for the solution \( u \) to (1.1). Namely, if \( T^* \) denotes the maximal existence time of \( u \), then for any boundary point \( (x,t) \in \partial\Omega \times [0,T^*) \),
\[
\begin{align*}
  u(x,t) &= 2 \int_{\Omega} \Phi(x-y,t) \ u_0(y) \, dy - 2 \int_0^t \int_{\partial\Omega} D_y [\Phi(x-y,t-\tau)] \cdot \nabla(y) \ u(y,\tau) \, dS(y) \, d\tau \\
  &\quad + 2 \int_0^t \int_{\Gamma_s} \Phi(x-y,t-\tau) \ u^q(y,\tau) \, dS(y) \, d\tau. \tag{A.1}
\end{align*}
\]
We want to remark that there is also representation formula for the inside point \((x,t) \in \Omega \times [0,T^*)\) (see Theorem 3.8 in [27]), but that formula is different from (A.1) in that there does not have the coefficients 2 in front of the integrals on the right hand side. The appearance of the coefficient 2 in (A.1) is due to the jump relation of the single-layer heat potential when \(x \in \partial \Omega\) (see e.g. [6], Sec. 2, Chap. 5).

The formula in (A.1) based on the initial data \(u_0(\cdot)\). Now for any \(T \in (0,T^*)\), we are asking that if regarding \(T\) to be the initial time and \(u(\cdot,T)\) to be the initial data, then do we still have the representation formula? It seems trivial by just shifting the time \(T\) regarding \(T^\ast\). But it should be careful since Corollary 3.9 in [27] deals with \(C^1(\Omega)\) initial data \(u_0\) but \(u(\cdot,T)\) is only in \(C^2(\Omega) \cap C(\overline{\Omega})\). The next lemma claims that as long as \(u\) is the solution to (1.1) with the assumption (1.2), then for any \(T \in [0,T^*)\), there also holds a representation formula similar to (A.1) but with initial data \(u(\cdot,T)\).

**Lemma A.1.** Assume (1.2). Let \(T^\ast\) be the maximal existence time and \(u\) be the maximal solution to (1.1). Then for any \(x \in \partial \Omega\), \(T \in [0,T^\ast)\) and \(t \in [0,T^\ast - T)\),

\[
\begin{align*}
\frac{\partial}{\partial t} u(x,T) &= \Delta v(x,t) \quad \text{in} \quad \Omega \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v(x,t) &= u^\theta(x,T + t) \quad \text{on} \quad \Gamma_1 \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v(x,t) &= 0 \quad \text{on} \quad \Gamma_2 \times (0,T^\ast - T), \\
v(x,0) &= u(x,T) \quad \text{in} \quad \Omega.
\end{align*}
\]

Proof. When \(T = 0\), (A.2) is just the representation formula (A.1) which has been proven in [27]. So we can assume \(T > 0\). Define \(v : \Omega \times [0,T^\ast - T) \rightarrow \mathbb{R}\) by

\[
v(x,t) = u(x,T + t).
\]

Then \(v \in C^{2,1}(\Omega \times (0,T^\ast - T)) \cap C(\overline{\Omega} \times [0,T^\ast - T))\) and satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} v(x,t) &= \Delta v(x,t) \quad \text{in} \quad \Omega \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v(x,t) &= u^\theta(x,T + t) \quad \text{on} \quad \Gamma_1 \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v(x,t) &= 0 \quad \text{on} \quad \Gamma_2 \times (0,T^\ast - T), \\
v(x,0) &= u(x,T) \quad \text{in} \quad \Omega.
\end{align*}
\]

Since \(u(\cdot,T)\) is continuous on \(\overline{\Omega}\) as a function in space variable \(x\), we can continuously extend \(u(\cdot,T)\) to \(\mathbb{R}^N\) and still denote it to be \(u(\cdot,T)\). Let \(\eta \in C^\infty_0(\mathbb{R}^N)\) be the standard mollifier. That is,

\[
\eta(x) = \begin{cases} 
C \exp \left(\frac{-1}{|x|^2} \right) & \text{if } |x| < 1, \\
0 & \text{if } |x| \geq 1,
\end{cases}
\]

where the positive constant \(C\) is selected so that \(\int_{\mathbb{R}^N} \eta(x) \, dx = 1\). Then for any \(j \geq 1\), there exists \(\epsilon_j > 0\) such that by defining

\[
\eta_j(x) = \epsilon_j^{-N} \eta(x/\epsilon_j) \quad \text{and} \quad g_j(x) = (\eta_j(\cdot) * u(\cdot,T))(x),
\]

then

\[
\max_{x \in \mathbb{R}^N} |g_j(x) - u(x,T)| \leq \frac{1}{j}.
\]

Since \(g_j \in C^1(\overline{\Omega})\), it follows from Theorem B.4 in [27] that there exists \(v_j \in C^{2,1}(\Omega \times (0,T^\ast - T)) \cap C(\overline{\Omega} \times [0,T^\ast - T))\) and satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} v_j(x,t) &= \Delta v_j(x,t) \quad \text{in} \quad \Omega \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v_j(x,t) &= u_j^\theta(x,T + t) \quad \text{on} \quad \Gamma_1 \times (0,T^\ast - T), \\
\frac{\partial}{\partial n} v_j(x,t) &= 0 \quad \text{on} \quad \Gamma_2 \times (0,T^\ast - T), \\
v_j(x,0) &= u_j(x,T) \quad \text{in} \quad \Omega.
\end{align*}
\]
\[0, T^* - T)\] such that
\[
\begin{aligned}
(v_j)_t(x,t) &= \Delta v_j(x,t) & \text{in } \Omega \times (0, T^* - T), \\
\frac{\partial w_j}{\partial n}(x,t) &= u^q(x,T + t) & \text{on } \Gamma_1 \times (0, T^* - T), \\
\frac{\partial w_j}{\partial n}(x,t) &= 0 & \text{on } \Gamma_2 \times (0, T^* - T), \\
v_j(x,0) &= g_j(x) & \text{in } \Omega.
\end{aligned}
\] (A.6)

Again due to the fact that \(g_j \in C^1(\overline{\Omega})\), we can apply the representation formula (A.1) to \(v_j\) so that for any \((x, t) \in \partial \Omega \times [0, T^* - T)\),
\[
v_j(x,t) = 2 \int_{\Omega} \Phi(x-y,t) g_j(y) \, dy - 2 \int_{0}^{t} \int_{\partial \Omega} D_y [\Phi(x-y,t-\tau)] \cdot \vec{n}(y) v_j(y,\tau) \, dS(y) \, d\tau + 2 \int_{0}^{t} \int_{\Gamma_1} \Phi(x-y,t-\tau) u^q(y,T+\tau) \, dS(y) \, d\tau.
\] (A.7)

Define \(w_j = v_j - v\). Then \(w_j \in C^{2,1}(\Omega \times (0, T^* - T)) \cap C(\overline{\Omega} \times [0, T^* - T))\) and satisfies
\[
\begin{aligned}
(w_j)_t(x,t) &= \Delta w_j(x,t) & \text{in } \Omega \times (0, T^* - T), \\
\frac{\partial w_j}{\partial n}(x,t) &= 0 & \text{on } \Gamma_1 \times (0, T^* - T), \\
\frac{\partial w_j}{\partial n}(x,t) &= 0 & \text{on } \Gamma_2 \times (0, T^* - T), \\
w_j(x,0) &= g_j(x) - u(x,T) & \text{in } \Omega.
\end{aligned}
\]

So it follows from the maximum principle and the Hopf lemma that for any \((x, t) \in \overline{\Omega} \times [0, T^* - T)\),
\[
|w_j(x,t)| \leq \max_{x \in \Omega} |g_j(x) - u(x,T)| \leq \frac{1}{j}.
\]

That is,
\[
|v_j(x,t) - v(x,t)| \leq \frac{1}{j}, \quad \forall (x, t) \in \overline{\Omega} \times [0, T^* - T). \quad (A.8)
\]

Now fixing any point \((x, t) \in \partial \Omega \times [0, T^* - T)\) and sending \(j \to \infty\) in (A.7), then it follows from (A.8), (A.9) and Lebesgue’s dominated convergence theorem that
\[
v(x,t) = 2 \int_{\Omega} \Phi(x-y,t) u(y,T) \, dy - 2 \int_{0}^{t} \int_{\partial \Omega} D_y [\Phi(x-y,t-\tau)] \cdot \vec{n}(y) v(y,\tau) \, dS(y) \, d\tau + 2 \int_{0}^{t} \int_{\Gamma_1} \Phi(x-y,t-\tau) u^q(y,T+\tau) \, dS(y) \, d\tau.
\]

Finally, (A.2) is verified by recalling the definition (A.3) for \(v\). \(\square\)

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