Normalizers and permutational isomorphisms in simply-exponential time

Daniel Wiebking
RWTH Aachen University
wiebking@informatik.rwth-aachen.de

Abstract

We show that normalizers and permutational isomorphisms of permutation groups given by generating sets can be computed in time simply exponential in the degree of the groups. The result is obtained by exploiting canonical forms for permutation groups (up to permutational isomorphism).

1 Introduction

Computational group theory deals with practical computations in groups. For example, today’s computer algebra systems (such as Magma and GAP) can handle permutation groups of degree in the hundreds of thousands. The key for handling such large permutation groups lies in a compact implicit representations of the groups. In fact, every group \( G \leq \text{Sym}(V) \) can be represented by a generating set of size polynomial in \( |V| \), whereas the order \( |G| \) of a group can be exponential in \( |V| \). It is a priori unclear if fundamental algorithmic tasks as determining the order of a group or testing membership in a group can be solved efficiently when a group is given by such a compact representation. In Sim’s pioneering work, he gives solutions for these tasks, which have later been shown to run in polynomial time \cite{Sim70, Sim71, FHL80}. His algorithms are not only of theoretical interest, but randomized versions as in \cite{BCFS91} turned out to be fast in practice and build the core of present computer algebra systems (such as Magma and GAP).

On the other side, there are central group theoretic tasks for which efficient solutions are unavailable so far. Important examples are the problems in the so-called Luks equivalent class, which are all shown to be polynomial-time equivalent \cite{Luk91}. The Luks class includes the setwise-stabilizer problem, the group-intersection problem, the centralizer problem and the string-isomorphism problem. The class plays an important role even outside computational group theory. In fact, Babai’s recent breakthrough in graph isomorphism is obtained by showing that the string-isomorphism problem (and thus the entire Luks class) can be solved in quasipolynomial time \cite{Bab16}.

However, there is one important notoriously difficult problem outside the Luks class that does not have a quasipolynomial-time solution yet. This is the computation of the normalizer \( \text{Norm}(G) = \{ h \in \text{Sym}(V) \mid h^{-1}Gh = G \} \) of a group \( G \leq \text{Sym}(V) \). This problem is also related to various isomorphism problems. For example, the natural isomorphism problem for permutation groups, which asks for a permutational isomorphism between two given groups, reduces to
the normalizer problem in polynomial time \cite{Luk91}. Similarly, the isomorphism problem for linear codes, also known as code equivalence, is polynomial time reducible to permutational isomorphism and thus to the normalizer problem \cite{BCGQ11}.

Since the normalizer computation in general is a tough challenge, researchers considered normalizers $\text{Norm}(G) \cap H$ for restricted groups $H \leq \text{Sym}(V)$. The first polynomial-time results were obtained for normalizers in nilpotent groups, solvable groups, and $p$-groups \cite{KL90, Luk92, LRW94}. One decade later a result was obtained for normalizers $\text{Norm}(G) \cap H$ where both groups $G$ and $H$ are restricted \cite{LM02}. Another decade passed when these results were generalized to normalizers where $H$ has restricted composition factors \cite{LM11}.

However, for normalizers without restrictions to the groups, the best time bound (obtained via permutational isomorphism) is $2^{O(|V|/\log(|V|))}$ \cite{BCQ12}. Measured in terms of the degree of the group, this is still no improvement over the brute force running time, since the order $|G|$ of the group can be as large as the factorial of $|V|$. It is a stated open problem whether the running time can be improved to $2^{O(|V|)}$ \cite{BCGQ11, Cod11}.

In this paper, we resolve this open problem and prove the following theorem.

**Theorem.** For groups $G \leq \text{Sym}(V)$ and $G' \leq \text{Sym}(V')$ given by generating sets, one can compute the following tasks in time $2^{O(|V|)}$.

1. Deciding permutational isomorphism, i.e., deciding whether there is a bijection $\varphi : V \to V'$ such that $\varphi^{-1}G\varphi = G'$.
2. Computing the normalizer $\text{Norm}(G) = \{ h \in \text{Sym}(V) \mid h^{-1}Gh = G \}$.
3. Computing a canonical labeling for $G$ (up to permutational isomorphisms).

The first problem reduces to the second, whereas the second task reduces to the third canonization problem. On the other hand, no polynomial-time reduction from canonization to isomorphism is known. In fact, a lot research in group isomorphism has been done in recent years and none of them seem to provide canonical forms \cite{BCQ12, LW12, RW15, GQ15, GQ17, BMW17}.

**Our Technique** Recently, we introduced a canonization framework to obtain canonical forms of various objects matching fastest known isomorphism algorithms \cite{SW18}. In this paper, we show that the canonization framework can be used to obtain isomorphism and normalizer algorithms that are even faster than the existing ones. The approach of canonization instead of isomorphism allows the use of the object replacement paradigm. In particular, our main algorithm combines an object replacement lemma with a decomposition technique of permutation groups into cosets to obtain an efficient recursion.

To handle permutation groups also when they are given by generating sets, we need to extend the recent canonization framework. We therefore expand the notion of combinatorial objects by a new type of atom. This allows the use of implicitly given permutation groups and combines permutational group theory with powerful canonization techniques.

**Organization of the Paper** In Section 3 we extend the notion of combinatorial objects in our recent framework. The main difference is that groups occurring in combinatorial objects are allowed to have implicit representations via generating sets.

In Section 4 the first algorithm shows how a canonical labeling for a subgroup can be shifted to a canonical labeling for a coset. The second algorithm finally gives a canonical labeling for permutation groups in simply-exponential time and thereby proves our main theorem.
2 Preliminaries

Group Theory For \( t \in \mathbb{N} \), we define \([t] := \{1, \ldots, t\}\). The composition of two functions \( f : V \to U \) and \( g : U \to W \) is denoted by \( fg \) and is defined as the function that first applies \( f \) and then applies \( g \). The symmetric group on a set \( V \) is denoted by \( \text{Sym}(V) \) and the symmetric group of degree \( t \in \mathbb{N} \) is denoted by \( \text{Sym}(t) \). A (permutation) group coset \( X \) over a set \( V \) is as set of permutations such that \( X = Hf = \{hf \mid h \in H\} \) for some subgroup \( H \leq \text{Sym}(V) \) and some permutation \( f \in \text{Sym}(V) \). Analogous to subgroups, we say that \( Hf \) is a subcoset of a coset \( Gf \), written \( Hf \leq Gf \), if \( Hf \) is a subset of \( Gf \) that again forms a coset. In the following let \( G \leq \text{Sym}(V) \) be a group. The normalizer of \( Hf \leq \text{Sym}(V) \) in \( G \) is denoted by \( \text{Norm}_G(Hf) := \{g \in G \mid g^{-1}Hfg = Hf\} \) for all \( g \in G \). The setwise stabilizer of \( A \subseteq V \) in \( G \) is denoted by \( \text{Stab}_G(A) := \{g \in G \mid g(a) = a\text{ for all } a \in A\} \). The pointwise stabilizer of \( A \subseteq V \) in \( G \) is denoted by \( \text{PointStab}_G(A) := \{g \in G \mid g(a) = a\text{ for all } a \in A\} \). In the case that \( G \) is the symmetric group on \( V \), the subgroups \( \text{Stab}(A)_{\text{Sym}(V)} \), \( \text{PointStab}(A)_{\text{Sym}(V)} \) and \( \text{Norm}_{\text{Sym}(V)}(Hf) \) are also denoted as \( \text{Stab}(A) \), \( \text{PointStab}(A) \) and \( \text{Norm}(Hf) \), respectively. A set \( A \subseteq V \) is called \( G \)-invariant if \( \text{Stab}_G(A) = G \). A partition \( V = V_1 \cup \ldots \cup V_t \) is called \( G \)-invariant if each part \( V_i \) is \( G \)-invariant. The finest \( G \)-invariant partition (with non-empty parts) is called the \( G \)-orbit partition. The parts of the orbit partition are also called \( G \)-orbits. A group \( G \leq \text{Sym}(V) \) is called transitive on a set \( A \subseteq V \) if \( A \) is a \( G \)-orbit.

Generating Sets and Polynomial-Time Library For the basic theory of handling permutation groups given by generating sets, we refer to [Ser03]. Indeed, most algorithms are based on strong generating sets. However, given an arbitrary generating set, the Schreier-Sims algorithm is used to compute a strong generating set (of size quadratic in the degree) in polynomial time. In particular, we will use that the following tasks can be performed efficiently when a group is given by a generating set.

1. Given a vertex \( v \in V \) and a group \( G \leq \text{Sym}(V) \), the Schreier-Sims algorithm can be used to compute the pointwise stabilizer \( \text{Stab}_G(v) \) in polynomial time.
2. Given a group \( G \leq \text{Sym}(V) \), a subgroup that has a polynomial time membership problem can be computed in time polynomial in the index and the degree of the subgroup.

3 Combinatorial Objects With Implicitly Given Group Cosets

We start to recall and extend our framework from [SW13]. The crucial difference is that group cosets occurring in combinatorial objects are allowed to have implicit representations via generating sets.

Labeling Cosets A labeling coset of a set \( V \) is set of bijections \( \Lambda \) such that \( \Lambda = \Delta \rho = \{\delta \rho \mid \delta \in \Delta\} \) for some subgroup \( \Delta \leq \text{Sym}(V) \) and some bijection \( \rho : V \to \{1, \ldots, |V|\} \). We write \( \text{Label}(V) \) to denote the labeling coset \( \text{Sym}(V)\rho = \{\sigma \rho \mid \sigma \in \text{Sym}(V)\} \) where \( \rho : V \to \{1, \ldots, |V|\} \) is an arbitrary bijection. Analogous to subgroups, a set \( \Theta \tau \) is called a labeling subcoset of \( \Delta \rho \), written \( \Theta \tau \leq \Delta \rho \), if the labeling coset \( \Theta \tau \) is a subset of \( \Delta \rho \). The restriction of \( \Delta \rho \leq \text{Label}(V) \) to a \( \Delta \)-invariant set \( A \subseteq V \) is defined as \( (\Delta \rho)\mid_A := \{\lambda\mid_A \mid \lambda \in \Delta \rho\} \). Observe that the restriction \( (\Delta \rho)\mid_A \) does not necessarily form a labeling coset since the \( \rho(A) \) might be a set of natural numbers that differs from \( \{1, \ldots, |A|\} \). To rectify this, let \( \kappa \) be the unique bijection from \( \rho(A) \) to \( \{1, \ldots, |A|\} \) that preserves the ordering “\( < \)” on the natural numbers. The induced labeling coset of \( \Delta \rho \) on \( A \subseteq V \) is defined as the labeling coset \( (\Delta \rho)\downharpoonright_A := (\Delta \rho)\downharpoonright_{|AK|} \).
Hereditarily Finite Sets and Combinatorial Objects  In contrast to the previous framework, we will model group cosets as a third kind of atom in order to represent them implicitly.

Inductively, we define hereditarily finite sets over a ground set $V$. Each vertex $X \in V$ and each labeling coset $Y = \Delta \rho \leq \text{Label}(V)$ and also each group coset $Z = Gf \leq \text{Sym}(V)$ is called an atom. Each atom is in particular a hereditarily finite set. Inductively, if $X_1, \ldots, X_t$ are hereditarily finite sets, then also $\mathcal{X} = \{X_1, \ldots, X_t\}$ and $\mathcal{X} = (X_1, \ldots, X_t)$ are hereditarily finite sets where $t \in \mathbb{N} \cup \{0\}$. A (combinatorial) object is a pair $(V, \mathcal{X})$ consisting of a ground set $V$ and a hereditarily finite set $\mathcal{X}$ over $V$. The ground set $V$ is usually apparent from context and the combinatorial object $(V, \mathcal{X})$ is identified with the hereditarily finite set $\mathcal{X}$. The set Objects($V$) denotes the set of all (combinatorial) objects over $V$. An object is called ordered if the ground set $V$ is linearly ordered. The linearly ordered ground sets that we consider are always subsets of natural numbers with their standard ordering “$<$”. An object is unordered if $V$ is a usual set (without a given order). Partially ordered objects in which some, but not all, atoms are comparable are not considered.

Representation of Objects With Implicitly Given Group Cosets  All labeling cosets but also all group cosets occurring as atoms will be represented concisely by generating sets. This is the precise reason why we model them as an atom rather than a hereditarily finite set.

For an object $\mathcal{X} \in \text{Objects}(V)$, the transitive closure of $\mathcal{X}$, denoted by $\text{TClosure}(\mathcal{X})$, is defined as all objects that recursively occur in $\mathcal{X}$, i.e., $\text{TClosure}(\mathcal{X}) := \{X\}$ for $X = v \in V$ or $X = \Delta \rho \leq \text{Label}(V)$ or $X = Gf \leq \text{Sym}(V)$. Inductively, the transitive closure is defined as $\text{TClosure}(\mathcal{X}) := \{\mathcal{X}\} \cup \bigcup_{i \in [t]} \text{TClosure}(X_i)$ for $\mathcal{X} = \{X_1, \ldots, X_t\}$ or $\mathcal{X} = (X_1, \ldots, X_t)$. All objects are efficiently represented as colored directed acyclic graphs over the elements in its transitive closure. Using this representation, the input size (with implicitly given group cosets) of an object $\mathcal{X}$ is polynomial in $|\text{TClosure}(\mathcal{X})| + |V| + t_{\text{max}}$ where $t_{\text{max}}$ is the maximal length of a tuple in $\text{TClosure}(\mathcal{X})$.

Applying Functions to Unordered Objects  Let $V$ be an unordered ground set and let $V'$ be a ground set that is either ordered or unordered. The image of an unordered object $\mathcal{X} \in \text{Objects}(V)$ under a bijection $\mu : V \rightarrow V'$ is an object $\mathcal{X}' \in \text{Objects}(V')$ that is defined as follows. The image $X^\mu$ of a vertex $X = v \in V$ is defined as $\mu(v)$. The image $Y^\mu$ of a labeling coset $Y = \Delta \rho \leq \text{Label}(V)$ is defined as $\mu^{-1} \Delta \rho$. The image $Z^\mu$ of a group coset $Z = Gf \leq \text{Sym}(V)$ is defined as $\mu^{-1} Gf \mu$. Inductively, the image $\mathcal{X}' \in \text{Objects}(V')$ of an object $\mathcal{X} = \{X_1, \ldots, X_t\}$ is defined as $(X_1^\mu, \ldots, X_t^\mu)$. Similarly, the image $\mathcal{X}' \in \text{Objects}(V')$ of an object $\mathcal{X} = (X_1, \ldots, X_t)$ is defined as $(X_1^\mu, \ldots, X_t^\mu)$. Notice that the way we apply functions to cosets does not depend on whether they are modeled as an atom or as a hereditarily finite set.

Isomorphisms and Automorphisms of Unordered Objects  The set of all isomorphisms from an object $\mathcal{X} \in \text{Objects}(V)$ to an object $\mathcal{X}' \in \text{Objects}(V')$ is denoted by $\text{Iso}(\mathcal{X}; \mathcal{X}') := \{\varphi : V \rightarrow V' \mid \varphi \circ \varepsilon = \mathcal{X}'\}$. The set of all automorphisms of an object $\mathcal{X}$, denoted by $\text{Aut}(\mathcal{X}) := \text{Iso}(\mathcal{X}; \mathcal{X})$. Both isomorphisms and automorphisms are defined for objects that are unordered only. For specific objects, the automorphism group $\text{Aut}(\mathcal{X})$ often leads to a familiar notion, e.g., $\text{Aut}((\Theta \tau, \Delta \rho)) = \Theta \cap \Delta$, $\text{Aut}((Gf, \Delta \rho)) = \text{Norm}_\Delta(Gf)$ $\text{Aut}((A, \Delta \rho)) = \text{Stab}_\Delta(A)$ and $\text{Aut}((a_1, \ldots, a_t, \Delta \rho)) = \text{PointStab}_\Delta(\{a_1, \ldots, a_t\})$ where $A \in V$, $\Theta \tau, \Delta \rho \leq \text{Label}(V)$ and $Gf \leq \text{Sym}(V)$.

For two unordered sets $V$ and $V'$, the set $\text{Iso}(V; V')$ is also used to denote the set of all bijections from $V$ to $V'$. This notation indicates and stresses that both $V$ and $V'$ have to be
unordered. Additionally, it is used in a context where $\varphi \in \text{Iso}(V; V')$ is seen as an isomorphism $\varphi \in \text{Iso}(X; X')$.

**The Linear Ordering of Ordered Objects** We define a linear ordering of objects that remains polynomial-time computable when group cosets are implicitly given.

For this paragraph, we assume objects to be ordered where $V \subseteq \mathbb{N}$ and we define a linear order $\prec$ on them. For two atoms $X, Y \in \mathbb{N}$, the natural ordering is adapted, i.e., $X < Y$ if $X < Y$.

For two sets $X = \{X_1, \ldots, X_n\}$ and $Y = \{Y_1, \ldots, Y_n\}$ on which the order $\prec$ is already defined for the elements $X_i$ and $Y_j$, the linear order is defined as follows. We say that $X < Y$ if $|X| < |Y|$ or if $|X| = |Y|$ and the smallest element in $X \setminus Y$ is smaller than the smallest element in $Y \setminus X$. For two tuples $X = (X_1, \ldots, X_s)$ and $Y = (Y_1, \ldots, Y_t)$ where $\prec$ is already defined for the entries, the linear order is defined as follows. We say that $X < Y$ if $s$ is smaller than $t$ or if $s = t$ and for the smallest position $i \in [t]$ for which $X_i \neq Y_i$, it holds that $X_i < Y_i$. We extend the order to permutations of natural numbers as follows. For two permutations $\sigma_1, \sigma_2 \in \text{Sym}(\{1, \ldots, |V|\})$ we say that $\sigma_1 < \sigma_2$ if there is an $i \in \{1, \ldots, |V|\}$ such that $\sigma_1(i) < \sigma_2(i)$ and $\sigma_1(j) = \sigma_2(j)$ for all $1 \leq j < i$. The definition is extended to labeling cosets $\Delta\rho, \Theta\tau \leq \text{Label}(\{1, \ldots, |V|\})$. Similar to the case of sets, we say that $\Delta\rho \prec \Theta\tau$ if $|\Delta\rho| \leq |\Theta\tau|$ or if $|\Delta\rho| = |\Theta\tau|$ and the smallest element of $\Delta\rho \setminus \Theta\tau$ is smaller than the smallest element of $\Theta\tau \setminus \Delta\rho$. The definition for group cosets $Gf_1f_2 \leq \text{Sym}(\{1, \ldots, |V|\})$ is analogous. We say $Gf_1f_2 < Gf_2f_2$ if $|Gf_1| \leq |Gf_2|$ or if $|Gf_1| = |Gf_2|$ and the smallest element of $Gf_1f_2 \setminus Gf_2f_2$ is smaller than the smallest element of $Gf_2f_2 \setminus Gf_1f_1$. Indeed, for two cosets $X, Y \leq \text{Sym}(\{1, \ldots, |V|\}) = \text{Label}(\{1, \ldots, |V|\})$ the ordering $\prec$ for $X$ and $Y$ can be computed in time polynomial in $|V|$ when the cosets are given by generating sets (GNSW18, Corollary 22). For completeness, we define $X < Y < Z < X' < Y'$ for all integers $X \in \mathbb{N}$, all labeling cosets $Y = \Delta\rho \leq \text{Label}(\{1, \ldots, |V|\})$, all group cosets $Z = Gf \leq \text{Sym}(\{1, \ldots, |V|\})$, all tuples $X$ and all sets $Y$. We say that $X < Y$ if $X = Y$ or $X < Y'$.

The previous paragraph shows that the linear ordering remains polynomial-time computable for objects that contain implicitly given group cosets.

**Lemma 1.** The ordering $\prec$ on pairs of ordered objects can be computed in polynomial time in the input size (with implicitly given group cosets).

We list the definitions and results obtained in the recent canonization framework.

**Definition 2** ([SW18]). Let $\mathcal{C} \subseteq \text{Objects}(V)$ be an isomorphisms-closed class of unordered objects. A **canonical labeling function** $\text{CL}$ is a function that assigns each unordered object $X \in \mathcal{C}$ a labeling coset $\text{CL}(X) = \Lambda \leq \text{Label}(V)$ such that:

\begin{enumerate}
\item[(CL1)] $\text{CL}(X) = \varphi \text{CL}(X^\varphi)$ for all $\varphi \in \text{Iso}(V; V')$ and,
\item[(CL2)] $\text{CL}(X) = \text{Aut}(X)\pi$ for some (and thus for all) $\pi \in \text{CL}(X)$.
\end{enumerate}

In this case, the labeling coset $\Lambda$ is also called a **canonical labeling** for $X$.

**Lemma 3** ([SW18], Object replacement lemma). Let $X = \{X_1, \ldots, X_t\}$ be an object and let $\text{CL}$ and $\text{CL}_{\text{set}}$ be canonical labeling functions. Define $X^\text{Set} := \{\Delta_1\rho_1, \ldots, \Delta_t\rho_t\}$ where $\Delta_i\rho_i := \text{CL}(X_i)$ is a canonical labeling for $X_i \in X$. Assume that $X_i^{\rho_i} = X_j^{\rho_j}$ for all $i, j \in [t]$. Then $\text{CL}_{\text{Object}}(X) := \text{CL}_{\text{set}}(X^\text{Set})$ defines a canonical labeling for $X$.

**Lemma 4** ([SW18], Lemma 5). **Canonical labelings for pairs** $(M, \Delta\rho) \in \text{Objects}(V)$ where $M \subseteq V_1 \times V_2$ is a matching and $\Delta\rho$ is a labeling coset of $V = V_1 \cup V_2$ can be computed in time $2^{O(k_2)}|V|^{O(1)}$ where $k_2$ is the size of the largest $\Delta$-orbit of $V_2 \subseteq V$.  

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we obtain $\phi_M$ for all $i, j$ defined at the beginning of Section 3).

$\{-\{M : \text{labeling coset } \Lambda \subseteq \text{Label}(V) \}$ such that:

- $\text{CL}_\text{Shift}(Gf, \Delta \rho) = \varphi \text{CL}_\text{Shift}(\varphi^{-1}Gf \varphi, \varphi^{-1}\Delta \rho)$ for all $\varphi \in \text{Iso}(V; V')$.

- $\text{CL}_\text{Shift}(Gf, \Delta \rho) = \text{Norm}_\Delta(Gf)\pi$ for some (and thus for all) $\pi \in \Delta$.

The reader is encouraged to take a moment to verify that for input objects $\mathcal{X} = (Gf, \Delta \rho)$, the Conditions (CL1) and (CL2) stated here agree with Conditions (CL1) and (CL2) that are defined for objects in general.

The requirement that $\Delta = \text{Norm}_\Delta(G)$ says that the labeling coset $\Delta \rho$ consists of canonical labelings for the group $G$. Thus, the problem can be seen as the task of shifting the canonical labeling $\Delta \rho = \text{Norm}_\Delta(G)\rho$ for the group $G$ to a canonical labeling $\Lambda = \text{Norm}_\Delta(Gf)\pi$ for the group coset $Gf$.

**Lemma 7.** A function $\text{CL}_\text{Shift}$ solving Problem 6 can be computed in time $2^O(k)|V|^{O(1)}$ where $k$ is the size of the largest $\Delta$-orbit of $V$.

An isomorphism-version for groups with restricted composition factors is stated as Problem P7(1) in [KL90]. We use some of the ideas. However, we make use of our framework to extend the algorithm to canonization which later is required. Additionally, we use the adequate exponential recurrence that can handle unrestricted groups.

**Proof.** Define $\tilde{V} := \{\tilde{v}_1, \ldots, \tilde{v}_{|V|}\}$ to be a set of size $|V|$ that is disjoint from $V$. The set $\tilde{V}$ can essentially be seen as a copy of the set $V$. Define $U := \tilde{V} \cup V$. Define $\Delta_U \rho_U \leq \text{Label}(U)$ to be the labeling coset where $\Delta_U := \{\delta_U \in \text{Sym}(U) \mid \exists g \in G, \delta \in \Delta : \text{for all } i, j \in \{1, \ldots, |V|\} \text{ we have } \delta_U(v_i) = \delta(g(v_i)) \text{ and } \delta_U(v_j) = \delta(v_j)\}$ and where $\rho_U \in \text{Label}(U)$ such that for all $i, j \in \{1, \ldots, |V|\}$ it holds $\rho_U(v_i) = \rho(v_i) + |V|$ and $\rho_U(v_i) = \rho(v_i)$. By the assumption that $\varphi^{-1}G\delta = G$ for all $\delta \in \Delta$, the set $\Delta_U$ defines indeed a group closed under composition. Define a matching $M_f := \{(\tilde{f}(v_i), v_i) \mid i \in \{1, \ldots, |V|\}\}$ by pairing the vertices according to $f$. Compute $\Lambda_U := \text{CL}_\text{Match}(M_f, \Delta_U \rho_U)$ using the algorithm from Lemma 4. We claim that the induced labeling coset $\Lambda := \Lambda_U \downarrow V$ defines a canonical labeling for $(Gf, \Delta \rho)$ (the induced labeling coset is defined at the beginning of Section 3).

(CL1.) Assume we have $\varphi^{-1}Gf \varphi, \varphi^{-1}\Delta \rho$ instead of $Gf, \Delta \rho$ as an input. By the construction, we obtain $\varphi^{-1}_U \Delta_U \rho_U$ instead of $\Delta_U \rho_U$ for some $\varphi_U$ with $\varphi_U|_V = \varphi$. Moreover, we obtain $M_f^{\varphi_U \varphi^{-1}_U}$ instead of $M_f$ and for some $g_U$ with $g_U|_V = \text{id}$ (the choices for $f \in Gf$ vary up to elements in $G$). By (CL1) of $\text{CL}_\text{Match}$ and since $\varphi^{-1}_U \Delta_U \rho_U = \varphi^{-1}_U \varphi_U^2 \Delta_U \rho_U$, we obtain $\varphi^{-1}_U \varphi_U^2 \Delta_U \rho_U$ instead of $\Delta_U$. Therefore, we obtain $(\varphi^{-1}_U \varphi_U^2 \Delta_U)\downarrow V = \varphi^{-1} \Lambda$ instead of $\Lambda$. 

**4 Canonization of Implicitly Given Permutation Groups**

Before we canonize group cosets in general, we consider canonical labeling functions for objects $\mathcal{X} = (Gf, \Delta \rho)$ consisting of a group coset and a labeling coset with a particular restriction to $G$ and $\Delta$.

**Problem 6.** Compute a function $\text{CL}_\text{Shift}$ with the following properties:

- Input $(Gf, \Delta \rho) \in \text{Objects}(V)$ where $Gf \leq \text{Sym}(V)$, $\Delta \rho \leq \text{Label}(V)$, $V$ is an unordered set and with the restriction that $\Delta = \text{Norm}_\Delta(G)$.

- Output A labeling coset $\text{CL}_\text{Shift}(Gf, \Delta \rho) = \Lambda \leq \text{Label}(V)$ such that:

  - $(\text{CL1}) \text{CL}_\text{Shift}(Gf, \Delta \rho) = \varphi \text{CL}_\text{Shift}(\varphi^{-1}Gf \varphi, \varphi^{-1}\Delta \rho)$ for all $\varphi \in \text{Iso}(V; V')$.

  - $(\text{CL2}) \text{CL}_\text{Shift}(Gf, \Delta \rho) = \text{Norm}_\Delta(Gf)\pi$ for some (and thus for all) $\pi \in \Delta$.

  The requirement that $\Delta = \text{Norm}_\Delta(G)$ says that the labeling coset $\Delta \rho$ consists of canonical labelings for the group $G$. Thus, the problem can be seen as the task of shifting the canonical labeling $\Delta \rho = \text{Norm}_\Delta(G)\rho$ for the group $G$ to a canonical labeling $\Lambda = \text{Norm}_\Delta(Gf)\pi$ for the group coset $Gf$.

**Lemma 7.** A function $\text{CL}_\text{Shift}$ solving Problem 6 can be computed in time $2^O(k)|V|^{O(1)}$ where $k$ is the size of the largest $\Delta$-orbit of $V$.
In order to verify (CL2), we show that \((\text{Aut}(M_f) \cap \Delta)_{|V} = \text{Norm}_\Delta(G_f)\). The inclusion \(\text{Norm}_\Delta(G_f) \subseteq (\text{Aut}(M) \cap \Delta)_{|V}\) already follows from Condition (CL1) of the problem. It remains to show the reversed inclusion also holds, i.e., \((\text{Aut}(M_f) \cap \Delta)_{|V} \subseteq \text{Norm}_\Delta(G_f)\). Let \(\alpha : \overline{V} \to V\) be the bijection \(s\) with \(f(\overline{v}_i) = v_i\). So let \(\delta_U \in \text{Aut}(M_f) \cap \Delta\). Since \(\delta_U \in \Delta\), there are some \(v_i \in \Delta, g \in G\) such that \(\alpha(\delta_U(\overline{v}_i)) = \delta(g(v_i))\) and \(\delta_U(v_i) = \delta(v_i)\) for all \(i \in \{1, \ldots, |V|\}\). Since \(\delta_U \in \text{Aut}(M_f)\), it holds that \(\alpha(\delta_U(\overline{v}_i)) = f(\delta_U(f^{-1}(v_i)))\) for all \(i \in \{1, \ldots, |V|\}\). Both together imply that \(\delta(g(v_i)) = \alpha(\delta_U(\overline{v}_i)) = f(\delta_U(f^{-1}(v_i))) = f(\delta(f^{-1}(v_i)))\) for all \(v_i \in V\). Thus \(\delta^{-1}f \delta = g\) for some \(g \in G\). By assumption, it holds \(\delta^{-1}G\delta = G\) and thus \(\delta^{-1}Gf\delta = \delta^{-1}G\delta\delta^{-1}f\delta = Gf\). This proves \(\delta_U|_V = \delta \in \text{Norm}_\Delta(G_f)\).

(Running time.) All steps are polynomial-time computable, except the computation of CLMatch. The used algorithm runs in time simply exponential in \(k_2\) where \(k_2\) is the size of the largest \(\Delta\)-orbit of \(V \subseteq U\). By the construction, \(k_2\) is equal to the size of the largest \(\Delta\)-orbit of \(V\).

Next, we will consider the general problem without that restriction to the group and the labeling coset.

**Problem 8.** Compute a function CLGroup with the following properties:

Input \((G_f, \Delta) \subseteq \text{Objects}(V)\), where \(G_f \subseteq \text{Sym}(V)\), \(\Delta \subseteq \text{Label}(V)\) and \(V\) is an unordered set.

Output A labeling coset \(\text{CLGroup}(G_f, \Delta) = \Lambda \subseteq \text{Label}(V)\) such that:

1. \(\text{CLGroup}(G_f, \Delta) = \varphi \text{CLGroup}(\varphi^{-1}Gf\varphi, \varphi^{-1}\Delta)\) for all \(\varphi \in \text{Iso}(V; V')\).

2. \(\text{CLGroup}(G_f, \Delta) = \text{Norm}_\Delta(G_f)\) for some \(\pi \in \Lambda\).

As usual, Conditions (CL1) and (CL2) given here coincide with the general Condition (CL1) and (CL2). For \(\Delta = \text{Sym}(V)\) and \(G_f = G\), the problem is equivalent to computing a canonical labeling for a permutation group (up to permutation isomorphism).

**Theorem 9.** A function CLGroup solving Problem 8 can be computed in time \(O\left(\frac{2^{|V|}}{k}\right)^{O(1)} \subseteq 2^{O(|V|)}\), where \(k\) is the size of the largest \(\Delta\)-orbit of \(V\).

**Intuition for the Permutation Group Algorithm** To solve Problem 8, we will maintain at any point in time a set \(A \subseteq V\) which is \(\Delta\)-invariant and \(G\)-invariant and for which we require Condition (A): \(\delta = \text{Norm}_{\Delta}(G_A)\) for \(G_A := \text{PointStab}_G(A)\). Intuitively, this means that the labeling coset \(\Delta\) consists of canonical labelings for the subgroup \(G_A\) which is obtained from \(G\) by a pointwise fixation of the set \(A\). This set \(A\) measures our progress in the sense the index of \(G_A\) in \(G\) is bounded by \(|A|!\) and thus if the set \(A\) is small, then we have already canonized a relatively large subgroup \(G_A\) of \(G\).

In the transitive case in which the set \(A\) is a \(\Delta\)-orbit, we decompose the labeling coset \(\Delta\) into (intransitive) labeling subcosets. Each labeling subcoset can be seen as an individualization of the permutation domain and can be handled recursively. Each recursive call leads to the case of intransitive labeling cosets which we explain next.

In the intransitive case where \(\Delta \subseteq \text{Stab}(A_1, A_2)\) and where \(G \subseteq \text{Stab}(A)\), we define a subgroup chain \(G_A = G_{A_1 \cup A_2} \leq G_{A_1} \leq G_{\{A_1\}} \leq G\). The subgroups are defined as \(G_{A_1} := \text{PointStab}_G(A_1)\) and \(G(A_1) := \text{Stab}_G(A_1)\). In a bottom up fashion, we will compute canonical labelings for all these groups. By Condition (A), a canonical labeling of the group \(G_A = G_{A_1 \cup A_2}\) is already available. Since \(G_{A_1 \cup A_2}\) is a subgroup of \(G_{A_1}\) obtained by a pointwise fixation of the set \(A_2\), a canonical labeling for \(G_{A_1}\) can be obtained recursively using the same algorithm with \(G_{A_1}\) in the role of \(G\) and \(A_2\) in the role of \(A\). After this recursive call, the canonical labeling for \(G_{A_1}\) is available. Similar, the group \(G_{A_1}\) is a subgroup of \(G_{\{A_1\}}\) obtained by pointwise fixation of
the set \( A_1 \), so also a canonical labeling for \( G_{(A_1)} \) can be obtained recursively. To compute a canonical labeling for \( G \), we use a different strategy that exploits that the index of \( G_{(A_1)} \) in \( G \) is simply-exponentially bounded. We decompose \( G = \bigcup_{g \in G} G_{(A_1)} g \) into right cosets of \( G_{(A_1)} \). Using the previous shifting algorithm, we can shift the canonical labeling for \( G_{(A_1)} \) (which is already computed) to all the cosets \( G_{(A_1)} g \). So far, the algorithm computed canonical labelings for all the cosets \( G_{(A_1)} g \). Finally, we make use of an object replacement paradigm together with the main algorithm of our recent framework to combine the collection of the canonical labelings. By doing so, this results in a canonical labeling for the entire group \( G \).

**Detailed Description of the Permutation Group Algorithm** Proving Theorem 9 we give a detailed description and analysis of the algorithm for permutation groups and cosets.

**Proof of Theorem 9.** For the purpose of recursion, we need an additional input parameter. Specifically, we use a subset \( A \subseteq V \) such that \( G, \Delta \leq \text{Stab}(A) \) and such that

\[
\Delta = \text{Norm}_\Delta(G_A) \text{ for } G_A := \text{PointStab}_G(A).
\]

Initially, we set \( A := V \).

An algorithm for \( \text{CLGroup}(G_f, A, \Delta \rho) \):

*If \( G_f = G \):*

  - Compute \( \Lambda_1 := \text{CLGroup}(G, A, \Delta \rho) \) recursively.
  - Compute and return \( \Lambda := \text{CLShift}(G_f, \Lambda_1) \) using the algorithm from Lemma 7.

*(Now, we achieved that \( G_f = G \leq \text{Sym}(V) \) is a group, rather than a proper coset.)*

*If \( |A| \leq 1 \):*

  - Return \( \Lambda := \Delta \rho \).

If \( \Delta \) is intransitive on \( A \):

  - Partition \( A = A_1 \cup A_2 \) where \( A_1 \subseteq A \) is a minimal size \( \Delta \)-orbit such that \( A_1^\rho \) is minimal. *(The minimality is w.r.t. the order “\( < \)” that is defined in Section 3.)*
  - Define \( G_{A_1} := \text{PointStab}_G(A_1) \) and define \( G_{(A_1)} := \text{Stab}_G(A_1) \).
  - (To compute the group \( G_{A_1} \), we use the Schreier-Sims algorithm and to compute the group \( G_{(A_1)} \), we use a membership test. The running times are specified in the preliminaries.)
  - Compute \( \Lambda_1 := \text{CLGroup}(G_{A_1}, A_2, \Delta \rho) \) recursively.
  - Compute \( \Lambda_2 := \text{CLGroup}(G_{(A_1)}, A_1, \Lambda_1) \) recursively.
  - Define \( \Lambda := \{ G_{(A_1)} g \mid g \in \Lambda \} \).
  - Compute \( \Delta X \rho X := \text{CLShift}(X, A_2) \) for all \( X = G_{(A_1)} g \in \Lambda \) using the algorithm from Lemma 7.
  - Define \( X_{\text{Set}} := \{ \Delta X \rho X \mid X = G_{(A_1)} g \in \Lambda \} \).
  - Define an ordered partition \( X_{\text{Set}}^\rho = X_{\text{Set}}^\rho_1 \cup \ldots \cup X_{\text{Set}}^\rho_s \) such that \( X_{\rho}^p < Y_{\rho}^q \), if and only if \( \Delta X \rho X \in X_{\text{Set}}^\rho_1 \) and \( \Delta Y \rho Y \in X_{\text{Set}}^\rho_q \) for some \( p, q \in [s] \) with \( p < q \).
  - Return \( \Lambda := \text{CLObject}(\{ X_1, \ldots, X_s \}, \Delta \rho) \) using the algorithm from Theorem 5.

If \( \Delta \) is transitive on \( A \):

  - Define \( A_{\text{Can}} := A^\rho \) and define \( \Delta_{\text{Can}} := (\Delta \rho)^\rho \).
  - Partition \( A_{\text{Can}} = A_{\text{Can}}^1 \cup A_{\text{Can}}^2 \) where \( A_{\text{Can}}^1 = \{ 1, \ldots, \lceil |A_{\text{Can}}| \rceil \} \).
  - Compute the subgroup \( \Psi_{\text{Can}} := \text{Stab}_\Delta_{\text{Can}}(A_{\text{Can}}^1) \).
  - (To compute the group \( \Psi_{\text{Can}} \), we use a membership test.)
Decompose $\Delta^\text{Can} = \bigcup_{i \in [s]} \delta_i^\text{Can} \Psi^\text{Can}$ as union of left cosets of the group $\Psi^\text{Can}$.

Compute $\Delta_{\rho_i} := \text{CL}_\text{Group}(G, A, \rho_i^\text{Can} \Psi^\text{Can})$ for each $i \in [s]$ recursively.

Rename the indices in $[s]$ such that:

$$(G, \Delta \rho^1)\Delta \rho^2 \ldots = (G, \Delta \rho)^{\rho_1} < (G, \Delta \rho)^{\rho_2} \ldots < (G, \Delta \rho)^{\rho_s}.$$  

We need to show that the algorithm outputs $\varphi^{-1} \Lambda$ instead of $\Lambda$.

In the case $G \neq \Lambda$, we obtain $\varphi^{-1} \Lambda_1$ instead of $\Lambda_1$ by induction. (By (CL1) of CL$_\text{Shift}$, we return $\varphi^{-1} \Lambda$ instead of $\Lambda$.)

The intransitive case is similar to the analysis used to proof Lemma 3, however for completeness, we will recall it. The ordered objects $A^\text{Can}$ and $\Delta^\text{Can}$ remain unchanged since $A^\varphi \varphi^{-1} = A^\varphi$ and $(\varphi^{-1} \Delta \rho)^{\varphi^{-1} \rho_1} = (\Delta \rho)^\rho$. Also the partition $A^\text{Can} = A^\text{Can}_1 \cup A^\text{Can}_2$ and the ordered group $\Psi^\text{Can}$ remains unchanged. We obtain cosets of the form $\varphi^{-1} \rho_i^\text{Can} \Psi^\text{Can}$ instead of $\rho_i^\text{Can} \Psi^\text{Can}$ since the indexing is arbitrary. The calls are of the form $\text{CL}_\text{Group}(\varphi^{-1} G \varphi, A^\varphi, \varphi^{-1} \Delta \rho\varphi^*$ instead of $\text{CL}_\text{Group}(G, A, \delta_j^\text{Can} \Psi^\text{Can})$. By induction, we obtain $\varphi^{-1} \Delta \rho_i$ instead of $\Delta \rho_i$. Therefore, we obtain $\varphi^{-1} \Lambda$ instead of $\rho_i$. However, the ordered sequence remains unchanged since $(G, \Delta \rho)^\rho = (\varphi^{-1} G \varphi, \varphi^{-1} \Delta \rho)^{\varphi^{-1} \rho_i}$. The computation of $\Lambda$ is known to be isomorphism invariant and therefore the algorithm returns $\varphi^{-1} \Lambda$ instead of $\Lambda$.

(0.1) In the case where $|A| \leq 1$, we have that $G_A = \text{Stab}_G(A) = G$. Combined with Condition (A), it follows that $\Delta = \text{Norm}_\Delta(G_A) = \text{Norm}_\Delta(G)$.

By object replacement (Lemma 3), $\Lambda$ defines a canonical labeling for $(X^\text{Set}_1, \ldots, X^\text{Set}_n, \Delta \rho)$. By object replacement (Lemma 3), $\Lambda$ defines a canonical labeling for $(X^\text{Set}_1, \ldots, X^\text{Set}_n, \Delta \rho)$ where $X = X_1 \cup \ldots \cup X_n$ such that $X \in X_i$, and if only if $\Delta \rho \varphi^* \in X_i^\text{Set}$. Since $(X^\text{Set}_1, \ldots, X^\text{Set}_n, \Delta \rho)$ is an ordered partition of $X$ defined in an isomorphism-invariant way, it holds that $\Lambda$ defines a canonical labeling for $(\Lambda, \Delta \rho)$.

Consider the intransitive case and observe that Condition (CL1) of the problem CL$_\text{Group}$ already implies that $\text{Norm}_\Delta(G) \pi \in \Lambda$ for some $\pi \in \Lambda$. It remains to show that the reversed inclusion also holds, i.e., $\Lambda \subseteq \text{Norm}_\Delta(G) \pi$. Equivalently, we need to show $\rho_i \rho_j^\pi \in \text{Norm}_\Delta(G)$ for all $i, j \in [r]$. The membership $\rho_i \rho_j^\pi \in \text{Norm}(G)$ follows from the equation $G^{\rho_i} = G^\rho$ and the membership $\rho_i \rho_j^\pi \in \Delta$ follows similarly from the equation $(\Delta \rho)^{\rho_i} = (\Delta \rho)^\rho$. (Running time.) First, we consider the number of recursive calls of this algorithm. In Case $G \neq \Lambda$, there is one single recursive call that leading to a different case, so this case can be
neglected in our analysis. It remains to consider the recursive calls of the other cases. Let \( A^* \subseteq A \) be a \( \Delta \)-orbit that is of maximal size. We claim that the maximum number of recursive calls \( R([A^*], |A|) \) is bounded by \( T := 2^{|A^*||A|^2} \). In the intransitive case, this can be seen by induction:

\[
R([A^*], |A|) \leq 1 + \sum_{j \in [2]} R([A^*], |A_j|) \leq 1 + 2^{|A^*|(|A_1|^2 + |A_2|^2)} \leq T.
\]

In the transitive case, it holds that \( A^* = A \) and \( s \leq 2|A| \) and which leads to

\[
R([A], |A|) \leq 1 + s \cdot R([|A|/2], |A|) \leq 1 + 2^{|A|+3|A|^2} \leq T.
\]

Next, we give a bound on the running time that is needed for one single call without recursive costs. Let \( k \) be the size of the largest \( \Delta \)-orbit for the initial instance. In Case \( Gf \neq G \), we use the algorithm from Lemma 7 which runs in time \( 2^O(k)|V|^{O(1)} \). In the intransitive case, we need to compute a group \( G_1(A_1) \). The index of \( G_1(A_1) \) in \( G \) is bounded by the \( G \)-orbit of \( A_1 \) which in turn is at most \( b := \binom{|V|}{k} \). The group \( G_1(A_1) \) can be computed in time polynomial in the index and \( |V| \), i.e., \( (|V|)^{O(1)} \). We have \( b \) calls to the algorithm from Lemma 7 that run in time \( 2^O(k)|V|^{O(1)} \) per instance. Similar, the representation size of \( \chi^{\text{Set}} \) is bounded by \( (|V|)^{O(1)} \) and therefore \( \Lambda \) can be computed in time \( 2^O(k)(|V|)^{O(1)} \). In the transitive case, the group \( \Psi^N \) is computed in time polynomial in the index and \( |V| \), i.e., \( 2^O(k)|V|^{O(1)} \).

In total, we have a running time of at most \( T \cdot 2^O(k)(|V|)^{O(1)} \leq \left( \frac{2|V|}{k} \right)^{O(1)} \).

**Corollary 10.** Canonical labelings for permutation groups and cosets (up to permutational isomorphism) can be computed in time \( \left( \frac{2|V|}{k} \right)^{O(1)} \) where \( V \) is the permutation domain and \( k \) is the size of the largest color class of \( V \).

**Proof.** We need to a compute canonical labelings for \((Gf, C_1, \ldots, C_t)\) where \( Gf \subseteq \text{Sym}(V) \) is a group coset over the domain \( V = C_1 \cup \ldots \cup C_t \). This is done by calling the previous algorithm with input \((Gf, \Delta \rho)\) where \( \Delta \rho = \{ \lambda \in \text{Label}(V) \mid \forall i, j \in [t], i < j \forall v_i \in C_i, v_j \in C_j : \lambda(v_i) < \lambda(v_j) \} \).

## 5 Outlook and Open Questions

We showed that canonicalization of permutation groups on a permutation domain \( V \) can be done in \( 2^O(|V|) \) regardless of the order \( |G| \) of the group. Our result generalizes the known time bound of \( 2^O(|V|)|G|^{O(1)} \).

However, in the setting of bounded color class size \( k \), we did not achieve a generalization. In the this work, we presented an algorithm for implicitly given permutation groups running in time \( |V|^{O(k)} \). However, for explicitly given groups \( G \), an algorithm is known that runs in time \( 2^O(k)|V|^{O(1)}|G|^{O(1)} \) [SW18]. Since the running times are orthogonal to each other, we ask for an unifying running time for canonization of permutation groups in \( 2^O(k)|V|^{O(1)} \).

With our result for permutational isomorphisms most of the studied isomorphism problems have simply-exponential time bounds now. However, the isomorphism problem for implicitly given group codes (also known as group code equivalence) does not have such a time bound. While the problem for linear codes reduces to permutational isomorphism in polynomial time, the situation for cyclic groups seems difficult. It is an open problem if group code equivalence for cyclic groups \( G \) of prime power order (such as \( \mathbb{Z}/p^2\mathbb{Z} \)) and codes of length \( |V| \) can be decided in \( 2^O(|V| \log(|G|))^{O(1)} \).
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