LUSTERNIK–SCHNIRELMANN THEORY AND CLOSED REEB ORBITS

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Abstract. We develop a variant of Lusternik–Schnirelmann theory for the shift operator in equivariant Floer and symplectic homology. Our key result is that the spectral invariants are strictly decreasing under the action of the shift operator when periodic orbits are isolated. As an application, we prove new multiplicity results for simple closed Reeb orbits on the standard contact sphere, the unit cotangent bundle to the sphere and some other contact manifolds. We also show that the lower Conley–Zehnder index enjoys a certain recurrence property and revisit and reprove from a different perspective a variant of the common jump theorem of Long and Zhu. This is the second, combinatorial ingredient in the proof of the multiplicity results.

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1. **Introduction and main results**

1.1. **Introduction.** In this paper we develop a variant of Lusternik–Schnirelmann theory for the shift operator in equivariant Floer and symplectic homology and show that the spectral invariants are strictly decreasing under the action of the shift operator when periodic orbits are isolated. As an application of this theory, we obtain new multiplicity results for simple (i.e., un-iterated) closed Reeb orbits on, e.g., $S^{2n-1}$ and $ST^*S^n$. We also establish, as a second, combinatorial ingredient in the proof of these multiplicity results, a recurrence property of the lower Conley–Zehnder index and use it to reprove from a different perspective a variant of the common jump theorem from [LZ]; see also [DLW].

1.1.1. **Lusternik–Schnirelmann theory.** This theory, [LS], hinges on the general principle that the minimax critical values associated with homology classes decrease under cohomology operations. Moreover, they decrease strictly when the critical sets are sufficiently small, e.g., when the critical points are isolated. This principle, which we sometimes call the Lusternik–Schnirelmann inequality, applies to a broad class of functionals and some (perhaps, many) cohomology operations. In fact, the authors are not aware of any instance where the principle would convincingly break down. An alternative way to think about the principle is that although the local (co)homology of an isolated critical point can be arbitrarily large, it does not support non-trivial cohomology operations such as the cup product or Massey products. The Lusternik–Schnirelmann inequality is usually applied to a chain of homology classes to establish a lower bound on the number of critical values, and hence critical points, of a functional. Such lower bounds do not require any non-degeneracy conditions, but are weaker than those coming from Morse theory.

Here, referring the reader to, e.g., [GG09a, Sec. 6] and [Vi97] for a more thorough treatment of the question and references, we only illustrate the point by a few simple examples.

The most elementary variant of the theory is the Lusternik–Schnirelmann theorem for smooth functions, giving a lower bound for the number of critical values of a function with isolated critical points via the cup-length of its domain. Ultimately, it is based on the fact that the minimax value associated with the intersection product $v \cap w$ of two homology classes is strictly smaller than the minimax values for both $v$ and $w$, provided of course that the critical points are isolated and neither $v$ nor $w$ is the fundamental class. Note that this variant of the theory entirely bypasses the notion of the Lusternik–Schnirelmann category. The classical Lusternik–Schnirelmann theorem on the existence of three simple geodesics on $S^2$ is another incarnation of the same principle applied now to the length functional, [Gr].

In Floer theory the minimax critical values are usually referred to as spectral invariants, and Lusternik–Schnirelmann theory is at the very heart of the proof of the known cases of the Arnold conjecture for degenerate Hamiltonians; see, e.g., [Fl89a, Fl89b, Ho, How13, LO, Sc]. Here the functional is the action functional...
and the (co)homology operation is the pair-of-pants product or the action of the quantum homology on the Floer homology. (See [GG09a, Sec. 6.2] for a detailed discussion of the Hamiltonian Arnold conjecture in this context and further references.)

When the underlying space is equipped with an $S^1$-action, the pairing with the generator of $H^2(\mathbb{C}P^\infty)$ gives rise to an operator $D$ of degree $-2$ on the $S^1$-equivariant homology. We call $D$ the shift operator. When the functional is $S^1$-invariant, one can expect a variant of Lusternik–Schnirelmann theory to hold for $D$; cf. [Gi14]. For instance, for a smooth $S^1$-invariant function, the critical values associated with equivariant homology classes over $\mathbb{Q}$ are strictly decreasing under the action of $D$ when the $S^1$-action is locally free; see Section 2.1. One can think of this fact as simply the Lusternik–Schnirelmann theorem for the quotient orbifold.

The shift operator $D$ in the equivariant symplectic homology is the connecting map in the Gysin exact sequence relating the equivariant and non-equivariant symplectic homology. It is defined in [BO13b] via counting Floer trajectories matching marked points on the orbits. One of the main results of the present paper (Theorem 1.1) is the Lusternik–Schnirelmann theorem for the equivariant symplectic homology. This result is an easy consequence of the Lusternik–Schnirelmann inequality for the equivariant Floer homology of autonomous Hamiltonians (Theorem 1.2). One point of independent interest that arises in the proof is that while the original definition of $D$ is well adapted for the construction of the Gysin exact sequence, it is not perfectly suited for the proof of the Lusternik–Schnirelmann inequality. Hence we give a different definition of $D$ via the intersection index with the “hyperplane-section” cycle, and prove the equivalence of the two definitions in Section 2.2.4. A local counterpart of these results is the fact that $D \equiv 0$ on the level of the local symplectic or Floer homology of an isolated closed orbit (non-constant in the Hamiltonian case); see Proposition 2.21 and Corollary 3.7.

Applying the Lusternik–Schnirelmann theorem to Reeb flows on the standard contact $S^{2n-1}$ and $ST^*S^n$, we obtain sequences of closed Reeb orbits with strictly decreasing actions, satisfying certain index constraints; see Theorems 1.3 and 1.4. These theorems are central to the proofs of our multiplicity results (Theorems 1.5 and 1.6) discussed in more detail below.

Lusternik–Schnirelmann theory for the shift operator in equivariant symplectic homology has some closely related predecessors which have also been used to obtain lower bounds on the number of simple periodic orbits. For instance, in the framework of the classical calculus of variations, we have the Lusternik–Schnirelmann theorem for the shift operator in a suitably defined equivariant homology for convex Hamiltonians on $\mathbb{R}^{2n}$, based on the Fadell–Rabinowitz index, [FR]; see, e.g., [Ek, Sec. V.3] and also [Lo02, Chap. 15]. Also, there is a similar result for the energy functional on the space of closed loops yielding lower bounds for the number of closed geodesics on, say, $S^n$; see [BTZ]. More recently, the Lusternik–Schnirelmann inequality for $D$ in ECH was used to prove the existence of at least two simple periodic Reeb orbits on every closed contact three-manifold, [CGH]. In this case, the nature of the inequality is particularly transparent because the shift operator is defined by counting holomorphic curves “passing” through a fixed point. Placing this point away from the relevant periodic orbits, one obtains a lower bound on the energy of the holomorphic curve by the monotonicity lemma.
1.1.2. Multiplicity results for closed Reeb orbits and index theory. As has been mentioned above, our main applications of Lusternik–Schnirelmann theory are to multiplicity results for simple closed Reeb orbits (Theorems 1.5 and 1.6). Arguably, in this class of questions beyond dimension three, the most interesting manifolds are the standard contact $S^{2n-1}$ and $ST^*S^n$, i.e., the boundaries of star-shaped domains in $\mathbb{R}^{2n}$ and $T^*S^n$. For these manifolds, on the one hand, we have strong benchmark multiplicity results for simple closed characteristics on convex hypersurfaces and closed geodesics of Riemannian or Finsler metrics, which we will discuss shortly, and, on the other, we have precise general conjectures on the minimal number of simple periodic orbits.

Namely, hypothetically, any Reeb flow on the standard contact $S^{2n-1}$ has at least $r_{\min} = n$ simple closed Reeb orbits. For $ST^*S^n$, we have $r_{\min} = n$ when $n$ is even and $r_{\min} = n+1$ when $n$ is odd. In both cases, the lower bounds, which we will refer to as the multiplicity conjectures, are sharp. For $S^{2n-1}$, the example is an irrational ellipsoid and for $ST^*S^n$ the Katok–Ziller Finsler metrics have exactly $r_{\min}$ simple closed geodesics, [Ka, Zì]. A closely related fact is that for both manifolds the positive equivariant symplectic homology is rather small. In particular, in contrast with most other cotangent bundles and some other contact manifolds, there is no homological growth which would imply the existence of infinitely many simple periodic orbits; see, e.g., [CH, GM, HM, McL]. Nor do we have in this case the contact Conley conjecture type phenomena; see [GGM, GG15].

The existence of at least two simple closed Reeb orbits on $S^3$ and $ST^*S^2$ is a very particular case of the theorem from [CGH] mentioned above. This result was also proved, specifically for $S^3$ and $ST^*S^2$, in [GH2M, GGo, LL] by different methods; see also [BL] for the case of Finsler metrics on $S^2$.

When $n \geq 3$, the multiplicity conjectures are completely open. In general, it is not even known if there are at least two simple closed Reeb orbits on $S^{2n-1}$ or $ST^*S^n$, i.e., $r_{\min} \geq 2$. (It is easy to show that this is true in the non-degenerate case, [Gü15, Rmk. 3.3].)

However, once certain convexity or index conditions are imposed on the contact form, the question becomes much more tractable. Of course, convexity is not invariant under symplectomorphisms, and in the context of symplectic topology it is usually replaced by dynamical convexity introduced in [HWZ] for $\mathbb{R}^4$. A contact form on $S^{2n-1}$ is said to be dynamically convex if every closed Reeb orbit has Conley–Zehnder index $\mu \geq n + 1$ or, without non-degeneracy assumptions, $\mu_- \geq n + 1$ where $\mu_-$ is the lower Conley–Zehnder index; see Section 4. This lower bound on $\mu_-$, which follows from geometrical convexity, appears to be a suitable replacement for convexity of hypersurfaces in $\mathbb{R}^{2n}$. For $ST^*S^n$, the requirement is that $\mu_- \geq n - 1$, which is a consequence of certain curvature pinching conditions if the contact form comes from a Riemannian or Finsler metric on $S^n$; see, e.g., [HP, Ra04, Wa12, DLW].

The multiplicity conjectures for non-degenerate contact forms on $S^{2n-1}$ and $ST^*S^n$ are proved in [AM15, DL2W, DLW, GuKa, Lo02, LZ, Wa13, WHL] under the above or even much weaker index (or convexity) requirements. The proofs are usually based on a Morse theoretic methods using the size of the relevant homology to obtain lower bounds on the number of simple orbits. We review these results in detail in the next section and, in fact, reprove some of them in Section 6.
However, our main interest in this paper is the general case of possibly degenerate contact forms, and this is where Lusternik–Schnirelmann theory becomes absolutely essential. We prove that when $\mu_- \geq n + 1$ for $S^{2n-1}$ or $\mu_- \geq n - 1$ for $ST^*S^n$, the Reeb flow has at least $r = \lceil n/2 \rceil + 1$ simple periodic orbits; see Theorems 1.5 and 1.6. Leaving a comparison with previous results to the next section, we only mention here that this lower bound for $S^{2n-1}$ is a generalization of the lower bounds established in [LZ, Wa16] for convex hypersurfaces in $\mathbb{R}^{2n}$. We also obtain some additional information about the orbits when the Reeb flow has only finitely many simple closed orbits. In particular, we show that then the orbits satisfy certain action–index resonance relations, generalizing the relations from [Gü15], and that at least one of the orbits is elliptic; see Section 6.

The key difficulty intrinsic in this class of problems is that neither Lusternik–Schnirelmann nor Morse theory can distinguish simple from iterated orbits. In other words, while both theories do provide lower bounds on the number of periodic orbits in a given action or index range, these orbits need not be geometrically distinct and can in general be iterations of just one simple orbit. To circumvent this difficulty, we use what is essentially a combinatorial argument closely related to the common jump theorems proved in [DLW, LZ]. (See Section 5 where we establish a certain index recurrence property (Theorems 5.1 and 5.2) and prove a variant of the common jump theorem.) There is a considerable overlap between Section 5 and the results from those papers. Our treatment of the problem is, however, self-contained and quite different from [DLW, Lo02, LZ], and the proofs are relatively straightforward.

1.2. Results. Now we are ready to state precisely some of the key results of the paper starting with Lusternik–Schnirelmann theory for equivariant symplectic and Floer homology; see Sections 2 and 3 for more details.

Let $(M^{2n-1}, \alpha)$ be a closed contact manifold and let $W$ be an exact strong symplectic filling of $M$ with $c_1(TW)|_{\pi_2(W)} = 0$; see Section 3.1.1. Abusing notation, let us denote the positive equivariant symplectic homology of $W$ over $\mathbb{Q}$ by $\text{SH}_W^{G,+} = \text{SH}_W^G$, where $G = S^1$, graded by the Conley–Zehnder index. Let $D : \text{SH}_W^{G,+}(W) \to \text{SH}_{W}^{G,\pm}(W)$ be the shift operator introduced in [BO13b]. There is a spectral invariant $c_{\text{w}}(\alpha) \in (0, \infty)$, a point in the action spectrum $\mathcal{S}(\alpha)$, associated to any non-zero element $w \in \text{SH}_W^{G,+}(W)$. (By definition, $c_0(\alpha) = -\infty$.)

Denote by $\mathcal{P}(\alpha)$ the collection of contractible in $W$ closed Reeb orbits (not necessarily simple) of $\alpha$. An orbit $x \in \mathcal{P}(\alpha)$ is said to be isolated if it is isolated in the extended phase space $M \times (0, \infty)$, i.e., there exists a tubular neighborhood $U$ of the image of $x$ in $M$ such that no other closed Reeb orbit with period sufficiently close to the period of $x$ intersects $U$.

One of our main results is

**Theorem 1.1** (Lusternik–Schnirelmann inequality; symplectic homology). Assume that all orbits in $\mathcal{P}(\alpha)$ are isolated. Then, for any non-zero element $w \in \text{SH}_W^{G,+}(W)$, we have $c_w(\alpha) > c_{D(w)}(\alpha)$.

This theorem, proved in Section 3.1.2, readily follows from a similar inequality for the equivariant Floer homology. Let $H$ be an autonomous Hamiltonian on a symplectic manifold $V$ which is symplectically aspherical and either closed or the symplectic completion of a compact domain $W$ with contact type boundary. (In the
latter case, \( H \) is also required to be admissible at infinity.) We denote the filtered \( G = S^1 \)-equivariant Floer homology of \( H \) over \( \mathbb{Q} \) for an interval \( I \) by \( HF^G_{*}I(H; \mathbb{Q}) \). In this case we also have a shift operator \( D: HF^G_{*}I(H; \mathbb{Q}) \to HF^G_{*}I(H; \mathbb{Q}) \) (see Section 2.2), and to every non-zero class \( w \in HF^G_{*}I(H; \mathbb{Q}) \) we can associate a spectral invariant \( c_w(H) \in \mathcal{S}(H) \cap I \), where \( \mathcal{S}(H) \) is the action spectrum of \( H \).

**Theorem 1.2** (Lusternik–Schnirelmann inequality; Floer homology). Assume that all contractible one-periodic orbits of \( H \) with action in \( I \) are non-constant and isolated. Then for any non-zero class \( w \in HF^G_{*}I(H; \mathbb{Q}) \), we have

\[
c_w(H) > c_{D(w)}(H).
\]

This theorem is proved in Section 2.4. Note also that in both Theorems 1.1 and 1.2, the non-strict inequalities hold without any additional assumptions on the periodic orbits and readily follow from the definitions.

As has been mentioned in the introduction, to prove Theorem 1.2 we give an alternative definition of \( D \) for which the proof of the Lusternik–Schnirelmann inequality is more natural and show in Section 2.2.4 that this definition is equivalent to the one from [BO13b]. Another consequence of the proofs of Theorems 1.1 and 1.2 is that the operator \( D \) vanishes in the local symplectic or Floer homology over \( \mathbb{Q} \) of an isolated non-constant periodic orbit; see Sections 2.3 and 3.1.2.

Next, let us apply these results to the equivariant symplectic homology of \( S^{2n-1} \) and \( ST^*S^n \). Let \( \alpha \) be a contact form on \( M = S^{2n-1} \) supporting the standard contact structure on \( M \). Then \( (M, d\alpha) \) can be symplectically embedded as a hypersurface in \( \mathbb{R}^{2n} \) bounding a star-shaped domain \( W \). As is well known, there exists a sequence of elements \( w_k \in SH_{n+2k-1}^G(W) \), \( k \in \mathbb{N} \), such that \( Dw_k = w_k \). Set \( c_k := c_{w_k} \) and denote by \( \hat{\mu}(y) \) the mean index of \( y \in \mathcal{P}(\alpha) \) and, when \( y \) is isolated, by \( SH_G^G(y; \mathbb{Q}) \) the local equivariant symplectic homology of \( y \); see Section 3.1.2. Applying Theorem 1.1 to the classes \( w_k \), we obtain

**Theorem 1.3.** Assume that all closed Reeb orbits of \( \alpha \) are isolated. Then

\[
c_1(\alpha) < c_2(\alpha) < c_3(\alpha) < \cdots.
\]

As a consequence, there exists an injection

\[
\psi: \mathbb{N} \to \mathcal{P}(\alpha), \quad k \mapsto y_k
\]

called a carrier map such that

\[
\mathcal{A}_\alpha(y_1) < \mathcal{A}_\alpha(y_2) < \mathcal{A}_\alpha(y_3) < \cdots
\]

and \( SH_G^G(y_k; \mathbb{Q}) \neq 0 \) in degrees \( * = n + 2k - 1 \). In particular,

\[
|\hat{\mu}(y_k) - (n + 2k - 1)| \leq n - 1.
\]

This theorem is a symplectic homology analog of the Lusternik–Schnirelmann inequalities for Clarke’s dual action functional; see [Ek, Lo02] and references therein. Its slightly more general version is proved in Section 3.2.1 as Corollary 3.9.

A similar result holds for \( ST^*S^n \) but the chain of inequalities has length \( n \). Namely, let \( \alpha \) be a contact form supporting the standard contact structure on \( M = ST^*S^n \). We can treat \( (M, d\alpha) \) as the boundary of a fiber-wise star-shaped domain \( W \) in \( T^*S^n \). By Proposition 3.12, for every \( j \in \mathbb{N} \) there exist \( n \) non-zero elements \( w_i \in SH_{2i + (2j-1)(n-1)}^G(W; \mathbb{Q}) \), \( i = 0, \ldots, n-1 \), such that \( Dw_{i+1} = w_i \).
Theorem 1.4. Assume that all orbits in \( P(\alpha) \) are isolated. Then, for every \( j \in \mathbb{N} \), there exist \( n \) periodic orbits \( y_0, \ldots, y_{n-1} \) of the Reeb flow of \( \alpha \) such that \( c_{w_1}(\alpha) = A_\alpha(y_i) \) and \( \text{SH}^{c_1}(y_i; \mathbb{Q}) \neq 0 \) in degrees \( * = (2j - 1)(n - 1) + 2i \). In particular,
\[
A_\alpha(y_0) < A_\alpha(y_1) < \cdots < A_\alpha(y_{n-1}),
\]
and
\[
|\hat{\mu}(y_i) - ((2j - 1)(n - 1) + 2i)| \leq n - 1.
\]

This result is proved, in a slightly more general form, in Section 3.2.2 as Corollary 3.14. The non-strict inequalities in both of the theorems hold without any assumptions on the orbits. Furthermore, these theorems and the multiplicity results below readily extend with straightforward modifications to several other classes of contact manifolds and Liouville domains. Among these are, for instance, displaceable (e.g., subcritical) Liouville domains; see Section 3.2.1 and Remark 6.11.

Let us now turn to the multiplicity results for simple closed Reeb orbits. Although our main focus is on the general setting where the contact form can be degenerate and Lusternik–Schnirelmann theory is essential, we also consider for the sake of completeness the non-degenerate case where stronger results can usually be obtained by other methods.

We start with lower bounds on the number of simple closed Reeb orbits on \( S^{2n-1} \).

Theorem 1.5. Let \( \alpha \) be a dynamically convex contact form on \( S^{2n-1} \) supporting the standard contact structure. Then the Reeb flow of \( \alpha \) has at least \( r = \lceil n/2 \rceil + 1 \) simple closed orbits. When \( \alpha \) is non-degenerate, \( r = n \). Moreover, assume in addition that there are only finitely many geometrically distinct closed orbits. Then these \( r \) simple orbits \( x \) can be chosen so that the ratios \( A(x)/\hat{\mu}(x) \) are the same for all of them (the resonance relations) and, when \( \alpha \) is non-degenerate, all \( r \) simple orbits are even.

This theorem is an immediate consequence of Theorem 6.1 (multiplicity) and Theorem 6.4 (resonance relations). The part of the theorem on the resonance relations generalizes the relations for perfect Reeb flows from [Gü15, Thm. 1.2].

Except for the resonance relations, the non-degenerate case of the theorem is not new and included only for the sake of completeness and because the proof requires no extra work. For strictly convex hypersurfaces in \( \mathbb{R}^{2n} \) the result goes back to [LZ] and [Lo02]. (Here convexity is understood in a very strong sense: the (outward) second fundamental form is negative definite at every point.) For dynamically convex hypersurfaces, the non-degenerate case of the theorem is proved in [AM15, GuKa] under the slightly less restrictive index condition that \( \mu \geq n - 1 \). More recently, in [DL2W], this index requirement is relaxed even further.

The degenerate case of the theorem is new. For convex hypersurfaces, a similar result is proved in [LZ] with the same lower bound when \( n \) is even and the lower bound \( \lceil n/2 \rceil + 1 \) when \( n \) is odd, and the case of a convex hypersurface in \( \mathbb{R}^6 \) is treated in [WHL]. Recently, again for convex hypersurfaces, the lower bound exactly as in Theorem 1.5 has been established in [Wa16] for all \( n \). The existence of at least two distinct orbits in the dynamically convex case is proved in [AM15].

To the best of our understanding, Theorem 1.5 incorporates essentially all, but one, results to date on the multiplicity of simple closed Reeb orbits on convex or dynamically convex hypersurfaces in \( \mathbb{R}^{2n} \) without additional assumptions such as non-degeneracy, pinching or symmetry. The exception is a theorem from [Wa13]
asserting the existence of four closed characteristics on a convex hypersurface in $\mathbb{R}^8$. This is the first dimension where the lower bound from Theorem 1.5, which for $n = 4$ is three, is below the multiplicity conjecture lower bound $n$.

Next, let us turn to contact forms $\alpha$ on $M = ST^* S^n$ supporting the standard contact structure. The pair $(M, \alpha)$ is then the boundary of a fiber-wise star-shaped domain in $T^* S^n$. In this setting, the right analog of the dynamical convexity condition is the requirement that $\mu_- \geq n - 1$ for all closed Reeb orbits of $\alpha$. For the unit sphere bundle of a Riemannian or Finsler metric, this requirement is satisfied, for instance, when the metric meets certain curvature pinching conditions; see, e.g., [AM14, DLW, HP, Ra04, Wa12]. Along the lines of Theorem 1.5, we have the following result.

**Theorem 1.6.** Let $\alpha$ be a contact form on $M = ST^* S^n$ supporting the standard contact structure. Assume that $\mu_-(y) \geq n - 1$ for every closed Reeb orbit $y$ on $M$. Then $M$ carries at least $r$ simple closed orbits, where $r = \lceil n/2 \rceil - 1$. When $\alpha$ is non-degenerate, we can take $r = n$ if $n$ is even and $r = n + 1$ if $n$ is odd.

This theorem is restated as Theorem 6.13 and proved in Section 6.2. Again, the non-degenerate case of the theorem is not new and included only for the sake of completeness. A much more general result is established in [AM15]. (However, the argument in Section 6.2 is self-contained.)

There are three main ingredients to the proofs of Theorems 1.5 and 1.6. The first and the major one is, of course, Lusternik–Schnirelmann theory discussed extensively above. The second one, having considerable overlap with the results in [DLW, LZ, Lo02], is the combinatorial index analysis from Section 5. These are sufficient to prove Theorem 1.6 and generalize the results of [LZ] to dynamically convex hypersurfaces, but not to refine those results. The refinement in Theorem 1.5 comes from an application of one of the key theorems from [GHFM]. Namely, it turns out that in every second dimension (odd $n$) a dynamically convex Reeb flow on $S^{2n-1}$ with exactly $\lfloor n/2 \rfloor + 1$ geometrically distinct closed orbits must have a simple closed orbit of a particular type, the so-called symplectically degenerate maximum (SDM). A Reeb flow with a simple SDM orbit necessarily has infinitely many periodic orbits, [GHFM]. Hence, for an odd $n$, there must be at least $\lfloor n/2 \rfloor + 2 = \lceil n/2 \rceil + 1$ closed Reeb orbits.

There are several ways, completely within the scope of our methods, to generalize Theorems 1.5 and 1.6 and other results of this type, which, although of interest, are either not considered here at all or only discussed very briefly. For instance, one can generalize Theorem 1.5 by making the lower bound depend on the “degree of non-degeneracy”. This is the approach taken in [LZ, Lo02]. Secondly, the dynamical convexity condition $\mu_- \geq n + 1$ in Theorem 1.5 and the condition $\mu_- \geq n - 1$ in Theorem 1.6 can also be quantified and replaced by the condition $\mu_- \geq q$ yielding, in general, weaker lower bounds on the number of simple closed orbits. (See Theorems 6.9 and 6.15. For $S^{2n-1}$ and $q = n - 1$, we recover [GuKa, Thm. 1.4]. In the non-degenerate case, conceptually stronger results are now available; see [DL2W, DLW].)

In the setting of Theorems 1.5 and 1.6, one can draw some conclusions on the existence of simple closed orbits of specific type (e.g., elliptic) when the number of simple closed orbits is finite; cf. [AM15, GuKa, LZ, Lo02]. We mainly leave this question aside; see, however, Remark 6.2. Generalizations of these theorems to some other contact manifolds are noted in Remark 6.11. Finally, we touch upon
the Ekeland–Lasry theorem from the perspective of Lusternik–Schnirelmann theory for the shift operator; see Section 6.1.3 and, in particular, Corollary 6.12.

1.3. Organization of the paper. The rest of the paper is organized as follows. In Section 1.4 we specify in detail our conventions and notation.

In Section 2 we outline the construction of the equivariant Floer homology and prove the Lusternik–Schnirelmann inequality for the shift operator. This section carries the most technical and conceptual load in the paper. We start by discussing the shift operator and the Lusternik–Schnirelmann inequality in the equivariant homology for $S^1$-actions in Section 2.1. Formally speaking, the results and constructions from this section are never used in the rest of the paper. However, on the conceptual level the difference between Morse theory and Floer theory is purely technical in this setting and we often refer to Section 2.1 when dealing with its Floer theoretic counterpart. In Section 2.2 we define equivariant Floer homology and the shift operator for autonomous Hamiltonians and state some of our main results. Section 2.3 is a digression on local equivariant Floer homology and other localization constructions. The main objective of Section 2.4 is the proof of the Lusternik–Schnirelmann inequality in Floer homology. This is the key technical part of the paper. Finally, in Section 2.5, we show that the equivariant Floer homology for Hamiltonians with non-constant periodic orbits can be interpreted as the homology of a complex generated by periodic orbits by proving that the underlying Morse–Bott spectral sequence collapses in the $E^2$-term. This construction comes handy in the proofs of multiplicity results in the non-degenerate case.

The main objective of Section 3 is to define the shift operator in the equivariant symplectic homology and prove the Lusternik–Schnirelmann inequalities. We do this in Section 3.1 where we also briefly discuss local equivariant symplectic homology and its properties. Examples and applications to $S^{2n-1}$, the unit cotangent bundle $ST^*S^n$ and some other contact manifolds are worked out in Section 3.2.

Sections 2.2 and 3.1 comprise an introduction to equivariant Floer and symplectic homology, following mainly [BO12, BO13b] with some modifications.

In Section 4 we introduce some basic concepts from the Conley–Zehnder index theory and (re)prove several auxiliary results. For instance, in Section 4.2 we give a simple proof of the fact that convexity implies dynamical convexity. In Section 5 we establish the index recurrence theorem and a variant of the common jump theorem, [DLW, LZ], which are both essential for deriving multiplicity results from the Lusternik–Schnirelmann inequalities. Sections 4 and 5 can be read independently of the rest of the paper, and Section 4 is intended to be a reasonably self-contained, although concise, introduction to index theory.

Finally, in Section 6, we prove the multiplicity theorems for simple closed Reeb orbits and the action-index resonance relations. We also discuss other related results including the Ekeland–Lasry theorem.

1.4. Conventions and notation. In this section we spell out the conventions and notation used throughout the paper.

We usually assume that the underlying symplectic manifold $(V,\omega)$ is symplectically aspherical, i.e., $\omega|_{\pi_2(V)} = 0 = c_1(TV)|_{\pi_2(V)}$, and either compact or $V$ is the symplectic completion of a Liouville domain $(\tilde{W},\omega = da)$. To be more specific, in the latter case, we have

$$V = \tilde{W} = W \cup_M (M \times [1,\infty))$$
with the symplectic form $\omega$ extended to the cylindrical part as $d(r\alpha)$, where $\alpha$ is a contact primitive of $\omega$ on $M = \partial W$ and $r$ is the coordinate on $[1, \infty)$. (See Section 3.1.1 for more details.) As noted in Remarks 2.15 and 3.8, these conditions can be modified or relaxed.

We denote by $\mathcal{P}(\alpha)$ the set of contractible in $W$ periodic orbits of the Reeb flow on $(M^{2n-1}, \alpha)$ and by $\mathcal{S}(\alpha)$ the period or action spectrum, i.e., the collection of their periods or, equivalently, contact actions.

The circle $S^1 = \mathbb{R}/\mathbb{Z}$ plays several different roles throughout the paper. We denote it by $G$ when we want to emphasize the role of the group structure on $S^1$.

The Hamiltonians $H$ on $V$ are always required to be one-periodic in time, i.e., $H: S^1 \times V \to \mathbb{R}$. In fact, most of the time the Hamiltonians we are actually interested in are autonomous, i.e., independent of time. The (time-dependent) Hamiltonian vector field $X_H$ of $H$ is given by the Hamilton equation $i_{X_H} \omega = -dH$. For instance, on the cylindrical part $M \times [1, \infty)$ with $\omega = d(r\alpha)$ the Hamiltonian vector field of $H = r$ is the Reeb vector field of $\alpha$.

We focus on contractible one-periodic orbits (or $k$-periodic with $k \in \mathbb{N}$) of $H$. Such orbits can be identified with the critical points of the action functional $\mathcal{A}_H: \Lambda \to \mathbb{R}$ on the space $\Lambda$ of contractible loops $x$ in $V$ given by

$$\mathcal{A}_H(x) = \mathcal{A}(x) - \int_{S^1} H(t, x(t)) \, dt,$$

where $\mathcal{A}(x)$ is the symplectic area bounded by $x \in \Lambda$, i.e., the integral of $\omega$ over a disk with boundary $x$. (Modifications needed to work with non-contractible orbits are discussed in Remarks 2.15 and 3.8.) When $V = \hat{W}$, we always require $H$ to be admissible, i.e., to have the form $H = \kappa r + c$, where $\kappa \notin \mathcal{S}(\alpha)$, outside a compact set. Under this condition the Floer homology of $H$ is defined; see, e.g., [Vi99]. Note, however, that the homology depends on $\kappa$.

The action spectrum of $H$, i.e., the collection of action values for all contractible one-periodic orbits of $H$, will be denoted by $\mathcal{S}(H)$. When $H$ is autonomous, a one-periodic orbit $y$ is said to be a reparametrization of $x$ if $y(t) = x(t + \theta)$ for some $\theta \in G = S^1$. Two one-periodic orbits are said to be geometrically distinct if one of them is not a reparametrization of the other. We denote by $\mathcal{P}(H)$ the collection of all geometrically distinct contractible one-periodic orbits of $H$ and by $\mathcal{P}(H, I)$ the collection of such orbits with action in an interval $I$.

With our sign conventions in the definitions of $X_H$ and $\mathcal{A}_H$, the Hamiltonian actions converge to the contact actions in the construction of the symplectic homology; see Section 3.1.1. In particular, $\mathcal{S}(H) \to \mathcal{S}(\alpha)$.

We normalize the Conley–Zehnder index, denoted throughout the paper by $\mu$, by requiring the flow for $t \in [0, 1]$ of a small positive definite quadratic Hamiltonian $Q$ on $\mathbb{R}^{2n}$ to have index $n$. More generally, when $Q$ is small and non-degenerate, the flow has index equal to $(\text{sgn} Q)/2$, where sgn $Q$ is the signature of $Q$; see Section 4.1.2. In other words, the Conley–Zehnder index of a non-degenerate critical point of a $C^2$-small autonomous Hamiltonian $H$ on $V^{2n}$ is equal to $n - \mu_{sl}$, where $\mu_{sl} = \mu_{sl}(H)$ is the Morse index of $H$. The Floer homology and symplectic homology are graded by the Conley–Zehnder index.

These action and index conventions differ by sign for both the action and the index from the conventions in most of our recent papers (see, e.g., [GG09c, GG10, GG14]) and to our taste are rather awkward to use when $V$ is closed, although we still have $\text{HF}_*(H) \cong H_{*+n}(V)$ in this case. (The reason is that $\mathcal{A}_H$ restricted
to the space $V \subset \Lambda$ of constant loops is $-H$. Thus, for a $C^2$-small autonomous Hamiltonian $H$, the Floer complex of $H$ graded by the Conley–Zehnder index is the Morse complex of $-H$ graded by $n - \mu_{\text{M}}(H) = \mu_{\text{M}}(-H) - n$; see [SZ].) However, we find these conventions convenient when $V$ is open, which is the main focus of this paper.

The differential in the Floer or Morse complex is defined by counting the downward Floer or Morse trajectories. As a consequence, a monotone increasing homotopy of Hamiltonians induces a continuation map preserving the action filtration in homology. (Clearly, $H \geq K$ on $S^1 \times V$ if and only if $A_H \leq A_K$ on $\Lambda$.) In many instances in this paper, the coefficient ring in Floer or symplectic homology has to have zero characteristic and then we take it to be $\mathbb{Q}$. When the coefficient ring is immaterial, it is omitted from the notation.

Our choice of signs in (1.1) also has effect on the Floer equation. The Floer equation is the $L^2$-anti-gradient flow equation for $A_H$ on $\Lambda$ with respect to a metric $\langle \cdot, \cdot \rangle$ compatible with $\omega$:

$$\partial_s u = -\nabla_{L^2} A_H(u),$$

where $u: \mathbb{R} \times S^1 \to V$ and $s$ is the coordinate on $\mathbb{R}$. Explicitly, this equation has the form

$$\partial_s u + J\partial_t u = \nabla H,$$

(1.2)

where $t$ is the coordinate on $S^1$. Here the almost complex structure $J$ is defined by the condition $\langle \cdot, \cdot \rangle = \omega(J \cdot, \cdot)$ making $J$ act on the first argument in $\omega$ rather than the second one, which is more common, to ensure that the left hand side of the Floer equation is still the Cauchy–Riemann operator $\bar{\partial}_J$. (In other words, $J = -J_0$, where $J_0$ is defined by acting on the second argument in $\omega$, and $X_H = -J\nabla H$.) Note, however, that now the right hand side of (1.2) is $\nabla H$ with positive sign.

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2. Shift operator: Morse and Floer homology

Our goal in this and the next sections is to develop Lusternik–Schnirelmann theory for the shift operator in equivariant Floer and symplectic homology. We do this with some redundancy in several steps starting with Morse theory and then moving on to Floer theory and, in Section 3, to symplectic homology.

Recall that we denote the circle $S^1 = \mathbb{R}/\mathbb{Z}$ by $G$ when it is treated as a group rather than a manifold. Furthermore, unless the coefficient ring is specified, a homology group can be taken with arbitrary coefficients. However, the choice of coefficients is essential in our strict Lusternik–Schnirelmann inequality results: Theorems 2.1, 2.12, 3.4, and Corollary 3.9. In this case, the coefficient field must have zero characteristic and we take to be $\mathbb{Q}$. 
2.1. Shift operator in equivariant Morse theory. To set the stage for studying the shift operator in equivariant Floer and symplectic homology, let us start with the toy model of Morse theory and ordinary equivariant homology.

2.1.1. Lusternik–Schnirelmann theory for the shift operator in equivariant homology. Recall that for a $G = S^1$-principal bundle $\pi: P \to B$ we have the Gysin exact sequence

$$
\ldots \to H_* (P) \xrightarrow{\pi^*} H_* (B) \xrightarrow{D} H_{*-2} (B) \xrightarrow{\pi^*} H_{*-1} (P) \to \ldots
$$

(2.1)

Here the operator $\pi^*$ sends a class $[C]$ in $B$, where $C$ is a cycle, to the class $[\pi^{-1} (C)]$ in $P$ and the operator $D$, the one we are interested in, is given by the paring with $-c_1 (\pi)$, the negative first Chern class of the principle $G$-bundle $\pi$. In other words, on the level of cycles, $D$ is the intersection product with the Poincaré dual of $-c_1 (\pi)$. (The negative sign here is not particularly important; its role is to make this definition of $D$ equivalent to its standard Morse theoretic counterpart; see Section 2.1.2.)

Next, consider a closed manifold $Y$ with an action of the circle $G$. The equivariant homology of $Y$ is by definition the homology $H_*^G (Y) = H_* (Y \times_G EG)$ where $Y \times G EG := (Y \times EG)/G$ and the product is equipped with the diagonal circle action; see, e.g., [GGK, Appen. C] for more details. We can take the limit of odd-dimensional spheres $S^{2m+1}$ with the Hopf action as $EG$ and $BG = \mathbb{CP}^\infty$. Since $EG$ is contractible, the Gysin exact sequence (2.1) for the principal bundle $\pi: Y \times EG \to Y \times_G EG$ takes the form

$$
\ldots \to H_* (Y) \xrightarrow{\pi^*} H_*^G (Y) \xrightarrow{D} H_{*-2}^G (Y) \xrightarrow{\pi^*} H_{*-1} (Y) \to \ldots
$$

(2.2)

As is easy to see, $c_1 (\pi)$ is the pull-back of the first Chern class of the universal bundle $EG \to BG = \mathbb{CP}^\infty$, the generator of $H^2 (\mathbb{CP}^\infty)$, under the natural projection $Y \times_G EG \to BG$, and $D$ is the paring with $-c_1 (\pi)$. We call $D$ the shift operator in equivariant homology. Alternatively, one can replace here the universal bundle by the Hopf bundle $S^{2m+1} \to \mathbb{CP}^m$ and then pass to the limit as $m \to \infty$.

To every non-zero $w \in H_*^G (Y)$ and a smooth $G$-invariant function $f: Y \to \mathbb{R}$ one can associate a spectral invariant or a critical value selector $c_w (f)$. This is done exactly as in the non-equivariant case. For instance, we can set

$$
c_w (f) = \inf \{ a \in \mathbb{R} \mid w \in \text{im} (i^a) \},
$$

where $i^a : \{ f \leq a \} \to Y$ is the natural inclusion. Alternatively, $c_w (f)$ can be defined using the standard minimax construction for cycles representing $w$. It is not hard to show that $c_w (f)$ is a critical value and $c_w$ has all the expected properties of critical value selectors; see, e.g., [GG09a, Sect. 6.1]. Finally, let us set $c_0 (f) = -\infty$.

**Theorem 2.1.** Assume that the $G = S^1$-action on $Y$ is locally free, i.e., the action has no fixed points, and that the critical sets of $f$ are isolated $G$-orbits and $w \neq 0$ in $H_*^G (Y; \mathbb{Q})$. Then

$$
c_w (f) > c_{D(w)} (f).
$$

(2.3)

This is the Lusternik–Schnirelmann inequality for the shift operator in the equivariant homology.

**Remark 2.2.** The conditions that the action is locally free and that the critical sets of $f$ are isolated $G$-orbits are essential; as is the assumption that the homology is taken over $\mathbb{Q}$. Also note that here only the strict inequality requires a proof.
For the non-strict inequality $c_w(f) \geq c_{D(w)}(f)$ is a general feature of critical value selectors, which holds without any restrictions on $f$ or on the coefficient ring; see [GG09a, Sect. 6.1].

Proof. First, note that since the sequence of graded vector spaces $H_*(Y \times_G S^{2m+1})$ converging to $H^*_G(Y)$ stabilizes in every degree, we can replace $EG$ in the construction by a sphere $S^{2m+1}$ of sufficiently high dimension. Let us pull-back $f$ to $P = Y \times S^{2m+1}$ and then push forward the resulting function to the smooth manifold $B = P/G$. We denote the push-forward by $\overline{f}$. Clearly, $\overline{f}$ is a smooth function on $B$ and the critical manifolds of $\overline{f}$ are lens spaces. (A critical circle $S$ of $f$ gives rise to the critical manifold $C = S \times_G S^{2m+1}$ of $\overline{f}$ diffeomorphic to the lens space $S^{2m+1}G$, where $G$ is the stabilizer of the orbit $S$ of $G$.)

To prove (2.3), it is sufficient to show that

$$c_w(\overline{f}) > c_{D(w)}(\overline{f})$$

for any $w \in H_*(B;\mathbb{Q})$.

Lemma 2.3. Let $\overline{f}$ be a smooth function on a closed manifold $B$ such that

$$c_w(\overline{f}) = c_{w \cap \text{PD}(v)}(\overline{f})$$

for some non-zero $w \in H_*(B)$ and $v \in H^*G(B)$, where PD stands for the Poincaré duality map and $\cap$ is the intersection product. Then the restriction of $v$ to the critical set $K$ of $\overline{f}$ with critical value $c_w(\overline{f})$ is non-zero in $H^*(K)$. (If $K$ is not smooth, $H^*(K)$ is the Alexander–Spanier cohomology of $K$.)

This lemma is central to Lusternik-Schnirelmann theory. For instance, as a consequence of the lemma, we have $c_w(\overline{f}) > c_{w \cap \text{PD}(v)}(\overline{f})$ when the critical points of $\overline{f}$ are isolated, and hence in this case the number of critical values is strictly greater than the cup-length of $B$. The lemma is not new (see, e.g., [Vi97, Thm. 1.1] or [GG09a, Rmk. 6.8]) and we omit its proof.

In the setting of Theorem 2.1, we have $D(w) = -w \cap \text{PD}(c_1(\pi))$, where $c_1(\pi) \in H^2(B;\mathbb{Q})$ is the first Chern class of the principal $G$-bundle $\pi: P \to B$. As has been mentioned above, the critical manifolds of $\overline{f}$ are the lens spaces $C$, and hence $K$ is a disjoint union of several such lens spaces. Since $H^2(C;\mathbb{Q}) = 0$, we have $H^2(K;\mathbb{Q}) = 0$, and therefore $c_1(\pi)|_K = 0$. Thus the strict inequality in (2.4) follows from the lemma.

Remark 2.4. The condition that the manifold $Y$ is closed can be relaxed. Namely, it is sufficient to assume that $Y$ is a manifold with boundary and the function $f: Y \to I = [a, b]$ is locally constant on $\partial Y$ and $f(\partial Y) \subset \{a, b\}$. In other words, without any assumption on $Y$, the results from this section hold for any proper function $f$ on $Y$ and the filtered Morse homology of $f$ with a range of values $I$, i.e., for $H_*(f^{-1}(I), \{f = a\};\mathbb{Q})$, such that all critical sets of $f$ within $I$ are isolated $G$-orbits with finite stabilizer.

A Floer theoretic analog of Lemma 2.3 for closed symplectic manifolds has been proved in [How12, How13] by using the pair-of-pants product. With the definition of the product from [AS], the argument readily extends to admissible Hamiltonians on the symplectic completion of a Liouville domain. Alternatively – and this would be a simpler approach – one can employ the action of (quantum) homology on the filtered Floer homology; see, e.g., [LO], and also [MS, Rmk. 12.3.3] and [GG14, Sect. 2.3] and [Li, Vi95].
One difficulty in extending Theorem 2.1 to the Floer theory setting lies in that the definition of the operator $D$ in Floer homology, given in [BO13b], is based on the standard Morse theoretic description of $D$ which is different from the one used above. Thus, before turning to the Floer theoretic setting, let us discuss the two definitions of the operator $D$ in Morse theory in more detail. The results from the next section are nowhere directly used in the paper and serve only as an illustration and motivation for the Floer theoretic arguments in Section 2.2.

### 2.1.2. Two definitions of the shift operator in equivariant Morse theory.

Let, as in Section 2.1.1, $\pi: P \to B$ be a principal $G = S^1$-bundle. We assume that $B$ is a closed manifold and $F: B \to \mathbb{R}$ is a Morse function on $B$ with critical points $x_i$. Then $\tilde{F} = F \circ \pi$ is a Morse-Bott function on $P$ with critical sets $S_i = \pi^{-1}(x_i)$. We denote by $\mu_{\mathcal{M}}(S_i)$ the transverse Morse index of $\tilde{F}$ at $S_i$, i.e., the Morse index of $F$ at $x_i$. Let us recall the description of the Morse (or rather Morse–Bott) homology $\text{HM}_*(\tilde{F})$ of $\tilde{F}$ using broken trajectories. On each circle $S_i$ we fix a Morse function $g_i$ with exactly one maximum $S^+_i$ and one minimum $S^-_i$. Let us also fix a $G$-invariant metric on $P$. To work over $\mathbb{Z}$ or an arbitrary ring, for every $x_i$ we also need to pick a co-orientation of the unstable manifold of $x_i$. The graded module $\text{CM}_*(\tilde{F})$, over any ground ring, is generated by $S^+_i$ with grading $\mu_{\mathcal{M}}$ determined by setting $\mu_{\mathcal{M}}(S^+_i) = \mu_{\mathcal{M}}(S_i) + 1$ and $\mu_{\mathcal{M}}(S^-_i) = \mu_{\mathcal{M}}(S_i)$.

The differential $\partial$: $\text{CM}_*(\tilde{F}) \to \text{CM}_{*+1}(\tilde{F})$ decomposes as $\partial = \partial_1 + \partial_2$. The term $\partial_1$ counts two-stage (or one-stage) broken Morse trajectories from $S_i$ to $S_j$ with $\mu_{\mathcal{M}}(S_j) = \mu_{\mathcal{M}}(S_i) - 1$, connecting minima to minima or maxima to maxima. For instance, a coefficient $\langle S^+_i, S^+_j \rangle$ in $\partial S^+_i$ is the number, with signs, of broken trajectories made of an anti-gradient trajectory $\eta: (-\infty, 0] \to S_i$ of $g_i$ starting at $S^+_i$ and then the anti-gradient trajectory of $\tilde{F}$ from $\eta(0)$ to $S^+_j$. The term $\partial_2$ counts one-stage trajectories of $\tilde{F}$ connecting minima to maxima with $\mu_{\mathcal{M}}(S^+_j) = \mu_{\mathcal{M}}(S_i) - 2$.

To be more precise, when $\mu_{\mathcal{M}}(S_j) = \mu_{\mathcal{M}}(S_i) - 2$, we have $\partial_2 S^+_i = 0$ and

$$\partial_2 S^+_i = \sum_j \langle S^-_i, S^+_j \rangle S^+_j,$$

where $\langle S^-_i, S^+_j \rangle$ is the number, with signs, of anti-gradient trajectories of $\tilde{F}$ from $S^-_i$ to $S^+_j$.

Note that, strictly speaking, the decomposition of $\partial$ should also include the term $\partial_0$ coming from the standard Morse differential for $g_i$, but this term is obviously zero since the Morse functions $g_i$ are perfect.

It is a standard fact that $\partial^2 = 0$ and that the homology of $(\text{CM}_*(\tilde{F}), \partial)$, called the Morse or Morse–Bott homology of $\tilde{F}$ is canonically isomorphic to $\text{H}_*(P)$. Note that that the module $\text{CM}_*(\tilde{F})$ is completely determined by $\tilde{F}$, but the differential $\partial$ depends on the auxiliary data including the functions $g_i$ and the metric on $P$.

Let $\mathcal{C}^\pm$ be the submodules of $\text{CM}_*(\tilde{F})$ generated by $S^\pm_i$. Then $\partial_1: \mathcal{C}^\pm \to \mathcal{C}^\mp$, i.e., the decomposition $\text{CM}_*(\tilde{F}) = \mathcal{C}^+ \oplus \mathcal{C}^-$ is preserved by $\partial_1$. Furthermore, $\partial_2(\mathcal{C}^+) = 0$ and $\partial_2(\mathcal{C}^-) = \mathcal{C}^-$. Hence, $(\mathcal{C}^+, \partial_1)$ is a subcomplex of $\text{CM}_*(\tilde{F})$ and the quotient complex $\text{CM}_*(\tilde{F})/\mathcal{C}^+$ is also isomorphic to $(\mathcal{C}^-, \partial_1)$. Essentially by definition, the homology of $(\mathcal{C}^+, \partial_1)$ and $(\mathcal{C}^-, \partial_1)$ is isomorphic to the Morse homology of $F$, i.e., to $\text{H}_*(B)$, up to the shift of grading. Then the exact sequence of complexes

$$0 \to (\mathcal{C}^+, \partial_1) \to (\text{CM}_*(\tilde{F}), \partial) \to (\mathcal{C}^-, \partial_1) \to 0$$
gives rise to a long exact sequence in homology with connecting map \( \delta \) induced by \( \partial_2 : C^- \to C^+ \) on the homology of these complexes. This is the Gysin sequence (2.1), although now we used a different definition of the connecting map.

**Lemma 2.5.** *The two definitions of the shift operator are equivalent: \( D = \delta \).*

**Proof.** Let us first translate the definition of \( D \) from Section 2.1.1 to the language of Morse homology. Fix a Morse–Smale metric on \( B \) and denote by \( \mathcal{M}(x_i, x_j) \) the moduli space of anti-gradient trajectories \( u \) of \( F \) from \( x_i \) to \( x_j \). This is a smooth manifold of dimension \( \mu_{\text{sd}}(x_i) - \mu_{\text{sd}}(x_j) \). We orient the moduli spaces \( \mathcal{M}(x_i, x_j) \) using coherent orientations. Denote by \( \text{ev} : \mathcal{M}(x_i, x_j) \to B \) the evaluation map \( u \to u(0) \) and let \( \Sigma \) be a cycle Poincaré dual to \( -c_1(\pi) \). (Since this class has degree two, \( \Sigma \) can be taken to be a co-oriented submanifold of \( B \).) For a generic choice of \( \Sigma \), this cycle is transverse to all evaluation maps \( \text{ev} \) and the number of intersections of \( \text{ev}(\mathcal{M}(x_i, x_j)) \) with \( \Sigma \) is finite when \( \mu_{\text{sd}}(x_j) = \mu_{\text{sd}}(x_i) - 2 \). Let \( \langle x_i, x_j \rangle_{\Sigma} \in \mathbb{Z} \) be the intersection index of \( \text{ev} \) and \( \Sigma \), and

\[
D_{\Sigma} x_i = \sum_j \langle x_i, x_j \rangle_{\Sigma} x_j. \tag{2.5}
\]

Then \( D_{\Sigma} \) commutes with the Morse differential on \( \text{CM}_{\ast}(F) \), and the operator \( D \) induced by \( D_{\Sigma} \) on the Morse homology is the shift operator from the Gysin sequence (2.1). In what follows, it will also be convenient to pick \( \Sigma \) so that every un-parametrized trajectory intersects \( \Sigma \) at most once; clearly this is true for a generic choice of \( \Sigma \).

Since \( \delta \) is induced by \( \partial_2 \), to prove the lemma, it suffices to show that

\[
\langle x_i, x_j \rangle_{\Sigma} = \langle S^-_i, S^+_j \rangle \tag{2.6}
\]

for a suitable lift of the metric to \( P \).

Consider a closed two-form \( \sigma \) supported in a small tubular neighborhood \( U \) of \( \Sigma \) and “Poincaré dual” to \( \Sigma \). (In particular, \([\sigma] = -c_1(\pi)\).) When \( \mu_{\text{sd}}(x_j) = \mu_{\text{sd}}(x_i) - 2 \), the pull back \( \text{ev}^* \sigma \) has compact support and

\[
\langle x_i, x_j \rangle_{\Sigma} = \int_{\mathcal{M}(x_i, x_j)} \text{ev}^* \sigma. \tag{2.7}
\]

Note that here we need to take \( U \) sufficiently small since for a single point \( x_i \) the evaluation map \( \text{ev} \) need not in any sense be a cycle.)

Therefore, to finish the proof of the lemma it remains to show that the right hand sides of (2.6) and (2.7) are equal. To this end, fix a connection on \( \pi \) with curvature \(-2\pi \sigma \) and consider the standard lift of the metric from \( B \) to \( P \) using this connection. (The authors are aware of and apologize for this clash of \( \pi \)'s; see, e.g., [GGK, Appen. A] for a discussion of the relevant curvature conventions.) Then the anti-gradient trajectories of \( \tilde{F} \) are the horizontal, i.e., tangent to the connection, lifts of the anti-gradient trajectories of \( F \). Let \( \mathcal{M} \) be a connected component of \( \mathcal{M}(x_i, x_j) \). When \( U \) is sufficiently small, \( \text{ev}^* \sigma|_{\mathcal{M}} \) is supported in small disjoint disks \( B_k \) centered at the inverse images of the intersections \( \text{ev}(\mathcal{M}) \cap \Sigma \) and the integral of \( \text{ev}^* \sigma \) over \( B_k \) is \( \pm 1 \) with the sign determined by the sign of the intersection. Let \( \tilde{\mathcal{M}} = \mathcal{M}/\mathbb{R} \) be the space of un-parametrized trajectories. Since every un-parametrized trajectory intersects \( \Sigma \) at most once, the images \( \tilde{B}_k \) of the disks \( B_k \) in \( \tilde{\mathcal{M}} \) do not overlap if \( U \) is small.
For every un-parametrized trajectory $u \in \hat{\mathcal{M}}$, denote by $\hat{u}$ its horizontal lift to $P$ starting at $S_j^-$. We can view $\hat{\mathcal{M}}$ as a one-parameter family of trajectories and, as $u$ varies through this family, the end-point $\hat{u}(\infty)$ will traverse the circle $S_j$. When $u$ passes through $\hat{B}_k$, the end-point will make exactly one revolution in $S_j$ in the direction given by the sign of the intersection. When $u$ is outside the union of the intervals $\hat{B}_k$, the end-point $u(\infty)$ does not move because the connection is flat outside $U$. Furthermore, for such a connection chosen generically, this point is different from $S_j^+$. Thus the number, with signs, of un-parametrized anti-gradient trajectories of $\hat{F}$ in $\hat{\mathcal{M}}$ connecting $S_i^-$ to $S_j^+$ is equal to the integral of $ev^*\sigma$ over $\mathcal{M}$. In other words, this integral is exactly the contribution of $\mathcal{M}$ to the right hand side $(S_i^-, S_j^+)$ of (2.6). Taking the sum over all connected components of all two-dimensional moduli spaces, we obtain (2.6).

Remark 2.6. The requirements that every trajectory intersects $\Sigma$ only once and that the intervals $\hat{B}_k$ do not overlap are imposed only to make the proof geometrically more transparent and are not really essential. By tracking the end-point $\hat{u}(\infty) \in S_j$ as $u$ varies through $\hat{\mathcal{M}}$, it is not hard to show that even without these conditions the contribution of $\hat{\mathcal{M}}$ to $(S_i^-, S_j^+)$ is equal to the integral of $ev^*\sigma$ over $\mathcal{M}$. It is essential, however, that the critical points of $F$ are outside $U$ and hence $ev^*\sigma$ is compactly supported. (The parts of the broken trajectories in the compactification of $\hat{\mathcal{M}}$ have relative index one and thus do not intersect $\Sigma$ by transversality.)

The definitions of $D$ and $\delta$, and Lemma 2.5 carry over by continuity to the setting when $F$ is not necessarily Morse and the homology of $B$ and $P$ is replaced by the filtered Morse homology. Furthermore, applying the construction of $\delta$ and the lemma to the principal $G$-bundle

$$\pi: P = Y \times S^{2m+1} \to Y \times_G S^{2m+1} = B$$

with $F = \hat{f}$ in the notation of the proof of Theorem 2.1, we arrive at the Morse theoretic definition of the shift operator in the equivariant homology $H^G_*(f) \cong H^*_*(Y)$ and a Morse theoretic proof of the Gysin sequence (2.2).

2.1.3. Equivariant homology as the Morse homology on the quotient. When the $G$-action on $Y$ is locally free, the quotient $Y/G$ is an orbifold and as is well known $H^G_*(Y; \mathbb{Q}) \cong H_*(Y/G; \mathbb{Q})$; see, e.g., [GGK]. Furthermore, assume that the critical sets $S$ of a $G$-invariant function $f$ are isolated Morse–Bott non-degenerate $G$-orbits. Then the equivariant Morse homology of $f$ over $\mathbb{Q}$ is isomorphic to $H^G_*(Y; \mathbb{Q})$ and can be interpreted as the Morse homology of the push-forward of $f$ to the orbifold $Y/G$. In other words, the homology $H^G_*(Y; \mathbb{Q}) \cong H_*(Y/G; \mathbb{Q})$ can be viewed as the homology of a certain complex generated by the critical sets $S$ of $f$.

To be more specific, consider a critical set $S$. By our assumptions, $S$ is a $G = S^1$-orbit $G/\Gamma$ where $\Gamma$ is a cyclic subgroup. Let $\nu$ be the determinant line bundle (i.e., the top wedge) of the unstable bundle of $S$. Clearly, $G$ acts on $\nu$, and hence we have a representation $\Gamma \to \mathbb{Z}_2 = \{\pm1\}$ on the fiber of $\nu$. Let us call $S$ good if this representation is trivial and bad otherwise. For instance, $S$ is automatically good when $|\Gamma|$ is odd. These are proto-good/bad orbits in equivariant symplectic or contact homology.
Proposition 2.7. The equivariant homology $H_*^G(Y; \mathbb{Q})$ is equal to the homology of a certain complex generated by the good critical sets $S$ of $f$, graded by the Morse index, i.e., the dimension of the unstable manifold of $S$, and filtered by $f$.

In Section 2.5 we will establish, by a rather similar argument, a Floer theoretic analog of this result. The proof of the proposition hinges on the following general, purely algebraic lemma that, roughly speaking, asserts that a spectral sequence starting with $E^r_0$ can be reassembled into a single complex.

Lemma 2.8. Let $E^r$ with $r \geq r_0$ be a spectral sequence of complexes over a field of zero characteristic converging in a finite number of steps. Then there exists a differential $\bar{\partial}$ on $E^r$ such that $H_*(E^r, \bar{\partial}) = E^\infty$. Moreover, fixing an inner product on $E^r_0$ makes the choice of $\bar{\partial}$ canonical and $E^\infty$ is then identified with the space of harmonic representatives for $H_*(E^r, \bar{\partial})$.

Proof. Let us fix an inner product on $E^r_0$. Then we can identify $E^{r_0+1}$ with the orthogonal complement to $\text{im} \partial_{r_0}$ in $\ker \partial_{r_0}$ and, proceeding inductively, $E^{r+1}$ with the orthogonal complement to $\text{im} \partial_r$ in $\ker \partial_r$. Thus we have a nested sequence of bi-graded spaces

$$E^{r_0} \supset E^{r_0+1} \supset \ldots \supset E^k,$$

where we have assumed that $E^r$ converges in $k - r_0$ steps, i.e., $E^k = E^\infty$. The differential $\partial_r$ is a map $E^r \to E^r$ such that $E^{r+1} \subset \ker \partial_r$ and, moreover,

$$\ker \partial_r = \text{im} \partial_r \oplus E^{r+1}.$$

Denote by $V_r$ the orthogonal complement to $\ker \partial_r$ in $E^r$. We have the following orthogonal decomposition

$$E^{r_0} = (V_{r_0} \oplus \text{im} \partial_{r_0}) \oplus \ldots \oplus (V_{k-1} \oplus \text{im} \partial_{k-1}) \oplus E^k. \quad (2.8)$$

Let us extend $\partial_r$ to $\bar{\partial}_r : E^{r_0} \to E^r$ by setting $\bar{\partial}_r \equiv 0$ on the orthogonal complement to $E^r$. Then

$$\bar{\partial}_r : V_r \overset{\cong}{\longrightarrow} \text{im} \partial_r$$

is an isomorphism and $\bar{\partial}_r \equiv 0$ on all terms of the decomposition (2.8) other than $V_r$. Let

$$\bar{\partial} = \bar{\partial}_{r_0} + \ldots + \bar{\partial}_{k-1}.$$

As is clear from (2.8), $\bar{\partial}^2 = 0$ and $H_*(E^{r_0}, \bar{\partial}) = E^k$. Furthermore, $E^k$ is the space of harmonic representatives for $H_*(E^{r_0}, \bar{\partial})$. □

Proof of Proposition 2.7. Recall that given a Morse–Bott function $F$ on some manifold $P$ with critical sets $C$, its Morse–Bott homology can be defined by fixing a Morse function $g_C$ on every $C$ and then taking a complex, called the Morse–Bott complex, generated by the critical points of $g_C$. The differential on this complex is obtained by counting broken anti-gradient trajectories of the functions $g_C$ and $F$ as in Section 2.1.2, although now the manifolds $C$ need not be circles. The resulting complex is graded by the total Morse index, i.e., by the sum of the Morse–Bott index of $F$ at $C$ and the Morse index of a critical point of $g_C$.

Furthermore, the complex is filtered by the value of $F$. To be more precise, let us fix a sequence of regular values $a_p$ of $F$ such that every interval $(a_{p-1}, a_p)$ contains exactly one critical value. Then the filtration is given by the Morse–Bott complexes of $F$ for the intervals $(-\infty, a_p)$. We will call the resulting spectral sequence $(E^{r}_{p,q}, \partial_r)$ associated with this filtration the Morse–Bott spectral sequence...
of \( F \). By construction, this spectral sequence converges to \( E^\infty = H_\ast(P; \mathbb{Q}) \) in a finite number of steps. (For the range of \( p \) is finite.) Its \( E^1 \)-page is the direct sum of the homology spaces \( H_\ast(C; \mathcal{N}) \) with the local coefficient system \( \mathcal{N} \) given by the determinant line bundle of the unstable bundle of \( C \). In other words,

\[
E^1_{p,q} = \bigoplus_C H_{p+q}(C; \mathcal{N}),
\]

where the sum is taken over all critical manifolds of \( F \) with critical value in \((a_{p-1}, a_p)\). We note that this construction is different from that in Section 2.1.2 where we used the index filtration rather than the filtration by the value of \( F \); see Remark 2.11.

In the setting of the proposition, let us consider the Morse–Bott spectral sequence of the function \( F \) induced on \( B = Y \times_G S^{2m+1} \) by \( f \). Then the critical sets \( C \) are the lens spaces \( S \times_G S^{2m+1} = S^{2m+1}/\Gamma \). The local coefficient system \( \mathcal{N} \) (over \( \mathbb{Q} \)) is trivial if and only if the critical set \( S \) is good. Thus we see that the contribution of \( S \) to the \( E^0 \)-page, over \( \mathbb{Q} \), is concentrated in degrees 0 and \( 2m + 1 \) if \( S \) is good and only in degree \( 2m + 1 \) when \( S \) is bad. Let us pass to the limit as \( m \to \infty \). It is not hard to see that the limit spectral sequence, still denoted by \( E' \), exists for a suitable “coherent” choice of the Morse–Bott data for \( F \) (the metric and the functions \( g_C \)) for all \( m \in \mathbb{N} \); cf. \([GH^2M, Appendix]\). Furthermore, the limit sequence converges to \( H^0_c(Y; \mathbb{Q}) \) in a finite number of steps and has the \( E^1 \)-term generated by the good critical sets \( S \) of \( f \). We take an inner product on \( E^1 \) for which this collection of critical sets is an orthonormal basis and apply Lemma 2.8 with \( r_0 = 1 \). Then \((E^1, \bar{\partial})\) is the required complex, graded by \( \mu_m \) and filtered by \( f \). Alternatively, one could apply the lemma to \( E^1 \) for a finite value of \( m \) and then pass to the limit as \( m \to \infty \). \( \square \)

We conclude the section by several remarks elaborating on various aspects of these constructions.

**Remark 2.9.** The differential \( \bar{\partial} \) on \( E^1 \) is completely determined by the Morse–Bott data for \( F \) (the metric and the functions \( g_C \) for all \( m \in \mathbb{N} \)). Furthermore, it is clear from the construction that the differential is “natural”, i.e., a monotone decreasing homotopy from \( f_0 \) to \( f_1 \), gives rise to a homomorphism of the filtered complexes. Likewise, different choices of the Morse–Bott data result in isomorphic complexes. Note also that rather than working with the Morse–Bott complex in the proof of Proposition 2.7 we could have taken a non-degenerate perturbation of \( F \), resulting in exactly the same complex \((E^1, \bar{\partial})\) for a suitable choice of the perturbation.

**Remark 2.10.** The key idea of the constructions from Lemma 2.8 and the proof of Proposition 2.7 is that the complex \((E^0, \bar{\partial})\) can be much smaller and more convenient to work with than the original filtered complex giving rise to the spectral sequence. It is worth keeping in mind that applying the lemma to the \( E^0 \)-term, which is isomorphic to the original complex as a vector space, may result in a complex \((E^0, \bar{\partial})\) different from the original one. For instance, consider the filtered Morse complex of a Morse function. This complex comes with a preferred basis and hence is canonically isomorphic to \( E^0 \). Yet, the differential \( \bar{\partial} \) on \( E^0 \) need not be equal to the Morse differential. Likewise, in the setting of Proposition 2.7 when the \( G \)-action on \( Y \) is free, the complex \((E^1, \bar{\partial})\) is not necessarily equal to the Morse complex of the function \( \tilde{f} \) induced by \( f \) on the quotient \( Y/G \). In fact, for a suitable
choice of the Morse–Bott data, the complex $(E^1, \partial)$ is isomorphic to the complex $(\hat{E}^0, \hat{\partial})$ for $\hat{f}$.

**Remark 2.11.** In the construction of the Morse–Bott complex it would be more convenient to use the Morse–Bott index of $C$, rather than the value of $F$, to produce a filtration as in Section 2.1.2. In other words, we would set $p$ to be the Morse–Bott index of $C$ and $q$ to be the Morse index of a critical point of $\gamma_C$ in $E^0_{p,q}$. However, this is not always possible, for the differential can increase the Morse–Bott index. There are, however, two relevant particular cases when the Morse–Bott index does give rise to a filtration. One is when $\dim C \leq 1$ for all $C$ as, for instance, when the critical manifolds are circles. The second one is when $F$ is the pull-back of a Morse function on the base to a fiber bundle and the metric is a lift of a metric on the base. Both cases include the pull-back of a Morse function to a principle $S^1$-bundle considered in Section 2.1.2.

### 2.2. Shift operator in equivariant Floer theory

In this section we extend the constructions from Section 2.1 to equivariant Floer homology.

#### 2.2.1. The setting and the main result

Let $(V^{2n}, \omega)$ be a symplectic manifold. As in Section 1.4, we assume that $V$ is symplectically aspherical and that $V$ is either closed or $V$ is the symplectic completion $\widehat{W}$ of a compact symplectic manifold $W^{2n}$ with contact type boundary. The condition that $V$ is symplectically aspherical can be significantly relaxed but is sufficient for our purpose and simplifies the exposition; see Remark 2.15.

Fix an interval $I = [a, b]$ and let $H$ be a Hamiltonian on $V$ such that the endpoints of $I$ are outside the action spectrum $S(H)$ of $H$. (The interval $I$ can be infinite or semi-infinite. For instance, if $I = \mathbb{R}$, we set $a = -\infty$ and $b = \infty$.) When $V = \widehat{W}$ we also require $H$ to be admissible at infinity in the standard sense; see Section 3.1 for the definition. The filtered Floer homology $HF^I_*(H)$ is defined by continuity even though the periodic orbits of $H$ can be degenerate. Namely, we set $HF^I_*(H) := HF^I_*(H')$ where $H'$ is $C^\infty$-small non-degenerate perturbation of $H$. Since by our assumptions $S(H)$ is nowhere dense and the end-points of $I$ are outside $S(H)$, the homology spaces $HF^I_*(H')$ are canonically isomorphic for all sufficiently small perturbations $H'$ of $H$.

Assume in addition that $H$ is autonomous. Then we also have the equivariant filtered Floer homology $HF^{G,I}_*(H)$, where $G = S^1$, defined for contractible one-periodic orbits of $H$ with action in $I$; see [BO13b, Vi99]. We will recall in more detail the definition and some other relevant constructions in Section 2.2.2. As is proved in [BO13b], the homology spaces $HF^{G,I}_*(H)$ fit into the exact sequence

\[ \ldots \to HF^I_*(H) \to HF^{G,I}_*(H) \to HF^{G,I}_{*-1}(H) \to \ldots, \]

which is a Floer theoretic analog of the Gysin sequence (2.2). We emphasize that no non-degeneracy assumption on $H$ is needed here.

Finally, for a non-zero element $w \in HF^{G,I}_*(H)$ the spectral invariant or the action selector $c_w(H)$ is defined in the standard way. Namely, we set

\[ c_w(H) = \inf \left\{ b' \in I \setminus S(H) \mid w \in \im (i^{b'}) \right\} \]

where $i^{b'}$ is the natural map $HF^{G,I'}_*(H) \to HF^{G,I}_*(H)$ for $[a, b'] = I' \subset I = [a, b]$. We also set $c_0(H) = a$. As in the non-equivariant setting, $c_w(H)$ is monotone and Lipschitz (with Lipschitz constant equal to one) in $H$. 

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Abusing terminology, we say that a one-periodic orbit \( x \) of an autonomous Hamiltonian is isolated if all one-periodic orbits \( y \) intersecting its sufficiently small neighborhood in \( V \) are its reparametrizations, i.e., \( y(t) = x(t + \theta) \) for some \( \theta \in G \). Denote by \( \mathcal{P}(H, I) \) the collection of geometrically distinct contractible one-periodic orbits of \( H \) with action in \( I \). The key result of this section is

**Theorem 2.12.** Assume that all one-periodic orbits in \( \mathcal{P}(H, I) \) are isolated and non-constant. Then, for any non-zero element \( w \in HF^G_*(H; \mathbb{Q}) \), we have

\[
c_w(H) > c_{D(w)}(H). \tag{2.11}
\]

This is Theorem 1.2 from the introduction. We prove it in Section 2.4.

**Remark 2.13.** As in the case of equivariant Morse theory, the non-trivial point of the theorem is that the inequality is strict. By (2.13), the non-strict inequality \( c_w(H) \geq c_{D(w)}(H) \) holds without any assumptions on \( H \) and with any coefficients.

We emphasize again that in this theorem we impose no non-degeneracy requirements on \( H \). (When \( H \) is non-degenerate, the result easily follows from the definitions and is not particularly useful.) However, under the conditions of the theorem, the collection \( \mathcal{P}(H, I) \) is finite. Similarly to the finite-dimensional case, the requirements that all orbits in \( \mathcal{P}(H, I) \) are non-constant and isolated are essential.

Let \( \text{gap}(H) > 0 \) be the minimal positive action gap in \( S(H) \cap I \), i.e.,

\[
\text{gap}(H) = \min |A_H(x) - A_H(y)|, \tag{2.12}
\]

where the minimum is taken over all pairs of geometrically distinct orbits \( x \) and \( y \) in \( \mathcal{P}(H, I) \). Then, since \( c_w(H) \) and \( c_{D(w)}(H) \) are both elements of \( S(H) \), as an immediate consequence of the theorem, we have

**Corollary 2.14.** Under the conditions of Theorem 2.11,

\[
c_w(H) \geq c_{D(w)}(H) + \text{gap}(H) > c_{D(w)}(H).
\]

**Remark 2.15 (Generalizations and variations, I).** In both cases, when \( V \) is closed and when \( V = \tilde{W} \), the results from this section carry over to the equivariant Floer homology for periodic orbits in a fixed, possibly non-trivial, free homotopy class of loops in \( V \). To be more precise, let us fix such a class \( f \) and, if \( f \neq 1 \), also require \( V \) to be atoroidal. Under this assumption, the index and the action of a periodic orbit of \( H \) are defined by fixing a reference loop with a reference trivialization of \( TV \) along the loop; see, e.g., [BO13b, Gü13]. With this in mind, one can extend word-for-word the proofs to the non-contractible setting.

In particular, when \( V = \tilde{W} \) is exact and \( c_1(TW) = 0 \), one has the equivariant Floer homology defined for all free homotopy classes, naturally filtered by the action and graded by the free homotopy class. It is clear that the analogs of Theorem 2.12 and Corollary 2.14 hold in this case.

Furthermore, one can significantly relax our assumptions on \( V \). For instance, when \( f = 1 \), it would be sufficient to require \( V \) to be weakly monotone (see [HS]) and rational. The latter condition is needed to ensure that spectral invariants take values in \( S(H) \); see [Us]. Now, however, the Floer homology becomes a module over a Novikov ring and periodic orbits must be equipped with cappings, making the geometrical meaning of the results somewhat less clear. Finally, even the weak monotonicity assumption can be dropped at the expense of relying on the virtual cycle machinery or one of its variants such as polyfolds.
Before turning to the proof of Theorem 2.12 in Section 2.4, we need to recall, following [BO13b], the construction of the equivariant Floer homology $HF^G(H)$, define the shift operator $D$ and also establish in Section 2.2.4 a Floer theoretic analog of Lemma 2.5.

2.2.2. Equivariant Floer homology. Throughout this section the symplectic manifold $V$ and the Hamiltonian $H$ are as in Section 2.2.1. However, in contrast with Theorem 2.12, we do not impose any additional assumption on one-periodic orbits of $H$. The action interval $I$ plays only a superficial role, and hence, for the sake of brevity, we will suppress $I$ in the notation while keeping in mind that all the orbits of $H$ considered here are required to be in $\mathcal{P}(H, I)$. For instance, we will write $\mathcal{P}(H)$ and $HF^G(H)$ for $\mathcal{P}(H, I)$ and $HF^G,I(H)$, etc.

Our first goal is to define the equivariant Floer homology $HF^G(H)$. To this end, consider parametrized Hamiltonians $\tilde{H} : S^1 \times V \times S^{2m+1} \to \mathbb{R}$ invariant with respect to the diagonal $G = S^1$-action, i.e., such that $\tilde{H}(t + \theta, z, t \cdot \zeta) = \tilde{H}(t, z, \zeta)$ where $(t, z, \zeta) \in S^1 \times V \times S^{2m+1}$ and the dot denotes the Hopf action of $\theta \in G$ on $\zeta$. (When $V = \mathbb{W}$, the Hamiltonians $\tilde{H}$ are required to meet a certain admissibility condition at infinity. For instance, it is sufficient to assume that at infinity $\tilde{H}$ is independent of $(t, \zeta)$ and admissible in the standard sense; see Section 3.1.1.) We will sometimes write $\tilde{H}_\zeta(t, z)$ for $\tilde{H}(t, z, \zeta)$. Let $\Lambda$ be the space of contractible loops $S^1 \to V$. The Hamiltonian $\tilde{H}$ gives rise to the action functional

$$A_{\tilde{H}} : \Lambda \times S^{2m+1} \to \mathbb{R},$$

$$A_{\tilde{H}}(x, \zeta) = A(x) - \int_{S^1} \tilde{H}(t, x(t), \zeta) \, dt,$$

where $A(x)$ is the symplectic area bounded by $x \in \Lambda$. This functional is just a parametrized version of the standard action functional (1.1).

The critical points of $A_{\tilde{H}}$ are the pairs $(x, \zeta)$ satisfying the condition:

the loop $x$ is a one-periodic orbit of $\tilde{H}_\zeta$ and $\int_{S^1} \nabla_\zeta \tilde{H}_\zeta(t, x(t)) \, dt = 0$.

One can introduce the notion of transverse non-degeneracy for such critical points and then show that transversely non-degenerate parametrized Hamiltonians form a second Baire category set in the space of all parametrized Hamiltonians (admissible at infinity); see [BO10]. Here we only point out that due to the $G$-invariance of $\tilde{H}$ the critical points of $A_{\tilde{H}}$ come in families, even when $\tilde{H}$ is non-degenerate. Each family $S$ is an orbit of $G$, called a critical orbit, and hence the Hessian $d^2 A_{\tilde{H}}$ necessarily has a kernel. Thus this notion of transverse non-degeneracy, requiring the kernel to be one-dimensional, is more similar to the maximal, in the obvious sense, non-degeneracy of non-constant periodic orbits of autonomous Hamiltonians rather than the non-degeneracy of time-dependent Hamiltonians. Under our assumptions on $V$ and $H$, there are only finitely many critical orbits.

By the transverse non-degeneracy condition, the equivariant (or rather the invariant) Floer complex $\mathcal{CF}^G_*(\tilde{H})$ of $\tilde{H}$ is generated by the critical orbits $S$ and the differential $\partial^G$ is defined via the parametrized Floer equation (2.13). This complex is filtered by the action functional $A_{\tilde{H}}$ and graded by a variant of the Conley–Zehnder index $\mu(S)$ of $S$; see, e.g., [BO13a]. The homology $HF^G_*(\tilde{H})$ of $(\mathcal{CF}^G_*(\tilde{H}), \partial^G)$ is by definition the equivariant Floer homology of $\tilde{H}$, [BO13b, Vi99].
Before proceeding to the definition of the operator $D$, let us give some more details on this construction and revisit spectral invariants. Recall that a point in $V$ is denoted by $z$ and $\zeta$ stands for a point in $S^{2m+1}$. Fix a $(t, \zeta)$-dependent and $G$-invariant $\omega$-compatible almost complex structure $J$ on $V$ and a $G$-invariant metric on $S^{2m+1}$, and consider pairs of functions $\tilde{u} := (u, \lambda)$, where $u: \mathbb{R} \times S^1 \to V$ and $\lambda: \mathbb{R} \to S^{2m+1}$, satisfying the parametrized Floer equation
\[
\partial_t u + J\partial_s u - \nabla_x H_\lambda = 0,
\]
and asymptotic in the standard sense to some critical points $(x, \zeta) \in S$ and $(x', \zeta') \in S'$ of $A_{\tilde{H}}$ on $\Lambda \times S^{2m+1}$ as $s \to \pm \infty$. One should think of these equations as a parametrized version of the standard Floer equation (1.2). When certain natural transversality conditions are met – and this is the case generically – the moduli space of solutions, modulo the $\mathbb{R} \times G$-action, has dimension $\mu(S) - \mu(S') - 1$; see [BO10, BO13b]. (Here the critical orbits $S$ and $S'$ are fixed, but the points $(x, \zeta) \in S$ and $(x', \zeta') \in S'$ are allowed to vary.) Then, as in all other versions of the oriented Floer or Morse theory,
\[
\partial^G S = \sum \langle S, S' \rangle S',
\]
where $\langle S, S' \rangle$ is the number of points in such a moduli space, taken with signs given by coherent orientations, when $\mu(S) - \mu(S') = 1$.

The original autonomous Hamiltonian $H$ can also be viewed as a parametrized Hamiltonian independent of $t$ and $\zeta$. Let us pick a transversely non-degenerate perturbation $\tilde{H}$ sufficiently $C^\infty$-close to $H$. (When $V = \tilde{W}$, we can furthermore assume that $\tilde{H} \equiv H$ at infinity.) The equivariant Floer homology $HF^G_*(H)$ is by definition the limit of the homology spaces $HF^G_*(\tilde{H})$ as $\tilde{H} \to H$ and then $m \to \infty$. In the first of these limits, the graded homology stabilizes due to the assumption that the end-points of $I$ are outside $S(H)$. In the second limit, the graded spaces stabilize in every fixed degree. Hence, in what follows, we will assume that $m$ is fixed and set, abusing notation, $HF^G_*(H) := HF^G_*(\tilde{H})$ where $\tilde{H}$ is sufficiently close to $H$. This truncated homology is literally equal to $HF^G_*(H)$ for any finite range of degrees when $m$ is large enough.

Having this definition settled, let us take a second look at the spectral invariants $c_w(H)$. For a chain $C = \sum \lambda_S S \in CF_*(\tilde{H})$, set
\[
A_{\tilde{H}}(C) = \max_S \{ A_{\tilde{H}}(S) \mid \lambda_S \neq 0 \}.
\]
Then, for $w \in HF^G_*(H) = HF^G_*(\tilde{H})$, we have
\[
c_w(H) := \min_{|C|=w} A_{\tilde{H}}(C) \text{ and } c_w(H) = \lim_{\tilde{H} \to H} c_w(\tilde{H}).
\]
It is easy to see that this “minimax” definition of $c_w(H)$ is equivalent to (2.10). (The spectral invariant $c_w(\tilde{H})$ can also be defined similarly to (2.10).)

Example 2.16. When $V$ is a closed symplectically aspherical manifold, the global (i.e., for $I = \mathbb{R}$) equivariant Floer homology $HF^G_*(H)$ is not a very interesting object. The homology is independent of $H$ and hence $HF^G_*(H) \cong H_{*-n}(V) \otimes H_*(CP^\infty)$. (To prove this, it suffices to take $H$ autonomous and $C^2$-small.) The shift map $D$ is the pairing with the generator of $H^2(CP^\infty)$ on the second factor.
The construction of continuation maps extends to equivariant Floer homology word-for-word. Namely, let $H_s$, $s \in [0, 1]$, be a monotone increasing (i.e., $\partial_s H_s \geq 0$) homotopy of autonomous Hamiltonians on $V$. When $V = \hat{W}$, we also assume that all $H_s$ are admissible at infinity. Note that with our conventions $A_{H_s}$ is a monotone decreasing family of functionals on $\Lambda$. Then the homotopy $H_s$ gives rise to a homomorphism $\text{HF}_{s}^{G,I}(H_0) \to \text{HF}_{s}^{G,I}(H_1)$ as long as the end points of $I$ are outside $S(H_0) \cup S(H_1)$. This homomorphism is, in fact, independent of the homotopy $H_s$ with the initial and terminal Hamiltonians $H_0 \leq H_1$ fixed. Thus, for instance, one can take the linear homotopy $H_s = (1-s)H_0 + sH_1$. When a homotopy is not increasing, it still gives rise to a map in the filtered Floer homology with an action shift depending on the homotopy in the same way as for the standard Floer homology; see, e.g., [Gi07, Sect. 3.2.2].

2.2.3. The shift operator via the Floer trajectory count. Let us next briefly recall the definition of the shift operator $D$ from [BO13b]. That definition is nearly identical to the definition of the operator $\delta$ in Section 2.1.2 and we deliberately recycle the argument and notation to emphasize the similarity. We apologize for the redundancy.

Consider a general non-degenerate parametrized Hamiltonian $\hat{H}$, not necessarily $C^\infty$-close to $H$, and denote by $S_i$ its critical $G$-orbits. Now we treat $\hat{H}$ as a Morse-Bott Hamiltonian and want to define its non-equivariant Morse-Bott Floer complex and homology with the differential counting broken trajectories. On each circle $S_i$ we fix a Morse function $g_i$ with exactly one maximum $S_i^+$ and one minimum $S_i^-$. Consider the vector space $\text{CF}_{\ast}(\hat{H})$ generated by $S_i^\pm$ and graded, with our conventions in mind, by setting $\mu(S_i^+) = \mu(S_i) + 1$ and $\mu(S_i^-) = \mu(S_i)$. The differential $\hat{\partial}: \text{CF}_{\ast}(\hat{H}) \to \text{CF}_{\ast-1}(\hat{H})$ comprises two terms $\hat{\partial}_1$ and $\hat{\partial}_2$. The term $\hat{\partial}_1$ counts two-stage (or one-stage) broken trajectories from $S_i$ to $S_j$ with $\mu(S_j) = \mu(S_i) + 1$ connecting $S_i^-$ to $S_j^-$ or $S_i^+$ to $S_j^+$ and the term $\hat{\partial}_2$ counts one-stage Floer trajectories connecting $S_i^-$ to $S_j^+$ with $\mu(S_j) = \mu(S_i) - 2$.

To be more specific, when $\mu(S_j) = \mu(S_i) - 1$, we have

$$\hat{\partial}_1 S_i^\pm = \sum_j \langle S_i^\pm, S_j^\pm \rangle S_j^\pm.$$

Here $\langle S_i^+, S_j^+ \rangle$ counts the number of elements, with signs, in the moduli space of broken trajectories made of an anti-gradient trajectory $\eta: (-\infty, 0] \to S_i$ of $g_i$ starting at $S_i^+$ and then the Floer trajectory, i.e., a solution $u = (u, \lambda)$ of (2.13) from $S_i$ to $S_j$ such that the line $\hat{u}(\cdot, 0)$ connects $\eta(0)$ and $S_j^+$. Likewise, $\langle S_i^-, S_j^- \rangle$ counts the number of elements, with signs, in the moduli space of broken trajectories made of a Floer trajectory from $S_i$ to $S_j$ with $\hat{u}(s, 0) \to S_j^-$ as $s \to -\infty$ and an anti-gradient trajectory of $g_j$ connecting $\lim_{s \to -\infty} \hat{u}(s, 0)$ to $S_j^-$. Furthermore, when $\mu(S_j) = \mu(S_i) - 2$, we set

$$\hat{\partial}_2 S_i^+ = 0$$

and

$$\hat{\partial}_2 S_i^- = \sum_j \langle S_i^-, S_j^+ \rangle S_j^+,$$

where $\langle S_i^-, S_j^+ \rangle$ is the number of Floer trajectories from $S_i$ to $S_j$ with the line $\hat{u}(\cdot, 0)$ connecting $S_i^-$ to $S_j^+$.

Let $\partial = \hat{\partial}_1 + \hat{\partial}_2$. (As in the Morse theoretic case, the differential $\partial$ should in general have an additional term $\hat{\partial}_0$ counting anti-gradient trajectories of $g_i$ within
each $S_i$, but due to our choice of $g_i$ as perfect Morse functions this term is obviously zero; cf. [BO13b, Prop. 5.2]). It is useful to keep in mind that, while the vector space $CF_*(\hat{H})$ is essentially determined by $\hat{H}$, the differential $\partial$ depends on the almost complex structure $J$, the metric on $S^{2m+1}$, the metrics on $S_i$ and the Morse functions $g_i$. We have $\partial^2 = 0$ and the homology $HF_*(\hat{H})$ of the complex $(CF_*(\hat{H}), \partial)$ is the ordinary Floer homology of $\hat{H}$, canonically isomorphic to the standard Floer homology $HF_*(H)$ when $\hat{H}$ is a sufficiently small perturbation of an autonomous Hamiltonian $H$ on $V$; we refer the reader to [BO09a] and [BO13b] for the proofs of these facts. Note that $HF_*(H)$ is the filtered Floer homology with an action interval $I$ suppressed in the notation and hence this space depends on $H$.

Consider the vector space decomposition $CF_*(\hat{H}) = C^+ \oplus C^-$ with $C^\pm$ generated by $S_i^\pm$. This decomposition is preserved by $\partial_1$, i.e., $\partial_1 : C^\pm \to C^\pm$ while $\partial_2(C^+) = 0$ and $\partial_2 : C^- \to C^+$. Hence, $(C^+, \partial_1)$ is a subcomplex of $CF_*(\hat{H})$ and the quotient complex $CF_*(\hat{H})/C^+$ is isomorphic to $(C^-, \partial_1)$. By the definitions of $\partial_1$ and $\partial^G$, the homology of $(C^+, \partial_1)$ or $(C^-, \partial_1)$ is $HF^G_*(\hat{H})$, up to the shift of grading by $+1$ in the former case. Then, as in the Morse theoretic case, the exact sequence of complexes

$$0 \to (C^+, \partial_1) \to (CF_*(\hat{H}), \partial) \to (C^-, \partial_1) \to 0$$

gives rise to a long exact sequence in homology which is nothing else but the Gysin sequence (2.9). The connecting map, i.e., shift operator $D$ as defined in [BO13b], is the map induced by $\partial_2 : C^- \to C^+$ on the homology of these complexes. This construction applies to any finite range of degrees when $m$ is sufficiently large. To define $D$ for $HF^G_*(\hat{H})$ for all degrees $*$, it remains to take the limit as $m \to \infty$. The construction extends by continuity to all autonomous Hamiltonians $H$.

As an immediate consequence of this definition and (2.16), we see that $D$ is action decreasing, although not necessarily strictly, i.e.,

$$c_w(H) \geq c_{D(w)}(H),$$

without any assumptions on $H$ and for any coefficient ring.

2.2.4. An alternative definition of the shift operator. The definition of the shift operator from the previous section is parallel to the definition of the operator $\delta$ in Section 2.1.2. In this section we give an alternative definition of $D$ similar to the one in Section 2.1.1 via the intersection index and prove an analog of Lemma 2.5 showing that the two definitions are equivalent. We keep the notation and conventions from Section 2.2.2. In particular, we suppress the interval $I$ in the notation.

Let $\hat{H} : S^1 \times V \times S^{2m+1} \to \mathbb{R}$ be a transversely non-degenerate $G$-invariant parametrized Hamiltonian. Fix a $G$-invariant metric on $S^{2m+1}$ and a $(t, \zeta)$-dependent $G$-invariant almost complex structure $J$ on $V$ such that all the transversality requirements are met. Then the moduli space $\mathcal{M}(S_i, S_j)$ of solutions $\tilde{u} = (u, \lambda)$ of the parametrized Floer equation (2.13), taken up to the natural $G$-action and asymptotic to $S_i$ and $S_j$, has dimension $\mu(S_i) - \mu(S_j)$ and the evaluation map

$$ev : \mathcal{M}(S_i, S_j) \to \mathbb{C}P^m, \quad \tilde{u} \mapsto \lambda(0)$$

is well defined. (Here $\lambda(0)$ is understood as a point in $\mathbb{C}P^m$ rather than in $S^{2m+1}$.) Recall also that every critical set $S_i$ of $\mathcal{A}_{\hat{H}}$ projects to one point in $\mathbb{C}P^m$. Let $\Sigma \subset \mathbb{C}P^m$ be a closed codimension-two co-oriented submanifold, Poincaré dual to the negative first Chern class of the Hopf bundle $\pi : S^{2m+1} \to \mathbb{C}P^m$. In addition,
we require $\Sigma$ not to meet any of the projections of $S_i$ to $\mathbb{C}P^m$ and be transverse to all evaluation maps $ev$ for moduli spaces of dimensions up to three. It is not hard to show that such a submanifold $\Sigma$ exists. When $\mu(S_j) = \mu(S_i) - 2$, let $\langle S_i, S_j \rangle_\Sigma \in \mathbb{Z}$ be the intersection index of $ev$ with $\Sigma$. Similarly to (2.5), we set

$$D_\Sigma S_i = \sum_j \langle S_i, S_j \rangle_\Sigma S_j.$$  

The same argument as in the standard Floer theory (see, e.g., [LO] and also [GG14, Li, Vi95] and [MS, Rmk. 12.3.3]) shows that $D_\Sigma$ commutes with $\partial^G$ and the resulting operator, which we temporarily denote by

$$[D_\Sigma]: \text{HF}^G_\ast(\tilde{H}) \to \text{HF}^G_{\ast-2}(\tilde{H}),$$

is independent of the auxiliary data and a particular choice of $\Sigma$. Passing to the limit as $\tilde{H} \to H$ and $m \to \infty$, where $H$ is an autonomous Hamiltonian on $V$, we obtain an operator $[D_\Sigma]$ on the filtered homology $\text{HF}^G_\ast(H)$.

**Lemma 2.17.** The two definitions of the shift operator in equivariant Floer theory are equivalent: $D = [D_\Sigma]$.

**Proof.** We reason as in the proof of Lemma 2.5. However, the logical structure of the argument is somewhat different now and first we need to specify the auxiliary data. Let as above $\Sigma$ be a closed codimension-two co-oriented submanifold in $\mathbb{C}P^m$, Poincaré dual to the negative first Chern class of the Hopf bundle $\pi$. (For instance, we can take a generic projective hyperplane $\mathbb{C}P^{m-1}$ as $\Sigma$ at this stage.) Furthermore, let $\sigma$ be a closed two-form “Poincaré dual” to $\Sigma$ and supported in a small tubular neighborhood $U$ of $\Sigma$. (Thus, in particular, $[\sigma] = -c_1(\pi).$) Let us fix a Riemannian metric on $\mathbb{C}P^m$ and a connection on the Hopf bundle with curvature $-2\pi\sigma$. Then we equip $S^{2m+1}$ with the standard $G$-invariant lift of the metric from $\mathbb{C}P^m$ to $S^{2m+1}$ using this connection.

Examining the argument from [BO10, Sect. 7], one can see that the transversality conditions are satisfied for a generic choice of the almost complex structure $J$, provided that $\tilde{H}$ also meets some further generic requirements. Moreover, we can ensure that all evaluation maps $ev$ for all moduli spaces are transverse to $\Sigma$ by replacing $\Sigma$ with its $C^\infty$-small perturbation without changing $U$ or $\sigma$.

The connection on the Hopf bundle is flat outside $U$ and thus can be used to fix a $G$-equivariant trivialization

$$S^{2m+1} \setminus \tilde{U} \cong (\mathbb{C}P^m \setminus U) \times S^1,$$  

(2.17)

where $\tilde{U}$ is the inverse image of $U$. All critical sets $S_i$ lie in $S^{2m+1} \setminus \tilde{U}$ and for each of them the projection onto $S^1$ is a $G$-equivariant diffeomorphism. Furthermore, for each solution $\tilde{u}$ of the Floer equation (2.13) its $S^{2m+1}$-component $\lambda(s)$ is the horizontal lift of its projection to $\mathbb{C}P^m$. (This follows readily from the observation that the integral of $H_\xi(t, x(t))$ over $S^1$ on the right hand side of (2.13) is a $G$-invariant function on $S^{2m+1}$.)

Assume now that $\mu(S_j) = \mu(S_i) - 2$. Then the pull-back $ev^* \sigma$ has compact support in $\mathcal{M}(S_i, S_j)$ and, to prove the lemma for $\tilde{H}$ and a fixed $m$, it suffices to show that

$$\langle S_i^-, S_j^+ \rangle = \int_{\mathcal{M}(S_i, S_j)} ev^* \sigma.$$  

(2.18)
For, similarly to (2.7), the right hand side is easily seen to be equal to $(S_i, S_j)_{S}$; cf. Remark 2.6. Let $\mathcal{M}$ be a connected component of $\mathcal{M}(S_i, S_j)$ and $\tilde{\mathcal{M}} = \mathcal{M}/\mathbb{R}$ be the space of un-parametrized trajectories.

For every element $\tilde{u} \in \tilde{\mathcal{M}}$ let us fix its unique lift to a map $\mathbb{R} \times S^1 \to V \times S^{2m+1}$, still denoted by $\tilde{u} = (u, \lambda)$, satisfying the Floer equation (2.13) and such that $\tilde{u}(s, 0) \to S_i^{-}$ as $s \to -\infty$. Then $\tilde{u}(\infty, 0) := \lim_{s \to \infty} \tilde{u}(s, 0)$ is a well-defined point in $S_J$. Denote by $\lambda(\infty)$ the projection of this point to the fiber $S^1$ via the identification (2.17). Viewing $\tilde{\mathcal{M}}$ as a one-parameter space of un-parametrized trajectories, we obtain a map $\tilde{\mathcal{M}} \to S^1$ sending (the equivalence class of) $\tilde{u}$ to $\lambda(\infty)$. Since $\lambda$ as a map to $S^{2m+1}$ is a horizontal lift of its projection to $\mathbb{C}P^m$ and since the connection is flat outside $U$, this map is constant outside the image of $\text{supp}(ev^* \sigma)$ in $\tilde{\mathcal{M}}$. Generically, the value of this map outside the image is a point different from the projection of $S_j^+ \to S^1$ by (2.17). The number of times the end-point $\lambda(0)$ traverses $S^1$ as $\tilde{u}$ varies through $\tilde{\mathcal{M}}$ is equal to the integral of $ev^* \sigma$ over $\tilde{\mathcal{M}}$. Since the map $S_j \to S^1$ is one-to-one, this is also equal to the number of revolutions $\tilde{u}(\infty, 0)$ makes in $S_j$ which is obviously equal to the contribution of $\tilde{\mathcal{M}}$ to $(S_i^{-}, S_j^+)$. Summing up over all connected components of $\mathcal{M}(S_i, S_j)$, we obtain (2.18). This proves the lemma for $\tilde{H}$ and a fixed $m$.

Finally, to show that $D = [D_S]$ on the filtered equivariant Floer homology of an autonomous Hamiltonian $H$ on $V$, it remains to pass to the limit as $\tilde{H} \to H$ and $m \to \infty$. \hfill \Box

2.3. Local homology. In this section we examine two local variants of equivariant Floer homology: the equivariant local Floer homology of an isolated orbit and the equivariant Morse--Bott Floer homology.

2.3.1. Equivariant local Floer homology. Just as in the non-equivariant case (cf., e.g., [Gi10, GG10]), the constructions from the previous sections can be easily adapted to define the local equivariant Floer homology and the shift operator on it. Namely, let $x$ be an isolated one-periodic orbit of an autonomous Hamiltonian $H$. (We do not need to impose any assumptions on $V$; in fact, it suffices to have $H$ defined only on a neighborhood of $x$.) Under a $C^2$-small perturbation $\tilde{H}$ of $H$, as above, the orbit $x$ breaks down into a collection of transversely non-degenerate critical $G$-orbits $S_i$ lying close to $C = Gx \times_G S^{2m+1}$ where $Gx$ is the orbit of $x$ in $\Lambda$. Let $\text{CF}_G^\tilde{G}(x)$ be the (relatively) graded vector space or module generated by $S_i$. In Section 2.12, we will show that the connecting Floer trajectories $\tilde{u}$ between $S_i$ and $S_j$ stay close to $C$. As a consequence, (2.14) gives rise to a differential on $\text{CF}_G^\tilde{G}(x)$. The resulting local equivariant Floer homology of $x$, denoted by $\text{HF}_G^\tilde{G}(x)$ or $\text{HF}_G^{\tilde{G}}(x, H)$ when we need to keep track of $H$, is independent of the perturbation $\tilde{H}$. (This construction readily extends from one isolated orbit to a compact, connected, isolated set of one-periodic orbits.)

Example 2.18 (Simple non-degenerate orbits). Assume that $x$ is a non-degenerate (or rather maximally non-degenerate in the obvious sense) non-constant orbit of an autonomous Hamiltonian. Then, when $x$ is simple, $\text{HF}_G^{\tilde{G}}(x)$ has rank one when $*$ is equal to the Conley--Zehnder index $\mu(x)$ and zero otherwise. We leave a detailed proof of this fact as an exercise for the reader. Here we only mention that one can use a $C^2$-small Morse function $f$ on the sphere $C$ to construct a perturbation $\tilde{H}$; see
[BO12] and Remark 2.26. Then the resulting homology is isomorphic to \( HM_*(f) \) up to a shift of degree by \( \mu(x) \). This Morse homology is non-zero only in degrees \( * = 0 \) and \( * = 2m - 1 \). Passing to the limit as \( m \to \infty \), we see that the homology has rank one in degree \( * = \mu(x) \) and \( HF^G_*(x) = 0 \) otherwise.

**Example 2.19 (Iterated non-degenerate orbits).** When \( x \) is a non-constant maximally non-degenerate iterated orbit, the situation is somewhat more involved. Recall that a simple orbit \( y \) is said to be odd if \( y \) has an odd number of Floquet multipliers in \((-1, 0)\) and even if the number of such Floquet multipliers is even.

An iterated orbit \( x = y^k \) is called bad if \( y \) is odd and \( k \) is even; otherwise, \( x = y^k \) is good. The calculation from Example 2.18 still holds over \( \mathbb{Q} \) when \( x \) is iterated and good: \( HF^G_*(x; \mathbb{Q}) = 0 \) unless \( * = \mu(x) \) when the homology is \( \mathbb{Q} \). When \( x \) is bad, \( HF^G_*(x; \mathbb{Q}) = 0 \) in all degrees. The reason is that in this case \( HF^G_*(x; \mathbb{Q}) \) is equal to the homology of the lens space \( C \) with coefficients in a non-trivial local coefficient system with fiber \( \mathbb{Q} \) and this homology is zero.

Denote by \( \bar{\mu}(x) \) the mean index of \( x \). As in the non-equivariant case, the equivariant local Floer homology is supported close to \( \bar{\mu}(x) \) with an error not exceeding \( \dim V/2 \); see, e.g., [Gi10, GG10, McL]. More precisely, let \( \mu_{\pm}(x) \) be the upper and lower Conley–Zehnder indices of an orbit \( x \); see Section 4. Then we have

**Proposition 2.20.** Let \( x \) be an isolated non-constant one-periodic orbit of an autonomous Hamiltonian. Then

\[
\text{supp } HF^G_*(x) \subseteq [\mu_-(x), \mu_+(x)] \subseteq [\bar{\mu}(x) - n + 1, \bar{\mu}(x) + n - 1]
\]

i.e., \( HF^G_*(x) \) can only be non-trivial for \( * \) in this range of degrees. Moreover, if at least one of the Floquet multipliers of \( x \) is different from 1, the end-points of the second interval can be excluded.

The proposition is an immediate consequence of the fact that, by (4.6), under a small non-degenerate perturbation \( x \) splits into periodic orbits with Conley–Zehnder index in \([\mu_-(x), \mu_+(x)]\).

In general, when \( x \) is isolated and non-constant, one can expect \( HF^G_*(x) \) to be isomorphic to the equivariant local Floer homology \( HF^G_*(\varphi) \), where \( \varphi \) is the Poincaré return map of \( x \) in the level containing \( x \) and \( \Gamma = \mathbb{Z}_k \) is the stabilizer of \( x \); cf. [GH2M].

The operator \( D \) vanishes in the equivariant local Floer homology. To be more precise, we have the following result established as a part of the proof of Theorem 2.12 in Section 2.4; see (2.26) and its proof.

**Proposition 2.21.** Let \( x \) be an isolated non-constant one-periodic orbit of an autonomous Hamiltonian. Then

\[
D \equiv 0 \text{ in } HF^G_*(x; \mathbb{Q}).
\]

Proposition 2.21 fits in a larger pattern mentioned in the introduction that under reasonable assumptions (co)homology operations (such as \( D \) or the pair-of-pants product) vanish in the local homology of an isolated orbit; cf. [GG10, Prop. 5.3] and [GoHi].

2.3.2. **Equivariant Morse–Bott Floer homology.** We will now consider another case of localization of equivariant Floer homology. Let as above \( H \) be an autonomous Hamiltonian on \( V \) and let \( P \) be a connected Morse–Bott non-degenerate manifold.
of one-periodic orbits of $H$ in the sense of [Po]. Since $H$ is autonomous, $P$ carries a natural $G = S^1$-action by reparametrizations. Clearly, the action functional $A_H$ is constant on $P$. Set $c := A_H(P)$ and assume that all one-periodic orbits of $H$ with action $c$ are contained in $P$.

**Proposition 2.22.** Let $I$ be any interval such that $c$ is the only point of $S(H)$ in $I$. Then, for any coefficient ring,

$$HF_*^{G,I}(H) \cong H_*^G(P)$$

up to a shift of degrees, and the isomorphism intertwines the operator $D$ in the equivariant Floer homology and in $H_*^G(P)$. In particular, when the $G$-action on $P$ is locally free and all orbits have the same stabilizer $\mathbb{Z}_k$, we have

$$HF_*^{G,I}(H; \mathbb{Q}) \cong H_*(P/G; \mathbb{Q}),$$

where now $D$ on the right comes from the Gysin exact sequence of the $G/\mathbb{Z}_k$-principal bundle $P \to P/G$. (The same is true without the stabilizer assumption, but then one needs to replace an ordinary principal bundle by an orb-bundle.)

The proof of the proposition is standard and we omit it for the sake of brevity. The non-equivariant case goes back to [Po]; see also [Bo, BO09a] for a different approach.

### 2.4. Proof of the Lusternik–Schnirelmann inequality in Floer homology.

In this section we prove Theorem 2.12 and hence Theorem 1.2. Throughout the proof we keep the notation and convention from Section 2.2.2. In particular, we continue suppressing the interval $I$ in the notation. We will use the definition of the operator $D$ from Section 2.2.4 rather than the original definition from [BO13b]. The two definitions are equivalent by Lemma 2.17.

Consider a Hamiltonian $H$ as in the statement of the theorem. In other words, $H$ is autonomous and, when $V = \tilde{W}$, admissible at infinity, and all orbits in $P(H)$ are non-constant and isolated. Our goal is to show that for a non-degenerate small perturbation $\tilde{H}$ of $H$ the operator $D$ on $HF_*^G(\tilde{H})$ decreases the action by at least $\text{gap}(H) - O(\|H - \tilde{H}\|_{C^0})$. To be more specific, we will prove that, in the notation from (2.12) and (2.16),

$$c_w(\tilde{H}) \geq c_{D(w)}(\tilde{H}) + \text{gap}(H) - O(\|H - \tilde{H}\|_{C^0})$$

(2.19)

for every $w \neq 0$ in $HF_*^G(\tilde{H}) = HF_*^G(H)$. Then the theorem will follow by passing to the limit as $\tilde{H} \to H$ and $m \to \infty$.

When we view $H$ as a degenerate parametrized Hamiltonian, the critical sets of $A_H$ in $\Lambda \times_G S^{2m+1}$ have the form $Gx \times_G S^{2m+1}$, where $x \in P(H)$ and $Gx$ is the orbit of $x$ in $\Lambda$. We denote these sets by $C_x$ or just $C$ when the role of $x$ is inessential. Note that $C_x = C_y$ when $y \in Gx$, i.e., $x = y$ up to a reparametrization. As in the finite-dimensional case considered in Section 2.1, $C$ is diffeomorphic to the lens space $S^{2m+1}/\Gamma$ where $\Gamma \subset G$ is the stabilizer of $x$, i.e., $\Gamma = \mathbb{Z}_k$ when $x$ is the $k$th iteration of a $1/k$-periodic orbit.

Let $\tilde{H}: S^1 \times V \times S^{2m+1} \to \mathbb{R}$ be a parametrized, transversely non-degenerate $G$-invariant perturbation of $H$. (When $V = \tilde{W}$, we can take $\tilde{H} \equiv H$ at infinity.) We require $\tilde{H}$ to be sufficiently $C^\infty$-close to $H$. Under this perturbation, the critical sets $C = C_x$ of $H$ break down into finite collections of one-dimensional families $S_{x,i}$ of critical points of $A_{\tilde{H}}$. The action functional $A_{\tilde{H}}$ is constant on $S_{x,i}$ and hence we
set $\mathcal{A}_H(S_{x,i})$ to be its value at an arbitrary point of $S_{x,i}$. Each $S_{x,i}$ is an isolated orbit of $G$ lying close to $C$ in $\Lambda \times S^{2m+1}$. In particular, when $(y,\zeta) \in S_{x,i}$, the loop $y$ is $C^\infty$-close to $x$, up to a parametrization, and $\mathcal{A}_H(S_{x,i})$ is close to $\mathcal{A}_H(x)$. Furthermore, recall that the spectrum $\mathcal{S}(H)$ (or, to be more precise, $\mathcal{S}(H) \cap I$) is finite due to our requirements on $H$. For every $c \in \mathcal{S}(H)$ pick a short interval $I_c$ centered at $c$ so that the intervals $I_c$ and $I_{c'}$ do not overlap when $c \neq c'$. Thus, when $\tilde{H}$ is sufficiently close to $H$, the action values $\mathcal{A}_H(S_{x,i})$ are in $I_c$ where $c = \mathcal{A}_H(x)$.

The complex $\text{CF}_c^G(\tilde{H})$ is generated by $S_{x,i}$, where $x$ runs through all geometrically distinct one-periodic orbits of $H$, and graded by $\mathcal{A}_H(S_{x,i})$. Let $I$ be one of the intervals $I_c$. The filtered Floer homology $\text{HF}_c^{G,\mathcal{I}}(H)$ is the homology of the complex $(\text{CF}_c^{G,\mathcal{I}}(\tilde{H}), \partial^G)$ generated by the critical points $S_{x,i}$ with action in $I$. The operator $D_{x,i}$ is defined on $\text{CF}_c^{G,\mathcal{I}}(\tilde{H})$ and $D$ is defined on $\text{HF}_c^{G,\mathcal{I}}(H)$.

It is not hard to see that to prove (2.19) it suffices to show that

$$D \equiv 0 \text{ on } \text{HF}_c^{G,\mathcal{I}}(H; \mathbb{R}) = \text{HF}_c^{G,\mathcal{I}}(H; \mathbb{R}).$$

Before proving (2.20), we need to recall several standard facts about the behavior of Floer trajectories, drawing from [Sa99, Sect. 1.5]. For a solution $\tilde{u} = (u, \lambda)$ of the parametrized Floer equation (2.13), consider the following two energy integrals

$$E(u) = \int_{-\infty}^{\infty} \|\partial_s u(s)\|^2_L^2 ds \quad \text{and} \quad E(\tilde{u}) = E(u) + \int_{-\infty}^{\infty} \|\lambda(s)\|^2 ds,$$

where we set $u(s) = u(\cdot, s)$. (Here and below it is convenient to interpret $\tilde{u}$ as literally a map to $V \times S^{2m+1}$, i.e., without taking a quotient by $G$; this should cause no confusion.) Then, when $\tilde{u}$ is asymptotic to $S_{x,i}$ at $-\infty$ and to $S_{y,j}$ at $\infty$, we have

$$E(u) \leq E(\tilde{u}) = \mathcal{A}_H(S_{x,i}) - \mathcal{A}_H(S_{y,j}).$$

(2.21)

Since $V$ is aspherical, $\|\partial_s u\|$ is point-wise bounded from above by a constant independent of $H$ as long as $H$ is $C^2$-close to $H$. Moreover, when $E(u)$ is sufficiently small, we have a sharper point-wise bound

$$\|\partial_s u\| \leq O\left(E(u)^{1/4}\right) \leq O\left((\mathcal{A}_H(S_{x,i}) - \mathcal{A}_H(S_{y,j}))^{1/4}\right).$$

(2.22)

This is a consequence of the fact that the energy density $\varphi = \|\partial_s u\|^2$ satisfies the differential inequality $\Delta \varphi \geq -a$ for some $a \geq 0$ independent of $H$ as long as $H$ is $C^2$-close to $H$. Indeed, first observe that

$$\Delta \varphi \geq -a'(1 + \varphi^2)(1 + \|\nabla \lambda\|)$$

with $a' \geq 0$. This inequality can be proved exactly in the same way as its counterpart (without the term $\|\nabla \lambda\|$) for the ordinary Floer equation; see [Sa90, p. 137–138]. As we have already seen, $\|\partial_s u\|$ and hence $\varphi$ are a priori bounded. Likewise, $\|\nabla \lambda\|$ is also a priori bounded as a consequence of (2.13), and therefore $\Delta \varphi$ is a priori bounded from below. Then (2.22) follows from this lower bound by a standard argument; see, e.g., [Sa90, Sect. 5] or [Sa99, Sect. 1.5].

Furthermore, recall that for any map $z : S^1 \to V$ and for some $\epsilon > 0$ independent of $z$, we have

$$\int_{S^1} \|\dot{z}(t) - X_H(z(t))\|^2 dt > \epsilon,$$

(2.23)

unless $z$ is sufficiently $C^0$-close to one of the periodic solutions in $P(H)$; see [Sa99, Exercise 1.22]. Clearly, (2.23) also holds with $\tilde{H}$ in place of $H$. 


These facts have two consequences important for our proof. First of all, we claim that for any two geometrically distinct orbits $x$ and $y$ such that there exists a solution $\tilde{u}$ asymptotic to $S_{x,i}$ and $S_{y,j}$, we have

$$A_R(S_{x,i}) - A_R(S_{y,j}) \geq e > 0,$$  \hspace{1cm} (2.24)

where the lower bound $e$ depends only on $H$. (We assume here that the background auxiliary structure is sufficiently close to a fixed almost complex structure on $V$ and a fixed metric on $S^{2m+1}$.) Indeed, by (2.22) and (2.23),

$$E(u) \geq e > 0,$$

where $e$ depends only on $H$ as long as $\tilde{H}$ is sufficiently close to $H$ and on the auxiliary structure. This, combined with (2.21), implies (2.24).

Secondly, for every $\tilde{u}$ asymptotic to $S_{x,i}$ and $S_{x,j}$ we have

$$d_{C^1}(u(s), x) \to 0 \text{ as } \|H - \tilde{H}\|_{C^2} \to 0$$  \hspace{1cm} (2.25)

uniformly in $s \in \mathbb{R}$ and $\tilde{H}$, where $d_{C^1}$ stands for the $C^1$-distance in the loop space $\Lambda$. Indeed, it readily follows from (2.22) and (2.23) that $u(s)$ is $C^1$-close to $x$ when $\|H - \tilde{H}\|_{C^2}$ is small. Then, by the Floer equation and again by (2.22), $u(s)$ is also $C^1$-close to $x$.

We are now in a position to prove (2.20) and thus finish the proof of the theorem.

Pick $c \in \mathcal{S}(H)$ and let as above $I = I_c$. We have a natural decomposition of vector spaces

$$\text{CF}^G_{\mathcal{I}}(\tilde{H}) = \bigoplus_x \text{CF}^G_{\mathcal{I}}(x),$$

where the sum is taken over all geometrically distinct periodic orbits $x$ of $H$ with $A_R(x) = c$ and $\text{CF}^G_{\mathcal{I}}(x)$ stands for the subspace generated by $S_{x,i}$. Let us choose interval $\mathcal{I}$ so short that its length is less than $e$ from (2.24). Then, by (2.24), this is also a decomposition of complexes, i.e., $\partial^G : \text{CF}^G_{\mathcal{I}}(x) \to \text{CF}^G_{\mathcal{I}}(x)$, and moreover this decomposition is also obviously preserved by $D$. (One can use here either of the definitions of $D$.) The homology $\text{HF}^G_{\mathcal{I}}(x)$ of $\text{CF}^G_{\mathcal{I}}(x)$ is the local Floer homology of $x$ discussed in Section 2.3.1, and the above argument fills in the technical details of the definition of $\text{HF}^G_{\mathcal{I}}(x)$ omitted in that section.

Now we see that it is sufficient to prove that

$$D \equiv 0 \text{ in } \text{HF}^G_{\mathcal{I}}(x).$$  \hspace{1cm} (2.26)

To this end, we will employ the second definition of $D$. Fix a $C^1$-small tubular neighborhood $\mathcal{U}$ of $C = C_x$ in $\Lambda \times_{G} S^{2m+1}$ with $\rho : \mathcal{U} \to C$ being the smooth tubular neighborhood projection. More explicitly, one can show that there exists a $G$-invariant $C^1$-small neighborhood $\tilde{\mathcal{U}}$ of $Gx$ in $\Lambda$ and a smooth, in the obvious sense, $G$-equivariant projection $\tilde{\rho} : \tilde{\mathcal{U}} \to Gx$. Then $\mathcal{U} = \tilde{\mathcal{U}} \times_{G} S^{2m+1}$ and $\rho$ is induced by $\tilde{\rho}$ on the first factor.

Suppressing $x$ in the notation, let us write $S_i$ for $S_{x,i}$. By (2.25), a solution $\tilde{u}$ asymptotic to $S_i$ and $S_j$ is entirely contained in $\mathcal{U}$. Thus we have a well-defined evaluation map

$$\tilde{ev} : \mathcal{M}(S_i, S_j) \to C, \quad \tilde{ev}(\tilde{u}) = (\rho(\tilde{u}(0), \lambda(0))).$$

The composition of $\tilde{ev}$ with the natural projection $C \to \mathbb{CP}^m$ is the evaluation map $ev$ from Section 2.2.4. Let us fix a submanifold $\Sigma \subset \mathbb{CP}^m$ as in Section 2.2.4 and let $\tilde{\Sigma}$ be its inverse image in $C$ under the submersion $C \to \mathbb{CP}^m$. Clearly, $\tilde{\Sigma}$ is a closed,
smooth, co-oriented submanifold of $C$ of codimension two, which is transverse to $\tilde{\ev}$ if and only if $\Sigma$ is transverse to $\ev$. Therefore, $\langle S_i, S_j \rangle_{\Sigma}$ can also be evaluated as the intersection index $\langle S_i, S_j \rangle_{\tilde{\Sigma}}$ of $\tilde{\ev}$ with $\tilde{\Sigma}$. Hence, in the complex $\CF_\ast^G(x)$, we have

$$D_{\Sigma}S_i = \sum_j \langle S_i, S_j \rangle_{\tilde{\Sigma}} S_j.$$ 

The cycle $\tilde{\Sigma}$ is homologous to zero in $C$ over $\mathbb{Q}$, since $C$ is a lens space and thus $\mathbb{H}^2(C; \mathbb{Q}) = 0$. Now the standard argument showing that the “intersection” action of the ordinary homology on the Morse or Floer homology is well-defined (see, e.g., [LO]) also shows that the operator $D$ induced by $D_{\Sigma}$ on the Floer homology $\HF_\ast^G(x)$ over $\mathbb{Q}$ is zero. This concludes the proof of (2.26) and of Theorem 2.12 (and hence of Theorem 1.2).

2.5. “Quotient” construction of equivariant Floer homology. Throughout this section we keep the notation and conventions from Section 2.2.2. In particular, we suppress the interval $I$ in the notation.

One shortcoming of the construction of equivariant Floer homology is that even when $H$ is, say, autonomous and its orbits are maximally non-degenerate, $\HF_\ast^G(H)$ is not defined as the homology of a complex generated by one-periodic orbits of $H$. This creates some inconvenience, admittedly minor, when the equivariant Floer or symplectic homology is used to produce lower bounds on the number of periodic orbits. A simple remedy for this is the following Floer theoretic analog of Proposition 2.7.

**Proposition 2.23.** Let $H$ be an autonomous Hamiltonian such that all one-periodic orbits of $H$ with action in $I$ are non-constant and maximally non-degenerate. Then $\HF_\ast^G(H)$ is the homology of a certain complex $\CF_\ast^G$ generated over $\mathbb{Q}$ by the good one-periodic orbits of $H$, graded by the Conley–Zehnder index, and filtered by the action.

**Proof.** We will argue as in the proof of Proposition 2.7. To this end, it is convenient to slightly modify the definition of $\HF_\ast^G(H)$ and use the standard Morse–Bott approach. Namely, denote by $x$ the periodic orbits of $H$ with action in $I$. Fixing $m$, we will treat $H$ as a $G$-invariant Hamiltonian $S^1 \times V \times S^{2m+1} \to \mathbb{R}$. As in the proof of Theorem 2.12, every $x$ gives rise to a critical set $C_x = Gx \times_G \Lambda$ of $A_H$ in $\Lambda \times G S^{2m+1}$ diffeomorphic to the lens space $S^{2m+1}/\Gamma$ where $\Gamma \subset G$ is the stabilizer of $x$, i.e., $\Gamma = \mathbb{Z}_k$ when $x$ is the $k$th iteration of a $1/k$-periodic orbit. (Furthermore, two orbits which differ by a parametrization give rise to the same critical set.) Moreover, since the orbits $x$ are maximally degenerate, the critical sets $C_x$ are now Morse–Bott non-degenerate.

Next, rather than taking a transversely non-degenerate perturbation $\tilde{H}$ of $H$, we can use the Morse–Bott construction to account for the contributions of the critical manifolds $C_x$ along the lines of, e.g., [BO09a]. Namely, let us equip each manifold $C_x$ with a Morse function $g_x$ and a Riemannian metric. Denoting the critical points of $g_x$ by $S_{x,i}$, we obtain a complex generated by $S_{x,i}$, for all $x$ and all $i$, equipped with the differential counting broken Morse–Floer trajectories in a manner similar to the definition of the Floer differential in Section 2.2.3, but with critical manifolds being lens spaces rather than circles. This complex is graded by the sum of the Conley–Zehnder index of $x$ and the Morse index of $S_{x,i}$ in $C_x$ and
filtered by the action of $x$. A standard (but non-trivial) argument shows that the homology of this complex converges to $HF^G_{*}(H)$ as $m \to \infty$; cf. [Bo, BO09a].

Consider the spectral sequence associated with the action filtration. Its $E_1$-page is the direct sum of the Morse homology spaces of the functions $g_x$ with possibly twisted coefficients. For every closed orbit $x$, this is the homology over $Q$ of $C_x$ with a trivial local coefficient system $Q$ when $x$ is good and non-trivially twisted local coefficients when $x$ is bad; see Example 2.19. Therefore, every good $x$ contributes $Q$ to the $E_1$-page in total degree $p + q = 0$ and $p + q = 2m + 1$ and every bad $x$ contributes only in degree $p + q = 2m + 1$. Clearly, the spectral sequence converges in a finite number of steps bounded from above by the length of the action spectrum (and thus independent of $m$) and, for a suitable choice of auxiliary data, stabilizes as $m \to \infty$ in any finite $(p, q)$-region.

Let us apply Lemma 2.8 with $r_0 = 1$ to this spectral sequence. As a result we obtain a complex $(E_1, \bar{\partial})$ whose homology is isomorphic to $E_\infty$. The $E_1$-term comes with a preferred set of generators, and hence the complex is canonically determined by the spectral sequence. Since the spectral sequence stabilizes as a function of $m$ in every finite $(p, q)$-range, the sequence of complexes $(E_1, \bar{\partial})$ converges as $m \to \infty$. Passing to the limit, we obtain a single complex $CF_*(H)^G := \lim E_1$ equipped with the limit differential, still denoted by $\bar{\partial}$, such that its homology is $\lim E_\infty = HF^G_*(H; Q)$. (Here we use the fact that the direct limit and the homology commute.) The complex $CF_*(H)^G$ is generated by the good orbits of $H$, graded by the Conley–Zehnder index and filtered by the action.

**Remark 2.24.** As is clear from the proof, the differential $\bar{\partial}$ can be described explicitly. However, such a description does not seem to be particularly useful; for the differential, as in all Floer-type constructions, would be very difficult to calculate except when it is obviously zero due to, say, lacuna in the complex. It is also clear that although the differential depends on the auxiliary data, different choices of such data result in isomorphic complexes. Furthermore, the differential is “natural” in the same sense as the Floer or Morse differential: a monotone increasing homotopy from $H_0$ to $H_1$ gives rise to a homomorphism of complexes; cf. Remark 2.9. Finally, this description of equivariant Floer homology carries over word-for-word to the local case, i.e., $HF^G_*(x; Q)$ is isomorphic to the homology of a certain complex graded by the Conley–Zehnder index and generated by the good orbits which $x$ breaks down into under a non-degenerate perturbation of $H$.

**Remark 2.25.** With Proposition 2.23 in mind one can replace contact homology by equivariant symplectic homology without affecting the rest of the argument everywhere in the proofs from [Gü15] and in some other instances.

**Remark 2.26.** Our choice to work with the Morse–Bott complex in the proof of Proposition 2.23 is mainly determined by expository considerations. Instead, we could have worked with a transversely non-degenerate parametrized perturbation of $H$ resulting in exactly the same complex $(CF_*(H)^G, \bar{\partial})$; cf. Remark 2.9. Furthermore, Proposition 2.23 and its proof also enable one to replace a part of the proof of Theorem 2.12 by a purely Morse-theoretic argument as in Section 2.1 with some minor simplifications, although conceptually the proof would remain the same. However, then the definition of $D$ via the Floer trajectory count in Section 2.2.3 requires an awkward from our perspective two-level Morse–Bott construction.
3. Shift operator: symplectic homology

3.1. Shift operator in equivariant symplectic homology. In this section we prove, essentially by passing to the limit, an analog of Theorem 2.12 for equivariant symplectic homology. To state the result, let us briefly recall the relevant definitions. Note that our conventions differ slightly from, e.g., [BO13b], although the resulting definitions are equivalent to the standard ones.

3.1.1. Conventions and requirements. We will assume that $(W^{2n}, \omega)$ is a compact exact symplectic manifold with a contact type boundary $(M, \alpha)$, i.e., $M = \partial W$ and $d\alpha = \omega|_M$, and in addition the orientation $\alpha \wedge (d\alpha)^{n-1}$ agrees with the boundary orientation of $M$, i.e., the Liouville vector field along $M$ points outward. In other words, $(W, \omega)$ is a strong symplectic filling of $(M, \alpha)$. Furthermore, we will require that $c_1(TW)|_{\pi_2(W)} = 0$. The symplectic completion $\hat{W} = W \cup_M (M \times [1, \infty))$ with the symplectic form $\omega$ extended as $d(r\alpha)$ to the cylindrical part. (Here $r$ is the coordinate on $[1, \infty)$.)

We will concentrate on contractible in $W$ closed Reeb orbits of $\alpha$ and denote by $P(\alpha)$ the collection of such orbits and by $S(\alpha)$ the action or period spectrum of $(M, \alpha)$, i.e.,

$$S(\alpha) = \{ A_\alpha(x) := \int_x^\infty \alpha \mid x \in P(\alpha) \}.$$  

Recall also that the Hamiltonian flow of $H \equiv r$ on the cylindrical part coincides with the Reeb flow.

Let us consider autonomous Hamiltonians $H$ on $\hat{W}$ meeting the following requirements:

(i) $H$ is $C^2$-small and negative on $W$;
(ii) $H = h(r)$, where $h: [1, \infty) \to \mathbb{R}$ is convex (i.e., $h'' \geq 0$) on the cylindrical part $M \times [1, \infty)$;
(iii) $H = \kappa r - c$ outside a compact set (i.e., when $r \geq r_0$ for some $r_0$), where $\kappa \notin S(\alpha)$ is positive.

We call such Hamiltonians admissible. (Note that for such Hamiltonians we necessarily have $c > 0$ in (iii).) When $H$ satisfies only (iii), we call it admissible at infinity. We emphasize that we do not impose here any non-degeneracy conditions on $H$. When $H$ is admissible at infinity, its Hamiltonian flow has no one-periodic orbits in the region $r \geq r_0$ and the global Floer homology of $H$ is defined and independent of $H$ as long as $\kappa$ is fixed.

Let $I = [a, b] \subset \mathbb{R}$ be a fixed interval with end-points outside $S(\alpha)$. This interval can be finite or semi-finite or infinite and in the last two cases the infinite end-points are not included in $I$. For any two Hamiltonians $H_0 \leq H_1$ admissible at infinity and such that the end points of $I$ are outside $S(H_0)$ and $S(H_1)$, we have a well-defined continuation map

$$HF^{G,I}_*(H_0) \to HF^{G,I}_*(H_1)$$

and we set

$$SH^{G,I}_*(W) := \lim_{H \to H} HF^{G,I}_*(H),$$  

where the limit is taken over all admissible Hamiltonians $H$ such that the end-points of $I$ are not in $S(H)$. For instance, when $I = [\delta, \infty)$ for a sufficiently small $\delta > 0$, 

we obtain the standard positive equivariant symplectic homology $\text{SH}_G^+(W)$. In fact, in the situation we are interested in we can always assume that $a > 0$.

Finally, note that by passing to the limit we obtain the shift operator

$$D : \text{SH}_G^I(W) \rightarrow \text{SH}_G^{I-2}(W).$$

Likewise, we have the spectral invariants $c_w(\alpha)$ for $w \in \text{SH}_G^I(W)$, which can be defined in two ways. First of all, as in (2.10), we can set

$$c_w(\alpha) = \inf \{ b' \in I \setminus \mathcal{S}(\alpha) \mid w \in \text{im} (i^{b'}) \}$$

where $i^{b'}$ is the natural map $\text{SH}_G^{I'}(W) \rightarrow \text{SH}_G^I(W)$ for $[a, b'] = I' \subset I = [a, b]$. (By definition, $c_0(\alpha) = a$.) Alternatively, we can take a cofinal sequence of Hamiltonians $H_j$ and a sequence $w_j \in \text{HF}_G^I(H_j)$ converging to $w$ in the obvious sense, and set

$$c_w(\alpha) = \lim_{j \rightarrow \infty} c_{w_j}(H_j).$$

It is not hard to show that these definitions are equivalent; cf. the proof of Theorem 3.4.

The shift map $D$ and the spectral invariants are “functorial”. To be more specific, consider a symplectic cobordism $Z$ with $\partial Z = M_0 \cup M_1$ from $(M_0, \alpha_0) = \partial W_0$, in obvious notation, to $(M_1, \alpha_1) = \partial W_1$, where $W_1 = W_0 \cup Z$, such that $W_1$ is exact and $c_1(TW_1)|_{\pi_2(W_1)} = 0$. Then $Z$ gives rise to a cobordism map, also known as the transfer map,

$$\Phi_Z : \text{SH}_G^{I'}(W_1) \rightarrow \text{SH}_G^I(W_0),$$

induced, before passing to the limit, by continuation maps in equivariant Floer homology; see Section 2.2.2. This map was originally introduced in [Vi99] in a slightly different setting and then studied in detail in [Gut15].

**Proposition 3.1.** The shift operator $D$ and the cobordism map $\Phi_Z$ commute and $c_{\Phi_Z(w)}(\alpha_0) \leq c_w(\alpha_1)$.

**Corollary 3.2.** The spectral invariant $c_w(\alpha)$ is Lipschitz (with Lipschitz constant equal to one) in $\alpha$.

The proofs of the proposition and the corollary are absolutely standard and we omit them. Furthermore, as a consequence of Proposition 2.23 we obtain, by passing to the limit, the following result.

**Proposition 3.3.** Assume that $I \subset (0, \infty)$ and that all contractible closed Reeb orbits of $\alpha$ with action in $I$ are non-degenerate. Then $\text{SH}_G^I(W; \mathbb{Q})$ is the homology of a certain complex generated over $\mathbb{Q}$ by the good closed Reeb orbits with action in $I$, graded by the Conley–Zehnder index, and filtered by the action.

We will give a detailed proof of Proposition 3.3 in Section 3.1.2. It is not clear to us if the complex from the proposition is necessarily isomorphic to the filtered contact homology complex of $(M, \alpha)$ when (if) the contact homology is defined; see [BO12]. However, in any event, the nature of the differential is not really essential for our purposes. As we have already pointed out in Remark 2.24, even if the differential is defined explicitly, it can rarely be calculated beyond some obvious cases. Furthermore, as in the case of equivariant Morse or Floer homology (see Remark 2.9), although the differential depends on the auxiliary data, different choices of such data result in isomorphic complexes. The differential is “natural”...
in the same sense as the Floer or Morse differential, i.e., a symplectic cobordism gives rise to a homomorphism of complexes.

3.1.2. The Lusternik–Schnirelmann inequality in equivariant symplectic homology.
With general definitions in place, we are ready to (re)state our main result, Theorem 1.1 from the introduction. Recall that a closed Reeb orbit $x$ is called isolated if there exists a tubular neighborhood $U$ of $x$ and an interval $I = (A_\alpha(x) - \epsilon, A_\alpha(x) + \epsilon)$ such that no periodic orbit with action in $I$ enters $U$.

**Theorem 3.4.** Assume that $I = [a, b] \subset (0, \infty)$ and that all closed Reeb orbits of $\alpha$ with action in $I$ are isolated. Then, for any non-zero element $w \in SH^{G,I}_S(W; \mathbb{Q})$, we have

$$c_w(\alpha) > c_{D(w)}(\alpha).$$  \hspace{1cm} (3.3)

We emphasize again that the main new point of this theorem is that the inequality is strict. The non-strict inequality holds without any assumptions on the orbits and for any coefficient ring; cf. Remark 2.13. Furthermore, it is essential that in this theorem, as in Theorem 2.12, we make no non-degeneracy assumptions on $\alpha$.

**Proof.** First, observe that it is sufficient to prove the theorem for an interval $I$ with a finite upper end-point $b$. (If the upper end of the interval is $\infty$, we can replace it by any $b > c_w(\alpha)$.) Then $S(\alpha) \cap I$ is a finite set and hence gap$(\alpha)$, which is by definition the infimum of positive action gaps in $I$, is strictly positive.

Consider a cofinal sequence of admissible Hamiltonians $H_j$ with the following properties:

- $H_j = \kappa_j r - c_j$ on $M \times [r_j, \infty)$ where $r_j \to 1$ and $\kappa_j \to \infty$,
- $h_j \leq 0$ on $[1, r_j]$.

Every $x \in P(\alpha)$ occurs as a one-periodic orbit of $H_j$ for a sufficiently large $j$ (depending on $x$) exactly once. This orbit, denoted by $x_j$, lies on the level $r = \rho_j \in (1, r_j)$, where $\rho_j = \rho_j(T)$ is uniquely determined by the condition $h_j'(\rho_j) = T$ with $T := A_\alpha(x)$. The Hamiltonian action of the resulting orbit $x_j$ is

$$A_{H_j}(x_j) = \rho_j(T) - h_j(\rho_j) =: f_j(T).$$  \hspace{1cm} (3.4)

Clearly, $\rho_j \to 1$ and $h_j(\rho_j) \to 0$ since the sequence $H_j$ is cofinal. Thus

$$A_{H_j}(x_j) \to A_\alpha(x) \text{ as } j \to \infty.$$

As a consequence, when $j$ is large enough, $S(H_j) \cap I$ is a finite set converging to $S(\alpha) \cap I$ as $j \to \infty$.

Furthermore, we also have convergence of the minimal action gaps:

$$\text{gap}(H_j) \to \text{gap}(\alpha) > 0,$$  \hspace{1cm} (3.5)

where in both cases we have intersected the action spectrum with $I$. (To prove this, it is enough to guarantee that gap$(H_j) \to 0$.) To establish (3.5), observe first that, by (3.4), $S(H_j) = f_j(S(\alpha))$, where $f_j(T)$ for any $T \in I$ is given by (3.4) with $\rho_j$ determined via $h_j'(\rho_j) = T$. A direct calculation shows that $f_j'(T) = \rho_j(T) \to 1$ uniformly on $I$ as $j \to \infty$, which implies (3.5).

Finally note that all one-periodic orbits of $H_j$ with action in $I$ are non-constant since $a > 0$. Now the theorem follows from Corollary 2.14."
Proof of Proposition 3.3. Throughout the proof we keep the notation from the proof of Theorem 3.4. Assume first that the interval I is finite and its end points are outside S(α). Let H_j be a cofinal, increasing sequence of admissible Hamiltonians as in the proof of Theorem 3.4. Then it is not hard to see that when j is large enough the generators x_j of the complex CF_*(H_j)^G from Proposition 2.23 are naturally in one-to-one correspondence with the good closed Reeb orbits x with action in I. Furthermore, again when j is large enough, a monotone homotopy from H_j to H_{j+1} induces an isomorphism CF_*(H_j)^G → CF_*(H_{j+1})^G. (This isomorphism has the form id + Φ, where Φ is strictly Hamiltonian action decreasing.) Furthermore, when the functions h_j are concave (i.e., h''_j ≤ 0), the Hamiltonian action filtration and the contact action filtration are equivalent: \mathcal{A}_H(x_j) ≥ \mathcal{A}_{H_j}(x'_j) if and only if \mathcal{A}_n(x) ≥ \mathcal{A}_n(x').

Thus we have a well-defined complex CF_*(α)^G := lim CF_*(H_j)^G, graded by the Conley–Zehnder index and filtered by the contact action, with homology equal to SH^G_I(W). In fact, this complex with its grading and filtration is isomorphic to CF_*(H_j)^G for a large j. For any, not necessarily finite, interval I, the complex CF_*(α)^G with the required properties is constructed by exhausting I by finite intervals and applying the diagonal process. (Here we again use the fact that the homology functor and the direct limit functor commute.)

The proof of Theorem 3.4 also lends itself readily for the definition of the local equivariant symplectic homology. Namely, let x be an isolated closed Reeb orbit on M. Then the corresponding orbit x_j of H_j is also isolated, although the size of the isolating neighborhood goes to zero in the r-direction as j → ∞. It is not hard to see that for any fixed degree * the equivariant Floer homology HF^G_*(x_j, H_j) stabilizes as j → ∞.

Definition 3.5. The equivariant local symplectic homology SH^G_*(x) of x is by definition the equivariant local Floer homology HF^G_*(x_j, H_j) where j is large enough.

Example 3.6 (Non-degenerate orbits). Assume that x is non-degenerate. Then by Examples 2.18 and 2.19, SH^G_*(x; Q) is one-dimensional and concentrated in degree equal to the Conley–Zehnder µ(x) when x is good and zero when x is bad.

By Proposition 2.20, SH^G_*(x; Q) is supported in [µ_-(x), µ_+(x)], which in turn is contained in [\tilde{µ}(x) - n + 1, \tilde{µ}(x) + n - 1]. The operator D obviously descends to SH^G_*(x). However, by Proposition 2.21, the resulting operator is trivial.

Corollary 3.7. The shift operator in the local equivariant symplectic homology of an isolated orbit is identically zero: D ≡ 0 in SH^G_*(x; Q).

The argument from [BO12] readily translates to the proof of the fact that SH^G_*(x; Q) is isomorphic, up to a shift of degree, to the local contact homology HC_*(x) introduced in [HM]. Alternatively, Proposition 3.3 carries over to the local case and SH^G_*(x; Q) is isomorphic to the homology of a certain complex graded by the Conley–Zehnder index and generated by the good orbits that x splits into under a non-degenerate perturbation (cf. Remark 2.24). When x is simple, HC_*(x) is isomorphic to the local Floer homology of the Poincaré return map φ in M; see [HM, GH^2M]. When x = y^k, with y simple, HC_*(x) is expected to be isomorphic to the equivariant local Floer homology HF^G_*(φ); see [GH^2M].
Remark 3.8 (Generalizations and variations, II). The results of this section readily extend to non-contractible periodic orbits, although the conditions on \( W \) cannot be relaxed to the same degree as in Remark 2.15. To be more specific, one can focus on closed Reeb orbits in a fixed free homotopy class \( \mathcal{f} \) of loops in \( W \). Then, for any \( \mathcal{f} \), the condition that \( \omega \) is exact can be replaced by that \( \omega \) is aspherical and, for instance, \( \pi_1(M) \to \pi_1(W) \) is a monomorphism. (The role of this condition is to ensure that the contact action in \( M \) is equal to the symplectic area in \( W \) giving rise to the filtration in symplectic homology. In general, the symplectic area can differ from the contact action as can be seen from the example of the pre-quantization disk bundle over a surface of genus \( g \geq 2 \).) Furthermore, when \( \mathcal{f} \neq 1 \) we need to assume \( c_1(TW) \) to be atoroidal. Note also that the condition \( I \subset (0, \infty) \) can be dropped in Proposition 3.3 and Theorem 3.4 when \( \mathcal{f} \neq 1 \).

When \( W \) is exact and \( c_1(TW) = 0 \), the equivariant symplectic homology of \( W \) is defined for all free homotopy classes. This homology is naturally filtered by the action and graded by the free homotopy class. It is clear that the analogs of the results from this section including Theorem 3.4 hold in this case; cf. Remark 2.15.

3.2. Examples and applications. In this section, having our main applications to dynamics in mind, we consider some simplest cases where Theorem 3.4 can be utilized to produce non-obvious results: the standard contact sphere \( S^{2n-1} \), the boundary of a displaceable Liouville domain, and the boundary of a Liouville domain in \( T^*S^n \) containing the zero section, e.g., the standard unit cotangent bundle \( ST^*S^n \).

3.2.1. The standard contact \( S^{2n-1} \) and displaceable Liouville domains. Let \( \alpha \) be a contact form on \( M = S^{2n-1} \) supporting the standard contact structure. Then, as is well known, \((M, d\alpha)\) can be embedded as a hypersurface in \( \mathbb{R}^{2n} \) bounding a star-shaped domain \( W \). We take \( I = [\delta, \infty) \) and work with the standard positive equivariant symplectic homology \( \text{SH}_*^{G, +}(W; \mathbb{Q}) \), where \( G = S^1 \). The homology is concentrated and equal to \( \mathbb{Q} \) in every second degree starting with \( n+1 \):

\[
\text{SH}_*^{G, +}(W; \mathbb{Q}) = \begin{cases} 
\mathbb{Q} & \text{for } * = n+1, n+3, \ldots, \\
0 & \text{otherwise;}
\end{cases}
\]

and moreover the shift operator

\[
D: \mathbb{Q} = \text{SH}_*^{G, +}(W; \mathbb{Q}) \to \text{SH}_{*-2}^{G, +}(W; \mathbb{Q}) = \mathbb{Q}
\]

is an isomorphism for \( * = n + 3, n + 5, \ldots \); see, e.g., [BO13b] and references therein. Therefore, there exists a sequence of non-zero homology classes \( w_k \in \text{SH}_{*-2k-1}^{G, +}(W; \mathbb{Q}) \), \( k \in \mathbb{N} \), such that \( Dw_{k+1} = w_k \).

In a similar vain but slightly more generally, we may assume that \((M, \alpha)\) is a restricted contact type hypersurface in \( \mathbb{R}^{2n} \) bounding a region \( W \). (This is automatically the case when, e.g., \( W \) is a simply connected Liouville domain in \( \mathbb{R}^{2n} \).) Then we have \( \lambda W \subset B_R \subset W \subset B_R \) for two balls in \( \mathbb{R}^{2n} \) and some \( \lambda > 0 \). Thus the cobordism map \( \text{SH}_*^{G, +}(B_R; \mathbb{Q}) \to \text{SH}_*^{G, +}(W; \mathbb{Q}) \) is an isomorphism, and we conclude that the image of \( w_k \) in \( \text{SH}_*^{G, +}(W; \mathbb{Q}) \) is non-zero. Hence, as above, we have a well-defined sequence of non-zero elements, which we still denote by \( w_k \), of degree \( n + 2k - 1 \) such that \( Dw_{k+1} = w_k \). Set \( c_k = c_{w_k} \), where \( c_{w_k} \) is defined by \((3.2)\).

As an immediate consequence of Theorem 3.4, we obtain
Corollary 3.9. Let \( (M, \alpha) \) be a restricted contact type hypersurface in \( \mathbb{R}^{2n} \). Then there exists a carrier map
\[
\psi : \mathbb{N} \to \mathcal{P}(\alpha), \quad k \mapsto y_k
\]
such that \( c_k(\alpha) = A_\alpha(y_k) \) and
\[
A_\alpha(y_1) \leq A_\alpha(y_2) \leq A_\alpha(y_3) \leq \cdots \quad \text{and} \quad \mu_-(y_k) \leq n + 2k - 1 \leq \mu_+(y_k). \tag{3.6}
\]
In particular,
\[
|\hat{\mu}(y_k) - (n + 2k - 1)| \leq n - 1.
\]
Furthermore, assume that all orbits in \( \mathcal{P}(\alpha) \) are isolated. Then \( \psi \) is an injection, \( \text{SH}^G_{\ast}(y_k) \neq 0 \) in degrees \( \ast = n + 2k - 1 \), and
\[
c_1(\alpha) < c_2(\alpha) < c_3(\alpha) < \cdots.
\]
or, in other words,
\[
A_\alpha(y_1) < A_\alpha(y_2) < A_\alpha(y_3) < \cdots.
\]
Finally, when \( \alpha \) is non-degenerate, the orbits \( y_k \) are good.

This corollary is a minor generalization of Theorem 1.3 from the introduction. Note that, in general, the carrier map is not unique. However, it becomes unique when, for instance, all closed characteristics on \( M \) have different actions.

**Proof.** Assume first that all orbits in \( \mathcal{P}(\alpha) \) are isolated. It is easy to show by arguing as in, e.g., [CGG] that for every \( k \in \mathbb{N} \) there exists an orbit \( y_k \) such that \( A_\alpha(y_k) = c_k(\alpha) \) and \( \text{SH}^G_{n+2k-1}(y_k; \mathbb{Q}) \neq 0 \). (If such an orbit is not unique, we just pick one of them.) Then the map \( k \mapsto y_k \) is an injection due to the fact that the inequalities in Corollary 3.9 are strict. The second inequality relating \( \hat{\mu}(y_k) \) and \( k \) holds by Proposition 2.20. The general case follows by continuity, but, of course, the strict inequalities and hence the injectivity of \( \psi \) are lost in the process. \( \square \)

**Example 3.10 (Ellipsoids).** Let \( M \) be the ellipsoid
\[
\sum_j |z_j|^2 = 1
\]
in \( \mathbb{C}^n \) with the standard contact form \( \alpha \). Let us combine \( n \) sequences \( \pi r_j^2 k, \; k \in \mathbb{N} \), into one monotone increasing sequence \( c_1 \leq c_2 \leq \cdots \). Then this is exactly the sequence of spectral invariants \( c_k(\alpha) \). The Reeb orbits on \( M \) are isolated if and only if \( r_j^2 = qr_j^2 \), with \( q \in \mathbb{Q} \) only when \( j = j' \), i.e., the sequences \( \pi r_j^2 k \) do not overlap or, equivalently, the sequence \( c_k \) is strictly increasing.

**Remark 3.11.** When \( W \) is convex, the spectral invariants \( c_k(\alpha) \) are believed to be equal to the Ekeland–Hofer capacities of \( W \), [EH89]. In this case, variants of the corollary are known in a form not relying on the machinery of Floer or symplectic homology; see [Ek, Lo02]. The only feature of the domain \( W \) essential for Corollary 3.9 is that there exists an infinite chain of non-zero classes \( w_k \), such that \( Dw_{k+1} = w_k \). Thus, whenever \( W \) is a Liouville domain with such a chain and \( c_1(TW)|_{Tz(W)} = 0 \), the corollary holds (for the trivial free homotopy class) with \( n + 2k - 1 \) in the index bounds replaced by the degree of \( w_k \). When the free homotopy class is non-trivial, it suffices to require in addition that \( c_1(TW) = 0 \); cf. Remark 3.8.
For instance, let $W$ be a subcritical Stein manifold with $c_1(TW) = 0$. Then the above results extend to $W$ word-for-word for the trivial free homotopy class. Indeed, by [BO13b, Cor. 1.3], there exists a sequence of non-zero homology classes $w_k \in \text{SH}_{n+2k-1}^G(W; \mathbb{Q})$, $k \in \mathbb{N}$, such that $Dw_{k+1} = w_k$ and thus Corollary 3.9 holds in this case exactly as stated. In fact, every non-zero element in $H_d(W; \partial W; \mathbb{Q})$ gives rise to such a sequence starting with degree $d + 1 - n$. However, it is not clear to us how to make use of these multiple sequences.

More generally, Corollary 3.9 holds when $W$ is a Liouville domain displaceable in $\hat{W}$ or even when $W$ is a displaceable Liouville subdomain in some other Liouville manifold, provided that $c_1(TW) = 0$. Indeed, as is well known, then the full ordinary symplectic homology of $W$ vanishes; see, e.g., [CFO, Ri]. As a consequence, we also have $\text{SH}^*_G(W) = 0$. Now, exactly as for $\mathbb{R}^{2n}$, the Gysin exact sequence implies that there exists a sequence of non-zero homology classes $w_k \in \text{SH}_{n+2k-1}^G(W; \mathbb{Q})$, $k \in \mathbb{N}$, such that $Dw_{k+1} = w_k$; cf. [BO13b].

3.2.2. Simple Reeb orbits on $ST^*S^n$. Our next objective is to extend some of the results from Section 3.2.1 to Liouville domains in $T^*S^n$ containing the zero section $S^n$. For the sake of simplicity we will assume throughout this section that $n \geq 3$.

Let $W$ be a such a domain with smooth boundary and $M = \partial W$. In other words, $M$ is a restricted contact type hypersurface $M \subset T^*S^n$ enclosing $S^n$. For instance, $M$ can be the unit cotangent bundle $ST^*S^n$ with respect to the round metric and $W$ is then the unit disk bundle $W_1$, or we can take as $W$ any compact fiberwise star-shaped domain with smooth boundary.

**Proposition 3.12.** We have

$$\dim \text{SH}_{k}^{G,+}(W; \mathbb{Q}) = \begin{cases} 0 & \text{for } k \equiv n \pmod{2}, \\ 2 & \text{for } k = j(n-1) \text{ for all } j > 1 \text{ when } n \text{ is odd}, \\ 2 & \text{for } k = j(n-1) \text{ for odd } j > 1 \text{ when } n \text{ is even}, \\ 1 & \text{for all other } k \equiv n-1 \pmod{2}. \end{cases}$$

For every $j \in \mathbb{N}$, there exist $n$ non-zero elements $w_i \in \text{SH}_{2i+(2j-1)(n-1)}^{G,+}(W; \mathbb{Q})$, $i = 0, \ldots, n-1$, such that $Dw_{i+1} = w_i$.

**Remark 3.13.** The first part of the proposition is standard and included only for the sake of completeness; see, e.g., [KvK]. The second part of Proposition 3.12 holds in a more general setting than considered here. Namely, with suitable modifications, it holds when $M$ is a pre-quantization $S^1$-bundle over a closed symplectic manifold $B^{2n}$ and $W$ is the disk bundle. Note also that in general $\text{SH}_{*}^{G,+}(W; \mathbb{Q})$ may have a much longer sequence of non-vanishing classes with $Dw_{i+1} = w_i$ than the sequences of length $n$ coming from the proposition. For instance, $S^{2n-1}$ admits such an infinite sequence; see Section 3.2.1. We do not know how long a sequence for $ST^*S^n$ can actually be.

**Proof.** First, we note that it is enough to prove the proposition for the unit disk bundle $W_1$. Indeed, it is easy to see from the inclusions $\lambda W \subset W_R \subset W \subset W_{R'}$, where $W_R$ and $W_{R'}$ are disk bundles and $\lambda > 0$ is sufficiently small, that by Proposition 3.1

$$\text{SH}_{*}^{G,+}(W; \mathbb{Q}) \cong \text{SH}_{*}^{G,+}(W_1; \mathbb{Q}).$$
The homology $\text{SH}^{G,+}(W_1; \mathbb{Q})$ can be easily calculated using the Morse–Bott techniques. Indeed, recall that $ST^*S^n$ is a $G = S^1$-principal bundle whose base $B$ is the Grassmannian $\text{Gr}^+(2, n+1)$ of oriented two-planes in $\mathbb{R}^n$. The Reeb flow on $ST^*S^n$ is the geodesic flow on $S^n$ with respect to the round metric. Hence, the flow is Morse–Bott in the sense of [Bo, Es, Po] and its “critical sets” $P_j$ are formed by $j$-iterated geodesics for all $j \in \mathbb{N}$. The set $P_j$ is equivariantly diffeomorphic to $ST^*S^n$ with the $G$-action obtained by combining the standard action with the $j$-fold covering $G \to G$. By Proposition 2.22 and since $H_*(B; \mathbb{Q})$ vanishes in odd degrees, $\text{SH}^{G,+}(W_1; \mathbb{Q})$ breaks down into the sum of infinite number of terms $H^G_*(P_j; \mathbb{Q})$ up to a shift of degree. The $G$-action on $P_j$ is locally free and $P_j/G = B$. Thus, $H^G_*(P_j; \mathbb{Q}) = H_*(B; \mathbb{Q})$.

In fact, $H_*(B; \mathbb{Q})$ has one generator $w'_i \neq 0$ in every even degree $i = 0, 2, \ldots, 2n - 2$ and also, when $n - 1$ is even, one extra generator in degree $n - 1$ (the middle of the range); see, e.g., [KvK] and references therein. In all other degrees the homology is zero.

With our conventions, the shift for $P_j$ is equal to $-(n - 1) + j\Delta$ where $\Delta$ is the mean index of a closed Reeb orbit in $P_1$ (a simple geodesic) or, equivalently, $\Delta = 2(c_1(ST^*S^n \to B), u)$, where $u$ is a suitably oriented generator of $\pi_2(B) \cong \mathbb{Z}$; cf. [Es, Ex. 8.2]. We have $\Delta = 2(n - 1)$ and thus

$$\text{SH}^{G,+}(W_1; \mathbb{Q}) = \bigoplus_j H_*(-(2j - 1)(n - 1))B; \mathbb{Q}),$$

where every term in the sum contributes to the homology in degrees of the same parity as $n - 1$ in the interval

$$[(2j - 1)(n - 1), \ldots, (2j + 1)(n - 1)]$$

centered at $2(n - 1)j$; see [KvK, Sect. 5.6].

The operator $D$ on $H^G_*(P_j; \mathbb{Q}) = H_*(B; \mathbb{Q})$ is Poincaré dual (up to a factor) to the multiplication by $c_1(ST^*S^n \to B)$, i.e., by the cohomology class of the standard symplectic structure on $\text{Gr}^+(2, n + 1)$. It follows that $Dw'_i = w'_i$ for a suitable choice of the generators $w'_i \in H_2(B; \mathbb{Q})$. Now, fixing $j \in \mathbb{N}$, we let $w_i$ be the image of $w'_i$ under the identification (3.7). Then, by Proposition 2.22, $Dw_{i+1} = w_i \neq 0$. \hfill $\square$

As an immediate consequence of Proposition 3.12, we obtain the following analog of Corollary 3.9 generalizing Theorem 1.4 from the introduction.

**Corollary 3.14.** Let $(M, \alpha)$ be a restricted contact type hypersurface in $T^*S^n$ enclosing the zero section and such that all periodic orbits of the Reeb flow are isolated. Then, in the notation of Proposition 3.12, for every $j \in \mathbb{N}$, there exist $n$ periodic orbits $y_0, \ldots, y_{n-1}$ such that $c_{w_i}(\alpha) = A_\alpha(y_i)$ and $\text{SH}^G_*(y_i; \mathbb{Q}) \neq 0$ in degrees $* = (2j - 1)(n - 1) + 2i$. In particular,

$$A_\alpha(y_0) < A_\alpha(y_1) < \cdots < A_\alpha(y_{n-1}),$$

and

$$|\hat{\mu}(y_i) - ((2j - 1)(n - 1) + 2i)| \leq n - 1.$$  

**Remark 3.15.** Without the assumption that the closed Reeb orbits of $\alpha$ are isolated, we still have $n$ orbits $y_i$ such that $c_{w_i}(\alpha) = A_\alpha(y_i)$ and (3.9) is satisfied, but now the action inequalities (3.8) are not necessarily strict:

$$A_\alpha(y_0) \leq A_\alpha(y_1) \leq \cdots \leq A_\alpha(y_{n-1}).$$
3.2.1 in a more formal context, we could have introduced action carriers for spectral invariants in the equivariant Floer or symplectic homology similarly to the carriers in [CGG]. Thus, for instance, when $W$ is exact with $c_1(TW)|_{\pi_2(W)} = 0$ and all orbits in $\mathcal{P}(\alpha)$ are isolated, we would have a map $SH_*^{S^1}(W; \mathbb{Q}) \setminus \{0\} \to \mathcal{P}(\alpha)$ such that $w$ and $Dw \neq 0$ are never mapped to the same orbit.

4. INDEX THEORY

4.1. PRELIMINARIES: DEFINITIONS AND BASIC FACTS. In this section we recall for the reader’s convenience some basic properties of the mean and Conley–Zehnder indices. We refer the reader to, e.g., [Lo02] or [SZ, Sect. 3] for a more thorough treatment; see also [Ab, Gut14] and, for a very quick introduction, [Sa99, Sect. 2.4].

4.1.1. Definitions. To every continuous path $\Phi: [0, 1] \to \text{Sp}(2m)$ beginning at $\Phi(0) = I$, one can associate the mean index $\hat{\mu}(\Phi) \in \mathbb{R}$, a homotopy invariant of the path with fixed end-points. To give a formal definition, recall first that a map $\hat{\mu}$ from a Lie group to $\mathbb{R}$ is said to be a quasimorphism if it fails to be a homomorphism only up to a constant, i.e.,

$$|\hat{\mu}(\Phi \Psi) - \hat{\mu}(\Phi) - \hat{\mu}(\Psi)| < \text{const},$$

where the constant is independent of $\Phi$ and $\Psi$. One can prove that there is a unique quasimorphism $\hat{\mu}: \text{Sp}(2m) \to \mathbb{R}$, on the universal covering $\tilde{\text{Sp}}(2m)$ of the symplectic group, which is continuous and homogeneous (i.e., $\hat{\mu}(\Phi^k) = k\hat{\mu}(\Phi)$) and satisfies the normalization condition:

$$\hat{\mu}(\Phi_0) = 2 \text{ for } \Phi_0(t) = \exp{(2\pi \sqrt{-1}t)} \oplus I_{2m-2}$$

with $t \in [0, 1]$, in the self-explanatory notation; see [BG]. In particular, $\hat{\mu}$ restricts to an isomorphism $\pi_1(\text{Sp}(2m)) \to 2\mathbb{Z}$. The quasimorphism $\hat{\mu}$ is the mean index. The continuity requirement holds automatically and is not necessary for the characterization of $\hat{\mu}$, although this is not immediately obvious. Furthermore, $\hat{\mu}$ is also automatically conjugation invariant, as a consequence of the homogeneity.

The mean index $\hat{\mu}(\Phi)$ measures the total rotation angle of certain unit eigenvalues of $\Phi(t)$ and can be explicitly defined as follows. Following [SZ], for an elliptic transformation $A \in \text{Sp}(2)$, let us say that an eigenvalue $\exp(\sqrt{-1}t) \in S^1$ of $A$ is of the first kind if $0 \leq \theta \leq \pi$ and $A$ is conjugate to the rotation in $\theta$ counterclockwise or if $-\pi < \theta < 0$ and $A$ is conjugate to the rotation in $\theta$ clockwise. (This rule unambiguously picks one of the two eigenvalues of an elliptic transformation $A \in \text{Sp}(2)$.)

We set $\rho(A)$ to be equal its eigenvalue of the first kind when $A$ is elliptic and $\rho(A) = \pm 1$ when $A$ is hyperbolic with the sign determined by the sign of the eigenvalues of $A$. Then $\rho: \text{Sp}(2) \to S^1$ is a Lipschitz (but not $C^1$) function, which is conjugation invariant and equal to det on $U(1)$. A matrix $A \in \text{Sp}(2m)$ with distinct eigenvalues, can be written as the direct sum of matrices $A_j \in \text{Sp}(2)$ and a matrix with complex eigenvalues not lying on the unit circle. We set $\rho(A)$ to be the product of $\rho(A_j) \in S^1$. Again, $\rho$ extends to a continuous function $\rho: \text{Sp}(2m) \to S^1$, which is conjugation invariant (and hence $\rho(AB) = \rho(BA)$) and restricts to det on $U(n)$; see, e.g., [SZ]. Finally, given a path $\Phi: [0, 1] \to \text{Sp}(2m)$, there is a continuous function $\theta(t)$ such that $\rho(\Phi(t)) = \exp{(\sqrt{-1}\theta(t))}$, measuring the total rotation of
the eigenvalues of the first type, and we set
\[ \hat{\mu}(\Phi) = \frac{\theta(1) - \theta(0)}{\pi}. \]

It is clear from the definition that \( \hat{\mu}(\Phi_s) = \text{const} \) for a family of paths \( \Phi_s \) as long as the eigenvalues of \( \Phi_s(1) \) remain constant.

Assume now that the path \( \Phi \) is non-degenerate, i.e., by definition, all eigenvalues of the end-point \( A = \Phi(1) \) are different from one. We denote the set of such matrices \( A \in \text{Sp}(2m) \) by \( \text{Sp}^*(2m) \) and also denote the part of \( \text{Sp}(2m) \) lying over \( \text{Sp}^*(2m) \) by \( \widetilde{\text{Sp}}^*(2m) \). It is not hard to see that \( A \) can be connected to a symplectic transformation with elliptic part equal to \(-I\) (if non-trivial) by a path \( \Psi \) lying entirely in \( \text{Sp}^*(2m) \). Concatenating this path with \( \Phi \), we obtain a new path \( \Phi' \). By definition, the Conley–Zehnder index \( \mu(\Phi) \in \mathbb{Z} \) of \( \Phi \) is \( \hat{\mu}(\Phi') \). One can show that \( \mu(\Phi) \) is well-defined, i.e., independent of \( \Psi \). The function \( \mu: \text{Sp}^*(2m) \to \mathbb{Z} \) is locally constant, i.e., constant on connected components of \( \widetilde{\text{Sp}}^*(2m) \). In other words, \( \mu(\Phi_s) = \text{const} \) for a family of paths \( \Phi_s \) as long as \( \Phi_s(1) \in \text{Sp}^*(2m) \).

Furthermore, let us call \( \Phi \) weakly non-degenerate if at least one eigenvalue of \( \Phi(1) \) is different from one and totally degenerate otherwise. A path is strongly non-degenerate if all its “iterations” \( \Phi^k \) are non-degenerate, i.e., none of the eigenvalues of \( \Phi(1) \) is a root of unity. The multiplicity of the generalized eigenvalue one of \( \Phi(1) \) is an even number, which we denote by \( 2 \nu(\Phi) \) and call \( \nu(\Phi) \) the nullity of \( \Phi \).

Following, e.g., [Lo90, Lo97], we define the upper and lower Conley–Zehnder indices as
\[ \mu_+(\Phi) := \limsup_{\Phi \to \Phi'} \mu(\Phi) \quad \text{and} \quad \mu_-(\Phi) := \liminf_{\Phi \to \Phi'} \mu(\Phi), \]

where in both cases the limit is taken over \( \Phi \in \widetilde{\text{Sp}}^*(2m) \) converging to \( \Phi \in \text{Sp}(2m) \). In fact, \( \mu_+(\Phi) \) is simply \( \max \mu(\Phi) \), where \( \Phi \in \widetilde{\text{Sp}}^*(2m) \) is sufficiently close to \( \Phi \) in \( \text{Sp}(2m) \); likewise, \( \mu_-(\Phi) = \min \mu(\Phi) \). (In terms of actual paths, rather than their homotopy classes, \( \Phi \) can be taken \( C^r \)-close to \( \Phi \) for any \( r \geq 0 \); the resulting definition of \( \mu_\pm(\Phi) \) is independent of \( r \) and equivalent to the one above due to homotopy invariance of \( \mu \).) Clearly, \( \mu(\Phi) = \mu_\pm(\Phi) \) when \( \Phi \) is non-degenerate. As readily follows from the definition, the indices \( \mu_\pm \) are the upper semi-continuous and, respectively, lower semi-continuous extensions of \( \mu \) from \( \text{Sp}^*(2m) \) to \( \text{Sp}(2m) \).

The upper and lower Conley–Zehnder indices are quasimorphisms \( \text{Sp}(2m) \to \mathbb{Z} \). These indices are of particular interest to us because they bound the support of the local Floer homology of an isolated periodic orbit; cf. Proposition 2.20 and [GG10].

4.1.2. Basic properties. Let us now list the properties of the Conley–Zehnder type indices, which are essential for our purposes. Most of these properties readily follow from the definitions and are well-known; see, e.g., [Lo92, SZ]. In what follows, all paths are required to begin at \( I \) and are taken up to homotopy, i.e., as elements of \( \text{Sp}(2m) \). Furthermore, we will tacitly assume the paths to be parametrized by \([0, 1]\) unless this is obviously not the case.

We start with three specific examples. For the path \( \Phi(t) = \exp (2\pi \sqrt{-1} \lambda t) \), \( t \in [0, 1] \), in \( \text{Sp}(2) \) we have
\[ \hat{\mu}(\Phi) = 2\lambda \text{ and } \mu(\Phi) = \text{sign}(\lambda) (2|\lambda| + 1) \text{ when } \lambda \notin \mathbb{Z}. \] (4.1)
Next, let $H$ be a non-degenerate quadratic form on $\mathbb{R}^{2m}$ with eigenvalues in the range $(-\pi, \pi)$. (Here, as is customary in Hamiltonian dynamics, the eigenvalues of a quadratic form $H$ on a symplectic vector space are by definition the eigenvalues of its Hamiltonian vector field $X_H = J \nabla H$, where $J$ is the matrix of the symplectic form; see, e.g., [Ar].) The path $\Phi(t) = \exp(JHt)$, $t \in [0, 1]$, is the linear autonomous Hamiltonian flow generated by $H$. Then, with our conventions,

$$\mu(\Phi) = \frac{1}{2} \text{sgn}(H),$$

where $\text{sgn}(H)$ is the signature of $H$, i.e., the number of positive squares minus the number of negative squares in the diagonal form of $H$ with $\pm 1$ and 0 on the diagonal. In addition, when $\Phi(1)$ is hyperbolic, we have

$$\mu(\Phi) = \hat{\mu}(\Phi).$$

Let us now list some “additive and multiplicative properties” which, combined with the examples above, allow one to calculate the indices in many cases. We start by observing that

$$\mu(\Phi^{-1}) = -\mu(\Phi)$$

for any non-degenerate path $\Phi$ and hence, in general,

$$\mu(\Phi^{-1}) = \mp \mu(\Phi).$$

When $\varphi$ is a loop, we have

$$\mu_{\pm}(\varphi \Phi) = \hat{\mu}(\varphi) + \mu_{\pm}(\Phi).$$

Finally, as readily follows from the definitions, $\hat{\mu}$ and $\mu$ are additive under direct sum. Namely, for $\Phi \in \tilde{\text{Sp}}(2m)$ and $\Psi \in \tilde{\text{Sp}}(2m')$, we have

$$\hat{\mu}(\Phi \oplus \Psi) = \hat{\mu}(\Phi) + \hat{\mu}(\Psi) \quad \text{and} \quad \mu(\Phi \oplus \Psi) = \mu(\Phi) + \mu(\Psi),$$

where in the second identity we assumed that both paths are non-degenerate. We will extend this additivity property to $\mu_{\pm}$ in Lemma 4.3.

The mean index and the upper and lower indices are related by the inequalities

$$\hat{\mu}(\Phi) - m \leq \mu_{-}(\Phi) \leq \mu_{+}(\Phi) \leq \hat{\mu}(\Phi) + m,$$

where $\Phi \in \tilde{\text{Sp}}(2m)$. (Moreover, at least one of the inequalities is strict when $\Phi$ is weakly non-degenerate.) As a consequence,

$$\lim_{k \to \infty} \frac{\mu_{\pm}(\Phi^k)}{k} = \hat{\mu}(\Phi),$$

hence the name “mean index” for $\hat{\mu}$.

The Conley–Zehnder index can also be evaluated as an intersection index with the discriminant $\Sigma = \{A \in \text{Sp}(2m) \mid \det(A - I) = 0\}$, leading to another extension of $\mu$ from $\tilde{\text{Sp}}(2m)$ to $\tilde{\text{Sp}}(2m)$ known as the Robbins–Salamon index. This index, introduced in [RS], is a quasimorphism $\tilde{\text{Sp}}(2m) \to \frac{1}{2} \mathbb{Z}$. Let us briefly recall its definition. For a path $\Phi$ in $\text{Sp}(2m)$, beginning at $I$ and generated by a quadratic time-dependent Hamiltonian $H_t$, i.e., $\dot{\Phi}(t) = JH_t \circ \Phi(t)$, let us call $\tau \in [0, 1]$ a crossing if $\Phi(\tau) \in \Sigma$. A crossing is non-degenerate if the quadratic form $Q_{\tau} := H_{\tau} | V_{\tau}$, where $V_{\tau} := \ker(\Phi(\tau) - I)$, is non-degenerate. Geometrically, that means that $\Phi(\tau)$ is not tangent to the stratum of $\Sigma$ through $\Phi(\tau)$. (Generically, all crossings $\tau$ are non-degenerate and the interior crossings $\tau \in (0, 1)$ are also simple, i.e.,
dim $V_\tau = 1$.) Non-degenerate crossings are isolated. Now, for a path $\Phi$ with only non-degenerate crossings, we set

$$
\mu_{RS}(\Phi) = \frac{1}{2} \text{sgn} \, Q_0 + \sum_{0 < \tau < 1} \text{sgn} \, Q_\tau + \frac{1}{2} \text{sgn} \, Q_1,
$$

where the last term is dropped when $1$ is not a crossing. Then $\mu_{RS}$ is homotopy invariant, $\mu_{RS}(\Phi) = \mu(\Phi)$ when $\Phi \in \tilde{\text{Sp}}^*(2m)$ (see [RS]), and $\mu_- \leq \mu_{RS} \leq \mu_+$. (In fact, $\mu_{RS} = (\mu_+ + \mu_-)/2$, but we do not need this relation.)

**Example 4.1 (Positive definite Hamiltonians).** Assume that $H_t > 0$ on $\mathbb{R}^{2m}$, i.e., $H_t$ is positive definite, for all $t \in [0, \infty)$, and let $\Phi_t$ be generated by $H_t$. Then, as readily follows from (4.7), $\hat{\mu}(\Phi|_{[0,T]})$ and $\hat{\mu}(\Phi|_{[0,T]})$ are increasing functions of $T > 0$ and $\mu_-(\Phi|_{[0,T]}) \geq m$ for all $T > 0$.

4.1.3. Further details: totally degenerate paths, additivity of $\mu_\pm$, and signature multiplicities $b_{0,\pm}$. In this subsection, we establish two additional properties of the indices $\mu_\pm$, somewhat less standard than the facts from the previous subsection, although still well-known to the experts. For the sake of completeness we provide detailed proofs. We also discuss some finer invariants of symplectic paths and their relations with the indices $\mu_\pm$.

**Lemma 4.2.** Assume that $A = \Phi(1)$ is totally degenerate, i.e., all eigenvalues of $A$ are equal to one. Then $\Phi$ is homotopic to the product of a loop and a path $\Psi$ such that for every $t$ all eigenvalues of $\Psi(t)$ are equal to one even if $A$ is not in $\exp(B)$. This proves the first assertion of the lemma.

To prove the second assertion, we can assume that $A \in \exp(B)$ and $Q$ is small. By (4.4) and since $\hat{\mu}(\Psi) = 0$, we have

$$
\hat{\mu}(\varphi) = \hat{\mu}(\Phi) \quad \text{and} \quad \mu_{\pm}(\Phi) = \hat{\mu}(\varphi) + \mu_{\pm}(\Psi).
$$

(4.9)

Set $\nu_0(Q) = \dim \ker Q$ and

$$
\text{sgn}_+(Q) := \max \text{sgn}(\tilde{Q}) = \text{sgn}(Q) + \nu_0
$$

and

$$
\text{sgn}_-(Q) := \min \text{sgn}(\tilde{Q}) = \text{sgn}(Q) - \nu_0,
$$

where max and min are taken over all small non-degenerate perturbations $\tilde{Q}$ of $Q$. The exponential mapping is a diffeomorphism from a small neighborhood of $Q$ onto a small neighborhood of $A$. Hence, by (4.2), we have

$$
\mu_{\pm}(\Psi) = \frac{1}{2} \text{sgn}_{\pm}(Q).
$$

(4.10)
Let us now apply this argument to the path $\Psi^k$ ending at $A^k$ for a fixed $k$. Again, without changing the index we can assume that $A^k = \exp(B)$ and $A \in \exp(B)$. We have $A^k = \exp(JH)$ and $A = \exp(JQ)$ where $H$ and $kQ$ are in $B$. Since $\exp$ is one-to-one on $B$, it readily follows that $H = kQ$. Thus

$$
\mu_{\pm}(\Psi^k) = \frac{1}{2} \sgn_{\pm}(kQ) = \frac{1}{2} \sgn_{\pm}(Q) = \mu_{\pm}(\Psi).
$$

This proves (4.8) for $\Psi$ since $\mu(\Psi) = 0$. The result for the original path $\Phi$ follows from (4.9) and the fact that $\pi_1(\Sp(2m))$ lies in the center of $\widetilde{\Sp}(2m)$.

The last general property of the index essential for us is the direct sum additivity for $\mu_{\pm}$ generalizing the second identity in (4.5).

**Lemma 4.3** ([Lo97]). The upper and lower Conley–Zehnder indices $\mu_{\pm}$ are additive with respect to direct sum, i.e., for $\Phi \in \widetilde{\Sp}(2m)$ and $\Psi \in \widetilde{\Sp}(2m')$, we have

$$
\mu_{\pm}(\Phi \oplus \Psi) = \mu_{\pm}(\Phi) + \mu_{\pm}(\Psi). \quad (4.11)
$$

This result, which is a part of [Lo97, Thm. 1.4], is also less known than most of the properties listed in Section 4.1.2 and not obvious. Since it plays a crucial role in our argument, we briefly outline the proof for the sake of completeness.

**Remark 4.4.** Note that the lemma is not a direct consequence of the definition of the indices $\mu_{\pm}$. Namely, it only follows from the definitions that $\mu_{\pm}$ is sup-additive and $\mu_{\pm}$ is sub-additive. However, by (4.3), it is enough to prove the lemma only for one of these indices.

**Proof.** The lemma holds for the direct sum of two non-degenerate paths by (4.5). From (4.9) and (4.10) and the fact that $\sgn_{\pm}$ are clearly direct sum additive, we observe that the lemma holds for the direct sum of paths with totally degenerate end-points.

Every path $\Phi \in \widetilde{\Sp}(2m)$ can be decomposed, up to homotopy, as the direct sum of a non-degenerate path $\Phi_1 \in \widetilde{\Sp}^\ast(2m_1)$ and a path $\Phi_0 \in \widetilde{\Sp}(2m_0)$ with totally degenerate $\Phi_0(1)$. Here $m = m_0 + m_1$. Thus it is sufficient to show that

$$
\mu_{\pm}(\Phi) = \mu(\Phi_1) + \mu_{\pm}(\Phi_0). \quad (4.12)
$$

Let $\tilde{\Phi}$ be a small non-degenerate perturbation of $\Phi$. The eigenvalues of $\tilde{\Phi}(1)$ can be broken down into two groups. The first group is formed by the eigenvalues close to one; the sum $V_0$ of their generalized eigenspaces has dimension $2m_0$ and is close to $\mathbb{R}^{2m_0}$. The second group is formed by the eigenvalues close to the eigenvalues of $\Phi(1)$ different from one; the sum $V_1$ of their generalized eigenspaces has dimension $2m_1$ and is close to $\mathbb{R}^{2m_1}$. It is not hard to see that $\tilde{\Phi}$ can be deformed, in a neighborhood of $\Phi \in \widetilde{\Sp}(2m)$, to a path $\tilde{\Phi}'$ such that $\tilde{\Phi}'(1)$ has the same eigenvalues as $\tilde{\Phi}(1)$ and $V_0 = \mathbb{R}^{2m_0}$ and $V_1 = \mathbb{R}^{2m_1}$, and moreover the eigenvalues of the end-point map remain constant in the process of deformation. Thus $\mu(\tilde{\Phi})$ also remains constant in the process of deformation, and in particular $\mu(\tilde{\Phi}') = \mu(\tilde{\Phi})$.

As an element of $\widetilde{\Sp}^\ast(2m)$, the path $\tilde{\Phi}'$ decomposes into the sum of a path $\tilde{\Phi}'_0 \in \widetilde{\Sp}^\ast(2m_0)$ close to $\Phi_0$ and a path $\tilde{\Phi}'_1 \in \widetilde{\Sp}^\ast(2m_1)$ close to $\Phi_1$. (This follows from the fact that the map from $\Sp(2m_0) \times \Sp(2m_1)$ to the part of $\widetilde{\Sp}(2m)$ lying above $\Sp(2m_0) \times \Sp(2m_1)$ is a covering map.) By non-degeneracy, $\mu(\tilde{\Phi}'_1) = \mu(\Phi_1)$. Hence

$$
\mu(\tilde{\Phi}') = \mu(\tilde{\Phi}') = \mu(\Phi_1) + \mu(\tilde{\Phi}'_0)
$$
and, therefore, $\mu_+(\Phi) \geq \mu(\Phi_1) + \mu_+ (\Phi_0)$. (For $\mu_-$, we have the opposite inequality.) Combining this with the fact that $\mu_+$ is sup-additive (see Lemma 4.3), we obtain (4.12) for $\mu_+$. For $\mu_-$ the result follows from a similar argument or from (4.3). This concludes the proof of the lemma. \hfill \Box

We finish this subsection by introducing certain invariants of $\Phi$, which we call signature multiplicities and the absolute nullity. These invariants play a central role in the index recurrence theorem (Theorem 5.2).

Consider first a totally degenerate operator $A \in \text{Sp}(2m)$. (In other words, we require all eigenvalues of $A$ to be equal to one.) Then, as we have seen in the proof of Lemma 4.2, $A = \exp(JQ)$ where all eigenvalues of $Q$ equal zero. The quadratic form $Q$ can be symplectically decomposed into a sum of terms of four types:

- the identically zero quadratic form on $\mathbb{R}^{2s_0}$,
- the quadratic form $Q_0 = p_1 q_2 + p_2 q_3 + \cdots + p_{d-1} q_d$ in Darboux coordinates on $\mathbb{R}^{2d}$, where $d \geq 1$ is odd,
- the quadratic forms $Q_{\pm} = \pm (Q_0 + q_2^2/2)$ on $\mathbb{R}^{2d}$ for any $d$.

(We find these normal forms, taken from \cite[App. 6]{AG}, more convenient to work with than the original Williamson normal forms; see \cite{Wi} and also, e.g., \cite[App. 6]{Ar}.) Clearly, $\text{dim ker } Q_0 = 2$ and $\text{sgn } Q_0 = 0$, and $\text{dim ker } Q_\pm = 1$ and $\text{sgn } Q_\pm = \pm 1$. Let $b_*(Q)$, where $* = 0, \pm$, be the number of the $Q_0$ and $Q_\pm$ terms in the decomposition. Let us also set $b_*(A) := b_*(Q)$ and $\nu_0(A) := \nu_0(Q)$. These are symplectic invariants of $Q$ and $A$. Then, arguing as in the proof of Lemma 4.2, we see that for a path $\Phi$ with $\Phi(1) = A$, we have

$$\mu_+(\Phi) = \hat{\mu}(\Phi) + b_0 + b_+ + \nu_0 \quad \text{and} \quad \mu_-(\Phi) = \hat{\mu}(\Phi) - b_0 - b_- - \nu_0. \quad (4.13)$$

These formulas readily extend to all paths. Namely, every $\Phi \in \widetilde{\text{Sp}}(2m)$ can be written (non-uniquely) as a product of a loop $\varphi$ and the direct sum $\Psi_0 \oplus \Psi_1$ where $\Psi_0 \in \widetilde{\text{Sp}}(2m_0)$ is a totally degenerate (for all $t$) path $\Psi_0(t) = \exp(JQT)$ and $\Psi_1 \in \widetilde{\text{Sp}}^*(2m_1)$. In particular, $m_0 = \nu(\Phi)$ and $m_0 + m_1 = m$. (Note that $\varphi$ can be absorbed into $\Psi_1$ unless $m_1 = 0$.)

**Definition 4.5.** The signature multiplicities and the absolute nullity of $\Phi$ are

$$b_*(\Phi) := b_*(\Psi_0) \quad \text{for } * = 0, \pm \quad \text{and} \quad \nu_0(\Phi) := \nu_0(\Psi_0).$$

One can show that these are symplectic invariants of $\Phi$, and

$$\mu_+(\Phi) = \hat{\mu}(\varphi) + \mu(\Psi_1) + b_0 + b_+ + \nu_0$$

and

$$\mu_-(\Phi) = \hat{\mu}(\varphi) + \mu(\Psi_1) - b_0 - b_- - \nu_0.$$

**4.2. Dynamical convexity.** The notion of dynamical convexity was originally introduced in \cite{HWZ} for Reeb flows on the standard contact $S^3$ and, somewhat in passing, for higher-dimensional contact spheres. In \cite{AM14, AM15} the definition was extended to other contact manifolds. Here we mainly focus on the linear algebra aspect of dynamical convexity, which is more essential for our purposes. Thus our entire approach is quite different from that in \cite{AM14, AM15}. It is convenient for us to adopt the following definition.

**Definition 4.6.** A path $\Phi \in \widetilde{\text{Sp}}(2m)$ is said to be **dynamically convex** (DC) if $\mu_-(\Phi) \geq m + 2$. 
Example 4.7. Assume that \( \Phi \) is generated by a positive definite Hamiltonian; see Example 4.1. Then \( \Phi \) need not be dynamically convex. However, by (4.7), \( \Phi \) is dynamically convex if it has at least two simple interior crossings or at least one interior crossing with multiplicity two.

Lemma 4.8. For any \( \Phi \in \tilde{Sp}(2m) \), we have

\[
\mu_-(\Phi^{k+1}) \geq \mu_-(\Phi^k) + (\mu_-(\Phi) - m)
\]

for all \( k \in \mathbb{N} \). In particular,

\[
\mu_-(\Phi^k) \geq (\mu_-(\Phi) - m)k + m.
\]

Assume furthermore that \( \Phi \) is dynamically convex. Then the function \( \mu_-(\Phi^k) \) of \( k \in \mathbb{N} \) is strictly increasing,

\[
\mu_-(\Phi^k) \geq 2k + m,
\]

and \( \hat{\mu}(\Phi) \geq \mu_-(\Phi) - m \geq 2 \). Thus \( \mu_-(\Phi^k) \geq m + 2 \) and all iterations \( \Phi^k \) are also dynamically convex.

The proof of the lemma is quite standard. Therefore, we will just briefly outline the argument; cf. [Lo02, SZ].

Proof. To prove the lemma, it is sufficient to establish (4.14). The rest of the assertion follows from that \( \mu_-(\Phi) \geq m + 2 \) by the definition of dynamical convexity.

Let us first assume that \( \Phi \) is strongly non-degenerate and its elliptic eigenvalues are distinct. Any such path \( \Phi \) is homotopic, i.e., equal as an element of \( \tilde{Sp}(2m) \), to a product of a loop \( \varphi \) and a path which is the direct sum of a path \( \Psi \) with hyperbolic end-point \( \Psi(1) \) and some number \( q \leq m \) of exponential paths in \( Sp(2) \) of the form \( \Phi_i(t) = \exp(2\pi \sqrt{1-\lambda_i}t) \) with \( 0 < \lambda_i < 1 \). (Using dynamical convexity of \( \Phi \) we can ensure that \( \hat{\mu}(\varphi) \geq 2 \) and \( \hat{\mu}(\Psi) \geq 0 \), but we do not need this fact.) By (4.1), (4.4) and (4.11),

\[
\mu(\Phi) = \hat{\mu}(\varphi) + \hat{\mu}(\Psi) + q,
\]

and therefore

\[
\Delta := \hat{\mu}(\varphi) + \hat{\mu}(\Psi) \geq \mu(\Phi) - m.
\]

Finally, note that, by (4.1), the sequences \( \mu(\Phi^k) \) are non-decreasing. Now we have

\[
\mu(\Phi^{k+1}) - \mu(\Phi^k) = \Delta + \sum_i (\mu(\Phi_i^{k+1}) - \mu(\Phi_i^k)) \geq \Delta \geq \mu(\Phi) - m,
\]

which proves (4.14) for \( \Phi \).

It is not hard to see that with \( \mu \) replaced by \( \mu_- \) this argument extends to the case where \( \Phi \) is still non-degenerate but its eigenvalues are not necessarily distinct and some iterations \( \Phi^k \) may be degenerate. (In essence, the reason is that the index is determined by the behavior and type (with multiplicity) of the eigenvalues of \( \Phi(t) \) rather than the map itself.)

In the case where \( \Phi(1) \) is totally degenerate, (4.14) is an immediate consequence of Lemma 4.2 together with (4.6). Finally, the general case follows from these two cases by additivity (Lemma 4.3).

Let now \( (M^{2n-1}, \xi) \) be a co-oriented contact manifold with \( c_1(\xi)\mid_{\xi_2(M)} = 0 \) and \( \alpha \) be a contact form supporting \( \xi \). We denote by \( \varphi_t \) its Reeb flow. For a contractible closed Reeb orbit \( x \), its linearized Poincaré return map \( \Phi = d\varphi_{1/2} \in \tilde{Sp}(2m) \) with \( m = n - 1 \) is defined in the standard way. Namely, we fix a capping of \( x \) (i.e., a
map from $D^2$ with boundary $x$) and trivialization of $\xi$ along the capping. With this trivialization, the linearized flow of $\varphi_t$ along $x$ becomes a path in $\text{Sp}(2m)$ starting at $I$. This path is well-defined as an element of $\tilde{\text{Sp}}(2m)$ (no tilde!) up to conjugation by a linear map from $\text{Sp}(2m)$. By definition $\mu_\pm(x)$, $\hat{\mu}(x)$, etc. are $\mu_\pm(\Phi)$, $\hat{\mu}(\Phi)$, etc. The condition that $c_1(\xi)$ vanishes on $\pi_2(M)$ guarantees that the indices are independent of the capping. The resulting indices inherit all the properties of their linear algebra counterparts from Section 4.1. For instance, the mean index $\hat{\mu}$ is homogeneous under iterations: $\hat{\mu}(x^k) = k\hat{\mu}(x)$.

In fact, a global trivialization of the complex determinant bundle $\text{det} \xi = \wedge^{n-1} \xi$ is sufficient to define the indices; see, e.g., \cite{Es}. Thus, when $c_1(\xi) = 0$ in $H^2(M; \mathbb{Z})$, one can instead use such a trivialization and, in this case, the orbits need not be contractible. Again, the indices have all the expected properties including the homogeneity of the mean index. (Note that the behavior of the indices under iterations is tied up with the relation between trivializations of $\xi$ along $x^k$. For arbitrary, unrelated trivializations, the mean index would not be homogeneous when $x$ is not contractible.)

A variant of this construction applies to a hypersurface $M$ in a symplectic manifold $(W^{2n}, \omega)$. In this case one can associate the linearized Poincaré return map and the indices to an oriented closed characteristic. This is done exactly as for contact manifolds but now using the symplectic normal $TM/TM^\omega$ to the characteristic foliation in place of $\xi$.

In either case, the notion of a dynamically convex Reeb orbit or a closed characteristic is well defined.

**Definition 4.9.** The Reeb flow on a $(2n-1)$-dimensional contact manifold is said to be dynamically convex (DC) if every closed Reeb orbit (or equivalently every simple closed Reeb orbit) is dynamically convex, i.e., $\mu_-(x) \geq n + 1$ for all Reeb orbits $x$.

Here, again, if needed one can fix a collection of free homotopy classes of closed Reeb orbits, which is closed under iterations. The definition extends to hypersurfaces in symplectic manifolds in an obvious way. The reader should keep in mind that this definition “makes sense” only for a rather narrow class of contact manifolds such as those strongly fillable by displaceable Liouville domains. For instance, it readily follows from Proposition 3.12 that the standard contact $ST^*S^n$ admits no dynamically convex contact forms in the sense of Definition 4.9. Furthermore, the “right” condition in this case is $\mu_- \geq n - 1$; see Section 6.2. (In fact, the authors are not aware of any examples of dynamically convex Reeb flows on contact manifolds other than the standard contact $S^{2n-1}$.) A much more general notion of dynamical convexity for pre-quantization circle bundles (e.g., for $ST^*S^n$) is introduced and studied in \cite{AM14, AM15}.

**Theorem 4.10** ([HWZ]). The Reeb flow on a strictly convex hypersurface in $\mathbb{R}^{2n}$ is dynamically convex.

**Remark 4.11.** There is a typo in a remark on pp. 222–223 in \cite{HWZ} concerning the higher-dimensional case of the theorem. As in Lemma 4.8, the index lower bound at the end of the remark should be $2k + n - 1$, not $nk + 1$.

For the sake of completeness we give a simple proof of Theorem 4.10.
Proof. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a convex Hamiltonian, homogeneous of degree two and such that $M = \{ H = 1 \}$. Then the restriction of the Hamiltonian flow $\varphi^t_H$ of $H$ to $M$ is the Reeb flow on $M$. In particular, there is a one-to-one correspondence between $\mathcal{P}(\alpha)$ and the periodic orbits of $H$ on $M$. Let $x$ be one of such orbits. Without loss of generality we may assume that $x$ is one-periodic. The linearized flow $d\varphi^t_H|_x$ along $x$ is a path in $\mathrm{Sp}(2n)$ starting at $I$ and generated by the positive definite time-dependent Hamiltonian $d^2H|_{T_{x(t)}\mathbb{R}^{2n}}$. (Here we have identified $T_{x(t)}\mathbb{R}^{2n}$ with $\mathbb{R}^{2n}$ itself.) Hence, by Example 4.1,

$$\mu_-(d\varphi^t_H|_x) \geq n.$$  

Let us now fix a trivialization of the contact structure $\xi$ on $M$ along a capping of $x$. This trivialization can be extended to a trivialization of $T\mathbb{R}^{2n}$ along the capping by adding to it the frame $\{ X_H, JX_H \}$. The path $d\varphi^t_H|_x$ in the new trivialization decomposes into the direct sum $\Phi \oplus I_2$, where $\Phi \in \mathrm{Sp}(2m)$ with $m = n - 1$ is the linearized Poincaré return map of $x$. The standard trivialization of $\mathbb{R}^{2n}$ and the new one are homotopic along the capping and hence along $x$. By the additivity of $\mu_-$ (Lemma 4.3), we have

$$n \leq \mu_-(\Phi \oplus I_2) = \mu_-(\Phi) - 1.$$  

In other words, $\mu_-(\Phi) \geq n + 1 = m + 2$. \[ \square \]

Remark 4.12. Note that in this proof we could have as well used the Robin–Salamon index or the Conley–Zehnder index, after passing to a small perturbation to obtain a lower bound on $\mu_-$. A minor modification of the argument also shows that the sequence $\mu_-(\Phi^k)$ is increasing and that $\mu_-(\Phi^k) \geq 2k + m$. However, as we have already seen in Lemma 4.8, these facts are formal consequences of dynamical convexity and even a stronger result, (4.14), holds.

Remark 4.13 (Other consequences of convexity). In addition to dynamical convexity, convexity of a hypersurface $M \subset \mathbb{R}^{2n}$ imposes other restrictions on the behavior of the indices of closed Reeb orbits $x$, although the exact scope of these restrictions is unclear to us. In light of the proof of Theorem 4.10, as a preliminary step one can examine the geometry of positive paths $\Phi$ in $\mathrm{Sp}(2n)$, i.e., paths generated by positive definite Hamiltonians. This question must have been extensively studied, but we could not pin-point exact references; see however [Ek, Lo02]. Here we would like to mention only some simple facts that go beyond Example 4.1:

- For every point $A \in \mathrm{Sp}(2n)$, there is a positive path, with possibly very high mean index, from $I$ to $A$.
- Assume that $\Phi(1)$ is totally degenerate. Then $\tilde{\mu}(\Phi) \geq 2(b_0 + \nu_0) + b_- + b_+$. The first observation is obvious when $A$ is close to $I$ and the general case follows from the fact that the product of positive paths is again positive. The second observation can be proved using (4.7). (It would be interesting and useful to understand how close to being sharp this inequality is.)

Furthermore, it is not hard to see that the Reeb flow on a convex hypersurface is index-positive in a very strong sense. Namely, the mean index $\tilde{\mu}$ of an orbit grows with the length $l$ of the orbit and more specifically $\tilde{\mu} \geq al - b$ for some $a > 0$ and $b$ independent of the orbit. In fact, once a suitable extra structure is fixed, this is true for all, not necessarily closed, orbits; see, e.g., [Es]. (This lower bound is an immediate consequence of a suitable version of the Sturm comparison
5. INDEX RECURSION

5.1. Index recurrence theorem and its consequences. Now we are in a position to state and prove our main combinatorial results concerning the behavior of the index under iterations. These results have non-trivial overlap with the treatment of the question in [LZ] and [DLW, Lo02], although our argument is self-contained and its logical structure is quite different.

To set the stage for the general case, let us first state a simpler version of the theorem which requires the paths to be strongly non-degenerate. This result is essentially contained, although in a different form, in [DLW, LZ] as a part of the “common jump theorem”.

**Theorem 5.1** (Index recurrence theorem, the non-degenerate case). Consider a finite collection of strongly non-degenerate elements \( \Phi_1, \ldots, \Phi_r \) in \( \text{Sp}(2m) \). Then for any \( \eta > 0 \) and any \( \ell_0 \in \mathbb{N} \), there exists an integer sequence \( d_j \to \infty \) and \( r \) integer sequences \( \tilde{k}_{ij} \), \( i = 1, \ldots, r \), at least one of which goes to infinity, such that for all \( i \) and \( j \), and all \( \ell \in \mathbb{Z} \) in the range \( 1 \leq |\ell| \leq \ell_0 \), we have:

(i) \( |\tilde{\mu}(\Phi_i^{\tilde{k}_{ij}}) - d_j| < \eta \), and

(ii) \( \mu(\Phi_i^{\tilde{k}_{ij} + \ell}) = d_j + \mu(\Phi_i^\ell) \).

Furthermore, when all mean indices \( \Delta_i := \tilde{\mu}(\Phi_i) \) are non-zero we can ensure that \( k_{ij} \to \pm \infty \) as \( j \to \infty \) for all \( i \), and that \( k_{ij} \to \infty \) when, in addition, all \( \Delta_i \) have the same sign. Moreover, for any \( N \in \mathbb{N} \) we can make all \( d_j \) and \( k_{ij} \) divisible by \( N \).

To illuminate this result, let us first consider the case where \( r = 1 \), i.e., the case of one strongly non-degenerate path \( \Phi \). Fix any “interval” \( L = [-\ell_0, \ell_0] \cap \mathbb{Z} \) and denote by \( \tilde{L} \) the punctured interval \( L \setminus \{0\} \). Then the theorem asserts, in particular, that up to a sequence of common shifts \( d_j \), the restricted pattern \( \mu|_L \) repeats itself for infinitely many shifted copies of \( \tilde{L} \). (Hence, the name of the theorem.) In other words, there exists an infinite sequence of shifts \( k_j \) in the argument direction and a sequence of shifts \( d_j \) in the \( \mu \) direction such that

\[
\mu|_{L + k_j} = d_j + \mu|_L.
\]

This is also true for \( r \) functions, where we now have \( r \) sequences \( k_{ij} \) of shifts in the argument direction but still only one shift sequence \( d_j \) in the \( \mu \) direction.

Without non-degeneracy, the theorem still holds for the upper and lower indices \( \mu_\pm \) when the interval \( L \) lies entirely in the positive domain, i.e., for \( L = \{1, \ldots, \ell_0\} \). However, for negative values of \( \ell \) the assertion is no longer literally true. The result holds only up to a correction term of the form \( b_+(\Phi(\tilde{L})) - b_-(\Phi(\tilde{L})) \), where \( b_\pm \) are the signature multiplicities defined in Section 4.1.3; see Definition 4.5.

**Theorem 5.2** (Index recurrence theorem, the degenerate case). Let \( \Phi_1, \ldots, \Phi_r \) be a finite collection of elements in \( \text{Sp}(2m) \). Then for any \( \eta > 0 \) and any \( \ell_0 \in \mathbb{N} \), there exists an integer sequence \( d_j \to \infty \) and \( r \) integer sequences \( k_{ij} \), \( i = 1, \ldots, r \), at least one of which goes to infinity, such that for all \( i \) and \( j \), and all \( \ell \in \mathbb{N} \) in the range \( 1 \leq \ell \leq \ell_0 \), we have
(i) $|\hat{\mu}(\Phi_{lj}^{k_{lj}}) - d_j| < \eta$,
(ii) $\mu_\pm(\Phi_{lj}^{k_{lj}+\ell}) = d_j + \mu_\pm(\Phi_{lj}^{\ell})$,
(iii) $\mu_\pm(\Phi_{lj}^{k_{lj}-\ell}) = d_j + \mu_\pm(\Phi_{lj}^{-\ell}) \pm (b_+(\Phi_{lj}^{\ell}) - b_-(\Phi_{lj}^{\ell}))$.

Furthermore, when all mean indices $\Delta_i := \hat{\mu}(\Phi_{lj})$ are non-zero we can ensure that $k_{lj} \to \pm \infty$ as $j \to \infty$ for all $i$, and that $k_{lj} \to \infty$ when, in addition, all $\Delta_i$ have the same sign. Moreover, for any $N \in \mathbb{N}$ we can make all $d_j$ and $k_{lj}$ divisible by $N$.

Clearly, this theorem reduces to Theorem 5.1 when all paths $\Phi_j$ are strongly non-degenerate, for in this case $b_\pm = 0$. (In fact, the proof of Theorem 5.1 is contained in the proof of Theorem 5.2 as a “subset”.) Note also that due to (4.3) we can replace the term $\mu_\pm(\Phi_{lj}^{-\ell})$ in (iii) by $-\mu_\pm(\Phi_{lj}^{\ell})$. In particular, since the correction term in (iii) obviously does not exceed $\nu(\Phi_{lj}^{\ell})$, we have the following result.

**Corollary 5.3.** In the setting of Theorem 5.2,

$$\mu_-(\Phi_{lj}^{k_{lj}+\ell}) = d_j + \mu_-(\Phi_{lj}^{\ell}),$$

and

$$\mu_+(\Phi_{lj}^{k_{lj}-\ell}) = d_j - \mu_-(\Phi_{lj}^{\ell}) + (b_+(\Phi_{lj}^{\ell}) - b_-(\Phi_{lj}^{\ell})) \leq d_j - \mu_-(\Phi_{lj}^{\ell}) + \nu(\Phi_{lj}^{\ell}).$$  \quad (5.1)

Combining these inequalities with Lemma 4.8, we obtain

**Corollary 5.4.** Assume in the setting of Theorem 5.2 that the paths $\Phi_1, \ldots, \Phi_r$ are dynamically convex. Then $d_i \to \infty$ and $k_{ij} \to \infty$ as $j \to \infty$ for all $i$, and

- $\mu_-(\Phi_{lj}^{k_{lj}+\ell}) \geq d_j + 2\ell + m$ for $1 \leq \ell \leq \ell_0$,
- $\mu_+(\Phi_{lj}^{k_{lj}-\ell}) \leq d_j - m - 2\ell + \nu(\Phi_{lj}^{\ell}) \leq d_j - 2\ell$ for $1 \leq \ell \leq \ell_0$.

In particular, $\mu_-(\Phi_{lj}^{k_{lj}+\ell}) \geq d_j + 2 + m$ and $\mu_+(\Phi_{lj}^{k_{lj}-\ell}) \leq d_j - 2 - m$ for all $\ell \in \mathbb{N}$ (or $\mu_+(\Phi_{lj}^{k_{lj}-\ell}) \leq d_j - 2 - m$ in the strongly non-degenerate case).

This corollary is a variant of the common jump theorem, [DLW, LZ]. In essence, the corollary asserts that there exists a sequence of sufficiently long intervals $L_j \subset \mathbb{N}$, containing $d_j$, such that the intervals $[\mu_-(\Phi_{lj}^{\ell}), \mu_+(\Phi_{lj}^{\ell})]$ can possibly overlap with $L_j$ only for $k = k_{lj}$. In other words, $[\mu_-(\Phi_{lj}^{\ell}), \mu_+(\Phi_{lj}^{\ell})] \cap L_j = \emptyset$ when $k \neq k_{lj}$.

More specifically, we have $L_j = [d_j - 1, d_j + m + 1]$ in general, and $L_j = [d_j - m - 1, d_j + n + 1]$ when all $\Phi_j$ are strongly non-degenerate. Thus the length of $L_j$ is $m + 2$ in the former case and $2(m + 1)$ in the latter.

We emphasize that none of these results give any new information about the index of $\Phi_{lj}^{k_{lj}}$. However, since the difference between $d_j$ and $\hat{\mu}(\Phi_{lj}^{k_{lj}})$ does not exceed $\eta$, we can conclude that $\mu_\pm(\Phi_{lj}^{k_{lj}})$ is in the range $[d_j - m, d_j + m]$ once $\eta < 1/2$.

### 5.2. Proof of the index recurrence theorem

We start the proof of Theorem 5.2 by focusing on the case of a single path $\Phi$, i.e., $r = 1$, and then show how to modify the argument for a finite collection of paths. Below, without loss of generality, we can assume that all paths are parametrized by $[0, 1]$.

#### 5.2.1. The case of $r = 1$

Let $\Phi = \Phi_1 \in \widetilde{Sp}(2m)$. Throughout the argument we suppress $i$ in the notation, i.e., we will write $k_{lj}$ for $k_{ij}$, etc. To establish the theorem in this setting, we will consider several subcases depending on the end-map $\Phi(1)$. Then the general case will be established by additivity. Fix $\eta > 0$ and $\ell_0 \in \mathbb{N}$. Without loss of generality we can assume that $\eta < 1/2$. 

...
**Subcase A:** $\Phi(1)$ *is hyperbolic.* Set $d_k = \tilde{\mu}(\Phi^k)$. (Here, and also in Subcases B and D below, it is more convenient to index $d$ by $k$ rather than $j$.) Clearly, (i) is automatically satisfied. Furthermore, $\Phi^k$ is non-degenerate for all $k \in \mathbb{N}$ and $\tilde{\mu}(\Phi^k) = \mu(\Phi^k)$. Hence, we have

$$
\mu(\Phi^{k+\ell}) = \tilde{\mu}(\Phi^k) + \tilde{\mu}(\Phi^\ell) = \mu(\Phi^k) + \mu(\Phi^\ell).
$$

Thus (ii) and (iii) hold for all $k$, i.e., with $k_j = j$, and all $\ell$. To ensure that $N | d_k$, it suffices to take $k$ divisible by $N$.

**Subcase B:** $\Phi(1)$ *is totally degenerate.* By Lemma 4.2, $\Phi$ as an element of $\widetilde{\text{Sp}}(2m)$ is the product of a loop $\varphi$ and a path $\Psi(t)$ such that all eigenvalues of $\Psi(t)$ for all $t \in [0, 1]$ are equal to one. Multiplication by $\varphi^k$ shifts $\mu_\pm(\Psi^k)$ by $k\tilde{\mu}(\varphi)$, i.e.,

$$
\mu_\pm(\Phi^k) = k\tilde{\mu}(\varphi) + \mu_\pm(\Psi^k).
$$

Furthermore, by Lemma 4.2 and (4.3), for any $k > 0$ we have

$$
\mu_\pm(\Psi^k) = \mu_\pm(\Psi) \quad \text{and} \quad \mu_\pm(\Psi^{-1}) = \mp\mu_\pm(\Psi).
$$

Set again $d_k = \tilde{\mu}(\Phi^k) = \tilde{\mu}(\varphi^k)$. Then, for any $\ell \in \mathbb{N}$,

$$
\mu_\pm(\Phi^{k+\ell}) = d_k + \tilde{\mu}(\varphi^\ell) + \mu_\pm(\Psi^{k+\ell})
= d_k + \tilde{\mu}(\varphi^\ell) + \mu_\pm(\Psi^\ell)
= d_k + \mu_\pm(\Psi^\ell).
$$

This proves (ii) for all $k \in \mathbb{N}$ and all $\ell \in \mathbb{N}$.

In the notation of Definition 4.5 and again by Lemma 4.2,

$$
\mu_+ (\Psi^\ell) = b_0 + b_+ + \nu_0 = -\mu_-(\Psi) + b_+ - b_-
$$

where for brevity we set $b_+ := b_+ (\Psi) = b_+ (\Psi^\ell)$.

Likewise, when $0 < \ell < k$, again by using (4.3) and Lemma 4.2, we see that

$$
\mu_+ (\Phi^{k-\ell}) = d_k - \tilde{\mu}(\varphi^\ell) + \mu_+ (\Psi^{k-\ell})
= d_k - \tilde{\mu}(\varphi^\ell) + \mu_+ (\Psi^\ell)
= d_k - \tilde{\mu}(\varphi^\ell) - \mu_- (\Psi^\ell) + b_+ - b_-
= d_k - \mu_- (\Psi^\ell) + b_+ - b_-
= d_k + \mu_+ (\Phi^{-\ell}) + b_+ - b_-.
$$

This proves (iii) for $\mu_+$. The case of $\mu_-$ is handled similarly. In other words, (iii) holds for all $k \in \mathbb{N}$ and all $\ell$ such that $0 < |\ell| < k$.

**Subcase C:** $\Phi(1)$ *is elliptic and strongly non-degenerate.* This is the first case in the argument which does not hold for all $k \in \mathbb{N}$ and all with $0 < |\ell| < k$, and we need to take a subsequence $k_j$ and limit the range of $\ell$. Let $\exp (\pm 2\pi \sqrt{-1}\lambda_q)$, $q = 1, \ldots, m$, be the eigenvalues of $\Phi(1) \in \text{Sp}(2m)$, where $|\lambda_q| < 1$, and at the moment the choice of the sign of $\lambda_q$ can be arbitrary. Since $\Phi(1)$ is strongly non-degenerate, all $\lambda_q$ are irrational. Set

$$
\epsilon_0 = \min_{0 < \ell \leq \ell_0} \min_{q} \| \lambda_q \ell \| > 0, \quad (5.2)
$$

where $\| \cdot \|$ stands for the distance to the nearest integer. Let $\epsilon > 0$ be so small that

$$
\epsilon \leq \epsilon_0 \quad \text{and} \quad m\epsilon < \eta.
$$
It is easy to see that there exists a sequence \( k_j \to \infty \) such that for all \( q \) we have
\[
\| \lambda_q k_j \| < \epsilon \leq \epsilon_0. \tag{5.3}
\]
Indeed, consider the semi-orbit \( \Gamma = \{ k\lambda \mid k \in \mathbb{N} \} \subset \mathbb{T}^m \) where \( \lambda \in \mathbb{T}^m \) is the collection of eigenvalues of \( \Phi(1) \). As is well known, the closure of \( \Gamma \) is a subgroup of \( \mathbb{T}^m \). Hence, \( \Gamma \) contains points arbitrarily close to the unit in \( \mathbb{T}^m \) and, in particular, there exist infinitely many points \( k_j \lambda \) in the \( 2\pi \epsilon \)-neighborhood of the unit. Then, by passing to a subsequence we can ensure that \( k_j \to \pm \infty \) and that, in fact, \( k_j \to \infty \) by changing if necessary the sign of all \( k_j \).

Let \( d_j \) be the nearest integer to \( \tilde{\mu}(\Phi^{k_j}) \). Then
\[
| d_j - \tilde{\mu}(\Phi^{k_j}) | \leq m\epsilon < \eta,
\]
and hence (i) is satisfied. This also shows that \( d_j \) is unambiguously defined. Clearly, for any \( N \in \mathbb{N} \) we can also make all \( k_j \) and \( d_j \) divisible by \( N \). (To see this, it suffices to replace the semi-orbit \( \Gamma \) by \( \{ kN\lambda \mid k \in \mathbb{N} \} \).

To prove (ii) and (iii), observe first that a small perturbation of \( \Phi \) does not affect individual terms in these inequalities for fixed \( k_j \) and \( \ell \). Thus, by altering \( \Phi \) slightly, we can ensure that all eigenvalues \( \lambda_q \) are distinct. Then we can write \( \Phi \), up to homotopy, as the product of a loop \( \varphi \) and the direct sum of paths \( \Psi_q = \exp(2\pi\sqrt{-1}\lambda_q t) \in \text{Sp}(2) \) for a suitable choice of signs of \( \lambda_q \); see, e.g., [SZ, Sect. 3].

The loop \( \varphi \) contributes \( k\tilde{\mu}(\varphi) \) to \( \mu(\Phi^k) \) and hence we only need to prove (ii) and (iii) when \( \varphi = I \).

Then, for any \( k \),
\[
\mu(\Phi^k) = \sum_q \mu(\Psi_q^k).
\]

Next, observe that by (5.2) and (5.3) we have
\[
d_j = \sum_q \lfloor \tilde{\mu}(\Psi_q^k) \rfloor,
\]
where \( \lfloor \cdot \rfloor \) denotes the nearest integer. Thus it suffices to prove (ii) and (iii) for each path \( \Psi_q \) individually when we set \( d_j = \lfloor \tilde{\mu}(\Psi_q^k) \rfloor \). However, with (5.2) and (5.3) in mind, (ii) and (iii) for \( \Psi_q \) easily follows from, e.g., (4.1).

**Subcase D:** \( \Phi(1) \) is non-degenerate, but \( \Phi(1)^N = I \) for some \( N \in \mathbb{N} \). This subcase is a combination of Subcases B and C.

Let us first assume that all eigenvalues of \( \Phi(1) \) are equal to each other, up to complex conjugation, and thus equal to \( \exp(\pm 2\pi\sqrt{-1}\lambda) \) where \( \lambda \) is a root of unity of degree \( N \). We claim that (i)–(iii) hold for all \( k \) divisible by \( N \) and all \( \ell \) with \( d_k = \tilde{\mu}(\Phi^k) \).

There are two cases to consider depending on whether \( \ell \) is divisible by \( N \) or not.

Assume first that \( N \nmid \ell \). Then \( \Phi^{k+\ell} \) is non-degenerate. All eigenvalues of \( \Phi^{k+\ell} \) are equal to one since \( N \nmid k \), and we can connect \( \Phi^{k}(1) \) to \( I \) by a path \( \Lambda(s) \), starting at \( \Phi^{k}(1) \) at \( s = 0 \) and ending at \( I \) at \( s = 1 \), such that all eigenvalues of \( \Lambda(s) \) are equal to one for all \( s \); cf. the proof of Lemma 4.2. Consider now the following deformation \( Z_s \) of the path \( \Phi^{k+\ell} \). Namely, \( Z(s) \) is the concatenation of two paths. The first one, ending at \( \Lambda(s) \), is itself the concatenation of \( \Phi^k \) and the path \( \Lambda(\tau) \) for \( \tau \in [0, s] \). The second one, starting at \( \Lambda(s) \), is the path \( \Phi^{\ell}\Lambda(s) \). The endpoint \( \Phi^{\ell}(1)\Lambda(s) \) of the path \( Z_s \) is non-degenerate for all \( s \) and hence \( \mu(Z_s) \) remains
constant. By construction, \( Z_0 = \Phi^{k+1} \) and \( Z_1 \) is the concatenation of a loop with the same mean index \( d_k \) as \( \Phi^k \) and the path \( \Phi^\ell \). We conclude that
\[
\mu(\Phi^{k+\ell}) = d_k + \mu(\Phi^\ell).
\]
This implies (ii). Clearly, this argument works not only for positive \( \ell \) but for any \( \ell \in \mathbb{Z} \) not divisible by \( N \). Recalling that \( \mu(\Phi^{-1}) = -\mu(\Phi) \), we obtain (iii).

The remaining case is when \( N \mid \ell \). The end-point of the path \( \Phi^N \) is totally degenerate. Hence, we can apply the argument from Subcase B to \( \Phi^N \) in place of \( \Phi \) with \( k \) and \( \ell \) replaced by \( k' = k/N \) and \( \ell' = \ell/N \). Then we have
\[
\mu(\Phi^{k+\ell}) = d_{k'} + \mu(\Phi^{\ell'})
\]
and
\[
\mu(\Phi^{k+\ell}) = d_k + \mu(\Phi^\ell) + r \in \mathbb{Z}
\]
which proves (ii) and (iii) in this case.

In general, we can decompose \( \Phi \), up to homotopy, into a direct sum of paths \( \Phi_q \), where \( \Phi_q(1) \) has only one eigenvalue, up to complex conjugation, and this eigenvalue is a root of unity of degree \( N_q \). Applying the above argument to each \( \Phi_q \) individually, when \( k \) is divisible by \( N = \text{lcm}\{N_q\} \) or any other \( N \) with \( \Phi^N = I \), we see that (ii) and (iii) hold for \( \Phi \) for all \( k \) divisible by \( N \) and all \( \ell \).

*Putting Subcases A–D together.* Let us decompose \( \Phi \) into the direct sum of four paths \( \Phi_A, \ldots, \Phi_D \) with each path as in one of Subcases A–D. Let \( N \) be such that \( \Phi^N = I \). We can chose \( k_j \) divisible by \( N \) so that (i), (ii) and (iii) are satisfied for \( \Phi^k \) with \( d_j = \lfloor \mu(\Phi^k) \rfloor \). Furthermore, in all cases but Subcase C, we have \( \mu(\Phi_{A,B,D}) = d_k \). Thus it is clear that (i) holds for \( \Phi \) for the sequence \( d_j = d_{k_j} \), which is the sum of such sequences for all four subcases. Likewise, since (ii) and (iii) hold in Subcases A, B, and C for all \( k \) divisible by \( N \), we conclude that (ii) and (iii) are satisfied for \( \Phi \) for the sequence \( k_j \). In addition, we can make \( k_j \) divisible by any other integer.

### 5.2.2. The general case: \( r \geq 1 \)

Let \( \Phi_1, \ldots, \Phi_r \) be a finite collection of elements in \( \text{Sp}(2m) \). If we apply the argument from Section 5.2.1 to each \( \Phi_i \) individually, we obtain \( r \) integer sequences \( k_{ij} \) and \( r \) integer sequences \( \Delta_i \) such that (i)–(iii) hold. Thus our goal is to show that \( k_{ij} \) can be chosen so that (i)–(iii) hold for the same sequence \( d_j \).

Denote by \( \exp \left( \pm 2\pi \sqrt{-1} \lambda_{ij} \right) \) the elliptic eigenvalues of \( \Phi_i \) with irrational \( \lambda_{ij} \) and set \( \Delta_i = \mu(\Phi_i) \). (The choice of the sign of \( \lambda_{ij} \) is immaterial at the moment.) Given \( \epsilon > 0 \), consider the system of inequalities
\[
||k_{ij}\lambda_{ij}|| < \epsilon \quad \text{for all } i \text{ and } q,
\]
\[
|k_1 \Delta_i - k_i \Delta_i| < \frac{1}{8} \quad \text{for } i = 2,\ldots,r,
\]
where we treat the integer vector \( \vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r \) as a variable. Introducing additional integer variables \( c_{iq} \), we can rewrite the first group of inequalities in the form
\[
|k_{ij}\lambda_{ij} - c_{ij}| < \epsilon.
\]
With this in mind, system (5.4) has one fewer equation than the number of variables. By Minkowski’s theorem (see, e.g., [Ca]), there exists an infinite sequence of distinct solutions \( \vec{k}_j = (k_{1j}, \ldots, k_{rj}) \) of (5.4).
Now, by passing to a subsequence and changing if necessary the signs of $k_{ij}$, we can ensure that at least one of the sequences $k_{ij}$ goes to $\infty$ as $j \to \infty$. Then the second group of inequalities implies that $k_{ij} \to \pm \infty$ for all $i$ when all mean indices $\Delta_i \neq 0$ and also that $k_{ij} \to \infty$ when all $\Delta_i$ have the same sign.

Moreover, we can make all $k_{ij}$ divisible by any fixed integer $N$. In particular, let $N$ be the least common multiple of the degrees of roots of unity among the eigenvalues of all $\Phi_i(1)$. We take the sequences $k_{ij}$ divisible by $N$ and by any other integer as required in the statement of the theorem.

Finally, fix $\ell_0$ and $\eta > 0$ which we assume to be sufficiently small (e.g., $\eta < 1/4$). Similarly to Subcase C, set

$$\epsilon_0 = \min_{0 < \varepsilon \leq \ell_0} \min_{i,q} \|\lambda_{iq}\ell\| > 0,$$

and let $\epsilon > 0$ be so small that again

$$\epsilon \leq \epsilon_0 \quad \text{and} \quad m\epsilon < \eta.$$

By the second series of inequalities in (5.4), we have

$$|k_i \Delta_i - k_{i'} \Delta_{i'}| < \frac{1}{4} \quad \text{for all} \quad i \text{ and } i',$$

and $\|k_{ij} \Delta_i\| < \eta$ by the first group of inequalities. Thus $k_{ij} \Delta_i$ is $\eta$-close, for all $i$, to the same integer

$$d_j = [k_{ij} \Delta_i].$$

In other words, (i) is satisfied for this choice of $d_j$. Note that for every $i$, decomposing $\Phi_i$ according to the Subcases A–D, we have in the obvious notation

$$d_j = [\hat{\mu}(\Phi_{i,C}^{k_{ij}}) + \hat{\mu}(\Phi_{i,A}^{k_{ij}}) + \hat{\mu}(\Phi_{i,B}^{k_{ij}})].$$

Furthermore, for every $i$, condition (5.3) is met for $\lambda_{iq}$ and, since all $k_{ij}$ are divisible by $N$, it readily follows as in Section 5.2.1 that (ii) and (iii) hold for all $i$. This concludes the proof of Theorem 5.2. \hfill $\Box$

### 6. Multiplicity results and other applications

In this section we combine the results from Lusternik–Schnirelmann theory for the shift operator and the index theory to establish our multiplicity results for simple closed Reeb orbits. These are Theorem 1.5, which is a direct consequence of Theorems 6.1 and 6.4, and also Theorem 1.6 proved in Section 6.2. As mentioned in the introduction, we focus on the standard contact $S^{2n-1}$ or, more generally, the boundary of a displaceable Liouville domain and $ST^*S^n$.

#### 6.1. Hypersurfaces in $\mathbb{R}^{2n}$ and displaceable Liouville domains

We start with the simplest and arguably the most interesting situation where $(M^{2n-1},\alpha)$ is a closed, restricted contact type, dynamically convex hypersurface in $\mathbb{R}^{2n}$. For instance, $M$ can be the boundary of a star-shaped domain, provided that the Reeb flow is dynamically convex. However, even when $M$ is convex, some of our results are new.
6.1.1. Multiplicity results for hypersurfaces in $\mathbb{R}^{2n}$. Let us call a simple orbit re-occurring if its iterations occur infinitely many times in the image of a carrier injection $\psi$ from Corollary 3.9. (This notion depends on the choice of $\psi$.) One can show that a generic Reeb flow has no reoccurring closed orbits, but the flows with only finitely many simple orbits necessarily do. Our main multiplicity result is the following theorem.

**Theorem 6.1.** Let $(M^{2n-1},\alpha)$ be a closed contact type, dynamically convex hypersurface in $\mathbb{R}^{2n}$ bounding a simply connected Liouville domain. Then $M$ carries at least $r$ simple closed characteristics $x_1,\ldots,x_r$, where $r = \lfloor n/2 \rfloor + 1$ in general and $r = n$ when $\alpha$ is non-degenerate. Moreover, assume that $\mathcal{P}(\alpha)$ is finite. Then the orbits $x_i$ can be chosen to be reoccurring and, if in addition $\alpha$ is non-degenerate, so that all $x_i$ are even and $\mu(x_i) \equiv n + 1 \pmod{2}$.

**Remark 6.2.** Similarly to the case of convex hypersurfaces considered in [LZ], at least one of the orbits $x_i$ is elliptic (two, in the non-degenerate case) when $M$ carries only finitely many simple periodic orbits. This can be easily seen from the proof of the theorem. (See also [AM15] for some relevant results.)

**Proof.** Without loss of generality we may assume that the Reeb flow of $\alpha$ has only finitely many simple closed orbits, which, as in the theorem, we denote by $x_1,\ldots,x_r$. The set of closed Reeb orbits $\mathcal{P}(\alpha)$ comprises all iterations $x_i^k$, $k \in \mathbb{N}$, of the orbits $x_i$. Let us first show that $r = \lfloor n/2 \rfloor + 1$ in general and $r = n$ when all closed orbits are non-degenerate.

Consider the map

$$
\psi: \mathcal{I} = \{n + 1, n + 3, n + 5, \ldots\} \to \mathcal{P}(\alpha), \quad d \mapsto y_d
$$

from Corollary 3.9, where we relabeled the domain of $\psi$ by the index. (In other words, this map is obtained by composing the map in the corollary with the bijection $d \mapsto (d + 1 - n)/2$ from $\mathcal{I}$ to $\mathbb{N}$.) Thus the orbits which were denoted in the corollary by $y_1, y_2, \ldots$ are now $y_{n+1}, y_{n+3}, \ldots$. We have

$$
\mu_-(y_d) \leq d \leq \mu_+(y_d).
$$

Let $\Phi_i \in \widetilde{Sp}(2m)$, $m = n - 1$, be the linearized Poincaré return map along $x_i$ (see Section 4.2); without loss of generality we can assume that the paths $\Phi_i$ are parametrized by $[0,1]$. Fixing a small parameter $\eta > 0$ and a sufficiently large $\ell_0 \in \mathbb{N}$, let us apply Corollary 5.4 to the paths $\Phi_i$, where we require the iterations $k_{ij}$ to be even and divisible by the degrees of the roots of unity among the eigenvalues of $\Phi_i(1)$. Then, for all $\ell \in \mathbb{N}$, we have

- $\mu_-(x_i^{k_{ij}+\ell}) \geq d_j + n + 1$,
- $\mu_+(x_i^{k_{ij}-\ell}) \leq d_j - 2$ and $\mu_+(x_i^{k_{ij}+\ell}) \leq d_j - n - 1$ when all orbits $x_i$ are strongly non-degenerate.

Denote by $L$ the index interval $[d_j - n, d_j + n] \cap \mathcal{I}$ in the non-degenerate case and set $L = [d_j - 1, d_j + n] \cap \mathcal{I}$ in general. Then for any $d \in L$ the orbit $y_d$ must have the form $x_i^{k_{ij}}$, and therefore at most one iteration of $x_i$ can occur as $y_d$ with $d \in L$. It follows that $r$, the number of simple orbits $x_i$, is greater than or equal to the number of points in $L$, i.e., $r \geq \#(L)$. In the non-degenerate case $\#(L) = n$ and in general $\#(L) = \lfloor n/2 \rfloor + 1$. Furthermore, in the non-degenerate case the orbits $x_i$ must be even since the iterations $k_{ij}$ are even; see Example 2.19. We then necessarily have $\mu(x_i) \equiv n + 1 \pmod{2}$.
Our next goal is to improve in the degenerate case this lower bound by one when \( n \) is odd which we will assume from now on. Then \( d_{\text{max}} = d_j - 2 \) is the largest point in \( \mathcal{I} \) before the interval \( L \). It readily follows from Corollary 5.4 that \( y_{d_{\text{max}}} \) can only have the form \( x_i^{k_{ij}} \) or \( x_i^{k_{ij} - 1} \). In the former case, \( x_i^{k_{ij}} \) does not contribute to the interval \( \mathcal{I} \) and hence, when \( r \geq \#(L) + 1 \).

We claim that in the latter case, i.e., when \( y_{d_{\text{max}}} = x_i^{k_{ij} - 1} \) and hence \( \ell = 1 \), we have infinitely many simple periodic orbits. To prove this, first note that

\[
\mu_+(x_i^{k_{ij} - 1}) \leq d_j - n - 1 + \nu(x_i) \leq d_{\text{max}} = d_j - 2
\]

since \( \nu(x_i) \leq n - 1 \), and

\[
\nu(x_i) = n - 1 \quad \text{and} \quad \mu_+(x_i^{k_{ij} - 1}) = d_j - 2.
\]

As a consequence, \( x_i \) is totally degenerate and \( d_j = \hat{\mu}(x_i^{k_{ij}}) \). It follows that \( x_i^{k_{ij} - 1} \) is the so-called symplectically degenerate maximum (SDM); see [GH²M]. Indeed, by (4.13) and, since \( M \) is dynamically convex, we must have \( \mu_-(x_i) = \hat{\mu}(x_i) = n + 1 \) and \( b_+(x_i) = 0 \) and \( b_-(x_i) = b_0(x_i) = \nu_0(x_i) = 0 \). Furthermore, since by construction \( k_{ij} \) is divisible by the degrees of the roots of unity among the eigenvalues of \( x_i \), the iteration \( k_{ij} - 1 \) is relatively prime with these degrees and hence \( k_{ij} - 1 \) is an admissible iteration. Therefore, \( x_i \) is also an SDM; see [GH²M, Prop. 3]. (Thus the local Floer homology of \( x_i \) is concentrated in degree \( \mu_+(x_i) = 2n \) which is the upper end point of its support.) Finally, as is shown in [GH²M], the presence of a simple SDM orbit implies that the Reeb flow of \( \alpha \) has infinitely many simple periodic orbits.

To summarize, we have \( r \geq \lceil n/2 \rceil + 1 \) when \( n \) is even and \( r \geq \lceil n/2 \rceil + 2 \) when \( n \) is odd. This is equivalent to that \( r \geq \lceil n/2 \rceil + 1 =: r \). This completes the proof of the first part of the theorem.

Next, observe that Corollary 5.4 provides an infinite sequence of the intervals \( L \), and hence, when \( M \) carries only finitely many simple periodic orbits, there exist \( r \) reoccurring simple orbits \( x_1, \ldots, x_r \), with \( r = \lceil n/2 \rceil + 1 \) in general and \( r = n \) when \( \alpha \) is non-degenerate.

**Remark 6.3.** When \( n \) is even, the point \( d_j - 2 \) is not in \( \mathcal{I} \). The largest point in \( \mathcal{I} \) before the interval \( L \) is \( d_{\text{max}} = d_j - 3 \). Consider the orbit \( y_{d_{\text{max}}} \). There are now two possible cases. One is that \( \nu(y_{d_{\text{max}}}) = n - 1 \) and \( \hat{\mu}(y_{d_{\text{max}}}) = n + 1 \). Then, exactly as in the proof, \( y_{d_{\text{max}}} \) is the \( k_{ij} \)th iteration of a simple SDM orbit \( x_i \) and \( (M, \alpha) \) carries infinitely many periodic orbits. However, there is a second possibility. This is that \( \nu(y_{d_{\text{max}}}) = n - 2 = b_+(y_{d_{\text{max}}}) \) with \( b_0 = b_0 = 0 \) and \( \hat{\mu}(y_{d_{\text{max}}}) \) is either \( n \) or \( n + 1 \). (Thus the linearized Poincaré return map along \( y_{d_{\text{max}}} \) is the direct sum of a totally degenerate map in dimension \( 2(n - 2) \) and an elliptic or negative hyperbolic map in dimension \( 2 \).) It is not clear to us how to rule out such an orbit \( y_{d_{\text{max}}} \).

6.1.2. **Resonance relations.** In a variety of settings the actions and/or indices of closed Reeb orbits satisfy certain resonance relations (in fact, more than one type), which have applications in Reeb dynamics; see, e.g., [EH87, GGo, GG09b, GK, Güi15, LL, LLW, Ra94, Vİ89]. This is also the case for the orbits \( x_1, \ldots, x_r \) from Theorem 6.1. Namely, set

\[
\hat{c}(x) = \frac{A_\mu(x)}{\mu(x)}.
\]
where $x \in \mathcal{P}(\alpha)$. (If $\hat{\mu}(x) = 0$, we set $\hat{c}(x) = \infty$.) This ratio, originally considered in [Gü15], is a contact analog of the augmented action from [CGG, GG09a]. It is clear that $\hat{c}(x) = \hat{c}(x^k)$ for all $k$.

**Theorem 6.4 (Resonance Relations).** Let $(M^{2n-1}, \alpha)$ be a closed contact type hypersurface in $\mathbb{R}^{2n}$ bounding a simply connected Liouville domain. Assume that the set $\{\hat{c}(x)\}$, where $x$ ranges over all reoccurring orbits, is discrete. Then, for any two reoccurring closed Reeb orbits $x$ and $y$, we have $\hat{c}(x) = \hat{c}(y)$, i.e.,

$$\frac{A_d(x)}{\hat{\mu}(x)} = \frac{A_d(y)}{\hat{\mu}(y)}.$$  

**Remark 6.5.** In general, the carrier map $\psi$ from Corollary 3.9 is not unique. Theorem 6.4 holds for any choice of $\psi$. One can show that the requirement on the reoccurring augmented action spectrum $\{\hat{c}(x)\}$ is satisfied if, for instance, the ordinary action spectrum $\mathcal{S}(\alpha)$ is discrete and $c_1(\xi) = 0$. It is also met for quasi-finite hypersurfaces introduced below, e.g., when $M$ carries only finitely many simple periodic orbits.

Together, Theorems 6.1 and 6.4 imply Theorem 1.5. Without additional assumption on $(M, \alpha)$, Theorem 6.4 gives little information. For, hypothetically, it is possible that the image of a carrier injection $\psi$ consists entirely of the iterations of a single simple orbit. Furthermore, it is also possible that there are no reoccurring orbits. (In fact, this should be true $C^\infty$-generically.) In this vein, one consequence of Theorem 6.4 is the $C^\infty$-generic existence of infinitely many simple periodic orbits on a restricted contact type hypersurface in $\mathbb{R}^{2n}$; see, e.g., [GG09b] for an applicable argument. The key example meeting the requirements of the theorem is the standard contact sphere $S^{2n-1}$. In this case, however, the $C^\infty$-generic existence of infinitely many periodic orbits is well-known; see [Vi89].

Another application of Theorem 6.4 concerns the behavior of the “normalized” spectral invariants $c_d(\alpha)/d$ in the notation from Section 3.2.1. Let us call a restricted contact type hypersurface $M \subset \mathbb{R}^{2n}$ *quasi-finite* if there exists a finite collection of simple periodic orbits $x_i$ such that the iterations of $x_i$ occur in the image of $\psi$ infinitely many times and cover all but a finite part of the image. In other words, all but a finite part of $\psi(I)$ lies in the union of the sets $\{x_i^k | k \in \mathbb{N}\}$ and each of the sets has infinite intersection with the image. This is an extremely non-generic condition. However, it is satisfied, for instance, when $M$ carries only finitely many simple closed characteristics. For a quasi-finite $M$, let us set $\hat{c}(\alpha) = A_\alpha(x_i)/\hat{\mu}(x_i) = \hat{c}(x_i)$. By Theorem 6.4, this ratio is independent of $x_i$.

**Corollary 6.6.** Assume that $(M, \alpha)$ is a quasi-finite contact type hypersurface in $\mathbb{R}^{2n}$ bounding a simply connected Liouville domain. Then

$$\lim_{d \to \infty} \frac{c_d(\alpha)}{d} = \hat{c}(\alpha).$$

**Proof.** Consider an infinite sequence $x_i^{k_{ij}} = \psi(d_j)$ in the image of $\psi$. Then

$$\frac{c_d(\alpha)}{d_j} = A_\alpha(x_i) \frac{k_{ij}}{d_j}.$$  

By (4.6), $k_{ij}/d_j \to \hat{\mu}(x_i)^{-1}$ and hence $c_d(\alpha)/d_j \to \hat{c}(\alpha)$. The sequence $c_d(\alpha)/d$ is a finite “union”, in the obvious sense, of the sequences $c_d(\alpha)/d_j$ converging to the
same limit $\hat{c}(\alpha)$, and the result follows. (Note that this argument breaks down if we omit the requirement that the collection $\{x_i\}$ is finite.) \hfill $\square$

**Example 6.7 (Ellipsoids).** Let $M$ be the ellipsoid $\sum_i |z_i|^2/r_i^2 = 1$ in $\mathbb{C}^n = \mathbb{R}^{2n}$ as in Example 3.10. Let us assume that the closed orbits are isolated and denote by $x_i$ the simple periodic orbit lying on the $z_i$-axis. As is shown in [Ba, Example 1.2],

$$\hat{\mu}(x_i) = 2r_i^2 \sum_j r_j^{-2},$$

and obviously $A_\alpha(x_i) = \pi r_i^2$. Therefore,

$$\hat{c}(\alpha) = \frac{\pi}{2} \sum_j r_j^{-2}.$$  

Curiously, it is not immediately obvious how to directly prove that the sequence $c_d(\alpha)/d$, explicitly written down in Example 3.10, converges to $\hat{c}(\alpha)$.

**Remark 6.8.** It is clear that the sequence $c_d(\alpha)/d$ lies in the interval $[\pi r^2, \pi R^2]$, where $r$ is the radius of a ball enclosed by $M$ and $R$ is the radius of a sphere enclosing $M$. It would be interesting to understand if or when this sequence converges and what the limit is in general. When $M$ is convex, this question appears to be related to some of the results from [EH87]. Also note that, by Corollary 6.6, $\hat{c}(W) := \hat{c}(\alpha)$ is a monotone function of $W$ with respect to inclusions and hence a capacity, as long as $\partial W$ is quasi-finite.

**Proof of Theorem 6.4.** It is sufficient to prove the theorem for simple periodic orbits. We will focus exclusively on simple orbits $z$ with $\hat{\mu}(z) > 0$, and hence with $\hat{c}(z) > 0$. Let us say that $d \in \mathcal{I}$ is represented by an orbit $z$ if $\psi(d)$ is an iteration of $z$. (Here, as in the proof of Theorem 6.1, we prefer to index the domain of $\psi$ by $\mathcal{I} = \{n + 1, n + 3, \ldots\}$.) Whenever $\psi(d) = z^k$, we have

$$A_\alpha(z^k) \approx \hat{c}(z)d \quad (6.1)$$

up to an error not exceeding the constant $C(z) = (n - 1)\hat{c}(z)$ which is independent of $d \in \mathcal{I}$. Indeed, since $|d - \hat{\mu}(z^k)| = |d - k\hat{\mu}(z)| \leq n - 1$ by Corollary 3.9, we have

$$\hat{c}(z)d = A_\alpha(z) \frac{d}{\hat{\mu}(z)} = kA_\alpha(z) + e\hat{c}(z),$$

where $|e| \leq n - 1$, and (6.1) follows.

To establish the theorem, it suffices to show that for any two simple reoccurring orbits $x$ and $y$ we necessarily have $\hat{c}(x) \geq \hat{c}(y)$ and thus, by symmetry, $\hat{c}(x) = \hat{c}(y)$. We prove this by contradiction.

First, let $x$ and $y$ be two simple, not necessarily reoccurring, orbits with $\hat{c}(x) < \hat{c}(y)$. Then there exists a constant $d(x, y)$ such that $x$ cannot represent $d + 2$ when $d \geq d(x, y)$ is represented by $y$. Indeed, let $d(x, y)$ be the first integer in $\mathcal{I}$ for which

$$\hat{c}(x)((d(x, y) + 2) + (n - 1)) < \hat{c}(y)(d(x, y) - (n - 1)).$$

Then $d + 2$ cannot be represented by $x$. For, if it were, the action $A_\alpha(\psi(d + 2))$ on the resulting iteration $x^{d+2}$ would be, by (6.1), strictly smaller than $A_\alpha(\psi(d))$. This is impossible by Corollary 3.9. Note that $d(x, y)$ is completely determined by $\hat{c}(x)/\hat{c}(y)$, and $d(x, y)$ is a decreasing function of this ratio.

Consider all simple orbits $x'$ with

$$\hat{c}(x) \leq \hat{c}(x') < \hat{c}(y).$$
By the conditions of the theorem, the set \( \{ \hat{c}(x') \} \) is finite. Assuming now that \( y \) is reoccurring, we can find \( d \geq \max d(x', y) \) represented by \( y \). Then \( d + 2 \) cannot be represented by any of the orbits \( x' \) including \( x \). We denote by \( y_1 \) a simple orbit representing \( d + 2 \).

By construction, \( \hat{c}(y) \leq \hat{c}(y_1) \); for otherwise \( y_1 \) would be one of the orbits \( x' \). Hence \( d(x', y_1) \leq d(x', y) \leq d + 2 \). Therefore, any of the orbits \( x \) or \( x' \) cannot represent \( d + 4 \), which is then represented by some simple orbit \( y_2 \) with \( \hat{c}(y) \leq \hat{c}(y_2) \). (But not necessarily \( \hat{c}(y_1) \leq \hat{c}(y_2) \).) Thus \( d(x', y_2) \leq d(x', y) \leq d + 4 \), and \( d + 6 \) is represented by some orbit \( y_3 \) different from \( x \) and \( x' \), and so on. Arguing inductively, we conclude that when \( y \) is reoccurring \( x \) cannot represent any \( d \geq d(x, y) \) and hence cannot be reoccurring. 

\[ \square \]

6.1.3. Generalizations, refinements and failures. The proof of Theorem 6.1 readily lends itself to several generalizations and refinements which we will now discuss.

The first of these results, generalizing [GuKa, Thm. 1.4], concerns the situation where the dynamical convexity lower bound \( \mu_-(x) \geq n + 1 \) is replaced by \( \mu_-(x) \geq q + 1 \) for some \( q \geq 0 \). Here, again, we have a lower bound \( r \), depending on \( q \), on the number of simple periodic orbits. Some examples where the lower bound \( \mu_- \geq n - 1 \) arises naturally are considered in [AM15].

**Theorem 6.9.** Let \((M^{2n-1}, \alpha)\) be a closed contact type hypersurface in \( \mathbb{R}^{2n} \) bounding a simply connected Liouville domain. Assume that \( \mu_-(y) \geq q \) with \( 0 < q \leq n \) for all, not necessarily simple, closed characteristics \( y \) on \( M \). Then \( M \) carries at least \( r \) simple closed characteristics, where

\[
  r = \begin{cases} 
    q + 1 - \lfloor n/2 \rfloor & \text{when } n \text{ and } q \text{ have the same parity,} \\
    q + 1 - \lceil n/2 \rceil & \text{otherwise.}
  \end{cases}
\]

(If the right hand side is negative or zero the result is void.) When \( \alpha \) is non-degenerate, we can take

\[
  r = \begin{cases} 
    q + 1 & \text{when } n \text{ and } q \text{ have the same parity,} \\
    q & \text{when } n \text{ and } q \text{ have opposite parity.}
  \end{cases}
\]

**Remark 6.10.** The main new point of the theorem is the lower bound on \( r \) in the degenerate case and then it is sufficient to only assume that \( \mu_-(x) \geq q \) for the orbits with \( \hat{\mu}(x) > 0 \). In the non-degenerate case, a stronger result has recently been established under different but conceptually less restrictive assumptions; see [DL²W]. When \( q = n - 2 \) and \( M \) is non-degenerate we obtain [GuKa, Thm. 1.4]. Similarly to Theorem 6.1, all \( r \) orbits in the general case and \( r - 1 \) orbits in the non-degenerate case can be chosen reoccurring when \( M \) carries only a finite number of simple periodic orbits. Furthermore, in the non-degenerate case, \( r - 1 \) orbits can be chosen even and with Conley–Zehnder index of the same parity as \( n + 1 \). The remaining orbit is either odd or has Conley–Zehnder index of the same parity as \( n \).

**Proof.** The general case of the theorem is derived from Theorem 5.2 exactly in the same way as the general case of Theorem 6.1 and we omit the argument; see also the proof of Theorem 6.15. Note, however, that in the present setting we cannot conclude that the orbit \( \hat{y}_{\text{max}} \) is an SDM and thus strengthen the result.

In the non-degenerate case, reasoning as in the proof of Theorem 5.2, we can take \( L \) to be the open interval \( (d_j - q, d_j + q) \). Then \( \#(L \cap I) = r - 1 \). Thus,
for $q < n$, we have found $r - 1$ simple even orbits $x_i$ such that $\mu(x_i)$ has the same parity as $n + 1$.

To find an extra orbit, we borrow an argument from the proof of [GuKa, Thm. 1.4]. Observe first that without loss of generality we can assume that the Reeb flow has an orbit, not necessarily simple, of index $q < n + 1$. Let us denote this orbit by $y^k$, where $y$ is simple. We can further assume that $y^k$ is good; for otherwise $y$ is odd and hence different from the orbits $x_i$. As a consequence, the parity of $\mu(y)$ is the same as $q$. It follows that if $q$ has parity different from $n + 1$, the orbit $y$ is different from any of the orbits $x_i$.

The remaining case is that of $q$, and hence $\mu(y)$, having the same parity as $n + 1$. It is easy to see that, since $\text{SH}_y^{G,+}(W) = 0$, there must be a good orbit with Conley–Zehnder index $q + 1$. Let us denote this orbit $z'$, where $z$ is simple. Now the parity of $\mu(z)$ is the same as that of $q + 1$, and therefore different from $n + 1$. Thus $z$ is different from any of the orbits $x_i$. □

Remark 6.11. Ultimately, on the side of spectral invariants, the proofs of Theorems 6.1, 6.4 and 6.9 and Corollary 6.6 depend only on Corollary 3.9. As was mentioned in Section 3.2.1, these corollaries carry over word-for-word to any simply connected Liouville domain $W$ displaceable in $\hat{W}$ or even to $W$ displaceable in some other Liouville manifold, provided that $c_1(TW) = 0$. Hence, for such $W$, the theorems and Corollary 6.6 also hold as stated.

We finish this section by briefly touching upon some related results. First, recall that by the Ekeland–Lasry theorem, [EL], a convex hypersurface $M^{2n-1} \subset \mathbb{R}^{2n}$ enclosing a sphere $S^{2n-1}_R$ and enclosed by the sphere $S^{2n-1}_{R'}$ with $R' = \sqrt{2}R$ carries at least $r = n$ closed characteristics. (Here we say that $M$ encloses $M'$ when $M'$ lies in the open domain bounded by $M$.) In fact, there is a similar lower bound for any $R'$ with $r = [n/\kappa]$, where $\kappa \in \mathbb{N}$ is the smallest positive integer such that $R' < \sqrt{\kappa + 1}R$, [AmMa]. It would be interesting to cast the Ekeland–Lasry theorem into the symplectic-topological framework (see, e.g., [AGH, Ke]) and, for instance, extend it to the dynamically convex hypersurfaces. What follows is an obvious observation along these lines, cf. [Gut15].

**Corollary 6.12.** Let $(M^{2n-1}, \alpha) \subset \mathbb{R}^{2n}$ be a contact type hypersurface bounding a simply connected Liouville domain, enclosing a sphere $S^{2n-1}_R$ and enclosed by the sphere $S^{2n-1}_{R'}$ with $R' = \sqrt{2}R$. Assume that

$$\min S(\alpha) \geq \pi R^2. \quad (6.2)$$

Then $M$ carries at least $r = n$ closed characteristics.

The corollary is an immediate consequence of Corollary 3.9, and a similar argument can also be used in the more general setting from [AmMa]. The proof of Corollary 6.12 ultimately relies on Theorem 3.4 in the same way as the proof of the Ekeland–Lasry theorem utilized Lusternik–Schnirelmann theory for convex Hamiltonians, [Ek]. Not surprisingly, Corollary 6.12 readily implies the Ekeland–Lasry theorem; for (6.2) is satisfied for convex hypersurfaces by the Croke–Weinstein theorem, [CW]. In fact, (6.2) for convex hypersurfaces is the assertion of the Croke–Weinstein theorem. Its proof, while non-trivial, is self-contained and does not use Morse or Lusternik–Schnirelmann theory. This lower bound fails easily for star-shaped hypersurfaces meeting any prescribed pinching condition, but not dynamically convex; cf. [HZ, Sect. 3.5]. Note also that in the corollary one of the two
“enclosures” need not be strict. For instance, it is sufficient to assume that \( S^{2n-1}_R \) lies in the closed domain bounded by \( M \) while \( M \) is in the open domain bounded by \( S^{2n-1}_R \), and the other way around.

Finally, recall that, as is proved in [LLZ] (see also [Lo02]), a convex hypersurface in \( \mathbb{R}^{2n} \), which is symmetric with respect to the involution \( x \mapsto -x \), necessarily carries at least \( n \) closed characteristics. One could expect a similar lower bound to also hold for dynamically convex symmetric hypersurfaces. However, it is not clear how prove such a generalization in the present framework. This is somewhat surprising because the results from [DDE], and more generally from [Arn] for \( \mathbb{Z}_k \)-symmetry, on the existence of elliptic orbits on symmetric convex hypersurfaces in \( \mathbb{R}^{2n} \), which seem to be closely related to [LLZ], can be generalized to symmetric dynamically convex hypersurfaces and to other classes of Reeb flows on pre-quantization contact manifolds; see [AM14].

6.2. Reeb flows on \( ST^*S^n \). Our next goal is to extend some of the results from Section 3.2.1 to Liouville domains \( W \) in \( T^*S^n \) containing the zero section \( S^n \). Let \( W \) be such a domain with smooth boundary \( M = \partial W \). Thus \( M \) is a restricted contact type hypersurface, equipped with a contact form \( \alpha \), in \( T^*S^n \) enclosing \( S^n \).

For instance, \( M \) can be the unit cotangent bundle \( ST^*S^n \) or, more generally, the boundary of any compact fiberwise star-shaped domain. Throughout this section we will assume that \( n \geq 2 \). When \( n = 2 \) and the hypersurface \( M \) in \( T^*S^2 \) is three-dimensional, much more general results are available; see [CGH, GGo] and also [BL] for the case of a Finsler metric on \( S^2 \).

In this setting the right analog of the dynamical convexity condition is the requirement that \( \mu_-(y) \geq n - 1 \) for all \( y \in P(\alpha) \). When \( W \) is the unit disk bundle of a Finsler metric, this requirement is satisfied, for instance, if the metric meets certain curvature pinching conditions; see, e.g., [Ra04, Wa12] and also [AM14, DLW] for further references and [HP] for the case of \( n = 2 \). Similarly to Theorem 6.1, we have the following result.

**Theorem 6.13.** Let \((M^{2n-1}, \alpha)\) be a closed contact type hypersurface in \( T^*S^n \) enclosing the zero section and bounding a simply connected Liouville domain. Assume that \( \mu_-(y) \geq n - 1 \) for every closed characteristic \( y \) on \( M \). Then \( M \) carries at least \( r \) simple closed characteristics, where \( r = \lfloor n/2 \rfloor - 1 \). When \( \alpha \) is non-degenerate, we can take \( r = n \) if \( n \) is even and \( r = n + 1 \) if \( n \) is odd.

This is Theorem 1.6 from the introduction. The non-degenerate case of the theorem is not new and included only for the sake of completeness. A much more general result is proved in [AM15]. However, the proof below is self-contained and relatively simple. We also emphasize that in this case the lower bound is sharp as the Katok–Ziller examples show; see [Ka, Zi].

**Proof.** The general case of the theorem is established similarly to that of Theorem 6.1. Assume that \((M, \alpha)\) carries only a finite number of closed characteristics and denote them by \( x_1, \ldots, x_r \). For \( j \in \mathbb{N} \), consider the range of indices \( I_j = [-n-1, n] + d_j = [d_j - (n-1), d_j + (n-1)] \) centered at \( d_j = 2j(n-1) \). By Corollary 3.14, for every \( j \in \mathbb{N} \) and \( d \in I_j \), there exists \( y_d = x_{ij}^d \in P(\alpha) \) with \( SH_0^\alpha(y_d) \neq 0 \) and \( c_d(\alpha) = A_\alpha(y_d) \), where \( i \) depends on \( j \) and \( d \). (Here it is more convenient again to relabel the orbits by \( I_j \).) Hence, in particular, \( |\hat{\mu}(y_{ij}) - d| \leq n - 1 \). By Theorem 5.2, when \( \mu_-(y) \geq n - 1 \) for all
$y \in \mathcal{P}(\alpha)$, the number of simple closed characteristics is bounded from below by 
\[ \#[(d_j, d_j + n - 1) \cap \mathcal{I}], \] 
where $r$ is the number of integers of the same parity as $n - 1$ in the open interval $(d_j, d_j + n - 1)$, where $d_j$ can be taken of the form $2j(n - 1)$ for some sequence $j \to \infty$. Thus $r = \lfloor n/2 \rfloor - 1$.

Let us now turn to the non-degenerate case. We use Proposition 3.3. Assume that $M$ carries only finitely many periodic orbits and denote these orbits by $x_i$. Let as above $d_j = 2j(n - 1)$ and

\[ L = (d_j - (n - 1), d_j + (n - 1)). \]

By Theorem 5.1, for some sequence $j \to \infty$ there exist $k_{ij} \in 2\mathbb{N}$ such that $\mu(x_i^{k_{ij}})$ is in the closed interval $\bar{L} = [d_j - (n - 1), d_j + (n - 1)]$ while $\mu(x_i^{k_{ij} \pm \ell})$ for all $\ell \geq 1$ is outside $L$. Hence, the number of even simple periodic orbits with Conley–Zehnder index of the same parity as $n - 1$ is bounded from below by

\[ r_0 = \sum_i \dim \text{SH}_i^{G,+}(W; \mathbb{Q}), \quad (6.3) \]

where $i \equiv n - 1 \pmod{2}$ is in $L$. The homology $\text{SH}_i^{G,+}(W; \mathbb{Q})$ is well known and calculated, for instance, in Proposition 3.12. For the relevant range of degrees, the homology is one-dimensional in every degree $i$ of the same parity as $n - 1$, except for $i = 2j(n - 1)$ if $n - 1$ is even where the homology is two-dimensional. When $i \equiv n \pmod{2}$, the homology is zero. Now it is easy to see that $r_0 = r - 2$.

Thus, to complete the proof, it suffices to find two more simple periodic orbits.

Consider good periodic orbits of index $d_\pm = d_j \pm (n - 1)$. There are at least two such orbits for both $d_-$ and $d_+$ because $\dim \text{SH}_{d_\pm}^{G,+}(W; \mathbb{Q}) = 2$. These orbits are either of the form $x_i^{k_{ij}}$ (the first type) or $x_i^{k_{ij} \pm \ell}$ for some $\ell \geq 1$ (the second type). If an orbit is of the first type, the simple orbit $x_i$ does not contribute to (6.3). Thus we can assume that for at least in one of the degrees $d_\pm$ all orbits have second type, for otherwise the proof is finished. (The same orbit $x_i^{k_{ij}}$ cannot contribute simultaneously to $d_-$ and $d_+$.) Observe that, by Theorem 5.1, $\mu(x_i^{k_{ij} - \ell}) = d_-$ if and only if $\mu(x_i^{k_{ij} + \ell}) = d_+$ and if and only if $\mu(x_i^\ell) = n - 1$. In fact, by Lemma 4.8, $\mu(x_i^\ell)$ is a non-decreasing function of $\ell$, and hence $\mu(x_i^\ell) = n - 1$ for all $1 \leq \ell' \leq \ell$. As a consequence, there are at least two good periodic orbits of index $n - 1$.

Next, note that if $M$ carries at least three, not necessarily simple, good periodic orbits of index $n - 1$, it must also carry at least two good periodic orbits of index $n$ since $\dim \text{SH}_n^{G,+}(W; \mathbb{Q}) = 1$. Under the conditions of the theorem, every good periodic orbit of index $n$ is necessarily simple. Thus, we have two extra orbits, not accounted for in (6.3) when there are three or more orbits of index $n - 1$.

If $M$ carries only one closed orbit of index $n - 1$, say $x_1$, then the only orbits of the second type are $x_1^{k_{ij} \pm 1}$. Therefore, for both degrees $d_-$ and $d_+$ there must be at least one orbit of the first type. This gives us two extra simple periodic orbits.

Focusing on the remaining case where there are exactly two good closed orbits (not necessarily simple) of index $n - 1$, we can in addition assume that there is exactly one good closed orbit, say $x_2$, of index $n$. (For otherwise there are extra two simple periodic orbits.) As has been pointed out above, the orbit $x_2$ is simple. Furthermore, we can also assume that there are no orbits of the first type because such an orbit together with $x_2$ would give us the required two simple orbits.
As a consequence, we have exactly two good periodic orbits of index $d_-$ and $d_+$. It is not hard to see that if there is a good orbit $y$ with $\mu(y) \in L$ and such that $\mu(y) \equiv n \pmod{2}$, there must be an extra simple orbit $x_i$ such that $x_i^{k_{ij}}$ “cancels” $y$. This is the case, for instance, when $x_2$ is even. Indeed, then we can take $y = x_2^{k_{ij}}$.

When $x_2$ is odd, $x_2^{k_{ij}}$ is bad and the above argument does not apply. To recover an extra simple closed orbit, observe first that by our assumptions we have no good periodic orbits of index $d_- + 1$, two good periodic orbits of index $d_-$ and one good closed orbit $x_2^{k_{ij}-1}$ of index $d_- - 1$. Since $\dim \text{SH}^{G, +}_{d_- - 2}(W; \mathbb{Q}) = 1$, there are at least two good closed orbits of index $d_- - 2$. Therefore, by Theorem 5.1, we have two good periodic orbits of index $n - 1$, one simple closed orbit $x_2$ of index $n$, and at least two good periodic orbits of index $n + 1$. Since $\dim \text{SH}^{G, +}_{n+1}(W; \mathbb{Q}) = 1$, there must be at least one good closed orbit $y$ of index $n + 2$. Arguing as in the proof of Lemma 4.8, it is not hard to show that $\mu(x_2^{k_{ij}}) \geq n + 3$ for $\ell > 1$ since $\mu(x_2) = n$ and $x_2$ is odd. Hence $y \neq x_2^{k_{ij}}$ for all $\ell \in \mathbb{N}$. Therefore, there exists an extra simple closed orbit with index of the same parity as $n$. This completes the proof of the non-degenerate case of the theorem. □

Remark 6.14. One might expect that as in Theorem 6.1 the lower bound $r = [n/2] - 1$ could be improved by one when $n - 1$ is even by showing that either the spectral invariant corresponding to the lower limit $d_i$ of the range of degrees is carried by one of the orbits $x_i^{k_{ij}}$ or there exists an SDM orbit and hence infinitely many simple periodic orbits, [GH2M]. However, in the latter case the SDM orbit need not be simple and the result from [GH2M] does not apply.

Theorem 6.15. Let $(M^{2n-1}, \alpha)$ be a closed contact type hypersurface in $T^*S^n$ enclosing the zero section and bounding a simply connected Liouville domain. Assume that $\mu_-(y) \geq q$, where $0 < q < n-1$, for all, not necessarily simple closed Reeb orbits $y$ on $M$. Then $M$ carries at least $r$ simple closed orbits, where

$$r = \begin{cases} q - [n/2] & \text{when } n \text{ and } q \text{ are odd}, \\ q + 1 - [n/2] & \text{otherwise}. \end{cases}$$

(If the right hand side is negative or zero the result is void.) When $\alpha$ is non-degenerate, we can take

$$r = \begin{cases} q + 1 & \text{when } n \text{ is odd or when } n \text{ and } q \text{ are both even}, \\ q & \text{when } n \text{ is even and } q \text{ is odd}. \end{cases}$$

Remark 6.16. In the degenerate case it is sufficient to assume only that $\mu_-(x) \geq q$ when $\hat{\mu}(x) > 0$ and when $\alpha$ is non-degenerate such an assumption yields the lower bound $r - 1$ rather than $r$. Furthermore, it follows from the proof that in the non-degenerate case $r - 1$ orbits can be chosen even and with Conley–Zehnder index of the same parity as $n - 1$, just as in Theorem 6.9. The remaining orbit is either odd or has Conley–Zehnder index of the same parity as $n$. For bumpy Finsler metrics on $S^n$ stronger lower bounds on the number of simple prime closed geodesics are established under less restrictive, at least on the conceptual level, assumptions in [DLW] where the index conditions are also related to certain curvature bounds; see also [BTZ, Wa13] and references therein.
Proof. By Corollary 3.14 and Theorem 5.2, we can take as $r$ the number of the integers of the same parity as $n-1$ in the open interval

$$L = (2j(n-1) - q + (n-1), 2j(n-1) + q).$$

Here $j$ is an unknown positive integer, which can be arbitrarily large. However, $r$ is independent of $j$. This proves the general case of the theorem.

Next, assume that $\alpha$ is non-degenerate. As in the proof of Theorem 6.13, we rely on Proposition 3.3. Due to Theorem 5.1, the number of simple periodic orbits is bounded from below by $r_0$ given by (6.3), where $i$ ranges over integers of the same parity as $n-1$ in the open interval

$$L = (2j(n-1) - q, 2j(n-1) + q)$$

and $W$ is the domain bounded by $M$ in $T^*S^n$. It is easy to see that $r_0 = q$ when $n$ is odd or when $n$ and $q$ are both even, and $r_0 = q - 1$ otherwise.

Finally, arguing exactly as in the proof of Theorem 6.9, one can show that $(M, \alpha)$ carries an extra simple periodic orbit and hence $r = r_0 + 1$. This proves the lower bound in the non-degenerate case. \hfill \Box

Remark 6.17. In the results from this section, the condition that the domain $W$ is simply connected can be relaxed or perhaps eliminated. For instance, it would be sufficient to assume that $\partial W$ has restricted contact type and $\pi_1(W)$ is torsion free.

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