PULLBACK ATTRACTORS FOR A WEAKLY DAMPED WAVE EQUATION WITH DELAYS AND SUP-CUBIC NONLINEARITY

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Abstract. In this paper, we consider the weakly damped wave equations with hereditary effects and the nonlinearity $f$ satisfying sup-cubic growth. Based on the recent extension of the Strichartz estimates to the case of bounded domains, we establish the global well-posedness of the Shatah-Struwe solutions for the non-autonomous case. Then, we prove the existence of the pullback $\mathcal{D}$-attractors in $C_{H^1(\Omega)} \times C_{L^2(\Omega)}$ for the solutions process $\{U(t, \tau)\}_{t \geq \tau}$ by applying the idea of contractive functions.

1. Introduction. As it is well known, delay differential equations arises as mathematical models to describe the dynamical behavior depending on past events. For this reason, delay differential equations have received extensive attention and a great deal of literature has emerged, see for instance [9, 14, 12, 13, 15, 7, 8, 18, 19, 20, 21, 22, 27, 29, 33, 32, 38, 39, 40] and [23, 24].

In this paper, our aim is to consider the global well-posedness and long-time behavior of solutions for the following weakly damped wave equation with delays on the bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary:

$$
\begin{aligned}
\partial_t u + \gamma \partial_t u - \Delta u + f(u) &= g(t, u_t) + k(t) & \text{in } \Omega \times (\tau, \infty), \\
u(x, t) &= 0 & \text{on } \partial \Omega \times (\tau, \infty), \\
u(x, t) &= \phi(x, t - \tau), & x \in \Omega, \ t \in [\tau - h, \tau], \\
\partial_t \nu(x, t) &= \partial_t \phi(x, t - \tau), & x \in \Omega, \ t \in [\tau - h, \tau],
\end{aligned}
$$

(1)

where $\tau \in \mathbb{R}$, the constant $\gamma > 0$, $g$ is a operator acting on the solutions containing some hereditary characteristics (assumptions on $g$ are given below), $\phi$ is the initial datum on the interval $[\tau - h, \tau]$, $h(> 0)$ is the length of the delay effects, and for

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each $t \geq \tau$, we denote by $u_t$ the function defined in $[-h, 0]$ with $u_t(\theta) = u(t + \theta)$, $\theta \in [-h, 0]$. For the nonlinearity $f$, we assume that $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(0) = 0$ and the following growth condition:

$$|f'(u)| \leq C(1 + |u|^{p-1}), \ p \in [1, 5), \ \forall u \in \mathbb{R}. \quad (2)$$

Denote

$$\mathcal{F}(u) := \int_0^u f(s) ds,$$

then from (2), there exists a constant $C > 0$ such that

$$|\mathcal{F}(u)| \leq C(1 + |u|^{p+1}), \ \forall u \in \mathbb{R}. \quad (3)$$

Similar to those in [37], we make the following classical assumptions on $f$ and $\mathcal{F}$:

$$\lim \inf_{|u| \to \infty} \frac{\mathcal{F}(u)}{u^2} \geq 0, \quad (4)$$

and

$$\lim \inf_{|u| \to \infty} \frac{f(u) - \mu \mathcal{F}(u)}{u^2} \geq 0, \quad (5)$$

where the constant $\mu > 0$.

For the time-dependent external forcing $k(\cdot) \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, we assume that

$$\int_{-\infty}^t e^{\alpha s} \|k(s)\|_2^2 ds < \infty \quad \text{for all } t \in \mathbb{R}, \quad (6)$$

where $\alpha$ will be characterized later (see Lemma 3.5 for details).

We will denote by $C_X$ the Banach space $C([-h, 0]; X)$, equipped with the sup-norm. For an element $u \in C_X$, its norm will be written as $\|u\|_{C_X} = \max_{t \in [-h, 0]} \|u(t)\|_X$.

Let $(X, \| \cdot \|_X)$, $(Y, \| \cdot \|_Y)$ be two Banach spaces such that the injection $X \subset Y$ is continuous. We denote by $C_{X,Y}$ the Banach space $C_X \cap C^1([-h, 0]; Y)$ with the norm $\| \cdot \|_{C_{X,Y}}$ defined by

$$\|\phi\|_{C_{X,Y}}^2 = \|\phi\|_{C_X}^2 + \|\partial_t \phi\|_{C_Y}^2 \quad \text{for all } \phi \in C_{X,Y}.$$

Let $H = L^2(\Omega)$ with norm $\| \cdot \|_2$ and inner product $(\cdot, \cdot)$, $V = H^1_0(\Omega)$ with norm $\| \nabla \cdot \|_2$ and inner product $(\nabla \cdot, \nabla \cdot)$. Let $A = -\Delta$ for any $u \in D(A)$, where $D(A) = \{ u \in V; \ Au \in H \} = H^1_0(\Omega) \cap H^2(\Omega)$.

For the operator $g(\cdot, \cdot)$, similarly as that in [9], we assume that $g : \mathbb{R} \times C_{L^2} \to L^2(\Omega)$, and it satisfies

(I) for all $\xi \in C_{L^2}$, the function $g(t, \xi) \in L^2(\Omega)$ is measurable w.r.t. $t \in \mathbb{R}$;

(II) $g(t, 0) = 0$ for all $t \in \mathbb{R}$;

(III) there exists $L_g > 0$ such that for all $t \in \mathbb{R}$ and $\xi, \eta \in C_{L^2}$, it holds

$$\|g(t, \xi) - g(t, \eta)\|_2 \leq L_g \|\xi - \eta\|_{C_{L^2}};$$

(IV) there exist $m_0 \geq 0$, $C_g > 0$ such that for all $u, v \in C([-\tau - h, \tau]; L^2(\Omega))$ and $m \in [0, m_0]$, it holds

$$\int_\tau^t e^{ms} \|g(s, u_s) - g(s, v_s)\|_2^2 ds \leq C_g^2 \int_{\tau - h}^t e^{ms} \|u(s) - v(s)\|_2^2 ds.$$

The well-posedness and long-time behavior of the weakly damped wave equations have been investigated extensively by many authors (see, e.g., [1, 2, 3, 25]), which depends crucially on the growth rate of the nonlinearity. For a long time, in a
bounded (or unbounded) smooth domain of $\mathbb{R}^3$ (or $\mathbb{R}^n$ ($n > 3$)), the global well-posedness of problem (1) holds only in the case of $p \leq 3$ (or $p \leq \frac{n}{n-2}$) for the growth rates of the nonlinearity $f(u)$, in which the uniqueness is usually guaranteed by the technology of the Sobolev embedding. When $p > 3$, the uniqueness of the usual energy solution is unknown (see, e.g., J.M. Ball[3] for details). Moreover, for the $H^1_0(\Omega)$-bounded subset $B$, $f(B)$ will not precompact in $L^2(\Omega)$ as $p > 3$, which will bring essential difficulty for deducing the asymptotic compactness of the corresponding solutions semigroup (or process). Therefore, for the nonlinearity $f(u)$, the growth exponent $p = 3$ (or $p = \frac{n}{n-2}$) has been regarded as the critical growth exponent associated with the natural energy space $\mathcal{E} = H^1_0(\Omega) \times L^2(\Omega)$ for the case of 3 (or $n$)-D domain (see, e.g., J. Arrieta, A.N. Carvalho and J.K. Hale[1] for details).

For the long-time behavior of the weakly damped wave equations, when the non-linear term $f(u)$ satisfying the critical exponential growth, A.V. Babin and M.I. Vishik in [2] have proved the existence of the global attractors in $\mathcal{E}$ by applying the decomposition technique. Furthermore, J. Arrieta, A.N. Carvalho and J.K. Hale in [1] have also obtained the global attractors in $\mathcal{E}$ by employing the decomposition technique and combining with the Alekseev formulation under more general conditions. In the following, in 2004, J.M. Ball in [3] has verified the existence of the global attractors in $\mathcal{E}$ by using the energy methods without decomposing the semigroup. When the nonlinearity $f(u)$ satisfying quintic growth, V. Kalantarov, A. Savostianov and S. Zelik in [25] have established the global well-posedness of the Shatah-Struwe solutions by utilizing the Strichartz type estimates for the linear wave equation in bounded domain of $\mathbb{R}^3$ (see [4, 5, 6] for details) and obtained the existence of the global attractors in $\mathcal{E}$. Moreover, they also put forward an open question for the non-autonomous case.

For the weakly damped wave equations with delays, there are also some references therein, see, e.g., [9, 38, 43]. In [9], T. Caraballo, P.E. Kloeden and J. Real have considered the dynamical behavior of the solutions for the weakly damped wave equations containing a delay forcing term, which covers the models of Sine-Gordon type, and obtained the uniform forward attractor in $C_{V,H}$ and the pullback attractor in $C_{D(A),V}$ respectively; in [38], the authors have proved the existence of unique pullback attractors in $C_{V,H}$ and $C_{D(A),V}$ for the multi-valued process in the linear case; in [43], we have considered the case of the nonlinearity $f(u)$ satisfying the critical nonlinear growth and obtained the pullback attractor in $C_{V,H}$.

In this paper, based on the recent extension of the Strichartz estimates to the case of bounded domains, we consider the global well-posedness and the long-time behavior of the Shatah-Struwe solutions (see [4, 5, 6] for details) for the weakly damped wave equations with delays and the nonlinearity $f(u)$ satisfying sup-cubic and sub-quintic growth condition.

For our problem, we will confront two main difficulties when we establish the global well-posedness of the solutions and verify the compactness of the process $\{U(t,\tau)\}_{t \geq \tau}$. One difficulty is the nonlinearity $f(u)$ satisfies sup-cubic nonlinearity growth, which lead to the fact that the uniqueness of the usual energy solution is unknown. The other difficulty is our problem contains delay term $g(t,u_t)$, which makes the Banach space $C_X \times C_Y$ as the phase space rather than $X \times Y$. In the Banach space $C_X \times C_Y$, the already existing methods and techniques for verifying the compactness of the process $\{U(t,\tau)\}_{t \geq \tau}$, such as the Sobolev embedding theorem, are no longer valid.
In order to overcome the difficulties mentioned above, we extend the Strichartz type estimates to the non-autonomous case and establish the global well-posedness of the Shatah-Struwe solutions. Then, we verify the pullback asymptotic compactness of the process \( \{U(t, \tau)\}_{t \geq \tau} \) by constructing the energy functional and combining with the idea of contractive functions (see [17, 26, 28, 34, 36, 41, 42]).

The outline of the paper is as follows. In Section 2, we recall some basic notions and abstract results about the Strichartz estimates. In Section 3, we establish the global well-posedness of the Shatah-Struwe solutions (see Theorems 3.4 and 3.6). In Section 4, we prove the existence of the pullback \( \mathcal{D} \)-attractor \( \hat{A} \) in \( C_{H^1_0(\Omega)} \times C_{L^2(\Omega)} \) for the process \( \{U(t, \tau)\}_{t \geq \tau} \) (see Theorem 4.3).

2. Preliminaries. In this section, we firstly recall the Strichartz type estimates for the following linear wave equation (see [4, 5, 6] for more details):

\[
\begin{aligned}
&\frac{\partial_t v + \gamma \partial_t v - \Delta v = G(t)}{v(x, t) = 0} \quad \text{in } \Omega \times (\tau, \infty), \\
&\left.\xi_v(t)\right|_{t \to \tau} = \xi_\tau, \\
&v(x, t) = 0 \quad \text{on } \partial\Omega \times (\tau, \infty), \\
&x \in \Omega, \ t \in [\tau - h, \tau],
\end{aligned}
\tag{7}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \), the initial data \( \xi_\tau(t) = \xi := (v_\tau, v'_\tau) \) is taken from the standard energy space \( \mathcal{E} \):

\[
\mathcal{E} := H^1_0(\Omega) \times L^2(\Omega), \quad \|\xi_v\|^2_2 := \|\nabla v\|^2_2 + \|\partial_t v\|^2_2.
\]

Lemma 2.1. Let \( \xi_\tau \in \mathcal{E} \), \( G \in L^1(\tau, T; L^2(\Omega)) \) and let \( v(t) \) be a solution of equation (7) such that \( \xi_\tau(t) \in C[\tau, T; \mathcal{E}] \). Then the following estimate holds:

\[
\|\xi_\tau(t)\|_\mathcal{E} \leq C \left( \|\xi_\tau\|_\mathcal{E} e^{-\beta(t - \tau)} + \int_\tau^t e^{-\beta(t - s)} \|G(s)\|_2 ds \right),
\tag{8}
\]

where the positive constants \( \beta, C \) are dependent of \( \gamma > 0 \), but are independent of \( t, \tau, \xi_\tau \) and \( G \).

Lemma 2.2. Let the assumptions of Lemma 2.1 hold. Then \( v \in L^4(\tau, T; L^{12}(\Omega)) \) and the following estimate holds:

\[
\|v\|_{L^4(\tau, T; L^{12}(\Omega))} \leq C \left( \|\xi_\tau\|_\mathcal{E} + \|G\|_{L^1(\tau, T; L^2(\Omega))} \right),
\tag{9}
\]

where \( C \) may depend on \( \tau \), \( T \) and \( |\Omega| \), but is independent of \( \xi_\tau \) and \( G \).

Remark 1. According to the energy estimate (8) and the interpolation inequality

\[
\|v\|_{L^5(\tau, T; L^{10}(\Omega))} \leq C \|v\|_{L^4(\tau, T; L^{12}(\Omega))} \|v\|_{L^\infty(\tau, T; H^1_0(\Omega))}^{\frac{4}{5}},
\tag{10}
\]

we can obtain the control of the \( L^5(\tau, T; L^{10}(\Omega)) \)-norm of the solution \( v \).

On the other hand, combining with (8) in Lemma 2.1 and (9) in Lemma 2.2, we can get that

\[
\|\xi_\tau(t)\|_\mathcal{E} + \|v\|_{L^4(\tau(t), T; L^{12}(\Omega))} \leq C \left( \|\xi_\tau\|_\mathcal{E} e^{-\beta(t - \tau)} + \int_\tau^t e^{-\beta(t - s)} \|G(s)\|_2 ds \right),
\tag{11}
\]

where \( \tau(t) = \max\{\tau, t - 1\} \) and the constants \( C, \beta > 0 \) are independent of \( t, \tau, \xi_\tau \) and \( G \), but \( C \) depends on \( |\Omega| \).

The following lemmas (see [31, 35] for details) will be used to verify the existence of the Global Shatah-Struwe solutions of equation (1).
\textbf{Lemma 2.3. (see [35])} Suppose that a function $y(\cdot) \in C[a, b]$ satisfies $y(a) = 0$, $y(s) \geq 0$ for any $s \in [a, b]$ and

$$y(s) \leq C_0(y(s))^{p_1} + \varepsilon$$

for some $p_1 > 1$, $0 < C_0 < \infty$ and $0 < \varepsilon < \frac{1}{2c_0} \frac{1}{p_1 - 1}$. Then

$$y(s) \leq 2\varepsilon \quad \text{for all } s \in [a, b].$$

\textbf{Lemma 2.4. (see [31])} Assume that $k(\cdot) \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ is translation bounded, i.e., $k(\cdot) \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$,

$$\|k\|^2_{L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|k(s)\|^2_2 ds < \infty.$$ 

Then for any $t \in \mathbb{R}$, $\tau < t$ and $c > 0$,

$$\int_\tau^t e^{-c(t-s)} \|k(s)\|^2_2 ds \leq \frac{1}{1 - e^{-c}} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|k(s)\|^2_2 ds.$$

\textbf{Proof.} For any $t \in \mathbb{R}$, $\tau < t$ and $c > 0$,

$$\int_\tau^t e^{-c(t-s)} \|k(s)\|^2_2 ds \leq e^{-c} \left( \int_t^{t+1} e^{cs} \|k(s)\|^2_2 ds + \int_{t-1}^{t} e^{cs} \|k(s)\|^2_2 ds + \cdots \right)$$

$$\leq e^{-c} \left( e^{c(1 + e^{-c} + e^{-2c} + \cdots)} \sup_{t \in \mathbb{R}} \int_t^{t+1} e^{cs} \|k(s)\|^2_2 ds \right)$$

$$\leq e^{-c} \left( e^{c(1 + e^{-c} + e^{-2c} + \cdots)} \sup_{t \in \mathbb{R}} \int_0^{t+1} e^{cs} \|k(s)\|^2_2 ds \right)$$

$$\leq \frac{1}{1 - e^{-c}} \sup_{t \in \mathbb{R}} \int_t^{t+1} \|k(s)\|^2_2 ds. \qed$$

For the convenience of the reader, we recall some basic notions and results about pullback attractors (see [16, 10, 11] for details).

Let $\{U(t, \tau)\}_{t \geq \tau}$ be a process (or a two-parameter semigroup) on a metric space $X$, i.e., a family $\{U(t, \tau); -\infty < \tau < t < +\infty\}$ of mappings $U(t, \tau) : X \to X$, such that $U(\tau, \tau) x = x$, $\forall x \in X$, and

$$U(t, \tau) = U(t, s)U(s, \tau) \quad \text{for all } \tau \leq s \leq t.$$

Let $\mathcal{D}$ be a nonempty class of parameterized sets $\hat{D} = \{D(t); t \in \mathbb{R}\} \subset \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the family of all nonempty subsets of $X$.

\textbf{Definition 2.5.} The process $\{U(t, \tau)\}_{t \geq \tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t \in \mathbb{R}$, any $\hat{D} \in \mathcal{D}$, any sequence $\{\tau_n\}_{n=1}^{\infty}$ with $\tau_n \to -\infty$ as $n \to +\infty$ and any sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}_{n=1}^{\infty}$ is precompact in $X$.

\textbf{Definition 2.6.} It is said that $\hat{B} \in \mathcal{D}$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t, \tau)\}_{t \geq \tau}$ if for any $t \in \mathbb{R}$ and any $\hat{D} \in \mathcal{D}$, there exists a $\tau_0 = \tau_0(t, \hat{D}) \leq t$ such that

$$U(t, \tau)D(\tau) \subset B(t) \quad \text{for all } \tau \leq \tau_0(t, \hat{D}).$$
A family $\hat{A} = \{A(t); \ t \in \mathbb{R}\} \subset \mathcal{P}(X)$ is said to be a pullback $D$-attractor for the process $\{U(t, \tau)\}_{t \geq \tau}$ in $X$ if

1. $A(t)$ is compact in $X$ for all $t \in \mathbb{R}$;
2. $\hat{A}$ is pullback $D$-attracting in $X$, i.e.,
   \[ \lim_{\tau \to -\infty} dist_X(U(t, \tau)D(\tau), A(t)) = 0, \]
   for all $\hat{D} \in D$ and all $t \in \mathbb{R}$;
3. $\hat{A}$ is invariant, i.e., $U(t, \tau)A(\tau) = A(t)$, for $-\infty < \tau \leq t < +\infty$.

In addition, $\hat{A}$ is minimal in the sense that if $\hat{C} = \{C(t); \ t \in \mathbb{R}\}$ is a family of nonempty sets such that $C(t)$ is a closed subset of $X$ and

\[ \lim_{\tau \to -\infty} dist_X(U(t, \tau)D(\tau), C(t)) = 0, \quad \forall t \in \mathbb{R}, \]

then $A(t) \subset C(t)$ for any $t \in \mathbb{R}$.

Moreover, we need the following definition and lemma, which are similar to those in [17, 26, 34, 36, 41] and will be used to verify the existence of pullback attractors later.

**Definition 2.8.** Let $(X, \| \cdot \|_X)$ be a Banach space and $\hat{B} = \{B(t); \ t \in \mathbb{R}\}$ be a subset of $X$. A function $\psi(\cdot, \cdot)$, defined on $X \times X$, is said to be a contractive function on $B(t) \times B(t)$ if for any sequence $\{x_n\}_{n=1}^\infty \subset B(t)$, there is a subsequence $\{x_{n_j}\}_{j=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that

\[ \lim_{i \to \infty} \lim_{j \to \infty} \psi(x_{n_i}, x_{n_j}) = 0. \]

We denote the set of all contractive functions on $B(t) \times B(t)$ by $\text{Contr}(B(t))$.

**Lemma 2.9.** Let $\{U(t, \tau)\}_{t \geq \tau}$ be a process on Banach space $X$ and have a pullback $D$-absorbing set $\hat{B} = \{B(t); \ t \in \mathbb{R}\}$. Moreover, suppose that for any $\varepsilon > 0$, there exist $T_1 = T_1(t, \varepsilon) = t - \tau$ and $\psi_{t, T_1}(\cdot, \cdot) \in \text{Contr}(B(\tau))$ such that

\[ \|U(t, t - T_1)x - U(t, t - T_1)y\|_X \leq \varepsilon + \psi_{t, T_1}(x, y), \quad \forall x, y \in B(\tau), \]

where $\psi_{t, T_1}(\cdot, \cdot)$ depends on $t$ and $T_1$. Then $\{U(t, \tau)\}_{t \geq \tau}$ is pullback $D$-asymptotically compact in $X$.

**Proof.** We need to prove that for any $\{x_n\}_{n=1}^\infty \subset X$ and any $\tau_n \to -\infty$ as $n \to \infty$,

the sequence $\{U(t, \tau_n)x_n\}_{n=1}^\infty$ is precompact in $X$.

In the following, we will prove that $\{U(t, \tau_n)x_n\}_{n=1}^\infty$ has a convergent subsequence via diagonal methods.

Selecting $\varepsilon_m > 0$ with $\varepsilon_m \to 0$ as $m \to \infty$.

Then for $\varepsilon_1 > 0$, by the assumptions, there exist $T_1 = T_1(t, \varepsilon_1) = t - \tau$ and $\psi_{t, T_1}(\cdot, \cdot) \in \text{Contr}(B(\tau))$ such that

\[ \|U(t, t - T_1)x - U(\tau, t - T_1)y\|_X \leq \varepsilon_1 + \psi_{t, T_1}(x, y), \quad \forall x, y \in B(\tau), \]

(13) for any $t \in \mathbb{R}$, where $\psi_{t, T_1}(\cdot, \cdot)$ depends on $t$ and $T_1$.

Since $\tau_n \to -\infty$, without loss of generality, we assume that $\tau_n \leq \tau$ such that $U(\tau, \tau_n)x_n \in B(\tau)$ for each $n \in \mathbb{N}$. Set $y_n = U(\tau, \tau_n)x_n$, then from (13) we have

\[
\begin{align*}
\|U(t, \tau_n)x_n - U(t, \tau_m)x_m\|_X &= \|U(t, \tau)U(\tau, \tau_n)x_n - U(t, \tau)U(\tau, \tau_m)x_m\|_X \\
&= \|U(t, \tau)y_n - U(t, \tau)y_m\|_X \\
&\leq \varepsilon_1 + \psi_{t, T_1}(y_n, y_m).
\end{align*}
\]

(14)
By Definition 2.8 and \( \psi_{t,T_1}(\cdot, \cdot) \in \text{Contr}(B(\tau)) \), we know that the sequence \( \{y_n\}_{n=1}^\infty \) have a subsequence \( \{y_n^{(1)}\}_{n=1}^\infty \) such that
\[
\lim_{j \to \infty} \lim_{i \to \infty} \psi_{t,T_1}(y_n^{(1)}, y_n^{(1)}) \leq \varepsilon_1. \tag{15}
\]

Similarly to [26, 36, 41], we have
\[
\lim_{i \to \infty} \sup_{q \in \mathbb{N}} \|U(t, \tau_{n_{i+q}}^{(1)})x_{n_{i+q}}^{(1)} - U(t, \tau_{n_{i}}^{(1)})x_{n_{i}}^{(1)}\|_X \leq \\
\leq \lim_{i \to \infty} \sup_{q \in \mathbb{N}} \lim_{j \to \infty} \|U(t, \tau_{n_{i+q}}^{(1)})x_{n_{i+q}}^{(1)} - U(t, \tau_{n_{i}}^{(1)})x_{n_{i}}^{(1)}\|_X \\
+ \lim_{i \to \infty} \sup_{q \in \mathbb{N}} \lim_{j \to \infty} \|U(t, \tau_{n_{i}}^{(1)})x_{n_{i}}^{(1)} - U(t, \tau_{n_{j}}^{(1)})x_{n_{j}}^{(1)}\|_X \\
\leq \varepsilon_1 + \lim_{i \to \infty} \sup_{q \in \mathbb{N}} \lim_{j \to \infty} \psi_{t,T_1}(y_{n_{i+q}}^{(1)}, y_{n_{j}}^{(1)}) + \varepsilon_1 + \lim_{i \to \infty} \psi_{t,T_1}(y_{n_{i}}^{(1)}, y_{n_{j}}^{(1)}),
\]
where we have used (14). Then combining with (15), we obtain that
\[
\lim_{i \to \infty} \sup_{q \in \mathbb{N}} \|U(t, \tau_{n_{i}}^{(1)})x_{n_{i}}^{(1)} - U(t, \tau_{n_{j}}^{(1)})x_{n_{j}}^{(1)}\|_X \leq 4\varepsilon_1.
\]

Therefore, there exists a \( K_1 \in \mathbb{N} \) such that
\[
\|U(t, \tau_{n_{i}}^{(1)})x_{n_{i}}^{(1)} - U(t, \tau_{n_{j}}^{(1)})x_{n_{j}}^{(1)}\|_X \leq 5\varepsilon_1, \quad \text{for all } i, j \geq K_1.
\]

By induction, we know that, there exists a subsequence \( \{U(t, \tau_{n_{i}}^{(m+1)})x_{n_{i}}^{(m+1)}\}_{i=1}^\infty \) of \( \{U(t, \tau_{n_{i}}^{(m)})x_{n_{i}}^{(m)}\}_{i=1}^\infty \) and certain \( K_{m+1} \) with \( m \geq 1 \), such that
\[
\|U(t, \tau_{n_{i}}^{(m+1)})x_{n_{i}}^{(m+1)} - U(t, \tau_{n_{j}}^{(m+1)})x_{n_{j}}^{(m+1)}\|_X \leq 5\varepsilon_{m+1}, \quad \text{for all } i, j \geq K_{m+1}.
\]

Now, consider the diagonal subsequence \( \{U(t, \tau_{n_{i}}^{(i)})x_{n_{i}}^{(i)}\}_{i=1}^\infty \). Since for each \( m \in \mathbb{N} \), \( \{U(t, \tau_{n_{i}}^{(i)})x_{n_{i}}^{(i)}\}_{i=1}^\infty \) is a subsequence of \( \{U(t, \tau_{n_{i}}^{(i)})x_{n_{i}}^{(i)}\}_{i=1}^\infty \), then
\[
\|U(t, \tau_{n_{i}}^{(i)})x_{n_{i}}^{(i)} - U(t, \tau_{n_{j}}^{(j)})x_{n_{j}}^{(j)}\|_X \leq 5\varepsilon_m, \quad \text{for all } i, j \geq \max\{m, K_m\},
\]
which combining with \( \varepsilon_m \to 0 \) as \( m \to \infty \), implies that \( \{U(t, \tau_{n_{i}}^{(i)})x_{n_{i}}^{(i)}\}_{i=1}^\infty \) is a Cauchy sequence in \( X \). This shows that \( \{U(t, \tau_{n})x_n\}_{n=1}^\infty \) is precompact in \( X \). □

Thanks to Theorems 3.3 and 3.5 in [30], we have the following theorem, which will be used to verify the existence of pullback attractors.

**Theorem 2.10.** Let \( \{U(t, \tau)\}_{t \geq \tau} \) be a continuous process on Banach space \( X \). Then \( \{U(t, \tau)\}_{t \geq \tau} \) has a pullback \( \mathcal{D} \)-attractor in \( X \) provided that the following conditions hold:

1. \( \{U(t, \tau)\}_{t \geq \tau} \) has a pullback \( \mathcal{D} \)-absorbing set \( \hat{B}_0 \) in \( X \);
2. \( \{U(t, \tau)\}_{t \geq \tau} \) is pullback \( \mathcal{D} \)-asymptotically compact in \( X \).

3. Existence and uniqueness of solutions. In this section, we will obtain the well-posedness and dissipation of the Shatah-Struwe solutions for equation (1). At first, we give the following definitions.

**Definition 3.1.** A function \( u(t) \) is said to be a weak (energy) solution of equation (1) if \( \xi_u := (u, \partial_t u) \in L^\infty(\tau, T; E) \) with the initial data \( (\phi, \partial_t \phi) \in L^\infty(\tau, T; E) \) and
(1) is satisfied in the sense of distribution, i.e.,
\[
- \int_\tau^T (\partial_t u, \partial_t \varphi) dt - \gamma \int_\tau^T (u, \partial_t \varphi) dt + \int_\tau^T (\nabla u, \nabla \varphi) dt
= \gamma(u, \varphi) + (u_t, \varphi) + \int_\tau^T (f(u), \varphi) dt + \int_\tau^T (g(t, u_t), \varphi) dt + \int_\tau^T (k(t), \varphi) dt
\]
for any \( T > \tau \) and any function \( \varphi \) comes from the class
\[
W_T = \{ \varphi \mid \varphi(x, t) = \varphi_1(x) \varphi_2(t), \ \varphi_1 \in C^\infty(\Omega), \ \varphi_2 \in C^\infty([\tau, T]) \text{ and } \varphi_2(T) = 0 \}.
\]
Here \((\cdot, \cdot)\) stands for the usual inner product in \( L^2(\Omega) \).

**Definition 3.2.** A weak solution \( u(t) \), \( t \in [\tau, T] \) is said to be a Shatah-Struwe (S-S) solution of equation (1) if the following additional regularity holds:
\[
u \in L^4(\tau, T; L^{12}(\Omega)) \quad \text{for any } T > \tau.
\]

For the Shatah-Struwe solutions of equation (1), we have the following result, which shows the local existence.

**Lemma 3.3.** Let \( f \) satisfy (2)-(5), \( g(t, u_t) \) be subject to assumptions (I)-(IV), \( k(\cdot) \in L^2_{loc}(\mathbb{R}; L^2(\Omega)) \) and \( \xi_\tau = (\phi, \partial_t \phi) \in C_{H^1(\Omega)} \times C_{L^2(\Omega)}. \) Then, for any \( \tau \in \mathbb{R}, \) we can find a \( T_2 = T_2(\varepsilon, \|\xi_\tau\|_{C^\infty(\Omega)}) \geq t - \tau, \) \( t \in [\tau, T] \) such that there exists a Shatah-Struwe solution \( u(t) \) for equation (1) on \([\tau, T]\). Moreover, the following estimate holds:
\[
\|\xi_u\|_{C^\infty(\Omega)} + \|u\|_{L^4(\tau, t; L^{12}(\Omega))} \leq Q_1(\|\xi_\tau\|_{C^\infty(\Omega)}) + Q_1(\int_\tau^t \|k(s)\|_2^2 \,ds),
\]
where \( Q_1 \) is an increasing function on \([0, +\infty)\) which depends on \( T_2 \) and \( |\Omega| \), but independent of \( \|\xi_\tau\|_{C^\infty(\Omega)} \) and \( \int_\tau^t \|k(s)\|_2^2 \,ds \).

**Proof.** We will construct the desired solution \( u(t) \) by using the Faedo-Galerkin method.

Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \leq \cdots \) be the eigenvalues of the operator \(-\Delta\) with homogeneous Dirichlet boundary conditions and \( e_1, e_2, \ldots, e_N, \ldots \) be the corresponding eigenfunctions. Then the eigenfunctions form an orthonormal base in \( L^2(\Omega) \) and they are also smooth: \( \{e_i\}_{i=1}^\infty \subset C^\infty(\Omega) \), since \( \Omega \) is smooth. And we can define the Galerkin approximations to problem (1) as follows:
\[
\begin{align*}
\partial_t u_N + \gamma \partial_t u_N - \Delta u_N + P_N f(u_N) &= P_N g(t, u_N) + P_N k(t), \quad u_N \in P_N L^2(\Omega), \\
\xi_{u_N}(\tau) &= \xi_N := P_N \xi_\tau \in |P_N L^2(\Omega)|^2.
\end{align*}
\]
where \( P_N : L^2(\Omega) \rightarrow L^2(\Omega) \) is the orthogonal projector from \( L^2(\Omega) \) to the linear subspace spanned by the first \( N \) eigenfunctions \( \{e_1, e_2, \ldots, e_N\} \).

The above approximation system is actually an ordinal differential system. In order to take the limit on \( N \), we need some uniform estimates for the \( L^4(\tau, T; L^{12}(\Omega)) \)-norm of the solutions \( u_N(t) \).

To this end, we decompose \( u_N(t) = P_N u(t) \) into the sum
\[
u_N(t) = v_N(t) + w_N(t),
\]
where \( \xi_{v_N}(t) = (v_N(t), \partial_t v_N(t)) \) is the solution of the linear equation
\[
\partial_t v_N + \gamma \partial_t v_N - \Delta v_N = P_N k(t), \quad \xi_{v_N}(\tau) = \xi_{u_N}(\tau)
\]
and \( \xi_{w_N}(t) = \left( w_N(t), \partial_t w_N(t) \right) \) is a remainder which satisfies

\[
\partial_t w_N + \gamma \partial_t w_N - \Delta w_N = -P_f(v_N + w_N) + P_N g(t, u_Nt), \quad \xi_{w_N}(\tau) = 0. \tag{18}
\]

From (2), the Hölder and Young inequalities we can deduce that

\[
\| P_N f(v_N + w_N) \|_{L^1(\tau, t; L^2(\Omega))} = \int_\tau^t \left( \int_\Omega |P_N f(v_N + w_N)|^2 \, dx \right)^{\frac{1}{2}} \, ds 
\leq C|\Omega|^\frac{1}{2}(t - \tau) + \| v_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p + \| w_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p 
\leq C'_{\gamma,|\Omega|}[\tau] \left( \| w_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p + \| w_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p \right) 
\leq C'_{\gamma,|\Omega|}[\tau] \left( \| w_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p + \| w_N \|_{L^p(\tau, t; L^{2p}(\Omega))}^p \right) \tag{19}
\]

where we used the Young inequality

\[
(t - \tau)^{1 - \frac{p}{2}} \| w_N \|_{L^p(\tau, t; L^{10}(\xi))}^p \leq \frac{(t - \tau)^{1 - \frac{p}{2}}}{\frac{5}{p}} + \| w_N \|_{L^p(\tau, t; L^{10}(\xi))}^p 
= (1 - \frac{p}{5})(t - \tau) + \frac{p}{5} \| w_N \|_{L^p(\tau, t; L^{10}(\xi))}^p,
\]

and the constant \( C_{\gamma,|\Omega|} \) depends only on \( p \) and \( |\Omega| \).

On the other hand, by the assumption (III) and the Hölder inequality, we have

\[
\| g(s, u_Ns) \|_{L^1(\tau, t; L^2(\Omega))} \leq L_g \int_\tau^t \| u_Ns \|_{C_{L^2(\Omega)}} \, ds 
\leq L_g (t - \tau)^{\frac{1}{2}} \left( \int_\tau^t \| u_Ns \|_{C_{L^2(\Omega)}}^2 \, ds \right)^{\frac{1}{2}}, \quad \forall t \geq \tau. \tag{20}
\]

Moreover, by (32) we know that

\[
\| u_Nt \|_{C_{t,v}}^2 + \| u_Nt \|^2_{C_{\xi t}} \leq (\| \phi \|_{C_{\xi t}}^2 + \| \partial_t \phi \|_{C_{\xi t}}^2) e^{-\alpha(t-h-\tau)} + C_1 e^{-\alpha(t-h-\tau)} 
+ \frac{C_4}{\alpha} + C_3 e^{-\alpha(t-h)} \int_\tau^t e^{\alpha s} \| k(s) \|_{L^2}^2 \, ds, \quad \forall t - h \geq \tau.
\]

Therefore,

\[
\| u_Nt \|_{C_{t,v}}^2 \leq \frac{1}{\lambda_1} (\| \phi \|_{C_{t,v}}^2 + \| \partial_t \phi \|_{C_{\xi t}}^2) e^{-\alpha(t-h-\tau)} + \frac{C_1}{\lambda_1} e^{-\alpha(t-h-\tau)} 
+ \frac{C_4}{\lambda_1 \alpha} + \frac{C_3}{\lambda_1} e^{-\alpha(t-h)} \int_\tau^t e^{\alpha s} \| k(s) \|_{L^2}^2 \, ds, \quad \forall t - h \geq \tau. \tag{21}
\]
By using the Young inequality to \((20)\), then combining with \((21)\), we obtain that
\[
\|g(s,u_Ns)\|_{L^2(\tau,T;L^2(\Omega))} \leq \frac{1}{2} L^2 g(t-t) + \frac{1}{2} \int_{\tau}^{t} \|u_Ns\|^2_{L^2(\Omega)} \, ds \\
\leq \frac{1}{2} L^2 g(t-t) + \frac{1}{2\lambda_1} \int_{\tau}^{t} (\|\phi\|^2_{C^2} + \|\partial_t\phi\|^2_{C^2}) e^{-\alpha(s-h-t)} \, ds \\
+ \frac{C_1}{2\lambda_1} \int_{\tau}^{t} e^{-\alpha(s-h-t)} \, ds + \frac{C_4}{2\lambda_1} \int_{\tau}^{t} \, ds \\
+ \frac{C_3}{2\lambda_1} \int_{\tau}^{t} \left( \int_{\tau}^{s} e^{\alpha(s_1)} \|k(s_1)\|^2_{L^2(\Omega)} \right) \, ds \\
\leq \frac{1}{2} L^2 g(t-t) + \frac{1}{2\lambda_1} \int_{\tau}^{t} (\|\phi\|^2_{C^2} + \|\partial_t\phi\|^2_{C^2}) e^{ah} + \frac{C_1}{2\lambda_1} e^{ah} \\
+ \frac{C_4}{2\lambda_1} (t-t) + \frac{C_3}{2\lambda_1} e^{ah}(t-t) \int_{\tau}^{t} \|k(s)\|^2_{L^2(\Omega)} \, ds. \tag{22}
\]

Then, applying the estimate \((11)\) to \((18)\) and exploiting the interpolation inequality \((10)\), then combining with \((19)\) and \((22)\), we deduce that
\[
\|\xi_{wN}(t)\|\varepsilon + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} \\
\leq C_{p,|\Omega|} \left( \|\xi_{\varepsilon}\|_{C^{1,\frac{1}{2}}} + (t-t)^{\frac{1}{2}} \right) \left( \int_{\tau}^{t} \|k(s)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{23}
\]

Therefore, by applying \((9)\) in Lemma \(2.2\) to \((17)\), we have
\[
\|v_N\|_{L^2(\tau,T;L^2(\Omega))} \leq C_{\tau,|\Omega|} \left( \|\xi_{\varepsilon}\|_{C^{1,\frac{1}{2}}} + (t-t)^{\frac{1}{2}} \right) \left( \int_{\tau}^{t} \|k(s)\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{24}
\]

for all \(t \leq T_2\). Hence,
\[
\|\xi_{wN}(t)\|\varepsilon + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} \\
\leq C_{p,|\Omega|} \left( \|\xi_{wN}(t)\|\varepsilon + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} \right)^{\frac{1}{2}} + C_1 e^{ah} \\
+ \frac{1}{2\lambda_1} \left( \|\phi\|^2_{C^2} + \|\partial_t\phi\|^2_{C^2} \right) e^{ah} + \varepsilon.
\]

Setting \(Y_N(t) = \|\xi_{wN}(t)\|\varepsilon + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} \right)^{\frac{1}{2}} + C_1 e^{ah} + C_3 e^{ah}, \) then using the Young inequality, we can get that
\[
Y_N(t) + \|w_N\|_{L^2(\tau,T;L^2(\Omega))} \leq C_{p,|\Omega|} Y_N^4(t) + Y_N(t) + \varepsilon \\
\leq C_{p,|\Omega|} Y_N^4(t) + \varepsilon + C_3 Y_N^3(t) + \varepsilon \\
\leq C_{p,|\Omega|} Y_N^4(t) + 2\varepsilon. \tag{24}
\]
For the given \( \varepsilon > 0 \), letting \( 0 < T_2 < 1 \) and applying Lemma 2.3 to (24), we get the uniform estimate

\[
\|\xi_{u_N}(t)\|_\varepsilon + \|w_N\|_{L^6(\tau,T;L^1(\Omega))} \leq C_5,
\]

where the constant \( C_5 > 0 \) depends on \( p, T_2, |\Omega|, \|\xi_0\|_\varepsilon, \lambda_1, \alpha, h, \|\phi\|_{C_V}^2 + \|\partial_t \phi\|_{C_H}^2 \) and \( \int_0^T \|k(s)\|_2^2 ds \).

From (17) we easily obtain that \( \|\xi_{u_N}(t)\|_\varepsilon \) is bounded. Then combining with (25) and (23), we can deduce that

\[
\|\xi_{u_N}(t)\|_\varepsilon + \|u_N\|_{L^6(\tau,T;L^2(\Omega))} \leq C_6,
\]

where the constant \( C_6 > 0 \) depends on \( p, T_2, |\Omega|, \|\xi_0\|_\varepsilon, \lambda_1, \alpha, h, \|\phi\|_{C_V}^2 + \|\partial_t \phi\|_{C_H}^2 \) and \( \int_0^T \|k(s)\|_2^2 ds \).

Therefore, by taking the weak limit \( N \to \infty \) in a standard way and then putting \( t + \theta \) instead of \( t \), we can get a Shatah-Struwe solution \( u(t) \) for equation (1) on \( [\tau,T] \).

The following theorem shows the uniqueness of the Shatah-Struwe solutions for equation (1).

**Theorem 3.4.** Under the assumptions of Lemma 3.3, the Shatah-Struwe solutions \( u(t) \) of equation (1) is unique. Moreover, for every two solutions \( u_1(t) \) and \( u_2(t) \) (with different initial data), the following Lipschitz continuity holds:

\[
\|\nabla \omega(t)\|_{C_{L^2(\Omega)}}^2 + \|\omega(t)\|_{C_{L^2(\Omega)}}^2 \leq C_1' \left( \|\nabla \omega(\tau)\|_{C_{L^2(\Omega)}}^2 + \|\omega(\tau)\|_{C_{L^2(\Omega)}}^2 \right) e^{C_2'(t-\tau)}, \quad \forall t \geq \tau,
\]

where \( \omega(t) = u_1(t) - u_2(t) \), the constant \( C_1' = C_1'(|\Omega|, \int_0^\tau \|u_1(s)\|_{L^2}^4 + \|u_2(s)\|_{L^2}^4 ds) \),

\( C_2' = C_2'(\lambda_1, \gamma, L_g, |\Omega|) \).

**Proof.** Denote \( \omega(t) = u_1(t) - u_2(t) \), then \( \omega(t) \) satisfies the equation

\[
\partial_t \omega + \gamma \partial_t \omega - \Delta \omega + f(u_1) - f(u_2) = g(t, u_{1t}) - g(t, u_{2t})
\]

with the initial data

\[
(\omega(x,t), \partial_t \omega(x,t)) = (\phi_1(x,t-\tau) - \phi_2(x,t-\tau), \partial_t \phi_1(x,t-\tau) - \partial_t \phi_2(x,t-\tau)),
\]

with \( t \in [\tau-h, \tau] \).

Multiplying (26) by \( \partial_t \omega(t) \) and integrating over \( x \in \Omega \), we arrive at

\[
\frac{d}{dt}(\|\nabla \omega\|_2^2 + \|\partial_t \omega\|_2^2) + \gamma \|\partial_t \omega\|_2^2 \\
\leq 2|f(u_1) - f(u_2), \partial_t \omega)| + \frac{1}{\gamma} \|g(t, u_{1t}) - g(t, u_{2t})\|_2^2,
\]

where we have used the Hölder and Young inequalities.
By (2), the Hölder and Young inequalities, we get that
\[
\left| (f(u_1) - f(u_2), \partial_t u) \right| \leq C \int_{\Omega} (1 + |u_1|^{p-1} + |u_2|^{p-1}) \omega \| \partial_t \omega \| dx \\
\leq C' \left( \int_{\Omega} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \int_{\Omega} \left( |u_2|^{p-1} \Omega + \frac{1}{\gamma} \right) \right) \omega \| \partial_t \omega \| dx \\
\leq C' \left( \int_{\Omega} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \int_{\Omega} \left( |u_2|^{p-1} \Omega + \frac{1}{\gamma} \right) \right) \omega \| \partial_t \omega \| dx \\
\leq C' \left( \int_{\Omega} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \int_{\Omega} \left( |u_2|^{p-1} \Omega + \frac{1}{\gamma} \right) \right) \omega \| \partial_t \omega \| dx. 
\] (28)

Combining with (27)-(28) and assumption (III), we know that
\[
\frac{d}{dt} \left( \| \nabla \omega \|_2^2 + \| \partial_t \omega \|_2^2 \right) + \gamma \| \partial_t \omega \|_2^2 \\
\leq 2C' \left( \int_{\Omega} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \int_{\Omega} \left( |u_2|^{p-1} \Omega + \frac{1}{\gamma} \right) \right) \omega \| \partial_t \omega \|_2^2 + \frac{1}{\gamma} \left( g(t, u(t)) - g(t, u_1) \right)_2^2 \\
\leq C' \left( \int_{\Omega} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \int_{\Omega} \left( |u_2|^{p-1} \Omega + \frac{1}{\gamma} \right) \right) \omega \| \partial_t \omega \|_2^2 + \frac{L_g^2}{\lambda_1 \gamma} \omega \| \partial_t \omega \|_2^2. 
\] (29)

Integrating (29) with respect to \( t \) in \([\tau, t]\), we deduce that
\[
\| \nabla \omega(t) \|_2^2 + \| \partial_t \omega(t) \|_2^2 \leq \left( \| \nabla \omega(\tau) \|_2^2 + \| \partial_t \omega(\tau) \|_2^2 \right) \\
+ C' \int_{\tau}^{t} \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) \omega \| \partial_t \omega \|_2^2 + \frac{L_g^2}{\lambda_1 \gamma} \omega \| \partial_t \omega \|_2^2 ds \\
+ \frac{L_g^2}{\lambda_1 \gamma} \int_{\tau}^{t} \omega \| \partial_t \omega \|_2^2 ds. 
\]

In particular, taking \( t + \theta \) instead of \( t \), we obtain that
\[
\| \nabla \omega(t) \|_2^2 c_{L^2(\Omega)}^\prime + \| \omega(t) \|_2^2 c_{L^2(\Omega)}^\prime \leq \left( \| \nabla \omega(\tau) \|_2^2 c_{L^2(\Omega)}^\prime + \| \omega(\tau) \|_2^2 c_{L^2(\Omega)}^\prime \right) \\
+ \int_{\tau}^{t} \left( C' \left( |u_1|^{p-1} \Omega + \frac{1}{\gamma} \right) + \frac{L_g^2}{\lambda_1 \gamma} \right) \\
\times \left( \| \nabla \omega \|_2^2 c_{L^2(\Omega)}^\prime + \| \omega(t) \|_2^2 c_{L^2(\Omega)}^\prime \right) ds. 
\] (30)

Applying the Gronwall lemma to (30), it yields
\[
\| \nabla \omega(t) \|_2^2 c_{L^2(\Omega)}^\prime + \| \omega(t) \|_2^2 c_{L^2(\Omega)}^\prime \leq \\
\leq \left( \| \nabla \omega(\tau) \|_2^2 c_{L^2(\Omega)}^\prime + \| \omega(\tau) \|_2^2 c_{L^2(\Omega)}^\prime \right) c C' \left( |u_1(\tau)| \Omega + \frac{1}{\gamma} \right) + \frac{L_g^2}{\lambda_1 \gamma} (t-\tau), 
\]
for all \( t \geq \tau \).

Since the \( L^4(\tau, t; L^2(\Omega)) \)-norms of \( u_1 \) and \( u_2 \) are finite by the definition of the Shatah-Struwe solutions, we obtain the uniqueness.

Thus, we can define the process \( \{ U(t, \tau) \}_{t \geq \tau} \) in \( C_{H_1^1(\Omega)} \times C_{L^2(\Omega)} \) as follows:
\[
U(t, \tau)(\phi, \partial_t \phi) = (u(t), \partial_t u(t)), \quad \forall t \geq \tau. 
\] (31)
Moreover, Theorem 3.4 shows that this process is Lipschitz continuous in \( C_{H_1^1(\Omega)} \times C_{L^2(\Omega)} \).
In order to obtain the global well-posedness of the Shatah-Struwe solutions and the pullback D-absorbing sets for equation (1), we need to obtain the following a priori estimate, which can be obtained in the meaning of Galerkin approximation. For the sake of simplicity, we only make formal estimate.

**Lemma 3.5.** Let $f$ satisfy (2)-(5), $g(t, u_t)$ be subject to assumptions (I)-(IV), $k(\cdot) \in L^2_{loc}(\mathbb{R}; L^2(\Omega))$, $\tau \in \mathbb{R}$ and $\xi_\tau = (\phi, \partial_t \phi) \in C^1_{H_0^1(\Omega)} \times C_L^1(\Omega)$. Then, the solution $\xi_u(t) = (u(t), \partial_t u(t))$ of equation (1) satisfies the following estimates:

$$
\|\xi_u\|_{L_{v, h}^2}^2 = \|u_t\|_{L_h^2}^2 + \|u_{tt}\|_{L_h^2}^2 \\
\leq (\|\phi\|_{L_v^2}^2 + \|\partial_t \phi\|_{L_h^2}^2)e^{-\alpha(t-h)} + C_1 e^{-\alpha(t-h)} + \frac{C_4}{\alpha} \\
+ C_3 e^{-\alpha(t-h)} \int_\tau^t e^{\alpha s}\|k(s)\|_{L^2_h}^2 ds, \quad \forall t - h \geq \tau, \tag{32}
$$

where $C_1 = \frac{2}{\lambda} (\frac{1}{5} + \delta) C^2_g \|\phi\|_{L^2}^2$, $C_2 = \frac{2}{\lambda_1 \lambda_2} \frac{1}{\gamma} C^2_g$, $C_3 = 2 (\frac{1}{5} + \frac{2}{\lambda})$, $C_4 = 2 (\varepsilon C_\delta + \frac{1}{c_\varepsilon} C_\varepsilon(\Omega))$, $\alpha = \lambda - C_2 > 0$, $\lambda = \min\{\frac{(2 \lambda_1 - 3 \delta_0)}{\lambda_1 (\gamma + 1 + \varepsilon)}, \frac{\gamma - 2 \varepsilon}{\gamma + 1 + \varepsilon}, \varepsilon \mu, \frac{\gamma}{\lambda_1}\}.$

**Proof.** Multiplying (1) by $\partial_t u$ and integrating in $\Omega$, we obtain that

$$
\frac{1}{2} \frac{d}{dt} (\|\nabla u\|_2^2 + \|\partial_t u\|_2^2) + \gamma \|\partial_t u\|_2^2 + (f(u), \partial_t u) = (g(t, u_t), \partial_t u) + (k(t), \partial_t u). \tag{33}
$$

On the other hand, taking the $L^2$-inner product between (1) and $2\epsilon u$ ($0 < \epsilon < \gamma$), we get that

$$
2\epsilon \left( \frac{d}{dt} (\|\partial_t u\|_2^2 + \|\partial_t u\|_2^2 + 2(\mathcal{F}(u), 1)) + \gamma \|\partial_t u\|_2^2 \right) \leq \frac{2}{\gamma} \|g(t, u_t)\|_2^2 + \frac{2}{\gamma} \|k(t)\|_2^2. \tag{34}
$$

From (5), we know there exist constant $\delta$: $0 < \delta \ll 1$ and $C_\delta > 0$ such that

$$
f(u)u - \mu \mathcal{F}(u) \geq -\delta u^2 - C_\delta, \quad \forall u \in \mathbb{R}. \tag{34}
$$

Then,

$$
(f(u), u) \geq \mu (\mathcal{F}(u), 1) - \delta \|u\|_2^2 - C_\delta |\Omega|,
$$

and

$$
\frac{d}{dt} (\gamma \epsilon \|u\|_2^2 + 2 \epsilon (\partial_t u, u)) + 2 (1 - \frac{\epsilon}{\lambda_1}) \epsilon \|\nabla u\|_2^2 - 2 \epsilon \|\partial_t u\|_2^2 + 2 \epsilon (\mathcal{F}(u), 1) \\
\leq 2 \epsilon (g(t, u_t), u) + 2 \epsilon (k(t), u) + 2 \epsilon C_\delta |\Omega|, \tag{35}
$$

Furthermore, by the Hölder and Young inequalities, we have

$$
\frac{d}{dt} (\gamma \epsilon \|u\|_2^2 + 2 \epsilon (\partial_t u, u)) + (2 - \frac{3 \delta}{\lambda_1}) \epsilon \|\nabla u\|_2^2 - 2 \epsilon \|\partial_t u\|_2^2 + 2 \epsilon (\mathcal{F}(u), 1) \\
\leq \frac{2 \epsilon}{\delta} \|g(t, u_t)\|_2^2 + \frac{2 \epsilon}{\delta} \|k(t)\|_2^2 + 2 \epsilon C_\delta |\Omega|, \tag{35}
$$

where the constant $\delta > 0$ is same as that in (34).
Combining with (33) and (35), we obtain that
\[
\frac{d}{dt} (\|\nabla u\|_2^2 + \|\partial_t u\|_2^2 + 2(F(u), 1) + \gamma \epsilon \|u\|_2^2 + \gamma \epsilon \|\partial_t u, u\|_2^2 + 2(\partial_t u, u))
\]
\[
+ (2 - \frac{3\delta}{\lambda_1}) \epsilon \|\nabla u\|_2^2 + (\gamma - 2\epsilon) \|\partial_t u\|_2^2 + 2\epsilon \mu (F(u), 1)
\]
\[
\leq 2(\frac{1}{\gamma} + \frac{\epsilon}{\delta}) \|g(t, u_t)\|_2^2 + 2(\frac{1}{\gamma} + \frac{\epsilon}{\delta}) \|k(t)\|_2^2 + 2\epsilon C_3 |\Omega|. \tag{36}
\]

Denoting \(\xi_u(t) = (u(t), \partial_t u(t))\) and
\[
E(\xi_u(t)) = \|\nabla u\|_2^2 + \|\partial_t u\|_2^2 + 2(F(u), 1) + \gamma \epsilon \|u\|_2^2 + \gamma \epsilon \|\partial_t u, u\|_2^2,
\]
then by (4) we know that there exist constants \(\delta > 0\) (same as that in (34)) and \(C'_3 > 0\) such that
\[
F(u) \geq -\delta u^2 - C'_3, \quad \forall u \in \mathbb{R},
\]
then there exists a constant \(c_1 > 0\) such that
\[
c_1 (\|\nabla u\|_2^2 + \|\partial_t u\|_2^2) - 2C'_3 |\Omega| \leq E(\xi_u(t)) \tag{37}
\]
and
\[
E(\xi_u(t)) \leq (1 + \frac{\gamma + 1}{\lambda_1}) \|\nabla u\|_2^2 + (1 + \epsilon) \|\partial_t u\|_2^2 + 2(F(u), 1). \tag{38}
\]

Taking \(\lambda = \min \{\frac{2\lambda_1 - 3\delta}{\lambda_1 + (\gamma + 1)\epsilon}, \frac{\gamma - 2\epsilon}{\gamma + \epsilon \mu, \frac{\epsilon}{\delta}}\} > 0\), we deduce from (36) and (38) that
\[
\frac{d}{dt} E(\xi_u(t)) + \lambda E(\xi_u(t)) \leq 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \|g(t, u_t)\|_2^2 + 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \|k(t)\|_2^2 + 2\epsilon C_3 |\Omega|. \tag{39}
\]

Therefore, multiplying (39) by \(e^{\lambda t}\), we can get that
\[
\frac{d}{dt} (E(\xi_u(t)) e^{\lambda t}) \leq 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \|g(t, u_t)\|_2^2 e^{\lambda t}
\]
\[
+ 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \|k(t)\|_2^2 e^{\lambda t} + 2\epsilon C_3 |\Omega| e^{\lambda t}. \tag{40}
\]

Integrating (40) in \([\tau, t]\), we have
\[
E(\xi_u(t)) e^{\lambda t} \leq E(\xi_u(t)) e^{\lambda \tau} + 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \int_{\tau}^{t} \|g(s, u_s)\|_2^2 e^{\lambda s} ds
\]
\[
+ 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \int_{\tau}^{t} \|k(s)\|_2^2 e^{\lambda s} ds + 2\epsilon C_3 |\Omega| \int_{\tau}^{t} e^{\lambda s} ds, \quad \forall t \geq \tau.
\]

Thanks to assumptions (II) and (IV), we obtain
\[
E(\xi_u(t)) e^{\lambda t} \leq E(\xi_u(t)) e^{\lambda \tau} + 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) C^2 g \int_{\tau-h}^{t} \|u(s)\|_2^2 e^{\lambda s} ds
\]
\[
+ 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \int_{\tau}^{t} \|k(s)\|_2^2 e^{\lambda s} ds + 2\epsilon C_3 |\Omega| \int_{\tau}^{t} e^{\lambda s} ds, \quad \forall t \geq \tau.
Furthermore, by (37), we can get that
\[
\begin{aligned}
E(\xi_k(t))e^{\lambda t} & \leq \frac{2}{\lambda}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2\|\phi\|_2^2 + \frac{2}{\lambda_1}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2 \int_\tau^t \|\nabla u(s)\|_2^2 e^{\lambda s} ds \\
& \quad + 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \int_\tau^t \|k(s)\|_2^2 e^{\lambda s} ds + 2\epsilon C_\delta |\Omega| \int_\tau^t e^{\lambda s} ds \\
& \leq E(\xi_k(\tau))e^{\lambda \tau} + \frac{2}{\lambda}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2\|\phi\|_2^2 + \frac{2}{\lambda_1}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2 \int_\tau^\tau E(\xi_k(s))e^{\lambda s} ds \\
& \quad + 2\left(\frac{1}{\gamma} + \frac{\epsilon}{\delta}\right) \int_\tau^\tau \|k(s)\|_2^2 e^{\lambda s} ds + 2(\epsilon C_\delta + \frac{1}{\epsilon} C_\delta') |\Omega| \int_\tau^\tau e^{\lambda s} ds, \quad \forall \tau \geq \tau. \quad (41)
\end{aligned}
\]

For brevity, we denote \(C_1 = \frac{2}{\lambda}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2\|\phi\|_2^2, C_2 = \frac{2}{\lambda_1}\left(1 + \frac{\epsilon}{\delta}\right)C_2^2, C_3 = 2\frac{1}{\gamma} + \frac{\epsilon}{\delta}, C_4 = 2(\epsilon C_\delta + \frac{1}{\epsilon} C_\delta') |\Omega|\). Then applying the Gronwall lemma to (41), it yields
\[
E(\xi_k(t))e^{\lambda t} \leq \frac{C_1}{\lambda_\beta} e^{\lambda t} + C_2 e^{\lambda t} + C_3 e^{\lambda t} \int_\tau^t \|k(s)\|_2^2 ds + C_4 e^{\lambda t} \int_\tau^t e^{\lambda t} ds, \quad \forall \tau \geq \tau.
\]

Therefore,
\[
E(\xi_k(t)) \leq E(\xi_k(\tau))e^{-(\lambda - C_2)(t-\tau)} + C_1 e^{-(\lambda - C_2)(t-\tau)} \\
+ \frac{C_3}{\lambda} e^{-(\lambda - C_2)t} \int_\tau^t e^{(\lambda - C_2)s} \|k(s)\|_2^2 ds, \quad \forall \tau \geq \tau.
\]

Setting \(\alpha = \lambda - C_2 > 0\), we have
\[
E(\xi_k(t)) \leq E(\xi_k(\tau))e^{-\alpha(t-\tau)} + C_1 e^{-\alpha(t-\tau)} + \frac{C_3}{\alpha} e^{-\alpha t} \int_\tau^t e^{\alpha s} \|k(s)\|_2^2 ds, \quad (42)
\]
for all \(t \geq \tau\).

In particular, taking \(t + \theta\) instead of \(t\), we complete the proof immediately. \(\Box\)

**Remark 2.** Noticing that \(\alpha = \lambda - C_2 > 0\) in (42) is similar to the corresponding assumption in [18], which, combining with (6), will be used to obtain the existence of pullback \(\mathcal{D}\)-absorbing sets.

In the following, we will establish the global existence of the Shatah-Struwe solutions \(u(t)\) for equation (1). In order to obtain the global existence, we assume further that \(k(\cdot) \in L_2^2(\mathbb{R}; L^2(\Omega))\).

**Theorem 3.6.** Let \(f\) satisfy (2)-(5), \(g(t, u_t)\) be subject to assumptions (I)-(IV), \(k(\cdot) \in L_2^2(\mathbb{R}; L^2(\Omega))\), \(\tau \in \mathbb{R}\) and \(\xi = (\phi, \partial_t \phi) \in C_{H_{\beta}(\Omega)} \times C_{L^2(\Omega)}\). Then there exists a global Shatah-Struwe solution \(u(t)\) for equation (1). Moreover, the following estimate holds:
\[
\left\|\xi_u\right\|_{C_{V,H}} + \|w\|_{L^4(\max\{\tau, t-1\}; L^2(\Omega)))} \leq Q_3\left(\left\|\xi_t\right\|_{C_{V,H}}\right) + Q_3\left(\|k\|_{L_2^2(\mathbb{R}; L^2(\Omega))}\right), \quad \text{for } t \geq \tau,
\]
where \(Q_3(\cdot)\) is an increasing function on \([0, +\infty)\) which depends on \(|\Omega|\), but independent of \(\left\|\xi_t\right\|_{C_{V,H}}\) and \(\|k\|_{L_2^2(\mathbb{R}; L^2(\Omega))}\).

**Proof.** In order to prove the global existence of the Shatah-Struwe solutions, we need to show its \(E\)-norm and \(L^4(\tau, t; L^2(\Omega))\)-norm can not blow up in any finite time.
In fact, by applying (11) to (17) and combining with Lemma 2.4, we have
\[
\|\xi_{v_N}(t)\|_\mathcal{E} + \|v_N\|_{L^4(\tau, t; L^2(\Omega))} \\
\leq C \left( \|\xi_t\|_{C_{v,N}} e^{-\beta(t-\tau)} + \int_\tau^t e^{-\beta(t-s)} \|P_N k(s)\|_2 ds \right) \\
\leq C \left( \|\xi_t\|_{C_{v,N}} + \frac{1}{\sqrt{\beta}} \left( e^{-\beta t} \int_\tau^t e^{\beta s} \|k(s)\|_2^2 ds \right)^{\frac{1}{2}} \right) \\
\leq C \left( \|\xi_t\|_{C_{v,N}} + \frac{1}{\sqrt{\beta}} \left( \frac{1}{1 - e^{-\beta \tau}} \sup_{t \in \mathbb{R}} \int_\tau^{t+1} \|k(s)\|_2^2 ds \right)^{\frac{1}{2}} \right). \tag{44}
\]

On the other hand, by applying (11) to (18), we obtain that
\[
\|\xi_{w_N}(t)\|_\mathcal{E} + \|w_N\|_{L^4(\tau, t; L^2(\Omega))} \\
\leq C \left( \int_\tau^t e^{-\beta(t-s)} \|f(u_N(s)) + g(s, u_{N_s})\|_2 ds \right) \\
\leq C \left( \int_\tau^t e^{-\beta(t-s)} \|f(u_N(s))\|_2 ds + \int_\tau^t e^{-\beta(t-s)} \|g(s, u_{N_s})\|_2 ds \right). \tag{45}
\]

In the following, we will deal with (45) according to \( p \in [1, 3] \) and \( p \in (3, 5) \). When \( p \in [1, 3] \), by (2) and the embedding \( H^1_0(\Omega) \hookrightarrow L^{2p}(\Omega) \), we deduce that
\[
\int_\tau^t e^{-\beta(t-s)} \|f(u_N(s))\|_2 ds \leq \\
\leq C \int_\tau^t e^{-\beta(t-s)} \left( \int_\Omega (1 + |v_N|^{2p} + |w_N|^{2p}) dx \right)^{\frac{1}{2}} ds \\
\leq C |\Omega|^{\frac{1}{2}} + C \int_\tau^t e^{-\beta(t-s)} \left( \int_\Omega |v_N|^{2p} dx \right)^{\frac{1}{2}} ds + C \int_\tau^t e^{-\beta(t-s)} \left( \int_\Omega |w_N|^{2p} dx \right)^{\frac{1}{2}} ds \\
\leq C |\Omega|^{\frac{1}{2}} + C \int_\tau^t e^{-\beta(t-s)} \|\nabla v_N(s)\|_p^p ds + C \int_\tau^t e^{-\beta(t-s)} \|\nabla w_N(s)\|_p^p ds \\
\leq C \left( |\Omega|^{\frac{1}{2}} + \|v_N\|_{L^\infty(\tau, t; H^1_0(\Omega))}^p + \|w_N\|_{L^\infty(\tau, t; H^1_0(\Omega))}^p \right), \tag{46}
\]

and by assumption (III), the Hölder inequality, Lemma 2.4 and (32) in Lemma 3.5, we can get that
\[
\int_\tau^t e^{-\beta(t-s)} \|g(s, u_{N_s})\|_2 ds \leq \\
\leq L_g \int_\tau^t e^{-\beta(t-s)} \|u_{N_s}\|_{C_{l,2}} ds \\
\leq L_g \left( \int_\tau^t e^{-\beta(t-s)} ds \right)^{\frac{1}{2}} \left( \int_\tau^t e^{-\beta(t-s)} \|u_{N_s}\|_{C_{l,2}}^2 ds \right)^{\frac{1}{2}} \\
\leq \frac{L_g}{\sqrt{\beta}} \left( \int_\tau^t e^{-\beta(t-s)} \|u_{N_s}\|_{C_{l,2}}^2 ds \right)^{\frac{1}{2}} \\
\leq \frac{L_g}{\sqrt{\beta}} \left( \frac{1}{\lambda_1} \int_\tau^t e^{-\beta(t-s)} \left( \|\phi\|_{C_{v,u}}^2 + \|\partial_t \phi\|_{C_{h}}^2 + C_1 + \frac{C_4}{\alpha} \right) ds \right)^{\frac{1}{2}}
\]
we can get that by assumption (III), the Hölder inequality, (10), we deduce that

\[
\int_{\tau}^{t} e^{-\beta(s-t)} \|f(u_N(s))\|_2 \, ds \leq \frac{L_g}{\sqrt{\lambda_1}} \left( \left\| \phi \right\|_{C^0}^2 + \|\partial_t \phi\|_{C^0}^2 + C_1 + \frac{C_4}{\alpha} + \frac{C_3 e^{\alpha h}}{1 - e^{-\alpha}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 \, ds \right)^{\frac{1}{2}}.
\]

When \( p \in (3,5) \), by (2), the Hölder inequality and the interpolation inequality (10), we deduce that

\[
\int_{\tau}^{t} e^{-\beta(s-t)} \|f(u_N(s))\|_2 \, ds \leq \frac{C}{\|f\|_p} \left( \int_{\tau}^{t} \left( 1 + |u_N|^{2p} + |w_N|^{2p} \right) dx \right)^{\frac{1}{2}} \leq C |\Omega|^{\frac{1}{2}} \left( t - \tau \right) + C \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |u_N|^{2p} dx \right)^{\frac{1}{2}} \, ds + C \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |w_N|^{2p} dx \right)^{\frac{1}{2}} \, ds
\]

\[
\leq C |\Omega|^{\frac{1}{2}} \left( t - \tau \right) + C |\Omega|^{\frac{5-p}{2p}} \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |u_N|^{10} dx \right)^{\frac{1}{10}} \, ds + C |\Omega|^{\frac{5-p}{2p}} \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |w_N|^{10} dx \right)^{\frac{1}{10}} \, ds
\]

\[
\leq C |\Omega|^{\frac{1}{2}} \left( t - \tau \right) + C |\Omega|^{\frac{5-p}{2p}} \left( \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |u_N|^{10} dx \right)^{\frac{1}{10}} \, ds \right)^{\frac{1}{2}} \left( \int_{\tau}^{t} \left( \int_{\Omega} |u_N|^{10} dx \right)^{\frac{1}{10}} \, ds \right)^{\frac{1}{2}}
\]

\[
\leq C |\Omega|^{\frac{1}{2}} \left( t - \tau \right) + C |\Omega|^{\frac{5-p}{2p}} \left( \int_{\tau}^{t} e^{-\beta(s-t)} \left( \int_{\Omega} |u_N|^{10} dx \right)^{\frac{1}{10}} \, ds \right)^{\frac{1}{2}} \left( \int_{\tau}^{t} \left( \int_{\Omega} |u_N|^{10} dx \right)^{\frac{1}{10}} \, ds \right)^{\frac{1}{2}}
\]

and by assumption (III), the Hölder inequality, Lemma 2.4 and (32) in Lemma 3.5, we can get that

\[
\int_{\tau}^{t} e^{-\beta(s-t)} \|g(s,u_{N,s})\|_2 \, ds \leq \frac{L_g}{\sqrt{\lambda_1}} \left( \left\| \phi \right\|_{C^0}^2 + \|\partial_t \phi\|_{C^0}^2 + C_1 + \frac{C_4}{\alpha} + \frac{C_3 e^{\alpha h}}{1 - e^{-\alpha}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 \, ds \right)^{\frac{1}{2}}.
\]

\[
\leq \frac{L_g}{\sqrt{\lambda_1}} \left( \left\| \phi \right\|_{C^0}^2 + \|\partial_t \phi\|_{C^0}^2 + C_1 + \frac{C_4}{\alpha} + \frac{C_3 e^{\alpha h}}{1 - e^{-\alpha}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 \, ds \right)^{\frac{1}{2}}.
\]
Combining with (45)-(49), we have

$$
\|w_N\|_{L^1(\tau,t;L^2(\Omega))} \leq \\
\leq \frac{C}{\beta} \left( |\Omega|^{\frac{1}{2}} + \|v_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}^p + \|w_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}^p \right) \\
+ \frac{L_g}{\sqrt{\lambda_1}} \left( \|\phi\|^2_{C^\infty_V} + \|\partial_t \phi\|^2_{C^\infty_H} + C_1 + C_4 \alpha \right) \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}},
$$

for all $p \in [1, 3]$, and

$$
\|w_N\|_{L^1(\tau,t;L^2(\Omega))} \leq \\
\leq C |\Omega|^{\frac{1}{2}} (t - \tau) + C |\Omega|^{\frac{\lambda_1}{\beta}} (t - \tau)^{1 - \frac{\beta}{2}} \|v_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}^p \|w_N\|_{L^1(\tau,t;L^2(\Omega))}^p \\
+ \frac{L_g}{(\lambda_1)^{\frac{1}{2}}} (t - \tau)^{\frac{1}{2}} \left( \|\phi\|^2_{C^\infty_V} + \|\partial_t \phi\|^2_{C^\infty_H} + C_1 + C_4 \alpha \right) \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}} \\
+ C |\Omega|^{\frac{\lambda_1}{\beta}} \left( \frac{1}{\beta} \right)^{1 - \frac{\beta}{2}} \|w_N\|_{L^\infty(\tau,t;H^1_0(\Omega))} \|w_N\|_{L^1(\tau,t;L^2(\Omega))}^p,
$$

for all $p \in (3, 5)$.

Therefore, combining with (44) and (50), we obtain that

$$
\|u_N\|_{L^1(\tau,t;L^2(\Omega))} \leq \\
\leq \frac{C}{\sqrt{\beta}} \left( \int_{t}^{t+1} \frac{1}{1 - e^{-\alpha}} \sup_{t \in \mathbb{R}} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}} \\
+ \frac{C}{\beta} \left( \|\xi\|^2_{C^\infty_V} + \|v_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}^p + \|w_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}^p \right) \\
+ \frac{C}{\sqrt{\lambda_1}} \left( \|\phi\|^2_{C^\infty_V} + \|\partial_t \phi\|^2_{C^\infty_H} + C_1 + C_4 \alpha \right) \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}},
$$

for all $p \in [1, 3]$.

Denote by $C_T = \frac{L_g}{(\lambda_1)^{\frac{1}{2}}} \left( \|\phi\|^2_{C^\infty_V} + \|\partial_t \phi\|^2_{C^\infty_H} + C_1 + C_4 \alpha \right) \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}} \\
+ C |\Omega|^{\frac{\lambda_1}{\beta}} \left( \frac{1}{\beta} \right)^{1 - \frac{\beta}{2}} \|u_N\|_{L^\infty(\tau,t;H^1_0(\Omega))} \left( \|\xi\|^2_{C^\infty_V} + \frac{1}{\beta} \left( \frac{1}{1 - e^{-\alpha}} \sup_{t \in \mathbb{R}} \int_{t}^{t+1} \|k(s)\|^2_2 ds \right)^{\frac{1}{2}} \right)^{\frac{4}{\beta}}$, $C_s = C |\Omega|^{\frac{\lambda_1}{\beta}} \left( \frac{1}{\beta} \right)^{1 - \frac{\beta}{2}} \|w_N\|_{L^\infty(\tau,t;H^1_0(\Omega))}$ in (51), then

$$
\|w_N\|_{L^1(\tau,t;L^2(\Omega))} \leq C_T (t - \tau)^{1 - \frac{\beta}{2}} + C_s \|w_N\|_{L^1(\tau,t;L^2(\Omega))},
$$

for any $t - \tau \in (0, 1)$ and $p \in (3, 5)$.

Setting $T_0 = t - \tau$, then for any $T_0 \in (0, 1)$, we can derive the above estimate (53) on shifted intervals of size $t - t_0 \leq T_0$ ($t_0 = \max\{\tau, t - T_0\}$) as following

$$
\|w_N\|_{L^1(t_0,t;L^2(\Omega))} \leq C_T (t - \tau)^{1 - \frac{\beta}{2}} + C_s \|w_N\|_{L^1(t_0,t;L^2(\Omega))}.
$$

By choosing $\varepsilon > 0$ small enough such that $\sigma_0 = \frac{\varepsilon}{C_T^{\frac{1}{\beta - 1}}} < 1$, then applying Lemma 2.3 to (54), we have

$$
\|w_N\|_{L^1(\max\{\tau,t - \sigma_0\},t;L^2(\Omega))} \leq 2C_T \sigma_0^{\frac{4}{\beta}}.
$$
Denote \( \tau(t) = \max\{\tau, t - 1\} \), \( N_0 = \left[ \frac{1}{\sigma_0} \right] \), then
\[
\|w_N\|_{L^1(\tau(t), t; L^2(\Omega))} \leq \sum_{i=0}^{N_0-1} \|w_N\|_{L^1(\tau(t+i\sigma_0), \tau(t+(i+1)\sigma_0); L^2(\Omega))} \\
+ \|w_N\|_{L^1(\tau(t+N_0\sigma_0), t; L^2(\Omega))} \\
\leq 2C_7\alpha^{\frac{3-p}{p}} \left(1 + \frac{1}{\sigma_0}\right) \\
= 2\varepsilon^{\frac{3-p}{p}} \left(1 + \frac{1}{\sigma_0}\right) \\
\leq 4\varepsilon^{\frac{3-p}{p}} \frac{1}{\sigma_0} \\
\leq 4\varepsilon^{\frac{3-p}{p}} C_7^{5/(5-p)} \\
\leq Q_2(|\xi_\tau|_{C_{V,H}}) + Q_3(\|k\|_{L^4(\mathbb{R}; L^2(\Omega))}^2).
\]

Therefore, combining with (52), (44) and (56), we obtain that
\[
\|u_N\|_{L^1(\tau(t), t; L^2(\Omega))} \leq Q_3(|\xi_\tau|_{C_{V,H}}) + Q_3(\|k\|_{L^4(\mathbb{R}; L^2(\Omega))}^2)
\]
for all \( p \in [1, 5) \).

On the other hand, according to the dissipative estimate obtained in Lemma 3.5, we know that the energy norm \( \|\xi_{u_t}\|_{C_{V,H}} \) does not blow up in any finite time.

Therefore, \( \|\xi_{u_t}\|_{C_{V,H}} \) and \( \|u\|_{L^1(\tau(t); L^2(\Omega))} \) can not blow up in any finite time, which shows the global existence of the Shatah-Struwe solutions.

\[ \square \]

4. **Pullback attractors.** In this section, we will prove the existence of the pullback \( \mathcal{D} \)-attractors in \( \mathcal{C}_{H_1^1(\Omega)} \times \mathcal{C}_{L^2(\Omega)} \) for the process \( \{U(t, \tau)\}_{t \geq \tau} \).

4.1. **Pullback \( \mathcal{D}_\alpha \)-absorbing sets.** In this subsection, we will obtain the pullback \( \mathcal{D} \)-absorbing sets in \( \mathcal{C}_{H_1^1(\Omega)} \times \mathcal{C}_{L^2(\Omega)} \) for the process \( \{U(t, \tau)\}_{t \geq \tau} \).

Now, we give the following definition.

**Definition 4.1.** For any \( \alpha > 0 \), we will denote by \( \mathcal{D}_\alpha \) the class of all families of nonempty subsets \( \bar{D} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{P}(C_{V,H}) \) such that
\[
\lim_{\tau \to -\infty} \left( e^{\alpha\tau} \sup_{u \in D(\tau)} \|u\|_{C_{V,H}}^2 \right) = 0.
\]

**Corollary 1.** Under the assumptions of Lemma 3.5, if moreover (6) and (42) are satisfied, then, the family \( \bar{D}_0 = \{D_0(t) : t \in \mathbb{R}\} \) with \( D_0(t) = \overline{B}_{C_{V,H}}(0, \rho(t)) \), the closed ball in \( C_{V,H} \) of center zero and radius \( \rho(t) \), where
\[
\rho^2(t) = 1 + \frac{C_4}{\alpha} + C_5 e^{-\alpha(t-h)} \int_{-\infty}^{t} e^{\alpha s} \|k(s)\|^2_2 ds,
\]

is pullback \( \mathcal{D}_\alpha \)-absorbing for the process \( \{U(t, \tau)\}_{t \geq \tau} \). Moreover, \( \bar{D}_0 \in \mathcal{D}_\alpha \).

**Proof.** That \( \bar{D}_0 \) is pullback \( \mathcal{D}_\alpha \)-absorbing for the process \( \{U(t, \tau)\}_{t \geq \tau} \) is a immediate consequence of (32) in Lemma 3.5.

Thanks to (58), we have \( e^{\alpha t} \rho^2(t) \to 0 \) as \( t \to -\infty \). Then \( \bar{D}_0 \) belongs to \( \mathcal{D}_\alpha \). \[ \square \]

4.2. **Pullback \( \mathcal{D}_\alpha \)-attractors.** In this subsection, we will prove the existence of the pullback \( \mathcal{D}_\alpha \)-attractors in \( \mathcal{C}_{H_1^1(\Omega)} \times \mathcal{C}_{L^2(\Omega)} \) for the process \( \{U(t, \tau)\}_{t \geq \tau} \).
4.2.1. A priori estimates. The main object of this subsection is to establish the energy inequalities (66) and (67), which will be used to obtain the necessary asymptotic compactness.

Let \((u^i(t), \partial_t u^i(t))\) be the Shatah-Struwe solution of equation (1) corresponding to initial data \((\phi^i(x, t - \tau), \partial_t \phi^i(x, t - \tau)) \in D_0(\tau) \times D_0(\tau)\) \((i = 1, 2, t \in [\tau - h, \tau])\). Denote \(w(t) = u^1(t) - u^2(t)\), then \(w(t)\) satisfies the equation

\[
\partial_t w + \gamma \partial_x w - \Delta w + f(u^1) - f(u^2) = g(t, u^1) - g(t, u^2) \quad \text{in} \quad \Omega \times (\tau, \infty),
\]

(59)

with initial data

\[(w(x, t), \partial_t w(x, t)) = (\phi^1(x, t - \tau) - \phi^2(x, t - \tau), \partial_t \phi^1(x, t - \tau) - \partial_t \phi^2(x, t - \tau))\]

for any \(x \in \Omega\) and \(t \in [\tau - h, \tau]\).

We also define an energy functional

\[E_w(t) = \frac{1}{2} \int_{\Omega} (|\nabla w(t)|^2 + |\partial_t w(t)|^2) dx.\]

Then, similar to the proof process of Theorem 4.5 in [43], we have

\[
E_w(t) \leq \frac{6}{\gamma(t - \tau)} e^{-\alpha(t-\tau)} E_w(\tau) + \frac{C_2}{\alpha} \left( \frac{1}{2\gamma} + \frac{2 + \alpha}{2\lambda_1} + \frac{3}{\gamma^2(t - \tau)} \right) \|\phi^1 - \phi^2\|_2 e^{-\alpha(t-\tau)} \]

\[
- \left( \frac{2}{t - \tau} + \alpha \right) (\partial_t w(t), w(t)) + \left( \frac{3\alpha - 2\gamma}{t - \tau} \right) e^{-\alpha t} \int_{\tau}^{t} e^{\alpha s} (\partial_t w(s), w(s)) ds
\]

\[
+ \frac{\alpha(\alpha - \gamma)}{(t - \tau)} e^{-\alpha t} \int_{\tau}^{t} \int_{s}^{t} e^{\alpha r} (\partial_t w(r), w(r)) dr ds + \frac{2}{t - \tau} e^{-\alpha(t-\tau)} \left( \partial_t w(\tau), w(\tau) \right)
\]

\[
+ \frac{6}{\gamma(t - \tau)} e^{-\alpha t} \int_{\tau}^{t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) \partial_t w(r) dx dr
\]

\[
+ \frac{1}{t - \tau} e^{-\alpha t} \int_{\tau}^{t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) \partial_t w(r) dx dr ds
\]

\[
+ \frac{\alpha}{t - \tau} e^{-\alpha t} \int_{\tau}^{t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) w(r) dx dr ds
\]

\[
+ \frac{2}{t - \tau} e^{-\alpha t} \int_{\tau}^{t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) w(r) dx dr
\]

\[
+ C_2 \left( \frac{1}{2\gamma} + \frac{2 + \alpha}{2\lambda_1} + \frac{3}{\gamma^2(t - \tau)} \right) \int_{\tau}^{t} \|u^1(r) - u^2(r)\|_2^2 dr.
\]

(60)

In the following, we will deal with the terms on the right of (60). At first, by (2) and the Hölder inequality, we have

\[
\frac{2}{t - \tau} e^{-\alpha t} \int_{\tau}^{t} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) w(r) dx dr
\]

\[
\leq \frac{2}{t - \tau} e^{-\alpha t} \left( \int_{\Omega} |f(u^1(r)) - f(u^2(r))|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |w(r)|^2 dx \right)^{\frac{1}{2}} dr
\]
\[ \leq \frac{2C}{t-\tau} e^{-at} \left( \int_{\tau}^{t} e^{\alpha r} \left| \int_{\Omega} (1 + |u^1(r)|^{2p} + |u^2(r)|^{2p}) dr \right| \right)^{\frac{\alpha}{\alpha-1}} \times \left( \int_{\tau}^{t} e^{\alpha r} \left| \int_{\Omega} |w(r)|^2 dx \right|^{\frac{5}{5-\alpha}} dr \right)^{\frac{5-\alpha}{5}} \]

\[ \leq \frac{2C}{t-\tau} \left( \frac{\Omega^\frac{5}{\alpha}}{\alpha} + \Omega \frac{5-\alpha}{5} \int_{\tau}^{t} (\|u^1(r)\|^5_{L^\infty(\Omega)} + \|u^2(r)\|^5_{L^\infty(\Omega)}) dr \right)^{\frac{\alpha}{\alpha}} \times \left( \int_{\tau}^{t} \|w(r)\|^2_{\frac{5}{5-\alpha}} dr \right)^{\frac{5-\alpha}{5}} . \] 

(61)

Secondly,

\[ \frac{\alpha}{t-\tau} e^{-at} \int_{\tau}^{t} \int_{s}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) w(r) dx \, dr \, ds \leq \alpha e^{-at} \int_{\tau}^{t} e^{\alpha r} \left( \int_{\Omega} |f(u^1(r)) - f(u^2(r))|^2 dx \right)^{\frac{1}{2}} \left( \int_{\Omega} e^{\alpha r} |w(r)|^2 dx \right)^{\frac{1}{2}} dr \leq C \alpha e^{-at} \left( \int_{\tau}^{t} e^{\alpha r} \left| \int_{\Omega} (1 + |u^1(r)|^{2p} + |u^2(r)|^{2p}) dx \right| \right)^{\frac{\alpha}{\alpha-1}} \times \left( \int_{\tau}^{t} e^{\alpha r} \left| \int_{\Omega} |w(r)|^2 dx \right|^{\frac{5}{5-\alpha}} dr \right)^{\frac{5-\alpha}{5}} \leq C \alpha \left( \frac{\Omega^\frac{5}{\alpha}}{\alpha} + \Omega \frac{5-\alpha}{5} \int_{\tau}^{t} (\|u^1(r)\|^5_{L^\infty(\Omega)} + \|u^2(r)\|^5_{L^\infty(\Omega)}) dr \right)^{\frac{\alpha}{\alpha}} \times \left( \int_{\tau}^{t} \|w(r)\|^2_{\frac{5}{5-\alpha}} dr \right)^{\frac{5-\alpha}{5}} . \] 

(62)

By the definition of the Shatah-Struwe solutions and Remark 1, we know that

\[ \left( \frac{\Omega^\frac{5}{\alpha}}{\alpha} + \Omega \frac{5-\alpha}{5} \int_{\tau}^{t} (\|u^1(r)\|^5_{L^\infty(\Omega)} + \|u^2(r)\|^5_{L^\infty(\Omega)}) dr \right)^{\frac{\alpha}{\alpha}} \leq C_{t,\tau} < \infty, \] 

(63)

for all \((\phi, \partial_t \phi) \in D_0(\tau) \times D_0(\tau)\), where \(C_{t,\tau}\) depends on \(t, \tau, \alpha\) and \(|\Omega|\), \(D_0(\tau)\) is given by Corollary 1.

Moreover, by the Hölder inequality and (58) in Corollary 1, we can get

\[ e^{-at} \int_{\tau}^{t} e^{\alpha s} (\partial_t w(s), w(s)) ds \leq \left( e^{-at} \int_{\tau}^{t} e^{\alpha s} |\partial_t w(s)|^2 ds \right)^{\frac{1}{2}} \left( e^{-at} \int_{\tau}^{t} e^{\alpha s} |w(s)|^2 ds \right)^{\frac{1}{2}} \leq C_{\frac{1}{2}, \rho(t)} \left( e^{-at} \int_{\tau}^{t} e^{\alpha s} |w(s)|^2 ds \right)^{\frac{1}{2}} \leq C_{\frac{1}{2}, \rho(t)} \left( \int_{\tau}^{t} |w(s)|^2 ds \right)^{\frac{1}{2}} , \] 

(64)
\[
e^{-\alpha t} \int_{\tau}^{t} \int_{s}^{t} e^{\alpha r} (\partial_t w(r), w(r)) dr ds \leq \]
\[
\leq (t - \tau) \left( e^{-\alpha t} \int_{\tau}^{t} e^{\alpha r} \| \partial_t w(r) \|_2^2 dr \right)^{\frac{1}{2}} \left( e^{-\alpha t} \int_{\tau}^{t} e^{\alpha r} \| w(r) \|_2^2 dr \right)^{\frac{1}{2}} \]
\[
\leq C \frac{1}{\gamma} \psi(t - \tau) \left( \int_{\tau}^{t} \| w(r) \|_2^2 dr \right)^{\frac{1}{2}}.
\]
Combining with (60)-(65), we set \( T_1 = t - \tau \) and
\[
\psi_{t, T_1}((\phi^1, \partial_t \phi^1), (\phi^2, \partial_t \phi^2)) = \]
\[
= \left( \frac{2}{t - \tau} + \alpha \right) (\partial_t w(t), w(t)) + (2 + \alpha) \tilde{C}_{t, \tau} \left( \int_{\tau}^{t} \| w(r) \|_2^2 dr \right)^{\frac{1}{2}} \]
\[
+ C_1 \left( \frac{2}{t - \tau} + \alpha \right) \left( \int_{\tau}^{t} \| w(r) \|_2^2 dr \right)^{\frac{1}{2}} \]
\[
+ C_2 \left( \frac{1}{t - \tau} + \frac{2}{t - \tau} \frac{\alpha}{2\lambda_1} + \frac{3}{\gamma^2 (t - \tau)} \right) \int_{\tau}^{t} \| u^1(r) - u^2(r) \|_2^2 dr \]
\[
+ \frac{6}{\gamma (t - \tau)} e^{-\alpha t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) \partial_t w(r) dx dr \]
\[
+ \frac{1}{t - \tau} e^{-\alpha t} \int_{\tau}^{t} e^{\alpha r} \int_{\Omega} (f(u^2(r)) - f(u^1(r))) \partial_t w(r) dx dr ds,
\]
where \( \tilde{C}_{t, \tau} \) is related to \( C \) in (61) and \( C_{t, \tau} \) in (63). Therefore,
\[
E_{w_t} \leq \frac{6}{\gamma (t - \tau)} e^{-\alpha (t - \tau)} E_{w_t} + \frac{2}{t - \tau} e^{-\alpha (t - \tau)} (\partial_t w(\tau), w(\tau)) \]
\[
+ C_2 \left( \frac{1}{\gamma} + \frac{2 + \alpha}{2\lambda_1} + \frac{3}{\gamma^2 (t - \tau)} \right) \| \phi^1 - \phi^2 \|_2^2 e^{-\alpha (t - \tau)} \]
\[
+ \psi_{t, T_1}((\phi^1, \partial_t \phi^1), (\phi^2, \partial_t \phi^2)).
\]

Hence, we can select a \( \tau_0 = \tau_0(t, D_0, \varepsilon) \) small enough such that
\[
E_{w_t} \leq \varepsilon + \psi_{t, T_1}((\phi^1, \partial_t \phi^1), (\phi^2, \partial_t \phi^2))
\]
for all \((\phi^i, \partial_t \phi^i) \in D_0(\tau) \times D_0(\tau)\).

4.2.2. Pullback asymptotic compactness. Next, we have the following theorem, which shows that the process \( \{U(t, \tau)\}_{t \geq \tau} \) is pullback \( D_\alpha \)-asymptotically compact in \( C_{H^2(\Omega)} \times C_{L^2(\Omega)} \).

Theorem 4.2. Let \( f \) satisfy (2)-(5), \( g(t, u_t) \) be subject to assumptions (I)-(IV), \( k(\cdot) \in L_{loc}^2(\mathbb{R}; L^2(\Omega)) \), \( \tau \in \mathbb{R} \) and \( \xi_\tau = (\phi, \partial_t \phi) \in C_{H^2(\Omega)} \times C_{L^2(\Omega)} \). Then the process \( \{U(t, \tau)\}_{t \geq \tau} \) is pullback \( D_\alpha \)-asymptotically compact in \( C_{H^2(\Omega)} \times C_{L^2(\Omega)} \).

Proof. Due to Lemma 2.9 and Theorem 3.6, we only need to verify that the function \( \psi_{t, T_1}((\phi^1, \partial_t \phi^1), (\phi^2, \partial_t \phi^2)) \) defined by (66) is a contractive function on \( D_0(\tau) \times D_0(\tau) \).
Let \((u^m, \partial_t u^m)\) be the Shatah-Struwe solutions of equation (1) corresponding to initial data \((\phi^m, \partial_t \phi^m) \in D_0(\tau) \times D_0(\tau) \ (m = 1, 2, \cdots)\). For any fixed \(t \in \mathbb{R}\), since \(D_0(t) \times D_0(t)\) is bounded in \(C_{H^1} \times C_{L^2}\), then
\[
\|(u^m(s), \partial_t u^m(s))\|_{C_{H^1} \times \mathcal{C}_{L^2}} \leq C_{t, \tau}' < +\infty \quad \text{for all } s \in [\tau, t] \text{ and } m \in \mathbb{N},
\]
where \(C_{t, \tau}'\) depends on \(t, \tau, \alpha, h \) and \(C_3, C_4\) in (32).

Then, for any \(p \in [1, 5]\), without loss of generality, we assume that
\[
\begin{align*}
&u^m \to u \quad \text{weak-star in } L^\infty(\tau, t; H^1_0(\Omega)), \\
&u^m \to u \quad \text{in } L^{p+1}(\tau, t; L^{p+1}(\Omega)), \\
&\partial_t u^m \to \partial_t u \quad \text{weak-star in } L^\infty(\tau, t; L^2(\Omega)), \\
&u^m \to u \quad \text{in } L^2(\tau, t; L^2(\Omega)) \text{ and } L^\infty(\tau, t; L^2(\Omega)),
\end{align*}
\]
where we used the compact embeddings \(H^1_0(\Omega) \hookrightarrow L^{p+1}(\Omega)\).

Now, we will deal with the terms in (66) one by one.

Firstly, from (71), we get that
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_\tau^t \|u^m(r) - u^m(r)\|_2^2 \, dr = 0
\]
(73)
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_\tau^t \|u^m(r) - u^m(r)\|_2^2 \, dr = 0.
\]
(74)

Secondly,
\[
\begin{align*}
\int_\tau^t e^{\alpha r} \int_\Omega (f(u^n(r)) - f(u^m(r))) (\partial_t u^n(r) - \partial_t u^m(r)) \, dx \, dr \\
= & \int_\tau^t e^{\alpha r} \int_\Omega f(u^n(r)) \partial_t u^n(r) \, dx \, dr \\
&+ \int_\tau^t e^{\alpha r} \int_\Omega f(u^m(r)) \partial_t u^m(r) \, dx \, dr \\
&- \int_\tau^t e^{\alpha r} \int_\Omega f(u^n(r)) \partial_t u^m(r) \, dx \, dr \\
&- \int_\tau^t e^{\alpha r} \int_\Omega f(u^m(r)) \partial_t u^n(r) \, dx \, dr \\
= & e^{\alpha t} \int_\Omega F(u^n(t)) - e^{\alpha t} \int_\Omega F(u^m(t)) + e^{\alpha t} \int_\Omega F(u^n(\tau)) - e^{\alpha t} \int_\Omega F(u^m(\tau)) \\
&- \int_\tau^t e^{\alpha r} \int_\Omega f(u^n(r)) \partial_t u^m(r) \, dx \, dr \\
&- \int_\tau^t e^{\alpha r} \int_\Omega f(u^m(r)) \partial_t u^n(r) \, dx \, dr \\
&- \alpha \int_\tau^t e^{\alpha r} \int_\Omega \mathcal{F}(u^n(r)) \, dx \, dr \\
&- \alpha \int_\tau^t e^{\alpha r} \int_\Omega \mathcal{F}(u^m(r)) \, dx \, dr.
\end{align*}
\]
(75)

Since \(f(u^m)\) is weakly convergent to \(f(u)\) in \(L^1(\tau, t; L^2(\Omega))\), by (2), (69), (70), (72), (75), and similar to the proof as that in (75), we can deduce that
\[
\begin{align*}
\lim_{n \to \infty} \lim_{m \to \infty} \int_\tau^t e^{\alpha r} \int_\Omega (f(u^n(r)) - f(u^m(r))) (\partial_t u^n(r) - \partial_t u^m(r)) \, dx \, dr \\
= & e^{\alpha t} \int_\Omega F(u(t)) - e^{\alpha t} \int_\Omega F(u(\tau)) + e^{\alpha t} \int_\Omega F(u(t)) - e^{\alpha t} \int_\Omega F(u(\tau))
\end{align*}
\]
\[-\int_t^\tau e^{\alpha r} \int_\Omega f(u(r))\partial_t u(r)dxdr - \int_t^\tau e^{\alpha r} \int_\Omega f(u(r))\partial_r u(r)dxdr \]
\[-\alpha \lim_{n \to \infty} \int_t^\tau e^{\alpha r} \int_\Omega F(u^n(r))dxdr - \alpha \lim_{n \to \infty} \int_t^\tau e^{\alpha r} \int_\Omega F(u^m(r))dxdr \]
\[= \alpha \int_t^\tau e^{\alpha r} \int_\Omega F(u(r))dxdr - \alpha \lim_{n \to \infty} \int_t^\tau e^{\alpha r} \int_\Omega F(u^n(r))dxdr \]
\[+ \alpha \int_t^\tau e^{\alpha r} \int_\Omega F(u(r))dxdr - \alpha \lim_{n \to \infty} \int_t^\tau e^{\alpha r} \int_\Omega F(u^m(r))dxdr \]
\[= 0. \quad (76)\]

Similarly, since \(|\int_\tau^t e^{\alpha r} \int_\Omega (f(u^n(r)) - f(u^m(r)))\partial_t u^n(r) - \partial_t u^m(r)dxdr|\) is bounded for each \(s \in [\tau, t]\), then by (76) and the Lebesgue dominated convergence theorem, we have
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_t^\tau \int_\mathbb{R}^n e^{\alpha r} \int_\Omega (f(u^n(r)) - f(u^m(r)))\partial_t u^n(r) - \partial_t u^m(r)dxdrds \]
\[= \int_t^\tau \left( \lim_{n \to \infty} \lim_{m \to \infty} \int_\mathbb{R}^n e^{\alpha r} \int_\Omega (f(u^n(r)) - f(u^m(r)))\partial_t u^n(r) - \partial_t u^m(r)dxdr \right) ds \]
\[= \int_t^\tau 0ds = 0. \quad (77)\]

Combining (73)-(77), we know that \(\psi_{1, r}(\cdot, \cdot)\) is a contractive function on \(D_0(\tau) \times D_0(\tau)\). Then, by taking \(t + \theta\) instead of \(t\) in (66), we conclude that the process \(\{U(t, \tau)\}_{t \geq \tau}\) is pullback \(D_\alpha\)-asymptotically compact in \(C_{H^1(\Omega)} \times C_{L^2(\Omega)}\).

Combining with (32) in Lemma 3.5, Corollary 1, Theorems 2.10 and 4.2, we have the main result of this paper.

**Theorem 4.3.** Let \(f\) satisfy (2)-(5), \(g(t, u_t)\) be subject to assumptions (I)-(IV), \(k(\cdot) \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))\), \(\tau \in \mathbb{R}\) and \(\xi_\tau = (\phi, \partial_t \phi) \in C_{H^1(\Omega)} \times C_{L^2(\Omega)}\). If moreover (6) and (42) are satisfied, then the process \(\{U(t, \tau)\}_{t \geq \tau}\) possesses a pullback \(D_\alpha\)-attractor \(\mathcal{A} = \{\mathcal{A}(t): t \in \mathbb{R}\}\) in \(C_{H^1(\Omega)} \times C_{L^2(\Omega)}\); that is, \(\mathcal{A}(t)\) is compact, \(\mathcal{A}\) is nonempty, invariant in \(C_{H^1(\Omega)} \times C_{L^2(\Omega)}\) and pullback attracts every set \(\hat{D} \in D_\alpha\) with respect to the \(C_{H^1(\Omega)} \times C_{L^2(\Omega)}\)-norm.

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