ON $q$-ALGEBRAIC EQUATIONS 
AND THEIR POWER SERIES SOLUTIONS

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Abstract. We study the existence of formal power series solutions to $q$-algebraic equations. When a solution exists, we give a sufficient condition on the equation for this solution to have a positive radius of convergence. We emphasize on the case where the solution is divergent, giving a sharp estimate on the growth of the coefficients. As a consequence, we obtain a bound on the $q$-Gevrey order of the formal solution, which is optimal in some cases. Various examples illustrate our main results.

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1. Introduction. The purpose of this paper is to develop a theory of $q$-algebraic equations and their power series solutions, meaning the following: a formal power series $f(z) = \sum_{n \geq 0} f_n z^n$ with complex coefficients $(f_n)$ is $q$-algebraic if there exists a complex polynomial of several variables $P(z, z_0, z_1, \ldots, z_k)$ and a complex number $q$ such that

$$P(z, f(z), f(qz), \ldots, f(q^k z)) = 0.$$  \hspace{1cm} (1.1)

Since the coefficients of the polynomial are free parameters in this paper, they may in fact depend on $q$ in any fashion that one wishes. These $q$-algebraic equations occur in various areas, including combinatorics, dynamical systems, knot theory, and mathematical physics.

There is no loss of generality in assuming that the polynomial $P$ is irreducible. However, in some applications, such as in combinatorics, one may derive a $q$-algebraic equation for a generating function, and the polynomial $P$ may not be irreducible.

Some important problems on $q$-algebraic equations parallel those in differential equations, and loosely speaking concern the existence, uniqueness, and some form of regularity of the solutions, all three problems being in terms of formal power series which may or may not be convergent. In this setting, the regularity problem is to be interpreted as that of the asymptotic behavior of the coefficients of the formal solutions. We will address these issues and emphasize on examples involving divergent power series solutions.
Before putting our results in context, their flavor is illustrated on the equation

\[
2f(z) = 4z^3 f(q^6z) + 5q^2 z^6 f(z) f(q^9 z) f(q^{10} z) \\
+ 18q^4 z^7 f(q^{14} z)^2 + 9z^{10} f(q^{-3} z) f(q^5 z) f(q^{14} z) f(q^{16} z) \\
+ 3z^{14} f(q^{-5} z) + (q^8 + 2q^{17}) z^{14} f(z) \\
+ 72z^{14} f(z) f(q^3 z) f(q^5 z) + 1 + 15z^7.
\]  

(1.2)

From one of our results and a few lines of simple calculations it follows that whenever \( q \) is a complex number of modulus greater than 1, for some positive constant \( c_q \) and for the positive root \( R_q \) of the equation \( z^3 + 9q^{-24} z^7 = 1 \),

\[
f_n \sim c_q q^{n(n-3)} R_q^{-n}
\]  

(1.3)

as \( n \) tends to infinity. The main interest of such formula is that \( c_q \) does not depend on \( n \), so that the asymptotic growth of the sequence \( (f_n) \) is entirely captured by the term \( q^{n(n-3)} R_q^{-n} \).

The significance of our results is related to the many applications of \( q \)-functional equations, and a section of the paper is devoted to examples coming from combinatorics, dynamical systems, knot theory and statistical physics.

One of our results on the growth of \( (f_n) \) is reminiscent of Maillet’s (1903) theorem on formal power series solutions of algebraic differential equations. Maillet’s result was extended to analytic non-linear differential equations by Malgrange (1989), and though our technique allows us to extend our result to some \( q \)-analytic functional equations, the needed assumptions on the analytic function are not particularly sightly and we choose to restrict our exposition to \( q \)-algebraic equations, a setting which seems suitable in most applications.

The Maillet-Malgrange theorem was extended to \( q \)-analytic functional equations by Zhang (1998), who showed that the solution belongs to some \( q \)-Gevrey class, meaning that the sequence \( (f_n) \) can grow at most as some \( q^c n^2 \) for some unspecified \( c \). This result was made more precise by Di Vizio (2009) and Cano and Fortuny Ayuso (2012), in the sense that an upper bound for \( c \) was provided in terms of the Newton polygon associated with a linearized version of the \( q \)-functional equation. While these results are important from a theoretical perspective since they are indeed comparable to the Maillet-Malgrange theorem, they are insufficient for many applications, in
particular in combinatorics and statistical physics where \((f_n)\) is the cardinality of some interesting set, and one is interested in calculating this cardinality in a nice closed form. Obtaining such an explicit form is usually impossible with our current knowledge, and one has to settle for less, such as having a good estimate on the growth of \(f_n\) with \(n\). Our result provides such estimate when \(f\) is a divergent series, for in many cases it provides an explicit asymptotic equivalent for \(f_n\) up to a multiplicative constant.

A different view on the existence and regularity questions was developed by Li and Zhang (2011) who gave sufficient conditions for an analytic nonlinear \(q\)-difference equation to have an analytic solution. In the case of \(q\)-algebraic equations, we provide an alternative sufficient condition.

It is worth noting that the study of linear \(q\)-difference equations has a long history spanning a century, from Carmichael (1912), Birkhoff (1913), Adams (1929, 1931), Trjitzinsky (1938) to the more recent works of Bézivin (1992), Ramis (1992), Sauloy (2000, 2003), Zhang (2002), Ramis, Sauloy and Zhang (2013). Nonlinear \(q\)-functional equations, which seem more prevalent in applied mathematics, have been studied systematically only in the past few years by the mentioned authors.

Differing from the approach taken in the previous papers, our results do not rely on the Newton polygon method, and the linearization of the equation that we will use is twofold: a partial one on the equation, and one developed from the recursion that the coefficients of the solution must satisfy. However, the full potential of our method is achieved only when combined with the result of Cano and Fortuny Ayuso (2012). Indeed, using their result, any \(q\)-algebraic equation which has a solution in the Hahn field of generalized power series can be transformed into one whose solution is a power series, and for which the result of this paper may then be applied.

This paper is organized as follows. Section 2 contains the main theoretical results. Section 3 discusses the computational aspects to implement our results. Section 4 is devoted to some examples. Section 5 contains the proofs of the results stated in section 2. An appendix contains the Maple code which we used in one of the examples in section 4. The second section is divided in various subsections, the first one introducing the mathematical objects which we will use, the second one defining and discussing the notion of
reduced equation, the third one studying the existence of solutions of a reduced equation and stating some basic regularity results. The fourth subsection provides an estimate of the growth of the coefficients for divergent solutions. The paper is essentially self-contained.

Throughout the paper, when \((u_n)\) is some sequence which does not vanish ultimately and \(c\) is a complex number, the relation \(f_n \sim cu_n\), means that \(\lim_{n \to \infty} f_n/u_n = c\). In particular, if \(c\) vanishes, this means \(f_n = o(u_n)\) as \(n\) tends to infinity.

If \(f(z) = \sum_{n \geq 0} f_n z^n\) is a formal power series, we write \([z^n]f(z)\) or \([z^n]f\) for the coefficient \(f_n\) of \(z^n\) in this power series. The map \([z^n]\) is linear on formal power series.

Throughout this paper, \(q\) is a complex number of modulus greater than 1.

2. Main results. In order to state our results, the usual notation (1.1) for \(q\)-algebraic equation is not suitable. We will need a different formalism which requires some definitions and notations which we will introduce in the first subsection. In the second subsection, we will describe a way to bring an algebraic \(q\)-functional equation into a suitable form. Our main results are in the third and fourth subsections.

2.1. Preliminaries. Refering to the polynomial involved in (1.1), our most basic object will be the monomials \(z_0^{a_0} z_1^{a_1} \cdots z_k^{a_k}\), but we need to think of them in a different way, closer to their contribution in equation (1.1). As we will see afterwards, the following definition gives our analogue of the monomials.

**Definition 2.1.1.**

(i) A \(q\)-factor \(A\) is a tuple \((a; \alpha_1, \ldots, \alpha_\ell)\) in \(\mathbb{N} \times \mathbb{Z}^\ell\) with \(\alpha_1 \leq \cdots \leq \alpha_\ell\) and \(\ell\) is a positive integer.

(ii) This \(q\)-factor is shifting if \(a\) is positive. A set of \(q\)-factors is shifting if all its elements are shifting.

When writing \(q\)-factors, a capital letter denotes the \(q\)-factor, the corresponding lower case the first component of the \(q\)-factor, and the corresponding Greek one the other components. The index \(\ell\) depends on the \(q\)-factor. For instance, if \(Q\) is a set of \(q\)-factors, a
series $\sum_{A \in \mathcal{Q}} q^a z^{\alpha_1} \cdots z^{\alpha_\ell}$ is the sum over all $A$ in $\mathcal{Q}$ of $q$ at the power the first component of $A$ and $z$ at the power the last component of $A$.

Definition 2.1.1 is introduced since to a $q$-factor $A$ corresponds an operator on formal power series $Af(z) = z^a f(q^{\alpha_1} z) \cdots f(q^{\alpha_\ell} z)$.

We see that when $A$ is shifting,

$$[z^n] Af(z) = [z^{n-a}] f(q^{\alpha_1} z) \cdots f(q^{\alpha_\ell} z)$$

involves only $[z^i] f$ with $i \leq n - a < n - 1$, hence the term ‘shifting’.

The set of $q$-factors can be viewed as a free $\mathbb{C}$-module, and to a finite weighted sum of $q$-factors, say $\sum r_i A_i$ corresponds the operator $f \mapsto \sum r_i A_i f$.

Let $\mathcal{Q}$ be a set of $q$-factors. In this paper we are primarily interested in the formal power series solutions of equation $s$ of (1.1), but written in the form

$$P(z) + \sum_{A \in \mathcal{Q}} r_A Af(z) = 0 \quad (2.1.1)$$

where $P(z)$ is a polynomial in $z$, the $r_A$ are some complex numbers, none of them being 0, and $\mathcal{Q}$ is a finite set of $q$-factors.

**Example.** Consider equation (1.2). The set of $q$-factors involved in that equation is

$$\mathcal{Q} = \{ (0; 0), (3; 1, 6), (6; 0, 9, 10), (7; 14, 14),
(10; -3, 5, 14, 16), (14; -5), (14; 0), (14; 0, 3, 5) \}. \quad (2.1.2)$$

This $\mathcal{Q}$ is not shifting since it contains the nonshifting $q$-factor $(0; 0)$.

In light of the interpretation of $q$-factors as operators, the following definition simply distinguishes the linear ones

**Definition 2.1.2.** A $q$-factor $(a; \alpha_1, \ldots, \alpha_\ell)$ is linear if $\ell$ is 1.

The nonshifting $q$-factors play a special role, which leads us to introduce the following notation.
Notation 2.1.3. Let $Q$ be a set of $q$-factors. We write

$$Q_0 = \{ A \in Q : a = 0 \},$$

its subset of nonshifting $q$-factors and

$$Q_+ = \{ A \in Q : a > 0 \},$$

its subset of shifting ones.

We set $\alpha(Q) = \max\{ \alpha_\ell : A \in Q \}$.

Consider a $q$-algebraic equation (2.1.1) without nonshifting $q$-factor. If $P(0) \neq 0$, the equation has no solution. If $P(0) = 0$, then we may factor $z$ out of the equation and simplify it by $z$. Therefore, for an equation to have a solution, one must be able to simplify it into one that has a nonshifting factor. Hence, there is no loss of generality in what follow to consider only $q$-algebraic equations with at least one nonshifting factor.

Considering the constant term in (2.1.1), we see that for $f$ to be a solution we must have

$$P_0 + \sum_{A \in Q_0} r_A f_0^\ell = 0. \quad (2.1.3)$$

The only way for this equation not to have a solution is for the polynomial $\sum_{A \in Q_0} r_A f_0^\ell$ to be degenerate and different than $-P_0$. Equation (2.1.3) makes clear that several solutions may exist whenever $Q$ contains at least one nonshifting $q$-factor for which $\ell$ is at least 2.

2.2. Reduced equations. The equations in which all the nonshifting $q$-factors are linear are important, and this motivates the following definition.

Definition 2.2.1. A $q$-algebraic equation is reduced if all its nonshifting $q$-factors are linear.

Of course, not every $q$-algebraic equation is reduced. However, most of them can be reduced as we will now explain. We will use an algorithm similar to that proposed in Cano and Fortuny Ayuso (2012) to bring an equation to a quasi-solved form. However, since we are dealing exclusively with power series, our algorithm is somewhat easier to describe.
We will write \( \mathcal{O}(z^k) \) to indicate a series in the ideal generated by \( z^k \). Thus, \( f(z) = \mathcal{O}(z^k) \) means that there exists a formal power series \( g(z) \) such that \( f(z) = z^k g(z) \).

Assume that (2.1.3) has at least one solution \( f_0 \). We make the change of function \( f(z) = f_0 + zg(z) \) in (2.1.1). If \( A \) is nonshifting, then

\[
Af(z) = \prod_{1 \leq i \leq \ell} (f_0 + q^{\alpha_i} z g(q^{\alpha_i} z)) = f_0^\ell + f_0^{\ell-1} z \sum_{1 \leq i \leq \ell} q^{\alpha_i} g(q^{\alpha_i} z) + \mathcal{O}(z^2).
\]

If \( A \) is shifting, then

\[
Af(z) = z^a f_0^\ell + \mathcal{O}(z^{a+1}).
\]

Thus,

\[
P(z) + \sum_{A \in \mathcal{Q}_+} r_A Af(z) = P_0 + \sum_{A \in \mathcal{Q}_0} r_A f_0^\ell + z \sum_{A \in \mathcal{Q}_0} r_A f_0^{\ell-1} \sum_{1 \leq i \leq \ell} q^{\alpha_i} g(q^{\alpha_i} z) + \sum_{A \in \mathcal{Q}_+} r_A z^a f_0^\ell + P_1 + \mathcal{O}(z^2).
\]

Taking into account (2.1.3), we see that the constant term in (2.2.1) vanishes. Thus, after simplifying by \( z \), we can rewrite (2.1.1) as

\[
\sum_{A \in \mathcal{Q}_0} r_A f_0^{\ell-1} \sum_{1 \leq i \leq \ell} q^{\alpha_i} g(q^{\alpha_i} z) + \sum_{A \in \mathcal{Q}_+} r_A f_0^\ell + P_1 + \mathcal{O}(z) = 0.
\]

The nonshifting part of this new equation is given by the operator

\[
\sum_{A \in \mathcal{Q}_0} r_A f_0^{\ell-1} \sum_{1 \leq i \leq \ell} q^{\alpha_i} g(q^{\alpha_i} z) + \sum_{A \in \mathcal{Q}_+} r_A f_0^\ell + P_1.
\]

It involves only linear \( q \)-factors; however, this nonshifting operator may be 0. In particular this is the case if \( f_0 = 0 \) and \( \ell > 1 \) for all nonshifting \( q \)-factors in the original equation. In this case, one may iterate this change of function, setting \( g(z) = g_0 + zh(z) \), and iterate further until one obtains a reduced equation. It seems that most equations encountered in applications can be reduced in one or
two changes of function. It is probably possible to characterize all equations that cannot be reduced by this change of function.

Note that for a reduced $q$-algebraic equation, condition (2.1.3) becomes $P_0 + \sum_{A \in Q_0} r_A f_0 = 0$, so that a necessary condition for such an equation to have a solution is that

$$P_0 = 0 \quad \text{or} \quad \sum_{A \in Q_0} r_A \neq 0. \quad (2.2.2)$$

If condition (2.2.2) holds, then $f_0$ is unique if $\sum_{A \in Q_0} r_A \neq 0$ and is arbitrary otherwise.

**Example.** $(qP_1)$ The $q$-difference first Painlevé equation, $qP_1$, is

$$\omega(qz)\omega\left(\frac{z}{q}\right) = \frac{1}{\omega(z)} - \frac{1}{z\omega^2(z)}.$$

This equation, or an equivalent after a change of function, was introduced in Ramani and Grammaticos (1996) and further studied in the context of classification of rational surfaces by Sakai (2001), and from a different perspective by Nishioka (2010) and Joshi (2012). The notation $\overline{\omega}$ and $\omega$ for $\omega(qz)$ and $\omega(z/q)$ is convenient for this example. This equation can be rewritten as

$$\overline{\omega}\omega^2 = \omega - \frac{1}{z}. \quad (2.2.3)$$

This suggests to set $f(z) = \omega(1/z)$ to transform (2.2.3) into

$$ff^z = f - z, \quad (2.2.4)$$

which is a $q$-algebraic equation. The nonshifting $q$-factors are $(0;-1,0,0,1)$ and $(0;0)$, and one of them, corresponding to the term $ff^2$ is nonlinear. Applying $[z^0]$ to both sides of (2.2.4), we must have $f_0^4 = f_0$. Therefore, $f_0$ may be $0, 1, e^{2\pi i/3}$ or $e^{4\pi i/3}$. Following the indicated procedure to reduce the equation, we set $f(z) = f_0 + zg(z)$.

**Case** $f_0 = 0$. In this case, $f(z) = zg(z)$ and (2.2.4) becomes

$$z^3gg^z = g - 1. \quad (2.2.5)$$

The only nonshifting $q$-factor is $(0;0)$, which corresponds to the linear term $g$. Thus, this equation is reduced.
Case $f_0 = 1$. We then have $f(z) = 1 + zg(z)$ and (2.2.4) becomes, after some calculation

$$
\frac{1}{q}q + qg + g + zgq + \frac{2}{q}zqg + 2qzgq + zg^2 + 2z^2 q^2 gq + 1 = 0. 
$$

(2.2.6)

The nonshifting $q$-factors are $(0; -1)$, $(0; 1)$ and $(0; 0)$, which are all linear. Thus, this equation is reduced.

Case $f_0 = e^{2\pi i/3}$ or $f_0 = e^{4\pi i/3}$. Set $f(z) = f_0 h(z/f_0)$. One easily checks that $h$ solves (2.2.4) and that $h_0 = 1$, bringing us back to the previous case.

Following the custom in $q$-algebraic equations, we consider the operator $\sigma$ and its powers $\sigma^n$ defined by

$$
\sigma^n f(z) = f(q^n z), \quad n \in \mathbb{Z}.
$$

If $A$ is a $q$-factor, then

$$
A \sigma^n f(z) = z^a f(q^{a_1+n}z) \cdots f(q^{a_\ell+n}z),
$$

so that $\sigma^n$ acts on $q$-factors to the right as

$$(a; \alpha_1, \ldots, \alpha_\ell) \sigma^n = (a; \alpha_1 + n, \ldots, \alpha_\ell + n).$$

Consequently, if $f$ solves (2.1.1), then $g(z) = \sigma^{-n} f(z)$ solves

$$
\sum_{A \in Q} r_A A \sigma^n g(z) + P(z) = 0.
$$

Therefore, taking $n = -\alpha(Q_0)$, there is no loss of generality at all in assuming that $\alpha(Q_0) = 0$.

2.3. Existence and convergence or divergence of the solutions. Not every reduced $q$-algebraic equation has a power series solution. For instance $zf(z) = 1$ has not. Our first theorem is a sufficient condition for a reduced equation to have a solution. Recall that throughout the paper we assume that $|q| > 1$.

**Theorem 2.3.1.** A sufficient condition for a reduced $q$-algebraic equation to have a power series solution is that for any $n \geq 0$,

$$
\sum_{A \in Q_0} r_A q^{\alpha_1+n} \neq 0. 
$$

(2.3.1)
If this condition holds, the solution is uniquely determined by \( f_0 \).

Condition (2.3.1) is analogous to the condition introduced by Cano and Fortuny Ayuso (2012) for an equation in quasi-solved form to be in solved form, and the second assertion of Theorem 2.3.1 may be seen as a rephrasing of their Theorem 2 in our context of power series.

**Example.** For equation (1.2), which is reduced, we have

\[
\sum_{A \in \mathcal{Q}_0} r_A q^{\alpha_1 n} = 2
\]

and condition (2.3.1) holds.

For the \( q \)-difference first Painlevé equations (2.2.5), the unique nonshifting power is \( (0; 0) \), which corresponds to the term \( g \) in that equation, so that (2.3.1) is \( 1 \neq 0 \).

For equation (2.2.6), condition (2.3.1) is

\[
\frac{1}{q^{n+1}} + q^{n+1} + 1 \neq 0.
\]

This condition is always satisfied when \( |q| > 1 \) since the polynomial \( z^2 + z + 1 \) has two roots of modulus 1.

Once a reduced equation has a solution under criterion (2.3.1), the question arises as to whether this solution has a positive radius of convergence or is a divergent series.

If \( \mathcal{Q}_+ \) is empty, a reduced equation has the form

\[
\sum_{A \in \mathcal{Q}_0} r_A f(q^{\alpha_1} z) + P(z) = 0.
\]

Applying \([z^n]\) to both sides of this equation,

\[
\left( \sum_{A \in \mathcal{Q}_0} r_A q^{\alpha_1 n} \right) f_n + P_n = 0.
\]

Under condition (2.3.1), we obtain \( f_n \) and since \( p \) is the degree of \( P \),

\[
f(z) = - \sum_{0 \leq n \leq p} \frac{P_n}{\sum_{A \in \mathcal{Q}_0} r_A q^{\alpha_1 n}} z^n
\]
is a polynomial in $z$. In what follows we will then assume that $Q_+$ is not empty. Our next result provides a simple test for convergence of the solutions. To avoid some trivial situations, we use the following definition.

**Definition 2.3.2.** We say that the $q$-factors involved in a $q$-algebraic equation have been collected if each $q$-factor occurs only once in the equation.

For instance, the $q$-factors $(0; 1)$ and $(0; 0)$ in the equation $f(qz) + f(z) - f(z) = 0$ have not been collected. There is of course no loss of generality to assume that the $q$-factors have been collected.

**Theorem 2.3.3.** Consider a reduced $q$-algebraic equation whose $q$-factors have been collected, such that (2.3.1) holds and $Q_+ \neq \emptyset$. If $\alpha(Q_0) \geq \alpha(Q_+)$ then the unique power series solution has a positive radius of convergence.

It is possible for the radius of convergence to be infinite, as in the equation $f(z) + qzf(z) - z - qz^2 = 0$ which is designed to have $f(z) = z$ for solution.

In light of Theorem 2.3.3, note that a $q$-algebraic equation may have both convergent and divergent solutions, because different initial values $f_0$ usually lead to different reduced equations.

**Example.** $(qP I)$ The $q$-factors involved in (2.2.6) are

$$
Q = \{ (0; -1), (0; 0), (0; 1), (1; -1, 1), (1; -1, 0), (1; 0, 1), \\
(1; 0, 0), (2; -1, 0, 1), (2; -1, 0, 0), (2; 0, 0, 1), (3; -1, 0, 0, 1) \}.
$$

Thus $\alpha(Q_0) = 1 = \alpha(Q_+)$ and Theorem 2.3.3 implies that the solution has a positive radius of convergence.

Theorem 2.3.3 raises the question of what can be said when $\alpha(Q_0) < \alpha(Q_+)$. This question is partially answered in the next subsection.

**2.4. Asymptotic behavior of the coefficients of the divergent solutions.** In this subsection we will derive an asymptotic equivalent for the coefficients of the solution of a reduced $q$-algebraic
equation satisfying (2.3.1). To state our results, we need to introduce further definitions related to $q$-factors.

**Definition 2.4.1.** Consider a set $Q$ of $q$-factors such that $Q_+ \neq \emptyset$ and $\alpha(Q_0) = 0$.

(i) The height of a shifting $q$-factor $A = (a; \alpha_1, \ldots, \alpha_\ell)$ is

$$H(A) = \frac{\alpha_\ell}{2a}.$$

(ii) If $Q$ is a finite set of $q$-factors, its height is the largest height of its shifting elements, that is

$$H(Q) = \max_{A \in Q_+} H(A).$$

(iii) The crest $\hat{Q}$ of $Q$ is

$$\hat{Q} = \{ A \in Q : 2aH(Q) = \alpha_\ell \}.$$

(iv) The co-height of $Q$ is

$$h(Q) = \min\{ a : A \in \hat{Q} \cap Q_+ \}.$$

(v) The scope of $A$ is the number of maximal $\alpha_i$,

$$s(A) = \sharp \{ i : \alpha_i = \alpha_\ell \}.$$

(vi) The $Q$-Borel transform of a formal power series $f(z) = \sum_{n \geq 0} f_n z^n$ is the formal power series

$$B_Q f(z) = \sum_{n \geq 0} q^{-H(Q)n(n-h(Q))} f_n z^n.$$

The crest $\hat{Q}$ coincides with the set of shifting $q$-factors of maximal height together with the set of nonshifting $q$-factors for which $\alpha_\ell$ vanishes. Given how the crest is defined, we see that $2aH(Q) - \alpha_\ell$ is positive for all $q$-factors not in $\hat{Q}$.

**Example.** The heights of the shifting $q$-factors in (2.1.2) are $1, 5/6, 1, 8/10, -5/28, 0, 5/28$. The largest height is $H(Q) = 1$. Since both $(3; 1, 6)$ and $(7; 14, 14)$ have height 1, the crest of $Q$
is then $\hat{Q} = \{ (0; 0), (3; 1, 6), (7; 14, 14) \}$. The co-height of $Q$ is $h(Q) = \min(3, 7) = 3$.

To each reduced $q$-functional equation of the form (2.1.1) for which $Q_+$ is not empty, we associate a polynomial as follows.

**Definition 2.4.2.** Assume that (2.1.1) is reduced and that $\alpha(Q_0) = 0$. The crest polynomial associated to (2.1.1) is

$$C_{q,t}(z) = \sum_{A \in \hat{Q}} r_A s(A) q^{-H(Q) a(a-h(Q))} z^a t^{\ell-1}.$$  

There is a slight abuse of terminology in Definition 2.4.2 for a $q$-algebraic equation can be divided by any nonzero complex number. So the crest polynomial is defined only up to a multiplicative constant. However, since we will be interested in the zeros of this polynomial, this abuse of terminology will not create any ambiguity.

We see that the crest polynomial is the $Q$-Borel transform of $\sum_{A \in \hat{Q}} r_A s(A) z^a t^{\ell-1}$.

From the Definition 2.4.1.iii, if $A$ is in $\hat{Q}$ and $a = 0$, then $\alpha_\ell = 0$, and $\ell = 1$ if the equation is reduced. Therefore, the constant term of the crest polynomial is

$$C_{q,t}(0) = \sum_{A \in Q_0 \cap \hat{Q}} r_A s(A).$$

If (2.1.1) is reduced, then $Q_0$ contains only linear $q$-factors, which forces $s(A) = 1$ whenever $A$ is nonshifting, and then

$$C_{q,t}(0) = \sum_{A \in Q_0 \at \alpha_\ell = 0} r_A.$$

Thus, condition (2.3.1) for $n = 0$ may be rewritten as $C_{q,t}(0) \neq 0$.

**Example.** To evaluate the crest polynomial associated to (1.2), we have the following table listing the parts of the equation relevant to the crest.

| $A$     | $(0; 0)$ | $(3; 1, 6)$ | $(7; 14, 14)$ |
|---------|----------|-------------|---------------|
| $r_A$   | $-2$     | $4$         | $18q^4$       |
| $s(A)$  | $1$      | $1$         | $2$           |
| $\ell$  | $1$      | $2$         | $2$           |
| $H(Q)a(a-h(Q))$ | $0$ | $0$ | $28$ |
The crest polynomial is then
\[ C_{q,t}(z) = -2 + 4z^3t + 36q^{-24}z^7t. \]

If \( A \) is in the crest of \( Q \), its height is that of \( Q \), so that 
\[ 2H(Q)a(a - h(Q)) = \alpha_\ell(a - h(Q)) \]
is an integer. Also, \( \ell \) is at least 1. Therefore, \( C_{q,t}(z) \) is a polynomial in \( (z, q^{-1/2}, t) \), provided we ignore a possible dependence in \( q \) in the coefficient \( r_A \).

If \( Q \) is shifting, the polynomial \( \sum_{A \in \widehat{Q} \setminus Q_0} r_A q^{-H(Q)a(a - h(Q))} z^a \) is in the ideal generated by \( z \). However, it is possible that this sum vanishes due to some cancelations. Consequently, a crest polynomial may be constant. If this is the case, we consider that it has a root at infinity for the following result to hold without further discussion.

Our main result is the following, and its implications will be discussed afterwards.

**Theorem 2.4.3.** Consider a reduced \( q \)-algebraic equation where the \( q \)-factors have been collected, such that (2.3.1) holds and \( 0 = \alpha(Q_0) < \alpha(Q+) \). Let \( f \) be a solution of (2.1.1). Let \( C_{q,t}(z) \) be the crest polynomial associated to (2.1.1), and let \( R_{q,f_0} \) be the smallest modulus of the zeros of \( C_{q,f_0} \).

(i) \( B_Q f(z) \) has radius of convergence \( R_{q,f_0} \).

(ii) There exists some positive \( \Theta \) such that \( C_{q,f_0}(z)B_Q f(z) \) is a convergent power series that has no singularities other than removable ones in the disk centered at 0 and of radius \( q^\Theta R_{q,f_0} \).

(iii) if
\[ \{ P_i : 0 \leq i \leq p \} \cup \{ r_A : A \in Q_+ \} \cup \{ -r_A : A \in Q_0 \} \]
is a set of real numbers all of the same sign, then \( (f_n) \) is a nonnegative sequence. Moreover, if \( Q_0 = \{ (0;0) \} \) then \( C_{q,f_0}(z)B_Q f(z) \) is a power series whose coefficients are nonnegative provided we choose \( r_{(0;0)} \) nonnegative, and these coefficient all vanish if and only if \( f = 0 \).

In the statement of Theorem 2.4.3, recall that \( \alpha(Q_0) = 0 \) is not restrictive at all, and that Theorem 2.3.3 asserts that \( \alpha(Q+) > 0 \) is the only case where we can have a divergent solution. The third assertion of Theorem 2.4.3 implies that the solution is indeed
divergent in equations that have coefficients of the proper signs and a single nonshifting $q$-factor. Recall that one should read $R_{q,f_0} = +\infty$ if $C_{q,f_0}$ is a nonzero constant polynomial and $R_{q,f_0} = 0$ if $C_{q,f_0}$ is the constant polynomial 0.

Our proof shows that $\Theta$ in statement (ii) of Theorem 2.4.3 may be taken to be the smallest of $2H(\mathcal{Q})$, $\max_{\mathcal{A} \in \mathcal{Q} \setminus \mathcal{Q}} (2Ha - \alpha_\ell)$ and 1.

A careful examination of the proof shows that the power series $(C_{q,f_0}B_{Q,f})(z)$ is almost a linear combination of tangled products in the sense of Garsia (1981) of $Q$-Borel transforms of $f$. However, since there is no simple way to calculate this $Q$-Borel transform, such expression does not appear useful to estimate $[z^n]f$.

The strength of Theorem 2.4.3 comes from the following. Set

$$U_{q,f_0}(z) = C_{q,f_0}(z)B_{Q,f}(z).$$

Theorem 2.4.3 asserts that $U_{q,f_0}$ has no singularity of modulus less than $q^\Theta R_{q,f_0}$. The coefficient

$$q^{-H(\mathcal{Q})n(n-h(\mathcal{Q}))}f_n = [z^n] \frac{U_{q,f_0}(z)}{C_{q,f_0}(z)}$$

(2.4.1)

can be evaluated via the Cauchy formula, and results on the asymptotic behavior of these coefficients are readily available through the singularity analysis of meromorphic functions, as explained in Flajolet and Sedgewick (2009; see in particular Chapter IV).

**Example.** For equation (1.2), $f_0 = 1/2$ and the crest polynomial is $C_{q,1/2}(z) = -2 + 2z^3 + 18q^{-24}z^7$ has a positive solution, call it $R$, which then satisfies

$$1 = R^3 + 9q^{-24}R^7.$$

To see that $R$ is the solution of smallest modulus, if $C_{q,1/2}(Rz) = 0$ had another solution $\zeta$ of modulus at most 1, then we would have

$$1 = R^3\zeta^3 + 9q^{-24}R^7\zeta^7$$

so that we would be able to write 1 as a convex combination of $\zeta^3$ and $\zeta^7$ which both belong to the convex unit disk centered at 0 and of radius 1 and have relatively prime exponents; this forces $\zeta = 1$.

We then write $C_{q,1/2}(z) = (1 - z/R)Q(z)$ where $Q$ is some polynomial of degree 6. It then follows from singularity analysis (see
Flajolet and Sedgewick, 2009; chapter IV) that, with the notation as in Theorem 2.4.3,

$$[z^n] \frac{U_{q,1/2}(z)}{C_{q,1/2}(z)} = [z^n] \frac{U_{q,1/2}(z)}{(1 - z/R)Q(z)} \sim R^{-n} \frac{U_{q,1/2}(R)}{Q(R)}$$  \hspace{1cm} (2.4.2)

as $n$ tends to infinity. Set $c_q = U_{q,1/2}(R)/Q(R)$. Theorem 2.4.3.iii implies that $U_{q,1/2} \neq 0$, hence $c_q$ does not vanish. This yields (1.3).

Following Bézivin (1992), Ramis (1992) and others, recall that a formal power series $f(z) = \sum_{n \geq 0} f_n z^n$ is of q-Gevrey order $s$ if the series $\sum_{n \geq 0} f_n q^{-s n^2/2} z^n$ has a positive radius of convergence. Since we consider $|q| > 1$, if $f$ has q-Gevrey order $s$, it also has q-Gevrey order any number greater than $s$. It is then of interest to find the smallest q-Gevrey order of a divergent power series. Clearly, Theorem 2.4.3 yields some information on the q-Gevrey order of the solution of (2.1.1), which is sharp when the assumptions in Theorem 2.4.3.iii are satisfied.

**Corollary 2.4.4.** Under the assumptions of Theorem 2.4.3, the power series solution of (2.1.1) has q-Gevrey order at least $2H(Q)$. Moreover, if assumptions of Theorem 2.4.3.iii hold, then $2H(Q)$ is the smallest q-Gevrey order of the power series solution.

**Remark.** The reduction algorithm described in section 2 changes the equation and, therefore, may change its height. In order for Theorem 2.4.3 and Corollary 2.4.4 to deliver a sharp result, we should reduce the equation in such a way that its height is as small as possible. Assume that the solution $f(z)$ of equation (2.1.1) is such that $f_0 = 0$. We then set $f(z) = zg(z)$. For a q-factor $A = (a; \alpha_1, \ldots, \alpha_\ell)$,

$$Af(z) = q^{\alpha_1 + \cdots + \alpha_\ell} z^\ell Ag(z).$$

We then set

$$\tilde{A} = (a + \ell - 1; \alpha_1, \ldots, \alpha_\ell)$$

and $\tilde{r}_A = q^{\alpha_1 + \cdots + \alpha_\ell} r_A$, so that $g$ is a solution of

$$P(z) + \sum_{A \in \mathcal{Q}} \tilde{r}_A z \tilde{A}g(z) = 0.$$
Since \( f_0 = 0 \), we have \( P(0) = 0 \) and we can simplify this equation by \( z \) to obtain
\[
\frac{P(z)}{z} + \sum_{A \in Q} \tilde{r}_A \tilde{A} g(z) = 0.
\]
We have
\[
H(\tilde{A}) = \frac{\alpha_\ell}{2(a + \ell - 1)} \leq H(A)
\]
and the inequality is strict whenever \( \ell > 1 \). Thus, when \( f_0 = 0 \), we should reduce the equation, in particular if the crest of \( Q \) contains nonlinear \( q \)-factors, that is factors for which \( \ell > 1 \).

The following corollary to Theorem 2.4.3 covers important applications and shows that some oscillatory behavior may occur and gives a sharp bound on the periodicity.

**Corollary 2.4.5.** Under the assumptions of Theorem 2.4.3, if the crest \( \tilde{Q} \) has a unique shifting element \( A = (a; \alpha_1, \ldots, \alpha_\ell) \), then there exist some complex numbers \( c_0, \ldots, c_{a-1} \) such that, as \( n \) tends to infinity,
\[
[z^n]f \sim q^{H(Q)n(n-h(Q))} \left( \frac{-r_{AS}(A)f_0^{\ell-1}}{r(0;0)} \right)^{n/h(Q)} c_m
\]
where \( m \) is the remainder in the Euclidean division algorithm of \( n \) by \( a \).

**Remark.** In the statement of Corollary 2.4.5, \( r_{AS}(A)f_0^{\ell-1}/r(0;0) \) may not be a positive real number. Thus, to take the fractional power \( n/h(Q) \) of this number requires one to choose a branch cut in the complex plane. The fractional power is then defined up to some \( e^{2i\pi k n/h(Q)} \) for some \( 0 \leq k < h(Q) \). Since \( h(Q) = a \), this indeterminacy may be absorbed in the constant \( c_m \).

As far as the oscillatory behavior of \([z^n]f \) is concerned, Corollary 2.4.5 shows that it may be decomposed into two parts: write \( r_{AS}(A)f_0^{\ell-1}/r(0;0) \) as \( \rho e^{2i\pi \theta} \) with \( \rho \) nonnegative and \( \theta \) in \( [0,1) \).

Corollary 2.4.5 asserts that
\[
[z^n]f \sim q^{H(Q)n(n-h(Q))} \rho^{n/h(Q)} e^{2i\pi \theta n/h(Q)} c_m
\]
as \( n \) tends to infinity. If \( \theta \) is irrational, the sequence \( (e^{2i\pi \theta n/H(Q)})_{n \in \mathbb{N}} \) is dense in the unit circle, making \([z^n]f \) with seemingly little regularity. If \( \theta \) is rational, say \( \theta = h(Q)p'/p \) with \( p' \) and \( p \) positive integers mutually prime, the sequence \( (e^{2i\pi \theta n/h(Q)})_{n \in \mathbb{N}} \) has periodicity
Thus, the sequence \((e^{2i\pi n\theta/h(Q)}c_m)_{n\in\mathbb{N}}\) has periodicity the least common multiple of \(p\) and \(a\). Example 5 of section 5 will illustrate this phenomenon with \(p = 4\) and \(a = 17\), leading to a periodicity of 68. However, because \((c_m)_{0\leq m<a}\) may have a periodicity a divisor of \(a\), it is possible that the sequence \((e^{2i\pi n\theta/h(Q)}c_m)_{n\in\mathbb{N}}\) has a period smaller than the least common multiple of \(p\) and \(a\).

**Proof.** Since the crest has a unique shifting element \(A\) and \(\alpha(Q_0) = 0\), the height of \(Q\) is \(\alpha/2a\) and its co-height is \(a\). Since \(\alpha(Q_0) = 0\), the crest contains exactly the elements \((0; 0)\) and \(A\). The crest polynomial is

\[C_{q,f_0}(z) = r_{(0;0)} + r_{A}s(A)z^{f-1}.\]

Its roots are some \(\zeta_0\) and \(\zeta_k = e^{2i\pi k/a}\zeta_0\), \(1 \leq k < a\). They all have the same modulus \(|r_{(0;0)}/r_{A}s(A)f^{k-1}/a|\). Through (2.4.1), Theorem 2.4.3 yields

\[\left[z^n\right]f \sim q^{(\alpha/2a)n(n-a)}/r_{(0;0)}\left[U_{q,f_0}(z)\prod_{0\leq k<a}(1-z/\zeta_k)\right].\]

But

\[
\frac{1}{\prod_{0\leq k<a}(1-z/\zeta_k)} = \sum_{0\leq k<a} \frac{1}{1-z/\zeta_k} \prod_{0\leq j<a, j\neq k} \frac{1}{1-\zeta_k/\zeta_j}
\]

\[= \sum_{0\leq k<a} \frac{1}{1-z/\zeta_k} \prod_{0\leq j<a, j\neq k} \frac{1}{1-e^{2i(k-j)\pi/a}}.\]

Thus,

\[\left[z^n\right]f \sim q^{H(Q)n(n-h(Q))}/r_{(0;0)} \sum_{0\leq k<a} \frac{U_{q,f_0}(\zeta_k)}{\prod_{0\leq j<a, j\neq k} 1-e^{2i(k-j)\pi/a}} \zeta_k^{-n}.\]

Note that \((\zeta_k/\zeta_0)^{-n}\) depends only on the remainder \(m\) in the Euclidean division algorithm of \(n\) by \(a\). We then set

\[\tilde{c}_m = \frac{1}{r_{(0;0)}} \sum_{0\leq k<a} \frac{U_{q,f_0}(\zeta_k)}{\prod_{0\leq j<a, j\neq k} 1-e^{2i(k-j)\pi/a}} \left(\frac{\zeta_0}{\zeta_k}\right)^n,\]
and $c_m = e^{2ik\pi/a}c_m$ for some $0 \leq k < a$ according to the choice of $\zeta_0$ among the $a$ roots of the crest polynomial.

**Example.** (qP$_1$, continued) Consider the $q$-functional equation (2.2.5). One can see by induction that $g$ is in fact a function of $z^3$ (see also Joshi, 2012). Thus, we set $g(z) = h(z^3)$ and rewrite (2.2.5) as

$$-h(z^3) + z^3h\left(\frac{z^3}{q^3}\right)h(z^3)^2h(q^3z^3) + 1 = 0.$$  

Setting $r = q^3$, and substituting $z$ for $z^3$, we have

$$-h(z) + zh(z/r)h(z)^2h(rz) + 1 = 0,$$

which, in the notation of $q$-factors means

$$(-(0; 0) + (1; -1, 0, 0, 1))h(z) + 1 = 0.$$

For this equation, $H(Q) = 1/2$ and $h(Q) = 1$. Corollary 2.4.5 yields

$$h_n \sim cq^{n(n-1)/2}$$

as $n$ tends to infinity. It follows that $g_{3n+1} = g_{3n+2} = 0$ and

$$g_{3n} = h_n \sim cq^{n(n-1)/2}.$$

Going back to equation (2.2.4), we then have $f_n = g_{n-1}$ which yields $f_{3n} = f_{3n+2} = 0$ and $f_{3n+1} \sim cq^{n(n-1)/2}$. Hence $\omega(z) = \sum_{n \geq 0} \omega_n z^{-n}$ with $\omega_{3n} = \omega_{3n+2} = 0$ and $\omega_{3n+1} \sim cq^{n(n-1)/2}$ as $n$ tends to infinity.

3. Computational aspects. The reduction procedure often increases the number of terms in a $q$-algebraic equation, as it may be seen for instance in comparing (2.2.4) with (2.2.6). In the next section, devoted to examples, we will consider an equation due to Cano and Fortuny Ayuso (2012) for which the reduction procedure needs to be applied 10 times, leading to an equation with 397 terms. Such an example makes clear that for our theory to be useful in applications, it has to be implemented on a computer algebra package. The goal of this section is not to describe an implementation for a specific computer algebra system, but to present algorithms
from which an efficient implementation is easily written. Since it reformulates our theory in terms of multivariate polynomials, it also gives it a more algebraic-geometric flavor.

To a $q$-factor $(a; \alpha_1, \ldots, \alpha_\ell)$ corresponds a monomial $z^a Y_{\alpha_1} \cdots Y_{\alpha_\ell}$. We will also consider monomials as operators, with then

$$(z^a Y_{\alpha_1} \cdots Y_{\alpha_\ell}) f(z) = z^a f(q^{\alpha_1} z) \cdots f(q^{\alpha_\ell} z).$$

In particular, following one of the many known ways of writing $q$-algebraic equations, (1.1) may be rewritten as

$$P(z, Y_0, \ldots, Y_k) f(z) = 0. \quad (3.1)$$

Recall that the total degree of a monomial is

$$\text{Deg}(z^a Y_0^{n_1} \cdots Y_k^{n_k}) = a + n_1 + \cdots + n_k,$$

and that the total degree of a polynomial is the largest degree of its monomials. We can now transcribe the meaning of a reduced equation.

**Proposition 3.1.** Equation (3.1) is reduced if and only if $\text{Deg}P(0, Y_0, \ldots, Y_k) = 1$.

**Proof.** Using the correspondence between monomials and $q$-factors,

$$[z^0]P(z, Y_0, \ldots, Y_k) = \sum_{A \in \mathcal{Q}} \sum_{a=0}^{r_A} r_A A + P(0, \ldots, 0) = P(0, Y_0, \ldots, Y_k).$$

Thus, $P(0, Y_0, \ldots, Y_k)$ has total degree 1 if and only if each $A$ in $\mathcal{Q}$ with $a = 0$ is of the form $(0; \alpha_\ell)$, and therefore is linear. $\blacksquare$

In the reduction steps, we set $f(z) = f_0 + zg(z)$. This corresponds to transforming the equation as follows.

**Proposition 3.2.** The power series $f(z) = f_0 + zg(z)$ solves (3.1) if and only if $P(0, f_0, \ldots, f_0) = 0$ and

$$P(z, f_0 + zq^0 Y_0, f_0 + zq^1 Y_1, \ldots, f_0 + zq^k Y_k) g(z) = 0.$$
Proof. We have

\[ [z^0]P(z, f(z), \ldots, f(q^k z)) = P(0, f_0, \ldots, f_0). \]

The second assertion follows from the fact that

\[ Y_j(f_0 + zg(z)) = f_0 + zq^j g(q^j z) = f_0 + zq^j Y_j g(z). \]

Propositions 3.1 and 3.2 yield the following algorithm for reducing a $q$-algebraic equation. We first define a procedure which removes trivial factors of a polynomial whenever possible. Recall that the order of a polynomial with respect to a variable $Y_j$ is the largest $n$ such that $Y_j^n$ divides the polynomial. We write $\text{ord}_{Y_j}$ for that order.

Procedure RemoveTrivialFactors(polynomial $R$)
return $R/(z^{\text{ord}_z} R_0^{\text{ord}_{Y_0}} R \cdots Y_k^{\text{ord}_{Y_k}} R)$

and the algorithm is

\[ P(z, Y_0, \ldots, Y_k) \leftarrow \text{RemoveTrivialFactors}(P(z, Y_0, \ldots, Y_k)) \]
while $\text{Deg}(P(0, Y_0, \ldots, Y_k)) > 1$ do
solve for $f_0$ in $P(0, f_0, \ldots, f_0) = 0$
\[ P(z, Y_1, \ldots, Y_k) \leftarrow \text{RemoveTrivialFactors}(P(z, f_0 + zq^0 Y_0, f_0 + zq^1 Y_1, \ldots, f_0 + zq^k Y_k)) \]

If this loops terminates, which it may not in rather unusual cases, then the equation $P(z, Y_0, \ldots, Y_k)f(z) = 0$ is reduced. Thus, from now on we assume that the equation is reduced.

Note that the simplicity of the algorithm as it is written hides that the equation $P(0, f_0, \ldots, f_0)$ may have multiple roots, so that if one would like to keep track of all the solutions, an extra tree-like structure is needed.

At the end of subsection 2.2, we mentioned the important aspect that there is no loss of generality to assume $\alpha(Q_0) = 0$, since we can make a change of function $g(z) = \sigma^{-n} f(z)$. This property is awkward on polynomials, because the indices of the variables $Y_j$ have meaning, and can be rewritten as follows.

Proposition 3.3. The formal power series $f = \sigma^n g$ solves (3.1) if and only if

\[ P(z, Y_n, \ldots, Y_{k+n})g(z) = 0. \]
Proof. We have $Y_i f(z) = f(q^i z) = g(q^{i+n} z) = Y_{i+n} g(z)$. 

We then see that we may need to use polynomials in variables with negative indices, which is a substantial distinction with the usual algebraic geometry where the names of variables are irrelevant. So instead of a polynomial in the $k + 2$ variables $z, Y_0, \ldots, Y_k$, we will consider one in $2k + 2$ variables yielding the equation

$$P(z, Y_{-k}, \ldots, Y_k) f(z) = 0, \quad (3.2)$$

and assume that this equation is reduced.

We set

$$P_0(Y_{-k}, \ldots, Y_k) = P(0, Y_{-k}, \ldots, Y_k) - P(0, \ldots, 0)$$

and

$$P_+(z, Y_{-k}, \ldots, Y_k) = P(z, Y_{-k}, \ldots, Y_k) - P_0(Y_{-k}, \ldots, Y_k) - P(z, 0, \ldots, 0).$$

The notation $P_0$ and $P_+$ indicates that these polynomials represent the nonshifting and the shifting part of the equation since

$$\sum_{A \in Q_0} r_A A f(z) = P_0(Y_{-k}, \ldots, Y_k) f(z)$$

and

$$\sum_{A \in Q_+} r_A A f(z) = P_+(z, Y_{-k}, \ldots, Y_k) f(z).$$

Then, $\alpha(Q_0)$ is the largest $j$ such that $P_0(Y_{-k}, \ldots, Y_k)$ does not depend on $Y_{j+1}, \ldots, Y_k$. It can be calculated with the following obvious algorithm.

\begin{verbatim}
  j ← k
  while \( \frac{\partial}{\partial Y_j} P_0(0, Y_{-k}, \ldots, Y_k) = 0 \) do j ← j − 1
  \( \alpha(Q_0) \leftarrow j \)
\end{verbatim}

Similarly, $\alpha(Q_+)$ is the largest $j$ such that $P_+(Y_{-k}, \ldots, Y_k)$ does not depend on $Y_{j+1}, \ldots, Y_k$ and may be calculated with the following algorithm.
Proposition 3.4. For the reduced equation (3.2), condition (2.3.1) is equivalent to
\[ P(0, q^{-kn}, \ldots, q^{kn}) \neq 0 \quad \text{for any } n \geq 0. \]  

Proof. Since the equation is reduced,
\[
\sum_{A \in Q_0} r_A f(q^{\alpha_z} z) = \sum_{-k \leq i \leq k} \sum_{A \in Q_0} r_A f(q^i z)
= \sum_{-k \leq i \leq k} [Y_i] P_0(Y_{-k}, \ldots, Y_k) f(q^i z)
= P_0(f(q^{-k} z), \ldots, f(q^k z)).
\]
The result follows by taking \( f(z) = z^n \).  

Proposition 3.5. Consider a reduced equation (3.2) such that (3.3) holds and \( \alpha(Q_0) \geq \alpha(Q_+) \). If \( P_0(0, \ldots, 0, Y_{\alpha(Q_0)}, 0, \ldots, 0) \neq 0 \), then the unique power series solution has a positive radius of convergence.

Proof. We have
\[
\sum_{A \in Q_0, \alpha_z = \alpha(Q_0)} r_A = \sum_{-k \leq i \leq k} [Y_i] P_0(Y_{-k}, \ldots, Y_k) \mathbb{1}\{ i = \alpha(Q_0) \}
\]
and since the nonshifting factors are linear, this is
\[
P(0, \ldots, 0, Y_{\alpha(Q_0)}, 0, \ldots, 0).
\]
The result then follows from Theorem 2.3.3.
To implement the results of section 2.4, we need an efficient way to calculate the maximal height of the shifting factors involved in $q$-algebraic equations. For this, we decompose the shifting part of the equation as

$$
\sum_{A \in \mathbb{Q}_+} r_A A = \sum_{-k \leq i \leq k} \sum_{A \in \mathbb{Q}_+} \sum_{\alpha_i = i} r_A A.
$$

Therefore,

$$
H(Q) = \max_{-k \leq i \leq k} \max_{A \in \mathbb{Q}_+} i/2a = \max_{A \in \mathbb{Q}_+} i/\left(2 \min_{\alpha_i = i} a\right).
$$

Since $\sum_{A \in \mathbb{Q}_+} r_A A$ corresponds to the part of $P_+$ which contains $Y_i$ but not $Y_{i+1}, \ldots, Y_k$, it corresponds to

$$
P_+(z, Y_{-k}, \ldots, Y_i, 0, \ldots, 0) - P_+(z, Y_{-k}, \ldots, Y_{i-1}, 0, \ldots, 0).
$$

We view this difference as a polynomial $R_i(z)$ in $z$. Then the height $H(\{ A \in \mathbb{Q}_+ : \alpha_i = i \})$ is $i$ divided by twice the order of $R(z)$. This leads to the following algorithm.

**for** $i$ **from** $-k$ **to** $k$ **by increment of 1**

$R_i(z) \leftarrow P_+(z, Y_{-k}, \ldots, Y_i, 0, \ldots, 0) - P_+(z, Y_{-k}, \ldots, Y_{i-1}, 0, \ldots, 0)$

$H_i \leftarrow i/2 \ord_z R_i(z)$

$H(Q) \leftarrow \max_{-k \leq i \leq k} H_i$

Then, the crest contains the shifting part of all monomials obtained by taking $i$ such that $\ord_z R_i(z) = i/2H(Q)$ and taking $[z^{i/2H(Q)}] R_i(z)$; it also contains a part obtained from the nonshifting factor $(0; 0)$, that is, the coefficient

$$
[Y_0]P_0(0, \ldots, 0, Y_0, 0, \ldots, 0) = P_0(0, \ldots, 0, 1, \ldots, 0)
$$

where 1 corresponds to $Y_0 = 1$.

We then view $[z^{i/2H(Q)}] R_i(z)$ as a polynomial in $Y_{-k}, \ldots, Y_i$. The scopes of the corresponding $q$-factors are given by the degree of that polynomials in $Y_i$, and $\ell$ is the total degree of that polynomial. Thus, the crest polynomial can be calculated as follows.
\( C(z) \leftarrow P_0(0, \ldots, 0, Y_0 = 1, 0, \ldots, 0) \)

for all \( i \) such that \( \text{ord}_z R_i(z) = i/2H(\mathcal{Q}) \) do

\( T(Y_{-k}, \ldots, Y_i) \leftarrow [z^{i/2H(\mathcal{Q})}] R_i(z) \)

for all the monomials \( M(z, Y_1, \ldots, Y_i) \) of \( T(Y_{-k}, \ldots, Y_i) \) do

\( r_A \leftarrow M(1, 1, \ldots, 1) \)

\( s(A) \leftarrow \text{ord}_Y M(Y_{-k}, \ldots, Y_i) \)

\( a \leftarrow i/2H(\mathcal{Q}) \)

\( \ell \leftarrow \text{Deg} M(Y_{-k}, \ldots, Y_i) \)

\( C(z) \leftarrow C(z) + r_A s(A) q^{-H(\mathcal{Q})a} a^\ell q^{a-\ell} \)

At the end of this procedure, \( C(z) \) is the crest polynomial \( C_{q, t}(z) \).

4. Further examples. The purpose of this section is to give further examples.

Example 1. Motivated by combinatorics of lattice paths, Drake (2009) considers a set of integers \( S_0 \) which does not contain 0 and the equation

\[
 f(z) = 1 + qz f(z) f(q^2 z) + \sum_{j \in S_0} q^j (j-1) z^j f(z) f(q^2 z) \cdots f(q^{2(j-1)} z).
\]

(4.1)

Though Drake (2009) allows for \( S_0 \) to be infinite, we restrict \( S_0 \) to be finite here (extending Theorem 2.4.3 to allow \( \mathcal{Q} \) to be infinite is possible in some cases, and in a subsequent paper, we will provide such an extension in the context of the \( q \)-Lagrange inversion). In our notation, Drake’s set \( S \) is \( S_0 \cup \{ 0 \} \).

Equation (4.1) is of the form (2.1.1) with \( P(z) = 1 \). Applying \([z^0]\) to both sides of (4.1), we have \( f_0 = 1 \). Since \( f(z) = (0; 0)f(z) \),

\[
 \mathcal{Q} = \{ (0; 0), (1; 0, 2) \} \cup \{ (j; 0, 2, \ldots, 2(j-1)) : j \in S_0 \}.
\]

The height of the \( q \)-factor \( (1; 0, 2) \) is 1, while that of \( (j; 0, 2, \ldots, 2(j-1)) \) is \((j-1)/j\), which is less than 1. Thus, the height of \( \mathcal{Q} \) is 1; its co-height is 1; and its crest is \( \hat{\mathcal{Q}} = \{ (0; 0), (1; 0, 2) \} \). This crest has a unique shifting \( q \)-factor whose coefficient in equation (4.1) is \( r_{(1; 0, 2)} = q \). Corollary 2.4.5 yields

\[
 [z^n] f \sim q^{n(n-1)} c q^n = c q^2.
\]

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as $n$ tends to infinity. Drake’s combinatorial argument, while less
general than ours, is more specific for that equation. Indeed,
equation (5) in Drake (2009) is our equation (4.1), so that his $r_n^{(S)}(q)$
is our $f_n$ and his $r_n^{(S)}(1/q)$ is our $q^{-n^2}f_n$. The constant $c$ is then given
in Drake’s (2009) Theorem 1 and is
\[ c = \prod_{i \geq 1} \frac{1}{1 - q^{-2i}} \left( 1 + \sum_{j \in S_0} q^{-j(2i-1)} \right). \]

**Example 2.** Drake (2009, display (12)) considers also the example
\[ f(z) = 1 + zf(z) + qz^2f(z)f(qz). \tag{4.2} \]
He shows that there exist positive constants $\tilde{c}_0$ and $\tilde{c}_1$, such that
\[ f_{2m} \sim q^{m^2} \tilde{c}_0 \quad \text{and} \quad f_{2m+1} \sim q^{m^2+m} \tilde{c}_1 \tag{4.3} \]
as $n$ tends to infinity. Note that if $n = 2m$, then $m^2 = n^2/4$, while
if $n = 2m + 1$, then $m^2 + m = (n^2/4) - 1/4$. Thus, setting $c_0 = \tilde{c}_0$
and $c_1 = q^{-1/4} \tilde{c}_1$, (4.3) can be rewritten as
\[ f_n \sim q^{n^2/4} c_m \tag{4.4} \]
as $n$ tends to infinity, with $m$ being 0 if $n$ is even and $m$ being 1 if $n$
is odd.

For the $q$-functional equation (4.2), the set of $q$-factors involved is
\[ Q = \{ (0; 0), (1; 0), (2; 0, 1) \}. \]

Its height is $1/4$ and co-height 2. The crest is $\{ (0; 0), (2; 0, 1) \}$ and
contains a unique shifting $q$-factor. The corresponding coefficient
$r_{(2,0,1)}$ in equation (4.2) is $q$. Corollary 2.4.5 implies that there exist
$c_0$ and $c_1$ such that
\[ [z^n]f(z) \sim q^{(1/4)n(n-2)} q^{n^2/2} c_m = q^{n^2/4} c_m \]
where $m$ is the remainder of the division of $n$ by 2. Therefore,
we recovered (4.4), however with the possibility that $c_0$ or $c_1$ may
vanish. To see that they do not vanish, we can either use the proof
of Corollary 2.4.5, or make a direct calculation. Indeed, since all the
coefficients involved in equation (4.2) are positive, Theorem 2.4.3.i

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asserts that the function $U_{q,f_0}$ in (2.4.1) has nonnegative coefficients. The crest polynomial is

$$C_q(z) = 1 - qz^2$$

and has root $1/\sqrt{q}$ and $-1/\sqrt{q}$. We then have, writing $U$ for $U_{q,f_0}$ in (2.4.1),

$$[z^n] \frac{U(z\sqrt{q})}{1 - qz^2} = \frac{1}{2} [z^n] \left( \frac{U(z\sqrt{q})}{1 - qz} + \frac{U(z\sqrt{q})}{1 + qz} \right).$$

Using singularity analysis, we obtain

$$[z^n] \frac{U(z\sqrt{q})}{1 - qz^2} \sim \frac{1}{2} \left( U(1) q^{n/2} + U(-1) (-1)^n q^{n/2} \right) \sim q^{n/2} \frac{1}{2} \left( U(1) + (-1)^n U(-1) \right).$$

So we have $c_0 = (U_q(1) + U_q(-1))/2$ and $c_1 = (U_q(1) - U_q(-1))/2$. Since $U_q$ has nonnegative coefficients, $U_q(1) - U_q(-1)$ does not vanish, and we have indeed $f_n \sim q^{n^2/4} c_m$ with $c_0$ and $c_1$ not being 0.

The combinatorial argument in Drake (2009) provides an explicit value for $c_0$ and $c_1$. Indeed, his $\tilde{m}_n(1) (q)$ is our $f_n$, so that his $m_{2n}^{(1)} (1/q)$ is our $q^{-n^2} f_{2n}$ and his $m_{2n+1}^{(1)} (1/q)$ is our $q^{-(n^2+n)} f_{2n+1}$. Since we have $q^{-n^2} f_{2n} \sim \tilde{c}_0$ and

$$q^{-(n^2+n)} f_{2n+1} = q^{1/4} q^{-(2n+1)^2/4} f_{2n+1} \sim \tilde{c}_1,$$

as $n$ tends to infinity, it follows from Drake’s (2005) Theorem 5 that $c_0$ is his $\Phi_{2}(1/q)$ and $c_1$ his $q^{-1/4} \Psi_{2}(1/q)$, that is, with $r = 1/q$,

$$c_0 = \frac{1}{(r; r)_{\infty} (r^2; r^{12})_{\infty} (r^9; r^{12})_{\infty} (r^{10}; r^{12})_{\infty}}$$

and

$$c_1 = \frac{q^{-1/4}}{(r; r^2)_{\infty} (r^4; r^{12})_{\infty} (r^6; r^{12})_{\infty} (r^8; r^{12})_{\infty} (r^{12}; r^{12})_{\infty}}.$$

Example 3. Gessel (1980) provides also interesting examples of $q$-functional equations, motivated by the $q$-Lagrange inversion. For instance, he considers the equation (Gessel, 1980, equation (10.16))

$$f(z) = 1 + q(1+s)zf(z) + q^3 s^2 h(z)h(gz).$$ (4.5)
The \( q \)-factors involved in this equation are \((0; 0)\), \((1; 0)\) and \((2; 0, 1)\). The crest is reduced to the \( q \)-factor \((0; 0)\) and the shifting one \((2; 0, 1)\). The corresponding coefficient in equation (4.5) is \(r_{(2,0,1)} = q^3 s\).

Corollary 2.4.5 yields

\[
[z^n] f \sim q^{(1/4)n(n-2)} q^{3n/2} s^n c_m \sim q^{(n^2/4)+n} s^n c_m
\]

with \(m\) being 0 or 1 according to the parity of \(n\). Again, as in the previous example, the coefficients \(c_0\) and \(c_1\) do not vanish when the coefficients in equation (4.5) are positive, that is when \(s\) is positive.

**Example 4.** Following Garoufalidis (2004), the colored Jones polynomials for the figure 8 knot are

\[
J_n(q) = \sum_{0 \leq k \leq n-1} q^{nk} \left( \frac{1}{q^{n+1};q} \right)_k \left( \frac{1}{q^{n+1};q} \right)_k
\]

with \(J_0(q) = 1\). Recall the operator \(\sigma f(z) = f(qz)\). Garoufalidis (2004) showed that the generating function \(J(z) = \sum_{n \geq 0} J_n z^n\) satisfies the \(q\)-algebraic equation

\[
C_0 J(z) + zC_1 J(z) + z^2 C_2 J(z) + z^3 C_3 J(z) = 0,
\]

with

\[
C_0 = q\sigma(q^2 + \sigma)(q^5 - \sigma^2)(1 - \sigma)
\]

\[
C_1 = -q^2 \sigma^{-1}(1 + \sigma)(q^4 - \sigma q^3(2q - 1) - q^3 \sigma^2(q^2 - q + 1) + q^4 \sigma(q - 2) + \sigma^4 q^4)(q^3 - \sigma^2)(1 - \sigma)
\]

\[
C_2 = q^7 \sigma^{-1}(1 - \sigma)(1 + \sigma)(1 - q^3 \sigma^2)(q\sigma(q - 2) + \sigma^2(-1 + q - q^2) - \sigma^3(2q - 1) + q\sigma^4)
\]

\[
C_3 = -q^{10} \sigma(1 - \sigma)(1 + q^2 \sigma)(1 - q^5 \sigma^2).
\]

The nonshifting \( q \)-factors come from \(C_0\). By considering \(C_0\) as a polynomial in \(\sigma\), we see that \(\alpha(Q_0) = \deg_\sigma C_0 = 5\). Thus, we consider \(f(z) = \sigma^5 J(z)\) as our new function. It solves the \(q\)-algebraic equation

\[
(D_0 + zD_1 + z^2 D_2 + z^3 D_3) f(z) = 0
\]

with \(D_i = C_i \sigma^{-5}\). The \(D_i\) are Laurent polynomials. This equation has now \(\alpha(Q_0) = 0\). It involves many \( q \)-factors, all of the form \((a; \alpha_1, \ldots, \alpha_\ell)\) for \(a = 0, 1, 2, 3\), the \( q \)-factor \((a; \alpha_1, \ldots, \alpha_\ell)\) coming
from $D_a$. Thus, $\max_{\alpha, \ell} \alpha_\ell$ is $\deg \sigma D_i$. For this equation, the following table then shows how to calculate efficiently the quantities involved in theorem 2.4.3.

| $i$ | 0   | 1   | 2   | 3   |
|-----|-----|-----|-----|-----|
| leading term of $C_i$ | $q\sigma^5$ | $-q^6\sigma^7$ | $q^{11}\sigma^7$ | $-q^{17}\sigma^5$ |
| leading term of $D_i$ | $q$ | $-q^6\sigma^2$ | $q^{11}\sigma^2$ | $-q^{17}$ |
| corresponding $r_A A$ | $q(0;0)$ | $-q^6(1;2)$ | $q^{11}(2;2)$ | $-q^{17}(3;0)$ |
| $H(A)$ | $q$ | $1$ | $1/2$ | $0$ |
| $r_A$ | $q$ | $-q^6$ | $q^{11}$ | $-q^{17}$ |

From this table, we read that the height of the set of $q$-factors involved in the equation for $f$ is 1 and the co-height is 1. The crest is

$$\hat{Q} = \{(0;0), (1;2)\}.$$  

Corollary 2.4.5 yields

$$[z^n]f(z) \sim cq^n(n-1)q^{5n}$$

as $n$ tends to infinity. Since $f_n = q^{5n}J_n$, we conclude that

$$J_n \sim cq^n(n-1)$$  \hspace{1cm} (4.6)

as $n$ tends to infinity.

Since the assumptions for the last assertion of Theorem 2.4.3 does not hold in this example, it is possible that $c$ is 0. This example is instructive since the asymptotic behavior of $J_n$ can be obtained directly from its expression. Indeed, we have

$$J_n = \sum_{0 \leq k \leq n-1} q^{nk} \frac{1 - 1/q^n}{(1 - 1/q^{n-k}) \cdots (1 - 1/q^{n+k})}$$

$$= \sum_{0 \leq j \leq n-1} q^{n(1-j)} \frac{1 - 1/q^n}{(1 - 1/q^{2j+1}) \cdots (1 - 1/q^{2n-1-j})}.$$

Therefore, isolating the term for which $j = 0$,

$$J_n = q^{n(1-1)/2n-1} \frac{1 - 1/q^n}{(1/q; 1/q)_{2n-1}}$$

$$+ q^{n(1-1)/2n} \sum_{1 \leq j \leq n-1} \frac{q^{-nj}}{(1/q^{j+1}; 1/q)_{2(n-j)-1}}.$$
Since $|q| > 1$, we have $0 \leq 1 - 1/|q|^j \leq |1 - 1/q|^j$ for any nonnegative integer $j$, and therefore,

$$
\left| \sum_{1 \leq j \leq n} \frac{q^{-n j}}{(1/q^{j+1}; 1/q)_{2(n-j)-1}} \right| \leq \sum_{1 \leq j \leq n} \frac{|q|^{-n j}}{(1/|q|; 1/|q|)_\infty}
\leq \frac{1}{(1/|q|; 1/|q|)_\infty} |q|^{-n}.
$$

Consequently,

$$
J_n \sim \frac{q^{n(n-1)}}{(1/q; 1/q)_\infty}
$$
as $n$ tends to infinity. We see that (4.6) is in fact sharp.

**Example 5.** To illustrate their results, Cano and Fortuny Ayuso (2012) consider the $q$-algebraic equation given by the polynomial

$$
4Y_4^4 - 9Y_0^2Y_1Y_2 + 2Y_3^2Y_2 + \frac{z}{q^4}Y_0Y_2 - z^3Y_0^2Y_0^2 - \frac{z^3}{q^4}Y_2 - z^3Y_0 + z^5. \tag{4.7}
$$

For this example, we need the imaginary unit $i$ such that $i^2 = -1$. Considering the particular value $q = 4$, they show among other things that this $q$-algebraic equation has a unique solution $f(z) = z^2 + 42\sqrt{2}z^{7/2} + o(z^{7/2})$ and that $f_n$ is of 4-Gevrey order $3/34$. The following result illustrates how Cano and Fortuny Ayuso’s results can be fruitfully used with ours to obtain information on the asymptotic behavior of the coefficients of generalized diverging power series of nearly any $q$-algebraic equation.

**Proposition 4.1.** For all but countably many values of $q$ of modulus greater than 1, (4.7) has a solution a power series $f(z)$ in $z^{1/2}$ such that

$$
f(z) = z^2 + \rho z^{7/2} + o(z^{7/2})
$$

with $\rho^2 = -q(q^2 - 2)(2q - 1)(2q + 1)$ and

$$
f_n \sim c_{q,m} q^{(3n^2 - 63n)/68} \left( \frac{-2}{\rho} \right)^{n/17}.
$$

for $m = n \mod 17$ and $0 \leq m < 17$. 

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In particular, Proposition 4.1 implies that $f_n$ has 4-Gevrey order $\frac{3}{34}$, as indicated by Cano and Fortuny Ayuso (2012).

Our proof shows that the conclusion of Proposition 4.1 is valid for the specific value $q = 4$. When $q = 4$ we have $\rho^2 = -42^2 \times 2$. So, we may take $\rho = 42i\sqrt{2}$, which leads to

$$f_n \sim c_{m}^{n/17} \frac{2^{(3n^2-64n)/34}}{21^{n/17}}$$

as $n$ tends to infinity. Since $i^{(n+17)/17} = i \frac{n}{17}$, we may rewrite (4.8) as

$$f_n \sim \tilde{c}_{m}^{2^{(3n^2-64n)/34}} \frac{21^{n/17}}{21^{n/17}}$$

with now $m = n \text{ mod } 68$ and the additional constraint that $\tilde{c}_{m+17} = i \tilde{c}_m$ for $0 \leq m < 51$. Using Maxima, José Cano computed exactly the first 390 coefficients of the solution and kindly shared and allowed us to use the result of this computation. An inspection of the coefficients reveals that $f_n$ is real if $n$ is even and purely imaginary if $n$ is odd, and that the sign of $f_n$ is the opposite of that of $f_{n+34}$, confirming the relation $\tilde{c}_{m+17} = i \tilde{c}_m$. The following plot shows $\log |f_n 21^{n/17} / 2^{(3n^2-64n)/34}|$.

This plots confirms our result, and could be used to evaluate the coefficients $c_m$ numerically. After Theorem 2.4.3, we made the remark that we can provide an estimate for $\Theta$. The positivity of $\Theta$ implies that the convergence of

$$\left(\frac{-\rho}{2}\right)^{n/17} q^{-(3n^2-63n)/68} f_n - \tilde{c}_m$$
to 0 is exponentially fast. This is confirmed by the plot which exhibits a complete stability for $n$ greater than about 70. While this plot supports that some $\tilde{c}_m$ do not vanish, it is unclear if those of magnitude less than $10^{-10}$ say are zero or not.

**Proof.** Cano and Fortuny-Ayuso’s (2012) method provides the crucial information that for $q = 4$, the equation has a solution which is a power series in $z^{1/2}$. Equipped with this information, we try in general to find a solution as a power series in $z^{1/2}$, which leads to the change of function $g(z) = f(z^2)$ and to define $r$ such that $r^2 = q$; thus $r$ is defined up to a sign. Substituting $z^2$ for $z$ in the equation given by (4.7), we see that if $f$ is a solution, then $g$ solves the equation given by

$$P = 4Y_1^4 - 9Y_0^2Y_1Y_2 + 2Y_0^3Y_2 + \frac{z^2}{r^8}Y_0Y_2 - z^6Y_0^4Y_5 - \frac{z^6}{r^8}Y_2 - z^6Y_0 + z^{10}.$$  

The study of this equation is made with the algorithms described in the previous section, and the Maple code that we used is in the appendix to this paper; this code is a very simple and primitive implementation, easy to read and to adapt to other equations; this is not a polished package.

The equation given by $P$ is not reduced since the nonshifting part contains the nonlinear terms $Y_1^4$, $Y_0^2Y_1Y_2$ and $Y_0^3Y_2$. We apply the reduction algorithm. In fact we need to apply the reduction step 4 times to obtain that $g_0 = g_1 = g_2 = g_3 = 0$ and $g_4 = 1$.

Setting $g(z) = z^4 + z^5h(z)$, the function $h$ solves an equation given by a polynomial made of 41 monomials. This equation is still not reduced. We then apply the reduction procedure 3 more times, to obtain $g_5 = g_6 = 0$ and $g_7$ solves

$$g_7^2 = r^2(2 - r^4)(2r^2 - 1)(2r^2 + 1).$$

Thus, $g_7$ is to be chosen up to a sign, and we may set $g_7 = \rho$. Setting $g(z) = z^2 + \rho z^7 + z^8h(z)$, we obtain that $h_0 = 0$. Thus, to obtain an estimate as accurate as possible, following the remark after Corollary 2.4.4, we apply the reduction step 3 more times, to obtain that $g_8 = g_9 = 0$ and that $g_{10}$ solves

$$(r^6 + 1)g_{10} + r^2(6 - 18r^4 + 2r^6 - 9r^7 - 9r^{10} + 16r^{11}) = 0.$$
Since $|r| > 1$, this equation has a nonvanishing solution $g_{10}$ except when $r$ is one of the 11 roots of the polynomial $6 - 18r^4 + 2r^6 - 9r^7 - 9r^{10} + 16r^{11}$. So we set

$$g(z) = z^4 + \rho z^7 + z^{10}h(z)$$

and obtain that $h_0 \neq 0$ and that $h$ satisfies a $r$-algebraic equation. After simplification, the polynomial corresponding to that equation is made of 397 monomials. The nonshifting factors are given by the polynomial

$$P_0 = \rho(r^6Y_2 + Y_0).$$

Thus, writing $Q$ for the set of $r$-factors involved in this equation for $h$, we have $\alpha(Q_0) = 2$. The shifting $r$-factors are contained in a polynomial $P_+$ which is made of 292 monomials. An application of the algorithm given in the previous section shows that $\alpha(Q_+) = 5$. To check the existence condition (3.3), a computation with Maple shows that $P(0,Y_0,\ldots,Y_3)$ is proportional to

$$Y_0 + r^6Y_2 + r^2(6 - 18r^4 + 2r^6 - 9r^7 - 9r^{10} + 16r^{11}).$$

Thus, this existence condition is that

$$r^{6+2n} + 1 + 6r^2 - 18r^6 + 2r^8 - 9r^9 - 9r^{12} + 16r^{13} \neq 0$$

for all $n$. For a given $n$, this condition fails for at most $(6 + 2n) \vee 13$ values of $r$. Therefore, this condition fails for at most countably many $r$. In particular, if $r = 2$, this condition becomes

$$64 \cdot 4^n + 88\,985 \neq 0$$

which is of course verified.

Having verified the condition, we need to transform the equation to one for which $\alpha(Q_0)$ vanishes so that we can apply Theorem 2.4.3. This is done by setting $h(z) = k(z/r^2)$. For the new equation in $k$, the algorithm of the previous section yields that the height of the equation is $3/34$ and the co-height is 17. The crest is given by the following part of the equation,

$$\rho r^6Y_0 - 2r^{64}z^{17}Y_3.$$
In particular, the crest has a unique shifting \( r \)-factor, \(-2r^{64}z^{17}Y_3\).

We then apply Corollary 2.4.5 to obtain

\[
k_n \sim c_{r,m}r^{3n(n-17)/34} \left( -\frac{2r^{64}}{\rho r^6} \right)^{n/17} \sim c_{r,m}r^{(3n^2+65n)/34} \left( -\frac{2}{\rho} \right)^{n/17}
\]
as \( n \) tends to infinity. Given our change of function, we have for any \( n \) at least 10,

\[
f_n = g_n = h_{n-10} = k_{n-10}/r^{2(n-10)}
\]
The result follows after some calculations.

5. Proof of the Theorems. The main tool is the recursion that the coefficients of the solution must obey. We consider a \( q \)-algebraic equation (2.1.1). We write \( P(z) = \sum_{0 \leq i \leq p} P_i z^i \). Applying \([z^n]\) to both sides of the equation, we obtain for any \( n \geq 0 \),

\[
\sum_{\mathcal{A} \in \mathcal{Q}} r_{\mathcal{A}} \sum_{n_1 + \ldots + n_\ell = n-a} q^{a_1 n_1 + \ldots + a_\ell n_\ell} f_{n_1} \cdots f_{n_\ell} + P_n = 0. \tag{5.1}
\]

5.1. Proof of Theorem 2.3.1. In (5.1) we distinguish according to the shifting and nonshifting \( q \)-factors. Since the nonshifting ones are linear we obtain

\[
- \sum_{\mathcal{A} \in \mathcal{Q}_0} r_{\mathcal{A}} q^{a_1 n} f_n = \sum_{\mathcal{A} \in \mathcal{Q}_+} r_{\mathcal{A}} \sum_{n_1 + \ldots + n_\ell = n-a} q^{a_1 n_1 + \ldots + a_\ell n_\ell} f_{n_1} \cdots f_{n_\ell} + P_n. \tag{5.1.1}
\]

By definition all \( q \)-factors in \( \mathcal{Q}_+ \) are shifting and therefore have \( a \geq 1 \). Thus, we see that under (2.3.1), we can calculate \( f_n \) inductively once \( f_0 \) is determined.

5.2. Proof of Theorem 2.3.3. Let \( L = \max_{\mathcal{A} \in \mathcal{Q}} \ell \). We first consider the sequence defined inductively by \( g_0 = 1 \) and for any \( n \geq 1 \),

\[
g_n = \sum_{n_1 + \ldots + n_L = n-1} g_{n_1} \cdots g_{n_L}. \tag{5.2.1}
\]

This sequence is nonnegative. Furthermore, considering the tuple \((n_1, \ldots, n_L) = (n-1, 0, \ldots, 0)\) in (5.2.1), we see that \( g_n \geq g_{n-1} \).

Lemma 5.2.1. The generating function \( g(z) = \sum_{n \geq 0} g_n z^n \) has a positive and finite radius of convergence.
Proof. The proof of this lemma is inspired by an argument in Fürlinger and Hofbauer (1985). Given recursion (5.2.1), considering \( g(z) \) as a formal power series, we have

\[
g(z) = 1 + zg(z)^L.
\]

Thus,

\[
g(z^{L-1}) = 1 + z^{L-1}g(z^{L-1})^L.
\]

Set \( k(z) = zg(z^{L-1}) \). Then \( k(z) \) is a formal power series and

\[
k(z) = \sum_{n \geq 1} k_n z^n = z \sum_{n \geq 0} g_n z^{n(L-1)}. \]

In particular the sequence \((k_n)\) is nonnegative. Multiplying both sides of (5.2.2) by \( z \), we have

\[
k(z) = z + k(z)^L. \tag{5.2.3}
\]

Setting \( y = k(z) \), (5.2.3) becomes

\[
z = y(1 - y^{L-1}).
\]

The relation \( k(z) = \sum_{n \geq 1} k_n z^n \) then becomes the relation between power series,

\[
y = \sum_{n \geq 0} k_n (y(1 - y^{L-1}))^n.
\]

Since \((k_n)\) is a nonnegative sequence, since \( 0 \leq 1 - y^{L-1} \leq 1 \) whenever \( 0 \leq y \leq 1 \), this relation leads to

\[
y \geq \sum_{n \geq 0} k_n y^n = k(y) \geq 0,
\]

in the range \( 0 \leq y \leq 1 \). This shows that the radius of convergence of \( k(z) \) is positive. Therefore, the radius of convergence of \( g \) is also positive.

Since we assume that \( \alpha(Q_0) \geq \alpha(Q_+), \) the discussion at the end of subsection 2.2 shows that we can assume without any loss of generality that \( \alpha(Q_0) = 0 \) so that \( \alpha(Q_+) \leq 0 \).

Lemma 5.2.2. Consider a reduced \( q \)-algebraic equation whose \( q \)-factors have been collected. If \( \alpha(Q_0) = 0 \) and (2.3.1) holds, then

\[
\inf_{n \in \mathbb{N}} \left| \sum_{A \in Q_0} r_A q^{\alpha_1 n} \right| \neq 0.
\]
Proof. Since (2.3.1) holds, \( \sum_{A \in Q_0} r_A q^{\alpha_1^n} \) does not vanish for any \( n \). Since \( \alpha(Q_0) = 0 \), all the \( \alpha_\ell \) pertaining to a nonshifting \( q \)-factor are nonpositive. Thus,

\[
\lim_{n \to \infty} \left| \sum_{A \in Q_0} r_A q^{\alpha_1^n} \right| = \sum_{\substack{A \in Q_0 \\ \alpha_\ell = 0}} r_A .
\]

(5.2.4)

Since the \( q \)-algebraic equation is reduced and its \( q \)-factors have been collected, there is a unique nonshifting \( q \)-factor with \( \alpha_\ell = 0 \), and \( r_{(0;0)} \neq 0 \). Hence the right hand side of (5.2.4) is \( r_{(0;0)} \) and does not vanish.

Let \((g_n)\) be defined in (5.2.1), with \( g_0 = 1 \). To prove Theorem 2.3.3, it suffices to prove that there exists some positive \( c \) and some \( \gamma \) greater than 1 such that \( |f_n| \leq c \gamma^n g_n \). The proof is by induction. We set \( c = 1 \lor \max_{0 \leq n \leq p} |f_n| \). We will see how to choose \( \gamma \) afterwards.

Given Lemma 5.2.2,

\[
r = \inf_{n \in \mathbb{N}} \left| \sum_{A \in Q_0} r_A q^{\alpha_1^n} \right|
\]

is positive.

Let \( n \) be some integer greater than \( p \). Suppose that we proved that \( |f_i| \leq c \gamma^i g_i \) for any \( 0 \leq i \leq n-1 \). Consider the recursion (5.1.1). If \( A \) is in \( Q_+ \) then all the \( \alpha_\ell \) are nonpositive and \( |q^{\alpha_1 n_1 + \cdots + \alpha_\ell n_\ell}| \leq 1 \) since \( |q| > 1 \). Thus, (5.1.1) yields

\[
r |f_n| \leq \sum_{A \in Q_+} |r_A| \sum_{n_1 + \cdots + n_\ell = n-a} |f_{n_1}| \cdots |f_{n_\ell}| .
\]

(5.2.5)

Using the induction hypothesis and the notation \( L = \max_{A \in Q_\ell} \), this is at most

\[
c^L \gamma^{n-1} \sum_{A \in Q_+} |r_A| \sum_{n_1 + \cdots + n_\ell = n-a} g_{n_1} \cdots g_{n_\ell} .
\]

Since \((g_n)\) is nondecreasing, we see that whenever \( a \) is positive, \( g_{n_1} \leq g_{n_1 + a - 1} \). Since a tuple \((n_1, \ldots, n_\ell)\) may be seen as tuples \((n_1, \ldots, n_\ell, 0, \ldots, 0)\) of length \( L \) and \( g_0 = 1 \),

\[
\sum_{n_1 + \cdots + n_\ell = n-1} g_{n_1} \cdots g_{n_\ell} \leq \sum_{n_1 + \cdots + n_L = n-1} g_{n_1} \cdots g_{n_L} .
\]

(5.2.6)

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Thus, (5.2.5) yields
\[ |f_n| \leq c^{L-1} \sum_{A \in Q_+} |r_A| g_n. \]

We choose \( \gamma \geq c^{L-1} \sum_{A \in Q_+} |r_A|/r \) to obtain \( |f_n| \leq c^{n}g_n \) for all \( n \) greater than \( p \). If \( \gamma \) is large enough, this inequality holds for all integers \( n \).

5.3. Proof of Theorem 2.4.3. The sequence
\[ d_n = H(Q)n(n - h(Q)) \]
is key to our proof. For simplicity of notation, we will write \( H \) for \( H(Q) \) and \( h \) for \( h(Q) \), so that \( d_n = Hn^2 - Hhn \). We also agree that \( d_n = 0 \) if \( n \) is negative. Recall that a sequence like \( (d_n) \) is strictly convex if \( (d_{n+1} - d_n) \) is an increasing sequence.

Throughout this proof, \( (n_1, \ldots, n_\ell) \) denotes a tuple of nonnegative integers.

**Lemma 5.3.1.** The sequence \( (d_n) \) is strictly convex.

**Proof.** Clearly, \( d_{n+1} - d_n = (2n + 1)H - Hh \) is increasing in \( n \). ■

For a \( q \)-factor \( A = (a; \alpha_1, \ldots, \alpha_\ell) \), set
\[ D_A(n_1, \ldots, n_\ell) = d_{n_1 + \cdots + n_\ell + a} - (d_{n_1} + \cdots + d_{n_\ell} + \alpha_1 n_1 + \cdots + \alpha_\ell n_\ell). \]

The following definition is stated to set the notation and recall some known terminology (see MacDonald, 1995, §I.1 for raising operators).

**Definition 5.3.2.** Given two positive integers \( i < j \), the following operators act on tuples of length at least \( j \):
(i) the transposition \( \tau_{i,j} \) permutes the \( i \)-th and \( j \)-th entries.
(ii) the raising operator \( R_{i,j} \) decreases the \( i \)-th entry by 1 and increasing the \( j \)-th entry by 1.

For instance \( R_{3,5}(1,2,3,4,5,6) = (1,2,3,4,6,6) \). Note that transpositions and raising operators leave invariant the sum of the entries of a tuple.
Lemma 5.3.3. Let \( A = (a; \alpha_1, \ldots, \alpha_\ell) \) be a \( q \)-factor and let \( 1 \leq i < j \leq \ell \) be some integers. Then

\[
D_A \circ \tau_{i,j}(n_1, \ldots, n_\ell) = D_A(n_1, \ldots, n_\ell) + (\alpha_j - \alpha_i)(n_j - n_i)
\]

and,

\[
D_A \circ R_{i,j}(n_1, \ldots, n_\ell) = D_A(n_1, \ldots, n_\ell) + d_{n_i - 1} - d_{n_j - 1} + \alpha_i - \alpha_j.
\]

Proof. Both assertions follow from some elementary calculation. 

Because it is a cumbersome quantity in our coming calculations, for a \( q \)-factor \( A = (a; \alpha_1, \ldots, \alpha_\ell) \) and a positive integer \( k \) at most \( \ell \) we set

\[
\theta(A, 1) = Ha^2 + (Hh - \alpha_\ell)a.
\]

and for \( 2 \leq k \leq \ell \),

\[
\theta(A, k) = H(a + k - 1)^2 + (Hh - \alpha_\ell)(a + k - 1) + (k - 1)H(1 - h) + (\alpha_{\ell-k+1} + \cdots + \alpha_{\ell-1}).
\]

Lemma 5.3.4. Let \( A \) be a \( q \)-factor. Let \( n \) be a nonnegative integer with \( n \geq a \). The smallest value of \( D_A(n_1, \ldots, n_\ell) \) when \( n_1 + \cdots + n_\ell = n - a \) and \( \sharp \{ i : n_i > 0 \} = k \) is

\[
n(2Ha - \alpha_\ell + 2H(k - 1)) - \theta(A, k),
\]

and is achieved at the unique tuple \((0, \ldots, 0, 1, \ldots, 1, n - a - k + 1)\).  

Proof. Consider a tuple \((n_1, \ldots, n_\ell)\) whose entries sum to \( n - a \). If \( i < j \) and \( n_i > n_j \), we apply a transposition \( \tau_{i,j} \) so that Lemma 5.3.3 yields

\[
D_A \circ \tau_{i,j}(n_1, \ldots, n_\ell) \leq D_A(n_1, \ldots, n_\ell).
\]

Thus, if we are given \((n_1, \ldots, n_\ell)\) up to a permutation, the smallest value of \( D_A(n_1, \ldots, n_\ell) \) is achieved when \( n_1 \leq \cdots \leq n_\ell \).

Consider such an ordered tuple. If \( n_i > 1 \) and \( i < j \leq \ell \), Lemma 5.3.1 yields

\[
d_{n_i} - d_{n_i - 1} - (d_{n_j + 1} - d_{n_j}) \leq 0,
\]
and, since $\alpha_i \leq \alpha_j$, we also have $\alpha_i - \alpha_j \leq 0$. Therefore, Lemma 5.3.3 yields

$$D_A \circ R_{i,j}(n_1, \ldots, n_\ell) \leq D_A(n_1, \ldots, n_\ell).$$

To prove the lemma, we start with an arbitrary tuple $(n_1, \ldots, n_\ell)$ with exactly $k$ positive entries. We order it, and apply several raising operators $R_{i,j}$, $i < j$, to bring it to the form $(0, \ldots, 0, 1, \ldots, 1, n - a - k + 1)$. Each time, $D_A$ decreases.

It follows that the minimum of $D_A$ over the given tuples is achieved at the unique tuple $(0, \ldots, 0, 1, \ldots, 1, n - a - k + 1)$ and is

$$d_n - (k-1)d_1 - d_{n-a-k+1} - (\alpha_{\ell-k+1} + \cdots + \alpha_{\ell-1} + (n-a-k+1)\alpha_\ell).$$

The conclusion of the lemma then follows from the definition of $d_n$ and an elementary calculation.

Lemma 5.3.4 has the following consequences.

**Lemma 5.3.5.** There exist some positive $\Theta$ and $\theta^*$ such that

(i) for any $A$ in $\hat{Q}$,

$$\min_{\substack{n_1 + \cdots + n_\ell = n-a \atop \sharp \{i : n_i > 0\} \geq 2}} D_A(n_1, \ldots, n_\ell) \geq \Theta n - \theta^*; \quad (5.3.1)$$

(ii) for any $A$ in $Q \setminus \hat{Q}$,

$$\min_{n_1 + \cdots + n_\ell = n-a} D_A(n_1, \ldots, n_\ell) \geq \Theta n - \theta^*; \quad (5.3.2)$$

(iii) If only the $i$-th entry of $(n_1, \ldots, n_\ell)$ does not vanish,

$$D_A(0, \ldots, 0, n - a, 0, \ldots, 0) = n(2aH - \alpha_i) - Ha^2 - Hha + \alpha_i a.$$

**Proof.** Set $\theta^* = \max_{A \in Q} \max_{1 \leq k \leq \ell} \theta(A, k)$ where the $\ell$ in the inner maximum pertains to the $A$ in the outer one. We take $\Theta$ to be the smallest of $2H$, $\min_{A \in \hat{Q} \setminus \hat{Q}} (2Ha - \alpha_\ell)$ and 1.

(i) If $A$ is in $\hat{Q}$, then $2Ha - \alpha_\ell = 0$. Thus, Lemma 5.3.4 implies that for $A$ in $\hat{Q}$ and if $(n_1, \ldots, n_\ell)$ has at least 2 positive entries, then

$$D_A(n_1, \ldots, n_\ell) \geq 2nH - \theta^*.$$
(ii) If \( A \) is in \( Q \setminus \tilde{Q} \), then \( 2Ha - \alpha_\ell \) is positive. A tuple \( (n_1, \ldots, n_\ell) \) with \( n_1 + \cdots + n_\ell = n - a \) has at least one positive entry. Lemma 5.3.4 implies that \( D_A(n_1, \ldots, n_\ell) \) is at least \( n(2Ha - \alpha_\ell) - \theta^* \).

(iii) This is \( d_n - d_{n-a} - \alpha_i(n - a) \).

Recall that throughout the paper we assume \(|q| > 1\). We define

\[
g_n = q^{-d_n} f_n. \tag{5.3.3}
\]

It is convenient to agree that \( g_n = 0 \) if \( n \) is negative. We write

\[
P(z) = \sum_{0 \leq i \leq p} P_i z^i
\]

the polynomial involved in (2.1.1).

**Lemma 5.3.6.** For any nonnegative integer \( n \),

\[
- \sum_{A \in Q_0} r_A q^{\alpha_1 n} g_n = q^{-d_n} P_n
\]

\[
+ \sum_{A \in Q_+} r_A \sum_{n_1 + \cdots + n_\ell = n-a} q^{-D_A(n_1, \ldots, n_\ell)} g_{n_1} \cdots g_{n_\ell}. \tag{5.3.4}
\]

**Proof.** Multiplying both sides of (5.1.1) by \( q^{-d_n} \), the result follows after some simple calculations.

**Lemma 5.3.7.** There exist some positive \( c \) and \( G \) such that \( |g_n| \leq c G^n \) for any \( n \geq 0 \).

**Proof.** The proof is by induction. We write \( D_A(0, \ldots, n - a, \ldots 0) \), for \( D_A(0, \ldots, 0, n - a, 0, \ldots, 0) \) where \( n - a \) is the \( i \)-th component. We set \( c = 1 \lor \max_{0 \leq i \leq p} |g_i| \) and we will see how to determine \( G \). Provided that \( G \geq 1 \), we have \( |g_i| \leq c G^i \) for any \( 0 \leq i \leq p \). Let \( n \) be an integer greater than \( p \). Assume that we have \( |g_i| \leq c G^i \) for any \( i \leq n - 1 \). We split the second term on the right hand side of
(5.3.4) as the sum of

\[ V_{1,n} = \sum_{A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}}} r_A \sum_{\substack{1 \leq i \leq \ell \\ \alpha_i = \alpha}} q^{-DA(0, \ldots, n-a, \ldots, 0)_i} g_{n-a} g_0^{\ell-1}, \]

\[ V_{2,n} = \sum_{A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}}} r_A \sum_{\substack{1 \leq i \leq \ell \\ \alpha_i \neq \alpha}} q^{-DA(0, \ldots, n-a, \ldots, 0)_i} g_{n-a} g_0^{\ell-1}, \]

\[ V_{3,n} = \sum_{A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}}} r_A \sum_{\substack{n_1 + \cdots + n_{\ell} = n-a \\ \forall i: n_i > 0}} q^{-DA(n_1, \ldots, n_{\ell})} g_{n_1} \cdots g_{n_{\ell}}, \]

\[ V_{4,n} = \sum_{A \in \mathcal{Q}_+ \setminus \hat{\mathcal{Q}}} r_A \sum_{n_1 + \cdots + n_{\ell} = n-a} q^{-DA(n_1, \ldots, n_{\ell})} g_{n_1} \cdots g_{n_{\ell}}. \]

Consider \( V_{1,n} \). Let \( A \) be in \( \mathcal{Q}_+ \cap \hat{\mathcal{Q}} \). If \( \alpha_i = \alpha \), then Lemma 5.3.5.iii yields

\[ DA(0, \ldots, n-a, \ldots, 0)_i = Ha(a-h). \]

Thus,

\[ V_{1,n} = \sum_{A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}}} r_A s(A) q^{-Ha(a-h)} g_{n-a} g_0^{\ell-1}. \quad (5.3.5) \]

Using the induction hypothesis, if \( A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}} \), then \( a \geq 1 \) and \( |g_{n-a}| \leq c G^{n-1} \). Thus, introducing

\[ C_1 = \sum_{A \in \mathcal{Q}_+ \cap \hat{\mathcal{Q}}} |r_A| s(A) |g_0|^{\ell-1}, \]

we have

\[ |V_{1,n}| \leq C_1 c G^{n-1}. \]

We now consider \( V_{2,n} \). If \( A \) is \( \mathcal{Q}_+ \cap \hat{\mathcal{Q}} \) and \( \alpha_i \neq \alpha \), then Lemma 5.3.5.iii yields

\[ DA(0, \ldots, n-a, \ldots, 0)_i = n(\alpha - \alpha_i) + Ha(a-h) + (\alpha_i - \alpha) a. \]

Moreover, if \( \alpha_i \neq \alpha \), then \( \alpha_i - \alpha_i \geq 1 \). Hence,

\[ DA(0, \ldots, n-a, \ldots, 0)_i \geq n + Ha(a-h) + (\alpha_i - \alpha) a. \quad (5.3.6) \]
Therefore, using the induction hypothesis, there exists $C_2$ such that

$$|V_{2,n}| \leq |q|^{-n} C_2 c G^{n-1}.$$  

We consider now $V_{3,n}$. In what follows, we will write $\ell^*$ for $\max\{ \ell : A \in Q \}$. If $A$ is in $Q_+ \cap \hat{Q}$ and two $n_i$ are positive, Lemma 5.3.5.i yields

$$D_A(n_1, \ldots, n_\ell) \geq \Theta n - \theta^*.$$  

Moreover, if $n_1 + \cdots + n_\ell = n - a$, then the largest $n_i$ is at most $n - a$, which is at most $n - 1$ since $A$ is shifting. Therefore, using the induction hypothesis, and bounding by $n^\ell$ the number of tuples $(n_1, \ldots, n_\ell)$ with at least two postive entries and $n_1 + \cdots + n_\ell = n - a$, we obtain

$$|V_{3,n}| \leq \sum_{A \in Q_+ \cap \hat{Q}} |r_A| \sum_{n_1 + \cdots + n_\ell = n - a} \sum_{\{i : n_i > 0\} \geq 2} |q|^{-n + \theta^*} c^\ell G^{n-a}$$

$$\leq \sum_{A \in Q_+ \cap \hat{Q}} |r_A||q|^{-n + \theta^*} c^\ell G^{n-1} n^\ell$$

$$\leq |q|^{-n} c^{\ell^*} \sum_{A \in Q_+ \cap \hat{Q}} |r_A||q|^{\theta^*} c^{\ell^*} G^{n-1}.$$  

Since the sequence $(|q|^{-n} c^{\ell^*})$ is bounded, there exists $C_3$ such that

$$|V_{3,n}| \leq C_3 c^{\ell^*} G^{n-1}.$$  

Finally, if $A$ is in $Q_+ \setminus \hat{Q}$, then Lemma 5.3.5.ii yields

$$D_A(n_1, \ldots, n_\ell) \geq \Theta n - \theta^*.$$  

Thus, using the induction hypothesis,

$$|V_{4,n}| \leq \sum_{A \in Q_+ \setminus \hat{Q}} |r_A| \sum_{n_1 + \cdots + n_\ell = n - a} |q|^{-n + \theta^*} c^\ell G^{n-a}$$

$$\leq q^{-n} c^{\ell^*} \sum_{A \in Q_+ \setminus \hat{Q}} |r_A||q|^{\theta^*} c^{\ell^*} G^{n-1}.$$  

Thus, there exists some constant $C_4$ such that

$$|V_{4,n}| \leq C_4 c^{\ell^*} G^{n-1}.$$  

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Combining all these upper bounds, we obtain

$$|V_{1,n} + V_{2,n} + V_{3,n} + V_{4,n}| \leq (C_1 + C_2 + C_3 + C_4) \epsilon^* G^{n-1}. $$

Considering (5.3.4) for $n \geq p + 1$,

$$\left| \sum_{A \in Q_0} r_A q^{\alpha_1 n} \right| |g_n| \leq (C_1 + C_2 + C_3 + C_4) \epsilon^* G^{n-1}. $$

Using Lemma 5.2.2, we then take $G$ to be any number we like greater than 1 and

$$\inf_{n \in \mathbb{N}} \left| \sum_{A \in Q_0} r_A q^{\alpha_1 n} \right|^\epsilon - 1$$

to obtain that indeed $|g_n| \leq c G^n$.

It follows from Lemma 5.3.7 that the generating function $g(z) = \sum_{n \geq 0} g_n z^n$ is finite in a neighborhood of the origin. Let us define

$$\tilde{g}(z) = \sum_{n \geq 0} |g_n| |z|^n. $$

The radius of convergence of $g$ and $\tilde{g}$ are identical, and we write $\rho$ for this common radius. We can now prove Theorem 2.4.3.

We consider identity (5.3.4), multiply both sides by $z^n$ and sum over $n$. The left hand side of (5.3.4) provides the term

$$- \sum_{A \in Q_0} r_A g(q^{\alpha_1} z) = - \sum_{A \in Q_0 \cap \hat{Q}} r_A g(z) - \sum_{A \in Q_0 \setminus \hat{Q}} r_A g(q^{\alpha_1} z). $$

Since $\alpha(Q_0) = 0$ in the assumption of Theorem 2.4.3, if $A$ is in $Q_0 \setminus \hat{Q}$, then $\alpha_1 \leq -1$, so that $g(q^{\alpha_1} z)$ has radius of convergence at least \( q \rho \).

For $n$ greater than the degree $p$ of $P$, the right hand side of (5.3.4) is decomposed into $V_{1,n}, V_{2,n}, V_{3,n}$ and $V_{4,n}$ as in the proof of Lemma 5.3.7. Given (5.3.5), we have

$$V_1(z) = \sum_{n \geq 0} V_{1,n} z^n = \sum_{A \in Q_0 \cap \hat{Q}} r_A s(A) q^{-H(a-h)} g_{\ell-1}^{a} g(z). $$

Thus,

$$V_1(z) + \sum_{A \in Q_0} r_A g(q^{\alpha_1} z) = C_{q,g_0}(z) g(z) + \sum_{A \in Q_0 \setminus \hat{Q}} r_A g(q^{\alpha_1} z). \quad (5.3.7)$$

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Next, the part from $V_{2,n}$ is

$$V_2(z) = \sum_{n \geq 0} V_{2,n} z^n$$

$$= \sum_{A \in Q_+ \cap \hat{Q}} r_A \sum_{1 \leq i \leq \ell} \sum_{n \geq 0} q^{-D_A(0, \ldots, n-a, \ldots, 0)} z^n g_{n-a} z^n g_{n-a-1}.$$

Given (5.3.6),

$$\sum_{n \geq 0} |q|^{-D_A(0, \ldots, n-a, \ldots, 0)} |g_{n-a}| |z^n| \leq C \sum_{n \geq 0} |q|^{-n} |g_{n-a}| |z|^n$$

$$= C |z/q|^{a(q/z)}.$$

Therefore, the radius of convergence of $V_2(z)$ is at least $q \rho$.

Similarly, the radius of convergence of $V_3(z) = \sum_{n \geq 0} V_{3,n} z^n$ is at least the smaller of the radiiuses of convergence of

$$V_{A,3}(z) = \sum_{n \geq 0} \sum_{n_1 + \cdots + n_\ell = n-a, \ell \geq 2} q^{-D_A(n_1, \ldots, n_\ell)} g_{n_1} \cdots g_{n_\ell} z^n$$

where $A$ runs over $Q_+ \cap \hat{Q}$. Using Lemma 5.3.5.i, $|V_{A,3}(z)|$ is at most

$$\sum_{n \geq 0} \sum_{n_1 + \cdots + n_\ell = n-a, \ell \geq 2} |q|^{-\Theta n + \Theta^*} |g_{n_1}| \cdots |g_{n_\ell}| |z|^{n_1 + \cdots + n_\ell + a}$$

$$\leq |q|^{\Theta^*} \sum_{n \geq 0} \sum_{n_1 + \cdots + n_\ell = n-a} |q^{-\Theta z} g_{n_1}| \cdots |q^{-\Theta z} g_{n_\ell}| |q|^{-\Theta a} |z|^a$$

$$\leq |q|^{\Theta^*} |z|^{a(q/z)^\ell}.$$

Therefore, the radius of convergence of $V_{A,3}$ is at least $q^\Theta \rho$ and so is that of $V_3(z)$.

Finally, the radius of convergence of $V_4(z) = \sum_{n \geq 0} V_{4,n} z^n$ is at least the smaller of the radiiuses of convergence of

$$V_{A,4}(z) = \sum_{n \geq 0} \sum_{n_1 + \cdots + n_\ell = n-a} q^{-D_A(n_1, \ldots, n_\ell)} g_{n_1} \cdots g_{n_\ell} z^n$$

where $A$ is in $Q_+ \setminus \hat{Q}$. Using Lemma 5.3.5.ii, we obtain that $|V_{A,4}(z)|$ is at most

$$\sum_{n \geq 0} \sum_{n_1 + \cdots + n_\ell = n-a} |q|^{-\Theta(n_1 + \cdots + n_\ell + a)} |z|^{n_1 + \cdots + n_\ell + a} |g_{n_1}| \cdots |g_{n_\ell}|$$

$$\leq |z|^a |q|^{\Theta^* - \Theta a} |g(z/q)^\ell|.$$
Therefore, the radius of convergence of $V_4(z)$ is at least $|q|^\Theta \rho$.

Combining (5.3.7) with our estimates on the radius of convergence of $V_1(z), \ldots, V_4(z)$, we see that the function

$$U(z) = \sum_{A \in Q_0 \setminus \hat{Q}} r_A g(q^{\alpha_1} z) + V_2(z) + V_3(z) + V_4(z) + \sum_{0 \leq i \leq p} P_i q^{-d_i} z^i$$

(5.3.8)

does not have a singularity of radius at least $q^{\min(\Theta, 1)}$ as $z \to 0$. Moreover, given (5.3.4) and that $g_0 = f_0$, we have

$$C_{q,f_0}(z)g(z) = -U(z).$$

(5.3.9)

Since $\rho$ is positive and $C_{q,f_0}$ is a polynomial in $z$, this relation forces $\rho$ to be the smallest modulus of the zeros of the crest polynomial. This proves the first assertion of Theorem 2.4.3. Since $U$ has radius of convergence $q^\Theta \rho$, identity (5.3.9) shows that $C_{q,f_0}(z)g(z)$ has removable singularities in the open disk centered at the origin and of radius $q^\Theta \rho$. This proves the second assertion of Theorem 2.4.3.

To prove the third assertion, consider the identity (5.3.4). If $\sum_{A \in Q_0} r_A q^{\alpha_1} n$ is negative, its left hand side is of the sign of $g_n$. If the $P_i$ are nonnegative as well as the $r_A$ for $A$ in $Q_+$, then we can use (5.3.4) to show inductively that the sign of $g_n$ is positive whenever that of $g_0$ is.

Finally, if $Q_0$ has a unique element, it must be $(0; 0)$ and it belongs to the crest. Thus, (5.3.8) becomes

$$U(z) = V_2(z) + V_3(z) + V_4(z) + \sum_{0 \leq i \leq p} P_i q^{-d_i} z^i.$$

(5.3.10)

Since $(f_n)$ is nonnegative, so is $(g_n)$. From their definition, we check that $V_{2,n}$, $V_{3,n}$, $V_{4,n}$ and $P_i$, $0 \leq i \leq p$, have all the same sign under all the assumptions of Theorem 2.4.3.iii. Thus $([z^n]U(z))$ is a sequence of constant sign, which is that of $r_A$, $A \in Q_+$. The only way for (5.3.10) to vanish is that $(g_n)$ is a sequence whose elements are all 0, that is $(f_n) = 0$ or $f = 0$.

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6. **Appendix.** We reproduce the **Maple** code which we used in
example 5 of section 4.

The first procedure, getCoeff(R) takes a polynomial $R$ and solves for $f_0$ in the equation $R(0, f_0, \ldots, f_0) = 0$.

> getCoeff:=proc(R):
> RootOf(factor(subs(z=0, Y0=c, Y1=c, Y2=c, Y3=c,
> Y4=c, Y5=c, R)), c);
> end proc:

The next procedure isReduced(R) takes a polynomial $R$ and returns true if the equation given by that polynomial is reduced.

> isReduced:=proc(R):
> evalb(degree(subs(z=0, R), [Y0,Y1,Y2,Y3,Y4,Y5]))
> = 1):
> end proc:

The next procedure reduceEquation(R,f0) takes a polynomial $R$; it makes the change of function $f(z) = f_0 + zg(z)$ in the equation $R(z,Y_0,\ldots,Y_5)f(z) = 0$ and returns the new polynomial that encodes the new equation in $g$.

> reduceEquation:=proc(R,f0):
> Q := simplify(subs(Y0=f0+z*Y0, Y1=f0+z*r*Y1,
> Y2=f0+z*r^2*Y2, Y3=f0+z*r^3*Y3,
> Y4=f0+z*r^4*Y4, Y5=f0+z*r^5*Y5,
> R));
> lz := ldegree(Q,z);
> simplify(Q/z^lz);
> end proc:

We input the polynomial in $r$ (not in $q$) accounting for the change of function $g(z) = f(z^2)$.

\[
P_0 := 4*Y1^4-9*Y0^2*Y1*Y2+2*Y0^3*Y2+2*Y0*Y2/r^8
-2*z^6*Y0^4*Y5-2-z^6*Y2/r^8+z^6*Y0+z^10
\]

Then, we proceed with the reduction steps.

> isReduced(P0);
At this step the system tells us that $g_1$ it is a root of
\[ g_1^2(1 + g_1^2 r^8 (r - 2)(4r - 1) + 1). \]

When $r = 2$ or $r = 1/4$, the only possibility is $g_1 = 0$, but in general one could choose $g_1$ differently. We follow Cano and Fortuny Ayuso's path and take $g_1 = 0$.

```plaintext
> g0 := getCoeff(P0); 0
> P1 := r^6*reduceEquation(P0, g0):
> isReduced(P1); false
> g1 := getCoeff(P1);

> g1 := 0:
> P2 := reduceEquation(P1, g1)/r^2:
> isReduced(P2); false
> g2 := getCoeff(P2);
> P3 := reduceEquation(P2, g2)/r^2:
> isReduced(P3); false
> g3 := getCoeff(P3);
> P4 := reduceEquation(P3, g3)/r^2:
> isReduced(P4); false
> g4 := getCoeff(P4); 1
> P5 := reduceEquation(P4, g4)/r^2:
> isReduced(P5); false
> g5 := getCoeff(P5);
> P6 := reduceEquation(P5, g5)/r^2:
> isReduced(P6); false
> g6 := getCoeff(P6);
```

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\[
g_{10} = \frac{-r^2(16r^{11} - 9r^{10} - 9r^7 + 2r^6 - 18r^4 + 6)}{r^6 + 1}
\]

We can check how many monomials are in \(P_{10}\):

\[
\text{nops}(P_{10})\quad 397
\]

Now we have a reduced equation with a solution whose constant term does not vanish. We calculate \(P_0\) and \(P_+\).

\[
P := P_{10};
P0 := \text{subs}(z=0, P) - \text{subs}(z=0, Y0=0, Y1=0, Y2=0, Y5=0, P);
PPlus := P - P0 - \text{subs}(Y0 = 0, Y1 = 0, Y2 = 0, Y5 = 0, P); \text{nops}(PPlus);
\]

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This gives $P_0 = a(Y_0 + r^6Y_2)$, while $P_+$ has 292 terms. We then calculate $\alpha(Q_0)$.

```maple
> diff(P0, Y5); diff(P0, Y2);
```

We obtain $\partial P_0/\partial Y_0 = 0$ and $\partial P_0/\partial Y_2 = r^6a$. Therefore $\alpha(Q_0) = 2$. We then calculate $\alpha(Q_+)$.

```maple
> evalb(diff(PPlus, Y5) = 0);
false
```

We then have $\alpha(Q_+) = 5$. We may then check condition (3.3). However, this only confirms the result of Cano and Fortuny Ayuso (2012).

```maple
> simplify(subs(z=0,P)/a);
```

We obtain

$$Y_0 + 2r^8 + 16r^{13} + 6r^2 - 9r^9 - 9r^{12} - 18r^6 + r^6Y_2.$$ 

To apply Theorem 2.4.3, we need $\alpha(Q_0) = 0$. This amounts to make the change of function $f(z) = g(z/(r^2))$. For the new unkown $g$, the equation is obtained by the change of variables $(Y_0, Y_1, ..., Y_5) \rightarrow (Y_{-2}, Y_{-1}, ..., Y_3)$.

```maple
> NPO := subs(Y0=Ym2, Y1=Ym1, Y2=Y0, Y5=Y3, P0);
> NPPPlus := subs(Y0=Ym2, Y1=Ym1, Y2=Y0, Y5=Y3, PPlus);
```

We then calculate the heights of $R_{-2}, ..., R_3$. The heights of $R_{-2}$ and $R_{-1}$ are negative, and cannot be maximal, while that of $R_0$ vanishes. So we need only the heights of $R_1$ and $R_2$.

```maple
> R1 := subs(Y2=0, Y3=0, NPPPlus)-subs(Y1=0, Y2=0, Y3=0, NPPPlus): H1:=1/(2*1degree(R1, z));
0
> R2 := subs(Y3=0, NPPPlus)-subs(Y2=0, Y3=0, NPPPlus): H2 := 2/(2*1degree(R2, z));
0
```
We obtain $H_1 = H_2 = 0$ and $H_3 = \frac{3}{34}$. Thus, the height is given by $H_3$, that is $\frac{3}{34}$. Since $\frac{3}{2H_3} = 17$, we need the term in $z^{17}$ in $R_3$.

We obtain $H_1 = H_2 = 0$ and $H_3 = \frac{3}{34}$. Thus, the height is given by $H_3$, that is $\frac{3}{34}$. Since $\frac{3}{2H_3} = 17$, we need the term in $z^{17}$ in $R_3$.

This coefficient is $-2r^{64}Y_3$. For this term, $r_A = -2r^{64}$, $s(A) = 1$, $a = 17$ and the height is also 17. To calculate the crest polynomial, we also need the term involving the $r$-factor $(0; 0)$.

which is $pr^{6}Y_0$. Thus, the crest polynomial is

$$C_{r,t}(z) = pr^{6} - 2r^{64}z^{17}.$$