CONSTRUCTION OF ROTA-BAXTER ALGEBRAS VIA HOPF MODULE ALGEBRAS

RUN-QIANG JIAN

Abstract. We propose the notion of Hopf module algebras and show that the projection onto the subspace of coinvariants is an idempotent Rota-Baxter operator of weight -1. We also provide a construction of Hopf module algebras by using Yetter-Drinfeld module algebras. As an application, we prove that the positive part of a quantum group admits idempotent Rota-Baxter algebra structures.

1. Introduction

Rota-Baxter operators originate from a work of G. Baxter on probability theory in 1960 [5]. In that paper, Baxter showed that some well-known identities in the theory of fluctuations for random variables can be deduced from a simple relation of operators on a commutative algebra. At the end of 1960s, based on this work and others, Rota introduced the notion of Baxter algebra, which is called Rota-Baxter algebra nowadays in honor of the contribution of Rota, in his fundamental papers [21]. Roughly speaking, a Rota-Baxter algebra is an associative algebra together with a linear endomorphism which is analogous to the integral operator. During the past four decades, this algebraic object has been investigated extensively by many mathematicians with various motivations. At present, Rota-Baxter algebras have become a useful tool in many branches of mathematics, such as combinatorics [9], Loday type algebras [8], pre-Lie and pre-Poisson algebras [13, 2], [1], multiple zeta values [11, 16], and so on. Besides their own interest in mathematics, Rota-Baxter algebras also have many important applications in mathematical physics. For instance, the theory of non-commutative Rota-Baxter algebras with idempotent Rota-Baxter operator is used to provide an algebraic setting in the Hopf algebraic approach to the Connes-Kreimer theory of renormalization in perturbative quantum field theory (cf. [6], [7], [12], [13]).

Recently, in the work [17], we provide examples of Rota-Baxter algebras from the so-called quantum quasi-shuffle algebras. It leads to the notion of braided Rota-Baxter algebra. This also brings our attention to the relation between Rota-Baxter algebras and quantum algebras. On one hand, this relation will enable one to use tools from Rota-Baxter algebras to study quantum algebras. On the other hand, one would expect to provide more interesting examples of Rota-Baxter algebras via quantized algebras. The second point of view is the aim of the present paper. We first attempt to construct idempotent Rota-Baxter operators for their importance.

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which has been mentioned at the end of the first paragraph. In other words, we need some sort of good algebras and projections. In order to provide such good things, we introduce the notion of Hopf module algebra which mixes Hopf module structure and algebra structure via some relevant compatibility conditions, and consider the projection onto the subspace of coinvariants. These new objects are very natural since they can be constructed from the bosonization of algebras in the category of Yetter-Drinfeld modules. A particular important example is the bosonization of Nichols algebras which is closely related to quantum groups. Hence, thanks to these relations, we can show that the positive part of a quantum group admits idempotent Rota-Baxter algebra structures.

This paper is organized as follows. In Section 2, we introduce the notion of Hopf module algebra and provide a construction of idempotent Rota-Baxter operators of weight -1 on these algebras. Consequently, we provide a construction of Hopf module algebras via Yetter-Drinfeld module algebras, as well as an application to quantum groups, in Section 3.

2. Hopf module algebras and the main construction

Throughout this paper, we fix a ground field $\mathbb{K}$ of characteristic 0 and assume that all vector spaces, algebras, coalgebras and tensor products are defined over $\mathbb{K}$. An algebra is always assumed to be associative, but not necessarily unital.

We always denote by $(H, \Delta, \varepsilon, S)$ a Hopf algebra. We adopt Sweedler’s notation for coalgebras and comodules: for any $h \in H$, denote

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)},$$

for a left $H$-comodule $(M, \delta_L)$ and any $m \in M$, denote

$$\delta_L(m) = \sum m_{(-1)} \otimes m_{(0)},$$

and for a right $H$-comodule $(N, \delta_R)$ and any $n \in N$, denote

$$\delta_R(n) = \sum n_{(0)} \otimes n_{(1)}.$$

Let $M$ and $N$ be two left $H$-modules. As usual, we consider $M \otimes N$ as a left $H$-module via the diagonal action $h \cdot (m \otimes n) = \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n$ for $h \in H$, $m \in M$ and $n \in N$. A similar convention is used for right modules, left comodules and right comodules.

**Definition 2.1** ([22]). A right (resp. left) $H$-Hopf module is a vector space $M$ equipped simultaneously with a right (resp. left) $H$-module structure and a right (resp. left) $H$-comodule structure $\delta_R$ (resp. $\delta_L$) such that $\delta_R(m \cdot h) = \delta_R(m) \Delta(h)$ (resp. $\delta_L(h \cdot m) = \Delta(h) \delta_L(m)$) whenever $h \in H$ and $m \in M$.

Let $M$ be a right $H$-Hopf module. We denote by $M^R$ the subspace of right coinvariants, i.e.,

$$M^R = \{ m \in M | \delta_R(m) = m \otimes 1_H \}.$$

It is well known that the map $P_R : M \to M$ given by $P_R(m) = \sum m_{(0)} \cdot S(m_{(-1)})$ for $m \in M$ is the projection from $M$ onto $M^R$ (cf. [22]). Similarly, for a left $H$-Hopf module, we can define the subspace $M^L$ of left coinvariants and have the projection formula $P_L(m) = \sum S(m_{(-1)}) \cdot m_{(0)}$. 
Now we consider algebras in the category of $H$-Hopf modules.

**Definition 2.2.** A right $H$-Hopf module algebra is a right $H$-Hopf module $M$ together with an associative multiplication $\mu : M \otimes M \to M$ such that for any $h \in H$ and $m, m' \in M$,

\[
\begin{cases}
    (mm') \cdot h = m(m' \cdot h), \\
    \delta_R(mm') = \sum m_{(0)}m'_{(0)} \otimes m_{(1)}m'_{(1)},
\end{cases}
\]

where $mm' = \mu(m \otimes m')$.

The compatibility conditions between $\mu$ and the Hopf module structure mean that $\mu$ is both a module morphism and a comodule morphism. We can define the left $H$-Hopf module algebra in a similar way.

In the following, we will construct Rota-Baxter operators on Hopf module algebras. We first recall the definition of Rota-Baxter algebras.

**Definition 2.3.** Let $\lambda$ be an element in $\mathbb{K}$. A pair $(R, P)$ is called a Rota-Baxter algebra of weight $\lambda$ if $R$ is an algebra and $P$ is a linear endomorphism of $R$ satisfying that for any $x, y \in R$,

\[ P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy). \]

The map $P$ is called a Rota-Baxter operator.

A useful observation is that if $P$ is a Rota-Baxter operator of weight $\lambda$ with $\lambda \neq 0$, then $\nu \lambda^{-1} P$ is a Rota-Baxter operator of weight $\nu$ for arbitrary $\nu \in \mathbb{K}$. So the weight is not "intrinsic" in some sense.

We can state our main construction now.

**Theorem 2.4.** Let $(M, \mu)$ be a right $H$-Hopf module algebra. Then $(M, P_R)$ is a Rota-Baxter algebra of weight $-1$.

**Proof.** For any $m, m' \in M$, we have

\[
P_R(mp_R(m')) + P_R(p_R(m)m') - P_R(mm')
= \sum (mp_R(m'))_{(0)} \cdot S((p_R(m)m')_{(1)}) + \sum (p_R(m)m')_{(0)} \cdot S((p_R(m)m')_{(1)})
- \sum (mm')_{(0)} \cdot S((mm')_{(1)})
= \sum (m_{(0)}p_R(m')_{(0)}) \cdot S(m_{(1)}p_R(m')_{(1)}) + \sum (p_R(m)(m')_{(0)}) \cdot S(p_R(m)(m')_{(1)})
- \sum (m_{(0)}m'_{(0)}) \cdot S(m_{(1)}m'_{(1)})
= \sum (m_{(0)}p_R(m') \cdot S(m_{(1)}) + \sum (p_R(m)m'_{(0)}) \cdot S(m'_{(1)})
- \sum (m_{(0)}m'_{(0)}) \cdot S(m_{(1)}m'_{(1)})
= p_R(m) \sum m'_{(0)} \cdot S(m'_{(1)})
+ \sum (m_{(0)}p_R(m') \cdot S(m_{(1)}) - \sum (m_{(0)}(m'_{(0)}) \cdot S(m'_{(1)}) \cdot S(m_{(1)}))
\]
\[ P_R(m)P_R(m'). \]

Remark 2.5. (i) One can show that if an algebra can be decomposed into a direct sum, as vector spaces, of two subalgebras, then the projection onto one of them is an idempotent Rota-Baxter operator of weight -1 (cf. [15]). So we can give another proof of the above theorem by showing that both \( \text{Im} P_R = M^R \) and \( \text{Ker} P_R \) are subalgebras.

(ii) Similarly, if \((M, \mu)\) is a left \( H \)-Hopf module algebra, then one can show that \((M, P_L)\) is a Rota-Baxter algebra of weight -1.

Example 2.6 ([19]). Let \( A \) be a bialgebra. Suppose there are two bialgebra maps \( i: H \to A \) and \( \pi: A \to H \) such that \( \pi \circ i = \text{id}_H \). Set \( \Pi = \text{id}_A \ast (i \circ S \circ \pi) \), where \( \ast \) is the convolution product on \( \text{End}(A) \). Then \( \Pi \) is an idempotent Rota-Baxter operator of weight -1. In fact, \( A \) is a right \( H \)-Hopf module algebra with the following right \( H \)-Hopf module structure: for any \( a \in A \) and \( h \in H \),

\[
\begin{cases} 
    a \cdot h = ai(h), \\
    \delta_R(a) = \sum a(1) \otimes \pi(a(2)).
\end{cases}
\]

It is not hard to see that the projection from \( A \) to \( A^R \) is just \( \Pi \). So by the above theorem, we get the conclusion.

The above construction contains many other interesting examples. Let \( H \) be a graded Hopf algebra. Then \( H \) can be written as a direct sum \( H = \bigoplus_{k \geq 0} H_k \) such that \( H_k H_l \subset H_{k+l} \) and \( \Delta(H_k) \subset \bigoplus_{r=0}^k H_r \otimes H_{k-r} \). Obviously, the 0-component \( H_0 \) is a Hopf algebra with the induced operations. Note that the inclusion \( i \) from \( H_0 \) into \( H \) and the projection \( \pi \) from \( H \) onto \( H_0 \) verify the required conditions. Another important example is the multi-brace cotensor Hopf algebra which is related to quantum groups (cf. [14]).

3. Construction of Hopf module algebras

In order to give a construction of \( H \)-Hopf module algebras, we need some necessary notions about Yetter-Drinfeld modules.

A left \( H \)-Yetter-Drinfeld module is a vector space \( V \) equipped simultaneously with a left \( H \)-module structure \( \cdot \) and a left \( H \)-comodule structure \( \rho \) such that whenever \( h \in H \) and \( v \in V \),

\[
\sum h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = \sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)}.
\]

The category of left \( H \)-Yetter-Drinfeld modules, denoted by \( H^H_{H} \mathcal{YD} \), consists of the following data: its objects are left \( H \)-Yetter-Drinfeld modules and morphisms are linear maps which are both module and comodule morphisms. An algebra \((V, \mu)\) in \( H^H_{H} \mathcal{YD} \) is an associative algebra such that the underlying space \( V \) is an object in \( H^H_{H} \mathcal{YD} \) and \((V, \mu)\) is both a module-algebra and a comodule-algebra. More precisely, if we denote \( vv' = \mu(v \otimes v') \) for any \( v, v' \in V \), then

\[
h \cdot (vv') = \sum (h_{(1)} \cdot v)(h_{(2)} \cdot v'),
\]
and
\[ \rho(vv') = \sum v_{(-1)}'v'_{(-1)} \otimes v_{(0)}v'_{(0)}. \]

Let \((V, \mu)\) be an algebra in \(\mathcal{H}YD\). The smash product \(V \# H\) of \(V\) and \(H\) is defined to be \(V \otimes H\) as a vector space and
\[ (v\#h)(v'\#h') = \sum v(h_{(1)} \cdot v')\#h_{(2)}h', \quad v, v' \in V, h, h' \in H, \]
where we use \# instead of \(\otimes\) to emphasize this new algebra structure.

**Theorem 3.1.** Let \((V, \mu)\) be an algebra in \(\mathcal{H}YD\).

(i) The algebra \(V \# H\) is a right \(H\)-Hopf module algebra with the following right \(H\)-module structure: for any \(h, x \in H\) and \(v \in V\),
\[ (v\#h) \cdot x = v\#hx, \]
and
\[ \delta_R(v\#h) = \sum (v\#h_{(1)}) \otimes h_{(2)}. \]
Hence the map \(P_R\) given by \(P_R(v\#h) = v\#\varepsilon(h)1_H\) is an idempotent Rota-Baxter operator of weight -1.

(ii) The algebra \(V \# H\) is a left \(H\)-Hopf module algebra with the following left \(H\)-module structure:
\[ x \cdot (v\#h) = \sum x_{(1)} \cdot v\#x_{(2)}h, \]
and
\[ \delta_L(v \otimes h) = \sum (v_{(-1)}h_{(1)}) \otimes (v_{(0)}\#h_{(2)}). \]
Hence the map \(P_L\) given by \(P_L(v\#h) = \sum S(v_{(-1)}h_{(2)}) \cdot v_{(0)}\#S(v_{(-2)}h_{(1)})h_{(3)}\) is an idempotent Rota-Baxter operator of weight -1.

**Proof.** The constructions of right and left Hopf module structures from left Yetter-Drinfeld modules given above are well-known. So the only thing to prove is the compatibility conditions.

(i) For any \(v, v' \in V\) and \(h, h', x \in H\), we have
\[ ((v\#h)(v'\#h')) \cdot x = \sum (v(h_{(1)} \cdot v')\#h_{(2)}h') \cdot x \]
\[ = \sum v(h_{(1)} \cdot v')\#h_{(2)}h'x \]
\[ = (v\#h)(v'\#h'x) \]
\[ = (v\#h)((v'\#h') \cdot x), \]
and
\[ \delta_R((v\#h)(v'\#h')) = \delta_R(\sum (v(h_{(1)} \cdot v')\#h_{(2)}h')) \]
\[ = \sum (v(h_{(1)} \cdot v')\#h_{(2)}h'_{(1)}) \otimes h_{(3)}h'_{(2)} \]
\[ = \sum (v\#h_{(1)})(v'\#h'_{(1)}) \otimes h_{(2)}h'_{(2)} \]
\[ = \sum (v\#h)(v'\#h'_{(0)}) \otimes (v\#h)(v'\#h')_{(1)}. \]
(ii) We have
\[ x \cdot ((v \# h)(v' \# h')) = \sum x \cdot (v(h(1) \cdot v') \# h(2) h') \]
\[ = \sum x_{(1)} \cdot (v(h(1) \cdot v') \# x(2) h(2) h') \]
\[ = \sum (x_{(1)} \cdot v)(x(2) h(1) \cdot v') \# x(3) h(2) h') \]
\[ = \sum ((x_{(1)} \cdot v) \# x(2) h)(v' \# h') \]
\[ = (x \cdot (v \# h))(v' \# h'), \]
and
\[ \delta_L((v \# h)(v' \# h')) = \delta_L(\sum v(h(1) \cdot v') \# h(2) h') \]
\[ = \sum (v(h(1) \cdot v'')(\cdot 1) h(2) h'(1) \otimes (v(h(1) \cdot v'))(0) \# h(3) h'(2) \]
\[ = \sum v(-1) h(1) \cdot v'(\cdot 1) h(2) h'(1) \otimes (v(0) h(1) \cdot v')(0) \# h(3) h'(2) \]
\[ = \sum v(-1) h(1) v'(-1) h(1) \otimes (v(0) h(2) \cdot v(0)) \# h(3) h'(2) \]
\[ = \sum v(-1) h(1) v'(-1) h(1) \otimes (v(0) \# h(2))(v'(0) \# h'(2)) \]
\[ = \sum (v \# h)(-1) (v' \# h')(\cdot 1) \otimes (v \# h)(0) (v' \# h')(0), \]
where the fourth equality follows from the Yetter-Drinfeld module condition. \( \square \)

We now give a concrete example of the above constructions which helps us to obtain many Rota-Baxter algebras.

**Example 3.2.** The algebra \( H \) is an algebra in \( \mathcal{H}_H \mathcal{YD} \) with the following left \( H \)-Yetter-Drinfeld module: for any \( x, h \in H \),
\[ x \cdot h = \sum x_{(1)} h S(x_{(2)}), \quad \rho(h) = \sum h_{(1)} \otimes h_{(2)}. \]
Then the smash product on \( H \# H \) is given by the following formula:
\[ (h^1 \otimes h^2)(h^3 \otimes h^4) = \sum h^1 h^2_{(1)} h^3 S(h^2_{(2)}) \otimes h^2_{(3)} h^3, h^i \in H. \]

Explicitly, the corresponding Rota-Baxter operators \( P_R \) and \( P_L \) are given by
\[ P_R(h \# h') = h \# \varepsilon(h') 1_H, \]
and
\[ P_L(h \# h') = \sum S(h_{(2)} h'_{(2)}) \cdot h_{(3)} \# S(h_{(1)} h'_{(1)}) h'_{(3)} \]
\[ = \sum S(h_{(3)} h'_{(3)}) h_{(4)} S^2(h_{(2)} h'_{(2)}) \# S(h_{(1)} h'_{(1)}) h'_{(4)} \]
\[ = \sum S(S(h_{(3)} h'_{(3)}) h'_{(3)}) \# S(h_{(1)} h'_{(1)}) h'_{(4)} \]

Finally, we apply our construction to Nichols algebras. Let \( V \) be a left \( H \)-Yetter-Drinfeld module and \( R(V) = \bigoplus_{n \geq 0} R(n) \) be a graded Hopf algebra in \( \mathcal{H}_H \mathcal{YD} \). We call \( R(V) \) a **Nichols algebra** of \( V \) if \( K \cong R(0) \) and \( V \cong R(1) \) in \( \mathcal{H}_H \mathcal{YD} \), and \( R(V) \) is generated as an algebra by \( R(1) \) and the set of primitive elements is exactly \( R(1) \).
These algebras play an important role in the classification of pointed Hopf algebras (cf. [4]). For more detailed information about these algebras, we refer the reader to [3]. Since $R(V)$ is an algebra in $\mathbb{H}YD$, we have Rota-Baxter algebra structures on $R(V)\#H$.

A particular important example is the quantum symmetric algebra. Assume that $A = (a_{ij})_{1 \leq i, j \leq N}$ is a symmetrizable generalized Cartan matrix, and $(d_1, \ldots, d_N)$ are positive relatively prime integers such that $(d_ia_{ij})$ is symmetric. Let $q \in \mathbb{C}^\times$ and define $q_{ij} = q^{d_ia_{ij}}$. Let $G = \mathbb{Z}^r \times \mathbb{Z}/l_1 \times \mathbb{Z}/l_2 \times \cdots \times \mathbb{Z}/l_p$ and $H = K[G]$ be the group algebra of $G$. We fix generators $K_1, \ldots, K_N$ of $G$ ($N = r + p$). Denote by $V$ the vector space over $\mathbb{C}$ with basis $\{e_1, \ldots, e_N\}$. It is known that $V$ is an $H$-Yetter-Drinfeld module with action and coaction given by $K_i \cdot e_j = q_{ij}e_j$ and $\delta_L(e_i) = K_i \otimes e_i$ respectively. The quantum shuffle product $*$ on $T(V)$ is defined by the following inductive formula:

$$
(e_{i_1} \otimes \cdots \otimes e_{i_n}) * (e_{j_1} \otimes \cdots \otimes e_{j_n}) = e_{i_1} \otimes ((e_{i_2} \otimes \cdots \otimes e_{i_n}) * (e_{j_1} \otimes \cdots \otimes e_{j_n})) + a_{i_1j_1} \cdots a_{i_nj_n} e_{j_1} \otimes ((e_{i_2} \otimes \cdots \otimes e_{i_n}) * (e_{j_2} \otimes \cdots \otimes e_{j_n})).
$$

Then the subalgebra $S(V)$ generated by $V$ is a Nichols algebra. By Theorem 15 in [20], $S_H(M) = S(V)\#H$ is isomorphic, as a Hopf algebra, to the sub Hopf algebra $U_q^+$ of the quantized universal enveloping algebra associated with $A$ when $G = \mathbb{Z}^N$ and $q$ is not a root of unity; $S_H(M)$ is isomorphic, as a Hopf algebra, to the quotient of the restricted quantized enveloping algebra $u_q^+$ by the two-sided Hopf ideal generated by the elements $(K_i - 1)$, $i = 1, \ldots, N$ when $G = (\mathbb{Z}/l)^N$ and $q$ is a primitive $l$-th root of unity. Thus we obtain that

**Proposition 3.3.** Both $U_q^+$ and $u_q^+$ possess idempotent Rota-Baxter algebra structures.

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School of Computer Science, Dongguan University of Technology, 1, Daxue Road, Songshan Lake, 523808, Dongguan, P. R. China

E-mail address: jian.math@gmail.com