On the rate of convergence of a regular martingale related to the branching random walk

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Let $M_n, n = 0, 1, \ldots$ be the supercritical branching random walk, in which the number of direct descendants of one individual may be infinite with positive probability. Assume that the standard martingale $W_n$ related to $M_n$ is regular, and $W$ is a limit random variable. Let $a(x)$ be a nonnegative function which regularly varies at infinity, with exponent greater than $-1$. The paper presents sufficient conditions of the almost sure convergence of the series $\sum_{n=1}^{\infty} a(n)(W_n - W)$. Also we establish a criterion of finiteness of $E W \log^+ W a(\log^+ W)$ and $E \log^+ |Z_\infty| a(\log^+ |Z_\infty|)$, where $Z_\infty := Q_1 + \sum_{n=2}^{\infty} M_1 \cdots M_n Q_{n+1}$, and $(M_n, Q_n)$ are independent identically distributed random vectors, not necessarily related to $M_n$.

1 Introduction and main results

Let $M$ be a point process on $\mathbb{R}$, i.e. random locally finite counting measure. Assume that $M \{ +\infty \} = 0$ and set $L := M(\mathbb{R})$. In this paper the variable $L$ may be deterministic or random, finite or infinite with positive probability.

By the branching random walk (BRW) we mean the sequence of point processes $M_n, n = 0, 1, \ldots$, where for any Borel set $B \subset \mathbb{R}$, $M_0(B) = 1_{\{0 \in B\}}$.

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†The original, Ukrainian variant of the paper was published in Ukrainian Mathematical Journal (2006), 58(3), 326-342.
\[ M_{n+1}(B) := \sum_r M_{n,r}(B) - A_{n,r}, n = 0, 1, \ldots \]

Here \(\{A_{n,r}\}\) are the points of \(M_n\), and \(\{M_{n,r}\}\) are independent copies of the \(M\). More detailed definition of the process can be found in [8, 7].

Notice that our definition of the BRW differs from the two previously known ones. The modern definition of the BRW introduced in [3] assumes that \(L < \infty\) a.s. Before the appearance of [3] by the BRW was meant the sequence (1) where an underlying point process \(M\) had independent and identically distributed points. Now the latter processes are sometimes called the homogeneous BRW.

In the paper we only consider the supercritical BRW. Therefore, if \(\mathbb{P}\{L < \infty\} = 1\), it is additionally assumed that \(\mathbb{E}L > 1\). Recall that supercriticality ensures the survival of a population with positive probability.

Let \(U := \bigcup_{n=0}^{\infty} \mathbb{N}^n\) be the set of all finite sequences \(u = i_1 \ldots i_n, i_k \in \mathbb{N}\) that contains the empty sequence \(\mathbb{N}^0 := \{\varnothing\}\). A tree \(T\) with root \(\varnothing\) is a subset of \(U\) that contains \(\varnothing\) and such that \(i_1 \ldots i_n \in T\) implies \(i_1 \ldots i_k \in T, k = 1, n - 1\); each element \(i_1 \ldots i_n \in T\) is assigned \(L_{i_1 \ldots i_n} \in [0, \infty]\), and \(i_1 \ldots i_n j \in T \iff j \in \{1, \ldots, L_{i_1 \ldots i_n}\}\). A tree \(T\) is called labelled if each \(u \in T\) is assigned a label \(A_u\).

For each realization of the BRW there is a labelled tree with root \(\varnothing\). The elements \(u\) of this tree are called individuals, \(\varnothing\)–the initial ancestor; the label \(A_u\) defines the position of the individual \(u\) on the real line, \(A_\varnothing = 0\). If \(u = i_1 \ldots i_n\), then \(n\) is called the generation of the individual \(u\) (notation: \(|u| = n (|\varnothing| = 0))\).

Assume that for some \(\gamma > 0\)

\[ m(\gamma) := \int_{-\infty}^{\infty} e^{\gamma x} M_1(dx) \in (0, \infty). \]

For \(n = 1, 2, \ldots\) denote by \(F_n := \sigma(M_1, \ldots, M_n)\) the \(\sigma\)-field generated by the point processes \(M_1, \ldots, M_n\) and set

\[ W_n := m^{-n}(\gamma) \int_{\mathbb{R}} e^{\gamma x} M_n(dx) = m^{-n}(\gamma) \sum_{|u|=n} e^{\gamma A_u}. \]

Under extra moment restrictions in [3] and [11] (for the case \(L < \infty\) a.s.) and in [12] conditions were given for the regularity (uniform integrability) of the non-negative martingale \(\{(W_n, F_n) : n = 1, 2, \ldots\}\). For the case
when $L$ can be infinite with positive probability, and without extra moment assumptions a criterion of regularity of the martingale was pointed out in Proposition 1.1 [9] (see [7] for a proof).

Recall that the regularity of arbitrary martingale $(U_n, G_n)$ ensures the existence of (equivalence class of) $G_\infty$–measurable random variable $U$ such that (a) $E U = E U_n$; (b) as $n \to \infty$ $U_n$ a.s. converges to $U$.

Denote by $W$ the limit random variable for the regular martingale $W_n$. Then $E W = 1$, and

$$W = m(\gamma)^{-n} \sum_{|u|=n} e^{\gamma A_u} W^{(u)},$$

where, given $F_n$, $\{W^{(u)} : |u| = n\}$ are conditionally independent copies of the $W$.

Set $Y_u := e^{\gamma A_u} / m|u|(\gamma)$. Let $(Z, S)$ be a random vector whose distribution is defined by the equality

$$\mathbb{E} \sum_{|u|=1} Y_u k(Y_u, \sum_{|v|=1} Y_v) = \mathbb{E} k(Z, S),$$

which is assumed to hold for any nonnegative bounded Borel function $k(x, y)$. For problems that the present paper is aimed at, a joint distribution of $(Z, S)$ does not matter, but knowledge of marginal distributions is essential. If $k$ does not depend on $x$, (3) implies the equality

$$\mathbb{P}\{S \in dy\} = y \mathbb{P}\{W_1 \in dy\}.$$  

Taking in (3) $k(x, y) = r(x)$ leads to

$$\mathbb{E} r(Z) = \mathbb{E} \sum_{|u|=1} Y_u r(Y_u),$$

or, more generally,

$$\mathbb{E} r(Z_1 \cdots Z_n) = \mathbb{E} \sum_{|u|=n} Y_u r(Y_u),$$

where $Z_1, Z_2, \ldots$ are independent copies of the $Z$. Notice that (4) holds for any Borel function $r$ with such a convention: if the right-hand side is infinite or does not exist the same is true for the left-hand side.

Let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be a function that regularly varies at $\infty$ with exponent $\alpha > -1$. If $\alpha = 0$, we additionally assume that $a$ does not decrease near $\infty$.  

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If (6) given below holds, $W_n$ converges to $W$ in mean (see Proposition 1.1 [9]). The paper provides sufficient conditions for the a.s. convergence of the series
\[ \sum_{n=0}^{\infty} a(n)(W - W_n) \] (5)
provided (6) holds. This result is a statement about the rate of the a.s. convergence of the regular martingale $W_n$ to its limit $W$.

**Theorem 1.1.** Let
\[ \mathbb{E} \log Z \in (-\infty, 0) \quad \text{and} \quad \mathbb{E}W_1 \log^+ W_1 < \infty, \] (6)
and the distribution of $\log Z$ is non-arithmetic. The conditions
\[ \mathbb{E}(\log^+ Z)^3 a(\log^+ Z) < \infty \quad \text{and} \quad \mathbb{E}W_1(\log^+ W_1)^2 a(\log^+ W_1) < \infty \] (7)
are sufficient for the a.s. convergence of (3).

The author thinks that the first inequality in (7) can be weakened to
\[ \mathbb{E}(\log^+ Z)^2 a(\log^+ Z) < \infty. \]
If the conjecture is correct then according to Theorem 1.2 the following equivalence should be true:
\[ \left| \sum_{n=0}^{\infty} a(n)(W - W_n) \right| < \infty \quad \Leftrightarrow \quad \mathbb{E}W \log^+ W a(\log^+ W) < \infty. \] (8)

The corollary given below proves the conjecture for two particular cases.

**Corollary 1.1.** Assume that (6) holds. If either $\mathcal{M}(-\infty, -\gamma^{-1} \log m(\gamma)) = 0$ a.s., and the distribution of $\log Z$ is non-arithmetic, or $W_n = \mathcal{M}_n(\mathbb{R})/(\mathbb{E}\mathcal{M}(\mathbb{R}))^n$, then (5) a.s. converges iff $\mathbb{E}W \log^+ W a(\log^+ W) < \infty$.

**Theorem 1.2.** If (6) holds then
\[ \mathbb{E}W \log^+ W a(\log^+ W) < \infty \quad \text{iff} \quad \mathbb{E}W_1(\log^+ W_1)^2 a(\log^+ W_1) < \infty. \]

**Remark 1.1.** Theorem 1.3(b) [9] provides a criterion of finiteness of $\mathbb{E}W f(W)$ for concave functions $f$ that grow more rapidly than any power of logarithm. Under the conditions of Theorem 1.2 this result cannot be used.
Proof of Theorem 1.1. We will use the idea of the proof of Theorem 4.1 [2].

On the set of extinction of the population (its probability measure may equal zero) the series contains finite number of non-zero terms, and hence it trivially converges. Therefore, not indicating this explicitly, we will investigate the convergence of the series on the set of survival and assume that \( W > 0 \).

Without loss of generality we can and do assume that \( m(\gamma) = 1 \). Indeed, the positions \( \{A_u, |u| = n\} \) of individuals in the \( n \)-th generation can be replaced by \( \{B_u := A_u - |u| \log m(\gamma), |u| = n\} \). However we keep all the previously introduced notation.

Put \( b(x) := x^a(x) \) and note that \( b(x) \) regularly varies at \( \infty \) with exponent \( \beta := \alpha + 1 > 0 \). For \( n = 0, 1, \ldots \) we define the sequences

\[
\tilde{W}_{n+1} := \sum_{|u|=n} e^{\gamma A_u W_1^{(u)}} 1_{\{b(n)e^{\gamma A_u W_1^{(u)}} \leq 1\}};
\]

\[
R_n := \mathbb{E}(W_n - \tilde{W}_{n+1}|\mathcal{F}_n) = \mathbb{E} \left( \sum_{|u|=n} e^{\gamma A_u W_1^{(u)}} 1_{\{b(n)e^{\gamma A_u W_1^{(u)}} > 1\}} |\mathcal{F}_n \right),
\]

where, given \( \mathcal{F}_n \), \( \{W_1^{(u)} : |u| = n\} \) are conditionally independent copies of the random variable \( W_1 \).

**Lemma 1.1.** Assume that \( (\mathcal{Z}) \) holds, and the distribution of \( \log Z \) is non-arithmetic. Then the conditions

\[
\mathbb{E}(\log^+ Z)^3 a(\log^+ Z) < \infty \quad \text{and} \quad \mathbb{E}W_1 \log^+ W_1 a(\log^+ W_1) < \infty \quad (9)
\]

are sufficient for convergence of the series

\[
\sum_{n=0}^{\infty} \mathbb{P}\{W_{n+1} \neq \tilde{W}_{n+1}\};
\]

\[
\sum_{n=0}^{\infty} \mathbb{D}(b(n)(\tilde{W}_{n+1} - W_n + R_n)).
\]

Thus, if \( (\mathcal{Z}) \) holds, the sequence

\[
\sum_{n=0}^{m} b(n)(\tilde{W}_{n+1} - W_n + R_n), m = 0, 1, \ldots
\]
is an $L_2$-bounded and hence regular martingale. Therefore, the series 
\[ \sum_{n=0}^{\infty} b(n)(W_{n+1} - W_n + R_n) \] is a.s. convergent. By Lemma 1.1 and Borel-Cantelli lemma the series \[ \sum_{n=0}^{\infty} b(n)(W_{n+1} - W_n + R_n) \] is a.s. convergent too. From (4.8) \[2\] it follows that the series \[ \sum_{n=1}^{\infty} a(n)(W - W_n + \sum_{k=n}^{\infty} R_k) \] converges a.s.

Consequently, the a.s. convergence of \[ \sum_{n=1}^{\infty} a(n)(W - W_n) \] is equivalent to that of \[ \sum_{n=1}^{\infty} a(n) \sum_{k=n}^{\infty} R_k \], which in its turn is equivalent to the a.s. convergence of the series \[ \sum_{n=1}^{\infty} b(n)R_n \]. The latter follows from that fact that \( R_n \geq 0 \) a.s., the equality

\[ \sum_{n=1}^{m} a_n \sum_{k=n}^{\infty} R_k = \left( \sum_{k=1}^{m} a_k \right) \sum_{n=m+1}^{\infty} R_n + \sum_{n=1}^{m} R_n \left( \sum_{k=1}^{n} a_k \right), \]

which holds for any \( m \in \mathbb{N} \), and Lemma 4.2 \[2\].

The next lemma completes the proof of Theorem 1.1.

**Lemma 1.2.** Assume that the conditions (7) and (10) hold, and the distribution of \( \log M \) is non-arithmetic. the series \[ \sum_{n=1}^{\infty} b(n)R_n \] is a.s. convergent iff

\[ \mathbb{E}W_1(\log^+ W_1)^2a(\log^+ W_1) < \infty. \] (10)

At this point it is appropriate to prove Corollary 1.1.

**Proof of Corollary 1.1.** Let \( M(-\infty, -\gamma^{-1}\log m(\gamma)) = 0 \) a.s., or equivalently \( Z \in [0, 1] \) a.s. In this case, the inequalities containing \( Z \) in Theorem 1.1 and Theorem 1.2 and Lemma 1.1 and Lemma 1.2 hold automatically. Assume that \( \mathbb{E}W^{\log^+ Wa(\log^+ W)} < \infty \). By Theorem 1.2 this is equivalent to

\[ \mathbb{E}W_1(\log^+ W_1)^2a(\log^+ W_1) < \infty. \]

By Theorem 1.1 the series (5) is a.s. convergent.

Let now the series (5) is a.s. convergent. If \( \mathbb{E}W_1^{\log^+ W_1a(\log^+ W_1)} < \infty \), Lemma 1.2 implies that \( \mathbb{E}W_1(\log^+ W_1)^2a(\log^+ W_1) < \infty \). Therefore, in view of Theorem 1.2 \( \mathbb{E}W^{\log^+ Wa(\log^+ W)} < \infty \). Assume that \( \mathbb{E}W_1^{\log^+ W_1a(\log^+ W_1)} = \infty \). Since by the assumption of the corollary \( \mathbb{E}W_1^{\log^+ W_1} < \infty \), then \( \alpha \geq 0 \). If \( \alpha > 0 \), then there exists a \( \delta \in [0, \alpha) \) such that \( \mathbb{E}W_1(\log^+ W_1)^{\delta+1} < \infty \) and \( \mathbb{E}W_1(\log^+ W_1)^{\delta+2} = \infty \). By Lemma 1.2 the series \[ \sum_{n=1}^{\infty} n^{\delta}(W - W_n) \] diverges. According to Abel’s criterion, for \( \epsilon \in (0, \alpha - \delta) \) the series \[ \sum_{n=1}^{\infty} n^{\alpha-\epsilon}(W - W_n) \] cannot converge. Hence the series (5) diverges. If \( \alpha = 0 \), then \( \mathbb{E}W_1^{\log^+ W_1} < \infty \) and \( \mathbb{E}W_1(\log^+ W_1)^2 = \infty \).
By Lemma 1.2 the series \( \sum_{n=1}^{\infty} (W - W_n) \) diverges. By the assumption at the beginning of the section \( a(x) \) does not decrease for large \( x \). This implies that the series \( (\ref{eq:series}) \) diverges. The proof of the corollary for Galton-Watson process follows a similar route. It suffices to remark that in this case we should take \( \gamma = 0 \) in \( (2) \) and that \( Z = (E_M(\mathbb{R}))^{-1} \) a.s. \( \Box \)

**Proof of Lemma 1.1.** Denote by \( F(x) \) the distribution function of the random variable \( W_1 \). Let \( S_n \) be a random walk, starting at the origin, with a step distributed like \((−\log Z)\). By the assumption of the lemma \( \mu := E_S1 \in (0, \infty) \). By Lemma 3.1(b) for \( x > 0 \)

\[
V(x) := \sum_{n=0}^{\infty} b(n)P\{S_n \leq \log b(n) + \log x\} < \infty. \quad (\text{11})
\]

For \( x > 0 \) define

\[
K(x) := \int_0^x ydV(y) = xV(x) - \int_0^x V(y)dy;
\]

\[
M(x) := \int_x^{\infty} y^{-1}dV(y) = -x^{-1}V(x) + \int_x^{\infty} y^{-2}V(y)dy.
\]

Since the function \( l(x) := \mu^{-\alpha-2}b(\log x) \) slowly varies at \( \infty \), and by Lemma 3.1 \( V(x) \) defined in (11) satisfies (28), this \( V \) belongs to de Haan’s class \( \Pi_l \).

By Theorem 3.7.1 \[4\]

\[
\lim_{x \to \infty} \frac{K(x)}{xb(\log x)} = \mu^{\alpha+2}, \quad \lim_{x \to \infty} \frac{xM(x)}{b(\log x)} = \mu^{\alpha+2}. \quad (\text{12})
\]

Further we have

\[
\sum_{n=0}^{\infty} P\{W_{n+1} \neq W_{n+1} \} = \sum_{n=0}^{\infty} P\{b(n)\sup_{|u|=n} e^{\gamma A_u} W_{1}^{(u)} > 1\} \leq \sum_{n=0}^{\infty} E \sum_{|u|=n} P\{b(n) e^{\gamma A_u} W_{1}^{(u)} > 1\mid F_n\} =
\]

\[
= \sum_{n=0}^{\infty} E \sum_{|u|=n} e^{\gamma A_u} \left( \int_{b^{-1}(n)e^{-\gamma A_u}}^{\infty} dF(x)e^{-\gamma A_u} \right) \leq \sum_{n=0}^{\infty} E e^{S_n} \int_{b^{-1}(n)e^{S_n}}^{\infty} dF(x) =
\]

\[
= \int_0^{\infty} E \left( \sum_{n=0}^{\infty} e^{S_n} 1_{\{e^{S_n} \leq b(n)x\}} \right) dF(x) = \int_0^{\infty} K(x) dF(x).
\]

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The latter integral converges in view of (12) and $\mathbb{E} W_1 b(\log^+ W_1) < \infty$.

Recall that the conditional variance is defined by the equality $\mathbb{D}(X|G) = \mathbb{E}(X^2|G) - (\mathbb{E}(X|G))^2$. Since $\mathbb{E}(\tilde{W}_{n+1}|\mathcal{F}_n) = W_n - R_n$ and $\mathbb{E}(\tilde{W}_{n+1} - W_n + R_n|\mathcal{F}_n) = 0$, then

$$\mathbb{D}(b(n) (\tilde{W}_{n+1} - W_n + R_n)) = b^2(n) \mathbb{E}(\mathbb{D}(\tilde{W}_{n+1}|\mathcal{F}_n)).$$

Further

$$\mathbb{D}(\tilde{W}_{n+1}|\mathcal{F}_n) = \sum_{|u|=n} \mathbb{D}(e^{\gamma A u} W_1^{(u)} 1_{\{b(n)e^{\gamma A u} W_1^{(u)} \leq 1\}}|\mathcal{F}_n) \leq \mathbb{E} \left( \sum_{|u|=n} e^{2\gamma A u} \mathbb{E}(W_1^{2} 1_{\{b(n)e^{\gamma A u} W_1 \leq 1\}}|\mathcal{F}_n) \right) = \mathbb{E} \left( \sum_{|u|=n} e^{2\gamma A u} \int_0^{b^{-1}(n)e^{-\gamma A u}} x^2 dF(x) \right).$$

Thus

$$\sum_{n=0}^{\infty} \mathbb{D}(b(n) (\tilde{W}_{n+1} - W_n + R_n)) = \sum_{n=0}^{\infty} b^2(n) \mathbb{E}(\mathbb{D}(\tilde{W}_{n+1}|\mathcal{F}_n)) \leq \sum_{n=0}^{\infty} b^2(n) \mathbb{E} \left( \int_0^{b^{-1}(n)e^{S_n}} x^2 dF(x) \right) \leq \int_0^{\infty} x^2 M(x) dF(x).$$

The latter integral converges in view of (12) and $\mathbb{E} W_1 b(\log^+ W_1) < \infty$. □

For each fixed $x \in \mathbb{R}$ consider the random variables

$$Q(x) := \sum_{n=1}^{\infty} b(n) \sum_{|u|=n} e^{\gamma A u} 1_{\{e^{\gamma A u} > e^{-x}\}};$$

$$\hat{Q}(x) := \sum_{n=1}^{\infty} b(n) \sum_{|u|=n} e^{\gamma A u} 1_{\{e^{\gamma A u} > e^{-x} b^{-1}(n)\}}.$$
If
\[\mathbb{E} \log Z \in (-\infty, 0) \text{ and } \mathbb{E}(\log^+ Z)^2 b(\log^+ Z) < \infty,\]
these are a.s. finite. This follows from Theorem 1 [1] that guarantees that
for all \( x \in \mathbb{R} \)
\[\mathbb{E} Q(x) = \sum_{n=1}^{\infty} b(n) \mathbb{P}\{S_n \leq x\} < \infty,\]
where \( S_n \) is the same random walk as in the proof of Lemma 1.1. That
\( \mathbb{E} \hat{Q}(x) < \infty \) follows from similar considerations and inequality (20).

**Lemma 1.3.** If (6) holds, and \( \mathbb{E}(\log^+ Z)^2 b(\log^+ Z) < \infty \), than a.s. on the
set of survival
\[\lim_{x \to \infty} Q(x) = \lim_{x \to \infty} \hat{Q}(x) = \frac{W}{(\beta + 1)(-\mathbb{E} \log Z)^{\beta+1}} > 0, \tag{14}\]
where \( \beta > 0 \) is the exponent of regular variation of \( b \).

**Proof** is similar to the proof of Theorem B [3]. Pick any \( 0 < a < \mu = -\mathbb{E} \log Z \). For each \( x > 0 \) there exists an integer \( N = N(x) > 0 \) such that
\((N - 1)^2 a \leq x < N^2 a\). For \( x > 0 \) define the random variables
\[Q_1(x) := \frac{1}{(N-1)^2 ab((N-1)^2 a)} \left( \sum_{n=1}^{N^2} b(n) W_n \right),\]
\[Q_2(N,x) := \frac{1}{(N-1)^2 ab((N-1)^2 a)} \left( \sum_{n=N^2}^{\infty} b(n) \sum_{|u|=n} e^{\gamma A u} 1_{\{e^{\gamma A u} > e^{-a n}\}} \right).\]
Recall that for almost all \( \omega \) from the set of survival \( W(\omega) > 0 \). Since, as \( m \to \infty \), \( \sum_{n=1}^{m} b(n) W_n \sim W \sum_{n=1}^{m} b(n) \) a.s.; \( \sum_{n=1}^{m} b(n) \sim (\beta + 1)^{-1} mb(m) \), then
\[\lim_{x \to \infty} Q_1(x) = \frac{W}{(\beta + 1)a^{\beta+1}} \text{ a.s.} \tag{15}\]
By Lemma 3.11 the series \( \sum_{n=1}^{\infty} b(n) \mathbb{P}\{S_n - an \leq 0\} \) converges. Hence
\[\mathbb{E} \sum_{N=2}^{\infty} Q_2(N,x) = \sum_{N=2}^{\infty} \frac{1}{(N-1)^2 ab((N-1)^2 a)} \left( \sum_{n=N^2}^{\infty} b(n) \mathbb{P}\{e^{-S_n} > e^{-an}\} \right) < \infty,\]

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which implies that
\[
\lim_{x \to \infty} Q_2(N, x) = 0 \quad \text{a.s.} \quad (16)
\]

By Theorem 1.5.3 [4] without loss of generality we can and do assume that
\[b(x)\] does not decrease for \(x > 0\). Therefore for \(x > 0\)
\[
\frac{Q(x)}{xb(x)} \leq Q_1(x) + Q_2(x).
\]
Now taking into account (15) and (16), and sending \(a \to \mu\) gives
\[
\limsup_{x \to \infty} \frac{Q(x)}{xb(x)} \leq \frac{W}{(\beta + 1)\mu^{\beta + 1}} \quad \text{a.s.} \quad (17)
\]

Pick now any \(a > \mu\). For each \(x \geq a\) there exists an integer \(N = N(x) > 0\)
such that \(Na \leq x < (N + 1)a\). For \(x \geq a\) consider the random variables

\[
Q_3(x) := \frac{1}{(N + 1)ab((N + 1)a)} \left( \sum_{n=1}^{N} b(n)W_n \right),
\]

\[
Q_4(x) := \frac{1}{(N + 1)ab((N + 1)a)} \left( \sum_{n=0}^{N} b(n) \sum_{|u|=n} e^{\gamma A_u} 1_{\{e^{\gamma A_u} \leq e^{-an}\}} \right).
\]

In a similar way as it was done for \(Q_1(x)\) we can prove that
\[
\lim_{x \to \infty} Q_3(x) = \frac{W}{(\beta + 1)a^{\beta + 1}} \quad \text{a.s.} \quad (18)
\]

The proof of the fact that
\[
\lim_{x \to \infty} Q_4(x) = 0 \quad \text{a.s.} \quad (19)
\]
almost coincides with the proof of a similar statement given on p. 35 [3]. By
Theorem 4.2 [13] \(r := \sum_{n=1}^{\infty} n^{-1} \mathbb{P}\{S_n > an\} < \infty\). Therefore
\[
\mathbb{E} \sum_{n=1}^{\infty} n^{-1} \sum_{|u|=n} e^{\gamma A_u} 1_{\{e^{\gamma A_u} \leq e^{-an}\}} = r < \infty.
\]

By Kronecker’s lemma
\[
\lim_{n \to \infty} (n + 1)^{-1} \sum_{k=1}^{n} \sum_{|u|=n} e^{\gamma A_u} 1_{\{e^{\gamma A_u} \leq e^{-an}\}} = 0 \quad \text{a.s.}
\]
From this and monotonicity of $b$ follows.

For large $x \frac{Q(x)}{xb(x)} \geq Q_3(x) - Q_4(x)$. Hence from (18) and (19) letting $a$ go to $\mu$ we get

$$\liminf_{x \to \infty} \frac{Q(x)}{xb(x)} \geq \frac{W}{(\beta + 1)\mu^{\beta+1}} \text{ a.s.}$$

Together with (17) the latter inequality proves the limit relation for $Q(x)$.

Now fix $\delta \in (0, \mu)$ and choose $r = r(\delta) > 0$ so that $\log b(n) \leq \delta n + r, n = 1, 2, \ldots$. We have the following

$$Q(x) \leq \hat{Q}(x) \leq \sum_{n=1}^{\infty} b(n) \sum_{|u|=n} e^{\gamma A_u} 1_{\{e^{\gamma A_u} > e^{-x - \delta n - r}\}}.$$

Using the same analysis as above allows us to check that the right-hand side of this inequality satisfies the same limit relation (14) as $Q(x)$.

**Proof of Lemma 1.2** The definition of $R_n$ implies the following representation

$$R_n = \sum_{|u|=n} e^{\gamma A_u} \int_{b^{-1}(n)e^{-\gamma A_u}}^{\infty} x dF(x),$$

where, as before, $F(x)$ is the distribution function of $W_1$. We have the (formal) equality

$$\sum_{n=1}^{\infty} b_n R_n = \int_{0}^{\infty} x dF(x) \sum_{n=1}^{\infty} b(n) \sum_{|u|=n} e^{\gamma A_u} 1_{\{\gamma A_u > -\log x - \log b(n)\}} = \int_{0}^{\infty} \hat{Q}(\log x) x dF(x).$$

By the assumptions of the lemma $\mathbb{E} \log Z \in (0, \infty)$ and $\mathbb{E}(\log^+ Z)^2 b(\log^+ Z) < \infty$. Therefore by Lemma 13 as $x \to \infty$, $\hat{Q}(\log x) \sim \text{const} \log x b(\log x)$ a.s.

Hence, the series with non-negative terms $\sum_{n=1}^{\infty} b(n) R_n$ converges iff (10) holds. □

**2 Moments of random series and the proof of Theorem 1.2**

Assume that $(M_1, Q_1), (M_2, Q_2), \ldots$ are independent copies of a random vector $(M, Q)$, not necessarily related to the BRW, and defined on a fixed prob-
ability space. Set

\[ \Pi_0 := 1 \quad \text{and} \quad \Pi_n := M_1 M_2 \cdots M_n, n = 1, 2, \ldots; \]

\[ Z_\infty := \sum_{k=1}^{\infty} \Pi_{k-1} Q_k \] (21)

Throughout this section we assume that

\[ \mathbb{P}\{M = 0\} = 0, \mathbb{P}\{Q = 0\} < 1, \]

and, provided the series in (21) is absolutely convergent, that the distribution of \( Z_\infty \) is non-degenerate.

Theorem 2.1 given below may be of some interest on its own, supplements the result of Theorem 1.6 [9] and is a key ingredient in proving Theorem 1.2. Recall that \( b(x) \) is a regularly varying function with exponent \( \beta > 0 \) and put \( c(x) := x b(x) \).

**Theorem 2.1.** If \( \mathbb{E} \log |M| \in (-\infty, 0) \) and \( \mathbb{E} \log^+ |Q| < \infty \), (22)

then

\[ \mathbb{E} b(\log^+ |Z_\infty|) < \infty \iff \mathbb{E} c(\log^+ |M|) < \infty, \quad \mathbb{E} c(\log^+ |Q|) < \infty. \] (23)

**Proof.** The functions \( b \) and \( c \) can be represented as follows: \( b(x) = x^\beta L(x) \), \( c(x) = x^{\beta+1} L(x) \), where \( L(x) \) slowly varies at \( \infty \). For \( y > 1 \) set \( \Lambda_\beta(y) := \frac{\log^{\beta-1} y L(\log y)}{\beta y} \). This function regularly varies at \( \infty \) with exponent \( (1) \).

The function \( \sup_{t \geq x} \Lambda_\beta(t) \) does not increase, and by Theorem 1.5.3 [4] \( \sup_{t \geq x} \Lambda_\beta(t) \sim \Lambda_\beta(x) \) (from here on the record \( F \sim G \) means that \( \lim_{x \to \infty} (F(x)/G(x)) = 1 \)). Changing variable and then appealing to Karamata’s theorem yield

\[ b(\log x) \sim \int_1^x \Lambda_\beta(y) dy \sim \int_1^x \sup_{t \geq y} \Lambda_\beta(t) dy =: \tilde{f}(x - 1). \] (24)

Analogously

\[ c(\log x) \sim \int_1^x \Lambda_{\beta+1}(y) dy \sim \int_1^x \sup_{t \geq y} \Lambda_{\beta+1}(t) dy =: \phi(x - 1). \]
The functions $\tilde{f}$ and $\phi$ are non-decreasing and concave on $\mathbb{R}^+$. Also these equal 0 at $x = 0$ and tend to $\infty$ as $x \to \infty$. In particular, $\phi$ is subadditive. Also, from (24) and Karamata’s theorem it follows that

$$(\beta + 1)^{-1}c(\log x) \sim \int_1^x (\tilde{f}(y)/y)dy =: \tilde{g}(x - 1).$$

The function $c(x)$ regularly varies at $\infty$ with exponent $\beta + 1 > 1$. By Theorem 1.5.3 it is equivalent at $\infty$ to a non-decreasing function. Therefore, according to Lemma 1(a) there exists a non-decreasing function $\psi(x) \sim c(\log x)$ such that $\psi(x) = 0$ for $x \leq 1$ and

$$\psi(xy) \leq a(\psi(x) + \psi(y)) \quad (25)$$

for all $x, y \in [1, \infty)$ and some positive constant $a$.

Thus we conclude that it suffices to prove the equivalence with $b(\log x)$ replaced by $\tilde{f}(x)$, and $c(\log x)$–by $\tilde{g}(x)$, $\phi(x)$ or $\psi(x)$.

To prove implication $\Leftarrow$ of the theorem we should use the fact that according to Theorem 2.1 the condition (22) ensures that $|Z_\infty| < \infty$ a.s.

First we assume that $|M| \in [0, 1]$ a.s. In that case the condition $E(c(\log^+ |M|)) < \infty$ holds automatically. Let $E(c(\log^+ |Q|)) < \infty$ or equivalently $E(\tilde{g}(|Q|)) < \infty$. By Theorem 1.6(a) $E(\tilde{f}(|Z_\infty|)) < \infty$. This is equivalent to $E(b(\log^+ |Z_\infty|)) < \infty$. The proof of the other way implication goes the same path and, in particular, appeals to the same Theorem 1.6(a).

Now we are ready to consider the general case. First assume that in (23) the inequalities for $|M|$ and $|Q|$ hold. Equivalently,

$$E(\tilde{g}(|M|)) < \infty, E(\tilde{g}(|Q|)) < \infty.$$

Consider the random variables

$$N_0 := 0, N_{i+1} := \inf \{n > N_i : |\Pi_n| < |\Pi_{N_i}| \}, i = 0, 1, \ldots$$

Since under the assumptions of the theorem $\Pi_n \to 0$ a.s. as $n \to \infty$, $E N_i < \infty, i = 1, 2, \ldots$. For $k = 1, 2, \ldots$ put

$$M'_k := |M_{N_k + 1}| \cdots |M_{N_k}|, \Pi'_0 := 1, \Pi'_k := M'_1 \cdots M'_k;$$

$$Q'_k := |Q_{N_k + 1}| + |M_{N_k + 1}| |Q_{N_{k+1}}| + \cdots + |M_{N_k + 1}| \cdots |M_{N_k}| |Q_{N_k}|.$$
The random vectors \{(M_k', Q_k'), k = 1, 2, \ldots\} are independent copies of 
\((|\Pi_{N_1}|, \sum_{k=1}^{N_1} |\Pi_{k-1}||Q_k|)\) and, furthermore,
\[
\sum_{k=1}^{\infty} |\Pi_{k-1}||Q_k| = \sum_{k=1}^{\infty} \Pi_{k-1}'Q_k'.
\]
If we could prove that
\[
\mathbb{E}\tilde{g}(\sum_{k=1}^{N_1} |\Pi_{k-1}||Q_k|) < \infty,
\]
then this implied that \(\mathbb{E}b(|Z_{\infty}|) < \infty\), and, therefore, one way of the theorem
would be established. Indeed, since \(|\Pi_{N_1}| \in (0, 1)\) a.s. and \(P\{|\Pi_{N_1}| = 1\} = 0\),
and \(26\) guarantees that \(\mathbb{E}\log^+\sum_{k=1}^{N_1} |\Pi_{k-1}||Q_k|) < \infty\), than the first part of
the proof applied on the vector \((|\Pi_{N_1}|, \sum_{k=1}^{N_1} |\Pi_{k-1}||Q_k|)\) instead of \((|M|, |Q|)\)
gives the wanted.

Let us check \(26\) with \(\tilde{g}\) replaced by \(\psi\). Since
\[
\sum_{k=1}^{N_1} |\Pi_{k-1}||Q_k| \leq N_1 \sup_{1 \leq k \leq N_1} |\Pi_{k-1}||Q_k| \leq N_1 \sup_{0 \leq k \leq N_1-1} |\Pi_k| \sum_{i=1}^{N_1} |Q_i|,
\]
then taking into account \(25\) allows us to conclude that to prove \(26\) it
suffices to establish three inequalities: 1) \(\mathbb{E}\psi(N_1) < \infty\);
2) \(\mathbb{E}\psi(\sup_{0 \leq k \leq N_1-1} |\Pi_k|) < \infty\); 3) \(\mathbb{E}\psi(\sum_{i=1}^{N_1} |Q_i|) < \infty\). Since \(\mathbb{E}N_1 < \infty\), and \(\psi\)
grows more slowly than the linear function, the first inequality holds true.
Further, \(\psi(e^x)\) regularly varies with exponent \(\beta + 1 > 1\). Therefore, according
to \(26\), the second inequality is implied by \(\mathbb{E}\psi(|M| \vee 1) < \infty\). The latter
is equivalent to \(\mathbb{E}\psi(|Q|) < \infty\). To check the third inequality we replace
\(\psi\) with \(\phi\). A benefit of the replacement is that \(\phi\) be subadditive. As the
random variables \(1_{\{N_1 \geq n\}}\) and \(|Q_n|\) are independent, we have
\[
\mathbb{E}\phi(\sum_{i=1}^{N_1} |Q_i|) \leq \mathbb{E} \sum_{i=1}^{N_1} \phi(|Q_i|) = \mathbb{E}N_1 \mathbb{E}\phi(|Q|) < \infty.
\]
Now assume that the left-hand side of \(27\) holds. This is equivalent to
\[
\mathbb{E}\tilde{f}(|Z_{\infty}|) < \infty. \quad (27)
\]
By Proposition 3.1 \cite{9}, either $\infty > \mathbb{E} \tilde{f}(\sup_{n \geq 0} |\Pi_n|)$ or $\infty > \mathbb{E} \tilde{f}(\sup_{n \geq 0} |\Pi_{2n}|)$.

Equivalently, either $\infty > \mathbb{E} f(\sup_{n \geq 0} S_n)$, or $\infty > \mathbb{E} f(\sup_{n \geq 0} \hat{S}_n)$, where $S_n := \log |\Pi_n|$, $\hat{S}_n := \log |\Pi_{2n}|$, $n = 0, 1, \ldots$ are random walks with steps distributed like $\log |M|$ and $\log |M_1 M_2|$ respectively. In view of (36), either $\mathbb{E} g(\log M) < \infty$ or $\mathbb{E} g(\log (M_1 M_2)) < \infty$. Clearly, both of these imply the next to last inequality.

On the other hand, by Proposition 3.1 \cite{9} (27) implies that either

$$\mathbb{E} \tilde{f}(\sup_{k \geq 1} \Pi^*_k |Q^*_k|) \leq \mathbb{E} \tilde{f}(\sup_{k \geq 1} |Q^*_k|) < \infty,$$

or

$$\mathbb{E} \tilde{f}(\sup_{k \geq 1} \hat{\Pi}^*_k |\hat{Q}^*_k|) \leq \mathbb{E} \tilde{f}(\sup_{k \geq 1} |\hat{Q}^*_k|) < \infty,$$

hold, where

$\hat{\Pi}_0 := 1$, \quad $\hat{\Pi}_n := \hat{M}_1 \hat{M}_2 \cdots \hat{M}_n$, $n = 1, 2, \ldots$,

the vectors

$$(\hat{M}_k, \hat{Q}_k) := (M_{2k-1} M_{2k}, M_{2k-1} Q_{2k} + Q_{2k-1}), \quad k = 1, 2, \ldots$$

are independent and identically distributed; $(M_n, Q_n) \overset{d}{=} (M_n, Q'_n)$, given $M_n, Q_n$ and $Q'_n$ are conditionally independent, $Q^*_n := Q_n - Q'_n$, $\hat{Q}^*_n$ and $\hat{Q}'_n$ have the same meaning, but are defined in terms of $\hat{M}_n$ and $\hat{Q}_n$; $\Pi^*_0 := 1$, $\Pi^*_k := M^*_1 \cdots M^*_k$, $M^*_k := |M_k| \wedge 1, k = 1, 2, \ldots$, and $\hat{\Pi}^*_k$ are defined similarly. Since $M^*_k, \hat{M}^*_k \leq 1$ a.s., and strictly smaller than one with positive probability, Corollary 3.1 \cite{9} implies that $\mathbb{E} g(|Q|) < \infty$. Hence, $\mathbb{E} g(\log^+ |Q|) < \infty$. \hfill $\square$

**Proof of Theorem 1.2.** Theorem 1.2 can be obtained from Theorem 2.1 by using the same approach that was exploited in \cite{9} to deduce Theorem 1.3 from Theorem 1.6. Theorem 2.1 applies to the random series generated by the vector $(Z, S)$. The latter was defined in \cite{3}. \hfill $\Box$

### 3 Appendix

Lemma 3.1 is a key ingredient in proving Lemma 1.1. Part b) of the lemma deals with *perturbed* random walks and generalizes a result of \cite{11} for random walks.
Lemma 3.1. Assume that a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ regularly varies at $\infty$ with exponent $\beta > 0$. Let $T_n, n = 0, 1, \ldots$ be a random walk starting at zero with $\mu := E T_1 \in (0, \infty)$. If $E (T_1^-)^2 \varphi(T_1^-) < \infty$, then

a) for any $\epsilon > 0$

$$I := \sum_{n=1}^{\infty} \varphi(n) P\{T_n > (\mu + \epsilon)n\} < \infty; \quad \sum_{n=1}^{\infty} \varphi(n) P\{T_n \leq (\mu - \epsilon)n\} < \infty;$$

b) for all $x \in \mathbb{R}$

$$V(x) := \sum_{n=1}^{\infty} \varphi(n) P\{T_n \leq \log \varphi(n) + \log x\} < \infty,$$

if, moreover, the distribution of $T_1$ is non-arithmetic, then for all $h > 0$

$$\lim_{x \to \infty} \frac{V(hx) - V(x)}{\mu^{-\beta-1} \varphi(\log x)} = \log h. \quad (28)$$

Proof. According to Theorem 1.5.3 [4] we can and do assume that $\varphi$ is non-decreasing on $\mathbb{R}^+$.

a) The sequence $\tilde{T}_n := -T_n + (\mu + \epsilon)n, n = 0, 1, \ldots$ is a random walk with $E\tilde{T}_1 = \epsilon \in (0, \infty)$. Therefore by Theorem 1(a) [1] $I = \sum_{n=1}^{\infty} \varphi(n) P\{\tilde{T}_n \leq 0\} < \infty$. The convergence of the second series can be established similarly.

(b) In what follows we will use the idea of the proof of Theorem 2 [10]. Fix $\delta \in (0, \mu)$ and pick $r = r(\delta) > 0$ such that $\log \varphi(n) \leq \delta n + r, n = 1, 2, \ldots$. The sequence $\hat{T}_n := T_n - \delta n$ is a random walk with $E\hat{T}_1 = \mu - \delta \in (0, \infty)$. Since $V(x) \leq \sum_{n=1}^{\infty} \varphi(n) P\{\hat{T}_n \leq \log x + r\}$, and the latter series converges by Theorem 1(a) [1], then the function $V(x)$ is finite for all $x > 0$. The relation (28) is tantamount to

$$\lim_{x \to \infty} \frac{U(x+h) - U(x)}{\mu^{-\beta-1} \varphi(\log x)} = \log h, \quad \text{for all } h \in \mathbb{R}, \quad (29)$$

where $U(x) := V(e^x)$. Actually it suffices to prove (29) for small positive $h$ from an interval $(h_0, h_1)$ (see, for example, Lemma 3.2.1 [1]). Fix such an $h$. For any $\epsilon \in (0, \mu/2)$ and large enough $x$ the inequality $\log \varphi(n) \leq c n$ holds for $n \geq N_2 = N_2(x) := \left[\frac{x+\hat{h}}{\mu - 2\epsilon} + 1\right]$. Set $N_1 = N_1(x) := \left[\frac{x+\hat{h}}{\mu + \epsilon}\right]$. By using part a) of the lemma, given $\rho > 0$ we can choose $m = m(\rho) > 0$ such that

$$\sum_{n=m+1}^{\infty} \varphi(n) P\{T_n > (\mu + \epsilon)n\} \leq \rho. \quad (30)$$
Write
\[ U(x + h) - U(x) = \sum_{n=1}^{\infty} \varphi(n) \mathbb{P}\{T_n \leq \log \varphi(n) + x\} = \]
\[ = \sum_{n=1}^{m} + \sum_{n=m+1}^{N_1} + \sum_{n=N_1+1}^{N_2-1} + \sum_{n=N_2}^{\infty} =: I_1(x) + I_2(x) + I_3(x) + I_4(x). \]

It is obvious that \( \lim_{x \to \infty} I_1(x) = 0. \)

If for large \( x \) and \( n \geq N_2(x) \) \( T_n \geq (\mu - \epsilon)n \), then \( T_n - \log \varphi(n) \geq (\mu - 2\epsilon)n. \)

Therefore, when \( x \to \infty \),
\[ I_4(x) \leq \sum_{n=N_2(x)}^{\infty} \varphi(n) \mathbb{P}\{T_n \leq (\mu - \epsilon)n\} \to 0 \]

by part a) of the lemma.

If for large \( x \) and \( n \in \{m + 1, \ldots, N_1(x)\} \) \( T_n \leq (\mu + \epsilon)n \), then
\[ T_n - \log \varphi(n) \leq (\mu + \epsilon)N_1 - h \leq x. \]

Hence
\[ I_2(x) \leq \sum_{n=m+1}^{N_1} \varphi(n) \mathbb{P}\{T_n - \log \varphi(n) > x\} \leq \sum_{n=m+1}^{N_1} \varphi(n) \mathbb{P}\{T_n > (\mu + \epsilon)n\} \leq \rho. \]

By Potter’s inequality (Theorem 1.5.6 [4]), for any positive \( q \) and \( \theta \) there exists an \( x_0 > 0 \) such that
\[ \log \varphi\left(\frac{x+h}{\mu-2\epsilon}\right) - \log \varphi\left(\frac{x+h}{\mu+\epsilon}\right) \leq (1+q) + (\beta + \theta)(\log(\mu + \epsilon) - \log(\mu - 2\epsilon)) := B(q, \theta). \]

Hence for \( x \geq x_0 \)
\[ I_3(x) \leq \sum_{n=N_1+1}^{N_2-1} \varphi(n) \mathbb{P}\{\log \varphi(N_1 + 1) + x < T_n \leq \log \varphi(N_2 - 1) + x + h\} \leq \]
\[ \sum_{n=1}^{\infty} \varphi(n) \mathbb{P}\{\log \varphi\left(\frac{x+h}{\mu-2\epsilon}\right) + x < T_n \leq \log \varphi\left(\frac{x+h}{\mu-2\epsilon}\right) + x + h + B(q, \theta)\}. \]
An appeal to Theorem 2 yields
\[ \limsup_{x \to \infty} \frac{I_3(x)}{\varphi(x)} \leq \frac{h + B(q, \theta)}{\mu^{\beta+1}}. \]

Letting \( q \) and \( \epsilon \) go to 0 results in
\[ \limsup_{x \to \infty} \frac{I_3(x)}{\varphi(x)} \leq \frac{h}{\mu^{\beta+1}}. \]

Thus, we have proved that
\[ \limsup_{x \to \infty} \frac{U(x + h) - U(x)}{\varphi(x)} \leq \frac{h}{\mu^{\beta+1}}. \]

We now intend to check that
\[ \liminf_{x \to \infty} \frac{U(x + h) - U(x)}{\varphi(x)} \geq \frac{h}{\mu^{\beta+1}}. \tag{31} \]

Put \( R_n := T_n - \log \varphi(n), n = 1, 2, \ldots \). For each \( \epsilon \in (0, \mu) \) we define \( N_3 = N_3(x) := \left[ \frac{x + h}{\mu - \epsilon} + 1 \right] \) and make use of the random variable \( N_1 \) defined above.

For any positive \( q \) and \( \theta \) such that \( \tau = \tau(q, \theta, \epsilon) := \log(1 + q) \left( \frac{\mu + \epsilon}{\mu - \epsilon} \right)^{\beta+\theta} < h_0 \), and large \( x \) Potter’s inequality \( \log \varphi(N_3(x) - 1) - \log \varphi(N_1(x)) \leq \tau \) holds. Moreover, we have the following
\[
U(x + h) - U(x) \geq \sum_{n=N_1+1}^{N_3-1} \varphi(n) \mathbb{P}\{ x < R_n \leq x + h \} \geq \\
\geq \sum_{n=N_1+1}^{N_3-N_1-1} \varphi(n) \mathbb{P}\{ \log \varphi(n) - \log \varphi(N_1) + x < R_{N_1} + T_n - T_{N_1} \leq x + h \} \geq \\
\geq \sum_{n=1}^{N_3-N_1-1} \varphi(n+1) \mathbb{P}\{ \tau + x - R_{N_1} < T_n \leq x - R_{N_1} + h \} \geq \\
\geq \varphi \left( \frac{x}{\mu + \epsilon} \right) \sum_{n=1}^{N_3-N_1-1} \mathbb{P}\{ \tau + x - R_{N_1} < T_n \leq x - R_{N_1} + h \} = 
\]
\[ \phi\left(\frac{x}{\mu + \epsilon}\right) \mathbb{E} g(x - R_{N_1(x)}), \]

where \( g(t) := \sum_{n=1}^{N_3 - N_1 - 1} \mathbb{P}\{\tau + t < T_n \leq t + \theta\}. \) We will show that a.s.

\[ \lim_{x \to \infty} g(x - R_{N_1(x)}) = \mu^{-1}(h - \tau). \] (32)

Blackwell’s theorem [5] implies that the function \( g(t) \) is bounded. Therefore from (32) it follows that

\[ \lim_{x \to \infty} \mathbb{E} g(x - R_{N_1(x)}) = \mu^{-1}(h - \tau). \]

Consequently, taking into account the regular variation of \( \phi \) allows us to conclude that

\[ \lim \inf_{x \to \infty} \frac{U(x + h) - U(x)}{\varphi(x)} \geq \frac{h - \tau(q, \theta, \epsilon)}{(\mu + \epsilon)^2 \mu}. \]

Sending \( q \) and \( \epsilon \) to 0 leads to (33).

By the strong law of large numbers, as \( x \to \infty \) we have \( R_{N_1(x)} = \mu N_1(x) + o(N_1(x)) \) a.s. Hence, as \( x \to \infty \) \( x - R_{N_1(x)} = \epsilon(\mu + \epsilon)^{-1}x + o(x) \) a.s. To prove (32) it suffices to verify that for arbitrary non-random function \( z(x) = \epsilon(\mu + \epsilon)^{-1}x + o(x) \)

\[ \lim_{x \to \infty} \sum_{n=1}^{N_3(x) - N_1(x) - 1} \mathbb{P}\{\tau + z(x) < T_n \leq z(x) + h\} = \mu^{-1}(h - \tau). \] (33)

If positive integer \( n \geq N_3 - N_1 \) and \( T_n > (\mu - \epsilon)n \), then for large \( x \) \( T_n > 2\epsilon x(\mu + \epsilon)^{-1} + h > z(x) + h \). Therefore

\[ \sum_{n=N_3(x)-N_1(x)}^{\infty} \mathbb{P}\{T_n \leq z(x) + h\} \leq \sum_{n=N_3(x)-N_1(x)}^{\infty} \mathbb{P}\{T_n \leq (\mu - \epsilon)n\}. \]

According to part a) of the lemma the last expression tends to 0 when \( x \to \infty \). By Blackwell’s theorem (33) holds and hence (32) holds too.

Let \( \xi_1, \xi_2, \ldots \) be independent copies of a random variable \( \xi \) with \( m := \mathbb{E} \xi \in (-\infty, 0) \). Define \( S_0 := 0, S_n := \xi_1 + \ldots + \xi_n, n = 1, 2, \ldots \). Then \( M_\infty := \sup_{n \geq 0} S_n < \infty \) a.s., and \( \mathbb{E} \tau_x^- < \infty \), where

\[ \tau_x^- := \inf\{n : S_n < -x\}, \quad x \geq 0. \]
Let \( f \) be a non-negative measurable function such that \( \lim_{x \to \infty} f(x) = \infty \) and there exists an \( x_0 \geq 0 \) such that \( f \) is increasing and concave for \( x \geq x_0 \). Define the new function \( g \) as follows:

\[
g(x) := \int_{x_0}^x \frac{f(y)}{y} \, dy \quad \text{for} \quad x \geq x_0; \quad g(x) := 0 \quad \text{for} \quad x < x_0.
\]

Set \( u(x) := f(e^x) \), \( v(x) := g(e^x) \). Assume that a function \( h \) regularly varies at \( \infty \) with exponent \( \beta > 0 \).

**Lemma 3.2.** For \( x \geq 0 \)

\[
\mathbb{E} u(M_\infty) < \infty \iff \mathbb{E} v\left( \sup_{0 \leq n \leq \tau^-} S_n \right) < \infty.
\]  
(34)

Each of these inequalities ensures that

\[
\mathbb{E} v(\bar{\xi}) < \infty.
\]  
(35)

Also the following equivalences hold:

\[
(\sup_{0 \leq n \leq \tau^-} S_n) h(\sup_{0 \leq n \leq \tau^-} S_n) < \infty \iff \mathbb{E} h(M_\infty) < \infty \iff \mathbb{E} \bar{\xi}^+ h(\bar{\xi}) < \infty.
\]  
(36)

**Proof.** Without loss of generality we can assume that \( f \) is increasing and concave on \( \mathbb{R}^+ \), \( f(0) = 0 \), \( \lim_{x \to \infty} f(x) = \infty \), and \( g(x) = \int_0^x (f(u)/u) \, du \). This follows from the fact that we can consider the function \( \hat{f}(x) = f(x + x_0) - f(x_0) \) in place of \( f \). This function possesses the properties listed above, and the ratio \( \hat{f}/f \) is bounded away from zero and bounded from the above. For fixed \( x \geq 0 \) define the random variables \( N_0 := 0 \),

\[
N_{i+1} := \inf\{n > N_i : S_n < S_{N_i} - x\}, \ i = 0, 1, \ldots.
\]

Notice that \( \tau^-_x = N_1 \) and \( N_i < \infty \) a.s. Put

\[
V_k := \sup\{S_{N_k}, S_{N_k+1}, \ldots, S_{N_{k+1}-1}\}, \ k = 0, 1, \ldots;
\]

\[
Z_{k+1} := \sup\{0, \xi_{N_k+1}, \ldots, \xi_{N_{k+1}} + \ldots + \xi_{N_{k+1}-1}\}, \ k = 0, 1, \ldots.
\]

Then \( V_k = S_{N_k} + Z_{k+1} \) and \( M_\infty = \sup_{k \geq 0} V_k \). Notice that \( Z_1, Z_2, \ldots \) are independent copies of \( Z := \sup_{0 \leq i \leq \tau^-} S_i \), and by Wald’s identity \( \mathbb{E}|Z| = \mathbb{E} \sum_{k=1}^{\tau^-} |\xi_k| = \)
\[ E_{\tau^x} |\xi| < \infty. \]
\[ \Rightarrow \text{in } (34). \] Since \( S_{N_k} < -kx \), and for fixed \( \epsilon > 0 \)
\[ \mathbb{P}\{ M_{\infty} > y \} \leq \mathbb{P}\{ \sup_{k \geq 0} (-kx + Z_{k+1}) > y \} \leq \]
\[ \leq \sum_{k=0}^{\infty} \mathbb{P}\{ Z_{k+1} > y + k(x + \epsilon) \} \leq \sum_{k=[y/(x+\epsilon)]}^{\infty} \mathbb{P}\{ Z > k(x + \epsilon) \} \leq \]
\[ \leq \int_{[y/(x+\epsilon)]}^{\infty} \mathbb{P}\{ Z > (x + \epsilon)y \} dy. \]
The latter integral converges, since \( E|Z| < \infty \). Thus,
\[ \infty > E_{u}(M_{\infty}) = \int_{0}^{\infty} u'(z) \mathbb{P}\{ M_{\infty} > z \} dz, \]
if
\[ \int_{-\infty}^{\infty} u'(z) \int_{z}^{\infty} \mathbb{P}\{ Z > y \} dy dz < \infty. \]
Since \( u(x) = v'(x) \), integrating by parts shows that the latter inequality is equivalent to
\[ \infty > \int_{-\infty}^{\infty} v'(z) \mathbb{P}\{ Z > z \} dz = \mathbb{E}v(Z) = \mathbb{E}v\left( \sup_{0 \leq n \leq \tau^x - 1} S_n \right). \]
\[ \Rightarrow \text{in } (34). \] \{\( S_{N_k}, k = 1, 2, \ldots \)\} is a random walk starting at zero and with a step distributed like \( S_{\tau^x} \). The random vectors \((S_{N_k} - S_{N_{k-1}}, Z_k), k = 1, 2, \ldots\) are independent and identically distributed, and \( \lim_{n \to \infty} S_{N_n} = -\infty \) a.s. Let \((\widetilde{M}_1, \widetilde{Q}_1), (\widetilde{M}_2, \widetilde{Q}_2), \ldots\) be independent copies of the vector \((\widetilde{M} := e^{S_{\tau^x}}, \widetilde{Q} := e^{Z})\). By construction \( \mathbb{P}\{ \widetilde{M} \leq 1 \} = 1 \) and \( \mathbb{P}\{ \widetilde{M} = 1 \} = 0 \).
Therefore, by Corollary 3.1 \[ \text{9} \] the inequality \( \mathbb{E}g(\widetilde{Q}) < \infty \) follows from \( \mathbb{E}f(\sup \widetilde{M}_1 \cdots \widetilde{M}_{k-1} \widetilde{Q}_k) < \infty \). It remains to note that \( \widetilde{Q} = \exp(Z) = \exp(\sup_{0 \leq i \leq \tau^x - 1} S_i) \). In a similar way \( \sup \widetilde{M}_1 \cdots \widetilde{M}_{k-1} \widetilde{Q}_k = \exp(\sup_{0 \leq n \leq \tau^x - 1} S_n) \) is implied by \( (34) \).
At the beginning of the proof of Theorem 2.1 it is shown that there exists a non-decreasing, concave on $\mathbb{R}^+$ function $f$ that additionally satisfies $f(0) = 0$, $\lim_{x \to \infty} f(x) = \infty$ and $h(x) \sim f(e^x)$. Therefore the first equivalence and implication $\Leftarrow$ in the second equivalence in (36) follow from (35). The rest can be deduced from Theorem 3 [1].

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