How Current Loops and Solenoids Curve Space-time

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The curved space-time around current loops and solenoids carrying arbitrarily large steady electric currents is obtained from the numerical resolution of the coupled Einstein-Maxwell equations in cylindrical symmetry. The artificial gravitational field associated to the generation of a magnetic field produces gravitational redshift of photons and gravitational acceleration of neutral massive particles. The strength of the generated gravitational field is extremely weak from what can be obtained through present technology, although it might be detectable with high-precision measurements such as atom interferometry.

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Physicists studying gravity are, in some sense, contemplative: we restrict ourselves to the study of natural, already existing, sources of gravitation. Generating artificial gravitational fields, that could be switched on or off at will, is a question captured or left to science-fiction. However, the equivalence principle, at the very heart of Einstein’s general relativity, states that all types of energy produce gravitation and in the same way. The equivalence principle implies that one also generates gravitational fields when generating electromagnetic fields. Gravitational field generation therefore does not rely on invoking exotic types of energies. In this letter, we present how space-time is curved around current loops and solenoids carrying arbitrarily large electric currents.

The so-called Einstein-Maxwell (EM) equations regroup the classical field theories of general relativity and electromagnetism in a covariant way, although without truly unifying them. In some sense, the idea of gravitational field generation with magnetic field can be attributed to Levi-Civita whose early analytical work describes the curvature of space-time completely filled by a uniform magnetic field. Subsequent works have established analytical solutions for space-time around an infinitely long straight wire carrying steady current. The main problem of these analytical solutions is that the associated metric is not asymptotically flat, due to the infinitely large current distribution, which makes these analytical solutions of poor interest for practical applications. Overcoming this problem requires considering current distributions of finite extent such as current loops and solenoids. Asymptotic space-time around current loops carrying steady current has been studied in [4]. An attempt to derive the full solution of EM equations around the current loop was realized a bit later, however this attempt lead to an unphysical solution due to an oversimplifying assumption. The case of an infinitely long solenoid was considered in [5] but only for weak perturbations of the metric in linearized general relativity. Therefore, the solutions of the full non-linear EM equations sourced by the steady currents carried by loops and solenoids remained so far unexplored until now.

The EM system models the interaction of gravitation and electromagnetism in the following way. Space-time is curved by the energy of the electromagnetic field as ruled by Einstein equations of general relativity [1]. In the same time, the electromagnetic field propagates in the non-trivial background it generated through Einstein’s equations [1], and this propagation is described by the covariant Maxwell equations in curved space-time [2]. This juxtaposition of gravitational and electromagnetic interactions is encompassed in the following coupled tensorial field equations (in S.I. units [10]):

\[ R_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}^{(em)} \]  
\[ \nabla_{\mu} F^{\mu\nu} = \mu_0 J^\nu \]  

where \( T_{\mu\nu}^{(em)} = -\frac{1}{\mu_0} \left( g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \) is the Maxwell stress-energy tensor, \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is the Faraday tensor of the electromagnetic field, \( g_{\mu\nu} \) and \( A_\mu \) are the metric and the four-vector potential, i.e. the fundamental fields describing gravitation and electromagnetism respectively, \( R_{\mu\nu} \) is the Ricci tensor and \( J^\nu \) the four-current density. Because we are interested in the additional gravitational field that is produced by the magnetic field, we neglect the mass of the current carriers (the wires for instance).

Since current loops and solenoids possess one axis of symmetry, we choose the so-called Weyl gauge for the metric field:

\[ ds^2 = c^2 e^{\rho(r,z)} dt^2 - e^{\Lambda(r,z)} (dr^2 + dz^2) - e^{-\rho(r,z)} r^2 d\phi^2. \]  

In this symmetry, the vector potential \( A_\mu \) trivially reduces to one non-vanishing magnetic component \( A_\phi = a(r,z)/r \). The advantage of the Weyl gauge is that the equations of motion directly exhibit usual Laplacian operators on flat background with cylindrical coordinates. Indeed, Eqs. [1] and [2] using Eq. [3] now read (see...
also Bonnor, 1960):

\[ \nabla^2 (r,z) = \frac{8\pi G e^\rho}{\epsilon^4 \mu_0 r^2} \left( (\partial_r a)^2 + (\partial_z a)^2 \right) \]  
(4)

\[ \nabla^2 (r,z) \lambda + (\partial_r \rho)^2 = \frac{8\pi G e^\rho}{\epsilon^4 \mu_0 r^2} \left( (\partial_r a)^2 - (\partial_z a)^2 \right) \]  
(5)

\[ \nabla^2 (r,z) a - \frac{2}{r} \partial_r a = - \left( \partial_r a \partial_r \rho + \partial_z a \partial_z \rho \right) - r \mu_0 J \]  
(6)

where \( \nabla^2 (r,z) = \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 \) the usual Laplacian in cylindrical coordinates and where \( J \) is the angular component of the current density.

Eq. (6) is the Maxwell equation on a curved spacetime described by cylindrical coordinates. For a flat Minkowski background \( \rho = \lambda = 0 \), we have that the non-relativistic part \( a_{nr} \) satisfies

\[ \nabla^2 (r,z) a_{nr} - \frac{2}{r} \partial_r a_{nr} = - r \mu_0 J. \]  
(7)

We can therefore decompose the total field \( a(r,z) \) into the sum of a non-relativistic part \( a_{nr} \) (solution of Eq. (7)) and a relativistic contribution \( a_{rel} \) by setting \( a = a_{nr} + a_{rel} \). The relativistic contribution \( a_{rel} \) will therefore be a solution of

\[ \nabla^2 (r,z) a_{rel} - \frac{2}{r} \partial_r a_{rel} = - \left\{ (\partial_r a_{nr} + \partial_r a_{rel}) \partial_r \rho + \cdots \right\} \]  
(8)

This avoids dealing with point-like sources representing the current loop and the solenoid in cylindrical coordinates. The source of the field equations now lies in the non-relativistic contribution \( a_{nr}(r,z) \).

A current loop of radius \( l \) corresponds to a current density located on an infinitely thin ring: \( J \sim \delta(z) \cdot \delta(r-l) \) while a solenoid of finite length \( L \) and of radius \( l \) corresponds to a current density located on an infinitely thin sheet located at \( r = l \) and \( z \in [-\frac{L}{2}, \frac{L}{2}] \). Analytical expressions of the vector potential \( A_\phi \) in both cases can be derived from the Biot-Savart law, expressions that of course verify Eq. (7). The non-relativistic solution \( a_{nr}^{\text{loop}} \) for the current loop is given by (8):

\[ a_{nr}^{\text{loop}}(r,z) = \frac{\mu_0 I}{2\pi} \sqrt{(l^2 + r^2) + z^2} \times \left\{ \frac{l^2 + r^2 + z^2}{(l^2 + r^2 + z^2)^2} \left( K(k^2) - E(k^2) \right) \right\} \]  
(9)

where \( I \) is the steady current carried by the wire, \( k^2 = 4rl \left( \frac{l + r}{2} \right)^{-1} \) and \( K(k^2) = \int_0^{\pi/2} (1 - k^2 \sin^2(\varphi))^{-1/2} d\varphi, \ E(k^2) = \int_0^{\pi/2} (1 - k^2 \sin^2(\varphi))^{1/2} d\varphi \) are the complete elliptic integrals of the first and second kind respectively. For the solenoid of finite length \( L \), the non-relativistic solution \( a_{nr}^{\text{sol}}(r,z) \) is given by (10):

\[ a_{nr}^{\text{sol}}(r,z) = \frac{\mu_0 n I}{4\pi} \sqrt{r} \times \left\{ \left( \frac{\kappa^2 + g^2 - g^2 k^2}{k^4 g^2} \right) K(k^2) - \left( \frac{E(k^2)}{k^2} + \left( \frac{g^2}{2} - 1 \right) \Pi(g^2, k^2) \right) \right\} \]  
(10)

where \( n \) is the number of wire loops per unit length, \( k^2 = 4rl \left( \frac{l + r}{2} + \xi^2 \right)^{-1} \), \( g^2 = 4rl(l + r)^{-2} \), \( \xi = z \pm \frac{L}{2} \) and \( \Pi(g^2, k^2) = \int_0^{\pi/2} (1 - g^2 \sin^2(\varphi))^{-1/2} d\varphi \) is the complete elliptic integral of the third kind. In order to be used as source terms in Eq. (5), the gradients of \( a_{nr}(r,z) \) can be obtained analytically from the formulae Eqs. (9-10) and the properties of complete elliptic functions.

If we set \( r = ul, z = vl \) and \( a \rightarrow a/(\mu_0 H) \) \((a \rightarrow a/(\mu_0 n H L) \) for the solenoid), Eqs. (9-10) now reduce to the following set of dimensionless equations

\[ \nabla^2 = C_l^\rho \left( \left( \partial_u (a_{nr} + a_{rel}) \right)^2 + \left( \partial_v (a_{nr} + a_{rel}) \right)^2 \right) \]  
(11)

\[ \nabla^2 \lambda + (\partial_v \rho)^2 = C_l^\rho \left( \left( \partial_u (a_{nr} + a_{rel}) \right)^2 - \left( \partial_v (a_{nr} + a_{rel}) \right)^2 \right) \]  
(12)

\[ \nabla^2 a_{rel} - \frac{2}{u} \partial_u a_{rel} = - \left( \partial_u (a_{nr} + a_{rel}) \partial_u \rho + \partial_x (a_{nr} + a_{rel}) \partial_x \rho \right) \]  
(13)

The dimensionless magneto-gravitational coupling for the current loop and the solenoid are given by

\[ C_l^{\text{loop}} = \frac{8\pi G}{c^4} \mu_0 I^2; \quad C_l^{\text{sol}} = \frac{8\pi G}{c^4} \mu_0 I^2 n^2 L^2. \]  
(14)

Solving the system (11-13) requires the specification of boundary conditions. Far away from the current loop or the solenoid \((u,v) \rightarrow (+\infty, \pm \infty)\), these devices behaves as magnetic dipoles and the space-time is asymptotically flat (see also (8)). The magnetic component \( a \) is then
ruled by Eq. (7) so that the relativistic term \( \partial_u a \partial_u \rho + \partial_l a \partial_l \rho \) vanishes. This condition is achieved if \( a_{ul} \) vanishes and \( \partial_l a_{ul} = u \partial_l \psi \) and \( \partial_u a_{ul} = -u \partial_u \psi \) with \( \psi \) an harmonic function called the scalar magnetic potential \([4, 5]\). For magnetic dipoles, the (dimensionless) scalar magnetic potential is given by \( \psi = \frac{u}{2} (u^2 + v^2)^{3/2} \). The metric functions \( \rho \) and \( \lambda \) are therefore ruled by the following equations at large distances \((u, v) \to (+\infty, \infty)\), \( \rho \ll 1 \) so that \( e^\rho \approx 1 \):

\[
\nabla^2 \rho = C_I \left( (\partial_u \psi)^2 + (\partial_v \psi)^2 \right) \\
\nabla^2 \lambda = C_I \left( (\partial_u \psi)^2 - (\partial_v \psi)^2 \right).
\]

The asymptotic behaviors for the metric fields around the current loop and the solenoid are therefore given by

\[
\rho \sim \frac{C_I}{32} \frac{u^2}{(u^2 + v^2)^3}, \\
\lambda \sim \frac{C_I}{16} \left( \frac{2u^2}{u^2 + v^2} - \frac{u^2}{2(u^2 + v^2)^3} - \frac{9u^4}{4(u^2 + v^2)^4} \right).
\]

Now that these boundary conditions have been found, we solve Eqs. (11)-(13) numerically by using a combination of relaxation and spectral methods\([11, 13]\). Figure 1 presents the metric functions \( \rho \) and \( \lambda \) obtained from the numerical resolution of Eqs. (11)-(13) with the boundary conditions Eqs. (14). Space-time around the current loop is singular at the loop location \((u = 1, v = 0)\) while it is everywhere regular around the solenoid. The electric current located on the loop and the solenoid generate potential wells with maximal depths around the center of the loop or on the \( z \)-axis inside the solenoid. These metric potential wells will produce light deflexion as well as gravitational redshift which will be maximal for a light source located at the origin of coordinates. For an observer located at spatial infinity (where \( g_{tt} \to 1 \)) and a source located at the origin of coordinates, the gravitational redshift is simply \( z = 1 - \exp(-\rho(0,0)/2) \), and is of the order of magnitude of the magneto-gravitational coupling \( C_I \) (see also Fig. 2). The precision achieved by optical lattice clocks in the measurement of a transition frequency is of the order \( 10^{-15} \)\([11]\). Achieving such a gravitational redshift with single-layered solenoids would require \( C_I \approx 10^{-15}, \) i.e. for an electric current of \( 1kA, n = 100 \) it would require a solenoid length of about \( 10^{13}m \). Gravitational redshift therefore does not seem to be appropriate to detect gravitational fields artificially generated by coils with current superconducting technology. Fig. 2 illustrates how the gravitational redshift \( z \) evolves with the magneto-gravitational coupling \( C_I \) Eq. (14). For \( C_I \ll 1 \), the redshift \( z \) varies linearly with \( C_I \) while a space-time singularity appears at the center of coordinates, \( \rho(0,0) \to -\infty \) \((z \to 1)\) when \( C_I \to \infty \).

In the weak field regime \((\rho, \lambda \ll 1)\), a massive particle moving in the \((u, v)\)-plane at low velocity will experience a newtonian potential \( V = \frac{c^2}{2} (\exp(\rho) - 1) \approx \frac{c^2}{2} \rho \). Therefore, the current loop and the solenoid generate an artificial gravitational force field on massive particles given by \( \vec{F} = -\frac{m c^2}{2} \vec{\nabla} \rho \) (with \( m \) the mass of the particle). Figure 2 illustrates the gravitational attraction of the slowly moving massive particles in the weak field regime through the representation of the force field and its integral curves (i.e., force field lines). On the left panel, we see that massive particles are attracted toward the current loop if they started from outside of the loop \((u > 1)\) while they are attracted toward the loop plane and then toward the loop location if they started with \( u < 1 \). On the central panel, we see a slightly different behavior for the force
field generated by a solenoid with $L = l/2$: the massive particles are attracted toward a central region ($v = 0$) inside the solenoid. Finally, the right panel of Figure 2 shows us that a long solenoid (here with $L = 10l$) attracts massive particles toward its central region $u = 0$ and $v \in [-L/2, L/2]$ with radial acceleration for particles starting with $|v| < L/2$ (see also [6]) and vertical acceleration with initial positions $|v| > L/2$.

The amplitude of space-times deformations generated by current loops and solenoids is ruled by the dimensionless parameter $C_I$ ([1]), which is extremely small from what can be achieved with present technology. As an example, the current world’s largest superconducting coil, the CMS magnet [12], for which $I \approx 20kA$, $n \approx 40$, $l = 3m$ and $L = 13m$ gives a value of the magneto-gravitational coupling of $C_I \approx 10^{-35}$, resulting in the generation of extremely weak gravitational redshift and gravitational attraction. In Maxwell theory, electrically neutral particles are not submitted to the Lorentz force and are therefore not affected by an external magnetic field. According to general relativity, this is in fact not true since the magnetic field energy curves space-time producing an extra gravitational force on these neutral particles. This effect, although extremely faint because of [14], is interesting from the experimental point of view since it could be used to construct null experiments of Maxwell theory that will be falsified due to general relativistic effects. Both the solenoid characteristics and the precision required for achieving such tests are extreme. However, atomic interferometry has allowed detecting extremely faint differential accelerations [13] and have been considered for the determination of the extremely faint change in the gravitational potential produced by small masses on matter waves [14]. We computed a chart of the amplitude of the extra gravitational acceleration generated by the magnetic field of a solenoid with characteristics within reach of present technology [13]: $I = 10^5A$, $n = 100m^{-1}$, $l = 1m$ and $L = 3m$. The generated gravitational acceleration is the most important around the solenoid ($r = l$, $z = 0$) and its edges ($|z| \approx L/2$, $r < l$) and is of the order $10^{-19}m/s^2$. In this example, the amplitude of the gravitational acceleration artificially generated by a single-layered solenoid is extremely weak but lies just a few orders of magnitude below the precision of atomic interferometry in the measurement of differential acceleration ($10^{-15}g$, [13]).

In conclusion, the generation of artificial gravitational fields with electric currents could be in principle detected as an interaction of magnetic fields with electrically neutral particles and photons. This effect does not invoke any new physics, as it is a consequence of the equivalence principle, but its weakness poses technological challenges for both the generation of the gravitational field with a coil and its subsequent detection. However, we claim that such detection would open new eras in experimental gravity. It would also allow new tests of the equivalence principles including in the quantum regime through the impact on Aharonov-Bohm effect. And would this technology be developed, it could lead to amazing applications like the controlled emission of gravitational waves with large alternative electric currents.

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[1] T. Levi-Civita, Gen. Rel. Grav. 43, 2307 (2011); B. Bertotti, Phys. Rev. 116, 1331 (1959); I. Robinson, Bull. Acad. Polon. Sci. 7, 351 (1959).

[2] B. Mukherji, Calcutta Mathematical Society Bulletin 30, 95 (1938).

[3] L. Witten, Centre Belge de Recherches Mathématiques, Colloque sur la Théorie de Relativité, p. 59, 1960 ; L. Witten in "Gravitation : an Introduction to current Research", chap. 9, edited by L. Witten (Wiley, New York, 1962).

[4] W.B. Bonnor, Proc. Phys. Soc. A 67, 225 (1954).

[5] W.B. Bonnor, Proc. Phys. Soc. A 76, 891 (1960).

[6] B. V. Ivanov, Mod. Phys. Lett. A 9, 1627 (1994).

[7] H. Weyl, Ann. Phys. 54, 117 (1917).

[8] J. D. Jackson, "Classical Electrodynamics" (Wiley, New York, 1998).

[9] L. Landau, E. Lifchitz & L.P. Pitaevskii, "Electrodynamics of Continuous Media" (Pergamon, New York, 1984).

FIG. 3: Newtonian gravitational force field ($\partial_\nu \rho$, $\partial_\mu \rho$) and its field lines in the $(u, v)$-plane along the current loop (upper left panel) and the solenoid (upper right panel $L = l/2$, lower left panel $L = l$, lower right panel $L = 10l$). Red dots indicate the position of the current loop and the solenoid.
The relevant fundamental constants of the EM system are $G$ as Newton’s constant, $c$ as the speed of light and $\mu_0$ as the (vacuum) magnetic permeability. We only keep two of Einstein equations, the others being redundant through Bianchi identities. We will detail this procedure in a forthcoming paper.