Equilibrium controls in time inconsistent stochastic linear quadratic problems

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November 9, 2018

Abstract

This paper deals with a class of time inconsistent stochastic linear quadratic (SLQ) optimal control problems in Markovian framework. Three notions, i.e., closed-loop equilibrium controls/strategies, open-loop equilibrium controls and their closed-loop representations, are characterized in unified manners. These results indicate clearer and deeper distinctions among these notions. For example, in particular time consistent setting, the open-loop equilibrium controls are fully characterized by first-order, second-order necessary optimality conditions, and become needlessly optimal, while the closed-loop equilibrium controls naturally reduce into closed-loop optimal controls.

Keywords. linear quadratic optimal control problems, time inconsistency, equilibrium controls, Riccati equations.

AMS Mathematics subject classification. 93E20, 49N10, 91B51, 60H10.

1 Introduction

Throughout this paper, \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})\) is a complete filtered probability space, on which one-dimensional standard Brownian motion \(W(\cdot)\) is defined. Here \(\mathbb{F} \equiv \{\mathcal{F}_t\}_{t \geq 0}\) is the natural filtration of \(W(\cdot)\) augmented by \(\mathbb{P}\)-null sets.

1.1 Formulation of time inconsistent optimal control problems

For any \(t \in [0, T)\), we consider the following stochastic differential equation (SDE):

\[
\begin{aligned}
    dX(s) &= \left[A(s)X(s) + B(s)u(s) + b(s)\right]ds \\
    &\quad + \left[C(s)X(s) + D(s)u(s) + \sigma(s)\right]dW(s), \quad s \in [t, T], \\
    X(t) &= \xi,
\end{aligned}
\]  

(1.1)

*The research was supported by the NSF of China under grant 11231007, 11401404 and 11471231.

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and the cost functional defined by
\[
J(t, \xi; u(\cdot)) = \frac{1}{2} \mathbb{E}_t \left\{ \int_t^T \left[ \langle Q(s)X(s), X(s) \rangle + 2 \langle S(s)X(s), u(s) \rangle \\
+ \langle R(s)u(s), u(s) \rangle \right] ds + \langle GX(T), X(T) \rangle \right\}.
\]
(1.2)

Here \( A, B, C, D, Q, S, R, G \) are suitable matrix-valued (deterministic) functions, \( b, \sigma \) are proper stochastic processes, and \( \mathbb{E}_t(\cdot) := \mathbb{E}[\cdot | \mathcal{F}_t] \) stands for conditional expectation operator. In the above, \( X(\cdot) \), valued in \( \mathbb{R}^n \), is called the state process, \( u(\cdot) \), valued in \( \mathbb{R}^m \), is called the control process, and \((t, \xi) \in \mathcal{D}\) is called the initial pair where
\[
\mathcal{D} := \left\{ (t, \xi) \mid t \in [0, T], \xi \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}[\xi]^2 < \infty \right\}.
\]

We denote the set of all control processes by
\[
\mathcal{U}[t, T] = \left\{ u : [t, T] \times \Omega \to \mathbb{R}^m \mid u \text{ is } \mathcal{F}_t\text{-progressively measurable, } \mathbb{E} \int_t^T |u(s)|^2 ds < \infty \right\}.
\]

Under some mild conditions on the coefficients, for any initial pair \((t, \xi)\) and a control \( u(\cdot) \in \mathcal{U}[t, T] \), the state equation (1.1) admits a unique solution \( X(\cdot) = X(\cdot; t, x, u(\cdot)) \), and the cost functional \( J(t, \xi; u(\cdot)) \) is well-defined. We pose the following stochastic linear quadratic (SLQ) optimal control problem.

**Problem (SLQ).** For any given \((t, \xi)\), find a \( \bar{u}(\cdot) \in \mathcal{U}[t, T] \) such that
\[
J(t, \xi; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, \xi; u(\cdot)) \overset{\Delta}{=} V(t, \xi).
\]
(1.3)

Any \( \bar{u}(\cdot) \in \mathcal{U}[t, T] \) satisfying (1.3) is called an optimal control for the given initial pair \((t, \xi)\), the corresponding state process \( \bar{X}(\cdot) \) is called an optimal state process for \((t, \xi)\), \((\bar{X}(\cdot), \bar{u}(\cdot))\) is called an optimal pair for \((t, \xi)\), and \( V(\cdot, \cdot) \) is called the value function of Problem (SLQ).

For above optimal control problem, it is reasonable to keep the state process stable with respect to possible variation of random factors. To this end, one effective way is to add the variation of \( X(\cdot) \), i.e.
\[
\text{Var}_t[X] := \mathbb{E}_t[X(T) - \mathbb{E}_tX(T)]^2 = \mathbb{E}_t[X(T)]^2 - [\mathbb{E}_tX(T)]^2
\]
into the cost functional (e.g., [3], [4], [11], [12], [13], [14], [22], [26], etc). Therefore, it is natural to propose the following general modified cost functional
\[
J(t, \xi; u(\cdot)) = \frac{1}{2} \mathbb{E}_t \left\{ \int_t^T \left[ \langle Q(s)X(s), X(s) \rangle + 2 \langle S(s)X(s), u(s) \rangle \\
+ \langle \bar{Q}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[X(s)] \rangle + 2 \langle \bar{S}(s)\mathbb{E}_t[X(s)], \mathbb{E}_t[u(s)] \rangle \\
+ \langle R(s)u(s), u(s) \rangle + \langle \bar{R}(s)\mathbb{E}_t[u(s)], \mathbb{E}_t[u(s)] \rangle \right] ds \\
+ \langle GX(T), X(T) \rangle + \langle \bar{G}\mathbb{E}_t[X(T)], \mathbb{E}_t[X(T)] \rangle + 2 \langle g, \mathbb{E}_tX(T) \rangle \right\}.
\]
(1.4)

Here \( \bar{S}, \bar{R}, \bar{G}, \bar{Q} \) are deterministic matrices-valued functions and \( g \) is a vector.

In this scenario, the optimal controls become time-inconsistent, i.e., the “optimal” control based on this moment may not keep optimality in future. We refer to [26] for some explicit examples.
1.2 Related literature

The study on time inconsistency by economists actually dates back to Strotz [12] in the 1950s. One possible way to treat time inconsistency is to discuss the pre-committed controls for which the solutions are verified to be optimal only at the initial time.

In this paper, we shall discuss above optimal control problem from another viewpoint. More precisely, we investigate the time inconsistency within a game-theoretic framework and analyze the time-consistent equilibrium solution (e.g., [16], [15], [10]). Recently, people began to treat the equilibrium controls using the ideas of stochastic control theories, and developed several different approaches in the existing papers. These methods range from dynamic programming principles and verification procedures to maximum principles and variational techniques.

⋄ In Björk-Murgoci [2], Björk et al [3], the authors examined a general class of time inconsistent problems under Markovian framework by equilibrium value functions. In the continuous case, they formally derived the extended HJB equations, and then rigorously proved the verification theorem by the conclusions of discrete time case, see Theorem 5.2 in [3]. They also present some special cases including a linear quadratic control problem in which equilibrium solutions are constructed. This method was also used to treat investment-reinsurance problems with mean-variance criterion, see e.g., [14], [28].

⋄ In Yong ([24], [26]), the author discussed a class of time inconsistent optimal control problems by multi-person differential games approach, where a new kind of equilibrium HJB equations/systems of Riccati equations were introduced. Unlike [2], [3], they started the investigations in continuous time setting, made partition on time intervals and used tricks of forward-backward stochastic differential equations (FBSDEs). Further study along this can be found in [20], [23], and so on.

⋄ In Ekeland and Lazrak ([9], [8]), they considered some financial problems such as investment and consumption model with time-inconsistency feature. They used the variational ideas to introduce certain feedback/closed-loop equilibrium controls, and spread out discussions via equilibrium value functions. Compared with the general situation in [2], [3], the particular form of equilibrium value functions were proposed according to the given cost functional, while the complex convergence arguments were avoided.

⋄ Inspired by the ideas of stochastic maximum principles in optimal control theories, Hu et al. [11] studied a class of time inconsistent SLQ problems in Markovian setting, introduced open-loop equilibrium controls and their closed-loop representations, derived general sufficient conditions through a flow of FBSDEs or systems of backward ordinary differential equations (ODEs). Just recently, the same authors continued to discuss the uniqueness of open-loop equilibrium controls in [12]. More related details can also be found in [7], [22], [21].

1.3 Unified approach and contributions

As to Problem (SLQ), in this article we propose a unified method to characterize the open-loop equilibrium controls, the closed-loop representations of open-loop equilibrium controls, closed-loop equilibrium controls/strategies. We combines the ideas from variational analysis, forward-backward stochastic differential equations and forward-backward decoupling procedures. In the following, we provide a brief outline of our approach.
For any \((\Theta_1, \Theta_2, \varphi) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2(0, T; \mathbb{R}^{m \times n}) \times L^2_F(0, T; \mathbb{R}^m)\), we start with control processes
\[
u := (\Theta_1 + \Theta_2) X + \varphi, \quad u^\varepsilon := \Theta_1 X^\varepsilon + \Theta_2 X + \varphi + v I_{[t, t+\varepsilon]},\] (1.5)

They can reduce into the required equilibrium controls and perturbed controls in various settings. More precisely, if \(\Theta_2 \equiv 0\), or \(\Theta_1 \equiv 0\), or \(\Theta_1 \equiv \Theta_2 \equiv 0\), \(u\) and \(u^\varepsilon\) play the important roles in obtaining closed-loop equilibrium controls/strategies, open-loop equilibrium controls, the closed-loop representation of open-loop equilibrium controls, respectively. We refer to Subsection 4.4 for more detailed discussions.

In view of the definitions for equilibrium controls, we proceed to consider the difference of the cost functional at \(u, u^\varepsilon\). To do so, given \(X\) and \(X^\varepsilon\), we introduce, respectively, backward stochastic differential equations (BSDEs) with conditional expectations. We point out that the one associated with \(X^\varepsilon\) appears for the first time in the literature. As a result, we obtain two forward-backward systems in which the terminal parts and generators of backward systems rely respectively on \(X, X^\varepsilon\).

To tackle the limit part in the definitions of both open-loop and closed-loop equilibrium controls (i.e., Definitions 2.1, Definition 2.3 next), we continue to decouple the above two forward-backward systems. More precisely, we make conjectures on the solutions of backward systems, formally obtain a class of systems of BSDEs merely depending on given coefficients, and then verify our arguments rigorously. At last we establish our characterizations with proper convergence procedures.

At this very moment, it is worth mentioning that the previous proposed approach demonstrates several new advantages on the treatment of both open-loop equilibrium controls, closed-loop equilibrium controls/strategies. Unlike \([2], [3], [24], [26]\), our procedures on closed-loop equilibrium strategy in continuous time drop the reliance on complex convergence arguments from discrete time to continuous case. Comparing with \([11], [12]\), our methodology on open-loop equilibrium controls neither requires any non-definite assumptions on the involved coefficients, nor directly uses the conclusions of stochastic maximum principles. Moreover, it can be adjusted into the random coefficients case, see \([22]\).

Even though both open-loop equilibrium controls and closed-loop equilibrium controls are widely investigated in the literature, there is no paper discussing their differences to our best. In this paper, we give a clear picture by the obtained characterizations. For example, in the classical SLQ setting, open-loop equilibrium controls are fully characterized by first-order, second-order necessary conditions. In other words, they are weaker than optimal controls (Remark 3.9). However, in the same situation, the closed-loop equilibrium controls happen to reduce exactly into closed-loop optimal controls (Remark 3.9). Eventually, we point out that the characterizations on open-loop, closed-loop equilibrium controls, respectively, include two different second-order equilibrium conditions, which are absent in nearly all the relevant articles.

1.4 Outline of the article

The remainder of this article is structured as follows. In Section 2, an overview of assumptions, notation used in the sequel is provided. In Section 3, the main conclusions of this article are gathered and some important remarks are demonstrated. In Section 4, the proofs of the main results in Section 3 are given. Section 5 concludes this article.
2 Preliminary notations

Given $H := \mathbb{R}^n, \mathbb{R}^{n \times n}, \mathbb{S}^{n \times n}$, etc, we introduce the following hypotheses on the coefficients of (1.1), (1.4).

\((H1)\) Suppose $A, B, C, D, R, \tilde{R}, Q, \tilde{Q}, S, \tilde{S} \in L^\infty(0, T; H), G, \tilde{G}, g \in H, b \in L^2_\mathcal{F}(\Omega; L^1(0, T; H)), \sigma \in L^2_\mathcal{F}(0, T; H)$.

For $0 \leq s \leq t \leq T$, we also define some involved spaces as follows.

$$L^2_\mathcal{F}(s, t; H) := \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}\text{-adapted, measurable,} \right.$$ \[ \mathbb{E} \int_s^t |X(r)|^2 \, dr < \infty \},$$

$$L^\infty(s, t; H) := \left\{ X : [s, t] \rightarrow H \mid X \text{ is deterministic, measurable, } \sup_{r \in [s, t]} |X(r)| < \infty \right\},$$

$$L^2_\mathcal{F}(\Omega; L^1(s, t; H)) := \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}\text{-adapted, measurable,} \right.$$ \[ \mathbb{E} \left[ \int_s^t |X(r)| \, dr \right]^2 < \infty \},$$

$$L^2_\mathcal{F}(\Omega; C([s, t]; H)) := \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathcal{F}\text{-adapted, measurable} \right.$$ \[ \text{continuous } \mathbb{E} \sup_{r \in [s, t]} |X(r)|^2 < \infty \right\}.$$

To begin with, we look at Problem (SLQ) from an open-loop equilibrium control viewpoint. The following definition is adapted from [11], [12].

**Definition 2.1** Given $X^*(0) = x_0 \in \mathbb{R}^n$, a state-control pair $(X^*, u^*) \in L^2_\mathcal{F}(\Omega; C([0, T]; \mathbb{R}^n)) \times L^2_\mathcal{F}(0, T; \mathbb{R}^m)$ is called an open-loop equilibrium pair if for any $t \in [0, T)$, small $\varepsilon > 0$, $\mathcal{F}_t$-measurable $\nu$ satisfying $\mathbb{E} \nu^2 < \infty$, the following holds:

$$\lim_{\varepsilon \to 0} \frac{J(t, X^*(t); u^{\nu, \varepsilon}(\cdot)) - J(t, X^*(t); u^*(\cdot))|_{[t, T]}}{\varepsilon} \geq 0, \quad (2.1)$$

where $u^{\nu, \varepsilon} = u^* + \nu 1_{[t, t+\varepsilon]}$. Here $u^*$ and $X^*$ are called open-loop equilibrium control and open-loop equilibrium state process.

Roughly speaking, the definition shows the dynamic local optimality in some manner. In this paper we will explore deeper properties of such equilibrium controls via their characterizations.

Due to our particular linear quadratic structure, we also introduce the closed-loop representation of open-loop equilibrium control $u^*$.

**Definition 2.2** An open-loop equilibrium control $u^* \in L^2_\mathcal{F}(0, T; \mathbb{R}^m)$ associated with $X^*(0) = x_0 \in \mathbb{R}^n$ is said to have a closed-loop representation if $u^* = \Theta^* X^* + \varphi^*$ where $X^*$ is the associated state process on $[0, T]$, and $(\Theta^*, \varphi^*) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2_\mathcal{F}(0, T; \mathbb{R}^m)$. Here they are called open-loop equilibrium strategy pair, which are independent of $x_0$. 


From the open-loop strategy viewpoint, we can capture more explicit expression of open-loop equilibrium control. However, this kind of strategy is distinctive from the following one.

**Definition 2.3** \((\Theta^*, \varphi^*) \in L^2(0, T; \mathbb{R}^{m \times m}) \times L^2(0, T; \mathbb{R}^m)\) is called a closed-loop equilibrium strategy, if for any initial state \(x_0 \in \mathbb{R}^n\), \(t \in [0, T]\), small \(\varepsilon > 0\), \(\mathcal{F}_t\)-measurable \(v\) satisfying \(\mathbb{E}|v|^2 < \infty\),

\[
\lim_{\varepsilon \to 0} \frac{J(t, X^*(t); u^\varepsilon(\cdot)) - J(t, X^*(t); u^\ast(\cdot)|_{[t,T]})}{\varepsilon} \geq 0, 
\]

where \(u^* := \Theta^* X + \varphi^*, u^\varepsilon := \Theta^* X^\varepsilon + v_{\varepsilon[t,t+\varepsilon]} + \varphi^*, X^*, X^\varepsilon\) are the state process on \([0, T]\) associated with \(u^*, u^\varepsilon\), respectively.

We emphasize that both open-loop equilibrium strategy and closed-loop equilibrium strategy are independent of initial state \(x_0\). However, the perturbed control \(u^{\varepsilon, \ast}\) in Definition 2.1 is actually different from \(u^\varepsilon\) in Definition 2.3. In this paper, we will demonstrate further connections between these two kinds of strategies.

In the following, let \(K\) be a generic constant which varies in different context and

\[
\mathcal{R} := R + \tilde{R}, \quad \mathcal{Q} := Q + \tilde{Q}, \quad \mathcal{G} := G + \tilde{G}, \quad \mathcal{S} = S + \tilde{S}. 
\]

### 3 Characterizations of equilibrium controls/strategies

In this part, we state the main results of this article. We start with the case of open-loop equilibrium controls. To this end, given \(u \in L^2(0, T; \mathbb{R}^m)\), we introduce

\[
\begin{align*}
    dP_1 &= -\left[ A^T P_1 + \mathcal{A}^T P_1 C - Q \right] ds, \\
    dP_2 &= -\left[ A^T P_2 - \tilde{Q} \right] ds, \\
    dP_3 &= -\left[ A^T P_3 + P_2 b + (P_2 B - \tilde{S}^T) u \right] ds + L_3 dW(s), \\
    dP_4 &= -\left[ A^T P_4 + C^T L_4 + C^T P_1 \sigma + P_1 b + (C^T P_1 D + P_1 B - \tilde{S}^T) u \right] ds + L_4 dW(s), \\
    P_1(T) &= -G, \quad P_2(T) = -\tilde{G}, \quad P_3(T) = 0, \quad P_4(T) = -g.
\end{align*}
\]

Here \(P_1, P_2\) do not rely on \(u\) while \(P_3, P_4\) do. It is easy to see the solvability, as well as the following regularities, of systems of equations (3.1),

\[
P_1, P_2 \in C([0, T]; \mathbb{R}^{n \times n}), \quad (P_3, \Lambda_3), (P_4, \Lambda_4) \in L^2_\mathcal{F}(\mathcal{Q}; C([0, T]; \mathbb{R}^n)) \times L^2_\mathcal{F}(0, T; \mathbb{R}^n).
\]

For \(X\) in (1.1), we define

\[
\begin{align*}
    M(s, t) &= P_1(s) X(s) + P_2(s) \mathbb{E}_t X(s) + \mathbb{E}_t P_3(s) + P_4(s), \quad s \in [t, T], \\
    N(s) &= P_1(s) (C(s) X(s) + D(s) u(s) + \sigma(s)) + L_4(s), \quad s \in [0, T].
\end{align*}
\]
**Theorem 3.1** Suppose (H1) holds, $P_1$ satisfies (3.1). Then $\bar{u}$ is an open-loop equilibrium control associated with initial state $\bar{X}(0) = x_0 \in \mathbb{R}^n$ if and only if
\[
\mathcal{R}(s) - D(s)^\top P_1(s) D(s) \geq 0, \quad s \in [0, T], \text{ a.e.}
\]
and given $(\bar{M}, \bar{N})$ in (3.2) associated with $\bar{u}$,
\[
\mathcal{R}(s)\bar{u}(s) + \mathcal{J}(s)\bar{X}(s) - B(s)^\top \bar{M}(s, s) - D(s)^\top \bar{N}(s) = 0, \quad s \in [0, T], \text{ a.e.}
\]

Above (3.3), (3.4) are named as first-order, second-order equilibrium conditions, which are comparable with classical first-order, second-order necessary optimality conditions (e.g., [5], [27]) in optimal control theories.

**Remark 3.1** As to $P_1$ in (3.3), it is indeed the unique solution of classical second-order adjoint equation in optimal control theories. That is to say, (3.3) can reduce into the traditional second-order necessary optimality condition if $\bar{R} = 0$. To our best, this point was not discussed seriously in [11], [12], and other related papers on open-loop equilibrium controls.

**Remark 3.2** For $X$ in (1.1), we see that $(M, N)$ satisfies
\[
\begin{aligned}
dM &= -\left[A^\top M + C^\top N - QX - S^\top u - \bar{Q}E_e X - \bar{S}^\top E_e u\right]dr + NdW(r), \\
M(T, t) &= -GX(T) - \bar{G}E_e X(T) - g.
\end{aligned}
\]
As a result, if $\bar{R} = \bar{Q} = \bar{S} = \bar{G} = 0$ and $u$ is optimal, (3.5) becomes the first-order adjoint equation. In other words, (3.4) degenerates into an equivalent form of first-order necessary condition.

**Remark 3.3** If $\bar{R} = \bar{S} = S = 0, R, Q, G$ are definite matrices, then (3.3) is obvious to see. In this scenario, a characterization of open-loop equilibrium control, which is different yet equivalent with (3.4), was given in Theorem 3.5 of [12]. However, there were no systems of equations (3.1) involved in their conclusion.

Next we characterize the closed-loop representation of open-loop equivalent control in the sense of Definition 2.2. For $(\Theta_2, \varphi) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2(0, T; \mathbb{R}^m)$ in above (1.5), we introduce system of equations
\[
\begin{aligned}
dP_1 &= -\left[P_1 A + A^\top P_1 + C^\top P_1 C + (P_1 B + C^\top P_1 D - S^\top)\Theta_2 - Q\right]ds, \\
dP_2 &= -\left[P_2 A + A^\top P_2 - \bar{Q} + (P_2 B - \bar{S}^\top)\Theta_2\right]ds, \\
dP_3 &= -\left[A^\top P_3 + (P_2 B - \bar{S}^\top)\varphi + P_2b\right]ds + \mathcal{L}_3dW(s), \\
dP_4 &= -\left[A^\top P_4 + C^\top \mathcal{L}_4 + C^\top P_1 \sigma + (C^\top P_1 D + P_1 B - S^\top)\varphi\\
&\quad + P_1b\right]ds + \mathcal{L}_4dW(s), \\
P_1(T) &= -G, \quad P_2(T) = -\bar{G}, \quad P_3(T) = 0, \quad P_4(T) = -g,
\end{aligned}
\]

and following-up processes $(\mathcal{M}, \mathcal{N})$ as follows,
\[
\begin{aligned}
\mathcal{M} &:= P_1 X + P_2 E_e X + E_e P_3 + P_4, \\
\mathcal{N} &:= P_1 (C + D \Theta_2) X + P_1 (D\varphi + \sigma) + \mathcal{L}_4.
\end{aligned}
\]

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Remark 3.4 Given \((\Theta_2, \varphi)\), if \(u := \Theta_2 X + \varphi\) where \(X\) is the associated state satisfying (1.1) on \([0, T]\), we see that \((\mathcal{M}, \mathcal{N})\) solves (3.5) as well. By the uniqueness of BSDEs, \((\mathcal{M}, \mathcal{N}) \equiv (\mathcal{M}, \mathcal{N})\). Consequently, we obtain two different representations, i.e., (3.2), (3.7), for the solutions of (3.5).

Theorem 3.2 Suppose \((H1)\) holds, \(P_1\) satisfies (3.1). Then for any \(X^*(0) = x_0 \in \mathbb{R}^n\), there exists equilibrium control \(u^*\) in the sense of Definition 2.2 if and only if (3.3) is true and there exist \(P_1^*, P_2^*, (P_3, L_3), (P_4, L_4)\) satisfying BSDEs (3.6) with \((\Theta_2, \varphi) \equiv (\Theta^*, \varphi^*)\) and

\[
\begin{align*}
\Theta^* &= \left[ \mathcal{R} - D^T P_1^* D \right] \Theta^* + B^T \left( P_1^* + P_2^* \right) + D^T P_1^* C - \mathcal{J}, \\
\varphi^* &= D^T \left( P_1^* \sigma + L_4^* \right) + B^T \left( P_3^* + P_4^* \right).
\end{align*}
\]

Remark 3.5 From (3.8), there exists \(\theta' \in L^2(0, T; \mathbb{R}^{n \times n})\), \(\varphi' \in L^2(0, T; \mathbb{R}^m)\) s.t.

\[
\begin{align*}
\Theta^* &= \left[ \mathcal{R} - D^T P_1^* D \right] \Theta^* + B^T \left( P_1^* + P_2^* \right) + D^T P_1^* C - \mathcal{J}, \\
\varphi^* &= D^T \left( P_1^* \sigma + L_4^* \right) + B^T \left( P_3^* + P_4^* \right) + \left\{ I - \left[ \mathcal{R} - D^T P_1^* D \right] \right\} \Theta' + \left\{ I - \left[ \mathcal{R} - D^T P_1^* D \right] \right\} \varphi'.
\end{align*}
\]

Moreover,

\[
\begin{align*}
\mathcal{R} \left( B^3 (P_1^* + P_2^*) + D^3 P_1^* C - \mathcal{J} \right) \subset & \mathcal{R} \left( \mathcal{R} - D^T P_1^* D \right), \text{ a.e.} \\
\left[ B^3 (P_1^* + P_2^*) + D^3 (P_1^* \sigma + L_4^*) \right] \subset & \mathcal{R} \left( \mathcal{R} - D^T P_1^* D \right), \text{ a.e. a.s.} \\
\left[ \mathcal{R} - D^T P_1^* D \right] \left[ B^3 (P_1^* + P_2^*) + D^3 (P_1^* \sigma + L_4^*) \right] & \in L^2(0, T; \mathbb{R}^{m \times n}), \\
\left[ \mathcal{R} - D^T P_1^* D \right] \left[ B^3 (P_1^* + P_2^*) + D^3 (P_1^* \sigma + L_4^*) \right] & \in L^2(0, T; \mathbb{R}^m).
\end{align*}
\]

In above, \(\mathcal{R}(A), A^\dagger\) is the range, pseudo-inverse of matrix \(A\), respectively. Therefore, we obtain one representation of open-loop equilibrium strategy pair \((\Theta^*, \varphi^*)\), as well as some intrinsic relations among coefficients in (3.10). Compared with open-loop equilibrium controls in Theorem 3.1, such closed-loop representations are advantageous in some sense and provide us more useful information.

At last, we give the characterizations of closed-loop equilibrium strategies. For \((\Theta_1, \varphi) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2(0, T; \mathbb{R}^m)\) in above (1.5), we introduce

\[
\begin{align*}
d\mathcal{P}_1 &= -\left[ \mathcal{P}_1 (A + B \Theta_1) + (A + B \Theta_1)^\top \mathcal{P}_1 + (C + D \Theta_1)^\top \mathcal{P}_1 (C + D \Theta_1) - [Q + \Theta_1^\top S + \Theta_1^\top \mathcal{R} \Theta_1 + S^3 \Theta_1] \right] ds, \\
d\mathcal{P}_2 &= -\left[ \mathcal{P}_2 (A + B \Theta_1) + (A + B \Theta_1)^\top \mathcal{P}_2 + \mathcal{P}_2 \sigma + \mathcal{P}_2 \bar{b} + \left( \mathcal{P}_2 \bar{b} - \bar{S}^\top - \Theta_1^\top \mathcal{R} \right) \varphi \right] ds + \mathcal{L}_2 dW(s), \\
d\mathcal{P}_3 &= -\left[ (A + B \Theta_1)^\top \mathcal{P}_3 + \mathcal{P}_3 \sigma + \mathcal{P}_3 \bar{b} + \left( \mathcal{P}_3 \bar{b} - \bar{S}^\top - \Theta_1^\top \mathcal{R} \right) \varphi \right] ds + \mathcal{L}_3 dW(s), \\
d\mathcal{P}_4 &= -\left[ (A + B \Theta_1)^\top \mathcal{P}_4 + (C + D \Theta_1)^\top \mathcal{L}_4 + (C + D \Theta_1)^\top \mathcal{P}_1 (D \varphi + \sigma) + \mathcal{P}_1 (B \varphi + b) - (S^\top + \Theta_1^\top \mathcal{R}) \varphi \right] ds + \mathcal{L}_4 dW(s), \\
\mathcal{P}_1(T) &= -G, \quad \mathcal{P}_2(T) = -\tilde{G}, \quad \mathcal{P}_3(T) = 0, \quad \mathcal{P}_4(T) = -g.
\end{align*}
\]
and following-up $\mathcal{M}$, $\mathcal{N}$ as follows,

$$
\begin{aligned}
\mathcal{M} := & \mathcal{P}_1X + \mathcal{P}_2E_tX + \mathcal{E}_t\mathcal{P}_3 + \mathcal{P}_4, \\
\mathcal{N} := & \mathcal{P}_1(C + D\Theta_1)X + \mathcal{P}_1(D\varphi + \sigma) + \mathcal{L}_4.
\end{aligned}
$$

(3.12)

**Theorem 3.3** A pair of $(\Theta^*, \varphi^*) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2(0, T; \mathbb{R}^m)$ is a closed-loop equilibrium strategy if and only if there exists $\mathcal{P}_i^*$ satisfies (3.11) with $(\Theta_1, \varphi) \equiv (\Theta^*, \varphi^*)$ such that

$$
\begin{aligned}
R - D^T\mathcal{P}_1^*D &\geq 0, \\
(R - D^T\mathcal{P}_1^*D)\Theta^* &= B^T(\mathcal{P}_1^* + \mathcal{P}_2^*) + D^T\mathcal{P}_1^*C - \mathcal{F}, \\
(R - D^T\mathcal{P}_1^*D)\varphi^* &= B^T(\mathcal{P}_3^* + \mathcal{P}_4^*) + D^T\mathcal{P}_1^*\sigma + D^T\mathcal{L}_4^*.
\end{aligned}
$$

(3.13)

For the closed-loop equilibrium strategy $(\Theta^*, \varphi^*)$, the first inequality in (3.13) is referred as the second-order equilibrium condition, while the other two conditions are named as first-order equilibrium condition.

**Remark 3.6** We make some comparisons among (3.1), (3.6), (3.11), from which we see the connections between open-loop equilibrium controls and their closed-loop representations, as well as that of closed-loop equilibrium controls and closed-loop representations.

- The later two systems reduce to the first one if $\Theta_1 = 0$, or $\Theta_2 = 0$, and $\varphi \equiv u$.
- The solutions of the first two equations in (3.1), (3.11) are symmetric, while the analogue of (3.6) are non-symmetric (see e.g., [26]).
- The first two equations in (3.1) merely depends on given coefficients, while the counterparts in (3.6) and (3.11) are determined by $\Theta_1$, or $\Theta_2$.
- The last two equations in (3.1) rely on control process $u$, while the analogue equations in (3.6) and (3.11) are determined by $\varphi$.

**Remark 3.7** To capture the new feature of time inconsistency, let $\tilde{G} = \tilde{S} = \tilde{Q} = \tilde{R} = 0$, $b = \sigma = g = 0$. Suppose there exists closed-loop representation of open-loop optimal control $u_1^* = \Theta_1^*X_1^*$ and closed-loop optimal control $u_2^* = \Theta_2^*X_2^*$, where $\varphi_1^* = \varphi_2^* = 0$. We claim that $\mathcal{P}_1^* = \mathcal{P}_2^*$. If furthermore $R - D^T\mathcal{P}_1^*D > 0$, a.e., $u_1^* = u_2^*$, and $\mathcal{P}_1^* \equiv \mathcal{P}_2^*$ satisfies the Riccati equations in classical stochastic linear quadratic problems. Actually, in this setting,

$$
\begin{aligned}
\mathcal{P}_2^* &= \mathcal{P}_3^* = \mathcal{L}_3^* = \mathcal{P}_4^* = \mathcal{L}_4^* = 0, \\
\mathcal{P}_2^* &= \mathcal{P}_3^* = \mathcal{L}_3^* = \mathcal{P}_4^* = \mathcal{L}_4^* = 0.
\end{aligned}
$$

and the last two conditions in (3.8), (3.13) become,

$$
\begin{aligned}
[R - D^T\mathcal{P}_1^*D]\Theta_1^* &= B^T\mathcal{P}_1^* + D^T\mathcal{P}_1^*C - S, \\
(R - D^T\mathcal{P}_1^*D)\Theta_2^* &= B^T\mathcal{P}_1^* + D^T\mathcal{P}_1^*C - S.
\end{aligned}
$$

(3.14)

Substituting the second expression into the first equation of (3.11) with $(\Theta_1, \varphi) \equiv (\Theta_2^*, 0)$, we have

$$
d\mathcal{P}_1^* = -\left[\mathcal{P}_1^*A + A^T\mathcal{P}_1^* + C^T\mathcal{P}_1^*C + (\mathcal{P}_1^*B + C^T\mathcal{P}_1^*D - S^T)\Theta_2^* - Q\right]ds.
$$
For $u \in L^2_T(0, T; \mathbb{R}^m)$, $\xi \in L^2_T(\Omega; \mathbb{R}^n)$, by Itô’s formula to $X^\top \mathcal{P}_1 X$,

$$
X(T)^\top G X(T) + \int_t^T [X^\top Q X + 2u^\top S X + u^\top R u] \, dr
= -\xi^\top \mathcal{P}_1^*(t) \xi + \int_t^T \mathcal{L}_2 \, dr + \int_t^T W_2 dW(r),
$$

where

$$
\mathcal{L}_2 := u^\top \hat{R}_2 u + 2u^\top \hat{S}_2 X - X^\top \hat{S}_2^\top \Theta_2 X,
W_2 := -2X^\top [\mathcal{P}_1^* C X + \mathcal{P}_1^* D u],
\hat{S}_2 := S - D^\top \mathcal{P}_1^* C - B^\top \mathcal{P}_1^*, \quad \hat{R}_2 := R - D^\top \mathcal{P}_1^* D.
$$

Thanks to the second equality of (3.14), as well as the symmetry of $R$, $\mathcal{P}_1^*$,

$$
-X^\top \hat{S}_2^\top \Theta_2 X = X^\top [\Theta_2^\top \hat{R}_2 \Theta_2^\top X], \quad u^\top \hat{S}_2 X = -u^\top \hat{R}_2 \Theta_2^\top X.
$$

As a result,

$$
\mathcal{L}_2 = (u - \Theta_2^\top X)^\top \hat{R}_2 (u - \Theta_2^\top X),
$$

and for optimal control $u_2^*$, one has,

$$
V(t, \xi) = J(t, \xi, u_2^*) = -\frac{1}{2} \xi^\top \mathcal{P}_1^*(t) \xi.
$$

Similarly we can deduce that

$$
V(t, \xi) = J(t, \xi, u_1^*) = -\frac{1}{2} \xi^\top \mathcal{P}_1^*(t) \xi.
$$

By the continuity of $\mathcal{P}_1^*$, $\mathcal{P}_1^*$, and the arbitrariness of $\xi$,

$$
P\{\omega \in \Omega; \; \mathcal{P}_1^*(t, \omega) = \mathcal{P}_1^*(t, \omega), \; \forall t \in [0, T]\} = 1.
$$

The equality of $u_1^* = u_2^*$ is easy to obtain.

In general, $\mathcal{P}_1^*$ is different from $\mathcal{P}_1^*$, not to mention the equality of $u_1^* = u_2^*$. For example, when $\tilde{G} \neq 0$, one can see that $\mathcal{P}_1^*$ is symmetric while $\mathcal{P}_1^*$ is not.

To sum up, the closed-loop optimal controls coincide with closed-loop representation of open-loop optimal controls under proper conditions. However, this relation breaks when time-inconsistency happens.

**Remark 3.8** For the second-order equilibrium conditions in Theorem 3.1, Theorem 3.2 and Theorem 3.3, we have the following comments.

- As to open-loop equilibrium controls, no matter it has closed-loop representations or not, we use $\mathcal{R} - D^\top \mathcal{P}_1^* D \geq 0$, where $\mathcal{P}_1^*$ satisfies the second-order adjoint equation in classical stochastic linear quadratic optimal control problems. This condition was missing in [11], [12], [22], [21].

- As to closed-loop equilibrium controls, we introduce $\mathcal{R} - D^\top \mathcal{P}_1^* D \geq 0$ where $\mathcal{P}_1^*$ satisfies one backward ordinary differential equation that contains Riccati equation as special case. Notice that this condition has not been discussed in [2], [3], [24], [26].
Remark 3.9 At this moment, we revisit the open-loop equilibrium controls and closed-loop equilibrium controls when \( G = \bar{S} = \bar{Q} = \bar{R} = g = 0 \).

From Remark 3.1, 3.2, the open-loop equilibrium controls under this framework are fully characterized by first-order, second-order necessary optimality conditions. This gives us a quantitative and clear picture of this kind of equilibrium control. Notice that the characterization of optimal controls includes first-order necessary condition and the following convexity condition (see [6])

\[
\mathbb{E}_t \int_t^T u^T [Ru + SX^0 - B^T Y^0 - D^T Z^0] \, dr \geq 0, \quad \forall u \in L^2_F(t, T; \mathbb{R}^m),
\]  

(3.15)

where \( X^0 \) satisfies (1.1) with \( \xi = 0 \), \( (Y^0, Z^0) \) solves (3.5) with \( G = \bar{S} = \bar{Q} = g = 0 \) and \( X \equiv X^0 \). Consequently, the exact difference between equilibrium controls and optimal controls in the open-loop sense is attributed to that between (3.3) and (3.15).

For closed-loop equilibrium controls/strategies in Theorem 3.3, their characterization (3.13) reduces to

\[
\begin{align*}
R - D^T \mathcal{P}^*_1 D & \geq 0, \\
(R - D^T \mathcal{P}^*_1 D)\Theta^* = B^T \mathcal{P}^*_1 + D^T \mathcal{P}^*_1 C - S, \\
(R - D^T \mathcal{P}^*_1 D)\sigma^* = B^T \mathcal{P}^*_1 + D^T \mathcal{P}^*_1 \sigma + D^T L^*_1.
\end{align*}
\]  

(3.16)

According to [18], [19], (3.16) is equivalent to the optimality of strategy pair \((\Theta^*, \varphi^*)\) or control variable \( u^* := \Theta^* X^* + \varphi^* \). In other words, our defined closed-loop equilibrium controls/strategies are natural extension of closed-loop optimal controls/strategies. This not only leads to one more essential distinction between open-loop, closed-loop equilibrium controls, but not illustrate the reasonability of introduced closed-loop equilibrium controls from the optimality viewpoint.

4 Proofs of the main results

In this section, we prove Theorem 3.1–3.3.

For \((\Theta_1, \Theta_2, \varphi) \in L^2(0, T; \mathbb{R}^{m \times m}) \times L^2(0, T; \mathbb{R}^{m \times m}) \times L^2(0, T; \mathbb{R}^m)\), we consider

\[
\begin{cases}
    dX = [AX + B(\Theta_1 + \Theta_2)X + B\varphi + b] \, ds \\
    \quad + [CX + D(\Theta_1 + \Theta_2)X + D\varphi + \sigma] \, dW(s), \quad s \in [0, T], \\
    X(0) = x_0.
\end{cases}
\]  

(4.1)

In the following, let

\[
u := \Theta_1 X + \varphi, \quad u^* := \Theta_1 X^* + \Theta_2 X + \varphi + v\mathbb{I}_{[t, t+\varepsilon]}.
\]  

(4.2)

Fix \( t \in [0, T) \), \( v \in \mathbb{R}^m \) and small \( \varepsilon > 0 \), let \( X^\varepsilon \) be the solution to the following perturbed system:

\[
\begin{cases}
    dX^\varepsilon = [(A + B\Theta_1)X^\varepsilon + B\Theta_2 X + B\varphi + b] \, ds \\
    \quad + [(C + D\Theta_1)X^\varepsilon + D\Theta_2 X + D\varphi + \sigma] \, dW(s), \\
    X^\varepsilon(0) = x_0,
\end{cases}
\]  

(4.3)
with \( s \in [0, T] \). Hence we see that \( X^\varepsilon_0 := X^\varepsilon - X \) satisfies
\[
\begin{align*}
\frac{dX^\varepsilon_0}{ds} &= [(A + B\Theta_1)X^\varepsilon_0 + Bw_I(t, t+\varepsilon)] ds \\
&+ [(C + D\Theta_1)X^\varepsilon_0 + Dw_I(t, t+\varepsilon)] dW(s), \\
X^\varepsilon_0(0) &= 0.
\end{align*}
\] (4.4)

**Remark 4.1** By Proposition 2.1 in [19], we have the following estimate of \( X^\varepsilon_0 \)
\[
\mathbb{E}_t \sup_{r \in [t, t+\varepsilon]} |X^\varepsilon_0(r)|^2 \leq K\varepsilon, \quad \text{a.s., } t \in [0, T).
\]

To begin with, we have the following difference of cost functional.

**Lemma 4.1** Suppose (H1) holds, \((\Theta_1, \Theta_2, \varphi)\) are given as above, \( u, u^\varepsilon \) are defined in (4.2). Then we have
\[
J(t, x, u^\varepsilon(\cdot)) - J(t, x, u(\cdot)) = J_1(t, x) + J_2(t, x) + \mathbb{E}_t \int^{t+\varepsilon}_t \langle (\mathcal{T}^T + \Theta_1^T \mathcal{R})v, X^\varepsilon_0 \rangle ds,
\] (4.5)
where \( \mathcal{R}, \mathcal{T} \) are defined in (2.3),
\[
\begin{align*}
J_1(t) := \mathbb{E}_t \int^{T}_t \left[ \langle F_1, X^\varepsilon_0 \rangle + \langle F_2, w_I(t, t+\varepsilon) \rangle \right] ds + \mathbb{E}_t \langle G(X(T) + \tilde{G}E_tX(T) + g, X^\varepsilon_0(T) \rangle,
\end{align*}
\]
and
\[
\begin{align*}
J_2(t) := \frac{1}{2} \mathbb{E}_t \int^{T}_t \langle F_1^T, X^\varepsilon_0 \rangle ds + \frac{1}{2} \mathbb{E}_t \langle GX^\varepsilon_0(T) + \tilde{G}E_tX^\varepsilon_0(T), X^\varepsilon_0(T) \rangle,
\end{align*}
\]

**Proof 4.1** By above definitions of \( X, X^\varepsilon \) and \( X^\varepsilon_0 \), we deal with the terms in the cost functional one by one. First let us treat the term associated with \( Q \),
\[
\langle QX^\varepsilon, X^\varepsilon \rangle - \langle QX, X \rangle = 2 \langle QX, X^\varepsilon_0 \rangle + \langle QX^\varepsilon_0, X^\varepsilon_0 \rangle.
\]

Then we look at the one with \( S \). From the definitions of \( u \) and \( u^\varepsilon \), we have
\[
\langle SX^\varepsilon, u^\varepsilon \rangle - \langle SX, u \rangle
= (S^T\Theta_1X^\varepsilon_0, X^\varepsilon_0) + (X^\varepsilon_0, S^T[(\Theta_1 + \Theta_2)X + vI_{[t, t+\varepsilon]} + \varphi])
+ (X^\varepsilon_0, S^T\Theta_1SX) + \langle SX, vI_{[t, t+\varepsilon]} \rangle.
\]

We also have
\[
\langle Ru^\varepsilon, u^\varepsilon \rangle - \langle Ru, u \rangle
= (\Theta_1^T R\Theta_1X^\varepsilon_0, X^\varepsilon_0) + 2 \langle RV_I_{[t, t+\varepsilon]}, \Theta_1X^\varepsilon_0 \rangle + \langle RV, vI_{[t, t+\varepsilon]} \rangle
+ 2 \langle R\Theta_1X^\varepsilon_0, (\Theta_1 + \Theta_2)X + \varphi \rangle + 2 \langle RV_I_{[t, t+\varepsilon]}, (\Theta_1 + \Theta_2)X + \varphi \rangle.
\]
Similarly one can obtain the terms involving $\bar{Q}$, $\bar{S}$, $\bar{R}$ as,

$$
\begin{align*}
&\left\{ \langle \bar{Q}e_t X^\varepsilon, E_t X^\varepsilon \rangle - \langle \bar{Q}e_t X, E_t X \rangle = 2 \langle \bar{Q}e_t X, E_t e_t X_0^\varepsilon \rangle + \langle \bar{Q}e_t X_0^\varepsilon, E_t e_t X_0^\varepsilon \rangle, \\
&\langle \bar{S}e_t X^\varepsilon, E_t u^\varepsilon \rangle - \langle \bar{S}e_t X, E_t u \rangle \\
&= \langle \bar{S}^\top \Theta_1 e_t X_0^\varepsilon, E_t X_0^\varepsilon \rangle + \langle E_t X_0^\varepsilon, \bar{S}^\top [(\Theta_1 + \Theta_2) E_t X + v I_{[t,t+\varepsilon]} + E_t \varphi] \rangle \\
&+ \langle E_t X_0^\varepsilon, \Theta_1^\top \bar{S}e_t X \rangle + \langle \bar{S}e_t X, v I_{[t,t+\varepsilon]} \rangle, \\
&\langle \bar{R}e_t u^\varepsilon, E_t u^\varepsilon \rangle - \langle \bar{R}e_t u, E_t u \rangle \\
&= \langle \Theta_1^\top \bar{R}\Theta_1 e_t X_0^\varepsilon, E_t X_0^\varepsilon \rangle + 2 \langle \bar{R}v I_{[t,t+\varepsilon]}, \Theta_1 e_t X_0^\varepsilon \rangle + \langle \bar{R}v, v I_{[t,t+\varepsilon]} \rangle \\
&+ 2 \langle \bar{R}\Theta_1 e_t X_0^\varepsilon, (\Theta_1 + \Theta_2) E_t X + E_t \varphi \rangle + 2 \langle \bar{R}v I_{[t,t+\varepsilon]}, (\Theta_1 + \Theta_2) E_t X + E_t \varphi \rangle.
\end{align*}
$$

At last we have the follows results on the terms associated with $G$ and $\bar{G}$,

$$
\begin{align*}
&\langle GX^\varepsilon(T), X^\varepsilon(T) \rangle - \langle GX(T), X(T) \rangle \\
&= 2 \langle GX(T), X_0^\varepsilon(T) \rangle + \langle GX_0^\varepsilon(T), X_0^\varepsilon(T) \rangle, \\
&\langle \bar{G}e_t X^\varepsilon(T), e_t X^\varepsilon(T) \rangle - \langle \bar{G}e_t X(T), e_t X(T) \rangle \\
&= 2 \langle \bar{G}e_t X(T), e_t X_0^\varepsilon(T) \rangle + \langle \bar{G}e_t X_0^\varepsilon(T), e_t X_0^\varepsilon(T) \rangle.
\end{align*}
$$

To sum up, we deduce above (4.5).

Next we spread out further study on $J_1(t)$ and $J_2(t)$ by making some equivalent transformations. In fact, from the definitions of equilibrium controls it is unavoidable to take certain convergence arguments. Fortunately, in above we derive the important and useful structure of $E_t \int_t^{t+\varepsilon} \langle F_3(v), v \rangle dr$. Consequently, we will derive similar expressions for other terms in $J_1(t)$, $J_2(t)$. This is the starting point for our later investigations.

### 4.1 A new decoupling result

Inspired by the decoupling tricks in the literature (e.g., [11], [25], etc), we present one conclusion which serves our purpose of this paper. It is interesting in its own right and may be potentially useful for (among others) various problems.

Given $t \in [0, T]$, we consider

$$
\begin{align*}
&dX = \left[ A_1 X + A_2 \right] dr + \left[ B_1 X + B_2 \right] dW(r), \quad r \in [t, T], \\
&dY = - \left[ C_1 Y + C_2 Z + C_3 X + C_4 E_t X + C_5 + E_t C_6 \right] dr + ZdW(r), \tag{4.6}
\end{align*}
$$

(H1) For $H := \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n$, etc, suppose $A_1, B_1, C_i \in L^2(0, T ; H), A_2, C_5 \in L^2(\Omega; L^1(0, T ; H)), B_2 \in L^2_0(0, T ; H), D_1, D_2, D_3, x \in H.$

For $t \in [0, T]$ and $s \in [t, T]$, suppose that

$$
Y(s, t) = P_1(s)X(s) + P_2(s)E_t X(s) + E_t P_3(s) + P_4(s), \tag{4.7}
$$

13
where $P_1$, $P_2$ are deterministic, $P_3$, $P_4$ are stochastic processes satisfying

$$
\begin{cases}
    dP_i(s) = \Pi_i(s)ds, & i = 1, 2, \ P_1(T) = D_1, \ P_2(T) = D_2, \\
    dP_j(s) = \Pi_j(s)ds + L_j(s)dW(s), & j = 3, 4, \ P_3(T) = 0, \ P_4(T) = D_3.
\end{cases}
$$

Here $\Pi_i$ are to be determined. It is easy to see

$$
d\mathbb{E}_t X = [A_1\mathbb{E}_t X + \mathbb{E}_t A_2]dr.
$$

Using Itô’s formula, we derive that

$$
\begin{align*}
   &d[P_1 X] = \left[\Pi_1 X + P_1 (A_1 X + A_2) \right] ds + P_1 (B_1 X + B_2)dW(s), \\
   &d[P_2 \mathbb{E}_t X] = \left\{\Pi_2 \mathbb{E}_t X + P_2 [A_1 \mathbb{E}_t X + \mathbb{E}_t A_2] \right\} ds.
\end{align*}
$$

As a result, we have

$$
\begin{align*}
   dY &= \left\{\Pi_1 + P_1 A_1\right\} X + (\Pi_2 + P_2 A_1) \mathbb{E}_t X \\
   &\quad + \mathbb{E}_t \left[\Pi_4 + P_2 A_2\right] + \Pi_4 + P_1 A_2\right\} ds + \left[P_1 B_1 X + P_1 B_2 + L_4\right] dW(s).
\end{align*}
$$

Consequently, it is necessary to see

$$
Z = P_1 B_1 X + P_1 B_2 + L_4. \tag{4.8}
$$

In this case, from (4.7), (4.8), we see that

$$
\begin{align*}
   \mathbb{E}_t Y &= (P_1 + P_2) \mathbb{E}_t X + \mathbb{E}_t \left[P_3 + P_4\right], \\
   \mathbb{E}_t Z &= P_1 B_1 \mathbb{E}_t X + \mathbb{E}_t \left[P_1 B_2 + L_4\right].
\end{align*}
$$

On the other hand,

$$
\begin{align*}
   &- \left[C_1 Y + C_2 Z + C_3 X + C_4 \mathbb{E}_t X + C_5 + \mathbb{E}_t C_6\right] \\
   &= - C_1 \left\{P_1 X + P_2 \mathbb{E}_t X + \mathbb{E}_t P_3 + P_4\right\} - C_2 \left[P_1 B_1 X + P_1 B_2 + L_4\right] \\
   &\quad - C_3 X - C_4 \mathbb{E}_t X - C_5 - \mathbb{E}_t C_6.
\end{align*}
$$

At this moment, we can choose $\Pi_i(\cdot)$ in the following ways,

$$
\begin{cases}
    0 = \Pi_1 + P_1 A_1 + C_1 P_1 + C_2 P_1 B_1 + C_3, \\
    0 = \Pi_2 + P_2 A_1 + C_1 P_2 + C_4, \\
    0 = \Pi_4 + P_1 A_2 + C_1 P_4 + C_2 [P_1 B_2 + L_4] + C_5, \\
    0 = \Pi_4 + P_2 A_2 + C_1 P_3 + C_6.
\end{cases}
$$

Next we make above arguments rigorous. Given the notations in (2.3), for $s \in [0, T]$, we consider the following systems of equations

$$
\begin{align*}
   dP_1 &= - \left[P_1 A_1 + C_1 P_1 + C_3 P_1 B_1 + C_3\right] ds, \\
   dP_2 &= - \left[P_2 A_1 + C_1 P_2 + C_4\right] ds, \\
   dP_3 &= - \left[C_1 P_3 + P_2 A_2 + C_5\right] ds + L_3 dW(s), \\
   dP_4 &= - \left[C_1 P_4 + C_3 L_4 + C_2 P_1 B_2 + P_1 A_2 + C_5\right] ds + L_4 dW(s), \\
   P_1(T) &= D_1, \ P_2(T) = D_2, \ P_3(T) = 0, \ P_4(T) = D_3.
\end{align*} \tag{4.9}
$$
From Proposition 2.1 in [19], under (H1) we see the following regularities,

\[ P_1, P_2 \in C([0, T]; \mathbb{R}^{n \times n}), \quad (P_3, L_3), (P_4, L_4) \in L_\mathcal{F}^2(\Omega; C([0, T]; \mathbb{R}^n)) \times L_\mathcal{F}^2(0, T; \mathbb{R}^n). \]

At this moment, for \( s \in [0, T] \), and \( t \in [0, s] \), we define a pair of processes

\[
M := P_1 X + P_2 \mathbb{E}_t X + \mathbb{E}_t P_3 + P_4, \quad N := P_1 B_1 X + P_1 B_2 + L_4.
\]

(4.10)

By the results of \( P_1 \), we can conclude that

\[
(M_s, N) \in L_\mathcal{F}^2(\Omega; C([0, T]; \mathbb{R}^n)) \times L_\mathcal{F}^2(0, T; \mathbb{R}^n)
\]

where \( M_s(s) = M(s, s) \) with \( s \in [0, T] \). We present the following result.

**Lemma 4.2** Given \((\Theta, \varphi) \in L^2(0, T; \mathbb{R}^{m \times n}) \times L^2(0, T; \mathbb{R}^m)\), suppose \((X, Y, Z)\) is the unique solution of (4.6) and \((M, N)\) are defined in (4.10). Then for any \( t \in [0, T] \),

\[
\begin{align*}
\mathbb{P}\{\omega \in \Omega; \ Y(s, t) &= M(s, t), \ \forall s \in [t, T]\} = 1, \\
\mathbb{P}\{\omega \in \Omega; \ Z(s, t) &= N(s)\} = 1, \ s \in [t, T]. \quad \text{a.e.}
\end{align*}
\]

**Proof 4.2** Given (4.10), it is easy to see that

\[
\mathbb{E}_t M = (P_1 + P_2) \mathbb{E}_t X + \mathbb{E}_t [P_3 + P_4], \quad \mathbb{E}_t N = P_1 B_1 \mathbb{E}_t X + P_1 \mathbb{E}_t B_2 + \mathbb{E}_t L_4.
\]

Using Itô’s formula, we know that

\[
\begin{align*}

d[P_1 X] &= \left[ - (C_1 P_1 + C_2 P_4 X + C_3) X + P_1 A_2 \right] ds + P_1 (B_1 X + B_2) dW(s), \\

d[P_2 \mathbb{E}_t X] &= \left\{ - \left[ C_1 P_2 + C_4 \right] \mathbb{E}_t X + P_2 \mathbb{E}_t A_2 \right\} ds.
\end{align*}
\]

Consequently, after some calculations one has

\[
dM = - \left[ C_1 M + C_2 N + C_3 X + C_4 \mathbb{E}_t X + C_5 + \mathbb{E}_t C_6 \right] dr + N dW(r).
\]

Considering \( P_1(T) \) in (4.9), we see that for any \( t \in [0, T] \), \((M, N) \in L_\mathcal{F}^2(\Omega; C([t, T]; \mathbb{R}^n)) \times L_\mathcal{F}^2(0, T; \mathbb{R}^n)\) satisfies the backward equation in (4.6). The conclusion is followed by the uniqueness of BSDEs.

### 4.2 A new expression of \( J_1 \)

In this part, we deal with \( J_1(t) \) in Lemma 4.1. For convenience, we rewrite the equation of \( X_0^\varepsilon := X^\varepsilon - X \) as

\[
\begin{align*}
\begin{cases}
    dX_0^\varepsilon = [A_\theta X_0^\varepsilon + B \mathbb{V} I_{[t, t+\varepsilon]}] ds + [C_\theta X_0^\varepsilon + D \mathbb{V} I_{[t, t+\varepsilon]}] dW(s), \\
    X_0^\varepsilon(0) = 0,
\end{cases}
\end{align*}
\]

(4.11)

where \( s \in [0, T] \), and

\[
A_\theta := A + B \Theta_1, \quad C_\theta := C + D \Theta_1.
\]

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We introduce
\[
\begin{cases}
    dY = -\left[ A_0^T Y + C_0^T Z - F_1 \right] dr + Z dW(r), & r \in [t, T], \\
    Y(T, t) = -G X(T) - \tilde{G} \mathbb{E}_t X(T) - g,
\end{cases}
\]
(4.12)
where \( X \) satisfies (4.1), \( F_1 \) is in Lemma 4.1. From Proposition 2.1 in [19], (4.12) is solvable with
\[
(Y, Z) \in L^2_\mathbb{F}(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_\mathbb{F}(t, T; \mathbb{R}^n), \quad t \in [0, T).
\]
By Itô’s formula on \([t, T]\), we have
\[
d\langle Y, X_0^r \rangle = -\langle A_0^T Y + C_0^T Z - F_1, X_0^r \rangle dr + \langle Z, X_0^r \rangle dW(r) + \langle Y, C_0 X_0^r + Dv I_{[t, t+\varepsilon]} \rangle dW(r) + \langle Z, C_0 X_0^r + Dv I_{[t, t+\varepsilon]} \rangle dr.
\]
From (4.12) we then arrive at
\[
\mathbb{E}_t \left( -G X(T) - \tilde{G} \mathbb{E}_t X(T) - g, X_0^r(T) \right) - \mathbb{E}_t \int_t^T \langle F_1, X_0^r \rangle dr \tag{4.13}
\]
(4.13)
= \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^T Y + D^T Z, v \rangle dr.

Inspired by Lemma 4.2, we introduce
\[
\begin{align*}
    dP_1 &= - \left[ P_1(A + B \Theta_1 + B \Theta_2) + (C + D \Theta_1)^T P_1(C + D \Theta_1 + D \Theta_2) \\
    &\quad + (A + B \Theta_1)^T \left[ Q + \Theta_1^T S + \Theta_1^T R(\Theta_1 + \Theta_2) + S^T (\Theta_1 + \Theta_2) \right] \right] ds,
    \\
    dP_2 &= - \left[ P_2(A + B \Theta_1 + B \Theta_2) + (A + B \Theta_1)^T P_2 - \left[ \tilde{Q} + \tilde{S} \tilde{S} \right] \\
    &\quad + \Theta_1^T \tilde{R}(\Theta_1 + \Theta_2) + \tilde{S}^T (\Theta_1 + \Theta_2) \right] ds,
    \\
    dP_3 &= - \left[ (A + B \Theta_1)^T P_3 + P_2(B \phi + b) - \left( \tilde{S}^T + \Theta_1^T \tilde{R} \right) \phi \right] ds + \mathcal{L}_3 dW(s),
    \\
    dP_4 &= - \left[ (A + B \Theta_1)^T P_4 + (C + D \Theta_1)^T \mathcal{L}_4 + (C + D \Theta_1)^T P_1(D \phi + \sigma) \\
    &\quad + \mathbb{P}_1(B \phi + b) - \left( \tilde{S}^T + \Theta_1^T \tilde{R} \right) \phi \right] ds + \mathcal{L}_4 dW(s),
    \\
    \mathbb{P}_1(T) = -G, \quad \mathbb{P}_2(T) = -\tilde{G}, \quad \mathbb{P}_3(T) = 0, \quad \mathbb{P}_4(T) = -g.
\end{align*}
\]
(4.14)
Moreover, the following equalities hold on \([t, T]\),
\[
Y = P_1 X + P_2 \mathbb{E}_t X + \mathbb{E}_t P_3 + P_4, \quad Z = P_1(C + D \Theta_1 + D \Theta_2) X + P_1(D \phi + \sigma) + \mathcal{L}_4.
\]
Consequently,
\[
B^T Y + D^T Z = [B^T P_1 + D^T P_1(C + D \Theta_1 + D \Theta_2)] X + B^T P_2 \mathbb{E}_t X
\]
\[
+ B^T \mathbb{E}_t P_3 + B^T P_4 + D^T P_1(D \phi + \sigma) + D^T \mathcal{L}_4.
\]
(4.14)
This shows that
\[
\begin{align*}
    \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^T Y + D^T Z, v \rangle dr &= \mathbb{E}_t \int_t^{t+\varepsilon} \left( [B^T (P_1 + P_2) + D^T P_1(C + D \Theta_1 + D \Theta_2)] X \\
    &\quad + B^T (P_3 + P_4) + D^T P_1(D \phi + \sigma) + D^T \mathcal{L}_4, v \right) dr.
\end{align*}
\]
By the definition of $J_1(t)$ and above (4.13), we see that

\[
J_1(t) = \mathbb{E}_t \int_t^{t+\varepsilon} \left\langle \left[ \mathcal{J} + \mathcal{R}(\Theta_1 + \Theta_2) - \left[ B^\top (\mathcal{P}_1 + \mathcal{P}_2) + D^\top \mathcal{P}_1 (C + D\Theta_1 + D\Theta_2) \right] \right] X \\
+ \frac{1}{2} \mathcal{R}v + \mathcal{R} \varphi - B^\top (\mathcal{P}_3 + \mathcal{P}_4) - D^\top \mathcal{P}_1 (D\varphi + \sigma) - D^\top \mathcal{L}_4, v \right\rangle dr.
\]

(4.15)

**Lemma 4.3** Suppose (H1) holds, $X$ solves (4.1) associated with $(\Theta_1, \Theta_2, \varphi)$, and $J_1(t)$ is defined in Lemma 4.1. Then (4.15) is true, where $\mathcal{P}_i$ satisfies (4.14).

### 4.3 A new expression of $J_2$

In the following, we turn to treating $J_2$. To this end, we introduce

\[
\begin{align*}
    dY_{\varepsilon}^0 &= -\left[ A_{\varepsilon}^T X_{\varepsilon}^0 + C_{\varepsilon}^T Z_{\varepsilon}^0 - F_1 \right] dr + Z_{\varepsilon}^0 dW(r), \quad r \in [t, T], \\
    Y_{\varepsilon}^0(T, t) &= -GX_{\varepsilon}^0(T) - \tilde{G}E_t X_{\varepsilon}^0(T),
\end{align*}
\]

where $F_1$ is defined in Lemma 4.1. From Proposition 2.1 in [19], we see that

\[(Y_{\varepsilon}^0, Z_{\varepsilon}^0) \in L^2_2(\Omega; C([t, T]; \mathbb{R}^n)) \times L^2_2(t, T; \mathbb{R}^n), \quad t \in [0, T).\]

Recall $X_{\varepsilon}^0$ in (4.11), we obtain the following by Itô's formula,

\[
\begin{align*}
    d \langle Y_{\varepsilon}^0, X_{\varepsilon}^0 \rangle &= -\langle A_{\varepsilon}^T Y_{\varepsilon}^0 + C_{\varepsilon}^T Z_{\varepsilon}^0 - F_1, X_{\varepsilon}^0 \rangle dr + \langle Z_{\varepsilon}^0, X_{\varepsilon}^0 \rangle dW(r) \\
    &+ \langle Y_{\varepsilon}^0, A_{\varepsilon} X_{\varepsilon}^0 + B v I_{[t, t+\varepsilon]} \rangle dr + \langle Y_{\varepsilon}^0, C_{\varepsilon} X_{\varepsilon}^0 + D v I_{[t, t+\varepsilon]} \rangle dW(r) \\
    &+ \langle Z_{\varepsilon}^0, C_{\varepsilon} X_{\varepsilon}^0 + D v I_{[t, t+\varepsilon]} \rangle dr.
\end{align*}
\]

As a result, we then have

\[
\begin{align*}
    \mathbb{E}_t \langle -GX_{\varepsilon}^0(T) - \tilde{G}E_t X_{\varepsilon}^0(T), X_{\varepsilon}^0(T) \rangle - \mathbb{E}_t \int_t^T \langle F_1, X_{\varepsilon}^0 \rangle dr \\
    &= \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^T Y_{\varepsilon}^0 + D^T Z_{\varepsilon}^0, v \rangle dr.
\end{align*}
\]

(4.16)

By the decoupling tricks in Lemma 4.2, we introduce

\[
\begin{align*}
    d\bar{P}_1 &= -\left( \bar{P}_1 (A + B\Theta_1) + (A + B\Theta_1)^\top \bar{P}_1 + (C + D\Theta_1)^\top \bar{P}_1 (C + D\Theta_1) \\
    &\quad - \left[ Q + S^\top \Theta_1 + \Theta_1^\top S + \Theta_1^\top R\Theta_1 \right] \right] ds, \\
    d\bar{P}_2 &= -\left( \bar{P}_2 (A + B\Theta_1) + (A + B\Theta_1)^\top \bar{P}_2 - \left[ \tilde{Q} + \tilde{S}^\top \Theta_1 + \Theta_1^\top \tilde{S} + \Theta_1^\top \tilde{R}\Theta_1 \right] \right] ds, \\
    d\bar{P}_3 &= -\left( \bar{P}_3 (A + B\Theta_1)^\top \bar{P}_3 + \bar{P}_2 B v I_{[t, t+\varepsilon]} \right] ds + \tilde{L}_3 dW(s), \\
    d\bar{P}_4 &= -\left( (A + B\Theta_1)^\top \bar{P}_4 + \left[ (C + D\Theta_1)^\top \bar{P}_4 D + \bar{P}_3 B \right] v I_{[t, t+\varepsilon]} \right) ds + \tilde{L}_4 dW(s), \\
    \bar{P}_1(T) &= -G, \quad \bar{P}_2(T) = -\tilde{G}, \quad \bar{P}_3(T) = 0, \quad \bar{P}_4(T) = 0.
\end{align*}
\]
Moreover, from Lemma 4.2, the following holds on \([t, T]\),
\[
Y_0^\varepsilon = \bar{P}_1X_0^\varepsilon + \bar{P}_2E_tX_0^\varepsilon + E_t\bar{P}_3 + \bar{P}_4, \quad Z_0^\varepsilon = \bar{P}_1(C + D\Theta_1)X_0^\varepsilon + \bar{P}_1DvI_{[t, t+\varepsilon]} + \bar{L}_4.
\]

At this moment, we take a closer look at \((\bar{P}_3, \bar{L}_3), (\bar{P}_4, \bar{L}_4)\). By the uniqueness of BSDEs in Proposition 2.1 of \([19]\), we have the following equalities
\[
\bar{P}_3(s) = \bar{P}_3(s)v, \quad \bar{L}_3(s) = 0, \quad \bar{P}_4(s) = \bar{P}_4(s)v, \quad \bar{L}_4(s) = 0, \quad s \in [t, T],
\]
where
\[
\begin{align*}
\begin{cases}
\quad d\bar{P}_3 = -\left[(A + B\Theta_1)^\top\bar{P}_3 + \bar{P}_2BI_{[t, t+\varepsilon]}\right]ds, & s \in [t, T], \\
\quad d\bar{P}_4 = -\left[(A + B\Theta_1)^\top\bar{P}_4 + \left((C + D\Theta_1)^\top\bar{P}_1D + \bar{P}_1B\right)I_{[t, t+\varepsilon]}\right]ds, & s \in [t, T],
\end{cases}

\bar{P}_3(T) = \bar{P}_4(T) = 0.
\end{align*}
\]

Consequently, on \([t, T]\) we conclude that
\[
B^\top Y_0^\varepsilon + D^\top Z_0^\varepsilon = [B^\top \bar{P}_1 + D^\top \bar{P}_1(C + D\Theta_1)]X_0^\varepsilon + B^\top \bar{P}_2E_tX_0^\varepsilon
\]
\[
+ B^\top E_t\bar{P}_3 + B^\top \bar{P}_4 + D^\top \bar{P}_1DvI_{[t, t+\varepsilon]}.
\]

As a result,
\[
\begin{align*}
\mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top Y_0^\varepsilon + D^\top Z_0^\varepsilon, v \rangle \, dr \\
= \mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top [\bar{P}_1 + \bar{P}_2 + D^\top \bar{P}_1(C + D\Theta_1)]X_0^\varepsilon + B^\top [\bar{P}_3 + \bar{P}_4] + D^\top \bar{P}_1Dv, v \rangle \, dr.
\end{align*}
\]

By the estimate of \(X_0^\varepsilon\), for almost \(t \in [0, T]\),
\[
\mathbb{E}_t \int_t^{t+\varepsilon} \langle B^\top [\bar{P}_1 + \bar{P}_2 + D^\top \bar{P}_1(C + D\Theta_1)]X_0^\varepsilon, v \rangle \, dr = o(\varepsilon).
\]

From the equations of \((\bar{P}_3, \bar{P}_4)\),
\[
\sup_{t \in [t, t+\varepsilon]} \left[[\bar{P}_3(t)]^2 + [\bar{P}_4(t)]^2\right] = o(\varepsilon).
\]

To sum up, by the definition of \(J_2\) and (4.16), for almost \(t \in [0, T]\) we deduce that
\[
J_2(t) = \frac{\varepsilon}{2} \left(D(t)\top \bar{P}_1(t)D(t)v, v \right) + o(\varepsilon). \tag{4.17}
\]

**Lemma 4.4** Suppose (H1) holds, \(X_0^\varepsilon\) is in (4.11) associated with \((\Theta_1, \Theta_2, \varphi)\), and \(J_2(t)\) is defined in Lemma 4.1. Then (4.17) is true.

### 4.4 Proofs of the main results

We are in the position to give the proofs of the main results in Section 3.

To begin with, we give the proof of Theorem 3.1.
In Lemma 4.1, Lemma 4.3, Lemma 4.4, we take \( \Theta_1 \equiv \Theta_2 \equiv 0 \). Hence for the notations in (4.2), \( u \equiv \varphi \) and

\[
\begin{aligned}
J_1(t) &= \mathbb{E}_t \int_t^{t+\varepsilon} \left\langle \left[ \mathcal{J} - \left[ B^\top (P_1 + P_2) + D^\top P_1 C \right] \right] X + \frac{1}{2} \mathcal{R} v \\
&\quad + \mathcal{R} u - B^\top (P_3 + P_4) - D^\top P_1(Du + \sigma) - D^\top L_4, v \right\rangle dr,
\end{aligned}
\]

where \( P_i, i = 1, 2, (P_j, L_j), j = 3, 4 \), satisfies (3.1). Moreover, for any \( t \in [0, T) \), by Remark 4.1,

\[
\mathbb{E}_t \int_t^{t+\varepsilon} \langle \mathcal{J}^\top v, \bar{X}_0 \rangle ds = o(\varepsilon).
\]

We set out to define \( \bar{X} \) the state process associated with \( \bar{u}, w^{\varepsilon} := \bar{u} + v I_{[t, t+\varepsilon]} \), and for any \( t \in [0, T) \)

\[
\begin{aligned}
\mathcal{D}_0(t) &:= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_t^{t+\varepsilon} \left[ \mathcal{R}(s) - D(s)^\top P_1(s) D(s) \right] ds, \\
\mathcal{K}_0(t) &:= \lim_{\varepsilon \to 0} \mathbb{E}_t \int_t^{t+\varepsilon} \left[ \mathcal{J}(s) \bar{X}(s) + \mathcal{R}(s) \bar{u}(s) - B(s)^\top \bar{M}(s, s) - D(s)^\top \bar{N}(s) \right] ds
\end{aligned}
\]

(4.18)

with \( (\bar{M}, \bar{N}) \) in (3.2) corresponding to \( \bar{u} \). To sum up, \( u \equiv \bar{u} = \varphi \) is an equilibrium control associated with \( x_0 \) if and only if for any \( t \in [0, T) \), \( v \in L^2_F(\Omega; \mathbb{R}^m) \),

\[
0 \leq \lim_{\varepsilon \to 0} \frac{J(t, \bar{X}(t); w^{\varepsilon}(\cdot)) - J(t, \bar{X}(t); \bar{u}(\cdot))}{\varepsilon} = \langle \mathcal{D}_0(t), v \rangle + \langle \mathcal{K}_0(t), v \rangle.
\]

Given \( t \in [0, T) \), this holds if and only if both \( \mathcal{K}_0(t) = 0 \) and \( \mathcal{D}_0(t) \geq 0 \). Since both \( \mathcal{R} \) and \( P_1 \) are bounded and deterministic, we thus know that

\[
0 \leq \mathcal{R}(t) - D(t)^\top P_1(t) D(t), \quad t \in [0, T]. \quad \text{a.e.}
\]

If \( \mathcal{K}_0(t) = 0 \), then by Lemma 3.4 in [12], above (3.4) holds. Conversely, if (3.4) is true, we immediately obtain \( \mathcal{K}_0(t) = 0 \).

Next we present the proof of Theorem 3.2.

**Proof 4.4** In Lemma 4.1, Lemma 4.3, Lemma 4.4, we take \( \Theta_1 \equiv 0 \). Hence for the notations in (4.2), we have \( u \equiv \Theta_2 \) \( + \varphi \) and

\[
\begin{aligned}
J_1(t) &= \mathbb{E}_t \int_t^{t+\varepsilon} \left\langle \left[ \mathcal{J} + \mathcal{R} \Theta_2 - \left[ B^\top (P_1 + P_2) + D^\top P_1(C + D\Theta_2) \right] \right] X + \frac{1}{2} \mathcal{R} v \\
&\quad + \mathcal{R} \varphi - B^\top (P_3 + P_4) - D^\top P_1(D\varphi + \sigma) - D^\top L_4, v \right\rangle dr,
\end{aligned}
\]

where \( P_i, i = 1, 2, (P_j, L_j), j = 3, 4 \), satisfies (3.6). Moreover, by Remark 4.1,

\[
\mathbb{E}_t \int_t^{t+\varepsilon} \langle \mathcal{J}^\top v, X_0^\varepsilon \rangle ds = o(\varepsilon), \quad t \in [0, T).
\]
For open-loop equilibrium strategy pair \((\Theta^*, \varphi^*)\) and associated equilibrium control \(u^*\), we define \(X^*\) the corresponding state process as,

\[
\begin{align*}
    dX^* &= [(A + B\Theta^*)X^* + B\varphi^* + b]ds + [(C + D\Theta^*)X^* + D\varphi^* + \sigma]dW(s), \\
    X^*(0) &= x_0,
\end{align*}
\]

and perturbed control \(u^*; \varepsilon := \Theta^*X^* + \varphi^* + \varepsilon I_{[t,t+\varepsilon]}\). Moreover, for \((\mathcal{M}^*, N^*)\) in (3.7) corresponding to \(u^*\), let

\[
\mathcal{H}_1(t) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \left[ \mathcal{K}(s)X^*(s) + \mathcal{K}(s)u^*(s) - B^T\mathcal{M}^*(s, s) - D^T\mathcal{N}^*(s) \right]ds.
\]

To sum up, \(u^* = \Theta^*X^* + \varphi^*\) is an equilibrium control associated with \(x_0 \in \mathbb{R}^n\) if and only if for any \(t \in [0,T]\), \(v \in L_2^2(\Omega; \mathbb{R}^m)\),

\[
0 \leq \langle \mathcal{D}_0(t)v, v \rangle + \langle \mathcal{H}_1(t), v \rangle, \tag{4.19}
\]

where \(\mathcal{D}_0\) is in (4.18). Given \(t \in [0,T]\), this holds if and only if both \(\mathcal{H}_1(t) = 0\) and \(\mathcal{D}_0(t) \geq 0\). Since both \(\mathcal{K}\) and \(P_1\) are bounded and deterministic,

\[
0 \leq \mathcal{H}_1(t) - D(t)^TP_1(t)D(t), \quad t \in [0,T]. \quad \text{a.e.}
\]

\(\Rightarrow\) If \(\mathcal{H}_1(t) = 0\), then by Lemma 3.4 in [12], for almost \(s \in [0,T]\), we have

\[
0 = \mathcal{K}X^* + \mathcal{K}u^* - B^T\mathcal{M}^* - D^T\mathcal{N}^*
\]

\[
= \left[ \mathcal{K} + \mathcal{K}\Theta^* - \left[ B^T(P_1^* + P_2^*) + D^TP_1^*(C + D\Theta^*) \right] \right]X^*
\]

\[
+ \mathcal{K}\varphi^* - B^T(P_3^* + P_4^*) - D^TP_1^*(D\varphi^* + \sigma) - D^TL^*.
\]

Notice that (4.20) holds for any \(x_0 \in \mathbb{R}^n\). We choose \(x_0 = 0\), and denote the state process by \(X_0^*\). As a result,

\[
\left[ (\mathcal{K} - D^T[P_1^* D])\Theta^* - B^T[P_1^* + P_2^*] - D^TP_1^*C + \mathcal{K} \right](X^* - X_0^*) = 0.
\]

At this moment, given \(I \in \mathbb{R}^{n \times n}\) the unit matrix, we consider the following equation

\[
\begin{align*}
    d\mathcal{X} &= (A + B\Theta^*) \mathcal{X} ds + (C + D\Theta^*) \mathcal{X} dW(s), \quad s \in [0,T], \\
    \mathcal{X}(0) &= I,
\end{align*}
\]

the solvability of which is easy to see. Moreover, \(\mathcal{X}^{-1}\) also exists. By the standard theory of SDEs,

\[
\mathbb{P}\{ \omega \in \Omega; \mathcal{X}(t,\omega)x = X^*(t,\omega) - X_0^*(t,\omega), \forall t \in [0,T] \} = 1.
\]

Using the existence of \(\mathcal{X}^{-1}\), it is easy to see above (3.8).

\(\Leftarrow\) In this case, it is easy to see (4.20) with \(u^* := \Theta^*X^* + \varphi^*\). Consequently, the conclusion is followed by (4.19), (3.3) and the fact of \(\mathcal{H}_1(t) = 0\).

At last, we show the proof of Theorem 3.3
Proof 4.5 In Lemma 4.1, Lemma 4.3, Lemma 4.4, we take \( \Theta_2 \equiv 0 \). Hence for the notations in (4.2), \( u \equiv \Theta_1 X + \varphi \) and

\[
\begin{align*}
J_1(t) &= E_t \int_t^{t+\xi} \langle J + \mathcal{R} \Theta_1 - [B^T(\mathcal{P}_1 + \mathcal{P}_2) + D^T \mathcal{P}_1 (C + D \Theta_1)] \rangle X + \frac{1}{2} \mathcal{R} v \\
&\quad + R \varphi - B^T(\mathcal{P}_3 + \mathcal{P}_4) - D^T \mathcal{P}_1 (D \varphi + \sigma) - D^T L_4(v) \rangle dr, \\
J_2(t) &= \frac{\varepsilon}{2} \langle D(t) \mathcal{P}_1(t) D(t)v, v \rangle + o(\varepsilon),
\end{align*}
\]

where \( \mathcal{P}_i, i = 1, 2 \), \( \mathcal{P}_j, j = 3, 4 \), satisfies (3.11). Moreover, in view of Remark 4.1, it is straightforward to get

\[
E_t \int_t^{t+\xi} \langle (J + \Theta_1^\dagger \mathcal{R}) v, X^*_0 \rangle dt = o(\varepsilon), \quad t \in [0, T).
\]

For closed-loop equilibrium strategy pair \( (\Theta^*, \varphi^*) \) in the sense of Definition 2.3 and associated equilibrium control \( u^* := \Theta^* X^* + \varphi^* \), we define \( X^* \) the corresponding state process as,

\[
\begin{align*}
dX^* &= [(A + B \Theta^*)X^* + B \varphi^* + b] ds + [(C + D \Theta^*)X^* + D \varphi^* + \sigma] dW(s), \\
X^*(0) &= x_0,
\end{align*}
\]

and perturbed control variable \( u^{\varepsilon} := \Theta^* X^{\varepsilon} + \varphi^* + \varepsilon I_{[t,t+\xi]} \). In addition, for \( (\mathcal{M}^*, \mathcal{N}^*) \) in (3.12) corresponding to \( u^* \), we denote by

\[
\begin{align*}
\mathcal{H}_2(t) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} E_t \int_t^{t+\xi} [\mathcal{J}(s) X^*(s) + \mathcal{R}(s) u^*(s) - B^T \mathcal{M}^*(s,s) - D^T \mathcal{N}^*(s)] ds, \\
\mathcal{D}_1(t) &= \lim_{\varepsilon \to 0} \frac{1}{2 \varepsilon} \int_t^{t+\xi} [\mathcal{R}(s) - D(s)^T \mathcal{P}_1(s) D(s)] ds.
\end{align*}
\]

To sum up, \( u^* := \Theta^* X^* + \varphi^* \) is a closed-loop equilibrium control associated with \( x_0 \in \mathbb{R}^n \) if and only if for any \( t \in [0, T], v \in L_2^{\mathcal{P}_1}(\Omega; \mathbb{R}^m), \)

\[
0 \leq \langle \mathcal{D}_1(t) v, v \rangle + \langle \mathcal{H}_2(t), v \rangle. \tag{4.22}
\]

Given \( t \in [0, T] \), this holds if and only if both \( \mathcal{H}_2(t) = 0 \) and \( \mathcal{D}_1(t) \geq 0 \).

\[\implies\] Given equilibrium strategy pair \( (\Theta^*, \varphi^*) \), we conclude that \( \mathcal{P}_1^* \) is bounded and deterministic. Recall the requirement on \( \mathcal{R} \), it is clear that

\[
0 \leq \mathcal{R}(t) - D(t)^T \mathcal{P}_1^*(t) D(t), \quad t \in [0, T]. \tag{4.23}
\]

If \( \mathcal{H}_2(t) = 0 \), then by Lemma 3.4 in [12], for almost \( s \in [0, T] \), we have

\[
0 = \mathcal{J} X^* + \mathcal{R} u^* - B^T \mathcal{M}^* - D^T \mathcal{N}^*
\]

\[
= [\mathcal{J} + \mathcal{R} \Theta^* - [B^T(\mathcal{P}_1 + \mathcal{P}_2) + D^T \mathcal{P}_1 (C + D \Theta^*)]] X^* + \mathcal{R} \varphi^* - B^T(\mathcal{P}_3 + \mathcal{P}_4) - D^T \mathcal{P}_1 (D \varphi^* + \sigma) - D^T L_4. \tag{4.24}
\]

Notice that (4.24) holds for any \( x_0 \in \mathbb{R}^n \). We choose \( x_0 = 0 \), and denote the state process by \( X^*_0 \). As a result,

\[
\left[ [\mathcal{R} - D^T \mathcal{P}_1 D] \Theta^* - B^T [\mathcal{P}_1 + \mathcal{P}_2] - D^T \mathcal{P}_1 C + \mathcal{J} \right] (X^* - X^*_0) = 0.
\]
As in Theorem 3.2, we introduce $\mathcal{X}$ satisfying (4.21), and therefore obtain (3.13) by following the same spirit of that in Theorem 3.2.

$\iff$ In this case, it is easy to see (4.20) with $u^* := \Theta^* X^* + \varphi^*$. Consequently, the conclusion is followed by (4.22), (4.23) and the fact of $\mathcal{H}_1(t) = 0$.

5 Concluding remarks

In the Markovian setting, a unified approach by variational idea is developed to build the characterizations for three notions, i.e., closed-loop equilibrium controls/strategies, open-loop equilibrium controls, as well as the closed-loop representations of open-loop equilibrium controls. The intrinsic differences among different equilibrium controls are also revealed clearly and deeply. Related studies with random coefficients or in mean-field setting are under consideration. We hope to do some relevant research in future.

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