A CLASS OF PHYSICALLY MOTIVATED CLOSURES FOR RADIATION HYDRODYNAMICS

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ABSTRACT

Radiative transfer and radiation hydrodynamics use the relativistic Boltzmann equation to describe the kinetics of photons. The radiative intensity, which is proportional to the photon distribution function, is a seven-dimensional hyper-surface embedded in eight-dimensional phase space (Misner et al. 1973). Unless the radiative intensity is in equilibrium, or the problem is highly symmetric, it is difficult to solve the radiative transfer equation either analytically or numerically (see standard textbook such as Chandrasekhar 1960; Mihalas & Mihalas 1984; Rybicki & Lightman 1986; Shu 1991; Peraiah 2001; Castor 2004; Graziani 2006). On the one hand, there is no successful theory to reduce the complexity of radiative transfer. On the other hand, although numerical algorithms such as direct discretization of the radiative transfer equation and Monte Carlo methods exist, they are computationally too expensive.

In astrophysics, very often we are interested in the radiative energy and flux instead of the intensity. If the radiation field is smooth (in terms of directions), we simply take angular moments of the radiative transfer equation and solve only for the frequency-dependent moment equations. The zeroth-, first-, and second-order angular moments of the intensity carry clear physical meanings. They are the radiative energy density, radiative flux, and radiative stress tensor, respectively. For many classic problems in astrophysics like stellar atmospheres, the global symmetry of the system is used to further reduce the degrees of freedom. Analytical or numerical solutions are then obtained by solving the reduced frequency-dependent moment equations. This approach is naturally extended to radiation hydrodynamics, which not only describes how the (moving) media radiates, but also how radiation feeds back to matter (Mihalas & Mihalas 1984; Mihalas 2001; Castor 2004; Krumholz et al. 2007).

The dynamic equations of the radiative stress tensor contain terms that are related to the third-order moment of the intensity, while the equations of the third-order moments depend on the fourth-order moments and so on. In order to close the system, we need to make approximations and truncate the moment hierarchy. This is known as the closure problem in radiative transfer. Popular closure schemes in astrophysics include variant forms of flux-limited diffusion (FLD; Chang & Cooper 1970; Levermore & Pomraning 1981; Pomraning 1983; Levermore 1984), $P_n$ approximations (sometimes with diffusion corrections; see Olson et al. 2000; Seibold & Frank 2009; McClaren et al. 2008; Ravishankar et al. 2010, and reference therein), the $M_1$ closure (equivalent to maximal entropy closure; see Janka et al. 1992; Smit et al. 2000; Farris et al. 2008), and variable Eddington factors (VEF; Feautrier 1964; Pomraning 1969; Auer & Mihalas 1970; Fukue 2008a, 2008b, 2009). They have been successfully applied to many astrophysics problems, yet their underlying assumptions (what is the optimal form of a flux limiter for a particular problem? why is the entropy of photons maximized? how many moments do we need?) are rather ad hoc and are not guaranteed to capture the correct physics.

In hydrodynamics, the Chapman–Enskog theory expands the particle distribution in power series of the Knudsen number, which is the ratio between the particle mean free path and the typical length scale in the problem. The zeroth-order expansion gives rise to ideal hydrodynamics, while the first-order terms allow us to calculate transport coefficients such as kinematic viscosity and thermal conductivity (Huang 1965; Chapman & Cowling 1970; Liboff 1979; Cercignani & Kremer 2002). We would like to apply a similar technique to radiative transfer in order to solve the closure problem. Unfortunately, photons do not self-interact. The photon mean free path is determined by the extinction coefficient, which is a material property. In the cases when the photon density is high but the media is optically thin, radiation dominates the energy budget but cannot be thermalized. The Chapman–Enskog series for photon distribution function fails to converge (Mihalas 2001).

In addition to the closure problem, as studies of astrophysical objects become more detailed, complicated sub-structures such as convection in stars and turbulence in accretion disks are always found. The radiating medium moves non-uniformly,
which makes the computation of spectra more complicated due to the spatial-dependent Doppler effect. The standard approach is to Taylor expand the extinction and emission coefficients and to derive moment equations to at least \( \mathcal{O}(v/c) \), where \( v \) and \( c \) are the speed of the media and the speed of light, respectively. The resulting equations have terms that are physically important even in the limit \( v/c \rightarrow 0 \) (Mihalas & Mihalas 1984). Keeping track of these higher order terms in different physical regimes is a challenging task (Krumholz et al. 2007). Moreover, there are astrophysical systems such as the inner regions of accretion disks and ultra-relativistic jets that move with relativistic speed so the described approach converges too slowly.

If radiative feedback is weak, we can solve the hydrodynamic equations and compute the radiative spectra using post-processing (see Noble et al. 2007; Chan et al. 2009). However, there is a large set of problems for which the feedback is important. We have to solve the frequency-dependent three-dimensional moment equations at every single time step. The full system of radiation hydrodynamics is highly nonlinear. Studies of turbulence show that reducing the number of spatial dimensions and enforcing symmetry in these systems can produce fundamentally wrong results (Kraichnan 1967, 1971).

To summarize, we have listed three major difficulties in using moment methods for radiative transfer and radiation hydrodynamics.

1. Because of the collisionless nature of photons, standard methods in kinetic theory such as the Chapman–Enskog theory fail to provide a physical closure.
2. When the non-uniform motion of media is considered, it is difficult to properly take the spatial-dependent Doppler effect into account.
3. If the radiative feedback is important, the naive approach is computationally too expensive to couple radiation back to hydrodynamics.

Hydrodynamics depends only on the frequency-integrated radiative force. Many existing codes, therefore, reduce the degrees of freedom and resolve problem (3) by using frequency-integrated equations (Stone et al. 1992; Turner & Stone 2001; Hayes & Norman 2003; González et al. 2007; Krumholz et al. 2007; Aubert & Teyssier 2008; Farris et al. 2008). If the radiative spectrum is needed, it is always possible to post-process the numerical solution.

Once we integrate over frequency, the moment equations are fully relativistic. The radiative energy density, momentum, and stress tensors together form a covariant stress-energy tensor. We can then ignore the complication of mixed frame formalism. This relativistic approach is originally developed to solve radiative transfer in general relativity (Anderson & Spiegel 1972; Schmid-Burgk 1978; Thorne 1981; Udey & Israel 1982). It is well adopted in numerical general relativistic hydrodynamics (Aguirre et al. 2005; Park 2006; Takahashi 2007; Farris et al. 2008). However, we believe that its true advantage is in the covariant equations, which trivially addresses problem (2).

Grad (1949) proposed a moment method that expands the ratio between the particle distribution and the equilibrium distribution in multi-dimensional polynomials of the momentum. Instead, it treats the moments as fundamental fields and keeps track of their evolutions. This method is independent of the mean free path. Once the reference frame and temperature are chosen, Grad’s coefficients are simply linear transforms of the momentum moments, which converge exponentially fast when the distribution function is well behaved.\(^3\) Because photon transport is linear, Grad’s moments method provides the most natural way to resolve problem (1).

The idea of applying Grad’s moment method to study radiation is not new. A classic paper by Thorne (1981) presented very detailed moment formalisms for relativistic radiative transfer by using projected, symmetric, trace-free tensors (Thorne 1980). Shortly after that, Udey & Israel (1982) derived a 14-field approximation for general relativistic radiative transfer based on Grad’s method. However, as far as we know, these attempts were only used in one-dimensional problems (e.g., Zampieri et al. 1996). On one hand, this is possibly due to the general concept that Newtonian formalisms are always simpler than relativistic formalisms. On the other hand, Thorne and Udey & Israel’s 14-field methods are indeed complicated compared with the standard \( \mathcal{O}(v/c) \) equations in Mihalas & Mihalas (1984). Their advantages only appear when we consider more physical regimes such as the ones described in Krumholz et al. (2007).

Based on the above points and some additional physical understanding of moment methods, we propose a new class of closure schemes for radiative transfer and radiation hydrodynamics in this paper. The paper is organized as the following. In the next section, we introduce our notations and review the standard radiation hydrodynamics equations. In Section 3, we describe Grad’s moment method and its ultra-relativistic generalization. We also summarize the linear transformations that we derive in the appendices. In Section 4, we study carefully ideal hydrodynamics and some existing closures for radiative transfer. They provide us important physical insight, which leads to a new class of closure schemes. Although the scheme is generic, we specifically look at the 14-field approximations and derive the closure equations in Section 5. Finally, we conclude this paper in Section 6.

2. NOTATIONS AND STANDARD EQUATIONS

We use the component notation in Misner et al. (1973) throughout this paper. Greek indices run from 0 to 3 and Latin indices run only from 1 to 3. The metric tensor is denoted by \( g^{\alpha \beta} \) and the metric signature is \((−, +, +, +)\). A point in space time is denoted by \( x^\alpha = (c t, x^i) \) with \( c \) being the speed of light. The Einstein summation convention is used unless specified otherwise.

We start from the standard radiative transfer equation. Let \( \nu \) be the photon frequency, \( n^i \) be a three-dimensional unit vector. We define \( I = I(t, x^i; \nu, n^i) \) be the specific radiative intensity, \( \eta_\nu \) be the total emission coefficient, and \( \chi_\nu \) be the total extinction coefficient. The intensity is then governed by the radiative transfer equation

\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + n^i \frac{\partial}{\partial x^i} \right) I_\nu = \eta_\nu - \chi_\nu I_\nu. \tag{1}
\]

We first describe the standard moment methods (Chandrasekhar 1960; Mihalas 1978; Mihalas & Mihalas 1984; Rybicki & Lightman 1986; Shu 1991; Periaiah 2001; Castor 2001), on convergent properties of series expansions.

\(^2\) Another fundamental difficulty of radiation hydrodynamics comes from the large separation between the radiative timescale and the (hydro-)dynamic timescale. Nevertheless, this difficulty arises not because of the moment method. It is a generic property of non-relativistic astrophysical systems. We will therefore leave it out from this paper.

\(^3\) It is a simple application of Darboux’s principle. See, for example, Boyd (2001), on convergent properties of series expansions.
Taking the zeroth and first angular momentum of the whole transfer Equation (1), we obtain two frequency-dependent moment equations,

$$\partial_t E_v + \partial_i F^i_v = -cG^0_v$$

(2)

$$\partial_t F^i_v + c^2 \partial_j P^{ij}_v = -c^2 G^i_v.$$  

(3)

The quantities on the left are the frequency-dependent radiative energy density, radiative flux, and radiative stress tensor,

$$E_v \equiv \frac{1}{c} \int d\Omega I_v,$$

(4)

$$F^i_v \equiv \int d\Omega I_v n^i,$$

(5)

$$P^{ij}_v \equiv \frac{1}{c} \int d\Omega I_v n^i n^j.$$  

(6)

The ones on the right are the radiative energy and moment inputs (to the radiating medium),

$$G^0_v \equiv \frac{1}{c} \int d\Omega (\chi_v I_v - \eta_v),$$

(7)

$$G^i_v \equiv \frac{1}{c} \int d\Omega (\chi_v I_v - \eta_v) n^i.$$  

(8)

Note that the four-tuple ($E_v$, $F^i_v$) is not a four-vector, so Equations (2) and (3) do not form a covariant equation. The quantities $E_v$, $F^i_v$, and $F^{ij}_v$, as well as $G^0_v$ and $G^i_v$, all depend on the reference frame. This is exactly the difficulty in solving angular moment equations when the radiating medium moves non-uniformly—we cannot Lorentz transform ($E_v$, $F^i_v$) to obtain their values in a different frame.

We follow Mihalas & Mihalas (1984) to derive the covariant formulae as we suggested in the introduction. Let $h$ be Planck’s constant and $n^a\equiv (1, n^i)$ be the “unit” null vector. The photon four-momentum is $p^a\equiv (h\nu/c)n^a$. The Lorentz invariant photon distribution function $f$ is related to the intensity by the equation

$$f(x^a, p^\beta) \equiv f(t, x; v, n^i) = \frac{c^2}{h^4 v^3} I(t, x; v, n^i).$$

We can now rewrite Equation (1) as a ultra-relativistic Boltzmann transport equation,

$$p^a f_{,a} = \epsilon - xf,$$

(10)

where the subscript $\alpha$ denotes covariant derivative. The Lorentz invariant emission coefficient

$$e(v) \equiv \frac{c}{h^4 v^3} \eta_v$$

(11)

and extinction coefficient

$$\chi(v) \equiv \frac{h v}{c} \chi_v$$

(12)

are usually assumed independent of $f$.

Using the covariant volume element, $d^3 p/p^0$, we can take four-momentum moments of Equation (10) to arbitrary order,

$$R^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} = -G^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}.$$  

(13)

The moment tensor on the left-hand side is defined by

$$R^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} \equiv c \int \frac{d^3 p}{p^0} f p^{\alpha_1} p^{\alpha_2} \ldots p^{\alpha_m},$$

(14)

while on the right-hand side, we have

$$G^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n} \equiv c \int \frac{d^3 p}{p^0} (x f - e) p^{\alpha_1} p^{\alpha_2} \ldots p^{\alpha_m}.$$  

(15)

We refer $G^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$ as the moment extinction term because $-G^{\alpha_1\alpha_2\ldots\alpha_m}_{\beta_1\beta_2\ldots\beta_n}$ is usually called the moment production term (Cercignani & Kremer 2002).

It is easy to connect the relativistic formulas with the non-relativistic ones. Using the first-order equation, the radiative stress-energy four-tensor is

$$R^{\alpha_1}_{\beta_1} \equiv \frac{1}{c} \int dv d\Omega I_v n^a n^\beta = \left[ \frac{E}{F^i/c} \frac{F^i}{P^j/c} \right].$$

The frame-dependent components are the frequency-integrated radiative energy density, radiative flux, and radiative stress tensor. Similarly, the radiative four-force is

$$G^\alpha \equiv \frac{1}{c} \int dv d\Omega (\chi_v I_v - \eta_v) n^\alpha = \left[ \frac{G^0}{G^i} \right].$$  

(17)

The temporal and spatial components are the frequency-integrated radiative energy and momentum inputs. We remark that higher angular moments of the intensity, even though they are frequency-integrated, do not have counterparts in Equation (13).

For completeness, we also write down the equations for relativistic hydrodynamics. We use $v_i$ to denote the fluid velocity. The four-velocity is $u^a = \gamma(c, v^i)$, where $\gamma$ is the Lorentz factor. The material four-momentum is given by

$$T^a = \rho u^a,$$

(18)

where $\rho$ is material density in the local Lorentz rest frame. For simplicity, we assume perfect fluid so that the material stress-energy tensor takes the simple form

$$T^{\alpha\beta} = \rho u^a u^b + p g^{ab}. $$

(19)

Here, $h = e/c^2 + p/\rho c^2$ is the specific enthalpy, $e$ is the specific internal energy density including the rest energy, and $p$ is the thermal pressure.

After all the definitions, standard radiation hydrodynamics can be summarized in three tensor equations, namely, the continuity equation,

$$T^{\alpha}_{;\alpha} = 0,$$

(20)

the covariant Euler equation,

$$T^{\alpha\beta}_{;\beta} = G^\alpha,$$

(21)

and the radiative stress-energy equation

$$R^{\alpha\beta}_{;\beta} = -G^\alpha.$$  

(22)

The above equations are Lorentz covariant. Nevertheless, we refer to them as “lab frame equations” in order to distinguish them from the radiation fiducial frame and the fluid comoving frame.
3. GRAD’S MOMENT METHOD

Because of the perfect fluid assumption, the material part of standard radiation hydrodynamics is closed by Equation (19) and an equation of state. We will refer to this as the ideal (relativistic) hydrodynamic closure. Conversely, there are nine unknowns in the trace-free symmetric radiative stress-energy tensor but only four independent equations. The transport part of radiation is subject to the closure problem. In addition, it is unclear how to relate the radiative four-force to other macroscopic quantities. Grad (1949) proposed evolving the moments as fundamental fields and use them to reconstruct the particle distribution function. The unknown components of the moments and the moment extinction terms can then be evaluated self-consistently. In this section, we will generalize Grad’s moment method to work with radiation.

3.1. Grad’s Expansion

We generalize Grad’s moment method in a covariant form and expand \( f \) as the following multi-dimensional power series

\[
f = f^{(0)}(\hat{a} + \hat{a}_{\alpha} p^\alpha + \hat{a}_{\alpha\beta} p^\alpha p^\beta + \cdots)
\]

at each point \( \lambda^\alpha \). In the above expansion, we choose

\[
f^{(0)} \equiv \frac{2}{h^3} \exp(-p^\alpha U_\alpha/\theta) - 1 \]

be the equilibrium (black body) photon distribution function, where \( U^\alpha \) is a four-velocity of some fiducial observer and \( \theta \) is the energy scale for the equilibrium distribution. The coefficients \( \hat{a}, \hat{a}_\alpha, \hat{a}_{\alpha\beta}, \ldots \) are totally symmetric with trace-free spatial parts (see Appendix A or Thorne 1980).

In the fiducial reference frame, \( -p^\alpha U_\alpha \) reduces to the photon energy \( h \nu \). Using the energy scale \( \theta \), we define a dimensionless quantity

\[
\xi \equiv h \nu/\theta.
\]

The equilibrium distribution reduces to \( f^{(0)} = (2/h^3) w(\xi) \), where

\[
w(\xi) \equiv \frac{1}{\exp(\xi) - 1}
\]

can be interpreted as a dimensionless weight. We can rewrite Grad’s expansion as

\[
f = \frac{2}{h^3} w(\xi)(a + \xi n^\alpha a^\alpha + \xi^2 a_{\alpha\beta} n^\alpha n^\beta + \cdots).
\]

The rescaled coefficients \( a, a^0, a_{\alpha\beta}, \ldots \) are dimensionless.

Note that the weight \( w(\xi) \) depends explicitly on direction after a Lorentz transformation—which is simply the Doppler effect. The coefficients in expansions (23) and (27) are specific for the (at the moment, arbitrarily) chosen energy scale and fiducial frame. Nevertheless, the infinite series contains full information of the original Lorentz invariant distribution \( f \). We can Taylor expand the anisotropic part of the weight and obtain another set of Grad’s coefficients in the other frame. The distribution is fiducial frame dependent only after we truncate the expansion at a certain order.

3.2. Solving for Grad’s Coefficients

In principle, we could derive evolution equations for Grad’s coefficients by substituting Grad’s expansion (27) into the ultra-relativistic Boltzmann equation (10) but we would then lose the physical intuition. Instead, we follow Grad (1949) and treat the moments as fundamental fields. The left-hand sides of the moment equations (13) suggest that we need to keep track of \( R^{(l+1)}_{\alpha\beta\cdots\lambda} \) (components that have at least one 0 in the indices). The flux terms \( R^{(l+1)}_{\alpha\beta\cdots\lambda} \) (components with no 0 in the indices) and the extinction terms \( G^{(l+1)}_{\alpha\beta\cdots\lambda} \) must be solved in terms of them.

Taking the moment is a linear operation. We can skip the distribution function \( f \) and directly write down a linear transformation between the coefficients and the moments. Let the Euler script

\[
\tilde{R}^{(l+1)}_{\alpha\beta\cdots\lambda} \equiv \frac{\lambda^3}{c} \left( \frac{\theta}{c} \right)^{l-1} R^{(l+1)}_{\alpha\beta\cdots\lambda}
\]

be the dimensionless moments, where \( \lambda \equiv hc/\theta \) is the photon mean separation. Using Appendix B, the relation between the dynamic variables and Grad’s coefficients can be summarized in the following hierarchy of matrix equations,

\[
\begin{align*}
\begin{bmatrix}
\tilde{R}^{(0)}_0 \\
\tilde{R}^{(0)0} \\
\tilde{R}^{(1)}_0 \\
\tilde{R}^{(2)}_0 \\
\vdots \\
\tilde{R}^{(l+1)}_0 \\
\end{bmatrix} &= \frac{8\pi}{3} \begin{bmatrix}
W_2 & W_3 & W_4 & \cdots & a \\
W_3 & W_4 & W_5 & \cdots & a_0 \\
W_4 & W_5 & W_6 & \cdots & a_1 \\
W_5 & W_6 & W_7 & \cdots & a_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
W_l & W_{l+1} & W_{l+2} & \cdots & a_l \\
\end{bmatrix}, \\
\frac{1}{3} \begin{bmatrix}
\tilde{R}^{(0)00} \\
\tilde{R}^{(1)0} \\
\tilde{R}^{(2)0} \\
\vdots \\
\tilde{R}^{(l+1)0} \\
\end{bmatrix} &= \frac{16\pi}{15} \begin{bmatrix}
W_6 & \cdots & \cdots & \cdots & \cdots \\
W_7 & \cdots & \cdots & \cdots & \cdots \\
W_8 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
W_{l+1} & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}.
\end{align*}
\]

where \( W_i \) is the shorthand of the integral \( W_i \equiv \int_0^\infty d\xi \, w(\xi) \xi^i \). Substituting our weighting function (26), the integral has the closed from solution \( W_i \equiv \Gamma(l + 1) \xi^{l+1} \). Some numerical values are listed in Equation (B4).

Of course, there are infinitely many equations in the hierarchy. We just present the ones that are useful for this paper. It is easy to solve the coefficients by inverting the above transforms. We will provide some examples in Sections 4 and 5.

3.3. Computing the Flux Terms

Once we obtain Grad’s coefficients, we can use them to compute the flux terms. Grad’s moment method is linear in the fiducial frame. The linearity naturally form a class of closure schemes. Since we fix the weight \( w(\xi) \), the only freedoms in the closures are the energy scale \( \theta \) and the fiducial reference frame corresponds to \( U^\alpha \).

We will discuss how to choose the fiducial velocity \( U^\alpha \) in Section 4. For now, we use the subscript \( t \) to indicate the radiation fiducial frame,

\[
R^{(l+1)}_{\alpha\beta\cdots\lambda} = N^{(l+1)}_{\alpha\beta\cdots\lambda} R^{(l+1)}_{\alpha\beta\cdots\lambda}.
\]

where \( N^{(l+1)}_{\alpha\beta\cdots\lambda} \) is the Lorentz transformation matrix. We use the result derived in Appendix B to compute the flux terms,

\[
\begin{align*}
\tilde{R}^{(1)} &= 8\pi \frac{1}{3} \begin{bmatrix} W_5 a_{ij} + 2W_4 a_{0ij} + 3W_3 a_{00ij} + \cdots \end{bmatrix}, \\
\tilde{R}^{(2)} &= 16\pi \frac{1}{15} \begin{bmatrix} W_5 a_{ij} + 2W_4 a_{0ij} + 3W_3 a_{00ij} + \cdots \end{bmatrix}.
\end{align*}
\]
\[ R_t^{ijk} = \frac{16\pi}{35} \left( W_2 \epsilon_{tijk} + \ldots \right) + \frac{1}{3} \left( \delta_t^{0k} \delta_t^{ij} + R_t^{00} \delta_t^{ij} + R_t^{00} \delta_t^{ij} \right). \]

To obtain the flux terms in the lab frame, we simply apply an inverse Lorentz transformation to \( R_t^{a_1 a_2 \cdots a_l} \).

Note that the dimensionless photon number density \( N_t^\alpha \) does not enter the closure equations, we can use it to choose the energy scale. We require \( \theta \) to approach \( k_B T \) in local thermal dynamic equilibrium, i.e., \( \theta = \frac{h^3 c^2 R_t^0}{8\pi W_2} \), which leads to

\[ \theta = \left( \frac{h^3 c^2 R_t^0}{8\pi W_2} \right)^{1/3}. \]  

### 3.4. Computing the Extinction Terms

Photon emission and absorption are fluid properties. They take their simplest form in the fluid comoving frame. Therefore, we use the fluid temperature \( k_B T \) and the fluid four-velocity \( u^\alpha \) to replace \( \theta \) and \( U^\alpha \) in Grad’s expansion (see Appendix C).

To compute the extinction terms, we apply the Lorentz transformation to boost the moments to the fluid comoving frame,

\[ G_{t}^{\alpha_1 \alpha_2 \cdots \alpha_l} = N_{\tau \rho_1 \cdots \rho_l} b_{\tau} p_{\rho_1} \cdots p_{\rho_l} G_{t}^{\beta_1 \beta_2 \cdots \beta_l}. \]

We follow the procedure described in Section 3.2 to obtain the comoving coefficients \( a_{t}, a_{t+</alpha}, a_{t<\alpha}, \ldots \). The subscript \( t \) here indicates the fluid comoving frame. Introducing the notation

\[ b_{Ia_1 a_2 \cdots a_l} = a_{Ia_1 a_2 \cdots a_l} - \delta_t^{I0} \]

and letting the Euler script

\[ G_{t}^{Ia_1 a_2 \cdots a_l} = \lambda^4 \left( \frac{c}{k_B T} \right)^4 G_{t}^{\alpha_1 \alpha_2 \cdots \alpha_l} \]

be the dimensionless comoving extinction terms, the derivation in Appendix C gives us the following linear transforms:

\[
\begin{bmatrix}
G_{t}^{I} \\
G_{t}^{I0} \\
G_{t}^{I00}
\end{bmatrix} = 8\pi \begin{bmatrix}
X_2 & X_3 & X_4 & \cdots & \cdots & b_I \\
X_4 & X_3 & X_4 & \cdots & \cdots & b_{I0} \\
X_4 & X_5 & X_6 & \cdots & \cdots & b_{I00}
\end{bmatrix}, 
\]

(40)

\[
\begin{bmatrix}
G_{t}^{Ij} \\
G_{t}^{I0j} \\
G_{t}^{I00j}
\end{bmatrix} = \frac{8\pi}{3} \begin{bmatrix}
X_4 & 2X_5 & \cdots & \cdots \\
X_5 & 2X_6 & \cdots & \cdots \\
X_4 & \cdots & \cdots & \cdots
\end{bmatrix}, 
\]

(41)

\[
\begin{bmatrix}
G_{t}^{Ij} \\
G_{t}^{I0j} \\
G_{t}^{I00j}
\end{bmatrix} - \frac{1}{3} \begin{bmatrix}
G_{t}^{I00} \\
G_{t}^{I00} \\
G_{t}^{I00}
\end{bmatrix} = 16\pi \begin{bmatrix}
X_5 & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots
\end{bmatrix} 
\]

(42)

There is no freedom left in the above equations because \( T \) and \( u^\alpha \) are both fixed as fluid properties. We can apply inverse Lorentz transform to boost the extinction terms back to the lab frame.

### 4. PHYSICAL IMPLICATIONS IN CLOSURE SCHEMES

It seems that we have all the equations to close the radiative moment hierarchy. Unfortunately, Grad’s moment method has some arbitrariness in the choices of the fiducial frame velocity \( U^\alpha \) as we remarked in Section 3.3. In order to get some physical insights to constrain this arbitrariness, we need to understand closure schemes at a more fundamental level.

#### 4.1. The Origin of Non-linearity

Let us think deeper about the very successful ideal hydrodynamic closure. It only requires the left hand side of the Boltzmann transport equation, which is linear, to derive the Euler equation. Taking moments is also a linear operation. So why is the resulting equation nonlinear?

Without loss of generality, we use non-relativistic hydrodynamics for this discussion. Let \( m \) be the particle mass,

\[ v^i \equiv \frac{1}{m} \int d^3 p f p^i \]

be the fluid velocity, and \( p^i \equiv p^i - m v^i \) be the particle momentum in the comoving frame. The particle distribution function \( f \) is always well approximated by a Maxwellian in the fluid comoving frame because of collisions. We can choose a linear closure in the comoving frame, such as \( p = (2/3) p \), yet the nonlinear inertial term will always appear in the lab frame,

\[ \frac{1}{m} \int d^3 p f p^i p^j = m v^i v^j \int d^3 p f + \frac{1}{m} \int d^3 p f p^i p^j \]

(43)

\[ = \rho v^i v^j + p g^{ij}. \]

(44)

Therefore, nonlinearity does not just come from the closure approximation. It is a direct consequence of Galilean transforming the second (or higher order) moment with a velocity that depends on the distribution. In other words, hydrodynamics is nonlinear because of Galilean symmetry and fluid particle self-interaction.

For radiation, we replace Galilean symmetry by Lorentz symmetry in the above reasoning. Photons do not self-interact so they can only be thermalized by external medium. In the diffusion regime, a few moments can fully describe the photon distribution function. It does not matter rather we use the fluid velocity \( u^\alpha \) or some radiative quantities to solve for the radiation fiducial velocity \( U^\alpha \). Both choices can lead to fully relativistic and well behaved closures.

In the free-streaming regime, however, the radiating medium does not contribute much. On the one hand, setting \( U^\alpha = u^\alpha \) is nonsense. On the other hand, employing the lab frame for linearity causes closures, such as the \( P_4 \) closures, violates the Lorentz symmetry. We believe that Lorentz symmetry is an important property\(^4\) so we ensure that it is satisfied in the proposed scheme.

#### 4.2. Unphysical Photon Self-interaction

It is educational to study the simplest Lorentz invariant closure. We need at least one variable for the energy scale and three variables for the fiducial velocity. By symmetry, the radiative stress-energy tensor must take the form

\[
R_{t}^{\alpha \beta} = \begin{bmatrix}
3P & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & P & 0 \\
0 & 0 & 0 & P
\end{bmatrix}
\]

(45)

\(^4\) There is an exception. If we need to implement a radiative transfer solver on a low resolution grid, the spatial discretization introduces fiducial frame dependence because of truncation error. It this case, it may not be a bad idea to just give up Lorentz invariance and use high order linear closures. In some sense, this is the philosophy behind Lattice Boltzmann methods (see, for example, Succi et al. 1993; He & Luo 1997; Lallemand & Luo 2000).
in the fiducial frame. Boosting it back to the lab frame, we obtain the standard stress-energy tensor for photon fluid,

\[ R^{\alpha\beta} = \frac{4}{c^2} P U^\alpha U^\beta + P g^{\alpha\beta}. \]  

(46)

Recalling Equation (16) and comparing different components, we obtain

\[ E = \left( \frac{4}{c^2} U^0 U^0 - 1 \right) P, \]  

(47)

\[ F^i = \frac{4}{c} U^0 U^i P, \]  

(48)

\[ P^{ij} = \left( \frac{4}{c^2} U^i U^j + g^{ij} \right) P. \]  

(49)

Let us treat the radiative energy \( E \) and flux \( F^i \) as the fundamental fields. The closure problem reduces to solving \( P^{ij} \) in terms of \( E \) and \( F^i \). The solution is simply

\[ P^{ij} = \frac{F^i F^j}{c^2 (E + P)} + P g^{ij}, \]  

(50)

with

\[ P = E \left( \frac{2}{3} \left( 1 - \frac{3 F^i F_i}{4 c^2 E^2} - 1 \right) \right). \]  

(51)

The above equations describe exactly the \( M_1 \) closure although the derivation is different from the standard ones (Janka et al. 1992; Smit et al. 2000; Farris et al. 2008). In fact, our derivation is probably more general since the only assumes are the number of fundamental fields, isotropy in the fiducial frame, and Lorentz invariance in the closure. There is no assumption about the spectrum.

The above derivation is fully relativistic so the information propagation speed is limited by the speed of light. The closure appears to have correct diffusion and free-streaming limits. However, the derivation implies that the photons always thermalize themselves even without interacting with the radiating medium.\(^5\) We can interpret the truncation errors as unphysical photon self-interaction. Our closure problem now reduces to minimizing this unphysical effect.

### 4.3. Moment Decomposition

There is no freedom left in the \( M_1 \) closure. We have to introduce more dynamic variables to reduce the unphysical photon self-interaction. The simplest trial is to include the zeroth moment of the radiative transfer equation,

\[ R^{\alpha\beta} = -G. \]  

(52)

The above equation and Equation (22) form a 5-field method. Compared with \( M_1 \), the extra information governed by Equation (52) allows us to go beyond the gray approximation. It is easy to derive the extinction terms by using Section 3.4 or Appendix C. For the purpose of this paper, we only focus at the closure problem.

The structure of the 5-field method is similar to relativistic hydrodynamics (20) and (21), where the fluid velocity is equal to the fiducial velocity; i.e., the fiducial frame is chosen so that the particle density flux vanishes. The heat flux, viscosity, etc are then defined in such a frame. There is a special name associated with this choice: the Eckart (1940) decomposition. Alternatively, Landau & Lifshitz (1987) proposed shifting the fiducial velocity so that the heat flux vanishes in the fiducial frame, \( T^{ij}_L = 0 \), where the subscript \( L \) denotes the Landau–Lifshitz frame.

The fiducial velocity in the Eckart decomposition describes the particle flow, while in the Landau–Lifshitz decomposition it describes the energy flow. When the radiation is not in local thermodynamic equilibrium, the two decompositions result different fiducial frames and have different level of unphysical photon self-interactions. However, the 5-field method is too restrictive to describe non-equilibrium effects. The photon density flux \( R^i \) is parallel to the photon energy flux \( R^0 \) in the fiducial frame,

\[ R^i = \frac{W_{ij} c}{W_{ij}} R^0. \]  

(53)

The two decompositions are therefore identical. We must use higher order moment methods to reduce unphysical photon self-interaction.

### 5. A PHYSICALLY MOTIVATED 14-FIELD METHOD

To describe non-equilibrium effects, we follow Grad (1949) to truncate the expansion (27) at second order. The photons are described by an ultra-relativistic Grad’s distribution function,

\[ f^{(2)} = \frac{2}{h^3} w(\xi)(\alpha + \xi a_{\alpha\beta} n^\beta + \xi^2 a_{\alpha\beta\gamma} n^\beta n^\gamma) \approx f. \]  

(54)

Because \( a_{\alpha\beta} \) is symmetric and has trace-free spatial parts, there are only nine independent components. Taking \( a \) and \( a_\alpha \) into account, the polynomial has fourteen independent coefficients. They can be solved by the fourteen time dependent fundamental fields \( R^0, R^{0\alpha}, \) and \( R^{\alpha\beta} \).

To derive our 14-field method, we first reduce the flux equations to include only the non-vanishing coefficients,

\[ T^0 = \frac{8\pi}{3} (W_3 a_{ij} + 2 W_4 a_{ij}). \]  

(55)

\[ T^{ij} = \frac{16 \pi}{15} (W_5 a_{ij}) + \frac{1}{3} \pi^{00} \delta^{ij}, \]  

(56)

\[ T^{ijk} = \frac{1}{5} (\pi^{000} \delta^{ijk} + \pi^{0ij} \delta^{ki} + \pi^{0kij} \delta^{0i}). \]  

(57)

The dimensionless moments in the above equations are rescaled by the energy scale \( \theta \) chosen in Equation (36). Note that \( a, a_\alpha, \) and \( a_{\alpha\beta} \) do not appear in the flux equations. Therefore, we can skip Equation (29) and just invert the reduced forms of Equations (30) and (31),

\[ \begin{bmatrix} \pi^{000} \\ \pi^{0ij} \end{bmatrix} = \begin{bmatrix} W_4 \\ W_5 \end{bmatrix} \begin{bmatrix} W_4 \\ 2 W_5 \end{bmatrix}^{-1} \begin{bmatrix} a_{ij} \\ \theta \end{bmatrix}. \]  

(58)

\[ \pi^{000} - \frac{1}{3} \pi^{0ij} \delta^{ij} = \frac{16 \pi}{15} W_6 a_{ij}. \]  

(59)
Eliminating Grad’s coefficients, we have

\[
\mathcal{R}_r^i = \frac{W_3 W_6 - W_4 W_5}{W_4 W_6 - W_3 W_5} \mathcal{R}_r^{0i} - \frac{W_3 W_5}{W_4 W_6 - W_3 W_5} \mathcal{R}_r^{00}, \tag{60}
\]

\[
\mathcal{R}_r^{ij} = \frac{W_5}{W_6} \left( \mathcal{R}_r^{0j} - \frac{1}{3} \mathcal{R}_r^{00} \delta_{ij} \right) + \frac{1}{3} \mathcal{R}_r^{00} \delta_{ij}. \tag{61}
\]

Equations (60), (61), and (57) almost form a closure relation. If we fixed the fiducial frame to the lab frame, it would become a linear (but frame dependent) closure similar to \( P_2 \).

Equation (60) shows that the Eckart and Landau–Lifshitz decompositions result different fiducial frames. In addition, we can introduce a third-order decomposition so that the fiducial frame is chosen to satisfies the requirement \( \mathcal{R}_r^{0i} = 0 \). Equation (60) becomes

\[
\mathcal{R}_r^{0i} = 0.1035 \mathcal{X}_r^{0i} \text{ in the Eckart frame,} \tag{62}
\]

\[
\mathcal{R}_r^{0i} = -0.0548 \mathcal{X}_r^{0i} \text{ in the Landau–Lifshitz frame, and} \tag{63}
\]

\[
\mathcal{R}_r^{ij} = 0.5299 \mathcal{X}_r^{0i} \text{ in the third-order frame,} \tag{64}
\]

respectively. We can refer the Eckart and Landau–Lifshitz frames as first- and second-order frames.

The sign difference between \( \mathcal{R}_r^{ij} \) and \( \mathcal{X}_r^{0i} \) in Equation (63) has interesting meaning. It indicates that the Landau–Lifshitz fiducial velocity lies between the other two. Appendix D demonstrates that higher order velocities are less sensitive to the distribution function. The nonlinear terms have small contribution to the overall dynamics. Therefore, we conjecture that higher order decompositions introduce less unphysical photon-self interaction. In order words, we propose using the third order decomposition for the 14-field method.

6. CONCLUSIONS

In this paper, we propose a class of physically motivated closures for radiative transfer and radiation hydrodynamics. We start by showing the advantages of frequency-integrated schemes, which are fully relativistic. The transport terms and extinction terms can be easily evaluated in different reference frames. We then apply Grad’s moment method to compute the flux terms and the extinction terms from the fundamental fields. The truncated Grad’s series is energy scale and fiducial frame dependent. For the extinction terms, the fluid comoving frame and temperature are the natural choices. For the flux terms, however, we propose using high order decomposition to reduce unphysical photon self-interaction as well as using photon number density to obtain the energy scale.

We believe that this paper clarifies some implicit assumptions in standard closures and points out the importance of fiducial frame. Although we only present the 14-field method, it is straightforward to derive arbitrarily high order closures from our formulas (see the appendices). With minimal modifications, our closures are also applicable to neutrino transport and relativistic rarefied gas dynamics. We expect the methods derived from our new framework to outperform existing ones.

Of course, there are many open questions associated with the proposed schemes such as limiting behavior and linear stability of the theory. We will address these issues in subsequent papers. We are also implementing the 14-field method and integrating it with some hydrodynamic solvers. We will perform verifications and validations before applying the algorithms to study astrophysical systems. As a final remark, moment methods are not optimal for solving all problems (Graziani 2006). For example, for situations with strong beaming, we are better off if we use other techniques such as ray tracing (see Abel & Wandelt 2002; Trac & Cen 2007; Finlator et al. 2009, and reference therein).

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APPENDIX A

GRAD’S EXPANSION AND SYMMETRIC TRACE-FREE TENSORS

Following the notations in Section 2, we use \( p^\alpha = p^0(1, n') \) to denote the four-momentum of a massless particle, where \( n' \) is some spatial unit three-vector. We assume there exists an energy scale \( \theta \) to describe the particles in some fiducial reference frame. Note that \( \theta \) is not necessary related to the temperature. Let \( U^\alpha \) be the four-velocity of the fiducial frame, we can define \( \xi \) as a dimensionless measurement of energy for each particle, \( \xi = -p^\alpha U_\alpha / \theta \). For photon with frequency \( \nu \), it reduces to \( \xi = h\nu / \theta \) in the fiducial frame, where \( h \) is Planck’s constant.

Considering a fixed position \( x^\alpha \) in the fiducial frame, we can expand the local particle distribution function in a weighted multi-dimensional polynomial of \( p^\alpha \),

\[
f = \frac{g_\xi}{h^\xi} w(\xi) (\hat{a} + \hat{a}_{\beta_1} p^\beta_1 + \hat{a}_{\beta_1\beta_2} p^\beta_1 p^\beta_2 + \hat{a}_{\beta_1\beta_2\beta_3} p^\beta_1 p^\beta_2 p^\beta_3 + \cdots). \tag{A1}
\]

In the above equation, \( g_\xi \) is the number of available states, which equals 2 for “photon gas.” The function \( w(\xi) \) is a dimensionless weight. When it is well chosen, the integral for each term in the polynomial is guaranteed to converge when we take moments. Bear in mind that \( \xi \) is subject to Doppler shift. Therefore, the above expansion depends on the choice of the fiducial frame, i.e., \( U^\alpha \), in addition to \( \theta \). The coefficient \( \hat{a}_{\beta_1\beta_2\cdots\beta_l} \) is a \( l \)-th rank tensor, which has dimension \( (c/\theta)^l \). It is implicit in the notations that these coefficients \( \hat{a}, \hat{a}_{\beta_1}, \hat{a}_{\beta_1\beta_2}, \cdots \) depend on the position \( x^\alpha \).

In the original Grad’s moment method for non-relativistic and non-degenerated gas, because of the Maxwellian distribution in equilibrium, the weighting function \( w(\xi) \) is chosen to be a multi-dimensional Gaussian, which turns out to be the weight for...
the orthogonal conditions of Hermite polynomials, resulting a Hermite series expansion. By the same token, a natural weight for relativistic non-degenerated gas is an exponentially decaying function because of the Maxwell–Jüttner distribution. It corresponds to the orthogonal condition of Laguerre polynomials. Using this choice, $\theta/k_B$ converges to temperature in the limit of local thermodynamic equilibrium. In this paper, however, we use $w(\xi) = [\exp(\xi) - 1]^{-1}$ for radiation so the polynomial expansion reduces to unity in the limit of black body radiation. It is not a weight for classical/standard orthogonal polynomials. Nevertheless, because of its asymptotic behaviors, it is easy to verify that all the useful integrals are guaranteed to converge.

From expansion (A1), it is clear that the tensor $\tilde{\alpha}_{\beta_1 \beta_2 \ldots \beta_l}$ is totally symmetry. In addition, it has trace-free spatial parts because $p^a$ has only three independent components. One way to verify this is to rewrite the above expansion as

$$ f = \frac{g_s}{h^3} w(\xi) (a + \xi a_{\beta i} n^{\beta i} + \xi^2 a_{\beta_1 \beta_2} n^{\beta_1 \beta_2} + \xi^3 a_{\beta_1 \beta_2 \beta_3} n^{\beta_1 \beta_2 \beta_3} n^{\beta 4} + \cdots )$$

(A2)

$$ = \frac{g_s}{h^3} w(\xi) [(a + \xi a_{0} + \xi^2 a_{000} + \cdots ) + \xi (a_{0i} + 2\xi a_{00i} + 3\xi^2 a_{000i} + \cdots ) n^{i1}$$

$$+ \xi^2 (a_{i1i2} + 3\xi a_{0i1i2} + \cdots ) n^{i2} n^{i3} + \xi^3 (a_{i1i2i3} + \cdots ) n^{i2} n^{i3} n^{i4} + \cdots ] .$$

(A3)

where the rescaled coefficients $a, a_{\beta i}, a_{\beta_1 \beta_2}, \ldots$ are all dimensionless. We can now employ the results in Thorne (1980) to conclude $a_{i1i2i3}$ must be a linear combination of symmetric trace-free tensors. Therefore, $\tilde{\alpha}_{\beta_1 \beta_2 \ldots \beta_l}$ has trace-free spatial parts. It is easy to see the close relationship between (A1) and hydrogen energy eigenstates by using the identity between spherical harmonics and the basis set of symmetric trace-free tensors.

Counting degrees of freedom is another way to verify the trace-free property. Recalling the zero moment of the ultra-relativistic Boltzmann equation yields one dynamic equation; the first moments yield four dynamic equations. However, the second moments yield only nine equations because photons are massless. In order to match the degrees of freedom at each order, $a_{\beta_1 \beta_2 \cdots \beta_l} \equiv \rho_{\beta_1 \beta_2 \cdots \beta_l}$ has to be totally symmetric with trace-free spatial parts.

APPENDIX B

MOMENTUM MOMENTS OF MASSLESS PARTICLES

We now derive the equations for computing four-momentum moments, $R^{a_1 a_2 \cdots a_l}$, from the ultra-relativistic one-particle distribution function $f$. First, we separate the $\xi$-dependence and angular-dependence of the particle moment $p^a$ by writing

$$ R^{a_1 a_2 \cdots a_l} \equiv c \int \frac{d^3p}{p^0} f \ p^{a_1} p^{a_2} \cdots p^{a_l}$$

$$= c \int p^0 dp^0 d\Omega \ f \ p^{a_1} p^{a_2} \cdots p^{a_l}$$

$$= c \left( \frac{\theta}{c} \right)^{l+2} \int d\xi \ d\Omega \ f \ \xi^{l+1} n^{a_1} n^{a_2} \cdots n^{a_l}$$

$$= \frac{g_s c}{h^3} \left( \frac{\theta}{c} \right)^{l+2} \int d\xi \ d\Omega \ w(\xi) (\xi^{l+1} n^{a_1} n^{a_2} \cdots n^{a_l}) .$$

(B1)

We are free to rearrange the indices because $R^{a_1 a_2 \cdots a_l}$ is totally symmetric. Let $m$ be the total number of non-zero indices, without loss of generality, the moment can be written in the form $R^{a_1 a_2 \cdots a_l} \equiv \rho_{a_1 a_2 \cdots a_l}$ by packing all the zeros to the beginning and leave the spatial indices at the end. We can separate all the $\xi$-dependence and angular dependence,

$$ R^{a_1 a_2 \cdots a_l} = R^{a_1 \cdots a_l a_{l+1} \cdots a_m}$$

$$= \frac{g_s c}{h^3} \left( \frac{\theta}{c} \right)^{l+2} \int d\xi \ d\Omega \ w_{l+1} (a + \xi a_{0} + \xi^2 a_{000} + \cdots ) n^{i1} n^{i2} \cdots n^{im}$$

$$+ w_{l+2} (a_{j1} + 2\xi a_{0j1} + 3\xi^2 a_{00j1} + \cdots ) n^{i1} n^{i2} \cdots n^{im} n^{j1}$$

$$+ w_{l+3} (a_{j1j2} + 3\xi a_{0j1j2} + 6\xi^2 a_{00j1j2} + \cdots ) n^{i1} n^{i2} \cdots n^{im} n^{j1} n^{j2}$$

$$+ w_{l+4} (a_{j1j2j3} + 4\xi a_{0j1j2j3} + 10\xi^2 a_{00j1j2j3} + \cdots ) n^{i1} n^{i2} \cdots n^{im} n^{j1} n^{j2} n^{j3} + \cdots ) .$$

(B2)

\[\text{The moment method proposed here is very general. When applying to neutrino transport, we can choose } w(\xi) = [\exp(\xi) + 1]^{-1}. \text{This results } W = (1 - 2\xi) \Gamma(t + 1) \xi(t + 1) \text{ instead of Equation (B3).}\]
where we have introduced the shorthand \( w_I \equiv w(\xi)\xi^I \). It turns out that there is a closed form solution for the \( \xi \)-integral,

\[
W_I \equiv \int_0^\infty d\xi \, w(\xi)\xi^I = \Gamma(l + 1)\,\zeta(l + 1) \text{ for photons,}
\]
where \( \Gamma \) is the Gamma function and \( \zeta \) is the Riemann zeta function. Some numerical values are given here:

\[
W_1 \approx 1.645, \quad W_2 \approx 2.404, \quad W_3 \approx 6.494, \quad W_4 \approx 24.89, \quad W_5 \approx 122.1, \quad W_6 \approx 726.0, \text{ etc, for photons. (B4)}
\]

Therefore, with our shorthands,

\[
\begin{align*}
R^{0 \cdots 0} & = \frac{g_s c}{\hbar^4} \left( \frac{\theta}{c} \right)^{4+2} \int d\Omega \left[ \left( C_0^0 W_{l+1} a + C_0^1 W_{l+2} a_0 + C_0^2 W_{l+3} a_0 a_0 + \cdots \right) n_i n_j \cdots n_v \\
+ \left( C_1^1 W_{l+3} a_j + C_1^2 W_{l+4} a_0 j_i + C_1^3 W_{l+4} a_0 j_i a_0 j_i + \cdots \right) n_i n_j \cdots n_v n_j \\
+ \left( C_2^2 W_{l+4} a_j j_i + C_2^3 W_{l+5} a_0 j_i j_i + C_2^4 W_{l+5} a_0 j_i j_i a_0 j_i + \cdots \right) n_i n_j \cdots n_v n_j n_j \\
+ \left( C_3^3 W_{l+5} a_j j_i j_i + C_3^4 W_{l+6} a_0 j_i j_i j_i + C_3^5 W_{l+6} a_0 j_i j_i j_i a_0 j_i j_i + \cdots \right) n_i n_j \cdots n_v n_j n_j n_j + \cdots \right] \\
& = \frac{g_s c}{\hbar^4} \left( \frac{\theta}{c} \right)^{4+2} \int d\Omega \sum_{p=0}^{\infty} \sum_{q=p}^{\infty} C_q^p W_{l+q+1} a_0 \cdots j_p n_i n_j \cdots n_v n_j + \cdots n_p \int d\Omega n_i n_j \cdots n_v n_j j_i \cdots n_p
\end{align*}
\]

where \( C_q^p \) denotes binomial coefficient \( C_q^p \equiv q! / p!(q - p)! \).

The integrands in Equation (B5) are products of different components of unit vectors. We can employ Thorne’s (1980) orthogonal condition to evaluate it:

\[
\frac{1}{4\pi} \int d\Omega n_i n_j \cdots n_v \equiv \frac{1}{n+1} \delta_{\Gamma}^{i_1 i_2 \cdots i_n},
\]

where we have defined \( \delta_{\Gamma}^{i_1 i_2 \cdots i_n} \) be the Thorne delta function of rank-\( n \). It carries interesting symmetric properties to help us simplify the evaluation of the integral. The function vanishes when \( n \) is odd. For even \( n \), it is defined to be the completely symmetrized product of Kronecker delta functions. For completeness, we copy the formulae from Thorne (1980):

\[
\delta_{\Gamma}^{i_1 i_2 \cdots i_n} = \delta^{i_1 j_1} \cdots \delta^{i_{n-1} j_{n-1}} \equiv \frac{1}{(n-1)!!} \sum_{k_1 k_2 \cdots k_n} \delta^{j_1 k_1} \delta^{j_2 k_2} \cdots \delta^{j_{n-1} k_{n-1}} \delta^{j_n k_n},
\]

where

1. \( k_1 \) is summed from 2 to \( n \);
2. \( k_1 \) is the smallest integer not equal to 1 or \( k_2 \);
3. \( k_2 \) is summed over all integers from 2 to \( n \), not equal to \( k_1 \) or \( k_3 \);
4. \( k_3 \) is the smallest integer not equal to 1 or \( k_2 \) or \( k_3 \) or \( k_4 \);
5. ...

We list all the non-zero Thorne deltas up to eight indices below:

\[
\delta_{\Gamma} = 1; \quad \delta^i_{\Gamma} = 1; \quad \delta^{ii}_{\Gamma} = 1; \quad \delta^{iii}_{\Gamma} = 1; \quad \delta^{ij}_{\Gamma} = \frac{1}{3}; \quad \delta^{iij}_{\Gamma} = 1; \quad \delta^{iiij}_{\Gamma} = \frac{1}{5}; \quad \delta^{iiijj}_{\Gamma} = \frac{1}{15};
\]

\[
\delta^{iiijj}_{\Gamma} = 1; \quad \delta^{iiijjj}_{\Gamma} = \frac{1}{7}; \quad \delta^{iiijjjj}_{\Gamma} = \frac{3}{35}; \quad \delta^{iiijjjjj}_{\Gamma} = \frac{1}{35}.
\]

Note that the Einstein summation convention is not applied in the above equations. The symbols \( i, j, k \) denote indices that are not equal to each other. With the help of the Thorne delta function, we obtain the expression for all moments:
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The first few momentum moments have important physical meanings. The first moment $R^α$ is the radiative four-flow. Based on Equation (B9),

$$R^0 = \frac{8\pi c}{h^3} \left( \frac{\theta}{c} \right)^3 \left( W_2 a + W_3 a_0 + W_4 a_{00} + \cdots \right) + \frac{1}{3} \left( W_4 a_{j1} + 3 W_5 a_{0j1} + 6 W_6 a_{00j1} + \cdots \right) \delta_{ij} + \cdots. \tag{B10}$$

Because $a_{α1α2\cdotsα_l}$ has trace-free spatial part, the above equation reduces to

$$R^0 = \frac{8\pi c}{\lambda^3} (W_2 a + W_3 a_0 + W_4 a_{00} + \cdots), \tag{B11}$$

where we have defined the length scale $\lambda \equiv hc/\theta$, which is proportional to the photon mean separation (i.e., $n^{-1/3}$, it is a purely radiation property and has nothing to do with the material-dependent photon mean free path). Similarly, the spatial components of the radiative four-flow take the form

$$R^i = \frac{8\pi c}{3\lambda^3} (W_2 a_i + 2 W_4 a_{0i} + 3 W_5 a_{00i} + \cdots). \tag{B12}$$

It becomes clear now, although a temperature is not well defined in non-equilibrium relativistic statistical mechanics, the notion of mean photon separation always exists. It is possible to use this physical interpretation to choose $θ$. The second moment $R^{αβ}$ is the radiation stress-energy tensor. Its components take the form

$$R^{0i} = \frac{16\pi c}{15\lambda^5} (W_5 a_{ij} + 3 W_6 a_{0ij} + 6 W_7 a_{00ij} + \cdots) + \frac{1}{3} R^{000} \delta_{ij}. \tag{B15}$$

For $R^{αβγ}$,

$$R^{000} = \frac{8\pi c}{\lambda^3} \left( \frac{\theta}{c} \right)^2 (W_2 a + W_3 a_0 + W_4 a_{00} + \cdots), \tag{B16}$$

$$R^{00i} = \frac{8\pi c}{3\lambda^3} \left( \frac{\theta}{c} \right)^2 (W_5 a_i + 2 W_6 a_{0i} + 3 W_7 a_{00i} + \cdots), \tag{B17}$$

$$R^{0ij} = \frac{16\pi c}{15\lambda^5} \left( \frac{\theta}{c} \right)^2 (W_7 a_{ij} + 3 W_8 a_{0ij} + 6 W_9 a_{00ij} + \cdots) + \frac{1}{3} R^{000} \delta_{ij}. \tag{B18}$$

$$R^{ijk} = \frac{16\pi c}{35\lambda^3} \left( \frac{\theta}{c} \right)^2 (W_7 a_{ijk} + 4 W_8 a_{0ijk} + 10 W_9 a_{00ijk} + \cdots) + \frac{1}{5} \left( R^{00i} \delta_{ij} + R^{00j} \delta_{ij} + R^{00k} \delta_{ij} \right). \tag{B19}$$

Higher order moments can easily be obtained by using Equation (B9). Fixing the energy scale $θ$ at each point $x^α$, the moments are simply linear transform of Grad’s coefficients. Hence, we can solve for the coefficients by applying an inverse linear transfer. Although the zeroth moment $R$ is never used in radiative transfer, for completeness, we give its expression here:

$$R = c \int \frac{d^3 p}{p^0} f = \frac{8\pi c^2}{\lambda^3} (W_1 a + W_2 a_0 + W_3 a_{00} + \cdots). \tag{B20}$$
APPENDIX C

GENERAL FORM OF THE MOMENT EXTINCTION TERMS

The right-hand side of the radiative transfer equation has the same mathematical form as the linearized collision term of the Bhatnagar–Gross–Krook model (see standard textbook such as Huang 1987; Cercignani & Kremer 2002). We start deriving its moments by defining

\[ G^{α_1α_2...α_l} ≡ c \int \frac{d^3p}{p_0} x(f - s)p^{α_1}p^{α_2}...p^{α_l} = c \left( \frac{k_B T}{c} \right)^{l+2} \int dξ dΩ x(f - s)ξ^{l+1}n^{α_1}n^{α_2}...n^{α_l}, \]  

(C1)

where \( s \) denotes the source distribution function and \( x \) is the Lorentz invariant extension coefficient. Similar to Appendix B, we define \( ξ ≡ hν/k_BT \) and replace the integral over \( ν \) by the integral over \( ξ \). We assume thermal radiation so

\[ s \equiv \frac{e}{x} = \frac{g_s/h^3}{e^{pα_1/k_BT} - 1} = \frac{g_s}{h^3}w(ξ), \]  

(C2)

where we further replace \( U^α \) by the fluid four-velocity \( u^α \) and \( ϑ \) by the fluid temperature \( k_BT \) for \( w(ξ) \). Substituting Grad’s series for \( f \), the extinction terms become

\[ G^{α_1α_2...α_l} ≡ \frac{8c}{k^3} \left( \frac{k_BT}{c} \right)^{l+2} \int dξ dΩ xw_{l+1}(a - 1 + ξa_{β_1}n^{β_1} + ξ^2a_{β_1β_2}n^{β_1}n^{β_2} + ...)n^{α_1}n^{α_2}...n^{α_l}. \]  

(C3)

The zeroth and first moments of the collision term have clear physical meanings. When \( l = 0 \), \( G \) is the photon extinction rate; while for \( l = 1 \), \( G \) is the radiative four-force. Higher order moment such as \( G^{α_1α_2} \) are simply called the extinction term of the third moment because of the balance equations. We evaluate \( G^{α_1α_2...α_l} \) in the fluid comoving frame so that \( u^α \) has only the temporal component.

The Lorentz invariant extension coefficient \( x \equiv (hν/c)X \), depends on the radiative process. Considering free–free, bound–free, or electron scatter, we notice that each of them is proportional to some power of temperature and frequency, \( χ_ν \propto T^φν^ψ \), in standard approximations. Therefore, we assume for the general form,

\[ χ_ν = \frac{τ(ξ)}{λ} \left( \frac{k_BT}{m_ec^2} \right)^{φ+ψ-1}, \]  

(C4)

where \( λ \equiv hc/k_BT \) and \( τ(ξ) \) is some dimensionless function in \( ξ \). The momentum moment of the collision term due to a particular radiative process becomes

\[ G^{α_1α_2...α_l} = \frac{8c}{λ^3} \left( \frac{k_BT}{c} \right)^{l+1} \int dξ dΩ τ(ξ)w_{l+2}(a - 1 + ξa_{β_1}n^{β_1} + ξ^2a_{β_1β_2}n^{β_1}n^{β_2} + ...)n^{α_1}n^{α_2}...n^{α_l}. \]  

(C5)

The expression is very similar to Equation (B1). We can further define

\[ b ≡ a - 1, \ b_{β_1} ≡ a_{β_1}, \ b_{β_1β_2} ≡ a_{β_1β_2}, \ldots \]  

(C6)

and the shorthand

\[ X_l ≡ \int_0^∞ dξ ξ^l. \]  

(C7)

Note that \( l \) is not necessarily an integer in the above definition. Fortunately, the values of \( X_l \) are well define as soon as \( χ_ν \) does not grow exponentially. Follow the same procedures in Appendix B, we can easily deduce the following expression

\[ G^{α_1α_2...α_l} = G^0...0_{l1l2...lm} = 4π \frac{8c}{λ^3} \left( \frac{k_BT}{c} \right)^{l+1} \sum_{p=0}^{∞} \sum_{q=0}^{p} \sum_{m+p+1}^{∞} C_q^p X_{l1q+2} b_0...0_{j_1j_2...j_p} X_1^{l1} X_2^{l2}...X_m^{lm}. \]  

(C8)

The zeroth-order moment is the photon extinction rate. It is simply

\[ G = \frac{8πc}{λ^3} \left( \frac{k_BT}{m_ec^2} \right)^{φ+ψ-1} (X_2b + X_3b_0 + X_4b_0 + \ldots). \]  

(C9)

The radiative 4-force takes the form

\[ G^0 = \frac{8πc}{λ^3} \left( \frac{k_BT}{m_ec^2} \right)^{φ+ψ-1} (X_3b + X_4b_0 + X_5b_0 + \ldots). \]  

(C10)
\[ G^i = \frac{8\pi c}{3\lambda^3} \left( \frac{k_B T}{m_e c^2} \right)^{\phi + \psi - 1} \left( X_4 b_i + 2X_5 b_{0i} + 3X_6 b_{00i} + \cdots \right). \]  

(C11)

For the second moment \( G^{ii} \), we have

\[ G^{ii} = \frac{8\pi c}{\lambda^4} \left( \frac{k_B T}{c} \right)^2 \left( \frac{k_B T}{m_e c^2} \right)^{\phi + \psi - 1} \left( X_4 b_i + X_5 b_0 + X_6 b_{00} + \cdots \right), \]

(C12)

\[ G^{ij} = \frac{16\pi c}{15\lambda^4} \left( \frac{k_B T}{c} \right)^2 \left( \frac{k_B T}{m_e c^2} \right)^{\phi + \psi - 1} \left( X_6 b_{ij} + 3X_7 b_{0ij} + 6X_8 b_{00ij} + \cdots \right) + \frac{1}{3} G^{00} \delta^{ij}. \]

(C14)

Other higher order moments can be obtained by using Equation (C8). If there are more than one radiative process in the problem, we can compute the extinction terms for each process and then sum over the results.

**APPENDIX D**

**SENSITIVITY OF FIDUCIAL VELOCITY**

We consider a very simple radiative transfer problem. Assuming there are only two streams of photons moving along the \( \hat{x}_1 \)-axis in opposite directions, the photon distribution function is

\[ f = \frac{8\pi}{h^3} (n_+ \delta(v - v_+) \delta^3(n^1 - \delta^1) + n_- \delta(v - v_-) \delta^3(n^1 + \delta^1)), \]

(D1)

where \( n_\pm \) and \( v_\pm \) are the density and frequency for the two streams, \( \delta(v) \) and \( \delta^3(n^1) \) are the one- and three-dimensional Dirac delta functions. It is trivial to evaluate its momentum moments. The non-vanishing terms are

\[ R^{0\cdots0} \propto (n_+v_+^{1+} + n_-v_-^{1+}) \quad \text{and} \quad R^{0\cdots1} \propto (n_+v_+^{1-} - n_-v_-^{1-}) \cdots \]

(D2)

For an \( l \)-th order decomposition, we choose the fiducial reference frame so that the \( l \)-th order flux \( R^{0\cdots0} \) vanish. This is equivalent to solving for \( \beta \equiv U^1 / c \) in the equation

\[ n_+ \left( v_+ \sqrt{\frac{1 - \beta}{1 + \beta}} \right)^{l+1} = n_- \left( v_- \sqrt{\frac{1 + \beta}{1 - \beta}} \right)^{l+1}. \]

(D3)

Suppose \( \beta_1 \) is a solution to the above equation. To see how the fiducial frame depends on the distribution function, we suppose there is a small change to the photon number density \( n_\pm \). The corresponding change in the fiducial velocity is related to the derivative

\[ \frac{d\beta}{dn_\pm} \bigg|_{\beta=\beta_1} = \pm \frac{1}{2n_\pm(1+1)}. \]

(D4)

In the limit \( l \to \infty \), the derivative \( d\beta / dn_\pm \to 0 \), which recovers the linear closure. Although the general situation is more complicated, it is sensible to conjecture that the fiducial velocity obtained from higher order decompositions are less sensitive to the distribution function. Hence, the non-linear Lorentz transforms should introduce less unphysical photon self-interaction.

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