FOUR-DIMENSIONAL RICCI-FLAT SPACE DEFINED BY THE KP EQUATION

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Abstract

Four-dimensional affinely connected Ricci-flat space dependent from solutions of the Kadomtsev-Petviashvili equation are constructed. Conditions of metrizability of corresponding connection are investigated. Their properties are discussed.

1 The Ricci-flat 8-dim metrics

An examples of the eight-dimensional Ricci-flat metrics of Riemann extensions of affinely-connected four-dimensional spaces $A^4$ dependent from solutions of the KP-equation were obtained recently by the author ([8]).

Here we give the example of four-dimensional Ricci-flat affinely connected space $E^4$ defined by solutions of the KP-equation which has Ricci-flat the Riemann extension also dependent from solutions of the KP-equation.

Proposition 1 Eight-dimensional Riemann space in local coordinates $(x, y, z, t, P, Q, U, V)$ equipped with the metric

$$\begin{align*}
ds^2 &= \left( -2 P H11(y, z, t) - 2 F \frac{\partial}{\partial t} F(y, z, t) - 2 H12(y, z, t) Q - 2 \Gamma_{11}^3(y, z, t) U + 2 H22(y, z, t) V \right) dx^2 + \\
&+2 \left( 2 H11(y, z, t) Q - 2 H21(y, z, t) V \right) dx dy + 2 \left( -2 \left( \frac{\partial}{\partial z} F(y, z, t) \right) V + 2 \left( \frac{\partial}{\partial t} F(y, z, t) \right) U \right) dx dz + \\
&+2 dx dP + (2 H11(y, z, t) U - 2 H31(y, z, t) V) dy^2 + 2 dy dQ + 2 dz dU + 2 dt dV
\end{align*}$$

is a Ricci-flat

$$R_{ij} = 0$$

if the conditions on the coefficients $H_{ij}(y, z, t)$

$$\begin{align*}
\frac{\partial}{\partial y} H12 (y, z, t) - \frac{\partial}{\partial t} H22 (y, z, t) &= 0 \\
- \frac{\partial}{\partial y} H11 (y, z, t) + \frac{\partial}{\partial t} H21 (y, z, t) &= 0 \\
- \frac{\partial}{\partial z} H11 (y, z, t) + \frac{\partial}{\partial t} H31 (y, z, t) &= 0
\end{align*}$$

are valid.
An arbitrary functions $\Gamma^3_{11}(y, z, t)$ and $F(y, z, t)$ in so doing satisfy the relation
\[
\frac{\partial}{\partial y} \Gamma^3_{11}(y, z, t) = 2 \left( \frac{\partial}{\partial t} F(y, z, t) \right)^2 + 2 H11(y, z, t) \frac{\partial}{\partial t} F(y, z, t) + 2 \left( H11(y, z, t) \right)^2
\]

or
\[
\Gamma^3_{11}(y, z, t) = \int 2 \left( \frac{\partial}{\partial t} F(y, z, t) \right)^2 + 2 H11(y, z, t) \frac{\partial}{\partial t} F(y, z, t) + 2 \left( H11(y, z, t) \right)^2 \, dz + \_F1(y, t). \quad (3)
\]

The metric (1) is an example of the Riemann extension of affinely-connected four-dimensional space with symmetrical connection $\Gamma^i_{jk}(x^l) = \Gamma^i_{kj}(x^l)$ depending from the local coordinates $x^l$.

In general case it is defined by the expression
\[
ds^2 = -2\Gamma^i_{jk}\xi_k dx^j dx^k + 2d\xi_k dx^k,
\]
where $\xi_k$ are an additional coordinates ([1]).

After the substitutions of the form
\[
H11(y, z, t) = -1/2 u(y, z, t), \quad H12(y, z, t) = -1/3 v(y, z, t),
\]
\[
H21(y, z, t) = -2/3 v(y, z, t) - 1/2 \frac{\partial}{\partial t} u(y, z, t),
\]
\[
H31(y, z, t) = -3/4 w(y, z, t) + 3/8 (u(y, z, t))^2 - \frac{\partial}{\partial t} v(y, z, t) - 1/2 \frac{\partial^2}{\partial t^2} u(y, z, t),
\]
\[
H22(y, z, t) = -1/2 w(y, z, t) + 1/2 (u(y, z, t))^2 - \frac{\partial}{\partial t} v(y, z, t) -
\]
\[
-1/2 \frac{\partial^2}{\partial t^2} u(y, z, t) + 1/2 \frac{\partial}{\partial y} u(y, z, t)
\]
from the conditions (2) the famous KP-equation is followed
\[
\frac{\partial^2}{\partial t \partial z} u(y, z, t) - 3/2 \left( \frac{\partial}{\partial t} u(y, z, t) \right)^2 - 3/2 u(y, z, t) \frac{\partial^2}{\partial t^2} u(y, z, t) - 1/4 \frac{\partial^4}{\partial t^4} u(y, z, t) - 3/4 \frac{\partial^2}{\partial y^2} u(y, z, t) = 0
\]
or
\[
\frac{\partial}{\partial t} \left( \frac{\partial u(y, z, t)}{\partial z} - 3/2 u(y, z, t) \frac{\partial^2}{\partial t^2} u(y, z, t) - 1/4 \frac{\partial^3}{\partial t^3} u(y, z, t) \right) = 3/4 \frac{\partial^2}{\partial y^2} u(y, z, t)
\]

Remark 1 A notion of the Riemann extensions of affinely-connected spaces ([1]) was used by author for the studying of geometrical problems in theory nonlinear dynamical systems and in General Relativity ([2]-[8]).

Remark 2 About presentation of the KP-equation in form of the system (2) see ([9]).

2 Four-dimensional subspace

Eight-dimensional metrics (4) is closely related with a properties of four-dimensional spaces in local coordinates $x^k$.

Let us consider an example.

The full system of geodesic of the metric (4) decomposes into two parts.

The first part has the form of the linear system of equations for the coordinates ($\xi_k = P, Q, U, V$)
\[
\frac{d^2 \xi_k}{ds^2} + A(x^i) \frac{d\xi_k}{ds} + B(x^i) \xi_k = 0
\]
and the second part for local coordinates \( x^i = (x, y, z, t) \) is defined by the system of equations

\[
\frac{d^2 x^k}{ds^2} + \Gamma^k_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0.
\]

In our case it takes the form

\[
\frac{d^2 x}{ds^2} + \left( \frac{d}{ds} x(s) \right)^2 H11(y, z, t) + \left( \frac{d}{ds} x(s) \right)^2 \frac{\partial}{\partial t} F(y, z, t) = 0,
\]

\[
\frac{d^2 y}{ds^2} + \left( \frac{d}{ds} x(s) \right)^2 H12(y, z, t) - 2 \frac{\partial}{\partial t} F(y, z, t) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) = 0,
\]

\[
\frac{d^2 z}{ds^2} + \Gamma^3_{11}(y, z, t) \left( \frac{d}{ds} x(s) \right)^2 - 2 \left( \frac{\partial}{\partial t} F(y, z, t) \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} z(s) - H11(y, z, t) \left( \frac{d}{ds} y(s) \right)^2 = 0,
\]

\[
\frac{d^2 t}{ds^2} - H22(y, z, t) \left( \frac{d}{ds} x(s) \right)^2 + 2 H21(y, z, t) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} y(s) + 2 \left( \frac{\partial}{\partial z} F(y, z, t) \right) \left( \frac{d}{ds} x(s) \right) \frac{d}{ds} z(s) + H31(y, z, t) \left( \frac{d}{ds} y(s) \right)^2 = 0.
\]

From these relations we find the coefficients of affine connection \( \Gamma^i_{jk} \) of the 4-dimensional subspace in local coordinates \((x^i = x, y, z, t)\)

\[
\begin{align*}
\Gamma^1_{11} &= H11 + \frac{\partial F}{\partial t}, \quad \Gamma^2_{11} = H12, \quad \Gamma^3_{11} = -H11, \\
\Gamma^3_{13} &= \Gamma^3_{11}, \quad \Gamma^3_{12} = -\frac{\partial F}{\partial t}, \quad \Gamma^3_{22} = -H11, \\
\Gamma^4_{11} &= -H22, \quad \Gamma^4_{12} = H21, \quad \Gamma^4_{13} = \frac{\partial F}{\partial z}, \quad \Gamma^4_{22} = H31.
\end{align*}
\]

Corresponding the Ricci tensor

\[
R_{ij} = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} + \Gamma^k_{ki} \Gamma^i_{kj} - \Gamma^k_{im} \Gamma^m_{kj}
\]

is symmetrical

\[
R_{ij} = R_{ji}
\]

and it is equal to zero \( R_{ik} = 0 \) if the conditions \((2), (3)\) are hold.

As the example of geodesic equation connected with an additional coordinates we refer to the expression on the coordinate \( V \)

\[
\begin{align*}
& \left( \left( \frac{d}{ds} x(s) \right)^2 \frac{\partial}{\partial t} H11(y, z, t) + \left( \frac{d}{ds} x(s) \right)^2 \frac{\partial^2}{\partial t^2} F(y, z, t) \right) P+
+ 2 \left( \frac{d}{ds} x(s) \right) \left( \frac{d}{ds} y(s) \right) \left( \frac{\partial}{\partial t} H11(y, z, t) \right) Q+
\end{align*}
\]

\[
\begin{align*}
& \frac{d^2}{ds^2} V(s) - \left( \frac{d}{ds} x(s) \right)^2 \left( \frac{\partial}{\partial t} H22(y, z, t) \right) V - \left( \frac{d}{ds} y(s) \right)^2 \left( \frac{\partial}{\partial t} H12(y, z, t) \right) U+
+ 2 \left( \frac{d}{ds} x(s) \right) \left( \frac{d}{ds} y(s) \right) \left( \frac{\partial}{\partial t} H21(y, z, t) \right) V+
\end{align*}
\]

\[
\begin{align*}
& - 2 \left( \frac{d}{ds} x(s) \right) \left( \frac{d}{ds} z(s) \right) \left( \frac{\partial^2}{\partial t^2} F(y, z, t) \right) U + \left( \frac{d}{ds} y(s) \right)^2 \left( \frac{\partial}{\partial t} H31(y, z, t) \right) V+
\end{align*}
\]
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