From a delayed constrained minimization to the harmonic map heat equation

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In the context of cell motility modelling and more particularly related to the Filament Based Lamelipodium Model \cite{15,7,8}, this work deals with a rigorous mathematical proof of convergence between an adhesion delay non-linear space-dependent problem and the corresponding friction limit. The convergence is performed with respect to the bond characteristic lifetime $\varepsilon$ whose inverse is also proportional to the stiffness of the bonds. The originality of this work is the extension of gradient flow techniques to our setting. Namely, the discrete finite difference term in the gradient flow energy is here replaced by a delay term which complicates greatly the mathematical analysis. Contrarily to the standard approach \cite{2,16}, compactness in time is not provided by the energy minimization process: a series of past times are taken into account in our discrete energy. A supplementary equation on the time derivative is obtained requiring uniform estimate with respect to $\varepsilon$ of the Lagrange multiplier and provides compactness. Due to the non-linearity induced by the constraint, a specific stability estimate useful in our previous works, is not at hand here. Numerical simulations even showed that this estimate does not hold. Nevertheless, transposing our delay operator, we succeed in proving convergence under slightly weaker hypotheses. The result relies on a careful initial layer analysis, extending \cite{13} to the space dependent setting.

Keywords: integral equations, memory effects, cell motility, parabolic equations, non-linear pointwise constraint, adhesion, gradient flow, Lagrange multiplier, harmonic map

1. Introduction

Cell motility is at heart of important biological/medical concerns (cancer metastasis, wound healing, etc.) \cite{3}. Among models describing spontaneous motion of cells, two types appear: those who heuristically mimic macroscopic features and models based on a microscopic description that are in some sense homogenized. The Filament Based Lamelipodium Model (FBLM) \cite{15} belongs to the second category and has reached a certain level of maturity \cite{7,8}.

Adhesion mechanisms are some of the pillars of the FBLM and appear as friction terms. In the pioneering paper \cite{15} they are obtained as formal limits of memory terms inside the Euler-Lagrange equations associated to a minimization process. This limit is interpreted as quasi-instantaneous with respect to a dimensionless parameter $\varepsilon$. Our work deals with the rigorous mathematical justification of this asymptotic.

Previously, we introduced simplifications that allowed to fully understand from the mathematical point of view either the delay model for fixed $\varepsilon$ or its convergence when $\varepsilon$ tends to zero \cite{9,10,11,13}. More specifically in \cite{9} and \cite{10} we studied the adhesion of a single point submitted to an external load and proved convergence. In \cite{11} we proved that a non-linear fully coupled model could either have global solutions or, if the forces were greater than the microscopic adhesions capacity, blow-up could occur. More recently \cite{13}, we extended these results adding space dependent adhesion and diffusion.

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In the previous works, Euler-Lagrange equations were considered and in some cases this is equivalent to the minimization of a convex energy. Here we consider the minimization process for which the energy functional contains adhesion terms and the Dirichlet energy. But it is set pointwisely on the sphere almost everywhere, leading to a non-linear saddle point problem at the Euler-Lagrange level. The mathematical tools previously introduced extend only very partially to this new problem.

Gradient flow techniques provide existence of solutions for complicated possibly non-linear energies complemented with a finite difference term in time. Here the delay term in the energy could be considered as a generalization of such a finite difference. Except that it provides neither existence of discrete solutions nor compactness in time. Thus we are forced to discretize the energy with respect to time and to age. Here the age accounts for delay. First, we obtain new energy estimates similar to the minimization principle in gradient flow theory, but there is then an extra amount of work in order to prove compactness i.e. boundedness of the time derivative in an appropriate space. This estimate is made possible thanks to a closed equation obtained for the discrete time derivative of the position. This equation appears when taking finite differences with respect to time of the Euler-Lagrange equations of the minimization process. In another estimate of the elongation provided extra compactness useful in the asymptotic of the variational formulation that is not at hand here. The reason will be made more precise below. Thus, we were forced to transpose the delay term in the Euler-Lagrange equations on the test function variational formulation that is not at hand here. The reason will be made more precise below. Thus, we were forced to transpose the delay term in the Euler-Lagrange equations on the test function.

To be more specific, we denote by $\Omega := (0, 1)$. The vector position in $\mathbb{R}^d$ of the moving binding site, $z_\varepsilon(x, t)$, minimizes at each time $t \geq 0$ an energy functional:

$$z_\varepsilon(x, t) = \arg \min_{w \in \mathcal{A}} \mathcal{E}_t(w), \quad (1.1)$$

where the minimization is performed on the set

$$\mathcal{A} := \{ w \in H^1(\Omega) \text{ s.t. } |w(x)|^2 = 1, \text{ a.e. } x \in \Omega \}.$$  

The energy is defined for every $w \in \mathcal{A}$ as

$$\mathcal{E}_t(w) := \frac{1}{2\varepsilon} \int_0^t \int_{\mathbb{R}^d} \frac{|w(x) - z_\varepsilon(x, t - \varepsilon a)|^2}{\varepsilon} \rho_\varepsilon(x, t, a) da dx + \frac{1}{2} \int_\Omega |\partial_x w|^2 dx. \quad (1.2)$$

Past positions are given by the function $z_\varepsilon(x, t) = z_\varepsilon(x, t)$ for $t < 0$. The age distribution $\rho_\varepsilon = \rho_\varepsilon(x, a, t)$ is the solution of the structured model:

$$\begin{cases}
\varepsilon \partial_t \rho_\varepsilon + \partial_a \rho_\varepsilon + \zeta \rho_\varepsilon = 0, & x \in \Omega, a > 0, t > 0, \\
\rho_\varepsilon(x, a = 0, t) = \beta_\varepsilon(x, t) (1 - \mu_{0, \varepsilon}(t, x)), & x \in \Omega, a = 0, t > 0, \\
\rho_\varepsilon(x, a, t = 0) = \rho_1(x, a), & x \in \Omega, a > 0, t = 0,
\end{cases} \quad (1.3)$$

where $\mu_{0, \varepsilon}(t, x)$ := $\int_0^\infty \rho_\varepsilon(x, \tilde{a}, t) d\tilde{a}$ and the on-rate of bonds is a given function $\beta_\varepsilon$ times a factor, that takes into account saturation of the moving binding site with linkages. When the off-rate $\zeta_\varepsilon$ is a prescribed function, we say that the problem is weakly coupled: first one exhibits $\rho_\varepsilon$ solving (1.3) which then becomes the weight in (1.2).

First, we discretize in time and age the minimization process (1.1) and the age structured system (1.3). For the transport problem (1.3) we use i) the upwind scheme inside the domain, ii) an implicit
discretization of the off-rates, iii) the non-local term is discretized using a piecewise constant approximation. This step provides, as in the gradient flow case (see for instance minimizing movements chap. 2), existence of a discrete pair of solutions \(((\rho_{\varepsilon}^n,z_{\varepsilon}^n))_{n\in\mathbb{N}}\). Then thanks to compactness arguments, we pass to the limit with respect to the discretization parameter $\Delta a$, and prove that there exists a unique couple $((\rho_{\varepsilon},z_{\varepsilon}))$. The bond population density $\rho_{\varepsilon}$ solves \((1.3)\), whereas $z_{\varepsilon}$ satisfies, almost everywhere in $(0,T)$, the minimization principle \((1.4)\) and solves as well the Euler-Lagrange equation

\[
\begin{align*}
\mathcal{L}_{\varepsilon} - \partial_{xx}z_{\varepsilon} + \lambda_{\varepsilon}z_{\varepsilon} &= 0, \quad \text{a.e. } (x,t) \in \Omega \times (0,T), \\
|z_{\varepsilon}(x,t)| &= 1 \quad \text{a.e.}(x,t) \in \Omega \times (0,T), \\
\partial_t z_{\varepsilon} &= 0 \quad \forall (x,t) \in \{0,1\} \times (0,T), \\
z_{\varepsilon}(x,t) &= z_p(x,t) \quad \text{a.e. } (x,t) \in \Omega \times \mathbb{R}_+,
\end{align*}
\]

(1.4)

where $\mathcal{L}_{\varepsilon}(x,t) := \int_{\mathbb{R}_+} \rho_{\varepsilon}(x,a,t)(z_{\varepsilon}(x,t) - z_{\varepsilon}(x,t - \varepsilon a))/\varepsilon da$ and $\lambda_{\varepsilon}$ is the Lagrange multiplier associated to the constraint $|z_{\varepsilon}(x,t)| = 1$. We prove that when $\varepsilon$ goes to zero, $((\rho_{\varepsilon},z_{\varepsilon}))$ the solutions of the previous minimization problem converge to $(\rho_0,z_0)$. These solve the limit problems reading:

- Find $z_0 \in C^0([0,T];H^1_+(\Omega)) \cap H^1((0,T);L^2(\Omega))$

  $$
  \begin{align*}
  \mu_{1,0}\partial_t z_0 - \partial_{xx}z_0 - |\partial_x z_0|^2 z_0 &= 0, \quad \text{a.e. } (x,t) \in \Omega \times (0,T), \\
  |z_0(x,t)| &= 1 \quad \text{a.e.}(x,t) \in \Omega \times (0,T), \\
  \partial_t z_0 &= 0 \quad \forall (x,t) \in \{0,1\} \times (0,T), \\
  z_0(x,0) &= z_0(x,0) \quad \text{a.e. } (x,t) \in \Omega \times \{0\},
  \end{align*}
  $$

(1.5)

- The function $\mu_{k,0} := \int_{\mathbb{R}_+} a^k \rho_0(x,a,t) da$ represents the moment of order $k$ of $\lim_{\varepsilon \to 0} \rho_{\varepsilon} =: \rho_0$ which solves in $C([0,T];L^1_{\text{a}}(\mathbb{R}_+;L^\infty(\Omega)))$

  $$
  \begin{align*}
  \partial_t \rho_0 + \zeta_0 \rho_0 &= 0, \\
  \rho_0(x,a = 0, t) &= \beta_0(x,t) (1 - \mu_{0,0}(x,t)), \quad x \in \Omega, \ a = 0, \ t > 0.
  \end{align*}
  $$

(1.6)

The article is structured as follows. In Section 2, we list the hypotheses used throughout the paper and set notations. In Section 3, we detail the discrete minimization process in age and time providing the discrete solutions \(((\rho_{\varepsilon}^n,z_{\varepsilon}^n))_{n\in\mathbb{N}}\). In the same section, we provide stability estimates in the appropriate functional spaces. We underline that most of the results obtained therein are uniform with respect to $\varepsilon$ and $\Delta a$, so that the same properties can be extended to the continuous model for fixed $\varepsilon$. This leads to study first this latter limit for $\varepsilon$ fixed and $\Delta a$ going to 0. This is done in Section 4. Then, when $\varepsilon$ tends to zero, we prove, in Section 5 that indeed convergence occurs towards the limit heat harmonic map equation \((1.5)\). As $\partial_t z_{\varepsilon}$ converges weakly in $L^2(\Omega \times (0,T))$ and $\rho_{\varepsilon}$ converges strongly in $L^1(\mathbb{R}_+ \times (0,T);L^\infty(\Omega))$, it is not possible to obtain directly the convergence of the delay term $\mathcal{L}_{\varepsilon}(x,t)$ towards $\mu_{1,0}\partial_t z_0$. Instead, as mentioned above, we transpose the delay operator on a test function and $\rho_{\varepsilon}$, and then we pass to the limit with respect to $\varepsilon$.

2. Notations and hypotheses

We set $Q_T := \mathbb{R}_+ \times (0,T)$ and $\overline{Q}_T := \mathbb{R}_+ \times [0,T]$. The domain $\Omega$ is open, bounded and regular. We set as well $\Omega_T := \Omega \times \mathbb{R}_+ \times (0,T)$.

Assumptions 2.1. The dimensionless parameter $\varepsilon > 0$ is assumed to induce two families of chemical
rate functions that satisfy:

(i) For limit functions \( \beta_0 \in W^{1,\infty}(Q_T) \) and \( \zeta_0 \in W^{1,\infty}(\Omega \times \mathbb{R}_+ \times [0,T]) \) it holds that

\[
\|\zeta_\varepsilon - \zeta_0\|_{L^\infty_{x,a,t}} \to 0 \quad \text{and} \quad \|\beta_\varepsilon - \beta_0\|_{L^\infty_{x,t}} \to 0
\]

as \( \varepsilon \to 0 \).

(ii) We also assume that there are upper and lower bounds such that

\[
0 < \zeta_{\min} \leq \zeta_\varepsilon(x,a,t) \leq \zeta_{\max} \quad \text{and} \quad 0 < \beta_{\min} \leq \beta_\varepsilon(x,t) \leq \beta_{\max}
\]

for all \( \varepsilon > 0, x \in \Omega, a \geq 0 \) and \( t > 0 \).

The initial data for the density model (1.3) satisfies some hypotheses that we sum up here.

Assumptions 2.2. The initial condition \( \rho_I \in L^\infty_{x,a}(\Omega \times \mathbb{R}_+) \) satisfies

- positivity and boundedness: there exists \( M > \beta_{\max} \), s.t.

\[
M \geq \rho_I(x,a) \geq 0, \quad \text{a.e. } (x,a) \in \Omega \times \mathbb{R}_+,
\]

moreover, one has also that the total initial population satisfies

\[
0 < \int_{\mathbb{R}_+} \rho_I(x,a) da < 1
\]

for almost every \( x \in \Omega \).

- boundedness from below of the zero order moment,

\[
0 < \mu_I := \int_{\mathbb{R}_+} \rho_I(x,a) da, \quad \text{for a.e. } x \in \Omega,
\]

- initial integrability with respect to the limit problem:

\[
\int_{\mathbb{R}_+} \sup_{x \in \Omega} \rho_I(x,a)^p da < \infty, \quad \text{for } p \in \{0,1,2\},
\]

- the derivative with respect to age satisfies as well:

\[
\limsup_{\sigma \to 0} \int_{\mathbb{R}_+} \sup_{x \in \Omega} |D_\sigma \rho_I(x,a)| da < \infty.
\]

Concerning the minimization problem (1.1), we assume

Assumptions 2.3. The past data satisfies:

i) for every time \( t \leq 0 \), we assume that \( z_p(\cdot,t) \) is in \( \mathcal{A} \),

ii) there exists a Lipschitz constant which is \( L^2 \) in space s.t.:

\[
|z_p(x,t_2) - z_p(x,t_1)| \leq C_{z_p}(x)|t_2 - t_1|, \quad \forall(t_2,t_1) \in (\mathbb{R}_-)^2
\]

for a.e. \( x \in \Omega \) where \( C_{z_p}(x) \in L^2(\Omega) \).

We define \( X_T := C^0(\Omega_T; L^1(\Omega)) \) to be the Banach space of continuous functions in age and time whose \( L^1 \) norm in space goes to zero when \( a \) goes to infinity. We endow \( X_T \) with the norm:

\[
\|f\|_{X_T} := \sup_{(a,t) \in \Omega_T} \|f(\cdot,a,t)\|_{L^1(\Omega)}.
\]
$X_T$ is a Banach space. It is also a closed subspace of $Z_T := C^0(\partial T; L^1(\Omega))$ which is a non-separable Banach space. We define $Y_T := L^1(\partial T; L^\infty(\Omega))$ which is also a Banach space endowed with the corresponding norm:

$$\|f\|_{Y_T} := \int_{\partial T} \|f(\cdot, a, t)\|_{L^\infty(\Omega)} \, da \, dt.$$  

We denote the discrete differences as

$$D_\tau^T f := f(x, a, t + \tau) - f(x, a, t), \quad D_\alpha^a f := \frac{f(x, a + \alpha, t) - f(x, a, t)}{\alpha},$$

and we define the space of Banach valued functions

$$U_T := \left\{ f \in Y_T \text{ s.t. } \limsup_{\sigma \to 0} \left( \|D_\sigma^T f\|_{Y(T-\sigma)} + \|D_\sigma^a f\|_{Y(T-\sigma)} \right) < \infty \right\}$$

and one endows $U_T$ with the norm:

$$\|f\|_{U_T} := \|f\|_{Y_T} + \limsup_{\sigma \to 0} \left( \|D_\sigma^T f\|_{Y(T-\sigma)} + \|D_\sigma^a f\|_{Y(T-\sigma)} \right).$$

If the same space is set on a time interval $(t_1, t_2)$ then the notation $U_{(t_1, t_2)}$ is well understood. In the rest of the paper we abbreviate the notation of function spaces writing the subscripts $t$ for function spaces on $t \in [0, T]$ and the subscript $x$ for function spaces on $x \in \Omega$, for instance $C_t^1 L^1_a L^\infty_x$ denotes $C([0, T]; L^1(\mathbb{R}_+; L^\infty(\Omega)))$.

### 3. Existence of minimizers and a priori estimates: the discrete scheme

We discretize both (1.3) and the minimization process (1.1) in time and age, but not in space. We set $\Delta a$ a small parameter denoting the age discretization step, while the time step is set to be $\Delta t = \varepsilon \Delta a$. We solve:

- for the $\rho_k$ model, we use a first order upwind scheme and treat the source term implicitly, so we define inside the mesh

$$\varrho_{\varepsilon,i}^{n+1}(x) := \varrho_{\varepsilon,i-1}^n(x) / \left( 1 + \frac{\Delta t}{\varepsilon} \zeta_{\varepsilon,i}^{n+1}(x) \right), \quad i \in \mathbb{N}^*, \quad n \in \mathbb{N}, \quad (3.1)$$

while on the boundary we set

$$\varrho_{\varepsilon,0}^{n+1} := \varrho_{\varepsilon,b}^n / \left( 1 + \frac{\Delta t}{\varepsilon} \zeta_{\varepsilon,0}^{n+1} \right), \quad n \in \mathbb{N}, \quad (3.2)$$

where

$$\varrho_{\varepsilon,b}^n := \rho_{n+1}^{\varepsilon,0}(1 - \mu_{n+1}^{\varepsilon}), \quad \mu_{n+1}^{\varepsilon} := \sum_{i=0}^{\infty} \varrho_{\varepsilon,i}^{n+1} \Delta a.$$  

This definition provides explicitly $\varrho_{\varepsilon,0}^{n+1}$,

$$\varrho_{\varepsilon,0}^{n+1} := \frac{\rho_{n+1}^{\varepsilon,0}}{1 + \Delta a (\rho_{n+1}^{\varepsilon,0} + \zeta_{\varepsilon,0}^{n+1})} \left( 1 - \sum_{i=1}^{\infty} \varrho_{\varepsilon,i}^{n+1} \Delta a \right).$$

The initial condition is defined as

$$\varrho_{\varepsilon,i}^{-1} := \frac{1}{\Delta a} \int_{(i+1)\Delta a}^{(i+1)\Delta a} \rho_I(a) \, da, \quad \forall i \in \mathbb{N}. \quad (3.3)$$
The zero order moment $\mu_{\varepsilon}^{n+1} := \Delta t \sum_{i \in \mathbb{N}} \bar{\varepsilon}_{i,n}^{n+1}$ can be expressed in an inductive way:

$$\mu_{\varepsilon}^{n+1} + \Delta a \sum_{i=0}^{\infty} \frac{\Delta t}{\varepsilon} \bar{\varepsilon}_{i,n}^{n+1} = \mu_{\varepsilon}^{n} + \Delta a \beta_{\varepsilon}^{n+1}(1 - \mu_{\varepsilon}^{n+1}).$$

(3.3)

We define a piecewise constant function

$$\rho_{\varepsilon,\Delta}(x, a, t) := \sum_{i,j \in \mathbb{N}^2} \varrho_{\varepsilon,j}^{n}(x) \chi((i\Delta a, (i+1)\Delta a) \times (j\Delta t, (j+1)\Delta t)) (a, t).$$

• whereas the minimization process is performed for each $n \in \mathbb{N}$

$$Z_i^n := \arg \min_{w \in \mathcal{A}} \mathcal{E}_n (w),$$

(3.4)

where the discrete energy functional reads:

$$\mathcal{E}_n (w) := \frac{1}{2} \int_\Omega |\partial_x w|^2 \, dx + \frac{\Delta a}{4 \varepsilon} \left\{ \int_\Omega (w - Z_i^{n-1})^2 \varrho_{\varepsilon,0} \, dx + \sum_{j=1}^{\infty} \int_\Omega ((w - Z_i^{n-j})^2 + (w - Z_i^{n-j-1})^2) \varrho_{\varepsilon,j} \, dx \right\}$$

for all $n \in \mathbb{N}$ and $Z_i^n := \int_{t \Delta t}^{(i+1)\Delta t} z_p(x, t) \, dt / \Delta t$ for all $i \in \mathbb{Z}, i < 0$, and we set $Z_i^n = Z_i^0$ for every $n < 0$. We define the piecewise constant function

$$z_{\varepsilon,\Delta}(x, t) := \sum_{n=-\infty}^{\infty} Z_i^n(x) \chi_{(n \Delta t, (n+1)\Delta t)}(t).$$

The piecewise linear extension reads:

$$z_{\varepsilon,\Delta} := \sum_{n \in \mathbb{N}} \left( Z_i^n + \left( \frac{t}{\Delta t} - n \right) \delta Z_{i+\frac{1}{2}} \right) \chi_{(n \Delta t, (n+1)\Delta t)}(t),$$

where $\delta Z_{i+\frac{1}{2}} := Z_i^{n+1} - Z_i^n$, while in what follows we denote as well $\delta \varrho_{\varepsilon,i+\frac{1}{2}} := \varrho_{\varepsilon,i+1} - \varrho_{\varepsilon,i}$ an so on.

### 3.1. Positivity and convergence of the discrete solution $\rho_{\varepsilon,\Delta}$

From Lemma 3.1 to Theorem 3.1 we extend results from previous works\[13\] to the discrete case. When needed, we characterize also some properties of $\rho_{\varepsilon}$, the continuous solution of (1.3).

**Lemma 3.1.** For almost every $x \in \Omega$, under the cf.L condition $\Delta t = \varepsilon \Delta a$, and for $\Delta a$ sufficiently small, under hypotheses 2.1 if $\varrho_{\varepsilon}^{-1} \geq 0$ and $\mu_{\varepsilon}^{-1} \leq 1$ then

$$\varrho_{\varepsilon}^{-1} \geq 0, \quad 0 \leq \mu_{\varepsilon} \leq 1, \quad \forall (n, i) \in \mathbb{N}^2.$$

Moreover if there exists a constant $0 < \mu_{0,\min} < \min(\mu_{\varepsilon}^{-1}, \beta_{\min}/(\beta_{\min} + \zeta_{\max})), then $\mu_{\varepsilon}^n > \mu_{0,\min}$ for every $n \in \mathbb{N}$.

**Proof.** The first result is proved by induction : by hypothesis, the claim is true for $k = -1$. We assume that for $k = n$, $\varrho_{\varepsilon,i}^n \geq 0$ and $\mu_{\varepsilon}^n \in [0, 1]$. Since $\varrho_{\varepsilon,i}^n \geq 0$ for $i \in \mathbb{N}$, it is straightforward that
Indeed, as $x$ is strictly positive, so is the left hand side. This shows the statement for $k = 0$ provided that (1 + $\Delta a_0^2 + \beta_0^2$) > 0 which is true for $\Delta a < (\beta_{\min} + \sqrt{\beta_{\min}^2 + \zeta_{\min}})/(2\zeta_{\max}\beta_{\max})$. This in turn proves that $\psi_{x,0}^n > 0$.

We prove the last claim by induction, under the hypothesis on $\mu_{0,\min}$, the claim is true for $k = -1$. We suppose that the claim is true for $k = n$. Using (3.3) gives

$$(\mu_{x,n+1}^n - \mu_{0,\min}^n)(1 + \Delta a(\zeta_{\max} + \beta_{x}^n)) \\ \geq (\mu_{x}^n - \mu_{0,\min})^n + \Delta a_0^2 \beta_0^2 (1 - \mu_{0,\min}^n) - \zeta_{\max} \Delta a_0 \mu_{0,\min} > (\mu_{x}^n - \mu_{0,\min})^n,$$

where the latter inequality holds since $\mu_{0,\min} \leq \beta_{\min}^{n+1}$ and $\beta_{\max} > 1$. Because the right hand side is strictly positive, so is the left hand side. This shows the statement for $k = n + 1$, and the recursion is complete.

Using the same Lyapunov functional $\mathcal{H}[u] := \int_{\mathbb{R}^+} |u(a)| \, da + \int_{\mathbb{R}^+} u(a) \, da$, as in [13], one proves that

**Proposition 3.1.** Under hypotheses [2.1] and [2.2] there exists a unique solution $\rho_x \in Y_T$, solving (1.3). Moreover:

$$\|D_t^r \rho_x\|_{Y_T} \leq C \left( \|D_t^r \rho_t\|_{L^1_\omega L^\infty} + \|\beta_x\|_{W^{1,\infty}} + \|\zeta_x\|_{W^{1,\infty}} + \|\rho_t(\cdot, 0) - \beta_x(\cdot, 0)\|_{L^\infty(\Omega)} \right),$$

where the constant is independent of $\varepsilon$ and on $\sigma$.

**Proof.** For the existence and uniqueness part, one proceeds as in Theorem 3.1 in [13] as $x$ is a mute parameter, for a.e. $x \in \Omega$, there exists a solution $\rho_x(x, \cdot, \cdot) \in C_t([0, T]; L^1_\omega(\mathbb{R}^+))$ Then using Duhamel’s formula in order to commute the supremum with respect to $x$ with the integrals, one obtains the result in $Y_T$. Combining results from the proof of Lemma 5.1, p. 16 [13] and from Theorem 3.2 [12], one gets:

$$\|D_t^r \rho_x\|_{L^\infty_\omega(\Omega; L^1_\omega(Q_T)))} \leq C.$$

Indeed, again, since $x$ is only a mute parameter, one obtains easily that

$$\mathcal{H}[D_t^r \rho_x(x, \cdot, t)] \leq C_1 \exp(-\zeta_{\min} t/\varepsilon) \varepsilon + C_2,$$
which then integrated in time and taking the ess-sup on $\Omega$ proves this first step. Then we use the metod of characteristics and write:

$$
D^\tau_t \rho_c(x,0,t-\varepsilon a) \exp \left( - \int_a^0 \zeta_c(x,a+s,t+\varepsilon s) ds \right) +
$$

$$
\begin{cases}
D^\tau_t \rho_c(x,a,t) := \\
+ \int_{-a}^{0} \exp \left( - \int_{t}^{0} \zeta_c(x,a+s,t+\varepsilon s) ds \right) \mathcal{R}_r(x,a+\tau,t+\varepsilon \tau) d\tau \\
+ \int_{-t/\varepsilon}^{0} \exp \left( - \int_{t}^{0} \zeta_c(x,a+s,t+\varepsilon s) ds \right) \mathcal{R}_r(x,a+\tau,t+\varepsilon \tau) d\tau
\end{cases}
$$

if $t \geq \varepsilon a$,

where $\mathcal{R}_r(x,a,t) := D^\tau_t \zeta_c(x,a,t) \rho_c(x,a,t)$. Now we define $q(a,t) := \esssup_{x \in \Omega} |D^\tau_t \rho_c(x,a,t)|$. One has

$$
\int_{0}^{t} q(a,t) da \leq \int_{0}^{t} q(0,t-\varepsilon a) da + C \int_{0}^{a} \esssup_{x \in \Omega} \int_{0}^{a} \exp(-\zeta_{\min} \tau) \rho_c(x,a+\tau,t-\varepsilon \tau) d\tau da
$$

$$
=: I_1 + I_2
$$

then

$$
I_2 \leq \frac{1}{\varepsilon} \int_{0}^{t} \int_{t-\varepsilon a}^{t} \exp(-\zeta_{\min}(t-\tilde{t})/\varepsilon) \esssup_{x \in \Omega} \rho_c(x,a-\tilde{t} / \varepsilon,\tilde{t}) d\tilde{t} da
$$

$$
= \frac{1}{\varepsilon} \int_{0}^{t} \int_{t-\varepsilon a}^{t} \exp(-\zeta_{\min}(t-\tilde{t})/\varepsilon) \esssup_{x \in \Omega} \rho_c(x,a-\tilde{t} / \varepsilon,\tilde{t}) d\tilde{t} d\tilde{t}
$$

$$
= \frac{1}{\varepsilon} \int_{0}^{t} \exp(-\zeta_{\min}(t-\tilde{t})/\varepsilon) \int_{0}^{\tilde{t}} \esssup_{x \in \Omega} \rho_c(x,a,\tilde{t}) d\tilde{t} < C.
$$

For the term $I_1$, one has $D^\tau_t \rho_c(x,0,t) = (D^\tau_t \beta_\varepsilon)(1-\mu_{0,\varepsilon}) - \beta_\varepsilon D^\tau_t \mu_{0,\varepsilon}$ which gives

$$
|q(0,s)| \leq \beta_{\max} \esssup_{x \in \Omega} \int_{\mathbb{R}_+} |D^\tau_t \rho_c(x,a,t)| da + |D^\tau_t \beta_\varepsilon| \leq \beta_{\max} \exp(-\zeta_{\min} t/\varepsilon) / \varepsilon + C,
$$

so that

$$
I_1 = \frac{1}{\varepsilon} \int_{0}^{t} q(0,s) \exp(-\zeta_{\min}(t-s)/\varepsilon) q(0,s) ds \leq C + t / \varepsilon \exp(-\zeta_{\min} t/\varepsilon).
$$

These two estimates guarantee that $\int_{0}^{T} \int_{0}^{t} q(a,t) dadt < C$. In a similar way one writes that

$$
\int_{0}^{\infty} q(a,t) da \leq \int_{0}^{\infty} \esssup_{x \in \Omega} \left| \rho_c(x,a-t/\varepsilon,\tau) - \rho_c(x,a-t/\varepsilon,0) \right| / \tau \exp(-\zeta_{\min} t/\varepsilon)
$$

$$
+ \int_{0}^{\infty} \int_{t/\varepsilon}^{0} \exp(\zeta_{\min} \tau) \esssup_{x \in \Omega} \rho_c(x,a+\tau,t+\varepsilon \tau) d\tau
$$

$$
\leq \int_{0}^{\infty} \esssup_{x \in \Omega} \left| \rho_c(x,a,\tau) - \rho_c(x,a,0) \right| / \tau \exp(-\zeta_{\min} t/\varepsilon)
$$

$$
+ \int_{0}^{\infty} \exp(-\zeta_{\min} \tau) \int_{\mathbb{R}_+} \esssup_{x \in \Omega} \rho_c(x,a,t-\varepsilon a \tau) d\tau d\tau,
$$
the latter term being under control, we focus on the first one, that we denote $I_3$.

$$I_3 = \exp(-\zeta_{\min} t/\varepsilon) \left\{ \left( \int_0^{\tau/\varepsilon} + \int_{\tau/\varepsilon}^{\infty} \right) \left| \frac{\rho_\varepsilon(x,a,\tau) - \rho_\varepsilon(x,a,0)}{\varepsilon} \right| da \right\} =: I_{3,1} + I_{3,2}.$$  

Since $\rho_\varepsilon$ is bounded uniformly in space and with respect to $\varepsilon$, the first term $I_{3,1}$ is smaller than $\exp(-\zeta_{\min} t/\varepsilon) M/\varepsilon$. Then using the method of characteristics, one splits $I_{3,2}$ in two parts:

$$I_{3,2} \leq \exp(-\zeta_{\min} t/\varepsilon) \left\{ \int_{\tau/\varepsilon}^{\infty} \sup_{x \in \Omega} |\rho_\varepsilon(x,a,\tau) - \rho_\varepsilon(x,a,0)|/\tau da \right\} 
+ \|\zeta_\varepsilon\|_{W^{1,\infty}(Q_T:L^\infty(\Omega))} \int_{\mathbb{R}^+} \sup_{x \in \Omega} \rho_\varepsilon(x,a) dada.$$  

This shows that $\int_0^T \int_{\mathbb{R}^+} q(a,t) dadt < C$ which ends the proof. \qed

Then using standard a priori estimates provides in a similar manner as in the previous proof:

**Proposition 3.2.** Under the previous hypotheses, one has as well that

$$\limsup_{\sigma \to 0} \|D^\sigma \rho_\varepsilon\|_{Y_T} \leq C,$$

where the constant is uniform with respect to $\varepsilon$. This result together with the previous proposition shows that $\rho_\varepsilon \in U_T$ uniformly with respect to $\varepsilon$.

One defines $C^n_j := (j\Delta a, (j+1)\Delta a) \times (n\Delta t, (n+1)\Delta t)$, and one sets

$$\mathcal{T}^n_{\varepsilon,j}(x) := \frac{1}{|C^n_j|} \int_{C^n_j} \rho_\varepsilon(x,a,t) dadt,$$

where $\rho_\varepsilon$ is the exact solution of (1.3) and

$$\mathcal{P}_{\varepsilon,\Delta}(x,a,t) := \sum_{(j,n) \in \mathbb{N}^2} \mathcal{T}^n_{\varepsilon,j}(x) \chi_{(j\Delta a,(j+1)\Delta a) \times (n\Delta t,(n+1)\Delta t)}(a,t).$$

With these notations, we compute error estimates for the upwind scheme:

**Lemma 3.2.** Under the same hypotheses as above, if $\rho_\varepsilon \in U_T$ solves (1.3), and $\mathcal{P}_{\varepsilon,\Delta}$ is its piecewise constant approximation computed using the upwind scheme (3.1) with the non-local boundary term (3.2), then one has

$$\|\rho_\varepsilon - \mathcal{P}_{\varepsilon,\Delta}\|_{Y_T} \leq O(\Delta a), \quad \|\mathcal{P}_{\varepsilon,\Delta} - \rho_\varepsilon\|_{Y_T} \leq O(\Delta a).$$
Proof. Using the method of characteristics one gets:

\begin{align*}
\frac{\partial^{n+1}}{\partial t^{n+1}} \phi^n_{\varepsilon,j} + \Delta a \phi^n_{\varepsilon,j+1} &=: e^{n+1}_j = \\
\frac{1}{C_j} \left( \left( \frac{\phi^n_{\varepsilon,j+1} - \phi^n_{\varepsilon,j}}{\varepsilon} \right) \Delta a + O(\Delta a^2) \right) \rho_{\varepsilon}(x,a,t) dt \\
\frac{1}{C_j} \left( \int_{C_j} \left( \frac{\phi^n_{\varepsilon,j+1} - \phi^n_{\varepsilon,j}}{\varepsilon} \right) \Delta a \exp \left( - \int_{C_j} \phi^n_{\varepsilon,j} (x,a+s,t+\varepsilon \tau) ds \right) \right) - 1 \right) \rho_{\varepsilon}(x,a,t) dt \\
\end{align*}

for all \( j \geq 0 \). In a similar fashion one derives for \( n \geq 1 \)

\[ e^n_0 := \left| (1 + \Delta a \phi^n_{\varepsilon,0}) \phi^n_{\varepsilon,0} - \beta^n_0 (1 - \phi^n_{\varepsilon,0}) \right| \leq \Delta a \| \beta^n \|_{W^{1,\infty}} (1 + \| \phi^n \|_{W^{1,\infty}}) , \]

while if \( n = 0 \),

\[ e^n_0 = \left| \phi^n_{0,0} (1 + \Delta a \phi^n_{0,0}) - \beta^n_0 (1 - \phi^n_{0,0}) \right| \leq \| \rho_{\varepsilon} \|_{L^{\infty}} (1 + \Delta a \phi^n_{\varepsilon,0}) + \beta_{\varepsilon} \leq C_0. \]

Setting \( E^n = \Delta a \sum_{j \in \mathbb{N}} | \phi^n_{\varepsilon,j} - \phi^n_{\varepsilon,j+1} | \), the previous estimates give for \( n \geq 1 \)

\[ E^{n+1} \leq \alpha (E^n + C_2 \Delta a^2) \]

and

\[ E^n \leq \alpha (E^{-1} + C_1 \Delta a + C_2 \Delta a^2) \leq C_3 \Delta a , \]

where \( \alpha := 1/(1 + \Delta a \phi^n_{\varepsilon,0}) \) and by definition \( E^{-1} = 0 \). Combining these estimates leads to

\[ E^{n+1} \leq C \left( \Delta a + \frac{\alpha}{1 - \alpha} \Delta a^2 \right) \leq C \Delta a , \]

which gives the first result. Using similar arguments as in Lemma \text{ Appendix B.2} one can show that

\[ \| \phi_{\varepsilon,\Delta} - \rho_{\varepsilon} \|_{Y_T} \leq \Delta a \| \rho_{\varepsilon} \|_{U_T} , \]

which gives the second result. \( \square \)

**Theorem 3.1.** Under hypotheses \( \text{2.1 and 2.2} \) one has

\[ (1 + \alpha) \rho_{\varepsilon,\Delta} \to (1 + \alpha) \rho_{\varepsilon} \]

strongly in \( Y_T \) when \( \Delta a \) goes to zero for \( \varepsilon \) fixed.

### 3.2. Existence, uniqueness and stability of the discrete solution \( z_{\varepsilon,\Delta} \)

Existence of minimizers relies on the convexity of the Dirichlet norm and is standard as the few properties listed below (see for instance Lemma 1 and 2, p. 973 [13]).

**Theorem 3.2.** Under hypotheses \( \text{2.1 and 2.2} \) for every \( n \geq 0 \) there exists a minimizer \( Z^n_{\varepsilon} \in A \) of \[ \text{3.4} \], i.e. there exists a minimizing subsequence \( (Z^n_{\varepsilon,k})_{k \in \mathbb{N}} \) s.t. as \( k \to \infty \),

1. \( Z^n_{\varepsilon,k} \to Z^n_{\varepsilon} \) weak in \( H^1(\Omega) \),
2. \( Z^n_{\varepsilon,k} \to Z^n_{\varepsilon} \) strong in \( L^2(\Omega) \),
3. \( Z^n_{\varepsilon,k} \to Z^n_{\varepsilon} \) a.e. \( x \in \Omega \),
4. \( Z^n_{\varepsilon} \in A \) and thus \( Z^n_{\varepsilon} \neq 0 \).
A way to insure convergence, when \( \varepsilon \) or \( \Delta t \) go to zero, is to obtain some control on \( \mathbf{z}_{\varepsilon, \Delta t} \), typically an \( \mathbf{L}^2_{x, \varepsilon} \)-bound is obtained in the case of a classical gradient flow directly from the minimization principle. Here the result is less immediate: first, in the next lemma, we obtain a dissipation term in the energy estimates. These estimates provide a uniform bound on the dissipation term. It then appears as a source term in a closed equation (3.8), on \( \delta \mathbf{Z}_{\varepsilon}^{n+1} \) that finally provides these key estimates (cf. Proposition (3.5)).

**Lemma 3.3.** If \((\phi^n_{j,n})_{l(\omega)\in \mathbb{N}^2}\) and \((\mathbf{Z}^n_{j})_{n \in \mathbb{N}}\), are defined as above, one has :

\[
E_{n+1}(\mathbf{Z}^{n+1}_{\varepsilon}) + \sum_{m=1}^{n} \Delta t D_n \leq E_0(\mathbf{Z}^0_{\varepsilon}), \quad \forall n \in \mathbb{N} \tag{3.5}
\]

where the dissipation term reads :

\[
D_n := \frac{\Delta a}{2} \int \sum_{j \in \mathbb{N}} |U^n_{j, \varepsilon}|^2 \varepsilon^{n+1}_{e,j+1} \varepsilon^{n+1}_{e,j+1} dx, \quad U^n_{j, \varepsilon} := \frac{1}{\varepsilon} \left( \mathbf{Z}^n_{\varepsilon} - \frac{|\mathbf{Z}^{n-j}_{\varepsilon} + \mathbf{Z}^{n-j+1}_{\varepsilon}|}{2} \right),
\]

and we denote by \( U^n_{j, \varepsilon} \) the discrete elongation variable for \((j, n) \in \mathbb{N}^2\).

**Proof.** By definition of the minimization process, one has

\[
E_{n+1}(\mathbf{Z}^{n+1}_{\varepsilon}) \leq E_{n+1}(\mathbf{Z}^n_{\varepsilon}),
\]

since \( \mathbf{Z}^{n+1}_{\varepsilon} \) minimises the energy at time step \( t = (n+1)\Delta t \). This reads

\[
E_{n+1}(\mathbf{Z}^{n+1}_{\varepsilon}) \leq \frac{\Delta a}{4\varepsilon} \int \sum_{j=1}^{\infty} (|\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n+1-j}_{\varepsilon}|^2 + |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n+1-j-1}_{\varepsilon}|^2) \varepsilon^{n+1}_{e,j} dx + \frac{1}{2} \int \partial_\varepsilon \mathbf{Z}^n_{\varepsilon} dx + \frac{1}{2} \int \varepsilon^n_{e,j} dx.
\]

Changing the indexes in the first summation of the latter right hand side provides

\[
E_{n+1}(\mathbf{Z}^{n+1}_{\varepsilon}) \leq \frac{\Delta a}{4\varepsilon} \int \sum_{j=1}^{\infty} (|\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-j}_{\varepsilon}|^2 + |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-j-1}_{\varepsilon}|^2) \varepsilon^n_{e,j} + \varepsilon^n_{e,0} |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-1}_{\varepsilon}|^2 dx -
\]

\[
- \frac{\Delta a \Delta t}{4\varepsilon^2} \int \sum_{j=1}^{\infty} \varepsilon^{n+1}_{e,j} (|\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n+1-j}_{\varepsilon}|^2 + |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-j}_{\varepsilon}|^2) dx +
\]

\[
+ \frac{1}{2} \int \partial_\varepsilon \mathbf{Z}^n_{\varepsilon} dx + \frac{1}{2} \int \varepsilon^n_{e,j} dx
\]

\[
\leq E_n(\mathbf{Z}^n_{\varepsilon}) - \frac{\Delta a \Delta t}{2} \int \sum_{j=0}^{\infty} |U^n_{j, \varepsilon}|^2 \varepsilon^{n+1}_{e,j+1} \varepsilon^{n+1}_{e,j+1} dx = E_n(\mathbf{Z}^n_{\varepsilon}) - \Delta t D_n,
\]

for all \( n \in \mathbb{N} \). In the last estimates we used the convexity of the square function, writing

\[
|U^n_{j, \varepsilon}|^2 = \frac{1}{\varepsilon^2} \left| \mathbf{Z}^n_{\varepsilon} - \frac{|\mathbf{Z}^{n-j}_{\varepsilon} + \mathbf{Z}^{n-j+1}_{\varepsilon}|}{2} \right|^2 \leq \frac{1}{2\varepsilon^2} \left( |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-j}_{\varepsilon}|^2 + |\mathbf{Z}^n_{\varepsilon} - \mathbf{Z}^{n-j+1}_{\varepsilon}|^2 \right), \tag{3.6}
\]
where $j \geq 1$, while for $j = 0$, one has simply $(U^n_{\epsilon, j})^2 \leq \left( \delta Z_{\epsilon}^{n-\frac{1}{2}} \right)^2 / (2\epsilon^2)$. For $Z^n_{\epsilon}$, one has simply that
\[
\mathcal{E}_0(Z^n_{\epsilon}) \leq \mathcal{E}_0(Z^{-1}_{\epsilon}) \leq \|Z^{-1}_{\epsilon}\|^2_{H^1(\Omega)} + \frac{\Delta t}{4\epsilon} \int_\Omega \sum_{j=1}^{\infty} g^0_{\epsilon,j} \left( |Z^{-1}_{\epsilon} - Z_{\epsilon}^{-j}|^2 + |Z_{\epsilon}^{-1} - Z_{\epsilon}^{-j-1}|^2 \right) dx.
\]
Using (2.1), one has that for almost every $x \in \Omega$ and $j > 1$
\[
|Z^{-1}_{\epsilon} - Z_{\epsilon}^{-j}| \leq \frac{C_{\epsilon,j}(x)}{\Delta t} \int_0^{\Delta t} |z_p(s) - z_p(s + (1 - j)\Delta t)| dt \leq C_{\epsilon,j}(x)\Delta t(j - 1),
\]
which gives then that
\[
\mathcal{E}(Z^n_{\epsilon}) \leq \epsilon \left( (1 + \epsilon/\delta) \rho_1 \|G_{\epsilon,j}\|_{L^2(\Omega)} + \|Z_{\epsilon}^{-1}\|^2_{H^1(\Omega)} + \|Z^n_{\epsilon}\|^2_{H^1(\Omega)} \right).
\]

For a.e. $x \in \Omega$, we denote by $L^n_{\epsilon}(x) := \Delta \sum_{j \in \mathbb{N}} g^0_{\epsilon,j} U^n_{\epsilon,j}$.

**Lemma 3.4.** For every time $t^n = n\Delta t$, $Z^n_{\epsilon}$ solves:
\[
(L^n_{\epsilon}, v) + \int_\Omega \lambda^n_{\epsilon} Z^n_{\epsilon} \cdot v \, dx + (\partial_x Z^n_{\epsilon}, \partial_x v) = 0,
\]
for all $v \in H^1(\Omega)$, and $\lambda^n_{\epsilon}(x) := -L^n_{\epsilon} \cdot Z^n_{\epsilon} - |\partial_x Z^n_{\epsilon}|^2$, is a $L^1(\Omega)$ function.

**Proof.** We take $v \in H^1(\Omega)$, and set
\[
v(\tau) := \frac{Z^n_{\epsilon} + \tau v}{|Z^n_{\epsilon} + \tau v|},
\]
because $Z^n_{\epsilon} \in \mathcal{A}$ for a $\tau$ small enough $|Z^n_{\epsilon} + \tau v|$ is strictly positive and bounded, thus on this interval $v(\tau) \in \mathcal{A}$. As $Z^n_{\epsilon}$ minimizes $\mathcal{E}_n$, $i(\tau) := \mathcal{E}_n(v(\tau))$ admits a minimum in $\tau = 0$. This leads to $i'(0) = 0$, as $\partial_\tau v(0) = (I - Z^n_{\epsilon} \otimes Z^n_{\epsilon}) v$, this gives
\[
(L^n_{\epsilon}, (I - Z^n_{\epsilon} \otimes Z^n_{\epsilon}) v) + (\partial_x Z^n_{\epsilon}, \partial_x ((I - Z^n_{\epsilon} \otimes Z^n_{\epsilon}) v)) = 0,
\]
where the parentheses denote the $L^2(\Omega)$ scalar product and $I$ the identity matrix in $\mathbb{R}^d$. As $\partial_x Z^n_{\epsilon} \cdot Z^n_{\epsilon} = 0$ for almost every $x \in \Omega$, the previous expression transforms into
\[
(L^n_{\epsilon}, (I - Z^n_{\epsilon} \otimes Z^n_{\epsilon}) v) + (\partial_x Z^n_{\epsilon}, \partial_x v) - \int_\Omega Z^n_{\epsilon} \cdot v |\partial_x Z^n_{\epsilon}|^2 dx = 0,
\]
for all $v \in H^1(\Omega)$. Denoting $\lambda^n_{\epsilon} := -L^n_{\epsilon} \cdot Z^n_{\epsilon} - |\partial_x Z^n_{\epsilon}|^2$, it is a Lagrange multiplier associated to the constraint. Thanks to Theorem 3.2 and Lemma 3.3, $\lambda^n_{\epsilon} \in L^1(\Omega)$. Thus, (3.7) together with the constraint $|Z^n_{\epsilon}| = 1$ is the Euler-Lagrange system associated to the discrete minimization problem (3.4). \hfill \Box

**Proposition 3.3.** Under the previous hypotheses, one has the estimate
\[
\forall n \in \mathbb{N}, \quad \|\lambda^n_{\epsilon}\|_{L^1(\Omega)} \leq C,
\]
where the constant does not depend neither on $\epsilon$ nor on $Z^n_{\epsilon}$.
Proof. As $Z^n_x \in L^\infty_t H^1_x$ uniformly in $\varepsilon$, it is already clear that $|\partial_t Z^n_x|^2$ belongs to $L^\infty_t L^1_x$. It remains to estimate $\|\mathcal{L}^n_x \cdot Z^n_x\|_{L^2(\Omega)}$. Since $Z^{n-j}_x \in A$ for all $j \in \mathbb{N}$ (this statement uses the first assumption in hypotheses 2.3 in the case when $n-j < 0$), a simple computation gives that, for every $x \in \Omega$,  

$$ (Z^n_x - Z^{n-j}_x) \cdot Z^n_x = \frac{1}{2} (Z^n_x - Z^{n-j}_x)^2 \geq 0, \quad \forall j \in \mathbb{N}. $$

This in turn suggests that  

$$ \frac{1}{4\varepsilon} \int_\Omega \left\{ \sum_{j=1}^\infty ((Z^n_x - Z^{n-j}_x)^2 + (Z^n_x - Z^{n-j-1}_x)^2) \psi^n_{\varepsilon,j} + (Z^n_x - Z^{n-1}_x)^2 \psi^n_{\varepsilon,0} \right\} \Delta x dx $$

$$ = \int_\Omega \mathcal{L}^n_x \cdot Z^n_x dx = \int_\Omega |\mathcal{L}^n_x \cdot Z^n_x| dx. $$

By Lemma 3.3, the first term is bounded for any $n \geq 0$. Thanks to the definition of $\lambda^n$, the claim follows. 

Remark 3.1. Proposition 3.3 shows as well that the energy minimization procedure provides a $L^\infty_t L^1_x$ bound, uniform in $\varepsilon$, on the Lagrange multiplier $\lambda_{\varepsilon, \Delta}$. Direct use of the energy estimates from Lemma 3.3 and Jensen’s inequality give $\|\mathcal{L}^n_x\|_{L^2(\Omega)} \lesssim \sqrt{\varepsilon} \|Z^n_x\|_{L^2(\Omega)} \lesssim \varepsilon^{-1/2}$ which provides only  

$$ \|\lambda^n_x\|_{L^1(\Omega)} \leq C \|\mathcal{L}^n_x\|_{L^2(\Omega)} + \|\partial_t Z^n_x\|_{L^2(\Omega)} \leq O(\varepsilon^{-\frac{1}{2}}). $$

Remark 3.2. The previous result shows that the delay operator $\mathcal{L}^n_x$ points out of the unit sphere since by convexity of the square function, $\mathcal{L}^n_x \cdot Z^n_x > 0$ for a.e. $x \in \Omega$. In the next proposition, we show that the scalar product is of order $\varepsilon$ with respect to the $L^1_x$ norm, which makes sense. Indeed, when $\varepsilon$ is small, $\mathcal{L}^n_x$ approximates $\mu_{1,0} \partial_t z_0$ which is tangent to the sphere, and thus orthogonal to $z_0$.

Proposition 3.4. Under hypotheses 2.1, 2.2 and 2.3 one can also show that  

$$ \|\mathcal{L} \cdot z_{\varepsilon, \Delta}\|_{L^1_x} = \Delta t \sum_{n \in \mathbb{N}} \int_\Omega \mathcal{L}^n_x \cdot Z^n_x dx \leq \varepsilon C, $$

where the constant does not depend on $\varepsilon$.

Proof. Using again the same idea as in the previous proof, one writes :  

$$ \Delta t \sum_{n \in \mathbb{N}} \int_\Omega \mathcal{L}^n_x \cdot Z^n_x dx = \Delta t \frac{\varepsilon}{\varepsilon C} \sum_{n \in \mathbb{N}} \int_\Omega \sum_{j=1}^\infty \psi^n_{\varepsilon,j} (|Z^n_x - Z^{n-j}_x|^2 + |Z^n_x - Z^{n-j-1}_x|^2) dx $$

$$ \leq \frac{\varepsilon C}{\varepsilon C}, $$

the latter estimate coming from the dissipation term in the proof of Lemma 3.3. 

Here we show one of the key estimates of the paper.

Proposition 3.5. Under hypotheses above, and for $\Delta t$ small enough, one has :  

$$ \sum_{n=1}^N \Delta t \left\{ \left\| \frac{Z^{n+1}_x - Z^n_x}{\Delta t} \right\|^2_{L^2(\Omega)} + \varepsilon \left\| \frac{\partial_z Z^{n+1}_x - \partial_z Z^n_x}{\Delta t} \right\|^2_{L^2(\Omega)} \right\} \leq C, $$
where the constant does not depend neither on $\varepsilon$ nor on $\Delta t$.

**Proof.** Recalling the definition of $U^n_{\varepsilon,j}$ one checks easily that

$$\varepsilon \delta U^{n+\frac{1}{2}}_{\varepsilon,j} + \Delta t \frac{\delta U^n_{\varepsilon,j}}{\Delta a} = \delta Z^{n+\frac{1}{2}}_{\varepsilon,j} \quad \forall j \geq 1,$$

while $U^n_{\varepsilon,0} = \delta Z^{n-\frac{1}{2}}_{\varepsilon}/(2\varepsilon)$. Equivalently, because of the specific c.f.L condition, $U^{n+1}_{\varepsilon,j+1} = U^n_{\varepsilon,j} + \delta Z^{n+\frac{1}{2}}_{\varepsilon,j}/\varepsilon$ for all $j \geq 1$. Setting $T^n_{\varepsilon,j} = \varrho^n_{\varepsilon,j} U^n_{\varepsilon,j}$ for $j \in \mathbb{N}$, one obtains using (3.1):

$$\varepsilon \delta T^{n+\frac{1}{2}}_{\varepsilon,j} + \Delta t \frac{\delta T^n_{\varepsilon,j}}{\Delta a} + \Delta t \zeta^{n+1}_{\varepsilon,j} T^{n+1}_{\varepsilon,j} = \varrho^n_{\varepsilon,j-1} \delta Z^{n+\frac{1}{2}}_{\varepsilon,j},$$

which, summing over $j \in \mathbb{N}^*$, gives

$$\varepsilon \sum_{j \geq 1} \delta T^{n+\frac{1}{2}}_{\varepsilon,j} \Delta a - \varepsilon \Delta a T^n_{\varepsilon,0} + \Delta t \sum_{j \geq 1} \zeta^{n+1}_{\varepsilon,j} T^{n+1}_{\varepsilon,j} \Delta a = \mu^n_{\varepsilon} \delta Z^{n+\frac{1}{2}}_{\varepsilon,j}.$$

By definition,

$$\varepsilon \Delta a T^{n+1}_{\varepsilon,0} \equiv \varepsilon \Delta a \varrho^{n+1}_{\varepsilon,0} U^{n+1}_{\varepsilon,0} = \varepsilon \Delta a \left( \varrho^{n+1}_{\varepsilon,b} U^{n+1}_{\varepsilon,0} - \Delta a \zeta^{n+1}_{\varepsilon,0} \varrho^{n+1}_{\varepsilon,0} U^{n+1}_{\varepsilon,0} \right)$$

$$= \varepsilon \Delta a \left( \varrho^{n+1}_{\varepsilon,b} \frac{\delta Z^{n+\frac{1}{2}}_{\varepsilon,j}}{2\varepsilon} - \frac{\Delta t}{\varepsilon} \zeta^{n+1}_{\varepsilon,0} T^{n+1}_{\varepsilon,0} \right).$$

Adding both equations gives:

$$\varepsilon \delta L^{n+\frac{1}{2}}_{\varepsilon,j} + \Delta t \sum_{j \in \mathbb{N}} \zeta^{n+1}_{\varepsilon,j} T^{n+1}_{\varepsilon,j} \Delta a = \left( \mu^n_{\varepsilon} + \frac{\Delta a}{2} \varrho^{n+1}_{\varepsilon,b} \right) \delta Z^{n+\frac{1}{2}}_{\varepsilon,j},$$

since $\sum_{j \in \mathbb{N}} T^{n+1}_{\varepsilon,j} \Delta a = \sum_{j \in \mathbb{N}} \varrho^{n+1}_{\varepsilon,j} U^{n+1}_{\varepsilon,0} \Delta a = \mathcal{L}^{n+1}_{\varepsilon,j}$. Now we take the discrete difference of (3.7) taken at time $n+1$ and $n$, in order to express $\delta L^{n+\frac{1}{2}}_{\varepsilon,j}$ as a function of $\delta Z^{n+\frac{1}{2}}_{\varepsilon,j}$. This reads:

$$\left( \delta L^{n+\frac{1}{2}}_{\varepsilon,j}, v \right) + \left( \partial_x \left( \delta Z^{n+\frac{1}{2}}_{\varepsilon,j} \right), \partial_x v \right) + \left( \delta (\lambda_{\varepsilon} Z_j), n+\frac{1}{2}, v \right) = 0$$

We now close the problem solved by $\delta Z^{n+\frac{1}{2}}_{\varepsilon,j}$:

$$\left( \left( \mu^n_{\varepsilon} + \frac{\Delta a}{2} \varrho^{n+1}_{\varepsilon,b} \right) \delta Z^{n+\frac{1}{2}}_{\varepsilon,j}, v \right) + \varepsilon \left( \partial_x \left( \delta Z^{n+\frac{1}{2}}_{\varepsilon,j} \right), \partial_x v \right)$$

$$+ \varepsilon \left( \int_{\Omega} \lambda^{n+1} Z_{\varepsilon}^{n+1} v dx - \int_{\Omega} \lambda^n Z_{\varepsilon}^{n} v dx \right) = \Delta t \left( \sum_{j \in \mathbb{N}} \zeta^{n+1}_{\varepsilon,j} T^{n+1}_{\varepsilon,j} \Delta a, v \right). \quad (3.8)$$

We rewrite the difference

$$J^{n+\frac{1}{2}}(v) := \int_{\Omega} \frac{1}{2} \left( \delta \lambda^{n+\frac{1}{2}} (Z^{n+1}_{\varepsilon} + Z_{\varepsilon}^{n}) + (\lambda^{n+1} + \lambda^n) (\delta Z^{n+\frac{1}{2}}_{\varepsilon,j}) \right) v dx.$$

Applying $J^{n+\frac{1}{2}}$ to $v = \delta Z^{n+\frac{1}{2}}_{\varepsilon,j}$ and using that both $Z^{n+1}_{\varepsilon}$ and $Z_{\varepsilon}^{n}$ satisfy the constraint, reduces to:

$$J^{n+\frac{1}{2}}(\delta Z^{n+\frac{1}{2}}_{\varepsilon,j}) := \frac{1}{2} \int_{\Omega} (\lambda^{n+1} + \lambda^n) \left| \delta Z^{n+\frac{1}{2}}_{\varepsilon,j} \right|^2 dx.$$
cancelling the term containing the finite differences \( \delta \lambda^{n+\frac{1}{2}} \). Next we use the crucial estimates from Proposition 3.3 indeed:

\[
J^{n+\frac{1}{2}}(\delta z^{n+\frac{1}{2}}) \leq \left( \| \lambda^n \|_{L^1(\Omega)} + \| \lambda^{n+1} \|_{L^1(\Omega)} \right) \| \delta z^{n+\frac{1}{2}} \|_{L^\infty(\Omega)}^2.
\]

In one space dimension, the Gagliardo-Nirenberg estimates (cf. [1], p. 140, Theorem 5.9) provide

\[
\| \delta z^{n+\frac{1}{2}} \|_{L^\infty(\Omega)}^2 \leq C \left( \| \delta z^{n+\frac{1}{2}} \|_{H^1(\Omega)}^2 + \| \partial_x \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 \right) \leq C \left( \epsilon^{-\frac{1}{2}} \| \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 + \epsilon \| \partial_x \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 \right).
\]

Thus setting \( v = \delta z^{n+\frac{1}{2}} \) in the weak formulation above gives finally:

\[
(\mu_{0, \min} - 2C\sqrt{\epsilon}) \| \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 + (\epsilon - C\epsilon^2) \| \partial_x \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 \leq \Delta t \sum_{j \in \mathbb{N}} t^{n+1} \Delta a |_{L^2(\Omega)} \| \partial_x \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2.
\]

Using Young’s inequality on the right hand side above, for \( \epsilon \) small enough, one has:

\[
\frac{1}{\Delta t} \sum_{n=0}^N \left\| \delta z^{n+\frac{1}{2}} \right\|_{L^2(\Omega)}^2 \leq \sum_{n=0}^N \Delta t \sum_{j \in \mathbb{N}} t^{n+1} \Delta a |_{L^2(\Omega)} \| \partial_x \delta z^{n+\frac{1}{2}} \|_{L^2(\Omega)}^2 \leq \Delta t \sum_{n=0}^N \int_{\Omega} \sum_{j \in \mathbb{N}} \theta_{x,j}^n (U_{x,j}^n)^2 \Delta a = \Delta t \sum_{n=0}^N \mathcal{D}_n \leq C.
\]

The previous argument provides uniqueness as well:

**Proposition 3.6.** Under hypotheses 2.1, 2.2 and 2.3 there exists a unique solution \( z_{\epsilon, \Delta} \in L^\infty((0, T); H^1(\Omega)) \cap H^1((0, T); L^2(\Omega)) \) solving (3.7).

**Proof.** We use induction arguments to show the claim. We suppose that there exists two solutions \( z_{\epsilon, \Delta, i} \) for \( i \in \{1, 2\} \). We denote by \( \delta z^k := z_{\epsilon, \Delta, 2}^k - z_{\epsilon, \Delta, 1}^k \), and we write the equation it satisfies for \( k = 0 \):

\[
\left( \sum_{j=1}^\infty \theta_{x,j}^0 + \theta_{x,0}^0 / 2 \right) \Delta a \delta z^0, v \right) + \epsilon (\partial_x \delta z^0, v) + \epsilon (\delta \lambda^0 z_{\epsilon, \Delta, 2}^0 + \lambda_{\epsilon, \Delta, 1}^0 \delta z^0, v) = 0.
\]

Thus choosing \( v = \delta z^0 \) and using the same arguments as above implies that \( \delta z^0 = 0 \). We suppose at this point that \( \delta z^k = 0 \) for \( k \leq n \). Then a careful decomposition of \( L_{\epsilon, \Delta, 2}^n - L_{\epsilon, \Delta, 1}^n \) leads to

\[
\left( \sum_{j=1}^\infty \theta_{x,j}^{n+1} + \theta_{x,0}^{n+1} / 2 \right) \Delta a \delta z^{n+1}, v \right) + \epsilon (\partial_x \delta z^{n+1}, v) + \epsilon (\delta \lambda^{n+1} z_{\epsilon, \Delta, 2}^n + \lambda_{\epsilon, \Delta, 1}^{n+1} \delta z^{n+1}, v) = 0,
\]

and then concludes.

\[ \square \]
Lemma 3.3, this leads to:

\[
\|\delta z^{n+1}\|_{L^2(\Omega)}^2 + \varepsilon (1 - c_2 \sqrt{\varepsilon}) \|\partial_t \delta z^{n+1}\|_{L^2(\Omega)}^2 \leq 0,
\]

proving the claim for \(\varepsilon\) small enough and \(k = n + 1\). This ends the proof since \(z_{\varepsilon,\Delta,2} = z_{\varepsilon,\Delta,1}\). \(\square\)

**Proposition 3.7.** Under hypotheses 2.1, 2.2 and 2.3, \(\tilde{z}_{\varepsilon,\Delta}\), the piecewise linear interpolation of \(Z^n_{\varepsilon}\) satisfies

\[
\tilde{z}_{\varepsilon,\Delta} \in C^{0,\gamma}(\Omega) \quad \text{for every } \gamma \in (0, 1),
\]

the bound is uniform with respect to \(\Delta t\) and \(\varepsilon\). Thus \(\tilde{z}_{\varepsilon,\Delta}\) converges strongly in \(C^0(\Omega \times [0, T])\) when \(\Delta t\) goes to zero. Moreover, \(z_{\varepsilon,\Delta}\) converges strongly in \(L^\infty((0, T); C(\Omega))\).

**Proof.** Thanks to Lemma 3.3, \(\tilde{z}_{\varepsilon,\Delta}\) belongs to \(L^\infty H^1_{\Delta}\) uniformly with respect to \(\varepsilon\), which shows weak-* convergence in this space. Weak convergence in \(H^1_{\Delta} L^2\) follows from Proposition 3.5. The interpolation inequality

\[
\|u\|_{C^{0,\gamma}(\Omega)} \leq c \|u\|_{H^{1/2}(\Omega)} \leq c \|u\|_{L^2(\Omega)}^{(1-\gamma)/2} \|u\|_{H^1(\Omega)}^{(1+\gamma)/2}
\]

holds for every \(u \in H^1(\Omega)\) and for every \(\gamma \in (0, 1)\). Combined with the \(L^\infty H^1_{\Delta}\) bound provided by Lemma 3.3, this leads to:

\[
\|\tilde{z}_{\varepsilon,\Delta}(t_2) - \tilde{z}_{\varepsilon,\Delta}(t_1)\|_{C^{0,\gamma}(\Omega)} \leq c |t_2 - t_1|^{(1-\gamma)/2}.
\]

We complete the convergence proof for \(\tilde{z}_{\varepsilon,\Delta}\) by an application of the Ascoli-Arzela theorem. \(\square\)

**Corollary 3.1.** Under the previous hypotheses, the same result can be derived for \(z_{\varepsilon} := \lim_{\Delta t \to 0} z_{\varepsilon,\Delta}\), i.e.

\[
z_{\varepsilon} \in C^{0,\gamma}(\Omega) \quad \text{for every } \gamma \in (0, 1),
\]

the bound is uniform with respect to \(\varepsilon\). This implies that \(z_{\varepsilon}\) converges to \(z_0\) strongly in \(C^0(\Omega \times [0, T])\) when \(\varepsilon\) goes to zero.

**Proof.** Considering \(\tilde{z}_{\varepsilon,\Delta}\), the piecewise continuous function in time, \(\partial_t \tilde{z}_{\varepsilon,\Delta}\) is bounded in \(L^2_{\Delta, t}\) uniformly with respect to \(\varepsilon\), thus \(\partial_t \tilde{z}_{\varepsilon,\Delta} \rightharpoonup \partial_t z_{\varepsilon}\) weakly in \(L^2_{\Delta, t}\) and one has that

\[
\|\partial_t z_{\varepsilon}\|_{L^2_{\Delta, t}} \leq \liminf_{\Delta t \to 0} \|\partial_t \tilde{z}_{\varepsilon,\Delta}\|_{L^2_{\Delta, t}} = \liminf_{\Delta t \to 0} \left(\Delta t \sum_{n\in\mathbb{N}} \|\delta Z^{n+1/2}_{\varepsilon}/\Delta t\|_{L^2(\Omega)}^2 \right)^{1/2}.
\]

A similar argument provides an \(L^\infty H^1_{\Delta}\) bound for \(z_{\varepsilon}\). One can then follow again the same steps as in the proof of Proposition 3.7. \(\square\)

4. Convergence when \(\varepsilon\) is fixed and \(\Delta a\) goes to 0.

Next, we consider the convergence of \(L_{\varepsilon,\Delta}(x, t) := \sum_{n=0}^N \chi_{(n,n+1)}(t) L^n_{\varepsilon}(x)\).

**Proposition 4.1.** Under hypotheses 2.1, 2.2 and 2.3, for every fixed \(\varepsilon > 0\), the discrete delay term converges to the continuous limit when \(\Delta a\) goes to zero, i.e.

\[
\int_0^T \int_{\Omega} L_{\varepsilon,\Delta}(x, t, \varphi)dxdt \to \int_0^T \int_{\Omega} L_{\varepsilon}(x, t, \varphi)dxdt
\]
for all \( \varphi \in C^0([0,T]; L^2(\Omega)) \) and \( \varphi_{\Delta}(x,t) := \sum_{n=0}^{N} x_{(n,n+1)\Delta t}(t)\varphi^n(x) \) where \( \varphi^n(x) := \int_{n\Delta t}^{(n+1)\Delta t} \varphi(x,t)dt/\Delta t \).

**Proof.** In what follows the terms that we handle are integrable on the domain \( \Omega \times \mathbb{R}_+ \times (0,T) \) so the systematic use of Fubini’s Theorem is implicitly assumed and we freely commute integrals with respect to space, age and time. We set \( I_\Delta := \int_{\Omega} \int_0^T \mathbf{L}_{\varepsilon}(x,t) \varphi_{\Delta}(x,t)dt dx \) that we split in two parts:

\[
I_\Delta := \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}_+} \rho_{\varepsilon,\Delta}(x,a,t) z_{\varepsilon,\Delta}(x,t) \cdot \varphi_{\Delta}(x,t) dt dx
\]

\[
- \frac{1}{\varepsilon} \int_{\Omega} \int_{\mathbb{R}_+} \Delta a \sum_{n=0}^{N} \sum_{j=0}^{(n+1)\Delta t} \left( \frac{Z^{n,j} + Z^{n,j-1}}{2} \right) \cdot \varphi^n_{\varepsilon,j} dt dx = \frac{1}{\varepsilon} (I_{1,\Delta} - I_{2,\Delta}).
\]

By Lemma 3.1 \( \mu_{0,\varepsilon,\Delta} \) is uniformly bounded with respect to \( \varepsilon, \Delta t \) and \( \Delta a, \mu_{0,\varepsilon,\Delta} \rightharpoonup \mu_{0,\varepsilon} \) in the weak-* topology in \( L^\infty((0,T) \times \Omega) \). Moreover since \( L^2((0,T); L^2(\Omega)) \) is a separable space, the step functions in time with values in \( L^2(\Omega) \) are dense. Thus \( \varphi_{\Delta} \) tends to \( \varphi \) strongly in \( L^2(\Omega) \) and the product \( z_{\varepsilon,\Delta} \varphi_{\Delta} \) converges strongly in \( L^1(\Omega \times (0,T)) \). All this gives:

\[
I_{1,\Delta} := \int_0^T \int_{\Omega} \mu_{0,\varepsilon,\Delta} z_{\varepsilon,\Delta} \cdot \varphi_{\Delta} dx dt \rightarrow \int_0^T \int_{\Omega} \mu_{0,\varepsilon} z_{\varepsilon} \cdot \varphi dx dt.
\]

For the second term, one first defines:

\[
C^n_j := \{(a,t) \in C^n_j \text{ s.t. } t > \varepsilon a + (n-j)\Delta t\}, \quad C^n_j := \{(a,t) \in C^n_j \text{ s.t. } t < \varepsilon a + (n-j)\Delta t\},
\]

and then one has:

\[
I_{2,\Delta} = \int_{\Omega} \sum_{n=0}^{N} \sum_{j=0}^{\infty} \int_{C^n_j} Z^{n,j} \cdot \varphi^n_{\varepsilon,j} da dt + \int_{C^n_j} Z^{n,j-1} \cdot \varphi^n_{\varepsilon,j} da dt
\]

\[
= \int_{\Omega} \sum_{n=0}^{N} \sum_{j=0}^{\infty} \int_{C^n_j} \varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta}(x,t) \rho_{\varepsilon,\Delta}(x,a,t) da dt dx
\]

\[
= \int_{\Omega} \int_{\mathbb{R}_+ \times (0,T)} \varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta}(x,t) \rho_{\varepsilon,\Delta}(x,a,t) da dx dt.
\]

We consider the convergence of the term \( \varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta}(x,t) \) on \( \Omega \times \{(a,t) \in \mathbb{R}_+ \times (0,T) \text{ s.t. } t > \varepsilon a\} \):

\[
\left| \int_{0}^{T} \int_{\Omega} \int_{0}^{T} (\varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta}(x,t) - \varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta}(x,t)) \rho_{\varepsilon,\Delta} da dx dt \right| \leq \frac{T}{\varepsilon} \left( \|\varphi_{\varepsilon,\Delta}\|_{L^\infty} \|Z_{\varepsilon,\Delta} - \varphi_{\varepsilon,\Delta}\|_{L^2_{a,t}} + \|\varphi_{\varepsilon,\Delta}\|_{L^\infty} \|\varphi_{\varepsilon,\Delta} - \varphi\|_{L^2_{a,t}} \right) \sim o_T(1)
\]

and thus in a similar manner as for \( I_{1,\Delta} \) one proves the convergence of \( I_{2,1,\Delta} := \int_0^T \int_{\Omega} \int_{0}^{T} \rho_{\varepsilon,\Delta}(x,a,t) \varphi_{\varepsilon,\Delta}(x,t-\varepsilon a) \cdot \varphi_{\varepsilon,\Delta} da dx dt \). On the other hand, on \( \Omega \times \{(a,t) \in \mathbb{R}_+ \times (0,T) \text{ s.t. } t < \varepsilon a\} \), one has
that:

\[ I_{2,2,\Delta} := \int_0^T \int_\Omega \int_0^\infty \rho_{\varepsilon,\Delta}(x,a,t) (z_{\varepsilon,\Delta}(x,t) - z_{\varepsilon}(x,t - \varepsilon a)) \cdot \varphi_{\Delta}(x,t) dx dt \]

\[ = \frac{1}{\varepsilon} \int_0^T \int_\Omega \int_{R^+} \rho_{\varepsilon,\Delta} \left( x, \frac{t - \bar{a}}{\varepsilon}, t \right) (z_{\varepsilon,\Delta}(x,\bar{a}) - z_{\varepsilon}(x,\bar{a})) \cdot \varphi_{\Delta}(x,t) dx dt \]

\[ = \frac{1}{\varepsilon} \int_0^T \sum_{n=0}^{(n+1)\Delta t} \operatorname{ess sup}_{x \in \Omega} \rho_{\varepsilon,\Delta} \left( \cdot, \frac{t - \bar{a}}{\varepsilon}, t \right) \| z_{\varepsilon}^{n+1} - z_{\varepsilon}(\cdot,\bar{a}) \|_{L^2} \| \varphi_{\Delta}(\cdot,t) \|_{L^2} d \bar{a} dt. \]

A simple computation shows that if \( \bar{a} \in (n, n + 1)\Delta t \) then

\[ \| z_{\varepsilon}^{n+1} - z_{\varepsilon}(\cdot,\bar{a}) \|_{L^2} \leq \Delta t \| \partial_t z_{\varepsilon} \|_{L^2}, \]

which gives then that

\[ |I_{2,2,\Delta}| \leq \Delta t \| \partial_t z_{\varepsilon} \|_{L^\infty(L^2(\Omega))} \| \varphi_{\Delta} \|_{L^\infty(L^2)} \int_{R^+ \times (0,T)} \operatorname{ess sup}_{x \in \Omega} \rho_{\varepsilon,\Delta}(x,a,t) d\alpha dtdt \]

\[ \leq C \Delta t \| \partial_t z_{\varepsilon} \|_{L^\infty(L^2(\Omega))} \| \varphi_{\Delta} \|_{L^\infty(L^2)} T. \]

In a similar way, one proves thanks to hypotheses \([2,3]\) that

\[ \left| \int_0^T \int_\Omega \int_0^\infty (\rho_{\varepsilon,\Delta}(x,a,t) - \rho_{\varepsilon}(x,a,t)) z_{\varepsilon}(x,t - \varepsilon a) \cdot \varphi_{\Delta}(x,t) dx dt \right| \]

\[ \leq \| (\rho_{\varepsilon,\Delta} - \rho_{\varepsilon})(1 + a) \|_{L^1(0,T)} \left( \| z_{\varepsilon}(\cdot,0) \|_{L^2} + \| C_{\varepsilon} \|_{L^2} \right) \| \varphi_{\Delta} \|_{L^\infty(L^2)} \sim o_{\Delta}(1). \]

For the last part, on \( \Omega \times \{(a,t) \in R^+ \times (0,T) \; s.t. \; t < \varepsilon a \} \), one has that:

\[ J_{2,3,\Delta} := \int_0^T \int_{R^+} \int_0^\infty \rho_{\varepsilon}(x,a,t) z_{\varepsilon}(x,t - \varepsilon a) \cdot (\varphi_{\Delta}(x,t) - \varphi(x,t)) dx dt \]

\[ \leq \sup_{t \in (0,T)} \int_{R^+} (1 + a) \operatorname{ess sup}_{x \in \Omega} \rho_{\varepsilon}(x,a,t) \left( \| z_{\varepsilon}(\cdot,0) \|_{L^2} + \| C_{\varepsilon} \|_{L^2} \right) \| \varphi_{\Delta} - \varphi \|_{L^1(L^2)} \]

\[ \sim o_{\Delta}(1), \]

which proves that

\[ \int_0^T \int_{R^+} \rho_{\varepsilon,\Delta}(x,a,t) z_{\varepsilon,\Delta}(t - \varepsilon a) \cdot \varphi_{\Delta}(x,t) - \rho_{\varepsilon}(x,a,t) z_{\varepsilon}(t - \varepsilon a) \cdot \varphi(x,t) dx da dt \to 0 \]

and ends the proof. \( \square \)

**Theorem 4.1.** Under hypotheses above, for almost every \( t \in (0,T) \), there exists a unique \( z_{\varepsilon} \in H^1((0,T);L^2(\Omega)) \cap L^\infty((0,T);H^1(\Omega)) \) solving

\[ (L_{\varepsilon}(\cdot,t),v) + (\partial_z z_{\varepsilon}(\cdot,t),\partial_z v) + \int_\Omega \lambda_{\varepsilon}(x,t) z_{\varepsilon}(x,t) \cdot v(x) dx = 0, \; \forall v \in H^1(\Omega), \quad (4.1) \]

where the brackets denote the \( L^2(\Omega) \) scalar product and the Lagrange multiplier \( \lambda_{\varepsilon} = -L_{\varepsilon} z_{\varepsilon} - |\partial_z z_{\varepsilon}|^2 \) is an \( L^\infty((0,T);L^2(\Omega)) \) function uniformly with respect to \( \varepsilon \). Moreover, everywhere in \( \overline{\Omega} \times [0,T], |z_{\varepsilon}(x,t)| = 1 \).
Proof. By Proposition 4.1, the convergence of $\mathcal{L}_{\varepsilon, \Delta}$ is proved. Since $\partial_x z_{\varepsilon, \Delta} \in L^\infty_t L^2_x$ uniformly with respect to $\varepsilon, \Delta \alpha$ and $\Delta t$, one has

$$
\int_0^T (\partial_x z_{\varepsilon, \Delta}, \partial_x \varphi) dt \to \int_0^T (\partial_x z_{\varepsilon}, \partial_x \varphi) dt,
$$

where again $\varphi \in C^0([0, T]; H^1(\Omega))$ and $\varphi_{\Delta}(x, t) := \sum_{n=0}^N \chi_{(n, n+1)\Delta t}(t) \varphi^n(x)$ where $\varphi^n(x) := \int_{n \Delta t}^{(n+1)\Delta t} \varphi(x, t)dt/\Delta t$. On the other hand,

$$
J_{\varepsilon} := \left| \int_0^T \int_\Omega \lambda_{\varepsilon, \Delta} \varphi_{\Delta}(x, t) dx dt - \int_0^T \int_\Omega z_{\varepsilon}(x, t) \varphi(x, t) \lambda_{\varepsilon} dx dt \right|
\leq \int_0^T \int_\Omega |\lambda_{\varepsilon, \Delta} \varphi_{\Delta} - \varphi| dx dt + \int_0^T \int_\Omega |\lambda_{\varepsilon, \Delta} \varphi_{\Delta} - \varphi_{\Delta} + \varphi_{\Delta} - z_{\varepsilon}| dx dt
+ \int_0^T \int_\Omega |\lambda_{\varepsilon, \Delta} z_{\varepsilon} - \varphi| dx dt
\leq \|\lambda_{\varepsilon, \Delta}\|_{L^\infty_t L^1_x} \|\varphi_{\Delta} - \varphi\|_{L^1_t L^\infty_x} + \|\lambda_{\varepsilon, \Delta}\|_{L^\infty_t L^1_x} \|\varphi_{\Delta}\|_{L^\infty_t L^\infty_x} + \|\lambda_{\varepsilon, \Delta} z_{\varepsilon} - \varphi\|_{L^\infty_t L^1_x}
+ \int_0^T \int_\Omega \lambda_{\varepsilon, \Delta} z_{\varepsilon} \varphi dx dt - < \lambda_{\varepsilon, \Delta} z_{\varepsilon}, \varphi >
\right|

The first term tends to zero thanks to the density of weakly subsequences in $L^1_t L^2_x$, the second term is small due to the strong convergence of $\lambda_{\varepsilon, \Delta}$ established above, the last one tends to zero thanks to the weak-* convergence of $\lambda_{\varepsilon, \Delta}$ in $L^\infty_t L^1_x$. At that point, the solution pair $(z_{\varepsilon}, \lambda_{\varepsilon})$ solves:

$$
\int_0^T \int_\Omega \mathcal{L}_{\varepsilon} \cdot \varphi + \partial_x z_{\varepsilon} \cdot \partial_x \varphi dx dt + < z_{\varepsilon}, \varphi_{\varepsilon}, \lambda_{\varepsilon} \rangle = 0,
$$

(4.2)

where the last brackets denote the duality bracket $(L^\infty_t L^1_x, L^1_t L^\infty_x)$. Setting $\varphi(x, t) = z_{\varepsilon}(x, t) \theta(x, t)$ with $\theta \in \mathcal{D}(\Omega \times (0, T))$, in (4.2), proves that for almost every $(x, t) \in \Omega \times (0, T)$, $\lambda_{\varepsilon} = -\mathcal{L}_{\varepsilon} \cdot z_{\varepsilon} = -|\partial_x z_{\varepsilon}|^2 < 0$ and the right hand side is a $L^\infty_t L^1_x$ function.

Taking now $\varphi(x, t) = v(x) \psi(t)$ for any $v \in H^1(\Omega)$ and $\psi \in \mathcal{D}(0, T)$ shows that (4.1) holds a.e. $t \in (0, T)$ for any $v \in H^1(\Omega)$.

An easy computation shows that

$$
0 \leq 1 - |\tilde{z}_{\varepsilon, \Delta}| \leq \sum_{n \in N} \chi_{(n, n+1)\Delta t} \frac{(t - n\Delta t)}{\Delta t} \|\delta Z_{\varepsilon}^{n+\frac{1}{2}}\|,\n$$

which gives that

$$
\|1 - |\tilde{z}_{\varepsilon, \Delta}|\|_{L^2_t}^2 \leq \Delta t \sum_{n=0}^N \frac{1}{\Delta t} \|\delta Z_{\varepsilon}^{n+\frac{1}{2}}\|_{L^2_x}^2 \leq \Delta t
$$

thanks to Proposition 3.5. Then a triangular inequality gives:

$$
\|z_{\varepsilon}\|_{L^2_t} \leq \|z_{\varepsilon} - \tilde{z}_{\varepsilon, \Delta}\|_{L^2_t} + \|1 - |\tilde{z}_{\varepsilon, \Delta}|\|_{L^2_t} \leq \|z_{\varepsilon} - \tilde{z}_{\varepsilon, \Delta}\|_{L^2_t} + \|1 - |\tilde{z}_{\varepsilon, \Delta}|\|_{L^2_t}.
$$

As the right hand side is arbitrary small, the left hand side is zero. Thus the constraint is fulfilled a.e. in $\Omega \times (0, T)$. Since $z_{\varepsilon}$ is a continuous function in time and in space, the result holds true everywhere.
Proposition 4.2. Under the previous hypotheses, one has
\[ \| \mathcal{L}_\epsilon \cdot z_\epsilon \|_{L^1_{x,t}} \leq C \epsilon, \]
where the constant is independent of \( \epsilon \).

**Proof.** Using the same arguments as in Proposition 4.1, one shows that \( \mathcal{L}_\epsilon \cdot z_\epsilon, \Delta \) tends to \( \mathcal{L}_\epsilon \cdot z_\epsilon \) in \( L^1_{x,t} \) as \( \Delta \to 0 \). Then using the estimate established in Proposition 3.4, one concludes. \( \square \)

5. Convergence when \( \epsilon \) goes to zero in the continuous framework

In \( \Omega \), we derived, uniformly with respect to \( \epsilon \), \( L^\infty_t L^1_x \) estimates for \( \partial_t z_\epsilon \). Here we were not able to obtain this uniformity with respect to \( \epsilon \), and numerical simulations showed that these estimates do not hold true here. Thus the rest of the paper deals with the asymptotic when \( \epsilon \) goes to zero when only \( L^1_{x,t} \) compactness for \( z_\epsilon \) is available.

Proposition 5.1. If \( f \) is in \( U_T \) then its weak derivatives \( \partial_a f \) and \( \partial_t f \) are in \( X'_T \). One defines the corresponding duality brackets as
\[ \langle \partial_a f, \varphi \rangle_{X'_T X_T} := \lim_{\sigma \to 0} \int_{\Omega_T} D^\sigma_a f \varphi dx da dt \]
for any \( \varphi \in X_T \).

Proposition 5.2. If \( f \) is in \( U_T \) then its weak derivatives \( \partial_a f \) and \( \partial_t f \) are in \( Z'_T \). One defines the corresponding duality brackets as
\[ \langle \partial_a f, \varphi \rangle_{Z'_T Z_T} := \lim_{\sigma \to 0} \int_{\Omega_T} D^\sigma_a f \varphi dx da dt \]
for any \( \varphi \in Z_T \).

For sake of conciseness, the proofs of these propositions are postponed in Appendix B. Using then these one shows :

Proposition 5.3. Under hypotheses 2.1 and 2.2, the previous convergence result can be extended to \( Z'_T \) where \( Z_T := C_b(Q_T; L^1(\Omega)) \). Namely for any \( \varphi \) in \( Z_T \), there exists a subsequence \( \tau_k \) s.t.
\[ \langle D^\tau_{t_k} \rho_\epsilon, \varphi \rangle \to \langle \partial_t \rho_\epsilon, \varphi \rangle \]
when \( k \to \infty \).

In order to identify the limit to which \( \partial_t \rho_\epsilon \) tends when \( \epsilon \) goes to zero, (part of the main ingredients were presented in Proposition 3.2 p.10, \( \Omega \), but the space variable was not taken in account), we define an initial layer, as in \( \Omega \). Setting \( \tilde{t} = t/\epsilon \), we look for \( \tilde{\rho}_0 \) solution of
\[
\begin{cases}
\partial_t \tilde{\rho}_0 + \partial_a \tilde{\rho}_0 + \zeta_0(x, a, 0) \tilde{\rho}_0 = 0, & (x, a, \tilde{t}) \in \Omega \times (\mathbb{R}^+)^2, \\
\tilde{\rho}_0(x, 0, \tilde{t}) = -\beta_0(x, 0) \int_{\mathbb{R}^+} \tilde{\rho}_0(x, a, \tilde{t}) da, & x \in \Omega, a = 0, \tilde{t} > 0, \\
\tilde{\rho}_0(x, a, 0) = \rho_I(x, a) - \rho_0(x, a, 0) =: \tilde{\rho}_I(x, a), & x \in \Omega, a > 0, \tilde{t} = 0
\end{cases}
\]
and we define \( \rho_{0,\epsilon}(x, a, t) := \tilde{\rho}_0(x, a, t/\epsilon) \). As in \( \Omega \), we obtain at the microscopic level global existence and a priori bounds :
Theorem 5.1. Under hypotheses 2.1 and 2.2, there exists a unique solution $\tilde{\rho}_0$ belonging to $C^0(\mathbb{R}^+; L^1_a(\mathbb{R}^+; L^\infty(\Omega))) \cap L^\infty(\Omega \times \mathbb{R}^+ \times \mathbb{R}^+)$. Moreover, one has

$$\int_{\mathbb{R}^+} \left( \sup_{x \in \Omega} |\tilde{\rho}_0(x, a, \ell)| da \right) \leq \exp(-c_{\text{min}} \ell), \quad \forall \ell \in \mathbb{R}^+,$$

and there exists a subsequence s.t. $D^T_{\ell} \tilde{\rho}_0 \rightharpoonup \partial_\sigma \tilde{\rho}_0$ weak-$*$ in the $\sigma(Z'_\infty, Z_\infty)$ topology.

Proof. The proof of the existence and uniqueness part is easy and follows the same ideas as in Theorem 2.2 where one shall only manage the $x$ dependence in addition. A priori estimates on $D^T_{\ell} \tilde{\rho}_0$ are obtained as in Theorem 2.2 p. 6 The last part follows the same ideas as in Propositions 5.1 and 5.2 below.

Corollary 5.1. Under the same hypotheses, one has the scaling

$$\langle \partial_\ell \tilde{\rho}_0, \varphi \rangle_{Z'_\tau, Z_\tau} = \langle \partial_\ell \tilde{\rho}_0, \varphi(\cdot, \cdot, \varepsilon) \rangle_{Z'_\tau, Z_\tau}. $$

Proof. We start from the change of variable $\ell = t/\varepsilon$, which gives

$$\int_{\Omega \times \mathbb{R}^+ \times (0, T)} D^T_{\ell} \tilde{\rho}_0(x, a, t) dx da dt = \int_{\Omega \times \mathbb{R}^+ \times (0, T/\varepsilon)} D^T_{\ell} \tilde{\rho}_0(x, a, \ell) \varphi(x, a, \ell) dx da d\ell,$$

where $\tilde{\tau} = \tau/\varepsilon$, then the right hand side (resp. left hand side) converges up to a subsequence to the right hand side (resp. left hand side) of the claim by the same arguments as in Propositions 5.1 and 5.2.

Theorem 5.2. Under hypotheses 2.1 and 2.2 one has for any $\varphi \in C_b(\mathbb{R}^+; L^1(\Omega))$,

$$\lim_{\varepsilon \to 0} \langle \partial_\ell \tilde{\rho}_0, \varphi \rangle_{Z'_\tau, Z_\tau} = -\int_{\Omega \times \mathbb{R}^+} \varphi(x, a) (\rho_t(x, a) - \rho_t(a, 0)) da dx,$$

and we underline that here $\varphi$ does not depend on time.

Proof. Using a priori estimates (5.3), one has

$$\sup_{\tau \in (0, \tau_0)} \int_0^T \left| D^T_{\ell} \int_{\Omega \times \mathbb{R}^+} \varphi(x, a, \tau) \tilde{\rho}_0(x, a, \tau) dx da \right| dt \leq \sup_{\tau \in (0, \tau_0)} \int_0^T \int_{\Omega \times \mathbb{R}^+} |\varphi(x, a)| |D^T_{\ell} \tilde{\rho}_0(x, a, \tau)| dx da dt$$

$$\leq \|\varphi\|_{L^\infty(\mathbb{R}^+; L^1(\Omega))} \|D^T_{\ell} \tilde{\rho}_0\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^+; L^\infty(\Omega))} < C,$$

which shows that $q(t) := \int_{\Omega \times \mathbb{R}^+} \varphi(x, a) \tilde{\rho}_0(x, a, t) dx da$ is a function of bounded variation. Thus there exists a signed Radon measure $\nu_{\partial \sigma}$ associated to the time derivative of $q$.

$$\int_0^{T/\varepsilon} d\nu_{\partial \sigma} = q(T/\varepsilon) - q(0) = \int_{\Omega \times \mathbb{R}^+} \varphi(x, a) \tilde{\rho}_0(x, a, T/\varepsilon) da - \int_{\Omega \times \mathbb{R}^+} \varphi(x, a) \tilde{\rho}_1(x, a) dx da.$$

Indeed, the integral $\int_0^{T/\varepsilon} d\nu_{\partial \sigma}$ coincides with the Riemann-Stieltjes integral, thus integration by parts holds. Moreover one has that

$$\int_0^{T/\varepsilon} D^T_{\ell}^{\gamma} q(t) dt \to \int_0^{T/\varepsilon} d\nu_{\partial \sigma}$$
as $\tau$ (up to a subsequence) goes to zero. On the other hand

$$\int_{Q_\tau} D_t^\tau \tilde{\rho}_{0,\varepsilon} \varphi \, dx \, dt \to <\partial_t \tilde{\rho}_{0,\varepsilon}, \varphi >_{Z'_T, Z_T}$$

in the weak-* topology $\sigma(Z'_T, Z_T)$ (as in the proof of Proposition 5.2). Because

$$\int_{Q_\tau} D_t^\tau \tilde{\rho}_{0,\varepsilon}(x,a,t) \varphi(x,a) \, dx \, dt = \int_0^{T/\varepsilon} D_t^\tau(\varepsilon l) \, dt$$

and the arguments above, one has finally :

$$<\partial_t \tilde{\rho}_{0,\varepsilon}, \varphi >_{Z'_T, Z_T} = q(T/\varepsilon) - q(0).$$

One concludes since $|q(T/\varepsilon)| \lesssim \exp(-\zeta_{\min}T/\varepsilon)$ thanks to (5.2).

\begin{proposition}
Under assumptions 2.1 and 2.2, one has

$$\limsup_{\tau \to 0} \| D_t^\tau (\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon}) \|_{Y_T} \sim o_\varepsilon(1)$$

which implies that :

$$\lim_{\varepsilon \to 0} | <\partial_t \rho_\varepsilon - \partial_t \rho_0 - \partial_t \tilde{\rho}_{0,\varepsilon}, \varphi >_{Z'_T, Z_T} | = 0$$

for all $\varphi \in Z_T$.
\end{proposition}

\begin{proof}
As $x$ is a mute variable in the $\rho_\varepsilon$ model, we first establish that :

$$\esssup_{x \in \Omega} \int_{Q_T} | D_t^\tau (\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon}) | \, dx \, dt \sim o_\varepsilon(1)$$

using exactly the same arguments as in Proposition 3.2 p.10. The method of characteristics gives, under the hypotheses above :

$$\int_{Q_T} \esssup_{x \in \Omega} | D_t^\tau (\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon}) | \, dx \, dt \sim o_\varepsilon(1)$$

as the bounds do not depend on $\tau$, the first claim follows. One writes then in the $Z'_T, Z_T$ duality pairing, that

$$| <\partial_t \rho_\varepsilon - \partial_t \rho_0 - \partial_t \tilde{\rho}_{0,\varepsilon}, \varphi > | \leq | <\partial_t \rho_\varepsilon - D_t^\tau \rho_\varepsilon, \varphi > | + | <\partial_t \rho_0 - D_t^\tau \rho_0, \varphi > |$$

$$+ | <\partial_t \tilde{\rho}_{0,\varepsilon} - D_t^\tau \tilde{\rho}_{0,\varepsilon}, \varphi > | + | <\partial_t \tilde{\rho}_{0,\varepsilon} - D_t^\tau \tilde{\rho}_{0,\varepsilon}, \varphi > |.$$ 

Thanks to Proposition 5.4, for any fixed $\delta > 0$ and any fixed $\varphi \in Z_T$, there exists $\varepsilon_0$ s.t. $\varepsilon < \varepsilon_0$ implies

$$| < D_t^\tau (\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon}), \varphi > | \leq \| D_t^\tau (\rho_\varepsilon - \rho_0 - \tilde{\rho}_{0,\varepsilon}) \|_{Y_2} \| \varphi \|_{Z_T} \leq \delta/2.$$ 

By Proposition 5.2 there exists $\tau_0$ s.t. $\tau < \tau_0$ implies

$$| <\partial_t \rho_\varepsilon - D_t^\tau \rho_\varepsilon, \varphi > | + | <\partial_t \rho_0 - D_t^\tau \rho_0, \varphi > | + | <\partial_t \tilde{\rho}_{0,\varepsilon} - D_t^\tau \tilde{\rho}_{0,\varepsilon}, \varphi > | \leq \delta/2,$$

which ends the proof.
\end{proof}

\begin{proposition}
Under the same hypotheses, there is a limit related to the initial layer : for any $\varphi \in Z_T$,

$$\lim_{\varepsilon \to 0} <\partial_t \tilde{\rho}_{0,\varepsilon}, \varphi(\cdot,\cdot) - \varphi(\cdot, 0) >_{Z'_T, Z_T} = 0.$$ 

\end{proposition}
Proof. We set $\psi(x, a, t) := \varphi(x, a, t) - \varphi(x, a, 0)$, and we use Corollary 5.1 giving that

$$
< \partial_t \tilde{p}_0, \psi >_{Z_T^x, Z_T} = < \partial_t \tilde{p}_0, \psi(\cdot, \cdot, \varepsilon) >_{Z_T^x, Z_T^x}.
$$

Next we write :

$$
< \partial_t \tilde{p}_0, \psi(\cdot, \cdot, \varepsilon) >_{Z_T^x, Z_T} = < \partial_t \tilde{p}_0 - D^\varepsilon_1 \tilde{p}_0, \psi(\cdot, \cdot, \varepsilon) >_{Z_T^x, Z_T^x} + < D^\varepsilon_1 \tilde{p}_0, \psi(\cdot, \cdot, \varepsilon) >_{Z_T^x, Z_T} := I_1 + I_2.
$$

We start with $I_2$ and write :

$$
|I_2| \leq \int_{Q_T} \| \psi(\cdot, a, \varepsilon t) \|_{L^1(\Omega)} \underset{x \in \Omega}{\overset{\text{ess sup}}{\text{ess sup}}} |D^\varepsilon_1 \tilde{p}_0| \, dadt.
$$

Since $\underset{x \in \Omega}{\overset{\text{ess sup}}{\text{ess sup}}} |D^\varepsilon_1 \tilde{p}_0|$ is a positive function in $L^1(Q_T)$, there exists $\nu$ a weak-* limit in $\sigma(M^1(\overline{Q_T}), C^0(\overline{Q_T}))$ of the measure $\nu_\varepsilon$ associated to it. Because $\nu_\varepsilon$ is tight with respect to $\tau$, this convergence extends to the weak-* topology in $\sigma(C_b(\overline{Q_T}), C_b(\overline{Q_T}))$.

Since $\| \psi(\cdot, a, \varepsilon t) \|_{L^1(\Omega)}$ is a continuous bounded function on $\overline{Q_T}$, converging pointwisely to $0$ a.e. $(a, t) \in Q_T$, there exists $\varepsilon_0$ s.t. for $\varepsilon < \varepsilon_0$,

$$
\int_{Q_T} \| \psi(\cdot, a, \varepsilon t) \|_{L^1(\Omega)} d\nu < \delta/3.
$$

From here until the end of the proof, $\varepsilon$ is fixed. Thanks to the previous tight convergence result, there exists a $\tau_0$, s.t.

$$
\left| \int_{Q_T} \| \psi(\cdot, a, \varepsilon t) \|_{L^1(\Omega)} d\nu - \int_{Q_T} \| \psi(\cdot, a, \varepsilon t) \|_{L^1(\Omega)} d\nu_\varepsilon \right| \leq \delta/3
$$

and finally there exists $\tau_1$ s.t. $\tau < \tau_1$ implies

$$
|I_1| < \delta/3
$$

thanks to the weak-* convergence in topology $\sigma(Z_T^x, Z_T^x)$. Summing the three terms ends the proof.

Theorem 5.3. Under hypotheses \[2.1\] and \[2.2\] one has

$$
< \partial_t \rho_\varepsilon, \varphi >_{Z_T^x, Z_T} \to \int_{Q_T} \partial_t \rho_0(x, a, t) \varphi(x, a, t) \, dx \, da dt - \int_{\Omega \times \mathbb{R}_+} (\rho_1(x, a) - \rho_0(x, a, 0)) \varphi(x, a, 0) \, dx \, da
$$

for every $\varphi \in Z_T$.

Proof. We set

$$
I := < \partial_t \rho_\varepsilon, \varphi > - \int_{Q_T} \partial_t \rho_0 \varphi \, dx \, da dt + \int_{\Omega \times \mathbb{R}_+} (\rho_1(x, a) - \rho_0(x, a, 0)) \varphi(x, a, 0) \, dx \, da
$$

and split this difference adding and subtracting extra terms :

$$
|I| \leq \left| \langle \partial_t \rho_\varepsilon - \partial_t \rho_0, \varphi \rangle_{Z_T^x, Z_T} \right| + \left| \langle \partial_t \tilde{p}_0, \varphi - \varphi(\cdot, \cdot, 0) \rangle_{Z_T^x, Z_T} \right| + \left| \langle \partial_t \tilde{p}_0, \varphi(\cdot, \cdot, 0) \rangle_{Z_T^x, Z_T} + \int_{\Omega \times \mathbb{R}_+} (\rho_1(x, a) - \rho_0(x, a, 0)) \varphi(x, a, 0) \, dx \, da \right| = \sum_{i=1}^{3} |I_i|.
$$
Now for every fixed $\delta$ (small), there exists $\varepsilon_0$ s.t. $\varepsilon < \varepsilon_0$ implies $I_1 < \delta/3$ thanks to Proposition 5.5 s.t. $I_2 < \delta/3$ thanks to Proposition 5.5 and s.t. $I_3 < \delta/3$ thanks to Theorem 5.2 which ends the proof.

We define

$$K_\varepsilon(x,a,t) := aD_1^{\alpha_0}\rho_\varepsilon = \frac{\rho_\varepsilon(x,a,t+\varepsilon a) - \rho_\varepsilon(x,a,t)}{\varepsilon},$$

a.e in $P_\varepsilon := \Omega \times \{(a,t) \in Q_T \text{ s.t. } a < \frac{T-t}{\varepsilon}\}$.

**Theorem 5.4.** Under hypotheses 2.1 and 2.2, $K_\varepsilon$ solves the weak problem: for all $\psi \in Z_T$

$$\int_\Omega \psi K_\varepsilon \, da \, dt \, dx = (\partial_\mu \rho_\varepsilon, \varphi_\varepsilon)_T, x_T, x_T - \int_\Omega \varphi_\varepsilon(x,a,t)(\delta a) D_1^{\alpha_0} \zeta_\varepsilon(x,a,t) \, da \, dt \, dx, \quad (5.4)$$

where we set

$$\varphi_\varepsilon(x,\varepsilon t) := \chi_{I_\varepsilon}(x,\varepsilon t) \int_{T-\varepsilon t}^{T-t} \exp \left( - \int_0^a \zeta_\varepsilon(x,s+t+\varepsilon s) \, ds \right) \psi(x,a) \, da. \quad (5.5)$$

**Proof.** In order to express the problem solved by $K_\varepsilon$, we regularize the data. It gives a pointwise meaning to an approximation of $\partial_\mu \rho_\varepsilon$. For this sake, we regularize the initial and boundary datum and the off-rate setting:

$$\beta_\delta^\varepsilon(x,t) := (1 - \chi_\delta(t))\beta_\varepsilon \ast \omega_\delta, \quad \rho_\delta^\varepsilon(x,a) := (1 - \chi_\delta(a))\rho_\varepsilon \ast \omega_\delta, \quad \zeta_\delta(x,a) := \zeta_\varepsilon \ast (\omega_\delta \ast \omega_\delta),$$

where the cut-off function $\chi_\delta$ is monotone and $C^\infty(\mathbb{R}_+)$ s.t.

$$\chi_\delta(a) := \begin{cases} 1 & \text{if } a < \delta, \\ 0 & \text{if } a > 2\delta, \end{cases}$$

and $\omega_\delta$ and $\omega_\delta$ are the standard mollifiers in the $a$ and $t$ variable. One solves (1.3) with initial, boundary and off-rate datum $(\rho_\delta^\varepsilon, \beta_\delta^\varepsilon, \zeta_\delta^\varepsilon)$, the solution is denoted $\rho_\varepsilon^\delta$. Together with assumptions 2.1 and 2.2, the time derivative $\partial_t \rho_\varepsilon^\delta$ solves:

$$\begin{aligned}
\begin{cases}
(\varepsilon \partial_t + \partial_a + \zeta_\delta) \partial_t \rho_\varepsilon^\delta &= \partial_t \zeta_\delta \rho_\varepsilon^\delta, &a > 0, t > 0, \\
\partial_t \rho_\varepsilon^\delta(x,0,t) &= \partial_t \beta_\delta^\varepsilon(1 - \rho_\delta^\varepsilon) - \beta_\delta^\varepsilon \partial_t \mu_0^\delta, &a = 0, t > 0, \\
\varepsilon \partial_t \rho_\varepsilon^\delta(x,a,0) &= - (\partial_a + \zeta_\delta(x,a,0)) \rho_\varepsilon^\delta, &a > 0, t = 0,
\end{cases} \quad (5.6)
\end{aligned}$$

where $\rho_\delta^\varepsilon(x,t) = \int_{\mathbb{R}_+} \rho_\delta^\varepsilon(x,a,t) \, da$. Since the data of (5.6) is regular, for a fixed $x \in \Omega$, existence results follow from Theorem 2.1 p. 488 and thanks to similar arguments as in Proposition 3.1, one proves as well that $\partial_t \rho_\varepsilon^\delta$ is a $C^\infty(0,T); L^1(\mathbb{R}_+; L^\infty(\Omega))$ function. One obtains a priori estimates, uniform in $\varepsilon$, leading to $\partial_t \rho_\varepsilon^\delta \in Y_T$. In the same way as in Propositions 5.1 and 5.2 there is a limit $\partial_\mu \rho_\varepsilon$ in the weak-* topology $\sigma(Z_T^*, Z_T^*)$, up to a subsequence. Moreover, using the Lyapunov functional $H[\cdot]$, one has also that $\rho_\varepsilon^\delta - \rho_\varepsilon \sim o_\delta(1)$ in $Y_T$. Now as $\partial_t \rho_\varepsilon^\delta$ is regular enough, one derives the ODE solved by $K_\varepsilon$,

$$\begin{aligned}
\begin{cases}
\partial_t K_\varepsilon^\delta + \zeta_\delta(x,a,t+\varepsilon a) K_\varepsilon^\delta &= \partial_t \rho_\varepsilon^\delta - (aD_1^{\alpha_0} \zeta_\varepsilon) \rho_\varepsilon^\delta, &\text{a.e. } (x,a,t) \in P_\varepsilon, \\
K_\varepsilon^\delta(x,0,t) = 0, &\text{a.e. } x \in \Omega, a = 0, \text{ a.e. } t > 0.
\end{cases} \quad (5.7)
\end{aligned}$$
This can be integrated and gives:

\[ K_c^\delta(x, a, t) := \chi_{P_c}(x, a, t) \times \int_0^a \exp \left( - \int_\delta^a \zeta_c^\delta(x, s, t + \varepsilon s) ds \right) \left\{ \partial_t \rho_c^\delta(x, \bar{a}, t) - (\bar{a} D_t^\zeta_c^\delta) \rho_c^\delta(x, \bar{a}, t) \right\} d\bar{a}. \]

Tested against \( \psi \in Z_T \) and integrated on \( \Omega_T \), this becomes:

\[
\int_{\Omega_T} \psi(x, a, t) K_c^\delta(x, a, t) dx da dt = - \int_{\Omega_T} \int_0^a \exp \left( - \int_\delta^a \zeta_c^\delta(s, t + \varepsilon s) ds \right) \left( \bar{a} D_t^\zeta_c^\delta \rho_c^\delta(\bar{a}, t) \right) d\bar{a} dx da dt + \int_{\Omega_T} \int_0^1 \int_0^{T-\varepsilon} \exp \left( - \int_\delta^a \zeta_c^\delta(x, s, t + \varepsilon s) ds \right) \psi(x, a, t) dx da \partial_t \rho_c^\delta(x, \bar{a}, t) d\bar{a} dx da dt.
\]

Setting

\[ \varphi_c^\delta(x, \bar{a}, t) := \chi_{P_c}(x, \bar{a}, t) \int_\delta^{T-\varepsilon} \exp \left( - \int_\delta^a \zeta_c^\delta(x, s, t + \varepsilon s) ds \right) \psi(x, a, t) da, \]

one recovers the regularized version of \( (5.3) \). Since \( \zeta_c^\delta \in W^{1,\infty}(\Omega_T) \), one has \( \zeta_c^\delta \rightarrow \zeta_c \) strongly in \( C(K) \) for any compact \( K \subset \Omega_T \). Thus one has : \( \varphi_c^\delta \rightarrow \varphi_c \) strongly in \( L^\infty_{\text{loc}} L^1_x \) when \( \delta \rightarrow 0 \). Since \( \varphi_c^\delta \) and \( \varphi_c \) are continuous and compactly supported, the strong convergence occurs as well in \( Z_T \). There exists a subsequence \( \partial_t \rho_c^\delta \) converging in the \( \sigma(Z_T, Z_T) \) topology to \( \varphi_c \) thus

\[
\int_{\Omega_T} \varphi_c^\delta \partial_t \rho_c^\delta d\bar{a} dx dt - \langle \partial_t \rho_c, \varphi_c \rangle _{Z_T, Z_T} = \int_{\Omega_T} (\varphi_c^\delta - \varphi_c) \partial_t \rho_c^\delta d\bar{a} dx dt + \int_{\Omega_T} \varphi_c \partial_t \rho_c^\delta d\bar{a} dx dt - \langle \partial_t \rho_c, \varphi_c \rangle _{Z_T, Z_T} \rightarrow 0,
\]

when \( \delta \) goes to zero. Now other arguments using the strong convergence of \( \rho_c^\delta \) justify the claim. Moreover, \( \varphi_c \) is bounded uniformly with respect to \( \varepsilon \). Indeed:

\[
\| \varphi_c(\cdot, \bar{a}, t) \|_{L^1(\Omega)} \leq \chi_{P_c} \int_\delta^{T-\varepsilon} \exp(-\zeta_{\min}(a - \bar{a})) \| \psi(\cdot, a, t) \|_{L^1(\Omega)} da,
\]

which gives after taking the sup over \( \mathcal{Q}_T \) that

\[
\| \varphi_c \|_{L^\infty} \leq \sup \left( \frac{\varphi_c(\cdot, \bar{a}, t)}{(\bar{a}, t) \in \mathcal{P}_c} \right) \int_\delta^{T-\varepsilon} \exp(-\zeta_{\min}a) da \| \psi \|_{Z_T} \lesssim \| \psi \|_{Z_T}. \tag{5.8}
\]

Corollary 5.2. Under the previous hypotheses, one has that \( \| \varphi(\cdot, a, t) - \varphi_0(\cdot, a, t) \|_{L^1_x} \) tends to zero when \( \varepsilon \) goes to zero, for every fixed \( (a, t) \in \mathcal{Q}_T \), where

\[ \varphi_0(x, \bar{a}, t) := \int_\delta^\infty \psi(x, a, t) \exp \left( - \int_\delta^a \zeta_0(x, s, t) ds \right) da. \]

Proposition 5.6. Under hypotheses 2.7 and 2.2 one has

\[
\langle \partial_t \rho_c, \varphi_c \rangle \rightarrow \int_{\Omega_T} \partial_t \rho_0 \varphi_0 dx da - \int_{\Omega_T \times \mathbb{R}_+} \varphi_0(x, a, 0)(\rho_1(x, a) - \rho_0(x, a, 0)) dx da,
\]

where \( \varphi_c \) is defined in \( (5.5) \).

Proof. We set \( \ell := \int_{\Omega_T \times \mathbb{R}_+} \varphi_0(x, a, 0)(\rho_1(x, a) - \rho_0(x, a, 0)) dx da \). As above one has

\[
|\langle \partial_t \rho_c, \varphi_c \rangle - \langle \partial_t \rho_0, \varphi_0 \rangle + \ell| \leq |\langle \partial_t \rho_c - \partial_t \rho_0 - \partial_t \rho_0, \varphi_c \rangle | + |\langle \partial_t \rho_0, \varphi_c - \varphi_0 \rangle | + |\langle \partial_t \rho_0, \varphi_c \rangle + \ell| \leq \varepsilon \ell + |\langle \partial_t \rho_0, \varphi_c \rangle + \ell| =: \varepsilon \ell + J_\ell,
\]

where \( \varepsilon \) is defined in \( (5.5) \).

\[ \|
\]
the first term in the right hand side is $o_\varepsilon(1)$ thanks to Proposition 5.4. We focus on the second one: thanks to Corollary 5.2 and as $\text{ess sup}_{x \in \Omega} |\partial_t \rho_0(x, a, t)|$ is an integrable function on $Q_T$,
\[
\left| \int_{Q_T} (\varphi_\varepsilon - \varphi_0) \partial_t \rho_0 \, dx \, dt \right| \leq \int_{Q_T} \left\| (\varphi_\varepsilon - \varphi_0) (\cdot, a, t) \right\|_{L^1(\Omega)} \text{ess sup}_{x \in \Omega} |\partial_t \rho_0(x, a, t)| \, dx \, dt.
\]

By Lebesgue’s Theorem, the right hand side tends to zero. Using Corollary 5.1, one writes then
\[
J_\varepsilon = \left| \langle \partial_t \rho_0, \varphi_\varepsilon (\cdot, \varepsilon, \cdot) \rangle \right|_{Z_T} \leq \left| \langle \partial_t \rho_0, (\varphi_\varepsilon - \varphi_0) (\cdot, \varepsilon, \cdot) \rangle \right|_{Z_T} + \left| \langle \partial_t \rho_0, \varphi_0 (\cdot, \varepsilon, 0) \rangle \right|_{Z_T} + \frac{3}{\varepsilon} J_{\varepsilon,2}.
\]

Since $\chi_0(0, \varepsilon, \cdot)(t) \|\varphi_\varepsilon - \varphi_0\|_{L^1(\Omega)}$ and $\chi_0(0, \varepsilon, \cdot)(t) \|\varphi_\varepsilon (\cdot, a, t) - \varphi_0 (\cdot, a, 0)\|_{L^1(\Omega)}$ tend to zero for a.e. $(a, t) \in (\mathbb{R}^+)^2$, by the same arguments as in the proof of Proposition 5.5, one concludes that $J_{\varepsilon,1}$ and $J_{\varepsilon,2}$ vanish when $\varepsilon$ goes to 0. For the last term we use Theorem 5.2 and one concludes.

Now we are in the position to prove

\begin{proposition}
Under hypotheses 2.1 and 2.2, when $\varepsilon$ goes to zero,
\[
\int_{Q_T} K_\varepsilon(x, a, t) \psi(x, a, t) \, dx \, dt - \int_{Q_T} a \partial_t \rho_0(x, a, t) \psi(x, a, t) \, dx \, dt - \int_{\Omega \times \mathbb{R}^+} \varphi_0(x, a, 0) (\rho_I(x, a) - \rho_0(x, a, 0)) \, dx \, da 
\]
for all $\psi \in C_b(\overline{\Omega_T})$.
\end{proposition}

\textbf{Proof}. Considering the first term in [5.4], Proposition 5.6 shows that:
\[
\langle \partial_t \rho_\varepsilon, \varphi_\varepsilon \rangle \rightarrow \int_{\Omega_T} \partial_t \rho_0 \varphi_0 \, dx \, dt - \int_{\Omega \times \mathbb{R}^+} \varphi_0(x, a, 0) (\rho_I(x, a) - \rho_0(x, a, 0)) \, dx \, da.
\]

On the other hand, hypotheses 2.1, standard arguments and the strong convergence of $\rho_\varepsilon$ imply that
\[
\int_{\Omega_T} \varphi_\varepsilon a(D_t^\alpha \zeta_\varepsilon) \rho_\varepsilon(x, a, t) \, dx \, dt \rightarrow \int_{\Omega_T} \varphi_0(x, a, t) a \partial_t \zeta_0(x, a, t) \rho_0(x, a, t) \, dx \, dt.
\]

So that finally, one has
\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} K_\varepsilon(x, a, t) \psi(x, a, t) \, dx \, dt - \int_{\Omega_T} \partial_t \rho_0 (a \partial_t \zeta_0 \rho_0) \varphi_0 \, dx \, dt - \int_{\Omega \times \mathbb{R}^+} \varphi_0(x, a, 0) (\rho_I(x, a) - \rho_0(x, a, 0)) \, dx \, da.
\]

As $K_0(x, a, t) := a \partial_t \rho_0(x, a, t)$ is solving
\[
(\partial_t + \zeta_0(x, a, t)) K_0 = \partial_t \rho_0 - a \partial_t \zeta_0 \rho_0, \quad K_0(x, 0, t) = 0,
\]

it is explicit and reads:
\[
K_0(x, a, t) = \int_0^a \exp \left( - \int_s^a \zeta_0(x, s, t) \, ds \right) (\partial_t \rho_0 - a \partial_t \zeta_0 \rho_0) \, da.
\]
It is then a matter of check to write
\[ \int_{\Omega} K_0 \psi \, d\mu \, dx = \int_{\Omega} \varphi_0 \left( \partial_t \rho_0 - a \partial_t \zeta_0 \rho_0 \right) \, d\mu \, dx, \]
which ends the proof. \( \square \)

We consider
\[ I_\varepsilon(\rho_\varepsilon, z_\varepsilon, \psi) := \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}^+} \rho_\varepsilon(x, a, t) \left( z_\varepsilon(x, t) - z_\varepsilon(x, t - \varepsilon a) \right) \, d\mu \psi(x, t) \, dt \, dx \]
and we want to express the limit of this operator when \( \varepsilon \) goes to 0.

**Theorem 5.5.** Under hypotheses [2.1, 3.2] and [2.3] when \( \varepsilon \) goes to zero, one has that
\[ I_\varepsilon(\rho_\varepsilon, z_\varepsilon, \psi) \to \left[ \int_{\Omega} \psi(x, t) \cdot g_0(x, t) \mu_1(x, t) \, dx \right]_{t=0}^{t=T} \quad \int_{\Omega \times (0, T)} \int_{\mathbb{R}^+} z_0(x, t) \cdot \partial_t (\mu_1 \psi) \, dt \, dx, \]
where \( \mu_1(x, t) := \int_{\mathbb{R}^+} a \rho_0(x, a, t) \, da \) and for any regular \( \psi \in W^{1, \infty}(Q_T). \)

**Proof.** Splitting the domain of integration,
\[ I_\varepsilon(\rho_\varepsilon, z_\varepsilon, \psi) = \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}^+} \rho_\varepsilon(x, a, t) z_\varepsilon(x, t) \cdot \psi(x, t) \, d\mu \, dt \, dx \]
\[ - \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}^+} \rho_\varepsilon(x, a, t) \left( \psi(x, t) \rho_\varepsilon(x, a, t + \varepsilon a) - \rho_\varepsilon(x, a, t) \right) \, d\mu \, dt \, dx \]
\[ - \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}^+} \rho_\varepsilon(x, a, t) \left( \psi(x, t + \varepsilon a) - \psi(x, t) \right) \rho_\varepsilon(x, a, t + \varepsilon a) \, d\mu \, dt \, dx \]
\[ - \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}^+} \rho_\varepsilon(x, a, t) \left( \psi(x, t - \varepsilon a) - \psi(x, t) \right) \rho_\varepsilon(x, a, t) \, d\mu \, dt \, dx =: \ell_1 - \sum_{i \in \{2, 3, 4\}} \ell_i. \]

Thanks to [Appendix A] \( \ell_1 \) and \( \ell_4 \) tend respectively to
\[ \ell_1 \to \int_{\Omega} z_0(x, T) \cdot \psi(x, T) \mu_1(x, T) \, dx, \quad \ell_4 \to \int_{\Omega} z_0(x, 0) \cdot \int_{\mathbb{R}^+} \varphi_0(x, a, 0) \rho_1(x, a) \, da \, dx, \]
where \( \varphi_0(x, a, t) := \psi(x, t) \int_0^\infty \exp \left( - \int_0^t \zeta_0(x, s, t) \, ds \right) \, da. \) By strong convergence results established for \( \rho_\varepsilon \) and \( z_\varepsilon, \) and the Lebesgue’s Theorem, one shows that
\[ \ell_3 \to \int_{\Omega \times \mathbb{R}^+ \times (0, T)} z_0(x, t) \cdot \partial_t \psi(x, t) a \rho_0(x, a, t) \, dt \, dx \, da. \]

Setting \( \tilde{z}_\varepsilon = z_\varepsilon - z_0, \) and rewriting \( \ell_2 \) gives :
\[ \ell_2 = \int_{\Omega} K(x, a, t) \tilde{z}_\varepsilon(x, t) \cdot \psi(x, t) \, d\mu \, dx \]
\[ = \int_{\Omega} \tilde{z}_\varepsilon(x, t) \cdot \psi(x, t) K(x, a, t) \, dt \, dx + \int_{\Omega} K(x, a, t) z_0(x, t) \cdot \psi(x, t) \, d\mu \, dx. \]

Thanks to Corollary 5.1 and Proposition 5.7 one concludes that :
\[ \ell_2 \to \int_{\Omega} a \partial_t \rho_0(x, a, t) z_0(x, t) \cdot \psi(x, t) \, dt \, dx - \int_{\mathbb{R}^+} \int_{\Omega} (\rho_1(x, a) - \rho_0(x, a, 0)) z_0(x, 0) \cdot \varphi_0(x, a, 0) \, da \, dx. \]
then gathering the terms provides that
\[ \mathcal{L}_z(\rho, z, \psi) \rightarrow \int_{\Omega \times \mathbb{R}_+^+} a\rho_0(x, a, T)z_0(x, a, T) \cdot \psi(x, t) \, dx \, dt - \int_{\Omega} a(\rho_0 z_0 \cdot \partial_t \psi + \partial_1 \rho_0 z_0 \cdot \psi) \, dt \, dx \]
\[ - \int_{\Omega \times \mathbb{R}_+} \rho_0(x, a, 0) \varphi_0(x, a, 0) \, da \cdot z_0(x, 0) \, dx. \]

The latter term can then be transformed into:
\[ \int_{\Omega} \int_{\mathbb{R}_+^+} \int_0^a \rho_0(x, a, 0) \exp \left( - \int_a^a \zeta_0(x, s, 0) \, ds \right) \, d\alpha \psi(x, 0) \cdot z_0(x, 0) \, dx \]
\[ = \int_{\Omega} \left\{ \int_{\mathbb{R}_+^+} a\rho_0(x, a, 0) \, da \right\} \psi(x, 0) \cdot z_0(x, 0) \, dx = \int_{\Omega} \psi(x, t) \cdot z_0(x, 0) \mu_{1, 0}(x, 0) \, dx, \]
since \( \partial_2(a\rho_0) + \zeta_0(a\rho_0) = \rho_0 \), one sees easily that the latter inner integral corresponds exactly to the integration of the latter ODE.

\[ \square \]

**Theorem 5.6.** Under hypotheses 2.1, 2.2 and 2.3, the unique solution \( z_t \) of the problem 4.1 converges towards the unique solution pair \( z_0 \in H^1((0, T); L^2(\Omega)) \cap L^\infty((0, T); H^1(\Omega)) \) satisfying:
\[ \int_{\Omega \times (0, T)} \mu_{1, 0} \partial_t z_0 \cdot \varphi + \partial_2 z_0 \cdot \partial_x \varphi - |\partial_x z_0|^2 z_0 \cdot \varphi \, dx \, dt = 0 \]
for every \( \varphi \in H^1((0, T); L^2(\Omega)) \cap L^\infty((0, T); H^1(\Omega)) \).

**Proof.** From Theorem 5.5 and stability results above one has that the solution \( z_t \) satisfying the weak formulation 4.2 converges strongly in \( C(\bar{\Omega} \times [0, T]) \) to \( z_0 \in H^1((0, T); L^2(\Omega)) \cap L^\infty((0, T); H^1(\Omega)) \) satisfying:
\[ \left[ \int_{\Omega} \psi(x, t) \mu_{1, 0}(x, t) z_0(x, t) \, dx \right]_{t=0}^{t=T} - \int_{Q_T} z_0(x, t) \partial_t (\mu_{1, 0} \psi) \, dt \, dx \\
+ \int_{\Omega \times (0, T)} \partial_2 z_0 \cdot \partial_x \varphi \, dx \, dt + \langle z_0 \cdot \varphi, \lambda_0 \rangle = 0 \]
\[ \text{together with the constraint } |z_0| = 1 \text{ everywhere in } \bar{\Omega} \times [0, T] \text{ obtained in the same way as in Theorem 4.1.} \]

Then since in the first line of the latter equation all terms are well defined, one can perform an integration by parts in time and obtain that \( z_0 \) solves:
\[ \int_{\Omega \times (0, T)} \mu_{1, 0} \partial_t z_0 \cdot \varphi + \partial_2 z_0 \cdot \partial_x \varphi \, dx \, dt + \langle z_0 \cdot \varphi, \lambda_0 \rangle = 0 \]
again as in Theorem 4.1 choosing the test function to be \( \varphi = z_0 \theta \), with \( \theta \in D(\Omega \times (0, T)) \), one proves that for almost every \( (x, t) \), \( \lambda_0 = -|\partial_x z_0|^2 \) which, because \( z_0 \) belongs to \( L^\infty_{x,t} H^1_{x,t} \), is an \( L^1_{x,t} \) function.

\[ \square \]

**Appendix A. Initial and final terms in the weak formulation**

**Proposition Appendix A.1.** Under hypotheses 2.1 and 2.2, one has
\[ \ell_4 = \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \int_{\mathbb{R}_+^+} \rho_\varepsilon(x, a, t) z_\varepsilon(x, t - \varepsilon a) \cdot \psi(x, t) \, d\varepsilon \, dt \, dx \rightarrow \int_{\Omega} z_0(x, 0) \cdot \int_{\mathbb{R}_+} \varphi_0(x, a, 0) \rho_1(x, a) \, da \, dx, \]
where $\ell_4$ and $\varphi_0$ are defined in the proof of Theorem 5.5.

**Proof.** Using the characteristics, one writes:

$$\ell_4 = \frac{1}{\varepsilon} \int_\Omega \int_0^T \int_0^\infty \rho_t(x, a - t/\varepsilon) \exp \left( - \int_0^{t/\varepsilon} \zeta_\varepsilon \left( x, a + s, -\frac{t}{\varepsilon}, \varepsilon s \right) ds \right) \times z_p(x, t - \varepsilon a) \psi(x, t) d\mu dt dx$$

$$= \frac{1}{\varepsilon} \int_\Omega \int_0^T \int_0^\infty \rho_t(x, \tilde{a}) \exp \left( - \int_0^{t/\varepsilon} \zeta_\varepsilon \left( x, \tilde{a} + s, \varepsilon s \right) ds \right) z_p(x, -\varepsilon \tilde{a}) \psi(x, \varepsilon t) d\mu dt dx$$

By the Lebesgue’s Theorem the latter term tends to

$$\lim_{\varepsilon \to 0} \ell_4 = \int_\Omega z_p(x, 0) \psi(x, 0) \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \rho_t(x, a) \left( - \int_0^t \zeta_\varepsilon (x, a + s, 0) ds \right) d\mu dt dx$$

where $\varphi_0(a, t)$ is defined as

\[\text{Proposition Appendix A.2. Under hypotheses 2.1 and 2.2, one has}\]

$$\ell_1 \to \int_\Omega z_0(x, T) \psi(x, T) \mu_{1,0}(x, T) dx,$$

where $\ell_4$ and $\varphi_0$ are defined in the proof of Theorem 5.5.

**Proof.** Dividing $(0, T)$ in two equal parts $(0, T/2)$ and $(T/2, T)$, gives two terms $\ell_{1,1}$ and $\ell_{1,2}$. For

![Diagram](attachment:image.png)

**Fig. 1. Partition of the age-time domain for the splitting of the $\ell_1$ terms.**
the first one, we have
\[
\ell_{1,1} = \frac{1}{\varepsilon} \int_{\Omega} \int_{0}^{T} \rho_{I}(x,a) \exp \left( - \int_{0}^{t} \zeta(x,a - s) ds \right) \times 
\]
\[\times z_{c}(x,t) \psi(x,t) \, dt \, dx,\]
\[= \frac{1}{\varepsilon} \int_{\Omega} \int_{0}^{T - \varepsilon} \rho_{I}(x,a) \exp \left( - \int_{0}^{t} \zeta(x,a + s) ds \right) z_{c}(x,t) \psi(x,t) \, dt \, dx,\]
\[= \int_{\Omega} \int_{0}^{T - \varepsilon} \rho_{I}(x,a) \exp \left( - \int_{0}^{t} \zeta(x,a + s) ds \right) z_{c}(x,t) \psi(x,t) \, dt \, dx.
\]

Now, we estimate the latter term as
\[
|\ell_{1,1}| \leq C \int_{\Omega} \int_{0}^{T - \varepsilon} \exp(-\zeta_{\min} t) \rho_{I}(x,a) \, dt \, dx
\]
\[= \int_{\Omega} \int_{0}^{T} \rho_{I}(x,a) \exp(-\zeta_{\min} t) \, dt \, dx
\]
\[+ \int_{\Omega} \int_{0}^{T} \rho_{I}(x,a) \exp(-\zeta_{\min} t) \, dt \, dx = \ell_{1,1,1} + \ell_{1,1,2},\]

applying Lebesgue’s Theorem shows that \(\ell_{1,1,2} \to 0\) when \(\varepsilon\) goes to zero.
\[
\ell_{1,1,1} = \int_{\Omega} \int_{0}^{T} \rho_{I}(x,a) \int_{t - \varepsilon}^{t} \exp(-\zeta_{\min} t) \, dt \, dx
\]
\[\leq \int_{\Omega} \int_{0}^{T} \rho_{I}(x,a) (\exp(-\zeta_{\min}(T - \varepsilon a)) - \exp(-\zeta_{\min}(T))) \, dt \, dx,
\]

which again by Lebesgue’s Theorem vanishes. The term \(\ell_{1,2}\) can be split in two parts:
\[
\ell_{1,2} := \int_{\Omega} \int_{0}^{T} \int_{t - \varepsilon}^{t} \rho_{c}(x,a,t) z_{c}(x,t) \psi(x,t) \, dt \, dx
\]
\[+ \int_{\Omega} \int_{0}^{T} \int_{t - \varepsilon}^{t} \rho_{c}(x,a,t) z_{c}(x,t) \psi(x,t) \, dt \, dx = \ell_{1,2,1} + \ell_{1,2,2},
\]

as above
\[
|\ell_{1,2,2}| = \left| \int_{\Omega} \int_{0}^{T} \int_{t - \varepsilon}^{t} \rho_{c}(x,a,t) \exp \left( - \int_{0}^{t} \zeta(x,a + s) ds \right) \, dt \, dx \right|
\]
\[\leq \exp \left( - \frac{T}{2 \varepsilon} \right) \|\rho_{c}\|_{L^{1}(\mathbb{R}^{+},L^{\infty}(\Omega))} \to 0.
\]

Finally, we split \(\ell_{1,2,1}\) in two terms
\[
\ell_{1,2,1} = \frac{1}{\varepsilon} \int_{\Omega} \left\{ \left( \int_{t - \varepsilon}^{t} \int_{0}^{T - \varepsilon} \rho_{c}(x,a,t) z_{c}(x,t) \psi(x,t) \, dt \, dx \right) \right\}
\]
\[= \ell_{1,2,1,1} + \ell_{1,2,1,2}.
\]
In a straightforward manner, the latter term goes to zero since

\[
\ell_{1,2,1,2} := \int_\Omega \int_T^T \int_0^\frac{t}{\epsilon} \rho_a(x,a,t)z_a(x,t)\psi(x,t)d\tau dx dt \\
= \frac{1}{\epsilon} \int_\Omega \int_T^T \int_0^\frac{t}{\epsilon} \rho_a(x,a,t)z_a(x,t)\psi(x,t)d\tau dx dt \\
\leq \frac{1}{\epsilon} \int_\Omega \int_T^T \exp(-\zeta_{\min}(a))(T - \epsilon a)da \exp(-\zeta_{\min}(T)/(2\epsilon)) \sim o_\epsilon(1),
\]

where we used that \(\rho_a(x,0,t - \epsilon a)\) is bounded with respect to \(\epsilon\). The only term remaining and which should converge to a non-zero limit is

\[
J := \ell_{1,2,1,1} = \frac{1}{\epsilon} \int_\Omega \int_T^T \int_0^\frac{t}{\epsilon} \rho_a(x,a,t)z_a(x,t)\psi(x,t)d\tau dx dt \\
= \frac{1}{\epsilon} \int_\Omega \int_0^T \int_{t-a}^T \rho_a(x,a,t)z_a(x,t)\psi(x,t)d\tau dx dt,
\]

where \(\rho_a = \rho_a - \rho_0\). Using Lemma 3.2, one has the estimate \(|J_1| \sim o_\epsilon(1)\). By Lemma 3.4 \[3.4\], \(|\rho_0(x,a,t)| \leq \exp(-\zeta_{\min}(a))\) and \(\|\hat{z}_a\|_{L^\infty((0,T);L^2(\Omega))} \sim o_\epsilon(1)\), one has

\[
|J_2| \leq \frac{\|\hat{z}_a\|_{L^\infty((0,T);L^2(\Omega))}}{\epsilon} \int_0^T \int_{t-a}^T dt \exp(-\zeta_{\min}(a))da \sim o_\epsilon(1).
\]

In order to prove the limit of \(J_3\) we define \(f(a,t) = a \int_\Omega \rho_0(x,a,t)\psi(x,t)z_0(x,t)dx\), and write:

\[
|f(a,t) - f(a,T)| \leq a \left\{ \exp(-\zeta_{\min}(a)) \left( \|z_0\|_{L^\infty((0,T);L^2(\Omega))} \sup_{x \in \Omega} |\psi(x,t) - \psi(x,T)| \right. \\
+ \|\psi\|_{L^\infty((0,T);L^2(\Omega))} \left. \sup_{x \in \Omega} |\rho_0(x,a,t) - \rho_0(x,a,T)| \right\}
\]

for all \(a \in \mathbb{R}_+\) and all \(\delta > 0\) there exists \(\eta_i\), \(i \in \{1,2,3\}\) s.t.

\[
\forall t \in (0,T) \text{ s.t. } |t - T| < \eta_1 \implies \sup_{x \in \Omega} |\psi(x,t) - \psi(x,T)| < \frac{\delta}{3(1+a)\epsilon \exp(-\zeta_{\min}(a))\epsilon L^2},
\]

\[
\forall t \in (0,T) \text{ s.t. } |t - T| < \eta_2 \implies \|z_0(\cdot,t) - z_0(\cdot,T)\|_{L^2(\Omega)} < \frac{\delta}{3(1+a)\epsilon \exp(-\zeta_{\min}(a))\epsilon L^2},
\]

\[
\forall t \in (0,T) \text{ s.t. } |t - T| < \eta_3 \implies \sup_{x \in \Omega} |\rho_0(x,a,t) - \rho_0(x,a,T)| < \frac{\delta}{3(1+a)\epsilon \exp(-\zeta_{\min}(a))\epsilon L^2},
\]

which means that \(\forall a \in \mathbb{R}_+, \forall \delta > 0\) there exists \(\eta(a,\delta) = \min_{i \in \{1,2,3\}} \eta_i > 0\) s.t.

\[
\forall t \in (0,T) \text{ s.t. } |t - T| < \eta \implies |f(a,t) - f(a,T)| < \delta.
\]
Now there exists $\varepsilon(a, \delta)$ small enough $\varepsilon < \eta(a, \delta)/(1 + a)$ s.t. for all $t \in (T - \varepsilon a, T)$, $|t - T| < \eta$, thus
\[
\frac{1}{\varepsilon a} \int_{T-\varepsilon a}^{T} |f(a, t) - f(a, T)| dt \leq \frac{1}{\varepsilon a} \int_{T-\varepsilon a}^{T} |f(a, t) - f(a, T)| dt \leq \delta.
\]
Setting $g_{\varepsilon}(a) = \frac{1}{\varepsilon a} \int_{T-\varepsilon a}^{T} f(a, t) dt$ and $g_{0}(a) = f(a, T)$, this shows that there is pointwise convergence for every fixed $a$, $g_{\varepsilon}(a) \rightarrow g_{0}(a)$. Moreover,
\[
g_{\varepsilon}(a) \lesssim (1 + a) \exp(-\zeta \min a) \in L^1(\mathbb{R}^+)
\]
and by the Lebesgue’s Theorem the convergence holds:
\[
\int_{\mathbb{R}^+} g_{\varepsilon}(a) da \rightarrow \int_{\mathbb{R}^+} a \int_{\Omega} \rho_{0}(x, a, T) z_{0}(x, T) \cdot \psi(x, T) dx da = \int_{\Omega} \mu_{1, 0}(x, T) z_{0}(x, T) \cdot \psi(x, T) dx
\]
when $\varepsilon$ goes to zero.

### Appendix B. Functionnal analysis in Banach valued spaces

**Proof.** [of Proposition 5.1] One has:
\[
\left| \int_{\Omega_T} D_{a}^{\varphi} f(x, a, t) dx dt \right| \leq \| D_{a}^{\varphi} f \|_{Y_T} \| \varphi \|_{X_T}
\]
for any $\varphi \in X_T$. This proves that $D_{a}^{\varphi}$ belongs to $X'_T$. $X_T$ is a separable Banach space. Thus according to Corollary 3.30 p. 76 [4] for any bounded sequence in $X'_T$ there exists a subsequence converging in the weak-* topology $\sigma(X'_T, X_T)$. As the bound $\| D_{a}^{\varphi} f \|_{Y_T}$ is uniform with respect to $\sigma$, there exists a weak limit in $X'_T$ denoted $g$ and we define the $X'_T, X_T$ duality pairing as
\[
\lim_{\sigma_k \rightarrow 0} \int_{\Omega_T} D_{a}^{\varphi} f dx da dt = \langle g, \varphi \rangle_{X'_T, X_T}, \quad \text{as } \tau \rightarrow 0.
\]
Moreover, $g = \partial_a f$ in the weak sense i.e. for any $\varphi \in \mathcal{D}(\Omega_T)$ one has
\[
\int_{\Omega_T} f(x, a, t) \partial_a \varphi(x, a, t) dx dt = -\langle g, \varphi \rangle_{X'_T, X_T}.
\]
The same results hold obviously for $\partial_t f$. $\square$

**Proof.** [of Proposition 5.2] By Proposition 5.1 there exists a limit $\partial_a f \in X'_T$ and a subsequence s.t.
\[
D_{a}^{\varphi} f \rightharpoonup \partial_a f \text{ weak-* in } X'_T.
\]
First, $D_{a}^{\varphi} f$ defines a linear continuous functional on $Z_T$:
\[
\left| \int_{\Omega_T} D_{a}^{\varphi} f dx \right| \leq \| D_{a}^{\varphi} f \|_{Y_T} \| \varphi \|_{Z_T}.
\]
Moreover, one has tightness, for any continuous positive $\theta_m$ s.t. $\theta_m \equiv 1$ on $[0, m]$ and $\text{supp} \theta_m \in [0, m + 1]$  

\[
|\langle D^n f, \varphi(1 - \theta_m) \rangle| \leq \int_{Q_T} |D^n f(x, a, t)\varphi(1 - \theta_m)(x, a, t)| dxdadt \\
\leq \int_{m}^{T} \int_{\mathbb{R}^+} \chi_{\text{supp}(1-\theta_m)}(x) \text{ess sup} |D^n f| dxdadt \|\varphi(1 - \theta_m)\|_{L^\infty(Q_T; L^1(\Omega))} \\
\leq \int_{0}^{T} \int_{m}^{\infty} \text{ess sup} |D^n f| dxdadt \|\varphi(1 - \theta_m)\|_{X_T},
\]

the latter term goes to zero as $m$ tends to infinity by the Lebesgue's Theorem. Now one writes for any $\varphi$ in $Z_T$ 

\[
|\langle D^n f, \varphi \rangle - \langle D^n f, \varphi \rangle| \leq |\langle D^n f, \varphi \theta_m \rangle - \langle D^n f, \varphi \theta_m \rangle| \\
+ |\langle D^n f, \varphi(1 - \theta_m) \rangle - \langle D^n f, \varphi(1 - \theta_m) \rangle|.
\]

For every $\delta > 0$ there exists $k_0$ s.t. for $k$ and $j$ greater than $k_0$ the first term on the right hand side is smaller than $\delta/3$ by weak-* convergence in $\sigma(X_T^*, X_T)^*$, while the two latter terms can be made smaller than $2\delta/3$ due to the tightness proved above. This implies, because $\mathbb{R}$ is complete, that there exists a limit $L$ s.t. 

\[
L_j := \langle D^n f, \varphi \rangle > |z^*_{T_j} \rightarrow L, \text{ when } j \rightarrow \infty.
\]

Since for every arbitrary fixed $\delta$ there exists $j_0$ s.t. $j > j_0$ implies 

\[
|L| \leq |L - L_j| + |L_j| \leq \delta + C\|\varphi\|_{Z_T}.
\]

$L$ is also a linear continuous form on $Z_T$ thanks to [B.1]. By similar arguments as above we identify this limit with the weak derivative $\partial_t f$ and we denote $\langle \partial_t f, \varphi \rangle > |z^*_{T_j} \rightarrow L(\varphi)$ for every $\varphi \in Z_T$. The same proof holds for the time derivative as well. 

**Lemma Appendix B.1.** If $f \in U_T$ then 

\[
\sup_{(\varphi_1, \varphi_2) \in \mathcal{D}(Q_T; L^1(\Omega)) \atop |\varphi_2(x, a, t)| \leq 1, i = 1, 2} \left| \int_{Q_T} f(\varphi_1 + \varphi_2)dx da dt \right| \leq \|f\|_{Y_T}.
\]

**Proof.** Taking the test function $\varphi_1 \in \mathcal{D}(Q_T; L^1(\Omega))$, one has 

\[
\left| \int_{Q_T} f\partial_t \varphi dx da dt \right| = \lim_{\sigma \rightarrow 0} \left| \int_{Q_T} f(x, a, t)\varphi(x, a, t + \sigma) \frac{\sigma}{\varphi} - \varphi(x, a, t) dx da dt \right| \\
= \lim_{\sigma \rightarrow 0} \left| \int_{\text{supp}\varphi + \sigma e} \varphi(x, a, t + \sigma) \frac{f(x, a, t + \sigma) - f(x, a, t)}{\sigma} dx da dt \right| \\
\leq \lim_{\sigma \rightarrow 0} \|D^n f\|_{Y_T} \|\varphi\|_{X_T} \leq \|f\|_{U_T}.
\]

**Lemma Appendix B.2.** If $f \in U_T$ then 

\[
\int_{0}^{T - \Delta t} \int_{\mathbb{R}^+} \text{ess sup}_{x \in \Omega} f(x, a + \Delta a, t + \Delta t) - \int_{0}^{1} f(x, a + s\Delta a, t + s\Delta t) ds dxdadt \\
\leq (\Delta a + \Delta t)\|f\|_{U_T}.
\]
One regularises $f$ by convolution with a standard mollifier in age and time setting $f^\delta := \omega_\delta * f$ where $\omega_\delta(a,t) := \omega(t/\delta) \omega(a/\delta) / \delta^2$ and $\omega$ is a standard approximation kernel. By density of regular $L^\infty(\Omega)$-valued functions in $Y_T$ (cf. p.12 Corollary 4.11), one has
\[
\int_0^{T-\Delta t} \int_{\mathbb{R}_+} \|f(\cdot, a + \Delta a, t + \Delta t) - f^\delta(\cdot, a + \Delta a, t + \Delta t)\|_{L^\infty(\Omega)} da dt \to 0
\]
as well as
\[
\int_0^{T-\Delta t} \int_{\mathbb{R}_+} \left\| \int_0^1 f(\cdot, a + s\Delta a, t + s\Delta t) - f^\delta(\cdot, a + s\Delta a, t + s\Delta t) ds \right\|_{L^\infty(\Omega)} da dt \to 0
\]
as $\delta \to 0$. One then has that
\[
f^\delta(x, a + \Delta a, t + \delta t) - \int_0^1 f^\delta(x, a + s\Delta a, t + s\Delta t) ds
\]
\[
= \left( \int_0^1 \left\{ \nabla_{a,t} f^\delta(x, a + s\Delta a, t + s\Delta t) - \int_0^1 \nabla_{a,t} f^\delta(x, a + s\tilde{s}\Delta a, t + s\tilde{s}\Delta t) ds \right\} ds \right) \cdot \left( \Delta a / \Delta t \right).
\]
Setting $J^\delta(x, a, t) := f^\delta(x, a + \Delta a, t + \delta t) - \int_0^1 f^\delta(x, a + s\Delta a, t + s\Delta t) ds$ and integrating in age and time gives that
\[
\|J^\delta\|_{Y_T} \leq 2(\Delta a + \Delta t) \|\nabla_{a,t} f^\delta\|_{Y_T} \leq 2(\Delta a + \Delta t) \|f^\delta\|_{U_T},
\]
because $f^\delta$ is regular and thus the limit of the finite differences is the $L^1_a L^\infty_x$-norm of the gradient. Next, thanks to the linearity of the convolution operator and its continuity in $L^1$ norm, one shows that $\|f^\delta\|_{U_T} \lesssim \|f\|_{U_T}$ and one concludes thanks to a triangular inequality. \qed

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