Dynamics of symmetric holomorphic maps on projective spaces

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Abstract

We consider complex dynamics of a critically finite holomorphic map from $\mathbb{P}^k$ to $\mathbb{P}^k$, which has symmetries associated with the symmetric group $S_{k+2}$ acting on $\mathbb{P}^k$, for each $k \geq 1$. The Fatou set of each map of this family consists of attractive basins of superattracting points. Each map of this family satisfies Axiom A.

1 Introduction

For a finite group $G$ acting on $\mathbb{P}^k$ as projective transformations, we say that a rational map $f$ on $\mathbb{P}^k$ is $G$-equivariant if $f$ commutes with each element of $G$. That is, $f \circ r = r \circ f$ for any $r \in G$, where $\circ$ denotes the composition of maps. Doyle and McMullen [4] introduced the notion of equivariant functions on $\mathbb{P}^1$ to solve quintic equations. See also [11] for equivariant functions on $\mathbb{P}^1$. Crass [2] extended Doyle and McMullen’s algorithm to higher dimensions to solve sextic equations. Crass [3] found a good family of finite groups and equivariant maps for which one may say something about global dynamics. Crass [3] conjectured that the Fatou set of each map of this family consists of attractive basins of superattracting points. Although I do not know whether this family has relation to solving equations or not, our results will give affirmative answers for the conjectures in [3].

In section 2 we shall explain an action of the symmetric group $S_{k+2}$ on $\mathbb{P}^k$ and properties of our $S_{k+2}$-equivariant map. In section 3 and 4 we shall show our results about the Fatou sets and hyperbolicity of our maps by using properties of our maps and Kobayashi metrics.
2 $S_{k+2}$-equivariant maps

Crass [3] selected the symmetric group $S_{k+2}$ as a finite group acting on $\mathbb{P}^k$ and found an $S_{k+2}$-equivariant map which is holomorphic and critically finite for each $k \geq 1$. We denote by $\mathcal{C} = \mathcal{C}(f)$ the critical set of $f$ and say that $f$ is critically finite if each irreducible component of $\mathcal{C}(f)$ is periodic or preperiodic. More precisely, $S_{k+2}$-equivariant map $g_{k+3}$ defined in section 2.2 preserves each irreducible component of $\mathcal{C}(g_{k+3})$, which is a projective hyperplane. The complement of $\mathcal{C}(g_{k+3})$ is Kobayashi hyperbolic. Furthermore restrictions of $g_{k+3}$ to invariant projective subspaces have the same properties as above. See section 2.3 for details.

2.1 $S_{k+2}$ acts on $\mathbb{P}^k$

An action of the $(k+2)$-th symmetric group $S_{k+2}$ on $\mathbb{P}^k$ is induced by the permutation action of $S_{k+2}$ on $\mathbb{C}^{k+2}$ for each $k \geq 1$. The transposition $(i,j)$ in $S_{k+2}$ corresponds with the transposition “$u_i \leftrightarrow u_j$” on $\mathbb{C}_u^{k+2}$, which pointwise fixes the hyperplane $\{u_i = u_j\} = \{u \in \mathbb{C}_u^{k+2} \mid u_i = u_j\}$. Here $\mathbb{C}^{k+2} = \mathbb{C}_u^{k+2} = \{u = (u_1,u_2,\ldots,u_{k+2}) \mid u_i \in \mathbb{C} \text{ for } i = 1,\ldots,k+2\}$.

The action of $S_{k+2}$ preserves a hyperplane $H$ in $\mathbb{C}_u^{k+2}$, which is identified with $\mathbb{C}^{k+1}$ by projection $A : \mathbb{C}_u^{k+2} \rightarrow \mathbb{C}^{k+1}$,

$$H = \left\{ \sum_{i=1}^{k+2} u_i = 0 \right\} \cong \mathbb{C}_x^{k+1} \text{ and } A = \begin{pmatrix} 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}. $$

Here $\mathbb{C}^{k+1} = \mathbb{C}_x^{k+1} = \{x = (x_1,x_2,\ldots,x_{k+1}) \mid x_i \in \mathbb{C} \text{ for } i = 1,\ldots,k+1\}$.

Thus the permutation action of $S_{k+2}$ on $\mathbb{C}_u^{k+2}$ induces an action of “$S_{k+2}$” on $\mathbb{C}^{k+1}$. Here “$S_{k+2}$” is generated by the permutation action $S_{k+1}$ on $\mathbb{C}_x^{k+1}$ and a $(k+1,k+1)$-matrix $T$ which corresponds to the transposition $(1, k+2)$ in $S_{k+2}$,

$$T = \begin{pmatrix} -1 & 0 & \ldots & 0 \\ -1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ -1 & 0 & \ldots & 1 \end{pmatrix}. $$

Hence the hyperplane corresponding to $\{u_i = u_j\}$ is $\{x_i = x_j\}$ for $1 \leq i < j \leq k+1$. The hyperplane corresponding to $\{u_i = u_{k+2}\}$ is $\{x_i = 0\}$ for $1 \leq i \leq k+1$. Each element in “$S_{k+2}$” which corresponds to some transposition in $S_{k+2}$ pointwise fixes one of these hyperplanes in $\mathbb{C}^{k+1}$.
The action of “$S_{k+2}$” on $C^{k+1}$ projects naturally to the action of “$S_{k+2}$” on $P^k$. These hyperplanes on $C^{k+1}$ projects naturally to projective hyperplanes on $P^k$. Here $P^k = \{ x = [x_1 : x_2 : \cdots : x_{k+1}] \mid (x_1, x_2, \cdots, x_{k+1}) \in C^{k+1} \setminus \{0\} \}$. Each element in the action of “$S_{k+2}$” on $P^k$ which corresponds to some transposition in $S_{k+2}$ pointwise fixes one of these projective hyperplanes. We denote “$S_{k+2}$” also by $S_{k+2}$ and call these projective hyperplanes transposition hyperplanes.

### 2.2 Existence of our maps

One way to get $S_{k+2}$-equivariant maps on $P^k$ which are critically finite is to make $S_{k+2}$-equivariant maps whose critical sets coincide with the union of the transposition hyperplanes.

**Theorem 1** ([3]). For each $k \geq 1$, $g_{k+3}$ defined below is the unique $S_{k+2}$-equivariant holomorphic map of degree $k + 3$ which is doubly critical on each transposition hyperplane.

$$g = g_{k+3} = [g_{k+3,1} : g_{k+3,2} : \cdots : g_{k+3,k+1}] : P^k \to P^k,$$

where $g_{k+3,l}(x) = x_l^3 \sum_{s=0}^{k} (-1)^s \frac{s + 1}{s + 3} x_l^s A_{k-s}$. $A_0 = 1$, and $A_{k-s}$ is the elementary symmetric function of degree $k-s$ in $C^{k+1}$.

Then the critical set of $g$ coincides with the union of the transposition hyperplanes. Since $g$ is $S_{k+2}$-equivariant and each transposition hyperplane is pointwise fixed by some element in $S_{k+2}$, $g$ preserves each transposition hyperplane. In particular $g$ is critically finite. Although Crass [3] used this explicit formula to prove Theorem 1, we shall only use properties of the $S_{k+2}$-equivariant maps described below.

### 2.3 Properties of our maps

Let us look at properties of the $S_{k+2}$-equivariant map $g$ on $P^k$ for a fixed $k$, which is proved in [3] and shall be used to prove our results. Let $L^{k-1}$ denote one of the transposition hyperplanes, which is isomorphic to $P^{k-1}$. Let $L^m$ denote one of the intersections of $(k-m)$ or more distinct transposition hyperplanes which is isomorphic to $P^m$ for $m = 0, 1, \cdots, k - 1$.

First, let us look at properties of $g$ itself. The critical set of $g$ consists of the union of the transposition hyperplanes. By $S_{k+2}$-equivariance, $g$ preserves each transposition hyperplane. Furthermore the complement of the critical set of $g$ is Kobayashi hyperbolic.
Next, let us look at properties of $g$ restricted to $L^m$ for $m = 1, 2, \cdots, k - 1$. Let us fix any $m$. Since $g$ preserves each $L^m$, we can also consider the dynamics of $g$ restricted to any $L^m$. Each restricted map has the same properties as above. Let us fix any $L^m$ and denote by $g|_{L^m}$ the restricted map of $g$ to the $L^m$. The critical set of $g|_{L^m}$ consists of the union of intersections of the $L^m$ and another $L^{k-1}$ which does not include the $L^m$. We denote it by $L^m_{k-1}$, which is an irreducible component of the critical set of $g|_{L^m}$. By $S_{k+2}$-equivariance, $g|_{L^m}$ preserves each irreducible component of the critical set of $g|_{L^m}$. Furthermore the complement of the critical set of $g|_{L^m}$ in $L^m$ is Kobayashi hyperbolic.

Finally, let us look at a property of superattracting fixed points of $g$. The set of superattracting points, where the derivative of $g$ vanishes for all directions, coincides with the set of $L^0$'s.

**Remark 1.** For every $k \geq 1$ and every $m$, $1 \leq m \leq k$, a restricted map of $g_{k+3}$ to any $L^m$ is not conjugate to $g_{m+3}$.

### 2.4 Examples for $k = 1$ and 2

Let us see transposition hyperplanes of the $S_3$-equivariant function $g_4$ and the $S_4$-equivariant map $g_5$ to make clear what $L^m$ is. In [3] one can find explicit formulas and figures of dynamics of $S_{k+2}$-equivariant maps in low-dimensions.

**2.4.1 $S_3$-equivariant function $g_4$ in $P^1$**

$$g_3([x : y]) = [x^3(-x + 2y) : x^2(2x - y)] : \mathbb{P}^1 \to \mathbb{P}^1,$$

$$C(g_3) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_1 = x_2\} = \{0, 1, \infty\} \text{ in } \mathbb{P}^1.$$ 

In this case "transposition hyperplanes" are points in $\mathbb{P}^1$ and $L^0$ denotes one of three superattracting fixed points of $g_3$.

**2.4.2 $S_4$-equivariant map $g_5$ in $P^2$**

$$C(g_5) = \{x_1 = 0\} \cup \{x_2 = 0\} \cup \{x_3 = 0\} \cup \{x_1 = x_2\} \cup \{x_2 = x_3\} \cup \{x_3 = x_1\} \text{ in } \mathbb{P}^2.$$ 

In this case $L^1$ denotes one of six transposition hyperplanes in $\mathbb{P}^2$, which is an irreducible component of $C(g_5)$. For example, let us fix a transposition hyperplane $\{x_1 = 0\}$. Since $g_5$ preserves each transposition hyperplane,
we can also consider the dynamics of \( g_5 \) restricted to \( \{ x_1 = 0 \} \). We denote by \( g_5|_{x_1=0} \) the restricted map of \( g_5 \) to \( \{ x_1 = 0 \} \). The critical set of \( g_5|_{x_1=0} \) in \( \{ x_1 = 0 \} \simeq \mathbb{P}^1 \) is

\[
C(g_5|_{x_1=0}) = \{ [0 : 1 : 0], [0 : 0 : 1], [0 : 1 : 1] \}.
\]

When we use \( L^0 \) after we fix \( \{ x_1 = 0 \} \), \( L^0 \) denotes one of intersections of \( \{ x_1 = 0 \} \) and another transposition hyperplane, which is a superattracting fixed point of \( g_5|_{x_1=0} \) in \( \mathbb{P}^1 \). The set of superattracting fixed points of \( g_5 \) in \( \mathbb{P}^2 \) is

\[
\{ [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 0], [1 : 1 : 1], [0 : 1 : 1] \}.
\]

In general \( L^0 \) denotes one of intersections of two or more transposition hyperplanes, which is a superattracting fixed point of \( g_5 \) in \( \mathbb{P}^2 \).

## 3 The Fatou sets of the \( S_{k+2} \)-equivariant maps

### 3.1 Definitions and preliminaries

Let us recall theorems about critically finite holomorphic maps. Let \( f \) be a holomorphic map from \( \mathbb{P}^k \) to \( \mathbb{P}^k \). The Fatou set of \( f \) is defined to be the maximal open subset where the iterates \( \{ f^n \}_{n \geq 0} \) is a normal family. The Julia set of \( f \) is defined to be the complement of the Fatou set of \( f \). Each connected component of the Fatou set is called a Fatou component. Let \( U \) be a Fatou component of \( f \). A holomorphic map \( h \) is said to be a limit map on \( U \) if there is a subsequence \( \{ f^n \}_{s \geq 0} \) which locally converges to \( h \) on \( U \). We say that a point \( q \) is a Fatou limit point if there is a limit map \( h \) on a Fatou component \( U \) such that \( q \in h(U) \). The set of all Fatou limit points is called the Fatou limit set. We define the \( \omega \)-limit set \( E(f) \) of the critical points by

\[
E(f) = \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty} f^n(C).
\]

**Theorem 2.** ([10, Proposition 5.1]) If \( f \) is a critically finite holomorphic map from \( \mathbb{P}^k \) to \( \mathbb{P}^k \), then the Fatou limit set is contained in the \( \omega \)-limit set \( E(f) \).

Let us recall the notion of Kobayashi metrics. Let \( M \) be a complex manifold and \( K_M(x,v) \) the Kobayashi quasimetric on \( M \),

\[
\inf \left\{ |a| \mid \varphi : D \to M : \text{holomorphic}, \varphi(0) = x, D\varphi \left( a \left( \frac{\partial}{\partial z} \right) \right) = v, a \in \mathbb{C} \right\}
\]
for $x \in M$, $v \in T_xM$, $z \in D$, where $D$ is the unit disk in $C$. We say that $M$ is Kobayashi hyperbolic if $K_M$ becomes a metric. Theorem 5 is a corollary of Theorem 3 and Theorem 4 for $k = 1$ and 2.

Theorem 3. (a basic result whose former statement can be found in [8, Corollary 14.5]) If $f$ is a critically finite holomorphic function from $P^1$ to $P^1$, then the only Fatou components of $f$ are attractive components of superattracting points. Moreover if the Fatou set is not empty, then the Fatou set has full measure in $P^1$.

Theorem 4. ([5, theorem 7.7]) If $f$ is a critically finite holomorphic map from $P^2$ to $P^2$ and the complement of $C(f)$ is Kobayashi hyperbolic, then the only Fatou components of $f$ are attractive components of superattracting points.

3.2 Our first result

Let us fix any $k$ and $g = g_{k+3}$. For every $m$, $2 \leq m \leq k$, we can apply an argument in [5] to a restricted map of $g$ to any $L^m_k$ because every $L^{m-1}_k$ is smooth and because every $L^m_k \cup C(g|_{L^m_k})$ is Kobayashi hyperbolic. We shall use this argument in Lemma 1, which is used to prove Proposition 1.

Proposition 1. For any Fatou component $U$ which is disjoint from $C(g)$, there exists an integer $n$ such that $g^n(U)$ intersects with $C(g)$.

Proof: We suppose that $g^n(U)$ is disjoint from $C(g)$ for any $n$ and derive a contradiction by using Lemma 1 and Remark 3 below. Take any point $x_0 \in U$. Since $E(g)$ coincides with $C(g)$, $g^n(x_0)$ accumulates to $C(g)$ as $n$ tends to $\infty$ from Theorem 2. Since $C(g)$ is the union of the transposition hyperplanes, there exists a smallest integer $m_1$ such that $g^n(x_0)$ accumulates to some $L^{m_1}$.

Let $h_1$ be a limit map on $U$ such that $h_1(x_0)$ belongs to the $L^{m_1}$. From Lemma 1 below, the intersection of $h_1(U)$ and the $L^{m_1}$ is an open set in the $L^{m_1}$ and is contained in the Fatou set of $g|_{L^{m_1}}$.

We next consider the dynamics of $g|_{L^{m_1}}$. If there exists an integer $n_2$ such that $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with $C(g|_{L^{m_1}})$, then $g^{n_2}(h_1(U) \cap L^{m_1})$ intersects with some $L^{m_1-1}$. In this case we can consider the dynamics of $g|_{L^{m_1-1}}$. On the other hand, if there does not exist such $n_2$, then there exists an integer $m_2$ and a limit map $h_2$ on $h_1(U) \cap L^{m_1}$ such that the intersection of $h_2(h_1(U) \cap L^{m_1})$ and some $L^{m_2}$ is an open set in the $L^{m_2}$ from Remark 3 below. Thus it is contained in the Fatou set of $g|_{L^{m_2}}$. Here $m_2$ is smaller than $m_1$. In this case we can consider the dynamics of $g|_{L^{m_2}}$.

We continue the same argument above. These reductions finally come to some $L^1$ and we use Theorem 5. One can find a similar reduction argument in the proof of Theorem 5. Consequently $g^n(x_0)$ accumulates to some
superattracting point $L^0$. So there exists an integer $s$ such that $g^s$ sends $U$ to
the attractive Fatou component which contains the superattracting point
$L^0$. Thus $g^s(U)$ intersects with $C(g)$, which is a contradiction.

**Remark 2.** Even if a Fatou component $U$ intersects with some $L^m$ and is disjoint
from any $L^{m-1}$, then the similar thing as above holds for the dynamics in the $L^m$.
In this case $U \cap L^m$ is contained in the Fatou set of $g|_{L^m}$ and there exists an integer
$n$ such that $g^n(U \cap L^m)$ intersects with $C(g|_{L^m})$.

**Lemma 1.** For any Fatou component $U$ which is disjoint from $C(g)$ and any
point $x_0 \in U$, let $h$ be a limit map on $U$ such that $h(x_0)$ belongs to some $L^m$ and
does not belong to any $L^{m-1}$. If $g^n(U)$ is disjoint from $C(g)$ for every $n \geq 1$,
then the intersection of $h(U)$ and the $L^m$ is an open set in the $L^m$.

**Proof:** Let $B$ be the complement of $C(g)$. Since $B$ is Kobayashi hyperbolic
and $B$ includes $g^{-1}(B)$, $g^{-1}(B)$ is Kobayashi hyperbolic, too. So we can
use Kobayashi metrics $K_B$ and $K_{g^{-1}(B)}$. Since $B$ includes $g^{-1}(B)$,

$$K_B(x,v) \leq K_{g^{-1}(B)}(x,v) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbb{P}^k.$$ 

In addition, since $g$ is an unbranched covering from $g^{-1}(B)$ to $B$,

$$K_{g^{-1}(B)}(x,v) = K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbb{P}^k.$$ 

From these two inequalities we have the following inequality

$$K_B(x,v) \leq K_B(g(x), Dg(v)) \text{ for all } x \in g^{-1}(B), v \in T_x \mathbb{P}^k.$$ 

Since the same argument holds for any $g^n$ from $g^{-n}(B)$ to $B$,

$$K_B(x,v) \leq K_B(g^n(x), Dg^n(v)) \text{ for all } x \in g^{-n}(B), v \in T_x \mathbb{P}^k.$$ 

Since $g^n$ is an unbranched covering from $U$ to $g^n(U)$ and $B$ includes $g^n(U)$
for every $n$, a sequence $\{K_B(g^n(x), Dg^n(v))\}_{n \geq 0}$ is bounded for all $x \in U$,
$v \in T_x \mathbb{P}^k$. Hence we have the following inequality for any unit vectors $v_n$
in $T_{x_0} U$ with respect to the Fubini-Study metric in $\mathbb{P}^k$,

$$0 < \inf_{|v|=1} K_B(x_0,v) \leq K_B(x_0,v_n) \leq K_B(g^n(x_0), Dg^n(x_0)v_n) < \infty.$$ 

That is, the sequence $\{K_B(g^n(x_0), Dg^n(x_0)v_n)\}_{n \geq 0}$ is bounded away from
$0$ and $\infty$ uniformly.

We shall choose $v_n$ so that $Dg^n(x_0)v_n$ keeps parallel to the $L^m$ and claim
that $Dh(x_0)v \neq 0$ for any accumulation vector $v$ of $v_n$. Let $h = \lim_{n \to \infty} g^n$
for simplicity. Let $V$ be a neighborhood of $h(x_0)$ and $\psi$ a local coordinate on $V$ so that $\psi(h(x_0)) = 0$ and $\psi(L^m \cap V) \subset \{y = (y_1, y_2, \ldots, y_k) \mid y_1 = \cdots = y_{k-m} = 0\}$. In this chart there exists a constant $r > 0$ such that a polydisk $P(0, 2r)$ does not intersect with any images of transposition hyperplanes which do not include the $L^m$. Since $\psi(g^n(x_0))$ converges to 0 as $n$ tends to $\infty$, we may assume that $\psi(g^n(x_0))$ belongs to $P(0, r)$ for large $n$. Let $\{v_n\}_{n \geq 0}$ be unit vectors in $T_{x_0}P^k$ and $\{w_n\}_{n \geq 0}$ vectors in $T_{\psi(g^n(x_0))}C^k$ so that $w_n$ keep parallel to $\psi(L^m)$ with a same direction and

$$Dg^n(x_0)v_n = |Dg^n(x_0)v_n| \frac{D\psi^{-1}(w_n)}{r}.$$ 

So we may assume that the length of $w_n$ is almost unit for large $n$. We define holomorphic maps $\varphi_n$ from $D$ to $P(0, 2r)$ as

$$\varphi_n(z) = \psi(g^n(x_0)) + rzw_n \quad \text{for } z \in D$$

and consider holomorphic maps $\psi^{-1} \circ \varphi_n$ from $D$ to $B$ for large $n$. Then

$$(\psi^{-1} \circ \varphi_n)(0) = g^n(x_0),$$

$$D(\psi^{-1} \circ \varphi_n) \left( \frac{|Dg^n(x_0)v_n|}{r} \frac{\partial}{\partial z} \right)_0 = Dg^n(x_0)v_n.$$ 

Suppose $Dh(x_0)v = 0$, then $Dg^n(x_0)v$ converges to 0 as $n$ tends to $\infty$ and so does $Dg^n(x_0)v_n$. By the definition of Kobayashi metric we have that

$$K_B(g^n(x_0), Dg^n(x_0)v_n) \leq \frac{|Dg^n(x_0)v_n|}{r} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$ 

Since this contradicts (1), we have $Dh(x_0)v \neq 0$. This holds for all directions which are parallel to $\psi(L^m)$. Consequently the intersection of $h(U)$ and the $L^m$ is an open set in $L^m$. \qed

Remark 3. The similar thing as above holds for the dynamics of any restricted map. Thus even if a Fatou component $g^n(U)$ intersects with $C(g)$ for some $n$, the same result as above holds. Because one can consider the dynamics in the $L^m$ when $g^n(U)$ intersects with some $L^m$.

Theorem 5. For each $k \geq 1$, the Fatou set of the $S_{k+2}$-equivariant map $g$ consists of attractive basins of superattracting fixed points which are intersections of $k$ or more distinct transposition hyperplanes.
**Proof:** This theorem follows from Proposition 1 and Remark 2 immediately. Let us describe details. Take any Fatou component $U$. From Proposition 1 there exists an integer $n_k$ such that $g^{n_k}(U)$ intersects with $C(g)$. Since $C(g)$ is the union of the transposition hyperplanes, $g^{n_k}(U)$ intersects with some $L^{k-1}$. By doing the same thing as above for the dynamics of $g$ restricted to the $L^{k-1}$, there exists an integer $n_{k-1}$ such that $g^{n_k+n_{k-1}}(U)$ intersects with some $L^{k-2}$ from Remark 2. We again do the same thing as above for the dynamics of $g$ restricted to the $L^{k-2}$. These reductions finally come to some $L^1$. That is, there exists integers $n_{k-2}, \cdots, n_2$ such that $g^{n_k+n_{k-1}+\cdots+n_2}(U)$ intersects with some $L^0$. Hence $g^{n_k+n_{k-1}+\cdots+n_1}$ sends $U$ to the attractive Fatou component which contains the superattracting fixed point $L^0$ in $P^k$. 

4 Axiom A and the $S_{k+2}$-equivariant maps

4.1 Definitions and preliminaries

Let us define hyperbolicity of non-invertible maps and the notion of Axiom A. See [6] for details. Let $f$ be a holomorphic map from $P^k$ to $P^k$ and $K$ a compact subset such that $f(K) = K$. Let $\hat{K}$ be the set of histories in $K$ and $\hat{f}$ the induced homeomorphism on $\hat{K}$. We say that $f$ is hyperbolic on $K$ if there exists a continuous decomposition $T_{\hat{K}} = E^u + E^s$ of the tangent bundle such that $D\hat{f}(E^u_{\hat{x}}) \subset E^u_{\hat{f}(\hat{x})}$ and if there exists constants $c > 0$ and $\lambda > 1$ such that for every $n \geq 1$,

$$|D\hat{f}^n(v)| \geq c\lambda^n|v| \text{ for all } v \in E^u \text{ and}$$

$$|D\hat{f}^n(v)| \leq c^{-1}\lambda^{-n}|v| \text{ for all } v \in E^s.$$ 

Here $|\cdot|$ denotes the Fubini-Study metric on $P^k$. If a decomposition and inequalities above hold for $f$ and $K$, then it also holds for $\hat{f}$ and $\hat{K}$. In particular we say that $f$ is expanding on $K$ if $f$ is hyperbolic on $K$ with unstable dimension $k$. Let $\Omega$ be the non-wandering set of $f$, i.e., the set of points for any neighborhood $U$ of which there exists an integer $n$ such that $f^n(U)$ intersects with $U$. By definition, $\Omega$ is compact and $f(\Omega) = \Omega$. We say that $f$ satisfies Axiom A if $f$ is hyperbolic on $\Omega$ and periodic points are dense in $\Omega$.

Let us introduce a theorem which deals with repelling part of dynamics. Let $f$ be a holomorphic map from $P^k$ to $P^k$. We define the $k$-th Julia set
$J_k$ of $f$ to be the support of the measure with maximal entropy, in which repelling periodic points are dense. It is a fundamental fact that in dimension 1 the 1st Julia set $J_1$ coincides with the Julia set $J$. Let $K$ be a compact subset such that $f(K) = K$. We say that $K$ is a repeller if $f$ is expanding on $K$.

**Theorem 6.** ([7]) Let $f$ be a holomorphic map on $\mathbb{P}^k$ of degree at least 2 such that the $\omega$-limit set $E(f)$ is pluripolar. Then any repeller for $f$ is contained in $J_k$. In particular,

$$J_k = \{\text{repelling periodic points of } f\}$$

If $f$ is critically finite, then $E(f)$ is pluripolar. We need the theorem above to prove our second result.

4.2 Our second result

**Theorem 7.** For each $k \geq 1$, the $S_{k+2}$-equivariant map $g$ satisfies Axiom A.

**Proof:** We only need to consider the $S_{k+2}$-equivariant map $g$ for a fixed $k$, because argument for any $k$ is similar as the following one. Let us show the statement above for a fixed $k$ by induction. A restricted map of $g$ to any $L^1$ satisfies Axiom A by using the theorem of critically finite functions (see [8, Theorem 19.1]). We only need to show that a restricted map of $g$ to a fixed $L^2$ satisfies Axiom A by symmetry. Argument for a restricted map of $g$ to any $L^m$, $3 \leq m \leq k$, is similar as for a restricted map of $g$ to the $L^2$. Let us denote $g|_{L^2}$, $\Omega(g|_{L^2})$, and $L^2$ by $g$, $\Omega$, and $\mathbb{P}^2$ for simplicity. 

We want to show that $g|_{L^2}$ is hyperbolic on $\Omega(g|_{L^2})$ by using Kobayashi metrics. If $g$ is hyperbolic on $\Omega$, then $\Omega$ has a decomposition to $S_i$,

$$\Omega = S_0 \cup S_1 \cup S_2,$$

where $i=0,1,2$ indicate the unstable dimensions. Since $C(g)$ attracts all nearby points, $S_0$ includes all the $L^0$'s and $S_1$ includes all the Julia sets of $g|_{L^1}$. We denote by $J(g|_{L^1})$ the Julia set of $g|_{L^1}$. Then $g$ is contracting in all directions at $L^0$ and is contracting in the normal direction and expanding in an $L^1$-direction on $J(g|_{L^1})$. Let us consider a compact, completely invariant subset in $\mathbb{P}^2 \setminus C$,

$$S = \{x \in \mathbb{P}^2 \mid \text{dist}(g^n(x), C) \to 0 \text{ as } n \to \infty\}.$$ 

By definition, we have $J_2 \subset S_2 \subset S$. If $g$ is expanding on $S$, then it follow that $S_0 = \cup L^0$, $S_1 = \cup J(g|_{L^1})$. Moreover $J_2 = S_2 = S$ holds from Theorem...
Since periodic points are dense in $J(g|_{L^1})$ and $J_2$, expansion of $g$ on $S$ implies Axiom A of $g$.

Let us show that $g$ is expanding on $S$. Because $f$ is attracting on $C$ and preserves $C$, there exists a neighborhood $V$ of $C$ such that $V$ is relatively compact in $g^{-1}(V)$ and the complement of $V$ is connected. We assume one of $L^1$'s to be the line at infinity of $P^2$. By letting $B$ be $P^2 \setminus V$ and $U$ one of connected components of $g^{-1}(P^2 \setminus V)$, we have the following inclusion relations,

$$U \subset g^{-1}(B) \subset B \subset C^2 = P^2 \setminus L^1.$$

Because $B$ and $U$ are in a local chart, there exists a constant $\rho < 1$ such that $K_B(x,v) \leq \rho K_U(x,v)$ for all $x \in U$, $v \in T_x C^2$.

In addition, since the map $g$ from $U$ to $B$ is an unbranched covering,

$$K_U(x,v) = K_B(g(x), Dg(v)) \text{ for all } x \in U, \ v \in T_x C^2.$$

From these two inequalities we have the following inequality

$$K_B(x,v) \leq \rho^n K_B(g^n(x), Dg^n(v)) \text{ for all } x \in g^{-n}(B), \ v \in T_x C^2.$$  

Since $g$ preserves $S$, which is contained in $g^{-n}(B)$ for every $n \geq 1$,

$$K_B(x,v) \leq \rho^n K_B(g^n(x), Dg^n(v)) \text{ for all } x \in S, \ v \in T_x C^2.$$  

Consequently we have the following inequality for $\lambda = \rho^{-1} > 1$,

$$K_B(g^n(x), Dg^n(v)) \geq \lambda^n K_B(x,v) \text{ for all } x \in S, \ v \in T_x C^2.$$  

Since $K_B(x,v)$ is upper semicontinuous and $|v|$ is continuous, $K_B(x,v)$ and $|v|$ may be different only by a constant factor. There exists $c > 0$ such that

$$|Dg^n(x)v| \geq c\lambda^n |v| \text{ for all } x \in S, \ v \in T_x C^2.$$  

Thus $g$ is expanding on $S$ and satisfies Axiom A.

**Remark 4.** Unlike the case when $k = 1$, it does not seem obvious that $S$ being a repeller implies $J_k = S$ when $k \geq 2$.

**Remark 5.** From [1, Theorem 4.11] and [9], it follows that the Fatou set of the $S_{k+2}$-equivariant map $g$ has full measure in $P^k$ for each $k \geq 1$.

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