A Note on the Convolution of Circle Impulses

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We present a formula for the convolution $\delta_{C_1} \ast \delta_{C_2}$ of two circle impulses. The derivation uses a classical addition theorem for Bessel functions.

Index Terms—Convolution, Impulses, Fourier transform, Hankel transform.

I. INTRODUCTION AND MAIN RESULT

The purpose of this short note is to derive a formula for the convolution of two circle impulses. To my knowledge such a result has not appeared. Appealing aspects of the derivation are the use of an old addition formula for Bessel functions, and at one point the algebra is rather remarkable.

We proceed with the formula.

For $i = 1, 2$ let $C_i$ be the circle $\|x - b_i\| = R_i$ centered at $b_i \in \mathbb{R}^2$ with radius $R_i$. Let $\delta_{C_i}$ be the impulse supported on $C_i$ (definition below) and let $\rho = \|x - (b_1 + b_2)\|$. The main result is that convolution results in the function

$$\delta_{C_1} \ast \delta_{C_2}(\rho) = \frac{4R_1R_2}{\sqrt{(\rho^2 - (R_1 - R_2)^2)((R_1 + R_2)^2 - \rho^2)}}, \quad (1)$$

if $|R_1 - R_2| < \rho < R_1 + R_2$, and $\delta_{C_1} \ast \delta_{C_2}(\rho) = 0$ if $0 < \rho < |R_1 - R_2|$ or $\rho > R_1 + R_2$. In particular, the formula expresses the fact that $\delta_{C_1} \ast \delta_{C_2}$ is radial with respect to the point $b_1 + b_2$. It is also symmetric in $C_1$ and $C_2$ as it should be.

II. DEFINITIONS AND BACKGROUND

One can define a circle impulse $\delta_{C}$ in several equivalent ways: via a limiting process, as for example in [1], or directly as a distribution paired with a smooth test function $\varphi$ that is of compact support on $\mathbb{R}^2$. The pairing is

$$\langle \delta_{C}, \varphi \rangle = \int_{C} \varphi \, ds, \quad (2)$$

where $ds$ is arclength along $C$.

A. Use of Pullbacks

It is useful for definitions and formulas to employ pullbacks of functions and distributions, see [2] for the general setup and [3] for special cases. For the circle $\|x - b\| = R$, the formula (1) in fact comes from the definition of $\delta_{C}$ as the pullback, $\delta_{C} = g^* \delta$, of the usual one-dimensional point $\delta$ by $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, $g(x) = \|x - b\| - R$. Keeping to the custom of evaluating $\delta$’s at points, this is usually expressed as $\delta_{C}(x_1, x_2) = \delta(g(x_1, x_2))$. Note that $\delta_{C}$ and subsequent formulas depend on the choice of the function $g$ describing $C$. Observe that for pullback we write $g^*$ instead of the more common $g^\ast$ so as not to clash, inevitably, with $\ast$ used for convolution.

Let $\tau_b(x) = x - b$ be the shift operator on $\mathbb{R}^2$. It is easy to show that if $C$ is the circle $\|x\| = R$ then $\tau_{b_1} \delta_{C}$ is the circle impulse for $\|x - b\| = R$. Moreover, as a shift of the convolution is the convolution of the shifts, if $C_i, i = 1, 2,$ are the circles $\|x\| = R_i$, we have

$$(\tau_{b_1+b_2})^\ast(\delta_{C_1} \ast \delta_{C_2}) = \tau_{b_1} \delta_{C_1} \ast \tau_{b_2} \delta_{C_2}, \quad (3)$$

so it suffices to prove (1) for circles concentric to the origin.

B. Radial Functions

To express a function (distribution) $f$ in polar coordinates is to form $\tilde{f}(r, \theta) = (Pf)(r, \theta)$ using the polar coordinate mapping $P(r, \theta) = (r \cos \theta, r \sin \theta)$. A function (distribution) is radial if $A^* f = f$ for any orthogonal transformation $A$, in which case we write the polar coordinate version simply as $\tilde{f}(r)$. The convolution of two radial functions (distributions) is radial and $\delta_{C}$ is radial for a circle centered at the origin. The Fourier transform of a radial function is also radial and

$$(\mathcal{F}f)^\sim(r) = \mathcal{F}\tilde{f}(r) = 2\pi \int_0^\infty \tilde{f}(\rho) J_0(2\pi r \rho) \, d\rho. \quad (4)$$

Here, $J_0$ is the zeroth order Bessel function of the first kind and $\mathcal{H}$ is the Hankel transform.

For the circle $C$, $\|x\| = R$, the Fourier transform of $\delta_{C}$ is the radial function $(\mathcal{F}\delta_{C})^\sim(r) = 2\pi R J_0(2\pi r R)$. This is a standard result. We can write it as

$$\mathcal{H}\delta_{C}(r) = 2\pi R J_0(2\pi r R) \quad (5)$$

See [4] for a general discussion of radial distributions.

III. CONVOLUTION AND THE NEUMANN ADDITION FORMULA

We consider the convolution $\delta_{C_1} \ast \delta_{C_2}$ for the concentric circles $\|x\| = R_i$. Invoking the convolution theorem and then expressing the result in polar coordinates, we have $$(\mathcal{F}(\delta_{C_1} \ast \delta_{C_2}))^\sim = (\mathcal{F}\delta_{C_1})^\sim(\mathcal{F}\delta_{C_2})^\sim,$$ or

$$\mathcal{H}(\delta_{C_1} \ast \delta_{C_2})^\sim(r) = \mathcal{H}\delta_{C_1}(r) \mathcal{H}\delta_{C_2}(r) = (2\pi)^2 R_1 R_2 J_0(2\pi r R_1) J_0(2\pi r R_2). \quad (4)$$

The Hankel transform is its own inverse, so we want to find $\mathcal{H}$ applied to the right-hand side.

An old result on the product of two Bessel functions, a consequence of the Neumann addition formula, comes to our aid. In the present setting it reads

$$J_0(2\pi R_1 r) J_0(2\pi R_2 r) = \frac{1}{2\pi} \int_0^{2\pi} J_0((R_1^2 + R_2^2 - 2 R_1 R_2 \cos \theta)^{1/2} 2\pi r) \, d\theta. \quad (5)$$

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See [5], Section 11.2. Temporarily drop the factor \((2\pi)^2 R_1 R_2\) in [4], and to further save space let
\[
\Psi(\theta) = (R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta)^{1/2}.
\]
Thus,
\[
\mathcal{H} \left( \frac{1}{2\pi} \int_0^{2\pi} J_0(\Psi(\theta)2\pi r) \, d\theta \right)(\rho) = \int_0^\infty \int_0^{2\pi} J_0(2\pi r \rho) J_0(\Psi(\theta)2\pi r) \, r \, d\theta \, dr.
\]
Interchange the order of integration and for the inner integral appeal to the “closure equation”:
\[
\int_0^{2\pi} J_0(2\pi r \rho) J_0(\Psi(\theta)2\pi r) \, r \, d\theta = \frac{\delta(\rho-\Psi(\theta))}{(2\pi)^2 \delta(\rho-\Psi(\theta))}.
\]
Finally, put the factor \((2\pi)^2 R_1 R_2\) back to obtain
\[
(\delta C_1 * \delta C_2)(\rho) = \frac{R_1 R_2}{\rho} \int_0^{2\pi} \delta(\rho-\Psi(\theta)) \, d\theta.
\]

A. \(\delta\) and Zeros

We can further analyze the right-hand side of (6) using the well known formula
\[
\delta(\Phi(\theta)) = \sum_n \frac{\delta(\theta-\theta_n)}{|\Phi'(\theta_n)|},
\]
where
\[
\Phi(\theta) = \rho - \Psi(\theta)
\]
and the \(\theta_n\) are the zeros of \(\Phi(\theta)\). This may be derived via pullbacks, or by other means, see, e.g., [3]. The formula holds so long as \(\Phi'(\theta_n) \neq 0\). The derivative is
\[
\Phi'(\theta) = -\frac{R_1 R_2 \sin \theta}{(R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta)^{1/2}}.
\]
This is zero at \(\theta = 0, \pi\) and \(2\pi\), and only there, and we will exclude these points by restricting \(\rho\).

The minimum of \(\Psi(\theta) = (R_1^2 + R_2^2 - 2R_1 R_2 \cos \theta)^{1/2}\) occurs for \(\theta = 0\) and \(\theta = 2\pi\) and has the value \(|R_1 - R_2|\).
The maximum occurs at \(\theta = \pi\) and has the value \(R_1 + R_2\). Thus if we restrict \(\rho\) to
\[
|R_1 - R_2| < \rho < R_1 + R_2,
\]
then within this range \(\Phi(\theta)\) has two zeros with nonzero derivatives, say at \(0 < \theta_1 < \pi\) and \(\pi < \theta_2 < 2\pi\). Evidently \(\theta_2 = 2\pi - \theta_1\) and the derivatives have equal absolute value. Strictly outside this range of \(\rho\), either \(0 < \rho < |R_1 - R_2|\) or \(\rho > R_1 + R_2\), there are no zeros and \(\delta(\Phi(\theta)) = 0\).

Now, \(\Phi(\theta) = 0\) when
\[
\cos \theta = \frac{R_1^2 + R_2^2 - \rho^2}{2R_1 R_2}.
\]

At such a value,
\[
\Phi'(\theta)^2 = \left(\frac{R_1 R_2}{\rho}\right)^2 \sin^2 \theta = \left(\frac{R_1 R_2}{\rho}\right)^2 (1 - \cos^2 \theta),
\]
and we can use (10) to write
\[
\Phi'(\theta)^2 = \left(\frac{R_1 R_2}{\rho}\right)^2 \left(1 - \left(\frac{R_1^2 + R_2^2 - \rho^2}{2R_1 R_2}\right)^2\right).
\]
This simplifies (!) to
\[
\Phi'(\theta)^2 = \frac{1}{4\rho^2}(\rho^2 - (R_1 - R_2)^2)((R_1 + R_2)^2 - \rho^2).
\]
Note how directly the condition (9) comes into play.

Going back to (7), again denoting the two zeros by \(\theta_1\) and \(\theta_2 = 2\pi - \theta_1\) we have \(|\Phi'(\theta_1)| = |\Phi'(\theta_2)|\) and
\[
\delta(\Phi(\theta)) = \frac{2\rho(\delta(\theta - \theta_1) + \delta(\theta - \theta_2))}{\sqrt{(\rho^2 - (R_1 - R_2)^2)((R_1 + R_2)^2 - \rho^2))}}.
\]
from which
\[
\frac{R_1 R_2}{\rho} \int_0^{2\pi} \delta(\Phi(\theta)) \, d\theta = \frac{4R_1 R_2}{\sqrt{(\rho^2 - (R_1 - R_2)^2)((R_1 + R_2)^2 - \rho^2))}}.
\]
This is (11) for circles concentric at the origin. The general result follows from this as in [3].

B. Sample Plots

Here is a surface plot of \(\delta C_1 * \delta C_2\) for the concentric circles \(|x| = 2\) (red) and \(|x| = 3\) (blue), together with the profile plot.

It is interesting to experiment with different configurations of circles.
IV. ADDITIONAL FORMULAS

Finally, it is natural to inquire about two other operations with impulses, namely multiplication and convolution with a function. Let $C$ be the circle $\|x\| = R$. For a function $f$ and for $x \neq 0$ let $f_C(x) = f(Rx/\|x\|)$. Then

$$f \delta_C = f_C \delta_C. \quad (11)$$

Note that if $f$ is radial then $f \delta_C = \tilde{f}(R) \delta_C$.

For convolution the result is

$$(f * \delta_C)(x_1, x_2) = \int_0^{2\pi} f(x_1 + R \cos \theta, x_2 + R \sin \theta) Rd\theta. \quad (12)$$

In words, forming $(f * \delta_C)(x)$ replaces $f(x)$ with an average of $f$ on the circle centered at $x$ of radius $R$. Thus $(f * \delta_C)(x_0)$ depends only on the values of $f$ on the circle $\|x - x_0\| = R$, a kind of “locally radializing” the function at each point.

The derivations of (11) and (12) are straightforward and we omit the details.

REFERENCES

[1] R. Bracewell, Two-Dimensional Imaging. Englewood Cliffs, NJ: Prentice Hall, 1995.
[2] G. Friedlander and M. Joshi, Introduction to the Theory of Distributions, 2nd edition. Cambridge, UK: Cambridge Univ. Press, 1998.
[3] B. Osgood, Lectures on the Fourier Transform and its Applications. Providence, RI: Amer. Math. Soc., 2019.
[4] L. Grafakos and G. Teschl, “On Fourier Transforms of Radial Functions and Distributions,” J. Fourier Anal. Appl., vol. 19, pp. 167–179, 2013.
[5] G. Watson, A Treatise on the Theory of Bessel Functions. Cambridge, UK: Cambridge Univ. Press, 1922.