Comparison of finite difference method and mesh analysis for state-space models of the cochlea

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Abstract: In this paper, we demonstrate that instabilities of the cochlear transmission-line model depend on numerical solutions. The transmission-line model approximates the fluid motion and the mechanical vibration in the cochlea. The mechanical vibration is enhanced by active cochlear feedback gain. For a realistic cochlea, spatial distribution of the feedback gain is varied randomly. However, most modeling studies set the gain to be constant as in an ideal cochlea because the spatial variation of the gain affects divergence of the calculation. To discretize the cochlear model for computation, the finite difference method and mesh analysis have been proposed. The finite difference method has been commonly used to solve the model represented in mechanical form; mesh analysis has been used to solve the model represented as an electro-acoustical circuit. This paper develops a state-space model of the cochlea for each of the two methods. The state-space formulation is well suited for testing instabilities of the model. As the result, both models show similar responses and stabilities under constant feedback gain (the ideal cochlea). On the other hand, with a randomly varied gain factor (the realistic cochlea), the model discretized by the finite difference method demonstrates greater instability than the model discretized by the mesh analysis.

Keywords: Cochlea, Instability, Transmission-line model, Finite difference method, Mesh analysis

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1. INTRODUCTION

The function of the cochlea is to decompose complex sound into various frequency components of neural information. Transmission-line models of the cochlea can predict spatial vibration patterns along the cochlea at a given frequency or its frequency response at a given position [1–3]. In these models, the propagation of an incoming sound wave can be described with partial differential equations of mechanical motion and can also be represented by electrical circuits. In order to produce numerical results, these transmission-line models are spatially discretized into segments along the cochlear length.

It has been known that modeling methods influence numerical results [4,5]. The WKB method is widely and classically used to solve the equations of the continuous cochlear transmission-line model [4]. On the other hand, the finite difference method has been commonly used to solve the one-dimensional (1D) transmission-line model of the cochlea represented by a partial differential equation of a mechanical system [3]. In addition, mesh analysis solves the electrical circuit of the 1D transmission-line model of the cochlea [6]. With both the finite difference method and the mesh analysis, the partial spatial derivatives are replaced by first-order approximations.

A frequency domain model refers to the way a system responds to a frequency and has been used to investigate input–output (IO) properties of the system. For the cochlear transmission-line model, the frequency domain model shows the spatial vibration patterns known as a traveling wave [3]. However, it is difficult for a frequency domain solution to determine whether a system being modeled is stable or not. Stability is a typical feature used to describe a system. For an unstable system, the result of the calculation is divergence. Moreover, cochlear mechanics is assumed to be active and nonlinear [7]; the active and nonlinear elements can easily generate an unstable response [8]. State-space formulation is a powerful tool for determining the stability of a system. Reference [9] applies this state-space formulation to the transmission-line model of the cochlea using the finite difference method and shows that
instability of the state-space model for cochlear mechanics depends on spatial gain factors via active elements. It is assumed that the source of the active elements is the outer hair cells (OHCs) that are aligned on the cochlear location \( x \) shown in Fig. 1 [10,11]. OHC density randomly varies with the cochlear location \( x \) among the hearing impaired. Most studies have found that OHC loss in humans and animals is caused by acoustic injury or is age-related [12–14]. Furthermore, for normal hearing, OHC density plays an important role in invasive measurements of cochlear function using oto-acoustic emission (OAE) [15]. To test the model for such cases, a random change in gain factor \( \gamma(x) \) is used [15,16].

The aim of this paper is to investigate how the two solutions, the finite difference method and mesh analysis, influence the numerical results for the transmission-line model of the cochlea. This paper develops two state-space models using the finite difference method and mesh analysis, respectively. In addition, two state-space models are established by the discrete cochlear models.

2. COCHLEAR MODELS

In this section, we introduce the active mechanics of the cochlea and two transmission-line models of the cochlea, one represented in mechanical form and the other by an electrical circuit. These models are discretized by the finite difference method and by mesh analysis, respectively. In addition, two state-space models are established by the discrete cochlear models.

2.1. Active Mechanics of Cochlea

The cochlea consists of upper and lower fluid-filled chambers, the scala vestibuli and scala tympani, with the basilar membrane (BM). The incoming sound wave vibrates the BM mechanically. Locations of the mechanical BM vibration depend on the sound frequency as shown in Fig. 1. To sense the BM vibration, two types of sensor cells, the inner hair cell (IHC) and the outer hair cell (OHC), line the BM. In the human cochlea, there are around 3,500 IHCs and 12,000 OHCs. The IHCs connect to the auditory nerve (AN) and transmit the sound information to the brain via the AN. The OHCs, on the other hand, enhance the BM motion as shown in Fig. 1. The mechanisms for this cochlear amplification by the OHCs is still unclear; however, active motilities of the OHC have been obtained in OHC isolation experiments and are assumed to be the mode of the cochlear amplification [10,11].

To simulate BM vibration, cochlear transmission-line models have been proposed [1–3]. Figure 2 illustrates the scheme of the transmission-line model. At each location, the BM vibrates at the natural frequency. In addition to this passive structure, the transmission-line model includes active elements such as the OHCs. Many models control the active feedback of the OHCs by introducing a gain factor, \( \gamma(x) \). The value of this gain factor varies with the location, \( x \), and can also be interpreted as the density of the OHCs at each location.

2.2. Mechanical Model Solved by Finite Difference Method

A state-space model of a discretized cochlea represented in mechanical form was developed in Ref. [9]. Adopting the long wavelength assumption, the differential equation describing 1D wave propagation along the cochlea is

\[
\frac{\partial^2 p(t)}{\partial x^2} - \frac{2\rho}{H} \ddot{w}(t) = 0,
\]

where \( p(t) \) is the pressure difference across the BM, \( w(t) \) denotes displacement of the BM, \( \rho \) is fluid density, and \( H \) is the height of the upper and lower chambers. Differentiation in time is denoted by subscript \( t \), and differentiation in space by subscript \( x \). The boundary conditions for the wave equation are...
The vectors and the matrix are set to $F$ respectively, and $p(t)|_{x=L} = 0$,

$$
\frac{\partial p(t)}{\partial x} \bigg|_{x=0} = 2 \rho \ddot{w}_s(t), \quad (2)
$$

$$
p(t)|_{x=0} = 0, \quad (3)
$$

where $\ddot{w}_s(t)$ is the acceleration of the stapes footplate and $L$ is the cochlear length.

Using finite difference approximations for the spatial derivatives in Eqs. (1) and (2), as originally proposed in Ref. [3], in which the cochlear length $L$ is divided into $N$ sections of length $\Delta$, the wave equation [Eq. (1)] can be written as

$$
p_{n-1}(t) - 2 p_n(t) + p_{n+1}(t) \frac{\Delta^2}{\Delta^2} = \frac{2 \rho}{H} \ddot{w}_s(t), \quad (4)
$$

for $2 \leq n \leq N - 1$, where $p_n(t)$ and $\ddot{w}_s(t)$ are the pressure difference and acceleration of the BM at the $n$th segment, respectively. The boundary conditions Eqs. (2) and (3) can also be approximated by

$$
p_{2}(t) - \frac{p_1(t)}{\Delta} = \frac{2 \rho}{H} \ddot{w}_s(t), \quad (5)
$$

$$
p_{N}(t) = 0. \quad (6)
$$

Equations (4)–(6) can be written in matrix form as

$$
F p(t) - \ddot{w}(t) = q(t), \quad (7)
$$

where $p(t)$, $\ddot{w}(t)$, and $q$ are the vectors of pressure differences, cochlear accelerations, and source terms, respectively, and $F$ denotes the finite-difference matrix.

The vectors and the matrix are set to

$$
F = \frac{H}{2 \rho \Delta^2} \begin{bmatrix}
- \frac{\Delta}{H} & \frac{\Delta}{H} & 0 \\
1 & -2 & 1 \\
0 & 1 & -2 & 1 \\
& & \ddots & \ddots & \ddots \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 0 & \frac{\Delta^2}{H}
\end{bmatrix}, \quad (8)
$$

$$
p(t) = \begin{bmatrix}
p_1(t) & \cdots & p_N(t) \end{bmatrix}^T, \quad (9)
$$

$$
\ddot{w}(t) = \begin{bmatrix}
\ddot{w}_1(t) & \cdots & \ddot{w}_{N-1}(t) & 0 \end{bmatrix}^T, \quad (10)
$$

$$
q(t) = \begin{bmatrix}
\dddot{w}_s(t) & 0 & \cdots & 0 
\end{bmatrix}^T. \quad (11)
$$

As shown in Fig. 3, to produce the sharp tuning seen in the cat’s auditory nerve, a cochlear micromechanical element is modeled as two lumped masses $m_n$ coupled by a spring $k$ and damper $c$, with each mass connected to a wall by a spring and damper [17]. The equations of motion for this micromechanical model at the $n$th segment are derived and can be written in state-space form

$$
\dot{x}_n(t) = A_n x_n(t) + B_n p_n(t) \quad (12)
$$

where $x_n(t)$ is the vector of state variables and is defined as

$$
x_n(t) = \begin{bmatrix} \dot{x}_1(t) & x_1(t) & \dot{x}_2(t) & x_2(t) \end{bmatrix}_n^T. \quad (14)
$$

The micromechanical model proposed in Ref. [17].

The outer hair cell is considered to be an active element in the cochlea.

$$
\dot{w}_n(t) = C_n x_n(t) \quad (13)
$$

where $x_n(t)$ is the vector of state variables and $C_n$ with gain factor $\gamma$ are defined as

$$
A_n = \begin{bmatrix}
-\frac{c_1 + c_2 + c_3}{m_1} & -\frac{k_{1} + k_{2}}{m_1} & \frac{k_{1} - y_{2}}{m_1} & \frac{k_{1} - y_{3}}{m_1} \\
1 & 0 & 0 & 0 \\
\frac{c_3}{m_2} & \frac{k_3}{m_2} & -\frac{c_1 + c_3}{m_2} & -\frac{k_2 + k_3}{m_2} \\
0 & 0 & 1 & 0
\end{bmatrix}_{n}, \quad (15)
$$

$$
B_n = \begin{bmatrix}
\frac{1}{m_1} & 0 & \cdots & 0 \end{bmatrix}_n^T, \quad (16)
$$

$$
C_n = \begin{bmatrix}
1 & 0 & \cdots & 0 \end{bmatrix}_n. \quad (17)
$$

The state-space form for the micromechanical elements is

$$
x(t) = A_F x(t) + B_F p(t) \quad (18)
$$

$$
\dot{w}(t) = C_F x(t), \quad (19)
$$

where the vectors are defined as

$$
x(t) = \begin{bmatrix}
x_1(t) & \cdots & x_N(t) \end{bmatrix}_n^T, \quad (20)
$$

$$
\dot{w}(t) = \begin{bmatrix}
\dot{w}_1(t) & \cdots & \dot{w}_{N-1}(t) & 0 \end{bmatrix}_n^T, \quad (21)
$$

$$
p(t) = \begin{bmatrix}
p_1(t) & \cdots & p_{N}(t) \end{bmatrix}_n^T \quad (22)
$$

and the matrices are defined as

$$
A_F = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
\cdots & \ddots & \ddots & \ddots \\
0 & A_N & \cdots & \cdots \\
0 & 0 & \cdots & B_N
\end{bmatrix}, \quad (23)
$$

$$
B_F = \begin{bmatrix}
B_1 & 0 & \cdots & 0 \\
\cdots & \ddots & \ddots & \ddots \\
0 & B_N & \cdots & \cdots \\
0 & 0 & \cdots & C_N
\end{bmatrix}.
The overall state-space equation including boundary conditions can be written as
\[ \dot{x}(t) = Ax(t) + Bu(t), \]
where matrices \( A \) and \( B \) are the matrices of state and input, respectively, and are defined as
\[ A = [I - B_{E} R^{-1} C_{E}]^{-1} A_{E}, \]
\[ B = [I - B_{E} R^{-1} C_{E}]^{-1} B_{E}, \]
\[ u = F^{-1} q. \]

The eigenvalues \( \lambda \) of the state matrix \( A \) are poles of the system and can be written in the form
\[ \lambda = \sigma + i\omega, \]
where \( \sigma \) is the real part and \( \omega \) is the imaginary part of the eigenvalue. A positive value for \( \sigma \) indicates that the system is linearly unstable. On the other hand, a negative value for \( \sigma \) shows a stable response obtained from the system.

### 2.3. Electrical Circuit Model Solved by Mesh Analysis

The transmission-line model of the cochlea can be represented by an electrical circuit. Figure 4 shows an electrical representation of the 1D transmission-line model using an electroacoustic analogy [18]. In this analogy, electrical voltage and current variables are analogous to pressure and volume velocity, respectively, and the electrical impedance of a circuit is equivalent to the acoustic impedance of a system.

Mesh analysis is a method that is used to solve circuits for the currents at any position and produces a set of equations by using Kirchhoff’s voltage law [19]. Reference [6] has applied this method to generate the discrete cochlear model.

In Fig. 4, a voltage source \( V_{1}(t) \) drives the transmission-line model, where \( V_{1}(t) \) is analogous to the stapes sound pressure. The BM is spatially discretized into \( N \) segments. To produce the sharp tuning seen in the cat’s auditory nerve, the micromechanical model was amplified by a voltage source \( V_{n}^{a}(t) \) representing the pressure generated by an active element [17].

The electrical impedances \( Z_{n}^{a}, Z_{n}^{p} \), and \( Z_{n}^{f} \) represent the acoustic impedances of the BM, the TM, and the hair bundles (HB), respectively. Inductance \( L \), resistance \( R \), and capacitance \( C \) represent the acoustical mass, resistance, and compliance, respectively:
\[ L = \frac{m}{b\Delta}, \quad R = \frac{c}{b\Delta}, \quad C = \frac{b\Delta}{k}, \]
where \( m, c, \) and \( k \) are mass, damping, and stiffness per area, respectively, and \( b \) and \( \Delta \) are the width of the BM and the segment length, respectively. Displacement of \( \xi_{n} \) at the \( n \)th segment is
\[ \xi_{n}(t) = \frac{1}{b\Delta} \int I_{n}(t) dt, \]
where \( I_{n}(t) \) is the brunch current at each segment.

The electrical impedance \( Z_{n}^{i} \) couples neighboring segments and represents the acoustic impedance of the fluid. The present transmission-line model assumes that the fluid is lossless and not compressive. Inductance \( L_{n}^{i} \) represents the acoustic mass of the fluid and is
\[ L_{n}^{i} = \frac{2\rho\Delta}{A}, \]
where \( \rho \) is the fluid density and \( A \) is the cross-sectional area of the scale.

In general, the circuit equation is defined as
\[ v(t) = L\dot{i}(t) + Ri(t) + Sq(t), \]
where \( v, i, \) and \( q \) are voltage, current, and electric charge vectors, and \( L, R, \) and \( S \) are inductance, resistance, and compliance matrices, respectively. (Compliance is the reciprocal of capacitance.) Furthermore, the state-space model is obtained from Eq. (32):
\[ \begin{bmatrix} \dot{i}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} L^{-1}R & L^{-1}S \\ I & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} L^{-1}0 \end{bmatrix} \begin{bmatrix} v(t) \\ 0 \end{bmatrix}. \]
(33)

As described for Eq. (28), the eigenvalues \( \lambda \) of the state matrix indicate the instability of the system. In Eq. (33), the state matrix \( A \) obtained from the mesh analysis is
\[ A = \begin{bmatrix} L^{-1}R & L^{-1}S \\ I & 0 \end{bmatrix}. \]
(34)

To apply mesh analysis for the Neely and Kim model shown in Fig. 5, voltage vector \( v \), current vector \( i \), and electric charge vector \( q \) are set as
\[ v(t) = [V(t) \ 0 \ \cdots \ 0]^{T}, \]
\[ i = [i_{1}(t) \ i_{2}(t) \ \cdots \ i_{N}(t)]^{T}, \]
\[ q = [q_{1}(t) \ q_{2}(t) \ \cdots \ q_{N}(t)]^{T}. \]
(35)

Inductance matrix \( L \), resistance matrix \( R \), and compliance matrix \( S \) are
where $\gamma$ is feedback gain; $L_{CP}$, $R_{CP}$, and $S_{CP}$ are the inductance, resistance, and compliance matrices of the passive elements; and $R_a$ and $S_a$ are the resistance and compliance matrices derived from the active elements. The mesh analysis sets the values of the matrices of the inductance, resistance, and compliance (see Appendix).

3. RESULT

3.1. Response for Pure Tone

The cochlear simulations were conducted using the parameters of the original Neely and Kim model, shown in Table 1. This model simulates the BM motion for the cat. Figure 6 shows BM displacement along the cochlear length at various frequencies of pure tones. Both the finite difference method and mesh analysis calculate similar patterns and the enhanced peaks of BM displacements by the feedback gain $\gamma$. These patterns are equivalent to those in Ref. [17]. For the value $\gamma = 1$ originally used by Neely and Kim, the maximum enhancement reached 140 dB. This enhancement is higher than the value obtained by experiment, which indicated a maximum enhancement of about 45 dB near the base and 20 dB near the apex in difference species [20]. To adjust the maximum enhancement, the gain factor $\gamma$ is chosen to be 0.8, and the BM displacement is amplified to 60 dB.

3.2. Instability of Models under Constant Gain Factor

To analyze instability in the cochlear model, the eigenvalues of the system matrices in Eqs. (24) and (34) are calculated by the “eig” routine in SciPy, which uses the QR algorithm. A positive real part for eigenvalue $\sigma$
indicates instability of the system. Figure 7 shows these eigenvalues for the systems with gains of $\gamma = 0.8$ and 1.0. The cochlear models calculated by both the finite difference method and mesh analysis are stable for gains of $\gamma = 0.8$ and 1.0. A gain factor lower than the original value affects the system stability because of the reduction in BM enhancement. The poles are distributed on two lines since the system includes the two-degree-of-freedom model shown in Fig. 3.

3.3. Instability of Models under Spatial Random Gain Factor

References [9,15] show that spatial variation of the gain factors affects the instability of the cochlear model and that roughly varying the gain factors produces more instability in the model than does smoothly varying the gain factors. To set smoothly varying gain factors, a low-pass filter was designed using a fifth-order Butterworth filter. As an example, Fig. 8 shows the amplitude response of the filter with the cutoff wavelength $\lambda_{3dB}$ set to 0.55 mm.

![Fig. 7](image1.jpg)

Fig. 7 Distribution of poles in the cochlear models discretized by the finite difference method and by mesh analysis for various values of feedback gain $\gamma$.

![Fig. 8](image2.jpg)

Fig. 8 Example of low-pass filter with cutoff wavelength $\lambda_{3dB} = 0.55$ mm.

Figure 9 shows pole distributions calculated by both the finite difference method and mesh analysis with random variation of feedback gain $\gamma$. With rough variation of gain $\gamma$ with the cutoff wavelength $\lambda_{3dB}$ set at 0.55 mm, unstable poles are produced by both the finite difference method and
mesh analysis, as shown in Figs. 9(a)–9(b). On the other hand, with smooth variation of gain $C_3$ with the cutoff wavelength $\lambda_{3dB}$ set at 1.7 mm, the unstable poles disappear.

To evaluate the instability of the model, the number of unstable poles were counted. Figure 10 shows how the average number of unstable poles varied with magnitude of the peak-to-peak random gain. The average numbers of unstable poles increased with the magnitude of the gain variation. For the condition of roughly varying gain distribution with $\lambda_{3dB} = 0.55$ mm, the number of unstable poles was greater than for the condition of smoothly varying gain distribution with $\lambda_{3dB} = 0.85$ mm. On the condition of roughly varying gain distribution, the finite difference method calculated a greater number of poles than did mesh analysis.

Figure 11 shows how the average number of unstable poles varied with the cutoff wavelength $\lambda_{3dB}$ of the filter. The number of unstable poles decreased as the cutoff wavelength $\lambda_{3dB}$ increased and differed between the finite difference method and mesh analysis. When the cutoff wavelength $\lambda_{3dB}$ was less than 0.85 mm, the finite difference method calculated a greater number of unstable poles than did mesh analysis.

### 4. DISCUSSION

It has been pointed out that the calculation result of the cochlear transmission-line model depends on the analytical and numerical solutions [4]. The distinctive feature in both solutions is the solving of the spatial derivative in Eq. (1), which leads to the different calculation results. However, Fig. 6 shows that the response and active enhancement of the frequency domain model derived from both methods match each other. These results are thought to occur by the first-order approximation of the spatial derivative in both the finite difference method in Eq. (4) and mesh analysis shown in Fig. 4.

Figure 7 shows that the pole distributions for a constant gain factor $\gamma$ are independent of the numerical solution. Furthermore, spatial variation of the feedback gain factor $\gamma$
influences the instability of the cochlear model [9]. The pole distributions when gain factor $\gamma$ is spatially varied seem to be similar, as shown in Fig. 9. However, the instability of the model depends on a positive value of the real part of the pole. Figures 10 and 11 show that the number of unstable poles is larger with the finite difference method than with mesh analysis. These results indicate that the robustness under spatial variation of the gain factor $\gamma$ is higher for the mesh analysis than for the finite difference method. These results were produced by using different values for the system matrix in Eqs. (24) and (33).

Spatial variation of the feedback gain factor $\gamma$ (and thus of the density of the damaged OHCs) is used to simulate hearing loss [16] and OAE [15]. In this paper, smooth and rough variations of the feedback gain $\gamma$ represent weak and serious damage in the cochlea, respectively. Figures 10 and 11 show the robustness of the mesh analysis for rough spatial variation of the feedback gain factor $\gamma$. This result suggests that mesh analysis is a useful approach for simulating hearing loss and OAE.

The finite difference method also has an advantage; this is to systematically set the matrices of the state space model as described in Sect. 2.2. In this formulation, the finite difference method is simply applied to other cochlear transmission-line models [21,22]. With the mesh analysis, on the other hand, it is difficult to systematically set the matrices, as is graphically shown in Fig. 5. In this case, the setting of the circuit matrices is more complicated (see Appendix).

5. CONCLUSION

This paper investigates the different results generated by the finite difference method and mesh analysis for the cochlear transmission-line model. In Sect. 2, we introduced the cochlear transmission-line model with gain factor $\gamma$ relating to healthy hearing and described the finite difference method and mesh analysis for the cochlear transmission-line model. In the numerical results, the frequency domain model shows similar responses with both the finite difference method and mesh analysis under constant gain factor in the ideal cochlea. However, under spatial variation of the gain factor $\gamma$ as in the realistic cochlea, the state-space models indicate that mesh analysis is more robust than the finite difference method. In future work, we will analyze the instability of the state-space models in the time domain.

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APPENDIX  SETTING OF CIRCUIT MATRICES

Inductance matrix $L_{CP}$, resistance matrix $R_{CP}$, and compliance matrix $S_{CP}$ are

$$L_{CP} = \begin{bmatrix} L^1_{CP} & 0 \\ \vdots & \ddots & \ddots \\ 0 & L^N_{CP} & 0 \end{bmatrix}, \quad R_{CP} = \begin{bmatrix} R^1_{CP} & 0 \\ \vdots & \ddots & \ddots \\ 0 & R^N_{CP} & 0 \end{bmatrix},$$

$$S_{CP} = \begin{bmatrix} S^1_{CP} & 0 \\ \vdots & \ddots & \ddots \\ 0 & S^N_{CP} & 0 \end{bmatrix},$$

where $L^i_{CP}$, $R^i_{CP}$, and $S^i_{CP}$ are matrices of inductance, resistance, and compliance, respectively, at each branch. On the base side ($n = 1$), the matrices $L^1_{CP}$, $R^1_{CP}$, and $S^1_{CP}$ are set to

$$L^1_{CP} = \begin{bmatrix} L^1_1 + L^1_2 & -L^2_1 & -L^1_1 \\ -L^1_1 & L^1_1 + L^1_2 & -L^1_1 \\ -L^1_2 & -L^1_1 & L^1_2 + L^1_3 \\ 0 & \cdots & 0 \end{bmatrix},$$

$$R^1_{CP} = \begin{bmatrix} R^1_1 + R^1_2 & -R^2_1 & -R^1_1 \\ -R^1_1 & R^1_2 + R^1_3 & -R^1_1 \\ -R^2_1 & -R^1_1 & R^2_1 + R^2_3 \\ 0 & \cdots & 0 \end{bmatrix},$$

$$S^1_{CP} = \begin{bmatrix} \frac{1}{C^1_1} + \frac{1}{C^1_2} & -\frac{1}{C^1_1} & -\frac{1}{C^1_1} \\ -\frac{1}{C^1_1} & \frac{1}{C^1_1} + \frac{1}{C^1_2} & -\frac{1}{C^1_1} \\ -\frac{1}{C^1_2} & -\frac{1}{C^1_1} & \frac{1}{C^1_2} \end{bmatrix}.$$
The voltage source $V^n_a$ represents pressure generated by the active element. The inductance matrix $L_a$, resistance matrix $R_a$, and compliance matrix $S_a$ are

$$R_a = \begin{bmatrix} R^a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R^a_N \end{bmatrix}, \quad S_a = \begin{bmatrix} S^a_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S^a_N \end{bmatrix},$$

where $L^a_n$, $R^a_n$, and $S^a_n$ are matrices of inductance, resistance, and compliance at each branch. On the base side ($n = 1$), the matrices $L^1_a$, $R^1_a$, and $S^1_a$ are set to

$$R^1_a = \begin{bmatrix} 0 & -R^1_1 & R^1_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^1_a = \begin{bmatrix} 0 & -S^1_1 & S^1_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

At a middle site ($2 \leq n \leq N - 1$), the matrices $L^2_a$, $R^2_a$, and $S^2_a$ are set to

$$R^2_a = \begin{bmatrix} 0 & R^2_{n-1} & -R^2_n & -R^2_n & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^2_a = \begin{bmatrix} 0 & S^2_{n-1} & -S^2_n & -S^2_n & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  

On the apex side, the matrices are set to

$$R^N_a = \begin{bmatrix} 0 & -R^N_{N-1} & R^N_{N-1} & R^N_N \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S^N_a = \begin{bmatrix} 0 & -S^N_{N-1} & S^N_{N-1} & S^N_N \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

The voltage source $V^n_a$ represents pressure generated by the active element. The inductance matrix $L_a$, resistance matrix $R_a$, and compliance matrix $S_a$ are

$$S^C_N = \begin{bmatrix} -\frac{1}{C_{n-1}} & -\frac{1}{C_{n-1}} & \frac{1}{C_{n-1}} + \frac{1}{C_n} + \frac{1}{C_n} & -\frac{1}{C_n} \\ 0 & 0 & \frac{1}{C_n} + \frac{1}{C_n} & \end{bmatrix}. \quad (A-10)$$