ANALYSIS OF A SPACE–TIME HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE ADVECTION–DIFFUSION PROBLEM ON TIME-DEPENDENT DOMAINS *

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Abstract. This paper presents the first analysis of a space–time hybridizable discontinuous Galerkin method for the advection–diffusion problem on time-dependent domains. The analysis is based on non-standard local trace and inverse inequalities that are anisotropic in the spatial and time steps. We prove well-posedness of the discrete problem and provide a priori error estimates in a mesh-dependent norm. Convergence theory is validated by a numerical example solving the advection–diffusion problem on a time-dependent domain for approximations of various polynomial degree.

Key words. Space–time, hybridized, discontinuous Galerkin, advection–diffusion equations, time-dependent domains.

AMS subject classifications. 65N12, 65M15, 65N30, 35R37, 35Q35

1. Introduction. Many important physical processes are governed by the solution of time-dependent partial differential equations on moving and deforming domains. Of particular importance are advection dominated transport problems, with applications ranging from multi-phase flows separated by evolving interfaces to incompressible flow problems arising from fluid-structure interaction. The presence of dynamic meshes introduces additional challenges in the design of numerical methods. Most notable of these challenges is the Geometric Conservation Law (GCL) [10], which requires that uniform flow solutions remain uniform under grid motion. Satisfaction of the GCL is not trivial, as observed for the popular Arbitrary Lagrangian-Eulerian (ALE) class of methods in which the time-varying domain is mapped to a fixed reference domain. By performing all computations on the reference domain, the ALE method allows the use of explicit or multi-step time-stepping schemes. However, additional constraints must be placed on the algorithm to satisfy the GCL that often are not met in practice for arbitrary mesh movements, see e.g., [14].

In contrast, the space–time discontinuous Galerkin (DG) method inherently satisfies the GCL, as shown in [10]. Rather than mapping to a fixed reference domain, the problem is recast into a space–time domain in which spatial and temporal variables are not distinguished. This space–time domain is then partitioned into slabs and each slab is discretized using discontinuous basis functions in both space and time. The result is a fully conservative scheme that automatically accounts for grid movement and can be made arbitrarily higher-order accurate in space and time. The space–time DG method is suitable for both hyperbolic and parabolic problems, and has been successfully applied to diffusion and advection–diffusion problems [3, 6, 7, 19], as well

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as compressible and incompressible flow problems [17, 20, 21].

We pause to mention two possible solution procedures for space–time methods. The first, which we do not pursue in this article, constructs a single global system for the solution in the whole space–time domain [13]. The advantage of solving the PDE in the space–time domain all-at-once is the applicability of parallel-in-time methods [12]. The second, outlined in [9, 11, 21], instead computes the solution slab-by-slab. We apply this second approach.

In the space–time DG method, a time-dependent $d$-dimensional partial differential equation is discretized by DG in $d + 1$ dimensions resulting in substantially more degrees-of-freedom in each element compared to traditional time-stepping approaches. To alleviate the computational burden of the space–time DG method, Rhebergen and Cockburn [15, 16] introduced the space–time Hybridizable DG (HDG) method, extending the HDG method of [4] to space–time.

The HDG method reduces the number of globally coupled degrees-of-freedom by first introducing approximate traces of the solution on element facets. Enforcement of continuity of the normal component of the numerical flux across element facets allows for the unique determination of these approximate traces. The resulting linear system may then be reduced through static condensation to a global system of algebraic equations for only these approximate traces. In essence, the dimension of the problem is reduced by one, since the number of globally coupled degrees of freedom is of order $O(p^d)$ instead of $O(p^{d+1})$, where $p$ denotes the polynomial order and $d$ is the spatial dimension of the problem under consideration.

The first space–time HDG methods for partial differential equations on time-dependent domains were introduced in [15, 16], however, to the authors’ knowledge, these methods have not been analyzed. The analysis on fixed domains of the space–time DG method, however, has been considered for both linear and nonlinear advection–diffusion problems in [6, 7], and for the space–time HDG method in [13]. Recently, for a time-dependent diffusion equation, [3] analyzed the heat equation on prismatic space–time elements.

The consideration of moving domains significantly alters the analysis of the method compared to fixed domains. In particular, moving meshes lack the tensor product structure necessary to use the space–time projection introduced in [6, 7], or the inverse and trace inequalities derived in [3] without modification. The first error analysis of a space–time DG method on moving and deforming domains for the linear advection–diffusion equation was performed in [18], and for the Oseen equations in [20], laying the groundwork for the error estimates in section 5.

In this paper, we analyze a space–time HDG method for the advection–diffusion equation on a time-dependent domain. In section 2 we discuss the scalar advection–diffusion problem in a space–time setting. Next, in section 3, we discuss the finite element spaces necessary to obtain the weak formulation of the advection–diffusion problem, which we subsequently introduce. Section 4 deals with the consistency and stability of the space–time HDG method. Theoretical rates of convergence of the space–time HDG formulation in a mesh-dependent norm on moving grids are derived in section 5. Finally, section 6 presents the results of a numerical example to support the theoretical analysis, and a concluding discussion is given in section 7.

2. The advection–diffusion problem. Let $\Omega(t) \subset \mathbb{R}^d$ be a time-dependent polygonal ($d = 2$) or polyhedral ($d = 3$) domain whose evolution depends continuously on time $t \in [t_0, t_N]$. Let $x = (x_1, \ldots, x_d)$ be the spatial variables and denote the spatial gradient operator by $\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$. We consider the time-dependent
advection–diffusion problem

\( \partial_t u + \nabla \cdot (\beta u) - \nu \nabla^2 u = f \quad \text{in } \Omega(t), \ t_0 < t \leq t_N, \)

with given advective velocity \( \beta \), forcing term \( f \) and constant and positive diffusion coefficient \( \nu \).

Before introducing the space–time HDG method in section 3, we first present the space–time formulation of the advection–diffusion problem (2.1). Let \( \mathcal{E} := \{(t, x) : x \in \Omega(t), \ t_0 < t < t_N \} \subset \mathbb{R}^{d+1} \) be a \((d+1)\)-dimensional polyhedral space–time domain. We denote the boundary of \( \mathcal{E} \) by \( \partial \mathcal{E} \), and note that it is comprised of the hyper-surfaces \( \Omega(t_0) := \{(t, x) \in \partial \mathcal{E} : t = t_0\}, \Omega(t_N) := \{(t, x) \in \partial \mathcal{E} : t = t_N\} \), and \( \mathcal{Q}_\mathcal{E} := \{(t, x) \in \partial \mathcal{E} : t_0 < t < t_N\} \). The outward space–time normal vector to \( \partial \mathcal{E} \) is denoted by \( n := (n_t, \bar{n}) \), where \( n_t \) and \( \bar{n} \) are the temporal and spatial parts of the space–time normal vector, respectively.

To recast the advection–diffusion problem in the space–time setting we introduce the space–time velocity field \( \beta := (1, \bar{\beta}) \) and the operator \( \nabla := (\partial_t, \nabla) \). The space–time formulation of (2.1) is then given by

\[ \nabla \cdot (\beta u) - \nu \nabla^2 u = f \quad \text{in } \mathcal{E}, \]

where \( f \in L^2(\mathcal{E}) \) and where \( \beta, \nabla \cdot \beta \in L^\infty(\mathcal{E}) \).

We partition the boundary of \( \Omega(t) \) such that \( \partial \Omega(t) = \Gamma_D(t) \cup \Gamma_N(t) \) and \( \Gamma_D(t) \cap \Gamma_N(t) = \emptyset \) and we partition the space–time boundary into \( \partial \mathcal{E} = \partial \mathcal{E}_D \cup \partial \mathcal{E}_N \), where \( \partial \mathcal{E}_D := \{(t, x) : x \in \Gamma_D(t), t_0 < t \leq t_N\} \) and \( \partial \mathcal{E}_N := \{(t, x) : x \in \Gamma_N(t) \cup \Omega(t_0), t_0 < t \leq t_N\} \).

Given a suitably smooth function \( g : \partial \mathcal{E}_N \rightarrow \mathbb{R} \), we prescribe the initial and boundary conditions

\begin{align*}
(2.3a) & \quad -\zeta u \beta \cdot n + \nu \nabla u \cdot \bar{n} = g \quad \text{on } \partial \mathcal{E}_N, \\
(2.3b) & \quad u = 0 \quad \text{on } \partial \mathcal{E}_D,
\end{align*}

where \( \zeta \) is an indicator function for the inflow boundary of \( \mathcal{E} \), i.e., the portions of the boundary where \( \beta \cdot n < 0 \). Note that (2.3a) imposes the initial condition \( u(x, 0) = g(x) \) on \( \Omega(t_0) \).

3. The space–time hybridizable discontinuous Galerkin method. In this section we introduce the space–time mesh, the space–time approximation spaces and the space–time HDG formulation for the advection–diffusion problem (2.2)–(2.3).

3.1. Description of space–time slabs, faces and elements. We begin this section with a description of the discretization of the space–time domain. First, the time interval \([t_0, t_N]\) is partitioned into the time levels \( t_0 < t_1 < \cdots < t_N \), where the \( n \)-th time interval is defined as \( I_n = (t_n, t_{n+1}) \) with length \( \Delta t_n = t_{n+1} - t_n \). For simplicity we will assume a fixed time interval length, i.e., \( \Delta t_n = \Delta t \) for \( n = 0, 1, \cdots, N - 1 \). For ease of notation, we will denote \( \Omega(t_n) = \Omega_n \) in the sequel. The space–time domain is then divided into space–time slabs \( \mathcal{E}^n = \mathcal{E} \cap (I_n \times \mathbb{R}^d) \). Each space–time slab \( \mathcal{E}^n \) is bounded by \( \Omega_n, \Omega_{n+1}, \) and \( \mathcal{Q}_\mathcal{E}^n = \partial \mathcal{E}^n \setminus (\Omega_n \cup \Omega_{n+1}) \).

We further divide each space–time slab into space–time elements, \( \mathcal{E}^n = \bigcup_j K_j^n \). To construct the space–time element \( K_j^n \), we divide the domain \( \Omega_n \) into non-overlapping spatial elements \( K_j \), so that \( \Omega_n = \bigcup_j K_j^n \). Then, at \( t_{n+1} \) the spatial elements \( K_j^{n+1} \) are obtained by mapping the nodes of the elements \( K_j^n \) into their new position via the transformation describing the deformation of the domain. Each space–time element
Fig. 3.1. An example of a space–time slab in a polyhedral \((1+1)\)-dimensional space–time domain.

\(K^n_j\) is obtained by connecting the elements \(K^n_j\) and \(K^{n+1}_j\) via linear interpolation in time.

The boundary of the space–time element \(K^n_j\) consists of \(K^n_j\), \(K^{n+1}_j\), and \(Q^n_j = \partial K^n_j \setminus (K^n_j \cup K^{n+1}_j)\). On \(\partial K^n_j\), the outward unit space–time normal vector is denoted by \(n^{K^n_j} = (\bar{n}^{K^n_j}, \bar{n}^{K^n_j})\), where \(\bar{n}^{K^n_j}\) and \(\bar{n}^{K^n_j}\) are, respectively, the temporal and spatial parts of the space–time normal vector. On \(K^n_j\), \(n^{K^n_j} = (-1, 0)\), while on \(K^{n+1}_j\), \(n^{K^n_j} = (1, 0)\). In the remainder of the article, we will drop the subscripts and superscripts when referring to space–time elements, their boundaries and outward normal vectors wherever no confusion will occur.

We complete the description of the space–time domain with the tessellation \(T^n_h\) consisting of all space–time elements in \(E^n\), and \(T_h = \bigcup_n T^n_h\) consisting of all space–time elements in \(E\). An illustration of a space–time domain is shown in the case of one spatial dimension in Figure 3.1.

Finally, an interior space–time facet \(S\) is shared by two adjacent elements \(K^L\) and \(K^R\), \(S = \partial K^L \cap \partial K^R\), while a boundary facet is a face of \(\partial K\) that lies on \(\partial E\). The set of all facets will be denoted by \(\mathcal{F}\), and the union of all facets by \(\Gamma\).

### 3.2. Approximation spaces

We define the Sobolev space \(H^s(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega) \text{ for } |\alpha| \leq s\}\), where \(D^\alpha v\) denotes the weak derivative of \(v\), \(\alpha\) is the multi-index symbol, \(s\) a non-negative integer, and \(\Omega \subset \mathbb{R}^n\) is an open domain with \(n = d\) or \(n = d+1\) (see e.g. [2]). The space \(H^s(\Omega)\) is equipped with the following norm and semi-norm:

\[
\|v\|_{H^s(\Omega)}^2 = \sum_{|\alpha| \leq s} \|D^\alpha v\|_{L^2(\Omega)}^2 \quad \text{and} \quad |v|_{H^s(\Omega)}^2 = \sum_{|\alpha| = s} \|D^\alpha v\|_{L^2(\Omega)}^2,
\]

where \(\|\cdot\|_{L^2(\Omega)}\) is the standard \(L^2\)-norm on \(\Omega\). In the sequel, we will simply write \(\|v\|_{\Omega} = \|v\|_{L^2(\Omega)}\).

Next, we introduce anisotropic Sobolev spaces on an open domain \(\Omega \subset \mathbb{R}^{d+1}\) [8]. For simplicity, we follow [19, 20] by restricting the anisotropy to the case where the Sobolev index can differ only between spatial and temporal variables. All spatial variables will have the same index. Let \((s_1, s_s)\) be a pair of non-negative integers,
with $s_t, s_s$ corresponding to the spatial and temporal Sobolev indices. For $\alpha_t, \alpha_s \geq 0$, $i = 1, \ldots, d$, we define the anisotropic Sobolev space of order $(s_t, s_s)$ on $\Omega \subset \mathbb{R}^{d+1}$ by

$$H^{(s_t, s_s)}(\Omega) = \{ v \in L^2(\Omega) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\Omega) \text{ for } \alpha_t \leq s_t, \alpha_s \leq s_s \},$$

where $\alpha_s = (\alpha_{s_1}, \ldots, \alpha_{s_d})$. The anisotropic Sobolev norm and semi-norm are given by, respectively,

$$\|v\|_{H^{(s_t, s_s)}}^2 = \sum_{\alpha_t \leq s_t, \alpha_s \leq s_s} \|D^{\alpha_t} D^{\alpha_s} v\|^2_{\Omega} \quad \text{and} \quad |v|_{H^{(s_t, s_s)}}^2 = \sum_{\alpha_t = s_t, \alpha_s = s_s} \|D^{\alpha_t} D^{\alpha_s} v\|^2_{\Omega}.$$

We assume that each space–time element $\mathcal{K}$ is the image of a fixed master element $\tilde{\mathcal{K}} = (-1,1)^{d+1}$ under two mappings. First, we construct an intermediate tensor-product element $\tilde{\mathcal{K}}$ from an affine mapping $F_\mathcal{K} : \tilde{\mathcal{K}} \rightarrow \mathcal{K}$ of the form $F_\mathcal{K} (\tilde{x}) = A_\mathcal{K} \tilde{x} + b$, where $A_\mathcal{K} = \text{diag} \left( \frac{\Delta t}{T}, \frac{h_2}{T}, \ldots, \frac{h_d}{T} \right)$. Here $h_i$ is the edge length in the $i$-th coordinate direction, $\Delta t$ the time-step, and $b \in \mathbb{R}^{d+1}$ is a constant vector.

Next, the space–time element $\mathcal{K}$ is obtained from $\tilde{\mathcal{K}}$ via the suitably regular diffeomorphism $\phi_\mathcal{K} : \tilde{\mathcal{K}} \rightarrow \mathcal{K}$. The mapping $\phi_\mathcal{K}$ determines the shape of the space–time element after the size of the element has been specified by $F_\mathcal{K}$. Following [8], we will assume that the Jacobian of the diffeomorphism $\phi_\mathcal{K}$ satisfies:

$$C_1^{-1} \leq |\text{det} J_{\phi_\mathcal{K}}| \leq C_1, \quad \|\text{det} J_{\phi_\mathcal{K} \backslash mn}\|_{L^\infty(\mathcal{K})} \leq C_2, \quad m, n = 0, \ldots, d, \quad \forall \mathcal{K} \in \mathcal{T}_h,$$

where $C_1$ and $C_2$ are constants independent of the edge lengths $h_i$ and the time-step $\Delta t$, and where $\text{det} J_{\phi_\mathcal{K} \backslash mn}$ denotes the $(m, n)$ minor of $J_{\phi_\mathcal{K}}$.

Following [19], we define the Sobolev space $H^{(s_t, s_s)}(\mathcal{K})$ as

$$H^{(s_t, s_s)}(\mathcal{K}) = \{ v \in L^2(\mathcal{K}) : D^{\alpha_t} D^{\alpha_s} v \in L^2(\mathcal{K}) \text{ for } \alpha_t \leq s_t, \alpha_s \leq s_s \}.$$

Furthermore, the Sobolev space $H^{(s_t, s_s)}(\mathcal{K})$ is defined as

$$H^{(s_t, s_s)}(\mathcal{K}) = \{ v \in L^2(\mathcal{K}) : v \circ \phi_\mathcal{K} \in H^{(s_t, s_s)}(\tilde{\mathcal{K}}) \}.$$
see [8, Definition 2.9].

For the analysis in section 4 we require the concept of a broken anisotropic Sobolev space. We assign to \( \mathcal{T}_h \) the broken Sobolev space

\[
H^{(s_1,s_s)}(\mathcal{T}_h) = \{ v \in L^2(\mathcal{E}) : v|_K \in H^{(s_1,s_s)}(K), \forall K \in \mathcal{T}_h \},
\]

which we equip with the broken anisotropic Sobolev norm and semi-norm, respectively,

\[
\| v \|_{H^{(s_1,s_s)}(\mathcal{T}_h)}^2 = \sum_{K \in \mathcal{T}_h} \| v \|_{H^{(s_1,s_s)}(K)}^2 \quad \text{and} \quad | v |_{H^{(s_1,s_s)}(\mathcal{T}_h)}^2 = \sum_{K \in \mathcal{T}_h} | v |_{H^{(s_1,s_s)}(K)}^2.
\]

For \( v \in H^{(1,1)}(\mathcal{T}_h) \), we define the broken (space–time) gradient \( \nabla_h v \) by \( (\nabla_h v)|_K = \nabla(v|_K) \), \( \forall K \in \mathcal{T}_h \).

Additionally, we will make use of the following (spatial) shape regularity assumption. Suppose \( K \in \mathcal{T}_h \) is constructed from the fixed reference element \( \hat{K} \) via the mappings \( F_K : \hat{K} \to K \) and \( \phi_K : K \to \hat{K} \). Let \( h_K \) and \( \rho_K \) denote the radii of the \( d \)-dimensional circumsphere and inscribed sphere of the brick \( h_1 \times \cdots \times h_d \), respectively. We assume the existence of a constant \( c_r > 0 \) such that

\[
\frac{h_K}{\rho_K} \leq c_r, \quad \forall K \in \mathcal{T}_h.
\]

For the HDG method, we require the finite element spaces

\[
V_h^{(p_t,p_x)} = \left\{ v_h \in L^2(\mathcal{E}) : v_h|_K \circ \phi_K \circ F_K \in Q_{(p_t,p_x)}(\hat{K}), \forall K \in \mathcal{T}_h \right\},
\]

\[
M_h^{(p_t,p_x)} = \left\{ \mu_h \in L^2(\Gamma) : \mu_h|_S \circ \phi_K \circ F_K \in Q_{(p_t,p_x)}(\hat{S}), \forall S \in \mathcal{F}, \mu_h = 0 \text{ on } \partial \mathcal{E}_D \right\},
\]

where \( Q_{(p_t,p_x)}(D) \) denotes the set of all tensor-product polynomials of degree \( p_t \) in the temporal direction and \( p_x \) in each spatial direction on a domain \( D \). Furthermore, we define \( V_h^* = V_h^{(p_t,p_x)} \times M_h^{(p_t,p_x)} \).

### 3.3. Weak formulation.

It will be convenient to introduce the bilinear forms

\[
a_h^t ((u, \lambda), (v, \mu)) = - \sum_{K \in \mathcal{T}_h} \int_K \beta u \cdot \nabla_h v \, dx + \int_{\partial \mathcal{E}_N} \frac{1}{2} (\beta \cdot n + |\beta \cdot n|) \lambda \mu \, ds \\
+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{1}{2} (\beta \cdot n(u + \lambda) + |\beta \cdot n|(u - \lambda)) (v - \mu) \, ds,
\]

\[
a_h^d ((u, \lambda), (v, \mu)) = \sum_{K \in \mathcal{T}_h} \int_K \nu \nabla_h u \cdot \nabla_h v \, dx + \sum_{K \in \mathcal{T}_h} \int_Q \frac{\nu \alpha}{h_K} (u - \lambda) (v - \mu) \, ds \\
- \sum_{K \in \mathcal{T}_h} \int_Q \left[ \nu (u - \lambda) \nabla_h v \cdot \bar{n} + \nu \nabla_h u \cdot \bar{n} (v - \mu) \right] \, ds,
\]

where \( \alpha > 0 \) is a penalty parameter. The space–time HDG method for (2.2)–(2.3) is then given by: find \((u_h, \lambda_h) \in V_h^*\) such that

\[
a_h ((u_h, \lambda_h), (v_h, \mu_h)) = \sum_{K \in \mathcal{T}_h} \int_K f v_h \, dx + \int_{\partial \mathcal{E}_N} g \mu_h \, ds \quad \forall (v_h, \mu_h) \in V_h^*,
\]

where \( a_h ((u, \lambda), (v, \mu)) = a_h^t ((u, \lambda), (v, \mu)) + a_h^d ((u, \lambda), (v, \mu)) \).
4. Stability and boundedness. In this section we prove stability and boundedness of the space–time HDG method (3.11). Our analysis will make repeated use of local trace and inverse inequalities valid on the finite element space \( V_h^{(p, p_s)} \). Using ideas from [8], the dependence on the spatial mesh size \( h_K \) and time-step \( \Delta t \) is made explicit in these inequalities. Motivated by the fact that these two parameters differ in general, this will allow us to derive error bounds in section 5 that are anisotropic in \( h_K \) and \( \Delta t \) as in [19, 20]. The local trace and inverse inequalities are summarized in the following lemma.

**Lemma 4.1.** Assume that \( K \) is a space–time element in \( \mathbb{R}^{d+1} \) constructed via the mappings \( \phi_K : \tilde{K} \to K \) and \( F_K : \tilde{K} \to \tilde{K} \) as defined in subsection 3.2. Assume further that the spatial shape regularity condition (3.7) holds. Then, for all \( v_h \in V_h^{(p, p_s)} \), the following local inverse and trace inequalities hold:

\[
\begin{align*}
(4.1a) \quad & \| \partial_t v_h \|_K \leq c_{I,t} \left( \Delta t^{-1} + h_K^{-1} \right) \| v_h \|_K, \\
(4.1b) \quad & \| \nabla_h v_h \|_K \leq c_{I,s} h_K^{-1} \| v_h \|_K, \\
(4.1c) \quad & \| v_h \|_Q \leq c_{T,Q} h_K^{-2} \| v_h \|_K, \\
(4.1d) \quad & \| v_h \|_{\partial K} \leq c_{T,\partial K} \left( \Delta t^{-\frac{3}{2}} + h_K^{-2} \right) \| v_h \|_K,
\end{align*}
\]

where \( c_{I,s}, c_{I,t}, c_{T,Q}, \) and \( c_{T,\partial K} \) are constants depending on the polynomial degrees \( p_h \) and \( p_s \), the spatial shape-regularity constant \( c_r \), and the Jacobian of the mapping \( \phi_K \), but independent of the spatial mesh size \( h_K \) and the time step \( \Delta t \).

**Proof.** Inequalities (4.1a)–(4.1d) are space–time variants of those found in [8, Corollary 3.54, Corollary 3.59].

Additionally, we will require the following discrete Poincaré inequality valid for \( (v_h, \mu_h) \in V_h^* \):

\[
\| v_h \|_E^2 \leq c_p^2 \left( \sum_{K \in T_h} \| \nabla_h v_h \|_K^2 + \sum_{K \in T_h} \frac{1}{h_K} \| v_h - \mu_h \|_Q^2 \right),
\]

where \( c_p > 0 \) is a constant independent of the spatial mesh size \( h_K \) and time-step \( \Delta t \).

Consider the following extended function spaces on \( E \) and \( \Gamma \):

\[
\begin{align*}
(4.3) \quad & V(h) = V_h^{(p, p_s)} + H^2(E), \quad M(h) = M_h^{(p, p_s)} + H^{3/2}(\Gamma),
\end{align*}
\]

where \( H^{3/2}(\Gamma) \) is the trace space of \( H^2(E) \). For notational purposes we also introduce \( V^*(h) = V(h) \times M(h) \). We define three norms on \( V^*(h) \). First, the “stability” norm is defined as

\[
\begin{align*}
(4.4) \quad & \| (v, \mu) \|_E^2 = \| v \|_E^2 + \| \beta_h^{1/2} \mu \|_{\partial E_N}^2 + \sum_{K \in T_h} \| \beta_h^{1/2} (v - \mu) \|_{\partial K}^2 \\
&\quad \quad + \sum_{K \in T_h} \| v \|_K^2 + \sum_{K \in T_h} \frac{v}{h_K} \| v - \mu \|_Q^2,
\end{align*}
\]

where for ease of notation we have defined \( \beta_n = |\beta \cdot n| \). Additionally, we introduce a stronger norm obtained by endowing the “stability” norm with an additional term
controlling the $L^2$-norm of time derivatives:

$$
\|(v, \mu)\|_s^2 = \|(v, \mu)\|_v^2 + \sum_{K \in T_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \|\partial_t v\|_K^2.
$$

To prove boundedness of the bilinear form in subsection 4.1 we introduce the following norm:

$$
\|(v, \mu)\|_{s,*}^2 = \|(v, \mu)\|_v^2 + \sum_{K \in T_h} \left( \beta^1/2 \right)^2 \|v\|_{K^+}^2 + \sum_{K \in T_h} \left( \beta^1/2 \right)^2 \|\mu\|_{\partial K^-}^2

+ \sum_{K \in T_h} h_K \nu \|\nabla_h v \cdot \bar{n}\|_Q^2 + \sum_{K \in T_h} \frac{\Delta t + h_K}{\Delta t h_K} \|v\|_K^2,
$$

where $\partial K^+$ denotes the outflow part of the boundary (where $\beta \cdot n > 0$) and where $\partial K^-$ denotes the inflow part of the boundary (where $\beta \cdot n \leq 0$). The additional terms are required since the inequalities in Lemma 4.1 are valid only on the discrete space $V_h^{(\mu, \nu)}$.

Let $u \in H^2(\mathcal{E})$ solve the advection–diffusion problem (2.2). Defining the trace operator $\gamma : H^2(\mathcal{E}) \to H^{3/2}(\Gamma)$, restricting functions in $H^2(\mathcal{E})$ to $\Gamma$, and letting $u = (u, \gamma(u))$, we have

$$
a_h(u, \gamma(u)) = \sum_{K \in T_h} \int_K f v_h \, dx + \int_{\partial \mathcal{E}_N} g \mu_h \, ds \quad \forall (v_h, \mu_h) \in V_h^*.
$$

This consistency result follows by noting that $u = \gamma(u)$ on element boundaries, integration by parts in space–time, single-valuedness of $\beta \cdot n$, $\nabla_h u \cdot \bar{n}$, $u$ and $\mu_h$ on element boundaries, the fact that $\mu_h = 0$ on $\partial \mathcal{E}_D$, and that $u$ solves (2.2)–(2.3). An immediate consequence of consistency is Galerkin orthogonality: Let $(u_h, \lambda_h) \in V_h^*$ satisfy (3.11), then

$$
a_h \left( (u, \gamma(u)) - (u_h, \lambda_h), (v_h, \mu_h) \right) = 0, \quad \forall (v_h, \mu_h) \in V_h^*.
$$

**4.1. Boundedness.** We now turn to the boundedness of the bilinear form.

**Lemma 4.2 (Boundedness).** There exists a $c_B > 0$, independent of $h_K$ and $\Delta t$, such that for all $u = (u, \gamma(u)) \in V^*(h)$ and all $(v_h, \mu_h) \in V_h^*$,

$$
|a_h(u, \gamma(u))| \leq c_B \|u\|_{s,*} \|(v_h, \mu_h)\|_{s,*}.
$$

**Proof.** We will begin by bounding each term of the advective part of the bilinear form, $a_h^a(u, v_h)$. We note that

$$
|a_h^a(u, (v_h, \mu_h))| \leq \sum_{K \in T_h} \int_K \beta u \cdot \nabla_h v_h \, dx + \int_{\partial \mathcal{E}_N} \frac{1}{2} (|\beta \cdot n| + |\beta \cdot n|) \gamma(u) \mu_h \, ds

+ \sum_{K \in T_h} \int_{\partial \mathcal{E}_N} \frac{1}{2} \left( |\beta \cdot n(u + \gamma(u)) + |\beta \cdot n||u - \gamma(u)| \right) (v_h - \mu_h) \, ds.
$$

To obtain a bound for the first term on the right hand side of (4.10), we first recall $\beta \cdot \nabla_h v_h = \partial_t v_h + \beta \cdot \nabla_h v_h$, so that

$$
\sum_{K \in T_h} \int_K \beta u \cdot \nabla_h v_h \, dx \leq \sum_{K \in T_h} \int_K |u \partial_t v_h| \, dx + \sum_{K \in T_h} \int_K |\beta u \cdot \nabla_h v_h| \, dx.
$$
Both terms on the right hand side may be bounded using the Cauchy–Schwarz inequality:

\[
\sum_{K \in T_h} \int_K |u \partial_t v_h| \, dx \leq \sum_{K \in T_h} \left( \frac{\Delta t + h_K}{\Delta t h_K^2} \right)^{\frac{1}{2}} \|u\|_K \left( \frac{\Delta t h_K^2}{\Delta t + h_K} \right)^{\frac{1}{2}} \|\partial_t v_h\|_K
\]

\[
\leq \|u\|_{s,s'} \|(v_h, \mu_h)\|_s,
\]

\[
\sum_{K \in T_h} \int_K \beta_n \cdot \nabla_h v_h \, dx \leq \|\beta\|_{L^\infty(\mathcal{E})} \sum_{K \in T_h} \nu^{-1/2} \|u\|_K \nu^{1/2} \|\nabla_h v_h\|_K
\]

\[
\leq \|\beta\|_{L^\infty(\mathcal{E})} \nu^{-1/2} \|u\|_{s,s'} \|(v_h, \mu_h)\|_s.
\]

The integral over the mixed boundary \( \partial \mathcal{E}_N \) in (4.10) may also be bounded via the Cauchy–Schwarz inequality:

\[
\left| \int_{\partial \mathcal{E}_N} \frac{1}{2} (\beta \cdot n + |\beta \cdot n|) \gamma(u) \mu_h \, ds \right| \leq \left\| \beta_n^{1/2} \gamma(u) \right\|_{\partial \mathcal{E}_N} \left\| \beta_n^{1/2} \mu_h \right\|_{\partial \mathcal{E}_N}
\]

\[
\leq \|u\|_{s,s'} \|(v_h, \mu_h)\|_s.
\]

For the final term appearing on the right hand side of (4.10), we have the bound

\[
\sum_{K \in T_h} \int_{\partial K} \frac{1}{2} \left( \beta \cdot n (u + \gamma(u)) + |\beta \cdot n| (u - \gamma(u)) \right) (v_h - \mu_h) \, ds
\]

\[
\leq \sum_{K \in T_h} \int_{\partial K^+} \beta_n (v_h - \mu_h) u \, ds + \sum_{K \in T_h} \int_{\partial K^-} \beta_n (v_h - \mu_h) \gamma(u) \, ds
\]

\[
\leq \sum_{K \in T_h} \left( \left\| \beta_n^{1/2} u \right\|_{\partial K^+} + \left\| \beta_n^{1/2} \gamma(u) \right\|_{\partial K^-} \right) \left\| \beta_n^{1/2} (v_h - \mu_h) \right\| \partial K
\]

\[
\leq \sqrt{2} \|u\|_{s,s'} \|(v_h, \mu_h)\|_s,
\]

where we used the triangle inequality for the first inequality, the Cauchy–Schwarz inequality for the second inequality, and finally combined the discrete Cauchy–Schwarz inequality with the fact that \((a + b)^2 \leq 2(a^2 + b^2)\). Collecting the above bounds we obtain, for all \( u \in V^*(h) \) and \((v_h, \mu_h) \in V^*_h\),

\[
|a_h^B(u, (v_h, \mu_h))| \leq c_{B,a} \|u\|_{s,s'} \|(v_h, \mu_h)\|_s,
\]

where \(c_{B,a} = 2 + \|\beta\|_{L^\infty(\mathcal{E})} \nu^{-1/2}\).

We now shift our focus to the diffusive part of the bilinear form, \(a_h^D(u, (v_h, \mu_h))\). We note that

\[
|a_h^D(u, (v_h, \mu_h))| \leq \sum_{K \in T_h} \int_K \nu \nabla_h u \cdot \nabla_h v_h \, dx + \sum_{K \in T_h} \int_Q \frac{\nu}{h_K} (u - \gamma(u)) (v_h - \mu_h) \, ds
\]

\[
+ \sum_{K \in T_h} \int_Q \left[ \nu(u - \gamma(u)) \nabla_h v_h \cdot \bar{n} + \nu \nabla_h u \cdot \bar{n} (v_h - \mu_h) \right] \, ds.
\]
By the Cauchy–Schwarz inequality, the first two terms on the right hand side of (4.16) can be bounded by $(1 + \alpha)||u||_{s,s}||v_h, \mu_h||_s$. To bound the remaining term of $a_h^d(u, (v_h, \mu_h))$, we note that

\begin{equation}
(4.17) \quad \sum_{K \in T_h} \int_Q \left| \nu(u - \gamma(u))\nabla_h v_h \cdot \bar{n} + \nu \nabla_h u \cdot \bar{n} (v_h - \mu_h) \right| \, ds \\
\leq \sum_{K \in T_h} \int_Q \nu(u - \gamma(u))\nabla_h v_h \cdot \bar{n} \, ds + \sum_{K \in T_h} \int_Q \nu \nabla_h u \cdot \bar{n} (v_h - \mu_h) \, ds.
\end{equation}

Application of the Cauchy–Schwarz inequality to the first term on the right hand side of (4.17), followed by the trace inequality (4.1c), yields

\begin{equation}
(4.18) \quad \sum_{K \in T_h} \int_Q \nu(u - \gamma(u))\nabla_h v_h \cdot \bar{n} \, ds \\
\leq c_{T,Q} \left( \sum_{K \in T_h} \frac{\nu}{h_K} ||u - \gamma(u)||_Q^2 \right)^{\frac{1}{2}} \left( \sum_{K \in T_h} ||\nabla_h v_h||_K^2 \right)^{\frac{1}{2}} \\
\leq c_{T,Q} ||u||_{s,s} ||v_h, \mu_h||_s.
\end{equation}

Finally, to bound the second term on the right hand side of (4.17), we apply the Cauchy–Schwarz inequality:

\begin{equation}
(4.19) \quad \sum_{K \in T_h} \int_Q \nu \nabla_h u \cdot \bar{n} (v_h - \mu_h) \, ds \leq ||u||_{s,s} ||v_h, \mu_h||_s.
\end{equation}

Therefore, for all $u \in V^*(h)$ and $(v_h, \mu_h) \in V_h^*$,

\begin{equation}
(4.20) \quad |a_h^d(u, (v_h, \mu_h))| \leq c_{B,d} ||u||_{s,s} ||v_h, \mu_h||_s,
\end{equation}

where $c_{B,d} = 2 + \alpha + c_{T,Q}$. Combining (4.15) with (4.20) yields the assertion with $c_B = c_{B,d} + c_{B,d}$.

In the sequel, we will also make use of the following bound valid for all $(u_h, \lambda_h), (v_h, \mu_h) \in V_h^*$:

\begin{equation}
(4.21) \quad |a_h^d((u_h, \lambda_h), (v_h, \mu_h))| \leq c_d ||u_h, \lambda_h||_e ||v_h, \mu_h||_v,
\end{equation}

which follows immediately from (4.20) using the equivalence of norms on finite dimensional spaces. However, to quantify the constant $c_d$ to ensure its independence of $h_K$ and $\Delta t$, we may simply repeat the proof of the bound on (4.16), instead applying the trace inequality (4.1c) to the term in (4.19) to obtain $c_d = 1 + \alpha + 2c_{T,Q}$.

**4.2. Stability.** Next we demonstrate that the method is stable in the norm (4.4) over the space $V_h^*$.

**Lemma 4.3 (Stability).** Let $\alpha$ be the penalty parameter appearing in (3.10b) which is such that $\alpha > c_{T,Q}^2$ where $c_{T,Q}$ is the constant from the local trace inequality (4.1c). Further, let $c_{\alpha} = (\alpha - c_{T,Q}^2)/(1 + \alpha)$ and suppose there exists a constant $\beta_0 > 0$ such that

\begin{equation}
(4.22) \quad \frac{c_{\alpha}(\alpha)}{c_T^2} + \inf_{x \in E} \nabla_h \cdot \bar{\beta} \geq \beta_0 > 0,
\end{equation}

where \(c_\nu\) is the constant from the discrete Poincaré inequality (4.2). Then there exists a constant \(c_c\), independent of \(h_K\) and \(\Delta t\), such that

\[
(4.23) \quad a_h((v_h, \mu_h), (v_h, \mu_h)) \geq c_c \| (v_h, \mu_h) \|_{v}^2, \quad \forall (v_h, \mu_h) \in V_h^*.
\]

**Proof.** By definition of the bilinear form \(a_h^\alpha(\cdot, \cdot)\) in (3.10a),

\[
(4.24) \quad a_h^\alpha((v_h, \mu_h), (v_h, \mu_h)) = \frac{1}{2} \sum_{K \in T_h} \int_K v_h^2 \nabla_h \cdot \beta \, dx - \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} \beta \cdot n v_h^2 \, ds \\
+ \int_{\partial E_N} \frac{1}{2} (\beta \cdot n + |\beta \cdot n|) \mu_h^2 \, ds + \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} \beta \cdot n (v_h + \mu_h)(v_h - \mu_h) \, ds \\
+ \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} |\beta \cdot n| (v_h - \mu_h)^2 \, ds,
\]

where we used that \(-2v_h \beta \cdot \nabla_h v_h = -\nabla_h (\beta v_h^2) + v_h^2 \nabla_h \cdot \beta\) and applied Gauss' Theorem. Expanding the fourth integral on the right hand side and using the fact that \(\beta \cdot n\) and \(\mu_h\) are single-valued on element boundaries, and that \(\mu_h = 0\) on \(\partial \mathcal{E}_D\), (4.24) reduces to

\[
(4.25) \quad a_h^\alpha((v_h, \mu_h), (v_h, \mu_h)) = \frac{1}{2} \sum_{K \in T_h} \int_K v_h^2 \nabla_h \cdot \beta \, dx + \int_{\partial E_N} \frac{1}{2} |\beta \cdot n| \mu_h^2 \, ds \\
+ \sum_{K \in T_h} \int_{\partial K} \frac{1}{2} |\beta \cdot n| (v_h - \mu_h)^2 \, ds.
\]

Next, by definition of the bilinear form \(a_h^d(\cdot, \cdot)\) in (3.10b),

\[
(4.26) \quad a_h^d((v_h, \mu_h), (v_h, \mu_h)) = \sum_{K \in T_h} \int_K \nu |\nabla_h v_h|^2 \, dx + \sum_{K \in T_h} \int_{\partial Q} \frac{\nu}{h_K} (v_h - \mu_h)^2 \, ds \\
- \sum_{K \in T_h} \int_{\partial Q} 2\nu \nabla_h v_h \cdot \bar{n} (v_h - \mu_h) \, ds.
\]

Applying the Cauchy–Schwarz inequality and the trace inequality (4.1c) to the third term on the right-hand side of (4.26),

\[
(4.27) \quad \left| 2 \sum_{K \in T_h} \int_{\partial Q} \nu \nabla_h v_h \cdot \bar{n} (v_h - \mu_h) \, ds \right| \leq 2\nu^{1/2} c_{T,Q} \| \nabla_h v_h \|_{K} \nu^{1/2} h_K^{-1/2} \| v_h - \mu_h \|_{Q}.
\]

Combining (4.26) and (4.27), and choosing \(\alpha > c_T^2\),

\[
(4.28) \quad a_h^d((v_h, \mu_h), (v_h, \mu_h)) \\
\geq \sum_{K \in T_h} \left( \nu \| \nabla_h v_h \|_{K}^2 - 2c_{T,Q} \nu \| \nabla_h v_h \|_{K} h_K^{-1/2} \| v_h - \mu_h \|_{Q} + \frac{\nu}{h_K} \| v_h - \mu_h \|_{Q}^2 \right) \\
\geq \sum_{K \in T_h} \frac{\alpha - c_T^2}{1 + \alpha} \left( \nu \| \nabla_h v_h \|_{K}^2 + \frac{\nu}{h_K} \| v_h - \mu_h \|_{Q}^2 \right).
\]
The second inequality follows from noting that for \( \alpha > \psi^2 \), with \( \psi \) a positive real number, it holds that \( x^2 - 2\psi xy + \alpha y^2 \geq (\alpha - \psi^2)(x^2 + y^2)/(1 + \alpha) \), for \( x, y \in \mathbb{R} \), and taking \( x = \nu^{1/2} \| \nabla_h v_h \|_{\mathcal{K}} \), \( y = \nu^{1/2} h_K^{-1/2} \| v_h - \mu_h \|_{\mathcal{Q}} \), and \( \psi = c_{\tau, \mathcal{Q}} \). Combining (4.25) and (4.28), and using that \( \nabla_h \cdot \beta = \nabla_h \cdot \bar{\beta} \),

\[
(4.29) \quad a_h((v_h, \mu_h), (v_h, \mu_h)) \geq \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_K v_h^2 \nabla_h \cdot \bar{\beta} \, dx + \frac{1}{2} \| \beta_h^{1/2} \mu_h \|_{\partial \mathcal{E}_N}^2 + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \| \beta_h^{1/2} (v_h - \mu_h) \|_{\partial \mathcal{K}}^2 + \sum_{K \in \mathcal{T}_h} c_{\alpha} \| \nabla_h v_h \|_{\mathcal{K}}^2 + \sum_{K \in \mathcal{T}_h} c_{\alpha} \frac{\nu}{h_K} \| v_h - \mu_h \|_{\mathcal{Q}}^2.
\]

Using the discrete Poincaré inequality (4.2) and (4.22), we obtain from (4.29):

\[
(4.30) \quad a_h((v_h, \mu_h), (v_h, \mu_h)) \geq \frac{1}{2} \| \beta_0 \|_{\mathcal{E}}^2 + \frac{1}{2} \| \beta_h^{1/2} \mu_h \|_{\partial \mathcal{E}_N}^2 + \frac{1}{2} \sum_{K \in \mathcal{T}_h} \| \beta_h^{1/2} (v_h - \mu_h) \|_{\partial \mathcal{K}}^2 + \frac{1}{2} c_{\alpha} \sum_{K \in \mathcal{T}_h} \nu \| \nabla_h v_h \|_{\mathcal{K}}^2 + \frac{1}{2} c_{\alpha} \sum_{K \in \mathcal{T}_h} \frac{\nu}{h_K} \| v_h - \mu_h \|_{\mathcal{Q}}^2.
\]

The result follows with \( c_c = \min(\beta_0, c_{\alpha})/2 \).

### 4.3. The inf-sup condition

Stability was proven in subsection 4.2 with respect to the norm \( \| (\cdot, \cdot) \|_v \). To obtain the error estimates in section 5, we instead consider a norm with additional control over the time derivatives of the solution. For this we prove an inf-sup condition with respect to the stronger norm (4.5) following ideas in [3, 5, 22]. We first state the inf-sup condition.

**Theorem 4.4 (The inf-sup condition).** There exists \( c_i > 0 \), independent of \( h_K \) and \( \Delta t \), such that for all \( (w_h, \lambda_h) \in V_h^* \)

\[
(4.31) \quad c_i \| (w_h, \lambda_h) \|_s \leq \sup_{(v_h, \mu_h) \in V_h^*} \frac{a_h((w_h, \lambda_h), (v_h, \mu_h))}{\| (v_h, \mu_h) \|_s}.
\]

The proof of the inf-sup condition follows after the following two intermediate results.

**Lemma 4.5.** Let \( (w_h, \lambda_h) \in V_h^* \) and let \( z_h = \frac{\Delta t h_K^2}{\Delta t + h_K} \partial_t w_h \). There exists a \( c_1 > 0 \), independent of \( h_K \) and \( \Delta t \), such that

\[
\| (z_h, 0) \|_s \leq c_1 \| (w_h, \lambda_h) \|_s.
\]

**Proof.** We bound each component of \( \| (z_h, 0) \|_s \) term-by-term. Using the inverse inequality (4.1a) and that \( h_K < 1 \), we have

\[
\| z_h \|_s^2 = \sum_{K \in \mathcal{T}_h} \left( \frac{\Delta t h_K^2}{\Delta t + h_K} \right)^2 \| \partial_t w_h \|_{\mathcal{K}}^2 \leq c_{i,t}^2 \| w_h \|_{\mathcal{E}}^2.
\]

Similarly, the inverse inequality (4.1a) and \( h_K < 1 \) yields

\[
\sum_{K \in \mathcal{T}_h} \nu \| \nabla_h z_h \|_{\mathcal{K}}^2 = \sum_{K \in \mathcal{T}_h} \nu \left( \frac{\Delta t h_K^2}{\Delta t + h_K} \right)^2 \| \partial_t (\nabla_h w_h) \|_{\mathcal{K}}^2 \leq c_{i,t}^2 \sum_{K \in \mathcal{T}_h} \nu \| \nabla_h w_h \|_{\mathcal{K}}^2.
\]
Next, the facet term arising from the advective portion of the norm may be bounded using the trace inequality (4.1d):

$$\sum_{K \in \mathcal{T}_h} \left\| \frac{\partial h}{\partial \mathbf{n}} \right\|_{2}^{2} \leq \beta \| \mathbf{\beta} \|_{L^\infty(\Sigma)} \sum_{K \in \mathcal{T}_h} \| z_h \|_{\partial K}^{2} \leq c_T^{2} \| \mathbf{\beta} \|_{L^\infty(\Sigma)} \sum_{K \in \mathcal{T}_h} \left( \frac{\Delta t h^2}{\Delta t + h_K} \right) \| \partial_t w_h \|_{K}^{2}.$$

The facet term diffusive portion of the norm may be bounded with an application of (4.1c) and (4.1a):

$$\sum_{K \in \mathcal{T}_h} \beta^{1/2} z_h^{2} \leq \beta \| \mathbf{\beta} \|_{L^\infty(\Sigma)} \sum_{K \in \mathcal{T}_h} \| z_h \|_{\partial K}^{2} \leq c_T^{2} \| \mathbf{\beta} \|_{L^\infty(\Sigma)} \sum_{K \in \mathcal{T}_h} \left( \frac{\Delta t h^2}{\Delta t + h_K} \right) \| \partial_t w_h \|_{K}^{2}.$$

For the remaining term, (4.1a) yields

$$\sum_{K \in \mathcal{T}_h} \frac{\nu}{h_K} z_h^{2} = \sum_{K \in \mathcal{T}_h} \left( \frac{\Delta t h^2}{\Delta t + h_K} \right) \| \partial_t w_h \|_{Q}^{2} \leq c_T^{2} \sum_{K \in \mathcal{T}_h} \| \partial_t w_h \|_{K}^{2}.$$

Collecting the above bounds, we obtain Lemma 4.5, with $c_1 = 3c_T^{2} + c_T^{2}, \| \mathbf{\beta} \|_{L^\infty(\Sigma)} + c_T^{2}, \| \mathbf{\beta} \|_{L^\infty(\Sigma)}$.

**Lemma 4.6.** Let $(w_h, \lambda_h) \in V_h^*$ and let $z_h = \frac{\Delta t h^2}{\Delta t + h_K} \partial_t w_h$. There exists a $c_2 > 0$, independent of $h_K$ and $\Delta t$, such that if $(v_h, \mu_h) = c_2(w_h, \lambda_h) + (z_h, 0) \in V_h^*$, then

$$\frac{1}{2} \| (w_h, \lambda_h) \|_{v}^{2} \leq a_h((w_h, \lambda_h), (v_h, \mu_h)).$$

**Proof.** Note that $a_h((w_h, \lambda_h), (z_h, 0)) = a_h^d((w_h, \lambda_h), (z_h, 0)) + a_h^d((w_h, \lambda_h), (z_h, 0))$.

Integrating by parts the volume integral of $a_h^d(\cdot, \cdot)$ we have the following decomposition:

$$\frac{\Delta t h^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^{2} = a_h((w_h, \lambda_h), (z_h, 0)) - a_h^d((w_h, \lambda_h), (z_h, 0))$$

$$- \sum_{K \in \mathcal{T}_h} \frac{\Delta t h^2}{\Delta t + h_K} \int_{K} w_h \nabla z_h \cdot \beta \partial_t w_h \, dx - \sum_{K \in \mathcal{T}_h} \frac{\Delta t h^2}{\Delta t + h_K} \int_{\partial K} \beta \cdot \nabla w_h \partial_t w_h \, dx$$

$$+ \frac{1}{2} \sum_{K \in \mathcal{T}_h} \frac{\Delta t h^2}{\Delta t + h_K} \int_{\partial K} (\beta \cdot n - |\beta \cdot n|) \partial_t w_h \, ds.$$

From the boundedness of the diffusive part of the bilinear form (4.21), and application of Young’s inequality, with $\epsilon_1 > 0$, we obtain the following bound for the second term on the right hand side of (4.33):

$$|a_h^d((w_h, \lambda_h), (z_h, 0))| \leq \frac{cd}{2\epsilon_1} \| (z_h, 0) \|_{v}^{2} + \frac{cd\epsilon_1}{2} \| (w_h, \lambda_h) \|_{v}^{2}$$

$$\leq \frac{cd}{2\epsilon_1} \| (w_h, \lambda_h) \|_{v}^{2} + \frac{cd\epsilon_1}{2} \| (w_h, \lambda_h) \|_{v}^{2}$$

$$\leq \frac{cd}{2\epsilon_1} \sum_{K \in \mathcal{T}_h} \frac{\Delta t h^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^{2} + \left( \frac{cd}{2\epsilon_1} + \frac{cd\epsilon_1}{2} \right) \| (w_h, \lambda_h) \|_{v}^{2}.$$
where we have used the fact that $|||\cdot|||_v \leq |||\cdot|||_s$ and applied Lemma 4.5 in the second
inequality, and the definition of $|||\cdot|||_s$ in the third inequality. Next, to bound the
third term on the right hand side of (4.33) we apply the Cauchy–Schwarz
inequality and (4.1a) to obtain

$$
\left| \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_K w_h \nabla h \cdot \beta \partial_t w_h \, dx \right| \leq c_{t, t} ||| \nabla h \cdot \beta |||_{L^\infty(\mathcal{E})} \sum_{K \in \mathcal{T}_h} h_K \|w_h\|_{K}^2.
$$

As for the fourth term on the right hand side of (4.33), we first apply the Cauchy–
Schwarz inequality, Young’s inequality with some $\epsilon_2 > 0$ and (4.1b),

$$
\left| \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_K \beta \cdot \nabla h w_h \partial_t w_h \, dx \right| \leq \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} ||| \nabla h \cdot \beta \||_{L^\infty(K)} \| \partial_t w_h \|_{K} \| \nabla h w_h \|_{K}
\leq ||| \beta |||_{L^\infty(\mathcal{E})} \left( \sum_{K \in \mathcal{T}_h} \frac{\epsilon_2}{2\epsilon_2} \|w_h\|_{K}^2 + \sum_{K \in \mathcal{T}_h} \epsilon_2 \frac{\Delta t h_K^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^2 \right).
$$

For the remaining term on the right hand side of (4.33), we use the Cauchy–
Schwarz inequality, Young’s inequality with some $\epsilon_3 > 0$, and apply the trace inequality (4.1d) to find:

$$
(4.34) \left| \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \int_{\partial K} \frac{1}{2} (\beta \cdot n - \beta \cdot n) \partial_t w_h \left( w_h - \lambda_h \right) \, ds \right|
\leq \frac{c_{T, \partial K}}{2\epsilon_3} \sum_{K \in \mathcal{T}_h} h_K \frac{\Delta t h_K^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^2 + \frac{\epsilon_3}{2} \| \beta \|_{L^\infty(\mathcal{E})} \sum_{K \in \mathcal{T}_h} \| \beta_{1/2}^n (w_h - \lambda_h) \|_{\partial K}^2.
$$

Combining all of the above estimates,

$$
(4.35) \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^2 \leq a_h ((w_h, \lambda_h), (z_h, 0))
+ \left( \frac{c_{d, \epsilon_1}^2}{2\epsilon_1} + \frac{\epsilon_2}{2} \| \beta \|_{L^\infty(\mathcal{E})} + \frac{c_{T, \partial K}^2}{2\epsilon_3} \right) \sum_{K \in \mathcal{T}_h} \frac{\Delta t h_K^2}{\Delta t + h_K} \| \partial_t w_h \|_{K}^2
+ \left( \frac{c_{d, \epsilon_1}^2}{2\epsilon_1} + \frac{c_{d, \epsilon_1}}{2} + c_{t, t} \| \nabla h \cdot \beta \|_{L^\infty(\mathcal{E})} + \frac{c_{T, \partial K}^2}{2\epsilon_2} \| \beta \|_{L^\infty(\mathcal{E})} \right)
+ \frac{\epsilon_3}{2} \| \beta \|_{L^\infty(\mathcal{E})} \right) ||| (w_h, \lambda_h) |||_{v}^2.
$$

Choosing $\epsilon_1 = 2c_{d, \epsilon_1}^2$, $\epsilon_2 = 1/(4\| \beta \|_{L^\infty(\mathcal{E})})$, and $\epsilon_3 = 4c_{T, \partial K}^2$, adding $\frac{1}{2} ||| (w_h, \lambda_h) |||_{v}^2$ to both sides, and rearranging yields

$$
(4.36) \frac{1}{2} ||| (w_h, \lambda_h) |||_{v}^2 \leq a_h ((w_h, \lambda_h), (z_h, 0))
+ \left( \frac{3}{4} + c_{d, \epsilon_1}^2 + c_{t, t} \| \nabla h \cdot \beta \|_{L^\infty(\mathcal{E})} + 2c_{T, \partial K}^2 \| \beta \|_{L^\infty(\mathcal{E})} \right) ||| (w_h, \lambda_h) |||_{v}^2.
$$

From the stability of $a_h (\cdot, \cdot)$, Lemma 4.3, we have the bound

$$
(4.37) \frac{1}{2} ||| (w_h, \lambda_h) |||_{v}^2 \leq a_h ((w_h, \lambda_h), (z_h, 0)) + c_2 a_h ((w_h, \lambda_h), (w_h, \lambda_h))
$$
where \( c_2 = c_e^{-1} \left( \frac{1}{4} + c_2^2 \right) + c_f ||| \nabla_h \cdot \beta |||_{L^\infty(\mathcal{E})} + 2c_f^2 ||| \beta |||_{L^\infty(\mathcal{E})} + 2c_{f,T,\beta K} ||| \beta |||_{L^\infty(\mathcal{E})} \). The result follows.

Combining Lemma 4.5 and Lemma 4.6 now yields the proof for the inf-sup condition stated in Theorem 4.4.

**Proof of Theorem 4.4.** Given any \((w_h, \lambda_h) \in V_h^\prime\), consider the linear combination \((v_h, \mu_h) = c_2 (w_h, \lambda_h) + (z_h, 0)\), with \( z_h = \frac{\Delta h^2}{\Delta t + h \kappa} \beta w_h \) and \( c_2 \) the constant from Lemma 4.6. An application of the triangle inequality and the combination of Lemma 4.5 and Lemma 4.6 yields

\[
|||(v_h, \mu_h)|||_s |||(w_h, \lambda_h)|||_s \leq |||(z_h, 0)|||_s |||(w_h, \lambda_h)|||_s + c_2 |||(w_h, \lambda_h)|||_s^2
\]

\[
\leq (c_1 + c_2) |||(w_h, \lambda_h)|||_s^2
\]

\[
\leq 2(c_1 + c_2) a_h ((w_h, \lambda_h), (v_h, \mu_h)),
\]

which implies the inf-sup condition with \( c_i = \frac{1}{4}(c_1 + c_2)^{-1} \). \( \square \)

5. **Error analysis.** We now turn to the error analysis of the space–time HDG method. The following Céa-like lemma will prove useful in obtaining the global error estimate in Lemma 5.3.

**Lemma 5.1 (Convergence).** If \( u = (u, \gamma(u)) \in H^2(\mathcal{E}) \times H^{3/2}(\Gamma) \), where \( u \) solves (2.1), and \((u_h, \lambda_h) \in V_h^\prime\) is the solution to the discrete problem (3.11), then

\[
(5.1) \quad |||u - (u_h, \lambda_h)|||_s \leq \left( 1 + \frac{c_B}{c_i} \right) \inf_{(v_h, \mu_h) \in V_h^\prime} |||u - (v_h, \mu_h)|||_s,\star.
\]

**Proof.** From inf-sup stability (Theorem 4.4), Galerkin orthogonality (4.8), and boundedness (Lemma 4.2), we have for any \((w_h, \omega_h) \in V_h^\prime\)

\[
c_i |||(u_h, \lambda_h) - (w_h, \omega_h)|||_s \leq \sup_{(v_h, \mu_h) \in V_h^\prime} \frac{a_h ((u_h, \lambda_h) - (w_h, \omega_h), (v_h, \mu_h))}{|||(v_h, \mu_h)|||_s}
\]

\[
= \sup_{(v_h, \mu_h) \in V_h^\prime} \frac{a_h (u - (w_h, \omega_h), (v_h, \mu_h))}{|||(v_h, \mu_h)|||_s}
\]

\[
\leq c_B \sup_{(v_h, \mu_h) \in V_h^\prime} \frac{|||u - (w_h, \omega_h)|||_{s,\star}}{|||(v_h, \mu_h)|||_s}
\]

\[
= c_B |||u - (w_h, \omega_h)|||_{s,\star}.
\]

The result follows after application of the triangle inequality to \(||u - (u_h, \lambda_h)|||_s\). \( \square \)

We next define the projections \( P : L^2(\mathcal{E}) \to V_h^{(p_1, p_2)} \) and \( P^\partial : L^2(\Gamma) \to M_h^{(p_1, p_2)} \) which satisfy

\[
(5.2) \quad \sum_{K \in T_h} \int_K (w - P w) v_h \, dx = 0, \quad \forall \, v_h \in V_h^{(p_1, p_2)},
\]

\[
(5.3) \quad \sum_{S \in F} \int_S (\lambda - P^\partial \lambda) \mu_h \, ds = 0, \quad \forall \, \mu_h \in M_h^{(p_1, p_2)}.
\]

These projections will be used to obtain interpolation estimates.
Lemma 5.2 (Interpolation estimates). Assume that $K$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_K$ and $F_K$, with $F_K : \hat{K} \to \hat{K}$ and $\phi_K : \hat{K} \to K$. Assume that the spatial shape-regularity condition (3.7) holds. Suppose $u|_K \in H^{(p_r+1,p_r+1)}(K)$ solves (2.2)-(2.3). Then, the error $u - P_u$, its trace at the boundary $\partial K$, and the error $u - P^\partial u$ on $\partial K$ satisfy the following error bounds:

\begin{align}
\|u - P_u\|_K^2 &\leq c \left( h^{2p_r+2} + \Delta t^{2p_r+2} \right) u_{H^{(p_r+1,p_r+1)}(K)}, \\
\|\nabla_h (u - P_u)\|_K^2 &\leq c \left( h^{2p_r} + \Delta t^{2p_r} \right) u_{H^{(p_r+1,p_r+1)}(K)}, \\
\|\partial_i (u - P_u)\|_K^2 &\leq c \left( h^{2p_r} + \Delta t^{2p_r} \right) u_{H^{(p_r+1,p_r+1)}(K)}, \\
\left\| \nabla_h (u - P_u) \cdot \bar{n} \right\|_Q^2 &\leq c \left( h^{2p_r-1} + h^{-1} \Delta t^{2p_r+2} \right) u_{H^{(p_r+1,p_r+1)}(K)}, \\
\|u - P_u\|_{\partial K}^2 &\leq c \left( h^{2p_r+1} + \Delta t^{2p_r+1} \right) u_{H^{(p_r+1,p_r+1)}(K)}, \\
\left\| u - P^\partial u \right\|_{\partial K}^2 &\leq c \left( h^{2p_r+1} + \Delta t^{2p_r+1} \right) u_{H^{(p_r+1,p_r+1)}(K)},
\end{align}

where $c$ depends only on the spatial dimension $d$, the polynomial degrees $p_r$ and $p_s$, the spatial shape-regularity constant $c_r$, and the Jacobian of the mapping $\phi_K$.

Proof. The bounds (5.4), (5.5) and (5.8) have been obtained previously in [19, Lemma 6.1 and Remark 6.2] by generalizing [8, Lemmas 3.13 and 3.17] to higher dimensions. We relax the assumption in [19, Remark 6.2] that all spatial edge lengths are equal through the spatial shape-regularity assumption (3.7). In doing so, the bound (5.6) may be obtained in an identical fashion to (5.5). The bound (5.7) is obtained as follows: we derive a bound for the spatial derivative of the interpolation error over each face $\partial K_i$, where $i = 1, \ldots, d$, generalizing [8, Lemma 3.20] to the space–time setting. Then, summing over the faces $i = 1, \ldots, d$ we obtain a bound of the spatial derivatives of the interpolation error over $Q = \partial K \setminus (K^n \cup K^{n+1})$, and sum over all of the spatial derivatives to obtain the result. Lastly, the bound (5.9) may be inferred from the bound (5.8) by the optimality of the $L^2$-projection $P^\partial$ on facets. 

With the interpolation estimates in place, we can now derive an error bound in the $\|\cdot\|_s$ norm:

Lemma 5.3 (Global error estimate). Suppose that $K$ is a space–time element in $\mathbb{R}^{d+1}$ constructed via two mappings $\phi_K$ and $F_K$, with $F_K : \hat{K} \to \hat{K}$ and $\phi_K : \hat{K} \to K$, and that the spatial shape-regularity condition (3.7) holds. Let $u = (u, \gamma(u))$, where $u|_K \in H^{(p_r+1,p_r+1)}(K)$ solves the advection–diffusion problem (2.2), and where $\gamma(u)$ denotes the trace of $u$ on $\partial K$. Furthermore, let $(u_h, \lambda_h) \in V_h^*$ be the solution to the discrete problem (3.11). Then, the following error bound holds:

\begin{align}
\|u - (u_h, \lambda_h)\|_s^2 &\leq C \left( h^{2p_r} + \Delta t^{2p_r+1} + \nu \left( h^{2p_r} + h^{-1} \Delta t^{2p_r+1} \right) \right) u_{H^{(p_r+1,p_r+1)}(E)},
\end{align}

where $h = \max_{K \in T_h} h_K$ is the spatial mesh size $\Delta t$ is the time-step and $C > 0$ a constant.

Proof. By Lemma 5.1, we may bound the discretization error $u - (u_h, \lambda_h)$ in the
Obtain the \( ||\cdot||_s \) norm by the interpolation error \( u - (P_u, P^\beta \gamma(u)) \) in the \( ||\cdot||_{s,*} \) norm:

\[
(5.11) \quad ||u - (u_h, \lambda_h)||_{s,*} \leq \left( 1 + \frac{c_R}{c_4} \right) ||u - (P_u, P^\beta \gamma(u))||_{s,*}.
\]

Thus, it suffices to bound each term of \( ||u - (P_u, P^\beta \gamma(u))||_{s,*} \) using the interpolation estimates in Lemma 5.2. For this we consider the rotating Gaussian pulse test case on a time-dependent domain as introduced in problem 4.3. We solve (2.2)–(2.3) with \( \tilde{\beta} = (-4x_2, 4x_1)^T \) and \( f = 0 \). The boundary and initial conditions are set such that the exact solution is given by

\[
(6.1) \quad u(t, x_1, x_2) = \frac{\sigma^2}{\sigma^2 + 2vt} \exp \left( -\frac{(\tilde{x}_1 - x_1 c)^2 + (\tilde{x}_2 - x_2 c)^2}{2\sigma^2 + 4vt} \right),
\]
where \( \tilde{x}_1 = x_1 \cos(4t) + x_2 \sin(4t), \tilde{x}_2 = -x_1 \sin(4t) + x_2 \cos(4t), \) \((x_{1c}, x_{2c}) = (-0.2, 0.1)\).

Furthermore, we set \( \sigma = 0.1. \)

The advection–diffusion problem is solved on a time-dependent domain. The deformation is based on a transformation of a uniform space–time mesh \((t, x_0^1, x_0^2) \in [0, t_N] \times [-0.5, 0.5]^2\) given by

\[
(6.2) \quad x_i = x_0^i + A \left( \frac{1}{2} - x_0^i \right) \sin \left( 2\pi \left( \frac{1}{2} - x_0^i + t \right) \right) \quad i = 1, 2,
\]

where \((x_1^*, x_2^*) = (x_2, x_1)\) and \(A = 0.1. \) We take \( t_N = 1. \)

This example was implemented using the Modular Finite Element Methods (MFEM) library [1] on unstructured hexahedral space–time meshes. The solution on the time-dependent domain is shown at different points in time in Figure 6.1.

In Table 6.1 we compute the rates of convergence in the \( \| (\cdot, \cdot) \|_s \) norm using polynomial degree \( p = p_t = p_s = 1, 2, 3. \) We consider both \( \nu = 10^{-2} \) and \( \nu = 10^{-6}. \)

Mesh refinement is done simultaneously in space and time. For the case that \( \nu = 10^{-2} \) we obtain rates of convergence of approximately \( p, \) as expected from Lemma 5.3, while for \( \nu = 10^{-6} \) we obtain slightly better rates of convergence, namely \( p + 1/2. \)

| Cells per slab | Nr. of slabs | \( p = 1 \) rates | \( p = 2 \) rates | \( p = 3 \) rates |
|----------------|--------------|-----------------|-----------------|-----------------|
| 64             | 8            | 8.00e-2         | -               | 1.52e-2         | -               |
| 256            | 16           | 3.15e-2         | 1.3             | 3.24e-3         | 2.2             | 2.92e-4         | 3.3             |
| 1024           | 32           | 1.30e-2         | 1.3             | 7.03e-4         | 2.2             | 3.21e-5         | 3.2             |
| 4096           | 64           | 5.95e-3         | 1.1             | 1.64e-4         | 2.1             | 3.80e-6         | 3.1             |
| 64             | 8            | 1.75e-1         | -               | 3.71e-2         | -               | 6.67e-3         | -               |
| 256            | 16           | 7.78e-2         | 1.2             | 6.23e-3         | 2.6             | 5.60e-4         | 3.6             |
| 1024           | 32           | 2.51e-2         | 1.6             | 1.03e-3         | 2.6             | 4.64e-5         | 3.6             |
| 4096           | 64           | 7.60e-3         | 1.7             | 1.76e-4         | 2.5             | 3.88e-6         | 3.6             |

Table 6.1

Rates of convergence when solving the advection–diffusion problem (2.2)–(2.3) on a time-dependent domain with mesh deformation satisfying (6.2) with \( \nu = 10^{-2} \) (top) and \( \nu = 10^{-6} \) (bottom).

7. Conclusions. In this paper, we presented and analyzed a space–time hybridizable discontinuous Galerkin method for the advection–diffusion equation on moving domains. We have shown the consistency, boundedness, and stability of the bilinear form, and the well-posedness of the method via an inf-sup condition. Further,
we demonstrated the convergence of the method and derived error estimates in a mesh dependent norm. Theory was validated by a numerical example.

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