Supergravity in the group-geometric framework: a primer

Leonardo Castellani

Dipartimento di Scienze e Innovazione Tecnologica
Università del Piemonte Orientale, viale T. Michel 11, 15121 Alessandria, Italy
INFN, Sezione di Torino, via P. Giuria 1, 10125 Torino, Italy
Arnold-Regge Center, via P. Giuria 1, 10125 Torino, Italy

Abstract

We review the group-geometric approach to supergravity theories, in the perspective of recent developments and applications. Usual diffeomorphisms, gauge symmetries and supersymmetries are unified as superdiffeomorphisms in a supergroup manifold. Integration on supermanifolds is briefly revisited, and used as a tool to provide a bridge between component and superspace actions. As an illustration of the constructive techniques, the cases of $d = 3, 4$ off-shell supergravities and $d = 5$ Chern-Simons supergravity are discussed in detail. A cursory account of $d = 10 + 2$ supergravity is also included. We recall a covariant canonical formalism, well adapted to theories described by Lagrangians $d$-forms, that allows to define a form hamiltonian and to recast constrained hamiltonian systems in a covariant form language. Finally, group geometry and properties of spinors and gamma matrices in $d = s + t$ dimensions are summarized in Appendices.

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leonardo.castellani@uniupo.it
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1 Introduction

Field theories with local symmetries are the theoretical tool “par excellence” to describe elementary particles and their interactions. Fields depend on spacetime coordinates $x$ and have specific transformation properties under local symmetries. For example the fields of the standard model transform under gauge symmetries as gauge or matter fields, and in a way dictated by the group representation they belong to. The fields in gravity theories also transform under diffeomorphisms according to their tensorial character. The essential difference between these two types of local transformations, in their infinitesimal versions, is that diffeomorphisms always contain a derivative of the field, which is absent in gauge transformations. This is simply due to the fact that general coordinate transformations relate fields at different spacetime points, whereas gauge transformations relate fields at the same spacetime point.

On the other hand, the idea of unifying gravity with gauge theories starting from their symmetry structure has an old and well-motivated history, culminating in the relatively recent AdS/CFT or gauge/gravity correspondences.
Even on the classical level we can provide a unified description of diffeomorphisms and gauge transformations. For this we need a group geometrical framework.

To set the stage we consider as basic fields of the theory the components of the vielbein one-form \( \sigma^A = \sigma(z)^A \Lambda dz^\Lambda \) on the manifold of a Lie group \( G \), \( A \) being an index in the \( G \) Lie algebra, and \( z^\Lambda \) the coordinates of the group manifold. This vielbein satisfies the Cartan-Maurer (CM) equations

\[
d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \tag{1.1}
\]

where \( C^A_{BC} \) are the structure constants of the \( G \) Lie algebra. The \( G \) vielbein \( \sigma^A(z) \) has a fixed dependence on the coordinates \( z \), and cannot therefore play the rôle of a dynamical object. We must consider then a “soft” group manifold, diffeomorphic to \( G \) and denoted by \( \tilde{G} \), with a vielbein \( \sigma^A \) not satisfying anymore the CM equations.

The amount of deformation from the original “rigid” group manifold is measured by the curvature two-form:

\[
R^A \equiv d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C \tag{1.2}
\]

Thus the soft \( \tilde{G} \) can fluctuate around the rigid \( G \) manifold (with \( R^A = 0 \)), in the same way spacetime of general relativity can fluctuate around flat Minkowski spacetime. Tangent vectors on \( \tilde{G} \), dual to the vielbein \( \sigma^A \), are denoted by \( t^A \), so that

\[
\sigma^A(t^B) = \delta^A_B.
\]

Diffeomorphisms along tangent vectors \( \varepsilon = \varepsilon^A t^A \) on \( \tilde{G} \) are generated by the Lie derivative \( \ell_\varepsilon \). When applied to the \( \tilde{G} \) vielbein, the variation under diffeomorphisms takes the suggestive form:

\[
\ell_\varepsilon \sigma^A = d\varepsilon^A + C^A_{BC} \varepsilon^B \varepsilon^C + \iota_\varepsilon R^A \tag{1.3}
\]

(\( \iota_\varepsilon \) is the contraction operator, see Appendix A) and one recognizes on the right-hand side the \( G \)-covariant derivative of the infinitesimal parameter \( \varepsilon^A \) plus a curvature term. When the curvature term vanishes, i.e. when \( \iota_\varepsilon R^A = 0 \), the diffeomorphism takes the form of a \textit{gauge transformation}, and the curvature is said to be \textit{horizontal} along the \( t^A \)'s entering the sum in \( \varepsilon = \varepsilon^A t^A \).

Thus in group manifold geometry \textit{gauge transformations} can be interpreted as \textit{particular} diffeomorphisms, along the directions on which the curvatures are horizontal.

To make the exposition more pedagogical, the group manifold approach will be developed within the basic examples of gravity and supergravity in \( d = 4 \).

The paper is organized as follows. Sections 2 and 3 deal with (first-order) gravity and supergravity in \( d = 4 \), and serve as an introduction to the group-geometric framework. The original references, where this approach was first proposed, are

\[\text{a short summary of group manifold geometry is given in Appendix A.}\]
given in [1]-[4]. Reviews can be found in [5]-[8]. In Section 4 we recall basic results in supermanifold integration (see for ex. [9] for a recent review, or [10] for a textbook), and new developments concerning integral forms, discussed in ref.s [11]-[15]. Section 5 summarizes the building rules of $d$-form Lagrangians, applied in subsequent examples. Section 6 extends the formalism to $p$-forms fields by considering Free Differential Algebras (FDA), first introduced in the context of supergravity theories in [16]. Off-shell $d = 3$ supergravity is recast in the group manifold setting in Section 7 and used in Section 8 to establish a bridge between the component and superspace actions, following [13]. Off-shell (new minimal) $d = 4$ supergravity in the group manifold setting is discussed in Section 9, based on ref. [17]. In Section 10 we provide selected examples of gauge supergravities, in odd dimensions (Chern-Simons supergravities, for a review see for example [18]) and in even dimensions (generalizations of the Mac Dowell-Mansouri action [19]). In these theories supersymmetry “lives on the fiber”, i.e. is part of a gauged superalgebra, and is not interpreted as a superdiffeomorphism. Finally, Section 11 recalls a covariant hamiltonian formalism well adapted to $d$-form Lagrangians [20]-[24], with an application to pure vierbein gravity first discussed in [20]. The Appendices contain a minireview on group geometry, and properties of spinors and gamma matrices in $d = s + t$ dimensions.

2 The first example: Poincaré gravity

2.1 Soft Poincaré manifold

Gravity in first order vierbein formalism can be recast in a group geometric setting as follows. Consider $G =$ Poincaré group, and denote the vielbein on the $\tilde{G}$ manifold as $\sigma^A = (V^a, \omega^{ab})$. The index $A = (a, ab)$ runs on the translations and Lorentz rotations of the Poincaré Lie algebra:

\[
[P_a, P_b] = 0
\]

\[
[M_{ab}, M_{cd}] = -\frac{1}{2} (\eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac})
\]

\[
[M_{ab}, P_c] = -\frac{1}{2} (\eta_{bc} P_a - \eta_{ac} P_b)
\]

$\eta$ being the flat Minkowski metric. The $V^a$ and $\omega^{ab}$ components of the $\tilde{G}$ vielbein are identified with the vierbein and the spin connection. The curvature two-form defined as in (1.2) becomes

\[
R^a = dV^a - \omega^{ab} \wedge V^b \eta_{bc}
\]

\[
R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega^{bd} \eta_{cd}
\]

and one recognizes the familiar expressions for the torsion and the Lorentz curvature. Taking the exterior derivative of these definitions yields the Bianchi identities.
\[ dR^a = -R^{ab} \wedge V^c \eta_{bc} + \omega^{ab} \wedge R^c \eta_{bc} \quad \rightarrow \quad D R^a = -R^{ab} \wedge V^c \eta_{bc} \]  

\[ dR^{ab} = (-R^{ac} \wedge \omega^{bd} + \omega^{ac} \wedge R^{bd}) \eta_{cd} \quad \rightarrow \quad D R^{ab} = 0 \]

where \( D \) is the Lorentz covariant exterior derivative.

At this stage all the fields depend on the \( \tilde{G} \) manifold coordinates, corresponding to the generators of the Lie algebra: thus \( V^a = V^a(x,y) \), \( \omega^{ab} = \omega^{ab}(x,y) \) where the coordinates \( x^a \), corresponding to the translations \( P_a \), describe usual spacetime, whereas \( y^{ab} \) are the coordinates in the “Lorentz directions”, corresponding to the \( SO(1,3) \) rotations generated by \( M_{ab} \). Moreover the one-forms \( V^a \), \( \omega^{ab} \) live on the whole \( \tilde{G} \), and therefore can be expanded as:

\[ V^a = V^a_\mu(x,y) dx^\mu + V^a_{\mu\nu}(x,y) dy^{\mu\nu} \]  

\[ \omega^{ab} = \omega^{ab}_\mu(x,y) dx^\mu + \omega^{ab}_{\mu\nu}(x,y) dy^{\mu\nu} \]

It would seem that we have an embarassment of riches, with unwanted extra fields \( V^a_{\mu\nu}, \omega^{ab}_{\mu\nu} \) and dependence of all the fields on extra coordinates \( y \).

### 2.2 Group manifold action

The overabundance of field components, and their dependence on \( y \) coordinates can be tamed by defining an appropriate action principle. To end up with a geometrical theory in four spacetime dimensions, we first construct a 4-form Lagrangian \( L \) made out of the \( \tilde{G} \) vielbein \( \sigma^A \) and its curvature \( R^A \), according to some building rules to be discussed later (Section 5). The Lagrangian for Poincaré gravity is given by:

\[ L = R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} \]  

We then define an action by integrating this Lagrangian on a 4-dimensional submanifold \( M^4 \) of the \( \tilde{G} \) manifold, spanned by the \( x \) coordinates.

Integration on submanifolds \( M^d \) of a \( d \)-form \( L \) that lives on a \( g \)-dimensional bigger space \( \tilde{G} \) can be performed as follows: we multiply \( L \) by the Poincaré dual of \( M^d \), a (singular) closed \( (g-d) \)-form \( \eta_{M^d} \) that localizes the Lagrangian on the submanifold \( M^d \), and integrate the resulting \( g \)-form on the whole \( \tilde{G} \). Thus the group manifold action is given by

\[ S = \int_{\tilde{G}} L \wedge \eta_{M^d} \]  

The fields of the theory are those contained in \( L \), i.e. the \( \tilde{G} \) vielbein components, and the embedding functions that define the \( M^d \) submanifold of \( \tilde{G} \), present in \( \eta_{M^d} \). We will see in Section 2.6 that the embedding functions do not enter the field equations obtained from the variation of \( S \).

In our example the group manifold action is the integral of a 10-form on \( \tilde{G} \) = soft Poincaré manifold:

\[ S = \int_{\tilde{G}} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd} \wedge \eta_{M^4} \]
2.3 Spacetime action

Consider now the action \( (2.12) \), but with a particular choice of \( \eta \) given by the 6-form

\[
\eta_{M^4} = \delta(y^{12})\delta(y^{13}) \cdots \delta(y^{34})dy^{12} \wedge dy^{13} \wedge \cdots \wedge dy^{34} \quad (2.13)
\]

Integration on the \( y \) coordinates reduces \( (2.12) \) to an integral on \( M^4 \), where the \( y \) dependence of all fields in \( L \) disappears because of the delta functions in \( \eta \), and the “legs” of \( L \) along \( dy \) differentials are killed by the product of all independent \( dy^{\mu \nu} \) in \( \eta \). Thus

\[
S = \int_{M^4} L|_{y=0,dy=0}
\]

is the spacetime action obtained from the group manifold action \( (2.12) \) with a specific choice of \( \eta_{M^4} \). It contains only the usual fields \( V^a_\mu(x) \) and \( \omega^{ab}_\mu(x) \) of Poincaré gravity, and reproduces the first order Einstein-Hilbert action. Indeed

\[
\epsilon_{abcd} R^{ab} \wedge V^c \wedge V^d|_{y=dy=0} = R^{ab}_{\mu \nu}(x)dx^\mu \wedge dx^\nu \wedge V^c_\rho(x)dx^\rho \wedge V^d_\sigma(x)dx^\sigma \epsilon_{abcd} =
R^{ab}_{\mu \nu}(x)V^e_\mu(x)V^f_\nu(x)dx^\mu \wedge dx^\nu \wedge V^c_\rho(x)dx^\rho \wedge V^d_\sigma(x)dx^\sigma \epsilon_{abcd} =
R^{ab}_{\mu \nu}(x)V^e_\mu(x)V^f_\nu(x)V^c_\rho(x)V^d_\sigma(x) \epsilon_{abcd} \epsilon^{\mu \nu \rho \sigma} d^4x = R^{ab}_{\mu \nu}(x) \det V \epsilon^{efcd} \epsilon_{abcd} d^4x
\]

where the volume 4-form \( d^4x \) is defined by \( dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \epsilon^{\mu \nu \rho \sigma} d^4x \).

Note: \( \eta \) is closed (because it contains “functions” depending on \( y \) multiplied by all the \( dy \) differentials) and not exact (because of the Dirac deltas \( \delta(y) \)), and thus belongs to a nontrivial de Rahm cohomology class. Deformations of the \( M^4 \) surface generated by diffeomorphisms leave the Poincaré dual \( \eta \) in the same cohomology class, since the Lie derivative commutes with the exterior derivative.

Discussion

Why go this roundabout way to obtain a well-known gravity action? The answer is at least fivefold:
- all fields have a group-geometric origin, even if they are not all gauge fields.
- all symmetries have a common origin as diffeomorphisms on \( \tilde{G} \), see Section 2.4.
- there is a systematic procedure based on group geometry to construct actions, invariant under diffeomorphisms, and under gauge symmetries closing on a subgroup of \( G \), see Section 5.
- supersymmetry is formulated in a very natural way as a diffeomorphism in Grassmann directions of a supermanifold.
- closer contact is maintained with the usual component action, whereas in the superfield formalism the action looks quite different. In fact the group manifold action interpolates between the component and the superfield actions of the same supergravity theory, see Section 8.
2.4 Symmetries

The action (2.12) with \( \eta \) given in (2.13) is the integral on \( \tilde{G} \) of a top form: it is clearly invariant under diffeomorphisms on \( \tilde{G} \). But what we are really interested in are the symmetries of the spacetime action as given in (2.14), where the variations are carried out only in the \( x \)-dependent fields in \( L|_{y=0,dy=0} \). The only symmetries guaranteed a priori are the 4-dimensional spacetime diffeomorphisms, the spacetime action being an integral of a 4-form on \( M^4 \).

Here resides most of the power of the group manifold formalism: if one considers the “mother” action (2.11) on \( \tilde{G} \), the guaranteed symmetries are all the diff.s on \( \tilde{G} \), generated by the Lie derivative \( \ell_\varepsilon \) along the tangent vectors \( \varepsilon = \varepsilon^A t_A \) of \( \tilde{G} \). But how do these symmetries transfer to the spacetime action ?

The variation of the group manifold action under diff.s generated by \( \ell_\varepsilon \) is

\[
\delta S = \int_{\tilde{G}} \ell_\varepsilon (L \wedge \eta) = \int_{\tilde{G}} (\ell_\varepsilon L) \wedge \eta + L \wedge \ell_\varepsilon \eta = 0 \tag{2.16}
\]

modulo boundary terms. One has to vary the fields \( \phi \) in \( L \) as well as the submanifold embedded in \( \tilde{G} \): the sum of these two variations gives zero on the group manifold action \( S \). But what we need in order to have a spacetime interpretation of all the symmetries of \( S \), is really

\[
\delta S = \int_{\tilde{G}} (\ell_\varepsilon L) \wedge \eta = 0 \tag{2.17}
\]

If this holds, varying the fields \( \phi \) inside \( L \) with the Lie derivative \( \ell_\varepsilon \) as in (1.3), and then projecting on spacetime \( (y = 0, dy = 0) \), yields spacetime variations

\[
\delta \phi(y = dy = 0) = \ell_\varepsilon \phi(x, y)|_{y=dy=0} \tag{2.18}
\]

that leave the spacetime action (2.14) invariant. We call them spacetime invariances. They originate from the diff. invariance of the group manifold action, and give rise to symmetries of the spacetime action (2.14) only when (2.17) holds. This happens if one of the following conditions is satisfied:

- the Lie derivative on \( \eta \) vanishes:

\[
\ell_\varepsilon \eta = 0 \tag{2.19}
\]

- the spacetime projection of the Lie derivative of \( L \) is exact:

\[
(\ell_\varepsilon L)|_{y=dy=0} = d\alpha \tag{2.20}
\]

\(^2\)Recall \( \ell_\varepsilon = \iota_\varepsilon d + d\iota_\varepsilon \) so that \( \ell_\varepsilon (\text{top form}) = d(\iota_\varepsilon \text{ top form}) \)

\(^3\)Since \( \ell_\varepsilon \) satisfies the Leibnitz rule, \( \ell_\varepsilon L \) can be computed by varying in turn all fields inside \( L \).

\(^4\)In the following the vanishing of action variations will always be understood modulo boundary terms.
In this case the variation \( \delta S = \int \tilde{G} (\ell_\varepsilon L) \wedge \eta = \int_{M^4} (\ell_\varepsilon L)|_{y=dy=0} \) (2.21) vanishes after integration by parts. The requirement (2.20) is equivalent to
\[
(\iota_\varepsilon dL)|_{y=dy=0} = d\alpha'
\] (2.22)
since \( l_\varepsilon = \iota_\varepsilon d + dt_\varepsilon \).

The Lagrangian \( L \) depends on the \( \tilde{G} \)-vielbein \( \sigma^A \) and its curvature \( R^A \), so that also \( dL \), after use of Bianchi identities, is expressed in terms of \( \sigma^A \) and \( R^A \). Then condition (2.22) translates into a condition on the contractions \( \iota_\varepsilon R^A \), i.e. a condition on the curvature components.

Let us see how this works for Poincaré gravity.

**Lorentz gauge transformations**

We choose \( \varepsilon = \varepsilon^{ab}t_{ab} \), with \( t_{ab} \) tangent vector on \( \tilde{G} \) dual to \( \omega^{ab} \), and compute \( \iota_\varepsilon dL \) in \( \tilde{G} \) (in \( M^4 \) we would have trivially \( dL = 0 \) since \( L \) is a 4-form). We find
\[
dL = [(dR^{ab})V^cV^d + 2R^{ab}(dV^c)V^d]\epsilon_{abcd} = 2R^{ab}R^cV^d\epsilon_{abcd}
\] (2.23)
using the Bianchi identities (2.6) and (2.7). The contraction along \( \varepsilon = \varepsilon^{cd}t_{cd} \)
\[
\iota_\varepsilon dL = 2(\iota_\varepsilon R^{ab})R^cV^d\epsilon_{abcd} + 2R^{ab}(\iota_\varepsilon R^c)V^d\epsilon_{abcd}
\] (2.24)
vanishes if the Poincaré curvatures satisfy the horizontality conditions
\[
\iota_{t_{cd}}R^a = 0
\] (2.25)

In this case \( \iota_{t_{cd}}dL = 0 \) and the spacetime action is invariant under transformations generated by \( \iota_{t_{cd}} \). The horizontality conditions (2.25) imply that the curvatures have no “legs” in the Lorentz directions: when expanded on a complete basis of 2-forms on \( \tilde{G} \) as in (2.44), (2.45), their \( V\omega \) and \( \omega\omega \) components vanish (horizontality in the Lorentz directions).

The transformations generated by \( \iota_{t_{cd}} \) are found by using the horizontality constraints (2.25) inside the general formula (1.3) and read:
\[
\ell_{t_{cd}}V^a = \varepsilon^a_b V^b
\] (2.26)
\[
\ell_{t_{cd}}\omega^{ab} = d\varepsilon^{ab} - \omega^a_c \varepsilon^{cb} + \omega^b_c \varepsilon^{ca} = D\varepsilon^{ab}
\] (2.27)
They are the usual local Lorentz rotations on the vierbein and the spin connection.

It is easy to check directly the invariance of the action under these transformations, recalling that \( R^{ab} = d\omega^{ab} - \omega^a_c \omega^b_c \) transforms homogeneously under (2.27):
\[
\ell_{t_{cd}}R^{ab} = \varepsilon^a_c R^{cb} - \varepsilon^b_c R^{ca}
\] (2.28)
5omitting the symbol \( \wedge \) for exterior products between forms.
Note: the constraints (2.25) will be derived as part of the equations of motion on $\tilde{G}$ in next Section.

The horizontality constraints can be used inside the action (2.11), i.e. we can consider the fields appearing in the action as satisfying the “partial shell” given by (2.25). Then the soft group manifold $\tilde{G}$ takes the structure of a principal fiber bundle with base space $G/H$ and fiber $H$, $H$ being the Lorentz group.

**Spacetime diffeomorphisms**

Diff.s along tangent vectors $\partial_\mu$ dual to $dx^\mu$ are known a priori to be invariances of the spacetime action, and we can verify that indeed (2.19) holds, i.e. $\ell_\varepsilon^\nu \partial_\nu \eta_M = 0$, since $\eta_M$ contains only $dy$ differentials. Diff.s along tangent vectors $t_a$ dual to $V^a$, i.e. generated by $\ell_\varepsilon$ with $\varepsilon = \varepsilon^a t_a$ are also spacetime invariances, when one uses the horizontality conditions (2.25). Indeed in this case we find $dL = 0$ ($dL$ is a 5-form, and cannot contain $5V$’s), and therefore also $\iota_{\varepsilon t_a} dL = 0$.

The diff.s along $\varepsilon = \varepsilon^a t_a$ act on the fields as:

$$\ell_\varepsilon V^a = D\varepsilon^a + \ell_\varepsilon R^a = D\varepsilon^a + 2R^a_{\ bc} \varepsilon^b V^c$$

$$\ell_\varepsilon \omega^{ab} = \ell_\varepsilon R^{ab} = 2R^{ab}_{\ cd} \varepsilon^c V^d$$

(2.29) (2.30)

Note: we can verify that the horizontality constraints (2.25) are consistent with the Bianchi identities (2.6), (2.7), by projecting the BI on the complete basis of 3-forms $VVV$, $VV\omega$, $V\omega\omega$, $\omega\omega\omega$.

**2.5 Variational principle and field equations**

The group manifold action (2.11) is a functional of $L$ and of the embedded submanifold $M$, and therefore varying the action means varying both $L$ and $M$. Varying $M$ corresponds to varying $\eta_M$. Then the variational principle reads:

$$\delta S[L, M] = \int_{\tilde{G}} (\delta L \wedge \eta_M + L \wedge \delta \eta_M) = 0.$$  

(2.31)

Any (continuous) variation of $M$ can be obtained by acting on $\eta_M$ with a diffeomorphism generated by a Lie derivative $\ell_\xi$. An arbitrary variation is generated by an arbitrary $\xi$ vector, and the variational principle becomes

$$\delta S[L, M] = \int_{\tilde{G}} (\delta L \wedge \eta_M + L \wedge \ell_\xi \eta_M) = 0.$$  

(2.32)

Since field variations in $L$ and variation of $M$ are independent, the two terms in (2.32) must vanish separately. From the vanishing of the first one we deduce

$$\int_{\tilde{G}} (\delta \phi \wedge \frac{\partial L}{\partial \phi} + d\delta \phi \wedge \frac{\partial L}{\partial (d\phi)}) \wedge \eta_M = 0$$  

(2.33)
where \( L = L(\phi, d\phi) \) is considered a function of the 1-form fields \( \phi \) and their “velocities” \( d\phi \). A summation on all fields is understood. Integrating by parts and recalling \( d\eta_M = 0 \) yields

\[
\int_{\tilde{G}} \delta\phi \wedge \left( \frac{\partial L}{\partial \phi} + d \frac{\partial L}{\partial (d\phi)} \right) \wedge \eta_M = 0
\]  

(2.34)

and since the \( \delta\phi \) are arbitrary we find

\[
\left( \frac{\partial L}{\partial \phi} + d \frac{\partial L}{\partial (d\phi)} \right) \wedge \eta_M = 0
\]  

(2.35)

This must hold for any \( \eta_M \) (i.e. for generic embedding functions): we arrive therefore at equations that hold on the whole \( \tilde{G} \), and are the form version of the Euler-Lagrange equations:

\[
\frac{\partial L}{\partial \phi} + d \frac{\partial L}{\partial (d\phi)} = 0
\]  

(2.36)

If \( L \) is a \( d \)-form, these equations are \((d-1)\)-forms. Their content can be examined by expanding them along a complete basis of \((d-1)\)-forms in \( \tilde{G} \).

Requiring the vanishing of the second term in the variation \( \delta L \) does not imply further equations besides the Euler-Lagrange field equations (2.36): indeed this term vanishes on the shell of solutions of Euler-Lagrange equations. To prove it, notice that

\[
\int_{\tilde{G}} L \wedge \ell_\xi \eta_M = - \int_{\tilde{G}} \ell_\xi L \wedge \eta_M = 0 \quad \text{(on shell)}
\]  

(2.37)

because \( \ell_\xi L \) is just a particular variation of \( L \), under which the action remains stationary on-shell.

Thus the group manifold variational principle leads to the field equations (2.36), holding as \((d-1)\)-form equations on the whole \( \tilde{G} \).

**Note 1:** The variational principle does not determine the embedding of \( M \) into \( \tilde{G} \).

**Note 2:** the field equations (2.36) are form equations, and therefore invariant under the action of a Lie derivative. More precisely, if \( \phi \) is a solution of (2.36), so is \( \phi + \ell_\xi \phi \): Lie derivatives generate symmetries of the field equations.

Finally, we have the following

**Theorem:** \( dL = 0 \) (on shell)

i.e. the Lagrangian, as a \( d \)-form on \( \tilde{G} \), is closed on shell. To prove it recall that \( \eta_M \) is closed, so that on shell we find, cf. (2.37):

\[
0 = \int_{\tilde{G}} L \wedge \ell_\xi \eta_M = \int_{\tilde{G}} L \wedge d\ell_\xi \eta_M = -(-)^d \int_{\tilde{G}} dL \wedge \ell_\xi \eta_M
\]  

(2.38)
ξ being arbitrary, this implies \( dL = 0 \) (on shell)\( \square \)

It is interesting to notice that in many cases \( dL = 0 \) holds only on a subset of the equations of motion, and in some cases, it holds completely off-shell, as we discuss later.

Let us apply the preceding discussion to the Poincaré gravity example. Varying \( \omega^{ab} \) and \( V^{c} \) in the action

\[
S[\omega, V, \eta] = \int_{\tilde{G}} R^{ab} V^{c} V^{d} \epsilon_{abcd} \eta_{M} \tag{2.39}
\]
yields respectively:

\[
\delta S = \int_{\tilde{G}} \mathcal{D}(\delta \omega^{ab}) V^{c} V^{d} \epsilon_{abcd} \eta_{M} = 2 \int_{\tilde{G}} (\delta \omega^{ab}) R^{c} V^{d} \epsilon_{abcd} \eta_{M} \tag{2.40}
\]
\[
\delta S = 2 \int_{\tilde{G}} R^{ab} \delta V^{c} V^{d} \epsilon_{abcd} \eta_{M} \tag{2.41}
\]

Imposing \( \delta S = 0 \) leads to the equations of motion

\[
R^{c} V^{d} \epsilon_{abcd} = 0 \tag{2.42}
\]
\[
R^{ab} V^{d} \epsilon_{abcd} = 0 \tag{2.43}
\]

These field equations are 3-form equations on \( \tilde{G} \). The curvatures \( R^{a} \) and \( R^{ab} \) can be expanded on a complete basis of 2-forms as

\[
R^{a} = R^{a}_{bc} V^{b} V^{c} + R^{a}_{bcd} V^{b} \omega^{cd} + R^{a}_{bcde} \omega^{bcd} \omega^{de} \tag{2.44}
\]
\[
R^{ab} = R^{ab}_{cd} V^{c} V^{d} + R^{ab}_{cde} V^{c} \omega^{de} + R^{ab}_{cd,ef} \omega^{cd} \omega^{ef} \tag{2.45}
\]

Substituting these expansions into the field equations (2.42), (2.43), and projecting on a complete basis of 3-forms in \( \tilde{G} \) yields:

\[
R^{c}_{e,f} V^{e} V^{f} V^{d} \epsilon_{abcd} = 0 \tag{2.46}
\]
\[
R^{ab}_{e,f} V^{e} V^{f} V^{d} \epsilon_{abcd} = 0 \tag{2.47}
\]

The three terms in each equation must vanish separately, since \( V V V, V \omega V, \omega \omega V \) are independent three-forms.

It is easy to see that the \( V \omega V \) and \( \omega \omega V \) projections of the first equation imply \( R^{a}_{b,cd} = R^{a}_{bc,de} = 0 \), i.e. horizontality of \( R^{a} \), while the \( V V V \) projection yields \( R^{a}_{b,c} = 0 \). Then \( R^{a} \) as a 2-form on \( \tilde{G} \) must vanish on shell. From

\[
R^{a}_{\mu \nu} = \partial_{\mu} V^{a}_{\nu} - \partial_{\nu} V^{a}_{\mu} - \omega^{a}_{\mu \rho} V^{b}_{\nu} V^{\rho}_{\mu} + \omega^{a}_{\nu \rho} V^{b}_{\mu} V^{\rho}_{\nu} = 0 \tag{2.48}
\]

one finds the spin connection in terms of \( V \):

\[
\omega_{ab,\mu} = V_{a}^{\rho} V_{b}^{\nu} \eta_{cd}(\partial_{\mu} V_{a}^{\rho} V_{\nu}^{d} - \partial_{\nu} V_{a}^{\rho} V_{\mu}^{d} + \partial_{\nu} V_{b}^{\nu} V_{\mu}^{d}) \tag{2.49}
\]

\(^6\) In fact, this is just Stokes theorem applied to a region of \( \tilde{G} \) bounded by two different hypersurfaces \( M \) and \( M' \).
where $V^\nu_a$ is the inverse vierbein.

Similarly the second field equation implies $R^{ab}_{cde} = R^{ab}_{cd,e} = 0$ (horizontality of $R^{ab}$), and the Einstein equations for the inner components $R^{ab}_{cd}$:

$$R^{ac}_{bc} - \frac{1}{2} \delta^a_b R^{cd}_{cd} = 0 \quad (2.50)$$

We can also check that $dL = 0$ on a subset of the field equations, since $dL = 2R^{ab} R^c V^d \epsilon_{abcd}$ (cf. (2.23)) vanishes when $R^a = 0$, or when using horizontality of the curvatures.

**Note 3:** The fact that the same horizontality conditions arise both from the request of the spacetime invariance under diffeomorphisms along the Lorentz directions, and from the field equations, should come as no surprise. Indeed $dL = 0$ on shell implies also $\iota_{\epsilon_{cd}} dL \vert_{y=dy=0} = 0$ on shell, so that the field equations imply conditions on the outer components of the curvatures similar to those requested by spacetime invariance. But there is an important difference: the conditions on the outer components of $R^A$ coming from (2.22) must hold off-shell, while those coming from the field equations are on-shell by definition, and may differ by field equations involving spacetime components of the curvatures.

**Note 4:** Lorentz gauge invariance of the action is really due to the absence of a bare connection $\omega$ in the Lagrangian (2.10). In this case also the field equations do not contain bare $\omega$'s, and their projections with at least one $\omega$ must then contain outer components of the curvatures. Horizontality follows, and Lorentz gauge transformations can be interpreted as diff.s in the Lorentz coordinates $y^{ab}$.

**Note 5:** the horizontality constraints arise as outer projections of the equations of motion, and, as noted in the preceding Section, can be used inside the group manifold action. The corresponding spacetime action (2.14) remains unchanged: this can be easily verified by substituting $R^{ab}$ with $R^{ab}_{cd} V^c V^d$ into (2.10).

## 3 Supergravity

In the group-geometric approach to supergravity theories, the “big” manifold $\tilde{G}$ is a (soft) supergroup manifold, and there are fermionic vielbeins $\psi$ (the gravitini) dual to the fermionic tangent vectors in $\tilde{G}$. 
### 3.1 Soft super-Poincaré manifold

The $N=1$ super-Poincaré Lie algebra is a superalgebra which extends the algebra given in (2.1)-(2.3) by means of a spinorial generator $\bar{Q}_\alpha$ satisfying:

\[
[P_a, \bar{Q}_\alpha] = 0 \tag{3.1}
\]
\[
[M_{ab}, \bar{Q}_\beta] = -\frac{1}{4} \bar{Q}_\alpha (\gamma_{ab})^{\alpha}_\beta \tag{3.2}
\]
\[
\{\bar{Q}_\alpha, \bar{Q}_\beta\} = -i (C\gamma^\alpha)_{\alpha\beta} P_a \tag{3.3}
\]

$C_{\alpha\beta}$ is the charge conjugation matrix, and the spinorial generator $\bar{Q}_\alpha \equiv Q^\beta C_{\beta\alpha}$ is a Majorana spinor, i.e. $Q^\beta C_{\beta\alpha} = Q^\dagger_{\beta} (\gamma^0)_{\beta\alpha}$. Thus the super-Poincaré manifold has 10 bosonic directions with coordinates $x^a$, $y^{ab}$, parametrizing translations and Lorentz rotations, and 4 fermionic directions with Grassmann coordinates $\theta^\alpha$, corresponding to the 4 supercharges $\bar{Q}_\alpha, \alpha = 1, \ldots, 4$.

The components of the vielbein of the $\tilde{G}$ (soft) super-Poincaré manifold are the vierbein $V^a$, the spin connection $\omega^{ab}$ and the gravitino $\psi^\alpha$, corresponding respectively to the generators $P_a$, $M_{ab}$ and $\bar{Q}_\alpha$.

The curvature (1.2) becomes, using the structure constants of the Lie superalgebra:

\[
R^a = dV^a - \omega^c V^e - \frac{i}{2} \bar{\psi} \gamma^a \psi \equiv D V^a - \frac{i}{2} \bar{\psi} \gamma^a \psi \tag{3.4}
\]
\[
R^{ab} = d\omega^{ab} - \omega^c \omega^{cb} \tag{3.5}
\]
\[
\rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi \equiv D \psi \tag{3.6}
\]

defining respectively the supertorsion, the Lorentz curvature and the gravitino field strength. $D$ is the Lorentz covariant exterior derivative.

As a consequence of the definitions (3.4)-(3.6), the following Bianchi identities hold:

\[
d R^a - \omega^b R^b + R^a b V^b - i \bar{\psi} \gamma^a \rho \equiv DR^a + R^a b V^b - i \bar{\psi} \gamma^a \rho = 0 \tag{3.7}
\]
\[
d R^{ab} - \omega^c R^{cb} + \omega^b R^{ca} \equiv DR^{ab} = 0 \tag{3.8}
\]
\[
d \rho - \frac{1}{4} \omega^{ab} \gamma_{ab} \rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi \equiv D \rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi = 0 \tag{3.9}
\]

### 3.2 The supergroup manifold action

The supergravity action is again the integral of a 4-form on a submanifold $M^4 \subset \tilde{G}$, diffeomorphic to Minkowski spacetime. In this case $\tilde{G}$ is the 14-dimensional super-Poincaré group manifold, and the action reads:

\[
S[V, \omega, \psi, \eta] = \int_{\tilde{G}} (R^{ab} V^c V^d \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_a \rho V^a) \eta_{M^4} \tag{3.10}
\]

with $\eta_{M^4} = $ Poincaré dual of $M^4$. Here $\eta_{M^4}$ is a (closed) “10-form” that localizes the Lagrangian on the submanifold $M^4$, to be discussed in Section 4 in the context of
integration on supermanifolds. The Lagrangian can be found by use of the building rules of Section 5; for a detailed derivation see for ex. [5, 8].

3.3 The spacetime action

The spacetime action is obtained by a specific choice of $\eta_{M^4}$ in (3.10). Its precise expression will be given in Section 4. Actually a piece of $\eta_{M^4}$ is the 6-form (2.13), that localizes the Lagrangian on $y = dy = 0$ once the integration on $y$ coordinates is carried out. We will always assume that this integration has been carried out, so that all fields depend only on $x$ and $\theta$ coordinates. Moreover all curvatures are taken to be horizontal in the Lorentz directions. As a consequence the theory lives in a superspace $M^{4|4}$ spanned by four bosonic coordinates $x^a$ and four fermionic coordinates $\theta^\alpha$.

3.4 Symmetries

The symmetries of the spacetime action (spacetime invariances) are those generated by a Lie derivative $\ell_\varepsilon$ such that $\iota_\varepsilon dL\big|_{\theta = \bar{\theta} = 0} = d\alpha'$, cf. (2.22). We need to compute $dL$. Using the Bianchi identities (3.8) and (3.9), and the definition of the torsion $R^a$ in (3.4) we find:

$$dL = 2R^{ab}R^c\psi^d\varepsilon_{abcd} + iR^{ab}\bar{\psi}\gamma^c\psi^d\varepsilon_{abcd} + 4\bar{\rho}\gamma_5\gamma_a\rho\psi^a +$$

$$+ \bar{\psi}\gamma_5\gamma_c\gamma_\alpha\theta\psi^d - 4\bar{\psi}\gamma_5\gamma_a\rho R^a - 2i\bar{\psi}\gamma_5\gamma_a\rho\bar{\psi}\gamma^a\psi$$

(3.11)

The gamma identity

$$\gamma_c\gamma_ab = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + i\varepsilon_{abcd}\gamma^d$$

(3.12)

implies $\bar{\psi}\gamma_5\gamma_c\gamma_\alpha\theta\psi = i\varepsilon_{abcd}\bar{\psi}\gamma^d\psi$, so that the second and the fourth term cancel in (3.11). Moreover from the Fierz identity in Appendix D one deduces

$$\gamma_a\bar{\psi}\gamma^a\psi = 0$$

(3.13)

and since $\bar{\psi}\gamma_5\gamma_a\rho = \bar{\rho}\gamma_5\gamma_a\psi$ also the last term in (3.11) vanishes due to (3.13). Therefore

$$dL = 2R^{ab}R^c\psi^d\varepsilon_{abcd} + 4\bar{\rho}\gamma_5\gamma_a\rho\psi^a - 4\bar{\psi}\gamma_5\gamma_a\rho R^a$$

(3.14)

Lorentz gauge transformations

It is immediate to see that if all curvatures are horizontal in the Lorentz directions (no “legs” along $\omega$) then indeed $\iota_\varepsilon_{ab\psi\theta}dL = 0$, and Lorentz transformations are a spacetime invariance of the supergravity action. This is essentially due to the absence of bare $\omega^{ab}$ in $L$. The general diffeomorphism formula (1.3) yields the usual
Lorentz transformations

\[ \ell_{\varepsilon} V^a = \varepsilon^a_b V^b \]  
(3.15)

\[ \ell_{\varepsilon} \omega^{ab} = d\varepsilon^{ab} - \omega^a_c \varepsilon^{cb} + \omega^b_c \varepsilon^{ca} = \mathcal{D}_\varepsilon^{ab} \]  
(3.16)

\[ \ell_{\varepsilon} \psi = \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \psi \]  
(3.17)

We can check directly the invariance of the action under these variations: again all curvatures and vierbeins appearing in (3.10) transform homogeneously.

**Spacetime diffeomorphisms**

Diff.s along tangent vectors \( \partial_{\mu} \) dual to \( dx^\mu \) are invariances of the spacetime action, since \( \ell_{\varepsilon} \partial_{\mu} \eta_M = 0 \) due to \( \eta_M \) containing only \( dy \) and \( d\theta \) differentials, see Section 4 on superintegration.

**Supersymmetry transformations**

Diff.s along tangent vectors \( t_\alpha \) dual to \( \psi^\alpha \) are spacetime invariances provided \( \epsilon dL |_{\theta = 0} = \text{total derivative} \) with \( \epsilon = \epsilon^\alpha t_\alpha \), that is to say

\[ \epsilon dL = 2(\epsilon R^{ab}) R^{cd} \varepsilon_{abcd} + 2R^{ab} (\epsilon R^c) \varepsilon_{abcd} + 8\bar{\rho} \gamma_5 \gamma_a (\epsilon \rho) V^a - 4\varepsilon \gamma_5 \gamma_a \rho R^a - 4\psi \gamma_5 \gamma_a (\epsilon \rho) R^a + 4\bar{\psi} \gamma_5 \gamma_a \rho (\epsilon R^a) = \text{tot. der.} \]  
(3.18)

with \( \theta = d\theta = 0 \). This is a condition for the contractions on the curvatures, and it is satisfied by:

\[ \epsilon R^a = 0 \]  
(3.19)

\[ \epsilon R^{ab} = -\epsilon^{abef} \tilde{\rho}_{ef} \gamma_5 \gamma_g \epsilon V^g - \varepsilon^{efg} [\tilde{\rho}_{ef} \gamma_5 \gamma_g \epsilon V^b] \equiv \tilde{\theta}^{ab}_c \epsilon V^c \]  
(3.20)

\[ \epsilon \rho = 0 \]  
(3.21)

Thus we have supersymmetry invariance of the spacetime action if the curvatures have the following parametrization on a basis of 2-forms:

\[ R^a = R^a_{\ bc} V^b V^c \]  
(3.22)

\[ R^{ab} = R^{ab}_{\ cd} V^c V^d + \tilde{\theta}^{ab}_c \epsilon V^c \]  
(3.23)

\[ \rho = \rho_{ab} V^a V^b \]  
(3.24)

where we have taken into account also horizontality in the Lorentz directions. The conditions (3.19)-(3.21) are called “rheonomic conditions”, and similarly (3.22)-(3.24) are called “rheonomic parametrizations” of the curvatures.

The diff.s along \( \epsilon = \epsilon^\alpha t_\alpha \) (supersymmetry transformations) act on the fields according to the general formula (1.3), where the contractions on the curvatures are given in (3.19)-(3.21):

\[ \ell_\epsilon V^a = i\bar{\epsilon} \gamma^a \psi \]  
(3.25)

\[ \ell_\epsilon \omega^{ab} = \tilde{\theta}^{ab}_c \epsilon V^c \]  
(3.26)

\[ \ell_\epsilon \psi = \mathcal{D}_\epsilon \equiv d\epsilon - \frac{1}{4} \omega^{ab} \gamma_{ab} \epsilon \]  
(3.27)
with $\tilde{\Theta}^{ab}_c$ defined in (3.20).

### 3.5 Bianchi identities

The Bianchi identities (3.7)-(3.9) are satisfied by the curvatures in (3.22)-(3.24), where the horizontality and rheonomic conditions are implemented, provided the following equations on their $VV$ components hold:

\begin{align*}
R^{a}_{bc} &= 0 \quad (3.28) \\
R^{ac}_{bc} - \frac{1}{2} \delta_b^a R^{cd}_{cd} &= 0 \quad (3.29) \\
\gamma^a \rho_{ab} &= 0 \quad (3.30)
\end{align*}

i.e. the zero torsion condition that allows to express $\omega$ as function of $V$ and $\psi$, and the Einstein and Rarita-Schwinger propagation equations for the vielbein and the gravitino, respectively. Thus the Bianchi identities for the curvatures parametrized as in (3.22)-(3.24) hold only on the shell of the propagation equations (3.28)-(3.30). As a consequence the superalgebra generated by the Lie derivatives closes only on-shell (see Appendix A). In other words, the transformations (3.25) - (3.27) can be interpreted as diffeomorphisms in $M^{d|4}$ only when they are applied on fields that are solutions of the field equations. On general field configurations the supersymmetry transformations (3.25) - (3.27) leave the action invariant, but their commutator cannot be expressed as a Lie derivative along a tangent vector of $\tilde{G}$. This situation can be cured by adding extra fields in the theory, called auxiliary fields, entering in the parametrization of the curvatures in such a way that the Bianchi identities do not imply propagation equations. The auxiliary fields are nondynamical fields, but their degrees of freedom (d.o.f.) are needed to ensure an equal number of off-shell d.o.f. of fermions and bosons. We will see how to achieve this for $d = 4$ supergravity in Section 9.

The rheonomic parametrizations (3.22)-(3.24) cannot be used inside the action, since it would amount to consider only on-shell fields and not all field configurations. They have been used exclusively in the transformation laws of the fields. In fact they have been determined in Section 3.4 by requiring supersymmetry invariance of the spacetime action.

**Note 1:** horizontality, rheonomic conditions and propagation equations will all be derived in next Section as field equations from the group manifold action.

**Note 2:** in deriving the rheonomic conditions from (3.18) we have tacitly assumed that (3.24) could be used in the uncontracted $\rho$ (and thus outside the expression of a field variation) in the fourth term of the $\iota_x dL$ expression. This in fact we can do, since the condition (3.18) only needs to hold with $\theta = d\theta = 0$, and $\rho = \rho_{\mu\nu} dx^\mu dx^\nu = \rho_{ab} V^a V^b$ when all quantities depend only on $x$ and $dx$.

**Note 3:** the spacetime action (3.10) and its invariance under the supersymmetry transformations (3.25)-(3.27) were first found in ref. [26] in second order formalism.
and in [27] in first order formalism, see also the standard references [28, 29] on supergravity.

### 3.6 Field equations

The variational equations for the group manifold action (3.10) read:

\[ 2 R^c V^d \epsilon_{abcd} = 0 \quad (3.31) \]
\[ 2 R^{ab} V_c \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_d \rho = 0 \quad (3.32) \]
\[ 8 \gamma_5 \gamma_a \rho V^a - 4 \gamma_5 \gamma_a \bar{\psi} R^a = 0 \quad (3.33) \]

The analysis proceeds as follows. We first expand the curvatures on a basis of 2-forms\(^7\)

\[ R^a = R^a_{bc} V^b V^c + \bar{\theta}^a_c \psi V^c + \bar{\psi} K^a \psi \quad (3.34) \]
\[ R^{ab} = R^{ab}_{cd} V^c V^d + \bar{\theta}^{ab}_c \psi V^c + \bar{\psi} K^{ab} \psi \quad (3.35) \]
\[ \rho = \rho_{ab} V^a V^b + H_c \psi V^c + \Omega_{\alpha \beta} \psi^\alpha \psi^\beta \quad (3.36) \]

and then insert them into the field equations (3.31)-(3.33). These, being 3-form equations, can be expanded on the basis \( \psi \psi \psi, \psi \psi V, \psi V V, V V V \). Their content is given below (the three lines correspond to the three eq.s of motion):

**\( \psi \psi \psi \) sector:**

\[ \Omega_{\alpha \beta} = 0 \quad (3.37) \]
\[ 0 = 0 \quad (3.38) \]
\[ K^a = 0 \quad (3.39) \]

**\( \psi \psi V \) sector:**

\[ 2 \bar{\psi} K^{ab} \psi V^c \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_d H_c \psi V^c = 0 \quad (3.40) \]
\[ 0 = 0 \quad (3.41) \]
\[ \bar{\theta}^a_c = 0 \quad (3.42) \]

**\( \psi V V \) sector:**

\[ 2 \bar{\psi}^{ab} \psi V^c V^e \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_d \rho_{ab} V^a V^b = 0 \quad (3.43) \]
\[ 0 = 0 \quad (3.44) \]
\[ \gamma_5 \gamma_a H_b \psi V^b V^a - 4 \gamma_5 \gamma_c \psi R^c_{ab} V^a V^b = 0 \quad (3.45) \]

**\( V V V \) sector:**

\[ R^a_{bc} = 0 \quad (3.46) \]
\[ R^{ac} - \frac{1}{2} \delta^a_b R^{cd} = 0 \quad (3.47) \]
\[ \gamma^a \rho_{ab} = 0 \quad (3.48) \]

\(^7\)assuming horizontality in the Lorentz directions. This amounts to consider configurations satisfying the Lorentz horizontality constraints on the curvatures.
Inserting $R^a_{bc} = 0$ into (3.45) yields $H_c = 0$, which used in (3.40) gives $K^{ab} = 0$. Thus the only nontrivial relation in the “outer” projections is (3.43), that determines $\theta^{ab,c}$ to be

$$\theta^{ab,c} = -\varepsilon^{abef} \bar{\rho}_{ef} \gamma_5 \gamma_c - \delta^{ab}_{[a} \varepsilon^{b]ef} \bar{\rho}_{ef} \gamma_5 \gamma_9$$

(3.49)

in agreement with the $\theta^{ab,c}$ obtained from the condition (3.20). Thus we arrive at the same curvature parametrizations (3.22)-(3.24) obtained in Sect. 2 by requiring spacetime supersymmetry invariance.

Finally, the VVV sector reproduces the torsion equation, and the propagation equations for the vierbein and the gravitino, already obtained from the Bianchi identities in (3.28)-(3.30).

**Note:** the torsion equation (3.31) has the same solution as in the pure gravity case:

$$R^a = 0$$

(3.50)

with a different definition of $R^a$, cf. (3.4), that now includes the gravitino field. In particular

$$2R^a_{\mu\nu} = \partial_\mu V^\nu_a - \partial_\nu V^\mu_a - \omega^a_{\mu\nu} V^b + \omega^a_{\mu\nu} V^b - i \bar{\psi}_\mu \gamma^a \psi_\nu = 0$$

(3.51)

allows to express the spin connection in terms of $V$ and $\psi$, recovering second order formalism:

$$\omega_{ab,\mu} = \omega_{ab,\mu} + \frac{i}{4} V^\mu_a V^\rho_b (\bar{\psi}_\mu \gamma_\nu \psi_\rho + \bar{\psi}_\nu \gamma_\rho \psi_\mu - \bar{\psi}_\rho \gamma_\mu \psi_\nu - (\nu \leftrightarrow \rho))$$

(3.52)

where $\omega_{ab,\mu}$ is the spin connection of pure gravity in second order formalism, given in (2.49).

4 Integration on supermanifolds: integral forms

We have defined the supergravity action (3.10) as an integral of a top form on the superPoincaré group manifold. We have given explicitly only the 4-form Lagrangian, postponing the precise expression of $\eta_M$ to the present Section. In fact in the supergravity case we have tacitly assumed typical properties of bosonic integration, as for ex. the existence of a top form and Stokes’ theorem. Here we want to justify these assumptions, and give a short account of superintegration theory.

The construction of actions invariant under diffeomorphisms is solved “ab initio” in ordinary integration theory by form integration. The integral of a $d$-form

$$\omega^{(d)} = \omega_{[\mu_1\cdots\mu_d]}(x) \ d\mu^1 \wedge \cdots \wedge d\mu^d$$

(4.1)

on a $d$-dimensional manifold $M^d$ is defined by

$$I = \int_{M^d} \omega^{(d)} \equiv \int_{M^d} \frac{1}{d!} \omega_{[\mu_1\cdots\mu_d]}(x) \epsilon^{\mu_1\cdots\mu_d} \ d^d x$$

(4.2)
i.e. by usual (Riemann-Lebesgue) integration on $M^d$ of the function $\frac{1}{\pi^d} \omega_{[\mu_1 \cdots \mu_d]}(x)e^{\mu_1 \cdots \mu_d}$, where $\epsilon^{\mu_1 \cdots \mu_d}$ is the Levi-Civita antisymmetric symbol in the coordinate basis, a tensor density of weight $-1$. Therefore

$$\epsilon^{\mu_1 \cdots \mu_d} dx = \epsilon^{\mu_1 \cdots \mu_d} dx^1 \wedge \cdots \wedge dx^d$$

(4.3)

is a tensor, and the integrand of (4.2) is a scalar.

As in the previous Sections, we can consider infinitesimal diffeomorphisms as active transformations, generated by the Lie derivative $\ell_\epsilon = \iota_\epsilon d + d\iota_\epsilon$. Then the form integral (4.2) transforms as

$$\delta I = \int_{M^d} \ell_\epsilon \omega^{(d)} = \int_{M^d} (\iota_\epsilon d + d\iota_\epsilon)\omega^{(d)} = 0$$

(4.4)

since $d\omega^{(d)} = 0$ ($\omega^{(d)}$ is a top form) and $\int_{M^d} d(\iota_\epsilon \omega) = 0$ for appropriate boundary conditions. Thus we have checked invariance of the form integral under infinitesimal diffs generated by the Lie derivative. Note that the existence of a top form, namely the fact that a $d$-form is closed on $M^d$, is crucial to ensure action invariance under diffs.

Can we generalize form integration to supermanifolds, and use it to construct actions automatically invariant under superdiffeomorphisms? The answer to both questions is affirmative.

In analogy with the bosonic case, integration on forms living on supermanifolds is defined via integration of functions in superspace. Consider a function $\Phi(x, \theta)$, defined on a supermanifold $M^{d|m}$ with $d$ bosonic coordinates $x$ and $m$ fermionic (anticommuting) coordinates $\theta^\alpha$. It is called a superfield, and can be expanded in the $\theta^\alpha$ coordinates:

$$\Phi(x, \theta) = \phi(x) + \phi_{\alpha_1}(x)\theta^{\alpha_1} + \phi_{\alpha_1 \alpha_2}(x)\theta^{\alpha_1} \theta^{\alpha_2} + \cdots + \phi_{\alpha_1 \cdots \alpha_m}(x)\theta^{\alpha_1} \cdots \theta^{\alpha_m}$$

(4.5)

The functions $\phi_{\alpha_1 \cdots \alpha_p}(x)$ are called superfield components, and have antisymmetrized indices due to the anticommuting $\theta$’s in the expansion (4.5). The integral of the superfield on $M^{d|m}$ is defined by Berezin integration:

$$\int_{M^{d|m}} \Phi(x, \theta) \ d^{d+m} \theta \equiv \int_{M^d} \frac{1}{m!} \phi_{\alpha_1 \cdots \alpha_m}(x) \epsilon^{\alpha_1 \cdots \alpha_m} \ dx$$

(4.6)

Only the highest component of $\Phi$ (corresponding to the maximal number of $\theta$’s) enters the integral on $M^d$.

Note the striking similarity between the two integrals (4.2) and (4.6). In fact we can define form integration in terms of Berezin integration. Consider the differentials $dx$ in the $d$-form (4.1) as anticommuting coordinates $\xi^\mu = dx^\mu$, so that $\omega^{(d)}$ becomes a function of $x$ and $\xi$:

$$\omega^{(d)}(x, \xi) = \omega_{[\mu_1 \cdots \mu_d]}(x) \xi^{\mu_1} \cdots \xi^{\mu_d}$$

(4.7)

Its Berezin integral on $M^{d|m}$ exactly yields the form integral (4.2). This observation is the key for a definition of superform integration on supermanifolds.
A natural generalization of a bosonic top form (4.1) is a \((d + m)\)-superform:

\[
\omega^{(d+m)}(x, \theta) = \omega[\mu_1 \ldots \mu_d; \alpha_1 \ldots \alpha_m](x, \theta) \ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge d\theta^{\alpha_1} \wedge \cdots \wedge d\theta^{\alpha_m} \quad (4.8)
\]

Note that the \(d\theta\) differentials commute (since the \(\theta\)'s are anticommuting), so that the indices \(\alpha_i\) are symmetrized. For this reason \(\omega^{(d+m)}\) cannot be a top form: a superform can have an arbitrary number of \(d\theta\) differentials, and its exterior derivative does not vanish. Let’s ignore for the moment this difficulty, and try to define a superform integral. Inspired by the observation in the preceding paragraph, we consider the superform \(\omega^{(d+m)}(x, \theta)\) as a function of \(x, \theta, dx, d\theta\), i.e. a function of the commuting variables \(x, d\theta\) and anticommuting variables \(\theta, dx\). Its integral can be defined by Berezin integration on \(\theta, dx\), and usual Riemann-Lebesgue integration on \(x, d\theta\). Here a second difficulty arises: the ordinary integration on the \(u = d\theta\) coordinates produces integrals of the type

\[
\int u^m \ d^m u \quad (4.9)
\]

and there is no algorithmic way to assign a \(C\)-number to it. For the integral on the even variables \(u = d\theta\) to make sense, the integrand must have compact support as a function of \(u\). For this reason we consider functions of the \(d\theta\)'s which are distributions in \(d\theta\) with support at the origin:

\[
\omega(x, \theta, dx, d\theta) = \omega[\mu_1 \ldots \mu_d](x, \theta) \ dx^{\mu_1} \cdots dx^{\mu_d} \delta(d\theta^1) \cdots \delta(d\theta^m) \quad (4.10)
\]

These “functions” can be integrated on the supermanifold \(M^{d+m|d+m}\) spanned by the \(d + m\) bosonic variables \(x, d\theta\) and \(d + m\) fermionic variables \(dx, \theta\). The integral

\[
\int_{M^{d+m|d+m}} \omega(x, \theta, dx, d\theta) \ d^d x \ d^m \theta \ d^d(dx) \ d^m(d\theta) \quad (4.11)
\]

is defined by Berezin integration on the odd variables \(dx, \theta\) and usual Riemann-Lebesgue integration on the even variables \(x, d\theta\). Carrying out integration on the variables \(dx\) and \(d\theta\) the integral becomes

\[
\int_{M^{d+m}} \omega[\mu_1 \ldots \mu_d](x, \theta) \epsilon^{\mu_1 \ldots \mu_d} \ d^d x \ d^m \theta \quad (4.12)
\]

This integral can also be seen as an integral of the form:

\[
\omega^{d|m} = \omega[\mu_1 \ldots \mu_d](x, \theta) \delta(u^1) \cdots \delta(u^m) \ dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge du^1 \wedge \cdots \wedge du^m \quad (4.13)
\]

where the even variables \(u\) are the differentials \(d\theta\). Indeed, let us integrate this form with the recipe of considering it a function of \(x, \theta, u\) and of the differentials \(dx, du\), and then using Berezin and Riemann integration according to the odd or even grading of the variables. The result coincides with (4.12).

Thus the form \(\omega^{d|m}\) can be integrated, even if it contains \(d\theta\) differentials. We achieve this by confining the \(d\theta\)'s inside delta functions, and in this way overcome...
the first difficulty encountered with the superforms (4.8). But can \( \omega^{d|m} \) overcome also the second difficulty, and be a top form? The answer is yes: the \( dx \) and \( d\theta \) differentials are all anticommuting, so that their number in \( \omega^{d|m} \) is already maximal, and multiplying it by \( d\theta \) differentials gives zero because of the presence of the deltas. Therefore \( d\omega^{d|m} = 0 \), and \( \omega^{d|m} \) is a bona fide top form. Since it can be integrated and it is a top form, \( \omega^{d|m} \) is called an integral top form.

Finally, using the notation

\[
\delta(u^1) \cdots \delta(u^m) \; du^1 \wedge \cdots \wedge du^m \equiv \delta(u^1) \wedge \cdots \wedge \delta(u^m) \tag{4.14}
\]

the integral top form can be rewritten (using \( u = d\theta \)):

\[
\omega^{d|m} = \omega_{[\mu_1 \ldots \mu_d]}(x, \theta) \; dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge \delta(d\theta^1) \wedge \cdots \wedge \delta(d\theta^m) \tag{4.15}
\]

or also

\[
\omega^{d|m} = \omega_{[\mu_1 \ldots \mu_d][\alpha_1 \ldots \alpha_m]}(x, \theta) \; dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_d} \wedge \delta(d\theta^{\alpha_1}) \wedge \cdots \wedge \delta(d\theta^{\alpha_m}) \tag{4.16}
\]

where indices \( \alpha \) are antisymmetrized since the \( \delta(d\theta^{\alpha}) \) anticommute, and

\[
m! \; \omega_{[\mu_1 \ldots \mu_d]} \equiv \omega_{[\mu_1 \ldots \mu_d][1 \ldots m]} \tag{4.17}
\]

In this notation \( \mu \) and \( \alpha \) indices play a similar role, and are both antisymmetrized. The numbers \( d, m \) are respectively called the form number and the picture number, and for integral top forms they coincide with the numbers of bosonic and fermionic dimensions of the supermanifold \( M^{d|m} \).

We call “superforms” the forms of the kind (4.8), with \( dx \) and \( d\theta \) differentials, without \( \delta(d\theta) \)’s. Thus superforms have a form number that counts the \( dx, d\theta \) differentials, and zero picture number. For example the Lagrangian in (3.10) is a superform \( L^{4|0} \).

**Integration on submanifolds of supermanifolds**

Supergravity actions on supergroup manifolds \( \tilde{G} \) are given by integrals of a d-form Lagrangian \( L \) on a d-dimensional (bosonic) submanifold \( M^d \) of \( \tilde{G} \). They can be written as integrals on the whole \( \tilde{G} \) of the Lagrangian multiplied by an appropriate Poincaré dual \( \eta_{M^d} \) of \( M^d \), such that \( L \wedge \eta_{M^d} \) becomes an integral top form. Let us see how this works for \( N = 1, \; d = 4 \) supergravity.

The supergravity Lagrangian in (3.10) is a \( (4|0) \) superform. For simplicity we now assume that fields satisfy the Lorentz horizontality constraints on all the curvatures, and thus effectively depend only on the superspace coordinates \( x^\mu, \theta^\alpha \), with \( \mu = 1, \ldots, 4 \), \( \alpha = 1, \ldots, 4 \). Then \( \tilde{G} \) is \( M^{4|4} \) superspace, and only integral top forms of type \( (4|4) \) can be integrated on \( M^{4|4} \). We therefore need a Poincaré dual of type \( (0|4) \), so that

\[
L^{4|0} \wedge \eta_{M^4}^{0|4} \tag{4.18}
\]
is an integral top form, i.e. of type \((4|4)\). For this purpose we choose:

\[
\eta_{M^4}^{0|4} = \varepsilon_{\alpha\beta\gamma\delta} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \varepsilon_{\alpha'\beta'\gamma'\delta'} \delta(d\theta^{\alpha'}) \wedge \delta(d\theta^{\beta'}) \wedge \delta(d\theta^{\gamma'}) \wedge \delta(d\theta^{\delta'}) \tag{4.19}
\]

so that

\[
\int_{M^4} L^{4|0} \wedge \eta_{M^4}^{0|4} = \int_{M^4} L^{4|0}(\theta = 0, d\theta = 0) \tag{4.20}
\]

and we obtain a spacetime action, where all fields depend only on \(x\)-coordinates (the terms containing \(\theta\)'s are annihilated by the presence of the 4 \(\theta\)'s in \(\eta\)) and have no “legs” \(d\theta\) because of the \(\delta(d\theta)\) in \(\eta\). Note that \(\eta_{M^4}^{0|4}\) is closed, and the explicit \(\theta\)'s prevent it to be exact.

Since multiplying by the Poincaré dual changes the picture number of the resulting form, \(\eta\) is also called Picture Changing Operator (PCO), a name borrowed from string theory and string field theory.

The Poincaré dual is by no means unique: we can orient the \(M^4\) surface inside \(\tilde{G}\) in many different ways. For example consider the PCO obtained by acting on \(\eta\) with an infinitesimal diffeomorphism in the \(\theta\) directions:

\[
\eta' = \eta + \ell_\epsilon \eta = \eta + d(\iota_\epsilon \eta) \tag{4.21}
\]

This is still a PCO, being closed and not exact\(^8\) and dual to a submanifold diffeomorphic to the original \(M^4\). Note also that the change in \(\eta\) is exact.

## 5 Building rules

### 5.1 The Lagrangian \(d\)-form

The group geometric approach provides a systematic set of building rules\(^5\) for constructing Lagrangians of supersymmetric theories:

1) Choose a Lie (super)algebra \(G\), containing generators \(P_a\) that can be associated to \(d\) spacetime directions, and a Lorentz-like subalgebra \(H\). Examples are the superPoincaré algebras in \(d\) dimensions or their uncontracted versions (orthosymplectic superalgebras \(OSp(N|2^{[d/2]})\)). The fields of the theory are the vielbein components of the soft group manifold \(\tilde{G}\).

2) Construct the most general \(d\)-form on \(\tilde{G}\), by multiplying (with exterior products) 1-form vielbein components \(\sigma^A\) and 2-form curvatures \(R^A\), without bare Lorentz connection and contracting indices with \(H\)-invariant tensors, so that the resulting Lagrangian is a Lorentz scalar.

3) Require that the variational equations admit the “vacuum solution” \(R^A = 0\), described by the vielbein of the rigid group manifold \(G\).

\(^8\)because \(\eta\) is closed and not exact, and \(d\) commutes with \(\ell_\epsilon\).
4) The construction is greatly helped by scaling properties of the fields, dictated by the structure of the Lie (super)algebra $G$, or equivalently by the Cartan-Maurer equations for the $G$ vielbein. Consider for example the superPoincaré algebra: it is invariant under the rescalings $P_a \rightarrow \lambda P_a, M_{ab} \rightarrow M_{ab}, \bar{Q}_\alpha \rightarrow \lambda^{\frac{1}{2}} \bar{Q}_\alpha$. Then the curvature definitions (3.4)-(3.6) are invariant under

$$V^a \rightarrow \lambda V^a, \quad \omega^{ab} \rightarrow \omega^{ab}, \quad \psi \rightarrow \lambda^{\frac{1}{2}} \psi$$

(5.1)

The field equations must be invariant under these rescalings, and therefore the action must scale homogeneously under (5.1). Since the Einstein-Hilbert term scales as $\lambda^2$, all terms must scale in the same way, and this restricts the candidate terms in the Lagrangian.

5) Finally, requiring that all terms have the same parity as the Einstein-Hilbert term further narrows the list of candidates.

Following the above rules, one arrives at the $d = 4$ supergravity action (3.10), see for ex. [5] for a detailed derivation.

5.2 The use of Bianchi identities

We have seen in the preceding Sections how a geometrical theory can be constructed, and its action found, starting from a Lie (super)algebra $G$.

In many cases, however, the field equations of the theory and their invariances can be derived directly without reference to an action, using only the Bianchi identities and rheonomic constraints on the curvatures. As discussed in Section 3.5, the Bianchi identities\(^9\) imply the field equations of $N = 1, d = 4$ supergravity when the rheonomy and horizontality constraints hold on the outer components of the curvatures.

The rheonomy constraints were deduced by requiring supersymmetry invariance of the spacetime action. How can we find them without an action?

We first observe that some constraints are needed on the curvatures: their outer components must not contain new fields, besides the ones present in the spacetime theory, to avoid unwanted new degrees of freedom. Then the outer components must be expressed in terms of the inner components, which are the ones involved in the spacetime theory. Implementing these constraints into the Bianchi identities determines the exact form of the outer components and, as we have seen in Section 3.5, may imply conditions also on the inner components. These extra conditions are the propagation equations.

In the $N = 1, d = 4$ supergravity case, the procedure runs as follows:

1) first expand the (soft) group manifold curvatures on a basis of 2-forms as

$$R^A = R^A_{ab} V^a V^b + R^A_{\alpha\beta} \psi^\alpha \psi^\beta + R^A_{\alpha\beta\gamma\delta} \psi^\alpha \psi^\beta \psi^\gamma \psi^\delta$$

(5.2)

\(^9\)The Bianchi identities are identities only if the curvatures are not constrained.
2) then require that the outer components $R^A_{ab}, R^A_{a\beta}$ be expressed only in terms of the inner ones $R^A_{ab}$. This is the requirement of “rheonomy”, and together with the scaling properties of the fields and curvatures determines most of the structure of the outer components.

3) Finally, inserting the rheonomic parametrizations of the curvatures into the Bianchi identities yields the precise form (and fixes coefficients) of the outer components, and moreover produces the equations for the inner components, i.e. the propagation equations.

This procedure yields a result for the outer components of the curvatures different from the one obtained in (3.22)-(3.24) by requiring action invariance. The difference resides in the expression for $\tilde{\theta}^{ab}_{c}$. From the Bianchi identities we find

$$\tilde{\theta}^{ab}_{c} = 2i\tilde{\rho}_{c}^{[a\gamma}b^{\gamma]} - i\tilde{\rho}^{ab}_{\gamma}c_{\gamma}$$  \hspace{1cm} (5.3)$$

which differs from the $\tilde{\theta}^{ab}_{c}$ found in [3.20] or in (3.49) by a term proportional to the gravitino propagation equation.

As a consequence the supersymmetry variations obtained from eq. [1.3] by using the $\tilde{\theta}^{ab}_{c}$ in (5.3) differ in the spin connection variation (3.26). Since the difference is proportional to a field equation, they are still invariances of the equations of motion.

### 5.3 On shell and off shell degrees of freedom

In most examples of supersymmetric theories there is a matching between bosonic and fermionic degrees of freedom. Therefore the choice of the starting super Lie algebra (or free differential algebra, see next Section) must take this matching into account. We summarize the counting of d.o.f. in Table 1, where $V^a_{\mu}$ is the vielbein, $\psi^\alpha_{\mu}$ a real (Majorana) gravitino, $\lambda$ a real spinor, $A_{\mu}$ a gauge vector, $A_{[\mu_1...\mu_p]}$ an antisymmetric tensor (components of a $p$-form). If the spinors are complex their d.o.f. are doubled, and if they are Majorana-Weyl their d.o.f. are halved. The spacetime dimensions and signatures for Majorana-Weyl fermions are discussed in Appendix B.

The counting of on-shell d.o.f. summarized in Table 1 is obtained by recalling that:

- only transverse components contribute to on-shell d.o.f. ($d \to d - 2$).
- the dimension of the spinor representation in $d$ dimensions is $2^{[d/2]}$, where $[\ ]$ indicates the integer part, and the Dirac equation reduces the d.o.f by a factor $1/2$.
- for the vielbein $V^a_{\mu}$ Lorentz gauge invariance reduces the d.o.f. to those of a symmetric tensor. Taking into account transversality and subtracting the spinless trace gives

$$\frac{(d-2)(d-1)}{2} - 1 = \frac{d(d-3)}{2}$$  \hspace{1cm} (5.4)$$
- in the case of the spin 3/2 gravitino $\psi^\alpha_\mu$ the gauge condition $\gamma^\mu \psi^\alpha_\mu = 0$ eliminates the spin 1/2 part. Hence the coordinate index $\mu$ contributes with a factor $(d-2)-1$ to the spinorial d.o.f.

The counting of off-shell d.o.f. is obtained by subtracting only gauge invariances, without using equations of motion. Then $d \to d-1$ for coordinate indices, and no halving occurs in spinorial d.o.f. counting.

Table 1: Off-shell and on-shell degrees of freedom

| field | off-shell d.o.f. | on-shell d.o.f. |
|-------|------------------|------------------|
| $V^\alpha_\mu$ | $\frac{d(d-1)}{2}$ | $\frac{d(d-3)}{2}$ |
| $\psi^\alpha_\mu$ | $(d-1)\ 2^{[d/2]}$ | $\frac{1}{2}(d-3)\ 2^{[d/2]}$ |
| $\lambda^\alpha$ | $2^{[d/2]}$ | $\frac{1}{2} 2^{[d/2]}$ |
| $A_\mu$ | $d-1$ | $d-2$ |
| $A_{[\mu_1...\mu_p]}$ | $\left(\begin{array}{c} d-1 \\ p \end{array}\right)$ | $\left(\begin{array}{c} d-2 \\ p \end{array}\right)$ |

6 Free differential algebras

The dual formulation of Lie algebras provided by the Cartan-Maurer equations (1.1) can be naturally extended to $p$-forms ($p > 1$):

$$d\sigma_{(p)}^i + \sum_{n=1}^p \frac{1}{n} C^i_{i_1...i_n} \sigma_{(p_1)}^{i_1} \wedge \ldots \wedge \sigma_{(p_n)}^{i_n} = 0, \quad p + 1 = p_1 + \ldots + p_n$$

(6.1)

where $p, p_1, \ldots, p_n$ are, respectively, the degrees of the forms $\sigma^i, \sigma^{i_1}, \ldots, \sigma^{i_n}$, the indices $i, i_1, \ldots, i_n$ run on irreps of the (super)group $G$, and $C^i_{i_1...i_n}$ are generalized structure constants satisfying generalized Jacobi identities. When $p = p_1 = p_2 = 1$ and $i, i_1, i_2$ belong to the adjoint representation of $G$, eq.s (6.1) reduce to the ordinary Cartan-Maurer equations. The (anti)symmetry properties of the indices $i_1, \ldots, i_n$ depend on the bosonic or fermionic character of the forms $\sigma^{i_1}, \ldots, \sigma^{i_n}$.

If the generalized Jacobi identities hold, eq.s (6.1) define a free differential algebra $[30, 16, 31, 5]$ (FDA). The possible FDA extensions $G'$ of a Lie algebra $G$ have been studied in ref.s $[30, 31, 5]$, and rely on the existence of Chevalley cohomology

\[^{10}\text{obtained by applying } d \text{ to (6.1) and requiring } d^2 = 0.\]
classes in $G$ \[32]. Suppose that, given an ordinary Lie algebra $G$, there exists a $p$-form:

\[
\Omega_i^{(p)}(\sigma) = \Omega_{A_1...A_p}^i \sigma^{A_1} \wedge ... \wedge \sigma^{A_p}, \quad \Omega_{A_1...A_p}^i = \text{constants, } i \text{ runs on a } G - \text{irrep}
\]

(6.2)

which is covariantly closed but not covariantly exact, i.e.

\[
\nabla \Omega_i^{(p)}(\sigma) \equiv d\Omega_i^{(p)}(\sigma) + \Omega_{A_1}^{(p)}(\sigma) \wedge D(T_{A_1})^i_j \Omega_{A_2...A_p}^j(\sigma) = 0, \quad \Omega_i^{(p)}(\sigma) \neq \nabla \Phi_i^{(p-1)}
\]

(6.3)

Then $\Omega_i^{(p)}(\sigma)$ is said to be a representative of a Chevalley cohomology class in the $D_i^j$ irrep of $G$. $\nabla$ is the boundary operator satisfying $\nabla^2 = 0$ ($\nabla^2$ would be proportional to the curvature 2-form on the soft group manifold). The existence of $\Omega_i^{(p)}(\sigma)$ allows the extension of the original Lie algebra $G$ to the FDA $G'$:

\[
d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0
\]

\[
\nabla \sigma_i^{(p-1)} + \Omega_i^{(p)}(\sigma) = 0
\]

(6.4)

where $\sigma_i^{(p-1)}$ is a new $p-1$-form, not contained in $G$. Closure of eqs. (6.4) is ensured because $\nabla \Omega_i^{(p)}(\sigma) = 0$.

It is clear that $\Omega_i^{(p)}(\sigma)$ differing by exact pieces $\nabla \Phi_i^{(p-1)}$ lead to equivalent FDA’s, via the redefinition $\sigma_i^{(p-1)} \rightarrow \sigma_i^{(p-1)} + \Phi_i^{(p-1)}$. What we are interested in are really nontrivial cohomology classes satisfying eq.s (6.3).

The whole procedure can be repeated on the free differential algebra $G'$ which now contains $\sigma^A$, $\sigma_i^{(p-1)}$. One looks for the existence of polynomials in $\sigma^A$, $\sigma_i^{(p-1)}$

\[
\Omega_i^{(q)}(\sigma^A, \sigma_i^{(p-1)}) = \Omega_{A_1...A_r}^{i_1...i_s} \sigma^{A_1} \wedge ... \wedge \sigma^{A_r} \wedge \sigma_i^{(p-1)} \wedge ... \wedge \sigma_i^{(p-1)}
\]

satisfying the cohomology conditions (6.3). If such a polynomial exists, the FDA of eqs. (6.4) can be further extended to $G''$, and so on.

In constructing $d$-dimensional supergravity theories we usually choose as starting superalgebra $G$ the superPoincaré Lie algebra, whose Cartan-Maurer equations can be read off the curvature definitions in eq.s (3.4)-(3.6). The possible $G'$ extensions to FDA’s depend on the spacetime dimension $d$. For example in $d = 11$ there is a cohomology class of the superPoincaré algebra in the identity representation:

\[
\Omega(V, \omega, \psi) = \frac{1}{2} \bar{\psi} \Gamma^{ab} \psi V^a V^b
\]

(6.5)

d$\Omega = 0$ holds because of the $d = 11$ Fierz identity

\[
\bar{\psi} \Gamma^{ab} \psi \bar{\psi} \Gamma_a \psi = 0
\]

(6.6)

This allows the extension of the algebra (3.4) by means of a three-form $A$:

\[
dA - \Omega(V, \omega, \psi) = 0
\]

(6.7)
**Note 1:** only nonsemisimple algebras can have FDA extensions in nontrivial $G$-irreps. Indeed a theorem by Chevalley and Eilenberg [32] states that there is no nontrivial cohomology class of $G$ in nontrivial $G$-irreps when $G$ is semisimple.

As for ordinary Lie algebras, we find a dynamical theory based on FDA’s by allowing nonvanishing curvatures. This means, for example, that $d = 11$ supergravity is based on a deformation of the fields $V, \omega, \psi, A$ such that the superPoincaré curvatures and the $A$-curvature defined by the l.h.s. of (6.7) are different from zero. The construction of the action proceeds along the same lines of Section 3, and we refer the reader to ref. [5] for an exhaustive treatment.

The next two Sections provide examples of FDA’s in $d = 3$ and $d = 4$. Other theories containing higher forms and obtained as gaugings of free differential algebras can be found in [33, 34, 35, 5, 36].

**Note 2:** a “resolution” of FDA’s in terms of larger Lie (super)algebras, by expressing the $p$-forms with $p > 1$ as products of 1-form fields involving new fields, has been considered already in the seminal reference [16] for $d = 11$ supergravity. Recent developments of this idea can be found in ref.s [37, 38, 39].

**Note 3:** a dual formulation of FDA’s, based on a generalized Lie derivative “along antisymmetric tensors” has been developed in ref.s [40, 41, 42, 43] and leads to nonassociative extensions of Lie (super)algebras.

### 7 Off-shell $N = 1, d = 3$ supergravity

Three dimensional supergravity is one of the simplest models of a consistent extension of general relativity that includes fermions and local supersymmetry. The superfield action (see for ex. [44, 45]), supplemented by *ad hoc* constraints consistent with the Bianchi identities, provides an off-shell formulation of $d = 3$ supergravity, local supersymmetry being realized as a diffeomorphism in the fermionic directions.

On the other hand, the construction of off-shell $d = 3$, $N = 1$ supergravity in the group geometric approach [13] provides an action which yields both the correct spacetime equations of motion, and the constraints on the curvatures. The action is written as a Lagrangian 3-form integrated over a bosonic submanifold of a supermanifold $M^{3|2}$.

As discussed in [13], the same action can be written as an integral over the whole supermanifold of an integral form, using the Poincaré dual that encodes the embedding of the 3-dimensional bosonic submanifold, see Section 8.

#### 7.1 Off shell degrees of freedom

The theory contains a vielbein 1-form $V^a$ with 3 off-shell degrees of freedom $(d(d - 1)/2$ in $d$ dimensions), and a gravitino $\psi^a$ with 4 off-shell degrees of freedom $((d -
1) \(2^{d/2}\) in \(d\) dimensions for Majorana or Weyl). The mismatch can be cured by an extra bosonic d.o.f., here provided by a bosonic 2-form auxiliary field \(B\).

### 7.2 The extended superPoincaré algebra

The algebraic starting point is the FDA that enlarges the \(d = 3\) superPoincaré Cartan-Maurer equations to include the auxiliary 2-form field \(B\). This extension of the superPoincaré algebra is possible due to the existence of the \(d = 3\) cohomology class \(\Omega = \bar{\psi}\gamma_a\psi V^a\), closed because of the \(d = 3\) Fierz identity \((C.12)\).

The FDA yields the definitions of the Lorentz curvature, the torsion, the gravitino field strength and the 2-form field strength:

\[
R^{ab} = d\omega^{ab} - \omega^a_c \omega^{cb} \tag{7.1}
\]
\[
R^a = dV^a - \omega^a_b V^b - \frac{i}{2} \bar{\psi}\gamma^a \psi \equiv D V^a - \frac{i}{2} \bar{\psi}\gamma^a \psi \tag{7.2}
\]
\[
\rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi \equiv D \psi \tag{7.3}
\]
\[
H = dB - \frac{i}{2} \bar{\psi}\gamma^a \psi V^a \tag{7.4}
\]

where \(D\) is the Lorentz covariant derivative. The generalized Cartan-Maurer equations are invariant under the rescalings

\[
\omega^{ab} \rightarrow \lambda^0 \omega^{ab}, \ V^a \rightarrow \lambda V^a, \ \psi \rightarrow \lambda^{\frac{1}{2}} \psi, \ B \rightarrow \lambda^2 B \tag{7.5}
\]

Taking exterior derivatives of both sides yields the Bianchi identities:

\[
\mathcal{D} R^{ab} = 0 \tag{7.6}
\]
\[
\mathcal{D} R^a + R^a_b V^b - \frac{i}{2} \bar{\psi}\gamma^a \rho = 0 \tag{7.7}
\]
\[
\mathcal{D} \rho + \frac{1}{4} R^{ab} \gamma_{ab} \psi = 0 \tag{7.8}
\]
\[
dH - \frac{i}{2} \bar{\psi}\gamma^a \rho V^a + \frac{i}{2} \bar{\psi}\gamma^a \psi R^a = 0 \tag{7.9}
\]

### 7.3 Curvature parametrizations

As explained in Section 3, the redundancy introduced by promoting each physical field to a superfield has to be tamed by imposing some algebraic constraints on the curvature parametrizations. They are known as conventional constraints in the superspace language and as rheonomic parametrizations in the group manifold approach. Carrying out the protocol of Section 5.2, we find the following parametrizations

\[
R^{ab} = R^{ab}_{\ c_d} V^c V^d + \bar{\theta}^{ab}_{\ c} \psi V^c + c_1 f \bar{\psi}\gamma^{ab} \psi \tag{7.10}
\]
\[
R^a = 0 \tag{7.11}
\]
\[
\rho = \rho_{ab} V^a V^b + c_2 \gamma_a \psi V^a \tag{7.12}
\]
\[
H = f V^a V^b V^c \epsilon_{abc} \tag{7.13}
\]
\[
df = \partial_a f V^a + \bar{\psi} \Xi \tag{7.14}
\]
with

$$\bar{\rho}_{c,\alpha} = c_3 (\bar{\rho} \gamma^c)_\alpha + c_4 (\rho^\alpha \gamma_c)$$

$$\Xi^\alpha = c_5 \epsilon^{abc} (\gamma_a \rho_{bc})^\alpha \quad (7.15)$$

The coefficients $c_1, c_2, c_3, c_4, c_5$ are fixed by the Bianchi identities to the values:

$$c_1 = \frac{3i}{2}, \quad c_2 = \frac{3}{2}, \quad c_3 = 2i, \quad c_4 = -i, \quad c_5 = -\frac{i}{3!} \quad (7.16)$$

The VVV component $f$ of $H$ scales as $f \to \lambda^{-1} f$, and is identified with the auxiliary scalar superfield of the superspace approach of ref [13]. Note that, thanks to the presence of the auxiliary field, the Bianchi identities do not imply equations of motion for the spacetime components of the curvatures.

### 7.4 The Lagrangian

Applying the building rules of Section 5 yields the Lagrangian 3-form

$$L^{3|0} = R^{ab} V^c \epsilon_{abc} + 2i \bar{\psi} \rho + \alpha (f H - \frac{1}{2} f^2 V^a V^b V^c \epsilon_{abc}) \quad (7.17)$$

It is obtained by considering the most general Lorentz scalar 3-form, given in terms of the FDA curvatures and fields, invariant under the rescalings discussed above, and such that the variational equations admit the vanishing curvatures solution

$$R^{ab} = R^a = \rho = H = f = 0 \quad (7.18)$$

The remaining parameter is fixed to $\alpha = 6$ by requiring $\iota_\varepsilon dL^{3|0} = 0$ up to exact terms, i.e. supersymmetry invariance of the spacetime action, cf. (2.22). In fact with $\alpha = 6$ we find

$$dL^{3|0} = 0 \quad (7.19)$$

on the (off-shell) field configurations satisfying the curvature parametrizations (7.10)-(7.14).

### 7.5 Off-shell supersymmetry transformations

The off-shell closure of the supersymmetry transformations is ensured because the Bianchi identities hold without recourse to the spacetime field equations. The action is invariant under these transformations, given by the Lie derivative of the fields along the fermionic directions:

$$\delta_\varepsilon V^a = -i \bar{\psi} \gamma^a \varepsilon \quad (7.20)$$

$$\delta_\varepsilon \psi = \mathcal{D} \varepsilon + \frac{3}{2} f \gamma_a \epsilon V^a \quad (7.21)$$

$$\delta_\varepsilon \omega^{ab} = \bar{\omega}_{ab} \varepsilon V^c - 3i f \bar{\psi} \gamma^{ab} \varepsilon \quad (7.22)$$

$$\delta_\varepsilon B = -i \bar{\psi} \gamma^a \varepsilon V^a \quad (7.23)$$

$$\delta_\varepsilon f = \bar{\epsilon} \Xi \quad (7.24)$$

and closing on all the fields without need of imposing the field equations.
7.6 Field equations

Varying $\omega^{ab}, V^a, \psi, B$ and $f$ leads to the equations of motion:

\begin{align*}
R^a &= 0 \quad (7.25) \\
R^{ab} &= 9f^2 V^a V^b + \frac{3i}{2} f \bar{\psi} \gamma^{ab} \psi \quad (7.26) \\
\rho &= \frac{3}{2} f \gamma_a \psi V^a \quad (7.27) \\
 df &= 0 \quad (7.28) \\
H &= f V^a V^b V^c \epsilon_{abc} \quad (7.29)
\end{align*}

In the next Section we relate the group manifold formulation of $N = 1, d = 3$ supergravity to its superspace formulation.

8 A bridge between superspace and component actions

We discuss here a technique to relate component to superspace actions, based on different choices for the Poincaré dual that describes the embedding of the spacetime surface inside the supergroup manifold.

As discussed in Section 4, the group manifold action for a $d$-dimensional supergravity can be written as the superintegral:

\[ S_{SG} = \int_{M^{d|im}} L^{d|0} \wedge \eta^{0|lm} \quad (8.1) \]

where the Lagrangian $L^{d|0}$ is found by using the building rules of Section 5.

Suppose now that

\[ dL^{d|0} = 0 \quad (8.2) \]

Then two Poincaré duals differing by a total derivative give rise to the same action when inserted into (8.1). As a consequence, the action (8.1) can be expressed in multiple ways, using different choices of $\eta$ all in the same cohomology class. This observation can be used to relate component and superspace actions, as we illustrate now in the case of $d = 3$ supergravity.

The Lagrangian $L^{3|0}$ for $d = 3$ supergravity is given in (7.17). It is a $(3|0)$-form and, as observed at the end of Section 7.4, is closed when restricted on fields satisfying the parametrizations (7.10)-(7.14). Such field configurations are not on-shell since Bianchi identities with parametrizations (7.10)-(7.14) do not imply propagation equations.

The group manifold action

\[ S_{3d} = \int_{M^{3|2}} L^{3|0} \wedge \eta^{0|2} \quad (8.3) \]
reproduces the usual component action if we choose \( \eta^{0|2} \) to be given by

\[
Y^{0|2} = \epsilon_{\alpha\beta} \theta^\alpha \theta^\beta \epsilon_{\gamma\delta} \delta(d\theta^\gamma) \delta(d\theta^\delta) \equiv \theta^2 \delta^2(d\theta) \tag{8.4}
\]

This Poincaré dual is closed and not exact, and is an element of the cohomology class \( H^{(0|2)}(d, M^{3|2}) \). The integration over the \( d\theta \) and the \( \theta \) leads to:

\[
S_{d=3} = \int_{M^3} L^{30}(\theta = 0, d\theta = 0) = \int_{M^3} R^{ab} V^c \epsilon_{abc} + 2i\bar{\psi}\rho + 6(fH - \frac{1}{2} f^2 V^a V^b V^c \epsilon_{abc}) \tag{8.5}
\]

where all forms depend now only on \( x \) and have only \( dx \) “legs” because of the two \( \theta \)'s and \( \delta(d\theta) \)'s in \( \eta^{0|2} \).

Another Poincaré dual can be chosen as follows

\[
Y^{0|2}_{ss} = V^a V^b (\gamma_{ab})^{\alpha\beta} \iota_{\alpha} \iota_{\beta} \delta^2(\psi) \tag{8.6}
\]

with

\[
\delta^2(\psi) \equiv \epsilon_{\gamma\delta} \delta(\psi^\gamma) \delta(\psi^\delta), \quad \iota_{\alpha} \equiv \frac{\partial}{\partial \psi^\alpha}, \quad (\gamma_{ab})^{\alpha\beta} = (C^{-1})^{\beta\gamma} (\gamma_{ab})^\gamma \tag{8.7}
\]

We use the charge conjugation matrix \( C_{\alpha\beta} = \epsilon_{\alpha\beta} \) and its inverse \( (C^{-1})^{\beta\gamma} \) to lower and raise spinor indices, with the “upper left to lower right” convention. We will prove that \( Y^{0|2}_{ss} \) is a bona fide Poincaré dual (closed and not exact) by proving the following

**Theorem:** \( Y^{0|2}_{ss} \) and \( Y^{0|2} \) are in the same cohomology class, i.e.

\[
Y^{0|2}_{ss} = Y^{0|2} + d\Omega \tag{8.8}
\]

If the theorem holds, also \( Y^{0|2}_{ss} \) is closed and not exact. Moreover, the action \( (8.3) \) computed with \( \eta^{0|2} = Y^{0|2}_{ss} \) is equal to the one with \( \eta^{0|2} = Y^{0|2} \), thanks to \( dL^{0|3} = 0 \).

**Proof:**

1) Recall that varying continuously the embedded surface \( M^d \subset M^{d|m} \) does not change the action \( (8.1) \) when \( dL^{0|0} = 0 \), since the change in \( \eta^{0|m} \) is a total derivative, see \( (4.21) \). Thus by continuously deforming the soft group manifold to its rigid limit, \( Y^{0|2}_{ss} \) gets continuously connected to its rigid limit \( Y^{0|2}_{rigid} \), obtained from \( Y^{0|2}_{ss} \) by expressing \( V \) and \( \psi \) with their values on the rigid supergroup manifold \( M^{3|2} \). These values are given by the left-invariant vielbein components \( V^a, \omega^{ab} \) and \( \psi \):

\[
V^a = 2dx^a + \frac{i}{2} \bar{\gamma}^a d\theta \tag{8.9}
\]

\[
\omega^{ab} = 0 \tag{8.10}
\]

\[
\psi = d\theta \tag{8.11}
\]
and satisfy the Cartan-Maurer equations, i.e. eqs (7.1)-(7.3) with curvatures $= 0$.

2) Substituting inside (8.6) yields the rigid Poincaré dual

$$Y_{\text{rigid}}^{0|2} = (2dx^a + \bar{\theta} \gamma^a d\theta)(2dx^b + \bar{\theta} \gamma^b d\theta)(\gamma_{ab})^{\alpha\beta} t_{\alpha\beta} \delta^2(d\theta)$$

(8.12)

describing the embedding of flat Minkowski space into the supergroup manifold $M^{3|2}$. With the help $d = 3$ gamma identities (see Appendix C), it is not difficult to show that

$$Y_{\text{rigid}}^{0|2} = \epsilon_{\alpha\beta} \theta^\alpha \theta^\beta \delta^2(d\theta) + d\Omega' = Y^{0|2} + d\Omega'$$

(8.13)

where

$$\Omega' = dx^a \bar{\theta} \gamma^b \bar{\theta} \gamma^c \epsilon_{abc} \delta^2(d\theta) + dx^a dx^b (\gamma_{ab})^{\alpha\beta} \theta^{\gamma} t_{\alpha\beta} \delta^2(d\theta)$$

(8.14)

up to constant factors. Thus $Y_{\text{rigid}}^{0|2}$ is in the same cohomology class of $Y^{0|2}$, and because of 1) also in the same cohomology class of $Y_{ss}^{0|2}$, which proves the Theorem.

Thanks to the above theorem, we have the equivalence:

$$S_{3d} = \int_{M^{3|2}} L^{3|0} \wedge Y^{0|2} = \int_{M^{3|2}} L^{3|0} \wedge Y_{ss}^{0|2}$$

(8.15)

since $dL^{(3|0)} = 0$.

Computing now the action with $Y_{ss}^{0|2}$, we see that only the first two terms of $L^{(3|0)}$ contribute, and using the curvature parametrizations for $R^{ab}$ and $\rho$ one finds:

$$S_{3d} = 6i \int_{M^{3|2}} f \epsilon_{abc} V^a V^b V^c \delta^2(\psi) = 6i \int [d^3x d^2\theta] f \ Sdet(E)$$

(8.16)

where $E^A = (V^a, \psi^\alpha)$ is the supervielbein in superspace and we have used

$$\text{Vol}^{(3|2)} = \epsilon_{abc} V^a \wedge V^b \wedge V^c \wedge \delta^2(\psi) = Sdet(E) \ d^3x \ \delta^2(d\theta)$$

(8.17)

Recalling that $f$ is identified with the scalar superfield $R$ we make contact with the superspace action for $d = 3$ supergravity. The equality (8.17) can be proven by recalling the formula for the superdeterminant of a supermatrix:

$$Sdet \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A - BD^{-1}C)(detD)^{-1}$$

(8.18)

applied to the (super)vielbein supermatrix:

$$E^A_A = \begin{pmatrix} V^a_{\mu} \\ \psi^\alpha_{\mu} \\ V^a_{\beta} \\ \psi^\alpha_{\beta} \end{pmatrix}$$

(8.19)
The supermatrix $E^A_A$ is defined by the expansion of $V^a$ and $\psi^\alpha$ on a coordinate basis:

\[ V^a = V^a_\mu dx^\mu + V^a_\alpha d\theta^\alpha \]  
(8.20)

\[ \psi^\alpha = \psi^\alpha_\mu dx^\mu + \psi^\alpha_\beta d\theta^\beta \]  
(8.21)

Substituting (8.21) into $\delta^2(\psi) \equiv \varepsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta)$ of (8.17) produces the identification

\[ d\theta^\beta = - (\psi^{-1})^\alpha_\beta \psi^\alpha_\mu dx^\mu \]  
(8.22)

Then the dreibein $V^a$ as expanded in (8.20) can be written in (8.17) as

\[ V^a = (V^a_\mu - V^a_\beta (\psi^{-1})^\beta_\alpha \psi^\alpha_\mu) dx^\mu \]  
(8.23)

and one recognizes the $A-BD^{-1}C$ structure of the $Sdet$. Finally the $(detD)^{-1}$ factor in the $Sdet$ arises as the inverse Jacobian $1/det(\psi^\alpha_\beta)$ necessary to express $\varepsilon_{\alpha\beta}\delta(\psi^\alpha)\delta(\psi^\beta)$ in terms of $\varepsilon_{\alpha\beta}\delta(\theta^\alpha)\delta(\theta^\beta)$.

In conclusion, the group-manifold Lagrangian $L^{(3|0)}$, integrated on superspace, yields both the usual spacetime $d = 3$, $N = 1$ supergravity action, and its superspace version.

9. **Off-shell $N = 1, d = 4$ supergravity (new minimal)**

9.1 Off shell degrees of freedom

The theory contains a vierbein 1-form $V^a$ with 6 off-shell degrees of freedom and a Majorana gravitino $\psi^\alpha$ with 12 off-shell degrees of freedom. We can match off-shell d.o.f. by adding an auxiliary bosonic 1-form $A$ (3 d.o.f.) and a auxiliary bosonic 2-form $T$ (3 d.o.f.). The theory with these auxiliary fields was first constructed in ref. [46], and recast in the group manifold formalism in ref. [17].

9.2 The extended superPoincaré algebra

The starting superalgebra is the superPoincaré algebra, extended with a 1-form $A$ and a 2-form $T$.

The deformed Cartan-Maurer equations for the extended soft superPoincaré manifold are

\[ R^{ab} = d\omega^{ab} - \omega^a_c \cdot \omega^b_c \]  
(9.1)

\[ R^a = dV^a - \omega^a_b \cdot V^b - \frac{i}{2} \bar{\psi} \gamma^a \psi \equiv DV^a - \frac{i}{2} \bar{\psi} \gamma^a \psi \]  
(9.2)

\[ \rho = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi - \frac{i}{2} \gamma_5 \psi A \equiv D\psi - \frac{i}{2} \gamma_5 \psi A \]  
(9.3)

\[ R^\square = dA \]  
(9.4)

\[ R^\otimes = dT - \frac{i}{2} \bar{\psi} \gamma_a \psi V^a \]  
(9.5)
where $\mathcal{D}$ is the Lorentz covariant derivative. These equations can be considered definitions for the Lorentz curvature, the (super)torsion, the gravitino field strength and the 1-form and 2-form field strengths respectively. The Cartan-Maurer equations are invariant under rescalings

$$\omega^{ab} \rightarrow \lambda^0 \omega^{ab}, \quad V^a \rightarrow \lambda V^a, \quad \psi \rightarrow \lambda^{\frac{1}{2}} \psi, \quad A \rightarrow \lambda^0 A, \quad T \rightarrow \lambda^2 T$$  (9.6)

Taking exterior derivatives of both sides yields the Bianchi identities:

$$\mathcal{D} R^{ab} = 0$$  (9.7)
$$\mathcal{D} R^a + R^a_b V^b - i \bar{\psi} \gamma^a \rho = 0$$  (9.8)
$$\mathcal{D} \rho + \frac{1}{2} \gamma_5 \rho A + \frac{1}{4} R^{ab} \gamma_{ab} \psi - i \frac{1}{2} \gamma_5 \psi R^{\Box} = 0$$  (9.9)
$$dR^\Box = 0$$  (9.10)
$$dR^{\otimes} - i \bar{\psi} \gamma_a \rho V^a + i \frac{1}{2} \bar{\psi} \gamma_a \psi R^a = 0$$  (9.11)

invariant under the rescalings (9.6).

### 9.3 Curvature parametrizations

According to the rheonomic approach, we parametrize the curvatures so that “outer” components (i.e. components along at least one fermionic leg) are related to inner components (i.e. components on bosonic legs). The most general parametrization compatible with the scalings (9.6) and $SO(3,1) \times U(1)$ gauge invariance is the following:

$$R^{ab} = R^{ab}_{\ cd} V^c V^d + \bar{\theta}^{ab}_{\ c} \bar{\psi} V^c + i c_1 \epsilon^{abcd} \bar{\psi} \gamma_c \psi f_d$$  (9.12)
$$R^a = 0$$  (9.13)
$$\rho = \rho_{ab} V^a V^b + i a \gamma_5 \psi f_a V^a - i c_2 \gamma_5 \gamma_{ab} \bar{\psi} V^a V^b$$  (9.14)
$$R^{\Box} = F_{ab} V^a V^b + \bar{\psi} \chi_a V^a + i c_3 \bar{\psi} \gamma_a \psi f^a$$  (9.15)
$$R^{\otimes} = f^a V^b V^c V^d \epsilon_{abcd}$$  (9.16)
$$\mathcal{D} f_a = (\mathcal{D}_b f_a) V^b + \bar{\psi} \Xi_a$$  (9.17)

The $VV$ component $F_{ab}$ of $F$, and the $VVV$ component $f_a$ of $R^{\otimes}$ scale respectively as $F_{ab} \rightarrow \lambda^{-2} F_{ab}$ and $f_a \rightarrow \lambda^{-1} f_a$. The Bianchi identities require that:

$$c_1 = c_2 = \frac{3}{2}, \quad c_3 = 3 - a$$  (9.18)

and

$$\bar{\theta}^{ab}_{\ c} = 2 i \bar{\psi} [a \gamma^b] - i \bar{\rho}^{ab} \gamma_c$$  (9.19)
$$\Xi^a = - \frac{i}{3!} \epsilon^{abcd} \gamma_b \rho_{cd}$$  (9.20)
$$\chi_a = 2 (\gamma_5 \gamma^b \rho_{ab} + \frac{i a}{3!} \epsilon_{abcd} \gamma^b \rho_{cd})$$  (9.21)

Note that, thanks to the presence of the auxiliary fields, the Bianchi identities do not imply equations of motion for the spacetime components of the curvatures.
9.4 The group manifold action

With the usual group-geometrical methods, the action is determined to be

\[ S_{d=4SG} = \int_{M^4} R^{ab} V^c V^d \epsilon_{abcd} + 4 \bar{\psi} \gamma_5 \gamma_a \rho V^a - 4 R^{\square} T + \alpha (f_a R^\otimes V^a + \frac{1}{8} f_e f^e V^a V^b V^c V^d \epsilon_{abcd}) \]

(9.22)

The action is obtained by taking for the Lagrangian \( L_4 \mid_0 \) the most general \( SO(3,1) \times U(1) \) scalar 4-form, invariant under the rescalings discussed above, and then requiring that the variational equations admit the vanishing curvatures solution

\[ R^{ab} = R^a = \rho = R^\otimes = R^{\otimes} = f_a = 0 \]

(9.23)

The remaining parameter \( \alpha \) is fixed by requiring the closure of \( L_4 \mid_0 \), i.e. \( dL_4 \mid_0 = 0 \). This yields \( \alpha = 4(4a - 3) \), and ensures off-shell closure of the supersymmetry transformations given below. Notice that \( a \) is essentially free, since the term \( i a \gamma_5 \psi f_a V^a \) in the parametrization of the gravitino curvature \( \rho \) can be reabsorbed into the definition of the \( SO(3,1) \times U(1) \)-covariant derivative on \( \psi \), by redefining \( A' = A + 2a f_a V^a \). Choosing \( a = \frac{3}{4} \) simplifies the action, reducing it to the first three terms, so that the 0-forms \( f_a \) do not appear.

9.5 Field equations

Varying \( \omega^{ab}, V^a, \psi, A, T \) and \( f \) in the action \( (9.22) \) leads to the equations of motion:

\[ 2 \epsilon_{abcd} R^c V^d = 0 \Rightarrow R^a = 0 \]

(9.24)

\[ 2 R^{bc} V^d \epsilon_{abcd} - 4 \bar{\psi} \gamma_5 \gamma_a \rho + \alpha (-f_a R^\square + \frac{1}{2} f_e f^e \epsilon_{abcd} V^b V^c V^d) = 0 \]

(9.25)

\[ 8 \gamma_5 \gamma_a \rho V^a - 4 \gamma_5 \gamma_a R^a - i \alpha \gamma_a \psi V^a f_b V^b = 0 \]

(9.26)

\[ R^\otimes = 0 \]

(9.27)

\[ -4 R^{\square} + \alpha (V^a D f_a - \frac{i}{2} f_a \bar{\psi} \gamma^a \psi - f_a R^a) = 0 \]

(9.28)

\[ R^{\otimes} = f^a V^b V^c V^d \epsilon_{abcd} \]

(9.29)

These equations are satisfied by the curvatures parametrized as in Section 9.3 and also imply:

\[ R^a = R^{\square} = R^\otimes = R^{\otimes} = f_a = 0 \]

(9.30)

\[ R^{ac} \delta^{bc}_{bc} - \frac{1}{2} \delta^{a}_{cd} R^{cd} = 0 \quad (Einstein eq.) \]

(9.31)

\[ \gamma^a \rho_{ab} = 0 \quad (Rarita - Schwinger eq.) \]

(9.32)

The theory has therefore the same dynamical content as the usual \( N = 1, d = 4 \) supergravity without auxiliary fields.
9.6 Off-shell supersymmetry transformations

Supersymmetry transformations are obtained by applying the Lie derivative along the fermionic directions (i.e. along tangent vectors dual to \( \psi \)):

\[
\delta_{\varepsilon} V^a = -i \bar{\psi} \gamma^a \varepsilon \tag{9.33}
\]

\[
\delta_{\varepsilon} \psi = D \varepsilon + \frac{i}{2} \gamma_5 A \varepsilon + i a \gamma_5 \varepsilon f_a V^a - \frac{3i}{2} \gamma_5 \gamma_{ab} \varepsilon V^a f^b \tag{9.34}
\]

\[
\delta_{\varepsilon} A = \bar{\varepsilon} (\frac{i a}{3} \epsilon^{abcd} \gamma_b \rho_{cd} - 2 \gamma_5 \gamma_b \rho^b a) V_a \tag{9.35}
\]

\[
\delta_{\varepsilon} \omega^{ab} = \tilde{\theta}^{ab c} \varepsilon V^c + 3 i \epsilon^{abcd} \bar{\psi} \gamma_c \varepsilon f_d \tag{9.36}
\]

\[
\delta_{\varepsilon} T = i \bar{\psi} \gamma_a \varepsilon V^a \tag{9.37}
\]

\[
\delta_{\varepsilon} f^a = \varepsilon \Xi^a \tag{9.38}
\]

and close on all the fields without need of imposing the field equations.

9.7 The superspace action

The action \([9.22]\) originates from a 4-form lagrangian \( L^{4|0} \) integrated on a 4-dimensional bosonic submanifold of the (soft) group manifold \( \tilde{G} = \text{superPoincaré} \) in \( d = 4 \). This group-manifold action can be written as an integral on the \( M^{4|4} \) superspace:

\[
I = \int_{M^4} L^{4|0} = \int_{M^{4|4}} L^{4|0} \wedge \eta^{0|4}_{M^4} \tag{9.39}
\]

where \( \eta^{0|4}_{M^4} \) is the Poincaré dual of the \( M^4 \) bosonic submanifold embedded into \( M^{4|4} \). To retrieve the usual spacetime action one chooses for the Poincaré dual the following \((0|4)\)-form:

\[
\eta^{0|4}_{M^4} = \theta^4 \delta(d\theta)^4 \tag{9.40}
\]

with

\[
\theta^4 = \epsilon_{\alpha \beta \gamma \delta} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta,
\delta(d\theta)^4 = \epsilon_{\alpha \beta \gamma \delta} \delta(d\theta^\alpha) \delta(d\theta^\beta) \delta(d\theta^\gamma) \delta(d\theta^\delta) \tag{9.41}
\]

Berezin integration in \((9.39)\) yields an ordinary spacetime action, integrated on \( M^4 \):

\[
\int_{M^4} L^{4|0}(\theta = 0, d\theta = 0) \tag{9.42}
\]

where all forms depend only on \( x \) because of the 4 \( \theta^i \)'s in \( \eta_{M^4} \), and have only \( dx \) legs because of the 4 \( \delta(d\theta)^i \)'s in \( \eta_{M^4} \).

Since the \((4|0)\)-form \((9.40)\) is closed and not exact, it is a representative of the de Rahm cohomology class \( H^{4|0} \).

Also in this case we can relate the component action \((9.42)\) to the superspace action discussed for example in ref.s \([44, 45, 47]\). Indeed \( dL^{0|4} = 0 \), and the same mechanism used in \( d = 3 \) supergravity can be exploited. For this we refer to \([48]\).
10  Gauge supergravities

We give in this Section a brief account of “gauge supergravities”, i.e. theories where the local supersymmetry is realized as part of a gauged superalgebra. These theories are gauge invariant under a supergroup of transformations, so that supersymmetry “lives” on the fiber, and does not mix with diffeomorphisms on the base space. The gauge supersymmetry paradigm has been explored since long ago \cite{49,19,50,51}. Here we treat separately the odd and even dimensional cases, as they involve different constructive procedures. Indeed all gauge supergravity actions are written in terms of (products of) connection and curvature of a supergroup $G$, but odd-dimensional actions are Chern-Simons actions invariant under the whole $G$, while even dimensional actions are invariant only under a subgroup $F$ of $G$. This subgroup may include also part of the supersymmetries of $G$, and the resulting theory is then locally supersymmetric.

Two explicit constructions are given in detail: the $d = 5$ Chern-Simons supergravity action \cite{52,53}, and the $d = 4$ Mac Dowell-Mansouri action \cite{19}. For other odd-dimensional CS supergravity actions we refer to \cite{54,18,55}, while even-dimensional $d = 10 + 2$ and $d = 2 + 2$ gauge supergravity actions have been constructed in \cite{56} and \cite{57} respectively.

10.1  Gauge supergravities in odd dimensions

Chern-Simons (CS) supergravities \cite{52,53,54,18,55} offer interesting alternatives to standard supergravities, since

- supersymmetry is realized as a gauge symmetry, part of a gauge supergroup $G$ under which the CS Lagrangian is invariant up to a total derivative. The superalgebra closes off-shell by construction, without need of auxiliary fields.
- the gauge supergroup contains the (anti)-De Sitter superalgebra, so that the theory is translation-invariant and does not have dimensionful coupling constants. Group contraction can be used to recover the Poincaré superalgebra. Retrieving the Einstein-Hilbert term in this limit is problematic, but there are techniques (S-expansion method \cite{58}) that allow to recover Poincaré gravity from CS gravity with a particular ”expanded” gauge algebra.
- CS gravities are also a particular example of Lovelock gravities \cite{59}, with at most second order field equations for the metric.
- there is no automatic matching between bosonic and fermionic degrees of freedom, at least off-shell. Indeed the matching results from superPoincaré spacetime symmetry, and fields transforming as vector multiplets under supersymmetry. These assumptions do not hold in CS supergravities: the spacetime symmetry is (anti) de Sitter, and the fields are part of a connection belonging to the adjoint representation of a superalgebra.

These features can be relevant for a consistent quantization of the theory \cite{18}, and may give arguments for supersymmetry even if phenomenology seems to rule
out the superpartners one expects from Bose-Fermi matching.

CS gravities and supergravities live only in odd dimensions \( D = 2n - 1 \), and contain, besides the usual Einstein-Hilbert term and its supersymmetrization, also a cosmological term (in the uncontracted version) and higher powers of the curvature 2-form \( R \) up to order \( n - 1 \).

### 10.1.1 Chern-Simons forms

We consider the Chern-Simons \((2n - 1)\)-forms \( L_{CS}^{(2n-1)} \) defined by

\[
dL_{CS}^{(2n-1)} = Tr(R^n) \tag{10.1}
\]

where \( R^n \equiv R \wedge R \wedge \cdots \wedge R \) (\( n \) times), and \( R = d\Omega - \Omega \wedge \Omega \) is the curvature 2-form. The CS form \( L_{CS}^{(2n-1)} \) contains (exterior products of) the \( G \) gauge potential one-form \( \Omega \) and its exterior derivative. The (super)trace \( Tr \) is taken on some representation of the (super)group \( G \).

Thus the CS action is related to a topological action in \( 2n \) dimensions via Stokes theorem:

\[
\int_{\partial M} L_{CS}^{(2n-1)} = \int_M Tr(R^n) \tag{10.2}
\]

Gauge transformations are defined by

\[
\delta_\varepsilon \Omega = d\varepsilon - \Omega \varepsilon + \varepsilon \Omega, \quad \Rightarrow \quad \delta_\varepsilon R = -R\varepsilon + \varepsilon R \tag{10.3}
\]

so that \( Tr(R^n) \) is manifestly gauge invariant. Therefore also the CS action is gauge invariant.

The CS Lagrangian is given in terms of \( \Omega \) and \( d\Omega \) (or \( R \)) by the following expressions \[60, 61\]:

\[
L_{CS}^{(2n-1)} = n \int_0^1 Tr[\Omega(t d\Omega - t^2 \Omega^2)^{n-1}] dt = n \int_0^1 t^{n-1} Tr[\Omega(R + (1 - t)\Omega^2)^{n-1}] dt \tag{10.4}
\]

For example:

\[
L_{CS}^{(3)} = Tr[R\Omega + \frac{1}{3} \Omega^3] \tag{10.5}
\]

\[
L_{CS}^{(5)} = Tr[R^2\Omega + \frac{1}{2} R\Omega^3 + \frac{1}{10} \Omega^5] \tag{10.6}
\]

\[
L_{CS}^{(7)} = Tr[R^3\Omega + \frac{2}{5} R^2\Omega^3 + \frac{1}{5} R\Omega^2 R\Omega + \frac{1}{5} R\Omega^5 + \frac{1}{35} \Omega^7] \tag{10.7}
\]

Considering \( L_{CS}^{(2n-1)} \) as a function of \( \Omega \) and \( R \), a convenient formula for its gauge variation is \[62\]:

\[
\delta_\varepsilon L_{CS}^{(2n-1)} = d(j_\varepsilon L_{CS}^{(2n-1)}) \tag{10.8}
\]

where \( j_\varepsilon \) is a contraction acting selectively on \( \Omega \), i.e.

\[
j_\varepsilon \Omega = \varepsilon, \quad j_\varepsilon R = 0 \tag{10.9}
\]

with the graded Leibniz rule \( j_\varepsilon(\Omega\Omega) = j_\varepsilon(\Omega)\Omega - \Omega j_\varepsilon(\Omega) = \varepsilon \Omega - \Omega \varepsilon \) etc.
10.1.2 $d = 5$ Chern-Simons supergravity

The relevant supergroup for $d = 5$ CS supergravity is $SU(2,2|N)$ (for a group-geometric construction of standard $D = 5$ supergravity see for ex. [5], p. 755). Indeed this supergroup must contain the Poincaré or the uncontracted de Sitter group in $d = 5$, i.e. $SO(2,4)$. We discuss here the uncontracted case: the supergroup extension with $N$ supersymmetries is then $SU(2,2|N)$ (recall the local isomorphism $SO(2,4) \approx SU(2,2)$).

We begin by writing the connection and curvature supermatrices. The gauge connection 1-form is given by:

$$\Omega \equiv \begin{pmatrix} \Omega^\alpha_{\beta} & \psi^\alpha_j \\ -\bar{\psi}^j_\beta & A^i_j \end{pmatrix}, \quad \Omega^\alpha_{\beta} \equiv \left( \frac{i}{4} \omega^{ab}\gamma_{ab} - \frac{i}{2} V^a \gamma_a + \frac{i}{4} bI \right)^\alpha_{\beta}, \quad A^i_j = \frac{i}{N} b\delta^i_j + a^i_j$$

(10.10)

where the bosonic $U(2,2)$ subgroup is gauged by the 1-forms $\omega^{ab}$ (spin connection), $V^a$ (vielbein) and $b$ ($U(1)$ gauge field); the antihermitian matrix-valued 1-forms $a^i_j$ ($i, j = 1...N$) gauge the $SU(N)$ bosonic subgroup; finally the $N$ gravitino 1-form fields $\psi_j$ gauge the $N$ supersymmetries. The Dirac conjugate is defined as $\bar{\psi} = \psi^+ \gamma_0$.

The corresponding curvature supermatrix 2-form is

$$R = d\Omega - \Omega \wedge \Omega \equiv \begin{pmatrix} R + \psi \wedge \bar{\psi}^i & \Sigma_j \\ -\Sigma^i & F^i_j + \bar{\psi}^i \wedge \psi_j \end{pmatrix}$$

(10.11)

with

$$R = d\Omega - \Omega \Omega \equiv \frac{1}{4} R^{ab}\gamma_{ab} - \frac{i}{2} R^a \gamma_a + \frac{i}{4} rI$$

(10.12)

$$\Sigma_j = d\psi_j - \Omega \psi_j - \psi_k A^k_j \equiv D\psi_j$$

(10.13)

$$\Sigma^i = d\bar{\psi}^i - \bar{\psi}^i \Omega - A^i_k \bar{\psi}^k \equiv D\bar{\psi}^i$$

(10.14)

$$F^i_j = dA^i_j - A^i_k A^k_j$$

(10.15)

Immediate algebra yields the components of the $U(2,2)$ curvature $R$:

$$R^{ab} = d\omega^{ab} - \frac{1}{2} \omega^c [a, \omega^b]_c + \frac{1}{2} V^a V^b$$

(10.16)

$$R^a = dV^a - \omega^a_b V^b$$

(10.17)

$$r = db$$

(10.18)

A direct consequence of the curvature definition (10.11) is the Bianchi identity

$$dR = -R \Omega + \Omega R$$

(10.19)

which becomes, on the supermatrix entries

$$dR = -R \Omega + \Omega R, \quad dF = -FA + AF,$$

(10.20)

$$d\Sigma = -R\psi + \Omega \Sigma - \Sigma A + \psi F,$$

(10.21)

$$d\Sigma = -\Sigma \Omega + \bar{\psi} R - F\bar{\psi} + A\Sigma$$

(10.22)

\[11\text{we omit wedge products between forms}\]
**SU(2,2|N) gauge transformations**

The gauge transformations (10.3) close on the Lie (super)algebra:

\[
[\delta \epsilon_1, \delta \epsilon_2] = \delta \epsilon_1 \epsilon_2 - \epsilon_2 \epsilon_1 \tag{10.23}
\]

In the case at hand the \(SU(2,2|N)\) gauge parameter is given by the supermatrix

\[
\epsilon \equiv \left( \begin{array}{cc} \varepsilon_{\alpha \beta} & \epsilon^a_j \\ -\bar{\varepsilon}_{\beta \alpha} & \eta^b_{j} \end{array} \right), \quad \varepsilon_{\alpha \beta} \equiv \left( \begin{array}{c} \frac{1}{4} \varepsilon^{ab} \gamma_{ab} - \frac{i}{2} \varepsilon^a \gamma_a + \frac{i}{4} \varepsilon I \end{array} \right)_{\beta}, \quad \eta^b_{i} = \frac{i}{N} \varepsilon \delta^i + \varepsilon^i \tag{10.24}
\]

and the gauge variations (10.3) on the block entries of \(\Omega\) read

\[
\delta \Omega = d\varepsilon - \Omega \varepsilon + \psi_i \bar{\varepsilon}^i + \epsilon_i \bar{\psi}^i \tag{10.25}
\]

\[
\delta \psi_i = d\varepsilon_i - \Omega \varepsilon_i + \epsilon_j A^j_i - \psi_j \eta^i_j + \varepsilon \psi_i \tag{10.26}
\]

\[
\delta \bar{\psi}^j_i = d\bar{\varepsilon}^j_i + \bar{\varepsilon}^i \Omega - A^i_j \bar{\varepsilon}^j + \eta^i_j \bar{\psi}^j - \bar{\psi} \varepsilon \tag{10.27}
\]

\[
\delta A^i_j = d\eta^i_j - A^i_k \eta^k_j + \eta^i_k A^k_j + \bar{\psi} \bar{\varepsilon}^i_j - \bar{\epsilon} \psi_j \tag{10.28}
\]

On the \(\Omega\) component fields they take the form

\[
\delta \omega^{ab} = d\varepsilon^{ab} - \omega^c |a \varepsilon^{bc} + \varepsilon |a \omega^{bc} + 2V |a \varepsilon |b| + \frac{1}{2} (\bar{\psi} |a \gamma_{ab} \varepsilon - \bar{\epsilon} |a \gamma_{ab} \psi) \tag{10.29}
\]

\[
\delta V^a = d\varepsilon^a - \omega^b |a \varepsilon |b + V^b \varepsilon |a - \bar{\psi} |a \varepsilon - \bar{\epsilon} |a \psi \tag{10.30}
\]

\[
\delta b = d\varepsilon - i(\bar{\psi} \bar{\varepsilon} - \bar{\epsilon} \psi) \tag{10.31}
\]

For \(N = 4\) the supergroup \(SU(2,2|N)\) is not simple anymore and the \(U(1)\) gauged by the \(b\) field becomes a central extension. Consider now the \(U(1)\) gauge variation of the gravitini, cf. (10.26):

\[
\delta \psi_i = i \left( \frac{1}{4} - \frac{1}{N} \right) \varepsilon \psi_i \tag{10.32}
\]

For \(N = 4\) we see that the gravitini become uncharged with respect to this \(U(1)\).

**The action**

Substituting \(\bf{R}\) and \(\Omega\) into (10.6), we obtain the \(d = 5\) CS action invariant under the \(SU(2,2|N)\) gauge variations of the preceding subsection. The result is

\[
\int \text{Str}(L^{(5)}_{CS}) = \int L_{U(2,2)} + L_A + L_{fermi} \tag{10.33}
\]

with

\[
L_{U(2,2)} = Tr[RR\Omega + \frac{1}{2} R\Omega^3 + \frac{1}{10} \Omega^5] \tag{10.34}
\]

\[
L_A = -Tr[FFA + \frac{1}{2} FA^3 + \frac{1}{10} A^5] \tag{10.35}
\]

\[
L_{fermi} = \frac{3}{2} \bar{\psi}(R\Sigma + \Sigma F) + \frac{3}{2} \bar{\Sigma}(R\psi + \psi F) + \bar{\psi}\psi(\bar{\psi}\Sigma + \Sigma \psi) \tag{10.36}
\]
This is the action discussed in refs. [52, 54, 18]. The $b$ field kinetic term has two contributions, from the $RR\Omega$ and the $FFA$ terms, and is proportional to:

$$\left(\frac{1}{16} - \frac{1}{N^2}\right)(db \, db \, b)$$

vanishing for $N = 4$.

We can obtain a slightly more explicit form for $\int L_{U(2,2)}$ by splitting the $U(2,2)$ connection in its “Lorentz + rest” parts as

$$\Omega = \omega + V, \quad \omega \equiv \frac{1}{4} \omega^{ab} \gamma_{ab}, \quad V \equiv -\frac{i}{2} V^a \gamma_a + \frac{i}{4} b I$$

and correspondingly the $U(2,2)$ curvature as

$$R = R + T - VV, \quad \mathcal{R} \equiv d\omega - \omega \omega, \quad T \equiv dV - \omega V - V \omega$$

Then we find, after some integrations by parts and use of the Bianchi identities (10.20)-(10.22):

$$\int L_{U(2,2)} = 3 \int \text{Tr} [\mathcal{R} \mathcal{R} V - \frac{2}{3} \mathcal{R} V^3 + \frac{1}{5} V^5]
+ \frac{1}{2} (T \mathcal{R} + \mathcal{R} T) V + \frac{1}{3} TTV - \frac{1}{2} TV^3]
+ \int \text{Tr} [\mathcal{R} \mathcal{R} \omega + \frac{1}{2} \mathcal{R} \omega^3 + \frac{1}{10} \omega^5]$$

The last line is the integral of the Lorentz CS form $L_{\text{Lor}}$. Its derivative gives the Pontryagin 6-form:

$$dL_{\text{Lor}} = \text{Tr} [\mathcal{R} \mathcal{R} \mathcal{R}]$$

This 6-form $\text{Tr} [\mathcal{R} \mathcal{R} \mathcal{R}]$ vanishes identically, so that the last line in (10.40) can be deleted by virtue of (10.2).

### 10.2 Gauge supergravities in even dimensions

#### 10.2.1 The $d = 4$ Mac Dowell-Mansouri action

This Section follows closely ref. [63]. The Mac Dowell-Mansouri action [19] is a $R^2$-type reformulation of (anti)de Sitter supergravity in $D = 4$. It is based on the supergroup $OSp(1|4)$, and the fields $V^a$ (vierbein), $\omega^{ab}$ (spin connection) and $\psi$ (Majorana gravitino) are 1-forms contained in the $OSp(1|4)$ connection $\Omega$, in a $5 \times 5$ supermatrix representation:

$$\Omega \equiv \left( \begin{array}{cc} \Omega & \psi \\ \bar{\psi} & 0 \end{array} \right), \quad \Omega \equiv \frac{1}{4} \omega^{ab} \gamma_{ab} - \frac{i}{2} V^a \gamma_a$$
The corresponding $OSp(1|4)$ curvature supermatrix is
\[ R = d\Omega - \Omega \wedge \Omega \equiv \left( \begin{array}{cc} R & \Sigma \\ \Sigma & 0 \end{array} \right) \] (10.43)
and straightforward matrix algebra yields:
\[ R = \frac{1}{4} R^{ab} \gamma_{ab} - \frac{i}{2} R^a \gamma_a \] (10.44)
\[ \Sigma = d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi + \frac{i}{2} V^a \gamma_a \psi \] (10.45)
\[ \Sigma = d\bar{\psi} - \frac{1}{4} \bar{\psi} \omega^{ab} \gamma_{ab} + \frac{i}{2} \bar{\psi} V^a \gamma_a \] (10.46)
\[ R^{ab} \equiv d\omega^{ab} - \omega^a_c \omega^{cb} + V^a V^b + \frac{1}{2} \bar{\psi} \gamma^{ab} \psi \] (10.47)
\[ R^a \equiv dV^a - \omega^a_b V^b - \frac{i}{2} \bar{\psi} \gamma^a \psi \] (10.48)

We have also used the Fierz identity for 1-form Majorana spinors:
\[ \psi \bar{\psi} = \frac{1}{4} (\bar{\psi} \gamma^a \psi \gamma_a - \frac{1}{2} \bar{\psi} \gamma^{ab} \psi \gamma_{ab}) \] (10.49)
(to prove it, just multiply both sides by $\gamma_c$ or $\gamma_{cd}$ and take the trace on spinor indices).

The Mac Dowell-Mansouri action can be written in terms of the $OSp(1|4)$ curvature $R$ as:
\[ S = 4 \int STr (R G R \Gamma) \] (10.50)
where $STr$ is the supertrace and $G, \Gamma$ are the following constant matrices:
\[ \Gamma \equiv \left( \begin{array}{cc} i \gamma_5 & 0 \\ 0 & 0 \end{array} \right), \quad G = 1 + \frac{\Gamma^2}{2} = \frac{1}{2} \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \] (10.51)

All boldface quantities are $5 \times 5$ supermatrices. Carrying out the supertrace, and then the spinor trace, leads to the familiar expression of the MacDowell-Mansouri action:
\[ S = 2i \int Tr (R \wedge R \gamma_5 + 2\Sigma \wedge \Sigma \gamma_5) = 2 \int \frac{1}{4} R^{ab} \wedge R^{cd} \varepsilon_{abcd} - 2i \Sigma \wedge \gamma_5 \Sigma \] (10.52)
After inserting the curvature definitions the action takes the form
\[ S = \int R^{ab} V^c V^d \varepsilon_{abcd} + 4 \bar{\rho} \gamma_a \gamma_5 \psi V^a + \frac{1}{2} (V^a V^b V^c V^d + 2 \bar{\psi} \gamma^{ab} \psi V^c V^d) \varepsilon_{abcd} \] (10.53)
with
\[ R^{ab} \equiv d\omega^{ab} - \omega^a_c \omega^{cb}, \quad \rho \equiv d\psi - \frac{1}{4} \omega^{ab} \gamma_{ab} \psi \equiv D\psi \] (10.54)
We have dropped the topological term $\mathcal{R}^{ab}\mathcal{R}^{cd}\epsilon_{abcd}$ (Euler form), and used the gravitino Bianchi identity
\[ \mathcal{D}\rho = -\frac{1}{4}\mathcal{R}^{ab}\gamma_{ab}\psi \] (10.55)
and the gamma matrix identity $2\gamma_{ab}\gamma_5 = i\epsilon_{abcd}\gamma^{cd}$ to recognize that $\frac{1}{2}\mathcal{R}^{ab}\overline{\psi}\gamma^{cd}\psi\epsilon_{abcd} - 4i\overline{\rho}\gamma_5\rho$ is a total derivative. The action (10.53) describes $N = 1, D = 4$ anti-De Sitter supergravity, the last term being the supersymmetric cosmological term. After rescaling the vielbein and the gravitino as $V^a \rightarrow \lambda V^a, \psi \rightarrow \sqrt{\lambda}\psi$ and dividing the action by $\lambda^2$, the usual (Minkowski) $N = 1, D = 4$ supergravity is retrieved by taking the limit $\lambda \rightarrow 0$. This corresponds to the Inonü-Wigner contraction of $OSp(1|4)$ to the superPoincaré group.

**Invariances**

As is well known, the action (10.50), although a bilinear in the $OSp(1|4)$ curvature, is not invariant under the $OSp(1|4)$ gauge transformations:
\[ \delta\epsilon\Omega = d\epsilon - \Omega\epsilon + \epsilon\Omega \implies \delta\epsilon\mathbf{R} = -\mathbf{R}\epsilon + \epsilon\mathbf{R} \] (10.56)
where $\epsilon$ is the $OSp(1|4)$ gauge parameter:
\[ \epsilon \equiv \left( \begin{array}{cc} \frac{1}{2}\epsilon^{ab}\gamma_{ab} & \frac{i}{2}\epsilon^{a}\gamma_{a} & \epsilon \\ \bar{\epsilon} & 0 \end{array} \right) \] (10.57)

In fact it is not a Yang-Mills action (involving the exterior product of $\mathbf{R}$ with its Hodge dual), nor a topological action $\int STr(\mathbf{R}\mathbf{R})$: the constant supermatrices $\mathbf{G}$ and $\mathbf{\Gamma}$ ruin the $OSp(1|4)$ gauge invariance, and break it to its Lorentz subgroup. Indeed the gauge variation of the action (10.50)
\[ \delta S = 4 \int STr(\mathbf{R}[\mathbf{G},\epsilon]\mathbf{\Gamma} + \mathbf{R}\mathbf{G}[\mathbf{\Gamma},\epsilon]) \] (10.58)
vanishes when $\epsilon$ commutes with $\mathbf{\Gamma}$ (and therefore with $\mathbf{G}$), and this happens only when $\epsilon$ in (10.57) has $\epsilon^a = \epsilon = 0$, so that only Lorentz rotations leave the action invariant.

Specializing the gauge parameter $\epsilon$ to describe supersymmetry variations (i.e. only $\epsilon \neq 0$ in (10.57)), eq. (10.58) yields the supersymmetry variation of the Mac Dowell-Mansouri action:
\[ \delta_{\text{susy}}S = 2i \int (\overline{\epsilon}\gamma_5, R)\Sigma + \Sigma[\gamma_5, R]\epsilon \] (10.59)
\[ = -4 \int R^a\Sigma\gamma_a\gamma_5\epsilon \] (10.60)
with $R$ defined in (10.44). This variation is proportional to the torsion $R^a$, since only $R^a\gamma_a$ in $R$ has a nonzero commutator with $\gamma_5$. Therefore in second-order formalism, i.e. using the torsion constraint $R^a = 0$ to express $\omega^{ab}$ in terms of $V^a$
and ψ, the action is indeed supersymmetric. Another way to recover supersymmetry is by modifying the supersymmetry variation of the spin connection, see for ex. [51]. In both cases supersymmetry is not part of a gauge superalgebra: off-shell closure of the supersymmetry transformations is not automatic, and indeed necessitates the introduction of auxiliary fields.

10.2.2 Gauge supergravity in \(d = 10 + 2\)

Here we give a short description of \(d = 10 + 2\) gauge supergravity, summarizing the results of ref. [56].

Supergravity theories in dimensions greater than \(d = 11\) are believed to be inconsistent, since their reduction to \(d = 4\) would produce more than \(N = 8\) supersymmetries, involving multiplets with spin \(\geq 2\), and it is known that coupling of gravity with a finite number of higher spins is problematic.

On the other hand a twelve dimensional theory with signature \((10,2)\) avoids this difficulty, since fermions can be both Majorana and Weyl in \(d = 10 + 2\), with 32 real components, and therefore giving rise to at most eight supercharges when reduced to \(d = 4\). This fact has encouraged over the years various attempts and proposals ([64] - [76]) for a twelve-dimensional field theory of supergravity.

A \(d = 10 + 2\) structure emerges also from string/brane theory, and has been named \(F\)-theory [77]. The \(OSp(1|32)\) superalgebra, a natural choice for the gauge algebra of a \(d = 10 + 2\) supergravity, is also called \(F\) algebra [76].

In \(d = 10 + 2\) dimensions we can write a geometrical \(R^6\)-type action that resembles the \(R^2\)-type \(d = 4\) Mac Dowell-Mansouri action:

\[
S_{d=10+2SG} = \int STr(R^6\Gamma)
\]

where \(R\) is now the \(OSp(1|64)\) curvature supermatrix two-form, and \(\Gamma\) is a constant supermatrix involving \(\gamma_{13}\) and breaking \(OSp(1|64)\) to a \(\tilde{F}\) subalgebra that includes the \(F\) algebra (see below). Contrary to the \(d = 4\) case, \(N = 1\) supersymmetry (with a Majorana-Weyl supercharge) survives as part of this subalgebra, and closes off-shell.

The “would be” gauge fields of \(OSp(1|64)\) are one-forms \(B^{(n)}\) with \(n=1,2,5,6,9,10\) antisymmetric Lorentz indices and a Majorana gravitino \(\psi\). The vielbein and the spin connection are identified with \(B^{(1)}\) and \(B^{(2)}\) respectively. These one-forms are organized into an \(OSp(1|64)\) connection, in an explicit \(65 \times 65\) dimensional supermatrix representation. The constant matrix \(\Gamma\) in (10.61) ensures that the action is not topological (similarly to the MDM action) and breaks \(OSp(1|64)\) to a subalgebra \(\tilde{F} = OSp(1|32) \oplus Sp(32)\), under which the action is invariant. Here part of the supersymmetry of \(OSp(1|64)\) survives, in contrast to the \(D = 4\) case. Supersymmetry is then a gauge symmetry, and closes off-shell. Twelve dimensional

\[\text{12}\] The \(\tilde{F}\) algebra contains the \(F\) algebra: in fact the \(F\) algebra is the \(OSp(1|32)\) part of \(\tilde{F}\).
Lorentz symmetry $SO(10,2)$ is also part of the $\tilde{F}$ gauge symmetry, so that the action is $SO(10,2)$ invariant.

Under the action of $\tilde{F}$, the $OSp(1|64)$ fields split into a gauge multiplet and a matter multiplet. The gauge multiplet contains the $\tilde{F}$ gauge fields: the spin connection $B^{(2)}$, a Majorana-Weyl gravitino $\psi_+$, and the other “even” one-form fields $B^{(6)}$ and $B^{(10)}$. The matter multiplet contains the remaining $OSp(1|64)$ fields: the “odd” one-form fields $B^{(1)}$ (the vielbein), $B^{(5)}$ and $B^{(9)}$, and a Majorana anti-Weyl gravitino $\psi_-$. 

11 Form hamiltonian

In this final Section we recall a hamiltonian formalism well-adapted to geometrical theories described by $d$-form Lagrangians. It goes under the name of “covariant canonical formalism” (CCF), and has been first proposed in ref.s \[20\]-[24]. Recent developments can be found in [25].

In fact this formalism is suggested by the form version of the Euler-Lagrange equations (2.36), discussed in Section 2. Considering the Lagrangian $d$-form as depending on 1-form fields $\phi$ and 2-form “velocities” $d\phi$ naturally leads to the definition of a $(d-2)$-form momentum:

$$\pi \equiv \frac{\partial L}{\partial (d\phi)}$$  \hspace{1cm} (11.1)

and a $d$-form Hamiltonian density

$$H \equiv \pi \wedge d\phi - L$$  \hspace{1cm} (11.2)

This Hamiltonian density does not depend on the “velocities” $d\phi$ since

$$\frac{\partial H}{\partial (d\phi)} = \pi - \frac{\partial L}{\partial (d\phi)} = 0$$  \hspace{1cm} (11.3)

Thus $H$ depends on $\phi$ and $\pi$:

$$H = H(\phi, \pi)$$  \hspace{1cm} (11.4)

and the form-analogue of the Hamilton equations reads:

$$d\phi = \frac{\partial H}{\partial \pi}, \hspace{0.5cm} d\pi = + \frac{\partial H}{\partial \phi}$$  \hspace{1cm} (11.5)

The first equation is equivalent to the momentum definition, and the second is equivalent to the Euler-Lagrange form equations (note the + sign due to the + sign in (2.36)).

Note 1: the derivative of a $p$-form $F$ with respect to a basic 1-form $\phi$ or momentum $d-2$ form $\pi$ is always defined by first bringing $\phi$ or $\pi$ to the left in $F$ (taking into account the sign changes due to the gradings) and then canceling it against the
derivative. In other words, we use the graded Leibniz rule, considering $\frac{\partial}{\partial \phi}$ to have the same grading as $\phi$, and similar for $\pi$.

As an easy exercise, let us apply the formalism to pure $d = 4$ gravity. The fields $\phi$ in this case are the vierbein $V^a$ and the spin connection $\omega^{ab}$. From:

$$L(\phi, d\phi) = R^{ab}V^cV^d\varepsilon_{abcd} = d\omega^{ab}V^cV^d\varepsilon_{abcd} - \omega^a_\epsilon \omega^{eb}V^cV^d\varepsilon_{abcd}$$  \hspace{1cm} (11.6)

we find the momenta:

$$\pi_a = \frac{\partial L}{\partial (dV^a)} = 0$$ \hspace{1cm} (11.7)

$$\pi_{ab} = \frac{\partial L}{\partial (d\omega^{ab})} = V^cV^d\varepsilon_{abcd}$$ \hspace{1cm} (11.8)

and the Hamiltonian density:

$$H = dV^a\pi_a + d\omega^{ab}\pi_{ab} - d\omega^{ab}V^cV^d\varepsilon_{abcd} + \omega^a_\epsilon \omega^{eb}V^cV^d\varepsilon_{abcd}$$ \hspace{1cm} (11.9)

Both momenta definitions are primary constraints:

$$\Phi_a \equiv \pi_a = 0, \quad \Phi_{ab} \equiv \pi_{ab} - V^cV^d\varepsilon_{abcd} = 0$$ \hspace{1cm} (11.10)

since they do not involve the “velocities” $dV^a$ and $d\omega^{ab}$. Therefore $dV^a$ and $d\omega^{ab}$ are undetermined at this stage. Requiring the “conservation” of $\Phi_a$ and $\Phi_{ab}$, i.e. their closure in the present formalism, leads to the secondary constraints:

$$d\Phi_c = 0 \quad \Rightarrow \quad R^{ab}V^d\varepsilon_{abcd} = 0$$ \hspace{1cm} (11.11)

$$d\Phi_{ab} = 0 \quad \Rightarrow \quad R^cV^d\varepsilon_{abcd} = 0$$ \hspace{1cm} (11.12)

after use of the Hamilton equations

$$d\pi_c = \frac{\partial H}{\partial V^c}, \quad d\pi_{ab} = \frac{\partial H}{\partial \omega^{ab}}$$ \hspace{1cm} (11.13)

and the definitions of the curvatures

$$R^a = dV^a - \omega^a_\epsilon V^\epsilon, \quad R^{ab} = d\omega^{ab} - \omega^a_\epsilon \omega^{eb}$$ \hspace{1cm} (11.14)

To derive (11.12) we also made use of the identity

$$F^a_{[a\varepsilon_{bcd}]\epsilon} = 0$$ \hspace{1cm} (11.15)

holding for any antisymmetric $F$. Thus the secondary constraints reproduce the field equations (2.42), (2.43) of vierbein gravity.

No tertiary constraints arise since the secondary constraints (11.11), (11.12), i.e. the l.h.s. of the field equations, are “conserved”. This can be checked by applying the exterior differential to the constraints, and using the Bianchi identities (2.6), (2.7).
Form brackets

The differential of any \( f \)-form \( F \) depending on the 1-form fields \( \phi \) and their conjugated \((d-2)\)-form momenta \( \pi \) can be expressed as

\[
dF(\phi, \pi) = d\phi \frac{\partial F}{\partial \phi} + d\pi \frac{\partial F}{\partial \pi} = \frac{\partial H}{\partial \phi} \frac{\partial F}{\partial \phi} + \frac{\partial H}{\partial \pi} \frac{\partial F}{\partial \pi}
\]

(11.16)

where the graded Leibniz rule for the differential has been taken care of by the definition of partial derivative given in Note 1 after eq.s (11.5), and we have used the Hamilton equations of motion.

This would suggest a form analogue of the Poisson bracket of the \( f \)-form \( F \) with the \( d \)-form \( H \):

\[
\{F, H\} \equiv \frac{\partial H}{\partial \phi} \frac{\partial F}{\partial \pi} + \frac{\partial H}{\partial \pi} \frac{\partial F}{\partial \phi}
\]

(11.17)

so that we recover the familiar looking formula:

\[
dF = \{F, H\}
\]

(11.18)

To make the definition (11.17) consistent for any \( a \)-form \( A \) and \( b \)-form \( B \) (with any degree \( a \leq d, b \leq d \)) it must be generalized to [25]:

\[
\{A, B\} \equiv \frac{\hat{\partial} B}{\partial \pi^i} \frac{\hat{\partial} A}{\partial \phi_i} - (-)^{p_i d} \frac{\hat{\partial} B}{\partial \phi_i} \frac{\hat{\partial} A}{\partial \pi^i}
\]

(11.19)

where \( \hat{\partial} \) indicates the derivative “acting from the left” as defined in the Note 1 after eq.s (11.5), and \( \hat{\partial} \) is defined in a specular way as “acting from the right”. A sum on the fields (labelled by the index \( i \)) is understood, and \( p_i \) is the degree of the form \( \phi_i \). Thus the definition (11.19) holds also in the case of fundamental fields \( \phi_i \) being forms of arbitrary degree \( p_i \), and the covariant Hamiltonian formalism can be applied to the free differential algebras of Section 6. By using the relations

\[
\frac{\hat{\partial} F}{\partial \phi_i} = (-)^{p_i(f+1)} \frac{\hat{\partial} F}{\partial \phi_i}, \quad \frac{\hat{\partial} F}{\partial \pi^i} = (-)^{(d-p_i-1)(f+1)} \frac{\hat{\partial} F}{\partial \pi^i}
\]

(11.20)

one can verify that (11.19) indeed reduces to (11.17) for \( F \) and \( H \).

**Note 2:** The form Poisson bracket between the \( a \)-form \( A \) and the \( b \)-form \( B \) is a \((a+b-d+1)\)-form, and canonically conjugated forms satisfy:

\[
\{\phi_i, \pi^j\} = \delta_i^j
\]

(11.21)

Using the definition (11.19), the following relations can be shown to hold:

\[
\{B, A\} = -(-)^{(a+d+1)(b+d+1)}\{A, B\}
\]

(11.22)

\[
\{A, BC\} = B\{A, C\} + (-)^{c(a+d+1)}\{A, B\}C
\]

(11.23)

\[
\{AB, C\} = \{A, C\}B + (-)^{a(c+d+1)}A\{B, C\}
\]

(11.24)

\[
(-)^{(a+d+1)(c+d+1)}\{A, \{B, C\}\} + cyclic = 0
\]

(11.25)

\[
(-)^{(a+d+1)(b+d+1)}\{\{B, C\}, A\} + cyclic = 0
\]

(11.26)
i.e. graded antisymmetry, derivation property, and form-Jacobi identities.

**Note 3:** a different definition of form Poisson bracket was given in ref. [20], based on postulated properties of the FPB rather than on the Legendre transformation that leads to the evolution equation (11.18). In fact the properties of the FPB in [20] differ from the ones given above, which are *deduced* from the definition (11.19).

Using the form bracket we find the constraint algebra:
\[
(\Phi_a, \Phi_b) = (\Phi_{ab}, \Phi_{cd}) = 0; \quad (\Phi_a, \Phi_{bc}) = -2\varepsilon_{abcd}V^d
\]
showing that the constraints are not all first-class. There is, however, a first-class combination of the constraints:
\[
R^{ab}\Phi_{ab} + R^{a}\Phi_{a}
\]  

One may continue the constraint analysis, separating first-class from second-class constraints, constructing the form analogue of the Dirac bracket, etc. In part this has been done in ref. [21], where the correspondence with the “usual” hamiltonian formalism for first order tetrad gravity of ref. [78] was established, and extended to canonical supergravity [22]. It could be worthwhile to recast in covariant hamiltonian language the algorithm for the construction of gauge generators of ref. [79]. For \(d = 3\) and \(d = 4\) gravity this has been done in ref. [25].

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**A Group manifold geometry**

This brief resumé is taken from Sec. 2 of [8]. We start from a Lie algebra \(\text{Lie}(G)\), with generators \(T_A\) satisfying the commutation relations
\[
[T_A, T_B] = C^C_{AB}T_C
\]  

For simplicity we consider only usual Lie algebras. The extension to superalgebras is straightforward and only necessitates extra phases (for ex. anticommutators for fermionic generators) due to gradings.

A generic group element \(g \in G\) connected with the identity \(^{13}\) can be expressed as
\[
g = \exp(y^AT_A) \equiv y
\]

\(^{13}\)Hereafter \(G\) indicates the part of the group connected with the identity.
where \( y^A \) are the (exponential) coordinates of the group manifold. Each element of \( G \) is labelled by the coordinates \( y^A \), and for notational economy we denote it simply by \( y \). Similarly \( yx \) stands for \( \exp(y^AT_A)\exp(x^BT_B) \), the product of two group elements, and by \((yx)^M\) we denote the corresponding coordinates.

Consider now \((yx)^M\) as a function of \( x \):

\[
(yx)^M = y^M + e_A^M(y)x^A + e_{AB}^M(y)x^Ax^B + \ldots \tag{A.3}
\]

For infinitesimal \( x \):

\[
(yx)^M = y^M + (x^At_A)y^M = (1 + x^At_A)y^M, \quad t_A \equiv e_A^N(y)\frac{\partial}{\partial y^N} \tag{A.4}
\]

so that the \( t_A \) are a differential representation of the abstract generators \( T_A \), and satisfy therefore the same algebra:

\[
[t_A, t_B] = C_{ABC} t_C \tag{A.5}
\]

The geometrical meaning of the components \( e_A^N(y) \) in eq. (A.3) is clear: consider the infinitesimal displacement \( \delta_Ay^M \) due to the (right) action of \( 1 + \varepsilon T_A \) (\( \varepsilon \) = infinitesimal parameter). Then

\[
\delta_Ay^M = \varepsilon e_A^M(y) \tag{A.6}
\]

and the \( \text{dim} \) \( G \) vectors \( e_A^M(y) \), \( A=1,\ldots,\text{dim} \) \( G \) are simply the tangent vectors at \( y \) in the direction of the displacements \( \delta_Ay^M \). It is customary to call tangent vector along the \( T_A \) direction the whole differential operator \( t_A \equiv e_A^N(y)\frac{\partial}{\partial y^N} \).

Note that \( e_A^M \) is an invertible matrix, since the map \( y \to yx \) is a diffeomorphism. The \( t_A(y) \) span the tangent space of \( G \) at \( y \): they form a contravariant basis. The “coordinate” basis given by the vectors \( \frac{\partial}{\partial y^N} \) is related to the \( t_A \) (the intrinsic basis) via the nondegenerate matrix \( e_A^N \). The indices \( A,B,\ldots \) are tangent space indices (“flat” indices) and are inert under \( y \) coordinate transformations. The indices \( M,N,\ldots \) are coordinate indices (“world” indices) and do transform under coordinate transformations in the usual way (see later). Next we define the one-forms \( \sigma^A(y) \) as the duals of the \( t_A \):

\[
\sigma^A(t_B) = \delta_A^B \tag{A.7}
\]

The \( \sigma^A \) are a covariant basis (the intrinsic vielbein basis) for the dual of the tangent space, called cotangent space (the space of 1-forms). The “coordinate” cotangent basis dual to the \( \frac{\partial}{\partial y^N} \) vectors is given by the differentials \( dy^M \) \( (dy^M(\frac{\partial}{\partial y^N}) = \delta_M^N) \).

The components of \( \sigma^A(y) \) on the coordinate basis are denoted \( e_M^A(y) \):

\[
\sigma^A(y) = e_M^A(y) \ dy^M \tag{A.8}
\]

From the duality of the tangent and cotangent bases:

\[
e_M^A e_B^M = \delta_A^B \tag{A.9}
\]

\[
e_A^M e_N^A = \delta_M^N \tag{A.10}
\]

\(^{14}\)Since \( G \) is a Lie group, this function is smooth.
Note 1: Substituting $t_A$ by $e_A^N(y)\frac{\partial}{\partial y^N}$ into the commutator (A.5) leads to the differential condition on $e_A^M(y)$:

$$-2e_A^N e_B^M \partial_N e_M^C = C^C_{AB}$$  \hspace{1cm} (A.11)

Note 2: computing the exterior derivative of $\sigma^A$, using eq.s (A.8) and (A.11) leads to the equations

$$d\sigma^A + \frac{1}{2}C^A_{BC}\sigma^B \wedge \sigma^C = 0$$  \hspace{1cm} (A.12)

These are called Cartan-Maurer equations, and provide a dual formulation of Lie algebras in terms of the one-forms $\sigma^A$. It is immediate to verify that the closure of the exterior derivative ($d^2 = 0$) is equivalent to the Jacobi identities for the structure constants:

$$C^A_{B[C}C^B_{DE]} = 0$$  \hspace{1cm} (A.13)

(apply $d$ to eq. (A.12)).

Note 3:

Defining $\sigma(y) \equiv \sigma^A(y)T_A$ the Cartan-Maurer eq.s (A.12) take the form

$$d\sigma + \sigma \wedge \sigma = 0$$  \hspace{1cm} (A.14)

The Lie-valued one-form $\sigma(y)$ can also be constructed directly from the group element $y$:

$$\sigma(y) = y^{-1}dy$$  \hspace{1cm} (A.15)

It is easy to verify that (A.15) satisfies the Cartan-Maurer equation (A.14) (use $dy^{-1} = -y^{-1}dy y^{-1}$). Moreover, it takes the same value as $e_M^A dy^M T_A$ at the origin $y = 0$. Indeed from the definition of $e_A^M$ in eq. (A.3) one sees that $e_A^M(y = 0) = \delta_A^M$, and therefore $e_M^A(0) dy^M T_A = dy^A T_A$. This value coincides with $y^{-1}dy|_{y=0}$ since $y^{-1}|_{y=0} = [\text{group unit}]$, and $dy|_{y=0} = dy^A T_A$ (from (A.2)). This observation suffices to conclude that $y^{-1}dy$ is equal to $e_M^A(y) dy^M T_A$.

Soft group manifold

Consider a smooth deformation $\tilde{G}$ of the group manifold $G$. Its vielbein field is given by the intrinsic cotangent basis, defined for any differentiable manifold:

$$\mu^A(y) = \mu_M^A(y)dy^M$$  \hspace{1cm} (A.16)

(In this Appendix we use the symbol $\mu$ for the “soft” vielbein). In general $\mu^A$ does not satisfy the Cartan-Maurer equations any more, so that

$$d\mu^A + \frac{1}{2}C^A_{BC}\mu^B \wedge \mu^C \equiv R^A \neq 0$$  \hspace{1cm} (A.17)

The extent of the deformation $G \rightarrow \tilde{G}$ is measured by the curvature two-form $R^A$. $R^A = 0$ implies $\mu^A = \sigma^A$ and viceversa.
Applying the external derivative $d$ to the definition (A.17), using $d^2 = 0$ and the Jacobi identities on $C^A_{BC}$, yields the Bianchi identities

$$(\nabla R)^A \equiv dR^A - C^A_{BC}R^B \wedge \mu^C = 0 \quad \text{(A.18)}$$

**Diffeomorphisms and Lie derivative**

First we discuss the variation under diffeomorphisms of the vielbein field $\mu^A(y)$:

$$\mu^A(y + \delta y) - \mu^A(y) = \delta[\mu_M^A(y)dy^M] =$$

$$= (\partial_N\mu_M^A)\delta y^N dy^M + \mu_M^A(\partial_N\delta y^M)dy^N =$$

$$= dy^N[\partial_N\delta y^A + \delta y^M(\partial_M\mu_N^A - \partial_N\mu_M^A)] =$$

$$= d\delta y^A - 2\mu^B\delta y^C(d\mu)^A_{BC} = d(\iota_{\delta y}\mu^A) + \iota_{\delta y}d\mu^A \quad \text{(A.19)}$$

where

$$\delta y^A \equiv \delta y^M\mu_M^A, \quad \delta y \equiv \delta y^M\partial_M, \quad d\mu^A \equiv (d\mu)^A_{BC}\mu^B \wedge \mu^C, \quad \text{(A.20)}$$

and the contraction $\iota_t$ along a tangent vector $t$ is defined on p-forms

$$\omega(p) = \omega_{B_1...B_p}\mu^{B_1} \wedge ... \wedge \mu^{B_p} \quad \text{(A.21)}$$

as

$$\iota_t \omega(p) = p^A t^A \omega_{AB_2...B_p}\mu^{B_2} \wedge ... \wedge \mu^{B_p} \quad \text{(A.22)}$$

Note that $\iota_t$ maps p-forms into $(p-1)$-forms. The operator

$$\ell_t \equiv d \iota_t + \iota_t d \quad \text{(A.23)}$$

is called the *Lie derivative* along the tangent vector $t$ and maps p-forms into p-forms. As shown in eq. (3.7), the Lie derivative of the one-form $\mu^A$ along $\delta y$ gives its variation under the diffeomorphism $y \rightarrow y + \delta y$. This holds true for any p-form.

We now rewrite the variation $\delta \mu^A$ of eq. (A.19) in a suggestive way, by adding and subtracting $C^A_{BC}\mu^B \delta y^C$:

$$\delta \mu^A = d\delta y^A + C^A_{BC}\mu^B \delta y^C - 2\mu^B \delta y^C(d\mu)^A_{BC} - C^A_{BC}\mu^B \delta y^C \quad \text{(A.24)}$$

$$(\nabla \delta y)^A \equiv (\nabla \delta y)^A + \iota_{\delta y}R^A \quad \text{(A.25)}$$

$$(\nabla \delta y)^A \equiv d\mu^A + C^A_{BC}\mu^B \delta y^A \quad \text{(A.26)}$$

where we have used the definition (A.17) for the curvature, and the $G$-covariant derivative $\nabla$ acts on $\delta y^A$ as

$$(\nabla \delta y)^A \equiv d\mu^A + C^A_{BC}\mu^B \delta y^A \quad \text{(A.27)}$$
The algebra of Lie derivatives

The algebra of diffeomorphisms is given by the commutators of Lie derivatives:
\[ \ell_{\varepsilon^A t_A} \ell_{\varepsilon^B t_B} = \ell_{\varepsilon^C t_C} \] (A.28)
with
\[ \varepsilon^C = \varepsilon^A \partial_A \varepsilon^C - \varepsilon^2 \partial_2 \varepsilon^C - 2 \varepsilon_1 \varepsilon^B \mathcal{R}_{AB} \] (A.29)
and
\[ \mathcal{R}_{AB}^C \equiv \mathcal{R}_{AB}^C - \frac{1}{2} C_{AB}^C \] (A.30)
The components \( R_{BC}^A \) are defined by \( R^A = R_{BC}^A \mu^B \wedge \mu^C \). The closure of the algebra requires the Bianchi identities (A.18), that we can rewrite in the form
\[ \partial_B R_{CD}^A + 2 \mathcal{R}_{E[B}^A \mathcal{R}_{CD]}^E = 0 \] (A.31)
To prove (A.28) just apply both sides of the equation to the basic (soft) vielbein \( \mu \).

B Spinors in \( d = s + t \) dimensions

- \( \eta_{ab} = (1,1,\cdots,1,-1,-1,\cdots-1) \)
- \( \{\gamma_a, \gamma_b\} = 2\eta_{ab} \)
- \( \gamma_{a_1\cdots a_n} \) = antisymmetrized product of \( n \) gamma matrices, with weight 1.
- A matrix representation of \( \gamma \)'s can be made unitary by choice of basis \( \rightarrow \) “time” \( \gamma_a \) are hermitian, “space” \( \gamma_a \) are antihermitian.
- Explicit representation:

\[
\begin{align*}
\gamma_1 &= \sigma_1 \otimes 1 \otimes \cdots \otimes 1 \\
\gamma_2 &= \sigma_2 \otimes 1 \otimes \cdots \otimes 1 \\
\gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \cdots \otimes 1 \\
\gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes \cdots \otimes 1 \\
&\vdots \\
\gamma_d &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2, \quad d = 2p \\
\gamma_d &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_3, \quad d = 2p + 1
\end{align*}
\]
\( \sigma_i = \) Pauli matrices, and multiply space \( \gamma \)'s by \( i \).
- the product of all gammas \( \gamma \equiv \gamma_1 \gamma_2 \cdots \gamma_d \) is proportional to the unit matrix when \( d = 2p + 1 \). When \( d = 2p \), \( \gamma \) anticommutes with every \( \gamma_a \). For any \( d \), \( \gamma^2 = (-1)^{s+p} 1 \).
• In \( d = 2p \) there is only 1 irrep of the Clifford algebra\(^{15} \) in \( d = 2p + 1 \) there are two inequivalent irreps (if \( \gamma_a \) is in one irrep, the other inequivalent irrep is given by \(-\gamma_a\)).

• \( \gamma_a, -\gamma_a, \gamma_a^T \) and \( \gamma_a^\dagger \) satisfy the same Clifford algebra. Thus in \( d = 2p \) their irreps are all equivalent, while in \( d = 2p + 1 \) they are equivalent up to a sign. Therefore in any \( d \) we have

\[
\gamma_a^T = \pm C_{\pm}^a C_{\pm}^{-1} \quad \text{(B.2)} \\
\gamma_a^\dagger = \pm A_{\pm}^a A_{\pm}^{-1} \quad \text{(B.3)}
\]

In the explicit representation, the solution for \( C \) and \( A \) matrices is unique (up to a factor) in \( d = 2p + 1 \) and twofold in \( d = 2p \):

\( d \) odd:

\[
C = \gamma_1 \gamma_3 \gamma_5 \cdots \gamma_d \quad \text{(B.4)} \\
A = \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_t \quad \text{(B.5)}
\]

\( C \) is either \( C_+ \) or \( C_- \), depending on \( s \) and \( t \). The same holds for \( A \).

\( d \) even:

\[
C_I = \gamma_1 \gamma_3 \gamma_5 \cdots \gamma_{d-1} \quad \text{(B.6)} \\
C_{II} = \gamma_2 \gamma_4 \gamma_6 \cdots \gamma_d \quad \text{(B.7)} \\
A_I = \gamma_1 \gamma_2 \gamma_3 \cdots \gamma_t \quad \text{(B.8)} \\
A_{II} = \gamma_{t+1} \gamma_{t+2} \gamma_{t+3} \cdots \gamma_d \quad \text{(B.9)}
\]

If \( C_I \) is \( C_+ \), then \( C_{II} \) is \( C_- \), and vice versa, depending on \( s \) and \( t \).

The same holds for \( A \). Note that \( A_I \) reproduces the usual \( \gamma_1 \) for \( t = 1 \). In the following, we will always use \( A = A_I \).

• Transposition properties of \( \gamma_a \) matrices can be deduced in the explicit representation directly from Pauli matrices (\( \sigma_1, \sigma_3 \) symmetric, \( \sigma_2 \) antisymmetric), so that \( \gamma_a \) is symmetric if \( a \) is odd, antisymmetric for \( a \) even. Consequently one has

\[
C^T = \xi C \quad \text{(B.10)}
\]

For \( d \) odd one finds

\[
\xi = (-1)^{[(d+1)/4]} \quad \text{(B.11)}
\]

where \([\ldots]\) denotes the integer part, and for \( d \) even:

\[
\xi_I = (-1)^{[(d+1)/4]}, \quad \xi_{II} = (-1)^{[(d+2)/4]} \quad \text{(B.12)}
\]

\(^{15}\)more precisely irrep of the finite group \( \Gamma(t, s) \) with elements \( \pm I, \pm \gamma_a, \pm \gamma_{ab} \cdots, \pm \gamma_{a_1 \cdots a_d} \)
• defining
\[ Q_{I,II} \equiv (A^{-1})^T C_{I,II} \] (B.13)
we find for \( \gamma_a^* \) a relation analogous to (B.2), (B.3):
\[ \gamma_a^* = \chi Q \gamma_a Q^{-1} \] (B.14)
with
\[ \chi_I = (-1)^{(s-t+1)/2}, \quad \chi_{II} = (-1)^{(t-s+1)/2} \] (B.15)
and
\[ Q_I Q_I^* = (-1)^{(s-t+1)/4}, \quad Q_{II} Q_{II}^* = (-1)^{(t-s+1)/4} \] (B.16)

• Applying a Lorentz transformation \( \Lambda_{a}^{\; b} \in SO(t, s) \) on the vector index of \( \gamma_b \)
yields
\[ (\Lambda \gamma)_a = \Lambda_{a}^{\; b} \gamma_b \] (B.17)
and since \( (\Lambda \gamma_a) \) is still a representation of the Clifford algebra we must have:
\[ (\Lambda \gamma)_a = S^{-1}(\Lambda) \gamma_a S(\Lambda) \] (B.18)
Taking \( \Lambda_{a}^{\; b} \) infinitesimal, i.e. \( \Lambda_{a}^{\; b} = \delta_{a}^{\; b} + \varepsilon_{a}^{\; b} \) we find
\[ S(\Lambda) = 1 + \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \] (B.19)
by using
\[ [\gamma_{ab}, \gamma_{c}] = 2 \eta_{bc} \gamma_a - 2 \eta_{ac} \gamma_b \] (B.20)
valid in any dimension. The set of matrices \( S(\Lambda) \) forms the Spin group \( Spin(t, s) \), and
\[ SO(t, s) = \frac{Spin(t, s)}{Z_2} \] (B.21)
The following relations are easy to prove:
\[ A = SAS^\dagger, \quad C = S^T CS, \quad \gamma S = S \gamma, \quad Q_{I,II} = S^* Q_{I,II} S^{-1} \] (B.22)

• By definition a spinor transforms under a Lorentz transformation \( \Lambda \) as
\[ \psi' = S(\Lambda) \psi \] (B.23)

• The \( S(\Lambda) \) are not unitary in general. The matrix representation \( S(\Lambda) \) is reducible in \( d = 2p \) since all \( S(\Lambda) \) commute with \( \gamma \); there are two distinct irreps for spinors, with dimension \( 2^{p-1} \). On the other hand, for \( d = 2p + 1 \) the spinor representation is irreducible. Both irreps of \( \Gamma(t, s) \), connected by \( \gamma_a \to -\gamma_a \), lead to the same spinor irrep of dimension \( 2^p \) since \( \gamma_{ab} \) is not changed by \( \gamma_a \to -\gamma_a \).

• The Dirac conjugate:
\[ \bar{\psi} \equiv \psi^\dagger A \] (B.24)
transforms as
\[ \bar{\psi}' = \bar{\psi} S^{-1}(\Lambda) \]  
(B.25)

• The currents
\[ j_{a_1 \cdots a_n} = \bar{\psi} \gamma_{a_1 \cdots a_n} \psi \]  
(B.26)
transform under (B.23) as tensors:
\[ j'_{a_1 \cdots a_n} = \Lambda_{a_1}^{b_1} \cdots \Lambda_{a_n}^{b_n} j_{b_1 \cdots b_n} \]  
(B.27)

• The charge conjugated spinor is defined by
\[ \psi^c \equiv C\bar{\psi}^T \]  
(B.28)
If \( \psi \) satisfies the Dirac equation
\[ (i\gamma^a \partial_a - e\gamma^a A_a - m)\psi = 0 \]  
(B.29)
then \( \psi^c \) satisfies
\[ (i\gamma^a \partial_a + e\gamma^a A_a - \chi m)\psi^c = 0 \]  
(B.30)
with a change of sign of the electric charge, and \( \chi \) as defined in (B.14). For this reason \( C \) is also called the charge conjugation matrix.

• A Majorana spinor is defined to satisfy:
\[ \psi^\dagger A = \psi^T C \]  
(B.31)
or equivalently
\[ \psi^* = \xi Q \psi, \]  
(B.32)
\[ \psi^c = \alpha \psi \]  
(B.33)
with \( \xi \) given in (B.10) and \( \alpha = \pm 1 \) defined by \( (C^{-1})^T = \alpha C \). Iterating (B.32) one finds the condition on \( Q \):
\[ QQ^* = 1 \]  
(B.34)
implying
\[ [(s - t + 1)/4] = 0 \pmod{2} \quad \text{for } Q_I \text{ Majorana spinors} \]  
(B.35)
\[ [(t - s + 1)/4] = 0 \pmod{2} \quad \text{for } Q_{II} \text{ Majorana spinors} \]  
(B.36)
cf. (B.16). Therefore, defining
\[ f \equiv t - s \]  
(B.37)
one has \( Q_I \) Majorana spinors for \( f = -2, -1, 0, 1 \pmod{8} \) and \( Q_{II} \) Majorana spinors for \( f = -1, 0, 1, 2 \pmod{8} \).

• Self-dual tensors:
\[ F_{a_1 \cdots a_p} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_p b_1 \cdots b_p} F^{b_1 \cdots b_p} \]  
(B.38)
Iterating (B.38) and using
\[
\varepsilon_{a_1 \cdots a_r c_1 \cdots c_q} \delta_{b_1 \cdots b_r} = (-1)^s \, r! \, q! \, \delta_{a_1 \cdots a_r}^{b_1 \cdots b_r}
\] (B.39)
implies
\[
F_{a_1 \cdots a_p} = (-1)^{p+q} \, F_{a_1 \cdots a_p} = (-1)^{p/2} \, F_{a_1 \cdots a_p}
\] (B.40)
Therefore self-dual (or anti-self-dual) tensors exist only if \( f = 0 \mod 4 \).

- **Weyl spinors.** Defining a \( d \)-dimensional analog of \( \gamma_5 \)
  \[
  \Gamma = (-1)^{f/4} \gamma, \quad \Gamma^2 = \mathbb{1}
  \] (B.41)
Weyl or anti-Weyl spinors are defined in any even dimension by:
\[
\psi = \pm \Gamma \psi
\] (B.42)
Spinors satisfying both the Majorana and the Weyl condition exist in even dimensions only if \( \psi^c \) has the same chirality as \( \psi \), since \( \psi^c = \alpha \psi \) for Majorana spinors. Using the explicit representation one can prove that if \( \psi \) has chirality +1 (-1), then \( \psi^c \) has chirality \( (-1)^{f/2} \) \( (-1)^{f/2} \). Therefore \( f = 0 \mod 4 \) is a necessary condition for Weyl spinors to be Majorana. Combining this condition with the conditions for the existence of \( Q_I \) or \( Q_{II} \) Majorana spinors given after eq. (B.37), one finds that MW spinors exist if and only if \( f = 0 \mod 8 \).

- **Transposition properties of the matrices** \( C_{\gamma(n)} \), where \( \gamma(n) \) is a shorthand notation for \( \gamma_{a_1 \cdots a_n} \), are important to know, since they determine which currents
  \[
  \bar{\psi}_{\gamma(n)} \psi = \psi^\alpha C_{\alpha \gamma(n)} \gamma_{\beta} \psi^\beta
  \] (B.43)
can exist for Majorana spinors \( \psi \). If \( \psi \) is a zero-form (one-form), the current \( \bar{\psi}_{\gamma(n)} \psi \) exists if the matrix \( C_{\gamma(n)} \) is antisymmetric (symmetric), since \( \psi^\alpha \psi^\beta \) is antisymmetric (symmetric) in \( \alpha, \beta \). In general
  \[
  (C_{\gamma(n)})^T = \eta^n (-1)^{n(n-1)/2} \xi \, C_{\gamma(n)}
  \] (B.44)
where \( \xi \) is given after (B.10) and \( \eta \) is +1 for \( C_+ \) and −1 for \( C_- \).

A table for Minkowski signature \( (t = 1, s = d - 1) \) follows. In computing \( (C_{\gamma(n)})^T \) we have chosen \( C_I \) for \( d = 2, 4, 10, 12 \) and \( C_{II} \) for \( d = 6, 8 \). The table also lists the properties of \( C_I, C_{II} \), and the types of Majorana spinors (\( Q_I \) and/or \( Q_{II} \)) in \( 2 \leq d \leq 12 \).
Table 2: properties of gamma matrices and spinors in $d$ dimensions with Minkowski signature ($t = 1$, $s = d - 1$)

| $d$ | Symm $C\gamma(n)$ for $n =$ | Antisymm $C\gamma(n)$ for $n =$ | $C_I$ | $C_{II}$ | Majorana |
|-----|-----------------------------|-----------------------------|------|---------|-----------|
| 2   | 0,1                         | 2                           | $C^I_+ = C_+$ | $C^-_+ = -C_-$ | $Q_I, Q_{II}$ |
| 3   | 1,2                         | 0                           | $C^I_+ = -C_-$ |                     | $Q_I$ |
| 4   | 1,2                         | 0,3,4                       | $C^I_+ = -C_-$ | $C^-_+ = -C_+$ | $Q_I$ |
| 5   | 2                           | 0,1                         | $C^I_+ = -C_+$ |                     |             |
| 6   | 0,3,4                       | 1,2,5,6                     | $C^I_+ = -C_+$ | $C^-_+ = C_-$ |             |
| 7   | 0,3                         | 1,2                         | $C^I_+ = C_-$ |                     |             |
| 8   | 0,1,4,5,8                   | 2,3,6,7                     | $C^I_+ = C_-$ | $C^-_+ = C_+$ | $Q_{II}$ |
| 9   | 0,1,4,5                     | 2,3                         | $C^I_+ = C_+$ |                     | $Q_I$ |
| 10  | 1,2,5,6,9,10                | 0,3,4,7,8                   | $C^I_+ = C_+$ | $C^-_+ = -C_-$ | $Q_I, Q_{II}$ |
| 11  | 1,2,5                       | 0,3,4                       | $C^I_+ = C_-$ |                     | $Q_I$ |
| 12  | 1,2,5,6,9,10                | 0,3,4,7,8,11,12             | $C^I_+ = -C_-$ | $C^-_+ = -C_+$ | $Q_I$ |

A nice summary of the properties of spinors in $d = t + s$ is given by the “spinor clock” designed by Tullio Regge in ref. [4], reproduced in the following Figure:
Fig. 2. The spinor clock. The short hand points at the value of $d$, the long one at the value of $f$. Abbreviations: R, real irrep(s) (only one if symbol is preceded by the number 1); I, imaginary irrep(s) (one if preceded by 1); M, Majorana spinors, further specified $Q_I$ and/or $Q_{II}$; W, Weyl spinors; M-W, Weyl spinors which are Majorana as well; sdT, self-dual tensors. If a species does not exist at the value of $f$ indicated, its symbol is crossed out.
C \boldsymbol{\gamma} \text{ matrices in } d = 2 + 1

We adopt the traditional numbering 0, 1, 2 instead of 1, 2, 3.

\begin{align*}
\gamma_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
\eta_{ab} &= (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab} = -2i\varepsilon_{abc}\gamma^c, \\
\varepsilon_{012} &= \varepsilon_{012} = 1, \\
\gamma^\dagger_a &= \gamma_0\gamma_a\gamma_0, \quad \gamma^T_a = -C\gamma_aC^{-1}
\end{align*}

(C.1)

\begin{align*}
C &= i\gamma_0\gamma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rightarrow \quad C_{\alpha\beta} = \varepsilon_{\alpha\beta}
\end{align*}

(C.5)

\text{C.1 Useful identities}

\begin{align*}
\gamma_a\gamma_b &= \gamma_{ab} + \eta_{ab} = -i\varepsilon_{abc}\gamma^c + \eta_{ab} \\
\gamma_{ab}\gamma_c &= \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abc} \\
\gamma_a\gamma_{ab} &= \eta_{ac}\gamma_b - \eta_{bc}\gamma_a - i\varepsilon_{abc} \\
\gamma_{ab}\gamma_c &= \eta_{ab}\gamma_c + \eta_{bc}\gamma_a - \eta_{ac}\gamma_b - i\varepsilon_{abc} \\
\gamma_{ab}\gamma_{cd} &= -4\delta_{[a}^{\alpha}\gamma_{b]} - 2\delta_{cd}^{ab}
\end{align*}

(C.6) (C.7) (C.8) (C.9) (C.10)

where \(\delta_{cd}^{ab} = \frac{1}{2}(\delta_{d}^{a}\delta_{c}^{b} - \delta_{d}^{b}\delta_{c}^{a})\), and index antisymmetrizations in square brackets have weight 1.

\text{C.2 Fierz identity for two Majorana one-forms}

\begin{align*}
\psi\bar{\psi} &= \frac{1}{2}(\bar{\psi}\gamma^a\psi)\gamma_a
\end{align*}

(C.11)

As a consequence

\begin{align*}
\gamma_a\psi\bar{\psi}\gamma^a\psi &= 0
\end{align*}

(C.12)

D \boldsymbol{\gamma} \text{ matrices in } d = 3 + 1

We use the traditional numbering 0, 1, 2, 3 instead of 1, 2, 3, 4.

\begin{align*}
\eta_{ab} &= (1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \quad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \\
\gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \varepsilon_{0123} = -\varepsilon_{0123} = 1, \\
\gamma^\dagger_a &= \gamma_0\gamma_a\gamma_0, \quad \gamma^T_5 = \gamma_5 \\
\gamma^T_a &= -C\gamma_aC^{-1}, \quad \gamma^T_5 = C\gamma_5C^{-1}, \quad C^2 = -1, \quad C^T = -C
\end{align*}

(D.1) (D.2) (D.3) (D.4)
D.1 Useful identities

\[\gamma^a \gamma^b = \gamma^{ab} + \eta^{ab}\] (D.5)
\[\gamma^{ab} \gamma^5 = -\frac{i}{2} \varepsilon^{abcd} \gamma^{cd}\] (D.6)
\[\gamma^{ab} \gamma^c = \eta^{bc} \gamma^a - \eta^{ac} \gamma^b + i \varepsilon^{abcd} \gamma^d\] (D.7)
\[\gamma^a \gamma^b \gamma^c = \eta^{ac} \gamma^b - \eta^{bc} \gamma^a + i \varepsilon^{abcd} \gamma^d\] (D.8)
\[\gamma^{ab} \gamma_{cd} = i \varepsilon^{ab}_{\phantom{ab}cd} \gamma^5 - 4 \delta^5_{[c} \gamma^a \gamma^{b]} - 2 \delta_{cd}^{ab}\] (D.9)

D.2 Charge conjugation and Majorana condition

Dirac conjugate \(\bar{\psi} \equiv \psi^\dagger \gamma_0\) (D.11)
Charge conjugate spinor \(\psi^c = C(\bar{\psi})^T\) (D.12)
Majorana spinor \(\psi^c = \psi \Rightarrow \bar{\psi} = \psi^T C\) (D.13)

D.3 Fierz identity for two spinor one-forms

\[\psi \bar{\chi} = \frac{1}{4} \left((\bar{\chi} \psi) 1 + (\bar{\chi} \gamma_5 \psi) \gamma_5 + (\bar{\chi} \gamma^a \psi) \gamma_a + (\bar{\chi} \gamma^a \gamma_5 \psi) \gamma_a \gamma_5 - \frac{1}{2} (\bar{\chi} \gamma^{ab} \psi) \gamma_{ab}\right)\] (D.14)

D.4 Fierz identity for two Majorana spinor one-forms

\[\psi \bar{\psi} = \frac{1}{4} \left((\bar{\psi} \gamma^a \psi) \gamma_a - \frac{1}{2} (\bar{\psi} \gamma^{ab} \psi) \gamma_{ab}\right)\] (D.15)

As a consequence

\[\gamma_a \psi \bar{\psi} \gamma^a \psi = 0, \quad \psi \bar{\psi} \gamma^a \psi - \gamma_b \psi \bar{\psi} \gamma^{ab} \psi = 0\] (D.16)

E \(\gamma\) matrices in \(d = 4 + 1\)

\[\eta_{ab} = (1, -1, -1, -1, -1), \quad \{\gamma_a, \gamma_b\} = 2 \eta_{ab}, \quad [\gamma_a, \gamma_b] = 2 \gamma_{ab}\] (E.1)
\[\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 = -1, \quad \varepsilon_{01234} = \varepsilon^{01234} = 1,\] (E.2)
\[\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0,\] (E.3)
\[\gamma_a^T = C \gamma_a C^{-1}, \quad C^2 = -1, \quad C^\dagger = C^T = -C\] (E.4)
E.1 Useful identities

\[ \gamma_a \gamma_b = \gamma_{ab} + \eta_{ab} \] (E.5)
\[ \gamma_{abc} = \frac{1}{2} \epsilon_{abcde} \gamma^{de} \] (E.6)
\[ \gamma_{abcd} = -\epsilon_{abcde} \gamma^e \] (E.7)
\[ \gamma_{ab} \gamma_c = \eta_{bc} \gamma_a - \eta_{ac} \gamma_b + \frac{1}{2} \epsilon_{abcde} \gamma^{de} \] (E.8)
\[ \gamma_{c} \gamma_{ab} = \eta_{ac} \gamma_{b} - \eta_{bc} \gamma_{a} + \frac{1}{2} \epsilon_{abcde} \gamma^{de} \] (E.9)
\[ \gamma_{ab} \gamma_{cd} = -\varepsilon_{abcde} \gamma^e - 4\delta^{[a}_{d} \delta^{b]}_{c} - 2\delta_{cd}^{ab} \] (E.10)

where \( \delta^{ab}_{cd} \equiv \frac{1}{2}(\delta^{a}_{d} \delta^{b}_{c} - \delta^{b}_{d} \delta^{a}_{c}), \delta^{rse}_{abc} \equiv \frac{1}{3!}(\delta^{r}_{a} \delta^{s}_{b} \delta^{e}_{c} + 5 \text{ terms}), \) and index antisymmetrization in square brackets has total weight 1.

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