In the present paper the determination of the \( pp \)-wave metric form the geometry of certain spacelike two-surfaces is considered. It has been shown that the vanishing of the Dougan–Mason quasi-local mass \( m_\Sigma \), associated with the smooth boundary \( \partial \Sigma \approx S^2 \) of a spacelike hypersurface \( \Sigma \), is equivalent to the statement that the Cauchy development \( D(\Sigma) \) is of a \( pp \)-wave type geometry with pure radiation, provided the ingoing null normals are not diverging on \( \partial \) and the dominant energy condition holds on \( D(\Sigma) \). The metric on \( D(\Sigma) \) itself, however, has not been determined. Here, assuming that the matter is a zero-rest-mass-field, it is shown that both the matter field and the \( pp \)-wave metric of \( D(\Sigma) \) are completely determined by the value of the zero-rest-mass-field on \( \partial \) and the two dimensional Sen–geometry of \( \partial \) provided a convexity condition, slightly stronger than above, holds. Thus the \( pp \)-waves can be characterized not only by the usual Cauchy data on a three dimensional \( \Sigma \) but by data on its two dimensional boundary \( \partial \) too. In addition, it is shown that the Ludvigsen–Vickers quasi-local angular momentum of axially symmetric \( pp \)-wave geometries has the familiar properties known for pure (matter) radiation.
1. Introduction

The present paper is the third in a series on the 2 dimensional Sen connection and its applications in general relativity. In the first [1] a covariant spinor formalism was developed which is the two dimensional counterpart of the usual (3 dimensional) Sen geometry. In the second paper [2] this formalism was used to find the ‘most natural’ spinor propagation law needed in the quasi-local energy-momentum expressions based on the Nester–Witten form. It turned out that the quasi-local energy-momentum that the two dimensional Sen operator determines is precisely the Dougan–Mason energy-momentum [3]. To characterize the zero-mass and zero energy-momentum spacetimes we proved the following theorem [2,4]:

**Theorem:** Let $\Sigma$ be a spacelike hypersurface, its boundary, $\$: $\partial \Sigma$, be a smooth topological 2-sphere, and suppose that the dominant energy condition is satisfied on the Cauchy development $D(\Sigma)$ of $\Sigma$. Suppose that the ingoing null normals to $\$ are not diverging on $\$ (or, in other words, the GHP spin coefficient $\rho'$ is nonnegative) and $\$ is generic (i.e. there exist precisely two linearly independent spinor fields that are anti-holomorphic with respect to the 2 dimensional Sen connection and span the spin space at each point of $\$). Then the following pairs of statements are equivalent:

1. the Dougan–Mason quasi-local mass, associated with $\$, is zero (the quasi-local energy-momentum is zero),
2. $D(\Sigma)$ is a pp-wave spacetime with pure radiation ($D(\Sigma)$ is flat),
3. there exists a Sen–constant spinor field (two Sen–constant spinor fields) on $\$.

In addition to the characterization of the zero-mass and zero energy-momentum spacetimes the equivalence of 2. and 3. shows that gravity, together with matter fields satisfying the dominant energy condition, is so ‘rigid’ a system that the information that $D(\Sigma)$ is flat/pp-wave with pure radiation is completely encoded not only into the Cauchy data on the three dimensional $\Sigma$, but into the Sen-geometry of the two dimensional $\$. However, while this theorem tells us in the zero energy-momentum case what the metric of $D(\Sigma)$ is, that is flat, in the zero-mass case we know only the class of the metric of $D(\Sigma)$: that belongs to the class of the pp-wave metrics with pure radiation. One may therefore ask whether all the information on the metric of a pp-wave Cauchy development is encoded into the Sen-geometry of $\$. The condition $\rho' \geq 0$ is usually interpreted as some weak form of the convexity of $\$ [3]. However this condition corresponds to the non-negativity of one of the two mean curvatures, while in the theory of surfaces [5,6] the convexity is defined by the positivity of the Gauss curvature. Here we show that the full information on the geometry of a pp-wave Cauchy development $D(\Sigma)$ is in fact encoded into the Sen-geometry of $\$, provided certain generalization of the convexity condition of the theory of surfaces for curved four dimensional embedding geometries holds.

In the first section we review the 2 dimensional Sen–geometry and define the holomorphic/anti-holomorphic spinor fields on $\$. Such a geometry is a quadruple ($\$, $\varepsilon_{AB}$, $\gamma_{AB}$, $\Delta_e$), where $\varepsilon_{AB}$ is a symplectic and $\gamma_{AB}$ is a complex metric on the spin spaces and $\Delta_e$ is the covariant derivation with respect to a Sen–connection. Apart from a common conformal factor of the two metrics ($\$, $\varepsilon_{AB}$, $\gamma_{AB}$) is analogous to the so-called universal structure of the geometry of null infinity while $\Delta_e$ represents the so-called first order structure on $\$. Although the 2 dimensional Sen connections can be introduced as connections on certain SL(2,C)-principle bundle (or on an associated complex vector bundle of rank 2) over a 2 dimensional orientable Riemannian manifold ($\$, $g_{ab}$) as a base space [7], for the sake of simplicity we consider the Sen connection as a structure derived from an imbedding of $\$ into a 4 dimensional Lorentzian spacetime. (The 3 dimensional Sen connection on a 3 dimensional Riemannian manifold ($\Sigma$, $h_{ab}$) was introduced in a similar...
abstract way without any imbedding only recently [8].) Then in section 3 the Sen–geometry of those 2-surfaces are investigated which are homeomorphic to $S^2$ and admit a Sen–constant spinor field $\lambda_A$. We find a function, $\Phi$, describing the deviation of the holomorphic and anti-holomorphic spin frames from each other, which plays the role of a potential for the Sen–curvature. In the fourth section we determine the geometry of the zero-mass Cauchy developments, specified by the Theorem above, in terms of the 2 dimensional Sen–geometry of $\Sigma$. A convexity condition for $\Sigma$ is found which ensures that both $\Sigma$ and the whole Cauchy development are topologically trivial. This convexity condition is slightly stronger than that of Dougan and Mason. Then it is shown that, for any given $(\Sigma, \varepsilon, \gamma^A)$, the spacetime curvature and the matter radiation in $D(\Sigma)$ are completely determined by two complex functions on $\Sigma$, the potential $\Phi$ and the value of the matter radiation provided our convexity condition holds. Then the metric of $D(\Sigma)$ will be determined. If there is a nonzero constant null vector field in the spacetime then it is relatively easy to find a local coordinate system $(v, \zeta, \bar{\zeta}, u)$ (‘canonical coordinates’) in which the line element contains merely a real unknown function, $H(\zeta, \bar{\zeta}, u)$, and Einstein’s equations reduce to a single Poisson equation for $H$ [9]. What we should do here is however to solve a boundary value problem for the metric of $D(\Sigma)$ if the Sen geometry of $\Sigma$ is fixed. In its spirit our treatment of the $pp$-waves is similar to that of Aichelburg [10].

He also considered the metric as a solution of a boundary value problem for the Poisson equation above. However the 2-surface in his treatment was assumed to be in a hypersurface lying in a totally geodesic space. In this paper the abstract way without any imbedding only recently [8]. Thus the eigenvalues of $\gamma^A_B$ are $\pm 1$, and the corresponding eigenspinors, defined by $\gamma^A_B\psi^B = \epsilon^A\psi$, and $\gamma^A_B\psi^B = -\epsilon^A\psi$, are invariant with respect to the conformal rescalings of the spacetime metric and the ‘boost gauge transformation’ $(t^a, v^a) \rightarrow (t^a \cosh \eta + v^a \sinh \eta, t^a \sinh \eta + v^a \cosh \eta)$. $\gamma^A_B$ characterizes the algebraic and conformal properties of $\Sigma$, since: a.)

$$\gamma^A_A = 0, \quad \gamma^A_B\gamma^B_C = \delta^A_C.$$  

(2.1)

Thus the eigenvalues of $\gamma^A_B$ are $\pm 1$ and the corresponding eigenspinors, defined by $\gamma^A_B\psi^B = \epsilon^A\psi$ and $\gamma^A_B\psi^B = -\epsilon^A\psi$, and normalized by $\partial_R t^R = 1$, form a GHP spinor dyad on $\Sigma$ [12]. The natural projection $T_pM \rightarrow T_p\Sigma$, $p \in \Sigma$, of the tangent spaces is given by $\Pi^a_b := \delta^a_b - t^a t_b + v^a v_b = \frac{1}{2}(\delta^a_b\delta^c_d - \gamma^A_B\gamma^A_D\gamma^D_C\gamma^B^f)$. The induced volume form on $\Sigma$ is $\varepsilon_{cd} := t^a v^b \varepsilon_{abcd} = \frac{1}{2}(\varepsilon_{C'D'}\gamma_{CD} - \varepsilon_{C'D} \bar{\gamma}(C'D'))$. $\gamma_{AB}$ can also be considered as a complex metric on the spin spaces. b.) $\Sigma$, together with the induced metric $g_{ab} := \Pi^{a}_A \Pi^{b}_B g_{ef}$, is a
Riemannian manifold whose conformal structure is equivalent to a complex structure on $S$. The projection of the complexified tangent spaces of $S$ to the subspaces of the $(1,0)$ and $(0,1)$ type vectors [7] are $\pi^{-a}b := \pi^{-A}B\bar{\pi}^+A'B'_B$ and $\pi^+a_b := \pi^+A_B\bar{\pi}^-A'B'_B$, respectively, where $\pi^{\pm A}B := \tfrac{1}{2}(\delta_B^A \pm \gamma^A_B)$ are the projections of the spin space to the subspace of the $\pm 1$ eigenspinors, respectively. $m^a := o^{A\epsilon^A}$ and $\bar{m}^a := \bar{o}^{A\epsilon^A}$ are $(1,0)$ and $(0,1)$ type vectors, respectively.

The two extrinsic curvatures, $\tau_{ab} := \Pi^*_a\Pi^*_b\nabla_c v_f$ and $\nu_{ab} := \Pi^*_a\Pi^*_b\nabla_c v_f$, can be given in a boost gauge independent manner by $Q^a_{\nu ab} := -\Pi^*_a\Pi^*_b\nabla_c v_f \Pi^*_b = \tau^a_{ab} - \nu^a_{ab} v_b$. Thus $Q_{ab} = Q_{(ca)b}$ and the expansion tensor of the out and ingoing null geodesics orthogonal to $\theta$ are complicated since the Gauss equation gives $\rho^a_{ab} = \Pi^*_a\Pi^*_b\nabla_c v_f = Q_{abbb}k$ and $\theta^a_{ab} := \Pi^*_a\Pi^*_b\nabla_c v_f = Q_{abbb}k$, respectively, where $l^a := o^A\bar{o}^A$ and $n^a := \nu^A\nu^A$. The corresponding mean curvatures are $q^{ab}\theta_{ab} = -2\rho$ and $q^{ab}\theta'_{ab} = -2\rho'$; and let us define $k := \det{||\theta^a_b||} = \frac{1}{2}(\theta_{ab}\theta_{cd} - \theta_{ac}\theta_{bd})q^{ac}q^{cd}$ and $k' := \det{||\theta^a_b||} = \frac{1}{2}(\theta'_{ab}\theta'_{cd} - \theta'_{ac}\theta'_{bd})q^{ac}q^{cd}$. In the theory of surfaces [5,6], when the imbedding geometry is a flat 3 dimensional Riemannian manifold, we have only one normal and, because of the Gauss equation, $k$ reduces to the (real) Gauss curvature $K$ of $S$. If however the imbedding geometry is curved then $k$ and $K$ do not coincide, furthermore in four dimensional $M$ the relationship between $k, k'$ and $K$ is much more complicated since the Gauss equation gives $\tfrac{1}{2}R_{abcd}q^{ac}q^{bd} = K + \frac{1}{2}(\theta_{ac}\theta'_{bd} + \theta'_{ac}\theta_{bd} - \theta_{ad}\theta'_{bc} - \theta'_{ad}\theta_{bc})q^{ac}q^{bd}$. In the theory of surfaces the convexity of a 2-surface in a flat 3-space is defined by the positivity of the principle curvature of the curves in the 2-surface. In higher dimensional and/or curved imbedding geometries these definitions are not equivalent. Our convexity condition that will be used in section 4 is formulated in terms of $k$ and $k'$, or in other words in terms of the principal curvature of the curves in $S$.

The two dimensional Sen operator is defined by $\Delta_e := \Pi^*_e\nabla_f$. The commutator of two Sen operators:

\[
(\Delta_e\Delta_d - \Delta_d\Delta_e) \phi = -2Q^e_{[cd]} \Delta_e\phi \tag{2.2}
\]

\[
(\Delta_e\Delta_d - \Delta_d\Delta_e) \xi^A = -2Q^e_{[cd]} \Delta_e\xi^A - R^A_{Be}f\Pi^*_e\Pi^*_f\xi^B. \tag{2.3}
\]

The curvature of $\Delta_e$ is therefore the pull back to $S$ of the anti-self-dual part of the spacetime curvature: $F^A_{Bcd} := R^A_{Be}f\Pi^*_e\Pi^*_f$; while its ‘torsion’, $T^e_{[cd]} := 2Q^e_{[cd]}$, is determined by the extrinsic curvatures. Expressing $R_{ABcd}$ by the Weyl and Ricci spinors and the A scalar $F_{ABcd}$ can be reexpressed as

\[
F_{ABcd} = -\frac{i}{2} \left( \psi_{ABEF}^{\gamma^{EF}} - \phi_{ABEF}^{\gamma^{EF}} - 2\Lambda_{\gamma^{AB}} \right) \varepsilon_{cd}. \tag{2.4}
\]

The spinor field $\xi^A_{B...}^{...} \xi^A_{B...}$ will be called holomorphic/anti-holomorphic if $\pi^{\pm c_f} \Delta_e \xi^A_{B...} := 0$. The quantity $Q^A_{\epsilon B} := \frac{1}{2} \Delta_e \gamma^{A}K^{A}K_{B}$ measures the ‘non-\gamma^{AB}-metricity’ of the Sen operator, and the extrinsic curvature tensor $Q^e_{ab}$ can be reexpressed by $Q^A_{\epsilon B}$ too:

\[
Q^e_{ab} = \frac{1}{2} \left( \delta^e_B^E Q^E_{aB} + \delta^e_B^E Q^E_{aB} + Q^E_{aR} \gamma^R_{B} \gamma^{E}K_{B} + \bar{Q}^E_{aR} \gamma^{R}_{B} \right). \tag{2.5}
\]

By (2.5) $Q_{\epsilon B}$ is just the anti-self-dual part of the ‘torsion’: $T_{E'E'A'AB'B'} = - (\varepsilon_{A'B'}Q_{AB'E'E'}B' + \varepsilon_{AB'Q_{A'AB'E'E'}}B')$, furthermore the GHP spin coefficients $\rho, \sigma$ and $\rho', \sigma'$ can also be expressed by $Q_{ARRB}$ too. For example $\rho := o^{A}\bar{o}^{B}l^{B}\bar{o}^{R}Q_{ARRB}$ and $\rho' := -o^{A}l^{B}\bar{o}^{R}Q_{ARRB}$. The induced spin connection is defined by

\[
\delta_{e}\lambda^{A} := \Delta_{e}\lambda^{A} - Q^{A}_{\epsilon B}\lambda^{B}, \tag{2.6}
\]

which for surface tensors is precisely the induced Levi-Civit\`a covariant differentiation. $\delta_{e}$ annihilates both $\varepsilon_{AB}$ and $\gamma_{AB}$, and its curvature can be defined by $\delta^R_{A'B'd}Q^{B'} := -(\delta_{e}\delta_{d} - \delta_{d}\delta_{e}) \lambda^{A}$. Under the boost gauge
transformation the 1-form field \( A_e := \Pi^I \nabla_I k^e \) transforms as an \SO(1,1)\ gauge field: \( A_e \mapsto A_e - \Pi^I \nabla_I \eta \).

By means of \( A_e \) and the curvature scalar \( R \) of the Levi-Civit\'a connection of \( R \), \( R \) can be reexpressed as

\[
R_{ABcd} = \frac{1}{2} \gamma_{AB} \left( (\delta_c A_d - \delta_d A_c) - \frac{R}{4} (\varepsilon_{CD}\gamma_{CD} - \varepsilon_{CD}\gamma_{CD}) \right).
\]

(2.7)

Its imaginary part is the curvature of the usual Levi-Livit\'a (\SO(2)–) connection, while its real part is the curvature of the \SO(1,1)–gauge field \( A_e \). With this extension of \( \delta_e \) from surface tensors to spinors we have extended the Levi-Civit\'a covariant differentiation \( \delta_e \) to arbitrary tensors.

To summarize, by a 2 dimensional Sen geometry we mean a quadruple \((\mathcal{S}, \varepsilon_{AB}, \gamma_{AB}, \Delta_e)\), where \( \varepsilon_{AB} \) is a symplectic and \( \gamma_{AB} \) is a complex inner product on a complex vector bundle \( S^4(\mathcal{S}) \) of rank 2 over \( \mathcal{S} \) and \( \Delta_e \) is a covariant derivation on \( S^4(\mathcal{S}) \) such that i. \( \gamma_{AB} \) satisfies (2.1); ii. the complexified tangent bundle \( T C \mathcal{S} \) of \( \mathcal{S} \) is isomorphic to the Whitney sum of the bundles of the elements \( o^A \gamma^{A'} \) and \( o^A \gamma^{A'} \), respectively; iii. \( \Delta_e \) annihilates \( \varepsilon_{AB} \); iv. the tensor \( Q_{eab} \) defined by the non-\( \gamma \)-metricity of \( \Delta_e \) according to (2.5) is symmetric in \( ea \) and, finally, v. the derivation \( \delta_e \) defined by (2.6) is the (symmetric) Levi-Civit\'a covariant derivation on \( T \mathcal{S} \) determined by \( q_{ab} := \frac{1}{2}(\varepsilon_{AB} \varepsilon_{A'B'} - \gamma_{AB} \gamma_{A'B'}) \).

### 3. Sen–constant spinor fields

Let \( \lambda_R \) be a smooth spinor field on \( \mathcal{S} \) which is constant with respect to the 2 dimensional Sen connection: \( \Delta_e \lambda_R = 0 \). First we prove the following lemma, which is the 2 dimensional counterpart of Lemma 2 of [13]:

**Lemma 3.1** If \( \lambda_R \) is Sen–constant on \( \mathcal{S} \) then either it is identically zero or nowhere zero on \( \mathcal{S} \).

**Proof:** Let \( t^{A'A'} \) be any positive definite hermitian inner product on the spinor spaces, e.g. a future directed timelike unit normal to \( \mathcal{S} \), and define \( h_{ab} := g_{ab} - t_a t_b \). \( h_{ab} \) is a negative definite metric on the 3 dimensional subspaces of vectors orthogonal to \( t^a \). Let \( \gamma : [a, b] \rightarrow \mathcal{S} \) be any smooth curve and let \( X^a \) be its unit tangent. Then since \( \lambda_R \) is Sen–constant we have

\[
|X^a \delta_e (t^{RR} \lambda_R \lambda_{RR})| = |(X^a \Delta_e t^a) h_{ab} (h^{bc} \lambda_c \lambda_{c'})| \leq |h_{ab} (X^a \Delta_e t^a) (X^b \Delta f^b)| \frac{1}{2} |h_{ab} \lambda^A \lambda^{A'} \lambda^B \lambda^{B'}|^{\frac{1}{2}} = |X^a \Delta_e t_a| (t^{RR} \lambda_R \lambda_{RR});
\]

i.e. \( \frac{d}{ds} (t^{RR} \lambda_R \lambda_{RR}) \leq |X^a \Delta_e t_a| (t^{RR} \lambda_R \lambda_{RR}) \), where \( s \) is the arch length parameter on \( \gamma \) and \( \|Y_e\| \) is the norm of \( Y_e \) in the metric \( h_{ab} \). From now on the proof is the same that of Lemma 2 of [13]: If \( C \geq 0 \) such that \( C \geq \max(\|X^a \Delta_e t_a\| (\gamma(s)) s \in [a, b]) \) then \( \frac{d}{ds} (t^{RR} \lambda_R \lambda_{RR}) \leq C t^{RR} \lambda_R \lambda_{RR} \) \( \lambda_R \lambda_{RR} \), \( s \in [a, b] \). If however \( f(s) \) is any \( C^1 \) function on \( [a, b] \) satisfying \( \frac{d}{ds} f(s) \leq C |f(s)| \) and \( f(s_0) > 0 \) for some \( s_0 \in [a, b] \) then \( f \) is positive on the whole interval \( [a, b] \). Thus if \( \lambda_R \) is zero at a point \( p \) of \( \mathcal{S} \) then it must be zero on any finite piece of any smooth curve through \( p \), and therefore on the whole \( \mathcal{S} \).

\( \square \)

In [2] we showed that there are at least two linearly independent holomorphic and two linearly independent anti-holomorphic spinor fields on \( \mathcal{S} \) provided \( \mathcal{S} \) is homeomorphic to \( S^2 \). A Sen–constant spinor field is holomorphic and anti-holomorphic at the same time, and hence one of the certainly existing two anti-holomorphic and two holomorphic spinor fields can be chosen to be the constant spinor field \( \lambda_R \). Let \( \mu_R \) be the other, nonconstant anti-holomorphic and holomorphic spinor fields, respectively. The inner products, \( \lambda_{RR} t^R \) and \( \lambda_{R} t^R \), are always constant on \( \mathcal{S} \), and they are zero if and only if both \( \lambda_R \) and \( \mu_R \), and both \( \lambda_R \) and \( \nu_R \) have a zero [2]. From Lemma 3.1 we can see, however, that \( \lambda_R \) cannot have a zero, and
hence both $\lambda_R \mu^R$ and $\lambda_R \nu^R$ are nonzero and can be, and in fact will be chosen to be unity. Therefore, in the terminology of [2], a topological 2-sphere $\mathbb{S}$ admitting a constant spinor field is generic; i.e. there are precisely two linearly independent anti-holomorphic and two linearly independent holomorphic spinor fields and both pairs $\lambda_R, \mu_R$ and $\lambda_R, \nu_R$ span the spin space at each point of $\mathbb{S}$. Obviously, $\mu_R$ and $\nu_R$ are unique up to a complex number times $\lambda_R$. In the rest of the present paper we assume that $\mathbb{S}$ is homeomorphic to $S^2$.

Let us define $\Phi := \mu_R \nu^R$. Then $\mu_R - \nu_R = \Phi \lambda_R$; i.e. the difference of the nonconstant anti-holomorphic and holomorphic spinor fields is proportional to the constant spinor field and the factor of proportionality is just $\Phi$. There is another interpretation of $\Phi$: If we consider $(\lambda_R, \mu_R)$ and $(\lambda_R, \nu_R)$ as two normalized spinor dyads then by

$$
(\lambda_R, \nu_R) = (\lambda_R, \mu_R) \begin{pmatrix} 1 & -\Phi \\ 0 & 1 \end{pmatrix}
$$

(3.2)

$\Phi$ is the globally well defined $\mathbb{S}$-dependent parameter of the SL(2, $\mathbb{C}$) spin transformation between the anti-holomorphic and the holomorphic spin frames. Or, in other words, $\Phi$ measures the deviation of the holomorphic and the holomorphic spin frames. Or, in other words, $\Phi$ measures the deviation of the holomorphic and anti-holomorphic spin frames from each other. Obviously $\Phi$ is unique up to an additive constant and at a given point $\Phi$ can be taken to be zero by a constant SL(2, $\mathbb{C}$)-transformation. Furthermore the following statements are equivalent: 1. $\Phi$ is constant; 2. $\mu_R$ is Sen–constant; 3. $\nu_R$ is Sen–constant; 4. under the conditions of the Theorem of the introduction $D(\Sigma)$ is flat.

Let us fix a globally defined smooth spinor field $\omega_R$ such that $\lambda_R \omega^R = 1$. Then since $\lambda_R$ is Sen–constant, $\Delta_e$ can be specified by the globally defined complex 1-form $\Gamma_a := \omega^R \Delta_a \omega_R$ on $\mathbb{S}$. Since $(\lambda_R, \omega_R)$ is a spin frame $\mu_R = \omega_R + \alpha \lambda_R$ and $\nu_R = \omega_R + \beta \lambda_R$ hold for some complex functions $\alpha$ and $\beta$; and hence $\Phi = \alpha - \beta$. Then since $\lambda_R$ is constant, $\nu_R$ is holomorphic and $\mu_R$ is anti-holomorphic, the two globally defined complex valued 1-forms

$$
\Gamma_a^+ := \pi^{+b}_a (\Delta_b \mu_R) \mu^R = \Gamma_a + \delta_a \alpha = \pi^{+b}_a \delta_b \Phi
$$

$$
\Gamma_a^- := \pi^{-b}_a (\Delta_b \nu_R) \nu^R = \Gamma_a + \delta_a \beta = -\pi^{-b}_a \delta_b \Phi
$$

(3.3)

represent the connection coefficients of the Sen connection in the anti-holomorphic and holomorphic frames, respectively. They have only one nonzero component, the other is zero because of the special holomorphic/-anti-holomorphic ‘gauge-choice’. If $\Gamma_a^+ = \pi^{+b}_a \delta_b \Phi = \pi^{+b}_a \delta_b \Phi'$ then $\pi^{+b}_a \delta_b (\Phi - \Phi') = 0$; i.e. $\Phi - \Phi'$ is holomorphic on $\mathbb{S}$ and hence constant. Thus $\Phi$, up to a constant, can be recovered from either $\Gamma_a^+$ or $\Gamma_a^-$, and hence, apart from constant SL(2, $\mathbb{C}$) transformations, the anti-holomorphic frame and $\Gamma_a^+$ determine the holomorphic frame. Therefore for any given $(\mathbb{S}, \epsilon_{AB}, \gamma_{AB})$ and $\lambda_R$ there is a 1–1 correspondence between the functions $\Phi$ modulo constants and the gauge equivalence classes of the 2 dimensional Sen–connections admitting $\lambda_A$ as a constant spinor field.

Next calculate the curvature, applying the commutator $\Delta_a \Delta_b - \Delta_b \Delta_a$ to $\lambda^R$, $\mu^R$ and $\nu^R$, respectively. We obtain from (2.3), (2.6) and (3.3)

$$
\lambda^S F_{S Rab} = 0
$$

$$
\mu^S F_{S Rab} = -\lambda_R (\delta_a \Gamma_b^+ - \delta_b \Gamma_a^+) = -\lambda_R \left( \delta_a^b \pi^{+f}_b - \delta_b^b \pi^{+f}_a \right) \delta_c \delta_f \Phi
$$

$$
\nu^S F_{S Rab} = -\lambda_R (\delta_a \Gamma_b^- - \delta_b \Gamma_a^-) = \lambda_R \left( \delta_a^b \pi^{-f}_b - \delta_b^b \pi^{-f}_a \right) \delta_c \delta_f \Phi.
$$

(3.4)

$\Gamma_a^\pm$ are therefore abelian vector potentials for the curvature and by $\Gamma_a^+ = \pi^{+b}_a \delta_b \Phi = (\pi^{+b}_a + \pi^{-b}_a) \delta_b \Phi - \pi^{-b}_a \delta_b \Phi = \delta_a \Phi + \Gamma_a^-$ they are gauge-equivalent. The only independent component of the curvature of the Sen connection is $\mu^R \mu^S F_{RSab} m^a m^b = \nu^R \nu^S F_{RSab} m^a m^b = \frac{1}{2} \Phi^{ab} \delta_a \delta_b \Phi$. Thus $\Phi$ is a potential for the curvature and by (2.4) we have
Geometrically [7] the constant spinor field \( \lambda_R \) defines a reduction of the pull back to \( \$ \) of the \( SL(2,C) \) spin frame bundle to a \( GL(1,C) \approx C - \{0\} \)-principle fibre bundle over \( \$ \) and defines a reduction of the \( SL(2,C) \)-connection to a \( GL(1,C) \)-connection.

The complex norm \( \gamma_{AB} \lambda^A \lambda^B \) of the constant spinor field can be interpreted tensorially e.g. by the anti-self-dual simple 2-form \( L_{ab} := \varepsilon_{AB} \lambda^A \lambda^B \), since \( L^{ab} \varepsilon_{ab} = i \gamma_{AB} \lambda^A \lambda^B \). Let us define \( z^a : = \Pi^a_B \lambda^B \), whose length is \( \|z^a\|^2 := -q_a z^a \cdot z^b = \frac{1}{2} |\gamma_{AB} \lambda^A \lambda^B|^2 \). Thus \( \lambda_A \) is null with respect to the complex metric \( \gamma_{AB} \) at \( p \in \$ \) iff \( z^a(p) = 0 \). Since \( z^a \) is a continuous vector field on a 2-sphere it must have a zero, and the set \( W := \{ p \in \$ | z^a(p) \neq 0 \} \) is obviously open in \( \$ \). Since \( \delta_a z_b = -Q_{abc} \lambda^E \lambda^E' \) and \( Q_{abc} = Q_{(abc)} \), the 1-form \( z_a \) is closed on \( \$ \), and hence by \( H^1(S^2) = 0 \) it is exact: \( z_a = \delta_a U \) for some smooth \( U : \$ \to \mathbb{R} \). For any fixed value \( U \) of the function \( U \) let us define \( C_U := \{ p \in \$ | U(p) = U \} \), which are closed sets in \( \$ \). Obviously \( z^a = 0 \) on \( \text{interior} C_U \), thus \( C_U \cap W = (C_U - \text{interior}\ C_U) \cap W \). Hence \( C_U \cap W \) is a smooth one dimensional submanifold of \( \$ \), i.e. any connected component of \( C_U \cap W \) is a curve \( \beta_U(w) \), which is orthogonal to \( z^a \). Consequently its tangent \( \dot{\beta}_U(w) \) is proportional to \( \varepsilon^{ab} \beta_b = \frac{1}{2}(\delta^E \gamma^A_B - \delta^E \gamma^A_B') \lambda^B \lambda^B' \); for some real function \( b \) depending on \( U \) and the parametrization \( w \beta_U^a = \frac{1}{\|z^a\|^2} \varepsilon^{ab} \lambda^b \lambda^B \). \( \beta_U(w) \) is either closed or its endpoints are zeros of \( z^a \).

4. pp-wave Cauchy developments

4.1 Zero-mass Cauchy developments

Let \( \lambda_A \) be nonzero constant spinor field on \( \$ \). Then by Lemma 3.1 \( \$ \) is generic, and hence the Dougan–Mason energy-momentum and mass are well defined [3,2]: if \( \lambda^j_A := \lambda_A \), \( \lambda^1_A := \mu_A \) then they are \( P^B_B := \frac{1}{\kappa} \int_{\$} (\overline{\lambda}_A^B \Delta_{BB} \lambda^A + \overline{\lambda}_A^B \Delta_{AA} \lambda^A) \) and \( m^2 := \varepsilon_{AB} \varepsilon_{CD} \Delta^A \Delta^C \Delta^B \Delta^D = 2 \left( P^0_B P^B_0 - P^0_B P^B_0 \right) \), respectively. Since \( \lambda^1_A \) is constant \( P^0_B \) is zero. Let \( \Sigma \) be a smooth, compact spacelike hypersurface with the smooth boundary \( \Sigma = \partial \Sigma \) and suppose that \( \rho' \geq 0 \). Let \( t^a \) be the unit future normal to \( \Sigma \), \( P^0_B = \delta^0_B - t^a t^a \) the projection to \( \Sigma \), \( h_{ab} := P^0_B P^0_D g_{ef} \) the induced metric, \( \varepsilon_{abc} := t^c \varepsilon_{abc} \) the induced 3-form, \( D_{e} \) the induced Levi-Civita covariant derivation and \( D_{e} := P^f_{ef} \nabla_f \) the 3 dimensional Sen operator [13] on \( \Sigma \). Obviously \( \Sigma \) fixes a boost gauge on \( \$ \) and on the domain where \( \beta_U^a \) is defined the outward unit normal of \( \$ \) in \( \Sigma \) can be recovered as \( n^a = (\|z^a\|^2)^{-1} \varepsilon_{bc} \beta_c \). If \( \lambda^A_R \) is the spinor field on \( \Sigma \) satisfying the Sen–Witten equation \( D_{R} \tilde{\lambda}^A_R = 0 \) with the boundary condition \( \pi^{+B} \g_{BB} |\tilde{\lambda}^A_R R - \lambda^A_B| = 0 \) then by the Reula–Tod form [14] of the Sen–identity [13] we obtain for \( \mathbf{A} = 0 \), \( \mathbf{A}^t = 0 \) and for \( \mathbf{A} = 1 \), \( \mathbf{A}^t = 1 \) that

\[
P^A_{\mathbf{A}^t} = \frac{2}{\kappa} \int_{\Sigma} \rho(n^a |\tilde{\lambda}^A_R - \lambda^A_R|^2 |\partial^a \lambda^A_R| - \frac{1}{2} \tilde{\lambda}^A_R \tilde{\lambda}^A_R \Delta^B \partial^a \lambda^B_R \partial^a \lambda^B_R ) d\Sigma ,
\]

where the spinor components are defined by \( \lambda^A_R := \lambda^A_R \sigma_R - \lambda^A_R \sigma_R \). If the dominant energy condition holds on \( \Sigma \) then this implies \( P^0_{\mathbf{A}^t} \geq 0 \) and \( P^0_{\mathbf{A}^t} = P^0_{\mathbf{A}^t} = 0 \); and, for \( \mathbf{A} = 0 \), \( \mathbf{A} = 0' \), that \( \lambda^A_R \sigma_R = \lambda^A_R \sigma_R \) and \( D_{e} \lambda^A_R = 0 \). By Lemma 2. of [13] \( \lambda^A_R \) is nowhere zero on \( \Sigma \). Foliating \( D(\Sigma) \) by a family \( \Sigma_t \) of spacelike Cauchy hypersurfaces (for which \( \partial^a \lambda^A_R = 0 \) necessarily holds) and assuming the dominant energy condition to hold on the whole \( D(\Sigma) \), we obtain a smooth nowhere vanishing spinor field \( \lambda^A_R \) on \( D(\Sigma) \). Since \( \lambda^A_R \) is an extension of \( \lambda^A_R \) form \( \$ \) to \( D(\Sigma) \), we leave the ‘hat’ and denote this extension simply by \( \lambda_A \) too. In [4] it was shown that \( \lambda^A_R \) is covariantly constant on \( D(\Sigma) \), \( \nabla_a \lambda^A_R = 0 \), and
\[
\psi_{ABCD} = \psi \lambda_A \lambda_B \lambda_C \lambda_D \\
\phi_{ABA'B'} = \phi \lambda_A \lambda_B \bar{\lambda}_{A'} \bar{\lambda}_{B'} \\
\Lambda = 0,
\]

i.e. the Weyl tensor has Petrov N type and the matter is pure radiation with common wave vector \( L^a := \lambda^A A^A \). Here \( \psi \) is a complex and \( \phi \) is a non-negative real function and \( L^a \) is a covariantly constant nowhere vanishing null vector field on \( D(\Sigma) \). Under the conditions of the Theorem of the introduction a Sen–constant spinor field \( \lambda_R \) on \( \$ \) can therefore be extended into a constant spinor field \( \lambda_R \) on \( D(\Sigma) \) in a unique way. As a consequence of (4.2) the Bianchi identities take the following form:

\[
\lambda_A \lambda_B \lambda_C \lambda^D \nabla_{DD'} \psi = \frac{1}{3} \left( \lambda_A \lambda_B \nabla_{CC'} \phi + \lambda_B \lambda_C \nabla_{AC'} \phi + \lambda_C \lambda_A \nabla_{BC'} \phi \right) \bar{\lambda}_D \bar{\lambda}^{D'}. \tag{4.3}
\]

This implies that \( L^a \nabla_e \psi = 0 = L^e \nabla_e \phi \). By the definition of \( D(\Sigma) \) at each point \( p \) of \( D(\Sigma) \) the geometrical properties are uniquely determined by the Cauchy data on \( \Sigma \). In particular, since each \( p \in D(\Sigma) \) is on an integral curve of \( L^a \) intersecting \( \Sigma \) at precisely one point, say \( p_0 \), the curvature components \( \psi \) and \( \phi \) are the same at \( p \) and \( p_0 \). On the other hand by the normalization \( \lambda_R \mu^R = 1 \) and eq. (4.2) it follows from (3.5) that on \$ \n\]

\[
q^{ab} \delta_a \delta_b \Phi = \psi \gamma_{AB} \lambda^A \lambda^B - \phi \bar{\gamma}_{A'B'} \bar{\lambda}^A \bar{\lambda}^{B'}. \tag{4.4}
\]

We will show that the curvature on \( \Sigma \) is determined, through (4.3) and (4.4), by \( q_{ab} \), \( \gamma_{AB} \lambda^A \lambda^B \), \( \Phi \) and, if matter radiation is present, a complex function representing the matter radiation field on \$. Then we determine the line element of \( D(\Sigma) \) in terms of the Sen geometry of \$. First, however, we should clarify the structure of \( \Sigma \).

### 4.2 The structure of \( \Sigma \)

Since \( L_a = \lambda_A \bar{\lambda}_{A'} \) is constant, it is a gradient: \( L_a = \nabla_a u \) for some smooth \( u : \text{int} D(\Sigma) \rightarrow \mathbb{R} \). Furthermore \( L^a \) is nowhere vanishing and null on the whole \( \Sigma \), therefore its projection to \( \Sigma \), \( Z^a := P_a^b L^b \), is nowhere vanishing and \( z^a = \Pi^a Z^b \). Thus \( Z^a \) is orthogonal to \$ at \( p \in \$ \) iff \( z^a \) has a zero at \( p \). \( Z_a \) is also a gradient, \( Z_a = D_a u \), and hence \( \text{int} \Sigma = \Sigma - \partial \Sigma \) can be foliated by the 2 dimensional maximal integral submanifolds \( S_u := \{ q \in \text{int} \Sigma | u(q) = u \} \). Since \( \Sigma \) is orientable the leaves \( S_u \) are also orientable, and hence the induced Riemannian metric defines a complex structure on each \( S_u \); i.e. the leaves are Riemann surfaces.

Let \( \{ E_1, E_2, E_3 \} \) be an orthonormal basis on (an open subset of) \( \Sigma \) such that \( E_2 := \frac{1}{\| Z^c \|} Z^a \) and \( E_1, E_3 \) are also tangent to \( \Sigma \), where \( \| Z^c \|^2 := -h_{ef} Z^e Z^f \). \( E_3 \) is therefore globally well defined but in general \( E_1 \) and \( E_2 \) are not. The orientation of this basis will be defined by \( \varepsilon_{abc} E_1^a E_2^b E_3^c = 1 \). Let \( N^a \) be the uniquely determined future null vector field on \( \Sigma \) that is orthogonal to \( S_u \) and normalized by \( L^a N_a = 1 \); and let \( \sqrt{2} M^a := E_1^a - i E_2^a \). \( M^a \) and \( \bar{M}^a \) are (1,0) and (0,1) type vectors, respectively, in the complex structure of the Riemann surfaces \( S_u \). A local complex coordinate system \( (\zeta, \bar{\zeta}) \) can always be chosen so that \( M^a = P(\zeta, \bar{\zeta}, u) (\frac{\partial}{\partial \zeta})^a \), where \( P \) is real and positive. \( \{ M^a, \bar{M}^a, E_3 \} \) is proportional to the so-called geometric triad of Frauendiener [15]. Let \( \chi_{ab} := P_a^c P_b^f \nabla_c t_f \), the extrinsic curvature of \( \Sigma \). Then by taking the \( D_a \)-derivative of \( L^a = Z^a + \| Z^c \| t^a \) we obtain

\[
D_a Z_b = -\| Z^c \| \chi_{ab}. \tag{4.5}
\]

This implies that the extrinsic curvature \( k_{ab} \) of \( S_u \) in \( \Sigma \) is just the projection of \( \chi_{ab} \) onto \( S_u \): \( k_{ab} := (\delta_a^c + E_3^a E_3^a)(\delta_b^f + E_3^b E_3^b) D_c(-E_{3f}) = (\delta_a^c + E_3^a E_3^a)(\delta_b^f + E_3^b E_3^b) \chi_{cf} \). (It is \( -E_3^a \) along that \( u \) is increasing,
thus it is natural to consider $-E_3^a$ as the normal of $S_u$.) Then by (4.2) and the Gauss equation for the curvature scalar $uR$ of the leaves $S_u$ we have $uR = -2R_{abcd}M^aM^bM^cM^d = 0$; i.e. the Riemann surfaces are locally flat Riemann geometries. The complex coordinates $(\zeta, \bar{\zeta})$ can therefore be chosen so that $P(\zeta, \bar{\zeta}, u) = 1$ and the remaining allowed transformation of them is $\zeta = \exp(ia(u))\zeta' + A(u)$ for real $a(u)$ and complex $A(u)$.

Next let us clarify the structure of the boundary of the Riemann surfaces $S_u$. Using the fact that $Z^a$ is a well defined nonzero smooth vector field on the whole $\Sigma$ one can show that each point $p \in $ belongs to the closure $\overline{S_u}$ of at most one level surface $S_u$, and if $p$ does not belong to any $\overline{S_u}$ then $Z^a$ is orthogonal to $\partial$. We can therefore extend $u$ from int$\Sigma$ to the whole $\Sigma$ in the following way: if $p \in \partial \overline{S_u}$ for some $u$ then let $u(p) := u$; and if $p$ does not belong to any $\overline{S_u}$ then let $\gamma : [0, \varepsilon] \rightarrow \Sigma$ be the integral curve of $Z^a$ such that $\gamma(0) = p$ and $\dot{\gamma}^a = \pm Z^a$. If $z$ is the parameter of $\gamma$ then let us define $u(p) := \lim_{z \rightarrow 0} u(\gamma(z))$. This limit always exists since $\pm \frac{\partial u}{\partial z} = Z^aD_au = h_{ab}Z^aZ^b$, whose right hand side is bounded and $C^\infty$ on $\Sigma$. Next, it is a standard exercise to show that $u$ extended in this way is a smooth function on the whole $\Sigma$. This result has important consequences. First, $z_a$ is the gradient of the restriction of $u$ to $\partial$, i.e. $U = u$ can be chosen. Then since $u : \Sigma \rightarrow \mathbb{R}$ is $C^\infty$ and $\Sigma$ is compact $u$ has a minimum $u_-$ and a maximum $u_+$ somewhere, which must be on $\partial$ because $Z_a$ is nowhere vanishing. Thus $u : \partial \rightarrow [u_-, u_+]$ is onto. Consequently, the topological boundary of any Riemann surface $S_u$, $B_u := \partial \overline{S_u}$ is nonempty and $B_u = C_u - \text{int}C_u$. Thus, as we saw at the end of the previous section, the piece of the topological boundary $B_u$ lying in the open domain where $z^a \neq 0$, i.e. $B_u \cap W$, consists of smooth curves $\beta_u(w)$ orthogonal to $z^a$. Without further conditions on the geometry of $\partial$ one cannot say anything about the structure of $B_u$ at the zeros of $z^a$. The next proposition shows that certain generalization of the convexity conditions of differential geometry, however, excludes the strange structures for $B_u$.

**Proposition 4.6** Let $k > 0$ and $k' > 0$, and the ingoing null normals be converging and the outgoing null normals be diverging somewhere on $\partial$. Then

1. $z^a$ has two isolated zeros, $p_{\pm}$, for which $u(p_{\pm}) = u_\pm$, $p_{\pm}$ do not belong to the closure of any $S_u$, and the topological boundary $B_u$ of any Riemann surface $S_u$, $u \in (u_-, u_+)$, consists of a single smooth closed curve $\beta_u(w)$;
2. each Riemann surface $S_u$ is homeomorphic to $\mathbb{R}^2$;
3. $\Sigma$ is homeomorphic to the closed three–ball $B^3$.

**Proof:** (1) First, on the contrary, suppose that for some $S_u$ there is a point $p \in \partial \overline{S_u}$ which is a zero of $z^a$. The normal $-E_3^a$ of $S_u$ can obviously be extended to $\overline{S_u}$ and $\partial$ and $\overline{S_u}$ are tangent at $p$ to each other: $E_3^a = \pm v^a$. (Here $v^a$ is the outward directed unit normal to $\partial$ in $\Sigma$.) Then the position of $\partial$ relative to $\overline{S_u}$ can be determined by comparing the principle curvature of the curves through $p$ lying in $\partial$ and in $\overline{S_u}$ (see e.g. [5,6]). Let therefore $X^a \in T_p\partial = T_p\overline{S_u}$ be a unit vector and $\gamma$ and $\gamma_u$ be smooth curves in $\partial$ and $\overline{S_u}$, respectively, whose tangent at $p$ is $X^a$ and whose principle normal at $p$ is proportional to $v^a$. Let $\kappa_X$ and $\kappa_u$ be the corresponding principle curvatures at $p$. Let the NP null tetrad \{$l^a, m^a, n^a, n^a\}$, adapted to $\partial$, be normalized by $\sqrt{2l^a} = l^a + v^a$ and $\sqrt{2n^a} = n^a - v^a$. Then a short calculation shows that the difference of the principle curvatures are determined by the quadratic form defined by the expansion tensors of $\partial$:

\[
\kappa_X - \kappa_u^a = -\left(\Pi_\alpha^a \Pi_\beta^b D_\alpha v_\beta - \Pi_\alpha^a \Pi_\beta^b D_\beta E_3^f \right) X^a X^b = -\sqrt{2\theta_{ab}} X^a X^b \quad \text{if} \quad E_3^a = v^a \quad (4.7a)
\]
\[
\kappa_X - \kappa_u^a = -\left(\Pi_\alpha^a \Pi_\beta^b D_\alpha v_\beta + \Pi_\alpha^a \Pi_\beta^b D_\beta E_3^f \right) X^a X^b = \sqrt{2\theta_{ab}} X^a X^b \quad \text{if} \quad E_3^a = -v^a. \quad (4.7b)
\]

By the Cayley–Hamilton equation we have $\theta_{ab}\theta_{cf} q^{cf} = q_{ab} + q^{cf} \theta_{ca} \theta_{fb}$, whose right hand side is negative definite for everywhere positive $k$. Hence for somewhere positive $\theta_{cf} q^{cf}$ this implies that $\theta_{ab}$ is negative
definite; i.e. $\kappa^X > \kappa^X$. However $\kappa^X \leq \kappa^X$ must hold, as otherwise $\bar{\omega}_{u^j}$ would be \textit{outwardly} tangent to $p$ to $\mathcal{S}$. Similarly the Cayley–Hamilton equation, $k^p > 0$ on $\mathcal{S}$ and $\theta_{ab}q^{ab} < 0$ somewhere on $\mathcal{S}$ imply the positive definiteness of $\theta_{ab}^\prime$, which would contradict $\kappa^X \leq \kappa^X$. Thus no point of $\mathcal{S} \cap \bar{\omega}_{u^j}$ can be a zero of $z^a$ and hence by the argumentation just before the present proposition each connected component of $B_u$ is a single closed smooth curve. Thus $z^a$ may have zeros only at the points $a \in \mathcal{S}$ where $u(a) = u_\pm$.

Let $A$ be the preimage of $u_-$ by $u$; i.e. $A := \{a \in \mathcal{S} | u(a) = u_- \}$. Obviously $A$ is closed. If $A$ were not connected, say $A = A' \cup A''$ for disjoint nonempty closed sets $A'$ and $A''$, then since $\mathcal{S}$ is connected, $u|_{A'} : \mathcal{S} \to \mathbb{R}$ is smooth and $u_-$ is the minimum value of $u$, there would be a point $p \in \mathcal{S} - A$ where $u(p) \in (u_-, u_+)$ and $z_a(p) = \delta_a u(p) = 0$. This however would contradict the first part of the present proof, thus $A$ must be connected and, since $\mathcal{S}$ is a manifold, path connected. Let $a \in A$ and suppose that $A - \{a\}$ is not empty. Then there is a series $\{a_n\}$ of points of $A - \{a\}$ converging to $a$. Let $\gamma_n$ be the geodesic in $\mathcal{S}$ through $a$ and $a_n$; and let $X_a^n$ be its unit tangent at $a$. Then there is a unit vector $X^a$ at $a$ such that $\{X^n_a\}$ (or at least a sub-series of it) converges to $X^a$. Then it is easy to prove that $X^a(\delta_a \delta_b u) = 0$ at $a \in \mathcal{S}$. Let $B$ be the geodesic in $\mathcal{S}$ through $a$ with tangent $X^a$. At the points of $A$ $E^n_a = v^a$, thus if $F_a$ is the 1-parameter family of local diffeomorphisms generated by $-\frac{1}{\|z\|}Z^a$ then for sufficiently small $\epsilon > 0 F_{a,\epsilon}(A) \subset S_{u_-, \epsilon}$.

Let $\gamma^\epsilon := F_{\epsilon} \circ \gamma$, whose tangent $F_{a,\epsilon}(X^a)$ is easily seen to lie in $S_{u_-, \epsilon}$. Finally let $\tilde{\gamma}^\epsilon$ be the smooth curve in $S_{u_-, \epsilon}$ through $F_{\epsilon}(s)$ whose projection along the orbits of $F_{a,\epsilon}$ to $\mathcal{S}$ is $\gamma$. Since in general $\gamma$ does not lie in $A$ $\gamma^\epsilon$ does not coincide with $\tilde{\gamma}^\epsilon$. However the tangent of $\tilde{\gamma}^\epsilon$ at $F_{\epsilon}(a)$ is just $F_{a,\epsilon}(X^a)$; furthermore by $X^a(\delta_a \delta_b u)(a) = 0$ the principle curvature of $\gamma^\epsilon$ and $\tilde{\gamma}^\epsilon$ are equal at $a$. But the $\epsilon \to 0$ limit of the difference of these principle curvatures is $-\sqrt{2} \gamma_{ab} X^a X^b > 0$, which is a contradiction. Thus $A$ consists of a single point, or, in other words, $u|_{A'} : \mathcal{S} \to \mathbb{R}$ takes its minimum value at a single point $p_-$. By a similar argumentation, it takes its maximal value at another single point $p_+$. Since however $z^a$ may have zeros only at the points where $u$ takes its minimal or maximal value, the two zeros of $z^a$ must be $p_\pm$, which are isolated. What remained to show is that $B_u$ is connected for any $u \in (u_-, u_+)$.

Let $U$ be any open neighbourhood of $p_-$ in $\Sigma$. Then for sufficiently small $\epsilon > 0 S_{u_- + \epsilon} := \{p \in \mathcal{S} | u(p) < u_- + \epsilon\} \subset U \cap \mathcal{S}$ and $\partial S_{u_- + \epsilon} = B_{u_- + \epsilon}$, which is connected. Since the only zeros of $z^a$ are $p_\pm$, $-\frac{1}{\|z\|}z^a$ defines a 1-parameter family $f_u$ of local diffeomorphisms of $\mathcal{S} - \{p_- , p_+ \}$ onto itself. Since the action of these diffeomorphisms is transitive and $f_{u_- - \epsilon}B_{u_-} = B_u$, any $B_u$ can be mapped into $B_{u_- + \epsilon}$. Thus $B_u$ is connected. (See also [16].)

(2) Let $Y^a$ be any smooth vector field on $\Sigma$ which is tangent to the leaves $S_u$, orthogonal to $B_u$ and nonzero on $\mathcal{S}$. Since $v^a$ is the unit normal to $\mathcal{S}$ it is a linear combination of $Y^a$ and $Z^a$ on $\mathcal{S}$. Then for any pair of smooth functions $\alpha, \beta : \Sigma - \{p_- , p_+ \} \to \mathbb{R}$ let us define $\tilde{z}^a := \alpha Y^a + \beta Z^a$. For everywhere nonzero $\beta$ $\tilde{z}^a$ is nowhere zero on $\Sigma - \{p_- , p_+ \}$, and $\tilde{z}^a D_u u = \beta h_{ab} Z^a Z^b$. Furthermore if the restriction of $\alpha$ and $\beta$ to $\mathcal{S}$ are chosen to satisfy $\|Y^a\|^2 \alpha |_\mathcal{S} = -(v^a Y_a)(v^1 Z_1)$ and $\|Z^a\|^2 \beta |_\mathcal{S} = \|Z^a\|^2 - (v^a Z_a)^2$, respectively, then $\tilde{z}^a |_\mathcal{S} = z^a$. With this choice $\tilde{z}^a$ is a nowhere vanishing extension of $z^a$ to $\Sigma - \{p_- , p_+ \}$. Thus if $\tilde{F}_u$ is the 1 parameter family of local diffeomorphisms of $\Sigma - \{p_- , p_+ \}$ generated by $-\frac{1}{\|z\|}\tilde{z}^a$ then it is transitive on $\Sigma - \{p_- , p_+ \}$ and $\tilde{F}_{u - \epsilon}S_{u - \epsilon} = S_u$. Since for sufficiently small $\epsilon > 0 S_{u_- + \epsilon}$ is homeomorphic to $\mathbb{R}^2$ this implies that $S_u$ is homeomorphic to $\mathbb{R}^2$ for any $u \in (u_-, u_+)$.

(3) is now a simple consequence of the result that $\Sigma - \{p_- , p_+ \}$ is homeomorphic to $B^2 \times (u_-, u_+)$ where $B^2$ is the closed 2-ball. 

An immediate consequence of this proposition is that the Riemann surfaces $S_u$ are connected, simply connected subsets of a flat plane and they form a global foliation of $\Sigma$. (Its lapse $n$ is given by $n^{-1} := (-E_a^3)D_u u = \|Z^a\|$.) Consequently the complex null coordinate system $(\zeta, \bar{\zeta})$ is globally defined on the Riemann surfaces $S_u$ and hence on the whole $\Sigma$. In this coordinate system the closed curves $\beta_a(w)$ can be given as $(\zeta(w, u), \bar{\zeta}(w, u), u)$. If the parameter $w$ is chosen such that $b = b(u)$ is $\frac{1}{2\pi}$ times the arclength of
$$\beta_u \text{ then } (w, u), w \in [0, 2\pi), u \in (u_-, u_+), \text{ is a coordinate system on } \mathcal{S} - \{p_{\pm}\} \approx S^1 \times (u_-, u_+). \text{ Note that this coordinate system is determined by the geometry of } \mathcal{S} \text{ alone.}
$$

Let $I_A$ be the smooth spinor field on $\Sigma$ for which $\lambda_A I^A = 1$ and the complex null vectors $\lambda^A I^A$, $I^A \lambda^A$ are tangent to the Riemann surfaces $S_u$. These conditions uniquely determine $I^A$, and $I^A I^A = N^u$, $\lambda^A I^A = \exp(i\omega) M^a$ hold for some smooth function $\omega : \Sigma \rightarrow \mathbb{R}$. Let $\nu$ be the affine parameter along the (maximally extended null geodesic) integral curves of $L^a$ measured from $\Sigma$ and Lie propagate $(\zeta, \bar{\zeta})$ along $L^a$. (u is automatically Lie propagated along $L^a$.) Then $D(\Sigma)$ is covered by the domain of the coordinate system $(v, \zeta, \bar{\zeta}, u)$. In this coordinate system the spacetime metric takes the form $ds^2 = 2d\nu du - 2d\zeta d\bar{\zeta} + 2(Gd\zeta + Gd\bar{\zeta})du + 2Hd\nu^2$, where $H$ is a real and $G$ is a complex function of $\zeta$, $\bar{\zeta}$ and $u$. Since $D_a \lambda_R = 0$ and $M^a = \exp(-i\omega) \lambda^A I^A$, we have $M^a D_a M_c = -iD_a \omega$. Its left hand side can be reexpressed by the Christoffel symbols of the spacetime metric and we obtain $-iD_a \omega = M^a D_a M_c = \frac{i}{2} (\frac{\partial G}{\partial \zeta} - \frac{\partial G}{\partial \bar{\zeta}}) D_a u$. This implies that $\omega$ depends only on $u$, and hence by the allowed transformation of the coordinates $(\zeta, \bar{\zeta})$ by $a(u) = -\omega(u)$ it can be taken zero. Thus in this coordinate system $M^u = \lambda^A I^A$, $M^a = I^A \lambda^A$ and

$$\frac{\partial G}{\partial \zeta} - \frac{\partial G}{\partial \bar{\zeta}} = 0. \quad (4.8)$$

In the rest of this paper the complex coordinates will be chosen to satisfy these properties. Obviously all the spinor and vector fields on $\Sigma$ can be extended onto the whole domain of the coordinate system by Lie propagation along $L^a$.

### 4.3 The curvature and the metric of $D(\Sigma)$

Returning to the spinor Bianchi identity let us contract (4.3) with $I^A I^B I^C \bar{I}^D$. We obtain

$$M^a \nabla_a \psi = M^a \nabla_a \phi. \quad (4.9)$$

If $\phi$ were zero then by (4.9) $\psi$ would be anti-holomorphic on each Riemann surface $S_u$, and, since topologically $S_u$ is trivial and its boundary $B_u$ is a single closed curve, $\psi$ on $S_u$ would completely be determined by its value on $B_u$, and hence by $g_{ab}, \gamma_{AB} \lambda^A \lambda^B$ and $\Phi$ (cf. eq. (4.4)). Similarly, $\psi$ could be determined from its boundary value for any given $\phi$ using e.g. the Green function method of Aichelburg [10]. Since however (4.9) is only one equation for $\psi$ and $\phi$, it does not determine them on $\Sigma$ from their boundary value on $B_u$ unless the field equation for the matter fields are specified.

For the massless complex scalar field $\varphi$ the energy-momentum tensor takes the form $T_{ab} = \nabla_a (\phi \nabla_b \phi) - \frac{1}{2} g_{ab} g^{cd} \nabla_c \varphi \nabla_d \varphi$, thus the condition $T_{ab} L^b = 0$ implies that $\nabla_a \varphi = f L_a$ for some complex function $f$. Therefore $M^a \nabla_a \varphi = 0$ and $M^a \nabla_a \phi = 0$; i.e. $\varphi$ is constant on each Riemann surface $S_u$ and hence $\varphi$ on $\Sigma$ is completely determined by its value on $\mathcal{S}$.

For the electromagnetic field the field strength is described by a symmetric spinor $\varphi_{AB}$ defined by $F_{ab} = \varepsilon^{A'B'} \varphi_{AB} + \varepsilon_{AB} \varphi_{A'B'}$. Maxwell’s source free equations take the form $\nabla_A \varphi_{AB} = 0$ and the energy-momentum tensor is $T_{ab} = 2 \varepsilon_{AB} \varphi_{A'B'}$. If algebraically $T_{ab}$ is pure radiation with wave vector $L^a$ then $\varphi_{AB} = \varphi \lambda_A \lambda_B$ for some complex function $\varphi$, the only nonzero component of the Ricci spinor is $\phi = \kappa \varphi \bar{\varphi}$ and Maxwell’s equations reduce to $\lambda^A \nabla_A \varphi = 0$. Contracting it with $I^A$ we obtain $M^a \nabla_a \phi = 0$. Thus $\varphi$ is anti-holomorphic on each Riemann surface $S_u$ and therefore both $\varphi$ and $\phi$ are completely determined on $\Sigma$ by the value of $\varphi$ on $\mathcal{S}$. Physically, $\varphi$ is the complex combination of the electric and magnetic field strengths defined by the normal $t^a$ of $\Sigma$: $E_a + iB_a = -\varphi ||Z^a|| M_a$. This implies that $E^a E_a = B^a B_a$, $E^a B_a = 0$ and $Z^a (E_a + iB_a) = 0$, which are the characteristic properties of plane electromagnetic fields with spatial wave vector proportional to $Z^a$. 

10
One can consider general 2th order symmetric spinor fields $\varphi_{AB\ldots D}$ satisfying the zero-rest-mass-field equation $\nabla_A A^i \varphi_{AB\ldots D} = 0$ with integer $s$. (For half-integer $s$ the energy-momentum tensor is not expected to satisfy the dominant energy condition. In fact, for the Weyl neutrino field $\varphi_A$ the energy-momentum tensor is $T_{ab} = i (\varphi_A \nabla_B \varphi_A - \varphi_A \nabla_B \bar{\varphi}_A + \varphi_B \nabla_A \bar{\varphi}_B - \varphi_B \nabla_A \varphi_B)$, which does not satisfy even the weak energy condition.) $\varphi_{AB\ldots D}$ is said to describe a pure radiation with wave vector $\lambda^A \lambda^B$ if $\varphi_{AB\ldots D} = \varphi_{\lambda \ldots \lambda D}$; and the energy-momentum tensor of this radiation field can be defined by $T_{ab} := 2 \varphi \bar{\varphi} \lambda_A \lambda_B \nabla_A \nabla_B$. Then

$$M^a \nabla_a \varphi = 0;$$

(4.10)
i.e. $\varphi$ is anti-holomorphic on the Riemann surfaces $S_a$ and hence $\phi$ on $\Sigma$ is determined by the value of $\varphi$ on $\mathcal{S}$. To summarize, if we assume that the matter field is a pure radiative massless complex scalar field or zero-rest-mass-field with integer helicity, then the field equations imply (4.10) and, for any solution $\varphi$ of (4.10) on $\Sigma$, the solution of (4.9) for $\psi$ can be specified by the value of the solution $\psi$ on $\mathcal{S}$. Therefore, for given $g_{ab}$ and $\gamma_{AB} \lambda^A \lambda^B$, the curvature on $\Sigma$ (and consequently along all the integral curves of $L^a$ crossing $\Sigma$) is determined by the complex functions $\Phi$ and $\varphi|\mathcal{S}$.

Finally determine the metric of $D(\Sigma)$ from the data on $\mathcal{S}$. The only nonzero components of the curvature are

$$\begin{align*}
\phi &= I^A I^B R_{AB\ldots N} M^d = \frac{\partial}{\partial \zeta} \left( \frac{\partial H}{\partial \zeta} - \frac{\partial G}{\partial u} \right), \\
\psi &= I^A I^B R_{AB\ldots N} M^d = \frac{\partial}{\partial \zeta} \left( \frac{\partial H}{\partial \zeta} - \frac{\partial G}{\partial u} \right).
\end{align*}$$

(4.11)

The integrability condition of (4.11) is just (4.9). To show that the Sen geometry of $\mathcal{S}$ determines the metric of $D(\Sigma)$ completely we should fix a gauge. For any smooth real valued function $V$ of $\zeta$, $\bar{\zeta}$ and $u$ the mapping $V : (v, \zeta, \bar{\zeta}, u) \mapsto (v + V(\zeta, \bar{\zeta}, u), \zeta, \bar{\zeta}, u)$ is a smooth diffeomorphism of the domain of the coordinate system onto itself. Geometrically $V$ shifts the zero of the affine parameter $v$ of $L^a$ by $V$; and let $\Sigma' := \mathcal{V}(\Sigma)$. Then the action of $V$ on the coordinate vectors is $\mathcal{V}_\nu(L^a) = L^a$, $M^a := \mathcal{V}_\nu(M^a) = M^a + \frac{\partial}{\partial \zeta} L^a$, $\bar{M}^a := \mathcal{V}_\nu(\bar{M}^a) = \bar{M}^a + \frac{\partial}{\partial \bar{\zeta}} L^a$ and $(\frac{\partial}{\partial \zeta}) a := \mathcal{V}_\nu((\frac{\partial}{\partial \zeta}) a) = (\frac{\partial}{\partial \zeta}) a + \frac{\partial}{\partial \zeta} L^a$. The pull back $g_{ab}' := \mathcal{V}^* (g_{ab})$ of the spacetime metric is $ds^2 = 2dudv - 2dzd\bar{z} + 2(\bar{G} + \frac{\partial}{\partial \bar{\zeta}})d\zeta + 2(\bar{G} + \frac{\partial}{\partial \zeta})d\bar{\zeta} + 2(\bar{H} + \frac{\partial}{\partial \zeta})du + 2(\bar{H} + \frac{\partial}{\partial \bar{\zeta}})d\bar{u}$. Thus in the coordinates $(u, \zeta, \bar{\zeta}, u)$ the metrics $g_{ab}$ and $g_{ab}'$ have the same form if $G^a := G + \frac{\partial}{\partial \bar{\zeta}}$ and $H^a := H + \frac{\partial}{\partial \zeta}$. Let $N^a$ be the future directed null vector field orthogonal to $M^a$ and $\bar{M}^a$, and normalized by $N_a N^a = 1$. In the new basis $N^a$ is given by $N^a = -(H^a + \bar{G} \bar{G}) L^a + (\frac{\partial}{\partial \bar{\zeta}})^a + \bar{G} M^a + \bar{M}^a$. Since in general $\Sigma'$ is not a Cauchy surface for $D(\Sigma)$ (it might become timelike or null somewhere and some portions of $\Sigma'$ may even be outside $D(\Sigma)$), $\Sigma$ is not necessarily spacelike with respect to $g_{ab}'$. Its normal is proportional to $N^a - (H^a + \bar{G} \bar{G}) L^a$. Thus $\Sigma$ is spacelike/null/timelike at a point $p \in \Sigma$ if $H^a + \bar{G} \bar{G}$ is negative/zero/positive there; and define $\Sigma_+$, $\Sigma_0$ and $\Sigma_-$ the spacelike, null and timelike pieces of $\Sigma$, respectively. On $\Sigma_\pm$ the length of the normal can be chosen to be unity: $t^a := \pm \frac{1}{\sqrt{2|H^a + \bar{G} \bar{G}|}} (N^a - (H^a + \bar{G} \bar{G}) L^a)$, and in the null case $t^a$ will be chosen to be $N^a$. On $\Sigma_\pm$ $P_a^b := \frac{\partial}{\partial \zeta}$ $t_a^b$ is the projection, and let $Z^a := P_a^b L^b = \frac{1}{2}(L^a + \frac{1}{2}(H^a + \bar{G} \bar{G}) N^a)$. Its norm is $|Z^a|^2 := \frac{1}{2} (H^a + \bar{G} \bar{G})$. Let $I_A'$ be the (uniquely determined) spinor field for which $\lambda \bar{A} I_A'^A = 1$ and $N_a' = I_A' \bar{I}_A$ hold and $I A \bar{A}'$, $\lambda \bar{A}'$ are tangent to the (flat) 2-surfaces $u = \text{const}$, $v + V(\zeta, \bar{\zeta}, u) = \text{const}$. Then it is easy to show that $I_A' = I_A + (\frac{\partial}{\partial \zeta}) \lambda A$; which implies that $M^a = \lambda A I_A'^A$, $\bar{M}^a = I A \bar{A}'$ and that $I_A'$ is Lie propagated along $L^a$. The only nonzero components of the curvature $R_{AB\ldots N}^a$ of $g_{ab}'$ in the basis $(\lambda A, I_A')$ are $\phi' := I^A I^B R_{AB\ldots N}^a M^d = I^A I^B R_{AB\ldots N}^a M^d = I^A I^B R_{AB\ldots N}^a M^d = \phi$ and $\psi' := I^A I^B R_{AB\ldots N}^a M^d = \psi$; i.e. (4.11) remains valid for the primed $H'$ and $G'$ on the right too. By (4.8) there exists a $V$ for which $G' = 0$ (‘canonical gauge’ [9,10]). In general however this $V$ is not zero on $\mathcal{S}$; i.e. such a transformation would deform the boundary $\mathcal{S}$ too. Since the boundary is fixed in our problem...
and the boundary values for the metric of $D(\Sigma)$ are given on $\Sigma$, we should find a weaker gauge condition for $G'$ such that the corresponding diffeomorphism $\nu$ leaves $\Sigma$ fixed.

A weaker gauge condition might be the requirement of the holomorphicity of $G'$. The condition of the existence of a transformation of the form above yielding holomorphic $G'$ is $\frac{\partial G^a}{\partial \nu^\lambda} = -\frac{\partial \zeta^a}{\partial \nu^\lambda}$. By (4.8) this is a Poisson equation for $V$ with real source, which always has a unique solution with the boundary condition $V|_\Sigma = 0$. Thus there exists a diffeomorphism $\nu$ such that $G'$ in $g'_{ab} = V^a(g_{ab})$ is holomorphic (‘holomorphic gauge’). In this gauge (4.11) reduces to $\phi = \frac{\partial H'}{\partial \nu^\lambda}$ and $\psi + \frac{\partial \zeta^a}{\partial \nu^\lambda} = \frac{\partial H'}{\partial \nu^\lambda}$. Since however $G'$ is holomorphic, it is determined by its value on $\Sigma$; and hence $\phi, \psi$ and the value of $H'$ and $G'$ on $\Sigma$ determine $H'$ and $G'$ on the whole coordinate domain. What remained to show is that $H'|_\Sigma$ and $G'|_\Sigma$ are determined by the Sen geometry of $\Sigma$ and the boost gauge on $\Sigma$ defined by $\Sigma$.

Since $V|_\Sigma = 0$ the 2-surface coordinate vectors can be expressed in terms of $M'^a, \bar{M}'^a$ and $(\frac{\partial}{\partial \nu^\lambda})^a$ too:

$$\frac{\partial}{\partial \nu^\lambda} \bar{G}' = \frac{\partial}{\partial \nu^\lambda} \bar{G}' = \frac{\partial}{\partial \nu^\lambda} M'^a + \frac{\partial}{\partial \nu^\lambda} \bar{M}'^a + (\frac{\partial}{\partial \nu^\lambda})^a \frac{\partial}{\partial \nu^\lambda} M'^a + \frac{\partial}{\partial \nu^\lambda} \bar{M}'^a.$$  

The first implies that $G' = \lambda A \bar{I}'_A \left( \frac{\partial}{\partial \nu^\lambda} \bar{G}' + \frac{\partial}{\partial \nu^\lambda} \bar{G}' \right)$ and $H' = I'_A \bar{I}'_A \left( \frac{\partial}{\partial \nu^\lambda} \bar{G}' \right) - G' \bar{G}'$. Thus we should show that $P^A$ is determined by the Sen geometry and the boost gauge. Since $\gamma_{AB} \lambda A \lambda B$ is nonzero on $\Sigma$, $\bar{A}_A \bar{I}'_A \lambda A \lambda B$ span the spin space at each point of $\Sigma = \{ \bar{p}_k \}$. Thus by the continuity of $P^A$ and by $\lambda A I^A$ is 1 the spinor $P^A$ on $\Sigma$ is determined by $\gamma_{AB} \lambda A \lambda B$. We will show that $\gamma_{AB} \lambda A \lambda B$ is determined by the boost gauge on $\Sigma$. Since $\left( \frac{\partial}{\partial \nu^\lambda} \bar{G}' \right) = \frac{\beta}{\bar{\beta}}_u - \frac{\bar{\beta}}{\beta}_a M'_{\lambda\beta} = \frac{\beta}{\bar{\beta}}_u M'_{\lambda\beta}$ and by $0 = \frac{\beta}{\bar{\beta}}_u N'_{a} = \frac{1}{2 \| z' \|^2} \left( \gamma_{AB} \lambda A \lambda B - \lambda A \lambda B \right)$ the complex scalar product $\gamma_{AB} \lambda A \lambda B$ is real. Since $z' = \frac{1}{2} (\lambda A \lambda B - \lambda A \lambda B)$ is $\| z' \|$ times the unit vector orthogonal to $\frac{\beta}{\bar{\beta}}_u M'_{\lambda\beta}$ it has the form $z' = \| z' \|^2 (N_{\lambda} + (H' + G' \bar{G}') L') + \frac{1}{2} (\lambda A \lambda B \lambda A \lambda B) + \gamma_{AB} \lambda A \lambda B (G' - \frac{\partial}{\partial \nu^\lambda} \bar{G}') (\bar{G}' - \frac{\partial}{\partial \nu^\lambda} \bar{G}')$. Its contraction with $M'_{\lambda\beta}$ and $N'_{a}$, respectively, are zero.

\begin{equation}
\gamma_{AB} \lambda A \lambda B \left( \frac{\partial}{\partial \nu^\lambda} \bar{G}' + \bar{G}' \right) + \gamma_{AB} \lambda A \lambda B \left( \frac{\partial}{\partial \nu^\lambda} \bar{G}' - \bar{G}' \right) = -2 \gamma_{AB} \lambda A \lambda B, \tag{4.12}
\end{equation}

\begin{equation}
-2 \| z' \|^2 (H' + G' \bar{G}') = 1 - \gamma_{AB} \lambda A \lambda B. \tag{4.13}
\end{equation}

By (4.13) a point $p \in \Sigma$ is in $\Sigma_+ := (\Sigma - \{ \bar{p}_k \}) \cap \Sigma_+ \iff \gamma_{AB} \lambda A \lambda B | < 1$ at $p, p \in \Sigma_0 := (\Sigma - \{ \bar{p}_k \}) \cap \Sigma_0 \iff \gamma_{AB} \lambda A \lambda B | < 1$ and $p \in \Sigma_- := (\Sigma - \{ \bar{p}_k \}) \cap \Sigma_- \iff \gamma_{AB} \lambda A \lambda B | > 1$. Taking into account (4.12) and (4.13) $\frac{\beta}{\bar{\beta}}_u$ and $z'$ will have the form

\begin{equation}
\frac{\beta}{\bar{\beta}}_u = \frac{1}{2 \| z' \|^2} \left( \gamma_{C'D'} \lambda C' \lambda D' M'_{\lambda} - \gamma_{CD} \lambda C \lambda D \bar{M}'_{\lambda} \right), \tag{4.14}
\end{equation}

\begin{equation}
z' = -\| z' \|^2 \left( N_{\lambda} + (H' + G' \bar{G}') L' \right) + \frac{1}{2} \gamma_{RS} \lambda R \lambda S \left( \gamma_{C'D'} \lambda C' \lambda D' M'_{\lambda} + \gamma_{CD} \lambda C \lambda D \bar{M}'_{\lambda} \right). \tag{4.15}
\end{equation}

Suppose first that $p \in \Sigma_+$. Then by means of the norm $\| Z' \|$ (4.13) can be rewritten as $1 - \gamma_{AB} \lambda A \lambda B |^2 = \pm \frac{| z' \|^2}{\| z' \|^2}$. Then the unit normal of $\Sigma_+ \in \Sigma_+$ is

\begin{equation}
v^a = \frac{1}{\| Z' \|} \left( \gamma_{CD} \lambda C' \lambda D' M'_{\lambda} \right) + \frac{1}{2} \left( \gamma_{RS} \lambda R \lambda S \right) \left( \gamma_{CD} \lambda C \lambda D \bar{M}'_{\lambda} \right). \tag{4.16}
\end{equation}

A straightforward calculation shows that $t' e c a b \varepsilon_{e a c b} \frac{\partial}{\partial \nu^\lambda} z' = b \| Z' \| v^a$. To determine the orientation of the spacelike normals $v^a$ on $\Sigma_+$ and $t^a$ on $\Sigma_-$ recall that the induced volume form $\varepsilon_{a b}$ on $\Sigma$ is defined by $X Y f' e c f_{a b}$ for any future directed unit timelike $X'$ and outward directed spacelike $Y'$ for which $X Y f = 0$, and observe that $v^a$ is future directed and timelike on $\Sigma_+$. Thus the spacelike normals on $\Sigma_+ \in \Sigma_+$ are outward directed, hence $(t^a, v^a)$ and $(v^a, t^a)$ are the proper frames representing the boost gauge on $\Sigma_+$ and $\Sigma_-$, respectively, defined by $\Sigma'$. Then it is easy to see that
\[
\frac{v'_a L^a}{L_b I^b} = -\gamma_{AB} \lambda^A I^B, \quad (4.17)
\]
i.e. \(\gamma_{AB} \lambda^A I^B\) is completely determined by the boost gauge on \(\Sigma\). Finally, as we saw, if \(p \in \Sigma_0\) then \(\gamma_{AB} \lambda^A I^B = \pm 1\) there, and it is easy to show that \(t^a = N^a\) is an ingoing null normal to \(\Sigma_0\) if \(\gamma_{AB} \lambda^A I^B = 1\), and \(t^a\) is outgoing if \(\gamma_{AB} \lambda^A I^B = -1\).

5. Discussion, conclusions and remarks

5.1

From the argumentation following eq. (4.7) it is clear that our convexity condition implies the convexity conditions of Dougan and Mason; i.e. the outgoing null normals are not converging and the ingoing null normals are not diverging on \(\Sigma\). Thus if the convexity condition of Dougan and Mason is replaced by our (stronger) condition in the Theorem of the introduction then the statements remain true, but in addition the line element of \(D(\Sigma)\) can be determined from the data given on \(\Sigma\). We would like to stress, however, that the stronger convexity condition was used in the proof of Proposition 4.6 only to show that the Riemann surfaces \(S_u\) and their boundary \(B_u\) have the simplest possible topological structure. Nevertheless an antiholomorphic function on a Riemann surface \(S_u\) is determined by its value on the boundary \(B_u\) of \(S_u\) even if \(B_u\) has much more complicated structure. Thus one might be able to determine the metric on \(D(\Sigma)\) even if this convexity condition is weakened.

5.2

Although the energy-momentum and angular momentum (and their Casimirs, the mass and the spin) are among the most important quantities of physics, in general relativity it is not obvious how they should be defined. For the (quasi-local) energy-momentum the expression of Dougan and Mason seems promising since among the most important quantities of physics, \(\gamma_{AB} \lambda^A I^B\) is the quasi-local energy-momentum and angular momentum of the matter fields associated with \(\Sigma\), respectively. If \(t^a\) is spacelike for some \(a \in \Sigma\) then \(\gamma_{AB} \lambda^A I^B = \pm 1\) there, and it is easy to show that \(t^a = N^a\) is an ingoing null normal to \(\Sigma_0\) if \(\gamma_{AB} \lambda^A I^B = 1\), and \(t^a\) is outgoing if \(\gamma_{AB} \lambda^A I^B = -1\).

For a moment suppose that \(m = 0\) then \(\gamma_{AB} \lambda^A I^B\) is completely determined by the boost gauge on \(\Sigma\). Finally, as we saw, if \(p \in \Sigma_0\) then \(\gamma_{AB} \lambda^A I^B = \pm 1\) there, and it is easy to show that \(t^a = N^a\) is an ingoing null normal to \(\Sigma_0\) if \(\gamma_{AB} \lambda^A I^B = 1\), and \(t^a\) is outgoing if \(\gamma_{AB} \lambda^A I^B = -1\).
a future directed nonspacelike vector and it is null iff the matter is pure radiation; i.e. $T^{ab} = tk^n k^b$ for some nonnegative function $t$ and constant null vector field $k^a$ (see [4]). Then for null $P^a_\Sigma$ we have $P^a_\Sigma = I_S k^a$ and $J^a_\Sigma P^a_\Sigma = -I_S J_\Sigma P^a_\Sigma$, where $k^a := k^n K_n^a$, $I_S := \int_{\Sigma} tk^n n_c d\Sigma$ and $J_\Sigma := \int_{\Sigma} tk^n e_k x^k d\Sigma$. Thus for pure radiation $P^a_\Sigma$ is null and is an eigenvector of the quasi-local angular momentum. Since in this argumentation we have not used any specific properties of the fields, e.g. the field equations, similar properties for the gravitational energy-momentum and angular momentum may also be expected.

For the sake of simplicity let us consider axially symmetric $pp$-wave Cauchy developments, suppose that the 2-surface $\$ lies in the $v = \text{const}$ hypersurface of the canonical coordinate system, the Killing vector $X^a$ of the axial symmetry is tangent to $\$ on $\$ and calculate the angular momentum according to (5.1). If we assume that $D(\Sigma)$ does not admit an additional timelike Killing symmetry then the constant null vector field $L^a$ must commute with $X^a$ and $L^a X^a = 0$ must hold [19]. Then $X^a = (\frac{\partial}{\partial w})^a$, $\gamma_{AB} \lambda^A \lambda^B = \sqrt{2} \|z^e\| h^{-1} \zeta = \sqrt{2} \|z^e\| \exp(-iw)$ and by $L_X \|z^e\|^2 = -L^a L^b L_X q_{ab} = 0$ the norm $\|z^e\|$ does not depend on $w$. This implies that $J^0_\Sigma$ is zero, and hence $J^0_\Sigma J^0_\Sigma = -P^a_\Sigma (J^0_\Sigma + J^0_\Sigma^{1\prime})$ and $S^a_\Sigma = P^a_\Sigma i (J^0_\Sigma - J^0_\Sigma^{1\prime})$. Thus for (axially symmetric) $pp$-waves the Ludvigsen–Vickers definition (5.1) together with the Dougan–Mason propagation law for the spinor fields $\lambda^A_X$ yields physically reasonable results.

5.3

One of the most important principles of (classical and quantum) physics is the locality [20]. In local quantum field theory one associates a net of $C^*$-algebras \( \{A(U_\alpha)\} \) of quantum observables with every covering \( \{U_\alpha\} \) of the spacetime manifold \( M \), where the subsets \( U_\alpha \) are open and have compact closure; and certain axioms are expected to hold for the net \( \{A(U_\alpha)\} \). Although the covering \( \{U_\alpha\} \) may otherwise be arbitrary, it seems natural to choose the subsets \( U_\alpha \) to be finite Cauchy developments \( \text{int} D(\Sigma_\alpha) \) of finite spacelike hypersurfaces \( \Sigma_\alpha \). In fact, any spacetime admits a countable covering consisting of such globally hyperbolic open domains; and one can construct the quasi-local phase space of the fields and gravity. Hence one may hope to be able to construct the quasi-local $C^*$-algebras $A(\text{int} D(\Sigma_\alpha))$. However in the light of the result of the present paper the plane wave configurations both in electromagnetism and Einstein’s gravity can be specified by certain fields on the smooth 2-surfaces \( \$ : = \partial \Sigma_\alpha \); and hence the quasi-local algebra $A(\text{int} D(\Sigma_\alpha))$ would in fact be associated with the 2-surface \( \$ \). This result would provide a new example for the distinguished role of 2-surfaces in fundamental physics [21,22].

5.4

Because of the nonlinearity of Einstein’s equations it is difficult to define the radiative modes of general relativity. It can be done in the weak field approximation [23], for $pp$-waves and at null infinity [24]; i.e. when the field equations become linear, and some absolute structure (flat background, the space of anti-holomorphic spinor fields and the universal structure, respectively) is available. It is, however, a remarkable property of the Dougan–Mason energy-momentum $P^a_\Sigma$ that it tends to the Bondi-Sachs energy-momentum $BS P^a_\$ if $\$ tends to the spherical cut $\$ of future null infinity provided the anti-holomorphic spinor fields are used to define $P^a_\$. If however the holomorphic spinor fields are used then in general $P^a_\$ tends to infinity, whilst in stationary spacetimes (i.e. in absence of radiation) it tends to the Bondi-Sachs energy-momentum [17]. Therefore both at null infinity and in the $pp$-wave case it is the deviation of the holomorphic and anti-holomorphic structures of 2-spheres that characterizes the presence of radiation. The deviation of these structures can however be defined for generic 2-spheres in generic spacetimes too, yielding the possibility of finding the unconstrained (i.e. radiative) modes of gravity at the quasi-local level.

A systematic and more detailed discussion of the quasi-local energy-momentum and angular momentum (5.2), the quasi-local phase space and a ‘quasi-local quantization’ both of electromagnetism and the $pp$-wave configurations of general relativity (5.3), and the quasi-local radiative modes of general relativity (5.4) will be published in separate papers.
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