Abstract. We provide a theoretical framework for neural networks in terms of the representation theory of quivers, thus revealing symmetries of the parameter space of neural networks. An exploitation of these symmetries leads to a model compression algorithm for radial neural networks based on an analogue of the QR decomposition. A projected version of backpropagation on the original model matches usual backpropagation on the compressed model.

1. Introduction

Recent work has shown that representation theory, the formal study of symmetry, provides the foundation for various innovative techniques in deep learning [CW16, KT18, RSP17, CW17]. Much of this previous work considers symmetries inherent to the input and output spaces, as well as distributions and functions that respect these symmetries. By contrast, in this paper, we expose a broad class of symmetries intrinsic to the parameter space of the neural networks themselves. We use these symmetries to devise a model compression algorithm that reduces the widths of the hidden layers, and hence the number of trainable parameters. Unlike representation-theoretic techniques in the setting of equivariant neural networks, our methods are applicable to deep learning models with non-symmetric domains and non-equivariant functions, and hence pertain to some degree to all neural networks.

Specifically, we formulate a theoretical framework for neural networks in terms of the theory of representations of quivers, a mathematical field with connections to symplectic geometry and Lie theory [KJ16, N+98]. This approach builds on that of Armenta and Jodoin [AJ20] and of Wood and Shawe-Taylor [WST96], but is arguably simpler and encapsulates larger symmetry groups. Formally, a quiver is another name for a directed graph, and a representation of a quiver is the assignment of a vector space to each vertex and a linear map to each edge, where the source and target of the linear map are the vector spaces assigned to the source and target vertices of the edge. Representations of quivers carry rich symmetry groups via change-of-basis transformations of the vector spaces assigned to the vertices.

Our starting point is to regard the vector space of parameters for a neural network\(^1\) with \(L\) layers as a representation of a specific quiver, namely, the neural quiver:

\[
\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet
\]

with \(L + 1\) vertices in the top row and a ‘bias vertex’ at the bottom. As such, the parameter space carries a change-of-basis symmetry group. Factoring out this symmetry leads to a reduced parameter space without affecting the feedforward function. The size of the symmetry group

\(^1\)Unless specified otherwise, the term ‘neural network’ exclusively refers to a multilayer perceptron (MLP).
is determined by properties of the activation functions. We focus on the case of radial activation functions, as in [WC19, SFH17, WGW+18, WHS18]; these interact favorably with certain QR decompositions, and, consequently, the model compression is significant compared to the more common pointwise (also known as 'local') activations. We refer to neural networks with radial activation functions as radial neural networks.

Given a radial neural network, our results produce a neural network with fewer neurons in each hidden layer and the same feedforward function, and hence the same loss for any batch of training data. Moreover, we prove that the value of the loss function after a step of gradient descent applied to the compressed model is the same as the value of the loss function after a step of projected gradient descent applied to the original model. As we explain, projected gradient descent is a version of gradient descent where one subtracts a truncation of the gradient, rather than the full gradient. Admittedly, training the original model often leads to better performance after fewer epochs; however, when the compression is significant enough, the compressed model takes less time per epoch to train and reaches local minima faster.

To state these results slightly more precisely, recall that the parameters of a neural network with layer widths \((n_0, n_1, \ldots, n_L)\) consist of an \(n_i \times (n_{i-1} + 1)\) matrix \(W_i\) of weights for each layer \(i\), where we include the bias as an extra column. These are grouped into a tuple \(W = (W_i \in \mathbb{R}^{n_i \times (n_{i-1} + 1)})_i\). We define the reduced widths recursively as \(n_i^{\text{red}} = \min(n_i, n_{i-1}^{\text{red}} + 1)\) for \(i = 1, \ldots, L - 1\), with \(n_0^{\text{red}} = n_0\) and \(n_L^{\text{red}} = n_L\). Note that \(n_i^{\text{red}} \leq n_i\) for all \(i\).

**Theorem 1.1** (Informal version of Theorems 4.6 and 5.4). Suppose a neural network has \(L\) layers with widths \((n_0, \ldots, n_L)\), parameters \(W\), and radial activation functions. Let \(f_W : \mathbb{R}^{n_0} \to \mathbb{R}^{n_L}\) be the feedforward function of the network.

1. There exists a reduced radial neural network with layer widths \((n_0, n_1^{\text{red}}, \ldots, n_{L-1}^{\text{red}}, n_L)\), parameters \(R\), and the same feedforward function \(f_R = f_W\).
2. Training \(f_R\) with gradient descent is an equivalent optimization problem to training \(f_W\) with projected gradient descent.

This theorem can be interpreted as a model compression result: the reduced (or compressed) neural network has the same accuracy as the original neural network, and there is an explicit relationship between the gradient descent optimization problems for the two neural networks. We now state a simplified version of the algorithm used to compute the reduced neural network, which amounts to layer-by-layer computations of QR decompositions:

**Algorithm 1: QR Dimensional Reduction (simplified version of Algorithm 2)**

- **input:** Neural network: (layer widths \((n_0, \ldots, n_L)\), weights \(W = (W_i \in \mathbb{R}^{n_i \times n_{i-1}})_{i=1}^L\))
- **for** \(i \leftarrow 1\) **to** \(L - 1\) **do**
  - \(Q_i, R_i \leftarrow \text{QR-decomp}(W_i)\) // QR decomposition of the matrix \(W_i\)
  - \(W_{i+1} \leftarrow W_{i+1}Q_i\) // Update \(W_{i+1}\)
- **end**
- \(R_L \leftarrow W_LQ_{L-1}\) // Define \(R_L\)
- **return:** Neural network: (layer widths \((n_0, n_1^{\text{red}}, \ldots, n_{L-1}^{\text{red}}, n_L)\), weights \(R = (R_i)^L_{i=1}\))
We view this work as a step in the direction of improving learning algorithms by exploiting symmetry inherent to neural network parameter spaces. As such, we expect our framework and results to generalize in several ways, including: (1) further reductions of the hidden widths, (2) incorporating certain non-radial activation functions, (3) encapsulating neural networks beyond MLPs, such as convolutional, recurrent, and graph neural networks, (4) integration of regularization techniques. We postpone a detailed discussion of future directions to the end of the paper (Section 7.3).

Our contributions are as follows:

1. We provide a theoretical framework for neural networks based on the representation theory of quivers.
2. We derive a QR decomposition for radial neural networks and prove its compatibility with (projected) gradient descent.
3. We implement a lossless model compression algorithm $W \mapsto R$ for radial neural networks.

1.1. Related work.

**Quiver representation theory and neural networks.** Armenta and Jodoin [AJ20] establish an approach to understanding neural networks in terms of quiver representations. Our work generalizes their approach as it (1) encapsulates both pointwise and non-pointwise activation functions, (2) taps into larger symmetry groups, and (3) connects more naturally to gradient descent. Jeffreys and Lau [JL21] also place quiver varieties in the context of machine learning, and define a Kähler metric in order to perform gradient flow. Manin and Marcolli [MM20] advance the study of neural networks using homotopy theory, and the “partly oriented graphs” appearing in their work are generalizations of quivers. In comparison to the two aforementioned works, our approach is inspired by similar algebro-geometric and categorical perspectives, but our emphasis is on practical consequences for optimization techniques at the core of machine learning.

Although the study of neural networks in the context of quiver representations emerged recently, there are a number of precursors. One is the study and implementation of the “non-negative homogeneity” (also known as “positive scaling invariance”) property of ReLU activation functions [DPBB17, NSS15, MZZ+19], which is a special case of the symmetry studied in this paper. Wood and Shawe-Taylor [WST96] regard layers in a neural network as representations of finite groups and restrict their attention to the case of pointwise activation functions; by contrast, our framework captures Lie groups as well as non-pointwise activation functions. Our quiver approach to neural networks shares similarities with the “algebraic neural networks” of Parada-Mayorga and Ribeiro [PMR], and special cases of their formalism amount to representations of quivers over base rings beyond $R$, such as the ring of polynomials $R[t]$. In a somewhat different data-analytic context, Chindris and Kline [CK21] use quiver representation theory in order to untangle point clouds, though they do not use neural networks or machine learning.

**Equivariant neural networks.** Previously, representation theory has been used to design neural networks which incorporate symmetry as an inductive bias. A variety of architectures such as G-convolutions, steerable CNNs, and Clebsch-Gordon networks are constrained by various weight-sharing schemes to be equivariant or invariant to various symmetry groups [CWWK19, WC19, CW16, CDM18, KT18, BS19, WGTB17, CW17, WHS18, DFK16, LW21, RSP17]. Our approach,
in contrast, does not rely on symmetry of the input domain, output space, or mapping. Rather, our method exploits symmetry of the parameter space and thus applies more generally to domains with no obvious symmetry.

**Model compression and weight pruning.** In general, model compression aims to exploit redundancy in neural networks; enormous models can be reduced and run on smaller systems with faster inference [BCNM06, CWZZ17, FC18, ZYZ+18]. Whereas previous approaches to model compression are based on weight pruning, quantization, matrix factorization, or knowledge distillation, we take a fundamentally different approach by exploiting symmetries of neural networks parameter spaces.

1.2. **Organization of the paper.** This paper is a contribution to the mathematical foundations of machine learning, and hence uses precise mathematical formalism throughout. At the same time, our results are motivated by expanding the applicability and performance of neural networks. We hope our work is accessible to both machine learning researchers and mathematicians – allowing the former to recognize the practical potential of representation theory, while giving the latter a glimpse into the rich structure underlying neural networks.

Section 2 consists of preliminary material. We review the necessary background on linear algebra and group theory in Section 2.1. We recall basic facts related to the QR decomposition in Section 2.2. Section 2.3 serves to establish notation used in the context of gradient descent, and to prove an interaction between gradient descent and orthogonal transformations (Proposition 2.5).

In Section 3, we delve into quiver representation theory. We first provide the basic definitions (Section 3.1) before focusing on a specific quiver, called the neural quiver (Section 3.2). Finally, we give a formulation of neural networks in terms of quiver representations (Definition 3.2 in Section 3.3), and explain the sense in which the backpropogation training algorithm can be regarded as taking place on the vector space of representations of the neural quiver.

Section 4 contains the first set of main results of this paper. We recall the definitions of radial functions and radial neural networks (Section 4.1). In Section 4.2, we introduce the notion of a reduced dimension vector for any dimension vector of the neural quiver. This reduced dimension vector features in a QR decomposition (Theorem 4.6) for radial neural networks; we also provide an algorithm to compute the QR decomposition (Algorithm 2).

Section 5 contains the second set of main results of this paper, which are related to projected gradient descent. We first introduce an ‘interpolating space’ that relates the space of representations with a given dimension vector to the space of representations with the reduced dimension vector (Section 5.1). Using the interpolating space, we define a projected version of gradient descent (Definition 5.3 in Section 5.2) and state a result on the relationship between projected gradient descent and the QR decomposition for radial neural networks (Theorem 5.4).

In Section 6, we summarize implementations that empirically verify our main results (Sections 6.1 and 6.2) and demonstrate that reduced models train faster (Section 6.3).

Section 7 is a discussion section that includes a version of our results for neural networks with no biases (Section 7.1), a generalization involving shifts in radial functions (Section 7.2), and an overview of future directions (Section 7.3).
Finally, in Appendix A, we include a formulation and generalization of our results using the language of category theory.

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2. Preliminaries

In this section, we first review basic notation and concepts from linear algebra, group theory, and representation theory. We then state a version of the QR decomposition. Finally, we formulate a definition of the gradient descent map with respect to a loss function. References include [PPL+12], [QSS07], and [DF03].

2.1. Linear algebra. For positive integers $m$ and $n$, let $\operatorname{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ or $\mathbb{R}^{m \times n}$ denote the vector space of $n$ by $m$ matrices with entries in $\mathbb{R}$, that is, the vector space of linear functions (also known as homomorphisms) from $\mathbb{R}^m$ to $\mathbb{R}^n$. The general linear group $\operatorname{GL}_n(\mathbb{R})$ consists of the set of invertible $n$ by $n$ matrices, with the operation of matrix multiplication. Equivalently, $\operatorname{GL}_n(\mathbb{R})$ consists of all linear automorphisms of $\mathbb{R}^n$, with the operation of composition. Such automorphisms are given precisely by change-of-basis transformations. The unit is the identity $n$ by $n$ matrix, denoted $\text{id}_n$. The orthogonal group $O(n)$ is the subgroup of $\operatorname{GL}_n(\mathbb{R})$ consisting of all matrices $Q$ such that $Q^TQ = \text{id}_n$. Such a matrix is called an orthogonal transformation.

Let $G$ be a group. An action (or representation) of $G$ on the vector space $\mathbb{R}^n$ is the data of an invertible $n$ by $n$ matrix $\rho(g)$ for every group element $g \in G$, such that (1) for all $g, h \in G$, the matrix $\rho(gh)$ is the product of the matrices $\rho(g)$ and $\rho(h)$, and (2) for the identity element $1_G \in G$, we have $\rho(1_G) = \text{id}_n$. In other words, an action amounts to a group homomorphism $\rho : G \to \operatorname{GL}_n(\mathbb{R})$. We often abbreviate $\rho(g)(v)$ as simply $g \cdot v$, for $g \in G$ and $v \in \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ (non-linear in general) is invariant for the action of $G$ if $f(g \cdot v) = f(v)$ for all $g \in G$ and $v \in \mathbb{R}^n$.

Suppose $m \leq n$. Then $\text{inc}_{m,n} : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ denotes the standard inclusion into the first $m$ coordinates:

$$(v_1, \ldots, v_m) \mapsto (v_1, \ldots, v_m, 0, \ldots, 0)$$

The standard projection $\pi_{n,m} : \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m$ onto the first $m$ coordinates is defined as:

$$(v_1, \ldots, v_m, v_{m+1}, \ldots, v_n) \mapsto (v_1, \ldots, v_m)$$

An affine map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is one of the form $x \mapsto Ax + b$ for a $n \times m$ matrix $A$ and a $n$-dimensional vector $b$. The set of affine maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be identified with the vector space $\operatorname{Hom}(\mathbb{R}^{m+1}, \mathbb{R}^n)$, as we now explain. First, given $A$ and $b$ as above, we form the $n \times (m+1)$ matrix $[b \ A]$. Conversely, the affine map corresponding to $W \in \operatorname{Hom}(\mathbb{R}^{m+1}, \mathbb{R}^n)$ is given by

$$\tilde{W} = W \circ \text{ext}_m : \mathbb{R}^m \to \mathbb{R}^n; \quad x \mapsto W(\text{ext}_m(x)),$$

where $\text{ext}_m$ is defined as:

$$\text{ext}_m : \mathbb{R}^m \to \mathbb{R}^{m+1}; \quad (v_1, v_2, \ldots, v_m) \mapsto (1, v_1, v_2, \ldots, v_m) \in \mathbb{R}^{m+1}. $$
2.2. The QR decomposition. In this section, we recall the QR decomposition and note several relevant facts. For integers $n$ and $m$, let $\text{Hom}^{\text{upper}}(\mathbb{R}^m, \mathbb{R}^n)$ denote the vector space of upper triangular $n$ by $m$ matrices.

**Theorem 2.1 (QR Decomposition).** The following map is surjective:

$$O(n) \times \text{Hom}^{\text{upper}}(\mathbb{R}^m, \mathbb{R}^n) \rightarrow \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

$$Q, R \rightarrow Q \circ R$$

In other words, any matrix can be written as the product of an orthogonal matrix and an upper-triangular matrix. When $m \leq n$, the last $n - m$ rows of any matrix in $\text{Hom}^{\text{upper}}(\mathbb{R}^m, \mathbb{R}^n)$ are zero, and the top $m$ rows form an upper-triangular $m$ by $m$ matrix. These observations lead to the following special case of the QR decomposition:

**Corollary 2.2.** Suppose $m \leq n$. The following map is surjective:

$$\mu : O(n) \times \text{Hom}^{\text{upper}}(\mathbb{R}^m, \mathbb{R}^m) \rightarrow \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$$

$$Q, R \rightarrow Q \circ \text{inc}_{m,n} \circ R$$

We make some remarks:

1. There are several algorithms for computing the QR decomposition of a given matrix. One is Gram–Schmidt orthogonalization, and another is the method of Householder reflections. The latter has computational complexity $O(n^2m)$ in the case of a $n \times m$ matrix with $n \geq m$. The package numpy includes a function `numpy.linalg.qr` that computes the QR decomposition of a matrix using Householder reflections.

2. The QR decomposition is not unique in general, or, in other words, the map $\mu$ is not injective in general. For example, if $n > m$, each fiber of $\mu$ contains a copy of $O(n - m)$.

3. The QR decomposition is unique (in a certain sense) for invertible square matrices. To be precise, let $B^+_n$ be the subset of $\text{Hom}^{\text{upper}}(\mathbb{R}^n, \mathbb{R}^n)$ consisting of upper triangular matrices with positive entries along the diagonal. If $n = m$, then both $B^+_n$ and $O(n)$ are subgroups of $\text{GL}_n(\mathbb{R})$, and the multiplication map $O(n) \times B^+_n \rightarrow \text{GL}_n(\mathbb{R})$ is bijective. However, the QR decomposition is not unique for non-invertible square matrices.

2.3. Gradient descent. In this section, we establish a gradient descent formalism that will appear throughout the paper. We explain the interaction between gradient descent and orthogonal transformations, which will be a key aspect of our main results.

Let $V = \mathbb{R}^p$ and $L : V \rightarrow \mathbb{R}$ be a differentiable function\(^2\). Semantically, these are the parameter space and loss function. We write elements of $V$ as tuples, e.g. $v = (v_1, \ldots, v_p)$, or as column vectors. We write elements of the dual vector space $V^*$ as row vectors. The differential and gradient of $L$ are defined as:

$$dL : V \rightarrow \text{Hom}(V, \mathbb{R}) = V^*$$

$$v \mapsto d_v L = \left[ \frac{\partial L}{\partial x_1} \bigg|_v \cdots \frac{\partial L}{\partial x_p} \bigg|_v \right]$$

$$\nabla L : V \rightarrow V$$

$$v \mapsto \nabla_v L = \left( \frac{\partial L}{\partial x_1} \bigg|_v, \ldots, \frac{\partial L}{\partial x_p} \bigg|_v \right)$$

\(^2\)This discussion extends easily to the case where $L$ is piece-wise differentiable.
Remark 2.3. Suppose that \( \mathcal{L} \) is in fact a linear map. Then \( \mathcal{L} \) belongs to \( V^* \) and \( d\mathcal{L} \) is the constant map taking each \( v \in V \) to \( \mathcal{L} \). That is, \( d_v \mathcal{L} = \mathcal{L} \) as maps \( V \to \mathbb{R} \).

A step of gradient descent with respect to \( \mathcal{L} \) is the function
\[
\gamma : V \to V \quad v \mapsto v - \nabla_v \mathcal{L}
\]
Hence, gradient descent updates the \( i \)-th coordinate \( v_i \) of \( v \) to \( v_i - \frac{\partial \mathcal{L}}{\partial x_i} \big|_{v} \), for \( i = 1, \ldots, p \). For any \( k \geq 0 \), the \( k \)-fold composition of \( \gamma \) is \( \gamma^k := \gamma \circ \gamma \circ \cdots \circ \gamma \), with \( \gamma^0 \) being the identity map on \( V \).

Remark 2.4. More generally, a step of gradient descent with respect to \( \mathcal{L} \) at learning rate \( \eta > 0 \) is the function \( \gamma_\eta : V \to V \) taking \( v \) to \( v - \eta \nabla_v \mathcal{L} \). In what follows, we consider only the case \( \eta = 1 \) for the sole reason of easing notation. All results hold for general \( \eta > 0 \) and can be obtained either by inserting the scalar \( \eta \) where necessary, or by rescaling the function \( \mathcal{L} \).

Proposition 2.5. Let \( Q \) be an orthogonal transformation of \( V = \mathbb{R}^p \) such that \( \mathcal{L} \circ Q = \mathcal{L} \). Then, for any \( v \in V \) and \( k \geq 0 \), we have:
\[
Q(\nabla_v \mathcal{L}) = \nabla_{Q(v)} \mathcal{L} \quad \text{and} \quad \gamma^k(Q(v)) = Q(\gamma^k(v)).
\]

Proof. The hypothesis \( \mathcal{L} \circ Q = \mathcal{L} \) implies that:
\[
\nabla_v \mathcal{L} = (d_v \mathcal{L})^T = (d_v (\mathcal{L} \circ Q))^T = (d_{Q(v)} \mathcal{L} \circ Q)^T = (d_{Q(v)} \mathcal{L} \circ Q)^T = Q^T(d_{Q(v)} \mathcal{L})^T = Q^T(\nabla_{Q(v)} \mathcal{L})
\]
where \( d_{Q(v)} Q = Q \) since \( Q \) is a linear map. The fact that \( Q \) is an orthogonal transformation implies that \( Q^{-1} = Q^T \). The first claim follows. For the second claim, we first compute:
\[
\gamma(Q(v)) = Q(v) - \nabla_{Q(v)} \mathcal{L} = Q(v) - Q(\nabla_v \mathcal{L}) = Q(v - \nabla_v \mathcal{L}) = Q(\gamma(v))
\]
Inductive reasoning shows that \( \gamma^k(Q(v)) = Q(\gamma^k(v)) \) for all \( k \geq 0 \).

Proposition 2.6. Suppose that a group \( G \) acts on \( V \) by orthogonal transformations and that \( \mathcal{L} \) is \( G \)-invariant. Then there is a well-defined gradient descent map on the set of \( G \)-orbits on \( V \):
\[
\widetilde{\gamma} : V/G \to V/G.
\]

Figure 1. If the loss is invariant with respect to an orthogonal transformation \( Q \) of the parameter space, then optimization of the neural network by gradient descent is also invariant with respect to \( Q \).

We state the following additional results; these will not be used in what follows. We omit the proofs (which are straightforward).

Proposition 2.6. Suppose that a group \( G \) acts on \( V \) by orthogonal transformations and that \( \mathcal{L} \) is \( G \)-invariant. Then there is a well-defined gradient descent map on the set of \( G \)-orbits on \( V \):
\[
\widetilde{\gamma} : V/G \to V/G.
\]
Proposition 2.7. Let $M$ be a linear endomorphism of $V = \mathbb{R}^p$ such that $L \circ M = L$.

(1) For any $v \in V$, we have: $\nabla_v L = M^T(\nabla_{M(v)} L)$.

(2) If $M = P$ is an orthogonal projection, i.e. $P^2 = P = P^T$, then, for any $v \in V$ and $k \geq 0$, we have:

$$\gamma^k(v) = v - P(v) + \gamma^k(P(v)).$$

3. Neural networks as quiver representations

We turn our attention to the theory of quiver representations, which, although long used in Lie theory and symplectic geometry, has only recently been connected to deep learning. Neural networks may be viewed as representations of a quiver with the additional data of activation functions; moreover, backpropogation can also be formulated in terms of quiver representations.

3.1. Quiver representations. A quiver $Q$ consists of a set $I$ of vertices and a set $E$ of directed edges. Thus, $Q$ includes the data of maps $s : E \to I$ and $t : E \to I$ indicating the source and target of each edge. A dimension vector for a quiver $Q$ is a map $d : I \to \mathbb{Z}_{\geq 0}$ sending each vertex $i$ to a non-negative integer $d_i = d(i)$. A representation of a quiver $Q$ with dimension vector $d$ consists of the data of a linear map $A_e : \mathbb{R}^{d(s(e))} \to \mathbb{R}^{d(t(e))}$ for each edge $e \in E$. Therefore, a representation is a tuple $A = (A_e)_{e \in E}$ indexed by the set of edges. The set $\text{Rep}(Q, d)$ of all possible representations of a quiver $Q$ with dimension vector $d$ is the direct sum of matrix spaces, and hence a vector space:

$$\text{Rep}(Q, d) = \bigoplus_{e \in E} \text{Hom}(\mathbb{R}^{d(s(e))}, \mathbb{R}^{d(t(e))}).$$

3.2. The neural quiver. We now focus on a specific quiver. Let $L$ be a positive integer. The neural quiver $Q_L$ is the following quiver with $L + 2$ vertices:

\[
\begin{array}{cccccccc}
& \bullet & \rightarrow & \bullet & \rightarrow & \cdots & \rightarrow & \bullet \\
& & & & & & & \\
& & & & & & & \\
& & & & & & & \\
\end{array}
\]

The vertices in the top row are indexed from $i = 0$ to $i = L$. The first vertex ($i = 0$) in the top row is called the input vertex, the last vertex ($i = L$) in the top row is the output vertex, and every other vertex ($1 \leq i \leq L - 1$) in the top row is a hidden vertex. The vertex at the bottom is called the bias vertex and is connected to each vertex in the top row except for $i = 0$.

Notation: We will exclusively consider dimension vectors whose value at the bias vertex is equal to 1. Hence, a dimension vector for $Q_L$ will refer to an $(L + 1)$-tuple of positive integers $n = (n_0, n_1, \ldots, n_L)$.

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3The terms ‘quiver’ and ‘directed graph’ are synonymous; we use ‘quiver’ to emphasize the connection to representation theory literature.
Lemma 3.1. A representation of $Q_L$ with dimension vector $n$ is equivalent to the data of an affine map $\mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i}$ for each $i = 1, \ldots, L$. There is an isomorphism of vector spaces:

$$\text{Rep}(Q_L, n) \simeq \bigoplus_{i=1}^{L} \text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i}).$$

Proof. By definition, a representation of $Q_L$ with dimension vector $n$ consists of the data of a matrix $A_i \in \text{Hom}(\mathbb{R}^{n_i-1}, \mathbb{R}^{n_i})$ and a vector $b_i \in \mathbb{R}^{n_i}$, for each $i = 1, \ldots, L$. For each $i$, the corresponding affine map $\tilde{W}_i : \mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i}$ is given by $x \mapsto A_i x + b_i$. Conversely, given the data of an affine map $\mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i}$, for each $i$, the linear part defines an element of $\text{Hom}(\mathbb{R}^{n_i-1}, \mathbb{R}^{n_i})$ and the translation part defines an element of $\mathbb{R}^{n_i}$. This proves the first claim.

The isomorphism of the second claim follows from the discussion in Section 2.1. Explicitly, given an affine map $x \mapsto A_i x + b_i$ from $\mathbb{R}^{n_i-1}$ to $\mathbb{R}^{n_i}$ for each $i = 1, \ldots, L$, one defines an element of $\bigoplus_{i=1}^{L} \text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i})$ whose $i$-th factor is the matrix $W_i = [b_i, A_i]$. Conversely, pre-composing a matrix in $\text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i})$ with the map $\text{ext}_{n_i-1} : \mathbb{R}^{n_i-1} \to \mathbb{R}^{1+n_i-1}$ (see Equation 2.1) gives an affine map $\tilde{W}_i$ from $\mathbb{R}^{n_i-1}$ to $\mathbb{R}^{n_i}$. \hfill \Box

Notation: We henceforth denote a representation of $Q_L$ as a tuple $W = (W_i)_{i=1}^{L}$, where each $W_i$ belongs to $\text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i})$, and denote the corresponding affine maps as $\tilde{W}_i : \mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i}$.

Recall the group $\text{GL}_n(\mathbb{R})$, consisting of all invertible transformations (equivalently, change-of-basis transformations) of the vector space $\mathbb{R}^n$. Given a dimension vector $n$ for the neural quiver, an element of the group

$$\text{GL}_n^{\text{hidden}} := \prod_{i=1}^{L-1} \text{GL}_{n_i}(\mathbb{R})$$

consists of the choice of a transformation of $\mathbb{R}^{n_i}$ for each hidden vertex $i$, and results in a corresponding transformation of the matrices $W_i$ appearing in any representation $W$ with dimension vector $n$. In other words, we have an action of $\text{GL}_n^{\text{hidden}}$ on $\text{Rep}(Q_L, n)$. To be explicit, a particular choice $g = (g_i)_{i=1}^{L-1}$ of transformations results in the following linear transformation of representations:

$$W = (W_i)_{i=1}^{L} \quad \mapsto \quad g \cdot W := \left( g_i \circ W_i \circ \begin{bmatrix} 1 & 0 \\ 0 & g_i^{-1} \end{bmatrix} \right)_{i=1}^{L},$$

where $g_0 = \text{id}_{n_0}$ and $g_L = \text{id}_{n_L}$.

3.3. Neural networks. In this section, we define neural networks and the backpropagation algorithm in terms of representations of the neural quiver $Q_L$.

Definition 3.2. A neural network\footnote{For each $i$, the matrix $\begin{bmatrix} 1 & 0 \\ 0 & g_i^{-1} \end{bmatrix}$ is block diagonal with one block of size $1 \times 1$ consisting of the single entry ‘1’, and one block of size $n_{i-1} \times n_{i-1}$ consisting of the matrix $g_i^{-1}$.} consists of the following:

1. Hyperparameters. A positive integer $L$ and a dimension vector $n = (n_0, n_1, n_2, \ldots, n_L)$ for the neural quiver $Q_L$.

2. Neural quiver $Q_L$. A directed graph whose vertices are the hidden neurons and the inputs/outputs, the edges are the transformation matrices $W_i$.

3. Neural network $W$. A tuple $W = (W_i)_{i=1}^{L}$, such that $W_i$ is a linear transformation of $\mathbb{R}^{n_i-1} \to \mathbb{R}^{n_i}$ for each $i = 1, \ldots, L$.

4. Neural network $g \cdot W$. A transformation of representations $g \cdot W = (g \circ W_i)_{i=1}^{L}$.

5. Neural network $\text{GL}_n^{\text{hidden}}$. A group of transformations $g$ of dimension vectors $n$.

6. Neural network $g \cdot \text{Rep}(Q_L, n)$. The action of $g$ on representations $g \cdot \text{Rep}(Q_L, n)$.

As a reminder, throughout this paper, we consider only the case of multilayer perceptrons (MLPs).
(2) **Trainable parameters.** A representation $W = (W_1, \ldots, W_L)$ of the quiver $Q_L$ with dimension vector $n$. So, for $i = 1, \ldots, L$, we have a matrix $W_i \in \text{Hom}(R^{1+n_{i-1}}, R^{n_i}).$

(3) **Activation functions.** Piece-wise differentiable functions $a_i : R^{n_i} \rightarrow R^{n_i}$ for $i = 1, \ldots, L$. These are grouped into a tuple $a = (a_1, \ldots, a_L)$.

We denote by $\text{Neur}(Q_L, n)$ the vector space of neural networks with $L$ layers and dimension vector $n$, and regard elements therein as pairs $(W, a)$. The **neural function** (or **feedforward function**) of a neural network $(W, a)$ is defined as:

$$f_{(W, a)} : R^{n_0} \rightarrow R^{n_L} \quad x \mapsto a_L \circ W_L \circ \cdots \circ a_2 \circ W_2 \circ a_1 \circ W_1(x)$$

where $W_i : R^{n_{i-1}} \rightarrow R^{n_i}$ is the affine map corresponding to $W_i$ (see Section 3.2).

There is an action of the group $GL_n^{\text{hidden}}$ on $\text{Neur}(Q_L, n)$ given as follows. The element $g = (g_i)_{i=1}^{L-1} \in GL_n^{\text{hidden}}$ transforms $(W, a) \in \text{Neur}(Q_L, n)$ as:

$$(W, a) \mapsto g \cdot (W, a) := \left( \left( g_i \circ W_i \circ \begin{bmatrix} 1 & 0 \\ 0 & g_i^{-1} \end{bmatrix} \right)_{i=1}^L, \left( g_i \circ a_i \circ g_i^{-1} \right)_{i=1}^L \right),$$

where $g_0 = \text{id}_{n_0}$ and $g_L = \text{id}_{n_L}$. The following lemma shows that the neural function is unchanged by the $GL_n^{\text{hidden}}$ action, that is, if two neural networks are related by the action of an element in $GL_n^{\text{hidden}}$, then their neural functions are the same.

**Lemma 3.3.** For any neural network $(W, a)$ in $\text{Neur}(Q_L, n)$ and any $g$ in $GL_n^{\text{hidden}}$, the neural functions of $(W, a)$ and $g \cdot (W, a)$ coincide:

$$f_g(W, a) = f(W, a)$$

**Proof.** The element $g = (g_i) \in GL_n^{\text{hidden}}$ transforms $W_i = W_i \circ \text{ext}_{n_{i-1}}$ to $g_i \circ W_i \circ \text{ext}_{n_{i-1}} \circ g_i^{-1}$ and $a_i$ to $g_i \circ a_i \circ g_i^{-1}$, for $i = 1, \ldots, L$ (where $g_0 = \text{id}_{n_0}$ and $g_L = \text{id}_{n_L}$). Thus, $g$ transforms the composition $a_i \circ W_i$ to the composition $g_i \circ a_i \circ W_i \circ g_i^{-1}$. The result now follows from the definition of the neural function. $\square$

We now define a certain subgroup of $GL_n^{\text{hidden}}$ that will feature in later discussion.

**Definition 3.4.** The **stabilizer** in $GL_n^{\text{hidden}}$ of a choice of activation functions $a$ is defined as:

$$Z(a) = \{ g = (g_i) \in GL_n^{\text{hidden}} : g_i \circ a_i \circ g_i^{-1} = a_i \text{ for all } i = 1, \ldots, L-1 \}. $$

Note that each $a_i$ is a function from $R^{n_i}$ to $R^{n_i}$, as is each $g_i$, so the equality $g_i \circ a_i \circ g_i^{-1} = a_i$ is as functions from $R^{n_i}$ to $R^{n_i}$. As a subgroup of $GL_n^{\text{hidden}}$, the stabilizer $Z(a)$ acts on $\text{Rep}(Q_L, n)$.

We now formulate the optimization problem for neural networks in the language of quiver representations. For the remainder of this section, we fix a dimension vector $n$ for the the neural quiver $Q_L$, and activation functions $a = (a_1, \ldots, a_L)$. To a batch of training data $X = \{(x_j, y_j)\} \subseteq \mathbb{R}^{n_0} \times \mathbb{R}^{n_L}$, there is an associated **loss function** on $\text{Rep}(Q_L, n)$ defined as

$$\mathcal{L} = \mathcal{L}_X : \text{Rep}(Q_L, n) \rightarrow \mathbb{R} \quad \mathcal{L}(W) = \sum_j C(f(W, a)(x_j), y_j)$$

where $C : \mathbb{R}^{n_0} \times \mathbb{R}^{n_L} \rightarrow \mathbb{R}$ is a cost function on $\mathbb{R}^{n_0}$. As in Section 2, there is a gradient descent map $\gamma : \text{Rep}(Q_L, n) \rightarrow \text{Rep}(Q_L, n)$. Hence:
Remark 3.3: The backpropagation algorithm can be regarded as taking place on the vector space $\text{Rep}(Q_L,n)$ of representations of the neural quiver $Q_L$.

We now show that the loss function is invariant for the action of the group $Z(a)$.

**Proposition 3.5.** Let $a$ be activation functions, and let $\mathcal{L}$ be the loss function associated to a batch of training data. Then, for all $g \in Z(a)$ and $W \in \text{Rep}(Q_L,n)$, we have:

$$\mathcal{L}(g \cdot W) = \mathcal{L}(W).$$

**Proof.** One first verifies using the definition of $Z(a)$ that:

$$(g \cdot W, a) = g \cdot (W, a)$$

for any $g \in Z(a)$ and $W \in \text{Rep}(Q_L,n)$, where the left-hand side invokes the action of $Z(a)$ on $\text{Rep}(Q_L,n)$, while the right-hand side invokes the action of $Z(a)$ on $\text{Neur}(Q_L,n)$. Using this fact and Lemma 3.3, we see that: $f_{(g \cdot W, a)} = f_{(W, a)}$ and hence, for any $g \in Z(a)$ and $W \in \text{Rep}(Q_L,n)$, we have:

$$\mathcal{L}(g \cdot W) = \sum_j C(f_{(g \cdot W, a)}(x_j), y_j) = \sum_j C(f_{(W, a)}(x_j), y_j) = \mathcal{L}(W).$$

$\square$

**Remark 3.6.** The loss function can be written as the composition

$$\text{Rep}(Q_L,n) \xrightarrow{s_a} \text{Neur}(Q_L,n) \xrightarrow{\mathcal{L}} \mathbb{R}$$

where the first map takes $W$ to $(W, a)$ (and hence is a section of the natural projection from $\text{Neur}(Q_L,n)$ to $\text{Rep}(Q_L,n)$), and the second map sends $(W, a)$ to the sum $\sum_j C(f_{(W, a)}(x_j), y_j)$. One verifies directly that $s_a$ is equivariant for the action of the group $Z(a)$, while the second map is invariant for $\text{GL}_n^{\text{hidden}}$, of which $Z(a)$ is a subgroup.

**Remark 3.7.** Suppose $a_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ is pointwise ReLU for each $i$. Then $Z(a)$ consists of those $g = (g_i) \in \text{GL}_n^{\text{hidden}}$ such that each $g_i \in \text{GL}_{n_i}(\mathbb{R})$ is a diagonal matrix with positive entries along the diagonal. In symbols:

$$Z(a) = \prod_{i=1}^{l-1} (\mathbb{R}_{>0})^{n_i}.$$

Hence, the “non-negative homogeneity” (or “positive scaling invariance”) property of pointwise ReLU activation functions appearing in [DPBB17, NSS15, MZZ+19] emerges as a special case of Proposition 3.5.

### 4. Radial neural networks and the QR decomposition

We proceed to apply the framework of the previous section in the context of radial neural networks, which we define in Section 4.1. Section 4.2 includes one of the main results of this paper, namely a QR decomposition for radial neural networks. We also present an algorithm to compute the decomposition.
4.1. **Radial functions.** In this section we formulate the definition of radial functions, and state some basic results about such functions. Let $h : \mathbb{R} \to \mathbb{R}$ be a piece-wise differentiable function such that $\lim_{r \to 0} \frac{h(r)}{r} = h(0)$ exists. For any $n \geq 1$, set:

$$h^{(n)} : \mathbb{R}^n \to \mathbb{R}^n, \quad h^{(n)}(v) := h(|v|) \frac{v}{|v|}.$$ 

A function $a : \mathbb{R}^n \to \mathbb{R}^n$ is called a **radial function** if $a = h^{(n)}$ for some $h : \mathbb{R} \to \mathbb{R}$. Hence, $a$ sends each input vector $v \in \mathbb{R}^n$ to a scalar multiple of itself, and that scalar depends only on the norm of the vector.

**Example 4.1.** (1) The squashing function, where $h(r) = \frac{r^2}{r^2 + 1}$. (2) Shifted ReLU, where $h(r) = \text{ReLU}(r - b)$ for a positive real number $b$. See [WC19] and the references therein for more examples and discussion of radial functions.

**Lemma 4.2.** Let $a = h^{(n)} : \mathbb{R}^n \to \mathbb{R}^n$ be a radial function on $\mathbb{R}^n$.

1. The radial function $a$ commutes with any orthogonal transformation of $\mathbb{R}^n$. In symbols:

   $$a \circ Q = Q \circ a$$

   for any $Q \in O(n)$.

2. If $m \leq n$ and $\text{inc}_{m,n} : \mathbb{R}^m \hookrightarrow \mathbb{R}^n$ is the standard inclusion, then:

   $$h^{(n)} \circ \text{inc}_{m,n} = \text{inc}_{m,n} \circ h^{(m)}.$$ 

**Proof.** Suppose $Q \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$. Since $Q$ is norm-preserving, we have $|Qv| = |v|$ for any $v \in \mathbb{R}^n$. Since $Q$ is linear, we have $Q(\lambda v) = \lambda Qv$ for any $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Using the definition of $a = h^{(n)}$ we compute:

$$a(Qv) = \frac{h(|Qv|)}{|Qv|} Qv = \frac{h(|v|)}{|v|} Qv = Q \left( \frac{h(|v|)}{|v|} v \right) = Q(a(v)).$$

The first claim follows. The second claim is an elementary verification. 

**Remark 4.3.** More generally, it is straightforward to show that the restriction of the radial function $a$ to a linear subspace of $\mathbb{R}^n$ is a radial function on that subspace.

Let $L$ be a positive integer and $\mathbf{n}$ be a dimension vector for the neural quiver $Q_L$. A neural network $(W, a) \in \text{Neur}(Q_L, \mathbf{n})$ is said to be a **radial neural network** if each activation function $a_i$ is a radial function on $\mathbb{R}^n$. Hence, for each $i$, we have $a_i = h^{(n_i)}$ for some $h_i : \mathbb{R} \to \mathbb{R}$. Let $O(\mathbf{n})$ denote the following product of the orthogonal groups:

$$O(\mathbf{n}) = O(n_1) \times O(n_2) \times \cdots \times O(n_{L-1}).$$

Note that $O(\mathbf{n})$ is a subgroup of $G_{\text{hidden}}^n$ and hence acts on $\text{Rep}(Q_L, \mathbf{n})$. The following proposition involves the loss function associated to a batch of training data as in Section 3.3.

**Proposition 4.4.** For any radial neural network in $\text{Neur}(Q_L, \mathbf{n})$ and any batch of training data, the loss function $L : \text{Rep}(Q_L, \mathbf{n}) \to \mathbb{R}$ is invariant for the action of $O(\mathbf{n})$.

**Proof.** Let $g = (g_1, \ldots, g_{L-1}) \in O(\mathbf{n})$. Lemma 4.2 implies that the orthogonal transformation $g_i \in O(n_i)$ commutes with the radial function $a_i = h^{(n_i)}$, for each $i$, or, equivalently: $g_i \circ a_i \circ g_i^{-1} = a_i$. Hence $O(\mathbf{n})$ is a subgroup of the centralizer $Z(a)$, and we apply Proposition 3.5. 

□
Radial functions have the flexibility of defining the activation functions of a neural network for any given representation of $Q_L$ with arbitrary dimension vector. To be more explicit:

**Procedure 4.5.** Given a tuple $h = (h_1, \ldots, h_L)$ of functions\(^6\) $h_i : \mathbb{R} \to \mathbb{R}$ we attach a radial neural network to every representation of $Q_L$ as follows. Let $n = (n_0, \ldots, n_L)$ be the dimension vector of a representation $W$. The corresponding radial neural network has activation functions $a_i = h^{(n_i)}$, that is, we set $a = \left(h_1^{(n_1)}, \ldots, h_L^{(n_L)}\right)$. By slight abuse of notation, we write $(W, h)$ for this radial neural network and $f_{(W,h)}$ for its neural function.

### 4.2. The QR decomposition

We now formulate the QR decomposition for representations of the neural quiver. Making this precise involves associating a ‘reduced dimension vector’ to each dimension vector for the neural quiver and relating the corresponding spaces of representations.

Let $n = (n_0, n_1, \ldots, n_L)$ be a dimension vector for $Q_L$. Its **reduction** $n_{\text{red}} = (n_0^{\text{red}}, n_1^{\text{red}}, \ldots, n_L^{\text{red}})$ is defined recursively by setting $n_0^{\text{red}} = n_0$, then $n_i^{\text{red}} = \min(n_i, n_{i-1}^{\text{red}} + 1)$ for $i = 1, \ldots, L - 1$, and finally $n_L^{\text{red}} = n_L$. We abbreviate $\text{Rep}(Q_L, n)$ by $\text{Rep}(n)$ and $\text{Rep}(Q_L, n_{\text{red}})$ by $\text{Rep}(n_{\text{red}})$.

Observe that $n_i^{\text{red}} \leq n_i$ for each $i$. Therefore, taking a representation with dimension vector $n_{\text{red}}$, i.e., a certain tuple of matrices, one can pad each matrix with the necessary number of rows of zeros below and columns of zeros on the right in order to produce a representation with dimension vector $n$. Thus we have an inclusion: $i : \text{Rep}(n^{\text{red}}) \hookrightarrow \text{Rep}(n)$. (We give a more explicit description of this inclusion in Section 4.3 below.)

Fix a tuple $h = (h_1, \ldots, h_L)$ of functions $h_i : \mathbb{R} \to \mathbb{R}$. As in Procedure 4.5, we attach a radial neural network (and hence a neural function) to every representation of $Q_L$.

**Theorem 4.6.** Let $n$ be a dimension vector for $Q_L$ and fix functions $h = (h_1, \ldots, h_L)$ as above. For any $W \in \text{Rep}(n)$ there exist:

\[
Q \in O(n), \quad R = (R_1, \ldots, R_L) \in \text{Rep}(n_{\text{red}}), \quad \text{and} \quad U \in \text{Rep}(n),
\]

such that:

1. The matrices $R_1, \ldots, R_{L-1}$ are upper triangular.

2. The following equality holds, where we identify $R$ with its image in $\text{Rep}(n)$ under $i$:

\[
W = Q \cdot R + U,
\]

(4.1)

3. The neural functions of the neural networks defined by $W$ and $R$ coincide:

\[
f_{(W,h)} = f_{(R,h)}.
\]

**Remark 4.7.** Since each matrix (except possibly the last) in $R$ is upper triangular and $Q$ is a tuple of orthogonal matrices, Part (2) of Theorem 4.6 is a quiver representation analogue of the usual QR decomposition (c.f. Section 2.2). In particular, the first two parts of the Theorem above are purely quiver-representation theoretic results; only the third part refers to neural networks.

We give a proof of Theorem 4.6 in Section 4.3. The proof relies on the following constructive algorithm, which proceeds layer by layer and computes a QR decomposition at each stage (see

\[\text{We assume each } h_i \text{ is piece-wise differentiable and the limit } \lim_{r \to 0} \frac{h_i(r)}{r} \text{ exists.}\]
below for notational clarifications).

Algorithm 2: QR Dimensional Reduction (QRDimRed)

\begin{verbatim}
input : W ∈ Rep(n)
output : Q, R, U from Theorem 4.6
Q, R, U ← [], [], [0]  // initialize output matrix lists
V_1 ← W_1  // next layer transformed weights
for i ← 1 to L − 1 do  // iterate through layers
    if n_i^red < n_i then
        Q_i, R_i ← QR-decomp(V_i, mode = complete)  // V_i = Q_i ◦ inc_i ◦ R_i
    else
        Q_i, R_i ← QR-decomp(V_i)  // V_i = Q_i ◦ R_i
    end
    Append Q_i to Q
    Append R_i to R  // reduced weights for layer i
    Append W_{i+1} ◦ \begin{bmatrix} 1 & 0 \\ 0 & Q_i \end{bmatrix} ◦ p_i ◦ \begin{bmatrix} 1 & 0 \\ 0 & Q_i^{-1} \end{bmatrix} to U  // matrix multiplication
    Set V_{i+1} ← W_{i+1} ◦ \begin{bmatrix} 1 & 0 \\ 0 & Q_i \end{bmatrix} ◦ inc_i  // transform next layer
end
Append V_L to Q
return Q, R, and U
\end{verbatim}

We explain the notation. The symbol ‘◦’ denotes matrix multiplication. We initialize U as a list with a single element: the zero n_1 × (1 + n_0) matrix. For i = 1, . . . , L − 1, we have the standard inclusions

\[
\text{inc}_i = \text{inc}_{n_i^\text{red}, n_i} : \mathbb{R}^{n_i^\text{red}} \hookrightarrow \mathbb{R}^{n_i} \quad \text{inc}_i = \text{inc}_{1 + n_i^\text{red}, 1 + n_i} : \mathbb{R}^{1 + n_i^\text{red}} \hookrightarrow \mathbb{R}^{1 + n_i}
\]

into the first n_i^\text{red} and 1 + n_i^\text{red} coordinates, respectively. As matrices, they have ones along the main diagonal and zeros elsewhere. For later reference, we note the projection maps:

\[
\pi_i : \mathbb{R}^{n_i} \twoheadrightarrow \mathbb{R}^{n_i^\text{red}} \quad \tilde{\pi}_i : \mathbb{R}^{1 + n_i^\text{red}} \twoheadrightarrow \mathbb{R}^{1 + n_i^\text{red}}
\]

onto the first n_i^\text{red} and 1 + n_i^\text{red} coordinates, respectively. The endomorphism p_i : \mathbb{R}^{n_i + 1} → \mathbb{R}^{n_i + 1} zeros out the first 1 + n_i^\text{red} coordinates and leaves the remaining coordinates unchanged. As a matrix, p_i is block diagonal with two square blocks; the top block is the zero matrix of size n_i^\text{red} + 1, and the bottom is the identity matrix of size n_i − n_i^\text{red}. We also have the identity:

\[
\pi_i = p_{i-1} + \text{inc}_{i-1} \circ \tilde{\pi}_{i-1}
\]

(4.2)

The definition of n_i^\text{red} implies that either n_i^\text{red} = n_{i-1}^\text{red} + 1 or n_i^\text{red} = n_i. In the former case, n_i^\text{red} + 1 ≤ n_i and the function QR-decomp with ‘mode = complete’ computes the QR decomposition of the n_i × (1 + n_i^\text{red}) matrix V_i as Q_i ◦ inc_i ◦ R_i where Q_i ∈ O(n_i) and R_i is upper-triangular of size n_i^\text{red} × n_i^\text{red}. In the latter, n_i^\text{red} + 1 ≥ n_i and the method QR-decomp computes the QR decomposition of the n_i × (1 + n_i^\text{red}) matrix V_i as Q_i ◦ R_i where Q_i ∈ O(n_i) and R_i is upper-triangular of size n_i × (1 + n_i^\text{red})}. (The two decompositions coincide if n_i = n_i^\text{red} + 1.)
Example 4.8. Suppose the dimension vector is \( \mathbf{n} = (1, 8, 16, 8, 1) \), so the original radial neural network model has \( \sum_{i=1}^{4} (n_{i-1} + 1)n_i = 305 \) trainable parameters. The reduced dimension vector is \( \mathbf{n}^{\text{red}} = (1, 2, 3, 4, 1) \). One computes that the compressed model has \( \sum_{i=1}^{4} (n_{i-1}^{\text{red}} + 1)(n_i^{\text{red}}) = 34 \) trainable parameters.

4.3. Proof of Theorem 4.6. In this section, we give a proof of the QR decomposition stated in Theorem 4.6. We adopt all notational conventions of the previous section. First, we collect the following basic identities related to the map \( \text{ext}_n : \mathbb{R}^n \to \mathbb{R}^{n+1} \) (see Equation 2.1) taking \((v_1, \ldots, v_n)\) to \((1, v_1, \ldots, v_n)\). Their proofs are elementary.

Lemma 4.9. For \( m \leq n \), we have: \( \text{ext}_n \circ \text{inc}_{m,n} = \text{inc}_{m+1,n+1} \circ \text{ext}_m \).

Lemma 4.10. For any \( n \times m \) matrix \( A \), we have\(^7\): \( \text{ext}_n \circ A = \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \circ \text{ext}_m \).

Next, recall the inclusion \( i : \text{Rep}(\mathbf{n}^{\text{red}}) \hookrightarrow \text{Rep}(\mathbf{n}) \). To describe this inclusion more explicitly, let \( X = (X_1, \ldots, X_L) \in \text{Rep}(\mathbf{n}^{\text{red}}) \). Then \( i(X)_i = \text{inc}_i \circ X_i \circ \tilde{\pi}_{i-1} \). Indeed, post-composing \( X_i \) with \( \text{inc}_i \) amounts to adding \( n_i - n_i^{\text{red}} \) rows of zeros on the bottom, while pre-composing \( X_i \) with \( \tilde{\pi}_{i-1} \) amounts to adding \( n_{i-1} - n_i^{\text{red}} \) columns of zeros on the right.

Proof of Theorem 4.6. We claim that the outputs \( \mathbf{Q}, \mathbf{R}, \) and \( \mathbf{U} \) of Algorithm 2 satisfy the required conditions. The matrices \( R_1, \ldots, R_{L-1} \) are upper-triangular by construction, so the first part of the theorem holds.

To show that the neural functions of \((\mathbf{W}, \mathbf{h})\) and \((\mathbf{R}, \mathbf{h})\) coincide, we first show that, for all \( i = 1, \ldots, L \), we have:

\[
(4.3) \quad h^{(n_i)} \circ \widetilde{W}_i \circ Q_{l-1} \circ \text{inc}_{i-1} = Q_i \circ \text{inc}_i \circ h^{(n_i^{\text{red}})} \circ \tilde{R}_i
\]

where \( Q_0 = \text{id}_{n_0} \) and \( Q_L = \text{id}_{n_L} \). The justification is a computation:

\[
\begin{align*}
&h^{(n_i)} \circ \widetilde{W}_i \circ Q_{l-1} \circ \text{inc}_{i-1} = h^{(n_i)} \circ W_i \circ \text{ext}_{n_{i-1}} \circ Q_{l-1} \circ \text{inc}_{i-1} \\
&= h^{(n_i)} \circ W_i \circ \begin{bmatrix} 1 & 0 \\ 0 & Q_{l-1} \end{bmatrix} \circ \text{inc}_{i-1} \circ \text{ext}_{n_{i-1}}^{\text{red}} \\
&= h^{(n_i)} \circ Q_i \circ \text{inc}_i \circ R_i \circ \text{ext}_{n_{i-1}}^{\text{red}} = Q_i \circ \text{inc}_i \circ h^{(n_i^{\text{red}})} \circ \tilde{R}_i.
\end{align*}
\]

The first equality uses the definition of \( \widetilde{W}_i \) in terms of \( W_i \); the second uses the commutativity properties of \( \text{ext} \) stated in Lemmas 4.9 and 4.10; the third uses the definitions of \( Q_i, R_i, \) and \( \tilde{V}_i \); and the fourth uses the commutativity properties of radial functions as well as the definition of \( \tilde{R}_i \). The fact that \( f_{(\mathbf{W}, \mathbf{h})} = f_{(\mathbf{R}, \mathbf{h})} \) now follows from the definition of the neural function and iterative applications of Equation 4.3, noting that \( Q_0 \circ \text{inc}_0 = \text{id}_{n_0} \) and \( Q_L \circ \text{inc}_L = \text{id}_{n_L} \).

\(^7\)The matrix \( \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} \) is \((n+1) \times (m+1)\), so the top right 0 abbreviates a row of \( m \) zeros and the bottom left 0 abbreviates a column of \( n \) zeros.
Finally, to show that \( W = Q \cdot \iota(R) + U \), we compute, for \( i = 1, \ldots, L \):
\[
(Q \cdot \iota(R) + U)_i = (Q \cdot \iota(R))_i + U_i
\]
\[
= Q_i \circ \iota_c \circ R_i \circ \tilde{\pi}_{i-1} \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] + W_i \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] \circ p_{i-1} \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right]
\]
\[
= W_i \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] \circ \iota_{c,i} \circ \tilde{\pi}_{i-1} \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] + W_i \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] \circ p_{i-1} \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right]
\]
\[
= W_i \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] \circ (\iota_{c,i} \circ \tilde{\pi}_{i-1} + p_{i-1}) \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right]
\]
\[
= W_i \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] \circ \text{id}_{1+n_{i-1}} \circ \left[ \begin{array}{cc} 1 & 0 \\ 0 & Q_{i-1} \end{array} \right] = W_i.
\]
The first two equalities follow from definitions; the third uses the definitions of \( Q_i, R_i \), and \( V_i \); the fourth uses the distributive law; and the fifth appeals to the Equation 4.2.

\[\square\]

5. Projected gradient descent

In this section, we use the QR decomposition of Theorem 4.6 to relate projected gradient descent for a representation with dimension \( n \) to (usual) gradient descent for the corresponding reduced representation with dimension vector \( n^{\text{red}} \).

We adopt all notational conventions from the previous section. In particular, we fix a tuple \( \mathbf{h} = (h_1, \ldots, h_L) \) of functions, where \( h_i : \mathbb{R} \to \mathbb{R} \) for \( i = 1, \ldots, L \). As in Procedure 4.5, these can be used to define a neural function \( f_{(\mathbf{W}, \mathbf{h})} \) (resp. \( f_{(X, \mathbf{h})} \)) for every representation \( \mathbf{W} \) in \( \text{Rep}(\mathbf{n}) \) (resp. \( X \) in \( \text{Rep}(\mathbf{n}^{\text{red}}) \)). Furthermore, we fix a batch of training data \( \{(x_i, y_i)\} \subseteq \mathbb{R}^{n_L} \times \mathbb{R}^L = \mathbb{R}^{n_L^{\text{red}}} \times \mathbb{R}^{L^{\text{red}}} \).

Using \( \mathbf{h} \), we have loss functions on \( \text{Rep}(\mathbf{n}) \) and \( \text{Rep}(\mathbf{n}^{\text{red}}) \) (see Section 3.3):
\[
\mathcal{L} : \text{Rep}(\mathbf{n}) \to \mathbb{R} \\
\mathcal{L}_{\text{red}} : \text{Rep}(\mathbf{n}^{\text{red}}) \to \mathbb{R}
\]
\[
\mathbf{W} \mapsto \sum_j C(f_{(\mathbf{W}, \mathbf{h})}(x_j), y_j) \\
\mathbf{X} \mapsto \sum_j C(f_{(X, \mathbf{h})}(x_j), y_j)
\]
for a cost function \( C : \mathbb{R}^{n_L} \times \mathbb{R}^L \to \mathbb{R} \). There are resulting gradient descent maps on \( \text{Rep}(\mathbf{n}) \) and \( \text{Rep}(\mathbf{n}^{\text{red}}) \) given by:
\[
\gamma : \text{Rep}(\mathbf{n}) \to \text{Rep}(\mathbf{n}) \\
\mathbf{W} \mapsto \mathbf{W} - \nabla_{\mathbf{W}} \mathcal{L}
\]
\[
\gamma_{\text{red}} : \text{Rep}(\mathbf{n}^{\text{red}}) \to \text{Rep}(\mathbf{n}^{\text{red}}) \\
\mathbf{X} \mapsto \mathbf{X} - \nabla_{\mathbf{X}} \mathcal{L}_{\text{red}}
\]

5.1. The interpolating space. In this section, we introduce a subspace \( \text{Rep}^{\text{int}}(\mathbf{n}) \) of \( \text{Rep}(\mathbf{n}) \), that, as we will later see, interpolates between \( \text{Rep}(\mathbf{n}^{\text{red}}) \) and \( \text{Rep}(\mathbf{n}) \).

Recall the standard inclusions and projections, for \( i = 0, 1, \ldots, L \):
\[
\text{inc}_i = \text{inc}_{n_i^{\text{red}}, n_i} : \mathbb{R}^{n_i^{\text{red}}} \hookrightarrow \mathbb{R}^{n_i} \\
\tilde{\text{inc}}_i = \text{inc}_{1+n_i^{\text{red}}, 1+n_i} : \mathbb{R}^{1+n_i^{\text{red}}} \hookrightarrow \mathbb{R}^{1+n_i}
\]
\[
\pi_i : \mathbb{R}^{n_i} \twoheadrightarrow \mathbb{R}^{n_i^{\text{red}}} \\
\tilde{\pi}_i : \mathbb{R}^{1+n_i} \twoheadrightarrow \mathbb{R}^{1+n_i^{\text{red}}}
\]

**Definition 5.1.** Let \( \text{Rep}^{\text{int}}(\mathbf{n}) \) denote the subspace of \( \text{Rep}(\mathbf{n}) \) consisting of those \( \mathbf{T} = (T_1, \ldots, T_L) \in \text{Rep}(\mathbf{n}) \) such that:
\[
(id_{n_i} - \text{inc}_i \circ \pi_i) \circ T_i \circ \tilde{\text{inc}}_{i-1} \circ \tilde{\pi}_{i-1} = 0
\]
for $i = 1, \ldots, L$. Write $\iota_1 : \text{Rep}^\text{int}(n) \hookrightarrow \text{Rep}(n)$ for the natural inclusion.

In other words, the space $\text{Rep}^\text{int}(n)$ consists of representations $T = (T_1, \ldots, T_L) \in \text{Rep}(n)$ for which the bottom left $(n_i - n_i^{\text{red}}) \times (1 + n_i^{\text{red}})$ block of $T_i$ is zero for each $i$:

$$T_i = \begin{bmatrix} \ast & \ast \\ 0 & \ast \end{bmatrix}.\) 

**Definition 5.2.** Define a map

$$q_1 : \text{Rep}(n) \to \text{Rep}^\text{int}(n)$$

by taking $T \in \text{Rep}(n)$ to the representation whose $i$-th slot is

$$W_i - (\text{id}_{n_i} - \text{inc}_i \circ \pi_i) \circ W_i \circ \widetilde{\text{inc}}_{i-1} \circ \pi_{i-1}.$$ 

It is straightforward to check that $q_1$ is well-defined, i.e., that the resulting representation belongs to $\text{Rep}^\text{int}(n)$ and that $q_1$ is a surjective linear map. The transpose of $q_1$ is the inclusion $\iota_1$. We summarize the situation in the following diagram:

$$(5.1) \quad \text{Rep}^\text{int}(n) \xrightarrow{\iota_1} \text{Rep}(n) \xleftarrow{q_1} \text{Rep}(n)$$

We observe that the composition $q_1 \circ \iota$ is the identity on $\text{Rep}^\text{int}(n)$, while the endomorphism $\iota_1 \circ q_1$ of $\text{Rep}(n)$ takes a representation $W \in \text{Rep}(n)$ and, for each $i$, zeros all entries in the bottom left $(n_i - n_i^{\text{red}}) \times (1 + n_i^{\text{red}})$ submatrix of $W_i$.

### 5.2. Projected gradient descent and the QR decomposition.

In this section, we define the projected gradient descent map and state a theorem relating projected gradient descent on $\text{Rep}(n)$ with usual gradient descent on $\text{Rep}(n^{\text{red}})$.

**Definition 5.3.** The projected gradient descent map on $\text{Rep}(n)$ with respect to $\text{Rep}^\text{int}(n)$ is defined as:

$$\gamma^\text{proj} : \text{Rep}(n) \to \text{Rep}(n)$$

$$W \mapsto W - \iota_1 \circ q_1(\nabla_W \mathcal{L})$$

Hence, while all entries of each matrix $W_i$ in the representation $W$ contribute to the computation of the gradient $\nabla_W \mathcal{L}$, only those not in the bottom left submatrix get updated under the projected gradient descent map $\gamma^\text{proj}$. We now state the main result of this section. To ease notation, we identify elements of $\text{Rep}(n^{\text{red}})$ with their images in $\text{Rep}(n)$ under $\iota$.

**Theorem 5.4.** Let $W \in \text{Rep}(n)$ and let $Q$, $R$, and $U$ be the outputs of Algorithm 2, so $W = Q \cdot R + U$.

1. The representation $Q^{-1} \cdot W = R + Q^{-1} \cdot U$ belongs to $\text{Rep}^\text{int}(n)$.

2. For any $k \geq 0$, we have

$$\gamma^k(W) = Q \cdot \gamma^k(Q^{-1} \cdot W).$$

3. For any $k \geq 0$, we have

$$\gamma^k_{\text{proj}}(Q^{-1} \cdot W) = \gamma^k_{\text{red}}(R) + Q^{-1} \cdot U.$$
We summarize this result in the following diagram. The left horizontal maps indicate the addi-
tion of $Q^{-1} \cdot U$, the right horizontal arrows indicate the action of $Q$, and the vertical maps are
various versions of gradient descent. The shaded regions indicate the (smallest) vector space to
which the various representations naturally belong.

![Diagram](image)

**Remark 5.5.** The map $\Rep(n) \to \Rep(n)$ taking $W$ to $Q \cdot \gamma_{\text{proj}}(Q^{-1} \cdot W)$ is a projected gradient
descent map on $\Rep(n)$ with respect to the subspace of $\Rep(n)$ formed by translating $\Rep^{\text{int}}(n)$
by the action of $Q$, i.e. the subspace $Q \cdot \Rep^{\text{int}}(n) = \{ Q \cdot T \mid T \in \Rep^{\text{int}}(n) \}$.

### 5.3. Proof of Theorem 5.4.

We begin by explaining the sense in which $\Rep^{\text{int}}(n)$ interpolates
between $\Rep(n)$ and $\Rep(n^{\text{red}})$. One extends Diagram 5.1 as follows:

![Diagram](image)

- The map $i_2 : \Rep(n^{\text{red}}) \hookrightarrow \Rep^{\text{int}}(n)$
takes $X$ to the representation whose $i$-th coordinate is $\text{inc}_i \circ X_i \circ \tilde{\pi}_{i-1}$. It is straightforward
to check that $i_2$ is a well-defined injective linear map.

- The map $q_2 : \Rep^{\text{int}}(n) \twoheadrightarrow \Rep(n^{\text{red}})$
takes $T$ to the representation whose $i$-th slot is $\pi_i \circ T_i \circ \text{inc}_{i-1}$. It is straightforward to
check that $q_2$ is a surjective linear map. The transpose of $q_2$ is the inclusion $i_2$.

**Lemma 5.6.** We have the following:

1. The inclusion $i : \Rep(n^{\text{red}}) \hookrightarrow \Rep(n)$ coincides with the composition $i_1 \circ i_2$, and commutes with
the loss functions:
(2) The following diagram commutes:

\[
\begin{array}{ccc}
\text{Rep}^{\text{int}}(n) & \xrightarrow{q_2} & \text{Rep}(n^{\text{red}}) \\
\downarrow{\text{id}} & & \downarrow{\mathcal{L}_{\text{red}}} \\
\text{Rep}(n) & \xrightarrow{\mathcal{L}} & \mathbb{R}
\end{array}
\]

(3) For any \( T \in \text{Rep}^{\text{int}}(n) \), we have: \( q_1 \left( \nabla_{\mathcal{L}_{\text{head}}(T)} \right) = q_2 \left( \nabla_{\mathcal{L}_{\text{red}}(T)} \right) \).

**Proof.** The identity \( t = t_1 \circ t_2 \) follows directly from definitions. To prove the commutativity of the diagram, it is enough to show that, for any \( X \) in \( \text{Rep}(n^{\text{red}}) \), the neural functions of \( X \) and \( t(X) \) coincide. This follows easily from the fact that, for \( i = 1, \ldots, L \), we have:

\[ \pi_i \circ h^{(n_i^{\text{red}})} \circ \text{inc}_i = \pi_i \circ \text{inc}_i \circ h^{(n_i^{\text{red}})} = h^{(n_i^{\text{red}})} \).

For the second claim, let \( T \in \text{Rep}^{\text{int}}(n) \). It suffices to show that \( t_1(T) \) and \( q_2(T) \) have the same neural function. The key computation is:

\[
\text{inc}_i \circ h^{(n_i^{\text{red}})} \circ \pi_i \circ \tilde{T}_i \circ \text{inc}_{i-1} = h^{(n_i)} \circ \text{inc}_i \circ \pi_i \circ T_i \circ \text{ext}_{n_{i-1}} \circ \text{inc}_{i-1}
\]

which uses the fact that \( (\text{id}_{n_i} - \text{inc}_i \circ \pi_i) \circ T_i \circ \tilde{c}_{i-1} = 0 \), or, equivalently, \( \text{inc}_i \circ \pi_i \circ T_i \circ \tilde{c}_{i-1} = T_i \circ \tilde{c}_{i-1} \). Applying this relation successively starting with the second-to-last layer \( (i = L - 1) \) and ending in the first \( (i = 1) \), one obtains the result. For the last claim, one computes \( \nabla_T(\mathcal{L} \circ t_1) \) in two different ways. The first way is:

\[
\nabla_T(\mathcal{L} \circ t_1) = (d_T(\mathcal{L} \circ t_1))^T = (d_{t_1(T)}(\mathcal{L} \circ d_T t_1))^T = (d_{t_1(T)}(\mathcal{L} \circ t_1))^T = q_1 \left( \nabla_{t_1(T)}(\mathcal{L}) \right)
\]

where we use the fact that \( t_1 \) is a linear map whose transpose is \( q_1 \). The second way uses the commutative diagram of the second part of the Lemma:

\[
\nabla_T(\mathcal{L} \circ t_1) = \nabla_T(\mathcal{L}_{\text{red}} \circ q_2) = (d_T(\mathcal{L}_{\text{red}} \circ q_2))^T = (d_{q_2(T)}(\mathcal{L}_{\text{red}} \circ d_T q_2))^T = q_2^T \left( (d_{q_2(T)}(\mathcal{L}_{\text{red}}))^T \right) = t_2 \left( \nabla_{q_2(T)}(\mathcal{L}_{\text{red}}) \right).
\]

We also use the fact that \( q_2 \) is a linear map whose transpose is \( t_2 \).

**Proof of Theorem 5.4.** The first claim follows easily from the definitions of \( Q_i, R_i, \) and \( U_i \). The action of \( Q \in O(n) \) on \( \text{Rep}(n) \) is an orthogonal transformation, so the second claim follows from Proposition 2.5. For the last claim, we proceed by induction. The base case \( k = 0 \) follows from Theorem 4.6. For the induction step, we set

\[ Z^{(k)} = \iota(\gamma^{(k)}_{\text{red}}(R)) + Q^{-1} \cdot U. \]
Each $Z^{(k)}$ belongs to $\text{Rep}^\text{int}(n)$, so $i_1(Z^{(k)}) = Z^{(k)}$. Moreover, $q_2(Z^{(k)}) = \gamma^k_{\text{red}}(R)$. We compute:

$$
\gamma^{k+1}_{\text{proj}}(Q^{-1} \cdot W) = \gamma_{\text{proj}}\left(\gamma^k_{\text{proj}}(Q^{-1} \cdot W)\right) \\
= \gamma_{\text{proj}}\left(i(\gamma^k_{\text{red}}(R)) + Q^{-1} \cdot U\right) \\
= i(\gamma^k_{\text{red}}(R)) + Q^{-1} \cdot U - t_1 \circ q_1 \left(\nabla_i(\gamma^k_{\text{red}}(R)) + Q^{-1} \cdot U \nabla \mathcal{L}\right) \\
= i(\gamma^k_{\text{red}}(R)) - t_1 \circ q_1 \left(\nabla_i(Z^{(k)}) \mathcal{L}\right) + Q^{-1} \cdot U \\
= i(\gamma^k_{\text{red}}(R)) - t_1 \circ t_2 \left(\nabla q_2(Z^{(k)}) \mathcal{L}_{\text{red}}\right) + Q^{-1} \cdot U \\
= i\left(\gamma^k_{\text{red}}(R) - \nabla \gamma^k_{\text{red}}(R) \mathcal{L}_{\text{red}}\right) + Q^{-1} \cdot U \\
= i\left(\gamma^{k+1}_{\text{red}}(R)\right) + Q^{-1} \cdot U
$$

where the second equality uses the induction hypothesis, the third invokes the definition of $\gamma_{\text{proj}}$, the fourth uses the relation between the gradient and orthogonal transformations, the fifth and sixth use Lemma 5.6 above, and the last uses the definition of $\gamma_{\text{red}}$. □

6. Experiments

In addition to the theoretical results in this work, we provide an implementation of QRDimRed as described in Algorithm 2. We perform experiments in order to (1) empirically validate that our implementation satisfies the claims of Theorems 4.6 and Theorem 5.4 and (2) quantify real-world performance. Our implementation is written in Python and uses the QR decomposition routine in NumPy [HMvdW+20]. We also implement a general class RadNet for radial neural networks using PyTorch [PGM+19]. Our code is available at https://github.com/ivganev/QR-decomposition-radial-NNs/.

6.1. Empirical verification of Theorem 4.6. We verify the claim using a small model and synthetic data. We learn the function $f(x) = e^{-x^2}$ using $N = 121$ samples $x_j = -3 + j/20$ for $0 \leq j < 121$. We test $f_W$ on a radial neural network with layer widths $n = (1, 6, 7, 1)$ and activation functions the radial shifted sigmoid $h(x) = 1/(1 + e^{-x+b})$. Applying QRDimRed gives a radial neural network $f_R$ with widths $n_{\text{red}} = (1, 2, 3, 1)$. Theorem 4.6 implies that the neural functions of $f_W$ and $f_R$ are equal. Over 10 random initializations of $W$, the mean absolute error $(1/N) \sum_j |f_W(x_j) - f_R(x_j)| = 1.31 \cdot 10^{-8} \pm 4.45 \cdot 10^{-9}$. Thus $f_W$ and $f_R$ agree up to machine precision.

6.2. Empirical verification of Theorem 5.4. Similarly, we verify the conclusions of Theorem 5.4 using synthetic data. The claim is that training $f_{Q^{-1}W}$ with objective $\mathcal{L}$ by projected gradient descent coincides with training $f_R$ with objective $\mathcal{L}_{\text{red}}$ by usual gradient descent. We verified this for 3000 epochs at learning rate 0.01. Over 10 random initializations of $W$, the loss functions match up to machine precision with $|\mathcal{L} - \mathcal{L}_{\text{red}}| = 4.02 \cdot 10^{-9} \pm 7.01 \cdot 10^{-9}$.

6.3. Reduced model trains faster. The goal of model compression algorithms is usually to provide smaller trained models with faster inference but similar accuracy to larger trained models.
Due to the relation between projected gradient descent of the full network $W$ and gradient descent of the reduced network $R$, our method may produce a smaller model class which also trains faster without sacrificing accuracy.

We test this hypothesis using a different set of synthetic data. We learn the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ sending $x = (t_1, t_2)$ to $(e^{-t_1^2}, e^{-t_2^2})$ using $N = 121^2$ samples $(-3 + j/20, -3 + k/20)$ for $0 \leq j, k < 121$. We test $f_W$ on a radial neural network with layer widths $n = (2, 16, 64, 128, 16, 2)$ and activation functions the radial sigmoid $h(r) = 1/(1 + e^{-r})$. Applying QRDimRed gives a radial neural network $f_R$ with widths $n^{\text{red}} = (2, 3, 4, 5, 6, 2)$. We trained both models until the training loss was $\leq 0.01$. Running on test system with an Intel i5-8257U CPU at 1.40GHz and 8GB of RAM and averaged over 10 random initializations, the reduced network trained in 15.32 ± 2.53 seconds and the original network trained in 31.24 ± 4.55 seconds, approximately 2.06 ± 0.24 times as long to reach the same loss.

7. Discussion and conclusions

In this section, we first consider how our framework specializes to the case of no-bias, and then how it generalizes to shifts within the radial functions. Finally, we gather some conclusions of this work and discuss future directions.

7.1. Special case: No-bias version. We now consider neural networks with only linear maps between successive layers, rather than the more general setting of affine maps. In other words, there are no bias vectors.

Let $L$ be a positive integer. The no-bias neural quiver $\bar{Q}_L$ is the following quiver with $L + 1$ vertices:

\[
\bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \rightarrow \bullet
\]

A representation of this quiver with dimension vector $n = (n_0, \ldots, n_L)$ consists of a linear map from $\mathbb{R}^{n_{i-1}}$ to $\mathbb{R}^{n_i}$ for $i = 1, \ldots, L$; hence $\text{Rep}(\bar{Q}_L, n) = \bigoplus_{i=1}^L \text{Hom}(\mathbb{R}^{n_{i-1}}, \mathbb{R}^{n_i})$. The corresponding no-bias reduced dimension vector $\bar{n}^{\text{red}} = (\bar{n}_0^{\text{red}}, \bar{n}_1^{\text{red}}, \ldots, \bar{n}_L^{\text{red}})$ is defined recursively by setting $\bar{n}_0^{\text{red}} = n_0$, then $\bar{n}_i^{\text{red}} = \min(n_i, \bar{n}_{i-1}^{\text{red}})$ for $i = 1, \ldots, L - 1$, and finally $\bar{n}_L^{\text{red}} = n_L$. Since $\bar{n}_i^{\text{red}} \leq n_i$ for all $i$, we have an obvious inclusion $\text{Rep}(\bar{Q}_L, \bar{n}) \hookrightarrow \text{Rep}(\bar{Q}_L, n)$ and identify $\text{Rep}(\bar{Q}_L, \bar{n}^{\text{red}})$ with its image in $\text{Rep}(\bar{Q}_L, n)$. Given functions $h = (h_1, \ldots, h_L)$, one adapts Procedure 4.5 to define a radial neural network for every representation of $\bar{Q}_L$, where the trainable parameters define linear maps rather than affine maps.

**Proposition 7.1.** Theorem 4.6 holds with the neural quiver replaced by the no-bias neural quiver $\bar{Q}_L$ and the reduced dimension vector replaced by the no-bias reduced dimension vector $\bar{n}^{\text{red}}$.

We illustrate an example of the reduction for no-bias radial neural networks in Figure 2. Versions of Algorithm 2 and Theorem 5.4 also hold in the no-bias case, where one uses projected gradient descent with respect to the subspace $\text{Rep}^{\text{int}}(\bar{Q}_L, n)$ of representations $T$ having the lower left $(n_i - \bar{n}_i^{\text{red}}) \times (\bar{n}_{i-1}^{\text{red}})$ block of each $T_i$ equal to zero.

7.2. Generalization: Radial neural networks with shifts. In this section we consider radial neural networks with an extra trainable parameter in each layer which shifts the radial function.
Let \( h : \mathbb{R} \rightarrow \mathbb{R} \) be a function. For any \( n \geq 1 \) and any \( t \in \mathbb{R} \), the corresponding \textbf{shifted radial function} on \( \mathbb{R}^n \) is given by:

\[
 h^{(n,t)} : v \mapsto \frac{h(|v| - t)}{|v|} v.
\]

The following definition is a modification of Definition 3.2.

\textbf{Definition 7.2.} A \textbf{radial neural network with shifts} consists of the following data:

1. \textbf{Hyperparameters.} A positive integer \( L \) and a dimension vector \( \mathbf{n} = (n_0, n_1, n_2, \ldots, n_L) \) for the neural quiver \( Q_L \).

2. \textbf{Trainable parameters.} 
   (a) A representation \( \mathbf{W} = (W_1, \ldots, W_L) \) of the quiver \( Q_L \) with dimension vector \( \mathbf{n} \). So, for \( i = 1, \ldots, L \), we have a matrix \( W_i \in \text{Hom}(\mathbb{R}^{1+n_{i-1}}, \mathbb{R}^{n_i}) \).
   (b) A vector of shifts \( \mathbf{t} = (t_1, t_2, \ldots, t_L) \in \mathbb{R}^L \).

3. \textbf{Radial activation functions.} A tuple \( \mathbf{h} = (h_1, h_2, \ldots, h_L) \), where \( h_i : \mathbb{R} \rightarrow \mathbb{R} \). The activation function in the \( i \)-th layer is given by \( a_i = h^{(n_i,t_i)} : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{n_i} \) for \( i = 1, \ldots, L \).

The \textbf{neural function} of a radial neural network with shifts is defined as:

\[
 f_{(\mathbf{W},\mathbf{t},\mathbf{h})} : \mathbb{R}^{n_0} \rightarrow \mathbb{R}^{n_L}, \quad x \mapsto h^{(n_L,t_L)} \circ \bar{W}_L \circ \cdots \circ h^{(n_1,t_1)} \circ \bar{W}_2 \circ h^{(n_1,t_1)} \circ \bar{W}_1(x)
\]

where \( \bar{W}_i = W_i \circ \text{ext}_{n_{i-1}} : \mathbb{R}^{n_{i-1}} \rightarrow \mathbb{R}^{n_i} \) is the affine map corresponding to \( W_i \). The trainable parameters form the vector space \( \text{Rep}(\mathbf{n}) \oplus \mathbb{R}^L \), and the loss function of a batch of training data is defined as

\[
 \mathcal{L} = \mathcal{L}_X : \text{Rep}(\mathbf{n}) \oplus \mathbb{R}^L \rightarrow \mathbb{R}; \quad (\mathbf{W}, \mathbf{t}) \mapsto \sum_j \mathcal{C}(f_{(\mathbf{W},\mathbf{t},\mathbf{h})}(x_j), y_j)
\]

We have the gradient descent map:

\[
 \gamma : \text{Rep}(\mathbf{n}) \oplus \mathbb{R}^L \rightarrow \text{Rep}(\mathbf{n}) \oplus \mathbb{R}^L
\]

which updates the entries of both \( \mathbf{W} \) and \( \mathbf{t} \). The group \( O(\mathbf{n}) = O(n_1) \times \cdots \times O(n_{L-1}) \) acts on \( \text{Rep}(\mathbf{n}) \) as usual (see Section 4.1), and on \( \mathbb{R}^L \) trivially. The neural function is unchanged by this action. We conclude that the \( O(\mathbf{n}) \) action on \( \text{Rep}(\mathbf{n}) \oplus \mathbb{R}^L \) commutes with gradient descent \( \gamma \).

---

We also assume \( h \) is piece-wise differentiable and exclude those \( t \) for which the limit \( \lim_{r \to 0} \frac{h(r-t)}{r} \) does not exist.
We now state a generalization of Theorem 4.6 for the case of radial neural networks with shifts. We omit a proof, as it uses the same techniques as the proof of Theorem 4.6.

**Theorem 7.3.** Let \( n \) be a dimension vector for \( Q_L \) and fix functions \( h = (h_1, \ldots, h_L) \) as above. For any \( W \in \text{Rep}(n) \) there exist:

\[
Q \in O(n), \quad R = (R_1, \ldots, R_L) \in \text{Rep}(n^{\text{red}}), \quad \text{and} \quad U \in \text{Rep}(n),
\]

such that:

1. The matrices \( R_1, \ldots, R_{L-1} \) are upper triangular.
2. The following equality holds: \( (W, t) = Q \cdot (R, t) + (U, 0) \).
3. The neural functions defined by \( (W, t, h) \) and \( (R, t, h) \) coincide: \( f_{(W, t, h)} = f_{(R, t, h)} \).

One can use the output of Algorithm 2 to obtain the \( Q, R, \) and \( U \) appearing in Theorem 7.3. Theorem 5.4 also generalizes to the setting of radial neural networks with shifts, using projected gradient descent with respect to the subspace \( \text{Rep}^{\text{int}}(n) \oplus R^L \) of \( \text{Rep}(n) \oplus R^L \).

**7.3. Conclusions and future work.** In this paper, we have adopted the formalism of quiver representation theory in order to establish a theoretical framework for neural networks. While drawing inspiration from previous work, our approach is novel in that (1) reveals a large group of symmetries of neural network parameter spaces, (2) leads to a version of the QR decomposition in the context of radial neural networks, and (3) precipitates an algorithm to reduce the widths of the hidden layers in such networks. Our main results, namely Theorems 4.6 and 5.4, may potentially generalize in several ways, which we now describe.

First, these theorems are only meaningful if \( n_i > n_{i-1} + 1 \) for some \( i \) (otherwise \( n^{\text{red}} = n \)). In particular, this leads to compression for networks which have any consecutive layers of increasing width. Many networks such as decoders, super-resolution mappings, and GANs fulfill this criteria, but others do not. We expect our techniques to prove useful in weakening the assumptions necessary for a meaningful reduction, and hence widening the applicability of our results.

One shortcoming of our results is the necessity of projected gradient descent \( \gamma_{\text{proj}} \) in Part 2 of Theorem 5.4, rather than usual gradient descent \( \gamma \). This shortcoming may be alleviated by the following speculations. First, preliminary experimental evidence suggests that our version of projected gradient descent is a first-order approximation of usual gradient descent. Second, it may be possible that, for a different choice of QR decomposition in Theorem 4.6, the equation stated in Part 2 of Theorem 5.4 holds with usual gradient descent \( \gamma \), rather than projected gradient descent \( \gamma_{\text{proj}} \). This flexibility stems from the non-uniqueness of the QR decomposition of an \( n \times m \) matrix with \( n \geq m \) (see Section 2.2).

The hypothesis that the activation functions are radial is crucial for our results, as they commute with orthogonal matrices (Lemma 4.2), and hence interact favorably with QR decompositions. However, similar dimensional-reduction procedures may well be possible for other activation functions which relate to other matrix decompositions.

Our techniques may also lead to the incorporation of parameter space symmetries to improve generalization. In particular, there may be enhancements of our main results that incorporate
regularization. For example, the loss function of a radial neural network with dimension vector $n$ remains invariant for the $O(n)$ group action after adding an $L^2$ regularization term.

The conceptual flexibility of quiver representation theory can encapsulate neural networks beyond MLPs, including convolutional neural networks (CNNs), equivariant neural networks, recurrent neural networks, graph neural networks, and others. For example, networks with skip connections, such as ResNet, require altering the neural quiver $Q_1$ by adding extra edges encoding skip connections. On the other hand, the case of CNNs would require working with representations of the neural quiver over the ring $\mathbb{R}[t, t^{-1}]$ of Laurent polynomials.

Finally, from a more algebro-geometric perspective, our work naturally leads to certain quiver varieties (i.e. moduli spaces of quiver representations); these may reveal a novel approach to the manifold hypothesis, building on that of Armenta and Jodoin [AJ20].

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A. Appendix: Categorical formulation

In this appendix, we summarize a category-theoretic approach toward the main results of the paper. While there is no substantial difference in the proofs, the language of category theory provides conceptual clarity that leads to generalizations of these results. References for category theory include [Pie91, DF03].

A.1. The category of quiver representations. In this section, we recall the category of representations of a quiver. Background references include [KJ16, N98].

As in Section 3.1, let $Q = (I, E)$ be a quiver with source and target maps $s, t : E \to I$, and let $d : I \to \mathbb{Z}_{\geq 0}$, $i \mapsto d_i$ be a dimension vector for $Q$. Recall that a representation of $Q$ with dimension vector $d$ consists of a tuple $A = (A_e)_{e \in E}$ of linear maps, where $A_e : \mathbb{R}^{d(s(e))} \to \mathbb{R}^{d(t(e))}$.

Let $A$ and $B$ be representations of the quiver $Q$, with dimension vectors $d = \dim(A)$ and $k = \dim(B)$. A morphism of representations from $A$ to $B$ consists of the data of a linear map $\alpha_i : \mathbb{R}^{d_i} \to \mathbb{R}^{k_i}$, for every $i \in I$, subject to the condition that the following diagram commutes for every $e \in E$:

$$
\begin{array}{ccc}
\mathbb{R}^{d(s(e))} & \xrightarrow{A_e} & \mathbb{R}^{d(t(e))} \\
\downarrow{\alpha_{s(e)}} & & \downarrow{\alpha_{t(e)}} \\
\mathbb{R}^{k(s(e))} & \xrightarrow{B_e} & \mathbb{R}^{k(t(e))}
\end{array}
$$

The resulting category $\mathcal{R}(Q)$ is known as the category of representations of $Q$.

A.2. The category $\mathcal{I}(Q_L)$. In this section, we define a certain subcategory $\mathcal{I}(Q_L)$ of the category $\mathcal{R}(Q_L)$. Its objects are the same as the objects of $\mathcal{R}(Q_L)$, that is, representations of the neural quiver, while morphisms in $\mathcal{I}(Q_L)$ are given by isometries.

Let $L$ be a positive integer, and recall the neural quiver $Q_L$ from Section 3.2:

![Neural Quiver](image)

As a reminder, the vertices in the top row are indexed from $i = 0$ to $i = L$, and we only consider dimension vectors whose value at the bias vertex is equal to 1. So a dimension vector for $Q_L$ will refer to a tuple $n = (n_0, n_1, \ldots, n_L)$. Recall the isomorphism

$$\text{Rep}(Q_L, n) \simeq \bigoplus_{i=1}^{L} \text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i})$$

from Lemma 3.1. We denote a representation of $Q_L$ as a tuple $W = (W_i)_{i=1}^{L}$, where each $W_i$ belongs to $\text{Hom}(\mathbb{R}^{1+n_i-1}, \mathbb{R}^{n_i})$. Let $X = (X_i)_{i=1}^{L}$ be a representation of $Q_L$ with dimension vector $m$. Tracing through the proof of Lemma 3.1 and the definitions in Section A.1, we see that a morphism $\alpha : X \to W$ in $\mathcal{R}(Q_L)$ consists of the following data:
• a linear map \( \alpha_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i} \), for \( i = 0, 1, \ldots, L \), and
• a scalar \( \alpha_b \in \mathbb{R} \) at the bias vertex,

making the following diagram commute, for \( i = 1, \ldots, L \):

\[
\begin{array}{c}
\mathbb{R}^{1+m_{i-1}} \xrightarrow{X_i} \mathbb{R}^{m_i} \\
\begin{bmatrix}
\alpha_i \\
0 & \alpha_{i-1}
\end{bmatrix}
\end{array}
\begin{array}{c}
\mathbb{R}^{1+n_{i-1}} \xleftarrow{W_i} \mathbb{R}^{n_i}
\end{array}
\]

**Definition A.1.** We define a subcategory \( \mathcal{I}(Q_L) \) of \( \mathcal{R}(Q_L) \) as follows. The objects of \( \mathcal{I}(Q_L) \) are the same as the objects of \( \mathcal{R}(Q_L) \), that is, representations of \( Q_L \). Let \( X \) and \( W \) be such representations, with dimension vectors \( m \) and \( n \), respectively. A morphism \( \alpha : X \rightarrow W \) in \( \mathcal{R}(Q_L) \) belongs to \( \mathcal{I}(Q_L) \) if the following hold:

• \( n_0 = m_0 \) and \( \alpha_0 \) is the identity on \( \mathbb{R}^{m_0} \),
• \( n_L = m_L \) and \( \alpha_L \) is the identity on \( \mathbb{R}^{m_L} \),
• for \( i = 1, \ldots, L - 1 \), the linear map \( \alpha_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i} \) is norm-preserving, i.e. \( |\alpha_i(v)| = |v| \) for all \( v \in \mathbb{R}^{m_i} \), and
• \( \alpha_b = 1 \).

**Remark A.2.** A norm-preserving map is called an *isometry*, which explains the notation \( \mathcal{I}(Q_L) \).

**Lemma A.3.** We have:

1. The category \( \mathcal{I}(Q_L) \) is a well-defined subcategory of \( \mathcal{R}_L \).
2. Let \( \alpha : X \rightarrow W \) be a morphism in \( \mathcal{I}(Q_L) \). For \( i = 0, 1, \ldots, L \), the linear map \( \alpha_i : \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i} \) is injective.

**Proof.** The claims follow from the facts that (1) the composition of two norm-preserving maps is norm-preserving, and (2) any linear norm-preserving map is injective. \( \square \)

**Definition A.4.** Let \( \alpha : X \rightarrow W \) be a morphism in \( \mathcal{I}(Q_L) \), and let \( m = \dim(X) \) and \( n = \dim(W) \) be the dimension vectors. An **orthogonal factorization** of \( \alpha_i \) is an element \( Q = (Q_1, \ldots, Q_{L-1}) \) of \( O(n) = O(n_1) \times \cdots \times O(n_{L-1}) \) such that

\[
\alpha_i = Q_i \circ \text{inc}_{m_i,n_i}
\]

for \( i = 1, \ldots, L - 1 \). The **correction term** corresponding to an orthogonal factorization is:

\[
U = W - Q \cdot X.
\]

The correction term belongs to \( \text{Rep}(Q_L, n) \).

**Remark A.5.** Orthogonal factorizations always exist, since any norm-preserving linear map \( \mathbb{R}^{m} \rightarrow \mathbb{R}^{n} \) can be written as the composition \( Q \circ \text{inc}_{m,n} \) for some orthogonal \( Q \in O(n) \).

**Remark A.6.** Suppose \( \alpha \) is a morphism in \( \mathcal{I}(Q_L) \). For \( i \in \{1, \ldots, L - 1\} \), the map \( \alpha_i \) is an isomorphism if and only if \( m_i = n_i \). In this case, the choice of \( Q_i \) is unique and \( U_{i+1} = 0 \). Conversely, \( \alpha_i \) is not an isomorphism if and only if \( m_i < n_i \). In this case, there are \( O(n_i - m_i) \) choices for \( Q_i \).
Fix functions $h_i : \mathbb{R} \to \mathbb{R}$ for $i = 1, \ldots, L$. Hence we obtain radial functions $h_i^{(n)} : \mathbb{R}^n \to \mathbb{R}^n$ for any $n \geq 1$. We group the $h_i$ into a tuple $h = (h_1, \ldots, h_L)$. Given $h$, we attach a neural network (and hence a neural function) to every object in $\mathcal{I}(\mathcal{Q}_L)$ as in Procedure 4.5.

**Proposition A.7.** Let $X$ and $W$ be a representations of $\mathcal{Q}_L$. Suppose there is a morphism in $\mathcal{I}(\mathcal{Q}_L)$ from $X$ to $W$. Then the neural functions of the radial neural networks $(X, h)$ and $(W, h)$ coincide:

$$f_{(X,h)} = f_{(W,h)}.$$ 

**Sketch of proof.** The key is to show that, for $i = 1, \ldots, L$, we have:

$$h_i^{(n_i)} \circ \tilde{W}_i \circ \begin{bmatrix} 1 & 0 \\ 0 & \alpha_{i-1} \end{bmatrix} = \alpha_i \circ h_i^{(m_i)} \circ \tilde{X}_i$$

where $\tilde{W}_i$ and $\tilde{X}_i$ are the affine map corresponding to $W_i$ and $X_i$. The first step in the verification of this identity is to choose an orthogonal factorization of $\alpha$. The rest of the proof proceeds along the same lines as the proof of Equation 4.3 in Section 4.3. \hfill \square

A.3. **Projected gradient descent: set-up.** In this section, we collect notation necessary to state results about projected gradient descent.

Let $m = (m_0, \ldots, m_L)$ and $n = (n_0, n_1, \ldots, n_L)$ be dimension vectors for $\mathcal{Q}_L$. We write $m \preceq n$ if:

- $m_0 = n_0$ and $m_L = n_L$,
- $m_i \leq n_i$ for $i = 1, \ldots, L - 1$.

Consequently, if $\alpha : X \to W$ is a morphism in $\mathcal{I}(\mathcal{Q}_L)$, then the dimension vectors satisfy $\dim(X) \preceq \dim(W)$. For $m \preceq n$, we make the following abbreviations, for $i = 0, 1, \ldots, L$:

$$\text{inc}_i = \text{inc}_{m_i,n_i} : \mathbb{R}^{m_i} \hookrightarrow \mathbb{R}^{n_i} \quad \text{inc}_i = \text{inc}_{1+m_i,1+n_i} : \mathbb{R}^{1+m_i} \hookrightarrow \mathbb{R}^{1+n_i},$$

$$\pi_i : \mathbb{R}^{n_i} \twoheadrightarrow \mathbb{R}^{m_i} \quad \tilde{\pi}_i : \mathbb{R}^{1+n_i} \twoheadrightarrow \mathbb{R}^{1+m_i}$$

Using these maps, one defines an inclusion $i : \text{Rep}(m) \hookrightarrow \text{Rep}(n)$ taking $X = (X_1, \ldots, X_L) \in \text{Rep}(m)$ to the representation with $i(X)_i = \text{inc}_i \circ X_i \circ \tilde{\pi}_{i-1}$.

Recall the functions $h = (h_1, \ldots, h_L)$. As in Procedure 4.5, these define activation functions $(h_1^{(n_1)}, \ldots, h_L^{(n_L)})$ (resp. $(h_1^{(m_1)}, \ldots, h_L^{(m_L)})$) for a representation with dimension $n$ (resp. $m$).

Finally, we fix a batch of training data $X = \{(x_j, y_j)\} \subseteq \mathbb{R}^{n_0} \times \mathbb{R}^{n_L} = \mathbb{R}^{m_0} \times \mathbb{R}^{m_L}$. Using the activation functions defined by $h$, we have loss functions on $\text{Rep}(n)$ and $\text{Rep}(m)$ (see Section 3.3):

$$L_n : \text{Rep}(n) \to \mathbb{R} \quad \quad \quad \quad \quad L_m : \text{Rep}(m) \to \mathbb{R}$$

$$W \mapsto \sum_j C(f_{(W,h)}(x_j), y_j) \quad \quad \quad \quad \quad X \mapsto \sum_j C(f_{(X,h)}(x_j), y_j)$$

There are resulting gradient descent maps on $\text{Rep}(n)$ and $\text{Rep}(m)$ given by:

$$\gamma_n : \text{Rep}(n) \to \text{Rep}(n) \quad \quad \quad \quad \quad \gamma_m : \text{Rep}(m) \to \text{Rep}(m)$$

$$W \mapsto W - \nabla_W L_n \quad \quad \quad \quad \quad X \mapsto X - \nabla_X L_m$$

The verification of the following lemma is analogous to the proof of Part 1 of Lemma 5.6.

**Lemma A.8.** We have that $L_n \circ i = L_m$. 

A.4. The interpolating space. We first define a space that interpolates between \( \text{Rep}(\mathbf{m}) \) and \( \text{Rep}(\mathbf{n}) \). The discussion of this section is completely analogous to that in Sections 5.1 and 5.2.

**Definition A.9.** Let \( \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \) denote the subspace of \( \text{Rep}(\mathbf{n}) \) consisting of those \( T = (T_1, \ldots, T_L) \in \text{Rep}(\mathbf{n}) \) such that, for \( i = 1, \ldots, L \), we have:

\[
(i \text{id}_{n_i} - \text{inc}_i \circ \pi_i) \circ T_i \circ \text{inc}_{i-1} \circ \pi_{i-1} = 0.
\]

Just as in Section 5.1, the space \( \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \) consists of representations \( T = (T_1, \ldots, T_L) \in \text{Rep}(\mathbf{n}) \) for which the bottom left \( (n_i - m_i) \times (1 + n_{i-1}^{\text{red}}) \) block of \( T_i \) is zero for each \( i \). Consider the maps:

\[
\begin{align*}
\text{Rep}(\mathbf{m}) & \xymatrix{
\overset{i_1}{\ar@{<->}} & \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) & \overset{i_2}{\ar@{<->}} & \text{Rep}(\mathbf{n})
} \\
\text{Rep}(\mathbf{m}) \xymatrix{
\overset{q_2}{\ar@{<->}} & \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) & \overset{q_1}{\ar@{<->}} & \text{Rep}(\mathbf{n})
}
\end{align*}
\]

- The map \( i_1 : \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \hookrightarrow \text{Rep}(\mathbf{n}) \) is the natural inclusion.
- The map \( q_1 : \text{Rep}(\mathbf{n}) \rightarrow \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \) takes \( T \in \text{Rep}(\mathbf{n}) \) to the representation whose \( i \)-th slot is

\[
W_i - (i \text{id}_{n_i} - \text{inc}_i \circ \pi_i) \circ W_i \circ \text{inc}_{i-1} \circ \pi_{i-1}.
\]

- The map \( i_2 : \text{Rep}(\mathbf{m}) \hookrightarrow \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \) is defined by taking \( X \) to the representation whose \( i \)-th coordinate is \( \text{inc}_i \circ X_i \circ \pi_{i-1} \).
- The map \( q_2 : \text{Rep}^{\text{int}}(\mathbf{n}) \twoheadrightarrow \text{Rep}(\mathbf{m}) \) is defined by taking \( T \) to the representation whose \( i \)-th slot is \( \pi_i \circ T_i \circ \text{inc}_{i-1} \).

**Definition A.10.** The projected gradient descent map on \( \text{Rep}(\mathbf{n}) \) with respect to \( \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \) is defined as:

\[
\tau_n : \text{Rep}(\mathbf{n}) \rightarrow \text{Rep}(\mathbf{n}) \quad \text{W} \mapsto \text{W} - i_1 \circ q_1(\nabla \text{W} \text{L})
\]

We now state the main result of this appendix. We omit a proof, as it can be proven in largely the same way as the proof of Theorem 5.4 given in Section 5.3. The main difference is that all appearances of \( \mathbf{n}^{\text{red}} \) need to be replaced by \( \mathbf{m} \).

**Theorem A.11.** Let \( \alpha : X \rightarrow W \) be a morphism in \( \mathcal{L}(Q_L) \). Let \( Q \) be an orthogonal factorization of \( \alpha \) and set \( U = W - Q \cdot X \).

1. The representation \( Q^{-1} \cdot W = X + Q^{-1} \cdot U \) belongs to \( \text{Rep}^{\text{int}}(\mathbf{m}, \mathbf{n}) \).
2. For any \( k \geq 0 \), we have

\[
\gamma^k_n(W) = Q \cdot \gamma^k_n(Q^{-1} \cdot W).
\]
3. For any \( k \geq 0 \), we have

\[
\tau^k_n(Q^{-1} \cdot W) = \gamma^k_m(X) + Q^{-1} \cdot U.
\]

A.5. Relation to Theorem 5.4. In this final section, we relate the general categorical results of this appendix to Algorithm 2. Recall that, in Section 4.2, we associated a reduced dimension vector \( \mathbf{n}^{\text{red}} \) to any dimension vector \( \mathbf{n} \) of the neural quiver \( Q_L \).

**Proposition A.12.** Let \( W \) be a representation of \( Q_L \) with dimension vector \( \mathbf{n} \). Then \( W \) admits a morphism in \( \mathcal{L}(Q_L) \) from a representation with dimension vector \( \mathbf{n}^{\text{red}} \).
Proof. We proceed by induction on the number $N$ of $i \in \{1, \ldots, L-1\}$ such that $n^\text{red}_i < n_i$. If $N = 0$, there is nothing to show. Otherwise, let $j$ be the smallest element of $\{1, \ldots, L-1\}$ such that $n^\text{red}_j < n_j$. Then $n^\text{red}_j = n_{j-1} + 1$. Let $W_j = Q \circ \text{inc}_{1+n_{j-1}, n_j} \circ R$ be a QR decomposition of $W$, where $Q \in O(n_j)$ and $R$ is an upper-triangular $n_{j-1} + 1$ by $n_{j-1} + 1$ matrix. Consider the representation of $Q_L$ given by:

$$X = \left( W_1, \ldots, W_{j-1}, R, W_{j+1} \circ \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \circ \text{inc}_{1+n_{j-1}, n_j}, W_{j+2}, \ldots, W_L \right)$$

Then $\alpha = (\text{id}_{n_1}, \ldots, \text{id}_{n_{j-1}}, Q \circ \text{inc}_{1+n_{j-1}, n_j}, \text{id}_{n_{j+1}}, \ldots, \text{id}_{n_L})$ is a morphism from $X$ to $W$. The dimension vector of $X$ is $(n^\text{red}_0, \ldots, n^\text{red}_{j-1}, n^\text{red}_j, n_{j+1}, \ldots, n_L)$ and has one less coordinate that $n$ not equal to $n^\text{red}$, so the induction hypothesis applies. \hfill $\square$

Consequently, Theorems 4.6 and 5.4 can be viewed as corollaries of Proposition A.7, Theorem A.11, and Proposition A.12.

Remark A.13. Let $Q$, $R$ and $U$ be the outputs of Algorithm 2 applied to a representation $W$ of dimension vector $n$. Then $R$ defines a representation of dimension vector $n^\text{red}$, and $Q$ defines a morphism in $\mathcal{I}(Q_L)$ from $R$ to $W$. Indeed, for $i = 1, \ldots, L$, Algorithm 2 provides the equality

$$W_i \circ \begin{bmatrix} 1 & 0 \\ 0 & Q_{i-1} \end{bmatrix} \circ \text{inc}_{i-1} = V_i = Q_i \circ \text{inc}_i \circ R_i,$$

(where $Q_0 = \text{id}_{n_0} = \text{inc}_0$ and $Q_L = \text{id}_{n_L} = \text{inc}_L$), which implies that the following diagram commutes:

$$
\begin{array}{ccc}
\mathbb{R}^{1+n^\text{red}_i} & \xrightarrow{R_i} & \mathbb{R}^{n^\text{red}_i} \\
\begin{bmatrix} 1 & 0 \\ 0 & Q_{i-1} \end{bmatrix} \circ \text{inc}_{i-1} & \downarrow & Q_i \circ \text{inc}_i \\
\mathbb{R}^{1+n_{i-1}} & \xrightarrow{W_i} & \mathbb{R}^{n_i}
\end{array}
$$