COMMENTS ON SPACE-TIME

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I. ABSTRACT

In this article we discuss a few aspects of the space-time description of fields and particles. In sections II and III we demonstrate that fields are as fundamental as particles. In section IV we discuss non-equivalence of the Schwarzschild coordinates and the Kruskal-Szekeres coordinates. In section V we discuss that it is not possible to define causal structure in discrete space-time manifolds. In App.B we show that a line is not just a collection of points and we will have to introduce one-dimensional line-intervals as fundamental geometric elements. Similar discussions are valid for area and volume-elements. In App.C and App.D we make a comparative study of Quantum Field Theory and Quantum Mechanics and contradictions associated with probabilistics interpretation of these theories with space-time dimensional analysis. In App.E and App.F we discuss the geometry of Robertson-Walker model and electrostatic behavior of dielectrics respectively. In Sup.I we discuss the regularity of Spin-Spherical harmonics and also derive an energy-spectrum which is free of back-reaction problem. In Sup.II we discuss that in general the integral version of Gauss’s divergence law in Electrodynamics is not valid and rederive Gauss’s law and Ampere’s law. We also show that under duality transformation magnetic charge conservation law do not remain time reversal symmetric. In Sup.III we derive the complete equation for viscous compressible fluids and make a few comments regarding some contradictions associated with boundary conditions for fluid dynamics. In Sup.IV we discuss a few aspects on double slit interference experiments. We conclude this article with a few questions in Sup.V.

II. OVERVIEW

Gravitational interaction which is universally attractive is described by the General Theory of Relativity, a theory of fields realized through the existence of space-time and some of its geometric properties, e.g., the curvature. The principle of equivalence leading to the fact that the guinea and the feather fall the same way in vacuum together with the fact that gravity violets the elementary quantum mechanical principles (as will be evident from this article) indicates that fields are as fundamental as particles. Fields and particles with their common and contradictory kinematical and dynamical features are the two fundamental constituents of nature. The attraction of opposite charges can only be explained through accepting the electric field as a fundamental entity of nature and not through only interaction mediating particle interaction (be the interaction mediating particles interacting with the sources or among themselves) although the repulsion between like charges may be explained in the later way with photons as the interaction mediator. Quantum mechanics is an incomplete approach to explain the Solar system microscopic physics purely in terms of wave-like properties whereas Quantum Field Theory is an incomplete approach to explain the Solar system microscopic physics purely in terms of particle-like properties. For the vacuum polarization explanation of the potential [18] in the Bhaba scattering process it is not obvious how, in terms of photon exchanges, the loops in vacuum will screen the charges of $e^-, e^+$. Also an electron-positron loop with two external photon lines can not explain charge screening in any process as the particle-antiparticle pairs are created and annihilated after the interaction mediating photon had been created at one real particle and before the interaction mediating photon has interacted with the other real particle. None of these two theories are in accordance with gravity as will be manifested in this article.

III. INTRODUCTION

During the last few decades a lot of efforts had been devoted to unify the general theory of relativity (describing the gravitational interaction) with quantum mechanics (describing the microscopic interactions of the elementary particles). Yet the conventional theory of quantum mechanics, based on unitarity and

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symmetries, is contradictory with general relativity in many respects, e.g., formally infinite zero point energy associated with canonical quantization scheme, ultraviolet divergent energy density associated with vacuum fluctuations [1] for collapsing physical systems, unitarity violation for black hole evolution.

To unify these two descriptions of nature we can proceed along two directions [2]:

For the Euclidean school space-time geometry is an abstract concept which exist irrespective of matter fields. For the non-Euclidean school space-time does not exist independent of matter fields. Space-time is form of existence of matter and can not be conceived without matter. This feature is even more transparent from the facts, among others, that the universe is compact and there is no well-defined stress-tensor for gravity which interwinds matter fields and space-time geometry [3].

In the context of the general theory of relativity the conflict between the two schools arise in the following way:

As soon as one derives the geodesic deviation equation from the principle of equivalence one “can” forget the source, ascribing the relative acceleration of the nearby geodesics to the space-time manifold. This, in contradiction to the philosophy of general relativity, may lead to think that the space-time manifold is more fundamental leading to the concept of quantum gravity irrespective of existence of the corresponding sources that will produce the fluctuating geometries (no source indicates no space-time geometry and observationally, quantum fluctuations in matter fields are negligible to produce significant alternations of the space-time geometry).

Quantum gravity also led to many space-time geometries which are physically non-existent. One such example is the extreme Reissner-Nordstrom black hole which cannot be obtained through any realistic gravitational collapse [4]. It also has vanishing Hawking temperature. This may be interpreted through the fact that Hawking radiation through pair production near the black hole event horizon is not possible as the metric do not change its signature across the horizon.

We should keep in mind that the Einsteins equations in the regime of its validity determine the space-time geometry: the geometry of space-time as a whole is determined by corresponding matter fields described either in terms of some classical models or by a proper quantum theory. This gives a particular cosmology (the closed or mathematically more properly compact universe picture) and if we view cosmology as a whole there is really no test body.

We will now consider some aspects of the black hole space-time geometry. In the process of gravitational collapse an event horizon, the black hole event horizon, is formed breaking the global CP invariance and giving rise to the Kerr-Newman families of black holes (the no hair theorems). The black hole event horizon may be defined as the causal boundary of the set of complete time-like geodesics which originates at the past time-like infinity and terminate at the future time-like infinity as classically nothing can come out off the horizon. The black hole space-time is usually described either in terms of the Schwarzschild coordinate system or in terms of the Kruskal-Szekers coordinate system. In the Schwarzschild coordinate system the black hole event horizon is a two dimensional fixed point set of the time-like Killing vector field across which some of the metric components change sign. In the Kruskal-Szekers coordinate system the event horizon is a two dimensional null surface across which the square of some of the coordinates change sign. We will now consider the non-equivalence of the two coordinate systems in detail.

IV. NON-EQUIVALENCE OF THE SCHWARZSCHILD AND THE KRUSKAL-SZEKERS COORDINATE SYSTEM

The Schwarzschild space-time is a Lorentz signature, static spherically symmetric solution of the Einstein equations when the Ricci tensor vanishes. This solution describes the exterior geometry of a static spherically symmetric star and has been used to verify the predictions of general relativity for the Solar system.

A space-time is said to be static if there exits a space-like hypersurface which is orthogonal to the orbits of the time-like Killing vector field. A space-time is said to be spherically symmetric if the space-like hypersurfaces contains $SO(3)$ as a subgroup of the group of isometries. The orbit spheres of $SO(3)$ are isometric to the unit two sphere. These features together with the condition of the asymptotic Newtonian limit give the well-known Schwarzschild solution in the spherical polar coordinates [3]:

$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2[d\theta^2 + \sin^2 \theta d\phi^2]$$  \hspace{1cm} (1)

According to the Birkhoff’s theorem [19] all spherically symmetric solutions with $R_{ab} = 0$ are static and the Schwarzschild space-time is the unique static spherically symmetric solution, upto diffeomorphisms, of the Einstein equations with $R_{ab} = 0$.  


The norm of the time-like Killing vector field and \((\nabla r)^{\alpha}\) in the orthonormal coordinates vanishes and some of the metric components are not well-behaved at \(r = 2M\) in the Schwarzschild coordinates. The proper acceleration of the constant \(r\) observers can be obtained from the geodesic equations in the Schwarzschild coordinates. This acceleration, \(a = (1 - 2M/r)^{-1/2}M/r^2\), is divergent at the horizon \((r = 2M)\).

The ill-behavedness of the Schwarzschild coordinates is not a coordinate singularity like that of the spherical polar coordinate system where the azimuthal angular coordinate \(\phi\) become ambiguous at the poles. All the ill-behavedness of the Schwarzschild coordinates at the horizon originate from that of the space-time metric. The curvature scalars calculated from the metric are well-behaved at the horizon unlike the same space-time manifold. Consequently, according to Birkhoff’s theorem, the space-time represented

\[ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2\]  

(2)

The Regge-Wheeler coordinate system is defined through the null-geodesics and is given by:

\[r_* = r + 2Mln(r/2M - 1)\]  

(3)

in this coordinate \(r \to 2M\) corresponds to \(r_* \to -\infty\). The null coordinates are defined as:

\[u = t - r_*, \quad v = t + r_*\]  

(4)

A regular metric is obtained through the following transformation,

\[U = -e^{-u/4M}, \quad V = e^{v/4M}\]  

(5)

The metric in these coordinates becomes:

\[ds^2 = -\frac{32M^3e^{-r/2M}}{r}dUdV\]  

(6)

As there is no longer a coordinate singularity at \(r = 2M\) (i.e. at \(U = 0\) or \(V = 0\)) one extends the Schwarzschild solution by allowing \(U, V\) to take all possible values. However the transformation coefficients \(dU/dr = -d[(r/2M - 1)^{1/2}e^{-3(u/4M)}]/dr\) and \(dV/dr = d[(r/2M - 1)^{1/2}e^{(v/4M)}]/dr\) are singular at \(r = 2M\) and the extension is not diffeomorphically equivalent. Consequently as discussed at the beginning of this section the Schwarzschild coordinate system and the \((U, V)\) coordinate system do not represent physically the same space-time manifold. Consequently, according to Birkhoff’s theorem, the space-time represented
by the \((U, V, \theta, \phi)\) coordinate system is not a solution of the Einstein equations for a spherically symmetric black hole.

Similar discussions are valid for the Kruskal-Szekers coordinate transformations which are obtained through the following transformations:

\[
T = \frac{(U + V)}{2}, \quad X = \frac{(V - U)}{2}
\]

and the metric becomes,

\[
ds^2 = \frac{32M^3e^{-r/2M}}{r}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

The relation between the \((T, X)\) and the \((t, r)\) coordinates are well known and in the physical regions of interests are given by [4],

\[
X = (r/2M - 1)^{1/2}e^{r/4M}\cosh(t/4M)
\]

valid for \(r > 2M\), and

\[
T = (r/2M - 1)^{1/2}e^{r/4M}\sinh(t/4M)
\]

valid for \(r < 2M\).

Again the transformation coefficients are not defined on the horizon and the Kruskal-Szekers coordinates do not give a proper diffeomorphic extension of the Schwarzschild coordinate system. Hence the Kruskal-Szekers coordinates is not a solution of the Einstein’s equations for a spherically symmetric black hole.

The Kruskal-Szekers coordinate system had been introduced to eliminate a particular singular function (the metric components) in the Schwarzschild coordinate system through a singular coordinate transformation. This does not ensure that all singular tensors can be made regular in the new coordinate system and also tensors which are regular in the \((t, r)\) coordinates can become singular in the \((T, R)\) coordinates. To illustrate these features we consider the implicit relations between the two coordinate systems [1]:

\[
(r/2M - 1)e^{r/2M} = X^2 - T^2
\]

\[
\frac{t}{2M} = \ln\left(\frac{T + X}{X - T}\right)
\]

The horizon in this coordinates are defined as \(X = \pm T\).

Firstly the proper acceleration of the curves in Kruskal-Szekers’s coordinate system which correspond to the constant \(r\) observers in the Schwarzschild coordinate system is given by \(a = (X^2 - T^2)^{-1/2}[e^{r/2M}M/r^2]\). This is also divergent on the horizon.

Secondly we consider the vector \(\left(\frac{dR}{dr}\right)^a, R^a\), the proper rate of change of the curvature scalar \(R\) obtained from \((dR)^a\) and the proper distance \(ds\) [i.e, the vector \(\left(\frac{dR}{dr}\right)\left(\frac{d}{ds}\right)\), Appendix:A]. The norm of this vector in the Schwarzschild coordinate system is \((dR/dr)^2\) and is finite on the horizon. Whereas the corresponding quantity in the \((T, X)\) coordinates can be obtained from the following relations [apart from normalizing factors: \(\left(\frac{dX}{dr}\right), \left(\frac{dT}{dr}\right) = [r^{3/2M}]^{1/2}\) :

\[
\frac{dR}{dT} = \frac{\partial R}{\partial T} \frac{\partial T}{\partial r}, \quad \frac{dR}{dX} = \frac{\partial R}{\partial r} \frac{\partial r}{\partial T}
\]
and from equ.(13),
\[
\frac{\partial r}{\partial X} = \frac{8M^2Xe^{-r/2M}}{r}, \quad \frac{\partial r}{\partial T} = -\frac{8M^2Te^{-r/2M}}{r}
\]
(16)

and we have \(|R'^a_{\mu\nu}|^2 = \frac{64M^2Xe^{-r/2M}(dT)^2[X^2 - T^2]}{r^2} = 0\) on the horizon although the \(r\)-dependent multiplying factor in front of the Kruskal-Szekeres metric is finite at \(r = 2M\).

The unit space-like normal vector to the \(r = \text{constant}\) surfaces, which can be defined apart from \(r = 0\), \(k^a = \left(\frac{dr}{dT}\right)^a\) has unit norm \((k^a_k = 1)\) on \(r = 2M\) although \(k^a \rightarrow 0\) as \(r \rightarrow 2M\) which for an outside observer \((r > 2M)\) may be interpreted as nothing can propagate radially outward at \(r = 2M\), consistent with the divergent acceleration for a radially infalling particle. Also no combination of the unit time-like normal and the unit space-like normal to the \(r = \text{const.}\) surfaces are possible whose norm is zero on the horizon but finite for \(r > 2M\).

For two metric spaces the definitions of continuity is as follows [16]:

Let \((S, d_S)\) and \((T, d_T)\) be metric spaces and let \(f : S \rightarrow T\) be a function from \(S\) to \(T\). The function \(f\) is said to be continuous at a point \(p\) in \(S\) if for every infinitesimal \(\epsilon > 0\) there is an infinitesimal \(\delta > 0\) such that

\[
d_T[f(x), f(p)] < \epsilon, \quad \text{whenever} \quad d_S[x, p] < \delta.
\]
(17)

If \(f\) is continuous at every point of \(S\) then \(f\) is continuous on \(S\).

The definition is in accordance with the intuitive idea that points close to \(p\) are mapped by \(f\) into points closed to \(f(p)\). From equ.(13),(14) we have,

\[
|dT|_{Sch} = \frac{X}{(X^2 - T^2)^{1/2}}|dT|_{KS}, \quad |dT|_{Sch} = -\frac{T}{(X^2 - T^2)^{1/2}}|dX|_{KS}
\]
(18)

and,

\[
|dr|_{Sch} = \frac{X}{(X^2 - T^2)^{1/2}}|dX|_{KS}, \quad |dr|_{Sch} = -\frac{T}{(X^2 - T^2)^{1/2}}|dT|_{KS}
\]
(19)

where | | denotes the norm in the respective coordinate systems and we find that the coordinate transformation, \((t, r) \rightarrow (T, X)\) is not continuous on the horizon as the multiplicative factors diverge on the horizon \((X = \pm T)\). Consequently the coordinate transformation \((t, r) \rightarrow (T, X)\) is not a homeomorphism and the two coordinate systems do not topologically represent the same space-time manifolds [3,17]. Hence we show that that the Kruskal-Szekeres coordinate system is not a proper extension of the Schwarzschild coordinate system and it is not a solution of the Einstein's equation for spherically symmetric black hole. We conclude this discussion with the following note:

For any coordinate system we have,

\[
g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} (g_{Sch.})_{\rho\sigma}
\]
(20)

Consequently it is not possible to find a coordinate system with a regular \(g'_{\mu\nu}\) without absorbing the singularities of \((g_{Sch.})_{\rho\sigma}\) at \(r = 2M\) into the transformation coefficients \(\frac{\partial x^\rho}{\partial x'^\mu}\) at \(r = 2M\) i.e, without breaking the diffeomorphic equivalence of the two coordinate systems. Thus, as also discussed in the preceding sections, the Kruskal-Szekeres coordinate system with a regular metric at the horizon can not be diffeomorphically equivalent to the Schwarzschild coordinate system and thus do not represent a static asymptotically flat solution of the Einstein's equations representing a black hole formed out of the gravitational collapse of an uncharged spherically symmetric asymptotically flat star. [see also Appendix:E].

In passing we note that the gravitational collapse to form black hole is associated with entropy decrease. The entropy of a star is proportional to its volume for \(r > 2M\) whereas the entropy becomes proportional to the area of the horizon, \(16\pi M^2\), as the star crosses the Schwarzschild radius to form a black hole.

It is not obvious how to describe the space-time evolution of the complete gravitational collapse of matter fields as a whole in terms of time-like curves as, for a Schwarzschild observer, the time-like curves suffer a discontinuity across the horizon and become space-like inside the black hole event horizon. It is wellknown that expressed in terms of the Schwarzschild coordinates the black hole event horizon has profound impact
on the quantum description of matter fields and black hole evaporation through Hawking radiation makes the space-time dynamic. Also Hamiltonian evolution of matter fields break down on the fixed point sets of the time-like Killing vector field [14]. The canonically conjugate momentums are not well-defined on the horizon as will be evident from the lagrangians of the matter fields.

V. DISCUSSION

In a gravitational collapse once the collapsing body crosses the horizon it collapses to form the space-time singularity breaking the description of space-time in terms of continuous manifolds and the local symmetries. We can only characterize the presence of of the space-time singularity in a diffeomorphism invariant way, in terms of the curvature invariants along the space-time curves which cross the event horizon and necessarily terminate along the space-time singularity. The formation of black hole event horizon can be characterized through the formation of trapped surfaces. The gravitational collapse and the cosmological evolution are the only two processes in nature through which a three dimensional physical system collapses to zero dimension (forming the space-time singularity). Here through zero dimension we mean a point or a collection of points. We will illustrate this aspect in Appendix:B.

Einstein’s equations break down at the space-time singularity. This is something similar to electrodynamics. We can determine the electric field for a point charge using the Maxwell’s equations. But the field strength diverges and classical electrodynamics break down at the point charge (the corresponding quantum theory QED is not a resolution to this problem. It has its troubles associated with the point-like interaction terms. However experimental observations confirm that all the elementary particles are of finite volume). The formation of space-time singularity is associated with finite volume to zero dimension transition for the corresponding collapsing body and the richest structure that we can attribute to zero dimension is that of an analogue of (compact) three dimensional generalization of the Cantor set [5,6] provided we generalize the description of the collapsing matter field through a proper quantum theory [a generalization of the Pauli exclusion principle].

There are two ways that one can reach zero dimension from finite volume breaking the continuous topology of space-time manifold. One is through the point contraction mapping which requires an infinite number of iterations which, together with the discussions in Appendix:B, is in accordance with the fact that time is a continuous parameter. The other one is through the formation of an analogue of the Cantor set (or any other discrete manifold with different cardinality) in which case the underlying physical processes to achieve zero dimension may be discontinuous. A discrete manifold may not always form a normed vector space, e.g., the set of points \((n + x)\) on the real line, where \(n\) is an integer and \(x\) is a fractional number, can not form a normed vector space as the difference between two points do not belong to the set. Also it is not physically obvious to talk of causal structure, defined through propagation of signals, in a discrete manifold unless the manifold is space-like and frozen in time (which is defined through physical processes). As discussed earlier according to the General Theory of Relativity charges associated with space-time transformation symmetries are global properties of a continuous space-time manifold as a whole whereas a manifold without continuous topology can only have space-time independent charge.

To describe cosmological evolution and black hole evolution we will have to generalize and geometrize conventional quantum mechanics in a suitable way. In these respects the principal aspects to be critically studied, as will be discussed later in this article, towards unifying the general theory of relativity and conventional quantum mechanics is the concept of diffeomorphism invariance associated with the general theory of relativity and unitarity associated with the conventional quantum mechanics.

The facts that the continuous topology of space-time break down at the space-time singularity [Appendix:B] (indicating no well-defined observables associated with spatial transformations for the cosmologically evolving or collapsing matter fields in the near zero dimension region) and that nature choses a particular cosmology lead us to conclude that diffeomorphism invariance (which for large scale structure of space-time is equivalent to invariance under coordinate transformations) is not of so important (as it is for solar system microscopic physics) concern for the corresponding physical laws. Rather the fact Schwarzschild coordinates and the Kruskal- Szekers coordinates are not diffeomorphically equivalent indicates that an appropriate choice of a suitable coordinate system is most important. However we can express the generalized quantum theory in covariant form. This will help us to compare the generalized quantum theory with solar system quantum physics where the physical laws are invariant under inertial transformations and are formulated in a covariant way under the corresponding coordinate transformations.

To generalize the conventional quantum mechanics we should take into account the following important aspects:

1. Special relativity made the concept of size for ordinary objects a relative one. The strong curvature
effects near the space-time singularity will spoil the concept of dimension for the elementary particles forming the space-time singularity.

2. Quantum mechanics is the mechanics of quantum states which do not exist independent of their realizations at least in principle, i.e., through interactions with other quantum states. For solar system microscopic physics the fact that the elementary particles or bound states formed by them are of finite volume has to be considered in the corresponding quantum state description as long as the continuous structure of the space-time manifold holds.

3. Every measurement process through state reduction is a non-unitary operation on the space of quantum states [7]. Near the space-time singularity the strong curvature will destroy the description of matter fields in terms of a unitary quantum theory.

As far as the cosmological evolution is concerned (zero dimension to finite volume and finite volume to zero dimension transitions) no observer physics is the exact description of the evolution of the universe in the near zero volume region. A proper generalization of quantum mechanics may be non-unitary in the sense that the evolutions of the possible quantum states (if the space-time description of matter is given by a particular family of space-time curves representing possible particle trajectories [5]) representing the collapsing physical systems may be non-unitary.

However, it is obvious that the collapsing physical system (which is a bound system through gravitational interaction) collapses to zero dimension violating the conventional quantum mechanics based on the uncertainty principle (e.g., the electrons in an atom obeying the quantum mechanical principles do not collapse on the positively charged nucleas) and follow the deterministic laws of general relativity.

In the context of the above discussions an important contradiction between the general theory of relativity (describing the gravitational interaction) with conventional quantum mechanics (describing the microscopic interactions of the elementary particles) is the following:

Positivity of the energy momentum stress-tensor together with the general theory of relativity leads to gravitational collapses [8] and space-time singularities [9] where a three dimensional physical system collapses to zero dimension (breaking the continuous space-time topology) whereas positivity of the energy-momentum tensor together with the canonical commutation relations lead to the Pauli exclusion principle (unless one introduces additional structures about the space-time singularity).

We, living beings, are characterized by the fact that we can control some terrestrial processes. But we can neither change the physical laws nor the cosmological evolution. Many descriptions we had made are either incomplete (unitary quantum mechanics) or approximations (point particles for microscopic physics). The discussions in the preceding paragraph together with the facts that black holes contain no scalar hair [11], that there is no physical explanation of only recombination for the virtual particles (which “interpret” real effects as in the Casimir effect) to form loops in quantum field theory [Physically, in vacuum, even in Feynmann’s summing over path scheme it is not obvious why particle pairs produced at one space-time point will only recombine at another space-time point. The otherwise should give abundances of particles and antiparticles. We will discuss this issue in some details in Appendix:C.] and that the universality of the minimum uncertainty relations are lost in the gravitational collapses and are questionable in the solar system microscopic physics [12] lead to conclude that the proper avenue towards unifying these two theories and thereby explaining the cosmological evolution completely will be understanding the space-time singularity and extending the conventional quantum theory as the position-momentum Canonical commutation relations are in accordance with the corresponding minimum uncertainty relations [13].

VI. APPENDIX:A

We can obtain the one-form \((d\phi)_a\) from a zero-form (a scalar field) \(\phi\) in an explicit coordinate variables notation:

\[
(d\phi)_a = \sum \frac{\partial \phi}{\partial x^\mu}[(dx)_a]^\mu
\]  

(21)

here the range of the summation is the dimension of space-time and it does not represent an infinitesimal change in \(\phi\). When expressed in a particular coordinate basis \([(dx)_a]^\mu\] will be just \(dx^\mu\), a coordinate one-form, and the \(\mu\)-th component of \((d\phi)_a\) is \(\frac{\partial \phi}{\partial x^\mu}\) as an arbitrary tensor \(T\), in its operator form, represented in a coordinate basis can be expressed as:

\[
T = T^{\alpha\beta...\mu\nu...}(\frac{\partial}{\partial x^\alpha})(\frac{\partial}{\partial x^\beta})...(dx^\mu)(dx^\nu)... 
\]  

(22)
Here \( \left( \frac{\partial}{\partial x^i} \right) \) are coordinate unit vecors and \( dx^{\mu} \) are coordinate one-forms.

In terms of a coordinate basis the covariant \( d\text{A}lembartian \) operator can be obtained from the invariant:

\[
(d\phi)^\mu(d\phi)_\mu = \sum \sum g^{\mu\nu} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu}
\]

here the explicit summations are again over \( \mu, \nu \) with the ranges same as above. When expressed explicitly in a coordinate basis the Lagrangian density of a massless scalar field is given by: \( g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \).

The infinitesimal change \( \delta \phi \) of the scalar field \( \phi \) can be interpreted as the scalar product of the one form \( (d\phi)_a \) and the infinitesimal vector line elements.

### VII. APPENDIX:B

In this section we will briefly discuss the relationship between the fundamental properties of the elementary particles and space-time geometry. We will show that a line-element is not just a collection of points. Line-elements, area-elements and volume-elements are as fundamental as points. We also demonstrate that three spatial dimensions is accordance with the finite volume and spin of the elementary particles to give unique dynamics. We will also discuss the consequences of these aspects for geometry and coordinatization.

A point is a dimensionless object. A line is a one dimensional object. Two lines intersect at a point. If two lines intersect each other at every point they are said to be coincident. Two non-coincident lines are said to be parallel if they never intersect each other. A plane is a two dimensional object and any point on the plane can be characterized through the choice of two lines on the plane and constructing a coordinate system in the usual way. There are two possible motions on a plane, translations and rotations. Rotations can be uniquely characterized only through their magnitude and the unique normal to the plane. In three dimension any infinitesimal rotation can be taken to take place on a plane uniquely characterized through its unique normal which is contained within the three dimensional spatial geometry. Higher dimensions greater than three will spoil the uniquness of the normal to the plane of rotation.

Thus three dimensional spacial geometry is self-complete both geometrically and to describe the dynamics of matter particles uniquely. Also the concept of orientation with a proper convention is an essential aspect to formulate the laws of Classical Electrodynamics consistent with the energy conservation law. The fact that in general, apart from the case when the plane of the loop is perpendicular to the magnetic field, a time-varying magnetic field always induces an electric current in a closed loop obeying Faraday’s law indicate that spacial geometry should be of three dimension.

A line is not just a collection of points as a collection of dimensionless objects (points), however large may be in cardinality, cannot give a dimensionful object. For example the total length of the deleted intervals to form the Cantor set (a collection of discrete points) is 1 although the cardinality of the Cantor set is same as the original unit length interval \([5]\) considered, conventionally, as a collection of points. On a line “two points separated by an interval (finite or infinitesimal)” or “two points coincident” have meaning but “two points adjacent” is not defined. We have to introduce line intervals, finite or infinitesimal, as fundamental mathematical entities to form lines giving them the continuous topology. We illustrate this feature in the context of coordinatization of the real line.

An arbitrary real rational number may expressed as \( r = n.x_1x_2x_3...........x_p \), where \( n \) is an integer, \( x_i = 0, \ldots, 9 \) for \( i < p \), \( x_j = 0 \) for \( j > p \), \( p \) may be arbitrarily large but finite and \( x_p \neq 0 \), i.e., the sequence of the decimal places is finite. Whereas an irrational number is given by: \( ir = n.x_1x_2x_3........... \) and the sequence of the decimal places runs to infinity. Here the definitions differ from the conventions. We will consider \( r, ir > 0 \) in the following section.

It is easy to find that between any two rational numbers there are infinite number of irrational numbers as any irrational number (with \( x_i \) same as that of \( r \) for all \( i < p \)) whose \( p \)-th decimal number is equal to \( x_p \) is greater than \( r \) and any irrational number (again with \( x_i \) same as that of \( r \) for all \( i < p \)) whose \( p \)-th decimal number is equal to \( (x_p - 1) \) is lower than \( r \). Similar arguments as above and discussed below, are valid for two irrational numbers as the sequence of decimal places for an irrational number is infinite.

We now prove the following proposition:

On a line the concept ”two points adjacent” is not well-defined. However close two points may be we will always find a point lying between them. Thus on a line only the statements ”two points separated by an interval” or ”two points coincident” are well-defined and line-intervals are fundamental geometric objects.

If someone say that the two points on the real line represented by a rational number \( r(\neq \text{an integer}) \) and an irrational number \( ir \) are adjacent; we can always find an irrational number, following the construction
in the paragraph before the last one with any \(x_i\) \((i > p)\) replacing \(x_p\), which is greater than or lower than \(ir\) and is nearer to \(r\).

If \(r\) is an integer \((> 0)\) and \(ir = r.x_1x_2x_3........\), with all but \(x_p\) \((= 1)\) are zero for \(p \to \infty\), the whole region infinity on \(R^3\) can be used through a mapping of the decimal position index \(p\) of \(ir\) for \(p \to \infty\) (with only \(x_p\) nonzero, \(= 1)\) to the integers in the region infinity of \(R^3\) to construct an irrational number less than \(ir\) and closer to \(r\) \([i.e.,\ the\ number\ of\ decimal\ places\ having\ value\ zero\ before\ x_p = 1\ for\ p \to \infty\ can\ be\ increased\ indefinitely]\). Similar will be the case for \(ir = (r - 1).999.....\), with all \(x_i = 9\) but \(x_j = 8\) for \(j \to \infty\) and \(j > i\), to construct an irrational number greater than \(ir\) and closer to \(r\) through a mapping of the decimal position index \(j\) of \(ir\) for \(j \to \infty\) (with all \(x_i = 9, x_j = 8\) for \(j > i\)) to the integers in the region infinity of \(R^3\), \([i.e.,\ the\ number\ of\ decimal\ places\ having\ value\ 9\ before\ x_j = 8\ for\ j \to \infty\ can\ be\ increased\ indefinitely]\).

Similar as above will be the arguments with \((n \geq 0)\ r = n.x_1x_2x_3........\), \(ir = n.x_1x_2x_3........\) \((x_1\ same\ for\ both\ for\ 1 \leq i \leq p)\) with all \(x_i (i > p)\) but \(x_j (= 1)\) are zero for \(j \to \infty\) and for \(ir = n.x_1x_2x_3........\) \(y_p.999...\) with the \(p\)-th decimal place \(y_p = x_p - 1\) and all \(x_i (i > p)\) but \(x_j (= 8, j > i)\) for \(j \to \infty\) are equal to 9.

For two irrational numbers \((ir)_1, (ir)_2\) we can always recordinatize \(R^3\) so that \((ir)_1\) become a rational number and whether \((ir)_2\) is a rational number or not the arguments in the above paragraphs show that there are non-denumerably infinite number of points between \((ir)_1\) and \((ir)_2\).

For \(r \leq 0\) the corresponding arguments to prove that “two points on \(R^3\) are adjuscent” is not defined are very similar as in the preceding paragraphs.

The above arguments lead us to consider line interval as the fundamental geometrical entity to form a line and the real line is not just an array of ordered points. Irrational numbers, as defined in this section, demonstrate the continuous topology (existence of neighbourhoods) of the real line.

Similarly, as we will illustrate later in this section, we cannot obtain a two dimensional plane from a collection of lines and a three dimensional hyperplane from a collection of planes. Thus one, two and three dimensional intervals are fundamental geometrical entities to have one, two and three dimensional geometries and give the continuous topology. One can not obtain points without spoiling the continuous topology. The three dimensional spatial geometry of the Universe is realized through the finite three dimensional volume of the fundamental particles and the finite three dimensional volumes of the fundamental particles can only lead to three space dimensions for the Universe.

In brief following the principles of the General Theory of Relativity: a consistent and unique dynamics of the the Universe is realized through the finite three dimensional volumes and spin, an intrinsic property distinguishing spatial directions and orientations, of the elementary particles and the polarizaton and extended nature of fields. Also as discussed earlier the concept of orientation is an essential aspect to formulate the laws of Classical Electrodynamics [24] which require that spacial geometry should be of three dimension.

We note as the real line cannot be defined as a collection of points we have to introduce the concept of collection of one-dimensional line elements as a fundamental mathematical entity which, at its most primitive level, can be characterized into four classes:

(i) one-dimensional line elements without boundaries
(ii) one-dimensional line elements with one boundaries
(iii) one-dimensional line elements with two boundaries
(iv) one-dimensional line elements with the two boundaries identified (e.g, a circle)

We can introduce additional structures like length for the elements belonging to this collection. For example we can define the length of a circle (perimeter) and introduce the concept of its radius as the perimeter over \(2\pi\). The values of the radius form an abstract one-dimensional line element and, as shown earlier, the notion of two everywhere adjuscent circles (i.e, two circles with adjacent values of radii with everything else remaining the same) is not defined. This feature again leads us to conclude that we cannot obtain a two-dimensional disk from a collection of one-dimensional circles. We have to introduce the two dimensional circular strips as fundamental mathematical objects to construct a two-dimensional disk in the above way. Similar arguments for two dimensional spheres (where the radius is now defined as the positive square root of the area of the two-dimensional sphere over \(4\pi\)) lead us to conclude that we can not have a three dimensional volume element from a collection of two dimensional spheres and we have to consider three dimensional volume elements as fundamental mathematical entities. These discussions can be extended to higher dimensions.

We now discuss a few aspects regarding coordinatization. As far as the number system is concerned the above discussions prove that for any given number it is not possible to define the concepts of the immediate previous number or the next number; only the concepts of earlier or later numbers are well-defined. Consequently the number system with it’s conventional interpretation as one-to-one correspondence between numbers and points cannot cover a one-dimensional line. We can illustrate this again through the
Cantor set. After we have removed all the intervals, whose total length is same as the original interval, the remaining set of points forming the Cantor set have continuum cardinality, i.e., the elements of the Cantor set, through the very definition of cardinality, can be put into a one-to-one correspondence with the Real number system again. To summarize, the number system with it’s conventional interpretation as one-to-one correspondence between numbers and points can only characterize intervals. However the facts that elementary particles are of finite volume, the non-localized character of fields, the kinematic equivalence of space and time through the principles of Relativity (valid apart from the space-time singularity) and the dynamical character of the Universe indicate that space-time intervals are fundamental aspects and thereby the conventional coordinatization (there is no problem, if required, in coordinatizing a particular point as origin, even it may be so that at the origin the space-time is not well-defined) scheme defined through the concepts of neighbourhoods works well as far as physical evolutions are concerned.

We conclude this section with a few discussions:

Firstly let us consider two sets each containing a single object: a closed line-element. The line-elements are intersecting but non-coincident everywhere. The intersection of these two sets is a set containing points which are fundamentally different from line elements: we can not say that a point is a line-element with one point as the arguments in this section prove that a line-element is not a collection of points. Also as a line-element is not just a collection of points a set-theoretic union to form a closed line-element out of a set of half-open line-elements, a set of open line-elements and a set of points is not well-defined as all the above mentioned sets are geometrically and even topologically different. Similar discussions remain valid in two dimensions when two area elements intersect along a line-element and in three dimensions when two closed volume-element intersect along a common boundary. These problems can be solved if we construct a universal set containing all possible geometric-elements including points.

Secondly, the discussions in this section will also have some profound significances in Mathematical Analysis.

To illustrate we make some comments regarding the first and second countability of $R^1$ discussed in Appendix:A of Wald [3]. A topological space $X$ is first-countable if for every point $p$ belonging to $X$ there is a countable collection of open sets such that every open neighbourhood of $p$ contains at least one member of the collection. Whereas $X$ is second countable if there is a countable collection of open sets such that every open set can be expressed as an union of open sets from this family. For $R^2$, the open balls with rational radii centered on points with rational coordinates form such a countable collection open sets.

When defined in the conventional way there are infinite number of irrational numbers between two rational numbers [6]. As far as first-countability is concerned, an open set $V$ centered on a rational number with deleted perimeter on one of these neighbouring irrational number cannot contain open balls with rational radii centered on points with rational coordinates.

As far as second-countability is concerned the above-mentioned open set can not have locally finite subcover as the radii of the set of open intervals centered on the given rational number with deleted perimeter less than that of $V$ form a line-interval and as is proved in this article the corresponding cardinality is not well-defined. Similar discussions can be extended to $R^n$.

VIII. APPENDIX:C

Quantum field theory is the quantum theory of fields. It gives the dynamics of fields, the quantum probability amplitudes of creation and annihilation of particles, in contrast to quantum mechanics which gives the dynamics of the particles themselves obeying quantum principle. For the same boundary conditions these two descriptions match in the form of their kinematic solutions. Only for the free particle boundary condition the conventional interpretation of propagators in Q.F.T as giving the probability of particle propagation is in accordance with reality as the quantum probabilities are nowhere vanishing in both the theories. For microscopic particle physics experiments the free particle boundary condition is a good approximation in practice but ideally the field $\phi$ (or the quantum mechanical wave function $\psi$) is spatially confined within the experimental apparatus. In one dimension it is meaningless to say that a particle is propagating from one point to another if the probability of finding (or creation of) the particle is vanishing at some intermediate points.

For loops in one dimension, the momentum-space calculations give the probability that a pair of particles with given four-momentums are created at one space-time point and a pair of particles are annihilated at another space-time point with the same four-momentums. This feature is transparent if one consider all possible space-time particle trajectories to form loops in one dimension which cannot be possible without the possible space-time particle-antiparticle (originated with given four-momentums and annihilated with the same four-momentums) trajectories crossing each other at least once in between any two given space-time
potential in a p-type semiconductor? In semiconductor physics, as charge carriers holes are fictitious objects introduced for simplifications. In reality, quantum mechanically in a p-type semiconductor the motion of holes are out of the movements of electron-hole pairs. Once produced the quantum mechanically allowed stationary state position-space wave functions that are available to the particle-antiparticle pairs are \( \psi_p(x) = \exp(-ip_1x) \) and \( \psi_{AP}(x) = \exp(-ip_2x) \) respectively where \( x \) denotes space-time points. The quantum mechanical joint probability that the particles produced at \( x_1, x_2 = 0 \) with two-momentums \( p_1 \) and \( p_2 \) can again coincide at a space-time point between \( x \) and \( x + dx \) is:

\[
P_x(p_1, p_2) = N_1 \frac{(dx)^2}{T^2 L^2}
\]

where \( N_1 \) is the relative pair creation probability at the space-time point \( x = 0 \).

\( P_x(p_1, p_2) \) is independent of \( x \) \((p_1, p_2) \) and \( \to 0 \) as for free particle approximation \( L \to \infty \) although the total probability of coincidence is unity when integrated over all space-time points. Hence quantum mechanically, a number of particle-antiparticle pairs should be observed in any microscopic experiment performed during finite time-interval if there would have been spontaneous pair creations in vacuum.

In passing we note that a space-time formed out of loops in vacuum, closed time-like curves, as the source of gravity. Let us illustrate this feature following the standard literature. We consider situations where free particle approximation hold. In vacuum at a given space-time point \( x \) \((in one dimension)\), the particle-antiparticle pair production probabilities with two-momentums \( p_1, p_2 \) are \(|\exp(-ip_1x_1)|^2 \) and \(|\exp(-ip_2x_2)|^2 \) respectively apart from normalizing factors. Once produced the quantum mechanically allowed stationary state position-space wave functions that are available to the particle-antiparticle pairs are \( \psi_p(x) = \exp(-ip_1x) \) and \( \psi_{AP}(x) = \exp(-ip_2x) \) respectively where \( x \) denotes space-time points. The quantum mechanical joint probability that the particles produced at \( x_1, x_2 = 0 \) with two-momentums \( p_1 \) and \( p_2 \) can again coincide at a space-time point between \( x \) and \( x + dx \) is:

\[
P_x(p_1, p_2) = N_1 \frac{(dx)^2}{T^2 L^2}
\]

here \( N_1 \) is the relative pair creation probability at the space-time point \( x = 0 \).

\( P_x(p_1, p_2) \) is independent of \( x \) \((p_1, p_2) \) and \( \to 0 \) as for free particle approximation \( L \to \infty \) although the total probability of coincidence is unity when integrated over all space-time points. Hence quantum mechanically, numerous amount of particle-antiparticle pairs should be observed in any microscopic experiment performed during finite time-interval if there would have been spontaneous pair creations in vacuum.

In passing we note that a space-time formed out of loops in vacuum, closed time-like curves, as the source of gravity. We next note that in non-relativistic quantum mechanics the total joint probability that two distinguishable particles with energy \( E \) and momentums \( k, -k; k = \frac{2\pi}{L}, |n| \gg 1 \) can coincident at some point \( x \) \((here x is position only)\) is:

\[
P(all) = \frac{4}{L^2} \int \int |\sin^2(kx_1)\delta(x_1 - x_2)\sin^2(kx_2)dx_2| |dx_1|^2
\]

where the integrals are performed over the interval \([-L/2, L/2]\). This expression turns out to be unphysical as the corresponding probability turns out to be unphysical \([P(all) = 3/2]\). Whereas classical mechanically the maximum value of \( P(all) \) can be nearly unity when the particles suffer impulsive elastic collisions to stick together and come to rest at some point \( x_0 \) \((the corresponding quantum mechanical probability density should have been \delta^2(x - x_0))\).

Similar features as discussed in this section in the contexts of equ.(24) and equ.(25) will appear in three dimensions.

In semiconductor physics, as charge carriers holes are fictitious objects introduced for simplifications. In reality, quantum mechanically in a p-type semiconductor the motion of holes are out of the movements of the valence band or the acceptor level electrons. What is the proper explanation of the polarity of the Hall potential in a p-type semiconductor?

**IX. APPENDIX:D**

In this section we first consider the action for the gravitational field [20]:

\[
S_g = -\frac{e^3}{16\pi k} \int G \sqrt{-g} d^4 x
\]

where \( G_{ab} \) is Einstein’s tensor. General Theory of Relativity interwinds inertial mass (in general energy-momentum) of matter with space-time through the principle of equivalence and the dimension of the coupling constant \( k \) \((k = 6.67 \times 10^{-8} \text{cm}^3 \text{gm}^{-1} \text{sec}^{-2})\) is completely determined in terms of only mass, space and time unlike, for example, in electrodynamics where one have another fundamental quantity (electric charge) to determine the dimension of the coupling constant. This feature is transparent if we compare the Newtonian-limit of the general theory of relativity with the Coulomb’s law. The consequence of this feature is the following:

If we set \( c, \frac{\hbar}{m} = 1 \) the dimension of \( k \) is length-squared \((L^2)\) and it is no longer possible to set \( k = 1 \) as this will make the concept of space-time dimensions meaningless. Alternatively we could have set \( c, k = 1 \) (see footnote: page no. 269, [20]) then Planck’s constant become dimensionful \((L^2)\).
However we can set Boltzmann constant \((k') = 1\) by giving temperature the dimension of energy. In the reduced units \(\frac{c}{2\pi}, k' = 1\) the gravitational action becomes:

\[
S_g = -K \int G \sqrt{-g} d^4 x
\]  

(27)

where \(K\) have dimension \([l^{-2}]\).

We now conclude our discussion on Quantum Field Theory. In the rest of this section we will use the convention that \(<\alpha|\alpha>\) gives us probability density.

We first consider non-relativistic quantum mechanics in one dimension. In position-space the normalized quantum mechanical wave function \(\psi\) gives us the probability amplitude. \(\langle\psi^* \psi\rangle dx\) gives us the probability of finding the particle within the infinitesimal length interval \(dx\). For a free particle one adopts the delta function normalization scheme for the quantum mechanical wave function:

\[
\int_{-\infty}^{\infty} \psi_{k_1}^*(x) \psi_{k_2}(x) dx = \delta(k_1 - k_2)
\]  

(28)

In this equation the left-hand side is dimensionless while the one-dimensional delta function has dimension of length \([l]\) as is obvious from its definition:

\[
\int_{-\infty}^{\infty} f(k) \delta(k - k_0) dk = f(k_0)
\]  

(29)

for a regular function \(f(k)\).

It would be appropriate to replace the free-particle boundary condition by periodic boundary condition which is a reasonable approximation in situations where free-particle boundary conditions hold as for a large length interval the spacing between the adjacent values of the momentum allowed by the periodic boundary condition is negligible.

In the reduced units \((c, \hbar = 1)\) the action is dimensionless. The action for a complex scalar field is given by:

\[
I = \int L d^4 x
\]  

(30)

where the covariant lagrangian density for a massive field is given by,

\[
L = \frac{1}{2} \frac{\partial \phi^*}{\partial x^\mu} \frac{\partial \phi}{\partial x^\mu} - \frac{m^2}{2} \phi^2 \phi^2
\]  

(31)

Consequently the dimension of \(\phi\) \((\phi^*)\) should be inverse of length \(([l]^{-1})\). In the second quantization scheme \(<\beta|\phi|\alpha>\) replaces the classical field \([21]\) and the expression \(<\alpha|\phi^* \phi|\alpha>\) gives the probability density of creation or annihilation of particles. For free-particle boundary conditions the Euclidean-space generating functional for a real scalar field is given by \([22]\):

\[
W_E[J] = N_E e^{-S_E[\phi_0, J]} \frac{1}{N} Tr \left[ \ln((-\partial^\mu \partial^\nu + m^2 + V''(\phi_0))_1 \delta(\bar{x}_1 - \bar{x}_2) \right]
\]  

(32)

The terms in the logarithm giving quantum corrections are not dimensionless and the third term is not of the same dimension as of the first two terms.

For a real scalar field confined within a finite volume box with periodic boundary condition and consistent with the second quantization scheme we have (equ.(3.28,[10])):

\[
\phi(x) = \frac{1}{L^{n/2}} \sum_{(n_1, n_2, n_3)} \phi^+_{n_1, n_2, n_3} \exp \left[ \frac{2\pi i}{L} (n_\mu x^\mu) \right] + \frac{1}{L^{n/2}} \sum_{(n_1, n_2, n_3)} \phi^-_{n_1, n_2, n_3} \exp \left[ - \frac{2\pi i}{L} (n_\mu x^\mu) \right]
\]  

(33)

in a relativistic theory a covariant normalization using four volume would be appropriate and the normalizing factor should have dimension \([l^{-2}]\). Clearly the dimension of \(\phi\) \(([l^{-5}])\) in equ.(33) is no longer \([l^{-1}]\) and equ.(31) no longer gives us a lagrangian density.
In other words the dimension of the scalar field $\phi$ as required from the action determining its space-time evolution does not match with the dimension required in the second quantization scheme in order that one can interpret $\langle \alpha | \phi^* \phi | \alpha \rangle$ as giving the probability density of creation or annihilation of particles. One can absorb the the normalizing factor into the the fock state by multiplying it by a factor with dimension $[l^{-1}]$ as in the second quantization scheme $\langle \beta | \phi | \alpha \rangle$ replaces the classical field. This will be in accordance with the probabilistic interpretation of the field $\phi$ as we have,

$$\langle \alpha | \phi^* \phi | \alpha \rangle = \langle \alpha | \phi^* | \beta \rangle \sum_\beta \langle \beta | \phi | \alpha \rangle.$$  \tag{34}

where the sum is taken over all possible states.

However this will violate the interpretation of the Fock states as quantum mechanically the normalization of probability density $\langle \alpha | \alpha \rangle$ in a relativistic theory requires that the each of the normalizing factors for the Fock states, whose number depends on the number of particles present in the Fock state, should have dimension $[l^{-2}]$.

We now consider fermions and electromagnetic fields.

The covariant lagrangian density for each component of the free fermion fields (e.g, electrons-positrons) which is formed out of their causal space-time motion is given by:

$$L = \frac{1}{2} \partial \psi^* \partial x^\mu \partial \partial x^\mu - \frac{m^2}{2} \psi^* \psi^2$$  \tag{35}

and dimension of $\psi$ is again $([l]^{-1})$. After linearization of the second order partial differential equation satisfied by $\psi$ we get the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$  \tag{36}

Hereafter the following lagrangian density is used to study Q.E.D:

$$L' = \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m)\psi - \frac{1}{4} F^{\mu \nu}(x) F_{\mu \nu}(x)$$  \tag{37}

with the dimensions of $\psi$ as determined above the dimension of the first part of $L'$ is no longer that of a Lagrangian density and the action formed out of it is not dimensionless in the reduced units. Also the current density, $e\bar{\psi}\gamma^\mu \psi$ (although the four divergence vanishes), do not involve any momentum operator and it is not obvious whether it is possible to have, in any approximation, the conventional interpretation of current density as charge-density times velocity from this expression.

The Lagrangian density of a charged scalar field which is similar to the quadratic Lagrangian density for Q.E.D is given by,

$$L_T = L_f + L_{em} + L_{int}$$  \tag{38}

This complete Lagrangian density for a charged scalar is gauge invariant only if we take $L_{int}$ to be,

$$L_{int} = -A_{\mu} j^\mu + e^2 A^2 \phi^* \phi$$  \tag{39}

the second term do not have a transparent interpretation unless we consider screening effects from classical electrodynamics similar to the corresponding discussions given in Appendix II of this article.

X. APPENDIX: E

In this section we will illustrate the discussions in the context of equ.(20) in section IV.

The metric of the two-sphere $S^2(\theta, \phi)$ is given by

$$ds^2 = \sqrt{A/4\pi}(d\theta^2 + \sin^2 \theta d\phi^2)$$  \tag{40}
Here $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. For the unit two-sphere we have,

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

(41)

with the ranges of $\theta, \phi$ same as above. This coordinate system have the following ill-behavednesses:

(i) The coordinate $\phi$ suffers a discontinuity along some direction from $2\pi$ to $0$.

(ii) $\phi$ is degenerate at the poles $\theta = 0, \pi$. In spherical polar coordinate system $(r, \theta, \phi)$ the point $(r = c, \theta = 0)$ where $c$ is a finite constant is obtained through identifying any two arbitrary points on a circle characterized only through distinct values of $\phi$. Similar construction is valid for the point $(r = c, \theta = \pi)$. This will be obvious if we construct a two-sphere $S^2(\theta, \phi)$ from a two-dimensional circular strip by identifying the inner-boundary and the outer-boundary to two distinct points [see the discussions below eqn.(50) regarding the reduction of eqn.(48) to eqn.(50) for points on the polar axis]. This construction can be generalized to higher dimension.

(iii) The metric is singular at the poles [see the above discussions and App.B (a point can’t be obtained from a one-dimensional line-element without breaking the corresponding continuous topology)].

When the $S^2(\theta, \phi)$ can be embedded in the three dimensional Euclidean space one can introduce Cartesian coordinate system $(x, y, z)$ through the coordinate transformation:

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta.$$  

(42)

here $x^2 + y^2 + z^2 = 1$. Although the metric is regular in the Cartesian coordinate system the transformation coefficients ($\frac{\partial x}{\partial \phi}$) are singular at the poles and also at some isolated points on the $x - y$ plane demonstrating the discussions below eqn.(20). On the other hand to obtain the Spherical metric, singular at two isolated points, from the regular Cartesian metric one has to introduce coordinate transformation with singular transformation coefficients.

We can also introduce two homeomorphic stereographic projections to coordinatize $S^2(\theta, \phi)$ embedded in $\mathbb{R}^3$. The first one is from the North pole $\theta = 0$ on the Equator plane to coordinatize the Southern hemisphere $\frac{\pi}{2} \leq \theta \leq \pi$. We have,

$$X = \cot \frac{\theta}{2} \cos \phi, \quad Y = \cot \frac{\theta}{2} \sin \phi$$

(43)

and this transformation is a homeomorphism at $\theta = \pi$. The second stereographic transformation is from the South pole $\theta = \pi$ on the Equator plane to coordinatize the Northern hemisphere $0 \leq \theta \leq \frac{\pi}{2}$. We have,

$$U = \tan \frac{\theta}{2} \cos \phi, \quad V = \tan \frac{\theta}{2} \sin \phi$$

(44)

and this transformation is a homeomorphism at $\theta = 0$.

The transformation between the $(X,Y)$ and $(U,V)$ coordinate systems is a diffeomorphism at their common domain $\theta = \frac{\pi}{2}$. The metric is also regular in these coordinate systems.

However the transformation coefficients between $(X,Y)$ and $(\theta, \phi)$ coordinates are singular at the South pole ($\frac{\partial X}{\partial \phi}, \frac{\partial Y}{\partial \phi} = 0$ at $\theta = \pi$), i.e, this transformation is not a diffeomorphism. Similarly the transformation $(\theta, \phi) \rightarrow (U,V)$ is not a diffeomorphism.

We now consider the Robertson-Walker cosmological model. The space-time metric in terms of comoving isotropic observers is:

$$ds^2 = -d\tau^2 + a^2(\tau)(d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2))$$

(45)

here $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The constant-time spatial three surfaces $\sum \tau$ are compact (topologically $S^3$) and there is no four-dimensional spatial geometry available to embed $\sum \tau$. The spatial metric is singular along the closed line elements $\theta = 0, \theta = \pi$ including the two point-poles $\psi = 0, \pi$.

The discussions in section IV [eqn.(20)] and in this section show that metric singularities cannot be removed by diffeomorphically equivalent coordinate transformations. Thus the black hole and the cosmological metric singularities are unavoidable aspects of nature.
XI. APPENDIX: F

In this section we consider the electrostatic potential of a polarized dielectric system. The electrostatic potential of a polarized dielectric system is given by,\

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\vec{P} \cdot \hat{R}}{R^2} dv$$ \hspace{1cm} (46)$$

here $\vec{P}$ is the polarization vector of the dielectric material and $\vec{R}$ is the vector joining the infinitesimal volume element $dv$ carrying a dipole moment $\vec{P}dv$ to the point of observation. It’s magnitude is given by eqns.(46). We can reduce equ.(55) to a simpler form consisting two terms: one from a bound surface charge density $\sigma_b$ and another from a bound volume charge density $\rho_b$,

$$V = \frac{1}{4\pi\varepsilon_0} \int_{\text{surface}} \frac{\vec{P} \cdot \hat{\nu}}{R} - \frac{1}{4\pi\varepsilon_0} \int_{\text{volume}} \vec{\nabla} \cdot \vec{P} dv$$ \hspace{1cm} (47)$$

Here $\sigma_b = \vec{P} \cdot \hat{\nu}$, $\hat{\nu}$ is the normal to the surface of the material and $\rho_b = -\vec{\nabla} \cdot \vec{P}$.

The total volume charge density in presence of a polarized dielectric medium is given by:

$$\rho = \rho_b + \rho_f + \rho_{sb}$$ \hspace{1cm} (48)$$

where we include $\sigma_b = \vec{P} \cdot \hat{\nu}$ in the free volume charge density as $\rho_{sb}$ through the introduction of a proper delta function. For example in the case of a dielectric sphere we have,

$$\rho_{sb}(r, \theta, \phi) = \sigma_b(\theta, \phi) \delta(r - r_s)$$ \hspace{1cm} (49)$$

Here $r, r_s$ are radial distance (not vectors) and the delta function have dimension inverse of length. We then have:

$$\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_b + \rho'_f$$ \hspace{1cm} (50)$$

where $\rho'_f = \rho_f + \rho_{sb}$ and the divergence of the electric displacement vector $\vec{D}$ is given by,

$$\vec{\nabla} \cdot \vec{D} = \rho'_f$$ \hspace{1cm} (51)$$

Everywhere apart from the surface of the dielectric we have $\rho'_f = \rho_f$ and the above equation (51) maches with the conventional expression for the divergence of $\vec{D}$:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$ \hspace{1cm} (52)$$

The effect of $\rho_{sb}$ should be taken into the boundary conditions for $\vec{D}$ . This will also have consequences to obtain the energy density of a given electrostatic configuration in presence of dielectric mediums [23] as,

$$\vec{\nabla} \cdot \vec{D} - \vec{\nabla} \cdot \vec{D}_0 = \rho_{sb}$$ \hspace{1cm} (53)$$

We conclude this section with a few comments regarding the electrostatic field energy in presence of dielectrics.

The electrostatic field energy in presence of dielectric mediums can approximately be considered to consist of three parts [24]:

$$W_{tot} = \frac{1}{2} \int D.E d\tau = \frac{\varepsilon}{2} \int E.E d\tau = W_{free} + W_{spring} + W_{bound}$$ \hspace{1cm} (54)$$
here \( \epsilon = \epsilon_0 (1 + \chi_e) \). We briefly explain the three terms considering the realistic case of a dielectric filled charged parallel-plate capacitor:

i) \( W_{\text{free}} \) is the energy to charge the capacitor to produce the configuration with a given electric field. We can regain this energy if we discharge the capacitor by connecting the two plates through a conductor.

ii) \( W_{\text{spring}} \) is the energy required to increase the atomic/molecular dipole moments or to polarize the atoms/molecules depending on, respectively, whether the atoms/molecules have permanent dipole moments or not. This energy will be regained as heat when we discharge the capacitor.

iii) \( W_{\text{bound}} \) is the energy required to polarize the dielectric as a whole. The dipole-dipole interaction energy for two dipoles with dipole moments \( \vec{p}_1 \) and \( \vec{p}_2 \) and separated by \( \vec{r} \) is:

\[
U = \frac{1}{4 \pi \epsilon_0} \frac{1}{r^3} [\vec{p}_1 \cdot \vec{p}_2 - 3(\vec{p}_1 \cdot \vec{r})(\vec{p}_2 \cdot \vec{r})] \tag{55}
\]

\( U \) is minimum when the dipoles are antiparallel and maximum when the dipoles are parallel. Consequently for any statistically infinitesimal volume of the dielectric (i.e., volume elements which are very small compared to the dimension of system but large enough to contain sufficient number of atoms/molecules so that microscopic fluctuations can be approximately averaged to zero) the orientation of the atomic/molecular dipoles will be as isotropic as possible. To polarize the dielectric we have to orient the atomic/molecular dipoles in near-parallel configuration in a given direction and supply energy to increase the electrostatic energy of the dielectric. This energy, \( W_{\text{bound}} \), will be regained as heat if we discharge the capacitor.

XII. SUPPLEMENT: I

We will now study the behaviour of of the spectrum of covariant Klein-Gordon equation in the near horizon limit.

We will first consider the spectrum of the covariant Klein-Gordon equation in the (3 + 1)-dimensional constant curvature black hole background which contains a one dimensional fixed point set of the time-like Killing vector field. This black hole space-time was obtained by Prof. M. Bannados, Prof. R. B. Mann and Prof. J. D. E. Creighton through the identification of points along the orbits of a discrete subgroup of the isometry group of the anti-de Sitter apce-time. They used a static coordinate system where the constant-time foliations become degenerate along a particular direction apart from the black hole event horizon giving a one-dimensional fixed point set of the time-like Killing vector. The metric in the Schwarzschild like coordinates is given by,

\[
d s^2 = \frac{l^4 f^2(r)}{r^2} [d\theta^2 - \sin^2 \theta (dt/l)^2] + \frac{dr^2}{f^2(r)} + r^2 d\phi^2 \tag{56}
\]

where \( f(r) = (r^2 - r_h^2)^{1/2} \). These coordinates are valid outside the horizon \( r > r_h \) for \( 0 \leq \theta \leq \pi \) and \( 0 \leq \phi < 2\pi \). It is clear that the constant-time foliation becomes degenerate along a particular direction apart from the black hole event horizon giving a one-dimensional fixed point set of the time-like Killing vector field.

The covariant wave equation of a minimally coupled massive scalar field is given by,

\[
\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \psi) - m^2 \psi = 0. \tag{57}
\]

The solution of the angular equation is given by,

\[
P_\nu^\mu(x) = \frac{1}{1 - \mu} \left( \frac{1 + x}{1 - x} \right)^{\frac{\mu}{2}} F(-\nu, \nu + 1; 1 - \mu; \frac{1 - x}{2}) + c.c \tag{58}
\]

where \( x = \cos \theta \), \( \mu = iEl \) and \( (\nu + \mu) \neq \) an integer. Here \( F(-\nu, \nu + 1; 1 - \mu; \frac{1 - x}{2}) \) is the hypergeometric function. This solution is \( C^1 \) throughout the angular range \( 0 \leq \theta \leq \pi \). Consequently the energy spectrum is continuous with divergent density of states.

We will now illustrate that the divergent density of states is a characteristic feature of the fixed point set of the time-like Killing vector field indicating the breakdown of the canonical formalism of the conventional quantum mechanics.
We will illustrate this feature in the context of the Schwarzschild black hole which contains a two dimensional fixed point set (the event horizon) of the time-like Killing vector field. Since the Hawking radiation through which the non-unitary black hole evaporation takes place originates mostly from the near horizon region we will consider the behaviour of the spectrum of the covariant K-G equation in the near horizon region. Since the space-time foliation is static we will consider the stationary states. We will consider the radial solution of the covariant K-G equation. The radial solution can be obtained through the WKB approximation. However we can not use the semi-classical quantization condition. For the Schwarzschild black hole the constant-time foliations become degenerate at the black hole event horizon and it is not possible to impose any consistent boundary condition on the horizon. To obtain the degeneracy of the energy eigenstates we will now consider the radial part of a covariant generalized probability current density equation for the low energy eigenstates. For a general state composed out of superposition of different energy-eigenstates we consider the cross-term taken between states with neighbouring energy eigenvalues $E$ and $E + \delta E$. This gives us the following relation between $E$ and $E + \delta E$:

$$\partial_s [\text{Re}(\psi_1^* \psi_2)] = \frac{1}{m} \partial_s [\text{Re}(\psi_1^* \partial_s \psi_2)]$$ (59)

where the derivatives are taken w.r.t proper time and proper distance. This expression is similar to the probability current density equation of unitary quantum mechanics in presence of damping potentials. This equation is used because of observed decay, using conventional quantum mechanics, of the density of states with the proper distance from the black hole event horizon.

We will obtain the density of states of the energy eigenfunctions by considering the consistency of the integrated form of the generalized probability current density equation term by term in an infinitesemally thin spherical shell surrounding the black hole with radius $2M + h$ and $2M + 2h$ where $h <\ll 2M$. We obtain the following expression for the density of states:

$$\frac{1}{\delta E} = \frac{mK}{E^2 s}$$ (60)

where $s$ is the proper distance between the horizon and the spherical shell. As $s \to 0$ the density of states diverges and the generalized covariant current density equation becomes consistent. This divergent density of states is a property of the fixed point set of the time-like Killing vector field and this density of states gives vanishing internal energy and entropy for the spectrum of the covariant K-G equation.

The continuous energy spectrum is also obtained when one considers the behaviour of matter fields in the Taub-NUT space-time which contains a zero dimensional fixed point (in the Euclidean sector) of the time like Killing vector field. In this case the angular solution (in the Lorentzian sector) satisfies the minimum regularity condition, i.e., the angular part of the generalized probability current density integrated over $S^2$ is finite. This angular solution is similar to the spin-spherical harmonics.

The non-unitarity (decay of density of spectrum with distance away from the horizon) discussed above is a characteristic aspect of both the black hole event horizon and the cosmological event horizon.

We now make some comments regarding relativistic quantum mechanics similar to App:D. For relativistically covariant normalization of the quantum mechanical wave function (or each component of the wave function for spinors) of we have,

$$\int \psi^* \psi d^4 x = 1$$ (61)

This indicates that in the reduced units ($c, \hbar = 1$) the dimension of $\psi$ is inverse length squared $[l^{-2}]$. While the lagrangian leading to the Klien-Gordon equation is given by:

$$L = \frac{1}{2} \frac{\partial \psi^*}{\partial x^\mu} \frac{\partial \psi}{\partial x^\nu} - \frac{m^2}{2} \psi^* \psi^2$$ (62)

The action determining the space-time evolution of $\psi$ is dimensionless in the reduced units. This gives the dimension of $\psi$ to be $[l^{-1}]$ in contradiction to that ($[l^{-2}]$) obtained from the normalization condition.

In passing we note that to have a consistent time orientation for any space-time manifold, which through the principle of equivalence is form of existence of matter fields, particles should follow a particular family of reparameterization invariant curves in the space-time manifold [5].
XIII. SUPPLEMENTII: COMMENTS ON CLASSICAL ELECTRODYNAMICS

In this section we will make a few comments regarding the basic laws of Classical Electrodynamics.

We first consider the electrostatic potential of an extended but localized charged distribution. We will consider points outside the source. Let $\vec{r}'(r', \theta', \phi')$ be the position vector for an infinitesimal volume element $dv'$ within the source which makes an angle $\theta'$ with the positive Z polar axis and an azimuthal angle $\phi'$ w.r.t. the positive X axis. Let $\vec{r}(r, \theta, \phi)$ be the position vector of the point of observation (P) making an angle $\theta$ with the polar axis and an angle $\phi$ with the positive X axis. The magnitude of the position vector $R$ between $dv'$ and P is then given by:

$$R^2 = r^2 + r'^2 - 2rr'\cos(\gamma)$$  \hspace{1cm} (63)

where,

$$\cos(\gamma) = \cos \theta' \cos \theta + \sin \theta' \sin \theta \cos(\phi - \phi')$$  \hspace{1cm} (64)

The electrostatic potential at P is given by,

$$V(P) = \frac{1}{4\pi\varepsilon_0} \int \rho(r', \theta', \phi') r'^2 \sin \theta dr' d\theta' d\phi' \left[ r^2 + r'^2 - 2rr'\cos(\gamma) \right]^{1/2}$$  \hspace{1cm} (65)

For points outside the source the denominator can be binomially expanded in terms of Legendre polynomials of $\cos(\gamma)$. Using the addition theorem for Legendre polynomials:

$$P_l(\cos \gamma) = P_l(\cos \theta)P_l(\cos \theta') + 2 \sum_{m=1}^{\infty} \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta)P_l^m(\cos \theta') \cos[m(\phi - \phi')]$$  \hspace{1cm} (66)

the denominator can be expanded in terms of products of Legendre polynomials of $\cos \theta$ and $\cos \theta'$. For a spherically symmetric charge distribution we have only the monopole term $\frac{Q}{4\pi\varepsilon_0 r}$ from this expansion for points with $r > r'$. However if we consider points near the South pole, the conventional binomial expansion [24] is not valid in general for $(\frac{r}{r'}) \geq \sqrt{2} - 1$ although the series expressed in terms of the Legendre polynomials converge for these points (This aspect are also partly discussed in [28]). The convergence of the series does not justify the binomial expansion. To illustrate let us consider the potential of a point charge, $q$, situated at $z = a$. The potential expanded in terms of Legendre polynomials for $r > a$ is given by:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \sum \frac{a^l}{r^{l+1}} P_l(\cos \theta)$$  \hspace{1cm} (67)

For points on the negative Z-axis and infinitesimally close to $z = -a$ we have:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 a} [1 - f(\delta)]$$  \hspace{1cm} (68)

here $\delta = \frac{z}{a}$, $r = a + \epsilon$ and we have kept only terms linear in $\delta$. Whereas Coulomb’s law give us the following expression for electrostatic potential at $z = -(a + \epsilon)$:

$$V(\vec{r}) = \frac{q}{8\pi\varepsilon_0 a} \left[ 1 - \frac{(\delta)}{2} \right] \hspace{1cm} (69)$$

here also we have kept terms linear in $\delta = \frac{z}{a}$. Clearly the potential obtained at $z = -a$ from expansion in terms of Legendre polynomials is not in accordance with Coulomb’s law. Whereas the potential obtained from eqn.(71) is given by,

$$V(\vec{r}) = \frac{q}{8\pi\varepsilon_0 a} \left[ 1 - \frac{(\delta)^2}{2} \right]$$  \hspace{1cm} (70)
We should also note that Legendre polynomials are either symmetric or antisymmetric around \( \theta = \frac{\pi}{2} \) and for a charge distribution on an arbitrary shaped conductor which can not be expressed as a sum of a symmetric and an antisymmetric part we can not expand the corresponding potential in terms of Legendre polynomials.

In general, for any point outside the source we can use two consecutive binomial expansion: the first one is factoring out \( (r^2 + r'^2) \) in the denominator of \( V(P) \) and performing a binomial series expansion in terms of \( 2r' \). The second one is in terms of \( \frac{r^2}{r'} \) as is evident from the following expression:

\[
V(P) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r')}{(r^2 + r'^2)^{1/2} \left[ 1 - 2 \frac{r'}{r^2 + r'^2} \cos(\gamma) \right]^{1/2}} \sin \theta' \sin \theta \sin \phi' \sin \phi \, dr' \, d\theta' \, d\phi'
\]

(71)

In this expression any power of \( (r^2 + r'^2) \) can be binomially expanded in terms of \( \frac{r^2}{r'} \) for all \( r > r' \). This expansion gives the usual results for the monopole potential term and the dipole potential term (which vanishes for a spherically symmetric source). However for a spherically symmetric charge distribution we get a non-trivial screening term. To the order of \( \frac{1}{r^5} \) this term is given by:

\[
V\left(\frac{1}{r^5}\right) = \frac{1}{4\pi\varepsilon_0} \int \left\{ \frac{-3(r')^4}{8r^5} + \frac{7(r')^5}{32r^5} \left[ \frac{2}{5} (P_2(\cos \theta))^2 + \frac{1}{4} \right] \right\} r^2 \sin \theta' \rho(r') dr' \sin \theta' d\phi'
\]

(72)

The \( \theta \)-dependent term of the corresponding electric field vanishes at \( \theta = \frac{\pi}{2} \) and the angular component is directed towards the equatorial plane. These aspects are valid for all the \( \theta \)-dependent terms for a spherical charge distribution.

The potential of a charged spherical shell for points infinitesimal close to the surface of the shell is:

\[
V(P) = \frac{r_s^2 \sigma}{4\pi\varepsilon_0} \int \frac{\sin \theta' \sin \phi' \sin \phi'}{[2(1 - \cos \gamma) + 2\gamma(1 - \cos \gamma)]^{1/2}} \sin \theta \sin \phi \, d\theta \, d\phi
\]

(73)

We know from the electrostatic properties of conductors that the the out-side electric field on the surface of the conductor is,

\[
\vec{E} = \frac{\sigma}{\varepsilon_0} \hat{\gamma}
\]

(74)

This leads us to the following expression:

\[
\int \frac{\sin \theta' \sin \phi' \sin \phi'}{[2(1 - \cos \gamma)]^{1/2}} = 8\pi
\]

(75)

We will also have two quadratures with value zero.

However the expansion corresponding to equation (71) differs from the conventional expansion in terms of Legendre polynomials and the uniqueness theorems of Electrostatics lead us to conclude that one of these expansions are valid.

Secondly the presence of the screening term indicate that the integral version of Gauss’s divergence law in Electrostatics is in general not valid. The non-validness of the integral version of Gauss’s divergence law in Electrostatics is also evident for the electric field of an infinite charged cone [23]. We can also construct some charge distribution with a conductor, e.g. two finite radial cone joined through a spherical knot with center at the origin. For such charge distributions the potential near the conductor cannot be expressed in terms of Legendre polynomials and it is expected that that the integral version of Gauss’s law for electric field will not be valid. Non-validness of the integral version of Gauss’s divergence theorem may be associated with the discontinuous nature of the source.

We will now rederive Maxwell’s equations without using the integral version of Gauss’s divergence law for fields with localized sources.

We first consider Gauss’s law for the electrostatics. For a point charge at the origin the surface integral of the static electric field, determined by an inverse square law, over a closed sphere centered at the origin is \( Q/\varepsilon_0 \). This is because the origin of the coordinate system coincides with the point charge and the screening
terms in eqn. (72) are vanishing. Consequently the charge density in spherical polar coordinates is given by: \( \rho(r) = \frac{Q}{4\pi r^2}\delta(r) \). Here \( \delta(x) \) is the Dirac-delta function and we have the following useful relation:

\[
[\nabla_\mathcal{F} \times (\frac{\mathbf{E}}{r^2})] = 4\pi \delta^3(\mathbf{r}) \tag{76}
\]

We now consider the divergence of \( \mathbf{E} \) inside source. To calculate the divergence at a point \( \mathbf{r} \) within the source we break the source into two parts: one is an infinitesimal spherical volume element of radius \( r' \), centered at \( \mathbf{r} \) and the other is the rest of the source. The electric field is sum of two parts: one due to the infinitesimal volume element \( \Delta v \), \( \mathbf{E}_{\Delta v} \), and the other due to the rest of the source, \( \mathbf{E}_{\text{rest}} \). To calculate the divergence of \( \mathbf{E}_{\Delta v} \) we can use a spherical coordinate system centered at \( \mathbf{r} \). The position vector is given by \( \mathbf{r}' - \mathbf{r} \). The boundary of \( \Delta v \) is given by \( |\mathbf{r}' - \mathbf{r}| < r' \). We also break the charge density into two parts:

\[
\rho(r') = \rho_{\Delta v}(r') + \rho_{\text{rest}}(r') \tag{77}
\]

\( \rho_{\Delta v}(r') \) is non-zero \( (= \rho(r')) \) for points within \( \Delta v \) \( (|\mathbf{r}' - \mathbf{r}| < r') \) while \( \rho_{\text{rest}}(r') \) is non-zero \( (= \rho(r')) \) for points not within \( \Delta v \) \( (|\mathbf{r}' - \mathbf{r}| \geq r') \).

We now have the following expression for the divergence of \( \mathbf{E}_{\Delta v} \):

\[
4\pi\varepsilon_0 \nabla_\mathcal{F} \cdot \mathbf{E}_{\Delta v} = \int_{\Delta v} \rho_{\Delta v}(r') |\nabla_\mathcal{F} \cdot (\frac{\mathbf{E}}{r^2})| (|\mathbf{r}' - \mathbf{r}|)^2 d|r' - \mathbf{r}|^2 \sin \theta'd\theta'd\phi'
\]

\[
= -\int_{\Delta v} \rho_{\Delta v}(r') |\nabla_\mathcal{F} \cdot (\frac{\mathbf{E}}{r^2})| (|\mathbf{r}' - \mathbf{r}|)^2 d|r' - \mathbf{r}|^2 \sin \theta'd\theta'd\phi'
\]

\[
= \int_{\Delta v} \rho_{\Delta v}(r') |\nabla_\mathcal{F} \cdot (\frac{\mathbf{E}}{r^2})| (|\mathbf{r}' - \mathbf{r}|)^2 d|r' - \mathbf{r}|^2 \sin \theta'd\theta'd\phi' \tag{78}
\]

Here \( \mathbf{R} = (\mathbf{r}' - \mathbf{r}) \) and to obtain the first expression we have used the fact that \( \rho \) do not depend on the unprimed coordinates. As we had discussed in the last section \( \nabla_\mathcal{F} \cdot (\frac{\mathbf{E}}{r^2}) |(\mathbf{r}' - \mathbf{r})| = 4\pi\delta^3(\mathbf{r}' - \mathbf{r}) \). It is now straightforward to show that the divergence of \( \mathbf{E}_{\text{rest}} \) vanishes for points within \( \Delta v \). Thus we have Poisson’s equation for points inside an extended source:

\[
\nabla_\mathcal{F} \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\varepsilon_0} \tag{79}
\]

For a volume charge density \( \rho(\mathbf{r}) \) the charge density should vanish at the surface of the source. Otherwise we will have a non-trivial surface charge density. For non-trivial surface and line charge densitist the divergence of \( \mathbf{E} \) can be found following the above procedure and the results are same as replacing the source through proper delta functions.

As usual the curl of \( \mathbf{E} \) is zero.

The no-work law can be regained for the electrostatic field of an extended charge distribution if we generalize the derivation properly. The line-integral of \( \mathbf{E} \) over a closed contour for each element of the source with the source-coordinate remaining fixed is given by:

\[
\int_{\text{contour}} d\mathbf{E} \cdot d\mathbf{l} = \int_{\text{contour}} \frac{\partial [dV(\mathbf{R})]}{\partial \mathbf{R}} d\mathbf{R} = 0 \tag{80}
\]

and the total work done for the whole source is obviously zero.

We now consider the divergence and curl of the magnetostatic field: \( \mathbf{B} \). The Biot-Savart law for the general case of a volume current density \( \mathbf{J} \) is given by:

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(r', \theta, \phi) \times \mathbf{R}}{R^2} r'^2 \sin \theta d\theta' d\theta d\phi \tag{81}
\]
where $R$ is given by eqn.(63).

It can easily be shown that $\nabla \times \vec{B} = 0$ following the conventional procedure [24].

The curl of $\vec{B}$ is given by,

$$\nabla \times \vec{B} = \mu_0 \vec{J}(r', \theta, \phi) - \frac{\mu_0}{4\pi} \int (\vec{J} \cdot \nabla) \frac{\hat{R}}{R^2} d\nu$$  \hspace{1cm} (82)

Here the integration is over the source volume. The first term arises from the source volume integrand (apart from a multiplicative factor): $\vec{J} \cdot \nabla (R/R^2)$. Following the same procedure as to obtain the $\nabla \times \vec{E}$ law we obtain the first term in the above equation.

We now consider the second term. We have, for the $x$-component,

$$\int (\vec{J} \cdot \nabla) \frac{(x - x')}{R^2} d\nu = \int \nabla \cdot \left[ \frac{(x - x')}{R^3} \vec{J} \right] d\nu - \int \frac{(x - x')}{R^3} (\nabla \cdot \vec{J}) d\nu$$  \hspace{1cm} (83)

The second term in the r.h.s of the above equation vanishes as $\vec{J}$ do not depend on the unprimed variables. The first term is given by,

$$\int \nabla \cdot \left[ \frac{(x - x')}{R^3} \vec{J} \right] d\nu = - \int \nabla' \cdot \left[ \frac{(x - x')}{R^3} \vec{J} \right] d\nu$$  \hspace{1cm} (84)

This is possible because for steady currents $\nabla \cdot \vec{J} = 0$. The integration gives terms dependent on $\vec{J}$ on the boundary of the source. This is apparent if we use the Cartesian coordinate system. As discussed in the context of $\nabla \times \vec{E}$ law, $\vec{J}$ should vanish on the boundary of the source otherwise we will have a non-trivial surface current density. It can easily be shown that for an arbitrary curved space with line element: $ds^2 = h_1(x_1, x_2, x_3)(dx_1)^2 + h_2(x_1, x_2, x_3)(dx_2)^2 + h_3(x_1, x_2, x_3)(dx_3)^2$ the above integral vanishes for the same boundary condition as above on $\vec{J}$.

We will now consider surface current density. Let us first consider a surface with a wedge, i.e, two surfaces joined along a curve making an angle which can, in general, vary along the curve. If a surface current density originates at one surface the current can not propagate to the second surface as the only way that the ideal surface current density can remain tangential to both the surfaces at the wedge is to flow along the wedge. The above arguments demonstrates that we can not have surface currents out of point charges flowing along an arbitrarily shaped surface as an arbitrary surface can be expressed as a collection of infinitesimal flat surfaces with non-parallel normals. Similar arguments as above demonstrates that we can not have currents in an arbitrary curve out of motion of point charge carriers.

We can have surface currents for an infinite plane surface and provided the surface current density is not divergent the boundary term in eqn.(84) vanishes as the factor $\frac{(x - x')}{R^3}$ vanishes for $x' \to \pm \infty$. We can also have steady currents only out of axial rotation of an azimuthally symmetric surface charge distribution. In this case, expressed in terms of cylindrical polar coordinates or spherical polar coordinates the boundary term in eqn.(84) vanishes out of uniqueness. Similar arguments show that for line-current densities, which can be either an infinite straight line current or azimuthally symmetric rotation of a charged ring, the boundary terms in eqn.(84) vanishes.

Thus we have,

$$\nabla \times \vec{B} = \mu_0 \vec{J}(r', \theta, \phi)$$  \hspace{1cm} (85)

For ideal surface and line current densities the results will same as replacing $\vec{J}(\vec{r})$ by suitable delta functions measured on the the source provided the sources satisfy the required regularity conditions as discussed above.

The well-known integral law for a physical line current density $\int_{\text{contour}} B.\vec{dl} = \mu_0 I$, where the contour is a closed circle concentric with the source and lies on a plane perpendicular to the physical line-source, can be easily derived following the procedure used to establish the no work law for the electrostatic field although for a physical line-source $\vec{B}$ will have a small non-vanishing radial component on the plane of the circle. To
illustrate a physical line current density can be regarded as a collection of parallel infinitesimal line-current density for each of which ampere’s circuital law is valid along their axes.

Faraday’s law, with the following convention (Section:7.1.3 [24])

for motional emf or induced electric field the direction of the current or the electric field along a closed loop and the orientation of the enclosed surface giving the magnetic flux are related by the right-hand thumb rule,

... together with the above discussions and the current density equation (differential version of the electric charge conservation law) reproduces Maxwell’s laws of Classical Electrodynamics. After that we have reestablished Maxwell’s equations we should note that the finite volume of the elementary charge carriers indicates that ideal line/surface charge densities cannot exist in nature unless the charge carriers can move with velocity $c$ in one or two directions. Also to have ideal line/surface current densities the charge carriers should move with speed greater than $c$. Point charges can not exist in nature and the electromagnetic self-energies of the elementary particles are not infinite. Also considered as a classical model the finite volumes of the elementary charges give rise to screening terms as discussed in the context of the electrostatic potentials of extended charged systems.

We conclude this article with a few comments on magnetic monopoles. It is straight forward to show that under a general electromagnetic duality transformation, eqn.(6.151, 6.152) [23],

\[
\vec{E} = \vec{E}' \cos \xi + \vec{H}' \sin \xi \\
\vec{H} = -\vec{E}' \sin \xi + \vec{H}' \cos \xi \\
\vec{D} = \vec{D}' \cos \xi + \vec{B}' \sin \xi \\
\vec{B} = -\vec{D}' \sin \xi + \vec{B}' \cos \xi
\]

... and

\[
\rho_e = \rho_e' \cos \xi + \rho_m' \sin \xi \\
\vec{J}_e = \vec{J}_e' \cos \xi + \vec{J}_m' \sin \xi \\
\rho_m = -\rho_e' \sin \xi + \rho_m' \cos \xi \\
\vec{J}_m = -\vec{J}_e' \sin \xi + \vec{J}_m' \cos \xi
\]

... the differential version of the magnetic charge conservation law does not remain time-reversal symmetric due to the pseudoscalar and pseudovector nature of magnetic monopole charge and magnetic current density vector respectively, i.e, if $\vec{\nabla} \cdot \vec{J}_m = -\partial \rho_m/\partial t$ is time-reversal symmetric then $\vec{\nabla} \cdot \vec{J}_m = -\partial \rho_m/\partial t$ no-longer remains time-reversal symmetric although the differential version of the electric charge conservation law remains time-reversal symmetric under electromagnetic duality transformation.

**XIV. SUPPLEMENTIII: COMMENTS ON HYDRODYNAMICS**

In this article we will review the laws of fluid dynamics. Our discussions will be based on mainly that of chapter 40, 41 of The Feynman Lectures on Physics, Vol.2 [25].

The dynamics of dry water is governed by eqn.(40.6) [25]:

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla} \phi 
\]

or using a vector analysis identity to the second term of the above equation:

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{\nabla} \times \vec{v}) \times \vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla} \phi 
\]

where $\vec{v}$ is the velocity of an fluid element for which such laws can be applicable, $p$ is the fluid pressure and $\phi$ is the potential per unit mass for any potential force present. We can derive some important laws from eqn.(86). The first one is the equation for vorticity ($\vec{\Omega} = \vec{\nabla} \times \vec{v}$) and is obtained by taking curl of eqn.(87):

\[
\frac{\partial \vec{\Omega}}{\partial t} + \vec{\nabla} \times (\vec{\Omega} \times \vec{v}) = 0
\]

The second one is Bernoulli’s theorems (40.12) and (40.14) [26]:

\[
\vec{v} \cdot \vec{\nabla} \left( \frac{p}{\rho} + \phi + \frac{1}{2} \vec{v}^2 \right) = 0
\]

i.e,
\[ \frac{p}{\rho} + \phi + \frac{1}{2}v^2 = \text{const (along streamlines)} \]  
(90)

valid for steady flow and

\[ \frac{p}{\rho} + \phi + \frac{1}{2}v^2 = \text{const (everywhere)} \]  
(91)

valid for steady and irrotational flow.

However in all these equations the variation of the fluid density, \( \rho \), is not considered while in deriving eqn.(40.17) [25] the variation of fluid density is not properly taken into account. The consideration for variation of the density of a nearly-incompressible fluid may become important through the facts that when unconstrained the shape of a fluid can be changed almost freely and sparsed away and through the facts that layers of fluids can be very easily spreaded or detached away although these properties vary from fluid to fluid. These features together with the local version of the conservation of mass law (assuming that there is no local source or sink in the region of interest):

\[ \nabla . (\rho \vec{v}) = 0 \]  
(92)

indicate that we should consider the possibility for variation of \( \rho \) properly as we will illustrate later that some ideal models can cause a finite variation of \( \rho \) and in reality the description of the motion should be changed. While the divergence of \( \vec{v} \) may become important in cases like Couette flow where the centrifugal forces imposes a finite and may even be large divergence of \( \vec{v} \).

We can derive a proper version of eqn.(86) by applying Newton’s second law to the fluid momentum per unit volume and we have:

\[
\frac{\partial (\rho \vec{v})}{\partial t} + [\vec{v} \cdot \nabla](\rho \vec{v}) = -\nabla p - \nabla (\rho \phi) 
\]  
(93)

This equation is in general a non-linear coupled [through eqn.(92)] partial differential equation for \( \vec{v} \).

Bernoulli’s theorems for fluid dynamics can only be established when \( \rho \) is constant:

\[ \vec{v} \cdot \nabla [p + (\phi \rho) + \frac{1}{2}(\rho v^2)] = 0 \]  
(94)

i.e,

\[ p + (\phi \rho) + \frac{1}{2}(\rho v^2) = \text{const (along streamlines)} \]  
(95)

valid for steady flow and

\[ p + (\phi \rho) + \frac{1}{2}(\rho v^2) = \text{const (everywhere)} \]  
(96)

valid for steady and irrotational flow.

In general, when \( \rho \) is varying, only the first of the Bernouilli’s theorems:

\[ \vec{v} \cdot \nabla [p + (\phi \rho) + \frac{1}{2}(\rho v^2)] = 0 \]  
(97)

remains to be valid provided \( (\vec{v} \cdot \nabla \rho) \) is vanishing or is approximately valid if \( |(\vec{v} \cdot \nabla \rho) \vec{v}| \) is negligible compared to the other terms in eqn.(93). To illustrate the significance of these comments, let us consider the ideal model to calculate the efflux-coefficient, fig. 40-7 [25]. After that the contraction of the cross-section of the emerging jet has stopped we have, from the coservation of mass law, the following equation for \( \rho v \) at two vertical points:

\[ \vec{v} \cdot \nabla [p + (\phi \rho) + \frac{1}{2}(\rho v^2)] = 0 \]  
(97)
\[ \rho_1 v_1 = \rho_2 v_2 \]  

In this case pressure is the atmospheric pressure and remains the same throughout the flow and thus even for the flow of a nearly-incompressible fluid \( \rho \) can vary as \( v \) changes with height. In reality the flow usually gets sparsed away after a distance which varies for different flows.

The viscous flow of a fluid is governed by the following two laws which are obtained from eqn.(93) and eqn.(41.15),[25]:

\[
\frac{\partial (\rho \vec{v})}{\partial t} + [\vec{v}, \nabla] (\rho \vec{v}) = -\nabla p - \nabla (\rho \phi) + \eta \nabla^2 \vec{v} + (\eta + \eta') \nabla (\nabla . \vec{v}) \tag{99}
\]

\[
\nabla . (\rho \vec{v}) = 0 \tag{100}
\]

supplemented by proper boundary conditions. To illustrate the significance of the boundary conditions we can consider the change of the shape of the surface of water in a bucket when the bucket is given a steady rotational motion about it’s axis. The surface of the water become paraboloidal when the bucket is rotating. This shape can not be obtained without a vertical component of fluid velocity along the bucket surface for a finite duration although the bucket surface only have an angular velocity.

In the above equations \( \eta \) is the “first coefficient of viscosity” or the “shear viscosity coefficient” and \( \eta' \) is the “second coefficient of viscosity”. This equation is extremely significant in the sense that this equation, not eqn.(41.16) [26], is the equation which contains all the terms relevent to describe the dynamics of viscous fluids, both nearly-incompressible and compressible. For compressible fluids \( \rho \) will also depend on pressure, \( p(\vec{r}) \). We can modify this equation only through varying the nature of the viscous force.

The equation for vorticity is given by:

\[
\frac{\partial \vec{\omega}}{\partial t} + \nabla \times (\vec{\omega} \times \vec{u}) - \vec{\omega} \times \nabla \times \vec{u} = \frac{\eta}{\rho} (\nabla^2 \vec{\omega}) - \frac{(\nabla p) \times (\nabla \rho)}{\rho^2} - \frac{\nabla (\rho \phi)}{\rho^2} + \frac{\eta \nabla^2 \vec{v} \times (\nabla \rho)}{\rho^2} + \frac{\eta + \eta'}{\rho^2} \nabla \times (\nabla \rho) \times (\nabla \vec{v}) \tag{101}
\]

We can obtain an equation similar to eqn.(41.17) [25] describing the motion of a viscous fluid past a cylinder provided we can neglect the terms involving \( \nabla \rho \) and it is given by:

\[
\frac{\partial \vec{\Omega}}{\partial t} + \nabla \times (\vec{\Omega} \times \vec{u}) - \vec{\Omega} \times \nabla \times \vec{u} = \frac{\eta}{\rho} (\nabla^2 \vec{\Omega}) \tag{102}
\]

Following the procedure in section 41-3,[26] we can rescale the variables to obtain an equation which has Reynolds number \( (R) \) as the only free parameter :

\[
\frac{\partial \vec{\omega}}{\partial \tau} + \nabla' \times (\vec{\omega}' \times \vec{u}') - \vec{\omega}' \times \nabla' \times \vec{u}' = \frac{\eta}{R} (\nabla'^2 \vec{\omega}') \tag{103}
\]

where the prime describe the scaled variables, \( \vec{u}' \) is the scaled velocity and \( R \) is given by the usual expression, \( R = \frac{4V}{\eta} \).

To conclude in this section we have derived the exact equation describing fluid dynamics. We considered the motion of both non-viscous and viscous fluids. We proved that in both the cases there are terms which are neglected in the conventional theory but may become significant in some ideal model and in reality the description of motion is changed. Some of these terms even change the dynamical laws of viscous fluid motions by violating the conventional theory established in term of the Reynold number and these terms are significant for the dynamics of compressible fluids like air.
XV. SUPPLEMENTIV: COMMENTS ON DOUBLE SLIT INTERFERENCE

In this section we make a few comments regarding double slit interference experiment following section 13.3 of [26].

In Figure 13F the wave fronts that reach at the screen point \( P \) simultaneously from the slits \( S_1 \) and \( S_2 \) should have originated from the two slits at different instants. The wavefront from \( S_2 \) should originate at time \( \frac{S_2}{c} \) earlier than the corresponding interfering wave front that has originated at \( S_1 \). The phase difference is: \( \delta = \omega \frac{S_2}{c} = \frac{2\pi}{\lambda} S_2 A \) and we can continue the corresponding analyses discussed in [27] to obtain the fringe pattern. Similarly, to reach a particular screen point \( P \) simultaneously the wavefronts from different parts of an aperture should have originated from wavefronts at the aperture at different instants of time differing in phases. This gives rise to diffraction. These discussions demonstrate that interference and diffraction are not only wave phenomena but also associated with finite velocity of waves and for electromagnetic waves in vacuum this velocity is \( c \) as is verified in interference and diffraction experiments. Following the above arguments the proper expression for fringe-shift in the Michelson-Morley experiments [26] is \( 2\omega d \frac{S^2}{c} \), where \( \omega \) is the frequency of light. We can not follow the procedure followed in [26] to obtain the path-difference as the velocity of light, as is assumed, is different in different directions and as discussed above it is not proper to calculate the phase-difference between the interfering waves by multiplying the path-difference by \( \frac{2\pi}{\lambda} \) directly. The fact that we can have the interference fringe system by allowing one photon to emerge from the source at each instant indicate that we have to introduce the corresponding electromagnetic wave description as fundamental and this description do not depend on the width of the slits.

These discussions can be continued for electron beam interference experiments. In this case the wave fronts, the position-space wave functions \( \psi \), are of the form \( N e^{i(k-r-\omega t)} \), and the interfering states are the states of the electron at the slits at two different instants of time. Only for the central maximum the two states at the two slits are the same.

\( \psi \) is the complete microscopic description of the electrons in the electron-beam interference experiments. It is so as there is no underlying ensemble representation of the probabilistic interpretation of \( \psi \). If we have to assume that, unlike the electromagnetic interference experiments, \( \psi \) is vanishing at the slits for slit width less than that of the diameter of the electrons then the fundamental reality of \( \psi \) is questionable. In other words \( \psi \) gives a microscopic description of the electron but below a certain length the kinematic reality of \( \psi \) should be replaced by a proper description. We can also illustrate this aspect with the following question:

What is the quantum mechanical description of a Radon atom in a rigid box when the distance of consecutive nodes and antinodes of \( \psi \) is equal to or less than the diameter of the atom?

XVI. SUPPLEMENT:V(A FEW QUESTION)

What happens to the entropy increase principle as the Universe evolve to form the big-crunch singularity? What happens to the uncertainty relations along the process of gravitational collapses? What is the quantum mechanical description of a Radon atom in a rigid box when the distance of consecutive nodes and antinodes of \( \psi \) is equal to or less than the diameter of the atom? What is the position-space wave function of two finite volume massive bosons if we take contact interaction into account? How a photon produce electron-positron pair with finite volume concentrate rest masses? What are the charges and masses of the electron-positron pairs forming loops in the vacuum? How two particles with three-momentums \( k_1, k_2 \) \( (k_1 \neq k_2) \) produced to form a loop at a space-time point always arrive at another spacial point simultaneously? What is the microscopic explanation in terms of particle exchanges of the force in the Casimir effect? What is the mechanism of the collapse of the momentum-space wave function of a particle knocking out an electron from an atom? What is meant by \( |\Psi> = c_1(t)|\Psi_{U,238} + c_2(t)|\Psi_{T,238} > \) ? Quantum mechanically the region between the rigid walls (which is equiprobable in classical mechanics) is non-homogeneous for a particle in a rigid box! A photon can not reproduce Maxwell’s equations apart from moving with velocity \( c \). How can a process involving only a few photons be described starting from the Maxwell’s equations? The large scale structure of the Universe is homogeneous.

What is the screen in our brains to view objects, as they are, of sizes larger than our brains?

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XIX. SAMAPTA

Any one, who had seriously disturbed the author academically or non-academically during the last five years, in particular through undue slanging and horning out of a dogging heritage while the article was getting prepared, and/or encouraged to do so is a descendant of Avatar of Dharmaraj.

Reference: Jessy's artificial hand.