RADIATIVE EFFECTS ON THE THERMOELECTRIC PROBLEMS

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Abstract. There are two main directions in this paper. One is to find sufficient conditions to ensure the existence of weak solutions to thermoelectric problems. At the steady-state, these problems consist by a coupled system of elliptic equations of the divergence form, commonly accomplished with nonlinear radiation-type conditions on at least on a nonempty part of the boundary of a $C^1$ domain. The model under study takes the thermoelectric Peltier and Seebeck effects into account, whose describe the Joule-Thomson effect. The proof method makes recourse of a fixed point argument. To this end, well-determined estimates are our main concern. The paper is in the second direction for the derivation of explicit $W^{1,p}$-estimates ($p > 2$) for solutions of nonlinear radiation-type problems, where the leading coefficient is assumed to be a discontinuous function on the space variable. In particular, the behavior of the leading coefficient is conveniently explicit on the estimate of any solution. This regularity result is sufficiently general to contribute to other problems, in which the dependence on the values of the involved constants is essential, instead of the problem under study only.

1. Introduction

This paper is concerned with a model on thermoelectric devices with radiative effects. We formulate the problem by coupling a thermal model with a electrical model. The work is two-fold. Firstly, the first part (and Appendix) is of physical nature. Secondly, Sections 2, 3, 4, and 5 are of mathematical nature and are devoted to obtaining the existence result for the proposed model (Section 6). We prove the higher integrability of the gradient of weak solutions to the boundary value problem under study. Although the techniques used in the paper are standard (see [1,3,5,7,11,14,18,25] and the references therein), the explicit expressions of the involved constants as function of the data are new. The derivation of $W^{1,p}$-estimates in [8, 28] makes use of the contradiction argument which invalidates the determination of explicit expressions of the involved constants as function on the data. Our final result (Theorem 1.1) is to derive sufficient conditions on the data to ensure the existence of at least one weak solution to the thermoelectric problem under study in two-dimensional real space. Similar work on the 3D existence remains an open problem. We mention to [21] the optimal elliptic regularity for the spatio-material model constellations in three-dimensional real space.

Let $\Omega$ be a bounded domain (that is, connected open set) in $\mathbb{R}^n$ ($n \geq 2$) according to Definition 2.1 representing a thermoelectric conductor material, which is a heterogeneous anisotropic solid. Assume that $\Omega$ is of class $C^1$ (cf. Definition 1.1).
Let us consider the following problem that extends the thermoelectric problems, which were introduced in [9,10], in the sense of that the thermoelectric coefficient is assumed to be a given but arbitrary nonlinear function. The electrical current density \( j \) and the energy flux density \( J = q + \phi j \), with \( q \) being the heat flux vector, satisfy

\[
\begin{cases}
\nabla \cdot j = 0 & \text{in } \Omega \\
-j \cdot n = g & \text{on } \Gamma_N \\
j \cdot n = 0 & \text{on } \Gamma
\end{cases}
\]

\[
\begin{cases}
\nabla \cdot J = 0 & \text{in } \Omega \\
J \cdot n = 0 & \text{on } \Gamma_N \\
-J \cdot n = f_\lambda(\theta)|\theta|^{\ell-2}\theta - \gamma(\theta)\theta^{\ell-1} & \text{on } \Gamma,
\end{cases}
\]

for \( \ell \geq 2 \). Here \( n \) is the unit outward normal to the boundary \( \partial \Omega \), \( g \) denotes the surface current source, \( f_\lambda \) is a temperature dependent function that expresses the radiation law depending on the wavelength \( \lambda \), and \( \gamma\theta^{\ell-1} \) stands for the external heat sources, with \( \theta \) being an external temperature. The Kirchhoff radiation law, its variants, and extensions are analyzed in [11,22,24,27] for real physical bodies. According to the Stefan–Boltzmann radiation law, \( \ell = 5 \), \( f_\lambda(T) = \sigma_{SB}\epsilon(T) \) and \( \gamma(T) = \sigma_{SB}\alpha(T) \), where \( \sigma_{SB} = 5.67 \times 10^{-8} \text{W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \) is the Stefan-Boltzmann constant for blackbodies. The parameters, the emissivity \( \epsilon \) and the absorptivity \( \alpha \), both depend on the spatial variable and the temperature function \( \theta \). If \( \ell = 2 \), the boundary condition corresponds to the Newton law of cooling with heat transfer coefficient \( f_\lambda = \gamma \).

The constitutive equations of state,

\[
q = -k\nabla \theta - \Pi \sigma \nabla \phi;
\]

\[
j = -\alpha_s \sigma \nabla \theta - \sigma \nabla \phi,
\]

are based on the principle of local thermodynamic equilibrium of physically small sub-systems (see [19] and the references therein). Here \( \theta \) denotes the absolute temperature, \( \phi \) is the electric potential, \( \alpha_s \) represents the Seebeck coefficient, and the Peltier coefficient \( \Pi(\theta) = \theta \alpha_s(\theta) \) is due to the first Kelvin relation [32]. The electrical conductivity \( \sigma \), and the thermal conductivity \( k = k_T + \Pi\sigma \), with \( k_T \) denotes the purely conductive contribution, are, respectively, the known positive coefficient of Ohm and Fourier laws. Both coefficients depend on the spatial variable and the temperature function \( \theta \) [26], which invalidates, for instance, the use of the Kirchhoff transformation.

For \( p > 1 \), if \( \text{meas}(\Gamma) = 0 \) let the reflexive Banach space

\[ V_p := \{v \in W^{1,p}(\Omega) : \int_\Omega vdx = 0 \} \]

endowed with the seminorm of \( W^{1,p}(\Omega) \).

For \( p > 1 \), and \( \ell \geq 1 \), if \( \text{meas}(\Gamma) > 0 \) let the reflexive Banach space [13]

\[ V_{p,\ell} := \{v \in W^{1,p}(\Omega) : v \in L^\ell(\Gamma) \} \]

endowed with the norm

\[ ||v||_{1,p,\ell} := ||\nabla v||_{p,\Omega} + ||v||_{\ell,\Gamma}. \]

For the sake of simplicity, we denote by the same designation \( v \) the trace of a function \( v \in W^{1,1}(\Omega) \). Observe that \( V_{p,\ell} \) is a Hilbert space equipped with the inner product only if \( p = \ell = 2 \). By trace theorem, \( V_{p,\ell} = W^{1,p}(\Omega) \) if \( 1 \leq \ell < p(n-1)/(n-p) \). Otherwise, \( V_{p,\ell} \subsetneq W^{1,p}(\Omega) \).

We formulate the problem under study as follows:
(P) Find the pair temperature-potential \((\theta, \phi)\) such that it verifies the variational problem:

\[
\int_{\Omega} (k(\cdot, \theta) \nabla \theta) \cdot \nabla v \, dx + \int_{\Gamma} f_\lambda(\cdot, \theta) |\theta|^{\ell-2} \theta v \, ds = \\
\int_{\Omega} \sigma(\cdot, \theta) \left( \alpha_s(\cdot, \theta)(\theta + \phi) \nabla \theta + \phi \nabla \phi \right) \cdot \nabla v \, dx + \int_{\Gamma} \gamma(\cdot, \theta) \theta^{\ell-1} v \, ds;
\]

\[
\int_{\Omega} (\sigma(\cdot, \theta) \nabla \phi) \cdot \nabla w \, dx = - \int_{\Omega} (\sigma(\cdot, \theta) \alpha_s(\cdot, \theta) \nabla \theta) \cdot \nabla w \, dx + \int_{\Gamma_N} gw \, ds,
\]

for every \(v \in V_{p', \ell}\) and \(w \in V_p\), where \(p'\) accounts for the conjugate exponent of \(p\): \(p' = p/(p - 1)\).

We emphasize that this solution verifies, in the distributional sense, the PDE written in terms of the Joule and Thomson effects:

\[
0 = \nabla \cdot q + \nabla \cdot j = -\nabla \cdot (k_N \nabla \theta) - \frac{|j|^2}{\sigma} + \mu \nabla \theta \cdot j,
\]

where the Thomson coefficient \(\mu\) is the thermoelectric coefficient directly measurable for individual materials that satisfies the second Kelvin relation: \(\mu(T) = T \frac{\partial \alpha}{\partial T}(T)\). The verification of the second law of thermodynamics is stated in Appendix.

We assume that

(H1) The Seebeck coefficient \(\alpha_s : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) is a Carathéodory function, i.e. measurable with respect to \(x \in \Omega\) and continuous with respect to \(T \in \mathbb{R}\), such that

\[
\exists \alpha_# > 0 : \ |\alpha_s(x, T)| \leq \alpha_#, \quad \text{a.e. } x \in \Omega, \quad \forall T \in \mathbb{R}.
\]

(H2) The thermal and electrical conductivities \(k, \sigma : \Omega \times \mathbb{R} \rightarrow \mathbb{R}_{n \times n}\) are Carathéodory tensors, where \(\mathbb{M}_{n \times n}\) denotes the set of \(n \times n\) matrices. Furthermore, they verify

\[
\exists k_# > 0 : \ k_{ij}(x, T) \xi_i \xi_j \geq k_# |\xi|^2;
\]

\[
\exists \sigma_# > 0 : \ \sigma_{ij}(x, T) \xi_i \xi_j \geq \sigma_# |\xi|^2, \quad \text{a.e. } x \in \Omega, \quad \forall T \in \mathbb{R}, \quad \xi \in \mathbb{R}^n,
\]

under the summation convention over repeated indices: \(Aa \cdot b = A_{ij}a_j b_i = b^\top Aa\); and

\[
\exists k_# > 0 : \ |k_{ij}(x, T)| \leq k_#;
\]

\[
\exists \sigma_# > 0 : \ |\sigma_{ij}(x, T)| \leq \sigma_# \quad \text{a.e. } x \in \Omega, \quad \forall T \in \mathbb{R},
\]

for all \(i, j \in \{1, \ldots, n\}\).

(H3) The boundary operators \(f_\lambda\) and \(\gamma\) are Carathéodory functions from \(\Gamma \times \mathbb{R}\) into \(\mathbb{R}\) such that

\[
\exists b_# > 0 : \ b_# \leq f_\lambda(x, T) \leq b_#;
\]

\[
\exists \gamma_# > 0 : \ |\gamma(x, T)| \leq \gamma_# \quad \text{a.e. } x \in \Gamma, \quad \forall T \in \mathbb{R}.
\]

(H4) \(\theta_b \in L^{(\ell-1)(2+\delta)}(\Gamma)\), and \(g \in L^{2+\delta}(\Gamma_N)\) such that \(\int_{\Gamma_N} g ds = 0\), for some \(\delta > 0\).

For the sake of simplicity, we assume \(\delta = 1\).

Observe that for any \(1 \leq p \leq 3\) \(L^3(\Gamma_N) \hookrightarrow L^{p(n-1)/n}(\Gamma_N)\) (which is the dual space of \(L^{p'(n-1)/(n-p')}((\Gamma_N))\)), then \(gw \in L^1(\Gamma_N)\) for all \(w \in W^{1,p'}(\Omega)\).

Finally, we are able to state the existence result in the two-dimensional space. Let us denote by \(C_\infty = n^{-1/p} \omega_n^{-1/n}[(p - 1)/(p - n)]^{1/p'}|\Omega|^{1/n-1/p}\) the continuity constant
of the Morrey-Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p > n$, where $\omega_n$ is the volume of the unit ball $B_1(0)$ of $\mathbb{R}^n$, that is, $\omega_n = \pi^{n/2}/\Gamma(n/2 + 1)$.

**Theorem 1.1.** Suppose that the assumptions (H1)-(H4) be fulfilled. Then, there exists at least one solution $(\theta, \phi) \in V_{p,\ell} \times V_p$ of $(P)$, for $2 < p < 2 + 1/(\nu - 1)$, where

$$v = 65 \times 2^{12} \left(2 \max\left\{\frac{\sqrt{A(\sigma^\#)^2 + \sigma^\#}}{\sigma^\#}, \frac{\sqrt{A(k^\#)^2 + k^\#}}{k^\#}\right\} + 1\right)^2,$$  

if provided by the data smallness

$$4C_\infty \sigma^\# \left[\left(\frac{|\Omega|^{1-\frac{2}{\ell}}}{b^\# k^\#}\right)^{1/\ell} + M_5\right] (\alpha^\#(1 + M) + M^2) < 1/c,$$

with $M = M_1 + M_2$,

$$c = M_3\|\theta_e\|_{L^p(\Gamma)}^{\ell/2} + M_4\|\theta_e\|_{L^{p-1}(\Gamma)}^{1/(\ell-1)} + \|\theta_e\|_{L^\infty(\Gamma)}^{1/(\ell-1)}\|\theta_e\|_{L^\infty(\Gamma)}.$$

where $M_1$, $M_2$, $M_3$, $M_4$, and $M_5$ are given in (62), (63), (64), (65), and (66), respectively.

### 2. Abstract main results

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain of class $C^1$ with the following characteristics.

**Definition 2.1.** Its boundary $\partial \Omega$ is constituted by two disjoint open $(n-1)$-dimensional sets, $\Gamma_N$ and $\Gamma$, such that $\partial \Omega = \bar{\Gamma}_N \cup \bar{\Gamma}$.

We consider $\Gamma_N$ over which the Neumann boundary condition is taken into account, and $\Gamma$ over which the radiative effects may occur. Each one, $\Gamma_N$ and $\Gamma$, may be alternatively of zero $(n-1)$-Lebesgue measure.

We study the following boundary value problem, in the sense of distributions,

$$-\nabla \cdot (A \nabla u) = f - \nabla \cdot f \quad \text{in} \quad \Omega;$$  

$$\quad (A \nabla u - f) \cdot n + b(u) = h \quad \text{on} \quad \Gamma;$$  

$$\quad (A \nabla u - f) \cdot n = g \quad \text{on} \quad \Gamma_N,$$

where $n$ is the unit outward normal to the boundary $\partial \Omega$. Whenever the $(n \times n)$-matrix of the leading coefficient is $A = aI$, where $a$ is a real function and $I$ denotes the identity matrix, the elliptic equation stands for isotropic materials. Our problem includes the conormal derivative boundary value problem if provided by $\Gamma = \partial \Omega$ (or equivalently $\Gamma_N = \emptyset$). The problem (10)-(12) is the so-called mixed Robin-Neumann problem if $b$ is linear in (11).

Assume

(A): $A = [A_{ij}]_{i,j=1,\ldots,n} \in [L^\infty(\Omega)]^{n \times n}$ is uniformly elliptic, and uniformly bounded:

$$\exists a^\# > 0, \quad A_{ij}(x)\xi_i \xi_j \geq a^\# |\xi|^2, \quad \text{a.e.} \quad x \in \Omega, \forall \xi \in \mathbb{R}^n;$$

$$\exists a^\# > 0, \quad \|A\|_{L^\infty(\Omega)} \leq a^\#.$$
(B): \( b : \Gamma \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that it is monotone with respect to the last variable, and it has \((\ell - 1)\)-growthness properties:
\[
\exists b_\# > 0, \quad b(x, T) \text{sign}(T) \geq b_\# |T|^\ell - 1; \tag{15}
\]
\[
\exists b_\# > 0, \quad |b(x, T)| \leq b_\# |T|^\ell - 1, \tag{16}
\]
for a.e. \( x \in \Gamma \), and for all \( T \in \mathbb{R} \).

**Remark 2.1.** If \( b(T) = |T|^{\ell - 2}T \), for all \( T \in \mathbb{R} \), the property of strong monotonicity occurs with \( b_\# = 2(2 - \ell) \) [13, Lemma 3.3].

Our main abstract results are stated as follows. We observe that (18) has not a standard format in order to avoid inflated involved constants.

**Theorem 2.1** (\( \text{meas}(\Gamma) > 0 \)). Let \( \delta > 0 \). Let \( f \in L^{2+\delta}(\Omega) \), \( f \in L^{2+\delta}(\Gamma_N) \), and \( h \in L^{2+\delta}(\Gamma) \). Under the assumptions (A)-(B), there exists a weak solution \( u \in V_{2,\ell} \) to (10)-(12), in the sense
\[
\int_\Omega (A \nabla u) \cdot \nabla v \, dx + \int_\Gamma b(u)v \, ds = \int_\Omega f \cdot \nabla v \, dx + \int_\Omega f v \, dx + \int_{\Gamma_N} g v \, ds + \int_\Gamma h v \, ds, \quad \forall v \in V_{2,\ell}, \tag{17}
\]
such that belongs to \( W^{1,2+\varepsilon}(\Omega) \) for any \( \varepsilon \in [0, \delta] \cap [0, 1/(\nu - 1)] \), where \( \nu = \nu_U \) that is given by (52). In particular,
\[
\| \nabla u \|_{2+\varepsilon, \Omega} \leq N \left[ \frac{8}{(r_\#)^n} \right]^{\varepsilon/2} Z_1(\nu) \| \nabla u \|_{2,\Omega}^{2+\varepsilon} + \\
+ (2^{3\varepsilon/2} Z_1(\nu) + Z_2(\nu)) \| \mathcal{F}(a_\#) \|_{2+\varepsilon, \Omega} \| \mathcal{H}(a_\#) \|_{2+\varepsilon, \partial \Omega}, \tag{18}
\]
where
\[
\mathcal{H}(a_\#) = \frac{2K_{2n/(n+1)}}{(a_\#)^{1/2}} \left( \frac{2}{a_\#} + 2^{-1/n} \right)^{1/2} |g| r_N + h|\Gamma|; \tag{19}
\]
\[
\mathcal{F}(a_\#) = (a_\#)^{-1/2} \left[ \left( \frac{2}{a_\#} + 2 \right) |f|^2 + \frac{1}{\nu_3} |f|^2 \right]^{1/2}; \tag{20}
\]
\[
Z_1(\nu) = \frac{4}{4 - (n + 2)(\nu - 1)\varepsilon} \times 2^{n(1+\varepsilon/2)}; \tag{21}
\]
\[
Z_2(\nu) = \frac{\nu(4 + (n + 2)\varepsilon)}{4 - (n + 2)(\nu - 1)\varepsilon} \times 2^{n(1+\varepsilon/2)}, \tag{22}
\]
with \( K_{2n/(n+1)} = [\Gamma(n)]^{1/(n+1)}[(\sqrt{n\pi})^{n-1}\Gamma((n + 1)/2)]^{-1/(n+1)} \) (see Remark 2.2), and \( N \in \mathbb{N} \) being dependent on the space dimension.
Remark 2.2. For $1 < q < n$, the best continuity constant of the embedding $W^{1,q}(\Omega) \hookrightarrow L^{q^*}(\partial\Omega)$, with $q_* = q(n-1)/(n-q)$ being the critical trace exponent, is \cite{6}

$$K_q = \pi^{(1-q)/2} \left(\frac{q-1}{n-q}\right)^{q-1} \left[\frac{\Gamma\left(\frac{q(n-1)}{2(q-1)}\right)}{\Gamma\left(\frac{n-1}{2(q-1)}\right)}\right]^{(q-1)/(n-1)},$$

where $\Gamma$ stands for the Gamma function.

Theorem 2.2 (meas($\Gamma$) = 0). Let the assumption (A) be fulfilled, and $\delta > 0$. If $f \in L^{2+\delta}(\Omega)$, $f \in L^{2+\delta}(\Omega)$, and $g \in L^{2+\delta}(\Gamma_N)$, verify the compatibility condition

$$\int_\Omega f \, dx + \int_{\Gamma_N} g \, ds = 0,$$

then the Neumann problem

$$\int_\Omega (A \nabla u) \cdot \nabla v \, dx = \int_\Omega f \cdot \nabla v \, dx + \int_\Omega fv \, dx + \int_{\Gamma_N} gv \, ds, \quad \forall v \in V_p', \quad (24)$$

admits a unique weak solution $u \in V_p$ satisfying (18) with $H$ being replaced by

$$G(a_{##}) = 2K_{2n/(n+1)} (\frac{2}{a_{##} + 2^{-1/n}})^{1/2} |g|.$$

Throughout this work, we adopt the standard notations:

- $Q(x)$ denotes the cubic interval (cubes with edges parallel to coordinate planes), that is, $Q(x)$ denotes the open ball of radius $R > 0$ centered at the point $x \in \mathbb{R}^n$, defined by

$$Q(x) = \{y \in \mathbb{R}^n : |y - z| := \max_{1 \leq i \leq n} |y_i - x_i| < R\}.$$

We call by $Q$ any cube that is an orthogonal transformation of a cubic interval.

- $A[v > k] = \{x \in A : v(x) > k\}$, where $v \in L^1(A)$, $v \geq 0$ in $A$, with the set $A$ being either $\Omega$, $\Gamma_N$, $\Gamma$, $\partial\Omega$ or $\bar{\Omega}$. Moreover, the significance of $|A|$ stands the Lebesgue measure of a set of $\mathbb{R}^n$, and also for the $(n-1)$-Lebesgue measure.

3. Reverse Hölder inequalities with increasing supports

In this section, a $C^{0,1}$ domain is sufficient to be assumed.

Let us recall a result on the Stieltjes integral in the form that we are going to use (for the general form see \cite{2}).

Lemma 3.1. Suppose that $q, t_0 \in ]0, \infty]$, and $a \in ]1, \infty]$. If $h, H : [t_0, \infty[ \rightarrow [0, \infty]$ are nonincreasing functions such that

$$\lim_{t \to \infty} h(t) = \lim_{t \to \infty} H(t) = 0,$$

and that

$$-\int_t^\infty \tau^q dh(\tau) \leq a[t^q h(t) + H(t)], \quad \forall t \geq t_0,$$

(27)
then, for \( \gamma \in [q, aq/(a - 1)] \)
\[
- \int_{t_0}^\infty t^\gamma dh(t) \leq \frac{q}{aq - (a - 1)\gamma} \left( - \int_{t_0}^\infty t^\gamma dh(t) \right) + \frac{a^\gamma}{aq - (a - 1)\gamma} \left( - \int_{t_0}^\infty t^{-\gamma} dH(t) \right). \tag{28}
\]

Next, let us recall the Calderon-Zygmund subdivision argument \[17\] p. 127.

**Lemma 3.2.** Let \( Q \) be a open cube in \( \mathbb{R}^n \), \( v \in L^1(Q) \), \( v \geq 0 \) in \( Q \), and \( \zeta > (v)_Q = \frac{1}{|Q|} \int_Q v(y)dy \). Then there exists a sequence of cubes \( \{Q_j\}_{j \geq 1} \), with sides parallel to the axes and with disjoint interiors, such that \( v \leq \zeta \) a.e. in \( Q \setminus (\cup_{j \geq 1} Q_j) \), and
\[
\zeta < \frac{1}{|Q_j|} \int_{Q_j} v(y)dy \leq 2^n \zeta.
\]

Now we prove the following versions with increasing supports of Gehring lemma. Their proofs are based on classical arguments \[4, 17, 29\].

**Proposition 3.1.** Let \( p > 1 \), \( \delta > 0 \), and nonnegative functions \( \Phi \in L^p(\Omega) \) and \( \Psi \in L^{p+\delta}(\Omega) \) satisfy the estimate
\[
\frac{1}{R^n} \int_{Q_{aR}(z)} \Phi^p dx \leq B \left( \frac{1}{R^n} \int_{Q_R(z)} \Phi dx \right)^p + \frac{1}{R^n} \int_{Q_R(z)} \Psi^p dx, \tag{29}
\]
for all \( z \in \Omega \), \( R < \min \{\text{dist}(z, \partial \Omega)/\sqrt{n}, R_0\} \) with some constants \( \alpha \in [1/2, 1] \), \( R_0 > 0 \), and \( B > 0 \). Then, \( \Phi \in L^{q}_{\text{loc}}(\Omega) \) for all \( p \leq q \leq p + \delta \) and \( q < p + (p - 1)/(a_1 - 1) \), with
\[
a_1 = (8^n + 1)2^{3np} \left( B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} + 1 \right)^p. \tag{30}
\]
In particular, for any cubic interval \( Q_r(x_0) \subset \subset \Omega \) such that \( r < R_0 \), we have
\[
\|\Phi\|^q_{q,\omega} \leq [\text{dist}(\omega, \partial Q_r(x_0))]^{-nq/p} \left[ \frac{a_1(q - 1)r^{nq/p}}{q - 1 - a_1(q - p)} \|\Psi\|^q_{q, Q_r(x_0)} + \right. \\
\left. + \frac{(p - 1)r^n}{q - 1 - a_1(q - p)} \left( \|\Phi\|^p_{p, Q_r(x_0)} + \|\Psi\|^p_{p, Q_r(x_0)} \right)^{q/p} \right], \tag{31}
\]
for any measurable set \( \omega \subset \subset Q_r(x_0) \).

**Proof.** Fix \( Q_r(x_0) \subset \subset \Omega \) with \( r < R_0 \). Let us transform the cubic interval \( Q_r(x_0) \) into \( Q = Q_{3/2}(0) \) by the passage to new coordinates system \( y = 3(x - x_0)/(2r) \). Setting \( M = \left[3/(2r)^{n/p}(\|\Phi\|^p_{p, Q_r(x_0)} + \|\Psi\|^p_{p, Q_r(x_0)})\right]^{1/p} \), the normalized functions \( \Phi(y) = \Phi(x_0 + 2ry/3)/M \) and \( \Psi(y) = \Psi(x_0 + 2ry/3)/M \) satisfy \( \max\{\|\Phi\|_{p, Q}, \|\Psi\|_{p, Q} \leq 1 \). Let us define \( \Phi_0(y) = \Phi(y)\text{dist}^{n/p}(y, \partial Q) \), and \( \Psi_0(y) = \Psi(y)\text{dist}^{n/p}(y, \partial Q) \).
For each $t \in [1, \infty]$, we introduce
\[ h(t) = \int_{Q[\phi_0 > t]} \Phi_0(y)dy, \]
\[ H(t) = \int_{Q[\psi_0 > t]} \Psi_0^p(y)dy. \]

Then, $h, H : [1, \infty] \to [0, \infty]$ are nonincreasing functions such that verify (26). In order to apply Lemma 3.1 it remains to prove that (27) is verified with $q = p - 1$, taking the relation
\[ \int_{Q[\phi_0 > t]} \Phi_0^p(y)dy = -\int_{t}^{\infty} r^{p-1}dh(r), \quad \forall p > 1, \]
into account. More exactly, we must prove that
\[ \int_{Q[\phi_0 > t]} \Phi_0^p(y)dy \leq a_1 \left( t^{p-1} \int_{Q[\phi_0 > t]} \Phi_0(y)dy + \int_{Q[\psi_0 > t]} \Psi_0^p(y)dy \right), \quad (32) \]
for any $t \geq 1$.

We decompose $Q = \bigcup_{k \in \mathbb{N}} C^{(k)}$, where $C^{(0)} = Q_{1/2}(0)$, and for each $k \geq 1$, $C^{(k)} = \{ y \in Q : 2^{-k} < \text{dist}(y, \partial Q) \leq 2^{-k+1} \}$. Each set $C^{(k)}$ is the finite union of disjoint cubic intervals of size $1/2^{k+2}$, namely $D_i^{(k)} = Q_{1/2^{k+1}}(w^{(i)})$. In particular, $|D_i^{(k)}| = 2^{-(k+2)n}$.

Fix $t \geq 1$. Since we have
\[ \frac{1}{|D_i^{(k)}|} \int_{D_i^{(k)}} \Phi_0^p(y)dy \leq 2^{3n} \| \Phi_0 \|_{L^p, Q}^p \leq 2^{3n}, \]
from Lemma 3.2 with $v = \Phi_0^p \in L^1(D_i^{(k)})$ and $\zeta = t^p \sqrt[n]{\lambda}$ with $\lambda > 2^{3np}$ defined in (37), there exists a disjoint sequence of cubic intervals $Q^{(k)}_{i,j} \subset D_i^{(k)}$ in the conditions of Lemma. Since $i \in \{1, \ldots, I\}$ with $I \in \mathbb{N}$, we in fact have a disjoint sequence of cubic intervals $Q^{(k)}_j = Q^{(k)}_{i,j}(y^{(k,j)}) \subset C^{(k)}$ such that $0 \leq t \sqrt[n]{\lambda}$ a.e. in $C^{(k)} \setminus \left( \bigcup_{j \geq 1} Q^{(k)}_j \right) := E$, and
\[ t^p \lambda < \frac{1}{|Q^{(k)}_j|} \int_{Q^{(k)}_j} \Phi_0^p(y)dy \leq 2^n t^p \lambda, \quad \forall j \geq 1. \quad (33) \]

Considering that $|E[\Phi_0 > t \sqrt[n]{\lambda}]| = 0$, we compute
\[ \int_{Q[\phi_0 > t \sqrt[n]{\lambda}]} \Phi_0^p dy \leq \sum_{k \geq 0} \sum_{j \geq 1} \int_{Q^{(k)}_j} \Phi_0^p dy \leq 2^n t^p \lambda \sum_{k \geq 0} \sum_{j \geq 1} |Q^{(k)}_j|. \quad (34) \]

Next, to estimate the above right hand side, let us prove, for all $k \geq 0$, and $j \geq 1$, there exists $R = R_{k,j} \in ]r^{(k)}_j, 2r^{(k)}_j]$ that verifies
\[ t(2R)^n < \int_{Q_R[\phi_0 > t]} \Phi_0 dy + t^{-p+1} \int_{Q_R[\psi_0 > t]} \Psi_0^p dy, \quad (35) \]
with the notation $Q_R = Q_R(y^{(k,j)})$. Since $R \leq 2r^{(k)}_j < 2^{-(k+1)}$, $Q_R$ only intersects the sets $C^{(k-1)}, C^{(k)}$, and $C^{(k+1)}$.
Fix $Q_j^{(k)} \subset C^{(k)}$ such that $r_j^{(k)} < 2^{-(k+3)}$. Let us choose $R \in [r_j^{(k)}, 2r_j^{(k)}]$ provided by $\alpha = r_j^{(k)}/R \in [1/2, 1]$. We use the first inequality in (33), obtaining

$$t^p \lambda < \frac{1}{|Q_j^{(k)}|} \int_{Q_j^{(k)}} \Phi^p_0 \, dy \leq \frac{1}{R^n} \int_{Q_{\alpha R}} \Phi^p_0 \, dy \leq \frac{2^{-(k-1)n}}{R^n} \int_{Q_{\alpha R}} \Phi^p \, dy,$$

with $Q_{\alpha R} = Q_{\alpha R}(y^{(k,j)}) \subset C^{(k)}$.

Rewriting (29) in terms of the new coordinates system, taking $z = x_0 + 2ry^{(k,j)}/3$, and dividing the resultant inequality by $\|\Phi\|^p_{p,Q_0(x_0)} + \|\Psi\|^p_{p,Q_0(x_0)}$, we deduce

$$\frac{1}{R^n} \int_{Q_{\alpha R}} \Phi^p \, dy \leq B \left( \frac{2R_0}{3} \right)^{n(p-1)} \left( \frac{1}{R^n} \int_{Q_R} \Phi^p \, dy \right)^p + \frac{1}{R^n} \int_{Q_R} \Psi^p \, dy,$$

considering that $r < R_0$.

Then, gathering the above two inequalities, we find

$$(tR^n)^p \lambda < 2^{-(k-1)n} \left[ B \left( \frac{2R_0}{3} \right)^{n(p-1)} \left( \frac{1}{R^n} \int_{Q_R} \Phi^p \, dy \right)^p + R^n(p-1) \int_{Q_R} \Psi^p \, dy \right].$$

(36)

On one hand, we have

$$(tR^n)^p \lambda < 2^n \left[ B \left( \frac{2R_0}{3} \right)^{n(p-1)} \left( \frac{1}{R^n} \int_{Q_R} \Phi^p \, dy \right)^p + R^n(p-1) \int_{Q_R} \Psi^p \, dy \right],$$

obtaining

$$tR^n \sqrt[p']{\lambda} < 2^{n/p} \left[ B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} \left( \frac{1}{R^n} \int_{Q_R[\Phi_0 > t]} \Phi^p_0 \, dy + t(2R)^n \right) + R^n(p-1) \left( \frac{1}{R^n} \int_{Q_R[\Psi_0 > t]} \Psi^p \, dy \right)^{1/p} + t2^{n/p} R^n \right].$$

By applying the Young inequality,

$$R^n(p-1) \left( \frac{1}{R^n} \int_{Q_R[\Psi_0 > t]} \Psi^p \, dy \right)^{1/p} \leq \frac{t}{p} R^n + \frac{t^{-(p-1)}}{p} \int_{Q_R[\Psi_0 > t]} \Psi^p \, dy,$$

and taking $p > 1$ and $2^{n/p} + 1/p' = 2^n$ into account, we find

$$tR^n \left[ \sqrt[p']{\lambda} - 2^{n+n/p} \left( B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} + 1 \right) \right] <$$

$$< 2^{n/p} \left[ B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} \left( \frac{1}{R^n} \int_{Q_R[\Phi_0 > t]} \Phi^p_0 \, dy + t^{-(p-1)} \int_{Q_R[\Psi_0 > t]} \Psi^p \, dy \right) \right].$$

Therefore, we choose

$$\lambda = 2^3 \left( B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} + 1 \right)^p,$$  

(37)
concluding (35).

On the other hand, applying the Hölder inequality in (36), and using \( \max\{\|\Phi\|_{p,Q}, \|\Psi\|_{p,Q}\} \leq 1 \), we have
\[
t'^n < \lambda^{-1/p} \left( B^{1/p} \left( \frac{2R_0}{3} \right)^{n(1-1/p)} + 1 \right) R^{n(p-1)/p},
\]
and consequently, using (37), we conclude
\[
t'^n < 2^{-3n}.p.
\]

According to the Vitali covering lemma, there exist \( \sigma \in ]3,4[ \) and a sequence of disjoint cubic intervals \( \{Q_{y(i)} \} \) from the collection \( \{Q_{y(k,j)} \} \) such that
\[
\cup_{k \geq 0} \cup_{j \geq 1} Q_{y(k,j)} \subset \cup_{i \geq 1} Q_{\sigma y(i)} \subset \cup_{i \geq 1} y(i).
\]
Hence,
\[
\sum_{k \geq 0} \sum_{j \geq 1} |Q_{y(k)}| \leq \sum_{k \geq 0} \sum_{j \geq 1} |Q_{y(k,j)}| \leq \sigma^n \sum_{i \geq 1} |Q_{y(i)}|.
\]
Combining the above with (34), and (35), we find
\[
\int_{Q_{[\Phi_0 > t]} \Psi} \Phi_0 \, dy \leq 2^n \lambda s n \left( t^{p-1} \int_{Q_{[\Phi_0 > t]} \Psi} \Phi_0 \, dy + \int_{Q_{[\Psi_0 > t]} \Phi_0} \Psi_0 \, dy \right). \tag{38}
\]
Now, observing that
\[
\int_{Q_{[\Phi_0 > t]} \Psi} \Phi_0 \, dy \leq \int_{Q_{[\Phi_0 > t]} \Psi} \Phi_0 \, dy + t^{p-1} \lambda \int_{Q_{[\Phi_0 > t]} \Psi} \Phi_0 \, dy,
\]
(38) implies (32). Therefore, Lemma 3.1 can be applied, concluding that, for any \( \gamma \) such that \( p \leq \gamma + 1 < p + (p-1)/(a_1 - 1) \), (28) implies
\[
\int_{Q_{[\Phi_0 > 1]} \Psi} \Phi_0^{\gamma+1} \, dy \leq \frac{p-1}{a_1(p-1) - (a_1 - 1)\gamma} \int_{Q_{[\Phi_0 > 1]} \Psi} \Phi_0^{\gamma+1} \, dy + \frac{a_1\gamma}{a_1(p-1) - (a_1 - 1)\gamma} \int_{Q_{[\Phi_0 > 1]} \Psi} \Psi_0^{\gamma+1} \, dy.
\]
The requirement of \( q = \gamma + 1 < p + \delta \) assures the finiteness of the last integral of the RHS of the above inequality. Since \( \Phi_0^{\gamma+1} \leq \Phi_0 \) a.e. in \( Q \setminus Q_{[\Phi_0 > 1]} \), for any \( \omega \subset Q \), we find
\[
\left[ \frac{3\text{dist}(\omega, \partial Q_{y(x_0)})}{2r} \right]^{nq/p} \int_{Q_0} \Phi_0 \, dy \leq \frac{p-1}{q - a_1(q-1)} \left( \frac{3}{2} \right)^n \int_{Q} \Phi_0 \, dy + \frac{a_1(q-1)}{q - a_1(q-1)} \left( \frac{3}{2} \right)^{nq/p} \int_{Q} \Psi_0^{\gamma+1} \, dy,
\]
keeping the same designation to the transformed set \( \omega \subset Q_{y(x_0)} \). Passing the above inequality to the initial coordinates system, we conclude (31). \[\Box\]
Corollary 3.1. Let \( x_0 \in \Omega \) and \( R_0 > 0 \) such that \( \sqrt{n}R_0 < \text{dist}(x_0, \partial \Omega) \). Let \( p > 1, \delta > 0 \) and nonnegative functions \( \Phi \in L^p(\Omega) \) and \( \Psi \in L^{p+\delta}(\Omega) \) satisfy the estimate
\[
\frac{1}{R^n} \int_{Q_{\delta R}(z)} \Phi^p dx \leq B \left( \frac{1}{R^n} \int_{Q_R(z)} \Phi^p dx \right)^{p/t} + \frac{1}{R^n} \int_{Q_R(z)} \Psi^p dx,
\]
for all \( z \in Q_{R_0}(x_0) \) and \( \sqrt{n}R < \text{dist}(z, \partial Q_{R_0}(x_0)) \), with some constants \( \alpha \in [1/2, 1] \), \( t < p \), and \( B > 0 \). Then \( \Phi \in L^p_{\text{loc}}(Q_{R_0}(x_0)) \) for all \( q \in [p, p+\delta] \cap [p, p+(p-t)/(a_1-1)] \), with \( a_1 \) being defined by (37). In particular, it verifies
\[
\|\Phi\|^q_{q, Q_{\rho/2}(x_0)} \leq \frac{2^{\eta q/p}}{q-t-a_1(q-p)} \left( (p-t)^{r-n+(q-p)/p} 2^{(q-p)/p} \|\Phi\|^q_{p, Q_{\rho}(x_0)} + (2^{(n+1)(q-p)/(p-t)} + a_1(q-t)) \|\Psi\|^q_{q, Q_{\rho}(x_0)} \right),
\]
for any \( 0 < \rho < R_0 \).

Proof. We write (39) as
\[
\frac{1}{R^n} \int_{Q_{\delta R}(z)} \Phi^p dx \leq B \left( \frac{1}{R^n} \int_{Q_R(z)} \Phi^p dx \right)^{p/t} + \frac{1}{R^n} \int_{Q_R(z)} \Psi^p dx,
\]
and we apply Proposition 3.1 with \( \omega = Q_{\rho/2}(x_0) \). Next, we use the Hölder inequality to get
\[
\|\Psi\|^p_{p, Q_{\rho}(x_0)} \leq (2r)^{n(q-p)/p} \|\Psi\|^q_{q, Q_{\rho}(x_0)},
\]
Then we apply the relation \((a^p + b^p)^{q/p} \leq 2^{q/p-1}(a^q + b^q)\) to conclude (40).

Next, the higher summability of \( \Phi \) near the flattened boundary is established due to the reverse Hölder inequality with a surface integral.

Proposition 3.2. Let \( z_0 = (z_0', 0) \in \mathbb{R}^n, R_0 > 0, \) and
\[
Q^+_{R_0}(z_0) := \{ z \in \mathbb{R}^n : |z' - z'_0| < R_0, \ z_n > 0 \};
\]
\[
\Sigma_{R_0}(z_0) = \{ z \in \mathbb{R}^n : |z' - z'_0| < R_0, \ z_n = 0 \};
\]
\[
\partial \Sigma_{R_0}(z_0) = \{ z \in \mathbb{R}^n : |z' - z'_0| = R_0, \ z_n = 0 \},
\]
where \( |z'| = \max_{1 \leq i \leq n-1} |z_i| \). For \( p > 1 \) and \( \delta > 0 \), we suppose that the nonnegative functions \( \Phi \in L^p(Q^+_{R_0}(z_0)), \Psi \in L^{p+\delta}(Q^+_{R_0}(z_0)), \) and \( \varphi \in L^{p+\delta}(\Sigma_{R_0}(z_0)) \) satisfy the estimate
\[
\frac{1}{R^n} \int_{Q^+_{\delta R}(z)} \Phi^p dx \leq B \left( \frac{1}{R^n} \int_{Q^+_{R}(z)} \Phi^p dx \right)^{p} + \frac{1}{R^n} \int_{Q^+_{R}(z)} \Psi^p dx + \frac{1}{R^{n-1} \int_{\Sigma_{R}(z)}} \varphi^p dx,
\]
for all \( z \in \Sigma_{R_0}(z_0), \) and all \( R < \min\{R_0, \text{dist}(z, \partial \Sigma_{R_0}(z_0))\} \), with some constants \( \alpha \in [1/2, 1] \), and \( B > 0 \). Then, \( \Phi \in L^{p+\varepsilon}(\omega \cap Q^+_{R_0}(z_0)) \), for all \( \varepsilon \in [0, \delta] \cap [0, (p-1)/(a_U-1)] \).
and measurable set \( \omega \subset Q_{R_0}(z_0) \), and it verifies
\[
\|\Phi\|_{p+\varepsilon, p+\varepsilon, \omega \cap Q_{R_0}^+}(z_0) \leq \frac{[\text{dist}(\omega, \partial Q_{R_0}(z_0))]^{n(1+\varepsilon/p)}}{p - 1 - (a_U - 1)\varepsilon} \times \]
\[
\frac{(p - 1)R_0^n}{2} \left( 2\|\Phi\|_{p, Q_{R_0}^+}(z_0) + 2\|\Psi\|_{p, Q_{R_0}^+}(z_0) + \frac{2R_0}{3}\|\varphi\|_{p, \Sigma_{R_0}^+}(z_0) \right)^{1+\varepsilon/p} +
\]
\[
a_U(p - 1 + \varepsilon)R_0^{n(1+\varepsilon/p)} \left( \|\Psi\|_{p+\varepsilon, \omega \cap Q_{R_0}^+}(z_0) + \|\varphi\|_{p+\varepsilon, \Sigma_{R_0}^+}(z_0) \right),
\]
where
\[
a_U = (8^n + 1)2^{3np} \left( B^{1/p} R_0^{n(1-1/p)} + 1 \right)^p.
\]

Proof. We prolong \( \Phi \) and \( \Psi \) as even functions with respect to \( \Sigma_{R_0}(z_0) \):
\[
\widetilde{\Phi}(z', z_n) = \left\{ \begin{array}{ll}
\Phi(z', z_n), & z_n > 0 \\
\Phi(z', -z_n), & z_n < 0
\end{array} \right.
\]
\[
\widetilde{\Psi}(z', z_n) = \left\{ \begin{array}{ll}
\Psi(z', z_n), & z_n > 0 \\
\Psi(z', -z_n), & z_n < 0.
\end{array} \right.
\]

Transforming \( Q_{R_0}(z_0) \) into \( Q = Q_{3/2}(0) \) by the passage to new coordinates system \( y = 3(z - z_0)/(2R_0) \), setting \( M = [3/(2R_0)]^{n/p}(\|\Phi\|_{p, Q_{R_0}^+}(z_0) + \|\Psi\|_{p, Q_{R_0}^+}(z_0) + 2R_0\|\varphi\|_{p, \Sigma_{R_0}(z_0)/3}^{1/p}) \), and defining \( \overline{\varphi}(y) = \widetilde{\Phi}(z_0+2R_0y/3)/M, \overline{\psi}(y) = \widetilde{\Psi}(z_0+2R_0y/3)/M, \overline{\varphi}(y) = \varphi(z_0 + 2R_0y/3)/M \), we define
\[
H(t) = \int_{Q[\overline{\varphi} > t]} \overline{\varphi}^p(y)dy + \int_{\Sigma[\overline{\varphi} > t]} \overline{\varphi}^p(y)ds_y, \quad \Sigma = Q_{3/2}^{(n-1)}(0) \times \{0\}.
\]

The argument of the proof of Proposition 3.1 remains valid until (56), which reads
\[
(tR^n)^p \lambda < 2^n B \left( \frac{2R_0}{3} \right)^{(n-1)} \left( \int_{Q_R} \Phi_0 dy \right)^p + R^{n(p-1)} \int_{Q_R} \overline{\varphi}^p dy +
\]
\[
+ R^{n(p-1)+1} \frac{3}{R_0} \int_{\Sigma_R} \overline{\varphi}^p ds_y.
\]
Evaluating the term
\[
\left( \frac{3R^{n(p-1)+1}}{R_0} \int_{\Sigma_R} \overline{\varphi}^p ds_y \right)^{1/p} \leq \left( \frac{3R^{n(p-1)}}{R_0} \int_{\Sigma_R[\overline{\varphi} > t]} \overline{\varphi}^p ds_y \right)^{1/p} + 2^{n-1} R^n,
\]
and proceeding as in the proof of Proposition 3.1 we obtain
\[
tR^n \left[ \sqrt{\lambda} - 2^n \left( 2^{n/p} B^{1/p} R_0^{n(1-1/p)} + 2 \right) \right] < 2^n B^{1/p} R_0^{n(p-1)/p} \int_{Q_R[\Phi_0 > t]} \Phi_0 dy +
\]
\[
+ t^{p-1} \left( \int_{Q_R[\overline{\varphi} > t]} \overline{\varphi}^p ds_y + \frac{3}{R_0} \int_{\Sigma_R[\overline{\varphi} > t]} \overline{\varphi}^p ds_y \right).
\]
Introducing the new definition
\[
\lambda = 2^{3np} \left( B^{1/p} R_0^{n(1-1/p)} + 1 \right)^p,
\]
we take $a_U$ as in (43). For any $0 < r < R_0$, and $\omega \subset Q = Q_{3/2}(0)$, keeping the same designation to the transformed set $\omega \subset Q_{R_0}(z_0)$, we find

$$\left[\frac{3\text{dist}(\omega, \partial Q_{R_0}(z_0))}{2R_0}\right]^{n(\gamma+1)/p} \int_\omega \Phi^{\gamma+1} dy \leq \frac{1}{a_U(p-1) - (a_U - 1)\gamma} \times$$

$$\times \left[ (p - 1) \left(\frac{3}{2}\right)^n \int_Q \Phi^p dy + a_U \gamma \left( \int_Q \Phi^{\gamma+1} dy + \frac{3}{R_0} \int_{\Sigma} \Phi^{\gamma+1} ds_y \right) \right],$$

for any $\gamma$ such that $p \leq \gamma + 1 < p + (p-1)/(a_U - 1)$ and $\gamma + 1 \leq p + \delta$. Therefore, setting $p + \varepsilon = \gamma + 1$ we conclude (42).

In a similar manner that we have Corollary 3.1 from Proposition 3.1, Proposition 3.2 ensures the following Corollary.

**Corollary 3.2.** Under the conditions of Proposition 3.2, if instead of (41),

$$\frac{1}{R^n} \int_{Q_{n/R}(z)} \Phi^p dx \leq B \left( \frac{1}{R^n} \int_{Q_{n/R}(z)} \Phi^t dx \right)^{p/t} + \frac{1}{R^n} \int_{Q_{n/R}(z)} \Psi^p dx +$$

$$+ \frac{1}{R^{n-1}} \int_{\Sigma_{R}(z)} \varphi^p ds,$$

holds for $t < p$, then $\Phi \in L^{p+\varepsilon}(Q^+_p(z_0))$ for all $\varepsilon \in [0, \delta] \cap [0, (p-t)/(a_U - 1)]$, with $a_U$ being defined by (43), and $r = R_0/2$, and it verifies

$$\|\Phi\|_{p+\varepsilon, Q^+_p(z_0)} \leq R_0^{-n\varepsilon/p} \frac{(p-t)2^{n(p+\varepsilon)/p}}{p - t - (a_U - 1)\varepsilon} \Phi(0, Q^+_p, R_0) +$$

$$+ \frac{2^{n(p+\varepsilon)/p}}{p - t - (a_U - 1)\varepsilon} \left( 2^{(n+1)\varepsilon/p}(p-t) + a_U(p-t+\varepsilon) \right) \Phi(0, Q^+_p, R_0) +$$

$$+ \frac{2^{n(p+\varepsilon)/p}}{p - t - (a_U - 1)\varepsilon} \left( R_0(p-t) + a_U(p-t+\varepsilon) \right) \Phi(0, Q^+_p, R_0).$$

4. **Local higher regularity of the gradient**

We separately study the interior and up to the boundary the local higher regularity of the gradient of any weak solution to (10)-(12).

4.1. **Interior higher regularity of the gradient.** Let us begin by establishing a technical result.

**Lemma 4.1.** For any $x \in \mathbb{R}^n$ and $0 < R < \epsilon/(2S_{2n/(n+2)})$, every $u \in H^1(Q_R(x))$ verifies

$$\|u\|_{2, Q_R(x)} \leq \frac{S_{2n/(n+2)}}{1 - \epsilon} \|\nabla u\|_{2, Q_R(x)},$$

where $S_{2n/(n+2)} = \pi^{-1/2} n^{(2-3n)/(2n)} (n-2)^{(n-2)/(2n)} [\Gamma(n)/\Gamma(n/2)]^{1/n}$ (see Remark 4.1).

**Proof.** Making use of Remark 4.1 with $q = 2n/(n+2) < 2$, and the Hölder inequality, we obtain

$$\|u\|_{2, Q_R(x)} \leq S_{2n/(n+2)} \left( \|\nabla u\|_{2, Q_R(x)} + |Q_R(x)|^{1/n} \|u\|_{2, Q_R(x)} \right).$$
Hence, we conclude (45).

Remark 4.1. For $1 < q < n$, the best continuity constant of the Sobolev embedding $W^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$, with $q^* = qn/(n-q)$ being the critical Sobolev exponent, is 

$$S_q = \pi^{-1/2} n^{-1/q} \left( \frac{q-1}{n-q} \right)^{1-1/q} \left( \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/q)\Gamma(1+n-n/q)} \right)^{1/n}.$$  

For $1^* = n/(n-1)$, there exists the limit constant $S_1 = \pi^{-1/2} n^{-1}[\Gamma(1+n/2)]^{1/n}$.

We show the interior local $W^{2+\varepsilon}$-estimate for any weak solution to the problems under study (regardless of whether or not the solution under consideration is subject to any boundary condition) if provided by data with higher integrability.

Proposition 4.1. If there exists $\delta > 0$ such that $f \in L^{2+\delta}(\Omega)$, and $f \in L^{2+\delta}(\Omega)$, then any function $u \in V_{2,\ell}$ solving (17) or (24) belongs to $W^{1,2+\varepsilon}_{loc}(\Omega)$, for all $\varepsilon \in [0, \delta] \cap [0, 1/(\nu_1 - 1)]$, where

$$v_1 = (8^n + 1) 2^{6n} \left[ \left( \frac{2}{a\#} \right)^{1/2} \left( \frac{4(a\#)^2 + 1 + \nu_3}{a\#} \right)^{1/2} + 1 \right]^2,$$

with $\nu_3 = \nu_3(f)$ being positive constant if $f \neq 0$, and $\nu_3(0) = 0$ otherwise. In particular, for every $x_0 \in \Omega$ and $0 < r < \min \{ (4S_{2n/(n+2)})^{-1}, \text{dist}(x_0, \partial \Omega)/\sqrt{n} \}$, we have

$$\|\nabla u\|_{2+\varepsilon, Q_{r/2}(x_0)}^2 \leq r^{-\nu_3/2} \frac{2^{\varepsilon/2}}{Z_1(v_1)} \|\nabla u\|_{2, Q_{r}(x_0)}^2 + (2^{n+1})^{\varepsilon/2} Z_1(v_1) + Z_2(v_1) \|F(a\#)\|_{2+\varepsilon, Q_{r}(x_0)},$$

with $F(a\#)$, $Z_1$ and $Z_2$ being the functions defined in (20), (27) and (22), respectively.

Proof. Let $\eta \in W^{1,\infty}(\mathbb{R}^n)$. Taking $v = u\eta^2 \in V_{2,\ell}$ as a test function in (17), applying the Hölder inequality, and using (13)-(14), we obtain

$$a\# \|\eta \nabla u\|_{2,\Omega}^2 \leq 2a\# \|\eta \nabla u\|_{2,\Omega} \|u \nabla \eta\|_{2,\Omega} + \|f\|_{2,\Omega} (\|\eta \nabla u\|_{2,\Omega} + 2 \|u \nabla \eta\|_{2,\Omega} + \|f\|_{2,\Omega} \|u \eta\|_{2,\Omega}).$$

Fix $z \in \Omega$ and $0 < r < R \leq \min \{ 1/(4S_{2n/(n+2)}), R_0 \}$ such that $Q_{R_0}(z) \subset\subset \Omega$. Choosing $\eta \equiv 1$ in $Q_r(z)$, $\eta \equiv 0$ in $\mathbb{R}^n \setminus Q_{R}(z)$, $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, and $|\nabla \eta| \leq 1/(R-r)$ a.e. in $Q_{R}(z)$. For any $\alpha \in [1/2, 1]$, choosing $r = \alpha R$, we have $R-r < 1$. Thus, applying the Young inequality, we get

$$a\# (1 - \nu_0 - \nu_1) \|\nabla u\|_{2, Q_{\alpha R}(z)}^2 \leq \frac{1}{(1-\alpha)^2} \frac{2^{\varepsilon/2}}{\nu_0 a\#} \left( \frac{(a\#)^2 + 1 + \nu_3}{2} \right) \|u\|_{2, Q_{R}(z)}^2 + \left( \frac{1}{4\nu_1 a\#} + \frac{1}{\nu_2} \right) \|f\|_{2, Q_{R}(z)}^2 + \frac{1}{2
u_3} \|f\|_{2, Q_{R}(z)}^2.$$

Next, taking $\alpha = 1/2$, applying Lemma 4.1 and dividing by $R^n$, we obtain

$$\frac{1}{R^n} \int_{Q_{R/2}(z)} |\nabla u|^2 dx \leq B \left( \frac{1}{R^n} \int_{Q_R(z)} |\nabla u|^{2n/(n+2)} dx \right)^{\frac{n+2}{n}} + \frac{1}{a\# (1 - \nu_0 - \nu_1)} \frac{1}{R^n} \left( \int_{Q_R(z)} \left( \frac{1}{4\nu_1 a\#} + \frac{1}{\nu_2} \right)|f|^2 dx + \int_{Q_R(z)} \frac{1}{2\nu_3} |f|^2 dx, \right)$$
with \( \nu_0 + \nu_1 < 1 \), and

\[
B = \frac{4}{a_{\#}(1 - \nu_0 - \nu_1)} \left( \frac{(a_{\#})^2 + \nu_2 + \nu_3}{\nu_0 a_{\#}} \right) (2S_{2n/(n+2)})^2.
\]

(48)

Applying Corollary 3.1 with \( \Phi = |\nabla u|, t = 2n/(n+2), p = 2, \)

\[
\Psi = \left( \frac{1}{(4\nu_1 a_{\#}) + 1/\nu_2} |f|^2 + |f|^2/(2\nu_3) \right)^{1/2} \in L^{2+\delta}(\Omega),
\]

and taking \( \nu_0 = \nu_1 = 1/4 \) and \( \nu_2 = 1 \), we conclude the claim. \( \square \)

4.2. Higher regularity up to the boundary of the gradient. Let us recall the general definition of \( C^{k,\lambda} \) domain.

**Definition 4.1.** We say that \( \Omega \) is a domain of class \( C^{k,\lambda} \) (or simply \( C^{k,\lambda} \) domain), \( k \in \mathbb{N}_0 \) and \( \lambda \in [0, 1] \), if \( \Omega \) is an open, bounded, connected, nonempty set of \( \mathbb{R}^n \) and it verifies the following:

\[
\exists M \in \mathbb{N} \ \exists \varrho, \nu > 0 : \ \partial \Omega = \bigcup_{m=1}^{M} \Gamma_m,
\]

(50)

with

1. \( \Gamma_m = O_m^{-1}(\{y = (y', y_n) \in Q_{\varepsilon}^{(n-1)}(0) \times \mathbb{R} : y_n = \varpi_m(y')\}) \),
2. \( O_m^{-1}(\{y = (y', y_n) \in Q_{\varepsilon}^{(n-1)}(0) \times \mathbb{R} : \varpi_m(y') < y_n < \varpi_m(y') + \nu\}) \subset \Omega, \)
3. \( O_m^{-1}(\{y = (y', y_n) \in Q_{\varepsilon}^{(n-1)}(0) \times \mathbb{R} : \varpi_m(y') - \nu < y_n < \varpi_m(y')\}) \subset \mathbb{R}^n \setminus \Omega, \)

where

\[
Q_{\varepsilon}^{(n-1)}(0) = \{y' = (y_1, \cdots, y_{n-1}) \in \mathbb{R}^{n-1} : |y_i| < \varrho, \ i = 1, \cdots, n-1\},
\]

and for each \( m = 1, \cdots, M, O_m : \mathbb{R}^n \to \mathbb{R}^n \) denotes a local coordinate system:

\[
y^{(m)} = O_m(x) = Ox + b, \quad O^{-1} = O^T, \quad \text{det } O = 1;
\]

and \( \varpi_m \in C^{k,\lambda}(Q_{\varepsilon}^{(n-1)}(0)). \)

**Proposition 4.2.** Let \( \Omega \) be a \( C^1 \) domain. If there exists \( \delta > 0 \) such that \( f \in L^{2+\delta}(\Omega), \)

\( g \in L^{2+\delta}(\Gamma_N) \) and \( h \in L^{2+\delta}(\Gamma) \), then for every \( x_0 \in \partial \Omega \) there exists a cube \( Q_0 \subset \mathbb{R}^n \) of side length \( 2R_0 \) centered at the point \( x_0 \) such that any function \( u \in V_{2,\ell} \) solving (17) verifies

\[
\|\nabla u\|_{2+\varepsilon, Q_r \cap \Omega}^{2+\varepsilon} \leq 2^{3\varepsilon/2}Z_1(v_U)\|\nabla u\|_{2, Q_r \cap \Omega}^{2+\varepsilon} + (2^{3\varepsilon/2}Z_1(v_U) + Z_2(\nabla u))\|\mathcal{F}(a_{\#})\|_{2+\varepsilon, Q_r \cap \Omega}^{2+\varepsilon} + (Z_1(v_U) + Z_2(\nabla u))\|\mathcal{H}(a_{\#})\|_{2+\varepsilon, Q_r \cap \Omega}^{2+\varepsilon},
\]

(51)

for any cube \( Q_r \subset Q_0 \) of radius \( r \) centered at \( x_0, \)

\[
\nu_U = (8^n + 1)2^{6\nu_3} \left[ \left( \frac{2}{a_{\#}} \right)^{1/2} \left( \frac{8(a_{\#})^2}{a_{\#}^2} + 2 + \frac{\nu_3}{2} \right)^{1/2} + 1 \right]^2,
\]

(52)

where \( \nu_3 = \nu_3(f) \) is a positive constant if \( f \neq 0, \) and \( \nu_3(0) = 0 \) otherwise. Here \( \mathcal{H}(a_{\#}), \mathcal{F}(a_{\#}), Z_1 \) and \( Z_2 \) are the functions defined in (19), (20), (21) and (22), respectively.
Proof. Let \( \eta \in W_0^{1, \infty}(\mathbb{R}^n) \) satisfy \( 0 \leq \eta \leq 1 \) in \( \mathbb{R}^n \). Taking \( v = u\eta^2 \in V_{2, \ell} \) as a test function in (17), applying the Hölder inequality, and making use of (13)-(15), we obtain

\[
a_\# \| \eta \nabla u \|_{2, \Omega}^2 + b_\# \| \eta \eta \|_{1, \Gamma}^\ell \leq 2a_\# \| \eta \nabla u \|_{2, \Omega} \| u \nabla \eta \|_{2, \Omega} + \\
+ \| \eta f \|_{2, \Omega} (\| \eta \nabla u \|_{2, \Omega} + 2 \| u \nabla \eta \|_{2, \Omega}) + \| \eta f \|_{2, \Omega} \| u \eta \|_{2, \Omega} + \\
+ \| \eta (g_\Gamma + h \chi \eta) \|_{2, \partial \Omega} \| u \eta \|_{2, \partial \Omega}.
\]

By Definition 4.1 there exist \( M \in \mathbb{N} \) and \( \varrho, \nu > 0 \) such that for any \( x_0 \in \partial \Omega \) there is \( m \in \{1, \ldots, M\} \) such that a local coordinate system \( y^{(m)} = O_m(x) \) and a local \( C^{1,1} \)-mapping \( \varphi_m \) verify

\[
x_0 \in \Gamma_m = O_m^{-1} \circ \varphi_m^{-1} (Q^{(n-1)}_\varrho(0) \times \{0\}),
\]

where \( \varphi_m : Q^{(n-1)}_\varrho(0) \times \mathbb{R} \to \mathbb{R}^n \) of class \( C^{1,1} \) is defined by

\[
\varphi_m(y) = \left( y_n - \varphi_m(y') \right).
\]

We use the notation \( y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1} \). For each \( m \in \{1, \ldots, M\} \), we consider the change of variables

\[
z \in Q^{(n-1)}_\varrho(0) \times [0, R] \to y = \varphi_m^{-1}(z) \to x = O^{-1}(y).
\]

Since the Jacobian of the transformation \( O_m^{-1} \circ \varphi_m^{-1} \) is equal to 1, let us denote by the same letter any function \( f = f \circ O_m^{-1} \circ \varphi_m^{-1} \).

Fix \( m \in \{1, \ldots, M\} \) such that \( x_0 \in \Gamma_m \) in accordance with (54), set \( z_0 = \varphi_m \circ O_m(x_0) \), and

\[
\Sigma_R(z_0) = \{ z \in Q^{(n-1)}_\varrho(0) \times [0, R] : |z' - z_0'| < R, \ z_n = 0 \},
\]

for any \( 0 < R \leq \min\{\varrho, \nu\} \).

Let \( 0 < r < R \leq R_0 = \min\{\varrho, \nu, \text{dist}(z_0, \partial Q^{(n-1)}_\varrho(0)), (4S_{2n/(n+2)})^{-1}\} \). We choose \( \eta \equiv 1 \) in \( Q_r(z_0) \), \( \eta \equiv 0 \) in \( \mathbb{R}^n \setminus Q_R(z_0) \), and \( |\nabla \eta| \leq 1/(R-r) \) a.e. in \( Q_R(z_0) \setminus Q_r(z_0) \).

By Remark 2.2 and \( u\eta \in W^{1,2n/(n+1)}(Q^+_R(z_0)) \) with \( 2n/(n+1) < 2 \leq n \), making use of the Hölder inequality, we get

\[
\| \eta u \|_{2, \Sigma_R(z_0)} \leq K_{2n/(n+1)} \| Q^+_R(z_0) \|_{\mathcal{H}^n} \left( \| \eta \nabla u \|_{2, Q^+_R(z_0)} + \| u \nabla \eta \|_{2, \Sigma_R(z_0)} + \| u \eta \|_{2, Q^+_R(z_0)} \right),
\]

Inasmuch as \( \Gamma_0 = O_m^{-1} \circ \varphi_m^{-1}(\Sigma_R(z_0)) \), different cases occur, namely \( \Gamma_0 \cap \Gamma \neq \emptyset \) and \( \Gamma_0 \cap \Gamma_N \neq \emptyset \); \( \Gamma_0 \subset \Gamma \), and \( \Gamma_0 \subset \Gamma_N \). Throughout the sequel, we refer to \( \| \cdot \|_{2, \Sigma_R(z_0)} \) including cases where the set is empty.

Thus, the transformed last term in (53) is analyzed as follows

\[
\| \eta (g_\Gamma + h \chi \eta) \|_{2, \Sigma_R(z_0)} \| u \eta \|_{2, \Sigma_R(z_0)} \leq \nu_4 a_\# \| \eta \nabla u \|_{2, Q^+_R(z_0)}^2 + \\
+ \frac{\nu_5}{(R-r)^2} \| u \|_{2, Q^+_R(z_0)} + \\
+ R (K_{2n/(n+1)})^2 \left( \frac{1}{2\nu_4 a_\#} + \frac{(n-1)/n}{\nu_5} \right) \| g_\Gamma + h \chi \eta \|_{2, \Sigma_R(z_0)}^2.
\]
From above and reorganizing the other terms in (53) as in the proof of Proposition 4.1, we have

\[
\frac{1}{a_\#(1-\nu_0-\nu_1-\nu_4)} \left[ \left( \frac{1}{4\nu_1a_\#} + \frac{1}{\nu_2} \right) \|f\|_{2,Q_R^+(z_0)}^2 + \frac{1}{2\nu_3} \|f\|_{2,Q_R^+(z_0)}^2 + \right. \\
+ R(K2n/(n+1))^2 \left( \frac{1}{2\nu_4a_\#} + \frac{2(n-1)/n}{\nu_5} \right) \|g\chi_N + h\chi_R\|_{2,\Sigma_R(z_0)}^2, 
\]

where instead of (48) we use

\[
B = \frac{4}{a_\#(1-\nu_0-\nu_1-\nu_4)} \left( \frac{a_\#}{\nu_0a_\#} + \frac{\nu_2 + \nu_3}{2} + \frac{\nu_5}{2} \right) (2S2n/(n+2))^2. 
\]

Employing Lemma 4.1 into the above inequality, we apply Corollary 3.2 with \( \Phi = |\nabla u| \), \( t = 2n/(n + 2) \), \( p = 2 \), and

\[
\Psi = \left( \frac{1}{(1/(4\nu_1a_\#) + 1/\nu_2)}[f]^2 + \frac{|f|^2/(2\nu_3)}{a_\#(1-\nu_0-\nu_1-\nu_4)} \right)^{1/2} \in L^{2+\delta}(Q_R^+(z_0)); \\
\varphi = \left( \frac{1}{(2\nu_4a_\#) + 2^{1-1/n}/\nu_5} \right)^{1/2} K_{2\nu_4} (g\chi_N + h\chi_R) \in L^{2+\delta(\Sigma_R(z_0))}. 
\]

Upon choosing \( \nu_1 = 1/4, \nu_0 = \nu_4 = 1/8, \nu_2 = \nu_3 = 1, \) and \( Q_0 = O_{m-1} \circ \phi_{m-1}^{-1} (Q_{R_0}(z_0)) \), the application of the passage to the initial coordinates system finishes the proof of Proposition 4.2. \( \square \)

**Proposition 4.3.** Under the conditions of Proposition 4.2 such that the compatibility condition (23) is verified, provided that \( \Gamma_N = \partial \Omega \), any function \( u \in V_2 \) solving (24) verifies (51) with \( H \) being replaced by \( G \) which is defined by (23).

**Proof.** Let \( \eta \in W^{1,\infty}_0(\mathbb{R}^n) \) satisfy \( 0 \leq \eta \leq 1 \) in \( \mathbb{R}^n \), and \( d = -\int_\Omega u\eta^2dx \). Taking \( v = u\eta^2 + d \in V_2 \) as a test function in (17), applying the H"older inequality, and making use of (23) and (13)-(14), we obtain

\[
a_\#|\eta \nabla u|_{2,\Omega}^2 \leq 2a_\# |\eta \nabla u|_{2,\Omega}^2 |u \nabla \eta|_{2,\Omega} + |\eta f|_{2,\Omega} |u \eta|_{2,\Omega} + \\
+ |\eta f|_{2,\Omega} (|\eta \nabla u|_{2,\Omega} + 2 |u \nabla \eta|_{2,\Omega}) + |\eta g|_{2,\Gamma_N} |u \eta|_{2,\Gamma_N}. 
\]

The argument of the proof of Proposition 4.2 may be reproduced, finishing the proof of Proposition 4.3. \( \square \)

5. Existence results

5.1. Preliminary results. Making recourse of the Poincaré inequality [7, Corollary 3]:

\[
\|v\|_{p,\Omega} \leq P_p \left( \sum_{i=1}^{n} \|\partial_i v\|_{p,\Omega} + |\Gamma|^{1/p-1} \left| \int_{\Gamma} vds \right| \right),
\]

where \( P_p \) is a constant depending only on \( p \) and \( n \).
we introduce $S_{q,\ell} = S_q \max \{1 + P_q 2^{(n-1)1/q}, P_q |\Gamma|^{1/q - 1/\ell} \}$ and $K_{q,\ell} = K_q \max \{1 + P_q 2^{(n-1)(1/q)}, P_q |\Gamma|^{1/q - 1/\ell} \}$ that verify
\[
\|v\|_{nq/(n-q),\Omega} \leq S_{q,\ell} \|v\|_{1,q,\ell};
\|v\|_{(n-1)q/(n-q),\partial \Omega} \leq K_{q,\ell} \|v\|_{1,q,\ell}.
\]

Let us recall two standard existence results that we apply later.

**Proposition 5.1** (meas(Γ) > 0). Let $f \in L^2(\Omega)$, $f \in L^2(\Omega)$, $g \in L^s(\Gamma_N)$, and $h \in L^{\ell'}(\Gamma)$, with $s, \ell \geq 2$. Under the assumptions (A)-(B), there exists $u \in V_{2,\ell}$ being a weak solution to (10)-(12), i.e. solving (17) for all $v \in V_{2,\ell}$. Moreover, the following estimate holds
\[
\frac{a_\#}{2} \|\nabla u\|_{2,\Omega}^2 + \frac{b_\#}{\ell} \|u\|_{\ell,\Gamma} \leq \frac{\ell - 1}{\ell b_\#} \left( \|h\|_{\ell,\Gamma} + \mathcal{E}(1,1) \right)^{\ell/(\ell - 1)} + \frac{1}{2a_\#} \left( \|f\|_{2,\Omega} + \mathcal{E}(|\Omega|^{1/n}, |\Omega|^{1/2+(1/n-1)/s}) \right)^2,
\]
where $\mathcal{E}(A, B) = AS_{2n/(n+2),\ell} \|f\|_{2,\Omega} + BK_{ns/[n(s-1)+1],\ell} \|g\|_{s,\Gamma_N}$.

**Proof.** Since the existence and uniqueness are classical, we only pay attention on the derivation of (57). Taking $v = u \in V_{2,\ell}$ as a test function in (17), and making use of (13) and (15), we obtain
\[
a_\# \|\nabla u\|_{2,\Omega}^2 + b_\# \|u\|_{\ell,\Gamma} \leq \|f\|_{2,\Omega} \|\nabla u\|_{2,\Omega} + \|h\|_{\ell,\Gamma} \|u\|_{\ell,\Gamma} + \|f\|_{2,\Omega} S_{2n/(n+2),\ell} \left( |\Omega|^{1/n} \|\nabla u\|_{2,\Omega} + \|u\|_{\ell,\Gamma} \right) + \|g\|_{s,\Gamma_N} K_{ns/[n(s-1)+1],\ell} \left( |\Omega|^{1/2+(1/n-1)/s} \|\nabla u\|_{2,\Omega} + \|u\|_{\ell,\Gamma} \right),
\]
applying the Hölder inequality, the Sobolev embedding $W^{1,2n/(n+2)}(\Omega) \hookrightarrow L^2(\Omega)$, the trace embedding $W^{1,ns/[n(s-1)+1]}(\Omega) \hookrightarrow L^{\ell'}(\partial \Omega)$. This completes the proof of Proposition 5.1. \(\square\)

**Remark 5.1.** From the definition of the Gamma function, $\Gamma(n/2 + 1) = (n/2)!$ if $n$ is even, and $\Gamma(n/2 + 1) = \pi^{1/2} 2^{-(n+1)/2} n(n-2)(n-4) \cdots 1$ if $n$ is odd, the two-dimensional constant $S_1$ is simply $\pi^{-1/4} 2^{-3/2}$.

**Proposition 5.2** (meas(Γ) = 0). Let $f \in L^2(\Omega)$, $f \in L^2(\Omega)$, $g \in L^s(\Gamma_N)$, with $s \geq 2$, such that the compatibility condition (23) is verified. Under the assumption (A), there exists $u \in H^1(\Omega)$ being the unique function such that $\int_{\Omega} u \, dx = 0$, solving (24) for all $v \in V_2$. Moreover, the following estimate holds
\[
\|\nabla u\|_{2,\Omega} \leq \frac{1}{a_\#} \left( \|f\|_{2,\Omega} + |\Omega|^{1/n} S_{2n/(n+2)} \|f\|_{2,\Omega} + |\Omega|^{1/2+(1/n-1)/s} K_{ns/[n(s-1)+1]} \|g\|_{s,\Gamma_N} \right).
\]
5.2. **Proof of Theorem 2.1.** Supposing that the conditions of Proposition 5.1 are fulfilled, there exists \( u \in V_{2,\ell} \) being a weak solution to (10)-(12), i.e. solving (17) for all \( v \in V_{2,\ell} \).

On the one hand, Proposition 4.1 ensures that for each point \( x \in \Omega \) it is associated a sequence of cubic intervals \( Q_{r(x)/2}(x) \), with side lengths \( r(x) > 0 \) tending to zero, such that (46) is verified. On the other hand, Proposition 4.2 ensures that for each point \( x \in \partial \Omega \) it is associated a sequence of cubic intervals \( Q_{r(x)/2}(x) \), with side lengths \( r(x) > 0 \) tending to zero, such that (51) is verified.

Let us denote the collection of the above balls by \( B \), i.e.
\[
B = \{Q_{r_k/2}(x)\}_{x \in \Omega, \ k \geq 1}.
\]
All radii of balls in \( B \) are totally bounded by \((8S_{2n/(n+2)})^{-1}\). According to the Besicovitch covering theorem [20, Theorem 1.2], there exists a sequence of cubic intervals \( \{Q_{m_r/2}(x^{(m)})\}_{m \geq 1} \) in \( B \) such that: \( \overline{\Omega} \subset \cup_{m \geq 1} Q_{m_r/2}(x^{(m)}) \); and every point of \( \mathbb{R}^n \) belongs to at most \( 2^n + 1 \) balls in \( \{Q_{m_r/2}(x^{(m)})\}_{m \geq 1} \).

Since \( \Omega \) is bounded, its closure \( \overline{\Omega} \) is compact. Hence it can be covered with finitely many cubic intervals \( Q_{m_r/2}(x^{(m)}) \), \( m = 1, \ldots, M \). Let us define
\[
r_{\#} = \min \{r_m : m = 1, \ldots, M\}.
\]
Setting \( C = \{Q_{r_m}(x^{(m)})\}_{m = 1, \ldots, M} \), we build the collection \( C_1 \) as being the union of \( Q_{1}^{(1)} = Q_{r_1}(x^{(1)}) \) with all pairwise disjoint cubes \( Q_{m}^{(1)} = Q_{r_m}(x^{(m)}) \in C \) such that \( Q_{1}^{(1)} \cap Q_{m}^{(1)} = \emptyset \). This selection process may be recursively repeated, by building \( C_k \) as being the union of \( Q_{1}^{(k)} \in C \setminus C_{k-1} \) with all pairwise disjoint cubes \( Q_{m}^{(k)} \in C \setminus C_{k-1} \) such that \( Q_{1}^{(k)} \cap Q_{m}^{(k)} = \emptyset \). Consequently, there exists a number \( N \), depending on the dimension of the space, such that for each \( k \in \{1, \ldots, N\} \), we collect pairwise disjoint cubes corresponding to half ones from \( B \).

For each \( k \in \{1, \ldots, N\} \), we split the set of indices as \( \mathcal{I}(k) \cup \mathcal{J}(k) \), where \( \mathcal{I}(k) \) contains the indices with \( x^{(i)} \in \Omega \), while \( \mathcal{J}(k) \) contains the indices with \( x^{(j)} \in \partial \Omega \). Hence, combining (46) and (51) with
\[
\|\nabla u\|^p_{p,\Omega} \leq \sum_{k=1}^N \left( \sum_{i \in \mathcal{I}(k)} \|\nabla u\|^p_{p,Q_{r/m}(x^{(i)})} + \sum_{j \in \mathcal{J}(k)} \|\nabla u\|^p_{p,Q_{r/m}(x^{(j)})} \right),
\]
we find (18).

5.3. **Proof of Theorem 2.2.** Applying Propositions 4.3 and 5.2 instead of Propositions 4.2 and 5.1 respectively, this proof is *mutatis mutandis* the proof of Theorem 2.1.

6. **Proof of Theorem 1.1**

We consider the operator \( T : V_{p,\ell} \rightarrow V_{p,\ell} \), for any \( p \in [2, 2 + 1/(\nu - 1)] \) and \( \ell \geq 2 \), defined by
\[
\theta \mapsto \phi = \phi(\theta) \mapsto \Theta,
\]
where \( \phi \in W^{1,p}(\Omega) \) is the unique solution, verifying \( \int_{\Omega} \phi \, dx = 0 \), to the auxiliary electric problem

\[
\int_{\Omega} \sigma(\theta) \nabla \phi \cdot \nabla w \, dx = - \int_{\Omega} \alpha_s(\theta) \sigma(\theta) \nabla \theta \cdot \nabla w \, dx + \int_{\Gamma_N} g w \, ds, \quad \forall w \in V_{p'}.
\]

and \( \Theta \in V_{p,\ell} \) is the unique solution to the auxiliary thermal problem

\[
\int_{\Omega} k(\theta) \nabla \Theta \cdot \nabla v \, dx + \int_{\Gamma} f_\lambda(\theta) |\Theta|^{\ell-2} \Theta v \, ds = \int_{\Gamma} \gamma(\theta) \theta^{\ell-1} v \, ds - \int_{\Omega} \sigma(\theta) (\alpha_s(\theta)(\theta + \phi) \nabla \theta + \phi \nabla \phi) \cdot \nabla v \, dx, \quad \forall v \in V_{p',\ell}.
\]

The existence and uniqueness of \( \phi \) and \( \Theta \) are ensured by

- Theorem 2.2 under \( A = \sigma(\theta), f = \alpha_s(\theta) \sigma(\theta) \nabla \theta \in L^p(\Omega) \), and \( g \in L^p(\Gamma_N) \).
- Theorem 2.1 under \( A = k(\theta), f = \alpha_s(\theta)(\theta + \phi) \sigma(\theta) \nabla \theta + \phi \sigma(\theta) \nabla \phi \in L^p(\Omega) \), and \( h = \gamma(\theta) \theta^{\ell-1} \in L^p(\Gamma) \). The uniqueness of \( \Theta \) is true under the strict monotone property according to Remark 2.1.

Next, let us prove that \( \mathcal{T} \) maps the closed ball \( \overline{B}_R(0) \) into itself. Let \( \theta \in \overline{B}_R(0) \), i.e. \( \theta \in V_{p,\ell} \) satisfies \( \|\nabla \theta\|_{p,\Omega} + \|\theta\|_{p,\ell} \leq R \). Consequently, we have

\[
\|\theta\|_{p,\Omega} \leq C_{\infty} R.
\]

Using the estimates (58) and (18) in accordance with Theorem 2.2 and taking

\[
R \geq \|g\|_{p,\Gamma_N},
\]

we deduce

\[
\|\nabla \phi\|_{2,\Omega} \leq \frac{1}{\sigma_{\#}} \left( \sigma_{\#}^2 \alpha_{\#} \|\nabla \theta\|_{2,\Omega} + |\Omega|^{1/(2p')} K_{2p/(2p-1)} \|g\|_{p,\Gamma_N} \right);
\]

\[
\|\nabla \phi\|_{p,\Omega} \leq \mathcal{M}_1 \|g\|_{p,\Gamma_N} + \mathcal{M}_2 \|\nabla \theta\|_{p,\Omega} \leq (\mathcal{M}_1 + \mathcal{M}_2) R,
\]

with

\[
\mathcal{M}_1 = \frac{2^{3/2} N |\Omega|^{1/(2p')} K_{2p/(2p-1)} (r_{\#})^{2/p-1} + [1 + v(p-1)]^{1/p} \sqrt{1 + \sigma_{\#}}}{\sigma_{\#} \left( p - 1 - v(p-2) \right)^{1/p}};
\]

\[
\mathcal{M}_2 = \frac{2^{3/2} \alpha_{\#} N |\Omega|^{1 - 1/p} (r_{\#})^{2/p-1} + [2^{3(p-2)/p} + v(p-1)]^{1/p} \sqrt{1 + \sigma_{\#}}}{\sigma_{\#} \left( p - 1 - v(p-2) \right)^{1/p}},
\]

considering that \( K_{4/3} = 1/\pi \), and \( 2 < p < 3 \).

Using the estimates (57) and (18), we deduce

\[
k_{\#} \frac{1}{2} \|\nabla \Theta\|_{2,\Omega}^2 + b_{\#} \ell \|\Theta\|_{\ell,\Gamma} \leq \frac{\gamma^{\ell'}}{\ell' (b_{\#})^{1/(\ell - 1)}} \|\theta_e\|_{\ell,\Gamma} + \left( \sigma_{\#} \right)^{1-2/p} \left( \alpha_{\#} (\|\theta\|_{\infty,\Omega} + \|\phi\|_{\infty,\Omega}) \|\nabla \theta\|_{p,\Omega} + \|\phi\|_{\infty,\Omega} \|\nabla \phi\|_{p,\Omega} \right)^2;
\]

\[
\|\nabla \Theta\|_{p,\Omega} \leq \mathcal{M}_3 \|\theta_e\|_{\ell,\Gamma} + \mathcal{M}_4 \|\theta_e\|_{(\ell-1)\Gamma} + \mathcal{M}_5 \sigma_{\#} \left( \alpha_{\#} (\|\theta\|_{\infty,\Omega} + \|\phi\|_{\infty,\Omega}) \|\nabla \theta\|_{p,\Omega} + \|\phi\|_{\infty,\Omega} \|\nabla \phi\|_{p,\Omega} \right),
\]

where \( \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5 \) are constants.
with
\[ M_3 = \frac{2N}{\sqrt{k_#}} \frac{2^{2-3/p}(r_#)^{2/p-1}(\gamma^#)^{\ell'/2}(b^#)^{-1/[2(\ell-1)]}}{(p - 1 - v(p - 2))^{1/p}}; \]
\[ M_4 = \frac{2N}{\sqrt{k_#}} \frac{[1 + v(p - 1)]^{1/p} \sqrt{2/k_# + 1}}{(p - 1 - v(p - 2))^{1/p}}; \]
\[ M_5 = \frac{2^{3/2}N}{k_#} \frac{(r_#)^{2/p-1}(|\Omega|^{\frac{4}{p}} + (2(p-2)/2 + v(p - 1))^{1/p} \sqrt{1 + k_#})}{(p - 1 - v(p - 2))^{1/p}}. \]

Hence, we conclude
\[
\| \nabla \Theta \|_{p, \Omega} + \| \Theta \|_{\ell, \Gamma} \leq M_3 \| \theta_e \|_{\ell/2}^{\ell/2} + M_4 \| \theta_e \|_{(\ell-1)p, \Gamma}^{\ell-1} + \left( \frac{\gamma^#}{b^#} \right)^{1/(\ell-1)} \| \theta_e \|_{\ell, \Gamma} + \left( \frac{\ell' |\Omega|^{\frac{1}{p�}}}{2b^#k_#} \right)^{1/\ell} (\sigma^# C_\infty)^{2/\ell} \left( (\alpha^#(1 + M_1 + M_2) + (M_1 + M_2)^2)^{2/\ell} R^{4/\ell} + M_5 C_\infty \sigma^# \right) \left( (\alpha^#(1 + M_1 + M_2) + (M_1 + M_2)^2) R^2 \right).
\]

Since \( \ell \geq 2 \), we consider the quadratic function
\[ Q(R) = C_\infty \sigma^# \left[ \left( \frac{(|\Omega|^{\frac{1}{p�}})}{b^#k_#} \right)^{1/\ell} + M_5 \right] \left( (\alpha^#(1 + M_1 + M_2) + (M_1 + M_2)^2) R^2 - R + c. \right)
\]

Considering that (9) assures that \( Q \) has two positive real roots, \( R_1 \) and \( R_2 \), for all \( R_1 < R < R_2, Q(R) < 0 \).

Our aim is to find a fixed point \( \Theta = T(\Theta) \), by applying the following Tychonoff extension to weak topologies of the Schauder fixed point theorem [15, pp. 453-456 and 470]:

**Theorem 6.1.** Let \( K \) be a nonempty compact convex subset of a locally convex space \( X \). Let \( T : K \to K \) be a continuous operator. Then \( T \) has at least one fixed point.

Set the reflexive Banach space \( X = V_{p, \ell} \) (\( p > n = 2 \)) endowed with the weak topology, and \( K = \overline{B}_R(0) \). It remains to prove that the operator \( T \) is continuous (for the weak topologies). Let \( \{\theta_m\}_{m \in \mathbb{N}} \) be a sequence in \( K \) such that \( \theta_m \to \theta \) in \( W^{1,p}(\Omega) \), and \( \Theta_m \) and \( \phi_m \) be the corresponding solutions of (59) and (60), respectively. Observe that \( \theta \in K \) because \( K \) is convex and closed, hence it is weakly closed in \( W^{1,p}(\Omega) \). From (61) and \( \Theta_m, \phi_m \in K \), there exists \( (\Theta, \phi) \in K \times V_p \) being a weak cluster point of \( \{(\Theta_m, \phi_m)\} \). Let then \( \{(\Theta_m, \phi_m)\} \) be a non-relabeled subsequence such that \( (\Theta_m, \phi_m) \rightharpoonup (\Theta, \phi) \) in \( [W^{1,p}(\Omega)]^2 \).

By appealing to the compactness embedding \( W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \) for \( p > n = 2 \), we have that \( \theta_m \to \theta \) in \( L^\infty(\Omega) \), and also pointwisely in \( \Omega \). Applying the Krasnoselski Theorem to the Nemytskii operators \( \sigma \) and \( \alpha_s \), we get
\[ \sigma(\theta_m) \nabla w \to \sigma(\theta) \nabla w, \quad (\alpha_s \sigma)(\theta_m) \nabla w \to (\alpha_s \sigma)(\theta) \nabla w \quad \text{in} \quad L^{p'}(\Omega), \]
making use of the Lebesgue’s dominated convergence theorem, with (4) and (5).
Then, the variational equality (59) for the solutions $\phi_m$ passes to the limit as $m$ tends to infinity, concluding (59).

Similar strong convergence holds for the leading coefficient $k$. The compactness embedding also implies that $\phi_m \rightarrow \phi$ in $L^\infty(\Omega)$, and $\theta_m \rightarrow \theta$ and $\Theta_m \rightarrow \Theta$ in $L^\infty(\partial\Omega)$. In particular, $\theta_m$ pointwisely converges to $\theta$ on $\Gamma$. Thus, we can proceed as above, with the aid of (6)-(7), by applying the Lebesgue dominated convergence theorem, obtaining $f_\lambda(\theta_m)|\Theta_m|^{l-2}\Theta_m v \rightarrow f(\theta)|\Theta|^{l-2}\Theta v$ and $|\gamma(\theta_m)v| \rightarrow |\gamma(\theta)v|$ in $L^1(\Gamma)$ and in $L^{p'}(\Gamma)$, respectively.

Then, the variational equality (60) for the solutions $\Theta_m$ passes to the limit as $m$ tends to infinity, concluding (60).

Therefore, Theorem 6.1 ensures the existence of at least one fixed point $\theta = T(\theta)$ concluding the proof of Theorem 1.1.

Appendix

Let us prove that $\sigma_s \geq 0$, with $\sigma_s$ representing the entropy production which verifies

$$\rho \partial_t s = -\nabla \cdot J_s + \sigma_s,$$

where $\rho$ denotes the density, $s$ denotes the specific entropy, and $J_s$ denotes the entropy flux.

In the absence of external forces, the conservation laws of energy and electric charge are, respectively,

$$\rho \partial_t e = -\nabla \cdot J; \quad (67)$$

$$\rho \partial_t q = -\nabla \cdot j. \quad (68)$$

Here $e$ denotes the specific internal energy, and $q$ is the specific electric charge. We multiply (67) by $1/\theta$ and (68) by $\phi/\theta$, with $\theta$ denoting the absolute temperature, and $\phi$ representing the electric potential. Gathering the obtained equations with the local form of the Gibbs equation [23]:

$$de = \theta ds + \phi dq,$$

we deduce

$$\begin{cases}
J_s = (J - \phi j)/\theta \quad (= q/\theta) \\
\sigma_s = J \cdot \nabla (1/\theta) - j \cdot \nabla (\phi/\theta)
\end{cases}$$

Substituting (2)-(3) into the above expression of $\sigma_s$ we find

$$\sigma_s = \frac{(\nabla \theta)^T k \nabla \theta}{\theta^2} + \frac{(\alpha_s \nabla \theta + \nabla \phi)^T \sigma (\alpha_s \nabla \theta + \nabla \phi)}{\theta},$$

if provided by a symmetric matrix $\sigma$, and $k = k_T + \Pi \alpha_s \sigma$. Therefore, we conclude $\sigma_s \geq 0$, under positive semidefinite matrices $k$ and $\sigma$.

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