EDGE STATES IN 4d AND THEIR 3d GROUPS AND FIELDS

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Abstract

It is known that the Lagrangian for the edge states of a Chern-Simons theory describes a coadjoint orbit of a Kac-Moody (KM) group with its associated Kirillov symplectic form and group representation. It can also be obtained from a chiral sector of a nonchiral field theory. We study the edge states of the abelian \(BF\) system in four dimensions (4d) and show the following results in almost exact analogy: 1) The Lagrangian for these states is associated with a certain 2d generalization of the KM group. It describes a coadjoint orbit of this group as a Kirillov symplectic manifold and also the corresponding group representation. 2) It can be obtained from with a “self-dual” or “anti-self-dual” sector of a Lagrangian describing a massless scalar and a Maxwell field [the phrase “self-dual” here being used essentially in its sense in monopole theory]. There are similar results for the nonabelian \(BF\) system as well. These shared features of edge states in 3d and 4d suggest that the edge Lagrangians for \(BF\) systems are certain natural generalizations of field theory Lagrangians related to KM groups.
In a previous paper [1], we studied the topological action

\[ S_{BF} = \int_{M^3 \times \mathbb{R}^1} B \wedge F, \]

where the spatial manifold \( M^3 \) is a manifold with boundary \( \partial M^3 \), such as a solid ball \( B^3 \) or a solid torus \( T^3 \). It was established that there are states localised on \( \partial M^3 \) which carry a representation of a certain Lie algebra. If the superscript \( (j) \) indicates \( j \) forms, this algebra can be described in terms of the following commutators in its representation \( \rho \):

\[ [\rho(\lambda^{(0)}), \rho(\mu^{(0)})] = [\rho(\lambda^{(1)}), \rho(\mu^{(1)})] = 0, \]

\[ [\rho(\lambda^{(0)}), \rho(\lambda^{(1)})] = i \int_{\partial M^3} \lambda^{(0)} d\lambda^{(1)}. \]  

(2)

[In ref. [1], \( \rho(\lambda^{(j)}) \) were written as integrals \( \int_{M^3} d\lambda^{(0)} B, \int_{M^3} d\lambda^{(1)} A \) involving \( B \) or \( A \) when \( d\lambda^{(j)} \) were nonzero on \( \partial M^3 \), \( \lambda^{(j)} \) here being extensions of \( \lambda^{(j)} \) in (1) to all of \( M^3 \).] The subalgebra with generators \( \rho(\lambda^{(0)}) \) is the algebra of the group of maps \( \partial M^3 \to U(1) \) [or \( \mathbb{R}^1 \)] while \( \rho(\lambda^{(1)}) \) are the generators of another abelian subalgebra. (2) describes an extension of the direct sum of these subalgebras by the abelian Lie algebra of reals.

We now recall that the abelian Chern-Simons (CS) action on \((\text{disk } M^2) \times \mathbb{R}^1\) also produces an algebra on the bounding circle \( \partial M^2 \) of \( M^2 \) [2]. [See also ref. [3] and [4] and references therein.] It is the \( U(1) \) KM algebra spanned by functions \( \Lambda \) on \( \partial M^2 \) (and a central charge \( k \)). If \( \varphi : \Lambda \to \varphi(\Lambda) \) describes an irreducible representation of this algebra, it is characterized by the commutators

\[ [\varphi(\Lambda), \varphi(\hat{\Lambda})] = i \frac{k}{2\pi} \int_{\partial M^2} \Lambda d\hat{\Lambda}, \]  

(3)

\( k \) having a constant value in the representation. In this note, we show that there are several features common to (2) and (3).
The properties of the algebra defined by (2) which are of interest here are the following:

1) It can be produced by canonically quantising the action

\[ \frac{k}{4\pi} \int dt \int_{\partial M^2} \partial_t \chi d\chi \]  

where \( d \) does not involve differentiation in \( t \). [Cf. references [3], [5] and citations therein.]

2) An element of the algebra described by (3) is a pair \((\Lambda, \xi)\), \( \xi \) being a real number. An element of its dual can be written as \((\sigma^{(1)}, \eta)^*, \sigma^{(1)} \) being a one form and \( \eta \) a real number, on introducing the pairing

\[ \langle (\sigma^{(1)}, \eta)^*, (\Lambda, \xi) \rangle = \int_{\partial M^2}^{(1)} \sigma + \eta \xi . \]  

Now there is a natural action of the KM group \( K = \{ h \} \) on \((\sigma^{(1)}, \eta)^*\) called the coadjoint action. If \((\Lambda, \xi) \rightarrow h(\Lambda, \xi)h^{-1}\) is the adjoint action, the coadjoint action \( Ad^* h \) of \( h \) is defined by

\[ \langle Ad^* h(\sigma^{(1)}, \eta)^*, h(\Lambda, \xi)h^{-1} \rangle = \langle (\sigma^{(1)}, \eta)^*, (\Lambda, \xi) \rangle . \]  

From a general result of Kirillov [7], it is known that an orbit of \( K \) for this coadjoint action (a “coadjoint” orbit) carries a \( K \)-invariant symplectic form \( \omega^{(2)} \). For the orbit through \((0, 1)^*\), a simple calculation [3][5] also shows that we can write \( \omega^{(2)} = d\omega^{(1)} \). We can thus contemplate forming the action

\[ \int \omega^{(1)} \]  

which is like the action \( pdq \) in particle mechanics. [See ref. [8] and references therein.]

(7) can be brought to the form (4) up to surface terms [5].

3) The current algebra defined by (3) can be obtained from the scalar field Lagrangian

\[ \frac{|k|}{8\pi} \int_{\partial M^2} d\theta [ (\partial_t \phi)^2 - (\partial_\theta \phi)^2 ] \]  

by imposing either of the constraints

\[ \partial_\pm \phi = 0, \quad \partial_\pm = \partial_t \pm \partial_\theta \]  

\[ \partial_\pm \phi = 0, \quad \partial_\pm = \partial_t \pm \partial_\theta \]
depending on the sign of $k$. [Here $\theta(\text{mod } 2\pi)$ is the coordinate on the circle while the speed for the field $\phi$ has been set equal to 1.] Any solution of the field equation for (8) is in fact the sum $\phi_+ + \phi_-$ where $\partial_- \phi_+ = \partial_+ \phi_- = 0$.

4) The Hamiltonian for (4) is zero whereas that is not the case for (8). The latter in fact evolves $\phi$ preserving the condition in (9). This evolution of left- and right-movers are also given by the nonlocal Lagrangians

$$
\frac{\pi}{k} \int_0^{2\pi} d\theta d\theta' \epsilon(\theta - \theta') \partial_\theta \epsilon(\theta' - \theta) = 1 \text{ if } \theta > \theta'.
$$

In this note, we will show that each of these features can be generalised to the algebra defined by (2). There are similar generalisations for the nonabelian problem as well as we shall later indicate. All this suggests that the 3d systems coming from 4d topological actions are certain natural generalisations of 2d systems associated with KM groups.

For simplicity of presentation when discussing the generalizations of 1) to 4) under items 1) to 4) below, it is convenient to assume that $M^3$ is the solid torus $T^3$. For $\partial M^3 = T^2$ (the two torus), we also choose the flat metric $(d\theta^1)^2 + (d\theta^2)^2$ [$\theta^i \text{ mod } 2\pi$ being the standard coordinates on $T^2$].

**Item 1**

For the current algebra (2), the Lagrangian is

$$
\int_{\partial M^3} (\partial_\theta \phi dA + d\phi \partial_\theta A),
$$

where

$$
A = A_j d\theta^j, \quad d\phi = \partial_j \phi d\theta^j, \quad dA = \partial_k A_j d\theta^k \wedge d\theta^j = \partial_k A_j d\theta^k d\theta^j.
$$

[Wedge symbols between differential forms will hereafter be omitted.] Note that as before
\( d \) does not differentiate time \( t \). Also we can equally well consider the negative of the Lagrangian (\( \Pi \)).

This result can be shown as follows. If \( \Pi \) and \( P^j \) are the momenta conjugate to \( \phi \) and \( A_j \), (\( \Pi \)) leads to the constraints

\[
\Pi - *dA := \Pi - \epsilon^j_\ell \partial_\ell A_j \approx 0 , \quad P_i + (*d\phi)_i := P_i + \epsilon_{ij} \partial^j \phi \approx 0 ,
\]

(13)

where \( \epsilon_{ij} = -\epsilon_{ji} \), \( \epsilon_{12} = 1 \), the spatial metric is \((1, 1)_{\text{diagonal}}\) and \( \approx \) denotes weak equality. The first class variables or observables with zero Poisson brackets (PB’s) with these constraints are functions of

\[
\Pi + *dA , \quad P_i - (*d\phi)_i .
\]

(14)

We can now set

\[
\rho(\lambda^{(0)}) = \int_{T^2} \lambda^{(0)}(*\Pi + dA) , \quad \rho(\lambda^{(1)}) = -\frac{1}{2} \int_{T^2} \lambda^{(1)}(*P + d\phi) ,
\]

\[
* \Pi := \Pi d^2 \theta , \quad *P := \epsilon_{ij} d\theta^i P^j ,
\]

(15)

as they have the commutators (\( c \)) in quantum theory.

**Item 2**

We first outline the general method to construct the Kirillov symplectic form and its associated one form. Let \( G = \{ g \} \) be a Lie group with Lie algebra \( \mathfrak{g} = \{ \alpha \} \). If \( \mathfrak{g}^* = \{ \beta^* \} \) is the dual of \( \mathfrak{g} \), we denote the pairing of \( \beta^* \) and \( \alpha \) by \( < \beta^*, \alpha > \). If \( \alpha \rightarrow g \alpha g^{-1} \) is the adjoint action of \( g \), its coadjoint action \( Ad^* g \) is defined analogously to (\( c \)) by requiring

\[
< Ad^* g \beta^* , g \alpha g^{-1} > = < \beta^* , \alpha > .
\]

(16)

With this action, \( G \) defines orbits (“coadjoint orbits”) in \( \mathfrak{g}^* \). The coadjoint orbit through \( K^* \) with stability group \( H \) can be identified with the coset space \( G/H \) in a well-known way.
Now consider the one form

\[ \Omega^{(1)} = \langle K^*, g^{-1}dg \rangle \]

on \( G \). Then as frequently explained elsewhere \[8\], the associated two form \( d\Omega^{(1)} \) projects down to a two form \( \Omega^{(2)} \) on \( G/H \) and that form is the Kirillov symplectic form on \( G/H \).

Furthermore, according to our previous work \[8\], and analogously to (7), the Lagrangian leading to this form is

\[ \langle K^*, g^{-1}\partial_t g \rangle. \]  

(18)

We now show that (18) is exactly (11) for a suitable choice of \( K^* \).

A general element of the Lie algebra \( \mathfrak{g} \) for (2) can be written as \( (\lambda, \lambda, \xi) \) \[ \xi \in \mathbb{R}^1 \]. The Lie bracket is given by

\[ \left[ (\lambda, \lambda, \xi), (\mu, \mu, \nu) \right] = (0, 0, i \int \lambda \, d\mu - i \int \mu \, d\lambda). \]  

(19)

A general element of the group \( G \) with Lie algebra \( \mathfrak{g} \) is \( g(\lambda, \lambda, \xi) = \exp(i\lambda^0)\exp(i\lambda^1)\exp(i\xi) \), \( \exp \) being the usual exponential map, and the indicated order of factors in writing \( g(\lambda, \lambda, \xi) \) will hereafter be adopted as a convention. The adjoint group action can be worked out using (19):

\[ g(\lambda, \lambda, \xi)(\mu, \mu, \nu)g(\lambda, \lambda, \xi)^{-1} = (\mu, \mu, \nu + \int d\lambda^0 \mu^1 + \int d\lambda^1 \mu^0). \]  

(20)

Let \( \mathfrak{g}^* \) be the dual of \( \mathfrak{g} \). Its elements can be written as \( (\sigma^2, \sigma^1, \eta)^*\), \( \sigma^j \) being \( j \) forms and \( \eta \in \mathbb{R}^1 \). The pairing between elements of \( \mathfrak{g}^* \) and \( \mathfrak{g} \) here is

\[ \langle (\sigma^2, \sigma^1, \eta)^*, (\mu, \mu, \nu) \rangle = \int_{\partial M^3} (\sigma^2 \mu^0 + \sigma^1 \mu^1 + \eta \nu). \]  

(21)

The coadjoint action now follows from (16):

\[ \text{Ad}^* \, g(\lambda, \lambda, \xi)(\sigma^2, \sigma^1, \eta)^* = (\sigma^2 - \eta d\lambda^1, \sigma^1 - \eta d\lambda^0, \eta). \]  

(22)
With the choice $K^* = (0, 0, 2)$, the Lagrangian $[\Pi]$ readily follows from (22) and (18) [with $\phi = \lambda^0$, $A = \lambda^1$] on noting that

$$g(\lambda^0, \lambda^1, \xi)^{-1}\partial_t g(\lambda^0, \lambda^1, \xi) = \imath(\partial_t \lambda^0, \partial_t \lambda^1, \partial_t \xi - \int \partial_t \lambda^0 d\lambda^1).$$

(23)

and discarding certain total derivatives.

The formulae for $\rho(\lambda^j)$ in (15) involve pairings like in (21), with $*\Pi+dA$ corresponding to $\sigma^{(2)}$ and $*P+d\phi$ corresponding to $\sigma^{(1)}$. Thus $\rho$ can be identified with an element $[*\Pi+dA, *P+d\phi, c]^* := [\Sigma^{(2)}, \Sigma^{(1)}, c]^* (c \in \mathbb{R}^1)$ of the dual of $\mathcal{G}$ with values in a representation (and not real numbers as for $\mathcal{G}^*$), the pairing being

$$(\Sigma^{(2)}, \Sigma^{(1)}, c)^*, (\lambda^0, \lambda^1, \xi)) = \int_{\partial M^3} (\Sigma^{(2)} \lambda^0 + \frac{1}{2} \Sigma^{(1)} \lambda^1) + 1c\xi,$$

(24)

Comparison of (2) and (19) also shows that $c = 1$ for our $\rho$. It is interesting that the relations (2), which mean that $\rho$ is a representation, can be stated as a closure property of $\rho$ or of $[\Sigma^{(2)}, \Sigma^{(1)}, c]^*$ in a certain cohomology $[\Pi]$. Thus, the quantum fields are operator valued distributions on $\mathcal{G}$ with a certain closure property signifying that they lead to a representation of $\mathcal{G}$.

**Item 3**

Consider the Lagrangian

$$-\int_{\partial M} \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$$

(25)

where the metric is $(-1, 1, 1)$ diagonal. It leads to the equations of motion

$$\partial_\mu \partial^\mu \phi = 0, \quad \partial_\mu F^{\mu\nu} = 0.$$

(26)
The momenta $\Pi$ and $P^i$ conjugate to $\phi$ and $A_i$ for (25) are given by

$$\Pi = \partial_0 \phi, \quad P_i = F_{0i},$$

(27)

$P_i$ being subject to the first class constraint

$$\partial_i P^i \approx 0.$$  \hspace{1cm} (28)

Analogous to the chiral constraints (9), we now consider the so-called “self-dual” or “anti-self-dual” constraint

$$\partial_\mu \phi = \pm \frac{1}{2} \epsilon_{\mu\nu\lambda} F^{\nu\lambda}$$

(29)

similar to the Bogomol’nyi-Prasad-Sommerfield equations [11], where we adopt the convention $\epsilon_{012} = +1$ for the Levi-Cevita symbol.

It is readily seen that (29) implies (26). If we now rewrite (29) for the plus sign in terms of $\Pi$ and $P_i$, we get exactly (13). As for the minus sign in (29), it corresponds in a similar way to the negative of the Lagrangian (11), the equations replacing (13) for the latter being

$$\Pi + *dA \approx 0, \quad P_i - \epsilon_{ij} \partial_j \phi \approx 0.$$ \hspace{1cm} (30)

We next show that any solution of (26) is the sum of two pieces \((\phi^{(\pm)}, \Pi^{(\pm)}, *dA^{(\pm)}, P^{(\pm)} := d\theta^i P_i^{(\pm)})\), the fields with the plus (minus) sign fulfilling (13) for the plus (minus) sign. For this purpose, first note that the field equations (26) and the identifications (27) give the following equations:

$$\partial_0 [\Pi \mp *dA] = \mp \partial_i \epsilon_{ij} [P_j \pm \epsilon_{jk} \partial^k \phi],$$

$$\partial_0 [P_i \pm \epsilon_{ij} \partial^j \phi] = \pm \epsilon_{ij} \partial_j [\Pi \mp *dA].$$

(31)

They show that the field equations (26) preserve the constraints (13) and (30) during time evolution without generating new constraints. Hence, it is sufficient to show that the (gauge invariant) initial data \((\phi, \Pi, *dA, P := d\theta^i P_i)\) at a fixed time $t_0$ can be written
as the sum of two pieces $(\phi(\pm), \Pi(\pm), *dA(\pm), P(\pm))$, the fields with the plus (minus) signs fulfilling (13) and (30).

Now, in view of (28), we can write

\[ P_i = \epsilon_{ij} \partial^j f(0) \quad \text{or} \quad P = d\theta \epsilon_{ij} \partial^j f(0) := *d f(0) \quad \text{at} \ t = t_0 \] (32)

for some function $f^{(0)}$.

Let us next consider the initial data

\[ (\phi^{(+)}, \Pi^{(+)}, *dA^{(+)}) = \Pi^{(+)}, P^{(+)}, = -*d\phi^{(+)}) + (\phi^{(-)}, \Pi^{(-)}, *dA^{(-)} = -\Pi^{(-)}, P^{(-)} = -*d\phi^{(-)}) \] (33)

at time $t_0$. The first bracket of fields fulfills (13) and the second bracket (30). In order that (33) equals $(\phi, \Pi, *dA, P)$ at $t_0$, it is enough to choose the fields in (33) to satisfy

\[ \phi^{(+)} + \phi^{(-)} = \phi, \quad \Pi^{(+)} + \Pi^{(-)} = \Pi, \]
\[ \phi^{(+)} - \phi^{(-)} = -f^{(0)}, \quad \Pi^{(+)} - \Pi^{(-)} = *dA \] (34)

at time $t_0$.

We thus see that any solution of (26) is the sum of solutions satisfying (13) and (30).

The preceding analysis is “local” and does not address issues that arise from possible global observables such as Wilson loop integrals and the possible nonexactness of the closed form $*P$.

**Item 4**

The three dimensional analogue of the Lagrangians (10) is local and is the first order form

\[ \int d^3x \mathcal{L}, \]
\[ \mathcal{L} = \epsilon^{\mu\nu\lambda} B_\mu \partial_\nu A_\lambda - \frac{1}{2} B_\mu B^\mu \] (35)
of the Maxwell Lagrangian. This is because the equations of motion for (35) are

\[ B_\mu = \frac{1}{2} \epsilon_{\mu\nu\lambda} F^{\nu\lambda}, \tag{36} \]

\[ \epsilon^{\mu\nu\lambda} \partial_\nu B_\lambda = 0. \tag{37} \]

As under item 3) for \(*P\), we now assume that \( B_\mu = \pm \partial_\mu \phi \) for some \( \phi \) in view of (37). Then (36) becomes equivalent to (29).

It is interesting that (36) and (37) together can be written as a 2 + 1 dimensional variant of the Duffin-Kemmer-Petiau equation \[12\]. For this purpose, we define

\[ \Psi = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ B_0 \\ B_1 \\ B_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \]

\[ \beta_\mu = \begin{pmatrix} \tilde{\beta}_\mu \\ 0 \\ \tilde{\beta}_\mu \end{pmatrix}, \quad (\tilde{\beta}_\mu)_\nu^\lambda = \epsilon_{\mu\nu\lambda}; \]

\[ \beta_\mu^\dagger = \eta_{\mu\nu} \beta_\mu \text{ (no \( \mu \) sum)}, \tag{38} \]

where 1 is the 3×3 unit matrix. The \( \beta \) matrices satisfy the Duffin-Kemmer-Petiau algebra since the \( \tilde{\beta} \) do so:

\[ \tilde{\beta}_\mu \tilde{\beta}_\nu \tilde{\beta}_\lambda + \tilde{\beta}_\lambda \tilde{\beta}_\nu \tilde{\beta}_\mu = \eta_{\mu\nu} \tilde{\beta}_\lambda + \eta_{\lambda\nu} \tilde{\beta}_\mu. \tag{39} \]

With (38),(36) and (37) are

\[ (\beta_\mu \partial^\mu + \alpha) \Psi = 0. \tag{40} \]

The Lagrangian density in (33) can now also be written in the Dirac form

\[ \mathcal{L} = -\frac{1}{2} \overline{\Psi} (\beta_\mu \partial^\mu + \alpha) \Psi, \]

\[ \overline{\Psi} = \Psi^\dagger \gamma, \quad \Psi^\dagger = (A^0, A^1, A^2, B^0, B^1, B^2), \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{41} \]

after discarding a surface term. Here \( \Psi \) of course is real.
There are certain generalizations of these considerations to the nonabelian case. They will only be sketched here, as we plan a more thorough treatment of the nonabelian problem elsewhere. Let $G$ be a simple compact Lie group thought of concretely as a group of unitary matrices. Let $\mathcal{G}$ be its Lie algebra with basis $\{T(\alpha) | \alpha = 1, 2, \ldots\}$, dimension $[\mathcal{G}]$ of $G$ which fulfills $[T(\alpha), T(\beta)] = ic_{\alpha\beta}^\gamma T(\gamma)$, $TrT(\alpha)T(\beta) = N\delta_{\alpha\beta}$ ($N = \text{constant}$) and $T(\alpha)^\dagger = T(\alpha)$. In the nonabelian generalization of (2), $\lambda^{(j)}, \mu^{(j)}$ become $\mathcal{G}$ valued $j$ forms on $\partial M^3$ with $\lambda^{(j)} = i\lambda_{\alpha}^{(j)} T(\alpha)$, $\lambda_{\alpha}^{(j)}$ being real valued forms, while the commutators in (2) become

$$[\rho(\lambda^{(0)}), \rho(\mu^{(0)})] = \rho([\lambda^{(0)}, \mu^{(0)}]),$$
$$[\rho(\lambda^{(0)}), \rho(\lambda^{(1)})] = \rho([\lambda^{(0)}, \lambda^{(1)}]) + i \int_{\partial M^3} Tr\lambda^{(0)} d\lambda^{(1)},$$
$$[\rho(\lambda^{(1)}), \rho(\mu^{(1)})] = 0. \quad (42)$$

This gives the generalization of (19) as well. The generalization of (11) involves a field $u$ valued in $G$ and a one form $W$ valued in $\mathcal{G}$ and reads up to an overall constant,

$$iT r \int_{\partial M^3} \left[ u \partial_t u^{-1} (dW + u du^{-1} W + W u du^{-1}) + u du^{-1} \partial_\nu W \right] \quad (43)$$

while (13,15) must be replaced by their natural nonabelian versions. The method using coadjoint orbits also leads to \((43)\). The Lagrangians replacing \((25)\) and \((35)\) are more involved and will be reported elsewhere.

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