Hopf Lemma and regularity results for quasilinear anisotropic elliptic equations

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Abstract
We consider a class of quasi-linear anisotropic elliptic equations, possibly degenerate or singular, which are of interest in several applications such as computer vision and continuum mechanics. We prove a Hopf Lemma as well as local and global regularity estimates for positive solutions, generalizing previous results known in the context of p-Laplacian equations.

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1 Introduction

We consider quasilinear elliptic partial differential equations, in a possibly anisotropic medium. From the mathematical viewpoint, the anisotropy is responsible for a much richer geometric structure than the usual Euclidean geometry. However, the main interest rests in concrete applications, since anisotropic media naturally arise in several real world phenomena.

In fact, anisotropic energies are widely used in computer vision (see for instance [1,13, 20,28,29]) and in continuum mechanics, in particular in presence of materials with distinct behavior with respect to different directions, typically due to the crystalline microstructure of the medium (see for instance [3,4,6,7,15,25] and the references therein).

Our main results are a Hopf Lemma at the boundary, as well as local and global regularity estimates for positive solutions.

Let us remark that, in a forthcoming paper, as a direct application of the results discussed above, we shall develop a moving plane procedure in the general context of this anisotropic and possibly singular/degenerate elliptic equations, in order to prove monotonicity and symmetry results in this Finsler geometry setting for positive solutions, both on bounded and unbounded domains, such as the whole space $\mathbb{R}^n$ or on half spaces.

For $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let us consider the following Wulff type functional:

$$ I(u) = \int_\Omega [B(H(\nabla u)) - F(u)] \, dx, \quad (1) $$

whose weak form of the Euler–Lagrange equation is given by

$$ \int_\Omega B'(H(\nabla u))\langle \nabla H(\nabla u), \nabla \psi \rangle = \int_\Omega f(u)\psi, \quad \forall \psi \in C^1_c(\Omega), \quad (2) $$

(where $f = F'$) as well as its strong form

$$ -\text{div} \left( B'(H(\nabla u))\nabla H(\nabla u) \right) = f(u). \quad (3) $$

We assume the following set of hypotheses on $B$, $H$ and $f$:

(i) $B \in C^{3,\beta}_{loc}((0, +\infty)) \cap C^1([0, +\infty))$, with $\beta \in (0, 1)$
(ii) $B(0) = B'(0) = 0$, $B(t), B'(t), B''(t) > 0 \forall t \in (0, +\infty)$
(iii) there exist $p > 1, k \in [0, 1], \gamma > 0, \Gamma > 0$ such that:

$$ \gamma(k + t)^{p-2}t \leq B'(t) \leq \Gamma(k + t)^{p-2}t $$
$$ \gamma(k + t)^{p-2} \leq B''(t) \leq \Gamma(k + t)^{p-2} \quad (4) $$

for any $t > 0$
(iv) $H \in C^{3,\beta}_{loc}(\mathbb{R}^n \setminus \{0\})$ is even and such that $H(\xi) > 0 \forall \xi \in \mathbb{R}^n \setminus \{0\}$
(v) $H(\xi t) = tH(\xi) \forall \xi \in \mathbb{R}^n \setminus \{0\}, \forall t > 0$
(vi) $H$ is uniformly elliptic, that means the set $\mathcal{B}_1^H := \{\xi \in \mathbb{R}^n : H(\xi) < 1\}$ is uniformly convex, i.e.

$$ \exists \lambda > 0 : \langle D^2H(\xi)v, v \rangle \geq \lambda |v|^2 \forall \xi \in \partial \mathcal{B}_1^H, \forall v \in \nabla H(\xi) \perp. $$

(vii) $f$ is a positive continuous function on $[0, \infty)$ which is locally Lipschitz continuous on $(0, \infty)$
(viii) there exists $g \in C^0([0, +\infty))$ non-decreasing on $(0, \delta)$, $\delta > 0$, satisfying $g(0) = 0$, $f + g \geq 0$ and either $g = 0$ in $[0, d]$, $d > 0$, or
\[ \int_0^\delta \frac{1}{L^{-1}(G(s))} ds = +\infty, \]
where $G(s) = \int_0^s g(t) dt$ and $L(s) = sB'(s) - B(s)$.

The assumptions (iv)–(v)–(vi) ensure that $H$ is a Finsler norm. For $H(\xi) = |\xi|$ we get the usual Euclidean norm and, if we take $B(t) = t^p$, the operator at left-hand side of (3) becomes the usual $p$-Laplacian operator.

Let us observe that, under the above hypotheses, the natural space for the existence of solutions is $W^{1, p}(\Omega) \cap L^\infty(\Omega)$. However, better regularity holds in general. If $k > 0$ in hypothesis (iii), then the operator is uniformly elliptic and, by standard elliptic regularity, solutions are classical and (3) is satisfied. On the other hand, for $k = 0$, the operator becomes degenerate or singular, and solutions are not classical. In fact, in [9] the authors show that to Eq. (3) it is possible to apply results in [12, 26] to ensure that the solutions belong to $C^{1, \alpha}(\Omega \setminus Z)$ for some $0 < \alpha < 1$, where $Z$ denotes the critical set, i.e. the set whet $\nabla u$ vanishes. Moreover, if $\Omega$ is smooth, we can apply the regularity results in [16] to deduce that the solutions are in fact $C^1$ up to the boundary. Thus, in order to cover the general case, throughout the paper we shall consider solutions belonging to $C^1(\Omega)$, which implies that the equation shall always be intended in the weak sense (2). Anyhow, we will see that, as a consequence of our regularity results, the critical set $Z$ is negligible and the strong equation (3) shall be satisfied almost everywhere.

In [10, 11] were firstly introduced useful tools to get regularity and qualitative properties (comparison principles, Harnack inequality, monotonicity and symmetries, etc.) of equations involving the $p$-Laplace operator. Then these techniques were widely developed to study several type of more general equations and systems (for instance with lower order terms or singular data) both in bounded and unbounded domains (see for instance [14, 17–19, 22–24] and the references quoted there). Using this framework, we prove local regularity estimates (Sect. 3) for the solutions of our anisotropic elliptic quasilinear equations, namely a weighted integral hessian estimate as well as the integrability of the inverse of the gradient. For these kind of equations we also prove a Hopf type Lemma (Sect. 4). Thanks to this result, the local regularity estimates are then extended to the global case (Sect. 5).

2 Notation and some geometrical tools

For $a, b \in \mathbb{R}^n$ we denote by $a \otimes b$ the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. We remark that, for $v, w \in \mathbb{R}^n$, there holds:
\[ \langle a \otimes b, v, w \rangle = \langle b, v \rangle \langle a, w \rangle. \] (5)

Given an $n \times n$ matrix $A$, we set: $|A| := \sqrt{\sum_{i,j=1}^n |A_{ij}|^2}$. For $x_0 \in \mathbb{R}^n$ and $r > 0$ we set $B_r(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| < r \}$.

We briefly recall some basic properties about Finsler geometry, which is the main tool to study anisotropic problems. We recall that Riemannian geometry is a particular case of the Finsler one and, in fact, also in this more general framework it is possible to define length of
curves, geodesics, curvatures, etc. For our purposes, we focus the attention on Finsler norms not depending on the position, that means invariant by translations.

**Definition 2.1** A function $H : \mathbb{R}^n \rightarrow [0, +\infty)$ is said a **Finsler norm** if it is continuous, even, convex and it satisfies:

$$H(\lambda \xi) = |\lambda| H(\xi), \quad \forall \lambda \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^n$$

and

$$\exists c > 0 : \quad H(\xi) \geq c |\xi|, \quad \forall \xi \in \mathbb{R}^n. \quad (7)$$

The dual norm $H^\circ : \mathbb{R}^n \rightarrow [0, +\infty)$ is defined as:

$$H^\circ(x) = \sup\{\langle \xi, x \rangle : H(\xi) \leq 1\}.$$  

It is easy to prove that $H^\circ$ is also a Finsler norm and it has the same regularity properties of $H$. In particular $H^\circ$ satisfies (7) with $c^{-1}$ in place of $c$. Moreover it follows that $H^\circ \circ = H$.

For $r > 0$ and $\bar{x} \in \mathbb{R}^n$ we define:

$$B^H_r(\bar{x}) = \{x \in \mathbb{R}^n : H(x - \bar{x}) \leq r\}$$

and

$$B^{H^\circ}_r(\bar{x}) = \{x \in \mathbb{R}^n : H^\circ(x - \bar{x}) \leq r\}.$$  

For simplicity, when $\bar{x} = 0$, we set $B^H_r = B^H_r(0)$, $B^{H^\circ}_r = B^{H^\circ}_r(0)$. In literature $B^H_r$ and $B^{H^\circ}_r$ are also called “Wulff shape” and “Frank diagram” respectively.

We remark that there holds:

$$H(\nabla H^\circ(x)) = 1 = H^\circ(\nabla H(x)). \quad (8)$$

For more details on Finsler geometry see for instance [2,5].

### 3 Local regularity estimates

The aim of this section is to present some integral regularity estimates for the hessian and for the inverse of the gradient of any (local) solution of (2). First of all, we shall need the following lemma about some structural bounds for the principal part of our divergence form operator.

**Lemma 3.1** There exist $\tilde{C}_1, \tilde{C}_2 > 0$ such that:

$$\left( [B''(H(\xi))\nabla H(\xi) \otimes \nabla H(\xi) + B'(H(\xi))D^2 H(\xi)] v, v \right) \geq \tilde{C}_1 (k + |\xi|)^{p-2} |v|^2 \quad (9)$$

and

$$|B''(H(\xi))\nabla H(\xi) \otimes \nabla H(\xi) + B'(H(\xi))D^2 H(\xi)| \leq \tilde{C}_2 (k + |\xi|)^{p-2} \quad (10)$$

for any $\xi \in \mathbb{R}^n \setminus \{0\}$ and $v \in \mathbb{R}^n$.

**Proof** See formulas (3.2) and (3.3) in [9].
Next, we shall be interested in the linearized equation of (2) at any fixed solution \( u \), which we can write as follows. Set \( Z = \{ x \in \Omega : \nabla u(x) = 0 \} \) and, for \( \varphi \in C_c^\infty (\Omega \setminus Z) \), taking \( \psi = \varphi \) in (2), we get:

\[
\int_{\Omega} B''(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u \rangle \langle \nabla H(\nabla u), \nabla \varphi \rangle + B'(H(\nabla u)) \langle D^2 H(\nabla u) \nabla u, \nabla \varphi \rangle = \int_{\Omega} f'(u) \varphi,
\]

which, taking in account (5), can also be written as:

\[
\int_{\Omega} \left[ B''(H(\nabla u)) \nabla H(\nabla u) \otimes \nabla H(\nabla u) \right] \langle \nabla u, \nabla \varphi \rangle + B'(H(\nabla u)) \langle D^2 H(\nabla u) \nabla u, \nabla \varphi \rangle = \int_{\Omega} f'(u) \varphi.
\] (11)

Let us remark that we could make sense of (11) under several regularity hypotheses on the solution as well as on the test functions. However, even if we will not pursue such generality, let us point out that the right space for the linearization and a full spectral theory for this equation, in the singular/degenerate case when \( k = 0 \), has been introduced in [10,11] and completed in [8].

**Remark 3.2** In the sequel, with a little abuse of notation, we will denote by \( \nabla u_i \) (and \( u_{ij} \) respectively) the second derivatives of \( u \) outside \( Z \) (thought extended equal to 0 on \( Z \)). Then at the end of the section we will recover the sufficient regularity to ensure that actually these derivatives coincide with the distributional second derivatives of \( u \) in the whole of \( \Omega \).

We are now ready to prove one of our main regularity results, namely a local weighted integral estimate for the Hessian.

**Proposition 3.3** (Local Hessian estimate) Let \( u \in C^1(\overline{\Omega}) \) be a solution to (2). Fix \( x_0 \in \Omega \) and \( r > 0 \) such that \( B_{2r}(x_0) \subset \Omega \). For \( \beta \in [0, 1) \) and \( \gamma < n - 2 \) (\( \gamma = 0 \) if \( n = 2 \)), there holds:

\[
\sup_{y \in \Omega} \int_{B_r(x_0)} \frac{(k + |\nabla u|)^{p-2-\beta} |u_{ij}|^2}{|x - y|^{\gamma}} \, dx \leq C
\] (12)

and

\[
\sup_{y \in \Omega} \int_{B_r(x_0)} \frac{(k + |\nabla u|)^{p-2-\beta} |D^2 u|^2}{|x - y|^{\gamma}} \, dx \leq C,
\] (13)

where \( C = C(x_0, r, \beta, \gamma, p, n, \|u\|_{W^{1,\infty}, f}) \).

**Proof** Let \( G_\varepsilon : \mathbb{R} \to \mathbb{R} \) be defined as:

\[
G_\varepsilon(s) = \begin{cases} 
    s & \text{if } |s| \geq 2\varepsilon, \\
    2\left[s - \varepsilon \frac{s}{|s|}\right] & \text{if } \varepsilon < |s| < 2\varepsilon, \\
    0 & \text{if } |s| \leq \varepsilon,
\end{cases}
\]

and let \( \psi \) be a cut-off function such that

\[
\psi \in C_c^\infty (B_{2r}(x_0)) \quad \psi \equiv 1 \text{ in } B_r(x_0) \quad \text{and} \quad |\nabla \psi| \leq \frac{2}{r}. \quad (14)
\]
with $2r < \text{dist}(x_0, \partial \Omega)$. Fix $\beta \in [0, 1)$ and $\gamma < n - 2$ (or $\gamma = 0$ if $n = 2$) and set:

$$\varphi(x) = T_\varepsilon(u_i(x))K_\delta(|x - y|)\psi^2(x)$$

where $T_\varepsilon(t) = \frac{G_\varepsilon(t)}{|t|^\beta}$ and $K_\delta(t) = \frac{G_\delta(t)}{|t|^{\gamma+1}}$. (15)

Substituting $\varphi$ in (11), we get:

$$\begin{align*}
\int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i \rangle^2 T_\varepsilon'(u_i)K_\delta(|x - y|)\psi^2 \\
+ \int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i \rangle \nabla K_\delta(|x - y|)T_\varepsilon(u_i)\psi^2 \\
+ \int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i \rangle \nabla \psi T_\varepsilon(u_i)K_\delta(|x - y|)2\psi \\
+ \int_\Omega B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla u_i \rangle T_\varepsilon'(u_i)K_\delta(|x - y|)\psi^2 \\
+ \int_\Omega B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla \psi \rangle T_\varepsilon(u_i)K_\delta(|x - y|)2\psi \\
\end{align*}$$

$$= \int_\Omega f'(u)T_\varepsilon(u_i)K_\delta(|x - y|)\psi^2.$$ (16)

Recalling (5), we have:

$$\langle \nabla H(\xi), \nabla v \rangle^2 = \langle \nabla H(\xi) \otimes \nabla H(\xi)v, v \rangle \quad \forall \xi \in \mathbb{R}^n, \forall v \in \mathbb{R}^n$$

and

$$\langle \nabla H(\xi), v \rangle \nabla H(\xi), w \rangle = \langle \nabla H(\xi) \otimes \nabla H(\xi)v, w \rangle \quad \forall \xi \in \mathbb{R}^n, \forall v, w \in \mathbb{R}^n.$$

Hence by (9) and (10) we have:

$$B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i \rangle^2 + B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla u_i \rangle \geq \tilde{C}_1(k + |\nabla u_i|)^{p-2}|\nabla u_i|^2$$ (17)

and

$$|B''(H(\nabla u))\nabla H(\nabla u) \otimes \nabla H(\nabla u) + B'(H(\nabla u))D^2 H(\nabla u)| \leq \tilde{C}_2(k + |\nabla u_i|)^{p-2}.$$ (18)

By (17) and Cauchy–Schwartz inequality we get:

$$\tilde{C}_1 \int_\Omega (k + |\nabla u_i|)^{p-2}|\nabla u_i|^2 T_\varepsilon'(u_i)K_\delta(|x - y|)\psi^2$$

$$\leq \int_\Omega \left| \left( B''(H(\nabla u))\nabla H(\nabla u) \otimes \nabla H(\nabla u) + B'(H(\nabla u))D^2 H(\nabla u) \right) \nabla u_i, \nabla K_\delta(|x - y|) \right| |T_\varepsilon(u_i)|\psi^2$$

$$+ \int_\Omega \left| \left( B''(H(\nabla u))\nabla H(\nabla u) \otimes \nabla H(\nabla u) + B'(H(\nabla u))D^2 H(\nabla u) \right) \nabla u_i, \nabla \psi \right| |T_\varepsilon(u_i)|K_\delta(|x - y|)2\psi$$
In particular, since $\gamma < 0$, we can send $\delta$ to $0$ because

$$\int_{\Omega} |\nabla u_i|^2 \leq C \epsilon.$$ 

We remark that, if $n < 2$ this is true both for $s = \gamma$ and $s = \gamma + 1$. Therefore, for fixed $\epsilon > 0$, we can send $\delta$ to $0$ in \eqref{eq:20} and, recalling the definition of $K_\delta$, by dominated convergence we get:

$$\int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i|^2 T_\epsilon(u_i) \psi^2}{|x-y|^{\gamma'}} \leq C \int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i||T_\epsilon(u_i)||\psi^2}{|x-y|^{\gamma'+1}} + C \int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i||\nabla \psi||T_\epsilon(u_i)||\psi}{|x-y|^{\gamma'}} + C \int_{\Omega} \frac{|f'(u)||T_\epsilon(u_i)||\psi^2}{|x-y|^{\gamma'}}.$$  

We remark that, if $n = 2$, the first term in the sum at the right hand-side of \eqref{eq:21} is equal to $0$ because $\nabla K_\delta = 0$ if $\gamma = 0$. If instead $n \geq 3$, recalling that $G_\epsilon$ is an odd function and that $\gamma < n - 2$, for a $0 < \theta < 1$ we have:

$$\int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i||T_\epsilon(u_i)||\psi^2}{|x-y|^{\gamma'+1}} \leq \theta \int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i|^2 G_\epsilon(u_i)}{|x-y|^{\gamma'} |u_i|^{\beta}} \psi^2 + C.$$ 

Since $|\nabla \psi| \leq \frac{2}{\xi}$, we have:

$$\int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i||\nabla \psi||T_\epsilon(u_i)||\psi}{|x-y|^{\gamma}} \leq \theta \int_{\Omega} \frac{(k + |\nabla u_i|)^{p-2} |\nabla u_i|^2 G_\epsilon(u_i)}{|x-y|^{\gamma'} |u_i|^{\beta}} \psi^2 + C.$$ 

(22)
Recalling that $\beta \in [0, 1)$ and that $|T_{\varepsilon}(u_i)| \leq |u_i|^{1-\beta}$, since $f$ is Lipschitz and $u$ is bounded we get:
\[
\int_{\Omega} \frac{|f'(u)||T_{\varepsilon}(u_i)|\psi^2}{|x-y|^{\gamma}} \leq C \int_{\Omega} \frac{1}{|x-y|^{\gamma}} \, dx \leq C.
\] (24)
For $s > 0$ we have
\[
T_{\varepsilon}'(s) = \frac{1}{|s|^\beta} \left[ G_{\varepsilon}'(s) - \beta \frac{G_{\varepsilon}(s)}{s} \right]
\] and by (21), (22), (23) and (24) we get:
\[
\int_{\Omega} \frac{(k + |\nabla u|)^{p-2}|\nabla u_{ij}|^2}{|u_i|^{\beta}|x-y|^{\gamma}} \left( G_{\varepsilon}'(u_i) - (\beta + \vartheta) \frac{G_{\varepsilon}(u_i)}{u_i} \right) \psi^2 \leq C.
\] (25)
We now choose $\vartheta$ small enough so that $\beta + \vartheta < 1$, so that $G_{\varepsilon}'(u_i) - (\beta + \vartheta) \frac{G_{\varepsilon}(u_i)}{u_i}$ is positive. By definition of $G_{\varepsilon}$ it follows that, for any $s > 0$, $G_{\varepsilon}'(s) - (\beta + \vartheta) \frac{G_{\varepsilon}(s)}{s}$ tends to $1 - (\beta + \vartheta)$ as $\varepsilon$ goes to 0, and hence by Fatou’s Lemma we get:
\[
\int_{\Omega \setminus \{u_i = 0\}} \frac{(k + |\nabla u|)^{p-2}|\nabla u_{ij}|^2}{|u_i|^{\beta}|x-y|^{\gamma}} \psi^2 \leq C
\] (26)
and, since $(k + |\nabla u|)^{\beta} \geq |\nabla u|^{\beta} \geq |u_i|^{\beta}$, we have:
\[
\int_{\Omega \setminus \{u_i = 0\}} \frac{(k + |\nabla u|)^{p-2-\beta}|\nabla u_{ij}|^2}{|x-y|^{\gamma}} \psi^2 \leq C
\] (27)
whence, recalling that $u_{ij} = 0$ on $\{u_i = 0\} \cap (\Omega \setminus Z)$, we see that:
\[
\int_{\Omega \setminus Z} \frac{(k + |\nabla u|)^{p-2-\beta}|\nabla u_{ij}|^2}{|x-y|^{\gamma}} \psi^2 \leq C,
\] (28)
where $C$ depends on $x_0, r, n, p, \beta, \gamma, f, \|u\|_{W^{1,\infty}}$ but it does not depend on $y$. By the properties of $\psi$ it follows:
\[
\sup_{y \in \Omega} \int_{B_r(x_0) \setminus Z} \frac{(k + |\nabla u|)^{p-2-\beta}|D^2 u|^2}{|x-y|^{\gamma}} \leq C
\]
and by standard arguments (see for instance [10,23,24]) we get the thesis.

In fact, inspired by Stampacchia’s Theorem, we first extend the (generalized) second derivatives of $u$ to be zero over the critical set $Z$ and we can actually write that
\[
\sup_{y \in \Omega} \int_{B_r(x_0)} \frac{(k + |\nabla u|)^{p-2-\beta}|D^2 u|^2}{|x-y|^{\gamma}} \leq C
\] (29)
Such an estimate then allows us to conclude that the extended generalized derivatives are the effective distributional derivatives (see e.g. [10] for details). \hfill \Box

At this point, before proving the integrability of the inverse of the gradient, we shall need another structural estimate.

**Lemma 3.4** There exists $C > 0$ such that:
\[
|B'(H(\xi))| \leq C (k + |\xi|)^{p-1}.
\] (30)
Proof Since $H$ is a norm equivalent to the euclidean one, there exist $\lambda_1, \lambda_2 > 0$ such that:
\begin{equation}
\lambda_1 |\xi| \leq H(\xi) \leq \lambda_2 |\xi| \quad \forall \xi \in \mathbb{R}^n.
\end{equation}

Hence by (4) we infer:
\begin{equation}
B'(H(\xi)) \leq C_2(k + H(\xi))^{p-2}H(\xi) \leq C_2(k + H(\xi))^{p-2}\lambda_2|\xi|.
\end{equation}

Moreover we have:
\begin{equation}
\begin{cases}
(k + H(\xi))^{p-2} \leq (k + \lambda_2|\xi|)^{p-2} & \text{if } p \geq 2 \\
(k + H(\xi))^{p-2} \leq (k + \lambda_1|\xi|)^{p-2} & \text{if } p < 2.
\end{cases}
\end{equation}

Let us consider first the case $p \geq 2$. For $t > 0$, if $\lambda_2 \leq 1$, we have:
\begin{equation}
k + \lambda_2 t \leq k + t,
\end{equation}
while, if $\lambda_2 > 1$, we have:
\begin{equation}
k + \lambda_2 t = \lambda_2 \left( \frac{k}{\lambda_2} + t \right) \leq \lambda_2(k + t).
\end{equation}

By (32), the first equation in (33), (34) and (35), we get:
\begin{equation}
B'(H(\xi)) \leq C_2(k + \lambda_2|\xi|)^{p-2}\lambda_2|\xi| \leq C_2(k + \lambda_2|\xi|)^{p-1} \leq C_2 \max\{1, \lambda_2\}^{p-1}(k + |\xi|)^{p-1}.
\end{equation}

Let us now consider the case $p < 2$. For $t > 0$, if $\lambda_1 \geq 1$, we have:
\begin{equation}
k + \lambda_1 t \geq k + t,
\end{equation}
while, if $\lambda_1 < 1$, we have:
\begin{equation}
k + \lambda_1 t = \lambda_1 \left( \frac{k}{\lambda_1} + t \right) > \lambda_1(k + t).
\end{equation}

Hence by (32), the second equation in (33), (37) and (38), we get:
\begin{equation}
B'(H(\xi)) \leq \frac{C_2\lambda_2}{\lambda_1}(k + \lambda_1|\xi|)^{p-2}\lambda_1|\xi| \leq \frac{C_2\lambda_2}{\lambda_1}(k + \lambda_1|\xi|)^{p-1} \leq \frac{C_2\lambda_2}{\lambda_1} \min\{1, \lambda_1\}^{p-1}(k + |\xi|)^{p-1}.
\end{equation}

\hfill $\square$

We are now in position to state and prove our second main regularity result, which deals with the local integrability of the inverse of the weight.

Proposition 3.5 (Local estimate of the weight) Let $u \in C^1(\overline{\Omega})$ be a solution to (2). Fix $t \in [0, p - 1)$ and $\gamma < n - 2$ ($\gamma = 0$ if $n = 2$). Then, for any $\Omega' \subset \subset \Omega$ there exists $C$ such that
\begin{equation}
\sup_{\gamma \in \Omega} \int_{\Omega'} \frac{1}{(k + |\nabla u|)^{t}|x - y|^{\gamma}} \, dx \leq C,
\end{equation}
where $C = C(\Omega', t, \gamma, n, p, \|u\|_{W^{1,\infty}}, f)$. 

\hfill $\square$ Springer
Proof We first prove inequality (70) on balls, then the thesis will follow by a covering argument. For \( x_0 \in \Omega \) we choose \( r > 0 \) such that \( B_{2r}(x_0) \) is contained in \( \Omega \). Let \( \psi \) and \( K_\delta \) be defined as in Proposition 3.3. For \( t = p - 2 + \beta < p - 1 \) and \( \epsilon > 0 \), we consider in (2) the test function:

\[
\varphi = \frac{1}{(k + \vert \nabla u \vert)^t + \epsilon} K_\delta(\vert x - y \vert) \psi^2
\]

and, noticing that \( f(u(x)) \geq C(x_0) > 0 \) for any \( x \in B_{2r}(x_0) \), we get:

\[
\begin{align*}
C(x_0) \int_{B_{2r}(x_0)} \frac{1}{(k + \vert \nabla u \vert)^t + \epsilon} K_\delta(\vert x - y \vert) \psi^2 \\
\leq -t \int_{B_{2r}(x_0)} \frac{B'(H(\nabla u))(k + \vert \nabla u \vert)^{t-1}}{[(k + \vert \nabla u \vert)^t + \epsilon]^2} \left( \nabla H(\nabla u), \frac{\nabla u}{\vert \nabla u \vert} D^2 u \right) K_\delta(\vert x - y \vert) \psi^2 \\
+ \int_{B_{2r}(x_0)} \frac{B'(H(\nabla u))\psi^2}{(k + \vert \nabla u \vert)^t + \epsilon} \langle \nabla H(\nabla u), \nabla K_\delta(\vert x - y \vert) \rangle \\
+ \int_{B_{2r}(x_0)} \frac{B'(H(\nabla u))2\psi K_\delta(\vert x - y \vert)}{(k + \vert \nabla u \vert)^t + \epsilon} \langle \nabla H(\nabla u), \nabla \psi \rangle.
\end{align*}
\]

(41)

Recalling that \( H \) is 1-homogeneous, we have that \( \nabla H \) is 0-homogeneous and hence we have:

\[
\nabla H(\xi) = \nabla H \left( \frac{\xi}{\vert \xi \vert} \right) = \nabla H \left( \frac{\xi}{\vert \xi \vert} \right) \quad \forall \xi \in \mathbb{R}^n.
\]

Since \( \nabla H \) is continuous, we infer that there exists \( M > 0 \) such that:

\[
\vert \nabla H(\xi) \vert \leq M \quad \forall \xi \in \mathbb{R}^n.
\]

(42)

By (41) and (42) and Lemma 3.4 we argue:

\[
\begin{align*}
& \int_{B_{2r}(x_0)} \frac{1}{(k + \vert \nabla u \vert)^t + \epsilon} K_\delta(\vert x - y \vert) \psi^2 \\
& \leq C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1} \cdot (k + \vert \nabla u \vert)^{t-1}}{[(k + \vert \nabla u \vert)^t + \epsilon]^2} \vert D^2 u \vert \vert K_\delta(\vert x - y \vert) \vert \psi^2 \\
& + C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1}}{(k + \vert \nabla u \vert)^t + \epsilon} \vert \nabla K_\delta(\vert x - y \vert) \vert \psi^2 \\
& + C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1}}{(k + \vert \nabla u \vert)^t + \epsilon} \vert K_\delta(\vert x - y \vert) \vert \vert \nabla \psi \vert \psi.
\end{align*}
\]

(43)

Sending \( \delta \) to 0, by dominated convergence we get:

\[
\begin{align*}
& \int_{B_{2r}(x_0)} \frac{1}{(k + \vert \nabla u \vert)^t + \epsilon} \frac{1}{\vert x - y \vert^{\gamma'}} \psi^2 \\
& \leq C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1} \cdot (k + \vert \nabla u \vert)^{t-1}}{[(k + \vert \nabla u \vert)^t + \epsilon]^2} \frac{1}{\vert x - y \vert^{\gamma'}} \psi^2 \\
& + C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1}}{(k + \vert \nabla u \vert)^t + \epsilon} \frac{1}{\vert x - y \vert^{\gamma' + 1}} \psi^2 \\
& + C \int_{B_{2r}(x_0)} \frac{(k + \vert \nabla u \vert)^{p-1}}{(k + \vert \nabla u \vert)^t + \epsilon} \frac{1}{\vert x - y \vert^{\gamma'}} \vert \nabla \psi \vert \psi.
\end{align*}
\]

(44)
Recalling that \( t = p - 2 + \beta \), using Proposition 3.3, for \( 0 < \theta < 1 \) we get:

\[
\int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{p-1} \cdot (k + |\nabla u|)^{t-1}}{(k + |\nabla u|)^t + \epsilon} |D^2 u| \frac{1}{|x - y|^\gamma} \psi^2 \\
\leq \theta \int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{3t}}{(k + |\nabla u|)^t + \epsilon} |x - y|^\gamma \psi^2 \\
+ \frac{1}{4\theta} \int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{p-2-\beta}}{|x - y|^\gamma} |D^2 u|^2 \psi^2 \\
\leq \theta \int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{p-1}}{|x - y|^\gamma} \psi^2 + \frac{C}{\theta}. \quad (45)
\]

Since \( t < p - 1 \), \( \frac{(k + |\nabla u|)^{p-1}}{(k + |\nabla u|)^t + \epsilon} \) is bounded and hence we have:

\[
\int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{p-1}}{(k + |\nabla u|)^t + \epsilon} \frac{1}{|x - y|^\gamma+1} \psi^2 \leq C \int_{B_{2r}(x_0)} \frac{1}{|x - y|^\gamma+1} \psi^2 \leq C. \quad (46)
\]

Moreover by properties of \( \psi \) it follows:

\[
\int_{B_{2r}(x_0)} \frac{(k + |\nabla u|)^{p-1}}{(k + |\nabla u|)^t + \epsilon} \frac{1}{|x - y|^\gamma} |\nabla \psi| \psi \leq C \int_{B_{2r}(x_0)} \frac{1}{|x - y|^\gamma} \psi^2 \leq C. \quad (47)
\]

Choosing \( \theta \) small enough, by (44), (45), (46) and (47) we get:

\[
\int_{B_{2r}(x_0)} \frac{1}{(k + |\nabla u|)^t + \epsilon} \frac{1}{|x - y|^\gamma} \psi^2 \leq C \quad (48)
\]

and, sending \( \epsilon \) to 0, by Fatou’s Lemma we get the thesis. \( \square \)

### 4 Hopf Lemma

In this section we shall prove a Hopf Lemma for solutions of our anisotropic quasilinear elliptic equation. We begin with a couple of structural bounds from below for our principal part and then we state a weak comparison principle for a solution and a super-solution of an equation related to ours.

**Lemma 4.1** For \( x \in \mathbb{R}^n \setminus \{0\} \), \( y \in \mathbb{R}^n \) and \( p > 1 \) there exists \( C > 0 \) such that:

\[
\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (B'(H(x))H_j(x)) y_i y_j \geq C |x|^{p-2} |y|^2. \quad (49)
\]

and

\[
(B'(H(x)) \nabla H(x) - B'(H(y)) \nabla H(y), x - y) \geq C (|x| + |y|)^{p-2} |x - y|^2. \quad (50)
\]

**Proof** First we remark that

\[
\frac{\partial}{\partial x_i} (B'(H(x)) H_j(x)) = B''(H(x)) H_i(x) H_j(x) + B'(H(x)) H_{ji}(x)
\]
and hence (49) immediately follows by (9). Moreover, assuming $|y| \geq |x|$, using (49), we have:

$$B'(H(x))H_j(x) - B'(H(y))H_j(y) = \int_0^1 \sum_{i=1}^n \frac{\partial}{\partial x_i} (B'(H(x))H_j(x))|_{y+t(x-y)} (x_i - y_i)\,dt$$

and hence

$$(B'(H(x))\nabla H(x) - B'(H(y))\nabla H(y), x - y) =$$

$$\int_0^1 \sum_{i=1}^n \frac{\partial}{\partial x_i} (B'(H(x))H_j(x))|_{y+t(x-y)} (x_i - y_i)(x_j - y_j)\,dt \geq$$

$$\geq C|x - y|^2 \int_0^1 |y + t(x - y)|^{p-2}dt.$$  \hspace{1cm} (52)

For $t \in [0, 1]$ it holds $|y + t(x - y)| \leq |x| + |y|$ and hence, if $1 < p < 2$, then (50) immediately follows. Else, if $p > 2$, then we need to prove that

$$\int_0^1 |y + t(x - y)|^{p-2}dt \geq C(|x| + |y|)^{p-2}. \hspace{1cm} (53)$$

We recall that we are assuming $|y| \geq |x|$ and hence, if $|x - y| \leq \frac{|y|}{2}$, then for $0 < t < 1$ it holds

$$|y + t(x - y)| \geq |y| - |x - y| \geq \frac{|y|}{2} \geq \frac{|x| + |y|}{4},$$

from which (53) follows. If instead $|x - y| > \frac{|y|}{2} > 0$, we set $t_0 = \frac{|y|}{|x - y|}$ so that $t_0 \in (0, 2)$ and we have:

$$|y + t(x - y)| \geq ||y| - t|x - y|| = |t - t_0||x - y| \geq$$

$$\geq |t_0 - t| \frac{|y|}{2} \geq |t_0 - t| \frac{|x| + |y|}{4}.$$  \hspace{1cm} (54)

Recalling that $p > 2$, we have $\int_0^1 |t_0 - t|^{p-2}dt \geq C$ and hence (54) implies (53). \qed

Thanks to the above lemma we can now deal with the following weak comparison principle.

**Lemma 4.2** Let $u, v \in C^1(\Omega)$ satisfy:

$$\begin{cases}
-\text{div}(B'(H(\nabla u))\nabla H(\nabla u)) + g(u) \geq 0 \text{ in } \Omega \\
-\text{div}(B'(H(\nabla v))\nabla H(\nabla v)) + g(v) = 0 \text{ in } \Omega \\
v \leq u \text{ on } \partial \Omega,
\end{cases}$$

where $g$ satisfies the assumption (viii) given in Sect. 1. Then there holds:

$$v \leq u \text{ in } \Omega.$$
Proof By the weak formulation of (55) we get:

$$
\int_{\Omega} \langle B'(H(\nabla v))\nabla H(\nabla v) - B'(H(\nabla u))\nabla H(\nabla u), \nabla \psi \rangle \leq \int_{\Omega} (-g(v) + g(u))\psi.
$$

(56)

Taking $\psi = (v - u)^+$ as test function in (56), using (50) and, recalling that $-g$ is non-increasing, we infer:

$$
\int_{\Omega} (\|\nabla u\| + \|\nabla v\|)^{p-2} \|\nabla (v - u)^+\|^2 \leq \int_{\Omega} (-g(v) + g(u))(v - u)^+ \leq 0.
$$

(57)

If $p > 2$, it follows that $\nabla u = \nabla v$ a.e. in $\Omega$ and hence $(v - u)^+$ is constant and, since $(v - u)^+ = 0$ on the boundary, we infer that $(v - u)^+ = 0$. If $1 < p < 2$, since $u, v \in C^1(\Omega)$ (57) implies:

$$
\int_{\Omega} \|\nabla (v - u)^+\|^2 \leq 0
$$

and hence as above we get again the thesis. 

\[ \square \]

Remark 4.3 If $u$ satisfies (3), by property (viii) of $f$ we infer that $u$ satisfies also the first inequality in (55).

At this point, in order to prove a Hopf Lemma at the boundary for any positive solution, exactly as in the classical semilinear case, we shall need to construct a radial barrier from below defined in an annulus.

Lemma 4.4 For $R > 0$ and $\bar{x} \in \mathbb{R}^n$ we consider the annulus

$$
A_R(\bar{x}) = B^H_R(\bar{x}) \backslash \overline{B^H_{\frac{R}{2}}(\bar{x})}.
$$

Let $g$ satisfy the assumption (viii) in Sect. 1. Then for every $m > 0$ there exists a non-negative function $v \in C^1(\overline{A_R})$ satisfying:

$$
\begin{cases}
- \text{div} \left( B'(H(\nabla v))\nabla H(\nabla v) \right) + g(v) = 0 & \text{in } A_R \\
v = 0 & \text{on } \partial B^H_R \\
v = m & \text{on } \partial B^H_{\frac{R}{2}} \\
\frac{\partial v}{\partial \nu} > 0 & \text{on } \partial B^H_R,
\end{cases}
$$

(58)

where $\nu$ denotes the inner unit normal to $B^H_R$.

Proof We look for radial solutions. The word “radial” has to be understood in the Finsler framework: from now on we will say that $v$ is radial if there exists $\bar{x}$ such that $v$ is constant on the boundary of the Finsler balls $B^H_R(\bar{x})$ for any $R > 0$. In the sequel for simplicity we assume $\bar{x} = 0$ and we set $A_R = A_R(0)$. If we assume that $v$ is radial (with $\bar{x} = 0$), then there exists $w : [0, +\infty) \to \mathbb{R}$ such that:

$$
v(x) = w(H^o(x)).
$$

(59)
Using (59) in (2) (where we take test functions of the form \( \psi(x) = \varphi(H^o(x)) \) with \( \varphi : \mathbb{R} \to \mathbb{R} \)), we get:

\[
\int_{A_R} B' \left( H(w'(H^o(x))\nabla H^o(x)) \right) \left[ \nabla H \left( w'(H^o(x))\nabla H^o(x) \right), \varphi'(H^o(x))\nabla H^o(x) \right] \text{d}x + \int_{A_R} g(w(H^o(x)))\varphi(H^o(x)) \text{d}x = 0. \tag{60}
\]

Using the positive 1-homogeneity of \( H \) (and hence the 0-homogeneity of \( \nabla H \)), we infer:

\[
\int_{A_R} B' \left( |w'(H^o(x))|H(\nabla H^o(x)) \right) \text{sign} \left( w'(H^o(x)) \right) \varphi'(H^o(x)) \text{d}x + \int_{A_R} g(w(H^o(x)))\varphi(H^o(x)) \text{d}x = 0. \tag{61}
\]

Recalling that \( \langle \nabla H(\xi), \xi \rangle = H(\xi) \) and using (8), (61) becomes:

\[
\int_{A_R} B' \left( |w'(H^o(x))| \right) \text{sign} \left( w'(H^o(x)) \right) \varphi'(H^o(x)) \text{d}x + \int_{A_R} g(w(H^o(x)))\varphi(H^o(x)) \text{d}x = 0. \tag{62}
\]

We now consider on \( A_R \) a system of Finsler polar coordinates in the following sense. Since \( B^H_1 \) is convex, we can find a map \( s : U \subset \mathbb{R}^{n-1} \to \mathbb{R}^n \) such that \( s(U) = B^H_1 \). Therefore for \( \theta = (\theta_1, \ldots, \theta_{n-1}) \in U \) we have \( H^o(s(\theta)) = 1 \). For \( \rho > 0 \) we have \( H^o(\rho s(\theta)) = \rho \) and hence \( \rho s(\theta) \) belongs to \( B^H_1 \). This allows us to consider the transformation \( S : \mathbb{R}^n \to \mathbb{R}^n \) defined as:

\[
S(\rho, \theta) = \rho s(\theta) = (\rho s^1(\theta), \ldots, \rho s^n(\theta))
\]

and the change of variables:

\[x = S(\rho, \theta)\]

with \( \rho \in \left[ \frac{R}{2}, R \right] \) and \( \theta \in U \). Denoted by \( DS \) the Jacobian matrix of \( S \), we set \( J(\rho, \theta) = |\det DS(\rho, \theta)| \) and we have that there exists a suitable function \( \Gamma(\theta) > 0 \) such that:

\[
J(\rho, \theta) = \rho^{n-1}\Gamma(\theta). \tag{63}
\]

Using this change of variables, (62) becomes:

\[
\int_U \Gamma(\theta) \text{d}\theta_1 \cdots \text{d}\theta_{n-1} \int_{\frac{R}{2}}^R B' \left( |w'(\rho)| \right) \text{sign} \left( w'(\rho) \right) \varphi(\rho)\rho^{n-1} \text{d}\rho + \int_U \Gamma(\theta) \text{d}\theta_1 \cdots \text{d}\theta_{n-1} \int_{\frac{R}{2}}^R g(w(\rho))\varphi(\rho)\rho^{n-1} \text{d}\rho = 0 \tag{64}
\]

and hence:

\[
\int_{\frac{R}{2}}^R B' \left( |w'(\rho)| \right) \text{sign} \left( w'(\rho) \right) \varphi(\rho)\rho^{n-1} \text{d}\rho + \int_{\frac{R}{2}}^R g(w(\rho))\varphi(\rho)\rho^{n-1} \text{d}\rho = 0. \tag{65}
\]

whose form is:

\[- \left( B' \left( |w'(\rho)| \right) \text{sign} \left( w'(\rho) \right) \rho^{n-1} \right)' + g(w(\rho))\rho^{n-1} = 0. \tag{66}\]
If we are interested in Finsler radial solutions in the annulus $A_R(x)$, we can choose $\rho = H(x - x)$ and for $\rho \in [0, R/2]$ we consider $q(\rho) := (R - \rho)^{n-1}$. In such a way $q$ satisfies the assumptions required to apply Proposition 4.2.1 and Proposition 4.2.2 in [21], which states that the problem:

$$\begin{cases} (\Phi(t) q(\rho))' - g(w(\rho))q(\rho) = 0 & \text{in } (0, T) \\ w(0) = 0, \quad w(T) = m > 0, \quad w'(0) > 0 \end{cases}$$

with $\Phi(t) = B'(t) \text{sign}(t)$, $q$ as defined above, $f = g$ and $T = R/2$ in our case, admits a $C^1$ solution satisfying $w' > 0$.

We close this section with the following theorem, which states that the Hopf Lemma holds also for our equation. Let us point out that this can be seen an extension to the anisotropic setting of a classical Hopf Lemma for quasilinear equations by Vazquez [27].

**Theorem 4.5 (Hopf Lemma)** Let $u$ be a $C^1(\Omega)$ solution of (2) satisfying $u > 0$ in $\Omega$ and $u = 0$ on $\partial \Omega$. Let $y \in \partial \Omega$ such that $\Omega$ satisfies the interior sphere condition\(^1\) at $y$ and let $\nu$ denote the inner unit normal to $\partial \Omega$ at $y$. Then there holds:

$$\frac{\partial u}{\partial \nu}(y) > 0.$$

**Proof** Let $R > 0$ be small enough such that there exists $x \in \Omega$ such that $B^H_{R}(x) \subset \Omega$ and $y \in \partial B^H_{R}(x)$. Let $\nu$ be the function given by Lemma (4.4), which clearly satisfies the second equality in (55). Then by Lemma 4.2 and Remark 4.3 we have $u \geq v$ in $A_R(x)$ and hence $\frac{\partial u}{\partial \nu}(y) \geq \frac{\partial v}{\partial \nu}(y) > 0$. $\Box$

**5 Global regularity estimates**

In this section, thanks to the Hopf Lemma we just proved, we will see how it is possible to easily extend the local Hessian regularity result of Sect. 3 to the global case of the whole of $\Omega$.

**Proposition 5.1 (Global Hessian estimate)** Let $u \in C^1(\overline{\Omega})$ be a solution to (2). Then for $\beta \in [0, 1)$ and $\gamma < n - 2$ ($\gamma = 0$ if $n = 2$), it holds

$$\sup_{y \in \Omega} \int_{\Omega} \frac{(k + |\nabla u|)^{p-2-\beta} |u_{ij}|^2}{|x - y|^\gamma} \, dx \leq C$$

and

$$\sup_{y \in \Omega} \int_{\Omega} \frac{(k + |\nabla u|)^{p-2-\beta} |D^2 u|^2}{|x - y|^\gamma} \, dx \leq C,$$

where $C = C(\beta, \gamma, p, n, \|u\|_{W^{1,\infty}}, f)$ and $k \geq 0$ is given in Sect. 1.

**Proof** Recall that, since $\Omega$ is smooth, then the boundary $\partial \Omega$ satisfies the interior sphere condition at each point. By Theorem 4.5 we then have that $\nabla u$ does not vanish on $\partial \Omega$. Therefore by compactness of $\Omega$, applying Theorem 3.3, we get the thesis. $\Box$

\(^1\) We remark that the “interior sphere condition” in the Euclidean sense is equivalent to that in the Finsler geometry framework, that means, if $\Omega$ satisfies this condition with classical euclidean balls, then it satisfies the same condition also if we consider the Finsler balls $B^H$ or $B^H_{R}$. 

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Arguing as in the proof of Proposition 3.3, using once again Hopf’s Lemma to ensure that \( \nabla u \) does not vanish on the boundary of \( \Omega \), we can extend also the local result in Proposition 3.5 to the following global result.

**Proposition 5.2** (Global estimate of the weight) Let \( u \in C^1(\overline{\Omega}) \) be a solution to (2). Fix \( t \in [0, p - 1) \) and \( \gamma < n - 2 \) (\( \gamma = 0 \) if \( n = 2 \)). Then there exists \( C > 0 \) such that

\[
\sup_{y \in \Omega} \int_{\Omega} \frac{dx}{(k + |\nabla u|)^\gamma |x - y|} \leq C,
\]

where \( C = C(\Omega, t, \gamma, n, p, \|u\|_{W^{1,\infty}}, f) \). Moreover, \( |Z| = 0 \), where \( Z \) is the critical set of \( u \).

We finish this section with a discussion about Sobolev regularity for our positive solutions, as a corollary of the above global results.

**Theorem 5.3** Let \( u \) be a \( C^1(\overline{\Omega}) \) solution to (2). There holds:

1. if \( p \in (1, 3) \), then \( u \in W^{2,2}(\Omega) \)
2. if \( p \in [3, +\infty) \), then \( u \in W^{2-q}(\Omega) \) with \( q \leq \frac{p-1}{p-2} \).

**Proof** For any \( p > 1 \), taking \( \gamma = 0 \) in (69) and (70) respectively, we immediately get:

\[
\int_{\Omega} (k + |\nabla u|)^{p-2-\beta} |D^2 u|^2 \, dx \leq C,
\]

as well as

\[
\int_{\Omega} \frac{1}{(k + |\nabla u|)^t} \, dx \leq C,
\]

for any \( t < p - 1 \). Now, for \( 1 < p < 3 \), choosing \( \beta < 1 \) such that \( p - 2 - \beta < 0 \) and recalling that \( \nabla u \) is bounded, from (71) we immediately get:

\[
\int_{\Omega} |u_{ij}|^2 \, dx \leq \sup_{\Omega} (k + |\nabla u|)^{\beta+2-p} \int_{\Omega} (k + |\nabla u|)^{p-2-\beta} |u_{ij}|^2 \, dx \leq C,
\]

which proves statement (1). On the other hand, for \( p \geq 3 \), by (71) and (72) we have:

\[
\int_{\Omega} |u_{ij}|^q \, dx = \int_{\Omega} |u_{ij}|^q (k + |\nabla u|)^{(p-2-\beta)\frac{q}{2}} \cdot \frac{1}{(k + |\nabla u|)^{(p-2-\beta)\frac{q}{2}}} \, dx \leq \left( \int_{\Omega} |u_{ij}|^2 (k + |\nabla u|)^{p-2-\beta} \, dx \right)^{\frac{q}{2}} \left( \int_{\Omega} \frac{1}{(k + |\nabla u|)^{(p-2-\beta)\frac{q}{2-q}}} \, dx \right)^{\frac{2-q}{2}} \leq C,
\]

where we used Holder’s inequality with exponents \( \frac{2}{q} \) and \( \frac{2}{2-q} \). Notice that, in order to apply (72), we need \( (p - 2 - \beta) > 0 \), which is true for \( p \geq 3 \), as well as \( (p - 2 - \beta)\frac{q}{2-q} < p - 1 \), and this is ensured taking \( q < \frac{p-1}{p-2} \) and recalling that \( \beta < 1 \). This proves the statement (2).

Finally, recalling that \( |Z| = 0 \) and closely following the steps in the proof of Proposition 2.2 in [10], we infer that the generalized derivatives of \( u_i \) coincide with the classical ones almost everywhere in \( \Omega \) and we get the thesis. \( \square \)
Remark 5.4 In case we cannot apply Hopf’s Lemma, the same regularity results stated in Theorem 5.3 can be locally obtained using the local estimates (12), (13) and (70).

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