Entanglement Cost of Three-Level Antisymmetric States

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Abstract

We show that the entanglement cost of the three-dimensional antisymmetric states is one ebit.

The concept of entanglement is the key for quantum information processing. To quantify the resource of entanglement, its measures should be additive, such as bits for classical information. One candidate for such additive measures is entanglement of formation. In [1], it is shown that the entanglement cost $E_c$ to create some state can be asymptotically calculated from the entanglement of formation. In this sense, the entanglement cost has an important physical meaning. Since the known results are, nevertheless, not so much [3, 4], we pay attention to antisymmetric states that are easy to deal with.

As is already shown [2], the entanglement of formation for two states in $S(H_-)$ is additive. Furthermore, the lower bound for entanglement cost of density matrices in $d$-level antisymmetric space, obtained in [3], is $\log_2 \frac{d}{d-1}$ ebit.

In this paper, we show that the entanglement cost of three-level antisymmetric states ($d = 3$) in $S(H_-)$ is exactly one ebit.

We first define the three-level antisymmetric states. Let us consider a bipartite qutrit system, $H_A = H_B = C^3$. The antisymmetric subspace $H_-^n$ on $H_A \otimes H_B$ is defined as follows:

$$H_- := \text{span}_C \{|01\} - |10\}, |12\} - |21\}, |20\} - |02\} \subset H_A \otimes H_B.$$

Then, the antisymmetric state on $H_-^n$ shared with Alice and Bob is, in general,

$$|\psi\rangle = \sum_{j_1, j_2, \ldots, j_n = 0}^{2} \alpha_{j_1, j_2, \ldots, j_n ; k_1, k_2, \ldots, k_n} |j_1, j_2, \ldots, j_n ; k_1, k_2, \ldots, k_n\rangle \ (1)$$

$$\in H_-^n \subset H_A^{(1)} \otimes H_A^{(2)} \otimes \cdots \otimes H_A^{(n)} \otimes H_B^{(1)} \otimes H_B^{(2)} \otimes \cdots \otimes H_B^{(n)}.$$

$$\alpha_{j_1, j_2, \ldots, j_n ; k_1, k_2, \ldots, k_n} := \left(\frac{1}{\sqrt{2}}\right)^n \sum_{i_1, i_2, \ldots, i_n = 0}^{2} a_{i_1, i_2, \ldots, i_n} \prod_{m=1}^{n} \epsilon_{i_m j_m k_m}. \ (2)$$

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where \( \mathcal{H}^{(i)}_{A(B)} \) means \( i \)th space of Alice (resp. Bob) and \( \epsilon \) the Levi-Civita symbol, i.e., \( \epsilon_{ijk} = 1 \) for \( (ijk) = (123) \) and its even permutations, \(-1\) for odd permutations and \(0\) otherwise. Henceforth, we identify the above coefficient \( \alpha_{j_1,\ldots,j_n; k_1,\ldots,k_n} \) with the entries of a matrix \( \alpha \in M(3^n; \mathbb{C}) \) with respect to the rows \( \{j_1,\ldots,j_n\} \) and the columns \( \{k_1,\ldots,k_n\} \) with lexicographical order.

The entanglement of formation \( E_f \) is defined as follows:

\[
E_f(\rho) = \inf \sum_j p_j E(|\psi_j\rangle),
\]

where \( p_j \) and \( |\psi_j\rangle \) are decompositions such that \( \rho = \sum_j p_j |\psi_j\rangle \langle \psi_j| \) and \( E \) is the entropy of entanglement

\[
E(|\psi\rangle) = S(\text{tr}_B |\psi\rangle \langle \psi|).
\]

The following lemma is well known:

**Lemma 1 (Subadditivity)**

Let \( \rho^{(i)} \) be density matrices on \( \mathcal{H}_A \otimes \mathcal{H}_B \), i.e., bipartite states. Then,

\[
E_f(\otimes_{i=1}^n \rho^{(i)}) \leq \sum_{i=1}^n E_f(\rho^{(i)}).
\]

**Proof**

Let the decomposition for \( E_f \) be

\[
\otimes_{i=1}^n \rho^{(i)} = \sum_j p_j |\psi_j\rangle \langle \psi_j| \in S(\mathcal{H}_A^\otimes n \otimes \mathcal{H}_B^\otimes n)
\]

and

\[
\rho^{(i)} = \sum_{j_i} p_{j_i}^{(i)} |\psi_{j_i}^{(i)}\rangle \langle \psi_{j_i}^{(i)}| \in S(\mathcal{H}_A \otimes \mathcal{H}_B) \text{ for all } i.
\]

\[
E_f(\otimes_{i=1}^n \rho^{(i)}) = \inf \sum_j p_j E(|\psi_j\rangle)
\]

\[
\leq \inf \sum_{j_1,\ldots,j_n} \left( \prod_{i=1}^n p_{j_i}^{(i)} \right) E\left( \otimes_{i=1}^n |\psi_{j_i}^{(i)}\rangle \right)
\]

\[
= \inf \sum_{j_1,\ldots,j_n} \left( \prod_{i=1}^n p_{j_i}^{(i)} \right) \sum_{i=1}^n E\left( |\psi_{j_i}^{(i)}\rangle \right)
\]

\[
= \inf \sum_{i=1}^n \sum_{j_i} p_{j_i}^{(i)} E\left( |\psi_{j_i}^{(i)}\rangle \right)
\]

\[
= \sum_{i=1}^n E_f(\rho^{(i)}).
\]

\( \square \)

Hereafter we use properties of antisymmetric states. In [5], it is shown that \( E_f(\rho) = 1 \) for any \( \rho \in S(\mathcal{H}_-) \). Using their result, we obtain the following:
Corollary 1
For any $\rho^{(i)} \in \mathcal{S}(\mathcal{H}_-)$,
\[ E_f \left( \bigotimes_{i=1}^{n} \rho^{(i)} \right) \leq n. \]

To prove $E_c = 1$, it is therefore sufficient that we show the superadditivity $E_f \left( \bigotimes_{i=1}^{n} \rho^{(i)} \right) \geq n$. For the states in $\mathcal{H}_-^{\otimes n}$, we can prove the following lemma:

Lemma 2
For any $|\psi\rangle \in \mathcal{H}_-^{\otimes n}$,
\[ E (|\psi\rangle) \geq n. \] (4)

We give a proof of this lemma in appendix. The following corollary immediately follows from this lemma because the definition of the entanglement of formation (4) is a linear combination of (4).

Corollary 2
For any $\rho \in \mathcal{S}(\mathcal{H}_-^{\otimes n})$,
\[ E_f (\rho) \geq n. \]

Theorem 1
For any $\rho^{(i)} \in \mathcal{S}(\mathcal{H}_-)$,
\[ E_f \left( \bigotimes_{i=1}^{n} \rho^{(i)} \right) = n. \]

Proof
From the corollaries 1 and 2, this theorem holds. \qed

Hence, as a corollary of this theorem, we obtain the main result:

Corollary 3 (Main Result)
For any $\rho \in \mathcal{S}(\mathcal{H}_-)$,
\[ E_f (\rho^{\otimes n}) = n. \]

Therefore,
\[ E_c (\rho) := \lim_{n \to \infty} \frac{1}{n} E_f (\rho^{\otimes n}) = 1. \]

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Appendix: Proof of Lemma 2
It is well known that the entanglement of pure states is defined by von Neumann entropy of the reduced density matrix $\rho_A = \text{Tr}_B |\psi\rangle \langle \psi| = \alpha \alpha^\dagger$, where $\alpha$ is $3^n \times 3^n$ matrix, which is defined in (1). Let $\lambda_i$ be the eigenvalues of $\rho_A$ and its elementary
symmetric functions

\[ s_1 := \sum \lambda_i = \text{Tr} \rho_A = 1 \]
\[ s_2 := \sum_{i<j} \lambda_i \lambda_j \]
\[ \vdots \]
\[ s_{3^n} := \prod \lambda_i = \text{det} \rho_A, \]

the power sum \( I_k(\rho_A) = \sum \lambda^k_i = \text{Tr} \rho_A^k \), respectively. Notice that \( \sqrt{s_2} \) is the generalized concurrence\(^9\) \( ^{10} \) \( ^{11} \). As we will see later, the value of this generalized concurrence is closely related to the entanglement of formation in our case.

**Proposition 1**

*Let \( \alpha \) be the coefficient of \(|\psi\rangle \in \mathcal{H}^\otimes n \) and \( \rho_A = \alpha \alpha^\dagger \). Then,*

\[ I_2(\rho_A) \leq \frac{1}{2^n}. \] (5)

**Proof** The calculation of \( I_2(\rho_A) \) is lengthy but straightforward. First, let us choose two rows \( J := (j_1, j_2, \ldots, j_n), J' := (j'_1, j'_2, \ldots, j'_n) \) and two columns \( K := (k_1, k_2, \ldots, k_n), K' := (k'_1, k'_2, \ldots, k'_n) \) for a \( 2 \times 2 \) minor of matrix \( \alpha \). Since \( s_k(\rho_A) \) is equal to the square sum of all \( k \times k \) minors of \( \alpha \) or Gramian, we
where we denote (see, e.g., [4])

\[
s_2(p_A) = \frac{1}{4} \sum_{\substack{j_1,j_2, \ldots, j_n=0
d_1,d_2, \ldots, d_n=0}} 2 \begin{vmatrix} \alpha_{j_1, \ldots, j_n; k_1, \ldots, k_n} \alpha_{j'_1, \ldots, j'_n; k'_1, \ldots, k'_n} \\ -\alpha_{j_1, \ldots, j_n; k'_1, \ldots, k'_n} \alpha_{j'_1, \ldots, j'_n; k_1, \ldots, k_n} \end{vmatrix}^2
\]

\[
= \frac{1}{4} \left( \frac{1}{2^n} \right)^2 \sum_{J,J'K,K'} \left| \left( \sum_{p_1, \ldots, p_n=0} 2 a_{p_1, \ldots, p_n} \prod_{m=1}^n \epsilon_{p_m,j_m,k_m} \right) \left( \sum_{p'_1, \ldots, p'_n=0} 2 a_{p'_1, \ldots, p'_n} \prod_{m'=1}^n \epsilon_{p'_m,j'_m,k'_m} \right) \\
- \left( \sum_{p_1, \ldots, p_n=0} 2 a_{p_1, \ldots, p_n} \prod_{m=1}^n \epsilon_{p_m,j_m,k_m} \right) \left( \sum_{p'_1, \ldots, p'_n=0} 2 a_{p'_1, \ldots, p'_n} \prod_{m'=1}^n \epsilon_{p'_m,j'_m,k'_m} \right) \right|^2
\]

\[
= \frac{1}{2^{2n+2}} \sum_{J,J'K,K'} \sum_{PP'} \left| \sum_{p_1, \ldots, p_n=0} 2 a_{p_1, \ldots, p_n} a_{p'_1, \ldots, p'_n} \right|^2
\]

\[
\times \left( \prod_{m=1}^n \epsilon_{p_m,j_m,k_m} \prod_{m'=1}^n \epsilon_{p'_m,j'_m,k'_m} - \prod_{m=1}^n \epsilon_{p_m,j_m,k_m} \prod_{m'=1}^n \epsilon_{p'_m,j'_m,k_m'} \right)^2
\]

\[
= \frac{1}{2^{2n+1}} \sum_{J,J'K,K'} \sum_{PP'QQ'} \left| \sum_{p_1, \ldots, p_n=0} 2 a_{p_1, \ldots, p_n} a_{p'_1, \ldots, p'_n} \right|^2
\]

\[
\times \left( \prod_{m_1} \epsilon_{p_{m_1},j_{m_1},k_{m_1}} \prod_{m_2} \epsilon_{p'_{m_2},j'_{m_2},k'_{m_2}} \prod_{m_3} \epsilon_{p_{m_3},j_{m_3},k_{m_3}} \prod_{m_4} \epsilon_{p'_{m_4},j'_{m_4},k'_{m_4}} \\
- \prod_{m_1} \epsilon_{p_{m_1},j_{m_1},k_{m_1}} \prod_{m_2} \epsilon_{p'_{m_2},j'_{m_2},k'_{m_2}} \prod_{m_3} \epsilon_{p_{m_3},j_{m_3},k_{m_3}} \prod_{m_4} \epsilon_{p'_{m_4},j'_{m_4},k'_{m_4}} \right)
\]

(6)

where we denote \( \sum_{P} \equiv \sum_{p_1, p_2, \ldots, p_n=0} \) and \( a_P \equiv a_{p_1, p_2, \ldots, p_n} \), etc., for simplicity.

Let us divide (6) into two parts:
1. First Term

\[
\sum_{J,J'K'} \left( \prod_{m_1=1}^{n} \epsilon_{p_{m_1}j_1k_1} \prod_{m_2=1}^{n} \epsilon_{p_{m_2}'j_2'k_2'} \prod_{m_3=1}^{n} \epsilon_{q_{m_3}j_3k_3} \prod_{m_4=1}^{n} \epsilon_{q_{m_4}'j_4'k_4'} \right) 
= \sum_{j_2,...,j_n=0}^{2} \sum_{J'K'} \left( \sum_{j_1=0}^{2} \epsilon_{p_{j_1}j_1k_1} \epsilon_{q_{j_1}j_1k_1} \right) 
\times \left( \prod_{m_1=2}^{n} \epsilon_{p_{m_1}j_1k_1} \prod_{m_2=1}^{n} \epsilon_{p_{m_2}'j_2'k_2'} \prod_{m_3=2}^{n} \epsilon_{q_{m_3}j_3k_3} \prod_{m_4=1}^{n} \epsilon_{q_{m_4}'j_4'k_4'} \right) 
= \sum_{K'} \left[ \prod_{m=1}^{n} (\delta_{k_mk'_m} \delta_{p_mq_m} - \delta_{k_m,p_m} \delta_{k_mq_m}) \right] 
\times \sum_{K'} \left[ \prod_{m=1}^{n} (\delta_{k'_m,k'_m} \delta_{q_m',q_m'} - \delta_{k'_m,p'_m} \delta_{k_mq_m}) \right] 
= 2^{2n} \prod_{m=1}^{n} (\delta_{p_mq_m} \delta_{p_m'q_m'} + \delta_{p_m'q_m} \delta_{p_mq_m'}) ,
\]

where we use the relation \( \sum_{j_1=0}^{2} \epsilon_{p_{j_1}j_1k_1} \epsilon_{q_{j_1}j_1k_1} = \delta_{k_1k_1} \delta_{p_1q_1} - \delta_{k_1p_1} \delta_{k_1q_1} \).

2. Second Term

\[
\sum_{J,J'K'} \left( \prod_{m_1=1}^{n} \epsilon_{p_{m_1}j_1k_1} \prod_{m_2=1}^{n} \epsilon_{p_{m_2}'j_2'k_2'} \prod_{m_3=1}^{n} \epsilon_{q_{m_3}j_3k_3} \prod_{m_4=1}^{n} \epsilon_{q_{m_4}'j_4'k_4'} \right) 
= \sum_{j_2,...,j_n=0}^{2} \sum_{J'K'} \left( \sum_{j_1=0}^{2} \epsilon_{p_{j_1}j_1k_1} \epsilon_{q_{j_1}j_1k_1} \right) 
\times \left( \prod_{m_1=2}^{n} \epsilon_{p_{m_1}j_1k_1} \prod_{m_2=1}^{n} \epsilon_{p_{m_2}'j_2'k_2'} \prod_{m_3=2}^{n} \epsilon_{q_{m_3}j_3k_3} \prod_{m_4=1}^{n} \epsilon_{q_{m_4}'j_4'k_4'} \right) 
= \sum_{K'} \left[ \prod_{m=1}^{n} (\delta_{k_mk'_m} \delta_{p_mq_m} - \delta_{k_m,p_m} \delta_{k_mq_m}) \right] 
\times \sum_{K'} \left[ \prod_{m=1}^{n} (\delta_{k'_m,k'_m} \delta_{q_m',q_m'} - \delta_{k'_m,p'_m} \delta_{k_mq_m}) \right] 
= \prod_{m=1}^{n} (\delta_{p_mq_m} \delta_{p_m'q_m'} + \delta_{p_m'q_m} \delta_{p_mq_m'}) .
\]

We summarize these terms and obtain the following.

\[
s_2(\rho_A) = \frac{1}{2^{2n+1}} \sum_{P,P'Q,Q'} \frac{a_P a_P^* a_Q a_Q^*}{a_P a_P^* a_Q a_Q^*} 
\times \left[ 2^{2n} \prod_{m=1}^{n} (\delta_{p_mq_m} \delta_{p_m'q_m'} - \delta_{p_m'q_m} \delta_{p_mq_m'}) \right] 
= \frac{1}{2} \sum_{P,P'Q,Q'} \frac{a_P a_P^* a_Q a_Q^*}{a_P a_P^* a_Q a_Q^*} \prod_{m=1}^{n} (\delta_{p_mq_m} \delta_{p_m'q_m'} + \delta_{p_m'q_m} \delta_{p_mq_m'}) ,
\]

6
and
\[
I_2(\rho_A) = \frac{1}{2^n} \sum_{PP'QQ'} a_P a_{P'} a_Q^* a_{Q'}^* \prod_{m=1}^{n} \left( \delta_{p_m q_m} \delta_{p'_m q'_m} + \delta_{p'_m q_m} \delta_{p_m q'_m} \right)
\]
\[
= \frac{1}{2^n} \sum_{PP'QQ'} \prod_{m=1}^{n} \left( \delta_{p_m q_m} \delta_{p'_m q'_m} + \delta_{p'_m q_m} \delta_{p_m q'_m} \right)
\times \frac{1}{2} \left[ -|a_P a_{P'} - a_Q a_{Q'}|^2 + |a_P a_{P'}|^2 + |a_Q a_{Q'}|^2 \right]
\]
\[
= \frac{1}{2^n} - \frac{1}{2^{2n+1}} \sum_{PP'QQ'} \prod_{m=1}^{n} \left( \delta_{p_m q_m} \delta_{p'_m q'_m} + \delta_{p'_m q_m} \delta_{p_m q'_m} \right) |a_P a_{P'} - a_Q a_{Q'}|^2
\]
\[
\leq \frac{1}{2^n}.
\]

We have thus proved the proposition.\[\square\]

The following theorem is important:

**Theorem 2 (Furuta; Special case of [7, 8])**
Let $A$ be invertible positive operator. Then for any positive $x \in \mathbb{R}$
\[
-A \log_2 A \geq (1 - \log_2 x) A - \frac{1}{x} A^2.
\]

For hermitian matrix $A$, zero eigenvalues do not affect the above theorem due to $0 \log 0 = 0$.

**Corollary 4**
Let $S(A) = -\text{Tr}(A \log_2 A)$ and $\rho_A$ a normalized density matrix (i.e. $\text{Tr}\rho_A = 1$). Then
\[
S(\rho_A) \geq -\log_2 I_2(\rho_A).
\]
Hence, $S(\rho_A) \geq n$ and this ends the proof of Lemma.\[\square\]

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