CURVATURE CALCULATIONS FOR A CLASS OF HOMOGENEOUS OPERATORS

GADADHAR MISRA AND SUBRATA SHYAM ROY

Abstract. For an operator $T$ in the class $B_n(\Omega)$, introduced in [4], the simultaneous unitary equivalence class of the curvature and the covariant derivatives up to a certain order of the corresponding bundle $E_T$ determine the unitary equivalence class of the operator $T$. In the paper [6], the authors ask if there exists some pair of inequivalent operators $T_1$ and $T_2$ for which the simultaneous unitary equivalence class of the curvature along with all the covariant derivatives coincide except for the derivative of the highest order. Here we show that some of the covariant derivatives are necessary to determine the unitary equivalence class of the operators in $B_n(\Omega)$. Our examples consist of homogeneous operators. For homogeneous operators, the simultaneous unitary equivalence class of the curvature and all its covariant derivatives are determined from the simultaneous unitary equivalence class of these at 0. This shows that it is enough to calculate all the invariants and compare them at just one point, say 0. These calculations are then carried out in number of examples.

1. Introduction

For an open connected subset $\Omega$ of $\mathbb{C}$ and a positive integer $n$, the class $B_n(\Omega)$, introduced in [4], consists of bounded operators $T$ with the following properties

(a) $\Omega \subset \sigma(T)$
(b) $\text{ran}(T - \omega) = \mathcal{H}$ for $\omega \in \Omega$
(c) $\bigvee_{\omega \in \Omega} \ker(T - \omega) = \mathcal{H}$ for $\omega \in \Omega$
(d) $\dim \ker(T - \omega) = n$ for $\omega \in \Omega$.

A complete set of unitary invariants for the operators in the class $B_n(\Omega)$ was obtained in [4] as well. It was shown in [4, proposition 1.11] that the eigenspaces for each $T$ in $B_n(\Omega)$ form a Hermitian holomorphic vector bundle $E_T$ over $\Omega$, that is,

$$E_T := \{ (\omega, x) \in \Omega \times \mathcal{H} : x \in \ker(T - \omega) \}$$

and there exists a holomorphic frame $z \mapsto \gamma(\omega) := (\gamma_1(\omega), \ldots, \gamma_n(\omega))$ with $\gamma_i(\omega) \in \ker(T - \omega)$, $1 \leq i \leq n$. The hermitian structure at $z$ is the one that $\ker(T - \omega)$ inherits as a subspace of the Hilbert space $\mathcal{H}$. In other words, the metric at $\omega$ is simply the grammian $h(\omega) = \left( \langle \gamma_j(\omega), \gamma_i(\omega) \rangle \right)_{i,j=1}^n$.

The curvature $\mathcal{K}_T(\omega)$ of the bundle $E_T$ is then defined to be $\frac{\partial}{\partial \omega_h}(h^{-1} \frac{\partial}{\partial \omega} h)(\omega)$ for $\omega \in \Omega$ (cf. [4, pp. 211]).

**Theorem 1.1** ([5], Page. 326). *Two operators $T, \tilde{T}$ in $B_1(\Omega)$ are unitarily equivalent if and only if $\mathcal{K}_T(\omega) = \mathcal{K}_{\tilde{T}}(\omega)$ for $\omega$ in $\Omega$.*

Thus the curvature of the line bundle $E_T$ is a complete set of unitary invariant for an operator in $B_1(\Omega)$. It is not hard to see (cf. [4, pp. 211]) that the curvature of a bundle $E$ transforms according to the rule

$$\mathcal{K}(fg)(\omega) = (g^{-1} \mathcal{K}(f)g)(\omega), \quad \omega \in \Delta,$$

where $f = (e_1, \ldots, e_n)$ is a frame for $E$ over an open subset $\Delta \subseteq \Omega$ and $g : \Delta \rightarrow GL(n, \mathbb{C})$ is a holomorphic map, that is, $g$ a holomorphic change of frame. Since $g$ is a scalar valued holomorphic function for a line bundle $E$, it follows from the transformation rule for the curvature that it is independent of the choice of a frame in this case. In general, the curvature of a bundle $E$ of rank

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n > 1 depends on the choice of a frame. Thus the curvature $K(\omega)$ itself cannot be an invariant for the bundle $E$. However, the eigenvalues of $K(\omega)$ are invariants for the bundle $E$. More interesting is the description of a complete set of invariants given in [4] involving the curvature and the covariant derivatives $K_{z\bar{z}j}$, $0 \leq i \leq j \leq i+j \leq n$, $(i, j) \neq (0, n)$, where rank of $E = n$. They showed, in a subsequent paper (cf. [6]), by means of examples that fewer covariant derivatives will not suffice to determine the class of the bundle $E$. The examples they constructed do not necessarily correspond to an operator of the class in $B_n(\Omega)$. In this paper we construct examples of operators $T$ in $B_2(\mathbb{D})$ and $B_3(\mathbb{D})$ to show that the eigenvalues of curvature alone does not determine the class of the bundle $E_T$. Our examples show, one will need at least derivatives of order $(1, 1)$. Our examples consists of bundles homogeneous on the open unit disc $\mathbb{D}$. We will say that a holomorphic Hermitian bundle $E$ over the unit disc $\mathbb{D}$ is homogeneous if every bi-holomorphic automorphism $\varphi$ of the unit disc lifts to an isometric isomorphism of the bundle $E$. These verifications are somewhat nontrivial and use the homogeneity of the bundle in an essential way. It is not clear if for homogeneous bundle the curvature along with its derivatives up to order $(1, 1)$ suffices to determine its equivalence class. Secondly the original question of sharpness of [4, Page. 214] and [6, page. 39], remains open, although our examples a provides partial answer.

Let $B(z, w) = (1 - zw)^{-1}$ be the Bergman kernel on the unit disc, the Hilbert space corresponding to the non-negative definite kernel $B^{\alpha/2}(z, w) = (1 - zw)^{-\lambda}$ be $A^{(\lambda)}(\mathbb{D})$ for $\lambda > 0$. We let $M^{\alpha} : A^{(\lambda)}(\mathbb{D}) \rightarrow A^{(\lambda)}(\mathbb{D})$ be the multiplication operator, that is, $(M^{\alpha}f)(z) = zf(z)$, $f \in A^{(\lambda)}(\mathbb{D})$, $z \in \mathbb{D}$. Following the jet construction of [7], we construct a Hilbert space $A^{(\alpha, \beta)}_k(\mathbb{D})$ for $\alpha, \beta > 0$ consisting of holomorphic functions defined on the open unit disc $\mathbb{D}$ taking values in $\mathbb{C}^{k+1}$ starting from the kernel $B^{(\alpha, \beta)}(z, w) = B^{\alpha/2}(z_1, w_1)B^{\beta/2}(z_2, w_2)$, $z = (z_1, z_2), w = (w_1, w_2) \in \mathbb{D}^2$. It turns that the reproducing kernel for $A^{(\alpha, \beta)}_k(\mathbb{D})$ is

$$B^{(\alpha, \beta)}_k(z, w) = \left( B^{\alpha/2}(z_1, w_1)\partial z_2\partial w_2 B^{\beta/2}(z_2, w_2) \right)_{0 \leq i, j \leq k} \in \mathbb{D},$$

that is, $z_1 = z = z_2$ and $w_1 = w = w_2$. The multiplicitaion operator on $A^{(\alpha, \beta)}_k(\mathbb{D})$ is denoted by $M^{(\alpha, \beta)}_k$.

For a suitably restricted class of operators, some times, the unitary equivalence class of the curvature $K_T$ determines the unitary equivalence class of the operator $T$. For instance, the curvature at 0 of the generalised Wilkins operators $M^{(\alpha, \beta)}_k$ is of the form diag $\{\alpha, \cdots, \alpha, \alpha + (k+1)\beta + k(k+1)\}$. Thus the unitary equivalence class of the curvature at 0 determines the unitary equivalence class of these operators within the class of the generalised Wilkins operators of rank $k + 1$ (cf. [12], [2, page 428]).

### 2. Examples from the Jet Construction

**Example 2.1.** Consider the operators $M := M^{(\lambda/2)} \oplus M^{(\mu/2)}$ and $M' := M^{(\lambda, \beta)}_1$ for $\lambda, \mu > 0$ and $\alpha, \beta > 0$. Wilkins [13] has shown that the operator $M^*$ is in $B_2(\mathbb{D})$ and that it is irreducible. This operator is also homogeneous, that is, $\varphi(M)$ is unitary equivalence to $M$ for all bi-holomorphic automorphisms $\varphi$ of the open unit disc $\mathbb{D}$ (cf. [2]). It is easy to see that the operators $M^{(\lambda/2)}$ and $M^{(\mu/2)}$ are both homogeneous and the adjoint of these operators are in the class $B_1(\mathbb{D})$. Consequently, the direct sum, namely, $M^*$ is homogeneous and lies in the class $B_2(\mathbb{D})$. Let

$$h(z) = \begin{pmatrix} B^{\lambda/2}(z, z) & 0 \\ 0 & B^{\mu/2}(z, z) \end{pmatrix}, \lambda, \mu > 0,$$

$$h'(z) = B_1^{(\alpha, \beta)}(z, z) = \begin{pmatrix} (1 - |z|^2)^2 & \beta \bar{z}(1 - |z|^2) \\ \beta z(1 - |z|^2) & \beta(1 + \beta |z|^2) \end{pmatrix} (1 - |z|^2)^{-\alpha - \beta - 2}, \alpha, \beta > 0, \text{ for } z \in \mathbb{D}.$$

We see that $h$ and $h'$ are the metrics for bundles corresponding to the operators $M^*$ and $M''$ respectively. To emphasize the dependence of the curvature on the metric, we will find it useful to also write $K_h := \bar{\partial}(h^{-1}\partial h)$. 

Choosing for any reproducing kernel $h$ and is therefore uniquely determined up to a conjugation by a unitary matrix. Let $K(z,w) = \sum_{k,l,\ell \geq 0} a_{kl} \bar{z}^k \bar{w}^\ell$ and $\tilde{K}(z,w) = \sum_{k,l,\ell \geq 0} \tilde{a}_{kl} \bar{z}^k \bar{w}^\ell$, where $a_{kl}$ and $\tilde{a}_{kl}$ are determined by the real analytic functions $K$ and $\tilde{K}$ respectively, $a_{kl}$ and $\tilde{a}_{kl}$ are in $\mathcal{M}(n,\mathbb{C})$, for $k,\ell \geq 0$. Since $\tilde{K}(z,w)$ is a normalized kernel, it follows that $\tilde{a}_{00} = I$ and $\tilde{a}_{k0} = \tilde{a}_{0\ell} = 0$ for $k,\ell \geq 1$. Let $K(z,w)^{-1} = \sum_{k,l,\ell \geq 0} b_{k\ell} \bar{z}^k \bar{w}^\ell$, where $b_{k\ell}$ is in $\mathcal{M}(n,\mathbb{C})$, for $k,\ell \geq 0$. Clearly, $K(z,w)^* = K(w,z)$ for any reproducing kernel $K$ and $z, w \in \mathbb{D}$. Therefore, $a_{k\ell} = a_{\ell k}, \tilde{a}_{k\ell} = \tilde{a}_{\ell k}$ and $b_{k\ell}^* = b_{k\ell}$ for $k,\ell \geq 0$, where $X^*$ denotes the conjugate transpose of the matrix $X$.

If we assume that the adjoint $M^*$ of the multiplication operator $M$ on the Hilbert space $(\mathbb{H}, K)$ is in $B_k(\mathbb{D})$ then it is not hard to see that the operators $M^*$ on the Hilbert space $\tilde{\mathbb{H}}$ determined by the normalized kernel $\tilde{K}$ is equivalent to $M^*$ on the Hilbert space $(\mathbb{H}, \tilde{K})$. Hence the adjoint of the multiplication operator $M$ on $(\mathbb{H}, \tilde{K})$ lies in $B_k(\mathbb{D})$ as well. Let $(E, \tilde{h})$ be the corresponding bundle, where $\tilde{h}(z) = \tilde{K}(z,z)^t, z \in \mathbb{D}$.

**Lemma 2.2.** If $\tilde{h}(z) = \tilde{K}(z,z)^t$, then $\tilde{\partial}^m \tilde{h}(0) = \partial^m \tilde{h}(0) = \partial \tilde{h}^{-1}(0) = 0$ for $m, n \geq 0$.

**Proof.** Let $\tilde{h}(z) = \sum_{m,n \geq 0} h_{mn} z^m \bar{z}^n$ and $\tilde{h}^{-1}(z) = \sum_{m,n \geq 0} h'_{mn} z^m \bar{z}^n$. As $\tilde{h}(z) = \tilde{K}(z,z)^t$, $h_{mn} = a_{mn}$ for $m, n \geq 0$, so $h_{00} = I$, $h_{m0} = h_{0n} = 0$ for $m, n \geq 0$. As $h_{mn} = \partial^m \partial^n \tilde{h}(0)$, $h'_{mn} = \partial^m \tilde{h}^{-1}(0)$, $h'_{0m} = \partial^m \tilde{h}^{-1}(0)$, $h'_{m0} = 0$ for $m, n \geq 0$. To prove the first assertion, it is enough to show that $h'_{m0} = h'_{0m} = 0$ for $m, n \geq 1$. As $h'_k = h'_{tk}$ it is enough to show that $h'_{m0} = 0$ for $m \geq 1$. It follows from $\tilde{h}(z)\tilde{h}^{-1}(z) = I$ that $h'_{00} = I$ and $\sum_{k=0}^m h_{m-k,0} h'_{k0} = 0$ for $m \geq 1$. So, $0 = \sum_{k=0}^m h_{m-k,0} h'_{k0} = \sum_{k=0}^{m-1} h_{m-k,0} h'_{k0} + h'_{m,0} = h'_{m,0}$ for $m \geq 1$, as $h'_{00} = 0$ for $\ell \geq 1$.

For the second assertion we note that $\tilde{h}(z)\tilde{h}^{-1}(z) = I$ implies $\partial \tilde{h}^{-1}(0) = h'_{11} = -h_{11} = -a_{11}$. Clearly $\tilde{\partial} \tilde{h}^{-1}(0) = h'_{11} = a_{11}^t$ and $\tilde{\partial}^2 \tilde{h}(0) = 4h_{22} = 4a_{22}^t$. \hfill $\square$

**Lemma 2.3.** The curvature $\mathcal{K}_{\tilde{h}}$ and the covariant derivative $(\mathcal{K}_{\tilde{h}})^{\circ n}$ of curvature of the bundle $(E, \tilde{h})$ at 0 is $a_{11}$ and $(n+1)! a_{1,n+1}$, that is, $\mathcal{K}_{\tilde{h}}(0) = a_{11}$ and $(\mathcal{K}_{\tilde{h}})^{\circ n}(0) = (n+1)! a_{1,n+1}$.

**Proof.** By [4, page. 211] $\mathcal{K}_{\tilde{h}} = \tilde{\partial}(\tilde{h}^{-1}\tilde{\partial}\tilde{h})$. Hence $\mathcal{K}_{\tilde{h}}(0) = \tilde{\partial}(\tilde{h}^{-1}(0))\tilde{\partial}\tilde{h}(0) + \tilde{h}^{-1}(0)\tilde{\partial}\tilde{h}(0) = a_{11}^t$.

For the second assertion, we know from [4, Proposition 2.17, page 211], $\mathcal{K}_{\tilde{h}}(0) = \tilde{\partial}^n \mathcal{K}(0) = \tilde{\partial}^{n+1}(\tilde{h}^{-1}\tilde{\partial}\tilde{h})(0)$. Using Leibnitz’s rule, this is the same as $\sum_{k=0}^{n+1} \binom{n+1}{k} \tilde{\partial}^{n+1-k}(\tilde{h}^{-1}(0))\tilde{\partial}^k(\tilde{h})(0) = \tilde{\partial}^{n+1}\tilde{h}(0) = (n+1)! h_{1,n+1} = (n+1)! a_{1,n+1}$, as $\tilde{\partial}^n \tilde{h}^{-1}(0) = 0$ for $\ell \geq 1$ from Lemma 2.2. \hfill $\square$
Lemma 2.4. If $K_h$ is the curvature of the bundle $(E,\hat{h})$ then $(K_h)_{zz}(0) = 2(2\bar{a}_{22} - \bar{a}_{11}^2)t$.

Proof. We know from [4] that for a bundle map of a Hermitian holomorphic vector bundle $\Theta : (E,\hat{h}) \rightarrow (E,\hat{h})$ the covariant derivatives $\Theta_z$ and $\Theta_{\bar{z}}$ of $\Theta$ with respect to holomorphic frame $f$ are $\Theta_z(f) = \partial \Theta(f) + [\hat{h}^{-1}\partial \hat{h}, \Theta(f)]$ and $\Theta_{\bar{z}}(f) = \bar{\partial} \Theta(f)$. So $(K_h)_{zz}(z) = \bar{\partial}(\partial(K_h)_z(z) + [\hat{h}^{-1}\partial \hat{h}, K_h](z)) = \partial \partial K_h(z) + [\hat{h}^{-1}\partial \hat{h}, \partial K_h](z) = \partial \partial K_h(z) + [\hat{h}^{-1}\partial \hat{h}, \partial K_h](z)$. Hence $(K_h)_{zz}(0) = \partial \partial (0)$, as $\partial (0) = h_{10} = 0$. Now by Leibnitz rule we see that $\partial \partial (0)\partial \hat{h}_0(0) + \bar{\partial}^2\bar{\partial}^2\hat{h}_0(0) = -2a_{11}^t a_{11}^t + 4a_{22}^t = 2(2\bar{a}_{22} - \bar{a}_{11}^2)t$. \hfill \qed

Lemma 2.5. The coefficient of $z^{k+1}\bar{w}^{\ell+1}$ in the power series expansion of $K(z,w)$ is

$$\tilde{a}_{k+1,\ell+1} = a_{00}^{1/2} \left( \sum_{s=1}^{k} \sum_{t=1}^{\ell} b_{s0} a_{k+1-s,\ell+1-t} b_{0t} + \sum_{s=1}^{k} b_{s0} a_{k+1-s,\ell+1-t} b_{0t} \right) a_{00}^{1/2}$$

for $k, \ell \geq 0$.

Proof. From the definition of $K(z,w)$ we see that for $k, \ell \geq 0$

$$\tilde{a}_{k+1,\ell+1} = a_{00}^{1/2} \left( \sum_{s=0}^{k+1} \sum_{t=0}^{\ell+1} b_{s0} a_{k+1-s,\ell+1-t} b_{0t} \right) a_{00}^{1/2}$$

as the coefficient of $z^{k+1}$ in $K(z,w)^{-1}(K(z,w) = \sum_{s=0}^{k+1} b_{s0} a_{k+1-s,0} = 0$ and the coefficient of $w^{\ell+1}$ in $K(z,w)K(z,w)^{-1} = \sum_{t=0}^{\ell+1} a_{0,t+1-t} b_{0t} = 0$ for $k, \ell \geq 0$. \hfill \qed
The following Theorem will be useful in the sequel. For }T in B_n(\Omega),\text{ recall that }K_T\text{ denotes the curvature of the bundle }E_T\text{ corresponding to }T.

**Theorem 2.6.** Suppose that }T_1\text{ and }T_2\text{ are homogeneous operators in }B_n(\mathbb{D})\text{ then }K_{T_1}(0)\text{ and }K_{T_1}(z)\text{ are simultaneously unitarily equivalent to }K_{T_2}(0)\text{ and }K_{T_2}(z)\text{ respectively if and only if }K_{T_1}(z)\text{ and }K_{T_1}(z)\text{ are simultaneously unitarily equivalent to }K_{T_2}(z)\text{ and }K_{T_1}(z)\text{ respectively for }z in \mathbb{D}.

Before going into the proof of 2.6 let us fix some notations. Let M"ob denote the group of biholomorphic automorphisms on the unit disc }\mathbb{D}\text{ in the complex plane, }c : M"ob \times \mathbb{D} \rightarrow \mathbb{C}\text{ be the function which is given by the formula }c(\varphi^{-1}, z) := (\varphi^{-1})'(z),\text{ where the prime stands for differentiation with respect to }z\text{. The function }c\text{ satisfies the following cocycle property:}

\[ c(\varphi^{-1}, \psi^{-1}, z) = c(\varphi^{-1}, \psi^{-1}(z))c(\psi^{-1}, z), \text{ for } \varphi \in \text{M"ob and } z \in \mathbb{D}.\]

This cocycle property can easily verified by chain rule.

**Lemma 2.7.** Suppose that }T in B_n(\mathbb{D})\text{ is homogeneous. Then}

(a) \[K_T(\varphi^{-1}(0)) = |c(\varphi^{-1}, 0)|^{-2}U_{\varphi}^{-1}K_T(0)U_{\varphi}\]

(b) \[(K_T)_z(\varphi^{-1}(0)) = |c(\varphi^{-1}, 0)|^{-2}c(\varphi^{-1}, 0)^{-1}U_{\varphi}^{-1}\left((K_T)_z(0) - c(\varphi^{-1}, 0)^{-1}(\varphi^{-1})'(0)K_T(0)\right)U_{\varphi}\]

for some unitary operator }U_{\varphi}, \varphi \in \text{M"ob}.

**Proof.** From [4] it follows that homogeneity of }T\text{ implies }K_{\varphi(T)}(z) = U_{\varphi,z}^{-1}K_T(z)U_{\varphi,z}\text{ for some unitary operator }U_{\varphi,z}, \varphi \in \text{M"ob and } z in \mathbb{D}.\text{ Note that an application of chain rule gives the formula}

\[(2.1) \quad K_{\varphi(T)}(z) = [(\varphi^{-1})'(z)]^2K_T((\varphi^{-1})(z)), \text{ for } \varphi \in \text{M"ob and } z \in \mathbb{D}.\]

Now, assuming that }K_{\varphi(T)}(z) = U_{\varphi,z}^{-1}K_T(z)U_{\varphi,z}\text{, using (2.1), amounts to }U_{\varphi,z}^{-1}K_T(z)U_{\varphi,z} = |c(\varphi^{-1}, z)|^2K_T((\varphi^{-1})(z)).\text{ Putting } z = 0\text{, we get }U_{\varphi,0}^{-1}K_T(0)U_{\varphi,0} = |c(\varphi^{-1}, 0)|^2K_T((\varphi^{-1})(0)).\text{ If we set } U_{\varphi,0} := U_{\varphi}, \text{ then we have } K_T(\varphi^{-1}(0)) = |c(\varphi^{-1}, 0)|^{-2}U_{\varphi}^{-1}K_T(0)U_{\varphi}\text{ for } \varphi \in \text{M"ob, } z \in \mathbb{D}.\text{ This proves (a).}

To prove (b), we recall (2.1) and differentiating with respect to }z\text{ we get

\[(2.2) \quad (\varphi^{-1})'(z)\overline{(\varphi^{-1})'(z)}(z)K_T((\varphi^{-1})(z)) + |(\varphi^{-1})'(z)|^2K_T((\varphi^{-1})(z))\partial K_T((\varphi^{-1})(z)).\]

Using (2.2) and (a), putting }z = 0\text{ and } U_{\varphi,0} = U_{\varphi}, \text{ we see that}

\[(2.3) \quad c(\varphi^{-1}, 0)^{-1}(\varphi^{-1})'(0)U_{\varphi}^{-1}K_T(0)U_{\varphi} + |c(\varphi^{-1}, 0)|^2U_{\varphi}^{-1}K_T(0)U_{\varphi} = c(\varphi^{-1}, 0)^{-1}(\varphi^{-1})'(0)K_T((\varphi^{-1})(0)).\]

So, \[\partial K_T((\varphi^{-1})(0)) = |c(\varphi^{-1}, 0)|^{-2}c(\varphi^{-1}, 0)^{-1}U_{\varphi}^{-1}\left(\partial K_T(0) - c(\varphi^{-1}, 0)^{-1}(\varphi^{-1})'(0)K_T(0)\right)U_{\varphi}.\]

As \[(K_T)_z = \partial K_T\] by [4], this proves (b). 

**Corollary 2.8.** Suppose that }T_1, T_2\text{ are homogeneous operators in }B_n(\mathbb{D}).\text{ Then}

1. \[U^{-1}K_{T_2}(0)U = K_{T_1}(0)\]
2. \[U^{-1}(K_{T_2})_z(0)U = (K_{T_1})_z(0)\]

for some unitary operator }U\text{ if and only if}

(i) }V_{\varphi}^{-1}K_{T_2}(z)V_{\varphi} = K_{T_1}(z)
(ii) }V_{\varphi}^{-1}(K_{T_2})_z(z)V_{\varphi} = (K_{T_1})_z(z)

for some unitary operator }V_{\varphi}, \varphi \in \text{M"ob, } z \in \mathbb{D}.
Proof. One part is obvious, let us prove the other part.

Take \( \varphi = \varphi_{t,z} \), where \( \varphi_{t,z}(z) = t \frac{z - a}{|z - a|^2} \) for \( a, z, \in \mathbb{D} \) and \( t \in \mathbb{T} \). Pick a unitary operator such that (a) and (b) of Lemma 2.7 are satisfied. We get from (1) and Lemma 2.7(a) that

\[
\mathcal{K}_{T_2}(z) = |c(\varphi^{-1}, 0)|^{-2} \mathcal{U}_{\varphi}^{-1} \mathcal{K}_{T_2}(0) U_{\varphi} + |c(\varphi^{-1}, 0)|^{-2} \mathcal{U}_{\varphi}^{-1} \mathcal{U}_{\varphi}^{-1} \mathcal{K}_{T_1}(0) U_{\varphi} \]

As the product of unitary operators is again a unitary operator, taking \( V_{\varphi} = U_{\varphi}^{-1} U_{\varphi} \) we have (i).

From (2) and Lemma 2.7(b)

\[
(\mathcal{K}_{T_2})z(z) = |c(\varphi^{-1}, 0)|^{-2} (\mathcal{K}_{T_2})z(0) - |c(\varphi^{-1}, 0)|^{-1} (\mathcal{K}_{T_2})(0) \mathcal{K}_{T_2}(0) \]

Taking \( V_{\varphi} = U_{\varphi}^{-1} U_{\varphi} \) as before, we have (ii).

Proof of Theorem 2.6. Combining Lemma 2.7 and Corollary 2.8 we have a proof of the Theorem 2.6.

For a positive integer \( m \) let \( S_m(c_1, \ldots, c_m) \) denote the forward shift on \( \mathbb{C}^{m+1} \) with weight sequence \( \{c_1, \ldots, c_m\} \), \( c_i \in \mathbb{C} \), that is, \( \{S_m(c_1, \ldots, c_m)\}_{\ell, p} = c_i\delta_{p+1, \ell} \), \( 0 \leq p, \ell \leq m \). We set \( S_m := S_m(1, \ldots, m) \).

Example 2.9. Consider the operators \( M_1 = M^{(\alpha/2)} \oplus M^{(\alpha, \beta)}_1 \) and \( M_2 = M^{(\alpha, \beta)}_2 \) for \( \alpha, \beta, \beta' > 0 \).

Wilkins [13] has shown that the adjoint of the operator \( M^{(\alpha, \beta)}_1 \) is in \( B_2(\mathbb{D}) \). This operator is also homogeneous. It is easy to see that the operator \( M^{(\alpha/2)} \) is homogeneous and its adjoint is in the class \( B_3(\mathbb{D}) \). Consequently, the direct sum, namely, \( M^{(\alpha, \beta)}_1 \) is homogeneous and lies in the class \( B_3(\mathbb{D}) \).

The operator \( M^{(\alpha/2)}_2 \) is in \( B_3(\mathbb{D}) \) by [7, Proposition 3.6] and is homogeneous by [2, Page. 428] and [12, Theorem 5.1]. Let \( h_1(z) = (1 - |z|^2)^{-\alpha} \oplus B^{(\alpha, \beta)}_1(z, z) \) and \( h_2(z) = B^{(\alpha, \beta)}_2(z, z) \). We see that \( h_1 \) and \( h_2 \) the metrics for the bundles \( E_1 \) and \( E_2 \) corresponding to the operators \( M_1^* \) and \( M_2^* \) respectively, where \( B^{(\alpha, \beta)}_1(z, w) = \left(\begin{array}{cc} (1 - z\bar{w})^2 & \beta'(1 - z\bar{w}) \\ \beta'(1 - z\bar{w}) & (1 + \beta z\bar{w}) \end{array}\right) \) and \( B^{(\alpha, \beta)}_2(z, w) = \left(\begin{array}{cc} (1 - z\bar{w})^4 & \beta(1 - z\bar{w})^2 \\ \beta(1 - z\bar{w})^2 & (1 + \beta z\bar{w}) \end{array}\right) \).

Lemma 2.10. The curvature at zero and the covariant derivatives of curvature at zero up to order for the bundles \( E_1 \) and \( E_2 \) respectively are

(a) \( K_1(0) = \text{diag}(\alpha, \alpha, \alpha + 2\beta + 2) \), \( (K_1)_{z\bar{z}}(0) = S_2(0, -2\sqrt{\beta}(\beta' + 1)) \) and \( (K_1)_{zz}(0) = 2 \text{diag}(\alpha, \alpha + \beta'(\beta' + 1), \alpha + \beta'(\beta' + 1) + 2) \)

(b) \( K_2(0) = \text{diag}(\alpha, \alpha, \alpha + 3\beta + 6) \), \( (K_2)_{z\bar{z}}(0) = S_2(0, -3\sqrt{\beta+1}(\beta + 2)) \) and \( (K_2)_{zz}(0) = \text{diag}(\alpha, \alpha + 3(\beta + 1) + 2, \alpha + 3\beta(\beta + 2)) \),

where \( K_i, (K_i)_{z\bar{z}} \) and \( (K_i)_{zz} \) are computed with respect to a metric normalized at 0 obtained from \( h_i \) for \( i = 1, 2 \), that is, with respect to an orthonormal basis at 0.


Proof. For any reproducing kernel $K$ with $K(z,w) = \sum_{m,n \geq 0} a_{mn} z^m \bar{w}^n$ and $K(z,w)^{-1} = \sum_{m,n \geq 0} b_{mn} z^m \bar{w}^n$ the identity $K(z,w)^{-1} K(z,w) = I$ implies that $b_{00} = a_{00}^{-1}$ and $\sum_{\ell=0}^{k} b_{0,k-\ell} a_{\ell 0} = 0$ for $k \geq 1$. For $k = 1$ we have $b_{10} = -a_{00}^{-1} a_{10} a_{00}^{-1}$, $b_{01} = (b_{10})^*$. We have by Lemma 2.5

$$\tilde{a}_{11} = a_{00}^{1/2} (b_{00} a_{11} b_{00} - b_{10} a_{00} b_{01}) a_{00}^{1/2}$$

$$= a_{00}^{-1/2} (a_{11} - a_{10} a_{00}^{-1} a_{01}) a_{00}^{-1/2}.$$  

(2.5)

For $k = 2$ we have $b_{20} = - (b_{01} a_{01} + b_{00} a_{02}) a_{00}^{-1} = a_{00}^{-1} (a_{01} a_{00}^{-1} a_{01} - a_{02}) a_{00}^{-1}$. We get from Lemma 2.5

$$\tilde{a}_{12} = a_{00}^{1/2} (b_{00} a_{11} b_{00} + b_{10} a_{12} b_{00} - b_{20} a_{00} b_{02}) a_{00}^{1/2}$$

$$= a_{00}^{-1/2} (a_{12} - (a_{11} - a_{10} a_{00}^{-1} a_{01}) a_{00}^{-1} a_{01} - a_{10} a_{00}^{-1} a_{02}) a_{00}^{-1/2}.$$  

(2.6)

Observing that $b_{20} = b_{02}^*$ we have $a_{00}^{-1} (a_{10} a_{00}^{-1} a_{10} - a_{20}) a_{00}^{-1}$, from Lemma 2.5 we have

$$\tilde{a}_{22} = a_{00}^{1/2} (b_{10} a_{11} b_{00} + b_{00} a_{21} b_{00} + b_{00} a_{02} b_{00} - b_{20} a_{00} b_{02}) a_{00}^{1/2}$$

$$= a_{00}^{-1/2} (a_{10} a_{00}^{-1} a_{11} a_{00}^{-1} a_{01} - a_{10} a_{00}^{-1} a_{12} - a_{21} a_{00}^{-1} a_{01} + a_{22})$$

$$- (a_{20} a_{00}^{-1} a_{10} - a_{20}) a_{00}^{-1} (a_{01} a_{00}^{-1} a_{01} - a_{02}) a_{00}^{-1/2}$$

$$= a_{00}^{-1/2} (a_{22} + (a_{20} a_{00}^{-1} a_{10} - a_{21} a_{00}^{-1} a_{01} - a_{20} a_{00}^{-1} a_{02})$$

$$- a_{10} a_{00}^{-1} (a_{12} - (a_{11} - a_{10} a_{00}^{-1} a_{01}) a_{00}^{-1} a_{01} - a_{10} a_{00}^{-1} a_{02})) a_{00}^{-1/2}.$$  

(2.7)

We get from $h_1$ that $a_{11} = \text{diag} (\alpha, \alpha + \beta', \beta' (\alpha + 2\beta' + 2), a_{00} = \text{diag} (1,1, \beta'))$, $a_{10} = S_2 (0, \beta'^t)$, $a_{12} = S_2 (0, \beta' (\alpha + \beta' + 1))$, $a_{22} = \text{diag} (a_{00} + \beta', a_{00} + 2\beta', a_{00} + 3\beta', 1)$. We get from Lemma 2.5 $a_{11} = \text{diag} (\alpha, \alpha + \beta' + 2, \beta' (\alpha + 2\beta' + 2))$, from Equation (2.5) we have $K_1(0) = \tilde{a}_{11} = \text{diag} (\alpha, \alpha + \beta' + 2)$. We get from Equation (2.6) $\tilde{a}_{12} = S_2 (0, -\sqrt{\beta'} (\beta' + 1))$, so from Lemma 2.3 we have $(K_1)_{z}(0) = 2 \tilde{a}_{12} = 2 S_2 (0, -\sqrt{\beta'} (\beta' + 1))^t$.

From Equation (2.7) $\tilde{a}_{22} = \text{diag} (a_{00} + \beta', a_{00} + 2\beta', a_{00} + 3\beta', 1)$, hence from Lemma 2.4 we get $(K_1)_{z}(0) = 2 (2 \tilde{a}_{22} - \tilde{a}_{11}^2)^t = 2 \text{diag} (\alpha, \alpha + \beta' (\beta' + 1), \alpha + \beta' (-\beta' + 1) + 2)$. This completes the proof of (a).

To prove (b) we get from $h_2$ that $a_{00} = \text{diag} (1, \beta, 2\beta (\beta + 1))$, $a_{10} = S_2 (\beta, 2\beta (\beta + 1))^t$, $a_{12} = S_2 (\beta (\alpha + \beta + 1), \beta (\beta + 1)(2\alpha + 3\beta + 6))$,

$$a_{02}$$

$$\begin{cases} 
\beta (\beta + 1), & \text{for } i = 3, j = 1; \\
0, \quad & \text{otherwise.}
\end{cases}$$

$a_{11}$ is a diagonal matrix with diagonal entries $\alpha + \beta, \beta (\alpha + 2\beta + 2)$ and $2\beta (\beta + 1)(\alpha + 3\beta + 6)$ respectively and $a_{22}$ is also a diagonal matrix with diagonal entries $\frac{\alpha + \beta}{2}, \frac{\alpha + \beta + 2}{2}(\alpha + 3\beta + 3)$ and $\beta (\beta + 1)(\alpha + \beta + 4)(\alpha + \beta + 5) + 4(\beta + 1)(\alpha + \beta + 4) + \beta (\beta + 1)$ respectively. Therefore $a_{11} - a_{10} a_{00}^{-1} a_{01} = \text{diag} (\alpha, \alpha \beta, 2 \beta (\beta + 1)(\alpha + 3\beta + 6))$, hence from Lemma 2.3 and Equation (2.5) we have $K_2(0) = \tilde{a}_{11} = \text{diag} (\alpha, \alpha + 3\beta + 6)$.

We get from Equation (2.6), $\tilde{a}_{12} = S_2 (0, -\frac{2}{\sqrt{2}} \sqrt{\beta} (\beta + 2))$. Now from Lemma 2.3 we have $(K_2)_{z}(0) = 2 \tilde{a}_{12} = S_2 (0, -3 \sqrt{2} (\beta + 1)(\beta + 2))^t$. From Equation (2.7), $\tilde{a}_{22} = \text{diag} (a_{00} + \beta', a_{00} + 2\beta', a_{00} + 3\beta, \alpha + 2\beta + 3\beta + 3)$, using Lemma 2.4 we get $(K_2)_{z}(0) = 2 (2 \tilde{a}_{22} - \tilde{a}_{11}^2)^t = 2 \text{diag} (\alpha, \alpha + 3(\beta + 1)(\beta + 2), \alpha - 3\beta (\beta + 2))$. □
We prove a sequence of lemmas which exhibits a unitary between the vector spaces $(E_1)_0, h_1(0))$ and $(E_2)_0, h_2(0)$ which intertwines $K_1(0), K_2(0)$ and $(K_1)_{z}(0), (K_2)_{z}(0)$, where $(E_1)_0$ and $(E_2)_0$ are the fibres over 0 of the corresponding bundles.

**Lemma 2.11.** $U_0 : (C^3, h_2(0)) \rightarrow (C^3, h_1(0))$, is a diagonal unitary with $U_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3$, if and only if $|\alpha_1|^2 = 1, |\alpha_2|^2 = \beta, |\alpha_3|^2 = \frac{2\beta(\beta+1)}{\beta'}$.

**Proof.** “only if” part: As $U_0$ is a unitary $U_0^* = U_0^{-1}$, where * denotes the adjoint of $U_0$. Now, from [7, p. 395]

$$U_0^* = h_2(0)^{-1}U_0^*h_1(0)$$

$$= \text{diag}(1, \beta^{-1}, (2\beta(\beta + 1))^{-1}) \text{diag}(\alpha_1, \alpha_2, \alpha_3) \text{diag}(1, 1, \beta')$$

$$= \text{diag}(\alpha_1^{-1}, \alpha_2^{-1}, \beta', \alpha_3^{-1})$$

This implies the desired equalities.

“if” part: Taking $\alpha_1 = 1, \alpha_2 = \sqrt{\beta}, \alpha_3 = \sqrt{\frac{2\beta(\beta+1)}{\beta'}}$, we see that $U_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$ is a unitary between the two given vector spaces.

The proof of the next lemma is just a routine verification.

**Lemma 2.12.** Suppose that $T$ and $\tilde{T}$ are in $M(3, \mathbb{C})$ such that $(T)_{ij} = \begin{cases} \eta, & \text{for } i = 2, j = 3; \\ 0, & \text{otherwise.} \end{cases}$ and $\tilde{T} = \begin{cases} \tilde{\eta}, & \text{for } i = 2, j = 3; \\ 0, & \text{otherwise.} \end{cases}$ are two matrices, $T$ and $\tilde{T}$ satisfies $AT = \tilde{T}A$ for some invertible diagonal matrix $A = \text{diag}(a_1, a_2, a_3)$ if and only if $\frac{\tilde{\eta}}{\eta} = \frac{a_2}{a_3}$.

**Lemma 2.13.** If $\beta' = \frac{3}{2}\beta + 2$ then $U_0^{-1}K_1(0)U_0 = K_2(0)$ and $U_0^{-1}(K_1)_{z}(0)U_0 = (K_2)_{z}(0)$, where $U_0 : (C^3, h_2(0)) \rightarrow (C^3, h_1(0))$, is a diagonal unitary with $U_0 = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i \in \mathbb{C}$ for $i = 1, 2, 3$.

**Proof.** By the choice of $\beta'$, $K_1(0) = K_2(0)$ by Lemma 2.10, so the first equality is clear.

Take $T = (K_2)_{z}(0)$ and $\tilde{T} = (K_1)_{z}(0)$. Also choose $\alpha_1 = 1, \alpha_2 = \sqrt{\beta}, \alpha_3 = \sqrt{\frac{2\beta(\beta+1)}{\beta'}}$, with $\beta' = \frac{3}{2}\beta + 2$. To complete the proof of the second equality, by Lemma 2.12 we only have to verify $\frac{\tilde{\eta}}{\eta} = \frac{a_2}{a_3}$, where $\eta = -3\sqrt{2(\beta+1)(\beta+2)}, \tilde{\eta} = -2\sqrt{\beta'}(\beta'+1)$ and $\alpha_1, \alpha_2, \alpha_3$ as above. Now $\frac{a_2}{a_3} = \sqrt{\frac{\beta\beta'}{2(\beta+1)}} = \frac{3\beta + 4}{2\sqrt{3(\beta+1)(\beta+2)}} = \frac{3\beta + 4}{2\beta}. \frac{1}{2} \sqrt{\frac{3\beta + 4}{\beta} \beta'}$. Hence we have proved the lemma.

As the operators $M_1$ and $M_2$ are homogeneous, Combing Lemma 2.13 with Theorem 2.6 we have the following

**Corollary 2.14.** There exists a unitary operator $U_\varphi$ such that $U_\varphi^{-1}K_1(z)U_\varphi = K_2(z)$ and $U_\varphi^{-1}(K_1)_{z}(z)U_\varphi = (K_2)_{z}(z)$ for $z$ in $\mathbb{D}$, where $\varphi = \varphi_{t,z}$ in Möb for $(t, z) \in T \times \mathbb{D}$.

**Lemma 2.15.** If $\beta' = \frac{3}{2}\beta + 2$ then $(K_1)_{z}(0)$ and $(K_2)_{z}(0)$ are not unitarily equivalent.

**Proof.** By Lemma 2.10 $(K_1)_{z}(0)$ is $\text{diag}(p_1, q_1, r_1)$ for $i = 1, 2$, where $p_1 = \alpha, q_1 = \alpha + \beta'(\beta' + 1), r_1 = \alpha + \beta'(-\beta' + 1)$ and $p_2 = \alpha, q_2 = \alpha + 3(\beta + 1)(\beta + 2), r_2 = \alpha - 3\beta(\beta + 2)$. As $\beta' = \frac{3}{2}\beta + 2, q_1 = \alpha + \frac{3}{2}(\beta + 2)(3\beta + 4)$ and $r_1 = \alpha - \frac{3}{4}(3\beta + 2)(3\beta + 4)$. So clearly $p_1 = p_2, q_1 > r_1$ and $q_2 > r_2$ As $(K_1)_{z}(0)$ and $(K_2)_{z}(0)$ are diagonal matrices, they are unitarily equivalent if and only if $p_1 = p_2, q_1 = q_2$ and $r_1 = r_2$. We see that $q_1 \neq q_2$ and $r_1 \neq r_2$, hence $(K_1)_{z}(0)$ and $(K_2)_{z}(0)$ are not unitarily equivalent.
The proof of (a) is now complete since the matrices $\lambda$ of the matrix $M_1^*$ is reducible whereas the other $M_2^*$ is irreducible. Irreducibility of $\tilde{M}^*$ and $M_2^*$ follows from \cite{12}. We are interested in constructing such examples within the class of irreducible operators in $B_\mathbb{D}$. The class of irreducible homogeneous operators in $B_\mathbb{D}$ cannot possibly possess such examples. Therefore, we consider a class of homogeneous operators in $B_3(\mathbb{D})$ discussed in \cite{11}.

Let $\lambda$ be a real number and $m$ be a positive integer such that $2\lambda - m > 0$. For brevity, we will write $2\lambda_j = 2\lambda - m + 2j$, $0 \leq j \leq m$. Let

$$L(\lambda)_{\ell j} = \begin{cases} \frac{(\ell - j)!}{(2\lambda)_{\ell-j}}, & \text{for } 0 \leq j \leq \ell \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

and $B = \text{diag}(d_0, d_1, \ldots, d_m)$. Now consider for $\mu = (\mu_0, \ldots, \mu_m)$ with $\mu_0 = 1$ and $\mu_\ell > 0$ for $\ell = 1, \ldots, m$

$$K(\lambda, \mu)(z, w) = (1 - z\bar{w})^{-2\lambda - m}D(z\bar{w}) \exp(\bar{w}S_m)B \exp(zS_m^*)D(z\bar{w}),$$

where $B_{\ell,\ell} = d_\ell = \sum_{j=0}^{\ell} \frac{(\ell-j)!}{(2\lambda)_{\ell-j}} \mu_j^2$ for $0 \leq \ell \leq m$, that is, $L(\lambda)\mu' = d$ for $\mu' = (\mu_0^2, \mu_1^2, \ldots, \mu_m^2)^t$ and $d = (d_0, d_1, \ldots, d_m)^t$. $D(z\bar{w}) = (1 - z\bar{w})^{m-t}\delta_{\ell,0}$ is diagonal and $S_m$ is the forward shift with weight sequence $\{1, \ldots, m\}$, that is, $(S_m)_{\ell,0} = \ell \delta_{p+1, \ell}$, $0 \leq p, \ell \leq m$. $X^t$ denotes the transpose of the matrix $X$. $K(\lambda, \mu)$ is the reproducing kernel for the Hilbert space $A(\lambda, \mu)(\mathbb{D})$ of $C^{m+1}$-valued holomorphic functions described in \cite{11}. Let $M(\lambda, \mu)$ denote the multiplication operator on the Hilbert space $A(\lambda, \mu)(\mathbb{D})$. In \cite{11} it is shown that $M(\lambda, \mu)$ is homogeneous and irreducible, moreover, $M'(\lambda, \mu)^* = B_{m+1}(\mathbb{D})$.

**Lemma 3.1.** For the reproducing kernel $K(\lambda, \mu)$

(a) $\tilde{a}_{11} = [B^{-1}S_mB, S_m^*] + (2\lambda + m)I_{m+1} - 2D_m$,

(b) $\tilde{a}_{12} = B^{1/2}((\frac{1}{2}(B^{-1}S_mB^*B^{-1}S_m^*) + S_mB^*B^{-1}S_m^2) + B^{-1}[D_m, S_m] - B^{-1}S_mB^*B^{-1}S_mB)B^{1/2}$.

where $I_k$ denotes the identity matrix of order $k$ and $D_m = \text{diag}(m, \ldots, 1, 0)$.

**Proof.** From Equation (2.5) in Lemma 2.10 we get $\tilde{a}_{11} = a_{00}^{-1/2}(a_{11} - a_{10}a_{00}^{-1}a_{01})a_{00}^{-1/2}$. Form the expansion of the reproducing kernel $K(\lambda, \mu)$ we see that $a_{00} = B$, $a_{10} = BS_m^*$, $a_{01} = S_mB$, $a_{11} = S_mB^*B + (2\lambda + m)B - 2D_mB$. So, $a_{11} - a_{10}a_{00}^{-1}a_{01} = S_mB^*B + (2\lambda + m)B - 2D_mB - BS_m^*B^{-1}S_mB$. The proof of (a) is now complete since the matrices $S_mB^*S_m^*, S_mB^{-1}S_m^*, B, B^{1/2}, B^{-1/2}$ are diagonal.

From Lemma 2.5, we have $\tilde{a}_{12} = a_{00}^{1/2}(b_{00}a_{11}b_{01} + b_{00}a_{12}b_{00} - b_{10}a_{00}b_{02})a_{00}^{1/2}$. Again, from the expansion of the reproducing kernel $K(\lambda, \mu)$ it is easy to see that $a_{12} = \frac{1}{2}S_mB^*S_m^* + (2\lambda + m)S_mB - D_mS_mB - S_mBD_m$, $b_{00} = B^{-1}$, $b_{10} = -S_mB^{-1}$, $b_{02} = \frac{1}{2}B^{-1}S_m^2$. To complete the proof of (b), it is enough to note that two diagonal matrices $B$ and $D_m$ commute. □

It will be convenient to let $K_{\lambda, \mu}$ denote the curvature $K_{\tilde{h}}(z) = \frac{\partial}{\partial z}(\tilde{h}^{-1}\frac{\partial}{\partial z}\tilde{h})(z)$, where $\tilde{h}(z) = \tilde{K}(\lambda, \mu)(z, z)^t$ for $z$ in $\mathbb{D}$. Recall that $\tilde{K}(\lambda, \mu)$ is the normalized reproducing kernel obtained from the reproducing kernel $K(\lambda, \mu)$. Now we specialize to the case $m = 2$. 

3. **Irreducible Examples and Permutation of Curvature Eigenvalues**

In the first example constructed above one of the two homogeneous operators $M^*$ is reducible while the other $\tilde{M}^*$ is irreducible. Similarly in the second example one of the two operators $M_1^*$ is reducible whereas the other $M_2^*$ is irreducible. Irreducibility of $\tilde{M}^*$ and $M_2^*$ follows from \cite{12}.
Lemma 3.2. The curvature at zero $K_{\lambda,\mu}(0)$ and the covariant derivative of curvature at zero $(K_{\lambda,\mu})_{z}(0)$ are

(a) $K_{\lambda,\mu}(0) = \text{diag} \left( a - b - 2, a + b - c, a + c + 2 \right)$,
(b) $(K_{\lambda,\mu})_{z}(0) = 2S_{2}(−\sqrt{b(1 + b - \frac{c}{2})}, -\sqrt{c(1 + c - \frac{b}{2})})^{t}$, where $a = 2\lambda, b = d_{1}^{-1}, c = 4d_{1}d_{2}^{-2}$.

Proof. $B^{-1}S_{2}B_{S}^{*} = \text{diag}(0, d_{1}^{-1}, 4d_{1}d_{2}^{-2})$, and $S_{2}B^{-1}S_{2}B = \text{diag}(d_{1}^{-1}, 4d_{1}d_{2}^{-2}, 0)$. Therefore by Lemma 3.1(a) we see that $\tilde{a}_{11} = \text{diag} \left( 2\lambda - d_{1}^{-1}, 2\lambda + d_{1}^{-1} - 4d_{1}d_{2}^{-2}, 2\lambda + 4d_{1}d_{2}^{-2} + 2 \right) = \text{diag} \left( a - b - 2, a + b - c, a + c + 2 \right)$. Hence $K_{\lambda,\mu}(0) = \tilde{a}_{11}^{t} = \text{diag} \left( a - b - 2, a + b - c, a + c + 2 \right)$.

For (b) we note that $B^{-1}S_{2}B_{S}^{*} = S_{2}(0, 2d_{1}^{-1}d_{2}^{-2}), S_{2}B^{-1}S_{2}^{2} = S_{2}(4d_{2}^{-1}, 0), B^{-1}[D_{2}, S_{2}] = S_{2}(-d_{1}^{-1}, -2d_{2}^{-2}), B^{-1}S_{2}B_{S}^{*}B_{S}^{*}B_{S} = S_{2}(d_{1}^{-} - 2d_{1}d_{2}^{-2})$ we have the desired conclusion from Lemma 2.3 and Lemma 3.1(b). □

If $\delta_{1}, \delta_{2}, \delta_{3}$ are the eigenvalues of $K(0)$ then we know from [4, Proposition 2.20] that $\delta_{i} > 0$ for $i = 1, 2, 3$, is a fixed ordered triple of positive numbers. Then there exists $K^{(\lambda,\mu)}$ with $\lambda > 1$ and $\mu_{\ell} > 0$ for $\ell = 1, 2$ such that $K_{\lambda,\mu}(0) = \text{diag}(\delta_{1}, \delta_{2}, \delta_{3})$, only if $\delta_{i}$’s satisfy the inequalities of Lemma 3.5 below.

Remark 3.3. We emphasize that the reproducing kernel $K^{(\lambda,\mu)}$ is computed from an ordered basis, that is, $K^{(\lambda,\mu)}(w, w) = \left( \langle \gamma_{i}(w), \gamma_{j}(w) \rangle \right)^{\frac{3}{2}, i,j=1}$, where $\{\gamma_{1}(w), \gamma_{2}(w), \gamma_{3}(w)\}$ is an ordered basis. Consequently, the eigenvalues of $K_{\lambda,\mu}(0)$, which is diagonal, appear in a fixed order. If one considers $\{\gamma_{1}(w), \gamma_{2}(w), \gamma_{3}(w)\}$, it will give rise to a different reproducing kernel $P_{\sigma}K^{(\lambda,\mu)}P_{\sigma}^{*}$, say $K_{\lambda,\mu}^{(\sigma)}$, where $\sigma \in S_{3}$, $S_{3}$ denotes the symmetric group of degree 3 and

$$(P_{\sigma})_{i,j} = \begin{cases} 1, & \text{for } (i, j) = (i, \sigma(i)) \; \text{and} \\ 0, & \text{otherwise}. \end{cases}$$

Hence, $K_{\lambda,\mu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$, where $h_{\sigma}(z) = \tilde{K}_{\lambda}(z, z)^{t}$. It follows that the curvature of the corresponding bundle as a matrix depends on the choice of the particular ordered basis. The set of eigenvalues of curvature at 0, which is diagonal in our case, will be thought of as an ordered tuple, namely the ordered set of diagonal elements of $K_{\lambda,\mu}(0)$.

Lemma 3.4. $(\lambda, \mu) = (\lambda', \nu')$ if and only if $(a, b, c) = (a', b', c')$ as ordered tuples, where $\mu = (1, \mu_{1}, \mu_{2}), \nu = (1, \nu_{1}, \nu_{2}), \mu_{\ell}, \nu_{i} > 0$ for $\ell = 1, 2, d = (1, d_{1}, d_{2})^{t}, d' = (1, d_{1}', d_{2}')^{t}$; for $2\gamma_{j} = 2\gamma - 2 + 2j, \gamma = \lambda, \lambda'$

$L(\gamma)_{\ell j} = \begin{cases} \left( \frac{\ell}{\gamma} \right)^{2} \left( \frac{\ell - j}{\ell \gamma} \right), & \text{for } 0 \leq j \leq \ell \leq m; \\ 0, & \text{otherwise}. \end{cases}$

$d = L(\lambda)\mu', d' = L(\lambda')\nu', \mu'(1, \mu_{1}^{2}, \mu_{2}^{2})^{t}, \nu'(1, \nu_{1}^{2}, \nu_{2}^{2})^{t} 0 \leq j \leq i \leq 2, a = 2\lambda, b = d_{1}^{-1}, c = 4d_{1}d_{2}^{-1}, a' = 2\lambda', b' = d_{1}'^{-1}, c' = 4d_{1}'d_{2}'^{-1}.$

Proof. One implication is clear, so prove the other implication. $a = a'$ implies that $\lambda = \lambda', b = b'$ and $c = c'$ imply that $d = d'$. Now invertibility of $L(\lambda)$ implies that $\mu' = \nu'$, that is, $\mu = \nu$. □

Lemma 3.5. If $(\delta_{1}, \delta_{2}, \delta_{3})$ is an ordered tuple of positive numbers such that $K_{\lambda,\mu}(0) = \text{diag} \left( \delta_{1}, \delta_{2}, \delta_{3} \right)$ then

$$\delta_{1} + \delta_{2} + \delta_{3} > 6$$

$$\delta_{2} + \delta_{3} - 2\delta_{1} > 6$$

$$2\delta_{3} - \delta_{1} - \delta_{2} > 6.$$
Equivalently, \( Ax = b \), where \( A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \), \( x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \), \( b = \begin{pmatrix} \delta_1 + 2 \\ \delta_2 \\ \delta_3 - 2 \end{pmatrix} \). Clearly, this system of linear equations admits \( x = \frac{1}{3} \begin{pmatrix} \delta_1 + 2 + 2 \delta_3 - 2 \delta_2 - 6 \\ \delta_2 + 2 \delta_3 - 2 \delta_2 - 6 \\ \delta_3 + 2 \delta_2 - 2 \delta_3 - 6 \end{pmatrix} \) as the only solution. Since \( a = 2 \lambda, b = d_1^{-1}, c = 4d_1d_2^{-1} \), it follows that a necessary conditions for \( \delta_1, \delta_2, \delta_3 \) to be eigenvalues of \( K_{\lambda, \mu}(0) \) is the inequalities in the statement of the Lemma.

**Corollary 3.6.** Suppose \( K^{(\lambda, \mu)} \) and \( K^{(\lambda', \nu)} \) are such that \( K_{\lambda, \mu}(0) = K_{\lambda', \nu}(0) \) as matrices, where \((\lambda, \mu), (\lambda', \nu)\) as are in Lemma 3.4. Then \((\lambda, \mu) = (\lambda', \nu)\) as ordered tuples.

**Proof.** Let \( K_{\lambda, \mu}(0) = K_{\lambda', \nu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3) \). Consider the system of linear equations \( Ax = b \) and \( Ax' = b' \), where \( A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \), \( x = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \), \( x' = \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \), \( b, c \) and \( a', b', c' \) are as in Lemma 3.4. As \( \det A = 3 \), \( A \) is invertible. Hence \( (a, b, c) = (a', b', c') \) and by Lemma 3.4, \((\lambda, \mu) = (\lambda', \nu)\).

Suppose \((\delta_1, \delta_2, \delta_3), \delta_i > 0 \) for \( i = 1, 2, 3, \) is given satisfying the inequalities above. Then let us find \( \lambda > 1, \mu_1, \mu_2 > 0 \) such that \( K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3) \) with \( \mu = (1, \mu_1, \mu_2) \). We know that

\[
L(\lambda)\mu' = d, \text{ so } \mu' = L(\lambda)^{-1}d = \begin{pmatrix} -\frac{1}{\lambda(\lambda-1)} & 0 & 0 \\ -\frac{1}{\lambda(\lambda-1)} & 1 & 0 \\ -\frac{1}{\lambda(\lambda-1)} & 0 & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} \frac{d_1}{\lambda} \\ d_2 - \frac{2d_1}{\lambda} \frac{1}{\lambda(\lambda-1)} \\ d_3 - \frac{2d_1}{\lambda} \frac{1}{\lambda(\lambda-1)} \end{pmatrix} 
\]

Thus \( \mu_1^2 = d_1 - \frac{1}{2(\lambda-1)} = \frac{1}{\lambda} - \frac{1}{2(\lambda-1)} = \frac{2(\lambda-1)-b}{2(\lambda-1)} \) and \( 2(\lambda-1) - b = \frac{\delta_1 + 2 \delta_3 - 2 \delta_2 - 6}{\delta_2 + 2 \delta_3 - 2 \delta_2 - 6} \).

Thus \( \mu_1 > 0 \). \( \mu_2^2 = d_2 - \frac{2d_1}{\lambda} + \frac{1}{\lambda(\lambda-1)} = \frac{4}{b} - \frac{2}{\lambda} + \frac{1}{\lambda(\lambda-1)} = \frac{2}{b} \left( \frac{1}{\lambda} - 1 \right) + \frac{1}{\lambda(\lambda-1)} = \frac{2(\lambda - c)(\lambda - 1) + bc}{2(\lambda - c)(\lambda - 1) + bc} \), where \( a, b, c \) are as in Lemma 3.2. Thus we have proved the following Theorem.

**Theorem 3.7.** There exists \( K^{(\lambda, \mu)} \) such that \( K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3), \delta_i > 0 \) for \( i = 1, 2, 3 \) if

\[
\begin{align*}
\delta_1 + \delta_2 + \delta_3 & > 6 \\
\delta_2 + \delta_3 - 2 \delta_1 & > 6 \\
2 \delta_3 - \delta_1 - 2 \delta_2 & > 6 \\
2(a - c)(a - 1) + bc & > 0
\end{align*}
\]

where \( a, b, c \) are as in Lemma 3.2.

**Proposition 3.8.** Suppose \( \delta_i > 0 \) for \( i = 1, 2, 3 \) are such that \( \delta_1 \neq \delta_2 \) and \( 2(\delta_1 + \delta_2) > \delta_3 - 6 > \max\{2\delta_1 - \delta_2, 2\delta_2 - \delta_1\} \). Then there exists reproducing kernels \( K^{(\lambda, \mu)} \) and \( K^{(\bar{\lambda}, \bar{\mu})} \) such that \( K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3) \) and \( K_{\bar{\lambda}, \bar{\mu}}(0) = \text{diag}(\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3) \), where \( \lambda, \bar{\lambda} > 1, \mu = (1, \mu_1, \mu_2), \bar{\mu} = (1, \bar{\mu}_1, \bar{\mu}_2) \), \( \mu_1, \bar{\mu}_1 > 0 \) for \( \ell = 1, 2, \).

**Proof.** Consider \((\delta_1, \delta_2, \delta_3), \delta_i > 0 \) for \( i = 1, 2, 3 \) such that there exists \( K^{(\lambda, \mu)} \) and \( K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3) \) for some \( \lambda > 1, \mu = (1, \mu_1, \mu_2) \) with \( \mu_1, \mu_2 > 0 \). So, \( \delta_1, \delta_2, \delta_3 \) satisfy the inequalities of Lemma 3.5. We now produce \( \bar{\lambda} > 1, \bar{\mu} = (1, \bar{\mu}_1, \bar{\mu}_2) \) with \( \bar{\mu}_1, \bar{\mu}_2 > 0 \) such that \( K_{\bar{\lambda}, \bar{\mu}}(0) = \text{diag}(\delta_2, \delta_1, \delta_3) \). We recall that \( K_{\bar{\lambda}, \bar{\mu}} \) is the curvature of the metric \( \bar{\lambda}, \bar{\mu}^2 \langle z, z \rangle \) and \( \bar{K}(\bar{\lambda}, \bar{\mu}) \) denotes the normalization of the reproducing kernel \( K^{(\bar{\lambda}, \bar{\mu})} \). By Lemma 3.2 and Remark 3.3 we need to consider the equations

\[
\begin{align*}
\bar{a} - \bar{b} - 2 &= \delta_2 \\
\bar{a} + \bar{b} - \bar{c} &= \delta_1 \\
\bar{a} + \bar{c} + 2 &= \delta_3 
\end{align*}
\]

where \( \bar{a} = 2 \bar{\lambda}, \bar{b} = \bar{d}_1^{-1}, \bar{c} = 4\bar{d}_1\bar{d}_2^{-1} \). This is same as \( A\bar{x} = \bar{b} \), where \( A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \), \( \bar{x} = \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{pmatrix} \), 

\( \bar{b} = \begin{pmatrix} \delta_1 + 2 \\ \delta_2 \\ \delta_3 - 2 \end{pmatrix} \). This system of linear equations has only one solution, namely, \( x = \frac{1}{3} \begin{pmatrix} \delta_1 + 2 + 2 \delta_3 - 2 \delta_2 - 6 \\ \delta_2 + 2 \delta_3 - 2 \delta_2 - 6 \\ \delta_3 + 2 \delta_2 - 2 \delta_3 - 6 \end{pmatrix} \).
We observe that $a = \hat{a}$ and $c = \hat{c}$ but $b \neq \hat{b}$ if $\delta_1 \neq \delta_2$. From Lemma 3.5 and Theorem 3.7 we know that there exists $K(\hat{\lambda}, \hat{\mu})$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ if
\[
\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_1 + \delta_3 - 2\delta_2 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
2(\hat{a} - \hat{c})(\hat{a} - 1) + \hat{b}\hat{c} &> 0.
\end{aligned}
\]
Hence there exists $K(\lambda, \mu)$ and $K(\hat{\lambda}, \hat{\mu})$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\lambda, \mu}(0) = \text{diag}(\delta_2, \delta_1, \delta_3)$ if
\[
\delta_1 + \delta_2 + \delta_3 > 6 \\
\delta_2 + \delta_3 - 2\delta_1 > 6 \\
2\delta_3 - \delta_1 - \delta_2 > 6 \\
2(\hat{a} - \hat{c})(\hat{a} - 1) + \hat{b}\hat{c} > 0.
\]
Suppose $\delta_i > 0$ for $i = 1, 2, 3$ are such that $\delta_1 \neq \delta_2$ and $2(\delta_1 + \delta_2) > \delta_3 - 6 > \max\{2\delta_1 - \delta_2, 2\delta_2 - \delta_1\}$. Then we observe that the last inequality implies that $\delta_2 + \delta_3 - 2\delta_1 > 6$ and $\delta_1 + \delta_3 - 2\delta_2 > 6$, adding these two inequalities we have $2\delta_3 - \delta_1 - \delta_2 > 12$. Also $a - c = \frac{1}{\delta}(\delta_1 + \delta_2 + \delta_3) - \frac{1}{\delta}(2\delta_3 - \delta_1 - \delta_2 - 6) = \frac{1}{\delta}(2(\delta_1 + \delta_2) - \delta_3) + 2$. As $a = \hat{a}$ and $c = \hat{c}$, $a - c = \hat{a} - \hat{c} > 0$ and the first inequality in the choice of $\delta_i$ for $i = 1, 2, 3$ implies that $a - c = \hat{a} - \hat{c} > 0$, so the last two inequalities are satisfied. The first inequality in the choice of $\delta_i$ for $i = 1, 2, 3$ also implies that $\delta_3 > 6$, so the first inequality follows. Hence all the required inequalities for the existence of $K(\lambda, \mu)$ and $K(\hat{\lambda}, \hat{\mu})$ are satisfied by this choice of $\delta_i > 0$ for $i = 1, 2, 3$.

**Proposition 3.9.** Suppose $\delta_i > 0$ for $i = 1, 2, 3$ are such that $\delta_3 > \delta_2 > 3 + \frac{\delta}{2}$ and $\delta_1 < \min\{2\delta_3 - \delta_2, 2\delta_2 - \delta_3\} - 6$. Then there exists reproducing kernels $K(\lambda, \mu)$ and $K(\hat{\lambda}, \hat{\mu})$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_3, \delta_2)$, where $\lambda, \hat{\lambda} > 1$, $\mu = (1, \mu_1, \mu_2)$, $\hat{\mu} = (1, \hat{\mu}_1, \hat{\mu}_2)$, $\mu_\ell, \hat{\mu}_\ell > 0$ for $\ell = 1, 2$.

**Proof.** We construct a reproducing kernel $K(\hat{\lambda}, \hat{\mu})$ such that $K_{\hat{\lambda}, \hat{\mu}}(0) = \text{diag}(\delta_1, \delta_3, \delta_2)$ for some $\hat{\lambda} > 1$, $\hat{\mu} = (1, \hat{\mu}_1, \hat{\mu}_2)$, $\hat{\mu}_\ell > 0$ for $\ell = 1, 2$. By Lemma 3.2 and Remark 3.3 we obtain $(\hat{a}, \hat{b}, \hat{c})$ from the following set of equations
\[
\begin{aligned}
\hat{a} - \hat{b} - 2 & = \delta_1 \\
\hat{a} + \hat{b} - \hat{c} & = \delta_3 \\
\hat{a} + \hat{c} + 2 & = \delta_2
\end{aligned}
\]
where $\hat{a} = 2\hat{\lambda}, \hat{b} = \frac{\delta}{\delta_1 - 1}, \hat{c} = 4\hat{\lambda} \frac{\delta}{\delta_1 - 1}$. This is same as $A\hat{x} = \hat{b}$, where $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, $\hat{x} = \begin{pmatrix} \frac{\delta}{\delta_1 - 1} \\ 0 \\ \frac{\delta}{\delta_1 - 1} \end{pmatrix}$, $\hat{b} = \begin{pmatrix} \delta_3 + 2 \\ 2 \delta_3 - 2 \delta_1 - 6 \\ 2 \delta_2 - \delta_1 - \delta_3 - 6 \end{pmatrix}$. The vector $x = \frac{1}{\delta}(\delta_1 + \delta_2 + \delta_3 - 6)$ is the only solution of this system of equations. From Lemma 3.5 and Theorem 3.7 we know that there exists $K(\hat{\lambda}, \hat{\mu})$ such that $K_{\hat{\lambda}, \hat{\mu}}(0) = \text{diag}(\delta_1, \delta_3, \delta_2)$ if
\[
\begin{aligned}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_2 + \delta_3 - 2\delta_1 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
2(\hat{a} - \hat{c})(\hat{a} - 1) + \hat{b}\hat{c} &> 0.
\end{aligned}
\]
If $(\delta_1, \delta_2, \delta_3)$, $\delta_i > 0$ for $i = 1, 2, 3$ are such that there exists $K(\lambda, \mu)$ and $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$. Then $\delta_i$’s for $i = 1, 2, 3$ satisfies the inequalities of Lemma 3.5. Hence there exists $K(\lambda, \mu)$ and $K(\hat{\lambda}, \hat{\mu})$. 


such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\tilde{\lambda}, \tilde{\mu}}(0) = \text{diag}(\delta_1, \delta_3, \delta_2)$ if

\[
\begin{align*}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_1 + \delta_2 - 2\delta_1 &> 6 \\
2\delta_2 - \delta_1 - \delta_3 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
2(\hat{a} - \hat{c})(\hat{a} - 1) + bc &> 0 \\
2(a - c)(a - 1) + \hat{b}\hat{c} &> 0
\end{align*}
\]

We observe that $a = \hat{a}$ and $b = \hat{b}$ but $c \neq \hat{c}$ if $\delta_2 \neq \delta_3$. Suppose $\delta_i > 0$ for $i = 1, 2, 3$ are such that $\delta_3 > \delta_2 > 3 + \frac{\delta_1}{2}$ and $\delta_1 < \min\{2\delta_3 - \delta_2, 2\delta_2 - \delta_3\} - 6$. Now the first inequality implies that $\delta_3 > 6$, hence the first inequality is satisfied. The last inequality implies that $2\delta_3 - \delta_1 - \delta_2 > 6$ and $2\delta_2 - \delta_1 - \delta_3 > 6$, adding these two inequalities we have $\delta_2 + \delta_3 - 2\delta_1 > 12$. So the first four out of the set of six inequalities are satisfied. The second, third and the second, fourth from the set of the six inequalities respectively imply that $\delta_3 - \delta_1 > 4$ and $\delta_2 - \delta_1 > 4$. An easy computation involving the expressions for $a, b, c$ and $\hat{a}, \hat{b}, \hat{c}$ in terms of $\delta_i$ for $i = 1, 2, 3$ shows that $2(\hat{a} - \hat{c})(\hat{a} - 1) + bc > 0$ and $2(a - c)(a - 1) + bc > 0$ are equivalent to $(\delta_1 + \delta_2)(2\delta_1 + \delta_2) + \delta_3(\delta_2 - \delta_1) + 6\delta_1 > 0$ and $(\delta_1 + \delta_3)(2\delta_1 + \delta_3) + \delta_2(\delta_3 - \delta_1) + 6\delta_1 > 0$. These are satisfied as $\delta_2 - \delta_1 > 4$ and $\delta_3 - \delta_1 > 4$. Hence all the required inequalities for the existence of $K(\lambda, \mu)$ and $K(\tilde{\lambda}, \tilde{\mu})$ are satisfied by this choice of $\delta_i > 0$ for $i = 1, 2, 3$. \qed

**Remark 3.10.** The set $\{\delta_1 > 0 : i = 1, 2, 3\}$ satisfying the inequalities of Proposition 3.8 is non-empty. For instance, take $\delta_1 = 1$, $\delta_2 = 2$ and any $\delta_3$ in the interval $(9, 12)$. Then $\{\delta_1, \delta_2, \delta_3\}$ meets the requirement. Similarly, taking any $\delta_1$ in the interval $(0, 1)$, $\delta_2 = 7.5$ and $\delta_3 = 8$, we find that $\{\delta_1, \delta_2, \delta_3\}$ satisfies the inequalities prescribed in Proposition 3.9. Thus the two sets which are obtained from Propositions 3.8 and 3.9 are not identical.

**Corollary 3.11.** In Proposition 3.8 and Proposition 3.9, $(\lambda, \mu) \neq (\tilde{\lambda}, \tilde{\mu})$ and $(\lambda, \mu) \neq (\hat{\lambda}, \hat{\mu})$.

**Proof.** By Lemma 3.4 it suffices to show that $(a, b, c) \neq (\hat{a}, \hat{b}, \hat{c})$ and $(a, b, c) \neq (\tilde{a}, \tilde{b}, \tilde{c})$. In Proposition 3.8 $b \neq \hat{b}$ as $\delta_1 \neq \delta_2$ and in Proposition 3.9 $c \neq \tilde{c}$ as $\delta_2 \neq \delta_3$. \qed

Recall that $M(\lambda', \nu)$ denotes the multiplication operator on the reproducing kernel Hilbert spaces with reproducing kernel $K(\lambda', \nu)$.

**Theorem 3.12** ([11], Theorem 6.2). The reproducing kernels $K(\lambda, \mu)$ and $K(\lambda', \nu)$ are equivalent that is, the multiplication operators $M(\lambda, \mu)$ and $M(\lambda', \nu)$ are unitarily equivalent if and only if $(\lambda, \mu) = (\lambda', \nu)$.

**Corollary 3.13.** Suppose that $K(\lambda, \mu)$, $K(\tilde{\lambda}, \tilde{\mu})$ and $K(\lambda, \mu)$, $K(\tilde{\lambda}, \tilde{\mu})$ are as in Proposition 3.8 and Proposition 3.9 respectively. Then

(a) the multiplication operators $M(\lambda, \mu)$ and $M(\tilde{\lambda}, \tilde{\mu})$ are not unitarily equivalent.

(b) the multiplication operators $M(\lambda, \mu)$ and $M(\tilde{\lambda}, \tilde{\mu})$ are not unitarily equivalent.

**Proof.** Proof follows immediately from Theorem 3.12 and Corollary 3.11. \qed

**Remark 3.14.** In Proposition 3.8 and Proposition 3.9, we have shown the following: Given a reproducing kernel $K(\lambda, \mu)$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ there exists a reproducing kernel $K(\tilde{\lambda}, \tilde{\mu})$ with $(\lambda, \mu) \neq (\tilde{\lambda}, \tilde{\mu})$ such that $K_{\tilde{\lambda}, \tilde{\mu}}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and given a reproducing kernel $K(\lambda, \mu)$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ there exists a reproducing kernel $K(\tilde{\lambda}, \tilde{\mu})$ with $(\lambda, \mu) \neq (\tilde{\lambda}, \tilde{\mu})$ such that $K_{\tilde{\lambda}, \tilde{\mu}}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$, where $\rho, \tau \in \Sigma_3$ with $\rho(1) = 2, \rho(2) = 1, \rho(3) = 3$, $\tau(1) = 1, \tau(2) = 3, \tau(3) = 2$. In the next Proposition we prove that there does not exist $K(\lambda', \nu)$ with $(\lambda, \mu) \neq (\lambda', \nu)$ such that $K_{\lambda', \nu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ if $\sigma \in \Sigma_3$ and $\sigma \neq \rho, \tau$. Obviously, there
exists $K^{(\ell, \mu)}_\sigma := P_{\sigma}K^{(\ell, \mu)}P_{\sigma}^*$ such that $K_{h_\sigma}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ for all $\sigma \in \Sigma_3$, where $P_{\sigma}$ is in $M(3, \mathbb{C})$ such that

$$(P_{\sigma})_{ij} = \begin{cases} 
1, & \text{for } (i, j) = (i, \sigma(i)), \\
0, & \text{otherwise};
\end{cases}$$

and $h_\sigma(z) = \tilde{K}_\sigma^{(\ell, \mu)}(z, z)^t$. As the reproducing kernels $K^{(\ell, \mu)}$ and $K^{(\ell, \mu)}_\sigma$ are equivalent, that is, the multiplication operators on the reproducing kernel Hilbert spaces with reproducing kernels $K^{(\ell, \mu)}$ and $K^{(\ell, \mu)}_\sigma$ are unitarily equivalent, we do not distinguish between them.

**Proposition 3.15.** Given a reproducing kernel $K^{(\ell, \mu)}$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ there does not exist reproducing kernel $K^{(\lambda', \nu)}$ with $(\lambda, \mu) \neq (\lambda', \nu)$ such that $K_{\lambda', \nu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ if $\sigma \in \Sigma_3$ and $\sigma \neq \rho, \tau$.

**Proof.** Case 1. Pick $\sigma \in \Sigma_3$ such that $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 1$.

The existence of two reproducing kernels $K^{(\ell, \mu)}$ and $K^{(\lambda', \nu)}$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\lambda', \nu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ would imply, by an application of Lemma 3.5 to the ordered triples $(\delta_1, \delta_2, \delta_3)$ and $(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$,

$$
\begin{align*}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_2 + \delta_3 - 2\delta_1 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
\delta_{\sigma(1)} + \delta_{\sigma(2)} + \delta_{\sigma(3)} &> 6 \\
\delta_{\sigma(2)} + \delta_{\sigma(3)} - 2\delta_{\sigma(1)} &> 6 \\
2\delta_{\sigma(3)} - \delta_{\sigma(1)} - \delta_{\sigma(2)} &> 6.
\end{align*}
$$

This set of inequalities are equivalent to

$$
\begin{align*}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_2 + \delta_3 - 2\delta_1 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
\delta_1 + \delta_2 - 2\delta_3 &> 6 \\
2\delta_1 - \delta_2 - \delta_3 &> 6.
\end{align*}
$$

Adding the third and the fourth from these inequalities gives $0 > 12$.

Case 2. Choose $\sigma \in \Sigma_3$ such that $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$.

As in the first case the existence of two reproducing kernels $K^{(\ell, \mu)}$ and $K^{(\lambda', \nu)}$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\lambda', \nu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ would imply, by an application of Lemma 3.5 to the ordered triples $(\delta_1, \delta_2, \delta_3)$ and $(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)}) (= (\delta_2, \delta_3, \delta_1))$,

$$
\begin{align*}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_2 + \delta_3 - 2\delta_1 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
\delta_1 + \delta_3 - 2\delta_2 &> 6 \\
2\delta_1 - \delta_2 - \delta_3 &> 6.
\end{align*}
$$

Adding second and fifth of these inequalities gives $0 > 12$.

Case 3. Take $\sigma \in \Sigma_3$ such that $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 2$.

Finally, continuing in the same manner in the previous two cases, the existence of two reproducing kernels $K^{(\ell, \mu)}$ and $K^{(\lambda', \nu)}$ such that $K_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$ and $K_{\lambda', \nu}(0) = \text{diag}(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)})$ would imply, by an application of Lemma 3.5 to the ordered triples $(\delta_1, \delta_2, \delta_3)$ and $(\delta_{\sigma(1)}, \delta_{\sigma(2)}, \delta_{\sigma(3)}) (= (\delta_3, \delta_1, \delta_2))$,

$$
\begin{align*}
\delta_1 + \delta_2 + \delta_3 &> 6 \\
\delta_2 + \delta_3 - 2\delta_1 &> 6 \\
2\delta_3 - \delta_1 - \delta_2 &> 6 \\
\delta_1 + \delta_2 - 2\delta_3 &> 6 \\
2\delta_2 - \delta_3 - \delta_1 &> 6.
\end{align*}
$$
Adding third and fourth inequalities from this set of inequalities we have $0 > 12$.

**Corollary 3.16.** There does not exist any multiplication operator $M(X,\nu)$ not equivalent to $M(\lambda,\mu)$ other than $M(\lambda_0,\mu)$ or $M(\lambda_0,\bar{\mu})$ such that $K_{X,\nu}(0) = K_{\lambda,\mu}(0)$ as sets of positive numbers, where and $K(\lambda,\mu)$, $K(\lambda,\bar{\mu})$, $K(\lambda_0,\mu)$ are as in Proposition 3.8 and Proposition 3.9.

*Proof.* Combining Corollary 3.13, Theorem 3.12, Corollary 3.6 and Proposition 3.15, we obtain a proof of this corollary.

**Remark 3.17.** We discuss the case $m = 1$. Proceeding as in Lemma 3.2 we see that $K_{\lambda,\mu}(0) = \text{diag}(a - b - 1, a + b + 1)$, where $\lambda > 1/2$, $\mu = (1, \mu_1)$, $\mu_1 > 0$, $a = 2\lambda$, $b = d_1^{-1}$, $d_1$ is defined as before. If $K_{\lambda,\mu}(0) = \text{diag}(\delta_1, \delta_2)$, $\delta_i > 0$ for $i = 1,2$, for some $\lambda > 1/2$ and $\mu = (1, \mu_1)$, $\mu_1 > 0$. Then arguing as in Lemma 3.5 one notes that $a = 2\lambda = \frac{\delta_1 + \delta_2}{2}$, $b = \frac{\delta_2 - \delta_1 - 2}{2}$. As $a = 2\lambda > 1$ and $b = d^{-1} > 0$ it follows that $\delta_1 + \delta_2 > 2$ and $\delta_2 - \delta_1 > 2$ are necessary conditions for existence of a reproducing kernel $K(\lambda,\mu)$ such that $K_{\lambda,\mu}(0) = \text{diag}(\delta_1, \delta_2)$. If $\delta_i > 0$ for $i = 1,2$, proceeding as in Theorem 3.7, one observes that $\delta_2 - \delta_1 > 2$, $\delta_1 + \delta_2 > 2$ and $d_1 > \frac{1}{2\delta_1 - 1} = \frac{2}{\delta_1 + \delta_2 - 2}$ are the sufficient conditions for existence of a reproducing kernel $K(\lambda,\mu)$ such that $K_{\lambda,\mu}(0) = \text{diag}(\delta_1, \delta_2)$. Conversely, if $\delta_i > 0$ for $i = 1,2$ and $\delta_2 - \delta_1 > 2$ then clearly $\delta_1 + \delta_2 > 2$ and $d_1 = \frac{2}{\delta_2 - \delta_1 - 2} > \frac{2}{\delta_1 + \delta_2 - 2}$. So $\delta_i > 0$ for $i = 1,2$ and $\delta_2 - \delta_1 > 2$ are the necessary and sufficient conditions for the existence of reproducing kernel $K(\lambda,\mu)$ such that $K_{\lambda,\mu}(0) = \text{diag}(\delta_1, \delta_2)$.

**Remark 3.18.** If $\delta_i > 0$ for $i = 1,2$ and $\delta_2 - \delta_1 > 2$ there does not exist a reproducing kernel $K(X,\nu)$ other than $K(\lambda,\mu)$ (up to equivalence as discussed in Remark 3.14) such that $K_{X,\nu}(0) = \text{diag}(\delta_2, \delta_1)$. If $K(X,\nu)$ exists satisfying the above requirements then from Remark 3.17 we see that both of $\delta_2 - \delta_1 > 2$ and $\delta_1 - \delta_2 > 2$ have to be simultaneously satisfied. This is impossible as they imply $0 > 4$. Hence there does not exist inequivalent multiplication operators $M(\lambda,\mu)$ and $M(X,\nu)$ such that $K_{\lambda,\mu}(0) = K_{X,\nu}(0)$ as sets.

**Theorem 3.19.** Suppose that $K(\lambda,\mu)$, $K(\lambda,\bar{\mu})$, $K(\lambda_0,\mu)$ are as in Proposition 3.8 and Proposition 3.9 respectively. Then

(i) the multiplication operators $M(\lambda,\mu)$ and $M(\lambda,\bar{\mu})$ are not equivalent although $K_{\lambda,\mu}(z)$ and $K_{\lambda,\bar{\mu}}(z)$ are unitarily equivalent for $z \in \mathbb{D}$.

(ii) the multiplication operators $M(\lambda,\mu)$ and $M(\lambda,\bar{\mu})$ are not equivalent although $K_{\lambda,\mu}(z)$ and $K_{\lambda,\bar{\mu}}(z)$ are unitarily equivalent for $z \in \mathbb{D}$.

*Proof.* From 3.8 we see that the curvatures of the associated bundles have the same set of eigenvalues at zero namely, $\{\delta_1, \delta_2, \delta_3\}$. Since curvature is self-adjoint the set of eigenvalues is the complete set of unitary invariants for the curvature. So, $K_{\lambda,\mu}(0)$ and $K_{\lambda,\bar{\mu}}(0)$ are unitarily equivalent. As the operators $M(\lambda,\mu)$ and $M(\lambda,\bar{\mu})$ are homogeneous, by an application of Theorem 2.6 we see that $K_{\lambda,\mu}(z)$ and $K_{\lambda,\bar{\mu}}(z)$ are unitary equivalent for $z \in \mathbb{D}$. Now (i) follows from part (a) of Corollary 3.13. The proof of (ii) of this theorem is similar.

The proof of the next Theorem will be completed after proving a sequence of Lemmas. We omit the easy proof of the first of these lemmas.

**Theorem 3.20.** Suppose that $M(\lambda,\mu)$ and $M(X,\nu)$ are not unitarily equivalent and the two curvatures $K_{\lambda,\mu}(z)$ and $K_{X,\nu}(z)$ are unitarily equivalent for $z \in \mathbb{D}$. Then there does not exist any invertible matrix $L$ in $\mathcal{M}(3,\mathbb{C})$ satisfying $LK_{\lambda,\mu}(0) = K_{X,\nu}(0)L$ for which $L(K_{\lambda,\mu})_z(0) = (K_{X,\nu})_z(0)L$ also. In other words, the covariant derivative of order $(0,1)$ detects the inequivalence.

**Lemma 3.21.** Suppose that $\Delta = \{k_{ij}\}_{i,j=1}^n$, $\Delta_\sigma = \{k_{\sigma(i)\delta_{ij}}\}_{i,j=1}^n$, $k_i \neq k_j$ if $i \neq j$ and $C = \{c_{ij}\}_{i,j=1}^n$ in $\mathcal{M}(n,\mathbb{C})$ is such that $C\Delta = \Delta_\sigma C$. Then $c_{ij} = c_{ij}\delta_{\sigma(i),j}$ for $i, j = 1, \ldots, n$, where $\sigma$ is in $S_n$, $S_n$ denotes the permutation group of degree $n$. 
Corollary 3.22. Suppose that there exists reproducing kernels $K^{(\lambda, \mu)}$, $K^{(\tilde{\lambda}, \tilde{\mu})}$ such that $\mathcal{K}_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$, $\mathcal{K}_{\tilde{\lambda}, \tilde{\mu}}(0) = \text{diag}(\delta_{\rho}(1), \delta_{\rho}(2), \delta_{\rho}(3))$, $\delta_1 \neq \delta_2$ and $C = (c_{ij})_{i,j=1}^{3}$ in $\mathcal{M}(3, \mathbb{C})$ is such that $\text{CK}_{\lambda, \mu}(0) = \mathcal{K}_{\tilde{\lambda}, \tilde{\mu}}(0)C$, then $c_{ij} = c_{ij} \delta_{\rho(i), j}$ for $i, j = 1, 2, 3$, where $\rho \in \Sigma_3$ is such that $\rho(1) = 2$, $\rho(2) = 1$, $\rho(3) = 3$.

Proof. The proof of this Corollary is immediate from Lemma 3.21 once we ensure that $\delta_1, \delta_2, \delta_3$ are distinct. Recalling notations from Lemma 3.2 we write $\delta_1 = a - b - 2$, $\delta_2 = a + b - c$, $\delta_3 = a + c + 2$. Clearly, $\delta_3 - \delta_1 = b + c + 4 > 0$. Recalling notations from Proposition 3.8 one has $\delta_2 = \tilde{a} - \tilde{b} - 2$, $\delta_1 = \tilde{a} + \tilde{b} - \tilde{c}$, $\delta_3 = \tilde{a} + \tilde{c} + 2$. So, $\delta_3 - \delta_2 = \tilde{b} + \tilde{c} + 4 > 0$. We have $\delta_3 > \delta_1$, $\delta_3 > \delta_2$ and $\delta_1 \neq \delta_2$ by hypothesis. Hence the proof is complete. \(\square\)

The proof of the next Corollary is similar.

Corollary 3.23. Suppose that there exists reproducing kernels $K^{(\lambda, \mu)}$, $K^{(\tilde{\lambda}, \tilde{\mu})}$ such that $\mathcal{K}_{\lambda, \mu}(0) = \text{diag}(\delta_1, \delta_2, \delta_3)$, $\mathcal{K}_{\tilde{\lambda}, \tilde{\mu}}(0) = \text{diag}(\delta_{\tau}(1), \delta_{\tau}(2), \delta_{\tau}(3))$, $\delta_2 \neq \delta_3$ and $C = (c_{ij})_{i,j=1}^{3}$ in $\mathcal{M}(3, \mathbb{C})$ is such that $\text{CK}_{\lambda, \mu}(0) = \mathcal{K}_{\tilde{\lambda}, \tilde{\mu}}(0)C$, then $c_{ij} = c_{ij} \delta_{\tau(i), j}$ for $i, j = 1, 2, 3$, where $\tau \in \Sigma_3$ is such that $\tau(1) = 1$, $\tau(2) = 3$, $\tau(3) = 2$.

Lemma 3.24. Suppose that $C = (c_{ij} \delta_{\sigma}(i), j)_{i,j=1}^{3}$ for $\sigma = \rho, \tau$ in $\Sigma_3$. Then $C$ is invertible if and only if $c_{i, \sigma(i)} \neq 0$ for $i = 1, 2, 3$ and $\sigma = \rho, \tau$ in $\Sigma_3$.

Proof. Since $\det C \neq 0$ if and only if $c_{i, \sigma(i)} \neq 0$ for $i = 1, 2, 3$ and $\sigma = \rho, \tau$ in $\Sigma_3$, the proof is complete. \(\square\)

The proof of the following Lemma is straightforward. We recall that $(S_m(c_1, \ldots, c_m))_{\ell,p} = c_{\ell,p+1,1}$, $0 \leq p, \ell \leq m$.

Lemma 3.25. Suppose that $C = (c_{ij} \delta_{\sigma(i), j})_{i,j=1}^{3}$, $c_{i, \sigma(i)} \neq 0$ for $i = 1, 2, 3$ and $\sigma = \rho, \tau$ in $\Sigma_3$ is such that $CS_2(c_1, c_2)^{\dagger} = S_2(\tilde{c}_1, \tilde{c}_2)^{\dagger}C$ for $c_i, \tilde{c}_i \in \mathbb{C}$ for $i = 1, 2$. Then $c_i = \tilde{c}_i = 0$ for $i = 1, 2$.

Lemma 3.26. $(\mathcal{K}_{\lambda, \mu})_{\bar{z}}(0)$ is not the zero matrix.

Proof. If possible let $(\mathcal{K}_{\lambda, \mu})_{\bar{z}}(0) = 0$. Then it follows from Lemma 3.2 that $-\sqrt{b}(1 + b - \frac{c}{2}) = -\sqrt{c}(1 + c - \frac{b}{2}) = 0$. Equivalently, $1 + b - \frac{c}{2} = 1 + c - \frac{b}{2}$, as $b$ and $c$ are positive. This implies that $b = c$. So, $(\mathcal{K}_{\lambda, \mu})_{\bar{z}}(0) = 0$ implies by an application of Lemma 3.2 that $-\sqrt{b}(1 + \frac{b}{2}) = 0$, which is impossible as $b$ is positive. \(\square\)

Proof of Theorem 3.20: We observe by applying Proposition 3.8, Proposition 3.9 and Proposition 3.15 that if $M_{\lambda, \mu}$ is an multiplication operator not unitarily equivalent to $M_{\lambda', \mu'}$ then $(\lambda', \mu') = (\tilde{\lambda}, \tilde{\mu})$. We reach the conclude by an straight forward application of Corollary 3.22, Corollary 3.23, Lemma 3.24, Lemma 3.25 and Lemma 3.26. \(\square\)

The comments below are for the class of homogeneous operators constructed in [11].

Remark 3.27. Unfortunately, although we are able to carry out similar calculations for these operators of rank $\geq 4$, it is not clear, if this would completely answer the question raised in [6, page 39]. Indeed, for rank 3, we have shown that the simultaneous unitary equivalence class of the curvature at 0 along with the covariant derivative of curvature at 0 of order $(0, 1)$ determines the unitary equivalence class of these operators. Similarly, for rank 2, the unitary equivalence class of the curvature at 0 determines the unitary equivalence class of the operator.

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Indian Statistical Institute, R. V. College Post, Bangalore 560 059,
E-mail address, Gadadhar Misra: gm@isibang.ac.in
E-mail address, Subrata Shyam Roy: ssroy@isibang.ac.in