Solving systems of diagonal polynomial equations over finite fields

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Abstract
We present an algorithm to solve a system of diagonal polynomial equations over finite fields when
the number of variables is greater than some fixed polynomial of the number of equations whose degree
depends only on the degree of the polynomial equations. Our algorithm works in time polynomial in the
number of equations and the logarithm of the size of the field, whenever the degree of the polynomial
equations is constant. As a consequence we design polynomial time quantum algorithms for two algebraic
hidden structure problems: for the hidden subgroup problem in certain semidirect product $p$-groups of
constant nilpotency class, and for the multi-dimensional univariate hidden polynomial graph problem
when the degree of the polynomials is constant.

Keywords: Algorithm, Polynomial equations, Finite fields, Chevalley–Warning theorem, Quantum com-
puting
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1 Introduction
Finding small solutions in some well defined sense for a system of integer linear equations is an important,
well studied, and computationally hard problem. Subset Sum, which asks the solvability of a single equation
in the binary domain is one of Karp’s original 21 NP-complete problems [18].
The guarantees of many lattice based cryptographic systems come from the average case hardness of
Short Integer Solution, dating back to Ajtai’s breakthrough work [2], where we try to find short nonzero
vectors in a random integer lattice. Indeed, this problem has a remarkable worst case versus average case
hardness property: solving it on the average is at least as hard as solving various lattice problems in the
worst case, such as the decision version of the shortest vector problem, and finding short linearly independent
vectors.

Turning back to binary solutions, deciding if there exists a nontrivial zero-one solution of the system of
linear equations

\[
a_{11}y_1 + \ldots + a_{1n}y_n = 0
\]
\[
\vdots \quad \vdots \\
\vdots \\
\]
\[
a_{m1}y_1 + \ldots + a_{mn}y_n = 0
\]

in the finite field $\mathbb{F}_q$, where $q$ is a power of some prime number $p$, is easy when $q = p = 2$. However, by
modifying the standard reduction of Satisfiability to Subset Sum [27] it can be shown that it is an NP-hard
problem for $q \geq 3$. 

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An extended abstract reporting on preliminary versions of the results has appeared in [16].
The system (1) is equivalent to the system of equations
\[ a_{11}x_1^{q-1} + \ldots + a_{1n}x_n^{q-1} = 0 \]
\[ \vdots \]
\[ a_{m1}x_1^{q-1} + \ldots + a_{mn}x_n^{q-1} = 0 \]
where we look for a nontrivial solution in the whole \( \mathbb{F}_q^n \).

In this paper we will consider finding a nonzero solution for a system of diagonal polynomial equations similar to (2), but where more generally, the variables are raised to some power \( d \geq 2 \). We state formally this problem.

**Definition 1.** The **System of Diagonal Equations** problem SDE is parametrized by a finite field \( \mathbb{F}_q \) and three positive integers \( n, m \) and \( d \).

\[
\text{SDE}(\mathbb{F}_q, n, m, d)
\]

*Input:* A system of polynomial equations over \( \mathbb{F}_q \):
\[
a_{11}x_1^d + \ldots + a_{1n}x_n^d = 0 \\
\vdots \\
a_{m1}x_1^d + \ldots + a_{mn}x_n^d = 0
\]

*Output:* A nonzero solution \( (x_1, \ldots, x_n) \neq \vec{0} \).

Here \( \vec{0} \) stands for the zero vector of length \( n \). (We will use this notation where we want to stress the distinction between the zero element of a field and the zero vector of a vector space.)

For \( j = 1, \ldots, n \), let us denote by \( v_j \) the column vector \( (a_{1j}, \ldots, a_{mj})^T \in \mathbb{F}_q^m \). Then the system of equations (3) is the same as
\[
\sum_{j=1}^{n} x_j^d v_j = \vec{0}.
\]

That is, solving SDE(\( \mathbb{F}_q, n, m, d \)) is equivalent to the task of representing the zero vector as a nontrivial linear combination of a subset of \( \{v_1, \ldots, v_n\} \) with \( d \)th power coefficients. We present our algorithm actually as solving this vector problem. The special case \( d = q - 1 \) is the vector zero sum problem where the goal is to find a non-empty subset of the given vectors with zero sum.

Under which conditions can we be sure that for system (3) there exists a nonzero solution? The elegant result of Chevalley [6] and Warning [29] states that the number of solutions of a general (not necessary diagonal) system of polynomial equations is a multiple of the characteristic \( p \) of \( \mathbb{F}_q \), whenever the number of variables is greater than the sum of the degrees of the polynomials. For diagonal systems (3) this means that when \( n > dm \), the existence of a nonzero solution is assured.

In general little is known about the complexity of finding another solution, given a solution of a system which satisfies the Chevalley-Warning condition. When \( q = 2 \), Papadimitriou has shown [22] that this problem is in the complexity class Polynomial Parity Argument (PPA), the class of NP search problems where the existence of the solution is guaranteed by the fact that in every finite graph the number of vertices with odd degree is even. This implies that it cannot be NP-hard unless \( \text{NP} = \text{co-NP} \). It is also unlikely that the problem is in P since Alon has shown [3] that this would imply that there are no one-way permutations.

Let us come back to our special system of equations (3). In the case \( m = 1 \), a nonzero solution can be found in polynomial time for a single equation which satisfies the Chevalley condition due to the remarkable work of van de Woestijne [28] where he proves the following.

**Fact 2.** In deterministic polynomial time in \( d \) and \( \log q \) we can find a nontrivial solution for
\[
a_1x_1^d + \ldots + a_{d+1}x_{d+1}^d = 0.
\]
In the case of more than one equation we don’t know how to find a nonzero solution for system (3) under just the Chevalley condition. However, if we relax the problem, and take much more variables than are required for the existence of a nonzero solution, we are able to give a polynomial time solution. Using van de Woestijne’s result for the one dimensional case, a simple recursion based on reducing one big system with \( m \) equations into \( d + 1 \) subsystems with \( m - 1 \) equations shows that if \( n \geq (d + 1)^m \) then SDE(\( F_q, n, m, d \)) can be solved in deterministic polynomial time in \( n \) and \( \log q \). The time complexity of this algorithm is therefore polynomial for any fixed \( m \). The case when \( d \) is fixed and \( m \) grows appears to be more difficult. To our knowledge, the only existing result in this direction is the case \( d = 2 \) for which it was shown in the paper [15] by the authors and Sanselme that there exists a (randomized) algorithm that, when \( n = \Omega(m^2) \), solves SDE(\( F_q, n, m, d \)) in polynomial time in \( n \) and \( \log q \). In the main result of this paper we generalize this result by showing, for every constant \( d \), the existence of a deterministic algorithm that, for every \( n \) larger than some polynomial function of \( m \), solves SDE(\( F_q, n, m, d \)) in polynomial time in \( n \) and \( \log q \).

**Theorem 3.** Let \( d \) be constant. For \( n > d^{d^2 \log d}(m + 1)^{d \log d} \), the problem SDE(\( F_q, n, m, d \)) can be solved in time polynomial in \( n \) and \( \log q \).

The large number of variables that makes a polynomial time solution possible, unfortunately also makes our algorithm most probably irrelevant for cryptographic applications. Nonetheless, it turns out that the algorithm is widely applicable in quantum computing for solving efficiently various algebraic hidden structure problems. We now explain this connection.

Simply speaking, in a hidden structure problem we have to find some hidden object related to some explicitly given algebraic structure \( A \). We have access to an oracle input, which is an unknown member \( f \) of a family of black-box functions which map \( A \) to some finite set \( S \). The task is to identify the hidden object solely from the information one can obtain by querying the oracle \( f \). This means that the only useful information we can obtain is the structure of the level sets \( f^{-1}(s) = \{ a \in A : f(a) = s \}, s \in S \), that is, we can only determine whether two elements in \( A \) are mapped to the same value or not. In these problems we say that the input \( f \) hides the hidden structure, the output of the problem. We define now the two problems for which we can apply our algorithm for SDE.

**Definition 4.** The hidden subgroup problem HSP is parametrized by a finite group \( G \) and a family \( \mathcal{H} \) of subgroups of \( G \).

\[
\text{HSP}(G, \mathcal{H})
\]

**Oracle input:** A function \( f \) from \( G \) to some finite set \( S \).

**Promise:** For some subgroup \( H \in \mathcal{H} \), we have

\[
 f(x) = f(y) \iff Hx = H y.
\]

**Output:** \( H \).

The hidden polynomial graph problem HPGP is parametrized by a finite field \( F_q \) and three positive integers \( n, m \) and \( d \).

\[
\text{HPGP}(F_q, n, m, d).
\]

**Oracle input:** A function \( f \) from \( F_q^n \times F_q^m \) to a finite set \( S \).

**Promise:** For some \( Q : F_q^n \to F_q^m \), where \( Q(x) = (Q_1(x), \ldots, Q_m(x)) \), and \( Q_i(x) \) is an \( n \)-variate degree \( d \) polynomial over \( F_q \) with zero constant term, we have

\[
f(x, y) = f(x', y') \iff y - Q(x) = y' - Q(x').
\]

**Output:** \( Q \).

While no classical algorithm can solve the HSP with polynomial query complexity even if the group \( G \) is abelian, one of the most powerful results of quantum computing is that it can be solved by a polynomial time
quantum algorithm for any abelian $G$. Shor’s factorization and discrete logarithm finding algorithms [26], and Kitaev’s algorithm [19] for the abelian stabilizer problem are all special cases of this general solution.

Extending the quantum solution of the abelian HSP to non abelian groups is an active research area since these instances include several algorithmically important problems. For example, efficient solutions for the dihedral and the symmetric group would imply efficient solutions, respectively, for several lattice problems [24] and for graph isomorphism. While the non abelian HSP has been solved efficiently by quantum algorithms in various groups [5, 11, 12, 13, 14, 20, 21], finding a general solution seems totally elusive.

An extension in a seemingly different (not “group theoretical”) framework was proposed by Childs, Schulman and Vazirani [7] who considered the problem where the hidden object is a polynomial. To recover it we have at our disposal an oracle whose level sets coincide with the level sets of the polynomial. Childs et al. [7] showed that the quantum query complexity of this problem is polynomial in the logarithm of the field size when the degree and the number of variables are constant. The first time-efficient quantum algorithm was given by the authors with Decker and Wocjan [10] for the case of multivariate quadratic polynomials over fields of constant characteristic.

The hidden polynomial graph problem HPGP was defined in [8] by Decker, Draisma and Wocjan. Here the hidden object is again a polynomial, but the oracle is more powerful than in [7] because it can also be queried on the graphs that are defined by the polynomial functions. They obtained a polynomial time quantum algorithm that correctly identifies the hidden polynomial when the degree and the number of variables are considered to be constant. In [10], this result was extended to polynomials of constant degree in a framework that reveals relationship to the hidden subgroup problem. The version of the HPGP we define here is more general than the one considered in [8] in the sense that we are dealing not only with a single polynomial but with a vector of several polynomials. The restriction on the constant terms of the polynomials is due to the fact that level sets of two polynomials are the same if they differ only in their constant terms, and therefore the value of the constant term can not be recovered.

It will be convenient for us to consider a slight variant of the hidden polynomial graph problem which we denote by HPGP'. The only difference between the two problems is that in the case of HPGP' the input is not given by an oracle function but by the ability to access random level set states, which are quantum states of the form

$$\sum_{x \in \mathbb{F}_q^n} |x\rangle|u + Q(x)\rangle,$$

where $u$ is a random element of $\mathbb{F}_q^n$. Given an oracle input $f$ for HPGP, a simple and efficient quantum algorithm can create such a random coset state. Therefore an efficient quantum algorithm for HPGP' immediately provides an efficient quantum algorithm for HPGP.

In [9] the authors with Decker and Hoyer showed that HPGP'($\mathbb{F}_q, 1, m, d$) is solvable in quantum polynomial time when $d$ and $m$ are both constant. Part of the quantum algorithm repeatedly solved instances of SDE($\mathbb{F}_q, n, m, d$) under such conditions. We present here a modification of this method which works in polynomial time even if $m$ is not constant. For simplicity, here we restrict ourselves to prime fields. This will be still sufficient for application to a hidden subgroup problem.

**Theorem 5.** Let $d$ be constant and $p$ be a prime. If SDE($\mathbb{F}_p, n, m, d$) is solvable in (randomized) polynomial time for some $n$, then HPGP'($\mathbb{F}_p, 1, m, d$) is solvable in quantum polynomial time.

Using Theorem 5 it is possible to dispense in the result of the authors with Decker and Hoyer [9] with the assumption that $m$ is constant.

**Corollary 6.** If $d$ is constant then HPGP'($\mathbb{F}_p, 1, m, d$) is solvable in quantum polynomial time.

Bacon, Childs and van Dam in [9] have considered the HSP in $p$-groups of the form $G = \mathbb{F}_p \times \mathbb{F}_p^m$ when the hidden subgroup belongs to the family $\mathcal{H}$ of subgroups of order $p$ which are not subgroups of the normal subgroup $0 \times \mathbb{F}_p^m$. They have found an efficient quantum algorithm for such groups as long as $m$ is constant. In [10], based on arguments from [5] the authors with Decker and Hoyer sketched how the HSP($G, \mathcal{H}$) can be translated into a hidden polynomial graph problem. For the sake of completeness we state here and prove the exact statement about such a reduction.
Proposition 7. Let \( d \) be the nilpotency class of a group \( G \) of the form \( \mathbb{F}_q \times \mathbb{F}_q^m \). There is a polynomial time quantum algorithm which reduces HSP\((G, \mathcal{H})\) to HPGL\((\mathbb{F}_q, 1, m, d)\).

Putting together Corollary 5 and Proposition 4 it is also possible to get rid of the assumption that \( m \) is constant in the result of 3.

Corollary 8. If the nilpotency class of the group \( G \) of the form \( \mathbb{F}_q \times \mathbb{F}_q^m \) is constant then HSP\((G, \mathcal{H})\) can be solved in quantum polynomial time.

We illuminate the main ideas of the proof of Theorem 3 by showing special cases of weaker (randomized) versions for \( d = 2, 3 \) in Section 2. Actually, randomization in these algorithms is only required to obtain quadratic and cubic nonresidues in \( \mathbb{F}_q \). We remark that assuming the Extended Riemann hypothesis, such nonresidues can be found even deterministically in \( \log q \), see 4. The proof of Theorem 3 will be given in Section 3. There we also show how necessity of having nonresidues can be got around. Finally the proof of Proposition 7 will be given in Section 4, and the proof of Theorem 5 in Section 5.

2 Warm-up: the quadratic and cubic cases

2.1 The quadratic case

Proposition 9. The problem SDE\((\mathbb{F}_q, (m + 1)^2, m, 2)\) can be solved by a randomized algorithm in time polynomial in \( \log q \) and \( m \).

Proof. We assume that \( p > 2 \) and that we have a non-square \( \zeta \) in \( \mathbb{F}_q \) at hand. Such an element can be efficiently found by a random choice. Actually, this is the only point of our algorithm where randomization is used. Assuming ERH, even a deterministic polynomial time method exists for finding a non-square. Also, as we will see in Section 3, one can even get around the necessity of nonresidues. As we present this proof and that for the cubic case for showing the main lines of our general algorithm, we do not address this issue here.

Our input is a set \( V \) of \( (m + 1)^2 \) vectors in \( \mathbb{F}_q^m \), and we want to represent the zero vector as a nontrivial linear combination of some vectors from \( V \) where all the coefficients are squares. The construction is based on the following. Pick any \( m + 1 \) vectors \( v_1, \ldots, v_{m+1} \) from \( V \). Since they are linearly dependent, it is easy to represent the zero vector as a proper linear combination \( \sum_{i=1}^{m+1} \alpha_i v_i = 0 \). Let \( J_1 = \{ i : \alpha_i \frac{p+1}{p} = 1 \} \) and \( J_2 = \{ i : \alpha_i \frac{p-1}{p} = -1 \} \). Using \( \zeta \), we can find in deterministic polynomial time in \( \log q \) by the Shanks-Tonelli algorithm 25 field elements \( \beta \) such that \( \alpha_i = \beta_i^2 \) for \( i \in J_1 \) and \( \alpha_i = \beta_i \zeta \) for \( i \in J_2 \). Let \( w_1 = \sum_{i \in J_1} \beta_i^2 v_i \) and \( w_2 = \sum_{i \in J_2} \beta_i v_i \). Then \( w_1 = -\zeta w_2 \). Notice that we are done if either of the sets \( J_1 \) or \( J_2 \) is empty.

What we have done so far, can be considered as a high-level version of the approach of our earlier work 13 with Sanselme. The method of 13 then proceeds with recursion to \( m - 1 \). Unfortunately, that approach is appropriate only in the quadratic case. Here we use a completely different idea which will turn to be extensible to more general degrees.

From the vectors in \( V \) we form \( m + 1 \) pairwise disjoint sets of vectors of size \( m + 1 \). By the construction above, we compute \( w_1(1), w_2(1), \ldots, w_1(m + 1), w_2(m + 1) \), where

\[
 w_1(i) = -\zeta w_2(i),
\]

for \( i = 1, \ldots, m + 1 \). Moreover, these \( 2m \) vectors are represented as linear combinations with nonzero square coefficients of \( 2m \) pairwise disjoint nonempty subsets of the original vectors.

Now \( w_1(1), \ldots, w_1(m + 1) \) are linearly dependent and again we can find disjoint subsets \( J_1 \) and \( J_2 \) and scalars \( \gamma_i \) for \( i \in J_1 \cup J_2 \) such that for \( w_{11} = \sum_{i \in J_1} \gamma_i^2 w_1(i) \) and \( w_{12} = \sum_{i \in J_2} \gamma_i^2 w_1(i) \) we have \( w_{11} = -\zeta w_{12} \). But then for \( w_{21} = \sum_{i \in J_1} \gamma_i^2 w_2(i) \) and \( w_{22} = \sum_{i \in J_2} \gamma_i^2 w_2(i) \), using equation 13 for all \( i \), we similarly have \( w_{21} = -\zeta w_{22} \). On the other hand, if we sum up equation 13 for all \( i \), we get \( w_{11} = -\zeta w_{21} \). Therefore

\[
 w_{11} = \zeta^2 w_{22} \quad \text{and} \quad w_{12} = w_{21} = -\zeta w_{22},
\]
By Fact 2, we can find field elements $\delta_{11}, \delta_{22}, \delta_{12}$, not all zero, such that $\zeta^2 \delta_{11}^2 - 2\zeta \delta_{12}^2 + \delta_{22}^2 = 0$, and therefore $(\zeta^2 \delta_{11}^2 - 2\zeta \delta_{12}^2 + \delta_{22}^2)w_{22} = 0$. But

$$(\zeta^2 \delta_{11}^2 - 2\zeta \delta_{12}^2 + \delta_{22}^2)w_{22} = \delta_{11}w_{11} + \delta_{12}^2(w_{12} + w_{21}) + \delta_{22}w_{22}.$$  

Then expanding $\delta_{11}w_{11} + \delta_{12}^2(w_{12} + w_{21}) + \delta_{22}w_{22} = 0$ gives a representation of the zero vector as a linear combination with square coefficients (squares of appropriate product of $\beta$s, $\gamma$s and $\delta$s) of a subset of the original vectors. \hfill \Box

2.2 The cubic case

**Proposition 10.** Let $n = (9m + 1)(3m + 1)(m + 1)$. Then SDE($\mathbb{F}_q, n, m, 3$) can be solved by a randomized algorithm in time polynomial in $m$ and $\log q$.

**Proof.** We assume that $q - 1$ is divisible by 3 since otherwise the problem is trivial. By a randomized polynomial time algorithm we can compute two elements $\zeta_2, \zeta_3$ from $\mathbb{F}_q$ such that $\zeta_1 = 1, \zeta_2, \zeta_3$ are a complete set of representatives of the cosets of the subgroup $\{q^i: x \in \mathbb{F}_q^*\}$ of $\mathbb{F}_q^*$. Let $V$ be our input set of $n$ vectors in $\mathbb{F}_q^m$. We assume that $\zeta_1$ is non-empty and they satisfy the 27 equations

$$(\zeta^2 \delta_{11}^2 - 2\zeta \delta_{12}^2 + \delta_{22}^2)w_{22} = \delta_{11}w_{11} + \delta_{12}^2(w_{12} + w_{21}) + \delta_{22}w_{22}.$$  

Then by concatenating $\delta_{11}w_{11} + \delta_{12}^2(w_{12} + w_{21}) + \delta_{22}w_{22}$, we can easily find a proper linear combination summing to zero, $\sum_{i=1}^{m+1} \alpha_i v_i = 0$. For $r = 1, 2, 3$, let $J_r$ be the set of indices such that $0 \neq \alpha_i = \beta_i^3 \zeta_r$. We know that at least one of these three sets is non-empty. For each $\alpha_i \neq 0$ we efficiently identify the coset of $\alpha_i$ and even find $\beta_i$ using the method of $[1]$. Let $w_r = \sum_{i \in J_r} \beta_i^3 v_i$. Then $\zeta_1 w_1 + \zeta_2 w_2 + \zeta_3 w_3 = 0$. Without loss of generality we can suppose that $J_1$ is non-empty since if $J_r$ is non-empty for $r \in \{2, 3\}$, we can just multiply the $\alpha_i$s simultaneously by $\zeta_1/\zeta_r$.

From any subset of size $(3m + 1)(m + 1)$ of $V$ we can form $3m + 1$ disjoint vectors. We can do the procedure outlined above. This way we obtain, for $k = 1, \ldots, 3m + 1$, and $r = 1, 2, 3$, pairwise disjoint subsets $J_r(k)$ of indices and vectors $w_r(k)$ such that

$$\zeta_1 w_1(k) + \zeta_2 w_2(k) + \zeta_3 w_3(k) = 0.$$  

(7)

For $k = 1, \ldots, 3m + 1$, we know that $J_1(k) \neq \emptyset$ and the vectors $w_r(k)$ are combinations of input vectors with indices form $J_r(k)$ having coefficients which are nonzero cubes. Let $W(k) \in \mathbb{F}_q^{3m}$ denote the vector obtained by concatenating $w_1(k), w_2(k)$ and $w_3(k)$ (in this order). Then we can find three pairwise disjoint subsets $M_1, M_2, M_3$ of $\{1, \ldots, 3m + 1\}$, and for each $k \in M_s$, a nonzero field element $\gamma_k$ such that

$$\sum_{s=1}^{3} \zeta_s \sum_{k \in M_s} \gamma_k^3 W(k) = 0.$$  

(8)

We can arrange $M_2$ is non-empty. For $s, t \in \{1, 2, 3\}$, set $J_{rs} = \bigcup_{k \in M_s} J_r(k)$ and $w_{rs} = \sum_{k \in M_s} \gamma_k^3 w_r(k)$. Then $w_{rs}$ is a linear combination of input vectors with indices from $J_{rs}$ having coefficients that are nonzero cubes. The equality (8) just states that $\zeta_1 w_{1s} + \zeta_2 w_{2s} + \zeta_3 w_{3s} = 0$, for $r = 1, 2, 3$. Furthermore, summing up the equalities (7) for $k \in M_s$, we get $\zeta_1 w_{1s} + \zeta_2 w_{2s} + \zeta_3 w_{3s} = 0$, for $s = 1, 2, 3$.

Continuing this way, from $(9m + 1)(3m + 1)(m + 1)$ input vectors we can make 27 linear combinations with cubic coefficients $w_{rst}$, for $r, s, t = 1, 2, 3$, having pairwise disjoint supports such that the support of $w_{123}$ is non-empty and they satisfy the 27 equations

$$\zeta_1 w_{1st} + \zeta_2 w_{2st} + \zeta_3 w_{3st} = 0 \ (s, t = 1, 2, 3);$$
$$\zeta_1 w_{rt1} + \zeta_2 w_{rst} + \zeta_3 w_{srt} = 0 \ (r, t = 1, 2, 3);$$
$$\zeta_1 w_{rs1} + \zeta_2 w_{rs2} + \zeta_3 w_{rs3} = 0 \ (r, s = 1, 2, 3).$$

From these we use the following 6 equations:
\[ \zeta_1 w_{123} + \zeta_2 w_{223} + \zeta_3 w_{323} = 0; \]
\[ \zeta_1 w_{132} + \zeta_2 w_{232} + \zeta_3 w_{332} = 0; \]
\[ \zeta_1 w_{213} + \zeta_2 w_{223} + \zeta_3 w_{233} = 0; \]
\[ \zeta_1 w_{312} + \zeta_2 w_{322} + \zeta_3 w_{332} = 0; \]
\[ \zeta_1 w_{231} + \zeta_2 w_{232} + \zeta_3 w_{233} = 0; \]
\[ \zeta_1 w_{321} + \zeta_2 w_{322} + \zeta_3 w_{332} = 0. \]

Adding these equalities with appropriate signs so that the terms with coefficients \( \zeta_2 \) and \( \zeta_3 \) cancel and dividing by \( \zeta_1 \), we obtain
\[ w_{123} + w_{231} + w_{312} - w_{132} - w_{213} - w_{321} = 0. \] (9)

Observing that \(-1 = (-1)^3\), this gives a representation of zero as a linear combination of the input vectors with coefficients that are cubes. (Note that the algorithm described in this proof does not rely on van de Woestijne's result Fact 2. This is because we were in a position to eliminate the \( \zeta_i \)'s and obtained a linear dependency with coefficients \( \pm 1 \) which are always cubes of themselves in \( \mathbb{F}_q \), independently of \( q \).)

\[
\zeta_1 w_{123} + \zeta_2 w_{223} + \zeta_3 w_{323} = 0; \\
\zeta_1 w_{132} + \zeta_2 w_{232} + \zeta_3 w_{332} = 0; \\
\zeta_1 w_{213} + \zeta_2 w_{223} + \zeta_3 w_{233} = 0; \\
\zeta_1 w_{312} + \zeta_2 w_{322} + \zeta_3 w_{332} = 0; \\
\zeta_1 w_{231} + \zeta_2 w_{232} + \zeta_3 w_{233} = 0; \\
\zeta_1 w_{321} + \zeta_2 w_{322} + \zeta_3 w_{332} = 0.
\]

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3 The general case

In this section we prove Theorem 3. First we make the simple observation that it is sufficient to solve SDE(\( \mathbb{F}_q, n, m, d \)) in the case when \( d \) divides \( q - 1 \). If it is not the case, then let \( d' = \gcd(d, q - 1) \). Then from a nonzero solution of the system
\[ \sum_{j=1}^n x_j'^d v_j = 0, \]
one can efficiently find a nonzero solution of the original equation. Indeed, the extended Euclidean algorithm efficiently finds a positive integer \( t \) such that \( td = u(q - 1) + d' \) for some integer \( u \). Then for any nonzero \( x \in \mathbb{F}_q \) we have \( (x^d)^d = x^{d'} \mod p \), and therefore \( (x_1^d, \ldots, x_n^d) \) is a solution of equation (4).

From now on we suppose that \( d \) divides \( q - 1 \). Our algorithm will consist of two major procedures. The first one is devoted to finding two disjoint subsets of the input vectors, not both empty, and \( d \)th power coefficients such that the linear combinations of the vectors from the two subsets give equal vectors. Notice that this part already does the job when one of the two sets happen to be empty or \( d \) is odd (or, more generally, a \( d \)th root of \(-1\) is at hand). The second procedure consists of iterative applications of the first algorithm to obtain a vector with sufficiently many representations as linear combinations with \( d \)th power coefficients with pairwise disjoint supports.

We will denote by \( C(d, m) \) the number of vectors (variables) used by our algorithm. For \( d = 1 \), we can obviously take \( C(1, m) = m + 1 \).

The basic idea of the first algorithm is – like in the cubic and quadratic case outlined in the previous section – getting linear dependencies and effectively putting the coefficients of these dependencies into cosets of the multiplicative group of the \( d \)th powers on nonzero field elements. In the first subsection, based on an idea borrowed from [28], we show how to do this without having nonresidues at hand.

3.1 Classifying field elements

During the procedures of this section, one of the basic tasks is the following. Given a nonzero field element \( \alpha \), one has to write \( \alpha = \zeta_i \beta^d \), where \( 1 = \zeta_1, \ldots, \zeta_d \) are fixed elements. Ideally, the \( \zeta_i \) form a complete system of representatives of the cosets of the subgroup of the \( d \)th powers in the multiplicative group \( \mathbb{F}_q^\ast \). Unfortunately, no deterministic polynomial time algorithm is known to find an element of a nontrivial coset (unless assuming the generalized Riemann hypothesis). Therefore, instead of the whole \( \mathbb{F}_q^\ast \), we consider (roughly speaking) the subgroup generated by nonzero field elements already seen and we classify elements according to the cosets of \( d \)th powers of this subgroup. The classification fails (essentially) when we encounter an element outside this group. Then the subgroup, the sub-subgroup of its \( d \)th powers as well as the coset...
representatives are updated and all the computations done so far are redone. Obviously, this can happen at most $\log q$ times, resulting a $\log q$ factor in complexity (but not in the bound on the number of input vectors necessary for success).

To describe the details, we need some notation. Let $\pi$ be the set of prime divisors of $d$ and $\pi'$ be the set of prime divisors of $q - 1$ outside $\pi$. Then the multiplicative group $\mathbb{F}_q^*$ is the (direct) product of two subgroups $H_\pi$ and $H_{\pi'}$, where $H_\pi$ consists of the elements of order having prime factors from $\pi$, while the element of $H_{\pi'}$ are those having an order whose prime factors are from $\pi'$. Note that the primes in $\pi$ can be computed in time $d^{O(1)}$ by factoring $d$. The primes in $\pi'$ do not need to be explicitly computed. Instead, by successively dividing $q - 1$ by the primes in $\pi$, we can efficiently (that is, in time polynomial in $\log q$) compute the order of the subgroup $H_\pi$, which is the largest divisor of $q - 1$ coprime to $d$. Given an element $\alpha \in \mathbb{F}_q^*$, one can find in time polynomial in $\log q$ the unique elements $\gamma_1 \in H_\pi$ and $\gamma' \in H_{\pi'}$ such that $\alpha = \gamma_1 \gamma'$. (Actually, $\gamma' = \gamma''$ where $r \equiv 1$ modulo the order of $H_{\pi'}$.)

Instead of $H_\pi$ we use the subgroup $H$ of the $\pi$-parts of the field elements given so far to the classification procedure as input. We assume that $H$ is given by a generator $\eta$. Elements $1 = \zeta_1, \ldots, \zeta_d \in H$ are also assumed to be given such that they form a polynomially redundant, but complete system of representatives of cosets of the subgroup $H^d$ consisting of the $d$th powers of $H$. Initially $\eta = 1 = \zeta_1 = \ldots = \zeta_d$. Given $\alpha = \gamma\gamma'$, we (attempt to) compute the $\eta$-base discrete logarithm of $\gamma$ using the method of Pohlig and Hellman.

This takes time polynomial in $d$ and $\log q$. In the case of success, we can use the logarithm to locate the coset of $\gamma$ and write $\gamma$ as $\delta^d \zeta$ where $\delta \in H$. Then $\alpha = \beta^d \zeta$, where $\beta = \delta \delta'$.

In the case of failure, we replace $\eta$ by a generator of the subgroup generated by $\gamma$ and $\eta$ and we replace $\zeta_1, \ldots, \zeta_d$ by $\eta, \ldots, \eta^{d-1}$ (repetitions may occur). We restart the whole algorithm with these new data.

### 3.2 Finding Colliding Representations

In this subsection we prove the following.

**Theorem 11.** Assume that $d|q - 1$ and put $G(d, m) = d^{\frac{d(d-1)}{2}}(m + 1)^d$. Then, given $G = G(d, m)$ input vectors $v_1, \ldots, v_G \in \mathbb{F}_q^m$, in time polynomial in $G$ and $\log q$, we can find two disjoint subsets $I$ and $J$ of \{1, \ldots, G\} with $I \neq \emptyset$ and nonzero field elements $\gamma_j \in \mathbb{F}_q^*$ ($j \in I \cup J$) such that $\sum_{i \in I} \gamma_i^d v_i = \sum_{j \in J} \gamma_j^d v_j$.

**Proof.** The algorithm follows the lines already presented in the proof of Proposition 10 for the cubic case. The main difference is that here we (possibly) need more rounds of iteration. For $\ell = 1, \ldots, d$, put $B_\ell(d, m) = d^{\frac{\ell(\ell-1)}{2}}(m + 1)^\ell$. For $a = (a_1, \ldots, a_d) \in \{1, \ldots, d\}^\ell$, for $s \in \{1, \ldots, d\}$ and for $1 \leq j \leq \ell$, set $a(j, s) = (a_1, \ldots, a_{j-1}, s, a_{j+1}, \ldots, a_d)$.

**Lemma 12.** From $B = B_\ell(d, m)$ input vectors $v_1, \ldots, v_B$, in time polynomial in $B$ and $\log q$, we can find $d^\ell$ pairwise disjoint subsets $J_a \subseteq \{1, \ldots, B\}$ and field elements $\beta_1, \ldots, \beta_B$ such that $J_{(1, \ldots, \ell)} \neq \emptyset$, and if we set $w_a = \sum_{i \in J_a} \beta_i^d v_i$, then we have $\sum_{s=1}^d \zeta_s w_a(j, s) = 0$, for every $a \in \{1, \ldots, d\}^\ell$ and $j = 1, \ldots, \ell$.

**Proof.** We prove it by recursion on $\ell$. If $\ell = 1$ then any $B_1(d, m) = m + 1$ vectors from $\mathbb{F}_q^m$ are linearly dependent. Therefore there exist $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}_q$, not all zero, such that $\sum_{i=1}^{m+1} \alpha_i v_i = 0$. Using the procedure of Subsection 3.1 we find subsets $J_1, \ldots, J_m$ of \{1, \ldots, $m+1$\} and field elements $\beta_i$ ($i \in J_1 \cup \cdots \cup J_m$), such that for $i \in J_r$ we have $\alpha_i = \zeta_r \beta_i$. At least one of the sets $J_r$ is non-empty. If $J_1$ is empty then we multiply the coefficients $\alpha_i$ simultaneously by $\zeta_1^{-1}$ where $J_r$ is nonempty to arrange that $J_1$ becomes nonempty.
To describe the recursive step, assume that we are given $B_{t+1}(d,m) = d^t(m+1)B$ vectors. Put $E = d^t(m + 1)$, and for convenience assume that the input vectors are denoted by $v_{ki}$, for $k = 1, \ldots , E$ and $i = 1, \ldots , B$. By the recursive hypothesis, for every $k \in \{1, \ldots , E\}$, there exist subsets $J_{\mathbf{a}}(k) \subseteq \{1, \ldots , B\}$ and field elements $\beta_i(k)$ such that $J_{1, \ldots , t}(k) \neq \emptyset$, and with $w_\mathbf{a}(k) = \sum_{i \in J_{\mathbf{a}}(k)} \beta_i(k)^d v_{ki}$, we have
\[
\sum_{s=1}^{d} \zeta_s w_\mathbf{a}(j,s)(k) = 0, \tag{10}
\]
for every $\mathbf{a} \in \{1, \ldots , d\}^t$ and $j = 1, \ldots , \ell$.

For every $k = 1, \ldots , E$, let $W(k)$ be the concatenation of the vectors $w_\mathbf{a}(k)$ in a fixed, say the lexicographic, order of $\{1, \ldots , d\}^t$. Then the $W(k)$’s are vectors of length $d^t m < E$. Therefore there exist field elements $\alpha(1), \ldots , \alpha(E)$, not all zero, such that $\sum_{k=1}^{E} \alpha(k) W(k) = 0$. For a $k$ such that $\alpha(k) \neq 0$, let $\alpha(k) = \zeta_r \gamma(k)^d$ for some $1 \leq r \leq d$ and $\gamma(k) \in \mathbb{F}_q^*$. The index $r$ and $\gamma(k)$ are computed by the procedure of Subsection 3.1. For $r = 1, \ldots , d$, let $M_r$ be the set of $k$’s such that $\alpha(k) = \zeta_r \gamma(k)^d$. We can arrange that $M_{\ell+1}$ is non-empty by simultaneously multiplying the $\alpha(k)$’s by $\zeta_{\ell+1}/\zeta_r$ for some $r$, if necessary. Observe that we have
\[
\sum_{s=1}^{d} \zeta_s \sum_{k \in M_r} \gamma(k)^d W(k) = 0. \tag{11}
\]
For $i \in \{1, \ldots , B\}$ and $k \in \{1, \ldots , E\}$ set $\beta_{ki} = \gamma(k) \beta_i(k)$. We fix $\mathbf{a}' = (a_1', \ldots , a_{\ell+1}')$ and we set $\mathbf{a} = (a_1', \ldots , a_{\ell}')$ and $r = a_{\ell+1}'$. We define $J_{\mathbf{a}'} = \{(k,i) : k \in M_r \text{ and } i \in J_{\mathbf{a}}(k)\}$ and $w_{\mathbf{a}'} = \sum_{(k,i) \in J_{\mathbf{a}'}} \beta_{ki}^d v_{ki}$. Then $w_{\mathbf{a}'} = \sum_{k \in M_r} \gamma(k)^d w_\mathbf{a}(k)$. This equality, together with the equalities (10) imply that for every $j = 1, \ldots , \ell$, we have
\[
\sum_{s=1}^{d} \zeta_s w_{\mathbf{a}'}(j,s) = 0.
\]
For $j = \ell + 1$ consider the equality (11), from which follows that
\[
\sum_{s=1}^{d} \zeta_s \sum_{k \in M_r} \gamma(k)^d w_\mathbf{a}(k) = 0.
\]
Expanding $w_\mathbf{a}(k)$ in the inner sum $\sum_{k \in M_r} \gamma(k)^d w_\mathbf{a}(k)$ gives that it equals $w_{\mathbf{a}'}(\ell+1,s)$. Thus also
\[
\sum_{s=1}^{d} \zeta_s w_{\mathbf{a}'}(\ell+1,s) = 0,
\]
finishing the proof of the lemma.

We apply the procedure of Lemma 12 for $\ell = d$. From $B = B_d(d,m) = d^{d(d-1)}(m+1)^d$ input vectors $v_1, \ldots , v_B$, we compute in time polynomial in $\log q$ and $B$ subsets $J_{\mathbf{a}_2}$ with $J_{12, \ldots , d} \neq \emptyset$, as well as nonzero elements $\beta_1, \ldots , \beta_B \in \mathbb{F}_q^*$ such that with $w_\mathbf{a} = \sum_{i \in J_{\mathbf{a}_2}} \beta_i^d v_{1i}$, we have
\[
\sum_{s=1}^{d} \zeta_s w_{\mathbf{a}}(j,s) = 0, \tag{12}
\]
for every $j = 1, \ldots , d$ and for every $\mathbf{a} \in \{1, \ldots , d\}^d$.

Tuples from $\{1, \ldots , d\}^d$ without repetitions are of special interest. We identify such a $d$-tuple $\mathbf{a} = (a_1, \ldots , a_d)$ with the permutation $i \mapsto a_i$ from the symmetric group $S_d$ on $\{1, \ldots , d\}$. With some abuse
of notation, we denote this permutation also by $a$. By $\text{sgn}(a)$ we denote the sign of $a$, considered as a permutation. The sign of $a$ is 1 if $a$ is even and $-1$ if $a$ is odd. We show that
\[
\sum_{a \in S_d} \text{sgn}(a) w_a = 0.
\] (13)

For $a \in S_d$, let $j_a$ be the position of 1 in $a$ and for every $s \in \{1, \ldots, d\}$, we denote by $a[s]$ the sequence obtained from $a$ by replacing 1 with $s$. Notice that $a[s] = a(j_a, s)$, therefore (12) implies
\[
\sum_{a \in S_d} \text{sgn}(a) \sum_{s=1}^{d} \zeta_s w_{a[s]} = 0.
\] (14)

We claim that
\[
\sum_{a \in S_d} \text{sgn}(a) \sum_{s=2}^{d} \zeta_s w_{a[s]} = 0.
\] (15)

To see this, observe that for $s > 1$ the tuple $a[s]$ has entries from $\{2, \ldots, d\}$, where $s$ occurs twice, while the others once. Any such sequence $a'$ can come from exactly two permutations which differ by a transposition: these are obtained from $a'$ by replacing one of the occurrences of $s$ with 1. Then (13) is just the difference of equalities (14) and (15).

Put
\[
I = \bigcup_{a \text{ even}} J_a, \quad J = \bigcup_{a \text{ odd}} J_a \quad \text{and} \quad \gamma_i = \beta_i \quad \text{for} \quad i \in I \cup J.
\]

(Here, $a$ even resp. $a$ odd abbreviates that $a$ is an even or an odd permutation, respectively.) Then (13) gives the desired pair of colliding representations.

3.3 Accumulating collisions

In this subsection we finish the proof of Theorem 3.

Proof of Theorem 3. We assume that $q - 1$ is divisible by $d$. By Theorem 11 from $G(d, m)$ input vectors we can select two disjoint subsets, not both empty, and find $d$th power coefficients such that the corresponding linear combinations represent the same vector. Notice that we are done if this is the zero vector.

When we have $G(d, m)^2$ input vectors, the procedure of Theorem 11 applied to $G(d, m)$ gives $G(d, m)$ vectors and two representations as linear combination with $d$th power coefficients for each. (These combinations have $2G(d, m)$ pairwise disjoint sets as support.) Applying the procedure again to the $G(d, m)$ vectors and multiplying the coefficients gives a vector with 4 representations as linear combinations having pairwise disjoint support and coefficients that are explicit $d$th powers. Iterating this, using $G(d, m)^\ell$ input vectors, we obtain a vector with $2^\ell$ representations as linear combinations having pairwise disjoint support and $d$th power coefficients.

Iterating this, using $G(d, m)^\ell$ input vectors, we obtain a vector with $2^\ell$ representations as linear combinations having pairwise disjoint support and coefficients that are explicit $d$th powers. When $2^\ell \geq d + 1$, we can use Fact 2 to find field elements $z_1, \ldots, z_{d+1}$, not all zero, such that $z_1^d + \cdots + z_{d+1}^d = 0$. Multiplying the coefficients of the $i$th representation by $z_i^d$ we obtain the desired representation of the zero vector. We have
\[
C(d, m) \leq G(d, m)^{\lceil \log_2(d+1) \rceil} \leq d^{d \log d} (m + 1)^{d \log d}.
\]

4 Application in Quantum computing

4.1 Reduction from the special HSP to HPGP’

In this part we give the details of a reduction from a special instance of the hidden subgroup problem in groups which are semidirect products of an elementary abelian $p$-groups by a group of order $p$. The arguments here are quite standard.
Proof of Proposition. A semidirect product group of the form \( \mathbb{F}_p \rtimes \mathbb{F}_m^p \) can be specified by an automorphism of \( \mathbb{F}_m^p \). The automorphisms of \( \mathbb{F}_m^p \) can be identified with nonsingular \( m \times m \) matrices \( B \) over \( \mathbb{F}_p \), such that \( B^p = I \). For such a matrix \( B \), the group \( G_B = \mathbb{F}_p \rtimes \mathbb{F}_m^p \) can be represented as the set of \((m + 1) \times (m + 1)\) matrices over \( \mathbb{F}_p \)

\[
\left\{ \begin{pmatrix} B^x & v \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p, \ v \in \mathbb{F}_m^p \right\}.
\]

We choose the quantum encoding \(|x\rangle|v\rangle\) for the matrix

\[
M_B(x, v) = \begin{pmatrix} B^x & v \\ 0 & 1 \end{pmatrix}.
\]

Let

\[
K = \left\{ \begin{pmatrix} B^x & 0 \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p \right\} \quad \text{and} \quad N = \left\{ \begin{pmatrix} I & v \\ 0 & 1 \end{pmatrix} : v \in \mathbb{F}_m^p \right\}.
\]

Then \( N \) is a normal subgroup of \( G \) of index \( p \) and \( K \cap N = \{1_G\} \). For every \( v \in \mathbb{F}_m^p \), consider the cyclic subgroup

\[
H_v = \left\langle \begin{pmatrix} B^x & v \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p \right\rangle = \left\{ \begin{pmatrix} B^x & v(x) \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p \right\},
\]

where

\[
v(x) = \begin{pmatrix} v_1(x) \\ \vdots \\ v_m(x) \end{pmatrix} = (B^{x-1} + \cdots + B^1 + B^0)v.
\]

Then \( \mathcal{H} \), the family of subgroups of \( G_B \) of order \( p \) which are not subgroups of \( N \) is exactly \( \{H_v : v \in \mathbb{F}_m^p\} \). The hidden function hides some member of \( \mathcal{H} \). Since \( B^p = I \) we also have \((B - I)^p = 0\). It can be seen that if the nilpotency class of \( G_B \) is \( d \) then \( d \) is the smallest integer such that \((B - I)^d = 0\). In fact, if we let \( A = \log B \) then the lower central series of \( G_B \) is the sequence consisting of the images of \( A, A^2, \ldots, A^{d-1} \).

Claim 13. The functions \( v_i(x) \) are polynomials with 0 constant term and of degree \( \leq d \), for \( i = 1, \ldots, m \).

Proof. We have

\[
A = \log B = \sum_{j=1}^{d-1} \frac{A^j - I}{j} (B - I)^j.
\]

Then

\[
B^k = e^{kA} = \sum_{j=0}^{d-1} \frac{A^j}{j!} k^j,
\]

since \( A^d = 0 \). Therefore

\[
v(x) = \sum_{k=0}^{x-1} B^k v = \sum_{k=0}^{x-1} \sum_{j=0}^{d-1} \frac{A^j}{j!} k^j v = \sum_{j=0}^{d-1} \frac{A^j}{j!} \sum_{k=0}^{x-1} k^j v = \sum_{j=0}^{d-1} \frac{A^j}{j!} p_j(x - 1),
\]

where

\[
p_j(x) = \sum_{k=0}^{x-1} k^j.
\]
where \( p_0(x-1) = x \), and \( p_j(x) \) is a degree \( j + 1 \) polynomial expressed by the Faulhaber’s formula, for \( j = 1, \ldots, d - 1 \). It is known \cite{17} that \( p_j(x) \) is divisible by \( x+1 \), for all \( j \). Therefore indeed \( v_i(x) \) is a degree \( \leq d \) polynomial with constant member zero, for \( i = 1, \ldots, m \).

Let us now suppose that our input \( f \) to \( \text{HSP}(G_B, \mathcal{H}) \) hides the subgroup

\[
H_v = \left\{ \begin{pmatrix} B^x & v(x) \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p \right\}.
\]

We can take as coset representatives

\[
N = \left\{ \begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{F}_p^m \right\}.
\]

Since

\[
\begin{pmatrix} I & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^x & v(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} B^x & u + v(x) \\ 0 & 1 \end{pmatrix},
\]

the left cosets of \( H_v \) are of the form

\[
\left\{ \begin{pmatrix} B^x & u + v(x) \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_p \right\} = \{ M_B(x, u + v(x)) : x \in \mathbb{F}_p \},
\]

for \( u \in \mathbb{F}_p^m \). By a standard efficient quantum procedure we can create, for a random \( u \in \mathbb{F}_p^m \), the coset state

\[
\sum_{x \in \mathbb{F}_p} |x\rangle|u + v(x)\rangle.
\]

But this is also a random level set state of the function

\[
f : \mathbb{F}_p \times \mathbb{F}_p^m \to \mathbb{F}_p^m, \quad f(x, y) = y - v(x),
\]

and therefore the input to \( \text{HPGP}'(\mathbb{F}_p, 1, m, d) \) hiding the polynomial \( v(x) \). From the solution \( v(x) \) we can recreate the solution of the HSP problem since \( v = v(1) \).

5 Proof of Theorem 5

In this part we outline a modified version of the method of our work \cite{9} with Decker and Høyer. A critical ingredient is solving systems of diagonal polynomial equations with sufficiently many variables. At the time of writing \cite{9} polynomial time algorithms (except for the cases \( d = 1, 2 \)) were available only for the case when the number of equations is constant.) Now we have a version which works in polynomial time even if \( m \) is not constant.

\textit{Proof of Theorem} \cite{9} (\textit{sketch}). A solution for constant \( p \) is given in \cite{10}. (Interestingly, that solution goes through a reduction to the variant of the hidden subgroup problem with coset states as input in a \( p \)-group of nilpotency class \( d + 1 \) and exponent \( p \). The latter problem is solved by the method of the paper \cite{12} by the authors with Friedl, Magniez and Shen, which works efficiently in groups of constant derived length and constant exponent.) Therefore we may assume that \( p > d \). Although this assumption is not essential, it simplifies presentation very much.

The input for \( \text{HPGP}' \) consists of uniform superpositions of random level sets states of the form \cite{5}, which, for the special case we have are states

\[
|x\rangle|u + \sum_{j=1}^{d} x^j w_j\rangle,
\]

where \( w_j \) are random constants in \( \mathbb{F}_p \).
for random (unknown) \( u \in \mathbb{F}_p \). To handle dependency on \( u \), we apply the Fourier transform of \( \mathbb{F}_p \) to the second register of such a state. The result is

\[
\omega \sum_{k=1}^{m} y_k u_k \sum_{x=0}^{p-1} \omega \sum_{j=1}^{d} x^j \sum_{k=1}^{m} y_k w_x^k |x\rangle |y\rangle = \omega \sum_{k=1}^{m} y_k u_k |\phi_y\rangle |y\rangle,
\]

where \( \omega = \sqrt{\frac{1}{n}} \) and

\[
|\phi_y\rangle = \sum_{x=0}^{p-1} \omega \sum_{j=1}^{d} x^j \sum_{k=1}^{m} y_k w_x^k |x\rangle.
\]

Measuring the second register we obtain, up to a global phase, the state \( |\phi_y\rangle \) with known \( y \). We drop the useless states \( |\phi_0\rangle \). It can be seen that each \( y \in \mathbb{F}_p \) occurs with equal probability, therefore \( |\phi_0\rangle \) occurs with probability \( \frac{1}{p} \).

We rewrite \( |\phi_y\rangle \) in a more general form suitable for recursion. For hidden parameters \( \eta_1, \ldots, \eta_\ell \in \mathbb{F}_p \) and for \( Y \in \mathbb{F}_p^{\ell \times \ell} \) let

\[
|\psi_Y\rangle := \sum_{x=0}^{p-1} \omega \sum_{j=1}^{d} x^j \sum_{k=1}^{\ell} Y_{jk} \eta_k |x\rangle.
\]

In words, the coefficient of \( x^j \) in the phase of the state \( |\psi_Y\rangle \) is a linear combination of the hidden parameters with known coefficients \( Y_{j1}, \ldots, Y_{j\ell} \). Then \( |\phi_y\rangle = |\psi_Y\rangle \), where \( \ell = dm \), \( \eta_1 \cdots \eta_d = w_{jk} \), \( Y_{j(j-1)d+k} = y_k \), and \( Y_{j(j')d+k} = 0 \), for \( j, j' = 1, \ldots, d \), \( j' \neq j \), \( k = 1, \ldots, m \). The goal is to determine the hidden parameters \( \eta_1, \ldots, \eta_\ell \).

Let \( n = n(\ell, d) \) be a positive integer such that for any positive integer \( d' \leq d \) nonzero solutions of systems of equations of the form

\[
\sum_{j=1}^{n} a_{ij} \xi_j^{d'} = 0, \quad \text{for } i = 1, \ldots, \ell,
\]

in the variables \( \xi_1, \ldots, \xi_n \) can be found in time polynomial in \( n / \log p \).

Using \( n \) level set superpositions, we obtain \( n \) states of the form \( |\psi_Y\rangle \) with various \( Y \). More precisely, up to a global phase we obtain a state

\[
|\psi_{Y_1}\rangle \ldots |\psi_{Y_n}\rangle = \sum_{x_1, \ldots, x_n=0}^{p-1} \omega \sum_{j=1}^{d} (x_1^j + \cdots + x_n^j) Y_{j1} \eta_1 + \cdots + (x_1^\ell + \cdots + x_n^\ell) Y_{j\ell} \eta_\ell |x_1, \ldots, x_n\rangle.
\]

If the degree \( d \) term is completely missing from the phase of state \( |\psi_{Y_1}\rangle \), that is, \( Y_{jk} = 0 \) for \( k = 1, \ldots, \ell \), then we take \( |\psi_{Y_1}\rangle \) and ignore all the other states. Otherwise we produce a similar state without degree \( d \) term as follows. (This is the point where the new algorithm differs from that of our earlier work [9] with Decker and Hoyer. Originally the degree \( d \) terms had to be eliminated one-by-one which caused an exponential blowup of the costs in \( m \). The main result of the present paper allows us to eliminate all the degree \( d \) terms simultaneously, in one step, saving the exponential blowup.)

We find a nonzero solution \( (\delta_1, \ldots, \delta_n) \in \mathbb{F}_p^n \) of the system of equations \( \sum_{i=1}^{n} \delta_i Y_{ij}^i = 0, \) for \( k = 1, \ldots, \ell \).

(We have to solve \( \ell \) homogeneous linear equations in \( \delta_1^1, \ldots, \delta_n^\ell \).) Then we add a fresh register initialized to \( \sum_{t=0}^{p-1} |t\rangle \), and subtract \( \delta_i x \) from the \( i \)th register. We obtain

\[
\sum_{x=0}^{p-1} \sum_{x_1, \ldots, x_n=0}^{p-1} \omega \sum_{j=1}^{d} ((x_1 + \delta_1 x)^j + \cdots + (x_n + \delta_n x)^j) (x_1^\ell + \cdots + x_n^\ell) Y_{j\ell} \eta_\ell |x_1, \ldots, x_n\rangle |x\rangle.
\]

Collecting the terms according to the degree of \( x \) in the phase, we can rewrite the state as

\[
\sum_{x=0}^{p-1} \sum_{x_1, \ldots, x_n=0}^{p-1} \omega \sum_{j=0}^{d} x^j \sum_{k=1}^{\ell} Z_{jk}(x_1, \ldots, x_n) \eta_k |x_1, \ldots, x_n\rangle |x\rangle.
\]
Here $Z_{jk}(x_1, \ldots, x_n)$ is a degree $d - j$ polynomial in $x_1, \ldots, x_n$. By the choice of $\delta_1, \ldots, \delta_n$, we have

$$Z_{dk}(x_1, \ldots, x_n) = \delta_1^{d-1}Y_{dk}^1 + \ldots + \delta_n^{d-1}Y_{dk}^n = 0.$$ 

We also have

$$Z_{d-1,k}(x_1, \ldots, x_n) = \delta_1^{d-1}Y_{dk}^1x_1 + \ldots + \delta_n^{d-1}Y_{dk}^nx_n + \delta_1^{d-1}Y_{d-1,k}^1 + \ldots + \delta_n^{d-1}Y_{d-1,k}^n.$$ 

We have $\delta_i \neq 0$, for at least one index $i$ from $1, \ldots, n$. As $Y_{dk}^i$ is nonzero for at least one $k$, the polynomial $Z_{d-1,k}$ contains the term $x_i$ with nonzero coefficient. Hence, for a random choice of $x_1, \ldots, x_n$, it will be nonzero with probability at least $\frac{d-1}{p}$. Therefore, if we measure the first $n$ registers, we obtain a state of the form

$$\sum_{x=0}^{p-1} \sum_{j=0}^{d-1} \omega^x \sum_{k=1}^{\ell} \eta_k |x\rangle,$$

where not all the vectors $Z_{jk}$ are zero.

Starting with $n^{d-1}$ states with degree $d$ phase (coming from $n^{d-1}$ level set states), applying this procedure to groups of size $n$ we obtain $n^{d-2}$ states with degree $d - 1$ phase, from which we can produce $n^{d-3}$ degree $d - 2$ states and so on. Eventually, with overall failure probability at most $n^\ell/p$, we obtain a state of the form

$$\sum_{x=0}^{p-1} \omega^x \sum_{k=1}^{\ell} z_k |x\rangle,$$

with known $z_1, \ldots, z_k$, not all zero. Applying the inverse Fourier transform of $\mathbb{F}_p$, we obtain the value for $\sum_{k=1}^{\ell} z_k \eta_k$, that is, a linear equation for $\eta_1, \ldots, \eta_\ell$. Using this equation, we can substitute a linear combination of the others (and a constant term) into one of the parameters, and we can do a recursion with $\ell - 1$ unknown parameters.

The whole procedure uses $\ell n^{d-1}$ level set superpositions, has overall failure probability $\ell n^{d-1}/p$ and requires $\text{poly}(\ell n^{d-1} \log p)$ time to determine the hidden coefficients $w_j$. For our task, we take $\ell = md$.

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