Finite-time dissipative synchronization of discrete-time semi-Markovian jump complex dynamical networks with actuator faults

N. Sakthivel, S. Pallavi, Yong-Ki Ma, V. Vijayakumar

Abstract
This paper is concerned with the problem of finite-time synchronization for a class of discrete-time semi-Markovian jumping complex dynamical networks (CDNs) with actuator faults based on reliable control. The main aim of this paper is to design a state feedback controller such that the resulting closed-loop system is finite-time synchronized under a prescribed dissipativity performance level even in the presence of actuator failures. Moreover, a stochastic nature followed by Bernoulli distribution is described in the considered networks due to the occurrence of probabilistic nature in time-varying delays. By composing a suitable Lyapunov–Krasovskii functional containing triple summation terms with the aid of Kronecker product properties, Lyapunov stability theory and free weighting matrix approach, sufficient criteria are established in terms of linear matrix inequalities that assure finite-time synchronization and meet the dissipativity performance to the addressed CDNs. The usefulness of the presented design scheme is finally verified by numerical examples.

Keywords
Finite-time synchronization · Complex dynamical networks · Dissipativity · Semi-Markovian jumping · Reliable control

1 Introduction
Complex network is a large-scale network composed of a large number of nodes joined by edges, and it can mirror several important structure characteristics and dynamic performances of real networks (Zhang et al. 2019; Dong et al. 2017). In recent years, the study on CDNs serves as one of the most active research field in control theory that have found successful applications in real-world disciplines such as supply chain and manufacturing networks, electricity power grids, transportation systems, water distribution systems, gas transmission, world wide web, social interaction, biological networks, co-authorship and citation of networks of scientists. Due to these utilizations, the researchers have been devoting their interest enormously to investigate the control problems and dynamical behaviors of CDNs (Zhang et al. 2019; Dong et al. 2017; Li et al. 2017; Hao et al. 2016).

On the other hand, in practice, synchronization is interesting and significant dynamic phenomena of CDNs compared to other dynamical performance such as auto-waves, spatio-temporal chaos, self-organization and spiral waves. Inspired by these advantages, the research communities have investigated synchronization of a CDNs presented in He et al. (2016), Li et al. (2017, 2018). Li et al. (2017) studied the synchronization problem for a class of discrete-time CDNs with partial mixed impulsive effects. But in practice, the networks might be expected to achieve synchronization in finite-time interval rather than asymptotically. To achieve faster convergence rate in time delay complex networks, an effective method called finite-time synchronization or finite-time boundedness can be used. In addition, finite-time control design has been proposed for the synchronization of different CDNs as authorized by various literatures (Han et al. 2017; Sheng et al. 2018; Li et al. 2019). The finite-horizon bounded $H_{\infty}$ synchronization and state estimation problem for a class of discrete-time com-
plex networks with missing measurements are reported in Han et al. (2017).

Moreover, few actuators may be faulty and uniform of inferior quality, the CDNs structure may be changed as well as the closed-loop system becomes unstable. To improve system reliability and security, it is most important to design a reliable controller so that the stability and the performance of the Takagi-Sugeno fuzzy closed-loop system can operate fine, even in the existence of some actuator faults in Shen et al. (2014). Consequently, the study on fault tolerant control or reliable control for dynamical systems has received many results, see for example (Sakthivel et al. 2014; Dong et al. 2015; Sakthivel et al. 2015). Naturally, the inclusion of time delays is inevitable phenomenon in all kinds of dynamical systems and it arises due to the finite propagation speed between nodes, traffic congestions and memory effects, and it has the capability to alter notoriously the dynamic behavior of the system such as stability. Therefore, much work has been done for synchronization of discrete-time networks with time delay (Qunjiao et al. 2019; Zhang et al. 2018; Park and Kwon 2017). The existence of time delay obeys stochastic fashion in some practical systems, it consequences the researchers have focused the synchronization analysis of CDNs with probabilistic time-varying delay (Yang et al. 2015; Cheng and Peng 2016; Cheng et al. 2019). The problem of cluster synchronization in finite-time complex networks with probabilistic coupling delays has been addressed in Yang et al. (2015).

In most of the practical networks, the abrupt variations often expose in system dynamics due to random failures, changes in the subsystem interconnections and information latching. Therefore, in order to tackle this issues, it is essential to describe CDNs with Markovian jumping parameters, where each state denotes a discrete-time system with finite discrete jumping mode governed by the Markovian process, and few interesting results regarding Markov jump systems have been addressed in Wang et al. (2016), Ren et al. (2018), Akbari et al. (2020). It is pointed that the above studies have dealt with transition rates of Markov process which are constants due to that the sojourn time between two jumps of the Markov chain is governed by exponential distribution. Also, it should be mentioned that the Markovian process might consist of time-varying transition rates when modeling practical systems, such kind of process is known as semi-Markovian process. Apparently, semi-Markovian jump CDNs are comparatively more general than the Markovian jump CDNs. Shen et al. (2015) obtained the finite-time $H_{\infty}$ synchronization criterion for complex networks with time-varying delays and semi-Markovian jump topology. The existence of sufficient conditions in Liang et al. (2018) and convex optimization technique which ensure the $L_2 - L_\infty$ synchronization for singularly perturbed complex networks subject to semi-Markovian jump topology.

Furthermore, the dissipativity theory was introduced by Willems (1972), and it serves as a powerful tool in control applications such as robotics, active vibration damping, electromechanical system and circuit theory. Generally speaking, dissipativity which postulates the quantity of energy supplied from the external source is not less than the energy lost inside the dynamic system (Peng et al. 2018). The main idea of dissipative theory is the dissipative quality of a system can keep the system internally stable. Also, the dissipativity system theory is a more general criterion when compared with passivity and $H_\infty$ performance. In Ma et al. (2020), a set of sufficient conditions is established by using Lyapunov functional methodology and completing square technique for dissipativity control of discrete nonlinear Markovian jump systems with discrete and distributed time delays. Wang et al. (2019) addresses the problem of generalized dissipativity-based synchronization for complex networks with semi-Markovian jump topology. In spite, up to now, the finite-time dissipative synchronization has not yet been reported for a class of discrete-time semi-Markovian jump CDNs.

Motivated by the above discussions, this paper analyzes finite-time synchronization problem for discrete-time semi-Markovian jumping CDNs with probabilistic time-varying delay components based on dissipative performance. The reliable control strategy and probability distribution of the time-varying delays are proposed. By constructing a new LKF, utilizing the Kronecker product technique and Lyapunov stability theory, delay-dependent sufficient conditions are derived under which the CDNs are synchronized in the given finite-time interval. The derived conditions depend not only on the size of the delay but also on the probability of the delay taking values in some intervals. Based on this derived condition, a design algorithm of the proposed state feedback controller which ensures the finite-time synchronization of the CDNs with actuator faults. Finally numerical examples are provided to illustrate the effectiveness of our theoretical results.

**Notation:** Throughout this paper, superscripts “$T$” and “$(-1)$” stand for matrix transposition and matrix inverse, respectively, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, and $\mathbb{R}^{n \times n}$ denotes the set of all $n \times n$ real matrices. $P > 0$ and $P < 0$ represent positive definite and negative definite, respectively. $L_2[0, \infty)$ stands for the space of $n$-dimensional square integrable function over $[0, \infty)$. $I$ and $0$ represent identity matrix and zero matrix with compatible dimensions. The asterisk “$*$” represents a term that is induced by symmetry. $\text{diag} \{\ldots\}$ stands for a block-diagonal matrix. The Kronecker product of matrices $S \in \mathbb{R}^{m \times n}$ and $T \in \mathbb{R}^{p \times q}$ is a matrix in $\mathbb{R}^{mp \times nq}$ and denoted as $S \otimes T$. 

\(\copyright\) Springer
2 Network model and preliminaries

Consider the following discrete-time semi-Markovian jumping CDNs consisting of N nonlinearly coupled nodes, with each node being of n-dimensional and having identical dynamic performance. It can be designated as

\[ x_i(t + 1) = A(\sigma(k))x_i(t) + B(\sigma(k))f(k, x_i(t)) \]

\[ + C(\sigma(k))f(k, x_i(k-d(t))) + \sum_{m=1}^{N} d_{im}(\sigma(k)) \Gamma_1 \]

\[ x_m(k) + \sum_{m=1}^{N} g_{lm}(\sigma(k)) \Gamma_2 x_m(k-\tau(k)) \]

\[ + R(\sigma(k))V_f^l(k) + w(k), \]

\[ z_l(k) = J(\sigma(k))x_l(k), \quad l = 1, 2, ..., N, \]

where \( x_i(k) \in \mathbb{R}^n \) denotes the state vector of the \( i^{th} \) node; \( A(\sigma(k)), B(\sigma(k)), C(\sigma(k)), R(\sigma(k)) \) and \( J(\sigma(k)) \) are constant matrices with suitable dimensions. The input of the \( d_{im}(\sigma(k)) \) are constant matrices with suitable dimensions at instant \( k \); \( f(k, x_i(k)) \) and \( f(k, x_i(k-d(t))) \) represent the nonlinear vector-valued functions without and with time delays, respectively. \( d_{im}(\sigma(k)) \) are the elements of the outer coupling configuration matrices \( D(\sigma(k)) \) and \( G(\sigma(k)) \) which describes the coupling structure of the CDNs. The positive diagonal matrices \( \Gamma_1 \) and \( \Gamma_2 \) denote inner coupling matrices with suitable dimension; \( V_f^l(k) \) is the control input of the \( l^{th} \) node. If there is a link from node \( l \) to node \( m \), then \( d_{im}(\sigma(k)) > 0 \); otherwise \( d_{im}(\sigma(k)) = 0 \).

Further, the diagonal elements of the outer coupling matrices are given as follows: \( g_{ll}(\sigma(k)) = -\sum_{m=1;l \neq m}^{N} g_{lm}(\sigma(k)); \)

\( w(k) \) denotes the disturbance input which satisfies \( w(t)w(k) \leq \delta ; z(l) \) is the objective output to be estimated.

Let \( \sigma(k), k \leq 0 \) be a discrete-time semi-Markovian process and take values from a finite set \( \psi = \{1, 2, ..., N\} \) with transition probabilities \( P(\sigma(k+1) = q|\sigma(k) = p) = \Pi_{pq} \), where \( 0 \leq \Pi_{pq} \leq 1 \) for all \( p, q \in \psi \) and \( \sum_{q \in \psi} \Pi_{pq} = 1 \). For notational simplicity, we take \( \sigma(k) = p \).

**Assumption 1** Considering the information of probability distribution of the time delays \( d(k) \) and \( \tau(k) \), we define

\[ P[d(k) \in [d_m, d_0]] = \beta_0, \quad P[d(k) \in (d_0, d_M)] = 1 - \beta_0, \quad P[\tau(k) \in [\tau_m, \tau_0]] = \alpha_0, \quad P[\tau(k) \in (\tau_0, \tau_M)] = 1 - \alpha_0, \]

where \( 0 \leq \alpha_0, \beta_0 \leq 1 \) are constants and \( d_0 \), \( d_0, \tau_0 \) are integers satisfying \( d_0 \leq d_m < d_M, \tau_m \leq \tau_0 < \tau_M \).

Therefore the stochastic variable \( \alpha(k), \beta(k) \) can be defined as

\[ \alpha(k) = \begin{cases} 1, & \tau(k) \in [\tau_m, \tau_0] \\ 0, & \tau(k) \in (\tau_0, \tau_M) \end{cases} \]

\[ \beta(k) = \begin{cases} 1, & d(k) \in [d_m, d_0] \\ 0, & d(k) \in (d_0, d_M) \end{cases} \]

**Assumption 2** \( \beta(k) \) and \( \alpha(k) \) are the Bernoulli distributed sequences with

\[ P[\beta(k) = 1] = P[d(k) \in [d_m, d_0]] = E[\beta(k)] = \beta_0, \]

\[ P[\beta(k) = 0] = P[d(k) \in (d_0, d_M)] = 1 - E[\beta(k)] = 1 - \beta_0, \]

\[ P[\alpha(k) = 1] = P[\tau(k) \in [\tau_m, \tau_0]] = E[\alpha(k)] = \alpha_0, \]

\[ P[\alpha(k) = 0] = P[\tau(k) \in (\tau_0, \tau_M)] = E[\alpha(k)] = \alpha_0. \]

where \( E[\beta(k)] \) and \( E[\alpha(k)] \) are the expectations of \( \beta(k) \) and \( \alpha(k) \), respectively.

From Assumption 1, it is easy to say that \( E[(\beta(k) - \beta_0)^2] = \beta_0(1 - \beta_0), E[\alpha(k) - \alpha_0] = 0, \)
\( E[(\alpha(k) - \alpha_0)^2] = \alpha_0(1 - \alpha_0). \)

Next, the discrete delays \( d_1(k), d_2(k), \tau_1(k) \) and \( \tau_2(k) \) are introduced in the following manner:

\[ d(k) = \begin{cases} d_1(k), & d(k) \in [d_m, d_0] \\ d_2(k), & d(k) \in (d_0, d_M) \end{cases} \]

\[ \tau(k) = \begin{cases} \tau_1(k), & \tau(k) \in [\tau_m, \tau_0] \\ \tau_2(k), & \tau(k) \in (\tau_0, \tau_M) \end{cases} \]

By using Assumptions 1 and 2, CDNs (1) can be rewritten as

\[ x_i(k + 1) = A_p x_i(k) + B_p f(k, x_i(k)) \]

\[ + \beta_0 C_p f(k, x_i(k-d_1(k))) \]

\[ + (1 - \beta_0) C_p f(k, x_i(k-d_2(k))) \]

\[ - C_p f(k, x_i(k-d_2(k))) \]

\[ \quad - \sum_{m=1}^{N} g_{lm}(\sigma(k)) \Gamma_2 x_m(k-\tau(k)) \]

\[ + \alpha_0 \sum_{m=1}^{N} g_{lm}(\sigma(k)) \Gamma_2 x_m(k-\tau_1(k)) \]

\[ + (1 - \alpha_0) \sum_{m=1}^{N} g_{lm}(\sigma(k)) \Gamma_2 x_m(k-\tau_2(k)) \]

\[ + \alpha_0(\alpha_0) \sum_{m=1}^{N} g_{lm}(\sigma(k)) \Gamma_2 x_m(k-\tau_2(k)) \]

\[ \times R P V_f^l(k) + w(k), \]

\[ z_l(k) = J_p x_l(k). \]
Assumption 3 For the nonlinear function $f(k, x(k), x(k - d_1(k)), x(k - d_2(k)))$, there exists the known real constant matrix $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{X}_3$ such that $\| f(k, x(k)) \| \leq \mathcal{X}_1 \| x(k) \|$, $\| f(k, x(k - d_1(k))) \| \leq \mathcal{X}_2 \| x(k - d_1(k)) \|$ and $\| f(k, x(k - d_2(k))) \| \leq \mathcal{X}_3 \| x(k - d_2(k)) \|$ for any $x(k) \in \mathbb{R}^n$.

Let $e_j(k) = x_j(k) - s(k)$ be the synchronization error, where $s(k)$ be a solution of an isolated node which is described as

$$s(k + 1) = A_p s(k) + B_p f(k, s(k)) + \beta_0 C_p f(k, s(k - d_1(k))) + (1 - \beta_0) C_p f(k, s(k - d_2(k))) + \beta(k - \beta_0)[C_p f(k, s(k - d_1(k))) - C_p f(k, s(k - d_2(k)))]$$

Then the corresponding error dynamics of the CDNs (1) can be obtained as follows:

$$e_j(k + 1) = A_p e_j(k) + B_p F(k, e_j(k)) + \beta_0 C_p f(k, e_j(k - d_1(k))) + (1 - \beta_0) C_p f(k, e_j(k - d_2(k))) + \beta(k - \beta_0)[C_p f(k, e_j(k - d_1(k))) - C_p f(k, e_j(k - d_2(k)))]$$

$$+ \sum_{m=1}^{N} d_{imp} \Gamma_1 e_m(k) \alpha_0 \sum_{m=1}^{N} g_{imp} \Gamma_2 e_m(k - \tau_1(k)) + \sum_{m=1}^{N} (1 - \alpha_0) \sum_{m=1}^{N} g_{imp} \Gamma_2 e_m(k - \tau_2(k)) + (\alpha(k) - \alpha_0)$$

$$\times \left[ \sum_{m=1}^{N} g_{imp} \Gamma_2 e_m(k - \tau_1(k)) - \sum_{m=1}^{N} g_{imp} \Gamma_2 e_m(k - \tau_2(k)) \right] + R_p V_f^T(k) + w(k),$$

where $F(k, e_j(k)) = f(k, x_j(k)) - f(k, s(k)).$

Now consider the actuator fault model, for the control input $V_f^T(k)$. We designate $V_f^T(k)$ to describe the signal sent from the actuator and satisfies

$$V_f^T(k) = \mathbb{G} U_f(k),$$

where $\mathbb{G}$ is the actuator fault matrix defined as $\mathbb{G} = \text{diag}(g_1, g_2, \ldots, g_m)$, $0 \leq g_i \leq g_i \leq \bar{g}_i \leq 1$, where $g_i$ and $\bar{g}_i$, $i = 1, 2, \ldots, m$ are given constants. $g_i = 0$ means that $i^{th}$ actuator fails, $g_i = 1$ means that $i^{th}$ actuator is normal, when $0 < g_i < 1$ denotes the $i^{th}$ actuator meets partial failures. In this paper, we design a state feedback controller of the form:

$$U_f(k) = K_p e_f(k),$$

where $K_p$ is the state feedback controller gain matrix to be determined. Then, by substituting (5) in (4) and using Kronecker product properties, the resulting closed-loop system can be written as

$$e(k + 1) = \bar{A} e(k) + \bar{B} F(k, e(k)) + \beta_0 \bar{C} F(k, e(k - d_1(k))) + (1 - \beta_0) \bar{C} F(k, e(k - d_2(k)))$$

$$+ \beta(k - \beta_0)[\bar{C} F(k, e(k - d_1(k))) - \bar{C} F(k, e(k - d_2(k)))] + \bar{D} e(k) + \alpha_0 \bar{G} e(k - \tau_1(k))$$

$$+ (1 - \alpha_0) [\bar{G}(e(k - \tau_2(k)) + (\alpha(k) - \alpha_0)[\bar{G}(e(k - \tau_1(k)) - \bar{G}(e(k - \tau_2(k)))]$$

$$+ \bar{R} \mathbb{G} K_p e(k) + w(k),$$

where $\bar{A} = (I \otimes A_p), \bar{B} = (I \otimes B_p), \bar{C} = (I \otimes C_p), \bar{D} = (D_p \otimes \Gamma_1), \bar{G} = (G_p \otimes \Gamma_2), \bar{R} = (I \otimes R_p), \bar{J} = (I \otimes J_p), e(t) = [e_1(t), e_2(t), \ldots, e_N(t)]^T, F(k, e(k)) = [F^N(k, e_1(k)), F^T(k, e_2(k)), \ldots, F^T(k, e_N(k))]^T,$

$$F(k, e(k - \tau_1(k))) = [F^T(k, e_1(k - \tau_1(k))), F^T(k, e_2(k - \tau_1(k)), \ldots, F^T(k, e_N(k - \tau_1(k)))]^T, F(k, e(k - \tau_2(k))) = [F^T(k, e_1(k - \tau_2(k))), F^T(k, e_2(k - \tau_2(k)), \ldots, F^T(k, e_N(k - \tau_2(k)))]^T.$$

Further, we introduce the following lemmas and definitions, which will be important for the derivations of the main results.

Lemma 1 (Schur complement Cheng et al. 2015) Given constant matrices $E_{11}$, $E_{12}$ and $E_{22}$ with appropriate dimensions where $E_{11}^T = E_{11}$ and $E_{22}^T = E_{22}$ then $E_{11} + E_{12} E_{22}^{-1} E_{12} < 0$ iff $\left[ \begin{array}{cc} E_{11} & E_{12}^T \\ E_{12} & -E_{22} \end{array} \right] < 0$.

Definition 1 (Cheng et al. 2015) The error dynamics (6) subject to an exogeneous disturbance $w(k)$ satisfying $\sum_{k=0}^{\infty} w(k) w(k) < \delta$ is said to be stochastically finite-time bounded (SFTB) with respect to $(c_1, c_2, S_p, N, \delta)$, where $S_p(p \in \psi)$ is a positive definite matrix, $0 < c_1 < c_2$, $\delta > 0$ if $E(e^T(k_1)(I \otimes S_p) e(k_1)) \leq c_1 \Longrightarrow E(e^T(k_2)(I \otimes S_p) e(k_2)) \leq c_2, \forall k_1 \in [-d, -d + 1, \ldots, 0], \forall k_2 \in [1, 2, \ldots, N], \text{where} \ d = \max(d_M, \tau_M)$.

Definition 2 (Zhang et al. 2014) The error dynamics (6) is said to be stochastically finite-time stable (SFTS) with respect to $(c_1, c_2, S_p, N)$, where $S_p$ is a positive definite
In this section, to design a state feedback controller which
matrix, $0 < c_1 < c_2$ and $w(k) = 0$ if $E[e^T(k_1)(I \otimes S_p)e(k_1)) ≤ c_1 \iff E[e^T(k_2)(I \otimes S_p)e(k_2)) ≤ c_2, \forall k_1 \in \{-d_M, -d_M + 1, \ldots, 0\}, \forall k_2 \in \{1, 2, \ldots, N\}$.

**Definition 3** (Song et al. 2017) The error dynamics (6) is
$(L, M, \mathbb{R})$-γ dissipative with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, where $0 < c_1 < c_2$, $\gamma > 0$ and $S_p$ is a positive definite matrix, if the system is stochastically finite-time bounded (SFTB) with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, and under the zero initial condition, the output $z(k)$ satisfies

$$
\sum_{k=0}^{N}[\psi^T(k)E(k) + 2e^T(k)Mw(k) + w^T(k)Rw(k)]
\geq \gamma \sum_{k=0}^{N}w^T(k)w(k),
$$

for any nonzero satisfies $\sum_{k=0}^{N}w^T(k)w(k) < \delta$, here $L$, $M$ and $R$ are real-valued matrices of appropriate dimensions with $L$ and $R$ are symmetric. Without loss of generality, it is assumed that $L \leq 0$, then we have $-L = (L^{1/2})^2$.

**Remark 1** In the view points of the above definitions, it should be noted that the SFTB of the error system (6) can guarantee the SFTB. That is, in the absence of exogenous disturbance inputs $w(k) = 0$, the concept of SFTB is reduced to SFTS. Thus, SFTB implies SFTS, but the converse is not true. Consequently, the system (1) is finite-time synchronized under the feedback control law (5).

## 3 Main results

In this section, to design a state feedback controller which ensures the stochastic synchronization in a finite-time interval of the CDSNs (1). First we find the stochastic finite-time boundedness for the closed-loop system (6). Next, we find the stochastic finite-time boundedness with dissipativity performance index.

### A. Stochastic finite-time boundedness

**Theorem 1** Under Assumptions 1 and 2, given scalars $\mu \geq 1, k_1 > 0 (i = 1, 2)$ and known actuator fault matrix $\mathbb{G}$, the system (6) is stochastic finite-time bounded with respect to $(c_1, c_2, S_p, \mathbb{N}, \delta)$, if there exist symmetric matrices $P_p > 0 (p \in \psi), Q_b > 0 (b = 1, 2, \ldots, 6), T_e > 0$ and $U_e > 0 (c = 1, 2), V_l > 0 (l = 1, 2, 3, 4)$ such that the following inequalities hold:

$$
\Psi = \begin{bmatrix}
\Gamma_{20 \times 20} & \Sigma_1 & \Sigma_2 \\
* & -I & 0 \\
* & * & \Sigma_3
\end{bmatrix} < 0,
$$

$$
\xi_1 + \lambda w \delta < \lambda_1 c_2 \mu^{-k},
$$

where $\Gamma_{1,1} = -\mu (I \otimes P_p) + (I \otimes Q_1) + (I \otimes Q_2) + (I \otimes Q_3) + (I \otimes Q_4) + (I \otimes Q_5) + (I \otimes Q_6) + (d_0 - d_m + 1)(I \otimes T_1) + (t_0 - t_m + 1)(I \otimes T_2) + (d_M - d_m + 1)(I \otimes U_1) + (\tau_m - t_0 + 1)(I \otimes U_2) + d_m(I \otimes V_1) + d_q(I \otimes V_2) + \tau_m(I \otimes V_3) + \tau_m(I \otimes V_4) + k_1(I \otimes P_p)(A + R\Gamma K_p + D - 1) + (A + R\Gamma K_p + D - 1)^T(I \otimes P_p)k_1$,

$$
\Gamma_{1,11} = k_1(1 - \alpha_0)(I \otimes P_p)\mathbb{G}, \Gamma_{1,12} = k_1(I \otimes P_p)\mathbb{B}, \Gamma_{1,13} = k_1\beta_0(I \otimes P_p)\mathbb{C}, \Gamma_{1,14} = k_1(1 - \beta_0)(I \otimes P_p)\mathbb{C} \Gamma_{1,19} = -k_1(I \otimes P_p)(A + R\Gamma K_p + D - 1) + (I \otimes P_p)k_2$,

$$
\sum_{k=0}^{N}[e^T(k)Le(k) + 2e^T(k)Mw(k) + w^T(k)Rw(k)]
\geq \gamma \sum_{k=0}^{N}w^T(k)w(k),
$$

for any nonzero satisfies $\sum_{k=0}^{N}w^T(k)w(k) < \delta$, here $L$, $M$ and $R$ are real-valued matrices of appropriate dimensions with $L$ and $R$ are symmetric. Without loss of generality, it is assumed that $L \leq 0$, then we have $-L = (L^{1/2})^2$.

**Proof** Construct the following LKF for the error system (6) as

$$
\mathcal{V}(k) = \sum_{b=1}^{5} \mathcal{V}_b(k),
$$

where

$$
\mathcal{V}_b(k) = \begin{cases}
\xi_1 + \lambda w \delta < \lambda_1 c_2 \mu^{-k},
\end{cases}
$$
where

\[ V_1(k) = e^T(k)(I \otimes P_p)e(k), \]

\[ V_2(k) = \sum_{j=k-d_m}^{k-1} e^T(j)(I \otimes Q_1)e(j) + \sum_{j=k-d_0}^{k-1} e^T(j)(I \otimes Q_2)e(j) + \sum_{j=k-d_M}^{k-1} e^T(j)(I \otimes Q_3)(k)e(j) + \sum_{j=k-t_0}^{k-1} e^T(j)(I \otimes Q_4)e(s) + \sum_{j=k-t_M}^{k-1} e^T(j)(I \otimes Q_5)e(j), \]

\[ V_3(k) = \sum_{j=k-d_1(k)}^{k-1} e^T(j)(I \otimes T_1)e(j) + \sum_{j=d_0}^{d_0-1} e^T(l)(I \otimes T_1)e(l) + \sum_{j=k-t_1(k)}^{k-1} e^T(j)(I \otimes T_2)e(j) + \sum_{j=t_0}^{t_0-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes T_2)e(l). \]

\[ V_4(k) = \sum_{j=k-d_2(k)}^{k-1} e^T(j)(I \otimes U_1)e(j) + \sum_{j=d_1}^{d_1-1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes U_1)e(l) + \sum_{j=k-t_2(k)}^{k-1} e^T(j)(I \otimes U_2)e(j) + \sum_{j=t_0+1}^{t_0+1} \sum_{l=k-j}^{k-1} e^T(l)(I \otimes U_2)e(l), \]

\[ V_5(k) = \sum_{j=k-d_m}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_1)e(l) + \sum_{j=k-d_M}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_2)e(l) + \sum_{j=k-t_M}^{k-1} \sum_{l=j}^{k-1} e^T(l)(I \otimes V_3)e(l). \]

Then, taking mathematical expectation of the forward difference formula \( \Delta V(k) = V(k+1) - V(k) \) along the trajectories of the system (6), we have

\[ \mathbb{E}[\Delta V_1(k)] = \mathbb{E}[V_1(k+1) - V_1(k)] = \mathbb{E}[e^T(k+1) - e^T(k)P_pe(k)] = \mathbb{E}[e^T(k)P_pe(k)]. \]

where \( \eta(k) = e(k+1) - e(k) \)

\[ \mathbb{E}[\Delta V_2(k)] = \mathbb{E}[V_2(k+1) - V_2(k)] = \mathbb{E}[e^T(k)\sum_{j=1}^{6}(I \otimes Q_j)e(k) \]

\[ - e^T(k)d_m(I \otimes Q_1)e(k-d_m) - e^T(k-d_0)(I \otimes Q_2) \]

\[ e(k-d_0) - e^T(k-d_M)(I \otimes Q_3)e(k-d_M) - e^T(k-t_0)(I \otimes Q_5)e(k-t_0) - e^T(k-t_M)(I \otimes Q_6)e(k-t_M). \]

\[ \mathbb{E}[\Delta V_3(k)] = \mathbb{E}[V_3(k+1) - V_3(k)] \]

\[ \leq \mathbb{E}[(d_0 - d_m + 1)e^T(k)(I \otimes T_1)e(k) \]

\[ - e^T(k-d_1(k))(I \otimes T_1)e(k-d_1(k)) \]

\[ + (t_0 - t_m + 1)e^T(k)(I \otimes T_2)e(k) - e^T(k-t_1(k))(I \otimes T_2)e(k-t_1(k))]. \]

\[ \mathbb{E}[\Delta V_4(k)] = \mathbb{E}[V_4(k+1) - V_4(k)] \]

\[ \leq \mathbb{E}[(d_M - d_0 + 1)e^T(k)(I \otimes U_1)e(k) \]

\[ - e^T(k-d_2(k))(I \otimes U_1)e(k-d_2(k)) \]

\[ + (t_M - t_0 + 1)e^T(k)(I \otimes U_2)e(k) - e^T(k-t_2(k))(I \otimes U_2)e(k-t_2(k))]. \]

\[ \mathbb{E}[\Delta V_5(k)] = \mathbb{E}[V_5(k+1) - V_5(k)] \]

\[ \leq \sum_{j=k-d_m}^{k-1} \sum_{l=j}^{k-1} e(k) - \frac{1}{d_m} \sum_{j=k-d_m}^{k-1} e^T(k)(I \otimes V_1) \]

\[ d_m e^T(k)(I \otimes V_1)e(k) - \frac{1}{d_m} \sum_{j=k-d_m}^{k-1} e^T(k)(I \otimes V_1) \]

\[ \sum_{j=k-d_M}^{k-1} e(k) - \frac{1}{d_M} \sum_{j=k-d_M}^{k-1} e^T(k)(I \otimes V_2) \]

\[ \sum_{j=k-t_M}^{k-1} e(k) - \frac{1}{d_M} \sum_{j=k-t_M}^{k-1} e^T(k)(I \otimes V_3) \]

\[ \sum_{j=k-d_M}^{k-1} e(k) - \frac{1}{d_M} \sum_{j=k-d_M}^{k-1} e^T(k)(I \otimes V_4)e(k). \]
Furthermore, we have
\[ \eta(k) = e(k+1) - e(k) \]
\[ = (\bar{A} + \bar{R}GK_p + \bar{D} - 1)e(k) + \bar{B}F(k, e(k)) + \bar{C}F(k, e(k-d_1(k))) + (1 - \beta_0)\bar{C}(k, e(k-d_2(k))) \]
\[ - \bar{C}F(k, e(k-d_1(k))) \]
\[ + \alpha_0\bar{G}e(k - \tau_1(k)) + (1 - \alpha_0)\bar{G}(e(k - \tau_2(k))) \]
\[ - \bar{G}(e(k - \tau_2(k))) \]  
\[ = (\bar{A} + \bar{R}GK_p + \bar{D} - 1)e(k) + \bar{B}F(k, e(k)) + \bar{C}F(k, e(k-d_1(k))) + (1 - \beta_0)\bar{C}(k, e(k-d_2(k))) \]
\[ + \bar{C}F(k, e(k-d_1(k))) \]
\[ + \alpha_0\bar{G}e(k - \tau_1(k)) + (1 - \alpha_0)\bar{G}(e(k - \tau_2(k))) \]
\[ - \bar{G}(e(k - \tau_2(k))) \]  
\[ = e^T(k) = [e^T(k) \eta^T(k)] \].

From Assumption 3, we can obtain the following inequalities
\[ e^T(k)(I \otimes W)(I \otimes W)^T e(k) - F^T(k, x(k)) F(k, x(k)) \geq 0, \]
\[ e^T(k - d_1(k))(I \otimes E)(I \otimes E)^T e(k - d_1(k)) \]
\[ - F^T(k, x(k - d_1(k))) F(k, x(k - d_1(k))) \geq 0, \]
\[ e^T(k - d_2(k))(I \otimes H)(I \otimes H)^T e(k - d_2(k)) \]
\[ - F^T(k, x(k - d_2(k))) F(k, x(k - d_2(k))) \geq 0. \]

Combining (9)–(18), we have
\[ \mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\tilde{W}w(k)] \]
\[ \leq \xi^T(k)[\Omega_{[20 \times 20]} + E_1^T E_1 + \mathbb{E}_2^T] \xi(k), \]
\[ \xi^T(k)[\Omega_{[20 \times 20]} + E_1^T E_1 + \mathbb{E}_2^T] \xi(k), \]

where \( \xi(k) = [e^T(k) \ e^T(k - d_1(k)) \ e^T(k - d_2(k)) \ e^T(\tau_0(k)) \ e^T(k - \tau_M(k)) \ e^T(k - \tau_1(k)) \ e^T(k - \tau_2(k))] \)
\[ + \sum_{k=d_1}^{k=d_2} e^T(k) \]
\[ + \sum_{k=\tau_0}^{k=\tau_M} e^T(k) \]
\[ + \sum_{k=\tau_1}^{k=\tau_M} e^T(k) \]
\[ + \sum_{k=\tau_2}^{k=\tau_M} e^T(k) \]
\[ w^T(k)], \]
\[ \xi^T(k)[\Omega_{[20 \times 20]} + E_1^T E_1 + \mathbb{E}_2^T] \xi(k), \]
\[ \mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\tilde{W}w(k)] \leq 0. \]

Further, by using Schur complement to the right hand side of (19), we get the required LMI in (7). If the matrix inequality in (7) holds, it is obvious that
\[ \mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\tilde{W}w(k)] \leq 0. \]

Further, if \( \mu \geq 1 \), it follows that
\[ \mathbb{E}[\Delta V(k) - (\mu - 1)V(k) - w^T(k)\tilde{W}w(k)] \leq 0. \]

Next, we define the following parameters:
\[ \mathcal{P}_p = S_p^{-\frac{1}{2}} P_p S_p^{-\frac{1}{2}}, \]
\[ \mathcal{Q}_{bp} = S_p^{-\frac{1}{2}} Q_b S_p^{-\frac{1}{2}}, \]
\[ b = 1, \ldots, 6, \]
\[ \mathcal{F}_{cp} = S_p^{-\frac{1}{2}} T_c S_p^{-\frac{1}{2}}, \ c = 1, 2, \]
\[ \mathcal{V}_{ip} = S_p^{-\frac{1}{2}} U_j S_p^{-\frac{1}{2}}, \ j = 1, 2, \]
\[ \mathcal{V}_{lp} = S_p^{-\frac{1}{2}} V_l S_p^{-\frac{1}{2}}, \ l = 1, 2, 3, 4. \]
Then, from (8), we can have
\[
\mathbb{E} \{ \mathcal{V}(0) \} = \mathbb{E} \{ e^T(0) (I \otimes S_p^1 \mathcal{P}_p S_p^1) e(0) \} \\
+ \sum_{j=-d_m}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{P}_p S_p^1) e(j) + \sum_{j=-d_0}^{-1} e^T(j) \\
\times (I \otimes S_p^1 \mathcal{P}_2 p S_p^1) e(j) \\
+ \sum_{j=-d_M}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{P}_3 p S_p^1) e(j) \\
+ \sum_{j=-\tau_m}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_4 p S_p^1) \\
\times e(j) + \sum_{j=-\tau_0}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_p S_p^1) e(j) \\
+ \sum_{j=-\tau_m}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_p S_p^1) e(j) + \sum_{j=-d_M}^{-1} e^T(j) \\
\times (I \otimes S_p^1 \mathcal{Q}_2 p S_p^1) e(j) \\
+ \sum_{j=-d_M}^{-1} \sum_{l=-j}^{d_0-1} e^T(l) (I \otimes S_p^1 \mathcal{Q}_1 p S_p^1) e(l) \\
+ \sum_{j=-d_1(0)}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_2 p S_p^1) e(l) \\
\times e(j) + \sum_{j=-\tau_0}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_2 p S_p^1) e(l) \\
+ \sum_{j=-d_M}^{-1} \sum_{l=-j}^{d_0-1} e^T(l) (I \otimes S_p^1 \mathcal{Q}_1 p S_p^1) e(l) \\
+ \sum_{j=-d_M}^{-1} \sum_{l=-j}^{d_1(0)} e^T(l) (I \otimes S_p^1 \mathcal{Q}_2 p S_p^1) e(l) \\
\times \sum_{j=-\tau_m}^{-1} e^T(j) (I \otimes S_p^1 \mathcal{Q}_2 p S_p^1) e(j) \tag{21}
\]

\[
\leq c_1 \left[ \lambda_2 + \lambda_3 d_m + \lambda_4 d_0 + \lambda_5 d_M + \lambda_6 \tau_m + \lambda_7 \tau_0 + \lambda_8 \tau M \\
+ \lambda_9 d_0 + \lambda_9 \left( (d_0 - d_m) (d_m + d_0 - 1) \right) \right] \\
+ \lambda_{10} \frac{2}{\tau_0 - \tau_m} \left( \tau_m + \tau_0 - 1 \right) \\
+ \lambda_{11} d_M + \lambda_{11} \frac{2}{d_0 + d_m} \left( d_m - d_0 + 1 \right) \right] \\
+ \lambda_{12} \frac{2}{\tau_0 + \tau_m} \left( \tau_m - \tau_0 + 1 \right) \right] \\
+ \lambda_{13} \frac{2}{\tau_0 - \tau_m} \left( \tau_m + \tau_0 - 1 \right) \right] \\
+ \lambda_{14} \frac{2}{\tau_0 - \tau_m} \left( \tau_m + \tau_0 - 1 \right) \right] \\
\leq \zeta_1 c_1. \tag{22}
\]

Hence \( \mathbb{E} \{ \mathcal{V}(k) \} \geq \lambda_1 \mathbb{E} \{ e^T(k) (I \otimes S_p) e(k) \} \).

By using the inequalities (20) and (21), we can obtain
\[
\mathbb{E} \{ e^T(k) (I \otimes S_p) e(k) \} \\
\leq \frac{1}{\lambda_1} (\zeta_1 c_1 + \lambda_1 \lambda \mu^k) \mu^k \\
< c_2, \quad \forall k \in \{1, 2, \ldots, \}
\]

From the above computations, we can conclude that the closed-loop error system (6) is stochastically finite-time bounded with respect to \((c_1, c_2, S_p, N, l)\).

\[ \square \]

**B. Stochastic finite-time dissipative**

A sufficient condition is given to ensure the finite-time boundedness with dissipative performance of CDNs is analyzed in the following theorem:

**Theorem 2** Under Assumptions 1 and 2, for given scalar \( \mu \geq 1 \), the error system (6) is stochastic finite-time dissipative with respect to \((c_1, c_2, S_p, N, l)\), if there exist symmetric matrices \( X_p \succ 0 \) \((p \in \psi)\), \( \hat{Q}_{bp} \succ 0 \) \((b = 1, 2, \ldots, 6)\), \( \hat{T}_{cp} > 0 \), \( \hat{U}_{cp} > 0 \) \((c = 1, 2)\), \( \hat{V}_{lp} > 0 \) \((l = 1, 2, 3, 4)\) such that the following conditions hold:

\[
\hat{\Psi} = \begin{bmatrix}
\hat{\Gamma}_{20} & \hat{\xi}_1 & \hat{\xi}_2 & \gamma \\
* & -I & 0 & 0 \\
* & * & \hat{\xi}_3 & 0 \\
* & * & * & -I
\end{bmatrix} < 0, \tag{23}
\]

\[
\hat{\xi}_1 c_1 + \lambda_1 \delta < c_2 \mu^{-k}, \tag{24}
\]

where \( \hat{\Gamma}_{11} = -\mu(I \otimes X_p) + (I \otimes \hat{Q}_1 p) + (I \otimes \hat{Q}_2 p) + (I \otimes \hat{Q}_3 p) + (I \otimes \hat{Q}_4 p) + (I \otimes \hat{Q}_5 p) + (I \otimes \hat{Q}_6 p) + (d_0 - d_m + 1)(I \otimes \hat{T}_p) + (\tau_0 - \tau_m + 1)(I \otimes \hat{T}_2 p) + (d_M - d_0 + 1)(I \otimes \hat{U}_1 p) + (\tau_M - \tau_0 + 1)(I \otimes \hat{U}_2 p) + d_m (I \otimes \hat{V}_1 p) + d_M (I \otimes \hat{V}_2 p) + d_m (I \otimes \hat{V}_2 p)
\]
\[ \dot{V}_3(t) + \tau_1 (I \otimes \dot{V}_3) + \tau_2 (I \otimes \dot{V}_4) + \tau_3 (I \otimes \dot{V}_5) + \tau_4 (I \otimes X_p) + \tau_5 (I \otimes X_p)^T + \tau_6 (I \otimes X_p)^T \leq 0. \]

Proof

In order to describe the dissipative performance of the closed-loop system \( \Sigma \), we consider the energy function \( \mathcal{J} \) as

\[ \mathcal{J} = \mathbb{E} \left\{ \sum_{k=0}^{N-1} \left[ e^T(k) \mathcal{L} e(k) + 2 e^T(k) \mathcal{M} w(k) + w^T(k) [\mathcal{R} - \gamma I] w(k) \right] \right\}. \]

The proof follows from Theorem 1, and it follows that

\[ \mathbb{E}(\Delta V(k)) - \mathbb{E} (\mu V(0) + \sum_{i=0}^{k-1} \mathbb{E} \left( \sum_{j=i+1}^{k-1} [e^T(j) \mathcal{L} e(j) + 2 e^T(j) \mathcal{M} w(j + 1) + w^T(j + 1) [\mathcal{R} - \gamma I] w(j + 1) ] \right) ] \]

By simple modifications, it is quite easy to get that

\[ \mathbb{E}[\Delta V(k)] \leq \mathbb{E}[\mu V(0)] + \sum_{i=0}^{k-1} \mathbb{E}[\sum_{j=i+1}^{k-1} e^T(j) \mathcal{L} e(j) + 2 e^T(j) \mathcal{M} w(j + 1) + w^T(j + 1) [\mathcal{R} - \gamma I] w(j + 1) ] \]

Under zero initial condition and the fact \( V(k) \geq 0, \forall k = 1, 2, \ldots, N \), we have

\[ \mathbb{E} \left\{ \sum_{i=0}^{k-1} \mathbb{E}[\sum_{j=i+1}^{k-1} e^T(j) \mathcal{L} e(j) + 2 e^T(j) \mathcal{M} w(j + 1) + w^T(j + 1) [\mathcal{R} - \gamma I] w(j + 1) ] \right\} \geq 0. \]

Further, if \( \mu \geq 1 \), the above inequality implies that

\[ \mathbb{E} \left\{ \sum_{i=0}^{k-1} \mathbb{E}[\sum_{j=i+1}^{k-1} e^T(j) \mathcal{L} e(j) + 2 e^T(j) \mathcal{M} w(j + 1) + w^T(j + 1) [\mathcal{R} - \gamma I] w(j + 1) ] \right\} \geq 0. \]

Then, from (26), it is easily to get the inequality in the Definition 3. To complete the proof of this theorem, let

\[ X_p = P_{p,1}, \quad \dot{Q}_{bp} = X_p Q_p X_p (b = 1, 2, \ldots, 6), \quad T_{cp} = X_p T_{cp} X_p, \quad U_{cp} = X_p U_{cp} X_p (c = 1, 2), \quad V_{bp} = X_p V_{bp} X_p (c = 1, 2, 3, 4), \]

Then performing the congruence transformations to (7) by

\[ \text{diag}(I \otimes X_p), \ldots, (I \otimes X_p), (I \otimes I), (I \otimes I), (I \otimes X_p), \ldots, (I \otimes X_p), (I \otimes I), (I \otimes I), \underline{U}, \]

where \( \underline{U} = -\text{diag}([I \otimes X_1, (I \otimes X_2), \ldots, (I \otimes X_N)] \) and Letting \( K_p X_p = Y_p \), we can easily to get the required LMI (23).

Further, \( \lambda_1 \leq (I \otimes S_p^{-1} P_{p} S_p^{-1} ) \leq \lambda_2, \quad 0 < (I \otimes S_p^{-1} Q_{1} S_p^{-1} ) \leq \lambda_3, \quad 0 < (I \otimes S_p^{-1} Q_2 S_p^{-1} ) \leq \lambda_4, \quad 0 < (I \otimes S_p^{-1} Q_3 S_p^{-1} ) \leq \lambda_5, \quad 0 < (I \otimes S_p^{-1} Q_4 S_p^{-1} ) \leq \lambda_6, \quad 0 < (I \otimes S_p^{-1} Q_5 S_p^{-1} ) \leq \lambda_7, \quad 0 < (I \otimes S_p^{-1} Q_6 S_p^{-1} ) \leq \lambda_8, \quad 0 < (I \otimes S_p^{-1} T_1 S_p^{-1} ) \leq \lambda_9, \quad 0 < (I \otimes S_p^{-1} T_2 S_p^{-1} ) \leq \lambda_{10}, \quad 0 < (I \otimes S_p^{-1} U_1 S_p^{-1} ) \leq \lambda_{11}, \quad 0 < (I \otimes S_p^{-1} U_2 S_p^{-1} ) \leq \lambda_{12}, \quad 0 < (I \otimes S_p^{-1} V_1 S_p^{-1} ) \leq \lambda_{13}, \quad 0 < (I \otimes S_p^{-1} V_2 S_p^{-1} ) \leq \lambda_{14}, \quad \lambda_{14} \leq \lambda_{13} \leq \lambda_{12} \leq \lambda_{11} \leq \lambda_{10} \leq \lambda_{9} \leq \lambda_{8} \leq \lambda_{7} \leq \lambda_{6} \leq \lambda_{5} \leq \lambda_{4} \leq \lambda_{3} \leq \lambda_{2} \leq \lambda_{1} \leq \lambda_0. \]
0 < (I \otimes S_p^{-\frac{1}{2}} V_3 S_p^{-\frac{1}{2}}) < \lambda_{15}, \text{ and } 0 < (I \otimes S_p^{-\frac{1}{2}} V_4 S_p^{-\frac{1}{2}}) < \lambda_{16}.

Then, according to the congruence transformations, these above relations can be altered into \( \lambda_2^{-1} (I \otimes S_p^{-1}) < (I \otimes X_p) < \lambda_1^{-1} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{1p}) < \lambda_1^{-2} \lambda_3 (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{2p}) < \lambda_1^{-2} \lambda_4 (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{4p}) < \lambda_1^{-2} \lambda_6 (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{5p}) < \lambda_1^{-2} \lambda_8 (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{6p}) < \lambda_1^{-2} \lambda_9 (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{7p}) < \lambda_1^{-2} \lambda_{10} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{8p}) < \lambda_1^{-2} \lambda_{11} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{Q}_{9p}) < \lambda_1^{-2} \lambda_{12} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{V}_{1p}) < \lambda_1^{-2} \lambda_{13} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{V}_{2p}) < \lambda_1^{-2} \lambda_{14} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{V}_{3p}) < \lambda_1^{-2} \lambda_{15} (I \otimes S_p^{-1}) \), \( 0 < (I \otimes \hat{V}_{4p}) < \lambda_1^{-2} \lambda_{16} (I \otimes S_p^{-1}) \), and it can easily receive the inequality (24). Hence, completes the proof. \( \square \)

**Remark 2** By using Definitions 2 and 3, it should be noted that if \( w(k) = 0 \), the concept of SFTB is reduced to SFTS. Also we can observe that if the error system (6) is stochastically finite-time stable, then the proposed CDNs (1) is finite-time synchronized under the control law (5). In CDNs (1), removing the nonlinear function and external disturbance and considering only one mode in operation, then it can be rewritten as follows:

\[
\begin{aligned}
\dot{x}_i(k+1) &= A x_i(k) + B f(k, x_i(k)) + C f(k, x_i(k)) \\
&+ \sum_{m=1}^{N} d_{im} \Gamma_1 x_m(k) + \sum_{m=1}^{N} g_{lm} \Gamma_2 x_m(k - \tau(k)) \\
&+ RV_i^f(k), \\
\dot{z}_i(k) &= J x_i(k), \quad l = 1, 2, ..., N.
\end{aligned}
\]

(27)

Then the closed-loop error system (6) becomes

\[
\begin{aligned}
\dot{e}(k+1) &= \hat{A} e(k) + \hat{B} F(k, e(k)) + \beta_0 \hat{C} F(k, e(k)) + \hat{D} e(k) \\
&+ \alpha_0 \hat{G} e(k - \tau_1(k)) + (1 - \alpha_0) \times \hat{G} e(k - \tau_2(k))) + \alpha_0 \hat{G} e(k - \tau_1(k))) \\
&- \hat{G} e(k - \tau_2(k))) + \hat{R} \hat{G} K e(k), \\
\dot{z}(k) &= \hat{J} e(k).
\end{aligned}
\]

(28)

**Corollary 1** For given matrices \( S > 0 \), and scalars \( \mu \geq 1, c_1 > 0, c_2 > 0, N \text{ and } \alpha_i \geq 0 (i = 1, 2) \) and under Assumptions 1 and 2, the error system (27) is stochastic finite-time synchronization with respect to \( (c_1, c_2, S, N) \), if there exist symmetric matrices \( X > 0, \hat{Q}_b > 0 \) \((b = 1, 2, 3)\), \( \hat{T}_c > 0, \hat{U}_c > 0 \) \((c = 1, 2, 3)\), \( \hat{V}_l > 0 \) \((l = 1, 2)\), such that the following inequalities hold:
\[
\sum_{k=1}^{\tau_m} e^T(k) - \sum_{k=1}^{\tau_M} e^T(k) \eta^T(k).
\]

Remaining proof follows from Theorem 2, we can obtain the required result. Thus, the proof is completed. \qed

4 Numerical simulations

This section provides a simulation examples to show the effectiveness and superiority of the established criteria for finite-time synchronization of the proposed CDNs. The schematic diagram of synchronization for the addressed system with actuator faults is represented in Fig. 1.

Example 1 Consider a class of CDNs in the form of (2) with three nodes, and dimension of the state vector of each node is two.

Mode 1:
\[
A_1 = \begin{bmatrix} -1.43 & 1.14 \\ 1.23 & -1.26 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.25 & 2.0095 \\ 0.15 & -1.45 \end{bmatrix},
C_1 = \begin{bmatrix} 0.54 & 0.05 \\ 0.42 & 0.23 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 1.01 & 0.02 \\ 0 & 0.01 \end{bmatrix}, \quad R_1 = \begin{bmatrix} 1.021 & 0 \\ 3.86 & 0.011 \end{bmatrix}
\]

Mode 2:
\[
A_2 = \begin{bmatrix} -0.79 & 0.26 \\ 0.31 & 0.13 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1.05 & 1.95 \\ -0.05 & 2.05 \end{bmatrix},
C_2 = 0.1 \begin{bmatrix} -2.14 & 0.05 \\ 0.042 & -1.23 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 1.1 & 0.01 \\ 0 & 1.2 \end{bmatrix},
R_2 = \begin{bmatrix} 0.052 & 1.96 \\ 0.01 & -0.1 \end{bmatrix}, \quad \alpha_0 = 0.6, \beta_0 = 0.4.
\]

The inner and outer coupling matrices of non-delayed and delayed terms are taken as
\[
\Gamma_1 = diag\{0.09, 0.09\}, \Gamma_2 = diag\{0.25, 0.25\}
\]
Moreover, the other parameters are considered as $c_\gamma = M = G$ the actuator failure matrix is estimated with the matrices $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}_3 = \text{diag}(0.4, 0.4)$ and also the noise signal is chosen as $w(k) = 0.05 \exp(-0.1k) \sin k$.

Further, the time delays are taken as $d_1(k) = 3.01 + 0.25 \sin(0.1k)$, $d_2(k) = 3.5 + 0.25 \sin(0.1k)$, $\tau_1(k) = 1.05 + 0.25 \sin(0.5k)$, $\tau_2(k) = 2.01 + 0.25 \sin(0.02k)$, and the actuator failure matrix is $G = \text{diag}(0.8, 0.8)$. The transition probability matrices are $\begin{bmatrix} 0.2 & 0.8 \\ 0.35 & 0.65 \end{bmatrix}$ and $\begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$. Moreover, the other parameters are considered as $c_1 = 0.01$, $c_2 = 13.8344$, $S_1 = S_2 = 0.2$, $N = 20$, $\delta = 0.1$, $\gamma = 0.4729$, $\mu = 1.1$. Also we choose $L = -1.15$, $M = \text{diag}(1, 1)$ and $R = \text{diag}(1.2, 1.2)$.

Now by using the aforementioned parameters and applying MATLAB toolbox to solve the LMIs formulated in Theorem 2, the following feedback control gain matrices are estimated

$$K_{11} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix},$$
$$K_{13} = \begin{bmatrix} 0.5860 & 3.3082 \\ -32.0133 & 16.1103 \end{bmatrix}, \quad K_{21} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix},$$
$$K_{22} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix}, \quad K_{23} = \begin{bmatrix} 174.2709 & -106.9181 \\ -3.3945 & 2.1245 \end{bmatrix}.$$

For the simulation purposes, the initial conditions for the states of the nodes and the isolated node are taken as $x_1(0) = [4 \quad 2]^T$, $x_2(0) = [-1 \quad 1]^T$, $x_3(0) = [2 \quad -4]^T$ and $s_1(0) = [1 \quad 2]^T$.

Based on the above values, simulation results are presented in Figs. 2, 3, 4, 5, 6, 7, and 8. Specifically, the state responses of the first, second and third nodes together with the isolated node are plotted in Fig. 2. It can be seen from this figure that the states of the nodes are exactly synchronized with the isolated node. The error responses with and without control are depicted in Fig. 3. It can be observed from Fig. 3a that the error state trajectories are synchronized...
within a finite-time period which exhibits the efficiency of the proposed control design. Figures 4 and 5 represent simulation of semi-Markovian jumping process and disturbance signal, respectively. In addition, the Bernoulli random variable $\alpha(k)$ with $\alpha_0 = 0.6$ and $\beta(k)$ with $\beta_0 = 0.4$ are plotted in Figs. 6 and 7. The time history of $e_T^i S_p e_i(k)(i = 1, \cdots, 6)$ is depicted in Fig. 8. From these figures, it can be realized that the CDNs(2) is finite-time bounded with respect to $(0.01, 13.8344, 0.2, 20, 0.1)$ even in the existence of the network-induced imperfections such as actuator fault and time-varying delays.

**Example 2** Consider the discrete-time system (27) with parameters as follows:

$$A = 2 \times \begin{bmatrix} 0.041 & -0.004 \\ -1.001 & 0.001 \end{bmatrix}, \quad B = 1.5 \times \begin{bmatrix} 0.0084 & 0 \\ 0 & 0.0001 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0.4106 & 0.984 \\ 0.9456 & 0.6766 \end{bmatrix}.$$

The inner and outer coupling matrices of non-delayed and delayed terms are taken as $\Gamma_1 = diag(0.09, 0.09)$, $\Gamma_2 = diag(0.25, 0.25)$ and $D_1 = G_1 = \begin{bmatrix} -0.4 & 0.4 & 0 \\ 0.1 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$
The nonlinear function \( f(x_i(k)) = \begin{bmatrix} \tanh(0.2x_{i1}(k)) \\ \tanh(0.2x_{i2}(k)) \end{bmatrix} \), and it can easily satisfy Assumption 3 with the matrices \( \mathcal{H}_1 = \text{diag}\{0.1, 0.1\} \).

Further, the time delays are taken as \( \tau_1(k) = 1.25 + 0.25\sin(0.5k) \), \( \tau_2(k) = 2.5 + 0.25\sin(0.02k) \) and the actuator failure matrix is \( \mathcal{G} = \text{diag}\{0.8, 0.8\} \). Moreover, the other parameters are considered as \( c_1 = 1.5, c_2 = 9.5379, S = 0.1, N = 20, \gamma = 0.4729, \mu = 1.1 \).

Now, by using the aforementioned parameters and applying MATLAB toolbox to solve the LMIs formulated in Corollary 1, the following feedback control gain matrices are estimated:

\[
K_{11} = K_{12} = K_{13} = \begin{bmatrix} 3.5056 & -0.6347 \\ -1.3886 & 0.4554 \end{bmatrix}.
\]

For the simulation purposes, the initial conditions for the states of the nodes and the isolated node are taken as \( x_1(0) = [3\ 2]^T \), \( x_2(0) = [-1\ 1]^T \), \( x_3(0) = [1\ 2]^T \), and \( s_1(0) = [1 - 4]^T \).

Based on the initial condition and system parameter values, simulation results are presented in Figs. 9, 10, and 11. Specifically, the state responses of the first, second and third nodes together with the isolated node are plotted in Fig. 9. It can be seen from this figure that the states of the nodes are exactly synchronized with the isolated node. The error responses with control are depicted in Fig. 10, and the time history of \( e_i^T Se_i(k) \) (\( i = 1, \ldots, 6 \)) is depicted in Fig. 11. From these figures, it can be realized that the CDNs (27) are finite-time synchronization with respect to \( (1.5, 9.5379, 0.1, 20) \) even in the existence of the network-induced imperfections such as actuator fault.

5 Conclusion

In this paper, the dissipative-based finite-time synchronization problem has been investigated for a discrete-time CDNs...
subject to semi-Markovian jumping parameters, probabilistic time-varying delays and actuator faults through the reliable control. By constructing suitable Lyapunov functional method and Kronecker product properties, the required criteria for ensuring the finite-time synchronization with dissipativity performance index of the CDNs have been obtained in the form of LMIs. Finally, two numerical examples have been exploited to show the effectiveness of the established proposed results.

Acknowledgements This work was financially supported by the UGC-BSR Research Start-Up Grant under Ministry of Human Resource Development, Government of India (Grant No. F.30-410/2018(BSR)). The work of Yong-Ki Ma was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2021R1F1A1048937).

Funding The authors have not disclosed any funding.

Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

Akbari Nasim, Sadr Ali, Kazemy Ali (2020) Exponential synchronization of Markovian jump complex dynamical networks with uncertain transition rates and mode-dependent coupling delay. Circuits Syst Signal Process 39:3875–3906

Cheng R, Peng M (2016) Adaptive synchronization for complex networks with probabilistic time-varying delays. J Franklin Inst 353:5099–5120

Cheng J, Zhu H, Zhong S, Zhang Y, Li Y (2015) Finite-time H control for a class of discrete-time Markovian jump systems with partly unknown time-varying transition probabilities subject to average dwell time switching. Int J Syst Sci 46:1080–1093

Cheng R, Peng M, Yu J, Li H (2019) Synchronization for discrete-time complex networks with probabilistic time delays. Physica A 525:1088–1101

Dong H, Wang Z, Ding SX, Gao H (2015) Finite-horizon reliable control with randomly occurring uncertainties and nonlinearities subject to output quantization. Automatica 52:355–362

Dong H, Hou N, Wang Z, Ren W (2017) Variance-constrained state estimation for complex networks with randomly varying topologies. IEEE Trans Neural Netw Learn Syst 29:2757–2768

Han F, Wei G, Ding D, Song Y (2017) Finite-horizon bounded H∞ synchronization and state estimation for discrete-time complex networks: local performance analysis. IET Control Theory Appl 11:827–837

Hao Y, Han J, Lin Y, Liu L (2016) Vulnerability of complex networks under three-level-tree attacks. Physica A 462:674–683

He G, Fang J, Zhang W, Li Z (2016) Synchronization of switched complex dynamical networks with non-synchronized subnetworks and stochastic disturbances. Neurocomputing 171:39–47

Li W, Jia Y, Du J (2017) Recursive state estimation for complex networks with random coupling strength. Neurocomputing 219:1–8

Li Z, Fang J, Huang T, Miao Q (2017) Synchronization of stochastic discrete-time complex networks with partial mixed impulse effects. J Franklin Inst 354:4196–4214

Li B, Wang Z, Ma L (2018) An Event-triggered pinning control approach to synchronization of discrete-time stochastic complex dynamical networks. IEEE Trans Neural Netw Learn Syst 29:5812–5822

Li Q, Guo J, Sun C, Wu Y, Ding Z (2019) Finite-Time Synchronization for a Class of Dynamical Complex Networks with Nonidentical Nodes and Uncertain Disturbance. J Syst Sci Complexity 32:818–834

Liang K, Dai M, Shen H, Wang J, Wang Z, Chen B (2018) \( L_2 − L_{\infty} \) synchronization for singularly perturbed complex networks with semi-Markov jump topology. Appl Math Comput 321:450–462

Ma L, Fang X, Yuan Y, Zhang J, Bo Y (2020) Dissipative control for nonlinear Markovian jump systems with mixed time-delays: the discrete-time case. Int J Robust Nonlinear Control. https://doi.org/10.1002/rnc.4906

Park MJ, Kwon OM (2017) Stability and stabilization of discrete-time T-S fuzzy systems with time-varying delay via Cauchy–Schwartz-based summation inequality. IEEE Trans Fuzzy Syst 25:128–140

Peng H, Lu R, Xu Y, Yao F (2018) Dissipative non-fragile state estimation for Markovian complex networks with coupling transmission delays. Neurocomputing 275:1576–1584

Qunjiao Z, Xiaojun W, Jie L (2019) Pinning synchronization of discrete-time complex networks with different time-varying delays. J Syst Sci Complexity 32:1560–1571

Ren H, Deng F, Peng Y (2018) Finite time synchronization of Markovian jumping stochastic complex dynamical systems with mix delays via hybrid control strategy. Neurocomputing 272:683–693

Sakthivel R, Santra S, Mathiyalagan K, Marshal Anthoney S (2014) Robust reliable sampled-data control for offshore steel jacket platforms with nonlinear perturbations. Nonlinear Dyn 78:1109–1123

Sakthivel R, Arunkumar A, Mathiyalagan K, Selvi S (2015) Robust reliable control for uncertain vehicle suspension systems with input delays. J Dyn Syst Meas Control, 137. https://doi.org/10.1115/1.4028776

Shen H, Park Ju H, Wu Z,G (2014) Finite-time reliable \( L_2 − L_{\infty}/H_{\infty} \) control for Takagi-Sugeno fuzzy systems with actuator faults. IET Control Theory Appl 8:688–696

Shen H, Park JH, Wu Z-G, Zhang Z (2015) Finite-time \( H_{\infty} \) synchronization for complex networks with semi-Markov jump topology. Commun Nonlinear Sci Numer Simul 24:40–51

Sheng S, Zhang X, Lu G (2018) Finite-time outer synchronization for complex networks with Markov jump topology via hybrid control and its application to image encryption. J Franklin Inst 355:6493–6519

Song J, Niu Y, Wang S (2017) Robust finite-time dissipative control subject to randomly occurring uncertainties and stochastic fading measurements. J Franklin Inst 354:3706–3723

Wang A, Dong T, Liao X (2016) Event-triggered synchronization strategy for complex dynamical networks with the Markovian switching topologies. Neural Netw 74:52–57

Wang J, Hu X, Wei Y, Wang Z (2019) Sampled-data synchronization of semi-Markov jump complex dynamical networks subject to generalized dissipativity property. Appl Math Comput 346:853–864

Willems JC (1972) Dissipative dynamical systems part I: general theory. Arch Ration Mech Anal 45:321–351

Yang X, Ho DWC, Lu J, Song Q (2015) Finite-time cluster synchronization of T-S fuzzy complex networks with discontinuous subsystems and random coupling delays. IEEE Trans Fuzzy Syst 23:2302–2316

Zhang Y, Shi P, Nguang SK, Karimi HR (2014) Observer-based finite-time fuzzy H1 control for discrete-time systems with stochastic jumps and time-delays. Sig Process 97:252–261
Zhang Q, Chen G, Wan L (2018) Exponential synchronization of discrete-time impulsive dynamical networks with time-varying delays and stochastic disturbances. Neurocomputing 309:62–69
Zhang H, Hu J, Liu H, Yu X, Liu F (2019) Recursive state estimation for time-varying complex networks subject to missing measurements and stochastic inner coupling under random access protocol. Neurocomputing 346:48–57

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.