Fast Approximation Schemes for Bin Packing
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Abstract
We present new approximation schemes for bin packing based on the following two approaches: (1) partitioning the given problem into mostly identical sub-problems of constant size and then construct a solution by combining the solutions of these constant size sub-problems obtained through PTAS or exact methods; (2) solving bin packing using irregular sized bins, a generalization of bin packing, that facilitates the design of simple and efficient recursive algorithms that solve a problem in terms of smaller sub-problems such that the unused space in bins used by an earlier solved sub-problem is available to subsequently solved sub-problems.

Key words: Bin Packing; Approximation Algorithms; Approximation Schemes; PTAS; Exact Algorithms; Heuristics; Design and Analysis of Algorithms.

1 Introduction
The Bin Packing problem is a classical combinatorial optimization problem that was first studied in the 1970’s by Garey, Graham and Ullman[10] and Johnson[14], and can be stated as follows:

Given a collection $B$ of unit capacity bins and a sequence $L = (a_1, a_2, ..., a_n)$ of $n$ items with their respective sizes $s_1, s_2, ..., s_n$ such that $\forall i s_i \in [0, 1]$, determine a packing of the items in $L$ that uses a minimum number of bins from $B$.

Bin Packing has a wide variety of applications[18] including cutting stock applications, packing problems in supply chain management, resource allocation problems in distributed systems. Algorithms for bin packing can be broadly classified as offline and online. Offline algorithms are algorithms that pack items with complete knowledge of the list $L$ of items prior to packing, whereas online algorithms need to pack items as they arrive without any knowledge of future. The bin packing problem even for the offline version is known to be NP-Hard[8] and hence has led researchers to the study of polynomial time approximation algorithms (i.e. provides near optimal solutions).

Most of the initial research in Bin Packing has been in the design of simple deterministic algorithms and their combinatorial analysis leading to tighter upper bounds on the performance of these algorithms and tighter lower bounds on estimating the optimal offline and online solutions. Subsequently, there has been significant work on probabilistic analysis of these deterministic algorithms as well as on the design of randomized algorithms and approximation schemes for Bin Packing. For a comprehensive survey of classical algorithms for Bin Packing from the perspective of
design and analysis of approximation algorithms, we refer the readers to Johnson’s Phd Thesis [15], Coffman et al. [5] and Hochbaum [13].

In Bin Packing problem, we are required to pack the items in $L$ using minimum number of bins in $B$. In this paper, we will refer to this classic version of Bin Packing as the regular bin packing problem. The offline version of regular bin packing problem is known to be NP-Hard [8] and hence most research efforts have focused on the design of fast online and offline approximation algorithms with good performance. The performance of an approximation algorithm is defined in terms of its worst case behavior as follows: Let $A$ be an algorithm for bin packing and let $A(L)$ denote the number of bins required by $A$ to pack items in $L$, and OPT denote the optimal algorithm for packing items in $L$. Let $L$ denote the set of all possible list sequences whose items are of sizes in $[0,1]$. For every $k > 1$, $R_A(k) = \sup_{L \in L} \{A(L)/k : OPT(L) = k\}$. Then the asymptotic worst case ratio is given by $R_A^\infty = \lim_{k \to \infty} R_A(k)$. This ratio is the asymptotic approximation ratio and measures the quality of the algorithms packing in comparison to the optimal packing in the worst case scenario. The second way of measuring the performance of an approximation algorithm is $\sup_{L \in L} \{A(L)/OPT(L)\}$ and this ratio is the absolute approximation ratio of the algorithm. In the case of online algorithms this ratio is often referred to as competitive ratio.

**Online Algorithms:** NEXT-FIT (NF), FIRST-FIT (FF) and BEST-FIT (BF) are the three most natural online algorithms for regular bin packing that has been widely studied in the literature. These three algorithms are a part of a larger class of algorithms called Any Fit (AF) Algorithms that at any time packs an item into an empty bin only if it does not fit into any already open bin. Johnson et al. [15, 16, 18] showed that both FF and BF have an asymptotic competitive ratio of 1.7. Johnson showed that no AF algorithm can improve upon FF. Yao’s REVISED-FF (RFF) [23] was the first non-AF online algorithm with an asymptotic competitive ratio of 5/3 and was based on FF but essentially classifies items into types based on their sizes and uses separate bins for different item types. Later Lee and Lee [20] generalized this idea and designed Harmonic – Fit$_k$ (HF$_k$) with asymptotic competitive ratio $\approx 1.69103$. There are many other variants of Harmonic and the best among them is the algorithm of Seiden [21] and has an asymptotic competitive ratio of 1.5889. More recently, Balogh et al. [2] settled this online problem by presenting an optimal online bin packing with absolute worst case competitive ratio of 5/3.

**Offline Algorithms:** The most natural offline algorithms first reorder the items and then employing other classical online algorithms like NF, FF, BF or other online algorithms to pack the items. This has resulted in three simple but effective offline algorithms; they are denoted by
NFD, FFD, and BFD, with the D standing for Decreasing”. The sorting needs $O(n \log n)$ time and so the total running time of each of these algorithms is $O(n \log n)$. Baker and Coffman[3] established the asymptotic approximation ratio for NFD to be $\approx 1.69103$, Johnson et al.[18] established FFD and BFD’s asymptotic approximation ratio to be $11/9$. Subsequently, Baker[1] and Yue[24], and Csirik[4] and Xu[22] presented simplified proofs. The first improvement over FFD was due to Yao’s Refined-First-Fit Decreasing (RFFD)[23] with asymptotic approximation ratio $= 11/9 - 10^{-7}$. This was an $O(n \log n)$ time algorithm. Garey and Johnson[9] then proposed Modified First Fit (MFFD), which essentially packs the items with sizes in $(1/6, 1/3]$ after packing all items $> 1/3$, and then proved $R_{MFFD} = 71/60 = 1.183333$. Friesen and Langsten[7] also proposed two simple algorithms Best-Two-Fit (B2F) and CombinedAlgorithm (CFB) that combines B2F and FFD with asymptotic approximation ratios 1.25 and 1.2 respectively.

Asymptotic Approximation Schemes: Fernandez de la Vega and Lueker[6] presented a PTAS that for any $\epsilon > 0$, designed an $C_\epsilon + Cn \log(1/\epsilon)$ time algorithm $A$ with asymptotic worst case ratio $R_A \leq 1 + \epsilon$, where $C_\epsilon$ and $C$ are constants that depend on $\epsilon$. Johnson[17] observed that if $\epsilon$ is allowed to grow slowly when compared to $OPT(L)$ then more efficient approximation schemes can be constructed. This was incorporated by Karmarkar and Karp[19] to obtain an approximation scheme where $A(L) \leq OPT(L) + O(OPT(L)^{1-\delta})$, for some positive constant $\delta$. Using the idea of dual approximation algorithms, Hochbaum and Shmoys[11, 12] present approximation schemes for bin packing developed using polynomial approximation scheme for makespan.

1.1 Our Results

In this paper, we present fast polynomial time approximation schemes for bin packing based on the following two approaches: (i) Near Identical Partitioning and (ii) Irregular Bin Packing. Our approximation schemes in some non-trivial special cases yield asymptotically optimal solutions.

Near Identical Partitioning Approach: In this approach, we present an algorithm that (i) for some real number $\delta \in (0, 1/2)$ partitions the input sequence $L$ into $l$ identical sub-sequences $L_c$ (except for the last sub-sequence) of length $c \in ([1/3], [2/3])$ (i.e. sum of sizes of items in these subsequences is $c$); (ii) determines the optimal packing for $L_c$ and the last sub-sequence using an existing polynomial time approximation scheme for regular bin packing; and (iii) constructs the packing for $L$ by concatenating the packing for the $l$ copies of $L_c$ and the packing of the last sub-sequence.

Irregular Bin Packing Approach: In this approach, we view the regular bin packing problem as a special case of irregular bin packing, a slight generalization, that can be defined as follows:
Irregular Bin Packing: Given a sequence $L = (a_1, a_2, ..., a_n)$ of $n$ items with their respective sizes $(s_1, s_2, ..., s_n)$ such that $\forall i \ s_i \in [0, 1]$, and a collection $B$ of $m$ bins with respective capacities $c_1, c_2, ..., c_m$ such that $\forall i \ c_i \in [0, 1]$, we need to design an algorithm to pack the items of $L$ among the bins in $B$ that minimizes the opening/use of regular bins, where a bin is regular if its capacity is 1 and irregular otherwise.

Notice if all the bins in $B$ are of unit capacity (i.e. regular ) then the irregular bin packing reduces to the regular bin packing problem. In irregular bin packing, if an item is assigned to an irregular bin (a bin with capacity < 1) then we do not charge the assigned item since that bin was already open. However, if an item is assigned to a regular bin then we charge the assigned item 1 unit since it opens that bin. This formulation facilitates the design of simple and efficient approximation schemes that recursively solve a problem in terms of smaller but similar sub-problems that exploit the unused space in bins used by an earlier solved sub-problem by subsequently solved sub-problems.

Paper Outline: The rest of this paper is organized as follows: In Section 2 we present a PTAS for bin packing through partitioning as described earlier, and in Section 3 we present a PTAS for bin packing through a dynamic program for irregular bin packing.

2 Bin Packing Through Near Identical Partitioning

In this section, we present an algorithm that given a real valued parameter $\epsilon \in (0, \frac{1}{2})$, partitions the input sequence $L$ into identical sub-sequences (except for the last sub-sequence) of length $c \in [1, \lceil \frac{2}{\epsilon} \rceil]$ (i.e. sum of sizes of items in these subsequences is $c$) and then packs the items in these $c$-length subsequences onto unit capacity bins with wastage (unused space) of at most $\delta \in (\epsilon, \frac{1}{2})$ using an existing polynomial time approximation scheme for regular bin packing. Now, we introduce some necessary terms and definitions and examples illustrating our key idea before presenting our approximation scheme and its analysis.

Definitions 2.1 The sequence $L = (a_1, a_2, ..., a_n)$ with $k$ distinct item sizes $\{s_1, s_2, ..., s_k\}$ can be viewed as a $k$ dimensional vector $\hat{d}(L) = (n_1 * s_1, n_2 * s_2, ..., n_k * s_k)$, where for $i \in [1..k]$, $n_i$ is the number of items of type $i$ (size $s_i$); we refer to $\hat{d}(L)$ as the distribution vector corresponding to $L$. For a given real number $c > 1$, let $\hat{d}_c(L)$ denote a $c$-length segment of $\hat{d}(L)$ (i.e. a vector that is parallel to $\hat{d}(L)$ and contains its initial segment such that its component sum equals $c$).

Definitions 2.2 For a real number $\delta \in (0, \frac{1}{2})$, the configuration of a unit capacity bin containing items whose sizes are $\{s_1, s_2, ..., s_k\}$ and has a wastage of at most $\delta$ can be specified by a
A \( k \)-dimensional vector whose \( i \)-th component, for \( i \in [1..k] \), is the sum of sizes of items of type \( i \) (size \( s_i \)) in that bin; and its length is in the interval \([1 − \delta, 1]\), where the length of a vector is defined to be the sum of its components. We refer to such a vector as a \((1 − \delta)\)-vector (bin configuration) consistent with \( L \); and we denote by \( e_\delta(L) \) the set of all \((1 − \delta)\)-vectors (bin configurations) consistent with \( L \).

Note: For certain sequences \( L \), the item sizes in \( L \) may be such that for some \( \delta \in (0, \frac{1}{2}) \) there are no \( 1 − \delta \) vectors consistent with \( L \) (i.e. \( e_\delta(L) \) is empty).

**Definitions 2.3** For a given sequence \( L \) and a real number \( \delta \in (0, 1/2) \), if \( e_\delta(L) \) is non-empty then we define

- a \( \delta \)-cover for \( \hat{d}(L) \) to be a minimal collection of \((1 − \delta)\)-vectors from \( e_\delta(L) \) such that for \( i \in [1..k] \), the sum of the \( i \)-th component of these collection of vectors is greater than or equal to the \( i \)-th component of \( \hat{d}(L) \);

- \( \text{min-cover}_\delta(\hat{d}(L)) \) to be a \( \delta \)-cover for \( \hat{d}(L) \) of the smallest size;

- \( \text{min-cover}(\hat{d}(L)) = \min_{\delta \in (0, \frac{1}{2})} \{ \text{min-cover}_\delta(\hat{d}(L)) \} \).

**Remark**: If the number of distinct sizes in \( L \) is not bounded by a constant \( k \), then we can still apply the above idea by partitioning the interval \([0, 1]\) into \( k \) distinct sizes \( 0, 1/k, 2/k, \ldots, 1 \) and round the item sizes in \( L \) to the nearest multiple of \( 1/k \) that is greater than or equal to the item size.

**Key Idea**: For an integer \( c^* \in [1, \lceil \frac{2}{\epsilon} \rceil] \), we partition the distribution vector \( \hat{d}(L) \) into many copies of \( \hat{d}_{c^*}(L) \), the \( c^* \) length segment of \( \hat{d}(L) \) (except for the last segment), where \( c^* \) is determined as follows: For each \( c \in [1, \lceil \frac{1}{\epsilon} \rceil] \), we determine \( \delta_c \) to be a real number \( \delta \in (\epsilon, \frac{1}{2}) \) for which \( \hat{d}_c(L) \) has the smallest \( \delta \)-cover (i.e. \( \text{min-cover}_{\delta_c}(\hat{d}_c(L)) = \min_{\delta \in (\epsilon, \frac{1}{2})} \text{min-cover}_{\delta}(\hat{d}_c(L)) \)). Then, we determine \( c^* \) to be an integer in \([1, \lceil \frac{2}{\epsilon} \rceil]\) that minimizes the packing ratio (i.e. \( \frac{\text{min-cover}_{\delta^*}(\hat{d}_{c^*}(L))}{c^*} = \min_{c \in (1, \lceil \frac{2}{\epsilon} \rceil)} \frac{\text{min-cover}_{\delta^*}(\hat{d}_c(L))}{c} \)).

**Example 1** Let us consider a sequence \( L \) of 3000 items consisting of 600 items of size 0.52, 600 items of size 0.29, 600 items of size 0.27 and 1200 items of size 0.21. Let \( \epsilon = 0.1 \) is the approximation ratio desired. For this instance the distribution vector \( \hat{d}(L) \) is a 4-dimensional vector \((0.21 \times 1200, 0.27 \times 600, 0.29 \times 600, 0.52 \times 600) = (252, 162, 174, 312) \) of length 900. Our algorithm attempts to partition \( \hat{d}(L) \) into a \( c \)-segment vector for some \( c \) between \((1, \lceil \frac{2}{\epsilon} \rceil) \). We can observe that we can partition \( \hat{d}(L) \) into 60 copies of the segment vector \((4.2, 2.7, 2.9, 5.2) =
(0.21 \ast 20, 0.27 \ast 10, 0.29 \ast 10, 0.52 \ast 10) of length 15. For \( \delta \leq 0.1 \), the minimum sized \( \delta \) cover for this segment vector of length 15 can be determined using any of the existing PTAS or exact algorithms for regular bin packing.

**Example 2** Let us consider a sequence \( L \) of 3000 items consisting of 1000 items of size 0.60, 1000 items of size 0.65, and 1000 items of size 0.75. Let \( \epsilon = 0.1 \) is the approximation ratio desired. For this instance the distribution vector \( \hat{d}(L) \) is a 3-dimensional vector \((0.60 \ast 1000, 0.65 \ast 1000, 0.75 \ast 1000) = (600, 650, 750) \) of length 2000. Our algorithm attempts to partition \( \hat{d}(L) \) into a segment vector for some \( c \) between \((1, \lceil \frac{2}{\epsilon} \rceil)\). We can observe that we can partition \( \hat{d}(L) \) into 100 copies of the segment vector \((6.0, 6.50, 7.50) = (0.60 \ast 10, 0.65 \ast 10, 0.75 \ast 10) \) of length 20. The minimum sized \( \delta \) cover for this segment vector of length 20 can be determined using any of the existing PTAS or exact algorithms for regular bin packing. In this instance there are no \( \delta \)-covers for \( \delta < 0.4 \).

**ALGORITHM B\((L, \epsilon)\)**

Input(s): (1) \( L = (a_1, a_2, ..., a_n) \) be the sequence of \( n \) items with their respective sizes 
\((s_1, s_2, ..., s_n)\) in the interval \([0, 1]\);
(2) \( \epsilon \in (0, \frac{1}{2}) \) be a user specified parameter;
Output(s): The assignment of the items in \( L \) to the bins in \( B \);

Begin

(1) Let \( \hat{d}(L) = (s_1 \ast n_1, s_2 \ast n_2, ..., s_k \ast n_k) \) be the distribution vector corresponding to \( L \);
(2) For \((c = 1; \; c \leq \lceil \frac{2}{\epsilon} \rceil; \; c = c + 1)\)
(2a) Let \( \hat{d}_c(L) = (s_1 \ast n_1^c, s_2 \ast n_2^c, ..., s_k \ast n_k^c) \) be the \( c \)-length segment of \( \hat{d}(L) \);
(2b) For \((\delta = \epsilon; \; \delta \leq \frac{1}{2}; \; \delta = \delta + 1)\)
\[\text{Cover}_{\delta}(\hat{d}_c(L)) = \begin{cases} \text{min-cover}_{\delta}(\hat{d}_c(L)) & \text{if } \epsilon_{\delta}(L) \neq \Phi \\ \Phi & \text{otherwise} \end{cases}\]
(2c) Let \( \delta_c \in (\epsilon, \frac{1}{2}) \) be a multiple of \( \epsilon \) such that \( |\text{Cover}_{\delta_c}(\hat{d}_c(L))| = \min_{\delta \in (\epsilon, \frac{1}{2})} |\text{Cover}_{\delta}(\hat{d}_c(L))| \)
(3) Let \( c^* \) be an integer in \((1, \lceil \frac{2}{\epsilon} \rceil)\) such that \( \frac{|\text{Cover}_{\delta_{c^*}}(\hat{d}_c(L))|}{c^{*}} = \min_{c \in (1, \lceil \frac{2}{\epsilon} \rceil)} \frac{|\text{Cover}_{\delta}(\hat{d}_c(L))|}{c} \);
(4) Let \( T = \hat{d}_{c^*}(L) \) and \( l = \frac{\hat{d}(L)}{c^{*}} \);
(5) Let \( \text{Cover}(\hat{d}(L)) = \bigcup_{i=1}^{l} \text{min-cover}(T) \cup \text{min-cover}(\hat{d}(L) - l \ast T) \);
(6) return \( \text{Cover}(\hat{d}(L)) \)

End

**Determining min-cover\((\hat{d}_c(L))\):** For determining the min-cover of \( \hat{d}_c(L) \), we need to determine a smallest sized collection of vectors from \( \epsilon_{\delta}(L), \; \delta \in [\epsilon, \frac{1}{2}] \), such that for \( i \in [1..k] \), the sum of
the $i$th components of these vectors is greater than or equal to the $i$th component of $\hat{d}_c(L)$. For this we make use of the PTAS result of Karmarkar and Karp\cite{19} to determine a $\delta$-cover that is of size $(1 + \epsilon)|\text{min-cover}(T)|$ in polynomial time. We now introduce some definitions that will help us present the analysis of Algorithm $B$.

**Definitions 2.4** For notational convenience, let $T = (t_1, t_2, ..., t_k)$ denote $\hat{d}_c(L)$, the $c$-length initial segment of $\hat{d}(L)$. Let $\delta \in [\epsilon, \frac{1}{2}]$ be a real number and $N = |e\delta(L)|$ denote the number of $1 - \delta$ configurations consistent with $L$. Let $C_1, C_2, ..., C_N$ denote the complete enumeration of the $1 - \delta$ vectors (bins) consistent with $L$, where $c_{ij}$ denotes the $i$th component of $C_j$.

Let $x_j$ denote the number of bins packed according to configuration $C_j$. Notice that the minimum $\delta$-cover for $T$ can be solved using the PTAS for regular bin packing originally due to Fernandez de la Vega and Leuker\cite{6}, and later improved by Karmarkar and Karp\cite{19}. In this PTAS, the bin packing problem is formulated as an integer program as follows:

$$\text{minimize } \sum_{j=1}^{N} x_j \quad (1)$$

subject to

$$\sum_{j=1}^{N} c_{ij}x_j \geq t_i \quad i = 1, ..., k \quad (2)$$

$$x_j \in N \quad j = 1, ..., N \quad (3)$$

**Definitions 2.5** Let Algorithm $B(L, \epsilon)$ partition $\hat{d}(L)$ into $l$ copies of $T = \hat{d}_c(L) = (s_1 \ast n'_1, s_2 \ast n'_2, ..., s_k \ast n'_k)$ (discarding the last segment), where $c$ is an integer in $[1, \lceil \frac{2}{\epsilon} \rceil]$ and $T$ is a $c$-length initial segment of $\hat{d}(L)$. Let $T^c = (s_1 \ast [n'_1], s_2 \ast [n'_2], ..., s_k \ast [n'_k])$ be the segment vector obtained by truncating for $i \in [1..k]$, the $i$th components of $T$ to the nearest integer multiple of $s_i$. Let $\text{Cover}(\hat{d}(L))$ be the $\delta$-cover determined by Algorithm $B$ for $\hat{d}(L)$.

**Theorem 1** $|\text{Cover}(\hat{d}(L))| \leq |\text{min-cover}(\hat{d}(L))| + k \ast l + 2c$.

The above theorem follows from Lemmas 3, 4 and 5 presented below.

**Corollary 2** If (i) $T = (n_1 \ast s_1, n_2 \ast s_2, ..., n_k \ast s_k)$, where for $i \in [1..k]$ the $i$th component is an integer multiple of $s_i$; OR (ii) $\sum_{i=1}^{k} s_i = o(c)$ OR $k = o(c)$, then Algorithm $B$ constructs an asymptotically optimal cover for $\hat{d}(L)$.
Lemma 3  \(|\min\text{-}cover(T)| \leq |\min\text{-}cover(T')| + |\min\text{-}cover(s_1, s_2, \ldots, s_k)| \leq |\min\text{-}cover(T')| + k\)

**Proof**  Notice that \(T'\) is obtained by truncating each component \(i \in [1..k]\), to the nearest multiple of \(s_i\). Therefore, the maximum difference between the length of \(T\) and \(T'\) is \(\sum_{i=1}^{k} s_i < k\). Therefore the size of the optimal \(\delta\)-cover for \(T\) cannot be more than the sum of the sizes of an optimal \(\delta\)-cover for \(T'\) and an optimal \(\delta\)-cover for \((s_1, s_2, \ldots, s_k)\). For \(i \in [1..k]\), the item sizes \(s_i \in (0, 1)\). Therefore the size of an optimal \(\delta\)-cover for \((s_1, s_2, \ldots, s_k)\) is at most \(k\). Hence the result. \(\square\)

Lemma 4  \(|\min\text{-}cover(\hat{d}(L))| \geq l \ast |\min\text{-}cover(T')|\)

**Proof**  Notice that \(T\) was constructed by partitioning \(\hat{d}(L)\) into identical segments of length \(c \in [1, \lceil \frac{2}{\epsilon} \rceil]\) with the minimum packing ratio. Suppose \(|\text{OPT}(\hat{d}(L))| < l \ast |\min\text{-}cover(T')|\) then this would imply that if we split \(\hat{d}(L)\) into \(l\) identical segment vectors then at least one of these segment vectors would have a packing ratio less than \(T\). A contradiction. \(\square\)

Lemma 5  \(|\text{Cover}(\hat{d}(L))| \leq l \ast |\min\text{-}cover(T)| + 2c.\)

**Proof**  The Algorithm \(B\) splits \(\hat{d}(L)\) into \(l\) copies of segment vectors \(T\) and a last segment vector \(\hat{d}(L) - l \ast T\) of length at most \(c\). Therefore, the size of the minimum \(\delta\)-cover of the last segment vector \(\hat{d}(L) - l \ast T\) is at most \(2c\). Now by concatenating the min \(\delta\)-covers of \(T\) \(l\) times along with the min \(\delta\)-cover of the last segment we get the result. \(\square\)

3 Bin Packing Through Irregular Bin Packing

In this section, we present a recursive algorithm that can be converted into a dynamic programming solution to the irregular bin packing problem. Our algorithm assumes (i) there exists a way of packing the items in \(L\) using at most \(m\) unit bins; and (ii) the sizes of all items in \(L\) are integer multiples of a small positive rational number \(\delta\) less than 1.

Let \(L = (a_1, a_2, \ldots, a_n)\) be a sequence of \(n\) items with their respective sizes \((s_1, s_2, \ldots, s_n)\) in the interval \([0, 1]\) and \(B = \{B_1, B_2, \ldots, B_m\}\) be a set of \(m\) unit capacity bins. Let \(\delta \in (0, 1]\) be a rational number such that every item size in \(L\) can be expressed as an integer multiple of \(\delta\). Let \(D = \lceil \frac{1}{\delta} \rceil\). At any given instance, bins in \(B\) are classified based on its level into one of \(D + 1\) types: a bin is of type \(i\) if its level is \(i/D\), where \(i\) is an integer in \([0..D]\). Initially (at instance 0), all \(m\) bins are of type 0. Our algorithm assigns the items in \(L\) one at a time onto bins in \(B\) in the order of their occurrence in \(L\). The state / configuration of our algorithm at instance \(i\) (i.e. after assigning
the \( i \)th item) is specified in terms of a \( D + 1 \) tuple \( C^i = (n^i_0, n^i_1, n^i_2, ..., n^i_D) \), where \( n^i_j, j \in [0,..D] \), denotes the number of bins of type \( j \).

**Note:** We classify bins based on their level and not based on their content. This reduces the number of possible bin configurations and thereby reducing the number of possible sub-problems that our dynamic program needs to consider while computing an optimal solution.

Now, we introduce definitions that are necessary for presenting our dynamic program.

**Definitions 3.1** For \( i \in [1..n] \), we define \( \text{type}(a_i) = \lceil \frac{i}{m} \rceil \) to be the type of item \( a_i \), and \( a_i \) can be assigned to a bin of type \( j \) only if \( s_i + j \delta \leq 1 \). For a given bin configuration \( C = (n_0, n_1, n_2, ..., n_D) \) and an item \( a_i \in L \), we define \( \text{Allow}(a_i, C) = \{ j : n_j \geq 1 \text{ and } s_i + j \delta \leq 1 \} \). That is, \( \text{Allow}(a_i, C) \) is the set of bin types to which \( a_i \) can be assigned without violating its capacity constraint. For a given a bin configuration \( C \), we define \( \text{cost}^C(a_i, j) \) to be 1 if \( a_i \) is assigned to an empty bin of type \( j \) in \( \text{Allow}(a_i, C) \) and 0 if it is assigned to a non-empty bin of type \( j \) in \( \text{Allow}(a_i, C) \). More formally,

\[
\text{cost}^C(a_i, j) = \begin{cases} 
1 & j \in \text{Allow}(a_i, C) \text{ and is of type } 0 \\
0 & j \in \text{Allow}(a_i, C) \text{ and not of type } 0 \\
\infty & \text{otherwise} 
\end{cases}
\] (4)

**Basic Description of Our Algorithm:** Our algorithm assigns the items in \( L \) to bins in \( B \) in order of their occurrence in \( L \). The state of our algorithm is defined in terms of the configuration of bins in \( B \); the configuration of a bin is defined in terms of its level and not its composition (the type of items it contains). That is, while assigning an item, our algorithm does not distinguish between bins that are filled to the same level but differ in their composition. Initially, (at instance 0), all \( m \) bins are of type 0 (empty), so the initial state \( C^0 \) of our algorithm is specified by the \( D + 1 \) tuple \((m, 0, 0, ..., 0)\). Suppose at instance \( i - 1 \), our algorithm is in state \( C^{i-1} = (n^i_1, n^i_2, ..., n^i_D) \), where \( n^i_j, j \in [0,..D] \), denotes the number of bins of type \( j \). The next item \( a_i \) can be assigned to any bin whose type is in \( \text{Allow}(a_i, C^{i-1}) \). If our algorithm chooses a bin of type \( j \in \text{Allow}(a_i, C^{i-1}) \) then it will end up in configuration \( C^i_j \) and would cost \( \text{cost}^{C^{i-1}}(a_i, j) \) plus the optimal number of regular bins required for assigning the items in \( L[i+1..n] \) starting in configuration \( C^i_j \). So, our algorithm assigns \( a_i \) to a bin type whose cost is \( \min_{j \in \text{Allow}(C^{i-1})(a_i)} \{ \text{Assign}(C^i_j, i + 1) + \text{Cost}^{C^{i-1}}(a_i, j) \} \)

**PROCEDURE** \( \text{Assign}(C, i) \)

Input(s): (1) \( C = (n_0, n_1, n_2, ..., n_D) \) - the initial configuration of the \( m \) bins in \( B \);

(2) \( i \) - the index of the next item in \( L \) that needs to be assigned.
Output(s): The minimum number of regular bins required for assigning items in \( L[i..n] \) to bins in \( B \) in configuration \( C \).

Begin

\[
\text{return } \min_{j \in \text{Allow}(a_i, C)} \{ \text{Assign}(C_j, i + 1) + \text{Cost}^C(a_i, j) \}, \text{ where } \]
\[
C_j = (n_0, n_1, \ldots, n_{j-1}, n_j - 1, \ldots, n_{o-1}, n_o + 1, \ldots, n_D), \text{ and } o = j + \text{type}(a_i).\]

End

Lemma 6 For regular bin packing problem, given any request sequence \( L = (a_1, a_2, \ldots, a_n) \) whose item sizes are in the interval \([0, 1]\) and are integer multiples of a small rational number \( \delta \in (0, 1) \) and a collection \( B \) of \( m \) unit capacity bins, the procedure \( \text{Assign}(C^0, 1) \) determines a \((1+\delta)\) approximate solution in approximately \( \frac{n}{\delta} m^{\left(\frac{1}{\delta} - 1\right)} \) time, where \( C^0 = (m, 0, 0, \ldots, 0) \) is a \( D + 1 \)-tuple denoting the initial bin configuration.

Proof Let \( D = \frac{1}{\delta} \). The run-time of \( \text{Assign}(C^0, 1) \) is proportional to the number of sub-problems, which in turn depends on the number of bin configurations that our formulation permits. This is the same as the number of ways we can partition \( m \) bins into \( \frac{1}{\delta} \) categories based on its level. This can be upper bounded by \((\frac{m+D-1}{D-1}) = \frac{n}{\delta} m^{\left(\frac{1}{\delta} - 1\right)} \). \( \square \)

Note: If we had defined the bin configuration in terms of the bin composition (as usually done for PTAS for bin packing), then the number of bin types would be bounded by \( R = \binom{M+K}{M} \), where \( K \) is the number of distinct item sizes and the number of different bin configurations is bounded by \( P = \binom{n+R}{R} \). So, defining the state of the algorithm in terms of its level instead of its composition results in a significant reduction in the number of states without impacting its approximation guarantee.

Theorem 7 For a real \( \epsilon \in (0, 1) \) and an integer \( c > 1 \), given a request sequence \( L = (a_1, a_2, \ldots, a_n) \) with item sizes in the interval \([0, 1]\) and a collection \( B \) of \( m \) unit capacity bins, the dynamic program \( A(L, B) \) determines a \((1 + \frac{\epsilon}{c} + \frac{1}{c})\) approximate solution for regular bin packing problem in approximately \( \frac{nc}{\epsilon} m^{\left(\frac{1}{\epsilon} - 1\right)} \) time.

Proof Without loss of generality we assume that all items in \( L \) are larger than \( \epsilon \). Let \( \delta = \epsilon/c \). First, we round the size of each item \( a_i \) in \( L \) to the smallest multiple of \( \delta \) greater than or equal to \( s_i \). Let \( L' \) be the modified instance of \( L \). This will induce a rounding error of at most \( \delta = \frac{\epsilon}{c} \) for each item. Since each item is at least \( \epsilon \) in size. The rounding error is at most \( \frac{1}{c} \). Now, if we run the invoke the algorithm \( A(L', B) \) it will determine an optimal solution for \( L' \). From Lemma 6,
we know that $A$ determines a $(1 + \delta)$ approximate solution for $L'$ in approximately $\frac{2}{\epsilon} m^{\left(\frac{1}{3} - 1\right)}$ time. Since the rounding error for converting $L$ to $L'$ is at most $\frac{\epsilon}{c}$. The approximation guarantee of its solution for $L$ would be $= (1 + \delta + \frac{1}{c}) = (1 + \frac{\epsilon}{c} + \frac{1}{c})$ and the run-time would be $\frac{nc}{c} m^{\left(\frac{c}{c} - 1\right)}$ time.

\[\square\]

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