Isotopic splittings and OPE in $B \rightarrow D$ semileptonic decays

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
and
Institute of Theoretical and Experimental Physics, Moscow, 117259

Abstract

It is shown that the kinematical difference in the decays $B^- \rightarrow D^0 \ell \bar{\nu}$ and $B^0 \rightarrow D^+ \ell \bar{\nu}$ due to the isotopic mass splittings of the $B$ and $D$ mesons is compensated in the total decay rate by an appropriate difference in the lepton spectra. Thus there is no effect on the total decay rates in the linear order in the isotopic mass splittings, as required by the general consideration based on the operator product expansion. Although phenomenologically the isotopic difference in the spectra amounts to at most about 1%, in the theoretical aspect this effect can be viewed as an additional illustration of how the general OPE results emerge from the properties of exclusive channels.
It has become a matter of common knowledge\textsuperscript{[1]} that the inclusive rates of weak decays for hadrons containing a heavy quark are governed by the short-distance QCD\textsuperscript{[2]} and are given by the decay rate of the heavy quark. The corrections to this leading behavior are suppressed by at least two powers of the inverse heavy quark mass, \(m_Q^{-2}\), or, in the situation where two heavy quarks are involved, like in the \(b \rightarrow c\) transitions, by the inverse second power of some combination of \(m_b\) and \(m_c\): \(m_{b,c}^{-2}\). Moreover, the dependence on the flavors of the light quarks in the hadron arises only starting with the subsequent order\textsuperscript{[2,3,4]} \(m_{b,c}^{-3}\). On the other hand, in the so-called SV limit\textsuperscript{[5]}: \(m_{b,c} \rightarrow \infty\), \(\Lambda_{QCD} \ll \Delta \equiv m_b - m_c \ll m_{b,c}\), the inclusive rate of the semileptonic decays \(B \rightarrow X_c \ell \bar{\nu}\) is saturated by just one exclusive decay channel: \(B \rightarrow D \ell \bar{\nu}\) for the vector part of the \(b \rightarrow c\) current and \(B \rightarrow D^* \ell \bar{\nu}\) for the axial part of the current. The latter property comes into effect due to that the form factor \(F\) of e.g. the vector \(b \rightarrow c\) current: \(\langle D | c^\dagger b | B \rangle\) is equal to one\textsuperscript{[5]} \(F = 1\), in the limit \(m_{b,c} \rightarrow \infty\) at zero recoil of the \(D\) meson, and the zero recoil limit is the only one relevant for calculating the total rate of the \(B \rightarrow D \ell \bar{\nu}\) decay if \(\Delta \ll m_c\). The relation \(F = 1\) is subject to small and calculable QCD radiative corrections\textsuperscript{[5,6]} that match the same corrections to the parton decay rate of the quark: \(b \rightarrow c \ell \bar{\nu}\), and at large but finite masses of the heavy quarks the mass corrections are also suppressed by at least\textsuperscript{[5]} \(m_{b,c}^{-2}\). The total rate of the decay is then given by the well known expression:

\[
\Gamma(B \rightarrow D \ell \bar{\nu}) = \frac{\eta G_F^2 |V_{cb}|^2 \Delta^5}{60 \pi^3},
\]

where \(\eta\) is the QCD radiative correction factor, and, for simplicity, the mass of the lepton is assumed to be small, \(m_\ell \rightarrow 0\). According to the general consideration of the heavy quark theory, the masses of the \(B\) and \(D\) mesons are heavier than the corresponding quark masses by equal amount (up to terms of order \(m_Q^{-2}\)): \(M_{B,D} = m_{b,c} + \Xi + \mathcal{O}(m_{b,c}^{-2})\), so that the energy release \(\Delta\) in eq.(1) is the same in the meson and in the quark decay: \(M_B - M_D = m_b - m_c + \mathcal{O}(m_{b,c}^{-2})\).

This agreement between the exclusive decay rate and the inclusive one comes into question if one takes into account the isotopic mass splittings of the \(B\) and \(D\) mesons. Indeed, the energy release in the decay \(B^- \rightarrow D^0 \ell \bar{\nu}\) is different from that in \(B^0 \rightarrow D^+ \ell \bar{\nu}\) by a small but non-zero amount

\[
\delta m = \left[M(B^-) - M(D^0)\right] - \left[M(B^0) - M(D^+)\right] = \left[M(D^+) - M(D^0)\right] - \left[M(B^0) - M(B^-)\right].
\]

\textsuperscript{1}For a recent review see e.g. Ref.\textsuperscript{[1]}. 
Thus naively applying the formula in eq.(1) to these exclusive decays would give a relative
difference in the rates of these decays $\delta \Gamma / \Gamma = 5 \delta m / \Delta$ scaling as inverse \textit{first} power of $\Delta$.
If identified with the total semileptonic decay rate in the SV limit, this scaling behavior of
the isotopic correction would contradict to the general short-distance OPE description of the
inclusive decay, since within the OPE there are no operators of the appropriate dimension.
Moreover, the dependence of the rates on the spectator quark flavor may arise in OPE only
starting from the terms scaling as inverse third power of the heavy quark masses. It is the
purpose of this paper to show how the isotopic difference in the energy release is reconciled
with the general results from the OPE. The final answer turns out to be that due to the
electromagnetic (in fact Coulomb) interaction of the charged lepton $\ell$ with the spectator
quark there also arises an isotopic difference in the lepton spectrum of the exclusive decays.
At energy of the lepton $E$, such that $E \gg \Lambda_{QCD}$, this difference is related to $\delta m$ as
\[
\frac{\delta d\Gamma/dE}{d\Gamma/dE} = -\delta m \frac{2E^2 - m_\ell^2}{E(E^2 - m_\ell^2)}.
\]
After integration over the energy this spectral correction completely cancels in the total
rate the correction due to the isotopic difference in the energy endpoint in the order $\delta m / \Delta$.
The cancellation, as expected, does not depend on either the charged lepton mass, or other
ingredients, e.g. the neutrino mass, that one might choose to introduce for the purpose of a
theoretical cross-check.

For the actual $B$ and $D$ mesons the discussed mass splitting is $8.43 \pm 0.31$ MeV,
thus at the typical energy of the charged lepton $E \approx 1$ GeV, the isotopic difference in
the lepton spectra amounts to about 0.9\% (at $m_\ell \to 0$) and is smaller at higher energy.
Towards lower energies the condition $E \gg \Lambda_{QCD}$, necessary for deriving eq.(3), starts to
be invalidated, and the spectral difference generally is not expressed in terms of $\delta m$, but
rather becomes sensitive to details of the electromagnetic form factor of the heavy mesons,
and therefore may be used for a study of this form factor, provided that sufficiently precise
experimental data may become available. In this paper however we are primarily concerned
with the theoretical aspect of accommodating the isotopic differences between exclusive
channels within the general OPE approach.

Proceeding to details of the argument, we first take a closer look into the isotopic mass
splittings within the heavy quark theory. The leading dependence of the meson mass on the
flavor of the light quark $q$ appears within the heavy mass expansion in the flavor dependence
of the parameter $\overline{\Lambda}$:
\[
M(Q\bar{q}) = m_Q + \overline{\Lambda}_q + \mathcal{O}(m_Q^{-2}) ,
\]
where the effects of the light quark mass and of the electromagnetic interaction within the meson can be parametrized in $\overline{\Lambda}_q$ as

$$\overline{\Lambda}_q = \overline{\Lambda}_0 + \mu_q - \alpha Q Q Q q \mu_C + \ldots$$  \hspace{1cm} (5)

In this equation $\overline{\Lambda}_0$ stands for the value of $\overline{\Lambda}$ in the limit of massless spectator quark and of zero electric charges of the quarks, $\mu_q$ is the shift of the meson mass due to the light quark mass:

$$\mu_q = m_q \partial M(Q \bar{q}) \over m_q = \langle (Q \bar{q}) | m_q (Q \bar{q}) \rangle Q Q q \mu_C \quad ,$$  \hspace{1cm} (6)

and the term with $\mu_C$ is due to the electromagnetic (essentially Coulomb) interaction between the quarks in the meson, where $\alpha$ is the QED fine structure constant, and $Q Q q q$ and $Q q q q$ are the electric charges of the quarks in units of $|e|$. Finally, the ellipses in eq.(5) stand for higher order terms in $\alpha$ and $m_q$ and those terms will be completely ignored in what follows. It is clear from eq.(5) that in the isotopic difference of the energy release in the $B^- \to D^0$ and $B^0 \to D^+$ transitions, $\delta m$ (cf. eq.(2)), the terms with $\mu_u$ and $\mu_d$ cancel, and the resulting effect is purely electromagnetic:

$$\delta m = \alpha (Q_c - Q_b) (Q_u - Q_d) \mu_C \quad .$$  \hspace{1cm} (7)

In the limit of heavy mass $m_Q$ the quantity $\mu_C$ can be expressed in terms of the form factor $f_q$ of the vector current of the light quark $j^{(q)}_\mu (\bar{q} \gamma_\mu q)$ for the meson:

$$\langle (Q \bar{q}) | j^{(q)}_\mu (\bar{q} \gamma_\mu q) \rangle Q Q q \mu = -f_q Q Q q \mu Q q \mu \delta_{\mu 0} \quad ,$$  \hspace{1cm} (8)

where the nonrelativistic normalization for the heavy states is used, and the condition $|q| \ll m_Q$ is implied, so that the spatial components of the current are small: $-f_q Q Q q \mu Q q \mu Q q Q q \mu$, and the recoil energy effects $\mathcal{O}(q^2/m_Q^2)$ can also be ignored. The shift of the energy proportional to $Q Q q q$ is then identified as the cross term between the electromagnetic currents of the heavy $Q$ quark and the light $q$ in the general formula for the electrostatic energy, so that one finally finds

$$\mu_C = 4\pi \int \frac{f_q Q Q q \mu Q q \mu Q q Q q \mu}{q^2} \frac{d^3 q}{(2\pi)^3} .$$  \hspace{1cm} (9)

In this equation it is taken into account, that at $|q| \ll m_Q$ the heavy quark has only electrostatic interaction in the leading order in $m_Q^{-1}$, and that the form factor of the heavy quark current is equal to one in this approximation. It should be also noticed that $f_q$ is defined here as the form factor for the spectator quark flavor and is normalized as $f_q(0) = 1$,}
thus the difference between the heavy mesons with different flavor of the light antiquark arises only through the overall factor of the spectator electric charge. (Taking into account flavor differences in the $f_q$ itself would lead to terms of higher order in the isospin, or the flavor SU(3), breaking.)

The assumption that it is sufficient to consider only the region of momenta such that $|q| \ll m_Q$, where the heavy quark is static, is consistent as long as the integral in eq.(9) is convergent. This definitely is the case, since the asymptotic behavior of the form factor $f_q$ at $q^2 \gg \Lambda^2_{QCD}$ (but still $q^2 \ll m_Q^2$) can be deduced from the short-distance QCD:

$$f_q(q^2) \sim \frac{\alpha_s(q^2)}{|q|^3}.$$  

It can be also noticed that the derivation of eq.(9) does not rely on the assumption (generally incorrect) that the overall electromagnetic form factor of the $Q\bar{q}$ meson is simply a sum of the electromagnetic form factors of the $Q$ quark and the $\bar{q}$ antiquark. It is only the light-flavor-dependent part of the electromagnetic energy of the meson, which is related by the equation (9) to the cross term between these form factors.

Returning to the semileptonic $B \to D$ decays, we discuss the new ingredients brought in by inclusion of the electromagnetic interaction, which breaks the heavy quark symmetry and results in a number of corrections. On the OPE side, i.e. in terms of the parton decay of the $b$ quark, there arise QED radiative corrections. These corrections are known and are determined only by $Q_b$ and $Q_c$, thus being insensitive to the isotopic charge splitting of the light quarks $Q_u - Q_d$. Moreover, these corrections match those for the exclusive $B \to D \ell \bar{\nu}$ decays and are not essential for the present discussion. Thus within the OPE approach it is required that the linear in the isotopic splitting effects should vanish in the total semileptonic decay rates of the $B$ mesons.

On the exclusive side, i.e. considering the modifications of the decay into exclusive channels, the difference in the electric charges generally violates the heavy quark symmetry relation $F = 1$ for the form factor of the weak $B \to D$ transitions in the SV limit. However, the effect on the $F$ due to the electrostatic mass shift is readily verified to be of the second order in the electrostatic energy, in agreement with the general Ademollo-Gatto theorem. Additionally, there arise in the first order in the electrostatic energy the non-vanishing amplitudes of transitions of the $B$ mesons into inelastic states, i.e. into the states different from the ground-state $D$ mesons. However, the effect of these transitions in the total rate is obviously of the second order in the isotopic splittings.

In order to find the sought linear in the isotopic splitting effect in the exclusive decay one should look into the light flavor dependent part of the QED radiative corrections in the
decays $B \rightarrow D \ell \bar{\nu}$. In the SV limit this part arises through the difference of the Coulomb interaction of the charged lepton $\ell$ with the produced $D$ meson, shown in Figure 1. Clearly, this difference is proportional to $Q_{\ell} (Q_u - Q_d)$, and the charge of the lepton is related to the charges of the heavy quarks by the charge conservation: $Q_{\ell} = Q_b - Q_c$, thus giving the correct parametric dependence of the effect on the quark electric charges. Furthermore, in the SV limit, and, as will be seen, given the convergence of the integral in eq.(9), it is sufficient to consider only static $B$ and $D$ mesons, which produce only the Coulomb field. The difference

$$\delta \frac{d\Gamma}{dE} = \frac{d\Gamma(B^- \rightarrow D^0 \ell \bar{\nu})}{dE} - \frac{d\Gamma(B^0 \rightarrow D^+ \ell \bar{\nu})}{dE}$$

(10)

of the spectra in the energy $E$ of the charged lepton is generated by the interference of the graph of Fig.1 with the ‘bare’ amplitude and is readily found in the form:

$$\delta \frac{d\Gamma}{dE} = \frac{d\Gamma}{dE} \frac{4\pi \alpha (Q_c - Q_b)(Q_u - Q_d)}{E^2} 2 \text{Re} \left[ \int \frac{f_q(q^2)}{q^2} \frac{2E^2 + (p \cdot q)}{q^2 + 2(p \cdot q)} d^3q \right]$$

(11)

where $p$ is the momentum of the charged lepton and no assumption is made about the mass of the neutrino or of the charged lepton, so that in particular $E = \sqrt{p^2 + m_\ell^2}$.

In calculating the integral in eq.(11) one can replace the factor with the angular dependence by its average over the angle between $p$ and $q$:

$$\left\langle \frac{2E^2 + (p \cdot q)}{q^2 + 2(p \cdot q)} \right\rangle = \frac{1}{2} + \frac{E^2 - q^2/4}{2 pq} \ln \frac{q + 2p}{q - 2p}$$

(12)

where $p = |p|$ and $q = |q|$. Let us now impose the condition that $p \gg \Lambda_{QCD}$ and thus $p$ is much larger than the characteristic values of $q$ in the form factor $f_q(q^2)$. This allows to consider the expansion of the expression in eq.(12) at small $q$. The leading term in the

Figure 1: The graph for the light-flavor-dependent part or the QED correction to the decay $B \rightarrow D \ell \bar{\nu}$. The dashed line denotes the photon, which in the SV limit is purely a Coulomb one carrying the spatial momentum $q$. 
expansion in $q$ is proportional to $q^{-1}$ and is purely imaginary, while the leading contribution to the real part is independent of $q$ and is given by:

$$2 \text{Re} \left\langle \frac{2 E^2 + (p \cdot q)}{q^2 + 2 (p \cdot q)} \right\rangle = \frac{2E^2 - m_t^2}{E^2 - m_t^2} + O \left( \frac{q^2}{p^2} \right).$$

(13)

The higher terms in this expansion result in corrections to the total rate scaling as higher powers of $\Delta^{-1}$ that have corresponding terms in the OPE. Thus of interest in the present discussion is only the leading term in eq. (13). Keeping only this leading term, one readily rewrites the spectral difference of eq. (11) in the form

$$\delta \frac{d \Gamma}{dE} = - \frac{d \Gamma}{dE} 4 \pi \alpha (Q_c - Q_b) (Q_u - Q_d) \frac{2E^2 - m_t^2}{E (E^2 - m_t^2)} \int \frac{f_q(q^2)}{q^2} \frac{d^3 q}{(2\pi^3)}.$$

(14)

By comparing this expression with the equations (7) and (9) one arrives at the final result in eq. (3) for the isotopic difference of the charged lepton energy spectra.

It can now be shown explicitly that the difference in the spectra exactly cancels the effect of the isotopic difference in the total energy release $\Delta$. Indeed, the total rate calculated as an integral over the ‘bare’ energy spectrum can be written as

$$\Gamma_0 = \eta \frac{G_F^2 |V_{cb}|^2}{2 \pi^3} \int_{m_\ell}^{E_{max}} E_\nu \nu p E p dE,$$

(15)

where $E_\nu = \Delta - E$ and $p_\nu = \sqrt{E_\nu^2 - m_\nu^2}$ are the energy and the momentum of the neutrino, and we allow for an arbitrary non-zero neutrino mass in order to illustrate the robustness of the discussed cancellation. The upper limit of integration is then $E_{max} = \Delta - m_\nu$. Notice also that the QCD correction factor in the SV limit is constant over the spectrum. Let us introduce the notation $N(E_\nu) = E_\nu \nu p$ and note that it enters in eq. (15) as $N(\Delta - E)$ and that $N(\Delta - E_{max}) = 0$. Due to the latter property the linear in $\delta m$ change in the rate under the shift of $\Delta$: $\Delta \rightarrow \Delta + \delta m$ is given by:

$$\delta \Gamma \eta = \frac{G_F^2 |V_{cb}|^2}{2 \pi^3} \delta m \int_{m_\ell}^{E_{max}} \left( \frac{dN(\Delta - E)}{d\Delta} \right) E p dE .$$

(16)

On the other hand, using the elementary relation

$$E p(E) \frac{2E^2 - m_t^2}{E (E^2 - m_t^2)} = \frac{d}{dE} E p(E),$$

This purely imaginary term develops into the difference of the Coulomb scattering phases, that is logarithmically divergent in the infrared and is proportional to $f_q(0) = 1.$
we find from eq. (3) for the change of the rate due to the isotopic difference in the lepton spectra the expression

\[ \delta_2 \Gamma = -\eta \frac{G_F^2 |V_{cb}|^2}{2 \pi^3} \delta m \int_{m_\ell}^{E_{max}} N(\Delta - E) \left[ \frac{d}{dE} E p(E) \right] dE = -\delta_1 \Gamma, \tag{17} \]

where the latter transition involves integration by parts (with the relations \( N(\Delta - E_{max}) = 0 \) and \( p|_{E=m_\ell} = 0 \) taken into account) and also noticing that \( dN(\Delta - E)/dE = -dN(\Delta - E)/d\Delta \). Thus we find that \( (\delta_1 + \delta_2) \Gamma = 0 \), which concludes our proof that the total semileptonic rate is not affected by the isotopic mass differences in the linear order in \( \delta m \), independently of the charged lepton or the neutrino masses, in full compliance with the OPE result. It should be also mentioned that although for notational definiteness the reasoning above is given for the decays of the \( B \) mesons into the pseudoscalar \( D \) mesons, all the formulas are fully applicable to the \( B \to D^* \) transitions, since in the SV limit the discussed effects are determined by the spin-independent electrostatic interaction.

The isotopic difference in eq. (3) is found in this paper in the SV limit, where the heavy mesons in the \( B \to D (D^*) \ell \bar{\nu} \) decays are strictly static. The theoretical parameter \( \xi = (m_b - m_c)^2/(m_b + m_c)^2 \) governing the deviation from this limit for the actual \( B \to D (D^*) \) transitions\(^5\) is not very small: \( \xi \approx 0.3 \). Also phenomenologically it is known that the exclusive decays \( B \to D (D^*) \ell \bar{\nu} \) saturate about 65% of the total semileptonic rate, rather than completely, as they should in the SV limit. Therefore a more elaborate consideration beyond the static SV approximation is desirable for more accurate predictions of the experimentally measurable isotopic difference in the lepton spectra, if such experimental study appears on the agenda. At this point, based on eq. (3) and on the reasonable smallness of \( \xi \), one can assert that the difference in the spectra is not hopelessly small and should amount to a sizeable fraction of 1% at the lepton energy about 1 GeV. At smaller energies, where the condition \( p \gg \Lambda_{QCD} \) cannot be used, the full formulas in eqs. (11) and (12) and their modification beyond the SV can be used for a study of the form factor \( f_q(q^2) \) in the spacelike region.

I am thankful to Arkady Vainshtein and Yuichi Kubota for enlightening discussions of the theoretical aspects and experimental possibilities. This work is supported in part by the DOE grant DE-FG02-94ER40823.

References
[1] I. Bigi, M. Shifman, and N. Uraltsev, Ann. Rev. Nucl. Part. Sci. 47 (1997) 591, [hep-ph/9703290].

[2] M.A. Shifman and M.B. Voloshin, (1981) unpublished, presented in the review V.A.Khoze and M.A. Shifman, Sov. Phys. Usp. 26 (1983) 387

[3] N. Bilic, B. Guberina and J. Trampetic, Nucl. Phys. B248 (1984) 261.

[4] M.A. Shifman and M.B. Voloshin, Sov. J. Nucl. Phys. 41 (1985) 120.

[5] M.B. Voloshin and M.A. Shifman, Sov. J. Nucl. Phys. 47 (1988) 511.

[6] A. Czarnecki, K. Melnikov, and N. Uraltsev, Phys. Rev. D57 (1998) 1769, [hep-ph/9706311].

A. Czarnecki and K. Melnikov, Phys. Rev. D56 (1997) 7216, [hep-ph/9706227].

[7] M.E. Luke, Phys. Lett. B252 (1990) 447.

[8] R.M. Barnett et.al., Particle Data Group, Phys. Rev. D54 (1996) 1.

[9] N. Isgur and M.B. Wise, Phys. Lett. B232 (1989) 113.

[10] D. Atwood and W.J. Marciano, Phys. Rev. D41 (1990) 1736.