Stability of Current Density Impedance Imaging

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Abstract

We study stability of reconstruction in current density impedance imaging (CDII), that is, the inverse problem of recovering the conductivity of a body from the measurement of the magnitude of the current density vector field in the interior of the object. Our results show that CDII is stable with respect to errors in interior measurements of the current density vector field, and confirm the stability of reconstruction which was previously observed in numerical simulations, and was long believed to be the case.

1 Introduction

The classical Electrical Impedance Tomography (EIT) aims to obtain quantitative information on the electrical conductivity \( \sigma \) of a conductive body from measurements of voltages and corresponding currents at its boundary. Mathematics of EIT has been extensively studied, and many interesting results have been obtained about uniqueness, stability and reconstruction algorithms for this problem. See [3, 4, 5, 6] for excellent reviews of the results. It is well known that that EIT is severely ill-posed, and provides images with very low resolution away from the boundary [9, 13].

A more recent class of Inverse Problems seeks to provide images with high accuracy and by using data obtained from the interior of the region. Such methods are referred to as Hybrid Inverse Problems or Coupled-physics methods, as they usually involve the interaction of two kinds of physical fields. In this paper we

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study stability of reconstruction in Current Density Impedance Imaging (CDII),
that is, the inverse problem of recovering the conductivity of a body from the
measurement of the magnitude of the current density vector field in the interior
of the object. Interior measurements of current density is possible by Magnetic
Resonance Imaging (MRI) due to the work of M. Joy and his collaborators [11, 12].
This problem has been studied in [18, 20, 22, 23, 24]. See also [25] for
a comprehensive review. While the uniqueness of the reconstruction in CDII is
established and a robust reconstruction algorithm is developed in [19], the stability
of CDII is still open. In this paper we aim to settle the stability of reconstruction
in CDII, and provide a detailed stability analysis.

Let \( \sigma \) be the isotropic conductivity of an object \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \), where \( \Omega \) is
a bounded open region in with connected boundary. Suppose \( J \) is the current
density vector field generated by imposing a given boundary voltage \( f \) on \( \partial \Omega \).
Then the corresponding voltage potential \( u \) satisfies the elliptic equation

\[
\nabla \cdot (\sigma \nabla u) = 0, \quad u|_{\partial \Omega} = f.
\]

By Ohm’s law \( J = -\sigma \nabla u \), and \( u \) is the unique minimizer of the weighted least
gradient problem

\[
I(w) = \min_{w \in BV_f(\Omega)} \int_\Omega a |\nabla w| dx,
\]

where \( a = |J| \), and \( BV_f(\Omega) = \{ w \in BV(\Omega), w|_{\partial \Omega} = f \} \), see [18, 20, 22, 23, 24].

Note that the weighted least gradient problem (2) is not strictly convex, and
hence in general it may not have a unique minimizer. See [10] where the second
author and his collaborators showed that for \( a \in C^{1,\alpha}(\Omega), 0 < \alpha < 1 \), the least
gradient problem (2) could have infinitely many minimizers. Since any stability result
trivially implies uniqueness, it is evident that one needs additional assumptions
to prove any stability result. Indeed stability analysis of CDII is a challenging
problem. In [21], Nashed and Tamasan showed the continuous dependence of the
minimizers of (2) with respect to \( a \). In [27], the authors proved two dimensional
stability results by reducing the problem to a two-point boundary value problem
for a second order 2-system. However, their approach does not yield stability for
conductivity \( \sigma \) (only for the potential \( u \)). In [15] Montalto and Stefanov proved
the following stability result for \( \sigma \).

**Theorem 1.1** ([15]). Let \( u \) solve equation (1) and let \( \tilde{u} \) solve equation (1) for \( \tilde{\sigma} \) with \( |\nabla \tilde{u}| > 0 \) in \( \overline{\Omega} \). For any \( 0 < \alpha < 1 \), there exists \( s > 0 \) such that if
\( ||\sigma||_{H^{s}(\Omega)} < L \) for some \( L > 0 \) then there is an \( \epsilon > 0 \) such that if
\[
||\sigma - \tilde{\sigma}||_{C^{2}(\Omega)} < \epsilon,
\]

\( \sigma \), \( \tilde{\sigma} \), and \( \Omega \).
then
\[ \| \sigma - \tilde{\sigma} \|_{L^2(\Omega)} < C \| | J | - | \tilde{J} | \|_{L^2(\Omega)} \]

Later in [14], Montalto and Tamasin proved the following stability result.

**Theorem 1.2 ([14]).** Let \( \sigma \in C^{1,\alpha}(\Omega) \), \( 0 < \alpha < 1 \), be positive in \( \Omega \). Let \( u \) solve equation (1) with \( | \nabla u | > 0 \) in \( \Omega \). There exists \( \epsilon > 0 \) depending on \( \epsilon \) and some \( C > 0 \) depending on \( \epsilon \) such that if \( \tilde{\sigma} \in C^{1,\alpha}(\Omega) \) with \( \tilde{u} \) solving (1) for \( \tilde{\sigma} \), \( u = \tilde{u} = f \) on \( \partial \Omega \), \( \sigma = \tilde{\sigma} \) on \( \partial \Omega \), and

\[ \| \sigma - \tilde{\sigma} \|_{C^{1,\alpha}(\Omega)} < \epsilon, \]

then
\[ \| \sigma - \tilde{\sigma} \|_{L^2(\Omega)} \leq C \| \nabla \cdot (\Pi_{\nabla u}(J - \tilde{J})) \|_{L^2(\Omega)}, \]

where \( \Pi_{\nabla u}(J - \tilde{J}) \) is the projection of \( J - \tilde{J} \) onto \( \nabla u \).

Note that both of the above results assume a priori that \( \sigma \) and \( \tilde{\sigma} \) are close, and a natural question which remains open is that whether there exists two distant conductivities \( \sigma \) and \( \tilde{\sigma} \) which could induce the corresponding currents \( J \) and \( \tilde{J} \) with \( | J | - | \tilde{J} | \) arbitrarily small. In this paper we address the this question and show that the answer is negative, and hence CDII is actually stable. Under some natural assumption, we shall prove that in dimensions \( n = 2, 3 \) the following stability result holds

\[ \| \sigma - \tilde{\sigma} \|_{L^2(\Omega)} \leq C \| | J | - | \tilde{J} | \|_{L^\infty(\Omega)}, \]

for some constant \( C \) independent of \( \tilde{\sigma} \) (see Theorems 4.5 and 4.6 for precise statements of the results).

The paper is organized as follows. In Section 2, under very weak assumptions, we will prove that the structure of level sets of the least gradient problem (2) is stable. In Section 3 we will provide stability results for minimizers of (2) in \( L^1 \). In Section 4 we will prove stability of minimizers of (2) in \( W^{1,1} \), and shall use them to prove Theorems 4.5 and 4.6 which are the main results of this paper.

## 2 Stability of level sets

In this section we show that the structure of the level sets of minimizers of the least gradient problem (2) is stable. Throughout the paper we will assume that \( a, \tilde{a} \in C(\Omega) \) with

\[ 0 < m \leq a(x), \tilde{a}(x) \leq M, \quad \forall x \in \Omega, \]

for some positive constants \( m, M \). The following theorem which was proved in [17] by the second author, shall play a crucial role in the proof of the results in this section.
Theorem 2.1 ([17]). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and assume that $a \in C(\overline{\Omega})$ is a non-negative function, and $f \in L^1(\partial \Omega)$. Then there exists a divergence free vector field $J \in (L^\infty(\Omega))^n$ with $|J| \leq a \text{ a.e. in } \Omega$ such that every minimizer $w$ of (2) satisfies
\[
J \cdot \frac{Dw}{|Dw|} = |J| = a, \quad |Dw| \text{ - a.e. in } \Omega,
\]
where $\frac{Dw}{|Dw|}$ is the Radon-Nikodym derivative of $Dw$ with respect to $|Dw|$.

Lemma 2.2. Let $f \in L^1(\partial \Omega)$, and assume $u$ and $\tilde{u}$ are minimizers of (2) with the weights $a$ and $\tilde{a}$, respectively. Then
\[
\left| \int_{\Omega} a|Du| dx - \int_{\Omega} \tilde{a}|D\tilde{u}| dx \right| \leq C\|a - \tilde{a}\|_{L^\infty(\Omega)},
\]
for some constant $C = C(m, M, \Omega, f)$ independent of $u$ and $\tilde{u}$.

Proof. First note that in view of (5) we have
\[
m \int_{\Omega} |D\tilde{u}| dx \leq \int_{\Omega} \tilde{a}|D\tilde{u}| dx \leq \int_{\Omega} \tilde{a}|Du| dx \leq M \int_{\Omega} |Dw|
\]
for any $w \in BV_f(\Omega)$. Thus $\int_{\Omega} |D\tilde{u}| \leq C$, and similarly $\int_{\Omega} |Du| < C$ for some constant $C$ which depends only on $m, M,$ and $\Omega$. Hence
\[
\max\{\int_{\Omega} |D\tilde{u}|, \int_{\Omega} |Du|\} \leq C,
\]
for some $C(m, M)$ independent of $\tilde{u}$ and $u$. Since $u, \tilde{u}$ are the minimizers of (2) with the weights $a$ and $\tilde{a},$
\[
\int_{\Omega} a|Du| dx - \int_{\Omega} \tilde{a}|D\tilde{u}| dx \leq \int_{\Omega} a|Du| dx - \int_{\Omega} \tilde{a}|D\tilde{u}| dx
\]
\[
\leq \int_{\Omega} a|D\tilde{u}| dx - \int_{\Omega} \tilde{a}|D\tilde{u}| dx.
\]
Thus
\[
\int_{\Omega} (a - \tilde{a})|Du| dx \leq \int_{\Omega} a|Du| dx - \int_{\Omega} \tilde{a}|Du| dx \leq \int_{\Omega} (a - \tilde{a})|D\tilde{u}| dx,
\]
and we get
\[
-\|a - \tilde{a}\|_{L^\infty(\Omega)} \|Du\|_{L^1(\Omega)} \leq \int_{\Omega} a|Du| dx - \int_{\Omega} \tilde{a}|Du| dx
\]
\[
\leq \|a - \tilde{a}\|_{L^\infty(\Omega)} \|D\tilde{u}\|_{L^1(\Omega)}.
\]
Hence (7) follows from (8).

Let \( \nu_\Omega \) denote the outer unit normal vector to \( \partial \Omega \). Then for every \( T \in (L^\infty(\Omega))^n \) with \( \nabla \cdot T \in L^n(\Omega) \) there exists a unique function \([T, \nu_\Omega] \in L^\infty(\partial \Omega)\) such that

\[
\int_{\partial \Omega} [T, \nu_\Omega] u \, dH^{n-1} = \int_\Omega u \nabla \cdot T \, dx + \int_\Omega T \cdot D u \, dx, \quad u \in C^1(\Omega). \tag{9}
\]

Moreover, for \( u \in BV(\Omega) \) and \( T \in (L^\infty(\Omega))^n \) with \( \nabla \cdot T \in L^n(\Omega) \), the linear functional \( u \mapsto (T \cdot Du) \) gives rise to a Radon measure on \( \Omega \), and (9) holds for every \( u \in BV(\Omega) \) (see [1, 2] for a proof). We shall need the weak integration by parts formula (9).

**Lemma 2.3.** Let \( f \in L^1(\partial \Omega) \), and assume \( u \) and \( \check{u} \) are minimizers of (2) with the weights \( a \) and \( \check{a} \), respectively. Let \( J \) and \( \check{J} \) be the divergence free vector fields guaranteed by Theorem 2.1. Suppose \( 0 \leq \sigma(x) \leq \sigma_1 \) in \( \Omega \) for some constant \( \sigma_1 > 0 \), where \( \sigma \) is the Radon-Nikodym derivative of \( |J|dx \) with respect to \( |Du| \). Then

\[
\int_\Omega |J| |\check{J}| - J \cdot \check{J} dx \leq C \|a - \check{a}\|_{L^\infty(\Omega)}, \tag{10}
\]

where \( C = C(m, M, \sigma_1, \Omega, f, u) \) is a constant independent of \( \check{a} \).

**Proof.** We have

\[
\int_\Omega |J||\check{J}| - J \cdot \check{J} dx = \int_\Omega \sigma |\check{J}| |Du| - \sigma \check{J} \cdot Du dx
\]

\[
\leq \sigma_1 \int_\Omega |\check{J}| |Du| - \check{J} \cdot Du dx
\]

\[
= \sigma_1 \int_\Omega |\check{J}| |Du| dx - \int_{\partial \Omega} f[\check{J}, \nu_\Omega] dx
\]

\[
= \sigma_1 \int_\Omega |\check{J}| |Du| - \check{J} \cdot D\check{u} dx
\]

\[
= \sigma_1 \int_\Omega |\check{J}| |Du| - |\check{J}| |D\check{u}| dx,
\]

where we have used (6) and the integration by parts formula (9). On the other hand it follows from lemma 2.2 that

\[
\sigma_1 \int_\Omega |\check{J}| |Du| - |\check{J}| |D\check{u}| dx = \sigma_1 \int_\Omega |\check{J}| |Du| - |J||Du| + |J||Du| - |\check{J}| |D\check{u}| dx
\]

\[
= \sigma_1 \left( \int_\Omega (a - \check{a}) |Du| dx + \int_\Omega a |Du| - \check{a} |D\check{u}| dx \right)
\]

\[
\leq \sigma_1 (\|Du\|_{L^1(\Omega)} \|a - \check{a}\|_{L^\infty(\Omega)} + C \|a - \check{a}\|_{L^\infty(\Omega)}),
\]

5
which yields the desired result. □

Roughly speaking, Lemma 2 implies that as \( a \to \tilde{a} \), \( \frac{Du}{Du} (x) \) becomes parallel to \( \frac{D\tilde{u}}{D\tilde{u}} (x) \) at points where the two gradients do not vanish. We are now ready to prove the main result of this section.

**Theorem 2.4.** Let \( f \in L^1(\partial \Omega) \), and assume \( u \) and \( \tilde{u} \) are minimizers of (2) with the weights \( a \) and \( \tilde{a} \), respectively. Let \( J \) and \( \tilde{J} \) be the divergence free vector fields guaranteed by Theorem 2.1. Suppose \( 0 \leq \sigma(x) \leq \sigma_1 \) in \( \Omega \) for some constant \( \sigma_1 > 0 \), where \( \sigma \) is the Radon-Nikodym derivative of \( |J|dx \) with respect to \( |Du| \). Then

\[
\|J - \tilde{J}\|_{L^1(\Omega)} \leq C\|a - \tilde{a}\|_{L^\infty(\Omega)}^{1/2},
\]  

(11)

where \( C = C(m, M, \sigma_1, \Omega, f, u) \) is a constant independent of \( \tilde{a} \).

**Proof.** We have

\[
\sqrt{|J - \tilde{J}|^2} = \sqrt{|J|^2 + |\tilde{J}|^2 - 2J \cdot \tilde{J}}
\]

\[
= \sqrt{|J - |\tilde{J}||^2 + 2(|J||\tilde{J}| - J \cdot \tilde{J})}
\]

\[
\leq |J - |\tilde{J}|| + \sqrt{2(|J||\tilde{J}| - J \cdot \tilde{J})}.
\]

Hence,

\[
\|J - \tilde{J}\|_{L^1(\Omega)} = \int_\Omega \sqrt{|J - |\tilde{J}||^2} \, dx
\]

\[
\leq \int_\Omega |J - |\tilde{J}|| + \sqrt{2(|J||\tilde{J}| - J \cdot \tilde{J})} \, dx
\]

\[
= \int_\Omega |a - \tilde{a}| \, dx + \sqrt{2(|J||\tilde{J}| - J \cdot \tilde{J})} \, dx
\]

\[
\leq |\Omega|\|a - \tilde{a}\|_{L^\infty(\Omega)} + |\Omega|^{1/2} \left( \int_\Omega 2(|J||\tilde{J}| - J \cdot \tilde{J}) \, dx \right)^{1/2}
\]

\[
\leq |\Omega|\|a - \tilde{a}\|_{L^\infty(\Omega)} + (2|\Omega|)^{1/2} \left( C\|a - \tilde{a}\|_{L^\infty(\Omega)} \right)^{1/2}
\]

\[
= \left( |\Omega|\|a - \tilde{a}\|_{L^\infty(\Omega)}^2 + (2|\Omega|)^{1/2} C^{1/2} \right) \|a - \tilde{a}\|_{L^\infty(\Omega)}^{1/2},
\]

where we have used the Holder’s inequality and Lemma 2.3. □

**Remark 2.5.** In view of Theorem 2.1, \( \frac{Du}{Du} \) and \( \frac{D\tilde{u}}{D\tilde{u}} \) are parallel to \( J \) and \( \tilde{J} \), respectively. So Theorem (2.4) implies that if \( \tilde{a} \) is close to \( a \), then the structure of level sets of \( \tilde{u} \) is close to that of \( u \).
3 $L^1$ stability of the minimizers

In this section we establish stability of minimizers of the least gradient problem (2) in $L^1$. In general (2) does not even have unique minimizers, so in order to prove any stability results further assumptions on the weights $a, \tilde{a}$, and on the corresponding minimizers are expected.

Definition 3.1. Fix the positive constants $\sigma_0, \sigma_1 \in \mathbb{R}$. We say that $u \in C^1(\bar{\Omega})$ is admissible if it solves the conductivity equation (1) for some $\sigma \in C(\Omega)$ with

$$0 < \sigma_0 < \sigma \leq \sigma_1,$$

and $m \leq |J| = |\sigma \nabla u| \leq M$, where $m$ and $M$ are positive constants as in (5). We shall denote the corresponding induced current by $J = -\sigma \nabla u$.

We will first prove our results in dimension $n = 2$ and then extend them to dimensions $n = 3$.

Let $u \in C^1(\Omega)$ with $|\nabla u| > 0$ in $\Omega$. Then it follows from the regularity result of De Giorgi (see, e.g, Theorem 4.11 in [7]) that all level sets of $u$ are $C^1$ curves. We will assume that the length of level sets of $u$ in $\Omega$ is uniformly bounded, i.e.

$$\sup_{t \in \mathbb{R}} \int_{\{u = t\} \cap \Omega} 1dS = L_M < \infty. \quad (12)$$

Theorem 3.2. Let $n = 2$, and suppose $u$ and $\tilde{u}$ are admissible with $u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f$, and corresponding current density vector fields $J$ and $\tilde{J}$, respectively. If $u$ satisfies (12), then

$$\|u - \tilde{u}\|_{L^1(\Omega)} \leq C\|J\| - \|\tilde{J}\|^{\frac{1}{2}}_{L^\infty(\Omega)}, \quad (13)$$

for some constant $C(m, M, \sigma_0, \sigma_1, f, u, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

Proof. Since $u$ is admissible,

$$|\nabla u(x)| = \frac{|J(x)|}{\sigma(x)} \geq \frac{m}{\sigma_1} > 0, \quad \forall x \in \Omega.$$

Using the coarea formula we get

$$\frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}|dx \leq \int_{\Omega} |\nabla u||u - \tilde{u}|dx = \int_{\mathbb{R}} \int_{\{u = t\} \cap \Omega} |u - \tilde{u}|dSdt. \quad (14)$$

Since $|\nabla u| > 0$ in $\Omega$, it follows from the regularity result of De Giorgi (Theorem 4.11 in [7]) that all level sets of $u$ are $C^1$ curves. Now let $\Gamma_t$ be a connected component
of \( \{ x \in \Omega : u(x) = t \} \subset \Omega \), and \( \gamma : [0, L] \to \Gamma_t \) to be a path parameterized by the arc length with \( \gamma(0) \in \partial \Omega \). Define
\[
h(s) := u(\gamma(s)) - \tilde{u}(\gamma(s)).
\]
Then \( h(\gamma(0)) = 0 \). Moreover since \( \nabla u(\gamma(s)) \cdot \gamma'(s) = 0 \) on \( \Gamma_t \),
\[
h'(s) = \nabla u(\gamma(s)) \cdot \gamma'(s) - \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s)
= \left( \frac{\sigma}{\tilde{\sigma}}(\gamma(s)) \nabla u(\gamma(s)) - \nabla \tilde{u}(\gamma(s)) \right) \cdot \gamma'(s).
\]
We can rewrite the above equality as
\[
h'(s) = \frac{J(\gamma(s)) - \tilde{J}(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s).
\]
Now let \( x^*_t \) be a point on \( \Gamma_t \) where the maximum distance between \( u \) and \( \tilde{u} \) along the path \( \gamma \) occurs, i.e.
\[
|u(x^*_t) - \tilde{u}(x^*_t)| = \max_{x \in \Gamma_t} |u(x) - \tilde{u}(x)|.
\]
Then \( x^*_t = \gamma(s_0) \) for some \( s_0 \in [0, L] \), and
\[
|u(x^*_t) - \tilde{u}(x^*_t)| = |h(s_0)| = \left| \int_0^{s_0} \frac{J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))}{\tilde{\sigma}(\gamma(\tau))} \cdot \gamma'(\tau) d\tau \right|
\leq \int_0^{s_0} \frac{1}{\tilde{\sigma}(\gamma(\tau))} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau
\leq \frac{1}{\sigma_0} \int_0^{s_0} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau.
\]
In particular for every \( x \in \Gamma_t \)
\[
|u(x) - \tilde{u}(x)| \leq |u(x^*_t) - \tilde{u}(x^*_t)| \leq \frac{1}{\sigma_0} \int_0^{L} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau,
\]
where \( L \) denotes the entire length of \( \Gamma_t \). Hence
\[
\int_{\Gamma_t} |u(x) - \tilde{u}(x)| dl \leq |u(x^*_t) - \tilde{u}(x^*_t)| \int_{\Gamma_t} 1 dl
\leq L_M |u(x^*_t) - \tilde{u}(x^*_t)|
\leq \frac{L_M}{\sigma_0} \int_0^{L} |J(\gamma(\tau)) - \tilde{J}(\gamma(\tau))| d\tau
= \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl,
\]
and therefore
\[ \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dt \leq \frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \Omega} |J - \tilde{J}| dl. \] (15)

Thus we have
\[ \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dtdt \leq \frac{L_M}{\sigma_0} \int_{\mathbb{R}} \int_{\{u=t\}} |J - \tilde{J}| dSdt \]
\[ = \frac{L_M}{\sigma_0} \int_{\Omega} |\nabla u||J - \tilde{J}| dx \]
\[ \leq \frac{L_M}{\sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \int_{\Omega} |J - \tilde{J}| dx \]
\[ \leq C \|a - \tilde{a}\|_{L^\infty(\Omega)}^{\frac{3}{2}}, \]
where we have used (15) and Theorem 2.4. □

Next we generalize Theorem 3.2 to dimension \( n = 3 \). In order to do this, we need the following additional assumption on level sets of \( u \).

**Definition 3.3.** Let \( u \in C^1(\bar{\Omega}) \) be admissible. We say that level sets of \( u \) can be foliated to one-dimensional curves if there exists a function \( g(x) \in C^1(\Omega) \) such that \( 0 < c_g \leq |\nabla g_t| \leq C_g \), for some constant \( c_g, C_g \in \mathbb{R} \), where \( g_t \) is the restriction of \( g \) to the level set \( \{u = t\} \) equipped with the metric induced from the Euclidean metric in \( \mathbb{R}^3 \). Moreover, every connected component of \( \{u = t\} \cap \{g = r\} \cap \Omega \) is a \( C^1 \) curve reaching the boundary \( \partial \Omega \) for all \( t, r \in \mathbb{R} \). Similar to the case \( n = 2 \), we assume that the length of connected components of \( \{u = t\} \cap \{g = r\} \cap \Omega \) are uniformly bounded by some constant \( L_M \).

**Theorem 3.4.** Let \( n = 3 \), and suppose \( u \) and \( \tilde{u} \) are admissible with \( u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f \) and corresponding current density vector fields \( J \) and \( \tilde{J} \), respectively. Suppose the level sets of \( u \) can be foliated to one-dimensional curves in the sense of Definition 3.3. Then
\[ \|u - \tilde{u}\|_{L^1(\Omega)} \leq C \|J\|_{L^\infty(\Omega)}^{\frac{3}{2}}, \] (16)
where \( C(m, M, \sigma_0, \sigma_1, f, u, L_M, c_g, C_g) \) is independent of \( \tilde{u} \) and \( \tilde{\sigma} \).

**Proof.** The proof is similar to the proof of Theorem 3.2, and we provide the details for the sake of the reader. Since \( u \) is admissible,
\[ \frac{m}{\sigma_1} \int_{\Omega} |u - \tilde{u}| dx \leq \int_{\Omega} |\nabla u||u - \tilde{u}| dx = \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \tilde{u}| dSdt. \] (17)
The level sets of $u$ can be foliated into one-dimensional curves by level sets of some function $g$ in the sense of Definition 3.3. Thus

$$
\int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \bar{u}| dS dt = \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} \frac{\nabla g_t}{|\nabla g_t|} |u - \bar{u}| dS dt
$$

$$
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} \frac{1}{|\nabla g_t|} |u - \bar{u}| dldr dt
$$

$$
\leq \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \bar{u}| dldr dt.
$$

Similar to the two dimensional case, we parameterize every connected component $\Gamma_t$ of $\{u = t\} \cap \{g = r\} \cap \Omega$ by arc length, $\gamma: [0, L] \to \Gamma_t$ with $\gamma(0) \in \partial \Omega$, and let $h(s) = u(\gamma(s)) - \bar{u}(\gamma(s))$. Let $x_t^*$ be the point that maximizes $|u - \bar{u}|$ on $\Gamma_t$ and suppose $\gamma(s_0) = x_t^*$ for some $s_0 \in (0, L)$, where $L$ is the length of $\Gamma_t$. Then by an argument similar to the one in the proof of Theorem 3.2 we get

$$
|u(x_m) - \bar{u}(x_m)| \leq \frac{1}{\sigma_0} \int_0^L |J(\gamma(\tau)) - J(\tilde{\gamma}(\tau))| d\tau,
$$

and consequently

$$
\int_{\Gamma_t} |u(x) - \bar{u}(x)| dl \leq \frac{L_M}{\sigma_0} \int_{\Gamma_t} |J - \tilde{J}| dl.
$$

Hence

$$
\int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \bar{u}| dl \leq \frac{L_M}{\sigma_0} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |J - \tilde{J}| dl. \tag{18}
$$

Using this estimate and the coarea formula we have

$$
\frac{m}{\sigma_1} \int_{\Omega} |u - \bar{u}| dx \leq \int_{\mathbb{R}} \int_{\{u=t\} \cap \Omega} |u - \bar{u}| dS dt
$$

$$
\leq \frac{1}{c_g} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |u - \bar{u}| dldr dt
$$

$$
\leq \frac{L_M}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\{u=t\} \cap \{g=r\} \cap \Omega} |J - \tilde{J}| dldr dt
$$

$$
= \frac{L_M}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\{u=t\}} |\nabla g_t| |J - \tilde{J}| dS dt
$$

$$
\leq \frac{L_M C_g}{c_g \sigma_0} \int_{\mathbb{R}} \int_{\{u=t\}} |J - \tilde{J}| dS dt
$$

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\begin{align*}
&= \frac{L_M C_g}{c_g \sigma_0} \int_\Omega |\nabla u||J - \tilde{J}| \, dx \\
&\leq \frac{L_M C_g}{c_g \sigma_0} \|\nabla u\|_{L^\infty(\Omega)} \|\tilde{J} - |\tilde{J}|^{1/2} \|_{L^\infty(\Omega)} \\
&\leq \frac{L_M C_g M}{c_g \sigma_0^2} \|\tilde{J} - |\tilde{J}|^{1/2} \|_{L^\infty(\Omega)},
\end{align*}

where we have applied Theorem 1.3. \(\square\)

4 \ W^{1,1} stability of the minimizers

In this section we prove stability of minimizers of (2) in \(W^{1,1}\). As mentioned in Section 3, in general (2) does not even have unique minimizers, so in order to prove stability results in \(W^{1,1}\), it is natural to expect stronger assumptions on the minimizers.

**Lemma 4.1.** Let \(n = 2, 3\), and suppose \(u\) and \(\tilde{u}\) are admissible with \(u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f \in L^\infty(\partial \Omega)\) and corresponding conductivities \(\sigma\) and \(\tilde{\sigma}\), and current density vector fields \(J\) and \(\tilde{J}\), respectively. Suppose \(\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})\) with

\[\|\sigma\|_{C^2(\Omega)}, \|\tilde{\sigma}\|_{C^2(\Omega)} \leq \sigma_2\]

for some \(\sigma_2 \in \mathbb{R}\). Let

\[G(x) := \frac{\tilde{J}(x) - J(x)}{\tilde{\sigma}(x)}, \quad x \in \Omega,\]

with \(G = (G_1, G_2)\) for \(n = 2\) and \(G = (G_1, G_2, G_3)\) for \(n = 3\). Then

\[\|\nabla G_i\|_{L^1(\Omega)} \leq C_1 \|J - \tilde{J}\|_{L^1(\Omega)}^{1/2},\]

for some constant \(C_1\) which depends only on \(\Omega, \sigma_0, \sigma_2\) and \(f\).

**Proof.** Since \(u\) and \(\tilde{u}\) satisfy (1), it follows from elliptic regularity that

\[\|u\|_{H^3(\Omega)}, \|\tilde{u}\|_{H^3(\Omega)} \leq C_1,\]

for some constant \(C_1\) depending only on \(\sigma_0, \sigma_2, f,\) and \(\Omega\). Now note that

\[G(x) = \nabla \tilde{u} - \frac{\sigma}{\tilde{\sigma}} \nabla u.\]

Thus it follows from (19) and (22) that

\[\|D^2 G_i\|_{L^1(\Omega)} \leq |\Omega|^{1/2} \|D^2 G_i\|_{L^2(\Omega)} \leq C, \quad 1 \leq i \leq n,\]
for some constant $C$ which only depends on $\sigma_0$, $\sigma_2$, $\Omega$ and $f$. On the other hand it follows from Gagliardo-Nirenberg interpolation inequality that
\[
\| \nabla G_i \|_{L^1(\Omega)} \leq C_2 \| D^2 G_i \|_{L^1(\Omega)}^{1/2} \| G_i \|_{L^1(\Omega)}^{1/2}, \tag{24}
\]
for some $C_2$ which only depends on $\Omega$. Combining (23), (24), and
\[
\| G_i \|_{L^1(\Omega)} \leq \| J - \tilde{J} \|_{L^1(\Omega)} / \sigma_0, \tag{25}
\]
we arrive at the inequality (21). □

Next we prove that $u$ and $\tilde{u}$ are close in $W^{1,1}(\Omega)$. In order to do so, we need additional assumptions on the structure of level sets of $u$.

**Definition 4.2.** Suppose $u$ is admissible, $n = 2$, and $x \in \Omega$. Pick $h \in \mathbb{R}^2$ with $|h| = 1$, and $t \in \mathbb{R}$ small enough such that $x + th \in \Omega$. Let $\Gamma$ and $\Gamma_t$ be the level sets of $u$ passing through $x$ and $x + th$, respectively. Parametrize $\Gamma$ and $\Gamma_t$ by the arc length such that $\gamma(0), \gamma_t(0) \in \partial \Omega$, and denote these parametrizations by $\gamma$ and $\gamma_t$, respectively.

Similarly in dimension $n = 3$, let $u$ be admissible and suppose level sets of $u$ can be foliated to one-dimensional curves in the sense of Definition 3.3. Pick $x \in \Omega$ and $h \in \mathbb{R}^3$ with $|h| = 1$, and choose $t$ small enough such that $x + th \in \Omega$. Let $\Gamma$ and $\Gamma_t$ be the unique curves in
\[
\{ \{ u = \tau \} \cap \{ g = r \} \mid \tau, r \in \mathbb{R} \}
\]
which pass through $x$ and $x + th$, respectively, and let $\gamma$ and $\gamma_t$ be the parametrization of these curves with respect to arc length with $\gamma(0), \gamma_t(0) \in \partial \Omega$.

We say that level sets of $u$ are well structured if the following conditions are satisfied

(a) There exists $K \geq 0$ such that
\[
\left| \frac{\gamma_i'(s) - \gamma'(s)}{t} \right| \leq K \tag{25}
\]
for every $s \in [0, L]$, $t \in \mathbb{R}$, $x \in \Omega$ and $h \in S^n$. In particular,
\[
\gamma_i'(s) \to \gamma'(s) \quad \text{as} \quad t \to 0, \tag{26}
\]
where $\gamma'(s) = \frac{d\gamma(s)}{ds}$ and $\gamma_i'(s) = \frac{d\gamma_i(s)}{ds}$. 

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(b) There exists a bounded function $F_{x,h}(s) = F(x, h; s) \in L^\infty(\Omega \times S^n \times [0, L_M])$ such that

$$\lim_{t \to 0} \frac{\gamma_t(s) - \gamma(s)}{t} = F_{x,h}(s)$$

for every $s \in [0, L]$, $x \in \Omega$ and $h \in S^n$.

**Theorem 4.3.** Let $n = 2$, and suppose $u$ and $\tilde{u}$ are admissible with $u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f$, corresponding conductivities $\sigma, \tilde{\sigma} \in C^2(\Omega)$, and current density vector fields $J$ and $\tilde{J}$, respectively. Suppose $\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})$ and satisfy (19). If $u$ satisfies (12), and the level sets of $u$ are well-structured in the sense of Definition 4.2, then

$$\|\nabla \tilde{u} - \nabla u\|_{L^1(\Omega)} \leq C\|J\|_{L^\infty(\Omega)},$$

for some constant $C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M)$ independent of $\tilde{u}$ and $\tilde{\sigma}$.

**Proof.** Fix $x \in \Omega$ and $h \in \mathbb{R}^n$ with $|h| = 1$. Then

$$\mathcal{L}(x, h) := (\nabla \tilde{u}(x) - \nabla u(x)) \cdot h = \lim_{t \to 0} \frac{[\tilde{u}(x + th) - u(x + th)] - [\tilde{u}(x) - u(x)]}{t}.$$

First we estimate the above limit. Since all level sets of $u$ reach the boundary $\partial \Omega$, there exist $z, z_t \in \partial \Omega$ such that

$$u(x) = u(z) = \tilde{u}(z),$$

$$u(x + th) = u(z_t) = \tilde{u}(z_t).$$

Thus

$$[\tilde{u}(x + th) - u(x + th)] - [\tilde{u}(x) - u(x)] = [\tilde{u}(x + th) - \tilde{u}(z_t)] - [\tilde{u}(x) - \tilde{u}(z)].$$

Let $\gamma$ and $\gamma_t$ be the curves passing through $x$ and $x + th$, described in Definition 4.2 with $\gamma(0) = z$ and $\gamma_t(0) = x + th$. Suppose $\gamma(s_0) = x$ and reparameterize $\gamma_t$ so that $\gamma_t(s_0) = x + th$. Then we have

$$[\tilde{u}(x + th) - \tilde{u}(z)] - [\tilde{u}(x) - \tilde{u}(z)] = [\tilde{u}(\gamma_t(s_0)) - \tilde{u}(\gamma_t(0))] - [\tilde{u}(\gamma(s_0)) - \tilde{u}(\gamma(0))]$$

$$= \int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s)\, ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s)\, ds.$$

Hence

$$\mathcal{L}(x, h) = \lim_{t \to 0} \frac{1}{t} \left(\int_0^{s_0} \nabla \tilde{u}(\gamma_t(s)) \cdot \gamma_t'(s)\, ds - \int_0^{s_0} \nabla \tilde{u}(\gamma(s)) \cdot \gamma'(s)\, ds\right).$$
Substituting $\nabla \tilde{u}$ by $\frac{\tilde{J}}{\tilde{\sigma}}$ and using the fact that $J$ is perpendicular to $\gamma'$ and $\gamma'_t$ we get

$$L(x, h) = \lim_{t \to 0} \frac{1}{t} \left( \int_0^\infty \frac{\tilde{J}(\gamma_t(s)) - J(\gamma_t(s))}{\tilde{\sigma}(\gamma_t(s))} \cdot \gamma'_t(s) ds - \int_0^\infty \frac{\tilde{J}(\gamma(s)) - J(\gamma(s))}{\tilde{\sigma}(\gamma(s))} \cdot \gamma'(s) ds \right).$$

Now define

$$G(x) := \frac{J(x) - J(\gamma(s))}{\tilde{\sigma}(x)}, \quad x \in \Omega.$$ 

Hence

$$L(x, h) = \lim_{t \to 0} \frac{1}{t} \left( \int_0^\infty G(\gamma_t(s)) \cdot \gamma'_t(s) ds - \int_0^\infty G(\gamma(s)) \cdot \gamma'(s) ds \right).$$

The expression in the right hand side can be rewritten as

$$\frac{1}{t} \int_0^\infty [G(\gamma_t(s)) - G(\gamma(s))] \cdot \gamma'_t(s) ds + \frac{1}{t} \int_0^\infty G(\gamma(s)) \cdot [\gamma'_t(s) - \gamma'(s)] ds.$$ 

It follows from the assumption (a) in Definition 4.2 that

$$\left| \frac{\gamma'_t(s) - \gamma'(s)}{t} \right| \leq K,$$

and hence

$$\left| \frac{1}{t} \int_0^\infty G(\gamma(s)) \cdot [\gamma'_t(s) - \gamma'(s)] ds \right| \leq \frac{K}{\sigma_0} \int_0^L |\tilde{J}(\gamma(s)) - J(\gamma(s))| ds.$$ 

Now we turn our attention to the first term in (29). Let $G = (G_1, G_2)$. Since

$$\lim_{t \to 0} \frac{\gamma_t(s) - \gamma(s)}{t} = F_{x,h}(s)$$

for $i = 1, 2$ we have

$$\lim_{t \to 0} \frac{G_i(\gamma_t(s)) - G_i(\gamma(s))}{t} = \lim_{t \to 0} \frac{G_i(\gamma(s) + tF(s)) - G_i(\gamma(s))}{t} = \nabla G_i(\gamma(s)) \cdot F(s).$$
Thus the first term of (29) can be rewritten as

$$\lim_{t \to 0} \frac{1}{t} \int_0^s [G(\gamma_t(s)) - G(\gamma(s))] \cdot \gamma'(s) dl = \int_0^s (\nabla G_1(\gamma(s)) \cdot F(s), \nabla G_2(\gamma(s)) \cdot F(s)) \cdot \gamma'(s) dl \leq \| F \|_{L^\infty} \int_0^s |\nabla G_1(\gamma(s))| + |\nabla G_2(\gamma(s))| dl$$

where we have used the assumption (b) in Definition 4.2. Combining (30) and (31) we obtain

$$|\nabla \tilde{u}(x) - \nabla u(x)| \leq \sup_{h \in \mathbb{R}^n, |h| = 1} \mathcal{L}(x, h) \leq \frac{K}{\sigma_0} \int_0^L |\bar{J}(\gamma(s)) - J(\gamma(s))| dl + \| F \|_{L^\infty} \int_0^L |\nabla G_1(\gamma(s))| + |\nabla G_2(\gamma(s))| dl.$$

Thus

$$\int_\Gamma |\nabla \tilde{u}(x) - \nabla u(x)| dl \leq \frac{KL_M}{\sigma_0} \int_\Gamma |\bar{J}(x) - J(x)| dl + L_M \| F \|_{L^\infty} \int_\Gamma |\nabla G_1(x)| + |\nabla G_2(x)| dl,$$

and consequently

$$\int_{\{u = \tau\} \cap \Omega} |\nabla \tilde{u}(x) - \nabla u(x)| dl \leq \frac{KL_M}{\sigma_0} \int_{\{u = \tau\} \cap \Omega} |\bar{J}(x) - J(x)| dl + L_M \| F \|_{L^\infty} \int_{\{u = \tau\} \cap \Omega} |\nabla G_1(x)| + |\nabla G_2(x)| dl. \tag{32}$$
Using (32) and the coarea formula we have

\[
\frac{m}{\sigma_1} \| \nabla \tilde{u} - \nabla u \|_{L^1(\Omega)} \leq \int_\Omega |\nabla u| |\nabla \tilde{u} - \nabla u| \, dx
\]

\[
= \int_\mathbb{R} \int_{\{u=\tau\} \cap \Omega} |\nabla \tilde{u} - \nabla u| \, dld\tau
\]

\[
\leq KL_M \frac{M}{\sigma_0} \int_\mathbb{R} \int_{\{u=\tau\} \cap \Omega} |\tilde{\mathbf{J}} - \mathbf{J}| \, dld\tau
\]

\[
+ L_M \| F \|_{L^\infty} \int_\mathbb{R} \int_{\{u=\tau\} \cap \Omega} |\nabla G_1| + |\nabla G_2| \, dld\tau
\]

\[
\leq \frac{KL_M M}{(\sigma_0)^2} \int_\mathbb{R} \int_{\{u=\tau\} \cap \Omega} \frac{|\tilde{\mathbf{J}} - \mathbf{J}|}{|\nabla u|} \, dld\tau
\]

\[
+ \frac{L_M \| F \|_{L^\infty} M}{\sigma_0} \int_\mathbb{R} \int_{\{u=\tau\} \cap \Omega} \frac{|\nabla G_1| + |\nabla G_2|}{|\nabla u|} \, dld\tau
\]

\[
= \frac{KL_M M}{(\sigma_0)^2} \int_\Omega |\tilde{\mathbf{J}} - \mathbf{J}| \, dx
\]

\[
+ \frac{L_M \| F \|_{L^\infty} M}{\sigma_0} \int_\Omega |\nabla G_1| + |\nabla G_2| \, dx
\]

\[
\leq \frac{KL_M M}{(\sigma_0)^2} \| \mathbf{J} - \tilde{\mathbf{J}} \|_{L^1(\Omega)}
\]

\[
+ \frac{2L_MC_1 \| F \|_{L^\infty} M}{\sigma_0} \| \mathbf{J} - \tilde{\mathbf{J}} \|_{L^1(\Omega)}^{\frac{1}{2}}
\]

where we have used (21) to obtain the last inequality. Applying Theorem 2.4, and noting that

\[
\| \mathbf{J} - \tilde{\mathbf{J}} \|_{L^1(\Omega)} \leq 2M,
\]

where \(M\) is defined in (5), we arrive at (28). \(\square\)

Now we prove three dimensional version of this theorem.

**Theorem 4.4.** Let \(n = 3\), and suppose \(u\) and \(\tilde{u}\) are admissible with \(u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f\), corresponding conductivities \(\sigma, \tilde{\sigma} \in C^2(\Omega)\), and current density vector fields \(\mathbf{J}\) and \(\tilde{\mathbf{J}}\), respectively. Suppose \(\sigma, \tilde{\sigma} \in C^2(\bar{\Omega})\) and satisfy (19). In addition suppose \(u\) satisfies (12), the level sets of \(u\) can be foliated to one-dimensional curves in the sense of Definition 3.4, and the level sets of \(u\) are well-structured in the sense of Definition 4.2. Then

\[
\| \nabla \tilde{u} - \nabla u \|_{L^1(\Omega)} \leq C \| a - \tilde{a} \|_{L^\infty(\Omega)}^{\frac{1}{2}},
\]

(33)
for some constant \(C(m, M, \sigma_0, \sigma_1, \sigma_2, u, f, L_M, c_g, C_g)\) is independent of \(\bar{u}\) and \(\bar{\sigma}\).

**Proof.** With an argument similar to the one used in the proof of Theorem 4.3 we get

\[
\int_{U_{\tau,r}} |\nabla \bar{u}(x) - \nabla u(x)| dl \leq \frac{KL_M}{\sigma_0} \int_{U_{\tau,r}} |\tilde{J}(x) - J(x)| dl + L_M \|F\|_{L^\infty} \int_{U_{\tau,r}} |\nabla G_1(x)| + |\nabla G_1(x)| + |\nabla G_3(x)| dl,
\]

where \(U_{\tau,r} := \{u = \tau\} \cap \{g = r\} \cap \Omega\) and \(G = (G_1, G_2, G_3)\) is defined in (20).

It follows follows from (34) and the coarea formula that

\[
\frac{m}{\sigma_1} \|\nabla \bar{u} - \nabla u\|_{L^1(\Omega)} \leq \int_{\Omega} |\nabla u||\nabla \bar{u} - \nabla u| dx
\]

\[
= \int_{\mathbb{R}} \int_{\{u = \tau\} \cap \Omega} |\nabla \bar{u} - \nabla u| dSdr
\]

\[
= \int_{\mathbb{R}} \int_{\{u = \tau\} \cap \Omega} \frac{|\nabla g|}{|\nabla g|} |\nabla \bar{u} - \nabla u| dSdr
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{1}{|\nabla g|} |\nabla \bar{u} - \nabla u| dldrd\tau
\]

\[
\leq \frac{KL_M}{\sigma_0 c_g} \int_{\mathbb{R}} \int_{U_{\tau,r}} |\tilde{J} - J| dl d\sigma d\tau + L_M \|F\|_{L^\infty} \int_{U_{\tau,r}} |\nabla G_1| + |\nabla G_2| + |\nabla G_3| dldrd\tau
\]

\[
\leq \frac{KL_M M c_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\tilde{J} - J|}{|\nabla u||\nabla g|} dldrd\tau + L_M M \|F\|_{L^\infty} C_g \int_{\mathbb{R}} \int_{U_{\tau,r}} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u||\nabla g|} dldrd\tau
\]

\[
= \frac{KL M M c_g}{(\sigma_0)^2 c_g} \int_{\mathbb{R}} \int_{\{u = \tau\} \cap \Omega} |\tilde{J} - J| dSdt + L_M M \|F\|_{L^\infty} C_g \int_{\mathbb{R}} \int_{\{u = \tau\} \cap \Omega} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u|} dSdt
\]

\[
= \frac{KL M M c_g}{(\sigma_0)^2 c_g} \int_{\Omega} |\tilde{J} - J| dx + L_M M \|F\|_{L^\infty} C_g \int_{\Omega} \frac{|\nabla G_1| + |\nabla G_2| + |\nabla G_3|}{|\nabla u|} dx.
\]
where we have used (21) to obtain the last inequality. Applying Theorem 2.4, and noting that
\[ \| J - \tilde{J} \|_{L^1(\Omega)} \leq 2M , \]
we obtain the inequality (28).

Now, we are ready to prove our main stability results.

**Theorem 4.5.** Let \( n = 2 \), and suppose \( u \) and \( \tilde{u} \) are admissible with \( u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f \), corresponding conductivities \( \sigma, \tilde{\sigma} \in C^2(\Omega) \), and current density vector fields \( J \) and \( \tilde{J} \), respectively. Suppose \( \sigma, \tilde{\sigma} \in C^2(\bar{\Omega}) \) and satisfy (19). If \( u \) satisfies (12) and level sets of \( u \) are well-structured in the sense of Definition 4.2, then
\[ \| \sigma - \tilde{\sigma} \|_{L^1(\Omega)} \leq C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)} , \]
for some constant \( C(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, f, L_M) \) independent of \( \tilde{\sigma} \).

**Proof.** Using Theorem 4.3 we have
\[
\int_\Omega |\sigma - \tilde{\sigma}| \, dx = \int_\Omega \left| \frac{|J|(|\nabla \tilde{u}| - |\nabla u|)}{\nabla u \cdot \nabla \tilde{u}} + \frac{|J| - |\tilde{J}|}{|\nabla \tilde{u}|} \right| \, dx \\
\leq \int_\Omega \frac{|J|}{\nabla u \cdot \nabla \tilde{u}} \| \nabla u - |\nabla \tilde{u}| \| \, dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \| |J| - |\tilde{J}| \| \, dx \\
\leq \int_\Omega \frac{|J|}{\nabla u \cdot \nabla \tilde{u}} \| \nabla u - \nabla \tilde{u} \| \, dx + \int_\Omega \frac{1}{|\nabla \tilde{u}|} \| |J| - |\tilde{J}| \| \, dx \\
\leq \frac{M \sigma_0^2 C}{m^2} \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)} + \frac{\sigma_1}{m} \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)} \\
\leq \left[ \frac{M \sigma_0^2 C}{m^2} + \frac{\sigma_1 (2M)^{\frac{3}{2}}}{m} \right] \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)} .
\]

**Theorem 4.6.** Let \( n = 3 \), and suppose \( u \) and \( \tilde{u} \) are admissible with \( u|_{\partial \Omega} = \tilde{u}|_{\partial \Omega} = f \), corresponding conductivities \( \sigma, \tilde{\sigma} \in C^2(\Omega) \), and current density vector fields \( J \) and \( \tilde{J} \), respectively. Suppose \( \sigma, \tilde{\sigma} \in C^2(\bar{\Omega}) \) and satisfy (19). If \( u \) satisfies (12), the
level sets of \( u \) can be foliated to one-dimensional curves in the sense of Definition 3.3, and the level sets of \( u \) are well-structured in the sense of Definition 4.2, then

\[
\| \sigma - \tilde{\sigma} \|_{L^1(\Omega)} \leq C \| |J| - |\tilde{J}| \|_{L^\infty(\Omega)},
\]

for some constant \( C(m, M, \sigma_0, \sigma_1, \sigma_2, \sigma, \sigma_0, \sigma_1, \sigma_2, \sigma, \sigma_0, \sigma_1, \sigma_2, \sigma) \) independent of \( \tilde{\sigma} \).

**Proof.** The proof follows from Theorem 4.4 and a calculation similar to that of the proof of Theorem 4.5. \( \square \)

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