CONSTRUCTION OF STABLE RANK 2 VECTOR BUNDLES ON $\mathbb{P}^3$ VIA SYMPLECTIC BUNDLES

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Abstract. In this article we study the Gieseker-Maruyama moduli space $B(e, n)$ of stable rank 2 algebraic vector bundles with Chern classes $c_1 = e \in \{-1, 0\}$, $c_2 = n \geq 1$ on the projective space $\mathbb{P}^3$. We construct two new infinite series $\Sigma_0$ and $\Sigma_1$ of irreducible components of the spaces $B(e, n)$ for $e = 0$ and $e = -1$, respectively. General bundles of these components are obtained as cohomology sheaves of monads, the middle term of which is a rank 4 symplectic instanton bundle in case $e = 0$, respectively, twisted symplectic bundle in case $e = -1$. We show that the series $\Sigma_0$ contains components for all big enough values of $n$ (more precisely, at least for $n \geq 146$). $\Sigma_0$ yields the next example, after the series of instanton components, of an infinite series of components of $B(0, n)$ satisfying this property.

2010 MSC: 14D20, 14E08, 14J60

Keywords: rank 2 bundles, moduli of stable bundles, symplectic bundles

1. Introduction

For $e \in \{-1, 0\}$ and $n \in \mathbb{Z}_+$ let $B(e, n)$ be the Gieseker-Maruyama moduli space of stable rank 2 algebraic vector bundles with Chern classes $c_1 = e$, $c_2 = n$ on the projective space $\mathbb{P}^3$. R. Hartshorne [9] showed that $B(e, n)$ is a quasi-projective scheme, nonempty for arbitrary $n \geq 1$ in case $e = 0$ and, respectively, for even $n \geq 2$ in case $e = -1$, and the deformation theory predicts that each irreducible component of $B(e, n)$ has dimension at least $8n - 3 + 2e$.

In case $e = 0$ it is known by now (see [9], [8], [2], [6], [13], [19], [20]) that the scheme $B(0, n)$ contains an irreducible component $I_n$ of expected dimension $8n - 3$, and this component is the closure of the smooth open subset of $I_n$ constituted by the so-called mathematical instanton vector bundles. Historically, $\{I_n\}_{n \geq 1}$ was the first known infinite series of irreducible components of $B(0, n)$ having the expected dimension $\dim I_n = 8n - 3$. In [9, Ex. 4.3.2] R. Hartshorne constructed a first infinite series $\{B_0(-1, 2m)\}_{m \geq 1}$ of irreducible components $B_0(-1, 2m)$ of $B(-1, 2m)$ having the expected dimension $\dim B_0(-1, 2m) = 16m - 5$.

The other infinite series of families of vector bundles of dimension $3k^2 + 10k + 8$ from $B(0, 2k + 1)$ was constructed in 1978 by W. Barth and K. Hulek [3], and G. Ellingsrud and S. A. Strømme in [8, (4.6)-(4.7)] showed that these families are open subsets of irreducible components distinct from the instanton components $I_{2k+1}$. Later in 1985-87 V. K. Vedernikov [22] and [23] constructed two infinite series of families of bundles from $B(0, n)$, and one infinite family of bundles from $B(-1, 2m)$. A more general series of rank 2 bundles depending on triples of integers $a, b, c$, appeared in 1984 in the paper of A. Prabhakar Rao [18]. Soon after that, in 1988, L. Ein [7] independently studied these bundles and proved that they constitute open parts of irreducible components of $B(e, n)$ for both $e = 0$ and $e = -1$. 
A new progress in the description of the spaces $B(0, n)$ was achieved in 2017 by J. Almeida, M. Jardim, A. Tikhomirov and S. Tikhomirov in [1], where they constructed a new infinite series of irreducible components $Y_a$ of the spaces $B(0, 1 + a^2)$ for $a \in \{2\} \cup \mathbb{Z}_{\geq 4}$. These components have dimensions $\dim Y_a = 4\left(\frac{a+3}{3}\right) - a - 1$ which is larger than expected. General bundles from these components are obtained as cohomology bundles of rank 1 monads, the middle term of which is a rank 4 symplectic instanton with $c_2 = 1$, and the lefthand and the righthand terms are $\mathcal{O}_{\mathbb{P}^3}(-a)$ and $\mathcal{O}_{\mathbb{P}^3}(a)$, respectively.

The aim of present article is to provide two new infinite series of irreducible components $\mathcal{M}_n$ of $B(e, n)$, one for $e = 0$ and another for $e = -1$ which in some sense generalizes the above construction from [1]. Namely, in case $e = 0$ we construct an infinite series $\Sigma_0$ of irreducible components $\mathcal{M}_n$ of $B(0, n)$, such that a general bundle of $\mathcal{M}_n$ is a cohomology bundle of a monad of the type similar to the above, the middle term of which is a rank 4 symplectic instanton with arbitrary second Chern class. The first main result of the article, Theorem 3, states that the series $\Sigma_0$ contains components $\mathcal{M}_n$ for all big enough values of $n$ (more precisely, at least for $n \geq 146$). The series $\Sigma_0$ is a first example, besides the instanton series $\{I_n\}_{n \geq 1}$, of the series with this property. (For all the other series mentioned above the question whether they contain components with all big enough values of second Chern class $n$ is open.)

In case $e = -1$ we construct in a similar way an infinite series $\Sigma_1$ of irreducible components $\mathcal{M}_n$ of $B(-1, n)$, such that a general bundle of $\mathcal{M}_n$ is a cohomology bundle of a monad of the type similar to the above, in which the lefthand and the righthand terms are $\mathcal{O}_{\mathbb{P}^3}(-a - 1)$ and $\mathcal{O}_{\mathbb{P}^3}(a)$, respectively, and the middle term is a twisted symplectic rank 4 bundle with first Chern class -2. The second main result of the article, Theorem 6, states that $\Sigma_1$ contains components $\mathcal{M}_n$ asymptotically for almost all big enough values of $n$. (A precise statement about the behaviour of the set of values of $n$ for which $\mathcal{M}_n$ is contained in $\Sigma_1$ is given in Remark 7).

Now give a brief sketch of the contents of the article. In Section 2 we study some properties of pairs $([\mathcal{E}_1], [\mathcal{E}_2])$ of mathematical instanton bundles and prove the vanishing of certain cohomology groups of their twists by line bundles $\mathcal{O}_{\mathbb{P}^3}(a)$ and $\mathcal{O}_{\mathbb{P}^3}(-a)$ (see Proposition 1). The direct sum $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ is then used in Section 3 as a test rank 4 symplectic instanton bundle. This bundle and its deformations are used as middle terms of anti-self-dual monads of the form $0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0$, the cohomology bundles of which provide general bundles of the components $\mathcal{M}_n$ of of the series $\Sigma_0$ (see Theorem 3). In Section 4 we study direct sums $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ of vector bundles, $\mathcal{E}_i$ are the bundles from the R. Hartshorne series $\{\mathcal{B}_0(-1, 2n)\}_{n \geq 1}$ mentioned above. We prove certain vanishing properties for cohomology of twists of $\mathcal{E}_i$ (see Proposition 4). These properties are then used in Theorem 6 in the construction of general vector bundles of cocomponents $\mathcal{M}_n$ of the series $\Sigma_1$. In Section 5 we give a list of components $\mathcal{M}_n \in \Sigma_0$ for $n \leq 20$ and of components $\mathcal{M}_n \in \Sigma_1$ for $n \leq 40$.

Conventions and notation.

- Everywhere in this paper we work over the base field of complex numbers $\mathbb{k} = \mathbb{C}$.
- $\mathbb{P}^3$ is a projective 3-space over $\mathbb{k}$.
- For a stable rank 2 vector bundle $E$ with $c_1(E) = e$, $c_2(E) = n$ on $\mathbb{P}^3$, we denote by $[E]$ its isomorphism class in $B(e, n)$. 
Acknowledgements. A. Tikhomirov was supported by the Academic Fund Program at the National Research University Higher School of Economics (HSE) in 2018-2019 (grant no. 18-01-0037) and by the Russian Academic Excellence Project "5-100". He also acknowledges the hospitality of the Max Planck Institute for Mathematics in Bonn, where this work was partially done during the winter of 2017.

2. Some properties of mathematical instantons

Let $a$ and $m$ be two positive integers, where $a \geq 2$, and let $\varepsilon \in \{0, 1\}$. In this section we prove the following proposition about mathematical instanton vector bundles which will be used in the proof of Theorem 3.

Proposition 1. A general pair

$$(1) (\mathcal{E}_1, \mathcal{E}_2) \in I_m \times I_{m+\varepsilon},$$

of instanton vector bundles satisfies the following conditions:

$$(2) \mathcal{E}_1 \neq \mathcal{E}_2;$$

for $i = 1, \ m \leq a + 1$, respectively, $i = 2, \ m + \varepsilon \leq a + 1$,

$$(3) h^1(\mathcal{E}_i(a)) = 0,$$

$$(4) h^2(\mathcal{E}_i(-a)) = 0, \ \text{if} \ a \geq 12;$$

for $i = 1, \ m \leq a - 4$, $a \geq 5$, respectively, $i = 2, \ m + \varepsilon \leq a - 4$, $a \geq 5$,

$$(5) h^2(\mathcal{E}_i(-a)) = 0;$$

for $j \neq 1$,

$$(6) h^j(\mathcal{E}_1 \otimes \mathcal{E}_2) = 0.$$

Proof. It is clearly enough to treat the case $i = 1$, as the case $i = 2$ is treated completely similarly. Consider two instanton vector bundles such that the condition $(2)$ can be evidently achieved. Show that the condition $(3)$ can also be satisfied for general bundles $\mathcal{E}_1 \in I_m$ and $\mathcal{E}_2 \in I_{m+\varepsilon}$. For this, consider a smooth quadric surface $S \in \mathbb{P}^3$, together with an isomorphism $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, and let $Y = \bigsqcup_{i=1}^{m+1} l_i$ be a union of $m + 1$ distinct projective lines $l_i$ in $\mathbb{P}^3$ belonging to one of the two rulings on $S$. Considering $Y$ as a reduced scheme, we have $\mathcal{I}_{Y,S} \cong \mathcal{O}_{\mathbb{P}^1}(-m-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}$. Thus the exact triple $0 \to \mathcal{I}_{S,\mathbb{P}^3} \to \mathcal{I}_{Y,\mathbb{P}^3} \to \mathcal{I}_{Y,S} \to 0$ can be rewritten as

$$(7) 0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{I}_{Y,\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^1}(-m-1) \boxtimes \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Tensor multiplication of $(7)$ by $\mathcal{O}_{\mathbb{P}^3}(a+1)$, respectively, by $\mathcal{O}_{\mathbb{P}^3}(a-3)$ yields an exact triple

$$(8) 0 \to \mathcal{O}_{\mathbb{P}^3}(a-1) \to \mathcal{I}_{Y,\mathbb{P}^3}(a+1) \to \mathcal{O}_{\mathbb{P}^1}(a-m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a+1) \to 0.$$

$$(9) 0 \to \mathcal{O}_{\mathbb{P}^3}(a-5) \to \mathcal{I}_{Y,\mathbb{P}^3}(a-3) \to \mathcal{O}_{\mathbb{P}^1}(a-4-m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a-3) \to 0.$$

By the Künneth formula $h^1(\mathcal{O}_{\mathbb{P}^1}(a-m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a+1)) = 0$ for $a \geq 2$ and $m \leq a + 1$, and $(8)$ implies that

$$(10) h^1(\mathcal{I}_{Y,\mathbb{P}^3}(a+1)) = 0.$$
Now consider an extension of $\mathcal{O}_{\mathbb{P}^3}$-sheaves of the form
\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{E}_1 \to \mathcal{I}_{Y,\mathbb{P}^3}(1) \to 0.
\end{equation}

Such extensions are classified by the vector space $V = \text{Ext}^1(\mathcal{I}_{Y,\mathbb{P}^3}(1), \mathcal{O}_{\mathbb{P}^3}(-1))$, and it is known that, for a general point $\xi \in V$ the extension sheaf $\mathcal{E}_1$ in (11) is a locally free instanton sheaf from $I_m$ (see, e. g., [16]) called a 't Hooft instanton.

Now tensoring the triple (11) with $\mathcal{O}_{\mathbb{P}^3}(a)$ and passing to cohomology, in view of (10) we obtain (3) for $i = 1$.

To prove (5), assume that $m \leq a - 4$; then similar to (10) we obtain using (9):
\begin{equation}
h^1(\mathcal{I}_{Y,\mathbb{P}^3}(a - 3)) = 0, \quad m \leq a - 4.
\end{equation}

Tensoring (11) with $\mathcal{O}_{\mathbb{P}^3}(a - 4)$ we obtain the triple
\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^3}(a - 5) \to \mathcal{E}_1(a - 4) \xrightarrow{\sim} \mathcal{I}_{Y,\mathbb{P}^3}(a - 3) \to 0.
\end{equation}

From (12) and (13) it follows that
\begin{equation}
h^1(\mathcal{E}_1(a - 4)) = 0, \quad m \leq a - 4.
\end{equation}

This together with Serre duality for $\mathcal{E}_1$ yields (5) for $i = 1$.

To prove (4), consider a smooth quadric surface $S' \subset \mathbb{P}^3$, together with an isomorphism $S' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and let $Z = \bigsqcup_{i=1}^d \tilde{L}_i$ be a union of $d$ distinct projective lines $\tilde{L}_i$ in $\mathbb{P}^3$, belonging to one of the two rulings on $S'$, where $1 \leq d \leq 5$. Considering $Z$ as a reduced scheme, we have $\mathcal{I}_{Z,S'} \simeq \mathcal{O}_{\mathbb{P}^1}(-d) \boxtimes \mathcal{O}_{\mathbb{P}^1}$. Without loss of generality we may assume that $\tilde{Z} \cap Y = \emptyset$ and that $Z$ intersects the quadric surface $S$ treated above in $2d$ distinct points $x_1, \ldots, x_{2d}$ such that the points $\text{pr}_2(x_i), \ i = 1, \ldots, 2d$, are also distinct, where $\text{pr}_2 : S \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection onto the second factor.

Tensoring the exact triple (7) with $\mathcal{O}_{\mathbb{P}^3}(a - 3)$ and restricting it onto $Z$ we obtain a commutative diagram of exact triples
\begin{equation}
\begin{array}{cccccc}
& 0 & \to & \mathcal{O}_{Z}(a - 5) & \to & \mathcal{O}_{Z}(a - 3) & \to & 0 \\
& \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \to & \mathcal{O}_{\mathbb{P}^3}(a - 5) & \to & \mathcal{I}_{Y,\mathbb{P}^3}(a - 3) & \to & \mathcal{O}_{\mathbb{P}^3}(a - m - 4) \boxtimes \mathcal{O}_{\mathbb{P}^3}(a - 3) & \to & 0,
\end{array}
\end{equation}

where $f$, $g$ and $h$ are the restriction maps. The sheaf $\ker f = \mathcal{I}_{Z,\mathbb{P}^3}(a - 5)$ similar to (8) satisfies the exact triple
\begin{equation}
0 \to \mathcal{O}_{\mathbb{P}^3}(a - 5) \to \ker f \to \mathcal{O}_{\mathbb{P}^3}(a - d - 5) \boxtimes \mathcal{O}_{\mathbb{P}^3}(a - 5) \to 0.
\end{equation}

Passing to cohomology of this triple we obtain in view of the conditions $1 \leq d \leq 5$ and $a \geq 12$ that $h^1(\ker f) = 0$, i. e.
\begin{equation}
h^0(f) : H^0(\mathcal{O}_{\mathbb{P}^3}(a - 5)) \to H^0(\mathcal{O}_{Z}(a - 5))
\end{equation}
is an epimorphism. On the other hand, we have: i) $a - m - 4 \geq 0$, ii) $a - 3 \geq 2d - 1$, since $a \geq 12$ and $d \leq 5$, and iii) the points $\text{pr}_2(x_i), \ i = 1, \ldots, 2d$, are distinct.
Therefore,
\[ h^0(h) : H^0(\mathcal{O}_{\mathbb{P}^3}(a - m - 4) \boxtimes \mathcal{O}_{\mathbb{P}^3}(a - 3)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2d_1) \boxtimes \mathcal{O}_{x_j}) \]
is also an epimorphism. Whence by the diagram (15) we obtain an epimorphism
\begin{equation}
(16) \quad h^0(g) : H^0(\mathcal{I}_{s,\mathbb{P}^3}(a - 3)) \to H^0(\mathcal{O}_S(a - 3)).
\end{equation}
Now consider the \( g \circ \varepsilon : \mathcal{E}_1(a - 4) \to \mathcal{O}_S(a - 3) \), where \( \varepsilon \) is the epimorphism in the triple (13) and set \( E := \ker(g \circ \varepsilon) \otimes \mathcal{O}_{\mathbb{P}^3}(4 - a) \). Thus, since \( \mathcal{O}_S = \mathcal{O}_l \), we have an exact triple:
\begin{equation}
(17) \quad 0 \to E(a - 4) \to \mathcal{E}_1(a - 4) \xrightarrow{\text{goc}} \mathcal{O}_l(a - 3) \to 0.
\end{equation}
From the triple (13) it follows that \( h^0(\varepsilon) : H^0(\mathcal{E}_1(a - 4)) \to H^0(\mathcal{I}_{s,\mathbb{P}^3}(a - 3)) \) is an epimorphism, hence by (16) \( h^0(\varepsilon) : H^0(\mathcal{E}_1(a - 4)) \to H^0(\mathcal{O}_l(a - 3)) \) is also an epimorphism. This together with (17) and (14) yields that
\begin{equation}
(18) \quad h^1(E(a - 4)) = 0.
\end{equation}
Note that from (17) it follows also that
\begin{equation}
(19) \quad c_2(E) = c_2(\mathcal{E}_1) + d = m + d \leq a + 1,
\end{equation}
since \( d \leq 5 \) and \( m \leq a - 4 \).

Now show that
\begin{equation}
(20) \quad [E] \in \mathcal{I}_{m+d},
\end{equation}
where \( \mathcal{I}_{m+d} \) is the closure of \( I_{m+d} \) in the Gieseker-Maruyama moduli scheme \( M(0, m + d, 0) \) of semistable rank 2 coherent sheaves with Chern classes \( c_1 = c_3 = 0 \) and \( c_2 = m + d \). (Recall that \( M(0, m + d, 0) \) is a projective scheme containing \( B(0, m + 1) \) as an open subscheme - see e. g., [9], [11].) It is enough to treat the case \( d = 2 \), since the argument for any \( d \leq 5 \) is completely similar. Consider the triple (17) and denote by \( E_0 \) the kernel of the composition
\[ \mathcal{E}_1 \xrightarrow{\text{goc}} \mathcal{O}_{l_1}(1) \oplus \mathcal{O}_{l_2}(1) \xrightarrow{\text{pr}_1} \mathcal{O}_{l_1}(1). \]
We then obtain an exact triple
\begin{equation}
(21) \quad 0 \to E \to E'_0 \xrightarrow{\varepsilon'} \mathcal{O}_{l_2}(1) \to 0.
\end{equation}
Now we invoke one of the main results of the paper [12] according to which the sheaf \( E'_0 \) lies in the closure \( I_{m+1} \) of \( I_{m+1} \) in the Gieseker-Maruyama moduli scheme \( M(0, m + 1, 0) \). This implies that there exists a punctured curve \( (C, 0) \in \mathcal{I}_{m+1} \) and a flat over \( C \) coherent \( \mathcal{O}_{\mathbb{P}^3 \times C} \)-sheaf \( \mathcal{E}' \) such that the sheaf \( E'_t := \mathcal{E}'_{|\mathbb{P}^3 \times \{t\}} \) is an instanton bundle from \( I_{m+1} \) for \( t \neq 0 \) and coincides with \( E'_0 \) for \( t = 0 \). Now, without loss of generality, after possible shrinking the curve \( C \), one can extend the epimorphism \( \varepsilon' \) in (21) to an epimorphism
\[ e : \mathcal{E}' \to \mathcal{O}_{l_2}(1) \boxtimes \mathcal{O}_C \]
such that \( e \otimes \mathcal{k}(0) = \varepsilon' \). Set \( \mathcal{E} = \ker e \) and denote \( E_t = \mathcal{E}_{|\mathbb{P}^3 \times \{t\}} \), \( t \in C \). As for \( t \neq 0 \) the sheaf \( E'_t \) is an instanton bundle from \( I_{m+1} \), and it fits in an exact triple \( 0 \to E_t \to E'_t \to \mathcal{O}_{l_1}(1) \to 0 \), the above mentioned result from [12] yields that \( [E_t] \in \mathcal{I}_{m+2} \) for \( t \neq 0 \). Hence, since \( [E_t] \in \mathcal{I}_{m+2} \) is projective, it follows that \( E_0 \in \mathcal{I}_{m+2} \). Now by construction \( E_0 \simeq E \). Thus, \( [E] \in \mathcal{I}_{m+2} \), i. e. we obtain the
desired result \((20)\) for \(d = 2\). Formula \((4)\) now follows from \((18)\) for a general \(E\) by Semicontinuity and Serre duality.

To prove the vanishing \((6)\), consider the triple \((11)\) twisted by \(E_2:\)
\[
0 \to E_2(-1) \to E_1 \otimes E_2 \to E_2 \otimes \mathcal{I}_{Y,p^3}(1) \to 0,
\]
and the exact triple \(0 \to E_2 \otimes \mathcal{I}_{Y,p^3}(1) \to E_2(1) \to \oplus_{i=1}^{m+1}(E_2|_{\mathcal{I}_i}) \to 0\). Since \(E_2\) is an instanton bundle, it follows that
\[
h^2(E_2(1)) = 0, \quad h^2(E_2(-1)) = 0.
\]

On the other hand, without loss of generality, by the Grauert-Mülich Theorem [14, Ch. 2] we may assume that \(E_2|_{\mathcal{I}_i} \simeq \mathcal{O}_{p^3}\). This together with the last exact triple and the first equality \((23)\) yields \(h^2(E_2 \otimes \mathcal{I}_{Y,p^3}(1)) = 0\). Therefore, in view of \((22)\) and the second equality \((23)\) we obtain the equality \((6)\) for \(j = 1\). Last, this equality for \(j = 0, 3\) follows from \((2)\) and the stability of \(E_1\) and \(E_2\).

**Remark 2.** Note that, under the conditions of Proposition 1, the equalities \((6)\) together with Riemann-Roch yield
\[
h^1(E_1 \otimes E_2) = 8m + 4\varepsilon - 4.
\]

3. **Construction of stable rank two bundles with even determinant**

We first recall the notion of symplectic instanton. By a symplectic structure on a vector bundle \(E\) on a scheme \(X\) we mean an anti-self-dual isomorphism \(\theta : E \overset{\sim}{\to} E^\vee, \quad \theta^\vee = -\theta\), considered modulo proportionality. Clearly, a symplectic vector bundle \(E\) has even rank:
\[
\text{rk } E = 2r, \quad r \geq 1,
\]
and, if \(X = \mathbb{P}^3\), vanishing odd Chern classes:
\[
c_1(E) = c_3(E) = 0.
\]

Following [1], we call a symplectic vector bundle \(E\) on \(\mathbb{P}^3\) a symplectic instanton if
\[
h^0(E(-1)) = h^1(E(-2)) = h^2(E(-2)) = h^3(E(-3)) = 0,
\]
and
\[
c_2(E) = n, \quad n \geq 1.
\]

Consider the instanton bundles \(E_1\) and \(E_2\) introduced in Section 2 (see Proposition 1). Since \(\det E_1 \simeq \det E_2 \simeq \mathcal{O}_{p^3}\), there are symplectic structures \(\theta_i : E_i \overset{\sim}{\to} E_i^\vee, \quad i = 1, 2\), which yield a symplectic structure on the direct sum \(E = E_1 \oplus E_2:\)
\[
\theta = \theta_1 \oplus \theta_2 : E = E_1 \oplus E_2 \overset{\sim}{\to} E_1^\vee \oplus E_2^\vee = E^\vee.
\]

Clearly, \(E\) is a symplectic instanton.

Now assume that \(E_1\) and \(E_2\) are chosen in such a way that there exist sections
\[
(s_i) : \mathcal{O}_{p^3} \to E_i(a), \quad \text{such that } \text{dim}(s_i)_0 = 1, \quad i = 1, 2; \quad (s_1)_0 \cap (s_2)_0 = \emptyset.
\]

Such \(E_1 \in \mathcal{I}_m, \quad [E_2] \in \mathcal{I}_{m+\varepsilon}\) always exist; for instance, two general ’t Hooft instantons \(E_1\) and \(E_2\) satisfy the property \((27)\) already for \(a = 1\), hence also for \(a \geq 2\). The condition \((27)\) implies that the section
\[
s = (s_1, s_2) : \mathcal{O}_{p^3}(-a) \to E
\]
is a subbundle morphism, hence its transpose
\[ \iota^s := s^\vee \circ \theta : E \to \mathcal{O}_{\mathbb{P}^3}(a) \]
is a surjection. As \( \theta \) in (26) is symplectic, the composition \( \iota^s \circ s : \mathcal{O}_{\mathbb{P}^3}(-a) \to \mathcal{O}_{\mathbb{P}^3}(a) \) is also symplectic. Since \( \mathcal{O}_{\mathbb{P}^3}(\pm a) \) are line bundles, it follows that \( \iota^s \circ s = 0 \). Therefore the complex
\[ K : 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \xrightarrow{s} E \xrightarrow{\iota^s} \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \]
is a monad and its cohomology sheaf
\[ E = \frac{\ker(\iota^s)}{\im(s)} \]
is locally free. Note that, since the instanton bundles \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are stable, they have zero spaces of global sections, hence also \( h^0(E) = 0 \), and (29) and (30) yield \( h^0(E) = 0 \), i.e. \( E \) as a rank 2 vector bundle with \( c_1 = 0 \) is stable. Besides, since \( c_2(E) = c_2(\mathcal{E}_1) + c_2(\mathcal{E}_2) = 2m + \varepsilon \), it follows from (29) that \( c_2(E) = 2m + \varepsilon + a^2 \).
Thus,
\[ [E] \in \mathcal{B}(0, 2m + \varepsilon + a^2), \]
and the deformation theory yields that, for any irreducible component \( \mathcal{M} \) of \( \mathcal{B}(0, 2m + \varepsilon + a^2) \),
\[ \dim \mathcal{M} \geq 1 - \chi(\text{End } E) = 8(2m + \varepsilon + a^2) - 3. \]
Note that, since \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) are instanton bundles, for \( a \geq 2 \) one has \( h^1(\mathcal{E}_i(-a)) = h^0(\mathcal{E}_i(a)) = 0 \), \( i = 1, 2 \), \( j = 2, 3 \), hence by (26)
\[ h^1(\mathcal{E}(-a)) = 0, \quad a \geq 2; \]
\[ h^j(\mathcal{E}(a)) = 0, \quad j = 2, 3, \quad a \geq 2. \]
Similarly, in view of (3)
\[ h^1(\mathcal{E}(a)) = 0, \quad m + \varepsilon \leq a + 1, \quad a \geq 2. \]
This together with (32) and Riemann-Roch yields:
\[ h^0(\mathcal{E}(a)) = \chi(\mathcal{E}(a)) = 4 \left( \frac{a + 3}{3} \right) - 2m(2m + \varepsilon)(a + 2), \quad m + \varepsilon \leq a + 1, \quad a \geq 2. \]
Next,
\[ \text{End } \mathcal{E} \simeq \mathcal{E} \otimes \mathcal{E} \simeq S^2 \mathcal{E} \oplus \wedge^2 \mathcal{E}, \]
and it follows from (26) that
\[ S^2 \mathcal{E} \simeq S^2 \mathcal{E}_1 \oplus (\mathcal{E}_1 \otimes \mathcal{E}_2) \oplus S^2 \mathcal{E}_2, \quad \wedge^2 \mathcal{E} \simeq \wedge^2 \mathcal{E}_1 \oplus (\mathcal{E}_1 \otimes \mathcal{E}_2) \oplus \wedge^2 \mathcal{E}_2. \]
Now, since \( \text{End } \mathcal{E}_i \simeq \mathcal{E}_i \otimes \mathcal{E}_i \simeq S^2 \mathcal{E}_i \oplus \wedge^2 \mathcal{E}_i \), \( \wedge^2 \mathcal{E}_i \simeq \mathcal{O}_{\mathbb{P}^3}, \quad i = 1, 2 \), it follows from [13] that \( h^1(\text{End } \mathcal{E}_1) \simeq h^1(S^2 \mathcal{E}_1) = 8m - 3, \quad h^1(\text{End } \mathcal{E}_2) \simeq h^1(S^2 \mathcal{E}_2) = 8m + 8\varepsilon - 3 \), and \( h^1(\text{End } \mathcal{E}_i) = h^1(S^2 \mathcal{E}_i) = 0 \), \( i = 1, 2 \), \( j \geq 2 \). This together with (35)-(36), (3) and (24) implies that
\[ h^1(\text{End } \mathcal{E}) = 32m + 16\varepsilon - 14, \quad h^1(S^2 \mathcal{E}) = 24m + 12\varepsilon - 10, \]
\[ h^1(\text{End } \mathcal{E}) = h^i(S^2 \mathcal{E}) = 0, \quad i \geq 2. \]
Now assume that
\[ 5 \leq a \leq 12, \quad 1 + \varepsilon \leq m + \varepsilon \leq a - 4, \quad \text{or} \quad a \geq 12, \quad 1 + \varepsilon \leq m + \varepsilon \leq a + 1. \]
It follows from (4), (5) and (26) that
\[ h^2(E(-a)) = 0. \]

Consider the total complex $T^\circ$ of the double complex $K^\circ \otimes K^\circ$, where $K^\circ$ is the monad (29):
\[ T^\circ : 0 \to \mathcal{O}_{\mathbb{P}^3}(-2a) \overset{d_2}{\longrightarrow} 2E(-a) \overset{d_1}{\longrightarrow} E \otimes E \oplus 2\mathcal{O}_{\mathbb{P}^3} \overset{d_0}{\longrightarrow} 2E(a) \overset{d_1}{\longrightarrow} \mathcal{O}_{\mathbb{P}^3}(2a) \to 0, \]
\[ E \otimes E = \frac{\ker(d_0)}{\im(d_{-1})}. \]

Following Le Potier [15], consider the symmetric part $ST^\circ$ of the complex $T^\circ$:
\[ ST^\circ : 0 \to E(-a) \overset{\alpha}{\longrightarrow} S^2E \oplus \mathcal{O}_{\mathbb{P}^3} \overset{\cdot a}{\longrightarrow} E(a) \to 0, \quad S^2E = \frac{\ker(t_\alpha)}{\im(a)}, \]
where $\alpha$ is the induced subbundle map. The inclusion of complexes $ST^\circ \hookrightarrow T^\circ$ induces commutative diagrams
\[ \begin{array}{cccccc}
0 & \longrightarrow & E(-a) & \longrightarrow & \ker(t_\alpha) & \longrightarrow & S^2E & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & & \\
0 & \longrightarrow & \im(d_{-1}) & \longrightarrow & \ker(d_0) & \longrightarrow & E \otimes E & \longrightarrow & 0,
\end{array} \]
\[ \begin{array}{cccccc}
0 & \longrightarrow & \ker(t_\alpha) & \longrightarrow & S^2E \oplus \mathcal{O}_{\mathbb{P}^3} & \longrightarrow & E(a) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \downarrow & & \\
0 & \longrightarrow & \ker(d_0) & \longrightarrow & \mathbb{E} \otimes \mathbb{E} \oplus 2\mathcal{O}_{\mathbb{P}^3} & \longrightarrow & \im(d_0) & \longrightarrow & 0,
\end{array} \]
and an exact triple
\[ 0 \to \mathcal{O}_{\mathbb{P}^3}(-2a) \overset{d_2}{\longrightarrow} 2E(-a) \to \im(d_{-1}) \to 0 \]

Passing to cohomology in (45)-(43) and using (31), (40), (33) and the equality $h^0(S^2E) = 0$, we obtain an equality $h^0(\ker\alpha) = 1$ and an exact sequence
\[ 0 \to H^0(E(a))/\mathbb{C} \to H^1(S^2E) \overset{\mu}{\longrightarrow} H^1(S^2E) \to 0, \]
which fits in a commutative diagram
\[ \begin{array}{cccccc}
0 & \longrightarrow & H^0(E(a))/\mathbb{C} & \longrightarrow & H^1(S^2E) & \overset{\mu}{\longrightarrow} & H^1(S^2E) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^1(E \otimes \mathbb{E}) & \longrightarrow & \mathbb{E} \otimes \mathbb{E}.
\end{array} \]

From (34), (37) and (46) it follows that
\[ h^1(S^2E) = h^0(E(a)) + 24m + 12\varepsilon - 11 = 4 \left(\frac{a + 3}{3}\right) + (2m + \varepsilon)(10 - a) - 11. \]

Note that, since $E$ is a stable rank-2 bundle, $H^1(\mathcal{E}_{nd} E) = H^1(S^2E)$ is isomorphic to the Zariski tangent space $T_{|E|} \mathcal{B}(0, 2m + \varepsilon + a^2)$:
\[ \theta_E : T_{|E|} \mathcal{B}(0, 2m + \varepsilon + a^2) \simeq H^1(\mathcal{E}_{nd} E) = H^1(S^2E). \]
(Here $\theta_E$ is the Kodaira-Spencer isomorphism.) Thus, we can rewrite (46) as

$$\dim T[U]:=(0,2m+\varepsilon+a^2)=\frac{a(a+3)}{3}+(2m+\varepsilon)(10-a)-11.$$

We will now prove the following main result of this section.

**Theorem 3.** Under the condition (39), there exists an irreducible family $\mathcal{M}_n(E) \subset \mathcal{B}(0,n)$, where $n=2m+\varepsilon+a^2$, of dimension given by the right hand side of (48) and containing the above constructed point $[E]$. Hence the closure $\mathcal{M}_n(E)$ in $\mathcal{B}(0,n)$ is an irreducible component of $\mathcal{B}(0,n)$. The set $\Sigma_0$ of these components $\mathcal{M}_n$ is an infinite series distinct from the series of instanton components $\{I_0\}_{n \geq 1}$ and from the series of components described in [7] and [1]. Furthermore, at least for each $n \geq 146$ there exists an irreducible component $\mathcal{M}_n$ of $\mathcal{B}(0,n)$ belonging to the series $\Sigma_0$.

**Proof.** According to J. Bingener [3, Appendix], the equality $h^2(\mathcal{E}nd\mathbb{E})=0$ (see (38)) implies that there exists (over $k=\mathbb{C}$) a versal deformation of the bundle $\mathbb{E}$, i.e., a smooth variety $B$ of dimension $\dim B=h^1(\mathcal{E}nd\mathbb{E})$, with a marked point $0 \in B$, and a locally free sheaf $\mathcal{E}$ on $\mathbb{P}^3 \times B$ such that $\mathcal{E}|_{\mathbb{P}^3 \times \{0\}} \simeq \mathbb{E}$ and the Kodaira-Spencer map $\theta: T[\mathbb{E}]B \to H^1(\mathcal{E}nd\mathbb{E})$ is an isomorphism. For $b \in B$ denote $E_b := \mathcal{E}|_{\mathbb{P}^3 \times \{b\}}$ and consider in $B$ a closed subset

$$U = \{b \in B \mid E_b \text{ is a symplectic instanton}\}.$$  

By definition, $U = \hat{U} \cap B^*$, where $\hat{U} = \{b \in B \mid E_b \text{ is a symplectic bundle}\}$ is a closed subset of $B$ and

$$B^* = \{b \in B \mid E_b \text{ satisfies the vanishing conditions (25) and the condition}$$

$$h^0(E_b) = h^1(E_b(-a)) = h^2(E_b(a)) = h^0(S^2E_b) = 0, \quad i = 1, 2, \quad j \geq 1, \quad k = 0, 2, 3,$$

is an open subset of $B$ by the Semicontinuity. (Here $a$ is taken from (39).) Since $\mathbb{E}$ is symplectic, so that $\mathcal{E}nd\mathbb{E} \simeq S^2\mathbb{E} \oplus \wedge^2\mathbb{E}$, it follows from [17] that the Kodaira-Spencer map $\theta$ yields an isomorphism $\theta: T[\mathbb{E}]U \simeq H^1(S^2\mathbb{E})$. Thus, $U$ is a smooth variety of dimension

$$\dim U = h^1(S^2\mathbb{E}) = 24m+12\varepsilon-10.$$

(We use Riemann-Roch and the vanishing of $h^i(S^2\mathbb{E})$, $i \neq 1$.)

Let $p: \mathbb{P}^3 \times B \to B$ be the projection. By the Base-Change and the vanishing conditions defining $B^*$, respectively, $U$ the sheaf $\mathcal{A} := p_*(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_U)$ is a locally free sheaf of rank $\chi(\mathbb{E}(a)) = h^0(\mathbb{E}(a))$ given by (34). Hence $\pi: \tilde{X} = \text{Proj}(\mathcal{S}_{\mathcal{O}_{\mathbb{P}^3}}\mathcal{A}^*) \to U$ is a projective bundle with the Grothendieck sheaf $\mathcal{O}_{\tilde{X}/U}(1)$ and a morphism $s: \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\tilde{X}/U}(-1) \to \tilde{\pi}^*(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_U)$ defined as the composition of canonical evaluation morphisms $\mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\tilde{X}/U}(-1) \to \tilde{\pi}^*\pi^*\mathcal{A} \to \tilde{\pi}^*(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^3}(a) \boxtimes \mathcal{O}_U)$, where $\tilde{\pi}: \tilde{\mathbb{P}}^3 \times \tilde{X} \to \mathbb{P}^3 \times U$ are the induced projections.

Let $X = \{x \in \tilde{X} \mid s^*|_{\mathbb{P}^3 \times \{x\}} \text{ is surjective}\}$. This is an open dense subset of the smooth irreducible variety $\tilde{X}$ since it contains the point $x_0 = (s: \mathcal{O}_{\mathbb{P}^3} \to \mathbb{E}(a))$ given in (28). Hence $X$ is smooth and irreducible. In addition, since $\mathcal{E}$ is a versal family of bundles, it follows that $X$ is an open subset of the Quot-scheme Quot$_{\mathbb{P}^3 \times B/H}(\mathcal{E}, P(n))$, where $P(n) := \chi(\mathcal{O}_{\mathbb{P}^3}(a+n))$. Therefore, by [11, Prop. 2.2.7] in view of (33) there is an exact triple

$$0 \to H^0(\mathbb{E}(a))/B \to T_{x_0}X \xrightarrow{d\pi} T[\mathbb{E}]B \to 0,$$

and
which is obtained as the cohomology sequence
\begin{equation}
0 \rightarrow H^0(E(a))/C \rightarrow H^1(\text{Hom}(F,E)) \rightarrow H^1(\text{End} E) \rightarrow 0
\end{equation}
of the exact triple \(0 \rightarrow \text{Hom}(F,E) \rightarrow \text{Hom}(E,E) \rightarrow E(a) \rightarrow 0\) obtained by applying the functor \(\text{Hom}(-,E)\) to the exact triple \(0 \rightarrow O_{\mathbb{P}^3}(-a) \rightarrow E \rightarrow F \rightarrow 0\), where \(F := \text{coker}(s)\).

Next, since \(E\) is a versal family of bundles, it follows that \(E = E|_{\mathbb{P}^3 \times U}\) is a versal family of symplectic instantons. Hence, denoting \(Y = U \times_B X\), we extend the exact triple (49) to commutative diagram
\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(E(a))/C & \rightarrow & T_{x_0}X & \rightarrow & T_{[\mathbb{P}^3]}B & \rightarrow & 0 \\
& & \uparrow & & \downarrow i_Y & & \downarrow i_U & & \\
0 & \rightarrow & H^0(E(a))/C & \rightarrow & T_{x_0}Y & \rightarrow & T_{[\mathbb{P}^3]U} & \rightarrow & 0,
\end{array}
\end{equation}
where \(i_Y\) and \(i_U\) are natural inclusions. (Note that, under the Kodaira-Spencer isomorphisms \(\theta : T_{[\mathbb{P}^3]}U \rightarrow H^1(S^2E)\) and \(T_{[\mathbb{P}^3]}B \rightarrow H^1(E \otimes E) \simeq H^1(\text{End} E)\) the rightmost inclusions in diagrams (45) and (51) coincide.) Consider the modular morphism
\[\Phi : Y \rightarrow B := B(0,2m+\varepsilon+a^2), \quad (b,s) \mapsto \frac{\text{Ker}(s)}{\text{Im}(s)},\]
where, as before, \(s : O_{\mathbb{P}^3}(-a) \rightarrow E_b\) is a subbundle morphism. Its differential \(d\Phi\) composed with the Kodaira-Spencer map \(\theta_E\) from (47) is a linear map
\[\phi = \theta_E \circ d\Phi : T_{x_0}Y \rightarrow H^1(S^2E) = H^1(E \otimes E).\]
Now from the functorial properties of the Kodaira-Spencer maps \(\phi\) and \(\theta\) it follows that the triple (44) and the lower triple in the diagram (51) fit in a commutative diagram
\begin{equation}
\begin{array}{ccccccccc}
0 & \rightarrow & H^0(E(a))/C & \rightarrow & H^1(S^2E) & \rightarrow & H^1(S^2E) & \rightarrow & 0 \\
& & \uparrow & & \downarrow \phi & & \downarrow \theta & & \simeq \\
0 & \rightarrow & H^0(E(a))/C & \rightarrow & T_{x_0}Y & \rightarrow & T_{[\mathbb{P}^3]U} & \rightarrow & 0.
\end{array}
\end{equation}
This implies that \(\phi\) is an isomorphism, so that, since \(Y\) is smooth at \(x_0\) and irreducible, \(\mathcal{M}_n(E) = \Phi(Y)\) is an open subset of an irreducible component \(\mathcal{M}_n\) of \(B(0,n)\), of dimension given by (48).

It is easy to check that the dimension \(\dim \mathcal{M}_n\) given by (48), with \(m, \varepsilon\) and \(a\) subjected to the condition (39), satisfies the strict inequality \(\dim \mathcal{M}_n > 8n - 3 = \dim I_n\). This shows that the series \(\Sigma_0\) is distinct from \(\{I_n\}_{n \geq 1}\). To distinguish \(\Sigma_0\) from the series of components described in [7], it is enough to see that the spectra of general bundles of these two series are different. (We leave to the reader a direct verification of this fact.)

Note that, for each \(a \geq 12\) we have \(1 \leq m \leq a + 1\) and \(0 \leq \varepsilon \leq 1\), so that \(n = 2m + \varepsilon + a^2\) ranges through the whole interval of positive integers \((a^2 + 2, (a + 1)^2 + 1) \subset \mathbb{Z}_+\). Hence, \(n\) takes at least all positive values \(\geq 12^2 + 2 = 146\). This shows that for each \(n \geq 146\) there exists an irreducible component \(\mathcal{M}_n \in \Sigma_0\).
CONSTRUCTION OF STABLE RANK 2 VECTOR BUNDLES ON \(\mathbb{P}^3\) VIA SYMPLECTIC BUNDLES

Last, remark that, for the series of components \(\mathcal{M}_n\) described in [1], \(n\) takes values \(n = 1 + k^2, \ k \in \{2\} \cup (4, \infty)\). Hence this series is distinct from \(\Sigma_0\). Theorem is proved. \(\square\)

4. Construction of stable rank two bundles with odd determinant

In this section we will construct an infinite series of stable vector bundles from \(\mathcal{B}(-1, 2m), m \in \mathbb{Z}_+\). It is known from [9, Example 4.3.2] that, for each \(m \geq 1\) there exists an irreducible component \(\mathcal{B}_0(-1, 2m)\) of \(\mathcal{B}(-1, 2m)\), of the expected dimension

\[
\dim \mathcal{B}_0(-1, 2m) = 16m - 5,
\]

which contains bundles \(\mathcal{E}_1\) obtained via the Serre constructions as the extensions of the form

\[
0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{E}_1 \to \mathcal{I}_Y \to 0,
\]

where \(Y\) is a union of \(m + 1\) disjoint conics in \(\mathbb{P}^3\).

Below we will need the following analogue of the Proposition 1.

**Proposition 4.** Let \(a, m \in \mathbb{Z}_+, a \geq 2,\) and let \(\varepsilon \in \{0, 1\}\). A general pair

\[
([\mathcal{E}_1], [\mathcal{E}_2]) \in \mathcal{B}_0(-1, 2m) \times \mathcal{B}_0(-1, 2(m + \varepsilon)),
\]

of vector bundles satisfies the following conditions:

\[
[\mathcal{E}_1] \neq [\mathcal{E}_2];
\]

for \(i = 1, \ a \geq 2m + 4,\) respectively, for \(i = 2, \ a \geq 2(m + \varepsilon) + 4,
\]

\[
h^1(\mathcal{E}_i(a)) = 0,
\]

\[
h^2(\mathcal{E}_i(-a)) = 0;
\]

\[
h^1(\mathcal{E}_i(-a)) = 0;
\]

\[
h^j(\mathcal{E}_1(1) \otimes \mathcal{E}_2) = 0, \quad j \neq 1.
\]

**Proof.** Let \(Y = \bigsqcup_{i=1}^{m+1} C_i\) be a disjoint union of conics \(C_i = l_i \cup l'_i\) decomposable into pairs of distinct lines \(l_i, l'_i\), such that

(i) there exist two smooth quadrics \(S \simeq \mathbb{P}^1 \times \mathbb{P}^1\) and \(S' \simeq \mathbb{P}^1 \times \mathbb{P}^1\) with the property that \(l_1, \ldots, l_{m+1}\), respectively, \(l'_1, \ldots, l'_{m+1}\) are the lines of one ruling on \(S\), respectively, on \(S'\); for instance, denoting \(Y_0 = l_1 \sqcup \ldots \sqcup l_{m+1}, \ Y' = l'_1 \sqcup \ldots \sqcup l'_{m+1}\), we may assume that

\[
\mathcal{O}_S(Y_0) \simeq \mathcal{O}_{\mathbb{P}^1}(m + 1) \boxtimes \mathcal{O}_{\mathbb{P}^1}, \quad \mathcal{O}_{S'}(Y') \simeq \mathcal{O}_{\mathbb{P}^1}(m + 1) \boxtimes \mathcal{O}_{\mathbb{P}^1};
\]

(ii) the set of \(m + 1\) distinct points \(Z = (Y' \cap S) \setminus (Y_0 \cap Y')\) satisfies the condition that \(pr_1(Z)\) is a union of \(m + 1\) distinct points, where \(pr_1 : S' \to \mathbb{P}^1\) is the projection.
We then have a diagram similar to (15):

\[
\begin{array}{ccl}
0 & \to & \mathcal{O}_{Y^*}(a-4) \\
0 & \to & \mathcal{O}_{Y^*}(a-3) \\
0 & \to & \mathcal{O}_Z(a-3) \\
\downarrow f & & \downarrow g \\
0 & \to & \mathcal{O}_{p^3}(a-4) \\
\to & & \mathcal{I}_{Y_0,p^3}(a-2) \\
\to & & \mathcal{O}_{p^3}(a-m-3) \otimes \mathcal{O}_{p^3}(a-2) \\
\end{array}
\]

Under the assumptions \(a \geq 2m + 4\) and \(m \geq 2\) the cohomology of the lower triple of this diagram yields

\[
h^1(\mathcal{I}_{Y_0,p^3}(a-2)) = 0.
\]

Next, similar to (8) we have an exact triple \(0 \to \mathcal{O}_{p^3}(a-6) \to \mathcal{I}_{Y,p^3}(a-4) \to \mathcal{O}_{p^1}(a-5-m) \otimes \mathcal{O}_{p^1}(a-4) \to 0\) which implies that \(h^1(\mathcal{I}_{Y,p^3}(a-4)) = 0\) since \(a-m \geq 0\) for \(a \geq 2m+4\) and \(m \geq 1\). Since \(\mathcal{I}_{Y,p^3}(a-4) = \ker f\), it follows that

\[
h^0(f) : H^0(\mathcal{O}_{p^3}(a-4)) \to H^0(\mathcal{O}_{Y^*}(a-4)) \text{ is surjective.}
\]

On the other hand, since \(a-3-m \geq m+1 = h^0(Z)\), from the above condition (ii) on \(Z\) it follows that \(h^0(h) : H^0(\mathcal{O}_{p^1}(a-m-3) \otimes \mathcal{O}_{p^1}(a-2)) \to H^0(\mathcal{O}_Z(a-3))\) is surjective. This together with (61) and diagram (59) yields that \(h^0(g) : H^0(\mathcal{I}_{Y_0,p^3}(a-2)) \to H^0(\mathcal{O}_{Y^*}(a-3))\) is surjective. Since \(\ker g \simeq \mathcal{I}_{Y,p^3}(a-2)\), it follows by (60) that

\[
h^1(\mathcal{I}_{Y,p^3}(a-2)) = 0.
\]

Now, twisting the triple (53) by \(\mathcal{O}_{p^3}(a-3)\) and using (62) we obtain \(h^1(\mathcal{E}_1(a-3)) = 0\), hence by Serre duality \(h^2(\mathcal{E}_1(-a)) = 0\). Besides, \(h^1(\mathcal{E}_1(a-3)) = 0\) clearly implies \(h^1(\mathcal{E}_1(a)) = 0\) since \(a \geq 2m+4\). Now, by Semicontinuity, this yields (55) and (5) for a general \([\mathcal{E}_1] \in \mathcal{B}_0(-1,2m)\). The same equalities are clearly true for \(i = 2\).

Next, since \(a \geq 2\), it follows that \(h^0(\mathcal{O}_{C_i}(1-a)) = 0\) for any conic \(C_i \subset Y\), hence the cohomology of the triple \(0 \to \mathcal{I}_{Y,p^3}(1-a) \to \mathcal{O}_{p^3}(1-a) \to \bigoplus_{i=1}^{m+1} \mathcal{O}_{C_i}(1-a) \to 0\) yields \(h^1(\mathcal{I}_{Y,p^3}(1-a)) = 0\); this together with (53) and the Semicontinuity yields (57) for \(i = 1\) and similarly for \(i = 2\).

Last, the equalities (58) are proved similarly to (6).

**Remark 5.** Note that, under the conditions of Proposition 4, the equalities (58) together with Riemann-Roch yield

\[
h^1(\mathcal{E}_1(1) \otimes \mathcal{E}_2) = 16m + 8\varepsilon - 6.
\]

Now, to construct new series of components of \(\mathcal{B}(-1,4m+2\varepsilon)\), we proceed along the same lines as in Section 3. We first introduce the notion of a twisted symplectic structure on a vector bundle. By a twisted symplectic structure on a vector bundle \(E\) on \(\mathbb{P}^3\) we mean an isomorphism \(\theta : E \to E^\vee(-1)\) such that \(\theta^2(1) = -\theta\), considered modulo proportionality. (Here by definition \(\theta^2(1) := \theta \otimes \text{id}_{\mathcal{O}_{\mathbb{P}^3}(1)}\).) Clearly, a vector bundle \(E\) with twisted symplectic structure has even rank: \(\text{rk } E = 2r, r \geq 1\).

Consider the vector bundles \(\mathcal{E}_1\) and \(\mathcal{E}_2\) introduced in Proposition 4. Since \(\text{det } \mathcal{E}_1 \simeq \text{det } \mathcal{E}_2 \simeq \mathcal{O}_{p^3}(-1)\), there are twisted symplectic structures \(\theta_i : \mathcal{E}_i \overset{\cong}{\to} \mathcal{E}_i^\vee(-1)\)
\( E_i^\vee(-1), \ i = 1, 2, \) which yield a twisted symplectic structure on the direct sum
\[ E = E_1 \oplus E_2: \]
\[ \theta = \theta_1 \oplus \theta_2 : \ E = E_1 \oplus E_2 \xrightarrow{\sim} E_1^\vee(-1) \oplus E_2^\vee(-1) = E^\vee(-1). \]

Now assume that \( E_1 \) and \( E_2 \) are chosen in such a way that there exist sections
\[ s_i : \mathcal{O}_{\mathbb{P}^3} \to E_i(a+1), \ s.t. \ \dim(s_i)_0 = 1, \ i = 1, 2, \ (s_1)_0 \cap (s_2)_0 = \emptyset. \]
(Such \( E_1 \in \mathcal{B}_0(-1,2m), \ [\mathcal{E}_2] \in \mathcal{B}_0(-1,2(m+\varepsilon)) \) always exist, since already for \( a = 1 \), hence also for \( a \geq 2 \) two general bundles of the form \( (53) \) satisfy the property \( (65) \).) The assumption \( (65) \) implies that the section \( s = (s_1, s_2) : \mathcal{O}_{\mathbb{P}^3}(-a-1) \to E \) is a subbundle morphism, hence its transpose \( ^t s : = s^\vee(-1) \circ \theta : \ E \to \mathcal{O}_{\mathbb{P}^3}(a) \) is an epimorphism. As \( \theta \) in \( (64) \) is twisted symplectic, the composition \( ^t s \circ s : \mathcal{O}_{\mathbb{P}^3}(-a-1) \to \mathcal{O}_{\mathbb{P}^3}(a) \) is also twisted symplectic. Therefore, since \( \mathcal{O}_{\mathbb{P}^3}(a) \) and \( \mathcal{O}_{\mathbb{P}^3}(-a-1) \) are line bundles, it follows that \( ^t s \circ s = 0, \ i.e. \) the complex
\[ K^\ast : \ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a-1) \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0, \ E = \ker(^t s)/\text{im}(s), \]
is a monad and its cohomology sheaf \( E \) is locally free. Note that, since the bundles \( E_1 \) and \( E_2 \) are stable, they have zero spaces of global sections, hence also \( h^0(E) = 0 \), and \( (66) \) yields \( h^0(E) = 0, \ i.e. \) \( E \) as a rank 2 vector bundle with \( c_1 = -1 \) is stable. Besides, since \( c_2(E) = c_2(E_1) + c_2(E_2) = 4m + 2\varepsilon \), it follows from \( (66) \) that \( c_2(E) = 4m + 2\varepsilon + a(a+1) \). Thus,
\[ [E] \in \mathcal{B}(-1,4m+2\varepsilon+a(a+1)), \]
and the deformation theory yields that, for any irreducible component \( \mathcal{M} \) of \( \mathcal{B}(-1,4m+2\varepsilon+a(a+1)), \)
\[ \dim \mathcal{M} \geq 1 - \chi(\text{End} \ E) = 8(4m + 2\varepsilon + a(a+1)) - 5. \]
Now, as in \( (3) \), consider the symmetric part of the total complex of the double complex \( K^\ast \otimes (K^\ast)\vee \), where \( K^\ast \) is the monad \( (66) \):
\[ 0 \to \mathbb{E}(-a) \xrightarrow{\alpha} S^2E(1) \oplus \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\alpha} \mathbb{E}(a+1) \to 0, \ S^2E(1) = \frac{\ker(\alpha)/\text{im}(\alpha)}{\text{im}(\alpha)}. \]
Here \( \alpha \) is the induced subbundle map and \( S^2E(1) \) is its cohomology sheaf. The monad \( (67) \) can be rewritten as a diagram of exact triples similar to \( (??) \):
\[ \begin{array}{ccc}
0 & \rightarrow & \mathbb{E}(-a) \\
\downarrow & & \downarrow \\
0 & \rightarrow & S^2E(1) \oplus \mathcal{O}_{\mathbb{P}^3} \\
\downarrow & & \downarrow \\
\mathbb{E}(a+1) & \rightarrow & \text{coker}o \\
\downarrow & & \downarrow \\
0 & & 0 \\
\downarrow & & \\
S^2E(1) & & \\
\downarrow & & \\
0 & & .
\end{array} \]
Note that, by (57) and (64) one has
\[ h^1(\mathbb{E}(a)) = 0, \quad a \geq 2, \]
and it follows from (64) that
\[
S^2\mathbb{E}(1) \simeq S^2\mathcal{E}_1(1) \otimes (\mathcal{E}_1(1) \otimes \mathcal{E}_2) \oplus S^2\mathcal{E}_2(1),
\]
and it follows from (64) that
\[
\wedge^2 \mathbb{E}(1) \simeq \wedge^2\mathcal{E}_1(1) \otimes (\mathcal{E}_1(1) \otimes \mathcal{E}_2) \oplus \wedge^2\mathcal{E}_2(1).
\]
Now, since \( \mathcal{E}_i(1) \otimes \mathcal{E}_i \simeq S^2\mathcal{E}_i(1) \oplus \wedge^2\mathcal{E}_i(1), \) \( \wedge^2\mathcal{E}_i \simeq \mathcal{O}_{\mathbb{P}^1}, \ i = 1, 2, \) it follows from [13] that
\[ h^1(\mathcal{E}_i(1)) = 16m - 5, \quad h^1(\mathcal{E}_i(1)) = 16(m + \varepsilon) - 5, \] and
\[ h^1(\mathcal{E}_i(1)) = h^1(S^2\mathcal{E}_i(1)) = 0, \quad i = 1, 2, \]
This together with (73)-(74), (55) and (63) implies that
\[ h^1(\mathcal{E}_i(1)) = 64m + 32\varepsilon - 22, \quad h^1(S^2\mathbb{E}(1)) = 48m + 24\varepsilon - 16, \]
\[ h^i(\mathcal{E}\mathcal{E}_i) = h^i(S^2\mathbb{E}(1)) = 0, \quad i \geq 2. \]
It follows from (56) and (64) that
\[ h^2(\mathbb{E}(a)) = 0. \]
Note that (69), (76) and (71), together with the diagram (68) yield an equality
\[ h^0(\text{coker}) = 1 \text{ and an exact sequence:} \]
\[ 0 \to H^0(\mathbb{E}(a + 1))/C \to H^1(S^2\mathbb{E}(1)) \xrightarrow{\theta_E} \to H^1(S^2\mathbb{E}(1)) \to 0, \]
hence by (72) and (75) we have
\[ h^1(S^2\mathbb{E}(1)) = h^0(\mathbb{E}(a + 1)) + 48m + 24\varepsilon - 17 = \]
\[
4 \left( \frac{a + 3}{3} \right) + 2 \left( \frac{a + 3}{2} \right) - (2m + \varepsilon)(2a - 19) - 17.
\]
Note that, since \( E \) is a stable rank-2 bundle, \( H^1(\mathcal{E}\mathcal{E}_i) = H^1(S^2\mathcal{E}(1)) \) is isomorphic to the Zariski tangent space \( T_{[E]}\mathcal{B}(-1, 4m + 2\varepsilon + a(a + 1)): \)
\[ \theta_E : T_{[E]}\mathcal{B}(-1, 4m + 2\varepsilon + a(a + 1)) \xrightarrow{\sim} H^1(\mathcal{E}\mathcal{E}_i) = H^1(S^2\mathcal{E}(1)). \]
(Here \( \theta_E \) is the Kodaira-Spencer isomorphism.) Thus, we can rewrite (46) as
\[ \dim T_{[E]}\mathcal{B}(-1, 4m + 2\varepsilon + a(a + 1)) = \]
\[ 4 \left( \frac{a + 3}{3} \right) + 2 \left( \frac{a + 3}{2} \right) - (2m + \varepsilon)(2a - 19) - 17. \]
Theorem 6. For \( m \geq 1, \, \varepsilon \in \{0, 1\} \) and \( a \geq 2(m+\varepsilon)+3 \), there exists an irreducible family \( \mathcal{M}_n(E) \subset \mathcal{B}(-1, n) \), where \( n = 4m + 2\varepsilon + a(a+1) \), of dimension given by the right hand side of (79) and containing the above constructed point \( [E] \). Hence the closure \( \mathcal{M}_n \) of \( \mathcal{M}_n(E) \) in \( \mathcal{B}(-1, n) \) is an irreducible component of \( \mathcal{B}(-1, n) \). The set \( \Sigma_1 \) of these components \( \mathcal{M}_n \) is an infinite series distinct from the series \( \{\mathcal{B}_0(-1, n)\}_{n \geq 1} \) and from the series of Ein components described in [7].

The proof of this Theorem is completely parallel to the proof of Theorem 3, with clear modifications due to the change from \( c_1(E) = 0 \) to \( c_1(E) = -1 \).

It is easy to check that the dimension \( \dim \mathcal{M}_n \) given by (79), with \( m, \varepsilon \) and \( a \) as in Theorem 6, satisfies the strict inequality \( \dim \mathcal{M}_n > 8n - 5 = \dim \mathcal{B}_0(-1, n) \) (cf. (52)). This shows that \( \Sigma_1 \) is distinct from \( \{\mathcal{B}_0(-1, n)\}_{n \geq 1} \). To distinguish \( \Sigma_1 \) from the series of Ein components, it is enough to see that the spectra of general bundles of these two series are different. (A direct verification of this fact is left to the reader.)

Remark 7. Let \( \mathcal{N} \) be the set of all values of \( n \) for which \( \mathcal{M}_n \in \Sigma_1 \), i.e.\n\[
\mathcal{N} = \{ n \in 2\mathbb{Z}_+ \mid n = 4m + 2\varepsilon + a(a+1), \text{ where } m \in \mathbb{Z}_+, \, \varepsilon \in \{0, 1\}, \, a \geq 2m+\varepsilon+3 \},
\]
Then one easily sees that\n\[
\lim_{r \to \infty} \frac{\mathcal{N} \cap \{2, 4, ..., 2r\}}{r} = 1.
\]

5. Examples of moduli components of stable vector bundles with small values of \( c_2 \)

The conditions imposed on the data \((m, \varepsilon, a)\) in Theorem 3, respectively, Theorem 6 may not be satisfied for small values of these data. However, for some of small values of \((m, \varepsilon, a)\) the equalities (3), (4), (5), respectively, (55), (56), (57) are still true. Hence, our construction of irreducible components \( \mathcal{M}_n \in \Sigma_0 \), where \( n = 2m + \varepsilon + a^2 \), respectively, \( \mathcal{M}_n \in \Sigma_1 \), where \( n = 4m + 2\varepsilon + a(a+1) \), given in Sections 3 and 4 is still true for these values of \((m, \varepsilon, a)\). A precise computation of these values is performed via using the Serre construction (11), respectively, (53) for the pairs \([E_1], [E_2]\) from (1), respectively, from (54). We thus provide the following list of irreducible components \( \mathcal{M}_n \in \Sigma_0 \) for \( n \leq 20 \) and, respectively, \( \mathcal{M}_n \in \Sigma_1 \) for \( n \leq 40 \).

5.1. Components \( \mathcal{M}_n \in \Sigma_0 \) for \( n \leq 20 \). By \( \text{Spec}(E) \) we denote the spectrum of a general bundle \( E \) from \( \mathcal{M}_n \). (Below we use a standard notation \( \text{Spec}(E) = (a^p, b^q, ...) \) for the spectrum \((a_1, a_2, b_1, b_2, ...)\).)

(1) \( n = 6, \, (m, \varepsilon, a) = (1, 0, 2) \). \( \mathcal{M}_6 \) is a component of the expected (by \( \text{Spec}(E) \) the deformation theory) dimension \( \dim \mathcal{M}_6 = 45 \), and \( \text{Spec}(E) = (-1, 0^4, 1) \). This corresponds to the case 6(2) of the Table 5.3 of Hartshorne-Rao [10].

(2) \( n = 7, \, (m, \varepsilon, a) = (1, 1, 2) \). \( \mathcal{M}_7 \) is a component of the expected dimension \( \dim \mathcal{M}_7 = 53 \), and \( \text{Spec}(E) = (-1, 0^3, 1) \) (cf. [10, Table 5.3, 7(2)]).

(3) \( n = 8, \, (m, \varepsilon, a) = (2, 0, 2) \). \( \mathcal{M}_8 \) is a component of the expected dimension \( \dim \mathcal{M}_8 = 61 \), and \( \text{Spec}(E) = (-1, 0^6, 1) \) (cf. [10, Table 5.3, 8(2)]).

(4) \( n = 9, \, (m, \varepsilon, a) = (2, 1, 2) \). \( \mathcal{M}_9 \) is a component of the expected dimension \( \dim \mathcal{M}_9 = 69 \), and \( \text{Spec}(E) = (-1, 0^7, 1) \).
(5) $n = 10$, $(m, \varepsilon, a) = (3, 0, 2)$. $\mathcal{M}_{10}$ is a component of the expected dimension $\dim \mathcal{M}_{10} = 77$, and $\text{Spec}(E) = (-1, 0^3, 1)$.

(6) $n = 11$, $(m, \varepsilon, a) = (3, 1, 2)$. $\mathcal{M}_{11}$ is a component of the expected dimension $\dim \mathcal{M}_{11} = 85$, and $\text{Spec}(E) = (-1, 0^3, 1)$.

(7) $n = 12$, $(m, \varepsilon, a) = (4, 0, 2)$. $\mathcal{M}_{12}$ is a component of the expected dimension $\dim \mathcal{M}_{12} = 93$, and $\text{Spec}(E) = (-1, 0^3, 1)$.

(8) $n = 18$, $(m, \varepsilon, a) = (1, 0, 4)$. $\mathcal{M}_{18}$ is a component of the expected dimension $\dim \mathcal{M}_{18} = 141$, and $\text{Spec}(E) = (-3, -2^2, -1^3, 0^6, 1^3, 2^2, 3)$.

5.2. Components $\mathcal{M}_n \in \Sigma_4$ for $n \leq 40$.

(1) $n = 24$, $(m, \varepsilon, a) = (1, 0, 4)$. $\mathcal{M}_{24}$ is a component of the expected dimension $\dim \mathcal{M}_{24} = 187$, and $\text{Spec}(E) = (-4, -3^2, -2^4, -1^6, 0^6, 1^3, 2^2, 3)$.

(2) $n = 34$, $(m, \varepsilon, a) = (1, 0, 5)$. $\mathcal{M}_{34}$ is a component of dimension $\dim \mathcal{M}_{34} = 281$ larger than expected, and $\text{Spec}(E) = (-5, -4^2, -3^3, -2^4, -1^7, 0^6, 1^4, 2^3, 3^2, 4)$.

(3) $n = 36$, $(m, \varepsilon, a) = (1, 1, 5)$. $\mathcal{M}_{36}$ is a component of dimension $\dim \mathcal{M}_{36} = 281$ larger than expected, and $\text{Spec}(E) = (-5, -4^2, -3^3, -2^4, -1^8, 0^6, 1^4, 2^3, 3^2, 4)$.

(4) $n = 38$, $(m, \varepsilon, a) = (2, 0, 5)$. $\mathcal{M}_{38}$ is a component of the expected dimension $\dim \mathcal{M}_{38} = 299$, and $\text{Spec}(E) = (-5, -4^2, -3^3, -2^4, -1^9, 0^6, 1^4, 2^3, 3^2, 4)$.

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