GENERALIZATION OF $\text{RAD-D}_{11}$-MODULE

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Abstract. This paper gives generalization of a notion of supplemented module. Here, we utilize some algebraic properties like supplemented, amply supplemented and local modules in order to obtain the generalization. Other properties that are instrumental in this generalization are $D_3$, SSP and SIP. If a module $M$ is $\text{Rad-D}_{11}$-module and has $D_3$ property, then $M$ is said to be completely-$\text{Rad-D}_{11}$-module ($C$-$\text{Rad-D}_{11}$-module). Similarly it is for $M$ with SSP property. We provide some conditions for a supplemented module to be $C$-$\text{Rad-D}_{11}$-module.

1. Introduction and Preliminaries

Throughout this paper all rings are unital and modules are considered to be right modules. A submodule $N$ of $M$ is small in $M$ ($N \ll M$) if for every submodule $L$ of $M$ with $N + L = M$, $L = M$. In [1], we have that any module $M$ is called hollow module if every proper submodule of $M$ is a small in $M$. The direct summand plays vital role in generalization of supplemented module. A submodule $N$ of $M$ is called supplement of $K$ in $M$ if $N + K = M$ and $N$ is minimal with respect to this property. A module $M$ is called supplemented if any submodule $N$ of $M$ has a supplement in $M$.

In [2], Y. Talebi and A. Mahmoudi studied $C$-$\text{Rad-D}_{11}$-module through $\text{Rad-⨁}$-supplemented modules and $D_3$ property to get $C$-$\text{Rad-D}_{11}$-module. Here we use other algebraic properties such as supplemented, amply supplemented and local modules to get generalization of $\text{Rad-D}_{11}$-module.

Also summand sum property (SSP) and summand intersection property (SIP) are very important in generalization of supplemented module (see Definition 3.1). From [2], if $M$ is a $\text{Rad-D}_{11}$-module and has SSP property, then $M$ is a $C$-$\text{Rad-D}_{11}$-module. If $M_1$ and $M_2$ are direct summands of $M$ with $M = M_1 + M_2$ and $M_1 \cap M_2$ is also direct summand of $M$ and so $M$ has $D_3$ property. Note that, the $D_3$-module with $\text{Rad-D}_{11}$-module already gives a $C$-$\text{Rad-D}_{11}$-module. There is another module called $T_1$-module which has relationship with $C$-$\text{Rad-D}_{11}$-module. If for every submodule $K$ of $M$ such that $M/K$ is isomorphic to a co-closed submodule of $M$ and every homomorphism $\mu : M \to M/K$ lifts to a homomorphism $\beta : M \to M$ in this case $M$ is called $T_1$-module [4]. If $M$ is a local module then it is a $C$-$\text{Rad-D}_{11}$-module. On the other hand, $M$ is $C$-$\text{Rad-D}_{11}$-module if it is projective, supplemented and has $D_3$ properties. There have been different notions

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of generalization of supplemented module conducted by many researchers. These
generalizations are motivated by different properties of supplemented module. How-
ever, in this study we try to give another notion of generalization for supplemented
module. It is interesting to note that several properties of supplemented module
have been harnessed to give important properties of the generalization considered.

In Section 2, we give some properties of $C$-$\text{Rad}$-$D_{11}$-module. We proved that
if $M$ is a projective and local module with $D_2$ property then it is a $C$-$\text{Rad}$-$D_{11}$-
module. Necessary condition for a supplemented module to have a generalization
of rank three is also given. In section 3, we study three properties (injective, SSP
and SIP) of supplemented module over $\text{Rad}$-$D_{11}$-module. An easy to follow proof
of the consequence of each property is provided. Using unique closure (UC) and
extending properties of a module, we give necessary and sufficient condition for
$\text{Rad}$-$D_{11}$-module to be $C$-$\text{Rad}$-$D_{11}$-module.

2. ALGEBRAIC PROPERTIES AND $C$-$\text{RAD}$-$D_{11}$ MODULE

In this section, we utilize some algebraic properties in order to obtain rank
three generalization of supplemented module. Let $M$ be an $R$-module. From [5], a
module is said to be a $D_i$-module) if it satisfies $D_i$ ($i=1, 2, 3$) condition. A module
$M$ is called $D_1$ if for every submodule $A$ of a module $M$, there is a decomposition
$M = M_1 \oplus M_2$ such that $M_1 \in A$ and $A \cap M_2 \ll M_2$.
Equivalently, any module $M$ is called lifting if for all $N$ submodule of $M$ there is a
decomposition

$$M = H \oplus G \ni H \in N \text{ and } N \cap H \leq M.$$ 

A module $M$ is called $D_2$ if $A \leq M$ such that $M/K$ is isomorphic to a summand
of $M$ and this implies that $A$ is a summand of $M$.
For $N$ and $L$ are submodules of $M$, $L$ is a radical supplement ($\text{Rad}$-supplement) of
$N$ in $M$ if

$$N + L = M \text{ and } N \cap L \ll \text{Rad}(L).$$

On the other hand from [6], $M$ is called $D_{11}$-module if every submodule of $M$
has supplement which is a direct summand of $M$. Therefore, any module $M$ is
called a $\text{Rad}$-$D_{11}$-module if every submodule of $M$ has a $\text{Rad}$-supplement that is
a direct summand of $M$. Also, a module $M$ is called semiperfect if every factor
module of $M$ has a projective cover.

**Definition 2.1.** A module $M$ is called $C$-$\text{Rad}$-$D_{11}$-module if every direct summand
of $M$ is $\text{Rad}$-$D_{11}$-module.

In [7], N.O. Ertas, gives the direct sum of additive Abelian groups $A \bigoplus B$, $A$
and $B$ are called direct summands. The map

$$\alpha_1 : A \to A \bigoplus B$$
defined by the rule $\alpha(a) = a \bigoplus 0$ is called the injection of the first summand and
the map

$$p_1 : A \bigoplus B \to A$$
defined by $p_1(a \bigoplus b) = a$ is called the projection onto the first summand. Similar
maps $\alpha_2, p_2$ are defined for the second summand $B$. Equivalently, the direct sum
of objects $A_\alpha$ with $\alpha \in I$ is denoted by $A = \bigoplus A_\alpha$ and each $A_\alpha$ is called direct summand of $A$.

Remark 2.2. Let $A$ and $B$ be two direct summands of an Abelian group $S$ such that $A + B = S$. Then the intersection of $A$ and $B$ is not a direct summand of $S$. An example is given by $S = K_2 \bigoplus K_8$, $A = \langle (1,1) \rangle$ and $B = \langle (0,1) \rangle$. Then $S$ is the direct sum of $A$ and $\langle (1,4) \rangle$ and of $B$ and $\langle (1,0) \rangle$. The intersection is $\langle (0,2) \rangle \cong K_4$, which is not a direct summand of $S$ because $S$ is not isomorphic to $K_4 \bigoplus K_4$ or to $K_2 \bigoplus K_2 \bigoplus K_4$. The group generated by $A$ and $B$ is $S$ because it contains $(0,1)$ and $(1,0) = (1,1) - (0,1)$.

From the definitions of $D_2$ and $D_3$ properties, it is obvious that $D_2$ implies $D_3$. Other properties like supplemented, amply supplemented and local modules are inherited by summands property. Also Rad-$D_{11}$-module take the same inherited property. It is a fact that there is an equivalence between supplemented and amply supplemented module. Finally, we say that:

\[ (*) \text{ If } M \text{ is a Rad-$D_{11}$-module with } D_2 \text{ property then it is a C-Rad-$D_{11}$-module.} \]

Lemma 2.3. Let $M$ be an $R$-module. If $M$ has largest submodule then it is a supplemented module.

The next theorem whose proof shall be given at the end of this section is one of the main results of this study.

Theorem 2.4. Let $M$ be an $R$-module, if $M$ satisfies the following conditions:
1. $M$ is a projective module,
2. $M$ is local module,
3. $M$ is $D_2$-module,
then $M$ is a C-Rad-$D_{11}$-module.

Lemma 2.5. Let $M$ be an $R$-module. If $M$ is a projective and supplemented then it is a Rad-$D_{11}$-module.

Proof. Since $M$ is a projective module with $M = N + K$ then $M$ is a $\pi$-projective. But $M$ is a supplemented module. Therefore $M$ is amply supplemented. Since a projective module with amply supplemented property is a semiperfect module, we infer that $M$ is a Rad-$D_{11}$-module.
\[ \Box \]

Theorem 2.6. Let $M$ be a projective and local module. If $M$ is $D_2$-module then it is a C-Rad-$D_{11}$-module.

Proof. Let $M$ be a projective and local module. Then from Lemma 2.5 we get $M$ is a Rad-$D_{11}$-module. Now we must show that $B$ is a Rad-supplemented. In other word, $B$ has a Rad-supplement in $A$ that is a direct summand of $A$. Let $A$ be a direct summand of $M$ and $B$, submodule of $A$. Since $M$ is a Rad-$D_{11}$-module, then there exists a direct summand $C$ of $M$ $\ni M = B + C$ and $B \cap C \leq \text{Rad}(C)$.

So,
\[ A = B + (A \cap C). \]
But from definition of $D_2$-module, if $A \leq M$ such that $M = A$ is isomorphic to a summand of $M$, then $A$ is a summand of $M$. Thus, $M$ has $D_3$ property and therefore $A \cap C$ is a direct summand of $M$. Hence $M_1 \cap M_2$ is also direct summand of $M$.

Thus,

\[ B \cap (A \cap C) = B \cap C, \quad B \cap C \leq \text{Rad}(M) \text{ and } B \cap C \leq A \cap C \]

So

\[ B \cap C \leq (A \cap C \cap \text{Rad}(M)) = \text{Rad}(A \cap C). \]

Consequently, by Definition 2.1, $M$ is a $C$-$\text{Rad}$-$D_{11}$-module. \hfill \Box

**Theorem 2.7.** ([8]) Let $R$ be a ring. Then the following statements are equivalent.

1. $R$ is a left perfect.
2. Every $R$-module $M$ is a supplemented.
3. Every projective $R$-module is amply supplemented.

The following theorem gives necessary condition for a supplemented module to possess a rank three generalization.

**Theorem 2.8.** Let $M$ be projective supplemented module with $D_2$ property. If every supplement submodule of $M$ is a direct summand, then $M$ has generalization of rank three.

**Proof.** Since every supplement submodule of $M$ is a direct summand, $M$ is a $D_{11}$-module. But $M$ is a supplemented module then it is a strongly-$D_{11}$-module and thus, $R$ is a perfect ring. From Theorem 2.7 we get $M$ is amply supplemented. But $M$ is projective module. Hence, $M$ is $\text{Rad}$-$D_{11}$-module. As a consequent of (*), $M$ has a generalization of rank three ($C$-$\text{Rad}$-$D_{11}$-module). \hfill \Box

**Theorem 2.9.** Let $M$ be a $\text{Rad}$-$D_{11}$-module with $D_1$ property. If $M = M_1 \bigoplus M_2$ is a direct sum of submodules $M_1$ and $M_2$, then $M_1$ and $M_2$ are relatively projective and so $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Proof.** By [9] and Lemma 2.5. \hfill \Box

**Theorem 2.10.** Let $M$ be an $R$-module. If $M$ satisfies the following conditions:

1. $M$ is a projective module;
2. $M$ is a semiperfect module;
3. $M$ is $D_3$-module;
then $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Proof.** Let $A \leq M$. Then by assumption, there exists a projective cover $\varphi : P \to M/K$ and there is an epimorphism $\varphi : M \to M/K$. Since $M$ is projective then there exists a homomorphism

\[ \mu : M \to P \ni \varphi \circ \mu = \beta. \]

Also since $\varphi$ is small and is an epimorphism, $\mu$ splits ($P$ is projective). We have a homomorphism

\[ g : P \to M \ni \mu \circ g = 1_P \text{ and } \varphi = \varphi \circ \mu \circ g = \delta \circ g. \]

Since

\[ M = \text{Ker}(\mu) \bigoplus g(P) \text{ and } \text{Ker}(\mu) \leq A; \text{ then } M = A + g(p). \]
Let $\alpha$ be the restriction of $\varphi$ to $g(p)$. Then $\varphi = \alpha$ and so $\alpha$ is an epimorphism. Also since $\varphi$ is small, $\alpha$ is also small. That is

$$Ker(\alpha) = A \cap g(p) \ll g(p)$$

then $g(p)$ is a supplement of $A$. Thus $M$ is $Rad_{D_{11}}$-module. But $M$ has $D_3$ property. Then if $M_1$ and $M_2$ are direct summands of $M$ with

$$M = M_1 + M_2$$

and $M_1 \cap M_2$, is also direct summand of $M$. Thus $M$ is a $C-Rad_{D_{11}}$-module.

The following is an example of matrix over $Rad_{D_{11}}$-module which gives $C-Rad_{D_{11}}$-module.

Example 2.11. Let $M_{4 \times 4}$ be a matrix over field $F$ such that it satisfies $D_2$ property.

$$M = \begin{bmatrix}
a & 0 & 0 & 0 \\
y & b & 0 & 0 \\
0 & 0 & b & 0 \\
0 & 0 & x & 0
\end{bmatrix}$$

such that a, $b$, $x$, $y$ in $F$.

Corollary 2.12. Let $A$ and $B$ submodules of projective $R$-module $M$. If $B$ is a minimal with respect to the property $A + B = M$ then $M$ is $C-Rad_{D_{11}}$-module.

Proof. Let $A$ and $B$ be submodules of $M$ and let $A$ be supplement of $B$ in $M$. Then $M$ is a supplemented module. We need to show that the homomorphism $\beta$ from $M$ into $M$ is a split homomorphism. Let $1_M$ be identity mapping. Let $M$ be projective module then there exists a mapping from $M$ into $A \oplus B$ such that $\beta \circ \gamma = 1_M$. This means $\beta$ is split ($M$ is a $\pi$-projective). Let $g$ belongs to endomorphism of $M$ such that $Im(g) \subset A$ and $Im(1-g) \subset B$ with $M = A + B$. Since $g(A) \subset A$, $M = A + g(B)$ and $g(A \cap B) = A \cap g(B)$ such that $n = g(b)$ then $b - n = (1-g)(b)$ and $b \in A$. Since $A \cap B \ll A$ and $A \cap g(A) \ll g(A)$ where $g(A)$ is a supplement of $A$ and $g(A) \subset B$, $M$ is $C-Rad_{D_{11}}$-module.

Corollary 2.13. Let $M$ be a $Rad_{D_{11}}$-module. If $M$ is a $T_1$-module then it is a $C-Rad_{D_{11}}$-module.

Remark 2.14. We are now ready to prove the main result of this section, Theorem 2.4. This theorem further emphasizes the role of $D_i$-modules in our rank three generalization of supplemented module.

Proof of Theorem 2.4. Let $M$ be a projective module with $M = N + K$. Then $M$ is $\pi$-projective. Since $M$ has largest submodule then $M$ is a local module. $M$ contains all proper submodules such that $A \subset Rad(M) \ll M$ and so $A \ll M$. Hence $M$ is a hollow module ($A$ submodule of $M$ then $A + M = M$). Again by definition of hollow module we get $A \cap M = A$. We have $A \ll M$ therefore $A$ supplement in $M$ and so $M$ is a supplemented module. Hence $M$ is amply supplemented, with projective property, implies that $M$ is semiperfect and so is a $Rad_{D_{11}}$-module. Since $M$ has $D_2$ property, then it is $D_{11}$-module. Thus by Lemma 2.5 $M$ is a $C-Rad_{D_{11}}$-module.
3. INJECTIVE, SSP AND SIP PROPERTIES OVER RAD-D_{11}-MODULE

In this section, our attention is drawn to three properties of supplemented modules; injective, SSP and SIP. Here we investigate these properties over Rad-D_{11}-module for the purpose of our notion of generalization of supplemented module.

**Definition 3.1.** A module $M$ is said to have the summand sum property SSP if the sum of any pair of direct summands of $M$ is a direct summand of $M$, i.e., if $N$ and $K$ are direct summands of $M$ then $N+K$ is also a direct summand of $M$.

**Lemma 3.2.** ([2]) Let $M$ be a Rad-D_{11}-module. If $M$ has SSP property then $M$ is a C-Rad-D_{11}-module.

Let $N$ be a submodule of left $R$-module $M$. Hence there exists submodule $L$ of $M$ where $M$ is the internal direct sum of $N$ and $L$. In other words, $N+L=M$ and $N\cap L=0$. This implies that $M$ is an injective module. Also, a module $P$ is called projective if and only if for every surjective module homomorphism $f : M \to P$ there exists a module homomorphism $h : P \to M$ such that $fh=id_P$.

**Theorem 3.3.** Let $M$ be an $R$-module. If $A$ and $B$ any two direct summands of $M$ such that $A\cap B$ is injective $R$-module then $M$ is a C-Rad-D_{11}-module.

**Proof.** Let $A$ and $B$ be direct summands of $M$. By hypothesis $M$ is injective module because $M\cap M=M$. Therefore any direct summand of $M$ is injective and so $A$ and $B$ are also injective. Again by hypothesis $M = \bigoplus K$ for some $K \leq M$. Hence,

$$A = A \cap B \bigoplus A \cap K.$$  

Also,

$$B = (A \cap B) \bigoplus (B \cap K).$$

Thus, $A \cap B$, $A \cap K$ and $B \cap K$ are injective.

We have

$$A + B = (A \cap B) \bigoplus (A \cap K \bigoplus B \cap K).$$

Then, it follows that $A + B$ is injective and so it is a direct summand of $M$, $M$ has SSP property. Thus, from Lemma 3.2, $M$ is a C-Rad-D_{11}-module. \qed

**Corollary 3.4.** Let $M$ be a projective and supplemented $R$-module. If $M = A \bigoplus B$ (direct summand of $M$) and $(A \cap B)$ is an injective $R$-module, then $M$ is a C-Rad-D_{11}-module.

**Lemma 3.5.** ([9]) Let $R$ be a ring. If $R$ is a semisimple then every $R$-module $M$ has SSP property.

**Definition 3.6.** Any module $M$ has $C_3$ property if $M_1$ and $M_2$ are summands of $M$ such that $M_1 \cap M_2 = 0$ then $M_1 \bigoplus M_2$ is a summand of $M$.

Recall that an $R$-module $M$ has the summand intersection property SIP if the intersection of two summands is again a summand. Let $M$ be a projective $R$-module then $M$ has the SIP property if and only if for any direct summands $A$ and $B$ of $M$, $A + B$ is a projective $R$-module.

**Theorem 3.7.** Let $M$ be $C_3$-module. If $M$ has the SIP then $M$ is a C-Rad-D_{11}-module.
Proof. Let $M$ be $C_3$-module and has the (SIP) property. Let $A$ and $B$ be a direct summands of $M$. We must show that $A + B$ is direct summand of $M$. Since $M$ has the SIP then there exists

$$D \leq M \ni A \cap B \bigoplus D = M.$$  

By modularity law, we obtain

$$A = A \cap B \bigoplus D \cap A$$

and

$$B = A \cap B \bigoplus D \cap A.$$  

Then we have

$$A + B = A \cap B + [D \cap A \bigoplus D \cap B].$$

Next we prove that

$$(A \cap B) \cap [D \cap A \bigoplus D \cap B] = 0.$$  

For if

$$x \in (A \cap B) \cap [D \cap A \bigoplus D \cap A],$$

then

$$x = n_1 + n_2$$

where $n_1 \in (D \cap A)$ and $n_2 \in D \cap B$.

We have

$$n_2 = x - n_1 \in [A \cap B + D \cap A] \cap D \cap B \bigoplus A \cap D \cap B = 0.$$  

Hence,

$$n_2 = 0$$

and

$$x = n_1.$$  

Now

$$x = n_1 \in A \cap B \cap D \cap A = A \cap B \cap D = 0.$$  

Thus,

$$A + B = A \cap B \bigoplus D \cap A \bigoplus D \cap B = B \bigoplus D \cap A.$$  

Since $M$ has the SIP property and $D$ and $A$ are direct summands, $D \cap A$ is a direct summand. From $C_3$ property it follows that $(A + B) = B \bigoplus D \cap A$ is a direct summand of $M$. Thus $M$ has $SSP$ property and from Lemma 3.2 $M$ is a $C$-$Rad-D_{11}$-module. $\Box$

**Corollary 3.8.** Any projective module $M$ with $C_3$ property over right hereditary ring $R$ is $C$-$Rad-D_{11}$-module.

*Proof. Suppose that $R$ is right hereditary and $M$ is any projective $R$-module. Since every submodule of a projective $R$-module over right hereditary is projective. Hence $M$ has the SIP. Thus from Theorem 3.7 $M$ is $C$-$Rad-D_{11}$-module. $\Box$

**Theorem 3.9.** Let $R$ be left hereditary ring. If $M$ is an injective and $Rad-D_{11}$-module then it is a $C$-$Rad-D_{11}$-supplemented module.

*Proof. Let $R$ be a left hereditary ring. We must prove that $M$ has SIP property. Factor module of every injective $R$-module is injective. Let $M$ be an injective module which has a decomposition $M = L \bigoplus N$. Let $f$ be a homomorphism from $L$ to $N$. Then $L$ is injective.

By assumption,

$$\text{Im}(h) \approx (L/\text{Ker}(h))$$

is injective.
Hence $\text{Im}(h)$ is direct summand of $N$. From [10] $M$ has the SIP and so has SSP property. But $M$ is $\text{Rad-}D_{11}$-module, thus, by Lemma 2.4 $M$ is a $C-\text{Rad-}D_{11}$-module.

**Corollary 3.10.** Let $M$ be $\text{Rad-}D_{11}$-module. If every injective $R$-module has the SIP property then $M$ is a $C-\text{Rad-}D_{11}$-module.

Let $\Lambda(M) = \{ k \in K : kM \subseteq M \}$. Note that $\Lambda(M)$ is a subring of $K$ containing $R$. For example, if $M$ is the $R$-module $R$, then $\Lambda(M) = R$. On the other hand, if $M$ is any subring of $K$ containing $R$ and $M$ is the $R$-module $S$ then $\Lambda(M) = S$. In particular, $\Lambda(R_K) = K$. For integral domain $R$, an $R$-module $M$ is called torsion free if $\text{Ann}(a) = 0$, for each $0 \neq a \in M$ and an $R$-module $M$ is called uniform if every non-zero submodule of $M$ is essential in $M$. According to [6], every finitely generated torsion-free uniform $R$-module is a $C-\text{Rad-}D_{11}$-module. Recall that a submodule $A$ of $M$ is called a fully invariant submodule if $g(A) \subseteq A$, for every $g \in \text{Hom}(M, M)$[10].

Moreover, in [11], a module $M$ is called a duo-module if every submodule of $M$ is fully invariant.

**Theorem 3.11.** Let $M$ be $\text{Rad-}D_{11}$-module. If a commutative domain $R$ is an integrally closed then every finitely generated torsion-free uniform $R$-module is a $C-\text{Rad-}D_{11}$-module.

**Proof.** Suppose that $R$ is integrally closed. Let $T$ be any finitely generated torsion-free uniform $R$-module. Let $k$ be an element in $\Lambda(T)$. Since $kT \subseteq T$ and $k$ is integral over $R$, then $k \in R$ and so, $\Lambda(T) = R$. By [11], $T$ is a duo module with $\text{Rad-}D_{11}$-module lead to $M$ is a $C-\text{Rad-}D_{11}$-module. □

**Lemma 3.12.** ([2, Lemma 3.4]). Let $M$ be a duo module. Then $M$ has the SIP property.

In [2], Y. Talebi and A. Mahmoudi calls a module $M$ a UC-module if every submodule of $M$ has a unique closure in $M$. A module $M$ is called extending if every closed submodule of $M$ is a direct summand of $M$. Therefore any UC-extending module has $D_3$ property.

**Theorem 3.13.** Let $M$ be UC-extending module. Then $M$ is a $\text{Rad-}D_{11}$-module if and only if $M$ is a $C-\text{Rad-}D_{11}$-module.

**Proof.** Sufficiency is clear. Conversely, assume that $M$ is $M$-supplemented module. From [1], $M$ has $D_3$ property. Hence $M$ is a completely-$\text{Rad-}D_{11}$-module. □

**Lemma 3.14.** ([12]) Let $M$ be an $R$-module with $\text{Rad}(M) = 0$. If $M$ is a closed weak supplemented module then $M$ is extending.

**Theorem 3.15.** For any ring $R$ the following are equivalent:

1. Every left $R$-module is a lifting.
2. Every left $R$-module is extending.

**Theorem 3.16.** Let $M$ be an $\text{Rad-}D_{11}$-module with the following conditions:

1. $M$ is UC-module;
2. $\text{Rad}(M)=0$;

3. Every nonsingular right $D_{11}$-module is projective;

then $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Proof.** Let $M$ be a nonsingular module and $N$ a closed submodule of $M$. Then $(M/K)$ is nonsingular. Since $M$ is a projective then $N$ is a direct summand of $M$. From [12], $M$ is closed weak supplemented with $\text{Rad}(M)=0$ implies that $M$ is extending module. Now from condition (1) with $\text{Rad}$-$D_{11}$-module we obtain $M$ is a $C$-$\text{Rad}$-$D_{11}$-module. (see Theorem 3.13). □

**Theorem 3.17.** Let $M$ be an $R$-module. If $M$ satisfies the following conditions:

1. $M$ is $UC$-module;
2. $M$ is $\text{Rad}$-$D_{11}$-module;
3. $M$ is lifting module;

then $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Corollary 3.18.** Let $M$ be an $R$-module. If $M$ is a local then $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Corollary 3.19.** Let $M$ be an $R$-module such that every direct summand of $M$ is a finite direct sum of hollow modules. If $M$ has $D_3$ property then $M$ is a $C$-$\text{Rad}$-$D_{11}$-module.

**Corollary 3.20.** Any duo module $M$ has $C3$ property is $C$-$\text{Rad}$-$D_{11}$-module.

**Remark 3.21.** We are now ready to prove the main result of this section, Theorem 3.7. The classification of $\text{Rad}$-$D_{11}$-module is very important in the process of generalization of supplemented module:

Let $S=\{s_1, \ldots, s_n\} \subset M$ be a set of generators for $M$ over $\text{End}_R(M)$. Since a direct sum of semisimple modules is also a sum of simple modules then it is semisimple. So every direct sum of semisimple modules is again semisimple. Hence $M^S$ is a semisimple module. Moreover, we have $R$-homomorphism $R \to M^S$ and $r \mapsto rs_1, \ldots, rs_n$ is injective: Suppose that $rs_i=0$ for all generators of $R$ as an $\text{End}_R M$-module.

Therefore we can write every

$$x \in M \text{ as } \beta_1(s_1) + \ldots + \beta_n(s_n) \text{ for } \beta_i \in \text{End}_RM.$$ 

So we have,

$$rx = r(\beta_1(s_1) + \ldots + \beta_n(s_n)) = \beta_1(rs_1) + \ldots + \beta_n(rs_n) = 0.$$

Also we see that $rx = 0 \forall x \in M$. By the faithfulness of $M$, we conclude that $r = 0$. This shows that $R$ is (as an $R$-module) isomorphic to a submodule of $M^S$. Hence $R$ is a semisimple and by Lemma 3.5 we obtain $M$ has $SSP$ property. But $M$ is $\text{Rad}$-$D_{11}$-module. Thus $M$ is a $C$-$\text{Rad}$-$D_{11}$-module. Theorem 3.7 is thus proved.
4. Conclusion

The supplemented module is very important in module theory specially when we study the generalization of supplemented module. In addition we obtained the third generalization of this module by use many concepts as injective module; semiperfect module and $D_1$-modules. Also we found if $M$ is a $\text{Rad}-D_{11}$-module with $D_2$ property gives $C-\text{Rad}-D_{11}$-module. Moreover; if $M$ is a $C_3$-module having the SIP and $C_3$ properties lead to $M$ is a $C-\text{Rad}-D_{11}$-module.

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