TRAJECTORY ATTRACTORS FOR 3D DAMPED EULER EQUATIONS AND THEIR APPROXIMATION

ALEXEI ILYIN\textsuperscript{1}, ANNA KOSTIANKO\textsuperscript{3,4}, AND SERGEY ZELIK\textsuperscript{1,2,3}

ABSTRACT. We study the global attractors for the damped 3D Euler–Bardina equations with the regularization parameter $\alpha > 0$ and Ekman damping coefficient $\gamma > 0$ endowed with periodic boundary conditions as well as their damped Euler limit $\alpha \to 0$. We prove that despite the possible non-uniqueness of solutions of the limit Euler system and even the non-existence of such solutions in the distributional sense, the limit dynamics of the corresponding dissipative solutions introduced by P. Lions can be described in terms of attractors of the properly constructed trajectory dynamical system. Moreover, the convergence of the attractors $A(\alpha)$ of the regularized system to the limit trajectory attractor $A(0)$ as $\alpha \to 0$ is also established in terms of the upper semicontinuity in the properly defined functional space.

Contents

1. Introduction
2. Preliminaries
3. Preliminaries II. The case $\alpha = 0$: dissipative solutions
4. The trajectory dynamical system
5. The trajectory attractor

References

2000 Mathematics Subject Classification. 35B40, 35B45, 35L70.

Key words and phrases. Regularized Euler equations, Bardina model, dissipative solutions, trajectory attractors.

This work was supported by Moscow Center for Fundamental and Applied Mathematics, Agreement with the Ministry of Science and Higher Education of the Russian Federation, No. 075-15-2019-1623 and by the Russian Science Foundation grant No.19-71-30004 (sections 2-4). The second author was partially supported by the Leverhulme grant No. RPG-2021-072 (United Kingdom).
1. Introduction

Being the central mathematical model in hydrodynamics, the Navier–Stokes and Euler equations permanently remain in the focus of both the analysis of PDEs and the theory of infinite dimensional dynamical systems and their attractors, see [1, 5, 11, 13, 14, 24, 25, 26, 33, 34, 35] and the references therein for more details. Most studied is the 2D case where reasonable results on the global well-posedness and regularity of solutions as well as the results on the existence of global attractors and their dimension are available. However, the global well-posedness in the 3D case remains a mystery and even listed by the Clay institute of mathematics as one of the Millennium problems. This mystery inspires a comprehensive study of various modifications/regularizations of the initial Navier-Stokes/Euler equations (such as Leray-α model, hyperviscous Navier-Stokes equations, regularizations via $p$-Laplacian, etc.), many of which have a strong physical background and are of independent interest, see e.g. [12, 15, 25, 28, 30] and the references therein.

In the present paper we shall be dealing with the following regularized 3D damped Euler system:

\[
\begin{aligned}
\frac{\partial}{\partial t}u + (\overline{u}, \nabla_x)\overline{u} + \gamma u + \nabla_x p &= g, \\
\text{div } \overline{u} &= 0, \\
\end{aligned}
\]

with forcing $g$ and Ekman damping term $\gamma u$, $\gamma > 0$ endowed with periodic boundary conditions ($x \in \mathbb{T} := [-\pi, \pi]^3$). The damping term $\gamma u$ makes the system dissipative and is important in various geophysical models [32]. Here and below $\overline{u}$ is a smoothed (filtered) vector field related with the initial velocity field $u$ by means of the solution of the Stokes problem

\[
\begin{aligned}
\frac{\partial}{\partial t}u - \alpha \Delta_x u + (\overline{u}, \nabla_x)u + \gamma (\overline{u} - \alpha \Delta_x u) + \nabla_x p &= g, \\
\end{aligned}
\]

\[
\overline{u} = (1 - \alpha \Delta_x)^{-1} u.
\]

System (1.1), (1.2) (at least in the conservative case $\gamma = 0$) is often referred to as the simplified Bardina subgrid scale model of turbulence, see [2, 3, 23] for the derivation of the model and further discussion, so in this paper we shall be calling (1.1) the damped Euler–Bardina equations. We also mention that rewriting (1.1) in terms of the variable $\overline{u}$ gives

\[
\begin{aligned}
\frac{\partial}{\partial t}u - \alpha \frac{\partial}{\partial t} \Delta_x \overline{u} + (\overline{u}, \nabla_x)u + \gamma (\overline{u} - \alpha \Delta_x \overline{u}) + \nabla_x p &= g
\end{aligned}
\]

which is a damped version of the so-called Navier–Stokes–Voight equations arising in the theory of viscoelastic fluids, see [22, 31] for the details. In the sequel, we will mainly use the equivalent equations (1.3) instead of (1.1).

Our main interest in the present paper is to study the limit $\alpha \to 0$ in terms of the attractors $\mathcal{A}(\alpha)$ of the corresponding regularized equations (1.3). We recall
that, in contrast to the classical Navier–Stokes approximations, the global well-posedness of Bardina–Euler equations is well-known, see [22, 23, 17] for more details. Moreover, it is also known that, if \( g \in L^2(T^3) \) and \( \alpha, \gamma > 0 \), problem (1.3) possesses a global attractor \( \mathcal{A}(\alpha) \) of finite fractal dimension in the phase space \( \mathcal{H}_\alpha := H^1(T^3) \cap \{ \text{div } \bar{u} = 0 \} \) endowed with the norm
\[
\|u\|_{\mathcal{H}_\alpha}^2 = \|u\|_{L^2}^2 + \alpha \|\nabla_x u\|_{L^2}^2.
\]
The fractal dimension of this attractor possesses the following explicit estimate
\[
\dim F \mathcal{A} \leq \frac{1}{12 \pi \alpha^{5/2} \gamma^4} \|g\|_{L^2}^2.
\]
and this estimate is sharp with respect to \( \alpha \) and \( \gamma \) in the sense that the lower bounds of the same order are attained on a family of specially constructed Kolmogorov flows, see [17, 18] for more details (see also [21, 18, 19, 20] for the analogous results for 2D case).

We however note that the above results do not give much information about the behavior of \( \mathcal{A}(\alpha) \) as \( \alpha \to 0 \) as well as about the limit attractor \( \mathcal{A}(0) \) because of the dependence of the above mentioned Kolmogorov flows on \( \alpha \), see [17] for the discussion. In particular, even in the relatively simple 2D case, the question about the finite-dimensionality of the limit attractor remains completely open.

Note also that, due to the simplified structure of vorticity equations in 2D, we have the uniform as \( \alpha \to 0 \) \( H^1 \)-estimate for the solutions of (1.3) and even can establish the uniqueness of slightly more regular solutions for the limit Euler equations using the Yudovich technique (see [37]). In particular, this technique gives us the uniqueness on the attractor \( \mathcal{A}(0) \) as well as the attraction to it in a strong topology of \( H^1 \), see [16, 6, 7, 8, 9] for more details.

In contrast to 2D case, the situation in much more complicated in 3D. Indeed, the presence of the so-called vorticity stretching term in the vorticity equations prevents us from obtaining good estimates of the \( H^1 \)-norm for the limit Euler equations, so we have only the control of the \( L^2 \)-norm of the solution which comes from the basic energy estimate. But, unfortunately, this control is not only insufficient for the uniqueness, but even the existence of weak solutions in the sense of distributions becomes non-trivial. For this reason, we have to use the notion of so-called dissipative solutions introduced by P. Lions in order to verify the solvability of the limit Euler equations, see [27] and the references therein. Note also that the distributional solutions for the classical 3D Euler equations can be constructed using the geometric integration technique, see [36] and references therein, but, in contrast to dissipative solutions, these type of solutions usually do not satisfy the energy inequality in a reasonable form and cannot be obtained as the limit of the corresponding solutions of the regularized system, so they look not very interesting from the point of view of attractors.

However, there is one more problem here. Namely, the class of dissipative solutions of the limit Euler equations is crucially not invariant with respect to time
shifts which prevents us to use the standard technique of trajectory attractors at least in the straightforward way. In order to overcome this problem, we introduce a wider (than the dissipative ones) class of solutions of the limit Euler equations

\[ K^+_0 \subset \Theta_+ := L^\infty_{w^*,\text{loc}}(\mathbb{R}^+, \mathcal{H}_0) \cap W^{1,\infty}_{w^*,\text{loc}}(\mathbb{R}^+, H^{-3}), \]

see sections \( \S \) and \( \S \) for more details. Roughly speaking, the trajectory phase space \( K^+_0 \) consists of all limits of solutions \( \bar{u}_{\alpha_n} \) of the regularized Bardina–Euler equations (1.3) as \( \alpha_n \to 0 \) in the topology of \( \Theta_+ \). In this case, the semigroup of time-shifts \( T(h) : \Theta_+ \to \Theta_+, h \geq 0, \) will act on \( K^+_0 \) and we may define the trajectory dynamical system \( (K^+_0, T(h)) \) associated with the limit Euler equation and may speak about the global attractor \( A_{tr}(0) \) of this DS (=trajectory attractor of equation (1.3) with \( \alpha = 0 \)).

Note that the space \( K^+_0 \) is not empty and contains all dissipative solutions of the limit Euler equation, but a priori may be larger. This may be connected with the well known fact that the weak limits of solutions of 3D Euler equations may not satisfy Euler equations, see \[10\] for more details.

Finally, in order to define the trajectory attractor \( A_{tr}(0) \), we need to specify the class of ”bounded” sets which will be attracted by this attractor. It is well-known that the topology on the space of initial data for problem (1.3) (with \( \alpha = 0 \), e.g. \( \bar{u}(0) \in H \)) is usually not appropriate for defining bounded sets of trajectories and the topology on the set of trajectories (\( \Theta_+ \) or its uniformly local analogues) should be used instead, see \[3, 29\] and references therein. However, in our case such a choice is also insufficient since it may lead to the situation when the \( \omega \)-limit set of \( K^+_0 \) will be outside \( K^+_0 \), so a bit more accuracy is necessary. We overcome this problem, following \[38\] (see also \[7\]) by introducing the so-called \( M \)-functional

\[ M_{\bar{u}}(t) := \inf_{\bar{u}_{\alpha_n} \to \bar{u}} \liminf_{n \to \infty} \| \bar{u}_{\alpha_n}(t) \|^2_{\mathcal{H}_{\alpha_n}}, \]

where the external inf is taken with respect to all sequences of solutions of (1.3) which converge to a given \( \bar{u} \in K^+_0 \), see section \( \S \) for more details. Note that for this quantity, we have only that \( M_{\bar{u}}(0) \geq \| \bar{u}(0) \|^2_{\mathcal{H}_0} \), but no upper bounds are a priori available. In particular, the value \( M_{\bar{u}}(0) \) depends not only on \( \bar{u}(0) \), but on the whole trajectory \( \bar{u}(t), t \geq 0 \). Then, we call a set \( B \subset K^+_0 \) bounded if

\[ \sup_{\bar{u} \in B} M_{\bar{u}}(0) < \infty. \]

We are now ready to state the main result of the paper.

**Theorem 1.1.** Let \( g \in L^2(\mathbb{T}) \) and let \( \gamma > 0 \) be fixed. Then, the limit damped Euler equation (which corresponds to (1.3) with \( \alpha = 0 \)) possesses a trajectory attractor \( A_{tr}(0) \) in the above described sense which attracts bounded sets of \( K^+_0 \) in the topology of \( \Theta_+ \). Moreover, the family of trajectory attractors \( A_{tr}(\alpha), \alpha > 0 \) of Bardina–Euler equations (1.3) converges as \( \alpha \to 0 \) to the limit attractor \( A_{tr}(0) \) in the sense of upper semicontinuity in the space \( \Theta_+ \).
Note that in the case \( \alpha > 0 \), we have the global well-posedness for problem (1.3) and even the existence of global attractors \( \mathcal{A}(\alpha) \) in the strong topology of \( \mathcal{H}_\alpha \), so the construction of the corresponding trajectory attractors is straightforward.

The paper is organized as follows. In section §2, we briefly discuss the known facts on the well-posedness of Bardina–Euler equations on the 3D torus, state the basic dissipative estimates as well as the results on the existence of global \( \mathcal{A}(\alpha) \) and trajectory \( \mathcal{A}_{tr}(\alpha) \) attractors for the case \( \alpha > 0 \).

In section §3, we recall the basic facts about dissipative solutions adapted to the case of the damped Euler equations and discuss the existence of dissipative solutions for this equation as well as their standard properties.

In section §4, we construct the trajectory dynamical system \( (K^+_0, T(h)) \) associated with the limit damped Euler equations, introduce the \( M \)-functional and study its basic properties.

Finally, the proof of the main theorem is given in section §5.

2. Preliminaries

We study the following 3D damped Bardina–Euler system:

\[
\begin{aligned}
\partial_t \bar{u} - \alpha \partial_t \Delta_x \bar{u} + (\bar{u}, \nabla_x) \bar{u} + \nabla_x p + \gamma (\bar{u} - \alpha \Delta_x \bar{u}) &= g, \\
\text{div } \bar{u} &= 0, \\
\bar{u} \bigg|_{t=0} &= \bar{u}_0
\end{aligned}
\]

(2.1)

on the 3D torus \( \mathbb{T} := (-\pi, \pi)^3 \). Here \( \bar{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) \) is an unknown velocity field, \( p \) is an unknown pressure, \( \Delta_x \) is the Laplacian with respect to the variable \( x \in \mathbb{T} \) and \( g \) are the given external forces. We assume that the parameters \( \gamma > 0 \) (Ekman damping) and \( \alpha > 0 \) (Bardina regularization) are given and will study the damped Euler limit \( \alpha \to 0 \). It is also natural to assume that

\[ g \in L^2(\mathbb{T}) \]

(2.2)

and take the initial data

\[ \bar{u}_0 \in \mathcal{H}_\alpha := \{ v \in [H^1(\mathbb{T})]^3 : \text{div } v = 0 \} \]

(2.3)

endowed with the norm

\[ ||v||^2_{\mathcal{H}_\alpha} := ||v||^2_{L^2} + \alpha ||\nabla_x v||^2_{L^2}. \]

(2.4)

The following result concerning the well-posedness of problem (2.1) is straightforward, see e.g. [22, 23, 17].

**Theorem 2.1.** Let \( g \in L^2(\mathbb{T}) \) and let \( \alpha, \gamma > 0 \). Then, for every \( \bar{u}_0 \in \mathcal{H}_\alpha \), problem (2.1) possesses a unique solution

\[ \bar{u} \in C^1(0, T; \mathcal{H}_\alpha), \quad T > 0 \]

and this solution satisfies the following dissipative estimate

\[ ||\bar{u}(t)||^2_{\mathcal{H}_\alpha} \leq ||\bar{u}_0||^2_{\mathcal{H}_\alpha} e^{-\gamma t} + \frac{1}{\gamma^2} ||g||^2_{L^2}. \]

(2.5)
Thus, problem \(2.1\) defines a dissipative solution semigroup \(S_\alpha(t)\) in the phase space \(\mathcal{H}_\alpha\) via:

\[
S_\alpha(t) : \mathcal{H}_\alpha \to \mathcal{H}_\alpha, \quad S_\alpha(t)\bar{u}_0 := \bar{u}(t),
\]

where \(\bar{u}(t)\) solves \(2.1\) with the initial data \(\bar{u}(0) = \bar{u}_0\).

Moreover, as proved e. g., in [17], this semigroup possesses a global attractor \(A_\alpha\) in the space \(\mathcal{H}_\alpha\) for every \(\alpha > 0\). The latter means that

1. \(A_\alpha\) is compact in \(\mathcal{H}_\alpha\);
2. It is strictly invariant: \(S_\alpha(t)A_\alpha = A_\alpha\) for all \(t > 0\);
3. It attracts the images of all bounded sets of \(\mathcal{H}_\alpha\), i.e., for any bounded set \(B \subset \mathcal{H}_\alpha\) and any neighbourhood \(\mathcal{O}(A_\alpha)\), there is time \(T = T(B, \mathcal{O})\) such that

\[
S_\alpha(t)B \subset \mathcal{O}(A_\alpha), \quad t \geq T.
\]

We also recall that the attractor \(A_\alpha\) is generated by all bounded complete trajectories of problem \(2.1\):

\[
A_\alpha = K_\alpha \mid_{t=0},
\]

where \(K_\alpha := \{u \in C_b(\mathbb{R}, \mathcal{H}_\alpha), \bar{u}(t)\text{ solves }2.1\text{ for all }t \in \mathbb{R}\}\).

Our aim is to study the limit \(\alpha \to 0\). However, in contrast to the case of \(2.1\), we do not know whether or not the solution of the limit damped Euler equations:

\[
\begin{aligned}
\partial_t \bar{u} + (\bar{u}, \nabla_x)\bar{u} + \nabla_x p + \gamma \bar{u} &= g, \\
\text{div} \bar{u} &= 0, \quad \bar{u}\mid_{t=0} = \bar{u}_0
\end{aligned}
\] (2.8)

is unique (and even do not have enough regularity to understand this solution in the sense of distributions, see section §3). For this reason, we briefly recall below

the so-called trajectory approach developed in [5] which allows us to overcome

the potential non-uniqueness.

**Definition 2.2.** Let \(K_\alpha^+ \subset C^1_{loc}(\mathbb{R}_+, \mathcal{H}_\alpha)\) be the set of all global solutions \(\bar{u} \in C^1_{loc}(\mathbb{R}_+, \mathcal{H}_\alpha)\) of problem \(2.1\) which correspond to all possible initial data \(\bar{u}_0 \in \mathcal{H}_\alpha\). This set is called the trajectory phase space associated with equation \(2.1\).

Obviously, the translation semigroup

\[
(T(h)\bar{u})(t) := \bar{u}(t+h), \quad t, h \geq 0
\]

acts on \(K_\alpha^+\):

\[
(T(h): K_\alpha^+ \to K_\alpha^+, \quad h \geq 0,
\]

so the dynamical system \((T(h), K_\alpha^+)\) is well-defined. We endow the set \(K_\alpha^+\) with the topology of \(C^1_{loc}(\mathbb{R}_+, \mathcal{H}_\alpha)\) and will refer to the constructed dynamical system \((T(h), K_\alpha^+)\) as the trajectory dynamical system associated with equation \(2.1\).

**Remark 2.3.** According to the above constructions, equation \(2.1\) generates two dynamical systems (DS): one of them is the classical DS \((S_\alpha(t), \mathcal{H}_\alpha)\) which is defined on a usual phase space \(\mathcal{H}_\alpha\) of the problem and the second one is the trajectory DS \((T(t), K_\alpha^+)\) which is generated by the translations of the corresponding
trajectories. In the case where the uniqueness is proved, these two approaches are equivalent. Indeed, as not difficult to see, the solution map

\[ S : H_\alpha \to K^+_\alpha, \quad S \bar{u}_0 := \bar{u}(\cdot) \]

is a homeomorphism and also maps bounded sets into bounded sets (if we define bounded sets in \( K^+_\alpha \) using the embedding \( K^+_\alpha \subset C_{loc}(\mathbb{R}_+, H_\alpha) \)). Moreover,

\[ T(t) = S \circ S_\alpha(t) \circ S^{-1}, \quad \text{on} \quad K^+_\alpha. \quad (2.9) \]

Thus, the trajectory dynamical system \((T(t), K^+_\alpha)\) also possesses a global attractor \( A_{tr}(\alpha) \) which is called the trajectory attractor of equation (2.1) and has the following structure:

\[ A_{tr}(\alpha) = S A_\alpha = K_\alpha |_{t \geq 0}, \quad (2.10) \]

see [5, 29] for more details. The advantage of the trajectory approach is that the uniqueness of a solution is not necessary for constructing the trajectory dynamical system and this will allow us to construct and study the limit attractor \( A_{tr}(0) \) of the damped Euler equations.

3. Preliminaries II. The case \( \alpha = 0 \): dissipative solutions

In this section, we recall the construction of the so-called dissipative solutions for the limit damped Euler equations (2.8) introduced in [27]. We remind that, according to estimate (2.5), we are able to control the \( L^\infty(\mathbb{R}_+, L^2(T)) \)-norm of the approximate solutions \( \bar{u} \) of (2.1) as \( \alpha \to 0 \), so it is natural to consider the solutions of the limit problem belonging to this space. We are even able to define distributional solutions in this space using the identity

\[ ((\bar{u}, \nabla_x)\bar{u}, \varphi) = -((\bar{u} \otimes \bar{u} : \nabla_x \varphi), \]

see [27] for more details. However, the obtained regularity \( u \in L^\infty(\mathbb{R}_+, H_0) \) is not sufficient for passing to the limit in the nonlinear term \( \bar{u} \otimes \bar{u} \), so the existence of distributional solutions for the 3D Euler equation remains an open problem. The above mentioned construction of dissipative solutions allows us to overcome this problem. Namely, let \( \varphi \) be a sufficiently smooth divergent free test function and let

\[ D_\alpha(\varphi) = D_\alpha(\varphi)(t) := \partial_t \varphi - \alpha \partial_t \Delta_x \varphi + \Pi(\varphi, \nabla_x) \varphi + \gamma(\varphi - \alpha \Delta_x \varphi) - \Pi g, \quad (3.1) \]

where \( \Pi \) is the Leray projector to divergence free vector fields. Then, if \( \bar{u} \) solves the approximation problem (2.1), we have the identity

\[ \partial_t(\bar{u} - \varphi) - \alpha \partial_t \Delta_x(\bar{u} - \varphi) + \gamma(\bar{u} - \varphi - \alpha \Delta_x(\bar{u} - \varphi)) + \Pi[(\bar{u}, \nabla_x)\bar{u} - (\varphi, \nabla_x)\varphi] + D_\alpha(\varphi) = 0. \quad (3.2) \]

Multiplying this identity by \( \bar{u} - \varphi \) and integrating in space, we arrive at

\[ \frac{1}{2} \frac{d}{dt} \|\bar{u} - \varphi\|_{H_\alpha}^2 + \gamma \|\bar{u} - \varphi\|_{H_\alpha}^2 + ((\bar{u} - \varphi, \nabla_x)\varphi, \bar{u} - \varphi) + (D_\alpha(\varphi), \bar{u} - \varphi) = 0 \quad (3.3) \]
Let us define the quantity
\[ e_\alpha(\varphi)(t) := \sup_{z \in H_\alpha} \frac{-(z, \nabla_x \varphi(t), z)}{\|z\|_{H_\alpha}^2}. \] (3.4)

Then equality (3.3) implies the desired differential inequality
\[ \frac{1}{2} \frac{d}{dt} \|\bar{u} - \varphi\|_{H_\alpha}^2 + (\gamma - e_\alpha(\varphi)(t))\|\bar{u} - \varphi\|_{H_\alpha}^2 + (D_\alpha(\varphi), \bar{u} - \varphi) \leq 0 \] (3.5)
and integrating this in time we finally arrive at the inequality
\[ \|\bar{u}(t) - \varphi(t)\|_{H_\alpha}^2 \leq \|\bar{u}(0) - \varphi(0)\|_{H_\alpha}^2 e^{-2\int_0^t (\gamma - e_\alpha(\varphi(s)) ds} - 2 \int_0^t e^{-2\int_0^s (\gamma - e_\alpha(\varphi(s))) ds} (D_\alpha(\varphi)(\tau), \bar{u}(\tau) - \varphi(\tau)) d\tau \] (3.6)

We are now ready to give the definition of dissipative solutions.

**Definition 3.1.** Let \( \alpha \geq 0 \) and let \( \bar{u}_0 \in H_\alpha \). Then the function
\[ \bar{u} \in L^\infty_{loc}(\mathbb{R}_+; H_\alpha) \cap C_{w,loc}(\mathbb{R}_+, H_\alpha) \] (3.7)
is a dissipative solution of problem (2.1) (or (2.8) if \( \alpha = 0 \)) if, for every smooth divergent-free test function \( \varphi \) and every \( t \geq 0 \), inequality (3.6) holds.

The next proposition shows that the concept of dissipative solutions gives nothing new on the level of the Bardina approximations.

**Proposition 3.2.** Let \( \alpha > 0 \) and let \( \bar{u}_0 \in H_\alpha \). Then the dissipative solution of (2.1) is unique and coincides with the solution \( \bar{u}(t) \) of this problem constructed in Theorem 2.1.

**Proof.** Indeed, let \( \bar{u}(t) \) be a dissipative solution of (2.1). Following [27], the idea of the proof is just to take \( \varphi = \bar{u} \). To this end we note that, due to the fact that \( \bar{u} \in C^1_{loc}(\mathbb{R}_+, H^1) \), we have \( D_\alpha(\bar{u}) \in C_{loc}(\mathbb{R}_+, H^{-1}) \) and \( e_\alpha(\bar{u}) \in L^1_{loc}(\mathbb{R}_+) \). Thus, inequality (3.6) is well-defined for \( \varphi = \bar{u} \) and can be justified by the standard density arguments (approximating \( \bar{u} \) by smooth divergence-free functions). Taking finally \( \varphi = \bar{u} \), we see that \( D_\alpha(\varphi) = 0 \) and \( \bar{u}(t) \equiv \bar{u}(t) \) for all \( t \). \( \square \)

**Remark 3.3.** In contrast to the case \( \alpha > 0 \), the regularity of the distributional solution \( \bar{u}(t) \) of (2.8) is not enough to verify that \( D_0(\bar{u}) \in L^1_{loc}(\mathbb{R}_+, L^2) \) and \( e_0(\bar{u}) \in L^1_{loc} \), therefore the above given proof does not work for \( \alpha = 0 \). However, it gives the so-called weak-strong uniqueness in the class of dissipative solutions. Namely, if a sufficiently regular solution \( \bar{u}(t) \) of damped Euler equations (2.8) is given (e.g., the solution which satisfies the Beale–Kato–Majda criterion), then this solution is unique in the class of dissipative solutions, see [27] for more details.

It is also not difficult to show that the dissipative solution \( \bar{u}(t) \) of the limit Euler equations solves (2.8) in the sense of distributions if it is regular enough. Indeed, let us take \( \varphi_\varepsilon(t) := \bar{u} + \varepsilon \theta \), where \( \varepsilon \in \mathbb{R}_+ \) is a small parameter and \( \theta \) is an
arbitrary smooth divergent free function. Then, dividing (3.3) by $\varepsilon$ and passing to the limit $\varepsilon \to 0$, we arrive at

$$\int_0^t e^{-2\int_0^\tau (\gamma - \varepsilon_0(\bar{u}(s))) ds} \langle D_0(\bar{u}(\tau)), \theta(\tau) \rangle d\tau = 0$$

for every test function $\theta$. This gives $D_0(\bar{u}) \equiv 0$ and $\bar{u}$ is a distributional solution of (2.8). Thus, despite a bit unusual form, the concept of a dissipative solution is a natural and convenient generalization of distributional solutions.

In addition, there is a natural connection between dissipative and distributional solutions. Indeed, as shown in [4] (for the case $\gamma = g = 0$, but the general case is analogous), any distributional solution $\tilde{u}$ of (2.8) which satisfies the energy inequality:

$$\|\tilde{u}(t)\|_{H_0}^2 \leq \|\tilde{u}(0)\|_{H_0}^2 e^{-2\gamma t} + 2\int_0^t e^{-2\gamma(t-s)} (g, \tilde{u}(s)) ds$$

is automatically a dissipative solution.

We conclude this section by establishing the existence of a dissipative solution.

**Proposition 3.4.** Let $\bar{u}_0 \in H_0$. Then there exists at least one dissipative solution of damped Euler equations (2.8).

**Proof.** Let $\alpha > 0$ and let $\bar{u}_0^\alpha \in H_\alpha$ be such that

$$\bar{u}_0 \to \bar{u}_0 \text{ in } H_0 \text{ and } \|\bar{u}_0^\alpha\|_{H_\alpha} \to \|\bar{u}_0\|_{H_0} \text{ as } \alpha \to 0.$$ 

In particular, one can take $\bar{u}_0^\alpha := (1 - \alpha \Delta_x)^{-1/2} \bar{u}_0$. Let $\bar{u}_\alpha(t) := S_\alpha(t)\bar{u}_0^\alpha$ be the corresponding solutions of damped Bardina–Euler equations (2.8). Then, according to Theorem (2.1) the functions $\bar{u}_\alpha$ are uniformly bounded in $C_b(\mathbb{R}, H_\alpha)$. Moreover, it is not difficult to see that

$$\|(\bar{u}_\alpha, \nabla_x \bar{u}_\alpha)\|_{H^{-3}(\mathbb{T})} \leq C\|\bar{u}_\alpha\|_{H_0}^2$$

and, therefore, $\partial_t \bar{u}_\alpha$ are uniformly bounded in $C_{loc}(\mathbb{R}^+, H^{-3}(\mathbb{T}))$. Thus, by Banach–Alaoglu theorem, we may assume without loss of generality that $\bar{u}_\alpha \to \bar{u}$, $\partial_t \bar{u}_\alpha \to \partial_t \bar{u}$ as $\alpha \to 0$ weakly star in $L^\infty_{loc}(\mathbb{R}^+, H_0)$ and $L^\infty_{loc}(\mathbb{R}^+, H^{-3})$ to some function $\bar{u}$. In particular, this convergence implies that $\bar{u} \in C_{w,loc}(\mathbb{R}^+, H_0)$, $\partial_t \bar{u} \in L^\infty_{loc}(\mathbb{R}^+, H^{-3})$ and that for every $t \in \mathbb{R}^+$ we have a weak convergence $\bar{u}_\alpha(t) \to \bar{u}(t)$ in $H_0$.

It only remains to check that $\bar{u}$ is a desired dissipative solution of (2.8). Let $\varphi$ be a smooth divergent free test function. Then, obviously, $e_\alpha(\varphi) \to e_0(\varphi)$ strongly in $L^1_{loc}$ and also $D_\alpha(\varphi) \to D_0(\varphi)$ strongly in $L^1_{loc}(\mathbb{R}^+, L^2)$. Writing down the variational inequality (3.3) for the dissipative solutions $\bar{u}_\alpha$ and passing to the limit $\alpha \to 0$, we now see that $\bar{u}$ satisfies the variational inequality for $\alpha = 0$ which finishes the proof of the proposition. 

$\square$
4. The trajectory dynamical system

In this section, we develop the trajectory approach for dissipative solutions of limit damped Euler equations (2.8). Following the general approach, we want to fix the trajectory phase space $K_+^+$ as a set of all dissipative solutions of (2.8) which correspond to $\bar{u}_0 \in H_0$. However, there is one more problem here, namely, the class of dissipative solutions is not invariant with respect to time shifts $T(t)$ since the variational inequality (3.6) is not invariant. As a result, the $\omega$-limit set of a subset of $K_+^0$ will not belong to $K_+^0$ in general and we will be unable to characterise the constructed attractor in terms of dissipative solutions of the considered problem. In order to overcome this difficulty, we restrict further (following e.g. [38], see also [7, 29]) the class of dissipative solutions by considering only those of them which can be obtained as weak limits of the corresponding solutions of damped Bardina–Euler equations (2.1) as $\alpha \to 0$. Namely,

**Definition 4.1.** The trajectory phase space

$$K_+^0 \subset L^\infty_{loc}(\mathbb{R}^+, H_0) \cap W^{1,\infty}_{loc}(\mathbb{R}^+, H^{-3})$$

(4.1)

is defined as follows: $\tilde{u}(t) \in K_+^0$ if and only if there exists a sequence $\alpha_n \to 0$, $\alpha_n > 0$ and a sequence of the initial data $\bar{u}_0^{\alpha_n}$ such that $\|\bar{u}_0^{\alpha_n}\|_{H_{\alpha_n}}$ is uniformly bounded in $H_{\alpha_n}$ and the corresponding solutions $\bar{u}_n(t) := S_{\alpha_n}(t)\bar{u}_0^{\alpha_n}$ of equations (2.1) converge weakly in $H_0$ to $\tilde{u}(t)$ for every $t \geq 0$. Then, obviously, the translation semigroup $T(h)$ acts on $K_+^0$. We endow the trajectory phase space $K_+^0$ with the *weak star topology* induced by embedding (4.1) and will refer to $(T(h), K_+^0)$ as the trajectory dynamical system associated with damped Euler equation (2.8).

**Remark 4.2.** Arguing as in the proof of Theorem 2.1, we see that the set $K_0^+$ is not empty and possesses a sufficiently large number of trajectories, namely, for any $\bar{u}_0 \in H_0$ there exists at least one element $\tilde{u} \in K_+^0$ such that $\tilde{u}(0) = \bar{u}_0$ (we may take a dissipative solution constructed in Theorem 2.1 as such a $\tilde{u}$).

However, in contrast to Theorem 2.1, we now unable to pass to the limit $\alpha_n \to 0$ and get that any $\tilde{u} \in K_+^0$ is a dissipative solution. The problem is that we now require only the weak convergence $\bar{u}_0^{\alpha_n} \to \bar{u}_0$ and cannot require the convergence of the norms ($\|\bar{u}_0^{\alpha_n}\|_{H_{\alpha_n}} \to \|\bar{u}_0\|_{H_0}$) since this norm convergence cannot be verified for $t > 0$ and would make $K_+^0$ not translation-invariant. For this reason, the concept of a dissipative solution should be modified. We will do this modification using the so-called $M$-functional introduced in [38] for study the supercritical damped wave equations, see also [7].

**Definition 4.3.** Let $\tilde{u} \in K_+^0$ and let $\varphi$ be a smooth divergent free test function. Define

$$M_{\tilde{u}, \varphi}(t) := \inf_{\bar{u}_{\alpha_n} \sim \tilde{u}} \liminf_{n \to \infty} \|\bar{u}_{\alpha_n}(t) - \varphi(t)\|^2_{H_{\alpha_n}},$$

(4.2)

where the external infimum is taken over all solutions $\bar{u}_{\alpha_n}$ of damped Bardina–Euler equations (2.1) such that $\alpha_n \to 0$ as $n \to \infty$, $\alpha_n > 0$, the sequence $\bar{u}_{\alpha_n}(0)$...
is uniformly bounded in $\mathcal{H}_{\alpha}$ and such that
\[
\bar{u}_{\alpha}(t) \rightharpoonup \bar{u}(t)
\] (4.3)
weakly in $\mathcal{H}_0$ for every $t \geq 0$.

The next proposition collects some straightforward properties of the introduced $M$-functional.

**Proposition 4.4.** Let $\bar{u} \in K_0^+$ and $\varphi$ be a smooth divergent free function. Then,
1) $\|\bar{u}(t) - \varphi(t)\|^2_{\mathcal{H}_0} \leq M_{\bar{u},\varphi}(t)$ for all $t \geq 0$;
2) $M_{\bar{u},\varphi}(t) = M_{\bar{u},0}(t) + \|\varphi\|^2_{\mathcal{H}_0} - 2(\bar{u}, \varphi)$ for all $t \geq 0$;
3) $M_{T(h)\bar{u}, T(h)\varphi}(t) \leq M_{\bar{u},\varphi}(t + h)$ for all $t, h \geq 0$;
4) The following analogue of variational inequality (3.6) holds:
\[
M_{\bar{u},\varphi}(t + \kappa) \leq M_{\bar{u},\varphi}(t)e^{-2\int_{t}^{t+\kappa}(\gamma - e_0(\varphi(s)))ds} \nonumber \]
\[
- 2 \int_{t}^{t+\kappa} e^{-2\int_{s}^{t+\kappa}(\gamma - e_0(\varphi(s)))ds}(D_0(\varphi)(\tau), \bar{u}(\tau) - \varphi(\tau)) d\tau
\] (4.4)
for all $t, \kappa \geq 0$.

**Proof.** Indeed, the first three statements are obvious, so we only need to check the fourth one. Let $\bar{u}_{\alpha}(t)$ be a sequence of solutions of Bardina–Euler approximations (2.1) which converges weakly to $\bar{u}(t)$ as $\alpha \to 0$ and let $\varphi$ be a smooth divergence free test function. Then, analogously to (3.6), we have the inequality
\[
\|\bar{u}_{\alpha}(t + \kappa) - \varphi(t + \kappa)\|^2_{\mathcal{H}_{\alpha}} \leq \|\bar{u}_{\alpha}(t) - \varphi(t)\|^2_{\mathcal{H}_{\alpha}} e^{-2\int_{t}^{t+\kappa}(\gamma - e_{\alpha}(\varphi(s)))ds} \nonumber \]
\[
- 2 \int_{t}^{t+\kappa} e^{-2\int_{s}^{t+\kappa}(\gamma - e_{\alpha}(\varphi(s)))ds}(D_{\alpha}(\varphi)(\tau), \bar{u}_{\alpha}(\tau) - \varphi(\tau)) d\tau
\] (4.5)
Thus, we only need to pass to the limit in (4.5) in a proper way. The passage to the limit in the last term in the RHS is immediate since we have a weak convergence of $\bar{u}_{\alpha}(t)$ to $\bar{u}(t)$ for all $t$. Taking $\liminf_{\alpha \to 0}$ followed by $\inf_{\alpha \to 0}$ from both sides of (4.5), we get (4.4) and finish the proof of the proposition. \qed

The next proposition is crucial for our construction of trajectory attractors.

**Proposition 4.5.** Let $\{\bar{u}^l\}_{l \in \mathbb{N}} \subset K_0^+$ be such that the sequence $M_{\bar{u}^l,0}(0)$ is uniformly bounded and
\[
\bar{u}^l \rightharpoonup \bar{u} \quad \text{weakly star in} \quad L^\infty_{\text{loc}}(\mathbb{R}_+, \mathcal{H}_0).
\] (4.6)
Then $\bar{u} \in K_0^+$ and
\[
M_{\bar{u},0}(t) \leq \liminf_{l \to \infty} M_{\bar{u}^l,0}(t)
\] (4.7)
for all $t \geq 0$. 
Proof. Taking $\varphi \equiv 0$ in (4.3) and using the assumed uniform boundedness of $M_{\tilde{u}_0,0}(0)$, we get that
\[ \| \tilde{u}'(t) \|^2_{H_0} \leq M_{\tilde{u}',0}(t) \leq C \] for all $t \geq 0$ and $l \in \mathbb{N}$. Moreover, by the definition of the $M$-functional, for any $l \in \mathbb{N}$, there exist a sequence $\alpha_{n,l} > 0$, $\lim_{n \to \infty} \alpha_{n,l} = 0$ and a sequence of initial data $\tilde{u}_0^{\alpha_{n,l}} \in \mathcal{H}_{\alpha_{n,l}}$ such that the corresponding solutions $\tilde{u}_{\alpha_{n,l}}(t) := S(t)\tilde{u}_0^{\alpha_{n,l}}$ of damped Bardina–Euler equations (2.1) converge weakly to $\tilde{u}'(t)$ in $\mathcal{H}_0$ for any $t \geq 0$. Moreover, since $M_{\tilde{u}',0}(0)$ are uniformly bounded, we may assume without loss of generality that
\[ \| \tilde{u}_0^{\alpha_{n,l}} \|_{\mathcal{H}_{\alpha_{n,l}}} \leq C, \] where $C$ is independent of $n$ and $l$. Arguing now as in Proposition 3.4, we see that
\[ \| \tilde{u}_{\alpha_{n,l}}(t) \|_{\mathcal{H}_{\alpha_{n,l}}} + \| \partial_t \tilde{u}_{\alpha_{n,l}}(t) \|_{H^{-3}} \leq C, \quad t \geq 0. \] In other words, the sequence $\tilde{u}_{\alpha_{n,l}}$ is uniformly bounded in the space $L^\infty(\mathbb{R}_+, \mathcal{H}_0) \cap W^{1,\infty}(\mathbb{R}_+, H^{-3})$ and the Banach–Alaoglu theorem now guarantees that
\[ \tilde{u} \in L^\infty(\mathbb{R}_+, \mathcal{H}_0) \cap W^{1,\infty}(\mathbb{R}_+, H^{-3}). \] To verify that $\tilde{u} \in \mathcal{K}_0^+$, we only need to construct a sequence of solutions of the approximating damped Bardina–Euler equations which is convergent to $\tilde{u}$. To this end, we note that any bounded set in \( L^\infty_{\text{loc}}(\mathbb{R}_+, \mathcal{H}_0) \cap W^{1,\infty}_{\text{loc}}(\mathbb{R}_+, H^{-3}) \) is metrizable in weak-star topology. Let $B$ be a bounded set of this space which contains all trajectories $\tilde{u}_{\alpha_{n,l}}$ and $\tilde{u}$ and let $d(\cdot, \cdot)$ be such a metric. Then, we have the convergence $\tilde{u}_{\alpha_{n,l}} \to \tilde{u}'$ as $n \to \infty$ in a metric space $(B,d)$ for every $l \in \mathbb{N}$ as well as the convergence $\tilde{u}' \to \tilde{u}$ as $l \to \infty$. Thus, there exists a sequence $\tilde{u}_{\alpha_{n,l}} \to \tilde{u}$ in $(B,d)$ and the approximating sequence is constructed. This proves the inclusion $\tilde{u} \in \mathcal{K}_0^+$.

The proof of the weak lower semicontinuity (4.7) is straightforward and we left it to the reader. □

Remark 4.6. Note that the properties of "solutions" $\tilde{u} \in \mathcal{K}_0^+$ are more delicate than the analogous properties of dissipative solutions discussed in Remark 3.3. Indeed, we are still able to prove that any sufficiently regular distributional solution is automatically in $\mathcal{K}_0^+$. However, by definition, $\mathcal{K}_0^+$ contains only those solutions which can be obtained as a limit of regularized solutions of damped Bardina–Euler equation and it is not clear whether or not any dissipative solutions can be obtained in such a way. Moreover, keeping in mind the oscillatory shear flow solutions of Euler equations constructed by Di Perna and Majda, see [10] as well as the result of Proposition 4.5, one may expect that there are regular trajectories $\tilde{u} \in \mathcal{K}_0^+$ which do not satisfy the Euler equation (2.8) in the sense of distributions. Actually, the main thing which we know is that $\mathcal{K}_0^+$ is not empty and, for every $\tilde{u}_0 \in \mathcal{H}_0$ there is a dissipative solution $\tilde{u} \in \mathcal{K}_0^+$ such
that \( \tilde{u}(0) = \tilde{u}_0 \). Note that this situation is somehow standard for the theory of trajectory attractors, see \([5]\) and the references therein.

5. The trajectory attractor

In this section, we construct a weak global attractor of the trajectory dynamical system \((T(h), K_0^+)\) associated with the damped Euler equation \((2.8)\). Recall that we have already introduced the topology on the trajectory phase space \(K_0^+\) as the topology induced by the embedding

\[
K_0^+ \subset \Theta_+ := L_{w^*, loc}^\infty(\mathbb{R}_+, \mathcal{H}_0) \cap W_{w^*, loc}^{1, \infty}(\mathbb{R}_+, H^{-3}).
\]

However, this topology is not convenient for defining the bornology (= the class of bounded sets) which is necessary for the attractor theory. Instead, following \([38]\) (see also \([7, 29]\)) we give the following definitions.

**Definition 5.1.** A set \( B \subset K_0^+ \) is bounded in \( K_0^+ \) if

\[
\sup_{\tilde{u} \in B} M_{\tilde{u}, 0}(0) \leq C_B < \infty.
\]

A set \( \mathcal{A}_{tr}(0) \subset K_0^+ \) is a weak global attractor of the trajectory dynamical system \((T(h), K_0^+)\) (= a (weak) trajectory attractor of equation \((2.8)\)) if

1. \( \mathcal{A}_{tr}(0) \) is a compact set in \( K_0^+ \);
2. \( \mathcal{A}_{tr} \) is strictly invariant: \( T(h)\mathcal{A}_{tr}(0) = \mathcal{A}_{tr}(0) \) for all \( h \geq 0 \);
3. \( \mathcal{A}_{tr}(0) \) attracts the images of all bounded sets of \( K_0^+ \), i.e., for every bounded set \( B \subset K_0^+ \) and every neighbourhood \( O(\mathcal{A}_{tr}(0)) \) of \( K_0^+ \) in \( \Theta_+ \), there exists \( T = T(B, O) \) such that

\[
T(t)B \subset O(\mathcal{A}_{tr}(0)) \quad \text{for all} \quad t \geq T.
\]

The following theorem can be considered as the main result of the paper.

**Theorem 5.2.** Let \( g \in \mathcal{H}_0 \). Then the damped Euler equation \((2.8)\) possesses a weak trajectory attractor \( \mathcal{A}_{tr}(0) \) in the sense of Definition 5.1. Moreover, this attractor is generated by complete bounded trajectories of this equation:

\[
\mathcal{A}_{tr}(0) = K_0 |_{t \geq 0},
\]

where \( \tilde{u} \in K_0 \subset L^\infty(\mathbb{R}, \mathcal{H}_0) \cap W^{1, \infty}(\mathbb{R}, H^{-3}) \) if and only if there exist sequences \( t_n \to -\infty, \alpha_n \to 0 \) and \( \tilde{u}_0^{\alpha_n} \in \mathcal{H}_{\alpha_n} \) such that the norms \( \| \tilde{u}_0^{\alpha_n} \|_{\mathcal{H}_{\alpha_n}} \) are uniformly bounded as \( n \to \infty \) and the corresponding solutions \( u_{\alpha_n}(t) := S(t + t_n)\tilde{u}_0^{\alpha_n}, t \geq -t_n \) of the corresponding damped Bardina–Euler equations \((2.1)\) converge weakly to the function \( \tilde{u} \):

\[
\tilde{u}_{\alpha_n}(t) \to \tilde{u}(t) \quad \text{in} \ \mathcal{H}_0 \quad \text{for all} \quad t \in \mathbb{R}.
\]

**Proof.** According to the abstract theorem on existence of global attractors, see \([1, 5, 34]\) and the references therein, we need to verify that

...
1. There exists a bounded absorbing set $B \subset K_0^+$ of the translation semigroup $T(h)$ which is a metrizable compact in $K_0^+$;

2. The operators $T(h)$ are continuous on $B$ for every fixed $h \geq 0$.

The representation formula (5.1) is then a standard corollary of this theorem.

Let us construct an absorbing set $B$ for the semigroup $T(h)$ with the desired properties. Indeed, taking $\varphi \equiv 0$ in (4.4), we have

$$ M_{\tilde{u},0}(t + \kappa) \leq M_{\tilde{u},0}(t) e^{-2\gamma \kappa} - 2 \int_{t}^{t+\kappa} e^{-2\gamma(t+\kappa-\tau)} (g, \tilde{u}(\tau)) d\tau \leq M_{\tilde{u},0}(t) e^{-2\gamma \kappa} + 2 \int_{t}^{t+\kappa} e^{-2\gamma(t+\kappa-\tau)} \|g\|_{H_0} \sqrt{M_{\tilde{u},0}(\tau)} d\tau \quad (5.2) $$

and the Gronwall inequality gives that

$$ M_{\tilde{u},0}(t + \kappa) \leq M_{\tilde{u},0}(t) e^{-2\gamma \kappa} + \frac{1}{\gamma^2} \|g\|^2_{H_0} \quad (5.3) $$

and, therefore,

$$ B := \{ \tilde{u} \in K_0^+ : M_{\tilde{u},0}(0) \leq \frac{2}{\gamma^2} \|g\|^2_{H_0} \} \quad (5.4) $$

is an absorbing set for the semigroup $T(h)$ in $K_0^+$. By the Banach-Alaoglu theorem, $B$ is precompact and metrizable in $\Theta$ and due to Proposition 4.5 $B$ is closed in $K_0^+$. Thus, $B$ is a metric compact in $K_0^+$. The continuity of $T(h)$ is obvious since the shift semigroup $T(h)$ is continuous on $\Theta$.

Thus, all the assumptions of the abstract attractors existence theorem are verified and the theorem is proved.

We conclude the section by proving the upper semicontinuity of the attractors $A_{tr}(\alpha)$ at $\alpha = 0$.

**Corollary 5.3.** Let $g \in H_0$ and let $A_{tr}(\alpha)$, $\alpha > 0$ and $A_{tr}(0)$ be the trajectory attractors of the damped Bardina–Euler equations (2.1) and the limit damped Euler equations (2.8), respectively. Then, for every neighbourhood of $O(A_{tr}(0))$ of the limit attractor $A_{tr}(0)$ in the topology of $\Theta^+$, there exists $\alpha(O) > 0$ such that

$$ A_{tr}(\alpha) \subset O(A_{tr}(0)) \quad \text{for all } \alpha \leq \alpha(O). \quad (5.5) $$

**Proof.** The statement of the corollary is almost tautological. Indeed, as we have verified before, the sets $K_\alpha$, $\alpha > 0$ of complete trajectories are bounded in the space $L^\infty(\mathbb{R}, H_0) \cap W^{1,\infty}(\mathbb{R}, H^{-3})$ uniformly with respect to $\alpha \to 0$ and by Banach-Alaoglu theorem, for any sequences $\alpha_n \to 0$ and $\tilde{u}_{\alpha_n} \in K_\alpha$ there is a subsequence (which we denote by $\tilde{u}_{\alpha_n}$ again) which is convergent in the space

$$ \Theta := L^{\infty}_{w^*,loc}(\mathbb{R}, H_0) \cap W^{1,\infty}_{w^*,loc}(\mathbb{R}, H^{-3}) $$

to some function $\tilde{u} \in \Theta$. To verify the upper semicontinuity, it is sufficient to prove that we necessarily have $\tilde{u} \in K_0$. This can be done exactly as in the proof of Proposition 4.3. Thus, the corollary is proved. \qed
References

[1] A. Babin and M. Vishik, Attractors of Evolution Equations. Studies in Mathematics and its Applications, vol 25. North-Holland Publishing Co., Amsterdam, 1992.

[2] J. Bardina, J. Ferziger, and W. Reynolds, Improved subgrid scale models for large eddy simulation, in Proceedings of the 13th AIAA Conference on Fluid and Plasma Dynamics, (1980).

[3] Y. Cao, E. M. Lunasin, and E.S. Titi, Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. Commun. Math. Sci., vol 4, no 4, (2006) 823–848.

[4] C. Bardos, E. S. Titi, Euler equations for incompressible ideal fluids, Uspeki Mat. Nauk, Volume 62, Issue 3(375) (2007), 5–46.

[5] V. V. Chepyzhov and M. I. Vishik, Attractors for Equations of Mathematical Physics. Amer. Math. Soc. Colloq. Publ., vol 49, Providence, RI: Amer. Math. Soc., 2002.

[6] V.V.Chepyzhov, A.A.Ilyin, S.V.Zelik, Vanishing viscosity limit for global attractors for the damped Navier–Stokes system with stress free boundary conditions, Physica D, vol 376–377, (2018) 31–38.

[7] V. Chepyzhov, M. Vishik and S. Zelik, Strong trajectory attractors for the dissipative Euler equations, Journal des Mathématiques Pures et Appliquées 96 (2011), no. 4, 395–407.

[8] V. Chepyzhov and S. Zelik, Infinite energy solutions for dissipative Euler equations in \( \mathbb{R}^2 \), J. Math. Fluid Mech. 17 (2015), no. 3, 513–532

[9] V. Chepyzhov, A. Ilyin and S. Zelik, Strong trajectory and global \( W^{1,p} \)-attractors for the damped-driven Euler system in \( \mathbb{R}^2 \), Discrete Contin. Dyn. Syst. Ser. B 22 (2017), no. 5, 1835–1855.

[10] R. DiPerna, and A. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys., 108(4) (1987), 667–689.

[11] C. Fefferman, Existence and smoothness of the Navier–Stokes equation, Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, (2006), 57–67.

[12] C. Foias, D. D. Holm, and E. S. Titi, The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory, Jour Dyn and Diff Eqns, vol 14, (2002) 1–35.

[13] C. Foias, O. Manely, R. Rosa, and R. Temam, Navier–Stokes Equations and Turbulence. Cambridge Univ. Press, Cambridge, 2001.

[14] U. Frisch, Turbulence. The legacy of A. N. Kolmogorov, Cambridge University Press, Cambridge, 1995.

[15] M. Holst, E. Lunasin, and G. Tsogtgerel, Analysis of a general family of regularized Navier–Stokes and MHD models, J. Nonlinear Sci. vol 20, no 5, (2010) 523–567.

[16] A.A. Ilyin, The Euler equations with dissipation, Sb. Math. 74 (1993) 475–485.

[17] A. Ilyin, A. Kostianko and S.Zelik, Sharp Upper and Lower Bounds of the Attractor Dimension for 3D Damped Euler–Bardina Equations, submitted.

[18] S. Zelik, A. Ilyin, and A. Kostianko, Sharp dimension estimates for the attractors of the regularized damped Euler system, Doklady Mathematics, vol. 104, no. 1 (2021) 169–172.

[19] A. A. Ilyin and E. S. Titi, Attractors to the two-dimensional Navier–Stokes–α models: an \( \alpha \)-dependence study, J. Dynam. Diff. Eqns, vol 15, (2003) 751–778.
[20] A. A. Ilyin, A. Miranville, and E. S. Titi, Small viscosity sharp estimates for the global attractor of the 2-D damped-driven Navier-Stokes equations, Commun. Math. Sci. vol 2, (2004) 403–426.

[21] A. A. Ilyin and S. V. Zelik, Sharp dimension estimates of the attractor of the damped 2D Euler-Bardina equations, In book: Partial Differential Equations, Spectral Theory, and Mathematical Physics, EMS Series of Congress Reports, EMS Press, Berlin, 2021, p. 209–229.

[22] V. K. Kalantarov and E. S. Titi, Global attractors and determining modes for the 3D Navier-Stokes–Voight equations, Chin. Ann. Math. vol 30B, no 6, (2009) 697–714.

[23] R. Layton and R. Lewandowski, On a well-posed turbulence model, Discrete Continuous Dyn. Sys. B, vol. 6, (2006) 111–128.

[24] O. A. Ladyzhenskaya, Attractors for Semigroups and Evolution Equations. Leizioni Lincei, Cambridge Univ. Press, Cambridge, 1991.

[25] J. Leray, Essai sur le mouvement d’un fluide visqueux emplissant l’espace, Acta Math. vol 63, (1934) 193–248.

[26] J. Lions, Quelques méthodes des problèmes aux limites non linéaires, Doud, Paris, 1969.

[27] P. Lions, Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models, Oxford Lecture Series in Mathematics and Its Applications, 1996.

[28] M. Lopes Filho, H. Nussenzveig Lopes, E. Titi, A. Zang, Convergence of the 2D Euler-α to Euler equations in the Dirichlet case: indifference to boundary layers, Phys. D, vol 292-293, (2015) 51–61.

[29] A. Miranville and S. Zelik, Attractors for dissipative partial differential equations in bounded and unbounded domains, In: Handbook of differential equations: evolutionary equations. Vol. IV, 103–200, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[30] E. Olson and E. Titi, Viscosity versus vorticity stretching: global well-posedness for a family of Navier-Stokes-α-like models, Nonlinear Anal. vol 66, no 11, (2007) 2427–2458.

[31] A. Oskolkov, The uniqueness and solvability in the large of boundary value problems for the equations of motion of aqueous solutions of polymers, Zap. Nauchn. Sem. LOMI, vol. 38, (1973) 98–136.

[32] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1979.

[33] T. Tao, Finite time blowup for an averaged three-dimensional Navier-Stokes equation, J. Amer. Math. Soc., vol 29 (2016), 601–674.

[34] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd ed. Springer-Verlag, New York 1997.

[35] R. Temam, Navier-Stokes equations and nonlinear functional analysis, vol. 66, Siam, 1995.

[36] E. Wiedemann, Existence of weak solutions for the incompressible Euler equations, Ann. I. H. Poincaré – AN 28 (2011) 727–730.

[37] V.I. Yudovich, Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid, Math. Res. Lett. 2 (1995) 27–38.

[38] S. Zelik, Asymptotic regularity of solutions of singularly perturbed damped wave equations with supercritical nonlinearities, Disc. Cont. Dyn. Sys. 11 (2004), no. 2-3, 351–392.
Email address: ilyin@keldysh.ru
Email address: a.kostianko@imperial.ac.uk
Email address: s.zelik@surrey.ac.uk

1 Keldysh Institute of Applied Mathematics, Moscow, Russia

2 University of Surrey, Department of Mathematics, Guildford, GU2 7XH, United Kingdom.

3 School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, P.R. China

4 Imperial College, London SW7 2AZ, United Kingdom.