AUTOMORPHISMS OF GROUPS AND A HIGHER RANK
JSJ DECOMPOSITION II: THE SINGLE ENDED CASE

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The JSJ decomposition encodes the automorphisms and the virtually cyclic splittings of a hyperbolic group. For general finitely presented groups, the JSJ decomposition encodes only their splittings.

In this sequence of papers we study the automorphisms of a hierarchically hyperbolic group that satisfies some weak acylindricity conditions. To study these automorphisms we construct objects that can be viewed as a higher rank JSJ decomposition.

In the first paper of the sequence we constructed a higher rank Makanin-Razborov diagram that encodes the automorphisms of an HHG. In the second paper we use this diagram, and techniques from the solution to Tarski’s problem on the elementary theory of a free group, to associate two groupoids with this automorphism group. The groupoids generalize the structure of the automorphism group of a hyperbolic group that follows from the existence of its JSJ decomposition.

In this paper we construct the groupoids for HHGs with single ended higher rank MR diagrams. In the next paper of the sequence we generalize the construction to HHGs with general higher rank diagrams.

The (canonical) JSJ decomposition of a torsion-free hyperbolic group describes the structure of the (outer) automorphism group, and is a key for understanding the dynamics of automorphisms of a hyperbolic group ([Se7], [Le]).

The construction of the JSJ was later generalized to general finitely presented groups (see [Gu-Le]). In this general setting, the JSJ encodes all the splittings of a f.p. group over a given family of subgroups (in a rather subtle way), but it is far from encoding the automorphism group nor the dynamics of individual automorphisms.

In this sequence of papers we use some of the JSJ concepts, to study automorphisms of hierarchically hyperbolic groups. Hierarchically hyperbolic groups and spaces were defined by Behrstock, Hagen and Sisto [BHS1]. The definition axiomatizes the hierarchical structure of the mapping class groups, that was defined and studied in the work of Masur and Minsky [Ma-Mi]. Automorphisms of families of HHGs were studied earlier by Fioravanti [Fi], and by Casals-Ruiz, Hagen and Kazachkov ([Ca-Ka], [CHK]).

In the first paper of the sequence, we showed how to associate a higher rank Makanin-Razborov diagram with the automorphism group of an HHG. The higher rank diagram contains a finite set of $m$-collections of (cover) resolutions, where $m$ is the number of orbits of projection spaces. The higher rank diagram is constructed using a compactness argument, so it can not be canonical, but it is universal. Every

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automorphism of the HHG factors through at least one of the $m$-collections in the higher rank diagram.

In the second paper our goal is to use the higher rank MR diagram that was constructed in the first paper, to construct objects that encode some canonical properties and structure of both the automorphism group and single automorphisms of an HHG.

For a one ended hyperbolic group, $\Gamma$, there exists a natural and canonical epimorphism from a finite index subgroup of $Out(\Gamma)$ onto a direct product of the mapping class groups of the 2-orbifolds that appear in the canonical JSJ decomposition of $\Gamma$, where the kernel is a f.g. virtually abelian group ([Se7],[Le]).

In section 1 of the paper we do prove a direct generalization of these results for HHGs, but only in the case in which each $m$-collection in the higher rank diagram contains at most a single non-trivial resolution. For general HHGs $G$ we prove a weaker structure theorem. We replace the existence of a homomorphism from $Out(G)$ into the direct product of finitely many mapping class groups of 2-orbifolds and outer automorphism groups of some f.g. virtually abelian groups, with the construction of two groupoids.

The objects in the first groupoid are some finite sets of QH and virtually abelian vertex groups in some of the resolutions in the $m$-collections in the higher rank MR diagram. The morphisms are associated with automorphisms in $Out(G)$, and each morphism is a set of isomorphisms between the QH and the virtually abelian vertex groups that are included in the domain and those in the target (one isomorphism for each QH or virtually abelian vertex group).

The objects in the second groupoid are some finite sets of virtually infinite cyclic edge groups in some of the resolutions in the $m$-collections in the higher rank diagram. The morphisms that are associated with automorphisms in $Out(G)$, are not isomorphisms, but rather uniform quasimorphisms of $\mathbb{Z}^n$, where $n$ is the number of virtually infinite cyclic edge groups in the domain and in the target.

As part of the definition of a groupoid, each morphism has an inverse or a quasi-inverse, and it is possible to compose morphisms in a way that agrees with compositions of outer automorphisms in $Out(G)$. However, we were only able to associate finitely many morphisms with each outer automorphism, and not a unique one as in the case of a hyperbolic group.

This lack of uniqueness is a major weakness of our structure theory. We do not know if and where the lack of uniqueness really occurs, and we hope that in many families of HHGs it doesn’t occur or it can be bypassed. e.g., RAAGs, (virtually) special cubulated groups etc. We see our results as an initial step, and we expect that it will be followed by quite a few strengthenings and refinements, as well as applications.

The construction of the groupoids from the higher rank MR diagram involves an analysis of limit quotients and sets of homomorphisms (or rather quasimorphisms) of HHGs that encode automorphisms via the higher rank MR diagram. Hence, it is natural that it requires tools that were defined and constructed for analyzing varieties and more generally the first order theory of groups, as part of the solution to Tarski’s problem ([Se1]-[Se6]) and its generalizations. We refer the reader to some notions, objects and constructions that appear in this sequence of papers, although we are aware that it makes the paper harder to follow. We do intend to elaborate in some parts in revised versions.

Throughout this sequence of papers we assume that the HHGs satisfy some weak
acylindricity assumption, that appears in definition 3.1 in [Se8], and there exists a finite index subgroup of the HHG, under which the orbits of the projection spaces are pairwise transversal. These two properties are known to hold for the mapping class groups (see [BBF1] and [BBFS]), and were already assumed in the construction of the higher rank MR diagram. The second property is closely connected to property QT in [BBF2].

For presentation purposes, we analyze groups that act on a product of (finitely many) hyperbolic spaces in the first two sections, and generalize the analysis and the constructions to HHGs that satisfy our weak acylindricity assumption in the third section.

The JSJ was constructed first for hyperbolic groups, and then generalized to f.p. groups. For f.p. groups the JSJ encodes some family of its automorphisms, but not all the automorphism group. In the end of the third section we state implications of our results for HHGs to all f.p. groups. The construction of our groupoids can be generalized to every f.p. group and its automorphism group, but similarly to the JSJ, the groupoids that we associate with the automorphism group of a general f.p. group detect only the part of the automorphism group that can be visualized by actions of the f.p. group on HHS that are uniformly weakly acylindrical.

In the second paper we also assume that the HHGs that we analyze have a higher rank MR diagram that is single ended. This means that the virtually abelian decompositions that are associated with the resolutions in the $m$-collections in the higher rank MR diagram contain no edges with finite (nor trivial) edge groups. The last assumption makes the arguments technically easier. The assumption is dropped in the next paper in the sequence using further concepts from the work on Tarski’s problem.

§1. A higher rank JSJ decomposition of a product I: higher rank MR diagrams with at most a single non-trivial resolution in each collection

In section 3 of the first paper in this sequence we studied the automorphism group of a group $G$ that acts properly and cocompactly on a product of $m$ hyperbolic spaces, and the induced action of the group on each of the projection spaces is weakly acylindrical (definition 3.1 in [Se8]). With the automorphism group of such a group we associated a higher rank Makanin-Razborov diagram (theorem 3.6 in [Se8]). The higher rank diagram is a finite set of $m$-collections of (cover) resolutions, where in each collection, each cover resolution is associated with one of the factors in the product.

The higher rank MR diagram is not canonical, but it is universal. Every automorphism of the group $G$ that acts properly and cocompactly on the product, factors through at least one of the $m$-collections of cover resolutions.

The group $G$ that acts properly and cocompactly on the product of $m$ hyperbolic spaces, may permute the $m$ projection spaces. To construct the higher rank diagram we needed to pass from the group $G$ to a finite index characteristic subgroup that we denote $H$, that fixes the projection spaces. The subgroup $H$ is going to be fixed for the rest of the section.

In this section we add a simplifying assumption on the higher rank MR diagram, and obtain a higher rank JSJ decomposition. Our goal is to prove that if each $m$-collection (of resolutions) in the higher rank MR diagram contains at most a single non-trivial resolution (i.e., at most a single resolution with more than a single
level), then the higher rank MR diagram can be replaced by $m$ graphs of groups that are associated with the $m$ projection spaces. The $m$ graphs of groups have virtually abelian edge groups, and their fundamental groups are quotients of the characteristic finite index subgroup $H$. The collection of these $m$ graphs of groups encodes the dynamics of individual automorphisms in $Aut(G)$ and can be used to study the algebraic structure of $Aut(G)$.

In this paper we do not discuss the canonical properties of the graphs of groups that we construct and are associated with the different factors, but we believe that the decompositions, or at least the parts of them that encode the dynamics of individual automorphisms, can be made canonical. Because of the similarity with the properties and the structure of the JSJ decomposition of a hyperbolic group, we call the $m$-collection of graphs of groups that we construct, a higher rank JSJ decomposition.

In the next section we treat products with general higher rank MR diagrams. In this more general case, our results are weaker than the results that we obtain in this section, i.e., in case every $m$-collection in the higher rank diagram has at most a single non-trivial resolution. In the last section we generalize our results for products to HHG that satisfy some further natural assumptions (that hold in the case of the mapping class group), that appear in section 4 in [Se8], where we construct a higher rank MR diagram for such groups.

Let $X$ be a HHS on which a f.g. group $G$ acts isometrically, properly discontinuously and cocompactly. Suppose that the top dimensional hyperbolic space that is associated with the HHS structure $X$ is bounded, and it dominates a (maximal) collection of orthogonal unbounded projection spaces $V_1 \perp \ldots \perp V_m$. We further assume that $S$, the top complexity subspace, and $V_1, \ldots, V_m$, are the only projection spaces that are associated with the HHS $X$. i.e., in particular, we assume that $X$ is quasi-isometric to the product of the spaces $V_j$, $1 \leq j \leq m$.

We assume that $G$ acts strongly acylindrically (definition 3.2 in [Se8]) on the HHS $X$. $G$ permutes the projection spaces $V_1, \ldots, V_m$. Hence, there exists a subgroup of finite index, $\hat{H}$, that fixes these hyperbolic spaces. We take $H$ to be the intersection of all the subgroups in $G$ that have the same index as $\hat{H}$. $H$ is a characteristic subgroup of $G$. By our assumptions $H$ acts weakly acylindrically on each of the hyperbolic spaces $V_j$, and $Aut(G)$ acts on $H$.

Theorem 3.6 in [Se8] associates a finite set of collections of $m$ cover resolutions with the action of $Aut(G)$ on $H$. In analyzing homomorphisms, the finite subset of collections of (cover) resolutions, i.e., the Makanin-Razborov diagram, is the finer structure that one can construct, and in our setting in which resolutions need to be replaced by covers, it is generally not canonical (unlike the JSJ decomposition of a hyperbolic group).

However, we are analyzing automorphisms, that are not just homomorphisms (or quasimorphisms). In particular, automorphisms are bi-Lipschitz maps, they can be composed and they have an inverse. In analyzing automorphisms, our goal is to use the finite set of collections of $m$ cover resolutions (theorem 3.6) to construct JSJ like objects, that will encode both the dynamics of individual automorphisms and algebraic properties of $Out(G)$.

We are going to construct the objects that encode the dynamics of automorphisms and algebraic properties of $Out(G)$ by gradually extending the types of cover resolutions that are contained in the higher rank Makanin-Razborov diagram.
that is associated with $\text{Aut}(G)$. For each possible type we use (weak) test sequences that were originally constructed to study formulas with 2 quantifiers in [Se2], complexity of resolutions that appear in [Se4], and framed resolutions that appear in [Se6], to construct these objects that replace the higher rank MR diagram. We start with a rather degenerate case.

**Lemma 1.1.** Suppose that $X$ is an HHS that is quasi-isometric to a product of $m$ (hyperbolic) projection spaces, $V_1, \ldots, V_m$, $G$ is a f.g. group that acts properly and cocompactly on $X$, and $H$ is a characteristic finite index subgroup of $G$ that does not permute the projection spaces $V_1, \ldots, V_m$.

With $\text{Aut}(G)$ and the HHS $X$ we associate a higher rank Makanin-Razborov diagram according to theorem 3.6 in [Se8]. Suppose that a collection of $m$ cover resolutions that is part of the higher rank Makanin-Razborov diagram that is associated with $\text{Aut}(G)$, contains only resolutions of length 1 (i.e., none of the resolutions contains a quotient map). Then only finitely many outer automorphisms of $G$ factor through that cover.

**Proof:** Let $h_1, \ldots, h_\ell$ be a generating set of $H$. All the $m$ cover resolutions have length 1, so there are bounds on the lengths of the projections of the twisted generators: $\varphi(h_i), 1 \leq i \leq \ell$, to all the spaces $V_j, 1 \leq j \leq m$, and for all the automorphisms $\varphi \in \text{Aut}(G)$ that factor through the collection of $m$ resolutions, after composing them with appropriate inner automorphisms.

This gives a global bound on the lengths of the elements $\varphi(h_i), 1 \leq i \leq \ell$, for all the automorphisms that factor through the given collection of $m$ cover resolutions, after we compose them with inner automorphisms. Since $H$ acts properly on the HHS $X$, there are only finitely many such outer automorphisms.

$\square$

**Proposition 1.2.** With the assumptions of lemma 1.1, suppose that $\text{Out}(G)$ is infinite, and all the (finitely many) collections of $m$ cover resolutions have the following properties:

(1) each collection contains at most one resolution that is not of a single level. i.e., at most one resolution with a quotient map.

(2) the resolution with a quotient map has an abelian decomposition with a single edge that has a virtually infinite abelian stabilizer, and no QH nor a virtually abelian vertex group.

Then:

(i) $H$ has a normal subgroup $N$, and a decomposition: $H = A \ast_C B$ or $H = A \ast_C$, where $N < C$, $C/N$ is virtually infinite cyclic, and $A/N$ and $B/N$ are f.g.

(ii) a finite index subgroup of $\text{Aut}(G)$ preserves $N$ and the conjugacy classes of $A, B, C$. Hence, it preserves the decomposition in (i).

(iii) $\text{Out}(G)$ is virtually cyclic. A finite index subgroup of $\text{Out}(G)$ acts as Dehn twists on the virtually cyclic decomposition of $L = H/N$: $L = A/N \ast_{C/N} B/N$ or $L = A/N \ast_{C/N}$. 

**Proof:** By lemma 1.1 only finitely many (outer) automorphisms factor through a collection of $m$ resolutions that does not contain a resolution with a quotient map.
We look at those cover resolutions from the collections of \( m \) cover resolutions that contain a quotient map. We denote them \( CRes_t, 1 \leq t \leq d \).

Let \( \Lambda_t \) be the virtually abelian decompositions that are associated with the top level limit groups of the cover resolutions, \( CRes_t, 1 \leq t \leq d \). We denote the fundamental group of \( \Lambda_t, L_t \). \( \Lambda_t \) contains one or two f.g. vertex groups, \( A_t \) and (possibly) \( B_t \), and a single edge with a virtually infinite abelian edge group, \( C_t \). Since the action of \( H \) on each of the hyperbolic spaces, \( V_j, 1 \leq j \leq m \), is assumed to be weakly acylindrical, and the edge groups are generated by universally bounded elements, the edge groups, \( C_t \), in the virtually abelian decompositions that are associated with the top level in the resolutions \( CRes_t \), have to be virtually cyclic.

With \( L_t \), there is a quotient map: \( \eta_t : H \rightarrow L_t \). In particular, there exist f.g. subgroups \( \hat{A}_t, \hat{B}_t \), that together generate \( H \), and are mapped by \( \eta_t \) to \( A_t \) and \( B_t \). By the construction of the higher rank Makanin-Razborov diagram, for every automorphism \( \varphi \in Aut(G) \) that factors through \( CRes_t \), there is a quasi-morphism: 

\[
\mu^\varphi_t : L_t \rightarrow Isom(V_j(t)).
\]

Furthermore, by the structure of the resolutions \( CRes_t \), and the properness of the action of \( H \) on the ambient space \( X \), and by the existence of uniform bounds on the stretching factors of the fixed generating sets of the terminal limit groups of these resolutions, for any fixed \( t \), there is a uniform bound on the number of (elementwise) conjugacy classes of the images: \( \varphi(\hat{A}_t), \varphi(\hat{B}_t) \), for all the automorphisms that factor through the resolution \( CRes_t \).

We look at all the possible (boundedly many) pairs of (elementwise) conjugacy classes of the images of the vertex groups in the virtually cyclic decompositions \( \Lambda_t, 1 \leq t \leq d \), for which there is an infinite sequence of (distinct) automorphisms that factor through the resolutions \( CRes_t \), so that the vertex groups in \( \Lambda_t \) are mapped to these pairs of conjugacy classes.

For each infinite sequence of automorphisms \( \{\varphi_s\} \), that factor through a fixed resolution, \( CRes_t, 1 \leq t \leq d \), and for which the conjugacy classes of the images of \( \varphi_s(\hat{A}_t) \) and \( \varphi_s(\hat{B}_t) \) are fixed, the stable kernel of the action of \( H \) on a limit tree that is obtained from any subsequence of the sequence \( \{\varphi_s\} \) is always the same. i.e., the stable kernel depends only on the conjugacy classes of the images of the subgroups, \( \hat{A}_t \) and \( \hat{B}_t \), under the convergent sequence of automorphisms \( \{\varphi_s\} \).

Since there are only finitely many resolutions \( CRes_t \), and only finitely many possibilities for the pairs of conjugacy classes of the images of the subgroups, \( \hat{A}_t \) and \( \hat{B}_t \), there are only finitely many possibilities for the stable kernels of convergent sequences of automorphisms \( \{\varphi_s\} \).

Let \( \{\varphi_s\} \) be a convergent sequence of automorphisms, and let \( \tau \in Aut(G) \). Then \( \{\varphi_s \circ \tau\} \) is a convergent sequence of automorphisms as well. Hence, every automorphism \( \tau \in Aut(G) \) permutes the finite set of possible stable kernels of convergent sequences of automorphisms. Therefore, we can pass to a finite index subgroup \( U_1 \) of \( Aut(G) \) (or \( Out(G) \)), that fixes these stable kernels.

We fix a convergent sequence of automorphisms \( \{\varphi_s\}, \varphi_s \in U_1 \), that factors through a fixed resolution, \( CRes_{t_0} \), where the images \( \varphi_s(\hat{A}_{t_0}) \) are fixed, and the images \( \varphi_s(\hat{B}_{t_0}) \) are in the same conjugacy class. Let \( \tau \in U_1 \). Then for each \( s \):

\[
\varphi_s(h) = \varphi_s \circ \tau \circ \tau^{-1}(h) = \psi_s(\tau^{-1}(h))
\]

where \( \psi_s = \varphi_s \circ \tau \).
By passing to a subsequence, we may assume that the sequence \( \{\psi_s\} \) converges, factors through the same resolution, \( CRes_\nu \), and maps the subgroups, \( \hat{A}_\nu \) and \( \hat{B}_\nu \), to fixed conjugacy classes. Since \( \psi_s(\tau^{-1}(\hat{A}_t)) = \varphi_s(\hat{A}_t) \), it follows that: \( \tau^{-1}(\hat{A}_t) < \hat{A}_\nu N \) or \( \tau^{-1}(\hat{A}_t) < \hat{B}_\nu N \), where \( N \) is the stable kernel. Similarly it follows that: \( \tau^{-1}(\hat{B}_t) < \hat{B}_\nu N \) or \( \tau^{-1}(\hat{B}_t) < \hat{A}_\nu N \) in correspondence.

Let \( \{\varphi_s\} \) be a convergent sequence of automorphisms that factor through \( CRes_{t_0} \), and let \( \tau_1, \tau_2 \in U_1 \) be distinct automorphisms. Suppose that after passing to subsequences we get two convergent sequences of automorphisms, \( \{\psi_s = \varphi_s \circ \tau_1\} \) and \( \{\eta_s = \varphi_s \circ \tau_2\} \), that factor through the same resolution \( CRes_\nu \), so that for every \( s \), \( \psi_s(\hat{A}_\nu) \) and \( \eta_s(\hat{A}_\nu) \) are all conjugate (elementwise), and \( \psi_s(\hat{B}_\nu) \) and \( \eta_s(\hat{B}_\nu) \) are also all conjugate.

Let \( a_1, \ldots, a_\ell \) be a generating set of \( \hat{A}_{t_0} \), and let \( a_1', \ldots, a_\ell' \) be a generating set of \( \hat{A}_\nu \). For each \( s, 1 \leq i \leq \ell \), \( \varphi_s(a_i) = \psi_s(\tau_1^{-1}(a_i)) = \eta_s(\tau_2^{-1}(a_i)) \). In addition, \( \tau_1^{-1}(\hat{A}_{t_0})N = \tau_2^{-1}(\hat{A}_{t_0})N = A_\nu N \).

Hence, \( \tau_1^{-1}(a_i) = w_i^1 \bar{w}_i^1 \), where \( w_i^1 \in \hat{A}_\nu \) and \( \bar{w}_i^1 \in N, 1 \leq i \leq \ell \). Since \( \psi_s(\tau_1^{-1}(a_i)) = \eta_s(\tau_2^{-1}(a_i)) \) and \( \psi_s(\hat{A}_\nu) \) is conjugate elementwise to \( \eta_s(\hat{A}_\nu) \), it follows that there exists some element \( u \in H \) such that: \( \tau_2^{-1}(a_i) = uw_i^1 u^{-1} \bar{w}_i^1 \), \( 1 \leq i \leq \ell \).

So: \( \tau_2 \circ \tau_1^{-1}(a_i) = va_i v^{-1} w_i, \) for \( v \in H \), and \( w_i \in N \), for 1 \( \leq i \leq \ell \).

The subgroup that is generated by all the elements \( \tau_2^{-1} \circ \tau_1 \), for all the pairs of automorphisms \( \tau_1, \tau_2 \in U_1 \), for which for a subsequence of the automorphisms \( \varphi_s \) (that factors through \( CRes_{t_0} \), the sequences \( \varphi_s \circ \tau_1 \) and \( \varphi_s \circ \tau_2 \) factor through the same resolution, \( CRes_\nu \), and maps the subgroups \( \hat{A}_\nu \) and \( \hat{B}_\nu \) to conjugate subgroups, is of finite index in \( U_1 \), since finitely many cosets of it cover \( U_1 \).

We denote the subgroups that is generated by all these compositions \( \tau_2 \circ \tau_1^{-1} \), \( U_2 \). \( U_2 \) is of finite index in \( Aut(G) \). Every automorphism \( \tau \in U_2 \) preserves the stable kernel \( N \), and maps \( \hat{A}_{t_0} \) and \( \hat{B}_{t_0} \) to their conjugates elementwise modulo the stable kernel \( N \). Hence, \( U_2 \) satisfies parts (i) and (ii) in the conclusion of the proposition.

Let \( \tau \in U_2 \). By composing \( \tau \) with an inner automorphism, we may assume that \( \tau(\hat{A}_{t_0}) = \hat{A}_{t_0} \) elementwise, modulo the kernel \( N \). Furthermore, \( \tau(\hat{B}_{t_0}) = \hat{B}_{t_0} \circ \tau \) modulo the kernel \( N \) for some element \( \hat{c} \in C_{t_0} \). If \( \tau_1 \tau_2 \in U_2 \), then \( \tau_2 \circ \tau_1(\hat{A}_{t_0}) = \hat{A}_{t_0} \) modulo \( N \), and \( \tau_2 \circ \tau_1(\hat{B}_{t_0}) = \tau_2 \circ \tau_1 \circ \tau_2 \circ \tau_1^{-1}(\hat{B}_{t_0}) = \hat{B}_{t_0} \circ \tau_2 \circ \tau_1^{-1}(\hat{B}_{t_0}) = \hat{B}_{t_0} \circ \tau_2 \circ \tau_1^{-1}(\hat{B}_{t_0}) = \hat{B}_{t_0} \) elementwise modulo the kernel \( N \).

Hence, there is a homomorphism from \( U_2 \) into the virtually cyclic subgroup \( C_{t_0} = \hat{C}_{t_0} / N \). Since there is only a single resolution in the higher rank MR diagram that contains a quotient map, images of a fixed set of generators of \( \hat{A}_{t_0} \) and \( \hat{B}_{t_0} \), under all the automorphisms \( \tau \in U_2 \) move a base point in all the other projection spaces a uniformly bounded distance. Since the action of \( H \) on \( X \) is proper, this implies that the kernel of the action from \( U_2 \) to \( C_{t_0} \) is finite. Therefore, \( U_2 \), and hence also \( Aut(G) \), are virtually cyclic, and we get part (iii).

\[ \square \]

Proposition 1.2 proves the existence of a higher rank JSJ decomposition in case of at most a single two steps resolution, with a virtually infinite cyclic edge group, and no QH nor virtually abelian vertex groups in all the \( m \)-collections of cover resolutions in the higher rank MR diagram. The next theorem generalizes the
construction in case there is still only at most one single two steps resolution, but the associated virtually cyclic abelian decomposition is only assumed not to have edges with finite (nor trivial) stabilizers.

**Theorem 1.3.** With the assumptions of lemma 1.1, suppose that in each of the (finitely many) collections of m cover resolutions in the higher rank Makanin-Razborov diagram, there is at most one resolution that has two levels (i.e., a single quotient map), and all the other resolutions in each collection have a single level. Furthermore, assume that the virtually abelian decomposition that is associated with the top level of each two steps resolution contains no edges with a finite (nor trivial) edge group.

Then:

(i) $H$ admits a graph of groups decomposition that is preserved by a finite index subgroup of $\text{Aut}(G)$. If $N$ is the (pointwise) stabilizer of the action of $H$ on the associated Bass-Serre tree, then the associated decomposition of $H/N$ contains $\text{QH}$, f.g. virtually abelian, and rigid vertex groups, and virtually (infinite) cyclic edge groups.

(ii) A finite index subgroup of $\text{Out}(G)$ maps into the direct sum of the mapping class groups of the QH vertex groups in the decomposition, together with outer automorphism groups of the f.g. virtually abelian vertex groups (that fix their virtually abelian edge groups). The kernel of this map is f.g. virtually abelian (cf. [CCV] and [CV] for a similar theorem in the case of RAAGs, and a stronger statement in [Le] for hyperbolic groups).

*Proof:* If there are no two steps resolutions in all the m-collections, $\text{Out}(G)$ is finite by lemma 1.1. Hence, we may assume that at least in one m-collection there is a two steps resolution (i.e., that $\text{Out}(G)$ is infinite).

Given a virtually abelian abelian decomposition with no edges with finite edge groups $\Lambda$, we define its complexity to be a tuple of integers that are ordered lexicographically. The tuple is composed of (in decreasing order):

1. the pairs of tuples: $(-\chi(O_i), g(O_i))$ of the orbifolds that are associated with the QH vertex groups in $\Lambda$, ordered lexicographically in non-increasing order. $\chi(O_i)$ is the Euler characteristic of the orbifold, and $g(O_i)$ is the genus of the underlying surface of each orbifold.
2. the number of virtually abelian vertex groups in $\Lambda$, followed by a non-increasing sequence of the ranks of the maximal rank free abelian group in each of these virtually abelian vertex groups.
3. the number of non-conjugate virtually cyclic edge groups in $\Lambda$ that connect between rigid vertex groups, followed by a non-increasing sequence of the numbers of edges between vertex groups in $\Lambda$ that each conjugacy class stabilizes.

Let $\{CRes_t\}, 1 \leq t \leq d$, be the two steps resolutions in the higher rank Makanin-Razborov diagram. Each automorphism in $\text{Aut}(G)$ factors through one of the resolutions in the higher rank MR diagram. For each rigid vertex group in the virtually cyclic decomposition that is associated with $CRes_t$, we fix a f.g. subgroup in $H$ that is mapped onto it. Each such f.g. subgroup is mapped to a finite number
of subgroups up to conjugacy, by all the automorphisms that factor through \( CRes_t \).

**Definition 1.4.** We say that a sequence of automorphisms in \( \text{Aut}(G) \) that factor through a resolution \( CRes_t \), and converge into an action of \( H \) on a real tree, such that the virtually abelian decomposition that is associated with the action is \( \Lambda_t \), is a weak test sequence of the resolution \( CRes_t \) (cf. theorem 1.18 in [Se2] for a test sequence).

Since we assumed that all the cover resolutions in the higher rank MR diagram have at most two levels, the construction of the diagram guarantees that in case of two levels the cover resolutions have weak test sequences that factor through them.

With each resolution \( CRes_t \), we look at the finitely many tuples of possible conjugacy classes of the images of the rigid vertex groups, and the virtually cyclic edge groups in \( \Lambda_t \), that extend to weak test sequences of \( CRes_t \). Since the limit group that is associated with a weak test sequence that factor through \( CRes_t \), is obtained from the rigid vertex groups by adding finitely many generators and relations, and since there are only finitely many possible conjugacy classes of images of f.g. subgroups of \( H \) that map onto each of the rigid vertex groups, there are only finitely many possibilities for the stable kernel \( N \) for all the weak test sequences that factor through a resolution \( CRes_t \), \( 1 \leq t \leq d \).

Let \( \{ \varphi_i \} \) be a weak test sequence for a maximal complexity resolution \( CRes_t \). For every automorphism \( \tau \in \text{Aut}(G) \), a subsequence of the sequence, \( \{ \varphi_i \circ \tau \} \), is a weak test sequence of some maximal complexity resolution \( CRes_t' \). Hence, \( \tau \) permutes the finite set of stable kernels of weak test sequences of maximal complexity resolutions. Therefore, a finite index subgroup \( U_1 < \text{Aut}(G) \) fixes these finite set of stable kernels elementwise.

Since the automorphisms \( \tau \in U_1 \) fix the stable kernels in the limit groups that are associated with convergent sequences \( \{ \varphi_i \circ \tau \} \), each such automorphism restricts to an isomorphism between these limit groups. Since for any subgroup \( D < H \), \( \varphi_s(D) = \varphi_s \circ \tau(\tau^{-1}(D)) \), it follows that the restriction of an automorphism \( \tau \) to an isomorphism of the limit groups of weak test sequences, permutes the conjugacy classes of edge groups, rigid vertex groups, virtually abelian vertex groups, and QH vertex groups.

As in the proof of proposition 1.2, this implies that there exists a finite index subgroup \( U_2 < U_1 \), that preserves the conjugacy classes of these edge and vertex groups in the limit group that is associated with the weak test sequence \( \{ \varphi_s \} \). We denote the limit group \( L \) and its virtually abelian decomposition \( \Lambda \).

Hence, every automorphism \( \tau \in U_2 \) restricts to outer automorphisms in the mapping class groups of the QH vertex groups in \( \Lambda \), and to outer automorphisms of the virtually abelian vertex groups in \( \Lambda \). Therefore, we get a map \( \eta \) from \( U_2 \) to the direct sum of the mapping class groups of QH vertex groups and the outer automorphism groups of virtually abelian vertex groups in \( \Lambda \).

We continue with the kernel of this map \( \eta \). Each automorphism from the kernel of \( \eta \) preserves the stable kernel \( N \) of the original sequence \( \{ \varphi_s \} \) and maps each rigid vertex group and each edge group in \( \Lambda \) to a conjugate modulo the stable kernel, and each QH vertex group and each virtually abelian vertex group are mapped to a conjugate modulo the kernel as well.

Therefore, the higher rank Makanin-Razborov diagram that is associated with the kernel of \( \eta \) contains finitely many \( m \)-collections of resolutions, where in each
There is at most a single resolution with a (single) quotient map. Furthermore, the abelian decompositions that are associated with these resolutions have (infinite) virtually cyclic edge groups, and no QH nor virtually abelian edge groups. Hence, by the proof of proposition 1.2, the kernel of $\eta$ is virtually abelian. Furthermore, there is exists a virtually cyclic decomposition of a quotient of $H$, for which a finite index subgroup of the kernel of $\eta$ preserves the splitting. 

Note that we chose a weak test sequence w.r.t. the higher rank virtually abelian decomposition in order to define the restrictions of each automorphism $\tau \in U_2$ to each of the QH and virtually abelian vertex groups in that decomposition. However, it is not difficult to verify that the conjugacy classes of the restrictions do not depend on the choices of the weak test sequence.

We further generalize theorem 1.3, and allow the single two step resolution in each $m$-collection to have a virtually abelian decomposition that includes free splittings and splittings over finite edge groups.

**Theorem 1.5.** With the assumptions of lemma 1.1, suppose that in each of the (finitely many) collections of $m$ cover resolutions in the higher rank Makanin-Razborov diagram, there is at most one resolution that has two levels (i.e., a single quotient map), and all the other resolutions in each collection have a single level. Then:

(i) $H$ admits a graph of groups decomposition that is preserved by a finite index subgroup of $\text{Aut}(G)$. If $N$ is the (pointwise) stabilizer of the action of $H$ on the associated Bass-Serre tree, then the associated decomposition of $H/N$ contains QH, f.g. virtually abelian, and rigid vertex groups, and virtually cyclic (possibly finite or trivial) edge groups.

(ii) A finite index subgroup of $\text{Out}(G)$ maps into the direct sum of the mapping class groups of the QH vertex groups in the decomposition, together with outer automorphism groups of the f.g. virtually abelian vertex groups (that fix their virtually abelian edge groups). If there exist QH or virtually abelian vertex groups in the decomposition, then the image has to be an infinite group.

(iii) When the invariant virtually abelian decomposition of $H/N$ contains no QH and no virtually abelian vertex groups, but contains virtually infinite cyclic edge groups, then a finite index subgroup of $\text{Out}(G)$ maps into the direct sum of the virtually cyclic edge groups in the decomposition, and the image is a virtually (infinite) f.g. abelian group.

**Proof:** If all the two steps resolutions in the higher rank Makanin-Razborov diagram, do not have associated virtually abelian decomposition with free decompositions nor with edges with finite edge groups, the conclusion of the theorem follows from theorem 1.3. Hence, we may assume that virtually abelian decompositions with free splittings or with edges with finite edge groups are associated with some of the resolutions in the higher rank MR diagram.

To deal with such virtually abelian splittings we need to generalize the definition of the complexity of a two step resolution that was previously defined in the course
of proving theorem 1.3.

**Definition 1.6.** In definition 2.1 in [Se4] we defined the taut structure and the rank of a taut resolution over a free group. We first modify the definition to a two step resolution in our setting.

Let \( L_1 \rightarrow L_2 \) be a two step resolution, and let \( \Lambda \) be the virtually abelian decomposition that is associated with \( L_1 \). Following definition 2.1 in [Se4] with the QH vertex groups in \( \Lambda \) we can associate finitely many taut structures. With a taut structure we can naturally associate a graph of groups, \( \Lambda_T \), by cutting the orbifolds that are associated with the QH vertex groups along s.c.c. that are associated with the taut structure. We assume that \( \Lambda_T \) is a reduced graph of groups. We define the Kurosh rank of the taut structure to be the number of edges with finite or trivial edge groups in \( \Lambda_T \).

Given an \( m \) collection of cover resolutions from the MR diagram, we associate with it finitely many taut structures of its associated virtually abelian decompositions, by enumerating all the possible taut structures of the QH vertex groups in the virtually abelian decompositions that are associated with the \( m \) resolutions.

We continue by associating a complexity of an \( m \)-collection of cover resolutions that generalizes the one that appears in the proof of theorem 1.3. We first define the complexity of each resolution, and then of the \( m \)-collection. The complexity of a resolution is a tuple of integers, that are ordered lexicographically. We start with the more significant terms and go down to the less significant ones.

1. the maximum of the Kurosh rank over all the finitely many possible taut structures of the virtually abelian decomposition that is associated with the resolution.
2. parts (1)-(3) in the definition of the complexity of a resolution as it appears in the proof of theorem 1.3.

The complexity of an \( m \)-collection of resolutions is simply the tuple of the complexities of the \( m \) resolutions in the collection in a non-increasing order.

The set of complexities of resolutions is well-ordered. In particular, the set of the complexities of the finitely many two steps resolutions in the higher rank MR diagram have a maximum. Let \( \{CRes_t\}, 1 \leq t \leq d \), be the collection of two steps resolutions in the higher rank Makanin-Razborov diagram (that have a weak test sequence that factors through them), that have maximal complexity.

We continue as in the proof of theorem 1.3. First, for each each rigid vertex group in the virtually cyclic decomposition that is associated with \( CRes_t \), we fix a f.g. subgroup of \( H \) that is mapped onto it. Each such f.g. subgroup is mapped to a finite number of subgroups up to conjugacy, for all the possible weak test sequences of \( CRes_t \).

With each maximal complexity taut resolution \( CRes_t \), we look at the finitely many tuples of possible conjugacy classes of the images of the rigid vertex groups, and the virtually cyclic edge groups in \( \Lambda_t \), that extend to weak test sequences of \( CRes_t \). Since the limit group that is associated with a weak test sequence that factor through \( CRes_t \), is obtained from the rigid vertex groups by adding finitely many generators an relations, and since there are only finitely many possible conjugacy classes of images of f.g. subgroups of \( H \) that map onto each of the rigid vertex groups, there are only finitely many possibilities for the stable kernel \( N \) for all the
weak test sequences that factor through a maximal complexity resolution $CRes_t$, $1 \leq t \leq d$.

Let $\{\varphi_s\}$ be a weak test sequence that is taut with respect to a maximal complexity resolution $CRes_{t_0}$. For every automorphism $\tau \in Aut(G)$, a subsequence of the sequence, $\{\varphi_s \circ \tau\}$, is a weak test sequence of some maximal complexity taut resolution $CRes_{t'}$. Hence, $\tau$ permutes the finite set of stable kernels of weak test sequences of maximal complexity resolutions. Therefore, a finite index subgroup $U_1 < Aut(G)$ fixes this infinite set of stable kernels elementwise.

The rest of the proof that there exists a virtually abelian splitting of a quotient of $H$ that is invariant under a finite index subgroup $U_2 < Out(G)$, is identical to the argument that appears in the proof of theorem 1.3. We denote the virtually abelian decomposition that $U_2$ preserves, $\Lambda$.

In particular, $U_2$ preserves the conjugacy classes of the QH vertex groups and the virtually abelian vertex groups, and the edge groups that are connected to them in the invariant graph of groups that $U_2$ preserves $\Lambda$. Hence, each automorphism in $U_2$ restricts to an outer automorphism of each of the QH vertex groups and each of the virtually abelian vertex groups in the invariant virtually abelian decomposition. Therefore, we get a map $\eta_1$ from $U_2$ into the direct sum of the outer automorphism groups of the QH and the virtually abelian vertex groups in $\Lambda$.

If $\Lambda$ contains no QH and no virtually abelian vertex groups, then automorphisms in $U_2$ restrict to automorphisms of the fundamental groups of connected subgraphs of $\Lambda$, that are obtained by deleting all the edges in $\Lambda$ that have finite or trivial edge groups. By the proof of theorem 1.3, this gives a map from $U_2$ into the direct sum of the virtually infinite edge groups in $\Lambda$, hence, we get a map from $U_2$ into a f.g. virtually abelian group.

Note that in case of free splittings or splittings over finite edge groups, the map of $U_2$ into the direct sum of the mapping class groups of orbifolds and of the outer automorphism groups of the f.g. virtually abelian vertex groups, is not guaranteed to have a f.g. virtually abelian kernel, because the kernel may contain automorphisms of multi-ended groups. It may also be the case that an MR diagram that is associated with automorphisms that belong to the kernel of this map may have resolutions with more than two levels, and at this stage we didn’t analyze such MR diagrams. Hence, the conclusion of theorem 1.5 is stated in a somewhat weaker form than the conclusion of theorem 1.3.

So far we have assumed that the $m$-collections of resolutions in the higher rank Makanin-Razborov diagram that we associated with the subgroup $H$, contain at most a single resolution with two levels, and all the other resolutions in each $m$-collection have a single level. In the next step we still assume that in each $m$-collection in the higher rank MR diagram at most one (cover) resolution is not of a single level, but this one resolution can have arbitrarily many levels.

In this case our goal is to show that the higher rank MR diagram can be replaced by another higher rank diagram in which in every $m$-collection of cover resolutions, all but possibly a single resolution have a single level, and the single resolution with more than one level has at most two levels (although the resolution with two levels may not admit a weak test sequence). Even though it may be that the new diagram with two steps resolutions will have no weak test sequence, it will be eventually possible to associate with such a higher rank MR diagram a higher rank
JSJ decomposition similar to the one that was constructed in theorem 1.5.

Note that such a goal can not be valid when analyzing homomorphisms. In fact the multi-layer structure of a general Makanin-Razborov diagram is basic in the whole theory, and distinguishes the general MR diagram from a JSJ decomposition. But in analyzing automorphisms it is possible to improve the MR diagram to have resolutions with at most two levels, and afterwards replace the diagram with a higher rank JSJ decomposition. This is the basic principle of our whole approach.

**Proposition 1.7.** With the assumptions of lemma 1.1, suppose that in each of the (finitely many) collections of m cover resolutions in the higher rank Makanin-Razborov diagram, there is at most one resolution that has more than a single level. Suppose further that:

(i) all the virtually abelian decompositions that are associated with the various levels of the resolutions that have more than a single level, contain only a single edge with virtually (infinite) abelian edge group, and no QH nor virtually abelian vertex groups.

(ii) all the quotient maps in these resolutions are proper quotients.

(iii) a virtually abelian edge group in the virtually abelian decomposition that is associated with one of the levels of these resolutions is not elliptic in the virtually abelian decomposition that is associated with the next level of the resolution, except for the terminal quotient map.

Then:

(1) all the resolutions in the higher rank MR diagram that have more than a single level, have exactly two levels, and the virtually abelian decomposition that is associated with them contains a single edge with a virtually (infinite) cyclic edge group.

(2) parts (i)-(iii) of theorem 1.5 hold for H and Aut(G). In particular, H admits a higher rank JSJ decomposition.

**Proof:** Let $\text{CRes}$ be one of the resolutions with more than a single level in one of the $m$-collections in the higher rank MR diagram. Let $L$ be the limit group that is associated with the resolution $\text{CRes}_t$. Then $L$ inherits a virtually abelian decomposition from the top level of the resolution. Each vertex group in this graph of groups decomposition inherits a virtually abelian decomposition from the virtually abelian decomposition that is associated with the second level of the resolution, and we continue iteratively.

Since all the edge groups in the virtually abelian decompositions that are associated with the various levels of $\text{CRes}$ are virtually cyclic, all the vertex groups and edge groups along this iterative inherited decompositions are f.g. Let the terminal vertex groups (after the iterative decompositions) be $A_1, \ldots, A_\ell$.

Since in the $m$-collection that contains $\text{CRes}, \text{CRes}$ is the only resolution with more than a single level, for each terminal vertex group $A_i$, there are only finitely many possible images in $H$, such that any automorphism that factors through $\text{CRes}$ sends $A_i$ to one of these images up to conjugation (elementwise).

Let $\{\varphi_s\}$ be a weak test sequence that factors through the $m$-collection that contains $\text{CRes}$, and converges to some quotient of $L$ that we denote $\hat{L}$. Let $N$ be the kernel of the quotient map from $H$ onto $\hat{L}$. Since the number of images of each of the groups $A_i$ is finite up to conjugation, there are only finitely many
possibilities for the kernel $N$ in $H$ for all the weak test sequences that factor through the $m$-collection that contains $CRes$.

We define the complexity of a resolution that satisfies properties (i)-(iii) in the statement of the proposition, to be the number of quotient maps along the resolution.

We look only at resolutions with maximal complexity in the diagram, $CRes_t$, $1 \leq t \leq d$. Let $\{\varphi_s\}$ be a weak test sequence for such a maximal complexity resolution $CRes_{t_0}$. Let $\tau \in Aut(G)$. Because of the universality of the higher rank MR diagram, a subsequence of the sequence: $\{\varphi_s \circ \tau\}$ factors through one of the $m$-collections in the diagram, and in particular through one of the resolutions $CRes$ in this $m$-collection.

Since the sequence $\{\varphi_s\}$ is a weak test sequence of the maximal complexity resolution $CRes_{t_0}$, it subconverges to an action of the completion $Comp(CRes_{t_0})$ on a real tree, where the subgroups that are associated with the various levels in $Comp(CRes_{t_0})$ act on corresponding real trees as well. The virtually abelian decompositions that are associated with these actions have two vertices and a single edge, where the edge group is virtually cyclic, and one of the two vertex groups is virtually abelian with a finite index subgroup isomorphic to a free abelian group of rank 2. Furthermore, by the assumptions of the proposition, the edge group in the abelian decomposition that is associated with a level is not elliptic in the virtually abelian decomposition that is associated with the next level.

We denote the limit quotient of $Comp(CRes_{t_0})$ that corresponds to the limit action by $T$. This limit group has a tower structure similar to the tower structure of the completion $Comp(CRes_{t_0})$, just the base group may be replaced by a proper quotient.

A subsequence of the sequence $\{\varphi_s \circ \tau\}$ factors through a resolution $CRes$. Hence, the subsequence extends to a subsequence of quasi-actions of the completion, $Comp(CRes)$, on the corresponding hyperbolic projection space $V_j$. As in the proof of the generalized Merzlyakov theorem in the first section in [Se2], by possibly shorten the (extended) quasimorphisms without changing the images of $H$ under the subsequence $\{\varphi_s \circ \tau\}$, the images of the quasimorphisms from $Comp(Res)$ converge into a subgroup of some closure of $T$, $Cl(T)$ (for the notion of a closure see the first section in [Se2]). The closure, $Cl(T)$, is obtained from $T$ by possibly adding roots to virtually abelian vertex groups, and possibly extend the terminal limit group of the completion. In particular, we get a map: $\eta : Comp(CRes) \rightarrow Cl(T)$.

Note that by construction the original f.g. group $H$ is mapped onto the limit groups that are associated with the top levels of the completions, $Comp(CRes)$ and $Comp(CRes_{t_0})$. Since $Cl(T)$ was constructed from subsequences of $\{\varphi_s\}$ and $\varphi_s \circ \tau$, it follows that for each $h \in H$, the image of $h$ in $Comp(CRes)$ is mapped by $\eta$ to the image of $\tau(h)$ in $Comp(CRes_{t_0})$.

Both of the completions, $Comp(CRes)$ and $Comp(CRes_{t_0})$, are towers that are built from a base (terminal) subgroup, to which a single virtually abelian vertex group is added in each level along a virtually (infinite) cyclic edge group. Furthermore, by our assumptions, each of the virtually cyclic edge groups in the various levels of the completion, $Comp(Res_{t_0})$, and hence in the tower $T$, are not elliptic in the virtually abelian decomposition that is associated with the next level.

$CRes_{t_0}$ is a maximal complexity resolution, i.e., it has the maximal number of levels among all the resolutions in the $m$-collections in the higher rank MR diagram. Therefore, the existence of the map $\eta : Comp(CRes) \rightarrow Cl(T)$ that maps the base
level and the top level limit groups in $\text{Comp}(CRes)$ into the base level and the top level limit groups in $Cl(T)$, implies that $CRes$ has to be of maximal complexity as well.

Therefore, the automorphisms in $\text{Aut}(G)$ permute the finitely many possible kernels in $H$ of all the possible weak test sequences of maximal complexity resolutions in the $m$-collections of the higher rank MR diagram. Hence, a finite index subgroup $U < \text{Aut}(G)$ preserves these kernels.

Let $N$ be the kernel in $H$ that is preserved by $U < \text{Aut}(G)$. Let $C_1, \ldots, C_r$ be the virtually cyclic subgroups that are the edge groups in the virtually abelian decompositions that are associated with the top levels of the maximal complexity resolutions in the higher rank MR diagrams. Then the automorphisms in $U$ permute the subgroups $C_1N, \ldots, C_rN$.

Therefore, for each fixed element $h$ in one of these subgroups, and for every $\tau \in U$, there is a global bound on the traces of the elements $\tau(h)$ when acting on the projection space that is associated with the maximal complexity resolution. Hence, the intersection between conjugates of the subgroups edge groups $C_1N, \ldots, C_rN$ and $H$ have to be elliptic in all the levels of a maximal complexity resolution. So by our assumptions, a maximal complexity resolution can not have more than two levels, and the proposition follows.

The next proposition generalizes proposition 1.7 to the case in which quotient maps are still proper, but edge groups in the virtually abelian decomposition that is associated with one level need not be elliptic in the next level, and there can be more than a single edge in each level.

**Theorem 1.8.** Suppose that only assumptions (i)-(ii) (but not necessarily assumption (iii)) in proposition 1.7 hold for the resolutions in the $m$-collections in the higher rank MR diagram. We also allow the number of edges with infinite virtually abelian edge groups in each level to be arbitrary (and not necessarily only 1). Then:

1. The resolutions with more than a single level in the higher rank MR diagram can be replaced by a resolution with two levels, in which each edge group is virtually (finite or infinite) cyclic, and vertex groups are either rigid or virtually f.g. abelian.
2. parts (i)-(iii) of theorem 1.5 hold for $H$ and $\text{Aut}(G)$. In particular, $H$ admits a higher rank JSJ decomposition.

**Proof:** By the construction of the higher rank MR diagram in [Se8], since we assumed that all the quotient maps in the resolutions in the $m$-collections in the diagram are proper quotients, for each $m$ collection of resolutions in the higher rank diagram there exists a weak test test sequence of automorphisms that factor through it. Furthermore, in case all the quotient maps are proper the resolutions are strict (definition 5.9 in [Se1]).

With each resolution with more than a single level in the higher rank diagram, $CRes$, we associate its completion, $\text{Comp}(CRes)$ (see section 1 in [Se2] for the construction of a completion). Note that by [Se2] completions can be associated with strict resolutions, and by our assumptions all the resolutions $CRes$ are strict.

By construction, the number of levels in the completion, $\text{Comp}(CRes)$, is identical with the number of levels (or the number of quotient maps plus 1) in the resolution $CRes$. At this stage we gradually modify the structure of the virtually
abelian decompositions that are associated with the various levels of the completion, \( \text{Comp}(CRes) \), without changing the (limit) group that is associated with \( \text{Comp}(Res) \).

If the completion has only two levels, we do not change it. We start with the virtually abelian decomposition that is associated with the third level of the completion (counting from its bottom terminal level). If a virtually abelian edge group in the virtually abelian decomposition that is associated with the third level of the resolution \( CRes \) is elliptic in the virtually abelian decomposition that is associated with the second level in \( CRes \), we move the virtually abelian vertex group and the edge group that is connected to it, and both are associated with that edge group in the third floor of the completion, to the second floor of the completion. Note that we can move these vertex and edge groups since the edge group is assumed to be elliptic in the virtually abelian decomposition that is associated with the second level in the resolution \( CRes \).

We continue iteratively, from bottom to top. At each step \( \ell \) we move all the pairs of a virtually abelian vertex group and its associated edge group that appear in the \( \ell \)-th level of the completion, that are associated with edge groups that are elliptic in the virtually abelian decomposition that is associated with level \( \ell - 1 \) in the modified completion \( \text{Comp}(CRes) \) (i.e., in the completion \( \text{Comp}(CRes) \) after we modified its bottom \( \ell - 1 \) levels), into the \( \ell - 1 \) level of the modified completion. And if the edge group is elliptic in further lower level virtually abelian decompositions, we further move the virtually abelian vertex group and the edge group that is connected to it, to the lowest possible level. i.e., to a level for which the edge group is not elliptic in the virtually abelian decomposition in the level below it.

At the end of the modification procedure, we did not change the group \( \text{Comp}(CRes) \), but we possibly modified the virtually abelian decompositions of the original completion by moving some virtually abelian vertex and edge groups to lower levels.

By construction, all the edge groups in the virtually abelian decompositions that are associated with the modified completion have (infinite) virtually cyclic edge groups. Furthermore, every edge group in a virtually abelian decomposition which is not the lowest one, is not elliptic in the virtually abelian decomposition that is associated with the next (lower) level.

Also, the existence of weak test sequences of the original completion guarantee that there exist weak test sequences for the modified completion.

So far we modified the completions of the resolutions so that a virtually cyclic edge group in a virtually abelian decomposition is not elliptic in the next level virtually abelian decomposition. However, it may still be that some of the virtually abelian vertex groups are not really essential, or that the degree of their maximal free abelian subgroups can be further decreased. To get rid of these redundancies, and possibly reduce the number of virtually abelian vertex groups, or reduce their ranks, we use another procedure, that is conceptually related to the auxiliary resolutions that were introduced in definition 2.1 in [Se5], although it is technically a different procedure.

With each of the resolutions, \( CRes \), in the higher rank MR diagram we associate its completion, \( \text{Comp}(CRes) \). The construction of the completion of a well-structured resolution appears in the first section of [Se2].

The limit group \( L \) (with which we started the original resolution \( CRes \)) is now mapped into the completion of the resolution. We denote by \( \nu(L) \) the image of
the limit group in the completion, \( \text{Comp}(CRes) \) Also, each resolution \( CRes \) has a weak test sequence, hence, each of the completions, \( \text{Comp}(CRes) \) has a weak test sequence, that extends a weak test sequence of the resolution \( CRes \).

For each of the completions, \( \text{Comp}(CRes) \), we define its complexity. The complexity of a completion, \( \text{Comp}(CRes) \), is a tuple of integers ordered lexicographically in the following order:

1. the number of levels in the completion.
2. a tuple of integers for each virtually abelian decomposition that appears along the completion, \( \text{Comp}(CRes) \), going from top to bottom. The tuple of integers consists of the number of edge groups in the virtually abelian decomposition that is associated with the level in the completion, \( \text{Comp}(CRes) \), followed by the ranks of the virtually abelian vertex groups in the virtually abelian decomposition that is associated with the level in \( \text{Comp}(CRes) \), ordered in a non-increasing order.
3. a tuple of integers for each virtually abelian decomposition that appears along the completion, \( \text{Comp}(CRes) \), going from top to bottom, that consists of the number of virtually infinite cyclic edge groups in each level of the completion, that can be conjugated into the image of the limit group \( \nu(L) \).

Note that complexities of resolutions are well-ordered, and the higher rank MR diagram has only finitely many resolutions, so the higher rank MR diagram contains resolutions of maximal complexity. These will be eventually used to construct the higher rank JSJ decomposition.

Let \( \{ \varphi_s \} \) be a weak test sequence of automorphisms in \( \text{Aut}(G) \) for the resolution \( CRes \). \( \{ \varphi_s \} \) extends to a weak test sequence of the completion, \( \text{Comp}(CRes) \). Let \( \tau \in \text{Aut}(G) \). A subsequence of \( \{ \varphi_s \circ \tau \} \) factors through one of the \( m \)-collection in the higher rank MR diagram. Let \( CRes_1 \) be the non-trivial resolution in this \( m \)-collection (by our assumption every \( m \)-collection contains at most one non-trivial resolution).

By the construction of formal solutions and formal limit groups in \([Se2]\), from the sequences \( \{ \varphi_s \} \) and \( \{ \varphi_s \circ \tau \} \) it is possible to pass to a further subsequence that converges to a map: \( \nu_\tau : \text{Comp}(CRes_1) \to \text{Cl}(CRes) \), where \( \text{Cl}(CRes) \) is a closure of the completion \( \text{Comp}(CRes) \) (for the notion of a closure see section 1 in \([Se2]\)).

Suppose that both \( CRes \) and \( CRes_1 \) are resolutions of maximal complexity. This means that both have the same number of levels, the same number of edges in each level, and the same ranks of virtually abelian groups in each of the levels. Suppose further that the image of \( \nu_\tau \) contains a conjugate of a finite index subgroup of each of the virtually abelian vertex group in the closure \( \text{Cl}(CRes) \).

In this case we can use the same argument that was used in the proof of proposition 1.7, and deduce that the map \( \nu_\tau \) maps edge groups and virtually abelian vertex groups in each level in \( \text{Comp}(CRes_1) \) into edge groups and finite index subgroups of virtually abelian groups in the same level in the closure, \( \text{Cl}(CRes) \).

Hence, suppose that there exists a maximal complexity resolution in the higher rank MR diagram, \( CRes \), for which there exists a weak test sequence, \( \{ \varphi_s \} \), such that for every automorphism \( \tau \in \text{Aut}(G) \), every subsequence of \( \{ \varphi_s \circ \tau \} \) that factor through a fixed resolution, factors through a maximal complexity resolution, \( CRes_{i(\tau)} \), and the map \( \nu_\tau : \text{Comp}(CRes_{i(\tau)}) \to \text{Comp}(CRes) \) (that is constructed from convergent subsequences of \( \{ \varphi_{s_i} \} \) and \( \{ \varphi_{s_i} \circ \tau \} \)) has an image that contains a conjugate of a finite index subgroup of every virtually abelian vertex group in the
completion $\text{Comp}(\text{Res})$.

In this case we look only at such sequences $\{\varphi_s\}$ w.r.t. all the maximal complexity resolutions in the higher rank MR diagram. If there exists an edge group in a level that is above the bottom two levels in one of these maximal complexity resolutions, that has a conjugate that intersects the limit group $L$ in a subgroup of finite index, then as in the proof of proposition 1.7, a finite index subgroup of $\text{Aut}(G)$ preserves the conjugacy class of the subgroup of $H$ that is mapped onto this edge group. This contradicts the assumption that the edge group is in a level that is above the bottom two levels, and is hyperbolic in the level below it, which is not the bottom level.

Hence, in this case the image of $L$ does not intersect any conjugate of an edge group in a level above the bottom two levels in a subgroup of finite index. Therefore, $L$ inherits splittings over finite edge groups from all the levels above the two bottom ones in the maximal complexity resolutions, so the maximal complexity resolutions in the higher rank MR diagram can be replaced by resolutions with at most two levels, and the conclusion of the theorem follows from theorem 1.5.

Therefore, for the rest of the argument, we assume that for every maximal complexity resolution, $\text{CRes}$, and every weak test sequence, $\{\varphi_s\}$, that factors through it, there is an automorphism $\tau \in \text{Aut}(G)$, such that a subsequence of $\{\varphi_s \circ \tau\}$ factors through some resolution $\text{CRes}_1$, and after passing to convergent subsequences, the image of the associated map: $\nu_\tau : \text{Comp}(\text{CRes}_1) \to \text{Cl}(\text{CRes})$ (for some closure $\text{Cl}(\text{CRes})$ of $\text{CRes}$), intersects the conjugates of some virtually abelian vertex group in $\text{Cl}(\text{CRes})$ in subgroups of conjugates of some fixed subgroup of infinite index of the virtually abelian vertex group in $\text{Cl}(\text{CRes})$.

In this case our aim is to show that the maximal complexity resolution $\text{CRes}$ can be replaced by finitely many resolutions with strictly smaller complexity. Since every decreasing sequence of complexities of resolutions terminates after finitely many steps, an iterative modification of the higher rank MR diagram, i.e., an iterative replacement of maximal complexity resolutions by strictly lower complexity ones, concludes the proof of the theorem.

Let $\text{CRes}$ be a maximal complexity resolution in the higher rank MR diagram, and suppose that for every weak test sequence of $\text{CRes}$, $\{\varphi_s\}$, there exists some automorphism $\tau \in \text{Aut}(G)$, that depends on the weak test sequence, such that a subsequence of $\{\varphi_s \circ \tau\}$ factor through a resolution $\text{CRes}_1$ from the higher rank diagram, and after passing to convergent subsequences the map: $\nu_\tau : \text{Comp}(\text{CRes}_1) \to \text{Cl}(\text{CRes})$ (for some closure $\text{Cl}(\text{CRes})$ of $\text{CRes}$), has an image that intersects conjugates of at least one of the virtually abelian vertex groups along the closure $\text{Cl}(\text{CRes})$ in subgroups of conjugates of a fixed infinite index subgroup of that virtually abelian vertex group.

Given a weak test sequence of $\text{CRes}$, an automorphism $\tau \in \text{Aut}(G)$, and the (proper) image of $\text{Comp}(\text{CRes}_1)$ in $\text{Cl}(\text{CRes})$, we look at the highest level in $\text{Cl}(\text{CRes})$ that contains a virtually abelian vertex group $VA$ such that $\nu_\tau(\text{Comp}(\text{CRes}_1))$ intersects conjugates of $VA$ in subgroups of conjugates of some fixed infinite index subgroup of $VA$, that we denote $\hat{VA}$.

By possibly enlarging $\hat{VA}$, we may assume that $\hat{VA}$ contains the edge group that is connected to $VA$ (even after this enlargement $\hat{VA}$ is of infinite index in $VA$). We set $\hat{\text{Cl}}$ to be the subgroup of $\text{Cl}(\text{CRes})$, that is obtained from $\text{Cl}(\text{CRes})$ by replacing the virtually abelian vertex group $VA$ by its subgroup $\hat{VA}$, and leaving
all the other vertex groups and virtually abelian decompositions along the various levels unchanged.

By our assumptions the subsequence of the weak test sequence \( \{ \varphi_s \} \) that was used to construct the closure, \( \hat{C}l(CRes) \), asymptotically factors through \( \hat{C}l \), i.e., all the automorphisms in the subsequence factor through \( \hat{C}l \) except at most finitely many. By construction, the complexity of \( \hat{C}l \) is strictly smaller than the complexity of \( CRes \), since we reduced the rank of one of its virtually abelian vertex groups.

We assumed that for every weak test sequence \( \{ \varphi_s \} \) of \( CRes \) it is possible to find an automorphism \( \tau \), that enables us to construct a modified closure, \( \hat{C}l \), with strictly smaller complexity. Clearly, there are only countable possible modified closures, \( \hat{C}l \). Hence, by ordering the countable set of possible modified closures, and applying the compactness argument that was used to construct the higher rank MR diagram in section 3 of [Se8], there exists finitely many modified closures: \( \hat{C}l_1, \ldots, \hat{C}l_r \), with the following properties:

1. all the modified closures have strictly smaller complexity than the maximal complexity of the resolutions in the original higher rank MR diagram.
2. every weak test sequence \( \{ \varphi_s \} \) of a maximal complexity resolution, \( CRes \), in the original higher rank MR diagram, has a subsequence that factors through one of the modified closures: \( \hat{C}l_1, \ldots, \hat{C}l_r \).

In order to replace the maximal complexity resolutions in the original higher rank MR diagram by finitely many resolutions with strictly smaller complexity, we need to further consider all the automorphisms \( \sigma \in Aut(G) \), that factor only through maximal complexity resolutions in the (original) higher rank MR diagram, and do not factor through any of the modified closures, \( \hat{C}l_1, \ldots, \hat{C}l_r \). W.l.o.g. we may consider such automorphisms that factor through a fixed maximal complexity resolution, \( CRes \).

Every automorphism \( \sigma \in Aut(G) \) that factors through \( CRes \), extends to a homomorphism of the completion, \( Comp(CRes) \), into the isometry group of the corresponding factor, \( X_j \), of the product space \( X \). We look at the set of all the extensions of the automorphisms \( \sigma \) to homomorphisms of \( CRes \), for automorphisms of \( \sigma \) that do do not factor through the modified closures and through the resolutions that are not of maximal complexity in the higher rank MR diagram.

The set of such extensions does not contain a weak test sequence of \( CRes \), since otherwise a subsequence of the weak test sequence factor through one of the modified closures.

We look at sequences of extension of such automorphisms, \( \sigma \in Aut(G) \), to homomorphisms of \( Comp(CRes) \). We can pass to a convergent subsequence. The convergent subsequence can not be a weak test sequence of \( Comp(CRes) \), so it must converge into a proper quotient of \( Comp(CRes) \), that we denote \( QC \). The map between \( Comp(CRes) \) and its proper quotient \( QC \) must have at least one of the following properties:

(i) the rank of the image in \( QC \) of some virtually abelian vertex group in \( Comp(CRes) \) is strictly smaller.

(ii) the images in \( QC \) of two non-conjugate virtually abelian vertex groups in \( Comp(CRes) \) are conjugate in \( QC \).

(iii) the image in \( QC \) of an edge group in \( Comp(CRes) \) can be conjugated in \( QC \) into the image of the subgroup that is associated with the lower levels in \( Comp(CRes) \). In that case, in the sequence of virtually abelian decom-
positions that are associated with $QC$, the virtually abelian vertex group that is connected to the image of that edge group can be pushed into a lower level.

The limit $QC$ can be given the structure of a completion, and in all the 3 cases, the complexity of this completion has to be strictly smaller than the complexity of $CRes$. There can be only countably many covers of such proper quotients $QC$ of $Comp(Res)$. Hence, by the compactness argument that was used in constructing the higher rank MR diagram in section 3 in [Se8], there exist finitely many such covers, $QC_1, \ldots, QC_h$, all proper quotients of $Comp(CRes)$, all with strictly smaller complexity than $Comp(CRes)$, with the property that all the automorphisms $\sigma \in Aut(G)$ that factor through $CRes$, but do not factor through the finitely many modified closures, $Cl_1, \ldots, Cl_r$, nor through any of the resolutions in the higher rank MR diagram that are not of maximal complexity, do factor through at least one of the quotients, $QC_1, \ldots, QC_h$.

Therefore, in case for every maximal complexity resolution, $CRes$, and every weak test sequence, $\{\varphi_s\}$, that factors through it, there is an automorphism $\tau \in Aut(G)$, such that a subsequence of $\{\varphi_s \circ \tau\}$ factors through some resolution $CRes_1$, and after passing to convergent subsequences, the image of the associated map: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$ (for some closure $Cl(CRes)$ of $CRes$), intersects the conjugates of some virtually abelian vertex group in $Cl(CRes)$ in subgroups of conjugates of some fixed subgroup of infinite index of the virtually abelian vertex group in $Cl(CRes)$, we can replace the original higher rank MR diagram with a new higher rank diagram, such that all the automorphisms in $Aut(G)$ factor through the new diagram, and the maximal complexity of a resolution in the new diagram is strictly smaller than the maximal complexity of a resolution in the original diagram.

The new diagram consists of the modified closures, $Cl_1, \ldots, Cl_r$, together with the quotient completions, $QC_1, \ldots, QC_h$, and the resolutions in the original higher rank diagram that do not have maximal complexity. By construction, the new diagram has the universal property for automorphisms in $Aut(G)$, and the complexities of all the (non-trivial) resolutions in it are strictly bounded by the maximal complexity of resolutions in the original higher rank MR diagram.

Iteratively replacing the higher rank MR diagram, and strictly reducing the maximal complexity of its resolutions, we get a higher rank diagram, for which a further complexity reduction is not possible. i.e., we get a diagram with maximal complexity resolutions and weak test sequences that factor through them, for which no automorphism $\tau \in Aut(G)$ enables one to apply the procedure for complexity reduction. As we argued earlier, in this case the maximal complexity resolutions can be replaced with resolutions with at most two levels, and the conclusion of the theorem follow by the proof of theorem 1.5.

Theorem 1.8 proves the conclusions of proposition 1.5 in case all the $m$-collections in the higher rank MR diagram contain at most a single non-trivial resolution, and all the virtually abelian decompositions that are associated with the various levels in this single resolution contain only infinite virtually abelian edge groups, and no QH nor virtually abelian vertex groups.

The next proposition generalizes theorem 1.8 by allowing the virtually abelian decompositions in the various levels of the single non-trivial resolution in each $m$-collection to have QH and virtually abelian vertex groups.
Theorem 1.9. With the assumptions of lemma 1.1, suppose that in each of the (finitely many) collections of m cover resolutions in the higher rank Makanin-Razborov diagram, there is at most one resolution that has more than a single level. Suppose further that:

(i) all the virtually abelian decompositions that are associated with the various levels of the resolutions that have more than a single level do not contain edges with finite (nor trivial) edge groups.

(ii) all the quotient maps in these resolutions are proper quotients.

Then conclusions (1) and (2) in proposition 1.8 hold. In particular, H admits a higher rank JSJ decomposition.

Proof: Since all the quotient maps in the resolutions in the higher rank MR diagram are assumed to be proper quotients, each m-collection in the diagram admits a weak test sequence, that restricts to a weak test sequence of the single resolution with more than a single level.

These resolutions may contain QH vertex groups, and the restrictions of the weak test sequences to these QH vertex groups may extend to finite index QH supergroups of these QH vertex groups. Clearly, given a QH vertex group, there are only finitely many finite index QH supergroups that contain it.

In definition 5 in [Se6] we have associated finitely many framed resolutions with a given (well-structured) resolution, precisely for that purpose. It contains all the finitely many possibilities to replace the QH vertex groups in a given resolution by finite index supergroups, such that the obtained resolution (still) admits a (weak) test sequence.

Since the extension of a (well-structured) resolution to each of its (finitely many possible) framed resolutions involved adding finitely many roots to its QH subgroups, the construction of the higher rank MR diagram can be easily modified to produce framed resolutions. Once the construction of the higher rank MR diagrams contains framed resolutions, we can assume that the restriction of the weak test sequence to a framed resolution in the higher rank diagram does not extend to a sequence of specializations of any framed resolution that strictly contains it (i.e., to a framed resolution with strictly bigger finite index QH supergroups).

Therefore, we continue by assuming that resolutions with more than a single level in the higher rank MR diagram are maximal framed resolutions. i.e., that their associated weak test sequences do not extend to a framed resolution that strictly contains them.

We start with the procedure that was used in the first part of the proof of theorem 1.8. Let $CRes$ be a resolution with more than 2 levels in one of the m-collections of resolutions in the higher rank MR diagram of H. We go along the levels of the completion, $Comp(Res)$, and push down QH and virtually abelian vertex groups, for which the virtually abelian edge groups that are connected to them that are elliptic in the lower level of the completion. After this procedure terminates, all the edge groups in the modified completion are virtually cyclic. An edge group that is not connected to a QH vertex group in a virtually abelian decomposition that is associated with a level above the two terminating ones in $CRes$, is not elliptic in the virtually abelian decomposition that is associated with the next level of $CRes$. Furthermore, at least one of the virtually cyclic edge groups that are connected to each of the QH vertex groups in a virtually abelian decomposition that is associated with a level above the two terminating ones in $CRes$, is not
elliptic in the virtually abelian decomposition that is associated with the next level.

As in the proof of theorem 1.8, it may still be that some QH or virtually abelian vertex groups in the virtually abelian decompositions along the modified resolutions are not really essential, or that the degrees of the maximal free abelian subgroups in virtually abelian vertex groups can be further decreased. To get rid of these redundancies, and possibly reduce the number of QH or virtually abelian vertex groups, or reduce their ranks, we use a procedure that generalizes the one that was used in the proof of theorem 1.8. Conceptually, these procedures are connected to auxiliary resolutions that are introduced in definition 2.1 in [Se5], though they are technically different.

With each of the non-trivial (framed) resolutions, $CRes$, in the higher rank MR diagram we associate its completion, $Comp(CRes)$. The limit group $L$ (with which we started the original resolution $CRes$) is now mapped into the completion of the resolution. We denote by $\nu(L)$ the image of the limit group in the completion, $Comp(CRes)$. Also, each resolution $CRes$ has a weak test sequence, hence, each of the completions, $Comp(CRes)$ has a weak test sequence, that extends a weak test sequence of the resolution $CRes$.

As in the proof of theorem 1.8, for each of the completions, $Comp(CRes)$, we define its complexity. This generalizes the complexity of resolutions with no QH vertex groups in the proof of theorem 1.8. The complexity of a completion, $Comp(CRes)$, is a tuple of integers ordered lexicographically in the following order:

1. with each QH vertex group $Q$ that appears in one of the levels of the completion, $Comp(CRes)$, we associate a pair $(-\chi(Q),g(Q))$, the Euler characteristic and the genus of the associated orbifold. The highest term in the complexity of $Comp(CRes)$ is the list of pairs that are associated with the QH vertex groups along $Comp(CRes)$, in a non-increasing lexicographical order.
2. the number of levels in the completion.
3. a tuple of integers for each virtually abelian decomposition that appears along the completion, $Comp(CRes)$, going from top to bottom. The tuple of integers consists of the list of pairs that are associated with the QH vertex groups in that level in a non-increasing lexicographical order, followed by the number of edge groups in the virtually abelian decomposition that is associated with the level in the completion, $Comp(CRes)$, followed by the ranks of the virtually abelian vertex groups in the virtually abelian decomposition that is associated with the level in $Comp(CRes)$, ordered in a non-increasing order.
4. a tuple of integers for each virtually abelian decomposition that appears along the completion, $Comp(CRes)$, going from top to bottom. The tuple consists of the pairs that are associated with QH vertex groups that contain finite index subgroups that can be conjugated into $\nu(L)$, in a non-increasing order, followed by the number of virtually abelian vertex groups that contain subgroups of rank at least 2 that can be conjugated into $\nu(L)$, followed by the number of virtually infinite cyclic edge groups that are connected to virtually abelian edge groups in each level of the completion, that can be conjugated into the image of the limit group $\nu(L)$.

As for the complexity that was used in the proof of theorem 1.8, complexities
of resolutions are well-ordered. We continue by modifying the argument that was used in proving theorem 1.8.

Let $CRes$ be a non-trivial resolution in some $m$-collection in the higher rank MR diagram. Let $\{\varphi_s\}$ be a weak test sequence of automorphisms in $Aut(G)$ for the resolution $CRes$. $\{\varphi_s\}$ extends to a weak test sequence of the completion, $Comp(CRes)$. Let $\tau \in Aut(G)$. A subsequence of $\{\varphi_s \circ \tau\}$ factors through one of the $m$-collection in the higher rank MR diagram. Let $CRes_1$ be the non-trivial resolution in this $m$-collection (by our assumption every $m$-collection contains at most one non-trivial resolution).

As in the proof of theorem 1.8, by the construction of formal solutions and formal limit groups in [Se2], from the sequences $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$ it is possible to pass to a further subsequence that converges to a map: $\nu_\tau : Comp(CRes_1) \rightarrow Cl(CRes)$, where $Cl(CRes)$ is a closure of the completion $Comp(CRes)$.

**Lemma 1.10.** Suppose that $CRes$ is a resolution of maximal complexity in the higher rank MR diagram. Then the closure, $Cl(CRes)$, can be modified to a closure with a similar structure, and the map $\nu_\tau$ can be modified accordingly, so that either:

1. $\nu_\tau(Comp(CRes_1))$ intersects some QH vertex group in $Cl(CRes)$ trivially, i.e. in products of conjugates of some of its boundary components, which is an infinite index subgroup in the QH vertex group.
2. $\nu_\tau$ maps the QH vertex groups in $Comp(CRes_1)$ isomorphically onto QH vertex groups in $Cl(CRes)$, and every QH vertex group in $Cl(CRes)$ has a conjugate that is a subgroup of $\nu_\tau(Comp(CRes_1))$. There exists some virtually abelian vertex group $VA$ in $Cl(CRes)$, such that the intersection between $\nu_\tau(Comp(CRes_1))$ and conjugates of $VA$ are contained in conjugates of some subgroup of infinite index: $\hat{VA} < VA$.
3. $CRes_1$ is a resolution of maximal complexity in the higher rank MR diagram, $\nu_\tau$ maps the QH vertex groups in $Comp(CRes_1)$ isomorphically onto conjugates of distinct QH vertex groups in $Cl(CRes)$, and virtually abelian vertex groups in $Comp(CRes_1)$ onto conjugates of finite index subgroups of distinct virtually abelian vertex groups in $Cl(CRes)$.

**Proof:** The argument that we use to prove lemma 1.10 is based on the analysis of quotient resolutions of minimal rank, that appears in section 1 of [Se4], and in section 1 in [Se5].

Let $\nu_\tau : Comp(CRes_1) \rightarrow Cl(CRes)$. If the maximal complexity resolution, $CRes$, contain no QH vertex groups, the lemma follows from theorem 1.8. If $CRes_1$ contains no QH vertex groups, the lemma follows from the proof of theorem 1.8.

Suppose that $CRes_1$, and hence the maximal complexity resolution $CRes$, contains QH vertex groups. For presentation purposes, suppose first that $CRes_1$ contains a QH vertex group $Q_1$ in its terminal virtually abelian decomposition. i.e., all the boundary components of $Q_1$ are contained in the terminal bounded group in $CRes_1$.

By construction, $\nu_\tau$ maps the boundary elements of $Q_1$ into the bounded subgroup in $Cl(CRes)$. Suppose that $\nu_\tau(Q_1)$ intersects conjugates of some QH vertex groups or some virtually abelian vertex groups along the levels of $Cl(CRes)$ non-trivially. i.e., in the case of QH vertex group, in subgroups that are not conjugate to free products of conjugates of boundary elements. Suppose that $Q_u$ is a QH vertex group in $Cl(CRes)$, that intersects non-trivially a conjugate of $\nu_\tau(Q_1)$, and
$Q_u$ is in the highest level in $Cl(CRes)$ for which there are such $QH$ or virtually abelian vertex groups.

Since the virtually abelian decompositions along the resolution $CRes_1$ contains no edges with finite edge groups ($CRes_1$ is of minimal rank following the definition in section 1 in [Se4]), a suborbifold of $Q_1$ is mapped by $\nu_\tau$ onto a finite index subgroup of a conjugate of $Q_u$. Therefore, the complexity (pair) of $Q_u$ is bounded by the complexity of $Q_1$, with equality if and only if $Q_u$ is a QH vertex group in the terminal virtually abelian decomposition of $Cl(CRes)$, and $\nu_\tau$ maps $Q_1$ isomorphically onto a conjugate of $Q_u$.

Let $i$ be the highest level in the closure, $Cl(CRes)$, for which $\nu_\tau(Q_1)$ intersects non-trivially a conjugate of a QH or a virtually abelian vertex group in that closure. Let $Cl(CRes)_{i+1}$ be the subgroup which the part of the closure, $Cl(CRes)$, that contains all the levels up to level $i + 1$ (the level below level $i$). Let $\eta_{i+1}$ be the retraction: $\eta_{i+1} : Cl(CRes) \to Cl(CRes)_{i+1}$.

We look at the image of $Q_1$ under the composition: $\eta_{i+1} \circ \nu_\tau$. The images of boundary components of $Q_1$ do not change by the composition with the retraction $\eta_{i+1}$. Suppose that the image of $Q_1$ under the composition intersects non-trivially a conjugate of a QH vertex group $Q_{i+1}^{i+1}$ in level $i + 1$ of $Cl(CRe)$, which is the highest level in the part of $Cl(CRes)$ that is the image of the retraction $\eta_{i+1}$.

By the same arguments that applied to the QH vertex group $Q_u$, the complexity of $Q_{i+1}^{i+1}$ is bounded by the complexity of $Q_1$, with equality if and only if $Q_{i+1}^{i+1}$ is a QH vertex group in the terminal virtually abelian decomposition of $Cl(CRes)$, and $\eta_{i+1} \circ \nu_\tau$ maps $Q_1$ isomorphically onto a conjugate of $Q_u$.

We continue iteratively, by composing with retractions to lower levels. By the same arguments we conclude that any QH vertex group that has a conjugate that is intersected non-trivially by the image of $Q_1$ under the composition of a retraction and the map $\nu_\tau$, has complexity that is bounded by the complexity of $Q_1$, with equality if and only if the QH vertex group is in the bottom level of $Cl(CRes)$, and the composition of the retraction and $\nu_\tau$ maps $Q_1$ isomorphically onto that QH vertex group.

The bounds on the complexities of the QH vertex groups in $Cl(CRes)$ that intersect non-trivially conjugates of the images of a QH vertex group $Q_1$ in the bottom virtually abelian decomposition of $CRes_1$, under compositions of retractions of $Cl(CRes)$ with the map $\nu_\tau$, enable us to analyze the image of $\nu_\tau$.

If the complexities of all the QH vertex groups in $Cl(CRes)$ that have conjugates that are intersected non-trivially have strictly smaller complexity than $Q_1$, we gain in the ambient complexity and we do not change anything. If there is a QH vertex group $Q$ in $Cl(CRes)$ that has a conjugate that is intersected non-trivially and has the same complexity as $Q_1$, $Q$ has to be in the terminal virtually abelian decomposition of $Cl(CRe)$.

If there are two QH vertex groups $Q_1$ and $Q_2$ in the terminal virtually abelian decomposition of $CRes_1$, such that the images of them under a composition of a retraction with $\nu_\tau$ intersect conjugates of the same QH vertex group $Q$ in the terminal virtually abelian decomposition of $CRes$, we also gain in the ambient complexity and we do not make any changes.

**Definition 1.11 (cf. definition 1.6 in [Se4]).** A QH vertex group $Q$ in the terminal virtually abelian decomposition of $CRes$ is called surviving orbifold if:

1. there exists a QH vertex group $Q_1$ in the terminal virtually abelian decom-
position of $CRes_1$ such that the composition of a retraction and $\nu_\tau$ maps $Q_1$ isomorphically onto a conjugate of $Q$.

(ii) the images of all the other QH vertex groups in the terminal level of $CRes_1$ under the composition of a retraction and $\nu_\tau$, intersect every conjugate of $Q$ trivially (i.e., in a free product of conjugates of boundary subgroups of $Q$).

Suppose that $Q$ is a surviving orbifold, and let $Q_1$ be the unique QH vertex group in the terminal virtually abelian decomposition of $CRes_1$, that is mapped by a composition of a retraction and $\nu_\tau$ isomorphically onto a conjugate of $Q$. We continue in a similar way to what was done in section 1 in [Se4] for bounding the complexity of quotient resolutions.

We rearrange QH and virtually abelian subgroups along the levels of the closure, $Cl(CRes)$, that are mapped by retractions isomorphically onto suborbifolds of $Q$, and modify accordingly the map $\nu_\tau$. We do that to guarantee that for each surviving orbifold $Q$ in the terminal virtually abelian decomposition of $Cl(CRes)$, the unique QH vertex group $Q_1$ that was mapped isomorphically onto a conjugate of $Q$ by a composition of a retraction and the map $\nu_\tau$, will be mapped isomorphically onto $Q$ by the modification of $\nu_\tau$ (with no composition with a retraction). This will imply that a QH vertex group $Q_1$ in $CRes_1$ that is mapped onto a conjugate of a surviving orbifold $Q$, intersects trivially conjugates of all the other QH vertex groups in $Cl(CRes)$. Hence, the contribution of $Q_1$ to the ambient complexity of $CRes_1$ will be the same as the contribution of surviving orbifold $Q$ to the complexity of $CRes$.

Let $Q^i, Q^{i+1}, \ldots, Q^{t-1} = Q$, be the sequence of (not necessarily connected, some of which possibly empty) 2-orbifolds and virtually abelian vertex groups in $Cl(CRes)$, from level $i$ down to the QH vertex group $Q$ in the terminal virtually abelian decomposition in level $t - 1$, such that the image of $Q_1$ under a composition of the corresponding retraction with $\nu_\tau$ intersects non-trivially a conjugate of them.

Since $Q_1$ is mapped isomorphically onto a conjugate of the surviving orbifold $Q$ by a composition of a retraction and $\nu_\tau$, suborbifolds of $Q_1$ are mapped isomorphically onto conjugates of the QH vertex groups in the sequence: $Q^i, \ldots, Q^{t-1}$, and s.c.c. in $Q_1$ are mapped onto finite index subgroups of conjugates of edge groups in virtually abelian vertex groups along this sequence. Furthermore, the final retraction maps QH vertex groups in $Q^1, \ldots, Q^{t-2}$ isomorphically onto conjugates of suborbifolds of $Q$, and finite index subgroups of edge groups of virtually abelian vertex groups onto s.c.c. in conjugates of $Q$. Since $Q$ is a surviving orbifold, the images of all the other QH vertex groups in the terminal level of $CRes_1$, under compositions of appropriate retractions with $\nu_\tau$, intersect trivially conjugates of all the QH and virtually abelian vertex groups in the sequence: $Q^i, \ldots, Q^{t-1}$.

These observations allow us to modify the structure of the closure $Cl(CRes)$ and the map $\nu_\tau$ in a similar way to the modification of quotient resolutions (in the minimal rank case) in the first sections of [Se4] and [Se5].

We modify the closure $Cl(CRes)$ by inverting the order of the levels of the QH and virtually abelian vertex groups that appear in the sequence: $Q^i, \ldots, Q^{t-2}$, leaving $Q^{t-1} = Q$ unchanged. i.e., we set it to be: $Q^{t-2}, \ldots, Q^i, Q^{t-1} = Q$. This order inversion is completed by changing accordingly the map $\nu_\tau$, that after modification maps $Q_1$ isomorphically onto a conjugate of $Q$ (without composing it with a retraction). For the detailed modification of the map $\nu_\tau$ see section 1 in [Se4]. We denote the modified map $\bar{\nu}_\tau$, and the modified closure, $\bar{Cl}(CRes)$. 25
Note that the images of the other QH and virtually abelian vertex groups in the terminal virtually abelian decomposition in Comp(CRes$_1$) are not affected by the modification of $\nu_\tau$, as these images intersect the QH and virtually abelian vertex groups in the sequence $Q^1, \ldots, Q^{t-1} = Q$ trivially. The images of QH and virtually abelian vertex groups in higher level virtually abelian decompositions in Comp(CRes$_1$) may be affected by the modification of the map $\nu_\tau$, and the same holds for the image of $L, \nu(L)$, in the modified closure $\hat{C}l(CRes)$.

We perform these modifications of the closure $Cl(CRes)$, and of the map $\nu_\tau$, for all the surviving orbifolds. Note that because of trivial intersections, these modifications can be conducted in parallel. Also note that because the edge groups that are connected to the virtually abelian vertex groups in the terminal virtually abelian decomposition of Comp(CRes$_1$) do all belong to the terminal bounded subgroup in Comp(CRes$_1$), $\nu_\tau$ and the modified map $\hat{\nu}_\tau$ map these virtually abelian vertex groups into the terminal bounded subgroup in $\hat{C}l(CRes)$, or into conjugates of virtually abelian vertex groups in the terminal virtually abelian decomposition in $\hat{C}l(CRes)$.

We continue iteratively by climbing along the levels of Comp(CRes$_1$), and possibly further modifying the structure of the closure $\hat{C}l(CRes)$ and the map $\hat{\nu}_\tau$. After completing the modification at level $i$, $2 \leq i \leq t - 1$, we set $M_i$ to be the subtower of the modified closure, $\hat{C}l(CRes)$, that contains the QH and virtually abelian vertex groups in $\hat{C}l(CRes)$ that intersect non-trivially images (under $\hat{\nu}_\tau$) of QH and virtually abelian vertex groups in the levels $i$ and below in the completion Comp(CRes$_1$). By construction, since $M_i$ is associated with a subtower, it is closed under the retractions of the ambient modified closure $\hat{C}l(CRes)$.

Note that the images under $\hat{\nu}_\tau$ of all the edge groups in the virtually abelian decomposition in level $i - 1$ in Comp(CRes$_1$), including all the boundary elements of QH vertex groups in this virtually abelian decomposition, are contained in $M_i$. Hence, if a QH vertex group $Q_{i-1}$ in the virtually abelian decomposition that is associated with level $i - 1$ in Comp(CRes$_1$), and $Q$ is a QH vertex group in the highest possible level in $\hat{C}l(Res)$ that is not in $M_i$ and intersects non-trivially a conjugate of $\hat{\nu}_\tau(Q_{i-1})$, then the complexity of $Q$ is bounded by the complexity of $Q_{i-1}$. Furthermore, equality in the complexities of $Q$ and $Q_{i-1}$ occurs if and only if $Q$ is a QH vertex group in some level of $\hat{C}l(CRes)$, and all the QH and virtually abelian vertex groups in the levels below it, that contain conjugates of its boundary components, are in the subtower of $\hat{C}l(CRes)$ that is associated with $M_i$.

We look at compositions of retractions of the modified tower, $\hat{C}l(CRes)$, and the modified map $\hat{\nu}_\tau$, precisely as we did in analyzing QH vertex groups in the terminal virtually abelian decomposition of CRes$_1$. We define surviving orbifolds precisely in the same way, and modify the levels of $\hat{C}l(CRes)$, and accordingly the map $\hat{\nu}_\tau$, precisely as we did in analyzing the terminal level.

The analysis is complete when we analyze the highest level in Comp(CRes). Since CRes was assumed to be of maximal complexity, the definition of the complexity of a completion implies that if all the QH vertex groups in $\hat{C}l(CRes)$ intersect non-trivially conjugates of $\nu_\tau(Comp(CRes))$, then all the QH vertex groups in Comp(CRes) are surviving orbifolds, and each of them is mapped by the modified map, $\hat{\nu}_\tau$, isomorphically onto a corresponding surviving orbifold in the modified closure $\hat{C}l(CRes)$.
Suppose that all the QH vertex groups in $\hat{Cl}(CRes)$ are surviving surfaces. Then each of the QH vertex groups in $CRes_1$ is mapped by $\hat{\nu}_\tau$ onto a QH vertex groups in $\hat{Cl}(CRes_1)$, precisely as we did in the proof of theorem 1.8. If there exists a virtually abelian vertex group in one of the levels of $\hat{Cl}(CRes_1)$ that intersect $\hat{\nu}_\tau(Comp(CRes_1))$ in a subgroups that can be conjugated into some fixed infinite index subgroup of the virtually abelian vertex group in $\hat{Cl}(CRes)$ we get part (2) of the lemma. If there isn’t such a virtually abelian vertex group, then by the proof of theorem 1.8, $\hat{\nu}_\tau$ maps the virtually abelian vertex groups in $Comp(CRes_1)$ isomorphically onto finite index subgroups of distinct virtually abelian vertex groups in $\hat{Cl}(CRes)$. In particular, $CRes_1$ has to be a maximal complexity resolution, and we get part (3) of the lemma.

We continue the proof of theorem 1.9 along the lines of the argument that was used to prove theorem 1.8. Let $CRes$ be a maximal complexity resolution in the higher rank MR diagram, and suppose that there exists a weak test sequence, $\{\phi_s\}$ in $Aut(G)$, such that for every automorphism $\tau \in Aut(G)$, all the convergent pairs of subsequences, $\{\phi_s\}$ and $\{\phi_s \circ \tau\}$, converge into maps: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$, that satisfy part (3) in lemma 1.10.

In this case, we look at the set of all the weak test sequences that satisfy this condition w.r.t. all the maximal complexity resolutions in the higher rank MR diagram. We argue as in the proofs of proposition 1.7 and theorem 1.8. If there exists an edge group in a level that is above the bottom two levels in one of these maximal complexity resolutions, that has a conjugate that intersects the image of the limit group $L$ in a subgroup of finite index, then a finite index subgroup of $Aut(G)$ preserves the conjugacy class of the subgroup of $H$ that is mapped onto this edge group. This contradicts the assumption that the edge group is in a level that is above the bottom two levels, and is hyperbolic in the level below it, which is not the bottom level.

Hence, in this case the image of $L$ does not intersect any conjugate of an edge group in a level above the bottom two levels in a subgroup of finite index. Therefore, $L$ inherits splittings over finite edge groups from all the levels above the two bottom ones in the maximal complexity resolutions, so the maximal complexity resolutions in the higher rank MR diagram can be replaced by resolutions with at most two levels, and the conclusion of the theorem follows from theorem 1.5.

For the rest of the argument, we assume that for every maximal complexity resolution, $CRes$, and every weak test sequence, $\{\phi_s\}$, that factors through it, there is an automorphism $\tau \in Aut(G)$, such that a subsequence of $\{\phi_s \circ \tau\}$ factors through some resolution $CRes_1$, and after passing to a convergent pair of subsequences, the image of the associated map: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$ satisfies properties (1) or (2) in lemma 1.10.

In this case we do not use a compactness argument similar to the one that was used in the proof of theorem 1.8, but we rather reconstruct a higher rank MR diagram, and use the above condition to guarantee that the complexities of the resolutions in the new diagram is strictly bounded by the maximal complexity of the resolutions in the previous diagram.

**Lemma 1.12.** Suppose that for every maximal complexity resolution $CRes$ in the higher rank MR diagram, and every weak test sequence $\{\phi_s\}$ that factors through
there exists some automorphism $\tau \in Aut(G)$, such that the sequence of pairs, $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$ has a convergent subsequence, that converges into a map: $\nu_\tau : \text{Comp}(CRes_1) \to Cl(CRes)$, where $CRes_1$ is some resolution in the higher rank diagram, and the map $\nu_\tau$ satisfies parts (1) or (2) in lemma 1.10.

Then the higher rank MR diagram can be replaced with a new higher rank MR diagram, in which every $m$-collection has a unique non-trivial resolution, and the maximal complexity of a resolution in the new diagram is strictly bounded by the maximal complexity of resolutions in the original higher rank diagram.

Proof: Let $CRes$ be the only non-trivial resolution in an $m$-collection in the higher rank MR diagram, and suppose that $CRes$ has maximal complexity. Let $\{\varphi_s\}$ be a weak test sequence w.r.t. $CRes$, and let $\tau \in Aut(G)$, such that the pair of sequences, $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$, has a subsequence that converges into a map: $\nu_\tau : \text{Comp}(CRes_1) \to Cl(CRes)$, where $CRes_1$ is a resolution in some $m$-collection in the higher rank MR diagram, and the map $\nu_\tau$ satisfies properties (1) or (2) in lemma 1.10.

Given the weak test sequence, $\{\varphi_s\}$, the ability to find an automorphism $\tau \in Aut(G)$, a resolution $CRes_1$, and a map $\nu_\tau$ that satisfies conclusions (1) or (2) in lemma 1.10, depends only on the structure of the tower that is the limit of the given weak test sequence.

**Lemma 1.13.** Let $T$ be the limit tower of the weak test sequence $\{\varphi_s\}$. Note that $T$ is obtained from $\text{Comp}(CRes)$ by replacing the terminal subgroup with some f.g. limit quotient. Conclusions (1) or (2) hold for $\{\varphi_s\}$ if and only if there exists some level $i$, $1 \leq i \leq t - 1$, and a subgroup $M < T$, with the following properties:

1. $M$ is generated by the terminal group in $T$, all the QH and virtually abelian vertex groups above and below level $i$, and either all the virtually abelian vertex groups and all the QH vertex groups in level $i$, except a single QH one, or all the QH vertex groups and all the virtually abelian vertex groups in level $i$, except a single virtually abelian vertex group that is replaced by a subgroup of infinite index.

2. $M$ has a tower structure, in which the QH and virtually abelian vertex groups are those that are described in (1). In particular, it has a similar structure as that of $T$ (or $\text{Comp}(CRes)$), except in level $i$ in which a QH vertex group is removed, or a virtually abelian vertex group is replaced by a subgroup of infinite index.

3. $M$ is embedded in $T$ in a natural way, hence, the retractions along the tower structure of $M$ are inherited from those of $T$.

4. the image of the limit group $L$ in $T$ is a subgroup of $M$.

5. all the edge groups in levels above level $i$ are contained in $M$.

Proof: Suppose that conclusions (1) or (2) hold for the weak test sequence $\{\varphi_s\}$. Then there exists some automorphism $\tau \in Aut(G)$, for which a subsequence of the pair of sequences, $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$, converges into a map $\nu_\tau : \text{Comp}(CRes_1) \to Cl(CRes)$, that by lemma 1.10 can be modified so that the image of $\text{Comp}(CRes_1)$ does not intersect either a conjugate of some QH vertex group in $T$, or there exists an infinite index subgroup of some virtually abelian vertex group $VA$ in some level of $Cl(CRes)$, such that the image of $\text{Comp}(CRes_1)$ intersects conjugates of $VA$ only in subgroups of conjugates of the infinite index subgroup.


In that case we look at the highest level $i$ in $T$ that contains such a QH or a virtually abelian vertex group. We define $M < T$ to be the subgroups that is generated by the terminal limit group of $T$, all the QH and virtually abelian vertex groups above and below $T$, and either all the virtually abelian groups and all the QH vertex groups in level $i$, except the one that doesn’t intersect conjugates of the image of $Comp(CRes_1)$, or all the QH vertex groups and all the virtually abelian vertex groups, except the one that can be replaced by an infinite index subgroup, in which case we add the infinite index subgroup.

By definition, $M$ satisfies (1). $M$ has a tower structure, after we modify the retractions only at the levels above level $i$. We do that by either composing the retractions from levels above level $i$, with the restriction of the retraction of level $i$ to the missing QH vertex group, or composing them with a retraction from the virtually abelian vertex group $VA$ to an infinite index subgroup (this may require replacing the infinite index subgroup of $VA$ in $M$, by a finite index supergroup).

The natural embedding of $M$ in $T$ follows from its construction. Parts (4) and (5) follow from the properties of the map $\nu_\tau$ and properties (1) and (2) in lemma 1.9.

Conversely, suppose that $\{\varphi_s\}$ converges to some tower $T$, and there exists a tower $M$ that satisfies properties (1)-(5) of the lemma. In this case, for any automorphism $\tau \in Aut(G)$ (including the identity), there exists a convergent subsequence of pairs, $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$, that converges into a map: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$, and the image of $\nu_\tau$ is a subgroup of $Cl(M) < Cl(CRes)$, i.e., a closure of $M$ that is obtained by adding roots to virtually abelian vertex groups, and possibly enlarging the terminal limit group.

Lemma 1.13 implies that the ability to reduce the complexity of the tower can be expressed as a basic condition and not as a Diophantine one. Let $\{\sigma_s\}$ be a sequence of automorphisms in $Aut(G)$. By section 3 in [Se8], we can pass to a subsequence that converges to an $m$-collection with at most a single non-trivial resolution that we denote $NRes$.

We continue with the assumption that all the quotient maps along $NRes$ are proper quotients. This assumption is going to be dropped in theorem 1.14. Hence, $NRes$ has a weak test sequence. Suppose that the complexity of $NRes$ is at least the maximal complexity of resolutions in the original higher rank MR diagram.

By the universality of the original higher rank MR diagram, there is a subsequence of $\{\sigma_s\}$ that factors through some resolution $CRes_1$ in the original diagram, and the subsequence converges to a map: $\rho : Comp(CRes_1) \to Cl(NRes)$. If the complexity of $CRes_1$ is strictly smaller than that of $NRes$, then the image of $\rho$ satisfies parts (1) or (2) in the conclusion of lemma 1.10. If the complexities of $CRes$ and $NRes$ are equal, then $CRes$ is a maximal complexity resolution in the original higher rank MR diagram, and either $\rho$ satisfies conclusions (1) or (2) of lemma 1.10, or conclusion (3) holds for $\rho$, and then $\{\sigma_s\}$ is a weak test sequence for $CRes$ as well. By our assumptions, it is then possible to find an automorphism $\tau \in Aut(G)$, and some resolution $CRes_1$ in the original higher rank diagram, together with a map: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$ that satisfies conclusions (1) or (2) of lemma 1.9. Therefore, $\nu_\tau \circ \rho : Comp(CRes_1) \to Cl(NRes)$ satisfies conclusion (1) or (2) in lemma 1.10.

In all these cases, by lemma 1.13, a subsequence of the original sequence $\{\sigma_s\}$ factor through the tower $M$ that embeds in $Cl(NRes)$, and the complexity of
M is strictly smaller than the complexity of $NRes$. Repeating this construction iteratively, we can deduce that there exists a subsequence of the sequence $\{\sigma_s\}$ that factors and converges to some $m$-collection with a single non-trivial resolution, and the complexity of this resolution is strictly bounded by the maximal complexity of the resolutions in the original higher rank MR diagram.

To complete the proof of theorem 1.9, we can now construct a new higher rank MR diagram with a strictly smaller bound on the complexity of its resolutions. We look at all the convergent sequences of automorphisms. Given a sequence we pass to a subsequence that converges to an $m$-collection with a single non-trivial resolution, where the complexity of the resolution is strictly smaller than the maximal complexity of the resolutions in the original higher rank MR diagram.

We look at a cover of such a resolution, which is f.p. with a f.p. terminal limit group. Clearly there are countably many such covers of $m$-collections, and covers of non-trivial cover resolutions. All these cover resolutions has strictly smaller complexity than the maximal complexity of the resolutions in the original higher rank MR diagram.

As in the construction of the higher rank MR diagram, in section 3 of [Se8], we order these cover $m$-collections and cover resolutions. We claim that finitely many of them can serve as a new higher rank MR diagram, i.e., there is a finite collection of these covers, such that every automorphism in $Aut(G)$ factors through at least one of these finitely many covers.

If finitely many do not suffice, there is a sequence of automorphisms $\{\sigma_s\}$ in $Aut(G)$, such that $\sigma_s$ does not factor through the first $s-1$ covers. By our procedure, the sequence $\{\sigma_s\}$ has a subsequence that converges and factors through some cover $m$-collection with a single non-trivial resolution, and with complexity that is strictly bounded by the maximal complexity of the resolutions in the original higher rank MR diagram. This cover $m$-collection appears in the list of covers that we started with. Hence, for large $s$, the automorphisms $\sigma_s$ factor through a cover that appears earlier in the list, a contradiction.

By lemma 1.12, in case there is no weak test sequence $\{\varphi_s\}$ w.r.t. a maximal complexity resolution, for which there is no automorphism $\tau \in Aut(G)$, such that a subsequence of pairs of: $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$, converge into a map: $\nu_\tau : Comp(CRes_1) \to Cl(CRes)$, that satisfies properties (1) or (2) in lemma 1.10, it is possible to replace the higher rank MR diagram with another higher rank MR diagram with strictly smaller maximal complexity resolutions.

Iteratively replacing the higher rank MR diagram, and strictly reducing the maximal complexity of its resolutions, we get a higher rank diagram, for which a further complexity reduction is not possible. i.e., we get a diagram with maximal complexity resolutions and weak test sequences that factor through them, for which no automorphism $\tau \in Aut(G)$ enables one to apply the procedure for complexity reduction. As we argued earlier, in this case the maximal complexity resolutions can be replaced with resolutions with at most two levels, and the conclusions of theorem 1.9 follow by the proof of theorem 1.5.

So far we have assumed that in each $m$-collection in the higher rank MR diagram there is at most one non-trivial resolution, and that the virtually abelian decompositions along non-trivial resolutions contain no edges with finite (nor trivial) edges groups. We further assumed that the quotient maps along the non-trivial resolu-
tions are proper quotients, which guarantee that each resolution (or \( m \)-collection) in the MR diagram has a weak test sequence of automorphisms that factor through it.

The next theorem drops the assumption for proper quotient maps. It still requires that the virtually abelian decompositions along the non-trivial resolutions contain no edges with finite (nor trivial) edge groups. This assumption is kept in this paper, and will be dropped only in the next paper in this sequence.

**Theorem 1.14 (cf. theorem 1.9).** With the assumptions of lemma 1.1, suppose that in each of the (finitely many) collections of \( m \) cover resolutions in the higher rank Makanin-Razborov diagram, there is at most one resolution that has more than a single level, and all the virtually abelian decompositions that are associated with the various levels of the resolutions that have more than a single level do not contain edges with finite (nor trivial) edge groups.

Then conclusions (1) and (2) in theorem 1.8 hold. In particular, \( H \) admits a higher rank JSJ decomposition.

**Proof:** In theorem 1.9 we assumed that in all the resolutions in the higher rank MR diagram, all the quotient maps along the non-trivial resolutions are proper quotients. By the construction of the higher rank MR diagram [Se8], this assumption guarantees that each resolution (or rather an \( m \)-collection) in the higher rank MR diagram has a weak test sequence of automorphisms that factor through it.

Since weak test sequences are used to prove theorem 1.9, to apply the argument that is used in the proof of theorem 1.9 to prove theorem 1.14, we need to modify the construction of the higher rank MR diagram that appears in section 3 of [Se8], to guarantee that resolutions, or \( m \)-collections, that appear in the higher rank diagram have weak test sequences, even if some of the quotient maps along the resolutions in an \( m \)-collection are not proper quotients.

The construction of resolutions in the higher rank MR diagram in [Se8] guarantees that resolutions, and hence \( m \)-collections, have weak test sequences that factor through them, only if all the quotient maps along the resolutions are proper quotients. Therefore, to guarantee that all the \( m \)-collections in the higher rank MR diagram have weak test sequences that factor through them, we need to modify the construction of the resolutions in the \( m \)-collections in the higher rank MR diagram. Indeed, a construction of resolutions that admit weak test sequences even when some of the quotient maps along the resolutions are isomorphisms (and not proper quotients), appears in our work on Makanin-Razborov diagrams of pairs, that analyze and encode varieties over a free semigroup [Se8].

The construction of resolutions that is used in [Se8], that involve infinite chain of decompositions that are eventually replaced by a finite resolution, construct finite resolutions with weak test sequences, where some of the quotient maps between consecutive levels are isomorphisms and some are proper quotients.

When we apply the construction of resolutions that is used in [Se8] (for pairs and semigroups), we start with a sequence of automorphisms, \( \{\varphi_s\} \) in \( Aut(G) \), and pass to a subsequence which is a weak test sequence w.r.t. some finite resolution of a limit quotient of \( H \) (the characteristic finite index subgroup in \( G \)).

The same compactness argument that is used in the construction of the higher rank MR diagram in section 3 in [Se8], implies that using the construction of resolutions in [Se8], there exists a higher rank MR diagram, that contains finitely
many $m$-collections of (framed) resolutions, and for each $m$-collection there is a weak test sequence of automorphisms that factor through it.

Once we replaced the higher rank MR diagram with a diagram in which every $m$-collection has a weak test sequence, the proof of theorem 1.14 follows by the same argument that was used in the proof of theorem 1.9.

§2. A higher rank JSJ decomposition of a product II: general diagrams and their groupoids

In the previous section we analyzed higher rank MR diagrams of products of hyperbolic spaces, in which every $m$-collection contains at most a single resolution with more than 2 levels. In this case we were able to associate with the product a higher rank JSJ decomposition, from which it is possible to extract information on the dynamics of individual automorphisms, and on the algebraic structure of $\text{Out}(G)$.

In this section we continue with products of hyperbolic spaces, but we analyze general higher rank MR diagrams. In this case we do not manage to get graphs of groups decompositions (of limit quotients) that are associated with the various factors, and are invariant under a finite index subgroup of $\text{Out}(G)$. The techniques that we use for analyzing general higher rank MR diagrams of products, does not give us an invariant graph of groups, but enable us to associate groupoids with finitely many objects with the higher rank MR diagram.

These groupoids replace the maps of $\text{Out}(G)$ into direct products of mapping class groups of orbifolds and outer automorphisms of virtually abelian groups and into virtually abelian groups that we got from the existence of the higher rank JSJ decomposition in the previous section. However, the information that they currently give on the dynamics of individual automorphisms and on the algebraic structure of $\text{Out}(G)$ is essentially weaker.

We continue with the notation of the previous section. Hence, $X$ is a product of $m$ hyperbolic spaces $X_1, \ldots, X_m$, and $G$ is an HHG that acts on $X$ properly and cocompactly. $H$ is a characteristic finite index subgroup of $G$ that preserves the factors $X_1, \ldots, X_m$ and acts on them weakly acylindrically and isometrically.

Following section 3 in the previous paper in this sequence, with the action of $H$ on $X$ we associate a higher rank MR diagram. The higher rank diagram contains finitely many $m$-collections of cover resolutions, and each automorphism in $\text{Aut}(G)$ factors through at least one of the $m$-collections in the higher rank MR diagram. As in the previous section we analyze general higher rank MR diagrams of products by gradually relaxing the assumptions on their structure.

**Theorem 2.1.** Suppose that in each of the the (finitely many) collections of $m$ cover resolutions in the higher rank Makanin-Razborov diagram, each resolution has at most two levels.

Definition 1.6 associates a complexity with a 2-level resolution. Given an $m$-collection in which all the resolutions have at most two levels, we set its complexity to be the sequence of complexities of its $m$ resolutions, ordered in a non-increasing order. The set of complexities of $m$-collections can be ordered by ordering the complexities in a lexicographical order.
Since we assumed that all the resolutions in the m-collections in the higher rank MR diagram have at most two levels, there are maximal complexity m-collections in the higher rank MR diagram.

With H and the QH and the virtually abelian groups in the virtually abelian decompositions that are associated with the m resolutions in the maximal complexity m-collections we associate a groupoid GD1. The objects in GD1 are all the possible ordered collections of QH and virtually abelian vertex groups in the maximal complexity m-collections, where QH vertex groups are equipped with markings for their boundaries. Hence, the groupoid GD1 has finitely many objects.

With each outer automorphism τ ∈ Aut(G) we associate finitely many maps (arrows) µτ between the QH and the virtually abelian vertex groups that are associated with vertices in GD1.

(i) A map µτ consists of isomorphisms from the QH and the virtually abelian vertex groups that appear in the virtually abelian decompositions that are associated with the collection in the source vertex in GD1 to the QH and virtually abelian vertex groups that are associated with the target vertex in GD1. The isomorphisms are defined up to compositions with inner automorphisms.

(ii) With each arrow µτ in GD1 there is an inverse arrow µτ−1.

(iii) if τ = τ1 ◦ τ2, then for each arrow µτ, there exist arrows µτ1 and µτ2 such that: µτ = µτ1 ◦ µτ2.

Proof: Let Coll be one of the maximal complexity m-collections in the higher rank MR diagram. Since the resolutions in the m-collection have at most two levels, they have a weak test sequence. Let {φs} be a weak test sequence w.r.t. the m-collection Coll. Note that this implies that the sequence {φs} restricts to weak test sequences of all the m resolutions in the collection Coll.

Let τ ∈ Aut(G). Since {φs} is a weak test sequence w.r.t. a maximal complexity collection, the sequence {φs ◦ τ} must contain a subsequence that is a weak test sequence of at least one of the maximal complexity collections in the higher rank MR diagram.

Proposition 2.2. Let {φs} be a weak test sequence of a maximal m-collection Coll, let τ ∈ Aut(G), and suppose that {φs ◦ τ} factors through a maximal collection Coll1 from the higher rank MR diagram. Then:

(1) Coll1 is a maximal complexity m-collection.
(2) the sequence {φs ◦ τ} contains a subsequence that is a weak test sequence w.r.t. Coll1.
(3) the convergent subsequence of the sequence {φs ◦ τ}, which is a weak test sequence of the collection Coll1, defines maps that we denote µτ from the QH vertex groups and the virtually abelian vertex groups in Coll1 onto the QH vertex groups and virtually abelian vertex groups in Coll. These maps are defined up to a composition with inner automorphisms.

Proof: the weak test sequence, {φs}, converges into limit groups with virtual abelian decompositions that are associated with the maximal complexity m-collection Coll.

Let τ ∈ Aut(G). A subsequence of {φs ◦ τ} factors through some m-collection Coll1. We still denote the subsequence, {φs ◦ τ}. Since τ ∈ Aut(G), {φs ◦ τ}
converges into limit groups with virtually abelian decompositions that have the same structure as the virtually abelian decompositions that are associated with the \( m \)-collection \( Coll \). Since the complexities of of these virtually abelian decompositions are bounded by the complexities of the virtually abelian decompositions that are associated with \( Coll_1 \), and since they have the complexities of the maximal complexity collection \( Coll \), \( Coll_1 \) must be a maximal complexity collection as well. Furthermore, the virtually abelian decompositions that are associated with the limit of the sequence, \( \{ \varphi_s \circ \tau \} \) must be the virtually abelian decompositions that are associated with the \( m \)-collection \( Coll_1 \), hence, \( \{ \varphi_s \circ \tau \} \) is a weak test sequence for the \( m \)-collection \( Coll_1 \).

Since \( \{ \varphi_s \} \) and \( \{ \varphi_s \circ \tau \} \) converge into limit groups with virtually abelian groups that have the same structure, the automorphism \( \tau \) naturally maps the QH and virtually abelian vertex groups in the virtually abelian decompositions that are associated with \( \{ \varphi_s \circ \tau \} \) onto the QH and virtually abelian vertex groups in the virtually abelian decompositions that are associated with \( \{ \varphi_s \} \). Since QH and virtually abelian groups have the Hopf property, the homomorphisms that are associated with \( \tau \), that are necessarily epimorphisms, must be isomorphisms.

\[ \square \]

Proposition 2.2 proves that from any weak test sequence \( \{ \varphi_s \} \) w.r.t. a maximal collection \( Coll \), and an automorphism \( \tau \in Aut(G) \), it is possible to pass to a subsequence for which the precomposed sequence \( \{ \varphi_s \circ \tau \} \) is a weak test sequence w.r.t. some maximal complexity collection \( Coll_1 \). the two convergent sequences \( \{ \varphi_s \} \) and \( \{ \varphi_s \circ \tau \} \), enable us to define a map \( \mu_\tau \) between the QH and virtually abelian vertex groups in the virtually abelian decompositions that are associated with the \( m \) resolution in \( Coll_1 \), to those in \( Coll \), where the map is well-defined up to composition with an inner automorphism.

The next proposition proves that for each \( \tau \in Aut(G) \) there can only be finitely many distinct maps \( \mu_\tau \) that are constructed from all the weak test sequences w.r.t. all the maximal complexity collections in the higher rank MR diagram.

**Proposition 2.3.** Let \( \tau \in Aut(G) \) and let \( Coll_1 \) and \( Coll_2 \) be two maximal complexity \( m \)-collections in the higher rank MR diagram. Up to composition with inner automorphisms, there exist only finitely many maps \( \mu_\tau \), that are constructed according to part (iii) of proposition 2.2 from weak test sequences, \( \{ \varphi_s \} \) w.r.t. \( Coll_1 \) and \( \{ \varphi_s \circ \tau \} \) w.r.t. \( Coll_2 \).

**Proof:** To prove the proposition we use a compactness argument that is similar to the one that was used in the construction of the higher rank MR diagram.

Let \( \tau \in Aut(G) \). We look at the collection of all the sequences, \( \{ \varphi_s \} \), that factor through \( Coll_1 \) and converge into limit groups with virtually abelian decompositions that have the same structure as the virtually abelian decompositions that are associated with \( Coll_1 \). Furthermore, we require that \( \{ \varphi_s \circ \tau \} \) factor through \( Coll_2 \) and converge to limit groups with virtually abelian decompositions that have the same structure as the virtually abelian decompositions that are associated with \( Coll_2 \). Note that we don’t require these sequences to be weak test sequences w.r.t. \( Coll_1 \) and \( Coll_2 \).

By the same argument that was used to deduce part (iii) of proposition 2.2 (for weak test sequences), each pair of such sequences gives maps from the collection of QH and virtually abelian vertex groups in the virtually abelian decompositions that
are associated with \( Coll_2 \) isomorphically onto the collection of QH and virtually abelian vertex groups in the virtually abelian decompositions that are associated with \( Coll_1 \).

If there are only finitely many such maps, the conclusion of proposition 2.3 follows. Hence, we assume that there are infinitely many such maps (that are defined up to compositions with inner automorphisms). Clearly there can only be countably many such maps, and we order them.

We start with the given sequence of sequences, \( \{ \varphi^1_s \}, \{ \varphi^2_s \}, \ldots \), and the maps \( \{ \mu^j_\tau \} \) between the QH and virtually abelian vertex groups in \( Coll_1 \) and \( Coll_2 \) that are associated with them. By our assumptions, the maps \( \mu^j_\tau \) are distinct even after compositions with inner automorphisms.

We construct a new sequence. For each index \( j \), we choose an automorphism \( \psi_j \in Aut(G) \), such that:

1. \( \psi_j \) factors through \( Coll_1 \) and \( \psi_j \circ \tau \) factors through \( Coll_2 \).
2. the pair \( \psi_j \) and \( \psi_j \circ \tau \) do not factor through the maps \( \mu^i_\tau \), for \( i = 1, \ldots, j \).
3. the projections to the various factors of the images of \( H \) under \( \psi_j \) and \( \psi_j \circ \tau \) satisfy the relations and the inequalities in the (cover) limit groups that are associated with \( Coll_1 \) and \( Coll_2 \) for all the elements in a ball of radius \( j \) in \( H \) (w.r.t. a fixed set of generators of \( H \)). Furthermore, elements in the ball of radius \( j \) in \( H \) that are not mapped to the bounded vertex groups in the projections to the various factors in \( Coll_1 \) and \( Coll_2 \), are mapped by \( \psi_j \) and \( \psi_j \circ \tau \) to elements of length at least \( j \) times the bound on the lengths of the images of the (fixed) generators of the bounded vertex groups in \( Coll_1 \) and \( Coll_2 \) for each of the factor spaces of \( X \).

Such automorphisms \( \psi_j \) exist for every index \( j \), since we assumed that there are infinitely many distinct maps \( \mu^j_\tau \), so we can pick \( \psi_j \) to be an automorphism from the sequence: \( \{ \varphi^j_s \} \), for large enough \( s \).

From the map \( \psi_j \) we can pass to a convergent subsequence. This subsequence converges into limit groups with virtually abelian decompositions that have the same structure as the virtually abelian decompositions that are associated with the maximal complexity \( m \)-collection \( Coll_1 \). Hence, with the convergent sequences (still denoted), \( \psi_j \) and \( \psi_j \circ \tau \), it is possible to associate maps \( \nu_\tau \) between the QH and virtually abelian vertex groups in \( Coll_2 \) to these in \( Coll_1 \). But this map must be one of the maps \( \mu^j_{\tau_0} \) that were associated with \( \tau \) in the ordered sequence. Therefore, for large \( j > j_0 \), the automorphisms \( \psi_j \) and \( \psi_j \circ \tau \), factor through \( \mu^j_{\tau_0} \), a contradiction to the way the automorphisms \( \psi_j \) were chosen. Hence, there can only be finitely many maps \( \mu^j_\tau \) and the proposition follows.

Propositions 2.2 and 2.3 enable us to complete the proof of theorem 2.1. The finitely many objects in the groupoid \( GD1 \), are associated with all the possibilities for collections of QH vertex groups (with marked boundary components) and virtually abelian vertex groups in all the virtually abelian decompositions that are associated with resolutions in the maximal complexity \( m \)-collections in the higher rank MR diagram.

With each automorphism \( \tau \in Aut(G) \) we associate the finitely many maps \( \mu^j_\tau \), that are constructed from weak test sequences \( \{ \varphi_s \} \) and \( \{ \varphi_s \circ \tau \} \) w.r.t. maximal complexity collections \( Coll_1 \) and \( Coll_2 \). Proposition 2.2 explains how to construct these maps, and proposition 2.3 proves that there are only finitely many of them.
for each $\tau \in \text{Aut}(G)$. By construction, the maps $\mu_\tau^j$ depend only on the class of $\tau$ in $\text{Out}(G)$.

By construction, for every map $\mu_\tau^j$ there is a map $\mu_{\tau \tau^{-1}}^j$, so that their compositions in both orders are the corresponding identity maps. Furthermore, if $\tau = \tau_1 \circ \tau_2$, then for each of the maps $\mu_\tau^j$, there exist maps, $\mu_{\tau_1}^j$ and $\mu_{\tau_2}^j$, such that: $\mu_\tau^j = \mu_{\tau_1}^j \circ \mu_{\tau_2}^j$. This concludes the proof of theorem 2.1 □

In case all the resolutions in the higher rank MR diagram have at most two levels, theorem 2.1 associates with the maximal complexity collections in the diagram a groupoid. The groupoid is non-trivial if there are QH or virtually abelian vertex groups in the virtually abelian decompositions that are associated with maximal complexity collections in the diagram. The next theorem associates another groupoid with higher rank MR diagrams, for which the virtually abelian decompositions that are associated with their maximal complexity collections contain no QH nor virtually abelian vertex groups.

**Theorem 2.4.** With the assumptions of theorem 2.1, suppose that the virtually abelian decompositions that are associated with maximal complexity collections in the higher rank MR diagram contain no QH nor virtually abelian vertex groups.

With $H$ and the infinite virtually cyclic edge groups in the virtually abelian decompositions that are associated with the $m$ resolutions in the maximal complexity $m$-collections we associate a groupoid $GD_2$. The objects in $GD_2$ are the ordered collections of virtually infinite cyclic edge groups in the maximal complexity $m$-collections in the higher rank MR diagram.

With each outer automorphism $\tau \in \text{Out}(G)$ we associate finitely many arrows, that we denote: $\nu_\tau^j$, between pairs of vertices in $GD_2$. With each arrow, $\nu_\tau^j$, we associate an element in $\mathbb{Z}^n$ that we denote $\ell_\tau^j$.

(i) With each arrow $\nu_\tau^j$ in $GD_2$ there is an inverse arrow $\nu_{\tau^{-1}}^j$, $\ell_\tau^j + \ell_{\tau^{-1}}^j$ is a uniformly bounded element in $\mathbb{Z}^n$.

(ii) if $\tau = \tau_1 \circ \tau_2$, then for each arrow $\nu_\tau^j$, there exist arrows $\nu_{\tau_1}^j$ and $\nu_{\tau_2}^j$ such that the composition of the arrows that are associated with $\nu_{\tau_1}^j$ and $\nu_{\tau_2}^j$ is the arrow $\nu_\tau^j$, and $\ell_\tau^j - \ell_{\tau_1}^j - \ell_{\tau_2}^j$ is a uniformly bounded element in $\mathbb{Z}^n$.

**Proof:** The argument is similar to the proof of theorem 2.1. We define the complexity of a virtually abelian decomposition with no QH nor virtually abelian vertex groups, and only two levels, to be the following tuples of numbers:

1. the number of edges with trivial or finite edge groups.
2. the orders of the finite edge groups, from the smallest order to the biggest one (counted with multiplicities).
3. the number of edges with infinite virtually cyclic edge groups.

We order the complexities lexicographically, a virtually abelian decomposition $\Lambda_1$ has bigger complexity than $\Lambda_2$, if the number of edges in (1) in $\Lambda_1$ is bigger, or if there is equality in (1) and the tuples of orders in (2) of $\Lambda_1$ is smaller in lexicographical order than that of $\Lambda_2$, or if the tuples in (1) and (2) for $\Lambda_1$ and $\Lambda_2$ are equal, and the number of edges in (3) in $\Lambda_1$ is bigger than that of $\Lambda_2$.

We set the complexity of an $m$-collection of resolutions with at most two levels, with no QH nor virtually abelian vertex groups, to be the $m$-tuple of complexities of the virtually abelian decompositions that are associated with the $m$ (cover)
resolutions in the \(m\)-collection ordered in a non-increasing order. The complexities of such \(m\)-collections are naturally ordered lexicographically.

Given a higher rank MR diagram where all the resolutions in the \(m\)-collections have at most two levels, and all the virtually abelian decompositions that are associated with the resolutions have no QH nor virtually abelian vertex groups, we look at the maximal complexity \(m\)-collections in the higher rank diagram. If \(\{\varphi_s\}\) is a weak test sequence of such a maximal complexity \(m\)-collection, and \(\tau \in \text{Aut}(G)\), then \(\{\varphi_s \circ \tau\}\) must have a subsequence that is a weak test sequence of some maximal complexity \(m\)-collection in the higher rank diagram as well.

Given two maximal \(m\)-collections \(\text{Coll}_1\) and \(\text{Coll}_2\) from the higher rank MR diagram, and \(\tau \in \text{Aut}(G)\), We look at all the weak test sequences, \(\{\varphi_s\}\) of \(\text{Coll}_1\), for which \(\varphi_s \circ \tau\) is a weak test sequence of \(\text{Coll}_2\).

A weak test sequence \(\{\varphi_s\}\) of \(\text{Coll}_1\) converges into limit groups with the virtually abelian decompositions that are associated with \(\text{Coll}_1\). Similarly \(\{\varphi_s \circ \tau\}\) converges into limit groups with the virtually abelian decompositions that are associated with \(\text{Coll}_2\). Hence, \(\tau\) maps the vertex and edge groups in the virtually abelian decompositions that are associated with \(\text{Coll}_2\) onto the vertex and edge groups in the virtually abelian decompositions that are associated with \(\text{Coll}_1\). We denote this map \(\nu_\tau\).

With each \(m\)-collection, \(\text{Coll}\), we fix (finite) generating sets of all the vertex groups, and a generator of the maximal cyclic subgroup in all the edge groups in the \(m\) virtually abelian decompositions that are associated with the \(m\)-collection. By construction, each automorphism \(\varphi \in \text{Aut}(G)\) that factors through an \(m\)-collection \(\text{Coll}\), maps the fixed set of generators of a vertex group into elements, that map some point in the corresponding projection space, a uniformly bounded distance, i.e., a distance that depends only on the fixed generating set and not on the specific automorphism \(\varphi\).

Let \(\{\varphi_s\}\) and \(\{\varphi_s \circ \tau\}\), be two convergent weak test sequences that factor through the maximal complexity \(m\)-collections \(\text{Coll}_1\) and \(\text{Coll}_2\). Then the vertex and edge groups in the virtually abelian decompositions that are associated with the limits of \(\{\varphi_s \circ \tau\}\) are mapped by a map that we denote \(\nu_\tau\) to the vertex and edge groups in the virtually abelian decompositions that are associated with the limits of \(\{\varphi_s\}\).

Given an edge \(E\) in one of the virtually abelian decompositions in the \(m\)-collection \(\text{Coll}_1\), where \(E\) has an infinite virtually cyclic edge group, we look at the points \(p^1_s, p^2_s\) that move minimally by the images under \(\{\varphi_s\}\) of the fixed generating sets of the vertex groups that are adjacent to the edge, and at the points \(q^1_s, q^2_s\) that move minimally by the images under \(\{\varphi_s \circ \tau\}\) of the vertex groups in the virtually abelian decompositions in the \(m\)-collection \(\text{Coll}_2\) that are mapped by \(\nu_\tau\) to these two vertex groups.

By possibly composing the automorphism \(\tau\) with an inner automorphisms, we can assume that the distance between \(p^1_s\) and \(q^1_s\) is uniformly bounded. Furthermore, there is some power \(\ell_E\), such that \(\varphi_s(g_E)(p^2_s)\) has a uniformly bounded distance from \(q^2_s\), where \(g_E\) is a fixed generator of the maximal cyclic group in the virtually cyclic edge group that stabilizes the edge \(E\). Note that the power \(\ell_E\) is not uniquely defined, but it is defined up to some universal constant (that does not depend on the automorphism \(\tau\)).

Hence, with the convergent weak test sequences, \(\{\varphi_s\}\) and \(\{\varphi_s \circ \tau\}\), it is possible to associate a tuple of integers with the edges in the virtually abelian decompositions that are associated with the \(m\)-collection \(\text{Coll}_1\). Therefore, with these two weak
test sequences it is possible to associate an element in $\mathbb{Z}^n$, where $n$ is the total number of edges with infinite virtually cyclic edge groups in the virtually abelian decompositions that are associated with a maximal complexity collection in the higher rank MR diagram. We denote this element in $\mathbb{Z}^n$, $\ell_\tau$. Note that $\ell_\tau$ is not uniquely defined, but it is defined up to a uniformly bounded element in $\mathbb{Z}^n$.

Properties (i) and (ii) of the elements $\ell_j^\tau$ that are associated with an automorphism $\tau \in \text{Aut}(G)$ and all the possible convergent weak test sequences, $\{\varphi_s\}$ and $\{\varphi_s \circ \tau\}$, follow from the construction of the elements $\ell_j^\tau$. It is left to prove that with $\tau$ it is possible to associate only finitely many arrows, i.e., finitely many such elements $\ell_j^\tau$ (for all the convergent weak test sequences).

Suppose that there is an automorphism $\tau \in \text{Aut}(G)$, with an infinite sequence of elements $\ell_j^\tau$. In this case we can pass to an unbounded subsequence of elements $\ell_j^\tau$ in $\mathbb{Z}^n$.

We look at pairs of convergent weak test sequences of maximal complexity $m$-collections $\text{Coll}_1$ and $\text{Coll}_2$: $\{\varphi_s^1\}$ and $\{\varphi_s \circ \tau\}$, with the unbounded sequence of elements $\ell_j^\tau \in \mathbb{Z}^n$. In each sequence $\{\varphi_s^1\}$ we pick an automorphism $\psi_j$, such that the sequences, $\{\psi_j\}$ and $\{\psi_j \circ \tau\}$, converge. Since the sequence of elements $\{\ell_j^\tau\}$ is unbounded, we can pass to a further subsequence, for which at least one of the virtually abelian decompositions that is associated with the pair of sequences is a proper refinement of the corresponding virtually abelian decomposition that is associated with the $m$-collections $\text{Coll}_1$ and $\text{Coll}_2$. Since these $m$-collections are of maximal complexity, we got a contradiction, since the $m$-collection of virtually abelian decompositions that are associated with the convergent subsequences, $\{\psi_j\}$ and $\{\psi_j \circ \tau\}$, have strictly bigger complexity. Therefore, with each automorphism $\tau \in \text{Aut}(G)$ there are at most finite many associated arrows $\nu_j^\tau$ in the groupoid $GD2$.

Theorems 2.1 and 2.4 analyze higher rank MR diagrams in which all the resolutions in the $m$-collections in the diagram have no more than two levels. We further apply the arguments that were used in the previous section to generalize the results to $m$-collections with resolutions with arbitrary many levels. As in the previous section, in this paper we still assume that all the virtually abelian decompositions along the various levels contain no edges with finite (nor trivial) edge groups. These will be analyzed in the next paper in this sequence.

**Theorem 2.5 (cf. theorems 1.9 and 1.14).** Suppose that in each of the (finitely many) collections of $m$ cover resolutions in the higher rank Makanin-Razborov diagram all the virtually abelian decompositions that are associated with the various levels of the resolutions that have more than a single level do not contain edges with finite (nor trivial) edge groups (and in particular that the taut structure of all the QH vertex groups that appear in these decompositions are trivial).

Then it is possible to associate with the higher rank diagram a groupoid, similar to either the groupoid $GD_1$ that was constructed in theorem 2.1, or $GD_2$ that was constructed in theorem 2.4.

**Proof:** We start by assuming that all the quotient maps along the resolutions in the higher rank MR diagrams are proper quotients. In this case all the $m$-collections in the diagram have weak test sequences. We follow the analysis of resolutions that we used in the previous section, i.e., in case each $m$-collection has at most a single non-trivial resolution (theorem 1.9), and apply it to the $m$ resolutions of the
As in the proof of theorem 1.9, we first replace the higher rank MR diagram, by requiring that all the resolutions in the \( m \)-collections to be framed. i.e., the \( m \)-collections in the (framed) diagram have the property that automorphisms that factor through them do not extend to any \( m \)-collection of framed resolutions that strictly contain (or dominate) the \( m \)-collections in the (framed) higher rank diagram.

Continuing as in theorem 1.9, we modify the resolutions in each \( m \)-collection from the higher rank diagram using the procedure that is applied in the proof of theorem 1.9, and push QH and virtually abelian vertex groups to lower levels, if the edge groups that are connected to these vertex groups are all elliptic in the level below them.

With the modified \( m \)-collections in the higher rank diagram we associated a complexity, which is the \( m \)-tuple of complexities of the resolutions in the \( m \)-tuple in a non-increasing order (the complexity of a resolution is the complexity that is used in the proof of theorem 1.9). The complexities of \( m \)-collections are well-ordered, and as in theorem 1.9 we continue by analyzing maximal complexity \( m \)-collections.

Let \( Coll \) be an \( m \)-collection in the modified higher rank MR diagram. Let \( \{\phi_s\} \) be a weak test sequence w.r.t. \( Coll \), and let \( \tau \in Aut(G) \). By the construction of formal solutions that appears in [Se2], there exists a subsequence of pairs: \( \{\phi_s\} \) and \( \{\phi_s \circ \tau\} \), that converge into \( m \) homomorphisms: \( \nu^i_\tau : Comp(CRes^1_i) \to Cl(CRes_i) \), where \( CRes_i, i = 1, \ldots, m \), are the \( m \) resolutions in the \( m \)-collection \( Coll \), \( CRes^1_i \), \( 1, \ldots, m \), are the \( m \) resolutions in some \( m \)-collection \( Coll_1 \) from the modified higher rank MR diagram, and \( Cl(CRes_i), 1, \ldots, m \), are closures of the resolutions \( CRes_i \).

Suppose that for every maximal complexity \( m \)-collections in the higher rank MR diagram, and every weak test sequence that factors through it, there exists some automorphism \( \tau \in Aut(G) \) (that depends on the sequence), such that the pair of sequences, the weak test sequence and the weak test sequence twisted by \( \tau \), have a convergent subsequence that satisfy parts (1) or (2) in theorem 1.10, for at least one of the maps \( \nu^i_\tau \), \( i = 1, \ldots, m \).

In that case we apply the argument that was used to prove lemma 1.12 to the given higher rank MR diagram, and replace it by another higher rank diagram with strictly smaller maximal complexity. Repeating this procedure iteratively, and using the descending chain condition for complexities of \( m \)-collections, we get a new higher rank MR diagram, with smaller maximal complexity \( m \)-collections, in which maximal complexity \( m \)-collections admit weak test sequences, such that for every automorphism \( \tau \in Aut(G) \), and every convergent subsequence of pairs, part (3) of lemma 1.10 holds. In that case, as in the proof of theorem 1.9, we continue with all the weak test sequences of maximal complexity \( m \)-collections that have this property.

Suppose first that there are QH or virtually abelian vertex groups along some of the resolutions in a maximal complexity \( m \)-collection, and that there exists an index \( i, i = 1, \ldots, m \), such that \( L_i \), the limit quotient of \( H \) that is mapped in \( Comp(CRes_i) \), intersects a conjugate of some QH vertex group in \( Comp(CRes_i) \) in a subgroup of finite index, or a conjugate of some virtually abelian vertex group in a subgroup of rank at least 2.

In this case we can associate a groupoid with \( Aut(G) \), that is similar to the groupoid \( GD1 \) that was constructed in theorem 2.1. The vertices in this groupoid are the collections of QH vertex groups (with marked boundaries) that intersect a...
conjugate of the corresponding limit quotient of $H$, $L_j$, in a subgroup of finite index, and of virtually abelian vertex groups that intersect a conjugate of a corresponding limit quotient of $H$ in a subgroup of rank at least 2.

By the same argument that was used in the proof of theorem 2.1, using the collection of all the weak test sequences of maximal complexity $m$-collections that satisfy part (3) in lemma 1.10, it is possible to associate with each (outer) automorphism $\tau \in Aut(G)$ finitely many maps between these collections, and the composition properties of these maps that were proved in theorem 2.1 hold in this general case by exactly the same arguments.

Suppose that maximal complexity resolutions in the higher rank diagram do not contain QH vertex groups that intersect conjugates of the associated limit group $L_j$ in a finite index subgroup, nor virtually abelian vertex groups that intersect conjugates of $L_j$ in a subgroup of rank at least 2. In that case we associate with the maximal complexity $m$-collections and their weak test sequences a groupoid similar to the groupoid $GD_2$ that was constructed in theorem 2.1, in case the resolutions in the higher rank MR diagram have at most two levels.

Let $MColl$ be a maximal complexity $m$-collection in the higher rank diagram. For presentation purposes, suppose first that the $m$ resolutions in the maximal complexity $m$-collection $MColl$ contains no QH and no virtually abelian vertex groups.

For each $i$, $1 \leq i \leq m$, we look at the sequence of virtually abelian decompositions that the limit group $L_i$ inherits from the virtually abelian graphs of groups that are associated with the $i$-th resolution in the $m$-collection, from top to bottom. i.e., we look at the virtually abelian decomposition that $L_i$ inherits from the top virtually abelian decomposition of the $i$-th resolution. We denote this decomposition, $\Delta^i_1$.

We continue with each of the vertex groups in $\Delta^i_1$, and look at the virtually abelian decomposition that it inherits from the virtually abelian decomposition that is associated with the virtually abelian decomposition that is associated with the second level of the $i$-th resolution in $MColl$. We get finitely many virtually abelian decompositions that are associated with the vertex groups in the first level. We continue with the vertex groups in these decompositions, and look at the virtually abelian decompositions that they inherit from the virtually abelian decomposition that is associated with the third level in the $i$-th resolution in $MColl$, and so on.

Following section 4 in [Se1], we call the iterative hierarchical set of virtually abelian decompositions that are obtained with the limit groups $L_i$, $i = 1, \ldots, m$, in the maximal complexity $m$-collection, $MColl$, its analysis lattice. Note that the analysis lattices that are associated with all the subset of maximal complexity $m$-collections in the higher rank diagram, on which the automorphism group of the hierarchical group $G$ acts transitively, have similar structure.

The objects in the groupoid of the second type, $GD_2$, are the ordered edges with virtually infinite cyclic edge groups along the analysis lattices of the maximal complexity resolutions in the higher rank MR diagram. The morphisms are arrows between these objects, and with each arrow there is an associate element in $\mathbb{Z}^n$, where $n$ is the number of edges in the objects, i.e., the number of edges with virtually infinite cyclic edge groups in the analysis lattice.

As in theorem 2.4, with element $\tau \in Aut(G)$ (or rather in $Out(G)$) we associate finitely many morphisms in the groupoid. We look at all the weak test sequences, $\{\phi_s\}$ of maximal complexity 4m4-collections, $MColl$, in the higher rank MR diagram. We further assume that for every automorphism $\mu \in Aut(G)$, and
every convergent subsequence of pairs, \(\{\varphi_s\}\) and \(\{\varphi_s \circ \mu\}\), where the automorphisms in the second subsequence factor through some \(m\)-collection \(\text{Coll}_1\) in the higher rank MR diagram, \(\text{Coll}_1\) is of maximal complexity, and the corresponding \(m\) maps: \(\nu^i_\tau : \text{Comp}(\text{CRes}^i_1) \to \text{Cl}(\text{CRes}_i)\) satisfy property (3) in lemma 1.10 for all \(i = 1, \ldots, m\).

To associate a morphism with \(\tau \in \text{Aut}(G)\), we start with such a weak test sequence of a maximal complexity \(m\)-collection, \(\text{MColl}\), and pass to a convergent subsequence of pairs, \(\{\varphi_s\}\) and \(\{\varphi_s \circ \tau\}\). The subsequence, \(\{\varphi_s \circ \tau\}\) factors through some maximal complexity \(m\)-collection, \(\text{Coll}_1\), and the pair of subsequences converges into \(m\) maps: \(\nu^i_\tau : \text{Comp}(\text{CRes}^i_1) \to \text{Cl}(\text{CRes}_i)\) that satisfy property (3) in lemma 1.10 for all \(i = 1, \ldots, m\).

The maps \(\nu^i_\tau\) restrict to map the virtually infinite cyclic edge stabilizers in the analysis lattice of each of the limit groups \(L_i\) in \(\text{Coll}_1\), and the vertex groups that are adjacent to these edges, to conjugates of the virtually infinite cyclic edge stabilizers in the analysis lattice of the closures, \(\text{Cl}(\text{CRes}_i)\), and the vertex groups that are adjacent to them, that are images of these edge stabilizers and the vertex groups that are adjacent to them in the analysis lattice of the maximal complexity collection, \(\text{MColl}\).

With the map \(\nu^i_\tau\) and each edge group with virtually infinite cyclic edge stabilizer in the analysis lattice of \(\text{Coll}_1\), and the vertex groups that are adjacent to it, we associate an element in \(Z\), precisely as we did in the construction of \(GD_2\) in theorem 2.4. Hence, with \(\nu^i_\tau\) and all the edges with virtually infinite cyclic edge groups in the analysis lattice of \(\text{Coll}_1\) we associate an element in \(Z^n\).

With the map \(\nu^i_\tau\) we associate a morphism between the set of edges with virtually infinite cyclic edge groups in the analysis lattice of \(\text{Coll}_1\) to the set of edges of virtually infinite cyclic edge groups in the analysis lattice of \(\text{MColl}\) (both sets of edges are ordered). With the morphism we have associated an element in \(Z^n\).

The set of morphisms that is associated with the automorphism \(\tau \in \text{Aut}(G)\), is the set of such morphisms and elements in \(Z^n\) that are associated with all the convergent pairs, \(\{\varphi_s\}\) and \(\{\varphi_s \circ \tau\}\), that are associated with weak test sequences \(\{\varphi_s\}\) of maximal complexity \(m\)-collections that satisfy the conditions above.

By the same argument that was used in the proof of theorem 2.4, there are only finitely many such morphisms that are associated with an automorphism \(\tau\). Furthermore, the elements in \(Z^n\) that are associated with the morphisms satisfy the quasi-morphism laws that are listed in theorem 2.4.

Suppose that the resolutions in maximal complexity \(m\)-collection in the higher rank MR diagram contain QH and virtually abelian vertex groups, but the limit groups that are associated with the resolutions do not intersect conjugates of QH vertex groups in subgroups of finite index, and do not intersect conjugates of virtually abelian vertex groups in subgroups of rank at least 2.

In this case we also look at the analysis lattices of the maximal complexity \(m\)-collections in the higher rank MR diagram, and the objects in the groupoid are also the ordered edges with virtually infinite cyclic edge groups in the analysis lattices, that connect between two non-virtually abelian vertex groups. Morphisms are defined in the same way that they were defined in the absence of QH and virtually abelian vertex groups, and with each morphism we associate an element in \(Z^n\), where \(n\) is the number of edges in an object.

Once again and as in theorem 2.4, with each automorphism \(\tau \in \text{Aut}(G)\) we
associate finitely many morphisms, each morphism has a quasi-inverse, and the elements in $Z^n$ that are associated with the morphisms satisfy the (uniform) quasi-morphism law that is listed in theorem 2.4.

So far we assumed that all the quotient maps along the resolutions in the higher rank MR diagram are proper quotients. In that case all the $m$-collections in the higher rank diagram have weak test sequences, that were used in the construction of the groupoids $GD_1$ and $GD_2$.

As in theorem 1.14, in general we modify the construction of the higher rank MR diagram according to the construction of the diagram for pairs (semigroups) in [Se9]. The construction in [Se9] guarantees that $m$-collections have weak test sequences also in cases in which some of the quotient maps along the resolutions in the collection are isomorphisms (and not proper quotients). And we further replace resolutions in the constructed $m$-collections with framed resolutions, as we did in the proper quotient case. The rest of the construction of the groupoids $GD_1$ and $GD_2$ is identical, and the conclusion of theorem 2.5 holds for all single ended higher rank diagrams.

Note that our construction of groupoids that are associated with higher rank MR diagrams that contain $m$-collections with more than a single non-trivial resolution (theorem 2.5) is considerably weaker than theorems 1.9 and 1.14. This raises several questions:

**Question 1.** Can the number of morphisms that are associated with each automorphism in $\text{Aut}(G)$ in theorems 2.1, 2.4 and 2.5 be uniformly bounded? we only proved that it is finite. Note that if there is only a single morphism that is associated with each automorphism in $\text{Aut}(G)$ and a source object in the groupoid, then in the case of the groupoid $GD_1$, it is possible to obtain a homomorphism from $\text{Out}(G)$ into direct products of mapping class groups and outer automorphism groups of virtually f.g. abelian groups, which is similar to what is known for hyperbolic groups and to theorems 1.9 and 1.14.

**Question 2.** In proving theorem 2.5, although the maps $\nu^i_\tau$ that we constructed map edge groups to edge groups, we were not able to prove that the higher rank diagram can be replaced by a diagram with at most two level resolutions, as we proved in theorems 1.9 and 1.14. Can the higher rank MR diagram be replaced by a diagram with only two levels resolutions?

## §2. A higher rank JSJ decomposition of (some) HHG

In the previous section we used the higher rank MR diagram that was constructed in [Se8], to construct groupoids that are associated with the automorphism group of a group that acts discretely and cocompactly on a product of finitely many hyperbolic spaces. We further required that the action of the group is strongly acylindrical, which means that the action of the group on each factor, via the corresponding projection, is weakly acylindrical (see definition 3.1 in [Se8]).

In [Se8] we first constructed a higher rank MR diagram for the automorphism group of a group that acts discretely and cocompactly on a product of hyperbolic spaces (section 3 in [Se8]), and then modified and generalized the construction of the higher rank MR diagram to HHGs that satisfy a weak acylindrical condition,
and for which there exists a finite index subgroup so that spaces in an orbit of a projection space under the action of the finite index subgroup are transverse. These conditions do not hold for all HHG (e.g. Burger-Mozes groups), but they both hold for the mapping class groups, due to seminal works of Bowditch [Bo] and Bestvina-Bromberg-Fujiwara [BBF1].

In section 4 of [Se8], we analyze resolutions of automorphisms of an HHG $G$ under two possible assumptions. Under both assumptions it is possible to generalize the analysis of automorphisms of HHG that acts on products that was used in the first two sections. We start with the stronger assumption from section 4 in [Se8].

**Theorem 3.1.** Let $G$ be a HHG, and suppose that there exists a finite index subgroup $H < G$, such that spaces in an orbit of a projection space under the action of $H$ are transverse, and the (finitely many) actions of $H$ on the quasi-trees of metric spaces that are constructed from the action of $H$ on these orbits, via the [BBF1] construction, are weakly acylindrical.

By theorem 4.3 in [Se8], with these assumptions we can associate with $\text{Aut}(G)$ a higher rank MR diagram. Suppose that all the virtually abelian decompositions along this diagram do not contain edges with finite (nor trivial) edge groups.

Then it is possible to associate with the higher rank diagram a groupoid, similar to either the groupoid $GD_1$ that was constructed in theorem 2.1, or $GD_2$ that was constructed in theorem 2.4, precisely as we associated these groupoids with HHGs that act on products in theorem 2.5.

**Proof:** Having a higher rank MR diagram, and the weakly acylindrical action of $H$ on the (finitely many) quasi-trees of metric spaces, all the analysis of automorphisms of HHGs that act on product spaces, as it appears in the first two sections of this paper, generalize to HHGs that satisfy the assumption of the theorem. In particular, the conclusions of theorems 2.1, 2.4 and 2.5 remain valid, and we can associate the groupoids $GD_1$ and $GD_2$ with such HHGs.

The second possible assumption in section 4 of [Se8] is weaker and requires finer analysis and somewhat weaker notion of a cover resolution in order to construct a higher rank MR diagram, and then apply the analysis of automorphisms that appear in the first two sections of this paper, to associate the groupoids $GD_1$ and $GD_2$ with such HHGs.

**Theorem 3.2.** Let $G$ be a HHG, and suppose that there exists a finite index subgroup $H < G$, such that spaces in an orbit of a projection space under the action of $H$ are transverse, $H$ acts weakly acylindrically on the (finitely many) projection complexes that are constructed from the actions of $H$ on these orbits (using [BBF1]), and the set stabilizers of each projection space acts weakly acylindrically on the projection space.

Using the techniques that appear in section 4 of [Se8], with these assumptions we can associate with $\text{Aut}(G)$ a higher rank MR diagram. Let $m$ be the number of orbits of projection spaces under the action of $H$. Each $m$-collection in the higher rank MR diagram, contains $m$ hybrid cover resolutions. Each hybrid cover resolution consists of a resolution of an ambient limit quotient of $H$, followed by finitely many resolutions of quotients of f.g. subgroups (cf. proposition 4.5 in [Se8]).

Suppose that all the virtually abelian decompositions along the hybrid resolutions in this diagram do not contain edges with finite (nor trivial) edge groups.
Then it is possible to associate with the higher rank diagram a groupoid, similar to either the groupoid $GD_1$ that was constructed in theorem 2.1, or $GD_2$ that was constructed in theorem 2.4, precisely as we associated these groupoids with HHGs that act on products in theorem 2.5.

**Proof:** Using the finitely many actions of the finite index subgroup $H$ on the orbits of the projection spaces, we construct finitely many projection complexes that we denote, $P^j_K$, $j = 1, \ldots, m$, by applying the construction in [BBF1] (see theorem 4.2 in [Se8]).

Let $\{\varphi_s\}$ be a sequence of automorphisms in $Aut(G)$. Since we assume that the actions of $H$ on the projection complexes, $P^j_K$, are weakly acylindrical, we pass to a convergent subsequence, and apply the construction that appears in section 3 in [Se8] to get an $m$-collection of resolutions, that are associated with the various projection complexes.

We continue with each of the $m$ resolutions in parallel. A resolution $Res$ from the $m$-collection terminates in a f.g. limit group $L$, with a subsequence of quasimorphisms of $H$, $\{\psi_s\}$, that are obtained from the automorphisms, $\{\varphi_s\}$, through the resolution $Res$, that converges into $L$.

We fix a generating set for the limit group $L$. Since we assumed that the virtually abelian decompositions along the resolutions have no edges with finite edge group, there is a uniform bound $b$, such that for every quasimorphism from the subsequence, $\{\psi_s\}$, that converges into $L$, there is some point $p_s$ in the corresponding projection complex, that moves to a point in a distance bounded by $b$ (in the projection complex) by the image under $\psi_s$ of each of the fixed set of (fixed preimages of) generators of the terminal limit group $L$.

In section 4 in [Se8] we continued by analyzing the action of $L$ on the quasi-tree of metric spaces that is constructed from the action of $H$ on its corresponding orbit of projection spaces, via the construction in [BBF1]. These are denoted $C_{jK}$. Note that the assumptions of theorem 3.2 do not imply that the action of $H$ on the quasi-tree of metric spaces is weakly acylindrical. The theorem assumes that the action of each set stabilizer of a projection space on the projection space is weakly acylindrical (which is a weaker assumption).

In proposition 4.5 in [Se8], we first associated a finite bipartite graph of groups with the action of $L$ on the constructed quasi-tree of metric spaces $C_{jK}$. A subset of vertex groups in this graph of groups $\Theta$ are set stabilizers of spaces in the orbit of the corresponding projection space, that we denoted $W^I_f$. For every finite collection of elements, $F_i$, in a vertex group that is not stabilized by one of the set stabilizers, $W^I_f$, there is some constant $b_{F_i}$, such that for every quasimorphism from the convergent subsequence $\{\psi_s\}$, there exists a point $p_{F_i}$ in some space in the orbit of the projection space $V$ (the point depends on the quasimorphism), that is moved a distance bounded by no more than $b_{F_i}$ by the images (under the quasimorphism $\psi_s$) of all the elements in the finite set $F_i$.

Furthermore, each set stabilizer, $W^I_f$, in $\Theta$, is generated by the finitely many edge groups that are connected to the vertex that it stabilizes in the graph of groups $\Theta$, $\{E^i\}$, together with finitely many elements.

In proposition 4.10 in [Se8] we further prove that it is possible to pass to a convergent subsequence of the sequence of quasimorphisms $\{\psi_s\}$, and obtain from it resolutions of the finitely many set stabilizers $W^I_f$. Note that $W^I_f$ are not f.g. in general, but it is f.g. relative to finitely many elliptic subgroups in it, and these
elliptic subgroups are only conjugated along the resolution.

Hence, by theorem 3.6 and propositions 4.5 and 4.10 in [Se8], given a sequence of automorphisms, \( \{\varphi_s\} \) in \( \text{Aut}(G) \), it is possible to pass to a subsequence that converges into an \( m \)-collection of hybrid resolution. Each hybrid resolution starts with a resolution of some limit quotient of the finite index subgroup \( H < G \), and continues with finitely many resolutions of the relative f.g. set stabilizers in the terminal limit group of the resolution we started with.

Hybrid resolutions are the main tool in the analysis of first order sentences and formulas in [Se4] and [Se6]. Indeed, they were used to construct an anvil and developing resolutions that are the main objects in constructing terminating procedures for the analysis of sentences and definable sets over free and other groups.

As in theorem 3.6 in [Se8], that deals with the case of a product space, our next goal is to use the \( m \)-collections of hybrid resolutions that are obtained from convergent sequences of automorphisms, to construct a higher rank MR diagram, i.e., to obtain finitely many \( m \)-collections of hybrid cover resolutions, such that every automorphism in \( \text{Aut}(G) \) factors through at least one of them.

To find such finitely many \( m \)-collections of hybrid cover resolutions, we use a compactness argument, similar to the one that was used in proving theorem 3.6 in [Se8] (in the product case). In hybrid resolutions, some of the groups along the resolutions are f.g. and not f.p. and even worse, the limit subgroups that stabilize setwise a given projection space are in general not f.g. nor are the elliptic vertex groups that are connected to these projection space stabilizers in the graph of groups \( \Theta \), nor the edge groups that connect the stabilizers of the projection spaces to the elliptic vertex groups. Furthermore, the finitely many resolutions of the stabilizers of projection spaces in each hybrid resolution, are resolutions of subgroups that are in general not f.g. - they are f.g. relative to finitely many elliptic subgroups. Hence, we need to find covers of hybrid resolutions that are f.p. objects, and that encode the entire geometry of the hybrid resolutions.

Let \( HbRes \) be a hybrid resolution. By construction, the top resolution in a hybrid resolution that we denote \( TPRes \), i.e., the resolution that was constructed from actions of \( H \) on one of the projection complexes, \( P^K_j, 1 \leq j \leq m \), is a resolution that has the same structure as the resolutions that were constructed in the case of product spaces. Hence, with the resolution \( TPRes \) we associate a cover in the same way that it was defined in [Ja-Se] and in definition 3.4 in [Se8]. i.e., the terminal limit group of the resolution \( TPRes \) is replaced by some f.p. approximation, and on top of this terminal f.p. group we construct a completion into which the covers of all the limit groups along the cover resolution are embedded (see definition 3.4 in [Se8]). Clearly, the whole completion is a f.p. group, that terminates in a f.p. limit group that we denote \( L_t \).

We continue with the graph of groups \( \Theta \), that is part of the hybrid resolution. The graph of groups \( \Theta \) is a bipartite graph of groups, where some of the vertices are elliptic (or bounded), and the others are limits of set stabilizers of projection spaces. Note that vertex and edge groups in \( \Theta \) are not necessarily finitely generated. Let \( D_t \) be the fundamental group of \( \Theta \), which is the terminal group of the top resolution \( TPRes \) in the hybrid resolution \( HbRes \). \( L_t \) is a f.p. cover of \( D_t \). Hence, there is a finite set of elements from the various vertex groups in \( \Theta \), that together with the finite set of Bass-Serre generators in \( \Theta \), generate \( D_t \). We fix this generating set of \( D_t \).

Let \( W^f \) be a vertex group which is a limit of sets stabilizers of a projection space.
By Proposition 4.5 in [Se8], $W^f$ is generated by finitely many elliptic subgroups (the edge groups that are connected to the vertex that is stabilized by $W^f$ in the graph of groups $\Theta$), and additional finitely many elements. By proposition 4.10 in [Se8] $W^f$ admits a finite resolution:

$$W^f/K^f = W^f_1/K^f_1 \rightarrow \ldots \rightarrow W^f_r/K^f_r$$

where $K^f_i$, $i = 1, \ldots, r$, are the stable kernels, i.e., the collection of elements that act stably quasi-trivially on the associated projection space under the corresponding sequence of quasimorphisms.

Under the assumptions of theorem 3.2, the virtually abelian decompositions that are associated with the various levels of the resolution do not contain edges with finite edge groups, and $W^f_r$ is a bounded group. i.e., for each finite set of elements $F_i$ in $W^f_r$, there is some bound $b_{F_i}$, such that for any quasimorphism from the sequence that converges to $W^f_r$, there exists a point in the projection space that is moved a distance bounded by $b_{F_i}$, by all the elements in the set $F_i$.

$W^f$ is not necessarily a f.g. group, but since it is generated by finitely many bounded subgroups in addition to finitely many elements, there exists a f.g. subgroup $R < W^f$, that we can assume contains the finitely many elements in $W^f$ that are part of the fixed generating set of $D_t$, such that $R$ obtains a resolution:

$$R/KR = R_1/KR_1 \rightarrow \ldots \rightarrow R_r/KR_r$$

where $R_i < W^f_i$, $KR_i = K_i \cap R_i$, the virtually abelian decompositions that are associated with the groups $R_i/KR_i$ have the same structure as those of $W^f_i/K_i$, and the quotient maps in the resolutions of $R/KR$ and $W^f/K$ together with the embedding of the groups $R_i$ into $W^f_i$ form a commutative diagram.

Altogether, we have a finite set of generators of $D_t$, the fundamental group of the graph of groups $\Theta$, to which we add finite sets of generators of the groups $R^f$ that approximate the limit stabilizers of projection spaces $W^f$ in $\Theta$ (and have resolutions with the same structure as those of $W^f$). We continue with this fixed finite set of generators of $D_t$, that are all elements in vertex groups in $\Theta$ and Bass-Serre generators in $\Theta$.

Recall that $D_t$ is a quotient of its f.p. approximation $L_t$. We further define an abstract f.p. group, $U_t$, that is generated by copies of the fixed set generators of $D_t$, is naturally a quotient of $L_t$ and $D_t$ is a quotient of it, and it is the fundamental group of a graph of groups with f.g. vertex and edge groups that is similar to $\Theta$. i.e., the quotient map from $U_t$ onto $D_t$ preserve the graphs of group structures of both - vertex and edge groups are mapped into vertex and edge groups.

Note that the relations between the fixed finite set of elements in $D_t$, that generate $D_t$ and are all contained in vertex groups in $\Theta$ or are Bass-Serre generators, can be taken to be relations in each of the vertex groups, and relations that correspond to foldings, i.e., relations that correspond to an enlargement of an edge group. Since these relations preserve the graph of groups structure, we can construct $U_t$ by adding finitely many such relations that will enable us to define a natural map from $L_t$ onto $U_t$, and will guarantee that $U_t$ is the fundamental group of a graph of groups that has a similar graph of groups structure as $\Theta$, just that its vertex and edge groups are all finitely generated. Furthermore, the map from $U_t$ onto $D_t$ respects the graph of groups structure, i.e., vertex and edge groups in the graph of groups decomposition of $U_t$ are mapped to vertex and edge groups in $\Theta$.

At this point we are ready to define a cover of the hybrid resolution, $HbRes$. 

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Definition 3.3. The cover $CHbRes$ of $HbRes$ is constructed from:

1. a f.p. cover of the completion of the top resolution, $TPRes$, that terminates with the f.p. group $L_t$.
2. the f.p. group $U_t$.
3. f.p. covers of the completions of the resolutions of the subgroups $R^f$, that are themselves f.g. approximations of the resolutions of the limit set stabilizers of the projection spaces, $W^f$ (that are vertex groups in the graph of groups $\Theta$).

$CHbRes$, the cover of the hybrid resolution $HbRes$, is the group that is generated by the f.p. covers from parts (1) and (2), and the f.p. group $U_t$, to which we add finitely many relations. We identify the fixed set of generators generators of $L_t$ (the terminal group in the cover of $TPRes$ with their image in $U_t$. We further identify the elements that are associated with the generators of each of the group $R^f$ in $U_t$, with the elements that are associated with these generators in the f.p. cover of the corresponding resolution of the group $R^f$ from part (3).

Clearly, the group that is associated with $CHbRes$ is f.p. and the convergent subsequence of automorphisms $\{\varphi_s\}$ that was used to obtain the hybrid resolution, $HbRes$, asymptotically factors through it. i.e., all the automorphisms from the convergent subsequence factor through $CHbRes$, except for at most finitely many of them.

Once we defined a (f.p.) cover of a hybrid resolution, we can apply a compactness argument and obtain a higher rank MR diagram for HHG that satisfy the conditions of theorem 3.2.

Proposition 3.4. Let $G$ be an HHG that satisfies the conditions of theorem 3.2. Then there exists a characteristic finite index subgroup $H < G$, and a finite collection of $m$-collections of (f.p.) covers of hybrid resolutions of limit quotients of $H$, such that every automorphism in $\text{Aut}(G)$ factors through at least one of these $m$-collections of covers of hybrid resolutions.

We call such a finite collection of $m$-collection of covers of hybrid resolutions, a higher rank MR diagram of the HHG $G$.

Proof: Covers of hybrid resolutions are f.p. so there are only countably many $m$-collections of such covers, and they can be ordered.

Now, we apply the compactness argument that was used to prove theorem 3.6 in [Se8], together with propositions 4.5 and 4.10 in [Se8] that construct hybrid resolutions from convergent sequences of automorphisms. The outcome is a finite $m$-collections of covers of hybrid resolutions, so that every automorphism in $\text{Aut}(G)$ factors through at least one of them.

With an $m$-collection of (covers of) hybrid resolutions we can associate its complexity.

Definition 3.5. Let $HbRes$ be a hybrid resolution. We set its complexity to be a tuple, in which the leading term is the complexity of its top resolution, $TPRes$ (i.e., the resolution that was constructed from the projection complexes $P^i_K$), followed by the sequence of complexities of the bottom resolutions of the subgroups $W^f$, ordered in a non-increasing lexicographical order. The complexity of a cover of a hybrid resolution is the complexity of the hybrid resolution that it covers.
The complexity of an \( m \)-collection of (covers of) hybrid resolutions is set to be the \( m \) complexities of the hybrid resolutions ordered in the \( m \)-collection, ordered in a non-increasing lexicographical order.

We continue with the proof of theorem 3.2, precisely as we did in the proofs of theorems 2.5 and 3.1. Each hybrid resolution can be extended in finitely many ways to a framed hybrid resolution, precisely as in the case of resolutions. Hence, the higher rank MR diagram can be replaced by a finite set of \( m \)-collections of covers of framed hybrid resolutions. Furthermore, by construction, each \( m \)-collection of covers of framed hybrid resolutions in the higher rank MR diagram has weak test sequences that factor through it and do not extended to framed hybrid resolutions that strictly extends the hybrid resolutions in the \( m \)-collection.

Once we have a higher rank MR diagram of \( m \)-collections of covers of framed hybrid resolutions, and their weak test sequences, the proof of theorem 3.2 continues precisely as the proofs of theorems 3.1 and 2.5.

The construction of the JSJ holds for all f.p. groups, although in this generality it does not encode the whole automorphism group, but rather all the splittings of the group over a family of edge groups, or equivalently all the actions of the group on simplicial trees with prescribed family of edge groups.

In a similar way our results for HHGs can be stated for all f.p. groups and their actions on HHS, if we require that the set of actions satisfy some uniform hyperbolicity and weak acylindricity properties, that are needed to prove theorems 3.1 or 3.2.

**Theorem 3.6.** Let \( G \) be a f.p. group, let \( g_1, \ldots, g_\ell \) be some fixed generating set of \( G \), and let \( c, \delta > 0 \), and \( m_0, n_0 \) be positive integers. With \( \text{Out}(G) \) it is possible to associate a higher rank Makanin-Razborov diagram, similar to the one that was constructed in theorem 3.6 in [Se8] (i.e., with hybrid resolutions), that is constructed from actions of \( G \) on all the HHS \( X \) with the following properties:

1. There exists a point \( x_0 \in X \) that moves a distance at most \( c \) by all the fixed set of generators: \( g_1, \ldots, g_\ell \).
2. The projection spaces in \( X \) are \( \delta \)-hyperbolic.
3. There are at most \( m_0 \) orbits of orbits of projection spaces in \( X \) under the action of \( G \).
4. There exists a finite index subgroup of index at most \( n_0 \), \( H < G \), such that the orbits of the projection spaces in \( X \) under the action of \( H \) are transversal.
5. The projection complexes that are constructed from the action of \( H \) on \( X \) (using [BBF]), are constructed uniformly for all \( H \) and \( X \), and the actions of \( H \) on these projection complexes are uniformly weakly acylindrical. Furthermore, the action of a subgroup in \( H \) that stabilize (setwise) a projection space in \( X \) is uniformly weakly acylindrical (i.e., with the same weakly acylindrical constants for all \( X \) and \( H \)).

If the higher rank MR diagram is single ended, then with \( \text{Out}(G) \) it is possible to associate groupoids \( GD_1 \) and \( GD_2 \), as in theorems 3.2.

**Proof:** Under the conditions of the theorem, the procedure for the construction of the higher rank diagram in section 3 in [Se8] is valid. If the diagram is single ended, then the proof of theorem 3.2 generalize to construct the groupoids.
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