ON THE NEGATIVE-ONE SHIFT FUNCTOR FOR FI-MODULES

WEE LIANG GAN

Abstract. We show that the negative-one shift functor $\tilde{S}_{-1}$ on the category of FI-modules is a left adjoint of the shift functor $S$ and a right adjoint of the derivative functor $D$. We show that for any FI-module $V$, the coinduction $QV$ of $V$ is an extension of $V$ by $\tilde{S}_{-1}V$.

1. Introduction

The shift functor $S$ and the derivative functor $D$ on the category of FI-modules have played an essential role in several recent works, for example, [1, 4, 5, 6, 7, 9]. The modest goal of this article is to explain how they are related to the negative-one shift functor $\tilde{S}_{-1}$ introduced in [2]. We also explain how the coinduction functor $Q$ defined in [3] is related to $\tilde{S}_{-1}$.

Let us begin by recalling some definitions.

Let $k$ be a commutative ring. Let $C$ be a small category. A $C$-module is a functor from $C$ to the category of $k$-modules. A homomorphism of $C$-modules is a natural transformation of functors. For any $C$-module $V$ and $X \in \text{Ob}(C)$, we write $V_X$ for $V(X)$.

Let $\text{FI}$ be the category whose objects are the finite sets and whose morphisms are the injective maps.

Fix a one-element set $\{\ast\}$ and define a functor $\sigma : \text{FI} \to \text{FI}$ by $X \mapsto X \sqcup \{\ast\}$ for each finite set $X$. If $f : X \to Y$ is a morphism in $\text{FI}$, then $\sigma(f) : X \sqcup \{\ast\} \to Y \sqcup \{\ast\}$ is the map $f \sqcup \text{id}_{\{\ast\}}$. Following [2, Definition 2.8], the shift functor $S$ from the category of FI-modules to itself is defined by $SV = V \circ \sigma$ for every FI-module $V$.

Suppose $V$ is an FI-module. For any finite set $X$, one has $(SV)_X = V_{X \sqcup \{\ast\}}$. There is a natural FI-module homomorphism $\iota : V \to SV$ whose components $V_X \to (SV)_X$ are defined by the inclusion maps $X \hookrightarrow X \sqcup \{\ast\}$. We denote by $DV$ the cokernel of $\iota : V \to SV$. Following [1, Definition 3.2], we call the functor $D : V \mapsto DV$ the derivative functor on the category of FI-modules.

For any finite set $X$, let

$$(\tilde{S}_{-1}V)_X := \bigoplus_{x \in X} V_{X \setminus \{x\}}.$$ 

Suppose $f : X \to Y$ is an injective map between finite sets. For any $x \in X$, the map $f$ restricts to an injective map $f|_{X \setminus \{x\}} : X \setminus \{x\} \to Y \setminus \{f(x)\}$. We let

$$f_* : (\tilde{S}_{-1}V)_X \to (\tilde{S}_{-1}V)_Y$$

by the $k$-linear map whose restriction to the direct summand $V_{X \setminus \{x\}}$ is the map

$$(f|_{X \setminus \{x\}})_* : V_{X \setminus \{x\}} \to V_{Y \setminus \{f(x)\}}.$$
This defines an FI-module $\tilde{S}_{-1}V$. We call the functor $\tilde{S}_{-1} : V \mapsto \tilde{S}_{-1}V$ the negative-one shift functor on the category of FI-modules. (In [2, Definition 2.19], a negative shift functor $\tilde{S}_{-a}$ is defined for each integer $a \geq 1$.)

After recalling some generalities in the next section, we shall show that the negative-one shift functor $\tilde{S}_{-1}$ is a left adjoint functor to the shift functor $S$ and a right adjoint functor to the derivative functor $D$; we shall also show that for any FI-module $V$, the coinduction $QV$ of $V$ is an extension of $V$ by $\tilde{S}_{-1}V$. These properties seem to have gone unnoticed; for instance, the paper [6] constructed a left adjoint functor of $S$ using the category algebra of $FI$ and called it the induction functor.

**Notations.** Suppose $C$ is a small category. For any $X, Y \in \text{Ob}(C)$, we write $C(X, Y)$ for the set of morphisms from $X$ to $Y$.

We write $C\text{-Mod}$ for the category of $C$-modules. Suppose $V, W \in C\text{-Mod}$. We write $\text{Hom}_{C\text{-Mod}}(V, W)$ for the $k$-module of all $C$-module homomorphisms from $V$ to $W$. If $\phi \in \text{Hom}_{C\text{-Mod}}(V, W)$, we write $\phi_X : V_X \rightarrow W_X$ for the component of $\phi$ at $X \in \text{Ob}(C)$.

2. **Generalities on adjoint functors**

Let $C$ and $C'$ be small categories, and $F : C\text{-Mod} \rightarrow C'\text{-Mod}$ a functor.

For any $X \in \text{Ob}(C)$, we define the $C$-module $M(X)$ by

$$M(X)_Y := \mathbb{k}C(X, Y)$$

for each $Y \in \text{Ob}(C)$, that is, $M(X)$ is the composition of the functor $C(X, -)$ followed by the free $k$-module functor.

For any morphism $f \in C(X, Y)$, we have a $C$-module homomorphism

$$\rho_f : M(Y) \rightarrow M(X), \quad g \mapsto gf,$$

where $g \in C(Y, Z)$ for any $Z \in \text{Ob}(C)$. Hence, we obtain a $C'$-module homomorphism

$$F(\rho_f) : F(M(Y)) \rightarrow F(M(X)).$$

**Definition 1.** Define a functor $F^\dagger : C'\text{-Mod} \rightarrow C\text{-Mod}$ by

$$(F^\dagger(V))_X := \text{Hom}_{C'\text{-Mod}}(F(M(X)), V)$$

for each $V \in C'\text{-Mod}$, $X \in \text{Ob}(C)$.

For any morphism $f \in C(X, Y)$, the map $f_* : (F^\dagger(V))_X \rightarrow (F^\dagger(V))_Y$ is defined by

$$f_*(\phi) := \phi \circ F(\rho_f) \
\text{for each } \phi \in (F^\dagger(V))_X.$$

**Proposition 2.** Let $C$ and $C'$ be small categories. Let $F : C\text{-Mod} \rightarrow C'\text{-Mod}$ be a right exact functor which transforms direct sums to direct sums. Then $F^\dagger$ is a right adjoint functor of $F$, that is, there is a natural isomorphism

$$\text{Hom}_{C'\text{-Mod}}(F(V), W) \cong \text{Hom}_{C\text{-Mod}}(V, F^\dagger(W)),$$

where $V \in C\text{-Mod}$ and $W \in C'\text{-Mod}$.

**Proof.** This is a well-known result for modules over rings with identity, see [10, Theorem 5.51]. For a proof in our setting, see [8, Proposition 1.3 and Theorem 2.1].

The proofs of the main results of this article are essentially exercises in applications of Proposition 2, nevertheless, they do not appear to be obvious.
Remark 3. In all our applications of Proposition \[2\], we have a pair of functors \( F,G : \text{C-Mod} \rightarrow \text{C-Mod} \) with the following properties. For each \( X \in \text{Ob}(C) \), there exists \( X_i \in \text{Ob}(C) \) for \( i \in I_X \) (where \( I_X \) is a finite indexing set) such that there is a \( C \)-module isomorphism

\[ \eta_X : \bigoplus_{i \in I_X} M(X_i) \rightarrow F(M(X)). \]

Moreover, for each \( V \in \text{C-Mod} \), there is a decomposition \( G(V)_X = \bigoplus_{i \in I_X} V_{X_i} \), such that for any \( \psi \in \text{Hom}_{\text{C-Mod}}(V,W) \), one has \( G(\psi)_X = \bigoplus_{i \in I_X} \psi_{X_i} \). It follows then that there is a \( k \)-module isomorphism

\[ \alpha_X : (F^\dagger(V))_X \rightarrow G(V)_X, \quad \phi \mapsto \sum_{i \in I_X} \phi_{X_i}((\eta_X)_{X_i}(\text{id}_{X_i})), \]

where \( \text{id}_{X_i} \in M(X_i)_X \). Therefore, if \( F \) is right exact and sends direct sums to direct sums, then by Proposition \[2\] to show that \( G \) is a right adjoint functor of \( F \), it suffices to verify that the collection of \( k \)-module isomorphisms \( \alpha_X \) for \( X \in \text{Ob}(C) \) are compatible with the \( C \)-module structures on \( F^\dagger(V) \) and \( G(V) \).

3. Left adjoint of the shift functor

Let \( X \) be a finite set. By \[2\] Lemma 2.17, there is an isomorphism

\[ \eta_X : M(X \sqcup \{\ast\}) \rightarrow \tilde{S}_{-1}M(X) \tag{1} \]

defined as follows. For any finite set \( Y \) and injective map \( f : X \sqcup \{\ast\} \rightarrow Y \), one has \( f \mid_X : X \rightarrow Y \setminus \{f(\ast)\} \). Let \( \eta_X(f) \) be the element \( f \mid_X \) of the direct summand \( M(X)_{Y \setminus \{f(\ast)\}} \) of \((\tilde{S}_{-1}M(X))_Y\).

**Theorem 4.** The functor \( \tilde{S}_{-1} \) is a left adjoint of the shift functor \( S : \text{FI-Mod} \rightarrow \text{FI-Mod} \).

**Proof.** The functor \( \tilde{S}_{-1} \) is exact and transforms direct sums to direct sums. By Proposition \[2\] we need to show that the functors \( \tilde{S}_{-1} \) and \( S \) are isomorphic.

Let \( V \) be an \( \text{FI} \)-module. Let \( X \) be any finite set. From Definition \[1\] and \([1]\), we have a \( k \)-module isomorphism

\[ \alpha_X : (\tilde{S}_{-1}V)_X \rightarrow (SV)_X, \quad \phi \mapsto \phi_{X\sqcup\{\ast\}}((\eta_X)_{X\sqcup\{\ast\}}(\text{id}_{X\sqcup\{\ast\}})). \]

We claim that this collection of \( k \)-module isomorphisms over all finite sets \( X \) are compatible with the \( \text{FI} \)-module structures on \( \tilde{S}_{-1}V \) and \( SV \).

To verify the claim, let \( f : X \rightarrow Y \) be an injective map between finite sets, and let \( \phi \in (\tilde{S}_{-1}V)_X \). Then one has:

\[
\begin{align*}
\alpha_Y(f_*(\phi)) &= f_*(\phi)_{Y\sqcup\{\ast\}}((\eta_Y)_{Y\sqcup\{\ast\}}(\text{id}_{Y\sqcup\{\ast\}})) = \phi_{Y\sqcup\{\ast\}}((S_{-1}\rho f)_{Y\sqcup\{\ast\}}((\eta_Y)_{Y\sqcup\{\ast\}}(\text{id}_{Y\sqcup\{\ast\}}))) \\
&= \phi_{Y\sqcup\{\ast\}}((\eta_X)_{X\sqcup\{\ast\}}(f \sqcup \text{id}_{\{\ast\}})) = \phi_{Y\sqcup\{\ast\}}((f \sqcup \text{id}_{\{\ast\}})_*(\eta_X)_{X\sqcup\{\ast\}}(\text{id}_{X\sqcup\{\ast\}})) \\
&= (f \sqcup \text{id}_{\{\ast\}})_*(\phi_{X\sqcup\{\ast\}}((\eta_X)_{X\sqcup\{\ast\}}(\text{id}_{X\sqcup\{\ast\}}))) = f_*(\alpha_X(\phi)).
\end{align*}
\]

\( \square \)
4. Right adjoint of the derivative functor

Let $X$ be a finite set. It is known that there is an isomorphism

$$\Theta_X : M(X) \oplus \left( \bigoplus_{x \in X} M(X \setminus \{x\}) \right) \rightarrow SM(X),$$

see [2, Proof of Proposition 2.12]. We need this isomorphism explicitly. The restriction of $\Theta$ to the direct summand $M(X)$ is $\iota : M(X) \rightarrow SM(X)$. To define the restriction of $\Theta$ to the direct summand $M(X \setminus \{x\})$, we shall use the following notation.

**Notation 5.** Suppose $f : X \rightarrow Y$ is a map between finite sets. Suppose $\{w\}$ and $\{z\}$ are any one-element sets. We write $f \sqcup (w \rightarrow z)$ for the map $X \sqcup \{w\} \rightarrow Y \sqcup \{z\}$ whose restriction to $X$ is $f$ and which sends $w$ to $z$.

Let $Y$ be a finite set. The map $\Theta_X : M(X \setminus \{x\})_Y \rightarrow SM(X)_Y$ is defined by

$$f \mapsto f \sqcup (x \rightarrow \ast)$$

for each injective map $f : X \setminus \{x\} \rightarrow Y$.

**Theorem 6.** The functor $\tilde{S}_{-1}$ is a right adjoint of the derivative functor $D : \text{FI-Mod} \rightarrow \text{FI-Mod}.$

**Proof.** The functor $D$ is right exact and transforms direct sums to direct sums. By Proposition [2] we need to show that the functors $D^\dagger$ and $\tilde{S}_{-1}$ are isomorphic.

Let $X$ be a finite set. It follows from above that we have an isomorphism

$$\theta_X : \bigoplus_{x \in X} M(X \setminus \{x\}) \rightarrow DM(X),$$

where $\theta_X$ is defined by the composition of $\Theta_X$ with the quotient map $\pi_X : SM(X) \rightarrow DM(X)$.

Let $V$ be an FI-module. From Definition [11] and (3), we have a $k$-module isomorphism

$$\beta_X : (D^1V)_X \rightarrow (\tilde{S}_{-1}V)_X, \quad \phi \mapsto \sum_{x \in X} \phi_{X \setminus \{x\}}((\theta_X)_{X \setminus \{x\}}(id_{X \setminus \{x\}})).$$

We claim that this collection of $k$-module isomorphisms over all finite sets $X$ are compatible with the FI-module structures on $D^1V$ and $\tilde{S}_{-1}V$.

To verify the claim, let $f : X \rightarrow Y$ be an injective map between finite sets, and let $\phi \in (D^1V)_X$. Then one has:

$$\beta_Y(f_* (\phi)) = \sum_{y \in Y} f_*(\phi_{Y \setminus \{y\}})((\theta_Y)_{Y \setminus \{y\}}(id_{Y \setminus \{y\}})) = \sum_{y \in Y} \phi_{Y \setminus \{y\}}(D(\rho_f)(\rho_f^{-1})(\theta_Y)_{Y \setminus \{y\}}(id_{Y \setminus \{y\}}))$$

$$= \sum_{y \in Y} \phi_{Y \setminus \{y\}}(\pi_X((id_{Y \setminus \{y\}} \sqcup (y \rightarrow \ast)) \circ f)) = \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((\theta_X)_{Y \setminus \{f(x)\}}(f_{|X \setminus \{x\}}))$$

$$= \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((f_{|X \setminus \{x\}})_*(\theta_X)_{X \setminus \{x\}}(id_{X \setminus \{x\}}))$$

$$= \sum_{x \in X}(f_{|X \setminus \{x\}})_*(\phi_{X \setminus \{x\}}((\theta_X)_{X \setminus \{x\}}(id_{X \setminus \{x\}}))) = f_*(\beta_X(\phi)).$$
5. Coinduction functor

The coinduction functor $Q$ on the category of FI-modules was defined in [3, Definition 4.1] as $S^\uparrow$. It is a right adjoint functor of $S$ by Proposition 2 (see [3, Lemma 4.2]). In this section, we give an explicit description of $Q$ in terms of $\widetilde{S}_{-1}$.

**Notation 7.** Suppose $f : X \to Y$ is a map between finite sets. If $y \in Y \setminus f(X)$, then define

$$\partial_y f : X \to Y \setminus \{y\}$$

by $\partial_y f(x) := f(x)$ for each $x \in X$.

**Notation 8.** Suppose $V$ is an FI-module. If $f : X \to Y$ is a morphism in FI, define the map

$$\partial f_* : V_X \to (\widetilde{S}_{-1} V)_Y, \quad v \mapsto \sum_{y \in Y \setminus f(X)} (\partial_y f)_*(v),$$

where $(\partial_y f)_* : V_X \to V_{Y \setminus \{y\}}$ is defined by the FI-module structure of $V$.

It is easily checked that if $f : X \to Y$ and $g : Y \to Z$ are morphisms in FI, then one has:

$$\partial (gf)_* = (\partial g)_* f_* + g_* (\partial f_*). \quad (4)$$

We define an FI-module $Q'V$ as follows. For each finite set $X$, let

$$(Q'V)_X := V_X \oplus (\widetilde{S}_{-1} V)_X. \quad (5)$$

If $f : X \to Y$ is a morphism in FI, then define $f_* : (Q'V)_X \to (Q'V)_Y$ to be

$$\begin{pmatrix} f_* & 0 \\ \partial f_* & f_* \end{pmatrix},$$

where we use column notation for the direct sum in (5). It follows from (4) that $Q'V$ is an FI-module.

**Theorem 9.** The coinduction functor $Q : \text{FI-Mod} \to \text{FI-Mod}$ is isomorphic to the functor $Q' : V \mapsto Q'V$.

**Proof.** Let $V$ be an FI-module. Let $X$ be any finite set. One has $QV = S^\uparrow V$ by definition of $Q$. From Definition 1 and 2, we have a $k$-module isomorphism

$$\gamma_X : (S^\uparrow V)_X \to (Q'V)_X, \quad \phi \mapsto \phi_X ((\Theta_X)_X (\id_X)) + \sum_{x \in X} \phi_{X \setminus \{x\}} ((\Theta_X)_{X \setminus \{x\}} (\id_{X \setminus \{x\}})).$$

We claim that this collection of $k$-module isomorphisms over all finite sets $X$ are compatible with the FI-module structures on $S^\uparrow V$ and $Q'V$. 
To verify the claim, let \( f : X \to Y \) be an injective map between finite sets, and let \( \phi \in (S^1V)_X \). Then one has:

\[
\gamma_Y(f_*(\phi)) = f_*(\phi)_Y((\Theta Y)_Y(\text{id}_Y)) + \sum_{y \in Y} f_*(\phi)_{Y \setminus \{y\}}((\Theta Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}}))
\]

\[
= \phi_Y(S(\rho_f)_Y((\Theta Y)_Y(\text{id}_Y))) + \sum_{y \in Y} \phi_Y(\{y\}((\Theta Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}}))
\]

\[
= \phi_Y((\Theta X)_Y(f)) + \sum_{y \in Y} \phi_Y(\{y\}((\Theta X)_{Y \setminus \{y\}}(\partial_y f))
\]

\[
+ \sum_{x \in X} \phi_Y(\{x\}((\Theta X)_{X \setminus \{x\}}(f |_{X \setminus \{x\}})))
\]

\[
= \phi_Y(f_*(\Theta X)_X(\text{id}_X)) + \sum_{y \in Y \setminus f(X)} \phi_Y(\{y\}((\Theta X)_X(\text{id}_X))
\]

\[
+ \sum_{x \in X} \phi_Y(\{x\}((f |_{X \setminus \{x\}})_*(\Theta X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}})))
\]

\[
= f_*(\phi_X((\Theta X)_X(\text{id}_X)) + \sum_{y \in Y \setminus f(X)} (\partial_y f)_*(\phi_X((\Theta X)_X(\text{id}_X))
\]

\[
+ \sum_{x \in X} (f |_{X \setminus \{x\}})_*(\phi_X(\Theta X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}}))) = f_*(\gamma_X(\phi)).
\]

\[\square\]

Let \( FB \) be the category whose objects are the finite sets and whose morphisms are the bijections. Then \( FB \) is a subcategory of \( FI \) and so there is a natural forgetful functor from \( FI\text{-Mod} \) to \( FB\text{-Mod} \) (see [1]).

**Corollary 10.** Let \( V \) be an \( FI\)-module. Then there is a short exact sequence

\[
0 \rightarrow \widetilde{S}_{-1}V \rightarrow QV \rightarrow V \rightarrow 0
\]

of \( FI\)-modules which splits after applying the forgetful functor to the category of \( FB\)-modules.

**Proof.** Immediate from Theorem 9 and the definition of \( Q' \). \[\square\]

From Corollary 10 and [1], we recover [3, Theorem 1.3] for \( FI \).

**Remark 11.** In [3], for any finite field \( \mathbb{F}_q \), Gan and Li also studied the coinduction functor for \( VI\)-modules where \( VI \) is the category whose objects are the finite dimensional vector spaces over \( \mathbb{F}_q \) and whose morphisms are the injective linear maps. Similarly to \( FI\)-modules, the coinduction functor for \( VI\)-modules is defined as \( S^1 \) where \( S \) is the shift functor for \( VI\)-modules. However, we do not know of analogues of the negative-one shift functor \( \widetilde{S}_{-1} \) and the derivative functor \( D \) for \( VI\)-modules.
I am grateful to the referee for making detailed suggestions to improve the exposition of the paper.

Acknowledgments

References

[1] T. Church, J. Ellenberg, Homology of FI-modules, arXiv:1506.01022v1.
[2] T. Church, J. Ellenberg, B. Farb, R. Nagpal, FI-modules over Noetherian rings. Geom. Top. 18-5 (2014), 2951-2984, arXiv:1210.1854v2.
[3] W.L. Gan, L. Li, Coinduction functor in representation stability theory. J. Lond. Math. Soc. (2) 92 (2015), no. 3, 689-711, arXiv:1502.06989v2.
[4] W.L. Gan, A long exact sequence for homology of FI-modules, arXiv:1602.08873v2.
[5] L. Li, Upper bounds of homological invariants of FI\(_G\)-modules. Arch. Math. (Basel) 107 (2016), no. 3, 201-211, arXiv:1512.05879v2.
[6] L. Li, E. Ramos, Depth and the Local Cohomology of FI\(_G\)-modules, arXiv:1602.04405v2.
[7] L. Li, N. Yu, Filtrations and Homological degrees of FI-modules, arXiv:1511.02977v2.
[8] J.F. Palmquist, D. Newell, Bifunctors and adjoint pairs. Trans. Amer. Math. Soc. 155 (1971) 293-303.
[9] E. Ramos, Homological Invariants of FI-modules and FI\(_G\)-modules, arXiv:1511.03964v3.
[10] J. Rotman, An introduction to homological algebra. Second edition. Universitext. Springer, New York, 2009.

Department of Mathematics, University of California, Riverside, CA 92521, USA
E-mail address: wlgan@math.ucr.edu