Epsilon-measure of coherence

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To estimate the difference between the expected state and the actually prepared state and quantify quantum coherence contained in actually prepared state, we discuss the smooth version of coherence quantifiers which depend on a precision parameter $\epsilon$. In this paper, we call it the $\epsilon$-measure of coherence, it can be interpreted as the minimal coherence guaranteed to present in an $\epsilon$ ball around given quantum state. We find that the $\epsilon$-measure of any coherence monotone is still a coherence monotone, but it does not satisfy monotonicity on average under incoherent operations. We show the $\epsilon$-measure of coherence is continuous even if the original coherence quantifier is not. In particular, we study some properties of the $\epsilon$-measure of distance-based coherence quantifier.

I. INTRODUCTION

Quantum coherent superposition is a fundamental property of quantum states of single or more systems, and it has been shown to be a useful resource in quantum information processing. Various ways have been builded to develop a resource-theoretic framework for understanding quantum coherence, its detection and its quantification are problems of fundamental relevance in quantum information.

In quantum information processing, we need to prepare the high quality coherence states in order to achieve useful tasks, for example, quantum algorithms, quantum metrology and so on. In fact, we know that any preparation apparatus has realistically only a certain degree of precision and reliability, that is to say, there is a certain distance between the expected state and the actually prepared state. Then, we can estimate their difference and quantify the coherence contained in actually prepared state. In entanglement theory, the $\epsilon$-measure of entanglement depends on a precision parameter $\epsilon$, and it quantifies the entanglement contained in a state which is only partially known, and it has been applied and further developed. We know that the coherence embodies the essence of entanglement in the multipartite system, thus we can discuss the $\epsilon$-version of every coherence quantifier. The $\epsilon$-measure of coherence aims to characterize the minimum guaranteed amount of coherence, given the promises that the state which has actually been prepared is within a distance $\epsilon$ from the expected state. The $\epsilon$-version of every coherence quantifier may be actually considered as a smoothed version of the coherence quantifier.

The paper is organized as follows. We give the definition of the $\epsilon$-measure of coherence in Sec. II and in Sec. III we discuss the properties of the $\epsilon$-measure of coherence. In Sec. IV, we provide the lower and upper bounds which establish a relation with the distance-based coherence quantifier. Sec. V is devoted to our conclusions.

II. THE DEFINITION OF $\epsilon$-MEASURES OF COHERENCE

We briefly give an account of the concepts that are required to derive our main results. We are concerned with the resource theory of coherence by described in Ref. 1, 2. Consider a finite $d$-dimensional Hilbert space $\mathcal{H}$ with a fixed reference basis $\{|i\rangle\}_{i=1}^{d}$, in which the set of incoherent states $\mathcal{I}$ is defined as the set of all the states of the form,

$$\delta = \sum_{i} \delta_{i} |i\rangle\langle i|,$$

where $\delta_{i}$ are probabilities, and $\sum_{i} \delta_{i} = 1$. Let $\mathcal{D}$ be the convex set of density operators acting on Hilbert space $\mathcal{H}$, we have $\mathcal{I} \subseteq \mathcal{D}$. Any state which cannot be written as above is defined as a coherent state, which means the coherence is basis-dependent. Baumgratz et al. proposed that any proper measure of the coherence $\mathcal{C}$ must satisfy the following conditions:

(C1) Nonnegativity: $\mathcal{C}(\rho) \geq 0$ for all quantum states $\rho$, and $\mathcal{C}(\rho) = 0$ if and only if $\rho \in \mathcal{I}$.

(C2) Monotonicity: $\mathcal{C}(\rho)$ is non-increasing under incoherent operation $\Lambda$, i.e., $\mathcal{C}(\rho) \geq \mathcal{C}(\Lambda(\rho))$.

(C3) Strong monotonicity: $\mathcal{C}(\rho)$ is non-increasing on average under selective incoherent operations, i.e.,

$$\mathcal{C}(\rho) \geq \sum_{k} p_{k} \mathcal{C}(\rho_{k}),$$

where $\rho_{k} = K_{k} \rho K_{k}^{\dag}/p_{k}$ and $p_{k} = \text{Tr}(K_{k}\rho K_{k}^{\dag})$ for all $\{K_{k}\}$ with $\sum_{k} K_{k}^{\dag} K_{k} = I$ and $K_{k} \mathcal{I} K_{k}^{\dag} \subseteq \mathcal{I}$.

(C4) Convexity: $\mathcal{C}(\rho)$ is a convex function of quantum states, i.e.,

$$\sum_{i} p_{i} \mathcal{C}(\rho_{i}) \geq \mathcal{C}(\sum_{i} p_{i} \rho_{i}),$$

for any ensemble $\{p_{i}, \rho_{i}\}$.

Note that conditions (C3) and (C4) automatically imply condition (C2). The condition (C3) is important because it allows for sub-selection based on measurement.

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outcomes, a process available in well controlled by quantum experiments \[1\]. It has been shown that the relative entropy and \(l_1\)-norm satisfy all conditions. However, the measures of coherence induced by the squared Hilbert-Schmidt norm satisfies conditions (C1), (C2), (C4), but not (C3). Recently, we have found that the measure of coherence induced by the fidelity does not satisfy condition (C3), and an explicit example is presented \[2\]. In general, if a quantity \(\mathcal{C}\) fulfills condition (C1) and either condition (C2) or (C3) (or both), we say this is a coherence monotone, and if a quantity \(\mathcal{C}\) fulfills conditions (C1)-(C4) we say this is a coherence measure.

We know that the distance \(D\) is a very important tool in metric theory, and it is applied with many respects in the information theory \[10, 11\]. The distance needs to satisfy non-negative, symmetric, and triangle inequality. We know that some functions could do not satisfy the fundamental conditions of the distance, but they are widely used in the information theory, e.g. the relative entropy. In this paper, we require that the distance \(D\) is also convex or jointly convex, that is,

\[
D \left( \sum_{i} \lambda_i \rho_i, \sum_{i} \lambda_i \tau_i \right) \leq \sum_{i} \lambda_i D(\rho_i, \tau_i),
\]

where \(\lambda_i \geq 0\) and \(\sum_{i} \lambda_i = 1\). Furthermore, we also require that the distance \(D\) is contractive under completely positive and trace preserving operations \(\Phi\), that is,

\[
D(\Phi(\rho), \Phi(\tau)) \leq D(\rho, \tau).
\]

Clearly, the trace distance meets our all requirements \[10, 11\], it is defined by

\[
D_{tr}(\rho, \tau) = \frac{1}{2} \text{Tr}|\rho - \tau|,
\]

where \(|X| = \sqrt{X^\dagger X}\). With the definition of coherence measure at hand, we present the definition of the \(\epsilon\)-measure of coherence with respect to the distance as follows.

**Definition 1.** For every coherence measure \(\mathcal{C}\), and any \(\epsilon \geq 0\), and fixed distance \(D\), the \(\epsilon\)-measure of coherence is defined as

\[
\mathcal{C}_\epsilon^{(D)}(\rho) = \min_{\tau \in B_\epsilon^{(D)}(\rho)} \mathcal{C}(\tau),
\]

where \(B_\epsilon^{(D)}(\rho) = \{\tau | D(\rho, \tau) \leq \epsilon\}\).

Note that the distance and the measure function of coherence may be the same, or they can also be different. Henceforth, we will omit the superscript \(D\) to avoid unnecessary misunderstanding. Intuitively, the \(\epsilon\)-measure of coherence \(\mathcal{C}_\epsilon\) describes the minimal the amount of coherence guaranteed to present in an \(\epsilon\) ball surround the given quantum state \(\rho\). This construction gives a technique to obtain a smooth function \(\mathcal{C}_\epsilon\) even when the original measure \(\mathcal{C}\) is not. This is consistent with the entanglement case \[4\].

Obviously, we have \(\mathcal{C}_0(\rho) = \mathcal{C}(\rho)\). If the parameter \(\epsilon\) is so large that there is a incoherent state in \(B_\epsilon(\rho)\), we find that \(\mathcal{C}_\epsilon(\rho)\) vanishes on all the set of states \(\mathcal{D}\). In general, for a fixed coherent state \(\rho\), we are typically interested in values of \(\epsilon\) that are smaller than the distance of the state \(\rho\) from the set of coherent states, that is, \(\epsilon \leq D(\rho, \delta)\), where \(\delta \in \mathcal{I}\). In general, we can give an order relation of the \(\epsilon\)-measure of coherence via the different parameters.

**Proposition 2.** For any \(\epsilon > \epsilon' \geq 0\), we have

\[
0 \leq \mathcal{C}_\epsilon(\rho) - \mathcal{C}_{\epsilon'}(\rho) \leq \frac{\epsilon - \epsilon'}{\epsilon} \mathcal{C}(\rho).
\]

**Proof.** Since \(\epsilon > \epsilon' \geq 0\), then we have \(B_\epsilon(\rho) \subseteq B_{\epsilon'}(\rho)\), this implies that

\[
\mathcal{C}_\epsilon(\rho) \leq \mathcal{C}_{\epsilon'}(\rho).
\]

Then, we will prove the most right hand of inequality. Suppose that the state \(\tau \in B_\epsilon(\rho)\) such that \(\mathcal{C}_\epsilon(\rho) = \mathcal{C}(\tau)\), then we denote \(\lambda = \epsilon'/\epsilon\) and take the state \(\tau_\lambda = \lambda \tau + (1 - \lambda)\rho\). Since the distance \(D\) is jointly convex, then we obtain

\[
D(\rho, \tau_\lambda) \leq \lambda D(\rho, \tau) \leq \epsilon'.
\]

This implies that the state \(\tau_\lambda \in B_{\epsilon'}(\rho)\). Further, we have

\[
\mathcal{C}_{\epsilon'}(\rho) \leq \lambda \mathcal{C}(\tau) + (1 - \lambda)\mathcal{C}(\rho) \leq \mathcal{C}_\epsilon(\rho) + (1 - \lambda)\mathcal{C}(\rho).
\]

Thus, we have

\[
\mathcal{C}_\epsilon(\rho) - \mathcal{C}_{\epsilon'}(\rho) \leq \frac{\epsilon - \epsilon'}{\epsilon} \mathcal{C}(\rho).
\]

Combining Eq. (7) with Eq. (10), we obtain the desired result.

**III. BASIC PROPERTIES**

In this section, we will list some basic properties of the \(\epsilon\)-measure of coherence.

**Proposition 3.** For any quantum state \(\rho\), \(\mathcal{C}_\epsilon(\rho)\) is non-increasing under incoherent operation \(\Lambda\), i.e.,

\[
\mathcal{C}_\epsilon(\rho) \geq \mathcal{C}_\epsilon(\Lambda(\rho))
\]

**Proof.** Suppose that the state \(\tau^* \in B_\epsilon(\rho)\) realizes the minimum for the associated \(\epsilon\)-measure \(\mathcal{C}_\epsilon(\rho)\), i.e.,

\[
\mathcal{C}_\epsilon(\rho) = \mathcal{C}(\tau^*).
\]

Since the distance \(D\) is contractive under incoherent operation \(\Lambda\), it follows that

\[
D(\Lambda(\rho), \Lambda(\tau^*)) \leq D(\rho, \tau^*) \leq \epsilon.
\]
Thus, we have
\[
C(\tau^*) \geq C(\Lambda(\tau^*))
\]
\[
\geq \min_{\tau \in B_+(\Lambda(\rho))} C(\hat{\tau})
\]
\[
= C_\epsilon(\Lambda(\rho)).
\] (14)
From Eq. (12), we obtain the desired result. \(\square\)

This shows that the \(\epsilon\)-measure of coherence \(C_\epsilon\) satisfies conditions (C1) and (C2), that is to say, it is a coherence monotone. But, the following result shows that this measure does not satisfy condition (C3).

**Proposition 4.** The \(\epsilon\)-measure of coherence \(C_\epsilon\) does not satisfy strong monotonicity.

**Proof.** Let us suppose a state \(\rho\), and
\[
\rho = \eta |0\rangle \langle 0| \otimes \rho_c + (1 - \eta) |1\rangle \langle 1| \otimes \delta,
\] (15)
where \(\eta \in (0,1)\), \(|0\rangle\) and \(|1\rangle\) are orthogonal states of a local qubit system. Here, we require that \(C_\epsilon(\rho_c) > 0\). Note that the state \(\delta\) is any incoherent state, we have \(C_\epsilon(\delta) = 0\). Without loss of generality, if one considers the trace distance, we have \(D_{\tau}(\rho, |1\rangle \langle 1| \otimes \delta) = \eta\). Thus, we say that the parameter \(\eta\) is small enough so that the incoherent state \(|1\rangle \langle 1| \otimes \delta \in B_+(\rho)\), this implies the following inequality, i.e.,
\[
C_\epsilon(\rho) = 0.
\] (16)
Then, one can perform an incoherent operation with the Kraus elements \(|0\rangle \langle 0| \otimes I\) and \(|1\rangle \langle 1| \otimes I\) on the state \(\rho\), the two outputs of such measurement are \(|0\rangle \langle 0| \otimes \rho_c\) and \(|1\rangle \langle 1| \otimes \delta\) with probabilities \(\eta\) and \(1 - \eta\), respectively. Then, we have
\[
\eta C_\epsilon(|0\rangle \langle 0| \otimes \rho_c) + (1 - \eta) C_\epsilon(|1\rangle \langle 1| \otimes \delta)
\]
\[
= \eta C_\epsilon(\rho_c) + (1 - \eta) C_\epsilon(\delta)
\]
\[
= \eta C_\epsilon(\rho_c).
\] (17)
From Eq. (16), we obtain our desired result. \(\square\)

Note that the first equality in Eq. (17) arises from the following relation, i.e.,
\[
C_\epsilon(|0\rangle \langle 0| \otimes \rho) = C_\epsilon(\rho).
\] (18)
In fact, for any states \(|0\rangle \langle 0| \otimes \tau, \tau' \otimes \tau \in B_+(|0\rangle \langle 0| \otimes \rho))\), we have
\[
C(\tau) = C(|0\rangle \langle 0| \otimes \tau) \leq C(\tau' \otimes \tau).
\] (19)
Thus, we have
\[
C_\epsilon(|0\rangle \langle 0| \otimes \rho) = \min_{\tau \in B_+(|0\rangle \langle 0| \otimes \rho)} C(\tau)
\]
\[
= \min_{\tau \in B_+(\rho)} C(|0\rangle \langle 0| \otimes \tau)
\]
\[
= \min_{\tau \in B_+(\rho)} C(\tau)
\]
\[
= C_\epsilon(\rho).
\] (20)

In the general case, we cannot obtain that the \(\epsilon\)-measure of coherence \(C_\epsilon\) satisfies the additive, i.e.,
\[
C_\epsilon(\rho \otimes \sigma) = C_\epsilon(\rho) + C_\epsilon(\sigma).
\] (21)

From the above result, according to the resource theory of coherence, although we have known that the \(\epsilon\)-measure of coherence \(C_\epsilon\) is not a coherence measure, the following result shows that it satisfies the convexity.

**Proposition 5.** The \(\epsilon\)-measure of coherence \(C_\epsilon\) is convex, i.e.,
\[
C_\epsilon \left( \sum_i \lambda_i \rho_i \right) \leq \sum_i \lambda_i C_\epsilon(\rho_i),
\] (22)
where \(\lambda_i \geq 0\) and \(\sum_i \lambda_i = 1\).

**Proof.** Suppose that the states \(\tau^*_i \in B_+(\rho_i)\) realize the minimum in \(C_\epsilon(\rho_i), i = 1, \ldots, n\). For every \(i\), then we have
\[
C_\epsilon(\rho_i) = C(\tau^*_i).
\] (23)
Since the distance \(D\) is joint convexity, we have
\[
D \left( \sum_i \lambda_i \rho_i, \sum_i \lambda_i \tau^*_i \right) \leq \sum_i \lambda_i D(\rho_i, \tau_i) \leq \epsilon.
\] (24)
This implies that the state \(\sum_i \lambda_i \tau^*_i \in B_+(\sum_i \lambda_i \rho_i)\). Thus, we have
\[
C_\epsilon \left( \sum_i \lambda_i \rho_i \right) \leq C \left( \sum_i \lambda_i \tau^*_i \right)
\]
\[
\leq \sum_i \lambda_i C_\epsilon(\tau^*_i)
\]
\[
= \sum_i \lambda_i C_\epsilon(\rho_i).
\] (25)
\(\square\)

The \(\epsilon\)-measure of coherence \(C_\epsilon\) is not a genuine coherence measure, but the convex property holds such that it is still a good coherence monotone. In the following we will prove that other possibly relevant properties of the original quantity \(C\), hold for \(C_\epsilon\) too.

**Proposition 6.** The \(\epsilon\)-measure of coherence \(C_\epsilon\) is continuous in \(\epsilon\) and \(\rho\), for all \(\epsilon\) and for all \(\rho\).

**Proof.** From Proposition 2 for fixed \(\rho\), we can directly prove that,
\[
|C_\epsilon(\rho) - C'_\epsilon(\rho)| \to 0,
\] (26)
as \(\epsilon' \to \epsilon\).
Then, we want to prove that for any \(\epsilon > 0\), and for any state \(\rho\),
\[
|C_\epsilon(\rho') - C_\epsilon(\rho)| \to 0,
\] (27)
as $D(\rho, \rho') \to \epsilon$. For fixed $\epsilon > 0$, and $\tau \in B_\epsilon(\rho)$ such that $C_\epsilon(\rho) = C(\tau)$. We denote $\eta = D(\rho', \rho)$, $\lambda = \epsilon/(\epsilon + \eta)$, and take the state $\tau_\lambda = \lambda \rho + (1 - \lambda)\rho'$, since the distance $D$ is convex, then we have

$$D(\rho', \tau_\lambda) \leq \lambda D(\rho', \tau) \leq \lambda (D(\rho', \rho) + D(\rho, \rho')) \leq \lambda (\epsilon + \eta) = \epsilon. \quad (28)$$

This implies that the state $\tau_\lambda \in B_\epsilon(\rho')$, and we have

$$C_\epsilon(\rho') \leq \lambda \epsilon C(\tau) + (1 - \lambda) C(\rho') = \lambda \epsilon C(\rho) + (1 - \lambda) C_\epsilon(\rho'). \quad (29)$$

Thus, we have

$$C_\epsilon(\rho') - C_\epsilon(\rho) \leq \frac{\eta}{\epsilon + \eta} (C(\rho') - C_\epsilon(\rho)). \quad (30)$$

By exchanging the role of $\rho$ and $\rho'$, we have

$$- \frac{\eta}{\epsilon + \eta} (C(\rho) - C_\epsilon(\rho')) \leq C_\epsilon(\rho') - C_\epsilon(\rho) \quad (31)$$

Taking $M = \max\{C(\rho) - C_\epsilon(\rho'), C(\rho') - C_\epsilon(\rho)\}$, we have

$$|C_\epsilon(\rho') - C_\epsilon(\rho)| \leq \frac{\eta}{\epsilon + \eta} M. \quad (32)$$

Thus, we obtain our desired result.

This shows that the $\epsilon$-measure of coherence $C_\epsilon$ is always a continuous function of the parameter $\epsilon$ and the state $\rho$, even though the original quantity $C$ is non-continuous or no one can presently prove it is continuous, (e.g., Robustness of coherence [12] and the Schmidt measure [13], to our knowledge, no one can prove they are continuous). This result holds for any choice of distance $D$ and for all bounded and convex coherence monotones. This contrasts with the case of entanglement [14], we say that a significant advantage of the $\epsilon$-generalization of quantum resource monotones is to allow one to transform non-continuous quantity into continuous ones.

**IV. DISTANCE-BASED COHERENT MEASURES**

Let us consider the family of distance-based coherence quantifier, as introduced in [1-3], they are defined by

$$C_D(\rho) = \inf_{\delta \in I} D(\rho, \delta), \quad (33)$$

where the infimum is taken over the set of incoherent states $I$. Clearly, any quantity defined in [38] fulfills condition (C1) for any distance $D$ which is non-negative and zero and only if $\rho = \delta$. The condition (C2) is also satisfied if the distance $D$ is contractive. Any distance-based coherence quantifier fulfills condition (C4) whenever the corresponding distance $D$ is jointly convex. There are already three important distance-based coherence quantifiers, the $l_1$ norm of coherence [1], geometric coherence via the fidelity [2-4], trace distance of coherence [14].

We know that the definitions of incoherent operations or free operations are not unique and there are different choices [2]. Here, we can define a particular incoherent operation $\Lambda_\epsilon^\delta$ with any incoherent state $\delta$ and the parameter $\rho \in [0, I]$, that is,

$$\Lambda_\epsilon^\delta(\rho) = (1 - \rho)\rho + p\delta. \quad (34)$$

With this particular incoherent operation at hand, we present an explicit evolutional relation of distance-based coherence quantifier.

**Proposition 7.** Suppose the distance $D$ is convex and contractive. Given any state $\rho$, and probability $p$, if $\delta^*$ realizes the optimal in (33), then

$$C_D(\Lambda_\epsilon^\delta(\rho)) = (1 - p)C_D(\rho). \quad (35)$$

**Proof.** By the convexity of the distance $D$, we have

$$C_D(\Lambda_\epsilon^\delta(\rho)) = \inf_{\delta \in I} D(\Lambda_\epsilon^\delta(\rho), \delta) \leq \inf_{\delta \in I} [(1 - p)D(\rho, \delta) + pD(\delta^*, \delta)] \leq (1 - p)C_D(\rho). \quad (36)$$

On the other hand, by the triangle inequality and the convexity of the distance $D$, we have

$$C_D(\Lambda_\epsilon^\delta(\rho)) = \inf_{\delta \in I} D(\Lambda_\epsilon^\delta(\rho), \delta) \geq \inf_{\delta \in I} [D(\rho, \delta) - D(\rho, \Lambda_\epsilon^\delta(\rho))] \geq \inf_{\delta \in I} D(\rho, \delta) - pD(\rho, \delta^*) = (1 - p)C_D(\rho). \quad (37)$$

We are now in the position to present lower and upper bounds for the $\epsilon$-measure of coherence $C_\epsilon$ via the original measure $C$ and the distance-based measure $C_D$.

**Proposition 8.** Let $C$ be a coherence measure, and the distance $D$ be a convex and contractive, then the $\epsilon$-measure of coherence $C_\epsilon$ satisfies

$$\min_{\tau \in \mathcal{S}^I} C(\tau) \leq C_\epsilon(\rho) \leq (1 - \frac{\epsilon}{C_D(\rho)}) C(\rho), \quad (38)$$

where $C_D(\rho)$ is the coherence quantifier associated to the distance $D$.

**Proof.** By definition, for any state $\tau \in B_\epsilon(\rho)$, we have

$$C_\epsilon(\rho) \leq C(\tau). \quad (39)$$
Since the coherence measure $C$ and the distance $D$ are convex, with the incoherent operation \( [\delta] \), we have
\[
C(\Lambda_p^\delta(\rho)) \leq (1 - p)C(\rho)
\] (40)
and
\[
D(\Lambda_p^\delta(\rho), \rho) \leq pD(\rho, \delta).
\] (41)

Since we require $D(\rho, \delta) \geq \epsilon$, then one can choose $p$ such that $p \leq \epsilon / D(\rho, \delta)$, we have $D(\Lambda_p^\delta(\rho), \rho) \leq \epsilon$. Therefore, we obtain
\[
C_\epsilon(\rho) \leq \min_{\Lambda_p^\delta(\rho), \rho \leq \epsilon} C(\Lambda_p^\delta(\rho))
\]
\[
\leq \min_{\Lambda_p^\delta(\rho), \rho \leq \epsilon} (1 - p)C(\rho)
\]
\[
\leq \min_{\delta} \left( 1 - \frac{\epsilon}{D(\rho, \delta)} \right) C(\rho)
\]
\[
\leq \left( 1 - \frac{\epsilon}{C_D(\rho)} \right) C(\rho),
\] (42)
where the minimum in the first and second inequalities are taken over all the incoherent states $\delta$ and the parameter $p$ via the condition $D(\Lambda_p^\delta(\rho), \rho) \leq \epsilon$, and the third inequality comes from a restriction of the minimum to the case where we fix the parameter $p = \epsilon / D(\rho, \delta)$.

Next, we prove the lower bound. Suppose that the state $\rho' \in B_\epsilon(\rho)$ realizes the minimum for the quantifier $C_\epsilon$, we have
\[
C_\epsilon(\rho) = C(\rho') \geq C(\Lambda_p^\delta(\rho')).
\] (43)

Without loss of generality, we assume that the incoherent state $\delta^*$ is optimal for $C_D(\rho')$, then we obtain
\[
C_D(\rho') = D(\rho', \delta^*)
\]
\[
\geq D(\rho, \delta^*) - D(\rho, \rho')
\]
\[
\geq C_D(\rho) - \epsilon.
\] (44)

We can take the parameter $s = 1 - (C_D(\rho) - \epsilon) / C_D(\rho')$. Obviously, we have $0 \leq s \leq 1$. From Proposition 8, we obtain
\[
C_D(\Lambda_s^\delta(\rho')) = (1 - s)C_D(\rho') = C_D(\rho) - \epsilon.
\] (45)

Therefore, we have
\[
C_\epsilon(\rho) = C(\rho')
\]
\[
\geq C(\Lambda_s^\delta(\rho'))
\]
\[
\geq \min_{\tau, C_D(\tau) = C_D(\rho) - \epsilon} C(\tau).
\] (46)

Note that the lower and upper bound depends on the fundamental properties of the distance, e.g., the symmetric and the triangle inequality. In particular, if one consider the distance-based coherence quantifier $C_D$, from Proposition 8, we give the relation between the $\epsilon$-measure of coherence $C_\epsilon$ and the coherence quantifier $C_D$, namely,
\[
C_D(\rho) = C_\epsilon(\rho) + \epsilon.
\] (47)

Note that the relative entropy is not a proper distance because it does not satisfy symmetric and triangle inequality, but the relative entropy of coherence $H$ is also viewed as distance-based coherence quantifiers, and there are many interesting properties [1], [2]. Next, we will discuss separately the $\epsilon$-generalization of relative entropy of coherence. We first give the definition of the relative entropy of coherence $H$, it is defined by
\[
C_\epsilon(\rho) = \min_{\delta \in \mathcal{I}} S(\rho|\delta),
\] (48)
where $S(\tau|\delta) = Tr\rho(\log_2 \rho - \log_2 \delta)$ is the relative entropy, and $supp(\rho) \subseteq supp(\tau)$. Then, we can rewrite the definition $H$ via the relative entropy as follows.

**Definition 9.** For the relative entropy of coherence $C_\epsilon$, and any $\epsilon \geq 0$, we define
\[
C_{\epsilon, \epsilon}(\rho) = \min_{\tau \in B_\epsilon(\rho)} C_\epsilon(\tau),
\] (49)
where $B_\epsilon(\rho) = \{ \tau | S(\rho|\tau) \leq \epsilon \}$, we call it as the $\epsilon$-measure relative entropy of coherence.

We hope that there does not exist the incoherent states in the $\epsilon$-ball of the given coherent state $\rho$, thus, for any state $\tau \in B_\epsilon(\rho)$, we require that $S(\rho|\tau) \leq \epsilon \leq C_\epsilon(\tau)$. Clearly, we know that the $\epsilon$-measure relative entropy of coherence is non-negative and convex, and it is also non-increasing under incoherent operation $\Lambda$. Since the relative entropy of coherence is continuous, it is easy to verify that the $\epsilon$-measure relative entropy of coherence is also continuous. We did not prove strong monotonicity, but we can give a weak strong monotonicity.

Suppose that the state $\tau \in B_\epsilon(\rho)$ realizes the minimum for the quantifier $C_{\epsilon, \epsilon}(\rho)$, we have
\[
C_{\epsilon, \epsilon}(\rho) = C_\epsilon(\tau).
\] (50)

Let $\rho_k = K_k \rho K_k^\dagger / p_k$ with $p_k = Tr(K_k \rho K_k^\dagger)$, and $\tau_k = K_k \tau K_k^\dagger / q_k$ with $q_k = Tr(K_k \tau K_k^\dagger)$ for all Kraus operators $\{ K_k \}$ with $\sum_k K_k^\dagger K_k = I$ and $K_k I K_k^\dagger \subseteq I$. From the properties of relative entropy [10], we have
\[
\sum_k p_k S(\rho_k||\tau_k) \leq S(\rho||\tau).
\] (51)

Since the state $\tau \in B_\epsilon(\rho)$, that is, $S(\rho|\tau_k) \leq \epsilon$ for all $k$, this implies that the states $\tau_k \in B_\epsilon(\rho_k)$. From the definition of the $\epsilon$-measure relative entropy of coherence, we have
\[
C_{\epsilon, \epsilon}(\rho_k) \leq C_\epsilon(\tau_k).
\] (52)

Since the relative entropy of coherence is non-increasing on average under selective incoherent operations, then we have
\[
\sum_k q_k C_{\epsilon, \epsilon}(\rho_k) \leq \sum_k q_k C_\epsilon(\tau_k) \leq C_\epsilon(\tau).
\] (53)
Combining Eq. (50) with Eq. (53), we obtain
\[
\sum_k q_k C_{r, \epsilon}(\rho_k) \leq C_{r, \epsilon}(\rho).
\] (54)

Note that if the probabilities \(q_k\) are replaced with the probabilities \(p_k\), this is a real strong monotonicity. This is the reason why we call weak strong monotonicity.

In particular, if one considers the incoherent operation (33), for the \(\epsilon\)-measure relative entropy of coherence, we have
\[
C_{r, \epsilon}({\Lambda}_p^\delta(\rho)) \leq (1 - p)C_{r, \epsilon}(\rho).
\] (55)

Further, using the jointly convexity of relative entropy, we have
\[
S(\rho||{\Lambda}_p^\delta(\rho)) \leq pS(\rho||\delta).
\] (56)

If we choose the parameter \(p = \epsilon/D(\rho, \delta)\), from Proposition 5 we obtain
\[
C_{r, \epsilon}(\rho) \leq C_r(\rho) - \epsilon.
\] (57)

V. CONCLUSIONS

We have introduced the \(\epsilon\)-measure of coherence via the original coherence quantifier. The \(\epsilon\)-measure of coherence of a given quantum state could be interpreted as the minimum guaranteed coherence contained in the actually prepared state, when we only know that it is \(\epsilon\)-close to the ideal target state. We have shown that the \(\epsilon\)-measure of any coherence monotone is still a coherence monotone. We have proved that the \(\epsilon\)-measure of coherence quantifier does not satisfy monotonicity on average under incoherent operations. In particular, we found that the \(\epsilon\)-measure of relative entropy of coherence satisfy a weak monotonicity on average under incoherent operations. We also proved that the \(\epsilon\)-measure a convex coherence monotone is continuous, and it can also be seen as a smooth version of the original non-continuous quantity. We have given a lower and upper bound of the \(\epsilon\)-measure of distance-based coherence quantifier. As potential next steps, our results could be extended to infinite dimensional system, or continuous setting. We also discuss the relationship between the \(\epsilon\)-measure and the smooth entropy measure \(\epsilon\). We believe that the newly quantifiers will play a significant role in the quantum resource theory.

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