Sampled-data control design for systems with quantized actuators
(Extended Version)
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Abstract: This paper deals with the problem of designing a sampled-data state feedback control law for continuous-time linear control systems subject to uniform input quantization. The sampled-data state feedback is designed in order to ensure uniform global asymptotic stability (UGAS) of an attractor surrounding the origin. The closed-loop system is rewritten as a hybrid dynamical system. To do this, an auxiliary clock variable triggering the occurrence of sampling events is introduced. A numerically tractable algorithm with feasibility guarantees, based on concave-convex decomposition, is then proposed allowing to minimize the size of the attractor. Theoretical results are illustrated in a numerical example.

Keywords: Quantized control, sampled-data systems, LMIs, stability, hybrid systems.

1. INTRODUCTION

The design of a new generation of control systems combining physical interactions with computational and communication abilities presents strong advantages in terms of scalability, maintenance, and computational resources, for example for transportation systems, autonomous robotics, and energy delivery systems (Murray et al. (2002)). This paper focuses on one of these digital elements, namely, quantization, which is typically used to reduce data traffic load in communication channels to comply with limited bandwidth constraints. As a side effect, quantization induces a nonlinear behavior into the closed loop. See, for example, Delchamps (1990), Ceragioli et al. (2011), Liberzon (2003b), Tarbouriech and Gouaisbaut (2012). Most of the works dealing with quantized control systems focus either on continuous-time systems (Brockett and Liberzon (2000), Liberzon (2003a), Fridman and Dambrine (2009), Ferrante et al. (2015), Ferrante et al. (2020)), or on discrete-time systems, (Picasso and Colaneri (2008), Fu and Xie (2009), Campos et al. (2018), Ichihara et al. (2018)). However, in many applications the actual setting consists of a physical plant that is controlled via a digital controller. Because the dynamics of the physical plants are naturally described by continuous-time systems, the resulting closed-loop system turns out to be a sampled-data system. Sampled-data systems have been extensively studied in the last twenty years; see Hetel et al. (2017) for a recent survey on sampled-data systems. One might argue that the impact of quantization in sampled-data systems may be studied by relying on a discretized model of the plant. However, by pursuing this approach the analysis of the intersample behavior is completely lost.

Despite the fact that quantization and sampling naturally coexist, surprisingly not much attention has been devoted to the control design of quantized sampled-data systems. The interplay of sampling and quantization has been analyzed in Nesic and Liberzon (2009). Therein, a unified framework for controller design in the presence of quantization and time scheduling based on an emulation approach is investigated. Stabilization of sampled-data systems with stochastic sampling times in the presence of logarithmic input quantization has been addressed in Shen et al. (2017) by relying on a discretized model of the closed-loop system. Recently, robust sampled-data stabilization in the presence of input/state quantization and uncertainties for a rather general class nonlinear systems has been studied in Di Ferdinando et al. (2021), where the authors provide sufficient conditions for semiglobal practical asymptotic stability in the presence of fast sampling and accurate quantization.

In this paper, we consider a setup in which a linear plant is controlled via a periodic uniformly quantized sampled-data state feedback controller. In contrast with Shen et al. (2017) and with the objective of fully characterizing the intersample behavior of the closed-loop system, we model the closed loop as a hybrid dynamical system in the framework of Goebel et al. (2012). Within this setting, we provide sufficient conditions for uniform global asymptotic stability of a compact set containing the origin of the plant substate. These conditions are later used to devise a design algorithm for the controller to mitigate the impact of input quantization on the closed-loop state. It is noteworthy to mention that, as opposed to Di Ferdinando et al. (2021), we assume that the quantization error bound and the
sampling time are prescribed and cannot be selected by the designer. In this sense, the approach we propose is fully constructive.

The remainder of the paper is organized as follows. Section 2 formalizes the stabilization problem we solve. Section 3 provides sufficient conditions in the form of matrix inequalities for closed-loop stability. Section 4 presents an optimal design algorithm for the controller. The applicability of the proposed results is illustrated in Section 5 via a numerical example.

**Notation**

The symbol $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}_{>0}$ is the set of positive integers, $\mathbb{R}$ represents the set of real numbers, $\mathbb{R}^n$ is the $n$-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the positive (open) orthant in $\mathbb{R}^n$, $\mathbb{R}^{n \times m}$ represents the set of $n \times m$ real matrices, $\mathbb{Z}$ is the set of relative integers, and $\mathbb{B}$ is the closed unitary ball in the Euclidean norm. Given $\delta > 0$, $\delta \mathbb{Z} := \{\delta k : k \in \mathbb{Z}\}$. The symbol $\mathbb{S}^n_{+}, (\mathbb{S}^n_{++})$ stands for the set of $n \times n$ symmetric positive semidefinite (definite) matrices, $\mathbb{D}^n_{+} (\mathbb{D}^n_{++})$ denotes the set of $n \times n$ diagonal positive semidefinite (definite) matrices. For a vector $x \in \mathbb{R}^n$ (a matrix $A \in \mathbb{R}^{n \times m}$) $x^\top (A^\top)$ denotes the transpose of $x \ (A)$. The notation $\|A\|$ indicates the spectral norm of the matrix $A$. The spectral radius of the matrix $A$ is denoted by $\rho(A)$. Given $A \in \mathbb{R}^{n \times n}$, $\text{He}(A) = A + A^\top$. The symbol $A \preceq 0 \ (A < 0)$ stands for negative (semi) definiteness of the symmetric matrix $A$. The symbol $*$ stands for symmetric blocks in symmetric matrices. Given a symmetric matrix $A$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ stand, respectively, for the largest and smallest eigenvalue of $A$. Given $A \in \mathbb{R}^{n \times n}$, the notation $\mathcal{E}(A) = \{x \in \mathbb{R}^n : x^\top A x \leq 1\}$ is used. The function sign: $\mathbb{R} \to \{-1, 1\}$ is defined for all $x \in \mathbb{R}$ as follows: $\text{sign}(x) = 1$ if $x \geq 0$ and $-1$ otherwise. The symbol $|x|_S := \inf_{y \in S} |x - y|$ denotes the distance of the point $x \in \mathbb{R}^n$ to the nonempty set $S \subset \mathbb{R}^n$. The symbol $[x]$ indicates the floor of the real number $x$. The notation $\mathcal{L}_V(c)$ stands for the $c$-sublevel set of the function $V, i.e., \mathcal{L}_V(c) := \{x \in \text{dom } V : V(x) \leq c\}$. Let $f : \mathcal{X} \to \mathcal{Y}$, and $x \in \mathcal{X}$, we denote by $Df(x) : S \to \mathcal{Y}$ the differential of $f$ at $x$. The symbol range $f$ indicates the image of the function $f$. The notation $\mathbb{B}$ indicates the closure of the set $S$.

## 2. PROBLEM STATEMENT AND MODELING

### 2.1 Problem setup

We consider the following continuous-time plant with quantized actuation:

\[
\dot{x}_p = A_p x_p + B_p q_\Delta(u_p)
\]

where $x_p \in \mathbb{R}^{n_p}$ and $u_p \in \mathbb{R}^{n_u}$ are, respectively, the plant state and control input. Matrices $A_p \in \mathbb{R}^{n_p \times n_p}$ and $B_p \in \mathbb{R}^{n_p \times n_u}$ are assumed to be known. The function $q_\Delta$ is the so-called uniform quantizer, which is defined next:

\[
u \mapsto q_\Delta(u) = (q_\delta(u_1), q_\delta(u_2), \ldots, q_\delta(u_{n_u})) \in \Delta \mathbb{Z}^{n_u}
\]

where $\Delta := (\delta_1, \delta_2, \ldots, \delta_{n_u}) \in \Delta \mathbb{Z}^{n_u}$ represents the vector of quantization levels of each channel, $\Delta \mathbb{Z}^{n_u} = \bigoplus_{i=1}^{n_u} \delta_i \mathbb{Z}$, and, for all $v_i \in \mathbb{R}$, $\delta_i \in \mathbb{R}_{>0}$,

\[
q_\delta(v_i) := \delta \text{sign}(v_i) \left\lfloor \frac{|v_i|}{\delta_i} \right\rfloor.
\]

We consider a scenario in which the state of plant (1) is controlled via the following sampled-data state feedback controller:

\[
u_c(t_k) = K_1 x_p(t_k) + K_2 u_c(t_{k-1})
\]

where $t_0 = 0$ and $t_1, t_2, \ldots$ denotes the sampling instances and $K_1 \in \mathbb{R}^{n_u \times n_p}$ and $K_2 \in \mathbb{R}^{n_u \times n_u}$ are two gains defining the controller dynamics to be designed. Within this setting, our goal is to design controller (2) ensuring closed-loop stability.

Throughout the paper, we assume that the sequence of sampling times is periodic, formally the following assumption is enforced:

**Assumption 1.** For the sequence $\{t_k\}^\infty_1$, there exist $T > 0$ such that

\[
0 \leq t_1 \leq T, \quad t_{k+1} - t_k = T, \quad \forall k \in \mathbb{N}_{>0}.
\]

Assumption 1 ensures that the sequence of sampling times is strictly increasing and unbounded. In particular, assuming $T > 0$ rules out the existence of accumulation points in $\{t_k\}^\infty_1$.

The interconnection of the continuous-time plant (1) and the sampled-data controller (2) is realized via the following “time-triggered” zero-order-holder device:

\[
\begin{align*}
\dot{\chi}(t) &= 0 & \text{when } t \notin \{t_k\}^\infty_1 \\
\chi(t) &= v(t) & \text{when } t \in \{t_k\}^\infty_1,
\end{align*}
\]

by setting $\chi = u_p$ and $v = u_c$.

### 2.2 Hybrid modeling

Due to the interplay of continuous-time dynamics and discrete-time updates in the closed-loop system (1)-(2)-(3), we recast it as a hybrid dynamical system in the framework in Goebel et al. (2012). To this end, we introduce an auxiliary clock variable $\tau$ triggering the occurrence of sampling events. In particular, we let $\tau$ increase as long as its value belongs to the interval $[0, T]$ and trigger a jump whenever $\tau = T$. After the jump, the value of $\tau$ is reset to 0. This leads to the following dynamics for $\tau$:

\[
\begin{cases}
\dot{\tau} = 1, & \tau \in [0, T] \\
\tau^+ = \tau, & \tau = T.
\end{cases}
\]

Having introduced this clock enables us to model the closed-loop system (1)-(2)-(3) as the following hybrid system with state $x := (x_p, \chi, \tau) = (\xi, \tau) \in \mathbb{R}^{n_p + n_u + 1}$:

\[
\begin{align*}
H_{cl} \left\{ \begin{array}{ll}
\dot{\xi} = A_d \xi + B_d \psi \Delta(\xi) \\
\dot{\tau} = G_d + J_d \psi \Delta(\xi)
\end{array} \right., & x \in \mathcal{C} \\
\begin{array}{ll}
\xi^+ = (G_d + J_d \psi \Delta(\xi)) \\
\tau^+ = \tau
\end{array} & x \in \mathcal{D}
\end{align*}
\]

where:

\[
A_d := \begin{bmatrix} A_p & B_p \end{bmatrix}, \quad G_d := \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad J_d := \begin{bmatrix} I \end{bmatrix},
\]

and

\[
K := \begin{bmatrix} K_1 & K_2 \end{bmatrix},
\]
\[ \mathcal{C} \coloneqq \mathbb{R}^{n_F} \times \Delta \mathbb{Z}^{n_u} \times [0, T], \quad \mathcal{D} \coloneqq \mathbb{R}^{n_F} \times \Delta \mathbb{Z}^{n_u} \times \{ T \}, \]

and

\[ \psi_\Delta(u) \coloneqq q_\Delta(u) - u, \quad \forall u \in \mathbb{R}^{n_u}. \]

**Remark.** As opposed to the original model, in the proposed hybrid model (5) the quantizer is placed in the jump map, thereby rendering the flow map affine. Notice that, due to the specific selection of the flow and jumps sets, this operation does not induce any modification of the closed-loop system behavior.

**2.3 Basic properties of \( \mathcal{H}_d \) and Krasovskii regularization**

In this subsection, we illustrate some structural properties of hybrid model \( \mathcal{H}_d \).

**Property 1.** For hybrid system \( \mathcal{H}_d \), the following properties hold:

(i) For any \( \xi \in \mathcal{C} \cup \mathcal{D} \), there exists a nontrivial solution \( \phi \) to \( \mathcal{H}_d \) with \( \phi(0, 0) = \xi \);

(ii) Any maximal solution to \( \mathcal{H}_d \) is complete;

(iii) Any complete solution \( \phi \) is \( t \)-complete, i.e., \( \sup_t \text{dom} \phi = \infty \).

**Proof.** The proof hinges upon direct application of (Goebel et al., 2012, Proposition 2.10, page 33). In particular, the viability condition (VC) in (Goebel et al., 2012, Proposition 2.10, page 33) trivially holds on \( \mathcal{C} \setminus \mathcal{D} = \mathbb{R}^{n_F} \times \mathbb{R}^{n_u} \times [0, T] \) due to the flow dynamics being affine and the dynamics of \( \tau \) allowing flows when \( \tau \in [0, T] \). This yields item (i). To show item (ii), still from (Goebel et al., 2012, Proposition 2.10, page 33), since no finite escape times are possible and \( \mathcal{D} \subset \mathcal{C} \), item (ii) follows directly. To conclude the proof, observe that \( g(\mathcal{D}) \subset (\mathcal{C} \setminus \mathcal{D}) \). Moreover, the timer dynamics (4) ensure that the length of any flow interval is lower bounded by \( T \). Thus, item (iii) follows.

The above result shows that hybrid system \( \mathcal{H}_d \) is well behaved in terms of existence and completeness of (maximal) solutions. However, the discontinuity of the map \( \psi_\Delta \) prevents \( \mathcal{H}_d \) from fulfilling the so-called hybrid basic conditions; see (Goebel et al., 2012, Assumption 6.5, page 120). Hybrid systems not fulfilling the hybrid basic conditions may be overly sensitive to small vanishing noise. Moreover, the fulfillment of the hybrid basic conditions is a key ingredient for the application of several important results. To overcome this problem, in this paper we consider the so-called Krasovskii regularization of \( \mathcal{H}_d \); see (Goebel et al., 2012, Definition 4.13, page 81). This regularization basically corresponds to the “smallest” hybrid dynamical systems containing \( \mathcal{H}_d \) fulfilling the hybrid basic conditions.

In the specific case of hybrid system \( \mathcal{H}_d \), such a regularization writes as:

\[ \tilde{\mathcal{H}}_d \colon \begin{cases} \dot{x} = f(x), & x \in \mathcal{C} \\ x^+ = G(x), & x \in \mathcal{D} \end{cases} \]

where for all \( x \in \mathcal{D} \)

\[ G(x) \coloneqq \bigcap_{\delta > 0} g((x + \delta \mathbb{Z}) \cap \mathcal{D}). \]

Because \( g \) is locally bounded on \( \mathcal{D} \), similarly as in (Bacciotti, 2004, Lemma 1), it is easy to see that for all \( x \in \mathcal{D} \):

\[ G(x) = \{ v \in \mathbb{R}^{n_u+n_v+1} : \exists \{x_k\} \subset \mathcal{D} \quad \text{s.t.} \ x_k \to x, g(x_k) \to v \}. \]

Using the above equivalent rewriting of \( G \), simple continuity arguments yield:

\[ G(x) = \begin{bmatrix} G_d + J_d K \xi + J_d \Phi(\xi) \\ 0 \end{bmatrix}, \quad \forall x \in \mathcal{D} \]

where for all \( \xi \in \mathbb{R}^{n_F+n_u} \):

\[ \Phi(\xi) \coloneqq \bigcap_{\delta > 0} \psi_\Delta(K(\xi + \delta \mathbb{Z})). \]

It is worth to observe that since \( g(\mathcal{D}) \subset G(\mathcal{D}) \), solutions to hybrid system (5) are solutions to (6). Therefore, stability properties established for (6), among others, carry over directly to (5). Moreover, completeness of maximal solutions to (6) is still ensured.

To simplify the analysis of the closed-loop system, we over-approximate the set-valued map \( \Phi \) via the following inclusion:

\[ \Phi(\xi) \subset \Psi_\Delta(\mathcal{K}_\xi), \quad \forall \xi \in \mathbb{R}^{n_F+n_u} \]

where \( \Psi_\Delta \) is the Krasovskii regularization of the mapping \( \psi_\Delta \), namely:

\[ \Psi_\Delta(u) \coloneqq \bigcap_{\delta > 0} \psi_\Delta(u + \delta \mathbb{Z}), \quad \forall u \in \mathbb{R}^{n_u}. \]

In particular, (8) can be proven directly by relying on the continuity of the function \( \xi \mapsto \mathcal{K}_\xi \). It is worth to observe that, since \( g(\mathcal{D}) \subset G(\mathcal{D}) \), solutions to hybrid system (5) are solutions to (6). Therefore, stability properties established for (6), among others, carry over directly to (5). Moreover, due to \( G(\mathcal{D}) \subset (\mathcal{C} \setminus \mathcal{D}) \), completeness and non-Zenoness of maximal solutions to (6) follows from similar arguments as in the proof of Property 1.

At this stage, it is important to observe that, due to the presence of input quantization, generally asymptotic stabilization of the origin of the \( \xi \)-substate cannot be achieved. Indeed, roughly speaking, due to \( \psi_\Delta \) being zero around zero, the closed-loop system behaves in open loop near the origin. Therefore, it is natural to turn the attention towards asymptotic stability of a “possibly minimal” compact set \( \mathcal{A} \subset \mathbb{R}^{n_F+n_u+1} \) containing the origin of the \( \xi \)-substate. This leads to the following statement for the stabilization problem we solve in this paper.

**Problem 1.** (Stabilization). Design a gain \( K \) such that there exists a compact set \( \mathcal{A} \subset \mathbb{R}^{n_F+n_u+1} \) such that the following hold:

(a) \( \{0_{n_p} \} \times \{0_{n_u} \} \times [0, T] \subset \mathcal{A} \);

(b) \( \mathcal{A} \) is uniformly globally asymptotically stable \( \text{UGAS} \) for hybrid system (6).

**3. SUFFICIENT CONDITIONS FOR STABILITY**

To address Problem 1, we make use of the following sector conditions for the set-valued map \( \Psi_\Delta \) in (9).

**Lemma 1.** Let \( S_1, S_2 \in \mathbb{R}^{n_u \times n_u} \) be diagonal positive definite matrices. Then, for all \( u \in \mathbb{R}^{n_u} \), \( v \in \Psi_\Delta(u) \) the following inequalities hold:
\[ v^\top S_1 v - \Delta^T S_1 \Delta \leq 0 \quad (10) \]
\[ v^\top S_2 (v + u) \leq 0. \quad (11) \]

The proof Lemma 1 follows closely the proof of (Ferrante et al., 2020, Lemma 4.2) and therefore it is omitted.

**Theorem 1.** Let \( T > 0 \) be given. Suppose that there exist \( S_1, S_2 \in \mathbb{D}^+_{n_+} \), \( \varrho \in (0, 1) \), \( P \in \mathbb{S}^n_{p^+T_{n^+}} \), \( K \in \mathbb{R}^{n_n \times (n_+ + n_+)} \) such that

\[ \Delta^T S_1 \Delta - \varrho \leq 0, \quad (12) \]
\[ M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} < 0 \quad (13) \]

where:

\[ M_{11} := (G_d + J_d K)^\top \exp(A_dT)P \exp(A_dT)(G_d + J_d K) + (\varrho - 1)P \]
\[ M_{12} := (G_d + J_d K)^\top \exp(A_dT)P \exp(A_dT)J_d - K^\top S_2 \]
\[ M_{22} := J_d^\top \exp(A_dT)P \exp(A_dT)J_d - S_1 - 2S_2. \]

Then, there exists \( \sigma^* > 0 \) small enough such that for all \( \sigma \in (0, \sigma^*] \) the set

\[ A := \{ x \in C : \xi^\top \exp(-\sigma \tau) \exp(A_d \tau)P \exp(A_d \tau) \xi \leq 1 \} \]

is UGAS for the hybrid system (6).

**Proof.** Let for all \( x = (\xi, \tau) \in \mathbb{R}^{n_+ + n_+} \times \mathbb{R} \)

\[ V(x) := \exp(-\sigma \tau) \Sigma(\tau)^\top P \Sigma(\tau) \xi \]

where \( \sigma > 0 \) is selected later and, for the sake of writing, we set \( \Sigma(\tau) := \exp(A_d (T - \tau)) \). The proof relies on Proposition 3. In particular, we show that under (12) and (13) all the assumptions in Proposition 3 in Appendix A are satisfied with the above selection of \( V \).

Observe that for all \( x = (\xi, \tau) \in C = \mathcal{C} \cup \mathcal{D} \cup G(\mathcal{D}) \)

\[ \omega_1 |\xi|^2 \leq V(x) \leq \omega_2 |\xi|^2 \quad (15) \]

where:

\[ \omega_1 := \min_{\tau \in [0, T]} \exp(-\sigma \tau) \lambda_{\min} \left( \Sigma(\tau)^\top P \Sigma(\tau) \right) \]
\[ \omega_2 := \max_{\tau \in [0, T]} \exp(-\sigma \tau) \lambda_{\max} \left( \Sigma(\tau)^\top P \Sigma(\tau) \right) \]

are well-defined strictly positive real numbers. In particular, the existence of the above lower bound ensures that (A.2a) holds.

Now we show the satisfaction of (A.2b). Straightforward calculations yield, for all \( \tau \in \mathbb{R} \):

\[ \frac{d}{d\tau} \Sigma(\tau) = -A_d \Sigma(\tau)^\top P \Sigma(\tau) - \Sigma(\tau) P A_d \Sigma(\tau) = -H \Sigma(\tau)^\top P A_d \Sigma(\tau) \]

where the second equality follows from the identity \( A_d \Sigma(\tau) = \Sigma(\tau) A_d \). Thus, using (16) and the definition of \( f \) in (5), one gets, for all \( x \in \mathcal{C} \)

\[ \langle \nabla V(x), f(x) \rangle = -\sigma V(x), \quad \forall x \in \mathcal{C}. \]

The latter shows that (A.2b) holds with \( \lambda_c = \sigma \).

To conclude the proof, next we show that the satisfaction of (12) and (13) ensures that (A.2c) and (A.2d) hold with \( \mu = 1 \) and some positive \( \lambda_d \). Let \( x = (\xi, \tau) \in \mathcal{D} \) and \( g \in G(x) \). From the definition of the jump set \( D \) one has that \( \tau = T \). Moreover, the definition of \( G \) and (8) ensure that there exists \( v \in \Psi_\Delta(K \xi) \) such that \( g = ((G_d + J_d K) \xi + J_d v, 0) \). From Lemma 1 one has

\[ V(y) \leq V(g) - v^\top S_1 v + \Delta^T S_1 \Delta - 2v^\top S_2 (v + K \xi) \]

Adointing to both sides \( -V(x) + g(V(x) - 1) \) and bearing in mind that \( x = (\xi, T) \) one gets:

\[ V(g) - V(x) + g(V(x) - 1) \leq \xi^\top \begin{bmatrix} M & \Delta^T S_1 \Delta + \varrho \\ \varrho & \xi^\top P (1 - \exp(-\sigma T)) \xi \end{bmatrix} \]

with \( M \) defined as in (13). Thus, thanks to (12) and (13), from (18) one gets:

\[ V(g) - V(x) + g(V(x) - 1) \leq \xi^\top (-\beta I + P(1 - \exp(-\sigma T))) \xi \]

with \( \beta := \lambda_{\max}(M) \). At this stage, let \( \sigma^* > 0 \) be small enough so that \( Q < 0 \). This selection is always possible. Pick any \( \sigma \in (0, \sigma^*] \), hence, using (15), (19) one gets

\[ V(g) - V(x) + g(V(x) - 1) \leq -\alpha \omega_2^{-1} V(x) \]

with \( \alpha := \lambda_{\max}(Q) \). Thus, if \( x \in \mathcal{D} \setminus L_V(1) \) one has \( V(g) - V(x) \leq -\alpha \omega_2^{-1} V(x) \), which leads to \( V(g) \leq e^{-\lambda_x V(x)} \), with \( \lambda_x := \max \{ 1 - \alpha \omega_2^{-1} \xi \} \), where \( \xi > 0 \) is an arbitrarily small scalar. Since the analysis holds for any \( x \in \mathcal{D} \setminus L_V(1) \), \( g \in G(x) \), (A.2c) holds with \( \mu = 1 \).

To conclude the proof, we show that (A.2d) follows from the satisfaction of (12) and (13). To this end, pick \( x \in \mathcal{D} \setminus L_V(1) \). Then, from (19) one has:

\[ 0 \geq V(g) - V(x) + g(V(x) - 1) = V(g) - (1 - g)(V(x) - 1) \]

Thus, since \( \varrho \in (0, 1) \) if \( x \in \mathcal{D} \setminus L_V(1) \), the latter implies \( V(g) \leq 1 \), which corresponds to (A.2d). This establishes the result.

Condition (13) is a nonlinear matrix inequality. Handling this type of expressions is generally difficult from a numerical standpoint. Next, we provide an equivalent condition to (13) that takes the form of a bilinear matrix inequality (BMI). This equivalent characterization is extensively exploited in Section 4.

**Proposition 1.** There exist \( S_1, S_2 \in \mathbb{D}^+_{n_+} \), \( \varrho \in (0, 1) \), \( P \in \mathbb{S}^n_{p^+T_{n^+}} \), \( K \in \mathbb{R}^{n_n \times (n_+ + n_+)} \) such that (13) holds if and only if

\[ \begin{bmatrix} (g - 1)P & -K^\top S_2 \\ * & -S_1 - 2S_2 \end{bmatrix} \begin{bmatrix} (G_d + J_d K)^\top \Gamma(P) \\ J_d^\top \Gamma(P) \end{bmatrix} < 0, \quad (20) \]

where \( \Gamma(P) := \exp(A_dT)P \exp(A_dT) \).

**Proof.** Consider the following equivalent rewriting of the matrix \( M \) in (13):

\[ M = \begin{bmatrix} (g - 1)P & -K^\top S_2 \\ * & -S_1 - 2S_2 \end{bmatrix} \begin{bmatrix} (G_d + J_d K)^\top \\ J_d^\top \end{bmatrix} \Gamma(P) \]

Then, from Schur complement lemma \( M \prec 0 \) and \( \Gamma(P) \succ 0 \) if and only if

\[ \begin{bmatrix} (g - 1)P & -S_2 K \\ * & -S_1 - 2S_2 \end{bmatrix} \begin{bmatrix} (G_d + J_d K)^\top \\ J_d^\top \end{bmatrix} < 0. \]
The latter, via a simple congruence transformation, is equivalent to (20). The result is established.

4. CONTROLLER DESIGN

In this section, we show how the results developed in Section 3 can be exploited to devise a design algorithm for the solution to Problem 1 based on semidefinite programming. In particular, the implicit objective of Problem 1 is to design a controller gain \( K \) minimizing the size of the attractor \( \mathcal{A} \) defined in (14). In this regard, it is worth to observe that for the attractor \( \mathcal{A} \) in (14) the following chain of inclusions holds:

\[
\mathcal{A} \subseteq \left( \bigcup_{\tau \in [0,T]} \mathcal{E} \left( \exp((A_{cl})^T P \exp(A_{cl} \tau)) \right) \right) \times [0,T] \quad (21)
\]

with

\[
\mathcal{E} \left( \varpi \lambda_{\min}(P) I \right) \times [0,T]
\]

This shows that, \( \varpi \) being fixed, the attractor \( \mathcal{A} \) can be shrunk by maximizing \( \lambda_{\min}(P) \). In particular, this naturally leads to formulate Problem 1 as the following optimization problem:

\[
\begin{align*}
\text{maximize} & \quad c \\
\text{subject to} & \quad P - cl \succeq 0, (12) - (13) \quad (22) \\
\vartheta & \geq 0, P \in S_+^{n_p+n_n}, S_1, S_2 \in D_+^{n_n} \\
K & \in \mathbb{R}^{n_n \times (n_p+n_n)}
\end{align*}
\]

It is important to observe that due to (20) being bilinear in the decision variables, (22) is numerically intractable. To overcome this problem, in this paper we rely on the approach proposed in Dinh et al. (2011) to suboptimally solve (22) via a sequence of semidefinite programming (SDP) problems, i.e., optimization problems with linear objective over linear matrix inequality constraints. To deploy this approach, as a first step we rewrite (20) in the following equivalent bilinear-linear decomposed form:

\[
\begin{align*}
& L(P,S_1,S_2) + \text{He}(X^T(g,K) Y(P,S_2)) < 0, \\
& L(P,S_1,S_2) := \begin{bmatrix} P & 0 & G_{cl} \Gamma(P) \\ 0 & -S_1 - 2S_2 J_{cl} \Gamma(P) \\ 0 & 0 & -\Gamma(P) \end{bmatrix}, \\
& X^T(g,K) := \begin{bmatrix} \frac{1}{2}I & K^T \\ K & 0 \\ 0 & 0 \end{bmatrix}, \\
& Y(P,S_2) := \begin{bmatrix} P & 0 & 0 \\ J_{cl} \Gamma(P) & 0 & -S_2 \\ 0 & 0 & -\Gamma(P) \end{bmatrix}.
\end{align*}
\]

The latter, dropping the dependency on the decision variables, can be equivalently rewritten in the following psd convex-concave decomposed form; see (Dinh et al., 2011, Definition 2.1):

\[
\begin{align*}
L + X^T \cdot X + Y^T \cdot Y - (X - \gamma Y)^T (X - \gamma Y) < 0
\end{align*}
\]

which, by Schur complement’s lemma, is equivalent to:

\[
\begin{bmatrix} L - X^T \cdot X - Y^T \cdot Y + \text{He}(X^T(g,K) Y(P,S_2)) & X^T \cdot Y^T \\ -Y^T \\ -I \end{bmatrix} < 0. \quad (23)
\]

Based on the equivalent rewriting of (20) in (23), the following “linear inner approximation” of optimization problem (22) around the point \( \eta^{(0)} = (\eta^{(0)}, P^{(0)}, K^{(0)}, S_2^{(0)}) \in \mathbb{R} \times S_+^{n_p+n_n} \times \mathbb{R}^{n_n \times (n_p+n_n)} \times D_+^{n_n} \) is considered:

\[
O_\eta^{(0)} : \begin{cases}
\text{maximize} & \quad c \\
\text{s.t.} & \quad M(P,S_1,S_2,K,\vartheta) \lambda(g(\vartheta)) \prec 0, \\
& \quad \Delta^\top S_1 \Delta - g \preceq 0, \quad \vartheta \geq 0, \quad \vartheta \in (0,1) \\
& \quad P - cl \succeq 0, \quad \vartheta \in (0,1) \\
& \quad P \in S_+^{n_p+n_n}, S_1, S_2 \in D_+^{n_n}, \\
& \begin{bmatrix} R(P,S_1,S_2,K,\vartheta) W(g,P,S_2,K) & -I \\
0 & -\Theta(W) \end{bmatrix} \prec 0
\end{cases}
\]

where

\[
\begin{align*}
M(P,S_1,S_2,K,\vartheta) := & \quad L(P,S_1,S_2,P) + Q(\vartheta(\vartheta)) \\
& + (DQ(\vartheta(\vartheta)))(\vartheta - \vartheta(\vartheta))P - P^{(0)}K - K^{(0)}S_2 - S_2^{(0)}W(g,P,S_2,K) \defeq [X^T(g,K) Y^T(P,S_2)],
\end{align*}
\]

in which

\[
(g,P,K,S_2) \Rightarrow Q(g,P,S_2,K) := -X^T X - Y^T Y + \text{He}(X^T Y).
\]

At this stage, provided that an initial feasible solution \( \eta(\vartheta) \) to optimization problem (22) is available, \( O_\eta^{(0)} \) is solved. This, due to the property of psd-convex functions; see (Dinh et al., 2011, Lemma 2.1, item (b)), generates another feasible solution to optimization problem (22). Then, the process is repeated, thereby giving rise to a sequence of SDP problems. The applicability of this approach requires the knowledge of an initial feasible solution to optimization problem (22), which is basically used to generate the sequence of SDP problems \( \{O^{(k)}\}_k \). To this end, we make use of the following result.

\begin{proposition}
Suppose that there exist \( W \in S_+^{n_p+n_n}, S_1 \in D_+^{n_n}, Y \in \mathbb{R}^{n_n \times (n_p+n_n)} \), and \( \vartheta \in (0,1) \) such that (12) holds and

\[
\begin{bmatrix} (\vartheta - 1) W & 0 & WG_{cl} + Y^T J_{cl}^T \\ * & -S_1 & J_{cl} \Gamma(P) \\ * & * & -\Theta(W) \end{bmatrix} \prec 0
\]

(24)
\end{proposition}

where

\[
\Theta(W) := \exp(-A_{cl} T) W \exp(-A_{cl} ^T T).
\]

Then, there exists \( S_2 \in D_+^{n_n} \) such that \( (g,S_1,S_2,P = W^{-1},K = W^{-1} Y) \) is a feasible solution to (12)-(20). Moreover, let the pair

\[
A_p := \exp(A_p T), \quad B_p := \int_0^T \exp(A_p \rho) B_p d\rho ds
\]

be discrete-time stabilizable. Then, (12) and (24) are feasible.

\begin{proof}
A simple congruence transformation shows that the satisfaction of (24) is equivalent to:

\[
\begin{bmatrix} (g - 1) P & 0 & (G_{cl} + J_{cl} K) \Gamma(P) \\ * & -S_1 & J_{cl} \Gamma(P) \\ * & * & -\Gamma(P) \end{bmatrix} \prec 0
\]

(26)
\end{proof}

which corresponds to (20) with \( S_2 = 0 \). Hence, standard perturbation arguments on the previous strict inequality
ensure the existence of $\epsilon > 0$ small enough such that for any $S_2 \in \mathbb{D}^n_{++}$ with $\|S_2\| \leq \epsilon$, $(g, S_1, S_2, P = W^{-1}, K = W^{-1}Y)$ fulfills (12)-(20). This concludes the first part of the proof. To conclude the proof, observe that from Proposition 4 in the Appendix, under the discrete-time stabilizability of the pair $(A^p_P, B^p_P)$, there exists a feasible solution to (12)-(26). Hence, bearing in mind the equivalence between (24) and (26) established above, the proof is concluded.

In light of Proposition 2, to select a feasible initial point to optimization problem (22) we proceed as follows. First, we determine a solution to the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad l \\
\text{subject to} & \quad (24) \\
& \quad \Delta^\top S_1 \Delta - \rho \leq 0 \\
& \quad W - I I \preceq 0, \; \rho \in (0, 1) \\
& \quad W \in \mathbb{S}^n_{++}, S_1 \in \mathbb{D}^n_{++}, Y \in \mathbb{R}^{n_x \times (n_p + n_u)}.
\end{align*}
\]

Proposition 2 ensures that (27) is always feasible provided that the pair $(A^0_P, B^0_P)$ defined in (25) is discrete-time stabilizable. Once a feasible solution to (27) is available, a feasible solution $(W, S_1, S_2, g, Y)$ to optimization problem (22) can be easily obtained by selecting $P$ and $K$ as indicated in Proposition 2 and $S_2 \in \mathbb{D}^n_{++}$ so to fulfill (20).

It is important to remark that whenever $g$ is fixed, (27) is an SDP problem. Thus a feasible solution to (27) can be easily computed by performing a line search on the variable $g$ in the interval $(0, 1)$.

Based on the steps presented so far and on the equivalence between (13) and (24) established in Proposition 2, our approach to solve optimization problem (22) is summarized in Algorithm 1.

**Algorithm 1: Optimal controller design**

**Input:** Matrices $A_p, B_p$, quantization levels vector $\Delta \in \mathbb{R}^{n_p}$, sampling-time $T > 0$, $k_{max} \in \mathbb{N}_{>0}$, and $\epsilon > 0$.

1. **Initial solution:** Solve (27) via a line search on $g \in (0, 1)$. Let $W^{(0)}$, $g^{(0)}$, $S_1^{(0)}$ and $Y^{(0)}$ be the values associated to the corresponding solution.
2. **Set $k = 0$ and $P^{(0)} = (W^{(0)})^{-1}, K^{(0)} = P^{(0)} Y^{(0)}$;**
3. **Fix $(P^{(0)}, S_1^{(0)}, g^{(0)}, K^{(0)})$ and solve LMI (20) with respect to $S_2$. Let $S_2^{(0)}$ be the corresponding solution;**
4. **while $k < k_{max}$ do**
5. **Solve SDP problem $O(g^{(k)}, P^{(k)}, K^{(k)}, S_2^{(k)});$**
6. **$g^{(k+1)} \leftarrow g^{(k)}, P^{(k+1)} \leftarrow P^{(k)}, K^{(k+1)} \leftarrow K^{(k)};$**
7. **if $|e^{(k+1)} - e^{(k)}| \leq \epsilon$**
8. **break;**
9. **end**
10. **$k \leftarrow k + 1;$**
11. **return $K$ and $P$**

![Fig. 1. Evolution of the plant state in closed loop with: optimal gain (black line) and LQR gain (blue line).](image)

5. **NUMERICAL SIMULATIONS**

We consider the following data for system (1):

\[
A_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Delta = 1
\]

and select $T = 0.5$. Setting $\epsilon = 10^{-4}$, Algorithm 1 terminates in 75 iterations and yields $^1$

\[
K = \begin{bmatrix} 0.5529 & -2.1873 & 0 \\ 2.5432 & 0.0046 & 0.0009 \\ -0.0009 & -0.0007 & 0.4916 \end{bmatrix}
\]

To show the effectiveness of Algorithm 1 in designing a controller inducing a “small” attractor, in Fig. 1 we report numerical simulation of the plant state in closed loop with the gain $K$ returned by Algorithm 1 and with a gain designed via LQR on the discretized version of the plant, with a generic (non necessarily optimal) tuning of the weighting matrices $Q$ and $R$. Fig. 1 conveys two important messages. First, it clearly shows that the selection of the control gain plays a relevant role on the response the closed-loop system. Second, it emphasizes the effectiveness of the proposed approach towards the objective of confining the plant state close to the origin.

Fig. 2 depicts the evolution of the $\xi$-component of the closed-loop state along with the projection of the outer estimate $\hat{O}$ of the attractor $A$ defined in (21).

6. **CONCLUSION**

The problem of designing a sampled-data state feedback control law for continuous-time linear control systems subject to uniform input quantization was addressed. The main goal was to design a sampled-data state feedback to ensure uniform global asymptotic stability (UGAS) of a compact set containing the origin of the plant substate. The closed loop was augmented with an auxiliary clock

---

$^1$ Numerical solutions to LMIs are obtained through the solver SDPT3 Tütüncü et al. (2003) and coded in MATLAB via YALMIP Lofberg (2004). Simulations of hybrid dynamical systems are performed in MATLAB via the Hybrid Equations Toolbox Sanfelice et al. (2013).
variable triggering the occurrence of sampling events and modeled as a hybrid dynamical system. Sufficient conditions for stability in the form of bilinear matrix inequalities were proposed. An iterative algorithm based on the use of the convex-concave decomposition approach was proposed to design an optimal controller with the objective of minimizing the size of the closed-loop attractor. Under some mild conditions on the open-loop system, the proposed design algorithm was shown to be deadlock free. The interest of the proposed methodology was illustrated through a numerical example.

This work paves the ways for future research on quantized sampled-data control design. Extensions to aperiodic sampled-data systems as well as to other classes of quantizers are currently part of our research.

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In this paper, we consider hybrid systems with state $x \in \mathbb{R}^n$ of the form:

$$\mathcal{H} \left\{ \begin{array}{ll} \dot{x} = f(x), & x \in \mathcal{C} \\
x^+ = g(x), & x \in \mathcal{D} \end{array} \right. \quad (A.1)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is the flow map, $\mathcal{C}$ is the flow set, $g : \mathbb{R}^n \to \mathbb{R}^n$ is the set-valued jump map, and $\mathcal{D}$ is the jump set. A set $E \subset \mathbb{R}^{n \times n}$ is a hybrid time domain if it is the union of a finite or infinite sequence of intervals $[t_j, t_{j+1}] \times \{j\}$, with the last interval (if existent) of the form $[t_J, T]$ with $T$ finite or $T = \infty$. Given a hybrid time domain $E$, we denote by $\text{sup} E = \text{sup}\{t \in \mathbb{R} ; 3j \in \mathbb{N}\} \cup \{T \in \mathbb{R} ; t_J \in E\}$. A function $\phi : \delta \to \mathbb{R}^n$ is a hybrid arc if $\phi(0) = \delta$. A hybrid arc $\phi$ is a solution to (A.1) if it satisfies the dynamics of (A.1). A solution $\phi$ to (A.1) is maximal if it cannot be extended and is complete if $\phi(t) = \delta$ is unbounded. In particular, given $S \subset \mathbb{R}^n$, we denote by $\mathcal{S}_h(S)$ the set of maximal solutions $\phi$ to $\mathcal{H}$ with $\phi(0, 0) \in S$; see Goebel et al. (2012) for more details on solutions to hybrid systems.

The following notion of asymptotic stability for compact sets is considered in this paper.

**Definition 1.** Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. We say that $\mathcal{A}$ is:

- stable for $\mathcal{H}$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that any $\phi \in \mathcal{S}(\mathcal{A} + \delta \mathbb{B})$ is such that $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ \forall (t, j) \in dom $\phi$;
- uniformly globally attractive (UGA) for $\mathcal{H}$, if all maximal solutions to $\mathcal{H}$ are complete and for any $\mu > 0, \varepsilon > 0$, there exists $T > 0$ such that for all $\phi \in \mathcal{S}(\mathcal{A} + \mu \mathbb{B})$ the following implication holds:

$$(t, j) \in \text{dom} \phi, t + j \geq T \implies |\phi(t, j)|_{\mathcal{A}} \leq \varepsilon;$$
- uniformly globally asymptotically stable (UGAS) for $\mathcal{H}$, if it is both stable and uniformly globally attractive.

**Definition 2.** Let $S \subset \mathbb{R}^n$, we say that $S$ is strongly forward invariant for $\mathcal{H}$ if any $\phi \in \mathcal{S}(S)$ is complete and such that range $\phi \subset S$.

**Proposition 3.** Let $\mathcal{H}$ satisfy the hybrid basic conditions. Suppose that there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable on an open neighborhood of $C$, and $\mu, \lambda, \lambda_\mathcal{C} > 0$ such that:

\[
\begin{align*}
\lim_{x \to +1} V(x) &= \infty \\
(\nabla V(x), f) &\leq -\lambda cV(x) \quad \forall x \in \mathcal{C} \setminus L_V(\mu), f \in F(x) \quad (A.2b) \\
V(g) &\leq \exp(-\lambda d)V(x) \quad \forall x \in D \setminus L_V(\mu), g \in G(x) \\
V(x) &\leq \mu \quad \forall x \in D \cap L_V(\mu), g \in G(x) \quad (A.2d)
\end{align*}
\]

Assume that $L_V(\mu)$ is nonempty. Then, if any maximal solution to $\mathcal{H}$ is complete, $L_V(\mu)$ is globally asymptotically stable for $\mathcal{H}$.

**Proof.** Now, observe that (A.2b), (A.2c), and (A.2d) ensure that for any $\rho \geq \mu$, the set $L_V(\mu)$ is strongly forward invariant for $\mathcal{H}$. Hence, since, thanks to (A.2a) sublevel sets of $V_{|\mathcal{V},d}$ are bounded, it turns out that any solution to $\mathcal{H}$ is bounded. To conclude the proof, we rely on the following claim, whose proof is reported next.

**Claim 1.** Consider hybrid system $\mathcal{H}$ in (A.1). Suppose that there exist $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuously differentiable on an open neighborhood of $C$ and $\mu, \lambda_\mathcal{C} > 0$ such that all the conditions in Proposition 3 hold. Define for all $\xi \in \mathbb{R}^n$

$$\Upsilon(\xi) := \begin{cases} \frac{1}{\mu} \ln \left( \frac{V(\xi)}{\mu} \right) & \text{if } V(\xi) > \mu \\ 0 & \text{else.} \end{cases} \quad (A.3)$$

Let $\phi$ be a complete solution to $\mathcal{H}$. Then, the following holds:

$$(t, j) \in \text{dom} \phi, t + j \geq \Upsilon(\phi(t, j)) \implies V(\phi(t, j)) \leq \mu.$$  

Hinging upon Claim 1, next we show that $L_V(\mu)$ is globally uniformly attractive for $\mathcal{H}$. Let $\Upsilon$ be defined as in (A.3). Observe that $\Upsilon$ is locally bounded on $\mathbb{R}^n$, in fact continuous. Now, pick $r > 0$ and let

$$T_r = \sup_{\xi \in L_V(\mu) + r \mathbb{B}} \Upsilon(\xi).$$

Notice that, due to $\Upsilon$ being continuous on $\mathbb{R}^n$ and $L_V(\mu)$ being compact and nonempty, $T_r$ is finite and strictly positive. Then, by the definition of $T_r$, it follows that for all $\phi \in \mathcal{S}(L_V(\mu) + r \mathbb{B})$, any $(t, j) \in \text{dom} \phi$ with $t + j \geq T_r$ implies that $\phi(t, j) \in L_V(\mu)$. This implies that $L_V(\mu)$ is UGA for $\mathcal{H}$. Bearing in mind that $\mathcal{H}$ satisfies the hybrid basic conditions and that $L_V(\mu)$ is strongly forward invariant and UGA for $\mathcal{H}$, by invoking (Goebel et al., 2012, Proposition 7.5, page 142), it follows that $L_V(\mu)$ is stable. This concludes the proof.

**Proof of Claim 1.** Under the considered assumptions, it follows that solutions to $\mathcal{H}$ cannot leave $L_V(\mu)$.
In this paper, we consider matrix valued functions of the form:

\[ X : S \to Y \]  

(\text{A.5})

where \( S \) is a finite dimensional real linear vector space and \( Y \subset \mathbb{R}^{n \times m} \).

**Definition 3.** (Differential) Let \( X \) be defined as in (A.5). We say that \( X \) is differentiable at \( x \in S \) if there exists a linear map \( DX(x) : S \to Y \) such that:

\[
\lim_{\|h\|_S \to 0} \frac{\|X(x+h) - X(x) - DX(x)h\|_Y}{\|h\|_S} = 0
\]

where \( \| \cdot \|_S \) and \( \| \cdot \|_Y \) are any norms, respectively, on \( S \) and \( Y \).

**Definition 4.** (Shapiro (1997)). Let \( C \subset S \) be convex and \( S^n \) be the set of \( n \times n \) symmetric matrices. A function \( X : C \to S^n \) is said to be positive semidefinite convex (psd-convex) on \( C \) if for all \( x, y \in C \) and \( t \in [0,1] \) the following holds:

\[ X(tx + (1-t)y) \preceq tX(x) + (1-t)X(y). \]

Furthermore, we say that \( X \) is positive semidefinite concave (psd-concave) if \(-X\) is psd-convex.

**Lemma 2.** (Dinh et al. (2011)) Let \( C \subset S \) be convex, \( S^n \) be the set of \( n \times n \) symmetric matrices, and \( X : C \to S^n \) be differentiable on an open neighborhood of \( C \). Then, \( X \) is psd-convex on \( C \) if and only if for all \( x, y \in C \)

\[ X(y) - X(x) \succeq DX(x)(y - x). \]

\[ \text{Proposition 4.} \text{ If the pair } (A_p^D, B_p^D) \text{ defined in (25) is discrete-time stabilizable, then the conditions of Theorem 1 are feasible. Moreover, for any } \epsilon \geq 0, \text{ there exists a solution to (12)-(13) with } \|S_2\| \leq \epsilon. \]

**Proof.** To show the result, we prove that (12) and (13) (which is equivalent to (20) of the virtue of Proposition 1) are feasible with \( S_2 = 0 \) and some \( P \in S_n^{p,p+nu} \), \( K \in (0,1) \), and \( S_1 \in D_{*+}^{n+p+nu} \). Notice that, the existence of such a tuple \((P, S_1, g, K)\), by continuity of the matrix \( M \) in (13) with respect to the entries of \( S_2 \) and the inequality in (13) being strict, ensures, for any \( \epsilon > 0 \), the existence of \( \epsilon^* \in [0, \epsilon] \) such that for any \( S_2 \in D_{*+}^{n+p+nu} \) with \( \|S_2\| \leq \epsilon^* \), \((P, S_1, S_2, g, K)\) fulfills (13), thereby establishing the result. To this end, pick \( K \) such that \( \text{spr}(exp(A_dT)(G_d + J_dK)) < 1 \). Notice that since

\[ E := \exp(A_dT)(G_d + J_dK) = \begin{bmatrix} A_d^D & 0 \\ 0 & B_d^D \end{bmatrix} \]

under the stabilizability of \((A_d^D, B_d^D)\) the above selection of \( K \) is feasible. Pick \( g \in (0,1) \) such that \( \sqrt{1-g} > \text{spr}(E) \), which is always possible due to \( \text{spr}(E) < 1 \). For this selection of \( g \), select \( S_1 \in D_{*+}^{n+p+nu} \) such that (12) holds. Let \( Q \in S_n^{p,p+nu} \) be any solution to the following matrix inequality:

\[ E^TQE + (g-1)Q < 0. \]

(A.6)

which is solvable due to the selection of \( g \) above. The latter, thanks to Schur complement lemma and a simple congruence transformation, is equivalent to:

\[ (g-1)Q E^TQ \begin{bmatrix} * & \ast \end{bmatrix} < 0. \]

(A.7)

At this stage, observe that (A.7) can be equivalently rewritten as:

\[ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (g-1)Q & 0 & E^TQ \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} < 0, \]

(A.8)

with \( F := \exp(A_dT)J_d \). Thus, from the Projection Lemma (Gahinet and Apkarian, 1994, Lemma 3.1), the satisfaction of (A.8) implies that there exists \( X \in \mathbb{R}^{n \times n} \) such that:

\[ \begin{bmatrix} (g-1)Q & 0 & E^TQ \\ * & \text{He}(X) & F^TQ \\ * & * & -Q \end{bmatrix} < 0. \]

(A.9)

Notice that, necessarily, \( \text{He}(X) < 0 \). Let

\[ g \leq \frac{\lambda_{\text{min}}(S_1)}{\lambda_{\text{max}}(-\text{He}(X))}. \]

(A.10)

observe that \( g > 0 \), due to \( S_1 > 0 \) and \( \text{He}(X) < 0 \). Therefore, by setting

\[ P := gQ, \quad Z := QX, \]

from (A.9), one has

\[ \begin{bmatrix} (g-1)P & 0 & E^TP \\ * & \text{He}(Z) & F^TP \\ * & * & -P \end{bmatrix} < 0. \]

At this stage, notice that by construction, \(-S_1 - \text{He}(Z) \preceq 0\), therefore (A.10) implies

\[ \begin{bmatrix} (g-1)P & 0 & E^TP \\ * & -S_1 & F^TP \\ * & * & -P \end{bmatrix} < 0 \]

which, up to a congruence transformation, is equivalent to (20) with \( S_2 = 0 \). Hence, the result is established.