On the stability of two-flavor and three-flavor quark stars

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Following our recently proposed self-consistent mean-field approximation approach, we have done some researches on the chiral phase transition of strong interaction matter within the framework of Nambu-Jona-Lasinio (NJL) model. The chiral susceptibility and equation of state (EOS) are computed in this work for both two-flavor and three-flavor quark matter for contrast. The Pauli-Villars scheme, which can preserve gauge invariance, is used in this paper. Moreover, whether the three-flavor quark matter is more stable than the two-flavor quark matter or not in quark stars is discussed in this work. In our model, when the bag constant are the same, the two-flavor quark matter has a higher pressure than the three-flavor quark matter, which is different from what Witten proposed in his pioneering work.

I. INTRODUCTION

It has long been believed that the quark stars or the materials in the core of neutron stars are made up of strange quark matter, with comparable amounts of u, d and s quarks. Some pioneering discussions can be found in Refs. [1,2,3]. With the lack of a first-principle understanding of the strong interaction dynamics, these discussions are based on the MIT bag model. For example, as Witten argues in his work Ref. [8], when exerting the same pressure, the Fermi momentum of the three-flavor quark matter is lower than the two-flavor quark matter. In 1999, studies on the quark matter in compact stars based on NJL-type models are performed [2,10]. Again, following the Witten’s conjecture, only three-flavor NJL model is considered in these works. However, in a recent study, the authors find that two-flavor quark matter generally has lower energy per baryon than normal nuclei and strange quark matter according to their model [11]. This result suggests that the stability of quark matter is model dependent, and the interaction plays an important role.

On the other hand, as is pointed out in the Refs. [12,13], there exist a contradiction between the picture of hadron degrees of freedom and the picture of quark degrees of freedom. Although from the picture of hadron degrees of freedom, it is generally believed that the system gradually changes from hadronic to quark matter in the region of $2n_0 - 7n_0$, where $n_0$ is the nuclear saturation density [14]. From the picture of quark degrees of freedom, the system will undergo the well-known chiral phase transition along with the continuous increase of the quark chemical potential to a critical value $\mu_c$. And in most effective quark models, the value of $\mu_c$ is 330-380 MeV that is corresponding to a baryon number density only around $2n_0$ [15,19]. To resolve this contradiction, a new self-consistent method of mean-field approximation is developed in Refs. [12,13] to calculate the phase diagram of quark matter and the mass-radius relation of quark stars. In that model, the critical chemical potential $\mu_c$ of the chiral phase transition can be greatly increased by changing the parameter $\alpha$ which reflects the weight of different interaction channels in the model. However, only two-flavor NJL model is discussed and the three momentum cutoff regularization is adopted in their model for simplicity. That means the this model is valid only if one assumes that the cutoff involved is much larger than all relevant momenta. So, the quark chemical potential cannot be greater than this cutoff, which will place a restriction on the maximal quark star mass based on the NJL model.

As an expansion of our previous work [13], both two-flavor and three-flavor quark matter are explored in this work based on the NJL model. While the three momentum cutoff scheme is widely used in the NJL-type models, the Pauli-Villars scheme that can preserve gauge invariance is used in this paper. Especially, whether the three-flavor quark matter is more stable than the two-flavor quark matter in the quark stars is discussed. In Sec. III we briefly introduce our self-consistent mean field theory of the NJL model, and work out the chiral susceptibility to specify the chiral phase transition point, then obtain the phase diagram for both two-flavor and three-flavor NJL models. In Sec. IV the equation of states of two-flavor and three-flavor NJL models are derived to discuss the stability of quark matter. The mass-radius relation of quark stars is also computed in this section to show the stiffness of the equation of state. Finally, Sec. V is the summary and discussion of our work.

II. NJL MODEL AND PHASE DIAGRAM

NJL model is first developed by Nambu and Jasinio in 1961 as a phenomenological model to describe hadronic interaction [20,21]. Then, in 1974, Eguchi and Sugawara introduced a two-flavor quark NJL model with up and down quarks [22]. In 1976, Kikawa generalized this model to three-flavor case [23]. However, as is pointed in the Ref. [24], the authors of the original NJL model failed to develop a self-consistent mean-field theory of their model. In this paper, as what we did in the Ref. [12], we combine both the original Lagrangian and the Fierz identity of it to construct an equivalent Lagrangian
that is proper for a self-consistent mean-field approximation:
\[ \mathcal{L}_R = (1 - \alpha) \mathcal{L} + \alpha \mathcal{L}_F, \]  

(1)

where \( \alpha \) is a free parameter that cannot be determined by the model itself. Then, the mean-field approximation is performed and the gap equation is obtained. More details of our model can be found in Refs. [12][13]. The results of two-flavor model and three-flavor model are shown in Part II A and Part II B of this section respectively.

A. TWO-FLAVOR NJL MODEL

The Lagrangian of two-flavor NJL model is given by
\[ \mathcal{L} = \bar{\psi} (i \partial - m) \psi + G \left[ \left( \bar{\psi} \psi \right)^2 + \left( \bar{\psi} i \gamma^5 \psi \right)^2 \right], \]

(2)

where \( m \) is current quark mass and \( G \) denote the coupling constant of four-fermion interaction. The Fierz identity of four-fermion interaction term is
\[ \mathcal{L}_F = \bar{\psi} (i \partial - m) \psi \]
\[ \quad + \frac{G}{8N_c} \left[ 2 (\bar{\psi} \gamma^5 \psi)^2 - 2 (\bar{\psi} \tau \psi)^2 \right] - 2 (\bar{\psi} \gamma^5 \psi)^2 - 4 (\bar{\psi} \gamma^\mu \gamma^5 \psi)^2 \]
\[ \quad + (\bar{\psi} \sigma^{\mu \nu} \psi)^2 - (\bar{\psi} \sigma^{\mu \nu} \tau \psi)^2 \].

\( N_c \) is the number of color. So, we construct our new Lagrangian and perform the mean field approximation:
\[ < \mathcal{L}_R > = (1 - \alpha) \langle \mathcal{L} \rangle + \alpha \langle \mathcal{L}_F \rangle \]
\[ = \bar{\psi} (i \partial - m) \psi + 2 (1 - \alpha) g \langle \bar{\psi} \psi \rangle \]
\[ \quad + \frac{\alpha g}{2N_c} \langle (\bar{\psi} \gamma^5 \psi)^2 \rangle + \text{constant}. \]

(4)

Here, we would like to point out that the combination of the original Lagrangian and the Fierz transformed Lagrangian indicates not only the contribution of the leading order term of the large \( N_c \) expansion but also the next leading term is considered in our model. The gap equation and expression of effective chemical potential are:
\[ M = m - 2 (1 - \frac{11}{12} \alpha) g \sum_{f=u,d} \langle \bar{\psi} \psi_f \rangle \]
\[ = m - 2 G \sum_{f=u,d} \langle \bar{\psi} \psi_f \rangle, \]

(5)

\[ \mu' = \mu - \frac{\alpha}{3} g \sum_{f=u,d} \langle \bar{\psi}_f \psi_f \rangle \]
\[ = \mu - \frac{\alpha}{3} \frac{12 G}{12 - 11 \alpha} \sum_{f=u,d} \langle \bar{\psi}_f \psi_f \rangle, \]

(6)

where \( f = u, d \), and \( m \) is the current quark mass. Our new coupling constant \( g \) should be obtained by fitting the coupling constant \( G \) at zero temperature and zero chemical potential. The relation between them is \( G = (1 - \frac{11}{12} \alpha) g \). From the above expressions, we can see that the quark condensate and number density is of essential importance to the computation of NJL model. It is well known that the NJL model is not renormalizable, but in most studies, the three-momentum cutoff is used as the regularization scheme. This regularization scheme will place a restriction on the quark chemical potential and thus place a restriction on the maximal quark star mass based on the NJL model [13]. So, the Pauli-Villars regularization is adopted in this paper. Pauli-Villars regularization can preserve gauge invariance and also leads to a higher maximal quark star mass. In the framework of Pauli-Villars regularization, the quark condensate of flavor \( f \) is [23]:
\[ \langle \bar{\psi}_f \psi_f \rangle = -4 M N_c \int \frac{d^4 \xi}{(2\pi)^4} \frac{1}{\xi^2 - M^2} \]
\[ \Rightarrow -4 M N_c \int \frac{d^4 \xi}{(2\pi)^4} \left( \frac{1}{\xi^2 - M^2} - \frac{a_1}{\xi^2 - \Lambda_1^2} - \frac{a_2}{\xi^2 - \Lambda_2^2} \right), \]

(7)

where \( f = (u,d) \), \( \xi = (\xi_0, \vec{\xi}) = (p_0 + \mu, \vec{p}) \) \( (\mu) \) is effective chemical potential, \( \Lambda_1 \) and \( \Lambda_2 \) are cutoff parameters. \( N_c \) represents the number of color and \( M \) is the effective quark mass of flavor \( f \) in the gap equation. By selecting appropriate values of \( a_1 \) and \( a_2 \) we can make quark condensate finite. The values are listed as
\[ \begin{cases} 
   a_1 = \frac{M^2 - \Lambda_1^2}{\Lambda_1^2 - \Lambda_2^2} \\
   a_2 = \frac{\Lambda_2^2 - M^2}{\Lambda_1^2 - \Lambda_2^2}.
\end{cases} \]

(8)

For simplicity, we can set the cutoff scales \( \Lambda_1 = \Lambda_2 = \Lambda \) after the subtraction. So only one cutoff parameter is left. In the finite temperature case, the final expression becomes:
\[ \langle \bar{\psi}_f \psi_f \rangle = -2 M N_c \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{E_p} (1 - f - T) - \frac{3 E_{\Lambda}^2 - E_{p}^2}{2 E_{\Lambda}^2} \right] \]
\[ + M N_c \int \frac{d^3 p}{(2\pi)^3} \frac{E_{p}^2 - 3 E_{\Lambda}^2}{E_{\Lambda}^2} (f_{\Lambda} + \bar{f}_{\Lambda}) \]
\[ + M N_c \int \frac{d^3 p}{(2\pi)^3} \frac{E_{p}^2 - 3 E_{\Lambda}^2}{E_{\Lambda}^2 T} (f_{\Lambda} + \bar{f}_{\Lambda}) \]
\[ - M N_c \int \frac{d^3 p}{(2\pi)^3} \frac{\Lambda^2 - M^2}{E_{\Lambda}^2 T} (f_{\Lambda}^2 + \bar{f}_{\Lambda}^2). \]

(9)

Based on the same method, the number density can be obtained:
\[ \langle \bar{\psi}_f \psi_f \rangle = 2 N_c \int \frac{d^3 p}{(2\pi)^3} \left[ (f - T) - (f_{\Lambda} - \bar{f}_{\Lambda}) \right], \]

(10)
ties with different the same. The quark number densities and chiral susceptibilities of u and d quarks are also equal to each other, so the quark number with a three-momentum cutoff regularization scheme [29, 30]. In these equations, $E = \sqrt{\vec{p}^2 + M^2}$ and $E_\Lambda = \sqrt{\vec{p}^2 + \Lambda^2}$. We first study the chiral phase transition under different values of $\alpha$ at zero temperature. The parameters we choose is listed in Table I. To determine the order of phase transition and the location of it, the chiral susceptibility is also computed [26].

$$\chi_m = \frac{\partial}{\partial m} \langle \bar{\psi} \psi \rangle.$$  \hspace{1cm} (15)

We assume the chemical potentials and current quark masses of u and d quarks are equal to each other, so the quark number densities and chiral susceptibilities of u and d quarks are also the same. The quark number densities and chiral susceptibilities with different $\alpha$ are shown respectively in Fig. 1 and Fig. 2. It can be seen from both Fig. 1 and Fig. 2, the quark number densities and chiral susceptibilities are non-trivial when the quark chemical potential is greater than around 308 MeV. This is a model-independent result that quarks can be excited from the vacuum only if the chemical potential is comparable with the constituent quark mass according to [28].

From Fig. 2 we can see that the chiral phase transition is a crossover under the Pauli-Villars regularization scheme no matter how large the $\alpha$ is, which is different from the three-cutoff situation [12]. That means there is no critical end point (CEP) according to our model. Although the CEP exists in most NJL-type models with a three-momentum cutoff regularization scheme [28, 30]. Ref. [31] shows that the CEP can disappear if another regularization scheme is implemented. Our result is in accordance with Refs. [27, 31]. Besides, the phase transition point can be remarkably postponed by increasing the parameter $\alpha$ in our model. The phase transition is the result of the competition between different interaction channels, and the parameter $\alpha$ in our model indicates the weights of different interaction channels. So, if we change the value of $\alpha$, the phase transition point will move accordingly. Here we would like to point out that although from the Lagrangian it seems that only the vector-isoscalar channel plays an important role, our model are different from those models where a vector-isoscalar term is added factitiously, because in physics the original NJL Lagrangian is damaged if there are other terms added by hand. The phase diagram is Fig. 3.

### Table I: Parameters in Pauli-Villars regularization (two flavor).

| $m_\pi$ (MeV) | $\Lambda$ (MeV) | $G$ (MeV$^{-2}$) |
|---------------|-----------------|-----------------|
| 5.0           | 1071            | $3.45 \times 10^{-6}$ |
B. THREE-FLAVOR NJL MODEL

The Lagrangian of three-flavor NJL model is

$$\mathcal{L}_{\text{NJL}} = \bar{\psi} (i \partial - m) \psi + G \sum_{i=0}^{8} \left( (\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma^5 \lambda^a \psi)^2 \right) - K \left[ \text{det} (1 + \gamma^5) \psi + \text{det} (1 - \gamma^5) \psi \right].$$

Where \( m \) is the current quark mass and G and K denote the coupling constant. \( \lambda^a \) is SU(3) generator in flavor space and \( \lambda^0 \equiv \sqrt{\frac{2}{3}} \mathbf{1} \). The determinant is also in flavor space. The Fierz identity of it is (according to Ref. [32], the contribution of Fierz identity of six-fermion interaction term to gap equation is the modification of coupling constant of K and it has nothing to do with effective chemical potential, so we can ignore the effect of it and just apply Fierz transformation to four-fermion interaction term only):

$$\mathcal{L}_F = \bar{\psi} (i \partial - m) \psi$$

$$-G \frac{1}{2} \sum_{i=0}^{8} \left[ (\bar{\psi} \gamma^\mu \lambda^i \psi)^2 - (\bar{\psi} i \gamma^5 \gamma^\mu \lambda^i \psi)^2 \right] - K \left[ \text{det} (1 + \gamma^5) \psi + \text{det} (1 - \gamma^5) \psi \right].$$

\( \lambda^C \) is the SU(3) generator in color space and \( \lambda^C_0 \equiv \sqrt{\frac{2}{3}} \mathbf{1} \). We just take color singlet into consideration to simply the computation, so in transformed four-fermion interaction, only the term with \( \lambda^C_0 \) will be reserved. Apply new self-consistent mean field approximation to three-flavor NJL model, we can get the final lagrangian:

$$\mathcal{L}_R = \bar{\psi} (i \partial - m_0) \psi + (1 - \alpha) g \frac{1}{2} \sum_{i=0}^{8} \left[ (\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i \gamma^5 \lambda^a \psi)^2 \right]$$

$$-\alpha g \frac{1}{2} \sum_{i=0}^{8} \left[ (\bar{\psi} \gamma^\mu \lambda^C \psi)^2 - (\bar{\psi} i \gamma^5 \gamma^\mu \lambda^C \psi)^2 \right]$$

$$- K \left[ \text{det} (1 + \gamma^5) \psi + \text{det} (1 - \gamma^5) \psi \right].$$

Apply mean field approximation to the Lagrangian and the only non-vanishing terms will be \( (\bar{\psi} \psi) \) and \( (\bar{\psi} \gamma^5 \psi) \). The gap equations are:

$$\begin{align*}
M_u &= m_u - 4G\sigma_u + 2K\sigma_d\sigma_s, \\
M_d &= m_d - 4G\sigma_d + 2K\sigma_u\sigma_s, \\
M_s &= m_s - 4G\sigma_s + 2K\sigma_u\sigma_d,
\end{align*}$$

where \( M_f (f = u, d, s) \) are constituent quark mass, \( G = (1 - \alpha) g \), \( \sigma_u = \langle \bar{\psi}_u \psi_u \rangle \), \( \sigma_d = \langle \bar{\psi}_d \psi_d \rangle \), \( \sigma_s = \langle \bar{\psi}_s \psi_s \rangle \). The expressions of effective chemical potential are

$$\mu_f' = \mu_f - \frac{2}{3} \frac{Ga}{1 - \alpha} \sum_{f = u,d,s} \langle \bar{\psi}_f \gamma^5 \psi_f \rangle.$$

The parameters are listed in Table II. We could derive the susceptibility at different chemical potentials of u quark, see Fig. 4. The phase transition keeps crossover as two-flavor condition. Carry on the steps we have done in two-flavor condition and connect the \( \mu_f \) at different temperature, we could draw the phase diagram of quark matter (see Fig. 5).
density-independent quantity and $M$ where $P$ is the gap equation. It is usually associated with the bag constant $m$ constant of pion respectively. $\eta$ denote the mass and decay constant of pion respectively.

\[
\sum n_i = m_u n_u + m_d n_d + m_s n_s + q \mu_n + q \mu_d + q \mu_s \sum n_i = 2 \bar{n}_u - \frac{1}{3} \bar{n}_d - \frac{1}{3} \bar{n}_s - n_e = 0, \tag{21}
\]

and

\[
\begin{align*}
\mu_d &= \mu_u + \mu_e, \\
\mu_d &= \mu_s.
\end{align*} \tag{22}
\]

From the expression of quark number densities, the pressure and energy densities can be obtained by \[35, 38\]

\[
P = P(\mu = 0, M) + \int_0^{\mu_\alpha} \rho(\mu) d\mu, \tag{23}
\]

\[
\epsilon = -P + \sum_i \mu_i \rho_i, \tag{24}
\]

where $P(\mu = 0, M)$ denotes the vacuum pressure, which is a density-independent quantity and $M$ is a solution of the quark gap equation. It is usually associated with the bag constant $B$. Although the bag constant can be calculated from the NJL model, the results depend on the model and parameters, and there is no reliable way from the first principle of QCD to calculate the value of the vacuum pressure. Detailed discussion on the bag constant can also be found in our previous work Ref. \[13\]. The bag constant has an empirical domain which ranges from $(100 \text{MeV})^4$ to $(200 \text{MeV})^4$ \[39, 40\]. So, it is in this paper considered as a free parameter in this domain. To discuss the stability, the pressure as a function of quark chemical potential is plotted in Fig. 6 and then the EOSs with different parameters of both two-flavor and three-flavor quark matter are separately plotted in Fig. 7 and Fig. 8. In our model, when the bag constant, $\alpha$ and the quark chemical potential are the same, two-flavor quark matter has a higher pressure than three-flavor quark matter, or, at the same pressure, the two-flavor matter has a lower chemical pressure than three-flavor matter.
also be seen from the Fig. 6 that the pressure is influenced by
the three-flavor quark matter in our model. However, it can
paper [8]. Witten estimates the relation between quark chem-
potential, which is different from what Witten expects in his
work [11]. This result indicates the significance of the
interaction between particles in the study of quark stars. So,
three-flavor models can construct a two-solar mass
neutron star. From the Lagrangian we can see that the vector-
isoscalar channel plays an important role in the Fierz trans-
formed Lagrangian after the mean-field approximation is per-
formed, and it is this term that increase the stiffness of EOSs.

\[ \alpha = 0.1, 0.3, 0.5, 0.7, 0.8 \] With a fixed
negative pressure of vacuum: \( B = (100\text{MeV})^4 \), the stiffness of EOS
increases along with \( \alpha \).

\[ \alpha = 0.9 \text{ and } B = (100\text{MeV})^4 \] and three-flavor condition with
\( \alpha = 0.8 \) and \( B = (120\text{MeV})^4 \) respectively. Masses are scaled by the
mass of sun: \( M_{\odot} \). According to the tidal deformability constrain,
the lower radius limit of a 1.6-solar-mass neutron star is 10.7km, the
upper radius limit of a 1.4-solar-mass neutron star is 13.2km.

In this paper, we utilize an NJL-type model to study the
phase diagram of two-flavor and three-flavor quark matter
based on the Pauli-Villars cutoff scheme. A parameter \( \alpha \)
that can change the position of chiral phase transition point is
introduced by our self-consistent mean-field approximation.
Our phase diagrams have no critical end point. Then, the
equation of states of both two-flavor quark matter and three-
flavor quark matter are calculated. We find that the two-flavor
quark matter is more stable than the three-flavor quark matter
in our model, which is different from what Witten expects
based on the MIT bag model, but is in accordance with a re-
cent work [11]. This result indicates the significance of the
interaction between particles in the study of quark stars. So,
whether the three-flavor quark matter is more stable than the
two-flavor quark matter is model dependent. Up to now there
is no reliable way to give the answer. Finally, we obtain the
mass-radius relation of quark stars that can be compared with
observations to optimize the parameters in our model.
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