Chains of End Elementary Extensions of Models of Set Theory

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Abstract

Large cardinals arising from the existence of arbitrarily long end elementary extension chains over models of set theory are studied here. In particular, we show that the large cardinals obtained that way (‘Unfoldable cardinals’) behave as a ‘boundary’ between properties consistent with ‘V=L’ and existence of indiscernibles. We also provide an ‘embedding characterisation’ of the unfoldable cardinals and study their preservation and destruction by various different forcings.

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0 Introduction.

Let \((M, E)\) be a model of set theory. A model of the same type \((N, F)\) end extends \((M, E)\) iff for every \(a \in M\), the sets \(a_E = \{b \in M|bEa\}\) and \(a_F = \{b \in N|bFa\}\) are the same. In other words, elements of \(M\) are not enlarged by the extension from \(M\) to \(N\). The structure \((\mathcal{E}_M, \prec_e)\) consists of
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all the non-trivial end elementary extensions (for short, eees) of $M$, ordered by the relation $\prec_e$. Properties related to the non-triviality of $(E_M, \prec_e)$ (i.e., the existence of eees of $M$) and its connection to the structure of $M$ were first studied by Keisler, Silver and Morley in [KeMo 68] and [KeSi 70]. Then, among others, Enayat in [En 84], Kaufmann in [Ka 83] and Villaveces in [Vi $\infty$] continued the study in that direction.

In this article, we concentrate on the study of the existence of various types of chains in $E_{\mathfrak{a}}$, where $\mathfrak{a}$ is an expansion of $\mathcal{R}(\kappa)$ by monadic predicates. Our analysis of chains in $E_{\mathfrak{a}}$ is based on the study of the following definition of a large cardinal property:

**Definition 1:** $\kappa$ inaccessible is **unfoldable** iff $\forall S \subset \kappa \forall \lambda \exists$ well-founded $N, \hat{S}$ such that $(\mathcal{R}(\kappa), \in, S) \prec_e (N, \in, \hat{S})$ and $o(N) \geq \lambda$.

We prove that unfoldable cardinals are exactly those at which $E_{(\mathcal{R}(\kappa), \in, S)}$ has arbitrarily long chains, for all $S \subset \kappa$ (Proposition 1.1).

We prove that unfoldable cardinals are in between weakly compact and Ramsey cardinals (Theorem 1.5). We compare the relative consistency strengths between unfoldables and several cardinal inside those bounds, in contexts compatible with $V = L$ and in the presence of large cardinals. Unfoldability relativises to $L$. When $V = L$ holds, unfoldables are quite big cardinals: they are beyond all the first $\Pi^m_1$-indescribable cardinals, and (if they exist) they are also beyond the first subtle cardinal. On the other hand, the consistency of the existence of subtle cardinals is stronger than that of unfoldable cardinals: we prove that if $\theta$ is subtle, then $\mathcal{R}(\theta) \models '\exists \kappa (\kappa$ unfoldable)' (Theorem 2.2).

In the presence of Ramsey cardinals, the situation changes rather dramatically: the strong indescribability properties of unfoldables disappear (in that context, unfoldable cardinals are $\Pi^1_2$-definable). The first unfoldable is strictly in between the first weakly compact and the first Ramsey cardinal. By contrast to the $V = L$ situation, unfoldables are now rather small as large cardinals: In addition to weak compactness, a partition property
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much weaker than Ramseyness already implies the existence of unfoldable cardinals.
We also provide the following characterisation of unfoldability in terms of elementary embeddings.

**Theorem 4.1.** Let $\kappa$ be inaccessible, $\lambda \geq \kappa$. Then $\kappa$ is $\lambda$-unfoldable iff

$$\forall M \left( M \textit{ transitive, } M \models ZF^-, |M| = \kappa \in M \right)$$

$$\Rightarrow \exists j, N \left[ N \textit{ transitive, } \lambda \in N, j : M \to N, \text{crit}(j) = \kappa, |j(\kappa)| = \lambda \right].$$

We then provide conditions on iterations that guarantee the preservation of unfoldability. We also provide examples of non preservation of unfoldability, and study the consistency strength of the failure of $GCH$ at an unfoldable cardinal. Using the embedding characterisation, we highlight certain analogies between a natural strengthening of unfoldability and strong cardinals.

The following table summarises several results on the relative strength of unfoldability.

| $V = L$ | 0 exists | Ramsey exist |
|--------|----------|--------------|
| unfoldable $\uparrow$ | no Ramseys | unfoldable ($\Pi^{1}_{2}$-def.) |
| $\pi^{n}_{m}$ ($\forall n \forall m$) $\uparrow$ | weakly compact | weakly compact |
| weakly compact $\uparrow ?$ | weakly compact |

$\uparrow$ means ‘the first $x$ cardinal is strictly less than the first $y$’.

The arrows in this table work as follows:

Some notational comments are important at this point: Following two different traditions, we freely switch between the two notations ‘$V_\kappa$’ and ‘$R(\kappa)$’ when we denote the set of objects of the universe of rank less than $\kappa$. Given a model $M$, $o(M)$ denotes the ordinal height of $M$. 
1 Unfoldable Cardinals.

1.1 Long Chains in $E^{wf}_{(R(\kappa), \in, S)}$.

The following definition is central in this paper. It is a definition of a large cardinal property that mentions objects of arbitrarily high rank in the universe. It turns out to capture many connections between the existence of arbitrarily long Chains in models of the form $E^{wf}_{(R(\kappa), \in, S)}$, and other large cardinal properties.

**Definition 1** A cardinal $\kappa$ is unfoldable if and only if it is inaccessible and given any ordinal $\alpha$ and any $S \subset R(\kappa)$, there exists a well-founded $M \in E_{(R(\kappa), \in, S)}$ such that $o(M) \geq \alpha$.

This definition looks quite strong (it mentions objects of arbitrarily high rank). Nevertheless, unfoldable cardinals arise quite soon in the hierarchy of large cardinals: all Ramsey cardinals are unfoldable, and depending on additional axioms, unfoldables can be found below smaller cardinals, as will soon be shown.

The structure of chains in $(E_{(R(\kappa), \in, S)}, \prec_e)$ is extremely rich whenever $\kappa$ is an unfoldable cardinal: not only does it contain objects of arbitrary height, but it also contains chains of arbitrary height (hence the name ‘unfoldable’ for such $\kappa$). More precisely, we have

**Proposition 1.1** Let $\kappa$ be an unfoldable cardinal. Then, for any $S \subset R(\kappa)$, for every ordinal $\alpha$, $(E^{wf}_{(R(\kappa), \in, S)}, \prec_e)$ contains chains of order type $\alpha$.

**Proof:** Fix $S \subset \kappa$. By the inaccessibility of $\kappa$, there exists a club $C \subset \kappa$ such that

$$\gamma \in C \implies (R(\gamma), \in, S \cap \gamma) \prec (R(\kappa), \in, S).$$
So, \((\mathcal{R}(\gamma), \in, S \cap \gamma)_{\gamma \in C}\) is an eee-chain. Let now \(\beta > \kappa\). By the unfoldability of \(\kappa\), there exist \(M\) a transitive model, \(\hat{S}\) and \(\hat{C}\) such that
\[
(\mathcal{R}(\kappa), \in, S, C) \prec_e (M, \in, \hat{S}, \hat{C}), \quad \text{and } o(M) > \beta.
\]
Clearly, \(\kappa \in \hat{C}\). Then, \((\mathcal{R}(\gamma)^M, \in, \hat{S} \cap \gamma)_{\gamma \in \hat{C} \setminus C}\) is a chain of order type \(o(M)\) in \((\mathcal{E}_{\mathcal{R}(\kappa), \in, S}, \prec_e)\).

The parameter \(S\) in the definition of unfoldable cardinals is crucial for them to be of any interest: as Enayat points out in [En \(\infty\)], given any model \(M\) of ZF such that \(\mathcal{E}_M \neq 0\) (say \(N \succ_e M\)), there exists (by a theorem of Schmerl) \(N' \prec_e N\) such that \(\text{cof}(N') = \omega\), and \(N' \in \mathcal{E}_M\). But then, by Keisler-Morley [KeMo 68], this \(N'\) has arbitrarily large eees. So, \(M\) also has arbitrarily large eees. Thus, the definition of unfoldable cardinals is nontrivial only when extending models together with arbitrary additional parameters is required. The consistency strength of the ‘lightface version’ of unfoldability is also quite low: if \(V = \mathcal{R}(\kappa)\), where \(\kappa\) is the first Mahlo, then there are many inaccessible cardinals \(\kappa\) in \(V\) such that \((\mathcal{E}_{\mathcal{R}(\kappa)}, \prec_e)\) has an eee-chain of length \(\text{ORD}\).
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On the other hand, by iterating the $M$-ultrapower construction over $M = \mathcal{R}(\kappa)$, for $\kappa$ weakly compact, we obtain arbitrarily long chains of eees of $M$. They need not all be well-founded: for example, if $V = L$, $\kappa^+$ is an upper bound on the length of chains of well-founded eees of $M$. As Ali Enayat explained to the author, a compactness argument, together with a forcing argument provides $2^\kappa$ non-isomorphic eees of $M$, all of them well-founded, of the same height, and of size $\kappa^+$. These form a $2^\kappa$-antichain in $\mathcal{E}^\text{wf}_M$. The argument is a natural generalisation of the following theorem, proved by Enayat in [En ∞], and used by him as part of the classification of countable models of complete extensions of ZF. He proved that if $\tau$ is a countable linear order which is not well-founded, and $T$ is a complete extension of ZF, if there exists a model of $T$ with order type $\tau$, then there exist $2^{\aleph_0}$ of them.

**Theorem 1.2** (Enayat [En ∞]) Let $M$ be a countable model of ZF. Then $M$ has continuum many non isomorphic eees.

**Sketch of the Proof:** first, one constructs a class $X$ in $M$ such that $(M, X) \models \text{ZFC}(X)$ and such that every ordinal of $M$ is definable in $(M, X)$ (by forcing piecewise approximations to $X$ in $M$). Then, one constructs further expansions $(M, X, Y_\alpha)$, for $\alpha < 2^\omega$, such that $Y_\alpha \neq Y_\beta$ implies $(M, X, Y_\alpha) \neq (M, X, Y_\beta)$. Finally, using Keisler-Morley ($M$ is countable!), one gets eees $N_\alpha \succ M$, such that $N_\alpha$ codes $(X, Y_\alpha)$, for $\alpha < 2^\omega$. Then the $2^\omega$ different $N_\alpha$s are pairwise non isomorphic.

The following summarises the previous discussion.

**Proposition 1.3** Let $\kappa$ be weakly compact. Then

1. $\mathcal{E}_{\mathcal{R}(\kappa)}$ has Ord-like chains,
2. $\mathcal{E}^\text{wf}_{\mathcal{R}(\kappa)}$ has chains of length $\alpha$, for each $\alpha < \kappa^+$,
3. $(\mathcal{E}^\text{wf}_{\mathcal{R}(\kappa)})/\sim$ has antichains of size $2^\kappa$. ∎
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The process of iterating ultrapowers of models of the form $(\mathcal{R}(\kappa), \in, S)$, for $S \subset \mathcal{R}(\kappa)$, whenever that is possible (for example, when $\kappa$ is measurable, or by taking various sorts of $\mathcal{R}(\kappa)$-ultrapowers, when $\kappa$ is weakly compact, completely ineffable, etc.) yields eee-chains of length $ORD$. This fact does not extend to every unfoldable cardinal.

**Fact 1** The unfoldability of $\kappa$ does not imply the existence of $ORD$-like chains in the structure $(\mathcal{E}_{w}^{\mathcal{R}(\kappa), \in, S}, \prec_e)$.

We establish this fact by making the following observations. But we need first the following definition.

**Definition 2** An inaccessible cardinal $\kappa$ is **long unfoldable** if and only if for every $S \subset \mathcal{R}(\kappa)$, the structure $\mathcal{E}_{w}^{\mathcal{R}(\kappa), \in, S}$ has $ORD$-like chains.

In other words, besides requiring $(\mathcal{R}(\kappa), \in, S)$ to have arbitrarily long (chains of) eees, here we ask that it actually have a long eee-chain. This requirement is not superfluous: for instance, if $V = L$ holds, then the only proper class is $L$. Suppose that there is an unfoldable cardinal, let $\gamma$ be the first one. If $\langle (L_\alpha, \in, S_\alpha) | \alpha \in ORD \rangle$ were an $ORD$-like chain in $(\mathcal{E}_{(L_\gamma, \in, S)}, \prec_e)$, then its union ($= L$) would be an elementary extension of $L_\gamma$. But this is impossible: $L_\gamma$ believes that there are no unfoldables, and $L$ knows that $\gamma$ is unfoldable. But $L \succ L_\gamma$!

On the other hand, in the presence of Ramsey cardinals, plus an additional condition on the kinds of chains we have, this situation changes strongly: if $\kappa$ is a Ramsey cardinal, and $\lambda < \kappa$ is unfoldable, then $\lambda$ is also long unfoldable.

To see this, we take any unfoldable cardinal $\lambda$, some $S \subset \lambda$, and we long unfold the model $(\mathcal{R}(\lambda), \in, S)$ in two steps: first, we consider some Ramsey cardinal $\kappa > \lambda$, and we unfold $(\mathcal{R}(\lambda), \in, S)$ **up to** $\kappa$ in the following strong sense:

\[
(*)_{\lambda}^\kappa \quad \begin{cases} 
\text{for every } S \subset \lambda, \text{ there is a chain } \langle (M_\alpha, \in, S_\alpha) | \alpha < \kappa \rangle \in \\
(\mathcal{E}_{w}^{\mathcal{R}(\lambda), \in, S}, \prec_e), \text{ such that for every } \alpha < \kappa, M_\alpha \in \mathcal{R}(\kappa). 
\end{cases}
\]
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Our second step corresponds to applying stretching to that chain, which by \((\ast)_\lambda^\kappa\) consists of objects from \(\mathcal{R}(\kappa)\). This way, we get at the end an ORD-like chain of set models in \((\mathcal{E}_{wf}^{\mathcal{R}(\lambda),\in,S}, \prec_e)\).

We have just proved the

**Proposition 1.4** If there are cofinally many Ramsey cardinals in the universe and for every unfoldable \(\lambda\) there exists a Ramsey \(\kappa > \lambda\) such that \((\ast)^\kappa_\lambda\), then all the unfoldable cardinals are long unfoldable. \(\Box\)

Thus, the consistency strength of ‘ZFC + unfoldability and long unfoldability are equivalent’ is not greater than that of the theory ‘ZFC + Ramsey cardinals are cofinal in the universe + for every unfoldable \(\lambda\) there exists a Ramsey \(\kappa > \lambda\) such that \((\ast)^\kappa_\lambda\).’

**Remark:** If \(\theta\) is an ineffable cardinal, then there exists \(\kappa < \theta, \mathcal{R}(\theta) \models \kappa\) is long unfoldable. The proof of this fact is very similar to that of Theorem 2.2 below, and is left to the reader. Thus, the consistency strength of the existence of long unfoldables is less than that of the existence of ineffable cardinals, and is thus still compatible with \(V = L\). In a later section, we study in greater detail the consistency strength of unfoldability under various different extensions of ZFC. The following chain of implications summarises the previous discussion:

\[
\text{Con (ZFC + \(\exists\) ineffable) } \implies \text{Con (ZFC + \(\exists\) long unfoldable) } \implies \text{Con (ZFC + \(\exists\) unfoldable).}
\]

Other facts worth noticing (parallel to those mentioned by Enayat in [En ∞] regarding intermediate extensions) about \((\mathcal{E}_M, \prec_e)\) (for general \(M\)) are

i it is closed upwards under chains: given any chain, one can take its union: it is clearly an element of \((\mathcal{E}_M, \prec_e)\), by Tarski’s elementary chain theorem.
it is also closed downward: let \( \langle M_\alpha | \alpha < \gamma \rangle \) be a sequence of eees of \( M \) with \( M_\beta \prec_e M_\alpha \) for \( \alpha < \beta < \gamma \) and let \( M_\gamma = \cap_{\zeta \in \gamma} M_\zeta \). Suppose that \( M_\alpha \models \exists x \varphi(\bar{a}, x) \), for some formula \( \varphi(y, x) \), and for a finite sequence \( \bar{a} \) from \( M_\gamma \). Then, since every \( M_\alpha \) contains the initial segment \( (R(\theta))^{M_0}_0 \), where \( \langle M_0, \bar{a} \rangle \models [\theta = \text{first } \zeta \text{ such that } \exists x \in R(\zeta) \varphi(\bar{a}, x)] \), we have that for some \( b \in M_\gamma \), \( M_\alpha \models \varphi(\bar{a}, b) \). So, \( M_\gamma \prec M_\alpha \), for each \( \alpha < \gamma \). Clearly, we also have \( M_\gamma \succ_e M \); hence \( M_\gamma \) is really a minimum for the chain \( \langle M_\alpha | \alpha < \gamma \rangle \) in \( (E_M, \prec_e) \).

It is reasonable to think that other structural features of the same structure should be obtained using the unfoldability assumption, maybe reinforced in some way.

We begin next the study of the consistency strength of unfoldability. As the next few sections show, this consistency strength may depend up to some point on the specific axioms of set theory considered.

### 1.2 How big are the unfoldable cardinals?

**Theorem 1.5** Let \( \kappa \) be a Ramsey cardinal. Then \( \kappa \) is unfoldable.

**Proof:** Let \( S \subset R(\kappa) \). Since \( \kappa \) is Ramsey, there are stationary many weakly compacts below it. Also, \( R(\kappa) \) supports theories of indiscernibles, on languages \( L_{\alpha, \omega} \), for \( \alpha < \kappa \). Our aim is to apply stretching to get arbitrarily high eees of the whole model \( (R(\kappa), \in, S) \). For this, it is enough to get a Silver theory of indiscernibles for the expanded model \( (R(\kappa), a \in R(\kappa)) \). This will guarantee endness of the elementary extensions. Let us denote by \( T \in \Sigma [\mathfrak{A}] \) the fact that \( T \) is a theory of indiscernibles for the structure \( \mathfrak{A} \), in the logic \( L_{[\mathfrak{A}], \omega} \). Now, in order to get the theory we need, we build a \( \kappa \)-tree \( T \) of theories of indiscernibles in the appropriate infinitary languages for elementary submodels of \( (R(\kappa), \in, S) \) by decreeing

\[
(T_\beta)_{\beta < \alpha} \in T
\]
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iff

\[ \forall \beta < \alpha \left[ T_\beta \in \Sigma \left[ (a, \in, S \cap |a|, a)_{a \in A} \right], A \prec \mathcal{R}(\kappa), |A| < \kappa \right]. \]

In this tree, there are branches of arbitrary lengths \(< \kappa\), since \(\kappa\) is a limit of weakly compacts, and thus given any \(\alpha < \kappa\) weakly compact, elementary submodels \(a\) of \(\mathcal{R}(\kappa)\) of height \(\alpha\) have eees, which implies the existence of a branch of length at least \(\alpha\) on top of the corresponding theory). But the ramification of \(T\) is small for \(\kappa\) (\(2^\lambda < \kappa\), since \(\kappa\) is strongly inaccessible). So, \(T\) is a \(\kappa\)-tree. Since \(\kappa\) is weakly compact, \(T\) must have a path (of length \(\kappa\)): this path corresponds to a theory of indiscernibles for the 'union of the supports’, \((\mathcal{R}(\kappa), a)_{a \in \mathcal{R}(\kappa)}\). So, we can apply stretching, and thus get arbitrarily long chains of eees over \((\mathcal{R}(\kappa), \in, S)\), for any \(S \subset \kappa\).

\[ \blacksquare \]

Also, by the Keisler-Morley characterisation of weakly compact cardinals, we have immediately

**Proposition 1.6** If \(\kappa\) is unfoldable, then it is weakly compact.

**Proof:** It suffices to notice that unfoldability requires the existence of arbitrarily high elements in \((\mathcal{E}^{wf}_{(\mathcal{R}(\kappa), \in, S)}, <e)\), whereas by Keisler-Morley [KeMo 68], \(\kappa\) weakly compact amounts to just having \((\mathcal{E}^{wf}_{(\mathcal{R}(\kappa), \in, S)}, <e) \neq 0\).

As we attempt to determine more precisely the consistency strength of unfoldability, the answer begins to depend on the particular model of set theory we work in. We will first study the case \(V = L\).

1.3 The Height Problem for Unfoldability.

In \([\text{Vi } \infty]\), the relation between the height of a model \(M\) and the existence of end elementary extensions of \(M\) was studied. Several possible situations were analysed there, and various independence results were obtained. The main result there was that it is consistent that in most cases, inner models
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$M \subset \mathcal{R}(\kappa)$ of height $\kappa$ do not have end elementary extensions even if $\mathcal{R}(\kappa)$ has them. More precisely, we proved the

**Theorem 1.7** The theory ‘$\text{ZFC} + \exists \lambda G(\lambda) + \forall \kappa (\kappa$ inaccessible not weakly compact $\rightarrow \text{NED}(\kappa))$’ is consistent relative to the theory ‘$\text{ZFC} + \exists \lambda G(\lambda)$’. □

![Figure 2: The Existence Height Problem](image)

Here, $\text{NED}(\kappa)$ means that there is a transitive $M_\kappa \models \text{ZFC}$ such that $\text{o}(M) = \kappa$ and $\mathcal{E}_M = 0$, and $G(\lambda)$ denotes any large cardinal property preserved under Easton-type extensions where the iteration process is not carried too often (e.g. measurability, etc.).

The corresponding ‘Height Problem for Unfoldability’ has a trivial answer:

**Fact 2** If $M$ is a transitive model of $\text{ZFC}$, and $\text{o}(M) = \kappa$, for $\kappa$ unfoldable, then given any $S \subset M$, the structure $(\mathcal{E}_{(M, \in, S)}, \prec_e)$ has elements of arbitrarily high rank.

![Figure 3: The Height Problem for unfoldables](image)
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To see this, just encode $M$ (as a subset of $\mathcal{R}(\kappa)$), and find (arbitrarily high ranked) ees of the corresponding expanded structure. As before, this provides the high ranked ees of $(M, \in, S)$.

This need not be true if the parameter is not required in the definition: observe that the ‘weakened’ form of Unfoldability resulting from omitting the mention to the parameter $S$ in Definition 3 amounts to the requirement $\mathcal{E}_{\mathcal{R}(\kappa)} \neq 0$. In this case, the situation is much more complicated; this was the main reason for the difficulty of theorem 1.7 in [Vi $\infty$].

2 Unfoldables in L.

2.1 Relativisations.

In this section, we study the relationship between Unfoldables in $V$ and Unfoldables in $L$. We compare the two following situations: first, properties such as relative consistency strength, structure of $(\mathcal{E}_{\mathcal{R}(\kappa), \in, S}, \prec_e)$, and definability of Unfoldables, in the presence of axioms such as $V = L$, and second, the same properties when certain large cardinals are present.

**Theorem 2.1** Let $\kappa$ be unfoldable in $V$. Then $\kappa$ is unfoldable in $L$.

**Proof:** Fix $S \subset \mathcal{R}(\kappa)$, $S \in L$. We need to build $S'$ in $L$, in such a way that $[(\mathcal{R}(\kappa), \in, S) \prec_e (M, \in, S')]^L$, for some $M$ high enough. Let $R \subset \kappa \times \kappa$ code the construction of $S$ in $L$ (so that $(\kappa, R) \approx (L(\xi), \in)$, for some $\xi$ such that $S \in L(\xi)$, $\xi < \kappa^+$. By the unfoldability of $\kappa$ in $V$, we can get $\gamma$ of uncountable cofinality, and sets $S', R' \subset L(\gamma)$ such that $(L(\kappa), \in, S) \prec_e (L(\gamma), \in, S', R')$. $R'$ is well-founded, since $\gamma$ was taken of uncountable cofinality. Then, it is possible to reconstruct $S'$ using $R'$. Hence, $(L(\gamma), \in, S')$ is an eee of $(L(\kappa), \in, S)$ in the sense of $L$. $\square$

We remind to the reader the
Definition 3  An inaccessible cardinal $\kappa$ is **subtle** if and only if given any sequence $(S_\alpha)_{\alpha<\kappa}$, where each $S_\alpha \subset \alpha$, and given any $C$ club in $\kappa$, there are $\beta < \gamma$, both in $C$, such that $S_\gamma \cap \beta = S_\beta$.

This is a natural weakening of the notion of an ineffable cardinal. Enayat [En 85] has obtained various results related to second-order characterisations of end elementary extendability in terms of completely ineffable cardinals. Subtlety is a $\Pi^1_1$ property; yet, the first subtle is greater than the first $\Pi^1_n$—indescribable, for each $n$. They are thus much larger than the first weakly compact.

**Theorem 2.2**  Let $\theta$ be a subtle cardinal. Then, there exists $\kappa < \theta$ such that $(\kappa$ is unfoldable)$^{R(\theta)}$.

**Proof:** Suppose by way of contradiction that the statement
\[
\forall S \subset R(\kappa) \forall \gamma < \theta \exists \text{ some } E_{(R(\kappa), \in, S)}\text{-chain of length } \gamma
\]
fails for each $\kappa < \theta$. Then we obtain the sequence $(S_\kappa, \gamma_\kappa, \Delta_\kappa|\kappa < \theta)$, where for every $\kappa < \theta$, $\gamma_\kappa < \theta$, $S_\kappa \subset R(\kappa)$, $\Delta_\kappa$ is the elementary diagram of $(R(\kappa), \in, S_\kappa)$, and where all chains of eees over $(R(\kappa), \in, S_\kappa)$ are of length at most $\gamma_\kappa < \theta$.

Consider now a club $C$ such that if $\kappa \in C$, then for all $\alpha < \kappa$, $\gamma_\alpha < \kappa$. The subtlety of $\theta$ guarantees that we can pick $\kappa_1 < \kappa_2$, both in $C$, such that
\[
(R(\kappa_1), \in, S_{\kappa_1}) \prec (R(\kappa_2), \in, S_{\kappa_2}).
\]

Now, this can be done in such a way that all the $\kappa$’s involved are inaccessible. But then the set
\[
C = \{\alpha < \kappa_2|(R(\alpha), \in, S_{\kappa_2} \cap \alpha) \prec (R(\kappa_2), \in, S_{\kappa_2})\}
\]
is a club in $\kappa_2$, and thus
\[
\{(R(\alpha), \in, S_{\kappa_2} \cap \alpha)\}_{\alpha \in C \setminus \kappa_1}
\]
is an eee-chain over $(R(\kappa_1), \in, S_{\kappa_1})$ of length $\kappa_2$. But this contradicts the fact that the maximum length of such chains is $\kappa_1 < \kappa_2$. 
\[\square\]
Proposition 2.3 If $\kappa$ is unfoldable, $\theta < \kappa$, and $(\theta$ is unfoldable $)^{\mathcal{R}(\kappa)}$, then $\theta$ is unfoldable in $V$.

Proof: Without loss of generality, take $\lambda > \kappa$, and let $S \subset \mathcal{R}(\theta)$. Since $\mathcal{R}(\kappa) \models \forall \alpha \exists M_\alpha (o(M_\alpha) \geq \alpha \land \mathcal{R}(\theta) \prec_e M_\alpha)$, and $\kappa$ is at least weakly compact, we can pick a sequence $C^\alpha = \langle c_\alpha^\alpha \rangle_{\alpha < \kappa}$ of ordinals in $\kappa$ which encodes the ‘witnessing sequence’ $\langle M_\alpha \rangle_{\alpha < \kappa}$ in $\mathcal{R}(\kappa)$. Now, we can pick a well founded eee of the structure $(\mathcal{R}(\kappa), \in, \mathcal{R}(\theta), S, C)$ of height $> \lambda$. Call this eee $(N, \in, \mathcal{R}(\theta)'^\prime, S'^\prime, C'^\prime)$. Then $C'^\prime$ encodes a chain of eees of $\mathcal{R}(\theta)'(= \mathcal{R}(\theta))$ of height $o(N) > \lambda$ in $(N, \in, \mathcal{R}(\theta), S)$. Without loss of generality, $N$ can have been picked high enough so that the last fact holds as well in $V$. Since $\lambda$ was arbitrary, this provides in $V$ eees of $(\mathcal{R}(\theta), \in, S)$. $\square$

So, the consistency strength of the existence of unfoldable cardinals lies strictly between those of the existence of weakly compact and of subtle cardinals. If $V = L$ holds, then the first weakly compact cardinal cannot be unfoldable: there are no chains of well-founded models of length $\kappa^+$ above $\mathcal{R}(\kappa)$ (otherwise, there would be such chains consisting of models of size $\kappa$; but then $\mathcal{R}(\kappa^+)$ would be an elementary extension of $\mathcal{R}(\kappa)$, which is clearly impossible). We also have that

Proposition 2.4 If $V=L$ holds, and there exist unfoldable and subtle cardinals, then the first unfoldable is bigger than the first subtle.

Proof: Let $\kappa$ be unfoldable, and let $\alpha$ be subtle. Suppose, without loss of generality, that $\alpha > \kappa$. Since $V = L$, there exists $\delta > \alpha^{++}$ such that $(L_\delta, \in, S'^\prime) \succ_e (L_\kappa, \in, S)$. But then, $L_\delta \models \exists \beta (\beta$ subtle), so $L_\kappa \models \exists \beta (\beta$ subtle). $\square$

Contrast this to Theorem 2.2: below the first subtle $\theta$, there are always unfoldables in the sense of $\mathcal{R}(\theta)$, yet are there NO real unfoldables below $\theta$. 
3 Unfoldables, Ramseys and $\Pi_1^2$-sets.

Not only does the consistency strength of unfoldability vary radically when we go from models of $V = L$ to models where larger cardinals are present: the definability properties also change. We first look at what happens in the extreme case $V = L$.

**Proposition 3.1** If $V = L$, and $\kappa$ is unfoldable, then for all $m$ and $n$, $\kappa$ is $\Pi_m^n$-indescribable.

**Proof:** Let $\varphi(v)$ be a $\Pi_m^n$ formula which holds ‘over’ $R(\kappa)$, with parameter $S$. Pick a high enough eee of $R(\kappa)$ (which necessarily is of the form $L(\lambda)$, by well-foundedness and condensation). By reflecting it down, we can get a club in $\kappa$ of inaccessibles at which $\varphi$ reflects down. Note that parameters in the definition of unfoldables are not used in this proof. $\square$

**Corollary 3.2** If $V = L$ holds, and there exist unfoldable and subtle cardinals, then the first unfoldable is bigger than $\pi_m^n$, for all $m$ and $n$.

($\pi_m^n$ denotes the first $\Pi_m^n$-indescribable cardinal.)

**Proof:** Just look at the previous proposition and at Proposition 2.4. $\square$

**Remark:** In the presence of Ramsey cardinals, the situation changes radically: unfoldable cardinals (below the first Ramsey) are $\Pi_1^2$-definable via the formula

$$\varphi_{unf}(\kappa) \equiv \left\{ \begin{array}{l} \forall S \subset \kappa \exists T [T \text{ is a theory of indiscernibles and} \\
(c_n)_{n<\omega} \text{ are indiscernibles and } T \text{ codes} \\
\text{ the stretching of } Th((R(\kappa), \in, S)) \text{ up to } \kappa ] \end{array} \right\}.$$ 

The formula $\varphi_{unf}$ is $\Pi_1^2$, and if $\kappa < \text{first Ramsey}$, then $\varphi_{unf}(\kappa)$ holds if and only if $\kappa$ is unfoldable, by

**Lemma 3.3** If $\theta$ is Ramsey and $E_{(R(\kappa), \in, S)}$ has elements arbitrarily high in $R(\theta)$, then $\kappa$ is unfoldable.
Proof: applying Proposition 1.1 to $E(\mathcal{R}(\kappa), \in, S)$ in $\mathcal{R}(\theta)$, get cofinal chains of ees of $(\mathcal{R}(\kappa), \in, S)$. Then stretch them to arbitrary heights by using the fact that $\theta$ is Ramsey, and a construction similar to the proof of Theorem 1.5. □

We are now in a position to compare the first unfoldable to the first Ramsey. This is the main result in this section.

**Theorem 3.4** Let $\kappa$ be the first unfoldable cardinal, and $\theta$ be the first Ramsey. Then $\kappa < \theta$.

But we actually prove the stronger

**Theorem 3.5** Let $\kappa$ be weakly compact such that $\kappa \rightarrow (\omega_1)^{<\omega}_\kappa$. Then the first unfoldable is less than $\kappa$.

(A Ramsey cardinal certainly satisfies the two hypotheses of this theorem; hence the previous Theorem.)

**Proof:** The proof will be divided into two steps. First, we will obtain an indiscernible theory at $\kappa$. For reasons analogous to those given in Theorem 1.5, getting this theory is enough to make $\kappa$ unfoldable. In a second step, we show how to get unfoldables below $\kappa$ from the result in the first step.

For each inaccessible cardinal $\alpha < \kappa$, we let $T_\alpha$ be a theory in $\omega_1$ indiscernibles for the $L_{\alpha, \omega}$-theory of $(\mathcal{R}(\kappa), \in, a)_{a \in \mathcal{R}(\alpha)}$. These theories are then pieced together by using the weak compactness of $\kappa$ in the same way we did in the case of Theorem 1.5.

We may now let $I$ be a set of indiscernibles for the model $\mathfrak{A} = (\mathcal{R}(\kappa), \in$, Skolem functions, $a)_{a \in \mathcal{R}(\kappa)}$, with a least $\omega$-th element among all the possible sets of indiscernibles for that model. By the partition property $\kappa \rightarrow (\omega_1)^{<\omega}_\kappa$, we know that we can take $I$ of order type $\omega_1$. Let now $M = \pi(\delta_\alpha(I))$ (the Mostowski transitive collapse of the Skolem hull of $I$ in $\mathfrak{A}$), and let $I^*$ be the set of indiscernibles for $M$ corresponding to $I$. Then $o(I^*) = \omega_1$, and $I^*$ is a club in (the true) $\omega_1$. Also,

$\alpha \in I^*$ implies [\alpha is unfoldable] $^M$:
if not, then none of the elements of $I^*$ would be unfoldable according to $M$. So, there would be counterexamples of the form $S_\alpha \subset \alpha$ for each $\alpha \in I^*$. But by Fodor’s Lemma, there exists a club $C$ of $I^*$ such that all the counterexamples corresponding to indiscernibles from $C$ necessarily cohere (by the original choice of $I$ as the set of indiscernibles with the least possible $\omega$-th element among them, and a remarkability argument). But then, by elementarity,

$$\alpha \in I \implies [\alpha \text{ is unfoldable }]^{\mathcal{R}(\kappa)}.$$ 

Up to now, we have cardinals $\alpha$ that unfold up to $\kappa$. By stretching the indiscernibles $I$ as in previous constructions, we observe that $\alpha$ is also unfoldable in $V$. □

The previous results give ‘upper bounds’ for consistency of Unfoldability. We now concentrate on the ‘lower bounds’. We already know that, under $V = \mathcal{L}$, the first unfoldable is at least the first completely indescribable cardinal (if they exist), and is thence much greater than the first weakly compact cardinal.

In general, as observed before, one cannot guarantee that the ‘gap’ between weakly compacts and unfoldables is as big as under $V = \mathcal{L}$: in the presence of Ramsey cardinals, unfoldables are $\Pi^2_1$-definable. Still, the fact that the first unfoldable is greater than the first weakly compact may hold under some additional large cardinal hypotheses. One important case is given by the following theorem.

**Theorem 3.6** If there exists a Ramsey cardinal, then the first unfoldable cardinal is greater than the first weakly compact. (And of course, these two exist!)

**Proof:** Let $\lambda$ be Ramsey, and let $\kappa < \lambda$ be unfoldable (the previous results show that this situation is possible). We prove that $\kappa$ is not the first weakly compact cardinal. To reach a contradiction, assume $\kappa = \text{first weakly compact}$. For such a $\kappa$, there is a chain in $(\mathcal{E}(\mathcal{R}(\kappa), \in, S), \prec_e)$ of length $\kappa$ that
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cannot be extended to a chain of length $\kappa^+$ in the same structure. (It could still have other chains of arbitrarily high length; we have not yet reached a contradiction). Now, let $\vec{M} = \langle (M_\alpha, \in, S_\alpha) \mid \alpha < \theta \rangle$ be one such chain, where $\theta$ is some ordinal less than $\kappa^+$ and $|M_\alpha| = \kappa$, for all $\alpha < \theta$. The chain $\vec{M}$, being of length $< \kappa^+$, can be coded by some $Q \subset \kappa$. By the unfoldability of $\kappa$, the model $(\mathcal{R}(\kappa), \in, S, Q)$ has some transitive end elementary extension $(N, \in, \tilde{S}, \tilde{Q})$ with $o(N) \geq \lambda$. But then, $\tilde{Q}$ codes an eee chain of length $o(N)$. Stretching the model $(N, \in, \tilde{S}, \tilde{Q})$: this provides eee-chains of arbitrary length over $(\mathcal{R}(\kappa), \in, S)$.

The following table summarises several of the results of the last sections.

| $V = L$ | $0^\sharp$ exists | Ramsey $\uparrow$ | Ramsey $\uparrow$ |
|---------|------------------|-----------------|------------------|
| unfoldable | no Ramseys | $\kappa$ w. comp. $\uparrow$ | $\kappa$ w. comp. $\uparrow$ |
| $\pi^m_n$ ($\forall n \forall m$) | unfoldable | $\kappa \rightarrow (\omega_1)^{<\omega_1}_{<\kappa}$ | weakly compact |
| weakly compact | weakly compact | weakly compact | weakly compact |

The arrows in this table work as follows:

$x \rightarrow y$ means ‘the first $x$ cardinal is strictly less than the first $y$’.

This table also reflects how the consistency strength of unfoldability ‘decreases’ when one moves from $V = L$ to the presence of larger cardinals. The question mark corresponds to the fact we do not know yet if the strict inequality between the first weakly compact and the first unfoldable that holds in the two ‘extreme’ cases also holds in the intermediate column.
4 Preservation of Unfoldability.

Our main tool in the study of the relationship between iterated forcing and unfoldability is the embedding characterisation from Section 4.1. We address the general problem of constructing iterations that preserve unfoldability, and then study the specific case of adding or destroying GCH. Several connections between unfoldable cardinals and strong cardinals are encountered on the way.

4.1 An embedding characterisation.

In this section, we provide a characterisation of unfoldability in terms of elementary embeddings. This is a useful version of the definition of these cardinals, and provides interesting variants. On the other hand, our characterisation highlights a sharp analogy between unfoldable and strong cardinals. This analogy suggests natural paths for further research. We intend to use master condition arguments in order to prove the preservation of the unfoldability of cardinals under various iterations. We will provide conditions under which elementary embeddings $j : M \to N$, where $M$ and $N$ model fragments of ZFC, lift to elementary embeddings $j : M[G] \to N[H]$, where $G$ is $P$-generic over $M$, for some $P \in M$, and $H$ is $j(P)$-generic over $N$. In what follows, we will refer to elementary embeddings just as ‘embeddings’, and in most cases, will abuse notation by using ‘$j$’ or ‘$i$’ for both the basic embedding and its lifting. Remember that a sufficient condition for the lifting to exist is that

$$\forall p \in G \ j(p) \in H.$$ 

Here is a reformulation of unfoldability in terms of elementary embeddings. James Cummings suggested this line of research.

**Theorem 4.1** Let $\kappa$ be inaccessible, $\lambda \geq \kappa$. Then $\kappa$ is $\lambda$-unfoldable iff

$$\forall M \left( M \text{ transitive}, M \models ZF^- , |M| = \kappa \in M \right)$$

$$\Rightarrow \exists j, N \left[ N \text{ transitive}, \lambda \in N, j : M \to N, \text{crit}(j) = \kappa, j(\kappa) \geq \lambda \right].$$
Proof: We first assume that $\kappa$ is $\lambda$-unfoldable. Start with $M$ a transitive model of a finite fragment of ZFC, such that $|M| = \kappa \in M$ and $M^{<\kappa} \subset M$. Code $(M, \in)$ by $(\kappa, E)$ in such a way that the Mostowski collapse $\pi$ of $(\kappa, E)$ is such that $\pi^*\kappa = M$, $\pi(0) = \kappa$. Then $E \subset \kappa^2 \subset V_\kappa$. By the $\lambda$-unfoldability of $\kappa$, get $(A, \in, \tilde{E}) \succ_e (\mathcal{R}(\kappa), \in, E)$, with $o(A) \geq \lambda$. Now interpret $\tilde{E}$ in $A$, and take its transitive collapse $\tilde{M}$, via $\tilde{\pi}$, so that $\tilde{\pi}(0) = \lambda$. Let also $j = \tilde{\pi} \circ \pi^{-1} : M \to \tilde{M}$. Then $j(\kappa) = \pi \circ \pi^{-1}(\kappa) = \tilde{\pi}(0) = \lambda$ and $|N| = |\tilde{M}| = \lambda$, and we get the next commuting diagram.

Let now $j = \tilde{\pi} \circ \pi^{-1} : M \to \tilde{M}$. Then $j(\kappa) = \pi \circ \pi^{-1}(\kappa) = \tilde{\pi}(0) = \lambda$ and $|N| = |\tilde{M}| = \lambda$.

For the converse, let $S \subset \mathcal{R}(\kappa)$, and take a model of enough set theory $M$ such that $|M| = \kappa$, $M \supset \mathcal{R}(\kappa)$, and both $\kappa$ and $S$ are elements of $M$. Obtain $\tilde{j}$ and $\tilde{N}$ as in the statement of the theorem. Define

$$(A, \in, \tilde{S}) = (N_{j(\kappa)}, \in, \tilde{j}(S)).$$
Then

i \((\mathcal{R}(\kappa), \in, S) \subset_e (\mathfrak{a}, \in, \tilde{S})\): membership in \(N\) is just \(\in\), and \(\tilde{S} = j(S) = S \cup T\), for some \(T\) above \(\mathcal{R}(\kappa)\) (that is, \(j(S) \cap \mathcal{R}(\kappa) = S\)).

ii \((\mathcal{R}(\kappa), \in, S) \prec (\mathfrak{a}, \in, \tilde{S})\): if \((\mathfrak{a}, \in, \tilde{S}) \models \exists x \varphi(x, a_1, \ldots, a_n)\), for \(a_i \in \mathcal{R}(\kappa)\), then

\[N \models [(\mathcal{R}(j\kappa), \in, j(S)) \models \exists x \varphi(x, j(a_1), \ldots, j(a_n))],\]

since \(j(a_i) = a_i\) for each \(i\). By elementarity, we have

\[M \models [(\mathcal{R}(\kappa), \in, S) \models \exists x \varphi(x, a_1, \ldots, a_n)].\]

Finally, since \(M_\kappa = \mathcal{R}(\kappa)\),

\[(\mathcal{R}(\kappa), \in, S) \models \exists x \varphi(x, a_1, \ldots, a_n).\]

iii \(o(\mathfrak{a}) \geq \lambda\), as \(|j(\kappa)| = \lambda\) and \(o(N_{j(\kappa)}) = j(\kappa)\).

These results are a generalisation of Keisler-Silver’s characterisation of weak compactness as the existence of non-trivial ees for all \(S \subset \kappa\), and Kunen’s characterisation of weak compactness of \(\kappa\) via the existence of an \(M\)-ultrafilter on \(\kappa\). (All these, for \(\kappa\) inaccessible!) A similar generalisation, but in the direction of \(\Pi^n_\alpha\)-indescribability was studied by Hauser in [Ha 91]. We quote his theorem next to observe the analogy, and also because we use it later.
Theorem 4.2 (Hauser [Ha 91]) Let $m \geq 1, n \geq 1$. An inaccessible cardinal $\kappa$ is $\Pi^n_m$-indecomposable iff

$$\forall M\left(M \text{ trans } \land M \models ZF^- \land |M| = \kappa \land \kappa \in M \land M^{<\kappa} \subset M\right)$$

$$\Rightarrow \exists j, N\left[N \text{ trans } \land |N| = |V_{\kappa+m-1}| \land N \text{ is } \Sigma^m_{n+1} \text{-correct for } \kappa \land j : M \to N \land \text{crit}(j) = \kappa\right].$$

Remarks: 1) Our characterisation of $\lambda$-unfoldability provides an analogy between these and $\lambda$-strong cardinals, parallel to the analogy observed by Kai Hauser in [Ha 91, p.445] for $\Pi^n_m$-indecomposable cardinals and $m$-hypermeasurable cardinals. In our case, the $\lambda$-unfoldability of a cardinal can be construed as an analogue of $\lambda$-strong cardinals in the same way as weakly compact cardinals relate to measurable cardinals. In the following table, the columns are the related pairs smaller/larger cardinal.

| $\Pi^n_m$-indecom. | weakly compact | $\lambda$-unfoldable |
|-------------------|---------------|----------------------|
| $m$-hypermeas.    | measurable    | $\lambda$-strong     |

2) Taking $\lambda$ arbitrarily large, the previous analogy extends to one between unfoldable and strong cardinals.

3) The proof of Theorem 4.1 suggests a strengthening of the definition of $\lambda$-unfoldable cardinals: in the characterisation via embeddings, we could require that $N$ contain the whole $R(\lambda)$. We get the following definition.

Definition 4 $\kappa$ inaccessible is $\lambda$-strongly-unfoldable if and only if

$$\forall M\left(M \text{ transitive, } M \models ZF^-, |M| = \kappa \in M, M^{<\kappa} \subset M\right)$$

$$\Rightarrow \exists j, N\left[N \text{ trans, } R(\lambda) \cup \{\lambda\} \subset N, j : M \to N, \text{crit}(j) = \kappa, j(\kappa) \geq \lambda\right].$$

the new ingredient
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It turns out that \( \lambda \)-strongly-unfoldable cardinals are \( \Pi^m_n \)-indescribable for all \( n \) and \( m \): since \( N \supset V_\lambda \), it is \( \Sigma^m_n \)-correct for all \( n \) and \( m \), and then Hauser’s characterisation of \( \Pi^m_n \)-indescribability works here.

If \( V = L \), then strong unfoldability and unfoldability are just the same. On the other hand, strong unfoldability must be stronger than unfoldability in the presence of Ramsey cardinals, since in that case, as observed right after Corollary 3.2, the formula

\[
\varphi_{\text{unf}}(\kappa) \equiv \begin{cases} 
\forall S \subset \kappa \exists T \text{ is a theory of indiscernibles and} \\
(c_n)_{n<\omega} \text{ are indiscernibles and } T \text{ codes} \\
\text{the stretching of } Th((\mathcal{R}(\kappa), \in, S)) \text{ up to } \kappa \end{cases}.
\]

4) This embedding characterisation of \( \lambda \)-unfoldability also provides an extender characterisation, which may be useful if finer approximations of the embedding in question are to be used: an inaccessible cardinal \( \kappa \) is \( \lambda \)-unfoldable if and only if for all transitive \( M \models ZF^- \), of size \( \kappa \), closed under \( (\kappa) \)-sequences, with \( \kappa \in M \), there exists an \( M \)-extender (the natural relativisation of the notion of an extender to \( M \)) with support \( V_\lambda \).

4.2 Iterations that preserve unfoldability.

To preserve unfoldability under iterations is not a trivial matter. Several of the iterations which preserve other large cardinal properties such as measurability, \( \Pi^1_n \)-indescribability, ineffability, fail to preserve unfoldability. Why? Let us first state a theorem due to Levinski, and then explain why unfoldability fails under iterations that work in other contexts.

**Theorem 4.3** (Levinski [Le 95]) Let \( V \models ZFC \), and let \( \mathbb{P}_\kappa \) be a Reverse Easton iteration of Lévy collapses \( L((a_\alpha)^+, 2^\alpha) \), where \( a_{\alpha+1} \) is forced over \( \mathbb{P}_{\alpha+1} \) to be \( (a_\alpha)^+ \), and for limit \( \alpha \), \( a_\alpha \) is forced over \( \mathbb{P}_\alpha \) to be the supremum of all the \( a_\xi \), for \( \xi < \alpha \).
Then, if $V \models ' \kappa \text{ is measurable (or } \Pi^1_n\text{-indescribable, or ineffable, or completely ineffable)'}$, then $V^{\mathbb{P}_\kappa} \models ' \kappa \text{ is measurable (or } \Pi^1_n\text{-indescribable, or ineffable, or completely ineffable)'}$.

One would expect an analogous result for unfoldability. In the case one does a Reverse Easton iteration of the usual forcings that ‘add new Cohen subsets’ to all inaccessibles below an unfoldable $\kappa$, it may well happen that $\kappa$’s unfoldability is not preserved. This is the content of the following counterexample to the preservation of unfoldability.

**Theorem 4.4** Suppose that $V = L$. Let $\kappa$ be the first unfoldable cardinal. Then, the Reverse Easton Iteration described next does not preserve $\kappa$’s unfoldability.

**Proof:** Since $V = L$ and $\kappa$ is the first unfoldable, $(\mathcal{R}(\kappa), \in, S)$ must unfold to well-founded models of the form $(\mathcal{R}(\beta), \in, \hat{S})$.

**Claim:** The previous $\beta$ cannot be an inaccessible cardinal. Actually, even more is true: $\beta$ cannot be a cardinal. The reasons are that $\kappa$ is the first unfoldable, and the elementarity of the extension.

Assume furthermore that $\lambda$ is larger than $\kappa$, and Easton-iterate the forcings that add new Cohen subsets to every inaccessible $\alpha < \lambda$, to get the forcing notion $\mathbb{P}$. Assume also that there is some inaccessible $\theta > \lambda$. Let $G$ be $\mathbb{P}$-generic over $V$.

Then, in $V[G]$, if $[\mathcal{R}(\kappa)]^{V[G]}$ is unfolded to some model $M$ of height $> 2^\theta$, $M$ will ‘think’ that a generic subset of $\theta$ has been added, since $\theta$ is inaccessible in $M$. But the forcing $\mathbb{P}$ is a $(\lambda+1)$-iteration, and thus could not have added new unbounded subsets to $\theta > \lambda$.

In the next section, we will encounter a situation similar to the one described in the last theorem.

On the other hand, there is a positive situation for preservation of unfoldability under iterations. We could roughly describe the next theorem as
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asserting that ‘(< κ)-support iterations of forcings of size < κ that satisfy additional lifting conditions preserve the unfoldability of cardinals κ’. Although the ‘lifting conditions’ will be lengthy to describe, they turn out to be quite natural, as the reader may check.

**Theorem 4.5** Suppose that κ is λ-unfoldable, there are inaccessibles above κ, and \( P = P_{κ+1} \) is an iteration of length κ+1. Suppose furthermore that P has (< κ)-supports, and that \(|P_κ| = κ\), and that P satisfies the lifting conditions described in what follows next. Then, \( V^P \models ‘κ is λ-unfoldable’ \).

We begin by describing our additional conditions.

**The Lifting Conditions for Theorem 4.5**

i  \( P \models \text{‘}\hat{S} \in R(κ)\text{’} \); γ is greater than the least inaccessible above κ and such that \( \text{cof}(γ) > κ \) and \( V_γ \models ZF^- \).

ii  \( P \) can be defined in \( R(\text{least inacc} > κ) \), so that \( P^{V_γ} = P \).

iii  (For \( M \models ZF^- \) transitive of size κ such that \( M^{< κ} \subset M, κ \in M \), and for \( i : M \to V_γ \) with critical point \((κ^+)^M \).) Let \( \hat{S}^M \) be such that \( i(\hat{S}^M) = \hat{S} \). Notice that \( \hat{S}^M \) is a \( P^M \)-name. We define \( i \) as a transitive collapse of \( M \) with \( \text{ran}(i) \subset V_γ \). This is possible by Mostowski’s theorem in \( V_γ \). Wlog, \( \hat{S} \in \text{ran}(i) \). As \( M^{< κ} \subset M \), the critical point of \( i \) is \((κ^+)^M \).

iv  (By the λ-unfoldability of κ, choose \( j \) with \(|N| = λ \in N, N \Sigma^1_0 \)-correct for κ, and \( j : M \to N \), \( \text{crit}(j) = κ \). Wlog, \( N \) can be picked in \( V_γ \).) There exist \( G \ni p, P\)-generic over \( V_γ \), \( H P^M \)-generic over \( M \) and \( K \) \((P_{j(κ)+1})^N\)-generic over \( N \) such that both \( i \) and \( j \) lift through the corresponding forcings and \( N[K] \) is \( \Sigma^1_0 \)-correct for λ in \( V[G] \). (See theorem 4.2).
Proof of Theorem 4.5: Working in $N[K]$, let

$$(\mathfrak{a}, \in, \hat{S}) = (V[K]_{j(\kappa)}, \in, j[(\hat{S}^M)^H]).$$

It is easy to check that $j[(\hat{S}^M)^H] = j[(\hat{S}^M)^G]$. Then, suppose that

$$N[K] \models [(\mathfrak{a}, \in, \hat{S}) \models \exists x \varphi(x, a_1, \ldots, a_n)],$$

where $a_1, \ldots, a_n \in [\mathcal{R}(\kappa)]^{V[K]}$. Then, since $\kappa$ is the critical point of $j$,

$$M[H] \models [(\mathcal{R}(\kappa), \in, (\hat{S})^G) \models \exists x \varphi(x, a_1, \ldots, a_n)].$$

But since whenever $\alpha \leq \kappa$, $M[H]_\alpha = V[G]_\alpha$, we have that

$$V[G] \models [(\mathcal{R}(\kappa), \in, (\hat{S})^G) \models \exists x \varphi(x, a_1, \ldots, a_n)].$$

But this implies that $\kappa$ is unfoldable in the extension by $\mathbb{P}$: we have just provided the embedding (in $V[G]$) required by our characterisation of unfoldability in Theorem 4.1.

The much nicer version of the theorem that would be obtained by removing the lifting conditions is unfortunately not true, as demonstrated by the previously given counterexample. Still, we think that those lifting conditions are natural enough as conditions for preservation of unfoldability.
5 Destroying or adding GCH at an unfoldable.

Preservation of large cardinal properties while forcing new subsets of them has played an important role in the development of iterated forcing notions. We look next at iterations that preserve unfoldability of a cardinal \( \kappa \) while adding large numbers of subsets. Kai Hauser proved in [Ha 91] that \( \Pi^1_n \)-indescribable cardinals \( \kappa \) are preserved under iterations that add Cohen subsets of all inaccessible cardinals below \( \kappa \). We are interested in studying the consistency strength of the failure of GCH at an unfoldable. Since we will be using iterations of forcings that add new Cohen subsets at various places, we begin by noticing the following useful fact.

**Fact 3** If the \( \lambda \)-unfoldability of \( \kappa \) is not destroyed by adding one Cohen subset to \( \kappa \), then it is not destroyed by adding any number \( \mu > \kappa \) of Cohen subsets to \( \kappa \).

We also remind the reader of the definition of ineffable cardinals.

**Definition 5** A cardinal \( \theta \) is *ineffable* iff for any sequence \( \langle S_\alpha \subset \alpha | \alpha < \theta \rangle \) there exists a ‘coherence’ set \( S \subset \theta \) such that \( \{ \alpha < \theta | S \cap \alpha = S_\alpha \} \) is stationary in \( \theta \).

The consistency strength of ineffables is way below that of Ramseys, but much stronger than that of weakly compacts: if \( \theta \) is ineffable, then there are \( \theta \) weakly compacts below it. On the other hand, ineffables are still in the ‘small team’ among large cardinals in the sense that they do not imply the existence of \( 0^\#$\): they actually relativise to \( L \).

Levinski’s theorem (Theorem 4.3 here) directly provides us with a way of adding GCH at unfoldable cardinals.

**Theorem 5.1** Let \( \kappa \) be ineffable. Then the iteration of collapses that adds GCH defined in Theorem 4.3 preserves the existence of unfoldables in \( \mathcal{R}(\kappa) \).
Proof: by Theorem 4.3, the ineffability of \( \kappa \) is preserved by \( P_\kappa \), and GCH is added all the way beneath \( \kappa \). Then, by theorem 2.2, there must be unfoldables in the sense of \( R(\kappa) \). Of course, GCH holds at those \( R(\kappa) \)-unfoldables. \( \Box \)

We provide next an upper bound for the consistency strength of the failure of GCH at an unfoldable cardinal.

**Theorem 5.2** \( \text{Con}[ZFC + \exists \theta \ (\theta \text{ is subtle})] \) implies \( \text{Con}[ZFC + \exists \kappa \ (\kappa \text{ is unfoldable } + 2^\kappa > \kappa)] \).

In some sense, this is not a very tight upper bound for the consistency strength of the failure of GCH at an unfoldable cardinal. But this is sharp enough to prove that the failure of GCH at an unfoldable cardinal does not imply the existence of \( 0^\sharp \). The sharper upper bound \( \text{Con}[ZFC + \exists \theta \ (\theta \text{ is unfoldable})] \) seems plausible, but would be obtained in a totally different way. Yet the proof using subtlety provides some additional insight into the relation between unfoldables and subtles (and ineffables). On the other hand, the next lemma is interesting independently of its connection to unfoldable cardinals.

**Proof of Theorem 5.2**: Start with a model of GCH that has a subtle cardinal \( \theta \). Let \( P_\theta \) be an iteration of length \( \theta \), such that the active stages correspond to the strongly inaccessible cardinals \( \alpha < \theta \), at which \( \alpha^{++} \) new Cohen subsets are added. Formally, define \( P_\theta \) as the following iteration: \( P_0 = \) the trivial forcing; for limit \( \alpha < \theta \), \( P_\alpha \) is the direct limit of \( \langle P_\beta \mid \beta < \alpha \rangle \) if \( \alpha \) is inaccessible, the inverse limit otherwise. Working on \( P_\alpha \), \( \dot{Q}_\alpha \in V^{P_\alpha} \) is such that \( \| -P_\alpha \dot{Q}_\alpha = \{0\} \) if \( \alpha \) is not inaccessible, \( \dot{Q}_\alpha = Add(\alpha^{++}, \alpha) \) otherwise; finally, \( P_{\alpha+1} = P_\alpha \ast \dot{Q}_\alpha \).

Our aim is to prove that the subtlety of \( \theta \) is preserved by this iteration: If this is the case, and \( G \) is \( P_\theta \)-generic, then, by Theorem 2.2, we know that the model \( [R(\theta)]^{V[G]} \) ‘thinks’ that there are unfoldable cardinals. But in particular, GCH has been destroyed at those unfoldables, and so \( [R(\theta)]^{V[G]} \models ‘ZFC + \exists \kappa \ (\kappa \text{ is unfoldable } + 2^\kappa > \kappa)’ \).
Lemma 5.3 The subtlety of $\theta$ is preserved under the iteration of forcings that add new Cohen subsets to inaccessibles below $\theta$.

Proof of the Lemma: We need to prove that, in $V[G]$, for every sequence $\langle S_\alpha | \alpha < \kappa \rangle$ such that $S_\alpha \subset \alpha$, for each $\alpha < \kappa$, and for any $C$ club in $\kappa$, there are at least two elements $\beta < \gamma$ of $C$ such that $S_\beta = S_\gamma \cap \beta$.

First, observe that clubs in $\kappa$ have to be in the ground model: no unbounded subset has been added to $\kappa$. So, fix $C$ a club in $\kappa$ (in $V$).

On the other hand, the sequence $\langle S_\alpha | \alpha < \kappa \rangle$ consists (in general) of elements of the extension; choose canonical names $\sigma_\alpha$ for each $S_\alpha$. This can be done in the following more canonical way: Let

$$ 1 \models_{\mathbb{P}_\theta} (\hat{F} \text{ is a function } \land \hat{F}(\alpha) \subset \alpha, \forall \alpha \in C). $$

We need to prove that the values of $\hat{F}$ cohere on at least two elements of $C$. Now, $\hat{F}(\alpha)$ depends only on $\alpha$ many generics (added at each level up to and including $\alpha$). For each $\alpha \in C$, choose a $\mathbb{P}_{\alpha+1}$-name $\tau_\alpha$, and choose a condition $p_\alpha \in \mathbb{P}_\theta$ such that

$$ p_\alpha \models_{\mathbb{P}_\theta} (\tau_\alpha = \hat{F}(\alpha)). $$

The $\theta$-cc of $\mathbb{P}_\theta$ implies that there must exist $\alpha < \beta$ from $C$ such that $p_\alpha$ and $p_\beta$ are compatible. Without loss of generality, these two conditions can be chosen in such a way that the supremum of the support of $p_\alpha$ is $\beta$. Moreover, without loss of generality, and by the subtlety of $\theta$ in the ground model, the restrictions $p_\alpha|\alpha$ and $p_\beta|\beta$ can be made coherent, so that $p_\alpha|\alpha = p_\beta|\alpha$.

Thus equipped, it only remains to make the $\tau$'s coherent at two places in $C$ as well. We cannot apply subtlety of $\theta$ in $V$ yet, since the $\tau_\alpha$'s do not live on the $\alpha$'s. Nevertheless, as each $\tau_\alpha$ only depends on $\alpha$ many of the generic subsets of $\alpha$, we can apply subtlety to subsets $\tilde{\tau}_\alpha$ of $\alpha$. These subsets represent isomorphism types of the possible $\tau_\alpha$'s.

The two previous applications of subtlety (to make two $p_\alpha$'s cohere and then to make two $\tilde{\tau}_\alpha$'s cohere) can really be made simultaneously, by coding the
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corresponding pairs. So, we get that

\[ \exists \alpha < \beta, \alpha, \beta \in C(p_\beta \models \tilde{\tau}_\beta \cap \alpha = \tau_\alpha) . \]

But then, \( \theta \) is also subtle in the generic extension. \( \square \)

With this, we finish the proof that GCH is destroyed at unfoldables, by looking at the model \( \mathcal{R}(\theta) \). \( \square \)

Remark: a crucial feature of the proof of Lemma 5.3 was the boundedness of the forcing; the fact that nothing was added at stage \( \theta \). The lemma shows that the subtlety of \( \theta \) plus the failure of GCH at all inaccessibles below \( \theta \) is not stronger consistencywise than the subtlety of \( \theta \). The related result that the subtlety of \( \theta \) plus the failure of GCH at \( \theta \) is not stronger consistencywise than the subtlety of \( \theta \) stems from the fact that \( \Pi^1_1 \) properties at \( \theta \) are not destroyed by \((< \theta)\)-closed forcings. (This last fact is due to Silver and Kunen).

Remark: the failure of GCH at a fixed unfoldable: the previous proof requires the existence of a subtle \( \theta \) and provides unfoldable cardinals beneath \( \theta \) at which the GCH fails. A natural question is whether the failure of GCH may be obtained at \( \kappa \), starting from hypotheses about \( \kappa \), and not about larger cardinals.

Conjecture Con(ZFC + \( \exists \kappa (\kappa \text{ strongly unfoldable}) \) \( \Rightarrow \) Con(ZFC + \( \exists \kappa (2^\kappa > \kappa \land \kappa \text{ unfoldable}) \)).

This would mean that the consistency strength of the failure of GCH at an unfoldable \( \kappa \) is not greater than the strong unfoldability of \( \kappa \). This would sharpen the previous theorem.

Here are some reasons for the previous conjecture: To start, fix \( V = L \), and \( \kappa \) a \( \lambda \)-strongly unfoldable cardinal. We plan to destroy GCH at \( \kappa \) while proving that the \( \lambda \)-unfoldability of \( \kappa \) is preserved in the corresponding forcing extension. The natural forcing would be an iteration of length \( \kappa + 1 \), where the active stages correspond to the strongly inaccessible cardinals \( \alpha < \kappa \), at which \( \alpha^{++} \) new Cohen subsets are added. Then we would use the observation in the
last section about the existence of a lifting for the corresponding embeddings to prove that the $\lambda$-unfoldability of $\kappa$ is preserved.

Formally, define $P_{\kappa+1}$ as the following iteration: $P_0 = \text{the trivial forcing}$; for limit $\alpha$, $P_\alpha$ is the direct limit of $\langle P_\beta | \beta < \alpha \rangle$ if $\alpha$ is inaccessible, the inverse limit otherwise. Working on $P_\alpha$, $\dot{Q}_\alpha \in V^{P_\alpha}$ is such that $\parallel -P_\alpha \dot{Q}_\alpha = \{0\}$ if $\alpha$ is not inaccessible, $\dot{Q}_\alpha = \text{Add}(\alpha^{++}, \alpha)$ otherwise; finally, $P_{\alpha+1} = P_\alpha * \dot{Q}_\alpha$.

Clearly, if $\mu \leq \kappa$ is inaccessible, then for all $\alpha < \mu$, $|P_\alpha| < \mu$. So, if $\mu \leq \kappa$ is Mahlo, the forcing $P_\mu$ is $\mu$-cc, since then $\{ \alpha < \mu | P_\alpha = \underset{\eta < \alpha}{\text{lim}} P_\eta \}$ is stationary in $\mu$. Also, $\parallel -P_\mu \mu$ is inaccessible’, and for all $\alpha < \mu$, $\parallel -P_\alpha P_\alpha,\mu$ is $(< \nu)$-closed’, where $\nu$ is the least inaccessible $\geq \alpha$ and $P_{\alpha,\mu}$ is the ‘remainder forcing’.

Fix $M$ a model of $ZF^-$ of size $\kappa$, where $\kappa \in M$ and $M^{<\kappa} \subset M$, and get $N$ and $j : M \rightarrow N$ a $\lambda$-unfoldability embedding as in Theorem 4.1.

The obstruction: To reach the conclusion that $\parallel -P_{\kappa+1} \kappa$ is $\lambda$-unfoldable’, we would need to produce $G \upharpoonright P_{\kappa+1}$-generic over $M$. A natural way to obtain $G$ would be by using the ‘lifting conditions’ mentioned in the last section. These, nevertheless, require an amount of closure for the portion of the forcing beyond $\kappa$ that is not yet met.

A Class Forcing and eees.

In this section, we consider what happens to models of $ZFC$ and their extendibility properties after various sorts of Class Forcing are applied. We first mention the observation made in [Vi $\infty$]

Fact 4 If $E_M \neq 0$, and $G$ is SET-generic over $M$, then, $E_{M[G]} \neq 0$.

When applying CLASS forcing, this situation may change dramatically: the fact that $M[G] \models ZFC$ is no longer guaranteed. Also, the proof that $E_{M[G]} \neq 0$ when $E_M \neq 0$ uses in a nontrivial way the fact that $G$ is $P$-generic, for some $P$ element of $M$. 
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We now restrict ourselves to the countable case, and provide an application of a theorem due to Aaron Beller from the famous book ‘Coding the Universe’ by Beller, Jensen and Welch [BeJeWe 83]. Using Keisler and Morley’s [KeMo 68], there is no problem here with the existence of eees. Nevertheless, the following fact gives insight into the situation we get into when applying class forcing.

**Proposition A.1** There exist countable $M \models ZFC + \text{‘unfoldable cardinals exist’}$, and $a \subset \omega$ such that $M[a] \models ZFC + \text{‘no unfoldable cardinals exist’}$.

**Proof:** Remember that $M[a]$ denotes the minimum model that contains $M \cup \{a\}$. In [BeJeWe 83], Aaron Beller proves that given any $\beta < \omega_1$, there is a real $a \subset \omega$ such that given any $\alpha < \beta$, $L_\alpha[a] \not\models ZFC$. So, pick $\beta < \omega_1$ and $a \subset \omega$ such that $L_\beta$ is a limit of a chain of eees, $L_\beta[a] \models ZFC$, and for any $\alpha < \beta$, $L_\alpha[a] \not\models ZFC$. Clearly, $L_\beta[a] \models \text{‘no unfoldable cardinals exist’}$. But since $\beta$ was chosen as a limit of a chain of eees, $L_\beta \models ZFC + \text{‘unfoldable cardinals exist’}$, yet the (class generic) extension $L_\beta[a]$ doesn’t. \qed

We finish by posing the following

**Question 1** Is there a model of $ZFC + \text{‘unfoldable cardinals exist’}$, such that for some $a \subset \omega$, $M[a] \models ZFC + \text{‘no unfoldable cardinals exist’}$, yet $M[a] \models ZFC + \text{‘weakly compact cardinals exist’}$?

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