HELICAL VORICES WITH SMALL CROSS-SECTION FOR 3D INCOMPRESSIBLE EULER EQUATION

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Abstract. In this article, we construct traveling-rotating helical vortices with small cross-section to the 3D incompressible Euler equations in an infinite pipe, which tend asymptotically to singular helical vortex filament evolved by the binormal curvature flow. The construction is based on studying a general semilinear elliptic problem in divergence form

$$\begin{cases}
-\varepsilon^2 \text{div}(K(x)\nabla u) = (u - q|\ln \varepsilon|)^p_x, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}$$

for small values of $\varepsilon$. Helical vortex solutions concentrating near several helical filaments with polygonal symmetry are also constructed.

Keywords: Incompressible Euler equation; Binormal curvature flow; Helical symmetry; Semilinear elliptic equations; Variational method.

1. Introduction and main results

The movement of an ideal incompressible flow confined in a 3D domain $D$ is governed by the following Euler equation

$$\begin{cases}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P, & D \times (0, T), \\
\nabla \cdot \mathbf{v} = 0, & D \times (0, T), \\
\mathbf{v}(\cdot, 0) = \mathbf{v}_0(\cdot), & D,
\end{cases}$$

where the domain $D \subseteq \mathbb{R}^3$, $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity field, $P$ is the scalar pressure and $\mathbf{v}_0$ is the initial velocity field. If $D$ has a boundary, the following impermeable boundary condition is usually assumed

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \partial D \times (0, T),$$

where $\mathbf{n}$ is the outward unit normal to $\partial D$.

Define the associated vorticity field $\mathbf{w} = (w_1, w_2, w_3) = \text{curl} \mathbf{v} = \nabla \times \mathbf{v}$, which describes the rotation of the fluid. Then $\mathbf{w}$ satisfies the vorticity equations

$$\begin{cases}
\partial_t \mathbf{w} + (\mathbf{v} \cdot \nabla) \mathbf{w} = (\mathbf{w} \cdot \nabla) \mathbf{v}, & D \times (0, T), \\
\mathbf{w}(\cdot, 0) = \nabla \times \mathbf{v}_0(\cdot), & D.
\end{cases}$$

For background of the 3D incompressible Euler equation, see the classical literature [32, 33].

Helmholtz [21] began the study of Euler equation in 1858, who first considered the vorticity equations of the flow and found that the vortex rings, which are toroidal regions in which the vorticity has small cross-section, translate with a constant speed along the
axis of symmetry. The translating speed of vortex rings was then studied by Kelvin and Hick [26] in 1899. Define the circulation of a vortex

$$c = \oint_{l} \mathbf{v} \cdot d\mathbf{l} = \iint_{\sigma} \mathbf{w} \cdot d\mathbf{\sigma},$$

(1.3)

where $l$ is any oriented curve with tangent vector field $\mathbf{t}$ that encircles the vorticity region once and $\sigma$ is any surface with boundary $l$. [26] showed that if the vortex ring has radius $r^*$, circulation $c$ and its cross-section $\varepsilon$ is small, then the vortex ring moves at the velocity

$$\frac{c}{4\pi r^*} \left( \ln \frac{8r^*}{\varepsilon} - \frac{1}{4} \right).$$

(1.4)

Then Da Rios [10] in 1906, and Levi-Civita [27] in 1908, formally found the general law of motion of a vortex filament with a small section of radius $\varepsilon$ and a fixed circulation, uniformly distributed around an evolving curve $\Gamma(t)$, which is well-known as the binormal curvature flow, or the localized induction approximation (LIA). Roughly speaking, under suitable assumptions on the solution, the curve evolves by the binormal flow, with a large velocity of order $|\ln \varepsilon|$. More precisely, if $\Gamma(t)$ is parameterized as $\gamma(s, t)$, where $s$ is the parameter of arclength, then $\gamma(s, t)$ asymptotically obeys a law of the form (see [29], p. 30, Eq. (8') and [35] p. 260 Eq. (62))

$$\partial_{t} \gamma = \frac{c}{4\pi} \ln \varepsilon |(\partial_{s} \gamma \times \partial_{ss} \gamma) = \frac{cK}{4\pi} \ln \varepsilon |b_{\gamma(t)},$$

(1.5)

where $c$ is the circulation of the velocity field on the boundary of sections to the filament, which is assumed to be a constant independent of $\varepsilon$, $b_{\gamma(t)}$ is the binormal unit vector and $K$ is its local curvature. If we scale $t = |\ln \varepsilon|^{-1}\tau$, then

$$\partial_{\tau} \gamma = \frac{cK}{4\pi} b_{\gamma(\tau)}.$$

(1.6)

Hence, under LIA vortex filaments move simply in the binormal direction with speed proportional to the local curvature and the circulation. It is worthwhile to note that, when $\Gamma$ is a circular filament, the leading term of (1.4) coincides with the coefficient of right hand side of (1.5) since in this case the local curvature $K = \frac{1}{r^*}$. The localized induction approximation found by Da Rios was applied to various physical problems, for instance, the induction due to electric currents in a wire [11], the gravitational effect associated with Saturnian rings [28] and vortex motion [29], for more detail, see the survey papers by Ricca [34, 35].

From mathematical justification, Jerrad and Seis [22] first gave a precise form to Da Rios’ computation under some mild conditions on a solution to (1.2) which remains suitably concentrated around an evolving vortex filament. Their result shows that under some conditions of a solution $w_\varepsilon$ of (1.2), there holds in the sense of distribution,

$$w_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \rightarrow c\delta_\gamma(t) t_{\gamma(\tau)}, \text{ as } \varepsilon \rightarrow 0,$$

(1.7)

where $\gamma(\tau)$ satisfies (1.6), $t_{\gamma(\tau)}$ is the tangent unit vector of $\gamma$ and $\delta_{\gamma(\tau)}$ is the uniform Dirac measure on the curve. See [23] for more results of this problem.
Until now the existence of a family of solutions to (1.2) satisfying (1.7), where \( \gamma(\tau) \) is a given curve evolved by the binormal flow (1.6), is still an open problem. This problem is well-known as the vortex filament conjecture, which is unsolved except for the filament being several kinds of special curves: the straight lines, the traveling circles and the traveling-rotating helices. For the problem of vortex concentrating near straight lines, it corresponds to the planar Euler equations concentrating near a collection of given points governed by the 2D point vortex model, see [5, 6, 12, 31, 33, 37] for example. When the filament is a traveling circle with radius \( r^* \), by (1.6) the curve is

\[
\gamma(s, \tau) = \left( r^* \cos \left( \frac{s}{r^*} \right), r^* \sin \left( \frac{s}{r^*} \right), \frac{c}{4\pi r^*} \tau \right)^t, \tag{1.8}
\]

where \( v^t \) is the transposition of a vector \( v \). Fraenkel [18] first gave a construction of vortex rings with small cross-section without change of form concentrating near a traveling circle satisfying (1.8) in sense of (1.7) and then many articles showed the desingularization results under a variety of conditions, such as constructing vortex rings in different kinds of domains with different vortex profiles, see [2, 4, 15, 19] for instance.

For vortex filament being a helix satisfying (1.6), the curve is parameterized as

\[
\gamma(s, \tau) = \left( r_s \cos \left( \frac{-s - a_1 \tau}{\sqrt{k^2 + r_s^2}} \right), r_s \sin \left( \frac{-s - a_1 \tau}{\sqrt{k^2 + r_s^2}} \right), \frac{ks - b_1 \tau}{\sqrt{k^2 + r_s^2}} \right)^t, \tag{1.9}
\]

where \( r_s > 0, k \neq 0 \) are constants characterizing the distance between a point in \( \gamma(\tau) \) and the \( x_3 \)-axis and the pitch of the helix, and

\[
a_1 = \frac{ck}{4\pi(k^2 + r_s^2)}, \quad b_1 = \frac{cr_s^2}{4\pi(k^2 + r_s^2)}. \]

Note that the local curvature and torsion of the helix are \( \frac{r_s}{k^2 + r_s^2} \) and \( \frac{k}{k^2 + r_s^2} \) respectively, and the parametrization (1.9) satisfies (1.6). It should be noted that the curve parameterized by (1.9) is a traveling-rotating helix. Let us define for any \( \theta \in [0, 2\pi] \)

\[
\tilde{R}_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \tilde{Q}_\theta = \begin{pmatrix} \tilde{R}_\theta & 0 \\ 0 & 1 \end{pmatrix}.
\]

One computes directly that

\[
\gamma(s, \tau) = \tilde{Q} \sqrt{a_1^2 \gamma(s, 0) + \left( 0, 0, -\frac{b_1 \tau}{\sqrt{k^2 + r_s^2}} \right)^t}.
\]

We can readily check that (1.9) with \( k > 0 \) and \( k < 0 \) correspond to the left-handed helix and the right-handed helix respectively (for consistency, throughout this paper we always choose a right-handed Cartesian reference centered at the origin). The problem of global well-posedness of solutions to the vorticity equation (1.2) with helical symmetry was studied in many articles, see [1, 3, 16, 24] for instance. For a helix \( \gamma(\tau) \) satisfying (1.9), there are a few results of existence of true solutions of (1.2) concentrating on this curve in sense of (1.7). The only result is shown by Dávila et al. [13], who considered...
traveling-rotating invariant Euler flows with right-handed helical symmetry concentrating near a single helix and multiple helices in the whole space $\mathbb{R}^3$. In their work, by considering

$$-\text{div}(K_H(x)\nabla u) = f_\varepsilon \left( u - \alpha |\ln \varepsilon| \left| \frac{x^2}{2} \right| \right) \text{ in } \mathbb{R}^2,$$

where $K_H$ is an elliptic operator in divergence form (defined by (2.8)), $f_\varepsilon(t) = \varepsilon^2 e^t$ and $\alpha$ is chosen properly, the authors construct solutions concentrating near a helix in the distributional sense. Note that by the choice of $f_\varepsilon$, the support set of vorticity is still the whole plane.

The aim of this paper is to construct traveling-rotating solutions to Euler equations (1.2) with helical symmetry in an infinite pipe, such that the support set of vortex is a helical tube with small cross-section $\varepsilon$ without change of form, which tends asymptotically to a traveling-rotating helix (1.9) in sense of (1.7). To get these result, we study the existence and asymptotic behavior of solutions to a general semilinear elliptic equations (see (2.16)). It should be noted that, Euler equations with helical symmetry can be regarded as the general case of 2D and 3D axisymmetric Euler equations. The cases $k \to +\infty$ and $k = 0$ correspond to the 2D Euler equations and 3D axisymmetric Euler equations, respectively.

In contrast to the 2D and 3D axisymmetric problems, the associated operator $L_H$ in vorticity equations (see (2.10)) is an elliptic operator in divergence form, which can bring essential difficulty in the construction of solutions. First, it seems impossible to reduce the second-order operator $L_H$ to the standard Laplace operator by means of a single change of coordinates. Second, lack of understanding the properties of the Green’s function of a general elliptic operator in divergence form is also a challenge. Moreover, since the eigenvalues of $K_H$ are different, solutions of the associated limiting equations are not radially symmetric functions, which is totally different from the 2D and 3D axisymmetric cases.

To state our results, we need to introduce some notations first. Since the helical vortices we are to construct is translating-rotating symmetric, domains of the flow must be helical domains with rotating symmetry about $x_3$ axis, which are the whole space and infinite pipes with circular cross section. For any $R^* > 0$, define $B_{R^*}(0) \times \mathbb{R} = \{(x_1, x_2, x_3) \mid (x_2, x_3) \in B_{R^*}(0), x_3 \in \mathbb{R}\}$ an infinite pipe in $\mathbb{R}^3$ whose section is a disc with radius $R^*$. For two sets $A, B$, define $\text{dist}(A, B) = \min_{x \in A, y \in B} |x - y|$ the distance between sets $A$ and $B$ and $\text{diam}(A)$ the diameter of the set $A$.

Our first result is concerned with the desingularization of traveling-rotating helical vortices in $B_{R^*}(0) \times \mathbb{R}$, whose support set has small cross-section $\varepsilon$ and concentrates near a single left-handed helix (1.9) in sense of (1.7).

**Theorem 1.1.** Let $k > 0$, $c > 0$ and $r_* \in (0, R^*)$ be any given numbers. Let $\gamma(\tau)$ be the helix parameterized by equation (1.9). Then for any $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$, there exists a classical solution pair $(v_\varepsilon, P_\varepsilon)(x, t) \in C^1(B_{R^*}(0) \times \mathbb{R} \times \mathbb{R}^+) \text{ of (1.1)}$ such that the support set of $w_\varepsilon$ is a topological traveling-rotating helical tube that does not change form and concentrates near the helix in sense of (1.7), that is for all $\tau$,

$$w_\varepsilon(\cdot, |\ln \varepsilon|^{-1}\tau) \to c\delta_{\gamma(\tau)}t_{\gamma(\tau)}, \quad \text{as } \varepsilon \to 0.$$
Moreover, one has

1. \( \mathbf{v}_\varepsilon \cdot \mathbf{n} = 0 \) on \( \partial B_{R^*}(0) \times \mathbb{R} \).
2. Define \( A_\varepsilon = \text{supp}(\mathbf{w}_\varepsilon) \cap \mathbb{R}^2 \times \{0\} \) the cross-section of \( \mathbf{w}_\varepsilon \). Then there are \( R_1, R_2 > 0 \) such that

\[
R_1 \varepsilon \leq \text{diam}(A_\varepsilon) \leq R_2 \varepsilon.
\]

**Remark 1.2.** By the physical meaning of \( k \), the sign of \( k \) determines two different helical structure. The curve parameterized by (1.9) with \( k > 0 \) and \( k < 0 \) correspond to the left-handed helical structure and right-handed helical structure, respectively. Theorem 1.1 shows the desingularization of a left-handed helix. For the case \( k < 0 \), results are similar.

One can also construct multiple traveling-rotating helical vortices in \( B_{R^*}(0) \times \mathbb{R} \) with polygonal symmetry. Let us consider the curve \( \gamma(\tau) \) parameterized by (1.9). For any integer \( m \), define for \( i = 1 \cdots, m \) the curves \( \gamma_i(\tau) \) parameterized by

\[
\gamma_i(s, \tau) = \bar{Q}_{2\pi(i-1)m} \gamma(s, \tau).
\]

The following result generalizes that of Theorem 1.1 to helical vortices concentrating near multiple helices with polygonal symmetry.

**Theorem 1.3.** Let \( k > 0, c > 0 \) and \( r_\ast \in (0, R^*) \) be any given numbers and \( m \geq 2 \) be an integer. Let \( \gamma_i(\tau) \) be the helix parameterized by (1.10). Then for any \( \varepsilon \in (0, \varepsilon_0] \) for some \( \varepsilon_0 > 0 \), there exists a classical solution pair \( (\mathbf{v}_\varepsilon, P_\varepsilon)(x, t) \in C^1(B_{R^*}(0) \times \mathbb{R} \times \mathbb{R}^+) \) of (1.1) such that the support set of \( \mathbf{w}_\varepsilon \) is a collection of \( m \) topological traveling-rotating helical tubes that does not change form and for all \( \tau \),

\[
\mathbf{w}_\varepsilon(\cdot, |\ln \varepsilon|^{-1} \tau) \to c \sum_{i=1}^{m} \delta_{\gamma_i(\tau)} t_{\gamma_i(\tau)}, \quad \text{as} \ \varepsilon \to 0.
\]

Moreover, one has

1. \( \mathbf{v}_\varepsilon \cdot \mathbf{n} = 0 \) on \( \partial B_{R^*}(0) \times \mathbb{R} \).
2. Define \( A_{i,\varepsilon} = \text{supp}(\mathbf{w}_\varepsilon) \cap B_{\bar{\rho}} \left( \bar{Q}_{2\pi(i-1)m}(r_\ast, 0) \right) \times \{0\} \) for some small constant \( \bar{\rho} > 0 \). Then there are \( R_1, R_2 > 0 \) such that

\[
R_1 \varepsilon \leq \text{diam}(A_{i,\varepsilon}) \leq R_2 \varepsilon.
\]

The paper is organized as follows. In section 2, we deduce the 2D vorticity-stream equations of left-handed helical solutions of (1.2) and the associated semilinear elliptic equations. A generalized toy model (see (2.15)) and the corresponding desingularization result (see Theorem 2.2) are introduced. In section 3, we show the approximate solutions and some basic estimates. In section 4 and section 5, we give proof of Theorem 2.2. The outline of proofs for Theorem 1.1 and Theorem 1.3 are given in section 6.
2. Solution with helical symmetry and a generalized model

Let us first define left-handed helical symmetric solutions and reduce (1.2) to a 2D vorticity-stream model, see [12, 16, 17]. Let \( k > 0 \). Define a one-parameter group \( \mathcal{G}_k = \{ H_\rho : \mathbb{R}^3 \to \mathbb{R}^3 \} \), where

\[
H_\rho(x_1, x_2, x_3)^t = (x_1 \cos \rho + x_2 \sin \rho, -x_1 \sin \rho + x_2 \cos \rho, x_3 + k\rho)^t.
\]

So \( H_\rho \) is a superposition of a rotation in \( x_1Ox_2 \) plane and a translation in \( x_3 \) axis, that is, \( H_\rho(x) = \tilde{Q}_\rho(x) + k\rho(0, 0, 1) \). Clearly, \( B_{R^*}(0) \times \mathbb{R} \) is invariant under the group \( \mathcal{G}_k \).

Define a vector field \( \vec{\zeta} = (x_2, -x_1, k)^t \).

Then \( \vec{\zeta} \) is the field of tangents of symmetry lines of \( \mathcal{G}_k \).

Let us define helical functions and vector fields. A scalar function \( h \) is called a helical function, if \( h(H_\rho(x)) = h(x) \) for any \( \rho \in \mathbb{R}, x \in B_{R^*}(0) \times \mathbb{R} \). By direct computations it is easy to see that a \( C^1 \) function \( h \) is helical if and only if

\[
\vec{\zeta} \cdot \nabla h = 0.
\]

A vector field \( \mathbf{h} = (h_1, h_2, h_3) \) is called a helical field, if \( \mathbf{h}(H_\rho(x)) = R_\rho \mathbf{h}(x) \) for any \( \rho \in \mathbb{R}, x \in B_{R^*}(0) \times \mathbb{R} \). Direct computation shows that a \( C^1 \) vector field \( \mathbf{h} \) is helical if and only if

\[
\vec{\zeta} \cdot \nabla \mathbf{h} = \mathcal{R} \mathbf{h},
\]

where \( \mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \) (see [17]). Helical solutions of (1.1) are then defined as follows.

**Definition 2.1.** A function pair \((v, P)\) is called a helical solution of (1.1) in \( B_{R^*}(0) \times \mathbb{R} \), if \((v, P)\) satisfies (1.1) and both vector field \( v \) and scalar function \( P \) are helical.

Throughout this paper, helical solutions also need to satisfy the orthogonality condition:

\[
v \cdot \vec{\zeta} = 0,
\]

that is, the velocity field and \( \vec{\zeta} \) are orthogonal.

Under the condition (2.1), one can check that the vorticity field \( w \) satisfies (see [17])

\[
w = \frac{w}{k} \vec{\zeta},
\]

where \( w = w_3 = \partial_{x_1}v_2 - \partial_{x_2}v_1 \), the third component of vorticity field \( w \), is a helical function. Moreover, the first equation of the vorticity equations (1.2) is equivalent to

\[
\partial_t w + (v \cdot \nabla)w + \frac{1}{k}w \mathcal{R}v = 0.
\]

As a consequence, \( w \) satisfies

\[
\partial_t w + (v \cdot \nabla)w = 0.
\]
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We now introduce a stream function and reduce the system (1.2) to a 2D vorticity-stream equation. Since \( \mathbf{v} \) is a helical vector field, we have \( \nabla \cdot \mathbf{v} = 0 \), which implies that

\[
x_2 \partial_{x_1} v_3 - x_1 \partial_{x_2} v_3 + k \partial_{x_3} v_3 = 0.
\]

(2.4)

The orthogonal condition shows that

\[
x_2 v_1 - x_1 v_2 + kv_3 = 0.
\]

(2.5)

It follows from the incompressible condition, (2.4) and (2.5) that

\[
0 = \partial_{x_1} v_1 + \partial_{x_2} v_2 + \partial_{x_3} v_3 = \partial_{x_1} v_1 + \partial_{x_2} v_2 - \frac{x_2}{k} \partial_{x_1} v_3 + \frac{x_1}{k} \partial_{x_2} v_3
\]

\[
= \partial_{x_1} v_1 + \partial_{x_2} v_2 - \frac{x_2}{k^2} \partial_{x_1} (-x_2 v_1 + x_1 v_2) + \frac{x_1}{k^2} \partial_{x_2} (-x_2 v_1 + x_1 v_2)
\]

\[
= \frac{1}{k^2} \partial_{x_1} [(k^2 + x_1^2) v_1 - x_1 x_2 v_2] + \frac{1}{k^2} \partial_{x_2} [(k^2 + x_1^2) v_2 - x_1 x_2 v_1].
\]

Since \( B_R(0) \) is simply-connected, we can define a stream function \( \varphi : B_R(0) \rightarrow \mathbb{R} \) such that \( \partial_{x_2} \varphi = \frac{1}{k^2} [(k^2 + x_1^2) v_1 - x_1 x_2 v_2] \cdot \partial_{x_1} \varphi = \frac{1}{k^2} [(k^2 + x_1^2) v_2 - x_1 x_2 v_1] \), that is,

\[
\left( \begin{array}{c}
\partial_{x_1} \varphi \\
\partial_{x_2} \varphi
\end{array} \right) = -\frac{1}{k^2} \left( \begin{array}{cc}
-x_1 x_2 & k^2 + x_1^2 \\
-k^2 - x_1^2 & x_1 x_2
\end{array} \right) \left( \begin{array}{c}
v_1 \\
v_2
\end{array} \right),
\]

or equivalently,

\[
\left( \begin{array}{c}
v_1 \\
v_2
\end{array} \right) = -\frac{1}{k^2 + x_1^2 + x_2^2} \left( \begin{array}{cc}
x_1 x_2 & -k^2 - x_1^2 \\
-x_1 x_2 & k^2 + x_1^2
\end{array} \right) \left( \begin{array}{c}
\partial_{x_1} \varphi \\
\partial_{x_2} \varphi
\end{array} \right).
\]

(2.6)

By the definition of \( w \) and (2.6), we get

\[
w = \partial_{x_1} v_2 - \partial_{x_2} v_1 = (-\partial_{x_2}, \partial_{x_1}) \left( \begin{array}{c}
v_1 \\
v_2
\end{array} \right)
\]

\[
= (-\partial_{x_2}, \partial_{x_1}) \left( \begin{array}{cc}
-x_1 x_2 & k^2 + x_1^2 \\
-x_1 x_2 & k^2 + x_1^2
\end{array} \right) \left( \begin{array}{c}
\partial_{x_1} \varphi \\
\partial_{x_2} \varphi
\end{array} \right);
\]

(2.7)

\[
= (-\partial_{x_2}, \partial_{x_1}) \left( \begin{array}{c}
-k^2 - x_1^2 & x_1 x_2 \\
-k^2 - x_1^2 & x_1 x_2
\end{array} \right) \left( \begin{array}{c}
\partial_{x_1} \varphi \\
\partial_{x_2} \varphi
\end{array} \right)
\]

\[
= \mathcal{L}H \varphi,
\]

where \( \mathcal{L}H \varphi = -\text{div}(K_H(x_1, x_2) \nabla \varphi) \) is a second order elliptic operator of divergence type with the coefficient matrix

\[
K_H(x_1, x_2) = \frac{1}{k^2 + x_1^2 + x_2^2} \left( \begin{array}{cc}
k^2 + x_2^2 & -x_1 x_2 \\
-x_1 x_2 & k^2 + x_1^2
\end{array} \right).
\]

(2.8)

Clearly from the definition of the matrix \( K_H \), one has

1. \( K_H \) is a positive definite matrix and \( (K_H(x))_{ij} \in C^\infty(B_R(0)) \) for \( i, j = 1, 2 \).
2. \( \mathcal{L}H \) is uniformly elliptic, namely, \( \lambda_1 = 1, \lambda_2 = \frac{k^2}{k^2 + |x|^2} \) are two eigenvalues of \( K_H \) which have positive lower and upper bounds.
From (2.3), (2.5) and (2.6), one has

\[ 0 = \partial_t w + v_1 \partial_{x_1} w + v_2 \partial_{x_2} w + v_3 \partial_{x_3} w \]

\[ = \partial_t w + v_1 \partial_{x_1} w + v_2 \partial_{x_2} w + \frac{1}{k} (-x_2 v_1 + x_1 v_2) \cdot \frac{1}{k} (-x_2 \partial_{x_1} w + x_1 \partial_{x_2} w) \]

\[ = \partial_t w + \frac{1}{k^2(v_1, v_2)} \left( k^2 + x_2^2 \right) \partial_{x_1} w \]

\[ = \partial_t w - \frac{1}{k^2(k^2 + x_1^2 + x_2^2)} \left( \partial_{x_1} \varphi, \partial_{x_2} \varphi \right) \left( \begin{array}{cc} x_1 x_2 & k^2 + x_2^2 \\ -k^2 - x_2^2 & -x_1 x_2 \end{array} \right) \left( \begin{array}{cc} k^2 + x_2^2 & -x_1 x_2 \\ -x_1 x_2 & k^2 + x_1^2 \end{array} \right) \]

\[ = \partial_t w - \left( \partial_{x_1} \varphi, \partial_{x_2} \varphi \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \partial_{x_1} w, \partial_{x_2} w \right) \]

\[ = \partial_t w + \nabla^\perp \varphi \cdot \nabla w, \]

(2.9)

where \( \perp \) denotes the clockwise rotation through \( \pi/2 \), i.e., \((a, b) \perp = (b, -a)\). As for the boundary condition of \( \varphi \), it follows from \( \mathbf{v} \cdot \mathbf{n} = 0 \) on \( \partial \mathcal{B}_{R^*} (0) \times \mathbb{R} \) that (see (2.66), [17]) \( \varphi \) is a constant on \( \partial \mathcal{B}_{R^*} (0) \). Without loss of generality, we set \( \varphi|_{\partial \mathcal{B}_{R^*} (0)} = 0 \). Thus the 2D vorticity-stream equations of (1.2) in \( B_{R^*} (0) \times \mathbb{R} \) is

\[
\begin{cases}
\partial_t w + \nabla^\perp \varphi \cdot \nabla w = 0, & \text{in } B_{R^*} (0), \\
w = \mathcal{L}_H \varphi, & \text{in } B_{R^*} (0), \\
\varphi = 0, & \text{on } \partial B_{R^*} (0).
\end{cases}
\] (2.10)

For a solution pair \( (w, \varphi) \) of (2.10), one can recover left-handed helical velocity field \( \mathbf{v} \) and vorticity field by \( \mathbf{w} \) of (1.2) by using (2.6), (2.5), \( \mathbf{v} (x, t) = \tilde{Q}_{x_2} \mathbf{v} \left( H_{-x_1} (x), t \right) \) and (2.2).

Let \( \alpha \) be a constant. To construct traveling-rotating helical solutions of (1.2), we look for solutions of (2.10) being of the form

\[ w(x', t) = W(\bar{R}_{-\alpha|\ln \varepsilon|} (x')), \quad \varphi(x', t) = \Phi(\bar{R}_{-\alpha|\ln \varepsilon|} (x')), \]

(2.11)

where \( x' = (x_1, x_2) \in B_{R^*} (0) \). Then one computes directly that \( (W, \Phi) \) satisfies

\[
\begin{cases}
\nabla W \cdot \nabla^\perp \left( \Phi - \frac{\alpha}{2} |x'|^2 |\ln \varepsilon| \right) = 0, \\
W = \mathcal{L}_H \Phi, \\
\Phi|_{\partial B_{R^*} (0)} = 0.
\end{cases}
\] (2.12)

So formally if

\[
\mathcal{L}_H \Phi = W = f_{\varepsilon} \left( \Phi - \frac{\alpha}{2} |x'|^2 |\ln \varepsilon| \right) \quad \text{in } B_{R^*} (0),
\]

(2.13)
for some function \( f_\varepsilon \), then (2.14) automatically holds. In the sequel we write \( x' \) as \( x = (x_1, x_2) \) and look for solutions of a semilinear elliptic equations

\[
\begin{align*}
-\text{div} \cdot (K_H(x) \nabla \Phi) &= \frac{1}{\varepsilon^2} \left( \Phi - \left( \frac{\alpha}{2} |x|^2 + \beta \right) |\ln \varepsilon| \right)_+, \quad x \in B_{R^*}(0), \\
\Phi(x) &= 0, \quad x \in \partial B_{R^*}(0),
\end{align*}
\]

(2.14)

where \( p > 1 \), \( \alpha, \beta \) are constants to be determined later. For a solution \( \Phi \) of (2.14), one can get a rotating-invariant solution pair \((w, \varphi)\) of (2.10) with angular velocity \( \alpha |\ln \varepsilon| \) by simply using (2.13) and (2.11).

Inspired by equations (2.14), let us study the existence of solutions concentrating around a couple of points to a more general model

\[
\begin{align*}
-\varepsilon^2 \text{div}(K(x) \nabla u) &= (u - q|\ln \varepsilon|)_+, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

(2.15)

where \( \Omega \subseteq \mathbb{R}^2 \) is a simply-connected bounded domain with smooth boundary, \( \varepsilon \in (0, 1) \) and \( p > 1 \). \( K = (K_{i,j})_{2 \times 2} \) is a positive definite matrix satisfying

(K1). \( K_{i,j}(x) \in C^\infty(\overline{\Omega}) \) for \( 1 \leq i, j \leq 2 \).

(K2). \( -\text{div}(K(x) \nabla \cdot) \) is a uniformly elliptic operator, that is, there exist \( \Lambda_1, \Lambda_2 > 0 \) such that

\[ q(x) \text{ is a function defined in } \overline{\Omega} \text{ satisfying } \]

(Q1). \( q(x) \in C^\infty(\overline{\Omega}) \) and \( q(x) > 0 \) for any \( x \in \overline{\Omega} \).

Denote \( \text{det}(K) \) the determinant of \( K \).

**Theorem 2.2.** Let \( K \) satisfy (K1)-(K2) and \( q \) satisfy (Q1). Then, for any given \( m \) distinct strict local minimum (maximum) points \( x_{0,j}(j = 1, \cdots, m) \) of \( q^2 \sqrt{\text{det}(K)} \) in \( \Omega \), there exists \( \varepsilon_0 > 0 \), such that for every \( \varepsilon \in (0, \varepsilon_0) \), (2.15) has a solution \( u_\varepsilon \). Moreover, the following properties hold

1. Define the set \( \bar{A}_{\varepsilon,i} = \left\{ u_\varepsilon > q \ln \frac{1}{\varepsilon} \right\} \cap B_{\bar{\rho}}(x_{0,i}) \), where \( \bar{\rho} \) is small. Then there exist \( (z_{1,\varepsilon}, \cdots, z_{m,\varepsilon}) \) and \( R_1, R_2 > 0 \) independent of \( \varepsilon \) satisfying

\[
\lim_{\varepsilon \to 0} (z_{1,\varepsilon}, \cdots, z_{m,\varepsilon}) = (x_{0,1}, \cdots, x_{0,m}),
\]

\[ B_{R_2}(z_{i,\varepsilon}) \subseteq \bar{A}_{\varepsilon,i} \subseteq B_{R_1}(z_{i,\varepsilon}). \]

2. Define \( \kappa_i(u_\varepsilon) = \frac{1}{\varepsilon^2} \int_{B_{R_1}(x_{0,i})} (u_\varepsilon - q \ln \frac{1}{\varepsilon})^+ \, dx \). Then

\[
\lim_{\varepsilon \to 0} \kappa_i(u_\varepsilon) = 2\pi q \sqrt{\text{det}(K)}(x_{0,i}).
\]

To obtain good estimates for approximate solutions, we change (2.13) to the following equivalent problem. Set \( \delta = \varepsilon |\ln \varepsilon|^{-\frac{1}{p-1}} \) and \( u = |\ln \varepsilon| w \), then (2.14) becomes

\[
\begin{align*}
-\delta^2 \text{div}(K(x) \nabla w) &= (w - q)_+^p, \quad x \in \Omega, \\
w &= 0, \quad x \in \partial \Omega.
\end{align*}
\]

(2.16)
We will construct multi-peak solutions of (2.16) in sections 3-5.

**Remark 2.3.** Results of Theorem 2.2 can be regarded as a generalization of the desingularization of classical planar vortex case (see [30, 37]) and the vortex ring case (see [15]). Note that the cases of planar vortices and vortex rings correspond to the coefficient matrix $K_H(x) = \text{Id}$ and $\frac{1}{x_1} \text{Id}$, respectively. In [15], by considering solutions of

$$
\left\{
\begin{array}{l}
-\text{div} \left( \frac{1}{b} \nabla u \right) = \frac{1}{\varepsilon^2} b \left( u - q \ln \frac{1}{\varepsilon} \right)^{p-1}, \quad x \in \Omega, \\
u = 0, \quad x \in \partial \Omega,
\end{array}
\right.
$$

where $b$ is a scalar function, the authors constructed a family of $C^1$ solutions $u_\varepsilon$ with nonvanishing circulation concentrating near a minimizer of $q^2/b$ as $\varepsilon \to 0$. Indeed, if we choose $K_H(x) = \frac{1}{b} \text{Id}$, then by Theorem 2.2 solutions will shrink to minimizers of $q^2 \sqrt{\det(K_H)} = q^2/b$, which coincides with the results in [15].

**Remark 2.4.** Recently, [8] considered desingularization of steady solutions to 3D Euler equation (1.2) with helical symmetry in helical domains. Using the critical point theory and the estimates of capacity, [8] proved the existence and asymptotic behavior of ground state solutions of (2.15) concentrating near a single point. While in this paper, by using finite-dimensional reduction method, we construct multi-peak solutions concentrating near a collection of given points, which extends the results in [8].

### 3. Approximate solutions

Our aim is to solve the following equations

$$
\left\{
\begin{array}{l}
-\delta^2 \text{div}(K(x) \nabla w) = (w - q)^p, \quad \text{in } \Omega, \\
w = 0, \quad \text{on } \partial \Omega.
\end{array}
\right.
$$

Note that $-\text{div}(K(x) \nabla \cdot)$ is a uniformly elliptic operator. Throughout this paper, we denote $C, C_1, C_2, \ldots$ positive constants independent of $\varepsilon$, whose values may change from line to line.

First, since $K$ is a $C^\infty$ positive definite matrix with all eigenvalues having uniformly positive lower and upper bounds, by the Cholesky decomposition one can find a matrix-valued function $T \in C^\infty(\Omega)$ such that for any $x \in \Omega$, $T(x)$ is invertible and

$$
(T(x)^{-1})(T(x)^{-1})^t = K(x).
$$

For simplicity, we denote $T_x = T(x)$.

Let $R > 1$ be a large constant satisfying $\Omega \subseteq T_x^{-1}(B_R(0)) + x$ for any $x \in \Omega$. Clearly by the positive definiteness of $K$, such $R$ exists.

Consider

$$
\left\{
\begin{array}{l}
-\delta^2 \Delta w = (w - a)^p, \quad \text{in } B_R(0), \\
w = 0, \quad \text{on } \partial B_R(0),
\end{array}
\right.
$$

(3.2)
where $a > 0$ is a constant. One computes directly that the unique $C^1$ positive solution of (3.2) is

$$W_{\delta,a}(x) = \begin{cases} a + \delta^{p-1} s_\delta^{-\frac{p}{p-1}} \phi \left(\frac{|x|}{s_\delta}\right), & |x| \leq s_\delta, \\ a \ln \frac{|x|}{R} / \ln \frac{s_\delta}{R}, & s_\delta \leq |x| \leq R, \end{cases}$$

where $\phi \in H^1_0(B_1(0))$ satisfies

$$-\Delta \phi = \phi^p, \quad \phi > 0 \text{ in } B_1(0),$$

and $s_\delta$ satisfy the relation

$$\delta^{\frac{2}{p-1}} s_\delta^{\frac{2}{p-1}} \phi'(1) = a / \ln \frac{s_\delta}{R}. \quad (3.3)$$

Hence (3.3) is uniquely solvable if $\delta > 0$ is sufficiently small and

$$\frac{s_\delta}{\delta \ln \frac{\delta^{\frac{2}{p-1}}}{\delta^{\frac{2}{p-1}}}} \to \left(\frac{|\phi'(1)|}{a}\right)^{\frac{p-1}{p}} \text{ as } \delta \to 0.$$ 

The Pohazaev identity implies

$$\int_{B_1(0)} \phi^{p+1} = \frac{\pi(p + 1)}{2} |\phi'(1)|^2, \quad \int_{B_1(0)} \phi^p = 2\pi |\phi'(1)|. \quad (3.4)$$

Now for any $\hat{x} \in \Omega, \hat{q} > 0$, let $V_{\delta,\hat{x},\hat{q}}$ be a $C^1$ positive solution of the following equations

$$\begin{cases} -\delta^2 \text{div}(K(\hat{x})\nabla v) = (v - \hat{q})_+^p, & \text{in } T_{\hat{x}}^{-1}(B_R(0)), \\ v = 0, & \text{on } \partial T_{\hat{x}}^{-1}(B_R(0)). \end{cases} \quad (3.5)$$

Thus one has $V_{\delta,\hat{x},\hat{q}}(x) = W_{\delta,\hat{q}}(T_{\hat{x}}x)$. Indeed, let $u(x) = v(T_{\hat{x}}^{-1}x) \in H^1_0(B_R(0))$. Then $u$ satisfies (3.2) with $a = \hat{q}$. So $u = W_{\delta,\hat{q}}$, which implies that $v(x) = W_{\delta,\hat{q}}(T_{\hat{x}}x)$. Clearly $V_{\delta,\hat{x},\hat{q}}$ has an explicit profile

$$V_{\delta,\hat{x},\hat{q}}(x) = \begin{cases} \hat{q} + \delta^{\frac{2}{p-1}} s_\delta^{-\frac{2}{p-1}} \phi \left(\frac{|T_{\hat{x}}x|}{s_\delta}\right), & |T_{\hat{x}}x| \leq s_\delta, \\ \hat{q} \ln \frac{|T_{\hat{x}}x|}{R} / \ln \frac{s_\delta}{R}, & s_\delta \leq |T_{\hat{x}}x| \leq R. \end{cases}$$

For any $z \in \Omega$, define

$$V_{\delta,\hat{x},\hat{q},z}(x) := V_{\delta,\hat{x},\hat{q}}(x - z), \quad \forall x \in \Omega.$$ 

Since $V_{\delta,\hat{x},\hat{q},z}$ is not 0 on $\partial \Omega$, we need to make a projection of $V_{\delta,\hat{x},\hat{q},z}$ on $H^1_0(\Omega)$. Let $PV_{\delta,\hat{x},\hat{q},z}$ be a solution of

$$\begin{cases} -\delta^2 \text{div}(K(\hat{x})\nabla v) = (V_{\delta,\hat{x},\hat{q},z} - \hat{q})_+^p, & \text{in } \Omega, \\ v = 0, & \text{on } \partial \Omega. \end{cases} \quad (3.6)$$

We claim that for $\delta$ sufficiently small,

$$PV_{\delta,\hat{x},\hat{q},z}(x) = V_{\delta,\hat{x},\hat{q},z}(x) - \frac{\hat{q}}{\ln \frac{R}{s_\delta}} g(x, T_{\hat{x}}x, T_{\hat{x}}z), \quad \forall x \in \Omega, \quad (3.7)$$

where $g(x, T_{\hat{x}}x, T_{\hat{x}}z)$ is sufficiently small.
where \( g_{x}(x, y) = 2\pi h_{x}(x, y) + \ln R \) for any \( x, y \in T_{x}(\Omega) \), and \( h_{x}(x, y) \) is the regular part of Green’s function of \(-\Delta\) on \( T_{x}(\Omega) \), namely for any \( y \in T_{x}(\Omega) \),

\[
\begin{cases}
-\Delta h_{x}(x, y) = 0, & x \in T_{x}(\Omega), \\
h_{x}(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|}, & x \in \partial T_{x}(\Omega).
\end{cases}
\]  
(3.8)

Note that the Green’s function \( G_{x}(x, y) \) of \(-\Delta\) in \( T_{x}(\Omega) \) with Dirichlet zero boundary condition has the decomposition

\[
G_{x}(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} - h_{x}(x, y), \quad \forall x, y \in T_{x}(\Omega).
\]  
(3.9)

Indeed, by (3.5) and (3.6) one has

\[
\begin{cases}
-\delta^{2} \text{div}(K(\hat{x})\nabla(V_{\delta, \hat{x}, \hat{q}, z} - PV_{\delta, \hat{x}, \hat{q}, z})) (x) = 0, & x \in \Omega, \\
V_{\delta, \hat{x}, \hat{q}, z} - PV_{\delta, \hat{x}, \hat{q}, z} = \hat{q} \ln \frac{\left| T_{x}(x - z) \right|}{\ln \frac{s_{\delta}}{R}}, & x \in \partial \Omega.
\end{cases}
\]

Define \( \bar{u}(y) = (V_{\delta, \hat{x}, \hat{q}, z} - PV_{\delta, \hat{x}, \hat{q}, z})(T_{x}^{-1}y), y \in T_{x}(\Omega) \). Then

\[
\begin{cases}
-\delta^{2} \Delta \bar{u}(y) = 0, & y \in T_{x}(\Omega), \\
\bar{u}(y) = \hat{q} \ln \frac{|y - T_{x}z|}{\ln \frac{s_{\delta}}{R}}, & y \in \partial T_{x}(\Omega).
\end{cases}
\]

So \( \bar{u}(y) = \frac{\hat{q}}{\ln \frac{s_{\delta}}{R}} (-2\pi h_{x}(y, T_{x}z) - \ln R) \), which implies that for any \( x \in \Omega \)

\[
(V_{\delta, \hat{x}, \hat{q}, z} - PV_{\delta, \hat{x}, \hat{q}, z})(x) = \bar{u}(T_{x}x) = \frac{\hat{q}}{\ln \frac{s_{\delta}}{R}} (-2\pi h_{x}(T_{x}x, T_{x}z) - \ln R) = \frac{\hat{q}}{\ln \frac{s_{\delta}}{R}} g_{x}(T_{x}x, T_{x}z).
\]

We get (3.7).

In the following, we will construct solutions of the form

\[
\sum_{j=1}^{m} PV_{\delta, \hat{x}_{j}, \hat{q}_{j}, z_{j}} + \omega_{\delta},
\]

where \( \Sigma_{j=1}^{m} PV_{\delta, \hat{x}_{j}, \hat{q}_{j}, z_{j}} \) is the main term and \( \omega_{\delta} \) is an error term. To make the norm of \( \omega_{\delta} \) as small as possible, we need to choose \( \hat{x}_{j} \) and \( \hat{q}_{j} \) suitably close to \( z_{j} \) and \( q(z_{j}) \).

Let \((x_{0,1}, \ldots, x_{0,m})\) be \(m\) distinct strict local maximum points (or minimum points) of \( q^{2}\sqrt{\det(K)} \) in \( \Omega \). Hence we can choose \( \tilde{\rho} > 0 \) sufficiently small such that

\[
B_{\rho}(x_{0,i}) \subseteq \Omega, \quad B_{\rho}(x_{0,i}) \cap B_{\rho}(x_{0,j}) = \emptyset, \quad \forall 1 \leq i \neq j \leq m.
\]

Define the admissible set \( \mathcal{M} \subseteq \mathbb{R}^{(2m)} \) satisfying

\[
\mathcal{M} = \{ Z = (z_{1}, z_{2}, \ldots, z_{m}) \in \mathbb{R}^{(2m)} | z_{i} \in B_{\rho}(x_{0,i}), \ i = 1, \ldots, m \}.
\]  
(3.10)

Let \( \hat{x}_{i} = z_{i} \)

and \( \hat{q}_{i} = \hat{q}_{\delta,i}(Z), \ i = 1, \cdots, m, \) be the solution the equations

\[
\hat{q}_{i} = q(z_{i}) + \frac{\hat{q}_{i}}{\ln \frac{s_{\delta}}{R}} g_{z_{i}}(T_{z_{i}}z_{i}, T_{z_{i}}z_{i}) - \Sigma_{j \neq i} \frac{\hat{q}_{j}}{\ln \frac{s_{\delta}}{R}} G_{z_{j}}(T_{z_{j}}z_{i}, T_{z_{j}}z_{j}),
\]  
(3.11)
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where \( \tilde{G}_{z_j}(x, y) = \ln \frac{R}{|x-y|} - g_{z_j}(x, y) = 2\pi G_{z_j}(x, y) \) for any \( x, y \in T_{z_j}(\Omega) \).

It follows from Lemma A.4 in Appendix that for any \( Z \) satisfying (3.10) and \( \delta \) sufficiently small, there exist \( \hat{q}_{\delta,i}(Z) \) satisfying (3.11). Moreover, one has

\[
\hat{q}_i = \frac{q(z_i) - \sum_{j \neq i} \frac{\hat{q}_{\delta,i}}{\ln s_{\delta,i}} \tilde{G}_{z_j}(T_{z_j}z_i, T_{z_j}z_j)}{1 - \frac{1}{\ln \frac{R}{\varepsilon}} g_{z_i}(T_{z_i}z_i, T_{z_i}z_i)}.
\]

For \( Z = (z_1, \cdots, z_m) \), denote

\[
V_{\delta,Z,j} = PV_{\delta,z_j,\hat{q}_{\delta,j},z_j}, \quad V_{\delta,Z} = \sum_{j=1}^{m} V_{\delta,Z,j}.
\]

Let \( s_{\delta,j} \) satisfy

\[
\delta^{\frac{2}{p+1}} \phi_j^{\frac{2}{p-1}} \phi'(1) = \hat{q}_{b,j}/\ln s_{\delta,j}.
\]

Then one can easily verify that

\[
\frac{1}{\ln \frac{R}{s_{\delta,j}}} = \frac{1}{\ln \frac{R}{\varepsilon}} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right). \quad (3.12)
\]

By the choice of \( \hat{x}_j, \hat{q}_{\delta,j} \), we claim that for any fixed constant \( L > 0 \) and \( x \in B_{Ls_{\delta,i}}(z_i) \),

\[
V_{\delta,Z}(x) - q(x) = V_{\delta,z_i,\hat{q}_{\delta,i},z_i}(x) - \hat{q}_{\delta,i} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right). \quad (3.13)
\]

Indeed, we have for any \( x \in B_{Ls_{\delta,i}}(z_i) \),

\[
V_{\delta,Z,i}(x) - q(x) = V_{\delta,z_i,\hat{q}_{\delta,i},z_i}(x) - \hat{q}_{\delta,i} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]
and for any $j \neq i$, $x \in B_{Ls_{\delta,i}}(z_i)$,

$$V_{\delta,Z,j}(x) = V_{\delta,z_i,\tilde{q}_{\delta,i},z_j}(x) - \frac{\tilde{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} g_{z_j}(T_{z_j}x, T_{z_j}z_j)$$

$$= \frac{\tilde{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} G_{z_j}(T_{z_j}x, T_{z_j}z_j)$$

$$= \frac{\tilde{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} G_{z_j}(T_{z_j}z_i, T_{z_j}z_j) + O\left(\frac{\ln |\ln \epsilon|}{|\ln \epsilon|^2} G_{z_j}(T_{z_j}z_i, T_{z_j}z_j)\right)$$

where we have used (3.12) and Lemma A.3 in Appendix. Adding up the above inequalities and using (3.11), we get

$$V_{\delta,Z}(x) - q(x) = V_{\delta,z_i,\tilde{q}_{\delta,i},z_i}(x) - \tilde{q}_{\delta,i} + O\left(\frac{\ln |\ln \epsilon|}{|\ln \epsilon|^2}\right), \quad \forall x \in B_{Ls_{\delta,i}}(z_i).$$

From (3.3), (3.11) and Lemma A.3, we get

$$\frac{\partial \tilde{q}_{\delta,i}}{\partial z_i,h} = O(1), \quad \frac{\partial s_{\delta,i}}{\partial z_i,h} = O(\delta |\ln \delta|^{|\frac{m-1}{2}|}). \quad (3.14)$$

Using the definition of $V_{\delta,z_i,\tilde{q}_{\delta,i},z_i}$, (3.14) and (3.15), we obtain

$$\frac{\partial V_{\delta,z_i,\tilde{q}_{\delta,i},z_i}(x)}{\partial z_i,h} = \left\{ \begin{array}{ll} -\frac{1}{s_{\delta,i}} \tilde{q}_{\delta,i} (\frac{\partial}{\partial z_i,h} \frac{s_{\delta,i}}{s_{\delta,i}}) + O(1), & |T_{z_i}(x - z_i)| \leq s_{\delta,i}, \\
\frac{\tilde{q}_{\delta,i}}{\ln \frac{R}{s_{\delta,i}}} \frac{T_{z_i}(x - z_i)}{|T_{z_i}(x - z_i)|^2} + O\left(\frac{\ln |\ln \epsilon|}{|\ln \epsilon|^2}\right), & |T_{z_i}(x - z_i)| > s_{\delta,i}, \end{array} \right.$$

where $(T_{z_i})_h$ is the h-th row of $(T_{z_i})^t$.

4. The Reduction

Now we find solution of (2.16) being of the form

$$V_{\delta,Z} + \omega_{\delta,i}.$$

First we prove that for any $Z$ satisfying (3.10), there exists $\omega_{\delta,Z}$ such that $V_{\delta,Z} + \omega_{\delta,Z}$ solves (2.16) in a co-dimensional 2m subspace of $H^1_0$. In the next section we choose proper $Z = Z(\delta)$ such that $V_{\delta,Z} + \omega_{\delta}$ is a solution.
Let us consider the following equation
\[- \Delta w = w^p_+, \quad \text{in } \mathbb{R}^2. \tag{4.1}\]
The unique \(C^1\) solution is
\[w(x) = \begin{cases} 
\phi(x), & |x| \leq 1, \\
\phi'(1) \ln |x|, & |x| > 1.
\end{cases} \]

By the classical elliptic equation theory, \(w \in C^{2,\alpha}(\mathbb{R}^2)\) for any \(\alpha \in (0, 1)\). The linearized equation of (4.1) at \(w\) is
\[- \Delta v - pw^p_+ v = 0, \quad v \in L^\infty(\mathbb{R}^2). \tag{4.2}\]
Clearly, \(\frac{\partial w}{\partial x_h}(h = 1, 2)\) are solutions of (4.2). It follows from [9] (see also [6]) that

**Proposition 4.1 (Non-degeneracy).** \(w\) is non-degenerate, i.e., the kernel of the linearized equation (4.2) is
\[\text{span}\{\frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_2}\}. \]

Denote
\[F_{\delta,Z} = \{u \in L^p(\Omega) | \int_\Omega \frac{\partial V_{\delta,Z,j}}{\partial z_j,h} u = 0, \quad \forall j = 1, \cdots, m, \ h = 1, 2\}, \tag{4.3}\]
and
\[E_{\delta,Z} = \{u \in W^{2,p} \cap H^1_0(\Omega) | \int_\Omega \text{div}(K(x)\nabla \frac{\partial V_{\delta,Z,j}}{\partial z_j,h}) u = 0, \quad \forall j = 1, \cdots, m, \ h = 1, 2\}. \tag{4.4}\]
So \(F_{\delta,Z}\) and \(E_{\delta,Z}\) are co-dimensional \(2m\) subspaces of \(L^p\) and \(W^{2,p} \cap H^1_0(\Omega)\), respectively.

For any \(u \in L^p(\Omega)\), define the projection operator \(Q_\delta : L^p \rightarrow F_{\delta,Z}\)
\[Q_\delta u := u - \sum_{j=1}^m \sum_{h=1}^2 C_{j,h} \frac{\partial}{\partial z_{j,h}}(-\delta^2\text{div}(K(z_j)\nabla V_{\delta,Z,j})), \tag{4.5}\]
where \(C_{j,h}(j = 1, \cdots, m, \ h = 1, 2)\) satisfies
\[\sum_{j=1}^m \sum_{h=1}^2 C_{j,h} \int_\Omega \frac{\partial}{\partial z_{j,h}}(-\delta^2\text{div}(K(z_j)\nabla V_{\delta,Z,j})) \frac{\partial V_{\delta,Z,i}}{\partial z_{i,h}} = \int_\Omega u \frac{\partial V_{\delta,Z,i}}{\partial z_{i,h}}, \quad \forall i = 1, \cdots, m, \ h = 1, 2. \tag{4.6}\]
By Lemma A.5, we know that $Q_\delta$ is a well-defined linear operator from $L^p$ to $F_{\delta,Z}$. Indeed, using (3.16) and Lemma A.3, the coefficient matrix

\[
\int_\Omega \frac{\partial}{\partial z_{j,h}} (-\delta^2 \text{div}(K(z_j) \nabla V_{\delta,z,j})) \frac{\partial V_{\delta,z,i}}{\partial z_{i,h}}
\]

\[
= p \int_\Omega (V_{\delta,z,j,\delta,j,z_j} - \hat{q}_{\delta,j}) \frac{\partial}{\partial z_{j,h}} (-\delta \frac{\partial q_{\delta,j}}{\partial z_{j,h}}) \frac{\partial V_{\delta,z,i}}{\partial z_{i,h}}
\]

\[
= p \int_\Omega (V_{\delta,z,j,\delta,j,z_j} - \hat{q}_{\delta,j}) \frac{\partial}{\partial z_{j,h}} \frac{\partial V_{\delta,z_i,\delta,i,z_i}}{\partial z_{i,h}} + O \left( \frac{\varepsilon}{|\ln \varepsilon|^p} \right)
\]

(4.7)

where $\hat{\delta}_{i,j} = 1$ if $i = j$; otherwise, $\hat{\delta}_{i,j} = 0$. $M_i$ are $m$ positive definite matrices and there exist positive constants $c_1, c_2$ independent of $\delta, Z$ such that all eigenvalues of $M_i$ belong to $(c_1, c_2)$. So there exists the unique $C_{j,h}$ satisfying (4.6). Note that for any $u \in L^p$, $Q_\delta u \equiv u$ in $\Omega \setminus \cup_{i=1}^m B_{Ls_\delta_i}(z_i)$ for some $L > 1$.

The linearized operator of (2.16) at $V_{\delta,Z}$ is

\[
L_{\delta,\omega} := -\delta^2 \text{div}(K(x) \nabla \omega) - p(V_{\delta,Z} - q)_+ \omega.
\]

We have the following estimates of $L_\delta$.

**Lemma 4.2.** There exist $\rho_0 > 0, \delta_1 > 0$ such that for any $\delta \in (0, \delta_1], Z$ satisfying (3.10), $u \in E_{\delta,Z}$ satisfying $Q_\delta L_{\delta} u = 0$ in $\Omega \setminus \cup_{j=1}^m B_{Ls_\delta_j}(z_j)$ for some $L > 1$ large, then

\[
||Q_\delta L_{\delta} u||_{L^p} \geq \frac{\rho_0 \varepsilon^2}{|\ln \varepsilon|^{p-1}} ||u||_{L^\infty}.
\]

**Proof.** We argue by contradiction. Suppose that there are $\delta_N \rightarrow 0$, $Z_N = (z_{N,1}, \ldots, z_{N,m}) \rightarrow (z_1, \ldots, z_m)$ satisfying (2.5) and $u_N \in E_{\delta_N,Z_N}$ with $Q_{\delta_N} L_{\delta_N} u_N = 0$ in $\Omega \setminus \cup_{j=1}^m B_{Ls_{\delta_N,j}}(z_{N,j})$ for some $L$ large and $||u_N||_{L^\infty} = 1$ such that

\[
||Q_{\delta_N} L_{\delta_N} u_N||_{L^p} \leq \frac{1}{N} \frac{\varepsilon^2}{|\ln \varepsilon|^{p-1}}.
\]

Let

\[
Q_{\delta_N} L_{\delta_N} u_N = L_{\delta_N} u_N - \sum_{j=1}^m \sum_{h=1}^2 C_{j,h,N} \frac{\partial}{\partial z_{j,h}} (-\delta_N^2 \text{div}(K(z_{N,j}) \nabla V_{\delta_N,z_{N,j}})).
\]

(4.8)

We now estimate $C_{j,h,N}$. For fixed $i = 1, \ldots, m$, $h = 1, 2$, multiplying (4.8) by $\frac{\partial V_{\delta_N,z_{N,i}}}{\partial z_{i,h}}$ and integrating on $\Omega$ we get

\[
\int_\Omega u_N L_{\delta_N} \left( \frac{\partial V_{\delta_N,z_{N,i}}}{\partial z_{i,h}} \right) = \int_\Omega L_{\delta_N} u_N \frac{\partial V_{\delta_N,z_{N,i}}}{\partial z_{i,h}}
\]

\[
= \sum_{j=1}^m \sum_{h=1}^2 C_{j,h,N} \int_\Omega \frac{\partial}{\partial z_{j,h}} (-\delta_N^2 \text{div}(K(z_{N,j}) \nabla V_{\delta_N,z_{N,j}})) \frac{\partial V_{\delta_N,z_{N,i}}}{\partial z_{i,h}}.
\]
We estimate $\int_{\Omega} u_N L \delta_N \left( \frac{\partial V_{\delta_N,z_i}}{\partial z_i} \right)$. Note that

\[
\int_{\Omega} u_N L \delta_N \left( \frac{\partial V_{\delta_N,z_i}}{\partial z_i} \right) = \int_{\Omega} u_N \left[ -\delta_N^2 \text{div}(K(x) \nabla \frac{\partial V_{\delta_N,z_i}}{\partial z_i}) - p(V_{\delta_N,z_i} - q)^{p-1} \frac{\partial V_{\delta_N,z_i}}{\partial z_i} \right]
\]

\[
= \int_{\Omega} u_N \left[ -\delta_N^2 \text{div}(K(z_i) \nabla V_{\delta_N,z_i}) \right] - \int_{\Omega} u_N \left( -\delta_N^2 \text{div} \left( \frac{\partial K(z_i)}{\partial z_i} \nabla V_{\delta_N,z_i} \right) \right)
\]

\[
+ \int_{\Omega} u_N \left( -\delta_N^2 \text{div} \left( (K(x) - K(z_i)) \nabla \frac{\partial V_{\delta_N,z_i}}{\partial z_i} \right) \right) - p \int_{\Omega} u_N (V_{\delta_N,z_i} - q)^{p-1} \frac{\partial V_{\delta_N,z_i}}{\partial z_i}
\]

\[
=: I_1 + I_2 + I_3 + I_4,
\]

where the matrix $\frac{\partial K(z_i)}{\partial z_i}$ is defined by

\[
\left( \frac{\partial K(z_i)}{\partial z_i} \right) = \left( \begin{array}{cc}
\frac{\partial K_{1,1}(Z)}{\partial z_i} & \frac{\partial K_{1,2}(Z)}{\partial z_i} \\
\frac{\partial K_{2,1}(Z)}{\partial z_i} & \frac{\partial K_{2,2}(Z)}{\partial z_i}
\end{array} \right).
\]

By (3.13), (3.14), Lemma A.3 and Lemma A.6 one has

\[
I_1 + I_4
\]

\[
=p \int_{\Omega} u_N \left( V_{\delta_N,z_i} \hat{\delta}_{\delta_N,z_i,i} - \hat{q}_{\delta_N,i} \right)^{p-1} \left( \frac{\partial V_{\delta_N,z_i}}{\partial z_i} \right) + O \left( \frac{|\ln| \ln \varepsilon_N|}{|\ln \varepsilon_N|^2} \right)
\]

\[
+ p \int_{\Omega} u_N \left( V_{\delta_N,z_i} \hat{\delta}_{\delta_N,z_i,i} \right)^{p-1} \frac{\partial V_{\delta_N,z_i}}{\partial z_i} + O \left( |\ln \varepsilon_N|^2 \right)
\]

\[
= O \left( |\ln \varepsilon_N|^2 \right)
\]

\[
= O \left( \frac{|\ln \varepsilon_N|}{|\ln \varepsilon_N|^2} \right).
\]
For $I_2$ and $I_3$, direct computation shows that

\[
I_2 = \int_{\Omega} u_N \left( \delta_N^2 \text{div} \left( \frac{\partial K(z_N,i)}{\partial z_{i,h}} \nabla V_{\delta N,z_N,i} \right) \right) \\
= \int_{\Omega} u_N \left( \delta_N^2 \text{div} \left( \frac{\partial K(z_N,i)}{\partial z_{i,h}} \nabla V_{\delta N,z_N,i,\delta_{N,i} z_N,i} \right) \right) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= \int_{|z_N,i-x-z_N,i| \leq s_{\delta N,i}} + \int_{|z_N,i-x-z_N,i| > s_{\delta N,i}} u_N \left( \delta_N^2 \text{div} \left( \frac{\partial K(z_N,i)}{\partial z_{i,h}} \nabla V_{\delta N,z_N,i,\delta_{N,i} z_N,i} \right) \right) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) + O(\delta_N^2) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= O \left( \frac{\varepsilon_N^2}{|\ln \varepsilon_N|^{p-1}} \right), \tag{4.11}
\]

and

\[
I_3 = \int_{\Omega} u_N \left( -\delta_N^2 \text{div} \left( (K(x) - K(z_N,i)) \nabla \frac{\partial V_{\delta N,z_N,i}}{\partial z_{i,h}} \right) \right) \\
= \int_{\Omega} u_N \left( -\delta_N^2 \text{div} \left( (K(x) - K(z_N,i)) \nabla \frac{\partial V_{\delta N,z_N,i,\delta_{N,i} z_N,i}}{\partial z_{i,h}} \right) \right) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= \int_{|z_N,i-x-z_N,i| \leq s_{\delta N,i}} + \int_{s_{\delta N,i} < |z_N,i-x-z_N,i| \leq \mu} + \int_{|z_N,i-x-z_N,i| > \mu} u_N \left( -\delta_N^2 \text{div} \left( (K(x) - K(z_N,i)) \nabla \frac{\partial V_{\delta N,z_N,i,\delta_{N,i} z_N,i}}{\partial z_{i,h}} \right) \right) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) + O(\delta_N^2) + O \left( \frac{\delta_N^2}{|\ln \varepsilon_N|} \right) \\
= O \left( \frac{\varepsilon_N^2}{|\ln \varepsilon_N|^{p-1}} \right). \tag{4.12}
\]

Here $\mu > 0$ is a small constant, and we have used (3.16), Lemmas A.3 and A.6. Taking (4.10), (4.11) and (4.12) into (4.9), we get

\[
\int_{\Omega} u_N L_{\delta N} \left( \frac{\partial V_{\delta N,z_N,i}}{\partial z_{i,h}} \right) = O \left( \varepsilon_N \ln |\ln \varepsilon_N| \right) \left| \ln \varepsilon_N \right|^{p+1}. \]

Combining with (4.7) we get

\[
C_{j,h,N} = O(\varepsilon_N \ln |\ln \varepsilon_N|).
\]
Hence
\[
\sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h,N} \frac{\partial}{\partial z_{j,h}} (-\delta_N^2 \text{div}(K(z_{N,i}) \nabla V_{\delta_N,z_{N,i}}))
\]
\[
=p \sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h,N}(V_{\delta_N,z_{N,i},q_{\delta_N,z_{N,i}},z_{N,i}} - \hat{q}_{\delta_N,z_{N,i}})^{p-1} \left( \frac{\partial V_{\delta_N,z_{N,i},q_{\delta_N,z_{N,i}},z_{N,i}}}{\partial z_{j,h}} - \frac{\partial \hat{q}_{\delta_N,z_{N,i}}}{\partial z_{j,h}} \right)
\]
\[
=O \left( \sum_{j=1}^{m} \sum_{h=1}^{2} \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right) + O \left( \sum_{j=1}^{m} \sum_{h=1}^{2} \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right)
\]
\[
=O \left( \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right), \text{ in } L^p(\Omega).
\]
So by the assumption and (1.8) we have
\[
L_{\delta_N} u_N = Q_{\delta_N} L_{\delta_N} u_N + \sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h,N} \frac{\partial}{\partial z_{j,h}} (-\delta_N^2 \text{div}(K(z_{N,i}) \nabla V_{\delta_N,z_{N,i}}))
\]
\[
=O \left( \frac{1}{N} \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right) + O \left( \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right)
\]
\[
=O \left( \frac{\epsilon_N^2}{|\ln \epsilon_N|^p} \right), \text{ in } L^p(\Omega).
\]
For any fixed \(i\), define \(\tilde{u}_{N,i}(y) = u_N(s_{\delta_N,i}y + z_{N,i})\) for \(y \in \Omega_{N,i} := \{y \in \mathbb{R}^2 \mid s_{\delta_N,i}y + z_{N,i} \in \Omega\}\). Define
\[
\tilde{L}_{N,i} u = -\text{div}(K(s_{\delta_N,i}y + z_{N,i}) \nabla u) - p \frac{s_{\delta_N,i}^2}{\delta_N^2}(V_{\delta_N,z_{N,i}}(s_{\delta_N,i}y + z_{N,i}) - q(s_{\delta_N,i}y + z_{N,i}))^{p-1} u.
\]
Then direct computation shows that
\[
||\tilde{L}_{N,i} \tilde{u}_{N,i}||_{L^p(\Omega_{N,i})} = \left( \int_{\Omega_{N,i}} \left( \tilde{L}_{N,i} \tilde{u}_{N,i} \right)^p dy \right)^{\frac{1}{p}}
\]
\[
= \left( \frac{1}{s_{\delta_N,i}^2} \int_{\Omega} \left( -s_{\delta_N,i}^2 \text{div}(K(x) \nabla u_N) - p \frac{s_{\delta_N,i}^2}{\delta_N^2}(V_{\delta_N,z_{N,i}} - q)^{p-1} u_N \right)^p dx \right)^{\frac{1}{p}}
\]
\[
= \frac{s_{\delta_N,i}^2}{s_{\delta_N,i}^2 \frac{s_{\delta_N,i}^2}{\delta_N^2}} ||L_{\delta_N} u_N||_{L^p(\Omega)},
\]
i.e., \(s_{\delta_N,i}^2 \frac{s_{\delta_N,i}^2}{\delta_N^2} ||\tilde{L}_{N,i} \tilde{u}_{N,i}||_{L^p(\Omega_{N,i})} = ||L_{\delta_N} u_N||_{L^p(\Omega)}\).
By the fact that $\frac{s^2_\delta}{\delta_N} = O\left(\frac{1}{\ln \varepsilon_N} \right)$ and $s_\delta \varepsilon_N = O(\varepsilon_N)$, we have

$$L_{\delta_N,i} u_{\delta_N,i} = o(1) \quad \text{in} \quad L^p(\Omega_{\delta_N,i}).$$

Since $\|u_{\delta_N, i}\|_{L^\infty(\Omega_{\delta_N, i})} = 1$, by the classical regularity theory of elliptic equations, $u_{\delta_N, i}$ is uniformly bounded in $W^{2,p}_{\text{loc}}(\mathbb{R}^2)$. Hence we may assume that $u_{\delta_N, i} \to u_i$ in $C^1_{\text{loc}}(\mathbb{R}^2)$.

We claim that $u_i \equiv 0$. On the one hand, by (3.13), the definition of $V_{\delta_N, z_N,i}, \hat{\delta}_{\delta_N, i}, z_N,i$ and the fact that $z_N,i \to z_i$ as $N \to \infty$, we obtain

$$\frac{s^2_\delta}{\delta_N} (V_{\delta_N, z_N}(s_\delta y + z_N,i) - q(s_\delta y + z_N,i))^{p-1}$$

$$= \frac{s^2_\delta}{\delta_N} \left( V_{\delta_N, z_N,i}, \hat{\delta}_{\delta_N, i}, z_N,i(s_\delta y + z_N,i) - \hat{\delta}_{\delta_N, i} + O\left(\frac{1}{\ln \varepsilon_N} \right)\right)^{p-1}$$

$$\to \phi(T_{z,i})^{p-1} \quad \text{in} \quad C^0_{\text{loc}}(\mathbb{R}^2) \quad \text{as} \quad N \to \infty.$$

So $u_i$ satisfies

$$-\text{div}(K(z_i) \nabla u_i(x)) - p\phi(T_{z,i})^{p-1} u_i(x) = 0, \quad x \in \mathbb{R}^2.$$

Let $\hat{u}_i(x) = u_i(T_{z,i}^{-1} x)$. Note that $T_{z,i}^{-1}(T_{z,i}^{-1})^t = K(z_i)$. Then

$$-\Delta \hat{u}_i(x) = -\text{div}(K(z_i) \nabla u_i) (T_{z,i}^{-1} x) = p\phi(x)^{p-1} \hat{u}_i(x), \quad \forall \ x \in \mathbb{R}^2.$$

Since $\hat{u}_i(x) \in L^\infty(\mathbb{R}^2)$, it follows from Proposition 4.11 that there are $c_1, c_2$ satisfying

$$\hat{u}_i = c_1 \frac{\partial \phi}{\partial x_1} + c_2 \frac{\partial \phi}{\partial x_2}.$$  (4.13)

On the other hand, since $u_N \in E_{\delta_N, z_N}$, we have

$$\int_\Omega -s^2_\delta \text{div} \left( K(x) \nabla \left( \frac{\partial V_{\delta_N, z_N,i}}{\partial z_{i,h}} \right) \right) u_N = 0, \quad \forall h = 1, 2,$$

which implies that

$$0 = \int_\Omega -s^2_\delta \text{div} \left( K(z_N,i) \nabla \left( \frac{\partial V_{\delta_N, z_N,i}}{\partial z_{i,h}} \right) \right) u_N + \int_\Omega -s^2_\delta \text{div} \left( (K(x) - K(z_N,i)) \nabla \left( \frac{\partial V_{\delta_N, z_N,i}}{\partial z_{i,h}} \right) \right) u_N$$

$$= \int_\Omega \frac{\partial}{\partial z_{i,h}} \left[ -s^2_\delta \text{div} (K(z_N,i) \nabla \delta_{\delta_N, i}) \right] u_N - \int_\Omega -s^2_\delta \text{div} \left( \frac{\partial K(z_N,i)}{\partial z_{i,h}} \nabla \delta_{\delta_N, i} \right) u_N$$

$$+ \int_\Omega -s^2_\delta \text{div} \left( (K(x) - K(z_N,i)) \nabla \left( \frac{\partial V_{\delta_N, z_N,i}}{\partial z_{i,h}} \right) \right) u_N.$$  (4.14)

By (4.11), we have

$$\int_\Omega -s^2_\delta \text{div} \left( \frac{\partial K(z_N,i)}{\partial z_{i,h}} \nabla \delta_{\delta_N, i} \right) u_N = O\left( \frac{\varepsilon^2_N}{\ln \varepsilon_N} \right).$$  (4.15)
By (4.12), we get
\[ \int \Omega - \delta_N^2 \text{div} \left( (K(x) - K(z_{N,i})) \nabla \frac{\partial V_{\delta_N, z_{N,i}}}{\partial z_{i,h}} \right) u_N = O \left( \frac{\varepsilon_N^2}{\ln \varepsilon_N^{p-1}} \right). \] (4.16)

Using the definition of \( V_{\delta_N, z_{N,i}} \), (3.14) and (3.16), we obtain
\[ \int \Omega \frac{\partial}{\partial z_{i,h}} \left[ -\delta_N^2 \text{div}(K(z_{N,i}) \nabla \delta_{N,i}) \right] u_N \]
\[ = p \int \Omega (V_{\delta_N, z_{N,i}} \delta_{N,i} - \delta_{N,i}^{p-1}) \left( \frac{\partial V_{\delta_N, z_{N,i}}}{\partial z_{N,i}} \right) u_N \]
\[ = p \int \Omega \left( \frac{\delta_N}{s_{\delta_N,i}} \right)^2 \phi \left( \frac{T_{z_{N,i}}(x - z_{N,i})}{s_{\delta_N,i}} \right)^{p-1} \left( \frac{\delta_N}{s_{\delta_N,i}} \right)^{\frac{2}{p-1}} \phi'(\frac{T_{z_{N,i}}(x - z_{N,i})}{s_{\delta_N,i}}) \frac{(T_{z_{N,i}})_{h} \cdot T_{z_{N,i}}(x - z_{N,i})}{|T_{z_{N,i}}(x - z_{N,i})|} u_N \]
\[ + O \left( \frac{\varepsilon_N^2}{\ln \varepsilon_N^{p-1}} \right). \] (4.17)

Hence taking (4.15), (4.16) and (4.17) into (4.14), we have
\[ 0 = p s_{\delta_N,i} \left( \frac{\delta_N}{s_{\delta_N,i}} \right)^{\frac{2}{p-1}} \int_{\mathbb{R}^2} \phi(T_{z_{N,i}}) u_N(y) dy + O \left( \frac{\varepsilon_N^2}{\ln \varepsilon_N^{p-1}} \right). \] (4.18)

Dividing both sides of (4.18) into \( p s_{\delta_N,i} \left( \frac{\delta_N}{s_{\delta_N,i}} \right)^{\frac{2}{p-1}} \) and passing \( N \) to the limit, we get for \( h = 1, 2 \)
\[ 0 = \int_{\mathbb{R}^2} \phi(T_{z_{i}}) u_N(y) dy + O \left( \frac{\varepsilon_N^2}{\ln \varepsilon_N^{p-1}} \right). \]
\[ = \int_{\mathbb{R}^2} \phi(x) \frac{\partial u_N}{\partial x} \frac{|x|}{\sqrt{\text{det}(K(z))}} dx, \]

which implies that
\[ 0 = \int_{B_1(0)} \frac{\partial u_N}{\partial x} \frac{|x|}{\sqrt{\text{det}(K(z))}} dx. \] (4.19)

Combining (4.13) with (4.19), we have \( c_1 = c_2 = 0 \). That is, \( u_i \equiv 0 \).

So we conclude that \( \hat{u}_{N,i} \rightarrow 0 \) in \( C^1(B_L(0)) \), which implies that
\[ ||u_N||_{L^\infty(B_{Ls_{\delta_N,i}}(z_{N,i}))} = o(1). \] (4.20)

By the assumption that \( Q_{\delta_N} u_N = 0 \) in \( \Omega \setminus \cup_{i=1}^m B_{Ls_{\delta_N,i}}(z_{N,i}) \) and the definition of \( Q_{\delta_N} \), we have for \( L \) large
\[ L_{\delta_N} u_N = 0 \text{ in } \Omega \setminus \cup_{i=1}^m B_{Ls_{\delta_N,i}}(z_{N,i}). \]
It follows from Lemma \[ A.6 \] that
\[
(V_{\delta,N,Z} - q)_+ = 0 \text{ in } \Omega \setminus \bigcup_{i=1}^m B_{L\delta_N,i}(z_{N,i}).
\]
So we get \(-\text{div}(K(x)\nabla u_N) = 0\) in \(\Omega \setminus \bigcup_{i=1}^m B_{L\delta_N,i}(z_{N,i})\).

Since \(u_N = o(1)\) on \(\bigcup_{i=1}^m B_{L\delta_N,i}(z_{N,i})\) and \(u_N = 0\) on \(\partial\Omega\), by the maximum principle, we get
\[
||u_N||_{L^\infty(\Omega \setminus \bigcup_{i=1}^m B_{L\delta_N,i}(z_{N,i}))} = o(1),
\]
which combined with (4.20) we have
\[
||u_N||_{L^\infty(\Omega)} = o(1).
\]
This is a contradiction since \(||u_N||_{L^\infty(\Omega)} = 1\).

Then we can get

**Proposition 4.3.** \(Q_\delta L_\delta\) is a one to one and onto map from \(E_{\delta,Z}\) to \(F_{\delta,Z}\).

**Proof.** If \(Q_\delta L_\delta u = 0\), by Lemma \[ A.2 \] \(u = 0\). So \(Q_\delta L_\delta\) is a one to one map from \(E_{\delta,Z}\) to \(F_{\delta,Z}\).

We prove that \(Q_\delta L_\delta\) is an onto map from \(E_{\delta,Z}\) to \(F_{\delta,Z}\). Denote
\[
\hat{E} = \{ u \in H^1_0(\Omega) \mid \int_\Omega \left( K(x) \nabla u \cdot \nabla V_{\delta,Z,i} \right) = 0, \quad i = 1, \ldots, m, \quad h = 1, 2 \}.
\]
Then \(E_{\delta,Z} = \hat{E} \cap W^{2,p}(\Omega)\). For any \(\hat{h} \in F_{\delta,Z}\), by the Riesz representation theorem there is a unique \(u \in H^1_0(\Omega)\) such that
\[
\delta^2 \int_\Omega (K(x)\nabla u \cdot \nabla \varphi) = \int_\Omega \hat{h} \varphi, \quad \forall \varphi \in H^1_0(\Omega).
\]
Since \(\hat{h} \in F_{\delta,Z}\), we have \(u \in \hat{E}\). Using the classical \(L^p\) theory, we conclude that \(u \in W^{2,p}(\Omega)\), which implies that \(u \in E_{\delta,Z}\). Thus \(-\delta^2 \text{div}(K(x)\nabla) = Q_\delta(-\delta^2 \text{div}(K(x)\nabla))\) is a one to one and onto map from \(E_{\delta,Z}\) to \(F_{\delta,Z}\).

For any \(h \in F_{\delta,Z}\), \(Q_\delta L_\delta u = h\) is equivalent to
\[
u = (Q_\delta(-\delta^2 \text{div}(K(x)\nabla)))^{-1} p Q_\delta(V_{\delta,Z} - q)^{p-1} u + (Q_\delta(-\delta^2 \text{div}(K(x)\nabla)))^{-1} h, \quad u \in E_{\delta,Z}.
\]
Note that \(T \nu := (Q_\delta(-\delta^2 \text{div}(K(x)\nabla)))^{-1} p Q_\delta(V_{\delta,Z} - q)^{p-1} u\) is a compact operator in \(E_{\delta,Z}\). By the Fredholm alternative, (4.22) is solvable if and only if
\[
u = (Q_\delta(-\delta^2 \text{div}(K(x)\nabla)))^{-1} p Q_\delta(V_{\delta,Z} - q)^{p-1} u
\]
has only trivial solution, which is true since \(Q_\delta L_\delta\) is one to one. The proof is thus complete. \(\square\)

Now consider solutions of (2.16) being the form of \(V_{\delta,Z} + \omega_k\). Note that by (2.16), one has
\[
-\delta^2 \text{div}(K(x)\nabla(V_{\delta,Z} + \omega_k)) - (V_{\delta,Z} + \omega_k - q)_+ = 0,
\]
which is equivalent to

\[ L_\delta \omega_\delta = l_{1,\delta} + l_{2,\delta} + R_\delta(\omega_\delta), \]  

(4.23)

where

\[ l_{1,\delta} = (V_{\delta,Z} - q)_+^{p} - \sum_{j=1}^{m} (V_{\delta,z_j,\delta z_j,z_j} - \delta_{\delta,j})_+^{p}, \]

\[ l_{2,\delta} = \delta^{2} \sum_{j=1}^{m} \text{div}((K(x) - K(z_j))\nabla V_{\delta,Z,j}), \]

\[ R_\delta(\omega_\delta) = (V_{\delta,Z} + \omega_\delta - q)_+^{p} - (V_{\delta,Z} - q)_+^{p} - p(V_{\delta,Z} - q)^{p-1}_{+} \omega_\delta. \]

We first solve the existence and uniqueness of \( \omega \in E_{\delta,Z} \) satisfying

\[ Q_\delta L_\delta \omega = Q_\delta l_{1,\delta} + Q_\delta l_{2,\delta} + Q_\delta R_\delta(\omega), \]

(4.24)

or equivalently,

\[ \omega = T_\delta(\omega) := (Q_\delta L_\delta)^{-1}Q_\delta l_{1,\delta} + (Q_\delta L_\delta)^{-1}Q_\delta l_{2,\delta} + (Q_\delta L_\delta)^{-1}Q_\delta R_\delta(\omega). \]

And then we can reduce (4.23) to a finite dimensional problem.

**Proposition 4.4.** There is \( \delta_0 > 0 \), such that for any \( 0 < \delta < \delta_0 \) and \( Z \) satisfying (3.10), (4.24) has the unique solution \( \omega_{\delta,Z} \in E_{\delta,Z} \) with

\[ ||\omega_{\delta,Z}||_{L^\infty(\Omega)} = O\left(\frac{\ln|\ln\varepsilon|}{|\ln\varepsilon|^2}\right). \]

**Proof.** It follows from Lemma A.6 in Appendix that for \( L \) sufficiently large and \( \delta \) small,

\[ (V_{\delta,Z} - q)_+ = 0, \quad \text{in } \Omega \setminus \bigcup_{i=1}^{m} B_{L_{\delta,s_{i}}}(z_i). \]

Let \( N = E_{\delta,Z} \cap \{ \omega \mid ||\omega||_{L^\infty(\Omega)} \leq \frac{1}{|\ln\varepsilon|^{\theta_{0}}} \} \) for some \( \theta_{0} \in (0, 1) \). Then \( N \) is complete under \( L^\infty \) norm and \( T_\delta \) is a map from \( E_{\delta,Z} \) to \( E_{\delta,Z} \). We now prove that \( T_\delta \) is a contraction map from \( N \) to \( N \).

First, we claim that \( T_\delta \) is a map from \( N \) to \( N \). For any \( \omega \in N \), by Lemma A.6 we get that for \( L > 1 \) large and \( \delta \) small,

\[ (V_{\delta,Z} + \omega - q)_+ = 0, \quad \text{in } \Omega \setminus \bigcup_{i=1}^{m} B_{L_{\delta,s_{i}}}(z_i). \]

So \( l_{1,\delta} = R_\delta(\omega) = 0 \) in \( \Omega \setminus \bigcup_{i=1}^{m} B_{L_{\delta,s_{i}}}(z_i) \). Note that for any \( u \in L^\infty(\Omega) \),

\[ Q_\delta u = u, \quad \text{in } \Omega \setminus \bigcup_{i=1}^{m} B_{L_{\delta,s_{i}}}(z_i). \]

Hence

\[ Q_\delta l_{1,\delta} + Q_\delta R_\delta(\omega) = 0, \quad \text{in } \Omega \setminus \bigcup_{i=1}^{m} B_{L_{\delta,s_{i}}}(z_i). \]

So, we can apply Lemma A.2 to obtain

\[ ||(Q_\delta L_\delta)^{-1}(Q_\delta l_{1,\delta} + Q_\delta R_\delta(\omega))||_{L^\infty} \leq C \left(\frac{|\ln\varepsilon|^{p-1}}{\varepsilon^{p}}\right) ||Q_\delta l_{1,\delta} + Q_\delta R_\delta(\omega)||_{L^p}. \]
By Lemma [A.5] we know that
\[
||Q_0 l_1,\theta + Q_0 R_0(\omega)||_{L^p} \leq C(||l_1,\theta||_{L^p} + ||R_0(\omega)||_{L^p}).
\]

It follows from (3.13), the definition of \(l_1,\theta\), \(R_0(\omega)\) and Lemma [A.6] that
\[
||l_1,\theta||_{L^p} = ||(V_0,\omega - q)^p - \sum_{j=1}^{m} (V_0,\omega, q_{0j}, \omega_j - \hat{q}_{0j})^p||_{L^p}
\leq C \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \sum_{j=1}^{m} ||(V_0,\omega, q_{0j}, \omega_j - \hat{q}_{0j})^p - \hat{q}_{0j}^p||_{L^p}
\leq C \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \sum_{j=1}^{m} \left(\frac{\delta}{s_{0j}}\right)^2 \delta j
\leq C \frac{\varepsilon ^2}{p} \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+1}},
\]
and
\[
||R_0(\omega)||_{L^p} = ||(V_0,\omega + \omega - q)^p - (V_0,\omega - q)^p - p(V_0,\omega - q)^p \omega||_{L^p}
\leq C ||(V_0,\omega - q)^p - \omega||^2_{L^p}
\leq C \frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|^{p-2}}{|\ln \varepsilon|^{p-1}} ||\omega||^2_{L^p}.
\]

So
\[
||(Q_0 L_\delta)^{-1}(Q_0 l_1,\theta + Q_0 R_0(\omega))||_{L^\infty} \leq C \frac{\varepsilon ^2}{p} \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+1}} (||l_1,\theta||_{L^p} + ||R_0(\omega)||_{L^p})
\leq C \frac{\varepsilon ^2}{p} \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+1}} \left(\frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|}{|\ln \varepsilon|^{p+1}} + \frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|^{p-2}}{|\ln \varepsilon|^{p-1}} ||\omega||^2_{L^p}\right).
\]

By the definition of \(l_2,\delta\), one computes directly that
\[
||(Q_0 L_\delta)^{-1} Q_0 l_2,\delta||_{L^\infty} \leq C ||Q_0 l_2,\delta||_{L^\infty} \leq \frac{C}{|\ln \delta|^2}.
\]

Hence the definition of \(N\),
\[
||T_\delta(\omega)||_{L^\infty} \leq C \frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|^{p-1}}{|\ln \varepsilon|^{p+1}} (||l_1,\theta||_{L^p} + ||R_0(\omega)||_{L^p}) + \frac{C}{|\ln \delta|^2}
\leq C \frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|^{p-1}}{|\ln \varepsilon|^{p+1}} \left(\frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|}{|\ln \varepsilon|^{p+1}} + \frac{\varepsilon ^2}{p} \frac{|\ln \varepsilon|^{p-2}}{|\ln \varepsilon|^{p-1}} ||\omega||^2_{L^p}\right) + \frac{C}{|\ln \delta|^2}
\leq \frac{1}{|\ln \varepsilon|^{2-\theta_0}}.
\]

So \(T_\delta\) is a map from \(N\) to \(N\).
Then we prove that $T_\delta$ is a contraction map. For any $\omega_1, \omega_2 \in N$, 

$$T_\delta(\omega_1) - T_\delta(\omega_2) = (Q_\delta L_\delta)^{-1}Q_\delta(R_\delta(\omega_1) - R_\delta(\omega_2)).$$

Note that $R_\delta(\omega_1) = R_\delta(\omega_2) = 0$ in $\Omega \setminus \cup_{i=1}^m B_{L_{\delta,i}}(z_i)$. By Lemma 4.2 and the definition of $N$, for $\delta$ sufficiently small

$$||T_\delta(\omega_1) - T_\delta(\omega_2)||_{L^\infty} \leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} ||R_\delta(\omega_1) - R_\delta(\omega_2)||_{L^p} 
\leq C \varepsilon^{-\frac{2}{p}} |\ln \varepsilon|^{p-1} \frac{1}{\varepsilon^2} \left( ||\omega_1||_{L^\infty} + ||\omega_2||_{L^\infty} \right) ||\omega_1 - \omega_2||_{L^\infty} 
\leq \frac{1}{2} ||\omega_1 - \omega_2||_{L^\infty}.$$

So $T_\delta$ is a contraction map.

To conclude, $T_\delta$ is a contraction map from $N$ to $N$ and thus there is a unique $\omega_{\delta,Z} \in N$ such that $\omega_{\delta,Z} = T_\delta(\omega_{\delta,Z})$. Moreover, from (4.25), we have $||\omega_{\delta,Z}||_{L^\infty(\Omega)} = O\left(\frac{|\ln |\ln \varepsilon||}{|\ln \varepsilon|^p}\right)$.

\begin{remark}
Indeed, since $K, q \in C^\infty$ and $p > 1$, we can also check that $\omega_{\delta,Z}$ is a $C^1$ map about $Z$, see [6, 7] for example.
\end{remark}

5. Proof of Theorem 2.2

From Proposition 4.1, we know that for any $\delta$ small and $Z$ satisfying (3.10), there exists the unique $\omega_{\delta,Z} \in E_{\delta,Z}$ satisfying

$$Q_\delta L_\delta \omega_{\delta,Z} = Q_\delta l_\delta + Q_\delta R_\delta(\omega_{\delta,Z}),$$

i.e., for some $C_{j,h}$ one has

$$L_\delta \omega_{\delta,Z} = l_\delta + R_\delta(\omega_{\delta,Z}) + \sum_{j=1}^m \sum_{h=1}^2 C_{j,h} \frac{\partial}{\partial z_{j,h}} (-\delta^2 \text{div}(K(z_j) \nabla V_{\delta,Z,j})).$$

In the following we find proper $Z = Z(\delta)$ such that $V_{\delta,Z} + \omega_{\delta,Z}$ is a solution of (2.16). Note that the associated functional of (2.16) is

$$I_\delta(u) = \frac{\delta^2}{2} \int_{\Omega} (K(x) \nabla u \nabla u) - \frac{1}{p+1} \int_{\Omega} (u - q)^{p+1}_+.$$

(5.1)

Denote

$$P_\delta(Z) = I_\delta(V_{\delta,Z} + \omega_{\delta,Z}).$$

Then by Remark 1.3, we know that $P_\delta(Z)$ is a $C^1$ function.

By the regularity of $K$ and $q$, it is not hard to check that if $Z$ is a critical point of $P_\delta$, then $V_{\delta,Z} + \omega_{\delta,Z}$ is a critical point of $I_\delta$, i.e., a solution of (2.16). We now prove that $P_\delta$ has a critical point.

\begin{proposition}
There holds

$$P_\delta(Z) = I_\delta(V_{\delta,Z}) + O\left(\frac{\varepsilon^2 |\ln |\ln \varepsilon||}{|\ln \varepsilon|^{p+2}}\right).$$
\end{proposition}
Proof. Note that
\[
P_\delta(Z) = I_\delta(V_{\delta,Z}) + \delta^2 \int_\Omega (K(x)\nabla V_{\delta,Z} | \nabla \omega_{\delta,Z}) + \frac{\delta^2}{2} \int_\Omega (K(x)\nabla \omega_{\delta,Z} | \nabla \omega_{\delta,Z}) - \frac{1}{p+1} \left( \int_\Omega (V_{\delta,Z} + \omega_{\delta,Z} - q)^{p+1}_+ - \int_\Omega (V_{\delta,Z} - q)^{p+1}_+ \right).
\]
By Proposition 4.3, we get
\[
\int_\Omega (V_{\delta,Z} + \omega_{\delta,Z} - q)^{p+1}_+ - \int_\Omega (V_{\delta,Z} - q)^{p+1}_+ = (p + 1) \sum_{j=1}^m \int_{B_{l_{\delta,j}}(z_j)} (V_{\delta,Z} - q)^p \omega_{\delta,Z} + O \left( \sum_{j=1}^m \int_{B_{l_{\delta,j}}(z_j)} (V_{\delta,Z} - q)^{p-1}_+ \omega_{\delta,Z}^2 \right)
\]
\[
= O \left( \sum_{j=1}^m \frac{s_{\delta,j}^2 || \omega_{\delta,Z} ||_{L^\infty}}{| \ln \varepsilon |^p} \right) + O \left( \sum_{j=1}^m \frac{s_{\delta,j}^2 || \omega_{\delta,Z} ||_{L^\infty}^2}{| \ln \varepsilon |^{p-1}} \right)
\]
\[
= O \left( \frac{\varepsilon^2 | \ln | \ln \varepsilon |}{| \ln \varepsilon |^{p+2}} \right).
\]
Using Proposition 4.4, we obtain
\[
\delta^2 \int_\Omega (K(x)\nabla V_{\delta,Z} | \nabla \omega_{\delta,Z}) = \sum_{j=1}^m \int_{B_{l_{\delta,j}}(z_j)} (V_{\delta,z_j} - \hat{q}_{\delta,j})^p_+ \omega_{\delta,Z} + \sum_{j=1}^m \delta^2 \int_\Omega ((K(x) - K(z_j))\nabla V_{\delta,Z} | \nabla \omega_{\delta,Z})
\]
\[
= O \left( \sum_{j=1}^m \frac{s_{\delta,j}^2 || \omega_{\delta,Z} ||_{L^\infty}}{| \ln \varepsilon |^p} \right) + O \left( \frac{\delta^2 || \omega_{\delta,Z} ||_{L^\infty}}{| \ln \varepsilon |} \right)
\]
\[
= O \left( \frac{\varepsilon^2 | \ln | \ln \varepsilon |}{| \ln \varepsilon |^{p+2}} \right).
\]
Now we calculate the term $\frac{\delta^2}{2} \int_\Omega (K(x)\nabla \omega_{\delta,Z} | \nabla \omega_{\delta,Z})$. Since $\omega_{\delta,Z} \in E_{\delta,Z}$, we get
\[
Q_\delta L\delta \omega_{\delta,Z} = Q_\delta (-\delta^2 \text{div}(K(x)\nabla \omega_{\delta,Z})) - Q_\delta (p(V_{\delta,Z} - q)^{p-1}_+ \omega_{\delta,Z})
\]
\[
= - \delta^2 \text{div}(K(x)\nabla \omega_{\delta,Z}) - Q_\delta (p(V_{\delta,Z} - q)^{p-1}_+ \omega_{\delta,Z}),
\]
which combined with $Q_\delta L\delta \omega_{\delta,Z} = Q_\delta I_\delta + Q_\delta R_\delta (\omega_{\delta,Z})$ yields
\[
- \delta^2 \text{div}(K(x)\nabla \omega_{\delta,Z}) = Q_\delta (p(V_{\delta,Z} - q)^{p-1}_+ \omega_{\delta,Z}) + Q_\delta I_\delta + Q_\delta R_\delta (\omega_{\delta,Z}).
\]
Hence by Lemmas A.5, A.6 and Proposition 4.4 we get

\[
\delta^2 \int_\Omega (K(x) \nabla \omega_{\delta,Z} \nabla \omega_{\delta,Z})
\]

\[
= \int_\Omega Q_\delta(p(V_{\delta,Z} - q)_{+}^{p-1} \omega_{\delta,Z}) \omega_{\delta,Z} + \int_\Omega Q_\delta l_\delta \omega_{\delta,Z} + \int_\Omega Q_\delta R_\delta(\omega_{\delta,Z}) \omega_{\delta,Z}
\]

\[
\leq ||Q_\delta(p(V_{\delta,Z} - q)_{+}^{p-1} \omega_{\delta,Z})||_{L^1} ||\omega_{\delta,Z}||_{L^\infty} + ||Q_\delta l_\delta||_{L^1} ||\omega_{\delta,Z}||_{L^\infty} + ||Q_\delta R_\delta(\omega_{\delta,Z})||_{L^1} ||\omega_{\delta,Z}||_{L^\infty}
\]

\[
\leq C(||p(V_{\delta,Z} - q)_{+}^{p-1} \omega_{\delta,Z})||_{L^1} + ||l_\delta||_{L^1} + ||R_\delta(\omega_{\delta,Z})||_{L^1} ||\omega_{\delta,Z}||_{L^\infty}
\]

\[
= O\left(\frac{\varepsilon^2 ||\omega_{\delta,Z}||_{L^\infty}^2}{|\ln \varepsilon|^{p-1}}\right) + O\left(\frac{\varepsilon^2 \ln |\ln \varepsilon| ||\omega_{\delta,Z}||_{L^\infty}}{|\ln \varepsilon|^{p+1}}\right) + O\left(\frac{\varepsilon^2 ||\omega_{\delta,Z}||_{L^\infty}^3}{|\ln \varepsilon|^{p-2}}\right)
\]

To conclude, we get \(P_\delta(Z) = I_\delta(V_{\delta,Z}) + O\left(\frac{\varepsilon^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^{p+2}}\right)\).

\(\square\)

**Lemma 5.2.** There holds

\[
I_\delta(V_{\delta,Z}) = \sum_{j=1}^{m} \frac{\pi \delta^2}{\ln R} q^2(z_j) \sqrt{\det(K(z_j))} + O\left(\frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right).
\]

**Proof.** Note that by the definition of \(V_{\delta,Z},\)

\[
I_\delta(V_{\delta,Z}) = \frac{\delta^2}{2} \int_\Omega (K(x) \nabla V_{\delta,Z} \nabla V_{\delta,Z}) - \frac{1}{p + 1} \int_\Omega (V_{\delta,Z} - q)_{+}^{p+1}
\]

\[
= \frac{1}{2} \sum_{j=1}^{m} \int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p} V_{\delta,Z,j} + \frac{1}{2} \sum_{j=1}^{m} \delta^2 \int_\Omega ((K(x) - K(z_j)) \nabla V_{\delta,Z,j} \nabla V_{\delta,Z,j})
\]

\[
+ \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p} V_{\delta,Z,i} + \frac{1}{2} \sum_{1 \leq i \neq j \leq m} \delta^2 \int_\Omega ((K(x) - K(z_j)) \nabla V_{\delta,Z,j} \nabla V_{\delta,Z,i})
\]

\[
- \frac{1}{p + 1} \int_\Omega (V_{\delta,Z} - q)_{+}^{p+1}.
\]

(5.2)

By (3.7), we have

\[
\int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p} V_{\delta,Z,j}
\]

\[
= \hat{q}_{\delta,j} \int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p} + \int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p+1} - \frac{\hat{q}_{\delta,j}}{\ln R} \int_\Omega (V_{\delta,z_j,q_{\delta,j},z_j} - \hat{q}_{\delta,j})_{+}^{p} g_{z_j}(T_{z_j} x, T_{z_j} z_j).
\]
By the definition of $V_{\delta,z,j}, \hat{q}_{\delta,j,z,j}$, the fact that $T_{z,j}^{-1}(T_{z,j}^{-1})^t = K(z_j)$ and (3.4), we get

$$
\hat{q}_{\delta,j} \int_\Omega (V_{\delta,z,j}, \hat{q}_{\delta,j,z,j} - \hat{q}_{\delta,j}) P_+ = \hat{q}_{\delta,j} s_{\delta,j}^2 \frac{\delta}{s_{\delta,j}} \int_{|T_{z,j}x| \leq 1} \phi(T_{z,j}x) P dx
$$

$$
= \hat{q}_{\delta,j} s_{\delta,j}^2 \frac{\delta}{s_{\delta,j}} \sqrt{\det(K(z_j))} \cdot 2\pi |\phi'(1)|
$$

$$
= \hat{q}_{\delta,j} \delta^2 |\phi'(1)|^{p-1} \left( \frac{\ln \frac{R}{s_{\delta,j}}}{\hat{q}_{\delta,j}} \right)^{p-1} |\phi'(1)|^{-p} \sqrt{\det(K(z_j))} \cdot 2\pi |\phi'(1)|
$$

$$
= \frac{2\pi \delta^2}{\ln \frac{R}{s_{\delta,j}}} \hat{q}_{\delta,j} \sqrt{\det(K(z_j))}.
$$

Similarly, we have

$$
\int_\Omega (V_{\delta,z,j}, \hat{q}_{\delta,j,z,j} - \hat{q}_{\delta,j}) P_+ = \frac{s_{\delta,j}^2}{\hat{q}_{\delta,j}} \left( \frac{\delta}{s_{\delta,j}} \right)^{2(p+1)} \sqrt{\det(K(z_j))} \cdot \frac{(p+1)\pi}{2} |\phi'(1)|^2
$$

$$
= \delta^2 |\phi'(1)|^{p-1} \left( \frac{\ln \frac{R}{s_{\delta,j}}}{\hat{q}_{\delta,j}} \right)^{p-1} |\phi'(1)|^{-(p+1)} \left( \frac{\ln \frac{R}{s_{\delta,j}}}{\hat{q}_{\delta,j}} \right)^{-(p+1)} \sqrt{\det(K(z_j))} \cdot \frac{(p+1)\pi}{2} |\phi'(1)|^2
$$

$$
= \frac{(p+1)\pi \delta^2}{2 \left( \ln \frac{R}{s_{\delta,j}} \right)^2} \hat{q}_{\delta,j} \sqrt{\det(K(z_j))},
$$

and

$$
\frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} \int_\Omega (V_{\delta,z,j}, \hat{q}_{\delta,j,z,j} - \hat{q}_{\delta,j}) P g_{z_j}(T_{z,j}x, T_{z,j}z_j)
$$

$$
= g_{z_j}(T_{z,j}z_j, T_{z,j}z_j) \frac{\hat{q}_{\delta,j}}{\ln \frac{R}{s_{\delta,j}}} \int_\Omega (V_{\delta,z,j}, \hat{q}_{\delta,j,z,j} - \hat{q}_{\delta,j}) P_+ + O \left( \frac{\varepsilon^3 |\nabla g_{z_j}(T_{z,j}z_j, T_{z,j}z_j)|}{|\ln \varepsilon|^{p+1}} \right)
$$

$$
= \frac{2\pi \delta^2 g_{z_j}(T_{z,j}z_j, T_{z,j}z_j)}{\left( \ln \frac{R}{s_{\delta,j}} \right)^2} \hat{q}_{\delta,j} \sqrt{\det(K(z_j))} + O \left( \frac{\varepsilon^3}{|\ln \varepsilon|^{p+1}} \right),
$$
where we have used Lemma A.3. Thus we get

\[
\int_{\Omega} (V_{\delta, z_j, \delta, j, z_j} - \hat{q}_{\delta, j})^p V_{\delta, z_j} = 2\pi \delta^2 \overline{q}_{\delta, j} \sqrt{\text{det}(K(z_j))} + \frac{(p + 1)\pi \delta^2}{2 \left( \overline{q}_{\delta, j} \sqrt{\text{det}(K(z_j))} \right)} \\
- \frac{2\pi \delta^2 g_{z_j}(T_{z_j} z_j, T_{z_j} z_j)}{\left( \overline{q}_{\delta, j} \right)^2} \overline{q}_{\delta, j} \sqrt{\text{det}(K(z_j))} + O \left( \frac{\epsilon^3}{\ln \epsilon|p+1|} \right).
\]

(5.3)

Similarly for \(1 \leq i \neq j \leq m\),

\[
\int_{\Omega} (V_{\delta, z_j, \delta, j, z_j} - \hat{q}_{\delta, j})^p V_{\delta, z_j} = \frac{\overline{q}_{\delta, i}}{\overline{q}_{\delta, j} \left( \overline{q}_{\delta, j} \right)^2} \frac{2\pi \delta^2 \overline{q}_{\delta, j}}{\left( \overline{q}_{\delta, j} \right)^2} \frac{\overline{q}_{\delta, i} \overline{q}_{\delta, j}}{\overline{q}_{\delta, j} \sqrt{\text{det}(K(z_j))}} + O \left( \frac{\epsilon^3}{\ln \epsilon|p+1|} \right).
\]

(5.4)

For any \(1 \leq j \leq m\), using (3.16) and Lemma A.3 we have

\[
\delta^2 \int_{\Omega} ((K(x) - K(z_j)) \nabla V_{\delta, z_j, \delta, j} \nabla V_{\delta, z_j}) = O \left( \frac{\delta^2 \epsilon}{|\ln \epsilon|^2} \right) + O \left( \frac{\delta^2}{|\ln \epsilon|^2} \right) + O \left( \frac{\delta^2}{|\ln \epsilon|^2} \right)
\]

\[
= O \left( \frac{\delta^2 \epsilon}{|\ln \epsilon|^2} \right).
\]

(5.5)

Similarly for \(1 \leq i \neq j \leq m\),

\[
\delta^2 \int_{\Omega} ((K(x) - K(z_j)) \nabla V_{\delta, z_j, \delta, j} \nabla V_{\delta, z_i}) = O \left( \frac{\delta^2}{|\ln \epsilon|^2} \right).
\]

(5.6)
Finally by Lemma A.6 and (3.13), we get
\[
\int_{\Omega} (V_{\delta,Z} - q)^{p+1} \leq \sum_{j=1}^{m} \int_{B_{Ls\delta,j}(z_j)} \left( V_{\delta,z_j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j} \right) + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right) \]
\[
= \sum_{j=1}^{m} \int_{\Omega} (V_{\delta,z_j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j})^{p+1} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \sum_{j=1}^{m} \int_{\Omega} (V_{\delta,z_j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j})^p \right)
\]
\[
= \sum_{j=1}^{m} \frac{(p+1)\pi \delta^2}{2} (q^2_{\delta,j}) \sqrt{\text{det}(K(z_j))} + O \left( \frac{\delta^2 |\ln \varepsilon|}{|\ln \varepsilon|^3} \right).
\] 
(5.7)

Taking (5.3), (5.4), (5.5), (5.6) and (5.7) into (3.2), one has
\[
I_\delta(V_{\delta,Z}) = \sum_{j=1}^{m} \frac{\pi \delta^2}{\ln R_{s\delta,j}} q^2_{\delta,j} \sqrt{\text{det}(K(z_j))} + \sum_{j=1}^{m} \frac{(p+1)\pi \delta^2}{4 (\ln R_{s\delta,j})^2} \hat{q}^2_{\delta,j} \sqrt{\text{det}(K(z_j))}
\]
\[
- \sum_{j=1}^{m} \frac{\pi \delta^2 g_{z_j}(T_{z_j} z_j, T_{z_j} z_j)}{(\ln R_{s\delta,j})^2} \hat{q}_{\delta,j} \sqrt{\text{det}(K(z_j))}
\]
\[
+ \sum_{1 \leq i \neq j \leq m} \frac{\pi \delta^2 \tilde{G}_{z_i}(T_{z_i} z_j, T_{z_i} z_i)}{\ln R_{s\delta,i} \ln R_{s\delta,j}} \hat{q}_{\delta,i} \hat{q}_{\delta,j} \sqrt{\text{det}(K(z_j))}
\]
\[
- \sum_{j=1}^{m} \frac{2 \pi \delta^2}{2} (q^2_{\delta,j}) \sqrt{\text{det}(K(z_j))} + O \left( \frac{\delta^2 |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]

The result follows from (3.12) and the fact that \( \hat{q}_{\delta,j} = q(z_j) + O \left( \frac{1}{|\ln \varepsilon|} \right) \).

Proof of Theorem 2.2: By Propositions 5.1 and 5.2, we obtain
\[
P_\delta(Z) = \sum_{j=1}^{m} \frac{\pi \delta^2}{\ln R_{s\delta,j}} q^2 \sqrt{\text{det}(K(z_j))} + O \left( \frac{\delta^2 |\ln \varepsilon|}{|\ln \varepsilon|^2} \right).
\]

If each \( x_{0,j} \) is a strict local maximum (minimum) point of \( q^2 \sqrt{\text{det}(K)} \) for every \( j = 1, \cdots, m \), then for \( \delta > 0 \) sufficiently small, there exist at least one point \( z_{i,\delta} \) near \( x_{0,i} \) such that \( Z_\delta = (z_{1,\delta}, \cdots, z_{m,\delta}) \) is a maximum (minimum) point of \( P_\delta \) and as \( \delta \to 0 \),
\[
(z_{1,\delta}, \cdots, z_{m,\delta}) \to (x_{0,1}, \cdots, x_{0,m}).
\]

Hence, we get a solution \( u_\delta = V_{\delta,Z} + \omega_{\delta,Z} \) of (2.16). Let \( u_\varepsilon = |\ln \varepsilon| u_\delta \) and \( \delta = \varepsilon |\ln \varepsilon|^{-\frac{1}{2}} \), we get solutions of (2.13). Define the vortex set \( \tilde{A}_{\varepsilon,i} = \{ u_\varepsilon > q \ln \frac{1}{\varepsilon} \} \cap B_\rho(x_{0,i}) \). From
Lemma A.6, we can find constants $R_1, R_2 > 0$ such that

$$B_{R_1} \subset \overline{A}_{\varepsilon, i} \subset B_{R_2} \subset B_{\rho}(x_0, i).$$

It suffices to calculate the circulation of $u_\varepsilon$. Define $\kappa_i(u_\varepsilon) = \frac{1}{\varepsilon^2} \int_{B_\rho(x_0, i)} (u \varepsilon - q \ln \frac{1}{\varepsilon})^p dx$. We have

**Lemma 5.3.** There holds

$$\lim_{\varepsilon \to 0} \kappa_i(u_\varepsilon) = 2\pi q \sqrt{\det(K)(x_0, i)}.$$

**Proof.** It follows from (3.11), (3.13) and Proposition 4.4 that

$$\frac{1}{\varepsilon^2} \int_{B_\rho(x_0, i)} (u \varepsilon - q \ln \frac{1}{\varepsilon})^p dx = \frac{\ln |\varepsilon|^p}{\varepsilon^2} \int_{B_{\rho}(x_0, i)} (w \varepsilon - q)^p dx$$

$$= \frac{\ln |\varepsilon|^p}{\varepsilon^2} \int_{B_{Ls}(z_{i, \delta})} (V_{\delta, z_{i, \delta}} - \hat{q}_{\delta, i} + O(\ln |\varepsilon|) \ln |\varepsilon|^p) dx$$

$$= \frac{\ln |\varepsilon|^p}{\varepsilon^2} \int_{B_{\rho}(x_0, i)} \phi(T_{z_i, \delta})^p dx + o(1)$$

$$= \frac{2\pi \hat{q}_{\delta, i} \ln |\varepsilon|}{\ln \frac{R}{s_{\delta, i}}} \sqrt{\det(K(z_{i, \delta}))} + o(1)$$

$$\rightarrow 2\pi q \sqrt{\det(K)(x_0, i)} \quad \text{as } \delta \to 0.$$

The proof of Theorem 2.2 is thus complete.

**6. PROOF OF THEOREM 1.1 AND 1.3**

Consider the problem

\[
\begin{cases}
-\varepsilon^2 \text{div}(K_H(x) \nabla u) = \left(u - \left(\frac{\alpha |x|^2}{2} + \beta \right) \ln \frac{1}{\varepsilon}\right)^p, & x \in B_{R^+}(0), \\
u = 0, & x \in \partial B_{R^+}(0),
\end{cases}
\]  

(6.1)

where $\alpha, \beta$ are any given constants satisfying $\min_{x \in B_{R^+}(0)} \frac{\alpha |x|^2}{2} + \beta > 0$. Let $v = u \ln |\varepsilon|$ and $\delta = \varepsilon |\ln \varepsilon|^{-\frac{1}{p-1}}$, then

\[
\begin{cases}
-\delta^2 \text{div}(K_H(x) \nabla v) = \left(v - \left(\frac{\alpha |x|^2}{2} + \beta \right) \right)^p, & x \in B_{R^+}(0), \\
v = 0, & x \in \partial B_{R^+}(0).
\end{cases}
\]  

(6.2)
Note that (6.2) coincides with (2.16) with $q = \frac{\alpha|z|^2}{2} + \beta$, $K = K_H$ and $\Omega = B_{R^*}(0)$. However, results of Theorem 4.1 can not be deduced directly from those of Theorem 2.2 since $q^2 \sqrt{\text{det}(K_H)}$ is a radial function and has no strict local extreme points in $B_{R^*}(0)$. In this case one can also use the reduction procedure to construct solutions of (6.2), by using the rotational symmetry of $K_H$, $q$ and the domain $B_{R^*}(0)$.

Let $h(r) = h(|x|) = q^2 \sqrt{\text{det}(K_H)}(x)$ for any $x \in B_{R^*}(0)$. We call $z^*$ is a strict local maximum (minimum) point of $q^2 \sqrt{\text{det}(K_H)}$ up to rotation in $B_{R^*}(0)$, if $|z^*|$ is a strict local maximum (minimum) point of $h$ in $(0, R^*)$.

Let $z_1$ be a strict local maximum (minimum) point of $q^2 \sqrt{\text{det}(K_H)}$ up to rotation. Define $\mathcal{N} = B_{\rho}(z_1)$. Then we can construct solutions of (6.2) being of the form $v_\delta = V_{\delta,z} + \omega_{\delta,z}$, where $z$ is near $z_1$.

Indeed, by Lemma 4.2 and Proposition 4.4, for any $z \in \mathcal{N}$ and $\delta$ sufficiently small there exists a unique $\omega_{\delta,z} \in E_{\delta,z}$ such that $Q_\delta \mathcal{L}_z \omega_{\delta,z} = Q_\delta l_\delta + Q_\delta R_\delta(\omega_{\delta,z})$. So the final step is to prove the existence of $z = z_\delta$ near $z_1$ satisfying $\nabla_z P_\delta(z_\delta) = 0$. We claim that

$$P_\delta(z) = \frac{\pi \delta^2}{\ln \frac{r}{\rho}} q^2 \sqrt{\text{det}(K_H)}(z) + N_\delta(z),$$

(6.3)

where $N_\delta(z)$ is a $O\left(\frac{\delta^2 \ln |\ln \varepsilon|}{|\ln \varepsilon|^2}\right)$ perturbation term which is invariant under rotation. In fact, we can choose $T_x^{-1} = \begin{pmatrix} \cos \theta_x & -\sin \theta_x \\ \sin \theta_x & \cos \theta_x \end{pmatrix} \begin{pmatrix} \delta^{k/2} & 0 \\ 0 & 1 \end{pmatrix}$ in (3.1), where $(|x|, \theta_x)$ is the polar coordinate of $x$. By the rotational symmetry of $q$ and the domain $B_{R}(0)$, one can prove that for every $\theta \in [0, 2\pi]$, if we define $z = R_\theta(z)$, then

$$V_{\delta,z}(R_\theta(x)) = V_{\delta,z}(x) \quad \text{and} \quad \omega_{\delta,z}(R_\theta(x)) = \omega_{\delta,z}(x) \quad \text{for any} \quad x \in B_{R}(0).$$

Hence one computes directly that $P_\delta(z) = P_\delta(z)$, i.e., $P_\delta$ is a radially symmetric function. Note that $q^2 \sqrt{\text{det}(K_H)}(z) = \left(\frac{\alpha|z|^2}{2} + \beta\right)^2 \cdot \frac{k}{\sqrt{k^2 + |z|^2}}$ is also radially symmetric. Thus by Proposition 5.1 and 5.2 we have (6.3).

Since $z_1$ is a strict local maximum (minimum) point of $q^2 \sqrt{\text{det}(K_H)}$ up to rotation, it is not hard to prove the existence of $z_\delta$ near $z_1$ satisfying $\nabla_z P_\delta(z_\delta) = 0$, which yields a solution $v_\delta$ of (6.2). Let $u_\varepsilon = v_\delta |\ln \varepsilon|$, then $u_\varepsilon$ is a solution of (6.1). Moreover, by Lemma 5.3 one has

$$\lim_{\varepsilon \to 0} \frac{1}{2} \int_{B_\rho(z_1)} \left(u_\varepsilon - q \ln \frac{1}{\varepsilon}\right)^p dx = 2\pi q(z_1) \left(\int_{B_{R}(0)} \sqrt{\text{det}(K_H(z_1))} \right) = \frac{k\pi(\alpha|z_1|^2 + 2\beta)}{\sqrt{k^2 + |z_1|^2}}.$$

To conclude, we have

**Theorem 6.1.** Let $\alpha, \beta$ be two constants satisfying $\min_{x \in B_{R^*}(0)} \left(\frac{\alpha|z|^2}{2} + \beta\right) > 0$ and $z_1 \in B_{R^*}(0)$ be a strict local maximum (minimum) point of $\left(\frac{\alpha|z|^2}{2} + \beta\right)^2 \cdot \frac{k}{\sqrt{k^2 + |z|^2}}$ up to rotation. Then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0]$, (6.1) has a solution $u_\varepsilon$ satisfying the following properties:
(1) Define \( A_\varepsilon = \{ u_\varepsilon > \left( \frac{\alpha |x|^2}{2} + \beta \right) \ln \frac{1}{\varepsilon} \} \). Then \( \lim_{\varepsilon \to 0} \text{dist}(A_\varepsilon, z_1) = 0 \).
(2) \( \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - \left( \frac{\alpha |x|^2}{2} + \beta \right) \ln \frac{1}{\varepsilon} \right)^p dx = \frac{k\pi(\alpha |x|^2 + 2\beta)}{\sqrt{k^2 + |x|^2}}. \)
(3) There exist \( R_1, R_2 > 0 \) satisfying
\[ R_1 \varepsilon \leq \text{diam}(A_\varepsilon) \leq R_2 \varepsilon. \]

Proof of Theorem 1.1 To prove Theorem 1.1 we define for every \( r_* \in (0, R^*) \), \( c > 0 \) and
\[ \alpha = \frac{c}{4\pi k \sqrt{k^2 + r_*^2}}, \quad \beta = \frac{c}{2(3r_*^2 + 4k^2)}. \]  
(6.4)
One computes directly that \((r_*, 0)\) is a strict minimum point of \( q^2 \sqrt{\det(K_H)}(x) = \left( \frac{\alpha |x|^2}{2} + \beta \right)^2 \).

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One computes directly that \((r_*, 0)\) is a strict minimum point of \( q^2 \sqrt{\det(K_H)}(x) = \left( \frac{\alpha |x|^2}{2} + \beta \right)^2 \).

Hence by Theorem 6.1 for any \( \varepsilon \) small there exists a solution \( u_\varepsilon \) of (6.1) concentrating near \((r_*, 0)\) with \( \alpha = \frac{c}{4\pi k \sqrt{k^2 + r_*^2}} \) and \( \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - \left( \frac{\alpha |x|^2}{2} + \beta \right) \ln \frac{1}{\varepsilon} \right)^p dx \) tending to \( c \). Define for any \((x_1, x_2, x_3) \in B_{R^*}(0) \times \mathbb{R}, t \in \mathbb{R} \),
\[ w_\varepsilon(x_1, x_2, x_3, t) = \frac{w_\varepsilon(x_1, x_2, x_3, t)}{k} \rightarrow \xi, \]
where \( w_\varepsilon(x_1, x_2, 0, t) = \frac{1}{\varepsilon^2} \left( u_\varepsilon(\bar{R}_- \alpha |x|^2 + 2\beta \ln \frac{1}{\varepsilon})^p \right). \]  
(6.5)
Direct computations show that \( w_\varepsilon(x_1, x_2, 0, t) \) satisfies (2.10) and \( w_\varepsilon \) is a left-handed helical vorticity field of (1.2). Moreover, \( w_\varepsilon(x_1, x_2, 0, t) \) rotates clockwise around the origin with angular velocity \( \alpha |x| \varepsilon |\). By (1.3), the circulation of \( w_\varepsilon \) satisfies
\[ \int \int_{A_\varepsilon} w_\varepsilon \cdot n ds = \frac{1}{\varepsilon^2} \int_{A_\varepsilon} \left( u_\varepsilon - \left( \frac{\alpha |x|^2}{2} + \beta \right) \ln \frac{1}{\varepsilon} \right)^p dx \rightarrow c, \quad \text{as} \ \varepsilon \rightarrow 0. \]

It suffices to prove that the vorticity field \( w_\varepsilon \) tends asymptotically to (1.9) in sense of (1.7). Define \( P(\tau) \) the intersection point of the curve parameterized by (1.9) and the \( x_1 Ox_2 \) plane. Note that the helix (1.9) corresponds uniquely to the motion of \( P(\tau) \). Taking \( s = \frac{b_1}{k} \) into (1.9), one computes directly that \( P(\tau) \) satisfies a 2D point vortex model
\[ P(\tau) = \bar{R}_{\alpha'\tau}((r_*, 0)), \]
where
\[ \alpha' = \frac{1}{\sqrt{k^2 + r_*^2}} \left( a_1 + \frac{b_1}{k} \right) = \frac{c}{4\pi k \sqrt{k^2 + r_*^2}}. \]
which is equal to \( \alpha \) in (6.4). Thus by the construction, we readily check that the support set of \( w_\varepsilon(x_1, x_2, 0, |\ln \varepsilon|^{-1} \tau) \) defined by (6.3) concentrates near \( P(\tau) \) as \( \varepsilon \to 0 \), which implies that, the vorticity field \( w_\varepsilon \) tends asymptotically to (1.9) in sense of (1.7). The proof of Theorem 1.1 is thus complete.

**Proof of Theorem 1.3** The proof of Theorem 1.3 is similar to that of Theorem 1.1 by constructing multiple concentration solutions \( u_\varepsilon \) of (6.1) (or equivalently, \( v_\varepsilon \) of (6.2)) with polygonal symmetry. Indeed, let \( m \geq 2 \) be an integer, \( r_* \in (0, R) \), \( c > 0 \) and \( \alpha, \beta \) satisfying (6.4). For any \( z_1 \in B_p((r_*, 0)) \), define \( z_i = \tilde{R}_{2i(\pi/\varepsilon)}(z_1) \) for \( i = 2, \ldots, m \). Our goal is to construct solutions of (6.2) being of the form \( \sum_{i=1}^{m} V_{\delta, z_i} + \omega_\delta \). Using the symmetry of \( K_H, q \) and \( B_{R^*}(0) \), one can prove that

\[
P_\delta(z_1, \ldots, z_m) = \frac{m \pi \delta^2}{\ln R} q \sqrt{\det(K_H)}(z_1) + N_\delta(z_1),
\]

where \( N_\delta(z_1) \) is a \( O\left(\frac{\delta^2 \ln|\ln \varepsilon|}{\ln \varepsilon^2}\right) \) perturbation term which is invariant under rotation. The rest of the proof is exactly the same as in that of Theorem 1.1 and we omit the details.

**7. Appendix: Some basic estimates**

In this Appendix, we give some results which have been repeatedly used before.

First, we give estimates of the Green’s function. Let \( G(x, y) \) be the Green’s function of \( -\Delta \) in \( \Omega \) with zero-Dirichlet boundary condition and \( \hat{h}(x, y) := \frac{1}{2\pi} \ln \frac{1}{|x-y|} - G(x, y) \) be the regular part of \( G(x, y) \), then

**Lemma A.1** (Lemma 4.1, [14]). For all \( x, y \in \Omega \), there hold

\[
h(x, y) \leq \frac{1}{2\pi} \ln \frac{1}{\max\{|x - y|, \text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}};
\]

\[
h(x, y) \geq \frac{1}{2\pi} \ln \frac{1}{|x - y| + 2 \max\{\text{dist}(x, \partial \Omega), \text{dist}(y, \partial \Omega)\}}.
\]

Note that the Green’s function \( G(x, y) \) is only determined by the domain. To get (3.14) and (3.16), one must get estimates of \( \partial_{z_i, \delta, i} G_{z_i, z_i}(T_{z_i, z_i}) \) and \( \partial_{z_i, \delta, i} G_{z_i, z_i}(T_{z_i, z_i}) \), which involve the \( C^1 \)-dependence of the Green’s function on the domain. The following Hadamard variational formula gives qualitative estimates of the first derivative of the Green’s function on the domain, see [20, 25, 36] for instance.

More precisely, let \( \Omega \) be a simply-connected bounded domain with smooth boundary and \( \Omega_\varepsilon \) be the perturbation of the domain \( \Omega \) whose boundary \( \partial \Omega_\varepsilon \) is expressed in such a way that

\[
\partial \Omega_\varepsilon = \{x + \varepsilon \rho(x) n_x \mid x \in \partial \Omega\},
\]

where \( \rho \in C^\infty(\partial \Omega) \) and \( n_x \) is the unit outer normal to \( \partial \Omega \). For all \( y, z \in \Omega \), we define

\[
\delta_\rho G(y, z) := \lim_{\varepsilon \to 0} \varepsilon^{-1}(G_\varepsilon(y, z) - G(y, z)),
\]

\[
\delta_\rho h(y, z) := \lim_{\varepsilon \to 0} \varepsilon^{-1}(h_\varepsilon(y, z) - h(y, z)).
\]
where \( G_\varepsilon(y, z) = -\frac{1}{\pi} \ln |y - z| - h_\varepsilon(y, z) \) is the Green’s function of \(-\Delta\) in \( \Omega_\varepsilon \) with Dirichlet boundary condition and \( h_\varepsilon(y, z) \) is the regular part of \( G_\varepsilon(y, z) \). We have

**Lemma A.2** (\cite{25}). It holds that

\[
\delta_\rho G(y, z) = -\delta_\rho h(y, z) = \int_{\partial \Omega} \frac{\partial G(y, x)}{\partial n_x} \frac{\partial G(z, x)}{\partial n_x} \rho(x) d\sigma_x, \tag{7.4}
\]

where \( d\sigma_x \) denotes the surface element of \( \partial \Omega \).

Let \( h_\varepsilon(x, y) \) be the regular part of Green’s function \( G_\varepsilon(x, y) \) of \(-\Delta\) in \( T_\varepsilon(\Omega) \), where \( T \in C^\infty \) satisfies \((T_\varepsilon^{-1})(T_\varepsilon)^t = K(\hat{x})\) for any \( \hat{x} \in \Omega \). Based on Lemmas A.1 and A.2, one can get the following estimates.

**Lemma A.3.** For any \( Z \in \mathcal{M} \) and \( \delta \) sufficiently small, there exists \( C > 0 \) independent of \( \varepsilon \) and \( Z \) satisfying

\[
|h_{zi}(T_{zi}, z_i, T_{zi}, z_j)| + |\nabla h_{zi}(T_{zi}, z_i, T_{zi}, z_j)| \leq C, \quad 1 \leq i, j \leq m, \tag{7.5}
\]

\[
|G_{zi}(T_{zi}, z_i, T_{zi}, z_j)| + |\nabla G_{zi}(T_{zi}, z_i, T_{zi}, z_j)| \leq C, \quad 1 \leq i \neq j \leq m, \tag{7.6}
\]

\[
\left| \frac{\partial h_{zi}(T_{zi}, z_i, T_{zi}, z_j)}{\partial z_{i,h}} \right| \leq C, \quad 1 \leq i, j \leq m, \quad h = 1, 2, \tag{7.7}
\]

and

\[
\left| \frac{\partial G_{zi}(T_{zi}, z_i, T_{zi}, z_j)}{\partial z_{i,h}} \right| \leq C, \quad 1 \leq i \neq j \leq m, \quad h = 1, 2. \tag{7.8}
\]

**Proof.** Since for any \( Z \in \mathcal{M} \) all the eigenvalues of \( T_{zi} \) have positive upper and lower bounds uniformly about \( Z \), we can find \( C_1 > 0 \) such that \( \text{dist}(T_{zi}, z_i, T_{zi}(\Omega)) \geq C_1 > 0 \). Hence by Lemma A.1, we have \( |h_{zi}(T_{zi}, z_i, T_{zi}, z_j)| \leq C \). By the interior gradient estimates for the harmonic functions, we get \( |\nabla h_{zi}(T_{zi}, z_i, T_{zi}, z_j)| \leq C \). So \( (7.5) \) holds. \( (7.6) \) follows from \( (7.5) \) and the definition of \( G_{zi} \). Indeed, using the interior gradient estimates, one can also get that for any integer \( l \geq 1 \) and \( Z \in \mathcal{M} \),

\[
|\nabla^l h_{zi}(T_{zi}, z_i, T_{zi}, z_j)| \leq C.
\]

Now we estimate \( \frac{\partial h_{zi}(T_{zi}, z_i, T_{zi}, z_j)}{\partial z_{i,h}} \). For \( 1 \leq i \neq j \leq m \), we have

\[
\frac{\partial h_{zi}(T_{zi}, z_i, T_{zi}, z_j)}{\partial z_{i,h}} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( h_{zi + \varepsilon \varepsilon h} (T_{zi + \varepsilon \varepsilon h}(z_i + \varepsilon \varepsilon h), T_{zi + \varepsilon \varepsilon h}(z_j)) - h_{zi}(T_{zi}, z_i, T_{zi}, z_j) \right)
\]

\[
= \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( h_{zi + \varepsilon \varepsilon h} (T_{zi + \varepsilon \varepsilon h}(z_i + \varepsilon \varepsilon h), T_{zi + \varepsilon \varepsilon h}(z_j)) - h_{zi + \varepsilon \varepsilon h}(T_{zi}, z_i, T_{zi}, z_j) \right)
\]

\[
+ \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( h_{zi + \varepsilon \varepsilon h} (T_{zi}, z_i, T_{zi}, z_j) - h_{zi}(T_{zi}, z_i, T_{zi}, z_j) \right),
\]

where \( \varepsilon h \) is the unit vector of \( x_h\)-axis. By \( (7.5) \),

\[
\left| \lim_{\varepsilon \to 0} \varepsilon^{-1} \left( h_{zi + \varepsilon \varepsilon h} (T_{zi + \varepsilon \varepsilon h}(z_i + \varepsilon \varepsilon h), T_{zi + \varepsilon \varepsilon h}(z_j)) - h_{zi + \varepsilon \varepsilon h}(T_{zi}, z_i, T_{zi}, z_j) \right) \right| \leq C.
\]

Note that \( T_{zi + \varepsilon \varepsilon h}(\Omega) \) is a perturbation of \( T_{zi}(\Omega) \). Since \( T \) is a \( C^\infty \) matrix-valued function and \( \partial \Omega \) is a smooth curve, we find that \( \rho_{i,h}(x) : T_{zi}(\Omega) \to \mathbb{R} \), the normal displacement
function defined by (7.3), is smooth about \( x \in T_z(\Omega) \). Hence by Lemmas A.1 and A.2 for any \( Z \in \mathcal{M} \)

\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} (h_{z_{\varepsilon} + \varepsilon b}(T_z z_i, T_z z_j) - h_{z_i}(T_z z_i, T_z z_j))
\]

\[
= \left| \int_{\partial(T_z(\Omega))} \frac{\partial G_z(T_z z_i, x) \partial G_z(T_z z_j, x)}{\partial n_x} \rho_{i,h}(x) d\sigma_x \right|
\]

\[
\leq C \max_{\partial(T_z(\Omega))} |\nabla G_z(T_z z_i, \cdot)| \max_{\partial(T_z(\Omega))} |\nabla G_z(T_z z_j, \cdot)| \max_{\partial(T_z(\Omega))} |\rho_{i,h}(\cdot)| \leq C.
\]

So we get \( \left| \frac{\partial h_z(T_z z_i, T_z z_j)}{\partial z_{i,h}} \right| \leq C. \) (7.8) can be proved similarly. Indeed, one can also get that for any integer \( l \geq 1 \) and \( Z \in \mathcal{M} \),

\[
\left| \nabla^l \frac{\partial h_z(T_z z_i, T_z z_j)}{\partial z_{i,h}} \right| \leq C.
\]

\[\square\]

The following lemma shows the existence of \( \hat{q}_i \) satisfying (3.11).

**Lemma A.4.** For any \( Z \in \mathcal{M} \) and \( \delta \) sufficiently small, there exist \( \hat{q}_i = \hat{q}_{\delta,i}(Z) \) satisfying

\[
\hat{q}_i = q(z_i) + \frac{\hat{q}_i}{\ln \frac{R}{|x-y|}} g_{z_i}(T_z z_i, T_z z_i) - \frac{1}{\ln \frac{R}{|x-y|}} \hat{q}_i G_{z_j}(T_z z_i, T_z z_j), \tag{7.9}
\]

where \( G_{z_j}(x, y) = \ln \frac{R}{|x-y|} - g_{z_j}(x, y) = 2\pi G_{z_j}(x, y) \) for any \( x, y \in T_z(\Omega) \).

**Proof.** By Lemma A.3 it is not hard to prove that there exists the unique \( \hat{q}_i \) satisfying (7.3). Moreover, we have \( \hat{q}_i = q(z_i) + O(\frac{1}{\ln \frac{R}{|x-y|}}) \) and \( \frac{\partial \hat{q}_i}{\partial z_{i,h}} = O(1) \) for \( h = 1, 2 \). \[\square\]

This lemma shows the well-definedness of \( Q_\delta \) defined by (3.5).

**Lemma A.5.** \( Q_\delta \) is well-defined. Moreover, for any \( q \in [1, +\infty), u \in L^q(\Omega) \) with \( \text{supp}(u) \subseteq \bigcup_{j=1}^m B_{Ls_{\delta,j}}(z_j) \) for some \( L > 1 \), there holds for some \( C > 0 \)

\[ ||Q_\delta u||_{L^q} \leq C ||u||_{L^q}. \]

**Proof.** By (4.7), we already know that there exists a unique \( C_{j,h} \) satisfying (4.6).

By the assumption, for any \( q \in [1, +\infty), l = 1, \cdots, m, h = 1, 2, \) one has

\[
C_{l,h} = O \left( \varepsilon^{\frac{1}{2}} \sum_{i=1}^m \sum_{h=1}^2 \int_{\bigcup_{j=1}^m B_{Ls_{\delta,j}}(z_j)} u \frac{\partial V_{\delta, z_i}}{\partial z_{i,h}} \right)
\]

\[
= O \left( \varepsilon^{\frac{1}{2}} \sum_{i=1}^m \sum_{h=1}^2 ||u||_{L^q(\Omega)} \left( \frac{\partial V_{\delta, z_i}}{\partial z_{i,h}} \right) ||u||_{L^q(\bigcup_{j=1}^m B_{Ls_{\delta,j}}(z_j))} \right)
\]

\[
= O \left( \varepsilon^{\frac{1}{2}} ||u||_{L^q(\Omega)} \sum_{i=1}^m \sum_{h=1}^2 \frac{s_{\delta,j}^{\frac{1}{2} - 1}}{\varepsilon} \right)
\]

\[= O(\varepsilon^{\frac{1}{2}} ||u||_{L^q(\Omega)}), \]
where \( q' \) denotes the conjugate exponent of \( q \). Hence we get
\[
\sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h} \frac{\partial}{\partial z_{j,h}} (-\delta^2 \text{div}(K(z_j) \nabla V_{\delta,z,j}))
\]
\[
\begin{align*}
&= \sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h} (V_{\delta,z,j,\hat{q}_{\delta,j},z_j} - \hat{q}_{\delta,j})^{p-1} \left( \frac{\partial V_{\delta,z,j,\hat{q}_{\delta,j},z_j}}{\partial z_{j,h}} - \frac{\partial \hat{q}_{\delta,j}}{\partial z_{j,h}} \right) \\
&= O \left( \sum_{j=1}^{m} \sum_{h=1}^{2} C_{j,h} \ln \varepsilon \right)^{- \frac{2}{p-1}} \left( \frac{S_{\hat{q}_{\delta,j}}^{q-1}}{\ln \varepsilon} \right)
\end{align*}
\]
\[
= O(1) \| u \|_{L^q(\Omega)} \text{ in } L^q(\Omega).
\]

Thus \( \| Q_\delta u \|_{L^q(\Omega)} \leq C \| u \|_{L^q(\Omega)} \).

**Lemma A.6.** There exists a constant \( L > 1 \) such that for \( \varepsilon \) small
\[
V_{\delta,Z} - q > 0, \quad \text{in } \bigcup_{j=1}^{m} \left( T_{z_j}^{-1} B_{(1-L \frac{\ln \ln \varepsilon}{\ln \varepsilon})} s_{\delta,j}(0) + z_j \right),
\]
\[
V_{\delta,Z} - q < 0, \quad \text{in } \Omega \setminus \bigcup_{j=1}^{m} \left( T_{z_j}^{-1} B_{Ls_{\delta,j}}(0) + z_j \right).
\]

**Proof.** The proof is similar to that of Lemma A.1 in [6]. If \( |T_{z_j}(x-z_j)| \leq \left( 1 - L \frac{\ln \ln \varepsilon}{\ln \varepsilon} \right) s_{\delta,j} \), then by (3.13) and \( \phi'(1) < 0 \) we have
\[
V_{\delta,Z} - q(x) = V_{\delta,z,j,\hat{q}_{\delta,j},z_j}(x) - \hat{q}_{\delta,j} + O \left( \frac{\ln \ln \varepsilon}{\ln \varepsilon} \right) > 0,
\]
if \( L \) is sufficiently large.

On the other hand, if \( \tau > 0 \) small and \( |T_{z_j}(x-z_j)| \geq s_{\delta,j}^\tau \) for any \( j = 1, \cdots, m \), then
\[
V_{\delta,Z} - q(x) = \sum_{j=1}^{m} \left( V_{\delta,z,j,\hat{q}_{\delta,j},z_j}(x) - \frac{\hat{q}_{\delta,j}}{R} g_{\tau}(T_{z_j}x, T_{z_j}z_j) \right) - q(x) \leq \sum_{j=1}^{m} \frac{\hat{q}_{\delta,j}}{R} s_{\delta,j}^\tau - C \leq \tau \sum_{j=1}^{m} \hat{q}_{\delta,j} - C < 0.
\]
If \( Ls_{\delta,j} \leq |T_{z_j}(x - z_j)| \leq s_{\delta,j}^\tau \), then by Lemma A.3 we have

\[
V_{\delta,Z} - q(x)
= V_{\delta,z_j,q_{\delta,j},z_j} - \frac{\hat{g}_{\delta,j}}{\ln R_{s_{\delta,j}}} g_{z_j}(T_{z_j}x, T_{z_j}z_j) - q(x) + \sum_{i \neq j} \left( \frac{\hat{g}_{\delta,i}}{\ln R_{s_{\delta,i}}} G_{z_i}(T_{z_j}z_j, T_{z_i}z_i) \right)
+ \sum_{i \neq j} \frac{\hat{q}_{\delta,i}}{\ln R_{s_{\delta,i}}} G_{z_i}(T_{z_j}z_j, T_{z_i}z_i)
+ O(\varepsilon^7)
= V_{\delta,z_j,q_{\delta,j},z_j} - \frac{\hat{g}_{\delta,j}}{\ln R_{\varepsilon}} g_{z_j}(T_{z_j}z_j, T_{z_j}z_j) + \sum_{i \neq j} \frac{\hat{q}_{\delta,i}}{\ln R_{\varepsilon}} G_{z_i}(T_{z_j}z_j, T_{z_i}z_i) + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right)
= V_{\delta,z_j,q_{\delta,j},z_j} - \frac{\hat{g}_{\delta,j}}{\ln R_{s_{\delta,j}}} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right)
\leq - \frac{\hat{q}_{\delta,j} \ln L}{\ln s_{\delta,j}} + O \left( \frac{\ln |\ln \varepsilon|}{|\ln \varepsilon|^2} \right) < 0,
\]
if we choose \( L \) sufficiently large.

\( \square \)

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