Supplement to a Shimura’s theorem on Eisenstein series

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1 Introduction

Eisenstein series is an important concept in the field of automorphic forms, and has been studied by various researchers.

In [5], Shimura considered the Eisenstein series for various types of groups and studied their analytic properties extensively. For example, he showed that the residue of an Eisenstein series is a power of \( \pi \) times a modular form with rational Fourier coefficients (see, Theorem below).

In this paper, we shall show that this modular form can be easily specified by using the functional equation of the corresponding Eisenstein series.

One of the Eisenstein series that we consider in this paper is the non-holomorphic Eisenstein series for the Siegel modular group \( \Gamma_n := \text{Sp}_n(\mathbb{Z}) \).

For \( n \in \mathbb{Z}_{>0} \) and \( k \in 2\mathbb{Z}_{\geq 0} \), we set

\[
E^{(n)}_{k}(Z, s) := \text{det}(\text{Im}(Z))^s \sum_{\{C, D\}} \text{det}(CZ + D)^{-k} |\text{det}(CZ + D)|^{-2s}.
\]

Here \( Z \) is a variable of the Siegel upper half space \( \mathbb{H}_n \) of degree \( n \), and \( s \) is a complex variable. The sum \( \{C, D\} \) runs over a complete set of representatives of the coprime symmetric pairs. The above series converges absolutely and with local uniformity on

\[
\{(Z, s) \mid Z \in \mathbb{H}_n, \Re(s) > \frac{n+1-k}{2}\}.
\]

As is well known from the Langlands theory, \( E^{(n)}_{k}(Z, s) \) has a meromorphic continuation to the whole complex \( s \)-plane. In [5], Shimura studied the analytic properties for various types of Eisenstein series, including the type mentioned here.

One of his results is as follows.

**Theorem.** (Shimura [5], Proposition 10.3) The Eisenstein series \( E^{(n)}_{\frac{n}{2}-1}(Z, s) \) has at most a simple pole at \( s = 1 \). The residue is \( \pi^{-n} \) times a modular form \( f \) of weight \( \frac{n-1}{2} \) for \( \Gamma_n \) with rational Fourier coefficients.

One of the aims of this paper is to specify the modular form \( f \) above.

First we prove the following result.
Theorem 1. Let \( n \) and \( m \) be integers with \( n > m \). The Eisenstein series \( E_{n-m}^{(n)}(Z,s) \) has at most a simple pole at \( s = \frac{m+1}{2} \), and the residue is given by

\[
\text{Res}_{s=\frac{m+1}{2}} E_{n-m}^{(n)}(Z,s) = c_{n,m} \cdot E_{n-m}^{(n)}(Z,0)
\]

for some \( c_{n,m} \in \mathbb{R} \).

Corollary. If \( \frac{n-m}{2} \equiv 2 \pmod{4} \), \( E_{n-m}^{(n)}(Z,s) \) is holomorphic at \( s = \frac{m+1}{2} \).

In the case that \( m = 1 \), we have the following result, which specifies the modular form \( f \) in Shimura’s theorem.

Theorem 2. The residue of \( E_{n-1}^{(n)}(Z,s) \) at \( s = 1 \) is given by

\[
\text{Res}_{s=1} E_{n-1}^{(n)}(Z,s) = \pi^{-n} \cdot c_n E_{n-1}^{(n)}(Z,0)
\]

where the constant \( c_n \) is explicitly given by

\[
c_n = (-1)^{\frac{n-5}{2}} \cdot 2^{-\frac{n+3}{2}} \cdot 3^{-1} \cdot \frac{(\frac{n-5}{2})!}{(\frac{n-1}{2})!} \cdot \frac{(\frac{n-5}{2})!!}{B_{\frac{n+3}{2}} B_{n+1} B_{n-1}}.
\]

Here \( B_m \) is the \( m \)-th Bernoulli number.

This theorem asserts that the modular form \( f \) in Shimura’s theorem is just \( c_n E_{n-1}^{(n)}(Z,0) \).

We discussed the case of Hermitian Eisenstein series (Eisenstein series on special unitary group) in the second part. The main results are Theorem 5.1, and 5.3. The content of this paper was already stated in [4] without proof. This paper gives the proof, including its amendments.

Part I

Eisenstein series for Siegel modular groups

2 Siegel modular forms

2.1 Fourier expansion

Let \( \Gamma_n = \text{Sp}_n(\mathbb{Z}) \) be the Siegel modular group of degree \( n \) and \( M_k(\Gamma_n) \) be the space of Siegel modular forms of weight \( k \) for \( \Gamma_n \). Any element \( F \) in \( M_k(\Gamma_n) \) has a Fourier expansion of the form

\[
F(Z) = \sum_{0 \leq T \in A_n} a_F(T) \exp(2\pi \sqrt{-1}\text{Tr}(TZ)), \ Z \in \mathbb{H}_n,
\]
where 
\[ \Lambda_n := \{ T = (t_{ij}) \in \text{Sym}_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z} \}. \]

For a subring \( R \subset \mathbb{C} \), we denote by \( M_k(\Gamma_n)_R \) the space consisting of modular forms \( F \) in \( M_k(\Gamma_n) \) whose Fourier coefficients \( a_F(T) \) lie in \( R \).

If \( k > n + 1 \), the Eisenstein series \( E_k^{(n)}(Z, 0) \) defined in Introduction is holomorphic in \( Z \), and it is a typical example of element in \( M_k(\Gamma_n)_Q \). We call here \( E_k^{(n)}(Z, s) \) the Siegel Eisenstein series.

### 2.2 Functional equation of Siegel Eisenstein series

For \( n \in \mathbb{Z}_{>0} \) and \( k \in 2\mathbb{Z}_{\geq 0} \), we define the function \( f_{n,k}(s) \) by

\[
f_{n,k}(s) := \frac{\Gamma_n(s + \frac{k}{2})}{\Gamma_n(s)} \cdot \xi(2s) \prod_{j=1}^{[n/2]} \xi(4s - 2j), \tag{2.1}
\]

where

\[
\Gamma_n(s) := \pi^{-n(n-1)/2} \prod_{j=0}^{n-1} \Gamma(s - \frac{j}{2}),
\]

\[
\xi(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi(1 - s).
\]

The Siegel Eisenstein series \( E_k^{(n)}(Z, s) \) has the functional equation of the form

\[
E_k^{(n)}(Z, s) = \frac{f_{n,k}\left(\frac{n+1}{2} - \frac{k}{2} - s\right)}{f_{n,k}\left(s + \frac{k}{2}\right)} E_k^{(n)}(Z, \frac{n+1}{2} - k - s) \tag{2.2}
\]

(e.g. cf. [2]).

### 2.3 Analytic property of \( E_k^{(n)}(Z, s) \) at \( s = 0 \)

The analytic property of \( E_k^{(n)}(Z, s) \) at \( s = 0 \) is studied by Shimura [5], Weissauer [6], and later by Haruki [1].

**Theorem 2.1.** (Weissauer [6], Haruki [1])

1. If \( k > 0 \), \( E_k^{(n)}(Z, s) \) is holomorphic in \( s \) at \( s = 0 \).
2. In the case (1), we define that \( E_k^{(n)}(Z) := E_k^{(n)}(Z, 0) \).
3. \( E_k^{(n)}(Z) \) is holomorphic except for the two cases
   \[ k = \frac{n+2}{2}, \frac{n+3}{2} \equiv 2 \pmod{4}. \]
4. In holomorphic case, \( E_k^{(n)}(Z) \) has rational Fourier coefficients.

**Remark 2.2.** If we take the theory of singular modular forms into account, it turns out that \( E_k^{(n)}(Z) \) vanishes when \( k < \frac{n}{2} \) and \( k \equiv 2 \pmod{4} \).
2.4 Shimura’s result

As stated in Introduction, Shimura proved the following result.

Theorem 2.3. (Shimura, \cite{Shimura}, Proposition 10.3) The Eisenstein series $E_{\frac{n-m}{2}}(Z, s)$ has at most a simple pole at $s = 1$. The residue is written as

$$\text{Res}_{s=1} E_{\frac{n-m}{2}}(Z, s) = \pi^{-n} \cdot f$$

with some $f \in M_{\frac{n-m}{2}}(\Gamma_n)$.\footnote{\textsuperscript{1}}

Remark 2.4. The Eisenstein series Shimura considered is

$$E_{k}^{(n)}(Z, s) = \sum \det(C Z + D)^{-k} |\det(C Z + D)|^{-s}.$$ 

Namely

$$E_{k}^{(n)}(Z, s) = \text{det}(\text{Im}(Z))^{-\frac{s}{2}} E_{k}^{(n)}(Z, \frac{s}{2})$$

in our notation. Theorem 2.1 is a translation of his original statement of Proposition 10.3 in \cite{Shimura}.

3 Main result

3.1 Residue of some Eisenstein series

To refine Shimura’s result, we shall prove the following theorem as a preparation.

Theorem 3.1. Let $n, m$ be integers satisfying $n > m \geq 1$ and $\frac{n-m}{2} \in 2\mathbb{Z}_{>0}$. Then the Eisenstein series $E_{\frac{n-m}{2}}(Z, s)$ has at most a simple pole at $s = \frac{m+1}{2}$ and

$$\text{Res}_{s=\frac{m+1}{2}} E_{\frac{n-m}{2}}(Z, s) = c_{n, m} \cdot E_{\frac{n-m}{2}}^{(n)}(Z, 0)$$

for some constant $c_{n, m} \in \mathbb{R}^\times$.

When $\frac{n-m}{2} \equiv 2 \pmod{4}$, $E_{\frac{n-m}{2}}(Z, 0)$ vanishes identically because of the theory of singular modular forms (Remark 2.2). Hence we have the following corollary.

Corollary 3.2. If $\frac{n-m}{2} \equiv 2 \pmod{4}$, $E_{\frac{n-m}{2}}^{(n)}(Z, s)$ is holomorphic at $s = \frac{m+1}{2}$.

To begin with, we remark that the holomorphy of $E_{\frac{n-m}{2}}^{(n)}(Z, s)$ at $s = 0$ is guaranteed by Theorem 2.1 (2).

The functional equation of $E_{\frac{n-m}{2}}^{(n)}(Z, s)$ can be written as

$$E_{\frac{n-m}{2}}^{(n)}(Z, s) = F_{n, m}(s) E_{\frac{n-m}{2}}^{(n)}(Z, \frac{m+1}{2} - s),$$

(3.2)
where
\[ F_{n,m}(s) = \gamma_{n,m}(s) \xi_{n,m}(s), \] (3.3)
\[ \gamma_{n,m}(s) := \frac{\Gamma_n \left( \frac{n+1}{2} - s \right)}{\Gamma_n \left( \frac{n+m+2}{4} - s \right)} \cdot \frac{\Gamma_n \left( s + \frac{n-m}{2} \right)}{\Gamma_n \left( s + \frac{n+2}{2} \right)}, \] (3.4)
\[ \xi_{n,m}(s) := \frac{\xi \left( \frac{n+1}{2} - 2s \right)}{\xi \left( 2s + \frac{n-m}{2} \right)} \prod_{j=1}^{\infty} \frac{\xi \left( (n+2) - 4s - 2j \right)}{\xi \left( 4s + (n-m) - 2j \right)} . \] (3.5)

To prove Theorem 3.1, it is sufficient to show that
\[ F_{n,m}(s) \] has a simple pole at \( s = \frac{m+1}{2} \), and the residue is in \( \mathbb{R} \times \), (3.6)
because \( E_{\frac{n}{2}}^{(n)}(\frac{n+1}{2}) \) is holomorphic at \( s = 0 \) (cf. Theorem 2.1). Namely, if we set
\[ c_{n,m} = \text{Res}_{s=\frac{m+1}{2}} F_{n,m}(s), \]
then the theorem is proved. In the following, we are going to proceed the proof of (3.6).

### 3.1.1 Analysis of \( \gamma \)-factor \( \gamma_{n,m}(s) \)

**Proposition 3.3.** The function \( \gamma_{n,m}(s) \) is holomorphic at \( s = \frac{m+1}{2} \) and \( \gamma_{n,m}(\frac{m+1}{2}) \in \mathbb{R} \times \).

**Proof.** The factors appearing in the definition of \( \gamma_{n,m}(s) \) can be simplified by the following formulas.

**Lemma 3.4.** (Cancelation law)
\[ \frac{\Gamma_n \left( \frac{n+1}{2} - s \right)}{\Gamma_n \left( \frac{n+m+2}{4} - s \right)} \cdot \frac{\Gamma_n \left( s + \frac{n-m}{2} \right)}{\Gamma_n \left( s + \frac{n+2}{2} \right)} = \frac{\Gamma_{\frac{n-m}{2}} \left( \frac{s-m}{2} \right)}{\Gamma_{\frac{n-m}{2}} \left( s + \frac{n-m}{2} \right)}. \]

This lemma shows that \( \gamma_{n,m}(s) \) can be rewritten as follows:
\[ \gamma_{n,m}(s) = \gamma_{n,m}^{(1)}(s) \cdot \gamma_{n,m}^{(2)}(s), \] (3.7)
\[ \gamma_{n,m}^{(1)}(s) = \frac{\Gamma_{\frac{n-m}{2}} \left( \frac{n+1}{2} - s \right)}{\Gamma_{\frac{n-m}{2}} \left( s + \frac{n-m}{2} \right)}, \quad \gamma_{n,m}^{(2)}(s) = \frac{\Gamma_{\frac{n-m}{2}} \left( s - \frac{n}{2} \right)}{\Gamma_{\frac{n-m}{2}} \left( \frac{s}{2} - \frac{n}{2} \right)}. \]

Calculation of \( \gamma_{n,m}^{(1)}(s) \):
We can write as
\[ \gamma_{n,m}^{(1)}(s) = \prod_{j=0}^{n-m-2} \frac{\Gamma \left( \frac{n+1}{2} - s - \frac{j}{2} \right)}{\Gamma \left( s + \frac{n-m}{2} - \frac{j}{2} \right)}. \]
For any integer $j$ with $0 \leq j \leq \frac{n-2m-2}{2}$, the functions $\Gamma \left( \frac{n+1}{2} - s - \frac{j}{2} \right)$ and $\Gamma \left( s + \frac{n-m}{2} - \frac{j}{2} \right)$ are holomorphic at $s = \frac{m+1}{2}$ and the values are non-zero real because $\Gamma(s)$ is holomorphic for real $s > 0$ and $\Gamma(s) > 0$ there. Thus $\gamma_{n,m}^{(1)}(s)$ is holomorphic at $s = \frac{m+1}{2}$ and $\gamma_{n,m}^{(1)} \left( \frac{m+1}{2} \right) \in \mathbb{R}^\times$.

Calculation of $\gamma_{n,m}^{(2)}(s)$:

We rewrite $\gamma_{n,m}^{(2)}(s)$ as

$$\gamma_{n,m}^{(2)}(s) = \prod_{j=0}^{n-m-4} \frac{\Gamma \left( s - \frac{m}{2} - j \right)}{\Gamma(-s-j)} \cdot \prod_{j=0}^{n-m-4} \frac{\Gamma \left( s - \frac{m+1}{2} - j \right)}{\Gamma(-s+\frac{1}{2} - j)}$$

First we assume that $m$ is even. Then the functions $s - \frac{m}{2} - j$ and $-s-j$ take half-integral values at $s = \frac{m+1}{2}$ for any $j$. Therefore both of $\Gamma \left( s - \frac{m}{2} - j \right)$ and $\Gamma(-s-j)$ are holomorphic at $s = \frac{m+1}{2}$ and their values are non-zero real. This implies that the first product of the right hand side is holomorphic at $s = \frac{m+1}{2}$ and their values are non-zero real. On the other hand, each of $\Gamma \left( s - \frac{m+1}{2} - j \right)$ and $\Gamma(-s+\frac{1}{2} - j)$ has a simple pole at $s = \frac{m+1}{2}$. Therefore the second product is also holomorphic at $s = \frac{m+1}{2}$ and the value is non-zero real.

In the case that $m$ is odd, the situation regarding two factors is just reversed.

Combining these results, we see that $\gamma_{n,m}(s)$ is holomorphic at $s = \frac{m+1}{2}$ and $\gamma_{n,m} \left( \frac{m+1}{2} \right) \in \mathbb{R}^\times$. This completes the proof of Proposition 3.3.

3.1.2 Analysis of $\xi$-factor $\xi_{n,m}(s)$

**Proposition 3.5.** The function $\xi_{n,m}(s)$ has a simple pole at $s = \frac{m+1}{2}$ and

$$\operatorname{Res}_{s=\frac{m+1}{2}} \xi_{n,m}(s) \in \mathbb{R}^\times.$$

We decompose $\xi_{n,m}(s)$ into two factors as follows:

$$\xi_{n,m}(s) = \xi_{n,m}^{(1)}(s) \xi_{n,m}^{(2)}(s),$$

$$\xi_{n,m}^{(1)}(s) = \frac{\xi \left( \frac{n+m+2}{2} - 2s \right)}{\xi \left( 2s + \frac{n-m}{2} \right)} \prod_{j=0}^{n-m} \xi \left( (n + m + 2) - 4s - 2j \right),$$

$$\xi_{n,m}^{(2)}(s) = \xi \left( 2m + 2 - 4s \right).$$

This decomposition means that the factor $\xi_{n,m}^{(1)}(s)$ is obtained by extracting the factor $\xi_{n,m}^{(2)}(s) = \xi \left( 2m + 2 - 4s \right)$ from $\xi_{n,m}(s)$.

**Lemma 3.6.** (1) The function $\xi_{n,m}^{(1)}(s)$ is holomorphic at $s = \frac{m+1}{2}$ and $\xi_{n,m}^{(1)} \left( \frac{m+1}{2} \right) \in \mathbb{R}^\times$.

(2) The function $\xi_{n,m}^{(2)}(s)$ has a simple pole at $s = \frac{m+1}{2}$ and

$$\operatorname{Res}_{s=\frac{m+1}{2}} \xi_{n,m}^{(2)}(s) \in \mathbb{R}^\times.$$
Proof. (1) The functions
\[
\frac{n+2}{2} - 2s, \quad 2s + \frac{n-m}{2}, \quad 4s + (n-m) - 2j \quad (1 \leq j \leq [\frac{n}{2}])
\]
have integral values with \(\geq 2\) at \(s = \frac{m+1}{2}\). Hence the functions
\[
\xi \left( \frac{n+2}{2} - 2s \right) \quad \xi \left( 2s + \frac{n-m}{2} \right) \quad \text{and} \quad \prod_{j=1}^{[\frac{n}{2}] \xi \left( 4s + (n-m) - 2j \right)}
\]
are holomorphic at \(s = \frac{m+1}{2}\) and their values at \(s = \frac{m+1}{2}\) are non-zero real. We consider the factors
\[
\xi((n + m + 2) - 4s - 2j) \quad (1 \leq j \leq [\frac{n}{2}], \quad j \neq \frac{n-m}{2}).
\]
For \(j\) with \(1 \leq j < \frac{n-m}{2}\), the function \((n + m + 2) - 4s - 2j\) has positive even values with \(\geq 2\) at \(s = \frac{m+1}{2}\). Hence \(\xi((n + m + 2) - 4s - 2j)\) is holomorphic at \(s = \frac{m+1}{2}\) and
\[
\xi((n + m + 2) - 4s - 2j))|_{s=\frac{m+1}{2}} \in \mathbb{R}^\times.
\]
For \(j\) with \(\frac{n-m}{2} < j \leq \frac{n}{2}\), the function \((n + m + 2) - 4s - 2j\) has negative even value with \(\leq -2\) at \(s = \frac{m+1}{2}\). Hence \(\xi((n + m + 2) - 4s - 2j)\) is also holomorphic at \(s = \frac{m+1}{2}\) in this case, and the value is non-zero real.

(2) Since \(\xi(s)\) has a simple pole at \(s = 0\), \(\xi_{n,m}^{(2)}(s) = \xi(2m + 2 - 4s)\) has a simple pole at \(s = \frac{m+1}{2}\) and
\[
\text{Res}_{s=\frac{m+1}{2}} \xi_{n,m}^{(2)}(s) \in \mathbb{R}^\times.
\]

From this lemma, we see that \(\xi_{n,m}(s) = \xi_{n,m}^{(1)}(s) \xi_{n,m}^{(2)}(s)\) has a simple pole at \(s = \frac{m+1}{2}\) and
\[
\text{Res}_{s=\frac{m+1}{2}} \xi_{n,m}(s) = \xi_{n,m}^{(1)} \left( \frac{m+1}{2} \right) \text{Res}_{s=\frac{m+1}{2}} \xi_{n,m}^{(2)}(s) \in \mathbb{R}^\times.
\]
This completes the proof of Proposition [3,5]

We return to the proof of Theorem 3.1. We recall the expression
\[
E_{n,m}^{(n)}(Z, s) = F_{n,m}(s) E_{n,m}^{(n)} \left( Z, \frac{m+1}{2} - s \right)
\]
\[
F_{n,m}(s) = \gamma_{n,m}(s) \xi_{n,m}(s).
\]
Since \(\gamma_{n,m}(s)\) is holomorphic at \(s = \frac{m+1}{2}\) (Proposition [3,3] and \(\xi_{n,m}(s)\) has a simple pole at \(s = \frac{m+1}{2}\) (Proposition [3,3]), we see that
\[
\text{Res}_{s=\frac{m+1}{2}} E_{n,m}^{(n)}(Z, s) = \frac{1}{\text{Res}_{s=\frac{m+1}{2}} F_{n,m}(s) \cdot E_{n,m}^{(n)}(Z, 0)}
\]
\[
= \gamma_{n,m} \left( \frac{m+1}{2} \right) \text{Res}_{s=\frac{m+1}{2}} \xi_{n,m}(s) \cdot E_{n,m}^{(n)}(Z, 0).
\]
Therefore, if we set
\[ c_{n,m} := \gamma_{n,m} \left( \frac{m+1}{2} \right) \text{Res}_{s=\frac{m}{2}} \xi_{n,m}(s) \in \mathbb{R}^+, \]
then
\[ \text{Res}_{s=\frac{m}{2}} E^{(n)}_{\frac{m}{2}} (Z, s) = c_{n,m} E^{(n)}_{\frac{m}{2}} (Z, 0) \]
and this completes the proof of Theorem 3.1.

### 3.2 Refinement of Shimura’s result

We prove the following theorem which is a refinement of Shimura’s result (Theorem 4.1).

**Theorem 3.7.** Assume that \( \frac{n-1}{2} \equiv 0 \pmod{2} \). Then we have
\[ \text{Res}_{s=1} E^{(n)}_{\frac{n-1}{2}} (Z, s) = \pi^{-n} \cdot c_n E^{(n)}_{\frac{n-1}{2}} (Z, 0). \]
where the constant \( c_n \) is given by
\[ c_n = (-1)^{n-1} 2^{\frac{n+13}{4}} \cdot 3^{\frac{n-5}{2}} \cdot \frac{(n+3)! (n+5)! (n+1)!}{(n-1)! (n-5)!!!} \cdot \frac{B_{\frac{n-1}{2}}}{B_{n+1} B_{n-1}}. \]
Here \( B_m \) is the \( m \)-th Bernoulli number.

**Remark 3.8.** The theorem above is considered as a special case \( (m=1) \) of Theorem 3.1. The theorem asserts that the constant \( c_{n,1} \) in Theorem 3.1 can be expressed as \( c_{n,1} = \pi^{-n} \cdot c_n \) with \( c_n \in \mathbb{Q}^+ \).

We use the notation in the previous section as \( m = 1 \).

We recall the functional equation (3.2)
\[ E^{(n)}_{\frac{n-1}{2}} (Z, s) = F_{n,1}(s) E^{(n)}_{\frac{n-1}{2}} (Z, 1-s). \]
Since \( E^{(n)}_{\frac{n-1}{2}} (Z, s) \) is holomorphic at \( s = 0 \), it is sufficient to prove that
\[ \text{Res}_{s=1} F_{n,1}(s) = \pi^{-n} \cdot c_n. \]
We look back the definition of \( F_{n,1}(s) \):
\[ F_{n,1}(s) = \gamma_{n,1}(s) \xi_{n,1}(s), \]
\[ \gamma_{n,1}(s) = \frac{\Gamma_n \left( \frac{n+1}{2} - s \right) \Gamma_n \left( s + \frac{n-1}{2} \right)}{\Gamma_n \left( \frac{n+3}{4} - s \right) \Gamma_n \left( s + \frac{n-1}{2} \right)}, \]
\[ \xi_{n,1}(s) = \frac{\xi \left( \frac{n+3}{4} - 2s \right)}{\xi \left( 2s + \frac{n-1}{2} \right)} \prod_{j=1}^{\frac{n-1}{2}} \xi(n + 3 - 4s - 2j) / \xi(4s + n - 1 - 2j). \]
3.2.1 Analysis of $\gamma$-part

Proposition 3.9.

$$\lim_{s \to 1} \gamma_{n,1}(s) = (-1)^{\frac{n-1}{2}} \frac{(n-1)!}{(n-1)!} \left( \frac{n-1}{2} \right)!$$

Proof. As in (3.7), we decompose $\gamma_{n,1}(s)$ as

$$\gamma_{n,1}(s) = \gamma_{n,1}^{(1)}(s) \gamma_{n,1}^{(2)}(s)$$

Calculation of $\gamma_{n,1}^{(1)}(s)$:

We decompose $\gamma_{n,1}^{(1)}(s)$ into two factors:

$$\gamma_{n,1}^{(1)}(s) = \prod_{s=0}^{\frac{n-5}{2}} \frac{\Gamma \left( \frac{n+1}{2} - s \right)}{\Gamma \left( \frac{n+1}{2} - s - n \right)} \cdot \prod_{j=0}^{\frac{n-5}{2}} \frac{\Gamma \left( \frac{n}{2} - s - j \right)}{\Gamma \left( \frac{n}{2} - s - j - n \right)}.$$ (3.9)

In the first product of the right hand side above, both of $\Gamma \left( \frac{n+1}{2} - s - j \right)$ and $\Gamma \left( \frac{n}{2} - s - j \right)$ take non-zero integral values at $s = 1$ (note that $n$ is an odd integer such that $\geq 5$). Namely we obtain

$$\gamma_{n,1}^{(1)}(1) = \prod_{s=0}^{\frac{n-5}{2}} \frac{\Gamma \left( \frac{n+1}{2} - j \right)}{\Gamma \left( \frac{n+1}{2} - j - n \right)} \cdot \prod_{j=0}^{\frac{n-5}{2}} \frac{\Gamma \left( \frac{n}{2} - j \right)}{\Gamma \left( \frac{n}{2} - j - n \right)} = \frac{\Gamma \left( \frac{n+3}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)} \cdot \frac{\Gamma \left( \frac{n+1}{2} \right)}{\Gamma \left( \frac{n+1}{2} \right)}$$

Calculation of $\gamma_{n,1}^{(2)}(s)$:

As in the case of $\gamma_{n,1}^{(1)}(s)$, we decompose $\gamma_{n,1}^{(2)}(s)$ into two factors:

$$\gamma_{n,1}^{(2)}(s) = \prod_{s=0}^{\frac{n-5}{2}} \frac{\Gamma \left( s - 1 - j \right)}{\Gamma \left( -s - j \right)} \cdot \prod_{j=0}^{\frac{n-5}{2}} \frac{\Gamma \left( s - \frac{1}{2} - j \right)}{\Gamma \left( -s + \frac{1}{2} - j \right)}.$$ (3.10)

In the first product above, both of $\Gamma(s - 1 - j)$ and $\Gamma(-s - j)$ have simple poles at $s = 1$. Moreover we have

$$\lim_{s \to 1} \frac{\Gamma \left( s - 1 - j \right)}{\Gamma \left( -s - j \right)} = j + 1 \quad (0 \leq j \leq \frac{n-5}{4}).$$

Hence

$$\lim_{s \to 1} \prod_{s=0}^{\frac{n-5}{2}} \frac{\Gamma \left( s - 1 - j \right)}{\Gamma \left( -s - j \right)} = \prod_{j=0}^{\frac{n-5}{2}} (j + 1) = \left( \frac{n-1}{2} \right)!.$$ (3.10)
We consider the second product. We see that

\[
\lim_{s \to 1} \prod_{j=0}^{n-5} \frac{\Gamma(s - \frac{1}{2} - j)}{\Gamma(-s + \frac{1}{2} - j)} = \prod_{j=0}^{n-5} \left(-\frac{1}{2} - j\right)
\]

\[
= (-1)^{n-5} 2^{-\frac{n-1}{4}} \left(\frac{n-3}{2}\right)!!.
\]  

(3.11)

From (3.10) and (3.11), we see that

\[
\lim_{s \to 1} \gamma_{n,1}^{(2)}(s) = (-1)^{\frac{n-1}{2}} 2^{-\frac{n-1}{4}} \left(\frac{n-3}{2}\right)!!.
\]  

(3.12)

Combining (3.9) and (3.12), we obtain

\[
\lim_{s \to 1} \gamma_{n,1}(s) = \gamma_{n,1}^{(1)}(1) \cdot \lim_{s \to 1} \gamma_{n,1}^{(2)}(s)
\]

\[
= (-1)^{\frac{n-1}{2}} 2^{-\frac{n-1}{4}} \left(\frac{n-3}{2}\right)!! \cdot \gamma_{n,1}^{(1)}(1).
\]  

(3.13)

3.2.2 Analysis of \(\xi\)-part

Proposition 3.10.

\[
\text{Res}_{s=1} \xi_{n,1}(s) = \pi^{-n} 2^{-3-2n} 3^{-1} \left(\frac{n+3}{2}\right)!! \left(\frac{n+1}{2}\right)! \left(\frac{n-1}{2}\right)!! \frac{B_{n+1}}{B_{n+3} B_{n+1} B_{n-1}}.
\]

Proof. We recall the decomposition of \(\xi_{n,m}(s)\) in (3.8). If we apply this decomposition to \(\xi_{n,1}(s)\), we get the following expression:

\[
\xi_{n,1}(s) = \rho_n(s) \xi(4-4s),
\]

\[
\rho_n(s) = \frac{\xi \left(\frac{n+1}{2} - 2s\right)}{\xi \left(\frac{n+1}{2} + 2s\right)} \prod_{j=1}^{n-3} \xi \left(n + 3 - 4s - 2j\right) \prod_{j=1}^{n-3} \xi \left(4s + n - 1 - 2j\right).
\]

First we shall prove that \(\rho_n(1) \in \pi^{-n} \mathbb{Q}^\times\). (The holomorphy of \(\rho_n(s)\) at \(s = 1\) is guaranteed by the result of §3.1.2.)

\[
\rho_n(1) = \frac{\xi \left(\frac{n+1}{2}\right)}{\xi \left(\frac{n+1}{2}ight)} \cdot \frac{\prod_{j=1}^{n-3} \xi \left(n - 1 - 2j\right) \prod_{j=1}^{n-3} \xi \left(n + 3 - 2j\right)}{\xi \left(\frac{n+1}{2}\right) \xi \left(\frac{n+1}{2}\right)}.
\]  

(3.14)
We rewrite the factors appearing in the last formula by using the Bernoulli numbers.

\[
\xi\left(\frac{n-1}{2}\right) = -2^{-n-1} \pi^{-1} \frac{(n+1)!(n+3)!(n-1)!}{(n+1)!}\frac{B_{n+1}}{B_{n+3}}
\]

and

\[
\frac{\xi(2)}{\xi(n+1)\xi(n-1)} = -\pi^{-n} \cdot 2^{-2n+1} \cdot 3^{-1} \frac{(n+1)(n+1)(n+1)!}{(n+1)!}\frac{B_{n+1}}{B_{n+3}}.
\]

Consequently

\[
\rho_n(1) = \pi^{-n} \cdot 2^{-1-2n} \cdot 3^{-1} \frac{(n+1)(n+1)(n+1)!}{(n+1)!}\frac{B_{n+1}}{B_{n+3}}.
\]

Next we consider \(\xi(4-4s)\). This function has a simple pole at \(s = 1\), and

\[
\mathrm{Res}_{s=1} \xi(4-4s) = \frac{1}{4}.
\]

Consequently we obtain

\[
\mathrm{Res}_{s=1} \xi_n(s) = \rho_n(1) \cdot \mathrm{Res}_{s=1} \xi(4-4s)
\]

\[
= \pi^{-n} \cdot 2^{-3-2n} \cdot 3^{-1} \frac{(n+1)(n+1)(n+1)!}{(n+1)!}\frac{B_{n+1}}{B_{n+3}}.
\]

\[
(3.15)
\]

Summarizing (3.13) and (3.15), we conclude that

\[
\mathrm{Res}_{s=1} F_{n,1}(s) = \lim_{s \to 1} \gamma_n(s) \cdot \mathrm{Res}_{s=1} \xi_n(s)
\]

\[
= \pi^{-n} \cdot (-1)^{\frac{n-3}{2}} 2^{-\frac{n+13}{2}} \cdot 3^{-1} \frac{(n+1)(n+1)(n+1)!}{(n+1)!}\frac{B_{n+1}}{B_{n+3}}.
\]

Here we used \((\frac{n-3}{2})!!/((\frac{n-5}{2})!! = 1/((\frac{n-5}{2})!!). This completes the proof of Theorem 3.7. Namely

\[
\mathrm{Res}_{s=1} E^{(n)}_{\frac{n}{2}}(Z, s) = \pi^{-n} \cdot c_n \cdot E^{(n)}_{\frac{n}{2}}(Z, 0)
\]

and the constant \(c_n\) is given as above.
Table 1: Numerical examples of $c_n$

| $n$  | $c_n$              |
|------|--------------------|
| 5    | $-70875/16$        |
| 9    | $983275/512$       |
| 13   | $-4150336875/2830336$ |
| 17   | $65602154293050375/20796793028608$ |

### 3.2.3 Constant $c_{n,m}$ (some rationality)

The statement of Theorem 3.7 asserts that the constant $c_{n,m}$ appeared in Theorem 3.1 has the expression

$$c_{n,1} = \pi^{-n} \cdot c_n \quad (c_n \in \mathbb{Q}^\times)$$

in the case $m = 1$.

It is expected that the constant $c_{n,m}$ has the expression

$$c_{n,m} = \pi^* \cdot t_{n,m} \quad (t_{n,m} \in \mathbb{Q}^\times).$$

However, this generally seems unsuccessful because the constant $c_{n,m}$ includes the factors $\zeta(2j + 1)$ ($\zeta$-value at positive odd integer) in general.

For example, we consider the case that $n = 6$ and $m = 2$. In this case, the constant $c_{6,2}$ has the following expression:

$$c_{6,2} = \lim_{s \to 3/2} \frac{\Gamma(-s + 3/2) \Gamma(s - 3)}{\Gamma(-s + 1/2) \Gamma(-s)} \cdot \lim_{s \to 3/2} \frac{\zeta(5 - 2s) \zeta(8 - 4s)}{\zeta(2s + 2) \zeta(4s + 2) \zeta(4s) \zeta(4s - 2)} \cdot \text{Res}_{s=3/2} \zeta(6 - 4s)$$

$$= \frac{\zeta(2)^2 \zeta(3)}{\zeta(4) \zeta(5) \zeta(6) \zeta(8)} \cdot \text{Res}_{s=3/2} \zeta(6 - 4s)$$

$$= \frac{496125}{16} \pi^{-6} \zeta(3) \zeta(5).$$
Part II

Eisenstein series for Hermitian modular groups

In this part, we treat the case of Hermitian Eisenstein series \( E_{k,K}(Z,s) \) (for the precise definition, see §4.1) and give the results analogous to that in Part 1. Namely we shall prove the following theorems under the condition that the class number \( h_K \) is one. The goal is to refine a Shimura’s result (Theorem 4.1) in the case of Hermitian modular forms.

**Theorem 1 (Hermitian case)** We assume that \( h_K = 1 \). Let \( n, m \) be integers satisfying \( n > m \geq 1 \) and \( n - m \in 2\mathbb{Z}_{>0} \). Then the Eisenstein series \( E_{n-m,K}(Z,s) \) has at most a simple pole at \( s = m \)

\[
\text{Res}_{s=m} E_{n-m,K}(Z,s) = c_{n,m,K} \cdot E_{n-m,K}(Z,0)
\]

for some constant \( c_{n,m,K} \in \mathbb{R}^\times \).

The following theorem is an analogous result of Theorem 3.7 in Part 1, and this theorem specifies the modular form \( f \) given by Shimura (cf. Theorem 4.1).

**Theorem 2 (Hermitian case)** We assume that \( h_K = 1 \) and \( n \equiv 1 \mod 2 \). Then we have

\[
\text{Res}_{s=1} E_{n-1,K}(Z,s) = \pi^{-n} \cdot c_{n,K} \cdot E_{n-1,K}(Z,0) \newline
\]

where the constant \( c_{n,K} \) is given by

\[
c_{n,K} = 2^{-2n} \cdot D_K^{\frac{n-1}{2}} \cdot n! \cdot \frac{B_{1,\chi_K}}{B_{n,\chi_K} \cdot B_{n+1}} \in \mathbb{Q}^\times,
\]

where \( B_m \) (resp. \( B_{m,\chi} \)) is the \( m \)-th Bernoulli (resp. generalized Bernoulli) number.

4 Hermitian modular forms

4.1 Definition

Let \( \mathcal{H}_n \) be the Hermitian upper half space of degree \( n \) defined by

\[
\mathcal{H}_n = \{ Z \in M_n(\mathbb{C}) \mid I(Z) := \frac{1}{2\sqrt{-1}}(Z - \bar{Z}) > 0 \}.
\]

The special unitary group \( SU(n,n) \) is realized by

\[
G_n := \{ M \in SL_{2n}(\mathbb{C}) \mid \begin{bmatrix} M & J_n \end{bmatrix} J_n M = J_n \}.
\]
where \( J_n = \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix} \). The group \( G_n \) acts on \( \mathcal{H}_n \) by the generalized linear fractional transformations.

Let \( K \) be an imaginary quadratic number field with discriminant \(-D_K\). We denote by \( \mathcal{O}_K, \mathfrak{d}_K \) the ring of integers in \( K \), the different ideal of \( K \). Let \( \chi_K \) be the Kronecker character of \( K \), and \( h_K \) the class number of \( K \). We define the \textit{Hermitian modular group} of degree \( n \) for \( K \) by

\[
\Gamma_{n,K} = G_n \cap M_{2n}(\mathcal{O}_K).
\]

We denote by \( M_{k}(\Gamma_{n,K}) \) the \( \mathbb{C} \)-vector space of Hermitian modular forms of weight \( k \) for \( \Gamma_{n,K} \).

It is known that each \( F \in M_{k}(\Gamma_{n,K}) \) admits a Fourier expansion of the form

\[
F(Z) = \sum_{0 \leq H \in \Lambda_n(K)} a_F(H) \exp(2\pi \sqrt{-1} \text{tr}(HZ)),
\]

where

\[
\Lambda_n(K) := \{ H = (h_{ij}) \in \text{Her}(K) \mid h_{ii} \in \mathbb{Z}, h_{ij} \in \mathfrak{d}_K^{-1} \}.
\]

As in the Siegel modular case, we define \( M_{k}(\Gamma_{n,K})_R \) for a subring \( R \subset \mathbb{C} \).

### 4.1.1 Hermitian Eisenstein series

We define a parabolic subgroup of \( G_n \):

\[
P_n := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n \mid C = 0_n \right\}.
\]

The Eisenstein series considered here is

\[
E_k^{(n)}(Z, s) := \det(I(Z))^s \sum_{(C, D) \in (\Gamma_{n,K}) \setminus \Gamma_{n,K}} \det(CZ + D)^{-k} |\det(CZ + D)|^{-2s},
\]

where \( (Z, s) \in \mathcal{H}_n \times \mathbb{C}, k \in 2\mathbb{Z}_{\geq 0} \). It is known that this series is absolutely, uniformly convergent if \( \text{Re}(s) + k > 2n \). Therefore \( E_k^{(n)}(Z) := E_k^{(n)}(Z, 0) \) becomes an element of \( M_{k}(\Gamma_{n,K}) \). Moreover it has rational Fourier coefficients (i.e. \( M_{k}(\Gamma_{n,K})_\mathbb{Q} \)).

We call \( E_k^{(n)}(Z, s) \) the \textit{Hermitian Eisenstein series} of degree \( n \) here.

In the following, we study the analytic property of Hermitian Eisenstein series.

### 4.1.2 Functional equation of Hermitian Eisenstein series

In the rest of this paper, we put the assumption

\[
h_K = 1.
\]

For \( n \in \mathbb{Z}_{>0} \) and \( k \in 2\mathbb{Z}_{\geq 0} \), we define the function \( g_{n,k,K}(s) \) by

\[
g_{n,k,K}(s) = \frac{\Gamma_n(z)^{(s+k)/2}}{\Gamma_n(z/2)} \prod_{j=0}^{n-1} \xi(s - j; \chi_K),
\]

(4.1)
where
\[ \Gamma_{n, \mathcal{C}}(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s-j), \]
\[ \xi(s; \chi_K^j) = \begin{cases} \pi^{-\frac{j}{2}} \Gamma(\frac{j}{2}) \zeta(s) & \text{if } j \text{ is even,} \\ D_K^s \pi^{-\frac{j}{2}} \Gamma(\frac{j+1}{2}) L(s; \chi_K) & \text{if } j \text{ is odd,} \end{cases} \]
and \( L(s; \chi) \) is the Dirichlet \( L \)-function. Under the above assumption, the Hermitian Eisenstein has the functional equation of the form
\[ E_{n, K}(Z, s) = g_{n, K}(2n-k-2s) E_{n, K}(Z, n-k-s) \] (4.2)
(e.g. cf. [3], Theorem 1.8).

4.1.3 Shimura’s result in the Hermitian case

Shimura’s result in the case of Hermitian Eisenstein series is as follows.

**Theorem 4.1.** (Shimura [5], Proposition 10.3) The Eisenstein series
\[ E_{n, K}(Z, s) \]
has at most a simple pole at \( s = 1 \). The residue is written as
\[ \text{Res}_{s=1} E_{n, K}(Z, s) = \pi^{-n} \cdot f \]
with some \( f \in M_{n-1}(\Gamma_{n, K})_Q \).

**Remark 4.2.** In his statement, we do not need the assumption on the class number.

5 Main result in the case of Hermitian Eisenstein series

5.1 Residue of some Hermitian Eisenstein series

The following theorem is a Hermitian version of Theorem 3.1.

**Theorem 5.1.** We assume that \( h_K = 1 \). Let \( n, m \) be integers satisfying \( n > m \geq 1 \) and \( n - m \in 2\mathbb{Z}_{>0} \). Then the Eisenstein series \( E_{n-m, K}(Z, s) \) has at most a simple pole at \( s = m \) and
\[ \text{Res}_{s=m} E_{n-m, K}(Z, s) = c_{n,m,K} \cdot E_{n-m, K}(Z, 0) \] (5.1)
for some constant \( c_{n,m,K} \in \mathbb{R}^\times \).

When \( n - m \equiv 2 \) (mod 4), \( E_{n-m, K}(Z, 0) \) vanishes identically because of the theory of singular modular forms. Hence we have the following corollary.
**Corollary 5.2.** If \( n - m \equiv 2 \text{ (mod 4)} \), \( E_{n-m,K}^{(n)}(Z, s) \) is holomorphic at \( s = m \).

We start the proof of Theorem 5.1. Considering (4.2), we set

\[
G_{n,m,K}(s) := \frac{g_{n,k,K}(2n - k - 2s)}{g_{n,k,K}(2s + k)} \bigg|_{k=n-m} = \frac{g_{n,n-m,K}(n + m - 2s)}{g_{n,n-m,K}(2s + n - m)}.
\]

If we use this notation, the functional equation (4.2) can be written as

\[
E_{n-m,K}^{(n)}(Z, s) = G_{n,m,K}(s)E_{n-m,K}^{(n)}(Z, m - s).
\]

In order to prove Theorem 5.1, it is enough to show the followings:

(i) \( E_{n-m,K}^{(n)}(Z, s) \) is at \( s = 0 \).

(ii) \( G_{n,m,K}(s) \) has a simple pole at \( s = m \), and the residue is in \( \mathbb{R}^\times \).

If these statements are proved, then we get

\[
\text{Res}_{s=m} E_{n-m,K}^{(n)}(Z, s) = \text{Res}_{s=m} G_{n,m,K}(s) \cdot E_{n-m,K}^{(n)}(Z, 0).
\]

The statement (i) is a consequence of Shimura’s result (cf. [5]).

We are going to prove (ii).

We use the following expression of \( G_{n,m,K}(s) \):

\[
G_{n,m,K}(s) = \gamma_{n,m,K}(s) \xi_{n,m,K}(s),
\]

where

\[
\gamma_{n,m,K}(s) := \frac{\Gamma_{n,C}(n - s)}{\Gamma_{n,C}(\frac{n+m}{2} - s)},
\]

\[
\xi_{n,m,K}(s) := \prod_{j=0}^{n-1} \frac{\xi(n + m - 2s - j; \chi^{j}_{K})}{\xi(2s + n - m - j; \chi^{j}_{K})}.
\]

### 5.1.1 Analysis of \( \gamma_{n,m,K}(s) \)

First we note the following cancelation law:

\[
\frac{\Gamma_{n,C}(\frac{n+m}{2} + k)}{\Gamma_{n,C}(\frac{n+m}{2} - k)} = \prod_{j=0}^{k} \frac{\Gamma\left(\frac{n}{2} + \frac{k}{2} - j\right)}{\Gamma\left(\frac{n}{2} + \frac{k}{2} - n - j\right)}.
\]

From this, we see that

\[
\gamma_{n,m,K}(s) = \prod_{j=0}^{n-m-2} \frac{\Gamma(n - s - j)}{\Gamma(-s - j)} \cdot \frac{\Gamma(s - m - j)}{\Gamma(s + n - m - j)}.
\]

We use the following decomposition of \( \gamma_{n,m,K}(s) \):

\[
\gamma_{n,m,K}(s) = \gamma^{(1)}_{n,m,K}(s) \cdot \gamma^{(2)}_{n,m,K}(s)
\]

\[
\gamma^{(1)}_{n,m,K}(s) := \prod_{j=0}^{n-m-2} \frac{\Gamma(s - m - j)}{\Gamma(-s - j)}, \quad \gamma^{(2)}_{n,m,K}(s) := \prod_{j=0}^{n-m-2} \frac{\Gamma(n - s - j)}{\Gamma(s + n - m - j)}.\]
Calculation of $\gamma_{n,m,K}^{(1)}(s)$:

Since the functions $s - m - j$ and $-s - j$ take non-positive integral values at $s = m$, each of functions $\Gamma(s - m - j)$ and $\Gamma(-s - j)$ has a simple pole at $s = m$. Hence the function $\Gamma(s - m - j)/\Gamma(-s - j)$ is holomorphic at $s = m$, and the value is in $\mathbb{R}^\times$.

Calculation of $\gamma_{n,m,K}^{(2)}(s)$:

Since the functions $n - s - j$ and $s + n - m - j$ take positive integral values at $s = m$, $\Gamma(n - s - j)/\Gamma(s + n - m - j)$ is holomorphic at $s = m$ and the value is in $\mathbb{R}^\times$.

From these facts, it can be seen that $\gamma_{n,m,K}(s) = \gamma_{n,m,K}^{(1)}(s) \cdot \gamma_{n,m,K}^{(2)}(s)$ is holomorphic at $s = m$, and the value is in $\mathbb{R}^\times$.

5.1.2 Analysis of $\xi_{n,m,K}(s)$

We recall

$$
\xi_{n,m,K}(s) := \prod_{j=0}^{n-1} \xi(n + m - 2s - j; \chi_j^k)
$$

The function $\xi(n + m - 2s - j; \chi_j^k)$ appeared in the numerator of the right-hand side of the above formula has a simple pole at $s = m$ only when $j = n - m$. On the other hand, the rest functions

$$
\xi(n + m - 2s - j; \chi_j^k) \quad (0 \leq j \leq n - 1, j \neq n - m),
$$

$$
\xi(2s + n - m - j; \chi_j^k) \quad (0 \leq j \leq n - 1)
$$

are holomorphic at $s = m$ and the values are in $\mathbb{R}^\times$.

From these facts, we conclude that $\xi_{n,m,K}(s)$ has a simple pole at $s = m$, and the residue is in $\mathbb{R}^\times$. This proves (ii), and completes the proof of Theorem 5.1.

5.2 Refinement of Shimura’s result: Hermitian case

We give a Hermitian version of Theorem 5.3 which gives the residue of $E_{n-1,K}^{(n)}(Z, s)$ at $s = 1$. This is the second main theorem in Part 2.

**Theorem 5.3.** We assume that $h_K = 1$ and $n \equiv 1 \pmod{2}$. Then we have

$$
\text{Res}_{s=1} E_{n-1,K}^{(n)}(Z, s) = \pi^{-n} \cdot c_{n,K} \cdot E_{n-1,K}^{(n)}(Z, 0),
$$

where the constant $c_{n,K}$ is given by

$$
c_{n,K} = 2^{-2n} \cdot D_K^{\frac{1}{2}} \cdot n \cdot (n + 1)! \frac{B_{1,\chi_K}}{B_{n,\chi_K} \cdot B_{n+1}} \in \mathbb{Q}^\times,
$$
where $B_m$ (resp. $B_{m,\chi}$) is the $m$-th Bernoulli (resp. generalized Bernoulli) number.

This result means that the constant $c_{n,1,\mathcal{K}}$ given in Theorem 5.1 can be written as

$$c_{n,1,\mathcal{K}} = \pi^{-n} \cdot c_{n,\mathcal{K}} \quad (c_{n,\mathcal{K}} \in \mathbb{Q}^\times).$$

We recall the decomposition

$$G_{n,1,\mathcal{K}}(s) = \gamma_{n,1,\mathcal{K}}(s) \xi_{n,1,\mathcal{K}}(s). \quad (\text{cf. (5.2)})$$

### 5.2.1 Calculation of $\gamma_{n,1,\mathcal{K}}(s)$

We apply the decomposition (5.3) in the case $m = 1$.

$$\gamma_{n,1,\mathcal{K}}(s) = \gamma_{n,1,\mathcal{K}}^{(1)}(s) \cdot \gamma_{n,1,\mathcal{K}}^{(2)}(s)$$

$$\gamma_{n,1,\mathcal{K}}^{(1)}(s) := \prod_{j=0}^{n-3} \frac{\Gamma(s - 1 - j)}{\Gamma(-s - j)}, \quad \gamma_{n,1,\mathcal{K}}^{(2)}(s) := \prod_{j=0}^{n-3} \frac{\Gamma(n - s - j)}{\Gamma(s + n - 1 - j)}.$$  

Calculation of $\gamma_{n,1,\mathcal{K}}^{(1)}(s)$:

$$\lim_{s \to 1} \gamma_{n,1,\mathcal{K}}^{(1)}(s) = \prod_{j=0}^{n-3} \left( \lim_{s \to 1} \frac{\Gamma(s - 1 - j)}{\Gamma(-s - j)} \right) = \prod_{j=0}^{n-3} (j + 1) = \frac{(n - 1)!}{(n - 1)!}.$$  

Calculation of $\gamma_{n,1,\mathcal{K}}^{(2)}(s)$:

$$\lim_{s \to 1} \gamma_{n,1,\mathcal{K}}^{(2)}(s) = \prod_{j=0}^{n-3} \frac{\Gamma(n - 1 - j)}{\Gamma(n - j)} = \frac{\Gamma(n - 1 - j)}{\Gamma(n)} = \frac{(n - 1)!}{(n - 1)!}.$$  

Therefore we obtain

$$\lim_{s \to 1} \gamma_{n,1,\mathcal{K}}(s) = \left\{ \frac{(n - 1)!}{(n - 1)!} \right\}^2 = \frac{n}{(n - 1)!}.$$

### 5.2.2 Calculation of $\xi_{n,1,\mathcal{K}}(s)$

We recall

$$\xi_{n,1,\mathcal{K}}(s) := \prod_{j=0}^{n-1} \frac{\xi(n + 1 - 2s - j; \chi^j_{\mathcal{K}})}{\xi(2s + n - 1 - j; \chi^j_{\mathcal{K}})}.$$  

We decompose the right-hand side as

$$\xi_{n,1,\mathcal{K}}(s) = \rho_{\mathcal{K}}(s) \xi(2 - 2s)$$

$$\rho_{\mathcal{K}}(s) := \frac{\prod_{j=0}^{n-2} \xi(n + 1 - 2s - j; \chi^j_{\mathcal{K}})}{\prod_{j=0}^{n-1} \xi(2s + n - 1 - j; \chi^j_{\mathcal{K}})}.$$  

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As in the case of general \( m \), \( \rho_{\mathbf{K}}(s) \) is holomorphic at \( s = 1 \) and \( \zeta(2 - 2s) \) has a simple pole \( s = 1 \). Namely

\[
\Res_{s=1} \xi_{n,1,\mathbf{K}}(s) = \rho_{\mathbf{K}}(1) \cdot \Res_{s=1} \xi(2 - 2s).
\]

First we note that \( \Res_{s=1} \xi(2 - 2s) = 1/2 \). Next we calculate \( \rho_{\mathbf{K}}(1) \). By the definition, we can write

\[
\rho_{\mathbf{K}}(1) = \frac{\prod_{j=0}^{n-2} \xi(n - 1 - j; \chi_j^\mathbf{K})}{\prod_{j=0}^{n-1} \xi(n + 1 - j; \chi_j^\mathbf{K})} = \frac{\xi(1; \chi_K)}{\xi(n + 1) \xi(n; \chi_K)}.
\]

We note the following formulas:

\[
\xi(1; \chi_K) = -\pi^{1/2} B_{1, \chi_K},
\]

\[
\xi(n; \chi_K) = (-1)^{n+1} D_{\mathbf{K}}^{1/2} 2^{n-1} \pi^{n/2} (\frac{n-1}{2})! \frac{B_{n, \chi_K}}{n!}.
\]

\[
\xi(n + 1) = (-1)^{n+1} 2^n \pi^{(n+1)/2} (\frac{n-1}{2})! \frac{B_{n+1}}{(n+1)!}.
\]

Then we have

\[
\rho_{\mathbf{K}}(1) = \pi^{-n} \cdot D_{\mathbf{K}}^{1/2} \cdot 2^{1-2n} \cdot \frac{n! \cdot (n + 1)!}{(\frac{n-1}{2})!} \frac{B_{1, \chi_K}}{B_{n, \chi_K} \cdot B_{n+1}}.
\]

Consequently we obtain

\[
\Res_{s=1} G_{n,1,\mathbf{K}}(s) = \gamma_{n,1,\mathbf{K}}(1) \Res_{s=1} \xi_{n,1,\mathbf{K}}(s)
\]

\[
= \gamma_{n,1,\mathbf{K}}(1) \cdot \rho_{\mathbf{K}}(1) \cdot \Res_{s=1} \xi(2 - 2s)
\]

\[
= \pi^{-n} \cdot 2^{-2n} \cdot \frac{D_{\mathbf{K}}^{1/2}}{\pi} \cdot \frac{n! \cdot (n + 1)!}{(\frac{n-1}{2})!} \frac{B_{1, \chi_K}}{B_{n, \chi_K} \cdot B_{n+1}}.
\]

This completes the proof of Theorem 5.3.

Table 2: Numerical examples of \( c_{n,\mathbf{K}} \)

| \( n \) | \( c_{n,\mathbf{Q}(\sqrt{-1})} \) | \( c_{n,\mathbf{Q}(\sqrt{-3})} \) |
|---|---|---|
| 3 | 45 | 405/8 |
| 5 | 189/2 | 8505/64 |
| 7 | 4725/61 | 18225/128 |
| 9 | 18711/554 | 68201595/828416 |

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