Inf-convolution of G-expectations

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Abstract

In this paper we will discuss the optimal risk transfer problems when risk measures are generated by G-expectations, and we present the relationship between inf-convolution of G-expectations and the inf-convolution of drivers G.

Keywords: inf-convolution, G-expectation, G-normal distribution, G-Brownian motion

1 Introduction

Coherent risk measures were introduced by Artzner et al. [1] in finite probability spaces and lately by Delbaen [8,9] in general probability spaces. The family of coherent risk measures was extended later by Föllmer and Schied [10,11] and, independently, by Frittelli and Rosazza Gianin [12,13] to the class of convex risk measures.

The notion of g-expectations was introduced by Peng [15] as solutions to a class of nonlinear Backward Stochastic Differential Equations (BSDE in short) which were first studied by Pardoux and Peng [14]. Financial applications were discussed in detail by El Karoui et al. [6].

Let us introduce the optimal risk transfer model we are concerned with. This model can be briefly described as follows:

Two economic agents A and B are considered, who assess the risk associated with their respective positions by risk measures $\rho_A$ and $\rho_B$. The issuer, agent A, with the total risk capital X, wants to issue a financial product F and sell it to agent B for the price $\pi$ in order to reduce his risk
exposure. His objective is to minimize $\rho_A(X - F + \pi)$ with respect to $F$ and $\pi$, while the interest of buyer $B$ is not to be exposed to a greater risk after the transaction:

$$\rho_B(F - \pi) \leq \rho_B(0).$$

Using the cash translation invariance property, this optimization problem can be rewritten in the simpler form

$$\inf_F \{\rho_A(X - F) + \rho_B(F)\}.$$

This problem was first studied by El Karoui and Pauline Barrieu [2,3,4] for convex risk measures, in particular those described by g-expectation.

Related with the pioneering paper [1] on coherent risk measures, sublinear expectations (or, more generally, convex expectations, see [10,11,13]) have become more and more popular for modeling such risk measures. Indeed, in any sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ a coherent risk measure $\rho$ can be defined in a simple way by putting $\rho(X) := \mathbb{E}[-X]$, for $X \in \mathcal{H}$.

The notion of a sublinear expectation named G-expectation was first introduced by Peng [17,18] in 2006. Compared with g-expectations, the theory of G-expectation is intrinsic in the sense that it is not based on a given (linear) probability space. A G-expectation is a fully nonlinear expectation. It characterizes the variance uncertainty of a random variable. We recall that the problem of mean uncertainty has been studied by Chen-Epstein through g-expectation in [5]. Under this fully nonlinear G-expectation, a new type of Itô’s formula has been obtained, and the existence and uniqueness for stochastic differential equation driven by a G-Brownian motion have been shown. For a more detailed description the reader is referred to Peng’s recent papers [17,18,19].

This paper focuses on the mentioned optimization problem where the g-risk measures are replaced by one dimensional G-expectations, i.e., the problem:

$$\hat{\mathbb{E}}_{G_1} \square \hat{\mathbb{E}}_{G_2}[X] := \inf_F \{\hat{\mathbb{E}}_{G_1}[X - F] + \hat{\mathbb{E}}_{G_2}[F]\}.$$

The main aim of this paper is to present the relationship between the above introduced operator $\hat{\mathbb{E}}_{G_1} \square \hat{\mathbb{E}}_{G_2}[\cdot]$ and the G-expectation $\hat{\mathbb{E}}_{G_1 \square G_2}[\cdot]$. More precisely, we show that both operators coincide if $G_1 \square G_2 \neq -\infty$.

In this paper we constrain ourselves to one dimensional G-expectation, the multi-dimensional case is much more complicated and we hope to study this case in a forthcoming publication.

Our approach is mainly based on the recent results by Peng [19] which allow to show that $\hat{\mathbb{E}}_{G_1} \square \hat{\mathbb{E}}_{G_2}[\cdot]$ constructed by inf-convolution of $\hat{\mathbb{E}}_{G_1}[\cdot]$ and $\hat{\mathbb{E}}_{G_2}[\cdot]$ satisfies the properties of G-expectation. To our best knowledge, this is the first paper that uses the results of Theorem 4.1.3 of [19] to prove that a given nonlinear expectation is a G-expectation.
This paper is organized as follows: while basic definitions and properties of G-expectation and G-Brownian Motion are recalled in Section 2, Section 3 states and proves the main result of this paper: If \( G_1 \square G_2 \neq -\infty \), then 
\[
\hat{\mathbb{E}}_{G_1} \square \hat{\mathbb{E}}_{G_2}[\cdot] = \hat{\mathbb{E}}_{G_1 \square G_2}[\cdot].
\]

## 2 Notation and Preliminaries

The aim of this section is to recall some basic definitions and properties of G-expectations and G-Brownian motions, which will be needed in the sequel. The reader interested in a more detailed description of these notions is referred to Peng’s recent papers [17,18,19].

Adapting Peng’s approach in [19], we let \( \Omega \) be a given nonempty fundamental space and \( \mathcal{H} \) be a linear space of real functions defined on \( \Omega \) such that:

1. \( 1 \in \mathcal{H} \).
2. \( \mathcal{H} \) is stable with respect to local Lipschitz functions, i.e. for all \( n \geq 1 \), and for all \( X_1, ..., X_n \in \mathcal{H}, \varphi \in C_{l,lip}(\mathbb{R}^n) \), it holds also \( \varphi(X_1, ..., X_n) \in \mathcal{H} \).

Recall that \( C_{l,lip}(\mathbb{R}^n) \) denotes the space of all local Lipschitz functions \( \varphi \) over \( \mathbb{R}^n \) satisfying

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad x, y \in \mathbb{R}^n,
\]

for some \( C > 0, m \in \mathbb{N} \) depending on \( \varphi \). The set \( \mathcal{H} \) is interpreted as the space of random variables defined on \( \Omega \).

**Definition 2.1** A sublinear expectation \( \hat{\mathbb{E}} \) on \( \mathcal{H} \) is a functional \( \mathcal{H} \to \mathbb{R} \) with the following properties: for all \( X, Y \in \mathcal{H} \), we have

1. **Monotonicity**: if \( X \geq Y \) then \( \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y] \).
2. **Preservation of constants**: \( \hat{\mathbb{E}}[c] = c \), for all reals \( c \).
3. **Sub-additivity (or property of self-dominacy)**:

\[
\hat{\mathbb{E}}[X] - \hat{\mathbb{E}}[Y] \leq \hat{\mathbb{E}}[X - Y].
\]

4. **Positive homogeneity**: \( \hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X], \forall \lambda \geq 0 \).

The triple \( (\Omega, \mathcal{H}, \hat{\mathbb{E}}) \) is called a sublinear expectation space. It generalizes the classical case of the linear expectation \( E[X] = \int_{\Omega} XdP \), \( X \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \), over a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). Moreover, \( \rho(X) = \hat{\mathbb{E}}[-X] \) defines a coherent risk measure on \( \mathcal{H} \).
**Definition 2.2** For arbitrary $n, m \geq 1$, a random vector $Y = (Y_1, Y_2, ..., Y_n) \in \mathcal{H}^n (= \mathcal{H} \times \mathcal{H} \times ... \times \mathcal{H})$ is said to be independent of $X \in \mathcal{H}^m$ under $\hat{E}[]$ if for each test function $\varphi \in C_{l, lip}(\mathbb{R}^{n+m})$ we have

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]|x = x].$$

**Remark:** In the case of linear expectation, this notion of independence is just the classical one. It is important to note that under sublinear expectations the condition $Y$ is independent to $X$ does not imply automatically that $X$ is independent to $Y$.

Let $X = (X_1, ..., X_n) \in \mathcal{H}^n$ be a given random vector. We define a functional on $C_{l, lip}(\mathbb{R}^n)$ by

$$\hat{F}_X[\varphi] := \hat{E}[\varphi(X)], \varphi \in C_{l, lip}(\mathbb{R}^n).$$

It’s easy to check that $\hat{F}_X[\cdot]$ is a sublinear expectation defined on $(\mathbb{R}^n, C_{l, lip}(\mathbb{R}^n))$.

**Definition 2.3** Given two sublinear expectation spaces $(\Omega, \mathcal{H}, \hat{E})$ and $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$, two random vectors $X \in \mathcal{H}^n$ and $Y \in \tilde{\mathcal{H}}^n$ are said to be identically distributed if for each test function $\varphi \in C_{l, lip}(\mathbb{R}^n)$

$$\hat{F}_X[\varphi] = \tilde{F}_Y[\varphi].$$

We now introduce the important notion of G-normal distribution. For this, let $0 \leq \underline{\sigma} \leq \overline{\sigma} \in \mathbb{R}$, and let $G$ be the sublinear function:

$$G(\alpha) = \frac{1}{2}(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-), \alpha \in \mathbb{R}.$$

As usual $\alpha^+ = max\{0, \alpha\}$ and $\alpha^- = (-\alpha)^+$. Given an arbitrary initial condition $\varphi \in C_{l, lip}(\mathbb{R})$, we denote by $u_\varphi$ the unique viscosity solution of the following parabolic partial differential equation (PDE):

$$\partial_t u_\varphi(t, x) = G(\partial^2_{xx} u_\varphi(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R},$$

$$u_\varphi(0, x) = \varphi(x), \quad x \in \mathbb{R}.$$

**Definition 2.4:** A random variable $X$ in a sub-expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G_{\underline{\sigma}, \overline{\sigma}}$-normal distributed, and we write $X \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, if for all $\varphi \in C_{l, lip}(\mathbb{R})$,

$$\hat{E}[\varphi(x + \sqrt{7}X)] := u_\varphi(t, x), \quad (t, x) \in [0, \infty) \times \mathbb{R}.$$
Remark: From [18], we have the following Kolmogrov-Chapman chain rule:
\[ u_\phi(t + s, x) = \hat{E}[u_\phi(t, x + \sqrt{s}X)], \quad s \geq 0. \]

In what follows we will take as fundamental space \( \Omega \) the space \( C_0(\mathbb{R}^+ \cup \{0\}) \) of all real-valued continuous functions \( (\omega_t)_{t \in \mathbb{R}^+} \) with \( \omega_0 = 0 \), equipped with the topology generated by the uniform convergence on compacts.

For each fixed \( T \geq 0 \), we consider the following space of local Lipschitz functionals:
\[
H_T = \text{Lip}(\mathcal{F}_T) := \left\{ X(\omega) = \phi(\omega_t_1, ..., \omega_{t_m}), t_1, ..., t_m \in [0, T], \phi \in C_{l, lip}(\mathbb{R}^m), m \geq 1 \right\}.
\]

Furthermore, for \( 0 \leq s \leq t \), we define
\[
H^s_t = \text{Lip}(\mathcal{F}^s_t) := \left\{ X(\omega) = \phi(\omega_t - \omega_{t_1}, ..., \omega_{t_{m+1}} - \omega_{t_m}), t_1, ..., t_{m+1} \in [s, t], \phi \in C_{l, lip}(\mathbb{R}^m), m \geq 1 \right\}.
\]

It is clear that \( H^s_t \subseteq H_t \subseteq \text{Lip}(\mathcal{F}_T) \), for \( s \leq t \leq T \). We also introduce the space
\[
H = \text{Lip}(\mathcal{F}) := \bigcup_{n=1}^{\infty} \text{Lip}(\mathcal{F}_n).
\]

Obviously, \( \text{Lip}(\mathcal{F}^s_t) \), \( \text{Lip}(\mathcal{F}_T) \) and \( \text{Lip}(\mathcal{F}) \) are vector lattices.

We will consider the canonical space and set
\[
B_t(\omega) = \omega_t, t \in [0, \infty), \quad \text{for } \omega \in \Omega.
\]

Obviously, for each \( t \in [0, \infty) \), \( B_t \in \text{Lip}(\mathcal{F}_t) \). Let \( G(a) = G_{\varphi}(\sigma(a)) = \frac{1}{2}(\varphi^2 a^+ - \varphi^2 a^-), a \in \mathbb{R} \). We now introduce a sublinear expectation \( \mathbb{E} \) defined on \( H_T = \text{Lip}(\mathcal{F}_T) \), as well as on \( H = \text{Lip}(\mathcal{F}) \), via the following procedure: For each \( X \in H_T \) with
\[
X = \phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}}),
\]

and for all \( \phi \in C_{l, lip}(\mathbb{R}^m) \) and \( 0 = t_0 \leq t_1 < ... < t_m \leq T, \ m \geq 1 \), we set
\[
\hat{E}[\phi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})] = \mathbb{E}[\phi(\sqrt{t_1 - t_0} \xi_1, ..., \sqrt{t_m - t_{m-1}} \xi_m)],
\]

where \( (\xi_1, ..., \xi_m) \) is an \( m \)-dimensional random vector in some sublinear expectation space \( (\hat{\Omega}, \hat{H}, \hat{E}) \), such that \( \xi_i \sim N(0; [\varphi^2, \sigma^2]) \) and \( \xi_{i+1} \) is independent of \( (\xi_1, ..., \xi_i) \), for all \( i = 1, ..., m-1 \), \( m \in \mathbb{N} \). The related conditional
expectation of $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})$ under $\mathcal{H}_{t_j}$ is defined by

$$\hat{E}[X|\mathcal{H}_{t_j}] = \hat{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, ..., B_{t_m} - B_{t_{m-1}})|\mathcal{H}_{t_j}]$$

$$= \psi(B_{t_1} - B_{t_0}, ..., B_{t_j} - B_{t_{j-1}})$$

where

$$\psi(x_1, ..., x_j) = \hat{E}[\varphi(x_1, ..., x_j, \sqrt{t_{j+1} - t_j \xi_{j+1}}, ..., \sqrt{t_m - t_{m-1} \xi_m})].$$

We know from [18,19] that $\hat{E}[\cdot]$ defines consistently a sublinear expectation on $\text{Lip}(\mathcal{F})$, satisfying (a)-(d) in Definition 2.1. The reader interested in a more detailed discussion is referred to [18,19].

**Definition 2.5** The expectation $\hat{E}[\cdot] : \text{Lip}(\mathcal{F}) \to \mathbb{R}$ defined through the above procedure is called $G_{\mathcal{F}_T}$-expectation. The corresponding canonical process $(B_t)_{t \geq 0}$ in the sublinear expectation is called a $G_{\mathcal{F}_T}$-Brownian motion on $(\Omega, \mathcal{H}, \hat{E})$.

At the end of this section we list some useful properties that we will need in Section 3.

**Proposition 2.6 ([18,19])** The following properties of $\hat{E}[\cdot|\mathcal{H}_t]$ hold for all $X, Y \in \mathcal{H} = \text{Lip}(\mathcal{F})$:

(a') If $X \geq Y$, then $\hat{E}[X|\mathcal{H}_t] \geq \hat{E}[Y|\mathcal{H}_t]$.

(b') $\hat{E}[\eta|\mathcal{H}_t] = \eta$, for each $t \in [0, \infty)$ and $\eta \in \mathcal{H}_t$.

(c') $\hat{E}[X|\mathcal{H}_t] - \hat{E}[Y|\mathcal{H}_t] \leq \hat{E}[X - Y|\mathcal{H}_t]$.

(d') $\hat{E}[\eta X|\mathcal{H}_t] = \eta^+ \hat{E}[X|\mathcal{H}_t] + \eta^- \hat{E}[-X|\mathcal{H}_t]$, for each $\eta \in \mathcal{H}_t$.

We also have

$$\hat{E}[\hat{E}[X|\mathcal{H}_t]|\mathcal{H}_a] = \hat{E}[X|\mathcal{H}_{t_\wedge a}],$$

and in particular, $\hat{E}[\hat{E}[X|\mathcal{H}_t]] = \hat{E}[X]$.

For each $X \in \text{Lip}(\mathcal{F}_T)$, $\hat{E}[X|\mathcal{H}_t] = \hat{E}[X]$, moreover, the properties (b') and (c') imply: $\hat{E}[X + \eta|\mathcal{H}_t] = \hat{E}[X|\mathcal{H}_t] + \eta$, whenever $\eta \in \mathcal{H}_t$.

We will need also the following two propositions, and for proofs the reader is referred to [18,19].

**Proposition 2.7** For each convex function $\varphi$ and each concave function $\psi$ with $\varphi(B_t)$ and $\psi(B_t) \in \mathcal{H}_t$, we have $\hat{E}[\varphi(B_t)] = \hat{E}[\varphi(\sigma W_t)]$ and $\hat{E}[\psi(B_t)] = \hat{E}[\psi(\sigma W_t)]$, where $(W_t)_{t \geq 0}$ is a Brownian motion under the linear expectation $\hat{E}$.

**Proposition 2.8** Let $\hat{E}_1[\cdot]$ and $\hat{E}_2[\cdot]$ be a $G_{\mathcal{F}_T}$- and a $G_{\mathcal{F}_T}$-expectation on the space $(\Omega, \mathcal{H})$, respectively. Then, if $[\mathcal{F}_1, \mathcal{F}_2] \subseteq [\mathcal{F}_2, \mathcal{F}_2]$, we have $\hat{E}_1[X] \leq \hat{E}_2[X]$ and $\hat{E}_1[X|\mathcal{H}_t] \leq \hat{E}_2[X|\mathcal{H}_t]$, for all $X \in \mathcal{H}$ and all $t \geq 0$. 
3 Inf-convolution of G-expectations

The aim of this section is to state the main result of this paper, that is the relationship between the inf-convolution $\hat{E}_{G_1} \square \hat{E}_{G_2}[:], 1$ and the G-expectation $\hat{E}_{G_1 \square G_2}[:]$. We begin with the definitions necessary for the understanding of these both expressions.

For given $0 \leq \sigma_i, \sigma_i \in \mathbb{R}, i=1,2$, let $G_i = G_{\sigma_i, \sigma_i}$ and we denote by $\hat{E}_i[:]$ the $G_i$-expectation $\hat{E}_{G_i}[:](\Omega, \mathcal{H}) (= (C_0(\mathbb{R}^+), Lip(\mathcal{F})))$. The inf-convolution of $\hat{E}_1[:]$ with $\hat{E}_2[:],$ denoted by $\hat{E}_1 \square \hat{E}_2[:]$ is defined as:

$$\hat{E}_1 \square \hat{E}_2[X] = \inf_{F \in \mathcal{H}} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \}, \quad X \in \mathcal{H}.$$  

Notice that $\hat{E}_1 \square \hat{E}_2[:]: \mathcal{H} \to \mathbb{R} \cup \{-\infty\}.$

In the same way we define

$$G_1 \square G_2(x) = \inf_{y \in \mathbb{R}} \{ G_1(x - y) + G_2(y) \}, \quad x \in \mathbb{R}.$$  

Observe also that $G_1 \square G_2(\cdot): \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. It is easy to check that $G_1 \square G_2(\cdot)$ has the following form:

$$G_1 \square G_2(x) = \begin{cases} 
-\infty, & \text{if } \sigma_1 \leq \sigma_2, \\
\frac{1}{2}(\sigma_2^+ - \sigma_1^-), & \text{if } \sigma_1 \cap [\sigma_2, \sigma_2] = \emptyset; \\
[\sigma_1, \sigma_1] \cap [\sigma_2, \sigma_2] = \emptyset, & \text{if } [\sigma_1, \sigma_1] \cap [\sigma_2, \sigma_2] = [\sigma_1, \sigma_2]. \end{cases}$$

If $G_1 \square G_2(\cdot) = -\infty$, then also $\hat{E}_1 \square \hat{E}_2[:] = -\infty$. More precisely, we have the following proposition:

**Proposition 3.1** If $[\sigma_1, \sigma_1] \cap [\sigma_2, \sigma_2] = \emptyset$, then $\hat{E}_1 \square \hat{E}_2[X] = -\infty$, for all $X \in \mathcal{H}$.

**Proof:** Without loss of generality we may suppose $\sigma_1 < \sigma_2$. Choosing $F = -\lambda B_t^2, \lambda > 0, t > 0$, we then have due to Proposition 2.7 that for all $X \in \mathcal{H}$,

$$\hat{E}_1[X - F] + \hat{E}_2[F] = \hat{E}_1[X + \lambda B_t^2] + \hat{E}_2[-\lambda B_t^2] \leq \hat{E}_1[X] + \hat{E}_1[\lambda B_t^2] + \hat{E}_2[-\lambda B_t^2] \leq \hat{E}_1[X] + \lambda \sigma^2 t - \lambda \sigma^2 t.$$  

Letting $\lambda \to \infty$, we obtain $\hat{E}_1 \square \hat{E}_2[X] = -\infty.$  

If $[\sigma_1, \sigma_1] \cap [\sigma_2, \sigma_2]$ is not empty we have the following theorem, which is the main result of this paper.

**Theorem 3.2** Let $\hat{E}_1[:]$ and $\hat{E}_2[:]$ be the two G-expectations on the space $(\Omega, \mathcal{H})$, which have been defined above. If $G_1 \square G_2(\cdot) \neq -\infty$, then
Let us first discuss Theorem 3.2 in the special case.

**Lemma 3.3** Let \([\sigma_1, \bar{\sigma}_1] \subseteq [\sigma_2, \bar{\sigma}_2]\). Then \(G_1 \Box G_2(\cdot) = G_1(\cdot)\), as well as \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\cdot] = \hat{\mathbb{E}}_1[\cdot]\).

**Proof:** We already know that \(G_1 \Box G_2(\cdot) = G_1(\cdot)\), so it remains only to prove that \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\cdot] = \hat{\mathbb{E}}_1[\cdot]\). For this we note that, firstly, by choosing \(F = 0\) in the definition of \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2\), we get \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2 \leq \hat{\mathbb{E}}_1, \ i = 1, 2\).

On the other hand, due to Proposition 2.8 we know that \(\hat{\mathbb{E}}_1 \leq \hat{\mathbb{E}}_2\). Thus, from the subadditivity of \(\hat{\mathbb{E}}_1[\cdot],\)

\[
\hat{\mathbb{E}}_1[X - F] + \hat{\mathbb{E}}_2[F] \geq \hat{\mathbb{E}}_1[X - F] + \hat{\mathbb{E}}_1[F] \geq \hat{\mathbb{E}}_1[X], \ F \in \mathcal{H}.
\]

Consequently, \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\cdot] = \hat{\mathbb{E}}_1[\cdot]\). Thus, Theorem 3.2 holds true in this special case.

The case \([\sigma_1, \bar{\sigma}_1] \supseteq [\sigma_2, \bar{\sigma}_2]\) can be treated analogously. □

The situation becomes more complicated if neither \([\sigma_1, \bar{\sigma}_1] \subseteq [\sigma_2, \bar{\sigma}_2]\) nor \([\sigma_2, \bar{\sigma}_2] \subseteq [\sigma_1, \bar{\sigma}_1]\). Without loss of generality, we suppose that \([\sigma_1, \bar{\sigma}_1] \cap [\sigma_2, \bar{\sigma}_2] = [\sigma_2, \bar{\sigma}_1]\). In this case

\[
G_1 \Box G_2(x) = \frac{1}{2}(\bar{\sigma}_1^2 x^+ - \sigma_2^2 x^-) = G_3(x), \ x \in \mathbb{R},
\]

where \(G_3 = G_{\sigma_2, \bar{\sigma}_1}\). By \(\hat{\mathbb{E}}_3[\cdot]\) we denote the G-expectation on \((\Omega, \mathcal{H})\) with driver \(G_3(\cdot)\). The above notations will be kept for the rest of the paper. Our aim is to prove that \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\cdot] = \hat{\mathbb{E}}_3[\cdot]\).

The proof is based on Theorem 4.1.3 in Peng’s paper [19]; this theorem characterizes the intrinsic properties of G-Brownian motions and G-expectations.

**Lemma 3.4 (see Theorem 4.1.3, Peng [19])** Let \((\bar{B}_t)_{t \geq 0}\) be a process defined in the sub-expectation space \((\Omega, \mathcal{H}, \tilde{\mathbb{E}})\) such that

(i) \(\bar{B}_0 = 0\);
(ii) For each \(t, s \geq 0\), the increment \(\bar{B}_{t+s} - \bar{B}_t\) has the same distribution as \(\bar{B}_s\) and is independent of \((\bar{B}_{t_1}, \bar{B}_{t_2}, \ldots, \bar{B}_{t_n})\), for all \(0 \leq t_1, \ldots, t_n \leq t, n \geq 1\).
(iii) \(\tilde{\mathbb{E}}[\bar{B}_t] = \tilde{\mathbb{E}}[-\bar{B}_t] = 0\), and \(\lim_{t \downarrow 0} \tilde{\mathbb{E}}(|\bar{B}_t|^3)t^{-1} = 0\).

Then \((\bar{B}_t)_{t \geq 0}\) is a \(G_{\sigma, \bar{\sigma}}\)-Brownian motion with \(\bar{\sigma}^2 = \tilde{\mathbb{E}}[\bar{B}_1^2]\) and \(\sigma^2 = -\tilde{\mathbb{E}}[-\bar{B}_1^2]\).

In the sequel, in order to prove Theorem 3.2 we will show that the inf-convolution \(\hat{\mathbb{E}}_1 \Box \hat{\mathbb{E}}_2[\cdot]\) is a sublinear expectation on \((\Omega, \mathcal{H})\). This will make
Lemma 3.4 applicable. More precisely, we will show that under the sublinear expectation \( \hat{E}_1 \square \hat{E}_2[\cdot] \) the canonical process \((B_t)_{t \geq 0}\) satisfies the assumptions of Lemma 3.4 for \( \sigma = \sigma_1, \sigma = \sigma_2 \). This has as consequence that \((B_t)_{t \geq 0}\) is a \( G_{\sigma_1, \sigma_2} \)-Brownian motion under \( \hat{E}_1 \square \hat{E}_2[\cdot] \), and implies that \( \hat{E}_1 \square \hat{E}_2[\cdot] = \hat{E}_3[\cdot] \).

**Proposition 3.5** Under the assumption \( [\sigma_1, \sigma_1] \cap [\sigma_2, \sigma_2] = [\sigma_2, \sigma_1] \), the inf-convolution \( \hat{E}_1 \square \hat{E}_2[\cdot] \) is a sublinear expectation on \((\Omega, \mathcal{H})\).

**Proof:** (a) Monotonicity: The monotonicity is an immediate consequence of that of the G-expectation \( \hat{E}_1[\cdot] \).

(b) Preservation of constants: From the preservation of constants property and the subadditivity of \( \hat{E}_1[\cdot] \), we have

\[
\hat{E}_1 \square \hat{E}_2[c] = \inf_{F \in \mathcal{H}} \{ \hat{E}_1[c - F] + \hat{E}_2[F] \} = c + \inf_{F \in \mathcal{H}} \{ \hat{E}_1[-F] + \hat{E}_2[F] \} \geq c + \inf_{F \in \mathcal{H}} \{ \hat{E}_3[-F] + \hat{E}_3[F] \} \geq c.
\]

The latter lines follow from the fact that \( \hat{E}_3 \leq \hat{E}_i, i = 1, 2 \), and the subadditivity of \( \hat{E}_3 \). Moreover, by taking \( F=0 \) in the definition of \( \hat{E}_1 \square \hat{E}_2[c] \) we get the converse inequality.

(c) Sub-additivity: Given arbitrary fixed \( X,Y \in \mathcal{H} \), in virtue of the subadditivity of \( \hat{E}_1[\cdot] \) and \( \hat{E}_2[\cdot] \), we have for all \( F_1, F_2 \in \mathcal{H} \)

\[
\hat{E}_1[X - Y - F_1] + \hat{E}_2[F_1] + \hat{E}_1[Y - F_2] + \hat{E}_2[F_2] \geq \hat{E}_1[X - (F_1 + F_2)] + \hat{E}_2[F_1 + F_2].
\]

Consequently,

\[
\hat{E}_1 \square \hat{E}_2[X - Y] + \hat{E}_1 \square \hat{E}_2[Y] = \inf_{F_1, F_2 \in \mathcal{H}} \{ \hat{E}_1[X - Y - F_1] + \hat{E}_2[F_1] + \hat{E}_1[Y - F_2] + \hat{E}_2[F_2] \} \geq \inf_{F_1, F_2 \in \mathcal{H}} \{ \hat{E}_1[X - F_1 - F_2] + \hat{E}_2[F_1 + F_2] \} = \hat{E}_1 \square \hat{E}_2[X].
\]

(d) Finally, the positive homogeneity is an easy consequence of that of \( \hat{E}_1[\cdot] \) and \( \hat{E}_2[\cdot] \).

The following series of statements has as objective to prove that the canonical process \((B_t)_{t \geq 0}\) satisfies under the sublinear expectation \( \hat{E}_1 \square \hat{E}_2[\cdot] \) the
Lemma 3.6: Let \( \varphi \) be a convex or concave function such that \( \varphi(B_t) \in \mathcal{H} \), then \( \hat{E}_1 \square \hat{E}_2[\varphi(B_t)] = \hat{E}_3[\varphi(B_t)] \).

Proof: We only prove the convex case, the proof for concave \( \varphi \) is analogous. If \( \varphi \) is convex we have according to Proposition 2.7,

\[
\hat{E}_3[\varphi(B_t)] = \mathbb{E}[^{\varphi}(\sigma_1 W_t)] = \hat{E}_1[\varphi(B_t)].
\]

By Proposition 2.8 we know that \( \hat{E}_i[\cdot] \geq \hat{E}_4[\cdot], i = 1, 2 \), and consequently, also \( \hat{E}_1 \square \hat{E}_2[\cdot] \geq \hat{E}_3[\cdot] \). On the other hand, since obviously, \( \hat{E}_1 \square \hat{E}_2[\cdot] \leq \hat{E}_1[\cdot] \), we get, for convex functions \( \varphi \), \( \hat{E}_1 \square \hat{E}_2[\varphi(B_t)] = \hat{E}_3[\varphi(B_t)] \). Similarly we can prove the concave case. \( \blacksquare \)

Remark: From Proposition 3.5 we know already that \( \hat{E}_1 \square \hat{E}_2[\cdot] \) is a sub-linear expectation. This implies \( \hat{E}_1 \square \hat{E}_2[0] = 0 \). From Lemma 3.6, we have that \( F^* = 0 \) is an optimal control when \( \varphi \) is convex, while the optimal control is \( F^* = \varphi(B_t) \) when \( \varphi \) is concave. Moreover,

\[
\hat{E}_1 \square \hat{E}_2[-B_t] = \hat{E}_1 \square \hat{E}_2[B_t] = 0 \\
\hat{E}_1 \square \hat{E}_2[B_t^2] = \sigma_1^2 t, \; \hat{E}_1 \square \hat{E}_2[-B_t^2] = -\sigma_1^2 t.
\]

Lemma 3.7: We have \( \hat{E}_1 \square \hat{E}_2[|B_t|^3] \to 0 \), as \( t \to 0 \).

Proof: Since \( \varphi(x) = |x|^3 \) is convex, we obtain due to Lemma 3.6 that:

\[
\hat{E}_1 \square \hat{E}_2[|B_t|^3] = \hat{E}_3[|B_t|^3] = \sigma_1^3 \mathbb{E}[|W_1|^3] t^{3/2},
\]

where \( (W_t)_{t \geq 0} \) is Brownian motion under the linear expectation \( \mathbb{E} \). The statement follows now easily.

Proposition 3.8: We have

\[
\hat{E}_1 \square \hat{E}_2[\varphi(B_t - B_s)] = \hat{E}_1 \square \hat{E}_2[\varphi(B_{t-s})], \; t \geq s \geq 0, \varphi \in C_{l,\text{lip}}(\mathbb{R}).
\]

The proof of Proposition 3.8 is rather technical. To improve the readability of the paper, the proof is postponed to the annex.

Lemma 3.9: For each \( t \geq s \), \( B_t - B_s \) is independent of \( (B_{t_1}, B_{t_2}, \ldots, B_{t_n}) \) under the sub-linear expectation \( \hat{E}_1 \square \hat{E}_2[\cdot] \), for each \( n \in \mathbb{N}, 0 \leq t_1, \ldots, t_n \leq s \), that is, for all \( \varphi \in C_{l,\text{lip}}(\mathbb{R}^{n+1}) \)

\[
\hat{E}_1 \square \hat{E}_2[\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}, B_t - B_s)] \\
= \hat{E}_1 \square \hat{E}_2[\hat{E}_1 \square \hat{E}_2[\varphi(x_1, \ldots, x_n, B_t - B_s)|_{(x_1, \ldots, x_n)=(B_{t_1}, \ldots, B_{t_n})}]].
\]

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We shift also the proof of Lemma 3.9 to the annex.

We are now able to give the proof of Theorem 3.2:

**Proof (of Theorem 3.2):** It is sufficient to apply Lemma 3.4. Due to the above statements, we know that the canonical process \((B_t)_{t \geq 0}\) is a G-Brownian motion under the sublinear expectation \(\hat{E}_1 \square \hat{E}_2[\cdot]\). Consequently \(\hat{E}_1 \square \hat{E}_2[\cdot]\) is a G-expectation on the space \((\Omega, \mathcal{H})\) and has the driver \(G_1 \square G_2 = G_{\sigma_1, \sigma_1} \).

Given \(n\) sublinear expectations \(\hat{E}_1, \ldots, \hat{E}_n\) we define iteratively

\[ \hat{E}_1 \square \hat{E}_2 \square \hat{E}_3 := (\hat{E}_1 \square \hat{E}_2) \square \hat{E}_3, \]

and

\[ \hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_k := (\hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_{k-1}) \square \hat{E}_k, \quad 3 \leq k \leq n. \]

Then from Theorem 3.2 it follows:

**Corollary 3.10:** Let \(0 \leq \sigma_i \leq \overline{\sigma}_i, \quad 1 \leq i \leq n\), and denote by \(\hat{E}_i[\cdot]\) the \(G_{\sigma_i, \sigma_i}\)-expectation on the space \((\Omega, \mathcal{H})\). Then under the assumption

\[ \bigcap_{i=1}^n [\sigma_i, \overline{\sigma}_i] \neq \emptyset, \quad \hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_n[\cdot] \]

also is a G-expectation and has the driver \(G_{\sigma_1, \sigma_1} \square G_{\sigma_2, \sigma_2} \square \ldots \square G_{\sigma_n, \sigma_n}\). Moreover, for any permutation \(i_1, \ldots, i_n\) of the natural numbers \(1, \ldots, n\) it holds:

\[ \hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_n[\cdot] = \hat{E}_{i_1} \square \hat{E}_{i_2} \square \ldots \square \hat{E}_{i_n}[\cdot]. \]

**Remark:** If \(\bigcap_{i=1}^n [\sigma_i, \overline{\sigma}_i]\) is empty, then \(\hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_n[\cdot] = -\infty\), otherwise \(\hat{E}_1 \square \hat{E}_2 \square \ldots \square \hat{E}_n[\cdot]\) is a \(G_{\sigma, \sigma}\)-expectation, where \([\sigma, \overline{\sigma}] = \bigcap_{i=1}^n [\sigma_i, \overline{\sigma}_i]\).

4  **Annex**

4.1 **Proof of Proposition 3.8**

We begin with the proof of Proposition 3.8. For this we need the following two lemmas.

**Lemma 4.1:** For all \(T > 0\) and all \(X \in \mathcal{H}_T\), we have

\[ \inf_{F \in \mathcal{H}_T} \{\hat{E}_1[X - F] + \hat{E}_2[F]\} = \inf_{F \in \mathcal{H}_T} \{\hat{E}_1[X - F] + \hat{E}_2[F]\}. \]
Proof: From $\mathcal{H}_T \subseteq \mathcal{H}$ we see that
\[
\inf_{F \in \mathcal{H}_T} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \} \geq \inf_{F \in \mathcal{H}} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \}.
\]
Thus it remains to prove the converse inequality.
First we notice that, due to Proposition 2.8 and the subadditivity of $\hat{E}_3$,
for any $F \in \mathcal{H}$,
\[
\hat{E}_2[F|\mathcal{H}_T] + \hat{E}_1[-F|\mathcal{H}_T] \geq \hat{E}_3[F|\mathcal{H}_T] + \hat{E}_3[-F|\mathcal{H}_T] \geq 0.
\]
Consequently, for all $X \in \mathcal{H}_T$ and all $F \in \mathcal{H}$,
\[
\hat{E}_1[X - F] + \hat{E}_2[F] = \hat{E}_1[\hat{E}_1[X - F|\mathcal{H}_T]] + \hat{E}_2[F]
\]
\[
= \hat{E}_1[X + \hat{E}_1[-F|\mathcal{H}_T]] + \hat{E}_2[F]
\]
\[
= \hat{E}_1[X - (-\hat{E}_1[-F|\mathcal{H}_T])] + \hat{E}_2[-\hat{E}_1[-F|\mathcal{H}_T]]
\]
\[
- \hat{E}_2[\hat{E}_1[-F|\mathcal{H}_T]] + \hat{E}_2[\hat{E}_2[F|\mathcal{H}_T]]
\]
\[
\geq \hat{E}_1[X - (-\hat{E}_1[-F|\mathcal{H}_T])] + \hat{E}_2[-\hat{E}_1[-F|\mathcal{H}_T]]
\]
\[
\geq \inf_{F \in \mathcal{H}_T} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \}.
\]
The statement now follows easily. ■

**Lemma 4.2:** For all $X \in \mathcal{H}_t^s$, $0 \leq s \leq t$, the following holds true:
\[
\inf_{F \in \mathcal{H}_t^s} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \} = \inf_{F \in \mathcal{H}_t^s} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \}.
\]

Proof: Firstly, from $\mathcal{H}_t^s \subseteq \mathcal{H}_t$, we have, obviously, for all $X \in \mathcal{H}_t^s$,
\[
\inf_{F \in \mathcal{H}_t^s} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \} \leq \inf_{F \in \mathcal{H}_t^s} \{ \hat{E}_1[X - F] + \hat{E}_2[F] \}.
\]
Secondly, for any $X \in \mathcal{H}_t^s$ and $F \in \mathcal{H}_t$, we can suppose without loss of generality that $X = \varphi(B_{t_1} - B_s, ..., B_{t_n} - B_s)$ and $F = \psi(B_{t'_1}, B_{t'_2}, ..., B_{t'_k}, B_{t_1} - B_s, ..., B_{t_n} - B_s)$, where $t'_1, ..., t'_k \in [0, s]$, $t_1, ..., t_n \in [s, t]$, $n, k \in \mathbb{N}$, $\varphi \in C_{t, i, p}(\mathbb{R}^n)$ and $\psi \in C_{t, i, p}(\mathbb{R}^{n+k})$.
To simplify the notation we put:
\[
Y_1 = (B_{t'_1}, B_{t'_2}, ..., B_{t'_k}), Y_2 = (B_{t_1} - B_s, ..., B_{t_n} - B_s), \mathbf{x} = (x_1, x_2, ..., x_k).
\]
Thus the proof of Proposition 3.8 is complete now.

Lemma 4.3: Let us come now to the proof of Lemma 3.9, which we split into a sequel holds:

\[ \mathbb{E}_1[X - F] + \mathbb{E}_2[F] \]

\[ = \mathbb{E}_1[\mathbb{E}_1[\varphi(Y_2) - \psi(Y_1, Y_2)|\mathcal{H}_s]] + \mathbb{E}_2[F] \]

\[ = \mathbb{E}_1[\mathbb{E}_1[\varphi(Y_2) - \psi(x, Y_2)|x = Y_1]] + \mathbb{E}_2[F] \]

\[ = \mathbb{E}_1[(\mathbb{E}_1[\varphi(Y_2) - \psi(x, Y_2)] + \mathbb{E}_2[\psi(x, Y_2)])|x = Y_1] + \mathbb{E}_2[F] \]

\[ \geq \mathbb{E}_1[\inf_{F \in \mathcal{H}_t^s} \{\mathbb{E}_1[X - F] + \mathbb{E}_2[F]\} - \mathbb{E}_2[\psi(x, Y_2)]|x = Y_1] + \mathbb{E}_2[F] \]

\[ = \inf_{F \in \mathcal{H}_t^s} \{\mathbb{E}_1[X - F] + \mathbb{E}_2[F]\} + \mathbb{E}_1[-\mathbb{E}_2[\psi(x, Y_2)]|x = Y_1] \]

\[ \geq \inf_{F \in \mathcal{H}_t^s} \{\mathbb{E}_1[X - F] + \mathbb{E}_2[F]\}. \]

Thus the proof is complete now.

Now we are able to prove Proposition 3.8.

Proof (of Proposition 3.8): For arbitrarily fixed \( s \geq 0 \), we put \( \tilde{B}_t = B_{t+s} - B_s \), \( t \geq 0 \). Then, obviously, \( \mathcal{H}_{t+s}^s = \mathcal{H}_t \), \( t \geq 0 \), where \( \mathcal{H}_t \) is generated by \( B_t \). Moreover, \( \tilde{B}_t \) is a G-Brownian Motion under \( \mathbb{E}_1 \) and \( \mathbb{E}_2 \).

According to the Lemmas 4.1 and 4.2, we have the following:

\[ \mathbb{E}_1 \square \mathbb{E}_2[\varphi(B_t - B_s)] \]

\[ = \inf_{F \in \mathcal{H}_t^s} \{\mathbb{E}_1[\varphi(B_t - B_s) - F] + \mathbb{E}_2[F]\} \]

\[ = \inf_{F \in \mathcal{H}_{t+s}} \{\mathbb{E}_1[\varphi(\tilde{B}_{t+s}) - F] + \mathbb{E}_2[F]\} \]

\[ = \inf_{F \in \mathcal{H}_{t+s}} \{\mathbb{E}_1[\varphi(B_{t+s}) - F] + \mathbb{E}_2[F]\} \]

\[ = \mathbb{E}_1 \square \mathbb{E}_2[\varphi(B_{t+s})]. \]

Thus the proof of Proposition 3.8 is complete now.

4.2 Proof of Lemma 3.9

Let us come now to the proof of Lemma 3.9, which we split into a sequel of lemmas.

Lemma 4.3: For all \( \varphi \in C_{t,lip}(\mathbb{R}^{n+1}), n \in \mathbb{N} \) and \( 0 \leq t_1, \ldots, t_n \leq s \leq t \), it holds:

\[ \mathbb{E}_1 \square \mathbb{E}_2[\varphi(B_{t_1}, B_{t_2}, \ldots, B_{t_n}, B_t - B_s)] \]

\[ \geq \mathbb{E}_1 \square \mathbb{E}_2[\mathbb{E}_1 \square \mathbb{E}_2[\varphi(x_1, \ldots, x_n, B_t - B_s)]|_{(x_1, \ldots, x_n) = (B_{t_1}, \ldots, B_{t_n})}]. \]
Proof: Let $X = \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)$. Without loss of generality we can suppose that $F \in \mathcal{H}$ has the form $\psi(B_{t_1}, B_{t_2}, ..., B_{t_k}, B_{t_{k+1}} - B_{t_s}, ..., B_{t_m} - B_s)$, where $0 \leq t_1, ..., t_n, t'_1, ..., t'_k \leq s$, $t'_{k+1}, ..., t'_m \geq s$, $m \geq k$, $m, k \in \mathbb{N}$, and $\varphi \in C_{l, lip}(\mathbb{R}^{n+1})$, $\psi \in C_{l, lip}(\mathbb{R}^m)$.

For simplifying the notation we put:

\[ x_1 = (x_1, ..., x_n), x_2 = (x'_1, ..., x'_k), Y_1 = (B_{t_1}, B_{t_2}, ..., B_{t_n}), \]
\[ Y_2 = (B_{t'_1}, ..., B_{t'_k}), Y_3 = (B_{t'_{k+1}} - B_s, ..., B_{t'_m} - B_s). \]

Then

\[ \hat{E}_1[X - F] + \hat{E}_2[F] = \hat{E}_1[\hat{E}_1[X - F|\mathcal{H}_s]] + \hat{E}_2[\hat{E}_2[F|\mathcal{H}_s]] \]
\[ = \hat{E}_1[\hat{E}_1[\varphi(x_1, B_t - B_s) - \psi(x_2, Y_3)|x_1=1,x_2=2]] + \hat{E}_2[\hat{E}_2[\psi(x_2, Y_3)|x_2=1]] \]
\[ = \hat{E}_1[(\hat{E}_1[\varphi(x_1, B_t - B_s) - \psi(x_2, Y_3)] + \hat{E}_2[\psi(x_2, Y_3)])|x_1=1,x_2=2] + \hat{E}_2[\hat{E}_2[\psi(x_2, Y_3)|x_2=1]] \]
\[ \geq \hat{E}_1[(\hat{E}_1[\hat{E}_2[\varphi(x_1, B_t - B_s)]|x_1=1] + \hat{E}_2[\hat{E}_2[\psi(x_2, Y_3)|x_2=1]])|x_1=1,x_2=2] \]
\[ = \hat{E}_1[\hat{E}_2[\psi(x_1, ..., x_n)|x_1=1]] \]
\[ \geq \hat{E}_1[\hat{E}_2[\varphi(x_1, ..., x_n, B_t - B_s)|x_1=1]][x_1=(B_{t_1}, ..., B_{t_n})] \]

Hence, we get

\[ \hat{E}_1[\hat{E}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \]
\[ \geq \hat{E}_1[\hat{E}_2[\varphi(x_1, ..., x_n, B_t - B_s)|x_1=(B_{t_1}, ..., B_{t_n})]] \]

The proof of the Lemma 4.3 is complete now. \qed

Let $Lip(\mathbb{R}^n), n \in \mathbb{N}$, denote the space of bounded Lipschitz functions $\varphi \in Lip(\mathbb{R}^n)$ satisfying:

\[ |\varphi(x) - \varphi(y)| \leq C|x - y| \quad x, y \in \mathbb{R}^n, \]

where $C$ is a constant only depending on $\varphi$.

The proof that

\[ \hat{E}_1[\hat{E}_2[\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)] \]
\[ \leq \hat{E}_1[\hat{E}_2[\hat{E}_1[\hat{E}_2[\varphi(x_1, ..., x_n, B_t - B_s)|x_1=(B_{t_1}, ..., B_{t_n})]] \]

is much more difficult than that of the converse inequality. For the proof we need the following statements.
Lemma 4.4: We assume that the random variable \( \varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) \), with \( t_i \leq t_{i+1}, i = 1, ..., n - 1, n \in \mathbb{N} \) and \( \varphi \in \text{Lip}(\mathbb{R}^n) \), satisfies the following assumption: there exist \( L, M \geq 0 \) s.t. \( |\varphi| \leq L \), and \( \varphi(x, y) = 0 \), for all \( (x, y) \in [-M, M]^c \times \mathbb{R}^{n-1} \).

We define
\[
\phi(x) = \hat{E}_1[\hat{E}_2[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]]
\]
\[
= \inf_{F \in \mathcal{H}^{1}} \{ \hat{E}_1[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - F] + \hat{E}_2[F] \}.
\]

Then we have the existence of an \( \epsilon \)-optimal \( \tilde{\varphi}(x) \) of the form \( \varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_{j+1}} - B_{t_1}) \), i.e., for any \( \epsilon > 0 \) we can find a finite dimensional function \( \tilde{\varphi}(x, \cdot) \in C_{l, \text{Lip}}(\mathbb{R}^l), l \geq 1 \), such that, for suitable \( t_2', ... , t_{l+1}' \geq t_1 \),
\[
\tilde{\varphi}(x) := \hat{E}_1[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \varphi(x, B_{t_2'} - B_{t_1}, ..., B_{t_{j+1}' - B_{t_1}})] + \hat{E}_2[\varphi(x, B_{t_2'} - B_{t_1}, ..., B_{t_{j+1}' - B_{t_1}})]
\]
satisfies
\[
|\tilde{\varphi}(x) - \phi(x)| \leq \epsilon.
\]

Proof: Since \( \varphi \in \text{Lip}(\mathbb{R}^n) \), we find for any \( \epsilon > 0 \) some sufficiently large \( J \geq 1 \) s.t. for all \( x, \tilde{x} \in \mathbb{R} \) with \( |x - \tilde{x}| \leq 2M \) holds \( |\varphi(x) - \varphi(\tilde{x}, y)| \leq \epsilon/6 \).

We then let \( -M = x_0 \leq x_1 \leq ... \leq x_J = M \), be such that \( |x_{j+1} - x_j| = \frac{2M}{J}, 0 \leq j \leq J - 1 \).

On the other hand, for every fixed \( j \) there are some \( m_j \geq 1 \), \( t_{i,j} \geq t_1 \) \((2 \leq i \leq m_j)\) and \( \psi^{x_j} \in C_{l, \text{Lip}}(\mathbb{R}^{m_j-1}) \), such that
\[
\phi(x_j) \leq \hat{E}_1[\varphi(x_j, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \varphi^{x_j}(B_{t_{2,j}} - B_{t_1}, ..., B_{t_{m_j,j} - B_{t_1}})] + \hat{E}_2[\varphi^{x_j}(B_{t_{2,j}} - B_{t_1}, ..., B_{t_{m_j,j} - B_{t_1}})]
\]
\[
\leq \phi(x_j) + \epsilon/6.
\]

Since there are only a finite number of \( j \) we can find a finite dimensional function denoted by \( \psi(x_j, y), y \in \mathbb{R}^l \), s.t. for each fixed \( j \), \( \psi(x_j, \cdot) \in C_{l, \text{Lip}}(\mathbb{R}^l) \) and
\[
\psi(x_j, B_{t_2'} - B_{t_1}, ..., B_{t_{j+1}' - B_{t_1}}) = \psi^{x_j}(B_{t_{2,j}} - B_{t_1}, ..., B_{t_{m_j,j} - B_{t_1}}),
\]
where \( \{t_2', ..., t_{l+1}'\} = \bigcup_{j=1}^{J} \{t_{2,j}, ..., t_{m_j,j}\} \).
With the convention \( \psi(x_0, y) = \psi(x_j, y) = 0, y \in \mathbb{R}^l \), we define

\[
\psi(x, y) := \begin{cases}  
\frac{x_{j+1} - x_j}{x_{j+1} - x_j} \psi(x_j, y) + \frac{x - x_j}{x_{j+1} - x_j} \psi(x_{j+1}, y), & x \in [x_j, x_{j+1}], \\
0, & \text{otherwise}.
\end{cases}
\]

Obviously, \( \psi(x, y) \in C_{l, l_p}(\mathbb{R}^{l+1}) \).

We now introduce \( \tilde{\psi}(x) : \)

\[
= \tilde{\mathbb{E}}_1[\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \psi(x, B_{t_2} - B_{t_1}, ..., B_{t_{j+1}} - B_{t_1})] + \tilde{\mathbb{E}}_2[\psi(x, B_{t_2} - B_{t_1}, ..., B_{t_{j+1}} - B_{t_1})].
\]

If \( x \notin [-M, M] \), \( \varphi(x, \cdot) = 0 \) and \( \psi(x, \cdot) = 0 \). Consequently, \( \tilde{\psi}(x) = 0 \). Moreover, from Proposition 3.5 we have that for \( x \notin [-M, M] \) also \( \tilde{\varphi}(x) = 0 \). Then \( \tilde{\psi}(x) = \varphi(x) = 0 \) when \( x \notin [-M, M] \), and we have also \( |\tilde{\psi}(x_j) - \varphi(x_j)| \leq \varepsilon/6 \) for each \( j \). We also recall that, for all \( 0 \leq j \leq J - 1 \) and all \( x \in [x_j, x_{j+1}] \),

\[
|\varphi(x, y) - \varphi(x_j, y)| \leq \varepsilon/6, \text{ for all } y \in \mathbb{R}^{n-1}.
\]

Our objective is to estimate

\[
|\tilde{\psi}(x) - \phi(x)| \leq |\tilde{\psi}(x) - \phi(x_j)| + |\phi(x_j) - \phi(x)|.
\]

For this end we notice that, with the notation:

\[
Y_1 = (B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}), Y_2 = (B_{t_2} - B_{t_1}, ..., B_{t_{j+1}} - B_{t_1}),
\]

we have from the definition of \( \phi(x) \) and \( \varphi(x_j) \) and from the properties of \( \tilde{\mathbb{E}}_1 \square \tilde{\mathbb{E}}_2 \) as sublinear expectation:

\[
|\phi(x) - \phi(x_j)| \leq \tilde{\mathbb{E}}_1 \square \tilde{\mathbb{E}}_2 [||\varphi(x, Y_1) - \varphi(x_j, Y_1)||] \leq \varepsilon/6.
\]

On the other hand, since \( |\varphi(x, Y_1) - \varphi(x_j, Y_1)| \leq \varepsilon/6 \),

\[
|\tilde{\psi}(x) - \phi(x_j)|
= |\tilde{\mathbb{E}}_1[\varphi(x, Y_1) - \psi(x, Y_2)] + \tilde{\mathbb{E}}_2[\psi(x, Y_2)] - \phi(x_j)|
\leq |\tilde{\mathbb{E}}_1[\varphi(x_j, Y_1) - \psi(x, Y_2)] + \tilde{\mathbb{E}}_2[\psi(x, Y_2)] - \phi(x_j)| + \varepsilon/6.
\]
Due to the definition of \( \phi(x_j) \), the latter expression without module is non-negative. Thus,

\[
|\tilde{\psi}(x) - \phi(x_j)| \\
\leq \tilde{E}_1[\varphi(x_j, Y_1) - \psi(x, Y_2)] + \tilde{E}_2[\psi(x, Y_2) - \phi(x_j) + \varepsilon/6] \\
\leq \tilde{E}_1[\frac{x_{j+1} - x}{x_{j+1} - x_j}(\varphi(x_j, Y_1) - \psi(x, Y_2)) + \frac{x - x_j}{x_j - x}(\varphi(x_j, Y_1) - \psi(x, Y_2)) - \psi(x_{j+1}, Y_2)] \\
+ \tilde{E}_2[\frac{x_{j+1} - x}{x_{j+1} - x_j}\psi(x, Y_2) + \frac{x - x_j}{x_j - x}\psi(x_j, Y_2)] - \phi(x_j) + 2\varepsilon/6 \\
\leq \frac{x_{j+1} - x}{x_{j+1} - x_j}\{\tilde{E}_1[\varphi(x_j, Y_1) - \psi(x, Y_2)] + \tilde{E}_2[\psi(x, Y_2) - \phi(x_j)]\} \\
+ \frac{x - x_j}{x_j - x_j}\{\tilde{E}_1[\varphi(x_j, Y_1) - \psi(x, Y_2)] + \tilde{E}_2[\psi(x_j, Y_2) - \phi(x_j)]\} + 2\varepsilon/6.
\]

Hence, due to the choice of \( \psi^{x_j} \) and \( \psi^{x_{j+1}} \),

\[
|\tilde{\psi}(x) - \phi(x_j)| \leq 5\varepsilon/6.
\]

This latter estimate combined with the fact that for \( |\phi(x) - \phi(x_j)| \leq \varepsilon/6 \) then yields

\[
|\tilde{\psi}(x) - \phi(x)| \leq \varepsilon.
\]

The proof of Lemma 4.4 is complete now. \( \blacksquare \)

Lemma 4.4 allows to prove the following:

**Lemma 4.5:** Let \( \varphi \in Lip(\mathbb{R}^n) \) be bounded and such that, for some real \( M > 0 \), \( \text{supp}(\varphi) \subset [-M, M] \times \mathbb{R}^{n-1} \). Then, for all \( 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \),

\[
\tilde{E}_1 \square \tilde{E}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] \\
= \tilde{E}_1 \square \tilde{E}_2[\tilde{E}_1 \square \tilde{E}_2[\varphi(x, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})|x=B_{t_1}]]).
\]

Proof: Firstly, it follows directly from Lemma 4.3 that:

\[
\tilde{E}_1 \square \tilde{E}_2[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] \\
\geq \tilde{E}_1 \square \tilde{E}_2[\tilde{E}_1 \square \tilde{E}_2[\varphi(x, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})|x=B_{t_1}]]. \tag{1}
\]

Secondly, from Lemma 4.4 we know that for any \( \varepsilon > 0 \) there is some \( \psi \in C_Lip(\mathbb{R}^{n+1}) \) such that \( |\tilde{\psi}(x) - \phi(x)| \leq \varepsilon \), for all \( x \in \mathbb{R} \), where \( \tilde{\psi}(x) \) and \( \phi(x) \) have been introduced in Lemma 4.4.

Due to Lemma 4.4, there is \( \tilde{\phi}(B_{t_1}, \ldots, B_{t_k'}) \in \mathcal{H}_{t_1}, 0 \leq t_1', \ldots, t_k' \leq t_1, k \in \mathbb{N}, \) such that

\[
|\tilde{E}_1[\tilde{\phi}(B_{t_1}) - \tilde{\phi}(B_{t_1'}, \ldots, B_{t_k'})] + \tilde{E}_2[\tilde{\phi}(B_{t_1'}, \ldots, B_{t_k'})] - \tilde{E}_1 \square \tilde{E}_2[\tilde{\phi}(B_{t_1})]| \leq \varepsilon.
\]

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For \( t'_2, \ldots, t'_{t+1} \geq t_1 \) from the definition of \( \tilde{\psi}(x) \) in Lemma 4.4 we put
\[
\psi'(x) = \hat{E}_2[\psi(x, B_{t'_2} - B_{t_1}, \ldots, B_{t'_{t+1}} - B_{t_1})]
\]
and
\[
F = \psi(B_{t_1}, B_{t'_2} - B_{t_1}, \ldots, B_{t'_{t+1}} - B_{t_1}) + \tilde{\phi}(B_{t''}, \ldots, B_{t''_k}) - \psi'(B_{t_1}).
\]
Notice that
\[
\hat{E}_2[F|\mathcal{H}_{t_1}] = \tilde{\phi}(B_{t''}, \ldots, B_{t''_k})
\]
and
\[
\hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) - F|\mathcal{H}_{t_1}] = \tilde{\psi}(B_{t_1}) - \tilde{\phi}(B_{t''}, \ldots, B_{t''_k}).
\]
Then, due to the choice of \( \tilde{\phi}(B_{t''}, \ldots, B_{t''_k}), \)
\[
\hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) - F] = \tilde{\psi}(B_{t_1}) - \tilde{\phi}(B_{t''}, \ldots, B_{t''_k})
\]
and
\[
\hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] - \hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] - \hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] - F]
\]
From the definition of \( \varphi \) in Lemma 4.4 and the arbitrariness of \( \varepsilon > 0 \) it follows then that
\[
\hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] - \hat{E}_1[\varphi(x, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})[x = B_{t_1}]]
\]
This together with (1) yields the wished statement. The proof of Lemma 4.5 is complete now. \( \blacksquare \)

In the next statement we extend Lemma 4.5 to general functions of \( Lip(\mathbb{R}^n) \).

**Lemma 4.6:** Let \( \varphi \in Lip(\mathbb{R}^n), n \geq 1, \) and \( t_n \geq t_{n-1} \geq \ldots \geq t_1 \geq 0. \) Then
\[
\hat{E}_1[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})] = \hat{E}_1[\varphi(x, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})[x = B_{t_1}]].
\]
Proof: Let $L > 0$ be such that $|\varphi| \leq L$. Given an arbitrarily large $M > 0$ we define, for all $y \in \mathbb{R}^{n-1}$,

$$\tilde{\varphi}(x, y) := \begin{cases} 
\varphi(x, y), & x \in [-M, M] \\
\varphi(-M, y)(M + 1 + x), & x \in [-M - 1, -M] \\
\varphi(M, y)(M + 1 - x), & x \in [M, M + 1] \\
0, & \text{otherwise.}
\end{cases}$$

Obviously, $\tilde{\varphi}$ satisfies the assumptions of Lemma 4.5.

Letting

$$\tilde{\varphi}'(x) = \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]$$

and

$$\phi(x) = \hat{E}_1 \Box \hat{E}_2 [\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})],$$

we have

$$|\phi(x) - \tilde{\varphi}'(x)| = |\hat{E}_1 \Box \hat{E}_2 [\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] - \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]| \leq \hat{E}_1 \Box \hat{E}_2 [|\varphi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \tilde{\varphi}(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})|] \leq 2L \frac{|x|}{M}.$$ 

Consequently,

$$|\hat{E}_1 \Box \hat{E}_2 [\phi(B_{t_1})] - \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}'(B_{t_1})]| \leq \hat{E}_1 \Box \hat{E}_2 [|\phi(B_{t_1}) - \tilde{\varphi}'(B_{t_1})|] \leq \hat{E}_1 \Box \hat{E}_2 [2L \frac{|B_{t_1}|}{M}] = 2L \frac{1}{M} \hat{E}_1 \Box \hat{E}_2 [|B_{t_1}|].$$

On the other hand, from the definition of $\tilde{\varphi}$ we also obtain

$$|\hat{E}_1 \Box \hat{E}_2 [\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] - \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]| \leq \hat{E}_1 \Box \hat{E}_2 [|\varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}) - \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})|] \leq 2L \frac{1}{M} \hat{E}_1 \Box \hat{E}_2 [|B_{t_1}|].$$

Thus, since due to Lemma 4.5

$$\hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})] = \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}'(B_{t_1})],$$

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we get by letting $M \mapsto +\infty$ the relation
\[
\hat{E}_1 \Box \hat{E}_2 [\phi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_n-1})]
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\phi(x, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_n-1})]|_{x=B_{t_1}}].
\]

The proof of Lemma 4.6 is complete.

Lemma 4.7: For all $\varphi \in Lip(\mathbb{R}^n)$, $n \geq 1$, and $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, we have
\[
\hat{E}_1 \Box \hat{E}_2 [\varphi(B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})]
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\varphi(y, B_{t_3} - B_{t_2}, ..., B_{t_n} - B_{t_{n-1}})]|_{y=B_{t_2} - B_{t_1}}]
\]

Proof: Lemma 4.2 allows to repeat the arguments of the Lemmas 4.3 to 4.6 in $\mathcal{H}_{t_1}$. The result of Lemma 4.7 then follows.

Finally, we have:

Lemma 4.8: Let $\varphi \in Lip(\mathbb{R}^{n+1})$, $n \geq 1$ and $0 \leq t_1, ..., t_n \leq s$. Then
\[
\hat{E}_1 \Box \hat{E}_2 [\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)]
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1, ..., x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}].
\]

Proof: Without any loss of generality we can suppose $0 \leq t_1 \leq t_2 \leq ... \leq t_n$. Then there is some $\tilde{\varphi} \in Lip(\mathbb{R}^{n+1})$ such that $\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) = \tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}, B_t - B_s) \in \mathcal{H}_{t_1}$. With the notation $x = (x_1, ..., x_n)$, and due to the Lemmas 4.1 to 4.7 we have
\[
\hat{E}_1 \Box \hat{E}_2 [\varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)]
= \hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}}, B_t - B_s)]
\]
\[
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(x, B_t - B_s)]|_{x=(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})}]
\]
\[
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\tilde{\varphi}(x_1, ..., x_n, B_t - B_s)]|_{x=(B_{t_1}, B_{t_2}, ..., B_{t_n})}]
\]
\[
= \hat{E}_1 \Box \hat{E}_2 [\hat{E}_1 \Box \hat{E}_2 [\varphi(x_1, ..., x_n, B_t - B_s)]|_{(x_1, ..., x_n)=(B_{t_1}, B_{t_2}, ..., B_{t_n})}].
\]

The proof of Lemma 4.8 is complete now.

Let us now come to the proof of Lemma 3.9.

Proof (of Lemma 3.9): In a first step, we will prove that for each $\varphi \in C_{l, lip}(\mathbb{R}^{n+1})$ there exists a sequence of bounded Lipschitz functions $\langle \varphi_N \rangle_{N \geq 1}$ such that
\[
\hat{E}_1 |\varphi_N(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) - \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)|
\rightarrow 0, \quad \text{as} \quad N \rightarrow \infty.
\]
For this end we put
\[ l_N(x) = (x \land N) \lor (-N), N \geq 1, \ x \in \mathbb{R}, \]
and
\[ \varphi_N(x_1, ..., x_{n+1}) = \varphi(l_N(x_1), ..., l_N(x_{n+1})), \]
and we notice that
\[ |x - l_N(x)| \leq \frac{|x|^2}{N}, \text{ for all } x \in \mathbb{R}. \]
Obviously, the functions \( \varphi_N \) are bounded and Lipschitz, and, moreover, 
\[
|\varphi_N(x_1, ..., x_{n+1}) - \varphi(x_1, ..., x_{n+1})| \\
= |\varphi(l_N(x_1), ..., l_N(x_{n+1})) - \varphi(x_1, ..., x_{n+1})| \\
\leq C(1 + |x_1|^m + ... + |x_{n+1}|^m) \frac{\sum_{i=1}^{n+1} |x_i|^4}{N^2} \\
= \frac{C(1 + |x_1|^m + ... + |x_{n+1}|^m) \sqrt{\sum_{i=1}^{n+1} |x_i|^4}}{N^2},
\]
where \( C \) and \( m \geq 0 \) are constants only depending on \( \varphi \). Then, in virtue of the finiteness of \( \mathbb{E}_1[(1 + |B_{t_1}|^m + ... + |B_{t_n}|^m + |B_t - B_s|^m)(\sum_{i=1}^{n+1} |B_{t_i}|^4 + |B_t - B_s|^4)^{\frac{1}{2}}] \), we get
\[
\mathbb{E}_1[|\varphi_N(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s) - \varphi(B_{t_1}, B_{t_2}, ..., B_{t_n}, B_t - B_s)|] \to 0, \text{ as } N \to \infty.
\]
Let \( x_1 = (x_1, ..., x_n) \) and \( Y_1 = (B_{t_1}, B_{t_2}, ..., B_{t_n}) \). Then, due to our above convergence result,
\[
|\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\varphi_N(Y_1, B_t - B_s)] - \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\varphi(Y_1, B_t - B_s)]]] \\
\leq \hat{\mathbb{E}}_1[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
\leq \hat{\mathbb{E}}_1[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
\to 0, \text{ as } N \to \infty,
\]
and, from Lemma 4.3,
\[
|\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\varphi_N(x_1, B_t - B_s)]]_{x_1=Y_1}]] \\
- \hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\hat{\mathbb{E}}_1[\hat{\mathbb{E}}_2[\varphi(x_1, B_t - B_s)]]_{x_1=Y_1}]]] \\
= \hat{\mathbb{E}}_1[|\varphi_N(x_1, B_t - B_s) - \varphi(x_1, B_t - B_s)|]_{x_1=Y_1} \\
\leq \hat{\mathbb{E}}_1[|\varphi_N(Y_1, B_t - B_s) - \varphi(Y_1, B_t - B_s)|] \\
\to 0, \text{ as } N \to \infty.
\]
On the other hand, from Lemma 4.8 we have

\[ \hat{E}_1 \square \hat{E}_2 [\varphi_N (B_{t_1}, B_{t_2}, \ldots, B_{t_n}, B_t - B_s)] \]
\[ = \hat{E}_1 \square \hat{E}_2 [\hat{E}_1 \square \hat{E}_2 [\varphi_N (x_1, \ldots, x_n, B_t - B_s)] | (x_1, \ldots, x_n) = (B_{t_1}, B_{t_2}, \ldots, B_{t_n})]. \]

Combining the above results we can conclude that

\[ \hat{E}_1 \square \hat{E}_2 [\varphi (B_{t_1}, B_{t_2}, \ldots, B_{t_n}, B_t - B_s)] \]
\[ = \hat{E}_1 \square \hat{E}_2 [\hat{E}_1 \square \hat{E}_2 [\varphi (x_1, \ldots, x_n, B_t - B_s)] | (x_1, \ldots, x_n) = (B_{t_1}, B_{t_2}, \ldots, B_{t_n})]. \]

The proof is complete now. ■

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