Casimir effect for parallel plates in de Sitter spacetime

E. Elizalde\textsuperscript{1}, A. A. Saharian\textsuperscript{2}, T. A. Vardanyan\textsuperscript{2}

\textsuperscript{1}Instituto de Ciencias del Espacio (CSIC) \\
and Institut d’Estudis Espacials de Catalunya (IEEC/CSIC) \\
Campus UAB, Facultat de Ciències, Torre C5-Parell-2a planta, \\
08193 Bellaterra (Barcelona) Spain

\textsuperscript{2}Department of Physics, Yerevan State University, \\
1 Alex Manoogian Street, 0025 Yerevan, Armenia

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Abstract

The Wightman function and the vacuum expectation values of the field squared and of the energy-momentum tensor are obtained, for a massive scalar field with an arbitrary curvature coupling parameter, in the region between two infinite parallel plates, on the background of de Sitter spacetime. The field is prepared in the Bunch–Davies vacuum state and is constrained to satisfy Robin boundary conditions on the plates. For the calculation, a mode-summation method is used, supplemented with a variant of the generalized Abel-Plana formula. This allows to explicitly extract the contributions to the expectation values which come from each single boundary, and to expand the second-plate-induced part in terms of exponentially convergent integrals. Several limiting cases of interest are then studied. Moreover, the Casimir forces acting on the plates are evaluated, and it is shown that the curvature of the background spacetime decisively influences the behavior of these forces at separations larger than the curvature scale of de Sitter spacetime. In terms of the curvature coupling parameter and the mass of the field, two very different regimes are realized, which exhibit monotonic and oscillatory behavior of the vacuum expectation values, respectively. The decay of the Casimir force at large plate separation is shown to be power-law (monotonic or oscillating), with independence of the value of the field mass.

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1 Introduction

The Casimir effect \cite{1} is now known to be common to systems of very different kind, involving fluctuating quantities on which external boundary conditions are imposed. It can have important implications on all scales, from subnuclear to cosmological. Imposing boundary conditions on a quantum field leads to a modification of the spectrum of zero-point fluctuations and results in the shifting in the vacuum expectation values for physical quantities, such as the energy density and stresses. In particular, the confinement of quantum fluctuations induces forces that act on the constraining boundaries. The particular features of the resulting vacuum forces depend on the nature of the quantum field, on the type of the spacetime manifold, the geometry of the boundaries, and on the specific boundary conditions imposed on the field.
An interesting topic in the investigation of the Casimir effect is its explicit dependence on the geometry of the background spacetime. As usual, the relevant information is encoded in the vacuum fluctuations spectrum and, not surprisingly, analytic solutions can be found for highly symmetric geometries only. In special, motivated by Randall–Sundrum type braneworld scenarios, investigations of the Casimir effect in anti-de Sitter (AdS) spacetime have attracted a great deal of attention. The braneworld corresponds to a manifold with boundaries and all fields which propagate in the bulk will give Casimir-type contributions to the vacuum energy and, as a result, to the vacuum forces acting on the branes. The Casimir effect provides in this context a natural mechanism for stabilizing the radion field in the Randall–Sundrum model, as required for a complete solution of the hierarchy problem. In addition, the Casimir energy gives a contribution to both the brane and bulk cosmological constants and, hence, it has to be taken into account in any self-consistent formulation of the braneworld dynamics. The Casimir energy and corresponding Casimir forces for two parallel branes in AdS spacetime have been evaluated in Refs. [2] by using either dimensional or zeta function regularization methods. Local Casimir densities were considered in Refs. [3, 4]. The Casimir effect in higher-dimensional generalizations of the AdS spacetime with compact internal spaces has been investigated in [5].

Another popular background in gravitational physics is de Sitter (dS) spacetime. Quantum field theory in this background has been extensively studied during the past two decades. Much of the early interest was motivated by questions related with the quantization of fields propagating on curved backgrounds. The dS spacetime has a high degree of symmetry and numerous physical problems are exactly solvable on this background. Importance of such theoretical work was increased with the appearance of the inflationary cosmology scenario [6]. In most inflationary models, an approximate dS spacetime is employed with the aim to solve a number of problems in standard cosmology. During the inflationary epoch, quantum fluctuations in the inflaton field introduce inhomogeneities which play a central role in the generation of cosmic structures from inflation. More recently, astronomical observations of standard-candle supernovae, galaxy clusters and the cosmic microwave background [7] have clearly indicated that at present our (local) universe is accelerating and can be well approximated by ΛCDM, FRW cosmology with a positive cosmological constant Λ. Now, if the universe, as it seems, is going to accelerate for ever, this cosmology will lead asymptotically to a dS universe. Another motivation for the investigation of dS-based quantum theories is related to the holographic duality known to hold between quantum gravity on dS spacetime and a quantum field theory living on its boundary, identified with the timelike infinity surface of the dS spacetime.

Motivated by the above considerations—and since they are ingredients that more full-fledged models will necessarily have to incorporate—we will here calculate the Casimir densities and forces arising for the geometry of two parallel plates on the background of (D + 1)-dimensional dS spacetime. Previously, the Casimir effect on the background of dS spacetime described in planar coordinates was investigated in Refs. [8] for a conformally coupled massless scalar field. In this last case the problem is conformally related to the corresponding problem in Minkowski spacetime and the vacuum characteristics are generated from those for the Minkowski counterpart, just by multiplying with the conformal factor. In particular, for the geometry of a single plate, the vacuum expectation value of the energy-momentum tensor vanishes. The Casimir density induced by a single plate for a massive scalar field with an arbitrary curvature coupling parameter has been considered in [9]. In [10] the vacuum expectation value of the energy–momentum tensor for a conformally coupled scalar field was investigated in dS spacetime with static coordinates in presence of curved branes, on which the field obeys the Robin boundary conditions with coordinate dependent coefficients. In those papers the conformal relation between dS and Rindler spacetimes and the results for the Rindler counterpart were used. More recently, the Casimir density in a dS spacetime with toroidally compactified spatial dimensions

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has been investigated in \cite{11}.

The outline of the paper is as follows. In the next section the positive frequency Wightman function will be evaluated for a scalar field with general curvature coupling parameter with Robin boundary conditions on two parallel plane boundaries in the background of dS spacetime. Among the most important quantities describing the local properties of a quantum field and the corresponding quantum back-reaction effects are the expectation values of the field squared and of the energy-momentum tensor. These quantities, in the region between the plates, will be investigated in Sects. 3 and 4. Simple asymptotic formulae are obtained both for small and for large plate separations. The Casimir forces acting on the plates are obtained in Sect. 5. Finally, Sect. 6 contains a summary of the work done together with an outlook.

2 The Wightman function in de Sitter spacetime with two parallel plates

Consider a quantum scalar field $\varphi(x)$ on a $(D+1)$-dimensional dS spacetime background, as coming from a cosmological theory, as described in Sect. 1, with positive cosmological constant, $\Lambda$. The corresponding line element can be written in planar (inflationary) coordinates, which are most appropriate for cosmological applications:

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{D} (dz_i)^2.$$  \hspace{1cm} (1)

Here, the parameter $\alpha$ is related to the cosmological constant through the expression $\alpha = D(D-1)/(2\Lambda)$. For the discussion to follow, in addition to the synchronous time coordinate, $t$, it is very convenient to introduce the conformal time, $\tau$, defined as $\tau = -\alpha e^{-t/\alpha}$, $-\infty < \tau < 0$. In terms of this coordinate the metric tensor takes the conformally flat form: $g_{ik} = (\alpha/\tau)^2 \text{diag}(1, -1, \ldots, -1)$.

The dynamics of a massive scalar field with curvature coupling parameter are governed by the Klein–Gordon equation

$$\left(\nabla_l \nabla^l + m^2 + \xi R \right) \varphi = 0,$$  \hspace{1cm} (2)

where $\nabla_l$ is the covariant derivative operator and $R = D(D+1)/\alpha^2$ is the Ricci scalar for dS spacetime. The special cases $\xi = 0$ and $\xi = \xi_D \equiv (D-1)/4D$ correspond to minimally and to conformally coupled fields, respectively. The importance of these two special cases comes from the fact that, in the massless limit, the corresponding fields mimic the behavior of gravitons and photons, respectively. Note that non-minimal coupling is required by the renormalizability condition for interacting theories in curved spacetime \cite{12}.

In this paper we will be interested in the study of the Casimir densities and of the mutual forces occurring for the geometry of two infinite, parallel plates in dS spacetime. The plates are located at $z_D = a_1$ and $z_D = a_2$, $a_1 < a_2$. As the most general set up, we assume that on these boundaries the scalar field obeys Robin boundary conditions (BCs)

$$(1 + \beta_j n^l \nabla_l) \varphi(x) = [1 + \beta_j (1)^{-1} \partial_D] \varphi(x) = 0, \quad z_D = a_j, \ j = 1, 2,$$  \hspace{1cm} (3)

with constant coefficients $\beta_j$. For $\beta_j = 0$ these BCs reduce to Dirichlet ones, and for $\beta_j = \infty$ to Neumann BCs. The choice of different boundary conditions on the plates corresponds, in physical terms, to using different materials for the same. The imposition of BCs leads to a modification of the vacuum expectation values (VEVs) for physical quantities, as compared with those in the situation without boundaries. In the discussion below we will assume that the quantum scalar field is prepared in a dS invariant Bunch–Davies vacuum state \cite{13}.
Among the most important characteristics of the vacuum state are the VEVs of the field squared and of the energy-momentum tensor. These VEVs are obtained from the corresponding positive frequency Wightman function \( W(x, x') \) in the coincidence limit of the arguments. The Wightman function is also of the essence for the consideration of the response of particle detectors at a given state of motion (see, for instance, [14]). Expanding the field operator over a complete set \( \{ \varphi_\sigma(x), \varphi_\sigma^* (x) \} \) of solutions to the classical field equation, satisfying the boundary conditions, the positive frequency Wightman function is best expressed as the mode-sum

\[
W(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle = \sum_{\sigma} \varphi_\sigma(x) \varphi_\sigma^*(x'),
\]

where the collective index \( \sigma \) labels the solutions.

In the region between the plates, \( a_1 < z^D < a_2 \), the eigenfunctions realizing the Bunch–Davies vacuum state and satisfying the BC on the plate at \( z^D = a_1 \), have the form

\[
\varphi_\sigma(x) = C_\sigma \eta^{D/2} H^{(1)}_\nu (\eta K) \cos|k_D(z^D - a_1) + \alpha_1(k_D)| e^{i k z},
\]

with the notations \( \eta = |\tau| \), \( K = \sqrt{k^2 + k_D^2} \), and

\[
e^{2i\alpha_1(x)} = \frac{i\beta_1 x - 1}{i\beta_1 x + 1}.
\]

In Eq. (5), \( z = (z^1, \ldots, z^{D-1}) \) is the position vector along the dimensions parallel to the plates, \( k = (k_1, \ldots, k_{D-1}) \), and the order of the Hankel function \( H^{(1)}_\nu (x) \) is given by

\[
\nu = \left[ D^2/4 - D(D + 1)\xi - m^2\alpha^2 \right]^{1/2}.
\]

Note that \( \nu \) is either real and nonnegative or purely imaginary. For a conformally coupled massless field \( \nu = 1/2 \) and the function \( H^{(1)}_\nu (x) \) in (5) is expressed in terms of elementary functions. From the boundary condition on the second plate \( z^D = a_2 \) we find that the eigenvalues for \( k_D \) are solutions of the transcendental equation

\[
(1 - b_1 b_2 y^2) \sin y - (b_1 + b_2) y \cos y = 0,
\]

where \( y = k_D a \) and \( b_j = \beta_j / a \), with \( a = a_2 - a_1 \) being the separation between the plates. In the discussion below we will assume that all zeros are real. In particular, this is the case for the conditions \( b_j \leq 0 \) (see [15]). The positive solutions of Eq. (8) will be denoted by \( y = \lambda_n \), \( n = 1, 2, \ldots \), and for the eigenvalues of \( k_D \) one has \( k_D = \lambda_n / a \). The eigenfunctions are specified by a set of eigenfunctions \( \sigma = (k, n) \).

The coefficient \( C_\sigma \) in (5) is determined from the orthonormalization condition

\[
\int dz \int_{a_1}^{a_2} dz^D \sqrt{\mid g \mid} g^{00} [\varphi_\sigma(x) \partial_\tau \varphi_\sigma^*(x) - \varphi_\sigma^*(x) \partial_\tau \varphi_\sigma(x)] = i \delta_{nn'} \delta(k - k').
\]

By using the Wronskian relation for the Hankel functions, we find

\[
C_\sigma^2 = \frac{(2\pi)^{2-D} \alpha^{1-D} e^{i(\nu - \nu^*) \pi/2}}{4a \{ 1 + \cos[\lambda_n + 2\alpha_1(\lambda_n / a)] \sin(\lambda_n) / \lambda_n \}},
\]

the star meaning complex conjugate.
we find

\begin{equation}
W(x, x') = \frac{4(\eta \eta')D/2}{(2\pi)^D a\alpha D-1} \int dk e^{ik(x-x')} \sum_{n=1}^{\infty} K_\nu(\eta k_n e^{-\pi i/2})K_\nu(\eta' k_n e^{\pi i/2})
\end{equation}

\[
\times \cos[\lambda_n(z^D - a_1)/a + \alpha_1(\lambda_n/a)] \cos[\lambda_n(z^{D'} - a_1)/a + \alpha_1(\lambda_n/a)]
\frac{1}{1 + \cos[\lambda_n + 2\alpha_1(\lambda_n/a)] \sin(\lambda_n)/\lambda_n},
\]

with \( k_n = \sqrt{k^2 + \lambda_n^2}/a^2 \) and where we have written the Hankel functions in terms of the modified Bessel function \( K_\nu(x) \). It is well known that in dS spacetime without boundaries the Bunch–Davies vacuum state is not a physically realizable state for \( \text{Re}\nu \geq D/2 \). The corresponding Wightman function contains infrared divergences arising from long-wavelength modes. In the presence of boundaries, the BCs on the quantized field may exclude these modes and the Bunch–Davies vacuum becomes a realizable state. An example of this type of situation is provided by the geometry of two parallel plates described above. In the region between the plates and for \( \lambda_j \leq 0, \beta_j \neq \infty \), there is a maximum wavelength, \( 2\pi a/\lambda_1 \), and the two-point function (11) contains no infrared divergences. Mathematically, this situation corresponds to the one where in the arguments of the modified Bessel functions we have \( k_n \geq \lambda_n/a \).

As we do not know the explicit expression for \( \lambda_n \) as a function of \( n \), and being the summand in (11) a strongly oscillating function for large values of \( n \), this formula is not convenient for the evaluation of the VEVs of the field squared and of the energy-momentum tensor. In order to obtain a useful alternative representation, we apply to the series on \( n \) the summation formula (15) [15, 16]

\[
\sum_{n=1}^{\infty} \frac{\pi \lambda_n f(\lambda_n)}{\lambda_n + \sin(\lambda_n) \cos[\lambda_n + 2\alpha_1(\lambda_n/a)]} = \frac{\pi}{2} \frac{f(0)}{1 - b_2 - b_1} + \int_0^{\infty} dz f(z)
\]

\[
+i \int_0^{\infty} dz \frac{f(iz) - f(-iz)}{(b_1 z + 1)(b_2 z + 1)e^{2z} - 1},
\]

(12)

being

\[
f(z) = K_\nu(\eta e^{-\pi i/2}\sqrt{k^2 + z^2/a^2})K_\nu(\eta' e^{\pi i/2}\sqrt{k^2 + z^2/a^2})
\]

\[
\times \cos[z(z^D - a_1)/a + \alpha_1(z/a)] \cos[z(z^{D'} - a_1)/a + \alpha_1(z/a)].
\]

(13)

In the case \( b_j > 0 \) this function has poles on the imaginary axis and the corresponding residue terms should be added to the right hand side of the summation formula. In order to easy the presentation, in the discussion below we will just consider the case \( b_j \leq 0 \), but a similar procedure is valid in the general case.

Use of the summation formula (12) with (13) allows us to write the Wightman function in the decomposed form

\[
W(x, x') = W_1(x, x') + \frac{2\alpha_1^{1-D}}{(2\pi)^D} \int dk e^{ik(x-x')}
\]

\[
\times \int_k du \cosh[u(z^D - a_1) + \tilde{\alpha}_1(u)] \cosh[u(z^{D'} - a_1) + \tilde{\alpha}_1(u)]
\]

\[
\times y^{-D} \left[ I_\nu(\eta y) \tilde{K}_\nu(\eta y) + I_\nu(\eta y) \tilde{K}_\nu(\eta' y) \right]_{y=\sqrt{u^2-k^2}},
\]

(14)

where the function \( \tilde{\alpha}_1(u) \) is defined by the relation \( e^{2\tilde{\alpha}_1(u)} = c_1(u) \) and we have introduced the notations

\[
\tilde{K}_\nu(y) = y^{D/2}K_\nu(y), \quad \tilde{I}_\nu(y) = y^{D/2} [I_\nu(y) + I_{-\nu}(y)],
\]

(15)
and
\[ c_j(u) = \frac{\beta_j u - 1}{\beta_j u + 1}. \] (16)

Note that one has \( c_j(u) = -1 \) in the case of Dirichlet BCs and \( c_j(u) = 1 \) for Neumann BCs. In Eq. (14),
\[
W_1(x, x') = \frac{8(\eta\eta')^{D/2}}{(2\pi)^{D+1}\alpha^{D-1}} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{z} - \mathbf{z}')} \int_0^\infty du \cos[u(z^D - a_1) + \alpha_1(u)] \\
\times \cos[u(z'^D - a_1) + \alpha_1(u)] K_\nu(\sqrt{2 + u^2 e^{-\pi i/2}}) K_\nu(\sqrt{2 + u^2 e^{\pi i/2}}) \] (17)
is the Wightman function corresponding to a single plate at \( z^D = a_1 \).

The VEVs for the geometry of a single plate were investigated in [9]. Denoting the Wightman function for the dS spacetime without boundaries by \( \tilde{W}_{\text{dS}}(x, x') \), the corresponding Wightman function for a plate located at \( z^D = a_j \) can be written in the form
\[ W_j(x, x') = \tilde{W}_{\text{dS}}(x, x') + W_j^{(1)}(x, x'), \] (18)
where the part induced by the plate is given by the expression
\[
W_j^{(1)}(x, x') = \frac{\alpha_{1-D}}{2(2\pi)^D} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{z} - \mathbf{z}')} \int_k^\infty du \frac{e^{-u|z^D + z'^D - 2a_j|}}{c_j(u)} \\
x \times y^{-D} \left[ \tilde{I}_\nu(\eta') \tilde{K}_\nu(\eta y) + \tilde{I}_\nu(\eta y) \tilde{K}_\nu(\eta') \right]_{y = \sqrt{u^2 - k^2}}. \] (19)
The two-point function in the dS spacetime without boundaries was investigated in [13, 18] (see also [14]). It is given by the formula
\[
W_{\text{dS}}(x, x') = \frac{\alpha_{1-D}}{2(2\pi)^D} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{z} - \mathbf{z}')} \int_{k_1}^\infty du \frac{e^{-u|z^D + z'^D - 2a_j|}}{c_j(u)} \\
x \times y^{-D} \left[ \tilde{I}_\nu(\eta') \tilde{K}_\nu(\eta y) + \tilde{I}_\nu(\eta y) \tilde{K}_\nu(\eta') \right]_{y = \sqrt{u^2 - k^2}}. \] (20)

where \( P_\nu^m(x) \) is the associated Legendre function of the first kind and
\[
u = -1 + \frac{\sum_{i=1}^D (z_i - z'_i)^2 - (\eta - \eta')^2}{2\eta\eta'}. \] (21)

By using the definition of \( \tilde{\alpha}_1(u) \), the Wightman function in the region between the plates can be written in the more symmetric form:
\[
W(x, x') = W_{\text{dS}}(x, x') + \sum_{j=1}^2 W_j^{(1)}(x, x') + \frac{\alpha_{1-D}}{2(2\pi)^D} \int d\mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{z} - \mathbf{z}')} \\
\times \int_k^\infty du \frac{2 \cosh[u(z^D - z'^D)] + \sum_{j=1,2} e^{-u|z^D + z'^D - 2a_j|/c_j(u)}}{c_1(u)c_2(u)e^{2au} - 1} \\
\times y^{-D} \left[ \tilde{I}_\nu(\eta') \tilde{K}_\nu(\eta y) + \tilde{I}_\nu(\eta y) \tilde{K}_\nu(\eta') \right]_{y = \sqrt{u^2 - k^2}}, \] (22)
where the last term on the rhs can be referred to as the interference part. This integral representation of the Wightman function is valid for \( \text{Re} \nu < 1 \). As it has been shown in Ref. [17], the quantized graviton field in \( D = 3 \) dS spacetime is equivalent to a pair of minimally coupled massless scalar fields. Thus the representation (22) does not apply to the graviton. In the region \( z^D < a_1 \ (z^D > a_2) \) the Wightman function coincides with the corresponding function for a single plate located at \( z^D = a_1 \ (z^D = a_2) \) and is given by the expression (18), with \( j = 1 \) \( (j = 2) \).

The results obtained in the present paper can be applied to a more general problem where the cosmological constant is different in separate regions \( z^D < a_1 \), \( a_1 < z^D < a_2 \), and \( z^D > a_2 \). In this case the plate can be considered as a simple model of a thin domain wall separating the regions with different dS vacua.
3 Vacuum expectation value of the field squared

Once we have the Wightman function, we can proceed to evaluate the VEV of the field squared, by taking the coincidence limit of the arguments. In this limit the Wightman function is divergent and some renormalization procedure is needed. The important point here is that for points far away from the boundaries the divergences are the same as for the dS spacetime without boundaries. As in our previous procedure we have already extracted, from the Wightman function, the part \( W_{\text{dS}}(x, x') \), the renormalization of the VEVs is just reduced to the renormalization of the part corresponding to the geometry without boundaries, which is already done in literature. For the further discussion of the VEVs in the coincidence limit it is convenient to use the notations

\[
H(x, y) = \frac{(x^2 - y^2)(D-3)/2}{c_1(x/\eta)c_2(x/\eta)e^{2ax/\eta} - 1},
\]

\[
g(\beta_j u, yu) = c_j(u)e^{2uy} + e^{-2uy}/c_j(u) + 2.
\]

Introducing also the function

\[
F_\nu(y) = y^D[I_\nu(y) + I_{-\nu}(y)] K_\nu(y),
\]

The VEV of the field squared can be expressed in the decomposed form

\[
\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{\text{dS}} + \sum_{j=1,2} \langle \varphi^2 \rangle_j + \Delta \langle \varphi^2 \rangle,
\]

where

\[
\Delta \langle \varphi^2 \rangle = \frac{A_D}{\alpha^{D-1}} \int_0^\infty dy y^{1-D} F_\nu(y) \int_y^\infty dx H(x, y) h(x/\eta, z_D),
\]

In Eq. (26), \( \langle \varphi^2 \rangle_{\text{dS}} \) is the renormalized VEV in dS spacetime without boundaries, and the part \( \langle \varphi^2 \rangle_j \) is induced by a single boundary located at \( z_D = a_j \). Note that, due to the dS invariance of the Bunch–Davies vacuum state, the VEV \( \langle \varphi^2 \rangle_{\text{dS}} \) does not depend on the spacetime point. The ratio \( |z_D - a_j|/\eta \) in the formulae for the VEV is the proper distance of the observation point from the plate at \( z_D = a_j \) measured in units of the curvature scale, \( \alpha \) (note that \( \alpha \) is the dS horizon size). The boundary induced VEV is a function of the combinations \( |z_D - a_j|/\eta \) and \( \beta_j/\eta \) only. This property follows from the maximal symmetry of the dS spacetime.

We can write the expression for the VEV of the field squared in a more symmetric form by using (22)

\[
\langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{\text{dS}} + \sum_{j=1,2} \langle \varphi^2 \rangle_j + \Delta \langle \varphi^2 \rangle,
\]

where the interference term is given by the expression

\[
\Delta \langle \varphi^2 \rangle = \frac{A_D}{\alpha^{D-1}} \int_0^\infty dy y^{1-D} F_\nu(y) \int_y^\infty dx H(x, y) h(x/\eta, z_D),
\]
with the notation
\[ h(x, z^D) = 2 + \sum_{j=1,2} e^{-2x|z^D - a_j|}/c_j(x). \]  

(31)

Note that the interference part is finite everywhere, including the points on the plates. The surface divergences in the VEV are contained in the single plate parts only. For points near the plates the total VEV is dominated by these contributions. In particular, near the plate at \( z^D = a_j \) one has \( \langle \varphi^2 \rangle \sim (|z^D - a_j|/\eta)^{1-D} \). The corresponding result for two parallel plates in Minkowski spacetime is obtained from (30) in the limit \( \alpha \to \infty \). In this limit one has \( \nu \approx \text{Im} \alpha \) and the modulus of the order of the modified Bessel function is large. In addition, we have \( \eta \to \alpha \). Details of the corresponding limiting transition are given in Appendix, for the explicit case of the vacuum forces acting on the plates.

For a conformally coupled massless scalar field \( (\xi = \xi_D, m = 0) \) one has \( \nu = 1/2 \) and \( F_\nu(y) = y^{D-1} \). In this case, for the interference term we find
\[ \Delta \langle \varphi^2 \rangle = \frac{(\eta/\alpha)^{D-1}}{(4\pi)^{D/2}\Gamma(D/2)} \int_0^\infty dx \frac{x^{D-2} h(x, z^D)}{c_1(x)c_2(x)e^{2ax} - 1}. \]  

(32)

This result (32) could also have been directly obtained by using the fact that, in the special case under consideration, the problem is conformally related to the corresponding one for Robin plates in Minkowski spacetime [15, 16]. From this relation, it follows that \( \Delta \langle \varphi^2 \rangle = a^{1-D}(\eta)\Delta \langle \varphi^2 \rangle^{(M)} \), with scale factor \( a(\eta) = \alpha/\eta \), which leads to the result (32).

Formula (30) is further simplified for Dirichlet and Neumann BCs. Using the expansion \( (e^z - 1)^{-1} = \sum_{n=1}^{\infty} e^{-nz} \) and explicitly integrating over \( x \), we find
\[ \Delta \langle \varphi^2 \rangle^{(D)} = \frac{2\alpha^{1-D}}{(2\pi)^{D/2+1}} \sum_{n=1}^{\infty} \int_0^\infty dy \frac{y^{-1} F_\nu(\eta y)}{f_D(y + 2nax - |z^D - a_j|)}, \]  

where \( f_\mu(x) = K_\mu(x)/x^\mu \), \( J=D,N \) for Dirichlet and Neumann boundary conditions, \( \delta^{(D)} = 1 \), \( \delta^{(N)} = -1 \). Note that, as in the case of the Minkowski bulk, the coordinate dependent parts in the VEV have opposite signs for Dirichlet and Neumann scalars. By taking into account that for real values \( 0 \leq \nu < 1 \) the function \( F_\nu(\eta y) \) is non-negative and the function \( f_\mu(x) \) is monotonically decreasing, from (33) we conclude that the interference term for Dirichlet and Neumann scalars is always positive.

Now we turn to the investigation of the interference part in the VEV of the field squared in the asymptotic regions of the ratio \( a/\eta \). This ratio is the proper distance between the plates in units of the dS curvature scale \( \alpha \). For small values of \( a/\eta \) the main contribution to (30) comes from large values of \( y \), for which we have \( F_\nu(y) \approx y^{D-1} \). As a result, at leading order the interference part in the VEV of the field squared coincides with the corresponding quantity for a conformally coupled massless field, and is given by Eq. (32). In particular, this part is positive for Dirichlet or Neumann BCs on both plates, and is negative for Dirichlet BC in one plate and Neumann on the other.

For large proper distances between the plates one has \( a/\eta \gg 1 \). In order to find the leading terms in the corresponding asymptotic expansion we introduce in (30) new integration variables \( u = ax/\eta \) and \( v = ay/\eta \). In these variables the argument of the function (25) becomes \( \eta y/a \) and in the limit under consideration it is small. By using the asymptotic formulae for the modified Bessel functions to the leading order, we have
\[ F_\nu(y) \approx \sigma_\nu \text{Re} \left[ \frac{\Gamma(\nu)y^{D-2\nu}}{\Gamma(1-\nu)} \right], \quad y \ll 1, \]  

(34)
where \( \sigma_\nu = 1 \) and \( \sigma_\nu = 2 \) for real and imaginary \( \nu \), respectively. By taking into account this relation, as the next step we apply the integration formula

\[
\int_0^\infty dv v^{1-2\nu} \int_\nu^\infty du (u^2 - v^2)^{(D-3)/2} f(u) = \frac{\Gamma(1-\nu)\Gamma((D-1)/2)}{2\Gamma((D+1)/2-\nu)} \int_0^\infty dr r^{D-2\nu-1} f(r),
\]

for a given function \( f(u) \). Assuming also that \(|\beta_j|/a \ll 1\), for the remaining integral we can use the expression

\[
\int_0^\infty dx x^{D-2\nu-1} e^{-x|z^D-a_j|/a} = \sum_{n=1}^\infty \frac{(\delta_1 \delta_2)^n \Gamma(D-2\nu)}{(n+|z^D-a_j|/a)^{D-2\nu}},
\]

where \( \delta_j = c_j(0) \). Note that \( \delta_j = -1 \) for non-Neumann BC \((|\beta_j| < \infty)\) on the plate at \( a_j \), while \( \delta_j = 1 \) if the BC is Neumann. Finally, by using the duplication formula for the gamma function in (36), we find the following asymptotic behavior

\[
\Delta \langle \varphi^2 \rangle \approx \frac{\sigma_\nu \alpha^{1-D}}{(4\pi)^{D/2+1}(a/\eta)^D} \text{Re}[(2a/\eta)^{2\nu} g_\nu(z^D)],
\]

where we have defined the function

\[
g_\nu(z) = \Gamma(\nu)\Gamma((D/2-\nu)) \sum_{n=1}^\infty (\delta_1 \delta_2)^n \left[ 2n^{2\nu-D} + \sum_{j=1,2} \delta_j (n+|z-a_j|/a)^{2\nu-D} \right].
\]

As it is seen from Eq. (37), for large separations of the plates the behavior of the interference term is qualitatively different for real and for imaginary values of the parameter \( \nu \). For positive values of \( \nu \), to leading order, we find

\[
\Delta \langle \varphi^2 \rangle \approx \frac{\alpha^{1-D} g_\nu(z^D)}{4\pi^{D/2+1}(2a/\eta)^{D-2\nu}},
\]

For imaginary \( \nu \), the asymptotic behavior at large distances is of the form

\[
\Delta \langle \varphi^2 \rangle \approx \frac{|g_\nu(z^D)| \cos[2|\nu| \ln(2a/\eta) + \phi]}{2\pi^{D/2+1}\alpha^{D-1}(2a/\eta)^D},
\]

with \( g_\nu(z^D) = |g_\nu(z^D)| e^{i\phi} \). Hence, in this case the decay of the interference part is oscillatory. At a given spatial point, the dependence on the synchronous time coordinate has the form \( \Delta \langle \varphi^2 \rangle \sim e^{-Dt/\alpha} \cos[2|\nu|t/\alpha + \psi] \). One gets similar oscillations for the single plate parts at large distances from the plate (see Ref. [9]). Note that, for a given \( a \), the limit under consideration corresponds to the one for the late stages of the cosmological expansion.

### 4 Vacuum expectation value of the energy-momentum tensor

For the evaluation of the VEV of the energy-momentum tensor in the region between the plates, we use

\[
\langle 0| T_{ik} | 0 \rangle = \lim_{x' \to x} \partial_i \partial_k' W(x, x') + \left[ (\xi - 1/4) g_{ik} \nabla_i \nabla^l - \xi \nabla_i \nabla_k - \xi R_{ik} \right] \langle \varphi^2 \rangle,
\]

where \( R_{ik} = D g_{ik}/\alpha^2 \) is the Ricci tensor for the dS spacetime. Taking advantage of the expressions for the Wightman function and for the VEV of the field squared from the previous
sections, the renormalized VEVs for the diagonal components of the energy-momentum tensor can be expressed in the form (no summation over $l$)

\[
\langle T^l_l \rangle = \langle T^l_l \rangle_{\text{dS}} + \langle T^l_l \rangle_j + \frac{A_D}{\alpha^{D+1}} \int_0^\infty dy y^{1-D} \int_y^\infty dx H(x, y) \times \left[ g(\beta_j x/\eta, |z^D - a_j| x/\eta) G_l(y) + 2G_l x^2 F_\nu(y) \right],
\]

(42)

where we have introduced the notations

\[
G_0(y) = \left[ (y^2/4) \partial^2_y - D(\xi + \xi_D) y \partial_y + D^2 \xi 
+ m^2 a^2 - y^2 + (1 - 4\xi) x^2 \right] F_\nu(y),
\]

\[
G_D(y) = \left\{ \left( \xi - \frac{1}{4} \right) y^2 \partial^2_y + \left[ (2 - D) + \frac{D - 1}{4} \right] y \partial_y - \xi D \right\} F_\nu(y),
\]

\[
G_l(y) = G_{D}(y) + \left[ \frac{y^2 - x^2}{D - 1} + (1 - 4\xi) x^2 \right] F_\nu(y), \quad l = 1, \ldots, D - 1,
\]

and

\[
G_D = 1, \quad G_l = 4\xi - 1, \quad l = 0, 1, \ldots, D - 1.
\]

(43)

Note that, though not explicitly written, the functions $G_l(y)$ with $l = 0, \ldots, D - 1$ do depend on $x$ as well. In Eq. (42), $\langle T^l_l \rangle_{\text{dS}}$ is the corresponding renormalized VEV in dS spacetime without boundaries. For points away from the plates, renormalization is strictly necessary for this part only. Owing to the dS invariance of the Bunch–Davies vacuum, the part $\langle T^l_l \rangle_{\text{dS}}$ is proportional to the metric tensor with a constant coefficient and has been well investigated in the literature [13, 18]. For the part induced by a single plate at $z^D = a_j$, one has [9] (no summation over $l$)

\[
\langle T^l_l \rangle_j = \frac{A_D}{\alpha^{D+1}} \int_0^\infty dy y^{1-D} \int_y^\infty dx (x^2 - y^2)^{(D-3)/2} \times \frac{e^{-2x|z^D - a_j| / \eta}}{c_j(x/\eta)} G_l(y).
\]

(45)

The last term on the rhs of Eq. (42) is induced by the presence of the second plate.

For the non-zero off-diagonal component, we have

\[
\langle T^l_0 \rangle = \langle T^l_0 \rangle_j - \text{sgn}(z^D - a_j) \frac{A_D}{2\alpha^{D+1}} \int_0^\infty dy y^{1-D} G_{0D}(y) \times \int_y^\infty dx H(x, y) \left[ c_j(x/\eta) e^{2x|z^D - a_j| / \eta} - e^{-2x|z^D - a_j| / \eta} / c_j(x/\eta) \right],
\]

(46)

where the part corresponding to the geometry of a single plate is given by

\[
\langle T^l_0 \rangle_j = \text{sgn}(z^D - a_j) \frac{2A_D}{\alpha^{D+1}} \int_0^\infty dy y^{1-D} G_{0D}(y) \times \int_y^\infty dx x(x^2 - y^2)^{(D-3)/2} e^{-2x|z^D - a_j| / \eta} / c_j(x/\eta).
\]

(47)

In these formulae we have defined the function

\[
G_{0D}(y) = [(4\xi - 1) y \partial_y + 4\xi] F_\nu(y).
\]

(48)

The off-diagonal component (46) corresponds to the energy flux along the direction perpendicular to the plates. This type of the energy flux also appears in the geometry of a cosmic string on
backgrounds of Friedmann–Robertson–Walker and dS spacetimes [19]. Depending on the values of the coefficients in the boundary conditions and of the field mass this flux can be positive or negative. As an additional check of the expressions for the energy-momentum tensor, it can be seen that the ones for the single plate contribution and for the second plate induced part fulfill the trace relation

$$\langle T^l_l \rangle = \left[ D(\xi - \xi_D)\nabla^l \nabla^l + m^2 \right] \langle \varphi^2 \rangle. \quad (49)$$

The boundary induced part in the VEV of the energy-momentum tensor is traceless for a conformally coupled massless scalar. The trace anomaly is contained in the boundary-free part only.

For a conformally coupled massless scalar field ($\xi = \xi_D, m = 0$) the single plate part in the VEV of the energy-momentum tensor vanishes and one finds

$$\langle T^l_k \rangle = \langle T^l_l \rangle_{dS} - \frac{(\eta/\alpha)^{D+1}}{(4\pi)^{D/2} \Gamma(D/2 + 1)} \text{diag}(1, \ldots, 1, -D) \times \int_0^\infty \! dx \frac{x^D}{c_1(x)c_2(x)e^{2ax} - 1}. \quad (50)$$

As in the case of the field squared, the boundary induced part in this formula could have been obtained from the corresponding result for the Casimir effect in Minkowski spacetime, by using the fact that the two problems are conformally related. The electromagnetic field in $D = 3$ is conformally invariant and the Casimir problem with two perfectly conducting parallel plates is reduced to the corresponding problem with two scalar modes with Dirichlet and Neumann BCs. In this case, the single plate parts vanish and for the interference part we have $\Delta \langle T^k_k \rangle = - (\pi^2/720)(\alpha a/\eta)^{-4} \text{diag}(1, 1, 1, -3)$. In the case $D > 3$, the electromagnetic field is not conformally invariant and we expect that the corresponding VEV will depend on the distance from the plates. However, this case requires further consideration.

The VEVs for the components of the energy-momentum tensor can be written in the more symmetric form

$$\langle T^l_k \rangle = \langle T^l_l \rangle_{dS} + \sum_{j=1,2} \langle T^l_k \rangle_j + \Delta \langle T^l_k \rangle, \quad (51)$$

where for the interference terms we have (no summation over $l$)

$$\Delta \langle T^l_j \rangle = \frac{A_D}{\alpha^{D+1}} \int_0^\infty \! dy \ y^{1-D} \int_y^\infty \! dx \ H(x, y)$$

$$\times \left[ h(x/\eta, z^D)G_l(y) + 2G_l x^2 F_\nu(y) \right], \quad (52)$$

$$\Delta \langle T^D_0 \rangle = \frac{A_D}{2\alpha^{D+1}} \int_0^\infty \! dy \ y^{1-D} \int_y^\infty \! dx \ xH(x, y)$$

$$\times \sum_{j=1,2} \text{sgn}(z^D - a_j) \frac{e^{-2x|z^D - a_j|/\eta}}{c_j(x/\eta)} G_{0D}(y). \quad (53)$$

Note that when the coefficients in the BCs are the same, $\beta_1 = \beta_2$, the energy flux vanishes at the point $z^D = (a_1 + a_2)/2$. Of course, this is a direct consequence of the symmetry of the problem.

In the special cases of Dirichlet and Neumann BCs, expressions similar to (53) can be obtained for the interference terms in the VEVs of the energy-momentum tensor. Here we give the corresponding formulas for the interference parts in the normal stress and in the off-diagonal
component:
\[ \Delta \langle T_{\nu}^D \rangle = \frac{4\alpha^{-D-1}}{(2\pi)^{D/2+1}} \sum_{n=1}^{\infty} \int_0^\infty dy \left\{ y \left[ (D-1)f_{D/2}(z) + f_{D/2-1}(z) \right] F_{\nu}(y) + \left[ f_{D/2-1}(z) - \frac{\delta(D)}{2} \sum_{j=1,2} f_{D/2-1}(z + 2y|z^D - a_j|/\eta) \right] \frac{G_{D}(y)}{y} \right\} z = 2y\eta a/\eta, \] (54)

\[ \Delta \langle T_{0}^D \rangle = -\frac{\delta(D)\alpha^{-D-1}}{(2\pi)^{D/2+1}} \sum_{n=1}^{\infty} \int_0^\infty dy \sum_{j=1,2} \text{sgn}(z^D - a_j) \times z f_{D/2}(z)|z = 2y(a_n + |z^D - a_j|/\eta) G_{0D}(y), \] (55)

with \( J=D,N \) and being the function \( f_{\mu}(z) \) defined after the formula (33). As for the case of the field squared, the coordinate dependent parts have opposite signs for Dirichlet and Neumann boundary conditions, respectively.

We now wish to examine the behavior of the VEV of the energy-momentum tensor in the asymptotic regimes of small and of large separations between the plates. At small separation, assuming that \( a/\eta \ll 1 \), we introduce in (52), (53) new integration variables \( u = ax/\eta \) and \( v = ay/\eta \). The arguments of the functions \( G_{D}(y) \) and \( F_{\nu}(y) \) are large and we can use the corresponding asymptotic formulae for the modified Bessel functions. In particular, one has \( F_{\nu}(y) \approx y^{\nu-1} \) and

\[ G_{0}(y) \approx \left[ (1 - 4\xi) x^2 - y^2 \right] y^{D-1}, \quad G_{D}(y) \approx D(\xi - \xi) y^{D-1}, \]
\[ G_{l}(y) \approx G_{0}(y) + \frac{Dy^2 - x^2}{D-1} y^{D-1}, \quad l = 1, \ldots, D - 1. \] (56)

For the further evaluation of the integrals, we use Eq. (55) with \( \nu = \pm 1/2 \). As a result, to leading order, one has (no summation over \( l \))

\[ \Delta \langle T_{l}^{\nu} \rangle \approx \frac{2(4\pi)^{-D/2}}{\Gamma(D/2)(\alpha/\eta)^{D+1}} \int_0^\infty dx x^{D/2} \left[ B_l h(x, z^D) + C_l \right] c_1(x) c_2(x) e^{2ax - 1}, \]
\[ \Delta \langle T_{0}^{D} \rangle \approx \frac{2(4\pi)^{-D/2} D(\xi - \xi) D}{\Gamma(D/2)(\alpha/\eta)^{D}} \sum_{j=1,2} \int_0^\infty dx \frac{\text{sgn}(z^D - a_j) x^{D-1} e^{-2|z^D - a_j|}}{c_1(x) c_2(x) e^{2ax - 1} c_j(x)}, \] (57)

where
\[ B_l = -2(\xi - \xi), \quad B_D = 0, \quad C_l = 4\xi - 1, \quad C_D = 1, \] (58)

with \( l = 0, 1, \ldots, D - 1 \). Note that \( \Delta \langle T_{0}^{D} \rangle \sim (a/\eta) \Delta \langle T_{l}^{\nu} \rangle \) and that, for a conformally coupled field, the leading terms in the diagonal components are homogeneous. For small separation, to leading order, the energy density is equal to the stresses along the directions parallel to the plates. The vacuum stress normal to the plates, \( \Delta \langle T_{0}^{D} \rangle \), is positive for Dirichlet and Neumann BCs, and is negative for Dirichlet BC on one plate and Neumann on the other.

Now let us discuss the asymptotics at large distances between the plates, \( a/\eta \gg 1 \), when the curvature effects are essential. The corresponding asymptotic formulae for the interference parts in the VEV of the energy-momentum tensor are found in a way similar to that already described for the VEV of the field squared in Sect. 3. For positive values of the parameter \( \nu \), the leading terms have the form (no summation over \( l \))

\[ \Delta \langle T_{l}^{\nu} \rangle \approx \frac{f_l \alpha^{-D-1} g_{\nu}(z^D)}{4\pi^{D/2+1} (2a/\eta)^{D-2\nu}}, \]
\[ \Delta \langle T_{0}^{D} \rangle \approx \frac{\alpha^{-D-1} g_{\nu}^{(0)}(z^D)}{4\pi^{D/2+1} (2a/\eta)^{D-2\nu+1}}, \] (59)
where the function \( g_{\nu}(z) \) is defined by Eq. \((38)\) and

\[
g^{(0)}_{\nu}(z) = \big[(4\xi - 1)(D - 2\nu) + 4\xi]\Gamma(D/2 - \nu + 1) \\
\times \Gamma(\nu) \sum_{n=1}^{\infty} \sum_{j=1,2} \frac{\text{sgn}(z - a_j)\delta_1\delta_2\delta_{j}}{(n + |z - a_j|/a)^{D-2\nu+1}}. \tag{60}
\]

The coefficients \( f_l \) for separate components of the energy-momentum tensor are defined as

\[
f_l = (2\nu/D)f_0 = -2\nu[\xi + (\xi - 1/4)(D - 2\nu)], \tag{61}
\]

with \( l = 1, \ldots, D \). Note that for minimally and conformally coupled massless fields \( f_0 = f_l = 0 \) and thus the leading terms vanish. As we see, in the limit under consideration the vacuum stresses are isotropic and \(|\Delta\langle T^D_0\rangle| \ll |\Delta\langle T^l_0\rangle|\). The corresponding equation of state is of barotropic type: \( \Delta\langle T^1_0\rangle \approx \cdots \approx \Delta\langle T^D_0\rangle \approx (2\nu/D)\Delta\langle T^0_0\rangle \). A similar relation holds for the single plate parts at large distances \([9]\). As a consequence, the equation of state parameter for the total boundary-induced part is equal to \(-2\nu/D\), and it is negative.

For imaginary \( \nu \) the leading terms at large distances are given by

\[
\Delta\langle T^1_0\rangle \approx \frac{\alpha^{-D-1}|g_{\nu}(z^D)||f_l|}{2\pi^{D/2+1}(2a/\eta)^D} \cos[2\nu|\ln(2a/\eta) + \phi + \phi_l|], \\
\Delta\langle T^D_0\rangle \approx \frac{\alpha^{-D-1}|g^{(0)}_{\nu}(z^D)|}{2\pi^{D/2+1}(2a/\eta)^{D+1}} \cos[2\nu|\ln(2a/\eta) + \phi_0^{(D)}|]. \tag{62}
\]

In these formulae \( f_l = |f_l|e^{i\phi_l} \) and \( g^{(0)}_{\nu}(z)=|g^{(0)}_{\nu}(z^D)|e^{i\phi_0^{(D)}} \). In this case, the damping of the interference parts as functions of the proper distance is oscillatory. In terms of the synchronous time coordinate, at a given spatial point we have damping oscillations in accordance with \( \Delta\langle T^1_0\rangle \sim e^{-Dt/\alpha}\cos(2\nu|t/\alpha + \psi_l|) \). From \((61)\) it follows that the oscillations in the energy density and in the vacuum stresses are shifted in phase by \( \pi/2 \).

In Fig.\((1)\) for the case of a \( D = 3 \) conformally coupled scalar field with Dirichlet BCs on both plates, we have plotted the energy flux as a function of \( z^D/\eta \), for given values of \( a/\eta = 1 \) and \( m\alpha = 1/4, 1 \) (left plot), and as a function of \( m\alpha \) for given values of \( a/\eta = 1 \) and \( z^D/\eta = 0.3 \) (right plot). For Neumann BCs the flux has the opposite sign.

## 5 Casimir force

In this section we consider the vacuum forces acting on the plates. The vacuum force acting per unit surface of the plate at \( z^D = a_j \) is determined by the \( D \) component of the vacuum energy-momentum tensor evaluated at this point. For the region between the plates, the corresponding effective pressures can be written as a sum of two terms, namely

\[
p^{(j)} = p^{(j)}_l + p^{(j)}_{(\text{int})}, \quad j = 1, 2. \tag{63}
\]

The first term on the rhs is the pressure for a single plate at \( z^D = a_j \), when the second plate is absent. This term is divergent due to the surface divergences in the subtracted vacuum expectation values and needs additional renormalization. The second term on the rhs of Eq. \((63)\) is the pressure induced by the presence of the second plate, and can be termed as an interaction force. This contribution is finite for all nonzero distances between the plates. In the regions \( z^D < a_1 \) and \( z^D > a_2 \) we have \( p^{(j)} = p^{(j)}_l \). As a result, the contributions to the vacuum force
Figure 1: Energy flux in the region between the plates as a function of \( z^D/\eta \) for \( a/\eta = 1, \) \( m\alpha = 1/4, 1 \) (left plot), and as a function of \( m\alpha \) for \( a/\eta = 1, \) \( z^D/\eta = 0.3 \) (right plot), for a \( D = 3 \) conformally coupled scalar field with Dirichlet boundary condition. On the left plot the figures near the curves are the values of the parameter \( m\alpha, \) and the scaling factor \( n \) is equal to 2 and 3, for \( m\alpha = 1 \) and \( m\alpha = 1/4, \) respectively.

coming from the term \( p^{(j)}_1 \) are the same from the left and from the right sides of the plate, so that there is no net contribution to the effective force.

The interaction force on the plate at \( z^D = a_j \) is obtained from the last term on the rhs of Eq. (42) for \( \langle T^D_{\parallel} \rangle \) (with minus sign) substituting \( z^D = a_j, \) that is

\[
p^{(j)}_{\text{(int)}} = -\frac{2A_D}{\alpha^{D+1}} \int_0^\infty dy \int_y^\infty dx \, x^2 H(x, y) \times \left[ \frac{2 (\beta_j/\eta)^2 G_D(y)}{(\beta_j x/\eta)^2 - 1} + F_\nu(y) \right].
\]

Depending on the values of the coefficients in the boundary conditions, the effective pressures (64) can be either positive or negative, leading to repulsive or to attractive forces, respectively. For \( \beta_1 \neq \beta_2 \) the Casimir forces acting on the left and on the right plates are different. In the Appendix we show that, as it must be, in the limit \( \alpha \to \infty \) the corresponding result for the geometry of two parallel plates in Minkowski spacetime is obtained.

The general formula is further simplified for the special cases of Dirichlet and of Neumann boundary conditions. For Dirichlet BCs on both plates, we find

\[
p^{(D)}_{\text{(int)}} = -\frac{4\alpha^{D-1}}{(2\pi)^{D/2+1}} \sum_{n=1}^\infty \int_0^\infty dy \, y F_\nu(y) \left[ (D - 1) f_{D/2}/(2\eta y) + f_{D/2-1}/(2\eta y) \right].
\]

For \( 0 \leq \nu < 1 \) the integrand in this formula is positive and \( p^{(D)}_{\text{(int)}} \) is negative, yielding an attractive force for all separations. In the case of Neumann BCs the corresponding expression is

\[
p^{(N)}_{\text{(int)}} = p^{(D)}_{\text{(int)}} - \frac{8\alpha^{D-1}}{(2\pi)^{D/2+1}} \sum_{n=1}^\infty \int_0^\infty dy \, \frac{G_D(y)}{y} f_{D/2-1}/(2\eta y),
\]

where the function \( G_D(y) \) is defined in Eq. (43).
Let us now investigate the asymptotic behavior of the vacuum forces. This can be done in the way similar to that already described in the case of the VEV for the energy-momentum tensor. In the limit of small proper distances between the plates, \( a/\eta \ll 1 \), the main contribution to the integral over \( y \) comes from large values of \( y, y \sim \eta/a \). By using the corresponding asymptotics for the modified Bessel functions, to leading order we find

\[
p_{(\text{int})}^{(j)} \approx -\frac{2(\eta/\alpha)^{D+1}}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty dx \frac{x^D}{c_1(x)c_2(x)e^{2ax} - 1}.
\]

(67)

If, in addition, \(|\beta_j|/a \gg 1\), one has

\[
p_{(\text{int})}^{(j)} \approx -\frac{D\Gamma((D + 1)/2)\zeta_R(D + 1)}{(4\pi)(D+1)/2(\alpha a/\eta)^{D+1}},
\]

(68)

and the corresponding force is attractive. In (68), \( \zeta_R(x) \) is the Riemann zeta function. The same result is obtained for Dirichlet BCs on both plates. In the case of Dirichlet BC on one plate and non-Dirichlet one on the other, the leading term is obtained from (68) with an additional factor \((2^{-D} - 1)\). In this case the vacuum force at small distance is repulsive.

Now we consider the large distance asymptotics, \( a/\eta \gg 1 \). The cases of real and imaginary \( \nu \) must be studied separately. For positive values of \( \nu \), one has

\[
p_{(\text{int})}^{(j)} \approx -\frac{2\alpha^{-D-1}g_{\nu}^{(j)}}{\pi^{D/2+1}(2a/\eta)^{D-2\nu+2}}, \quad |\beta_j| < \infty,
\]

\[
p_{(\text{int})}^{(j)} \approx -\frac{\alpha^{-D-1}g_{\nu}^{N(j)}}{\pi^{D/2+1}(2a/\eta)^{D-2\nu}}, \quad \beta_j = \infty,
\]

(69)

where we have introduced the notations

\[
g_{\nu}^{(j)} = [(D + 1)/2 - \nu] \Gamma(D/2 - \nu + 1)\Gamma(\nu) \times [1 - 2(\beta_j/\eta)^2 f_D] \sum_{n=1}^\infty \frac{(\delta_1 \delta_2)^n}{n^{D-2\nu+2}}, \quad \delta_1 \delta_2 = \frac{a_1}{a_2}
\]

\[
g_{\nu}^{N(j)} = \Gamma(D/2 - \nu)\Gamma(\nu)f_D \sum_{n=1}^\infty \frac{(\delta_1 \delta_2)^n}{n^{D-2\nu+2}}
\]

(70)

and \( f_D \) is defined by Eq. (61). Note that \( \beta_j = \infty \) corresponds to Neumann BC. In the case of non-Neumann BCs we have assumed that \( |\beta_j|/a \ll 1 \). As it is seen from (69), when \( f_D \neq 0 \), at large distances the ratio of the Casimir forces acting on the plate with Neumann and non-Neumann BCs is of the order \((a/\eta)^2\). Note that in neither of these cases does the force depend on the specific value of Robin coefficient in the BC on the second plate. For Dirichlet BC on the plate at \( z^D = a_j \) (\( \beta_j = 0 \)), at large separations the Casimir force acting on that plate is repulsive (attractive) for Neumann (non-Neumann) BCs on the other plate. The nature of the force acting on the plate with Neumann BC depends on the sign of \( f_D \) and can be either repulsive or attractive, in function of the curvature coupling parameter and of the field mass. For minimally and conformally coupled massive scalar fields one has \( f_D = \nu(D/2 - \nu) \) and \( f_D = \nu(1/2 - \nu)/D \), respectively, and this parameter is positive. The corresponding force is attractive (repulsive) for Neumann (non-Neumann) BC on the second plate. Note that for the geometry of parallel plates in the Minkowski bulk the Casimir forces at large distances are repulsive for Neumann BC on one plate and for non-Neumann BC on the other plate. For all other cases of BCs the forces are attractive.
At large separations between the plates and for imaginary \( \nu \), the leading order terms in the corresponding asymptotic expansions take the form

\[
\begin{align*}
\tilde{p}^{(j)}_{\text{int}}(\nu) & \approx -\frac{4\alpha^{-D}}{\pi^{D/2+1}(2a/\eta)^{D+1}} \cos[2\nu \ln(2a/\eta) + \phi(j)], \ |\beta_j| < \infty, \\
\tilde{p}^{(j)}_{\text{int}}(\nu) & \approx -\frac{2\alpha^{-D}}{\pi^{D/2+1}(2a/\eta)^{D+1}} \cos[2\nu \ln(2a/\eta) + \phi_{\nu}(j)], \ \beta_j = \infty,
\end{align*}
\]

where we have defined \( g_{\nu}^{(j)} = |g_{\nu}^{(j)}| e^{\phi_{\nu}(j)} \) and \( g_{\nu}^{N(j)} = |g_{\nu}^{N(j)}| e^{i\phi_{\nu}^{N}(j)} \). In such case the decay of the vacuum forces is oscillatory. In terms of the synchronous time coordinate one gets the behavior

\[
\tilde{p}^{(j)}_{\text{int}} \sim \exp[(D + 2 - 2\delta_{D,1}/\beta_j)t/\alpha] \cos[2\nu t/\alpha + \psi_p].
\]

In Figs. 2 and 3 we have plotted the Casimir force for a \( D = 3 \) scalar field, conformally and minimally coupled to gravity, respectively, as a function of the proper distance between the plates, measured in units of the dS curvature scale, \( \alpha \). The left (resp. right) plots are for Dirichlet (resp. Neumann) BCs on both plates. The figures near the curves correspond to the values of the parameter \( m_{\alpha} \). Values are taken in a way so to have both possibilities, with positive and purely imaginary values of the parameter \( \nu \), with corresponding monotonic and oscillatory behavior of the forces at large distances, respectively. Note also, in particular, the plots corresponding to a massless field in the two cases, corresponding to a photon-like contribution.

Figure 2: Interaction forces between the plates for a \( D = 3 \) conformally coupled scalar field with Dirichlet (left plot) and Neumann (right plot) BCs. The figures near the curves correspond to the values of the parameter \( m_{\alpha} \). Note in particular the plots corresponding to a massless field in the two cases.

In Fig. 4 the dependence of the Casimir force on the parameter \( m_{\alpha} \) for a given separation corresponding to \( a/\eta = 3 \) is depicted. Conformally coupled scalar fields with Dirichlet and Neumann BCs, respectively, are considered.

From the discussion given above it follows that for the proper distances between the plates larger than the curvature radius of the dS spacetime, \( \alpha a/\eta \gtrsim \alpha \), the gravitational field essentially changes the behavior of the Casimir forces compared with the case of the plates in Minkowski spacetime. In particular, the forces may become repulsive at large separations between the plates. In particular, for real values \( \nu \) and for Neumann BC on both plates, Casimir forces are repulsive at large separations, in the range of parameters for which \( f_D < 0 \) [see Eqs. (69), (70)]. Recall that, for the geometry of parallel plates on the background of Minkowski spacetime,
Figure 3: Same as in Fig. 2 for a minimally coupled scalar field.

Figure 4: Interaction force between the plates for $a/\eta = 3$ as a function of the field mass, for a $D = 3$ conformally coupled scalar field with Dirichlet and Neumann BCs.
the only case with repulsive Casimir forces at large distances corresponds to Neumann BC on one plate and non-Neumann BC on the other. A remarkable feature of the influence of the gravitational field is the oscillatory behavior of the Casimir forces at large distances, which appears in the case of imaginary \( \nu \). In this case, the values of the plate distance yielding zero Casimir force correspond to equilibrium positions. Among them, the positions with negative derivative of the force with respect to the distance are locally stable.

In the discussion above we have considered the expectation values assuming that the field is prepared in the Bunch–Davies vacuum state. This corresponds to the effect of vacuum polarization by boundary conditions. In states containing particles the expectation values of physical observables will receive additional contributions. For example, the expectation value of the field squared has the form \( \langle \varphi^2 \rangle = \langle \varphi^2 \rangle_{\text{BD}} + 2 \sum_i n_i \varphi_i(x) \varphi_i^* (x) \), where \( n_i \) is the number of particles with the set of quantum numbers \( \sigma \) and \( \langle \varphi^2 \rangle_{\text{BD}} \) is the expectation value in the Bunch–Davies vacuum state. On the base of this relation, we can consider the effects of the boundaries at finite temperature assuming that the field is in thermodynamical equilibrium. In this case, the VEVs are changed by the thermodynamical expectation values. However, it should be noted that, owing to the time dependence of the background spacetime, we can talk about thermodynamical equilibrium in the adiabatic limit only. In dS spacetime this corresponds to the conditions \( \eta K \gg m \alpha \) and \( \eta K \gg 1 \), for the modes with wave number \( K \) (see also the discussion in [14]). By taking into account that, at temperature \( T \), the main contribution to the thermodynamic expectation values comes from the region \( K \lesssim T \), we obtain the conditions \( T_c \gg 1/\alpha \) and \( T_c \gg m \), where \( T_c = \eta T / \alpha \) is the comoving temperature. The dominant contribution to the boundary induced expectation values comes from the fluctuations with \( K \lesssim 1/a \). Combining this with the estimates given above, we conclude that the adiabatic approximation for the boundary induced expectation values corresponds to the limit \( a/\eta \ll 1 \). As has been shown above, in this limit the leading terms in the VEVs induced by the plates coincide with the corresponding quantities for the geometry of parallel plates in Minkowski spacetime. The same is true for the thermal corrections.

6 Conclusion

Amongst the most interesting topics in the investigation of the Casimir effect is the dependence of the characteristics of the vacuum fluctuations on the background geometry. In the present paper we have considered the classical geometry of two parallel plates on the background of dS spacetime for a scalar field with Robin boundary conditions on the plates. The general case has been investigated when the constants in the Robin boundary conditions are different for the two separate plates. In the region between the plates, the Wightman function has been obtained and displayed under the form of a mode sum involving series over zeros of the function defined by Eq. (8). For the summation of this series we have made use of expression (12). This has allowed us to extract, from the Wightman function, the part coming from a single plate, and to express the additional part in terms of integrals, which are exponentially convergent in the coincidence limit. The single plate contribution was investigated previously, in Ref. [9]. The contribution induced by the second boundary has been presented in two alternative forms, as given by Eqs. (14) and (22). By using the expression of the Wightman function, we have evaluated the VEVs of the field squared and of the energy-momentum tensor, in the region between the plates. These VEVs are decomposed into a boundary-free dS, a single plate-induced and an interference contribution, respectively. The last one, for the cases of the field squared and energy-momentum tensor, is given by Eqs. (30) and (52), (53), respectively. The vacuum energy-momentum tensor is non-diagonal, with the off-diagonal component corresponding to the energy flux along the direction normal to the plates. In the case of a conformally coupled
massless field, the total single plate contribution to the VEV of the energy-momentum tensor vanishes and the interference part is obtained from the corresponding result for the Minkowski bulk, by standard conformal transformation.

Various limiting cases have been studied. In the limit of small distances between the plates the interference part in the VEV of the field squared coincides, to leading order, with the corresponding quantity for a conformally coupled massless field, and is given by Eq. (52). The leading terms of the interference parts of the VEV for the energy momentum tensor are given by expressions (57). For a conformally coupled scalar field, the leading term of the off-diagonal component vanishes, and the leading terms of the diagonal components are homogeneous. In the opposite asymptotic limit of large separations between the plates, the behaviors of the interference parts crucially depend on the value of the parameter \( \nu \), defined by Eq. (7). For positive values of this parameter, the leading terms of the corresponding asymptotic expansions are given by Eqs. (39) and (59), for the field squared and the energy-momentum tensor, respectively. The interference contributions for the field squared and the diagonal components of the energy-momentum tensor decay as \( 1/(a/\eta)^{D-2\nu} \), whereas the off-diagonal component decays like \( 1/(a/\eta)^{D-2\nu+1} \). To leading order, the vacuum stresses are isotropic and the boundary induced VEV of the energy-momentum tensor corresponds to a gravitational source of barotropic type, with equation of state parameter equal to \(-2\nu/D\). At large separations between the plates and for imaginary values of the parameter \( \nu \), the asymptotic behavior of the interference parts for the field squared and for the energy-momentum tensor are given by Eqs. (40) and (62), respectively. The corresponding behavior is damping oscillatory and the VEVs decay as \((a/\eta)^{-D} \cos[2|\nu| \ln (2a/\eta) + \psi]\), for the field squared and the diagonal components of the energy-momentum tensor. For the off-diagonal component the amplitude decays as \((a/\eta)^{-D-1}\).

The vacuum forces acting on the plates are determined by the \( P_D \)-component of the stress. They have been studied in Sect. 5. The normal stresses on the plates are presented as sums of single plate and interaction contributions. The contributions to the vacuum force coming from the single plate terms are the same from the left and from the right sides of the plate and thus give no contribution to the effective force. The interaction forces per unit surface are determined by formula (64). This expression is further simplified in the special cases of Dirichlet and Neumann BCs, yielding Eqs. (65) and (66), respectively. For small distances between the plates, to leading order the vacuum forces are given by Eq. (67). If, in addition, \(|\beta_j|/a \gg 1\), the vacuum forces are attractive at small distances, except for the case of Dirichlet BC on one plate and non-Dirichlet on the other, in which case the force turns out to be repulsive. At large distances between the plates and for positive values of \( \nu \), the force acting on the plate decays monotonically as \( 1/(2a/\eta)^{D-2\nu+2} \), for non-Neumann BCs, and as \( 1/(2a/\eta)^{D-2\nu} \), in the case of Neumann BCs [see Eqs. (69)]. For imaginary values of \( \nu \) the behavior of the vacuum forces is damping oscillatory, in the leading order described by Eqs. (71). Having in mind that spectral properties of spin 2, 1, 0 operators for dS spacetime are known, the current study can be now extended to the calculation of the Casimir force due to quantum gravity (for the one-loop effective action of arbitrary quantum gravity in dS spacetime see Ref. [21]). This will be considered elsewhere.

From the analysis carried out above, it follows that the curvature of the background spacetime decisively influences the behavior of boundary induced VEVs at distances larger than the curvature scale. As we have seen, when the background is dS spacetime the decay of the VEVs at large separations between the plates is power-law (monotonical or oscillating), independently of the field mass. This is quite remarkable and clearly in contrast with the corresponding features of the same problem in a Minkowski bulk. To wit, the interaction forces between two parallel plates in Minkowski spacetime at large distances decay as \( 1/a^{D+1} \) for massless fields and these forces are exponentially suppressed for massive fields by a factor of \( e^{-2ma} \). For the geometry of
two parallel plates, in AdS spacetime the decay of the vacuum forces at large separations is also exponential (see Ref. [1]), with the suppression factor being determined by the AdS curvature scale. In a way very much similar to the procedure described in Ref. [22] (see also Ref. [23] for finite temperature effects), we are able to treat here the more general case of dS spacetime with compact internal subspaces and piston-like geometries. Note that this calculation can be extended to a self-interacting scalar field theory too, in which case mass becomes an effective mass, proportional to the background scalar. In this way, our results and procedures here can be used to the study of curvature-induced phase transitions of the in-in effective potential in the same way as it was proposed for the out-in effective potential in Ref. [24].

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A Minkowski spacetime limit

In this appendix we will explicitly show the limiting transition of the situations considered above to the geometry of two parallel Robin plates in Minkowski spacetime, for the vacuum interaction forces. In this limit $\alpha \to \infty$ and the modulus of the order of the modified Bessel functions is large, $\nu \approx ima$. In addition, we have $\eta \to \alpha$. Changing the integration variables to $x = u\eta$, $y = v\eta$, we see that the arguments of the modified Bessel functions are large, too. Hence, we make use of the uniform asymptotic expansions for these functions for imaginary values of the order with large modulus. The leading terms in these expansions have the form (see, for example, [20])

$$K_{i\mu}(\mu z) \sim \sqrt{\frac{2\pi}{\mu}}e^{-\mu z/2}\cos[\mu f(z/\mu) - \pi/4],$$

$$I_{i\mu}(\mu z) + I_{-i\mu}(\mu z) \sim -\frac{2e^{\mu z/2}}{\sqrt{2\pi\mu}}\sin[\mu f(z/\mu) - \pi/4],$$

for $z < 1$ and

$$K_{i\mu}(\mu z) \sim \sqrt{\frac{\pi}{2\mu(z^2 - 1)^{1/4}}}e^{-\mu g(z)},$$

$$I_{i\mu}(\mu z) + I_{-i\mu}(\mu z) \sim \frac{2e^{\mu z/2}}{\sqrt{2\pi\mu(z^2 - 1)^{1/4}}}e^{\mu g(z)},$$

for $z > 1$. The functions in these formulas are defined as

$$f(z) = \ln\left(\frac{1 + \sqrt{1 - z^2}}{z}\right) - \sqrt{1 - z^2},$$

$$g(z) = -\arcsin z + \sqrt{z^2 - 1}.\tag{74}$$
From Eqs. (72) and (73) it follows that

\[ \tilde{I}_\nu(y) \tilde{K}_\nu(y) \sim \frac{y^D}{m\alpha} \cos[2m\alpha f(y/m\alpha)], \quad y < m\alpha, \]

\[ \tilde{I}_\nu(y) \tilde{K}_\nu(y) \sim \frac{1}{m\alpha} \frac{y^D}{\sqrt{(y/m\alpha)^2 - 1}}, \quad y > m\alpha. \]

(75)

The main contribution to the force \[ \mathcal{F} \] comes from the region \((m\alpha, \infty)\) of the integration over \(y\); to leading order, we find

\[ p^{(j)}_{\text{(int)}} \approx -\frac{2(4\pi)^{-D/2}}{\Gamma(D/2)} \int_m^\infty du \frac{u^2(u^2 - m^2)^{D/2-1}}{(\beta_1 u^{-1})(\beta_2 u^{-1})} e^{2au} - 1. \]

(76)

This result coincides with the corresponding formula for parallel plates in the Minkowski bulk. In a similar way, the limiting transition for the other quantities can also be checked.

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