GENERALIZATION OF A THEOREM OF WALDSPURGER TO NICE REPRESENTATIONS

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The goal of this paper is to present some examples of equalities of integrals over local fields predicted by the stationary phase approximation. These examples should be considered in the context of the algebraic integration theory proposed by the first author in [6]. From the point of view of this theory, our main theorem gives examples of pairs of algebro-geometric data over a given local field $E$ producing equal integrals over arbitrary finite extensions of $E$. Another motivation for this work is the theory of Sato’s functional equations associated with prehomogeneous vector spaces over local fields. As an application of our techniques we find an explicit form of these equations for the action of $GL_n$ on symmetric $n \times n$ matrices in the case when $n$ is odd (this is a generalization of a particular case of the equations obtained by W. J. Sweet Jr. in [24]).

Let $G$ be a (connected) simply connected semisimple algebraic group over a local field $E$ of characteristic zero, $\rho : G \to \text{Aut}(V)$ be a rational representation defined over $E$.

Definition. A representation $\rho$ is called nice if the generic stabilizer subgroup $H$ is connected and reductive.

1 Note that the adjoint representation of $G$ is nice. Other interesting examples can be found using Elashvili’s tables of representations with positive-dimensional generic stabilizers (see [2],[3]).

Let $V_0 \subset V$ be a non-empty Zariski open affine subset such that the stabilizer of any point in $V_0$ is conjugate to $H$ over $E$, $V_0^\vee \subset V^\vee$ be a similar subset in the dual representation. We are interested in the subspaces $D^{st}_H$, $D^{st}_\epsilon$ of $G(E)$-invariant distributions on $V(E)$ consisting of stable and $\epsilon$-stable distributions. To define these spaces we have to introduce some notations.

Recall that for a non-degenerate quadratic form $q$ over a local field $E$ the Hasse-Witt invariant $\epsilon(q) = \pm 1$ is defined as follows: choose coordinates $x_1, \ldots, x_n$ in such a way that $q = a_1x_1^2 + \ldots + a_nx_n^2$ and set $\epsilon(q) = \prod_{i<j}(a_i, a_j)$ where $(\cdot, \cdot)$ denotes the Hilbert symbol (if rank of $q$ is equal to 1 we set $\epsilon(q) = 1$). Note that if $E = \mathbb{R}$ then $\epsilon(q) = (-1)^{(i-1)/2}$ where $i$ is the number of negative squares in $q$ (for $E = \mathbb{C}$ we have $\epsilon(q) = 1$). For a pair of non-degenerate quadratic forms $q$ and $q'$ we define the relative Hasse-Witt invariant $\epsilon(q, q') := \epsilon(q)\epsilon(q')$.

Let $\mathfrak{g}$ be the Lie algebra of $G$, $Q$ be the Killing form on $\mathfrak{g}$. For every point $x \in V_0(E)$ we denote by $H_x \subset G$ the stabilizer of $x$, by $\mathfrak{h}_x \subset \mathfrak{g}$ its Lie algebra. It is easy to see that the form $Q_x := Q|_{\mathfrak{h}_x}$ is still non-degenerate (see lemma [1.1.4]). Now for every $G$-orbit $O \subset V_0$ we define the relative sign function on $O(E) \times O(E)$

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1 This notion is slightly more general than the one defined in section 7 of [1]
by
\[ \epsilon(x, x') := \epsilon(Q_x, Q_{x'}). \]
Sometimes we will also use the absolute sign function
\[ \epsilon(x) := \epsilon(Q_x) \]
such that \( \epsilon(x, x') = \epsilon(x)\epsilon(x') \). Let us denote by \( p : V_0 \to V_0/G \) the natural projection to the geometric quotient by the action of \( G \).

**Definition.** Let \( \delta \) be a distribution which is given by a locally \( L^1 \) function \( f : \delta = f |dv| \) where \( |dv| \) is the Haar measure on \( V(E) \).

a) We say that \( \delta \) is stable if the function \( f \) is constant on the fibers of the natural projection \( p(E) : V_0(E) \to (V_0/G)(E) \).

b) We say that \( \delta \) is \( \epsilon \)-stable if for (almost) any pair \( x, x' \in V_0(E) \) such that \( p(E)(x) = p(E)(x') \) we have \( f(x) = \epsilon(x, x')f(x') \).

By the definition, the space of all stable (resp. \( \epsilon \)-stable) distributions on \( V(E) \) is the closure of the space of locally \( L^1 \) stable (resp. \( \epsilon \)-stable) distributions (another definition will be given in section 2.3).

Our main result is that in the case when \( E \) is \( p \)-adic, \( \rho \) is nice and either \( G \) is simple or \( H \) is semisimple, the notions of stable and \( \epsilon \)-stable distributions get switched by the Fourier transform. Note that in the case when \( \rho \) is the adjoint representation we have \( \epsilon \equiv 1 \). Thus, in this case our result is that stability is preserved under Fourier transform. This was proven previously by J.-L. Waldspurger (see [23], Cor. 1.6). We conjecture that similar result holds in the case \( E = \mathbb{R} \). We also introduce the “complementary” notions of antistable and \( \epsilon \)-antistable distributions and prove that they get switched by the Fourier transform in the case when \( \rho \) is nice, \( E \) is \( p \)-adic and \( H \) is semisimple.

The proof combines some local computations with global considerations. Our main local tool is the stationary phase approximation applied to linear functionals on \( G \)-orbits. The main global tool we use is the stabilization techniques introduced by R. Kottwitz in [11] to stabilize the elliptic semisimple part of the trace formula for \( G \). More precisely, we prove an analogue of the Kottwitz stabilization formula (Theorem 9.6 of [11]) for arbitrary nice representations over number field.

Using the same method we prove the following result concerning stable distributions and inner forms (sketched in [6] in the case \( \epsilon \equiv 1 \)). Let \( G' \to \text{Aut}(V') \) be an inner form of a nice representation \( G \to \text{Aut}(V) \) over \( E \). Then \( G \)-orbits on \( V \) are in bijection with \( G' \)-orbits on \( V' \). For any \( \bar{x} \in V/G(E) \) we denote by \( \mathcal{O}_{\bar{x}} \subset V, \mathcal{O}'_{\bar{x}} \subset V' \) the preimages of \( \bar{x} \). For any \( x \in \mathcal{O}_{\bar{x}}(E), x' \in \mathcal{O}'_{\bar{x}}(E) \) we define as before \( \epsilon(x, x') = \epsilon(Q_x, Q'_{x'}) \) (where \( Q'_{x'} \) is the restriction of the Killing form \( Q' \) on \( g' \) to \( h_{x'} \)).

**Definition.** a) Let \( \delta = f |dv|, \delta' = f' |dv'| \) be stable distributions on \( V(E), V'(E) \) given by locally \( L^1 \) functions. We write \( \delta \sim \delta' \) if for any \( \bar{x} \in V/G(E), x \in \mathcal{O}_{\bar{x}}(E), x' \in \mathcal{O}'_{\bar{x}}(E) \) we have \( f(x) = f'(x') \).

b) Let \( \delta = f |dv|, \delta' = f' |dv'| \) be \( \epsilon \)-stable distributions on \( V(E), V'(E) \) given by locally \( L^1 \) functions. We write \( \delta \sim_\epsilon \delta' \) if for any \( \bar{x} \in V/G(E), x \in \mathcal{O}_{\bar{x}}(E), x' \in \mathcal{O}'_{\bar{x}}(E) \) we have \( f(x) = f'(x')\epsilon(x, x') \).

We extend these definitions to arbitrary stable (resp. \( \epsilon \)-stable) distributions and prove that in the case when \( E \) is \( p \)-adic and \( G \) is simple, for any stable distributions \( \delta \) and \( \delta' \) on \( V(E) \) and \( V'(E) \) respectively, such that \( \delta \sim_\epsilon \delta' \), we have \( F(\delta) \sim_\epsilon F(\delta') \).

2 The sketch of the proof in the case \( \epsilon \equiv 1 \) was given in [11].
and $W_q$oring $\kappa(G,G')\mathcal{F}(\delta')$ where $\kappa(G,G')$ is the relative Hasse-Witt invariant of Killing forms on $g$ and $g'$. The latter sign coincides with the sign considered by Kottwitz in $[4]$.

Finally, we apply our main theorem to derive the following equation for distributions on the space $\text{Sym}_n(E)$ of symmetric $n \times n$ matrices over a $p$-adic field $E$ in the case when $n$ is odd:

$$\mathcal{F}(\chi(\det)) = c(\chi) \cdot \epsilon \cdot (\det, -1)^{\frac{n-1}{2}} \cdot |\det|^{-\frac{n+1}{2}} \chi^{-1}(\det) \quad (0.0.1)$$

where $\chi$ is a generic multiplicative character of a local field $E$, $\chi(\det)$ is considered as a distribution on $\text{Sym}_n(E)$ (defined by analytic continuation), $\epsilon$ is the function on the complement to the hypersurface $(\det = 0)$ in $\text{Sym}_n(E)$ that assigns to a symmetric matrix the Hasse-Witt invariant of the corresponding quadratic form, $c(\chi)$ is a non-zero constant. In the case when $\chi = |\cdot|^{\delta}$, where $\delta$ is a character of $E^*/(E^*)^2$, the equation $(0.0.1)$ follows from Proposition 4.8 of $[24]$. We show that in the case $E = \mathbb{R}$ the equation $(0.0.1)$ follows from computations of Shintani in $[22]$. This confirms the conjecture that our main theorem holds also in the real case.

The appearance of the sign $\epsilon$ in the above results is due to the connection of the relative Hasse-Witt invariant with Weil constants for quadratic forms. First, let us introduce some more notation concerning quadratic forms. Considering a non-degenerate quadratic form $q$ on an $E$-vector space $L$ as a symmetric isomorphism $L \to L^\vee$ we can define $\det(q) \in \det(L)^{\otimes(-2)}$, where $\det(L) = \bigwedge^{\dim L} L$. By choosing a trivialization of $\det(L)$ we get an element of $E^*$. The corresponding element in $E^*/(E^*)^2$ does not depend on a choice of trivialization (we call it determinant of $q$ modulo squares). If $E$ is $p$-adic, then two non-degenerate quadratic forms $q$ and $q'$ are equivalent over $E$ if and only if $\text{rk}q = \text{rk}q'$, $\det(q) \equiv \det(q') \mod (E^*)^2$ and $\epsilon(q,q') = 1$. More generally, if $E \neq \mathbb{C}$, then these invariants have the following $K$-theoretic meaning. Let $W_E$ be the Witt ring of $E$. By the definition, $W_E$ is the abelian group generated by pairs $(L,q)$ where $L$ is a finite-dimensional $E$-vector space and $q$ is a nondegenerate quadratic form on $L$ and the product in $W_E$ comes from the operation of tensor product $(L',q') \cdot (L'',q'') := (L,q)$ where $L = L' \otimes L''$. Let $W_E^1 \subset W_E$ be the ideal of forms of even dimension, $W_E^2 := (W_E^1)^\perp$. We define $W_E := W_E^1/W_E^{i+1}$. As is well-known $W_E^0 = \mathbb{Z}/2\mathbb{Z}$, $W_E^1 = E^*/(E^*)^2$, $W_E^2 \simeq K_2(E)/2K_2(E) \cong \mathbb{Z}/2\mathbb{Z}$. For a pair of non-degenerate quadratic forms $(q,q')$ of the same rank, we have $(q) - (q') \in W_E^1$ and its image in $W_E^0$ is identified with $\det(q)/\det(q') \mod (E^*)^2$. If the latter element is trivial, then we have $(q) - (q') \in W_E^2$ and its image in $W_E^0$ can be identified with the relative Hasse-Witt invariant $\epsilon(q,q')$. On the other hand, for a fixed (non-trivial) additive character $\psi : E \to U(1)$ and a non-degenerate quadratic form $q$, A. Weil introduced in $[20]$ a constant $\gamma(q,\psi)$ which is a root of unity of order 8 depending only on the equivalence class of $q$ (and on $\psi$). Note that the character $\psi$ defines a canonical Haar measure on $E$ which is self-dual for the Fourier transform. Let $V$ be a vector space over $E$ on which $q$ is defined. For a non-zero top-degree form $\nu$ on $V$ we have the induced Haar measure $|\nu|$ on $V$ (corresponding to the self-dual Haar measure on $E$). The Weil’s constant is defined by the functional equation for distributions

$$\mathcal{F}(\psi(q)) = \gamma(q,\psi) \cdot |\det(q)/\nu|^2 \cdot |\frac{1}{2} \psi(-q^\vee)|, \quad (0.0.2)$$

where $q^\vee$ is the dual quadratic form on $V^\vee$, $\psi(q)$ and $\psi(-q^\vee)$ are considered as distributions on $V$ and $V^\vee$. By the definition, the map $q \mapsto \gamma(q,\psi)$ extends to
a homomorphism from $W_E$ to roots of unity of order 8. As shown in [28] (nos. 25–28), for quadratic forms $q$ and $q'$ such that $(q) - (q') \in W_E^2$ one has
\[
\gamma((q) - (q'), \psi) = \epsilon(q, q').
\]

Here is the plan of the paper. In section 1 we gather some algebraic facts about nice representations. In particular, we prove that the generic stabilizers of a nice representation and of its dual are conjugate to each other (proposition 1.1.3). In section 2 we draw consequences from the stationary phase approximation in the case when the ground field is $p$-adic. In section 3 we prove an analogue of the stable trace formula for nice representations over number fields and combine it with local information from section 2 to derive the main result. Finally, in section 4 we derive the equation (0.0.1).

Notation. All our fields have characteristic zero. $E$ always denotes a local field, while $F$ always denotes a number field. By $p$-adic field we mean a finite extension of $\mathbb{Q}_p$. For a vector space $V$ over a local field $E$ and an open subset $U \subset V$ we denote by $S(U)$ the space of functions with support in $U$ which belong to the Schwartz-Bruhat space of $V$. When we work over a local field $E$, we fix a non-trivial additive character $\psi : E \to U(1)$. We fix the Haar measure on $E$ which is self-dual with respect to the Fourier transform defined in terms of $\psi$. For a smooth variety $X$ over $E$ and a non-vanishing top-degree form $\omega$ on $E$, we denote by $|\omega|$ the measure on $X(E)$ corresponding to $\omega$ and to the above Haar measure on $E$. For a variety $X$ defined over a field $k$ and an extension of fields $k \subset k'$, we denote by $X_{k'}$ the variety over $k'$ obtained from $X$ by the extension of scalars. For an algebraic group $H$ we denote by $Z(H)$ its centre. The group $G$ is assumed to be (connected) semisimple and simply connected, $\mathfrak{g}$ denotes the Lie algebra of $G$, $Q$ is the Killing form on $\mathfrak{g}$. When $G$ acts on a vector space $V$, for every point $x \in V$ we denote by $O_x \subset V$ the $G$-orbit of $x$ and by $H_x \subset G$ (resp. $\mathfrak{h}_x \subset \mathfrak{g}$) the stabilizer subgroup (resp. subalgebra) of $x$. For a field $k$ we denote by $\Gamma_k$ the Galois group $\text{Gal}(k/k)$. For an algebraic group $H$ defined over $k$ we set $H^i(k, H) = H^i(\Gamma_k, H(k))$ (where $i \leq 1$ if $H$ is noncommutative).

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1. Algebraic results

Throughout this section $G$ is a simply connected semisimple group over a field $k$ of characteristic zero, $\rho : G \to \text{Aut}(V)$ is a rational representation defined over $k$. In this section we gather results we need that can be proven after passing to an algebraic closure of $k$.

1.1. Representations with reductive generic stabilizer. The first restriction we impose in order for a representation $\rho$ to be nice is the reductivity of $\mathfrak{h}$. In this subsection we discuss some consequences of this condition.

It is well-known (see [13], [18]) that there exists a non-empty $G$-invariant Zariski open subset $V_0 \subset V$ and a subgroup $H \subset G$ such that for every point $x \in V_0(k)$ the stabilizer subgroup $H_x$ of $x$ is conjugate to $H$. Furthermore, clearly we can assume that $V_0$ is invariant under the natural $\mathbb{G}_m$-action on $V$. The following lemma can be found in [28] (see also [27], Theorem 7.3).
Lemma 1.1.1. For every $x \in V_0$ one has

$$V = V^{hx} + gx.$$

Proof. This follows immediately from the surjectivity of the map $G/H_x \times (V_0^{hx}) \to V_0$, where $V_0^{hx} = V^{hx} \cap V_0$. \hfill \square

Let $h \subset g$ be the Lie algebra of $H$. We will call $H$ (resp. $h$) the generic stabilizer subgroup (resp. subalgebra) for $\rho$. Henceforth, we always assume that $h$ is reductive. The theorem of V. L. Popov [16] states that in this case generic $G$-orbits in $V$ are closed.

Proposition 1.1.2. Let $\rho^\vee : G \to \text{Aut}(V^\vee)$ be the dual representation to $\rho$. Then the generic stabilizer subalgebra for $\rho^\vee$ is conjugate to $h$ over $k$.

Proof. Clearly, we can assume that $k = C$. Let $C \subset G(C)$ be a maximal compact subgroup. It is well-known that there exists a $C$-invariant positive-definite Hermitian form $H$ on $V$. Let $O \subset V$ be a generic orbit. Let us consider the restriction of the function $x \mapsto H(x, x)$ to $O(C)$. Since $O$ is closed, there exists a vector $x \in O(C)$ minimizing this function. In particular, we have $H(gx, x) = 0$. Let us consider the functional $x^\vee \in V^\vee(C)$ given by $x^\vee = H(?, x)$. We claim that the stabilizer subalgebra of $x^\vee$ coincides with $h_x$. Indeed, first we note that $x^\vee \in (gx)^\perp$. According to lemma 1.1.1, $h_x$ acts trivially on $V/gx$. Hence, it also acts trivially $(gx)^\perp$, so $h_x x^\vee = 0$. Let us define the $C$-bilinear form on $g/h_x$ by setting

$$B(\xi_1, \xi_2) = \langle \xi_1 x^\vee, \xi_2 x \rangle = -H(\xi_1 \xi_2 x, x)$$

where $\xi_1, \xi_2 \in g/h_x$. Note that since $H([\xi_1, \xi_2], x) = 0$, the form $B$ is symmetric. Now let $c \subset g$ be the Lie algebra of $C$. We claim that the restriction of $B$ to the real subspace $c/c \cap h_x \subset g/h_x$ is $R$-valued and positive-definite. Indeed, if $\xi \in c$ then using the fact that $H$ is $c$-invariant we get

$$B(\xi, \xi) = H(\xi x, \xi x) \in \mathbb{R}.$$  

Furthermore, this implies that $B(\xi, \xi) > 0$ for $\xi \notin c \cap h_x$. Since the subspace $c/c \cap h_x$ generates $g/h_x$ over $C$ this implies that the form $B$ on $g/h_x$ is non-degenerate. In particular, the stabilizer subalgebra of $x^\vee$ is equal to $h_x$. \hfill \square

Remark. The idea to look at vectors of minimal length on the orbit goes back to the work of Kempf and Ness [7]. One can give an alternative proof of proposition 1.1.2 using the theorem of Mostow [14] on self-adjoint groups.

For nice representations the assertion of the above theorem holds also for generic stabilizer subgroups.

Proposition 1.1.3. Assume that $\rho$ is a nice representation. Then the dual representation $\rho^\vee$ is also nice. In this case the generic stabilizer subgroups for $\rho$ and $\rho^\vee$ are conjugate over $\overline{k}$.

Proof. The generic stabilizer subgroup for $\rho^\vee$ has form $\sigma(H)$ where $\sigma$ is an automorphism of $G$ which restricts to the map $t \mapsto t^{-1}$ on some maximal torus. This immediately implies that $\rho^\vee$ is nice. The second assertion follows from proposition 1.1.2. \hfill \square
Let $Q$ denote the Killing form on $\mathfrak{g}$. The following lemma is well-known but we include the proof since we couldn’t find the reference (in its statement $\mathfrak{h}$ can be replaced by the Lie algebra of any reductive subgroup in $G$).

**Lemma 1.1.4.** The restriction of $Q$ to $\mathfrak{h}$ is non-degenerate.

**Proof.** We can assume that $k = \mathbb{C}$. Let $C \subset G(\mathbb{C})$ be a maximal compact subgroup containing a maximal compact subgroup in $H(\mathbb{C})$, and let $\mathfrak{c}$ be its Lie algebra. Then $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{c} + i\mathfrak{c} \cap \mathfrak{c}$. It remains to use the fact that the restriction of $Q$ to $\mathfrak{c}$ is $\mathbb{R}$-valued and negative definite. \hfill $\square$

**Lemma 1.1.5.** Let us denote $W = V^H$, $W_0 = V_0 \cap W$.

(a) For every point $x_0 \in W_0$ there exists a linear subspace $L \subset W_0$ and a Zariski open neighborhood of zero $S \subset L$ such that the natural morphism

$$a : G/H \times (S + x_0) \to V_0 : (gH, x) \mapsto gx$$

is étale.

(b) Let $\nu_V$ be a non-zero translation-invariant top-degree form on $V$, $\omega$ be a non-zero $G$-invariant top-degree form on $G/H$. Then we have $a^*\nu_V = \omega \wedge \nu_L$ for some translation-invariant top-degree form $\nu_L$ on $L$.

**Proof.** (a) Note that we have an isomorphism

$$G/H \times N(H)/H \to W_0$$

where $N(H)$ is the normalizer of $H$ in $G$. Let $L$ to be a complement in $W_0$ to the tangent space to $N(H)x_0$ at $x_0$. Then our assertion follows from Luna’s results in [13] (in this situation $S + x_0$ is an étale slice for the action of $N(H)/H$ on $W$).

(b) Since $G$ is semisimple and connected, the form $\nu_V$ is $G$-invariant. Therefore, the pull-back $a^*\nu_V$ is also $G$-invariant. On the other hand, the map $a$ is linear in the second argument, hence $a^*\nu_V$ is invariant with respect to translations on $L$. This implies our assertion. \hfill $\square$

**Lemma 1.1.6.** Assume that $G$ is simple, $\rho$ is nice. Then the generic stabilizer $H$ is either semisimple or commutative.

**Proof.** This follows immediately from Tables 1 and 2 of [3]. \hfill $\square$

### 1.2. Critical points.

In this subsection we study critical points of the restriction of a generic linear functional $x^\vee \in V_0^\vee$ to a generic orbit $O$ in a representation $\rho$ whose generic stabilizer subalgebra is reductive. Recall that a critical point $x$ of a function $\phi$ is called non-degenerate if the quadratic form of second derivatives of $\phi$ at $x$ is non-degenerate.

**Lemma 1.2.1.** Let $x$ be a critical point of $x^\vee|_O$. Then:

(a) $x$ has the same stabilizer subalgebra as $x^\vee$;

(b) $x$ is non-degenerate if and only if $\mathfrak{g}x \cap (\mathfrak{g}x^\vee)^\perp = 0$.

**Proof.** (a) Without loss of generality we can assume that $x^\vee \in W_0^\vee$. The condition that $x$ is a critical point of $x^\vee|_O$ is equivalent to $x^\vee(\mathfrak{g}x) = 0$, i.e., $x \in (\mathfrak{g}x^\vee)^\perp$. On
the other hand, by lemma \[1.1.3\] the group $H$ acts trivially on $V^\vee /gx^\vee$. It follows that $(gx^\vee)^\perp$ is contained in $V^\vee = W$, hence, $x$ is stabilized by $H$.

(b) The second derivative of $x^\vee|_O$ at a critical point $x$ is the following symmetric bilinear form on the tangent space $T_xO = gx$: $B_{x,x^\vee}(\xi_1 x, \xi_2 x) = \langle x^\vee, \xi_1 \xi_2 x \rangle$, where $\xi_1, \xi_2 \in g$. The kernel of this form is $gx \cap (gx^\vee)^\perp$.

Let $I \subset V_0 \times V_0^\vee$ be the subvariety consisting of $(x, x^\vee)$ such that $x$ is a critical point of $x^\vee|_O$. Let us consider the natural morphism

$$f : I \to V_0 / G \times V_0^\vee.$$  

By the definition, $f^{-1}(O, x^\vee)$ consists of critical points of $x^\vee|_O$.

**Proposition 1.2.2.** The morphism $f$ is dominant and there exists a non-empty $G$-invariant open subset $U \subset V_0 / G \times V_0^\vee$ (where $G$ acts on the second factor) such that $f$ is étale over $U$. For every pair $(O, x^\vee)$ in $U$ the set of critical points of $x^\vee|_O$ is finite and non-empty. Furthermore, all these critical points are non-degenerate.

**Proof.** By the definition, the subvariety $I \subset V_0 \times V_0^\vee$ consists of $(x, x^\vee)$ such that $\langle x^\vee, gx \rangle = 0$. Let $p_1 : I \to V_0, p_2 : I \to V_0^\vee$ be the natural projections. Note that the fibers of both these projections are open subsets in linear spaces of dimension $\dim V - \dim O = \dim V / G$. Hence, $I$ is smooth and irreducible of dimension $\dim V / G + \dim V$ (if non-empty). Thus, it suffices to prove that the morphism $f$ is dominant. We can assume that $k = \mathbb{C}$. Let $(x, x^\vee)$ be a pair constructed in the proof of proposition \[1.1.3\]. Then $(x, x^\vee) \in I$. We claim that the tangent map to $f$ at $(x, x^\vee)$ is an isomorphism. Indeed, the relative tangent space of the projection $p_2 : I \to V_0^\vee$ at $(x, x^\vee)$ is $(gx^\vee)^\perp$. Therefore, we have to check that the linear map $(gx^\vee)^\perp \to V / gx$ is an isomorphism. It remains to note that the non-degeneracy of the form $B$ constructed in the proof of proposition \[1.1.3\] implies that $gx \cap (gx^\vee)^\perp = 0$. This proves our claim. The last assertion of the proposition follows from lemma \[1.2.3\] (b) and from the fact that $f$ is generically étale.

1.3. Spin-coverings. In this subsection we are going to define certain double coverings of generic stabilizers which will play an important role later. We assume that $\rho$ is a nice representation.

Let $x \in V_0, x^\vee \in V_0^\vee$ be points such that $x$ is a non-degenerate critical point of $x^\vee|_O$. Let $B_{x,x^\vee}$ be the corresponding quadratic form of second derivatives of $x^\vee|_O$. Let $B_{x,x^\vee}$ be the corresponding quadratic form of second derivatives of $x^\vee|_O$. Then the action of the stabilizer $H_x$ of $x$ on $T_x$ preserves $B_{x,x^\vee}$, hence, we get a homomorphism $H_x \to \text{Aut}(T_x, B_{x,x^\vee})$. Since $H_x$ is connected, we obtain a homomorphism to the corresponding special orthogonal group

$$\iota_{x,x^\vee} : H_x \to \text{SO}(T_x, B_{x,x^\vee}).$$

Recall that for every vector space $T$ equipped with a non-degenerate quadratic form $B$ one can define the spin-covering $\text{Spin}(T, B) \to \text{SO}(T, B)$. It is defined in the standard way when $\dim T \geq 3$ (see e.g. \[1\]). If $\dim T < 3$ then we define the spin-covering as the restriction of the standard spin-covering of $\text{SO}(T \oplus U)$, where $U$ is an orthogonal space of large dimension. The group $\text{Spin}(T, B)$ is a central extension of $\text{SO}(T, B)$ by $\{ \pm 1 \}$. In the case $\dim(T) < 3$ this extension is trivial unless $\dim T = 2$ and the quadratic form $B$ is anisotropic. In the latter case $\text{SO}(T, B)$ is a 1-dimensional torus of the form $R^{(1)}_{k'/k}(G_m)$ for some quadratic
extension \( k \subset k' \), \( \text{Spin}(T, B) \) is isomorphic to \( \text{SO}(T, B) \) and the map \( \text{Spin}(T, B) \to \text{SO}(T, B) \) is identified with the map \( t \to t^2 \).

In proposition 1.3.1 below we compute the pull-back of the spin-covering \( \text{Spin}(T_x, B_{x,x'}) \to \text{SO}(T_x, B_{x,x'}) \) by \( \iota_{x,x'} \). Recall that by lemma 1.1.4 the restriction of \( Q \) to \( \mathfrak{h}_x \) is non-degenerate. Consider the homomorphism

\[
t_x : H_x \to \text{SO}(\mathfrak{h}_x, Q|_{\mathfrak{h}_x})
\]

induced by the adjoint action of \( H_x \). Let \( \tilde{H}_x \to H_x \) be the pull-back of the spin-covering by \( \iota_x \). Note that if \( H_x \) is commutative then \( \iota_x \) is trivial, so the extension \( \tilde{H}_x \to H_x \) splits in this case.

**Proposition 1.3.1.** There is a unique isomorphism of the following two central extensions of \( H_x \) by \( \{ \pm 1 \} \): the pull-back of the spin-covering by \( \iota_{x,x'} \) and \( \tilde{H}_x \).

**Proof.** We can reformulate the statement as follows: there exists a unique lifting of \( \iota_{x,x'} \) to a homomorphism \( \tilde{H}_x \to \text{Spin}(T_x, B_{x,x'}) \) which maps \( \{ \pm 1 \} \subset H_x \) to \( \{ \pm 1 \} \subset \text{Spin}(T_x, B_{x,x'}) \) identically.

First, let us prove the uniqueness. Indeed, two such liftings differ by a homomorphism \( H_x \to \{ \pm 1 \} \). Since \( H_x \) is connected such a homomorphism should be trivial. Now by uniqueness it suffices to prove the existence of a lifting over \( \bar{k} \). Therefore, we can assume that \( k = \mathbb{C} \).

Assume first that \( \dim(T_x) \geq 3 \) and the twofold covering \( \tilde{H}_x \to H_x \) is non-trivial. Then \( H_x \) is connected. We claim that the homomorphism

\[
\pi_1(H_x) \to \pi_1(\text{SO}(T_x, B_{x,x'})) = \{ \pm 1 \}
\]

induced by \( \iota_{x,x'} \) is surjective with the kernel \( \pi_1(\tilde{H}_x) \subset \pi_1(H_x) \). Indeed, let \( X \) be the space of \( H_x \)-invariant non-degenerate symmetric forms on \( T_x \). We have a continuous family of homomorphisms \( H_x \to \text{SO}(T_x, B) \) parametrized by \( B \in X \). Since \( X \) is connected, the induced homomorphism of fundamental groups \( \pi_1(H_x) \to \pi_1(\text{SO}(T_x, B)) \) does not depend on \( B \). Therefore, in the above claim we can replace \( B_{x,x'} \) by any other form in \( X \). Now using the natural identification of \( T_x \) with \( \mathfrak{g}/\mathfrak{h}_x = \mathfrak{h}_x^\perp \subset \mathfrak{g} \) (the orthogonal complement to \( \mathfrak{h}_x \) in \( \mathfrak{g} \) with respect to the Killing form \( Q \)) we take \( B = Q|_{\mathfrak{h}_x^\perp} \). Consider the following commutative diagram of homomorphisms

\[
\begin{array}{ccc}
H_x & \longrightarrow & \text{SO}(\mathfrak{h}_x, Q|_{\mathfrak{h}_x}) \times \text{SO}(\mathfrak{h}_x^\perp, Q|_{\mathfrak{h}_x^\perp}) \\
\downarrow & & \downarrow \\
G & \longrightarrow & \text{SO}(\mathfrak{g}, Q)
\end{array}
\]

(1.3.2)

This diagram implies that the following composition

\[
\pi_1(H_x) \to \pi_1(\text{SO}(\mathfrak{h}_x, Q|_{\mathfrak{h}_x})) \times \pi_1(\text{SO}(\mathfrak{h}_x^\perp, Q|_{\mathfrak{h}_x^\perp})) \ni \pi_1(\text{SO}(\mathfrak{g}, Q))
\]

is trivial (since it factors through \( \pi_1(G) = 1 \)). The assumption that the covering \( \tilde{H}_x \to H_x \) is non-trivial implies that \( H_x \) has a non-trivial semisimple component (in particular, \( \dim(\mathfrak{h}_x) \geq 3 \)) and that the map \( \pi_1(H_x) \to \pi_1(\text{SO}(\mathfrak{h}_x, Q|_{\mathfrak{h}_x})) \) is surjective with the kernel \( \pi_1(\tilde{H}_x) \subset \pi_1(H_x) \). On the other hand, both the maps

\[
\pi_1(\text{SO}(\mathfrak{h}_x, Q|_{\mathfrak{h}_x})) \to \pi_1(\text{SO}(\mathfrak{g}, Q))
\]
and
\[ \pi_1(\text{SO}(h^+_x, Q|_{h^+_x})) \to \pi_1(\text{SO}(g, Q)) \]
are isomorphisms. Our claim immediately follows from this. This implies that the homomorphism \( \tilde{\mathbf{H}}_x \to \text{SO}(T_x, B_{x,x'}) \) lifts to a homomorphism \( \tilde{\mathbf{H}}_x \to \text{Spin}(T_x, B_{x,x'}) \).

It is easy to see that the latter homomorphism is non-trivial on \( \{ \pm 1 \} \subset \tilde{\mathbf{H}}_x \). Indeed, otherwise it would factor through \( \mathbf{H}_x \) which contradicts to non-triviality of the homomorphism \([3.3]([13.3])\). This finishes the proof in this case.

Now assume that the covering \( \tilde{\mathbf{H}}_x \to \mathbf{H}_x \) is trivial (and \( \dim(T_x) \geq 3 \)). Then the map \( \pi_1(\mathbf{H}_x) \to \pi_1(\text{SO}(h_x, Q|_{h_x})) \) is trivial (when \( \dim h_x \leq 2 \) this follows from commutativity of \( \mathbf{H}_x \)). As above we deduce from this that the map \( \pi_1(\mathbf{H}_x) \to \pi_1(T_x, B_{x,x'}) \) is also trivial. Therefore, the homomorphism \( \mathbf{H}_x \to \text{SO}(T_x, B_{x,x'}) \) factors through \( \text{Spin}(T_x, B_{x,x'}) \).

Finally, if \( \dim(T_x) < 3 \) then we replace the orthogonal space \( (T_x, B_{x,x'}) \) by its direct sum with a fixed orthogonal space of large dimension (on which \( G \) acts trivially) and apply the same argument as above.

In the case when \( k = E \) is a local field, the exact sequence \( 1 \to \{ \pm 1 \} \to \tilde{\mathbf{H}}_x \to \mathbf{H}_x \to 1 \) gives a map of Galois cohomologies
\[ \epsilon_{\mathbf{H}_x} : H^1(E, \mathbf{H}_x) \to H^2(E, \{ \pm 1 \}) \simeq \{ \pm 1 \}. \]  
This is the sign function which will play an important role below.

2. Nice representations over local fields

In this section \( \rho : G \to \text{Aut}(V) \) denotes a nice representation over a local field \( E \). We formulate our main theorems about the behaviour of \( G \)-stable functions and distributions on \( V(E) \) under the Fourier transform. Also, we analyze the Fourier transform of certain stable functions using the stationary phase approximation (in the case when \( E \) is \( p \)-adic).

2.1. \( G \)-inner forms and stable \( G \)-equivalence. Let \( x \) be a point in \( V_0(E) \).

Note that since \( \mathbf{H}_x \) is reductive, the set \( H^1(E, \mathbf{H}_x) \) is finite (see e.g. \([13.3]\)). Consider the subset \( P_x \subset G(E) \) consisting of the elements \( g \) such that \( g^{-1} \sigma(g) \in \mathbf{H}_x \) for all \( \sigma \in \Gamma_E \). For every \( g \in P_x \) the 1-cocycle
\[ \epsilon_g(\sigma) = g^{-1} \sigma(g) \]
gives a class in \( \mathbf{H}_x \). It is easy to see that \( P_x \) is a union of right \( H_x(E) \)-cosets and of left \( G(E) \)-cosets, and the assignment \( g \mapsto \epsilon_g \) defines a bijection
\[ G(E) \backslash P_x / H_x(E) \simeq \text{ker}(H^1(E, \mathbf{H}_x) \to H^1(E, G)). \]

On the other hand, we have a natural bijection
\[ P_x / H_x(E) \simeq O_x(E) : g \mapsto gx. \]
In particular, we can identify the set of \( G(E) \)-orbits on \( \mathbf{O}_x(E) \) with \( \text{ker}(H^1(E, \mathbf{H}_x) \to H^1(E, G)) \). In the case when \( E \) is \( p \)-adic, we have \( H^1(E, G) = 0 \) since \( G \) is simply connected (see \([13.3]\)). Therefore, in this case we have a bijection between \( G(E) \backslash \mathbf{O}_x(E) \) and \( H^1(E, \mathbf{H}_x) \).

Let us call two points \( x, x' \in V(E) \) stably \( G \)-equivalent if there exists an element \( g \in G(E) \) such that \( gx = x' \), i.e., if \( \mathbf{O}_x = \mathbf{O}_{x'} \). Let \( K, K' \subset G \) be subgroups defined over \( E \). Let us say that \( K' \) is a \( G \)-inner form of \( K \) if there exists an element \( g \in G(E) \) such that \( gK(E)g^{-1} = K'(E) \) and \( g^{-1} \sigma(g) \in K(E) \) for every
\[ \sigma \in \Gamma_E. \] It is easy to see that this defines an equivalence relation between subgroups of \( G \) defined over \( E \). This definition is motivated by the following lemma.

**Lemma 2.1.1.** Let \( x \in V(E) \).

(a) If \( x' \in V(E) \) is stably \( G \)-equivalent to \( x \), then \( H_{x'} \) is a \( G \)-inner form of \( H_x \).

(b) Conversely, if \( H' \) is a \( G \)-inner form of \( H_x \), then \( H' \) is the stabilizer of some \( E \)-point which is stably \( G \)-equivalent to \( x \).

(c) The image of the natural map
\[
a : G/H(E) \times W_0(E) \to V_0(E)
\]
consists of all points in \( V_0(E) \) whose stabilizer is a \( G \)-inner form of \( H \).

The proof is straightforward.

**Remark.** It is easy to see that if \( K' \) is a \( G \)-inner form of \( K \) then \( K' \) is an inner form of \( K \) in the usual sense. On the other hand, if \( K \) is semisimple and simply connected, \( E \) is \( p \)-adic, then \( K' \) is a \( G \)-inner form of \( K \) if and only if \( K' \) is conjugate to \( K \) over \( E \).

Let \( x \) and \( x' \) be a pair of stably \( G \)-equivalent points in \( V_0(E) \) and let \( \mathcal{O} = \mathcal{O}_x = \mathcal{O}_{x'} \) be the corresponding orbit. Then \( G(E) \)-orbits on \( \mathcal{O}(E) \) can be identified with a subset in \( H^1(E, H_x) \) and with a subset in \( H^1(E, H_{x'}) \). On the other hand, \( H_{x'} \) is obtained from \( H_x \) by twisting with a cohomology class in \( H^1(E, H_x) \), so we have a canonical identification \( H^1(E, H_{x'}) \cong H^1(E, H_x) \). It is easy to see that these three identifications are compatible.

### 2.2. Local Kottwitz invariant and local sign function.

For every connected reductive group \( H \) over a field \( k \) let us denote by \( Z(H) \) the centre of the Langlands dual group (equipped with an action of the Galois group \( \Gamma_k \)). Note that \( H \) is defined canonically up to an inner conjugation, hence, \( Z(H) \) is defined canonically and carries an action of \( \Gamma_k \). Furthermore, an isomorphism \( i : H \to H' \) over \( k \), such that \( i^{-1} \sigma(i) \) is inner for all \( \sigma \in \Gamma_k \) (an inner twisting), induces an isomorphism of \( \mathcal{O}(Z(H)) \) with \( \mathcal{O}(H') \) as \( \Gamma_k \)-modules. Following Kottwitz we define
\[
A(H/k) := \pi_0(Z(H)^{\Gamma_k})^D
\]
where for a finite group \( A \) we denote by \( A^D \) the dual group. When \( k = E \) is a local field, the local Kottwitz invariant is a functorial map
\[
\text{inv} = \text{inv}_E : H^1(E, H) \to A(H/E)
\]
constructed in \([1]\). The definition of \( \text{inv} \) is a generalization of the isomorphism derived from Tate-Nakayama duality in the case when \( H = T \) is a torus. Indeed, this duality gives an isomorphism \( H^1(E, T) \cong H^1(E, X^*(T))^D \), where \( X^*(T) \) is the module of characters of \( T \). Now from the exact sequence of Galois modules
\[
0 \to X^*(T) \to X^*(T) \otimes \mathbb{C} \to X^*(T) \otimes \mathbb{C}^* \to 0
\]
one gets an isomorphism
\[
H^1(E, X^*(T)) \cong \text{coker}((X^*(T) \otimes \mathbb{C})^{\Gamma_E} \to (X^*(T) \otimes \mathbb{C}^*)^{\Gamma_E}).
\]
The latter group can be immediately identified with \( \pi_0((X^*(T) \otimes \mathbb{C}^*)^{\Gamma_E}) \). In the general case (when \( H \) is not necessarily a torus), Kottwitz showed in \([1]\) that \( \text{inv}_E \) is an isomorphism for \( p \)-adic \( E \). In particular, for such \( E \) we obtain the structure of abelian group on \( H^1(E, H) \).
For any 1-cocycle \( e : \Gamma_E \to H(E) \) we can consider the \( E \)-group \( H^e \) obtained from \( H \) by inner twisting with \( e \). By definition \( H^e(E) = H(E) \) while the action of an element \( \sigma \in \Gamma_E \) on \( H^e(E) \) differs from its action on \( H(E) \) by the inner automorphism associated with \( e(\sigma) \). In particular, we have \( Z(H^e) = Z(H) \), hence \( A(H^e/E) = A(H/E) \). According to lemma 1.4 of [11], the following diagram is commutative

\[
\begin{array}{ccc}
H^1(E, H^e) & \xrightarrow{\text{inv}} & A(H/E) \\
i_e & & \downarrow t_{\text{inv}(e)} \\
H^1(E, H) & \xrightarrow{\text{inv}} & A(H^e/E) = A(H/E)
\end{array}
\]

where \( t_{\text{inv}(e)} : A(H/E) \to A(H/E) \) is the translation by \( \text{inv}(e) \in A(H/E) \), \( i_e : H^1(E, H^e) \to H^1(E, H) \) is the canonical identification induced by \( e \). Thus, in the case when \( E \) is \( p \)-adic, the isomorphism \( i_e \) does not respect the group structures on the sets \( H^1(E, H^e) \) and \( H^1(E, H) \), but rather respects the structures of principal homogeneous spaces over \( A(H/E) = A(H^e/E) \).

Now assume that \( O \subset V_0 \) is an orbit. Then we have a system of compatible isomorphisms between the groups \( A(H_x/E) \) for \( x \in O(E) \). Let us denote the corresponding group isomorphic to all \( A(H_x/E) \) by \( A(O/E) \). Assume for a moment that \( E \) is \( p \)-adic. Then the set of \( G(E) \)-orbits on \( O(E) \) has a natural structure of a principal homogeneous space over \( A(O/E) \). Thus, for every pair of points \( x, x' \in O(E) \) we can define an element \( \text{inv}(x, x') \in A(O/E) \) such that \( G(E)x' \) is obtained from \( G(E)x \) by the action of \( \text{inv}(x, x') \). This definition extends to the case of archimedian \( E \) as follows:

\[
\text{inv}(x, gx) = \text{inv}(e_g)
\]

where \( g \in P_x \), \( e_g \) is the corresponding cohomology class in \( H^1(E, H_x) \), \( \text{inv}(e_g) \) is its local Kottwitz invariant in \( A(H_x/E) \simeq A(O/E) \). It is easy to see that the following properties are satisfied:

\[
\begin{align}
\text{inv}(x, x') + \text{inv}(x', x) &= 0, \\
\text{inv}(x, x') + \text{inv}(x', x'') &= \text{inv}(x, x''), \\
\text{inv}(tx, tx') &= \text{inv}(x, x')
\end{align}
\]

where \( g \in G(E), t \in E^* \), \( e_g \) is the 1-cocycle of \( \Gamma_E \) with values in \( H_x \) defined above.

Let us fix a point \( x \in V_0(E) \) and set \( H = H_x \). Recall that in section 1.3 we have defined a map

\[
\epsilon_H : H^1(E, H) \to H^2(E, \{\pm 1\}) = \{\pm 1\}
\]

induced by the central extension \( 1 \to \{\pm 1\} \to H \to H \to 1 \) (the pull-back of the spin covering associated with \( Q_{|B_x} \)). We are going to construct a character

\[
\text{sign} = \text{sign}_H : A(H/E) \to \{\pm 1\}
\]

such that \( \epsilon_H = \text{sign}_H \circ \text{inv}_E \). For this we note that the above central extension is induced by the similar extension \( 1 \to \{\pm 1\} \to H_{ad} \to H_{ad} \to 1 \) of the adjoint

\footnote{For two points \( x, x' \in O(E) \) such that \( H_x = H_{x'} \), the corresponding isomorphism \( A(H_x/E) \to A(H_{x'}/E) \) is not necessarily the identity: it corresponds to the action of some element in the normalizer of \( H_x \).}
Let $u : H_{sc} \to H$ be the universal covering of $H$ (it is defined over $E$), $C$ be the kernel of $u$. Then the homomorphism $u$ lifts uniquely to a homomorphism $\tilde{u} : H_{sc} \to \tilde{H}$. The restriction of $\tilde{u}$ to $C$ gives a homomorphism $\chi : C \to \{\pm 1\}$ defined over $E$. We can consider $\chi$ as an element of order 2 in $X^*(C)^{\Gamma_E}$. It remains to notice that there is an isomorphism $X^*(C) \simeq \mathbb{Z}(\tilde{H})$ of $\Gamma_E$-modules, so we can consider $\chi$ as a character $\chi : A(\tilde{H}/E) \to \{\pm 1\}$. The following result shows that this is the character we were looking for.

**Lemma 2.2.1.** One has the following equality of maps from $H^1(E,H)$ to $\{\pm 1\}$:

$$\epsilon_H = \text{sign}_H \circ \text{inv}_E.$$  

**Proof.** By duality for finite groups we have an isomorphism

$$H^2(E,C) \simeq H^0(E,X^*(C))^D \simeq A(\tilde{H}/E).$$

According to Lemma 1.8 of [11], under this isomorphism the map $\text{inv}_E : H^1(E,H) \to A(\tilde{H}/E)$ can be identified with the map $H^1(E,H) \to H^2(E,C)$ coming from the exact sequence $1 \to C \to H_{sc} \to H \to 1$. On the other hand, the character $\chi$ of $H^2(E,C)$ corresponding to $\text{sign}_H$ is the homomorphism on $H^2$ induced by the homomorphism $\chi : C \to \{\pm 1\}$. Therefore, the composition $\text{sign}_H \circ \text{inv}_E$ coincides with the map $H^1(E,H) \to H^2(E,\{\pm 1\})$ coming from the exact sequence $1 \to \{\pm 1\} \to \tilde{H} \to H \to 1$, which is the definition of $\epsilon_H$.

Recall that for every pair of points $x, x' \in O(E)$, where $O \subseteq V_0$ is a $G$-orbit, we have defined the sign $\epsilon(x,x') = \epsilon(Q_{|_{B'_x}}, Q_{|_{B'_x'}})$. It is well-known that if a quadratic form $B'$ is obtained from a non-degenerate quadratic form $B$ by the twist with an element $e \in H^1(E,SO(B))$ then the relative Hasse-Witt invariant $\epsilon(B,B')$ is equal to the image of $e$ under the coboundary homomorphism $H^1(E,SO(B)) \to H^2(E,\{\pm 1\}) = \{\pm 1\}$ coming from the spin-covering (see e.g. [22]). This implies the following relation between $\epsilon$ and the sign function (1.3.3) defined in 1.3:

$$\epsilon(x,gx) = \epsilon_{B_x}(e_g)$$

(2.2.3)

where $x \in V_0(E)$, $g \in P_x$. Comparing the definition of $\text{inv}(\cdot, \cdot)$ with (2.2.3) and using lemma 2.2.1, we get the following formula:

$$\epsilon(x,x') = \text{sign}_{B_x}(\text{inv}(x,x')).$$

2.3. **Critical points and stable equivalence.** Assume that we have a pair of points $x \in V_0(E), x' \in V_0(E)$ such that $x$ is a critical point of $x'_{|_{O_x}}$. Proposition 1.3.3 implies that the following diagram is commutative

$$
\begin{array}{ccc}
H^1(E,H_x) & \xrightarrow{\epsilon_{H_x}} & \{\pm 1\} \\
\downarrow & & \downarrow \text{id} \\
H^1(\epsilon_{x,x'}) & \xrightarrow{id} & \{\pm 1\}
\end{array}
$$

(2.3.1)

where the lower horizontal arrow is the coboundary homomorphism associated with the spin-covering of $SO(T_x,B_{x,x'})$.

This observation leads to the following result.
Lemma 2.3.1. Let $\omega$ be a $G$-invariant top-degree form on $O_x$. Recall that for every $g \in P_x$ one has $gx \in V_0(E)$ and $gx^y \in V_0'(E)$.

(a) One has $\det(B_{x,x^y})/\omega_x^2 = \det(B_{gx,gx^y})/\omega_{gx^y}^2$.

(b) The quadratic forms $B_{x,x^y}$ and $B_{gx,gx^y}$ have the same determinant modulo squares. Their relative Hasse-Witt invariant is given by

$$\epsilon(B_{gx,gx^y}, B_{x,x^y}) = \epsilon(x, gx) = \epsilon_H(e_g).$$

Proof. (a) This follows from the $\mathbb{F}$-isomorphism of data $(T_v, B_{v,v'}, \omega_v)$ and $(T_{gv}, B_{gv,gv'}, \omega_{gv})$ given by the action of $g$.

(b) Let $e_g \in H^1(E, \mathcal{H}_v)$ be the cohomology class defined by the cocycle $\sigma \mapsto g^{-1}\sigma(g)$. Then the quadratic form $B_{gv,gv'}$ is obtained from $B_{v,v'}$ by twisting with the class $H^1(t_{v,v'})(e_g) \in H^1(E, SO(T_v, B_{v,v'}))$. Now the diagram (2.3.1) implies that $\epsilon(B_{gx,gx^y}, B_{x,x^y}) = \epsilon_H(e_g)$.

2.4. Stable and antistable functions and distributions. Let us denote by $S(V(E))_{G(E)}$ the space of $G(E)$-coinvariants in $S(V(E))$. We have the natural projection $S(V(E)) \to S(V(E))_{G(E)} : \phi \mapsto \overline{\phi}$. On the other hand, for every $\phi \in S(V(E))$ we can define a function $I(\phi)$ on $G(E) \backslash V(E)$ by the formula

$$I(\phi)(y) = \int_{x \in G(E)\backslash y} \phi(x) |\omega_y|$$

where $\omega_y$ is a $G$-invariant top-degree form on $O_y$ (the integral is convergent since the orbit $O_y$ is closed in $V$). It is clear that $I(\phi)$ depends only on $\overline{\phi}$, so we will denote $I(\overline{\phi}) = I(\phi)$. Although we will not need this fact, it is worth mentioning that for a pair of functions $\phi, \phi' \in S(V_0(E))$ one has $\overline{\phi} = \overline{\phi'}$ if and only if $I(\phi) = I(\phi')$ (see [1]).

Definition.

(i) An element $\overline{\phi} \in S(V(E))_{G(E)}$ is called stable if for every $G$-orbit $O \subset V_0$, the restriction of the function $I(\overline{\phi})$ to $G(E) \backslash O(E)$ is constant. We denote by $S(V(E))_{\text{st}} \subset S(V(E))_{G(E)}$ the subspace of stable elements. Similarly, we define a subspace of antistable elements $S(V(E))_{\text{as}} \subset S(V(E))_{G(E)}$. By the definition, an element $\overline{\phi} \in S(V(E))_{G(E)}$ is antistable if for every $G$-orbit $O \subset V_0$ the total sum of the function $I(\overline{\phi})$ over $G(E) \backslash O(E)$ is zero.

(ii) Let $D(V(E))$ denote the space of distributions on $V(E)$, i.e., functionals on $S(V(E))$. Note that a $G(E)$-invariant distribution $\alpha \in D(V(E))$ descends to a functional on $S(V(E))_{G(E)}$. Now a $G(E)$-invariant distribution $\alpha$ is called stable (resp. antistable) if $\alpha(S(V(E))_{\text{as}}) = 0$ (resp. $\alpha(S(V(E))_{\text{st}}) = 0$). We denote by $D(V(E))_{\text{st}} \subset D(V(E))$ (resp. $D(V(E))_{\text{as}} \subset D(V(E))$) the subspace of stable (resp. antistable) distributions.

For a $G$-orbit $O \subset V_0$ and a non-zero $G$-invariant top-degree form $\omega$ on $O$ defined over $E$, we can define a stable distribution $\delta_{O,\omega} \in D(V(E))$ by the formula

$$\delta_{O,\omega}(\phi) = \int_{O(E)} \phi |\omega|$$

where $\phi \in S(V(E))$. When $O(E) = \emptyset$ we set $\delta_{O,\omega} = 0$. When the choice of $\omega$ is clear or is not important we will abbreviate $\delta_{O,\omega}$ to $\delta_O$. By the definition, an element $\overline{\phi} \in S(V(E))_{G(E)}$ is antistable if and only if $\delta_O(\overline{\phi}) = 0$ for all $G$-orbits $O \subset V_0$. 

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More generally, for every point $x \in V_0(E)$ and for a character $\kappa : A(O_x/E) \to \mathbb{C}^*$ we can define a $G(E)$-invariant distribution $\delta^\varphi_{O_x,\omega} = \delta^\varphi_{O_x} \omega$ by the formula

$$\delta^\varphi_{O_x,\omega}(\phi) = \int_{x \in O_x(E)} \kappa(\text{inv}(x,x')) |\phi(x')| |\omega(x')|.$$

Note that if $y$ is stably equivalent to $x$ then

$$\delta^\varphi_{O_y,\omega} = \kappa(\text{inv}(y,x)) \cdot \delta^\varphi_{O_x,\omega}.$$

In the case when $E$ is $p$-adic, an element $\varphi \in S(V(E))_{G(E)}$ is stable if and only if $\delta^\varphi_{O_x}(\varphi) = 0$ for all $x \in V_0(E)$ and all non-trivial characters $\kappa$ of $A(O_x/E)$.

Also, for every $E$-orbit $O \subset V_0$ we define the distribution

$$\delta^\varphi_{O,\omega}(\phi) = \int_{y \in O(E)} \epsilon(y) |\phi(y)| |\omega(y)|$$

where $\epsilon : V_0(E) \to \{\pm 1\}$ is the function defined in the introduction. The results of section 2.2 show that this distribution corresponds to some character of $A(O/E)$ as above. More precisely, for any point $x \in V_0(E)$ we have

$$\delta^\varphi_{O_x,\omega} = \epsilon(x) \cdot \delta^\text{sign}_{O_x,\omega}.$$

The Fourier transform (associated with some choice of a non-trivial additive character $\psi$) induces a well-defined operator

$$F : S(V(E))_{G(E)} \to S(V^\vee(E))_{G(E)}.$$

We want to describe the images of the subspaces $S(V(E))_{\text{st}}$ and $S(V(E))_{\text{as}}$ under $F$. This is equivalent to describing the images of the spaces of distributions $D(V(E))_{\text{st}}$ and $D(V(E))_{\text{as}}$.

**Definition.**

(i) An element $\varphi \in S(V(E))_{G(E)}$ is called $\epsilon$-stable if for every orbit $O \subset V_0$ and for every pair of points $y,y' \in O(E)$, one has $I(\varphi)(y') = \epsilon(y,y') I(\varphi)(y)$. We denote by $S(V(E))_{\text{st}}^\epsilon$ the subspace of $\epsilon$-stable elements in $S(V(E))_{G(E)}$. Similarly we define the subspace $S(V(E))_{\text{as}}^\epsilon \subset S(V(E))_{G(E)}$ of $\epsilon$-antistable elements. By the definition, an element $\varphi \in S(V(E))_{G(E)}$ is $\epsilon$-antistable if for every $y \in V_0(E)$ one has $\sum_{y' \in G(E) \setminus O_x(E)} \epsilon(y,y') I(\varphi')(y') = 0$.

(ii) Dually, we define the subspace $D(V(E))_{\text{as}}^\epsilon \subset D(V(E))_{G(E)}$ (resp. $D(V(E))_{\text{st}}^\epsilon \subset D(V(E))_{G(E)}$) of $\epsilon$-stable (resp. $\epsilon$-antistable) distributions, so that $D(V(E))_{\text{as}}^\epsilon$ is the annihilator of $S(V(E))_{\text{st}}^\epsilon$ (resp. $D(V(E))_{\text{st}}^\epsilon$ is the annihilator of $S(V(E))_{\text{as}}^\epsilon$).

Note that the distributions $\delta^\varphi_{O_x}$ are $\epsilon$-stable, and an element $\varphi \in S(V(E))$ is $\epsilon$-antistable if and only if $\delta^\varphi_{O_x}(\varphi) = 0$ for all $G$-orbits $O \subset V_0$.

We say that a function $\phi \in S(V(E))$ is stable (resp. antistable, $\epsilon$-stable, $\epsilon$-antistable) if this is true for the corresponding element $\varphi \in S(V(E))_{G(E)}$. Clearly, a function $\phi \in S(V_0(E))$ is $\epsilon$-stable (resp. $\epsilon$-antistable) if and only if $\epsilon \cdot \phi$ is stable (resp. antistable).

Our main result is the following theorem.

**Theorem 2.4.1.** Let $\rho : G \to \text{GL}(V)$ be a nice representation of a simply connected semisimple group $G$ over a $p$-adic field $E$. Assume that either $G$ is simple or the generic stabilizer is semisimple. Then $F(D(V(E))_{\text{st}}^\epsilon) = D(V^\vee(E))_{\text{as}}^\epsilon$ (equivalently, $F(S(V(E))_{\text{as}}^\epsilon) = S(V^\vee(E))_{\text{st}}^\epsilon$).
Another result concerns the Fourier transform of stable functions (equivalently, antistable distributions).

**Theorem 2.4.2.** Let \( \rho : G \to GL(V) \) be a nice representation of a simply connected semisimple group \( G \) over a \( p \)-adic field \( E \), such that the generic stabilizer is semisimple. Then \( \mathcal{F}(\mathcal{D}(V(E))^{st}) = \mathcal{D}(V^\vee(E))^{st} \) (equivalently, \( \mathcal{F}(\mathcal{S}(V(E)))^{st} = \mathcal{S}(V^\vee(E)))^{st} \).

**Remark.** In the case of the adjoint representation the theorem 2.4.1 is due to Waldspurger (for arbitrary connected reductive group \( G \)). In the case of the adjoint representation the theorem 2.4.1 is due to Theorem 2.4.1 will be deduced from the more general theorem 2.7.1, which will be proven along with theorem 2.4.2 in section 3.3.

**Remark.** Theorem 2.4.1 will be deduced from the more general theorem 2.7.1, which will be proven along with theorem 2.4.2 in section 3.3.

The main local ingredient of these results is the analysis of the Fourier transform of some explicit stable functions with the help of the stationary phase principle.

2.5. **Stationary phase.** In this subsection we assume that \( E \) is \( p \)-adic. Let \( \psi : E \to \mathbb{C}^\ast \) be a non-trivial additive character. We will use the following easy version of the stationary phase principle over \( E \).

**Lemma 2.5.1.** Let \( X \) and \( S \) be smooth varieties over \( E \), \( f : X \times S \to \mathbb{A}^1 \) be a morphism, such that for every \( s \in S(E) \) the function \( f_s = f|_{X \times \{s\}} \) on \( X \) has finitely many non-degenerate critical points. Let \( U \subseteq X(E) \) and \( P \subseteq S(E) \) be compact open subsets, and \( \omega \) a non-vanishing top-degree form on \( X \). For every \( x_0 \in Cr(f_\gamma) \) (where \( \gamma \in P \)) we denote by \( q_{x_0}(x - x_0) \) the quadratic form on the tangent space \( T_{x_0}X \) approximating \( f_\gamma(x) - f_\gamma(x_0) \) near \( x_0 \) (the Hessian of \( f_\gamma \) at \( x_0 \)). Then there exists a positive constant \( C \) such that for all \( t \in (E^\ast)^2 \) with \( |t| > C \) and all \( \gamma \in P \), we have

\[
\int_U \psi(tf_\gamma(x))|\omega| = \sum_{x_0 \in Cr(f_\gamma)} \psi(tf_\gamma(x_0))|t|^{-n/2} \cdot c(q_{x_0}, \omega_{x_0}, \psi)
\]

where \( n = \dim X \), \( c(q, \nu, \psi) = \gamma(q, \psi) \cdot |\det(q)/\nu^2|^{-1/2} \), and \( \gamma(q, \psi) \) is the Weil constant associated with \( q \) and \( \psi \).

**Proof.** Let us assume first that \( S \) is a point, so that we have a function \( f \) on \( X \) with a finite number of non-degenerate critical points. If \( f \) has no critical points on \( U \), then we can find a finite covering \( \{U_i\} \) of \( U \) by compact open subsets, such that on each \( U_i \) there exists an analytic system of coordinates \((x_1, \ldots, x_n)\) with \( x_1 = f \) and \( \omega = \lambda \cdot dx_1 \wedge \ldots \wedge dx_n \), where \( \lambda \in E^\ast \). Furthermore, we can assume that the subsets \( U_i \) are disjoint. Therefore, the statement reduces in this case to the vanishing of the integral

\[
\int_V \psi(tx_1) dx_1 \ldots dx_n
\]

for a compact open subset \( V \subseteq E^n \) and for sufficiently large \( t \), which is clear. Now let \( c_1, \ldots, c_k \) be critical points of \( f \) contained in \( U \). By Morse lemma, for each point \( c_i \) there exists a neighborhood \( V_i \) of \( c_i \) and a system of coordinates \((x_1, \ldots, x_n)\) on \( V_i \), such that \( f - f(c_i) = q(x_1, \ldots, x_n) \) for some non-degenerate quadratic form \( q \).
and $\omega = \lambda \cdot dx_1 \wedge \ldots \wedge dx_n$ on $V_i$. Let $B_i \subset V_i$ be a small ball around $c_i$ in this coordinate system. As we have shown above,

$$\int_{U \setminus \cup_i B_i} \psi(tf)|\omega| = 0$$

for sufficiently large $t$. Therefore, the statement is reduced to the case when $U$ is an open compact subgroup in a $E$-vector space $V$, $\omega$ is translation-invariant, and $f = q$ is a non-degenerate quadratic form. Then for any $a \in E^*$ we have

$$F(\delta_{at}) = |a|^n \cdot \text{vol}(U) \cdot \delta_{a^{-1}U^\perp},$$

where $U^\perp \subset V^\omega$ is the orthogonal complement to $U$ (with respect to $\psi$). Combining this with the equation (1.0.2) we obtain

$$\int_U \psi(a^2 q)|\omega| = |a|^{-n} \cdot \int_{aU} \psi(q)|\omega| = \text{vol}(U) \cdot c(q, \omega, \psi) \cdot \int_{a^{-1}U^\perp} \psi(-q^\omega)|\omega|^\omega$$

where $|\omega|^\omega$ is the dual measure on $V^\omega$. For sufficiently large $a$ we have

$$\int_{a^{-1}U^\perp} \psi(-q^\omega)|\omega|^\omega = |a|^{-n} \cdot \text{vol}(U^\perp).$$

By involutivity of the Fourier transform, we have $\text{vol}(U) \cdot \text{vol}(U^\perp) = 1$, hence we get

$$\int_U \psi(a^2 q)|\omega| = c(q, \omega, \psi) \cdot |a|^{-n}.$$

One can deal with the case of a general family of functions parametrized by a compact set $P \subset S(E)$ as follows. The subvariety $\text{Cr}(f) \subset X \times S$ of critical points of $f$ in $X$-direction is étale over $S$. Thus, for every point $p \in P$ the above argument works uniformly for all $p'$ in sufficiently small neighborhood of $p$. Now our assertion follows from compactness of $P$.

To analyze the result of the stationary phase approximation, it is convenient to use the following lemma.

**Lemma 2.5.2.** Let $f_1, \ldots, f_n$ be analytic functions on a ball $B$ in $E^N$ centered at 0. Assume that the differentials at zero $df_1, \ldots, df_n$ are linearly independent. Then there exists a constant $a \in E^*$ such that for all $t \in E^*$ with $|t|$ sufficiently large, the functions $\psi(tf_1), \ldots, \psi(tf_n)$ on $t^{-1}aB$ are linearly independent.

**Proof.** Without loss of generality we can assume that $f_i(0) = 0$. Let $C \subset E^N$ be a sufficiently large open compact containing 0, so that the restrictions of the functions $\psi(df_1), \ldots, \psi(df_n)$ to $C$ are linearly independent. Now for sufficiently large $t$ we have $t^{-1}C \subset B$ and $\psi(tf_i(t^{-1}x)) = \psi(df_i(x))$ for all $x \in C$. It remains to take $a$ such that $C \subset aB$. 

**2.6. Local computation.** The field $E$ is still assumed to be $p$-adic. We fix one of stabilizer subgroups $H \subset G$ and set $W = V^H, W_0 = W \cap V_0, W^\omega = (V^\omega)^H, W^\omega_0 = W_0^\omega \cap V_0^\omega$. Let us consider the subset of $H$-fixed points in the variety $I$:

$$I^H = I \cap (W_0 \times W_0^\omega) = I \cap (V_0 \times W_0^\omega)$$

(the last equality follows from lemma [1.2.1]). Let $U \subset V_0/G \times V_0^\omega$ be a non-empty $G$-invariant open subset such that the morphism $f : I \to V_0/G \times V_0^\omega$ is étale over $U$ (see proposition [1.2.2]). We have $I^H = f^{-1}(V_0/G \times W_0^\omega)$. Since $U$ is $G$-invariant, it has a non-empty intersection with $V_0/G \times W_0^\omega$. Therefore, the open subset
\( f^{-1}(U) \cap I^H \subset I^H \) is non-empty. Note that since \( I^H \) is an open subset in a vector bundle over \( W_0^V \), the set of \( E \)-rational points in \( I^H \) is dense in Zariski topology. Therefore, there exists a point \( (x_0, x_0') \in f^{-1}(U) \cap I(E) \). Set \( O = O_{x_0} \). Then \( x_0 \) is a critical point of \( x_0' \) and \( (O, x_0') \in U \).

Recall that according to lemma 1.1, there exists a linear subspace \( L \subset W \) such that the morphism \( p: x_0 + L \to V_0/G \) is étale near \( x_0 \). Let \( D \subset O(E) \) be a small ball centered at zero, \( U^\vee \subset V_0^\vee \) be a compact open neighborhood of \( x_0' \). We assume that \( D \) and \( U^\vee \) are small enough, so that the following two conditions are satisfied:

1. The restriction of \( p \) to \( x_0 + D \) is an isomorphism onto \( p(x_0 + D) \subset V_0/G(E) \);
2. \( p(x_0 + D) \times U^\vee \subset U(E) \) and \( f^{-1}(p(x_0 + D) \times U^\vee) \) is analytically isomorphic to a disjoint union of open subsets mapping identically to \( p(x_0 + D) \times U^\vee \).

Let \( O(E) = O_1 \cup \ldots \cup O_r \) be a partition of \( O(E) \) into \( G(E) \)-orbits. Let us choose non-empty open compact subsets \( K_i \subset O_i, i = 1, \ldots, r \) such that \( \text{vol}(K_i) \) does not depend on \( i \) (where the volume is computed using a \( G \)-invariant top-degree form on \( O \)). In addition we assume that for every \( i = 1, \ldots, r \) there exists an analytic isomorphism of \( K_i \) with a ball in \( E^d \), where \( d = \dim O \). We set \( K = K_1 \cup \ldots \cup K_r \subset O(E) \). Using the identification \( G/H \to O : yH \to gx_0 \) we can consider \( K \) as a subset of \( G/H(E) \). By our choice of \( D \), the restriction of the map \( G/H(E) \times L(E) \to V_0(E) \) to \( K \times (x_0 + D) \) is an isomorphism onto an open compact subset \( K(x_0 + D) \subset V_0(E) \). Now for every \( t \in E^* \) with \( |t| > 1 \) we consider a compact open subset

\[
U(t) = K(tx_0 + D) = tK(x_0 + t^{-1}D) \subset tK(x_0 + D).
\]

Note that by lemma 2.1.1, stabilizers of all points in \( U(t) \) are \( G \)-inner forms of \( H \). For every open compact set \( C \subset V(E) \) we denote by \( \delta_C \) the characteristic function of \( C \). It is clear that for every \( t \in E^* \) with \( |t| > 1 \) the function \( \delta_U(t) \in \mathcal{S}(V_0(E)) \) is stable. More generally, for every character \( \kappa : A(H/E) \to \mathbb{C}^* \) we define a function \( \delta_{U(t)}^\kappa \) supported on \( U(t) \) by

\[
\delta_{U(t)}^\kappa(k(x_0 + d)) = \kappa(\text{inv}(x_0, k)) \quad (2.6.1)
\]

where \( k \in K, d \in D \).

**Lemma 2.6.1.** There exists a compact set \( C \subset V^\vee(E) \) such that for all \( t \in E^* \) with \( |t| > 1 \) and all \( \kappa \in A(H/E)^D \) the support of \( F(\delta_{U(t)}^\kappa) \) is contained in \( C \).

**Proof.** Set \( U_i(t) = K_i(tx_0 + D), i = 1, \ldots, r \). Since \( \delta_{U(t)}^\kappa \) is a linear combination of \( \delta_{U_i(t)}^\kappa \), it suffices to prove that there exists an open compact neighborhood of zero \( B \subset V(E) \), such that \( U_i(t) + B = U_i(t) \) for all \( t \in E^* \) with \( |t| > 1 \) and all \( i \). Let us consider the norm on \( V(E) \) for which the unit ball is an integer lattice in \( V(E) \). Similarly, using an isomorphism of \( K_i \) with a ball in \( E^d \) and the norm on \( L(E) \) for which \( D \) is the unit ball, we get an (ultra)metric on \( K_i \times (x_0 + D) \) such that \( d((k, x), (k', x')) = \max(||k - k'||, ||x - x'||) \). Since the isomorphism \( a : K_i \times (x_0 + D) \to U_i = K_i(x_0 + D) \) is analytic we have \( d(a^{-1}(y), a^{-1}(y')) \leq c \cdot ||y - y'|| \) for some constant \( c > 0 \), where \( y, y' \in U_i \). Now for \( t \in E^* \) with \( |t| > 1 \) the subset \( t^{-1}U_i(t) = K_i(x_0 + t^{-1}D) \subset U_i \) consists of points \( y \in U_i \) such that \( d(a^{-1}(y), K_i \times x_0) \leq |t|^{-1} \) (since \( D \) is the unit ball for the norm on \( L(E) \)). Now let \( B \subset V(E) \) be a ball of radius \( < 1^{-1} \) centered at zero such that \( U_i + B = U_i \). We claim that \( U_i(t) + B = U_i(t) \). Indeed, we have to check that for every \( y \in t^{-1}U_i(t) \) one has \( y + t^{-1}B \subset t^{-1}U_i(t) \). Let \( y' \in y + t^{-1}B \). Then \( y' \in U_i \) since \( U_i + t^{-1}B = U_i \).
Also, $||y - y'|| \leq c^{-1}|t|^{-1}$, hence $d(a^{-1}(y), a^{-1}(y')) \leq |t|^{-1}$. By ultrametric triangle inequality this implies that $d(a^{-1}(y'), K_t \times x_0) \leq |t|^{-1}$, i.e., $y' \in t^{-1}U_t(t)$. 

Proposition 2.6.2. (i) There exists a constant $A > 0$ such that for all $t \in (E^*)^2$ with $|t| > A$, all $x^\vee \in U^\vee$ and $y^\vee \in \mathcal{O}_{x^\vee}(E)$ one has

$$I(\mathcal{F}(\delta_{U(t)}))(y^\vee) = \epsilon(x^\vee, y^\vee) \cdot I(\mathcal{F}(\delta_{U(t)}))(x^\vee).$$

(ii) There exists a ball $D^\vee \subset \mathcal{W}^\vee(E)$ centered at zero, such that for every sufficiently small ball $D$ as above, the restriction of the function $y \mapsto \delta_{\mathcal{O}_y}^{-1} \cdot \text{sign}(\mathcal{F}(\delta_{U(t)}))$ to $x_0^\vee + t^{-1}D^\vee$ is not identically zero provided that $t \in (E^*)^2$ is large enough.

Proof. (i) For $x^\vee \in \mathcal{V}_0^\vee(E)$ we have

$$\mathcal{F}(\delta_{U(t)})(x^\vee) = \int_{U(t)} \psi((x^\vee, x)) dx = |t|^\dim \mathcal{V} \cdot \int_{t^{-1}U(t)} \psi(t(x^\vee, x)) dx$$

where $dx$ is the Haar measure on $\mathcal{V}(E)$ corresponding to a top-degree form defined over $E$. By lemma 1.1.3(b) we can rewrite this integral as follows:

$$\int_{t^{-1}U(t)} \psi(t(x^\vee, x)) dx = \int_{x \in x_0 + t^{-1}D} \int_{K} \psi(t(x^\vee, ks)) |\omega(k)| \cdot |\nu(s)|$$

where $\omega$ is a $G$-invariant top-degree form on $G/H$, $\nu$ is a top-degree form on $L$. Now the inner integral has form

$$\int_{x \in K} \psi(t(x^\vee, x)) |\omega|$$

so we can apply the stationary phase principle to compute it. More precisely, by lemma 2.6.1 we know that $\mathcal{F}(\delta_{U(t)})(x^\vee) = 0$ for $x^\vee \notin C$, where $C$ is a compact in $\mathcal{V}^\vee(E)$. Also we know that for $x^\vee \in p^{-1}(p(U^\vee))$ and $s \in x_0 + D$, the function $x^\vee|_{\mathcal{O}_x(E)}$ has a finite number of non-degenerate critical points. Finally, we observe that the subset $p^{-1}(p(U^\vee)) \subset \mathcal{V}^\vee(E)$ is closed, hence, $C \cap p^{-1}(p(U^\vee))$ is compact. Thus, applying lemma 2.5.1 we derive that there exists a constant $A > 0$ such that for $s \in x_0 + D$, $x^\vee \in C \cap p^{-1}(p(U^\vee))$, $t \in (E^*)^2$, $|t| > A$ one has

$$\int_{x \in K} \psi(t(x^\vee, x)) |\omega| = |t|^{-\dim G \cdot \frac{1}{2}} \cdot \sum_{x \in K \cap \mathcal{V}} \psi(t(x^\vee, x)) c(B_{x, x^\vee, \omega_x, \psi}).$$

We claim that enlarging $C$ if necessary we can achieve that the RHS is zero for $x^\vee \in p^{-1}(p(U^\vee)) \setminus C$. Indeed, this follows immediately from the fact that $I(E) \cap (K(x_0 + D) \times p^{-1}(p(U^\vee)))$ is compact (as a preimage of the compact set $K(x_0 + D) \times p(U^\vee)$ under the map $I \to \mathbf{V}_0 \times \mathbf{V}_0^\vee / G$). Thus, if $t \in (E^*)^2$ is large enough then for all $x^\vee \in p^{-1}(p(U^\vee))$ we have

$$\mathcal{F}(\delta_{U(t)})(x^\vee) = |t|^\dim \mathcal{V} \cdot \frac{-\dim G}{2} \cdot \sum_{x \in K \cap \mathcal{V}} \psi(t(x^\vee, x)) c(B_{x, x^\vee, \omega_x, \psi}) |\nu(s)|.$$

Now let us substitute $x^\vee$ by another point $gx^\vee \in \mathcal{O}_{x^\vee}(E)$ and make the change of variables $x \mapsto gx$ (this makes sense since $x$ has the same stabilizer as $x^\vee$). Then we obtain

$$\mathcal{F}(\delta_{U(t)})(gx^\vee) = |t|^\dim \mathcal{V} \cdot \frac{-\dim G}{2} \cdot \sum_{x \in K \cap \mathcal{V}} \psi(t(gx^\vee, gx)) c(B_{gx, gx^\vee, \omega_{gx}, \psi}) |\nu(s)|.$$
Using lemma 2.3.1 we can rewrite the inner sum as follows:

\[
\sum_{x \in g^{-1}K \circ C(x^\vee)} \psi(t(x^\vee, x))e_H(e_g)c(B_{x,x^\vee}, \omega_x, \psi),
\]

where \(e_g \in H^1(E, H)\) is the cohomology class of the cocycle \(\sigma \mapsto g^{-1}\sigma(g)\). Hence, we obtain

\[
\mathcal{F}(\delta_U(t))(g x^\vee) = \left| t^{\dim V - \dim E} \cdot \epsilon_H(e_g) \cdot \int_{s \in x_0 + t^{-1}D} \mathcal{G}(E)/(Hg^{-1})(E) \left| \omega(g_g) \right| \sum_{x \in g^{-1}K \circ C(x^\vee)} \psi(t(x^\vee, x))c(B_{x,x^\vee}, \omega_x, \psi) \right| \nu(s)
\]

Now to calculate \(I(\mathcal{F}(\delta_U(t)))\) at \(g x^\vee\), we have to replace \(g\) by \(g_1 g\) in the above formula where \(g_1 \in G(E)/(Hg^{-1})(E)\) and integrate over \(g_1\). We get

\[
I(\mathcal{F}(\delta_U(t)))(g x^\vee) = \left| t^{\dim V - \dim E} \cdot \epsilon_H(e_g) \right| \mathcal{G}(E)/(Hg^{-1})(E) \left| \omega(g_g) \right| \sum_{x \in g^{-1}K \circ C(x^\vee)} \psi(t(x^\vee, x))c(B_{x,x^\vee}, \omega_x, \psi) \right| \nu(s)
\]

where volumes are computed using \(\omega\). By our choice of \(K\) we have \(\mathcal{G}(E)g x \cap K_s = \mathcal{G}(E)x_0 \cap K\). Hence, we conclude that

\[
I(\mathcal{F}(\delta_U(t)))(g x^\vee) = \left| t^{\dim V - \dim E} \cdot \epsilon_H(e_g) \right| \mathcal{G}(E)/(Hg^{-1})(E) \left| \omega(g_g) \right| \sum_{x \in g^{-1}K \circ C(x^\vee)} \psi(t(x^\vee, x))c(B_{x,x^\vee}, \omega_x, \psi) \right| \nu(s),
\]

Since \(\epsilon_H(e_g) = \epsilon(x^\vee, g x^\vee)\) this finishes the proof of the part (i) of the proposition.

(ii) Arguing as above, we obtain that

\[
I(\mathcal{F}(\delta_U(t)))(g x^\vee) = \left| t^{\dim V - \dim E} \cdot \epsilon_H(e_g) \right| \mathcal{G}(E)/(Hg^{-1})(E) \left| \omega(g_g) \right| \sum_{x \in g^{-1}K \circ C(x^\vee)} \psi(t(x^\vee, x))c(B_{x,x^\vee}, \omega_x, \psi) \right| \nu(s),
\]

for sufficiently large \(t \in (E^*)^2\), where \(x^\vee, g x^\vee \in p^{-1}(p(U^\vee))\).

By our choice of \(D\) and \(U^\vee\) there exists a collection of analytic maps

\[
x_i : (x_0 + D) \times U^\vee \rightarrow V(E), i = 1, \ldots, n,
\]

such that for every \(s \in x_0 + D\) the points \(x_1(s, x^\vee), \ldots, x_n(s, x^\vee)\) are disjoint and constitute the set of critical points of \(x^\vee\). Let us set \(x_i = x_i(x_0, x_0^\vee)\). Renumbering these maps if necessary we can assume that \(x_1 = x_0\). We claim that the differentials at \(x_0^\vee\) of the functions \(x^\vee \mapsto \l(x^\vee, x_i(x_0, x^\vee))\), where \(x^\vee \in U^\vee \cap W^\vee(E)\), \(i = 1, \ldots, n\), are linearly independent. Note that by lemma 2.1(a) for \(x^\vee \in U^\vee \cap W^\vee(E)\) we have \(x_i(x_0, x^\vee) \in W(E). Let L_i \in \text{Hom}(W^\vee, W)\) be the differential of \(x_i(x_0, x^\vee)|_{U^\vee \cap W^\vee(E)}\) at \(x_0^\vee\). Differentiating the condition \(x_i(x_0, x^\vee) \in O_{x_0}\), we get that \(L_i(W^\vee) \subset g x_0\). Now the differential at \(x_0^\vee\) of the function \(\l(x^\vee, x_i(x_0, x^\vee))\) on \(U^\vee \cap W^\vee\) is the functional

\[
w^\vee \mapsto \l(w^\vee, x_i(x_0, x_0^\vee)) + \l(x_0^\vee, L_i(w^\vee)) = \l(w^\vee, x_i(x_0, x_0^\vee))
\]
then setting $c$ and large enough $t$ the functions
\[ x^\nu \mapsto \psi(t(x^\nu, x_i(x_0, x^\nu))), \quad i = 1, \ldots, n, \] (2.6.5)
are linearly independent on $x_0^\nu + t^{-1}D^\nu$.

Since the functions $x_i$ are analytic, we can choose $D$ sufficiently small so that
\[ \psi(t(x^\nu, x_i(s, x^\nu))) = \psi(t(x^\nu, x_i(x_0, x^\nu))), \]
for $s \in x_0 + t^{-1}D$, $x^\nu \in U^\nu$, $i = 1, \ldots, n$. On the other hand, if $t$ is large enough then setting $c_i(s, x^\nu) = c(B_{x_i(s, x^\nu), x_i}, \omega_{x_i(s, x^\nu)}, \psi)$ for $i = 1, \ldots, n$, we get
\[ c_i(s, x^\nu) = c_i(x_0, x_0^\nu), \]
\[ \kappa(\mathrm{inv}(s, gx_i(s, x^\nu))) = \kappa(\mathrm{inv}(x_0, gx_i)) \]
for $s \in x_0 + t^{-1}D$, $x^\nu \in x_0^\nu + t^{-1}D^\nu$, $i = 1, \ldots, n$. Finally, we have
\[ \kappa(\mathrm{inv}(x_0, gx_i)) = \kappa(\mathrm{inv}(x_0, x_i)) \cdot \kappa(\tau_i(\mathrm{inv}(e_g))) \]
where $\tau_i : A(\mathbf{H}/E) \to A(\mathbf{H}/E)$ is the automorphism induced by the action of an element $g_i \in N(\mathbf{H})/\mathbf{H}$ such that $g_i x_i = x_0$. Note that since by our assumption $x_1 = x_0$, we have $\tau_1 = \mathrm{id}$. Thus, applying the formula (2.6.4) to $x^\nu \in x_0^\nu + t^{-1}D^\nu$ and large enough $t \in (E^*)^2$, and using Lemma (2.2.1), we get
\[
I(F(\delta_{U(t)}))(gx^\nu) = |t|^{\dim V - \dim \mathbf{E}} \cdot \mathrm{vol}(G(E)x_0 \cap K) \times \\
\int_{s \in x_0 + t^{-1}D} \sum_{i=1}^n \kappa(\mathrm{inv}(x_0, x_i))((\kappa \circ \tau_i) \mathrm{sign})(\mathrm{inv}(e_g)) \psi(t(x^\nu, x_i(s, x^\nu))) c_i(x_0, x_0^\nu) |\nu(s)| = \\
|t|^{\dim V - \dim \mathbf{E}} \cdot \mathrm{vol}(G(E)x_0 \cap K) \mathrm{vol}(t^{-1}D) \times \\
\sum_{i=1}^n \kappa(\mathrm{inv}(x_0, x_i))((\kappa \circ \tau_i) \mathrm{sign})(\mathrm{inv}(e_g)) c_i(x_0, x_0^\nu) \psi(t(x^\nu, x_i(x_0, x^\nu))).
\]
Hence, we have
\[
\delta_{O_{x_0^\nu}}^{-\sign}(F(\delta_{U(t)})) = c(t) \cdot \sum_{gx^\nu \in G(E) \setminus O_{x_0^\nu}(E)} \sum_{i=1}^n \kappa(\mathrm{inv}(x_0, x_i))((\kappa \circ \tau_i) \mathrm{sign})(\mathrm{inv}(e_g)) c_i(x_0, x_0^\nu) \psi(t(x^\nu, x_i(x_0, x^\nu))),
\]
where $c(t) \in \mathbb{C}^*$ is a constant depending on $t$. Interchanging two summations and noting that $\mathrm{inv}(e_g)$ runs through the entire group $A = A(O_{x_0^\nu}) = A(O_{x_0^\nu})$, we obtain
\[
\delta_{O_{x_0^\nu}}^{-\sign}(F(\delta_{U(t)})) = c(t) |A| \cdot \sum_{x \in G \cap \tau \in K} \sum_{i=1}^n \kappa(\mathrm{inv}(x_0, x_i)) c_i(x_0, x_0^\nu) \psi(t(x^\nu, x_i(x_0, x^\nu)))
\]
(2.6.6)
for $x^\nu \in x_0^\nu + t^{-1}D^\nu$ and large enough $t \in (E^*)^2$. Now our assertion follows from linear independence of the functions (2.6.5) on $x_0^\nu + t^{-1}D^\nu$ and the fact that $\tau_1 = \mathrm{id}$. \qed
2.7. Inner forms. Let $Z \subset G$ be a central subgroup defined over $E$, $\alpha \in H^1(E, G/Z)$ be a cohomology class. Then $\alpha$ defines an inner form $G'$ of $G$. Note that since $G$ is simply connected, the homomorphism
\[ d : H^1(E, G/Z) \to H^2(E, Z) \]
is an isomorphism when $E$ is $p$-adic, so in this case $\alpha$ is uniquely determined by $d(\alpha) \in H^2(E, Z)$. Assume that $\rho(Z) = 1$. Then we can twist $\rho$ by $\alpha$ to get a nice representation $\rho' : G' \to \text{Aut}(V')$. Let $H \subset G$ (resp. $H' \subset G'$) be a generic stabilizer subgroup, $V_0 \subset V$ (resp. $V'_0$) be the open subset consisting of points with the stabilizer conjugated to $H$ (resp. $H'$) over $\bar{E}$. The isomorphism $i : V \to V'$ defined over $\mathbb{F}$ induces a bijection between $G$-orbits on $V$ defined over $E$ and $G'$-orbits on $V'$ defined over $E$. Let $O \subset V_0$ be a $G$-orbit, $O' = i(O) \subset V'_0$ be the corresponding $G'$-orbit. If $\omega$ is a $G$-invariant non-zero top degree form on $O$ defined over $E$ then $i_* \omega$ is a $G'$-invariant top degree form on $O'$, also defined over $E$.

**Definition.** For a pair of functions $\phi \in S(V(E))$, $\phi' \in S(V'(E))$ we say that $\phi \sim \phi'$ (resp. $\phi \sim_{\epsilon} \phi'$) if for every $G$-orbit $O \subset V_0$ defined over $E$ one has $\delta_O \omega(\phi) = \delta_{O'} \omega(\phi')$ (resp. $\delta'_O \omega(\phi) = \delta'_{O'} \omega(\phi')$). For a pair of distributions $\delta \in D(V(E))$, $\delta' \in D(V'(E))$ we say that $\delta \sim \delta'$ (resp. $\delta \sim_{\epsilon} \delta'$) if for every pair of functions $\phi \in S(V(E))$, $\phi' \in S(V'(E))$ such that $\phi \sim \phi'$ (resp. $\phi \sim_{\epsilon} \phi'$) one has $\delta(\phi) = \delta'(\phi)$.

Note that we have $\phi \sim 0$ (resp. $\phi \sim_{\epsilon} 0$) if and only if $\phi$ is antistable (resp. $\epsilon$-antistable). Therefore, if $\delta \sim \delta'$ (resp. $\delta \sim_{\epsilon} \delta'$) then distributions $\delta$ and $\delta'$ are necessarily stable (resp. $\epsilon$-stable). Also, by the definition we have $\delta_O \omega \sim_{\epsilon} \delta_{O'} \omega$ (resp. $\delta'_O \omega \sim_{\epsilon} \delta'_{O'} \omega$).

To formulate our result on inner forms we have to introduce certain sign associated with $G$ and $G'$. Let $Q$ (resp. $Q'$) be the Killing form on $g$ (resp. $g'$). We set
\[ \kappa(G, G') = \epsilon(Q, Q'). \]
Note that $Q$ and $Q'$ have the same determinant modulo squares. Thus, in the case of $p$-adic $E$ the difference between equivalence classes of these quadratic forms is measured by the sign $\kappa(G, G')$.

**Theorem 2.7.1.** Let $\rho : G \to \text{GL}(V)$ be a nice representation of a simply connected semisimple group $G$ over a $p$-adic field $E$, $Z \subset G$ be a central subgroup acting trivially on $V$. Let $(G', V')$ be a twist of $(G, V)$ by a class $\alpha \in H^1(E, G/Z)$. Assume that either $G$ is simple or the generic stabilizer is semisimple. Then for every pair of functions $\phi \in S(V(E))$, $\phi' \in S(V'(E))$ one has $\phi \sim \phi'$ if and only if $\mathcal{F}(\phi) \sim_{\epsilon} \kappa(G, G') \mathcal{F}(\phi')$. The same result holds for distributions.

The proof will be given in 3.3. The remainder of this section consists of various local ingredients of the proof.

Theorem 2.4.3 is an immediate consequence of theorem 2.7.1. Indeed, we can take $(G', V') = (G, V)$ and $\phi' = 0$. Then the condition $\phi \sim 0$ means that $\phi$ is antistable, while the condition $\mathcal{F}(\phi) \sim_{\epsilon} 0$ means that $\mathcal{F}(\phi)$ is $\epsilon$-antistable.

**Lemma 2.7.2.** (a) Let $O \subset V_0$ be an orbit, $x \in O(E)$ be an $E$-point on $O$. Then $O'(E)$ is non-empty if and only if the class $\alpha$ belongs to the image of the map $H^1(E, H_x/Z) \to H^1(E, G/Z)$.
(b) For every $\phi \in S(V'(E))$ the functions $x \mapsto \delta_{O', i, \omega}(\phi)$ and $x \mapsto \delta'_{O', i, \omega}(\phi)$ on $V_0(E)$ are locally constant.
Proof. The proof of (a) is straightforward. To prove (b) let us fix a point \( x \in V_0(E) \). Set \( H = H_x \), \( W = V^H \), \( W_0 = W \cap V_0 \). The morphism \( \alpha : G/H \times W_0 \to V_0 \) is smooth, hence \( U = \alpha(G/H(E) \times W_0(E)) \) is an open subset in \( V_0(E) \). Assume first that \( \alpha \) does not belong to the image of the map \( H^1(E, H/\mathbb{Z}) \to H^1(E, G/\mathbb{Z}) \). Then for every point \( y \in U \) we have \( \mathcal{O}_y^*(E) = \emptyset \). Thus, we can assume that \( \alpha \) belongs to the image of this map. Then there exists an isomorphism \( i : (G, V) \to (G', V') \) over \( \overline{E} \), such that the subgroup \( i(H) \subset G' \) and the morphism \( i|_W \) are defined over \( E \). Therefore, the functions \( y \mapsto \delta_{\mathcal{O}_y,i*,\omega}(\phi) \) and \( y \mapsto \delta_{\mathcal{O}_y,i*,\omega}(\phi) \) on \( W_0(E) \) is locally constant. Let \( U_0 \subset W_0(E) \) be a neighborhood of \( x \) on which it is constant. Then \( a(G/H(E) \times U_0) \) is an open neighborhood of \( x \) in \( V_0(E) \) on which a similar function is constant.

\( \Box \)

Lemma 2.7.3. Assume that \( E \) is \( p \)-adic, \( G \) is simple, the generic stabilizer for \( \rho \) is commutative and all irreducible components of \( \rho_{\overline{E}} \) are defined over \( E \). Then there exists a point \( x \in V_0(E) \) such that \( H_x \) is an anisotropic.

Proof. In the case when \( \rho \) is the adjoint representation, we can use the existence of an anisotropic maximal torus \( H_x \) in \( G \) defined over \( E \) (see e.g., Theorem VI.21 of [1]). It turns out that in all other cases the action of a sufficiently big subgroup of \( G \) on \( V \) essentially reduces to the adjoint representation. Here is a more precise statement.

Claim. For every \( x \in V_0(E) \) there exists a semisimple subgroup \( G' \subset G \) defined over \( E \) and a \( G' \)-invariant decomposition \( V = V' \oplus V'' \), such that the following two conditions hold:

(i) the representation of \( G' \) on \( V'' \) is equivalent to the adjoint representation of \( G' \);
(ii) let \( x = x' + x'' \) where \( x' \in V' \), \( x'' \in V'' \), then \( G'x' = x' \) and \( H_x \) coincides with the stabilizer of \( x'' \) in \( G' \) (so by (i), \( H_x \) is a maximal torus in \( G' \)).

Our statement can be deduced from this claim as follows. The subset \( V_0 \cap (x' + V'') \subset x' + V'' \) is non-empty and Zariski open. Therefore, we can choose \( x'' \in V''(E) \) such that \( \tilde{x} = x' + x'' \in V_0(E) \) and the stabilizer of \( \tilde{x}'' \) in \( G' \) is anisotropic. Then \( H_{\tilde{x}} \) contains an anisotropic maximal torus in \( G' \). Since \( \tilde{x} \) is contained in \( V_0 \), this inclusion is in fact an equality. The proof of the Claim follows from Elashvili’s classification of representations of simple groups with generic stabilizers of positive dimension (see [2]). Here are the cases relevant for our situation:

(i) \( G_{\overline{E}} = SL(W) \), \( V_{\overline{E}} = V' \oplus V'' \), where \( V' \) and \( V'' \) are irreducible representations of \( SL(W) \) in \( S^2W \) and \( \wedge^2 W \) or in \( S^2W^* \) and \( \wedge^2 W^* \) (or dual to these). The stabiliser \( G' \) of a generic point in \( V' \) is the special orthogonal group \( SO(W) \). It is well-known that the representation of \( SO(W) \) in \( \wedge^2 W \) is isomorphic to the adjoint representation.

(ii) \( G_{\overline{E}} = SL(W) \), where \( \dim W = 4 \), \( V_{\overline{E}} \) is the direct sum of 4 copies of \( \wedge^2 W \). The stabilizer in \( SL(W) \) of a generic point in \( (\wedge^2 W)^2 \) is conjugate to the subgroup \( SL_2 \times SL_2 \) corresponding to a decomposition \( W = W_1 \oplus W_2 \), where \( \dim W_1 = \dim W_2 = 2 \) (see Table 1 of [3]). The generic stabilizer in \( (\wedge^2 W)^3 \) is the subgroup \( SL_2 = \{ (g, g^{-1}), g \in SL_2 \} \) in \( SL_2 \times SL_2 \) (which corresponds to choosing an isomorphism \( W_1 \simeq W_2 \)). Decomposing the 4-th factor as a representation of \( SL(W_1) \times SL(W_2) \):

\[
\wedge^2 W = 1 \oplus 1 \oplus W_1 \oplus W_2,
\]
we see that the action of the above subgroup on $\bigwedge^2 W$ is equivalent to the direct sum of the adjoint representation of $\text{SL}_2$ with 3 trivial representations. Thus, the above claim holds if we take $G'$ to be the stabilizer of the first three components of $x$, $V'$ to be the sum of $(\bigwedge^2 W)^3$ and of 3 trivial representations.

(iii) $G_{\mathbb{E}} = \text{SL}(W)$, where $\dim W = 6$, $V_{\mathbb{E}}$ is the direct sum of 2 copies of $\bigwedge^3 W$. The stabilizer in $\text{SL}(W)$ of a generic point in $\bigwedge^3 W$ is conjugate to the subgroup $\text{SL}_3 \times \text{SL}_3$ corresponding to a decomposition $W = W_1 \oplus W_2$, where $\dim W_1 = \dim W_2 = 3$ (see Table 1 of [3]). We can decompose the second copy of $\bigwedge^3 W$ as a representation of $\text{SL}(W_1) \times \text{SL}(W_2)$:

$$\bigwedge^3 W = 1 \oplus 1 \oplus W_1' \oplus W_2 \oplus W_2' \oplus W_1.$$

Note that the last two factors are non-isomorphic non-trivial irreducible representations of $\text{SL}(W_1) \times \text{SL}(W_2)$, so for any $E$-form of the pair $(\text{SL}(W_1) \times \text{SL}(W_2), \bigwedge^3 W)$ the similar decomposition takes place. Now the stabilizer in $\text{SL}(W_1) \times \text{SL}(W_2)$ of a generic point in one of the non-trivial factors is isomorphic to $\text{SL}_3$, and the action of this subgroup on the second non-trivial factor is the sum of the adjoint representation and of the trivial representation. Using this we can easily construct $V'$ and $V''$ (with $G'$ being the form of $\text{SL}_3$).

(iv) $G_{\mathbb{E}} = \text{SL}(W)$, where $\dim W = 8$, $V_{\mathbb{E}}$ is either $\bigwedge^3 W \oplus W$ or $\bigwedge^3 W \oplus W^\vee$. The stabilizer in $\text{SL}(W)$ of a generic point in $\bigwedge^3 W$ is isomorphic to $\text{SL}_3$. The embedding of $\text{SL}_3$ in $\text{SL}(W)$ corresponds to the identification of $W$ with the adjoint representation of $\text{SL}_3$ (see Table 1 of [3]). Therefore, we can take $G'$ to be the stabilizer of the first component of $x$ with respect to the above decomposition of $V$.

Remark. Note that if $G$ is split over $E$, then all irreducible components of $\rho_{\mathbb{E}}$ are defined over $E$.

**Lemma 2.7.4.** Assume that $E$ is $p$-adic and that either $H$ is semisimple, or $G$ is simple and all irreducible components of $\rho_{\mathbb{E}}$ are defined over $E$. Then there exists a point $x \in V_0(E)$, such that the map $H^1(E, H_x/\mathbb{Z}) \to H^1(E, G/\mathbb{Z})$ is surjective.

**Proof.** Note that the natural map $H^1(E, G/\mathbb{Z}) \to H^2(E, \mathbb{Z})$ is an isomorphism (since $E$ is $p$-adic). Therefore, we have to prove that the natural map

$$H^1(E, H_x/\mathbb{Z}) \to H^2(E, \mathbb{Z})$$

is surjective for some $x \in V_0(E)$.

Assume first that $H$ is semisimple. Then we claim that this surjectivity holds for every point $x \in V_0(E)$. Indeed, let $\tilde{H} \to H$ be the universal covering of $H$, $\mathbb{Z} \subset \tilde{H}$ be the preimage of $\mathbb{Z} \subset H$. Then we have an isomorphism

$$H^1(E, H/\mathbb{Z}) = H^1(E, \tilde{H}/\mathbb{Z}) \cong H^2(E, \mathbb{Z}).$$

Since the cohomological dimension of $E$ is equal to 2, the map $H^2(E, \mathbb{Z}) \to H^2(E, \mathbb{Z})$ is surjective, which finishes the proof in this case (the same argument can be applied to any $H_x$).

Now let us assume that $H$ is commutative. Then by lemma 2.7.3 there exists a point $x \in V_0(E)$ such that $H_x$ is an anisotropic torus. By Tate-Nakayama duality, for such a torus we have $H^2(E, H_x) = 0$, which implies the surjectivity we want. 

\[23\]
In the remainder of this section we will keep the assumptions of lemma 2.7.4. Furthermore, we choose a point \( x_0 \in V_0(E) \) as in this lemma and set \( H = H_{x_0} \), \( \mathbf{W} = \mathbf{V}^H, \mathbf{W}_0 = \mathbf{W} \cap V_0 \), etc. We also fix a cohomology class \( \alpha_0 \in H^1(E, H; \mathbf{Z}) \). This allows us to choose an isomorphism \( i : (G, \mathbf{V}) \rightarrow (G', \mathbf{V}') \) over \( \mathbf{E} \), such that the subgroup \( H' = i(H) \subset G' \) and the morphism \( i|_{W} \) are defined over \( E \). We denote \( \mathbf{W}' = (\mathbf{V}')^H' = i(\mathbf{W}) \). Also we denote by \( i' : \mathbf{V}' \rightarrow (\mathbf{V}')^\vee \) the \( \mathbf{E} \)-isomorphism induced by \( i \). Note that \( i'|_{W^\vee} \) is also defined over \( E \).

The following lemma (generalizing lemma 2.3.1) computes the sign that in the \( p \)-adic case measures the difference between the quadratic form \( B_{x,x^\vee} \) introduced in [13] and \( B_{i(x),i'(x^\vee)} \). Recall that for \( x \in \mathcal{O}, x' \in \mathcal{O'} \) we denote \( \epsilon(x, x') = \epsilon(Q_x, Q'_x) \).

**Lemma 2.7.5.** Let \( x \in \mathbf{W}_0 \) be a critical point of \( x^\vee|_{\mathcal{O}_x} \), where \( x^\vee \in \mathbf{W}^\vee_0 \). Let \( \omega \) be a non-zero \( G \)-invariant top-degree form on \( \mathcal{O}_x \) defined over \( E \).

(a) One has \( \det(B_{i(x),i'(x^\vee)})/\omega_{i(x)}^2 = \det(B_{x,x^\vee})/\omega_x^2 \).

(b) The quadratic forms \( B_{i(x),i'(x^\vee)} \) and \( B_{x,x^\vee} \) have the same determinant modulo squares. Their relative Hasse-Witt invariant is given by

\[
\epsilon(B_{i(x),i'(x^\vee)}, B_{x,x^\vee}) = \kappa(G, G')\epsilon(x, i(x)) = \kappa(G, G')\epsilon(x^\vee, i(x^\vee)).
\]

**Proof.** The proof of (a) is straightforward. To prove (b) we note that there is a natural homomorphism \( \iota : H/\mathbf{Z} \rightarrow SO(T_x, B_{x,x^\vee}) \) which induces a map

\[
\iota_* : H^1(E, H/\mathbf{Z}) \rightarrow H^1(E, SO(T_x, B_{x,x^\vee})).
\]

It is easy to see that the quadratic form \( B_{i(x),i'(x^\vee)} \) is equivalent to the twist of \( B_{x,x^\vee} \) by \( \iota_*(\alpha_0) \). In particular, these forms have the same determinant modulo squares. Let

\[
\delta : H^1(E, SO(T_x, B_{x,x^\vee})) \rightarrow H^2(E, \{\pm 1\}) \simeq \{\pm 1\}
\]

be the map induced by the spin-covering of \( SO(T_x, B_{x,x^\vee}) \). Then the Hasse-Witt invariants of \( B_{x,x^\vee} \) and of its twist by \( \iota_*(\alpha_0) \) differ by \( \delta(\iota_*(\alpha_0)) \). It remains to prove that

\[
\delta(\iota_*(\alpha_0)) = \kappa(G, G')\epsilon(x, i(x)). \tag{2.7.1}
\]

Recall that by proposition 1.3.1 the homomorphism \( \iota \) lifts to a homomorphism \( \tilde{H} \rightarrow \text{Spin}(T_x, B_{x,x^\vee}) \)

where \( \tilde{H} \rightarrow H \) is the pull-back of the spin-covering of \( SO(h, Q_h) \). Let \( \tilde{Z} \subset \tilde{H} \) be the preimage of \( Z \subset H \). Then we have an induced homomorphism

\[
\chi : \tilde{Z} \rightarrow \{\pm 1\}.
\]

On the other hand, we have a natural map of cohomologies

\[
\tilde{d} : H^1(E, H/\mathbf{Z}) = H^1(E, \tilde{H}/\mathbf{Z}) \rightarrow H^2(E, \tilde{Z}).
\]

Now it is easy to see that

\[
\delta \circ \iota_* = \chi_* \circ \tilde{d}
\]

where \( \chi_* : H^2(E, \tilde{Z}) \rightarrow H^2(E, \{\pm 1\}) \) is the homomorphism induced by \( \chi \). Therefore, we have

\[
\delta(\iota_*(\alpha_0)) = \chi_*(\tilde{d}(\alpha_0)) \tag{2.7.2}
\]
On the other hand, we have the natural commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{} & \text{Spin}(g, Q) \\
\downarrow & & \downarrow \\
G/Z & \xrightarrow{} & \text{SO}(g, Q)
\end{array}
\]

(2.7.3)

so we get a homomorphism \( \chi'_G : \mathbb{Z} \to \{ \pm 1 \} \) such that

\[
\kappa(G, G') = \chi'_{G,*}(d(\alpha)).
\]

(2.7.4)

Composing it with the natural projection \( \bar{Z} \to \mathbb{Z} \) we get a homomorphism \( \chi_G : \bar{Z} \to \{ \pm 1 \} \) such that

\[
\kappa(G, G') = \chi_{G,*}(\bar{d}(\alpha_0)).
\]

(2.7.5)

Also, by the definition of \( \bar{H} \) we have the following commutative diagram

\[
\begin{array}{ccc}
\bar{H} & \xrightarrow{} & \text{Spin}(h, Q_{|h}) \\
\downarrow & & \downarrow \\
\bar{H}/\bar{Z} & \xrightarrow{} & \text{SO}(h, Q_{|h})
\end{array}
\]

(2.7.6)

which gives the homomorphism \( \chi_H : \bar{Z} \to \{ \pm 1 \} \) such that

\[
\epsilon(x, i(x)) = \chi_{H,*}(\bar{d}(\alpha_0)).
\]

(2.7.7)

It remains to prove the equality

\[
\chi \cdot \chi_G \cdot \chi_H = 1
\]

of homomorphisms from \( \bar{Z} \) to \( \{ \pm 1 \} \). Indeed, then the equation (2.7.1) would follow from (2.7.2), (2.7.5) and (2.7.7). To prove the equality of algebraic homomorphisms we can pass to the algebraic closure of \( E \). Hence, it suffices to do this over \( \mathbb{C} \). Then we can proceed similarly to the proof of proposition 1.3.1. Namely, as in that proof we deduce that \( \chi \) coincides with the homomorphism induced by the homomorphism \( \bar{H} \to \text{Spin}(h^\perp, Q_{|h^\perp}) \) (the existence of such a homomorphism also follows from the proof of proposition 1.3.1). Now our statement follows from the commutativity of the diagram

\[
\begin{array}{ccc}
\bar{H} & \xrightarrow{} & \text{Spin}(h, Q_{|h}) \times \text{Spin}(h^\perp, Q_{|h^\perp}) \\
\downarrow & & \downarrow \\
G & \xrightarrow{} & \text{Spin}(g, Q)
\end{array}
\]

(2.7.8)

where the left vertical arrow factors through \( H \).
**Remark.** It is well-known that the spinor representation of $G$ is a multiple of the irreducible representation $V_\rho$ corresponding to the half-sum of positive roots $\rho$ (see [6]). It follows that the character $\chi_\Omega$ coincides with the restriction to $Z$ of $\rho$ considered as a character of the maximal torus of $G$. Together with the equation (2.7.4) this implies that the sign $\kappa(G, G')$ coincides with the sign introduced by Kottwitz in [8].

The following proposition shows how to construct pairs of functions $\phi \in \mathcal{S}(V_0(E))$, $\phi' \in \mathcal{S}(V'_0(E))$ with $\phi \sim \phi'$ and with some control over the Fourier transforms of $\phi$ and $\phi'$.

**Proposition 2.7.6.** Keep the assumptions of lemma 2.7.4. Then there exist $\phi \in \mathcal{S}(V_0(E))$ and $\phi' \in \mathcal{S}(V'_0(E))$ such that $\phi \sim \phi'$, while

$$\delta_{O_{x',\Omega}}(\mathcal{F}(\phi)) = \kappa(G, G')\delta_{O_{x',\Omega}}(\mathcal{F}(\phi')) \neq 0$$

for some $x' \in V'_0(E)$.

**Proof.** We use the notation of section 2.6. Let us set

$$\phi = \frac{1}{\text{vol}_\omega(G(E)x_0 \cap K)} \cdot \delta_U(t)$$

for $t \in (E^\ast)^2$ sufficiently large. Recall that $U(t) = K(tx_0 + D) \subset V_0(E)$. Let us choose some compact open subset $K' \subset O_i(x_0)(E)$ intersecting $G'(E)$-orbits at subsets of equal volumes, and let $U'(t) = K'(ti(x_0) + i(D))$ be the corresponding compact open subset of $V'(E)$. Now let us set

$$\phi' = \frac{1}{\text{vol}_{i,\omega}(G'(E)i(x_0) \cap K')} \cdot \delta_{U'}(t).$$

Then clearly we have $\phi \sim \phi'$. On the other hand, using the formula (2.6.3) we obtain that for $x' \in U' \cap W_0'(E)$ one has

$$\delta_{O_{x',\Omega}}(\mathcal{F}(\phi)) = |t|^\dim V - \dim \Omega \cdot |H^1(E, H)| \cdot \epsilon(x') \times$$

$$\int_{s \in x_0 + t^{-1}D} \sum_{x \in \text{Cr}(x') \cap \mathcal{O}_x} \psi(t(x', x))c(B_{x,x'}, \omega_x, \psi)|\nu(s)|,$$

$$\delta_{O_{i(x')}\Omega}(\mathcal{F}(\phi')) = |t|^\dim V - \dim \Omega \cdot |H^1(E, H')| \cdot \epsilon(i(x')) \times$$

$$\int_{s \in x_0 + t^{-1}D} \sum_{x' \in \text{Cr}(i(x') \cap \mathcal{O}_{i(x')})} \psi(t(i(x'), x'))c(B_{x',i(x')}, \omega_{x'}, \psi)|\nu(s)|,$$

where $H' = i(H)$. Since $H'$ is an inner form of $H$, we have $|H^1(E, H')| = |H^1(E, H')|$. On the other hand, the isomorphism $i: W$ sends $\text{Cr}(x' \cap \mathcal{O}_x)$ to $\text{Cr}(i(x') \cap \mathcal{O}_{i(x')})$. Hence, applying lemma 2.7.3 we get the equality (2.7.3). It remains to note that according to proposition 2.6.2(ii), we will also have $\delta_{O_{x',\Omega}}(\mathcal{F}(\phi)) \neq 0$ for some $x' \in U' \cap W_0'$. □
3. Global methods

In this section \( F \) is a number field, \( \mathbb{A} \) is the corresponding ring of adeles. For every finite set of places \( S \) we denote by \( \mathbb{A}^S \) (resp. \( \mathbb{A}_S \)) the restricted product (resp. the usual product) of \( F_v \) over places \( v \notin S \) (resp. \( v \in S \)) and by \( a \mapsto a^S \) (resp. \( a \mapsto a_S \)) the corresponding projection from \( \mathbb{A} \). We fix an algebraic closure \( \overline{F} \) of \( F \) and denote by \( \Gamma \) the Galois group of \( \overline{F} \) over \( F \). For every place \( v \) of \( F \) we denote by \( F_v \) the corresponding local field and by \( \Gamma_v \) the local Galois group at \( v \). For a reductive group \( H \) over \( F \) we denote by \( \ker_1^1(F, H) \) the preimage of the trivial element under the natural map

\[
\ker_1^1(F, H) \to \oplus_v H^1(F_v, H).
\]

The Hasse principle states that for \( H \) semisimple and simply connected, \( \ker_1^1(F, H) \) is trivial (see e.g., [15]). More generally, for arbitrary connected reductive group \( H \), Kottwitz constructed a bijection

\[
\ker_1^1(F, H) \simeq \ker_1^1(F, Z(\hat{H}))^D
\]

(3.0.10)

where \( Z(\hat{H}) \) is the centre of the Langlands dual group (see [10], (4.2.2)). We denote by \( S(V(\mathbb{A})) \) the space of Schwartz-Bruhat functions on \( V(\mathbb{A}) \). Let \( \theta \) be the distribution on \( S(V(\mathbb{A})) \) defined by

\[
\theta(\phi) = \sum_{x \in V(F)} \phi(x),
\]

\( \theta^\vee \) be the similar distribution on \( S(V^\vee(\mathbb{A})) \). We set

\[
\Theta(\phi) = \int_{g \in G(\mathbb{A})/G(F)} \theta^g(\phi) |dg|
\]

where \( \theta \mapsto \theta^g \) denotes the action of \( g \) on the distribution \( \theta \) (provided that the integral converges). Note that if \( G \) is anisotropic over \( F \) then \( G(\mathbb{A})/G(F) \) is compact, so \( \Theta(\phi) \) is always well-defined in this case. By Poisson summation formula we have \( \mathcal{F}(\theta) = \theta^\vee \), where \( \mathcal{F} \) is the Fourier transform. Hence, for a function \( \phi \in S(V(\mathbb{A})) \) we have

\[
\Theta(\mathcal{F}(\phi)) = \Theta(\phi)
\]

provided that the integral defining \( \Theta(\phi) \) converges. In this section we will apply this equality to compare information about orbital integrals of a function and of its Fourier transform. In this way we will obtain global proofs of Theorems 2.4.2 and 2.7.4.

3.1. Global Kottwitz invariant. Following Kottwitz we are going to rewrite the distribution \( \Theta \) evaluated on sufficiently nice functions in stably invariant terms. The main ingredient required for this is the global Kottwitz invariant defined in [10], [11].

Let \( H \) be a connected reductive group \( H \) over \( F \). For every place \( v \) of \( F \) there is a map

\[
\text{inv}_v : H^1(F_v, H) \to A(H/F_v) = \pi_0(Z(\hat{H})^F_v)^D
\]

(see 2.2). Now if we set \( A(H/F) = \pi_0(Z(\hat{H})^F)^D \), then for every place \( v \) we have a natural homomorphism \( r_v : A(H/F_v) \to A(H/F) \) induced by the embedding \( Z(\hat{H})^F \to Z(\hat{H})^F_v \). Thus, we can define a canonical map

\[
\text{inv} : \oplus_v H^1(F_v, H) \to A(H/F)
\]
by setting \( \text{inv}((c_v)) = \prod_v r_v \text{inv}_v(c_v) \).

The main result about the map \( \text{inv} \) is the exactness of the sequence

\[
H^1(F, \mathbb{H}) \to \bigoplus_v H^1(F, \mathbb{H}) \xrightarrow{\text{inv}} A(\mathbb{H}/F).
\]

**Lemma 3.1.1.** Assume that either \( \mathbb{H} \) is semisimple or \( \mathbb{H}_{F_v} \) is anisotropic. Then the homomorphism \( r_v : A(\mathbb{H}/F_v) \to A(\mathbb{H}/F) \) is surjective.

**Proof.** This follows from the fact that in both cases \( Z(\hat{\mathbb{H}})^{\Gamma_v} \) is finite. \( \square \)

Now let \( V \) be a nice representation of \( G \) defined over \( F \). Then for every \( x \in V_0(F) \) we can identify

\[
G(A) \backslash \mathcal{O}_x(A)
\]

with the kernel of the map

\[
H^1(\Gamma, \mathbb{H}_x(\overline{\mathbb{K}})) \to H^1(\Gamma, G(\overline{\mathbb{K}}))
\]

where \( \overline{\mathbb{K}} = A \otimes_F \overline{F} \). Thus, we can restrict the global Kottwitz invariant \( \text{inv} : H^1(\Gamma, \mathbb{H}_x(\overline{\mathbb{K}})) = \bigoplus_v H^1(F_v, \mathbb{H}) \to A(\mathbb{H}/F) \) to a map

\[
\text{inv}(x, \cdot) : G(\hat{A}) \backslash \mathcal{O}_x(\hat{A}) \to A(\mathbb{H}_x/F).
\]

We claim that this function takes value \( 0 \in A(\mathbb{H}_x/F) \) precisely on the set of \( G(A) \)-orbits of \( F \)-rational points in \( \mathcal{O}_x \). Indeed, consider the following commutative diagram

\[
\begin{array}{ccc}
G(F) \backslash \mathcal{O}_x(F) & \longrightarrow & H^1(F, \mathbb{H}_x) \\
\downarrow & & \downarrow i_G \\
G(\hat{A}) \backslash \mathcal{O}_x(\hat{A}) & \longrightarrow & H^1(\Gamma, \mathbb{H}_x(\overline{\mathbb{K}}))
\end{array}
\]

with exact central vertical column. The Hasse principle for \( G \) (which is simply connected) implies that the map \( i_G \) is injective. Now our claim follows by an easy diagram chase.

Let \( \mathbb{H} = \mathbb{H}_x \) for some \( x \in V_0(F) \). Recall that we have defined in section 2.2 a homomorphism \( \text{sign}_v : A(\mathbb{H}/F_v) \to \{ \pm 1 \} \) for every place \( v \) such that \( \epsilon_{\mathbb{H}_{F_v}} = \text{sign}_v \circ \text{inv}_v \). We claim that there exists a canonical homomorphism

\[
\text{sign}_F : A(\mathbb{H}/F) \to \{ \pm 1 \}
\]

such that \( \text{sign}_v = \text{sign}_F \circ r_v \) for every place \( v \). Indeed, all the local homomorphisms \( \text{sign}_v \) factor through \( A(\mathbb{H}_{ad}/F_v) \), it suffices to define \( \text{sign}_F \) in the case when \( \mathbb{H} \) is semisimple. This is done in absolutely the same way as in the local case. Namely, we consider the exact sequence \( 1 \to C \to \mathbb{H}_{sc} \to \mathbb{H} \to 1 \) defined over \( F \), where \( \mathbb{H}_{sc} \) is simply connected. The unique morphism of this exact sequence to \( 1 \to \{ \pm 1 \} \to \mathbb{H} \to \mathbb{H} \to 1 \) induces a homomorphism \( C \to \{ \pm 1 \} \) defined over \( F \), that can be
considered as a character \( A(\mathbb{H}/F) \rightarrow \{ \pm 1 \} \). This is our \( \varepsilon_f \). The compatibility with the local construction is obvious.

3.2. Stabilization.

**Theorem 3.2.1.** Assume that \( \textbf{G} \) is a simply connected semisimple group over \( F \), \( \rho : \textbf{G} \rightarrow \text{Aut}(\textbf{V}) \) is a nice representation defined over \( F \). Assume also that either \( \textbf{G} \) is anisotropic over \( F \), or the generic stabilizer \( \mathbb{H} \) is semisimple. Let \( f \in \mathcal{S}(\textbf{V}(\mathbb{A})) \) be a function such that \( \text{supp}(f) \cap \textbf{G}(\mathbb{A}) \textbf{V}(F) \subset \textbf{V}_0(\mathbb{A}) \). Then

\[
\Theta(f) = \sum_{x \in \mathbf{G}(F) \backslash \textbf{V}_0(F), \kappa \in \text{Ad}(\mathbb{H}_x/F)} \int_{a \in \mathcal{O}_x(\mathbb{A})} \kappa(\text{inv}(x,a)) f(a)|\omega_x|, \tag{3.2.1}
\]

where \( \mathbf{G}(F) \backslash \textbf{V}_0(F) \) denotes a set of representatives of stable \( \textbf{G} \)-equivalence classes of points in \( \textbf{V}_0(F) \); for every \( x \in \textbf{V}_0(F) \) we choose a \( \textbf{G} \)-invariant top-degree form \( \omega_x \) on \( \mathcal{O}_x \) defined over \( F \).

**Proof.** For brevity we will omit \( |\omega_x| \) in the integrals below. First, we claim that the infinite sum in the right-hand side of (3.2.1) is absolutely convergent. Indeed, for every point \( x \in \textbf{V}_0(F) \) there exists a Zariski neighborhood \( \textbf{U}_x \subset \mathbf{L}_x \) in a linear subspace \( \mathbf{L}_x \subset \textbf{V}^{\mathbb{H}_x} \) such that the map \( \textbf{G}/\mathbb{H}_x \times \textbf{U}_x \rightarrow \textbf{V}_0 \) is étale. In particular, the subsets \( \textbf{G}_x \textbf{U}_x \) form a Zariski open covering of \( \textbf{V}_0 \), so a finite number of them cover \( \textbf{V}_0 \). Therefore, it suffices to prove the absolute convergence of

\[
\sum_{u \in \mathbf{U}_x(F)} \int_{a \in \mathcal{O}_x(\mathbb{A})} f(a).
\]

Now from the fact that the map \( \textbf{G}/\mathbb{H}_x \times \textbf{U}_x \rightarrow \textbf{V} \) is linear in the second argument it is easy to deduce that the function \( u \mapsto \int_{a \in \mathcal{O}_x(\mathbb{A})} f(a) \) on \( \textbf{U}_x(\mathbb{A}) \) is rapidly decreasing at infinity, which implies our claim.

For every \( x \in \textbf{V}_0(F) \) we have

\[
\sum_{\kappa \in \text{Ad}(\mathbb{H}_x/F)} \int_{a \in \mathcal{O}_x(\mathbb{A})} \kappa(\text{inv}(x,a)) f(a) = |A(\mathbb{H}_x/F)| \cdot \int_{a \in \textbf{G}(\mathbb{A}) \mathcal{O}_x(F)} f(a)
\]

since \( \text{inv}(x,a) = 0 \) if and only if \( a \in \textbf{G}(\mathbb{A}) \mathcal{O}_x(F) \). Now we have a finite covering

\[
\bigsqcup_{y \in \mathbf{G}(F) \backslash \mathcal{O}_x(F)} \textbf{G}(\mathbb{A})y \rightarrow \textbf{G}(\mathbb{A}) \mathcal{O}_x(F).
\]

We claim that the degree of this covering is equal to \( |\ker^1(F, Z(\mathbb{H}))| \). Indeed, this degree is equal to the number of \( y \in \mathbf{G}(F) \backslash \mathcal{O}_x(F) \) such that \( \textbf{G}(\mathbb{A})y = \textbf{G}(\mathbb{A})y \). Such \( y \)'s correspond to the classes in \( \ker(H^1(F, \mathbb{H}_x) \rightarrow H^1(F, \textbf{G})) \) that have trivial restriction at every place. Since \( \textbf{G} \) is simply connected, by the Hasse principle the elements in \( H^1(F, \textbf{G}) \) with trivial restrictions at all places are trivial. Therefore, our set of cohomology classes coincides with \( \ker^1(F, \mathbb{H}_x) \). Using (3.0.10) we obtain that \( |\ker^1(F, \mathbb{H}_x)| = |\ker^1(F, Z(\mathbb{H}_x))| \). It follows that

\[
\sum_{\kappa \in \text{Ad}(\mathbb{H}_x/F)} \int_{a \in \mathcal{O}_x(\mathbb{A})} \kappa(\text{inv}(x,a)) f(a) = \frac{|A(\mathbb{H}_x/F)|}{|\ker^1(F, Z(\mathbb{H}_x))|} \sum_{y \in \mathbf{G}(F) \backslash \mathcal{O}_x(F)} \int_{a \in \textbf{G}(\mathbb{A})y} f(a).
\]

Now for \( y \in \mathbf{G}(F) \) we have

\[
\int_{a \in \textbf{G}(\mathbb{A})y} f(a) = \int_{g \in \textbf{G}(\mathbb{A})/\mathbb{H}_x(\mathbb{A})} f(gy) = \tau(\mathbb{H}_x)^{-1} \int_{g \in \textbf{G}(\mathbb{A})/\mathbb{H}_x(F)} f(gy).
\]
where \( \tau(H_g) = \text{vol}(H_g(\mathbb{A})/H_g(F)) \) is the Tamagawa number of \( H_g \). As Kottwitz showed, the Tamagawa number does not change if we pass to an inner form. More precisely, we have

\[
\tau_{H_g} = \frac{|A(H_g/F)|}{|\ker^1(F, Z(H_g))|}
\]

(see [4], (5.1.1) or the introduction to [12]). Since \( H_g \) is an inner form of \( H_x \), we can replace \( H_g \) by \( H_x \) in the RHS. Hence, we obtain

\[
\sum_{\kappa \in A(H_x/F) \backslash A(H_x/\mathbb{Q})} \int_{a \in O_x(\mathbb{A})} \kappa(\text{inv}(x, a)) f(a) = \sum_{y \in G(F) \backslash G(\mathbb{A})} \int_{g \in G(\mathbb{A})/H_g(F)} f(gy).
\]

Thus, the RHS of the formula (3.2.1) can be rewritten as follows:

\[
\sum_{x \in G(F) \backslash V_0(F)} \sum_{y \in G(F) \backslash O_x(F)} \int_{g \in G(\mathbb{A})/H_g(F)} f(gy) = \sum_{y \in G(F) \backslash V_0(F)} \int_{g \in G(\mathbb{A})/H_g(F)} f(gy) = \int_{g \in G(\mathbb{A})/G(F)} \sum_{x \in G(F) \backslash V_0(F)} f(gx) = \int_{g \in G(\mathbb{A})/G(F)} \sum_{x \in G(F) \backslash V_0(F)} f(gx)
\]

which is precisely the LHS of (3.2.1).

3.3. Global proofs. We are going to combine the obtained local and global information about nice representations with the stabilization formula (3.2.1) to prove theorems 2.7.1 and 2.4.2.

Proof of Theorem 2.7.1. We will only prove that if \( \phi \sim \phi' \) then \( \mathcal{F}(\phi) \sim_\kappa \mathcal{F}(\phi') \). The proof of the converse statement is absolutely analogous. For convenience we divide the proof into several steps. In step 1 we will extend our local data to the global one in an appropriate way. In the (crucial) step 2 we apply the stabilization formula to deduce the analogue of our statement for the product of local fields at two places of our global field. Finally, in step 3 we will deduce the statement for one local field. Let us rename the data \((G, \mathbb{Z}, V, \alpha, G', V')\) into \((G_E, \mathbb{Z}_E, V_E, \alpha_E, G'_E, V'_E)\) to reflect the fact that they are defined over \( E \).

STEP 1. We start by choosing a number field \( F \), the data \((G, \mathbb{Z}, V)\) defined over \( F \), and a place \( v_0 \) of \( F \) such that \( F_{v_0} = E \), \( G_{v_0} = G_E \), \( \mathbb{Z}_{v_0} = \mathbb{Z}_E \), \( V_{v_0} = V_E \). In addition we can assume that there are two finite places \( v_1, v_2 \) (different from \( v_0 \)) and a real place \( v_\infty \) of \( F \) such that \( G_{v_1} \) and \( G_{v_2} \) are split and \( G(F_{v_\infty}) \) is compact. Note that the fact that \( V_E \) is a nice representation implies that the representations \( V \) and \( V_v \) for all places \( v \) are nice.

Let us choose a point \( x_{v_0} \in V_{v_0}(F_{v_0}) \) such that the cohomology class \( \alpha_E \) comes from a cohomology class \( \beta_E \in H^1(\Gamma_{v_0}, H_{x_{v_0}}/\mathbb{Z}_{v_0}) \) (if no such point exist then the statement of the theorem is empty). In the case when the generic stabilizer is commutative we can also choose a point \( x_{v_1} \in V_{v_1}(F_{v_1}) \) such that \( H_{x_{v_1}} \) is anisotropic (this is possible by lemma 2.7.3). We can choose a global point \( x \in V_0(F) \) that approximates \( x_{v_0} \) and \( x_{v_1} \) (resp. \( x_{v_2} \) if the generic stabilizer is non-commutative) well enough, so that for \( H = H_x \) we have a class \( \beta_E \in H^1(\Gamma_{v_0}, H_{x_{v_0}}/\mathbb{Z}_{v_0}) \) inducing \( \alpha_E \), and in the case \( H \) is commutative we also have that \( H^0_{v_1} \) is anisotropic. Let us set \( K = H/\mathbb{Z} \). Then \( K_{v_1} \) is either semisimple or an anisotropic torus, hence, by lemma 3.1.3 the homomorphism \( A(K/F_{v_1}) \to A(K/F) \) is surjective. Recall that we have an exact sequence

\[
H^1(F, K) \to \bigoplus_v H^1(F_v, K) \to A(K/F).
\]
By surjectivity of the map $H^1(F_v, K) \to A(K/F)$ there exists an element $\beta \in H^1(F, K)$ such that $\beta_{v_0} = \beta_E$ and $\beta_v = 0$ for $v \neq v_0, v_1$. Let $\alpha \in H^1(F, G/\mathbb{Z})$ be the class induced by $\beta$, and let $(G', V')$ be the twist of $G$ by $\alpha$. Then $\alpha_{v_0} = \alpha_E$ while the restrictions of $\alpha$ to all places other than $v_0$ and $v_1$ are trivial. Thus we have $(G'_{v_0}, V'_{v_0}) = (G'_E, V'_E)$ and $(G'_{v}, V'_{v}) = (G_v, V_v)$ for $v \neq v_0, v_1$. In particular, $G'_{v_2} = G_{v_2}$ is split over $F_{v_2}$ and $G'_v(F_{v_2}) = G(F_{v_2})$ is compact. The latter condition implies that both groups $G$ and $G'$ are anisotropic over $F$, so the distribution $\Theta$ is defined on all functions in $S(V(A))$ (resp. $S(V'(A))$ and $S((V')'(A))$).

**STEP 2.** Set $S = \{v_0, v_1\}$. We are going to show that for every pair of functions $\phi_S \in S(V(A_S))$, $\phi'_S \in S(V'(A_S))$ such that $\phi_S \sim \phi'_S$, one has $F(\phi_S) \sim F(\phi'_S)$. For this we have to check that for every $x^y_S \in V'_y(A_S)$ one has

$$\delta_{\mathcal{O}_{x^y_S}}(F(\phi_S)) = \delta_{\mathcal{O}_{x^y_S}}(F(\phi'_S)).$$

Since $F(f_S) \in S(V(A_S))$ the functions $y_S \mapsto \delta_{\mathcal{O}_{y_S}}(F(\phi_S))$ and $y_S \mapsto \delta_{\mathcal{O}_{y_S}}(F(\phi'_S))$ on $V'_y(A_S)$ are locally constant (see lemma 2.7.3). Let $U_S$ be an open neighborhood of $x^y_S$ in $V'_y(A_S)$ on which these two functions are constant.

Since $G_{v_2}$ is split over $F_{v_2}$, we can apply the construction of section 2.6 and proposition 2.6.4 to construct a stable function $\phi_{v_2} \in S(V_0(F_{v_2}))$ and a non-empty open subset $U_{v_2} \subset V_0(F_{v_2})$ such that the restriction of $I(F(\phi_{v_2}))$ to $\bigcup_{y \in U_{v_2}} \mathcal{O}_y(F_{v_2})$ is $\epsilon$-stable and everywhere non-vanishing. In particular, for $y \in U_{v_2}$ we have $\delta_{\mathcal{O}_y}(F(\phi_{v_2})) \neq 0$. If $H$ is commutative, then by lemma 2.7.3 (and the remark after it) we can in addition assume that all points in $U_{v_2}$ and in the support of $\phi_{v_2}$ have anisotropic stabilizer.

We can find a point $x^y \in V'_y(F)$ such that $x^y \in U_S$ and $x^y \in U_{v_2}$. Then we have

$$\delta(\mathcal{O}_{x^y}, \mathcal{O}_S)(F(\phi_S)) = \delta(\mathcal{O}_{x^y}, \mathcal{O}_S)(F(\phi'_S)),
\delta(\mathcal{O}_{x^y}, \mathcal{O}_S)(F(\phi'_S)) = \delta(\mathcal{O}_{x^y}, \mathcal{O}_S)(F(\phi'_S)),$$

$$\delta(\mathcal{O}_{x^y}, \mathcal{O}_y)(F(\phi_{v_2})) \neq 0.$$

Let us set $S' = S \cup \{v_2\}$. We claim that for every function $\phi^{S'} \in S(V(A^{S'}))$ one has

$$\Theta(\phi_S \circ \phi_{v_2} \circ \phi^{S'}) = \Theta(\phi'_S \circ \phi_{v_2} \circ \phi^{S'}).$$

Indeed, since $\phi_{v_2}$ is supported on points in $V_0(F_{v_2})$ we can apply formula (3.2.1) to compute $\Theta(\phi_S \circ \phi_{v_2} \circ \phi^{S'})$. The RHS of this formula is the sum over $x \in V_0(F)$ and over $\kappa \in A(H_{v_2}/F)^D$ of terms

$$\int_{a \in \mathcal{O}_{x}(A)} \kappa(\text{inv}(x, a)) \phi_S(a_S) \phi_{v_2}(a_{v_2}) \phi^{S'}(a^{S'}) |\omega_x| = 0.$$

Since $\text{inv}(x, a)$ is the product of local terms, this integral is equal to the product of the corresponding local integrals. We claim that if $\kappa \neq 0$ the local integral at $v_2$ is zero. Indeed, if $\mathcal{O}_x(F_{v_2})$ does not intersect the support of $\phi_{v_2}$ this is clear. Otherwise, by lemma 3.1.1 the character $\kappa \circ r_{v_2}$ of $A(H_{v_2}/F_{v_2})$ is non-trivial (here we use the fact that the stabilizers of points in the support of $\phi_{v_2}$ are either semisimple or anisotropic). Hence,

$$\int_{a_{v_2} \in \mathcal{O}_x(F_{v_2})} (\kappa \circ r_{v_2})(a_{v_2}) \phi_{v_2}(a_{v_2}) |\omega_x| = 0.$$
since $\phi'_{v_2}$ is stable. Thus, the formula (3.2.1) in this case takes form
\[
\Theta(\phi_S \otimes \phi_{v_2} \otimes \phi'^{S'}) = \sum_{x \in G(F) \setminus V_0(F)} \int_{a \in O_x(A)} \phi_S(\alpha_S) \phi_{v_2}(\alpha_{v_2}) \phi'^{S'}(\alpha'^{S'}) |\omega_x|.
\]
Similar formula holds for $\Theta(\phi'_S \otimes \phi_{v_2} \otimes \phi'^{S'})$. Now our claim follows immediately from the condition $\phi_S \sim \phi'_S$.

Applying the Fourier transform, we obtain
\[
\Theta(F(\phi_S) \otimes F(\phi_{v_2}) \otimes F(\phi'^{S'})) = \Theta(F(\phi'_S) \otimes F(\phi_{v_2}) \otimes F(\phi'^{S'})).
\]
(3.3.1)

Let us choose one more finite place $v_3 \not\in S'$ and a function $\phi_{v_3}$ in $S(V(F_{v_3}))$ such that $F(\phi_{v_3})$ is supported on $V_0^\prime(F_{v_3})$ and $\delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi_{v_3}) \neq 0$. One can define such $\phi_{v_3}$ by setting
\[
F(\phi_{v_3}) = \epsilon \cdot \delta_{KU_0}
\]
where $U_0$ is a small compact neighborhood of $x^\prime$ in $F_{v_3}$-points of a linear slice for $G$-action, $K$ is a non-empty compact in $O_x(F_{v_3})$ intersecting $G(F_{v_3})$-orbits by the sets of equal volumes (see section 2.6 for similar constructions).

Let $S'' = S' \cup \{v_3\}$. Since the function $F(\phi_{v_3})$ is supported on $V_0^\prime(F_{v_3})$ we can apply formula (3.2.1) to calculate $\Theta(F(\phi))$ and $\Theta(F(\phi'))$ where
\[
\phi = \phi_S \otimes \phi_{v_2} \otimes \phi_{v_3} \otimes \phi^{S''},
\]
\[
\phi' = \phi'_S \otimes \phi_{v_2} \otimes \phi_{v_3} \otimes \phi'^{S''},
\]
for some $\phi'^{S''} \in S(V(\mathbb{A}^{S''}))$. Now the idea is to choose $\phi'^{S''}$ in such a way that all the terms in the RHS of (3.2.1) (applied to $F(\phi)$ and $F(\phi')$) corresponding to points of $V_0^\prime(F)$ which are not stably equivalent to $x^\prime$ vanish. Indeed, let $C \subset V^\prime(\mathbb{A}^{S''})$ (resp. $C' \subset V^\prime(\mathbb{A}^{S''})$) be the support of $F(\phi_S \otimes \phi_{v_2} \otimes \phi_{v_3})$ (resp. of $F(\phi'_S \otimes \phi_{v_2} \otimes \phi_{v_3})$). Set $D = V^\prime / G(F) \cap p_{\mathbb{A}^{S''}}(C \cup C')$, where $p : V^\prime \rightarrow V^\prime / G(F)$ is the natural projection (intersection is taken in $V^\prime / G(\mathbb{A}^{S''})$). Since $V^\prime / G(F)$ is discrete in $V^\prime / G(\mathbb{A})$ and $C \cup C'$ is compact, it follows that $D$ is discrete in $V^\prime / G(\mathbb{A}^{S''})$. Note that $p(x^\prime)$ belongs to $D$. Therefore, we can choose $\phi'^{S''}$ in such a way that its support is disjoint from $p^{-1}_L (D \setminus p(x^\prime))$, while $\delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi'^{S''}) \neq 0$.

Now applying (3.2.1) we get
\[
\Theta(F(\phi)) = \sum_{\kappa \in A(H_{x^\prime} / F)^D} \int_{a \in O_{x^\prime}(A)} \kappa(\text{inv}(x^\prime, a)) F(\phi_S)(\alpha_S) F(\phi_{v_2})(\alpha_{v_2}) F(\phi_{v_3})(\alpha_{v_3}) F(\phi'^{S''})(\alpha'^{S''}) |\omega_x|.
\]
Since $H_{x^\prime}$ is either semisimple or anisotropic over $F_{v_2}$, while the function $I(F(\phi_{v_2}))$ is $\epsilon$-stable on $O_{x^\prime}(F_{v_2})$, it follows that the local integral at $v_2$ vanishes unless $\kappa = \epsilon_F$. The same computation works for $\Theta(F(\phi'))$. Hence, the equality (3.3.1) for $\phi'^{S''} = \phi_{v_2} \otimes \phi'^{S''}$ reduces to
\[
\delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S)) \cdot \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S')) \cdot \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S')) \cdot \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S')) = \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S)) \cdot \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S')) \cdot \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S')).
\]
Therefore, we get
\[
\delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S)) = \delta_{(G \setminus \mathbb{A})_{F_{v_3}}}(\phi(S'))
\]
which implies our statement.
STEP 3. Applying proposition $2.7.6$ for the place $v_1$ we construct functions $\phi_{v_1} \in S(V_0(F_{v_1}))$ and $\phi'_{v_1} \in S(V'_0(F_{v_1}))$ such that $\phi_{v_1} \sim \phi'_{v_1}$, while

$$\delta_{\mathcal{O}_{x_{v}}}(F(\phi_{v_1})) = \kappa(G, G')\delta_{\mathcal{O}_{x_{v}}}(F(\phi'_{v_1})) \neq 0$$

for some orbit $\mathcal{O} \subset (V_0)_{v_1}$. Now let $\phi_0 \in S(V(F_{v_0}))$, $\phi'_0 \in S(V'(F_{v_0}))$ be a pair of functions such that $\phi_0 \sim \phi'_0$. Then Step 2 applied to $\phi_0 \otimes \phi_{v_1}$ and $\phi_0 \otimes \phi'_{v_1}$ implies that $F(\phi_0) \sim_\kappa(G, G')F(\phi'_0)$.

Remark. To generalize the above proof to the case when $G$ is not necessarily simple, it would be enough to prove the following version of lemma $2.7.3$ for $G$: if $G$ is split and $\rho$ is nice, then there exists a point $x \in V_0(E)$ such that the connected component of $Z(H_x)$ is anisotropic. Indeed, the lemmas $2.7.4$ and $3.1.1$ used in the proof can be easily generalized to the case when $Z(H_x)$ is anisotropic. The rest of the proof does not use the assumption that $G$ is simple.

Proof of theorem $2.4.3$. We will only prove the inclusion $F(S^{st}) \subset S^{st}$. The proof of the inverse inclusion is absolutely analogous. Thus, we have to prove that if $\phi_0$ is a stable function on $V(E)$ then for any $x_{E}^{\vee} \in V_{0}(E)$ the function $I(\epsilon \cdot F(\phi_0))$ on $\mathcal{O}_{x_{E}^{\vee}}(E)$ is constant. We will split the proof into two steps which are similar to the first two steps of the previous proof: step 1 consists of constructing an appropriate global setup, while step 2 is an application of the stabilization formula and of theorem $2.4.1$. As before we rename our data $(G, V)$ into $(G_E, V_E)$. We denote by $H_E \subset G_E$ the stabilizer of $x_{E}^{\vee}$.

STEP 1. Let $F$ be a global field, $v_0$ be a place of $F$ such that $F_{v_0} = E$. We want to construct the data $(G, V, H)$ over $F$ such that $G_{v_0} = G_E$, $V_{v_0} = V_E$, $H_{v_0}$ is $G(E)$-conjugate to $H_E$ and such that in addition the natural homomorphism

$$A(H/E) \to A(H/F)$$

is an isomorphism. First, we can find the data $(G, V)$ over $F$ such that $G_{v_0} = G_E$, $V_{v_0} = V_E$. Now let $x_{E}^{\vee} \in V'_{0}(F)$ be a global point sufficiently close to $x_{E}^{\vee}$, and let $H$ be the stabilizer of $x_{E}^{\vee}$. Then $H_{v_0}$ is $G(E)$-conjugate to $H_E$. Let $\pi \subset \text{Aut}(Z(H))$ be the quotient of the local Galois group $\Gamma_{v_0}$, through which it acts on $Z(H)$. Let us denote by $\Gamma' \subset \Gamma$ the preimage of $\pi$ under the natural homomorphism $\Gamma \to \text{Aut}(Z(H))$, and let $F' \supset F$ be the finite extension corresponding to $\Gamma'$. Then $\Gamma'$ contains $\Gamma_v$ (which is considered as a subgroup in $\Gamma$ via some fixed extension of the valuation $v_0$ to $F$). Hence, there is an extension of $v_0$ to a place $v'_{0}$ of $F'$ such that $F'_{v_0} = E$. Furthermore, by construction we have $A(H/E) = A(H/F')$. It remains to replace $(F, v_0)$ by $(F', v'_{0})$.

STEP 2. This step is very similar to the step 2 in the previous proof. Let $\kappa_E$ be a non-trivial character of $A(H/E)$. We have to prove that $\delta_{\mathcal{O}_{x_{E}^{\vee}}}^{\text{sign}}(F(\phi_0)) = 0$. Let us denote $W^{\vee} = (V^{\vee})^H$. Since the function $y \mapsto \delta_{\mathcal{O}_{y}}^{\text{sign}}(F(\phi_0))$ of $y \in W^{\vee}(E)$ is locally constant, we can choose a neighborhood $U_{v_0}$ of $x_{E}^{\vee}$ on which this function is constant. By our assumption $\kappa_E$ is induced by some character $\kappa$ of $A(H/F)$. For every place $v$ we denote by $\kappa_v$ the induced character of $A(H/F_{v_0})$.

Let us choose a finite place $v_1$ of $F$ (different from $v_0$). By proposition $2.6.2$ (ii) for a function of the form $\phi_{v_1} = \delta_{U(t)}^{v_1} \in S(V_0(F_{v_1}))$ where $t \in (E^*)^2$ is large
Indeed, since $S_x$ is non-empty open subset. Also since the character $\kappa_v$ is non-trivial by lemma 3.1.1, the function $\phi_{v_1}$ is antistable.

Let us choose a global point $x^{v} \in \mathcal{W}_{0}(F)$ such that $x^{v} \in U_{v_1}$ and $x^{v} \in U_{v_1}$. Then we have

$$\delta_{\mathcal{O}_{x^{v}}}^{\kappa_v \text{sign}}(\mathcal{F}(\phi_{v_1})) \neq 0$$

for $y^{v} \in U_{v_1}$, where $U_{v_1} \subset \mathcal{W}_{0}(F_{v_1})$ is a non-empty open subset. Also since the character $\kappa_v$ is non-trivial by lemma 3.1.1, the function $\phi_{v_1}$ is antistable.

Let $v_2$ be one more finite place of $F$. We can construct a function $\phi_{v_2} \in S(\mathcal{V}(F_{v_2}))$ such that $\mathcal{F}(\phi_{v_2})$ has support in $\mathcal{V}_{0}(F_{v_2})$ and such that

$$\delta_{\mathcal{O}_{x^{v}}^{v}}^{\kappa_v \text{sign}}(\mathcal{F}(\phi_{v_2})) = 0$$

for $\kappa \neq \kappa_{v_2}$ sign, while

$$\delta_{\mathcal{O}_{x^{v}}^{v}}^{\kappa_v \text{sign}}(\mathcal{F}(\phi_{v_2})) \neq 0.$$

Indeed, it suffices to take $\mathcal{F}(\phi_{v_2})$ to be the function of the form $\delta_{U(t)}^{\kappa_{v_2} \text{sign}}$ as in section 2.6.

Let us denote $S = \{v_0, v_1, v_2\}$. We claim that for every function $\phi^{S} \in S(\mathcal{V}(\mathbb{A}^{S}))$ one has

$$\Theta(\phi_{0} \otimes \phi_{v_1} \otimes \phi_{v_2} \otimes \phi^{S}) = 0.$$ 

Indeed, since $\phi_{v_1}$ has support in $\mathcal{V}_{0}(F_{v_1})$, we can apply formula (3.2.1). Now let $x \in \mathcal{V}_{0}(F)$ and let $\kappa'$ be a character of $A(\mathbb{H}_{x}/F)$. If $\kappa'$ is non-trivial then by lemma 3.1.1 the induced character $\kappa'_{v_2}$ of $A(\mathbb{H}_{x}/E)$ is non-trivial. Since $\phi_{0}$ is stable, we get

$$\int_{a \in \mathcal{O}_{x}(E)} \kappa'(\text{inv}(x, a))\phi_{0}(a)|\omega_{x}| = 0,$$

so the corresponding term in the RHS of (3.2.1) vanishes. On the other hand, if $\kappa' = 1$ then $\kappa'_{v_1} = 1$, so the corresponding term vanishes by antistability of $\phi_{v_1}$.

Applying the Fourier transform, we obtain

$$\Theta(\mathcal{F}(\phi_{0}) \otimes \mathcal{F}(\phi_{v_1}) \otimes \mathcal{F}(\phi_{v_2}) \otimes \mathcal{F}(\phi^{S})) = 0.$$ 

Furthermore, since $\mathcal{F}(\phi_{v_2})$ has support in $\mathcal{V}_{0}(F_{v_2})$ we can apply formula (3.2.1) again. As in the proof of theorem 2.4.1 we can choose $\phi^{S}$ in such a way that all the terms in the RHS of (3.2.1) corresponding to points of $\mathcal{V}_{0}(F)$ which are not stably conjugate to $x^{v}$ vanish while

$$\delta_{\mathcal{O}_{x^{v}}^{v}}(\mathcal{F}(\phi^{S})) \neq 0.$$ 

Now applying (3.2.1) we get

$$0 = \Theta(\mathcal{F}(\phi_{0}) \otimes \mathcal{F}(\phi_{v_1}) \otimes \mathcal{F}(\phi_{v_2}) \otimes \mathcal{F}(\phi^{S})) =$$

$$\sum_{\kappa' \in A(\mathbb{H}_{x}/F)^{\mathbb{A}}} \int_{a \in \mathcal{O}_{v}(\mathbb{A})} \kappa'(\text{inv}(x', a))\mathcal{F}(\phi_{0})(a_{v_0})\mathcal{F}(\phi_{v_1})(a_{v_1})\mathcal{F}(\phi_{v_2})(a_{v_2})\mathcal{F}(\phi^{S})(a^{S})|\omega_{x}|.$$ 

By our choice of $\phi_{v_2}$ the local integral at $v_2$ vanishes unless $\kappa'_{v_2} = \kappa_{v_2}$ sign. By lemma 3.1.1 this condition is equivalent to $\kappa' = \kappa \text{sign}_{F}$. Hence, we obtain that

$$\delta_{\mathcal{O}_{x^{v}}^{v}}^{\kappa \text{sign}_{F}}(\mathcal{F}(\phi_{0})) = 0$$

as required. \qed
4. Sign function for the space of symmetric matrices

In this section we will compute the sign function $\epsilon(\cdot, \cdot)$ for the space of symmetric $n \times n$ matrices $\text{Sym}_n$ over a local field $E$ considered as a representation of $\text{SL}_n$, where $g \in \text{SL}_n$ acts on $\text{Sym}_n$ by $X \mapsto gXg'$. As an application we will derive the formula (1.0.1) in the case of odd $n$.

Let us denote by $\text{Sym}_n' \subset \text{Sym}_n$ the complement to the hypersurface $(\text{det} = 0)$. By definition the sign $\epsilon(A, A')$, where $A, A' \in \text{Sym}_n'(E)$, is defined when $A$ and $A'$ belong to one $\text{SL}_n(E)$-orbit, i.e., when $\text{det}(A) = \text{det}(A')$. 

**Proposition 4.0.1.** For a pair of symmetric $n \times n$ matrices $A, A'$ with $\text{det}(A) = \text{det}(A') \neq 0$ one has

$$\epsilon(A, A') = \epsilon(q_A, q_{A'})^n$$

where $q_A$ denotes the quadratic form with matrix $A$.

**Proof.** By definition $\epsilon(A, A')$ is the product of Hasse-Witt invariants of quadratic forms obtained by restricting the Killing form $Q$ on $\text{sl}_n$ to stabilizer subalgebras $\mathfrak{h}_A$ and $\mathfrak{h}_{A'}$ of $A$ and $A'$. Since $A$ and $A'$ belong to one $\text{SL}_n(E)$-orbit, the forms $Q|_{\mathfrak{h}_A}$ and $Q|_{\mathfrak{h}_{A'}}$ have the same determinant modulo squares. Therefore, their relative Hasse-Witt invariant will not change if we replace $Q$ by its scalar multiple. Thus, we can do calculation with $Q(X) = \frac{1}{2} \text{Tr}(X^2)$. We claim that

$$\epsilon(Q|_{\mathfrak{h}_A}) = c(\text{det} A) \cdot \epsilon(q_A)^n$$

where $c$ is a sign depending only on $\text{det} A$ modulo squares. To prove this formula we notice that both sides do not change if we replace $A$ by $gAg'$ where $g \in \text{GL}_n(E)$. Thus, we can assume that $A$ is diagonal. Let $(a_1, \ldots, a_n)$ be diagonal entries of $A$. The subalgebra $\mathfrak{h}_A \subset \text{sl}_n$ consists of matrices $X$ such that $XA + AX^t = 0$. Thus, if $X = (x_{ij})$ then we should have $x_{ii} = 0$ while $x_{ji} = -\frac{a_i}{a_j} x_{ij}$. Thus, the quadratic form $Q|_{\mathfrak{h}_A}$ has diagonal matrix in the natural basis on $\mathfrak{h}_A$ and we have

$$\epsilon(Q|_{\mathfrak{h}_A}) = \prod_{i<j, k<l; (i,j) < (k,l)} \left( -\frac{a_j}{a_i}, -\frac{a_k}{a_i} \right)$$

where $(i, j) < (k, l)$ denotes the lexicographical order. A straightforward calculation shows that the RHS is equal to

$$\prod_{i<j} (a_i, a_j)^n \cdot \prod_i (a_i, -1)^{p(n)} \cdot (-1, -1)^{q(n)}$$

where $p(n), q(n)$ are some polynomials in $n$. Thus, we get

$$\epsilon(Q|_{\mathfrak{h}_A}) = \epsilon(q_A)^n \cdot (\text{det} A, -1)^{p(n)} \cdot (-1, -1)^{q(n)}$$

as required. \hfill $\square$

Now we can derive the formula (1.0.1) in the $p$-adic case. It is well-known (see [19, 20, 3]) that one has an equation of the form

$$F(\chi(\text{det})) = \tau_N \cdot (\cdot | \prod \chi^{-1})$$

where $\tau_N$ is some $(\text{GL}_n/\{\pm 1\})(E)$-invariant function on $\text{Sym}_n'(E)$. The stabilizer subgroup $H \subset (\text{GL}_n/\{\pm 1\})$ of a point in $\text{Sym}_n'(E)$ is the group $O_n/\{\pm 1\} \simeq \text{SO}_n$ (here we use the fact that $n$ is odd). Therefore, the set of $(\text{GL}_n/\{\pm 1\})(E)$-orbits on $\text{Sym}_n'(E)$ can be identified with $\ker(H^1(E, \text{SO}_n) \to H^1(E, \text{GL}_n/\{\pm 1\})).$ Note that the homomorphism $\text{SO}_n \to \text{GL}_n/\{\pm 1\}$ factors through $\text{SL}_n$ (since $n$ is
Then one has the following equation:

\[ t = \text{denote by } V \]\n
Recall the result of Shintani's computation in the form convenient for us. Let \( i \) \( n \) \( \Gamma \)-matrix for \( \text{Sym}_n(E) \).

\[ \text{It follows that the function } \chi \mapsto \epsilon(q_A) \cdot (\det A, -1) \frac{n-1}{2} \text{ is } (GL_n/\{\pm 1\})(E)-invariant, \]

so we should have

\[ \tau_\chi(A) = c(\chi) \cdot \epsilon(q_A) \cdot (\det A, -1) \frac{n-1}{2} \]

which is equivalent to the equation (0.0.1). The explicit value of the constant \( c(\chi) \) can be found in [24]. Note that it can also be determined using the stationary phase approximation as in section 2.3.

In conclusion, let us show that for \( E = \mathbb{R} \) the formula (0.0.1) is still true. Indeed, we have either \( \chi(x) = |x|^s \) or \( \chi(x) = \text{sgn}(x)|x|^s \), so our formula is equivalent to the set of two equalities

\[ \int_{A \in \text{Sym}_n(E)} |\det(A)|^s \hat{f}(A) dA = c_1(s) \int_{B \in \text{Sym}_n(E)} (-1)^{\frac{i \mu(s)}{2}} |\det(B)|^{-s - \frac{n+1}{2}} f(B) dB, \]

\[ \int_{A \in \text{Sym}_n(E)} \text{sgn}(\det(A)) |\det(A)|^s \hat{f}(A) dA = c_2(s) \int_{B \in \text{Sym}_n(E)} (-1)^{\frac{i \mu(s)}{2}} |\det(B)|^s - \frac{n+1}{2} f(B) dB, \]

where \( i_B \) is the number of negative eigenvalues of \( B \), \( f \) is a function from the Schwartz space of \( \text{Sym}_n(E) \), \( \hat{f} \) is its Fourier transform, \( c_1 \) and \( c_2 \) are some meromorphic functions of \( s \). These formulas can be deduced from the explicit form of \( \Gamma \)-matrix for \( \text{Sym}_n(E) \) computed by T. Shintani in Lemma 15 of [21]. Indeed, let us recall the result of Shintani's computation in the form convenient for us. Let us denote by \( V \) the connected component of \( \text{Sym}_n(E) \) consisting of matrices with exactly \( t \) positive eigenvalues.

Let us denote \( \Phi_\ell(f, s) = \int_{A \in V} |\det(A)|^s \hat{f}(A) dA \).

Then one has the following equation:

\[ \Phi_\ell(f, s) = c(s) \cdot \sum_{j=0}^n v_{ij}(s) \Phi_j(f, -s - \frac{n+1}{2}) \]

where

\[ v_{ij}(s) = \sum_{(v_1, \ldots, v_n)} \exp\left(\frac{\pi \sqrt{-1}}{2} \left[ \sum_{k=1}^j (k + s) \epsilon_k - \sum_{k=j+1}^n (k - j + s) \epsilon_k \right] \right) \]
where the summation is taken over all \( n \)-tuples \((\epsilon_1, \ldots, \epsilon_n) = (\pm 1, \ldots, \pm 1)\) such that exactly \( i \) of the \( \epsilon \)'s are \(+1\). Now we have

\[
c_j := \sum_{i=0}^{n} v_{ij} = 2^n \cdot \prod_{k=1}^{j} \cos\left(\frac{\pi}{2}(k + s)\right) \cdot \prod_{k=1}^{n-j} \cos\left(\frac{\pi}{2}(k + s)\right).
\]

Hence, for odd \( n \) we have

\[
\frac{c_j}{c_{j-1}} = \frac{\cos\left(\frac{\pi}{2}(j + s)\right)}{\cos\left(\frac{\pi}{2}(n + 1 - j + s)\right)} = (-1)^{\frac{n+1}{2} + j}.
\]

Therefore, the vector \((c_j)_{j=0,\ldots,n}\) is proportional to \((-1)^{j(n-j)}\) which is equivalent to the equation \([4.0.2]\). Similarly, we have

\[
c'_j := \sum_{i=0}^{n} (-1)^{n-i} v_{ij} = \sum_{(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n} \prod_{k=1}^{j} \epsilon_k \cdot \exp\left(\frac{\pi}{2}(k + s)\right) - \sum_{k=j+1}^{n} (k - j + s) \epsilon_k) =
\]

\[
(2\sqrt{-1})^n \cdot (-1)^{n-j} \cdot \prod_{k=1}^{j} \sin\left(\frac{\pi}{2}(k + s)\right) \cdot \prod_{k=1}^{n-j} \sin\left(\frac{\pi}{2}(k + s)\right).
\]

Hence, for odd \( n \) the vector \((c'_j)_{j=0,\ldots,n}\) is proportional to \((-1)^{(j+1)(n-j)}\) which is equivalent to \([4.0.3]\).

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