STRONG SOLUTIONS OF QUASILINEAR EQUATIONS IN BANACH SPACES NOT SOLVABLE WITH RESPECT TO THE HIGHEST-ORDER DERIVATIVE

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ABSTRACT. By means of the Mittag-Leffler function existence and uniqueness conditions are obtained for a strong solution of the Cauchy problem to quasilinear differential equation in a Banach space, solved with respect to the highest-order derivative. The results are used in the study of quasilinear equations with degenerate operator at the highest-order derivative. Some special restrictions for nonlinear operator in the equation are used here. Existence conditions of a unique strong solution for the Cauchy problem and generalized Showalter–Sidorov for degenerate quasilinear equations were found. The obtained results are illustrated by an example of initial-boundary value problem for a quasilinear system of equations not resolved with respect to the highest-order time derivative.

1. Introduction. Consider the quasilinear differential equation

\[
\frac{d^m}{dt^m} L x(t) = M x(t) + N(t, x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t), t \in (t_0, T),
\] (1.1)

where \( m \in \mathbb{N}, X, Y \) are Banach spaces, operators \( L \in \mathcal{L}(X; Y) \) (linear and continuous from \( X \) to \( Y \)), \( M \in \mathcal{C}(X; Y) \) (linear, closed and densely defined in \( X \), acting to \( Y \)) and nonlinear operator \( N : (t_0, T) \times X^m \to Y \). We consider the case of degenerate operator \( L \) at the derivative, i.e. \( \ker L \neq \{0\} \). Such equations will be called as degenerate evolution equations. Many partial differential equations and systems not solvable with respect to the highest-order time derivative can be reduced to the initial problem for equation (1.1). Such systems are often found in natural and engineering sciences [3, 14, 12]. See also works [2, 11, 6] and others who are close to the research topic.

In the study of optimal control problems for distributed systems described by degenerate evolution equations, the concept of a strong solution is essential [8, 5, 7]. This work is devoted to conditions research for existence and uniqueness of a
strong solution to Cauchy problem and generalized Showalter–Sidorov problem for equation (1.1). The study is carried out using methods of the theory of degenerate semigroups of operators, developed in monographs [3, 14] in the study of linear degenerate first order evolution equations.

Using the condition of \( (L, p) \)-boundedness of operator \( M \), we can reduce equation (1.1) to a system of two equations. The first of these is an equation solved with respect to the highest-order derivative. The second of these has nilpotent operator at this derivative, which facilitates its study. This technique in the study of classical solutions of first order degenerate semilinear equations was used in [13, 15, 1, 4] and in [16] for incomplete degenerate semilinear high order equation in the case of \( p = 0 \).

In this paper we consider a complete quasilinear equation with an arbitrary \( p \geq 0 \) that means degeneration of the equation not only on the kernel of operator at the derivative, but also on its chains \( M\)-adjoint vectors of height at most \( p \).

First, using the Mittag-Leffler function we found existence conditions for a unique strong solution of the Cauchy problem for a quasilinear evolution equations solved with respect to the highest derivative. Then the results are used to the study of the solvability in the sense of the strong solution of degenerate quasilinear evolution equations with special conditions on a nonlinear operator \( N \). There are matching condition data of the problem in the case of considering the Cauchy problem for the degenerate equation.

The obtained results are illustrated in the model example of the initial-boundary value problem for a system of partial differential equations not resolvable with respect to the time highest-order derivative. For the problem function spaces and operators are chosen so that it is reduced to the problem of generalized Showalter–Sidorov problem for degenerate quasilinear equation (1.1) in a Banach space. It is shown that in this situation the operator \( M \) is \( (L, 1) \)-bounded. This example demonstrates a sense of the conditions for the nonlinear operator in abstract theorems.

2. Inhomogeneous Cauchy problem for the linear equation. Define the Mittag-Leffler function for further research

\[
E_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta > 0,
\]

define Sobolev spaces also \( W^m_q(0, T; \mathcal{Z}) = \{ f \in L_q(0, T; \mathcal{Z}) : f^{(m)} \in L_q(0, T; \mathcal{Z}) \} \) for \( q \in (1, \infty) \).

Consider the Cauchy problem for the inhomogeneous differential equation

\[
z^{(k)}(0) = z_k, \quad k = 0, 1, \ldots, m - 1, \tag{2.1}
\]

\[
z^{(m)}(t) = A z(t) + f(t), \quad t \in (0, T), \tag{2.2}
\]

Here \( m \in \mathbb{N} \), \( A \) is a bounded operator on a Banach space \( \mathcal{Z} \) (i.e. \( A \in \mathcal{L}(\mathcal{Z}) \)), \( f : (0, T) \rightarrow \mathcal{Z} \) for a given \( T > 0 \). The function \( z \in W^m_q(0, T; \mathcal{Z}) \) is called a strong solution of problem (2.1), (2.2), if it satisfies the conditions (2.1) and (2.2) is valid a.e. on \( (0, T) \).

**Theorem 2.1.** Let \( A \in \mathcal{L}(\mathcal{Z}), f \in L_q(0, T; \mathcal{Z}), q \in (1, \infty) \). Then for every \( z_k \in \mathcal{Z}, \) \( k = 0, 1, \ldots, m - 1 \), there exists a unique strong solution \( z \in W^m_q(0, T; \mathcal{Z}) \) of problem
(2.1), (2.2). The solution has a form
\[
z(t) = \sum_{k=0}^{m-1} t^k E_{m,k+1}(A t^m) z_k + \int_0^t (t-s)^{m-1} E_{m,m}(A(t-s)^m) f(s) ds. \tag{2.3}
\]

**Proof.** For each \( k = 1, 2, \ldots, m - 1, \ l = 1, 2, \ldots, k \) we have
\[
\frac{d^l}{dt^l} t^k E_{m,k+1}(A t^m) = \sum_{n=0}^{\infty} \frac{A^n t^{m+k-l}}{(mn + k - l)!} = t^{k-l} E_{m,k+1-l}(A t^m), \tag{2.4}
\]
\[
\frac{d^m}{dt^m} t^k E_{m,k+1}(A t^m) = \sum_{n=1}^{\infty} \frac{A^n t^{m(n-1)+k}}{(m(n-1) + k)!} = t^k A E_{m,k+1}(A t^m). \tag{2.5}
\]

Therefore for \( l = 1, 2, \ldots, m - 1 \)
\[
\frac{d^l}{dt^l} \sum_{k=0}^{m-1} t^k E_{m,k+1}(A t^m) = \sum_{k=0}^{l-1} t^{m+k-l} A E_{m,m+k+1-l} + \sum_{k=l}^{m-1} t^{k-l} E_{m,k+1-l}(A t^m),
\]
\[
\frac{d^l}{dt^l} \sum_{k=0}^{m-1} t^k E_{m,k+1}(A t^m) z_k |_{t=0} = E_{m,1}(A t^m) z_l |_{t=0} = z_l. \tag{2.6}
\]

Using formulas (2.4), (2.5), we get a.e. on \((0, T)\)
\[
\frac{d^m}{dt^m} \int_0^t (t-s)^{m-1} E_{m,m}(A(t-s)^m) f(s) ds =
\]
\[
= \frac{d^{m-1}}{dt^{m-1}} \left[ (t-s)^{m-1} E_{m,m}(A(t-s)^m) f(s) |_{s=t} \right] +
\]
\[
+ \frac{d^{m-1}}{dt^{m-1}} \int_0^t \frac{d}{dt} \left[ (t-s)^{m-1} E_{m,m}(A(t-s)^m) \right] f(s) ds =
\]
\[
= \frac{d^{m-2}}{dt^{m-2}} \left[ (t-s)^{m-2} E_{m,m-1}(A(t-s)^m) f(s) |_{s=t} \right] +
\]
\[
+ \frac{d^{m-2}}{dt^{m-2}} \int_0^t (t-s)^{m-3} E_{m,m-2}(A(t-s)^m) f(s) ds = \cdots =
\]
\[
= \frac{d}{dt} \left[ (t-s) E_{m,2}(A(t-s)^m) f(s) |_{s=t} \right] + \frac{d}{dt} \int_0^t E_{m,1}(A(t-s)^m) f(s) ds =
\]
\[
= f(t) + A \int_0^t (t-s)^{m-1} E_{m,m}(A(t-s)^m) f(s) ds. \tag{2.7}
\]

By Hölder’s inequality we have
\[
\int_0^T \left\| \int_0^t (t-s)^{m-1} E_{m,m}(A(t-s)^m) f(s) ds \right\|_q^q dt \leq
\]
\[
\leq \left( \frac{q-1}{mq-1} \right)^{q-1} T^{mq} \| E_{m,m}(A) \|_{L^q(0,T)} \| f \|_{L^q(0,T)}^q.
\]
Therefore, function (2.3) belongs to the space $W^m_q(0, T; Z)$ and function (2.3) is a strong solution of problem (2.1), (2.2) as (2.5), (2.6) and (2.7) are satisfied.

If there are solutions $z_1$ and $z_2$ of problem (2.1), (2.2), then their difference $z = z_1 - z_2$ is a solution of the Cauchy problem (2.1) with initial data $z_k = 0$, $k = 0, 1, \ldots, m - 1$, for the homogeneous equation $z^{(m)}(t) = A z(t)$. Then

$$z(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} A z(s) \, ds \, dt_2 dt_1,$$

and $z \in C([0, T]; Z)$ because the space $W^m_q(0, T; Z)$ is continuously embedded in the space of continuous functions. Using the boundedness of the operator $A$ we can show that the integral operator defined by the right-hand side of (2.8) is a contraction operator in space $C([0, t_A]; Z)$ for $t_A < \|A\|_{\mathcal{L}(Z)}^{-1/m}$. Therefore $z \equiv 0$ on $[0, t_A]$. Proceed to the segment $[t_A, t_{2A}]$, and so on we will obtain uniqueness of the zero solution for the homogeneous Cauchy problem on $[0, T]$.

**Corollary 2.2.** Let $f \in L_q(0, T; Z)$. Then for any $z_k \in Z$, $k = 0, 1, \ldots, m - 1$, there exists a unique strong solution of problem (2.1) for equation $z^{(m)}(t) = f(t)$ on $(0, T)$ and it has a form

$$z(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} z_k + \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} f(s) \, ds.$$

**Proof.** Take in theorem 2.1 operator $A = 0$. \hfill \Box

3. The Cauchy problem for quasilinear equation. Let $Z$ be a Banach space, $z_k \in Z$, $k = 0, 1, \ldots, m - 1$, $A \in \mathcal{L}(Z)$, $m \in \mathbb{N}$. We will assume that the nonlinear operator $B : (t_0, T) \times Z^m \to Z$ is Carathéodory mapping, i. e. it defines a measurable mapping on $(t_0, T)$ for all $z_0, z_1, \ldots, z_{m-1} \in Z$ and it is continuous with respect to $z_0, z_1, \ldots, z_{m-1} \in Z$ for almost all $t \in (t_0, T)$.

Consider the Cauchy problem for a quasilinear equation

$$z^{(k)}(t_0) = z_k, \quad k = 0, 1, \ldots, m - 1,$$

$$z^{(m)}(t) = A z(t) + B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)).$$

**Remark 3.1.** It seems more natural to consider equation (3.2) with the operator $A = 0$ since the operator $B$ depends on unknown function $z$. However, for using in further considerations of a quasilinear degenerate equation we need have just such a form of equation (3.2).

The function $z \in W^m_q(t_0, T; Z)$ is called a strong solution of the problem (3.1), (3.2) on the interval $(t_0, T)$, if it satisfies (3.1) and equality (3.2) holds a. e. on the interval $(t_0, T)$.

**Lemma 3.2.** Let $A \in \mathcal{L}(Z)$, $B : (t_0, T) \times Z^m \to Z$ be Carathéodory mapping, the estimate

$$\|B(t, y_0, y_1, \ldots, y_{m-1})\|_Z \leq a(t) + c \sum_{k=0}^{m-1} \|y_k\|_Z,$$
Theorem 3.3. holds for all \( y_0, y_1, \ldots, y_{m-1} \in \mathfrak{F} \) and for almost all \( t \in (t_0, T) \), where \( a \in L_q(t_0, T; \mathbb{R}) \), \( c > 0 \), initial values \( z_0, z_1, \ldots, z_{m-1} \in \mathfrak{F} \). Then a function \( z \in W_q^m(t_0, T; \mathfrak{F}) \) is a strong solution of problem (3.1), (3.2) if and only if an equality

\[
z(t) = \sum_{k=0}^{m-1} (t - t_0)^k E_{m,k+1}(A(t - t_0)^m)z_k + \\
+ \int_{t_0}^{t} (t - s)^{m-1} E_{m,m}(A(t - s)^m)B(s, z(s), z^{(1)}(s), \ldots, z^{(m-1)}(s))ds.
\]

is valid for \( z \in W_q^{m-1}(t_0, T; \mathfrak{F}) \) a.e. on the interval \((t_0, T)\).

Proof. Let \( z \) be a solution of (3.1), (3.2), then \( z \in W_q^m(t_0, T; \mathfrak{F}) \). By (3.3) the operator \( B \) is bounded and continuous operator from \( W_q^m(t_0, T; \mathfrak{F}) \) to \( L_q(t_0, T; \mathfrak{F}) \). Arguing as in the proof of Theorem 2.1, we obtain that \( z \) satisfies equation (3.4).

Let \( z \in W_q^{m-1}(t_0, T; \mathfrak{F}) \) satisfies equation (3.4) a.e. on \((t_0, T)\), then the function \( B(\cdot, z(\cdot), \ldots, z^{(m-1)}(\cdot)) \in L_q(t_0, T; \mathfrak{F}) \) and as in Theorem 2.1 we can verify that \( z \) is a solution of (3.1), (3.2).

Hereinafter, the bar over a symbol indicates that this is an ordered set of \( m \) elements with indexes from zero to \( m - 1 \), for example, \( \overline{y} = (z_0, z_1, \ldots, z_{m-1}) \).

Mapping \( B : (t_0, T) \times \mathfrak{F}^m \to \mathfrak{F} \) will be called uniformly Lipschitz continuous in \( \overline{y} \), if there exists \( l > 0 \) such that for all \((t, \overline{y}), (t, \overline{z}) \) from \((t_0, T) \times \mathfrak{F}^m \) the inequality

\[
\|B(t, \overline{y}) - B(t, \overline{z})\|_3 \leq l \sum_{k=0}^{m-1} \|y_k - z_k\|_3
\]

is valid.

Theorem 3.3. Let \( A \in \mathcal{L}(\mathfrak{F}) \), \( m \in \mathbb{N} \), \( B : (t_0, T) \times \mathfrak{F}^m \to \mathfrak{F} \) be a Carathéodory mapping, uniformly Lipschitz continuous in \( \overline{y} \), for some \( \overline{y} \in \mathfrak{F}^m \, N(\cdot, \overline{y}) \in L_q(t_0, T; \mathfrak{F}) \). Then for any \( z_0, z_1, \ldots, z_{m-1} \in \mathfrak{F} \) problem (3.1), (3.2) has a unique strong solution on \((t_0, T)\).

Proof. For any \( \overline{y} \in \mathfrak{F}^m \) the inequality

\[
\|B(t, \overline{y})\|_3 \leq \|B(t, \overline{y})\|_3 + l \sum_{k=0}^{m-1} \|v_k\|_3 + l \sum_{k=0}^{m-1} \|y_k\|_3
\]

follows from the uniform Lipschitz condition. Therefore, the condition (3.3) is satisfied with \( a(t) = \|B(t, \overline{y})\|_3 + l \sum_{k=0}^{m-1} \|v_k\|_3 \), \( c = l \).

According to Lemma 3.2 it suffices to show that (3.4) has a unique solution \( z \in W_q^{m-1}(t_0, T; \mathfrak{F}) \). Define the operator \( G \) in \( W_q^{m-1}(t_0, T; \mathfrak{F}) \) by the equality

\[
G(y)(t) = \sum_{k=0}^{m-1} (t - t_0)^k E_{m,k+1}(A(t - t_0)^m)z_k + \\
+ \int_{t_0}^{t} (t - s)^{m-1} E_{m,m}(A(t - s)^m)B(s, y(s), y^{(1)}(s), \ldots, y^{(m-1)}(s))ds.
\]

By Theorem 2.1 \( G : W_q^{m-1}(t_0, T; \mathfrak{F}) \to W_q^{m-1}(t_0, T; \mathfrak{F}) \).
By $G^r$ $r$-th power of the operator $G$ is denoted, $r \in \mathbb{N}$. If $T - t_0 < 1$ then we will replace $T - t_0$ by 1 in the subsequent discussion. Prove by induction inequality

$$\| (G^r(y))^{(n)}(t) - (G^r(z))^{(n)}(t) \|_3 \leq \frac{K^r(t - t_0)^{r(1 - 1/q)}}{[r!]^{1 - 1/q}} \| y - z \|_{W_q^{m-1}(t_0, T; 3)} \quad (3.5)$$

for almost all $t \in (t_0, T)$. Let

$$\| G \|_{L(3)}$$

be Carathéodory mapping, uniformly Lipschitz continuous in $K$ for almost all $t \in (t_0, T)$. By 1 in the subsequent discussion. Prove by induction inequality

$$\| G(y) \|_{m,n} - \| G(z) \|_{m,n} \leq E_{m,n}((T - t_0)^{m} \| A \|_{L(3)}) \times$$

$$\times \int_{t_0}^{t} \min(1, t - s)^{n-1} \| B(\cdot, y(\cdot, s), \ldots, y^{(m-1)}(\cdot)) - B(\cdot, z(\cdot, s), \ldots, z^{(m-1)}(\cdot)) \|_3 \, ds \leq$$

$$\leq l(T - t_0)^{m} E_{m,n}((T - t_0)^{m} \| A \|_{L(3)}) (t - t_0)^{1 - 1/q} \| y - z \|_{W_q^{m-1}(t_0, T; 3)}.$$  

Assuming that (3.5) with $r = 1$ is satisfied, we obtain

$$\| (G^r(y))^{(n)}(t) - (G^r(z))^{(n)}(t) \|_3 \leq$$

$$\leq \frac{K}{m(T - t_0)} \int_{t_0}^{t} \sum_{k=0}^{m-1} \| (G^{r-1}(y))^{(k)}(s) - (G^{r-1}(z))^{(k)}(s) \|_3 \, ds \leq$$

$$\leq \frac{K}{T - t_0} \int_{t_0}^{t} \frac{K^{r-1}(s - t_0)^{(r-1)(1 - 1/q)}}{[r - 1]^{1 - 1/q}} \| y - z \|_{W_q^{m-1}(t, T; 3)} \, ds \leq$$

$$\leq \frac{K(t - t_0)^{r(1 - 1/q)}}{(T - t_0)^{1 - 1/q}} (T - t_0)^{1/q} \| y - z \|_{W_q^{m-1}(t_0, T; 3)}.$$  

From (3.5) it follows that for $r \in \mathbb{N}$

$$\| (G^r(y) - G^r(z)) \|_{W_q^{m-1}(t, T; 3)} \leq \frac{mK^r(T - t_0)^{r - 1}}{[r!]^{1 - 1/q(r(q - 1) + 1)^{1/q}} \| y - z \|_{W_q^{m-1}(t_0, T; 3)}.$$  

Therefore, if $r$ is sufficiently large, then $G^r$ is a contracting mapping in $W_q^{m-1}(t_0, T; 3)$ so it has a unique fixed point. A fixed point of the mapping $G$ is a fixed point for the mapping $G^r$ also, so it is unique. This point is a unique solution of equation (3.4) in the space $W_q^{m-1}(t_0, T; 3)$, and hence it is a unique strong solution of problem (3.1), (3.2) on the interval $(t_0, T)$.

**Corollary 3.4.** Let $m \in \mathbb{N}$, $B : (t_0, T) \times \mathbb{R}^m \to \mathbb{R}$ be Carathéodory mapping, uniformly Lipschitz continuous in $y$, for some $\bar{v} \in \mathbb{R}^m N(\cdot, \bar{v}) \in L_q(t_0, T; 3)$. Then for any $z_0, z_1, \ldots, z_{m-1} \in \mathbb{R}$ problem (3.1) for the equation

$$z^{(m)}(t) = B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t))$$

has a unique strong solution on $(t_0, T)$.

Next, we consider the equation with a mapping $B$ which is sufficiently smooth by the totality of all the variables. So there is a sense to consider the equation

$$z^{(m)}(t) = Az(t) + B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)) + f(t) \quad (3.6)$$

with function $f$ from a Lebesgue space in a general case.
Theorem 3.5. Let $A \in \mathcal{L}(3)$, $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $B \in C^n([t_0, T] \times 3; \mathfrak{F})$ be uniformly Lipschitz continuous in $z$, $f \in W_q^n(t_0, T; \mathfrak{F})$. Then for any $z_0, z_1, \ldots, z_{m-1} \in \mathfrak{F}$ there exists a unique strong solution $z \in W_q^{m+n}(t_0, T; \mathfrak{F})$ of problem (3.1), (3.6).

Proof. Since $B$ is continuous by the totality of all variables, then for every $\tau \in \mathfrak{F}$ $B(\cdot, \tau) + f \in L_q(t_0, T; \mathfrak{F})$. Therefore, the conditions of Theorem 3.3 are fulfilled.

Take the solution $z \in W_q^m(t_0, T; \mathfrak{F})$ whose existence was proved in the previous theorem. Since $B \in C^1([t_0, T] \times 3; \mathfrak{F})$ the function $t \mapsto B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t))$ has a derivative a. e. on $(t_0, T)$. Taking into account (3.2), we obtain

\[ z^{(m+1)}(t) = Az^{(1)}(t) + D_t B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)) + f^{(1)}(t). \]

Here $D_t B$ is complete derivative on $t$. By the conditions on operator $B$ and $z \in W_q^m(t_0, T; \mathfrak{F})$ $D_t B$ belongs $L_q(t_0, T; \mathfrak{F})$. So $z \in W_q^{m+1}(t_0, T; \mathfrak{F}) \subset C^m([t_0, T]; \mathfrak{F})$ by embedding theorem. As $D_t^r$ with $r \in \mathbb{N}$ we will denote $r$-th complete derivative. Therefore, when $n > 1$ the function $D_t B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t))$ is differentiable and there exists the derivative $z^{(m+2)} \in L_q(t_0, T; \mathfrak{F})$ and so on.

If $n \geq m$ then we have equality

\[ z^{(2m)}(t) = A z^{(m)}(t) + D_t^m B(t, z(t), z^{(1)}(t), \ldots, z^{(m)}(t)) = A^2 z(t) + AB(t, z(t), z^{(1)}(t), \ldots, z^{(m)}(t)) + D_t^m B(t, z(t), z^{(1)}(t), \ldots, z^{(m)}(t)) + f^{(m)}(t). \]

For $r \in \mathbb{N}$ we obtain

\[ z^{(rm)}(t) = A^r z(t) + \sum_{k=0}^{r-1} A^k D_t^{(r-1-k)m} B(t, z(t), z^{(1)}(t), \ldots, z^{(m)}(t)) + f^{((r-1)m)}(t). \]

At each step, the order of the derivative of $z$ in the left-hand side is one plus the order of the highest derivative of $z$ in the right-hand side of the equality. This fact allows us to continue a chain of reasoning, differentiating the right-hand side of this equality.

The expressions on the right will contain continuous derivatives of $B$, the derivatives of $z$, that are continuous by the embedding theorem, and only one derivative of order, one less than the order of the derivative in the left-hand side of equality, in the first degree. Above proved that it belongs to $L_q(t_0, T; \mathfrak{F})$ already. Therefore, at each step the right-hand side of equality will be in $L_q(t_0, T; \mathfrak{F})$.

The chain breaks when the highest derivative of $B$ in the right-hand side has the order $n$. In the left-hand side function $z$ has a derivative of the order $m + n$ at this step.

Corollary 3.6. Let $m \in \mathbb{N}$, $n \in \mathbb{N} \cup \{0\}$, $B \in C^n([t_0, T] \times 3; \mathfrak{F})$ be uniformly Lipschitz continuous in $\mathfrak{F}$, $f \in W_q^n(t_0, T; \mathfrak{F})$. Then for any $z_0, z_1, \ldots, z_{m-1} \in \mathfrak{F}$ problem (3.1) for the equation

\[ z^{(m)}(t) = B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)) + f(t) \]

has a unique solution $z \in W_q^{m+n}(t_0, T; \mathfrak{F})$.

4. Degenerate linear inhomogeneous equation. To study the degenerate equation we will use results of the theory of degenerate operator semigroups whose proofs can be found in [14].

Let $\mathfrak{X}, \mathfrak{Y}$ be Banach spaces, $\mathcal{L}(\mathfrak{X}; \mathfrak{Y})$ be the Banach space of linear continuous operators from $\mathfrak{X}$ to $\mathfrak{Y}$. Denote with $\mathcal{C}(\mathfrak{X}; \mathfrak{Y})$ the set of linear, closed and densely defined in $\mathfrak{X}$ operators, acting to $\mathfrak{Y}$. If $\mathfrak{Y} = \mathfrak{X}$ then the corresponding notations
will have the form $L(\mathcal{X})$ and $\text{Cl}(\mathcal{X})$, respectively. As $D_M$ we will denote the domain with graph norm $\|\cdot\|_{D_M} = \|\cdot\| + \|M\cdot\|_{\mathcal{Y}}$. Let $L \in L(\mathcal{X}; \mathcal{Y})$, $M \in \text{Cl}(\mathcal{X}; \mathcal{Y})$.

We introduce the denotations $\rho^k(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in L(\mathcal{Y}; \mathcal{X})\}$, $R^k_M(\mu) = (\mu L - M)^{-1}L$, $L^k_M = L(\mu L - M)^{-1}$. Operator $M$ is called $(L, \sigma)$-bounded, if

$$\exists a > 0 \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^k(M)).$$

**Lemma 4.1.** [14, p. 89, 90] Let $M$ be $(L, \sigma)$-bounded operator, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$. Then operators

$$P = \frac{1}{2\pi i} \int_{\gamma} R^L_{\mu}(M) \, d\mu \in L(\mathcal{X}), \quad Q = \frac{1}{2\pi i} \int_{\gamma} L^L_{\mu}(M) \, d\mu \in L(\mathcal{Y}) \quad (4.1)$$

are projectors.

Let $\mathcal{X}^0 = \ker P$, $\mathcal{Y}^0 = \ker Q$; $\mathcal{X}^1 = \text{im} P$, $\mathcal{Y}^1 = \text{im} Q$. Denote as $L_k$ $(M_k)$ the restrictions of $L$ $(M)$ on $\mathcal{X}^k (D_{M_k} = D_M \cap \mathcal{X}^k)$, $k = 0, 1$.

**Theorem 4.2.** [14, p. 90, 91] Let an operator $M$ $(L, \sigma)$-bounded. Then

(i) $M_1 \in L(\mathcal{X}^1; \mathcal{Y}^1)$, $M_0 \in \text{Cl}(\mathcal{X}^0; \mathcal{Y}^0)$, $L_k \in L(\mathcal{X}^k; \mathcal{Y}^k)$, $k = 0, 1$;

(ii) there exist operators $M_0^{−1} \in L(\mathcal{Y}^0; \mathcal{X}^0)$, $M_1^{−1} \in L(\mathcal{Y}^1; \mathcal{X}^1)$.

Denote $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $H = M_0^{−1}L_0$. For $p \in \mathbb{N}_0$ an operator $M$ is called $(L, p)$-bounded, if it is $(L, \sigma)$-bounded, $H^p \neq \emptyset$, $H^{p+1} = \emptyset$.

Using the resolvent identity, that is fair for operator-functions $R^L_{\mu}(M)$, $L^L_{\mu}(M)$ and by means of Cauchy theorem we can easily obtain the following statement.

**Lemma 4.3.** Let $m, n \in \mathbb{N}$, an operator $M$ be $(L, \sigma)$-bounded, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$, $X_{m,n}(t) = \frac{1}{2\pi i} \int_{\gamma} R^L_{\mu}(M)E_{m,n}(\mu t^{m}) \, d\mu$, $t \geq 0$. Then for any $t \geq 0$$X_{m,n}(t)P = PX_{m,n}(t) = X_{m,n}(t)$.

Obviously we can get a result.

**Corollary 4.4.** Let $m, n \in \mathbb{N}$, an operator $M$ be $(L, \sigma)$-bounded. Then for any $t \geq 0$ $\mathcal{X}^0 \subset \ker X_{m,n}(t)$, im$X_{m,n}(t) \subset \mathcal{X}^1$.

**Lemma 4.5.** Let $m, n \in \mathbb{N}$, an operator $M$ be $(L, \sigma)$-bounded. Then for any $t \geq 0$$X_{m,n}(t) = E_{m,n}(L_1^{−1}M_1 t^{m})P$.

**Proof.** By Theorem 4.2 $S_1 = L_1^{−1}M_1 \in L(\mathcal{X}^1)$. For $t \geq 0$ by Lemma 4.3 and Cauchy’s integral formula, we have

$$X_{m,n}(t) = X_{m,n}(t)P = \frac{1}{2\pi i} \int_{\gamma} (\mu I - S_1)^{−1}PE_{m,n}(\mu t^{m}) \, d\mu =$$

$$= \frac{1}{2\pi i} \int_{\gamma} \sum_{k=0}^{\infty} \mu^{−k−1}S_k \sum_{l=0}^{\infty} \frac{\mu^{l}t^{ml}}{(ml + n − 1)!} \, d\mu = E_{m,n}(S_1 t^{m})P.$$

Let $f \in L_q(0,T;\mathcal{Y})$. Function $x \in W_q^{m−1}(0,T;\mathcal{X}) \cap L_q(0,T;D_M)$ is called a strong solution of the Cauchy problem

$$\frac{d^m}{dt^m}Lx(t) = Mx(t) + f(t), \quad t \in (0,T), \quad (4.2)$$
\[x^{(k)}(0) = x_k, \quad k = 0, 1, \ldots, m - 1,\] (4.3)
if \(Lx \in W_q^m(0,T;Y)\), conditions (4.3) are satisfied and for almost all \(t \in (0,T)\) equality (4.2) holds.

**Lemma 4.6.** Let an operator \(H \in \mathcal{L}(X)\) be nilpotent of degree \(p \in \mathbb{N}_0\), \(H^k g \in W_q^{m(k+1)}(0,T;X)\), \(k = 0, 1, \ldots, p\). Then there exists a unique strong solution of the equation
\[
\frac{d^m}{dt^m} H x(t) = x(t) + g(t).
\] (4.4)
It has a form
\[
x(t) = -\sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k g(t).
\] (4.5)

**Proof.** Let \(x\) be a strong solution of equation (4.4). Acting on both sides of (4.4) by the operator \(H\), we obtain \(H \frac{d^m}{dt^m} H x(t) = H x(t) + H g(t)\). Since, by the continuity operator \(H\) the right side of this equation belongs to \(W_q^m(0,T;X)\), the same can be said about its left side, and
\[
\frac{d^2}{dt^2} H^2 x = \frac{d^m}{dt^m} H x + \frac{d^m}{dt^m} H g = x + g + \frac{d^m}{dt^m} H g.
\]
Continuing arguments since \(H\) is nilpotent on \(p\)-th step we arrive at the equality
\[
0 = \frac{d^{m(p+1)}}{dt^{m(p+1)}} H^{p+1} x = x + \sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k g.
\]
It follows that (4.5) is true, which means the uniqueness of solution. \(\square\)

**Theorem 4.7.** Let an operator \(M\) be \((L,p)\)-bounded, \(Qf \in L_q(0,T;Y)\), functions \(H^k M_0^{-1} (I - Q)f \in W_q^{m(k+1)}(0,T;X)\) for \(k = 0, 1, \ldots, p\),
\[
\sum_{k=0}^{p} \frac{d^{mk+t}}{dt^{mk+t}} H^k M_0^{-1} (I - Q)f(0) = -(I - P)x_l, \quad l = 0, 1, \ldots, m - 1.
\] (4.6)
Then there exists a unique strong solution of problem (4.2), (4.3), and it has a form
\[
x(t) = \sum_{k=0}^{m-1} t^k X_{m,k+1}(t)x_k + \int_0^t (t-s)^{m-1} X_{m,m}(t-s)L_1^{-1} Qf(s) \, ds - \sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k M_0^{-1} (I - Q)f(t).
\] (4.7)

**Proof.** Denote projections \(w(t) \equiv (I - P)x(t)\), \(v(t) \equiv Px(t)\). Then by Theorem 4.2 we have
\[
\frac{d^m}{dt^m} H w(t) = w(t) + M_0^{-1} (I - Q)f(t), \quad H \equiv M_0^{-1} L_0,
\] (4.8)
\[
\frac{d^m}{dt^m} v(t) = S_1 v(t) + h(t), \quad S_1 \equiv L_1^{-1} M_1, \quad h(t) = L_1^{-1} Qf(t).
\] (4.9)
By Lemma 4.6 there exists a unique solution of equation (4.8)
\[
w(t) = -\sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k M_0^{-1} (I - Q)f(t).
\]
It follows that matching conditions (4.6) must be satisfied.
By Theorem 4.2 an operator $S_1 \in \mathcal{L}(\mathcal{X}^1)$. Therefore, Theorem 2.1 implies the existence of a unique solution for the Cauchy problem $v^{(k)}(0) = P x_k$, $k = 0, 1, \ldots, m - 1$, to equation (4.9). And it has a form

$$v(t) = \sum_{k=0}^{m-1} t^k E_{m,k+1}(S_1 t^m) P x_k + \int_0^t (t-s)^{m-1} E_{m,m}(S_1 (t-s)^m) h(s) ds.$$

Link to Lemma 4.5 completes the proof.

For equation (4.2) we consider also a generalized Showalter–Sidorov problem which naturally arises for degenerate evolution equations [9, 10]

$$P(x^{(k)}(0) - x_k) = 0, \ k = 0, 1, \ldots, m - 1. \quad (4.10)$$

**Theorem 4.8.** Let an operator $M$ be $(L, p)$-bounded, $Q f \in L_q(0, T; \mathfrak{M})$, functions $H^k M_0^{-1}(I - Q)f \in W_q^{m(k+1)}(0, T; \mathcal{X})$ if $k = 0, 1, \ldots, p$. Then there exists a unique solution of problem (4.2), (4.10), and it has a form (4.7).

The proof is similar to the previous one. Feature of the generalized Showalter–Sidorov condition is that at the initial time values are taken into account only for the projections of solution $x$ on the subspace $\mathfrak{X}^1$ and its derivatives, and thus, there is no need to perform matching conditions (4.6).

5. **Quasilinear degenerate equation of high order.** Consider the equation

$$\frac{d^m}{dt^m} L x(t) = M x(t) + N(t, x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t)) + f(t) \quad (5.1)$$

with operators $L \in \mathcal{L}(\mathcal{X}; \mathfrak{Y})$, ker $L \neq \{0\}$, $M \in \mathcal{Cl}(\mathcal{X}; \mathfrak{Y})$, $N : (t_0, T) \times \mathcal{X}^m \rightarrow \mathfrak{Y}$ and with function $f : (t_0, T) \rightarrow \mathfrak{Y}$.

A function $x \in W_q^{m-1}(t_0, T; \mathfrak{X})$, $q \in (1, \infty)$, is called a strong solution of the equation (5.1) on interval $(t_0, T)$ if $L x \in W_q^m(t_0, T; \mathfrak{Y})$, $x \in L_q(t_0, T; D_M)$ and equality (5.1) holds a. e. on $(t_0, T)$.

A strong solution of the generalized Showalter–Sidorov problem (4.10) for equation (5.1) on interval $(t_0, T)$ is a such function $x \in W_q^m(t_0, T; \mathfrak{X})$, $q \in (1, \infty)$, that is a strong solution of (5.1) and satisfies (4.10).

**Theorem 5.1.** Let $p \in \mathbb{N}_0$, an operator $M$ be $(L, p)$-bounded, mapping $N : [t_0, T] \times \mathcal{X}^m \rightarrow \mathfrak{Y}$ such, that $QN \in C^{m(p+1)-1}([t_0, T] \times \mathcal{X}^m; \mathfrak{Y})$ be uniformly Lipschitz continuous in $\mathfrak{v} = (v_0, \ldots, v_{m-1})$, $H^k M_0^{-1}(I - Q)N \in C^{m(k+1)}([t_0, T] \times \mathcal{X}^m; \mathfrak{X})$ for $k = 0, 1, \ldots, p$, for all $(t, \mathfrak{v}) \in [t_0, T] \times \mathcal{X}^m$

$$N(t, v_0, v_1, \ldots, v_{m-1}) = N(t, P v_0, P v_1, \ldots, P v_{m-1}), \quad (5.2)$$

$Q f \in W_q^{m(p+1)-1}(t_0, T; \mathfrak{Y})$, $H^k M_0^{-1}(I - Q)f \in W_q^{m(k+1)}(t_0, T; \mathcal{X})$ hold for $k = 0, 1, \ldots, p$. Then for any $x_0, \ldots, x_{m-1} \in \mathfrak{X}$ problem (4.10), (5.1) has a unique strong solution on the interval $(t_0, T)$.

**Proof.** Alternately multiply equation (5.1) by continuous operators $L_1^{-1} Q$ and $M_0^{-1}$ $(I - Q)$ on the left and obtain a problem

$$v^{(m)}(t) = S_1 v(t) + L_1^{-1} Q \left[ N(t, v(t), v^{(1)}(t), \ldots, v^{(m-1)}(t)) + f(t) \right], \quad (v^{(k)}(t_0) = P x_k, \ k = 0, 1, \ldots, m - 1, \quad (5.3)$$

$$\frac{d^m}{dt^m} H w(t) = w(t) + M_0^{-1}(I - Q) \left[ N(t, v(t), v^{(1)}(t), \ldots, v^{(m-1)}(t)) + f(t) \right]. \quad (5.4)$$
for a pair of functions \( v(t) = Px(t), w(t) = (I-P)x(t) \). Here, as above, we use the notation \( S_1 = L_1^{-1}M_1, H = M_0^{-1}L_0. \)

The operator \( S_0 \) is bounded by Theorem 4.2, and \( L_1^{-1}QN \in C_m^{(p+1)-1}([t_0, T] \times \mathcal{X}^m; \mathcal{Y}) \) is Lipschitz mapping in \( \mathcal{Y} \) by the conditions of this theorem. Then problem (5.3) by Theorem 3.5 has a unique strong solution \( Qf \).

Knowing \( v \), using a nilpotent operator \( H \) and the formula (4.5), we can find the solution

\[
    w(t) = -\sum_{k=0}^{p} \frac{d^mk}{dt^mk} H^k M_0^{-1}(I - Q) \left[ N(t,v(t),v^{(1)}(t),\ldots,v^{(m-1)}(t)) + f(t) \right] \tag{5.5}
\]

of equation (5.4), where \( \frac{d^mk}{dt^mk} H^k M_0^{-1}(I - Q)N(t,v(t),v^{(1)}(t),\ldots,v^{(m-1)}(t)) \) is a complete derivative with respect to \( t \) of \( mk \)-th order for corresponding mapping \( t \to H^k M_0^{-1}(I - Q)N(t,v(t),v^{(1)}(t),\ldots,v^{(m-1)}(t)) \).

In this case, as in the proof of Theorem 3.5, the continuity of all derivatives of \( H^k M_0^{-1}(I - Q)N \) and of all derivatives of \( v \), except the highest-order derivative, which may be only in the first degree, is taken into account. This implies that all expressions that obtained after differentiation belong to \( W_q^m(t_0,T;\mathcal{X}) \).

Contrary to the generalized problem Showalter–Sidorov, the Cauchy problem

\[
    x^{(k)}(t_0) = x_k, \quad k = 0,1,\ldots,m-1, \tag{5.6}
\]

requires matching conditions of initial data and of right-hand side of the equation as in the linear case. As a strong solution of problem (5.1), (5.6) on interval \((t_0,T)\) we mean a strong solution \( x \in W_q^m(t_0,T;\mathcal{X}) \) of equation (5.1) which satisfies (5.6).

**Theorem 5.2.** Let \( p \in \mathbb{N}_0 \), an operator \( M \) be \((L,p)\)-bounded, mapping \( N : [t_0,T] \times \mathcal{X}^m \to \mathcal{Y} \) be such, that \( QN \in C_m^{(p+1)-1}([t_0,T] \times \mathcal{X}^m;\mathcal{Y}) \) be uniformly Lipschitz continuous in \( \tau = (v_0,\ldots,v_{m-1}) \), for \( k = 0,1,\ldots,p \) \( H^k M_0^{-1}(I - Q)N \in C_m^{(k+1)}([t_0,T] \times \mathcal{X}^m;\mathcal{X}) \), for \((t,\tau) \in [t_0,T] \times \mathcal{X}^m \) equality

\[
    N(t,v_0,v_1,\ldots,v_{m-1}) = N(t,Pv_0,Pv_1,\ldots,Pv_{m-1})
\]

holds, \( Qf \in W_q^{m(p+1)-1}(t_0,T;\mathcal{Y}) \), \( H^k M_0^{-1}(I - Q)f \in W_q^{m(k+1)}(t_0,T;\mathcal{X}) \) for \( k = 0,1,\ldots,p, \) for \( x_0,\ldots,x_{m-1} \in \mathcal{X} \) and \( n = 0,1,\ldots,m-1 \) equalities \((I-P)x_n = 0\) are valid, where \( v \in W_q^{m(p+2)-1}(t_0,T;\mathcal{X}) \) is a solution of problem (5.3). Then problem (5.1), (5.6) has a unique strong solution on the interval \((t_0,T)\).

**Proof.** The proof is almost the same as the previous one. We note only that the matching condition is condition (5.6) at initial time for solution (5.5) of equation (5.4).

**Theorem 5.3.** Let \( p \in \mathbb{N}_0 \), let an operator \( M \) be \((L,p)\)-bounded, let \( N : (t_0,T) \times \mathcal{X}^m \to \mathcal{Y} \) be a Carathéodory mapping and uniformly Lipschitz continuous in \( \tau \), let for some \( \tau \in \mathcal{X}^m \) \( N(\cdot,\tau) \in L_q(t_0,T;\mathcal{Y}), \) im\( N \subset \mathcal{Y}^1, \) \( Qf \in L_q(t_0,T;\mathcal{Y}), \) \( H^k M_0^{-1}(I - Q)f \in W_q^{m(k+1)}(t_0,T;\mathcal{X}) \) for \( k = 0,1,\ldots,p. \) Then for any \( x_0,x_1,\ldots,x_{m-1} \in \mathcal{X} \) problem (4.10), (5.1) has a unique strong solution on the interval \((t_0,T)\).
Proof. If \( \text{im}N \subset \mathcal{Y}^4 \), then \((I - Q)N \equiv 0\), \(QN \equiv N\). In this case, equation (5.1) after the action on both sides of the operator \( M_0^{-1}(I - Q) \) takes the form
\[
\frac{d^m}{dt^m} Hw(t) = w(t) + M_0^{-1}(I - Q)f.
\]
Since operator is nilpotent, then this equation has a unique strong solution
\[
w(t) = -\sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k M_0^{-1}(I - Q)f(t)
\]
by Lemma 4.6.

It remains to use Theorem 3.3 to show the unique solvability of the problem
\[
v^{(m)}(t) = S_1v(t) + L_1^{-1}Qf(t) + \sum_{k=0}^{p} \frac{d^{mk}}{dt^{mk}} H^k M_0^{-1}(I - Q)f(t),
\]
\[
 v^{(k)}(t_0) = P_{x_k}, \quad k = 0, 1, \ldots, m - 1.
\]

It is obtained from the problem \((4.10), (5.1)\) after the action of the operator \(L_1^{-1}Q\).

Here we use the fact that the right-hand side of the equation is a nonlinear operator
\[
B(t, v(t), v^{(1)}(t), \ldots, v^{(n)}(t)) = L_1^{-1}N(t, v(t) + w(t), v^{(1)}(t) + w^{(1)}(t), \ldots, v^{(m-1)}(t) + w^{(m-1)}(t)) + L_1^{-1}Qf(t)
\]
satisfies the conditions of Theorem 3.3. \(\square\)

**Theorem 5.4.** Let \( p \in \mathbb{N}_0 \), an operator \( M \) be \((L, p)\)-bounded, \( N : (t_0, T) \times \mathcal{X}^m \rightarrow \mathcal{Y} \) be Carathéodory mapping and uniformly Lipschitz with respect to \( \tau \), for some \( \tau \in \mathcal{X}^m \), \( N(\cdot, \tau) \in L_q(t_0, T; \mathcal{Y}) \), \( \text{im}N \subset \mathcal{Y}^4 \), \( Qf \in L_q(t_0, T; \mathcal{Y}) \), for \( k = 0, 1, \ldots, p \), \( H^k M_0^{-1}(I - Q)f \in W_q^{m(k+1)}(t_0, T; \mathcal{X}) \), \( x_0, \ldots, x_{m-1} \in \mathcal{X} \) satisfy the equalities
\[
(I - P)x_n = -\sum_{k=0}^{p} \frac{d^{mk+n}}{dt^{mk+n}} |_{t=t_{0}} H^k M_0^{-1}(I - Q)f(t), \quad n = 0, 1, \ldots, m - 1. \quad (5.7)
\]

Then problem (5.1), (5.6) has a unique strong solution on the interval \((t_0, T)\).

**Proof.** The proof is similar to the previous proof. But in this case additional compatibility conditions (5.7) will appear as in the proof of Theorem 5.2. \(\square\)

**Remark 5.5.** In applications where operator \( M \) is \((L, p)\)-bounded, often it is bounded \((M \in \mathcal{L}(\mathcal{X}; \mathcal{Y}))\). In this case, it would be natural to consider the equation
\[
\frac{d^m}{dt^m} Lx(t) = R(t, x(t), x^{(1)}(t), \ldots, x^{(m-1)}(t)). \quad (5.8)
\]

However, in Theorems 5.1–5.4 the structure of the degenerate equation and, accordingly, the conditions of the theorems on solvability of initial value problems for it are determined by the presence of \((L, p)\)-bounded \( M \), that is the correspondence between the two operators \( L \) and \( M \). Therefore, the proposed method for the study of equation (5.8) is not appropriate. But on the other hand, it is possible to study this equation as equation (5.1) using a selection of an appropriate operator \( M \in \mathcal{L}(\mathcal{X}; \mathcal{Y}) \), with a nonlinear operator \( N(t, y_0, y_1, \ldots, y_{m-1}) = R(t, y_0, y_1, \ldots, y_{m-1}) - My_0 \).
6. Example. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega$ of the class $C^\infty$, $\nu \in \mathbb{R}$.
Consider the initial-boundary value problem

$$\frac{\partial^k}{\partial t^k} x_1(s,t_0) = x_1^{k_0}(s), \quad s \in \Omega, \quad k = 0, 1, \ldots, m-1,$$

(6.1)

$$x_i(s,t) = 0, \quad (s,t) \in \partial \Omega \times (t_0, T), \quad i = 1, 2, 3,$$

(6.2)

$$\frac{\partial^m}{\partial t^m} \Delta x_1 = x_1 + g_1 \left( s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right), \quad (s,t) \in \Omega \times (t_0, T),$$

$$\frac{\partial^m}{\partial t^m} \Delta x_3 = x_2 + g_2 \left( s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right), \quad (s,t) \in \Omega \times (t_0, T),$$

$$0 = \Delta x_3 + g_3 \left( s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right), \quad (s,t) \in \Omega \times (t_0, T).$$

(6.3)

Here functions $g_i, \ i = 1, 2, 3,$ depend on the unknown functions $x_1 = x_1(s,t), x_2 = x_2(s,t), x_3 = x_3(s,t)$ and their derivatives with respect to $t$ of order 1, 2, \ldots, $m-1$.

Let $A$ be Laplace operator with domain $H^2_0(\Omega) = \{ z \in H^2(\Omega) : z(s) = 0, s \in \partial \Omega \} \subset L_2(\Omega)$, $\{ \varphi_k \}$ be the orthonormal in $L_2(\Omega)$ system of its eigenfunctions corresponding to the system $\{ \lambda_k \}$ of eigenvalues of operator $A$, indexed in decreasing order of their multiplicity.

For reducing problem (6.1)-(6.3) to problem (4.10), (5.1) choose spaces $\mathcal{X} = H^2_0(\Omega) \times L_2(\Omega) \times H^2_0(\Omega)$, $\mathcal{Y} = (L_2(\Omega))^3$ and operators

$$L = \begin{pmatrix} \triangle & 0 & 0 \\ 0 & 0 & \triangle \\ 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \triangle \end{pmatrix},$$

then

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \left( \frac{1}{\mu \lambda_k - 1} 0 0 \\ 0 -1 -\mu \\ 0 0 -\frac{1}{\lambda_k} \right) \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k.$$ 

It is easy to verify that $(\mu L - M)^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ if $|\mu| > |\lambda_1|^{-1}$. Formulas (4.1) imply that projectors has a form

$$P = Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that is why $\mathcal{X}^1 = H^2_0(\Omega) \times \{ 0 \} \times \{ 0 \}, \mathcal{X}^0 = \{ 0 \} \times L_2(\Omega) \times H^2_0(\Omega), \mathcal{Y}^1 = L_2(\Omega) \times \{ 0 \} \times \{ 0 \}, \mathcal{Y}^0 = \{ 0 \} \times L_2(\Omega) \times L_2(\Omega), H = \begin{pmatrix} 0 & \lambda_k \\ \lambda_k & 0 \end{pmatrix} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k.$ Therefore $H^2 = \mathcal{O}$ and the operator $M$ is $(L, 1)$-bounded.

It is clear that initial condition (6.1) in this case is equivalent to giving the generalized Showalter–Sidorov conditions (4.10). Theorem 5.1 in this case can be used when considering the problem (6.1), (6.2) for the system of equations

$$\frac{\partial^m}{\partial t^m} \Delta x_1 = x_1 + g_1 \left( s, t, x_1, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right),$$

$$\frac{\partial^m}{\partial t^m} \Delta x_3 = x_2 + g_2 \left( s, t, x_1, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right), \quad (s,t) \in \Omega \times (t_0, T),$$

$$0 = \Delta x_3 + g_3 \left( s, t, x_1, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_1 \right),$$

Here, the nonlinear part depends only on $x_1$ and its derivatives with respect to $t$ to order 1, 2, \ldots, $m-1$ (conformity of condition (5.2)).
Theorem 5.3 concerns to the case of system
\[
\begin{align*}
\frac{\partial^m}{\partial t^m} \triangle x_1 &= x_1 + g_1(s, t, x_1, x_2, x_3, \frac{\partial}{\partial t} x_1, \ldots, \frac{\partial^{m-1}}{\partial t^{m-1}} x_3), \\
\frac{\partial^m}{\partial t^m} \triangle x_3 &= x_2 + f_2(s, t), \\
0 &= \triangle x_3 + f_3(s, t),
\end{align*}
\]
with a nonlinear function only in the first equation, depending on \(x_1, x_2, x_3\) and their derivatives with respect to \(t\) of order 1, 2, \ldots, \(m - 1\) (conformity of condition \(\text{im}N \subset \mathbb{N}^1\)).

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