AN ANISOTROPIC INVERSE MEAN CURVATURE FLOW FOR SPACELIKE GRAPHIC HYPERSURFACES WITH BOUNDARY IN LORENTZ-MINKOWSKI SPACE $\mathbb{R}^{n+1}_1$

YAO GAO, JING MAO

Abstract. In this paper, we consider the evolution of spacelike graphic hypersurfaces defined over a convex piece of hyperbolic plane $\mathcal{H}^n(1)$, of center at origin and radius 1, in the $(n+1)$-dimensional Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$ along an anisotropic inverse mean curvature flow with the vanishing Neumann boundary condition, and prove that this flow exists for all the time. Moreover, we can show that, after suitable rescaling, the evolving spacelike graphic hypersurfaces converge smoothly to a piece of hyperbolic plane of center at origin and prescribed radius, which actually corresponds to a constant function defined over the piece of $\mathcal{H}^n(1)$, as time tends to infinity. Clearly, this conclusion is an extension of our previous work [2].

Keywords: Anisotropic inverse mean curvature flow, spacelike hypersurfaces, Lorentz-Minkowski space, Neumann boundary condition.

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1. Introduction

Given a smooth convex cone in the Euclidean $(n+1)$-space $\mathbb{R}^{n+1}$ ($n \geq 2$), Marquardt [15] considered the evolution of strictly mean convex hypersurfaces with boundary, which are star-shaped with respect to the center of the cone and which meet the cone perpendicularly, along the inverse mean curvature flow (IMCF for short), and showed that this evolution exists for all the time and the evolving hypersurfaces converge smoothly to a piece of a round sphere as time tends to infinity. The perpendicular assumption implies that the flow equation therein has the zero Neumann boundary condition (NBC for short). This interesting result has been improved (by Mao and his collaborator [11]) to the situation that the IMCF was replaced by an anisotropic IMCF with the anisotropic factor $|X|^{-\alpha}$, $\alpha \geq 0$, where $|X|$ denotes the Euclidean norm of the position vector of the evolving hypersurface contained in the convex cone in $\mathbb{R}^{n+1}$. Very recently, Gao and Mao [2] have firstly investigated the evolution of strictly mean convex, spacelike graphic hypersurfaces (contained a time cone) along the IMCF with zero NBC in the $(n+1)$-dimensional $(n \geq 2)$ Lorentz-Minkowski space, and obtained the long-time existence and the asymptotical behavior of the flow equation (after suitable rescaling).

Here, we try to transplant the successful experience on the anisotropic flow to our previous work [2], and luckily, we are successful. In order to state our main conclusion clearly, we need to give several notions first.

Throughout this paper, let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional $(n \geq 2)$ Lorentz-Minkowski space with the following Lorentzian metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 + \cdots + dx_n^2 - dx_{n+1}^2.$$ 

In fact, $\mathbb{R}^{n+1}_1$ is an $(n+1)$-dimensional Lorentz manifold with index 1. Denote by

$$\mathcal{H}^n(1) = \{(x_1, x_2, \cdots, x_{n+1}) \in \mathbb{R}^{n+1}_1 | x_1^2 + x_2^2 + \cdots + x_n^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\},$$

* Corresponding author.
Then we have: unique embedding $\Sigma$ defined along $\partial M$. Here is just to emphasize the relation between $\Sigma$. According to Lemma 3.3, the Lorentzian inner product $\langle L \rangle_{\partial M} = 0$, $\langle \mu X, \nu \circ X \rangle = 0$ on $\partial M \times (0, \infty)$, satisfying the following property:

- For any $x \in \partial M$, $\mu(x) \in T_x M^n$, $\mu(x) \notin T_x \partial M^n$, and moreover, $\mu(x) = \mu(rx)$.

Then we have:

(i) There exists a family of strictly mean convex spacelike hypersurfaces $M^n_t$ given by the unique embedding

$$X \in C^{2+\gamma, 1+\frac{\gamma}{2}}(M^n \times [0, \infty), \mathbb{R}^{n+1}) \cap C^\infty(M^n \times (0, \infty), \mathbb{R}^{n+1})$$

with $X(\partial M^n, t) \subset \Sigma^n$ for $t \geq 0$, satisfying the following system

$$\frac{\partial X}{\partial t} = \frac{1}{\|X\|} H^\nu \quad \text{in } M^n \times (0, \infty)$$

where $H$ is the mean curvature of $M^n := X(M^n, t) = X_t(M^n)$, $\nu$ is the past-directed timelike unit normal vector of $M^n$, and $|X| := |\langle X, X \rangle_L|^{1/2}$ is the norm of the point $X(t, \nu)$ induced by the Lorentzian metric $\langle \cdot, \cdot \rangle_L$. Moreover, the Hölder norm on the parabolic space $M^n \times (0, \infty)$ is defined in the usual way (see, e.g., [7], Note 2.5.4).

(ii) The leaves $M^n_t$ are spacelike graphs over $M^n$, i.e.,

$$M^n_t = \text{graph}_{M^n} u_t$$

\[1\] The reason why we call $\mathcal{H}^n(1)$ a hyperbolic plane is that it is a simply-connected Riemannian $n$-manifold with constant negative curvature and is geodesically complete.

\[2\] As usual, $T_x M^n$, $T_x \partial M^n$ denote the tangent spaces (at $x$) of $M^n$ and $\partial M^n$, respectively. In fact, by the definition of $\Sigma^n$ (i.e., a time cone), it is easy to have $\Sigma^n \cap \partial M^n = \partial M^n$, and we insist on writing as $\Sigma^n \cap \partial M^n$ here just to emphasize the relation between $\Sigma^n$ and $\mu$. Since $\mu$ is a vector field defined along $\partial M^n$, which satisfies $\mu(x) \in T_x \partial M^n$, $\mu(x) \notin T_x \partial M^n$ for any $x \in \partial M^n$, together with the construction of $\Sigma^n$, it is feasible to require $\mu(x) = \mu(rx)$. The requirement $\mu(x) = \mu(rx)$ makes the assumptions $\langle \mu \circ X, \nu \circ X \rangle_{\partial M} = 0$, $\langle \nu X, \nu \circ X \rangle = 0$ on $\partial M \times (0, \infty)$ are reasonable, which can be seen from Lemma 2.1 below in details. Besides, since $\nu$ is timelike, the vanishing Lorentzian inner product assumptions on $\mu, \nu$ implies that $\mu$ is spacelike.

\[3\] Since the evolving hypersurface $M^n_t$ is spacelike, which will be shown in the gradient estimate below (i.e., Lemma 3.3), the Lorentzian inner produce $\langle X, X \rangle_L$ will not degenerate, which implies that the anisotropic term $|X|^{-\alpha}$ in our setting is meaningful.
(iii) Moreover, the evolving spacelike hypersurfaces converge smoothly after rescaling to a piece of $\mathcal{H}^n(r_\infty)$, where $r_\infty$ satisfies

$$
\frac{1}{\sup_{M^n} a_0} \left( \frac{\mathcal{H}^n(M^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{\alpha}} \leq r_\infty \leq \frac{1}{\inf_{M^n} a_0} \left( \frac{\mathcal{H}^n(M^n)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{\alpha}},
$$

where $\mathcal{H}^n(\cdot)$ stands for the $n$-dimensional Hausdorff measure of a prescribed Riemannian $n$-manifold, $\mathcal{H}^n(r_\infty) := \{ r_\infty y : y \in \mathcal{H}^n(1) \}$.

**Remark 1.1.** (1) In fact, $M^n$ is some convex piece of the spacelike hypersurface $\mathcal{H}^n(1)$ implies that the second fundamental form of $\partial M^n$ is positive definite w.r.t. the vector field $\mu$ (provided its direction is suitably chosen).

(2) In [2, Remark 1.1], we have briefly announced the conclusion of Theorem 1.1 already. However, therein we used $|(X,X)_L|^{-\alpha}$, $\alpha \leq 0$, as the anisotropic factor to clearly emphasize that the flow was considered in $\mathbb{R}^{n+1}_1$ not in $\mathbb{R}^{n+1}$. But essentially there is no difference between the flow equation (1.4) in [2] Remark 1.1 and the one in the system (1.1). The purpose that we write the anisotropic factor as $|X|^{-\alpha}$ in (1.1) here is just for the convenience of calculation.

(3) It is easy to check that all the arguments in the sequel are still valid for the case $\alpha = 0$ except some minor changes should be made. For instance, if $\alpha = 0$, then the expression (3.1) below becomes $\varphi(t) = -\frac{1}{n} t + c$. However, in this setting, one can also get the $C^0$ estimate as well. Clearly, when $\alpha = 0$, the flow (1.1) degenerates into the parabolic system with the vanishing NBC in [2, Theorem 1.1], and correspondingly, our main conclusion here covers [2, Theorem 1.1] as a special case.

(4) The geometry of $\mathbb{R}^{n+1}_1$ leads to the fact that if we want to extend the main conclusion in [2, Theorem 1.1] for the IMCF with zero NBC to the anisotropic situation here, then the assumption $\alpha < 0$ should be imposed. However, this is totally different from the Euclidean setting where the precondition $\alpha > 0$ should be made – see [11, Theorem 1.1] for details.

(5) For simplicity, we will directly use notations, the summation convention, the agreement on symbols, and identities (i.e., structure equations, the Laplacian of the second fundamental forms, etc) for spacelike graphic hypersurfaces in $\mathbb{R}^{n+1}_1$ introduced in [2, Section 2] (see also [1, Section 2]).

(6) The lower dimensional case (i.e., $n = 1$) of the system (1.1), which actually describes the evolution of spacelike graphic curves defined over a convex piece of $\mathcal{H}^1(1)$ (in the Lorentz-Minkowski plane $\mathbb{R}^2_1$) along the anisotropic IMCF with zero NBC, has also been solved by us recently (see [3]). As also announced in [2, Remark 1.1], if the ambient space $\mathbb{R}^{n+1}_1$ in Theorem 1.1 was replaced by an $(n+1)$-dimensional Lorentz manifold $M^n \times \mathbb{R}$, with $M^n$ a complete Riemannian $n$-manifold with nonnegative Ricci curvature, then interesting conclusion can be expected (see [3]). Besides, if the speed $1/(|X|^\alpha H)$ in the RHS of the flow equation in (1.1) was replaced by $K^{-\frac{1}{\alpha}}$, with $K$ the Gaussian curvature of the evolving spacelike hypersurface $M^n$, then the long-time existence and the asymptotical behavior (after rescaling) of the new flow (i.e., inverse Gauss curvature flow) can be obtained provided the initial hypersurface $M^n_0$ satisfies more stronger convex assumption (see [4]).

This paper is organized as follows. In Section 2, we will show that using the spacelike graphic assumption, the flow equation (which generally is a system of PDEs) changes into a single scalar second-order parabolic PDE. In Section 3, several estimates, including $C^0$, time-derivative and gradient estimates, of solutions to the flow equation will be shown in details. Estimates of

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4 This result has been announced by the corresponding author, Prof. J. Mao, in an invited talk in Wuhan University on 16th, May 2021 and also in an invited online talk in Universität Konstanz on 20th, May 2021.
higher-order derivatives of solutions to the flow equation, which naturally leads to the long-time existence of the flow, will be investigated in Section 4. In the end, we will clearly show the convergence of the rescaled flow in Section 5.

2. The scalar version of the flow equation

Since the spacelike \( C^2,\gamma \)-hypersurface \( M_0^n \) can be written as a graph of \( M^n \subset \mathcal{H}^n(1) \), there exists a function \( u_0 \in C^2,\gamma(M^n) \) such that \( X_0 : M^n \rightarrow \mathbb{R}_1^{n+1} \) has the form \( x \mapsto G_0 := (x, u_0(x)) \). The hypersurface \( M_t^n \) given by the embedding
\[
X(\cdot, t) : M^n \rightarrow \mathbb{R}_1^{n+1}
\]
at time \( t \) may be represented as a graph over \( M^n \subset \mathcal{H}^n(1) \), and then we can make ansatz
\[
X(x, t) = (x, u(x, t))
\]
for some function \( u : M^n \times [0, T) \rightarrow \mathbb{R} \). The following formulae are needed.

**Lemma 2.1.** Under the same setting\(^5\) as \([2, \text{Lemma 3.1}]\), we have the following formulas:

(i) The tangential vector on \( M_t^n \) is
\[
X_i = \partial_i + u_i \partial_r,
\]
and the corresponding past-directed timelike unit normal vector is given by
\[
\nu = -\frac{1}{v} \left( \partial_r + \frac{1}{u^2} u^i \partial_j \right),
\]
where \( u^j := \sigma^{ij} u_i \), and \( v := \sqrt{1 - u^{-2} |Du|^2} \) with \( Du \) the gradient of \( u \).

(ii) The induced metric \( g \) on \( M_t^n \) has the form
\[
g_{ij} = u^2 \sigma_{ij} - u_i u_j,
\]
and its inverse is given by
\[
g^{ij} = \frac{1}{u^2} \left( \sigma^{ij} + \frac{u^i u^j}{u^2 v^2} \right).
\]

(iii) The second fundamental form of \( M_t^n \) is given by
\[
h_{ij} = \frac{1}{v} \left( \frac{2}{u} u_i u_j - u_{ij} - u \sigma_{ij} \right),
\]
and
\[
h^i_j = g^{ik} h_{kj} = -\left( \frac{1}{uv} \delta^i_j + \frac{1}{uv} \tilde{\sigma}^{ik} \varphi_{jk} \right), \quad \tilde{\sigma}^{ij} = \sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2}.
\]

Naturally, the mean curvature\(^6\) is given by
\[
H = \langle \nu, \nu \rangle_L \sum_{i=1}^n h_i^i = \frac{1}{uv} \left( n + (\sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2}) \varphi_{ij} \right),
\]
where \( \varphi = \log u \).

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\(^5\) This means the conceptions and notations in \([2, \text{Lemma 3.1}]\) should be directly used here.

\(^6\) Different from our treatment in \([2] \) for the mean curvature \( H \), here we use López’s definition of \( H \) in \([10] \) – the mean curvature \( H \) of a surface in \( \mathbb{R}_1^3 \) satisfies \( H = \langle \nu, \nu \rangle_L \cdot \text{tr}(A) = \epsilon \text{tr}(A) \), where clearly \( \epsilon = -1 \) if the surface is spacelike while \( \epsilon = 1 \) if the surface is timelike, and \( \text{tr}(A) \) stands for the trace of \( A \). However, for the system \((1.1) \), there is no essential difference between our previous definition for \( H \) (used in \([2] \)) and López’s, since here the direction of the timelike unit normal vector is chosen to be past-directed, which is exactly in the opposite direction with the one we have used in \([2, \text{Theorem 1.1}]\).
(iv) Let \( p = X(x,t) \in \Sigma^n \) with \( x \in \partial M^n \), \( \dot{\mu}(p) \in T_xM^n \), \( \dot{\mu}(p) \notin T_p\partial M^n \), \( \mu = \mu_i(x_p)\partial_i(x) \) at \( x \), with \( \partial_i \) the basis vectors of \( T_xM^n \). Then

\[
\langle \dot{\mu}(p), \nu(p) \rangle_L = 0 \iff \mu_i(x)u_i(x,t) = 0.
\]

**Proof.** Our Lemma 2.1 here is actually [2, Lemma 3.1]. Readers can check the corresponding proof therein, and, for convenience, we prefer to write down the above formulae here since they would be used often in the sequel. ✷

Using techniques as in Ecker [12] (see also [6, 7, 15]), the problem (1.1) can be degenerated into solving the following scalar equation with the corresponding initial data and the NBC

\[
\begin{cases}
\frac{\partial u}{\partial t} = -\frac{v}{u^{\alpha H}} & \text{in } M^n \times (0, \infty) \\
\nabla \mu u = 0 & \text{on } \partial M^n \times (0, \infty) \\
u(\cdot, 0) = u_0 & \text{in } M^n.
\end{cases}
\]

By Lemma 2.1 define a new function \( \varphi(x,t) = \log u(x,t) \) and then the mean curvature can be rewritten as

\[
H = \langle \nu, \nu \rangle_L \sum_{i=1}^{n} h^i_i = \frac{e^{-\varphi}}{v} \left( n + (\sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2}) \varphi_{ij} \right).
\]

Hence, the evolution equation in (2.1) can be rewritten as

\[
\frac{\partial \varphi}{\partial t} = -e^{-\alpha \varphi}(1 - |D\varphi|^2) \frac{1}{n + (\sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2})} := Q(\varphi, D\varphi, D^2\varphi).
\]

Thus, the problem (1.1) is again reduced to solve the following scalar equation with the NBC and the initial data

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} = Q(\varphi, D\varphi, D^2\varphi) & \text{in } M^n \times (0, T) \\
\nabla \mu \varphi = 0 & \text{on } \partial M^n \times (0, T) \\
\varphi(\cdot, 0) = \varphi_0 & \text{in } M^n,
\end{cases}
\]

where

\[
\left( n + (\sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2}) \varphi_{0,ij} \right)
\]

is positive on \( M^n \), since \( M_0 \) is strictly mean convex. Clearly, for the initial spacelike graphic hypersurface \( M_0^n \),

\[
\frac{\partial Q}{\partial \varphi_{ij}} |_{\varphi_0} = \frac{1}{u^{2+\alpha}H^2} \left( \sigma^{ij} + \frac{\varphi_0^i \varphi_0^j}{v^2} \right)
\]

is positive on \( M^n \). Based on the above facts, as in [6, 7, 15], we can get the following short-time existence and uniqueness for the parabolic system (1.1).

**Lemma 2.2.** Let \( X_0(M^n) = M_0^n \) be as in Theorem 1.1. Then there exist some \( T > 0 \), a unique solution \( u \in C^{2+1,\frac{1}{2}}(M^n \times (0, T]) \cap C^\infty(M^n \times (0, T]) \), where \( \varphi(x,t) = \log u(x,t) \), to the parabolic system (2.2) with the matrix

\[
\left( n + (\sigma^{ij} + \frac{\varphi^i \varphi^j}{v^2}) \varphi_{ij} \right)
\]
positive on $M^n$. Thus there exists a unique map $\tau : M^n \times [0,T] \to M^n$ such that $\tau(\partial M^n, t) = \partial M^n$ and the map $\hat{X}$ defined by

$$\hat{X} : M^n \times [0,T) \to \mathbb{R}^{n+1}_1 : (x,t) \mapsto X(\tau(x,t), t)$$

has the same regularity as stated in Theorem 1.1 and is the unique solution to the parabolic system (1.1).

Let $T^*$ be the maximal time such that there exists some $u \in C^{2+\gamma,1+\frac{\gamma}{2}}(M^n \times [0,T^*)) \cap C^\infty(M^n \times (0,T^*))$ which solves (2.2). In the sequel, we shall prove a priori estimates for those admissible solutions on $[0,T]$ where $T < T^*$.

3. $C^0$, $\dot{\varphi}$ AND GRADIENT ESTIMATES

**Lemma 3.1 (C^0 estimate).** Let $\varphi$ be a solution of (2.2), and then for $\alpha < 0$, we have

$$c_1 \leq u(x,t) \Theta^{-1}(t,c) \leq c_2, \quad \forall x \in M^n, t \in [0,T]$$

for some positive constants $c_1$, $c_2$, where $\Theta(t,c) := \begin{cases} -\frac{\alpha}{n} t + e^{\alpha c} \end{cases}$ with

$$\inf_{M^n} \varphi(\cdot,0) \leq c \leq \sup_{M^n} \varphi(\cdot,0).$$

**Proof.** Let $\varphi(x,t) = \dot{\varphi}(t)$ (independent of $x$) be the solution of (2.2) with $\varphi(0) = c$. In this case, the first equation in (2.2) reduces to an ODE

$$\frac{d}{dt} \varphi = -\frac{1}{n} e^{-\alpha \varphi}.$$ 

Therefore,

$$\varphi(t) = \frac{1}{\alpha} \ln \left( -\frac{\alpha}{n} t + e^{\alpha c} \right), \quad \text{for } \alpha < 0.$$ 

Using the maximum principle, we can obtain that

$$\frac{1}{\alpha} \ln \left( -\frac{\alpha}{n} t + e^{\alpha \varphi_1} \right) \leq \varphi(x,t) \leq \frac{1}{\alpha} \ln \left( -\frac{\alpha}{n} t + e^{\alpha \varphi_2} \right),$$

where $\varphi_1 := \inf_{M^n} \varphi(\cdot,0)$ and $\varphi_2 := \sup_{M^n} \varphi(\cdot,0)$. The estimate is obtained since $\varphi = \log u$. $\Box$

**Lemma 3.2 ($\dot{\varphi}$ estimate).** Let $\varphi$ be a solution of (2.2) and $\Sigma^n$ be the boundary of a smooth, convex domain defined as in Theorem 1.1, then for $\alpha < 0$

$$\min \left\{ \inf_{M^n} (\dot{\varphi}(\cdot,0) \cdot \Theta(0)^\alpha), -\frac{1}{n} \right\} \leq \dot{\varphi}(x,t) \Theta(t)^\alpha \leq \max \left\{ \sup_{M^n} (\dot{\varphi}(\cdot,0) \cdot \Theta(0)^\alpha), -\frac{1}{n} \right\}.$$ 

**Proof.** Set

$$\mathcal{M}(x,t) = \dot{\varphi}(x,t) \Theta(t)^\alpha.$$ 

Differentiating both sides of the first evolution equation of (2.2), it is easy to get that

$$\begin{aligned}
&\frac{\partial \mathcal{M}}{\partial t} = Q^{ij} D_{ij} \mathcal{M} + Q^k D_k \mathcal{M} - \alpha \Theta^{-\alpha} \left( \frac{1}{n} + \mathcal{M} \right) \mathcal{M} \quad \text{in } M^n \times (0,T) \\
&\nabla \mathcal{M} = 0 \quad \text{on } \partial M^n \times (0,T) \\
&\mathcal{M}(\cdot,0) = \dot{\varphi}_0 \cdot \Theta(0)^\alpha \quad \text{on } M^n,
\end{aligned}$$

where $Q^{ij} := \frac{\partial Q}{\partial \varphi_{ij}}$ and $Q^k := \frac{\partial Q}{\partial \varphi_k}$. Then the result follows from the maximum principle. $\Box$
Lemma 3.3 (Gradient estimate). Let \( \varphi \) be a solution of (2.2) and \( \Sigma^n \) be the boundary of a smooth, convex domain described as in Theorem 1.1. Then for \( \alpha < 0 \) we have,

\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)| < 1, \quad \forall \, x \in M^n, \, t \in [0, T].
\]

Proof. Set \( \psi = \frac{|D\varphi|^2}{2} \). By differentiating \( \psi \), we have

\[
\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t} \varphi_m \varphi^m = \dot{\varphi}_m \varphi^m = Q_m \varphi^m.
\]

Then using the evolution equation of \( \varphi \) in (2.2) yields

\[
\frac{\partial \psi}{\partial t} = Q^{ij} \varphi_{ijm} \varphi^m + Q^k \varphi_{km} \varphi^m - \alpha |D\varphi|^2.
\]

Interchanging the covariant derivatives, we have

\[
\psi_{ij} = D_j (\varphi_{mi} \varphi^m)
\]

\[
= \varphi_{mi} \varphi^m + \varphi_{mi} \varphi_j^m
\]

\[
= (\varphi_{ij} + R^l_{imj} \varphi^l) \varphi^m + \varphi_{mi} \varphi_j^m.
\]

Therefore, we can express \( \varphi_{ijm} \varphi^m \) as

\[
\varphi_{ijm} \varphi^m = \psi_{ij} - R^l_{imj} \varphi^l - \varphi_{mi} \varphi_j^m.
\]

Then, in view of the fact \( R_{ijkl} = \sigma_{il} \sigma_{jm} - \sigma_{im} \sigma_{jl} \) on \( \mathcal{H}^n(1) \), we have

\[
\frac{\partial \psi}{\partial t} = Q^{ij} \psi_{ij} + Q^k \psi_k - Q^{ij} (\varphi_i \varphi_j - \sigma_{ij} |D\varphi|^2)
\]

\[
- Q^{ij} \varphi_{mi} \varphi_j^m - \alpha |D\varphi|^2.
\]

Since the matrix \( Q^{ij} \) is positive definite, the third and the fourth terms in the RHS of (3.5) are non-positive. Since \( M^n \) is convex, using a similar argument to the proof of [15, Lemma 5] (see page 1308) implies that

\[
\nabla_\mu \psi = - \sum_{i,j=1}^{n-1} \lambda_{ij} \nabla_{e_i} \varphi \nabla_{e_j} \varphi \leq 0 \quad \text{on} \quad \partial M^n \times (0, T),
\]

where an orthonormal frame at \( x \in \partial M^n \), with \( e_1, \ldots, e_{n-1} \in T_x \partial M^n \) and \( e_n := \mu \), has been chosen for convenience in the calculation, and \( \lambda_{ij} \) is the second fundamental form of the boundary \( \partial M^n \subset \Sigma^n \). So, we can get

\[
\begin{cases}
\frac{\partial \psi}{\partial t} \leq Q^{ij} \psi_{ij} + Q^k \psi_k & \text{in} \ M^n \times (0, T), \\
\nabla_\mu \psi \leq 0 & \text{on} \ \partial M^n \times (0, T), \\
\psi(\cdot, 0) = \frac{|D\varphi(\cdot, 0)|^2}{2} & \text{in} \ M^n.
\end{cases}
\]

Using the maximum principle, we have

\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)|,
\]

Since \( G_0 = \{(x, u(x, 0)) \mid x \in M^n \} \) is a spacelike graph of \( \mathbb{R}^{n+1}_1 \), so we have

\[
|D\varphi| \leq \sup_{M^n} |D\varphi(\cdot, 0)| < 1, \quad \forall \, x \in M^n, \, t \in [0, T].
\]

Our proof is finished.
Remark 3.1. The gradient estimate in Lemma 3.3 makes sure that the evolving graphs \( G_t := \{(x, u(x, t)) | x \in M^n, 0 \leq t \leq T \} \) are spacelike.

Combing the gradient estimate with \( \dot{\varphi} \) estimate, we can obtain

Corollary 3.4. If \( \varphi \) satisfies (2.2), then we have

\[
0 < c_3 \leq H\Theta \leq c_4 < +\infty,
\]

where \( c_3 \) and \( c_4 \) are positive constants independent of \( \varphi \).

4. Hölder Estimates and Convergence

Set \( \Phi = \frac{1}{|X|}H, \ w = \langle X, \nu \rangle_L \) and \( \Psi = \frac{\Phi}{w} \). We can get the following evolution equations.

Lemma 4.1. Under the assumptions of Theorem 1.1, we have

\[
\frac{\partial}{\partial t} g_{ij} = 2\Phi h_{ij},
\]

\[
\frac{\partial}{\partial t} g^{ij} = -2\Phi h^{ij},
\]

\[
\frac{\partial}{\partial t} \nu = \nabla \Phi,
\]

\[
\partial_t h^j_i - \Phi H^{-1} \Delta h^j_i = -\Phi H^{-1}|A|^2 h^j_i - \frac{2\Phi}{H^2} H_i H^j + 2\Phi h_{ik} h^{kj} - 2\alpha \Phi u^{-1} H^{-1} u_i H^j + \alpha \Phi u^{-1} u_j - \alpha (\alpha + 1) \Phi u^{-2} u_i u_j,
\]

and

\[
\frac{\partial \Psi}{\partial t} = \text{div}_g (u^{-\alpha} H^{-2} \nabla \Psi) - 2H^{-2} u^{-\alpha} u^{-1}|\nabla \Psi|^2
\]

\[
+ \alpha \Psi^2 + \alpha \Psi^2 u^{-1} \nabla u \langle X, X \rangle_L - \alpha u^{-\alpha - 1} H^{-2} \nabla_i u \nabla^i \Psi.
\]

Proof. It is easy to get the first three evolution equations, and we omit here. Using the Gauss formula (see [2]), we have

\[
\partial_t h_{ij} = \nabla^2_{ij} \Phi + \Phi h_{ik} h^k_j.
\]

Direct calculation results in

\[
\nabla^2_{ij} \Phi = \Phi \left( -\frac{1}{H} H_{ij} + \frac{2H_i H_j}{H^2} \right) + 2\alpha \Phi u^{-1} H^{-1} u_i H_j - \alpha \Phi u^{-1} u_j + \alpha (\alpha + 1) \Phi u^{-2} u_i u_j.
\]

Since

\[
\Delta h_{ij} = -H_{ij} + H h_{ik} h^k_j + h_{ij} |A|^2,
\]

so

\[
\nabla^2_{ij} \Phi = \Phi H^{-1} \Delta h_{ij} - \Phi h_{ik} h^k_j - \Phi H^{-1} |A|^2 h_{ij} + \frac{2H_i H_j \Phi}{H^2}
\]

\[
+ 2\alpha \Phi u^{-1} H^{-1} u_i H_j - \alpha \Phi u^{-1} u_j + \alpha (\alpha + 1) \Phi u^{-2} u_i u_j.
\]

Thus,

\[
\partial_t h_{ij} - \Phi H^{-1} \Delta h_{ij} = -\Phi H^{-1} |A|^2 h_{ij} + \frac{2\Phi}{H^2} H_i H_j
\]

\[
+ 2\alpha \Phi u^{-1} H^{-1} u_i H_j + \alpha (\alpha + 1) \Phi u^{-2} u_i u_j - \alpha \Phi u^{-1} u_j.
\]
Then
\[
\partial_t H = -\partial_t g^{ij} h_{ij} - g^{ij} \partial_t h_{ij} = u^{-\alpha} H^{-2} \Delta H - 2u^{-\alpha} H^{-3} |\nabla H|^2 + u^{-\alpha} H^{-1} |A|^2 - 2\alpha u^{-\alpha-1} H^{-2} \nabla_i u \nabla^i H + \alpha u^{-\alpha-1} H^{-1} \Delta u - \alpha (\alpha + 1) u^{-\alpha-2} H^{-1} |\nabla u|^2.
\]
Clearly,
\[
\partial_t w = -\Phi - \alpha \Phi u^{-1} \nabla^i u \langle X, X_i \rangle_L - \Phi H^{-1} \nabla^i H \langle X, X_i \rangle_L,
\]
and using the Weingarten equation, we have
\[
w_{ij} = h^k_i \langle X, X_k \rangle_L + h_{ij} + h^k_i h_{kj} \langle X, X \rangle_L = h_{ij,k} \langle X, X^k \rangle_L + h_{ij} + h^k_i h_{kj}.w.
\]
Thus,
\[
\Delta w = -H - \nabla^i H \langle X, X_i \rangle_L + |A|^2 w.
\]
and
\[
\partial_t w = u^{-\alpha} H^{-2} \Delta w - u^{-\alpha} H^{-2} w |A|^2 - \alpha u^{-\alpha-1} H^{-1} \nabla^i u \langle X, X_i \rangle_L.
\]
Hence,
\[
\frac{\partial \Psi}{\partial t} = \alpha u^1 + \alpha H w^{-1} \nabla^i u \langle X, X_i \rangle_L - \frac{1}{w^{-1} H^2} \nabla_i H - \frac{1}{u^{-1} H w^2} \nabla_i w = \alpha u^{-2} H^{-2} w^{-1} + \alpha (\alpha + 1) u^{-2} H^{-3} w^{-1} |\nabla u|^2 + 2u^{-2} H^{-5} w^{-1} |\nabla H|^2 + 2\alpha u^{-2} H^{-4} w^{-1} \nabla_i u \nabla^i H - \alpha u^{-2} H^{-3} w^{-1} \Delta u - \alpha u^{-2} H^{-4} w^{-1} \Delta H - u^{-2} H^{-3} w^{-1} \Delta w + \alpha u^{-2} H^{-2} w^{-1} \nabla^i u \langle X, X_i \rangle_L.
\]
In order to prove (1.1), we calculate
\[
\nabla_i \Psi = -\alpha u^{-\alpha-1} H^{-1} w^{-1} \nabla_i u - u^{-\alpha} H^{-2} w^{-1} \nabla_i H - \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla_i w.
\]
and
\[
\nabla^2 \Psi = \alpha (\alpha + 1) u^{-\alpha-2} H^{-1} w^{-1} \nabla_i u \nabla_j u + \alpha u^{-\alpha-1} H^{-2} w^{-1} \nabla_i u \nabla_j H + \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla_i u \nabla_j w - \alpha u^{-\alpha-1} H^{-1} w^{-1} \nabla^2 i u + \alpha u^{-\alpha-1} H^{-2} w^{-1} \nabla_i H \nabla_j u + 2u^{-\alpha} H^{-3} w^{-1} \nabla_i H \nabla_j H + u^{-\alpha} H^{-2} w^{-1} \nabla_i H \nabla_j u + \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla_i w \nabla_j u + u^{-\alpha} H^{-2} w^{-2} \nabla_i w \nabla_j H + 2u^{-\alpha} H^{-1} w^{-3} \nabla_i w \nabla_j w - \alpha u^{-\alpha-1} H^{-1} w^{-2} \nabla^2 i w.
\]
Thus,
\[
u^{-\alpha} H^{-2} \Delta \Psi = \alpha (\alpha + 1) u^{-\alpha-2} H^{-3} w^{-1} |\nabla u|^2 + u^{-\alpha} H^{-5} w^{-1} |\nabla H|^2 + 2u^{-\alpha} H^{-3} w^{-3} |\nabla w|^2 + 2\alpha u^{-\alpha-1} H^{-4} w^{-1} |\nabla^2 u| + 2u^{-\alpha-1} H^{-3} w^{-2} |\nabla_i u | \nabla^i H + 2u^{-\alpha} H^{-4} w^{-2} \nabla_i H \nabla^i w - \alpha u^{-\alpha-1} H^{-3} w^{-1} \Delta u - u^{-\alpha} H^{-4} w^{-1} \Delta H - \alpha u^{-\alpha-1} H^{-3} w^{-1} \Delta H - u^{-\alpha} H^{-3} w^{-2} \Delta w.
\]
So, we have
\[
\text{div}(u^{-\alpha} H^{-2} \nabla \Psi) = -\alpha u^{-\alpha-1} H^{-2} \nabla^2 \Psi \nabla^i u - u^{-\alpha} H^{-3} \nabla \Psi \nabla^i H + u^{-\alpha} H^{-2} \Delta \Psi = (2\alpha^2 + \alpha) u^{-\alpha-2} H^{-3} w^{-1} |\nabla u|^2 + 5\alpha u^{-\alpha-1} H^{-4} w^{-1} \nabla_i u \nabla^i H + 3\alpha u^{-\alpha-1} H^{-3} w^{-2} \nabla_i u \nabla^i w + 4u^{-\alpha} H^{-5} w^{-1} |\nabla H|^2 + 4u^{-\alpha} H^{-4} w^{-2} \nabla_i w \nabla^i H + 2u^{-\alpha} H^{-3} w^{-3} |\nabla w|^2 - \alpha u^{-\alpha-1} H^{-3} w^{-1} \Delta u - u^{-\alpha} H^{-4} w^{-1} \Delta H - u^{-\alpha} H^{-3} w^{-2} \Delta w.
\]
and
\[ 2H^{-1}w|\nabla \Psi|^2 = 2\alpha^2 u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + 2u^{-2\alpha}H^{-5}w^{-1}|\nabla H|^2 + 2u^{-2\alpha}H^{-3}w^{-3}|\nabla w|^2 + 4\alpha u^{-2\alpha-1}H^{-4}w^{-1}\partial^i u \nabla^i H + 4\alpha u^{-2\alpha-1}H^{-2}w^{-2}\partial^i u \nabla^i w + 4u^{-2\alpha}H^{-4}w^{-2}\partial^i H \nabla^i w. \]

As above, we have
\[
\frac{\partial \Psi}{\partial t} = \text{div}(u^{-\alpha}H^{-2}\nabla \Psi) + 2H^{-1}w|\nabla \Psi|^2
= \alpha u^{-2\alpha}H^{-2}w^{-2} + \alpha u^{-2\alpha-1}H^{-2}w^{-2}\nabla^i u(X,X)_L + \alpha^2 u^{-2\alpha-2}H^{-3}w^{-1}|\nabla u|^2 + \alpha^{-2\alpha-1}H^{-4}w^{-1}\partial^i u \nabla^i H + \alpha u^{-2\alpha-1}H^{-3}w^{-2}\partial^i u \nabla^i w
= \alpha \Psi^2 + \alpha \Psi^2 u^{-1}\nabla^i u(X,X)_L - \alpha u^{-\alpha-1}H^{-2}\partial^i u \nabla^i \Psi.
\]

The proof is finished. \hfill \square

Now, we define the rescaled flow by
\[ \bar{X} = X\Theta^{-1}. \]
Thus,
\[ \bar{u} = u \Theta^{-1}, \quad \bar{\varphi} = \varphi - \log \Theta, \]
and the rescaled mean curvature is given by
\[ \bar{H} = H\Theta. \]

Then, the rescaled scalar curvature equation takes the form
\[ \frac{\partial}{\partial t} \bar{u} = -\frac{v}{u^\alpha H} \Theta^{-\alpha} + \frac{1}{n} \bar{u} \Theta^{-\alpha}. \]
Define \( t = t(s) \) by the relation
\[ \frac{dt}{ds} = \Theta^\alpha \]
such that \( t(0) = 0 \) and \( t(S) = T \). Then \( \bar{u} \) satisfies
\[
\begin{cases}
\frac{\partial}{\partial s} \bar{u} = -\frac{v}{u^\alpha H} + \frac{\bar{u}}{n} & \text{in } M^n \times (0,S) \\
\nabla_\mu \bar{u} = 0 & \text{on } \partial M^n \times (0,S) \\
\bar{u}(\cdot,0) = \bar{u}_0 & \text{in } M^n.
\end{cases}
\]

**Lemma 4.2.** Let \( X \) be a solution of (1.1) and \( \bar{X} = X\Theta^{-1} \) be the rescaled solution. Then
\[
D\bar{u} = Du\Theta^{-1}, \quad D\bar{\varphi} = D\varphi, \quad \frac{\partial \bar{u}}{\partial s} = \frac{\partial u}{\partial t} \Theta^{\alpha-1} + \frac{1}{n} u \Theta^{-1},
\]
\[ \bar{g}_{ij} = \Theta^{-2}g_{ij}, \quad \bar{g}^{ij} = \Theta^2 g^{ij}, \quad \bar{h}_{ij} = h_{ij} \Theta^{-1}. \]

**Proof.** These relations can be computed directly. \hfill \square

**Lemma 4.3.** Let \( u \) be a solution to the parabolic system (2.2), where \( \varphi(x,t) = \log u(x,t) \), and \( \Sigma^n \) be the boundary of a smooth, convex domain described as in Theorem 1.1. Then there exist some \( 0 < \beta < 1 \) and some \( C > 0 \) such that the rescaled function \( \bar{u}(x,s) := u(x,t(s))\Theta^{-1}(t(s)) \) satisfies
\[
[D\bar{u}]_\beta + \left[ \frac{\partial \bar{u}}{\partial s} \right]_\beta + [\bar{H}]_\beta \leq C(||u_0||_{C^{2,\gamma,1+\gamma}(M^n)}^n, n, \beta, M^n), \]

where \([f]_{\beta} := [f]_{x,\beta} + [f]_{s,\beta} \) is the sum of the Hölder coefficients of \(f\) in \(M^n \times [0, S]\) with respect to \(x\) and \(s\).

**Proof.** We divide our proof in three steps.

**Step 1:** We need to prove that
\[
[D\tilde{u}]_{x,\beta} + [D\tilde{u}]_{s,\beta} \leq C(||u_0||_{C^{2+\gamma,1+\bar{\gamma}}(M^n)} n, \beta, M^n).
\]
According to Lemmas 3.1, 3.2 and 3.3 it follows that
\[
|D\tilde{u}| + \left|\frac{\partial \tilde{u}}{\partial s}\right| \leq C(||u_0||_{C^{2+\gamma,1+\bar{\gamma}}(M^n)} M^n).
\]
Then we can easily obtain the bound of \([\tilde{u}]_{x,\beta}\) and \([\tilde{u}]_{s,\beta}\) for any \(0 < \beta < 1\). Fix \(s\) and the equation (2.2) can be rewritten as an elliptic PDE with the corresponding NBC
\[
(4.4) - \text{div}_{\tilde{g}} \left( \frac{\tilde{D}\tilde{\varphi}}{\sqrt{1 - |\tilde{D}\tilde{\varphi}|^2}} \right) = \frac{n}{\sqrt{1 - |\tilde{D}\tilde{\varphi}|^2}} + e^{-\alpha\tilde{\varphi}} \sqrt{1 - |\tilde{D}\tilde{\varphi}|^2}.
\]
So we can get the interior estimate and boundary estimate of \([\tilde{u}]_{x,\beta}\) by using a similar argument to that of the proof of [2, Lemma 5.3] (of course, analysis techniques introduced in [8, Chap. 3; Theorem 14.1; Chap. 10, §2] should be used in the argument). A similar but more detailed explanation of this estimate (for the case \(\alpha = 0\) can also be found in [11].

**Step 2:** The next thing to do is to show that
\[
\left[\frac{\partial \tilde{u}}{\partial s}\right]_{x,\beta} + \left[\frac{\partial \tilde{u}}{\partial s}\right]_{s,\beta} \leq C(||u_0||_{C^{2+\gamma,1+\bar{\gamma}}(M^n)} n, \beta, M^n).
\]
As \(\frac{\partial}{\partial s}\tilde{u} = \tilde{u}\left(-\frac{\nu}{\tilde{\nu}^{\alpha+1}} + \frac{1}{\tilde{\nu}}\right)\), it is sufficient to bound \(\left[\frac{\partial \tilde{u}}{\partial s}\right]_{s,\beta}\). Set \(\tilde{w}(s) := \frac{\nu}{\tilde{\nu}^{\alpha+1}} = \Theta^\alpha \Psi\). Let \(\tilde{\nabla}\) be the Levi-Civita connection of \(\tilde{M}_s := \tilde{X}(M^n, s)\) w.r.t. the metric \(\tilde{g}\). Combining (4.1) with Lemma 4.2 we get
\[
(4.5) \frac{\partial \tilde{w}}{\partial s} = \text{div}_{\tilde{g}}(\tilde{u}^{-\alpha}\tilde{H}^{-2}\tilde{\nabla}\tilde{w}) - 2\tilde{H}^{-2}\tilde{u}^{-\alpha}\tilde{w}^{-1} |\tilde{\nabla}\tilde{w}|^2_{\tilde{g}}
\]
\[-\frac{\alpha}{n}\tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w}^2 P - \alpha \tilde{u}^{-\alpha-1}\tilde{H}^{-2}\tilde{\nabla}_i \tilde{u} \tilde{\nabla}^i \tilde{w},
\]
where \(P := u^{-1}\nabla^i u(X, X_i)_L\). Applying Lemmas 3.1 and 3.3 we have
\[
|P| \leq |\nabla u|_{\tilde{g}} \leq C.
\]
where \(C\) depends only on \(\sup_{M^n}|Du(\cdot, 0)|\), \(c_1\) and \(c_2\). The weak formulation of (4.5) is
\[
\int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial \tilde{w}}{\partial s} \eta d\mu_s ds = \int_{s_0}^{s_1} \int_{\tilde{M}_s} \text{div}_{\tilde{g}}(\tilde{u}^{-\alpha}\tilde{H}^{-2}\tilde{\nabla}\tilde{w})\eta - 2\tilde{H}^{-2}\tilde{u}^{-\alpha}\tilde{w}^{-1} |\tilde{\nabla}\tilde{w}|^2_{\tilde{g}}\eta d\mu_s ds
\]
\[+ \int_{s_0}^{s_1} \int_{\tilde{M}_s} \left(-\frac{\alpha}{n}\tilde{w} + \alpha \tilde{w}^2 + \alpha \tilde{w}^2 P - \alpha \tilde{u}^{-\alpha-1}\tilde{H}^{-2}\tilde{\nabla}_i \tilde{u} \tilde{\nabla}^i \tilde{w}\right)\eta d\mu_s ds.
\]
Since \(\nabla \tilde{\varphi} = 0\), the boundary integrals all vanish, the interior and boundary estimates are basically the same. We define the test function \(\eta := \xi^2 \tilde{w}\), where \(\xi\) is a smooth function with values in \([0, 1]\) and is supported in a small parabolic neighborhood. Then
\[
(4.7) \int_{s_0}^{s_1} \int_{\tilde{M}_s} \frac{\partial \tilde{w}}{\partial s} \xi^2 \tilde{w} d\mu_s ds = \int_{s_0}^{s_1} \int_{\tilde{M}_s} \xi^2 \tilde{w}^2 d\mu_s ds,
\]
\[= \frac{1}{2} ||\tilde{w}||^2_{L^2(\tilde{M}_s)} \Big|_{s_0}^{s_1} - \int_{s_0}^{s_1} \int_{\tilde{M}_s} \xi^2 \tilde{w}^2 d\mu_s ds,
\]
\[\text{where } C \text{ is given by (4.6).} \]
where $\dot{\xi} := \frac{\partial \xi}{\partial t}$. Using the divergence theorem and Young’s inequality, we can obtain

$$
\int_{s_0}^{s_1} \int_{M_s} \text{div}_g(\overline{u}^{-\alpha} \overline{H}^{-2} \overline{\nabla} \overline{w}) \dot{\xi}^2 \overline{w} d\mu_s ds

= - \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha} \overline{H}^{-2} \dot{\xi}^2 \overline{\nabla} \overline{w} \overline{\nabla} \overline{v} d\mu_s ds

- 2 \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha} \overline{H}^{-2} \dot{\xi} \overline{\nabla} \overline{w} \overline{\nabla} \overline{v} d\mu_s ds

\leq \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha} \overline{H}^{-2} |\overline{\nabla} \dot{\xi}|^2 \overline{w}^2 d\mu_s ds
$$

and

$$
\int_{s_0}^{s_1} \int_{M_s} \left( \frac{\alpha}{n} \overline{\nabla} \overline{w} + \alpha \overline{w}^2 + \alpha \overline{w}^2 P - \alpha \overline{u}^{-\alpha-1} \overline{H}^{-2} \overline{\nabla} \overline{u} \overline{\nabla} \overline{v} \right) \dot{\xi}^2 \overline{w} d\mu_s ds

\leq C|\alpha| \int_{s_0}^{s_1} \int_{M_s} \xi^2 (\overline{w}^2 + |\overline{w}|^3) d\mu_s ds

+ \int_{s_0}^{s_1} \int_{M_s} |\alpha| \overline{u}^{-\alpha-1} \overline{H}^{-2} |\overline{\nabla} \overline{u}| |\overline{\nabla} \overline{w}| \dot{\xi}^2 \overline{w} d\mu_s ds

\leq C|\alpha| \int_{s_0}^{s_1} \int_{M_s} \xi^2 (\overline{w}^2 + |\overline{w}|^3) d\mu_s ds

+ \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha-2} \overline{H}^{-2} |\overline{\nabla} \overline{u}| \dot{\xi}^2 \overline{w}^2 d\mu_s ds.
$$

Combining (4.7), (4.8) and (4.9), we have

$$
\frac{1}{2} \| \overline{w} \xi \|^2_{2, M_s} \mid_{s_0}^{s_1} + (2 + \frac{\alpha}{2}) \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha} \overline{H}^{-2} |\overline{\nabla} \overline{w}|^2 \dot{\xi}^2 d\mu_s ds

\leq \int_{s_0}^{s_1} \int_{M_s} \xi^2 |\overline{w}|^2 d\mu_s ds

+ \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha} \overline{H}^{-2} |\overline{\nabla} \dot{\xi}|^2 \overline{w}^2 d\mu_s ds

+ \frac{|\alpha|}{2} \int_{s_0}^{s_1} \int_{M_s} \overline{u}^{-\alpha-2} \overline{H}^{-2} |\overline{\nabla} \overline{u}| \dot{\xi}^2 \overline{w}^2 d\mu_s ds,
$$

which implies that

$$
\frac{1}{2} \| \overline{w} \xi \|^2_{2, M_s} \mid_{s_0}^{s_1} + \frac{(2 + \frac{\alpha}{2})}{\max(\overline{u}^\alpha \overline{H}^2)} \int_{s_0}^{s_1} \int_{M_s} |\overline{\nabla} \overline{w}|^2 \dot{\xi}^2 d\mu_s ds

\leq (1 + \frac{1}{\min(\overline{u}^\alpha \overline{H}^2)}) \int_{s_0}^{s_1} \int_{M_s} \overline{w}^2 (|\xi|^2 + |\overline{\nabla} \dot{\xi}|^2) d\mu_s ds

+ \frac{|\alpha|}{2} \left( C + \frac{\max(|\overline{\nabla} \overline{u}|^2)}{2 \min(\overline{u}^2 + \alpha \overline{H}^2)} \right) \int_{s_0}^{s_1} \int_{M_s} \xi^2 \overline{w}^2 + \dot{\xi}^2 |\overline{w}|^3 d\mu_s ds.
$$

This means that $\overline{w}$ belong to the De Giorgi class of functions in $M^n \times [0, S)$. Similar to the arguments in [2], Chap. 5, §1 and §7], there exist constants $0 < \beta < 1$ and $C$ such that

$$
[\overline{w}]_\beta \leq C \| \overline{w} \|_{L^\infty(M^n \times [0, S))} \leq C \| \overline{u} \|_{C^{2+\gamma, 1+\overline{\gamma}}(M^n)} n, \beta, M^n).
$$

**Step 3:** Finally, we have to show that

$$
[\overline{H}]_\beta \leq C \| \overline{u} \|_{C^{2+\gamma, 1+\overline{\gamma}}(M^n)} n, \beta, M^n).
$$

This follows from the fact that

$$
\overline{H} = \frac{\sqrt{1 - |D\overline{\phi}|^2}}{\overline{u}^{\alpha+1} \overline{w}}
$$

together with the estimates for $\overline{u}$, $\overline{w}$, $D\overline{\phi}$.

\[\square\]
Then we can obtain the following higher-order estimates:

**Lemma 4.4.** Let \( u \) be a solution to the parabolic system (2.2), where \( \varphi(x, t) = \log u(x, t) \), and \( \Sigma^n \) be the boundary of a smooth, convex domain described as in Theorem 1.1. Then for any \( s_0 \in (0, S) \), there exist some \( 0 < \beta < 1 \) and some \( C > 0 \) such that

\[
|\tilde{u}|_{C^{2+\beta,1+\frac{\beta}{2}}(M^n \times [s_0, S])} \leq C\left( |u_0|_{C^{2+\gamma,1+\frac{\gamma}{2}}(M^n)} \right)^n, \beta, M^n
\]

and for all \( k \in \mathbb{N} \),

\[
|\tilde{u}|_{C^{2k+\beta,k+\frac{\beta}{2}}(M^n \times [s_0, S])} \leq C\left( |u_0(\cdot, s_0)|_{C^{2k+\beta,k+\frac{\beta}{2}}(M^n)} \right)^n, \beta, M^n.
\]

**Proof.** By Lemma 2.1, we have

\[
uvH = n + \left( \sigma^{ij} + \varphi^{i} \varphi^{j} \right) \varphi_{ij} = n + u^2 \Delta_g \varphi.
\]

Since

\[
u^2 \Delta_g \varphi = \tilde{u}^2 \Delta_g \tilde{\varphi} = -|\nabla \tilde{u}|^2 + \tilde{u} \Delta_g \tilde{u},
\]

then

\[
\frac{\partial \tilde{u}}{\partial s} = \frac{\partial u}{\partial t} \Theta^{a-1} + \frac{1}{n} \tilde{u}
\]

\[
= \frac{uvH}{u^{1+\alpha} H^2} \Theta^{a-1} - \frac{2v}{u^\alpha H} \Theta^{a-1} + \frac{1}{n} \tilde{u}
\]

\[
= \frac{\Delta_g \tilde{u}}{\tilde{u}^\alpha H^2} - \frac{2v}{\tilde{u}^\alpha H} + \frac{1}{n} \tilde{u} + \frac{n - |\nabla \tilde{u}|^2}{\tilde{u}^{1+\alpha} H^2},
\]

which is a uniformly parabolic equation with Hölder continuous coefficients. Therefore, the linear theory (see [13, Chap. 4]) yields the inequality (4.11).

Set \( \tilde{\varphi} = \log \tilde{u} \), and then the rescaled version of the evolution equation in (4.2) takes the form

\[
\frac{\partial \tilde{\varphi}}{\partial s} = -e^{-\alpha \tilde{\varphi}} \left[ n + \left( \sigma^{ij} + \frac{\varphi^i \varphi^j}{\varphi^2} \right) \tilde{\varphi}_{ij} \right] + \frac{1}{n},
\]

where \( v = \sqrt{1 - |D \tilde{\varphi}|^2} \). According to the \( C^{2+\beta,1+\frac{\beta}{2}} \)-estimate of \( \tilde{u} \) (see Lemma 4.3), we can treat the equations for \( \frac{\partial \tilde{\varphi}}{\partial s} \) and \( D_i \tilde{\varphi} \) as second-order linear uniformly parabolic PDEs on \( M^n \times [s_0, S] \). At the initial time \( s_0 \), all compatibility conditions are satisfied and the initial function \( u(\cdot, t_0) \) is smooth. We can obtain a \( C^{3+\beta,3+\frac{\beta}{2}} \)-estimate for \( D_i \tilde{\varphi} \) and a \( C^{2+\beta,2+\frac{\beta}{2}} \)-estimate for \( \frac{\partial \tilde{\varphi}}{\partial s} \) (the estimates are independent of \( T \)) by Theorem 4.3 and Exercise 4.5 in [13 Chapter 4]. Higher regularity can be proven by induction over \( k \).

**Theorem 4.5.** Under the hypothesis of Theorem 1.1 we conclude

\[
T^* = +\infty.
\]

**Proof.** The proof of this result is quite similar to the corresponding argument in [15] Lemma 8 and so is omitted. \( \square \)
5. Convergence of the rescaled flow

We know that after the long-time existence of the flow has been obtained (see Theorem 4.5), the rescaled version of the system (2.2) satisfies

\[
\begin{align*}
\frac{\partial \tilde{\varphi}}{\partial s} &= \tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) \quad \text{in } M^n \times (0, \infty) \\
\nabla_{\mu} \tilde{\varphi} &= 0 \quad \text{on } \partial M^n \times (0, \infty) \\
\tilde{\varphi}(\cdot, 0) &= \tilde{\varphi}_0 \quad \text{in } M^n,
\end{align*}
\]

where

\[
\tilde{Q}(\tilde{\varphi}, D\tilde{\varphi}, D^2\tilde{\varphi}) := -e^{-\alpha \tilde{\varphi}} \left[ \frac{v^2}{n + \left( \sigma^{ij} + \frac{\tilde{\varphi}^2}{v^2} \right) \tilde{\varphi}_{ij}} \right] + \frac{1}{n}
\]

and \(\tilde{\varphi} = \log \tilde{u}\). Similar to what has been done in the \(C^1\) estimate (see Lemma 3.3), we can deduce a decay estimate of \(\tilde{u}(\cdot, s)\) as follows.

**Lemma 5.1.** Let \(u\) be a solution of (2.1), then we have

\[
|D\tilde{u}(x, t)| \leq \lambda \sup_{M^n} |D\tilde{u}(\cdot, 0)|,
\]

where \(\lambda\) is a positive constant depending on \(c_1\) and \(c_2\).

**Proof.** Set \(\tilde{\psi} = \frac{|D\tilde{\varphi}|^2}{2}\). Similar to the argument in Lemma 3.3 we can obtain

\[
\frac{\partial \tilde{\psi}}{\partial s} = \tilde{Q}^{ij} \tilde{\psi}_{ij} + \tilde{Q}^k \tilde{\psi}_k - \tilde{Q}^{ij} (\tilde{\varphi}_{ij} - \sigma_{ij}|D\tilde{\varphi}|^2) - \tilde{Q}^{ij} \tilde{\varphi}_{mi} \tilde{\varphi}_j^m - \alpha \tilde{Q}|D\tilde{\varphi}|^2,
\]

with the boundary condition

\[
D_{\mu} \tilde{\psi} \leq 0.
\]

So we have

\[
\begin{align*}
\frac{\partial \tilde{\psi}}{\partial s} &\leq \tilde{Q}^{ij} \tilde{\psi}_{ij} + \tilde{Q}^k \tilde{\psi}_k \quad \text{in } M^n \times (0, \infty) \\
D_{\mu} \tilde{\psi} &\leq 0 \quad \text{on } \partial M^n \times (0, \infty) \\
\tilde{\psi}(\cdot, 0) &= \frac{|D\tilde{\varphi}(\cdot, 0)|^2}{2} \quad \text{in } M^n.
\end{align*}
\]

Using the maximum principle and Hopf’s lemma, we can get the gradient estimates of \(\tilde{\varphi}\), and then the inequality (5.2) follows from the relation between \(\tilde{\varphi}\) and \(\tilde{u}\). \(\square\)

**Lemma 5.2.** Let \(u\) be a solution of the flow (2.1). Then,

\(\tilde{u}(\cdot, s)\)

converges to a real number as \(s \to +\infty\).

**Proof.** Set \(f(t) := \mathcal{H}^n(M^n_t)\), which, as before, represents the \(n\)-dimensional Hausdorff measure of \(M^n_t\) and is actually the area of \(M^n_t\). According to the first variation of a submanifold, see
e.g. [16], and the fact $-\text{div}_{M^n_t} \nu = H$, we have
\begin{equation}
\begin{aligned}
f'(t) &= \int_{M^n_t} \text{div}_{M^n_t} \left( \frac{\nu}{|X|^\alpha H} \right) dH^n \\
&= \int_{M^n_t} \sum_{i=1}^n \left\langle \nabla e_i \left( \frac{\nu}{|X|^\alpha H} \right), e_i \right\rangle dH^n \\
&= -\int_{M^n_t} |u|^{-\alpha} dH^n,
\end{aligned}
\end{equation}
where \( \{e_i\}_{1 \leq i \leq n} \) is some orthonormal basis of the tangent bundle \( TM^n_t \). We know that (3.2) implies
\[
\left( -\frac{\alpha}{n} t + e^{\alpha \varphi_2} \right)^{-1} \leq u^{-\alpha} \leq \left( -\frac{\alpha}{n} t + e^{\alpha \varphi_1} \right)^{-1},
\]
where \( \varphi_1 = \inf_{M^n} \varphi(\cdot, 0) \) and \( \varphi_2 = \sup_{M^n} \varphi(\cdot, 0) \). Hence
\[
-\left( -\frac{\alpha}{n} t + e^{\alpha \varphi_2} \right)^{-1} f(t) \leq f'(t) \leq -\left( -\frac{\alpha}{n} t + e^{\alpha \varphi_1} \right)^{-1} f(t).
\]
Combining this fact with (5.4) yields
\[
\frac{\left( -\frac{\alpha}{n} t + e^{\alpha \varphi_2} \right)^{\frac{n}{\alpha}} \mathcal{H}^n(M^n_0)}{e^{n \varphi_2}} \leq f(t) \leq \frac{\left( -\frac{\alpha}{n} t + e^{\alpha \varphi_1} \right)^{\frac{n}{\alpha}} \mathcal{H}^n(M^n_0)}{e^{n \varphi_1}}.
\]
Therefore, the rescaled hypersurface \( \tilde{M}_s = M^n_t \Theta^{-1} \) satisfies the following inequality
\[
\frac{\mathcal{H}^n(M^n_0)}{e^{n \varphi_2}} \leq \mathcal{H}^n(\tilde{M}_s) \leq \frac{\mathcal{H}^n(M^n_0)}{e^{n \varphi_1}},
\]
which implies that the area of \( \tilde{M}_s \) is bounded and the bounds are independent of \( s \). Together with (4.11), Lemma 5.1 and the Arzelà-Ascoli theorem, we conclude that \( \tilde{u}(\cdot, s) \) must converge in \( C^\infty(M^n) \) to a constant function \( r_\infty \) with
\[
\frac{1}{e^{n \varphi_2}} \left( \frac{\mathcal{H}^n(M^n_0)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}} \leq r_\infty \leq \frac{1}{e^{n \varphi_1}} \left( \frac{\mathcal{H}^n(M^n_0)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}},
\]
i.e.,
\begin{equation}
\begin{aligned}
\frac{1}{\sup_{M^n} u_0} \left( \frac{\mathcal{H}^n(M^n_0)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}} \leq r_\infty \leq \frac{1}{\inf_{M^n} u_0} \left( \frac{\mathcal{H}^n(M^n_0)}{\mathcal{H}^n(M^n)} \right)^{\frac{1}{n}}.
\end{aligned}
\end{equation}
This completes the proof.

So, we have

**Theorem 5.3.** The rescaled flow
\[
\frac{d\tilde{X}}{ds} = \frac{1}{|X|^\alpha H} \nu + \frac{1}{n} \tilde{X}
\]
exists for all time and the leaves converge in \( C^\infty \) to a piece of hyperbolic plane of center at origin and radius \( r_\infty \), i.e., a piece of \( \mathcal{H}^n(r_\infty) \), where \( r_\infty \) satisfies (5.5).
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Faculty of Mathematics and Statistics, Key Laboratory of Applied Mathematics of Hubei Province, Hubei University, Wuhan 430062, China.

Email address: Echo-gaoya@outlook.com, jiner120@163.com