Introduction — The paradigmatic setting underpinning quantum optics is that of quantum emitters (QEs) coupled to a (photonic) bath. Since the discovery of the Purcell effect led to the general realization that modifying the bosonic mode structure through a cavity effects the emission profile in the weak-coupling regime [1], structured baths have received a great deal of attention from both the classical [2, 4] and quantum [5, 7] communities. Spurred on by recent technological advances in coupling at the micro- and nano-scale [8, 13], many theoretical proposals for realising exotic phenomena by coupling QEs to structured baths have been discussed particularly in the last decade [14–21], offering potential applications in quantum information [22–24] and simulation of many-body physics [25–29]. Central to all dynamics is the dispersion relation and dimensionality of the underlying bath, which determines the emission profile of a weakly coupled QE in space.

Generically, resonant emission into the band of a periodic bath is understood to consist of a travelling part and a rapidly decaying evanescent part [20], whose natures are determined by the geometry of the bath dispersion relation at the resonant frequency [16, 17, 30, 31]. In the one-dimensional case, or for isotropic higher-dimensional cases, the Green’s function, which determines the dynamics, may be decomposed exactly within a tight-binding formulation into contributions from travelling waves (real poles) and evanescent waves (complex poles). Excluding specific analyses [12, 13, 20, 23], an exact characterisation of the travelling part in generic higher-dimensional environments is currently unknown. In this work, we present a comprehensive picture of the asymptotic dynamics of linear wave propagation in local periodic baths and illustrate fundamental results using examples in the square tight-binding lattice. We use differential geometry to prove an asymptotic decomposition of wave propagation in a generic local d-dimensional (dD) bath as a sum of resonant contributions that follow a power-law decay, and evanescent waves that decay faster than any polynomial. Our analysis yields an approximation of the Green’s function as an integral over the resonant level set (Fermi surface) that converges up to exponential errors with increasing source-receiver separation. Through the geometry of resonant level sets, we show how excitations of diverging transverse mass produce caustics beyond which interactions are evanescently decaying, yet oscillating, ‘ghost waves’ [32, 33]. A universal scaling of ghost wave decay with respect to the off-caustic angle can be derived using saddle point analysis, and deep into the ghost wave regime we find that spin-spin exchanges effected through the bath may be purely incoherent. We probe the associated open orbits [35] in quasimomentum space via an effective magnetic field on the lattice. Quasi-1D emission with non-zero transport and macroscopic bath dependence may be realised, in contrast to isotropic magnetic orbits [27]. Simulating out-of-equilibrium dynamics, we find the emission to periodically and robustly refocus down to the single site level in the plane transverse to propagation. Consequently, QEs located at the refocusing points strongly interact to form high-fidelity bound states in the continuum using only a few QEs. The topological protection of open orbits enables preservation of transport in the presence of local obstructions and moderate global disorder.

Formalism — We consider a local and periodic dD tight-binding Hamiltonian $\hat{H}$, which is linear in the bosonic operators $\hat{a}_i^{(1)}$, so that (with $\hbar = 1$) $\hat{H} = \sum_{ij} J_{ij} \hat{a}_i^{\dagger} \hat{a}_j$, with $[\hat{a}_i^{\dagger}, \hat{a}_j] = \delta_{ij}$ and each site is coupled to only finitely many neighbours. Propagation in periodically structured baths can be analysed through the Green’s function $G(\Delta) = (\Delta - \hat{H})^{-1}$ with off-diagonal elements given by $G(r, r', \Delta) = \langle r | \hat{G} | r' \rangle$ and expressed as an integral over the Brillouin zone (BZ) of the momentum space kernel when $\hat{H}$ is hermitian

$$G(\rho, \Delta) = \mathcal{A} \sum_{\nu} \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \frac{\psi_\nu(k, \rho) \psi^*_\nu(k, \rho')}{\Delta - \omega_\nu(k) + i\epsilon}. \quad (1)$$

Here $\mathcal{A}$ is the volume of a unit cell of the structure,
\( \omega_\nu(k) \) describes the dispersion of band \( \nu \), and \( \psi_\nu(k, r) = \langle r | \psi_\nu(k) \rangle \) is the wavefunction in position space for the diagonal Bloch wave \( \psi_\nu(k) \). Translational invariance gives dependence only on the separation vector \( \rho = r - r' \). In the following, we exploit linearity to treat only a single band, and drop the band subscript \( \nu \). Away from a Van Hove singularity, the Sokhotski-Plemelj theorem permits a decomposition of \( G \) into (in)coherent terms mediated through off-resonant (resonant) excitations as \( G = \Omega - \frac{i}{2} \Gamma \) with

\[
\Gamma = -2\text{Im}[G] = A \int_S \frac{d^{d-1}k}{(2\pi)^{d-1}} \psi^*(k, r') \frac{\psi(k, r)}{v(k)},
\]

\[
\Omega = 2\text{Re}[G] = A \int_{BZ} \frac{d^d k}{(2\pi)^d} \psi^*(k, r') \frac{\psi(k, r)}{\Delta - \omega(k)},
\]

where \( S = S(\Delta) = \{ k : \omega(k) = \Delta \} \) is the resonant set at \( \Delta \) and \( \nabla \omega(k) = v(k) \) is the group velocity with \( v = |v| \). Under our assumptions (see Appendix A 1), \( S = S(\Delta) \) is a (union of) smooth \( (d - 1)D \) dimension manifold(s) \[52\]. In artificial light-matter systems \( S \) is highly controllable due to the tunability of \( \Delta \) through the QEs \[14\] \[17\] \[21\] \[26\] \[28\], and can be probed using the same QEs. We consider QEs of resonant frequency \( \Delta \), each coupled to a single site of the bath \( r_i \), with strength \( |g| \ll |J_{ij}| \), the weakly coupled regime \[14\] \[51\].

In the thermodynamic bath limit one makes the Born-Markov approximation to obtain the effective spin-spin coupling elements between spins \( i \) and \( j \) of the standard master equation as \[16\] \[41\]

\[
H_{ij} = g^2 G(\rho_{ij}, \Delta).
\]

The emitter dynamics are therefore entirely determined through \( G \), and chiefly through \( \omega(k) \) and \( \Delta \). Whilst we present our results in the context of QEs coupled to tight-binding lattices, we note that qualitative phenomena are observable \[41\] and systematically reproducible \[42\] in continuum media, including ultracold gases \[23\] \[43\], photonic crystals \[14\] \[48\], and acoustic meta-crystals \[44\] \[48\]. In particular we emphasize that our investigation applies generally in linear and periodic media.

In the current study we investigate transport of emission in a one-band tight-binding anisotropic square lattice. The nearest neighbour coupling strength is given by \( J_{x(y)} \) for coupling in the \( x(y) \) direction along \( a_x = a\hat{x} \) \( (a_y = a\hat{y}) \) as in Figure 2(b). In the Bloch basis we have \( \omega(k) = (2J_x \cos(k_x a_x) + J_y \cos(k_y a_y)) \). We take \( -\Delta = J_y \) \( = J_x \) well away from the Van Hove singularity at \( k = 0 \) or at the edges of the BZ (Figure 1(a)).

Asymptotic decomposition —The off-resonant interactions are unclear beyond one dimension \[13\] \[49\]. However, a treatment as a rapidly oscillating integral is permitted as \( \rho \) is increased commensurate with the lattice in a fixed direction. Standard methods of complex analysis used in 1D \[12\] \[14\] \[51\] or employed with Wannier functions \[52\] \[53\] can not be readily applied to the strongly singular manifold in \[3\] when \( d > 1 \). However, when \( S \) is smooth, results of differential geometry can be employed. Using a partition of unity, we may localize the contributions near (away from) \( S \), where the integrand is strongly singular (analytic), as in Figure 1(a). We apply Weyl’s tube formula \[52\] \[54\] to asymptotically obtain \[50\] the dominant contributions from a tubular region of \( S \). Up to evanescent corrections \[57\], we obtain an approximation for \( \Omega \), and thus \( G \), as a regular sum of waves resonant as \( S \) in the limit of large \( \rho = |\rho| \):

\[
G \sim -iA \int_S \frac{d^{d-1}k}{(2\pi)^{d-1}} \Pi_\rho(\hat{v} \cdot \hat{\rho}) \frac{\psi^*(k, r') \psi(k, r)}{v(k)}.
\]

Here \( \hat{v}(\hat{\rho}) \) is the normalized group velocity (separation) direction \( \hat{v}/v(\rho/\rho) \) at \( k \in S \), and \( \Pi_\rho \) is an amplitude function, defined in \[50\], that smoothly varies between 0 and 1 as the angle \( \hat{v} \cdot \hat{\rho} \) varies. Crucially, \( \Pi_\rho(t) \to H(t) \) as \( \rho \to \infty \), where \( H \) is the Heaviside function. With the decomposition \( \Pi_\rho = \frac{1 + \Theta_\rho}{2} \), containing (un)scattered part \( 1/2 \) \( (\Theta_\rho/2) \) corresponding to \( \Gamma \) \( (\Omega) \), Equation 5 generalizes the dominant plane wave contribution \( e^{ikx} \) from scattering in 1D \[10\] \[58\] \[60\] \[63\], and places a fundamental restriction on the form of travelling waves at infinity. For comparison, the classical approximation obtained \[50\] by a stationary phase argument (see equation 7), \[60\] \[63\], is valid up to the leading inverse power of \( \rho \). Through Equation 5 power-law decays of all orders are retained for \( \Omega \), which can be computed on the same footing as \( \Gamma \) through a regular surface integral whilst converging to evanescent order. In Fig. 1(b), for separation vector \( \rho = n(a_x + a_y) \) along the square diagonal, the evanescent versus power-law decay of the error from approximants 5 and 7 can be seen in the inset. Equation 5 offers particular utility when strongly subradiant \[64\] \[67\] spin-spin dynamics are present in a large system, where
additional corrections present in [13] and beyond those obtained by heuristic arguments [15, 58, 59, 63] have the potential to observably modify dynamics [67].

**Diverging transverse mass extending long range excitations.**—The standard dichotomy of emitters weakly coupled to baths in the Markovian approximation is between emission into travelling waves for resonant frequency within the band [13, 20, 29], and evanescent waves forming bound states outside of the band [21, 26, 65, 74]. However, in $dD$ with $d > 2$, both regimes of decay may occur at a single frequency, offering greater tunability in existing setups. The availability of travelling waves is quantified through the density of states. In a similar manner, one may quantify all the all the resonant contributions to $G$ with group velocity flux through a local solid angle. In [56], we map a sum over $S$ in $k$-space to a sum over the unit sphere in $\hat{v}$-space, and arrive at a cross section with respect to direction of propagation:

$$\sigma(\hat{v}, \Delta) \sim \frac{1}{v(k)|K(k)|} = |\det[m^T(k)]|v^{-d-2}(k)$$ (6)

where $K$ is the (intrinsic) Gaussian curvature of $S(\Delta)$, and an implicit sum is taken over all $k$ in $S$ with propagation direction $\hat{v}$. The transverse effective mass tensor $m^T$ restricts the usual effective mass tensor $m^{-1}(k)\psi_j = \frac{\partial \psi_j}{\partial k_i}\psi_j$ to the tangent space of $S$ at $k$, so that only the transverse response to transverse impulses determines long-range propagation. Physically, a weakly coupled emitter produces a long-time bath population distribution localized around $S(\Delta')$ with population distribution $\sigma(\hat{v}, \Delta')$, (see [56]) to lowest order. Here $\Delta'$ is the bath-modified emitter frequency. (Normalizing and) integrating over the cross section additionally gives the normalized density of states at $\Delta$ (conservation law for resonant scattering [17, 65, 71]), so that [60] also admits an interpretation as density of resonant waves with respect to propagation direction. Through this analysis we obtain a correspondence between intrinsic curvature on $S$, transverse effective mass $m^T$ of excitations, and and scattering cross section. Naturally, the influence of the effective mass is also present in the Green’s function elements. Application [56] of the classical method of stationary phase to [5], results in the further approximation

$$G \sim i\theta x^2 \frac{\psi(k_0, r)\psi^*(k_0, r')}{v(k_0)} \left(\frac{1}{2\pi \rho}\right)^{(d-1)/2}$$ (7)

plus terms of order $O(\rho^{-(1+d)/2})$ in the limit, where a implicit discrete and finite sum is over the wavevectors $k_0$ that satisfy the resonance condition $k_0 \in S$, with $v(k_0)$ parallel to $\rho$ and $v(k_0) \cdot \rho > 0$. The phase factor $\theta$ is given in Appendix [13]. Through [56], a divergence of transverse mass (vanishing of curvature) gives a singularity in [7], invalidating analysis and allowing one to move past the generic $\rho^{1-d/2}$ scaling along the caustic directions [56, 62, 63] corresponding to group velocities of massive excitations. Beyond the caustic, $\sigma$ drops to zero, as in Figure 2(b). In this region, oscillating and evanescently decaying waves are the dominant means of propagation, whose properties can be inferred from local massive excitations when close to the caustic.

**Ghost waves**—Beyond the caustic direction, the Green’s function elements decay evanescently with oscillations — these so-called ghost waves [32, 33] typically accompany hyperbolic dispersion in structured media [72]. Adjusting $\rho$ beyond the caustic allows one to modulate the exponential decay between QEs akin to standard dynamics near a band edge [13, 21, 26]. Specifically, in Figure 2(c), the exponential decay length of $G$ is tuned between zero and a maximum value determined by bath microscopics as $\rho$ is varied. Whilst dynamics deep into the ghost wave regime are dependent upon specifics of the bath, a universal scaling behaviour near the caustic can generally be obtained for $\Gamma$. When the separation vector is perturbed beyond a global caustic corresponding to a critical angle $\theta_{c}$, the stationary point of $k \cdot \rho$ on $S$ bifurcates into the complex plane. Using saddle point analysis [56], one then arrives at a generic decay in 2D:

$$\Gamma \sim e^{-\kappa |\rho|^{3/2}}, \quad \kappa \propto a|\theta - \theta_c|^{3/2}$$ (8)

as observed in Figure 2(d), which is consistent with a
We take the effective magnetic field \( B = \frac{\Phi}{2\pi} = B\hat{z} \) pointing out of the lattice, corresponding to the physical effect of a gauge field \( A(r) = \frac{x\hat{z}}{2\pi} \) effecting phase accumulation \( J_{ij} \to J_{ij} e^{i\phi_{ij}} \), with \( \phi_{ij} = \int_{r_i}^{r_j} A(r) \cdot dr \) and \( \Phi = \sum \phi_{ij} \). Conservation of \( \omega(k) \) through (11) dictates that the wave packet traces along \( S \) in \( k \) space, so that the geometry of \( S \) will be imprinted into evolution in \( r \) space. In particular, circular Landau orbits are obtained for closed orbits of isotropic regions \([27, 73]\) but the zero winding number \( \alpha \) of open orbits results in an average transport of wave packets over time. We may characterise the spatial (temporal) period \( \tau \) of orbit trajectories through the dimensionless strength of the magnetic field, \( \alpha = \frac{\Phi}{2\pi} \), together with fundamental bath quantities:

\[
\mathbf{1} = \int d\mathbf{r} = \frac{1}{\mathcal{B}} \int_{S} d\mathbf{v(k)} \frac{\mathbf{v(k)}}{v(k)} = \frac{\phi_{\mathbf{z}}}{\alpha},
\]

\[
\tau = \int dt = \frac{1}{\mathcal{B}} \int_{S} d\mathbf{k} \frac{1}{v(k)} = \frac{1}{2\pi},
\]

The transverse extent of the orbit may be calculated similarly. Provided that lattice constant is fixed, \( \mathbf{1} \) is invariant under microscopic variations of the lattice until a Van Hove singularity is encountered, so that such quasi-1D orbits are expected to be present generally when band transitions and Berry curvature can be neglected \([73]\). Phenomena due to non-zero \( \mathbf{I} = ||\mathbf{1}|| \) are manifest when a QE is coupled to the bath, as emission is dominated by wavepackets localized on \( S \) (as in Equation (5)) that evolve as Equations (10-13). In Figure 3 the population plots (a) and (b) reveal quasi-1D transport. Notably, periodicity of the travelling orbits combined with common initial position at the QE demands periodic refocussing of emission. In Figure 3(c), the population in a cross-sectional slice of the bath is given at the QE position, and after a single period. Excepting the QE position where local evanescent fields contribute, refocussing at subsequent sites occurs almost down to the single site level, and independently of bath length scale. Exploiting this single site resolution can in particular be used effect cavities within the lattice. The Markovian decay of a single emitter is strongly modified when more than one emitter is present, due to strong coupling at the refocussing points. Separating emitters by integer multiples of \( \tau \) in the \( y \) direction, an effective cavity can be formed from QEs prepared in a dark state \([36, 38]\). A snapshot of bath population in Figure 3(e) is shown at \( t = 10\tau \) for two QEs, and in Figure 3(d), the collective decay of one, two, and three QEs is shown, prepared in dark states in each case. Significant emitter and localized bath population between emitters is then sustained orders of magnitude beyond the single-emitter lifetime due to destructive interference forming bound states in the continuum \([36, 38]\). However, the cavities modes inherently comprise a continuum of \( \mathbf{k} \) modes in 2D (lying on \( S \)), whilst population nodes do

\[
\mathbf{r} = \mathbf{v(k)},
\]

\[
\mathbf{k} = -\mathbf{v(k)} \times \mathbf{B}.
\]

For a single closed orbit, \( n = 1 \). This quantity does not change with variations in parameters until a Van Hove singularity with zero group velocity is encountered and \( S \) self-intersects. In the case of Figure 2(a), we find \( n = 0 \), corresponding to an open orbit. To probe the orbit, we subject the lattice to an effective magnetic field. With the absence of Berry curvature, the semi-classical equations of motion for a wavepacket are \([73, 74]\):

\[
\mathbf{r} = \mathbf{v(k)},
\]

\[
\mathbf{k} = -\mathbf{v(k)} \times \mathbf{B}.
\]

The two-atom (three atom) subradiant state for two emitters separated a distance \( 3l \) is shown at \( t = 6\tau \) for an emitter prepared in excited state \( |\alpha\rangle \) and located at the central white pixel. In the case of Figure 2(a), we find a Van Hove singularity with zero group velocity is encountered, so that such quasi-1D orbits are expected to be present generally when band transitions and Berry curvature can be neglected \([73]\). Phenomena due to non-zero \( \mathbf{I} = ||\mathbf{1}|| \) are manifest when a QE is coupled to the bath, as emission is dominated by wavepackets localized on \( S \) (as in Equation (5)) that evolve as Equations (10-13). In Figure 3 the population plots (a) and (b) reveal quasi-1D transport. Notably, periodicity of the travelling orbits combined with common initial position at the QE demands periodic refocussing of emission. In Figure 3(c), the population in a cross-sectional slice of the bath is given at the QE position, and after a single period. Excepting the QE position where local evanescent fields contribute, refocussing at subsequent sites occurs almost down to the single site level, and independently of bath length scale. Exploiting this single site resolution can in particular be used effect cavities within the lattice. The Markovian decay of a single emitter is strongly modified when more than one emitter is present, due to strong coupling at the refocussing points. Separating emitters by integer multiples of \( \tau \) in the \( y \) direction, an effective cavity can be formed from QEs prepared in a dark state \([36, 38]\). A snapshot of bath population in Figure 3(e) is shown at \( t = 10\tau \) for two QEs, and in Figure 3(d), the collective decay of one, two, and three QEs is shown, prepared in dark states in each case. Significant emitter and localized bath population between emitters is then sustained orders of magnitude beyond the single-emitter lifetime due to destructive interference forming bound states in the continuum \([36, 38]\). However, the cavities modes inherently comprise a continuum of \( \mathbf{k} \) modes in 2D (lying on \( S \)), whilst population nodes do

\[
\mathbf{r} = \mathbf{v(k)},
\]

\[
\mathbf{k} = -\mathbf{v(k)} \times \mathbf{B}.
\]
not occur at the edges of the cavity, in contrast to the recently established correspondence between coupled QEs and site vacancies \cite{70}. These atypical cavities will be investigated in a separate work.

**Topological robustness.**—We finally investigate the robustness of the quasi-1D transport in the system. Elastic scattering of wavepackets preserves energy, so that waves scattered from local perturbations are also restricted to quasi-1D. In Figure 4(a) we observe bidirectional scattering, together with periodic refocussing of the beam after impinging on an obstruction, similar to the ‘healing’ observed in Bessel and Airy beams \cite{27,28}, albeit effected by a different mechanism. In addition to local obstructions, quasi-1D propagation is also robust to global lattice perturbations through invariant \cite{9}. Introducing onsite energy disorder $XJ$, where $X$ is a random variable sampled uniformly in $[-\chi, \chi]$, for the disorder parameter $\chi$ and averaging over 500 realizations, the average bath populations are given in Figure 4(b-f). The mean bath population retains its behaviour in Fig 4(b), with only slight background noise. In (c-f) a cross-section is taken at $y = 2l$, with the solid line showing the average and the shadow denoting a standard deviation width. The periodic refocussing is seen to be qualitatively preserved, with bath population maintaining its peak by many orders of magnitude at the central site. The fluctuations in the weakly populated sites are additionally many orders of magnitude smaller than the central site population, which is well preserved with relatively small fluctuations up to moderate disorder $\chi = 0.5$. In Fig 4(d), the subsequent evanescent decay rate is additionally shared across all disorders. Even up to moderate frequency disorders probing the vicinity Van Hove singularities, invariant \cite{9} protects quasi-1D emission in an manifestation of the Fermi surface topology \cite{30,39,79} in condensed matter physics. Phenomena induced by Fermi surface topology concern the behaviour of $k$ on $S$ rather than wavefunctions on the entire band, and Van Hove singularities in contrast to degeneracies. Whilst studies are generally limited to 2D, it would be interesting to explore the topological effects afforded by energy landscape in higher dimensions, possibly in conjunction with standard wavefunction induced topological effects.

**Conclusion.**—To summarize, we have studied the emission of quantum emitters into structured media, exploring the consequences of (quasi-)breaking of isotropy and periodicity of the effective bath. We have proved a decomposition of linear wave propagation in arbitrary periodic local lattices as a sum of a travelling part and evanescent part. Long-range wave transport can be dominantly characterised by effective mass of resonant excitations through the geometry of the level set. Inducing open orbits in the dispersion relation can effect ghost waves as the dominant transport mechanism, for which we found a universal long-range scaling near caustics. Finally, breaking periodicity with an effective magnetic field resulted in topologically protected quasi-1D transport in direct contrast to the usual Landau orbits, offering robust bidirectional transport and atypical bound states in the continuum. This work offers general insights into the regimes of wave propagation in generic systems, with potential applications in emerging metamaterials and quantum optics. Of particular interest would more detailed studies including transport around Van Hove singularities and degeneracies, whilst studying interacting non-linear many-body dynamics may prove particularly fruitful.

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Appendix A: Derivation of the long-range behaviour of the Green’s function

1. Proving that degeneracies contribute only evanescent decay

When site couplings are local (i.e. a site only couples to finitely many neighbours), the dispersion relation is analytic on the real axis \( S \) away from degeneracies. Away from Van Hove singularities \( \nabla \omega_{\nu}(k) \neq 0 \), the implicit function theorem applies, and the resonant set \( S_{\nu} \) is a smooth manifold of co-dimension \((d-1)\), allowing one to apply the full power of differential geometry. However, at a degeneracy, it seems that the integrand of

\[
I = \sum_{\nu} \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{\psi_{\nu}(k, r)\psi_{\nu}^*(k, r')}{\Delta - \omega_{\nu}(k)}
\]  

(A1)

becomes non-smooth \([84]\), contributing a power-law decay. Assuming the degeneracy does not coincide with the resonant set \( S \), we show that summing over all bands removes any non-analyticity in individual bands. This allows us to consider only a single band in the main text when the resonant set is away from degeneracies. The derivation follows in a manner similar to \([49]\). Recall the unitary transformation between eigenfunctions \( \psi \) and the Bloch wave corresponding to an individual sublattice:

\[
|\psi_{\nu}(k)\rangle = \sum_{j} U_{\nu j}(k)|k_j\rangle,
\]

(A2)

where \( |k\rangle_i = \frac{1}{\sqrt{N}} \sum_{r_i} e^{ik \cdot r_i} |r_i\rangle \) with \( |r_i\rangle \) the localized excitation at any lattice site \( r_i \) in lattice \( i \). In the thermodynamic limit we conversely have \( |r_i\rangle = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{-ik \cdot r_i} |k\rangle \), and may rewrite the inner product wavefunctions:

\[
\psi_{\nu}(k, r_i) = \langle r_i | \psi_{\nu}(k) \rangle = \sum_{j} \int_{BZ} \frac{d^d k'}{(2\pi)^d} e^{ik \cdot r_i} U_{\nu j}(k) \langle k' | k_j \rangle = e^{i k \cdot r_i} U_{\nu j}(k).
\]

(A3)

Assuming \( r \) and \( r' \) belong to lattices \( i \) and \( i' \) respectively, we rewrite the integrand

\[
I = \sum_{\nu} \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot r} U_{\nu j}(k) U_{\nu j}(k)}{\Delta - \omega_{\nu}(k)}
\]

(A4)

or more succinctly in matrix notation

\[
I = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{ik \cdot r} \left[ (U^\dagger(k)(\Delta - D(k))^{-1} U(k))_{i'i} \right].
\]

(A5)

We note the diagonalization \( h(k) = U^\dagger(k) D(k) U(k) \) and recognize the inner element as an inverse of the finite dimensional resolvent

\[
I = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{ik \cdot r} (\Delta - h(k))_{i'i}^{-1} = \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} e^{ik \cdot r} (\Delta - h(k))_{i'i}^{-1} = \int_{BZ} \frac{d^d k}{(2\pi)^d} e^{ik \cdot r} (\Delta - h(k))_{i'i}^{-1}
\]

(A6)

where \( h(k) \) is now the finite dimensional matrix with analytic elements describing coupling between each of the Bloch sublattices. By standard resolvent analysis \([84]\), the resolvent is analytic in both, \( k \) and \( \Delta \), whenever \( \Delta \) is not equal to any of the eigenvalues of \( h(k) \), i.e. not equal to any of the energy bands at \( k \). Thus, when the resonant level set energy \( \Delta \) does not lie at a degeneracy, the integrand is analytic with respect to integration variable \( k \), and localization of the full Green’s function integral around a degeneracy but away from the resonant level produces an analytic integrand with can be treated with the asymptotic analysis as in the following sections.

2. Long range behaviour of the coherent part of the Green function in \( d \) dimensions

Consider the Green’s function describing propagation through a single mode \( \nu \) (whose subscript we omit in the following) of a periodic continuum bath:

\[
G(\rho, \Delta) = A \int_{BZ} \frac{d^d k}{(2\pi)^d} \frac{\psi(k, r)\psi^*(k, r')}{\Delta - \omega(k) + i0}
\]

(A7)
The integral is understood through the limiting absorption principle, taking a small imaginary part in the denominator to zero. Whilst the wave function $\psi$ is in this case a scalar, we note that similar expressions arise for $\psi$ a tensor, i.e. when polarization is available in an electromagnetic system. With time reversal symmetry of the bath assumed, we extract the strongly singular continuum integral corresponding to coherent wave propagation:

$$\Omega = AP \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \frac{\psi(k, r) \psi^*(k, r')}{\Delta - \omega(k)}, \quad (A8)$$

and proceed with an asymptotic large $\rho = r - r'$ treatment, where the direction of $\rho$ is kept fixed. Having shown analyticity around degeneracies in Appendix A1, we now address the (near) resonant contributions from $S$. Under the assumption of locality in Appendix A1, we may take $S$ to be a smooth manifold of codimension one, and note that $\omega(k)$ is analytic for real $k$ away from degeneracies. With this we may use a partition of unity (or localization argument). That is, we may choose smooth bump functions [85] with support on arbitrarily small regions near $S$ and away from $S$ respectively. Away from $S$ the integrand is smooth, and so evanescent decay of the integrand is obtained.

Let us first factorize the Bloch waves:

$$\Omega = AP \int_{\text{BZ}} \frac{d^d k}{(2\pi)^d} \frac{\Phi(k) e^{i\rho \cdot k}}{\Delta - \omega(k)}, \quad (A9)$$

where we increase the separation with $r, r'$ fixed modulo the lattice, so that $\Phi(k) = \psi(k, r) \psi^*(k, r') e^{-i\rho \cdot k}$, $\Phi$ is unchanging with $\rho$ and can be considered purely as a function of $k$. We may parametrise the vicinity of the resonant set $S = S(\Delta) = \{k : \omega(k) = \Delta\}$ using canonical (Fermi) coordinates in a tubular neighbourhood of $S$:

$$k = (t, \xi) = \xi + t\mathbf{n}(\xi), \quad (A10)$$

where $\xi$ parameterizes the level set $S$, and $\mathbf{n}$ is the unit normal, which we take facing outwards. As the level set is a smooth surface of codimension 1, it is orientable and such a normal always exists. Note that as group velocity is assumed non-vanishing on the surface, the velocity vector always points inside or outside. We assume without loss of generality that the group velocity points outside, although a similar result is obtained otherwise. The above parametrisation is smooth when $t$ is smaller than any of the radii of curvature anywhere on $S$, and we choose $\epsilon$ such that $|t| < \epsilon$ satisfies this condition.

We now use the partition of unity to isolate the contribution from $S$. For this we consider the integral:

$$R = AP \int_{\delta S} \frac{d^d k}{(2\pi)^d} \phi_{\delta S}(k) \frac{\Phi(k) e^{i\rho \cdot k}}{\Delta - \omega(k)}, \quad (A11)$$

where $\delta S$ is a thin tubular region surrounding $S$, with $|t| < \epsilon$. This is smoothly facilitated by the bump function $\phi_{\delta S} \in C^\infty$ forming one of our partition of unity. The bump function can be given entirely through dependence on $t$:

$$\phi_{\delta S}(k) = \phi_\epsilon(t), \quad (A12)$$

where $\phi_\epsilon(t)$ is the usual one-dimensional bump function, normalized to unity at zero:

$$\phi_\epsilon(t) = \begin{cases} \exp\left(\frac{-t^2}{\epsilon^2}\right) & \text{if } |t| < \epsilon \\ 0 & \text{otherwise} \end{cases} \quad (A13)$$

We now change coordinates systems, giving the volume element obtained in the derivation of Weyl’s tube formula [52, 53];

$$d^d k = \det(I + tK) d\xi dt, \quad (A14)$$

where $K$ is the second fundamental form of the surface $S$. In the case of an implicit surface we have [86]

$$K = K(\xi) = -\frac{H_T(\xi)}{v(\xi)}, \quad (A15)$$

for the Hessian

$$[H(k)]_{ij} = \frac{\partial^2 \omega(k)}{\partial k_i \partial k_j}, \quad (A16)$$
which we restrict to a linear map $H_T(\xi)$ acting on the tangent space of $S$ at $\xi$. We obtain

$$R = A \mathcal{P} \int_S \frac{e^{i\xi \cdot r} d^{d-1} \xi}{(2\pi)^{d-1}} \int_{-\epsilon}^{\epsilon} \frac{dt}{2\pi} \phi_t(t) \frac{\Phi(t, \xi) e^{it \cdot \mathbf{n}}}{\Delta - \omega(t, \xi)} \det(I + tK), \quad (A17)$$

We note that the only non-smoothness is at $t = 0$ and rewrite

$$R = A \int_S \frac{e^{i\xi \cdot r} d^{d-1} \xi}{(2\pi)^{d-1}} \mathcal{P} \int_{-\epsilon}^{\epsilon} \frac{dt}{2\pi} L(t, \xi) e^{it \cdot \mathbf{n}} \frac{\phi_t(t)}{t} \det(I + tK), \quad (A18)$$

Here, $L(t, \xi)$ defines a smooth function within $\delta S$ (local expansion in small $t$ reveals that smoothness can be extended to $t = 0$). Additionally, the determinant may be expanded, upon which we need only the lowest order term (the higher order terms cancel $ts$ and produce evanescent decay). We can separate out only the contribution at $t = 0$, and discard small contributions to obtain

$$\Omega \sim R \sim A \int_S \frac{e^{i\xi \cdot r} d^{d-1} \xi}{(2\pi)^{d-1}} \mathcal{P} \int_{-\epsilon}^{\epsilon} \frac{dt}{2\pi} L(0, \xi) e^{it \cdot \mathbf{n}} \phi_t(t) + O(\rho^{-\infty}), \quad (A19)$$

as $\rho \to \infty$. We use $O(\rho^{-\infty})$ to denote decay faster than any polynomial which in the main text we term ‘evanescent decay’. The remainder we subtracted formed a smooth function in $\delta S$, and for $|t| < \epsilon/2$ the surface element is also smooth, so that the remainder will decay evanescently by the principal of non-stationary phase$[6]$

Thus, replacing $L(0, \xi)$ we may write (ignoring evanescent corrections)

$$\Omega = \frac{iA}{2} \int_S \phi(\rho \cdot \mathbf{n}(k)) \frac{\psi(k, r) \psi^*(k, r')}{v(k)(2\pi)^{d-1}} d^{d-1}k + O(\rho^{-\infty}), \quad (A20)$$

where we have relabelled $S \ni \xi$ with $k \in S$ and where we have the phase function

$$\Theta_{\rho}(x) = \frac{\int_{-\infty}^{\infty} dt \sin(xy) \phi_{\rho x}(y)}{\pi y} \quad (A21)$$

$\Theta_{\rho}$ acts as a sigmoid function, varying smoothly from -1 at $-\infty$ to 1 at $\infty$ (see Figure 5), and as the integral of the Fourier transform of the bump function$[8]$, has the asymptotic behaviour

$$\Theta_{\rho}(x) = \text{Sign}(x) + O(\text{Erf}(|\rho x|^{1/2})) = \text{Sign}(x) + O(|\rho x|^{-1/2} \exp(-|\rho x|^{1/2})) \quad (A22)$$

with $\epsilon > 0$ and $x \to \pm \infty$. We thus have the full Green’s function approximant of the main text, converging to $G$ with evanescent errors as $\rho \to \infty$:

$$G \sim \frac{iA}{2} \int_S (1 + \Theta_{\rho}(\rho \cdot \mathbf{n})) \frac{\psi(k, r) \psi^*(k, r')}{v(k)(2\pi)^{d-1}} d^{d-1}k, \quad (A23)$$

where the amplitude of the main text is given by

$$\Pi_{\rho}(\rho \cdot \mathbf{n}) = \frac{(1 + \Theta_{\rho}(\rho \cdot \mathbf{n}))}{2} \quad (A24)$$

We now briefly discuss when the above approximation can be expected to be accurate. Most notably, the tubular radius $\epsilon$ should be large enough to contain a significant fraction of the band, whilst the dispersion relation and Bloch wavefunctions should not vary significantly throughout the band. If this is not the case, our approximation will perform poorly at short and intermediate ranges due to insufficient capturing of the entire band dynamics away from $S$. Physically, this will occur when strong resonances exist away from $S$, when $S$ has regions of large curvature, or when $S$ lies close to a Van Hove singularity or degeneracy. In any of these cases $\Omega$ can be expected to dominate at shorter distances$[10, 12, 31, 87]$

3. **Calculation of the exact Green’s function**

The strongly singular manifold present in$[1]$ of the main text presents significant calculational difficulty, so that to obtain exact results either recurrence relations$[8, 88]$ or a small imaginary part $\bar{\theta}i \to ei$ are typically used. However, the former suffers from instabilities for large $\rho$, whilst the latter suffers an increase in computational cost due to


FIG. 5. The Fourier transform $\Theta_\rho(x)$ for different values of $\rho$. Due to the identical scaling of $\Theta_\rho$ with $\epsilon$ (in Equation (A21)), we fix $\epsilon = 1$ here. $\Theta_\rho$ smoothly interpolates $-1$ and $1$ over the real line, with $\Theta_\rho(0) = 0$. $\Theta_\rho(x)$ pointwise converges to $\text{Sign}(x)$ with evanescent corrections as $\rho \epsilon \to \infty$. We see that the sign changes according to the sign of $\hat{v}$ projected onto $\hat{\rho}$, or the angle between the two. Inset shows the variation of $\Theta_\rho$ as one traverses an isotropic level set $k(t) = (\cos(2\pi t), \sin(2\pi t))$.

the divergence of the integrand near $S$. To navigate around these difficulties we apply a higher-dimensional variant of integration by parts for $\Omega$ (with only divergent derivative). The integrand then permits standard quadrature techniques, whilst $\Gamma$ is obtained through the regular surface integral (2). In one dimension repeated integration by parts on periodic functions $f, \omega$ gives

$$\mathcal{P} \int \frac{df(k)}{\Delta - \omega(k)} = -\mathcal{P} \int \log|\Delta - \omega(k)| \frac{f(k)}{v(k)} \frac{d\omega(k)}{dv} = \mathcal{P} \int \frac{df(k)}{\Delta - \omega(k)} \left[ \log|\Delta - \omega(k)| - 1 \right] \frac{1}{v(k)} \left[ \frac{f(k)}{v(k)} \right]' \left[ \frac{f(k)}{v(k)} \right]' - \frac{1}{v^2(k)},$$

when the group velocity $v$ is non-vanishing on the support of $f$. We note that endpoint contributions cancel via periodicity, and the log divergences at either side of singularities can also be shown to cancel. If $v$ vanishes on the support of $f$, we can decompose $f(k) = f(k)\phi_\epsilon(\Delta - \omega(k)) + f(k)(1 - \phi_\epsilon(\Delta - \omega(k)))$ for the bump function $\phi_\epsilon$, such that $\epsilon$ is small enough that the support of the first term does not contain any points of vanishing group velocity (i.e. as satisfied by the tubular region). The first term is then amenable to integration by parts via (A25), and the second term is a regular integrand. In higher dimensions we accordingly make use of the divergence theorem

$$\int_S \nabla \cdot \mathbf{F} d\mathbf{k} = \int_{\partial S} \mathbf{F} \cdot \hat{n} dS,$$

noting again that periodic boundary conditions and divergences of opposite signs cancel on boundaries to see

$$\mathcal{P} \int \frac{df(k)}{\Delta - \omega(k)} = -\mathcal{P} \int \log|\Delta - \omega(k)| \nabla \cdot \left( \frac{v(k)f(k)}{v^2(k)} \right),$$

and applying integration by parts again for the final result. The bump function may be used here to regularize if necessary.

Appendix B: Characteristics of anisotropic emission

1. Calculating the directional density of states

We here characterise emission properties in real space through the distribution of waves on $S$ with respect to the propagation direction. When $S$ is non-convex (when $S$ has a point of zero curvature) there are multiple wavevectors
contributing to a given direction. In the following we assume that $S$ is convex, whereby the result is generally obtained by partitioning into regions where the sign of the curvature is unchanging. For bath quantities considered as a distribution in $k$-space, we may instead consider them as a distribution over the propagation direction $\mathbf{v}(k) = \mathbf{v}(k)/v(k)$, which maps onto the $(d-1)$-sphere. The mapping $\mathbf{k} \rightarrow \mathbf{v}$ is formally known as the Gauss map, whose Jacobian is given as the Gaussian curvature. The change of coordinates is then

$$d^{d-1}k = \frac{d\Omega}{|K(k)|},$$

for the intrinsic Gaussian curvature of $S$ at $k$. We may then decompose the formula for density of states:

$$D(\Delta) = \int_{S^{d-1}} \frac{d\Omega}{(2\pi)^d v(k)|K(k)|},$$

so that the number of states with propagation direction through an infinitesimal cross section $d\Omega$ is obtained by introducing the unit cell area (the argument $\Delta$ may be dropped when the resonant set $S(\Delta)$ is understood):

$$D(\hat{v}, \Delta) = \frac{A}{(2\pi)^d v(k)|K(k)|},$$

This can further be treated via the introduction of $m^{-1}(k)$, the inverse of the effective mass tensor of excitations in the periodic media, given by

$$[m^{-1}]_{ij} = \frac{\partial^2 \omega(k)}{\partial k_i \partial k_j}.$$  

In particular, examination of a perturbation $\mathbf{n}(k) \rightarrow \mathbf{n}(k + dk)$ reveals that impulses directed in (perpendicular to) $S$ result in changes in propagation direction remaining in (perpendicular to) $S$. We have the explicit form of the Jacobian

$$J(\mathbf{n}) = \frac{H(k)[I - \mathbf{n} \mathbf{n}]}{v(k)},$$

which equals

$$J_{\mathbf{n}}(k) = \frac{H(k)}{v(k)} = \frac{1}{m^T(k)v(k)},$$

when restricted to $S$. The effective mass tensor is now restricted transversely to $m^T(k)$, a function acting in the tangent space of $k$ where the inverse is generally well defined.

The Gaussian curvature for the implicitly defined resonant set $S$ is then given through

$$K(k) = -\det[J_{\mathbf{n}}(k)],$$

leading to the state density with respect to the cross section:

$$D(\hat{v}, \Delta) = \frac{A}{(2\pi)^d v(k)|K(k)|} \left| \frac{\det[v(k)m^T(k)]}{(2\pi)^d v(k)} \right|,$$

which in the following analysis we find proportional to the scattering cross section in $\hat{v}$ space.

### 2. Long time population of bath modes

We begin with the long time population of bath modes as $t \rightarrow \infty$:

$$C_k \rightarrow -\frac{ge^{-i\omega(k)t}}{\omega(k) - \Delta' + i\gamma/2},$$

where the frequency shift has been absorbed into $\Delta \rightarrow \Delta'$, and $\gamma = \frac{g^2}{\Gamma}$. The bath population is maximized when $\omega(k) = \Delta'$. In particular, the $k$-space cross section is obtained by

$$\sigma(k, \Delta) = \frac{\gamma|C_k|^2}{v(k)} = \frac{1}{v(k)} \frac{g^2\gamma}{(\omega(k) - \Delta')^2 + \frac{\gamma^2}{4}}.$$
The long-range interactions are composed of the waves resonant with the emitter for \(|g| \ll |J|\), and so the emission pattern in the far-field is dominated by the angular distribution induced on \(S(\Delta') \sim S(\Delta)\), obtained through the coordinate transform:

\[
\sigma(\hat{\psi}, \Delta) = \frac{4g^2}{v(k)|K(k)| \gamma},
\]

(B11)

whose scaling with \(v, K\) is shared by the cross sectional density of states \((B8)\). The bath population \(|C_k|^2\) scales as \(1/|K(k)|\).

3. Stationary phase approximation on the resonant level set

To evaluate the asymptotics

\[
G \sim \frac{iA}{2} \int_S (1 + \Theta_\rho(\hat{\rho} \cdot \hat{n})) \frac{\psi(k,r)\psi^*(k,r')}{v(k)(2\pi)^{d-1}} d^{d-1}k,
\]

(B12)

in the long-range geometric optics limit, we make use of the stationary phase argument. Whilst the phase function \(\Theta_\rho\) is also rapidly oscillating with \(\rho\), integration by parts shows that for our choice of \(\epsilon\) small enough, the oscillations due to the exponent are dominant, and the leading contribution to the integral come from points where the exponential argument is at an extremum on \(S\). The integral rapidly converges to a typical exponential integral, and we use the general stationary phase formula on a manifold\([6]\):

\[
\int_S d^{d-1}k f(k) e^{i\lambda(k)} = \frac{2\pi}{\lambda} \alpha^{(d-1)/2} \sum_k f(k_0) e^{i\text{sgn}(H(k_0))\pi/2} e^{i\lambda(k_0)} \sqrt{|\det[H(k_0)]|} + O(\lambda^{(d-3)/2}),
\]

(B13)
as \(\lambda \to \infty\), where the sum is over (assumed) discrete critical points \(k_0\) of \(\theta\) on \(S\). The Hessian \(H\) is also computed as a Hessian on the manifold \(S\). When applied to \(A23\), we have simply the phase function \(\theta(k) = k \cdot \hat{\rho}_\theta\), the height function with respect to \(\hat{\rho}\). According to \([8]\), the Hessian of the height function restricted to \(S\) and at a critical point is simply given as the second fundamental form, \(H(k_0) = -K(k_0)\), so that the denominator is simply the square root of \(|K(k_0)|\). Plugging in \(A23\) gives the final long-range approximation:

\[
G_R = \frac{iA}{2} \frac{\psi(k_0,r)\psi^*(k_0,r')}{v(k_0)\sqrt{|K(k_0)|}} e^{i\text{sgn}(\theta(k_0))\pi/2} \left( \frac{1}{2\pi \rho} \right)^{(d-1)/2} + O(\rho^{(d-3)/2}).
\]

(B14)

We can arrive at the same conclusion by locally parametrising the surface and using the classical stationary phase formula in Euclidean space:

\[
\int_{\mathbb{R}^n} d\rho(t) e^{ig(t)X} = \sum_{t_0} \rho(t_0) e^{i\theta(t_0)X + \text{sgn}(H_t(t_0))\pi/4} \left( \frac{2\pi}{X} \right)^{n/2} + O(X^{-n/2}).
\]

(B15)

4. Asymptotics beyond the critical angle

We approximate the exponential decay rate just of \(\Gamma\) close to but beyond the critical angle corresponding to the caustic using asymptotic analysis. Near a Van Hove singularity or when the dispersion varies rapidly, significant contributions to \(\Omega\) may come from regions away from the resonant level set. However, as in the main text, it is expected that at infinity \(\Omega\) and \(\Gamma\) become equal up to phase. We perform calculations in two dimensions for simplicity, but note that concepts may be extended to higher dimensions. We begin with

\[
\Gamma = A \int_S \frac{d|k|\psi(k,r)\psi^*(k,r')}{v(k)} = \int_S \frac{d|k|\Phi(k)e^{ik\cdot\rho}}{v(k)}.
\]

(B16)

We suppose that \(\rho_\infty\) lies exactly at a simple caustic corresponding to the wavevector \(k_\infty\). By simple we mean that \(K(k_\infty) \neq 0\). If we then consider a unit length direction \(\hat{\rho} = (\hat{\rho}_\infty + \hat{\epsilon})\) just beyond the caustic with \(\epsilon = |\hat{\epsilon}| \ll 1\), we have
\[ \dot{\epsilon} \cdot \mathbf{v}(k_\infty) = 0, \quad \text{(B17)} \]
\[ m^T(k_\infty) \epsilon = 0, \quad \text{(B18)} \]
\[ \rho_\infty \parallel \mathbf{v}(k_\infty). \quad \text{(B19)} \]

That is, \( \rho_\infty \) is parallel to the caustic, and \( \dot{\epsilon} \) travels orthogonally beyond the caustic. As in the main text, the caustic is assumed global such that there is no travelling wave with real vector \( k \) such that \( \dot{\rho} \parallel \mathbf{v}(k) \). Take a periodic parametrisation \( k(t) \) of \( S \) with \( k_\infty = k(t_\infty) \), and consider the extension for \( t \) in the complex plane, with small imaginary part. We seek the stationary point
\[ (\dot{\rho} + \dot{\epsilon}) \cdot k'(t) = 0. \quad \text{(B20)} \]

For small \( \epsilon \), we expect the stationary point to comprise a correspondingly small correcting complex part \( t' \) in \( t \) and thus \( k \). We expand
\[ 0 = \dot{\rho} \cdot k'(t) = \dot{\rho} \cdot k'(t_\infty + t') = (\rho_\infty + \dot{\epsilon}) \left[ k'(t_\infty) + k''(t_\infty)t' + \frac{t'^2}{2} k'''(t_\infty) \right]. \quad \text{(B21)} \]

To lowest order in \( \epsilon \) and \( t' \), we obtain
\[ 0 = \dot{\epsilon} \cdot k'(t_\infty) + \frac{t'^2}{2} \rho_\infty \cdot k'''(t_\infty). \quad \text{(B22)} \]

Under assumption of simple caustic, the third derivative is non-zero and the above equation is balanced for purely imaginary correction \( t' \): \( i \sqrt{\epsilon} \sim t' \). The normalized argument of the exponent at the new stationary point has a real and imaginary part to leading order
\[ \Re[i \dot{\rho} \cdot k(t_\infty + t')] \sim -\epsilon^{3/2}, \quad \text{(B23)} \]
\[ \Im[i \dot{\rho} \cdot k(t_\infty + t')] \sim \dot{\rho} \cdot k(t_\infty). \quad \text{(B24)} \]

Now that the saddle point is obtained, we proceed to deform the contour of integration. As \( k \) is periodic with respect to \( t \), the same period is respected when extended into the complex plane, in a small strip around the \( t \) axis. We may then shift the contour into the complex plane where the boundary contributions cancel due to periodicity when the endpoints have the same imaginary part. We deform to a contour passing through the new stationary point with the real part of the exponent constant and scaling as \( -\epsilon^{3/2} \). Stationary phase then applies to the rapidly varying imaginary part, producing the usual inverse square root contribution in two dimensions. The difference is now that a small real part is contained in the exponential, and we obtain the long-range behaviour:
\[ \Gamma \sim \frac{\exp \left[ -\epsilon^{3/2} \rho + i \dot{\rho} \cdot k_\infty \right]}{\sqrt{\rho}}. \quad \text{(B25)} \]

In two dimensions in particular, a small rotation of \( \rho_\infty \) by an angle \( \epsilon \) corresponds to a perpendicular shift by \( \epsilon \) to leading order, so that the \( \epsilon^{3/2} \) scaling of decay rate is also observed with variance of angle. The above approximation provides the fit in Figure 2 of the main text. In higher dimensions, one may heuristically extend the above line of reasoning and consider the higher dimensional analogue of steepest descent to arrive at a scaling in the generic case:
\[ \Gamma \sim \frac{\exp \left[ -\epsilon^{3/2} \rho + i \dot{\rho} \cdot k_\infty \right]}{\rho^{(d-1)/2}}. \quad \text{(B26)} \]

However, the geometry of saddle points is far more complex in higher dimensions, so that when the saddle point is degenerate, or multiple \( \epsilon \) are available, the scaling of both \( \rho \) and \( \epsilon \) can change, and the above is limited to a heuristic argument when \( d > 2 \). We additionally note that the above analysis is limited to small perturbations, as inherent branch points of the parametrisation will dominate if the perturbations are too large.

Appendix C: Supplementary numerics

1. Green’s function elements in the honeycomb lattice

Whilst the square lattice offers the clearest insight into dynamics, we here present calculations in the honeycomb lattice to further verify our theoretical results, and show that phenomena observed are not strictly limited to the
square lattice of the main investigation. In particular, we show that degeneracies do not typically contribute to long-range interactions, and show how dominantly incoherent interactions can be present in the honeycomb lattice also.

The honeycomb tight-binding lattice (see Figure 7(c)) Hamiltonian exhibits generic Dirac-point degeneracies between two bands. We consider QEs of resonant frequency $\Delta$, coupled to sites in the same sub-lattice. Assuming a nearest neighbor coupling of magnitude $J_1 = J$ and next-nearest-neighbor coupling $J_2 = TJ$, we obtain the off-diagonal term:

$$G = \frac{A}{2} \sum_{\pm} \int_{BZ} \frac{d^2k}{(2\pi)^2} \frac{e^{i\mathbf{k}\cdot\rho}}{\Delta - \omega_{\pm}(\mathbf{k})},$$

(C1)

where $\omega_{\pm}(\mathbf{k})$ gives the eigenenergy in the Bloch wave basis for the upper (lower) band, and can be found for $T = 0$ in (say) [31]. We take $\Delta = 2J$, where the Born-Markov approximation for two spins is accurate [31]. The results with $T = 0$ can be seen in Fig 6(b), whereby the calculation of the exact expression (3) is compared with the real part of (5). As in the main text, we find convergence up to evanescent contributions to the true value of $G$, so that the degeneracy does not contribute. We note that when near a degeneracy, the interactions may become dominantly coherent [31], or even perfectly coherent [16] due to vanishing density of states. In this case, the radius of the tubular region would necessarily shrink to zero as we tune closer to the emitter, so that dominant dynamics require a perturbative expansion localized at the degeneracy, and beyond the Markov approximation.

The investigation corresponding to Figure 2 of the main text is given in Figure 7, where the lattice is now taken with $T = 0.25$. In this case an emitter with $\Delta = -1.5J$ couples to higher-order massive particles, displayed by the vanishing of both curvature and its derivative of $S$ as in Figure 7(a). This results in a higher-order divergence in the usual directional density of states in (c) for the corresponding vertical direction, and a $n^{-1/4}$ power-law decay in (b), where the error in (d) is evanescently decaying as predicted. In (e), the ghost wave decay is confirmed also beyond the global caustics of the honeycomb lattice. Notably, along the $x$ direction of maximum decay, the coherent interactions are vanishingly small compared to incoherent interactions. Near the caustic, this is no longer the case.
FIG. 7. Wave transport in the anisotropic honeycomb lattice, obtained by effecting next-nearest-neighbour interactions with hopping strength $0.25J$ (denoted by the blue lines in (c), whilst nearest neighbour interactions are denoted by orange.) (a) Dispersion relation of the lower branch of the anisotropic honeycomb lattice. The resonant wavevectors corresponding to the directions investigated are given by red dots, whilst the (higher order) massive excitation is given by white dot $k_\infty(k_{\text{HO}})$. (b) Decay of the Green’s function elements along the two directions corresponding the divergences of the transverse effective mass. The approximation $\approx$ is used beyond the bounds of (d), where exact calculations of $\approx$ become too expensive to compute. (c) The directional cross section. (d) The decay of the relative error between our approximation and the exact result. As mentioned in (b), the error is computed as far as the exact computation is possible. (e) Ghost wave decay for the $x$ direction (with maximal decay), and for the lattice vector $a_1$. 
In this work an evanescent wave is understood in the sense of non-propagating: technically, we consider the wave decays superpolynomially.