Riemann boundary value problem with piecewise constant matrix.
Part I. General algorithm

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Abstract

The vector-matrix Riemann boundary value problem for the unit disk with piecewise constant matrix is constructively solved by a method of functional equations. By functional equations we mean iterative functional equations with shifts involving compositions of unknown functions analytic in mutually disjoint disks. The functional equations are written as an infinite linear algebraic system on the coefficients of the corresponding Taylor series. The compactness of the shift operators implies justification of the truncation method for this infinite system. The unknown functions and partial indices can be calculated by truncated systems.

1 Introduction

Solution to the vector-matrix Riemann boundary value problem (\(\mathbb{C}\)-linear conjugation problem in terminology [20]) and estimation of its partial indices is a fundamental problem of complex analysis. Though the general theory concerning solvability and existence of partial indices can be considered as completed [25] various special problems have been constructively solving in applications, see papers beginning from [13]. One can find a review and extensive references devoted to constructive solution to the problem in [21], [2].

The vector-matrix Riemann boundary value problem for the unit disk with piecewise constant matrix attracts special attention as a model fundamental problem. It follows from the Riemann-Hilbert monodromy problem (Hilbert 21st problem) where the partial indices play the crucial role [6]-[9]. Exact solutions to the problem in special cases were discussed in [3], [4], [11], [17], [21], [26].

The aim of this paper is a construction solution to this problem for an arbitrary matrix dimension and an arbitrary finite number of discontinuity points. The main idea of the method is the reduction of the problem to a system of functional equations in complex domain. By functional equations we mean iterative functional equations involving compositions of unknown functions with shifts without integral terms discussed in [20]. The functional equations for analytic functions are constructed for a circular multiply connected domain. The compactness of the corresponding operators is established. Using the method of series we can find the unknown functions analytic in disks by their Taylor series. The coefficients of the Taylor series satisfy an infinite linear algebraic system. The compactness implies justification of the truncation method for this infinite system. The partial indices can be calculated by the truncated systems.
In this first part, a general constructive procedure to determine the partial indices and to construct an approximate formula for the canonical matrix is described. The second part will concern numerical implementation.

Let \( n \) points \( \zeta_k \) divide the unit circle \(|\zeta| = 1\) onto \( n \) arcs \( L_k = (\zeta_{k-1}, \zeta_k)\). It is convenient to assume that \( k \) takes the values \( 1, 2, \ldots, n \) which belong to the group \( \mathbb{Z}_n \) (integers modulo \( n \)) where \( n \equiv 0 \). Let \( D^+ \) be the unit disk \(|\zeta| < 1\) and \( D^- \) be its exterior, i.e., \(|\zeta| > 1\). We introduce a topology on the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) in such a way that the closures of \( D^\pm \) do not intersect.

Let a constant matrix \( G_k \) of dimension \( N \times N \) \((N > 1)\) be given on each arc \( L_k \). The following restrictions are imposed on the matrices \( G_k \) \((k = 1, 2, \ldots, n)\):

**Restriction 1.** Each matrix \( G_k \) is invertible.

**Restriction 2.** Each matrix \( G_k^{-1}G_{k-1} \) has different eigenvalues.

These restrictions reduce the number of different cases which should be separately discussed following the same lines for complete solution to the problem.

By the order at infinity of a scalar function meromorphic in \(|\zeta| > 1\) we call the order of pole if it has a pole at \( \infty \), and the order of zero with the sign minus, if it is analytic at \( \infty \). In particular, the order is equal to zero, if the function has a non-zero value at \( \infty \). By the order of a vector function at infinity we call the maximum of the orders of its components.

The Riemann (\( \mathbb{C} \)-linear conjugation) problem in the theory of boundary value problems is stated as follows. To find a vector function \( \Phi(\zeta) = (\Phi_1(\zeta), \Phi_2(\zeta), \ldots, \Phi_N(\zeta))^T \) which is analytic in \( D^+ \cup D^- \), continuous in the closures of \( D^+ \) and \( D^- \) except the points \( V = \{\zeta_1, \zeta_2, \ldots, \zeta_n\} \) where it is almost bounded (see the definition in the next section), with the following conjugation conditions

\[
\Phi^+(\zeta) = G_k\Phi^-(\zeta), \quad \zeta \in L_k, \quad k = 1, 2, \ldots, n. \tag{1.1}
\]

Moreover, \( \Phi(\zeta) \) has a finite order at infinity. In relation (1.1), \( \Phi^+(\zeta) \) denotes the boundary limit value \( \lim_{\varsigma \to \zeta} \Phi(\varsigma) \), where \(|\varsigma| < 1\) and \( \Phi^-(\zeta) = \lim_{\varsigma \to \zeta} \Phi(\varsigma) \) where \(|\varsigma| > 1\). It is consistent with the positive orientation of the unit circle. Vector functions having the boundary values \( \Phi^+(\zeta) \) and \( \Phi^-(\zeta) \) from the left and the right sides of \(|\zeta| = 1\) can be also considered as two separate functions analytic in \(|\varsigma| < 1\) and in \(|\varsigma| > 1\), respectively.

The following general designations \[11\,22\] are used. Let \( D \) be a domain on the complex plane \( \hat{\mathbb{C}} \) and \( U \) be a finite set of the boundary points on \( \partial D \). Introduce the class of vector functions \( \mathcal{S}_1(D, U) \) analytic in \( D \), continuous in \( D \setminus U \) and almost bounded at \( U \). If the domain \( D = \hat{D} \cup \{\infty\} \) is unbounded, we distinguish the class of vector functions \( \mathcal{S}_1(\hat{D}, U) \) having a finite order at infinity. Moreover, we distinguish one more class \( \mathcal{S}_0(D, U) \) of vector functions which vanish at infinity. Hence, the problem (1.1) has been stated above in the class \( \mathcal{S}_1(\mathbb{D}^+ \cup \mathbb{D}^-, V) \). The class \( \mathcal{S}_1(D, U) \) is introduced for \( N \)-dimensional vector functions. The same designation are used for \( 2N \)-dimensional and scalar functions.
2 Classic theory

2.1 Scalar problem

In the present section, the well–known scalar Riemann (\(C\)–linear conjugation) problem with discontinuous coefficients are summarized due to F. D. Gakhov [11] and N. I. Muskhelishvili [22]. We consider only the case of piecewise constant coefficient. Despite the results are known, they are presented in non–traditional form which is needed in further investigations.

A scalar function \(f(\zeta)\) analytic in \(D\), continuous in its closure except a point \(\zeta_0 \in \partial D\) is called the almost bounded at \(\zeta = \zeta_0\) if \(\lim_{\zeta \to \zeta_0} |\zeta - \zeta_0|^{-\epsilon} f(\zeta) = 0\) for arbitrary positive \(\epsilon\). We introduce the class of functions \(G(D, U)\) which differs from the class \(H(D, U)\) by the almost boundness instead of the boundness at \(U\).

Let \(\nu_k (k = 1, 2, \ldots, n)\) be nonzero numbers, \(b(\zeta)\) and \(b'(\zeta)\) be functions Hölder continuous on \(|\zeta| = 1\) except a finite set of points \(V\) where they have one–sided limits. The scalar Riemann problem is stated as follows. To find \(\Phi \in G_0(D^+ \cup D^-, V)\) with the \(C\)–conjugation condition

\[
\Phi^+(\zeta) = \nu_k \Phi^-(\zeta) + b(\zeta), \quad \zeta \in L_k, \quad k = 1, 2, \ldots, n.
\]  

The problem (2.1) will be discussed simultaneously with the problem

\[
\Omega^+(\zeta) = \nu_k^{-1} \Omega^-(\zeta) + b'(\zeta), \quad \zeta \in L_k, \quad k = 1, 2, \ldots, n.
\]  

We begin the study from two homogeneous problems in \(G_0(D^+ \cup D^-, V)\)

\[
\Phi^+(\zeta) = \nu_k \Phi^-(\zeta), \quad \zeta \in L_k, \quad k = 1, 2, \ldots, n.
\]  

(2.3)

\[
\Omega^+(\zeta) = \nu_k^{-1} \Omega^-(\zeta), \quad \zeta \in L_k, \quad k = 1, 2, \ldots, n.
\]  

(2.4)

Let \(\theta_k \in (-\pi, \pi]\) be the argument of \(\nu_k^{-1} \nu_k^{-1}\), i.e., for a positive \(\rho_k\),

\[
\nu_k \nu_k = \rho_k e^{i\theta_k}.
\]  

(2.5)

Introduce the logarithm

\[
\gamma_k = \frac{1}{2\pi i} \ln \nu_k^{-1} \nu_k^{-1} = \frac{\theta_k}{2\pi} - i \frac{\chi_k}{2\pi} \ln \rho_k,
\]  

(2.6)

where the integer \(\chi_k\) is chosen in such a way that

\[
0 \leq \frac{\theta_k}{2\pi} - \chi_k < 1.
\]  

(2.7)

Along similar lines \(\chi'\) is chosen by the inequality

\[
0 \leq -\frac{\theta_k}{2\pi} - \chi' < 1.
\]  

(2.8)

Note that \(-\theta_k\) is the argument of \(\nu_k \nu_k^{-1}\). Let \(\chi = \sum_{k=1}^n \chi_k\) and \(\chi' = \sum_{k=1}^n \chi'\) be the indices of the problems (2.3) and (2.4), respectively.

\(^1\)Here, the prime is not a derivative.
Divide the set $V$ onto two disjoint subsets $V = \{ \zeta_k \in V : \theta_k \neq 0 \}$ and $W = \{ \zeta_k \in V : \theta_k = 0 \}$ ($V = V \cup W$). It follows from (2.10)–(2.13) that $\chi_k + \chi_k' = -1$ for $\zeta \in V$ and $\chi_k + \chi_k = 0$ for $\zeta \in W$. Let $\#(V)$ be the number of points in $V$, then

$$-(\chi_k + \chi_k) = \#(V). \tag{2.9}$$

According to [11] (p.437) the non–homogeneous problems (2.1) and (2.2) are solvable in $\mathcal{H}_0(\mathbb{D}^+ \cup \mathbb{D}^-, V)$ if and only if $\#(V)$ linearly independent conditions on the right parts are fulfilled

$$\int_{|\tau|=1} [X^+(\tau)]^{-1} \prod_{k=1}^n (\tau - \zeta_k)^{-\gamma_k} b(\tau) \tau^{j-1} d\tau = 0, \quad j = 1, 2, \ldots, -\chi, \tag{2.10}$$

$$\int_{|\tau|=1} [X^- (\tau)]^{-1} \prod_{k=1}^n (\tau - \zeta_k)^{-\gamma_k} b'(\tau) \tau^{j-1} d\tau = 0, \quad j = 1, 2, \ldots, -\chi', \tag{2.11}$$

where

$$X^+_1(\zeta) = \exp \Gamma^+(\zeta), \quad X^-_1(\zeta) = z^{-\gamma} \exp \Gamma^-(\zeta), \tag{2.12}$$

$$\Gamma(\zeta) = \frac{1}{2\pi i} \sum_{k=1}^n \int_{|\tau|=1} \ln \left[ \nu_k \tau^{-\frac{1}{2}} \sum_{m=1}^n (\rho_m - \ln \rho_m) \right] \frac{d\tau}{\tau - \zeta}, \quad \zeta \in \mathbb{D}^+ \cup \mathbb{D}^-. \tag{2.13}$$

If (2.10) is fulfilled, the unique solution of (2.1) has the form

$$\Phi^+(\zeta) = \prod_{k=1}^n (\zeta - \zeta_k)^{-\gamma_k} X^+_1(\zeta) \Psi^+_1(\zeta), \quad \Phi^-(\zeta) = \prod_{k=1}^n \left( \frac{\zeta - \zeta_k}{\zeta} \right)^{\gamma_k} X^-_1(\zeta) \Psi^-_1(\zeta), \tag{2.14}$$

where

$$\Psi^+_1(\zeta) = \frac{1}{2\pi i} \int_{|\tau|=1} \prod_{k=1}^n (\tau - \zeta_k)^{-\gamma_k} [X^+_1(\tau)]^{-1} b(\tau) \frac{d\tau}{\tau - \zeta}. \tag{2.15}$$

The solution of (2.2) has analogous form.

It follows from (2.12)–(2.15) that $\lim_{\zeta \to \zeta_k} \Phi^+(\zeta) = 0$ for $\zeta_k \in V$. If $\zeta_k \in W$, then $\Re \gamma_k = 0$ and $|\zeta - \zeta_k|^{-\gamma_k}$ is bounded as $\zeta$ tends to $\zeta_k$. Moreover, $\Psi_1(\zeta)$ has a logarithmic singularity at $\zeta = \zeta_k$, since $|\zeta - \zeta_k|^{-\gamma_k}$ is bounded at $\zeta = \zeta_k$ and $b(\zeta)$ is discontinuous at $\zeta = \zeta_k$. Similar asymptotic formulae are valid for $\Omega^\mp(\zeta)$. Thus, we have

**Lemma 1.** The non–homogeneous problems (2.1) and (2.2) in $\mathcal{H}_0(\mathbb{D}^+ \cup \mathbb{D}^-, V)$ are solvable if and only if

i) $\#(V)$ linearly independent conditions (2.10) and (2.11) are fulfilled,

ii) $b(\zeta)$ and $b'(\zeta)$ are continuous at the points of $W$.

**Remark.** The condition ii) can be written in the form $b(\zeta_k - 0) = b(\zeta_k + 0)$, $k = 1, 2, \ldots, \#(V)$. This implies that the problems (2.1) and (2.2) in $\mathcal{H}_0(\mathbb{D}^+ \cup \mathbb{D}^-, V)$ are solvable if and only if $n = \#(V) + \#(W)$ conditions are fulfilled.
2.2 N. P. Vekua’s theory

The exhaustive theory of solvability of the vector–matrix problem (1.1) is presented in the book [25]. Though [25] does not contain general exact formulas for \( \Phi(\zeta) \), Vekua’s theory is useful in our constructive method. In the present section, some results from [25] adopted to our investigations are shortly outlined.

Let the invertible matrix function \( G(\zeta) \) be Hölder continuous on the unit circle except the set \( V = \{ \zeta_1, \zeta_2, \ldots, \zeta_n \} \) where \( G(\zeta) \) has one–sided limits. Let \( b(\zeta) \) be Hölder continuous on the unit circle except \( V \) where it has one–sided limits. Consider the non–homogeneous Riemann (\( \mathbb{C} \)–linear conjugation problem) in the class \( \mathcal{H}_0(\mathbb{D}^+ \cup \mathbb{D}^-, V) \)

\[
\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta) + b(\zeta), \ |\zeta| = 1,
\]

and the homogeneous problem in the class \( \mathcal{H}(\mathbb{D}^+ \cup \mathbb{D}^-, V) \)

\[
\Phi^+(\zeta) = G(\zeta)\Phi^-(\zeta), \ |\zeta| = 1.
\]

The order at infinity, \( M \), of a solution of (2.17) is considered here as a control parameter. If for each \( k = 1, 2, \ldots, n \) the matrix \( G(\zeta_k + 0)^{-1}G(\zeta_k - 0) \) has different eigenvalues, the problem (2.17) is solvable for sufficiently large \( M \) (see Section 18, [25]).

By the linear independence of solutions of (2.17) we mean the linear independence with polynomial coefficients. Since for sufficiently large \( M \) the problem (2.17) is solvable, hence among its solutions there exist some with the lowest possible \( M \). Among all solutions, we take one, say \( X_1(\zeta) \). The next solution \( X_2(\zeta) \) has the lowest possible order \( -\kappa_1 \) from all solution of (2.17) linearly independent with \( X_1(\zeta) \). From the remaining solutions, consider all those are linearly independent with \( X_1(\zeta), X_2(\zeta) \). Among these solutions, we take one, say \( X_3(\zeta) \), with the lowest possible order. And so forth. At the \( N \)–step we have a solution \( X_N(\zeta) \) of order \( -\kappa_N \). The matrix \( X(\zeta) = (X_1(\zeta), X_2(\zeta), \ldots, X_N(\zeta)) \) is called the fundamental matrix of the problem (2.17). Vekua [25] shown that the fundamental matrix \( X(\zeta) \) has the following properties. The determinant of \( X(\zeta) \) is not equal to zero for all \( \zeta \in \mathbb{C} \). Among all fundamental matrices, it is possible to pick one for which \( \lim_{\zeta \to \infty} \zeta^m X_{jm}(\zeta) = \delta_{jm} \), where \( \delta_{jm} \) is the Kronecker symbol, \( X_{jm}(\zeta) \) is the \( m \)–th coordinate of the vector function \( X_j(\zeta) \). The numbers \( \kappa_1, \kappa_2, \ldots, \kappa_N \) are called the partial indices. By construction they are ordered as follows \( \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_N \). The sum \( \kappa = \sum_{j=1}^N \kappa_j \) is called the total index (total winding number).

Each solution of (2.17) in the class \( \mathcal{H}(\mathbb{D}^+ \cup \mathbb{D}^-, V) \) is represented in the form

\[
\Phi(\zeta) = X(\zeta)p(\zeta), \ \zeta \in (\overline{\mathbb{D}^+} \cup \overline{\mathbb{D}^-}) \setminus V,
\]

where \( p(\zeta) \) is a polynomial vector function. The partial indices are uniquely determined. In particular, they are independent of the choice of \( X(\zeta) \). These and other fundamental properties of the fundamental matrix and of the partial indices are established in [25]. We describe one of the most important Vekua’s result in the following proposition.

**Theorem 2** (Vekua [25], p. 121). Let \( X(\zeta) \) be a fundamental matrix of the problem (2.10) and \( \kappa_j \) (\( j = 1, 2, \ldots, N \)) be its partial indices. Then, the problem (2.10) is solvable in the class \( \mathcal{H}_0(\mathbb{D}^+, V) \) iff

\[
\int_{|\tau|=1} q(\tau)[X^+(\tau)]^{-1}b(\tau)d\tau = 0,
\]
where
\[ q(\tau) = (q_{-\kappa_1-1}(\tau), q_{-\kappa_2-1}(\tau), \ldots, q_{-\kappa_N-1}(\tau)), \quad (2.20) \]

\[ q_\nu(\tau) \text{ is arbitrary polynomial of degree } \nu \ (q_\nu(\zeta) \equiv 0 \text{ if } \nu < 0). \]

If (2.19) is fulfilled, the general solution of the problem (2.16) has the form
\[ \Phi(\zeta) = \frac{1}{2\pi i} X(\zeta) \int_{|\tau|=1} \frac{[X^+(\tau)]^{-1}b(\tau)}{\tau - \zeta} d\tau + X(\zeta)p(\zeta), \quad (2.21) \]

where
\[ p(\zeta) = (p_{\kappa_1-1}(\zeta), p_{\kappa_2-1}(\zeta), \ldots, p_{\kappa_N-1}(\zeta)). \quad (2.22) \]

Here \( p_\nu(\zeta) \) is an arbitrary polynomial of degree \( \nu \geq 0 \), and \( p_\nu(\zeta) \equiv 0 \) for \( \nu < 0 \).

In this theorem, Vekua’s notations are applied. For instance, the scalar product of vectors is used in the integrand from (2.19).

3 Reduction of the Riemann problem to functional equations

3.1 Reduction of the Riemann problem to the Riemann–Hilbert problem

In the present section we reduce the \( \mathbb{C} \)-linear conjugation problem (1.1) to a boundary value problem. Introduce the vector function \( \phi(z) \) of dimension \( 2N \)
\[ \phi(\zeta) := \left( \begin{array}{c} \Phi(\zeta) \\ \Phi(1/\zeta) \end{array} \right), \quad |\zeta| \leq 1, \quad (3.1) \]

where \( \Phi(1/\zeta) = \overline{\Phi(\zeta)} \). The vector function \( \phi(\zeta) \) is analytic in \( 0 < |\zeta| < 1 \), continuous in \( 0 < |\zeta| \leq 1 \) except the points of \( V \) where it is bounded. Moreover, the components \( \phi_{N+1}(\zeta), \phi_{N+2}(\zeta), \ldots, \phi_{2N}(\zeta) \) of the vector \( \phi(\zeta) \) admit a pole of order \( M \) at the point \( \zeta = 0 \), the components \( \phi_1(\zeta), \phi_2(\zeta), \ldots, \phi_N(\zeta) \) are analytic at \( \zeta = 0 \). The relation (1.1) can be written in the form
\[ \phi(\zeta) = g_k \overline{\phi(\zeta)}, \quad \zeta \in L_k, \ k = 1, 2, \ldots, n, \quad (3.2) \]

where
\[ g_k = \left( \begin{array}{cc} 0 & G_k \\ G_k & 0 \end{array} \right). \quad (3.3) \]

By substitution of (3.1) and (3.3) in (3.2) one can easily see that (3.2) is equivalent to (1.1). Hence, the relation (3.1) defines an isomorphism between the problems (1.1) and (3.2) in the described classes. One can consider the solution of the problem (1.1) as a pair of the \( N \)-dimensional vector functions which constitute by (3.1) a \( 2N \)-dimensional vector function.

Let \( n > 2 \). Consider a bounded circular polygon \( D^+ \) with zeroth angles bounded by touching positively oriented circles \( l_k = \{ t \in \mathbb{C} : |t - a_k| = r_k \} \ (k = 1, 2, \ldots, n). \) We assume
that all points of touching $w_k$ belong to the unit circle. Let $D_k = \{ z \in \mathbb{C} : |z - a_k| < r_k \}$, $D^-$ be the complement of $(\bigcup_{k=1}^{n} D_k) \cup D^+$ to the extended complex plane. Divide each circle $l_k$ onto two arcs $l_k^+ = \partial D^+ \cap l_k$ and $l_k^- = \partial D^- \cap l_k$. Let $f : \mathbb{D}^+ \to D^+$ be a conformal mapping of $\mathbb{D}^+$ onto $D^+$. The parameters $a_k$ and $r_k$ are chosen in such a way that $l_k^+ = f(L_k)$. In particular, the discontinuity set $V$ of the problem (3.2) is transformed onto the touching points of the circles $l_k$, the set $W = f(V) = \{ w_1, w_2, \ldots, w_n \}$, where $w_k = f(\zeta_k)$. The conformal mapping

$$z = f(\zeta)$$

(3.4)
can be presented in the form of the Schwarz-Christoffel integral [23]. Without loss of generality one can assume that $0 \in D^+$ and take $f(0) = 0$. The symmetry principle implies that

$$z = \left[ f \left( \frac{1}{\zeta} \right) \right]^{-1}$$

(3.5)
is the conformal mapping of $\mathbb{D}^-$ onto $D^-$ which maps $\zeta_k$ onto $w_k$.

**Remark 3.** In the case $n = 1$, the domain $D^+$ coincides with the unit disk, i.e., $f(\zeta) = \zeta$. In the case $n = 2$, $D^+$ is the strip $-1 < \text{Im} z < 1$, i.e., $f(\zeta)$ is a composition of the logarithm and Möbius transformations. The study of these cases is much easier than the general case $n > 2$. Below, we consider only the case $n > 2$.

Applying the conformal mapping $z = f(\zeta)$ to (3.2) we arrive at the Riemann–Hilbert
problem
\[ \varphi(t) = g_k \varphi(t), \quad t \in l_k^+, \quad k = 1, 2, \ldots, n, \quad (3.6) \]
with respect to the vector function \( \varphi = \phi \circ f^{-1} \) analytic in \( D^+ \), continuous in \( D^+ \setminus W \) and bounded at the points of \( W \). Moreover, the components \( \varphi_{N+1}(z), \varphi_{N+1}(z), \ldots, \varphi_{2N}(z) \) of the vector \( \varphi(z) \) admit a pole of order \( M \) at the point \( z = 0 \).

The inversion \( z \mapsto \frac{1}{z} \) transforms the arcs \( l_k^+ \) onto \( l_k^- \). Consider the auxiliary boundary value problem
\[ \varphi(t) = g_k \varphi(t), \quad t \in l_k^-, \quad k = 1, 2, \ldots, n, \quad (3.7) \]
with respect to the vector function \( \varphi \in \mathcal{H}(D^+, W) \). The problems (3.6) and (3.7) are related in the following way. Let \( \psi(z) = \varphi\left(\frac{1}{z}\right), \quad z \in D^+ \). Then, it follows from (3.7) that
\[ \psi(t) = g_k^{-1} \psi(t), \quad t \in l_k^+, \quad k = 1, 2, \ldots, n. \quad (3.8) \]

In accordance with [25], the problem (3.8) is called the accompanying problem to (3.6). Since \( g_k g_k = I \), \( g_k^{-1} \) in (3.8) can be replaced with \( g_k \).

### 3.2 Reduction of the Riemann–Hilbert problem to the R–linear problem

Consider the relations (3.6) and (3.7) as one boundary value problem with respect to \( \varphi \in \mathcal{H}(D^+ \cup D^-, W) \) except the components \( \varphi_{N+1}(w), \varphi_{N+1}(w), \ldots, \varphi_{2N}(w) \) which admit a pole of order \( M \) at the point \( z = 0 \).

Consider the matrix \( g_k \) defined by (3.3). Introduce the matrix (compare to [19])
\[ \lambda_k = \begin{pmatrix} I & -G_k \\ -iI & -iG_k \end{pmatrix}, \]
where \( I \) is the identity matrix. The matrix \( \lambda_k \) is invertible, since \( \det \lambda_k = -2i \det G_k \neq 0 \). The inverse matrix has the form
\[ \lambda_k^{-1} = \frac{1}{2} \begin{pmatrix} I & iI \\ -G_k^{-1} & iG_k^{-1} \end{pmatrix}. \]

By direct calculations one can check the following equality
\[ \lambda_k^{-1} \lambda_k = -g_k. \quad (3.9) \]

Using (3.9) we write (3.6) and (3.7) in the form
\[ \overline{\lambda_k \varphi(t)} + \lambda_k \varphi(t) = 0, \quad t \in l_k, \quad k = 1, 2, \ldots, n, \quad (3.10) \]
where \( l_k = l_k^+ \cup l_k^- \cup \{w_{k-1}\} \cup \{w_k\} \).

The boundary value problem (3.10) can be written as the \( \mathbb{R} \)–linear conjugation problem
\[ \overline{\lambda_k \varphi(t)} = \overline{\lambda_k \varphi(t) - \lambda_k \varphi_k(t)}, \quad t \in l_k, \quad k = 1, 2, \ldots, n, \quad (3.11) \]
where \( \varphi_k \in \mathcal{H}(D_k, \{w_{k-1}, w_k\}) \).
We now demonstrate that the problems (3.10) and (3.11) are equivalent. If \( \varphi(z) \) and \( \varphi_k(z) \) satisfy (3.11), then \( \varphi(z) \) satisfies (3.10). It is easily seen by taking the real part from (3.11). Conversely, let \( \varphi(z) \) satisfies (3.10). Then, we can construct \( \varphi_k(z) \) solving the simple Riemann-Hilbert problem for the disk \( D_k \)

\[
2\text{Im} \overline{\lambda_k} \varphi_k(t) = \text{Im} \overline{\lambda_k} \varphi(t), \quad t \in l_k.
\]

(3.12)

One can see that (3.12) is decomposed onto \( 2N \) scalar problems with respect to the components of the vector function \( \overline{\lambda_k} \varphi_k(z) \) and hence are written explicitly through \( \varphi(z) \). Then \( \varphi_k(z) \) is uniquely determined by \( \varphi(z) \) up to an additive constant vector (for details see [11]). Therefore, we have proved that the problems (3.10) and (3.11) are equivalent. It is worth noting that \( \varphi_k(z) \) are almost bounded at \( w_k \) and \( w_{k-1} \), e.g., they can have logarithmic singularities at these points. This follows from solution to the problem (3.12) with respect to \( \varphi_k(z) \). The right hand part of (3.12) can have finite jumps at \( w_k \) and \( w_{k-1} \). Hence, \( \varphi_k(z) \) are almost bounded at \( w_k \) and \( w_{k-1} \).

Multiplying (3.11) by \( \overline{\lambda_k}^{-1} \) and using (3.9) we obtain the following \( \mathbb{R} \)-linear problem

\[
\varphi(t) = \varphi_k(t) + g_k \overline{\varphi_k(t)}, \quad t \in l_k, \; k = 1, 2, \ldots, n.
\]

(3.13)

### 3.3 Reduction of the \( \mathbb{R} \)-linear problem to functional equations

We now proceed to reduce the problem (3.13) to a system of functional equations.

Introduce the inversion with respect to the circle \( l_k \)

\[
z_k^* = \frac{r_k^2}{z - a_k} + a_k.
\]

(3.14)

It is known that if \( \varphi_k(z) \) is analytic in \( |z - a_k| < r_k \), then \( \varphi_k(z_k^*) \) is analytic in \( |z - a_k| > r_k \). The functions \( \varphi_k(z) \) and \( \overline{\varphi_k(z_k^*)} \) have the same boundary behavior on \( l_k \) (continuity, boundness), since \( t_k = t \) on \( l_k \).

The orientation of each \( l_k \) is counter clockwise. The path along the curve \( l \) is defined as follows. It is assumed that the curve \( l \) is closed and consists of the sequent arcs \( l_1^+, l_2^+, \ldots, l_n^+, l_1^-, l_2^-, \ldots, l_n^- \). Then, the curve \( l \) divides the complex plane onto two domains \( \mathcal{D}^+ = \bigcup_{k=1}^n D_k \) and \( \mathcal{D}^- = \mathcal{D}^+ \cup \mathcal{D}^- \).

We now proceed to use Sochocki’s formula\(^2\) for the closed curve \( l \) separating the domains \( \mathcal{D}^\pm \). Let \( f(t) \) be a scalar function Hölder continuous separately in each \( l_k \) (\( k = 1, 2, \ldots, n \)) but having in general a jump at the touching points \( w_k \) when \( t \) passes from \( l_k \) to \( l_{k+1} \). For fixed \( m \), introduce the operators

\[
S_m f(z) = \frac{1}{2\pi i} \int_{t_m} \frac{f(t)}{t - z} dt, \quad z \in \bigcup_{k=1}^n D_k \quad (m = 1, 2, \ldots, n)
\]

(3.15)

The function \( S f(z) = \sum_{m=1}^n S_m f(z) \) is analytic in the domain \( \mathcal{D}^+ = \bigcup_{k=1}^n D_k \), continuous in its closure except the touching points where it is almost bounded, more precisely, can have

\(^2\)also known as Sokhotsky’s and Plemelj’s formulae, see for a historical review \(^22\)

\(^3\)z can belong to \( D^+ \cup D^- \)
logarithmic singularities [11], [22]. The function \( Sf(z) \) satisfies Sochcki’s formulae

\[
Sf(t) = \lim_{z \to t} Sf(z) = \frac{1}{2} f(t) + \sum_{m=1}^{n} \frac{1}{2\pi i} \int_{l_m} \frac{f(\tau)}{\tau - t} \, d\tau, \quad t \in \bigcup_{m=1}^{n} l_m \setminus W.
\]  

Introduce the Banach space \( \mathfrak{B}(l, W) \) of vector function \( h(t) = (h_1(t), h_2(t), \ldots, h_{2N}(t)) \) H"older continuous on each \( l_k \) with the exponent \( 0 < \mu \leq 1 \) endowed with the norm

\[
\|h\| = \left( \sum_{j=1}^{2N} \|h_j\|^2 \right)^{1/2}, \quad \|h_j\| = \max_{t \in l} |h(t)| + \max_{1 \leq k \leq n} \sup_{t_1 \neq t_2} \frac{|h(t_1) - h(t_2)|}{|t_1 - t_2|^{\mu}}.
\]

Here, it is assumed that the limit values of \( h(t) \) at \( w_k \in W \) are the same if \( t \to w_k \) with \( t \in l_k \), but they can be different if \( t \to w_k \) with \( t \in l_k \cup l_{k+1} \). Thus, the space \( \mathfrak{B}(l, W) \) can also be considered as the direct sum of the spaces of H"older continuous vector functions. Vector functions \( h \in \mathfrak{B}(l, W) \) analytically continued into all disks \( D_k \) generate a closed subspace of \( \mathfrak{B}(l, W) \) which we denote by \( \mathfrak{B}(\bigcup_{k=1}^{n} D_k, W) \). The operator \( S \) is bounded in \( \mathfrak{B}(\bigcup_{k=1}^{n} D_k, W) \) [10]. Instead of \( \mathfrak{B}(l, W) \) one can consider the operators in the space \( L_2(l) \) [10].

The operator \( S \) transforms a function \( f(t) \) into a function analytic in \( \bigcup_{k=1}^{n} D_k \). Apply the operator \( S \) to the conjugation condition (3.13). First, we have

\[
S\varphi(z) = \omega(z), \quad z \in D_k, \quad k = 1, 2, \ldots, n,
\]  

where

\[
\omega(z) = \left( \frac{\gamma_{s-1}}{0} \right) + \sum_{s=1}^{M} \left( 0 \right) z^{-s}.
\]  

Here, \( \gamma_s \) is a constant vector of dimension \( N \). The numeration of \( \gamma_s \) and the complex conjugation \( \overline{\gamma_s} \) is taking for unification of the formulæ given below. The vector-function \( \omega(z) \) is calculated by residues of \( \varphi(z) \) at \( z = 0 \). More precisely, the components \( \omega_{N+1}(z), \omega_{N+1}(z), \ldots, \omega_{2N}(z) \) of the vector function \( \omega(z) \) can have poles of order \( M \) at the point \( z = 0 \), since \( \omega(z) \) has the same behavior as \( \varphi(z) \) at zero. We now fix the vector function \( \omega(z) \). Equations on the undetermined vectors \( \gamma_s \) (\( s = -1, 0, \ldots, M \)) will be written latter.

Using the properties of the Cauchy integral we obtain \( S\varphi_k(z) = \varphi_k(z) \), \( z \in D_k \), and \( S\varphi_m(z) = 0 \), \( z \in D_k \), for \( m \neq k \). Therefore, applying the operator \( S \) to (3.18) we arrive at the system of integral equations

\[
\varphi_k(z) = \sum_{m=1}^{n} g_m \frac{1}{2\pi i} \int_{l_m} \frac{\varphi_m(t)}{t - z} \, dt + \omega(z), \quad z \in D_k \quad (k = 1, 2, \ldots, n),
\]  

where the matrix \( g_m \) is multiplied by the corresponding vector expressed in terms of the Cauchy integral.

The integrals in (3.19) can be calculated by residues. The vector-function \( \overline{\varphi_m(t)} = \overline{\varphi_m(t_m)} \) (see (3.14)) can be analytically continued into \( |z - a_m| > r_m \). Hence,

\[
\frac{1}{2\pi i} \int_{l_m} \frac{\varphi_m(t)}{t - z} \, dt = \begin{cases} \frac{\varphi_k(z_k^*) - \varphi_k(a_k)}{z_k - a_k}, & \text{for } m = k, \\ 0, & \text{for } m \neq k. \end{cases}
\]
Substitution of (3.20) into (3.19) yields the following system of functional equations with respect to \( \varphi_k(z) \)

\[
\varphi_k(z) = \sum_{m \neq k} g_m \varphi_m(z^{*}_{(m)}) + \omega(z), \quad z \in \overline{D_k} \setminus \{w_{k-1}, w_k\} \quad (k = 1, 2, \ldots, n). \tag{3.21}
\]

Here, the constant vectors \( \varphi_k(a_k) \) are included for shortness in the undetermined vectors \( \gamma_{-1} \) and \( \gamma_0 \) of \( \omega(z) \). Let \( \varphi_k(z) \) be a solution of (3.21). Then the vector function \( \varphi(z) \) is given by the formula

\[
\varphi(z) = \sum_{m=1}^{n} g_m \varphi_m(z^{*}_{(m)}) + \omega(z), \quad z \in D^+ \cup D^- \tag{3.22}
\]

The special structure (3.3) of the matrix \( g_k \) yields the decomposition of the system (3.21)

\[
\Psi_k(z) = \sum_{m \neq k} G_m \varphi_m(z^{*}_{(m)}) + \gamma_{-1}, \tag{3.23}
\]

\[
\tilde{\Psi}_k(z) = \sum_{m \neq k} G_m^{-1} \Psi_m(z^{*}_{(m)}) + \Omega(z), \quad z \in \overline{D_k} \setminus w_k, w_{k+1} \quad (k = 1, 2, \ldots, n) \tag{3.24}
\]

where the \( 2N \)-dimensional vector functions \( \varphi_k(z) \) are presented through two \( N \)-dimensional vector functions \( \Psi_k(z) \) and \( \tilde{\Psi}_k(z) \)

\[
\varphi_k(z) = \begin{pmatrix} \Psi_k(z) \\ \tilde{\Psi}_k(z) \end{pmatrix} \tag{3.25}
\]

and

\[
\Omega(z) = \sum_{s=0}^{M} \gamma_s z^{-s}. \tag{3.26}
\]

The vector functions \( \tilde{\Psi}_k(z) \) can be eliminated from equations (3.23)–(3.24)

\[
\Psi_k(z) = \sum_{m \neq k} \sum_{\ell \neq m} G_m G_{\ell}^{-1} \Psi_\ell(z^{*}_{(m)}) + F_k(z), \quad z \in \overline{D_k} \setminus w_k, w_{k+1} \quad (k = 1, 2, \ldots, n) \tag{3.27}
\]

where

\[
\tilde{z}^{*}_{(m)} = \left(z^{*}_{(m)}\right)^{*}_{(\ell)} = \frac{r^2_{\ell}(z - a_m)}{r^2_{m} - (a_\ell - a_m)(z - a_m)} + a_\ell
\]

denotes the compositions of inversions with respect to the \( m \)-th and the \( \ell \)-th circles and

\[
F_k(z) = \gamma_{-1} + \sum_{m \neq k} G_m \Omega\left(z^{*}_{(m)}\right) = \gamma_{-1} + \sum_{s=0}^{M} \sum_{m \neq k} G_m \gamma_s \left(z^{*}_{(m)}\right)^{-s}. \tag{3.28}
\]

One can see that each \( F_k(z) \) is the linear combination of the functions \( \{1, (z^{*}_{(m)})^{-s}\} \) with the coefficients consisting of the components of \( \gamma_s \). Hence, \( F_k(z) \) is analytic in \( D_k \).
Write equations (3.27) in the form
\[
\Psi_k(z) = G_{k-1}G_k^{-1}\Psi_k(z_{(k,k-1)}) + G_{k+1}G_k^{-1}\Psi_k(z_{(k,k+1)}) + \sum_{(m,\ell) \in \mathcal{N}_k} G_mG^{-1}_\ell\Psi_\ell(z^{*}_{(\ell m)}) + F_k(z),
\]
where \( k - 1 = n \) for \( k = 1 \). The set \( \mathcal{N}_k \) consists of the pairs \((m, \ell)\) with not equal elements \( m, \ell \in \mathbb{Z}_n \); the pairs with \( m = k \) and two pairs \((k, k-1), (k, k+1)\) are also eliminated. The shifts \( z^{*}_{(\ell m)} \) are Möbius transformations of elliptic type \([5]\). For \((m, \ell) \in \mathcal{N}_k\), every \( z^{*}_{(\ell m)} \) transforms the disks \( |z - a_k| \leq r_k \) and \( |z - a_\ell| \leq r_\ell \) into the disk \( |z - a_\ell| < r_\ell \). The map \( z^{*}_{(\ell m)} \) has a unique attractive fixed point \( z_a \) in \( |z - a_\ell| < r_\ell \) \([5, 20]\).

The Möbius transformation \( z^{*}_{(k,k-1)} \) and \( z^{*}_{(k,k+1)} \) are of parabolic type \([5]\) with the neutral fixed points \( w_{k-1} \) and \( w_k \), respectively, since the \((k - 1)\)th, \( k\)th circles touch at \( w_{k-1} \) and \((k + 1)\)th, \( k\)th at \( w_k \).

Equations (3.29) can be considered in the Hardy type space \( \mathcal{H}_2(\bigcup_{k=1}^n D_k) \) associated with \( L_2 \). Let \( \Psi_k(z) \) belongs to the Hardy space \( \mathcal{H}_2(D_k) \), i.e., \( \Psi_k(z) \) is analytic in \( D_k \) and the norm is defined by formula
\[
\|\Psi_k\|_2 = \sup_{0 < r < r_k} \left[ \frac{1}{2\pi} \int_0^{2\pi} |\Psi_k(re^{i\theta})|^2 d\theta \right]^{1/2}.
\]
A function \( \Psi(z) \) belongs to \( \mathcal{H}_2(\bigcup_{k=1}^n D_k) \) if \( \Psi \in \mathcal{H}_2(D_k) \) for all \( k = 1, 2, \ldots, n \) and \( \|\Psi\|_2 = \max_{1 \leq k \leq n} \|\Psi\|_2 \). The norm of the vector functions is introduced by the norms of components as \( \|\Psi\|_2 = \left( \sum_{j=1}^N \|\Psi_j\|_2^2 \right)^{1/2} \). It was shown in \([20]\) that the operator \( \sum_{(m,\ell) \in \mathcal{N}_k} G_mG^{-1}_\ell\Psi_\ell(z^{*}_{(\ell m)}) \) is compact in the spaces \( \mathcal{F}(\bigcup_{k=1}^n D_k, W) \) and \( \mathcal{H}_2(\bigcup_{k=1}^n D_k) \).

**Theorem 4.** Let \( \Psi \in \mathcal{H}_2(\bigcup_{k=1}^n D_k) \) where \( \Psi(z) = \Psi_k(z) \) in \( D_k \), i.e., the vector function \( \Psi(z) \) is separately defined in all the disks \( D_k \). The operators \( A \Psi(z) = G_{k-1}G_k^{-1}\Psi_k(z^{*}_{(k,k-1)}) \) and \( B \Psi(z) = G_{k+1}G_k^{-1}\Psi_k(z^{*}_{(k,k+1)}) \), \( z \in D_k \) \((k = 1, 2, \ldots, n)\) are compact in \( \mathcal{H}_2(\bigcup_{k=1}^n D_k) \).

Proof. The operators \( A \) and \( B \) consist of the compositions of the multiplications by matrices and of the shift operators. The multiplication operators are bounded. It is sufficient to investigate the scalar shift operators. For definiteness, consider the operator
\[
F f(z) = f(z^{*}_{(k,k-1)}).
\]

The general form of the map \( z^{*}_{(k,k-1)} \) can be reduced by translations and rotations to the case \( w_k = 0, a_k = r_1 > 0, a_{k-1} = -r_2 < 0 \). The compactness property of \( F \) does not change after such transformations. In this case,
\[
z^{*}_{(k,k-1)} = \frac{z}{1 + \left( \frac{1}{r_1} + \frac{1}{r_2} \right) z} =: \alpha(z).
\]

Compactness of the operator (3.31) follows from the Hilbert-Schmidt Theorem for composition operators \([24]\, page 26\). It is sufficient to check the boundedness of the integral
\[
\int_{-\pi}^{\pi} \frac{d\theta}{1 - \frac{1}{r_1}\alpha[r_1(1 + e^{i\theta})]} = 2\pi \frac{r^2 + 3r + 2 + \frac{1}{\sqrt{4r^2 + 12r + 13}}}{r^2 + 3r + 3},
\]
where \( r = \frac{r_1}{r_2} \). The integral (3.33) is computed with Mathematica®. The theorem is proved.

4 Solution to functional equations and calculation of partial indices

The space \( H_2(\cup_{k=1}^{n} D_k) \) of scalar functions is isomorphic to the space \( h_2 \) of the sequences \( \{\psi_{kj}\}_{k=1}^{n} \) with the norm

\[
\|\{\psi_{kj}\}\|_2 = \sum_{k=1}^{n} \left( \sum_{j=0}^{\infty} |\psi_{kj}|^2 \right)^{1/2}
\]

where the Taylor expansion of \( \Psi_k(z) \) is used

\[
\Psi_k(z) = \sum_{j=0}^{\infty} \psi_{kj}(z - a_k)^j, \quad |z - a_k| < r_k.
\]  

(4.1)

Here, \( \psi_{kj} = \left. \frac{1}{j!} \psi_{kj}(z) \right|_{z=a_k} \) denotes the derivative of order \( j \) of the function \( \Psi_k \) at \( z = a_k \).

The spaces of vector functions are introduced by coordinates. It is worth noting that \( H_2(\cup_{k=1}^{n} D_k, W) \subset H_2(\cup_{k=1}^{n} D_k) \) and given by (3.28) the terms \( F_k(z) \) in equations (3.29) belong to \( H_2(\cup_{k=1}^{n} D_k, W) \). It will be shown later that any solution in \( H_2(\cup_{k=1}^{n} D_k) \) with such \( F_k(z) \) belongs to the space \( H_2(\cup_{k=1}^{n} D_k, W) \).

Using the general properties of equations with compact operators in the Hilbert space \( h_2 \) [11, 12] we can develop the following constructive algorithm to solve the system (3.29) and to calculate the partial indices of the corresponding Riemann-Hilbert problem. We have

\[
\Psi_k(z^*_l) = \sum_{j=0}^{\infty} \psi_{lj} \left[ \frac{r_l^2(z - a_m)}{r_m^2 - (a_l - a_m)(z - a_m)} \right]^j, \quad |z - a_k| < r_k.
\]  

(4.2)

The discrete form of the functional equations is obtained after substitution of (4.1)–(4.2) into (3.29)

\[
\sum_{j=0}^{\infty} \psi_{kj}(z - a_k)^j = \sum_{m \neq k, \ell \neq m} G_m G_\ell^{-1} \sum_{j=0}^{\infty} \psi_{lj} \left[ \frac{r_l^2(z - a_m)}{r_m^2 - (a_l - a_m)(z - a_m)} \right]^j + \gamma - 1
\]

\[
+ \sum_{s=0}^{M} \sum_{m \neq k} G_m \gamma_s \left( \frac{r_m^2}{z - a_m} + \frac{a_m}{z - a_m} \right)^{-s}, \quad z \in D_k \setminus \{w_k, w_{k+1}\} \quad (k = 1, 2, \ldots, n).
\]

(4.3)

All the series in (4.3) can be expand in \( (z - a_k)^j \) in the disk \( |z - a_k| < r_k \). Comparison of the coefficients on \( (z - a_k)^j \) yields an infinite system of linear algebraic equations on the coefficients \( \psi_{lj} \) and parameters of \( \omega(z) \). The infinite system can be solved by the truncation
method \cite{12}. Taking a partial sum of Taylor series (4.1) with $Q$ first items and collecting the coefficients on the same powers of $z - a_k$ we arrive at the finite system on $\psi_{kj}$

$$
\psi_{kj} = \sum_{m \neq k} \sum_{l \neq m}^{Q} \mu_{ml}^{(kij)} G_{m} G_{l}^{-1} \psi_{li} + F_{k}^{(j)}(a_k), \quad k = 1, \ldots, n, \; j = 0, \ldots, Q, \tag{4.4}
$$

where the scalar

$$
\mu_{ml}^{(kij)} = \frac{1}{j!} \left[ \left( \frac{r_{m}^{2}(z - a_{m})}{r_{m}^{2} - (a_{l} - a_{m})(z - a_{m})} \right)^{i} \right]^{(j)} \bigg|_{z = a_{k}}, \tag{4.5}
$$

and the vectors

$$
F_{k}^{(j)}(a_k) = \frac{1}{j!} \left[ \gamma_{-1} + \sum_{s=0}^{M} \sum_{m \neq k}^{n} G_{m} \gamma_{s} \left( \frac{r_{m}^{2}}{z - a_{m}} + a_{m} \right)^{-s} \right]^{(j)} \bigg|_{z = a_{k}}. \tag{4.6}
$$

The vectors $\gamma_{s}$ can be also considered as unknowns. Then, (4.4) can be considered as a homogeneous system of equations on $\Psi = \{\psi_{kj}\}_{k=1,\ldots,n; j=0,\ldots,Q}$ and $\gamma = \{\gamma_{-1}, \gamma_{0}, \cdots \gamma_{M}\}$. The left hand part of (4.4) determines a compact operator $A$ in the space $h_{2}$. Then, the system (4.4) can be shortly written as the operator equation in $h_{2}$

$$
\Psi = A \Psi + F \gamma, \tag{4.7}
$$

where the finite rank operator $F$ is determined by (4.6). The truncated system is denoted as

$$
\Psi_{Q} = A_{Q} \Psi_{Q} + F \gamma, \tag{4.8}
$$

where $A_{Q}$ is the truncated matrix of the dimension $nN(Q+1) \times nN(Q+1)$, $\Psi_{Q} = \{\psi_{kj}\}_{k=1,\ldots,n; j=0,\ldots,Q}$ is a column vector with $nN(Q+1)$ entries. Solution to (4.8) approximates solution to the infinite system (4.7) for sufficiently large $Q$ \cite{11,12}. This fact is summarized in the following theorem which relates solution to the Riemann-Hilbert problem with linear systems.

**Theorem 5.** Let $G_{k}$ be invertible matrices ($k \in \mathbb{Z}_{n}$, $n > 2$) and each matrix $G_{k}^{-1}G_{k-1}$ has different eigenvalues. The problem (1.1) in $\mathcal{S}(\mathbb{D}^{+} \cup \mathbb{D}^{-}, V)$ with the prescribed order at infinity $M$ is solvable if and only if the linear algebraic system (4.7) on the $N$–vectors $\psi_{kj}$ and $\gamma_{s}$ ($s = -1, 0, \ldots, M$) is solvable.

Solvability of the system (4.7) is reduced to solvability of the system (4.8) for sufficiently large $Q$. Its solution can be approximated by solution to the system (4.8).

Let the general solution to the finite system (4.8) depend on the arbitrary vectors $\gamma_{s}$ ($s = r, r + 1, \ldots, M$), i.e., the rank of the system (4.4) is equal to $r+1$. Then the problem (1.1) has $M-r+1$ linearly independent solutions approximated by polynomial corresponding to the truncated series (4.4).

Applying directly the definition of partial indices described in Sec.2.2 and Theorem 5 to the problem (1.1) one can calculate the partial indices. Considering $M$ as a control parameter we investigate the linear system (4.8) for sufficiently large $Q$. 

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Let $M = 0$. If the system (4.7) has non-zero solutions, then $\kappa_1 = 0$ and the first column $X_1(\zeta)$ of the fundamental matrix $X(\zeta)$ is the constant vector $\gamma_{-1}$ in $|\zeta| \leq 1$ and $\gamma_0$ in $|\zeta| \geq 1$. Such a case is possible since (1.1) is reduced to

$$\gamma_{-1} = G_k \gamma_0, \quad k = 1, 2, \ldots, n$$

(4.9)

with $\gamma_0 = (1, 0, \ldots, 0)^T$. Among all matrices $G_k$ at least two ones are different, since $n > 2$ and $G_k^{-1}G_{k-1}$ has different eigenvalues. Equations (4.9) on $\gamma_{-1}$ and $\gamma_0$ can have non-trivial solutions. For instance, $G_k = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k = 1, 2, \ldots, n$ and $\gamma_{-1} = \gamma_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ satisfy (4.9).

Here, one can see the principal difference between the scalar ($N = 1$) and vector ($N > 1$) cases, because for $n > 1$ the scalar problem (1.1) always has only zero solution.

If the system (4.8) has only zero solution, we increase $M$ up to 1 and consider the corresponding system with $\gamma_{-1}$, $\gamma_0$ and $\gamma_1$. If this system has non-zero solutions, the partial index $\kappa_1 = -1$ and $X_1(\zeta)$ can be approximately found. If the system (4.8) with $M = 1$ has only zero solution, we again increase $M$ and so forth. Therefore, the partial indices are precisely calculated by the ranks of the system (4.8) for different $M$. It follows from the general theory this process of increasing $M$ will be always finite.

5 Discussion

In the present paper, the vector-matrix Riemann boundary value problem ($\mathbb{C}$–linear conjugation problem) for the unit disk with piecewise constant matrix has been solved in Hölder and Hardy spaces by the method of functional equations.

First, the conformal mapping (3.4) $z = f(\zeta)$ of the unit disk onto a circular polygon is not constructed here. It is worth noting that this problem differs from the classic problem of inverse mapping. Our problem is easier, since the points $\zeta_k$ are known. What is more, we suggest that any Christoffel-Schwarz transformation of the unit disk onto a circular polygon bounded by externally touching circles, where all touching points lie on one circle, can be taken as $f(\zeta)$.

The solution of the problem can be found in the form of the series. The second part will contain numerical examples of the above constructive scheme.

Acknowledgments

I am grateful to Prof. Frank-Olme Speck for valuable discussion during my visit in 2006 supported by Centro de Matemática e Aplicações of Instituto Superior Técnico, Lisboa. I thank Ekaterina Pesetskaya and Gia Giorgadze for fruitful discussions and preliminary numerical examples not presented in this paper.

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