Topological Sectors and a Dichotomy in Conformal Field Theory

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Abstract

Let $\mathcal{A}$ be a local conformal net of factors on $S^1$ with the split property. We provide a topological construction of soliton representations of the $n$-fold tensor product $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$, that restrict to true representations of the cyclic orbifold $(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^\mathbb{Z}_n$. We prove a quantum index theorem for our sectors relating the Jones index to a topological degree. Then $\mathcal{A}$ is not completely rational iff the symmetrized tensor product $(\mathcal{A} \otimes \mathcal{A})^{\text{flip}}$ has an irreducible representation with infinite index. This implies the following dichotomy: if all irreducible sectors of $\mathcal{A}$ have a conjugate sector then either $\mathcal{A}$ is completely rational or $\mathcal{A}$ has uncountably many different irreducible sectors. Thus $\mathcal{A}$ is rational iff $\mathcal{A}$ is completely rational. In particular, if the $\mu$-index of $\mathcal{A}$ is finite then $\mathcal{A}$ turns out to be strongly additive. By [31], if $\mathcal{A}$ is rational then the tensor category of representations of $\mathcal{A}$ is automatically modular, namely the braiding symmetry is non-degenerate. In interesting cases, we compute the fusion rules of the topological solitons and show that they determine all twisted sectors of the cyclic orbifold.

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1 Introduction

The main theme of this paper, topological sectors in Conformal Quantum Field Theory, has been the subject of interest by the authors for different reasons.

One motivation came from the study of the sector structure in the cyclic orbifold associated with rational models, where the operator algebraic methods go beyond the analysis by infinite Lie algebras, in particular by using the structure results in [31]. As we shall see, a quantum index theorem by the Jones index captures an essential part of information here.

Another motivation came in relation to irrational Conformal Field Theory, where most of the underlying structure is still to be uncovered. Also in this case, the algebraic approach is essential and leads to a surprising finite/uncountable dichotomy concerning the set of all irreducible sectors in the rational/irrational case.

Before stating further results and consequences of our work, and explaining in more detail the above mentioned issues, we recall the notion of complete rationality [31] which is at the basis of our analysis.

In all the present paper we shall deal with diffeomorphism covariant (irreducible) local nets of von Neumann algebras on $S^1$, called conformal nets, and we explain our results in this framework, although weaker assumptions would be sufficient.

Complete rationality. Let then $\mathcal{A}$ be a local conformal net on $S^1$. $\mathcal{A}$ is called completely rational if

- $\mathcal{A}$ is split,
- $\mathcal{A}$ is strongly additive,
- The $\mu$-index $\mu_\mathcal{A}$ is finite.

The first two conditions are, in a certain sense, one another dual. If $I_1, I_2$ are intervals of $S^1$, the split property states that the local von Neumann algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ “maximally decouple” if $I_1$ and $I_2$ have disjoint closures, namely $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$; while strong additivity requires that $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$ “maximally interact” if $I_1$ and $I_2$ have a common boundary point, namely $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I$ is the union of $I_1, I_2$ and the boundary point, see e.g. [42] and refs. therein.

In the last condition, $\mu_\mathcal{A}$ is the Jones index [28] of the inclusion of factors $\mathcal{A}(E) \subset \mathcal{A}(E')$ were $E \subset S^1$ and its complement $E'$ are union of two proper disjoint intervals.

In [58] it was shown that $\mu_\mathcal{A} < \infty$ when $\mathcal{A}$ is associated with $SU(N)$ loop group models. The general theory of complete rationality was developed in [31]. To check complete rationality one may use the fact that this property equivalently holds for finite-index subnet [42].

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1 We shall later see that, in the diffeomorphism covariant case, strong additivity follows from the other two conditions.

The symbol “$\vee$” denotes the von Neumann algebra generated.
One of the main points is that, if $\mathcal{A}$ is completely rational, then

$$\mu_{\mathcal{A}} = \sum_i d(\rho_i)^2,$$

(1)

the $\mu$-index equals the global index i.e. the sum of the indeces (= squares of dimensions) of all irreducible sectors; thus $\mathcal{A}$ is rational and indeed the representation tensor category is even modular.

One issue in this paper is to extend the above equality to non rational nets. This will lead in particular to a general characterization of rational nets.

A look at the basic models constructed by positive-energy representations of the diffeomorphism group, the Virasoro nets $\text{Vir}_c$, gives insight here. If the central charge $c$ is less then one then $\text{Vir}_c$ is completely rational, as is indeed the case of all conformal nets with $c < 1$ [29]. By contrast, if $c > 1$ then $\text{Vir}_c$ is not even strongly additive [11] and has uncountably many sectors as is known, see e.g. [14]. The boundary case, $\text{Vir}_c$ at $c = 1$, has uncountably many sectors and has recently been shown in [59] to be strongly additive. Moreover in the case $c \geq 1$ there are plenty of infinite index sectors [14]. We shall see that the structure manifested by Virasoro nets undergoes a general phenomenon. But, before this, we give a general picture of our mentioned dichotomy.

The dichotomy. Dichotomies concerning the cardinality of various structures appear in Mathematics. One simple example concerns a $\sigma$-algebra: it is either finite or uncountable. This is an immediate consequence of the basic Cantor-Bernstein theorem to the effect that $2^\mathbb{N}$ is uncountable.

Also elementary is the statement that a Hamel basis for a Banach space is either finite or uncountable. This is due to Baire category theorem. The dichotomy holds because limit points are included in the structure.

One further example is provided by a compact group. Again it is either finite or uncountable. Here the statement follows at once by the existence of a finite Haar measure, a structure property of global nature.

As a final example consider the case of a separable, simple, unital $C^*$-algebra $\mathfrak{A}$ and denote by Irr$\mathfrak{A}$ the set of equivalence classes of irreducible representations of $\mathfrak{A}$. Then either Irr$\mathfrak{A}$ consists of a single element ($\mathfrak{A}$ is a matrix algebra) or Irr$\mathfrak{A}$ is uncountable. This fact is a consequence of the deep theorem of Glimm on the classification of type $I$ $C^*$-algebras [17] (if $\mathfrak{A}$ has a representation not of type $I$ then uncountably many irreducibles have to appear in its disintegration).

The dichotomy in this paper is more similar in the spirit to this last example: a high degree of understanding of the structure is necessary to get it.

The statement is the following. Let $\mathcal{A}$ be a local conformal net with the split property. Assume that every irreducible sector of $\mathcal{A}$ has a conjugate sector. Then either $\mathcal{A}$ is completely rational or $\mathcal{A}$ admits uncountably many different irreducible sectors. We shall later return on the consequences of this fact.

Now, to exhibit uncountably many sectors in the irrational case, some new construction of representations has to appear at some stage. This is indeed one of the most interesting points. These representations are constructed topologically, as we
now explain. (Note that, in higher dimension spacetimes, charges of topological nature and wide localization have long been known and are natural in particular in quantum electrodynamics, see [9]).

**Topological sectors.** Let’s start with a simple observation. Let $\mathcal{A}$ be a conformal net and $\mathcal{A}_0$ its restriction to the real line $\mathbb{R} = S^1 \setminus \{\zeta\}$ obtained by removing a point $\zeta$ from the circle. If $h : \mathbb{R} \to S^1$ is a smooth, injective, positively oriented map, we get a representation $\Phi_h$ of the $C^\ast$-algebra $\bigcup_I \mathcal{A}_0(I)$ (union over all bounded intervals of $\mathbb{R}$) by setting

$$\Phi_h(x) \equiv U(k_I) x U(k_I)^*, \quad x \in \mathcal{A}_0(I),$$

where $k_I : S^1 \to S^1$ is any diffeomorphism of $S^1$ that coincides with $h$ on the interval $I$, and $U$ is the covariance projective unitary representation of $\text{Diff}(S^1)$. Assuming $h$ to be smooth also at $\pm\infty$, then $\Phi_h$ is a soliton, namely it is normal on the algebras associated with half-lines.

Incidentally, this gives an elementary and model independent construction of type III representations, see [20, 12] for constructions of type III representations in models.

Let now $f : S^1 \to S^1$ be a smooth, locally injective map of degree $n = \deg f \geq 1$. Then $f$ has exactly $n$ right inverses $h_i$, $i = 0, 1, \ldots, n-1$, namely there are $n$ injective smooth maps $h_i : S^1 \setminus \{\zeta\} \to S^1$ such that $f(h_i(z)) = z$, $z \in S^1 \setminus \{\zeta\}$. The $h_i$’s are smooth also at $\pm\infty$. For the moment we make an arbitrary choice of order $h_0, h_1, h_2, \ldots$.

As just explained, we have $n$ solitonic representations $\Phi_{h_i}$ of $\mathcal{A}$, hence one (reducible) soliton $\Phi_f \equiv \Phi_{h_0} \otimes \cdots \otimes \Phi_{h_{n-1}}$ of $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$.

Now, if $I \subset \mathbb{R}$ is an interval, the intervals $I_i \equiv h_i(I) \subset S^1$ have pairwise disjoint closures hence, by the split property that we now assume, there is a natural identification

$$\chi_I : \mathcal{A}(I_0) \otimes \cdots \otimes \mathcal{A}(I_{n-1}) \to \mathcal{A}(I_0) \lor \cdots \lor \mathcal{A}(I_{n-1})$$

therefore we get an irreducible solitonic representation $\pi_f$ of $\mathcal{A} \otimes \cdots \otimes \mathcal{A}$ by gluing together the range of $\Phi_{f,I}$ by $\chi_I$, namely

$$\pi_{f,I} \equiv \chi_I \cdot \Phi_{f,I}.$$ 

Let’s say now that we choose the $h_i$’s so that the sequence of intervals $I_0, I_1, \ldots, I_{n-1}$ is counter-clockwise increasing (this requirement does not depend on $I$). This fixes the order of the $h_i$’s up to a cyclic permutation.

If we go to the cyclic orbifold $(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^\mathbb{Z}_n$ the dependence on the cyclic permutations disappears and we can easily verify that

$$\tau_f \equiv \pi_f \upharpoonright (\mathcal{A} \otimes \cdots \otimes \mathcal{A})^\mathbb{Z}_n$$

is indeed a well-defined **DHR representation** with exactly $n$ irreducible components. We have thus generated a family of twisted sectors for the cyclic orbifold.

We shall further see that $\pi_f$ itself does not depend, up to unitary equivalence, on the ordering of the $h_i$’s, by choosing the $I_i$’s increasing as above, namely the
soliton sector $[\pi_f]$ is an intrinsic object. In other words, if we denote by $\pi_{f,p}$ the sector corresponding to another other ordering, where $p \in \mathbb{P}_n$ is the permutation rearranging the $h_i$'s, then $[\pi_{f,p}]$ depends only on the cosets $\mathbb{P}_n/\mathbb{Z}_n$. The conjugate sector of $\pi_f$ corresponds to the clockwise ordering of the $h_i$'s.

A quantum index theorem. The soliton representation $\pi_f$, and its DHR restriction $\tau_f$, depend on $f$ only up to unitary equivalence. In a sense these topological sectors play a role similar to the Toeplitz operators (see e.g. [1]) in the framework of Fredholm linear operators, where the analytical index coincides with the degree $\deg f$.

As explained in [41], Doplicher-Haag-Roberts localized endomorphisms [18] may be viewed as a Quantum Field Theory analog of elliptic operators, in the context of a quantum index theorem. The topological sectors provide a good illustration of this point. Denoting by $\tau_f^{(i)}$, $i \in \mathbb{Z}_n$, the $n$ direct summands of $\tau_f$, we have

$$\text{Index}(\tau_f^{(i)}) = \text{Index}(\pi_f) = \mu_{A}^{n-1}.$$ 

Here the index is the Jones index [28], the analog of the Fredholm index [36, 37], $\mu_A$ is the above structure constant for $A$, and $n = \deg f$ is the topological index, which is manifestly deformation invariant.

As we shall see, more general topological sectors arise from non-vacuum representations. The index and further structure of these sectors will be determined.

Most of the results in this paper depend, maybe implicitly, on the above index formulas.

The structure of the sectors. At the infinitesimal level, the twisted sectors of the cyclic orbifold have already been considered in the papers [2, 7] in the framework of Kac-Moody Lie algebras. To study the structure of the tensor category of topological sectors, it is however necessary to have the sectors in global exponential form and a general theory at one's disposal, as provided by our approach.

In Section 8 we shall determine all the twisted irreducible sectors of the $n$-cyclic orbifold and give a detailed account of the fusion rules of the topological solitons, in the cases $n = 2, 3, 4$, for a general completely rational net. The method of $\alpha$-induction [44, 54, 4, 6] is here essential.

What undergoes the structure of sectors is the covariance symmetry. The Lie algebra of $\text{Diff}(S^1)$, namely the Virasoro algebra at $c = 0$ with generators $L_n$ and relations

$$[L_m, L_n] = (m - n)L_{m+n},$$

has an endomorphism, for each positive integer $k$, given by

$$L_n \mapsto \frac{1}{k}L_{kn},$$

see [7, 51]. As we shall see, this corresponds to an embedding of the $k$-cover of $\text{Diff}(S^1)$ into $\text{Diff}(S^1)$. The covariance projective unitary representations are obtained.
by composing with this embedding the original representation (of the vacuum or of a non-trivial sector).

Rationality, modularity, strong additivity, sectors with infinite index. The above described dichotomy has the following corollary. A conformal net $\mathcal{A}$ with the split property is rational, in the sense that the representation tensor category has only finitely many inequivalent irreducible objects and all have a conjugate, if and only if $\mathcal{A}$ is completely rational.

All results obtained for completely rational nets [31] then immediately apply to rational conformal nets. Among them we only mention that a rational conformal net has a modular representation tensor category, namely the braiding symmetry is automatically non-degenerate. The modularity property is at the basis of most of the analysis in Conformal QFT and is often taken for granted or implicitly conjectured to hold, see e.g. [23].

Note that our work shows in particular that a conformal net with the split property and finite $\mu$-index is automatically strongly additive or, equivalently, Haag dual on the real line. To understand the interest of this result, note that the strong additivity property is crucial for many results and often one of the hardest point to prove, see [56, 42, 59, 14]. As suggested by Y. Kawahigashi, strong additivity can be thought as an amenability property; our result supports this view. Our proof makes use of basic properties of simple subfactors [35].

A further consequence is that the equality (1) between the $\mu$-index and the global index holds true for any diffeomorphism covariant local net with the split property (regardless $\mu_\mathcal{A}$ is finite or infinite), a non-trivial useful result at the basis of our work.

Last, we state the following characterization of being not complete rationality: a conformal net $\mathcal{A}$ with the split property is not completely rational if and only if the 2-orbifold net $(\mathcal{A} \otimes \mathcal{A})^{\text{flip}}$ admits an irreducible sector with infinite index.

General properties of sectors with infinite index were studied in [24], but first examples were constructed by Fredenhagen in [22]. Indeed, as mentioned, Carpi [13, 14] has recently shown that irreducible sectors with infinite index appear in the Virasoro nets $\text{Vir}_c$ if $c \geq 1$, as suggested by Rehren in [48]. (By contrast notice that, in QFT on Minkowski spacetime, all irreducible DHR sectors with an isolated mass shell have finite dimension [9]).

Our general construction of infinite index irreducible sectors is natural and surprising. Consider indeed the case of the net $\mathcal{A}$ associated with the $U(1)$-current algebra. All sectors are known in this case [10], the irreducible ones all have index 1 and form a one-parameter family. Thus, by [31], $\mathcal{A} \otimes \mathcal{A}$ has only a two-parameter family of irreducible sectors, all with index 1. Yet, the “trivial” passage to the index 2 subnet $(\mathcal{A} \otimes \mathcal{A})^{\text{flip}}$ makes infinite index sectors to appear. Note that one of Fredenhagen’s examples is similar in the spirit, but concerns the infinite index subnet $(\mathcal{A} \otimes \mathcal{A})^{SO(2)}$.

At this point we close our expository part and refer to the rest of the paper for a detailed account and further results. See [27, 53] as reference books.

6
2 On the symmetry groups

We shall denote by Diff($S^1$) the group of orientation preserving smooth diffeomorphisms of $S^1 \equiv \{ z \in \mathbb{C} : |z| = 1 \}$. Diff($S^1$) is an infinite dimensional Lie group whose Lie algebra is Vect($S^1$), the Lie algebra of smooth vector fields on the circle. The complexification Vect$_C(S^1)$ of Vect($S^1$) has a basis with elements $L_n \equiv -z^{n+1} \frac{d}{dz}$, $n \in \mathbb{Z}$, satisfying the relations

$$[L_m, L_n] = (m - n)L_{m+n}. \quad (2)$$

We shall consider Vect$_C(S^1)$ as a Lie algebra with involution

$$L^*_n = L_{-n}.$$ 

SU(1,1) is the group of $2 \times 2$ matrices defined by:

$$SU(1,1) \equiv \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \bigg| \alpha, \beta \in \mathbb{C}, \ |\alpha|^2 - |\beta|^2 = 1 \right\}. \quad (3)$$

SU(1,1) acts on $S^1$ by linear fractional transformations:

$$g(z) \equiv \frac{\alpha z + \beta}{\beta z + \bar{\alpha}}, \quad (4)$$

where $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$. This action factors through a faithful action of PSU(1,1) $\equiv SU(1,1)/\{\pm 1\}$ on $S^1$.

The corresponding diffeomorphisms $z \mapsto g(z)$ constitute a subgroup of Diff($S^1$), the M"obius group M"ob, isomorphic to PSU(1,1). SU(1,1) is a double cover of PSU(1,1) and SU(1,1) are locally isomorphic, they have the same Lie algebra $\mathfrak{s\ell}(2,\mathbb{R})$. The complexified Lie algebra $\mathfrak{s\ell}(2,\mathbb{C})$ of $\mathfrak{s\ell}(2,\mathbb{R})$ has generators $L_{-1}, L_0, L_1$ satisfying the relations

$$[L_1, L_{-1}] = 2L_0, \quad [L_{\pm 1}, L_0] = \pm L_{\pm 1}. \quad (5)$$

Therefore the elements $L_{-1}, L_0, L_1$ of Vect$_C(S^1)$ exponentiate to a subgroup of Diff($S^1$) locally isomorphic to M"ob. As $\exp(2\pi L_0)$ is the identity of Diff($S^1$), this group is indeed isomorphic to M"ob. Vect$_C(S^1)$ contains infinitely many further copies of $\mathfrak{s\ell}(2,\mathbb{C})$; for a fixed integer $n > 0$ we get a copy generated by the elements $L_{-n}, L_0, L_n$. Setting

$$L'_m \equiv \frac{1}{|n|}L_m, \quad m = n, -n, 0, \quad (5)$$

we have indeed the relations

$$[L'_n, L'_{-n}] = 2L'_0, \quad [L'_{\pm n}, L'_0] = \pm L'_{\pm n}. \quad (6)$$
The Lie subgroup of $\text{Diff}(S^1)$ corresponding to $L_{-n}, L_0, L_n$ is thus a cover of $\text{Möb}$. As $\exp(2\pi n L'_0) = \exp(2\pi L_0)$ is the identity, this group is then isomorphic to $\text{Möb}^{(n)}$, the $n$-cover group of $\text{Möb}$. Thus there is a natural embedding

$$M^{(n)} : \text{Möb}^{(n)} \hookrightarrow \text{Diff}(S^1).$$

With $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1, 1)$ we shall see that $M^{(n)}_g \in \text{Diff}(S^1)$ is formally given by

$$M^{(n)}_g(z) \equiv \sqrt[n]{\frac{\alpha z^n + \beta}{\beta z^n + \bar{\alpha}}},$$

if we locally identify $SU(1, 1)$ and $\text{Möb}^{(n)}$.

Denote by $g \mapsto \overline{g}$ the quotient map $\text{Möb}^{(n)} \rightarrow \text{Möb}$.

**Proposition 2.1.** There is a unique continuous isomorphism $M^{(n)}$ of $\text{Möb}^{(n)}$ into $\text{Diff}(S^1)$ such that the following diagram commutes for every $g \in \text{Möb}^{(n)}$

$$\begin{array}{ccc}
S^1 & \xrightarrow{M^{(n)}_g} & S^1 \\
\downarrow z^n & & \downarrow z^n \\
S^1 & \xrightarrow{M^{(n)}_g} & S^1 \\
\end{array}$$

i.e. $M^{(n)}_g(z)^n = M^{(n)}_{\overline{g}}(z^n)$ for all $z \in S^1$.

Denote by $\sqrt[n]{z}$ the $n$-th-root function on the cut plane $\mathbb{C} \setminus (-\infty, 0]$. For a fixed $g \in PSU(1, 1)$, the map $f_g : z \in S^1 \mapsto \frac{\alpha z^n + \beta}{\beta z^n + \bar{\alpha}} \in S^1$ has winding number $n$. The Riemann surface $\Sigma_n$ associated with the function $\sqrt[n]{z}$ is a $n$-cover of $\mathbb{C} \setminus \{0\}$, we may thus lift $f_g$ to a one-to-one map $\tilde{f}_g$ from $S^1$ to the elements of $\Sigma_n$ projecting onto $S^1$ on $\mathbb{C} \setminus \{0\}$. The lift is uniquely determined as soon as we specify the value $\tilde{f}_g(1)$ among the $n$ elements of $\Sigma_n$ projecting onto $f_g(1)$.

Let $\mathcal{V}$ be a connected neighborhood of the identity in $PSU(1, 1)$ such that $\tilde{f}_g(1) \in S^1 \setminus \{-1\}$ for all $g \in \mathcal{V}$. Then we define $\tilde{f}_g$ for $g \in \mathcal{V}$ by requiring that $\tilde{f}_g(1) = f_g(1) \in \mathbb{C} \setminus (-\infty, 0] \subset \Sigma_n$.

We then set

$$M^{(n)}_g(z) \equiv \sqrt[n]{\tilde{f}_g(z)}, \quad g \in \mathcal{V}.$$ 

(9)

Choosing a neighborhood $\mathcal{V}_0$ of the identity in $PSU(1, 1)$ such that $\mathcal{V}_0 \cdot \mathcal{V}_0 \subset \mathcal{V}$ we then have

$$M^{(n)}_{gh} = M^{(n)}_g M^{(n)}_h, \quad g, h \in \mathcal{V}_0,$$

namely we have a local isomorphism of $\mathcal{V}_0 \subset PSU(1, 1)$ into $\text{Diff}(S^1)$, and this extends to a global isomorphism of $\text{Möb}^{(n)}$ into $\text{Diff}(S^1)$, still denoted by $M^{(n)}$.

Clearly $M^{(n)}_g(z)^n = M^{(n)}_{\overline{g}}(z^n)$ for all $z \in S^1$ if $g \in \mathcal{V}$.
Note that for any \( g \in PSU(1, 1) \) and \( g \in Möb^{(n)} \) projecting onto \( g \), we have \( n \) diffeomorphisms
\[
R(\frac{2k\pi}{n})M^{(n)}_g, \quad k = 0, 1, \ldots n - 1.
\]
corresponding to the other possible choices of \( g \). Here \( R \) is the rotation one-parameter subgroup of Möb. Thus \( M^{(n)}_g(z)^n = M^{(n)}_g(z^n) \) for all \( z \in S^1 \) and all \( g \in Möb^{(n)} \).

Concerning the uniqueness of \( M^{(n)} \) note that \( M^{(n)}_I = I \) because \( M^{(n)} \) is an isomorphism. By continuity \( M^{(n)}_g(z) \in S^1 \setminus \{-1\} \) for \( g \) in a neighborhood \( U \) of \( I \) and this determines \( M^{(n)} \) on \( U \), hence on all Möb^{(n)}. □

Of particular interest is the case \( n = 2 \). Möb^{(2)} is isomorphic to \( SU(1, 1) \) and we thus have an isomorphism
\[
M^{(2)} : SU(1, 1) \hookrightarrow \text{Diff}(S^1).
\]
We shall often identify \( PSU(1, 1) \) with Möb and \( SU(1, 1) \) with Möb^{(2)}.

We now extend the above proposition to general diffeomorphisms.

Denote by \( \text{Diff}^{(n)}(S^1) \) the \( n \)-central cover group of \( \text{Diff}(S^1) \). The group \( \text{Diff}^{(n)}(S^1) \) is obtained from \( \text{Diff}^{(n)}(S^1) \) similarly as Möb^{(n)} is obtained from Möb (the 1-torus rotation subgroup lifts to its \( n \)-cover), but we shall soon give an explicit realization of \( \text{Diff}^{(n)}(S^1) \).

The universal cover group \( \text{Diff}^{(∞)}(S^1) \) of \( \text{Diff}(S^1) \) is the projective limit
\[
\text{Diff}^{(∞)}(S^1) = \lim_{\longleftarrow \, n \in \mathbb{N}} \text{Diff}^{(n)}(S^1).
\]

If \( n \in \mathbb{N} \), the map
\[
\text{Vect}_C(S^1) \to \text{Vect}_C(S^1), \quad L_m \mapsto \frac{1}{n}L_{nm}
\]
defines an injective endomorphism of \( \text{Vect}_C(S^1) \). Its inverse corresponds to a an embedding
\[
M^{(n)} : \text{Diff}^{(n)}(S^1) \hookrightarrow \text{Diff}(S^1)
\]
that extends the one in (8) (still denoted by the same symbol).

Denote by \( g \mapsto g \) also the quotient map \( \text{Diff}^{(n)}(S^1) \to \text{Diff}(S^1) \). We then have:

**Proposition 2.2.** There is a unique continuous isomorphism \( M^{(n)} \) of \( \text{Diff}^{(n)}(S^1) \) into \( \text{Diff}(S^1) \) such that the diagram (8) commutes for every \( g \in \text{Diff}^{(n)}(S^1) \), namely \( M^{(n)}_g(z)^n = M^{(n)}_g(z^n) \) for all \( z \in S^1 \) and \( g \in \text{Diff}(S^1) \).

\( M^{(n)} \) is the unique isomorphism of \( \text{Diff}^{(n)}(S^1) \) into \( \text{Diff}(S^1) \) such that \( M^{(n)} \restriction Möb^{(n)} \) is given in Prop. 2.1.

**Proof** The proof is analogous to the proof of Prop. 2.1. □
The Virasoro algebra is the infinite dimensional Lie algebra generated by elements \( \{ L_n | n \in \mathbb{Z} \} \) and \( c \) with relations

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}.
\]

and \([L_n, c] = 0\). It is the (complexification of) the unique, non-trivial one-dimensional central extension of the Lie algebra of \( \text{Vect}(S^1) \).

The elements \( L_{-1}, L_0, L_1 \) of the Virasoro algebra are clearly a basis of \( sl(2, \mathbb{C}) \). The Virasoro algebra contains infinitely many further copies of \( sl(2, \mathbb{C}) \), generated by the elements \( L'_{-n}, L'_0, L'_n, n > 1 \), where

\[
L'_n = \frac{1}{|n|}L_n, \quad n \neq 0,
\]

\[
L'_0 = \frac{1}{n}L_0 + \frac{c}{24}(n^2 - 1)\frac{(n^2 - 1)}{n}.
\]

For any fixed integer \( n > 0 \) we have

\[
[L'_n, L'_{-n}] = 2L'_0, \quad [L'_{\pm n}, L'_0] = \pm L'_{\pm n}
\]

which are indeed the relations for the usual generators in \( sl(2, \mathbb{C}) \).

There is a one-to-one correspondence between projective irreducible unitary representations of \( \text{Diff}(S^1) \) and irreducible unitary representations of \( \text{Diff}^{(\infty)}(S^1) \).

We shall be interested in positive energy \((L_0 \geq 0)\) representations of the Virasoro algebra which are unitary (i.e. preserving the involution). They correspond to projective unitary representations of \( \text{Diff}(S^1) \) with positive energy.

Given a projective unitary representation \( U \) of \( \text{Diff}(S^1) \) and a fixed \( n \in \mathbb{N} \), we obtain a projective unitary representation \( U^{(n)} \) of \( \text{Diff}^{(n)}(S^1) \)

\[
U^{(n)} \equiv U \cdot M^{(n)}.
\]

(There is an analogous passage from unitary representations of \( \text{Mob} \) to unitary representations of \( \text{Mob}^{(n)} \).)

Starting with a positive energy, unitary representation \( U \) of the Virasoro algebra with central charge \( c \), it can be easily seen that the above construction (16) gives a positive energy, unitary representation \( U^{(n)} \) of the Virasoro algebra with central charge \( nc \). This will also be clear by the content of this paper.

3 Conformal nets on \( S^1 \)

We denote by \( I \) the family of proper intervals of \( S^1 \). A net \( \mathcal{A} \) of von Neumann algebras on \( S^1 \) is a map

\[
I \in I \to \mathcal{A}(I) \subset B(\mathcal{H})
\]

from \( I \) to von Neumann algebras on a fixed Hilbert space \( \mathcal{H} \) that satisfies:
A. **Isotony.** If \( I_1 \subset I_2 \) belong to \( \mathcal{I} \), then
\[
\mathcal{A}(I_1) \subset \mathcal{A}(I_2).
\]

If \( E \subset S^1 \) is any region, we shall put \( \mathcal{A}(E) \equiv \bigvee_{E \supset I \in \mathcal{I}} \mathcal{A}(I) \) with \( \mathcal{A}(E) = \mathbb{C} \) if \( E \) has empty interior (the symbol \( \vee \) denotes the von Neumann algebra generated).

The net \( \mathcal{A} \) is called **local** if it satisfies:

B. **Locality.** If \( I_1, I_2 \in \mathcal{I} \) and \( I_1 \cap I_2 = \emptyset \) then
\[
[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},
\]
where brackets denote the commutator.

The net \( \mathcal{A} \) is called **Möbius covariant** if in addition satisfies the following properties C,D,E,F:

C. **Möbius covariance.** There exists a strongly continuous unitary representation \( U \) of \( \text{Mob} \) on \( \mathcal{H} \) such that
\[
U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Mob}, \quad I \in \mathcal{I}.
\]

D. **Positivity of the energy.** The generator of the one-parameter rotation subgroup of \( U \) (conformal Hamiltonian) is positive.

E. **Existence of the vacuum.** There exists a unit \( U \)-invariant vector \( \Omega \in \mathcal{H} \) (vacuum vector), and \( \Omega \) is cyclic for the von Neumann algebra \( \bigvee_{I \in \mathcal{I}} \mathcal{A}(I) \).

The above axioms imply Haag duality (see [8]):
\[
\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I},
\]
where \( I' \) is the interior of \( S^1 \setminus I \).

F. **Irreducibility.** \( \bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H}) \). Indeed \( \mathcal{A} \) is irreducible iff \( \Omega \) is the unique \( U \)-invariant vector (up to scalar multiples), and iff the local von Neumann algebras \( \mathcal{A}(I) \) are factors. In this case they are \( \text{III}_1 \)-factors (unless \( \mathcal{A}(I) = \mathbb{C} \) identically), see [25].

By a **conformal net** (or diffeomorphism covariant net) \( \mathcal{A} \) we shall mean a Möbius covariant net such that the following holds:

G. **Conformal covariance.** There exists a projective unitary representation \( U \) of \( \text{Diff}(S^1) \) on \( \mathcal{H} \) extending the unitary representation of \( \text{PSU}(1,1) \) such that for all \( I \in \mathcal{I} \) we have
\[
U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Diff}(S^1),
\]
\[
U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I), \quad g \in \text{Diff}(I'),
\]
\[
U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I), \quad g \in \text{Diff}(I').
\]

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where $\text{Diff}(I)$ denotes the subgroup of smooth diffeomorphisms $g$ of $S^1$ such that $g(z) = z$ for all $z \in I$.

A representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each $I$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_I|\mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \in \mathcal{I}.$$ 

$\pi$ is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation $U_\pi$ of $\text{Mob}$ (resp. $\text{Diff}^{(\infty)}(S^1)$) on $\mathcal{H}$ such that

$$\pi_{gI}(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*$$

for all $I \in \mathcal{I}$, $x \in \mathcal{A}(I)$ and $g \in \text{Mob}$ (resp. $g \in \text{Diff}^{(\infty)}(S^1)$). Note that if $\pi$ is irreducible and diffeomorphism covariant then $U$ is indeed a projective unitary representation of $\text{Diff}(S^1)$.

Following [18], given an interval $I$ and a representation $\pi$ of $\mathcal{A}$, there is an endomorphism of $\mathcal{A}$ localized in $I$ equivalent to $\pi$; namely $\rho$ is a representation of $\mathcal{A}$ on the vacuum Hilbert space $\mathcal{H}$, unitarily equivalent to $\pi$, such that $\rho_I = \text{id}|\mathcal{A}(I')$. We refer to [25] for basic facts on this structure, in particular for the definition of the dimension $d(\rho)$, that turns out to equal the square root of the Jones index [36]. The reader will also find basic notions concerning sectors of factors at the beginning of Sect. 8 or in [32].

### 3.0.1 Restriction to the real line

Denote by $\mathcal{I}_0$ the set of open, connected, non-empty, proper subsets of $\mathbb{R}$, thus $I \in \mathcal{I}_0$ iff $I$ is an open interval or half-line (by an interval of $\mathbb{R}$ we shall always mean a non-empty open bounded interval of $\mathbb{R}$).

Given a net $\mathcal{A}$ on $S^1$ we shall denote by $\mathcal{A}_0$ its restriction to $\mathbb{R} = S^1 \setminus \{-1\}$. Thus $\mathcal{A}_0$ is an isotone map on $\mathcal{I}_0$, that we call a net on $\mathbb{R}$.

A representation $\pi$ of $\mathcal{A}_0$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I}_0 \mapsto \pi_I$ that associates to each $I \in \mathcal{I}_0$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_I|\mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \in \mathcal{I}_0.$$ 

A representation $\pi$ of $\mathcal{A}_0$ is also called a soliton\(^3\).

Clearly a representation $\pi$ of $\mathcal{A}$ restricts to a soliton $\pi_0$ of $\mathcal{A}_0$. But a representation $\pi_0$ of $\mathcal{A}_0$ does not necessarily extend to a representation of $\mathcal{A}$.

### 3.1 Normality for $\alpha$-induction

Let $\mathcal{A}$ be a Möbius covariant net and $\mathcal{B}$ a subnet. Given a bounded interval $I_0 \in \mathcal{I}_0$ we fix canonical endomorphism $\gamma_{I_0}$ associated with $\mathcal{B}(I_0) \subset \mathcal{A}(I_0)$. Then we can choose

\(^3\)There are more general soliton sectors, namely representations normal on left (resp. right) half-lines, but non-normal on right (resp. left) half-lines. These will not be considered in this paper.
for each $I \subset \mathcal{I}_0$ with $I \supset I_0$ a canonical endomorphism $\gamma_I$ of $\mathcal{A}(I)$ into $\mathcal{B}(I)$ in such a way that $\gamma_I \upharpoonright \mathcal{A}(I_0) = \gamma_{I_0}$ and $\lambda_I$, is the identity on $\mathcal{B}(I_1)$ if $I_1 \in \mathcal{I}_0$ is disjoint from $I_0$, where $\lambda_I \equiv \gamma_I \upharpoonright \mathcal{B}(I)$.

We then have an endomorphism $\gamma$ of the $C^*$-algebra $\mathfrak{A} \equiv \overline{\bigcup_I \mathcal{A}(I)}$ ($I$ bounded interval of $\mathbb{R}$).

Given a DHR endomorphism $\rho$ of $\mathcal{B}$ localized in $I_0$, the $\alpha$-induction $\alpha_\rho$ of $\rho$ is the endomorphism of $\mathfrak{A}$ given by

$$\alpha_\rho \equiv \gamma^{-1} \cdot \text{Ad}_\varepsilon(\rho, \lambda) \cdot \rho \cdot \gamma,$$

where $\varepsilon$ denotes the right braiding unitary symmetry (there is another choice for $\alpha$ associated with the left braiding). $\alpha_\rho$ is localized in a right half-line containing $I_0$, namely $\alpha_\rho$ is the identity on $\mathcal{A}(I)$ if $I$ is a bounded interval contained in the left complement of $I_0$ in $\mathbb{R}$. Up to unitarily equivalence, $\alpha_\rho$ is localizable in any right half-line thus $\alpha_\rho$ is normal on left half-lines, that is to say, for every $a \in \mathbb{R}$, $\alpha_\rho$ is normal on the $C^*$-algebra $\mathfrak{A}(-\infty, a) \equiv \overline{\bigcup_{I \subset (-\infty, a)} \mathcal{A}(I)}$ ($I$ bounded interval of $\mathbb{R}$), namely $\alpha_\rho \upharpoonright \mathfrak{A}(-\infty, a)$ extends to a normal morphism of $\mathcal{A}(-\infty, a)$.

We now show that $\alpha_\rho$ is normal on right half-lines. To this end we use the fact that our nets on $\mathbb{R}$ are restrictions of nets on $S^1$.

**Proposition 3.1.** $\alpha_\rho$ is a soliton endomorphism of $\mathcal{A}_0$.

**Proof.** It is convenient to use the circle picture, thus $I_0 \subset S^1 \setminus \{-1\}$, say $I_0 = (a, b)$ where $a, b \in S^1 \setminus \{-1\}$, and $b > a$ in the counterclockwise order (intervals do not contain $-1$). Let $a_n, b_n \in S^1 \setminus \{-1\}$ with $a < b < a_n < b_n$ and $\rho_n$ an endomorphism of $\mathcal{B}$ equivalent to $\rho$ and localized in $(a_n, b_n)$. With $u_n \in \mathcal{B}(a, b_n)$ a unitary such that $\rho_n = \text{Ad} u_n \cdot \rho$, we have

$$\alpha_\rho \upharpoonright \mathcal{A}(c, a_n) = \text{Ad} u_n^* \upharpoonright \mathcal{A}(c, a_n),$$

for every $c < a$. Going to the limit $c \to -1^-$, $b_n \to -1^+$ the above gives the definition of $\alpha_\rho$ on the $C^*$-algebra $\mathfrak{A}$ originally given in [50].

We want to show that $\alpha_\rho \upharpoonright \mathfrak{A}(d, -1)$ extends to a normal map of $\mathcal{A}(-1, d)$ for any given $d \neq -1$.

Now, as $\mathcal{B}$ is defined on $S^1$, we may push the interval $(a_n, b_n)$ even beyond the point $-1$. Namely we may choose an interval $(a', b')$ with $-1 < a' < b' < a$, an endomorphism $\rho'$ of $\mathcal{B}$ equivalent to $\rho$ localized in $(a', b')$, and a unitary $u \in \mathcal{B}(a, b')$ such that $\rho' = \text{Ad} u \cdot \rho$. Then $\alpha_\rho \upharpoonright \mathfrak{A}(a, -1) = \text{Ad} u^* \upharpoonright \mathfrak{A}(a, -1)$, showing that $\alpha_\rho$ extends to a normal morphism of $\mathcal{A}(a, -1)$. Of course we may take a smaller $a$ in the definition of $I_0$, thus $\alpha_\rho$ is normal on all right half-lines. \(\square\)

### 3.2 CMS property

In this section $\mathcal{A}$ is a Möbius covariant local net on $S^1$. We shall say that $\mathcal{A}$ has *property CMS* if it admits at most countably many different irreducible (DHR) sectors and all of them have finite index.
Let $\beta$ be a vacuum preserving, involutive automorphism of $\mathcal{A}$ and $\mathcal{B} = \mathcal{A}^\beta \subset \mathcal{A}$ the fixed-point subnet. The restriction of $\mathcal{A}$ and $\mathcal{B}$ to $\mathbb{R} = S^1 \setminus \{-1\}$ are denoted by $\mathcal{A}_0$ and $\mathcal{B}_0$ as above.

We denote by $[\sigma]$ the sector of $\mathcal{B}$ dual to $\beta$. Choosing an interval $I_0 \subset \mathbb{R}$ there is a unitary $v \in \mathcal{A}(I)$, $v^* = v$, $\beta(v) = -v$.

Then $\sigma \equiv \text{Ad}v \upharpoonright \mathcal{B}$ is an automorphism of $\mathcal{B}$ localized in $I_0$. We have $d(\sigma) = 1$ and $\sigma^2 = 1$.

Given a DHR endomorphism $\mu$ of $\mathcal{B}$ localized in an interval $I_0 \subset \mathbb{R}$, we denote as above by $\alpha_\mu$ the right $\alpha$-induction of $\mu$ to $\mathcal{A}_0$. Recall that in general $\alpha_\mu$ is a soliton sector of $\mathcal{A}_0$. With $\varepsilon(\mu, \sigma)$ the right statistics operator, the condition for $\alpha_\mu$ to be localized in a bounded interval of $\mathbb{R}$, i.e. to be a DHR endomorphism of $\mathcal{A}$, is that the monodromy operator $\varepsilon(\mu, \sigma)\varepsilon(\sigma, \mu) = 1$. If $\mu$ is localized left to $\sigma$, then $\varepsilon(\sigma, \mu) = 1$, so we have:

**Proposition 3.2.** Let $\mu$ be localized in an interval $I \subset \mathbb{R}$ in the left complement of $I_0$ in $\mathbb{R}$. Then $\alpha_\mu$ is a DHR sector of $\mathcal{A}$ iff $\varepsilon(\mu, \sigma) = 1$.

**Proof** Let $I_1$ be an interval of $\mathbb{R}$ in the right complement of $I_0$, $\mu'$ an endomorphism of $\mathcal{B}$ localized in $I_1$ and $u \in \text{Hom}(\mu, \mu')$ a unitary. Then $\mu(x) = \text{Ad}u^*(x)$ for all $x \in \mathcal{B}(I_2)$ if $I_2$ is an interval left to $I_1$.

We then have $\alpha_\mu(x) = \text{Ad}u^*(x)$ if $x \in \mathcal{A}(I_2)$. It follows that $\alpha_\mu$ is localized in $I_1$ iff $\alpha_\mu$ acts trivially on $\mathcal{A}(I_0)$. As $\mathcal{A}(I_0)$ is generated by $\mathcal{B}(I_0)$ and $v$, this is the case iff

$$\alpha_\mu(v) = v \iff u^*vu = v \iff \varepsilon(\mu, \sigma) = u^*\sigma(u) = 1.$$ 

Let $\mu$ be an irreducible endomorphism localized left to $\sigma$. As $\varepsilon(\mu, \sigma) \in \text{Hom}(\mu\sigma, \sigma\mu)$ and $\sigma$ and $\mu$ commute, it follows that $\varepsilon(\mu, \sigma)$ is scalar. Denoting by $\iota$ the identity sector, by the braiding fusion relation we have

$$1 = \varepsilon(\mu, \iota) = \varepsilon(\mu, \sigma^2) = \sigma(\varepsilon(\mu, \sigma))\varepsilon(\mu, \sigma) = \varepsilon(\mu, \sigma)\varepsilon(\mu, \sigma),$$

thus $\varepsilon(\mu, \sigma) = \pm 1$.

If $\mu$ is not necessarily irreducible, we shall say that $\mu$ is $\sigma$-Bose if $\varepsilon(\mu, \sigma) = 1$ and that $\mu$ is $\sigma$-Fermi if $\varepsilon(\mu, \sigma) = -1$. As we have seen, if $\mu$ is irreducible then $\mu$ is either $\sigma$-Bose or $\sigma$-Fermi.

**Corollary 3.3.** Let $\mu, \nu$ be DHR sectors of $\mathcal{B}$. If $\mu, \nu$ are both $\sigma$-Fermi, then $\alpha_{\mu\nu}$ is a DHR sector of $\mathcal{A}$.

**Proof** We may assume that both $\mu$ and $\nu$ are localized left to $\sigma$. By the braiding fusion relation we have

$$\varepsilon(\mu\nu, \sigma) = \mu(\varepsilon(\nu, \sigma))\varepsilon(\mu, \sigma) = \varepsilon(\nu, \sigma)\varepsilon(\mu, \sigma) = 1.$$ 

\[\square\]
Lemma 3.4. Let $\mu$ be a $\sigma$-Bose sector of $B$. Then $\mu$ has a direct integral decomposition into irreducible $\sigma$-Bose sectors.

Proof $\alpha_\mu$ is a $\sigma$-Bose sector of $A$, thus $\alpha_\mu$ has a direct integral decomposition into irreducibles [31], say $\alpha_\mu = \int_\mathbb{T} \pi_t dm(t)$. Since $B \subset A$ is a finite-index subnet, the restriction of $\pi_t$ to $B$ is the sum of finitely many irreducible $\sigma$-Bose representations, so the restriction of $\alpha_\mu$ to $B$ has a direct integral decomposition into irreducible $\sigma$-Bose sectors. By Frobenius reciprocity (cf. Th. B.2) $\mu$ is contained in the restriction of $\alpha_\mu$ to $B$ and we are done.

Corollary 3.5. Assume $A$ to have property CMS, then $B$ has property CMS.

Proof First suppose that $B$ has an irreducible $\sigma$-Bose sector $\mu$ with $d(\mu) = \infty$. Then $\alpha_\mu$ is a DHR sector of $A$ with $d(\alpha_\mu) = d(\mu) = \infty$.

As $A$ has property CMS, there is an irreducible finite-index DHR sector $\lambda$ of $A$ with $\lambda \bowtie \alpha_\mu$.

By Frobenius reciprocity we have the equality between the the dimensions of the intertwiners spaces $\langle \alpha_\mu, \lambda \rangle = \langle \mu, \gamma \lambda \upharpoonright B \rangle$, thus $\gamma \lambda \upharpoonright B \bowtie \mu$. As $d(\gamma \lambda \upharpoonright B) < \infty$ then $d(\mu) < \infty$ and this shows that $B$ has no irreducible $\sigma$-Bose sector with infinite dimension.

Suppose now that $B$ has uncountably many $\sigma$-Bose irreducible sectors $\{\mu_i\}$ with finite dimension. As $A$ has property CMS there must be an irreducible finite dimensional DHR sector $\lambda$ such that $\alpha_{\mu_i} \bowtie \lambda$ for uncountably many $i$. By Frobenius reciprocity $\mu_i \bowtie \gamma \lambda \upharpoonright B$, thus $d(\gamma \lambda \upharpoonright B) = \infty$, which is not possible because $d(\lambda) < \infty$. Thus $A$ admits at most countably many inequivalent irreducible $\sigma$-Bose sectors and all have finite dimension.

Suppose now that $\mu$ an is irreducible, $\sigma$-Fermi and infinite dimensional sector of $B$. Then $\bar{\mu}\mu$ is $\sigma$-Bose. Now $B$ inherits the split property from $D$ (this is rather immediate, see [42]) so $\bar{\mu}\mu$ has a direct integral decomposition into irreducible sectors that must be almost everywhere $\sigma$-Bose because $\bar{\mu}\mu$ is $\sigma$-Bose.

By what we have proved above, $\bar{\mu}\mu$ is then a direct sum of finite dimensional $\sigma$-Bose sectors, and analogously the same is true for $\mu\bar{\mu}$, and this entails $d(\mu) < \infty$ as in of Lemma 3.6.

It remains to show that $B$ cannot have uncountably many $\sigma$-Fermi irreducible sectors $\{\mu_i\}$ with finite dimension. On the contrary for a given $i_0$ there should exist uncountably many $i$ and a fixed finite dimensional irreducible sector $\lambda$ of $B$ such that $\mu_i \bowtie \lambda$ because we have already proved that there are at most countably many finite dimensional $\sigma$-Bose irreducible sectors. By Frobenius reciprocity then $\lambda \mu_i \bowtie \bar{\mu}_i$, which is not possible because $d(\mu_i) < \infty$.

This concludes our proof. □

Lemma 3.6. Let $M$ be a factor and $\rho \in \text{End}(M)$ an irreducible endomorphism. If there are $\sigma, \sigma' \in \text{End}(M)$ such that $\rho\sigma \bowtie \mu$ and $\sigma'\rho \bowtie \mu'$ with $\mu, \mu'$ finite index subsectors, then $d(\rho) < \infty$. 

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Proof. With \( \rho' \equiv \sigma \bar{\sigma} \bar{\rho} \in \text{End}(M) \), we have \( \rho \rho' = \rho \sigma \bar{\sigma} \bar{\rho} \succ \mu \bar{\mu} \succ \iota \). Analogously there is \( \rho'' \in \text{End}(M) \) such that \( \rho'' \rho \succ \iota \), thus \( \rho \) has finite index by the criterion on the existence of the conjugate sector in [37]. \( \square \)

4 Canonical representation of \( A_0 \otimes A_0 \)

For simplicity we shall now consider the 2-fold tensor product, which is however sufficient for most of the applications. We shall return on this point in later sections and have a more general analysis in the case of arbitrary \( n \)-fold tensor product.

We shall say that a set \( E \subset S^1 \) is a symmetric 2-interval if \( E = I_1 \cup I_2 \) where \( I_1, I_2 \in \mathcal{I} \) are interval of with length less than \( \pi \) and \( I_2 = R(\pi)I_1 = -I_1 \). The set of all symmetric 2-intervals is denoted by \( \mathcal{I}^{(2)} \).

Given an interval \( I \in \mathcal{I} \), then

\[
E \equiv \sqrt{I} = \{ z \in S^1 | z^2 \in I \}
\]

is a symmetric 2-interval. Conversely, given a symmetric 2-interval \( E = I_1 \cup I_2 \), then \( I \equiv I_1^2 = I_2^2 \) is an interval and \( E = \sqrt{I} \), thus there is a bijection between \( \mathcal{I} \) and \( \mathcal{I}^{(2)} \).

In the following \( A \) denotes a diffeomorphism covariant, local net of von Neumann algebras on \( S^1 \). We denote by \( U \) the associated projective unitary representation of \( \text{Diff}(S^1) \).

We assume the split property.

Given \( \zeta \in S^1 \), we shall denote by \( \sqrt{\cdot} \) the square root function on \( S^1 \) with a discontinuity in \( \zeta \), namely \( z \in S^1 \mapsto \sqrt{z} \in S^1 \) is the unique function such that \( (\sqrt{z})^2 = z \), \( \sqrt{1} = 1 \), \( \sqrt{\cdot} \) is continuous at all \( z \neq \zeta \) and continuous from the right (counterclockwise) at \( z = \zeta \).

Let \( I \subset S^1 \) be an interval and set \( E = \sqrt{I} \in \mathcal{I}^{(2)} \), \( E = I_1 \cup I_2 \). Given \( \zeta \in \mathcal{T} \) we choose the two components of \( E \) so that \( I_1 = \sqrt{\mathcal{T}}, I_2 = -\sqrt{\mathcal{T}} = R(\pi)I_1 \).

Let \( h \in \text{Diff}(S^1) \) be such that \( h(z) = \sqrt{z}, z \in I \) (cf. [52]) and set \( \bar{h}(z) = -\sqrt{z}, z \in I \). Setting

\[
\Phi_I^{(c)}(h) \equiv \text{Ad}U(h) | A(I), \tag{17}
\]

\[
\Phi_I^{(c)}(\bar{h}) \equiv \text{Ad}U(\bar{h}) | A(I), \tag{18}
\]

by diffeomorphism covariance \( \Phi_I^{(c)} \), \( \Phi_I^{(c)}(\bar{h}) \) are isomorphisms of \( A(I) \) with \( A(I_1) \) and with \( A(I_2) \).

Proposition 4.1. Let \( I \in \mathcal{I} \) and \( \zeta \notin I \). We have:

(a) \( \Phi_I^{(c)}, \Phi_I^{(c)} \) do not depend on the choice of \( h \).

(b) \( \Phi_I^{(c)} = \text{Ad}U(R(\pi)) \cdot \Phi_I^{(c)} \).
(c) If $\zeta' \notin I$, then $\Phi^{(c)}_I = \Phi^{(c)}_I$ or $\Phi^{(c)}_I = \Phi^{(c)}_I$. Denote by $[\zeta, \zeta']$ the interval of $S^1$ in the counterclockwise order and assume $(\zeta, \zeta') \notin I$. Then $\Phi^{(c)}_I = \Phi^{(c)}_I$ iff $1 \notin (\zeta, \zeta')$.

**Proof** (a): Let $k \in \text{Diff}(S^1)$ be such that $k \mid I = (\sqrt{\mathbf{i}}).$ Then $k^{-1} \cdot h \mid I$ is the identity, thus $V \equiv U(k^{-1} \cdot h) \in A(I')$ and $\text{Ad}(h) \mid A(I) = \text{Ad}(U(k)V \mid A(I)) = \text{Ad}(U(k) \mid A(I))$.

(b): We have

$$\Phi^{(c)}_I = \text{Ad}(U(h)) \mid A(I) = \text{Ad}(U(R(\pi) \cdot h) \mid A(I))$$

$$= \text{Ad}(U(R(\pi))) \text{Ad}(U(h) \mid A(I)) = \text{Ad}(U(R(\pi))) \cdot \Phi^{(c)}_I. \quad (19)$$

(c): The restriction of $h$ to $I$ does not vary as long we choose another $\zeta' \notin I$ such that $(\sqrt{\mathbf{i}}) = (\sqrt{\mathbf{i}})$ for all $z \in I$, thus $\Phi^{(c)}_I = \Phi^{(c)}_I$ for such $\zeta'$. Otherwise $h(z)$ changes to $-h(z)$, $z \in I$, and then $\Phi^{(c)}_I = \Phi^{(c)}_I$. The rest is now clear. □

We now set $\pi^{(c)}_I \equiv \chi_I \cdot (\Phi^{(c)}_I \otimes \Phi^{(c)}_I)$, where $\chi_I$ is the canonical isomorphism of $A(I_1) \otimes A(I_2)$ with $A(I_1) \vee A(I_2)$ given by the split property. In other words $\pi^{(c)}_I$ is the unique isomorphism of $A(I) \otimes A(I)$ with $A(I_1) \vee A(I_2)$ such that

$$\pi^{(c)}_I(x_1 \otimes x_2) = \Phi^{(c)}_I(x_1)\Phi^{(c)}_I(x_2), \quad x_1, x_2 \in A(I). \quad (20)$$

**Proposition 4.2.** Let $I \subset \bar{I}$ be intervals and $\zeta \notin \bar{I}$. Then $\pi^{(c)}_I \mid A(I) \otimes A(I) = \pi^{(c)}_I$.

**Proof** Immediate by the above Proposition 4.1. □

**Corollary 4.3.** Let $I$ be an interval and $\zeta, \zeta' \notin I$. Then either $\pi^{(c)}_I = \pi^{(c)}_I$ or $\pi^{(c)}_I = \pi^{(c)}_I \cdot \alpha$, where $\alpha$ is the flip automorphism of $A \otimes A$.

The first alternative holds iff $\zeta, \zeta'$ both belong or both do not belong to the closure of the connected component of $I \setminus \{1\}$ intersecting the upper half plane.

**Proof** If $\Phi^{(c)}_I = \Phi^{(c)}_I$, then also $\Phi^{(c)}_I = \Phi^{(c)}_I$ and then clearly $\pi^{(c)}_I = \pi^{(c)}_I$.

Otherwise $\Phi^{(c)}_I = \Phi^{(c)}_I$, thus $\Phi^{(c)}_I = \Phi^{(c)}_I$, and we have

$$\pi^{(c)}_I(\alpha(x_1 \otimes x_2)) = \Phi^{(c)}_I(x_2)\Phi^{(c)}_I(x_1) = \Phi^{(c)}_I(x_1)\Phi^{(c)}_I(x_2)$$

$$= \pi^{(c)}_I(x_1 \otimes x_2), \quad x_1, x_2 \in A(I). \quad (21)$$

The rest follows by Prop. 4.1. □

In the following we shall denote the net $A \otimes A$ by $D$.

As usual we may identify $S^1 \setminus \{-1\}$ with $\mathbb{R}$ by the stereographic map. Let $A_0$ be the net on $\mathbb{R}$ obtained by restricting $A$ to $S^1 \setminus \{-1\}$. We denote by $\pi$ the restriction of $\pi^{(-1)}$ to $D_0 = A_0 \otimes A_0$. 

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Proposition 4.4. \( \pi \) is a representation of \( \mathcal{D}_0 \). Indeed \( \pi \) is an irreducible soliton.

Proof That \( \pi \) is a soliton representation follows from Prop. 4.2 and the fact that \( \pi^{(C)} \) is normal on \( \mathcal{D}(I) \) for every interval \( I \) not containing \( \zeta \), including the case \( \zeta \in \overline{I} \) (half-lines).

Now \( \pi(\mathcal{D}_0(I)) = \mathcal{A}(E) \) where \( E = \sqrt{I} \), thus

\[
\bigvee_{\zeta \notin I \in I} \pi(\mathcal{D}_0(I)) = \bigvee_{i \notin E \in I^{(2)}} \mathcal{A}(E) = \mathcal{A}(S^1 \setminus \{i, -i\}) = B(\mathcal{H})
\]

because \( \mathcal{A} \) is 2-regular by Haag duality and the factoriality of the local von Neumann algebras, so \( \pi \) is irreducible. \( \square \)

By Prop. 4.4 \( \pi \) is a representation of \( \mathcal{D}_0 \), namely \( \pi \) is consistently defined on all von Neumann algebras \( \mathcal{D}_0 \) with \( I \subset \mathbb{R} \) either an interval or an half-line. However \( \pi \) is not a DHR representation of \( \mathcal{D}_0 \) namely, given an interval \( I_0 \subset \mathbb{R} \), \( \pi \) is not normal on the \( C^* \)-algebra \( \mathcal{D}(I_0') \equiv \bigcup_{I \subset I_0} \mathcal{D}(I) \) (\( I \) interval of \( \mathbb{R} \)).

As \( \mathcal{D}_0 \) satisfies half-line duality, namely

\[
\mathcal{D}_0(-\infty, a') = \mathcal{D}_0(a, \infty), \quad a \in \mathbb{R},
\]

by the usual DHR argument \( [18] \) \( \pi \) is unitarily equivalent to a representation \( \rho \) of \( \mathcal{D}_0 \) on \( \mathcal{H} \otimes \mathcal{H} \) which acts identically on \( \mathcal{D}_0(-\infty, 0) \), thus \( \rho \) restricts to an endomorphism of \( \mathcal{D}_0(0, \infty) \).

Proposition 4.5. Setting \( M \equiv \mathcal{D}_0(0, \infty) \), the inclusion \( \rho(M) \subset M \) is isomorphic to the 2-interval inclusion \( \mathcal{A}(E) \subset \hat{\mathcal{A}}(E) \).

Proof In the circle picture with \( \zeta = -1 \), setting \( I = S^+ \) (the upper semicircle) and \( E \equiv \sqrt{I} \), we have \( M = \mathcal{D}(I) \) and

\[
\mathcal{A}(E) = \pi^{(C)}_I(\mathcal{D}(I)), \quad \mathcal{A}(E') = \pi^{(C)}_{I'}(\mathcal{D}(I')),
\]

thus we have an equality of inclusions:

\[
\{ \mathcal{A}(E) \subset \hat{\mathcal{A}}(E) \} = \left\{ \pi^{(C)}_I(\mathcal{D}(I)) \subset \pi^{(C)}_{I'}(\mathcal{D}(I'))' \right\}.
\]

As \( \pi \) is unitarily equivalent to \( \rho \) and \( \rho_I \) is the identity on \( \mathcal{D}(I') \), the second inclusion is isomorphic to

\[
\{ \rho_I(\mathcal{D}(I)) \subset \rho_{I'}(\mathcal{D}(I'))' \} = \{ \rho_I(\mathcal{D}(I)) \subset \mathcal{D}(I')' \} = \{ \rho_I(\mathcal{D}(I)) \subset \mathcal{D}(I) \}.
\]

\( \square \)

18
4.1 Canonical representation of $(A \otimes A)^{\text{flip}}$

We shall denote by $B \equiv (A \otimes A)^{\alpha}$ the fixed-point subnet of $D$ with respect to the flip symmetry $\alpha$.

By Prop. 4.3 $\pi_I^{(C)} \mid B(I) = \pi_I^{(C)} \mid B(I)$ for all $\zeta, \zeta' \notin I$, therefore

$$\tau_I \equiv \pi_I^{(C)} \mid B(I)$$

is independent of $\zeta \notin I$ and thus well defined.

Recall that the spin of a Möbius covariant representation is the lowest eigenvalue of the conformal Hamiltonian $L_0$ in the representation space.

**Corollary 4.6.** $\tau : I \mapsto \tau_I$ is a (DHR) diffeomorphism covariant representation of $B$ (with positive energy). The covariance unitary representation is given by

$$U^{(2)} \equiv U \cdot M^{(2)}$$

(see Sect. 2), where $U$ is the covariance unitary representation associated with $A$.

$\tau$ is direct sum of two irreducible diffeomorphism covariant representations with spin $c/16$ and $1/2 + c/16$.

**Proof** It follows by Prop. 4.4 that $\tau$ is a representation.

We shall show that the projective unitary representation $U^{(2)} \equiv U \cdot M^{(2)}$ of $\text{Diff}^{(2)}(S^1)$ implements the covariance of $\tau$, namely, setting $\bar{U}(g) \equiv U(g) \otimes U(g)$,

$$\tau_{gI}(\bar{U}(g)x\bar{U}(g)^*) = U^{(2)}(g)\tau_I(x)U^{(2)}(g)^*, \ \ I \in I, x \in B(I), g \in \text{Diff}^{(2)}(S^1).$$

The above formula will follow if we show that

$$\pi_{gI}^{(C)}(\bar{U}(g)x\bar{U}(g)^*) = U^{(2)}(g)\pi_I^{(C)}(x)U^{(2)}(g)^*, \ \ I \in I, x \in D(I), g \in \text{Diff}^{(2)}(S^1),$$

for some $\zeta \notin I, \zeta' \notin gI$, and indeed it will suffice to verify this for $x = x_1 \otimes 1$ or $x = 1 \otimes x_2, x_1, x_2 \in A(I)$. Suppose $x = x_1 \otimes 1$:

$$U^{(2)}(g)\pi_I^{(C)}(x)U^{(2)}(g)^* = \text{Ad}U^{(2)}(g)\Phi_I^{(C)}(x_1) = \text{Ad}U^{(2)}(g)U(h)(x_1)$$

$$= \text{Ad}U(h_g)U(g)(x_1) = \pi_{gI}^{(g\zeta)}(U(g)x_1U(g)^*) = \pi_{gI}^{(g\zeta)}(\bar{U}(g)x\bar{U}(g)^*) \quad (23)$$

where $h(z) = \sqrt{z}$ on $I$ and $h_g(z) = \sqrt{z}$ on $gI$ (see Prop. 2.2).

The computation in the case $x = 1 \otimes x_2$ is analogous.

Concerning the last statement, set $B_0(\mathbb{R})$ for the $C^*$-algebra $\bigcup_{I}B(I)$ (I bounded interval) and analogously for $D_0$ and note that

$$C = \pi(D_0(\mathbb{R}))' = \{\tau(B_0(\mathbb{R})), \pi(v)\}'$$

Thus $\text{Ad}\pi(v)$ acts ergodically on $\tau(B_0(\mathbb{R}))'$. Since $v^2 = 1$, $\dim(\tau(B_0(\mathbb{R})))' \leq 2$. As $U^{(2)}(R(2\pi)) = U(R(\pi))$ belongs to $\tau(B_0(\mathbb{R}']]'$, we have $\dim(\tau(B_0(\mathbb{R})))' = 2$, thus $\tau$ has exactly two irreducible direct summands.

The spin of these two representations is now soon computed by formula (14). \qed
**Lemma 4.7.** Let $\mathcal{A}$ be a local, split conformal net with the CMS property. Then $\mu_\mathcal{A} < \infty$.

**Proof** The CMS property holds for $\mathcal{D}$ by [31] (irreducible sectors of $\mathcal{D}$ are tensor product of irreducible sectors of $\mathcal{A}$). Thus for $\mathcal{B}$ by Cor. 3.5.

Now $\tau$ is the sum of two irreducible representations, thus, by the CMS property, $\tau$ has finite index.

With $I = S^+$ and $E = \sqrt{I}$ we have:

$$\mathcal{A}(E) = \pi_I(\mathcal{D}(I)) \supset \tau_I(\mathcal{B}(I))$$  \hspace{1cm} (24)

$$\mathcal{A}(E') = \pi_{I'}(\mathcal{D}(I')) \supset \tau_{I'}(\mathcal{B}(I'))$$  \hspace{1cm} (25)

thus

$$\tau_I(\mathcal{B}(I)) \subset \mathcal{A}(E) \subset \hat{\mathcal{A}}(E) \subset \tau_{I'}(\mathcal{B}(I'))$$

but $\tau_I(\mathcal{B}(I)) \subset \tau_{I'}(\mathcal{B}(I'))$ has finite index and this entails $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)] < \infty$. \hfill $\square$

Recall that a local net $\mathcal{A}$ is said to be $n$-regular if $\mathcal{A}(S^1 \setminus F)$ is irreducible if $F \subset S^1$ is a finite set with $n$ points, namely $(\vee_{I,F=\emptyset} \mathcal{A}(I))' = \mathcal{C}$ ($I \in \mathcal{T}$).

It is immediate that, if $\mathcal{A}$ is conformal, $n$-regularity does not depend on the choice of the $n$-point $F$ set and

$\mathcal{A}$ is 2n-regular $\iff \mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is irreducible  \hspace{1cm} (26)

where $E$ is any $n$-interval.

**Lemma 4.8.** Let $\mathcal{A}$ be a local, split conformal net. If $\mu_\mathcal{A} < \infty$ then the 2-interval inclusion $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$ is irreducible. Thus $\mathcal{A}$ is 4-regular.

**Proof** By Prop. 4.5 we have to show that $\rho_I(\mathcal{D}(I))' \cap \mathcal{D}(I) = \mathcal{C}$. This would follow from the theorem on the equivalence between local and global intertwiners [25], but $\rho$ is not a DHR representation and that theorem does not apply here directly, but it will nevertheless give the result.

Let $T \in \rho_I(\mathcal{D}(I))' \cap \mathcal{D}(I) = (\rho_I(\mathcal{D}(I)) \vee \rho_I(\mathcal{D}(I')))'$. Then $T \in \theta_I(\mathcal{B}(I)) \vee \theta_{I'}(\mathcal{B}(I'))$, thus $T \in \theta(\mathcal{B})'$ due to the equivalence between local and global intertwiners, because $\rho | \mathcal{B}$ is a covariant, finite-index representation.

On the other hand $T \rho_I(v) = \rho_I(v)T$, thus $T$ commutes with $\{\rho_I(\mathcal{B}(I)), \rho_I(v)\}'' = \rho_I(\mathcal{D}(I))$ for all intervals $I \supset I$ and $T$ is a scalar because $\rho$ is irreducible. \hfill $\square$

We now state and begin to prove the dichotomy.

**Theorem 4.9.** Let $\mathcal{A}$ be a local conformal net with the split property. Assume that every irreducible sector of $\mathcal{A}$ is finite dimensional. We then have the following dichotomy: Either

(a) $\mathcal{A}$ is completely rational or
(b) \( \mathcal{A} \) has uncountably many different irreducible sectors.

**Proof** Assuming that \( \mathcal{A} \) has the CMS property, we have to show that \( \mathcal{A} \) is completely rational. By Lemma 4.7 \( \mu_\mathcal{A} < \infty \), thus we have to show that, for a local conformal net with the split property, the implication “\( \mu_\mathcal{A} < \infty \Rightarrow \) strong additivity” holds. This will be the content of Section 5. \( \square \)

### 4.2 The canonical endomorphism of the \( n \)-interval inclusion

In this section \( \mathcal{A} \) is again a local conformal net with the split property and \( \mathcal{D} = \mathcal{A} \otimes \mathcal{A} \). Our results have direct extension to the case of a general \( n \)-fold tensor product, but we deal with the case \( n = 2 \) for simplicity, but in the last corollary.

We keep the above notations, thus \( \pi \) is the canonical representation of \( \mathcal{D}_0 \) and \( \rho \) is a soliton endomorphism of \( \mathcal{D}_0 \) equivalent to \( \pi \) and localized in \( S^+ \). The conjugate sector \( \bar{\rho} \) of \( \rho \) is given by \( [\bar{\rho}] = [\cdot j \cdot \rho \cdot j] \) where \( j = \text{Ad} J \) with \( J \) the modular conjugation of \( (\mathcal{A}(S^+), \Omega) \) [25]. Note that \( j \cdot \rho \cdot j \) is localized in the lower semicircle \( S^- \) but, as \( \bar{\rho} \) is normal on \( \mathcal{A}(S^-) \), we can choose, in the same unitary equivalence class of \( j \cdot \rho \cdot j \), an endomorphism \( \bar{\rho} \) localized in \( \mathcal{A}(S^+) \).

**Proposition 4.10.** \( \bar{\rho} \rho \) is a soliton of \( \mathcal{D}_0 \) localized in \( S^+ \).

**Proof** The statement is clear by the above comments, as both \( \rho \) and \( \bar{\rho} \) are solitons localized in \( S^+ \). \( \square \)

Denote by \( \lambda_\mathcal{E} \) the dual canonical endomorphism associated with the inclusion \( \mathcal{A}(\mathcal{E}) \subset \hat{\mathcal{A}}(\mathcal{E}) \).

**Proposition 4.11.** Let \( \rho \) be localized in the right half-line \( I \subset \mathbb{R} \approx S^1 \setminus \{-1\} \). If \( S^1 \setminus \{-1\} \supset \tilde{I} \supset I \) is a half-line, the two squares of inclusions

\[
\begin{align*}
\mathcal{D}(I) & \subset \mathcal{D}(\tilde{I}) & \hat{\mathcal{A}}(E) & \subset \hat{\mathcal{A}}(\tilde{E}) \\
\cup & \cup & \cup & \cup \\
\rho_I(\mathcal{D}(I)) & \subset \rho_I(\mathcal{D}(\tilde{I})) & \mathcal{A}(E) & \subset \mathcal{A}(\tilde{E})
\end{align*}
\]

are isomorphic, where \( E = \sqrt{I} \), \( \tilde{E} = \sqrt{\tilde{I}} \).

If \( \bar{\rho} \) is also localized in \( I \) the isomorphism \( \pi_{\tilde{I}} : \mathcal{D}(\tilde{I}) \to \mathcal{A}(\tilde{E}) \), interchanges \( [\rho_I \rho_I | \mathcal{D}(I)] \) and \( [\lambda_E] \).

**Proof** Let \( U \) be a unitary from \( \mathcal{H} \) to \( \mathcal{H} \otimes \mathcal{H} \) such that \( \pi_{\tilde{I}'} = \text{Ad} U | \mathcal{D}(I') \). Then we can assume \( \rho_I = \text{Ad} U^* \cdot \pi_I \). The isomorphism \( \pi_I : \mathcal{D}(I) \to \mathcal{A}(E) \) is thus the composition

\[
\mathcal{D}(I) \xrightarrow{\rho_I} \rho_I(\mathcal{D}(I)) \xrightarrow{\text{Ad} U} \mathcal{A}(E)
\]

\( \text{Ad} U \) maps \( \rho_I(\mathcal{D}(I)) \) onto \( \mathcal{A}(E) \) and \( \mathcal{D}(I) \) onto \( \hat{\mathcal{A}}(E) \) as in Prop. 4.5. As \( \tilde{I}' \subset I' \), we also have \( \pi_{\tilde{I}'} = \text{Ad} U | \mathcal{D}(\tilde{I}') \), therefore \( \text{Ad} U \) maps \( \rho_I(\mathcal{D}(\tilde{I})) \) onto \( \mathcal{A}(\tilde{E}) \) and \( \mathcal{D}(\tilde{I}) \) onto \( \hat{\mathcal{A}}(\tilde{E}) \), thus \( \text{Ad} U \) implements an isomorphism between the two squares.
In particular \( \text{Ad}U \) will interchange \( \lambda_E \) with the dual canonical endomorphism associated with \( \rho_I(\mathcal{D}(I)) \subset \mathcal{D}(I) \), which is \( \rho_I \bar{\rho}_I \mid \rho_I(\mathcal{D}(I)) \) (here \( \bar{\rho}_I \) is the conjugate of \( \rho_I \) as sectors of \( \mathcal{D}(I) \)). Then \( \rho_I \) will interchange the latter with \( \rho_I^{-1} \rho_I \bar{\rho}_I \rho_I = \bar{\rho}_I \rho_I \).

It will follow from the results in Sect. 6 that, in the case \( n = 2 \), \( \rho \) is self-conjugate, as both \( \rho \) and \( \bar{\rho} \) are associated with a degree 2 map on \( S^1 \). In the case of the \( n \)-fold tensor product this fact is not any longer true and we shall have a formula for \( \bar{\rho} \) in Prop. 6.1 which gives

\[
\bar{\rho} \simeq \beta^{-1} \cdot \rho \cdot \beta_p ,
\]

where \( \beta \) is the natural action of \( \mathbb{P}_n \) on \( A \otimes \cdots \otimes A \) and \( p \in \mathbb{P}_n \) is the inverse map on the group \( \mathbb{Z}_n \).

As a corollary of Prop. 4.11 we now show that in the completely rational case \( \rho \bar{\rho} \) is a true representation and we can express it explicitly. Here, the structure is better understood by dealing with the case of an arbitrary \( n \)-fold tensor product.

**Corollary 4.12.** Suppose \( \mathcal{A} \) is completely rational, \( \mathcal{D} = A \otimes \cdots \otimes A \) (\( n \)-fold tensor product) and let \( \rho \) be a soliton endomorphism equivalent to \( \pi \) (see also Sect. 6).

Then \( [\rho \bar{\rho}] = [\bar{\rho} \rho] \) is a DHR sector, and we have the equality (as sectors)

\[
\bar{\rho} \rho = \bigoplus_{i_0,i_1,\ldots,i_{n-1}} N^0_{i_0,i_1,\ldots,i_{n-1}} \rho_{i_0} \otimes \rho_{i_1} \otimes \cdots \otimes \rho_{i_{n-1}} ,
\]

where \( N^0_{i_0,i_1,\ldots,i_{n-1}} \) is the multiplicity of the identity sector in the product \( \rho_{i_0} \cdot \rho_{i_1} \cdots \rho_{i_{n-1}} \) and the sum is taken over all irreducible sectors of \( \mathcal{A} \).

**Proof** Formula (29) for \( \bar{\rho} \rho \) follows immediately by Prop. 4.11, which gives \( \bar{\rho} \rho \) in terms of the formula for the canonical endomorphism of the \( n \)-interval inclusion given in [31] in the completely rational case.

To show that \( \bar{\rho} \rho \) is equivalent to \( \rho \bar{\rho} \) note that by eq. (28) we have, setting \( \beta \equiv \beta_p = \beta^2 \),

\[
\rho \bar{\rho} = \rho \beta \rho \beta = \beta (\beta \rho \beta) \beta = \beta (\bar{\rho} \rho) \beta ,
\]

that, combined with formula (29) gives

\[
\rho \bar{\rho} = \bigoplus_{i_0,i_1,\ldots,i_{n-1}} N^0_{i_0,i_1,\ldots,i_{n-1}} \rho_{i_0} \otimes \rho_{i_1} \otimes \cdots \otimes \rho_{i_{n-1}}
\]

\[
= \bigoplus_{i_0,i_1,\ldots,i_{n-1}} N^0_{i_{p-1(0)}i_{p-1(1)}\ldots,i_{p-1(n-1)}} \rho_{i_0} \otimes \rho_{i_1} \otimes \cdots \otimes \rho_{i_{n-1}}
\]

which coincides with formula (29) because the \( \rho_i \)'s form a commuting family. \( \square \)

Note in particular the special case \( n = 2 \) in Cor. 4.12 gives the formula

\[
\rho^2 = \bar{\rho} \rho = \bigoplus_i \rho_i \otimes \bar{\rho}_i .
\]
5 Split & \( \mu_A < \infty \) imply strong additivity

Before deriving the strong additivity property from the finite \( \mu \)-index assumption, we recall some basic facts about simple subfactors [35]. Let \( M \) be a factor in a standard form on a Hilbert space \( \mathcal{H} \) with modular conjugation \( J \). A subfactor \( N \subset M \) is simple if

\[
N \vee JNJ = B(\mathcal{H}) .
\]

In other words \( N \) is a simple subfactor iff \( N' \cap M_1 = \mathbb{C} \) where \( M_1 \equiv JN'J \) is the basic extension in the sense of Jones [28]; in particular \( N' \cap M = \mathbb{C} \).

If \( N \) is a simple subfactor and there exists a normal conditional expectation \( \varepsilon \) from \( M \) onto \( N \), then \( N = M \). Indeed the expectation is faithful and the Takesaki-Jones projection implementing \( \varepsilon \) belongs to \( N' \cap M_1 = \mathbb{C} \), thus \( \varepsilon \) is the identity. In particular

\[
N \subset M \text{ simple } \& \ [M : N] < \infty \Rightarrow N = M ,
\]

which is the implication we are going to use.

We now return to a local conformal net \( \mathcal{A} \). We shall denote by \( \mathcal{A}^d \) the dual net of \( \mathcal{A} \) on \( \mathbb{R} \), namely \( \mathcal{A}^d(I) \equiv \mathcal{A}(\mathbb{R} \setminus I)' \).

**Lemma 5.1.** Let \( \mathcal{A} \) be a local Möbius covariant net. If \( I \subset \mathbb{R} \) is a bounded interval and \( I_1, I_2 \) the intervals obtained by removing a point from \( I \), we have:

(a) \( \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}(I) \subset \mathcal{A}^d(I) \) is a basic extension. In particular

\[
[\mathcal{A}^d(I) : \mathcal{A}(I)] = [\mathcal{A}(I) : \mathcal{A}(I_1) \vee \mathcal{A}(I_2)] .
\]

(b) \( \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}(I) \) is a simple subfactor \( \iff \mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}^d(I) \) is irreducible \( \iff \mathcal{A} \) is 4-regular.

**Proof** (a): By dilation-translation covariance we can assume that \( I_1 = (-1, 0) \), \( I_2 = (0, 1) \), \( I = (-1, 1) \). The modular conjugation \( J \) of \( M \equiv \mathcal{A}(-1, 1) \) is associated with the ray inversion map \( t \rightarrow -1/t \). With \( N = \mathcal{A}(-1, 0) \vee \mathcal{A}(0, 1) \) we then have:

\[
M_1 \equiv JN'J = J(\mathcal{A}(-1, 0) \vee \mathcal{A}(0, 1))'J = (\mathcal{A}(-\infty, -1) \vee \mathcal{A}(1, \infty))' = \mathcal{A}(-1, \infty) \cap \mathcal{A}(-\infty, 1) = \mathcal{A}^d(-1, 1) \quad (31)
\]

(b): This follows because

\[
N \vee JNJ = (\mathcal{A}(-1, 0) \vee \mathcal{A}(0, 1)) \vee J(\mathcal{A}(-1, 0) \vee \mathcal{A}(0, 1))J
\]

\[
= (\mathcal{A}(-1, 0) \vee \mathcal{A}(0, 1)) \vee (\mathcal{A}(-\infty, 0) \vee \mathcal{A}(0, \infty)) \quad (32)
\]

which is equal to \( B(\mathcal{H}) \) iff \( \mathcal{A} \) is 4-regular. \( \square \)
Lemma 5.2. Let $\mathcal{A}$ be a local M"obius covariant net. If $I \subset \mathbb{R}$ is a bounded interval and $I_1, I_2$ the intervals obtained by removing a point from $I$. Assume $[\mathcal{A}^d(I) : \mathcal{A}(I)] < \infty$. We have:

$\mathcal{A}$ is 4-regular $\Rightarrow \mathcal{A}$ is strongly additive.

Proof If $\mathcal{A}$ is 4-regular then $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) \subset \mathcal{A}(I)$ is a simple subfactor by Lemma 5.1. On the other hand there exists a normal expectation $\mathcal{A}(I) \to \mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ by the finite index assumption and Lemma 5.1. But there is no normal expectation onto a simple subfactor, unless the inclusion is trivial. Thus $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$, i.e. $\mathcal{A}$ is strongly additive. □

Theorem 5.3. Let $\mathcal{A}$ be a local conformal net with the split property. If $\mu_\mathcal{A}$ is finite, then $\mathcal{A}$ is strongly additive (thus completely rational).

Proof If $\mu_\mathcal{A} < \infty$, then the 2-interval inclusion is irreducible by Lemma 4.8, hence $\mathcal{A}$ is 4-regular. By the following Lemma 5.4 and Lemma 5.2 we get the thesis. □

Let $\mathcal{A}$ be a split local conformal net. If $\mu_\mathcal{A} < \infty$ we shall denote by $\varepsilon_E : \hat{\mathcal{A}}(E) \to \mathcal{A}(E)$ the conditional expectation associated with the 2-interval $E$ (unique by Lemma 4.8). The following lemma is contained in [31].

Lemma 5.4. Assume that the $\mu$-index of $\mathcal{A}$ is finite. Given a bounded interval $I \in \mathcal{I}$, there is a finite index expectation $\varepsilon_I : \mathcal{A}^d(I) \to \mathcal{A}(I)$.

Proof Consider a decreasing sequence of 2-intervals $E_n \equiv I \cup I_n$ where $-1 \in I_n$ and $\cap_n I_n = \{-1\}$. As shown in [31]

$$ \mathcal{A}(E_n) \searrow \mathcal{A}(I), \quad \hat{\mathcal{A}}(E_n) \searrow \mathcal{A}^d(I). $$

As in Prop. 2 of [31], any weak limit point $\varepsilon_I$ of $\varepsilon_{E_n} |\hat{\mathcal{A}}^d(I)$ (as a map $\mathcal{A}^d(I) \to \mathcal{A}(E_1)$) is a finite index expectation from $\mathcal{A}^d(I)$ to $\mathcal{A}(I)$. □

6 Topological sectors and an index theorem

In this section we generalize the previous construction to the case of cyclic orbifold based on a local conformal net $\mathcal{A}$ with the split property.

Let $\zeta$ be a point of $S^1$ and $h : S^1 \setminus \{\zeta\} \simeq \mathbb{R} \to S^1$ a smooth injective map which is smooth also at $\pm \infty$, namely the left and right limits $\lim_{z \to \zeta^\pm} \frac{dh}{dz}$ exist for all $n$.

The range $h(S^1 \setminus \{\zeta\})$ is either $S^1$ minus a point or a (proper) interval of $S^1$.

With $I \in \mathcal{I}$, $\zeta \notin I$, we set

$$ \Phi_{h,I}^{(\zeta)} \equiv \text{Ad}U(k), $$

where $k \in \text{Diff}(S^1)$ and $k(z) = h(z)$ for all $z \in I$ and $U$ is the projective unitary representation of $\text{Diff}(S^1)$ associated with $\mathcal{A}$. 

Then $\Phi_{h,I}^{(\zeta)}$ does not depend on the choice of $k \in \text{Diff}(S^1)$ and

$$\Phi_{h,I}^{(\zeta)} : I \mapsto \Phi_{h,I}^{(\zeta)}$$

is a well defined soliton of $A_0 \equiv A \upharpoonright \mathbb{R}$.

Clearly $\Phi_{h,I}^{(\zeta)}(A_0(\mathbb{R}))'' = A(h(S^1 \setminus \{\zeta\}))''$, thus $\Phi_{h,I}^{(\zeta)}$ is irreducible if the range of $h$ is dense, otherwise it is a type III factor representation. It is easy to see that, in the last case, $\Phi_{h,I}^{(\zeta)}$ does not depend on $h$ up to unitary equivalence.

Let now $f : S^1 \to S^1$ be a smooth, locally injective map of degree $\deg f = n \geq 1$. Choosing $\zeta \in S^1$, there are $n$ right inverses $h_i$, $i = 0, 1, \ldots n - 1$, for $f$; namely there are $n$ injective smooth maps $h_i : S^1 \setminus \{\zeta\} \to S^1$ such that $f(h_i(z)) = z$, $z \in S^1 \setminus \{\zeta\}$. The $h_i$’s are smooth also at $\pm \infty$.

Note that the ranges $h_i(S^1 \setminus \{\zeta\})$ are $n$ pairwise disjoint intervals of $S^1$, thus we may fix the labels of the $h_i$’s so that these intervals are counterclockwise ordered, namely we have $h_0(-\zeta) < h_1(-\zeta) < \cdots < h_{n-1}(-\zeta) < h_0(-\zeta)$.

Of course any other possible choice for the $h_i$’s is associated with an element $p$ of the permutation group $\mathbb{P}_n$ on $\mathbb{Z}_n$, namely we can consider the sequence $h_{p(0)}, h_{p(1)}, \ldots$.

For any interval $I$ of $\mathbb{R}$, we set

$$\pi_{f,I}^{(\zeta)} \equiv \chi_I \cdot (\Phi_{h_{0,I}}^{(\zeta)} \otimes \Phi_{h_{1,I}}^{(\zeta)} \otimes \cdots \otimes \Phi_{h_{n-1,I}}^{(\zeta)}) , \quad (33)$$

where $\chi_I$ is the natural isomorphism from $A(I_0) \otimes \cdots \otimes A(I_{n-1})$ to $A(I_0) \lor \cdots \lor A(I_{n-1})$ given by the split property, with $I_k \equiv h_k(I)$. Clearly $\pi_{f,I}^{(\zeta)}$ is a soliton of $D_0 \equiv A_0 \otimes A_0 \otimes \cdots \otimes A_0$ ($n$-fold tensor product).

If we order the right inverses $h_i$’s according to the permutation $p$ as above, we shall denote the corresponding soliton by $\pi_{f,p}$, thus $\pi_f \equiv \pi_{f,\text{id}}$. Clearly

$$\pi_{f,p} = \pi_f \cdot \beta_p$$

where $\beta$ is the natural action of $\mathbb{P}_n$ on $D$.

**Proposition 6.1.** Fix $\zeta = -1$ and denote $\pi_{f,I}^{(\zeta)}$ simply by $\pi_f$.

(a): If $f_0$ has $\deg f_0 = \deg f$, then $\pi_{f_0}$ is unitary equivalent to $\pi_{f,p}$ for some $p \in \mathbb{P}_n$.

(b): $\pi_{f,p}$ depends only on $\deg f$ and $p$ up to unitary equivalence.

(c): $\text{Index}(\pi_f) = \mu_{A^{-1}}^{n-1}$.

(d): The conjugate of $\pi_f$ is given by

$$\pi_f = \pi_{f,p}$$

where $\bar{f}(z) \equiv f(\bar{z})$ and $p$ is the inverse automorphism $m \mapsto -m$ of $\mathbb{Z}_n$.

**Proof** (a): If $f_0 : S^1 \to S^1$ is an injective smooth map and $\deg f_0 = \deg f$, there exists a $h \in \text{Diff}(S^1)$ such that $f_0 = f \cdot h$. Then the $h^{-1} \cdot h_i$ are right inverses for $f_0$ and we have $\Phi_{h^{-1} \cdot h_i}^{(\zeta)} = \text{Ad} U(h)^* \cdot \Phi_{h_0^{p(i)}}^{(\zeta)}$ for some $p \in \mathbb{P}_n$, so $U(h)$ implements a unitary equivalence between $\pi_{f,p}^{(\zeta)}$ and $\pi_{f_0}^{(\zeta)}$.  

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(b): This is clear from the proof of (a).

(c): An obvious extension of Prop. 4.11 shows that the index of $\pi_I$ is equal to the index of the $n$-interval inclusion, therefore by [31] we have $\text{Index}(\pi_I) = \mu_{\mathbb{A}}^{n-1}$.

(d): if $\rho$ is a soliton endomorphism of $\mathcal{D}$ localized $S^+$, the formula in [24, Th. 4.1] gives $\tilde{\rho} = j \cdot \rho \cdot j$, where $j = \text{Ad}J$ with $J$ the modular conjugation of $(\mathcal{D}(S^+), \Omega)$. As we are interested in $\tilde{\pi}_I$ up to unitary equivalence, we then have

$$\tilde{\pi}_I = j_0 \cdot \pi_I \cdot j$$

where $j_0 \equiv \text{Ad}J_0$ with $J_0$ any unitary involution on the Hilbert space $\mathcal{H}$ of $\mathcal{A}$. Let $J_0$ then be the modular conjugation of $(\mathcal{A}(S^+), \Omega)$, thus $j = j_0 \otimes \cdots \otimes j_0$. With the above notations let $x_0, x_1, \ldots, x_{n-1} \in \mathcal{A}(\tilde{I})$ where $\tilde{I}$ denotes here the conjugate interval of $I \subset S^1 \setminus \{-1\}$. We have

$$\tilde{\pi}_{f,I}(x_0 \otimes \cdots \otimes x_{n-1}) = j_0(\pi_{f,I}(j_0(x_0) \otimes \cdots \otimes x_{n-1}))$$

$$= j_0(\pi_{f,I}(j_0(x_0) \otimes \cdots \otimes j_0(x_{n-1})))$$

$$= j_0(\Phi_{\tilde{I}}^{(n)}(j_0(x_0)) \cdots \Phi_{\tilde{I}}^{(n)}(j_0(x_{n-1})))$$

$$= \chi_I \cdot (j_0 \cdot \Phi_{\tilde{I}}^{(n)} \cdot j_0 \otimes \cdots \otimes j_0 \cdot \Phi_{\tilde{I}}^{(n)})(x_0 \otimes \cdots \otimes x_{n-1})$$

$$= \chi_I \cdot (\Phi_{\tilde{I}}^{(n)} \otimes \cdots \otimes \Phi_{\tilde{I}}^{(n)})(x_0 \otimes \cdots \otimes x_{n-1})$$

$$= \pi_{f,I}(x_0 \otimes \cdots \otimes x_{n-1}).$$

Here $p \in \mathbb{P}_n$ is the re-labeling of the right inverses $h_i$ of $f$ associated with the map $z \mapsto \bar{z}$ on the circle. It can be checked immediately in the case $f(z) = z^n$ that $p(k) = n - k.$

We shall now see the sector $[\pi_I]$ is independent of the choice of the initial interval in the counterclockwise order associated with the $h_i$'s. Thus $[\pi]$ and $[\tilde{\pi}_I]$ are the unique sectors associated respectively with any counterclockwise/clockwise ordering of the $h_i$'s.

**Proposition 6.2.** (a): If $p \in \mathbb{P}_n$ is a cyclic permutation, then $\pi_I$ is unitarily equivalent to $\pi_{f,p}$.

(b): $\pi_f$ is irreducible if and only if $\mathcal{A}$ is $n$-regular.

**Proof** (a): It suffices to consider the case $f(z) = z^n$. With the choice of the $n$-th root function $\sqrt[n]{z}$ with discontinuity at $-1$, we may order counterclockwise the right inverses by setting $h_\ell \equiv e^{\frac{2\pi i}{n}}h_0$, $\ell \in 0, 1, \ldots, n-1$.

Thus for any $j \in \mathbb{Z}_n$, $h_{\ell+j} = R^j \cdot h_\ell$, for all $\ell \in \mathbb{Z}_n$, where $R \equiv R(\frac{2\pi}{n})$ denotes the rotation on $S^1$ of angle $\frac{2\pi}{n}$ and so $U(h_{\ell+j}) = U(R^j)U(h_\ell)$ (up to a phase factor).

If $p$ is the cyclic permutation $\ell \mapsto \ell + j$ on $\mathbb{Z}_n$, it follows that

$$\pi_{f,p,I} = \chi_I \cdot (\Phi_{h_1,I}^{(n)} \otimes \cdots \otimes \Phi_{h_{n-1},I}^{(n)})$$

$$= \chi_I \cdot (\text{Ad}U(R^1) \cdot \Phi_{h_0,I}^{(n)} \otimes \cdots \otimes \Phi_{h_{n-1},I}^{(n)})$$

$$= \text{Ad}U(R^1) \cdot \chi_I \cdot (\Phi_{h_0,I}^{(n)} \otimes \cdots \otimes \Phi_{h_{n-1},I}^{(n)})$$

$$= \text{Ad}U(R^1) \cdot \pi_{f,I}$$

$$= \text{Ad}U(R^1) \cdot \pi_{f,I}$$

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(b): As \( I \) varies in the intervals of \( S^1 \setminus \{-1\} \), \( \pi_f(D(I)) = A(I_0) \vee \cdots A(I_{n-1}) \) generates \( A(S^1 \setminus F) \) where \( F \) is the set of \( n \) points obtained by removing \( \cup_i h_i(S^1 \setminus \{-1\}) \) from \( S^1 \), hence the thesis. \( \square \)

Remark. As already said, 2-regularity is automatic for any Möbius covariant local net; but there are examples of Möbius covariant local nets that are not 3-regular \[26\]. We conjecture that every diffeomorphism covariant local net is automatically \( n \)-regular for any \( n \).

As \( \zeta \) varies, the \( \Phi_k^{(i)} \)'s undergo permutations among them, indeed cyclic permutations that, with a proper labeling, correspond to the cyclic permutations on \((0, 1, \ldots, n-1)\). The restriction
\[
\tau_f \equiv \pi_f^{(i)} \upharpoonright (A \otimes A \cdots \otimes A)^{Z_n}
\]
is therefore a DHR representation of \((A \otimes A \cdots \otimes A)^{Z_n}\), independent of \( \zeta \) up to unitary equivalence.

In the following we shall denote by \( \mathcal{I}^{(n)} \) the set of all \( n \)-intervals of \( S^1 \), not necessarily symmetric (union of \( n \) intervals with pairwise disjoint closures).

**Theorem 6.3.** (a): \( \tau_f \) depends only on \( n = \deg f \) up to unitary equivalence.

(b): \( \tau_f \) is diffeomorphism covariant; the corresponding projective unitary representation of \( \text{Diff}^{(\infty)}(S^1) \) is unitary equivalent to the projective unitary representation \( U^{(n)} = U \cdot M^{(n)} \) of \( \text{Diff}^{(n)}(S^1) \).

(c): The following formula for the index holds:
\[
\text{Index}(\tau_f) = n^2 \mu_A^{n-1}.
\]

(d): \( \tau_f \) is direct sum of \( n \) diffeomorphism covariant representations \( \tau_f^{(0)}, \tau_f^{(1)}, \ldots, \tau_f^{(n-1)} \) of \((A \otimes A \cdots \otimes A)^{Z_n}\). Each \( \tau_f^{(i)} \) is irreducible.

(e) We may choose our labels so that, for every \( i = 0, 1, \ldots, n-1 \),
\[
\text{spin}(\tau_f^{(i)}) = \frac{i}{n} + \frac{n^2 - 1}{24n} c,
\]
\[
\text{Index}(\tau_f^{(i)}) = \mu_A^{n-1},
\]
where, in the last equation, we assume \( \mu_A < \infty \).

**Proof**  (a): Immediate by (a) of Prop. 6.1.

(b): Because of the above point, it suffices to consider the case \( f(z) = z^n \). Then the covariance follows by the characterization of the map \( M^{(n)} \) in Prop. 2.2 expressed by the commutativity of the diagram (8).

(c): Analogously as in Proposition 4.5, the inclusion \( \pi_f(M) \subset \pi_f(M') \), \( M = (A \otimes \cdots \otimes A)(0, \infty) \), is isomorphic to the \( n \)-interval inclusion \( A(E) \subset \hat{A}(E) \), \( E \in \mathcal{I}^{(n)} \). If \( \mu_A < \infty \), then \( A \) is completely rational and the index formula in [31] gives
\[
\text{Index}(\pi_f) = [\hat{A}(E) : A(E)] = \mu_A^{n-1}.
\]
As $\tau_f$ is the restriction of $\pi_f^{(i)}$ to a $n$-index subnet we then have

$$\text{Index}(\tau_f) = \left[(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^{\mathbb{Z}_n} : \mathcal{A} \otimes \cdots \otimes \mathcal{A} \right] \cdot \text{Index}(\pi_f) = n^2 \mu_{\mathcal{A}}^{n-1}.$$  

(d): Fix an interval $I_0$ and a unitary $v \in \mathcal{D}(I_0)$, $v^n = 1$, that implements the action on $\mathcal{B}$ dual to cyclic permutations. Then $\mathcal{D}(I) = \{\mathcal{B}(I), v\}''$ for all intervals $I \supset I_0$, hence

$$\left\{ \bigvee_{\zeta \notin I} \tau_{f,I}(\mathcal{B}(I)), \pi_f^{(i)}(v) \right\}'' = \left( \bigvee_{\zeta \notin I} \pi_{f,I}(\mathcal{D}(I)) \right)'' = \mathcal{A}(S^1 \setminus F),$$

where $F$ is an $n$-point subset of $S^1$ (the complement of $\cup_i h_i(S^1 \setminus \{\zeta\})$), that depends on $\zeta$.

Now $\tau_f$ is a DHR representation, so we may vary the point $\zeta$ and get

$$\left\{ \bigvee_{I \in \mathcal{I}} \tau_{f,I}(\mathcal{B}(I)), \pi_f^{(i)}(v) \right\}'' = \bigvee_{I \in \mathcal{I}} \pi_{f,I}(\mathcal{D}(I)) = \mathcal{A}(S^1) = \mathcal{B}(\mathcal{H}),$$

where $\zeta \notin \bar{I}$ varies with $I$. As $\pi_f^{(i)}(v)$ normalizes $\bigvee_{I \in \mathcal{I}} \tau_{f,I}(\mathcal{B}(I))$, it follows as in Cor. 4.6 for the case $n = 2$ that the latter is the commutant of $\pi_f(v)$ and $\tau_f$ has exactly $n$ irreducible components.

(e): As in the case $n = 2$, the covariance of $\tau_f$ is given by a unitary representation of $\text{Diff}(S^1)$ equivalent to $U^{(n)} = U \cdot M^{(n)}$. Thus the conformal Hamiltonian $L_0'$ in the representation $\tau_f$ is unitarily equivalent to the one given by formula (14), and this readily implies that the spin of the $\tau_f^{(i)}$’s are as stated, by a suitable choice of the index labels. We will have additional information on these labels in Sect. 8.3 after (46). Concerning the formula for the index, by (44) we have $d(\tau_f^{(i)}) = d(\pi_f)$. By point (c) we have $d(\pi_f) = \sqrt{\mu_{\mathcal{A}}^{n-1}}$, thus $\text{Index}(\tau_f^{(i)}) = \mu_{\mathcal{A}}^{n-1}$. □

6.1 Extension to non-vacuum representation case

The construction given above in Sect. 6 extends to the case where one replaces the vacuum representation with another covariant representation $\lambda$ (cf. [24, 34] and Appendix A for the covariance condition). This extension generates new sectors and will be later used. Here we merely outline the construction, but all the above results have natural extensions in this setting.

Let $\lambda$ be a covariant representation of $\mathcal{A}$. Given an interval $I \subset S^1 \setminus \{\zeta\}$, we set

$$\pi_{f,\lambda}(x) = \lambda_J(\pi_{f,I}(x)), \quad x \in \mathcal{D}(I),$$

where $\pi_{f,I} \equiv \pi_f^{(i)}$ is defined as in (33), and $J$ is any interval which contains $I_0 \cup I_1 \cup \ldots \cup I_{n-1}$.

**Proposition 6.4.** The above definition is independent of the choice of $J$, thus $\pi_{f,\lambda}$ is a well defined soliton of $\mathcal{D}$. 

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We can choose an interval $I$ with $\zeta$ as a boundary point of $I$ such that $\pi_f$, $\pi_{f_h}$ and $\lambda$ are localized on $I$. Denote by $\tilde{\pi}_f$, $\tilde{\pi}_{f_h}$ and $(\lambda, 1, 1, \ldots, 1) := \lambda \otimes t \otimes u \cdots \otimes u \mid \mathcal{D}(I)$ respectively the corresponding endomorphisms of $\mathcal{D}(I)$. Then as sectors of $\mathcal{D}(I)$ we have

$$[	ilde{\pi}_{f_h}] = [\tilde{\pi}_f \cdot (\lambda, 1, 1, \ldots, 1)].$$

**Proof** If $J_1$ is another interval which contains $I_0 \cup I_1 \cup \cdots \cup I_{n-1}$, we need to show that $\pi_{\lambda,J_1}(x) = \pi_{\lambda,J}(x), \forall x \in \mathcal{A}(I) \otimes \cdots \otimes \mathcal{A}(I)$. It is sufficient to prove this for $x = x_0 \otimes \cdots \otimes x_{n-1}, x_i \in \mathcal{A}(I), 0 \leq i \leq n - 1$. By isotony, we have

$$\pi_{f_h,J_1}(x_0 \otimes \cdots \otimes x_{n-1}) = \lambda_{J_1}(\Phi_{h_0,I}(x_0)) \cdots \lambda_{J_1}(\Phi_{h_{n-1},I}(x_{n-1}))$$

$$= \lambda_{J}(\Phi_{h_0,I}(x_0)) \cdots \lambda_{J}(\Phi_{h_{n-1},I}(x_{n-1})) = \pi_{f_h,J}(x_0 \otimes \cdots \otimes x_{n-1}).$$

This shows that the above definition is independent of the choice of $J$.

As for the last formula, we may assume that $\zeta = -1, I = S^+$ (the upper half circle), $f(z) = z^n$, $h_0$ is the $n^{th}$-root function on $I$ with $h_0(1) = 1$ so that $I_0 \subset I$, and $h_0 \in \text{Diff}(J_0)$ for some interval $J_0 \supset I$, i.e. $h_0 \in \text{Diff}(S^1)$ and $h_0$ acts identically on $J_0'$. We may further assume that $\lambda$ is localized in $I_0$. By our assumption $U(h_0) \in \mathcal{A}(J_0)$, and we claim that

$$\lambda_{J_0}(U(h_0)\lambda_{J_1}(x)\lambda_{J_0}(U(h_0))^* = \lambda_{h_0,J_1}(U(h_0)xU(h_0)^*), \forall x \in \mathcal{A}(J_1), \forall J_1 \in \mathcal{I}. \quad (38)$$

This can be checked as follows: If $J_0 \cup J_1 \neq S^1$, then we can find an interval $J_2$ such that $J_0 \cup J_1 \subset J_2$, and in this case

$$\lambda_{J_0}(U(h_0)\lambda_{J_1}(x)\lambda_{J_0}(U(h_0))^* = \lambda_{J_2}(U(h_0)xU(h_0)^*), \forall x \in \mathcal{A}(J_1);$$

note that $U(h_0)xU(h_0)^* \in \mathcal{A}(h_0(J_1))$, and $h_0(J_1) \subset J_2$, so by isotony we have

$$\lambda_{J_2}(U(h_0)xU(h_0)^* = \lambda_{h_0,J_1}(U(h_0)xU(h_0)^*).$$

In general we cover $J_1$ by a set of sub-intervals $J_k \subset J_1, 2 \leq k \leq m$ such that $J_k \cup J_0 \neq S^1$. By additivity of conformal nets we have $\mathcal{A}(J_1) = \bigvee_{2 \leq k \leq m} \mathcal{A}(J_k)$, and since the equation (38) is true for any $x \in \mathcal{A}(J_k), 2 \leq k \leq m$, it follows that we have proved equation (38).

Define $z_\lambda(h_0) := \lambda_{J_0}(U(h_0))^*U(h_0)^*$. From (38) we have

$$\lambda_{h_0,J_1}(\text{Ad}U(h_0)(x)) = z_\lambda(h_0)\text{Ad}U(h_0)(\lambda_{J_1}(x))z_\lambda(h_0)^*, \forall x \in \mathcal{A}(J_1), \forall J_1.$$

Set $J_1 = I'$, we conclude from the above equation that $z_\lambda(h_0) \in \mathcal{A}(I') = \mathcal{A}(I_0)$. It follows that for all $x_0 \otimes x_1 \cdots \otimes x_{n-1} \in \mathcal{D}(I)$, we have

$$\lambda_{I_0}(\text{Ad}U(h_0)(x_0)) \otimes \text{Ad}U(h_1)(x_1) \otimes \cdots \otimes \text{Ad}U(h_{n-1})(x_{n-1})$$

$$= \text{Ad}z_\lambda(h_0) \cdot \text{Ad}(U(h_0(\lambda_I(x_0)) \otimes \text{Ad}U(h_1)(x_1) \otimes \cdots \otimes \text{Ad}U(h_{n-1})(x_{n-1}))$$

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where $h_1, \ldots, h_{n-1}$ are defined as in (33). Therefore on $\mathcal{D}(I)$

$$\pi_{f, I} = \text{Ad}z_{\lambda}(h_0) \cdot \pi_{f, I} \cdot (\lambda, 1, 1, \ldots). \quad (39)$$

Let $U' : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ ($n$-tensor factors) be a unitary such that $U'\pi_{f, I'}(\cdot)U'^* = \text{id}$ on $\mathcal{D}(I')$. Then both $\tilde{\pi}_f := U'\pi_{f, I}(\cdot)U'^*$ and $\tilde{\pi}_{f, \lambda} := U'\pi_{f, I, \lambda}(\cdot)U'^*$ are endomorphisms of $\mathcal{D}(I)$, and we have $\tilde{\pi}_{f, \lambda} = \text{Ad}U'z_{\lambda}(h_0)U'^* \cdot \tilde{\pi}_f \cdot (\lambda, 1, 1, \ldots, 1)$ by (39). Therefore, as sectors of $\mathcal{D}(I)$, we have

$$[\tilde{\pi}_{f, \lambda}] = [\tilde{\pi}_f \cdot (\lambda, 1, 1, \ldots, 1)]$$

since $U'z_{\lambda}(h_0)U'^* \in U'\mathcal{A}(I_0)U'^* \subset \mathcal{D}(I')' = \mathcal{D}(I)$.

\[ \square \]

7 Some consequences

We now discuss a few consequences of our results. The first two ones follow immediately from the implication “rationality $\Rightarrow$ complete rationality” because of the corresponding results in [31] in the completely rational case.

7.1 Rationality implies modularity

The first consequence concerns the invertibility of the matrices $T$ and $S$ in a rational model, see [49]. This property has long been expected and is at the basis most analysis, in particular concerning Topological QFT, cf. for example [23].

We shall say that a local conformal net $\mathcal{A}$ is rational if there are only finitely many irreducible sectors and all of them have a conjugate sector, i.e. they have finite index [36, 25]. Assuming the split property, then every sector is direct sum of irreducible sectors, cf. [31].

In the paper [31] the modularity has been proved for a completely rational local Möbius covariant net. By our results, complete rationality is equivalent to rationality for a local conformal net with the split property. Hence we have:

**Theorem 7.1.** Let $\mathcal{A}$ be a conformal net with the split property. If $\mathcal{A}$ is rational, then the tensor category of representations of $\mathcal{A}$ is modular, i.e. the braiding symmetry is non-degenerate.

7.2 The $\mu$-index is always equal to the global index

The equality of the $\mu$-index with the global index has been proved in [31] in the completely rational case. The extension of this equality to the case of infinite $\mu$-index is not covered by that work, in particular there was no argument to show that if there is no non-trivial sector then Haag duality holds for multi-connected regions. This is given here below.
Theorem 7.2. Let $\mathcal{A}$ be a conformal net with the split property. Then

$$
\mu_A = \sum_i d(\rho_i)^2,
$$

where the sum is taken over all irreducible sectors or, equivalently, over all the irreducible sectors that are diffeomorphism covariant with positive energy.

Proof If $\mu_A < \infty$ then $\mathcal{A}$ is completely rational by Theorem 5.3, thus the formula holds by [31].

If $\mu_A = \infty$ either there exists an irreducible sector with infinite index and formula (40) obviously holds, or by Th. 5.3 there are (uncountably) infinitely many irreducible sectors, thus (40) holds because $d(\rho_i) \geq 1$. □

Corollary 7.3. Let $\mathcal{A}$ be a conformal net with the split property. The following are equivalent:

(i) $\mathcal{A}$ has no non-trivial representation,

(ii) Haag duality holds for some $n$-intervals $E$ for some $n \geq 2$: $\mathcal{A}(E)' = \mathcal{A}(E')$,

(iii) Haag duality holds for all $n$-intervals: $\mathcal{A}(E)' = \mathcal{A}(E')$ for all $E \in \mathcal{I}^{(n)}$, $\forall n \in \mathbb{N}$.

Proof By eq. (40), (i) holds iff $\mu_A = 1$, namely iff (ii) holds with $n = 2$. In this case $\mathcal{A}$ is completely rational by Th. 40 and the formula $[\hat{A}(E) : \mathcal{A}(E)] = \mu_A^{n-1}$, $E \in \mathcal{I}^{(n)}$, in [31] shows that also (iii) holds.

It remains to show that (ii) $\Rightarrow$ (iii). Assume that $\mathcal{A}(E)' = \mathcal{A}(E')$ for some $n$-interval $E$, $n \geq 2$. Then $\mathcal{A}(E)' = \mathcal{A}(E')$ for all $n$-intervals $E$ by diffeomorphism covariance. Fix $E \in \mathcal{I}^{(n)}$ and $I$ one of its connected components. By considering a decreasing sequence of intervals $I \supset I_1 \supset I_2 \supset \cdots$ shrinking to a point, it is rather immediate to check, by the split property, that Haag duality $\mathcal{A}(E)' = \mathcal{A}(E')$ holds for $n - 1$-intervals. By iteration we get Haag duality for a 2-interval and then conclude our proof as above. □

7.3 Sectors with infinite statistics

General properties of sectors with infinite dimension were studied in [24] (see also [3]), yet first examples have been constructed by Fredenhagen in [22], see below. A natural family of infinite dimensional irreducible sectors has recently been pointed out by Carpi [14] in the Virasoro nets with $c > 1$, following a conjecture by Rehren [48].

The following theorem gives a natural and general construction of irreducible sectors with infinite dimension, as a consequence of the index formula in Sect. 6.

Theorem 7.4. Let $\mathcal{A}$ be a conformal net with the split property. The following are equivalent:

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(i) $\mathcal{A}$ is not completely rational;

(ii) $(\mathcal{A} \otimes \mathcal{A})^{\text{flip}}$ has an irreducible sector with infinite dimension;

(iii) $(\mathcal{A} \otimes \cdots \otimes \mathcal{A})^{\mathbb{Z}_n}$ has an irreducible sector with infinite dimension and diffeomorphism covariant with positive energy, any $n \geq 2$.

Proof Clearly (ii) or (iii) imply that $\mathcal{A}$ is not completely rational (complete rationality if hereditary for finite-index subnets $[59, 42]$). On the other hand, if $\mathcal{A}$ is not completely rational, the topological sector $\tau_f$ of the cyclic $n$-orbifold has infinite index by the index formula in Th. 6.3. So one of the $n$ direct summands $\tau_f^{(i)}$ must have infinite index. □

7.3.1 Example

Let $\mathcal{A}$ be the local conformal net on $S^1$ associated with the $U(1)$-current algebra. In the real line picture $\mathcal{A}$ is given by

$$\mathcal{A}(I) \equiv \{W(f) : f \in C^\infty_c(\mathbb{R}), \supp f \subset I\}''$$

where $W$ is the representation of the Weyl commutation relations

$$W(f)W(g) = e^{-i\int fg'}W(f+g)$$

associated with the vacuum state $\omega$

$$\omega(W(f)) \equiv e^{-||f||^2}, \quad ||f||^2 \equiv \int_0^\infty |\hat{f}(p)|^2pd$$

where $\hat{f}$ is the Fourier transform of $f$.

The superselection structure of $\mathcal{A}$ is completely described in [10]. There is a one parameter family $\{\alpha_q, q \in \mathbb{R}\}$ of irreducible sectors and all have index 1. We can choose a representative of $\alpha_q$ as

$$\alpha_q(W(f)) \equiv e^{2i\int Ff'}W(f), \quad F \in C^\infty, \quad \int F = q.$$ 

Now consider $\mathcal{A} \otimes \mathcal{A}$. By the argument in [31] all irreducible sectors of $\mathcal{A} \otimes \mathcal{A}$ are tensor product sectors, namely have the form $\alpha_q \otimes \alpha_{q'}$, in particular they have index 1.

Yet, the index 2 subnet $(\mathcal{A} \otimes \mathcal{A})^{\text{SO}(2)}$ has an irreducible sector with infinite index, by Th. 7.4 because $\mathcal{A}$ is not completely rational.

Fredenhagen [22] had shown that the subnet $(\mathcal{A} \otimes \mathcal{A})^{SO(2)} \subset \mathcal{A} \otimes \mathcal{A}$ admits an infinite dimensional irreducible sector. In his case the subnet $(\mathcal{A} \otimes \mathcal{A})^{SO(2)} \subset \mathcal{A} \otimes \mathcal{A}$ has infinite index.
8 Topological twisted sectors in the completely rational case

In this section we assume that $\mathcal{D}$ is a completely rational conformal net and $\mathcal{B}$ is the fixed point subnet of $\mathcal{B}$ under the proper action of $\mathbb{Z}_n$ on $\mathcal{D}$ (cf. 2 of [56]).

We note that we will be interested in the special case when $\mathcal{D} := A \otimes A \ldots \otimes A$ ($n$-fold tensor product) and $\mathcal{B} := (A \otimes A \ldots \otimes A)^{\mathbb{Z}_n}$ the fixed point subnet of $\mathcal{D}$ under the action of cyclic permutations in Sect. 8.

By Th. 2.9 of [56], $\mathcal{B}$ is completely rational with $\mu_\mathcal{B} = n^2 \mu_\mathcal{D}$. So $\mathcal{B}$ has finitely many inequivalent irreducible representations and the question is how to construct these representations from those of $\mathcal{D}$. This question can be raised for the case of a general orbifold. An answer to this question is given in an example of $\mathbb{Z}_2$ orbifold of a lattice by identifying the orbifold with a coset whose irreducible representations are known (cf. Sect. 3 of [56]). Partially motivated by this question for the case of cyclic permutations, [58] and [2], we were led to the constructions of Sect. 2 and 6.

We will see that the topological construction of Sect. 6 and its generalizations lead to a satisfying answer to the question for $n = 2, 3, 4$ and plays an important role in the general description of cyclic orbifold.

In this section we will make use of computations of sectors extensively as in [54]. Let us first recall some preliminaries about sectors. See [36], [37] and [38] for more details. Let $M$ be an infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of $M$. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$. We denote by $[\rho]$ the image of $\rho \in \text{End}(M)$ in $\text{Sect}(M)$.

It follows from [37] that $\text{Sect}(M)$ is endowed with a natural involution $\theta \mapsto \bar{\theta}$; moreover, $\text{Sect}(M)$ is a semiring.

Let $\rho \in \text{End}(M)$ and $\varepsilon$ be a normal faithful conditional expectation $\varepsilon : M \to \rho(M)$. We define a number $d_\varepsilon \geq 1$ (possibly $\infty$) by:

$$d_\varepsilon := \max\{t \in [0, +\infty) | \varepsilon(t m_+) \geq t m_+, \forall m_+ \in M_+\}$$

(Pimsner-Popa inequality in [47]).

We define

$$d(\rho) = \min_{\varepsilon} \{d_\varepsilon\},$$

where the minimum is taken over $\varepsilon$ with $d_\varepsilon < \infty$ (otherwise we put $d(\rho) = \infty$). $d(\rho)$ is called the dimension of $\rho$. It is clear from the definition that the dimension of $\rho$ depends only the sector $[\rho]$.

The properties of the dimension can be found in [36], [37] and [38], see also [32]. We recall that $d(\rho) < \infty$ is equivalent to the existence of a conjugate sector.

For $\lambda, \mu \in \text{End}(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from $\lambda$ to $\mu$, i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a \lambda(x) = \mu(x) a$ for any $x \in M$. Assuming the dimension of $\lambda$ and $\mu$ to be finite, $\text{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu \lambda, \mu \rangle = \langle \lambda, \nu \mu \rangle$, $\langle \nu \lambda, \mu \rangle = \langle \nu, \mu \lambda \rangle$ which follows from Frobenius duality (see
We will also use the following notation: if \( \mu \) is a subsector of \( \lambda \), we will write as \( \mu \prec \lambda \) or \( \lambda \succ \mu \). A sector is said to be irreducible if it has only one subsector. Usually we will use Greek letters to denote sectors, but we will denote the identity sector by 1 if no confusion arises.

Fix an interval \( J_0 \). Let \( \gamma : \mathcal{D}(J_0) \to \mathcal{B}(J_0) \) be the canonical endomorphism from \( \mathcal{D}(J_0) \) to \( \mathcal{B}(J_0) \) and let \( \gamma_B := \gamma \mid \mathcal{B}(J_0) \). Note \( [\gamma] = [1] + [g] + \ldots + [g^{n-1}] \) as sectors of \( \mathcal{D}(J_0) \) and \( [\gamma_B] = [1] + [\sigma] + \ldots + [\sigma^{n-1}] \) as sectors of \( \mathcal{B}(J_0) \). Here \([g^i]\) denotes the sector of \( \mathcal{D}(J_0) \) which is the automorphism induced by \( g^i \) and \( \sigma \) is a DHR representation of \( \mathcal{B} \) with \([\sigma^n] = [1]\) where \([1]\) denotes the identity sector. We note that the notation \([g^i]\) is an exception to our rule of using Greek letters to denote sectors. All the sectors considered in the rest of Sect. 8 will be sectors of \( \mathcal{D}(J_0) \) or \( \mathcal{B}(J_0) \) as should be clear from their definitions. All DHR representations will be assumed to be localized on \( J_0 \) and have finite statistical dimensions. For simplicity of notations, for a DHR representation \( \sigma_0 \) of \( \mathcal{D} \) or \( \mathcal{B} \) localized on \( J_0 \), we will use the same notation \( \sigma_0 \) to denote its restriction to \( \mathcal{D}(J_0) \) or \( \mathcal{B}(J_0) \) and we will make no distinction between local and global intertwiners (cf. Appendix A) for DHR representations localized on \( J_0 \) since they are the same by the strong additivity of \( \mathcal{D} \) and \( \mathcal{B} \).

### 8.1 Non-twisted sectors in general case

We will denote by \( \lambda \) the irreducible DHR representations of \( \mathcal{D} \) and by \( \lambda_B \) its restriction to \( \mathcal{B} \). \( \lambda_B \) and its irreducible summands will be called non-twisted representations (relative to \( \mathcal{D} \)). An irreducible DHR representation of \( \mathcal{B} \) is twisted if it is not non-twisted. Our goal in this section is to characterize the nature of non-twisted representations.

Let \( \sigma_1 \) be a DHR representation of \( \mathcal{B} \) localized on \( J_0 \). Recall from §3.1 the definition of \( \alpha_{\sigma_1} \). When restricted to \( \mathcal{D}(J_0) \), \( \alpha_{\sigma_1} \) is an endomorphism of \( \mathcal{D}(J_0) \) (cf. (1) of Th. 3.1 in [54] or Cor. 3.2 of [4]), and we use the same notation \( \alpha_{\sigma_1} \) to denote this endomorphism. For the rest of Sect. 8, \( \alpha_{\sigma_1} \) will always be understood as the endomorphism of \( \mathcal{D}(J_0) \). The following lemma which follows essentially from [54] (also cf. [4]) will be used repeatedly:

**Lemma 8.1.** Let \( \sigma_1, \sigma_2 \) (resp. \( \lambda, \mu \)) be DHR representations of \( \mathcal{B} \) (resp. \( \mathcal{D} \)) localized on \( J_0 \). Then:

1. \( [\alpha_{\sigma_1}] = [\alpha_{\sigma_1}] \) as sectors of \( \mathcal{D}(J_0) \) and \( d(\alpha_{\sigma_1}) = d(\sigma_1) \);
2. \( \langle \alpha_{\sigma_1}, \alpha_{\sigma_2} \rangle = \langle \sigma_1 \gamma_B, \sigma_2 \rangle, \langle \alpha_{\sigma_1}, \lambda \rangle = \langle \sigma_1, \lambda_B \rangle \);
3. \( [g\alpha_{\sigma_1}] = [\alpha_{\sigma_1}g], [\lambda_{\alpha_{\sigma_1}}] = [\alpha_{\sigma_1} \lambda] \);
4. \( [g^i \alpha_{\sigma_1}, g^j \alpha_{\sigma_2}] = \delta_{ij} \langle \alpha_{\sigma_1}, \alpha_{\sigma_2} \rangle, [g^i \alpha_{\sigma_1}, g^j \lambda] = \delta_{ij} \langle \alpha_{\sigma_1}, \lambda \rangle, [g^i \mu, g^j \lambda] = \delta_{ij} \langle \mu, \lambda \rangle, \)

\( 0 \leq i, j \leq n-1 \).

**Proof** (1) follows from Cor. 3.5 of [54], (2) follows from Th. 3.3 of [54], (3) follows from Th. 3.6 of [54], and (4) follows from Lemma 3.5 of [54]. \( \square \)

Note that \( \mathbb{Z}_n \) acts on \( \lambda \) naturally by \( g\lambda g^{-1} \): this is a DHR representation of \( \mathcal{D} \) localized on the fixed interval \( J_0 \) and whose restriction to \( \mathcal{D}(J_0) \) is simply \( g \cdot \lambda \cdot g^{-1} \).
Assume that the stabilizer of such an action on $\lambda$ is generated by $g^{n_1}$ with $n_1k_1 = n$. Then:

**Proposition 8.2.** $\lambda_B$ decomposes into $k_1$ different irreducible pieces denoted by $(\lambda; \sigma^i)$, $0 \leq i \leq k_1 - 1$. Moreover $[\alpha_{(\lambda; \sigma^i)}] = \bigoplus_{0 \leq i \leq n_1 - 1} [g^i \lambda g^{-i}]$, $d((\lambda; \sigma^i)) = n_1d(\lambda)$, and if $[(\lambda; \sigma^i)] = [(\mu; \sigma^j)]$ then there exists an integer $l$ such that $\mu = g^l \lambda g^{-l}$.

**Proof**  Let $\rho_1$ be an endomorphism of $D(J_0)$ such that $\rho_1(D(J_0)) = B(J_0)$ and $\rho_1 \bar{\rho}_1 = \gamma$. By [44] as sectors of $B(J_0)$ we have $[\lambda_B] = [\gamma \lambda | B(J_0)]$, it follows that

$$\text{Hom}(\lambda_B, \lambda_B)_{B(J_0)} \simeq \text{Hom}(\bar{\rho}_1 \lambda \rho_1, \bar{\rho}_1 \lambda \rho_1)_{D(J_0)}$$

By Frobenius duality we have

$$\langle \lambda_B, \lambda_B \rangle = \langle \lambda, \gamma \lambda \gamma \rangle$$

For $0 \leq i, j \leq n_1 - 1$, note that $g^i \lambda g^{-j}$ is a DHR representation of $D$, and by (4) of Lemma 8.1 we have $\langle \lambda, g^i \lambda g^{-j} \rangle = \langle \lambda, g^{i-j} g^j \lambda g^{-j} \rangle = \delta_{ij} \langle \lambda, g^j \lambda g^{-j} \rangle$. It follows that $\langle \lambda, \gamma \lambda \gamma \rangle = k_1$.

Notice that $[g \rho_1] = [\rho_1], [\bar{\rho}_1 g] = [\bar{\rho}_1 g]$. If we set $\nu_1 = \bar{\rho}_1, \nu = g^{n_1}, \nu_2 = \lambda \rho_1$, we have $[\nu_1, \nu] = [\tau_1], [\nu, \nu_1] = [\nu_2], \nu$ and has order $k_1$. Now apply Lemma 2.1 of [57] where $a, b, c$ of [57] correspond to our $\nu_1, \nu, \nu_2$ respectively, we have shown that $\text{Hom}(\lambda_B, \lambda_B)$ is an abelian algebra with dimension $k_1$ and it follows that $\lambda_B$ decomposes into a direct sum of $k_1$ irreducible pieces, denoted by $\sigma^i$, $0 \leq i \leq k_1 - 1$.

From $[\gamma \alpha_{\lambda_B}] = [\gamma \lambda \gamma]$ we have:

$$\langle \gamma \alpha_{\lambda_B}, [g^i \lambda g^{-i}] \rangle = k_1, \ 0 \leq i \leq n_1 - 1.$$

Note that by (4) of Lemma 8.1 we have

$$\langle \gamma \alpha_{\lambda_B}, g^i \lambda g^{-i} \rangle = \langle \alpha_{\lambda_B}, g^i \lambda g^{-i} \rangle.$$

It follows that $\alpha_{\lambda_B} \succ k_1 \bigoplus_{0 \leq i \leq n_1 - 1} [g^i \lambda g^{-i}]$. On the other hand

$$d(\alpha_{\lambda_B}) = d(\lambda_B) = nd(\lambda).$$

It follows that $[\alpha_{\lambda_B}] = \bigoplus_{0 \leq i \leq n_1 - 1} k_1 [g^i \lambda g^{-i}]$. So we must have $\alpha_{\sigma^i} \succ g^i \lambda g^{-i}$ for some $i$ where $0 \leq j \leq k_1 - 1$. By (3) of Lemma 8.1, $[g \alpha_{\sigma^j}] = [\sigma_{\sigma^j} g]$, so we must have $\alpha_{\sigma^j} \succ \bigoplus_{0 \leq i \leq n_1 - 1} [g^i \lambda g^{-i}]$. In particular $d(\sigma_j) \geq n_1d(\lambda)$. Since $\sum_{0 \leq j \leq k_1 - 1} d(\sigma_j) = k_1 n_1 d(\lambda)$, it follows that

$$[\alpha_{\sigma^j}] = \bigoplus_{0 \leq i \leq n_1 - 1} [g^i \lambda g^{-i}]$$

and $\langle \alpha_{\sigma^0}, \alpha_{\sigma^i} \rangle = n_1$. By (2) of Lemma 8.1 we have $\langle \alpha_{\sigma^0}, \alpha_{\sigma^i} \rangle = \langle \sigma^0 \sigma, [1] + [\sigma] + \ldots + [\sigma^{n-1}] \rangle = n_1$, it follows that the set $\{\sigma_0, \sigma_1, \ldots, \sigma_{k_1 - 1}\}$ is the same as $\{\sigma_0, \sigma \sigma_0, \ldots, \sigma^{k_1 - 1} \sigma_0\}$. We will use $(\lambda; \sigma^i), 0 \leq i \leq k_1 - 1$ to denote $\sigma^i \sigma_0$ in the following. It follows from formula (41) and (1) of Lemma 8.1 that $d((\lambda; \sigma^i)) = n_1d(\lambda)$.

The last part follows from formula (41) for $\alpha_{\sigma_j}$. □
The following simple lemma will be used in §8.4 and 8.5.

**Lemma 8.3.** Let $\mu$ be an irreducible DHR representation of $\mathcal{B}$. Let $i$ be any integer. Then:

1. $G(\mu, \sigma^i) := \varepsilon(\mu, \sigma^i)\varepsilon(\sigma^i, \mu) \in \mathbb{C}$, $G(\mu, \sigma^i) = G(\mu, \sigma^i)$. Moreover $G(\mu, \sigma)^n = 1$;
2. If $\mu_1 < \mu_2 \mu_3$ with $\mu_1, \mu_2, \mu_3$ irreducible, then $G(\mu_1, \sigma^i) = G(\mu_2, \sigma^i)G(\mu_3, \sigma^i)$;
3. $\mu$ is untwisted if and only if $G(\mu, \sigma) = 1$;
4. $G(\mu, \sigma^i) = G(\mu, \sigma^i)$.

**Proof** We have $G(\mu, \sigma^i) \in \text{Hom}(\sigma^i \mu, \sigma^i \mu) \simeq \mathbb{C}$ since $\sigma^i \mu$ is irreducible, and also $G(\mu, \sigma^i) = \varepsilon(\mu, \sigma^i)\varepsilon(\sigma^i, \mu) = G(\mu, \sigma^i)$, so $G(\mu, \sigma)^n = 1$ since $[\sigma^n] = [1]$. If $\mu_1 < \mu_2 \mu_3$ with $\mu_1, \mu_2, \mu_3$ irreducible, then $G(\mu_1, \sigma^i) = G(\mu_2, \sigma^i)G(\mu_3, \sigma^i)$ by the Braiding-Fusion equations (cf. [49]). For the second part, by Prop. 8.2 $\mu$ is untwisted if and only if $\alpha_{\mu}$ is a DHR representation of $\mathcal{D}$. By the remark before Prop. 3.2 this is true if and only if $G(\mu, \sigma) = 1$. The third part follows from (2) and $G(1, \sigma^i) = 1$.

Denote by $W$ the vector space whose basis consists of irreducible components of all $\alpha_{\mu}$ where $\mu$ are irreducible DHR representations of $\mathcal{B}$, and $W_0$ (resp. $W_t$) the subspaces whose bases consist of irreducible components of $\alpha_{\mu}$ where $\mu$ are irreducible non-twisted (resp. twisted) DHR representations of $\mathcal{B}$ (relative to $\mathcal{D}$). The elements in the basis of $W_t$ are also called twisted solitonic sectors. We note that $W_0$ has a natural ring structure where the product is the composition of sectors. Applying Prop. 8.4 and Th.3.6 of [6] we have

$$\dim W_t = \sum_{\lambda} (k_1 - 1)$$

(42)

So each $\lambda$ with nontrivial stabilizer contributes to the twisted solitonic sectors.

### 8.2 Non-twisted sectors for the cyclic permutations

For the rest of §8, we will consider the case of cyclic permutations, i.e., we assume that $\mathcal{D} := \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ ($n$-fold tensor product) and $\mathcal{B} := (\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A})^{\mathbb{Z}_n}$ the fixed point subnet of $\mathcal{D}$ under the action of cyclic permutations. Since we assume that $\mathcal{D}$ is completely rational, this is equivalent to assuming that $\mathcal{A}$ is completely rational. We will denote by $(\lambda_1, \ldots, \lambda_n) = \lambda_1 \otimes \cdots \otimes \lambda_n$ the irreducible product representation of $\mathcal{D}$ associated with the irreducible representations $\lambda_1, \ldots, \lambda_n$ of $\mathcal{A}$ and by $(\lambda_1, \ldots, \lambda_n)_B$ its restriction to $\mathcal{B}$. Note that $\mathbb{Z}_n$ acts on product sectors $(\lambda_1, \ldots, \lambda_n)$ naturally by cyclic permutations and $[g(\lambda_1, \ldots, \lambda_n)g^{-1}] = [(\lambda_{g(1)}, \ldots, \lambda_{g(n)})]$. Assume that the stabilizer of $[(\lambda_1, \ldots, \lambda_n)]$ is generated by $g^{n_1}$ with $n_1 k_1 = n$. Then by Prop. 8.2 we have:

**Corollary 8.4.** $(\lambda_1, \ldots, \lambda_n)_B$ decomposes into $k_1$ different irreducible pieces denoted by $(\lambda_1, \ldots, \lambda_i; \sigma^i), 0 \leq i \leq k_1 - 1$. Moreover $[\alpha_{(\lambda_1, \ldots, \lambda_n; \sigma^i)}] = \bigoplus_{0 \leq k \leq n_1} [g^k(\lambda_1, \ldots, \lambda_n)g^{-k}]$, and if $[(\lambda_1, \ldots, \lambda_n; \sigma^i)] = [(\mu_1, \ldots, \mu_n; \sigma^i)]$ then there exists an integer $l$ such that $\mu_k = \lambda_{g^l(k)}$, $1 \leq k \leq n$. 

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8.3 Topological twisted sectors for cyclic permutations

Let us first determine the relevant ring structures of the topological twisted sectors from Sect. 6. Choose $\zeta$ to be the right boundary point of the fixed interval $J_0$ in the anti-clockwise direction on the circle. We can assume that $J_0$ is the interval $I$ as chosen in Prop. 6.4. Since $\pi_f^{(c)}$ is a soliton, by the usual DHR argument [18], we can choose a soliton which is unitarily equivalent to $\pi_f^{(c)}$ and restricts to an endomorphism of $D(J_0)$ (also cf. the paragraph before Prop. 4.5). We will denote this endomorphism of $D(J_0)$ by $\pi$. We note that $\pi_f$ is a DHR representation of $B$ and we will denote by $\tau$ a DHR representation of $B$ localized on the fixed interval $I$ which is unitarily equivalent to $\pi_f$ and the corresponding endomorphism of $B(J_0)$ obtained by restriction to $B(J_0)$. (Notations differ here from the previously used ones: $\pi$ and $\tau$ are sectors of factors).

Note that by [44] we have $[\pi] = [\gamma \pi | B(J_0)]$ as sectors of $B(J_0)$. By (d) of Th. 6.3 we have $\langle \pi, \tau \rangle = n$, in fact $[\pi] = [\tau^{(0)}] + ... + [\tau^{(n-1)}]$. So $\langle \gamma \pi | B(J_0), \gamma \pi | B(J_0) \rangle = n$.

As in the beginning of the proof of Prop. 8.4, we have

$$\langle \gamma \pi | B(J_0), \gamma \pi | B(J_0) \rangle = \langle \pi, \gamma \pi \gamma \rangle = n.$$ (43)

By definition (cf. §3.1) $[\gamma \alpha_t] = [\tau \gamma] = [\gamma \pi \gamma]$. We get $\langle \gamma \alpha_t, \pi \rangle = \langle \gamma \pi \gamma, \pi \rangle = n$. Since $[\gamma \alpha_t] = [\alpha_t] + ... + [g^{n-1} \alpha_t]$ and $\langle g^i \alpha_t, g^j \alpha_t \rangle = \delta_{ij}[\alpha_t, \alpha_t], \forall 0 \leq i, j \leq n - 1$ (cf. (4) of Lemma 8.1), it follows that there exists an integer $0 \leq i \leq n - 1$ such that $\langle g^i \alpha_t, \pi \rangle = n$. On the other hand since $d(\alpha_t) = d(\tau) = nd(\pi)$, we must have $[g^i \alpha_t] = n[\pi]$. Since $[\alpha_t] = [\alpha_t^{(0)}] + ... + [\alpha_t^{(n-1)}]$, and $g^{-i} \pi$ is irreducible, we conclude that, for any $0 \leq j \leq n - 1$, we have $[\alpha_t^{(j)}] = [g^{-i} \pi]$.

Since $\alpha_t^{(j)} \pi$ are solitons localized on $J_0$ (cf. Prop. 3.1), using the next lemma we conclude that

$$[\alpha_t^{(j)}] = [\pi], 0 \leq j \leq n - 1.$$ (44)

Lemma 8.5. Let $\pi_1, \pi_2$ be two solitons of $D_0$ (The restriction of $D$ to $S^1 \setminus \{\zeta\}$, cf. §3.0.1) localized on $J_0$. If $[\pi_1] = [g^{-i} \pi_2]$ as sectors of $D(J_0)$ for some integer $i$, then $g^{-i}$ as a group element is the identity and $[\pi_1] = [\pi_2]$ as sectors of $D(J_0)$.

Proof. It is enough to prove that $g^{-i}$ as a group element is the identity. Let $J_1 \subset J_0, J_1 \neq J_0$ be an interval with $\zeta$ as a boundary point. Let $J_2 := J_0 \cap J_1$. Assume that $v$ is a unitary in $D(J_0)$ such that $\pi_1 = \text{Ad} v \cdot (g^{-i} \pi_2)$ on $D(J_0)$.

Consider $\pi_1, \pi_2$ on $D(J_1)$. Since $\pi_1, \pi_2$ are solitons, and $D(J_1)$ is a type III factor, we can find unitaries $v_1, v_2$ such that on $D(J_1)$ we have $\pi_1 = \text{Ad} v_1, \pi_2 = \text{Ad} v_2$. Since $\pi_1, \pi_2$ are localized on $J_0$, it follows that $v_1 \in D(J_0), v_2 \in D(J_0)$.

So on $D(J_2)$ we have $\text{Ad} v_1 = \text{Ad} v \cdot \text{Ad} g^{-i} v_2 \cdot g^{-i}$. Define $w := g^{-i} v_2^* v^* v_1$. Note that $w \in D(J_0)$, and $w x w^* = g^{-i}(x), \forall x \in D(J_2)$. It follows that $w \in D(J_0) \cap B(J_2)'$. By (2) of Lemma 3.6 in [59] the pair $B \subset D$ is strongly additive (cf. Definition 3.2 of [59]) since we assume that $D$ is strongly additive, and so $D(J_0)' \vee B(J_2) = D(J_1)'$ which implies by Haag duality $D(J_0) \cap B(J_2)' = D(J_1)$. Therefore $w \in D(J_1), g^{-i}(x) = x, \forall x \in D(J_2)$, and so $g^{-i}$ as a group element is the identity since one checks easily that the action of the cyclic group on $D(J_2)$ is faithful.

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Note that by Corollary 4.12 we have as sectors of $D(J_0)$:

$$[\bar{\pi}\pi] = \bigoplus_{\lambda_1, \ldots, \lambda_n} \langle \lambda_1 \cdots \lambda_n, 1 \rangle [(\lambda_1, \ldots, \lambda_n)] .$$

(45)

where $\bar{\pi}$ is the conjugate sector of $\pi$. From $[\alpha_{\tau(j)}] = [\pi]$ we have

$$[\alpha_{\tau(j)} \alpha_{\tau(j)}] = \bigoplus_{\lambda_1, \ldots, \lambda_n} \langle \lambda_1 \cdots \lambda_n, 1 \rangle [(\lambda_1, \ldots, \lambda_n)].$$

(46)

where we have also used $[\alpha_{\tau(j)}] = [\bar{\alpha}_{\tau(j)}]$ (cf. (1) of Lemma 8.1).

Recall that the spins of $\tau(j)$ are given in $(e)$ of Th. 6.3. By (44) $[\alpha_{\tau(j)}] = [\alpha_{\tau(0)}]$, by (2) of Lemma 8.1 we have

$$\sum_{0 \leq l \leq n-1} \langle \tau(j), \sigma^l \tau(0) \rangle = 1 .$$

Since both $\tau(j)$ and $\sigma^l \tau(0)$ are irreducible, we must have that $[\tau(j)] = [\sigma^{k(j)} \tau(0)]$ where $k(\cdot)$ is a map from $\mathbb{Z}_n$ to itself. $k(\cdot)$ is also one to one (hence onto) since if $[\sigma^{l_1} \tau(0)] = [\tau(0)]$ for some $0 < l_1 \leq n - 1$, then by (2) of Lemma 8.1 again $\langle \alpha_{\tau(0)}, \alpha_{\tau(0)} \rangle = \sum_{0 \leq l \leq n-1} \langle \tau(0), \sigma^l \tau(0) \rangle \geq 2$ contradicting the fact that $\alpha_{\tau(0)}$ is irreducible. We claim that in fact $k(j) = j k(1), 0 \leq j \leq n - 1$. This follows essentially by the grading Lemma 8.3: by definition of the monodromy, $G(\sigma^{j k(1)} \tau(0), \sigma^{k(1)} \tau(0)) = G(\tau(0), \sigma^{k(1)} \tau(0))$ because all $\sigma^j$'s have integer spins and are automorphisms. From the monodromy equation (cf. [49]) we have

$$G(\sigma^{j k(1)} \tau(0), \sigma^{k(1)} \tau(0)) = e^{2\pi i (\text{spin}(\sigma^{(j+1) k(1)} \tau(0)) - \text{spin}(\sigma^{k(1)} \tau(0)))} ,$$

(47)

hence, modulo integers, $\text{spin}(\sigma^{j(1+k)} \tau(0)) - \text{spin}(\sigma^{k(1)} \tau(0))$ is a constant independent of $0 \leq j \leq n - 1$. Since $[\sigma^{k(1)} \tau(0)] = [\tau(1)]$, $\text{spin}(\sigma^{j+1 k(1)} \tau(0)) - \text{spin}(\sigma^{k(1)} \tau(0))$ is equal to $\frac{1}{2}$ modulo integers. It follows that $\text{spin}(\sigma^{j k(1)} \tau(0))$ is equal to the spin of $\tau(j)$ modulo integers. We conclude that

$$[\sigma^{j k(1)} \tau(0)] = [\tau(j)] \text{ and } j k(1) = k(j), \ 0 \leq j \leq n - 1 .$$

Since $k(\cdot)$ is one to one, the greatest non-negative common divisor of $k(1)$ and $n$ must be 1.

In the following we define

$$G(\mu) := G(\mu, \sigma^{k(1)})$$

and will refer to $G(\mu)$ as the grading of $\mu$. Note that by definition $G(\tau(0)) = e^{2\pi i} .$

Let $\lambda$ be a covariant representation of $\mathcal{A}$ and $\tau_{\lambda} = \pi_{\lambda} | \mathcal{B}$ (cf. Prop. 6.4) the DHR representation of $\mathcal{B}$ obtained by restriction of $\pi_{\lambda}$. As in the beginning of this section, we denote by $\pi_{\lambda}$ the endomorphism of $D(J_0)$ obtained from the restriction to $D(J_0)$ of a soliton unitarily equivalent to $\pi_{\lambda}^{(\xi)}$. Note that an analogue of $(d)$ of Th. 6.3 holds and $\tau_{\lambda}$ is a direct sum of $n$ DHR representations $\tau_{\lambda}^{(0)}, \ldots, \tau_{\lambda}^{(n-1)}$.
Note that $[\pi_\lambda] = [\pi \cdot (\lambda, 1, ..., 1)]$ by Prop. 6.4, and it follows that $[\gamma \alpha_\tau] = [\gamma \pi(\lambda, 1, ..., 1) \gamma]$. By the same argument as in the case when $\lambda = 1$ above we have $[\alpha_\tau] = n [g^k \pi(\lambda, 1, ..., 1)]$ for some $0 \leq k \leq n - 1$, and by (3), (4) of Lemma 8.1 again we have $k = 0$ and

$$[\alpha_\tau] = [\alpha_{\tau(0)}] = [(\lambda, 1, ..., 1) \alpha_{\tau(0)}] = [\pi \lambda], \ 0 \leq j \leq n - 1. \ (48)$$

From these equations we can prove the following:

**Theorem 8.6.** (1) $[\pi_\lambda] = [\pi_\mu]$ as sectors of $\mathcal{D}(J_0)$ iff $\lambda \simeq \mu$ as DHR representations of $\mathcal{A}$;

(2) $[\tau_\lambda^{(l)}] = [\tau_\mu^{(l)}]$ iff $\lambda \simeq \mu$ as DHR representations of $\mathcal{A}$ and $l = j$.

**Proof** (1): Since

$$[\pi_\lambda] = [\pi \cdot (\lambda, 1, ..., 1)], \ [\pi_\mu] = [\pi \cdot (\mu, 1, ..., 1)]$$

we have $[\pi_\lambda] = [\pi_\mu]$ iff $[\pi \lambda(1, 1, ..., 1)] = [\pi(\mu, 1, ..., 1)]$. It follows by Frobenius duality and equation (45)

$$\langle \pi \pi, (\lambda_\mu, 1, ..., 1) \rangle = 1 = \langle 1, \lambda_\mu \rangle.$$ 

It follows that $[\lambda] = [\mu]$ as sectors of $\mathcal{A}(J_0)$. Since $\mathcal{A}$ is strongly additive, it follows that $\lambda \simeq \mu$ as DHR representations of $\mathcal{A}$.

(2): It is sufficient to show that if $[\tau_\lambda^{(l)}] = [\tau_\mu^{(l)}]$ then $\lambda \simeq \mu$ as DHR representations of $\mathcal{A}$. Assume that $[\tau_\lambda^{(l)}] = [\tau_\mu^{(l)}]$. By equation (48) we have

$$[\alpha_\tau^{(l)}] = [\pi_\lambda], \ [\alpha_\tau^{(j)}] = [\pi_\mu]$$

and the proof follows from point (1). \qed

We note that Th. 8.6 is similar to the main theorems (Th. 3.9 and Th. 4.4) of [2] if one identifies $\pi_\lambda$ with the twisted module in the sense of [2]. Th. 8.6 supplies a class of twisted representation of the cyclic orbifold. In the next few sections we will show that these representations and their variations give all the twisted representations in the case $n = 2, 3, 4$.

### 8.4 Case $n = 2$

When $n = 2$, by (42) $\dim W_\ell$ is the same as the cardinality of the set $\{\lambda\}$. By (2) of Th. 8.6 and (48) $W_\ell$ has a basis $\{\alpha_{\tau \ell}^{(0)}\}$. If $\tau_\sigma$ is an irreducible twisted representation of $\mathcal{B}$, it follows that $\alpha_{\tau_\sigma} = \bigoplus_\lambda C_\lambda \alpha_{\tau_\ell}^{(0)}$, where $C_\lambda$ are positive integers. By eq. (47) and (2) of Lemma 8.1 it follows that $\tau_\sigma$ must be some $\tau_\lambda^{(l)}$. One can also prove this by computing index of all known DHR representations of $\mathcal{B}$ and check that they add up to $\mu_\mathcal{B} = 4 \mu_\mathcal{D}$. Hence we have proved the following:
Proposition 8.7. When \( n = 2 \) all the irreducible twisted representations of the fixed point net \( \mathcal{B} \) are \( \{\tau_\lambda^{(i)}\} \).

When \( n = 2 \) we can determine completely the fusion rules of \( \alpha_{\tau_\lambda^{(0)}} \) as follows:

**Proposition 8.8.** (1) \([\bar{\alpha}_{\tau_\lambda^{(0)}}] = [\alpha_{\tau_\lambda^{(0)}}] \);

(2) \([\mu_1, \mu_2]\alpha_{\tau_\lambda^{(0)}} = \bigoplus_\delta \langle \mu_1 \mu_2 \lambda, \delta \rangle [\alpha_{\tau_\lambda^{(0)}}] \);

(3) \( [\alpha_{\tau_\lambda^{(0)}} \alpha_{\tau_\mu^{(0)}}] = \bigoplus_{\lambda_1, \lambda_2} \langle \lambda \mu_1, \lambda \rangle [\lambda_2, \lambda_1] \).

**Proof** (1): Note that \([\alpha_{\tau_\lambda^{(0)}}] = [(\lambda, 1)\alpha_{\tau(0)}] \), so it is sufficient to show that \([\bar{\alpha}_{\tau(0)}] = [\alpha_{\tau(0)}] \). Here we give two different proofs. Since \( W_t \) is spanned by \( \{\alpha_{\tau_\lambda^{(0)}}\} \), we must have that \([\bar{\alpha}_{\tau(0)}] = [\alpha_{\tau_\mu^{(0)}}] = [(\mu, 1)\alpha_{\tau(0)}] \) for some \( \mu \) (cf. (48)). From this we have \( d(\mu) = 1 \). So \( \mu \lambda \) is irreducible for any \( \lambda \).

\[
[\bar{\alpha}_{\tau_\lambda^{(0)}}] = [(\mu \lambda, 1)\alpha_{\tau(0)}], [\bar{\alpha}_{\tau_\mu^{(0)}}] = [\alpha_{\tau_\mu^{(0)}}]
\]

we have

\[
\langle [\tau_\lambda^{(0)}], [\tau_\mu^{(0)}] + [\tau_\mu^{(1)}] \rangle = 1
\]

and therefore \( \tau_\lambda^{(0)} \) is either \( \tau_\mu^{(0)} \) or \( \tau_\mu^{(1)} \). In any case the univalence \((\doteq \exp(2\pi i \cdot \text{spin}))\) \( \omega_{\tau_\lambda^{(0)}} \) (cf. [24]) of \( \tau_\lambda^{(0)} \) must be the same as that of \( \tau_\mu^{(0)} \) or \( \tau_\mu^{(1)} \). Note that by (14) we have

\[
\omega_{\tau_\lambda^{(0)}} \cdot \omega_{\tau_\mu^{(0)}} = \omega_{\tau_\lambda^{(1)}} \cdot \omega_{\tau_\mu^{(1)}}, \omega_{\tau_\lambda^{(0)}} \cdot \omega_{\tau_\mu^{(0)}} = \omega_{\lambda} \cdot \omega_{\lambda} = 1,
\]

and therefore \( \omega_{\tau_\lambda^{(0)}} = \omega_{\mu \lambda}, \forall \lambda \). It follows that \( \mu \) is degenerate (cf. [49]) and therefore \( \mu \) is the vacuum representation since \( \mathcal{A} \) is modular (cf. [31]). This completes the first proof of \([\bar{\alpha}_{\tau(0)}] = [\alpha_{\tau(0)}] \).

For the second proof of \([\bar{\alpha}_{\tau(0)}] = [\alpha_{\tau(0)}] \), note that by (48) \([\bar{\pi}] = [\alpha_{\tau(0)}] \). By the remark after Prop. 4.10, we have \([\bar{\pi}] = [\pi] \). So we have \([\alpha_{\tau(0)}] = [\bar{\alpha}_{\tau(0)}] \).

(2): By (48) we have \([\alpha_{\tau_\lambda^{(0)}}] = [(\lambda, 1)\alpha_{\tau(0)}] \). So

\[
[(\mu_1, \mu_2)\alpha_{\tau_\lambda^{(0)}}] = \bigoplus_{\delta_1} \langle \mu_1 \lambda, \delta_1 \rangle [(\delta_1, \mu_2)\alpha_{\tau(0)}].
\]

Note that \([(\delta_1, \mu_2)\alpha_{\tau(0)}] = [(\delta_1, 1)(1, \mu_2)\alpha_{\tau(0)}] \). We claim that \([(1, \mu_2)\alpha_{\tau(0)}] = [(\mu_2, 1)\alpha_{\tau(0)}] \).

In fact by (46) and Frobenius duality we have:

\[
\langle (1, \mu_2)\alpha_{\tau(0)}, (1, \mu_2)\alpha_{\tau(0)} \rangle = \langle (\mu_2, \mu_2), \bar{\alpha}_{\tau(0)} \rangle = \langle (\bar{\mu}_2, \mu_2), \alpha_{\tau(0)} \rangle = \langle (\mu_2, 1), \alpha_{\tau(0)} \rangle = \langle (\mu_2, 1), \alpha_{\tau(0)} \rangle = \langle (\bar{\mu}_2, 1), \alpha_{\tau(0)} \rangle = \langle (\mu_2, 1), \alpha_{\tau(0)} \rangle = \langle (\bar{\mu}_2, 1), \alpha_{\tau(0)} \rangle = 1 \quad (49)
\]

It follows that \([(1, \mu_2)\alpha_{\tau(0)}] = [(\mu_2, 1)\alpha_{\tau(0)}] \). Hence \([(\delta_1, 1)(1, \mu_2)\alpha_{\tau(0)}] = [(\delta_1, 1)(\mu_2, 1)\alpha_{\tau(0)}] = \bigoplus_{\delta} \langle \delta_1 \mu_2, \delta \rangle [\alpha_{\tau_\delta^{(0)}}] \). So we have

\[
[(\mu_1, \mu_2)\alpha_{\tau_\lambda^{(0)}}] = \bigoplus_{\delta_1, \delta} \langle \mu_1 \lambda, \delta_1 \rangle [\delta_1 \mu_2, \delta \rangle [\alpha_{\tau_\delta^{(0)}}] = \bigoplus_{\delta} \langle \mu_1 \mu_2 \lambda, \delta \rangle [\alpha_{\tau_\delta^{(0)}}].
\]
(3): We have
\[
[\alpha_{\tau_0}(0)\alpha_{\tau_0}(0)] = [(\lambda, 1)(\mu, 1)\alpha_{\tau_0}(0)\alpha_{\tau_0}(0)] = \bigoplus_{\lambda_1}[(\lambda, 1)(\mu, 1)(\lambda_1, \bar{\lambda}_1)] = \bigoplus_{\lambda_1, \lambda_2}(\lambda \mu \lambda_1, \lambda_2)[(\lambda_2, \bar{\lambda}_1)]
\]
where we have used (48) in the first equality, the first part of the proposition and (46) in the second equality.

Before concluding this subsection, we note that \(\pi_\mu\) can be defined also for a reducible sector \(\mu\) of \(A\) and we clearly have
\[
\pi_\mu = \bigoplus_\delta(\mu, \delta)\pi_\delta,
\]
where \(\delta\) runs on the irreducible sectors of \(A\).

Hence Proposition 8.8 can be equivalently formulated, with the notations in Sect. 6, as follow:

1. \(\bar{\pi}_\lambda \simeq \pi_\lambda\),
2. \((\mu_1 \otimes \mu_2) \cdot \pi_\lambda \simeq \pi_{\mu_1\mu_2\lambda}\),
3. \(\pi_\lambda \pi_\mu \simeq \bigoplus_\delta \lambda \delta \otimes \mu \delta\),

where \(\lambda, \mu, \mu_1, \mu_2\) and \(\delta\) are irreducible.

(1) is proved in Prop. 6.1, (2) follows from the equality \((\mu \otimes \iota) \cdot \pi_\lambda = (\iota \otimes \mu) \cdot \pi_\lambda = \pi_{\mu\lambda}\) and (3) follows by Cor. 4.12. Note that the composition of two twisted solitons is a DHR sector.

8.5 Case \(n = 3\)

By (42) when \(n = 3\) \(\text{dim} W_t\) is twice the cardinality of the set \(\{\lambda\}\). We claim that in this case (unlike the case \(n = 2\)) \([\alpha_{\tau_0}] \neq [\alpha_{\tau_0}^2]\). If not, by Frobenius duality, (48) and (2) of Lemma 8.1 we have
\[
\langle (\bar{\lambda} \bar{\mu}, 1, 1), \alpha_{\tau_0}^2 \rangle = \langle (\bar{\lambda} \bar{\mu}, 1, 1) \uparrow B(J_0), \tau_0^2 \rangle = 1.
\]
It follows that \(\tau_0^2\) contains some untwisted DHR representation of \(B\). Note that \(G(\tau_0^2) = e^{2\pi i} = -1\), so by Lemma 8.3 we have arrived at a contradiction. Hence by counting we conclude that \(W_t\) is spanned by \(\{\alpha_{\tau_0}^{(i)}, \alpha_{\bar{\tau}_0}^{(i)}\}\), and by the same argument as in the proof of Prop. 8.7 we have:

**Proposition 8.9.** All the irreducible twisted representations of \(B\) in the case \(n = 3\) are \(\tau_\lambda^{(i)}\) and \(\bar{\tau}_\lambda^{(i)}, 0 \leq i \leq 2\).
8.6 Case $n = 4$

By (42) in this case $\dim W_i = |\{(\lambda_1, \lambda_2, \lambda_1, \lambda_2), \lambda_1 \neq \lambda_2, \} + 3|\{(\lambda, \lambda, \lambda)\}|$. One question is how to construct additional sectors corresponding to $(\lambda_1, \lambda_2, \lambda_1, \lambda_2)$. We notice that there is an intermediate fixed point net $\mathcal{C}$ between $\mathcal{B}$ and $\mathcal{D}$ such that $\mathcal{C}$ is the fixed point subnet of $\mathcal{B}$ under the action of $g^2$. In fact $\mathcal{C}$ is fixed point subnet of $\mathcal{D} = (\mathcal{A} \otimes \mathcal{A}) \otimes (\mathcal{A} \otimes \mathcal{A})$ under the natural cyclic $\mathbb{Z}_2$ action. So we can apply the results of §8.4 to the pair $\mathcal{C} \subset \mathcal{D}$. Now the representations of $\mathcal{A} \otimes \mathcal{A}$ are labeled by $(\lambda_1, \lambda_2)$, and so we label the solitons for the pair $\mathcal{C} \subset \mathcal{D}$ by $\pi(\lambda_1, \lambda_2)$ and its restriction to $\mathcal{C}$ (a DHR representation of $\mathcal{C}$) by $\tau(\lambda_1, \lambda_2)$. Recall from §8.3 that $\tau(\lambda_1, \lambda_2)$ is a direct sum of four irreducible DHR representations denoted by $\tau(0)_{(\lambda_1, \lambda_2)}$ and $\tau(1)_{(\lambda_1, \lambda_2)}$. We will denote by $\tau(i)_{(\lambda_1, \lambda_2)}$ the DHR representations of $\mathcal{B}$ obtained by restricting $\tau(i)_{(\lambda_1, \lambda_2)}$ to $\mathcal{B}$, $i = 0, 1$. Note that $\mathcal{C}$ is invariant under the automorphism induced by cyclic permutation $g$ and the $\mathcal{B}$ is the fixed point subnet under this action. Applying Prop. 8.2 to $\mathcal{B} \subset \mathcal{C}$ we have

\[
[\alpha_{\tau(0)}_{(\lambda_1, \lambda_2), \mathcal{B}}] = [\tau(0)_{(\lambda_1, \lambda_2)}] + [g^{\tau(0)}_{(\lambda_1, \lambda_2)} g^{-1}]
\]

where $\mathcal{B} \uparrow \mathcal{C}$ indicates the induction from $\mathcal{B}$ to $\mathcal{C}$ (note that an horizontal arrow has been used in [59]). By Lemma 3.3 of [56] we have

\[
[\alpha_{\tau(0)}^{\mathcal{B} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2), \mathcal{B}}] = [\alpha_{\tau(0)}^{\mathcal{C} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2)}] + [\alpha_{g^{\tau(0)}_{(\lambda_1, \lambda_2)} g^{-1}}]
\]

By (3) of Lemma 8.1 as sectors $\alpha_{\tau(0)}^{\mathcal{B} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2), \mathcal{B}}$ commutes with $g$ since $[g]$ is a subsector of the canonical endomorphism $\gamma$ from $\mathcal{D}$ to $\mathcal{B}$. So we must have

\[
[\alpha_{\tau(0)}^{\mathcal{B} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2), \mathcal{B}} g^{-1}] = [\alpha_{\tau(0)}^{\mathcal{C} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2)} g^{-1}] or [\alpha_{\tau(0)}^{\mathcal{C} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2)} g^{-1}]
\]

As in the proof of (43) and using (48) we have

\[
\langle \tau(\lambda_1, \lambda_2), \mathcal{B}, \tau(\lambda_1, \lambda_2), \mathcal{B} \rangle = \langle \alpha_{\tau(0)}^{\mathcal{C} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2)} \tau(\lambda_1, \lambda_2), \gamma \alpha_{\tau(0)}^{\mathcal{C} \uparrow \mathcal{D}}_{(\lambda_1, \lambda_2)} \gamma \rangle
\]

By using (52), (53) we conclude that $\tau(\lambda_1, \lambda_2)$ is a direct sum of four distinct irreducible pieces iff

\[
[\alpha^{\mathcal{B} \uparrow \mathcal{D}}_{\tau(0)}_{(\lambda_1, \lambda_2), \mathcal{B}} g^{-1}] = [\alpha^{\mathcal{C} \uparrow \mathcal{D}}_{\tau(0)}_{(\lambda_1, \lambda_2)}],
\]

and a direct sum of two distinct irreducible pieces iff

\[
[\alpha^{\mathcal{B} \uparrow \mathcal{D}}_{\tau(0)}_{(\lambda_1, \lambda_2), \mathcal{B}} g^{-1}] = [\alpha^{\mathcal{C} \uparrow \mathcal{D}}_{\tau(0)}_{(\lambda_1, \lambda_2)} g^{-1}] \neq [\alpha^{\mathcal{C} \uparrow \mathcal{D}}_{\tau(0)}_{(\lambda_1, \lambda_2)}].
\]

On the other hand, applying Prop. 8.2 to the pair $\mathcal{B} \subset \mathcal{C}$, we know that $\tau(\lambda_1, \lambda_2), \mathcal{B}$ is a direct sum of four irreducible pieces iff $[\tau(i)_{(\lambda_1, \lambda_2), \mathcal{B}} g^{-1}] = [\tau(i)_{(\lambda_1, \lambda_2), \mathcal{B}}], i = 0, 1$, and a direct sum of two distinct irreducible pieces iff $[\tau(i)_{(\lambda_1, \lambda_2), \mathcal{B}} g^{-1}] \neq [\tau(i)_{(\lambda_1, \lambda_2), \mathcal{B}}], i = 0, 1$.
0, 1, and \([g_{\tau(i)}(\lambda_1, \lambda_2), B g_{\tau(i)}^{-1}] \neq [\tau(1)_{\lambda_1, \lambda_2}]\). So we have that \([g \alpha_{\tau(i)}(0)_{\lambda_1, \lambda_2} g_{\tau(i)}^{-1}] = [\alpha_{\tau(i)} g_{\tau(i)}^{-1}]\) iff \([g_{\tau(i)}(\lambda_1, \lambda_2), B g_{\tau(i)}^{-1}] = [\tau(i)_{\lambda_1, \lambda_2}, i = 0, 1, \text{ and } [\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [g \alpha_{\tau(i)} g_{\tau(i)}^{-1}]\). In particular

\[
[\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [g \alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [g \alpha_{\tau(0)} g_{\tau(0)}^{-1} g(\lambda_1, \lambda_2, 1, 1) g^{-1}]
\]

Note that \([\alpha_{\tau(i)} g^{-1}] = [g \alpha_{\tau(i)} g^{-1}], \text{ and so } g_{\tau(i)}(1, 1, 1) g^{-1} \text{ is a twisted DHR representation of } C \text{ (relevant to } D). \text{ Applying Prop. 8.7 to the pair } C \subset D \text{ we have}

\[
[\alpha_{\tau(i)} g^{-1}] = [\alpha_{\tau(i)}(\sigma_1, \sigma_2, 1, 1)]
\]

for some \((\sigma_1, \sigma_2)\). By (54) we have \([\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [\alpha_{\tau(i)}(\sigma_1, \sigma_2, 1, 1)]\), and by (2) of Prop. 8.8 we have \([\alpha_{\tau(i)}(\sigma_1, \sigma_2, 1, 1)] = [\alpha_{\tau(i)}(\sigma_1, \sigma_2, \lambda_1)]\). Hence

\[
[\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [\alpha_{\tau(i)}(\sigma_1, \sigma_2, \lambda_1)]
\]

By (2) of Lemma 8.1 we have that \(g_{\tau(i)}(\lambda_1, \lambda_2) g^{-1} \simeq \tau(i)_{\lambda_1, \lambda_2}, i = 0 \text{ or } i = 1\), as DHR representations of \(C\). Notice that \(\omega_{\tau(i)} g_{\tau(i)}^{-1} = \omega_{\tau(i)}\), which can be checked directly from the definition of univalence (cf. [25]). Alternatively one can prove this as follows. First if \(g_{\tau(i)}(\lambda_1, \lambda_2) g^{-1} \simeq \tau(i)_{\lambda_1, \lambda_2}\) then we have nothing to prove. If \([g_{\tau(i)}(\lambda_1, \lambda_2) g^{-1}] \neq [\tau(i)_{\lambda_1, \lambda_2}]\), applying Prop. 8.4 to the pair \(B \subset C\) we know that \(g_{\tau(i)}(\lambda_1, \lambda_2) g^{-1}\) and \(\tau(i)_{\lambda_1, \lambda_2}\) restricts to the same DHR representation of \(B\), and so they must have the same univalence by Lemma 6.1 of [5].

So we have

\[
\omega_{\tau(i)} g_{\tau(i)}^{-1} = \omega_{\tau(i)} = \omega_{\tau(i)_{\lambda_1, \lambda_2}}
\]

where \(i = 0 \text{ or } 1\). As in the first proof of (1) of Prop. 8.8, from (57) we have \(\omega_{\sigma_1, \sigma_2, \lambda_1} = \omega_{\lambda_1, \lambda_2}, \forall (\lambda_1, \lambda_2)\). It follows that \((\sigma_1, \sigma_2)\) is degenerate. Therefore \((\sigma_1, \sigma_2) = (1, 1)\), and

\[
[\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [g \alpha_{\tau(i)} g_{\tau(i)}^{-1}].
\]

By (56) we have

\[
[\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [\alpha_{\tau(i)} g_{\tau(i)}^{-1}];
\]

and by (48) we have

\[
[\alpha_{\tau(i)} g_{\tau(i)}^{-1}] = [\alpha_{\tau(i)} g_{\tau(i)}^{-1}], \quad i = 0, 1.
\]

If \(\lambda_1 = \lambda_2\), by (58) and the remark before (54) we must have \(g_{\tau(i)}(\lambda_1, \lambda_2) g^{-1} \simeq \tau(i)_{\lambda_1, \lambda_2}, i = 0, 1\). Apply Prop. 8.2 to the pair \(B \subset C\) we know that \(\tau(i)_{\lambda_1, \lambda_1}\) is a direct sum of two distinct irreducible pieces denoted by \(\tau(i,j)_{\lambda_1, \lambda_1,B}; i, j = 0, 1\).
If $\lambda_1 \neq \lambda_2$, then from (58) and (2) of Lemma 8.1 we have that $g^{\tau(0)}_{(\lambda_1, \lambda_2)} g^{-1} \simeq \tau_{(\lambda_2, \lambda_1)}$ where $0 \leq j \leq 1$, and $[g^{\tau(i)}_{(\lambda_1, \lambda_2)} g^{-1}] \neq [\tau(i)_{(\lambda_2, \lambda_1)}]$. We may choose our labeling so that $g^{\tau(i)}_{(\lambda_1, \lambda_2)} g^{-1} \simeq \tau(i)_{(\lambda_2, \lambda_1)}$. Apply Prop. 8.2 to the pair $B \subset C$ we know that $\tau(i)_{(\lambda_1, \lambda_1), B}$ is an irreducible DHR representation of $B$, and $\tau(i)_{(\lambda_1, \lambda_2), B}$ are isomorphic to $\tau(i)_{(\lambda_2, \lambda_1), B}$ as DHR representations of $B$, $i = 0, 1$. By definitions we have $G(\tau(i)_{(\lambda_1, \lambda_2), B})^2 = G(\tau(i,j)_{(\lambda_1, \lambda_1), B})^2 = 1$, since $\tau(i)_{(\lambda_1, \lambda_2), B}$ and $\tau(i,j)_{(\lambda_1, \lambda_1), B}$ are non-twisted representations of $B$ relevant to $C$, and $B$ is the fixed point subnet of $C$ under the $\mathbb{Z}_2$ action. So these representations are different from $\tau(i)_{\lambda}$ whose grading is $e^{2\pi i}$ or $\tau(i)_{\lambda}$ whose grading is $e^{6\pi i}$.

We note that by applying Prop. 8.2 to $B \subset C$ we have

$$d(\tau(i,j)_{(\lambda_1, \lambda_1), B}) = d(\tau(i)_{(\lambda_1, \lambda_1)}), \quad d(\tau(i,j)_{(\lambda_1, \lambda_2), B}) = 2d(\tau(i)_{(\lambda_1, \lambda_2)}), \quad \lambda_1 \neq \lambda_2.$$ 

Applying (48) and (c) of Th. 6.3 to $C \subset D$ we have

$$d^2(\tau(i)_{(\lambda_1, \lambda_1)}) = 4d^2((\lambda_1, \lambda_1))\mu_A^2, \quad d^2(\tau(i,j)_{(\lambda_1, \lambda_2)}) = 4d^2((\lambda_1, \lambda_1))\mu_A^2.$$ 

Hence we know the indices of these known twisted representations $\tau(i,j)_{(\lambda_1, \lambda_1), B}$, $\tau(i,j)_{(\lambda_1, \lambda_2), B}$ of $B$ (relevant to $D$). By Prop. 8.4 we also know the indices of non-twisted representations of $B$ relevant to $D$. One can check easily that the sum of these indices add up to $\mu_B = 16\mu_D = 16\mu_A$. By [31] we have therefore identified all the irreducible DHR representations of $B$. In particular we have proved the following:

**Proposition 8.10.** All the irreducible twisted DHR representations of $B$ (relevant to $D$) are $\tau(i)_{\lambda}, \tau(i)_{\lambda}, 0 \leq i \leq 3$, $\tau(i,j)_{(\lambda, \lambda), B}, 0 \leq i, j \leq 1$, and $\tau(i,j)_{(\lambda_1, \lambda_2), B}, \lambda_1 \neq \lambda_2, 0 \leq i \leq 1$, where as DHR representations $\tau(i)_{(\lambda_1, \lambda_2), B}$ are isomorphic to $\tau(i,j)_{(\lambda_2, \lambda_1), B}$.

We note that our construction of $\tau(i)_{(\lambda_1, \lambda_2), B}$ and $\tau(i,j)_{(\lambda, \lambda), B}$ can be generalized to non-prime $n$ case.

### 8.7 Comments on the case of a general $n$

To motivate our discussion let us first consider the case when $A$ is holomorphic, i.e. when $\mu_A = 1$. In this case $D$ is also holomorphic, and $D$ has only one irreducible representation (the vacuum) labeled by $(1, ..., 1)$. In this case $\dim W = n - 1$. Note that $\alpha_{\tau(0)} \in W$ is a periodic automorphism, and we let $k \geq 1$ be the least integer such that $[\alpha^{(k)}] = [1]$. By Lemma 8.3 we must have $n|k$. On the other hand we must have $k \leq n$ since $\dim W = n$. So we conclude that $k = n$, and $W$ is spanned by $\{\alpha_{\tau(0)}, 0 \leq i \leq n - 1\}$ and all the irreducible representations of $B$ are given by $\sigma^j \tau(i)_{(0)}, 0 \leq i, j \leq n - 1$. So in the holomorphic case all twisted representations of $B$ are generated by $\tau(0)$ and $\sigma$ via fusion. This example shows that it is an interesting question to determine the nature of “composed” sectors $\alpha_{\tau(0)}^k (k \in \mathbb{N})$ in the general case as we have done for the case $n = 2$ in §8.3.
For general completely rational $\mathcal{A}$, we note that the grading $G(\tau^{(0)}) = e^{2\pi i n}$ by the remark after the definition of grading in §8.3. Now if $\sigma_1$ is an irreducible twisted DHR representation of $\mathcal{B}$, by Lemma 8.3 the grading $G(\sigma_1)$ is a complex number such that $G(\sigma_1)^n = 1$. Assume that $G(\sigma_1) = e^{-2\pi i k}$, $1 \leq k \leq n - 1$. Let $\sigma_2$ be any irreducible DHR representation of $\mathcal{B}$ such that $\tau^{(0)k} \succ \sigma_2$. By (1) of Lemma 8.3 $G(\sigma_2) = e^{2\pi i k}$ and if $\mu \prec \sigma_1 \sigma_2$ is an irreducible DHR representation of $\mathcal{B}$, then $G(\mu) = 1$. It follows from Lemma 8.3 that $\mu$ is non-twisted whose nature is determined in Cor. 8.4. By using Frobenius duality, we conclude that $\bar{\sigma}_1 \prec \bar{\mu}\tau^{(0)k}$. This observation shows once again the importance of $\tau^{(0)}$ and suggests that it is an interesting question to determine the nature of $\tau^{(0)k}$ ($k \in \mathbb{N}$) in general case. This question is related to the question in the previous paragraph by Lemma 8.1.

9 Generalizations and the case of two-dimensional nets

Results and proofs in this paper remain valid with weaker assumptions. We replace axiom D by the following ones:

- **Reeh-Schlieder property**: $\Omega$ is cyclic for $\mathcal{A}(I)$, $I \in \mathcal{I}$.

- **Modular PCT**: The modular conjugation of $(\mathcal{A}(S^+), \Omega)$ corresponds to the reflection $z \mapsto \bar{z}$ of $S^1$. (By Möbius covariance the modular conjugations associated with all intervals have then a geometric meaning.)

- **Factoriality**: $\mathcal{A}(I)$ is a factor for all $I \in \mathcal{I}$.

- **Equivalence between local and global intertwiners**: If $\mu, \nu$ are finite-index endomorphisms localized in the interval $I$, then $\text{Hom}(\mu, \nu) = \text{Hom}(\mu_I, \nu_I)$ as in [25].

If $\mathcal{C}$ is a local conformal net on the two-dimensional Minkowski spacetime $\mathbb{R}^2$ (see [30]), let $\mathcal{A}$ be the restriction of $\mathcal{C}$ to the time-zero axis: $\mathcal{A}(I) \equiv \mathcal{C}(O)$ where $O$ is the double cone with basis $I$. Then $\mathcal{A}$ satisfies all the above properties hence our results do apply. In particular we then have:

**Theorem 9.1.** If $\mathcal{C}$ is a local conformal net on the two-dimensional Minkowski spacetime. The following are equivalent:

(i) $\mathcal{A}$ is not completely rational;

(ii) $\sum_i d(\rho_i) = \infty$ (sum over all irreducible sectors);

(iii) $(\mathcal{A} \otimes \mathcal{A})^{\text{flip}}$ has an irreducible sector with infinite dimension.

The rest of our results have analogous extensions.
A Möb\(^{(n)}\) covariance in the strongly additive case

Let \(E\) be a symmetric \(n\)-interval of \(S^1\), namely \(E \equiv \sqrt{I}\) for some \(I \in \mathcal{I}\). With \(I_0, I_1, \cdots I_{n-1}\) the \(n\) connected component of \(E\), by the split property we have a natural isomorphism

\[\chi_E : \mathcal{A}(I_0) \otimes \mathcal{A}(I_1) \otimes \cdots \otimes \mathcal{A}(I_{n-1}) \to \mathcal{A}(I_0) \vee \mathcal{A}(I_1) \vee \cdots \vee \mathcal{A}(I_{n-1}) = \mathcal{A}(E).\]

A state of the form

\[\varphi \equiv (\varphi_0 \otimes \varphi_1 \otimes \cdots \otimes \varphi_{n-1}) \cdot \chi_E^{-1}\]

on \(\mathcal{A}(E)\), where \(\varphi_k\) is a normal faithful state on \(\mathcal{A}(I_k)\) and \(\varphi_k = \varphi_0 \cdot \text{Ad}(U(R(2k\pi/n)))\), is called a rotation invariant product state.

We state here a formula for the modular group of \(\mathcal{A}(E)\), that extends to the general case the formula by Schroer and Wiesbrock [51] in the example of the U(1)-current algebra, see [43].

**Proposition A.1.** There is a rotation invariant product state \(\varphi\) on \(\mathcal{A}(E)\) such that the corresponding modular group \(\sigma^\varphi\) of \(\mathcal{A}(E)\) is given by

\[\sigma^\varphi_t = \text{Ad}U^{(n)}(\Lambda^I(-2\pi t))|\mathcal{A}(E)\]

where \(\Lambda^I\) is the the lift to Möb\(^{(n)}\) of one parameter subgroup of Möb of generalized dilation associated with \(I\) (see [25]) and \(U^{(n)} = U \cdot M^{(n)}\) is the unitary representation of Möb\(^{(n)}\).

**Corollary A.2.** Let \(\mathcal{A}\) be a strongly additive local conformal net on \(S^1\) with the split property. Then every representation of \(\mathcal{A}\) with finite index is Möb\(^{(n)}\)-covariant with positive energy, for all \(n \in \mathbb{N}\).

**Proof** As \(\mathcal{A}\) is strongly additive, every finite index sector is Möb\(^{(n)}\)-covariant with positive energy by [24].

Fix \(n\) and let \(\text{Ad}U^{(n)}(g)\) be the action of Möb\(^{(n)}\) on \(\mathcal{A}\) given in Sect. 2. Let \(\rho\) be a finite-index localized endomorphism. We may assume \(\rho\) to be localized in an interval which is a connected component of a symmetric \(n\)-interval \(E = \sqrt{I}\).

With \(\{\Lambda^I_t\}_{t} \subset \text{Möb}^{(n)}\) the one-parameter dilation subgroup, denote by \(\alpha_t \equiv \text{Ad}U^{(n)} \cdot \Lambda^I_t(-2\pi t)\) the corresponding rescaled action on \(\mathcal{A}\).

We have to show that \(\rho_t \equiv \alpha_t \cdot \rho \cdot \alpha_t^{-1}\) is equivalent to \(\rho\) for every \(t \in \mathbb{R}\), namely that there is a unitary \(z_t \in \mathcal{A}(E)\) such that

\[\rho = \text{Ad}z_t \cdot \alpha_t \cdot \rho \cdot \alpha_t^{-1}; \tag{59}\]

having the covariance with respect to \(\Lambda^I_t\), by changing the interval \(I\) we then get the covariance with respect to Möb\(^{(n)}\).

By the Prop. A.1 \(\alpha\) restricts to the modular automorphism group of \(\mathcal{A}(E)\) with respect to \(\varphi\). With \(\Phi_\rho\) the left inverse of \(\rho |\mathcal{A}(E)\), by [40] the Connes [15] cocycle \(z_t = (D\varphi : \Phi_\rho : \varphi)_t \in \mathcal{A}(E)\) satisfies

\[\rho(x) = \text{Ad}z_t \cdot \alpha_t \cdot \rho \cdot \alpha_t^{-1}(x), \quad x \in \mathcal{A}(E),\]

hence we obtain eq. (59) by strong additivity. \(\blacksquare\)
B Frobenius reciprocity for global intertwiners

In this section we show that Th. 3.21 of [4] (also cf. (4) of Lemma 8.1) holds for global intertwiners when $\mathcal{N}$ is a conformal subnet of conformal net $\mathcal{M}$ with finite index. Note that we do not assume strong additivity conditions for the net $\mathcal{N}$ as in [4], but we consider global intertwiners. We will use the notations in §3 of [4] and refer the reader to [4] for unexplained notations. Fix an interval $I_0$. Let $\mathcal{M}$ be a conformal net on a Hilbert space $\mathcal{H}$ and $\lambda_1, \lambda_2$ be two DHR representations of $\mathcal{M}$ localized on $I_0$. Define

$$\text{Hom}(\lambda_1, \lambda_2) := \{x \in B(\mathcal{H}) | x_{\lambda_1,J}(m) = \lambda_{2,J}(m)x, \forall m \in \mathcal{M}(J), \forall J\}.$$  

$\text{Hom}(\lambda_1, \lambda_2)$ will be called the space of global intertwiners from $\lambda_1$ to $\lambda_2$. Its dimension will be denoted by $\langle \lambda_1, \lambda_2 \rangle$. The elements of $\text{Hom}(\lambda_1, \lambda_2)$ are referred to as local intertwiners from $\lambda_1$ to $\lambda_2$ (localized on $I_0$). Note that by Haag duality one obviously has $\text{Hom}(\lambda_1, \lambda_2) \subset \text{Hom}(\lambda_{1,k}, \lambda_{2,k})$. The following simple lemma tells us when a local intertwiner is global.

Lemma B.1. Let $I$ be an open interval which contains the closure of $I_0$. If $x \in \mathcal{M}(I_0) \cap \text{Hom}(\lambda_{1,I}, \lambda_{2,I})$, then $x \in \text{Hom}(\lambda_1, \lambda_2)$.

Proof By definition we can cover any interval $J$ by $I$ and a finite number of intervals $I_k$ such that $I_k \in I_0$. By the additivity of $\mathcal{M}$ we have $\mathcal{M}(J) \subset \mathcal{M}(I) \vee (\forall k, \mathcal{M}(I_k))$ and the lemma follows from the definitions. \qed

Now let $\lambda, \beta$ be DHR representations of $\mathcal{N}$ and $\mathcal{M}$ respectively localized in $I_0$, and $\sigma_\beta$ be the DHR representation of $\mathcal{N}$ localized on $I_0$ obtained from restriction of $\beta$ to $\mathcal{N}$. Assume that $\sigma_\lambda$ is a DHR representation of $\mathcal{M}$. We have the following theorem:

Theorem B.2. $\langle \sigma_\lambda, \beta \rangle_\mathcal{M} = \langle \lambda, \sigma_\beta \rangle_\mathcal{N}$.  

Proof We will adapt the proof of Th. 3.21 of [4]. Choose an interval $I$ as in the Lemma B.1. We can choose a $Q$-system $(\gamma_I, v, w)$ for the inclusion $\mathcal{N}(I) \subset \mathcal{M}(I)$ so that $\gamma_I$ extends to a canonical endomorphism of $\mathcal{M}(I)$ into $\mathcal{N}(I)$ for all intervals $\tilde{I} \supset I$ so that $(\gamma_{\tilde{I}}, v, w)$ $Q$-system for $\mathcal{N}(\tilde{I}) \subset \mathcal{M}(\tilde{I})$.

First we show the inequality $\leq$. Let $t \in \text{Hom}(\sigma_\lambda, \beta)$. By Haag duality we have $t \in \mathcal{M}(I_0)$ and $r = \gamma(t)w \in \mathcal{N}(I_0)$. The argument on Page 25 of [4] shows that $r \in \text{Hom}(\lambda_I, \sigma_\beta_I)$. By Lemma B.1 we have $r \in \text{Hom}(\lambda, \sigma_\beta)$. By Lemma 3.4 of [4] the map $t \mapsto r$ is injective, thus $\leq$ is proved.

We now turn to prove $\geq$. Suppose that $r \in \text{Hom}(\lambda, \sigma_\beta)$ is given. By Haag duality $r \in \mathcal{N}(I_0)$, and so $t = v^*r \in \mathcal{M}(I_0), s = \gamma(t) \in \mathcal{N}(I_0)$. Clearly $s \in \text{Hom}((\sigma_\lambda)_I, (\sigma_\beta)_I)$ since $r$ is a global intertwiner. It follows by Lemma B.1 that $s$ is also a global intertwiner, and so Lemma 3.20 of [4] applies. The rest of the proof is exactly the same as the proof on Page 26 of [4]. \qed
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