Complete classification of local conservation laws for generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation

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Abstract
In this paper, we consider nonlinear multidimensional Cahn–Hilliard and Kuramoto–Sivashinsky equations that have many important applications in physics and chemistry, and a certain natural generalization of these two equations to which we refer to as the generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation. For an arbitrary number of spatial independent variables, we present a complete list of cases when the latter equation admits nontrivial local conservation laws of any order, and for each of those cases, we give an explicit form of all the local conservation laws of all orders modulo trivial ones admitted by the equation under study. In particular, we show that the original Kuramoto–Sivashinsky equation admits no nontrivial local conservation laws, and find all nontrivial local conservation laws for the Cahn–Hilliard equation.

KEYWORDS
Cahn–Hilliard equation, conservation laws, Kuramoto–Sivashinsky equation
1 | INTRODUCTION

Below we study the following partial differential equation (PDE) in \( n + 1 \) independent variables \( t, x_1, ..., x_n \) and one dependent variable \( u \):

\[ u_t = a \Delta^2 u + b(u) \Delta u + f(u) |\nabla u|^2 + g(u), \tag{1} \]

where \( a \) is a nonzero constant, \( b, f, g \) are smooth functions of \( u \), \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \) is the Laplace operator, \( |\nabla u|^2 = \sum_{i=1}^{n} (\partial u / \partial x_i)^2 \), and \( n \) is an arbitrary natural number.

Equation (1), hereinafter referred to as the generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation, is a natural generalization of two well-known equations, the Cahn–Hilliard equation,

\[ u_t = c_1 \Delta (u^3 - u + c_2 \Delta u), \tag{2} \]

obtained from (1) by setting \( a = c_1 c_2 \), \( b = c_1 (3u^2 - 1) \), \( f = 6c_1 u \), and \( g = 0 \) (\( c_1 \) and \( c_2 \) are constants), and the Kuramoto–Sivashinsky equation,

\[ u_t + \Delta^2 u + \Delta u + |\nabla u|^2 / 2 = 0, \tag{3} \]

obtained from (1) by setting \( a = -1 \), \( b = -1 \), \( f = -1/2 \), and \( g = 0 \).

Equations (2) and (3) arise in a variety of physical, chemical, and biological contexts, with the Cahn–Hilliard equation (2) describing inter alia the process of phase separation in a binary alloy and having applications in many areas such as complex fluids, interfacial fluid flow, polymer science, spinodal decomposition, and tumor growth simulation, see, for example, Refs. [4, 20, 30], and references therein, and the Kuramoto–Sivashinsky equation (3) describing inter alia flame propagation, reaction–diffusion systems, and unstable drift waves in plasmas, see, for example, Refs. [13, 16], and references therein. Let us also note that Equation (3) exhibits chaotic behavior and is an important model in the study of chaotic phenomena, cf., for example, Refs. [16, 27].

While many authors have considered various generalizations of Equations (3) and (2), see, for example, Refs. [7, 21] for Equation (2) and Refs. [5, 23, 29] for Equation (3), there is no generally accepted definition of the term “generalized” in relation to these equations. The particular choice of (1) as a generalization of (2) and (3) studied in this paper is motivated by the desire to cover a reasonably large class of equations at once while keeping the classification results manageable.

Our goal is to present a complete description of nontrivial local conservation laws for (1) for all cases when they exist. To the best of our knowledge, this was not yet done, especially for an arbitrary number \( n \) of the space variables, although, for example, some partial results on conservation laws of several different generalizations of the Kuramoto–Sivashinsky equation for \( n = 1, 2 \) are known, see, for example, Refs. [5, 14, 29].

Recall that obtaining a complete description of nontrivial local conservation laws for a given PDE, rather than finding a few low-order ones, is a difficult goal that was achieved only for a rather small number of examples, cf., for example, Refs. [15, 18, 26, 32, 34], and references therein.

In closing, note that conservation laws have numerous important applications, cf., for example, Refs. [1, 3, 6, 12, 17, 25, 31, 33], including, for instance, improving numerically solving the PDE under study using the discretizations respecting known conservation laws, see, for example, Refs. [2, 11] and references therein, and construction of symmetries using Hamiltonian structures, Noether theorem, and other methods, see, for example, Refs. [22, 24, 25, 31], and references therein.
The rest of the paper is organized as follows. In Section 2 we recall some necessary definitions and set up notation, in Section 3 the main results are presented, while their proofs are given in Section 4.

2 | PRELIMINARIES

In this section, we just recall standard definitions and introduce some notation mostly following Ref. [25] but adapted to the case of the equation under study, that is, (1).

From now on, \( t, x_1, \ldots, x_n \), and \( u_I \) will be viewed as coordinates on a suitable jet space; here \( I = (i_0, i_1, \ldots, i_n) \in \mathbb{Z}^{n+1}_+ \) is a multi-index and

\[
u_I = \frac{\partial^{||I||} u}{\partial t^{i_0} \partial x_1^{i_1} \ldots \partial x_n^{i_n}},
\]

cf., for example, Ref. [25] for more details (note that \( u_{0\ldots0} \equiv u \)).

First of all, we define a differential function as a smooth function depending at most on \( t, x_1, \ldots, x_n \) and finitely many of \( u_I \).

We also need the operators of total derivatives

\[
D_{x_j} = \frac{\partial}{\partial x_j} + \sum_{I \in \mathbb{Z}^{n+1}_+} u_{Ij} \frac{\partial}{\partial u_I}, \quad D_I = \frac{\partial}{\partial t} + \sum_{I \in \mathbb{Z}^{n+1}_+} u_{It} \frac{\partial}{\partial u_I},
\]

where \( Ij = (i_0, \ldots, i_j + 1, \ldots, i_n) \) and \( It = (i_0 + 1, i_1, \ldots, i_n) \), see, for example, Refs. [18, 22, 25] for more details on those.

From now on, we assume the Laplace operator to be defined as \( \Delta = \sum_{i=1}^n D_{x_i}^2 \), where \( D_{x_i}^2 = D_{x_i} \circ D_{x_i} \).

A local conservation law for (1) is a differential expression

\[
D_I(T) + \text{Div} X,
\]

which vanishes modulo (1) and its differential consequences. Here, \( X = (X_1, \ldots, X_n) \), \( \text{Div} X = \sum_{i=1}^n D_{x_i}(X_i) \) stands for total divergence, and \( T, X_1, \ldots, X_n \) are differential functions.

The function \( T \) in (4) is called a density and the vector function \( X \) is called a flux for the conservation law under study.

It is immediate that a linear combination with constant coefficients of two local conservation laws for (1) also is a local conservation law for (1), so local conservation laws for (1) form a vector space.

A local conservation law of the form (4) for (1) is said to be trivial if either \( T \) and \( X \) themselves vanish modulo (1) and its differential consequences or \( D_I(T) + \text{Div} X \equiv 0 \) holds identically no matter whether (1) and its differential consequences hold.

Two local conservation laws are said to be equivalent if their difference is a trivial conservation law.

In what follows, we assume without loss of generality, cf., for example, [25, p. 204], that \( X_i \) and \( T \) do not depend on the \( t \)-derivatives of \( u \) or mixed derivatives of \( u \) involving derivatives with
respect to both $x_i$ and $t$, as disallowing such dependence only removes certain trivial parts from
the conservation laws under study because of the evolutionary nature of (1).

A local conservation law (4) for (1) is in characteristic form if

$$D_t(T) + \text{Div} X = Q \cdot (u_t - (a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u))),$$

where the differential function $Q$ is called a characteristic of the conservation law in question.

It can be shown that any local conservation law for (1) is equivalent to a local conservation law for (1) in characteristic form, see, for example, chapter 4 of Ref. [25] for further details.

Note also that a local conservation law for (1) is nontrivial if and only if there exists an equivalent local conservation law for (1) in characteristic form that has a nonzero characteristic, cf., for example, Ref. [25].

Below we deal only with local conservation laws, so, whenever a conservation law is mentioned, we mean a local conservation law unless explicitly stated otherwise.

We also tacitly assume, again without loss of generality (cf. the discussion above), that the conservation laws we consider below are in characteristic form, and thus classifying and describing nontrivial local conservation laws for (1) is essentially tantamount to doing the same for their characteristics.

3 | MAIN RESULTS

We now state the main results of this paper and discuss some of their implications. The proofs will be given in the next section.

**Theorem 1.** Let Equation (1) with $a \neq 0$ and smooth functions $b(u), f(u),$ and $g(u)$ satisfy one of the following sets of conditions:

1. $f \neq \frac{\partial b}{\partial u}$,
2. $f = 0, \frac{\partial^2 g}{\partial u^2} \neq 0$ and $b$ is constant,
3. $f = \frac{\partial b}{\partial u} \neq 0, \frac{\partial^2 g}{\partial u^2} \neq 0$, and $\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} - \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2} \neq 0$.

Then Equation (1) admits no nontrivial local conservation laws.

In particular, we immediately get the following important corollary of the above theorem:

**Corollary 1.** The Kuramoto–Sivashinsky equation (3) has no nontrivial local conservation laws.

Thus, the original Kuramoto–Sivashinsky equation (3) and, more broadly, Equation (1) satisfying the conditions of Theorem 1, admit no nontrivial local conservation laws.

This result has a number of important consequences. For one, as there are no nontrivial local conservation laws in the cases under study, while discretizing (1) under the assumptions of Theorem 1 or (3) in order to solve the equations in question numerically, one does not have to worry about consistency of the chosen discretization with the conservation laws, cf. the discussion in the introduction and in Refs. [15] and [10].
Corollary 2. The generalized Cahn–Hilliard equation (cf. eq. (1.7) in Ref. [7]),

\[ u_t + \Delta^2 u - \Delta f(u) + g(u) = 0 \]  

obtained from (1) by setting \( a = -1, b = \frac{\partial f}{\partial u}, f = \frac{\partial^2 f}{\partial u^2}, g = -\bar{g} \) admits no nontrivial local conservation laws provided one of the following sets of conditions holds:

1. \( \frac{\partial^2 f}{\partial u^2} = 0 \) and \( \frac{\partial^2 g}{\partial u^2} \neq 0 \),
2. \( \frac{\partial^2 f}{\partial u^2} \neq 0, \frac{\partial^2 g}{\partial u^2} \neq 0 \) and \( \frac{\partial^2 g}{\partial u^2} \frac{\partial^3 f}{\partial u^3} - \frac{\partial^2 f}{\partial u^2} \frac{\partial^3 g}{\partial u^3} \neq 0 \).

The following theorem provides a complete list of cases when Equation (1) admits nontrivial local conservation laws.

Theorem 2. If \( f = \frac{\partial b}{\partial u} \) then (1) with \( a \neq 0 \) and smooth functions \( b(u), f(u), \) and \( g(u) \) admit nontrivial local conservation laws, all of which are listed below modulo trivial ones, only in the following three cases:

I. Let \( f = 0, b = c_1, \) and \( g = c_2 u + c_3 \), where \( c_1, c_2, c_3 \) are arbitrary constants. Then, the densities of all nontrivial local conservation laws for (1) have the form

\[ T^I = \left( u + \frac{c_3}{c_2} \right) Q^I \]  

for \( c_2 \neq 0 \), and

\[ T^I = (u - c_3 t)Q^I \]  

for \( c_2 = 0 \); here \( Q^I = Q^I(t, x_1, \ldots, x_n) \) is any smooth solution of the linear PDE with constant coefficients

\[ \frac{\partial Q^I}{\partial t} + a\Delta^2 (Q^I) + c_1\Delta(Q^I) + c_2 Q^I = 0. \]  

II. Let \( f = \frac{\partial b}{\partial u} \neq 0 \) and \( g = c_2 u + c_3 \), where \( c_2 \) and \( c_3 \) are arbitrary constants. Then, the densities of all nontrivial local conservation laws for (1) have the form

\[ T^{II} = e^{-c_2 t} \left( u + \frac{c_3}{c_2} \right) \tilde{Q}^{II} \]  

for \( c_2 \neq 0 \), and

\[ T^{II} = (u - c_3 t)\tilde{Q}^{II} \]  

for \( c_2 = 0 \); here \( \tilde{Q}^{II} = \tilde{Q}^{II}(x_1, \ldots, x_n) \) is any smooth solution of the linear Laplace equation

\[ \Delta \tilde{Q}^{II} = 0. \]
Let $f = \frac{\partial b}{\partial u} \neq 0$, $\frac{\partial^2 g}{\partial u^2} \neq 0$, and $\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} = \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2}$. Then, there exist constants $c_4 \neq 0$ and $c_5$ such that $\frac{\partial g}{\partial u} = c_4 b + c_5$, and the densities of all nontrivial local conservation laws for (1) have the form

$$T_{III} = e^{-(a c_4^2 + c_5) t} \tilde{Q}_{III} u,$$

(12)

where $\tilde{Q}_{III} = \tilde{Q}_{III}(x_1, ..., x_n)$ is any smooth solution of a linear PDE with constant coefficients

$$\Delta \tilde{Q}_{III} + c_4 \tilde{Q}_{III} = 0.$$

(13)

Note that the densities of the conservation laws from Theorem 2 in general nontrivially depend on $t$ and $x_i$; while in particular for the densities given by (7) and (10), it is possible to find equivalent conservation laws whose densities do not depend explicitly on $t$, the fluxes of these equivalent conservation laws would have far more complicated and difficult-to-use expressions than those listed in Corollary 3 below.

Given the diversity of potential applications that was already mentioned in the introduction while discussing the original Cahn–Hilliard and Kuramoto–Sivashinsky equations, physical interpretation of even the simplest of the above conservation laws should be done on the case-by-case basis, thus being quite naturally beyond the scope of this paper.

Nevertheless, the following general interpretation can be given: If $u$ is a (physical or other) quantity whose evolution is described by a particular case of (1) and $\xi u$, where $\xi(x_1, ..., x_n, t)$ is a smooth function, is a conserved density for the equation in question, then, assuming that the integral converges, the quantity

$$\int_{\mathbb{R}^n} \xi u \, dx_1 \cdots dx_n$$

is, up to a proper normalization, a weighted space average of the quantity $u$ with the (possibly explicitly time-dependent) weight $\xi$.

The assumption that $\xi u$ is a conserved density for the equation in question then implies that the above integral, and thus the said average, in fact does not depend on $t$.

In the special cases I and II of Theorem 2 with additional assumptions that $g \equiv 0$, Equation (1) admits inter alia conservation laws with the densities $u$ and $x_i u$ with $i = 1, ..., n$, which implies, as per the above, that, assuming that the integrals exist, the quantities

$$\int_{\mathbb{R}^n} u \, dx_1 \cdots dx_n \quad \text{and} \quad \int_{\mathbb{R}^n} x_i u \, dx_1 \cdots dx_n, \quad i = 1, ..., n,$$

that is, the (nonnormalized) space average of $u$ and the (nonnormalized) first momenta of $u$, do not depend on the time $t$.

Let us also point out that some of the conservation laws from Theorem 2 could possibly be applied for proving existence results, estimates for solutions etc. for the equations under study in the spirit of, for example, Refs. [8, 9, 19], and references therein, although this again might have to be done ad hoc and for this and other reasons will not be discussed further in this article.

The following results are readily verified by straightforward computation assuming the validity of Theorem 2.
Corollary 3. The fluxes for the conservation laws listed in Theorem 2 have the following form, up to the obvious trivial contributions:

I. If \( f = 0, b = c_1, \) and \( g = c_2 u + c_3, \) where \( c_1, c_2, c_3 \) are constants, then the fluxes associated with the conservation laws with densities of the form \( T^I \) read

\[
X^I_i = c_1 \left( u + \frac{c_3}{c_2} (\partial Q^I_i - u x_i Q^I) \right) + a \left( u + \frac{c_3}{c_2} \right) \frac{\partial Q^I}{\partial x_i} - u x_i \Delta Q^I + \frac{\partial Q^I}{\partial x_i} \Delta u - Q^I \Delta u x_i \tag{14}
\]

for \( c_2 \neq 0, \) and

\[
X^I_i = c_1 \left( (u - c_3 t) \frac{\partial Q^I_i}{\partial x_i} - u x_i Q^I \right) + a \left( (u - c_3 t) \right) \frac{\partial Q^I}{\partial x_i} - u x_i \Delta Q^I + \frac{\partial Q^I}{\partial x_i} \Delta u - Q^I \Delta u x_i \tag{15}
\]

for \( c_2 = 0. \)

II. If \( f = \frac{\partial b}{\partial u} \neq 0 \) and \( g = c_2 u + c_3, \) where \( c_2, c_3 \) are constants, then the fluxes associated with the conservation laws with densities of the form \( T^II \) are

\[
X^{II}_i = e^{-c_2 t} \left( \frac{\partial Q^{II}_i}{\partial x_i} (\bar{b} + a \Delta u) - \bar{Q}^{II}_i (b u x_i + a \Delta u x_i) \right). \tag{16}
\]

where \( \bar{b} = \bar{b}(u) \) is such that \( \frac{\partial b}{\partial u} = b. \)

III. If \( f = \frac{\partial b}{\partial u} \neq 0, \frac{\partial^2 g}{\partial u^2} \neq 0, \) and \( \frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} = \frac{\partial^3 g}{\partial u^3} \frac{\partial^2 b}{\partial u^2}, \) so \( \frac{\partial g}{\partial u} = c_4 b + c_5 \) for suitable constants \( c_4 \) and \( c_5, \) with \( c_4 \neq 0, \) then the fluxes associated with the conservation laws with densities of the form \( T^{III} \) are

\[
X^{III}_i = e^{-(ac_2 + c_1)t} \left( \frac{\partial Q^{III}_i}{\partial x_i} \left( a \Delta u + \frac{g}{c_4} \right) - \frac{c_3}{c_4} u - ac_4 u \right) + Q^{III}_i \left( a c_4 u x_i + \frac{c_3}{c_4} u x_i - \frac{1}{c_4} \frac{\partial g}{\partial u} u x_i - a \Delta u x_i \right). \tag{17}
\]

Corollary 4. The only nontrivial local conservation laws for the Cahn–Hilliard equation (2)

\[
u_t = c_1 \Delta (u^3 - u + c_2 \Delta u)
\]

with \( c_1 \neq 0 \) and \( c_2 \neq 0 \) are, modulo trivial ones, those with the densities of the form \( Q(x_1, \ldots, x_n) u \) where \( Q \) satisfies the Laplace equation \( \Delta Q = 0. \)

Note that local conservation laws of generalized Cahn–Hilliard equation (5) whenever they exist can be obtained by substituting

\[
a = -1, \ b = \frac{\partial f}{\partial u}, \ f = \frac{\partial^2 f}{\partial u^2}, \ g = -\tilde{g}
\]

into general results from Theorem 2 and Corollary 3.
PROOF OF THE MAIN RESULTS

Since the proofs of both (1) and (2) are based on the analysis of the same determining equation for the characteristics of conservation laws for (1), we will prove the theorems in question simultaneously.

Proof of Theorems 1 and 2. Recall that a nontrivial local conservation law for (1) has a nonzero characteristic, so to classify such conservation laws, it is enough to do that for their characteristics, cf., for example, Refs. [1, 25].

Let a differential function $Q$ be a characteristic of a local conservation law for (1). Just like for $X_i$ and $T$, cf. (2), we assume without loss of generality that $Q$ depends only on $t, x_i, u$, and finitely many $x$-derivatives of $u$ but do not depend on $t$-derivatives of $u$ and derivatives of $u$ involving differentiation with respect to both $t$ and $x_i$, cf., for example, [25, chapter 4].

Then, for $Q$ to be a characteristic of a local conservation law for (1), it is necessary that (see [25, p. 330])

$$
\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} D_{x_i x_j x_j} (Q) + b \sum_{i=1}^{n} D_{x_i} (Q) + \left(2 \frac{\partial b}{\partial u} - 2f\right) \sum_{i=1}^{n} (u_{x_i} Q + u_{x_i} D_{x_i} (Q)) \\
+ \left(\frac{\partial^2 b}{\partial u^2} - \frac{\partial f}{\partial u}\right) \sum_{i=1}^{n} u_{x_i}^2 Q + \frac{\partial g}{\partial u} Q = 0.
$$

(18)

First of all, observe that Equation (1) satisfies the conditions of Theorem 6 from Ref. [18] with $N = -1$ and hence in fact $Q$ can depend at most on $t, x_1, \ldots, x_n$.

It is easily seen that because of this we can assume without loss of generality that the density $T$ of the associated conservation law depends at most on $t, x_1, \ldots, x_n$ and $u$ and we have (cf., for example, [25, p. 349])

$$
\frac{\partial T}{\partial u} = Q,
$$

(19)

whence we can readily find $T$ if given $Q$.

With this in mind it is readily verified that $Q$ is a characteristic of conservation law for (1) if and only if it satisfies the following simplification of condition (18):

$$
\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\delta x_i^2 \delta x_j^2} + b \sum_{i=1}^{n} \frac{\partial^2 Q}{\delta x_i^2} + \left(2 \frac{\partial b}{\partial u} - 2f\right) \sum_{i=1}^{n} (u_{x_i} Q + u_{x_i} \frac{\partial Q}{\partial x_i}) \\
+ \left(\frac{\partial^2 b}{\partial u^2} - \frac{\partial f}{\partial u}\right) \sum_{i=1}^{n} u_{x_i}^2 + \frac{\partial g}{\partial u} Q = 0.
$$

(20)

As $Q$ is independent of $u$ and its derivatives, applying $\delta / \delta u_{x_i x_i}$ to (20) for any $i$, we get

$$
\left(2 \frac{\partial b}{\partial u} - 2f\right) Q = 0,
$$

(21)
which means that we must have
\[ f = \frac{\partial b}{\partial u} \quad (22) \]
otherwise there exist no nontrivial conservation laws. This establishes part 1 of Theorem 1.

For the rest of the proof, we assume that (22) holds.

Using (22), one can simplify Equation (20) to
\[ \frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + \frac{\partial g}{\partial u} Q + b \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (23) \]

Upon applying \( \frac{\partial}{\partial u} \) to (23), we get
\[ \frac{\partial^2 g}{\partial u^2} Q + \frac{\partial b}{\partial u} \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (24) \]

We can split the analysis of (24) into three cases labeled as A, B, and C.

Case A: \( \frac{\partial b}{\partial u} = 0 \), meaning that \( b \) is constant and \( f = 0 \).

Then, from (24) we get
\[ \frac{\partial^2 g}{\partial u^2} Q = 0, \quad (25) \]
which means that we must have
\[ \frac{\partial^2 g}{\partial u^2} = 0 \quad (26) \]
otherwise there exist no nontrivial local conservation laws. This establishes part 2 from Theorem 1.

Setting \( b = c_1 \) and \( g = c_2 u + c_3 \), where \( c_1, c_2, \) and \( c_3 \) are arbitrary constants, we can simplify (23) to
\[ \frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + c_2 Q + c_1 \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (27) \]

Now using (19), we can readily find the associated \( T \), and thus establish part I from Theorem 2. For convenience, we denote in Theorem 2 the relevant \( Q \) and \( T \) as \( Q^I \) and \( T^I \) to indicate their relation to part I of the theorem in question, and adopt similar notation for parts II and III. Note that the way we used the residual freedom in the choice of the form of \( T^I \), making it inhomogeneous in \( u \), is motivated by the desire to keep the form of the associated flux components \( X^I_i \) reasonably simple.

Case B: \( \frac{\partial b}{\partial u} \neq 0 \) but \( \frac{\partial^2 g}{\partial u^2} = 0 \).

Then from (24) we get
\[ \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (28) \]
Setting $g = c_2 u + c_3$ and using Equation (28), we can simplify (23) to

$$\frac{\partial Q}{\partial t} + c_2 Q = 0. \quad (29)$$

Solving these we get $Q = e^{-c_2 t} \hat{Q}$, where $\hat{Q}$ is a function of independent variables $x_1, \ldots, x_n$, which needs to satisfy the Laplace equation $\Delta(\hat{Q}) = 0$, establishing part II from Theorem 2 upon another application of (19).

Case C: $\frac{\partial b}{\partial u} \neq 0$ and $\frac{\partial^2 g}{\partial u^2} \neq 0$.

Then, we can differentiate (24) with respect to $u$, which gives

$$\frac{\partial^3 g}{\partial u^3} Q + \frac{\partial^2 b}{\partial u^2} \sum_{i=1}^n \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (30)$$

Equations (24) and (30) constitute a linear system for $Q$ and $\Delta Q$, and for it to have a nontrivial solution (and thus nonzero $Q$), its determinant must vanish, that is,

$$\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} = \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2}, \quad (31)$$

otherwise there exist no nontrivial conservation laws. This establishes part 3 from Theorem 1 and completes the proof of the latter.

Assuming that (31) holds, we readily find that

$$\frac{\partial g}{\partial u} = c_4 b + c_5, \quad (32)$$

where $c_4 \neq 0$ and $c_5$ are arbitrary constants; note that we have to impose the condition $c_4 \neq 0$, otherwise the assumptions of Case C will be violated.

Now we can simplify (24) to

$$c_4 Q + \sum_{i=1}^n \frac{\partial^2 Q}{\partial x_i^2} = 0. \quad (33)$$

Using (32) and (33), we can simplify (23) to

$$\frac{\partial Q}{\partial t} + (ac_4^2 + c_5)Q = 0. \quad (34)$$

Hence, $Q = e^{-(ac_4^2 + c_5)t} \hat{Q}$, where $\hat{Q}$ is a (smooth) function of independent variables $x_1, \ldots, x_n$, which needs to satisfy $c_4 \hat{Q} + \Delta \hat{Q} = 0$. Making use of (19) yields the associated density $T$, thus establishing part III of Theorem 2 and completing the proof of this theorem. □

Note that while for $n > 1$ in all three cases listed in Theorem 2, Equation (1) admits infinitely many nontrivial local conservation laws, for $n = 1$ the situation is strikingly different: In Case I, there are still infinitely many nontrivial local conservation laws, while for Cases II and III, there are just two, because for $n = 1$, Equations (11) and (13) become linear ordinary differential
equations of second order while (8) remains a linear *partial* differential equation in two independent variables. Notice also that for \( n = 1 \) the results of Case I of Theorem 2 readily follow, up to a shift of \( u \) by a suitable constant to turn the special case of (1) under study into a linear homogeneous PDE, from Theorem 3 of Ref. [28].

In fact, the presence of infinitely many nontrivial local conservation laws in Case I of Theorem 2 is not unexpected, because this is a degenerate case of sorts, when (1) becomes a linear inhomogeneous partial differential equation. Even for \( n > 1 \), there still is a significant difference between the cases in Theorem 2 as it is readily verified that for Case I the associated infinite set of local conservation laws is parameterized by an arbitrary smooth function of \( n \) independent variables \( x_1, \ldots, x_n \) while for Cases II and III, the associated infinite sets of associated local conservation laws are parameterized by pairs of arbitrary smooth functions of \( n - 1 \) variables.

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**DATA AVAILABILITY STATEMENT**

Data sharing is not applicable to this paper as no data sets were generated or analyzed during this study.

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