CHARACTERIZING A SURFACE BY INVARIANTS

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Abstract. Canonical principal parameters are introduced for surfaces in $\mathbb{R}^3$ without umbilical points. It is proved that in these parameters the surface is determined (up to position in space) by a pair of invariants satisfying a partial differential equation equivalent to the Gauss equation. As such a pair of invariants we may use the principal curvatures or the Gauss and the mean curvature.

1. Introduction

An important problem in differential geometry is to characterize a geometric object by its invariants. For example, it is well known that any curve in $\mathbb{R}^3$ is determined (up to position in space) by its curvature and torsion as functions of its natural parameter.

For the surfaces in $\mathbb{R}^3$ the situation is more complicated. According to the classical Bonnet’s theorem, a surface is determined (up to position in space) by six functions – the coefficients of the first and the second fundamental forms satisfying the equations of Gauss and Codazzi. Of course the coefficients of the fundamental forms are not invariant functions, unlike the curvature and torsion of a curve, although these coefficients rest unchanged in motions. Nevertheless, the above Bonnet’s theorem helps us in studying the determination of a surface by invariants. Note that some differential equations between the invariants of the surfaces arise in a natural way as a result of the equations of Gauss and Codazzi. The so-called Lund-Regge problem here is to find the minimum possible invariants and relations between them that characterize a surface, see \cite{7}, \cite{8}. When trying to reduce the number of invariants and the compatibility conditions, it is common to search for special parameters, just as in the case of the curves and their natural parameters.

An important progress in this direction is made in \cite{5} – a work that actually inspired the present paper. Namely in \cite{5} it is proved that a regular surface is determined (up to position in space) by four invariants – the principal curvatures $\nu_1$, $\nu_2$ and the geodesic curvatures $\gamma_1$, $\gamma_2$ of the principal lines. These invariants satisfy three partial differential equations equivalent to the Gauss and Codazzi equations. In particular, for the class of Weingarten surfaces the authors introduce some special parameters that they call geometric and they prove that in these parameters the surface is determined by only one invariant function and two other functions. These three functions are closely related to the principal curvatures and are subjects to a single partial differential equation equivalent to the Gauss equation.

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In this paper, we introduce canonical parameters for any surface in \( \mathbb{R}^3 \) without umbilical points and we prove that in these parameters the surface is locally determined up to position in space by just two invariant functions related by just one partial differential equation equivalent to the Gauss equation. These two invariant functions are the principal curvatures or the Gauss curvature and the mean curvature. It is clear that the surface cannot be determined by just one of these invariant functions – for example there exist many surfaces with the same constant Gauss or mean curvature. So it appears that our results solve the Lund-Regge problem for surfaces without umbilical points.

For similar investigations about surfaces in some upper dimensional spaces of constant curvature \( c \), we mention \([9]\), where some special isothermal parameters are used in the case of minimal non-superconformal surfaces in \( Q^4_c \) and it is proved that the surface is determined by the Gauss curvature and the normal curvature, which satisfy a system of two partial differential equations; see also \([4]\).

### 2. Preliminaries

Let a regular surface in \( \mathbb{R}^3 \) be given by the parametric equation \( S : x = x(u,v) \). We denote by \( E, F, G, \) resp. \( L, M, N \) the coefficients of the first, resp. the second fundamental form. A point of \( S \) is called \textit{umbilical} if the two fundamental forms are proportional at that point. The Gauss curvature \( K \) and the mean curvature \( H \) of \( S \), which are the most important invariants of the surface, are expressed with these coefficients respectively by

\[
K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{EN - 2FM + GL}{2(EG - F^2)}.
\]

Moreover the coefficients of the two fundamental forms satisfy the equation of Gauss

\[
K = \frac{1}{2W} \left\{ \left( \frac{E_v - F_u}{W} \right)_v + \left( \frac{G_u - F_v}{W} \right)_u \right\} - \frac{1}{4W^4} \begin{vmatrix}
E & F & G \\
L & M & N \\
E_u & F_u & G_u
\end{vmatrix},
\]

and the equations of Codazzi

\[
2W^2(L_v - M_u) = (EN - 2FM + GL)(E_v - F_u) + \begin{vmatrix}
E & F & G \\
L & M & N \\
E_u & F_u & G_u
\end{vmatrix},
\]

\[
2W^2(M_v - N_u) = (EN - 2FM + GL)(F_v - G_u) + \begin{vmatrix}
E & F & G \\
L & M & N \\
E_v & F_v & G_v
\end{vmatrix},
\]

where \( W = \sqrt{EG - F^2} \). The classical theorem of Bonnet \([1]\) states that conversely, given six functions \( E, F, G, L, M, N \) (\( E > 0, EG - F^2 > 0 \)) that satisfy these equations, then locally there exists a unique (up to position in space) surface, having \( E, F, G \) as coefficients of the first fundamental form and \( L, M, N \) as coefficients of the second fundamental form; see also e.g. \([2]\), p. 236.

Suppose a curve \( c \) on \( S \) be defined by

\[
c : \quad u = u(s), \quad v = v(s),
\]
where \( s \) is the natural parameter of \( c \). Then the Frenet formulas are

\[
\begin{align*}
t' &= \gamma p + \nu l \\
p' &= -\gamma t + \alpha l \\
l' &= -\nu t - \alpha p
\end{align*}
\]

where \( t \) is the unit tangent vector field of \( c \), \( l \) is the unit normal vector field of \( S \) and \( p = l \times t \). The functions \( \gamma \), \( \nu \), \( \alpha \) are respectively the geodesic curvature, the normal curvature and the geodesic torsion of \( c \) on \( S \), respectively. The normal curvature of \( c \) is given by

\[
\nu = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\ddot{u}^2 + 2F\ddot{u}\dot{v} + G\ddot{v}^2}.
\]

Actually at each point of \( c \) the normal curvature \( \nu \) depends not on the curve itself, but on the direction of its tangent vector at that point, so we can speak about normal curvature of a direction in any point. The maximal and the minimal values of the normal curvatures at a point are called \textit{principal curvatures} and the corresponding directions and vectors – \textit{principal directions} and \textit{principal vectors}. A curve on \( S \) is called \textit{principal} if its tangent vector is principal at any point. When the surface has no umbilical points, the parameters \((u, v)\) can be chosen such that the parametric lines are principal. Then the parameters \((u, v)\) of \( S \) are called \textit{principal}. In terms of the coefficients of the fundamental forms this means that \( F = M = 0 \) on \( S \). In this case the geodesic torsions of the parametric lines vanish identically. On the other hand, the geodesic curvatures of the parametric lines are

\[
\begin{align*}
\gamma_1 &= -\frac{E_v}{2E\sqrt{EG}} , \\
\gamma_2 &= \frac{G_u}{2G\sqrt{EG}} .
\end{align*}
\]

Let \( \nu_1 \) and \( \nu_2 \) be the principal curvatures of \( S \). Then the classical definition of the Gauss curvature and the mean curvature becomes

\[
K = \nu_1 \nu_2 , \quad H = \frac{1}{2}(\nu_1 + \nu_2) .
\]

3. Determining non-umbilical surfaces

Suppose that \( S \) has no umbilical points and the parametric lines are principal, i.e. \( F = M = 0 \) on \( S \). Then the equation of Gauss is

\[
K = \nu_1 \nu_2 = -\frac{1}{2\sqrt{EG}} \left\{ \left( \frac{E_v}{\sqrt{EG}} \right)_v + \left( \frac{G_u}{\sqrt{EG}} \right)_u \right\}
\]

and the equations of Codazzi take the form

\[
2EGL_v = (EN + GL)E_v , \quad 2EGN_u = (EN + GL)G_u .
\]

On the other hand, the principal curvatures \( \nu_1, \nu_2 \) are given by

\[
\nu_1 = \frac{L}{E} , \quad \nu_2 = \frac{N}{G} .
\]
Since the surface has no umbilical points, the difference \( \nu_1 - \nu_2 \) cannot vanish. Hence it is easy to see that the equations of Codazzi may be written as

\[
\frac{E_v}{2E} = -\frac{(\nu_1)_v}{\nu_1 - \nu_2}, \quad \frac{G_u}{2G} = \frac{(\nu_2)_u}{\nu_1 - \nu_2}.
\]

Let us fix a point \( (u_0, v_0) \). The last equations imply that there exist two functions \( \varphi_1(u) \) and \( \varphi_2(v) \), such that

\[
\sqrt{E} = \varphi_1(u) e^{\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv}, \quad \sqrt{G} = \varphi_2(v) e^{\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du}.
\]

In other words, for any functions \( \phi_1(u), \phi_2(v) \), the function

\[
\phi_1(u) \sqrt{E} e^{\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv}
\]

does not depend on \( v \) and the function

\[
\phi_2(v) \sqrt{G} e^{\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du}
\]

does not depend on \( u \). Now we introduce new parameters \( (\bar{u}, \bar{v}) \) by the formulas

\[
\bar{u} = \frac{1}{\sqrt{E(u_0, v_0)}} \int_{u_0}^{u} \sqrt{E} e^{\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv} + \int_{u_0}^{u} \frac{(\nu_1)_u}{\nu_1 - \nu_2} (u, v_0) du \, du + \bar{u}_0,
\]
\[
\bar{v} = \frac{1}{\sqrt{G(u_0, v_0)}} \int_{v_0}^{v} \sqrt{G} e^{\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du} - \int_{v_0}^{v} \frac{(\nu_2)_v}{\nu_1 - \nu_2} (u_0, v) dv + \bar{v}_0.
\]

for some constants \( \bar{u}_0, \bar{v}_0 \). The parameters \( (\bar{u}, \bar{v}) \) are also principal. Moreover we have

\[
\frac{\sqrt{E}}{\sqrt{E(\bar{u}_0, \bar{v}_0)}} e^{\int_{\bar{u}_0}^{\bar{u}} \frac{(\bar{\nu}_1)_{\bar{u}}}{\bar{\nu}_1 - \bar{\nu}_2} d\bar{u}} + \int_{\bar{v}_0}^{\bar{v}} \frac{(\bar{\nu}_1)_{\bar{v}}}{\bar{\nu}_1 - \bar{\nu}_2} (\bar{u}, \bar{v}_0) d\bar{u} = 1
\]
\[
(3.4)
\frac{\sqrt{G}}{\sqrt{G(\bar{u}_0, \bar{v}_0)}} e^{\int_{\bar{u}_0}^{\bar{u}} \frac{(\bar{\nu}_2)_{\bar{u}}}{\bar{\nu}_2} d\bar{u}} - \int_{\bar{v}_0}^{\bar{v}} \frac{(\bar{\nu}_2)_{\bar{v}}}{\bar{\nu}_1 - \bar{\nu}_2} (\bar{u}_0, \bar{v}) d\bar{v} = 1
\]

We shall call *canonical principal parameters* any principal parameters \( (\bar{u}, \bar{v}) \) satisfying (3.4) for certain constants \( (\bar{u}_0, \bar{v}_0) \).

We can see by a straightforward check that if \( (u, v) \) are also canonical principal parameters, then

\[
\bar{u} = \lambda u + c_1 \quad \text{or} \quad \bar{u} = \lambda v + c_1
\]
\[
\bar{v} = \mu v + c_2
\]

or

\[
\bar{u} = \lambda u + c_2
\]
\[
\bar{v} = \mu v + c_1
\]
for some constants $\lambda$, $\mu$, $c_1$, $c_2$ ($\lambda \neq 0$, $\mu \neq 0$). More precisely, if $\overline{u}_0 = \overline{u}(u_0)$, $\overline{v}_0 = \overline{v}(v_0)$, then
\[
(\overline{u} - \overline{u}_0)\sqrt{E(\overline{u}_0, \overline{v}_0)} = (u - u_0)\sqrt{E(u_0, v_0)} \quad \text{and} \quad (\overline{v} - \overline{v}_0)\sqrt{G(\overline{u}_0, \overline{v}_0)} = (v - v_0)\sqrt{G(u_0, v_0)}.
\]

In the following we assume that the surface is parametrized with canonical principal parameters $(u, v)$. Then the coefficients $E$ and $G$ of the first fundamental form satisfy
\[
E = a e^{-2\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv} - 2\int_{u_0}^{u} \frac{(\nu_1)_u}{\nu_1 - \nu_2} (u, v_0) du
\]
\[
G = b e^{2\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du} + 2\int_{v_0}^{v} \frac{(\nu_2)_v}{\nu_1 - \nu_2} (u_0, v) dv,
\]
where $a = E(u_0, v_0)$, $b = G(u_0, v_0)$. In this case the Gauss equation (3.1) can be written in the following equivalent form
\[
\nu_1 \nu_2 \Psi_1 \Psi_2 = \frac{1}{b} \left( \frac{(\nu_1)_v}{\nu_1 - \nu_2} \Psi_1 \right)_v - \frac{1}{a} \left( \frac{(\nu_2)_u}{\nu_1 - \nu_2} \Psi_2 \right)_u
\]
where the functions $\Psi_1$ and $\Psi_2$ are defined by
\[
\Psi_1 = e^{-2\int_{v_0}^{v} \frac{(\nu_1)_v}{\nu_1 - \nu_2} dv} - \int_{u_0}^{u} \frac{(\nu_1)_u}{\nu_1 - \nu_2} (u, v_0) du
\]
\[
\Psi_2 = e^{2\int_{u_0}^{u} \frac{(\nu_2)_u}{\nu_1 - \nu_2} du} + \int_{v_0}^{v} \frac{(\nu_2)_v}{\nu_1 - \nu_2} (u_0, v) dv.
\]

Conversely, consider two differentiable functions $\nu_1$, $\nu_2$ that satisfy the equation (3.6) for some positive constants $a$, $b$, the functions $\Psi_1$, being defined by (3.7) (of course we suppose that the difference $\nu_1 - \nu_2$ never vanishes). With these functions $\nu_1$, $\nu_2$ we define $E$ and $G$ by (3.5) and after that $L$ and $N$ by (3.2). Then using the theorem of Bonnet we obtain:

**Theorem 1.** Let $a$ and $b$ be positive constants and two differentiable functions $\nu_1(u, v)$, $\nu_2(u, v)$ be given. Define $\Psi_1$, $\Psi_2$ by (3.7) and suppose that (3.6) is satisfied. Then locally there exists a unique (up to position in space) surface $S$, such that $\nu_1$ and $\nu_2$ are the principal curvatures of $S$ in canonical principal parameters. For this surface $E(u_0, v_0) = a$, $G(u_0, v_0) = b$.

Note that the integrability condition (3.6) (which is a form of the Gauss equation) is expressed only by the two invariants $\nu_1$ and $\nu_2$ – the principal curvature functions of the surface in canonical principal parameters.

Note also that the above theorem and the Gauss integrability equation (3.6) can be put in a different form in terms of the Gauss curvature and the normal curvature instead of the principal curvatures $\nu_1$, $\nu_2$. Indeed according to (2.2) we have (up to numeration)
\[
\nu_1 = H + \sqrt{H^2 - K}, \quad \nu_2 = H - \sqrt{H^2 - K}.
\]
In this case the condition that \( \nu_1 - \nu_2 \) never vanishes is replaced by the condition that \( H^2 - K \) never vanishes. As a result, the surface is determined up to position in space by its Gauss and mean curvature. More precisely, we obtain

**Theorem 2.** Let \( K(u, v), H(u, v) \) be differentiable functions such that the equation

\[
\frac{2K}{\sqrt{H^2 - K}} \frac{\Phi_1 \Phi_2}{a} = \frac{1}{b} \left( \frac{\Phi_1 (H + \sqrt{H^2 - K})}{\sqrt{H^2 - K}} \right)_v - \frac{1}{a} \left( \frac{\Phi_2 (H - \sqrt{H^2 - K})}{\sqrt{H^2 - K}} \right)_u
\]

where

\[
\Phi_1 = e^{-\int_{v_0}^{v} \frac{H_u}{2\sqrt{H^2 - K}} dv - \int_{u_0}^{u} \frac{H_v}{2\sqrt{H^2 - K}} (u, v_0) du}
\]

\[
\Phi_2 = e^{\int_{u_0}^{u} \frac{H_u}{2\sqrt{H^2 - K}} du + \int_{v_0}^{v} \frac{H_v}{2\sqrt{H^2 - K}} (u_0, v) dv}
\]

is satisfied for some positive constants \( a, b \). Then locally there exists a unique (up to position in space) surface, such that \( K \) and \( H \) are respectively its Gauss curvature and mean curvature in canonical principal parameters. For this surface \( (E\sqrt{(H^2 - K)})(u_0, v_0) = a, (G\sqrt{(H^2 - K)})(u_0, v_0) = b \).

Having two functions \( \nu_1, \nu_2 \) satisfying the conditions of Theorem 1 (or, what is the same, two functions \( K, H \) satisfying the conditions of Theorem 2), we determine the coefficients \( E, G \) of the first fundamental form of the induced surface \( S \) by \( (\nu_1, \nu_2) \). Now we can find the geodesic curvatures \( \gamma_1, \gamma_2 \) of the principal lines of the surface using \( (2.1) \). A geometric method to construct the surface with invariants \( \nu_1, \nu_2, \gamma_1, \gamma_2 \) is obtained in \([5]\).

### 4. Particular cases

The surface \( S : x = x(u, v), (u, v) \in D \) is called **strongly regular Weingarten surface** if

\[
(\nu_1(u, v) - \nu_2(u, v))\gamma_1(u, v)\gamma_2(u, v) \neq 0, \quad (u, v) \in D
\]

and there exist two differentiable functions \( f(t), g(t) \) defined on an interval \( I \) and a function \( \nu(u, v) \), defined on \( D \), such that

\[
(4.1) \quad f(t) - g(t) > 0, \quad f'(t)g'(t) \neq 0, \quad t \in I,
\]

\[
(4.2) \quad \nu_\alpha(u, v)\nu_\beta(u, v) \neq 0, \quad (u, v) \in D.
\]

\[
(4.3) \quad \nu_1 = f(\nu), \quad \nu_2 = g(\nu).
\]

Theorem 1 implies that given three functions \( f(t), g(t), \nu(u, v) \) with the properties \( (4.1), \ (4.2) \) and satisfying the equation

\[
(4.4) \quad A \left\{ f'\nu_{\alpha\nu} + \left( f'' - \frac{2f'^2}{f - g} \right) \nu_\alpha \right\} e^{2\int_{v_0}^{v} \frac{g'}{g - f} dt} = \begin{cases} \frac{1}{B} \left\{ g'\nu_{\alpha\nu} + \left( g'' + \frac{2g'^2}{f - g} \right) \nu_\alpha \right\} e^{2\int_{v_0}^{v} \frac{f'}{f - g} dt} & \text{if } f - g > 0, \\ f = g > 0. \end{cases}
\]

\[
\frac{2K}{\sqrt{H^2 - K}} \Phi_1 \Phi_2 = \frac{1}{b} \left( \frac{\Phi_1 (H + \sqrt{H^2 - K})}{\sqrt{H^2 - K}} \right)_v - \frac{1}{a} \left( \frac{\Phi_2 (H - \sqrt{H^2 - K})}{\sqrt{H^2 - K}} \right)_u
\]
for two positive constants $A$, $B$ and $\nu_0 = \nu(u_0, v_0)$ for $(u_0, v_0) \in D$, then there exists a unique (up to position in space) Weingarten surface $S$ with principal curvatures in canonical principal parameters given by \((4.3)\). This is one of the main results in \cite{5}. Note that in this case our canonical principal parameters coincide with the geometric principal parameters, defined in \cite{5}.

For the form of the Gauss equation \((4.4)\) for some important subclasses of Weingarten surfaces, e.g. surfaces of constant mean curvature, see \cite{5}.

It is more interesting to consider the surfaces of constant mean curvature $H$ by another point of view. Namely, according to Theorem 2, such a surface is uniquely determined by its Gauss curvature. More precisely Theorem 2 (with $a = b = 1$) implies that for a real number $H$ and a differentiable function $K$ satisfying $K < H^2$ and the differential equation

$$\Delta(\log(H^2 - K)) = \frac{4K}{\sqrt{H^2 - K}},$$

where $\Delta$ is the Laplace operator, there exists a unique (up to position) surface with Gauss curvature $K$ and constant mean curvature $H$.

In particular, for minimal surfaces ($H = 0$) this equation reduces to

$$\Delta\left( \log \sqrt{-K} \right) + 2\sqrt{-K} = 0,$$

or, if $\nu = \sqrt{-K}$ is the positive principal curvature,

$$(4.5) \quad \Delta(\log \nu) + 2\nu = 0.$$

According to \((3.5)\), in this case $E = G$ and since $F = 0$, the canonical principal parameters $(u, v)$ are isothermal. When we consider minimal surfaces, it is very common to use complex coordinates; in real coordinates this gives isothermal parameters. A method to obtain canonical principal parameters for a minimal surface from arbitrary isothermal ones is found in \cite{6}. In \cite{3} the equation \((4.5)\) is named natural partial differential equation of minimal surfaces.

The flat surfaces, i.e. the surfaces with vanishing Gauss curvature $K$, are well studied – they are general cylinders, general cones and tangent developable surfaces. When the surface has no umbilical points (for example for a tangent developable surface the torsion of the directrix must not vanish) the mean curvature $H$ can not vanish. It follows from Theorem 2 that these surfaces are characterized by

$$\left(\frac{1}{H}\right)_{vv} = 0 \quad \text{or} \quad H = \frac{1}{f(u)v + g(u)}$$

in canonical principal parameters for some functions $f(u), g(u)$.

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