An Acceleration of Fixed Point Iterations for M/G/1-type Markov Chains by Means of Relaxation Techniques

Gemignani, Luca
Dipartimento di Informatica
Università di Pisa
luca.gemignani@unipi.it

Meini, Beatrice
Dipartimento di Matematica
Università di Pisa
beatrice.meini@unipi.it

Abstract
We present some accelerated variants of fixed point iterations for computing the minimal non-negative solution of the unilateral matrix equation associated with an M/G/1-type Markov chain. These schemes derive from certain staircase regular splittings of the block Hessenberg M-matrix associated with the Markov chain. By exploiting the staircase profile we introduce a two-step fixed point iteration. The iteration can be further accelerated by computing a weighted average between the approximations obtained in two consecutive steps. The convergence of the basic two-step fixed point iteration and of its relaxed modification is proved. Our theoretical analysis along with several numerical experiments show that the proposed variants generally outperform the classical iterations.

Keywords: M-matrix, Staircase Splitting, Nonlinear Matrix Equation, Hessenberg Matrix, Markov Chain.

1 Introduction

The transition probability matrix of an M/G/1-type Markov chain is a block Hessenberg matrix \( P \) of the form

\[
P = \begin{bmatrix}
B_0 & B_1 & B_2 & \cdots \\
A_{-1} & A_0 & A_1 & \cdots \\
 & A_{-1} & A_0 & \cdots \\
 & & & \ddots & \ddots
\end{bmatrix},
\]

The authors are partially supported by INDAM/GNCS and by the project PRA 2020,61 of the University of Pisa.
with $A_i, B_i \in \mathbb{R}^{n \times n} > 0$ and $\sum_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} A_i$ stochastic matrices.

In the sequel, given a real matrix $A = (a_{ij})_{ij} \in \mathbb{R}^{m \times n}$, we write $A \geq 0$ ($A > 0$) if $a_{ij} \geq 0$ ($a_{ij} > 0$) for any $i, j$. A stochastic matrix is matrix $A \geq 0$ such that $A e = e$, where $e$ is the column vector having all the entries equal to 1.

The computation of the steady state vector of $P$ such that
\[
\pi^T P = \pi^T, \quad \pi^T e = 1, \quad \pi \geq 0,
\]
is related with the solution of the unilateral matrix equation
\[
X = A_{-1} + A_0 X + A_1 X^2 + A_2 X^3 + \ldots.
\]

Under some mild assumptions this equation has a componentwise minimal non-negative solution $G$ which determines, by means of Ramaswami’s formula \([12]\), the vector $\pi$. Formally, \((3)\) can be rewritten as the following block Toeplitz block Hessenberg linear system
\[
\begin{bmatrix}
A_0 - I_n & A_1 & A_2 & \ldots \\
A_{-1} & A_0 - I_n & A_1 & \ldots \\
& A_{-1} & A_0 - I_n & \ddots \\
& & & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
\end{bmatrix}
\begin{bmatrix}
X \\
X^2 \\
X^3 \\
\vdots \\
\vdots \\
\end{bmatrix} =
\begin{bmatrix}
-A_{-1} \\
0 \\
0 \\
\vdots \\
\vdots \\
\end{bmatrix}
\]
which yields the fixed point problem
\[
\hat{X} = \hat{P}\hat{X} + EA_{-1} \iff H\hat{X} = EA_{-1}, \quad H = I - \hat{P},
\]
where $\hat{X}^T = [X^T, X^2, \ldots]^T$. $\hat{P}$ is the matrix obtained from $P$ by removing its first block row and column and $E = [I_n, 0, \ldots]^T$.

Fixed point iterations for solving \((3)\) originate from classical iterative linear solvers applied to \((4)\). The \textit{natural partitioning} where $M = I$ and $N = \hat{P}$ leads to the \textit{natural fixed point iteration} $X_{k+1} = \sum_{i=1}^{\infty} A_i X_{k+1}$. The \textit{Jacobi partitioning} where $M = I \otimes (I_n - A_0)$ and $N = M - H$ leads to the \textit{traditional fixed point iteration}
\[
(I_n - A_0)X_{k+1} = A_{-1} + \sum_{i=1}^{\infty} A_i X_{k+1}^i, \quad k \geq 0.
\]
The \textit{anti-Gauss-Seidel partitioning} where $M$ is the block upper triangular portion of $H$ determines the \textit{U-based fixed point iteration}
\[
(I_n - \sum_{i=0}^{\infty} A_i X_i^0)X_{k+1} = A_{-1}, \quad k \geq 0.
\]
The convergence properties of these three fixed point iterations are analyzed in \([2]\). It turns out that fixed-point iterations exhibit arbitrarily slow convergence for problems which are close to singularity. Specifically, for positive recurrent Markov chains having a “drift” \([11]\) close to zero the convergence slows down and the number of iterations becomes arbitrarily large \([2]\).
In this paper we consider a different non-standard partitioning of the matrix $H$. This splitting—referred to as the \textit{staircase partitioning} of $H$—is defined by:

$$M = \begin{bmatrix}
    A_0 - I_n & A_0 - I_n & A_1 \\
    A_{-1} & A_0 - I_n & A_1 \\
    A_{-1} & A_0 - I_n & A_1 \\
    \times & \times & \times \\
    \times & \times & \times
\end{bmatrix}, \quad N = M - H. \quad (7)$$

The splitting has attracted interest for applications in parallel computing environments \cite{9,7}. Recently in \cite{3} a comparative analysis has been performed for the asymptotic convergence rates of some regular splittings of a non-singular block upper Hessenberg $M$-matrix of finite size. The conclusion is that the staircase splitting is faster than the anti-Gauss-Seidel splitting that in turn is faster than the Jacobi splitting. The second result is classical, while the first one is somehow surprising since the matrix $M$ in the staircase splitting is much more sparse than the corresponding matrix in the anti-Gauss-Seidel partitioning and the splittings are not comparable. This unexpected result provides the motivation of the present work where we are going to apply the staircase partitioning for solving $\mathbf{4}$.

In principle the alternating structure of the matrix $M$ in (7) suggests several different iterative schemes. From one hand, the computation of the odd components of $\hat{X}_{k+1}$ yields the traditional fixed point iteration. From the other hand, the computation of the even components of $\hat{X}_{k+1}$ leads to the implicit scheme $-A_{-1} + (I_n - A_0)X_{k+1} - A_1X_{k+1}^2 = \sum_{i=2}^{\infty} A_iX_{k+1}^{i+1}$ recently introduced in \cite{3}. Differently, by looking at the structure of the matrix $M$ on the whole we can devise a composite two-stage iteration of the form:

\begin{align*}
    & (I_n - A_0)Y_k = A_{-1} + \sum_{i=1}^{\infty} A_iX_{k}^{i+1}; \\
    & -A_{-1} + (I_n - A_0)X_{k+1} - A_1Y_k^2 = \sum_{i=2}^{\infty} A_iX_{k}^{i+1}, \quad k \geq 0, \quad (8)
\end{align*}

or, equivalently,

\begin{align*}
    & (I_n - A_0)Y_k = A_{-1} + \sum_{i=1}^{\infty} A_iX_{k}^{i+1}; \\
    & X_{k+1} = Y_k + (I_n - A_0)^{-1} A_1(Y_k^2 - X_k^2), \quad k \geq 0, \quad (9)
\end{align*}

starting from an initial approximation $X_0$. At each step $k$, this scheme consists of a traditional fixed point iteration that computes $Y_k$ from $X_k$, followed by a cheap correction step for computing the new approximation $X_{k+1}$.

The contribution of this paper is aimed at highlighting the properties of (9). We show that, if $X_0 = 0$, the sequence $\{X_k\}_k$ defined in (9) converges to $G$ faster than the traditional fixed point iteration. Moreover, at each iteration the scheme (9) determines two approximations which can be combined by using the relaxation technique, that is, the approximation computed at the $k$-th step takes the form of a weighted average between $Y_k$ and $X_k$:

$$X_{k+1} = \omega_k Y_k + (I_n - A_0)^{-1} A_1(Y_k^2 - X_k^2), \quad k \geq 0,$$

In matrix terms, the resulting relaxed variant of (9) can be written as

\begin{align*}
    & (I_n - A_0)Y_k = A_{-1} + \sum_{i=1}^{\infty} A_iX_{k}^{i+1}; \\
    & X_{k+1} = Y_k + \omega_k (I_n - A_0)^{-1} A_1(Y_k^2 - X_k^2), \quad k \geq 0. \quad (10)
\end{align*}
For $\omega_k = 0$, $k \geq 1$, the relaxed scheme reduces to the traditional fixed point iteration \([5]\). For $\omega_k = 1$, $k \geq 1$, the relaxed scheme coincides with \([9]\). Values of $\omega_k$ greater than 1 can speed up the convergence of the iterative scheme. Conditions under which over-relaxed variants of \([9]\) are still guaranteed to converge to the minimal nonnegative solution are provided. A formal analysis of the asymptotic convergence rate of the relaxed variants is performed. Moreover, an adaptive strategy for choosing the value of $\omega_k$ is described.

In the case where the starting matrix $X_0$ of \([10]\) is any column stochastic matrix and $G$ is also column stochastic, we prove that for $0 \leq \omega_k = \omega \leq 1$ the sequence $\{X_k\}_{k \in \mathbb{N}}$ still converges to $G$. Moreover, by comparison of the mean asymptotic rates of convergence, we conclude that \([9]\) is asymptotically faster than the traditional fixed point iteration.

The results of extensive numerical experiments confirm the effectiveness of the proposed variants which generally outperform the $U$-based fixed point iteration for nearly singular problems. In particular, the over-relaxed scheme \([10]\) with $X_0 = 0$ combined with the adaptive strategy for parameter estimation is capable to significantly accelerate the convergence without increasing the computational cost.

The paper is organized as follows. In Section 2 we set up the theoretical framework, briefly recalling some preliminary properties and definitions, and we prove convergence results for our iterative schemes \([9]\), \([10]\). In Section 3 a comparative analysis of the asymptotic convergence rates of relaxed variants is carried out. Adaptive strategies for the choice of the relaxation parameter are discussed in Section 4, together with their cost analysis under some simplified assumptions. Finally, the results of extensive numerical experiments are presented in Section 5 whereas conclusions and future work are the subjects of Section 6.

### 2 Theoretical Setup and Convergence Results

The convergence analysis of the sequences generated by \([9]\) and \([10]\) can be carried out by relying upon some structural properties of the sequence $\{A_i\}_{i \in \mathbb{N}}$, $A_i \in \mathbb{R}^{n \times n}$ which are here summarized. Let us introduce the following assumption:

- **A1.** $A_i \geq 0$, $i \geq -1$ and $A = \sum_{i=-1}^{\infty} A_i$ is irreducible and row stochastic, that is, $Ae = e$, $e = [1, \ldots, 1]^T$;

According to the results of [2, Chapter 4], condition [A1] implies that \([3]\) has a unique componentwise minimal nonnegative solution $G$; moreover, $I_n - A_0$ is an invertible $M$-matrix and, hence, $(I - A_0)^{-1} \succeq 0$.

Under assumption [A1], in view of the Perron Frobenius Theorem, there exists a unique vector $v$ such that $v^TA = v^T$ and $v^Te = 1$, $v > 0$. Let us denote $w = \sum_{i=-1}^{\infty} t_i A_i e \in \mathbb{R}^n$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm \infty\}$. Another relevant condition is the following:

- **A2.** The drift $\eta = v^Tw$ satisfies $\eta < 0$. In particular, this implies that $w \in \mathbb{R}^n$.

Under assumptions [A1] and [A2], the componentwise minimal nonnegative solution $G$ of equation \([3]\) is stochastic, $Ge = e$ \([2]\). Moreover $G$ is the only stochastic solution. Throughout this paper we assume that assumption [A1] always satisfied. Condition [A2] is explicitly required when necessary.

The first result is concerned with the convergence of \([9]\) with $X_0 = 0$.

**Proposition 1.** Set $X_0 = 0$. Then, the sequence $\{X_k\}_{k \in \mathbb{N}}$ generated by \([9]\) converges monotonically to $G$. 

Proof. We show by induction on \( k \) that \( 0 \leq X_k \leq Y_k \leq X_{k+1} \leq G \) for any \( k \geq 0 \). For \( k = 0 \) we verify easily that

\[
X_1 \geq Y_0 = (I_n - A_0)^{-1} A_{-1} \geq 0 = X_0, \quad (I_n - A_0)X_1 \leq A_{-1} + A_1 G^2 \leq (I_n - A_0)G,
\]

which gives \( G \geq X_1 \). Suppose now that \( G \geq X_k \geq Y_{k-1} \geq X_{k-1}, \ k \geq 1 \). We find that

\[
(I_n - A_0)X_k = A_{-1} + \sum_{i=1}^{\infty} A_i X_{k+1}^i \geq A_{-1} + A_1 Y_{k-1}^2 + \sum_{i=2}^{\infty} A_i G^{i-1} = (I_n - A_0)X_k,
\]

and

\[
(I_n - A_0)Y_k = A_{-1} + \sum_{i=1}^{\infty} A_i X_{k+1}^i \leq A_{-1} + \sum_{i=1}^{\infty} A_i G^{i-1} = (I_n - A_0)G.
\]

By multiplying both sides by the inverse of \( I - A_0 \) we obtain that \( G \geq Y_k \geq X_k \). This also implies that \( Y^{2}_{k} - X^{2}_{k} \geq 0 \) and therefore \( X_{k+1} \geq Y_{k} \). Since

\[
(I_n - A_0)X_{k+1} = A_{-1} + A_1 Y_{k}^2 + \sum_{i=2}^{\infty} A_i X_{k}^{i+1} \leq (I_n - A_0)G,
\]

we prove similarly that \( G \geq X_{k+1} \). It follows that \( \{X_k\}_{k \in \mathbb{N}} \) is convergent, the limit solves (3) by continuity and, hence, the limit coincides with the matrix \( G \), since \( G \) is the minimal nonnegative solution. \( \square \)

Remark 2. The proof of Proposition 1 can immediately be generalized to prove the convergence of the sequence \( \{X_k\}_k \) defined (10), with \( X_0 = 0 \), for any \( \omega_k = \omega \), \( k \geq 1 \), such that \( 0 \leq \omega \leq 1 \). Moreover, let \( \{X_k\}_{k \in \mathbb{N}} \) and \( \{\hat{X}_k\}_{k \in \mathbb{N}} \) be the sequences generated by (10) for \( \omega_k = \omega \) and \( \omega_k = \bar{\omega} \) with \( 0 \leq \omega \leq \bar{\omega} \leq 1 \), respectively. It is shown easily that \( G \geq \hat{X}_k \geq X_k \) for any \( k \) and, hence, that the iterative scheme (9) converges faster than (10) if \( 0 \leq \omega_k = \omega < 1 \).

A similar result also holds for the case where \( X_0 \) is a stochastic matrix, assuming that \( [A2] \) holds, so that \( G \) is stochastic.

Proposition 3. Let \( X_0 \) be a stochastic matrix. Suppose condition \([A2]\) is fulfilled. Then, the sequence \( \{X_k\}_{k \in \mathbb{N}} \) generated by (9) converges to \( G \).

Proof. From (8) we obtain that

\[
\begin{cases}
(I_n - A_0)Y_k = A_{-1} + \sum_{i=1}^{\infty} A_i X_{k+1}^i; \\
(I_n - A_0)X_{k+1} = A_{-1} + A_1 Y_k^2 + \sum_{i=2}^{\infty} A_i X_k^{i+1}
\end{cases}
\]

which gives that \( X_k \geq 0 \) and \( Y_k \geq 0 \), for any \( k \in \mathbb{N} \), since \( X_0 \geq 0 \). By assuming that \( X_0 e = e \), we may easily show by induction that \( Y_k e = X_k e = e \) for any \( k \geq 0 \). Therefore, all the matrices \( X_k \) and \( Y_k \), \( k \in \mathbb{N} \), are stochastic. Let \( \{X_k\}_{k \in \mathbb{N}} \) be the sequence generated by (9) with \( X_0 = 0 \). We can easily show by induction that \( X_k \geq \hat{X}_k \) for any \( k \in \mathbb{N} \). Since \( \lim_{k \to \infty} \hat{X}_k = G \), then any convergent subsequence of \( \{X_k\}_{k \in \mathbb{N}} \) converges to a stochastic matrix \( S \) such that \( S \geq G \). Since \( G \) is also stochastic, it follows that \( S = G \) and therefore, by compactness, we conclude that the sequence \( \{X_k\}_{k \in \mathbb{N}} \) is also convergent to \( G \). \( \square \)

The convergence analysis of the modified scheme (10), for \( \omega_k > 1 \) is much more involved since the choice of a relaxation parameter \( \omega_k > 1 \) can destroy the monotonicity and the nonnegativity of the approximation sequence, which is at the core of the proofs of Proposition 1 and Proposition 3. In order to maintain the convergence properties of the modified scheme we introduce the following definition.
**Definition 4.** The sequence \( \{\omega_k\}_{k \geq 1} \) is eligible for the scheme (10) if \( \omega_k \geq 0, k \geq 1 \), and the following two conditions are satisfied:

\[
\omega_{k+1} A_1(Y_k^2 - X_k^2) \leq A_1(X_{k+1}^2 - X_k^2) + \sum_{i=2}^{\infty} A_i(Y_i^{i+1} - X_i^{i+1}), \quad k \geq 0.
\] (11)

and

\[
X_{k+1}e = Y_k e + \omega_{k+1}(I_n - A_0)^{-1}A_1(Y_k^2 - X_k^2)e \leq e, \quad k \geq 0.
\] (12)

It is worth noting that condition (11) is implicit since the construction of \( X_{k+1} \) also depends on the value of \( \omega_{k+1} \). By replacing \( X_{k+1} \) in (11) with the expression in the right-hand side of (10) we obtain a quadratic inequality with matrix coefficients in the variable \( \omega_{k+1} \). Obviously \( \omega_k = \omega, 0 \leq \omega \leq 1, k \geq 1 \), are eligible sequences.

The following generalization of Proposition 4 holds.

**Proposition 5.** Set \( X_0 = 0 \) and let condition [A2] be satisfied. If \( \{\omega_k\}_{k \geq 1} \) is eligible then the sequence \( \{X_k\}_{k \in \mathbb{N}} \) generated by (10) converges monotonically to \( G \).

**Proof.** We show by induction that \( 0 \leq X_k \leq Y_k \leq X_{k+1} \leq G \). For \( k = 0 \) we have

\[
(I_n - A_0)X_1 \geq (I_n - A_0)Y_0 = A_{-1} \geq 0 = X_0
\]

which gives immediately \( X_1 \geq Y_0 \geq 0 \). Moreover, \( X_1 e \leq e \). Suppose now that \( X_k \geq Y_{k-1} \geq X_{k-1} \geq 0, k \geq 1 \). We find that

\[
(I_n - A_0)X_k = A_{-1} + A_1(X_{k-1}^2 + \omega_k(Y_{k-1}^2 - X_{k-1}^2) + \sum_{i=2}^{\infty} A_i X_{i-1}^{i+1}) \leq A_{-1} + A_1 X_{k-1}^2 + A_1(X_k^2 - X_{k-1}^2) + \sum_{i=2}^{\infty} A_i(Y_{i-1}^2 - X_{i-1}^2) + \sum_{i=2}^{\infty} A_i X_{i-1}^{i+1} \leq A_{-1} + A_1 X_k^2 + \sum_{i=2}^{\infty} A_i Y_{i-1}^{i-1} \leq A_{-1} + \sum_{i=1}^{\infty} A_i X_k^{i+1} = (I_n - A_0)Y_k
\]

from which it follows \( Y_k \geq X_k \geq 0 \) for all \( k \geq 0 \) and therefore the sequence of approximations is upper bounded and it has a finite limit \( H \). By continuity we find that \( H \) is the unique stochastic solution, then \( H = G \).

**Remark 6.** As previously mentioned condition [11] is implicit since \( X_{k+1} \) also depends on \( \omega_{k+1} \). An explicit condition can be derived by noting that

\[
A_1(X_{k+1}^2 - X_k^2) \geq A_1((Y_k^2 - X_k^2) + \omega_{k+1}(Y_k \Gamma_k + \Gamma_k Y_k)),
\]

with \( \Gamma_k = (I_n - A_0)^{-1}A_1(Y_k^2 - X_k^2) \). There follows that (11) is fulfilled whenever

\[
\frac{\omega_{k+1} - 1}{\omega_k} A_1((Y_k^2 - X_k^2) \leq (Y_k \Gamma_k + \Gamma_k Y_k) + \omega_{k+1} \sum_{i=2}^{\infty} A_i(Y_i^{i+1} - X_i^{i+1})).
\]

Let \( \omega \in [1, \hat{\omega}] \) be such that

\[
\frac{\omega - 1}{\omega} A_1((Y_k^2 - X_k^2) \leq (Y_k \Gamma_k + \Gamma_k Y_k) + \hat{\omega}^{-1} \sum_{i=2}^{\infty} A_i(Y_i^{i+1} - X_i^{i+1}).
\] (13)

Then we can impose that

\[
\omega_{k+1} = \max\{\omega: \omega \in [1, \hat{\omega}] \land (13) \text{ holds}\}.
\] (14)

From a computational viewpoint the strategy based on (13) and (14) for the choice of the value of \( \omega_{k+1} \) can be too much expensive and some weakened criterion would be considered (compare with Section 2 below).
3 Asymptotic Convergence Analysis of Relaxed Iterative Schemes

Relaxation techniques are usually aimed at accelerating the convergence speed of frustratingly slow iterative solvers. Such inefficient behavior is typically exhibited when the solver is applied to a nearly singular problem. Incorporating some relaxation parameter into the iterative scheme (3) can greatly improve its convergence rate. Preliminary insights on the effectiveness of relaxation techniques applied for the solution of the fixed point problem (4) come from the classical analysis for stationary iterative solvers and are developed in Section 3.1. A precise convergence analysis is presented in Section 3.2.

3.1 Finite dimensional convergence analysis

Suppose that $H$ in (4) is block tridiagonal of finite size $m = n\ell$, $\ell$ even. We are interested in comparing the iterative algorithm based on the splitting (7) with other classical iterative solvers for the solution of a linear system with coefficient matrix $H$. As usual, we can write $H = D - P_1 - P_2$, where $D$ is block diagonal, while $P_1$ and $P_2$ are staircase matrices with zero block diagonal. The eigenvalues $\lambda_i$ of the Jacobi iteration matrix satisfy

\[ 0 = \det(\lambda_i I - D^{-1}P_1 - D^{-1}P_2). \]

Let us consider a relaxed scheme where the matrix $M$ is obtained from (7) by multiplying the off-diagonal blocks by $\omega$. The eigenvalues $\mu_i$ of the iteration matrix associated with the relaxed staircase regular splitting are such that

\[ 0 = \det(\mu_i I - (\mu_i \omega + (1 - \omega))D^{-1}P_1 - D^{-1}P_2). \]

and, equivalently,

\[ 0 = \det(\mu_i I - (\mu_i \omega + (1 - \omega))D^{-1}P_1 - D^{-1}P_2). \]

By using a similarity transformation induced by the matrix $S = I_{\ell/2} \otimes \text{diag}[I_n, \alpha I_n]$ we find that

\[ \det(\mu_i I - (\mu_i \omega + (1 - \omega))D^{-1}P_1 - D^{-1}P_2) = \det(\mu_i I - \alpha(\mu_i \omega + (1 - \omega))D^{-1}P_1 - \frac{1}{\alpha}D^{-1}P_2). \]

There follows that

\[ \mu_i \alpha = \lambda_i \]

whenever $\alpha$ fulfills

\[ \alpha(\mu_i \omega + (1 - \omega)) = \frac{1}{\alpha}. \]

Therefore, the eigenvalues of the Jacobi and relaxed staircase regular splittings are related by

\[ \mu_i^2 - \lambda_i^2 \mu \omega + \lambda_i^2 (\omega - 1) = 0. \]

For $\omega = 0$ the staircase splitting reduces to the Jacobi partitioning. For $\omega = 1$ we find that $\mu_i = \lambda_i^2$ which yields the classical relation between the spectral radii of Jacobi and Gauss-Seidel methods. Observe that it is well known that the asymptotic convergence
rates of Gauss-Seidel and the staircase iteration coincide when applied to a block tridiagonal matrix \[1\]. For \(\omega > 1\) the spectral radius of the relaxed staircase scheme can be significantly less than the spectral radius of the same scheme for \(\omega = 1\). In Figure 1 we illustrate the plot of the function
\[
\rho_S(\omega) = \frac{\lambda^2 \omega + \sqrt{\lambda^4 \omega^2 - 4\lambda^2(\omega - 1)}}{2}, \quad 1 \leq \omega \leq 2,
\]
for a fixed \(\lambda = 0.999\). For the best choice of \(\omega = \omega^* = 2 + \sqrt{1 - \lambda^2}/\lambda^2\) we find
\[
\rho_S(\omega^*) = 1 - \sqrt{1 - \lambda^2} = \frac{\lambda^2}{1 + \sqrt{1 - \lambda^2}}.
\]

### 3.2 Asymptotic Convergence Rate

A formal analysis of the asymptotic convergence rate of the relaxed variants \[10\] can be carried out by using the tools described in \[8\]. Hereafter it is assumed that [A2] holds.

#### 3.2.1 The case \(X_0 = 0\)

Let us introduce the error matrix \(E_k = G - X_k\), where \(\{X_k\}_{k \in \mathbb{N}}\) is generated by \[10\] with \(X_0 = 0\). We also define \(E_{k+1/2} = G - Y_k\), for \(k = 0, 1, 2, \ldots\). Recall that, if \(X_0 = 0\), the sequence \(\{X_k\}_k\) converges monotonically to \(G\) and \(E_k \geq 0, E_{k+1/2} \geq 0\). Since \(E_k \geq 0\) and \(\|E_k\|_\infty = \|E_k e\|_\infty\), we analyze the convergence of the vector \(e_k = E_k e, k \geq 0\).

We have
\[
(I_n - A_0)E_{k+1/2} = \sum_{i=1}^{\infty} A_i (G^{i+1} - X^{i+1}_k) = \sum_{i=1}^{\infty} A_i \sum_{j=0}^i G^j E_k^{i-j}.
\]

Similarly, for the second equation of \[10\], we find that
\[
(I_n - A_0)E_{k+1} = (I_n - A_0)E_{k+1/2} - \omega_{k+1} A_1 ((G^2 - X^2_k) - (G^2 - Y^2_k)),
\]
which gives
\[
(I_n - A_0)E_{k+1} = (I_n - A_0)E_{k+1/2} - \omega_{k+1} A_1 (E_k G + X_k E_k) + \omega_{k+1} A_1 (G E_{k+1/2} + E_{k+1/2} Y_k).
\]

Fig. 1: Plot of \(\rho_S(\omega)\) for \(\omega \in [1, 2]\) and \(\lambda = 0.999\)
Denote by $R_k$ the matrix on the right hand side of (15), i.e.,

$$R_k = \sum_{i=1}^{\infty} A_i \sum_{j=0}^{i} G^j E_k X_{k}^{i-j}.$$ 

Since $Ge = e$, equation (16), together with the monotonicity, yields

$$(I_n - A_0)E_{k+1} e \leq R_k e - \omega_{k+1} A_1 (I_n + X_k) E_k e + \omega_{k+1} A_1 (I_n + G)(I_n - A_0)^{-1} R_k e.$$ \hspace{1cm} (17)

Observe that $R_k e \leq WE_k e$, where

$$W = \sum_{i=1}^{\infty} A_i \sum_{j=0}^{i} G^j,$$

hence

$$\epsilon_{k+1} \leq P(\omega_{k+1}) \epsilon_k, \quad k \geq 0, \hspace{1cm} (18)$$

with

$$P(\omega_{k+1}) = (I_n - A_0)^{-1} W - \omega_{k+1} (I_n - A_0)^{-1} A_1 (I_n + G)(I_n - (I_n - A_0)^{-1} W), \quad k \geq 0.$$ 

The matrix $P(\omega_{k+1})$ can be written as $P(\omega_{k+1}) = M^{-1} N(\omega_{k+1})$ where

$$M = I_n - A_0, \quad M - N(\omega_{k+1}) = A(\omega_{k+1}),$$

and

$$A(\omega_{k+1}) = (I_n + \omega_{k+1} A_1 (I_n + G)(I_n - A_0)^{-1})(I_n - A_0 - W).$$

When $\omega_{k+1} = 0$, for $k \geq 0$, we find that $A(0) = I_n - A_0 - W = I_n - V$, where $V = \sum_{i=0}^{\infty} A_i \sum_{j=0}^{i} G^j$. According to Theorem 4.14 in [2], $I - V$ is a nonsingular M-matrix and therefore, since $N(0) \geq 0$ and $M^{-1} \geq 0$, $A(0) = M - N(0)$ is a regular splitting. Hence, the spectral radius $\rho_0$ of $P(0)$ is less than 1. Now suppose that $\omega_{k+1} = \omega$, $k \geq 0$, $0 \leq \omega \leq \hat{\omega}$, is an eligible sequence according to Definition 2 with $\hat{\omega} \geq 1$ such that

$$\hat{\omega} A_1 (I_n + G)(I_n - A_0)^{-1}(I_n - V) \leq W.$$ 

This condition ensures that $A(\omega)^{-1} \geq 0$ and $N(\omega) \geq 0$, so that $A(\omega) = M - N(\omega)$ is a regular splitting and, hence, $\rho(P(\omega)) < 1$. Denote by $\rho_\omega$ the spectral radius of $P(\omega)$. From (15) the asymptotic rate of convergence of the sequence $\{X_k\}_{k \in \mathbb{N}}$ is bounded from above by the spectral radius of the matrix $P(\omega)$. The following result gives a comparison between $\rho_\omega$ and $\rho_0$.

**Theorem 7.** Let $\omega$ be such that $0 \leq \omega \leq \hat{\omega}$. Assume that the Perron eigenvector $v$ of $P(0)$ is positive. Then we have

$$\rho_0 - \omega(1 - \rho_0) \sigma_{\text{max}} \leq \rho_\omega \leq \rho_0 - \omega(1 - \rho_0) \sigma_{\text{min}} \hspace{1cm} (19)$$

where $\sigma_{\text{min}} = \min_i \frac{u_i}{v_i}$ and $\sigma_{\text{max}} = \max_i \frac{u_i}{v_i}$, with $u = (I_n - A_0)^{-1} A_1 (I + G)v$. Moreover, $0 \leq \sigma_{\text{min}}, \sigma_{\text{max}} \leq \rho_0$.

**Proof.** In view of the classical Collatz-Wielandt formula (see [10], Chapter 8), if $P(\omega)v = w$, where $v > 0$ and $w \geq 0$, then

$$\min_i \frac{w_i}{v_i} \leq \rho_\omega \leq \max_i \frac{w_i}{v_i}.$$
Observe that

\[ w = P(\omega)v = P(0)v - \omega(I_n - A_0)^{-1}A_1(I_n + G)(I - P(0))v = \rho_0v - \omega(1 - \rho_0)(I_n - A_0)^{-1}A_1(I_n + G)v = \rho_0v - \omega(1 - \rho_0)u, \]

which leads to (19), since \( u \geq 0 \). Moreover, since \( A_1(I_n + G) \leq W \), then

\[ u = (I_n - A_0)^{-1}A_1(I_n + G)v \leq (I_n - A_0)^{-1}Wv = \rho_0v, \]

hence \( u_i/v_i \leq \rho_0 \) for any \( i \).

Remark 8. For the sake of illustration we consider a quadratic equation associated with a block tridiagonal Markov chain taken from [6]. We set \( A_{-1} = W + \delta I \), \( A_0 = A_1 = W \) where \( 0 < \delta < 1 \) and \( W \in \mathbb{R}^{n \times n} \) has zero diagonal entries and all off-diagonal entries equal to a given value \( \alpha \) determined so that \( A_{-1} + A_0 + A_1 \) is stochastic. We find that for \( \omega_k = \omega \in [0, 6] N(\omega) \geq 0 \). In Figure 2 we plot the spectral radius of \( P = P(\omega) \). The linear plot is in accordance with Theorem 7.

### 3.2.2 The Case \( X_0 \) stochastic

In this section we analyze the convergence of the iterative method (10) starting with a stochastic matrix \( X_0 \), that is, \( X_0 \geq 0 \) and \( X_0e = e \). Suppose that the value of \( \omega_k \) is determined so that \( X_k \geq 0 \) for any \( k \geq 0 \). This happens for \( 0 \leq \omega_k \leq 1, k \geq 1 \). Observe that the property \( X_k e = e, k \geq 0 \) is automatically satisfied. Hence, all the approximations generated by the iterative scheme are stochastic matrices. Proposition 3 can be extended in order to prove that the sequence \( \{X_k\}_{k \in \mathbb{N}} \) is convergent to \( G \).

The analysis of the speed of convergence follows from relation (16). Let us denote as \( \text{vec}(A) \in \mathbb{R}^{n^2} \) the vector obtained by stacking the columns of the matrix \( A \in \mathbb{R}^{n \times n} \) on top of one another. Recall that \( \text{vec}(ABC) = (C^T \otimes A) \text{vec}(B) \) for any \( A, B, C \in \mathbb{R}^{n \times n} \).
By using this property we can rewrite (16) as follows. We have:

\[
[I_n \otimes (I_n - A_0)] \text{vec}(E_{k+1}) = \omega_{k+1}[I_n \otimes A_1 G(I_n - A_0)^{-1} A_1 G] \text{vec}(E_k) + \omega_{k+1}[X_k^T \otimes A_1 G(I_n - A_0)^{-1} A_1 G] \text{vec}(E_k) + \omega_{k+1}Y_k^T \otimes A_1 (I_n - A_0)^{-1} A_1 G] \text{vec}(E_k) + \omega_{k+1}Y_k \otimes A_1 (I_n - A_0)^{-1} A_1 G] \text{vec}(E_k) + (1 - \omega_{k+1})[I_n \otimes A_1 G] \text{vec}(E_k) + \sum_{i=0}^{\infty} \sum_{j=0}^{N}(X_k^{T,i-j} \otimes A_1 G(I_n - A_0)^{-1} A_1 G'] \text{vec}(E_k) + \omega_{k+1}\sum_{i=0}^{\infty} \sum_{j=0}^{N}(Y_k^{T,i-j} \otimes A_1 (I_n - A_0)^{-1} A_1 G') \text{vec}(E_k),
\]

(20)

for \( k \geq 0 \). The convergence of \{\text{vec}(E_k)\} depends on the choice of \( \omega_{k+1}, k \geq 0 \). In the sequel we compare the cases \( \omega_k = 0 \), which corresponds with the traditional fixed point iteration (5), and \( \omega_k = 1 \) which reduces to the staircase fixed point iteration (9).

For \( \omega_{k+1} = 0, k \geq 0 \), we find that

\[
\text{vec}(E_{k+1}) = \sum_{i=0}^{\infty} \sum_{j=0}^{N}(X_k^{T,i-j} \otimes (I_n - A_0)^{-1} A_1 G') \text{vec}(E_k), \quad k \geq 0.
\]

Let \( U_k^H X_k^T U_k = T_k \) be the Schur form of \( X_k^T \) and set \( W_k = (U_k^H \otimes I_n) \). Then

\[
W_k \sum_{i=1}^{\infty} \sum_{j=0}^{N}(X_k^{T,i-j} \otimes (I_n - A_0)^{-1} A_1 G')W_k^{-1} = \sum_{i=1}^{\infty} \sum_{j=0}^{N}(T_k^{T,i-j} \otimes (I_n - A_0)^{-1} A_1 G')
\]

which means that

\[
H_k = \sum_{i=1}^{\infty} \sum_{j=0}^{N}(X_k^{T,i-j} \otimes (I_n - A_0)^{-1} A_1 G')
\]

is similar to the matrix

\[
\sum_{i=1}^{\infty} \sum_{j=0}^{N}(T_k^{T,i-j} \otimes (I_n - A_0)^{-1} A_1 G').
\]

There follows that the eigenvalues of \( H_k \) belong to the set

\[
\cup_{\lambda} \{ \mu \mid \mu \text{ is eigenvalue of } \sum_{i=1}^{\infty} \sum_{j=0}^{N} (\lambda^{T,i-j} (I_n - A_0)^{-1} A_1 G') \}
\]

with \( \lambda \) eigenvalue of \( X_k^T \). Observe that \( \| X_k^T \|_1 = 1 \) and therefore \( |\lambda| \leq 1 \). Since

\[
\sum_{i=1}^{\infty} \sum_{j=0}^{N} \lambda^{T,i-j} (I_n - A_0)^{-1} A_1 G' \leq \sum_{i=1}^{\infty} \sum_{j=0}^{N} (I_n - A_0)^{-1} A_1 G' = P(0)
\]

we conclude that \( \rho(H_k) \leq \rho(P(0)) \) for any \( k \geq 0 \).

For \( \omega_{k+1} = \omega > 0, k \geq 0 \), we analyze the mean asymptotic rate of convergence of the sequence (20). Observe that this relation can be expressed as

\[
\text{vec}(E_{k+1}) = H_k \text{vec}(E_k), \quad k \geq 0,
\]

where

\[
H = [I_n \otimes (I_n - A_0)^{-1}] \left\{ (1 - \omega)[I_n \otimes A_1 G] + (1 - \omega)[G \otimes A_1] + \sum_{i=0}^{\infty} \sum_{j=0}^{N}(G^{T,i-j} \otimes A_1 G') + \omega \sum_{i=0}^{\infty} \sum_{j=0}^{N}(G^{T,i-j} \otimes A_1 G(I_n - A_0)^{-1} A_1 G') + \omega \sum_{i=0}^{\infty} \sum_{j=0}^{N}(G^{T,i-j+1} \otimes A_1 (I_n - A_0)^{-1} A_1 G') \right\}.
\]
Let $U^H G^T U = T$ be the Schur form of $G^T$ and set $W = (U^H \otimes I_n)$. By the same arguments as above we find that, for $\omega = 1$, the eigenvalues of $H$ belong to the set
\[
\bigcup \{ \mu : \mu \text{ is eigenvalue of } (I_n - A_0)^{-1} N(\lambda) \},
\]
with $\lambda$ eigenvalue of $G^T$, and
\[
N(\lambda) = \sum_{i=-2}^{\infty} \sum_{j=0}^{i} (\lambda^{i-j} A_i G^j) + \sum_{i=1}^{\infty} \sum_{j=0}^{i} (\lambda^{i-j} A_i G(I_n - A_0)^{-1} A_i G^j) + \\
\sum_{i=1}^{\infty} \sum_{j=0}^{i} (\lambda^{i-j} A_i (I_n - A_0)^{-1} A_i G^j).
\]
Since
\[
| (I_n - A_0)^{-1} N(\lambda) | \leq P(1)
\]
we conclude that $\rho(H) \leq \rho(P(1))$ in view of the Wielandt theorem [10].

Numerical results shown in Section 5 exhibit a very rapid convergence profile which might be explained with the dependence of the asymptotic convergence rate on the second eigenvalue of the corresponding iteration matrices as reported in [8].

4 Adaptive Strategies and Efficiency Analysis

The efficiency of fixed point iterations depends on both speed of convergence and complexity properties. Markov chains are generally defined in terms of sparse matrices. To take into account this feature we assume that $\gamma n^2$, $\gamma = \gamma(n)$, multiplications/divisions are sufficient to perform the following tasks:

1. to compute a matrix multiplication of the form $A_i \cdot W$, where $A_i, W \in \mathbb{R}^{n \times n}$;
2. to solve a linear system of the form $(I - A_0) Z = W$, where $A_0, W \in \mathbb{R}^{n \times n}$.

We also suppose that the transition matrix $P$ in (1) is banded, hence $A_i = 0$ for $i > q$. This is always the case in numerical computations where the matrix power series $\sum_{i=-1}^{\infty} A_i X_k^{i+1}$ has to be approximated by some finite partial sum $\sum_{i=-1}^{q} A_i X_k^{i+1}$. Under these assumptions we obtain the following cost estimates per step:

1. the traditional fixed point iteration (5) requires $qn^3 + 2\gamma n^2 + O(n^2)$ multiplicative operations;
2. the $U$-based fixed point iteration (6) requires $(q + 4/3)n^3 + \gamma n^2 + O(n^2)$ multiplicative operations;
3. the staircase-based (S-based) fixed point iteration (7) requires $(q + 1)n^3 + 4\gamma n^2 + O(n^2)$ multiplicative operations.

Observe that the cost of the S-based fixed point iteration is comparable with the cost of the $U$-based iteration, which is the fastest among classical iterations (2). Therefore, in the cases where the $U/S$-based fixed point iterative schemes require significantly less iterations to converge, these algorithms are more efficient than the traditional fixed point iteration.

Concerning the relaxed versions (10) of the S-based fixed point iteration for a given fixed choice of $\omega_k = \omega$ we get the same complexity as the unmodified scheme (9) obtained with $\omega_k = \omega = 1$. The adaptive selection of $\omega_k$ exploited in Proposition 5 and Remark 6 with $X_0 = 0$ requires more care.
5 Numerical Results

The strategy (13) is computationally unfeasible since it needs the additional computation of \( \sum_{i=2}^q A_i Y_k^{i+1} \). To approximate this quantity we recall that

\[
A_i (Y_k^{i+1} - X_k^{i+1}) = A_i \sum_{j=0}^i Y_k^j (Y_k - X_k) X_k^{i-j} \geq A_i \sum_{j=0}^i X_k^j (Y_k - X_k) X_k^{i-j}.
\]

Let \( \theta_{k+1} \) be such that

\[
Y_k - X_k \geq \frac{X_k - X_{k-1}}{\theta_{k+1}}.
\]

Then condition (13) can be replaced with

\[
\frac{\omega_{k+1} - 1}{\omega_{k+1}} A_i ((Y_k^2 - X_k^2) \leq (Y_k \Gamma_k + \Gamma_k Y_k) + (\hat{\omega} \theta_{k+1})^{-1} \sum_{i=2}^q A_i (X_k^{i+1} - X_{k-1}^{i+1})).
\] (21)

The iterative scheme (10) complemented with the strategy based on (21) for the selection of the parameter \( \omega_{k+1} \) requires no more than \((q + 3)n^3 + 4q n^2 + O(n^2)\) multiplicative operations. The efficiency of this scheme will be investigated experimentally in the next section.

5 Numerical Results

In this section we present the results of some numerical experiments which confirm the effectiveness of our proposed schemes. All the algorithms have been implemented in Matlab and tested on a PC i9-9900K CPU 3.60GHz \( \times 8 \).

5.1 Synthetic Examples

The first test concerns with the validation of the analysis performed above, regarding the convergence of fixed point iterations. We consider the block tridiagonal Markov chain of Remark 8. Observe that the drift of the Markov chain is exactly equal to \(-\delta\). In Table 1 we show the number of iterations for different iterative schemes with \( n = 100 \). Specifically we compare the traditional fixed point iteration, the \( U \)-based fixed point iteration, the \( S \)-based fixed point iteration (9) and the relaxed fixed point iterations (10). For the latter case we consider the \( S_\omega \)-based iteration where \( \omega_{k+1} = \omega \) is fixed a priori and the \( S_{\omega(k)} \)-based iteration where the value of \( \omega_{k+1} \) is dynamically adjusted at any step according to the strategy (21) complemented with condition (12).

The relaxed stationary iteration is applied with \( \hat{\omega} = 10 \). The iterations are stopped when the residual error \( \| \sum_{i=1}^q A_i X_k^{i+1} \|_\infty \) is smaller than \( 10^{-13} \).

The first four columns of Table 1 confirms the theoretical comparison of asymptotic convergence rates of classical fixed point iterations applied to a block tridiagonal matrix. Specifically, the \( U \)-based and the \( S \)-based iterations are twice faster than the traditional iteration. Also, the relaxed stationary variants greatly improve the convergence speed. An additional remarkable improvement is obtained by adjusting dynamically the value of the relaxation parameter. Also notice that the \( S_{\omega(k)} \)-based iteration is guaranteed to converge differently to the stationary \( S_\omega \)-based iteration.

The second test aims to compare the convergence speed of the traditional, \( U \)-based, \( S \)-based and \( S_{\omega(k)} \)-based fixed point iterations applied on the synthetic examples of \( M/G/1 \)-type Markov chains described in [3]. These examples are constructed in such a
5 Numerical Results

| δ        | trad. | 1.0e-2 | 731 | 724 | 515 | 496 | 479 | 65 |
|----------|-------|--------|-----|-----|-----|-----|-----|----|
|          |       |        |     |     |     |     |     |    |
| 1.0e-4   |       | 84067  | 42046 | 42037 | 30023 | 28976 | 28027 | 11771 |
| 1.0e-6   |       | 2310370 | 1154998 | 1155352 | 825121 | 796693 | 770250 | 329843 |

Tab. 1: Number of iterations for different values of $\delta$ and $\omega$.

Fig. 3: Residual errors generated by the four fixed point iterations applied to the synthetic example with drift $\mu = -0.1$ and $\mu = -0.005$.

way that the drift of the associated Markov chain is close to a given nonnegative value. We do not describe in detail the construction, as it would take some space, but refer the reader to [3, Sections 7.1].

In Figure 3 we report the semilogarithmic plot of the residual error in the infinity norm generated by the four fixed point iterations for two different values of the drift. Observe that the adaptive relaxed iteration is about twice faster than the traditional fixed point iteration. The observation is confirmed in Table 2 where we indicate the speed-up in terms of CPU-time with respect to the traditional fixed point iteration for different values of the drift $\mu$. In Figure 3 we repeat the set of experiments of Figure 3 with a starting stochastic matrix $X_0 = ee^T / n$. Here the adaptive strategy is basically the same as used before where we select $\omega_{k+1}$ in the interval $[0, \omega_k]$ as the maximum value which maintains the nonnegativity of $X_{k+1}$. In this case the adaptive strategy seems to be quite effective in reducing the number of iterations. Other results are not so favorable and we believe that in the stochastic case the design of a general efficient strategy for the choice of the relaxation parameter $\omega_k$ is still an open problem.
Tab. 2: Speed-up in terms of CPU-time w.r.t. the traditional fixed point iteration for different values of the drift $\mu$.

| $\mu$   | $U$-based | $S$-based | $S_{\omega(k)}$-based |
|---------|-----------|-----------|-----------------------|
| -0.1    | 1.6       | 1.5       | 2.1                   |
| -0.05   | 1.5       | 1.4       | 2.0                   |
| -0.01   | 1.6       | 1.5       | 2.1                   |
| -0.005  | 1.6       | 1.5       | 2.1                   |
| -0.001  | 1.6       | 1.4       | 2.2                   |
| -0.0005 | 1.6       | 1.5       | 2.2                   |
| -0.0001 | 1.7       | 1.6       | 2.4                   |

Fig. 4: Residual errors generated by the four fixed point iterations applied to the synthetic example with drift $\mu = -0.1$ and $\mu = -0.005$. 

5.2 Application Examples

In the third set of numerical experiments we consider several cases of PH/PH/1 queues introduced in [3] for testing purposes. Again, we do not describe in detail the construction, as it would take some space, but refer the reader to [3, Sections 7.1]. The construction depends on a parameter $\rho$ with $0 \leq \rho \leq 1$. In this case the drift is $\mu = 1 - \rho$.

In Tables 3-5 and 6 we report the number of iterations for different values of $\rho$ with $\omega = 2$. For comparison in Table 6 we show the number of iterations on Example $(8,3)$ starting with $X_0$ a stochastic matrix. We compare the traditional, $U$-based and $S$-based iterations. We observe a rapid convergence profile and the fact that the number of iterations is independent of the drift value.

For a more challenging example from applications, we take the generator matrix $Q$ from the queuing model described in [4]. This is a complex queuing model, a BMAP/PHF/1/$N$ model with retrial system with finite buffer and non-persistent customers. We do not describe in detail the construction of this matrix, but refer the reader to [4, Sections 4.3 and 4.5]. In Table 7 we indicate the number of iterations for different values of the capacity $N$.

6 Conclusion and Future Work

In this paper we have introduced a novel fixed point iteration for solving M/G/1-type Markov chains. It is shown that this iteration complemented with suitable adaptive relaxation techniques is generally more efficient than other classical iterations. Incorporating relaxation techniques into other different inner-outter iterative schemes as the
6 Conclusion and Future Work

| ρ       | trad. | U-based | S-based |
|---------|-------|---------|---------|
| 0.99    | 52    | 45      | 31      |
| 0.999   | 51    | 45      | 30      |
| 0.9999  | 51    | 45      | 30      |

Tab. 6: Number of iterations for different values of ρ on Example (8,3) with \( X_0 \) a stochastic matrix.

| N    | trad. | U-based | S-based | \( S_ω \)-based | \( S_{ω(\ell)} \)-based |
|------|-------|---------|---------|-----------------|------------------------|
| 20   | 4028  | 3567    | 3496    | 3089            | 2826                   |
| 30   | 4028  | 3567    | 3496    | 3089            | 2826                   |
| 40   | 4028  | 3567    | 3496    | 3089            | 2826                   |

Tab. 7: Number of iterations for different values of \( N \) on Example described in [4].

ones introduced in [3] is an ongoing research.

References

[1] P. Amodio and F. Mazzia. A parallel Gauss-Seidel method for block tridiagonal linear systems. *SIAM J. Sci. Comput.*, 16(6):1451–1461, 1995.

[2] D. A. Bini, G. Latouche, and B. Meini. *Numerical methods for structured Markov chains*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005. Oxford Science Publications.

[3] D. A. Bini, G. Latouche, and B. Meini. A family of fast fixed point iterations for M/G/1-type Markov chains. *IMA Journal of Numerical Analysis*, 42(2):1454–1477, 02 2021.

[4] S. Dudin, A. Dudin, O. Kostyukova, and O. Dudina. Effective algorithm for computation of the stationary distribution of multi-dimensional level-dependent Markov chains with upper block-Hessenberg structure of the generator. *J. Comput. Appl. Math.*, 366:112425, 17, 2020.

[5] L. Gemignani and F. Poloni. Comparison theorems for splittings of M-matrices in (block) Hessenberg form. *BIT Numerical Mathematics*, 2022.

[6] G. Latouche and V. Ramaswami. A logarithmic reduction algorithm for quasi-birth-death processes. *J. Appl. Probab.*, 30(3):650–674, 1993.

[7] H. Lu. Stair matrices and their generalizations with applications to iterative methods. I. A generalization of the successive overrelaxation method. *SIAM J. Numer. Anal.*, 37(1):1–17, 1999.

[8] B. Meini. New convergence results on functional iteration techniques for the numerical solution of M/G/1 type Markov chains. *Numer. Math.*, 78(1):39–58, 1997.
[9] G. Meurant. Domain decomposition preconditioners for the conjugate gradient method. *Calcolo*, 25(1-2):103–119 (1989), 1988.

[10] C. Meyer. *Matrix analysis and applied linear algebra*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. With 1 CD-ROM (Windows, Macintosh and UNIX) and a solutions manual (iv+171 pp.).

[11] M. F. Neuts. *Matrix-geometric solutions in stochastic models*, volume 2 of *Johns Hopkins Series in the Mathematical Sciences*. Johns Hopkins University Press, Baltimore, Md., 1981. An algorithmic approach.

[12] V. Ramaswami. A stable recursion for the steady state vector in Markov chains of $M/G/1$ type. *Comm. Statist. Stochastic Models*, 4(1):183–188, 1988.