Note

Acceleration effects on atomic clocks

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Abstract
We consider a free massive particle inside a box that is dragged by Rindler observers. Admitting that the particle obeys the Klein–Gordon equation, we find the frequencies of the stationary states of this system. Transitions between the stationary states are employed to set a standard frequency for a toy atomic clock. Comparing the energy spectrum of the accelerated system with the energy spectrum of an identical system in an inertial frame, we determine the influence of instantaneous acceleration on the rate of atomic clocks. We argue that our result does not violate the clock hypothesis.

Keywords: Non-inertial frame, clock hypothesis, locality principle, atomic clocks

1. Introduction

According to the Theory of Relativity the flux of time depends on the motion state of observers. Lorentz transformations allow us to compare measurements of the elapsed time between events when they are performed by inertial observers. The predicted time dilation has been confirmed experimentally in different contexts—the Doppler shift of the energy spectrum produced by atomic beams, the lifetime of unstable particles, the rate of atomic clocks—and with an increasing high accuracy [1, 2]. However, the temporal rhythm of an accelerated clock cannot be deduced from the two fundamental postulates of the Theory of Relativity. Indeed, discussion of this issue demands an additional assumption. The clock hypothesis, according to which the rate of a clock is not affected either by its instantaneous acceleration or higher order derivatives of velocity, is usually adopted in this context. Although it is widely accepted (for a critical approach, see [3]), there are speculations about empirical implications of alternative hypotheses [4, 5].

The physical basis of the clock hypothesis is the so-called principle of locality that establishes a local equivalence between an accelerated observer and an instantaneously comoving inertial observer [6]. The idea is that both observers momentarily share the same
position and velocity and, therefore, have the same physical state from the viewpoint of classical mechanics. Hence, those observers would be locally equivalent, implying, according to the principle of locality, that they would have the same notion of time, instantaneously. Mathematically this is translated into the well-known relation between the proper time ($d\tau$) of the accelerated observer and the time ($dt$) as measured in an inertial frame: $d\tau = dt/\gamma$, where $\gamma$ is the Lorentz factor.

So far the clock hypothesis has been confirmed experimentally. Measurements of the lifetime of unstable particles moving in circular orbits or when they are submitted to longitudinal acceleration are in agreement with the clock hypothesis [7, 8]. Within the accuracy of the instruments, these tests found no evidence of the influence of the acceleration on the decay rate, despite those particles having been subjected to very high acceleration: $10^{18}g$ (for the centripetal acceleration [7]) and $10^{15}g$ (for the average longitudinal acceleration, with peaks of $a = 10^{22}g$ [8]), where $g$ is the Earth gravitational acceleration.

On the other hand, it is worth mentioning that the principle of locality relies on the definition of the physical state established by classical mechanics. However, as nature has a quantum character, the locality principle has a conceptual limitation [4, 9, 10]. Because of this, it has been suggested that the rate of any actual clock would be influenced by its instantaneous acceleration or could even be affected by its past world line [4, 9]. Thus, the relation between $d\tau$ and $dt$ should be modified when the wave behavior of the system is taken into account. For instance, based on the study of the lifetime of artificial muonic particles in circular motion [11], it was proposed that, regarding the muon proper time, the new relation would be: $d\tau = (1 + 2/3(\lambda/L)^2)dt/\gamma$, where $\lambda$ is the muon Compton wavelength, $\lambda = \lambda/2\pi$ and $L = c^2/a$ defines a characteristic length scale where the particle state changes significantly [10]. Under the conditions in which the muon experiment was performed $a \sim 10^{18}g$ [7], which corresponds to $L \approx 1\text{ cm}$. The muon Compton wavelength is of the order of $10^{-12}\text{ m}$, then $(\lambda/L)^2$ is less than $10^{-25}$. This small correction could not be detected within the accuracy of that experiment, so these effects of the acceleration cannot be ruled out and the question remains open.

In this paper, we want to address this issue investigating the behavior of an accelerated atomic clock. Actually, for the sake of simplicity, our atomic clock is described by a toy model that consists of a particle in a box that obeys the Klein–Gordon (KG) equation. The acceleration of the box is implemented, within our scheme, assuming that the walls are dragged by Rindler observers. Solving the KG equation in the Rindler frame and imposing the appropriate boundary conditions, we determine the frequency of the stationary states of the system. Comparing this spectrum with the spectrum of an identical system in an inertial system, we can determine a relation between the ticking rate of the accelerated clock and the rate of an inertial atomic clock.

1.1. Accelerated atomic clock

The world line of a uniformly accelerated observer in a Minkowski spacetime is described, in terms of the rectangular coordinates ($t, x$) of an inertial frame, by the following parametric curve:

$$t(\tau) = \frac{c}{a} \sinh (a\tau/c), \quad (1)$$

$$x(\tau) = \frac{c^2}{a} \cosh (a\tau/c), \quad (2)$$

where $\tau$ is the observer’s proper time (defined in terms of the length of the curve) and $a$ is its proper acceleration. The world line corresponds to a hyperbola in the spacetime diagram of
the inertial frame. For the sake of simplicity, we are considering a spacetime of (1+1) dimensions.

The accelerated observer can construct a coordinate system adapted to its motion using the locality principle. At a particular instant of time $\tau$, the observer, with the help of the instantaneously comoving inertial frame $S_{\tau}$, determines the events that are simultaneous. All of them are labeled with the same temporal coordinate $\tau$. The spatial coordinate, $\xi$, of these simultaneous events is established using the spatial coordinate of $S_{\tau}$. From this definition, it follows that the transformation between $((t, x),)$ and the Rindler coordinates $((\tau, \xi),)$ is given by

$$ct = \left( c^2/a + \xi^2 \right) \sinh (a \tau/c),$$

$$x = \left( c^2/a + \xi^2 \right) \cosh (a \tau/c).$$

It is well known that this transformation is defined only in a region corresponding to the right-hand side of the light cone whose vertex is at the origin of the inertial frame. Each coordinate line constant $\xi$ is a hyperbola in the spacetime diagram of $S$ (in $(t, x)$-coordinates) and represents the world line of a uniformly accelerated observer with proper acceleration equal to $a/(1 + a^2 c^2)$ [12]. The set of these observers constitutes the Rindler frame. The Minkowski metric written in Rindler coordinates assumes the following form:

$$dx^2 = -c^2 \left( 1 + \xi^2/c^2 \right)^2 dt^2 + d\xi^2.$$

Consider now a particle with rest mass $m$ confined in a box that is dragged by Rindler observers. This means the walls are found at rest with respect to the Rindler frame. Let us assume that the walls are localized at positions $\xi$ and $\xi + \ell$ (where $\ell$ is the proper length of the box as measured in the Rindler frame) around the central Rindler observer (i.e., the observer at $\xi = 0$). Inside the box, the particle is free. Admitting that its behavior is governed by the KG equation, then, in the Rindler coordinates, the particle’s wave function $\phi(\tau, \xi)$ satisfies the following equation:

$$-\frac{1}{c^2 \left( 1 + \xi^2/c^2 \right)^2} \frac{\partial^2 \phi}{\partial \tau^2} + \frac{1}{\left( 1 + \xi^2/c^2 \right) \frac{\partial}{\partial \xi}} \left( \frac{1}{1 + \xi^2/c^2} \frac{\partial}{\partial \xi} \right) \phi = 0.$$

At this point it is important to emphasize that although the interpretation of $\phi$ as a wave function has some limitations, this does not affect the purpose of our discussion, since we are dealing with a toy model and we are basically concerned with the mathematical problem of finding the stationary states of the KG equation and their respective frequencies.

The solution can be written as $\phi = e^{-i \omega \tau} \psi$, where $\psi$ is independent of $\tau$ and satisfies the modified Bessel equation:

$$\rho^2 \frac{d^2 \psi}{d\rho^2} + \rho \frac{d\psi}{d\rho} - \left[ \rho^2 - \frac{\omega^2 c^2}{a^2} \right] \psi = 0,$$

where $\rho = (mc/h)(c^2/a + \xi)$. Here we can identify three length scales: the particle’s reduced Compton wavelength $\lambda = h/mc$, the size $\ell$ of the box and the acceleration length $L = c^2/a$. The general solution of equation (7) is a linear combination of the modified Bessel function of the first kind $I_\nu(\rho)$ and of the second kind $K_\nu(\rho)$ of the order of $\nu = i (mc/a)$. As the order is purely imaginary, $I_\nu(\rho)$ is complex on the positive real axis and, hence, a better companion for $K_\nu(\rho)$ is the function $L_\nu(x) = [I_\nu(x) + i K_\nu(x)]/2$, which is also a real function for $x > 0$ in the case of a pure imaginary order [13, 14]. Thus, the general solution can be appropriately written as $\psi(\rho) = C_1 L_\nu(\rho) + C_2 K_\nu(\rho)$, where $C_1$ and $C_2$ are arbitrary complex numbers. The stationary states in the box and their corresponding frequencies are determined by the
boundary conditions $\psi = 0$ at the points $\rho_1 = (mc/h)(c^2/a + \zeta_1)$ and $\rho_2 = (mc/h)(c^2/a + \zeta_2 + \ell)$. At least one of the coefficients must be nonzero, otherwise the wave function would be identically null. It happens that there exists a non-trivial solution for $C_1$ and $C_2$ only if the elements $L_n(\rho_1)$, $K_n(\rho_1)$, $L_n(\rho_2)$ and $K_n(\rho_2)$ constitute a matrix with a null determinant. This condition is equivalent to the equation

$$K_{n} \left( L/\hbar + (\zeta_1 + \ell)/\hbar \right) = L_{n+1} \left( L/\hbar + (\zeta_1 + \ell)/\hbar \right).$$

We want to analyze the above equation in the small acceleration regime. For this purpose, it is appropriate to consider the asymptotic expansion of the modified Bessel functions [13, 14]. Taking the first terms, we have [13]

$$e^{ax/\hbar}K_{n}(ax) \approx \frac{1}{\sqrt{2\pi}} \left[ \cos \left( a\theta(x) + \frac{\pi}{4} \right) I_{n} \left( \frac{a\theta(x) + \frac{\pi}{4}}{\alpha} \right) \right],$$

$$e^{-ax/\hbar}L_{n}(ax) \approx -\frac{1}{2\sqrt{2\pi}} \left[ \left( a\theta(x) + \frac{\pi}{4} \right) I_{n} \left( \frac{a\theta(x) + \frac{\pi}{4}}{\alpha} \right) \right],$$

where $a = oc/a$, $\theta(x) = \sqrt{1-x^2} - \ln \left( (1+\sqrt{1-x^2})x \right)$ and approximate expressions for $I_{n}$ and $I_{n+1}$ are

$$I_{n} \sim 2\frac{\sqrt{\pi}}{\sqrt{a}} \left[ 1 - x^2 \right]^{1/4} \sum_{s=0}^{\infty} \frac{V_{2s}}{\alpha^{2s}} \left[ i \left[ 1 - x^2 \right]^{-1/2} \right],$$

$$I_{n+1} \sim 2\frac{\sqrt{\pi}}{\sqrt{a}} \left[ 1 - x^2 \right]^{1/4} \sum_{s=0}^{\infty} \frac{iV_{2s+1}}{\alpha^{2s+1}} \left[ i \left[ 1 - x^2 \right]^{-1/2} \right],$$

where

$$V_{0}(q) = 1, V_{s+1}(q) = \frac{1}{2} q^2 \left[ q^2 + 1 \right] V_{s}(q) + \frac{1}{8} \int_{0}^{q} V_{s}(t) \left( 1 + 5t^2 \right) dt$$

for $s = 0, 1, 2, ...$ [13, 14]. Using this approximation in equation (8), we obtain the frequency, or speaking loosely, we find the energy $\omega$ of the stationary states. In the second order of $a$, the spectrum of positive energy as measured in the Rindler frame is given by:

$$E_{s} = E_{0}^{(0)} \left\{ 1 + \left( \frac{\ell}{2\ell} + \frac{\xi_{1}}{\ell} \right) + \left[ -\frac{1}{12} + \frac{1}{8n^2\pi^2} \left( \frac{1}{1 + n^2\pi^2\lambda^2/\ell^2} \right) \right] \right\} \left( \frac{\ell}{\lambda} \right)^2$$

$$+ \frac{1}{8} \left( \frac{5}{n^2\pi^2} + \frac{1}{3n^2\pi^2} \right) \left( \frac{\ell^2}{\lambda^2} \right)^2 \right\}. $$

where $E_{0}^{(0)} = [m^2c^4 + (n^2\pi^2\hbar^2c^2/\ell^2)^2]^{1/2}$ is the energy of the quantum level $n$ of an identical system (a mass $m$ in a box of proper length $\ell$) at rest in an inertial frame. Equation (14) is valid for $n^2 > (1/\pi^2)\ell^2/(\lambda^2)$ (see appendix A). Note that the linear term relative to the acceleration $a$ depends on the position of the box with respect to the central Rindler observer. In our analogy with an actual atom, let us assume that the path of this observer (at $\xi = 0$) corresponds to the trajectory of the atomic nucleus. Thus, a symmetric configuration of the walls around the nucleus (which seems to be the most natural choice, otherwise the atom would have a spontaneous electric dipole) corresponds to $\xi = -\ell/2$. In this case, the linear term vanishes and, therefore, the leading correction is quadratic. If we write the length $\ell$ of the
box, the wavelength $\lambda$ and the acceleration $a$ in terms of the Bohr radius ($r_0$), the electron mass ($m_e$) and $g$, respectively, we find the following estimates:

$$\frac{\ell}{\mathcal{L}} \sim 10^{-27} \left( \frac{\ell}{r_0} \right) \left( \frac{a}{g} \right),$$

$$\frac{\lambda}{\mathcal{L}} \sim 10^{-29} \left( \frac{m_e}{m} \right) \left( \frac{a}{g} \right).$$

For $a = 10^{18}g$, the quadratic correction produces a relative shift of the order of $10^{-19}(\ell/r_0)^2$.

In analogy with actual atomic processes, let us assume that, in a transition between two stationary states of our system, a quantum of some field with a well-defined frequency is emitted. Some device endowed with a counter of the cycles of the standard frequency defines our toy clock.

Now let us consider a transition from a certain state $f$ to the quantum level $i$. The emitted quantum has a frequency $\omega_f$ that is employed as the standard frequency of the accelerated clock. When measured in terms of the parameter $\tau$, the frequency $\omega_f = (E_f - E_i)/\hbar$ may be determined from equation (14). On its turn, for an identical system at rest in the inertial system $S$, the frequency of the emitted quantum in the corresponding transition from state $f$ to $i$ is given by $\omega_f^0 = (E_f^0 - E_i^0)/\hbar$. Note that $\omega_f^0$ is measured in terms of the time coordinate of the frame $S$. Having this in mind, now consider two close events $E_1$ and $E_2$ on the path of the central accelerated observer, with labels $\tau_1$ and $\tau_2$, respectively. If $\Delta T$ is the number of oscillations of the standard wave with frequency $\omega_f$ that happen during the elapsed time between $E_1$ and $E_2$, then

$$\Delta T = (\tau_2 - \tau_1)\omega_f/2\pi.$$

On the other hand, from the point of view of $S$, during the interval $\Delta \tau = \tau_2 - \tau_1$, the number of cycles of the inertial atomic clock running in the standard frequency $\omega_f^0$ is

$$\Delta \tau = (c/a)\sinh (a\tau_2/c) - \sinh (a\tau_1/c)]\omega_f^0/2\pi,$$

according to equation (1). Therefore, in the limit $\Delta \tau \to 0$, the instantaneous ratio between the ticking rates of the clocks is

$$\frac{dT}{dt} = \sqrt{1 - \nu^2/c^2} \left( \frac{\Delta E_f}{\Delta E_f^0} \right)$$

where $\nu$ is the relative velocity between the observers at the instant $\tau_1$ (see appendix B). The correction factor depends on the frequency of the atomic clock, i.e., on the particular transition ($f \to i$) that is chosen to set the standard frequency. However, in transitions with higher quantum numbers, the relation reduces to

$$\frac{dT}{dt} = \sqrt{1 - \nu^2/c^2} \left[ 1 - \frac{1}{12} \left( \frac{\ell}{\mathcal{L}} \right)^2 \right].$$

Therefore, the rate of an accelerated atomic clock depends on its instantaneous acceleration. Moreover, the leading term of corrections is quadratic with respect to the acceleration. This result is similar to that suggested in [10, 11], which predicts a correction of the order of $(\lambda/\mathcal{L})^2$ for the muon lifetime. Nevertheless, it is important to stress that, in comparison with the study of the muon proper time [11], our result differs in some aspects. In [11], the influence of a magnetic field on the decay rate of muons was determined. As this magnetic field is responsible for keeping the muons in circular motion, its influence can be written in terms of
the centripetal acceleration [10]. On the other hand, here we have studied the influence of a longitudinal acceleration over the energy levels of an atom. Therefore, the systems are physically distinct and are accelerated in different ways. Besides, the approaches are also different: [11] follows the second quantization approach, which is the appropriate formalism for dealing with corrections of Compton wavelength order, since it is expected that, on this scale, the quantum field theory effects of particle creation and annihilation take place. In contrast, our method is based on the first quantization approach. We find that the dominant correction does not depend on the Compton wavelength of the particle, but on the size ℓ of the atom, justifying, in this way, the use of the much simpler formalism of the first quantization. So, the results of [11] and ours are independent and, let us say, complementary since they contemplate different aspects of the problem.

The acceleration effect on the rhythm of atomic clocks may be much greater than the effect on the muon lifetime, since the correction depends on $(\ell/\mathcal{L})^2$ rather than $(\lambda/\mathcal{L})^2$. Indeed, if the size of the box is of the order of the Bohr radius ($\geq 10^{-11} \text{m}$), then $(\ell/\mathcal{L})^2$ is slightly greater than $10^{-18}$ when $a = 10^{18} \text{g}$. Thus, the acceleration effects on the proper time of an atomic clock will be, in a conservative estimate, $10^6$ times greater than those that were predicted for a muon circulating with the same acceleration\footnote{Concerning the decay rate of unstable particles, in order to have a better estimate for the particle lifetime, it is important to take into account the modifications of the formula of the survival probability for the late time decay. These modifications, which has quantum nature, arise even when the particle is not accelerated.} [11, 15]. Furthermore, the fact that the systems are accelerated in different ways may have experimental implications. In the study of longitudinally accelerated particles, the acceleration reaches peaks of $10^{22} \text{g}$ [8]. In this order of magnitude, the effect of acceleration will be approximately $(\ell/\mathcal{L})^2 \sim 10^{-10}$, which is close to the current accuracy ($\sim 10^{-8}$) of empirical tests of the time dilation [2].

The instantaneous acceleration influences the ticking rate of the clock and, as a consequence, it also affects the Doppler shift. According to (14), in transitions with higher quantum numbers, the frequency of the emitted quantum by the accelerated source is

$$\omega_0^L (1 - \ell^2/12 \mathcal{L}^2),$$

where $\omega_0^L$ is the frequency for the same transition when it happens in an inertial system. If the accelerated source emits forward and backward signals, then the frequency as measured by an inertial receiver is

$$\omega_k^0 = \gamma \left(1 \pm \frac{v}{c}\right) \left[1 \pm \frac{\lambda_0}{2 \mathcal{L}} \left(1 + \frac{1}{12} \left(\frac{\lambda_0}{\mathcal{L}}\right)^2 - \frac{1}{12} \left(\frac{\ell}{\mathcal{L}}\right)^2\right)\right] \omega_0^L,$$

where $v$ is the relative velocity at the moment of emission and $\lambda_0$ is the wavelength of the corresponding signal emitted in an inertial frame (see appendix C). The signs $+$ or $-$ are valid when the observers are approaching or receding, respectively. The acceleration modifies the Doppler-shift formula in two different ways: the term $\lambda_0/\mathcal{L}$ is associated with the variation of the source’s velocity during a complete cycle, while $(\ell/\mathcal{L})^2$ is a new contribution that arises due to the change of the ticking rate of the clock caused by its acceleration.

1.2. Discussion and conclusion

We showed that the rate of an accelerated clock is affected by its instantaneous acceleration; however, this result should not be seen as a violation of the clock hypothesis. Actually, the Rindler frame is built with the help of this hypothesis and even the usual form of the KG equation assumes the accelerated frame is somehow based on the principle of locality. The influence of the instantaneous acceleration on the rhythm of an atomic clock is just the expression of the fact that the internal dynamics of a system of finite size is affected by
acceleration. This is true even in the context of the classical mechanics, as we can check by calculating the period of an accelerated pendulum or the period of oscillations of a light beam between accelerated mirrors [16]. In view of this, a better interpretation of our result is that the rate of the accelerated actual clock deviates from the rate of an ideal point-like clock [17]. In this sense, we can say that equation (20) is actually compatible with the clock hypothesis, since, according to it, the rate of the atomic clock does not depend on its instantaneous acceleration in the limit $\ell \to 0$ and $\lambda \to 0$. Nevertheless, as $\ell$ and $\lambda$ are non-null for the actual physical system, the instantaneous acceleration produces some effects on the ticking rate of accelerated clocks. Perhaps the most important result of our study is the fact that, with the help of this simple toy model, we can identify the relevant parameters that contribute to these effects and estimate their magnitude order using equation (20). As we have seen, the influence of the acceleration on the proper time is very tiny; however, as the experiments are getting more accurate, it is important to take them into account in order to make a correct interpretation of empirical data. Recently, an Ives–Stiwell type experiment tested the time dilation factor within an accuracy of $10^{-9}$ [2]. If the same precision could be achieved in experiments involving acceleration of the order of greater than $10^{23} g$ then the acceleration effects on the ticking rate of an atomic clock would become detectable. As we have already mentioned, in the experiment involving longitudinal acceleration [8], the acceleration has peaks of $10^{22} g$.

Our toy model is a very simplified system. To deal with a real atom, first we have to devise an acceleration mechanism. In the case of a neutral atom, we should employ a non-uniform electric field to produce longitudinal acceleration. Thus, in a realistic model, we have to consider the interaction of the external field with the atomic particles whose behavior is governed by the Dirac equation in (3+1) dimensions. This problem is, of course, much more complicated. The effects of acceleration over the energy levels would appear indirectly as consequence of the interaction between the external field and the atom. Nevertheless, if the external field is not so strong in comparison with the strength of the internal interaction, the atom will remain as a bound system, with an accelerated center of mass. The fundamental characteristic of our model is that it represents an accelerated bound system. Then, in this sense, a particle in an accelerated box can be considered a toy model for an accelerated real atom. However, we must bear in mind that this study is a preliminary work and that our results need to be improved in light of more realistic models.

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Appendix A.

The expansion of the functions $L_{\alpha\nu}(\alpha x)$ and $K_{\alpha\nu}(\alpha x)$ given by equations (9) and (10) are valid when the argument $\alpha x$ is less than the order of $\alpha$, i.e., for $x < 1$ [13, 14]. To use these formulas in the equation (8) we have to make $\alpha x = L/\hbar + \xi/\hbar$ or $\alpha x = L/\hbar + (\xi + \ell)/\hbar$, where $\xi$ is the position of the first wall. Taking $\xi = -L/2$ (corresponding to a symmetric configuration of the box) and considering that $\alpha = \omega c/\hbar$, we find that the maximum value of $x$ is $(c/\hbar + \omega a/2\hbar c)/\omega$. Thus, the condition $x < 1$ implies that

$$h\omega > mc^2 + \frac{1}{2}ma\ell,$$  \hspace{1cm} (22)
recalling that $\hbar = h/mc$. As $\omega$ is the frequency of a certain stationary state, then $\hbar \omega$ is the corresponding energy. The energy of the unperturbed system is given by $[m^2c^4 + (n^2\pi^2\hbar^2c^2)/\ell^2]^{1/2}$. Thus, in the first order of $a$, we have the condition

$$n^2 > \frac{1}{\pi^2} \frac{\ell^3}{\hbar^2 L}.$$  \hfill (23)

This condition may be interpreted in two different ways. If the system (a particle of mass $m$ confined in the box of size $\ell$) is moving with a known proper acceleration $a$, then the above inequality establishes the quantum levels that can be used to define a standard frequency for the atomic clock. On the other hand, if the quantum levels were chosen previously, equation (23) gives the maximum acceleration, which is consistent with our approximation scheme. To make some estimates, let us consider that $\ell \approx 0.5 \times 10^{-10}$ m (Bohr radius) and that $m$ is the electron mass, which corresponds to $\lambda \approx 2.4 \times 10^{-12}$ m. Thus, $\mathcal{L} > 10^{-7}/m^2$. Therefore, even for the level $n = 1$, the bound on the acceleration, $a < 10^{22}g$, is not stringent.

Appendix B

According to equation (17), $\Delta T$ is the number of ticks between events $E_1$ and $E_2$ as measured by the accelerated clock. On the other hand, $\Delta t$ in equation (18) gives the number of ticks as measured by an inertial frame. The ratio between these quantities is

$$\frac{\Delta t}{\Delta T} = \frac{\text{cosh} (\alpha \tau_2/c) - \text{cosh} (\alpha \tau_1/c)}{\text{sinh} (\alpha \tau_2/c) - \text{sinh} (\alpha \tau_1/c)} \frac{\omega_f}{\omega_i}.$$  \hfill (24)

where we can write $\tau_2 = \tau_1 + \Delta \tau$. In the limit when $\Delta \tau \to 0$, we have

$$\frac{\text{d} t}{\text{d} T} = \text{cosh} (\alpha \tau_1/c) \omega_f/\omega_i.$$  \hfill (25)

From the motion equations of the accelerated observer, equations (1) and (2), we can verify that the instantaneous velocity of the accelerated observer with respect the inertial system is $v = \text{d} x/\text{d} t = c \text{ tanh} (\alpha \tau_1/c)$. Thus, writing $\text{cosh} (\alpha \tau_1/c)$ as $1/\sqrt{1 - (v/c)^2}$, we obtain equation (19).

Appendix C

Consider that the accelerated observer emits signals at times $\tau_1$ and $\tau_2$. The coordinates of the emission events can be determined from equations (1) and (2). The signals travel at the light velocity and reach an inertial observer fixed at $x = 0$ at instants $t_1$ and $t_2$, respectively. Therefore, we can write

$$t_1 - t(\tau_1) = \frac{1}{c} (x(\tau_1) - 0),$$  \hfill (26)

$$t_2 - t(\tau_2) = \frac{1}{c} (x(\tau_2) - 0).$$  \hfill (27)

It follows that

$$t_2 - t_1 = t(\tau_2) - t(\tau_1) + \frac{1}{c} (x(\tau_2) - x(\tau_1)).$$  \hfill (28)
if we write \( \tau_2 = \tau_1 + \Delta \tau \), and using equations (1) and (2), we find

\[
\Delta t = \frac{c}{a} \exp \left( a \tau_1 / c \right) \cdot \exp \left( a \Delta \tau / c \right) - 1.
\]

(29)

where \( \Delta t = t_2 - t_1 \). Now, expanding the above equation up to the second order of \( a \), we obtain

\[
\Delta t = \exp \left( a \tau_1 / c \right) \cdot \left[ 1 + \frac{1}{2} (c \Delta \tau / \mathcal{L}) + \frac{1}{6} (c \Delta \tau / \mathcal{L})^2 \right] \Delta \tau.
\]

(30)

If \( \Delta \tau \) is the wave period as measured by the accelerated observer, then \( \Delta t \) is the period of the wave received by the inertial observer. Therefore, the relation between the frequencies is

\[
\omega_0^E = \exp \left( -a \tau_1 / c \right) \cdot \left[ 1 - \frac{1}{2} (c \Delta \tau / \mathcal{L}) + \frac{1}{12} (c \Delta \tau / \mathcal{L})^2 \right] \omega_0^E.
\]

(31)

As we have already mentioned, the frequency emitted is \( \omega_0^E = \omega_0^E (1 - \ell^2 / 12 \mathcal{L}^2) \). Therefore, recalling that \( \Delta \tau = 2 \pi / \omega_0^E \), we can write \( (c \Delta \tau / \mathcal{L}) \), up to the second order of \( a \), as \( \lambda_0 / \mathcal{L} \), where \( \lambda_0 = 2 \pi c / \omega_0^E \) would be the wavelength of the signal if it had been emitted by an inertial frame. On its turn, from the fact that instantaneous relative velocity is \( c = \tan (a \tau_1 / c) \), we can write \( \exp (a \tau_1 / c) = \gamma (1 - \nu / c) \). If follows that

\[
\omega_0^E = \gamma \left( 1 - \frac{\nu}{c} \right) \left[ 1 - \frac{1}{2} (\lambda_0 / \mathcal{L}) + \frac{1}{12} (\lambda_0 / \mathcal{L})^2 - \frac{1}{12} (\ell / \mathcal{L})^2 \right] \omega_0^E.
\]

(32)

For \( \tau_1 > 0 \), the velocity is positive and the accelerated observer is running away from the inertial receiver located at \( x = 0 \).

Now, if we consider a receiver at a certain fixed position \( x = D \) at the right-hand side of the accelerated observer, we can show, following the same reasoning, that

\[
\omega_0^R = \gamma \left( 1 + \frac{\nu}{c} \right) \left[ 1 + \frac{1}{2} (\lambda_0 / \mathcal{L}) + \frac{1}{12} (\lambda_0 / \mathcal{L})^2 - \frac{1}{12} (\ell / \mathcal{L})^2 \right] \omega_0^E
\]

(33)

In this case, for \( \tau_1 > 0 \), the observers are approaching.

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