EXTENDING BINARY OPERATIONS TO FUNCTOR-SPACES

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ABSTRACT. Given a continuous monadic functor $T : \text{Comp} \to \text{Comp}$ in the category of compacta and a discrete topological semigroup $X$ we extend the semigroup operation $\varphi : X \times X \to X$ to a right-topological semigroup operation $\Phi : T\beta X \times T\beta X \to T\beta X$ whose topological center $\Lambda_\Phi$ contains the dense subsemigroup $T_fX$ consisting of elements $a \in T\beta X$ that have finite support in $X$.

One of powerful tools in the modern Combinatorics of Numbers is the method of ultrafilters based on the fact that each binary operation $\varphi : X \times X \to X$ defined on a discrete topological space $X$ can be extended to a right-topological operation $\Phi : \beta X \times \beta X \to \beta X$ on the Stone-Čech compactification $\beta X$ of $X$, see [13], [16]. The extension of $\varphi$ is constructed in two step. First, for every $x \in X$ extend the left shift $\varphi_x : X \to X$, $\varphi_x : y \mapsto \varphi(x,y)$, to a continuous map $\bar{\varphi}_x : \beta X \to \beta X$. Next, for every $b \in \beta X$, extend the right shift $\bar{\varphi}^b : X \to \beta X$, $\varphi^b : x \mapsto \bar{\varphi}_x(b)$, to a continuous map $\Phi^b : \beta X \to \beta X$ and put $\Phi(a,b) = \Phi^b(a)$ for every $a \in \beta X$. The Stone-Čech extension $\beta X$ is the space of ultrafilters on $X$. In [11] it was observed that the binary operation $\varphi$ extends not only to $\beta X$ but also to the superextension $\lambda X$ of $X$ and to the space $\lambda X$ of all inclusion hyperspaces on $X$. If $X$ is a semigroup, then $\lambda X$ is a compact Hausdorff right-topological semigroup containing $\lambda X$ and $\beta X$ as closed subsemigroups.

In this note we show that an (associative) binary operation $\varphi : X \times X \to X$ on a discrete topological space $X$ can be extended to an (associative) right-topological operation $\Phi : T\beta X \times T\beta X \to T\beta X$ for any monadic functor $T$ in the category $\text{Comp}$ of compact Hausdorff spaces. So, for the functors $\beta, \lambda$, or $G$, we get the extensions of the operation $\varphi$ discussed above.

1. Monadic functors and their algebras

Let us recall [14] VI, [17] §1.2 that a functor $T : \mathcal{C} \to \mathcal{C}$ in a category $\mathcal{C}$ is called monadic if there are natural transformations $\eta : \text{Id} \to T$ and $\mu : T^2 \to T$ making the following diagrams commutative:

\[
\begin{array}{ccc}
T & \xrightarrow{\eta} & T^2 \\
\downarrow{\eta} & & \downarrow{1_T} \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\quad
\begin{array}{ccc}
T^3 & \xrightarrow{\mu^T} & T^2 \\
\downarrow{T\mu} & & \downarrow{\mu} \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\]

In this case the triple $\mathbb{T} = (T,\eta,\mu)$ is called a monad, the natural transformations $\eta : \text{Id} \to T$ and $\mu : T^2 \to T$ are called the unit and multiplication of the monad $\mathbb{T}$, and the functor $T$ is the functorial part of the monad $\mathbb{T}$.

A pair $(X,\xi)$ consisting of an object $X$ and a morphism $\xi : TX \to X$ of the category $\mathcal{C}$ is called a $\mathbb{T}$-algebra if $\xi \circ \eta_X = \text{id}_X$ and the square

\[
\begin{array}{ccc}
T^2X & \xrightarrow{T\xi} & TX \\
\downarrow{\mu} & & \downarrow{\xi} \\
TX & \xrightarrow{\xi} & X
\end{array}
\]

is commutative. For every object $X$ of the category $\mathcal{C}$ the pair $(TX,\mu)$ is a $\mathbb{T}$-algebra called the free $\mathbb{T}$-algebra over $X$.

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For two $\mathbb{T}$-algebras $(X, \xi_X)$ and $(Y, \xi_Y)$ a morphism $h : X \to Y$ is called a morphism of $\mathbb{T}$-algebras if the following diagram is commutative:

\[
\begin{array}{ccc}
TX & \xrightarrow{T \eta} & TY \\
\downarrow_{\xi_X} & & \downarrow_{\xi_Y} \\
X & \xrightarrow{h} & Y
\end{array}
\]

The naturality of the multiplication $\mu : T^2 \to T$ of the monad $\mathbb{T}$ implies that for any morphism $f : X \to Y$ in $\mathcal{C}$ the morphism $Tf : TX \to TY$ is a morphism of free $\mathbb{T}$-algebras.

Each morphism $h : TX \to Y$ from the free $\mathbb{T}$-algebra into a $\mathbb{T}$-algebra $(Y, \xi)$ is uniquely determined by the composition $h \circ \eta$.

**Lemma 1.1.** If $h : TX \to Y$ is a morphism of a free $\mathbb{T}$-algebra $TX$ into a $\mathbb{T}$-algebra $(Y, \xi)$, then $h = \mu \circ T(h \circ \eta) = \mu \circ Th \circ T\eta$.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\eta} & TX & \xrightarrow{h} & Y \\
\downarrow_{\eta} & & \downarrow_{\mu} & \downarrow_{\xi} & \\
TX & \xrightarrow{T \eta} & T^2X & \xrightarrow{T \eta} & TY \\
\downarrow_{T(h \circ \eta)} & & & & \\
& & & &
\end{array}
\]

and observe that

\[h = h \circ \mu \circ \eta_T = \xi \circ Th \circ \eta_T = \xi \circ Th \circ T\eta \circ \mu \circ \eta_T = \xi \circ T(h \circ \eta).\]

\[\square\]

By a topological category we shall understand a subcategory of the category $\textbf{Top}$ of topological spaces and their continuous maps such that

- for any objects $X, Y$ of the category $\mathcal{C}$ each constant map $f : X \to Y$ is a morphism of $\mathcal{C}$;
- for any objects $X, Y$ of the category $\mathcal{C}$ the product $X \times Y$ is an object of $\mathcal{C}$ and for any object $Z$ of $\mathcal{C}$ and morphisms $f_X : Z \to X$ and $f_Y : Z \to Y$ the map $(f_X, f_Y) : Z \to X \times Y$ is a morphism of the category $\mathcal{C}$.

A discrete topological space $X$ is called discrete in $\mathcal{C}$ if $X$ is an object of $\mathcal{C}$ and each function $f : X \to Y$ into an object $Y$ of the category $\mathcal{C}$ is a morphism of $\mathcal{C}$. It is clear that any bijection $f : X \to Y$ between discrete objects of the category $\mathcal{C}$ is an isomorphism in $\mathcal{C}$.

From now on we shall assume $(\mathbb{T}, \eta, \mu)$ is a monad in a topological category $\mathcal{C}$ such that for any discrete objects $X, Y$ in $\mathcal{C}$ the product $X \times Y$ is discrete in $\mathcal{C}$.

2. Binary operations and their $\mathbb{T}$-extensions

By a binary operation in the category $\mathcal{C}$ we understand any function $\varphi : X \times Y \to Z$ where $X, Y, Z$ are objects of the category $\mathcal{C}$. For any $a \in X$ and $b \in Y$ the functions

\[\varphi_a : Y \to Z, \quad \varphi_a : y \mapsto \varphi(a, y)\]

and

\[\varphi^b : X \to Z, \quad \varphi^b : x \mapsto \varphi(x, b),\]

are called the left and right shifts, respectively.

A binary operation $\varphi : X \times Y \to Z$ is called right-topological if for every $y \in Y$ the right shift $\varphi^y : X \to Z$, $\varphi^y : x \mapsto \varphi(x, y)$, is continuous. The topological center of a right-topological binary operation $\varphi : X \times Y \to Z$ is the set $\Lambda_\varphi$ of all elements $x \in X$ such that the left shift $\varphi_x : Y \to Z$ is continuous.

**Definition 2.1.** Let $\varphi : X \times Y \to Z$ be a binary operation in the category $\mathcal{C}$. A binary operation $\Phi : TX \times TY \to TZ$ is defined to be a $\mathbb{T}$-extension of $\varphi$ if
(1) $\Phi(\eta_X(x), \eta_Y(y)) = \eta_Z(\varphi(x, y))$ for any $x \in X$ and $y \in Y$;
(2) for every $b \in TY$ the right shift $\Phi^b : TX \to TZ$, $\Phi^b : x \mapsto \Phi(x, b)$, is a morphism of the free $T$-algebra $TZ$;
(3) for every $x \in X$ the left shift $\Phi_{\eta(x)} : TY \to TZ$, $\Phi_{\eta(x)} : y \mapsto \Phi(\eta(x), y)$, is a morphism of the free $T$-algebra $TX, TZ$.

This definition implies that for any binary operation $\varphi : X \times Y \to Z$ its $T$-extension $\Phi : TX \times TY \to TZ$ is a right-topological binary operation whose topological center $\Lambda_\Phi$ contains the set $\eta(X) \subset TX$.

**Theorem 2.2.** Let $\varphi : X \times Y \to Z$ be a binary operation in the category $\mathcal{C}$.

1. The binary operation $\varphi$ has at most one $T$-extension $\Phi : TX \times TY \to TZ$.
2. If $X, Y$ are discrete in $\mathcal{C}$, then $\varphi$ has a unique $T$-extension $\Phi : TX \times TY \to TZ$.

**Proof.** 1. Let $\Phi, \Psi : TX \times TY \to TZ$ be two $T$-extensions of the operation $\varphi$. By the condition (3) of Definition 2.1, for every $x \in X$ and $a = \eta_X(x) \in TX$, the left shifts $\Phi_a, \Psi_a : TY \to TZ$ are morphisms of the free $T$-algebras.

By the condition (1) of Definition 2.1

$$\Phi_a \circ \eta_Y = \eta_Z \circ \varphi_x = \Psi_a \circ \eta_Y.$$  

Then Lemma 1.1 implies that

$$\Phi_a = \mu \circ T(\Phi_a \circ \eta_X) = \mu \circ T(\eta_Z \circ \varphi_x) = \mu \circ T(\Psi_a \circ \eta_X) = \Psi_a.$$  

The equality $\Phi = \Psi$ will follow as soon as we check that $\Phi^b = \Psi^b$ for every $b \in TY$. Since $\Phi^b, \Psi^b : TX \to TZ$ are morphisms of the free $T$-algebras $TX$ and $TZ$, the equality $\Phi^b = \Psi^b$ follows from the equality

$$\Phi^b \circ \eta(x) = \Phi_{\eta(x)}(b) = \Psi_{\eta(x)}(b) = \Psi^b \circ \eta(x), \quad x \in X$$

according to Lemma 1.1.

2. Now assuming that the spaces $X, Y$ are discrete in $\mathcal{C}$, we show that the binary operation $\varphi : X \times Y \to Z$ has a $T$-extension. For every $x \in X$ consider the left shift $\varphi_x : Y \to Z$. Since $Y$ is discrete in $\mathcal{C}$, the function $\varphi_x$ is a morphism of the category $\mathcal{C}$. Applying the functor $T$ to this morphism, we get a morphism $T\varphi_x : TY \to TZ$. Now for every $b \in TY$ consider the function $\varphi^b : X \to TZ$, $\varphi^b : x \mapsto T\varphi_x(b)$. Since the object $X$ is discrete, the function $\varphi^b$ is a morphism of the category $\mathcal{C}$. Applying to this morphism the functor $T$, we get a morphism $T\varphi^b : TX \to T^2Z$. Composing this morphism with the multiplication $\mu : T^2Z \to TZ$ of the monad $T$, we get the function $T\varphi^b = \mu \circ T\varphi^b : TZ \to TZ$. Define a binary operation $\Phi : TX \times TY \to TZ$ letting $\Phi(a, b) = T\varphi^b(a)$ for $a \in TX$.

**Claim 2.3.** $\Phi(\eta(x), b) = T\varphi_x(b)$ for every $x \in X$ and $b \in TY$.

**Proof.** The commutativity of the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\varphi^b} & TZ \\
\eta \downarrow & & \downarrow \eta \\
TX & \xrightarrow{T\varphi^b} & T^2Z
\end{array}$$

implies the desired equality

$$\Phi(\eta(x), b) = \mu \circ T\varphi^b(\eta(x)) = \varphi^b(x) = T\varphi_x(b).$$

Now we shall prove that $\Phi$ is a $T$-extension of $\varphi$.

i) For every $x \in X$ and $y \in Y$ we need to prove the equality

$$\Phi(\eta_X(x), \eta_Y(y)) = \eta_Z \circ \varphi(x, y).$$

By Claim 2.3

$$\Phi(\eta_X(x), \eta_Y(y)) = T\varphi_x \circ \eta_Y(y) = \eta_Z \circ \varphi_x(y) = \eta_Z \circ \varphi(x, y).$$
The latter equality follows from the naturality of the transformation $\eta : \text{Id} \to T$.

ii) The definition of $\Phi$ implies that for every $b \in TY$ the right shift $\Phi^b = \mu_Z \circ T\varphi^b$ is a morphism of free $\mathcal{T}$-algebras, being the compositions of two morphisms $T\varphi^b : TX \to T^2Z$ and $\mu_Z : T^2Z \to TZ$ of free $\mathcal{T}$-algebras.

iii) Claim 2.3 guarantees that for every $x \in X$ the left shift $\Phi_{\eta(x)} = T\varphi_x : TY \to TZ$ is a morphism of the free $\mathcal{T}$-algebras.

Proposition 2.4. Let $\varphi : X \times Y \to Z$, $\psi : X' \times Y' \to Z'$ be two binary operations in $\mathcal{C}$, $\Phi : TX \times TY \to TZ$, $\Psi : TX' \times TY' \to TZ'$ be their $\mathcal{T}$-extensions, and $h_X : X \to X'$, $h_Y : Y \to Y'$, $h_Z : Z \to Z'$ be morphisms in $\mathcal{C}$. If $\psi(h_X \times h_Y) = h_Z \circ \phi$, then $T\Phi(Th_X \times Th_Y) = Th_Z \circ \Phi$.

Proof. Observe that for any $x \in X$ and $x' = h_X(x)$, the commutativity of the diagrams

$$
\begin{array}{ccc}
Y & \xrightarrow{\varphi_x} & Z \\
\downarrow{h_Y} & & \downarrow{h_Z} \\
Y' & \xrightarrow{\psi_{x'}} & Z'
\end{array}
\quad
\begin{array}{ccc}
TY & \xrightarrow{T\varphi_x} & TZ \\
\downarrow{T\psi_x} & & \downarrow{T\psi_{x'}} \\
TY' & \xrightarrow{Th_Z} & TZ'
\end{array}
$$

imply that $Th_Z \circ T\varphi_x(b) = T\psi_{x'}(b')$ for every $b \in TY$ and $b' = Th_Y(b) \in TY'$.

It follows from Lemma 2.3 that $\Phi_{\eta(x)} = T\varphi_x : TY \to TZ$ and $\Psi_{\eta(x')} = T\psi_{x'} : TY' \to TZ'$. Consequently,

$$Th_Z \circ \Phi^b(\eta(x)) = Th_Z \circ \Phi_{\eta(x)}(b) = Th_Z \circ T\varphi_x(b) = T\psi_{x'}(b') = \Psi_{\eta(x')}(b') = \Psi^b(\eta(x'))$$

and hence

$$Th_Z \circ \Phi^b \circ \eta = \Psi^b \circ \eta \circ h_X.$$ 

Applying the functor $T$ to this equality, we get

$$T^2h_Z \circ T(\Phi^b \circ \eta) = T(\Psi^b \circ \eta) \circ Th_X.$$ 

Since $\Phi^b : TX \to TZ$ and $\Psi^b : TX' \to TZ'$ are homomorphisms of the free $\mathcal{T}$-algebras, we can apply Lemma 1.1 and conclude that $\Phi^b = \mu \circ T(\Phi^b \circ \eta)$ and hence

$$Th_Z \circ \Phi^b = Th_Z \circ \mu_Z \circ T(\Phi^{b} \circ \eta) = \mu_Z' \circ T^2h_Z \circ T(\Phi^b \circ \eta) = \mu_Z' \circ T(\Psi^b \circ \eta) \circ Th_X = \Psi^b \circ Th_X.$$ 

Then for every $a \in TX$ we get

$$Th_Z \circ \Phi(a, b) = Th_Z \circ \Phi^b(a) = \Psi^b \circ Th_X(a, b) = \Psi(Th_X(a), Th_Y(b)).$$

3. Binary operations and tensor products

In this section we shall discuss the relation of $\mathcal{T}$-extensions to tensor products. The tensor product is a function $\otimes : TX \times TY \to T(X \times Y)$ defined for any objects $X, Y \in \mathcal{C}$ such that $X$ is discrete in $\mathcal{C}$.

For every $x \in X$ consider the embedding $i_x : Y \to X \times Y$, $i_x : y \mapsto (x, y)$. The embedding $i_x$ is a morphism of the category $\mathcal{C}$ because the constant map $c_x : Y \to \{x\} \subset X$ and the identity map $id : Y \to Y$ are morphisms of the category and $\mathcal{C}$ contains products of its objects. Applying the functor $T$ to the morphism $i_x$, we get a morphism $Ti_x : TY \to T(X \times Y)$ of the category $\mathcal{C}$. Next, for every $b \in TY$ consider the function $Ti^b_x : x \mapsto Ti_x(b)$. Since $X$ is discrete in $\mathcal{C}$, the function $Ti^b_x$ is a morphism of the category $\mathcal{C}$. Applying the functor $T$ to this morphism, we get a morphism $TTi^b_x : TX \to T^2(X \times Y)$. Composing this morphism with the multiplication $\mu : T^2(X \times Y) \to T(X \times Y)$ of the monad $\mathcal{T}$, we get the morphism $\otimes^b = \mu \circ TTi^b_x : TX \to T(X \times Y)$. Finally define the tensor product $\otimes : TX \times TY \to T(X \times Y)$ letting $a \otimes b = \otimes^b(a)$ for $a \in TX$.

The following proposition describes some basic properties of the tensor product. For monadic functors in the category $\text{Comp}$ of compact Hausdorff spaces those properties were established in [17, 3.4.2].
Proposition 3.1.\( (1) \) The diagram \( X \times Y \overset{T X \times T Y}{\longrightarrow} T(X \times Y) \) is commutative for any discrete object \( X \) and any object \( Y \) of \( \mathcal{C} \); \( (2) \) the tensor product is natural in the sense that for any morphisms \( h_X : X \to X' \), \( h_Y : Y \to Y' \) of \( \mathcal{C} \) with discrete \( X, Y \), the following diagram is commutative:
\[
\begin{array}{ccc}
T X \times T Y & \otimes & T(X \times Y) \\
\downarrow{\scriptstyle h_X \times h_Y} & & \downarrow{\scriptstyle T(h_X \times h_Y)} \\
T X' \times T Y' & \otimes & T(X' \times Y')
\end{array}
\]
\( (3) \) the tensor product is associative in the sense that for any discrete objects \( X, Y, Z \) of \( \mathcal{C} \) the diagram
\[
\begin{array}{ccc}
T X \times T Y \times T Z & \otimes \text{id} & T(X \times Y) \times T Z \\
\downarrow{\scriptstyle \text{id} \times \otimes} & & \downarrow{\scriptstyle \otimes} \\
T X \times T(Y \times Z) & \otimes & T(X \times Y \times Z)
\end{array}
\]
is commutative, which means that \( (a \otimes b) \otimes c = a \otimes (b \otimes c) \) for any \( a \in TX, b \in TY, c \in TZ \).

Proof. 1. Fix any \( y \in Y \) and consider the element \( b = \eta_Y(y) \in TY \). The definition of the right shift \( \otimes^b \) implies that the following diagram is commutative:
\[
\begin{array}{ccc}
X & \overset{T^b}{\longrightarrow} & T(X \times Y) \\
\downarrow{\scriptstyle \eta} & & \uparrow{\scriptstyle \mu} \\
T X & \overset{T T^b \otimes^b}{\longrightarrow} & T^2(X \times Y)
\end{array}
\]
Consequently, for every \( x \in X \) we get
\[
\eta(x) \otimes^b \eta(y) = \otimes^b \circ \eta(x) = T_{i_z}^b \circ \eta(x) = T_{i_z}^b \circ \eta(y) = \eta(i_z(y)) = \eta(x, y).
\]
The latter equality follows from the diagram
\[
\begin{array}{ccc}
Y & \overset{i_x}{\longrightarrow} & X \times Y \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
TY & \overset{T_{i_x}}{\longrightarrow} & T(X \times Y)
\end{array}
\]
whose commutativity follows from the naturality of the transformation \( \eta : \text{Id} \to T \).

2. Let \( h_X : X \to X' \) and \( h_Y : Y \to Y' \) be any functions between discrete objects of the category \( \mathcal{C} \). Let \( Z = X \times Y, Z' = X' \times Y' \) and \( h_Z = h_X \times h_Y : Z \to Z' \). Given any point \( b \in TY \), consider the element \( b' = h_Y(b) \in TY' \). The statement \( (2) \) will follow as soon as we check that \( T h_Z \circ \otimes^b = \otimes^{b'} \circ T h_X \). By Lemma \( 1.1 \) this equality will follow as soon as we check that \( T h_Z \circ \otimes^b \circ \eta_X = \otimes^{b'} \circ T h_X \circ \eta_X \). The last equality follows from the naturality of the transformation \( \eta : \text{Id} \to T \). As we know from the proof of the preceding item, \( \otimes^{b'} \circ \eta_{X'}(x') = T i_{x'}(b') \) for any \( x' \in X' \). For every \( x \in X \) and \( x' = h_X(x) \) we can apply the functor \( T \) to the commutative diagram
and obtain the equality $Th_Z \circ Ti_x = Ti_{x'} \circ Th_Y$ which implies the desired equality:

$$\otimes^b \circ \eta_X \circ h_X(x) = \otimes^b \circ \eta_{X'}(x') = Ti_{x'}(b') = Th_Z \circ Ti_x(b) = Th_Z \circ \otimes^b \circ \eta(x).$$

3. The proof of the associativity of the tensor product can be obtained by literal rewriting the proof of Proposition 3.4.2(4) of [17].

**Theorem 3.2.** Let $\varphi : X \times Y \to Z$ be a binary operation in the category $C$ and $\Phi : TX \times TY \to TZ$ be its $T$-extension. If $X$ is a discrete object in $C$, then $\Phi(a, b) = T\varphi(a \otimes b)$ for any elements $a \in TX$ and $b \in TY$.

**Proof.** Our assumptions on the category $C$ guarantee that the product $X \times Y$ is a discrete object of $C$ and hence $\varphi : X \times Y \to Z$ is a morphism of the category $C$. So, it is legal to consider the morphism $T\varphi : T(X \times Y) \to TZ$. We claim that the binary operation

$$\Psi : TX \times TY \to TZ, \quad \Psi(a, b) = T\varphi(a \otimes b),$$

is a $T$-extension of $\varphi$.

1. The first item of Definition 2.1 follows Proposition 3.1(1) and the naturality of the transformation $\eta : \text{Id} \to T$:

$$\Psi(\eta_X(x), \eta_Y(y)) = T\varphi(\eta_X(x) \otimes \eta_Y(y)) = T\varphi \circ \eta_{X \times Y}(x, y) = \eta_Z \circ \varphi(x, y).$$

2. For every $b \in TY$ the morphism

$$\Psi^b = T\varphi \circ \otimes^b = T\varphi \circ \mu \circ TTi^b$$

is a morphism of the free $T$-algebras $TX$ and $TZ$.

3. For every $x \in X$ we see that

$$\Psi_{\eta(x)}(b) = T\varphi(\otimes^b(\eta(x))) = T\varphi \circ \mu \circ TTi^b \circ \eta(x) = T\varphi \circ \mu \circ \eta \circ Ti^b(x) = T\varphi \circ Ti^b(x)$$

is a morphism of the free $T$-algebras $TY$ and $TZ$.

Thus $\Psi$ is a $T$-extension of the binary operation $\varphi$. By the Uniqueness Theorem 2.2(1), $\Psi$ coincides with $\Phi$ and hence $\Phi(a, b) = \Psi(a, b) = T\varphi(a \otimes b)$. $\square$

4. THE TOPOLOGICAL CENTER OF $T$-EXTENDED OPERATION

Definition 2.1 guarantees that for a binary operation $\varphi : X \times Y \to Z$ in $C$ any $T$-extension $\Phi : TX \times TY \to TZ$ of $\varphi$ is a right-topological operation whose topological center $\Lambda_\varphi$ contains the subset $\eta_X(X)$. In this section we shall find conditions on the functor $T$ and the space $X$ guaranteeing that the topological center $\Lambda_\Phi$ is dense in $TX$.

We shall say that the functor $T$ is *continuous* if for each compact Hausdorff space $K$ that belongs to the category $C$ and any object $Z$ of $C$ the map $T : \text{Mor}(K, Z) \to \text{Mor}(TK, TZ)$, $T : f \mapsto Tf$, is continuous with respect to the compact-open topology on the spaces of morphisms (which are continuous maps).

**Theorem 4.1.** Let $\varphi : X \times Y \to Z$ be a binary operation in $C$ and $\Phi : X \times Y \to Z$ be its $T$-extension. If the object $X$ is finite and discrete in $C$, $TX$ is locally compact and Hausdorff, and the functor $T$ is continuous, then the operation $\Phi$ is continuous.

**Proof.** Since the space $X$ is discrete, the condition (2) of Definition 2.1 implies that the map $\Phi_\eta : X \times TY \to TZ$, $\Phi_\eta : (x, b) \mapsto \Phi(\eta(x), b)$ is continuous. Since $X$ is finite, the induced map

$$\Phi_\eta^{(1)} : TY \to \text{Mor}(X, TZ), \quad \Phi_\eta^{(1)} : b \mapsto \Phi_\eta^b$$

where $\Phi_\eta^b : x \mapsto \Phi(\eta(x), b)$, is continuous. By the continuity of the functor $T$, the map $T : \text{Mor}(X, TZ) \to \text{Mor}(TX, T^2Z)$, $T : f \mapsto Tf$, is continuous and so is the composition $T \circ \Phi_\eta^{(1)} : TY \to \text{Mor}(TX, T^2Z)$. Since $TX$ is locally compact and Hausdorff, we can apply [9 3.4.8] and conclude that the map

$$T\Phi_\eta^{(1)} : TX \times TY \to T^2Z, \quad T\Phi_\eta^{(1)} : (a, b) \mapsto T\Phi_\eta^b(a)$$

is continuous and so is the composition $\Psi = \mu \circ T\Phi_\eta^{(1)} : TX \times TY \to TZ$. Using the Uniqueness Theorem 2.2(1), we can prove that $\Psi = \Phi$ and hence the binary operation $\Phi$ is continuous. $\square$
Let $X$ be an object of the category $C$. We say that an element $a \in FX$ has discrete (finite) support if there is a morphism $f : D \to X$ from a discrete (and finite) object $D$ of the category $C$ such that $a \in Ff(FD)$. By $T_dX$ (resp. $T_fX$) we denote the set of all elements $a \in TX$ that have discrete (finite) support. It is clear that $T_fX \subset T_dX \subset TX$.

**Theorem 4.2.** Let $\varphi : X \times Y \to Z$ be a binary operation and $\Phi : TX \times TY \to TZ$ be a $T$-extension of $\varphi$. If the functor $T$ is continuous, and for every finite discrete object $D$ of $C$ the space $TD$ is locally compact and Hausdorff, then the topological center $\Lambda_\varphi$ of the binary operation $\Phi$ contains the subspace $T_fX$ of $TX$.

*Proof.* We need to prove that for every $a \in T_fX$ the left shift $\Phi_a : TY \to TZ$, $\Phi_a : b \mapsto \Phi(a, b)$, is continuous. Since $a \in T_fX$, there is a finite discrete object $D$ of the category $C$ and a morphism $f : D \to X$ such that $a \in Ff(FD)$. Fix an element $d \in FD$ such that $a = Ff(d)$.

Consider the binary operations
\[
\psi : D \times Y \to Z, \quad \psi : (x, y) \mapsto \phi(f(x), y),
\]
and
\[
\Psi : TD \times TY \to TZ, \quad \Psi : (a, b) \mapsto \Phi(Ff(a), b).
\]
It can be shown that $\Psi$ is a $T$-extension of $\psi$.

By Theorem 4.1 the binary operation $\Psi$ is continuous. Consequently, the left shift $\Psi_d : TY \to TZ$, $\Psi_d : b \mapsto \Psi(d, b)$, is continuous. Since $\Psi_d = \Phi_a$, the left shift $\Phi_a$ is continuous too and hence $a \in \Lambda_\varphi$. □

### 5. The associativity of $T$-extensions

In this section we investigate the associativity of the $T$-extensions. We recall that a binary operation $\varphi : X \times X \to X$ is associative if $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ for any $x, y, z \in X$. In this case we say that $X$ is a semigroup.

A subset $A$ of a set $X$ endowed with a binary operation $\varphi : X \times X \to X$ is called a subsemigroup of $X$ if $\varphi(A \times A) \subset A$ and $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ for all $x, y, z \in A$.

**Lemma 5.1.** Let $\varphi : X \times X \to X$ be an associative operation in $C$ and $\Phi : TX \times TX \to TX$ be its $T$-extension.

1. For any morphisms $f_A : A \to X$, $f_B : B \to X$ from discrete objects $A, B$ in $C$, the map $\varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X$ is a morphism of $C$ such that $\Phi(Tf_A(a), Tf_B(b)) = T\varphi_{AB}(a \otimes b)$ for all $a \in TA$ and $b \in TB$;
2. $\Phi(T_dX \times T_fX) \subset T_dX$ and $\Phi(T_fX \times T_fX) \subset T_fX$;
3. $\Phi((a, b), c) = \Phi(a, \Phi(b, c))$ for any $a, b, c \in T_dX$.

*Proof.* 1. Let $f_A : A \to X$, $f_B : B \to X$ be morphisms from discrete objects $A, B$ of $C$ and $\varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X$. By our assumption on the category $C$, the product $A \times B$ is a discrete object in $C$ and hence $\varphi_{AB}$ is a morphism in $C$. Consider the binary operation $\Phi_{AB} : TA \times TB \to TX$ defined by $\Phi_{AB}(a, b) = \Phi(Tf_A(a), Tf_B(b))$. The following diagram implies that $\Phi_{AB}$ is a $T$-extension of $\varphi_{AB}$:

![Diagram](image-url)

By Theorem 3.2
\[
\Phi(Tf_A(a), Tf_B(b)) = \Phi_{AB}(a, b) = T\varphi_{AB}(a \otimes b)
\]
for all \( a \in TA \) and \( b \in TB \).

2. Given elements \( a, b \in T_{\delta} X \), we need to show that the element \( \Phi(a, b) \in TX \) has discrete support. Find discrete objects \( A, B \) in \( C \) and morphisms \( f_A : A \to X \), \( f_B : B \to X \) such that \( a \in f_A(TA) \) and \( b \in f_B(TB) \). Fix elements \( \tilde{a} \in FA, \tilde{b} \in FB \) such that \( a = f_A(\tilde{a}) \) and \( b = f_B(\tilde{b}) \). Our assumption on the category \( C \) guarantees that \( A \times B \) is a discrete object in \( C \).

Consider the binary operations \( \psi : A \times B \to X \) and \( \Psi : FA \times FB \to FZ \) defined by the formulae

\[
\psi = \phi \circ (f_A \times f_B) \quad \text{and} \quad \Psi = \Phi \circ (Tf_A \times Tf_B).
\]

Let \( \tilde{c} = \tilde{a} \otimes \tilde{b} \in T(A \times B) \). By the first statement, \( \Phi(a, b) = T\psi(\tilde{a} \otimes \tilde{b}) = T\psi(\tilde{c}) \in T\psi(A \times B) \) witnessing that the element \( \Phi(a, b) \) has discrete support and hence belongs to \( T_{\delta} X \).

By analogy, we can prove that \( \Phi(T_{\delta} X \times T_{\delta} X) \subseteq T_{\delta} X \).

3. Given any points \( a, b, c \in T_{\delta} X \), we need to check the equality

\[
\Phi(\Phi(a, b), c) = \Phi(a, \Phi(b, c)).
\]

Find discrete objects \( A, B, C \) in \( C \) and morphisms \( f_A : A \to X \), \( f_B : B \to X \), \( f_C : C \to X \) such that \( a \in Tf_A(TA) \), \( b \in Tf_B(TB) \), and \( c \in Tf_C(TC) \). Fix elements \( \tilde{a} \in TA, \tilde{b} \in TB, \) and \( \tilde{c} \in TC \) such that \( a = Tf_A(\tilde{a}), b = Tf_B(\tilde{b}), \) and \( c = Tf_C(\tilde{c}) \).

Consider the morphisms \( \varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X, \varphi_{BC} = \varphi(f_B \times f_C) : B \times C \to X \) and \( \varphi_{ABC} = \varphi(\varphi_{AB} \times f_C) = \varphi(f_A \times \varphi_{BC}) : A \times B \times C \to X \). Consider the following diagram:

\[
\begin{array}{ccc}
TX \times TX \times TX & \xrightarrow{\Phi \times \text{id}} & TX \times TX \\
\downarrow{Tf_A \times Tf_B \times Tf_C} & & \downarrow{T\varphi_{AB} \times Tf_C} \\
TA \times TB \times TC \xrightarrow{\otimes \times \text{id}} (A \times B \times C) \times (A \times B \times C) & \xrightarrow{\otimes} & T(A \times B \times C) \\
\downarrow{\text{id} \times \Phi} & & \downarrow{\Phi} \\
TX \times TX \xrightarrow{\otimes \text{id} \times T\varphi_{BC}} & \xrightarrow{T\varphi_{ABC}} & TX \\
\end{array}
\]

In this diagram the central square is commutative because of the associativity of the tensor product \( \otimes \). By the item (1) all four margin squares also are commutative. Now we see that

\[
\begin{align*}
\Phi(\Phi(a, b), c) &= \Phi(\Phi(Tf_A(\tilde{a}), Tf_B(\tilde{b})), Tf_C(\tilde{c})) \\
&= \Phi(T\varphi_{AB}(\tilde{a} \otimes \tilde{b}), Tf_C(\tilde{c})) \\
&= T\varphi_{ABC}((\tilde{a} \otimes \tilde{b}) \otimes \tilde{c}) = T\varphi_{ABC}(\tilde{a} \otimes (\tilde{b} \otimes \tilde{c})) \\
&= \Phi(Tf_A(\tilde{a}), \Phi(Tf_B(\tilde{b}), Tf_C(\tilde{c}))) = \Phi(a, \Phi(b, c)).
\end{align*}
\]

Combining Lemma 5.1 with Theorem 4.2, we get the main result of this paper:

**Theorem 5.2.** Assume that the monadic functor \( T \) is continuous and for each finite discrete space \( F \) in \( C \) the space \( TF \) is Hausdorff and locally compact. Let \( \varphi : X \times X \to X \) be an associative binary operation in \( C \) and \( \Phi : X \times X \to X \) be its \( \mathbb{T} \)-extension. If the set \( T_{\delta} X \) of elements with finite support is dense in \( TX \), then the operation \( \Phi \) is associative.

**Proof.** By Theorem 4.2 the set \( T_{\delta} X \) lies in the topological center \( \Lambda_{\Phi} \) of the operation \( \Phi \) and by Lemma 5.1 \( T_{\delta} X \) is a subsemigroup of \( (TX, \Phi) \). Now the associativity of \( \Phi \) follows from the following general fact. \( \square \)

**Proposition 5.3.** A right topological operation \( \cdot : X \times X \to X \) on a Hausdorff space \( X \) is associative if its topological center contains a dense subsemigroup \( S \) of \( X \).

**Proof.** Assume conversely that \( (xy)z \neq x(yz) \) for some points \( x, y, z \in X \). Since \( X \) is Hausdorff, the points \( (xy)z \) and \( x(yz) \) have disjoint open neighborhoods \( O((xy)z) \) and \( O(x(yz)) \) in \( X \). Since the right shifts in \( X \) are continuous, there are open neighborhoods \( O(xy) \) and \( O(x) \) of the points \( xy \) and \( x \) such that
O(xy) · z ∈ O((xy)z) and O(x) · (yz) ∈ O(xy). We can assume that O(x) is so small that O(x) · y ⊂ O(xy).
Take any point a ∈ O(x) ∩ S. It follows that a(yz) ∈ O(x(yz)) and ay ∈ O(xy).
Since the left shift \( l_a : \beta S \to \beta S, l_a : y \mapsto ay \), is continuous, the points yz and y have open neighborhoods \( O(yz) \) and \( O(y) \) such that \( a \cdot O(yz) \subset O(xy) \) and \( a \cdot O(y) \subset O(xy) \).
We can assume that the neighborhood \( O(y) \) is so small that \( O(y) \cdot z \subset O(yz) \).
Choose a point \( b \in O(y) \cap S \) and observe that \( b \cdot O(z) \subset O(yz) \) and \( ab \cdot O(z) \subset O(xy) \).
Finally take any point \( c \in S \cap O(z) \). Then \( (ab)c = ab \cdot O(z) \subset O((xy)z) \) and \( abc \subset a \cdot O(yz) \subset O(x(yz)) \) belong to disjoint sets, which is not possible as \( (ab)c = a(bc) \).

\[ \square \]

6. \( \mathcal{T} \)-extension for some concrete monadic functors

In this section we consider some examples of monadic functors in topological categories. Let \( \mathcal{Tych} \)

denote the category of Tychonov spaces and their continuous maps and \( \mathbf{Comp} \) be the full subcategory of

the category \( \mathcal{Tych} \), consisting of compact Hausdorff spaces.

Discrete objects in the category \( \mathcal{Tych} \) are discrete topological spaces while discrete objects in the category

\( \mathbf{Comp} \) are finite discrete spaces.

Consider the functor \( \beta : \mathcal{Tych} \to \mathbf{Comp} \) assigning to each Tychonov space \( X \) its Stone-


dech compactification and to a continuous map \( f : X \to Y \) between Tychonov spaces its continuous extension \( \beta f : \beta X \to \beta Y \).
The functor \( \beta \) can be completed to a monad \( \mathbb{T}_\beta = (\beta, \eta, \mu) \) where \( \eta : X \to \beta X \) is the canonical embedding

\( \eta : X \to \beta X \) is the canonical embedding and \( \mu : \beta(\beta X) \to \beta X \) is the identity map.

A pair \( (X, \xi) \) is a \( \mathbb{T}_\beta \)-algebra if and only if \( X \) is a compact space and \( \xi : \beta X \to X \) is the identity map.

Combining Theorems 2.1, 2.2, 5.2 we get the following well-known

**Corollary 6.1.** Each binary right-topological operation \( \phi : X \times Y \to Z \) in \( \mathcal{Tych} \) with discrete \( X \) can be extended to a right-topological operation \( \Phi : \beta X \times \beta Y \to \beta Z \) containing \( X \) in its topological center \( \Lambda_\Phi \).

If \( X = Y = Z \) and the operation \( \phi \) is associative, then so is the operation \( \Phi \).

Now let \( \mathbb{T} = (T, \eta, \mu) \) be a monad in the category \( \mathbf{Comp} \). Taking the composition of the functors

\( \beta : \mathcal{Tych} \to \mathbf{Comp} \) and \( T : \mathbf{Comp} \to \mathbf{Comp} \), we obtain a monadic functor \( T\beta : \mathcal{Tych} \to \mathbf{Comp} \).

**Theorem 6.2.** Each binary right-topological operation \( \phi : X \times Y \to Z \) in the category \( \mathcal{Tych} \) with discrete \( X \) can be extended to a right-topological operation \( \Phi : T\beta X \times T\beta Y \to T\beta Z \) that contain the set \( \eta(X) \subset \beta X \) in its topological center \( \Lambda_\Phi \).

If the functor \( T \) is continuous, then the set \( T\beta X \) of elements \( a \in T\beta X \) with finite support is dense in \( T\beta X \) and lies in the topological center \( \Lambda_\Phi \) of the operation \( \Phi \).

Moreover, if \( X = Y = Z \) and the operation \( \phi \) is associative, then so is the operation \( \Phi \).

**Proof.** By Theorem 2.2 the binary operation \( \phi \) has a unique \( \mathbb{T} \)-extension \( \Phi : TX \times TY \to TZ \). By Definition 2.1 the set \( \eta(X) \subset \beta X \) lies in the topological center \( \Lambda_\phi \) of \( \phi \).

Now assume that the functor \( T \) is continuous. First we show that the set \( T\beta X \) is dense in \( T\beta X \).
Fix any point \( a \in F\beta X \) and an open neighborhood \( U \subset T\beta X \) of \( a \).
Then \( [a, U] = \{ f \in \text{Mor}(F\beta X, F\beta X) : f(a) \in U \} \)
is an open neighborhood of the identity map \( id : F\beta X \to F\beta X \) in the function space \( \text{Mor}(F\beta X, F\beta X) \)
edowed with the compact-open topology. The continuity of the functor \( T \) yields a neighborhood \( U(id_{\beta X}) \) of the identity map \( id_{\beta X} \in \text{Mor}(\beta X, \beta X) \) such that \( Tf \in [a, U] \) for any \( f \in U(id_{\beta X}) \).
It follows from the definition of the compact-open topology, that there is an open cover \( U \) of \( \beta X \) such that a map \( f : \beta X \to \beta X \) belongs to \( U(id_{\beta X}) \) if \( f \) is \( U \)-near to \( id_{\beta X} \) in the sense that for every \( x \in \beta X \) there is a set \( U \subset U \) with \( \{ x, f(x) \} \subset U \). Since \( \beta X \) is compact, we can assume that the cover \( U \) is finite.
Since \( X \) is discrete, the space \( \beta X \) has covering dimension zero \([9, 7.1.17]\). So, we can assume that the finite cover \( U \) is disjoint.
For every \( U \subset U \) choose an element \( x_U \in U \cap X \). Those elements compose a finite discrete subspace \( A = \{ x_U : U \subset U \} \) of \( X \).
Let \( i : A \to X \) be the identity embedding and \( f : X \to A \) be the map defined by \( f^{-1}(x_U) = U \) for \( U \subset U \).
It follows that \( i \circ f \in U(id_{\beta X}) \) and thus \( T(i \circ f) \in [a, U] \) and \( Ti \circ Tf(a) \in U \).
Now we see that \( b = Tf(a) \in TA \) and \( c = Ti(b) \in T\beta X \cap U \), so \( T\beta X \) is dense in \( \beta X \).

By Theorem 1.2 the set \( T\beta X \) lies in the topological center \( \Lambda_\Phi \) of \( \Phi \).

Now assume that the operation \( \phi \) is associative. By Lemma 5.1 \( T\beta X \) is a subsemigroup of \( (X, \Phi) \).
Since \( T\beta X \) is dense and lies in the topological center \( \Lambda_\Phi \), we may derive the associativity of \( \Phi \) from Proposition 5.3.

\[ \square \]
Problem 6.3. Given a discrete semigroup $X$ investigate the algebraic and topological properties of the compact right-topological semigroup $T\beta X$ for some concrete continuous monadic functors $T : \text{Comp} \to \text{Comp}$.

This problem was addressed in \cite{9}, \cite{10} for the monadic functor $G$ of inclusion hyperspaces, in \cite{2}–\cite{5} for the functor of superextension $\lambda$, in \cite{1}, \cite{12}, \cite{15} for the functor $P$ of probability measures and in \cite{6}, \cite{7}, \cite{8}, \cite{18} for the hyperspace functor $\exp$.

In \cite{19} it was shown that for each continuous monadic functor $T : \text{Comp} \to \text{Comp}$ any continuous (associative) operation $\varphi : X \times Y \to Z$ in $\text{Comp}$ extends to a continuous (associative) operation $\Phi : TX \times TY \to TZ$.

Problem 6.4. For which monads $T = (T, \eta, \mu)$ in the category $\text{Comp}$ each right-topological (associative) binary operation $\varphi : X \times Y \to Z$ in $\text{Comp}$ extends to a right-topological (associative) binary operation $\Phi : TX \times TY \to TZ$? Are all such monads power monads?

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\end{enumerate}

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