Linear stability of black holes with static scalar hair in full Horndeski theories: generic instabilities and surviving models

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In full Horndeski theories, we show that the static and spherically symmetric black hole (BH) solutions with a static scalar field $\phi$ whose kinetic term $X$ is nonvanishing on the BH horizon are generically prone to ghost/Laplacian instabilities. We then search for asymptotically Minkowski hairy BH solutions with a vanishing $X$ on the horizon free from ghost/Laplacian instabilities. We find that models with regular coupling functions of $\phi$ and $X$ result in no-hair Schwarzschild BHs in general. On the other hand, the presence of a coupling between the scalar field and the Gauss-Bonnet (GB) term $R^2_{GB}$ even with the coexistence of other regular coupling functions, leads to the realization of asymptotically Minkowski hairy BH solutions without ghost/Laplacian instabilities. Finally, we find that MB hairy BH solutions in power-law $F(R_{GB})$ gravity are plagued by ghost instabilities. These results imply that the GB coupling of the form $\xi(\phi)R^2_{GB}$ plays a prominent role for the existence of asymptotically Minkowski hairy BH solutions free from ghost/Laplacian instabilities.

I. INTRODUCTION

General Relativity (GR) has been tested by numerous experiments in the Solar System. While gravity can be well described by GR on the weak gravitational background in our local Universe [1], the dawn of gravitational-wave (GW) astronomy [2] and black hole (BH) shadow measurements [3] have started to allow us to probe the physics of extremely compact objects like BHs and neutron stars [4,7]. On the other hand, we also know that the Universe recently entered a phase of accelerated expansion [8,9]. While the cosmological constant is the simplest candidate for accelerations [25–30], although such a new scalar degree of freedom potentially manifests itself in the Solar System, fifth forces mediated by the scalar field can be screened [31–34] by Vainshtein [33] or chameleon [36] mechanisms around a compact body on the weak gravitational background. In the vicinity of a BH, on the other hand, a nonvanishing charge of the scalar field, i.e., scalar hair, gives rise to a nontrivial field profile affecting the background geometry. This offers an interesting possibility for probing the possible deviation from GR in strong gravity regimes.

In GR, the vacuum, asymptotically flat, static, and spherically symmetric solution is uniquely characterized by the Schwarzschild geometry containing the mass of a compact body [2]. The search for hairy BH solutions endowed with nontrivial field profiles has been performed for several subclasses of Horndeski theories. It has been recognized that there is no scalar hair for a minimally coupled, canonical scalar field [32,33] and k-essence [40] as well as for a nonminimally coupled scalar field with the Ricci scalar of the form $G_4(\phi)R$ [41–44]. If the scalar field is coupled to a Gauss-Bonnet (GB) curvature invariant $R^2_{GB}$ [see Eq. (5.1)] as $\xi(\phi)R^2_{GB}$ [45–47], where $\xi(\phi)$ is a function of $\phi$, there are asymptotically Minkowski hairy BH solutions for the dilatonic coupling $\xi(\phi) \sim e^{-\phi}$ [48,58], linear coupling $\xi(\phi) \propto \phi$ [59,60], and models for spontaneous scalarization of BHs where $\xi(\phi) \propto \sum_{j \geq 1} c_j \phi^j$ with $c_j$...
being constant \(61, 68\). The nonminimal derivative coupling \(\phi G_{\mu\nu} \nabla^\mu \nabla^\nu \phi\) to the Einstein tensor \(G_{\mu\nu}\), where \(\nabla\) is the covariant derivative operator, gives rise to non-asymptotically Minkowski BH solutions \(37, 69, 72, 73\) with the static background scalar field, but it was recently recognized that they are unstable against linear perturbations \(74\).

The purpose of this paper is to elucidate the class of Horndeski theories giving rise to asymptotically Minkowski BH solutions endowed with scalar hair which are free from ghost or Laplacian instabilities. We will focus on a time-independent background scalar field on the static and spherically symmetric background. In this case, we can exploit conditions for the absence of ghost/Laplacian instabilities against odd- and even-parity perturbations derived in Refs. \(75\). In shift-symmetric Horndeski theories where the coupling functions \(G_{2,3,4,5}\) depend on the canonical kinetic term \(X\) only, Hui and Nicolis \(78\) showed the absence of asymptotically Minkowski hairy BH solutions under several assumptions based on the properties of a conserved Noether current associated with the shift symmetry (see also Ref. \(74\) for a detailed review). Such a no-hair argument based on the Noether current cannot be applied to full Horndeski theories containing both \(\phi\)- and \(X\)-dependence in the coupling functions and hence breaking the shift symmetry. A systematic approach to the search for hairy BHs in such more general shift-symmetry-breaking Horndeski theories would be challenging.

Our approach is first to show that the BH solutions where \(X\) is an analytic function with a nonvanishing value on the horizon (\(X_s \neq 0\)) can be excluded by linear instability. This is a generalization of the result recognized for shift-symmetric Horndeski theories \(74\). We then search for hairy BHs with \(X_s = 0\). Assuming that the deviation from GR with a canonical scalar field is controlled by a single coupling constant, we perform consistent expansions of the metric and scalar field in terms of the coupling constant. Imposing that the metric is asymptotically Minkowski together with a vanishing radial scalar-field derivative as the boundary condition at spatial infinity, we will show that couplings of the form \(G_I \supset \alpha_I(\phi)F_I(X)\) with \(I = 2, 3, 4, 5\), where \(\alpha_I(\phi)\) and \(F_I(X)\) are analytic functions of \(\phi\) and \(X\) respectively, do not generally give rise to hairy asymptotically Minkowski BH solutions \(6\).

As an extension of the results in shift-symmetric Horndeski theories \(53, 92\), there is a possibility for evading the no-hair property of BHs in non-shift-symmetric theories with a particular choice of the coupling functions \(G_2 \supset \alpha_2(\phi)\sqrt{-X}, G_3 \supset \alpha_3(\phi)\ln |X|, G_4 \supset \alpha_4(\phi)\sqrt{-X},\) and \(G_5 \supset \alpha_5(\phi)\ln |X|\), which are no longer analytic in \(X\). Among them, however, we will see that only the quintic coupling \(G_5 \supset \alpha_5(\phi)\ln |X|\) allows us for realizing asymptotically Minkowski hairy BHs free from ghost or Laplacian instabilities. In particular, the scalar field coupled to the GB term of the form \(\alpha_5(\phi)R_{\text{GB}}^2\) can be embedded in Horndeski theories, where the corresponding action contains the quintic coupling of this type. For the GB coupling, we will derive hairy BH solutions by using expansions with respect to a small dimensionless coupling constant. Furthermore, we will show that they can satisfy all the conditions for the absence of ghost/Laplacian instabilities against odd- and even-parity perturbations. We note that, in Refs. \(64, 95, 93–98\), the linear stability of hairy BHs for scalar-GB theories has also been investigated.

In the presence of positive power-law functions of \(\phi\) or \(X\) in \(G_{2,3,4,5}\) besides the GB couplings, we also find new classes of BH solutions endowed with additional hair which are free from ghost or Laplacian instabilities. Since all these hairy BH solutions disappear without the coupling \(\alpha_5(\phi)\ln |X| \supset G_5\), the presence of such a logarithmic quintic interaction is crucial for realizing asymptotically Minkowski BHs with scalar hair. We will also show that \(F(R_{\text{GB}}^2)\) gravity \(94, 102\) rewritten in terms of Horndeski theories gives rise to hairy BHs, which, however, are subject to a ghost instability of even-parity perturbations. The extension of this \(F(R_{\text{GB}}^2)\)-equivalent Horndeski theory to that containing a canonical kinetic term of the scalar field leads to no-hair asymptotically Minkowski BH solutions.

The rest of this paper is organized as follows. In Sec. \(11\) we present the background equations of motion and conditions for the absence of ghost/Laplacian instabilities for static and spherically symmetric BHs in full Horndeski theories. In Sec. \(11\) we prove the generic instability of BHs for theories in which \(X\) is analytic and has a nonvanishing value on the horizon. In Sec. \(14\) we show the absence of hairy asymptotically Minkowski BHs for theories containing coupling functions \(G_I \supset \alpha_I(\phi)F_I(X)\), where \(\alpha_I(\phi)\) and \(F_I(X)\) are analytic functions of \(\phi\) and \(X\) respectively. Then, we search for the possibility for realizing hairy BH solutions in full Horndeski theories. In Sec. \(17\) we investigate the existence of hairy BHs and the issues of ghost/Laplacian instabilities for the GB coupling \(\alpha_5(\phi)R_{\text{GB}}^2\). We then extend the analysis to theories in which positive power-law functions of \(\phi\) or \(X\) in \(G_{2,3,4,5}\) are present besides the GB couplings. In Sec. \(18\) we study BH solutions in \(F(R_{\text{GB}}^2)\) gravity and its extensions. Section \(18\) is devoted to conclusions.

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3 We note that shift-symmetric theories admit asymptotically Minkowski BH solutions with a time-dependent background scalar field \(\phi = qt + \Phi(r)\) \(79, 81\), which we will not address in this paper.

4 Here, \(G_I \supset \alpha_I(\phi)F_I(X)\) means that \(G_I\) contains only the term \(\alpha_I(\phi)F_I(X)\) besides the canonical kinetic term of the scalar field in \(G_2\) and the Einstein-Hilbert term in \(G_4\). Likewise, in the present paper, we use the symbol “\(\supset\)” when we incorporate additional terms on top of those of primary interest (i.e., the canonical kinetic term of the scalar field and the Einstein-Hilbert term in Sec. \(14\) and the scalar-GB coupling \(\alpha_5(\phi)R_{\text{GB}}^2\) in Sec. \(17\)).
II. BACKGROUND EQUATIONS AND LINEAR STABILITY CONDITIONS

We consider full Horndeski theories \[^{[14]}^{[17]}\] given by the action

\[
S = \int d^4x \sqrt{-g} \mathcal{L}_H,
\]

(2.1)

where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \) and

\[
\mathcal{L}_H = G_2(\phi, X) - G_3(\phi, X) \nabla^2 + G_4(\phi, X) R + G_{4,x}(\phi, X) \left[ (\nabla^2 \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \right] + G_5(\phi, X) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi
\]

(2.2)

\[
- \frac{1}{6} G_{5,XX}(\phi, X) \left[ (\nabla^2 \phi)^3 - 3(\nabla^2 \phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) + 2(\nabla^\mu \nabla_\alpha \phi)(\nabla^\alpha \nabla_\beta \phi)(\nabla^\gamma \nabla_\gamma \phi) \right],
\]

with \( R \) and \( G_{\mu\nu} \) being the Ricci scalar and Einstein tensor associated with the metric \( g_{\mu\nu} \), respectively. The four functions \( G_I \)’s \((I = 2, 3, 4, 5)\) depend on the scalar field \( \phi \) and its canonical kinetic term \( X = -g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi / 2 \). We also use the notations \( \nabla^2 \equiv \nabla_\mu \nabla^\mu \) and \( G_{I,\phi} \equiv \partial G_I / \partial \phi \), \( G_{I,X} \equiv \partial G_I / \partial X \), \( G_{I,\phi X} \equiv \partial^2 G_I / (\partial X \partial \phi) \), etc.

We assume a static and spherically symmetric background metric and scalar field

\[
ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(2.3)

\[\phi = \phi(r),\]

(2.4)

where \( f(r) \), \( h(r) \), and \( \phi(r) \) are functions of the radial coordinate \( r \). On this background, the scalar-field kinetic term is given by \( X = -h^2 \phi'^2 / 2 \), where the prime represents the derivative with respect to \( r \). The \( tt \), \( rr \), \( \theta\theta \)-components of gravitational field equations are given, respectively, by \(^{[75]}^{[77]}\)

\[
\mathcal{E}_{tt} \equiv \left( A_1 + \frac{A_2}{r} + \frac{A_3}{r^2} \right) \phi'' + \left( \frac{\phi'}{2h} A_1 + \frac{A_4}{r} + \frac{A_5}{r^2} \right) h' + A_6 + \frac{A_7}{r} + \frac{A_8}{r^2} = 0,
\]

(2.5)

\[
\mathcal{E}_{rr} \equiv - \left( \frac{\phi'}{2h} A_1 + \frac{A_4}{r} + \frac{A_5}{r^2} \right) h f'' + A_9 - \frac{2\phi'}{r} A_1 - \frac{1}{r^2} \left[ \frac{\phi'}{2h} A_2 + (h - 1) A_4 \right] = 0,
\]

(2.6)

\[
\mathcal{E}_{\theta\theta} \equiv \left\{ \left[ A_2 + \frac{2h(2h - 1) A_2 - \phi' A_3}{2h^2} \right] f'' + \left( \frac{A_7}{4} + \frac{A_8}{r} \right) f' + \left( \frac{\phi'}{h} A_1 + \frac{A_7}{r} \right) f' + \left( \frac{\phi'}{h} A_1 + \frac{A_7}{r} \right) f' + \left( \frac{\phi'}{h} A_1 + \frac{A_7}{r} \right) f' \right\} = 0,
\]

(2.7)

where

\[
\begin{align*}
A_1 &= -h^2(G_{3,X} - 2G_{4,\phi X})\phi'^2 - 2G_{4,\phi} h, \\
A_2 &= 2h^2(2G_{4,XX} - G_{5,\phi X})\phi'^3 - 4h^2(G_{4,X} - G_{5,\phi})\phi', \\
A_3 &= -h^4G_{5,XX}\phi'^4 + h^2G_{5,X}(3h - 1)\phi'^2, \\
A_4 &= h^2(2G_{4,XX} - G_{5,\phi X})\phi'^4 + h(3G_{5,\phi} - 4G_{4,X})\phi'^2 - 2G_4, \\
A_5 &= -\frac{1}{2} \left[ G_{5,XX} h^3 \phi'^5 - h G_{5,X} (5h - 1) \phi'^3 \right], \\
A_6 &= h(G_{3,\phi} - 2G_{4,\phi})\phi'^2 + G_2, \\
A_7 &= -2h^2(2G_{4,XX} - G_{5,\phi X})\phi'^3 - 4G_{4,\phi} h \phi', \\
A_8 &= G_{5,XX} h^3 \phi'^4 - h(2G_{4,X} h - G_{5,\phi} h - G_{5,\phi}) \phi'^2 - 2G_4(h - 1), \\
A_9 &= -h(G_{2,X} - G_{3,\phi})\phi'^2 - G_2, \\
A_{10} &= \frac{1}{2} G_{5,XX} h^2 \phi'^4 - \frac{1}{2} h^2(2G_{4,X} - G_{5,\phi})\phi'^2 - G_4 \phi.
\end{align*}
\]

(2.8)

Varying the action \[^{[2.7]}^{[1]}\] with respect to \( \phi \), it follows that

\[
\frac{1}{r^2} \sqrt{\frac{h}{f}} \left( r^2 \sqrt{\frac{f}{h}} J' \right)' + P_\phi = 0,
\]

(2.9)

where

\[
J' = h \phi' \left[ G_{2,X} - \left( \frac{2}{r} + \frac{f'}{2f} \right) h \phi' G_{3,X} + 2 \left( \frac{1 - h}{r^2} - \frac{hf'}{rf} \right) G_{4,X} + 2h \phi'^2 \left( \frac{h}{r^2} + \frac{hf'}{rf} \right) G_{4,XX} \right].
\]
The coefficients $\lambda_1$–$\lambda_{12}$ in $P_\phi$ are given in Appendix A. We note that Eq. (2.9) is equivalent to the following equation:

$$E_\phi = -\frac{2}{\phi'} \left[ \frac{f'}{2f} E_{tt} + E'_{rr} + \left( \frac{f'}{2f} + \frac{2}{r} \right) E_{rr} + \frac{2}{r} E_{\theta\theta} \right] = 0.$$  \label{eq:Ephi}

This shows that the scalar-field equation is not independent of other Eqs. (2.5)–(2.7), which is always the case for theories with general covariance \cite{104}.

In Refs. \cite{75 76}, the odd- and even-parity perturbation theories about the static and spherically-symmetric solutions have been formulated in full Horndeski theories. In the following, we will briefly summarize conditions for the absence of ghost/Laplacian instabilities derived in these papers (see also Ref. \cite{74} for a brief summary of linear stability conditions in shift-symmetric Horndeski theories). Readers who are interested in the derivation of them should refer to Refs. \cite{75 77}.

The stability against odd-parity perturbations is ensured under the following three conditions \cite{77}:

$$\mathcal{F} \equiv 2G_4 + h\phi'^2 G_{5,\phi} - h\phi'^2 \left( \frac{1}{2} h' \phi' + h\phi'' \right) G_{5,X} > 0, \quad \mathcal{G} \equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 \left( G_{5,\phi} + \frac{f'h\phi' G_{5,X}}{2f} \right) > 0, \quad \mathcal{H} \equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 G_{5,\phi} - \frac{h^2\phi'^2 G_{5,X}}{r} > 0.$$  \label{eq:conditions}

The ghost is absent under the inequality (2.14). The squared propagation speeds of odd-parity perturbations along the radial and angular directions are given, respectively, by

$$c^2_{r,\text{odd}} = \frac{\mathcal{G}}{\mathcal{F}}, \quad c^2_{\theta,\text{odd}} = \frac{\mathcal{G}}{\mathcal{H}},$$  \label{eq:prop speeds odd}

which are both positive under the conditions (2.13)–(2.15).

In the even-parity sector, the no-ghost condition is quantified as \cite{76}

$$K \equiv 2\mathcal{P}_1 - \mathcal{F} > 0,$$  \label{eq:K cond}

with

$$\mathcal{P}_1 \equiv \frac{h\mu}{2fr^2 h^2} \left( \frac{fr^2 h^2}{\mu^2 h} \right)' \quad \text{and} \quad \mu \equiv \frac{2(\phi' a_1 + r \sqrt{f h} \mathcal{H})}{\sqrt{f h}}.$$  \label{eq:P1 def}

where $a_1$ is given in Appendix A. For the multipoles $\ell \geq 2$, the even-parity sector consists of two dynamical degrees of freedom. One is the perturbation of the scalar field $\delta \phi$, while the other, which we denote by $\psi$, can be regarded as the gravitational perturbation (see Refs. \cite{76 77 103} for the definition of $\psi$). In the limit of high frequencies, the conditions for the absence of Laplacian instabilities of $\psi$ and $\delta \phi$ along the radial direction are given, respectively, by

$$c^2_{r,\text{even}} = \frac{\mathcal{G}}{\mathcal{F}} > 0, \quad c^2_{\phi,\text{even}} = \frac{2\phi'[4r^2 (fh)^{3/2} h c_4(2\phi' a_1 + r \sqrt{f h} \mathcal{H}) - 2a_1 f^{3/2} h^{1/2} h' \phi' \mathcal{G} + (a_1 f' + 2c_2 f) r^2 h \mathcal{H}]}{f^{5/2} h^{1/2} (2\mathcal{P}_1 - \mathcal{F})^2 \mu^2} > 0,$$  \label{eq:prop speeds even}

where $c_2$ and $c_4$ are presented in Appendix A. Since $c^2_{r,\text{even}}$ is the same as $c^2_{r,\text{odd}}$, only the second propagation speed squared $c^2_{\phi,\text{even}}$ provides an additional stability condition.

For the monopole mode ($\ell = 0$), there is no propagation for the gravitational perturbation $\psi$, while the scalar-field perturbation $\delta \phi$ propagates with the same radial velocity as Eq. (2.20). For the dipole mode ($\ell = 1$), there is a gauge degree of freedom for fixing $\delta \phi = 0$, under which the perturbation $\psi$ propagates with the same radial speed squared as Eq. (2.20).
In the limit of large multipoles $\ell$, the conditions associated with the squared angular propagation speeds of even-parity perturbations are [77]

\[ c_{1\pm}^2 = -B_1 \pm \sqrt{B_1^2 - B_2} > 0, \] (2.21)

where we present the explicit form of $B_1$ and $B_2$ in Appendix E. These conditions are satisfied if

\[ B_1^2 \geq B_2 > 0 \quad \text{and} \quad B_1 < 0. \] (2.22)

The conditions (2.13), (2.14), (2.15), (2.17), (2.20), and (2.22) ensure the existence of consistent hyperbolic evolution of perturbations, which are essential for formulating the well-posed initial value problems. For the complete proof of the stability of BHs, as in the case of GR and conventional scalar-tensor theories, on top of these conditions, we further need to clarify the absence of mode instabilities and the stability at the nonlinear level, which will not be addressed in this paper.

### III. GENERIC INSTABILITIES OF BLACK HOLES WITH NONVANISHING SCALAR KINETIC TERM ON THE HORIZON

In this section, we will show the presence of ghost or Laplacian instabilities of BHs for a nonvanishing kinetic term on the BH horizon,

\[ X_s \equiv X(r_s) \neq 0, \] (3.1)

where $r_s$ denotes the radius of the BH horizon. We assume that $X$ is an analytic function of $r$ around $r = r_s$. Here, we will focus on the asymptotically Minkowski BHs and assume that the metric solution contains only a single horizon, namely the BH event horizon. Since the metric component $h$ vanishes at $r = r_s$, $X_s$ can be finite only when the derivative of the scalar field $\phi'$ diverges on the horizon. In shift-symmetric Horndeski theories like $G_4 \supset X$ or $G_4 \supset (-X)^{1/2}$ with $G_2 = \eta X - \Lambda$, where $\eta$ and $\Lambda$ are constants, there are some exact non-asymptotically Minkowski BHs with $X_s \neq 0$ [37, 63, 71, 92].

Since the sign of the scalar-field kinetic term is $X < 0$ for $r > r_s$ and $X > 0$ for $r < r_s$, the scalar field is spacelike outside the horizon and timelike inside the horizon, respectively. This means that $X$ undergoes a sudden change of the sign across the horizon, so the BH horizon corresponds to a singular hypersurface. Then, we can define a BH solution only outside the horizon in which the scalar field is spacelike. In Ref. [74], it was shown that BH solutions with $X_s \neq 0$ generically suffer from either ghost or Laplacian instabilities in the domain where the solution can exist, and hence such solutions could not be realistic. We will show that the same instability problem persists in full Horndeski theories with analytic coupling functions when $X_s \neq 0$.

#### A. Generic instabilities

We expand the background metric components around $r = r_s$ as

\[ f = f_1(r - r_s) + f_2(r - r_s)^2 + \cdots, \] (3.2)

\[ h = h_1(r - r_s) + h_2(r - r_s)^2 + \cdots, \] (3.3)

where $f_j$ and $h_j$ ($j = 1, 2, 3, \cdots$) are constants. Here and in the following, we focus on the standard case in which $h$ and $f$ simultaneously approach 0 as $r \to r_s$. Note that there are some spherically symmetric solutions where $f$ does not vanish as $h \to 0$ in specific Lorentz violating scalar-tensor theories [102], but we will not consider such cases in the context of Horndeski theories. Since both $f$ and $h$ are positive outside the horizon, we have $f_1 > 0$ and $h_1 > 0$. We are assuming that $X$ is an analytic function of $r$ around the horizon, so we can expand $X$ and $\phi$ in the forms

\[ X = X_s + X_1(r - r_s) + X_2(r - r_s)^2 + \cdots, \] (3.4)

\[ \phi = \phi_s + \phi_1(r - r_s)^{1/2} + \phi_2(r - r_s)^{3/2} + \cdots, \] (3.5)

where $X_j$, $\phi_j$, and $\phi_s$ are constants, with

\[ X_s = -\frac{1}{8} h_1 \phi_1^2. \] (3.6)
The expansion of the scalar field is valid only outside the horizon \((r > r_s)\), in which regime \(X_s < 0\) for \(\phi_1 \neq 0\). As we already mentioned, the BH solution can be defined only in the domain outside the horizon. Hence, it is enough to show instabilities only outside the horizon in order to exclude these BH solutions. Since the expansion of the scalar field is valid for \(X_s \neq 0\), in this section we focus on the solution with \(X_s < 0\).

In the vicinity of \(r = r_s\), the left-hand sides of the background equations reduce, respectively, to

\[
\mathcal{E}_{tt} = G_2 - 2X_sG_{3,\phi} + \frac{2(1 - h_1r_s)}{r_s}G_4 + \frac{4h_1X_s}{r_s}G_{4,X} + 4X_sG_{4,\phi,\phi} - \frac{2(1 + h_1r_s)}{r_s}G_{5,\phi} + \mathcal{O}((r - r_s)^{1/2}) ,
\]

\[
\mathcal{E}_{rr} = \sqrt{2}h_1X_s \left( -X_sG_{3,X} + G_{4,\phi} + 2X_sG_{4,\phi,\phi} - \frac{X_s}{r_s^2}G_{5,X} \right) (r - r_s)^{-1/2} + \mathcal{O}((r - r_s)^{0}) ,
\]

\[
\mathcal{E}_{\theta\theta} = -\frac{\sqrt{2}h_1X_s}{2} \left( 2G_{4,\phi} - 4X_sG_{4,\phi,\phi} + \frac{h_1X_s}{r_s}G_{5,X} + 2X_sG_{5,\phi,\phi} \right) (r - r_s)^{-1/2} + \mathcal{O}((r - r_s)^{0}) ,
\]

where the coupling functions \(G_{2,3,4,5}\) and their derivatives should be evaluated on the horizon. Then, the leading-order terms obey the following relations:

\[
G_2 - 2X_sG_{3,\phi} + \frac{2(1 - h_1r_s)}{r_s}G_4 + \frac{4h_1X_s}{r_s}G_{4,X} + 4X_sG_{4,\phi,\phi} - \frac{2(1 + h_1r_s)}{r_s}G_{5,\phi} = 0 ,
\]

\[
\sqrt{2}h_1X_s \left( -X_sG_{3,X} + G_{4,\phi} + 2X_sG_{4,\phi,\phi} - \frac{X_s}{r_s^2}G_{5,X} \right) = 0 ,
\]

\[
\sqrt{2}h_1X_s \left( 2G_{4,\phi} - 4X_sG_{4,\phi,\phi} + \frac{h_1X_s}{r_s}G_{5,X} + 2X_sG_{5,\phi,\phi} \right) = 0 .
\]

For given functions \(G_{2,3,4,5}\), the values of \(h_1, \phi_s, \) and \(X_s\) are fixed by solving Eqs. \(3.10\)–\(3.12\) in general. Since we are now interested in solutions with \(X_s \neq 0\), we will focus on the case in which the terms inside the parentheses of Eqs. \(3.11\) and \(3.12\) vanish. Depending on the coupling functions \(G_{3,4,5}\), there are specific cases in which the left-hand sides of Eqs. \(3.11\) or \(3.12\) vanish identically. In such cases, we need to compute their next-to-leading-order terms. For example, the next-order contributions to \(\mathcal{E}_{rr}\) are given by

\[
\mathcal{E}_{rr}^{(2)} = \frac{1}{r_s^2} \left\{ G_2r_s^2 + 2G_4(1 - h_1r_s) + 2X_s[G_{5,\phi} - 3G_{5,\phi}h_1r_s - (G_{2,X} - G_3,\phi - 2G_{4,\phi,\phi})r_s^2 - 2G_{4,X}(1 - 2h_1r_s)] - 4X_s^2[G_{5,\phi,\phi} - (2G_{4,XX} - 2G_{4,\phi,\phi})r_s] + (G_{3,\phi,\phi} - 2G_{3,\phi,\phi}r_s^2) \right\} .
\]

Later, we will consider specific theories in which the equation \(\mathcal{E}_{rr}^{(2)} = 0\) needs to be used.

Around the BH horizon, the quantities in Eqs. \(2.13\)–\(2.16\) are expanded as

\[
\mathcal{F} = 2(G_4 - X_sG_{5,\phi}) + \mathcal{O}((r - r_s)^{1/2}) ,
\]

\[
\mathcal{G} = -\sqrt{2}h_1[(X_s)^{3/2}G_{5,X}(r - r_s)^{-1/2} + 2(G_4 - X_sG_{4,X} + X_sG_{5,\phi} - 2X_s^2G_{5,\phi,\phi}) + \mathcal{O}((r - r_s)^{1/2}) ,
\]

\[
\mathcal{H} = 2(G_4 - X_sG_{4,X} + X_sG_{5,\phi}) + \mathcal{O}((r - r_s)^{1/2}) .
\]

Provided that \(G_{5,X}(\phi_s, X_s) \neq 0\), we have \(c_{r,odd}^2 = c_{r,even}^2 = G/\mathcal{F} \rightarrow \infty\) in the limit of \(r \to r_s\). Since this signals the strong coupling, we require the condition

\[
G_{5,X}(\phi_s, X_s) = 0 ,
\]

to realize finite values of \(c_{r,odd}^2\) and \(c_{r,even}^2\). We exploit the condition \(3.17\) in the following discussion. We note that the unusual divergence of \(\mathcal{G}\) for \(G_{5,X}(\phi_s, X_s) \neq 0\) arises from the assumption of \(X_s \neq 0\) with the scalar field expansion \(3.5\).

For the computation of \(c_{r,even}^2\), we resort to the expansions \(3.10\)–\(3.15\) around \(r = r_s\) as well as the leading-order background Eqs. \(3.11\) and \(3.12\) with \(X_s \neq 0\) to eliminate the terms \(G_{4,\phi,\phi}(\phi_s, X_s)\) and \(G_{5,\phi,\phi}(\phi_s, X_s)\). Then, the radial propagation speed squared of the scalar field perturbation \(\delta\phi\) reads

\[
c_{r,even}^2 = \frac{2h_1X_s\kappa_r}{\zeta(r - r_s)} + \mathcal{O}((r - r_s)^0) ,
\]

where we have defined

\[
\kappa_r = X_s^2(2X_sG_{3,XX} - G_{3,X}) + r_s^2(3G_{4,\phi} - 4X_s^2G_{4,\phi,XX}) + 2X_s^2G_{5,XX} ,
\]
The product $K_{r_2,\text{even}}$ is expanded as

$$K_{r_2,\text{even}}^2 = \frac{\sqrt{2}X_s h_1^{3/2} r_s^2 (G_4 - 2X_s X_4 + G_5,\phi X_4)^2 \kappa_r}{2\zeta^2 (r - r_s)^{3/2}} + O((r - r_s)^{-1/2}),$$

(3.21)

where

$$\zeta \equiv 2G_3,\phi X_s r_s^2 + r_s (G_4 h_1 - 2G_4,\phi X_s r_s) - 4G_4,XX h_1 - 4G_4,XY X_s h_1 - 4G_4,\phi,\phi X_s^2 r_s) + X_s [3G_5,\phi h_1 r_s + 2G_5,\phi X_s (1 + h_1 r_s)].$$

The necessary condition for avoiding ghost or Laplacian instabilities of even-parity perturbations is that the leading-order contribution to $K_{r_2,\text{even}}$ is positive. Since the term $G_4 - 2X_s X_4 + G_5,\phi X_4$ in Eq. (3.21) corresponds to the leading-order term of $\mathcal{H}/2$ on the horizon, we require the condition $G_4 - 2X_s X_4 + G_5,\phi X_4 > 0$. Then, the positivity of $K_{r_2,\text{even}}$ amounts to the inequality $\kappa_r > 0$, under which there is the divergence of $c_{r_2,\text{even}}$ on the horizon, which signals the strong coupling problem. For $\kappa_r < 0$, there is either ghost or Laplacian instability along the radial direction. Then, so long as $\kappa_r \neq 0$ on the horizon, we encounter either strong coupling or ghost/Laplacian instability of even-parity perturbations. If $\kappa_r = 0$, then the leading-order contribution to $c_{r_2,\text{even}}$ in Eq. (3.18) vanishes, in which case it may be possible to avoid the strong coupling problem. Even in this case, however, we further require that the next-to-leading-order terms of Eqs. (3.18) and (3.21) are both positive for the absence of ghost and Laplacian instabilities.

Around $r = r_s$, the product $\mathcal{F}K_2$ can be expanded as

$$\mathcal{F}K_2 = \frac{4 h_1^2 X_s^2 r_s^4 \kappa^2}{\zeta^2 (r - r_s)^2} + O((r - r_s)^{-1}),$$

(3.23)

where

$$\kappa \equiv G_4 G_4,XX + G_4^2 - G_5,\phi (G_4 - X_s G_4,XX) - G_5,\phi (2G_4,XX + X_s G_4,XX - G_5,\phi).$$

(3.24)

As long as $\kappa \neq 0$ on the horizon, the leading-order term of Eq. (3.23) is negative, i.e.,

$$\mathcal{F}K_2 < 0 \quad \text{as} \quad r \to r_s.$$  

(3.25)

Since one of the quantities $\mathcal{F}$, $K$, and $B_2$ must be negative, we cannot avoid either ghost or Laplacian instabilities. The angular propagation of even-parity perturbations plays a crucial role for reaching the conclusion of generic instabilities of BH solutions with $X_s \neq 0$ and $\kappa \neq 0$. For $\kappa = 0$, there is a possibility for avoiding the above instability, in which case the next-to-leading-order term of the product $\mathcal{F}K_2$ must be positive.

In summary, as long as $\kappa_r \neq 0$ or $\kappa \neq 0$, the BH solutions with $X_s \neq 0$ in Horndeski theories having analytic coupling functions are subject to either intrinsic ghost/Laplacian instabilities or strong coupling problems.

### B. Concrete models having BH solutions with $X_s \neq 0$

Let us proceed to the discussion of concrete models giving rise to BH solutions with $X_s \neq 0$. In such cases, the BH solutions with $X_s \neq 0$ are excluded by the instability around the horizon.

We first discuss the following example in the framework of shift-symmetric Horndeski theories:

$$G_2 = \eta X, \quad G_3 = \alpha_3 X, \quad G_4 = \frac{M_{pl}^2}{2} + \alpha_4 X, \quad G_5 = \alpha_5 X,$$

(3.26)

where $\eta$ and $\alpha_{3,4,5}$ are nonvanishing constants and $M_{pl}$ is the reduced Planck mass. If all $\alpha_3$, $\alpha_4$, and $\alpha_5$ are nonzero, there is no solution with $X_s \neq 0$ for Eqs. (3.11) - (3.12). The same conclusion persists if either $\alpha_3$ or $\alpha_5$ is vanishing. On the other hand, if both $\alpha_3$ and $\alpha_5$ are zero in Eq. (3.26), the left-hand sides of Eqs. (3.11) and (3.12) vanish identically, so we exploit the next-to-leading-order equation $\mathcal{E}_{\tau_{\tau}}^{(2)} = 0$ together with Eq. (3.10). Then, there is the following solution:

$$X_s = \frac{\eta M_{pl}^2 r_s^2}{4\alpha_4 (\eta r_s^2 + 2\alpha_4)}, \quad h_1 = \frac{\eta r_s^2 + 2\alpha_4}{2\alpha_4 r_s},$$

(3.27)
which corresponds to non-asymptotically Minkowski hairy BHs studied in Refs. \[37, 69, 71\]. Since the coupling functions \(G_2\) and \(G_4\) are both analytic functions of \(X\), we can resort to the analytic expansion of \(X\) used in Eq. \((3.4)\) around the horizon. The quantities \(\kappa_r\) and \(\kappa\) are given, respectively, by

\[
\kappa_r = 0, \quad \kappa = \alpha_4^2.
\]

As long as \(\eta \neq 0\) and \(\alpha_4 \neq 0\), we have \(X_s \neq 0\) and \(\kappa \neq 0\). Then, this solution inevitably suffers from the ghost or Laplacian instability around the horizon. This conclusion agrees with what was found for shift- and reflection-symmetric Horndeski theories containing the functional dependence \(G_2(X)\) and \(G_4(X)\) with \(G_3 = G_5 = 0\) \[74\].

As our second example, let us consider the following model:

\[
G_2 = \eta X, \quad G_3 = \alpha_3 X, \quad G_4 = \frac{M^2}{2} + \alpha_4 \phi^2 + \beta_4 \phi^2 X, \quad G_5 = \alpha_5 \phi^2,
\]

where \(\eta, \alpha_3, \alpha_4,\) and \(\beta_4\) are nonzero constants. We note that all the coupling functions in Eq. \((3.29)\) are analytic functions of \(\phi\) and \(X\). Solving the leading-order background equations for \(X_s, \phi_s,\) and \(h_1\), we find that there is the solution

\[
X_s = \frac{\alpha_4(\alpha_3 + 2\alpha_5)}{\beta_4(\alpha_3 - 6\alpha_5)}, \quad \phi_s = \frac{\alpha_3 + 2\alpha_5}{8\beta_4},
\]

which exists for \(\beta_4 \neq 0\) and \(\alpha_3 - 6\alpha_5 \neq 0\). Here, the explicit form of \(h_1\) is not shown due to its complexity. For this solution, the quantities \(\kappa_r\) and \(\kappa\) reduce, respectively, to

\[
\kappa_r = \frac{\alpha_4(\alpha_3 + 2\alpha_5)r^2}{2\beta_4}, \quad \kappa = \frac{(\alpha_3 + 2\alpha_5)^2(\alpha_3 - 14\alpha_5)^2}{4096\beta_4^2}.
\]

The existence of the solution with \(X_s \neq 0\) requires that \(\alpha_4(\alpha_3 + 2\alpha_5) \neq 0\), under which \(\kappa_r \neq 0\). Also, we have \(\kappa \neq 0\) unless \(\alpha_3 = 14\alpha_5\). We note that, even when \(\alpha_5 = 0\), there are solutions with \(X_s \neq 0\) plagued by the instability problem.

IV. THEORIES WITH NO-HAIR SCHWARZSCHILD BLACK HOLES

Given the existence of instabilities for the BH solutions with \(X_s \neq 0\), we are now interested in solutions with \(X_s = 0\). For example, the models \((3.20)\) and \((3.29)\) give rise to the branch \(X_s = 0\) besides the branch \(X_s \neq 0\) discussed in Sec. III. For the solution with \(X_s = 0\), there are in general two possibilities. One is a trivial field profile with \(\phi(r) = \phi_s = \text{constant}\) at any radius. The other is a hairy solution where the scalar field varies as a function of \(r\). For this hairy solution with \(X_s = 0\), the scalar field regular around the horizon can be expanded as

\[
\phi(r) = \phi_s + \phi_1(r - r_s) + \phi_2(r - r_s)^2 + \cdots.
\]

Since this is different from Eq. \((3.5)\), we need to handle this case separately for the discussion of ghost/Laplacian instabilities. Furthermore, the expansion \((4.1)\) is insufficient to ensure the BH stability throughout the horizon exterior. Thus, in this and subsequent sections, we will derive perturbative BH solutions with respect to a small parameter arising from the coupling functions \(G_{2,3,4,5}\) and explore the issue of ghost/Laplacian instabilities of BHs in the region outside the horizon.

In the horizon limit \(r \to r_s\), the hairy BH solutions with \(X_s = 0\) should satisfy the following boundary conditions:

\[
f(r) \to 0, \quad h(r) \to 0 \quad \text{with} \quad \frac{h(r)}{f(r)} \to \text{finite}, \quad \text{and} \quad |\phi'(r)| \to |\phi_1| < \infty.
\]

On the other hand, at spatial infinity \(r \to \infty\), we impose that the metric is asymptotically Minkowski and the scalar-field derivative vanishes, i.e.,

\[
f(r) \to \text{constant}, \quad h(r) \to 1, \quad \phi'(r) \to 0.
\]

This means that the scalar field \(\phi(r)\) approaches a constant value. Here, we do not impose that the asymptotic constant value of the scalar field is a specific value, e.g., zero, but we allow an arbitrary constant value as long as it does not conflict with the asymptotically Minkowski metric. We will construct hairy BH solutions under the boundary conditions \((4.2)\) and \((4.3)\).
Now, we search for the possibility of BH solutions with $X_s = 0$ which are not prone to the generic problem of ghost/Laplacian instabilities found in the previous section. We recall that the scalar-field equation of motion is given by Eq. (2.9). Outside the horizon, the solution to Eq. (2.9) can be expressed in an integrated form

$$J^r = \frac{Q}{r^2} \sqrt{\frac{h}{f}} - \frac{1}{r^2} \sqrt{\frac{h}{f}} \int_{r_s}^r r'^2 \sqrt{\frac{f}{h}} P_\phi \, dr,$$

where $Q$ is an integration constant and $J^r$ is the radial current component defined in Eq. (2.10), which we repeat here for the reader’s convenience:

$$J^r = h\phi' \left[ G_{2,X} \left( \frac{2}{r} + \frac{f'}{2f} \right) h\phi' G_{3,X} + 2 \left( \frac{1-h}{r^2} - \frac{hf'}{rf} \right) G_{4,X} + 2h\phi'^2 \left( \frac{h}{r^2} + \frac{hf'}{rf} \right) G_{4,XX} 
- \frac{f'}{2r^2 f} (1-3h)h\phi' G_{5,X} - \frac{f' h^3 \phi'^3}{2r^2 f} G_{5,XX} \right].$$

(4.5)

In shift-symmetric Horndeski theories, we have $P_\phi = 0$, and hence the second term on the right-hand side of Eq. (4.4) vanishes. In non-shift-symmetric Horndeski theories containing the $\phi$-dependence in $G_{2,3,4,5}$, the integral containing $P_\phi$ contributes to $J^r$. Throughout the discussion below, we include the kinetic term $\eta X$ in $G_2$ and the Einstein-Hilbert term $M_{Pl}^2/2$ in $G_4$, i.e.,

$$G_2 \supset \eta X, \quad G_4 \supset \frac{M_{Pl}^2}{2},$$

(4.6)

with $\eta$ being a constant.

### A. Shift-symmetric theories

In shift-symmetric theories, the radial current component is given by

$$J^r = \frac{Q}{r^2} \sqrt{\frac{h}{f}}.$$  

(4.7)

Let us first consider the simplest case of GR with a linear kinetic term of the scalar field, i.e.,

$$G_2 = \eta X, \quad G_3 = 0, \quad G_4 = \frac{M_{Pl}^2}{2}, \quad G_5 = 0.$$  

(4.8)

Since $J^r = \eta h \phi'$ in this case, the scalar-field derivative is given by

$$\phi'(r) = \frac{Q}{\eta h^2} \frac{1}{\sqrt{fh}}.$$  

(4.9)

As $r$ approaches the horizon radius $r_s$, there is the divergence of $\phi'$. To avoid this behavior, we require that $Q = 0$ and hence

$$\phi'(r) = 0,$$  

(4.10)

which corresponds to a no-hair solution.

We can generalize the above argument to more general theories where the coupling functions $G_{2,3,4,5}$ are analytic functions of $X$, i.e.,

$$G_I(X) = \sum_{p \geq 0} (\alpha_I)_p (-X)^p \quad (I = 2, 3, 4, 5),$$  

(4.11)

where $(\alpha_I)_p$ are constants and the sum is taken over all integers $p \geq 0$. Taking into account the terms in Eq. (4.6), we consider the following coupling functions:

$$G_2 = \eta X + \sum_{p_2 \geq 2} (\alpha_2)_{p_2} (-X)^{p_2}, \quad G_3 = \sum_{p_3 \geq 1} (\alpha_3)_{p_3} (-X)^{p_3},$$

$$G_4 = \frac{M_{Pl}^2}{2} + \sum_{p_4 \geq 1} (\alpha_4)_{p_4} (-X)^{p_4}, \quad G_5 = \sum_{p_5 \geq 1} (\alpha_5)_{p_5} (-X)^{p_5}.$$  

(4.12)
We drop constant terms in each $G_I$ apart from $M^2_{Pl}/2$ in $G_4$, as they are irrelevant to the existence of asymptotically Minkowski hairy BH solutions. Using the expansions (4.2) and (4.3) around the horizon, it follows that the metric functions $f$, $h$, and their derivatives do not cause divergences for the terms appearing in the square brackets in Eq. (4.9). This is also the case at spatial infinity where $f$ and $h$ approach constants with $\phi'(r) \to 0$ for asymptotically Minkowski solutions. The radial current component (4.5) can be written in the form

$$f = \frac{1}{\phi'[\eta + F(\phi')]} \frac{Q}{r^2} \sqrt{\frac{h}{f}}.$$  

(4.13)

To realize $h = 0$ on the horizon for a finite or vanishing value of $\phi'$, we require that $Q = 0$. Then, we obtain

$$J^r = h\phi'[\eta + F(\phi')] = 0.$$  

(4.14)

Under the boundary condition $\phi'(\infty) = 0$, the function $F(\phi')$ approaches zero at spatial infinity. Then, so long as $\eta \neq 0$, we have to choose the branch $\phi'(r) = 0$. This means that, for theories with the coupling functions of the form (4.12), we end up with no-hair BH solutions. This fact was already recognized in Ref. [78].

In the above discussion, the main reason for reaching the no-hair conclusion is that the dominant contribution to $J^r$ in the limit $\phi' \to 0$ is the linear term in $\phi'$. This behavior can be avoided by considering the following nonanalytic coupling functions [59, 62]:

$$G_2 = \eta X + \alpha_2 \sqrt{-X}, \quad G_3 = \alpha_3 \ln |X|, \quad G_4 = \frac{M^2_{Pl}}{2} + \alpha_4 \sqrt{-X}, \quad G_5 = \alpha_5 \ln |X|,$$

(4.15)

with $\alpha_{2,3,4,5}$ being constants. For $\phi' > 0$, we have

$$J^r = \eta h \phi' - \frac{h}{2} \alpha_2 + \left(\frac{f'}{f} + \frac{4}{r}\right) h \alpha_3 - \frac{\sqrt{2h}}{r^2} \alpha_4 - \frac{f' h (h-1)}{f r^2} \alpha_5.$$  

(4.16)

Apart from the first term, there are no $\phi'$-dependent terms. For any coupling functions with stronger divergence as $X \to 0$ than the choice (4.15), $J^r$ diverges as $\phi' \to 0$. In such theories, we cannot take the proper Minkowski limit at large distances [107]. In this sense, the choice of the coupling functions (4.15) is unique for the realization of hairy BH solutions in shift-symmetric Horndeski theories. From Eq. (4.17), the scalar-field derivative is now expressed as

$$\phi' = \frac{1}{\eta h} \left[ \frac{h}{2} \alpha_2 - \left(\frac{f'}{f} + \frac{4}{r}\right) h \alpha_3 + \frac{\sqrt{2h}}{r^2} \alpha_4 + \frac{f' h (h-1)}{f r^2} \alpha_5 + \frac{Q}{r^2} \sqrt{\frac{h}{f}} \right].$$  

(4.17)

The terms associated with $\alpha_2$ and $\alpha_4$ are dominated over the term $(Q/r^2)\sqrt{h/f}$ around the horizon.

In the case of $\alpha_3 = \alpha_5 = 0$, to avoid the divergence of $\phi'$ induced by the term $(Q/r^2)\sqrt{h/f}$ for $\alpha_2 \neq 0$ or $\alpha_4 \neq 0$, we need to set $Q = 0$. However, even under $Q = 0$, $\phi'(r)$ still diverges on the horizon. Indeed, the scalar-field kinetic term has the following dependence:

$$X = -\frac{\alpha_3^2}{4\eta^2}, \quad \alpha_2 \neq 0, \quad \alpha_4 = 0,$$

(4.18)

$$X = -\frac{\alpha_4^2}{\eta^2}, \quad \alpha_4 \neq 0, \quad \alpha_2 = 0.$$  

(4.19)

In both cases, $X$ is an analytic function of $r$ which is nonvanishing at $r = r_s$. Hence, these BH solutions are excluded by the intrinsic instability problem discussed in Sec. III.

In the case of $\alpha_2 = \alpha_4 = 0$, the terms associated with $\alpha_3$ and $\alpha_5$ as well as the term $(Q/r^2)\sqrt{h/f}$ in the square brackets in Eq. (4.17) approach constants as $r \to r_s$. In such cases, we can choose $Q$ such that $\phi'$ becomes regular on the horizon. Using the expansions (3.2) and (3.3), the values of $Q$ and $\phi'$ at $r = r_s$ are given by

$$Q = \alpha_3 \sqrt{f_1 h}, \quad \phi'(r_s) = -\frac{\alpha_3 (f_1 h_2 r_s + 3f_2 h_1 r_s + 12f_1 h_1)}{2\eta f_1 h_1 r_s} \quad (\alpha_3 \neq 0, \quad \alpha_5 = 0),$$

(4.20)

$$Q = \alpha_5 \sqrt{f_1 h}, \quad \phi'(r_s) = \frac{\alpha_5 (2f_1 h_1^2 - f_1 h_2 - 3f_2 h_1)}{2\eta f_1 h_1 r_s^2} \quad (\alpha_5 \neq 0, \quad \alpha_3 = 0).$$  

(4.21)
In both cases, the scalar-field derivative is finite on the horizon and hence \( X_s = 0 \). As we will show in Sec. IV C the BHs present for the cubic coupling case \((4.20)\) are not asymptotically Minkowski. On the other hand, the quintic coupling case \((4.21)\) realizes asymptotically Minkowski hairy BH solutions. Indeed, this is equivalent to the scalar field linearly coupled to the Gauss-Bonnet term studied in Refs. [59, 60].

### B. Non-shift-symmetric theories

Let us investigate the possibility for realizing hairy BH solutions in non-shift-symmetric theories containing the dependence of \( \phi \) as well as \( X \) in \( G_{2,3,4,5} \). For the \( X \)-dependent part of the couplings, we take one of the powers \( (-X)^{pi} \) in Eq. \((4.12)\) for each \( I = 2, 3, 4, 5 \) for simplicity. Multiplying such terms with \( \phi \)-dependent analytic functions \( \alpha_I(\phi) \), we can consider the following couplings:

\[
G_2 = \eta X + \alpha_2(\phi)(-X)^{p_2}, \quad G_3 = \alpha_3(\phi)(-X)^{p_3}, \quad G_4 = \frac{M^2_{Pl}}{2} + \alpha_4(\phi)(-X)^{p_4}, \quad G_5 = \alpha_5(\phi)(-X)^{p_5}, \quad (4.22)
\]

where \( p_{2,3,4,5} \geq 0 \) are integers. Since \( \alpha_{2,3,4,5}(\phi) \) are analytic functions of \( \phi \), they can be expanded around some \( \phi_0 \) as

\[
\alpha_I(\phi) = \sum_{q_I \geq 0} (\alpha_I)_{q_I} (\phi - \phi_0)^{q_I} \quad (I = 2, 3, 4, 5), \quad (4.23)
\]

where \( (\alpha_I)_{q_I} \) are constants. For \( p_{2,3,4,5} = 0 \), the purely \( \phi \)-dependent couplings in \( G_{2,3,4,5} \) can be accommodated.

Analogous to the discussion in shift-symmetric theories \((7.1)\), the radial current component for the couplings \((4.22)\) can be expressed in the form \( J^r = h\phi' [\eta + \tilde{F}(\phi', \phi)] \), where \( \tilde{F}(\phi', \phi) \) is an analytic function containing the positive power-law dependence of \( \phi' \) and \( \phi \). Assuming that \( P_\phi \) is finite in the limit of the horizon, which will be confirmed in the perturbative approach later, the last integral in Eq. \((4.3)\) vanishes as \( r \rightarrow r_s \). Then, on the horizon, there is the relation

\[
h = \frac{1}{\phi_0'[\eta + \tilde{F}(\phi'_s, \phi_s)]} \frac{Q}{r_s^2} \sqrt{\frac{h}{f}} \quad (4.24)
\]

with \( \phi'_s \equiv \phi'(r_s) \). Since \( h \) is vanishing at \( r = r_s \) with finite values of \( \phi'_s \) and \( \phi_s \), we require that \( Q = 0 \). Then, from Eq. \((4.3)\), we obtain

\[
h\phi'[\eta + \tilde{F}(\phi', \phi)] = -\frac{1}{r_s^2} \sqrt{\frac{h}{f}} \int_{r_s}^r \sqrt{\frac{f}{h}} P_{\phi'} dr. \quad (4.25)
\]

Regarding the finiteness of \( P_\phi \) in the horizon limit \( r \rightarrow r_s \), we note that \( P_\phi \) contains the second scalar-field derivative \( \phi'' \) as well as the contributions from \( f, h \), and their derivatives [see Eq. \((2.11)\) with Eq. \((A1)\)]. This means that we need to integrate Eq. \((2.9)\) together with the other background equations \((2.5)\) and \((2.6)\) to determine the value of \( P_\phi \). We will explicitly see this by using perturbative solutions in the small coupling regime.

For the purpose of deriving perturbative BH solutions, we write the coupling functions \( \alpha_I (I = 2, 3, 4, 5) \) in Eq. \((4.22)\) as

\[
\alpha_I(\phi) = \alpha \hat{\alpha}_I(\phi), \quad (4.26)
\]

where \( \alpha \) is a dimensionless coupling constant and \( \hat{\alpha}_{2,3,4,5}(\phi) \) are analytic functions of \( \phi \). This ansatz allows us to control the deviation from GR with a canonical scalar field by a single parameter \( \alpha \). Since we are considering BH solutions with the vanishing kinetic term on the event horizon, \( X_s = 0 \), the scalar-field derivative \( \phi'(r) \) (and the scalar field itself) is finite and regular on the horizon. We perform the perturbative expansions of \( f, h \), and \( \phi \) with respect to the small coupling constant \( \alpha \) around the Schwarzschild background given by the metric components \( f = h = 1 - 2m/r \), where \( m \) is a constant. Namely, we consider the metric and scalar field given by

\[
f(r) = \left(1 - \frac{2m}{r}\right) \left[1 + \sum_{j \geq 1} \hat{f}_j(r) \alpha^j \right]^2, \quad h(r) = \left(1 - \frac{2m}{r}\right) \left[1 + \sum_{j \geq 1} \hat{h}_j(r) \alpha^j \right]^{-2}, \quad (4.27)
\]

\[
\phi(r) = \phi_0(r) + \sum_{j \geq 1} \hat{\phi}_j(r) \alpha^j, \quad (4.28)
\]
functions \( \tilde{f}_j \), \( \tilde{h}_j \), \( \phi_0 \), and \( \phi_j \) are functions of \( r \). We note that, for the perturbative ansätze \( 4.27 \) and \( 4.28 \), the horizon distance \( r_+ \) is given by \( r_+ = 2m \). We require the validity of the perturbative expansion \( 4.27 \) and \( 4.28 \) with respect to a small coupling constant \( |\alpha| \ll 1 \) and impose that the coefficients \( \tilde{f}_j(r) \), \( \tilde{h}_j(r) \), and \( \phi_j(r) \) are regular \( r = 2m \). Moreover, since \( \phi \) is a scalar quantity, its value does not depend on the choice of the coordinates. If \( \phi \) is divergent at \( r = 2m \) in the original coordinates, it would also diverge at the corresponding position in the new coordinates. On the other hand, a divergence of the metric at \( r = 2m \) might be removed by an appropriate coordinate transformation. However, since the metric and scalar field are coupled in our system, a choice of the integration constants different from the below would result in the divergence of the scalar field. Thus, the constructed coordinate transformation. However, since the metric and scalar field are coupled in our system, a choice of the new coordinates. On the other hand, a divergence of the metric at \( r = 2m \) might be removed by an appropriate coordinate transformation. However, since the metric and scalar field are coupled in our system, a choice of the integration constants different from the below would result in the divergence of the scalar field. Thus, the constructed perturbative solution \( 4.27 \) and \( 4.28 \) indeed exists, irrespective of the choice of the coordinates. The coupling functions \( \tilde{\alpha}_I(\phi) \) \( (I = 2, 3, 4, 5) \) can be expanded as

\[
\tilde{\alpha}_I(\phi) = \tilde{\alpha}_I(\phi_0) + \sum_{n \geq 1} \tilde{\alpha}^{(n)}_I(\phi_0) \frac{(\phi - \phi_0)^n}{n!},
\]

where \( \tilde{\alpha}^{(n)}_I(\phi_0) \equiv d^n\tilde{\alpha}_I/d\phi^n|_{\phi=\phi_0} \). We derive the BH solutions perturbatively by using the dimensionless parameter \( \alpha \) arising from each coupling \( \alpha_{2,3,4,5} \). By construction, the no-hair argument given below is valid in the regime of the small coupling constant, \( |\alpha| \ll 1 \).

In Eqs. \( 4.15 \) and \( 2.11 \), the zeroth-order terms in \( \alpha \) are \( J^r = \eta h \phi' = (1 - 2m/r)\eta \phi' \) and \( P^r = 0 \). From Eq. \( 2.9 \), the zeroth-order scalar field obeys the differential equation

\[
\left[ r^2 \left( 1 - \frac{2m}{r} \right) \phi'_0(r) \right]' = 0,
\]

whose solution is given by

\[
\phi_0(r) = \tilde{\phi}_0 + \frac{D_0}{2m} \ln \left( 1 - \frac{2m}{r} \right),
\]

where \( \tilde{\phi}_0 \) and \( D_0 \) are integration constants. To avoid the divergent behavior of \( \phi_0(r) \) on the horizon at \( r = 2m \), we require that \( D_0 = 0 \) and hence \( \phi_0(r) = \tilde{\phi}_0 = \text{constant} \). In the following, we discuss two different cases: 1) \( p_2 \neq 0 \) and 2) \( p_2 = 0 \), in turn, where \( p_2 \) is the power appearing in the coupling function \( G_2 \) in Eq. \( 4.22 \).

1. \( p_2 \neq 0 \)

For \( p_2 \neq 0 \), the first-order expanded solutions in \( \alpha \) are given by

\[
\tilde{\phi}_1(r) = \tilde{\phi}_1 + \frac{D_1}{2m} \ln \left( 1 - \frac{2m}{r} \right), \quad \tilde{h}_1(r) = \frac{C_1}{r - 2m}, \quad \tilde{f}_1(r) = -\frac{C_1}{r - 2m} + C_2,
\]

where \( D_1, C_1, C_2 \) are integration constants. For the regularity at \( r = 2m \), we require that \( D_1 = 0 \) and \( C_1 = 0 \), and we also set \( C_2 = 0 \) by a suitable time reparametrization. Then, the first-order solution is given by \( \tilde{\phi}_1(r) = \tilde{\phi}_1, \tilde{h}_1(r) = 0, \) and \( \tilde{f}_1(r) = 0 \). One can show that the \( j \)th-order perturbative solutions \( (j \geq 2) \) are of the same form as Eq. \( 4.32 \).

Then, the solutions regular on the horizon are

\[
\tilde{\phi}_j(r) = \tilde{\phi}_j, \quad \tilde{h}_j(r) = 0, \quad \tilde{f}_j(r) = 0,
\]

where \( \tilde{\phi}_j \) are constants. Since these relations hold for all integers \( j \), this corresponds to the no-hair BH solution \( \text{i.e., } \phi(r) = \text{constant} \) with the Schwarzschild metric components \( f(r) = h(r) = 1 - 2m/r \).

2. \( p_2 = 0 \)

Let us consider the theories with \( p_2 = 0 \), i.e., the coupling \( \alpha_2(\phi) = \alpha \tilde{\alpha}_2(\phi) \) in \( G_2 \). The first-order solutions of \( h \) and \( f \) regular at \( r = 2m \) are given by

\[
\tilde{h}_1(r) = -\frac{4m^2}{6M_p^2} \tilde{\alpha}_2(\phi_0), \quad \tilde{f}_1(r) = \frac{(2m + r)}{6M_p^2} \tilde{\alpha}_2(\phi_0),
\]
where we have set the integration constant in $f_1(r)$ to be zero. To realize the asymptotically Minkowski metric, we require that $\alpha_2(\phi_0) = 0$. The first-order solution to the scalar field is given by

$$\hat{\phi}_1(r) = \hat{\phi}_1 + \frac{D_1}{2m} \ln \left(1 - \frac{2m}{r}\right) - \frac{[r^2 + 4mr + 8m^2 \ln(r/2m)]}{6\eta} \hat{\alpha}_2^{(1)}(\phi_0),$$

(4.35)

with $D_1$ being an integration constant. We impose $\hat{\alpha}_2^{(1)}(\phi_0) = 0$, as otherwise the last term leads to the divergence of $\hat{\phi}_1(r)$ at spatial infinity and the condition 1.33 [i.e., $\hat{\phi}_1(\infty) = 0$] is not respected. Then, the regularity of $\hat{\phi}_1(r)$ at $r = 2m$ requires that $D_1 = 0$, and hence $\hat{\phi}_1(r) = \hat{\phi}_1 = \text{constant}$. To avoid the divergences of $\hat{\phi}_1(r)$, $\hat{h}_j(r)$, and $f_j(r)$ at spatial infinity for higher-order solutions ($j \geq 2$), we require that $\hat{\alpha}_2^{(n)}(\phi_0) = 0$ for all $n \geq 1$. Hence, we end up with the no-hair solution characterized by $\hat{\phi}_j(r) = \hat{\phi}_1 = \text{constant}$ and $\hat{\alpha}_2(\phi) = \hat{\alpha}_2(\phi_0) = 0$.

The above discussion shows that, in theories with the coupling functions 1.22, the asymptotically Minkowski BH solutions respecting the regularity on the horizon are restricted to be no-hair solutions with $\phi(r) = \text{constant}$. Stiulating $\phi'(r) = 0$ and $\phi''(r) = 0$ into the expression of $P_\phi$ given in Eq. 2.13 and using the property that the $\phi$- and $X$-derivatives of the couplings $G_{2,3,4,5}$ do not contain negative powers of $\phi'$, it follows that $P_\phi = G_{2,\phi} + \lambda_4 G_{4,\phi}$, where $\lambda_4$ is defined in Eq. A1. Since the background geometry is the Schwarzschild metric, the quantity $\lambda_4$ vanishes. The coupling $\alpha_2(\phi)(-X)^{p_2}$ with $p_2 = 0$ gives rise to $\phi$-dependent contributions in $G_{2,\phi}$, but they vanish due to the property $\hat{\alpha}_2^{(n)}(\phi_0) = 0$ derived above. Then, we have $P_\phi = 0$, so the right-hand side of Eq. 4.26 vanishes. Since $\hat{F}(\phi_s, \phi_s)$ does not contain negative powers of $\phi'$, the no-hair solution with $\phi'(r) = 0$ is consistent with Eq. 4.26 everywhere outside the horizon.

The above results show that, for the theories characterized by the coupling functions 1.22 with 1.28, there are no asymptotically Minkowski BHs with scalar hair. Such theories include couplings of the forms $G_I \supset \phi^{q_I}(-X)^{p_I}$ ($I = 2, 3, 4, 5$), with integers $q_I \geq 0$ and $p_I \geq 0$. The no-hair property persists for the product of two analytic functions $\phi_1(\phi)$ and $F(X)$, i.e., $G_I \supset \phi_1(\phi) F_1(X)$.

As in the case of shift-symmetric theories, the possibility for evading the no-hair property of BHs is to choose couplings with specific non-analytic functions of $X$. In shift-symmetric theories, for the coupling functions 1.10, the $X$-dependences in $G_{2,3,4,5}$ are uniquely fixed in such a way that they give rise to terms without containing the $\phi'$-dependence in $J^\nu$, except for $\eta \phi'_1$. This can be straightforwardly extended to non-shift-symmetric theories by multiplying analytic functions of $\phi$ to each non-analytic functions of $X$ as

$$G_2 = \eta X + \alpha_2(\phi) \sqrt{-X}, \quad G_3 = \alpha_3(\phi) \ln |X|, \quad G_4 = \frac{M^2}{2} + \alpha_4(\phi) \sqrt{-X}, \quad G_5 = \alpha_5(\phi) \ln |X|,$$

(4.36)

where $\alpha_I(\phi)$'s ($I = 2, 3, 4, 5$) are analytic functions of $\phi$. From Eq. 4.33, we obtain

$$\phi' = \frac{1}{\eta \hbar} \left[ \sqrt{\frac{\hbar}{2}} \alpha_2(\phi) - \left(\frac{f'}{2} + \frac{4}{r}\right) h_3(\phi) + \frac{\sqrt{2h}}{r^2} \alpha_4(\phi) + \frac{f'(h-1)}{f r^2} \alpha_5(\phi) + \frac{Q + Q_\phi \hbar}{r^2} \sqrt{\frac{\hbar}{2}} \right],$$

(4.37)

where

$$Q_\phi = - \int_{r_h}^r r^2 \sqrt{\frac{f}{2} \hbar \phi} \, dr.$$

(4.38)

Around the horizon, we can use the expansions 4.23 and 4.33 of the metric components and the expansion 4.11 of the scalar field which is valid for $X_s = 0$. Then, the leading-order terms of $P_\phi$ for the couplings $G_2, G_3, G_4, G_5$ in Eq. 4.36 are proportional to $\sqrt{r - r_s}$, $\ln(r - r_s)$, $1/\sqrt{r - r_s}$, and $\ln(r - r_s)$ in the vicinity of $r = r_s$, respectively. Even in those cases, however, the integral 4.38 vanishes for $r \rightarrow r_s$, i.e., $Q_\phi = 0$. This means that the discussion performed in shift-symmetric theories can be applied to the present $\phi$-dependent couplings as well.

For the quadratic and quartic couplings, we need to choose $Q = 0$ as in shift-symmetric theories, but the scalar-field kinetic term on the horizon reduces to $X_s = -\alpha_3^2(\phi_s)/(4\eta^3)$ and $X_s = -\alpha_5^2(\phi_s)/(\eta^2 r_s^4)$, respectively. The results of Sec. 1.11 show that these solutions suffer from ghost or Laplacian instabilities around the horizon. We note that, for $X_s \neq 0$ where we employ the expansion 4.33, the leading-order terms of $P_\phi$ for the couplings $G_2$ and $G_4$ in Eq. 4.36 are proportional to $(r - r_s)^0$ and $1/\sqrt{r - r_s}$ in the vicinity of $r = r_s$, respectively. As in the case of $X_s = 0$, all these contributions lead to $Q_\phi = 0$ in the limit of $r \rightarrow r_s$.

For the cubic and quintic couplings, the charge $Q$ should be chosen to realize the regular behavior of $\phi'$ on the horizon. Indeed, we obtain the same expressions of $Q$ and $\phi'(r_s)$ as those given by Eqs. 4.20 and 4.21 for the cubic and quintic couplings, respectively, with the replacements $\alpha_3 \rightarrow \alpha_3(\phi_s)$ and $\alpha_5 \rightarrow \alpha_5(\phi_s)$. At least in the vicinity
of the horizon, these solutions have scalar hair characterized by finite values of $\phi'(r_s)$ and $X_s (=0)$, so they are not subject to the instability problem discussed in Sec. [III]. However, this is not enough to ensure the existence of asymptotically Minkowski hairy BH solutions throughout the horizon exterior.

In Sec. [IVC] we will study hairy BH solutions for the cubic coupling $G_3 = \alpha_3(\phi) \ln |X|$ and show that they are not asymptotically Minkowski in general. For the quintic coupling $G_5 = \alpha_5(\phi) \ln |X|$, there exist asymptotically Minkowski hairy BH solutions for constant $\alpha_5$. In non-shift-symmetric theories, we need other couplings besides $G_5 = \alpha_5(\phi) \ln |X|$ for the realization of regular BH solutions with scalar hair. We will address these issues in Secs. [IVD] and [V].

C. Cubic logarithmic couplings

We study the possibility for realizing asymptotically Minkowski BH solutions in theories containing cubic logarithmic couplings given by

$$G_2 = \eta X, \quad G_3 = \alpha_3(\phi) \ln |X|, \quad G_4 = \frac{M^2_{Pl}}{2}, \quad G_5 = 0,$$

where $\alpha_3(\phi)$ is an analytic function of $\phi$. In what follows, we write $\alpha_3(\phi) = \alpha \gamma(\phi)$, where $\alpha$ is a constant which we assume to be small. Performing the expansions (4.27) and (4.28) with respect to the small parameter $\alpha$, the leading-order solution to the scalar field is $\phi_0(r) = \phi_0$ = constant. The first-order solutions in $\alpha$ are given by

$$\hat{h}_1(r) = \frac{C_1}{r - 2m}, \quad \hat{f}_1(r) = - \frac{C_1}{r - 2m} + C_2, \quad \hat{\phi}_1(r) = \hat{\phi}_1 + \frac{C_3}{2m} \ln \left(1 - \frac{2m}{r}\right) - \frac{4\gamma(\phi_0)}{\eta} \ln \left(\frac{r}{2m}\right),$$

where $C_{1,2,3}$ and $\hat{\phi}_1$ are integration constants. The regularity of $\hat{h}_1(r)$, $\hat{f}_1(r)$, and $\hat{\phi}_1(r)$ at $r = 2m$ imposes that $C_1 = 0$ and $C_3 = 0$, and a suitable time reparametrization allows us to choose $C_2 = 0$. We then obtain

$$\hat{h}_1(r) = 0, \quad \hat{f}_1(r) = 0, \quad \hat{\phi}_1(r) = \hat{\phi}_1 - \frac{4\gamma(\phi_0)}{\eta} \ln \left(\frac{r}{2m}\right).$$

At second order in $\alpha$, the integrated solutions of the metric components are given by

$$\hat{h}_2(r) = - \frac{4\gamma(\phi_0)^2}{\eta M^2_{Pl} (r - 2m)} \ln \left(\frac{r}{2m}\right), \quad \hat{f}_2(r) = - \frac{4\gamma(\phi_0)^2 (r - m)}{\eta M^2_{Pl} (r - 2m)} \ln \left(\frac{r}{2m}\right).$$

In the following, we will discuss shift-symmetric and non-shift-symmetric theories separately.

1. Shift-symmetric theories

We first consider the case

$$\gamma(\phi) = \gamma_0 = \text{constant}.$$

Then, the metric components have the asymptotic behavior $\hat{h}_2(r) \to 0$ and $\hat{f}_2(r) \to - \left[4 \gamma_0^2 / (\eta M^2_{Pl}) \right] \ln[r/(2m)]$ at spatial infinity. The logarithmic divergence of $\hat{f}_2(r)$ can be eliminated by imposing $\gamma_0 = 0$, but in this case we end up with the no-hair Schwarzschild solution.

2. Non-shift-symmetric theories

Let us next proceed to the case in which $\gamma(\phi)$ is a nontrivial analytic function of $\phi$. Analogous to Eq. (4.29), we can expand $\gamma(\phi)$ around $\phi = \phi_0$ as

$$\gamma(\phi) = \gamma(\phi_0) + \sum_{n \geq 1} \gamma^{(n)}(\phi_0) \frac{\phi - \phi_0)^n}{n!},$$

where $\gamma^{(n)}(\phi_0) \equiv \frac{d^n \gamma}{d\phi^n}|_{\phi=\phi_0}$. From Eq. (4.42), the metric can be asymptotically Minkowski only if

$$\gamma(\phi_0) = 0.$$
Deriving the higher-order solutions in $\alpha$, we find that the regularity of $\hat{\phi}_j$ ($j \geq 2$) on the horizon requires the conditions
\begin{equation}
\gamma^{(n)}(\phi_0) = 0 \quad (n \geq 1).
\end{equation}
Namely, the $G_3$ term in the action must be absent. In this case, we obtain the following regular solutions
\begin{equation}
\hat{h}_j(r) = 0, \quad f_j(r) = 0, \quad \hat{\phi}_j(r) = \tilde{\phi}_j = \text{constant},
\end{equation}
for all $j$. Then, we obtain the no-hair Schwarzschild solution with $\phi = \text{constant}$ due to the absence of the $G_3$ term in the action.

We thus showed that, for both the shift-symmetric and non-shift-symmetric forms of the cubic logarithmic couplings \ref{4.39}, asymptotically Minkowski hairy BH solutions cannot be realized.

### D. Quintic logarithmic couplings

Let us finally discuss the model with the quintic logarithmic coupling only,
\begin{equation}
G_2 = \eta X, \quad G_3 = 0, \quad G_4 = \frac{M^2_{\text{pl}}}{2}, \quad G_5 = \alpha_5(\phi) \ln |X|,
\end{equation}
where $\alpha_5(\phi)$ is a regular function of $\phi$. Except for shift-symmetric theories with $\alpha_5 = \text{constant}$, this model generally gives rise to terms of the form $(-X)^p \ln |X|$ ($p \geq 1$) in the background equations of motion. Although they themselves vanish in the limit $X \rightarrow 0$, higher-order $X$-derivatives of them diverge for $X \rightarrow 0$. In this case, we may have the problem of instability at the level of higher-order perturbations. In the process of deriving perturbative solutions with the expansions \ref{4.27} and \ref{4.28}, we also encounter terms like $\alpha^p \ln |\alpha|$, so the power-law expansions with respect to $\alpha$ lose their validity. The quintic couplings like $G_5 = [1 + \sum_{p=1}^{4} C_p (-X)^p] \alpha_5(\phi) \ln |X|$, which reduce to $G_5 \rightarrow \alpha_5(\phi) \ln |X|$ in the limit $X \rightarrow 0$, again generate terms of the form $(-X)^p \ln |X|$ in the background equations.

As we will see in Sec. \ref{V} the scalar-GB coupling $\xi(\phi) R^2_{\text{GB}}$, which amounts to the coupling functions of the form \ref{5.2}, corresponds to the only exceptional case in which $\ln |X|$-dependent terms completely disappear from the background equations. In this case, since we do not face the aforementioned problem, we can resort to the expansions \ref{4.27} and \ref{4.28} to derive BH solutions perturbatively. As long as all the coupling functions in Eq. \ref{5.2} are present, we can add other regular functions like $(-X)^p$ and $\phi^q$ in the coupling functions to see how the structure of hairy BH solutions is modified. We will also address this issue in Sec. \ref{V.3}

### V. BHS IN THE PRESENCE OF THE GAUSS-BONNET COUPLING

In this section, we study the existence of asymptotically Minkowski BH solutions related to the GB coupling $\xi(\phi) R^2_{\text{GB}}$, where $\xi(\phi)$ is a function of $\phi$ and $R^2_{\text{GB}}$ is the GB term defined by
\begin{equation}
R^2_{\text{GB}} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu},
\end{equation}
with $R_{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}$ being the Ricci and Riemann tensors associated with the metric $g_{\mu\nu}$, respectively. In the language of Horndeski theories, the Lagrangian $\xi(\phi) R^2_{\text{GB}}$ is equivalent to the combination of the following couplings \ref{10, 68}:
\begin{align}
G_2 &= 8\xi^{(4)}(\phi) X^2 (3 - \ln |X|), \quad G_3 = 4\xi^{(3)}(\phi) X (7 - 3 \ln |X|), \\
G_4 &= 4\xi^{(2)}(\phi) X (2 - \ln |X|), \quad G_5 = -4\xi^{(1)}(\phi) \ln |X|,
\end{align}
where $\xi^{(n)}(\phi) \equiv d^n \xi(\phi)/d\phi^n$ \footnote{Note that the Horndeski Lagrangian with $G_2 = -8\xi^{(4)}(\phi) X^2$, $G_3 = -12\xi^{(3)}(\phi) X$, $G_4 = -4\xi^{(2)}(\phi) X$, and $G_5 = -4\xi^{(1)}(\phi)$ is a total derivative for any smooth function $\xi(\phi)$, and hence a simultaneous multiplication of a constant in each logarithmic function in Eq. \ref{5.2} does not matter. Therefore, it is legitimate to replace $\ln |X| \rightarrow \ln |X/X_0|$ with $X_0$ being a constant of mass dimension four to make the argument of the logarithmic function dimensionless.}. We note that the form of $G_5$ is identical to that given in Eq. \ref{4.30}. The linear GB coupling $\xi(\phi) = \alpha \phi$, where $\alpha$ is constant, corresponds to the quintic interaction $G_5 = -4\alpha \ln |X|$ with $G_{2,3,4} = 0$. In this class of shift-symmetric theories, it is known that there exist asymptotically Minkowski BHs endowed with scalar hairy \ref{59, 60}. The recent analysis of Ref. \ref{74} showed that these BH solutions satisfy the conditions for the absence of
ghost and Laplacian instabilities of odd- and even-parity perturbations. Moreover, the propagation speeds of all the perturbation modes approach unity in the asymptotic infinity \((r \to \infty)\).

For power-law GB couplings given by \(\xi(\phi) = \alpha \phi^n\) with \(n \geq 2\), which no longer respect the shift symmetry, asymptotically Minkowski hairy BHs have been obtained numerically \cite{1906.01164}. For more general GB couplings where \(\xi(\phi)\) is a generic analytic function of \(\phi\), we will construct solutions by using the method of perturbative expansions \eqref{eq:427} and \eqref{eq:428} valid for the small dimensionless coupling constant \(|\alpha| \ll 1\). Although general couplings accommodate models of BH spontaneous scalarization \cite{0809.4719,1711.08953} and nonlinear scalarization \cite{1801.02374} as well, our construction with the ansätze \eqref{eq:427} and \eqref{eq:428} does not incorporate such scalarized BHs which can be realized only in a nonperturbative regime with \(|\alpha| = \mathcal{O}(1)\). Our purpose is rather to address the issue of ghost/Laplacian instabilities for hairy BHs realized as the consequence of perturbative deviation from the Schwarzschild solution in non-shift-symmetric Horndeski theories.

A. General GB couplings and BH stabilities

We first consider scalar-GB theories in the presence of a kinetic term \(\eta X\) and the Einstein-Hilbert term in the action, i.e.,

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_{Pl}^2}{2} R + \eta X + \alpha \xi(\phi) R_{GB}^2 \right],
\]

where \(\xi(\phi)\) is an analytic function of \(\phi\) and \(\alpha\) is a dimensionless coupling constant which we assume to be small. We perform the expansions \eqref{eq:427} and \eqref{eq:428} with respect to \(\alpha\). Analogous to Eq. \eqref{eq:429}, the GB coupling function \(\xi(\phi)\) is Taylor-expanded around a constant scalar field value \(\phi_0\). The first-order solutions in \(\alpha\), which are regular on the BH horizon at \(r = 2m\), are given by

\[
\hat{h}_1(r) = 0, \quad \hat{f}_1(r) = 0, \quad \hat{\phi}_1(r) = \hat{\phi}_1 + \frac{2(3\hat{r}^2 + 3\hat{r} + 4)\xi^{(1)}(\phi_0)}{3\eta m^2 \hat{r}^3},
\]

where \(\hat{\phi}_1\) is a constant and we have introduced the dimensionless coordinate \(\hat{r} \equiv r/m\). At spatial infinity, \(\hat{\phi}_1(r)\) approaches a constant \(\hat{\phi}_1\) with the derivative \(\hat{\phi}_1'(r)\) proportional to \(\hat{r}^{-2}\).

Similarly, the regular second-order solutions are

\[
\hat{h}_2(r) = \hat{h}_{2GB}(r) \equiv \frac{(147\hat{r}^5 + 174\hat{r}^4 + 228\hat{r}^3 - 1624\hat{r}^2 - 3488\hat{r} - 7360)\xi^{(1)}(\phi_0)^2}{120\eta m^4 M_{Pl}^4 \hat{r}^8},
\]

\[
\hat{f}_2(r) = \hat{f}_{2GB}(r) \equiv -\frac{(147\hat{r}^5 + 294\hat{r}^4 + 548\hat{r}^3 + 56\hat{r}^2 - 416\hat{r} - 1600)\xi^{(1)}(\phi_0)^2}{120\eta m^4 M_{Pl}^4 \hat{r}^8},
\]

\[
\hat{\phi}_2(r) = \hat{\phi}_{2GB}(r) \equiv \hat{\phi}_2 + \frac{2(3\hat{r}^2 + 3\hat{r} + 4)\xi^{(2)}(\phi_0)\hat{\phi}_1}{3\eta m^2 \hat{r}^3}
\]

\[
+ \frac{\xi^{(1)}(\phi_0)\xi^{(2)}(\phi_0)[1095(\hat{r}^5 + \hat{r}^4) + 1460\hat{r}^3 + 2190\hat{r}^2 + 1344\hat{r} + 800]}{450\eta^2 m^4 \hat{r}^6},
\]

where \(\hat{\phi}_2\) is a constant. Since \(\hat{h}_2(r) \propto \hat{r}^{-1}\), \(\hat{f}_2(r) \propto \hat{r}^{-1}\), and \(\hat{\phi}_2(r) \propto \hat{r}^{-2}\) as \(\hat{r} \to \infty\), the first-order solutions are consistent with the asymptotically Minkowski metric. For the linear coupling \(\xi(\phi) \propto \phi\), we have \(\xi^{(2)}(\phi_0) = 0\), and hence \(\hat{\phi}_2(r) = \hat{\phi}_2\).

The regular third-order solutions are

\[
\hat{h}_3(r) = \hat{h}_{3GB}(r) \equiv (66319\hat{r}^8 + 86648\hat{r}^7 + 127306\hat{r}^6 - 174628\hat{r}^5 - 1046036\hat{r}^4 - 2874280\hat{r}^3 - 680960\hat{r}^2 - 5948320\hat{r}
\]

\[
- 4659200) \xi^{(1)}(\phi_0)^2 \xi^{(2)}(\phi_0) + \frac{(147\hat{r}^5 + 174\hat{r}^4 + 228\hat{r}^3 - 1624\hat{r}^2 - 3488\hat{r} - 7360)\xi^{(1)}(\phi_0)\xi^{(2)}(\phi_0)}{60\eta m^4 M_{Pl}^4 \hat{r}^8},
\]

\[
\hat{f}_3(r) = \hat{f}_{3GB}(r) \equiv -(66319\hat{r}^8 + 132638\hat{r}^7 + 249946\hat{r}^6 + 285272\hat{r}^5 + 199852\hat{r}^4 - 989206\hat{r}^3 - 981200\hat{r}^2 - 847840\hat{r}
\]

\[
- 716800) \xi^{(1)}(\phi_0)^2 \xi^{(2)}(\phi_0) + \frac{(147\hat{r}^5 + 294\hat{r}^4 + 548\hat{r}^3 + 56\hat{r}^2 - 416\hat{r} - 1600)\xi^{(1)}(\phi_0)\xi^{(2)}(\phi_0)}{60\eta m^4 M_{Pl}^4 \hat{r}^8},
\]

\[
\hat{\phi}_3(r) = \hat{\phi}_{3GB}(r) \equiv \hat{\phi}_3 + \frac{\varphi_{3GB}(\hat{r})}{23814000 \eta^3 m^6 M_{Pl}^2 \hat{r}^8},
\]
where $\hat{\phi}_3$ is a constant and $\varphi_{3\text{GB}}(\hat{r})$ is an eighth-degree polynomial of $\hat{r}$. For the linear coupling $\xi(\phi) \propto \phi$, $\hat{h}_3(r)$ and $\hat{f}_3(r)$ vanish identically, which is consistent with the result of [53, 60]. Deriving higher-order solutions, we find that, in the limit $r \to \infty$, the metric components and scalar field behave as

$$\hat{h}_j(r) \propto \hat{r}^{-1}, \quad \hat{f}_j(r) \propto \hat{r}^{-1}, \quad \hat{\phi}'_j(r) \propto \hat{r}^{-2} \quad (j \geq 2). \quad (5.11)$$

Hence, the metric is asymptotically Minkowski at all orders. We note that the constant parts of the scalar field $\phi_0$, $\hat{\phi}_1, \cdots$ are determined by the boundary conditions at spatial infinity.

For the linear GB coupling which respects the shift symmetry, it follows that $\xi^{(1)}(\phi_0) = \text{constant}$ and $\xi^{(n)}(\phi_0) = 0$ for $n \geq 2$. Then, the $\phi_0$-dependence disappears from all the expressions of $\hat{h}_j(r)$, $\hat{f}_j(r)$, and $\hat{\phi}'_j(r)$ with $j \geq 1$. This reflects the property of shift-symmetric theories in which the field value itself does not matter, so that we can set $\hat{\phi}_j = 0$ ($j \geq 1$). In this case, we realize asymptotically Minkowski hairy BH solutions where only the even-order ($j = 2, 4, \cdots$) terms of metric components and the odd-order ($j = 1, 3, \cdots$) terms of scalar field are nonvanishing [60, 74].

For general non-shift-symmetric GB couplings, the $\phi_0$-dependence remains in the metric components and scalar field. For positive power-law couplings $\xi(\phi) \propto \phi^n$ with integer $n \geq 2$, in the limit $\phi_0 \to 0$, both $\hat{h}_j$ and $\hat{f}_j$ vanish for all $j \geq 1$ and the Schwarzschild solution with a constant scalar field is recovered. Thus, in contrast to the case of $n = 1$, a nonvanishing value of $\phi_0$ is necessary to realize hairy BH solutions. Provided that $\phi_0 \neq 0$, the metric components $\hat{h}_j$ and $\hat{f}_j$ ($j \geq 2$) are subject to deviations from those in the Schwarzschild metric with the nonvanishing field derivative $\hat{\phi}'_j(r)$.

The quantities associated with the conditions for the absence of ghost or Laplacian instabilities of odd-parity perturbations are estimated as

$$\mathcal{F} = M_{Pl}^2 + \frac{16(2\hat{r}^3 + \hat{r}^2 + 2\hat{r} - 36)\xi^{(1)}(\phi_0)^2}{\eta m^4 \hat{r}^6} \alpha^2 + \mathcal{O}(\alpha^3), \quad (5.12)$$

$$\mathcal{G} = M_{Pl}^2 + \frac{16(\hat{r}^2 + 2\hat{r} + 4)\xi^{(1)}(\phi_0)^2}{\eta m^4 \hat{r}^6} \alpha^2 + \mathcal{O}(\alpha^3), \quad (5.13)$$

$$\mathcal{H} = M_{Pl}^2 + \frac{16(\hat{r}^3 - 8)\xi^{(1)}(\phi_0)^2}{\eta m^4 \hat{r}^6} \alpha^2 + \mathcal{O}(\alpha^3). \quad (5.14)$$

The next-to-leading-order terms of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are at most of order $\xi^{(1)}(\phi_0)^2 \alpha^2/(\eta m^4)$. Provided that

$$\frac{\xi^{(1)}(\phi_0)^2}{\eta m^4 M_{Pl}^2} \alpha^2 \ll 1, \quad (5.15)$$

there are neither ghost nor Laplacian instabilities in the odd-parity sector due to the dominance of the term $M_{Pl}^2$ in $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

In the even-parity sector, the quantity associated with the no-ghost condition is estimated as

$$\mathcal{K} = \frac{2(\hat{r}^2 + 2\hat{r} + 4)^2\xi^{(1)}(\phi_0)^2}{\eta m^4 \hat{r}^6} \alpha^2 + \mathcal{O}(\alpha^3), \quad (5.16)$$

and hence the ghost is absent for $\eta > 0$. The radial propagation speed squared $c_{r, \text{even}}^2$ of the gravitational perturbation, which is equivalent to $c_{r, \text{odd}}^2 = \mathcal{G}/\mathcal{F}$ in the odd-parity sector, is given by

$$c_{r, \text{even}}^2 = c_{r, \text{odd}}^2 = 1 + \frac{32(\hat{r} - 2)(\hat{r}^2 + 3\hat{r} + 8)\xi^{(1)}(\phi_0)^2}{\eta m^4 M_{Pl}^2 \hat{r}^6} \alpha^2 + \mathcal{O}(\alpha^3). \quad (5.17)$$

On the horizon ($\hat{r} = 2$), the next-to-leading-order term of Eq. (5.17) vanishes, so $c_{r, \text{even}}^2$ is close to the (squared) speed of light. In the vicinity of the horizon, $c_{r, \text{even}}^2$ deviates from unity, but it quickly decreases as $|c_{r, \text{even}}^2 - 1| \propto \hat{r}^{-3}$. Under the condition $\xi^{(1)}(\phi_0) \neq 0$, it is possible to satisfy the bound of speed of GWs given in Refs. [108, 110]. For the GB couplings satisfying $\xi^{(2)}(\phi_0) \neq 0$, the squared propagation speed of the scalar field perturbation $\delta \phi$ in the even-parity sector is generally of order unity

$$c_{r, \text{even}}^2 = 1 + \mathcal{O}(\alpha^2). \quad (5.18)$$

The linear coupling $\xi(\phi) \propto \phi$ gives rise to further suppression for the deviation of $c_{r, \text{even}}^2$ from unity, such that $c_{r, \text{even}}^2 = 1 + \mathcal{O}(\alpha^4) [74]$. The squared angular propagation speeds in the even-parity sector can be estimated as

$$c_{\Omega, \pm}^2 = 1 \pm \frac{24\xi^{(1)}(\phi_0)}{m^2 M_{Pl}^3} \sqrt{\frac{\eta}{\xi^{(1)}(\phi_0)}} + \mathcal{O}(\alpha^2). \quad (5.19)$$
where the double signs are in the same order and we have used the no-ghost condition \( \eta > 0 \). The above results show that, in the limit \(|\alpha| \ll 1\) with \( \eta > 0 \), all the conditions for the absence of ghost/Laplacian instabilities against odd- and even-parity sectors are consistently satisfied for asymptotically Minkowski hairy BH solutions present for the models of the form (5.3).

As mentioned previously, the above solutions (5.4)–(5.10) accommodate asymptotically Minkowski BH solutions for the linear coupling \( \xi(\phi) \propto \phi \) in the small coupling limit \(|\alpha| \ll 1\) [59, 60]. On the other hand, for the couplings of the form \( \xi(\phi) = \sum_{j \geq 1} c_j \phi^{2j} \) with \( c_j \) being constants, including \( \xi(\phi) \propto c_2 \phi^2 + c_4 \phi^4 \) (with \( c_2 > 0 \)) [62, 64] or \( \xi(\phi) \propto 1 - e^{-k\phi^2} \) \((k > 0)\), the solutions (5.4)–(5.10) do not incorporate BHs realized as the consequence of spontaneous scalarization, by reflecting the fact that scalarized BHs can be obtained only nonperturbatively for \(|\alpha| = \mathcal{O}(1)\).

### B. GB couplings with other interactions

We also study how the hairy BH solutions discussed in Sec. [V A] are subject to modifications by taking into account power-law coupling functions \((-X)^p\) or \(\phi^q\) to the GB theory given by Eq. (5.3). For simplicity, we consider lowest-order power-law functions in most cases, but we will also study theories containing the couplings \( G_3 \supset \gamma_3 \ln(-X) \) and \( G_4 \supset \gamma_4 \sqrt{-X} \). In scalar-GB theories with the quadratic potential \( V(\phi) = \mu_2 \phi^2 \) in \( G_2 \), the presence of hairy BH solutions was numerically confirmed in Ref. [111]. Hence, we do not analyze the same model here. In the presence of the term \( \alpha_3 \phi^2 \) in \( G_2 \), using the expansions (4.27) and (4.28) with respect to the small coupling constant \( \alpha \) shows that there are no corrections to \( \hat{h}_j, \hat{f}_j, \) and \( \hat{\phi}_j \) derived in GB theories [18] up to the order \( j = 3 \). Similarly, in the presence of \( G_2 \supset \alpha_2 \phi^n \) \((n \geq 3)\), the nontrivial corrections do not appear up to the order \( j = n + 1 \).

#### 1. Cubic and GB couplings

The \( \phi \)-dependent cubic coupling \( G_3(\phi) \) is equivalent to the term \(-2XG_{3,\phi} \) in \( G_2 \) [17], so adding the linear coupling \( \alpha \mu_3 \phi \) to \( G_3 \) does not modify the structure of the theory (5.3). We then consider

\[
G_3(\phi) \supset \alpha \mu_3 \phi^2, \tag{5.20}
\]

with \( \mu_3 \) being a nonvanishing constant, which is equivalent to \(-4\alpha \mu_3 \phi X \) in \( G_2 \). We perform the expansions (4.27) and (4.28) in terms of the small coupling constant \( \alpha \). The first-order solutions regular on the horizon are equivalent to those in Eq. (5.21), while the second- and third-order solutions are

\[
\begin{align*}
\hat{h}_2(r) &= \hat{h}_{2\text{GB}}(r), & \hat{f}_2(r) &= \hat{f}_{2\text{GB}}(r), & \hat{\phi}_2(r) &= \hat{\phi}_{2\text{GB}}(r) + \frac{8(3\hat{r}^2 + 3\hat{r} + 4)\phi_0 \xi(1)(\phi_0) \mu_3}{3\eta^2 m^2 \hat{r}^3}, \\
\hat{h}_3(r) &= \hat{h}_{3\text{GB}}(r) + \frac{(147\hat{r}^5 + 174\hat{r}^4 + 228\hat{r}^3 - 1624\hat{r}^2 - 3488\hat{r} - 7360)\phi_0 \xi(1)(\phi_0)^2 \mu_3}{30\eta^2 m^3 M_{\text{Pl}}^2 \hat{r}^6}, \\
\hat{f}_3(r) &= \hat{f}_{3\text{GB}}(r) + \frac{(147\hat{r}^5 + 294\hat{r}^4 + 548\hat{r}^3 + 56\hat{r}^2 - 416\hat{r} - 1600)\phi_0 \xi(1)(\phi_0)^2 \mu_3}{30\eta^2 m^4 M_{\text{Pl}}^2 \hat{r}^6}, \\
\hat{\phi}_3(r) &= \hat{\phi}_{3\text{GB}}(r) + \frac{\mu_3}{\eta^4 m^6 \hat{r}^5} \varphi_3(\hat{r}),
\end{align*}
\]

where \( \varphi_3(\hat{r}) \) is the fifth degree polynomial of \( \hat{r} \). The cubic coupling (5.20) gives rise to modifications in \( \hat{\phi}_2(r), \hat{h}_3(r), \hat{f}_3(r), \) and \( \hat{\phi}_3(r) \) in comparison to those derived for the GB couplings. At large distances, these new terms have the same radial dependence as their leading-order terms.

Using the expanded solutions with \( |\phi_0| \) at most of order \( M_{\text{Pl}} \), it follows that the conditions for the absence of ghost/Laplacian instabilities against odd- and even-parity perturbations are also satisfied for \(|M_{\text{Pl}} \mu_3| \lesssim 1\), \(|\alpha| \ll 1\), and \( \eta > 0 \). Then, the cubic coupling \( G_3(\phi) \supset \alpha \mu_3 \phi^2 \) besides the GB coupling \( \alpha \xi(\phi) G \) leads to the existence of hairy BH solutions free from ghost or Laplacian instabilities. Similarly, for more general \( \phi \)-dependent cubic coupling \( G_3 \supset \alpha \mu_3 \phi \), nontrivial corrections to the metric functions and those to the scalar field show up at the orders \( j = 3 \) and \( j = 2 \), respectively.

Second, we discuss the case in which the cubic Galileon coupling

\[
G_3(X) \supset \alpha \alpha_3 X, \tag{5.25}
\]
with $\alpha_3$ being a nonvanishing constant, is present besides GB coupling $\alpha \xi(\phi) R_{GB}^2$. Then, the corrections to the BH solutions in GB theories, up to the order $j = 3$, appear only in $\hat{\phi}_3(r)$ as

$$
\hat{\phi}_3(r) = \hat{\phi}_{3GB}(r) - \frac{2(21\hat{r}^5 + 42\hat{r}^4 + 84\hat{r}^3 - 24\hat{r}^2 - 84\hat{r} - 224)\xi^{(1)}(\phi_0)^2\alpha_3}{21\hat{r}^3m^3\hat{r}^9}.
$$

At large distances, the correction to $\hat{\phi}_3(r)$ arising from the cubic Galileon is proportional to $\hat{r}^{-5}$, which decays faster than $\hat{\phi}_{3GB}(r) \propto \hat{r}^{-2}$. Since the metric components are not modified up to the order $j = 3$, the cubic Galileon does not induce strong modifications to hairy GB BHs in comparison to the coupling $G_3(\hat{\phi}) \supset \alpha_4 \chi^2$. The absence of ghost/Laplacian instabilities of BHs is also ensured for $|M_\nu \alpha_3/m^2| \lesssim 1$, $|\alpha| \ll 1$, and $\eta > 0$. Similarly, in the presence of $G_3 \supset \alpha_3 X^n$ ($n \geq 3$), nontrivial corrections to the scalar field show up at the order $j = n + 2$.

The next example is the cubic logarithmic interaction given by

$$
G_3 \supset \alpha \gamma_3 \ln(-X),
$$

with $\gamma_3$ being a nonvanishing constant, which belongs to the couplings in Eq. (1.15). The first-order solutions in $\alpha$ regular on the horizon ($\hat{r} = 2$) are

$$
\hat{h}_1(r) = 0, \quad \hat{f}_1(r) = 0, \quad \hat{\phi}_1'(r) = -\frac{2}{\eta\hat{m}^3\hat{r}^4} \left[ (\hat{r}^2 + 2\hat{r} + 4)\xi^{(1)}(\phi_0) + 2m^2\hat{r}^3\gamma_3 \right].
$$

At large distances, the leading-order contributions to $\hat{\phi}_1'(r)$ arise from the cubic logarithmic coupling. The second-order solutions regular on the horizon are given by

$$
\hat{h}_2(r) = \hat{h}_{2GB}(r) + \frac{\gamma_3}{2\eta M_{Pl}^2} \left[ \frac{(3\hat{r}^2 + 10\hat{r} - 40)\xi^{(1)}(\phi_0)}{m^2\hat{r}^3} + \frac{8\gamma_3}{3\hat{r} - 2} \ln \left( \frac{\hat{r}}{\sqrt{2}} \right) \right],
$$

$$
\hat{f}_2(r) = \hat{f}_{2GB}(r) - \frac{\gamma_3}{6\eta M_{Pl}^2} \left[ \frac{(9\hat{r}^2 + 18\hat{r} - 88)\xi^{(1)}(\phi_0)}{m^2\hat{r}^3} + 24\gamma_3 \frac{\hat{r} - 1}{3\hat{r} - 2} \ln \left( \frac{\hat{r}}{\sqrt{2}} \right) \right],
$$

$$
\hat{\phi}_2'(r) = \hat{\phi}_{2GB}(r) - \frac{64\xi^{(2)}(\phi_0)\gamma_3}{\eta^2\hat{m}^3\hat{r}^4(3\hat{r} - 2)} \ln \left( \frac{\hat{r}}{\sqrt{2}} \right).
$$

At spatial infinity, the metric component $\hat{f}_2(r)$ exhibits the logarithmic divergence

$$
\hat{f}_2(r) \to -\frac{4\gamma_3^2}{\eta M_{Pl}^2} \ln \hat{r} \quad \text{as} \quad \hat{r} \to \infty.
$$

This means that, even in the presence of the GB couplings, the cubic logarithmic coupling prevents the realization of asymptotically Minkowski hairy BH solutions.

2. Quartic and GB couplings

We proceed to the model of a linear nonminimal coupling

$$
G_4 \supset \alpha \mu_4 \phi,
$$

with $\mu_4$ being a nonvanishing constant, besides the GB coupling $\alpha \xi(\phi) R_{GB}^2$. Then, we find that the first-order solutions in $\alpha$ regular on the horizon are the same as those derived in Eq. (5.4). The second-order solutions are given by

$$
\hat{h}_2(r) = \hat{h}_{2GB}(r) - \frac{(\hat{r} + 4)(\hat{r} + 10)\xi^{(1)}(\phi_0)\mu_4}{6\eta m^2 M_{Pl}^2 \hat{r}^3},
$$

$$
\hat{f}_2(r) = \hat{f}_{2GB}(r) - \frac{(23\hat{r}^2 + 22\hat{r} + 24)\xi^{(1)}(\phi_0)\mu_4}{6\eta m^2 M_{Pl}^2 \hat{r}^3},
$$

$$
\hat{\phi}_2(r) = \hat{\phi}_{2GB}(r).
$$

The linear nonminimal coupling affects $\hat{h}_2(r)$ and $\hat{f}_2(r)$, while its effect does not appear in $\hat{\phi}_2(r)$. In comparison to the cubic-order interactions discussed in Sec. 5B.1, the modifications to the background geometry arising from the quartic coupling $\alpha \mu_4 \phi$ in $G_4$ already appear at the order of $j = 2$. Higher-order solutions of $\hat{h}_j(r)$, $\hat{f}_j(r)$, and $\hat{\phi}_j(r)$
(j ≥ 3) also receive corrections from the linear nonminimal coupling. Since all \( \hat{f}_j(r), \hat{h}_j(r), \) and \( \hat{\phi}_j(r) \) \( (j \geq 1) \) vanish at spatial infinity, the resulting hairy BH solutions are asymptotically Minkowski. Similarly, for the \( \phi \)-dependent coupling function \( G_4 \geq \alpha \mu_4(\phi) \), nontrivial corrections to the metric components show up at the order \( j = 2 \).

Using the expanded solutions, the quantities associated with odd-parity perturbations are given by

\[
\mathcal{F} = M_{\text{Pl}}^2 + 2\phi_0\mu_4\alpha + O(\alpha^2), \quad G = M_{\text{Pl}}^2 + 2\phi_0\mu_4\alpha + O(\alpha^2), \quad \mathcal{H} = M_{\text{Pl}}^2 + 2\phi_0\mu_4\alpha + O(\alpha^2),
\]

so that the linear nonminimal coupling gives rise to corrections of order \( \alpha \). We can avoid the ghost and Laplacian instabilities under the condition

\[
M_{\text{Pl}}^2 + 2\phi_0\mu_4\alpha > 0.
\]

As long as \( |\alpha| \ll 1 \) and \( |\mu_4/M_{\text{Pl}}| \ll 1 \), this condition is satisfied for \( |\phi_0| \lesssim M_{\text{Pl}} \). We note that the radial and angular propagation speed squares for the odd modes, \( c^2_{r,\text{odd}} = \mathcal{G}/\mathcal{F} \) and \( c^2_{\Omega,\text{odd}} = \mathcal{G}/\mathcal{H} \), are both \( 1 + O(\alpha^2) \).

Up to the order of \( \alpha^2 \), the quantity \( K \) is the same as that given in Eq. (5.10). The propagation speed squared of the scalar field perturbation \( \delta \phi \) in the even-parity sector is estimated as \( c^2_{r,\text{even}} = 1 + O(\alpha^2) \) for theories with \( \xi(\phi) \neq 0 \).

For the linear GB coupling \( \xi(\phi) \propto \phi \), we have \( c^2_{r,\text{even}} = 1 + O(\alpha^3) \), where the terms of order \( \alpha^3 \) arise from \( \alpha \mu_4(\phi) \) in \( G_4 \). Up to the linear order in \( \alpha \), the squared angular propagation speeds \( c^2_{\Omega,\pm} \) in the even-parity sector are identical to those in Eq. (5.19). These discussions show that, provided \( |\alpha| \ll 1, |\mu_4/M_{\text{Pl}}| \ll 1, \eta > 0 \), and \( |\phi_0| \lesssim M_{\text{Pl}} \), there are neither ghost nor Laplacian instabilities for hairy BHs discussed above.

Instead of the linear nonminimal coupling of the form \( \beta_3 \phi R \), we can also consider nonminimal couplings with higher-order powers, i.e., \( \alpha \mu_4 \phi^p R \) with \( p \geq 2 \), besides the GB couplings (5.12). As in the case of \( p = 1 \), the contributions to \( f \) and \( h \) from \( \mu_4 \) appear at the order of \( j = 2 \), while the scalar-field derivative starts to receive corrections from the third order. In such models, we can also realize asymptotically Minkowski hairy BHs satisfying all the conditions for the absence of ghost/Laplacian instabilities.

Let us next study the model of quartic derivative couplings of the form

\[
G_4 \supset \alpha \gamma_4 X,
\]

with \( \alpha_4 \) being a nonvanishing constant, besides the GB coupling. Performing the expansions (4.27) and (4.28), we obtain the same first- and second-order regular solutions as those given in Eqs. (5.4) and (5.5)–(5.7). The third-order solutions are

\[
\hat{h}_3(r) = \hat{h}_{3\text{GB}}(r) + \frac{4(\hat{r}^2 + 2\hat{r} + 4)^2(\hat{r} - 2)\xi^{(1)}(\phi_0)^2\alpha_4}{\eta^2m^6M_{\text{Pl}}^4\hat{r}^3}, \quad \hat{f}_3(r) = \hat{f}_{3\text{GB}}(r), \quad \hat{\phi}_3(r) = \hat{\phi}_{3\text{GB}}(r).
\]

At this order, the effect of the quartic derivative coupling appears only in the expression of \( \hat{h}_3(r) \). This correction vanishes on the horizon, with the asymptotic behavior \( \hat{h}_3(r) \sim \hat{h}_{3\text{GB}}(r) \propto \hat{r}^{-4} \) at spatial infinity.

We recall that the BH solutions with \( X_s \neq 0 \) realized by quartic derivative interactions without GB couplings are prone to the instability problem around the horizon [74]. On the other hand, the GB coupling besides the term \( \alpha \gamma_4 X \) in \( G_4 \) gives rise to asymptotically Minkowski hairy BH solutions. The leading-order corrections due to \( \alpha \) to \( \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{K}, c^2_{r,\text{even}}, \) and \( c^2_{\Omega,\pm} \) are the same as those given in Eqs. (5.12), (5.13), (5.14), (5.16), (5.18), and (5.19), respectively, so these hairy solutions can satisfy all the conditions for the absence of ghost/Laplacian instabilities.

For quartic derivative interactions with higher-order powers, i.e., \( G_4 \supset \alpha \gamma_4 X^n \ (n \geq 2) \), nontrivial corrections to the metric components or the scalar field show up at the order \( j = n + 2 \). In such models there are BH solutions consistent with conditions for the absence of ghost/Laplacian instabilities and strong coupling problems, but it is difficult to distinguish them from those realized by GB couplings alone.

We also study the model given by

\[
G_4 \supset \alpha \gamma_4 \sqrt{-X},
\]

with \( \gamma_4 \) being a nonvanishing constant, which belongs to couplings in Eq. (4.19). In the absence of the GB coupling, this model gives rise to an exact BH solution (92), but it is unstable due to the property \( X_s \neq 0 \) on the horizon [74]. The first-order solutions with respect to the GB coupling constant \( \alpha \) are

\[
\hat{h}_1(r) = 0, \quad \hat{f}_1(r) = 0, \quad \hat{\phi}_1(r) = \frac{C_0m^2r^3 + m^2\sqrt{2(\hat{r} - 2)}\gamma_4 + 16\xi^{(1)}(\phi_0)}{\eta m^4r^4(\hat{r} - 2)},
\]

(4.22)
where \( C \) is an integration constant. The numerator of \( \hat{\phi}_1'(r) \) needs to vanish for its regularity at \( r = 2m \), which gives \( C = -2\xi^{(1)}(\phi_0)/(\eta m) \). Then, we obtain the following solution

\[
\hat{\phi}_1'(r) = -\frac{2(r^2 + 2\hat{r} + 4)\xi^{(1)}(\phi_0)}{\eta m^3 r^4} + \frac{\sqrt{2} \gamma_4}{\eta m^2 r^{3/2}} \sqrt{r - 2}.
\]

(5.43)

For \( \gamma_4 \neq 0 \), there is still the divergence of \( \hat{\phi}_1'(r) \) at \( \hat{r} = 2 \). The leading-order term of \( X \) on the horizon is a nonvanishing constant given by \( X_s = -\gamma_4^2/(16\eta^2 m^4) \). Hence, even in the presence of the GB term, the quartic coupling \( \alpha \gamma_4 \sqrt{-X} \) in \( G_4 \) violates the conditions for the absence of ghost/Laplacian instabilities of hairy BH solutions.

3. Quintic and GB couplings

The linear coupling \( \mu_5 \phi \) in \( G_5 \) is equivalent to the quartic derivative coupling \( -\mu_5 X \) in \( G_4 \) \[16\], so we already studied such a case in Sec. \[VI B 2\]. Let us then consider the coupling

\[
G_5 \supset \alpha \mu_5 \phi^2,
\]

(5.44)

with \( \mu_5 \) being a nonvanishing constant, besides the GB coupling. The first- and second-order regular solutions are equivalent to those derived in Eqs. \[5.3\] and \[5.5\]–\[5.7\], while the third-order solutions are given by

\[
\hat{h}_3(r) = \hat{h}_{3\text{GB}}(r) - \frac{8(r^2 + 2\hat{r} + 4)^2(\hat{r} - 2)\phi_0 \xi^{(1)}(\phi_0) \mu_5}{\eta^2 m^6 M_{\text{Pl}} r^{12}}, \quad \hat{f}_3(r) = \hat{f}_{3\text{GB}}(r), \quad \hat{\phi}_3(r) = \hat{\phi}_{3\text{GB}}(r).
\]

(5.45)

They are similar to those derived for the quartic derivative coupling \( G_4 \supset \alpha \mu_5 \phi \) [see Eq. \[5.40\]]. Up to the order \( j = 3 \), the quintic coupling \( \alpha \mu_5 \phi^2 \) affects only \( \hat{h}_3(r) \). Similarly, for more general \( \phi \)-dependent quintic coupling \( G_5 \supset \alpha \mu_5 (\phi) \), nontrivial corrections to the metric component \( h(r) \) show up at the order \( j = 3 \). Using these expanded solutions and computing the quantities associated with the linear stability of perturbations, it follows that the hairy BH solutions can satisfy all the conditions for the absence of ghost/Laplacian instabilities for \( |\mu_5 M_{\text{Pl}}/m^2| \lesssim 1 \), \( |\alpha| \ll 1 \), and \( \eta > 0 \).

Finally, we consider the quintic derivative coupling given by

\[
G_5(X) \supset \alpha \alpha_5 X,
\]

(5.46)

with \( \alpha_5 \) being a nonvanishing constant, besides the GB coupling. The corrections to BH solutions in GB theories, up to the order \( j = 3 \), arise only in \( \hat{\phi}_3(r) \) as

\[
\hat{\phi}_3(r) = \hat{\phi}_{3\text{GB}}(r) + \frac{(88\hat{r}^5 + 77\hat{r}^4 - 1232\hat{r}^2 - 1792\hat{r} - 2464)\xi^{(1)}(\phi_0) \alpha_5}{7\eta^2 m^6 \hat{r}^{12}}.
\]

(5.47)

Similarly, in the presence of the term \( \alpha \alpha_5 X^n \) \( (n \geq 2) \) in \( G_5 \), nontrivial corrections to the metric components or the scalar field show up at higher order. This property is similar to that for the cubic derivative coupling \( G_3 \supset \alpha \alpha_3 X \) [see Eq. \[5.20\]]. In comparison to the cubic coupling, the scalar-field derivative \( \hat{\phi}_3'(r) \) is more strongly suppressed at large distances \( [\hat{\phi}_3'(r) \propto r^{-8}] \). Although there are hairy BH solutions satisfying all the conditions for the absence of ghost/Laplacian instabilities for \( |\alpha_3 M_{\text{Pl}}/m^4| \lesssim 1 \), \( |\alpha| \ll 1 \), and \( \eta > 0 \), it would be challenging to distinguish them from those present for the pure GB theories.

VI. BLACK HOLES IN \( F(R_{\text{GB}}^2) \) GRAVITY

In this section, we explore the BH solutions in gravitational theories where the Lagrangian contains an arbitrary function \( F(R_{\text{GB}}^2) \) of the GB curvature invariant \( R_{\text{GB}}^2 \), besides the Einstein-Hilbert term. As was pointed out in \[16\], \( F(R_{\text{GB}}^2) \) gravity can be embedded in Horndeski theories. We then generalize this \( F(R_{\text{GB}}^2) \)-equivalent Horndeski theory by adding a canonical kinetic term of the scalar field.

A. \( F(R_{\text{GB}}^2) \) gravity

Let us consider theories given by the Lagrangian

\[
\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R + F(R_{\text{GB}}^2),
\]

(6.1)
which can be equivalently expressed as

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R + F_\phi R_{\text{GB}}^2 - V(\phi),$$  \hfill (6.2)

where $\phi$ is a new scalar degree of freedom associated with the GB term and we have defined

$$V(\phi) \equiv F_{,\phi} \phi - F(\phi).$$  \hfill (6.3)

Indeed, varying the Lagrangian \((6.2)\) with respect to $\phi$ leads to

$$(\phi - R_{\text{GB}}^2) F_{,\phi\phi} = 0.$$  \hfill (6.4)

Therefore, provided that $F_{,\phi\phi} \neq 0$, we have $\phi = R_{\text{GB}}^2$, and hence the Lagrangian \((6.2)\) reduces to the original one in Eq. \((6.1)\). From Eq. \((6.2)\), we find that a scalar field $\phi$ with the potential $V(\phi)$ couples to the GB term of the form $F_{,\phi} R_{\text{GB}}^2$.

We introduce the following quantities:

$$\phi \equiv M_{\text{Pl}} m^4 \phi, \quad \xi(\phi) \equiv F_{,\phi},$$  \hfill (6.5)

where $2m$ corresponds to the horizon radius of a BH solution (if it exists). The quantity $\phi$ has mass dimension one, which we identify as the scalar field in Horndeski theories. Then, the Lagrangian for $F(R_{\text{GB}}^2)$ gravity is equivalent to

$$\mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R + \xi(\phi) R_{\text{GB}}^2 - V(\phi), \quad \text{where} \quad V(\phi) = \xi \phi - F.$$  \hfill (6.6)

For a given function $F(R_{\text{GB}}^2)$, the GB coupling $\xi(\phi)$ and the scalar potential $V(\phi)$ are fixed by using the correspondence \((6.5)\) with $\phi = R_{\text{GB}}^2$. In the language of Horndeski theories, the theory \((6.6)\) corresponds to the following choice of the coupling functions \([16, 98]\):

$$G_2 = -V(\phi) + 8 \xi^{(4)}(\phi) X^2 (3 - \ln |X|), \quad G_3 = 4 \xi^{(3)}(\phi) X (7 - 3 \ln |X|),$$

$$G_4 = \frac{M_{\text{Pl}}^2}{2} + 4 \xi^{(2)}(\phi) X (2 - \ln |X|), \quad G_5 = -4 \xi^{(1)}(\phi) \ln |X|.$$  \hfill (6.7)

In the following, for concreteness, we consider the power-law $F(R_{\text{GB}}^2)$ models given by

$$F(R_{\text{GB}}^2) = \beta (R_{\text{GB}}^2)^n,$$  \hfill (6.8)

where $\beta$ and $n$ are constants. Introducing the dimensionless coupling $\alpha = 2 m^2 - 4 n M_{\text{Pl}}^2 \beta$ and performing the expansions \((4.27)\) of metric components with respect to the small parameter $|\alpha| \ll 1$, the scalar potential is given by

$$V(\phi) = \alpha (n - 1) \frac{M_{\text{Pl}}^2}{2 m^2} \left( \frac{\phi}{M_{\text{Pl}}} \right)^n.$$  \hfill (6.9)

Apart from the specific powers $n = 0$ and $n = 1$, the scalar field has a nonvanishing effective mass squared $M_\phi^2 \equiv V_{,\phi\phi}$. For $n = 0$, we have $F = \beta$ constant and hence the resulting solution is the Schwarzschild–(anti-)de Sitter spacetime. When $n = 1$, we have $\xi = \beta$ constant and $V = 0$, so we end up with the no-hair Schwarzschild solution. Thus, we will focus on integer powers with $n \geq 2$.

The scalar-field equation at first order in $\alpha$ gives the relation

$$n(n - 1) \left[ r^6 \phi_0(r) - 48 m^6 M_{\text{Pl}} \right] \phi_0(r)^{n-2} = 0.$$  \hfill (6.10)

When $n = 2$, we have only the following solution:

$$\phi_0(r) = \frac{48 m^6 M_{\text{Pl}}}{r^6},$$  \hfill (6.11)

which corresponds to the GB term in the Schwarzschild spacetime. In this case, we obtain

$$\hat{h}_1(r) = \frac{5 r(\hat{r}^2 + 2)(\hat{r}^2 + 4)(\hat{r}^4 + 16) - 17152}{8 \hat{r}^9},$$
\[ \hat{f}_1(r) = -\frac{5\hat{r}(\hat{r} + 2)(\hat{r}^2 + 4)(\hat{r}^4 + 16) - 2816}{8\hat{r}^9}, \]
\[ \hat{\phi}_1(r) = \frac{12(5\hat{r}^9 - 101376\hat{r} + 217088)M_{\text{Pl}}}{\hat{r}^{15}}, \]

where we recall that \( \hat{r} = r/m \). Deriving the higher-order solutions as well, we find that the metric is asymptotically Minkowski at all orders. Using such expanded solutions for \( n = 2 \), the dominant terms in \( F, \ G, \) and \( H \) are \( M_{\text{Pl}}^2 \) with the corrections of order \( \alpha \), and hence the BH is free from ghost/Laplacian instabilities against odd-parity perturbations. However, the quantity associated with the no-ghost condition in the even-parity sector is given by

\[ K = -\frac{15925248M_{\text{Pl}}^2}{\hat{r}^{18}} \alpha^2 + \mathcal{O}(\alpha^3). \]

Since the leading-order term in \( K \) is negative, there is the ghost instability for even-parity perturbations. While \( \xi_{\text{even}}^2 = 1 + \mathcal{O}(\alpha) \), the quantities associated with the angular propagation speeds in the even-parity sector are \( B_1 = 1/2 + \mathcal{O}(\alpha) \) and \( B_2 = 1 + \mathcal{O}(\alpha) \), so the conditions (2.22) are also violated.

For \( n \geq 3 \), we have the same branch as Eq. (6.11) besides the branch \( \phi_0(r) = 0 \). In such cases, the ghost arises in the even-parity sector, with a similar behavior of \( B_1 \) and \( B_2 \) as in the case of \( n = 2 \). When \( n = 3 \), for example, we find

\[ K = -\frac{33025542528M_{\text{Pl}}^2}{\hat{r}^{30}} \alpha^2 + \mathcal{O}(\alpha^3), \]

whose leading-order term is negative, and hence the hairy BH solutions with \( \phi_0(r) = 48n^6M_{\text{Pl}}/\hat{r}^6 \) are prone to ghost instabilities. When \( n \geq 3 \), Eq. (6.10) admits the other branch \( \phi_0(r) = 0 \). In this case, the \( j \)th-order expanded solutions \((j \geq 1)\) of Eqs. (12) and (14) are given by

\[ \hat{h}_j(r) = 0, \quad \hat{f}_j(r) = 0, \quad \hat{\phi}_j(r) = 0, \]

which correspond to no-hair BHs.

We have thus shown that the power-law \( F(R^2_{\text{GB}}) \) models with (6.8) do not give rise to nontrivial BH solutions with scalar hair satisfying all the conditions for the absence of ghost/Laplacian instabilities. We also studied the logarithmic model with \( F(R^2_{\text{GB}}) = \beta \ln |R^2_{\text{GB}}| \) and reached the same conclusion.

**B. \( F(R^2_{\text{GB}}) \)-equivalent Horndeski theories with a scalar-field kinetic term**

We also study BHs in theories where the kinetic term \( \eta X \) is added to the \( F(R^2_{\text{GB}}) \)-equivalent Horndeski theories (6.6), i.e.,

\[ \mathcal{L} = \frac{M_{\text{Pl}}^2}{2} R + \eta X + \xi(\phi)R^2_{\text{GB}} - V(\phi), \]

where \( \xi(\phi) \) and \( V(\phi) \) are given by Eqs. (6.3) and (6.6), respectively.

Considering the power-law coupling functions of the form \( F(\phi) = \beta\phi^n \) analogous to Eq. (6.8) and introducing the dimensionless parameter \( \alpha = 2m^{2-n}\beta \), the scalar potential is given by the same form as that in Eq. (6.9). We first investigate the case of \( n = 2 \), which corresponds to the quadratic potential \( V(\phi) = \alpha\phi^2/(2m^2) \) with the linear GB coupling \( \xi(\phi) = \alpha m^2\beta M_{\text{Pl}}\phi \). Although \( V(\phi) \) and \( \xi(\phi) \) are similar to those in the model studied in Ref. 111, both \( V(\phi) \) and \( \xi(\phi) \) are proportional to \( \alpha \) in our case. This fact affects the resulting BH solutions derived by using the expansions (14.27) and (14.28) with respect to \( \alpha \).

As discussed in Sec. 111, the zeroth-order equation for the scalar field reduces to the differential equation (14.30). The solution to \( \phi_0(r) \) regular on the horizon is \( \phi_0(r) = \phi_0 = \) constant. The first-order regular solutions are given by

\[ \hat{h}_1(r) = \frac{(\hat{r}^2 + 2\hat{r} + 4)\phi_0^2}{12M_{\text{Pl}}^2}, \quad \hat{f}_1(r) = -\frac{\hat{r}(\hat{r} + 2)\phi_0^2}{12M_{\text{Pl}}^2}, \quad \hat{\phi}_1(r) = \frac{\phi_0\hat{r}^3 - 6M_{\text{Pl}}(\hat{r}^2 + 2\hat{r} + 4)}{3\eta m\hat{r}^4}. \]

The asymptotically Minkowski metric can be realized only if \( \phi_0 = 0 \), under which both \( \hat{h}_1(r) \) and \( \hat{f}_1(r) \) vanish with the dependence \( \hat{\phi}_1(r) \propto \hat{r}^{-2} \) at large distances.
At second order, the metric components \( \hat{h}_2(r) \) and \( \hat{f}_2(r) \) are equivalent to those derived by taking the limit of \( \xi^{(1)}(\phi_0)^2/(m^4M_{\text{Pl}}^2) \rightarrow 1 \) in Eqs. (5.5) and (5.6), respectively. The scalar-field derivative consistent with the regularity on the horizon (\( \hat{r} = 2 \)) yields

\[
\phi_2' = \frac{M_{\text{Pl}}}{3\eta^2m|\hat{r} - 2|} \left[ 3(\hat{r} + 4)(\hat{r} - 2) + 8 \ln \left( \frac{\hat{r}}{2} \right) \right],
\]

which has the dependence \( \phi_2'(\hat{r}) \rightarrow M_{\text{Pl}}/(\eta^2m) \) as \( \hat{r} \rightarrow \infty \). Then, the scalar field does not satisfy the boundary condition \( \phi'(r) \rightarrow 0 \) at spatial infinity [see Eq. (4.3)].

For \( n \geq 3 \), again the asymptotically Minkowski metric is realized only if \( \phi_0 = 0 \), under which \( \hat{h}_1(r) = 0 \) and \( \hat{f}_1(r) = 0 \). The first-order solution to the scalar-field derivative regular on the horizon is \( \phi_1'(r) = 0 \), so that \( \phi_1(r) = \tilde{\phi}_1 = \text{constant} \) for arbitrary \( r \). When \( n = 3 \), the second-order regular solutions are given by

\[
\hat{h}_2(r) = 0, \quad \hat{f}_2(r) = 0, \quad \hat{\phi}_2(r) = \frac{2(3\hat{r}^2 + 3\hat{r} + 4)}{\eta^3} \phi_1,
\]

where \( \hat{\phi}_2 \) is a constant. Similarly, the metric components of third-order solutions are \( \hat{h}_3(r) = 0 \) and \( \hat{f}_3(r) = 0 \), while the leading-order contribution to \( \hat{\phi}_3'(r) \) at spatial infinity is \( \hat{\phi}_2^2\hat{r}/(\eta m M_{\text{Pl}}) \). To satisfy the condition \( \phi_3'(\hat{r}) \rightarrow 0 \) as \( \hat{r} \rightarrow \infty \), we require that \( \phi_3 \equiv 0 \). We also find that the solutions compatible with the boundary conditions (4.3), \( \hat{\phi}_3'(\hat{r}) \rightarrow 0 \) at spatial infinity, are given by

\[
\hat{h}_j(r) = 0, \quad \hat{f}_j(r) = 0, \quad \hat{\phi}_j(r) = 0 \quad (j \geq 1).
\]

The same conclusion holds also for \( n \geq 4 \). Thus, for \( n \geq 3 \), we only have the Schwarzschild BH solutions without scalar hair. These results show the absence of asymptotically Minkowski hairy BH solutions, at least as long as the perturbative ansätze (4.27) and (4.28) are valid in the small coupling limit. This does not exclude the possibility for the existence of asymptotically Minkowski hairy BH solutions beyond the perturbative regime.

For a massive scalar field with the potential \( V(\phi) = M_{\text{Pl}}^2\phi^2/2 \), the property of BHs was studied in Ref. [11] for the linear GB coupling \( \xi(\phi) = \alpha \phi \). Since in this case the mass \( M_{\phi} \) is not related to the GB coupling \( \alpha \), the resulting BH solution is different from that discussed above for \( n = 2 \). Indeed, the second-order differential equation for \( \phi_1(r) \) contains a mass term \(-M_{\phi}^2\phi_1(r)\). In such a case, we do not have an analytic solution for \( \phi_1(r) \), so it requires numerical integration as performed in Ref. [11]. At spatial infinity, the scalar field solution is approximately given by the form \( \phi_1(r) \simeq C_1 e^{M_{\phi}r}/r + C_2 e^{-M_{\phi}r}/r \), with \( C_1 \) and \( C_2 \) being constants. In order to satisfy the boundary condition (4.3), namely \( \phi_1'(\infty) = 0 \), we need to choose the coefficient \( C_1 \) to be zero. The existence of hairy BH solutions was numerically confirmed for the case of quadratic potential \( V(\phi) = M_{\phi}^2\phi^2/2 \) with several different choices of GB couplings including the linear coupling \( \xi(\phi) = \alpha \phi \).

VII. CONCLUSIONS

In this paper, we scrutinized the existence and linear stability of static and spherically symmetric BH solutions with a static scalar field in full Horndeski theories without imposing the shift symmetry from the outset. For this purpose, we employed a perturbative method of deriving BH solutions, which is valid in the regime of small coupling constant(s). We then exploited the conditions for the absence of ghost/Laplacian instabilities against odd- and even-parity perturbations derived in Refs. [77]–[79], which are summarized in Sec. II. In particular, the angular propagation speed of even-parity perturbations plays an important role for ruling out some of the BH solutions by the Laplacian instability around the BH horizon. In shift-symmetric Horndeski theories, it was shown in Ref. [74] that hairy BH solutions present for theories with the k-essence Lagrangian \( G_2(X) \) and a nonminimal derivative coupling \( G_4(X)R \) [37] are subject to this generic instability around the horizon.

In Sec. III we extended the linear stability analysis for BHs in shift-symmetric theories performed in [74] to full Horndeski theories. For hairy BHs where the scalar-field kinetic term \( X \) is an analytic function of \( r \) with a nonvanishing value on the horizon (\( X_s \neq 0 \)), the product \( \mathcal{F}K\mathcal{B}_2 \) was shown to be negative for nonzero values of \( \kappa \) defined by Eq. (3.24). This implies that the linear stability conditions summarized in Sec. III cannot be satisfied simultaneously, and hence BHs with \( X_s \neq 0 \) are generally subject to either ghost or Laplacian instability. We also found that, as long as \( X_s \neq 0 \) with \( \kappa_s \neq 0 \), there is the divergence of the radial propagation speed of scalar-field perturbations. In Sec. III we presented examples of theories that give rise to the branch of unstable hairy BH solutions with \( X_s \neq 0 \). Our results show that, even in full Horndeski theories, BH solutions free from instabilities
should not have a nonvanishing $X_\ast$ in general. Under this condition, there is also a jump of $X$ across the horizon, so the BH solutions are physically unacceptable.

Given the generic instability for BHs with $X_\ast \neq 0$, for the search of hairy BHs which are free from ghost or Laplacian instabilities, we focused on theories leading to the solutions with $X_\ast = 0$, i.e., a finite scalar-field derivative $\phi'(r_\ast)$ on the horizon. Using the scalar-field equation of motion for theories containing the scalar-field kinetic term $\eta X$ and the Einstein-Hilbert term $M_\text{Pl}^2/2$ in the action, we discussed the possibility for realizing such hairy BHs in both shift-symmetric and non-shift-symmetric Horndeski theories in Sec. [V]. For the couplings of the form $G_I \supset \alpha_1(\phi) F_I(X)$ ($I = 2, 3, 4, 5$) with $\alpha_1(\phi)$ and $F_I(X)$ being arbitrary regular functions, the asymptotically Minkowski solutions respecting the regularity on the horizon are restricted to be no-hair solutions with $\phi'(r) = 0$. There are possibilities for evading this no-hair feature of BHs in theories given by the coupling functions [130, 139], which are not analytic at $X = 0$. However, the couplings $G_2 \supset \alpha_2(\phi) \sqrt{-X}$ and $G_4 \supset \alpha_4(\phi) \sqrt{-X}$ result in BH solutions with $X_\ast \neq 0$, so they are excluded in terms of the gradient or Laplacian instabilities around the horizon. For the cubic logarithmic coupling $G_3 \supset \alpha_3(\phi) \ln |X|$, where $\alpha_3(\phi)$ is an analytic function of $\phi$, we showed the absence of asymptotically Minkowski hairy BH solutions.

The remaining theories allowing for the existence of hairy BH solutions should possess a quintic coupling of the form $G_5 = \alpha_5(\phi) \ln |X|$. With this coupling only, the background equations contain terms like $X^p \ln |X|$, which causes the breakdown of our perturbative analysis by using the expansions [127] and [128] with respect to a small coupling parameter. The scalar field coupled to the GB curvature invariant, $\xi(\phi) R^2_{GB}$, which is equivalent to the Horndeski functions [122], is the only exceptional case in which $\ln |X|$-dependent terms disappear from the background equations due to the presence of the other specific couplings $G_{2,3,4}$ besides $G_5 = \alpha_5(\phi) \ln |X|$.

In Sec. [V A] we derived the solutions to the metric components and the scalar field for the GB term $\alpha_5(\phi) R^2_{GB}$ with an analytic function $\xi(\phi)$ by resorting to the expansions with respect to the small GB coupling $\alpha$. We also showed that ghost and Laplacian instabilities of hairy BHs against odd- and even-parity perturbations are absent for the small GB coupling with $\eta > 0$. In Sec. [V B] we implemented positive power-law functions of $\phi$ or $X$ in $G_{2,3,4,5}$ besides the GB coupling $\alpha_5(\phi) R^2_{GB}$ and obtained new classes of hairy BH solutions free from ghost or Laplacian instabilities. Since all such hairy BH solutions disappear in the absence of the term $\alpha_5(\phi) \ln |X|$ in $G_5$, the existence of this form of quintic couplings is crucial for realizing asymptotically Minkowski hairy BH solutions. We also found that, even in the presence of the GB term $\alpha_5(\phi) R^2_{GB}$, the couplings $G_3 \supset \gamma_3 \ln(-X)$ or $G_4 \supset \gamma_4 \sqrt{-X}$ prevent the existence of asymptotically Minkowski solutions with scalar hair which do not suffer from ghost or Laplacian instabilities.

In Sec. [V I] we studied whether BH solutions free from ghost or Laplacian instabilities are present in $F(R^2_{GB})$ gravity where $F$ is a regular function of the GB term $R^2_{GB}$. For power-law couplings $F(R^2_{GB}) = \beta (R^2_{GB})^n$ with $n \geq 2$, we found a new class of asymptotically Minkowski BH solutions where the GB term plays a role of the new scalar degree of freedom. However, they are prone to the ghost instability of even-parity perturbations. Although we also studied $F(R^2_{GB})$-equivalent Horndeski theories with a scalar-field kinetic term $\eta X$, there are no hairy BH solutions with asymptotically Minkowski metric. These results show that the presence of the GB coupling $\alpha_5(\phi) R^2_{GB}$ besides the kinetic term $\eta X$ play a prominent role for realizing asymptotically Minkowski hairy BHs free from ghost or Laplacian instabilities in full Horndeski theories.

In Table we summarize the existence of hairy BH solutions and ghost/Laplacian instabilities for the theories studied in this paper. Theories (F) and (G) are the examples leading to asymptotically Minkowski BHs that can satisfy all the conditions for the absence of ghost or Laplacian instabilities against odd- and even-parity perturbations. It should be noted that both these theories contain the scalar-GB coupling $\alpha_5(\phi) R^2_{GB}$ It will be of interest to compute the sensitivity parameters as well as the quasinormal modes for such surviving BH solutions for the purpose of detecting signatures of the modification of gravity in future observations of GWs.

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[6] Our result is reminiscent of that in Ref. [113], where the authors showed that scalar-GB gravity is effectively the only theory within the Horndeski class that accommodates spontaneous scalarization of Schwarzschild BHs.
TABLE. Existence and the linear stability of asymptotically Minkowski hairy BHs for nine subclasses of Horndeski theories. Except for the theory (H), we assumed the presence of the canonical kinetic term \( \eta X \) in \( G_2 \) and the Einstein-Hilbert term \( M^2 \) in \( G_4 \). In the third column, “AM” means “asymptotically Minkowski.”

| Theory | Coupling functions | Hairy BHs | Stability of hairy BHs |
|--------|-------------------|-----------|-----------------------|
| (A) \( G_{2,3,4,5}(X) \) \( F_1(X) \) with regular \( F_1(X) \) | – | – |
| (B) \( G_{2,3,4,5}(\phi, X) \) \( \alpha_1(\phi) F_1(X) \) with regular \( \alpha_1(\phi) \) and \( F_1(X) \) | – | – |
| (C) \( G_2(\phi, X) \) \( \alpha_2(\phi) \sqrt{-X} \) | \( X_\pm \neq 0 \) | Unstable around the horizon |
| (D) \( G_3(\phi, X) \) \( \alpha_3(\phi) \ln |X| \) | \( X_\pm = 0 \), Non-AM | – |
| (E) \( G_4(\phi, X) \) \( \alpha_4(\phi) \sqrt{-X} \) | \( X_\pm \neq 0 \) | Unstable around the horizon |
| (F) GB coupling \( \alpha(\xi(\phi) R^2_{\text{GB}}) \) with regular \( \xi(\phi) \) | \( X_\pm = 0 \), AM | No ghost/Laplacian instability |
| (G) \( \alpha(\rho) R^2_{\text{GB}} \) plus regular functions of \( \phi \) and/or \( X \) in \( G_{2,3,4,5} \) | \( X_\pm = 0 \), AM | No ghost/Laplacian instability |
| (H) \( F(R^2_{\text{GB}}) \propto (R^2_{\text{GB}})^n \) gravity with \( n \geq 2 \) | AM | Ghost instability |
| (I) \( F(R^2_{\text{GB}}) \)-equivalent Horndeski theories with \( G_2 \propto \eta X \) | – | – |

Appendix A: Coefficients appearing in the background scalar-field equation

The coefficients \( \lambda_1 \sim \lambda_{12} \) in Eq. (2.9) are given by

\[
\lambda_1 = - \left( h' + \frac{4h}{r} + \frac{f'h}{f} \right) \phi' - 2h\phi'' , \quad \lambda_2 = -h\phi'^2 , \quad \lambda_3 = \frac{1}{2}h\phi'^2 (h'\phi' + 2h\phi'') , \\
\lambda_4 = \frac{2}{r^2} \left( 1 - h - r'h \right) + \frac{f'h^2}{2f^2} - \frac{r \left( 2f''h + f'h' \right) + 4f'h}{2fr} , \\
\lambda_5 = h\phi' \left[ \frac{8h'}{r} + \frac{6h}{r^2} - \frac{f'^2h}{2f^2} + \frac{(f''r + 6f'h + 2rf'h')}{fr} \right] \phi' + 3h \left( \frac{f'}{f} + \frac{4}{r} \right) \phi'' , \\
\lambda_6 = h^2\phi'^3 \left( \frac{f'}{f} + \frac{4}{r} \right) \left( h'\phi' + 2h\phi'' \right) , \\
\lambda_7 = -\frac{1}{2}h^2\phi'^3 \left( \frac{f'}{f} + \frac{4}{r} \right) (h'\phi' + 2h\phi'') , \\
\lambda_8 = \frac{1}{r^2} \left[ h'(3h - 1)\phi' + 2h(h - 1)\phi'' \right] - \frac{f'^2h^2\phi'}{f^2} + \frac{1}{fr^2} \left[ (2f''r + 3f')h^2\phi' + f'h(3rh' - 1)\phi' + 2f'h^2r\phi'' \right] , \\
\lambda_9 = \frac{h\phi'^2}{fr^2} \left[ f(h - 1) + f'hr \right] , \\
\lambda_{10} = \frac{h\phi'^2}{2r^2} \left[ 10h^2\phi'' + h(7h'\phi' - 2\phi'') - h'\phi' \right] + \frac{f'^2h^3\phi'^3}{2f^2r} - \frac{h^2\phi'^2}{2f^2r} \left[ (2f''r + 4f'h)\phi' + 10f'h\phi'' + 7f'h'r\phi' \right] , \\
\lambda_{11} = -\frac{h^3\phi'^4}{fr^2} (rf' + f) , \quad \lambda_{12} = \frac{h^3\phi'^4}{2f^2r} (rf' + f)(h'\phi' + 2h\phi'').
\]

Appendix B: Coefficients associated with perturbations

The quantities \( a_1, c_2, \) and \( c_4 \) in Eqs. (2.18) and (2.20) are given by

\[
a_1 = \sqrt{fh} \left[ G_{4,0} + \frac{1}{2}h(G_{3,0} - 2G_{4,0}X)\phi'^2 \right] r^2 + 2h\phi' \left[ G_{4,0} - G_{5,0} - \frac{1}{2}h(2G_{4,0}X - G_{5,0}X)\phi'^2 \right] r \\
+ \frac{1}{2}G_{5,0}XXh^3\phi'^4 - \frac{1}{2}G_{5,0}Xh(3h - 1)\phi'^2 \right] , \\
\]

\[
c_2 = \sqrt{fh} \left[ \left( \frac{1}{2}f - \frac{1}{2}h(3G_{3,0} - 8G_{4,0}X)\phi'^2 + \frac{1}{2}h^2(G_{3,0}XX - 2G_{4,0}XX)\phi'^4 - G_{4,0} \right) r^2 \\
- \frac{h\phi'}{f} \left( \frac{1}{2}h^2(2G_{4,0}XX - G_{5,0}XX)\phi'^4 - \frac{1}{2}h(12G_{4,0}XX - 7G_{5,0}XX)\phi'^2 + 3(G_{4,0}X - G_{5,0}X) \right) r \\
+ \frac{h\phi'^2}{4f} \left( G_{5,0}XXh^3\phi'^4 - G_{5,0}XXh(10h - 1)\phi'^2 + 3G_{5,0}(5h - 1) \right) \right] f' \\
+ \phi' \left[ \frac{1}{2}G_{2,0}X - G_{3,0}X - \frac{1}{2}h(G_{2,0}XX - G_{3,0}XX)\phi'^2 \right] r^2 \\
\]
The quantities $B_1$ and $B_2$ in Eq. (2.21) are
\[ B_1 = \frac{r^3}{4\sqrt{h}}[4h(\phi' a_1 + r\sqrt{\phi' h})\beta_1 + \beta_2 - 4\phi a_1 a_3] - 2fhG_r(\sqrt{\phi' h} - 2\phi')H(2\phi' a_1 + r\sqrt{\phi' h}) + 2\phi^2 a^2_1 P_1], \]
\[ B_2 = -r^2 h\beta_1[2fhFG(\phi' a_1 + r\sqrt{\phi' h}) + r^2 \beta_2] - r^4 \beta_2 \beta_3 - fhFG(\phi' hFG) a_1 + 2r^3 \sqrt{h}(\phi' h), \]
where
\[ \beta_1 = \frac{1}{\sqrt{\phi'}} \sqrt{\phi' h} e_4 - \phi'(\sqrt{\phi' h})\frac{\partial}{\partial \phi} \frac{\partial}{\partial h} \left[ \frac{\partial f}{\partial h} + \frac{f'}{f} \right] c_4 + \frac{\sqrt{\phi' h}}{\sqrt{f}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] \]
\[ \beta_2 = \frac{\sqrt{\phi' h}}{\sqrt{f}} \frac{\partial}{\partial h} \left[ \frac{\partial f}{\partial h} + \frac{f'}{f} \right] e_4 + \frac{\sqrt{\phi' h}}{\sqrt{f}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] \]
\[ \beta_3 = \frac{\sqrt{\phi' h}}{\sqrt{f}} \frac{\partial}{\partial h} \left[ \frac{\partial f}{\partial h} + \frac{f'}{f} \right] e_4 + \frac{\sqrt{\phi' h}}{\sqrt{f}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right], \]
with
\[ e_4 = \frac{1}{\sqrt{\phi'}} \frac{\partial}{\partial h} \left[ \frac{\partial f}{\partial h} + \frac{f'}{f} \right] e_4 + \frac{1}{\sqrt{\phi' h}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] c_4 + \frac{\sqrt{\phi' h}}{\sqrt{f}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] \]
\[ + r^2 h\beta_1[2fhFG(\phi' a_1 + r\sqrt{\phi' h}) + r^2 \beta_2] - r^4 \beta_2 \beta_3 - fhFG(\phi' hFG) a_1 + 2r^3 \sqrt{h}(\phi' h), \]
\[ d_3 = \frac{1}{r^2} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] \]
\[ + \frac{\sqrt{\phi' h} f}{\sqrt{f}} \frac{\partial}{\partial h} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right] e_4 + \frac{\sqrt{\phi' h} f}{\sqrt{f}} \left[ (f' - f) - f \frac{\partial f}{\partial h} \right]. \]

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