The geometry of the Kustaanheimo-Stiefel mapping

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This paper details the geometry of the Kustaanheimo-Stiefel mapping, which regularizes the Hamiltonian of the Kepler problem. It leans heavily on the paper [2] by J.-C. van der Meer. We use the theory of differential spaces, see [1, chpt VII §3] or [4].

1 A $\mathbb{T}^2$ action on $T\mathbb{R}^4$

Let $T\mathbb{R}^4 = \mathbb{R}^8$ be the tangent bundle of $\mathbb{R}^4$ with coordinates $(q, p)$ and standard symplectic form $\omega = \sum_{i=1}^{4} dq_i \wedge dp_i$. Let $\langle , \rangle$ be the Euclidean inner product on $\mathbb{R}^4$.

Consider a Hamiltonian action of the 2-torus $\mathbb{T}^2$ on $T\mathbb{R}^4$, which is generated by the flow $\varphi_t^{H_2}$ of the Hamiltonian vector field $X_{H_2}$ of the harmonic oscillator

$$H_2(q, p) = \frac{1}{2}(\langle p, p \rangle + \langle q, q \rangle)$$

and the flow $\varphi_s^\Xi$ of the Hamiltonian vector field $X_{\Xi}$ of the Hamiltonian

$$\Xi(q, p) = q_1 p_2 - q_2 p_1 + q_3 p_4 - q_4 p_1.$$

The momentum mapping of this $\mathbb{T}^2$ action is

$$\mathcal{J} : \mathbb{R}^8 \rightarrow \mathbb{R}_{\geq 0} \times \mathbb{R} : (q, p) \mapsto \left( H_2(q, p), \Xi(q, p) \right).$$

For every $(h, \xi)$ in the image of $\mathcal{J}$ we determine the $\mathbb{T}^2$ reduced space $\mathcal{J}^{-1}(h, \xi)/\mathbb{T}^2$. We do this in stages. First we find the reduced space $\Xi^{-1}(\xi)/S^1$ of the $S^1$ action generated by $\varphi_s^\Xi$. The algebra of polynomials on $\mathbb{R}^8$, which are invariant under the action $\varphi_s^\Xi$, is generated by

$$\pi_1 = q_1^2 + q_2^2, \quad \pi_2 = q_3^2 + q_4^2, \quad \pi_3 = p_1^2 + p_2^2,$$
$$\pi_4 = p_3^2 + p_4^2, \quad \pi_5 = q_1 p_1 + q_2 p_2, \quad \pi_6 = q_3 p_3 + q_4 p_4,$$
$$\pi_7 = q_1 p_2 - q_2 p_1 \quad \pi_8 = q_3 p_4 - q_4 p_3, \quad \pi_9 = q_1 q_4 - q_2 q_3,$$
$$\pi_{10} = q_1 q_3 + q_2 q_4, \quad \pi_{11} = p_1 p_4 - p_2 p_3, \quad \pi_{12} = p_1 p_3 + p_2 p_4,$$
$$\pi_{13} = q_1 p_4 - q_2 p_3, \quad \pi_{14} = q_1 p_3 + q_2 p_4, \quad \pi_{15} = q_4 p_1 - q_3 p_2,$$
$$\pi_{16} = q_3 p_1 + q_4 p_2.$$

The image of the orbit map

$$\Pi : \mathbb{R}^8 \rightarrow \mathbb{R}^{16} : (q, p) \mapsto \left( \pi_1(q, p), \ldots, \pi_{16}(q, p) \right)$$

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is the orbit space $\mathbb{R}^8/S^1$ of the $S^1$ action $\varphi^\mathbb{R}_{S^1}$, which is a semialgebraic subset of $\mathbb{R}^{16}$ that will be explicitly described below. $\mathbb{R}^8/S^1$ is a locally compact subcartesian differential space with differential structure $C^\infty(\mathbb{R}^8/S^1)$, where $f \in C^\infty(\mathbb{R}^8/S^1)$ if and only if $\Pi^* f \in C^\infty(\mathbb{R}^8)^{S^1}$, the space of smooth functions on $\mathbb{R}^8$ that are invariant under the $S^1$ action $\varphi^\mathbb{R}_{S^1}$.

We now show that $C^\infty(\mathbb{R}^8/S^1)$ has a Poisson structure. The quadratic polynomials $\pi_1, \ldots, \pi_{16}$ generate a Poisson subalgebra of the Poisson algebra of quadratic polynomials $\mathcal{Q}$ on $T\mathbb{R}^4$, using the Poisson bracket $\{ , \}$ associated to the symplectic form $\omega$, that is, for every $f, g \in \mathcal{Q}$ one has $\{ f, g \}_{\mathcal{Q}}(q, p) = \omega(q, p)(X_g(q, p), X_f(q, p))$ for every $(q, p) \in T\mathbb{R}^4$. Since every smooth function on the orbit space $\mathbb{R}^8/S^1$ is a smooth function of $\pi_1, \ldots, \pi_{16}$, the structure matrix $W_{C^\infty(\mathbb{R}^8/S^1)}$ of the Poisson bracket

$$\{ f, g \}_{\mathbb{R}^8/S^1} = \sum_{i,j=1}^{16} \frac{\partial g}{\partial \pi_i} \frac{\partial f}{\partial \pi_j} \{ \pi_i, \pi_j \}_{\mathcal{Q}}$$

for $f, g \in C^\infty(\mathbb{R}^8/S^1)$ is $W_\mathcal{Q}$.

Next we find an explicit description of the orbit space $\mathbb{R}^8/S^1$ as a semialgebraic variety in $\mathbb{R}^{16}$. For this we need a new set of generators of the algebra of invariant polynomials. Let

$$K_1 = -(\pi_{10} + \pi_{12}) = -(q_1 q_3 + q_2 q_4 + p_1 p_3 + p_2 p_4)$$
$$K_2 = -(\pi_9 + \pi_{11}) = -(q_1 q_4 - q_2 q_3 + p_1 p_4 - p_2 p_3)$$
$$K_3 = \frac{1}{2}(\pi_2 + \pi_4 - \pi_1 - \pi_3) = \frac{1}{2}(q_3^2 + q_4^2 + p_3^2 + p_4^2 - q_2^2 - p_1^2 - p_2^2)$$
$$L_1 = \pi_{15} - \pi_{13} = q_4 p_1 - q_3 p_2 + q_2 p_3 - q_1 p_4$$
$$L_2 = \pi_{14} - \pi_{16} = q_1 p_3 + q_2 p_4 - q_3 p_1 - q_4 p_2$$
$$L_3 = \pi_8 - \pi_7 = q_3 p_4 - q_2 p_3 + q_2 p_1 - q_1 p_2$$
$$H_2 = \frac{1}{2}(\pi_1 + \pi_2 + \pi_3 + \pi_4) = \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 + p_1^2 + p_2^2 + p_3^2 + p_4^2)$$
$$\Xi = \pi_7 + \pi_8 = q_1 p_2 - q_2 p_1 + q_3 p_4 - q_4 p_3$$
$$U_1 = -(\pi_5 + \pi_6) = -(q_1 p_4 + q_2 p_2 + q_3 p_3 + q_4 p_1)$$
$$U_2 = \pi_{10} - \pi_{12} = q_1 q_3 + q_2 q_4 - p_1 p_3 - p_2 p_4$$
$$U_3 = \pi_9 - \pi_{11} = q_1 q_4 - q_2 q_3 + p_2 p_3 - p_1 p_4$$
$$U_4 = \frac{1}{2}(\pi_1 - \pi_2 + \pi_3 - \pi_4) = \frac{1}{2}(q_1^2 + q_2^2 - q_3^2 - q_4^2 + p_1^2 + p_2^2 - p_3^2 - p_4^2)$$
$$V_1 = \frac{1}{2}(\pi_1 + \pi_2 - \pi_3 - \pi_4) = \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 - p_1^2 - p_2^2 - p_3^2 - p_4^2)$$
$$V_2 = \pi_{14} + \pi_{16} = q_1 p_3 + q_2 p_4 + q_3 p_1 + q_4 p_2$$
$$V_3 = \pi_{13} + \pi_{15} = q_1 p_4 - q_2 p_3 + q_4 p_1 - q_3 p_2$$
$$V_4 = \pi_5 - \pi_6 = q_1 p_1 + q_2 p_2 - q_3 p_3 - q_4 p_4.$$

The map

$$\mathbb{R}^{16} \to \mathbb{R}^{16} : (\pi_1, \ldots, \pi_{16}) \mapsto (K, L, H_2, \Xi; U, V)$$
is linear and invertible with inverse

$$
\begin{align*}
\pi_1 &= \frac{1}{2}(H_2 - K_3 + U_4 + V_1) \\
\pi_2 &= \frac{1}{2}(H_2 + K_3 - U_4 + V_1) \\
\pi_3 &= \frac{1}{2}(H_2 - K_3 - U_4 - V_1) \\
\pi_4 &= \frac{1}{2}(H_2 + K_3 + U_4 - V_1) \\
\pi_5 &= \frac{1}{2}(V_4 - U_1) \\
\pi_6 &= -\frac{1}{2}(U_1 + V_4) \\
\pi_7 &= \frac{1}{2}(\Xi - L_3) \\
\pi_8 &= \frac{1}{2}(\Xi + L_3) \\
\pi_9 &= \frac{1}{2}(U_3 - K_2) \\
\pi_{10} &= \frac{1}{2}(U_2 - K_1) \\
\pi_{11} &= -\frac{1}{2}(U_3 - K_2) \\
\pi_{12} &= -\frac{1}{2}(U_2 + K_1) \\
\pi_{13} &= \frac{1}{2}(V_3 - L_1) \\
\pi_{14} &= \frac{1}{2}(V_2 + L_2) \\
\pi_{15} &= \frac{1}{2}(V_3 + L_1) \\
\pi_{16} &= \frac{1}{2}(V_2 - L_2).
\end{align*}
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So \((K, L, H_2, \Xi; U, V)\) is another set of generators of the algebra of polynomials on \(\mathbb{R}^8\), which are invariant under the \(S^1\) action \(\varphi^T\). Instead of giving the structure matrix of the Poisson bracket on \(C^\infty(\mathbb{R}^8/S^1)\) we list the Poisson vector fields \(Y_G\) on \(\mathbb{R}^8/S^1\) induced from the Hamiltonian vector fields \(X_G\) on \((T\mathbb{R}^4, \omega)\), where \(G\) is one of the coordinate functions \((K, L, H_2, \Xi; U, V)\) listed in (3).
\begin{align*}
Y_2 &= 0 \\
Y_{U_1} &= 2U_2 \frac{\partial}{\partial K_1} + 2U_3 \frac{\partial}{\partial K_2} + 2U_4 \frac{\partial}{\partial K_3} - 2V_1 \frac{\partial}{\partial H_2} \\
&\quad + 2K_1 \frac{\partial}{\partial U_2} + 2K_2 \frac{\partial}{\partial U_3} + 2K_3 \frac{\partial}{\partial U_4} - 2H_2 \frac{\partial}{\partial V_1} \\
Y_{U_2} &= -2U_1 \frac{\partial}{\partial K_1} - 2U_4 \frac{\partial}{\partial L_2} + 2U_3 \frac{\partial}{\partial L_3} - 2V_2 \frac{\partial}{\partial H_2} \\
&\quad - 2K_2 \frac{\partial}{\partial U_1} + 2L_1 \frac{\partial}{\partial U_3} - 2L_2 \frac{\partial}{\partial U_4} - 2H_2 \frac{\partial}{\partial V_2} \\
Y_{U_3} &= -2U_1 \frac{\partial}{\partial K_2} + 2U_4 \frac{\partial}{\partial L_1} + 2U_2 \frac{\partial}{\partial L_3} - 2V_3 \frac{\partial}{\partial H_2} \\
&\quad - 2K_2 \frac{\partial}{\partial U_1} - 2L_1 \frac{\partial}{\partial U_2} - 2V_3 \frac{\partial}{\partial U_4} - 2H_2 \frac{\partial}{\partial V_4} \\
Y_{U_4} &= -2U_1 \frac{\partial}{\partial K_1} - 2U_5 \frac{\partial}{\partial L_1} + 2U_2 \frac{\partial}{\partial L_2} - 2V_4 \frac{\partial}{\partial H_2} \\
&\quad - 2K_3 \frac{\partial}{\partial U_1} + 2L_2 \frac{\partial}{\partial U_2} + 2V_3 \frac{\partial}{\partial U_3} - 2H_2 \frac{\partial}{\partial V_4} \\
Y_{V_1} &= 2V_2 \frac{\partial}{\partial K_1} - 2V_4 \frac{\partial}{\partial L_2} + 2V_3 \frac{\partial}{\partial L_3} + 2U_2 \frac{\partial}{\partial H_2} \\
&\quad + 2H_2 \frac{\partial}{\partial U_2} + 2K_1 \frac{\partial}{\partial V_2} + 2K_2 \frac{\partial}{\partial V_3} + 2U_4 \frac{\partial}{\partial V_4} \\
Y_{V_2} &= -2V_1 \frac{\partial}{\partial K_1} - 2V_4 \frac{\partial}{\partial L_2} + 2V_3 \frac{\partial}{\partial L_3} + 2U_2 \frac{\partial}{\partial H_2} \\
&\quad - 2H_2 \frac{\partial}{\partial U_2} - 2K_1 \frac{\partial}{\partial V_1} + 2L_3 \frac{\partial}{\partial V_3} - 2L_2 \frac{\partial}{\partial V_4} \\
Y_{V_3} &= -2V_1 \frac{\partial}{\partial K_2} + 2V_4 \frac{\partial}{\partial L_1} - 2V_2 \frac{\partial}{\partial L_3} + 2U_3 \frac{\partial}{\partial H_2} \\
&\quad + 2H_2 \frac{\partial}{\partial U_3} - 2L_3 \frac{\partial}{\partial U_1} - 2L_2 \frac{\partial}{\partial U_3} + 2L_1 \frac{\partial}{\partial V_4} \\
Y_{V_4} &= -2V_1 \frac{\partial}{\partial K_3} - 2V_5 \frac{\partial}{\partial L_1} + 2V_2 \frac{\partial}{\partial L_2} + 2U_4 \frac{\partial}{\partial H_2} \\
&\quad + 2H_2 \frac{\partial}{\partial U_4} - 2U_4 \frac{\partial}{\partial V_1} + 2L_2 \frac{\partial}{\partial V_2} - 2L_1 \frac{\partial}{\partial V_3}.
\end{align*}

Table 1. List of Hamiltonian vector fields of coordinate functions on $\mathbb{R}^8/S^1$ induced from $(T\mathbb{R}^4, \omega)$.

The orbit space $\mathbb{R}^8/S^1$ of this $S^1$ action is the 7 dimensional semialgebraic variety in $\mathbb{R}^{16}$ with coordinates $(K, L, H_2, \Xi; U_i, V_i)$ defined by

\begin{align*}
\langle U, U \rangle &= U_1^2 + U_2^2 + U_3^2 + U_4^2 = H_2^2 - \Xi^2 \geq 0, \quad H_2 \geq 0 \\
\langle V, V \rangle &= V_1^2 + V_2^2 + V_3^2 + V_4^2 = H_2^2 - \Xi^2 \geq 0, \quad (4a) \\
\langle U, V \rangle &= U_1 V_1 + U_2 V_2 + U_3 V_3 + U_4 V_4 = 0,
\end{align*}
of the preceding polynomials to the orbit space $R_S$ polynomials, which are invariant under the $S_j = 1$ vector field $R$ space $C_L X$ vector field on $R$ all time, since $X$ and $L$ on $R$ $H T S$ whose image is the orbit space is linear, Schwartz' theorem $[3]$ shows that every smooth $S$ monic oscillator vector field $R$ leaves invariant the ideal of polynomials, whose zeroes define the orbit space $H$ $2$ leaves invariant the ideal of polynomials, whose zeroes define the orbit $H$ 2, $Ξ = 2$ the polynomials $U_i$, $V_i$ for $i = 1, \ldots, 4$. Thus $Ξ^{-1}(ξ)/S^1$ is defined by setting $Ξ = ξ$ in the relations (4a), (4b), and (4c). This completes the first stage of studying the $T^2$ action.

The second stage in determining $J^{-1}(h, ξ)/T^2$ begins by looking at the harmonic oscillator vector field $X_{H_2} = \sum_{i=1}^4 (p_i \frac{∂}{∂q_i} - q_i \frac{∂}{∂p_i})$ on $T R^4$. $X_{H_2}$ induces the vector field

$$Y_{H_2} = 2 \sum_{i=1}^4 (V_i \frac{∂}{∂U_i} - U_i \frac{∂}{∂V_i})$$

(6)

on $R$, since

$$L_{X_{H_2}} H_2 = L_{X_{H_2}} Ξ = L_{X_{H_2}} K_j = L_{X_{H_2}} L_j = 0 \text{ for } j = 1, 2, 3$$

and

$$L_{X_{H_2}} U_i = 2V_i \quad L_{X_{H_2}} V_i = -2U_i \text{ for } i = 1, 2, 3, 4.$$

$L_{X_{H_2}}$ leaves invariant the ideal of polynomials, whose zeroes define the orbit space $R^8/S^1$, see equation (4). The induced vector field $Y_{H_2}$ is a derivation of $C^∞(R^8/S^1)$, each of whose integral curves is the image of an integral curve of $X_{H_2}$ on $R^8$ under the orbit map $Π$. The integral curves of $Y_{H_2}$ are defined for all time, since $X_{H_2}$ is a complete vector field. Consequently, $Y_{H_2}$ is a complete vector field on $R^8/S^1 \subseteq R$. Its flow is the restriction to $R^8/S^1$ of the flow of the vector field $Y_{H_2}$ on $R^8$ given by

$$φ_u^{Y_{H_2}} (K, L, Ξ, H_2; U, V) = (0, 0, 0, 0; 2U \cos u + 2V \sin u, -2U \sin u + 2V \cos u).$$

This defines a smooth $S^1$ action on the subcartesian space $R^8/S^1 \subseteq R$. For $j = 1, 2, 3$ the polynomials $K_j$, $L_j$, $Ξ$, and $H_2$ on $R^8$ generate the algebra of polynomials, which are invariant under the $S^1$ action $φ_u^{Y_{H_2}}$. Thus the restriction of the preceding polynomials to the orbit space $R^8/S^1$ generate the algebra...
$C^\infty(\mathbb{R}^8/S^1)^{S^1}$ of smooth functions on $\mathbb{R}^8/S^1$, which are invariant under the flow $\varphi^{Y_{H_2}}_{\mu_2}([\mathbb{R}^8/S^1])$ of the vector field $Y_{H_2}([\mathbb{R}^8/S^1])$.

We now determine the orbit space of the $S^1$ action on $\mathbb{R}^8/S^1$ generated by $\varphi^{Y_{H_2}}_{\mu_2}([\mathbb{R}^8/S^1])$. Substituting the equations (9a), (9b) and (10) into the identity

$$\sum_{1 \leq i < j \leq 4} (U_i V_j - U_j V_i) + (U, V)^2 = (U, U)(V, V)$$

(7)

gives the identity

$$(H_2^2 - \Xi^2)^2 = (K_1^2 + K_2^2 + K_3^2 + K_4^2 + \Xi^2)(H_2^2 + \Xi^2)
- 4(K_1 K_2 + K_2 K_3 + K_3 K_4)\Xi H_2,$$

which holds if

$$0 \leq K_1^2 + K_2^2 + K_3^2 + K_4^2 + \Xi^2 = H_2^2 + \Xi^2, \quad 0 \leq |\Xi| \leq H_2$$

(8a)

$$K_1 K_2 + K_2 K_3 + K_3 K_4 = \Xi H_2.$$  
(8b)

Using equation (3), a calculation shows that equations (8a) and (8b) indeed hold. Adding and subtracting $\frac{1}{2}$ times equation (8b) from $\frac{1}{4}$ times equation (8a) gives

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{1}{4}(H_2 + \Xi)^2, \quad 0 \leq |\Xi| \leq H_2$$

(9a)

$$\eta_1^2 + \eta_2^2 + \eta_3^2 = \frac{1}{4}(H_2 - \Xi)^2,$$

(9b)

where $\xi_j = \frac{1}{2}(K_j + L_j)$ and $\eta_j = \frac{1}{2}(K_j - L_j)$ for $j = 1, 2, 3$. Equations (9a) and (9b) define the orbit space $(\mathbb{R}^8/S^1)/S^1$ of the $S^1$ action $\varphi^{Y_{H_2}}_{\mu_2}([\mathbb{R}^8/S^1])$ on $\mathbb{R}^8/S^1$. The orbit space $(\mathbb{R}^8/S^1)/S^1$ is the orbit space $\mathbb{T}^2$ of the $T^2$ action on $\mathbb{R}^8$ generated by $\varphi^{Y_{H_2}}_{\mu_2}$ and $\varphi^{H_2}_{\mu_2}$. The orbit map of the $S^1$ action $\varphi^{Y_{H_2}}_{\mu_2}([\mathbb{R}^8/S^1])$ on $\mathbb{R}^8/S^1$ induced from the $S^1$ action $\varphi^{H_2}_{\mu_2}$ on $\mathbb{R}^8$ is

$$\varphi : (\mathbb{R}^8/S^1, C^\infty(\mathbb{R}^8/S^1)) \to (\mathbb{R}^8/T^2, C^\infty(\mathbb{R}^8/T^2)),$$

(10)

which is the restriction to $\mathbb{R}^8/S^1$ of the smooth map

$$\mathbb{R}^6 \to \mathbb{R}^8 : (K, L, H_2, \Xi; U, V) \mapsto (\xi, \eta, H_2, \Xi) = (\frac{1}{2}(K + L), \frac{1}{2}(K - L), H_2, \Xi),$$

and thus is a smooth mapping of locally compact subcartesian differential spaces.

From equation (10) it follows that the range of the momentum map $J$ (11) of the $T^2$ action on $\mathbb{R}^8$ is the closed wedge $W = \{(h, \xi) \in \mathbb{R}_{>0} \times \mathbb{R} : 0 \leq |\xi| \leq h\}$.

When $(h, \xi) \in \text{int } W$, the reduced space $J^{-1}(h, \xi)/T^2$, defined by equations (9a) and (9b) with $H_2 = h$ and $\Xi = \xi$, is diffeomorphic to $S^2_{\frac{1}{2}(h+\xi)} \times S^2_{\frac{1}{2}(h-\xi)}$. When $(h, \xi) \in \partial W \setminus \{(0,0)\}$, that is $\pm \xi = h > 0$, the reduced space $J^{-1}(h, \pm h)/T^2$ is diffeomorphic to $S^2_h$. When $(h, \xi) = (0,0)$ the reduced space $J^{-1}(0,0)/T^2$ is the point $(0,0)$ in $T\mathbb{R}^4$.  

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Since the momentum map $J$ (11) is invariant under the $S^1$ action $\varphi^\Xi$, it induces a smooth map

$$J : \mathbb{R}^8/S^1 \subseteq \mathbb{R}^{16} \to W \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} : (K, L, H_2, \Xi; U, V) \mapsto (H_2, \Xi), \quad (11)$$

which is surjective. We now determine the geometry of the fibration defined by $J$. Observe that $J = j \circ \phi$, where

$$j : \mathbb{R}^8/\mathbb{T}^2 \subseteq \mathbb{R}^8 \to \mathbb{W} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R} : (\xi, \eta, H_2, \Xi) \mapsto (H_2, \Xi)$$

is induced by the momentum mapping of the $\mathbb{T}^2$ action on $T\mathbb{R}^4$. Since $\text{int} W$ is simply connected, the fibration

$$J|_{J^{-1}(\text{int} W)} : J^{-1}(\text{int} W) \subseteq \mathbb{R}^8/S^1 \to \text{int} W \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$$

is trivial. Hence for each $(h, \xi) \in \text{int} W$ the fiber $J^{-1}(h, \xi)$ is diffeomorphic to the fiber $J^{-1}(h, 0)$, where $(h, 0) \in \text{int} W$. The fiber $J^{-1}(h, 0)$ is defined by

$$\langle U, U \rangle = h^2 = \langle V, V \rangle, \quad \langle U, V \rangle = 0, \quad h > 0$$

$$h^{-1}(U_2V_1 - U_1V_2) = K_1, \quad h^{-1}(U_4V_3 - U_3V_4) = L_1$$
$$h^{-1}(U_2V_1 - U_1V_2) = K_2, \quad h^{-1}(U_4V_3 - U_3V_4) = L_2$$
$$h^{-1}(U_2V_1 - U_1V_2) = K_3, \quad h^{-1}(U_4V_3 - U_3V_4) = L_3. \quad (12)$$

Hence $J^{-1}(h, 0)$ is diffeomorphic to

$$M_{h,0} = \{(U, V) \in \mathbb{R}^4 \times \mathbb{R}^4 \mid \langle U, U \rangle = h^2 = \langle V, V \rangle \& \langle U, V \rangle = 0\},$$

because $J^{-1}(h, 0)$ is the graph of the smooth mapping

$$M_{h,0} \subseteq \mathbb{R}^8 \to \mathbb{R}^8 : (U, V) \mapsto (K(U, V), L(U, V), h, 0),$$

where $(U, V) \mapsto K(U, V)$ and $(U, V) \mapsto L(U, V)$ are smooth functions defined by equation (12). $M_{h,0}$ is diffeomorphic to $T_hS^3_1$, the tangent $h$ sphere bundle to the unit 3 sphere $S^3_1$, via the mapping $(U, V) \mapsto (h^{-1}U, V)$. Since $(\partial W)^\pm = \{(h, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \mid \pm \xi = h > 0\}$ is simply connected, the fibration

$$J|_{J^{-1}(\partial W)^\pm} : J^{-1}(\partial W)^\pm \subseteq \mathbb{R}^8/S^1 \to (\partial W)^\pm \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$$

is trivial. Hence for each $(h, \pm h) \in (\partial W)^\pm$ with $h > 0$ the fiber $J^{-1}(h, \pm h)$ is defined by

$$\langle U, U \rangle = \langle V, V \rangle = h^2, \quad \langle U, V \rangle = 0, \quad h > 0$$

$$h^{-1}(U_2V_1 - U_1V_2) = \pm h^{-1}(U_4V_3 - U_3V_4) = \pm (L_1 - K_1) = \mp \eta_1$$
$$h^{-1}(U_2V_1 - U_1V_2) = \pm h^{-1}(U_4V_3 - U_3V_4) = \pm (L_2 - K_2) = \mp \eta_2$$
$$h^{-1}(U_2V_1 - U_1V_2) = \pm h^{-1}(U_4V_3 - U_3V_4) = \pm (L_3 - K_3) = \mp \eta_3. \quad (13)$$

Hence $J^{-1}(h, \pm h)$ is diffeomorphic to $M_{h,0}$ because it is the graph of the smooth mapping

$$M_{h,0} \subseteq \mathbb{R}^8 \to \mathbb{R}^8 : (U, V) \mapsto (\pm (L - K)(U, V), h, \pm h)$$

where $\pm (L - K)$ is the smooth function defined by equation (13). When $(h, \xi) = (0, 0) \in \mathbb{W}$, the fiber $J^{-1}(0, 0)$ is the point $(0, 0, 0, 0; 0, 0) \in \mathbb{R}^{10}$, since $h = 0$ implies $q = p = 0$, which gives $K = L = U = V = \Xi = 0$. 

7
2 The Kustaanheimo-Stiefel mapping

In this section we define the Kustaanheimo-Stiefel mapping and detail its relation to regularizing the Kepler vector field.

For \((h, 0) \in \text{int } W\), the mapping

\[
\wp_{h,0} : T_h S^3 = J^{-1}(h, 0)/S^1 \subseteq T\mathbb{R}^4 \rightarrow S^2 \times S^2 = J^{-1}(h, 0)/T^2 \subseteq \mathbb{R}^6 : (U, V) \mapsto (\xi(U, V), \eta(U, V)) = (\frac{1}{2}(K(U, V) + L(U, V)), \frac{1}{2}(K(U, V) - L(U, V)))\]

where \(\wp_{h,0}(U, V) = \wp(\xi(U, V), \eta(U, V), h, 0; U, V)\), is the orbit map of the pre-regularized Kepler Hamiltonian \((18)\). We describe its relation to the Kepler vector field in more detail.

On \(T_0 \mathbb{R}^4 = (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3\) with coordinates \((x, y)\) and standard symplectic form \(\sum_{i=1}^3 dx_i \wedge dy_i\), the Kepler Hamiltonian on the negative energy level set \(-\frac{1}{2}k^2\) with \(k > 0\) is

\[
K(x, y) = \frac{1}{2}|y|^2 - \frac{1}{|x|} = -\frac{1}{2}k^2. \quad (14)
\]

Here \(|\cdot|\) is the norm associated to the Euclidean inner product on \(\mathbb{R}^3\). Consider

\[
\tilde{K}(x, y) = \frac{|x|}{k}(K(x, y) + \frac{1}{2}k^2) + \frac{1}{k^2} = \frac{1}{2k}|x|(\frac{|y|^2}{k^2} + 1). \quad (15)
\]

The Hamiltonian vector field \(X_{\tilde{K}}\) has integral curves which satisfy

\[
\frac{dx}{ds} = \frac{|x|}{k} \frac{\partial K}{\partial y}, \quad \frac{dy}{ds} = -\frac{|x|}{k} \frac{\partial K}{\partial x} - (K(x, y) + \frac{1}{2}k^2) \frac{\partial |x|}{\partial x}. \quad (16)
\]

On the level set \(\tilde{K}^{-1}(\frac{1}{2})\) they satisfy

\[
\frac{dx}{ds} = \frac{|x|}{k} \frac{\partial K}{\partial y}, \quad \frac{dy}{ds} = -\frac{|x|}{k} \frac{\partial K}{\partial x}. \quad (17)
\]

With \(\frac{dx}{ds} = \frac{|x|}{k}\) a solution to equation \((17)\) is a time reparametrization of an integral curve of the Kepler vector field on the level set \(K^{-1}(-\frac{1}{2}k^2)\). The pre-regularized Kepler Hamiltonian on the level set \(K^{-1}(1)\) is

\[
K(x, y) = \frac{1}{2}|x|(\frac{|y|^2}{k^2} + 1), \quad (18)
\]

which is obtained from the Hamiltonian \(\tilde{K}\) \((15)\) using the symplectic coordinate change \((x, y) \mapsto (\frac{1}{k}x, ky)\).

On \(T_* \mathbb{R}^4 = T\mathbb{R}^4 \setminus \{q = 0\}\) consider the mapping

\[
ks : T_* \mathbb{R}^4 \rightarrow T_0 \mathbb{R}^3 : (q, p) \mapsto (x, y), \quad (19)
\]
where

\[
\begin{align*}
    x_1 &= 2(q_1q_3 + q_2q_4) = U_2 - K_1 \\
    x_2 &= 2(q_1q_4 - q_2q_3) = U_3 - K_2 \\
    x_3 &= q_1^2 + q_2^2 - q_3^2 - q_4^2 = U_4 - K_3 \\
    y_1 &= \frac{1}{(q, q)}(q_1p_3 + q_2p_4 + q_3p_1 + q_4p_2) = (H_2 + V_1)^{-1}V_2. \\
    y_2 &= \frac{1}{(q, q)}(q_1p_4 - q_2p_3 - q_3p_2 + q_4p_1) = (H_2 + V_1)^{-1}V_3 \\
    y_3 &= \frac{1}{(q, q)}(q_1p_1 + q_2p_2 - q_3p_3 - q_4p_4) = (H_2 + V_1)^{-1}V_4.
\end{align*}
\]

(20a)

Since the map \( ks \) has components with are invariant under \( \varphi_\alpha^\Xi \), it sends an orbit of the vector field \( X_\Xi \) on the level set \( \Xi^{-1}(\xi) \cap T_\alpha \mathbb{R}^4 \) to a point in \( T_\alpha \mathbb{R}^3 \).

A calculation shows that \( ks \) is a Poisson map, that is, \( (ks)^*\{f, g\}_{\mathbb{T}^3} = \{(ks)^*f, (ks)^*g\}_{\mathbb{T}^4} \) for every \( f, g \in C^\infty(T\mathbb{R}^3) \). In particular, the map \( ks \) pulls back the structure matrix

\[
W_{\mathbb{T}_3}(x, y) = \begin{pmatrix} (x_1, x_j)_{\mathbb{T}^3} & (x_1, y_j)_{\mathbb{T}^3} \\ (y_1, x_j)_ {\mathbb{T}^3} & (y_1, y_j)_ {\mathbb{T}^3} \end{pmatrix} = \begin{pmatrix} 0 & 2I_3 \\ -2I_3 & 0 \end{pmatrix}
\]

of the Poisson bracket \( \{ , \}_{\mathbb{T}^3} \) on \( T\mathbb{R}^3 \) to the structure matrix

\[
W_{\mathbb{T}_4}(x, y) = \begin{pmatrix} (x_1, x_j)_ {\mathbb{T}^4} & (x_1, y_j)_ {\mathbb{T}^4} \\ (y_1, x_j)_ {\mathbb{T}^4} & (y_1, y_j)_ {\mathbb{T}^4} \end{pmatrix} = \begin{pmatrix} 0 & I_4 \\ -I_4 & 0 \end{pmatrix}
\]

of the Poisson bracket \( \{ , \}_{\mathbb{T}^4} \) on \( T\mathbb{R}^4 \) associated to \( \omega = \sum_{i=1}^4 dq_i \wedge dp_i \).

The following calculation determines the pull back of the preregularized Kepler Hamiltonian \( \mathcal{K}_{\Xi} \) \((18)\) by the \( ks \) map on the level set \( K^{-1}(1) \). Using equation \((20a)\) we get

\[
|x|^2 = x_1^2 + x_2^2 + x_3^2 \\
= 4(q_2q_3 + q_2q_4)^2 + 4(q_1q_4 - q_2q_3)^2 + (q_1^2 + q_2^2 - q_3^2 - q_4^2)^2 \\
= (q, q)^2 (H_2 + V_1)^2, \\
\]

\[
|y|^2 = y_1^2 + y_2^2 + y_3^2 = \frac{1}{(H_2 + V_1)^2} (V_1^2 + V_2^2 + V_3^2 + V_4^2 - V_1^2) \\
= \frac{1}{(H_2 + V_1)^2} (H_2 - \Xi^2 - V_1^2).
\]

So

\[
(ks)^*\mathcal{K} = \frac{1}{2}(H_2 + V_1) \left[ \frac{1}{(H_2 + V_1)^2} (H_2 - \Xi^2 - V_1^2) \right] + \frac{1}{2}(H_2 + V_1) \\
= H_2 - \frac{1}{2} \frac{1}{H_2 + V_1} \Xi^2. \tag{21}
\]
Because $ks$ is a Poisson map, it follows that the vector fields $X_{H_2}$ on $T_x\mathbb{R}^4$ and $2X_K$ on $T_0\mathbb{R}^3$ are $ks$ related on $\Xi^{-1}(0) \cap T_x\mathbb{R}^4$.

Restricting the mapping $ks$ (19) to $\Xi^{-1}(0) \cap T_x\mathbb{R}^4$ gives the Kustaanheimo-Stiefel mapping

$$KS : \Xi^{-1}(0) \cap T_x\mathbb{R}^4 \to T_0\mathbb{R}^3 : (q,p) \mapsto (x,y),$$

(22)

where $(x, y)$ are given in equation (20). $KS$ is a Poisson map, because the map $ks$ (19) is. From equation (21) it follows that $KS$ pulls back the preregularized Hamiltonian $K$ (18) on $T_0\mathbb{R}^3$ to the harmonic oscillator Hamiltonian $H_2$ on $\Xi^{-1}(0)$ and sends an integral curve of the harmonic oscillator vector field $X_{H_2}$ on $\Xi^{-1}(0) \cap H^{-1}_2(1)$ to an integral curve of the vector field $2X_K$ on $K^{-1}(1)$, which is just a reparametrization of an integral curve of the Kepler vector field $X_K$ on $K^{-1}(-\frac{1}{2})$. Below we show that those integral curves of $X_{H_2}$ on $\Xi^{-1}(0)$ that intersect the plane $\{q = 0\}$ are sent by the KS map to integral curves of the Kepler vector field which are collision orbits that reach $\{x = 0\}$ in finite time, see equation (23), the KS map regularizes the Kepler vector field on the level set $K^{-1}(-\frac{1}{2})$.

First we find the pull back of the integrals of angular momentum $J$ and eccentricity vector $e$ of the preregularized Kepler vector field on the level set $K^{-1}(1)$ by the KS map and determine their relation to the integrals $L_j$ and $K_j$ for $j = 1, 2, 3$ of the harmonic oscillator vector field on $J^{-1}(1,0) = H^{-1}_2(1) \cap \Xi^{-1}(0)$. We use equations (19) and (20) and the definition of the KS map (20a) and (20b). We start with $L_3$, the third component of $L$. On $J^{-1}(1,0)$ we have

$$L_3 = U_2 V_3 - U_3 V_2 = U_2 (1 + V_1) y_2 - U_3 (1 + V_1) y_2.$$

So

$$(1 + V_1)^{-1} L_3 = (K_1 + x_1) y_2 - (K_2 + x_2) y_3 = K_1 y_2 - K_2 y_3 + x_1 y_2 - x_2 y_1$$

$$= (U_1 V_2 - U_2 V_1) \frac{1}{1 + V_1} V_3 - (U_1 V_3 - U_3 V_1) \frac{1}{1 + V_1} V_2 + x_1 y_2 - x_2 y_1$$

$$= -\frac{1}{1 + V_1} V_1 (U_2 V_3 - U_3 V_2) + x_1 y_2 - x_2 y_1$$

$$= -\frac{1}{1 + V_1} V_1 L_3 + x_1 y_2 - x_2 y_1.$$

Thus on $J^{-1}(1,0)$

$$L_3 = [(1 + V_1)^{-1} + V_1 (1 + V_1)^{-1}] L_3 = KS^* (x_1 y_2 - x_2 y_1) = KS^* (J_3).$$

A similar argument shows that $L_1$ and $L_2$ on $J^{-1}(1,0)$ are equal to the pull back by KS of $J_1$ and $J_2$, respectively, on $K^{-1}(1)$. Before dealing with $e_3$, the third component of the eccentricity vector, we compute the pull back of the Euclidean inner product $\langle x, y \rangle$ of $x$ and $y$ on $J^{-1}(1,0)$ as follows.
Suppose that $\cos(\gamma_{q,p}, \mu) = 1$. We first compute the pull back by $KS$ of the eccentricity vector $e = (x_1, x_2, x_3)$ at the time $t = 0$.

We show that

$$J^{-1}(1,0) \subset C,$$

where $C = \{ q = 0 \}$ in $T^4$. We show that $C = J^{-1}(1,0) \cap L^{-1}(0)$, that is,

$$C = \{ (q,p) \in J^{-1}(1,0) \mid L_1(q,p) + L_2(q,p) + L_3(q,p) = 0 \}. \quad (23)$$

**Proof.** Suppose that $\gamma_{q,p} : \mathbb{R} \to J^{-1}(1,0) \subseteq T^4$ is an integral curve of $X_{H_2}$ which starts at $(q,p) \in J^{-1}(1,0)$ and passes through the collision set $C$, that is, $(q,p) \in C$. Then there is a positive time $\tau$ such that $q \cos \tau + p \sin \tau = 0$, since

$$\varphi_{H_2}^{\tau}(q,p) = \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$  

Suppose that $\cos \tau \neq 0$. Then $q = -p \tan \tau = \lambda p$. From the definition of the functions $L_j$ for $j = 1, 2, 3$ we get

$$L_1(\lambda p, p) = (\lambda p_1)p_1 - (\lambda p_2)p_3 + (\lambda p_2)p_3 - (\lambda p_1)p_4 = 0$$
$$L_2(\lambda p, p) = (\lambda p_1)p_1 + (\lambda p_2)p_4 - (\lambda p_3)p_2 - (\lambda p_4)p_2 = 0$$
$$L_3(\lambda p, p) = (\lambda p_1)p_4 - (\lambda p_2)p_3 + (\lambda p_2)p_1 - (\lambda p_1)p_2 = 0.$$  

Thus $(q,p) \in J^{-1}(1,0) \cap L^{-1}(0)$. If $\cos \tau = 0$, then $\sin \tau \neq 0$. So $p = -q \cot \tau = \mu q$. Calculating as above shows that $L(q,\mu) = L_2(q,\mu q) = L_3(q,\mu q) = 0$. So $(q,p) \in L^{-1}(0) \cap J^{-1}(1,0) = C$. 

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Conversely, suppose that \((q, p) \in C\). Then \((q, p) \in J^{-1}(1, 0) \cap L^{-1}(0)\). Let \((x, y) = \text{KS}(q, p)\). Since \((\text{KS})^* (J|K^{-1}(1)) = L|J^{-1}(1, 0)\), it follows that \(x \times y = 0\), because \(L(q, p) = 0\). Let \(\Gamma_{(x, y)}\) be an integral curve of the vector field \(2X_K\) of energy \(-1\) starting at \((x, y)\), whose angular momentum \(J\) vanishes. Then \(\Gamma_{(x, y)} = r(t)e(x, y) = r(t)\frac{\vec{x}}{|\vec{x}|}\) is a ray such that \(r = r(t) > 0\) and \(r\) satisfies

\[
\dot{r}^2 - \frac{2}{r} = -1. \tag{24}
\]

From equation (24) we see that \(0 \leq r \leq 2\). Starting at \(r(0) = r_0\) with \(0 < r_0 \leq 2\) and \(\dot{r}(0) = 0\) there is a finite positive time \(\tau_0\) such that \(r(\tau_0) = 0\). To see this separate variables in equation (24) and integrate. Using the change of variables \(s^2 = \frac{2}{r} - 1\) we get

\[
\tau_0 = \int_0^{\tau_0} dt = \int_{r_0}^{0} \frac{dr}{\sqrt{2r^{-1} - 1}}
= 4 \int_0^{s_0} \frac{ds}{(s^2 + 1)^2}, \text{ where } s_0 = \sqrt{2r_0^{-1} - 1} \geq 0
< 4 \int_0^{\infty} \frac{ds}{(s^2 + 1)^2} = 4 \int_0^{\pi/2} \cos^2 u \, du, \text{ using } s = \tan u
= \pi.
\]

Since \(|x| = (q, q)\), the integral curve \(\gamma_{(q, p)}\) of \(X_{H_2}\) starting at \((q, p) \in J^{-1}(1, 0) \cap L^{-1}(0)\), whose image under the KS mapping is \(\Gamma_{(x, y)}\), reaches the collision set \(C\) at the finite positive time \(\tau_0\). Hence \((q, p) \in C\).

We now determine the structure matrix \(W_{\mathbb{R}^8/\mathbb{T}^2}\) of the Poisson bracket \(\{\ , \\}_{{\mathbb{R}^8}/\mathbb{T}^2}\) on the \(\mathbb{T}^2\) orbit space \(\mathbb{R}^8/\mathbb{T}^2\). The smooth surjective mapping

\[
\varphi : \mathbb{R}^8/\mathbb{S}^1 \to \mathbb{R}^8/\mathbb{T}^2 : (K, L, H_2, \Xi; U, V) \mapsto (\xi, \eta, H_2, \Xi)
\]

is a Poisson mapping, that is, \(\varphi^* \{f, g\}_{{\mathbb{R}^8}/\mathbb{T}^2} = \{\varphi^* f, \varphi^* g\}_{{\mathbb{R}^8}/\mathbb{S}^1}\), for every \(f, g \in C^\infty(\mathbb{R}^8/\mathbb{T}^2)\). Thus we need only determine the Poisson brackets \(\{\ , \\}_{{\mathbb{R}^8}/\mathbb{T}^2}\) of the functions \(K_j\) and \(L_j\) for \(j = 1, 2, 3\) on \((\mathbb{T}^2)\). A straightforward computation gives

\[
\{K_i, K_j\} = 2 \sum_{k=1}^3 \epsilon_{ijk} L_k, \quad \{L_i, L_j\} = 2 \sum_{k=1}^3 \epsilon_{ijk} L_k, \quad \{K_i, L_j\} = 2 \sum_{k=1}^3 \epsilon_{ijk} K_k,
\]

for \(i, j = 1, 2, 3\). Using \(\xi_j = \frac{i}{2}(K_j + L_j)\) and \(\eta_j = \frac{i}{2}(K_j - L_j)\) for \(j = 1, 2, 3\) the above equations become

\[
\{\xi_i, \xi_j\} = 3 \sum_{k=1}^3 \epsilon_{ijk} \xi_k, \quad \{\eta_i, \eta_j\} = -3 \sum_{k=1}^3 \epsilon_{ijk} \eta_k, \text{ and } \{\xi_i, \eta_j\} = 0. \tag{25}
\]
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