Percentiles of the run-length distribution of the Exponentially Weighted Moving Average (EWMA) median chart

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ABSTRACT. Quality control is crucial in a wide variety of fields, as it can help to satisfy customers’ needs and requirements by enhancing and improving the products and services to a superior quality level. The EWMA median chart was proposed as a useful alternative to the EWMA $\bar{X}$ chart because the median-type chart is robust against contamination, outliers or small deviation from the normality assumption compared to the traditional $\bar{X}$-type chart. To provide a complete understanding of the run-length distribution, the percentiles of the run-length distribution should be investigated rather than depending solely on the average run length (ARL) performance measure. This is because interpretation depending on the ARL alone can be misleading, as the process mean shifts change according to the skewness and shape of the run-length distribution, varying from almost symmetric when the magnitude of the mean shift is large, to highly right-skewed when the process is in-control (IC) or slightly out-of-control (OOC). Before computing the percentiles of the run-length distribution, optimal parameters of the EWMA median chart will be obtained by minimizing the OOC ARL, while retaining the IC ARL at a desired value.

1. Introduction
To fulfill customers’ requirements, manufacturers should maintain or even improve their products and services to an acceptable quality level. In practice, no process can maintain its stability forever. Therefore, there is a need to implement Statistical Process Control (SPC) techniques. Control chart is an effective method well known in SPC to maintain or enhance the quality of products and services. The main purpose of control chart is to identify variability in a production or service process.

The median ($\bar{X}$) chart is an useful substitute for the mean ($\bar{X}$) chart. First, the $\bar{X}$ chart is easier to apply compared to the $\bar{X}$ chart. Moreover, the $\bar{X}$ chart is robust against small variation from normality assumption. This characteristic is crucial for the $\bar{X}$ chart that is applied to monitor an ongoing process running for some time. Under this scenario, the $\bar{X}$ chart which is applied without identifying the outliers, may give unreliable conclusions. It is well-known that the EWMA $\bar{X}$ chart is capable of identifying small to moderate mean shifts in the process, however, Human et al. [1] proved that the EWMA $\bar{X}$ chart
is not suitable for the case where dataset contain outliers. Therefore, a better substitute for the EWMA $\bar{X}$ chart would be the EWMA $\tilde{X}$ chart as demonstrated by Castagliola et al[2].

The average run length (ARL) criterion is widely applied in examining the effectiveness of various control charts. The application of ARL as a sole performance measure has been criticized by many researchers, see, for example, [3, 4, 5]. As the run-length distribution follows the geometric distribution, it is highly skewed to the right for the IC or slightly OOC process [6]. Inference based on ARL alone can be confusing and may not be a suitable performance measure in practice due to excessive variability in the run-length distribution [7]. Hence, Zhou et al. [8] suggested that a better alternative performance measure for the ARL is the percentiles of the run-length distribution, as the percentiles of the run-length distribution can summarize the entire run-length distribution. Chakraborti [9] showed that the 50th percentile of the run-length distribution, which represents the median run-length (MRL), is a more appropriate performance measure for a control chart.

The 50th percentile of the run-length distribution gives more information than the ARL. For example, when the IC 50th percentile of the run-length distribution is 500, practitioners are confident that the control chart, in half of the time, is able to detect a false alarm by the 500th sample. The percentiles of the run-length distribution give a complete information on the characteristic of the process. For the IC process, the lower percentiles of the run-length distribution are able to give information on the early false alarm rate, while for the OOC process, the higher percentiles of the run-length distribution are able to provide valuable information on the sensitivity of the chart to identify an OOC signal. Furthermore, the information on the spread of the run-length distribution is also contained in the percentiles of the run-length distribution [10].

In this paper, we study the percentiles of the run-length distribution of the EWMA median chart. Note that, the optimal chart’s parameters are computed with a predefined IC ARL (ARL₀) value of 500. Then, based on these optimal chart’s parameters, the percentiles of the run-length distribution are computed.

2. EWMA $\tilde{X}$ chart

The procedure to implement the EWMA median chart is as follows:

Step 1. A sample of size $n$ is collected. Let the sample $\{Y_{i1}, ..., Y_{in}\}, i = 1, 2, ...$ be $n$ independent normal random variables, where $i$ is the subgroup number. Then, $\{Y_{i(1)}, Y_{i(2)}, ..., Y_{i(n)}\}$ is the $i$-th subgroup arranged in ascending order. The sample median of subgroup $i$, $\hat{Y}_i$ can be computed using

$$\hat{Y}_i = \begin{cases} 
Y_{i(\frac{n+1}{2})}, & \text{if } n \text{ is odd} \\
y_{i(\frac{n}{2})} + y_{i(\frac{n}{2}+1)} \times \frac{1}{2}, & \text{if } n \text{ is even}
\end{cases} \quad (1)$$

Step 2. The upper control limit (UCL) and lower control limit (LCL) of the EWMA $\bar{X}$ chart are computed as

$$\text{UCL} = \mu_0 + K\sigma_0 \quad (2)$$

and

$$\text{LCL} = \mu_0 - K\sigma_0, \quad (3)$$

respectively, where $\mu_0$ is the IC mean, $K > 0$ is a constant and $\sigma_0$ is the IC standard deviation.

Step 3. The EWMA sequence based on $\hat{Y}_i$, for $i = 1, 2, ...$, is computed as

$$Z_i = (1 - \lambda)z_{i-1} + \lambda\hat{Y}_i, \quad (4)$$

where $\lambda \in (0,1]$ is the smoothing constant and $z_0 = \mu_0$.

Step 4. If $Z_i$ is located between LCL and UCL, i.e., LCL < $Z_i$ < UCL, the process is considered to be IC and the procedure returns to Step 1. Otherwise, the process is classified as OOC and the procedure proceeds to Step 5.
Step 5. Signal an OOC condition. Necessary action is taken to investigate and eliminate the assignable cause(s). After that, the procedure returns to Step 1.

In this study, since we are dealing with sample median, without loss of generality, \( n \) is selected to be an odd value. Castagliola et al. [2] used the Markov chain method suggested by Brook and Evans [11] to represent the run-length characteristic of the EWMA \( \bar{X} \) chart, where the interval between \([LCL, UCL]\) is divided into \(2m+1\) subintervals, centered at \( H_j = \frac{LCL+UCL}{2} + 2j\delta\), where \( 2\delta = \frac{UCL-LCL}{2m+1} \).

**Figure 1.** Interval between LCL and UCL divided into \(2m+1\) subintervals of width \(2\delta\).

Here, \( H_j \), for \( j = -m, ..., 0, ..., +m \), represent the middle point of the \( j \)th subinterval. Each subinterval denotes a transient state of the Markov chain. The Markov chain is considered to be in the transient state, if \( Z_i \in (H_j - \delta, H_j + \delta] \), where \( j \in \{-m, ..., 0, ..., +m\} \) for sample \( i \). If \( Z_i \notin (H_j - \delta, H_j + \delta] \), the Markov chain reaches the absorbing state \((-\infty, LCL] \cup [UCL, +\infty)\). When the subinterval \(2m+1\) is sufficiently large, this method allows the run-length characteristic to be evaluated accurately. Let \( Q \) be the transition probability matrix corresponding to the \(2m+1\) transient states and \( q \) is the \((2m+1, 1)\) vector of initial probabilities related to the \(2m+1\) transient states, then the ARL is computed by

\[
ARL = q^T (I - Q)^{-1} 1,
\]

where \( I \) is known as identity matrix and \( 1 = (1, 1, ..., 1)^T \).

3. Optimal design of the EWMA \( \bar{X} \) chart and its run-length properties

An optimal design of the EWMA \( \bar{X} \) chart is applied to compute an optimal set of parameters that minimize the ARL, where the ARL\(_0\) is predefined by the user. There are two parameters for the EWMA \( \bar{X} \) chart, which are the \( \lambda \) and \( K \). In this study, the parameters are obtained through the optimization program written in ScicosLab. The sample size, \( n \), size of mean shift, \( \delta \) and desired ARL\(_0\), need to be determined before the computation of the values of the optimal \( \lambda \) and \( K \) parameters.

In the optimal design, \( n \in \{3, 5, 7, 9\} \), \( \delta_{opt} = 1.0 \) and ARL\(_0 = 500 \) are considered and the optimal parameters that satisfy these conditions are obtained. The optimization procedures start by initializing \( \lambda \) as 0.001. Then the program will search for the \( K \) value so that the ARL\(_0\) is equal to a predefined value, i.e. 500. Based on these \( \lambda \) and \( K \) values, the OOC ARL (ARL\(_1\)) is computed. The procedures are repeated where \( \lambda \) is increased incrementally by 0.001 up to 1 and the corresponding ARL\(_1\) is computed. The parameters \( \lambda \) and \( K \) that produce the minimum ARL\(_1\) will be identified by the program as the optimal parameters.

After the optimal parameters \( \lambda \) and \( K \) are obtained, the percentiles of the run-length distribution are computed with Monte Carlo simulation, which is written in the ScicosLab software. As the run-length distribution is known to be skewed to the right, the study of the full run-length distribution gives more information regarding the performance of the EWMA \( \bar{X} \) chart. In this study, only the increasing mean shifts are considered, i.e. when the process increases from \( \mu \) to \( \mu + \delta \sigma \).
In this study, $n \in \{3, 5, 7, 9\}, \delta \in \{0.00, 0.25, 0.50, 0.75, 1.00, 1.50, 2.00\}$ and $\text{ARL}_0 = 500$ are considered. 500000 simulation trials are repeated to estimate the percentiles of the run-length distribution of the EWMA median chart. The $5^{\text{th}}$, $10^{\text{th}}$, $25^{\text{th}}$, $30^{\text{th}}$, $40^{\text{th}}$, $50^{\text{th}}$, $60^{\text{th}}$, $70^{\text{th}}$, $75^{\text{th}}$, $80^{\text{th}}$, $90^{\text{th}}$ and $95^{\text{th}}$ percentiles of the run-length distribution are calculated for both the IC and OOC processes. The IC and OOC observations were simulated by setting $\delta = 0$ and $\delta > 0$, respectively, where the underlying distribution is the normal distribution.

4. Results and Findings

Table 1 shows the optimal parameters computed using the ScicosLab program, for various sample size $n$, when $\delta_{\text{opt}} = 1.0$ and $\text{ARL}_0 = 500$. From Table 1, when the sample size $n$ increases, the corresponding value of the optimal parameter $\lambda$ also increases. By using simulation program written in ScicosLab software, these optimal parameters $\lambda$ and $K$ are applied to calculate the percentiles of the run-length distribution of the EWMA $\bar{X}$ chart. Tables 2, 3, 4 and 5 present the results of the ARL, percentiles of the run-length distribution and standard deviation of the run length (SDRL) and, based on $n = 3, 5, 7$ and 9, respectively.

### Table 1. Optimal $\lambda$ and $K$ parameters for $\delta_{\text{opt}} = 1.0$ and $\text{ARL}_0 = 500$.

| $n$ | $\lambda$ | $K$ |
|-----|-----------|-----|
| 3   | 0.2365    | 0.7352 |
| 5   | 0.3241    | 0.7165 |
| 7   | 0.4073    | 0.7113 |
| 9   | 0.4932    | 0.7185 |

From Tables 2 to 5, it is observed that when the process is IC, i.e. $\delta = 0$, the $\text{ARL}_0 = 500$ falls between the $60^{\text{th}}$ and $70^{\text{th}}$ percentiles of the run-length distribution. This indicates that the IC run-length distribution of EWMA median chart is skewed to the right. For example, from Table 2, when the process is IC, i.e. $\delta = 0$, it is observed that the $\text{ARL}_0 = 500$ is located between the $60^{\text{th}}$ and $70^{\text{th}}$ percentiles of the run-length distribution; while the value of the $50^{\text{th}}$ percentile of the run-length distribution is 348. Therefore, the performance indicator for central tendency is better represented by the $50^{\text{th}}$ percentile of the run-length distribution compared to the ARL. Palm [12] stated that the ARL is only useful to determine the expected value of the run length and it cannot be viewed as “half of the time”. Therefore, practitioners may falsely interpret the chart’s performance based on the ARL value. For example, from Table 3, when $n = 5$ and $\delta = 0.50$, practitioners may falsely interpret that an OOC signal will be observed by the $15^{\text{th}}$ sample ($\text{ARL}_1 = 14.64$) with 50% chance, but actually this happens faster, i.e. by the $11^{\text{th}}$ sample (50th percentile of the run-length distribution = 11). However, the difference between ARL and the $50^{\text{th}}$ percentile of the run-length distribution decreases when the mean shift, $\delta$ increases. From Tables 2 to 5, we observe that all the percentiles of the run-length distribution are close to the $50^{\text{th}}$ percentile when $\delta \geq 1.0$. This shows that as the mean shift, $\delta$ increases the skewness of the run-length distribution decreases.

Furthermore, when the process is IC, it is useful to evaluate the lower percentiles of the run-length distribution (i.e. $5^{\text{th}}$ and $10^{\text{th}}$ percentiles), as they provide information regarding the early false alarm. For example, from Table 3, for the case of $n = 5$, there is a 5% chance that a false alarm signal will take place by the $28^{\text{th}}$ sample, even though the process is IC. Moreover, the higher percentiles of the run-length distribution (i.e. $80^{\text{th}}$ and $90^{\text{th}}$ percentiles) when the process is OOC provide valuable information about the speed of the chart in detecting a process shift to the practitioners. For example, from Table 3, for the case of $n = 5$, practitioners will have 95% confidence that a mean shift of size $\delta = 0.50$ will be detected by the $37^{\text{th}}$ sample.

Table 2: ARLs, SDRLs and percentiles of the run-length distribution based on simulation for the EWMA
The results obtained from the simulation program written in the ScicosLab software indicate that the IC chart when $n = 3, \lambda = 0.2365, K = 0.7352$

\[ \bar{X} \text{ chart when } n = 3, \lambda = 0.2365, K = 0.7352 \]

| $\delta$ | ARL | SDRL | Percentiles of the run-length distribution |
| --- | --- | --- | --- |
| 0.00 | 500.51 | 497.08 | 29 56 114 146 181 257 348 459 602 692 802 1148 1495 |
| 0.25 | 84.93 | 79.83 | 9 13 23 28 34 46 61 78 101 116 134 189 244 |
| 0.50 | 19.91 | 15.48 | 4 6 8 9 10 13 15 19 23 26 29 40 51 |
| 0.75 | 8.94 | 5.37 | 3 4 5 5 6 7 8 9 10 11 12 16 19 |
| 1.00 | 5.52 | 2.65 | 2 3 3 4 4 4 5 6 7 7 9 11 |
| 1.50 | 3.17 | 1.10 | 2 2 2 2 3 3 3 3 4 4 4 5 5 |
| 2.00 | 2.30 | 0.64 | 2 2 2 2 2 2 2 3 3 3 3 3 3 |

**Table 3.** ARLs, SDRLs and percentiles of the run-length distribution based on simulation for the EWMA $\bar{X}$ chart when $n = 5, \lambda = 0.3241$ and $K = 0.7165$.

| $\delta$ | ARL | SDRL | Percentiles of the run-length distribution |
| --- | --- | --- | --- |
| 0.00 | 500.51 | 498.81 | 28 55 113 145 180 256 347 458 601 693 803 1152 1499 |
| 0.25 | 69.40 | 65.73 | 7 11 18 23 27 37 49 64 83 95 110 155 200 |
| 0.50 | 14.64 | 11.36 | 3 4 6 7 7 9 11 14 17 19 22 29 37 |
| 0.75 | 6.37 | 3.75 | 2 3 3 4 4 5 5 6 7 8 9 11 14 |
| 1.00 | 3.92 | 1.81 | 2 2 2 3 3 3 4 4 4 5 5 6 7 |
| 1.50 | 2.30 | 0.75 | 1 2 2 2 2 2 2 3 3 3 3 3 4 |
| 2.00 | 1.69 | 0.54 | 1 1 1 1 1 2 2 2 2 2 2 2 2 |

**Table 4.** ARLs, SDRLs and percentiles of the run-length distribution based on simulation for the EWMA $\bar{X}$ chart when $n = 7, \lambda = 0.4073$ and $K = 0.7113$.

| $\delta$ | ARL | SDRL | Percentiles of the run-length distribution |
| --- | --- | --- | --- |
| 0.00 | 499.47 | 496.71 | 28 55 114 146 180 257 347 458 600 691 803 1146 1491 |
| 0.25 | 61.47 | 58.57 | 6 9 16 20 24 33 44 57 74 84 97 138 178 |
| 0.50 | 11.99 | 9.40 | 3 3 5 5 6 8 9 11 14 16 18 24 31 |
| 0.75 | 5.04 | 2.96 | 2 2 3 3 3 4 4 5 6 6 7 9 11 |
| 1.00 | 3.09 | 1.40 | 1 2 2 2 2 3 3 3 3 4 4 5 6 |
| 1.50 | 1.82 | 0.64 | 1 1 1 1 2 2 2 2 2 2 2 3 3 |
| 2.00 | 1.29 | 0.46 | 1 1 1 1 1 1 1 1 1 2 2 2 2 |

**Table 5.** ARLs, SDRLs and percentiles of the run-length distribution based on simulation for the EWMA $\bar{X}$ chart when $n = 9, \lambda = 0.4932$ and $K = 0.7185$.

| $\delta$ | ARL | SDRL | Percentiles of the run-length distribution |
| --- | --- | --- | --- |
| 0.00 | 500.52 | 500.27 | 27 54 113 144 179 256 347 458 602 692 804 1154 1497 |
| 0.25 | 57.91 | 55.69 | 5 8 15 18 22 31 41 53 69 79 92 130 169 |
| 0.50 | 10.51 | 8.39 | 2 3 4 5 5 7 8 10 12 14 16 21 27 |
| 0.75 | 4.27 | 2.55 | 2 2 2 2 3 3 4 4 5 5 6 8 9 |
| 1.00 | 2.58 | 1.19 | 1 1 2 2 2 2 2 2 3 3 3 3 4 5 |
| 1.50 | 1.49 | 0.57 | 1 1 1 1 1 1 1 2 2 2 2 2 2 |
| 2.00 | 1.09 | 0.29 | 1 1 1 1 1 1 1 1 1 1 1 1 1 2 |

5. Conclusions

The results obtained from the simulation program written in the ScicosLab software indicate that the IC ARL locates in between the 60th and 70th percentiles of the run-length distribution when the process is IC and when the mean shift, $\delta$ is small; whereas when the mean shift increases, the OOC ARL approaches the 50th percentile of the run-length distribution. This clearly indicates that the IC run-length distribution is right-skewed and as the size of the mean shift, $\delta$ changes, the skewness of the run length...
distribution changes accordingly. ARL does not give a full and clear understanding regarding the run-length distribution of the EWMA $\bar{X}$ chart. This study using the percentiles of the run-length distribution instead of the ARL, provides a more in-depth understanding and analysis of the EWMA $\bar{X}$ chart’s performance. Since the 50th percentile of the run length distribution, i.e. the median run length provides more suitable performance measure of the control chart’s performance than the ARL, future research work should consider the optimal design of the EWMA $\bar{X}$ chart based on the MRL, instead of the ARL.

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References
[1] Human S W Kritzinger P and Chakraborti S 2011 J. Appl. Stat. 38 2071-2087
[2] Castagliola P Maravelakis P E and Figueiredo F O 2015 The EWMA median chart with estimated parameters IIE Trans. 48 66-74
[3] Gan F F 1993 J. Stat. Comput. Simul. 45 169-184
[4] Teoh W L and Khoo M B C 2012 Int. J. Chem. Eng. Appl. 3 303-306
[5] Teoh W L, Khoo M B C, Castagliola P and Chakraborti S 2014 Optimal design of the double sampling $\bar{X}$ chart with estimated parameters based on median run length Comput. Ind. Eng. 67 104-115
[6] Khoo M B C, Wong V H, Wu Z and Castagliola P 2011 Optimal designs of the multivariate synthetic chart for monitoring the process mean vector based on median run length Qual. Reliab. Eng. Int. 27 981-997
[7] Castagliola P 2001 Int. J. Reliab. Qual. Saf. Eng. 8 123-135
[8] Zhou Q, Zou C, Wang Z and Jiang W 2012 J. Am. Stat. Assoc. 107 1049-1062
[9] Chakraborti S 2007 Run length distribution and percentiles: the Shewhart $\bar{X}$ chart with unknown parameters Qual. Eng. 19 119-127
[10] Teoh W L, Khoo M B C, Castagliola P and Lee M H 2016 The exact run length distribution and design of the Shewhart $\bar{X}$ chart with estimated parameters based on median run length Commun. Stat. Simul. Comp. 45 2081-2103
[11] Brook D and Evans D A 1972 An approach to the probability distribution of CUSUM run length Biometrika 59 539-549
[12] Palm A C 1990 J. Qual. Technol. 22 289-298