Detailed proofs of paper [1]
Slepian-Bangs formula and Cramér Rao bound for circular and non-circular complex elliptical symmetric distributions

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I. USEFUL RELATIONS AND LEMMA

A. Useful relations

We will make use of the following well known relations which hold for any conformable matrices \( A, B, C \) and \( D \).

\[
\text{vec}(ABC) = (C^T \otimes A)\text{vec}(B),
\]

\[
(A \otimes B)(C \otimes D) = AC \otimes BD,
\]

\[
\text{Tr}(AB) = \text{vec}^H(A^H)\text{vec}(B),
\]

\[
\text{Tr}(ABCD) = \text{vec}^H(A^H)(D^T \otimes B)\text{vec}(C),
\]

\[
\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B),
\]

\[
\text{Tr}[K(A \otimes B)] = \text{Tr}(AB),
\]

where \( K \) is the vec-permutation matrix which transforms \( \text{vec}(C) \) to \( \text{vec}(C^T) \) for any square matrix \( C \),

\[
(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1},
\]

where \( A, C \) and \( C^{-1} + DA^{-1}B \) are assumed invertible.

B. Useful lemma for the proof of Result 2

**Lemma 1:** Let \( \tilde{A} = \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_1^* \end{pmatrix} \) and \( \tilde{B} = \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_1^* \end{pmatrix} \) be two \( 2M \times 2M \) partitioned matrices with \( A_1 \) and \( B_1 \) are \( M \times M \) Hermitian matrices, \( A_2 \) and \( B_2 \) are \( M \times M \) complex symmetric matrices, and suppose that \( y \sim \mathbb{C}N_M(0, I) \). Then

\[
\mathbb{E}[(\tilde{y}^H \tilde{A} \tilde{y})(\tilde{y}^H \tilde{B} \tilde{y})] = \text{Tr}(\tilde{A})\text{Tr}(\tilde{B}) + 2\text{Tr}(\tilde{A}\tilde{B}),
\]

where \( \tilde{y} \stackrel{\text{def}}{=} (y^T, y^H)^T \).

**Proof:**

We get from (6) then (2)

\[
\mathbb{E}[(\tilde{y}^H \tilde{A} \tilde{y})(\tilde{y}^H \tilde{B} \tilde{y})] = \text{Tr}[(\tilde{A}^T \otimes \tilde{B})\mathbb{E}(\tilde{y}^* \tilde{y}^T \otimes \tilde{y}\tilde{y}^H)],
\]

where from e.g. [2] Appendix B

\[
\mathbb{E}(\tilde{y}^* \tilde{y}^T \otimes \tilde{y}\tilde{y}^H) = I \otimes I + K(J' \otimes J')(I \otimes I) + \text{vec}(I)\text{vec}^T(I),
\]

where \( K \) is the vec-permutation matrix which transforms \( \text{vec}(C) \) to \( \text{vec}(C^T) \) for any square matrix \( C \).
where \( J' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Plugging (10) in (9), we get:

\[
E[(\tilde{y}^H \tilde{A} \tilde{y})(\tilde{y}^H \tilde{B} \tilde{y})] = \text{Tr}[(\tilde{A}^T \otimes \tilde{B})(I \otimes I)] + \text{Tr}[(\tilde{A}^T \otimes \tilde{B})K(J' \otimes J')(I \otimes I)] \\
+ \text{Tr}[(\tilde{A}^T \otimes \tilde{B}) \text{vec}(I) \text{vec}^T(I)],
\]

where we have successively

\[
\text{Tr}[(\tilde{A}^T \otimes \tilde{B})(I \otimes I)] = \text{Tr}(\tilde{A}) \text{Tr}(\tilde{B})
\]

from (2) and (5).

\[
\text{Tr}[(\tilde{A}^T \otimes \tilde{B})K(J' \otimes J')(I \otimes I)] = \text{Tr}(\tilde{A} \tilde{B})
\]

from (2), (6) and (14). Consequently (13) reduces to

\[
\text{Tr}[(\tilde{A}^T \otimes \tilde{B}) \text{vec}(I) \text{vec}^T(I)] = \text{Tr}(\tilde{A} \tilde{B})
\]

from (4). Plugging these three expressions in (11), (8) follows.

\section*{II. Proof of Result 1 and Eq. (5) of (1)}

Since a linear transform in \( \mathbb{R}^{2M} \) is tantamount to \( \mathbb{R} \)-linear transform in \( \mathbb{C}^M \), the definition of GCES given in (3) is equivalent to saying that

\[
z = \mu + \Psi z_0 + \Phi z^*_0,
\]

where \( \Psi \) and \( \Phi \) are \( M \times M \) fixed complex-valued matrices and \( z_0 \) is a complex spherical distributed r.v. with stochastic representation \( z_0 = d R u \) [4, th. 3]. Since \( E(uu^H) = \frac{1}{M} I \) and \( E(uu^T) = 0 \) [4, lemma 1b], we get if \( E(R^2) < \infty \),

\[
\Sigma = AA^H = \frac{E(R^2)}{N \sigma_c} (\Psi \Psi^H + \Phi \Phi^H) \quad \text{and} \quad \Omega = A \Delta_c A^T = \frac{E(R^2)}{N \sigma_c} (\Psi \Phi^T + \Phi \Psi^T),
\]

where \( \sigma_c \) is defined by \( E[(z-\mu)(z-\mu)^H] = \sigma_c \Sigma \) and \( E((z-\mu)(z-\mu)^T) = \sigma_c \Omega \) whose value is \( E(R^2)/N \) [4, (14)]. Consequently (13) reduces to

\[
AA^H = \Psi \Psi^H + \Phi \Phi^H \quad \text{and} \quad A \Delta_c A^T = \Psi \Phi^T + \Phi \Psi^T.
\]

By the one to one change of variable (because \( A \) is nonsingular): \( \Psi' = A \Psi \) and \( \Phi' = A \Phi \), (14) is equivalent to:

\[
I = \Psi' \Psi'^H + \Phi' \Phi'^H \quad \text{and} \quad \Delta_c = \Psi' \Phi'^T + \Psi' \Phi'^T.
\]

It is clear that the solution of (15) is not unique, but we can look for solutions in real-valued diagonal form \((\Psi, \Phi) = (\Delta_1, \Delta_2)\) with

\[
I = \Delta_1^2 + \Delta_2^2 \quad \text{and} \quad \Delta_c = 2 \Delta_1 \Delta_2,
\]

whose solutions are \( \Delta_1 = \frac{\Delta_+ + \Delta_-}{2} \) and \( \Delta_2 = \frac{\Delta_+ - \Delta_-}{2} \) where \( \Delta_+ = \sqrt{\Delta_+ \Delta_-} \) and \( \Delta_- = \sqrt{\Delta_1 \Delta_2} \).

Consequently

\[
z = d \mu + R[\Psi u + \Phi u^*] = \mu + RA[\Delta_1 u + \Delta_2 u^*]
\]

(17).

If \( E(R^2) \) is not finite, the scatter and pseudo-scatter matrices of \( z \) given by (17) are also \( \Sigma = AA^H \) and \( \Omega = A \Delta_c A^T \), respectively.

\section*{3. From the eigenvalue decomposition}

From the eigenvalue decomposition

\[
\begin{pmatrix} I & \Delta_c \\ \Delta_c & I \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I + \Delta_c & 0 \\ 0 & I + \Delta_c \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \]

we deduce from \( \tilde{\Gamma} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} I & \Delta_c \\ \Delta_c & I \end{pmatrix} \begin{pmatrix} A^H & 0 \\ 0 & A^T \end{pmatrix} \) that \( \tilde{\Gamma}^{1/2} = \begin{pmatrix} A & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & \Delta_1 \end{pmatrix} \). Consequently,

\[1\text{Note that if } \Phi = 0, z \text{ is C-CES distributed.}
the stochastic representation \( z = d \mu + \mathcal{R} A v \) is equivalent to

\[
\tilde{z} = d \bar{\mu} + \mathcal{R} \tilde{\Gamma}^{1/2} \bar{u}
\]  

(18)

with \( \bar{u} \overset{\text{def}}{=} (u^T, u^H)^T \). It follows directly \( \frac{1}{2}(\tilde{z} - \bar{\mu})^H \tilde{\Gamma}^{-1}(\tilde{z} - \bar{\mu}) = d \frac{1}{2} \mathcal{R}^2 \| \bar{u} \|^2 = Q \). \( \blacksquare \)

III. PROOF OF RESULT 2

To prove this result, we follow the different steps of [5 sec. 3]. First, we check that the p.d.f. \( p(z; \alpha) \) satisfies the “regularity” condition

\[
E \left( \frac{\partial \log p(z; \alpha)}{\partial \alpha_k} \right) = 0.
\]  

(19)

Taking the derivative of the p.d.f. \( [1] (1) \) w.r.t. \( \alpha_k \), yields

\[
\frac{\partial \log p(z; \alpha)}{\partial \alpha_k} = -\frac{1}{2} \text{Tr}(\tilde{\Gamma}^{-1} \tilde{\Gamma}_k) + \phi(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k}.
\]  

(20)

It follows from the definition of \( \tilde{\eta} \) that

\[
\frac{\partial \tilde{\eta}}{\partial \alpha_k} = -\text{Re} \left( \mu_k^H \tilde{\Gamma}^{-1}(\tilde{z} - \bar{\mu}) \right) - \frac{1}{2} \left( \tilde{z} - \bar{\mu} \right)^H \tilde{\Gamma}_k \tilde{\Gamma}^{-1}(\tilde{z} - \bar{\mu}),
\]  

(21)

where \( \tilde{\mu}_k \overset{\text{def}}{=} \frac{\partial u}{\partial \alpha_k} \) and \( \tilde{\Gamma}_k \overset{\text{def}}{=} \frac{\partial \Gamma}{\partial \alpha_k} \). Making use of the extended stochastic representation (18), the second term of (21) is given by

\[
\frac{1}{2} \left( \tilde{z} - \bar{\mu} \right)^H \tilde{\Gamma}_k \tilde{\Gamma}^{-1}(\tilde{z} - \bar{\mu}) = d \frac{1}{2} Q \bar{u}^H \tilde{H}_k \bar{u}
\]  

(22)

where \( \tilde{H}_k \overset{\text{def}}{=} \tilde{\Gamma}^{-1/2} \tilde{\Gamma}_k \tilde{\Gamma}^{-1/2} \). Thus using \( \tilde{\eta} = d Q [1] (5) \), we get:

\[
E \left( \phi(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \right) = -E \left( Q^{1/2} \phi(Q) \text{Re}(\mu_k^H \tilde{\Gamma}^{-1/2} \bar{u}) \right) - \frac{1}{2} E[Q \phi(Q) \bar{u}^H \tilde{H}_k \bar{u}].
\]  

(23)

Since \( Q \) and \( u \) are independent, \( Q \) and \( \bar{u} \) are also independent. It follows then from \( E(\bar{u}) = 0 \), \( E(\bar{u} \bar{u}^H) = \frac{1}{M} I \) and \( E(\phi(Q)) = -M [5] (11) \) that

\[
E \left( Q^{1/2} \phi(Q) \text{Re}(\mu_k^H \tilde{\Gamma}^{-1/2} \bar{u}) \right) = 0
\]

and

\[
E[Q \phi(Q) \bar{u}^H \tilde{H}_k \bar{u}] = E[Q \phi(Q)] \text{Tr}[\tilde{H}_k E(\bar{u} \bar{u}^H)] = -\text{Tr}(\tilde{H}_k) = -\text{Tr}(\tilde{\Gamma}^{-1} \tilde{\Gamma}_k).
\]

Thus

\[
E \left( \phi(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \right) = \frac{1}{2} \text{Tr}(\tilde{\Gamma}^{-1} \tilde{\Gamma}_k),
\]  

(24)

which proves (19).

Now, we evaluate the elements of the FIM. It follows from (20), using (24), that

\[
\left[ \text{FIM} \right]_{k,l} = E \left( \frac{\partial \log p(z; \alpha)}{\partial \alpha_k} \frac{\partial \log p(z; \alpha)}{\partial \alpha_l} \right) = -\frac{1}{4} \text{Tr}(\tilde{\Gamma}^{-1} \tilde{\Gamma}_k \tilde{\Gamma}^{-1} \tilde{\Gamma}_l) + E \left( \phi^2(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \frac{\partial \tilde{\eta}}{\partial \alpha_l} \phi(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \frac{\partial \tilde{\eta}}{\partial \alpha_l} \phi(\tilde{\eta}) \phi(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \frac{\partial \tilde{\eta}}{\partial \alpha_l} \phi(\tilde{\eta}) \phi(\tilde{\eta}) \phi(\tilde{\eta}) \phi(\tilde{\eta}) \right). \]  

(25)

It follows from (18) that \( \tilde{\Gamma}^{-1/2}(\tilde{z} - \bar{\mu}) = d \sqrt{Q} \bar{u} \) and hence from (21) we get

\[
\phi^2(\tilde{\eta}) \frac{\partial \tilde{\eta}}{\partial \alpha_k} \frac{\partial \tilde{\eta}}{\partial \alpha_l} = d Q \phi^2(Q) \text{Re} \left( \mu_k^H \tilde{\Gamma}^{-1/2} \bar{u} \right) \text{Re} \left( \mu_l^H \tilde{\Gamma}^{-1/2} \bar{u} \right) + \frac{1}{2} Q^3/2 \phi^2(Q) \text{Re} \left( \mu_k^H \tilde{\Gamma}^{-1/2} \bar{u} \right) \left[ \bar{u}^H \tilde{H}_k \bar{u} \right] + \frac{1}{2} Q^3/2 \phi^2(Q) \text{Re} \left( \mu_l^H \tilde{\Gamma}^{-1/2} \bar{u} \right) \left[ \bar{u}^H \tilde{H}_l \bar{u} \right] + \frac{1}{4} Q^2 \phi^2(Q) \left[ \bar{u}^H \tilde{H}_k \bar{u} \right] \left[ \bar{u}^H \tilde{H}_l \bar{u} \right].
\]  

(26)
The first term of (26) can be further simplified as
\[
\text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1/2} \tilde{u} \right) \text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1/2} \tilde{u} \right) = \frac{1}{2} \text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1/2} \tilde{u} \tilde{u}^H \tilde{\Gamma}^{-1/2} \mu_k \right) + \frac{1}{2} \text{Re}\left(\mu_k^T \tilde{\Gamma}^{-1/2} \tilde{u}^* \tilde{u}^H \tilde{\Gamma}^{-1/2} \mu_k \right),
\]
and thanks to the independence between \(Q\) and \(\tilde{u}\), the expected value of the first term of (26) is given by
\[
\mathbb{E}[\phi^2(Q)]E\left(\mu_k^H \tilde{\Gamma}^{-1/2} \tilde{u} \right) \text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1/2} \tilde{u} \right) = \frac{\mathbb{E}[\phi^2(Q)]}{2M} \text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1} \mu_k \right) + \frac{\mathbb{E}[\phi^2(Q)]}{2M} \text{Re}\left(\mu_k^{T^*} \tilde{\Gamma}^{-1} \mu_k \right) = \frac{\mathbb{E}[\phi^2(Q)]}{M} \text{Re}\left(\mu_k^H \tilde{\Gamma}^{-1} \mu_k \right), (27)
\]
using \(\mathbb{E}(\tilde{u}^H) = \frac{1}{M} I\) and \(\mathbb{E}(\tilde{u}^* \tilde{u}^H) = \frac{1}{M} J\), \(\tilde{\Gamma}^{-1/2} J' \tilde{\Gamma}^{-1/2} = \tilde{\Gamma}^{-1} J'\) and \(J' \tilde{\mu}_l = \tilde{\mu}_l^T\). The expected value of the second and third terms of (26) are zero because the third-order moments of \(u\) are zero. Because \(y =_d \|y\| u\), where \(\|y\|\) and \(u\) are independent when \(y \sim \mathbb{C} N_M(0, I)\), we get
\[
\mathbb{E}[\tilde{u}^H \tilde{\Gamma}_k \tilde{u}] = \mathbb{E}[\tilde{y}^H \tilde{\Gamma}_k \tilde{y}] = \frac{1}{\mathbb{E}(\|y\|^4)} \mathbb{E}[(\tilde{y}^H \tilde{H}_k \tilde{y})(\tilde{y}^H \tilde{H}_l \tilde{y})].
\]
Noting that \(\tilde{H}_k\) and \(\tilde{H}_l\) are structured as \(\tilde{A}^2\) and \(\tilde{B}\) of the Lemma [1] this lemma applies to the couples \((\tilde{H}_k, \tilde{H}_l)\) and \((I, I)\) giving \(\mathbb{E}[(\tilde{y}^H \tilde{H}_k \tilde{y})(\tilde{y}^H \tilde{H}_l \tilde{y})] = \text{Tr}(\tilde{H}_k) \text{Tr}(\tilde{H}_l) + 2\text{Tr}(\tilde{H}_k \tilde{H}_l)\) and \(\mathbb{E}[\|y\|^4] = M(M + 1)\). Consequently the expected value of the last term of (26) is given by
\[
\mathbb{E}\left(\frac{1}{4} Q^2 \phi^2(Q) \|u\|^2 \right) = \frac{\mathbb{E}[\phi^2(Q)]}{4M(M + 1)} \left(\text{Tr}(\tilde{H}_k) \text{Tr}(\tilde{H}_l) + 2\text{Tr}(\tilde{H}_k \tilde{H}_l)\right) = \frac{\mathbb{E}[\phi^2(Q)]}{4M(M + 1)} \left(\text{Tr}(\tilde{H}_k \tilde{H}_l^{-1}) \text{Tr}(\tilde{H}_l \tilde{H}_k^{-1}) + 2\text{Tr}(\tilde{H}_k \tilde{H}_l^{-1} \tilde{H}_l \tilde{H}_k^{-1})\right) (28)
\]
Gathering (27) (28) in (25) concludes the proof.

IV. PROOF OF EQ. (9) OF [11]

Using that [11] (4) is a p.d.f. with \(\int_0^\infty \delta_{M, g}^{-1} M g(Q) dQ = 1\) and that \(\mathbb{E}(Q) = (\mathbb{E}(Q^2)) < \infty\), we get
\[
\mathbb{E}(Q^2(Q)) = \int_0^\infty \delta_{M, g}^{-1} M g(Q) dQ = \left[\delta_{M, g}^{-1} M g(Q)\right]_0^\infty - M \int_0^\infty \delta_{M, g}^{-1} M^{-1} g(Q) dQ = -M. (29)
\]
It follows from Cauchy-Schwarz inequality that
\[
M^2 = (\mathbb{E}(Q \phi(Q)))^2 \leq \mathbb{E}(Q) \mathbb{E}(Q^2(Q)) = \mathbb{E}(Q) M \xi_1. (30)
\]
Next, note that
\[
\mathbb{E}(Q) = \int_0^\infty \delta_{M, g}^{-1} M g(Q) dQ = \delta_{M, g}^{-1} M_{g=1} \int_0^\infty \delta_{M+1, g}^{-1} M g(Q) dQ = \delta_{M, g}^{-1} \delta_{M+1, g} = M. (31)
\]
Plugging (31) in (30) proves Eq. (9) of [11].

V. PROOF OF RESULT 4

Because \(\xi_2 = 1\) for Gaussian distributions, we get for NC-CES distributions:
\[
\tilde{\Gamma}_\text{CES}^{\text{NC}}(\alpha_2) - \tilde{\Gamma}_\text{CN}^{\text{NC}}(\alpha_2) = \frac{\xi_2 - 1}{2} \left(\frac{d\text{vec}(\tilde{\Gamma})}{d\alpha_2}\right)^H \left(\tilde{\Gamma}^{-T} \otimes \tilde{\Gamma}^{-1} + \frac{1}{2} \text{vec}(\tilde{\Gamma}^{-1}) \text{vec}^H(\tilde{\Gamma}^{-1})\right) \frac{d\text{vec}(\tilde{\Gamma})}{d\alpha_2} (32)
\]
where \(\tilde{\Gamma}^{-T} \otimes \tilde{\Gamma}^{-1} + \frac{1}{2} \text{vec}(\tilde{\Gamma}^{-1}) \text{vec}^H(\tilde{\Gamma}^{-1})\) is positive definite. Replacing \(\tilde{\Gamma}\) by \(\Gamma\), the proof is identical for C-CES distributions.
VI. PROOF OF RESULT 5

We note first that the general expressions of the SCRB proved here is valid for arbitrary parameterization of $A_\theta$ if the real-valued parameter of interest $\theta \in \mathbb{R}^L$ is characterized by the subspace generated by the columns of the full column rank $M \times K$ matrix $A_\theta$ with $K < M$. It can be applied for example to near or far-field DOA modeling with scalar or vector-sensors for an arbitrary number of parameters per source $s_t,k$ (with $s_t = (s_{t,1}, ..., s_{t,K})^T$) and many other modelings as the SIMO and MIMO modelings. Let us start with the circular case for which $\Omega = 0$ and thus $\widetilde{\Omega} = \text{Diag}(\Sigma, \Sigma^*)$ where $\Sigma = A_\theta R_s A_\theta^H + \sigma_n^2 I$. The SCRB form for this case can be then written through the compact expression of the general FIM given in Result 2, using (1) and (2), as follows:

$$
\frac{1}{T} \text{SCRB}^{-1}_{\text{CES}}(\alpha) = G^H \Pi_{\Delta} G,
$$

(34)

with $G = T_i^{1/2}(\Sigma^{-T/2} \otimes \Sigma^{-1/2}) \frac{\partial \text{vec} (\Sigma)}{\partial \sigma}$ and $\Delta = T_i^{1/2}(\Sigma^{-T/2} \otimes \Sigma^{-1/2}) \frac{\partial \text{vec} (\Sigma)}{\partial \sigma}$ where

$$
T_i \overset{\text{def}}{=} \xi_2 I + (\xi_2 - 1) \text{vec}(I) \text{vec}^T(I).
$$

(35)

Let’s further partition the matrix $\Delta$ as $\Delta = T_i^{1/2}(\Sigma^{-T/2} \otimes \Sigma^{-1/2}) \left[ \frac{\partial \text{vec} (\Sigma)}{\partial \rho} | \frac{\partial \text{vec} (\Sigma)}{\partial \sigma_n} \right] \overset{\text{def}}{=} [V | u_n]$. In the sequel, the proofs presented here follow the lines of the proof presented in [6] for circular Gaussian distributed observations. It follows from [6, rel. (14)] that

$$
\Pi_{\Delta}^{-1} = \Pi_V^{-1} - \frac{\Pi_V^{-1} u_n u_n^H \Pi_V^{-1}}{u_n^H \Pi_V^{-1} u_n}.
$$

(36)

Using $\frac{\partial \text{vec} (\Sigma)}{\partial \sigma_n} = \text{vec}(I)$, we obtain

$$
u_n = T_i^{1/2} \text{vec}(\Sigma^{-1}).
$$

(37)

Consequently using (34) and (36), if $g_k$ denotes the $k$th column of $G$, the $(k,l)$ element of $\text{SCRB}^{-1}_{\text{CES}}(\alpha)$ can be written elementwise as

$$
\frac{1}{T} [\text{SCRB}^{-1}_{\text{CES}}(\theta)]_{k,l} = g_k^H \Pi_V g_l - \frac{g_k^H \Pi_V u_n u_n^H \Pi_V g_l}{u_n^H \Pi_V u_n}.
$$

(38)

Let us proceed now to determine the expression of $g_k$. Letting $A'_{b_k} = \frac{\partial A_{b_k}}{\partial \theta}$, we get

$$
\frac{\partial \Sigma}{\partial \theta_k} = A'_{b_k} R_s A_{b_k}^H + A_{b_k} R_s A'_{b_k}^H.
$$

(39)

Hence, using (1), the $k$th column of $G$ in (38) is given by

$$
g_k = T_i^{1/2} \text{vec}(Z_k + Z_k^H) \quad \text{where} \quad Z_k \overset{\text{def}}{=} \Sigma^{-1/2} A_{b_k} R_s A_{b_k}^H \Sigma^{-1/2}.
$$

(40)

Next, we determine $V$ and then $\Pi_V$. Since $R_s$ is a Hermitian matrix, it can be then factorized as

$$
\text{vec}(R_s) = J \rho\n$$

(41)

where $J$ is a $K^2 \times K^2$ constant nonsingular matrix. It follows, using (1), that $V$ can be be expressed as

$$
V = T_i^{1/2}(\Sigma^{-T/2} A_{\theta}^* \otimes \Sigma^{-1/2} A_{\theta}) J = T_i^{1/2} W J.
$$

(42)
Note from (38) that the SCRB depends on $V$ only via $\Pi_V^\perp$, that can be expressed as

$$\Pi_V^\perp = I - V(V^HV)^{-1}V^H = I - T_i^{1/2}W(W^HT_i)W^{-1}W^HT_i^{1/2}. \tag{42}$$

After some algebraic manipulation, using (1) and (2), we obtain

$$W^HT_iW = \xi_2(U^* \otimes U) + (\xi_2 - 1)\text{vec}(U)\text{vec}^H(U),$$

where $U \triangleq A_\theta \Sigma^{-1}A_\theta$ is a $K \times K$ Hermitian nonsingular matrix. It follows from matrix inverse lemma (given by (7)), that its inverse can be expressed as

$$(W^HT_iW)^{-1} = \frac{1}{\xi_2}(U^{-*} \otimes U^{-1}) - \eta\text{vec}(U^{-1})\text{vec}^H(U^{-1})$$

where $\eta \triangleq \frac{\xi_2 - 1}{\xi_2(1 + (\xi_2 - 1)K)}$ can be simplified, using (4), as $\eta \triangleq \frac{\xi_2 - 1}{\xi_2(1 + (\xi_2 - 1)K)}$. Thus, using (1) and (2), we obtain

$$W(W^HT_iW)^{-1}W^H = \frac{1}{\xi_2}(H_i^* \otimes H_1) - \eta\text{vec}(H_1)\text{vec}^H(H_1) \triangleq B, \tag{43}$$

where $H_1 \triangleq \Sigma^{-1/2}A_\theta U^{-1}A_\theta^H \Sigma^{-1/2}$. Therefore, (42) becomes

$$\Pi_V^\perp = I - T_i^{1/2}BT_i^{1/2}. \tag{44}$$

Now let us show that $u_n^H\Pi_V^\perp g_k = 0$. It follows from (37) and (40), using (44), that

$$u_n^H\Pi_V^\perp g_k = \text{vec}^H(\Sigma^{-1})T_i\text{vec}(Z_k + Z_k^H) = \text{vec}^H(\Sigma^{-1})T_iB\text{vec}(Z_k + Z_k^H). \tag{45}$$

It follows, after some algebraic manipulation, using (1), (3) and (45) that

$$T_iBT_i = \xi_2(H_i^* \otimes H_1) - \xi_2^2\eta\text{vec}(H_1)\text{vec}^H(H_1) + \frac{(\xi_2 - 1)(1 - K\eta)(\eta\text{vec}(H_1)\text{vec}^H(H_1) + \text{vec}(H_1)\text{vec}^T(I))}{\xi_2} + \frac{(\xi_2 - 1)^2K}{\xi_2}(1 - K\eta)(\text{vec}(I)\text{vec}^T(I)), \tag{46}$$

using $H_1^2 = H_1$ and Tr($H_1$) = $K$. Using the definition (35) for $T_i$ and (3), the first term of (45) can be expressed as

$$\text{vec}^H(\Sigma^{-1})T_i\text{vec}(Z_k + Z_k^H) = \xi_2\text{Tr}(\Sigma^{-1}(Z_k + Z_k^H)) = \xi_2\text{Tr}(\Sigma^{-1}(Z_k^H)) = \xi_2\text{Tr}(\Sigma^{-1}(Z_k^H)) = 2\text{Re}(\text{Tr}(\Sigma^{-1}A_\theta R_sA_\theta^H)) + 2(\xi_2 - 1)\text{Tr}(\Sigma^{-1})\text{Re}(\text{Tr}(\Sigma^{-1}A_\theta R_sA_\theta^H)) \tag{47}$$

using $\text{Tr}(\Sigma^{-1}(Z_k + Z_k^H)) = 2\text{Re}(\text{Tr}(\Sigma^{-1}A_\theta R_sA_\theta^H))$ and $\text{Tr}(Z_k + Z_k^H) = 2\text{Re}(\text{Tr}(\Sigma^{-1}A_\theta R_sA_\theta^H))$. After simple algebraic manipulations, using (46), (1) and (3), and that $\text{Tr}(Z_k + Z_k^H) = \text{Tr}(Z_k^H) = \text{Tr}(Z_k^H) = \text{Tr}(H_1(Z_k^H + Z_k^H)H_1) = \text{Tr}(\Sigma^{-1}(Z_k^H))$ and $\text{Tr}(\Sigma^{-1}H_1^2) = \text{Tr}(\Sigma^{-1}H_1)$, the second term of (45) can be simplified as

$$\text{vec}^H(\Sigma^{-1})T_iB\text{vec}(Z_k + Z_k^H) = \xi_2\text{Tr}(\Sigma^{-1}H_1(Z_k + Z_k^H)H_1) = \frac{(\xi_2 - 1)(1 - K\eta)(\text{vec}(I)\text{vec}^T(I) + \text{vec}(H_1)\text{vec}^T(I)))}{\xi_2} + \frac{(\xi_2 - 1)^2K}{\xi_2}(1 - K\eta)(\text{vec}(I)\text{vec}^T(I)), \tag{48}$$

where the first term in the last line is obtained using $A_\theta U^{-1}A_\theta^H = \Sigma^{-1}A_\theta$. It follows, therefore, from (45), (47) and (48) that

$$u_n^H\Pi_V^\perp g_k = 0.$$
This identity together with \(40\) and \(44\) allows us to rewrite the individual elements of \(38\) as
\[
\frac{1}{T} \left[ \text{SCR}_{\text{CES}}(\theta) \right]_{k,l} = g_k^H \Pi \tilde{g}_l
\]
\[
= \text{vec}^H(Z_k + Z_k^H) T_i \text{vec}(Z_l + Z_l^H) - \text{vec}^H(Z_k + Z_k^H) T_i B T_i \text{vec}(Z_l + Z_l^H). \tag{49}
\]
After simple algebraic manipulations, using the definition \(35\) for \(T_i\), \(1\) and \(3\), the first term in \(49\) can be simplified as
\[
\text{vec}^H(Z_k + Z_k^H) T_i \text{vec}(Z_l + Z_l^H) = \xi_2 \text{Tr}((Z_k + Z_k^H)(Z_l + Z_l^H)) + (\xi_2 - 1) \text{Tr}(Z_k + Z_k^H) \text{Tr}(Z_l + Z_l^H)
\]
\[
= 2\xi_2 \left[ \text{Re}(\text{Tr}((\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H))) \right]
\]
\[
+ \text{Re}(\text{Tr}((\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)(\Sigma^{-1} R_s A_{\theta_i}^H)))
\]
\[
+ 4(\xi_2 - 1) \text{Re}(\text{Tr}(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)) \text{Re}(\text{Tr}(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)). \tag{50}
\]
Similarly, after some algebraic manipulations, using \(46\), \(1\) and \(3\), the second term in \(49\) can be simplified as
\[
\text{vec}^H(Z_k + Z_k^H) T_i B T_i \text{vec}(Z_l + Z_l^H) = 2\xi_2 \left[ \text{Tr}(\text{Re}((\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H))) \right]
\]
\[
+ \text{Tr}(\text{Re}((\Sigma^{-1} A U^{-1} A^H \Sigma^{-1} A_{\theta_i} R_s A_{\theta_i}^H)(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)))
\]
\[
+ 4(\xi_2 - 1) \text{Tr}(\text{Re}(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)) \text{Tr}(\text{Re}(\Sigma^{-1} A_\theta R_s A_{\theta_i}^H)). \tag{51}
\]
It follows then from \(50\) and \(51\) that \(49\) can be simplified as
\[
\frac{1}{T} \left[ \text{SCR}_{\text{CES}}(\theta) \right]_{k,l} = 2\xi_2 \text{Re} \left( \text{Tr} \left[ (\Sigma^{-1} - \Sigma^{-1} A U^{-1} A^H \Sigma^{-1}) (A_{\theta_i} R_s A_{\theta_i}^H \Sigma^{-1} A_\theta R_s A_{\theta_i}^H) \right] \right)
\]
\[
= \frac{2\xi_2}{\sigma_n^2} \text{Re} \left( \text{Tr} \left[ (\Pi A_{\theta_i}) (A_{\theta_i} R_s A_{\theta_i}^H \Sigma^{-1} A_\theta R_s A_{\theta_i}^H) \right] \right)
\]
\[
= \frac{2\xi_2}{\sigma_n^2} \text{Re} \left( \text{Tr} \left[ \Pi A_{\theta_i} A_{\theta_i}^H \Sigma_{\theta_i} \right] \right), \tag{52}
\]
where the second equality is obtained using \(\Sigma^{-1} - \Sigma^{-1} A U^{-1} A^H \Sigma^{-1} = \frac{1}{\sigma_n^2} \Pi A \) thanks to \(A U^{-1} A^H \Sigma^{-1} = A (A^H A)^{-1} A^H\). Using \(4\), we can write \(52\) in matrix form as is shown in Result 5.

In the noncircular case, the proof follows the similar above steps by replacing \(T_i\) by \(\Phi_i \equiv \xi_2 I + \xi_2 \vec{\text{vec}}(I) \vec{\text{vec}}^T(I)\), and \(\Sigma\) by \(\Sigma\) where \(39\) is replaced by \(\frac{\partial \delta}{\partial \theta_{\alpha}} = A_{\theta_{\alpha}} R_s \tilde{A}_{\theta_{\alpha}}^H + \tilde{A}_{\theta_{\alpha}} R_s \tilde{A}_{\theta_{\alpha}}^H\) with \(\tilde{A}_{\theta_{\alpha}} \equiv \text{Diag}(A_{\theta_{\alpha}}, A_{\theta_{\alpha}}^*)\) and \(A_{\theta_{\alpha}} \equiv \text{Diag}(A_{\theta_{\alpha}}, A_{\theta_{\alpha}}^*)\).

VII. PROOF OF RESULT 6

The proof of this result follows similar steps as the proof of Result 5 based on \([7\) th. 1\] by replacing \(\Sigma\) by \(\tilde{\Sigma}\) and \(\Sigma^{-1} A U^{-1} A^H \Sigma^{-1}\) by \(\tilde{\Sigma}^{-1} A U^{-1} A^H \tilde{\Sigma}^{-1}\) thanks to \(A U^{-1} A^H \tilde{\Sigma}^{-1} = \tilde{A} A^H (\tilde{A} A^H)^{-1} A^H\). Using \(\tilde{\Sigma}\), we can write \(52\) in matrix form as \(52\).
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