Quantum chaos in quantum Turing machines

Ilki Kim and Günter Mahler

Institut für Theoretische Physik I, Universität Stuttgart
Pfaffenwaldring 57, 70550 Stuttgart, Germany
phone: ++49-(0)711 685-5100, FAX: ++49-(0)711 685-4909
email: ikim@theo.physik.uni-stuttgart.de

PACS: 03.67.Lx, 05.45.Mt

Abstract

We investigate a 2-spin quantum Turing architecture, in which discrete local rotations $\alpha_m$ of the Turing head spin alternate with quantum controlled NOT-operations. We demonstrate that a single chaotic parameter input $\alpha_m$ leads to a chaotic dynamics in the entire Hilbert-space.

Key words: quantum chaos, quantum Turing machine, Fibonacci-like sequence

1 Introduction

Chaotic behaviour as an exponential sensitivity to initial conditions in a classical non-linear system has attracted a great deal of attention. The deterministic chaos, which occurs in non-dissipative systems, can typically be found starting from regular states as a function of some external control parameter. On the other hand, there seems to be no direct analogue to chaos in the quantum world, because the Schrödinger equation is linear in time, and the scalar product between different initial states (as a measure of distance) is conserved under unitary evolution. Accordingly, the semiclassical quantum chaology [1] has been constrained to studying some quantum-mechanical “fingerprints of chaos” (like spectral properties), and non-trivial transitions from the quantum - to classical domain and vice versa (e.g., Bohr’s correspondence principle).

Experimental progress in mesoscopic physics, e.g. the transport of electrons through so-called “chaotic quantum dots” [2], has allowed to study a quantum-mechanical system in a random potential, the results of which give numerical evidence for weak chaos (indicated by level repulsion) [3].

Recent theoretical and experimental studies in quantum information theory and quantum computation (QC) [4] should shed new light also on the
basic understanding of quantum mechanics itself. In QC one tries to utilize the quantum-mechanical superposition and (non-classical) entanglement to solve certain classes of problems in a potentially very powerful way. While most models of QC have been based on networks of quantum gates, which are reminiscent of classical integrated circuits, quantum Turing machines (QTM) [5,6] follow a different line but have not shown much potential for future applications up to now. In both cases the complexity of the computation is characterized by sequences of unitary transformations (or the corresponding Hamiltonians $\hat{H}$ acting during finite time interval steps).

The investigation of quantum chaos based on quantum gate networks has so far been proposed e.g. by an implementation of quantum baker’s map on a 3-qubit NMR quantum computer [7], or by realizing a quantum-mechanical delta-kicked harmonic oscillator in an ion trap [8]. In both cases some sort of sensitivity has been located with respect to parameters specifying the dynamics (e.g., the respective Hamiltonian). In this letter we address an iterative map which, though based on standard gates, can be thought to be realized as a QTM architecture: Local transformations of the Turing head controlled by a Fibonacci-like sequence of rotation angles alternate with a quantum-controlled NOT-operation with a second spin on the Turing tape. This type of control can generate a chaotic quantum propagation (Lyapunov exponent, $\ln \frac{\sqrt{\phi}}{2} > 0$) in the “classical” regime [9] which is defined here as the Turing head being restricted to an entanglement-free state sequence (“primitive”) [10]. It will be shown that chaos in local Bloch-vector space of the Turing head can be found also in the quantum-mechanical superposition of those primitives, implying entanglement between head and tape as a genuine quantum feature (see Fig 1). Due to this quantum correlation, we can observe a chaotic propagation even in the reduced subspace of the Turing tape (“chaos swapping”).

2 Chaotically driven quantum-Turing machine

The quantum network [11] to be considered in detail is composed of 2 spins $|p^{(\mu)}\rangle; p = -1, 1; \mu = S, 1$ (Turing-head $S$, Turing-tape spin 1) so that its network-state $|\psi\rangle$ lives in the 4-dimensional Hilbert-space spanned by the product wave-functions $|j^{(S)}k^{(1)}\rangle = |jk\rangle$. Correspondingly, any (unitary) network-operator can be expanded as a sum of product-operators, which may be based on the $SU(2)$-generators, Pauli matrices $\hat{\sigma}_j^{(\mu)}$, $j = 1, 2, 3$, with $\mathbb{1}^{(\mu)}$.

The initial state $|\psi_0\rangle$ will be taken to be a product of the Turing-head and tape wave-functions. For the discretized dynamical description of this externally driven system we identify the unitary operators $\hat{U}_n$, $n = 1, 2, 3, \ldots$ with the local unitary transformation on the Turing-head $S$, $\hat{U}^{(S)}_{\hat{a}_m}$, and the quantum-controlled-NOT (QCNOT) on $(S, 1)$, $\hat{U}^{(S,1)}$, respectively as follows:
\[ \hat{U}_{2m-1} = \exp\left(-i\hat{\sigma}_1^{(S)}\alpha_m/2\right) \tag{1} \]
\[ \hat{U}_{2m} = \hat{U}^{(S,1)} = \hat{P}_{-1,-1}^{(S)}\hat{\sigma}_1^{(1)} + \hat{P}_{1,1}^{(S)}\hat{1}^{(1)} = \left(\hat{U}^{(S,1)}\right)^\dagger, \tag{2} \]

where \( \alpha_{m+1} = \alpha_m + \alpha_{m-1}, \alpha_0 = 0, \) and \( \hat{P}_{j,j}^{(S)} = |j\rangle\langle j|^{(S)}j\rangle |j\rangle \) is a (local) projection operator. The \( m \)th Fibonacci number \( \alpha_m \) is given by
\[ \alpha_m = \frac{\alpha_1}{\sqrt{5}} (\beta^m - \gamma^m), \tag{3} \]

where \( \beta := \frac{1 + \sqrt{5}}{2}, \gamma := \frac{1 - \sqrt{5}}{2}. \) It is useful for later calculations to note that \( \beta^{m+1} = \beta^m + \beta^{m-1}, \gamma^{m+1} = \gamma^m + \gamma^{m-1}. \)

We restrict ourselves to the reduced state-space dynamics of the head \( S \) and tape-spin 1, respectively,

\[ \sigma_j^{(S)}(n) = \text{Tr}\left(\hat{\rho}_n^{(S)}\hat{\sigma}_j^{(S)}\right) = \langle\psi_n|\hat{\sigma}_j^{(S)}\otimes\hat{1}^{(1)}|\psi_n\rangle, \]
\[ \sigma_k^{(1)}(n) = \text{Tr}\left(\hat{\rho}_n^{(1)}\hat{\sigma}_k^{(1)}\right) = \langle\psi_n|\hat{1}^{(S)}\otimes\hat{\sigma}_k^{(1)}|\psi_n\rangle. \tag{4} \]

Due to the entanglement between the head and tape, both will, in general, appear to be in a “mixed-state”, which means that the length of the Bloch-vectors in (4) is less than 1. However, for specific initial states \( |\psi_0\rangle \) the state of head and tape will remain pure: As \( |\pm\rangle^{(1)} := \frac{1}{\sqrt{2}} (|1\rangle^{(1)} \pm |0\rangle^{(1)}) \) are the eigenstates of \( \hat{\sigma}_1^{(1)} \) with \( \hat{\sigma}_1^{(1)}|\pm\rangle^{(1)} = \pm|\pm\rangle^{(1)} \), the QCN_T-operation \( \hat{U}^{(S,1)} \) cannot create any entanglement, irrespective of the head state \( |\varphi^{(S)}\rangle \), i.e.

\[ \hat{U}^{(S,1)}|\varphi^{(S)}\rangle \otimes |+\rangle^{(1)} = |\varphi^{(S)}\rangle \otimes |+\rangle^{(1)} \]
\[ \hat{U}^{(S,1)}|\varphi^{(S)}\rangle \otimes |-\rangle^{(1)} = \hat{\sigma}_3^{(S)}|\varphi^{(S)}\rangle \otimes |-\rangle^{(1)}. \tag{5} \]

As a consequence, for the initial product-states \( |\psi_0\rangle = |\varphi_0^{(S)}\rangle \otimes |\pm\rangle^{(1)} \) with \( |\varphi_0^{(S)}\rangle = \exp\left(-i\hat{\sigma}_1^{(S)}\varphi_0/2\right) |1\rangle^{(S)} \) the state \( |\psi_n\rangle \) remains a product-state at any step \( n \) and the Turing-head then performs a pure-state trajectory (“primitive”) on the Bloch-circle \( (\sigma_1^{(S)}(n) = 0) \)
\[ |\psi_n^{\pm}\rangle = |\varphi_n^{\pm}\rangle^{(S)} \otimes |\pm\rangle^{(1)}, \quad \left(\sigma_2^{(S)}(n)\right)^2 + \left(\sigma_3^{(S)}(n)\right)^2 = 1. \tag{6} \]

It is easy to verify that the Fibonacci relation and the property (5) give for \( \sigma_j^{(S)}(n) \) (see eq. (4)) of \( |\varphi_n^{\pm}\rangle^{(S)} \otimes |+\rangle^{(1)} \), \( n = 2m, \)
\[ \sigma_2^{(S)}(2m|+) = \sin C_{2m}(+), \quad \sigma_3^{(S)}(2m|+) = -\cos C_{2m}(+), \tag{7} \]
where \( C_{2m}(+) := \sum_{j=1}^{m} \alpha_j \), and for \( n = 2m - 1 \), \( \sigma_k^{(S)}(2m - 1|+) = \sigma_k^{(S)}(2m|+) \).

In order to find the corresponding expression of \( \sigma_k^{(S)}(n|-) \) for \( |\varphi_n^{(S)}\rangle \otimes |-\rangle^{(1)} \), we utilize the following recursion relations for the cumulative rotation angle \( C_n(-) \) up to step \( n \)

\[
C_{2m}(-) = -C_{2m-1}(-), \quad C_{2m-1}(-) = \alpha_m + C_{2m-2}(-). \tag{8}
\]

Then \( C_{2m}(-), C_{2m-1}(-) \) are rewritten, respectively, as

\[
C_{2m}(-) = -C_{2m-2}(-) - \alpha_m = (-1)^{m-1} \sum_{j=1}^{m} (-1)^j \alpha_j
\]

\[
C_{2m-1}(-) = -C_{2m-3}(-) + \alpha_m = (-1)^{m} \sum_{j=1}^{m} (-1)^j \alpha_j, \tag{9}
\]

yielding \( \sigma_2^{(S)}(n|-) = \sin C_n(-), \sigma_3^{(S)}(n|-) = -\cos C_n(-) \) (cf. (7)). The Fibonacci property implies that \( |\varphi_n^{(S)}\rangle \otimes |-\rangle^{(1)} \) is also chaotically driven as \( |\varphi_n^{(+)}\rangle \otimes |+\rangle^{(1)} \) is.

From any initial state \( |\psi_0\rangle = a^{(+)}|\varphi_0^{(S)}\rangle^{(+)} + a^{(-)}|\varphi_0^{(S)}\rangle^{-} \), we then find at step \( n \)

\[
|\psi_n\rangle = a^{(+)}|\varphi_n^{(+)}\rangle^{(+)} + a^{(-)}|\varphi_n^{(-)}\rangle^{-} \tag{10}
\]

and, observing the orthogonality of the \( |\pm\rangle^{(1)} \),

\[
\sigma_k^{(S)}(n) = |a^{(+)}|^2 \sigma_k^{(S)}(n|+) + |a^{(-)}|^2 \sigma_k^{(S)}(n|-) . \tag{11}
\]

This trajectory of the Turing-head  \( S \) thus appears, for fixed \( n \), as a decomposition into two Bloch-vectors corresponding to non-orthogonal pure states, a consequence of the superposition as a quantum feature. By using (9), (11) (with \( a^{(+)} = a^{(-)} = 1/\sqrt{2} \)) we thus obtain for \( |\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)} \)

\[
\left( \sigma_2^{(S)}(2m), \sigma_3^{(S)}(2m) \right) = \cos A_m \cdot (\sin B_m, -\cos B_m) \\
\left( \sigma_2^{(S)}(2m-1), \sigma_3^{(S)}(2m-1) \right) = \cos B_m \cdot (\sin A_m, -\cos A_m), \tag{12}
\]

where \( A_m := \alpha_m + \alpha_{m-2} + \cdots, B_m := \alpha_{m-1} + \alpha_{m-3} + \cdots \). The equation (12) shows that the local dynamics of the Turing head is controlled by a “chaotic” driving force (“input”), because the sequences in \( A_m \) and \( B_m \), namely \( \{\alpha_{2m}\} \) or \( \{\alpha_{2m-1}\} \), are in fact both chaotic as \( \{\alpha_m\} \) is. The Bloch-vector \( \vec{\sigma}^{(S)}(n) \) can alternatively be calculated directly from the initial state (here: \( |-1, -1\rangle \)) and for any control angle \( \alpha_1 \) by using the relations
\[ A_m = \begin{cases} \frac{\alpha_1}{\sqrt{5}} (\beta^{m+1} - \gamma^{m+1}) & m = \text{odd} \\ \frac{\alpha_1}{\sqrt{5}} (\beta^{m+1} - \gamma^{m+1} - \sqrt{5}) & m = \text{even} \end{cases} \]

\[ B_m = \begin{cases} \frac{\alpha_1}{\sqrt{5}} (\beta^m - \gamma^m) & m = \text{odd} \\ \frac{\alpha_1}{\sqrt{5}} (\beta^m - \gamma^m - \sqrt{5}) & m = \text{even}. \end{cases} \] (13)

### 3 Instability with respect to perturbations

Now we show that the periodic orbits on the plane \( \{0, \sigma_2^S, \sigma_3^S\} \) are unstable, which means that the dynamics of the Turing head (“output”) is indeed chaotic. It is enough to check the periodicity only for step \( n = 2m \): Periodic orbits for \( |\psi_0\rangle = |-1\rangle \otimes |-1\rangle \) must obey \( C_{2m}(+) = C_{2m}(-) = 2\pi p, p \in \mathbb{Z} \) and \( \alpha_{m+1} = \alpha_1 \pmod{2\pi} \) (one concludes that \( \alpha_1 \) must be a rational multiple of \( \pi \)).

By using the Fibonacci numbers (3), we obtain \( C_{2m}^{\text{per}}(+) \) in (7) and \( C_{2m}^{\text{per}}(-) \) in (9), respectively, for period = \( 2m \) as

\[ C_{2m}^{\text{per}}(+) = \frac{\alpha_1}{\sqrt{5}} (\beta^{m+2} - \gamma^{m+2} - \sqrt{5}) \]

\[ C_{2m}^{\text{per}}(-) = \frac{\alpha_1}{\sqrt{5}} (-\beta^{m-1} + \gamma^{m-1} + (-1)^m \sqrt{5}) . \] (14)

Now let us consider a small perturbation \( \delta \) of the initial phase angle \( \alpha_0 = 0 \), implying \(|\varphi_0^S\rangle = \exp(-i\hat{\sigma}_1^S\delta/2) | -1\rangle^S \) and a perturbed Fibonacci-like sequence \{\( \alpha'_m \)\}:

\[ \alpha'_0 = \delta, \alpha'_1 = \alpha_1, \alpha'_2 = \alpha_1 + \delta, \ldots . \] (15)

Similarly to (14), one finds \( C'_{2m}(\pm) = C_{2m}^{\text{per}}(\pm) + \Delta C_{2m}(\pm) \), respectively, where

\[ \Delta C_{2m}(+) = \frac{\delta}{\sqrt{5}} (\beta^{m+1} - \gamma^{m+1}) \]

\[ \Delta C_{2m}(-) = -\frac{\delta}{\sqrt{5}} (\beta^{m-2} - \gamma^{m-2}) . \] (16)

By using (16) for \( |\psi_0\rangle = |-1\rangle \otimes |-1\rangle \) we represent the evolution of the perturbation at the \( 2m \)-th step:

\[ \begin{pmatrix} \Delta \sigma_2^S(2m) \\ \Delta \sigma_3^S(2m) \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{pmatrix} \Delta \sigma_2^S(0) \\ \Delta \sigma_3^S(0) \end{pmatrix} , \] (17)
where \( \Delta \sigma_2^{(S)}(0) = \sin \delta, \Delta \sigma_3^{(S)}(0) = -\cos \delta; \Delta \sigma_2^{(S)}(2m) = \cos(\delta \alpha_m) \sin (\delta \alpha_{m-1}), \Delta \sigma_3^{(S)}(2m) = -\cos(\delta \alpha_m) \cos(\delta \alpha_{m-1}); M_{11} = \cos(\delta \alpha_m) \cos(\delta \alpha_{m-1}) / \sin \delta, M_{22} = \cos(\delta \alpha_m) \cos(\delta \alpha_{m-1}) / \cos \delta \), respectively. One easily shows

\[
\lim_{\delta \to 0} M_{11} = \frac{1}{\sqrt{5}} (\beta^{m-1} - \gamma^{m-1}) , \quad \lim_{\delta \to 0} M_{22} = 1 ,
\]

which means that \( M_{11} \) grows exponentially (note that \(|\beta| > 1, |\gamma| < 1|\)), and the periodic orbit is thus unstable to a small perturbation \( \delta \) in the external control (e.g., for period \( n = 40 \), \( \lim_{\delta \to 0} M_{11} = 4181 \gg 1 \), and see Fig 2).

Strikingly enough, the local dynamics of the Turing tape also shows the exponential sensitivity to initial conditions \((\sigma_1^{(1)}(n) = \sigma_2^{(1)}(n) = 0)\):

\[
\sigma_3^{(1)}(n) = \begin{cases} 
- \cos \left( \alpha_{[\frac{n}{4}]} + 1 + \delta \text{Fib}_{[\frac{n}{4}]} \right) & n = 0, 1 \mod 4 \\
\cos \left( \alpha_{[\frac{n}{4}]} + \delta \text{Fib}_{[\frac{n}{4}]} \right) & n = 2, 3 \mod 4 
\end{cases}
\]

where \( \delta \text{Fib} := \frac{\delta}{\sqrt{5}} (\beta^m - \gamma^m) \); \([a] := n, a = n + r, n \in \mathbb{Z}, 0 \leq r < 1\). Similarly to the Turing-head case, it follows for \( \delta \to 0 \) at step \( n = 2m + 2 \), period \( 2m = 0 \mod 4 \)

\[
\Delta \sigma_3^{(1)}(2m + 2) = M \cdot \Delta \sigma_3^{(1)}(2); \lim_{\delta \to 0} M = \frac{1}{\sqrt{5}} (\beta^{m+1} - \gamma^{m+1}) \frac{\sin(\alpha_{m+2})}{\sin(\alpha_1)} ,
\]

where \( \Delta \sigma_3^{(1)}(2) = \cos(\alpha_1 + \delta) - \cos(\alpha_1), \Delta \sigma_3^{(1)}(n) = \cos(\alpha_{m+2} + \delta_{m+1}) - \cos(\alpha_{m+2}) \), confirming the exponential instability of the periodic orbit; it is easily shown that there is no periodic orbit with period \( 2m = 2 \mod 4 \). Note that the Turing tape can exhibit chaos only by means of the entanglement with the head (“chaos swapping”), not as a result of a chaotic driving force. The chaotic sequence of Fibonacci-type can be interpreted as temporal random (chaotic) “potential”, in analogy to 1-dimensional “chaotic quantum dots” in real space [12]. It is also interesting to compare this machine with a regular QTM [10] which is controlled by a fixed \( \alpha \) for local transformations of the Turing-head by using the Bures metric [13]:

\[
D_{\rho \rho'}^2 := \text{Tr} \left\{ (\hat{\rho} - \hat{\rho}')^2 \right\} .
\]

This distance between density matrices, \( \hat{\rho} \) and \( \hat{\rho}' \), lies, independent of the dimension of the Liouville space, between 0 and 2 [see Fig 3; the maximum (squared) distance of 2 applies to pure orthogonal states, \( D^2 = 2 \left(1 - \right)\].
\[ \langle \psi | \psi' \rangle^2 \]. For \( \alpha_m = \alpha \) and any \( \delta \) the distance remains constant; for the Fibonacci-like sequence we recognize an initial exponential sensitivity, which is eventually constrained, though, by \( D^2 \leq 2 \).

The source of the considered chaotic behaviour can be traced back to any small perturbation \( \delta \) of the initial state \( \langle \psi_0(\delta) \rangle \) which is directly connected with a perturbed unitary evolution, \( \hat{U}(\delta) \). This implies that the scalar product between different initial states (as a measure of distance) is no longer conserved under these evolutions:

\[
O' := |\langle \psi_0(\delta) | \hat{U}^\dagger(\delta) \hat{U}(0) | \psi_0(0) \rangle|^2, \quad D^2 = 2(1 - O').
\] (21)

Thus the initial state is directly correlated to its unitary evolution, which can lead to the exponential sensitivity to initial condition, whereas there is no chaos in a generic quantum system evolving by a fixed \( \hat{U} \) even if characterized by chaotic input parameters. This \( O' \) reminds us immediately of the test function \( O = \langle \psi | \hat{V}^\dagger(t) \hat{U}(t) | \psi \rangle \) [14], where \( \hat{U}, \hat{V} \) are specified by slightly different external parameters ("Peres test"): The corresponding parameter-sensitivity has been proposed as a measure to distinguish quantum chaos from regular quantum dynamics. The origin of chaos in our QTM may thus be alternatively ascribed to a perturbed \( \hat{V} = \hat{U}(\delta) \) in the control (see also the comment by R. Schack [15]).

4 Summary

In conclusion, we have studied the quantum dynamics of a chaotically driven QTM based on a decoherence-free Hamiltonian. We have found quantum chaos as a dynamical feature and cumulative loss of control in a pure quantum regime. This might be contrasted with the usual quantum chaology, which is concerned essentially with quantal spectrum analysis of classically chaotic systems (e.g., level-spacing, spectral rigidity). As quantum features we utilized the superposition principle and the physics of entanglement. Our dynamical chaos occurs as a result of the superposition and entanglement of a pair of "classical" (i.e., unentangled) chaotic state-sequences. Due to the entanglement, we can observe the chaos in any local Bloch-plane. This indicates that patterns in reduced Bloch-spheres (a quantum version of a Poincaré-cut, Fig 2) should be useful to characterize quantum chaos in a broad class of quantum networks. It is worth noting that this kind of control loss is completely different from the typical control limit of a quantum network resulting from the exponential blow-up of Hilbert-space dimension in which the state evolves [16]. It is natural to expect that a QTM architecture with an arbitrary number of spins on the Turing tape would also exhibit chaos under the same type of driving.
We would like to thank C. Granzow, M. Karremann, A. Otte and P. Pangritz for stimulating discussions.

References

[1] M. V. Berry, Proc. R. Soc. London A 400 (1985) 229; ibid. 413 (1987) 183; Physica Scripta 40 (1989) 335.

[2] L. P. Kouwenhoven et al., in Mesoscopic Electron Transport, edited by L. L. Sohn et al., NATO ASI Series E345 (Kluwer, Dordrecht, 1997).

[3] D. L. Shepelyansky, Phys. Rev. Lett. 73 (1994) 2607; X. Waintal and J-L. Pichard, Eur. Phys. J. B 6 (1998) 117; X. Waintal, D. Weinmann and J-L. Pichard, ibid. 7 (1999) 451.

[4] A. Steane, Reports Progr. Phys. 61 (1998) 117.

[5] P. Benioff, Phys. Rev. Lett. 48 (1982) 1581; Phys. Rev. A 54 (1996) 1106; Fortschr. Physik 46 (1998) 423.

[6] D. Deutsch, Proc. R. Soc. London A 400 (1985) 97; ibid. 425 (1989) 73.

[7] R. Schack, Phys. Rev. A 57 (1998) 1634; T. Brun and R. Schack, quant-ph/9807050.

[8] S. A. Gardiner, J. I. Cirac and P. Zoller, Phys. Rev. Lett. 79 (1997) 4790; Erratum, ibid. 80 (1998) 2968.

[9] R. Blümel, Phys. Rev. Lett. 73 (1994) 428.

[10] I. Kim and G. Mahler, Phys. Rev. A 60 (1999) 692.

[11] G. Mahler and V. A. Weberruss, Quantum Networks: Dynamics of Open Nanostructures (2nd ed. Springer, New York, 1998).

[12] Y. Avishai and D. Berend, Phys. Rev. B 43 (1991) 6873; A. Bovier and J-M. Ghez, J. Phys. A 28 (1995) 2313; E. Maciá and F. Domínguez-Adame, Phys. Rev. Lett. 76 (1996) 2957; see also P. Benioff, Phys. Rev. Lett. 78 (1997) 590; Physica D 120 (1998) 12.

[13] M. Hübner, Phys. Lett. A 163 (1992) 293; ibid. 179 (1993) 226.

[14] A. Peres, in Quantum Chaos, edited by H. A. Cerdeira et al. (World Scientific, Singapore, 1991); Quantum Theory: Concepts and Methods (Kluwer, Dordrecht, 1993).

[15] R. Schack, Phys. Rev. Lett. 75 (1995) 581.

[16] R. P. Feynman, Int. J. theor. Phys. 21 (1982) 467.
Fig. 1: Input-output-scheme of our quantum Turing machine (QTM).

Fig. 2: Turing-head patterns \( \{0, \sigma_2(n), \sigma_3(n)\} \) for initial state \( |\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)} \). Left: \( \alpha_1 = \frac{2}{5} \pi \) (periodic), right: \( \alpha_1 = \frac{2}{5} \times 3.141592654 \) (aperiodic) and total step number \( n = 10000 \).

Fig. 3: Evolution of the distance \( D_{\rho\rho'}^2 \) between Turing-head state with \( (\hat{\rho}') \) and without \( (\hat{\rho}) \) perturbation \( \delta \). \( \alpha_1 = \frac{2}{5} \pi \), \( |\psi_0\rangle = |-1\rangle^{(S)} \otimes |-1\rangle^{(1)} \) for \( \hat{\rho} \), and \( \left( \exp \left( -i \hat{\sigma}_1^{(S)} \frac{\delta}{2} \right) |-1\rangle^{(S)} \otimes |-1\rangle^{(1)} \right) \), \( \delta = 0.001 \) for \( \hat{\rho}' \). Left: chaotic input according to eq. (3) (inset shows initial behavior in more detail), right: \( \alpha_m = \alpha \left( D^2 \approx 0, \text{solid line} \right) \) and \( \alpha_{m+1} = 2\alpha_m - \alpha_{m-1} \) (Lyapunov exponent \( = 0 \)) (dotted line); the respective distances \( D_{\rho\rho'}^2 \) for tape-spin 1 and for total network state \( |\psi_n\rangle \) are similar to those shown.
chaotic input \( \{ \alpha_m \} \)

QTM \( \hat{H} \)

chaotic output \( \{ \sigma_i^{(S)}, \sigma_j^{(1)} \} \)