LOCAL DERIVATIONS ON SUBALGEBRAS OF $\tau$-MEASURABLE OPERATORS WITH RESPECT TO SEMI-FINITE VON NEUMANN ALGEBRAS

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ABSTRACT. This paper is devoted to local derivations on subalgebras on the algebra $S(M, \tau)$ of all $\tau$-measurable operators affiliated with a von Neumann algebra $M$ without abelian summands and with a faithful normal semi-finite trace $\tau$. We prove that if $A$ is a solid $*$-subalgebra in $S(M, \tau)$ such that $p \in A$ for all projection $p \in M$ with finite trace, then every local derivation on the algebra $A$ is a derivation. This result is new even in the case standard subalgebras on the algebra $B(H)$ of all bounded linear operators on a Hilbert space $H$. We also apply our main theorem to the algebra $S_0(M, \tau)$ of all $\tau$-compact operators affiliated with a semi-finite von Neumann algebra $M$ and with a faithful normal semi-finite trace $\tau$.

1. INTRODUCTION

Given an algebra $A$, a linear operator $D : A \to A$ is called a derivation, if $D(xy) = D(x)y +xD(y)$ for all $x, y \in A$ (the Leibniz rule). Each element $a \in A$ implements a derivation $D_a$ on $A$ defined as $D_a(x) = [a, x] = ax - xa, x \in A$. Such derivations $D_a$ are said to be inner derivations. If the element $a$, implementing the derivation $D_a$, belongs to a larger algebra $B$ containing $A$, then $D_a$ is called a spatial derivation on $A$. A well known direction in the study of the local action of derivations is the local derivation problem. Recall that a linear map $\Delta$ of $A$ is called a local derivation if for each $x \in A$, there exists a derivation $D : A \to A$, depending on $x$, such that $\Delta(x) = D(x)$. This notion was introduced in 1990 independently by Kadison [19] and Larson and Sourour [20]. In [19] it was proved that every norm continuous local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. In [20] the same result was obtained for the algebra of all bounded linear operators acting on a Banach space.

In the last decade the structure of derivations and local derivations on the algebra $LS(M)$ of all locally measurable operators affiliated with a von Neumann algebra $M$ and on its various subalgebras have been investigated by many authors (see [2, 6, 8, 16, 18]). In [4] local derivations have been investigated on the algebra $S(M)$ of all measurable operators with respect a von Neumann algebra $M$. In particular, it was proved that, for finite type I von Neumann algebras without abelian direct summands, every local derivation on $S(M)$ is a derivation. Moreover, in the case of abelian von Neumann algebra $M$ necessary and sufficient conditions have been obtained for the algebra $S(M)$ to admit local derivations which are not derivations. In [18] local derivations have been investigated on the algebra $S(M)$ for an arbitrary von Neumann algebra $M$ and it was proved that for a von Neumann algebras without abelian direct summands every local derivation on $S(M)$ is a derivation. It should be noted that the proofs of the main result in the paper [18] are essentially based on the fact that the von Neumann algebra $M$ is a subalgebra in the considered algebras. Local and 2-local maps have been studied on different operator algebras by many authors [4, 7, 10, 11, 17-20].
The present paper is devoted to local derivations on subalgebras of algebra $S(M, \tau)$ of all $\tau$-measurable operators affiliated with a von Neumann algebra $M$ with a faithful normal semi-finite trace $\tau$. Since in general case we do not assumed that these subalgebras contain the von Neumann algebra $M$, one cannot directly apply the methods of the papers [18] in this setting. Moreover, in our setting description of local derivations is an open problem. Therefore the aim of this paper to solve such a problem.

In Section 2 we give preliminaries from the theory of $\tau$-measurable operators affiliated with a von Neumann algebra $M$. In section 3 we consider a von Neumann algebra with a faithful normal semi-finite trace $M$, if derivations is an open problem. Therefore the aim of this paper to solve such a problem. Moreover, in our setting description of local derivations is an open problem. Therefore the aim of this paper to solve such a problem.

In Section 2 we give preliminaries from the theory of $\tau$-measurable operators affiliated with a semi-finite von Neumann algebra $M$. In section 3 we consider a von Neumann algebra $M$ and let $\mathcal{A}$ be the subalgebra of $S(M, \tau)$ such that $p \in \mathcal{A}$ for all projection $p \in M$ with a finite trace, then every local derivation $\Delta$ on the algebra $\mathcal{A}$ is a derivation.

In section 4 we apply the main theorem of the previous section to the Arens algebra and the algebra of all $\tau$-compact operators affiliated with a semi-finite von Neumann algebra $M$ and with a faithful normal semi-finite trace $\tau$.

2. ALGEBRAS OF $\tau$-MEASURABLE OPERATORS

Let $B(H)$ be the $*$-algebra of all bounded linear operators on a Hilbert space $H$, and let $1$ be the identity operator on $H$. Consider a von Neumann algebra $M \subset B(H)$ with the operator norm $\| \cdot \|$ and with a faithful normal semi-finite trace $\tau$. Denote by $P(M) = \{ p \in M : p = p^2 = p^* \}$ the lattice of all projections in $M$ and $P_s(M) = \{ p \in P(M) : \tau(p) < +\infty \}$.

A linear subspace $\mathcal{D}$ in $H$ is said to be affiliated with $M$ (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for every unitary $u$ from the commutant

$$M' = \{ y \in B(H) : xy = yx, \forall x \in M \}$$

of the von Neumann algebra $M$.

A linear operator $x : \mathcal{D}(x) \to H$, where the domain $\mathcal{D}(x)$ of $x$ is a linear subspace of $H$, is said to be affiliated with $M$ (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$.

A linear subspace $\mathcal{D}$ in $H$ is said to be strongly dense in $H$ with respect to the von Neumann algebra $M$, if

1. $\mathcal{D}\eta M$;
2. there exists a sequence of projections $\{ p_n \}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow 1$, $p_n(H) \subset \mathcal{D}$ and $p_n^\perp = 1 - p_n$ is finite in $M$ for all $n \in \mathbb{N}$.

A closed linear operator $x$ acting in the Hilbert space $H$ is said to be measurable with respect to the von Neumann algebra $M$, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in $H$.

Denote by $S(M)$ the set of all linear operators on $H$, measurable with respect to the von Neumann algebra $M$. If $x \in S(M)$, $\lambda \in \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, then $\lambda x \in S(M)$ and the operator $x^*$, adjoint to $x$, is also measurable with respect to $M$ (see [23]). Moreover, if $x, y \in S(M)$, then the operators $x + y$ and $xy$ are defined on dense subspaces and admit closures that are called, correspondingly, the strong sum and the strong product of the operators $x$ and $y$, and are denoted by $x + y$ and $x * y$. It was shown in [23] that $x + y$ and $x * y$ belong to $S(M)$ and these algebraic operations make $S(M)$ a $*$-algebra with the identity $1$ over the field $\mathbb{C}$. Here, $M$ is a $*$-subalgebra of $S(M)$. In what
follows, the strong sum and the strong product of operators $x$ and $y$ will be denoted in the same way as the usual operations, by $x + y$ and $xy$.

It is clear that if the von Neumann algebra $M$ is finite then every linear operator affiliated with $M$ is measurable and, in particular, a self-adjoint operator is measurable with respect to $M$ if and only if all its spectral projections belong to $M$.

Let $\tau$ be a faithful normal semi-finite trace on $M$. Then $\tau$ is called the right support of the element $x$, and the projection

$$r(x) = \inf\{p \in P(M) : xp = x\}$$

is called the right support of the element $x$, and the projection

$$l(x) = \inf\{p \in P(M) : px = x\}$$

is called its left support. The projection $s(x) = r(x) \lor l(x)$ is called the support of the element $x$. For $\tau$-subalgebra $\mathcal{A} \subset S(M, \tau)$ denote

$$\mathcal{F}_{\tau}(\mathcal{A}) = \{x \in \mathcal{A} : s(x) \in P_{\tau}(M)\}.$$ 

The following properties of $\mathcal{F}_{\tau}(\mathcal{A})$ directly follow from the definition.

**Lemma 3.2.** Let $\mathcal{A}$ be a $\tau$-subalgebra in $S(M, \tau)$. Then the following assertions are equivalent:

1. $x \in \mathcal{F}_{\tau}(\mathcal{A})$;
2. $\exists p \in P_{\tau}(M)$ such that $px = x$;
3. $\exists p \in P_{\tau}(M)$ such that $xp = x$;
4. $\exists p \in P_{\tau}(M)$ such that $xp = x$. 

**Theorem 3.1.** Let $M$ be a semi-finite von Neumann algebra without abelian direct summands and let $\tau$ be a faithful normal semi-finite trace on $M$. Suppose that $\mathcal{A}$ is a solid $\tau$-subalgebra in $S(M, \tau)$ such that $p \in \mathcal{A}$ for all $p \in P_{\tau}(M)$. Then every local derivation $\Delta$ on the algebra $\mathcal{A}$ is a derivation.
By construction of the projection $x$ one can find a derivation $D$ on $A$ such that $\Delta(x) = D(x)$. It is clear that
\[
l(D(x)s(x)) \leq s(x),
\]
\[
r(xD(s(x))) \leq s(x),
\]
where $p \leq q$ means that $p$ is equivalent to a subprojection of the projection $q$. Since
\[
D(x) = D(xs(x)) = D(x)s(x) + xD(s(x))
\]
we have
\[
\tau(s(D(x))) = \tau(s(D(x)s(x) + xD(s(x)))) \leq \\
\leq \tau(s(x) \lor l(D(x)s(x)) \lor r(xD(s(x)))) \leq \\
\leq \tau(s(x)) + \tau(s(x)) + \tau(s(x)) = 3\tau(s(x)),
\]
i.e.
\[
\tau(s(D(x))) \leq 3\tau(s(x)).
\]
Thus $D(x) \in \mathcal{F}_\tau(A)$, so $\Delta(x) \in \mathcal{F}_\tau(A)$.

Therefore, $\Delta$ maps $\mathcal{F}_\tau(A)$ into itself. \quad \square

Let $p \in A$ be a projection. One can see that the mapping
\[
D^{(p)} : x \rightarrow pD(x)p, \quad x \in pAp
\]
is a derivation on $pAp$. Now let $\Delta$ be a local derivation on $A$. Take an element $x \in A$ and a derivation $D$ on $A$ such that $\Delta(px) = D(px)$. Then
\[
p\Delta(px)p = pD(px)p = D^{(p)}(px).
\]
This means that the mapping $\Delta^{(p)}$ defined similar to (3.1) is a local derivation on $pAp$.

**Lemma 3.5.** If $\Delta$ is a local derivation on $A$, then the restriction $\Delta|_{\mathcal{F}_\tau(A)}$ is a derivation.

**Proof.** Let $x, y \in \mathcal{F}_\tau(A)$. Denote
\[
p = s(x) \lor s(y) \lor s(xy) \lor s(\Delta(x)) \lor s(\Delta(y)) \lor s(\Delta(xy))
\]
Since $\mathcal{F}_\tau(A)$ is an ideal in $A$ and $\Delta$ maps $\mathcal{F}_\tau(A)$ into itself, we obtain that the projection $p \in P_r(M)$. Consider the local derivation $\Delta^{(p)}$ on $pAp$. Since $p \in A$ and $A$ is a solid *-subalgebra in $S(M, \tau)$ we get $pMp \subseteq A$. Furthermore, $A$ has no abelian direct summands, and therefore [18, Theorem 1] implies that $\Delta^{(p)}$ is a derivation. Taking into account that $x, y \in pAp$ we obtain
\[
\Delta^{(p)}(xy) = \Delta^{(p)}(x)y + x\Delta^{(p)}(y).
\]
By construction of the projection $p$ we have
\[
\Delta(xy) = \Delta^{(p)}(xy) = \Delta^{(p)}(x)y + x\Delta^{(p)}(y) = \\
= \Delta(x)y + x\Delta(y).
\]
This means that $\Delta$ is a derivation on $\mathcal{F}_\tau(A)$. This completes the proof. \hfill $\square$

**Remark 3.6.** Let $y \in A$ and $yp = 0$ for all $p \in P_\tau(M)$. Since the map $x \mapsto yxy^*$ is positive and monotone continuous, taking $p \uparrow 1$ in $yyp^* = 0$, we obtain that $yy^* = 0$. Therefore $y = 0$.

**Proof of Theorem 3.1.** We shall show that $\Delta(xy) = \Delta(xyp) = \Delta(x)yp + x\Delta(yp)$ for all $x, y \in A$. We consider the following two cases.

**Case 1.** Let $x \in \mathcal{F}_\tau(A)$ and $y \in A$. Since $\mathcal{F}_\tau(A)$ is an ideal in $A$ and $\Delta$ maps $\mathcal{F}_\tau(A)$ into itself, we obtain that the projection

$$p = s(xy) \lor s(\Delta(xy)) \lor s(\Delta(x)y) \lor s(x\Delta(y))$$

has a finite trace. Taking into account the equalities $xyp = xy, \Delta(xy)p = \Delta(xy)$ and Lemma 3.5, we obtain

$$\Delta(xy) = \Delta(xyp) = \Delta(xy)p + xy\Delta(p) = \Delta(xy) + xy\Delta(p),$$

i.e. $xy\Delta(p) = 0$.

Further

$$x\Delta(yp)p = x\Delta(ypp) - xyp\Delta(p) = x\Delta(yp) - xy\Delta(p) = x\Delta(yp),$$

i.e.

$$(3.2) \quad x\Delta(yp)p = x\Delta(yp).$$

Now taking into account (3.2), the equalities $xy(1 - p) = 0, x\Delta(y)p = x\Delta(y)$ and the linearity of $\Delta$ we have

$$x\Delta(y) = x\Delta(y)p = x\Delta(ypp + y(1 - p))p = x\Delta(ypp) + x\Delta(y(1 - p))p = x\Delta(yp)p + xD(y(1 - p)p) - xy(1 - p)D(p) = x\Delta(yp),$$

where $D$ is a derivation on $A$ such that $\Delta(y(1 - p)) = D(y(1 - p))$. Consequently

$$x\Delta(yp) = x\Delta(y).$$

Finally we obtain that

$$\Delta(xy) = \Delta(xyp) = \Delta(x)yp + x\Delta(yp) = \Delta(xy)p = \Delta(xy) + x\Delta(y).$$

Similar as above we can check the case $x \in A$ and $y \in \mathcal{F}_\tau(A)$.
CASE 2. Let \( x, y \in A \) be arbitrary elements. Take an arbitrary \( q \in P_\tau(M) \). By the case 1 we have

\[
\Delta(y)q = \Delta(yq) - y\Delta(q).
\]

Taking into account this equality and the case 1 we obtain

\[
\Delta(xy)q = \Delta(xyq) - xy\Delta(q) = \Delta(x)q + x\Delta(yq) - xy\Delta(q) = \Delta(x)q + x[\Delta(yq) - y\Delta(q)] = \Delta(x)q + x\Delta(y)q,
\]

i.e.

\[
\Delta(xy)q = [\Delta(x)y + x\Delta(y)]q
\]

for all \( q \in P_\tau(M) \). Taking into account Remark 3.6 we obtain

\[
\Delta(xy) = \Delta(x)y + x\Delta(y).
\]

The proof is complete. \( \square \)

We stress that Theorem 3.1 is new even in the case of type I\(_\infty\) von Neumann factors.

For a Hilbert space \( H \) by \( F(H) \) we denote the algebra of all finite rank operators in \( B(H) \). Recall that a standard operator algebra is any subalgebra of \( B(H) \) which contains \( F(H) \).

Theorem 3.1 implies the following result.

Corollary 3.7. Let \( H \) be a Hilbert space and let \( \mathcal{U} \) be a standard algebra in \( B(H) \). Then any local derivation \( \Delta : \mathcal{U} \to \mathcal{U} \) is a spatial derivation and implemented by an element \( a \in B(H) \).

Remark 3.8. A similar result for local derivations on \( B(X) \), where \( X \) is a Banach space, has been obtained in [17, Corollary 3.7] under the additional assumption of continuity of the map with respect to the weak operator topology.

Remark 3.9. We note that if one replaces \( S(M, \tau) \) with \( S(M) \) all the results will remain true. In this case \( F_\tau(A) \) is replaced by the set of finite projections of \( A \) and instead of \( \tau \) is used the dimension function.

4. LOCAL DERIVATIONS ON ALGEBRA \( \tau \)-COMPACT OPERATORS AND ARENS ALGEBRAS

In this section we shall consider a local derivations on algebras \( \tau \)-compact operators and on Arens algebras, respectively.

4.1. algebra of \( \tau \)-compact operators. In this subsection we shall consider an algebra of \( \tau \)-compact operators.

In the algebra \( S(M, \tau) \) consider the subset \( S_0(M, \tau) \) of all operators \( x \) such that given any \( \varepsilon > 0 \) there is a projection \( p \in P(M) \) with \( \tau(p^\perp) < \infty \), \( xp \in M \) and \( \|xp\| < \varepsilon \). The elements of \( S_0(M, \tau) \) is called \( \tau \)-compact operators with respect to \( M \) and \( \tau \). It is known [21] that \( S_0(M, \tau) \) is a solid *-subalgebra in \( S(M, \tau) \) and a bimodule over \( M \), i.e. \( ax, xa \in S_0(M, \tau) \) for all \( x \in S_0(M, \tau) \) and \( a \in M \). Note that if \( M = B(H) \) and \( \tau = tr \), where \( tr \) is the canonical trace on \( B(H) \), then \( S_0(M, \tau) = K(H) \), where \( K(H) \) is the ideal of compact operators from \( B(H) \).

The following properties of the algebra \( S_0(M, \tau) \) are known (see [24]):
Let \( M \) be a von Neumann algebra with a faithful normal semi-finite trace \( \tau \). Then

1) \( S(M, \tau) = M + S_0(M, \tau) \);
2) \( S_0(M, \tau) \) is an ideal in \( S(M, \tau) \).

Note that if the trace \( \tau \) is finite then

\[ S_0(M, \tau) = S(M, \tau) = S(M). \]

It is well-known \( [24] \) \( S_0(M, \tau) \) equipped with the measure topology is a complete metrizable topological *-algebra.

It is clear that \( p \in S_0(M, \tau) \) for all \( p \in P_\tau(M) \).

Theorem 3.1 implies the following result.

**Theorem 4.1.** Let \( M \) be a semi-finite von Neumann algebra without abelian direct summands and let \( \tau \) be a faithful normal semi-finite trace on \( M \). Then every local derivation \( \Delta \) on the algebra \( S_0(M, \tau) \) is a derivation.

**Remark 4.2.** If \( M \) is an abelian von Neumann algebra with a faithful normal semi-finite trace \( \tau \) such that the lattice \( P(M) \) of projections in \( M \) is not atomic, then the algebra \( S_0(M, \tau) \) admits a local derivation which is not a derivation (see \( [4, \text{Theorem 3.2}] \)).

In \( [9, \text{Theorem 4.9}] \) it was proved that in the case when \( M \) is a properly infinite von Neumann algebra with a faithful normal semi-finite trace \( \tau \), then any derivation \( D \) on \( S_0(M, \tau) \) is a spatial derivation and implemented by an element \( a \in S(M, \tau) \). Therefore Theorem 4.1 implies that

**Theorem 4.3.** If \( M \) is a properly infinite von Neumann algebra with a faithful normal semi-finite trace \( \tau \), then any local derivation \( \Delta : S_0(M, \tau) \to S_0(M, \tau) \) is a spatial derivation and implemented by an element \( a \in S(M, \tau) \).

### 4.2. Arens algebras

Now we are going to consider Arens algebras associated with a von Neumann algebra and a semi-finite faithful normal trace.

Let \( M \) be a von Neumann algebra with a faithful normal semi-finite trace \( \tau \).

Take \( x \in S(M, \tau), x \geq 0 \) and let \( x = \int_0^{\infty} \lambda \, d\epsilon_\lambda \) be its spectral resolution. Denote \( \tau(x) = \sup_{n \geq 1} \int_0^n \lambda \, d\tau(\epsilon_\lambda) \).

Given \( p \geq 1 \) put \( L^p(M, \tau) = \{ x \in S(M, \tau) : \tau(|x|^p) < \infty \} \). It is known \( [21] \) that \( L^p(M, \tau) \) is a Banach space with respect to the norm

\[ \|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(M, \tau). \]

Consider the intersection

\[ L^\omega(M, \tau) = \bigcap_{p \geq 1} L^p(M, \tau). \]

It is proved in \( [11] \) that \( L^\omega(M, \tau) \) is a locally convex complete metrizable *-algebra with respect to the topology \( t \) generated by the family of norms \( \{\|\cdot\|_p\}_{p \geq 1} \). The algebra \( L^\omega(M, \tau) \) is called a (non commutative) Arens algebra.

Note that \( L^\omega(M, \tau) \) is a solid *-subalgebra in \( S(M, \tau) \) and if \( \tau \) is a finite trace then \( M \subset L^\omega(M, \tau) \).

Further consider the following spaces

\[ L^\omega_2(M, \tau) = \bigcap_{p \geq 2} L^p(M, \tau). \]
and

\[ M + L_2^\omega(M, \tau) = \{x + y : x \in M, y \in L_2^\omega(M, \tau)\}. \]

It is known [2] that \( L_2^\omega(M, \tau) \) and \( M + L_2^\omega(M, \tau) \) are a *-algebras and \( L_2^\omega(M, \tau) \) is an ideal in \( M + L_2^\omega(M, \tau) \).

Note that if \( \tau(1) < \infty \) then \( M + L_2^\omega(M, \tau) = L_2^\omega(M, \tau) = L^\omega(M, \tau) \).

It is known [2, Theorem 3.7] that if \( M \) is a von Neumann algebra with a faithful normal semi-finite trace \( \tau \) then any derivation \( D \) on \( L^\omega(M, \tau) \) is spatial, moreover it is implemented by an element of \( M + L_2^\omega(M, \tau) \), i.e.

\[ D(x) = ax - xa, \quad x \in L^\omega(M, \tau) \]

for some \( a \in M + L_2^\omega(M, \tau) \). In particular, if \( M \) is abelian, then any derivation on \( L^\omega(M, \tau) \) is zero.

Note that \( p \in L^\omega(M, \tau) \) for all \( p \in P_\tau(M) \).

We need the following auxiliary result.

**Lemma 4.4.** Let \( M \) be a semi-finite von Neumann algebra with a faithful normal semi-finite trace \( \tau \) and with the center \( Z(M) \). Then every local derivation \( \Delta \) on the algebra \( L^\omega(M, \tau) \) is necessarily \( P(Z(M)) \)-homogeneous, i.e.

\[ \Delta(zx) = z\Delta(x) \]

for any central projection \( z \in P(Z(M)) = P(M) \cap Z(M) \) and for all \( x \in L^\omega(M, \tau) \).

**Proof.** Take \( z \in P(Z(M)) \) and \( x \in L^\omega(M, \tau) \). For the element \( zx \) by the definition of the local derivation \( \Delta \) there exists a derivation \( D_a \) on \( L^\omega(M, \tau) \) of the form (4.1) such that \( \Delta(zx) = D_a(zx) \). Since the projection \( z \) is central, one has that

\[ D_a(zx) = [a, zx] = z[a, x] = zD_a(x). \]

Multiplying the equality \( \Delta(zx) = D_a(zx) \) by \( z \) we obtain

\[ z\Delta(zx) = zD_a(zx) = zD_a(x) = D_a(zx) = \Delta(zx), \]

i.e.

\[ (1 - z)\Delta(zx) = 0. \]

Replacing \( z \) by \( 1 - z \) one finds

\[ z\Delta((1 - z)x) = 0. \]

Therefore by the linearity of \( \Delta \) we have

\[ z\Delta(x) = z\Delta(zx) + z\Delta((1 - z)x) = z\Delta(zx) = \Delta(zx), \]

and thus \( z\Delta(x) = \Delta(zx) \). The proof is complete. \( \square \)

**Theorem 4.5.** Let \( M \) be a semi-finite von Neumann algebra with a faithful normal semi-finite trace \( \tau \). Then any local derivation \( \Delta \) on the algebra \( L^\omega(M, \tau) \) is a spatial derivation of the form (4.1).

**Proof.** Let \( M \) be a semi-finite von Neumann algebra. There exist mutually orthogonal central projections \( z_1, z_2 \) in \( M \) with \( z_1 + z_2 = 1 \) such that

- \( z_1M \) is abelian;
- \( z_2M \) has no abelian summands.
By Lemma 4.4, the operator $\Delta$ maps $z_iL^\omega(M, \tau) \equiv L^\omega(z_iM, \tau_i)$ into itself for $i = 1, 2$, where $\tau_i$ is the restriction of $\tau$ on $z_iM(i = 1, 2)$. As it was mentioned above $z_1\Delta$ is zero. By Theorem 3.1 we obtain that $\Delta = z_2\Delta$ is a derivation. The proof is complete.

Note that if the trace $\tau$ is finite, Theorem 4.5 is given in [11, Theorem 2.1].

Acknowledgments

The second named author (K.K.) acknowledges the MOHE grant ERGS13-024-0057 for support, and International Islamic University Malaysia for kind hospitality.

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