A Heisenberg Uncertainty Relation for Three Canonical Observables

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Uncertainty relations provide fundamental limits on what can be said about the properties of quantum systems. For a quantum particle, the commutation relation of position and momentum observables entails Heisenberg’s uncertainty relation. A third observable is presented which satisfies canonical commutation relations with both position and momentum. The resulting triple of pairwise canonical observables gives rise to a Heisenberg-type uncertainty relation for the product of three standard deviations. We derive the smallest possible value of this bound and determine the specific squeezed state which saturates the triple uncertainty relation. Quantum optical experiments are proposed to verify our findings.

Introduction

In quantum theory, two observables  and  are canonical if they satisfy the commutation relation

\[ [\hat{p}, \hat{q}] = \frac{\hbar}{i}, \tag{1} \]

with the momentum and position of a particle being a well-known and important example. The non-vanishing commutator expresses the incompatibility of the Schrödinger pair  of observables since it imposes a lower limit on the product of their standard deviations, namely

\[ \Delta q \Delta p \geq \frac{\hbar}{2}. \tag{2} \]

In 1927, Heisenberg [1] analysed the hypothetical observation of an individual electron with photons and concluded that the product of the measurement errors should be governed by a relation of the form (2). His proposal inspired Kennard [2] and Weyl [3] to mathematically derive Heisenberg’s uncertainty relation, thereby turning it into a constraint on measurement outcomes for an ensemble of identically prepared systems. Schrödinger’s [4] generalization of (2) includes a correlation term, and Robertson [5, 6] derived a similar relation for any two non-commuting Hermitean operators. Recently claimed violations of (2) do not refer to Kennard and Weyl’s preparation uncertainty relation but to Heisenberg’s error-disturbance relation (cf. [7–9]). However, these claims have been criticized strongly [10, 11].

Uncertainty relations are now understood to provide fundamental limits on what can be said about the properties of quantum systems. Imagine measuring the standard deviations  and  separately on two ensembles prepared in the same quantum state. Then, the bound (2) does not allow one to simultaneously attribute definite values to the observables  and .

In this contribution, we will consider a Schrödinger triple  consisting of three pairwise canonical observables [12], i.e.

\[ [\hat{\rho}, \hat{\eta}, \hat{\tau}] = [\hat{\eta}, \hat{\tau}] = [\hat{\tau}, \hat{\rho}] = \frac{\hbar}{i}, \tag{3} \]

and establish an uncertainty relation associated with it. In a system of units where both  and  carry physical dimensions of  , the observable  is given by a suitably rotated and rescaled version of the position operator  – as well as its generalization due to Robertson and Schrödinger—applies to each pair which immediately results in a limit on the product of three uncertainties,

\[ \Delta \rho \Delta \eta \Delta \tau \geq \left( \tau \frac{\hbar}{2} \right)^{3/2}. \tag{5} \]

where the number  is the triple constant with value

\[ \tau = \csc \left( \frac{2\pi}{3} \right) \equiv \sqrt[3]{\frac{4}{3}} \simeq 1.16. \tag{6} \]

The bound (5) is found to be tight, and we will show that the state which saturates it is unique up to rigid translations in phase space.

To appreciate the bound (5), let us evaluate the triple uncertainty \( \Delta \rho \Delta \eta \Delta \tau \) in two instructive cases. (i) Since each of the pairs  ,  ,  , and  is canonical, the inequality (2) as well as its generalization due to Robertson and Schrödinger—applies to each pair which immediately results in a limit on the product of three uncertainties,

\[ \Delta \rho \Delta \eta \Delta \tau \geq \left( \frac{\hbar}{2} \right)^{3/2}. \tag{7} \]

However, this argument provides no insight whether a state exists in which the triple uncertainty takes this
value. Our main result actually states that no such state exists. (ii) In a standard coherent state \(|0\rangle\), i.e. in the ground state of a harmonic oscillator with unit mass and unit frequency, the triple uncertainty takes the value
\[
\Delta p \Delta q \Delta r = \sqrt{2} \left( \frac{\hbar}{2} \right)^{3/2}.
\]
(8)
The exceed by a factor of \(\sqrt{2}\) in comparison with [7] has an intuitive explanation: while the vacuum state \(|0\rangle\) successfully minimizes the product \(\Delta p \Delta q\), it does not simultaneously minimize the uncertainty associated with the pairs \((\hat{q}, \hat{r})\) and \((\hat{r}, \hat{p})\). Thus, no coherent state will achieve the minimum of the inequality [5].

The observations (i) and (ii) suggest that the bound [5] on the triple uncertainty is not an immediate consequence of Heisenberg’s inequality for canonical pairs, Eq. (2). Furthermore, the invariance groups of the triple uncertainty relation, of Heisenberg’s uncertainty relation, and of the inequality by Schrödinger and Robertson are different, because they depend on two, three and four (cf. [14]) continuous parameters, respectively.

Extremal triple uncertainty To determine the states which minimize the left-hand-side of Eq. (5), we associate an uncertainty functional with each state \(|\psi\rangle \in \mathcal{H}\) (cf. [15]),
\[
J_\lambda[\psi] = \Delta p[\psi] \Delta q[\psi] \Delta r[\psi] - \lambda (\langle \psi | \psi \rangle - 1),
\]
using the standard deviations \(\Delta_x[\psi] \equiv \Delta x \equiv (\langle \psi | \hat{x}^2 | \psi \rangle - \langle \psi | \hat{x} | \psi \rangle^2)^{1/2}, x = p, q, r\), while the term with the Lagrange multiplier \(\lambda\) takes care of normalization.

To find the extremals of the functional \(J_\lambda[\psi]\), change its argument from \(|\psi\rangle\) to the state \(|\psi + \epsilon \rangle\), where \(|\epsilon\rangle = \epsilon |e\rangle\), with a normalized state \(|e\rangle = 1\) and a real parameter \(\epsilon \ll 1\), leading to
\[
J_\lambda[\psi + \epsilon] = J_\lambda[\psi] + \epsilon J_\lambda^{(1)}[\psi] + \mathcal{O}(\epsilon^2).
\]
(10)
The first-order variation \(J_\lambda^{(1)}[\psi]\) vanishes only if \(|\psi\rangle\) is an extremum of the functional \(J_\lambda[\psi]\) or, equivalently, if
\[
\frac{1}{3} \left( \frac{(\hat{p} - \langle \hat{p} \rangle)^2}{\Delta p^2} + \frac{(\hat{q} - \langle \hat{q} \rangle)^2}{\Delta q^2} + \frac{(\hat{r} - \langle \hat{r} \rangle)^2}{\Delta r^2} \right) |\psi\rangle = 0
\]
(11)
holds, which follows from generalizing a direct computation carried out in [16] to determine the extremals of the product \(\Delta p \Delta q\).

Eq. (11) is non-linear in the unknown state \(|\psi\rangle\) due to the expectation values \(\langle \hat{p} \rangle, \Delta p^2\), etc. Its solutions can be found by initially treating these expectation values as constants which will be determined only later in a self-consistent way. By means of the unitary operator \(U_{a,b,\gamma} = \hat{T}_a \hat{G}_b \hat{S}_\gamma\), the quadratic expression in the operators \(\hat{p}\) and \(\hat{q}\) on the LHS of (11) can be transformed into a well-known operator,
\[
\frac{1}{2} (\hat{p}^2 + \hat{q}^2) |\psi_{a,b,\gamma}\rangle = \frac{3}{2c} |\psi_{a,b,\gamma}\rangle,
\]
(12)
defining \(|\psi_{a,b,\gamma}\rangle = \hat{U}_{a,b,\gamma}^\dagger |\psi\rangle\), and \(c\) being a real constant. The unitary \(\hat{U}_{a,b,\gamma}\) consists of a rigid phase-space translation by \(\alpha \equiv (q_0 + ip_0)/\sqrt{2\hbar} \in \mathbb{C}\),
\[
\hat{T}_a = \exp \left( i(p_0q - q_0p)/\hbar \right),
\]
(13)
followed by a gauge transformation in the momentum basis
\[
\hat{G}_b = \exp \left( ibq^2/2\hbar \right), \quad b \in \mathbb{R},
\]
(14)
and a specific case of a general squeezing transformation,
\[
\hat{S}_\xi = \exp \left( [\xi(a^2 - \xi^2)/2] \right), \quad \xi = r e^{i\theta}, r > 0,
\]
(15)
where the annihilation operator and its adjoint \(a^\dagger\) are defined through 
\[
a = (\hat{q} + i\hat{p})/\sqrt{2\hbar} \quad \text{and} \quad \hat{a}^\dagger = (\hat{q} - i\hat{p})/\sqrt{2\hbar}.
\]
Here we consider \(\hat{S}_\gamma \equiv \exp [i(\gamma(\hat{p}\hat{q} + \hat{q}\hat{p}))/\sqrt{2\hbar}] \) corresponding to \(\xi \equiv \gamma, i.e. \theta = 0\).

According to (12), the states \(|\psi_{a,b,\gamma}\rangle\) coincide with the eigenstates \(|n\rangle, n \in \mathbb{N}_0\), of a harmonic oscillator with unit mass and frequency,
\[
|n; \alpha, b, \gamma \rangle \equiv \hat{T}_a \hat{G}_b \hat{S}_\gamma |n\rangle, \quad n \in \mathbb{N}_0,
\]
(16)
where we have suppressed irrelevant constant phase factors; for consistency, the quantity \(3/2\pi c^2\) in (12) must only take the values \(\hbar(n + 1/2)\) for \(n \in \mathbb{N}_0\), as a direct but lengthy calculation confirms. The parameters \(b\) and \(\gamma\) must take specific values for (12) to hold, namely
\[
b = \frac{1}{2} \quad \text{and} \quad \gamma = \frac{1}{2} \log \tau; \quad (17)
\]
we will denote the restricted set of states obtained from Eq. (16) by \(|n; \alpha\rangle\). There are no constraints on the parameter \(\alpha\), which means that we are free to displace the states \(|n\rangle\) in phase space without affecting the values of the variances. The variances of the observables \(\hat{p}, \hat{q}\), and \(\hat{r}\) are found to be equal, taking the value
\[
\Delta^2[n; \alpha] = \tau \hbar \left( n + \frac{1}{2} \right), \quad x = p, q, r,
\]
(18)
with the triple constant \(\tau\) introduced in (6). Inserting these results into Eq. (11) we find that
\[
\frac{1}{3} (\hat{p}^2 + \hat{q}^2 + \hat{r}^2) |n; \alpha\rangle = \tau \hbar \left( n + \frac{1}{2} \right) |n; \alpha\rangle,
\]
(19)
where
\[
|n; \alpha\rangle = \hat{G}_b \hat{S}_\gamma^{1/2} \hat{T}_a^{1/2} |n\rangle, \quad n \in \mathbb{N}_0, \alpha \in \mathbb{C}.
\]
(20)
Thus, for each value of \(\alpha\), the extremals of the uncertainty functional [14] form a complete set of normalized, orthogonal states,
\[
\sum_{n=0}^{\infty} |n; \alpha\rangle |n; \alpha\rangle = 1,
\]
(21)
since the oscillator eigenstates \(|n\rangle, n \in \mathbb{N}_0\), have this property.
States of minimal triple uncertainty  At its extremals the uncertainty functional \( J_\lambda[n;\alpha] \) takes the values
\[
J_\lambda[n;\alpha] = \left[ \tau \hbar \left( n + \frac{1}{2} \right) \right]^{3/2}, \quad n \in \mathbb{N}_0,
\]
according to Eq. (18), with the minimum occurring for \( n = 0 \). Thus, the two-parameter family of states \(|0;\alpha\rangle\), \( \alpha \in \mathbb{C} \), which we will denote by
\[
|\Xi_\alpha\rangle = \hat{T}_\alpha \left( \hat{G}_{1/2} \hat{S}_{\frac{1}{2} \log_3 |0\rangle} \right),
\]
minimize the triple uncertainty relation (5).

The states \(|\Xi_\alpha\rangle\) are displaced generalized squeezed states, with a squeezing direction along a line different from the position or momentum axes. To show this, it is sufficient to consider the state \(|\Xi_0\rangle\), which satisfies (19) with \( n = 0 \) and \( \alpha = 0 \). The action of the product of unitaries on the vacuum \(|0\rangle\) is easier to understand if one rewrites it as
\[
\hat{G}_b \hat{S}_\gamma = \hat{S}_\ell \hat{R}_\chi,
\]
with the operator \( \hat{S}_\ell \) defined in Eq. (15), while the unitary \( \hat{R}_\chi = \exp(i\chi a^d a) \) is a counterclockwise rotation by \( \chi \) in phase space. A standard Baker-Campbell-Hausdorff (BCH) calculation (using result from Sec. 6 of [17]) reveals that the values \( r = \log \sqrt{3}, \theta = \pi/2 \), and \( \chi = -\pi/12 \) turn Eq. (24) into an identity for the values of \( b \) and \( \gamma \) given in (17). With the state \(|0\rangle\) being invariant under rotations, we conclude that the state of minimal triple uncertainty is indeed a generalized squeezed state,
\[
|\Xi_0\rangle = \hat{S}_{\frac{1}{2}} \log_3 |0\rangle,
\]
ignoring an irrelevant phase factor. The state \(|\Xi_0\rangle\) is obtained by contracting the vacuum \(|0\rangle\) along the main diagonal in phase space by an amount characterized by \( \log \sqrt{3} < 1 \), at the expense of a dilation along the minor diagonal.

To visualize this result, we determine the Wigner function of the state \(|\Xi_0\rangle\). For \( z = 0 \) and \( re^{i\theta} = i \log \sqrt{3} \), the position representation of the most general squeezed state [18] reduces to
\[
\langle q |\Xi_0\rangle = \frac{1}{\sqrt{\sqrt{3} \pi}} \exp \left( -\frac{1}{2} e^{-i \pi \gamma} q^2 \right),
\]
suppressing an irrelevant factor \( e^{-i \pi \gamma} \). The Wigner function associated with the state \(|\Xi_0\rangle\) minimizing the triple uncertainty relation reads
\[
W_{\Xi_0}(q,p) = \frac{1}{\pi} \exp \left( -\frac{\tau}{\hbar} (q^2 + p^2 + qp) \right),
\]
which is non-negative. Plotting its contour lines in phase space confirms that we deal with a squeezed state aligned with the minor diagonal.

Fig. 1 shows the pair and triple uncertainties for squeezed states with \( r = \log \sqrt{3} \), rotated away from the position axis by an angle \( \varphi \in [0, \pi] \). The pair uncertainty \( \Delta p \Delta q \Delta r \) starts out at its minimum value of \( h/2 \) which is achieved again for \( \varphi = \pi/2 \) and \( \varphi = \pi \) (dashed line). The triple uncertainty has period \( \pi \), reaching its minimum for \( \varphi = 3\pi/4 \) for the state \(|\Xi_0\rangle\) (full line).

Threefold symmetry  The commutation relations [3] are invariant under the cyclic shift \( \hat{p} \to \hat{q} \to \hat{r} \to \hat{p} \), implemented by a unitary operator \( \hat{C} \)
\[
\hat{C} \hat{p} \hat{C}^\dagger = \hat{q}, \quad \hat{C} \hat{q} \hat{C}^\dagger = \hat{r}, \quad \hat{C} \hat{r} \hat{C}^\dagger = \hat{p},
\]
with the third equation being a consequence of the first two. The third power of \( \hat{C} \) obviously commutes with both \( \hat{p} \) and \( \hat{q} \) so it must be a scalar multiple of the identity, \( \hat{C}^3 \propto 1 \).

Acting with the operator \( \hat{C} \) on Eqs. (19) from the left, we find that not only the state \(|n;\alpha\rangle \) but also \( \hat{C} |n;\alpha\rangle \) must be a solution, with the same eigenvalue \( \tau \hbar (n+1/2) \). However, none of the eigenvalues is degenerate which implies that each of the states \(|n;\alpha\rangle \) must be an eigenstate of \( \hat{C} \) as well, with eigenvalue 1 or \( \pm \exp(2\pi i/3) \).

To determine the explicit form of the operator \( \hat{C} \) we first note that its action (28) on the triple \( (\hat{p}, \hat{q}, \hat{r}) \) is achieved by a clockwise rotation by \( \pi/2 \) in phase space, followed by a gauge transformation in the position basis:
\[
\hat{C} = \exp \left( -\frac{i}{2\hbar} q^2 \right) \exp \left( -\frac{i\pi}{4\hbar} (p^2 + q^2) \right).
\]
A BCH-type calculation enables us to express this product of exponentials in terms of a single one:
\[
\hat{C} = \exp \left( -\frac{i\pi}{3\sqrt{3}} (p^2 + q^2 + r^2) \right).
\]
Interestingly, the generator of \( \hat{C} \) is proportional to the operator which determines the extremals of the triple uncertainty functional (cf. [19]). Using (30) for \( \alpha = 0 \), the
diagonal form of the operator $\hat{C}$ is obtained as

$$\hat{C} = \sum_{n=0}^{\infty} e^{-2\pi n/3} |n; 0\rangle \langle n; 0|,$$  \hspace{1cm} (31)

where an overall phase factor $\exp(-i\pi/3)$ has been removed to achieve $\hat{C}^3 = \hat{I}$. Thus, the states $|n; 0\rangle$ (as well as the set $|n; \alpha\rangle$ with eigenvalues $\exp(-2\pi in/3)$, $n \in \mathbb{N}$, fall into three disjoint sets, corresponding to the subspaces associated, respectively, with the countably infinite degenerate eigenvalues 1 and $\exp(\pm 2\pi i/3)$ of the cyclic permutation $\hat{C}$.

The operator $\hat{C}$ cycles the elements of the Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$ just as a Fourier transform operator swaps position and momentum of the Schrödinger pair $(\hat{p}, \hat{q})$ (apart from a sign). If one introduces a unitarily equivalent symmetric form of the Schrödinger triple with operators $(\hat{P}, \hat{Q}, \hat{R})$ associated with an equilateral triangle in phase space, the metaplectic operator $\hat{C}$ simply acts as a rotation by $2\pi/3$, i.e., as a fractional Fourier transform.

Furthermore, denoting the factors of $\hat{C}$ in (29) by $\hat{A}$ and $\hat{B}$ (with suitably chosen phase factors), respectively, we find that $B^2 = \hat{I}$ and $(\hat{A}\hat{B})^3 \equiv \hat{C}^3 = \hat{I}$. These relations establish a direct link between the threefold symmetry of the Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$ and the modular group $SL_2(\mathbb{Z})/\{\pm 1\}$ which $\hat{A}$ and $\hat{B}$ generate [19].

Experiments To confirm the triple uncertainty relation [5], we propose an experiment based on optical homodyne detection. We exploit the fact that the state $|\Xi_0\rangle$ is a generalized coherent state, also known as a correlated coherent state [20]: such a state is obtained by squeezing the vacuum state $|0\rangle$ along the momentum axis followed by a suitable rotation in phase space.

The basic scheme for homodyne detection consists of a beam splitter, photodetectors and a reference beam, called the local oscillator, with which the signal is mixed; by adjusting the phase of the local oscillator one can probe different directions in phase space. If $\theta$ is the phase of the local oscillator, a homodyne detector measures the probability distribution of the observable

$$\hat{x}(\theta) = \frac{1}{\sqrt{2}} (a^e^{i\theta} + ae^{-i\theta}) = \hat{q} \cos \theta + \hat{p} \sin \theta$$  \hspace{1cm} (32)

along a line in phase space defined by the angle $\theta$; here $\hat{q}$ and $\hat{p}$ denote the quadratures of the photon field while the operators $a^\dagger$ and $a$ create and annihilate single photons [21]; note that $\hat{r} \equiv \sqrt{2} \hat{x}(5\pi/4)$.

The probability distributions of the observables $\hat{q}, \hat{p}$ and $\hat{r}$, corresponding to the angles $\theta = 0, \pi/2, 5\pi/4$, can be measured upon preparing a large ensemble of the state $|\Xi_0\rangle$. The resulting product of their variances may then be compared with the value of the tight bound given in Eq. (3). Under rigid phase-space rotations of the triple $(\hat{q}, \hat{p}, \hat{r})$ by an angle $\varphi$ the triple uncertainty will vary as indicated by the full line in Fig. [1]. A related experiment has been carried out successfully in order to directly verify Heisenberg- and Schrödinger-Robertson-type uncertainty relations [22, 23].

Summary and Discussion The results obtained in this paper should be seen in the context of earlier attempts to obtain uncertainty relations for more than two observables. In 1934, Robertson studied constraints which follow from the positive semi-definiteness of the covariance matrix for $N$ observables [6] but the resulting inequality trivializes for an odd number of observables. Shirokov obtained another inequality [24] which contains little information about the canonical triple considered here.

We have established a tight inequality [5] for the triple uncertainty associated with a Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$ of pairwise canonical observables. Ignoring rigid translations in phase space, there is only one state $|\Xi_0\rangle$ which minimizes the triple uncertainty, shown in Eq. (23). The state $|\Xi_0\rangle$ is an eigenstate of the operator $\hat{C}$ in (31) which describes the fundamental threefold cyclic symmetry of the Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$. An important conceptual link between the triple uncertainty and the one derived by Schrödinger and Robertson is due to the fact that both incorporate the correlation $\hat{C} \hat{q} \hat{p} = (\hat{p} \hat{q} + \hat{q} \hat{p})/2$, be it explicitly or indirectly via the expression $r^2$.

A derivation similar to the one leading to (2) establishes a tight bound for an additive uncertainty relation associated with the Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$:

$$(\Delta \hat{p})^2 + (\Delta \hat{q})^2 + (\Delta \hat{\tau})^2 \geq \frac{3\hbar}{2},$$  \hspace{1cm} (33)

also saturated by the state $|\Xi_0\rangle$ given in Eq. (23). Interestingly, this observation clashes with the relation between the additive and the multiplicative uncertainty relations for Schrödinger pairs $(\hat{p}, \hat{q})$: according to [14], the states which saturate the inequality $(\Delta \hat{p})^2 + (\Delta \hat{q})^2 \geq \hbar$ are a proper subset of those minimising Heisenberg’s product inequality [2].

The smallest possible value of the product $\Delta \hat{p} \Delta \hat{q} \Delta \hat{\tau}$ is noticeably larger than the unachievable value $(\hbar/2)^{3/2}$, which follows from inequality (2) applied to each of the Schrödinger pairs $(\hat{p}, \hat{q})$, $(\hat{q}, \hat{\tau})$, and $(\hat{p}, \hat{\tau})$. At the same time, the true minimum undercuts the value of the triple uncertainty in the vacuum state $|0\rangle$ by more than 10% (cf. Eq. (3)). The experimental verification of these results is within reach given current quantum optical technology.

An uncertainty relation for pairs of canonical observables also exists for the Shannon entropies $S_p$ and $S_q$ of their probability distributions [25, 26]. We conjecture that the relation $S_p + S_q \geq (3/2) \ln(e \pi)$ holds for the Schrödinger triple $(\hat{p}, \hat{q}, \hat{\tau})$, the minimum being achieved by the state $|\Xi_0\rangle$. This bound is tighter than $(3/2) \ln(e \pi)$, the value which follows from applying the bound $\ln(e \pi)$ for pairwise entropies to the triple.

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