Optimistic Policy Optimization with Bandit Feedback

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Abstract
Policy optimization methods are one of the most widely used classes of Reinforcement Learning (RL) algorithms. Yet, so far, such methods have been mostly analyzed from an optimization perspective, without addressing the problem of exploration, or by making strong assumptions on the interaction with the environment. In this paper we consider model-based RL in the tabular finite-horizon MDP setting with unknown transitions and bandit feedback. For this setting, we propose an optimistic trust region policy optimization (TRPO) algorithm for which we establish $\tilde{O}(\sqrt{SAH^3K})$ regret for stochastic rewards. Furthermore, we prove $\tilde{O}(\sqrt{SAH^3K}^{2/3})$ regret for adversarial rewards. Interestingly, this result matches previous bounds derived for the bandit feedback case, yet with known transitions. To the best of our knowledge, the two results are the first sub-linear regret bounds obtained for policy optimization algorithms with unknown transitions and bandit feedback.

1. Introduction
Policy Optimization (PO) is among the most widely used methods in Reinforcement Learning (RL) (Peters & Schaal, 2006; 2008; Deisenroth & Rasmussen, 2011; Lillicrap et al., 2015; Levine et al., 2016; Gu et al., 2017). Unlike value-based approaches, e.g., Q-learning, these types of methods directly optimize the policy by incrementally changing it. Furthermore, PO methods span wide variety of popular algorithms such as policy-gradient algorithms (Sutton et al., 2000), natural policy gradient (Kakade, 2002), trust region policy optimization (TRPO) (Schulman et al., 2015) and soft actor-critic (Haarnoja et al., 2018).

Due to their popularity, there is a rich literature that provides different types of theoretical guarantees for different PO methods (Scherrer & Geist, 2014; Abbasi-Yadkori et al., 2019; Agarwal et al., 2019; Liu et al., 2019; Bhandari & Russo, 2019; Shani et al., 2019; Wei et al., 2019) for both the approximate and tabular settings. However, previous results, concerned with regret or PAC bounds for the RL setting when the model is unknown and only bandit feedback is given, provide guarantees which critically depend on ‘concentrability coefficients’ (Kakade & Langford, 2002; Munos, 2003; Scherrer, 2014). These coefficients might be infinite and are usually small only for highly stochastic domains.

Recently, Cai et al. (2019) established an $\tilde{O}(\sqrt{K})$ regret bound for an optimistic PO method in the case of an unknown model and assuming full-information feedback on adversarially chosen instantaneous costs, where $K$ is the number of episodes seen by the agent. In this work, we eliminate the full-information assumption on the cost, as in most practical settings only bandit feedback on the cost is given, i.e., the cost is observed through interacting with the environment. Specifically, we establish regret bounds for an optimistic PO method in the case of an unknown model and bandit feedback on the instantaneous cost in two regimes:

1. For stochastic cost, we establish an $\tilde{O}(\sqrt{SAH^3K})$ regret bound for a PO method (Section 6).
2. For adversarially chosen cost, we establish an $\tilde{O}(\sqrt{SAH^3K}^{2/3})$ regret bound for a PO method. The regret bound matches the $\tilde{O}(K^{2/3})$ upper bound obtained by Neu et al. (2010a) for PO methods which have an access to the true model and observe bandit adversarial cost feedback (Section 7).

2. Preliminaries
Stochastic MDPs. A finite horizon stochastic Markov Decision Process (MDP) $\mathcal{M}$ is defined by a tuple $(S, \mathcal{A}, H, \{p_h\}_{h=1}^H, \{c_h\}_{h=1}^H)$, where $S$ and $\mathcal{A}$ are finite state and action spaces with cardinality $S$ and $\mathcal{A}$, respectively, and $H \in \mathbb{N}$ is the horizon of the MDP. On time step $h$, and state $s$, the agent performs an action $a$, transitions to the next state $s'$ according to a time-dependent
transition function $p_h(s' | s, a)$, and suffers a random cost $C_h(s, a) \in [0, 1]$ drawn i.i.d from a distribution with expectation $c_h(s, a)$.

A stochastic policy $\pi : S \times [H] \rightarrow \Delta_A$ is a mapping from states and time-step indices to a distribution over actions, i.e., $\Delta_A = \{ \pi \in \mathbb{R}^A : \sum_a \pi(a) = 1, \pi(a) \geq 0 \}$. The performance of a policy $\pi$ when starting from state $s$ at time $h$ is measured by its value function, which is defined as

$$V^\pi_h(s) = \mathbb{E} \left[ \sum_{h' = h}^{H} c_{h'}(s_{h'}, a_{h'}) \mid s_h = s, \pi, p \right], \quad (2.1)$$

where the expectation is with respect to the randomness of the transition function, the cost function and the policy. The $Q$-function of a policy given the state action pair $(s, a)$ at time-step $h$ is defined by

$$Q^\pi_h(s, a) = \mathbb{E} \left[ \sum_{h' = h}^{H} c_{h'}(s_{h'}, a_{h'}) \mid s_h = s, a_h = a, \pi, p \right]. \quad (2.2)$$

The two satisfy the following relation:

$$Q^\pi_h(s, a) = c_h(s, a) + p_h(s' \mid s, a)V^\pi_{h+1},$$

$$V^\pi_h(s) = \langle Q^\pi_h(s, \cdot), \pi_h(\cdot \mid s) \rangle,$$  \quad (2.3)

with $p_h(s' \mid s, a)V = \sum_{s'} p_h(s' \mid s, a)V(s')$ for $V \in \mathbb{R}^S$, and $\langle \cdot, \cdot \rangle$ is the dot product.

An optimal policy $\pi^*$ minimizes the value for all states $s$ and time-steps $h$ simultaneously (Puterman, 2014), and its corresponding optimal value is denoted by $V^*_{h}(s) = \min_{\pi} V^\pi_{h}(s)$, for all $h \in [H]$. We consider an agent that repeatedly interacts with an MDP in a sequence of $K$ episodes such that the starting state at the $k$-th episode, $s^k_1$, is initialized by a fixed state $s^*_1$. The agent does not have access to the model, and the costs are received by bandit feedback, i.e., the agent only observes the costs of encountered state-action pairs. At the beginning of the $k$-th episode, the agent chooses a policy $\pi_k$ and samples a trajectory $\{s^k_h, a^k_h, C^k_h(s^k_h, a^k_h)\}_{h=1}^H$ by interacting with the stochastic MDP using this policy, where $(s^k_h, a^k_h)$ are the state and action at the $h$-th time-step of the $k$-th episode.

The performance of the agent for stochastic MDPs is measured by its regret relatively to the value of the optimal policy, defined as $\text{Regret}(K') = \sum_{k=1}^{K'} V^{\pi^*_k}(s^k_1) - V^{\pi^*_k}(s^k_1)$ for all $K' \in [K]$, and $\pi_k$ is the policy of the agent at the $k$-th episode.

**Adversarial MDPs.** Unlike stochastic MDP, in adversarial MDP, we let the cost to be determined by an adversary at the beginning of every episode, whereas the transition function is fixed. Thus, we denote the MDP at the $k$-th episode by $M^k = (S, A, H, \{p^k_h\}_{h=1}^H, \{c^k_h\}_{h=1}^H)$.

As in (2.1), (2.2), we define the value function and $Q$-function of a policy $\pi$ at the $k$-th episode by

$$V^{k,\pi}_h(s) = \mathbb{E} \left[ \sum_{h' = h}^{H} c^k_{h'}(s_{h'}, a_{h'}) \mid s_h = s, \pi, p \right],$$

$$Q^{k,\pi}_h(s, a) = \mathbb{E} \left[ \sum_{h' = h}^{H} c^k_{h'}(s_{h'}, a_{h'}) \mid s_h = s, a_h = a, \pi, p \right].$$

\(^*\)For simplicity we fix the initial state, but the results hold when it is drawn from a fixed distribution.
Notably, \( V_{h}^{k,\pi} \) and \( Q_{h}^{k,\pi} \) satisfy the relations in relation (2.3).

We consider an agent which repeatedly interacts with an adversarial MDP in a sequence of \( K \) episodes. Each episode starts from a fixed initial state, \( s_1^k = s_1 \). As in the stochastic case, at the beginning of the \( k \)-th episode, the agent chooses a policy \( \pi_k \) and samples a trajectory \( \{ s_h^k, a_h^k, c_h^k(s_h^k, a_h^k) \}_{h=1}^{H} \) by interacting with the adversarial MDP. In this case, the performance of the agent is measured by its regret relatively to the value of the best policy in hindsight. The objective is to minimize \( \text{Regret}(K') = \max_{\pi} \sum_{k=1}^{K'} V_{1}^{k,\pi_k}(s_1) - V_{1}^{k,\pi}(s_1) \) for all \( K' \in [K] \).

**Notations and Definitions.** The filtration \( \mathcal{F}_k \) includes all events (states, actions, and costs) until the end of the \( k \)-th episode, including the initial state of the \( k+1 \) episode. We denote by \( n_{h}^{k}(s,a) \), the number of times that the agent has visited state-action pair \( (s,a) \) at the \( h \)-th step, and by \( X_k \), the empirical average of a random variable \( X \). Both quantities are based on experience gathered until the end of the \( k \)-th episode and are \( \mathcal{F}_k \) measurable. We also define the probability to visit the state-action pair \( (s,a) \) at the \( k \)-th episode at time-step \( h \) by \( w_{h}^{k}(s,a) = \Pr(s_h^k = s, a_h^k = a | s_1^k, \pi_k, p) \). Since \( \pi_k \) is \( \mathcal{F}_{k-1} \) measurable, so is \( w_{h}^{k}(s,a) \). In what follows, we refer to \( w_{h}^{k}(s,a) \) as the *state-action occupancy measure*. Furthermore, we define the state visitation frequency of a policy \( \pi \) in state \( s \) given a transition model \( p \) as \( d_{h}^{s}(s;p) = \mathbb{E}[\mathbb{I}\{ s_h=s \} | s_1, \pi, p] \). By the two definitions, it holds that \( d_{h}^{s}(s,a) = d_{h}^{s}(s;p)\pi_k(a | s) \).

We use \( \tilde{O}(X) \) to refer to a quantity that depends on \( X \) up to a poly-log expression of a quantity at most polynomial in \( S, A, K, H \) and \( \delta^{-1} \). Similarly, \( \lesssim \) represents \( \leq \) up to numerical constants or poly-log factors. We define \( X \vee Y := \max\{X, Y\} \).

**Mirror Descent.** The mirror descent (MD) algorithm (Beck & Teboulle, 2003) is a proximal convex optimization method that minimizes a linear approximation of the objective together with a proximity term, defined in terms of a Bregman divergence between the old and new solution estimates. In our analysis we choose the Bregman divergence to be the Kullback–Leibler (KL) divergence, \( d_{KL} \). If \( \{ f_k \}_{k=1}^{K} \) is a sequence of convex functions \( f_k : \mathbb{R}^d \rightarrow \mathbb{R} \), and \( C \) is a constraints set, the \( k \)-th iterate of MD is the following:

\[
x_{k+1} \in \arg \min_{x \in C} \{ t_K \langle \nabla f_k(x_k), x - x_k \rangle + d_{KL}(x \| x_k) \},
\]

where \( t_K \) is a stepsize. In our case, \( C \) is the unit simplex \( \Delta \), and thus the optimization problem has a closed-form solution,

\[
\forall i \in [d], \quad x_{k+1}(i) = \frac{x_k(i) \exp(-t_K \nabla_i f_k(x_k))}{\sum_j x_k(j) \exp(-t_K \nabla_j f_k(x_k))}.
\]

The MD algorithm ensures \( \text{Regret}(K') = \sum_{k=1}^{K'} f(x_k) - \min_{x} f(x) \in O(\sqrt{K}) \) for all \( K' \in [K] \).

**3. Related Work**

**Approximate Policy Optimization:** A large body of work addresses the convergence properties of policy optimization algorithms from an optimization perspective. In Kakade & Langford (2002), the authors analyzed the Conservative Policy Iteration (CPI) algorithm, an RL variant of the Frank-Wolfe algorithm (Scherrer & Geist, 2014; Vieillard et al., 2019), and showed it approximately converges to the global optimal solution. Recently, Liu et al. (2019) established the convergence of TRPO when neural networks are being used as the function approximators. Furthermore, Shani et al. (2019) showed that TRPO (Schulman et al., 2015) is in fact a natural RL adaptation of the MD algorithm, and established convergence guarantees. In (Agarwal et al., 2019), the authors obtained convergence results for policy gradient based algorithms. However, all of the aforementioned works rely on the strong assumption of a finite concentrability coefficient, i.e., \( \max_{x,s,h} d_{h}^{s}(s;p)/d_{h}^{s}(s;p) < \infty \). This assumption bypasses the need to address exploration (Kakade & Langford, 2002), and leads to global guarantees based on the local nature of the policy gradients (Scherrer & Geist, 2014).

**Mirror Descent in Adversarial Reinforcement Learning:** There are two different methodologies for using MD updates in RL. The first and more practical one, is using MD-like updates directly on the policy. The second is based on optimizing over the space of state-action occupancy measures, that is, visitation frequencies for state-action pairs. An occupancy measure represents a policy implicitly. For convenience, previous results for regret minimization using MD approaches are summarized in Table 1.

Following the policy optimization approach, and assuming bandit feedback and known dynamics, Neu et al. (2010b) (OMDP-BF) established \( \tilde{O}(K^{2/3}) \) regret for the average reward criteria. Alternatively, by assuming full information on the reward functions, unknown dynamics and further assuming both the reward and transition dynamics are linear in some \( d \)-dimensional features, Cai et al. (2019) established \( \tilde{O}(d^{3/2} H^{4} K) \) regret for their OPPO algorithm. This setting generalizes to the tabular case when \( d = SA \).

Instead of directly optimizing the policy, Zimin & Neu (2013) proposed optimizing over the space of state-action
occupancy measures with the Relative Entropy Policy Search (O-REPS) algorithm. The O-REPS algorithm implicitly learns a policy by solving an MD optimization problem on the primal linear programming formulation of the MDP (Altman, 1999; Neu et al., 2017). Considering full information and unknown transitions, Rosenberg & Mansour (2019b) obtained $\tilde{O}(\sqrt{S^2AH^2K})$ regret for their UC-O-REPS algorithm. Later, Rosenberg & Mansour (2019a) extended the algorithm to bandit feedback and obtained a regret of $\tilde{O}(K^{3/4})$. Recently, by considering an optimistically biased importance sampling estimator, Jin et al. (2019) established $\tilde{O}(\sqrt{S^2AH^2K})$ for their UOB-REPS algorithm. The O-REPS variants’ updates constitute solving a convex optimization problem with $HS^2A$ variables on each episode, instead of the closed form solution updates of the direct policy optimization variants.

Value-based Regret Minimization in Episodic RL: As opposed to Policy-based methods, there is an extensive literature about regret minimization in episodic MDPs using value-based methods. The works of (Azar et al., 2017; Dann et al., 2017; Jin et al., 2018; Zanette & Brunskill, 2019; Efroni et al., 2019) use the optimism in face of uncertainty principle to achieve near-optimal regret bounds. Jin et al. (2018) also establish a lower bound of $\Omega(\sqrt{S^2AH^2K})$.

4. Mirror Descent for MDPs

Algorithm 1 POMD with Known Model

Require: $t_K$, $\pi_1$ is the uniform policy.
for $k = 1, ..., K$ do
    # Policy Evaluation
    for $h = 1, ..., H$ do
        $Q_h(s, a) = c_h(s, a) + p_h(\cdot | s, a)V_{h+1}^{\pi_h}$
    end for
    # Policy Improvement
    for $h = 1, ..., H$ do
        $\pi_{h+1}(a | s) = \frac{\pi_h(a | s) \exp(-t_KQ_h(s, a))}{\sum_{a'} \pi_h(a' | s) \exp(-t_KQ_h(s, a'))}$
    end for
end for

The empirical success of TRPO (Schulman et al., 2015) and SAC (Haarnoja et al., 2018) had motivated recent study of MD-like update rules for solving MDPs (Geist et al., 2019) when the model of the environment is known. Although not explicitly discussed in (Geist et al., 2019), such an algorithm can also provide guarantees – by similar proof technique – for the case where the cost function is adversarially chosen on each episode.

Policy Optimization by Mirror Descent (POMD) (see Algorithm 1) is conceptually similar to the Policy Iteration (PI) algorithm (Puterman, 2014). It alternates between two stages: (i) policy evaluation, and (ii) policy improvement. Furthermore, much alike PI, POMD updates its policy on the entire state space, given the evaluated $Q$-function. However, as oppose to PI, the policy improvement stage is ‘soft’. Instead of updating according to the greedy policy, the algorithm applies soft update that keeps the next policy ‘close’ to the current one due to the KL-divergence term.

Similarly to standard analysis of the MD algorithm, Geist et al. (2019) established $\tilde{O}(\sqrt{K})$ bound on the regret of Algorithm 1. In the next sections, we apply the same approach to problems with unknown model and bandit feedback.

5. Extended Value Difference Lemma

The analysis of both stochastic and adversarial cases is built upon a central lemma which we now review. The lemma is a variant of (Cai et al., 2019)[Lemma 4.2], which generalizes classical value difference lemmas. Rewriting it in the following form, enables us to establish our results (proof in Appendix D).

Lemma 1 (Extended Value Difference). Let $\pi, \pi'$ be two policies, and $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \{p_h\}_{h=1}^H, \{c_h\}_{h=1}^H)$ and $\mathcal{M}' = (\mathcal{S}, \mathcal{A}, \{p'_h\}_{h=1}^H, \{c'_h\}_{h=1}^H)$ be two MDPs. Let $\hat{Q}_{\pi,\mathcal{M}}(s, a)$ be an approximation of the $Q$-function of policy $\pi$ on the MDP $\mathcal{M}$ for all $h, s, a$, and let $\hat{V}_{\pi,\mathcal{M}}(s) = \langle \hat{Q}_{\pi,\mathcal{M}}(\cdot, s), \pi(\cdot | s) \rangle$. Then, $\hat{V}_{\pi,\mathcal{M}}(s) - \hat{V}_{\pi',\mathcal{M}'}(s) = \sum_{h=1}^H \mathbb{E} \left\{ \left. \left( \hat{Q}_{\pi,\mathcal{M}}(s_h, \cdot) - \hat{Q}_{\pi',\mathcal{M}'}(s_h, \cdot) \right) \cdot | s_h, \pi(h | s_h) \right\} \cdot | s_1, \pi', \pi' \right\} + \sum_{h=1}^H \mathbb{E} \left\{ \left. \left( \hat{V}_{\pi,\mathcal{M}}(s_h, a_h) - \hat{V}_{\pi',\mathcal{M}'}^\prime(s_h, a_h) \right) \cdot | s_h, a_h \right\} \cdot | s_1, \pi', \pi' \right\}$

where $V_{\pi',\mathcal{M}'}$ is the value function of $\pi'$ in the MDP $\mathcal{M}'$.

This lemma generalizes existing value difference lemmas. For example, in (Kearns & Singh, 2002; Dann et al., 2017) the term $V_{\pi,\mathcal{M}^\prime}(s) - V_{\pi',\mathcal{M}^\prime}(s)$ is analyzed, whereas in (Kakade & Langford, 2002) the term $V_{\pi,\mathcal{M}^\prime}(s) - V_{\pi',\mathcal{M}^\prime}(s)$ is analyzed. In next sections, we will demonstrate how Lemma 1 results in a simple analysis of the POMD algorithm. Importantly, the resulting regret
bounds do not depend on concentrability coefficients (see Section 3) nor on any other structural assumptions.

6. Policy Optimization in Stochastic MDPs

We are now ready to analyze the optimistic version of POMD for stochastic environments (see Algorithm 2). Instead of using the known model as in POMD, in Algorithm 2 we use the empirical model to estimate the Q-function of an empirical optimistic MDP, with the empirical transition function \( \hat{p} \) and an optimistic cost function \( \hat{c} \). The empirical transition function \( \hat{p} \) and empirical cost function \( \hat{c} \) are computed by averaging the observed transitions and costs, respectively, that is,

\[
\hat{p}_k^h(s' \mid s, a) = \frac{\sum_{k'=1}^k \mathbb{1}(s'^h_k = s, a'^h_k = a) \mathbb{1}(s'^h_{k+1} = s')}{\sum_{k'=1}^k \mathbb{1}(s'^h_k = s, a'^h_k = a) \vee 1}
\]

\[
\hat{c}_h^k(s, a) = \frac{\sum_{k'=1}^k C_{h}^{k'}(s, a) \mathbb{1}(s'^h_k = s, a'^h_k = a)}{\sum_{k'=1}^k \mathbb{1}(s'^h_k = s, a'^h_k = a) \vee 1},
\]

for every \( s, a, s', h, k \).

**Algorithm 2** Optimistic POMD for Stochastic MDPs

**Require:** \( k, \pi_k \) is the uniform policy.

**for** \( k = 1, \ldots, K \) **do**

1. Rollout a trajectory by acting \( \pi_k \)
2. Policy Evaluation
   \( \forall s \in S, V^k_{H+1}(s) = 0 \)
   **for** \( h = H, \ldots, 1 \) **do**
   1. **for** \( \forall s, a \in S \times A \) **do**
      1. \( \hat{c}_h^k(s, a) = \frac{1}{c_{h-1}^k(s, a) - b_{h-1}^k(s, a)} \), Eq. (6.1)
      2. \( \hat{Q}_h^k(s, a) = \hat{c}_h^k(s, a) + \hat{p}_h^k(s, a|s', h) \hat{V}_{h+1}^k \)
      3. \( Q_h^k(s, a) = \max \{ Q_h^k(s, a), 0 \} \)
   **end for**
2. **for** \( \forall s \in S \) **do**
   1. \( V_h^k(s) = \langle Q_h^k(s, \cdot), \pi_h^k(\cdot | s) \rangle \)
   **end for**
**end for**
3. Policy Improvement
   **for** \( \forall h, s, a \in [H] \times S \times A \) **do**
   1. \( \pi_h^{k+1}(a|s) = \frac{\pi_h^k(a|s) \exp(-t_k \hat{Q}_h^k(s, a))}{\sum_{a'} \pi_h^k(a'|s) \exp(-t_k \hat{Q}_h^k(s, a'))} \)
   **end for**
4. Update counters and empirical model, \( n_k, \hat{c}, \hat{p} \)
**end for**

The optimistic cost function \( \hat{c} \) is obtained by adding a bonus term which drives the algorithm to explore, i.e., \( \hat{c}_h^k(s, a) = c_h^k(s, a) - b_{h-1}^k(s, a) \), and we set

\[
b_h^{k-1}(s, a) = b_h^{k-1}(s, a) + b_h^{p,k-1}(s, a).
\]

The two bonus terms compensate on the lack of knowledge of the true costs and transition model, and are

\[
b_h^{c,k-1}(s, a) = \hat{O} \left( \frac{1}{\sqrt{n_h^{k-1}(s, a)}} \right),
\]

\[
b_h^{p,k-1}(s, a) = \hat{O} \left( \frac{\sqrt{S}}{\sqrt{n_h^{k-1}(s, a)}} \right).
\]

The following theorem bounds the regret of Algorithm 2.

**Theorem 1.** For any \( K' \in [K] \) the regret of Algorithm 2 is bounded by

\[
\text{Regret}(K') \leq \hat{O} \left( \sqrt{S^2AH^2K} \right)
\]

for \( t_K = \hat{O}(H^{-1}K^{-1/2}) \).

**Proof Sketch.** We start by decomposing the regret into three terms according to Lemma 1, and then bound each term separately to get our final regret bound. For any \( \pi \),

\[
\text{Regret}(K') = \sum_{k=1}^{K'} V_1^\pi(s_1^k) - V_1^\pi(s_1^1)
\]

\[
= \sum_{k=1}^{K'} V_1^\pi(s_1^k) - V_1^\pi(s_1^k) + \sum_{k=1}^{K'} V_1^k(s_1^k) - V_1^\pi(s_1^k)
\]

\[
= \sum_{k} V_1^\pi(s_1) - V_1^k(s_1)
\]

\[
+ \sum_{k,h} \mathbb{E} \left[ \langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot \mid s_h) \rangle - \pi_h^k(\cdot \mid s_h) \mid s_1, \pi, p \right]
\]

\[
+ \sum_{k,h} \mathbb{E} \left[ Q_h^k(s_h, a_h) - c_h(s_h, a_h) - p_h(\cdot \mid s_h, a_h) \hat{V}_{h+1}^k \mid s_1, \pi, p \right]
\]

**Term (i): Bias of \( V_h^k \).** Term (i) is the bias between the estimated value \( V_h^k \) and the value of \( \pi_k \) in the true MDP. Applying Lemma 1 on this term while using that

\[
\mathbb{E} [X(s_h, a_h) \mid s_1, \pi, p] = \mathbb{E} [X(s_h^k, a_h^k) \mid \mathcal{F}_{k-1}]
\]

for any \( \mathcal{F}_{k-1} \)-measurable function \( X \in \mathbb{R}^{S \times A} \), we bound Term (i) by

\[
\sum_{k,h} \mathbb{E} [\Delta c_h^k(s_h^k, a_h^k) + H \| \Delta p_h^k(\cdot \mid s_h^k, a_h^k) \|_1] \mid \mathcal{F}_{k-1}
\]

\[
+ \sum_{k,h} \mathbb{E} [b_h^{c,k-1}(s_h^k, a_h^k) + b_h^{p,k-1}(s_h^k, a_h^k) \mid \mathcal{F}_{k-1}]
\]
Here $\Delta c_h^{k-1}(s, a) = c_h(s, a) - c_h^{k-1}(s, a)$ and $\Delta p_h^{k-1}(\cdot \mid s, a) = p_h(\cdot \mid s, a) - p_h^{k-1}(\cdot \mid s, a)$, are the differences between the true cost and transition model. Applying Hoeffding’s bound and $L_1$ deviation bound (Weissman et al., 2003) we get that w.h.p. for any $s, a$

$$\Delta c_h(s, a) \leq \tilde{O}\left(\frac{1}{\sqrt{n_h^{k-1}(s, a)}}\right) = b_h^s(s, a),$$

$$\|\Delta p_h(\cdot \mid s, a)\|_1 \leq \tilde{O}\left(\frac{\sqrt{S}}{\sqrt{n_h^{k-1}(s, a)}}\right) = b_h^p(s, a).$$

Thus, w.h.p., we get

$$(i) \lesssim \sum_{k=1}^{K'} \sum_{h=1}^{H} \mathbb{E} \left[\frac{H \sqrt{S}}{\sqrt{n_h^{k-1}(s_h, \cdot)}} \mid \mathcal{F}_{k-1}\right],$$

which can be bounded by $\tilde{O}\left(\sqrt{S^2AH^4K}\right)$ using standard techniques (e.g., Dann et al. (2017)).

**Term (ii): OMD Analysis.** Term (ii) is the linear approximation used in our MD optimization procedure. We bound it using an analysis of OMD. By applying usual OMD analysis (see Lemma 16) we have that for any policy $\pi$ and $s, h,$

$$\sum_{k=1}^{K} \langle Q_h^{k}(\cdot \mid s), \pi_h^{k}(\cdot \mid s) - \pi_h(\cdot \mid s) \rangle$$

$$\leq \frac{\log A}{t_K} + \frac{t_K}{2} \sum_{k=1}^{K} \sum_{a} \pi_h^{k}(a \mid s)(Q_h^{k}(s, a))^2.$$

We plug this back to Term (ii) and use the fact that $0 \leq Q_h^{k}(s, a) \leq H$, to obtain

Term (ii) =

$$= \sum_{h=1}^{H} \mathbb{E} \left[\sum_{k=1}^{K'} \langle Q_h^{k}(s_h, \cdot), \pi_h^{k}(\cdot \mid s_h) - \pi_h(\cdot \mid s_h) \rangle \mid s_1, \pi, p\right]$$

$$\leq \frac{H \log A}{t_K} + \frac{t_K H^3 K}{2}.$$

By choosing $t_K = \sqrt{2 \log A}/(H^2 K)$, we obtain

Term (ii) $\leq \sqrt{2H^4K \log A}.$

**Term (iii): Optimism.** We choose our exploration bonuses in Eq. (6.2) such that Term (iii) is non-positive.

**Remark 6.1.** The choice of the bonus term $b_h^p(s, a)$ is smaller than in (Cai et al., 2019) by a factor of $\sqrt{S}$. This translates to an improved regret bound by this factor, although (Cai et al., 2019) assumes full-information feedback on the cost function.

**Remark 6.2 (Bonus vs. Optimistic Model).** Instead of using the additive exploration bonus $b^p$ – which compensate on the lack of knowledge of transition model – one can use an optimistic model approach, as in UCRL2 (Jaksch et al., 2010). Following analogous analysis as of Theorem 1 one can establish the same guarantee $\tilde{O}(\sqrt{S^2AH^4K})$. However, the additive bonus approach results in an algorithm with reduced computational cost.

**Remark 6.3 (Optimism of POMD).** Unlike value-based algorithms (e.g., Jaksch et al. (2010)) $V^K$, the value-function by which POMD improves upon, is not necessarily optimistic relatively to $V^*$. Instead, it is optimistic relatively to the value of $\pi_k$, i.e., $V^K \leq V^{\pi_k}$.

### 7. Policy Optimization in Adversarial MDPs

**Algorithm 3 Optimistic POMD for Adversarial MDPs**

Require: $t_K, \gamma, \pi_1$ is the uniform policy.

for $k = 1, \ldots, K$ do

Rollout a trajectory by acting $\pi_k$

for all $h, s$ do

Compute $u_h^k(s)$ by $\pi_k, \mathcal{P}_h^{k-1}$, Eq. (7.1)

end for

# Policy Evaluation

forall $s \in S$, $V_{h+1}^k(s) = 0$

forall $h = H, \ldots, 1$ do

forall $s, a \in S \times A$ do

$\hat{c}_h^k(s, a) = \frac{c_h^k(s, a) + \gamma}{u_h^k(s) \pi_h^k(a \mid s) + \gamma}$

$\hat{p}_h^k(\cdot \mid s, a) = \arg \min \hat{p}_h(\cdot \mid s, a) V_{h+1}^k$

end for

forall $s \in S$ do

$V_h^k(s) = \langle Q_h^k(s, \cdot), \pi_h^k(\cdot \mid s) \rangle$

end for

# Policy Improvement

forall $h, s, a \in [H] \times S \times A$

$\pi_{h+1}^k(a \mid s) = \frac{\pi_h^k(a \mid s) \exp(-t_K Q_{h+1}^k(s, a))}{\sum_{a'} \pi_h^k(a' \mid s) \exp(-t_K Q_{h+1}^k(s, a'))}$

end for

Update counters and model, $n_k, \hat{p}_h^k$

end for

In this section, we turn to analyze an optimistic version of POMD for adversarial environments (Algorithm 3). Similarly to the stochastic case, Algorithm 3 follows the POMD...
Optimistic Policy Optimization with Bandit Feedback

scheme, and alternates between policy evaluation, and, soft policy improvement, based on MD-like updates.

Unlike POMD for stochastic environments, the policy evaluation stage of Algorithm 3 uses different estimates of the instantaneous cost and model. The instantaneous cost is evaluated by a biased importance-sampling estimator, originally suggested by (Neu, 2015), and recently generalized to adversarial RL settings by (Jin et al., 2019),

\[
\tilde{c}_h^k(s, a) = \frac{c_h^k(s, a) \mathbb{1}\{s = s_h, a = a_h\}}{u_h^k(s)\pi_h^k(a | s) + \gamma},
\]

where \( u_h^k(s) = \max_{\hat{p} \in \mathcal{P}^{k-1}} d_h^{\pi_k}(s; \hat{p}) \).

Here \( \mathcal{P}^{k-1} \) is the set of transition functions obtained by using confidence intervals around the empirical model (see Appendix C.1.2).

In Algorithm 3 of Jin et al. (2019), the authors suggest a computationally efficient dynamic programming based approach for calculating \( u_h^k(s) \) for all \( h, s \). The motivation for such an estimate lies in the EXP3 algorithm (Auer et al., 2002) for adversarial bandits, which uses an unbiased importance-sampling estimator \( \tilde{c}(a) = \frac{c(a)}{\pi(a)} \) that motivates exploration can also be used in this setting. Generalizing the latter estimator to the adversarial RL setting requires to use the estimator

\[
\tilde{c}_h^k(s, a) = \frac{c_h^k(s, a) \mathbb{1}\{s = s_h, a = a_h\}}{d_h^{\pi_k}(s, \hat{p})\pi_h^k(a | s) + \gamma}.
\]

However, since the model is unknown, Jin et al. (2019) uses \( u_h^k(s) \) as an upper bound on \( d_h^{\pi_k}(s; \hat{p}) \) which further drives exploration.

Instead of using the empirical model and subtracting a bonus term, Algorithm 3 uses an optimistic model (Jaksch et al., 2010) for the policy evaluation stage. The model by which \( Q^\pi \) is evaluated is the one results in the smallest loss among possible models,

\[
\tilde{p}_h^k(\cdot | s, a) \in \arg\min_{\hat{p}_h(\cdot | s, a) \in \mathcal{P}^{k-1}(s, a)} \tilde{p}_h(\cdot | s, a)V_h^{\pi_k}.
\]

The solution to this optimization problem can be computed efficiently (see, e.g., Jaksch et al. (2010)).

Remark 7.1 (Optimistic Model vs. Additive Exploration Bonus). Replacing the optimistic model with additive bonuses, we were able to establish \( \tilde{O}(K^{3/4}) \) regret bound. It is not clear whether this approach can attain a \( \tilde{O}(K^{2/3}) \) regret bound, as achieved when using an optimistic model.

The following theorem bounds the regret of Algorithm 3. A full proof is found in Appendix C.2.

Theorem 2. For any \( K' \in [K] \) the regret of Algorithm 3 is bounded by

\[
\text{Regret}(K') \leq O\left(\sqrt{K'HS^2AK'^{2/3}}\right),
\]

for \( \gamma = \tilde{O}(A^{-1/2}K^{-1/3}) \) and \( t_K = \tilde{O}(H^{-1}K^{-2/3}) \).

Central to the analysis are the following claims, formally established in Appendix C. The first is proved in (Jin et al., 2019)[Lemma 11], based upon (Neu, 2015)[Lemma 1].

Claim 1 (Jin et al. (2019), Lemma 11). Let \( \alpha^1, ..., \alpha^K \) be a sequence of \( F_{k-1} \) measurable functions such that \( \alpha^k \in [0, 2\gamma]^{S \times A} \). Then, for any \( h \) and \( K' \in [K] \), with high probability,

\[
\sum_{k=1}^{K'} \sum_{s,a} \alpha^k(s, a)(\tilde{c}_h^k(s, a) - c_h^k(s, a)) \leq \tilde{O}(1).
\]

Claim 2. Let \( \alpha^1, ..., \alpha^K \) be a sequence of \( F_{k-1} \) measurable functions such that \( \alpha^k \in [0, 2\gamma] \). For any \( s, h \) and \( K' \in [K] \), with high probability,

\[
\sum_{k=1}^{K'} \alpha^k(V_h^k(s) - V_h^{\pi_k}(s)) \leq \tilde{O}(H).
\]

Claim 2 (see Lemma 7 in the appendix) allows us to derive improved upper bound on \( \sum_{k=1}^{K'} V_h^k(s) \) which is crucial to derive the \( \tilde{O}(K^{2/3}) \) regret bound. Naively, we can bound \( V_h^k(s) \) by recalling it is a value function of an MDP with costs bounded by \( 1/\gamma \). This leads to the naive bound

\[
\sum_{k=1}^{K'} V_h^k(s) \leq K'H/\gamma.
\]

However, a tighter upper bound can be obtained by applying Claim 2 with \( \alpha^k = 2\gamma \) for all \( k \in [K'] \). We have that

\[
\sum_{k=1}^{K'} V_h^k(s) \leq \sum_{k=1}^{K'} V_h^{\pi_k}(s) + \frac{H}{\gamma} \leq HK' + \frac{H}{\gamma},
\]

where in the last relation we used the fact that for any \( s, h \), \( V_h^{\pi_k}(s) \leq H \). In the following proof sketch we apply the later upper bound and demonstrate its importance.

Proof Sketch. We decompose the regret as in Theorem 1 to (i) Bias term, (ii) OMD term, and (iii) Optimism term. We bound both the Bias and Optimism terms in the appendix while relying on both Claim 1 and Claim 2.

Term (ii): OMD Analysis. Similarly to the stochastic case, we utilize the usual OMD analysis (Lemma 16), which ensures that for any policy \( \pi \) and \( s, h \),
where the second relation holds since

\[ \sum_{k=1}^{K'} (Q_h^k(s, a), \pi_h^k(s, a) - \pi_h(s, a)) \]

\[ \leq \frac{\log A}{t_K} + \frac{t_K}{2} \sum_{k=1}^{K'} \sum_{a} \pi_h^k(a | s)(Q_h^k(s, a))^2 \]

\[ \leq \frac{\log A}{t_K} + \frac{t_K H}{2\gamma} \sum_{k=1}^{K'} \sum_{s} \pi_h^k(a | s)Q_h^k(s, a) \]

\[ \leq \frac{\log A}{t_K} + \frac{t_K H}{2\gamma} (H K' + \frac{H}{\gamma}), \]

where the second relation holds since \( 0 \leq Q_h^k(s, a) \leq \frac{H}{\gamma} \), and the third relation holds by applying Eq. (7.3).

Plugging this in Term (ii) we get

\[ \text{Term (ii)} = \sum_{h=1}^{H} \mathbb{E} \left[ \sum_{k=1}^{K'} (Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h(\cdot | s_h)) \mid s_1, \pi, p \right] \]

\[ \leq \frac{H \log A}{t_K} + \frac{t_K H^2}{2\gamma} (H K' + \frac{H}{\gamma}). \]

\[ \square \]

8. Discussion

On-policy vs. Off-policy. There are two prevalent approaches for policy optimization in practice, on-policy and off-policy. On-policy algorithms, e.g., TRPO (Schulman et al., 2015), update the policy based on data gathered following the current policy. This results in updating the policy only in observed states. However, in terms of theoretical guarantees, the convergence analysis of this approach requires the strong assumption of finite concentrability coefficient (Kakade & Langford, 2002; Scherrer & Geist, 2014; Agarwal et al., 2019; Liu et al., 2019; Shani et al., 2019). The assumption arises from the need to acquire global guarantees from the local nature of policy gradients.

The approach taken in this work, is fundamentally different than such on-policy approaches. In each episode, instead of updating the policy only at visited states, the policy is updated over the entire state space, by using all the historical data (in the form of the empirical model). Thus, the analyzed approach bears resemblance to off-policy algorithms, e.g., SAC (Haarnoja et al., 2018). There, the authors i) estimate the \( Q \)-function of the current policy by sampling from a buffer, which contains historical data, and ii) apply an MD-like policy update to states sampled from the buffer.

The uniform updates of policy-based methods analyzed in this work are in stark contrast to value-based algorithms, such as in (Jin et al., 2018; Efroni et al., 2019), where only observed states are updated. It remains an important open question, whether such updates could also be implemented in a provable policy based algorithm. In the case of stochastic POMD, this may be achieved by using optimistic \( Q \)-function estimates, instead of estimating the model with UCB-bonus, similarly to (Jin et al., 2019). There, the authors keep the estimates optimistic with respect to the optimal \( Q \)-function, \( Q^* \). However, in approximate policy optimization, the policy improvement is done with respect to \( Q^{\pi_k} \), as described in Algorithm 1. Therefore, differently than in (Jin et al., 2019), such off-policy version would require learning an optimistic \( Q^{\pi_k} \) estimator, instead of \( Q^* \).

Policy vs. State-Action Occupancy Optimization. In our work, we proposed algorithms which directly optimize the policy. In this scenario, the policy is updated independently at each time step \( h \) and state \( s \). That is, an optimization problem is solved over the action space in each \( h, s \). Therefore, this method requires solving \( HS \) optimization problems of size \( A \), where each has a closed form solution in the tabular setting.

Alternatively, algorithms based on the O-REPS framework (Zimin & Neu, 2013), follow a different approach and optimize over the state-action occupancy measures instead of directly on policies. In the case of unknown transition model, taking such an approach requires solving a constrained convex optimization problem, later relaxed to a convex optimization problem with only non-negativity constraints (Rosenberg & Mansour, 2019b) of size \( HS^2 A \), in each episode. Unlike the policy optimization approach, this optimization problem does not have a closed form solution. Thus, the computational cost of optimizing over the state-action occupancy measures is much worse than the policy optimization one.

Another significant shortcoming in applying the O-REPS framework is the difficulty to scale it to the function approximation setting. Specifically, in case the state-action occupancy measure is represented by a non-linear function, it is unclear how to solve the constrained optimization problem as defined in (Rosenberg & Mansour, 2019b). Differently than the O-REPS framework, the policy optimization approach scales naturally to the function approximation setting, e.g., Haarnoja et al. (2018). In this important aspect, policy optimization is preferable.

Interestingly, our work establishes \( \tilde{O}(\sqrt{K}) \) regret when using POMD for the stochastic case, suggesting that policy-based methods are sufficient for solving stochastic MDPs, and thus preferable, compared to the O-REPS framework, as they also enjoy better computational properties. How-
ever, in the adversarial case, Jin et al. (2019) recently established $\tilde{O}(\sqrt{K})$ regret for the UOB-REPS algorithm, where the adversarial variant of POMD only obtains $\tilde{O}(K^{2/3})$ regret. Hence, it is of importance to understand whether it is possible to bridge this gap between policy and occupancy measure based methods, or alternatively to show that this gap is in fact a true drawback of policy optimization methods in the adversarial case.

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A. Additional Notation

We denote, \( \bar{c} \) and \( \bar{p} \), the empirical estimators for \( c, p \) respectively. In the adversarial case, we denote \( \hat{c} \) as the importance sampling estimator for the costs and \( \hat{p} \) as the optimistic model. When referring to the estimated MDP, we always denote \( \hat{M} \), regardless of the estimation method. When using the notation \( Q^{\pi, p, c}_{h} \) and \( V^{\pi, p, c}_{h} \), for some policy \( \pi \), transition model \( p \) and costs \( c \), we refer to the expected Q-function and value function at the \( h \)-th step, of following the policy \( \pi \) on the MDP defined by the transitions \( p \) and costs \( c \).

B. Stochastic MDPs

First, we restate here Algorithm 2 for readability:

**Algorithm 2** Optimistic POMD for Stochastic MDPs

**Require:** \( t_K, \pi_1 \) is the uniform policy.

\[
\text{for } k = 1, \ldots, K \text{ do}
\]
\[
\text{Rollout a trajectory by acting } \pi_k
\]
\[
\text{# Policy Evaluation}
\]
\[
\forall s \in \mathcal{S}, \ V_{h+1}^k(s) = 0
\]
\[
\text{for } \forall h = H, \ldots, 1 \text{ do}
\]
\[
\text{for } \forall s, a \in \mathcal{S} \times \mathcal{A} \text{ do}
\]
\[
\hat{c}_{h}^{k-1}(s, a) = \hat{c}_{h}^{k-1}(s, a) - b^{k-1}(s, a), \text{ Eq. (6.1)}
\]
\[
Q_{h}^{k}(s, a) = \hat{c}_{h}^{k-1}(s, a) + \hat{p}_{h}^{k-1}(s|s, a)V_{h+1}^k
\]
\[
Q_{h}^{k}(s, a) = \max\{Q_{h}^{k}(s, a), 0\}
\]
\[
\text{end for}
\]
\[
\text{for } \forall s \in \mathcal{S} \text{ do}
\]
\[
V_{h}^k(s) = \langle Q_{h}^k(s, \cdot), \pi_h^{k}(\cdot | s) \rangle
\]
\[
\text{end for}
\]
\[
\text{end for}
\]
\[
\text{# Policy Improvement}
\]
\[
\text{for } \forall h, s, a \in [H] \times \mathcal{S} \times \mathcal{A} \text{ do}
\]
\[
\pi_{h}^{k+1}(a|s) = \frac{\pi_{h}^{k}(a|s) \exp(-t_K Q_{h}^{k}(s, a))}{\sum_{a'} \pi_{h}^{k}(a'|s) \exp(-t_K Q_{h}^{k}(s, a'))}
\]
\[
\text{end for}
\]
\[
\text{Update counters and empirical model, } n_k, \hat{c}_k, \hat{p}_k
\]
\[
\text{end for}
\]

In the stochastic case, we use the empirical model:

\[
\hat{c}_{h}^{k}(s, a) = \sum_{k'=1}^{k} I\{s_{h}^{k'} = s, a_{h}^{k'} = a\}c_{h}^{k'}(s, a)
\]
\[
\frac{n_{h}^{k}(s, a)}{n_{h}^{k}(s, a)}
\]
\[
\hat{p}_{h}^{k}(s'|s, a) = \frac{\sum_{k'=1}^{k'} I\{s_{h}^{k'} = s, a_{h}^{k'} = a, s_{h+1}^{k'} = s'\}}{\sum_{s', s''} \sum_{k'=1}^{k'} I\{s_{h}^{k'} = s, a_{h}^{k'} = a, s_{h+1}^{k'} = s''\}}
\]

The bonus term in Algorithm 2 is made of a bonus term dedicated to the uncertainty in the rewards and a second term dedicated to the uncertainty in the transition model (see (6.1)),

\[
b_{h}^{k-1}(s, a) = b_{h}^{c,k-1}(s, a) + b_{h}^{p,k-1}(s, a).
\]
We choose the additive bonus terms as follows (this choice is guided by the need to keep the term in Lemma 5 negative):

\[ b^{k,c}_h(s,a) = \sqrt{\frac{2 \ln 2S\bar{A}HT}{n^{k-1}_h(s,a)} + 1} \]

\[ b^{k,pv}_h(s,a) = H \sqrt{\frac{4S \ln 2S\bar{A}HT}{n^{k-1}_h(s,a)} + 1} \]

**Remark B.1** (Bounded Q and value estimators). For any \( k, h, s, a \), \( Q^k_h(s,a) \in [0, H] \) and \( V^k_h(s) \in [0, H] \). To see that, first note that by the update rule, we have that for any \( k, h, s, a \), \( Q^k_h(s,a) \geq 0 \). Moreover, using negative bonuses, \( Q^k_h \) is always smaller than \( \overline{Q}^{k:\pi,c}_h \). Therefore, it is always upper bounded by \( H \).

In the next section, B.1, we deal with all the failure events that can happen while running algorithm 2, and show that they happen with small probability. Then, in section B.2, we prove Theorem 1 which establishes the convergence of Algorithm 2.

**B.1. Failure Events**

Define the following failure events.

\[ F^c_k = \left\{ \exists s, a, h : |c_h(s,a) - \bar{c}^k_h(s,a)| \geq \sqrt{\frac{2 \ln 2S\bar{A}HT}{n^{k-1}_h(s,a)} + 1} \right\} \]

\[ F^p_k = \left\{ \exists s, a, h : \|p_h(\cdot | s, a) - \bar{p}^k_h(\cdot | s, a)\|_1 \geq \sqrt{\frac{4S \ln 2S\bar{A}HT}{n^{k-1}_h(s,a)} + 1} \right\} \]

\[ F^N_k = \left\{ \exists s, a, h : n^{k-1}_h(s,a) \leq \frac{1}{2} \sum_{j<k} w_j(s,a,h) - H \ln \frac{S\bar{A}HT}{\delta'} \right\} \]

Furthermore, the following relations hold.

- Let \( F^c = \bigcup_{k=1}^{K} F^c_k \). Then \( \Pr\{F^c\} \leq \delta' \), by Hoeffding’s inequality, and using a union bound argument on all \( s, a, \) and all possible values of \( n_k(s,a) \) and \( k \). Furthermore, for \( n(s,a) = 0 \) the bound holds trivially since \( R \in [0,1] \).

- Let \( F^p = \bigcup_{k=1}^{K} F^p_k \). Then \( \Pr\{F^p\} \leq \delta' \), holds by (Weissman et al., 2003) while applying union bound on all \( s, a, \) and all possible values of \( n_k(s,a) \) and \( k \). Furthermore, for \( n(s,a) = 0 \) the bound holds trivially.

- Let \( F^N = \bigcup_{k=1}^{K} F^N_k \). Then, \( \Pr\{F^N\} \leq \delta' \). The proof is given in (Dann et al., 2017) Corollary E.4.

**Lemma 2** (Good event of the stochastic case). Setting \( \delta' = \frac{4}{3} \) then \( \Pr\{F^c \cup F^p \cup F^N\} \leq \delta \). When the failure events does not hold we say the algorithm is outside the failure event, or inside the good event \( G \).

**B.2. Regret Analysis - Proof of Theorem 1**

By conditioning our analysis on the good event which was formalized in the previous section (see Lemma 2), we are ready to prove the following theorem, which establishes the convergence of Algorithm 2.

**Theorem 1.** For any \( K' \in [K] \) the regret of Algorithm 2 is bounded by

\[ \text{Regret}(K') \leq \tilde{O}\left(\sqrt{SAHT^2K}\right), \]

for \( t_K = \tilde{O}\left(H^{-1}K^{-1/2}\right) \).
Proof. First, we decompose the regret in the following way,

$$
\sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^\pi(s_1) = \sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^k(s_1) + V_1^k(s_1) - V_1^\pi(s_1)
$$

\[= \sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^k(s_1) \] (i)

\[\quad + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[Q_h^k(s_h, \cdot) - \pi_h(\cdot | s_h)] \mid s_1 = s, \pi, P] \] (ii)

\[\quad + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[Q_h^k(s_h, a_h) - c_h(s_h, a_h) - p_h(\cdot | s_h, a_h)V_{h+1}^{k} \mid s_1 = s, \pi, P], \] (iii)

where the second relation holds by using the extended value difference lemma (Lemma 1).

By applying Lemmas 3, 4 and 5 to bound each of the above three terms, respectively, we get that conditioned on the good event, for any $K' \in [K]$ and any $\pi$

$$
\sum_{k=1}^{K'} V_1^{\pi_k}(s_1) - V_1^\pi(s_1) \leq \tilde{O}(\sqrt{S^2AH^4K}) + \sqrt{2H^4K\log A} + 0 \leq \tilde{O}(\sqrt{S^2AH^4K})
$$

In what follows we will analyze the each of the three terms separately: Term (i) is a bias term between the value of the current policy and the estimation of that value, which we bound in Lemma 3. Term (ii) is the linear approximation term used in the OMD optimization problem. This term will be bounded by the OMD analysis (see Lemma 4). Term (iii) is an optimism term. It represents the error of our $Q$-function estimation w.r.t. to the $Q$-function obtained by having the real model, and thus, applying the true 1-step Bellman operator. By the optimistic nature of our estimators, this term is negative given the good event (see Lemma 5).

Lemma 3 (Bias Term of the Stochastic Case). Conditioned on the good event, we have that

$$
\text{Term (i)} = \sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^k(s_1) \leq O(S^2AH^3T).
$$

Proof. By the extended value difference lemma (Lemma 1), we get

\[\sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^k(s_1) \]
\[= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[c_h(s_h, a_h) + p_h(\cdot | s_h, a_h)V_{h+1}^{k} - Q_{h+1}^k(s_h, a_h) \mid s_1 = s, \pi_h, M] \]
\[= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}[c_h(s_h, a_h) + p_h(\cdot | s_h, a_h)V_{h+1}^{k} \mid s_1 = s, \pi, M] \]
\[\quad - \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\max\left\{c_h^k(s_h, a_h) - b_h^k(s_h, a_h) + p_h^{k-1}(\cdot | s_h, a_h)V_{h+1}^{k} - b_h^{k-1}(s_h, a_h)\right\} \mid s_1 = s, \pi, M\right]. \quad (B.1)\]
where the second relation follows from the update rule of $Q_{k+1}^h$.

First, observe that for any $(k, h, s, a)$

\[
\begin{align*}
c_h(s, a) + p_h(\cdot | s, a)V_{h+1}^k &- \min \left\{c_h^k(s, a) - b_h^{k,c}(s, a) + \hat{p}_h^{k-1}(\cdot | s, a)V_{h+1}^k - b_h^{k,pv}(s, a)\right\} \\
&= c_h(s, a) + p_h(\cdot | s, a)V_{h+1}^k - \min \left\{-c_h^k(s, a) + b_h^{k,c}(s, a) - p_h^{k-1}(\cdot | s, a)V_{h+1}^k + b_h^{k,pv}(s, a), 0\right\} \\
&\leq c_h(s, a) - \hat{c}_h^k(s, a) + b_h^{k,c}(s, a) + p_h(\cdot | s, a)V_{h+1}^k - p_h^{k-1}(\cdot | s, a)V_{h+1}^k + b_h^{k,pv}(s, a),
\end{align*}
\]

(B.2)

where the second relation is by the definition of minimum between two terms.

Conditioning on the good event, we have that for any $(h, k, s, a)$

\[
c_h(s, a) - \hat{c}_h^k(s, a) + b_h^{k,c}(s, a) \leq 2b_h^{k,c}(s, a),
\]

(B.3)

and

\[
p_h(\cdot | s, a)V_{h+1}^k + b_h^{k,pv}(s, a) \\
= (p_h(\cdot | s, a) - \hat{p}_h^{k-1}(\cdot | s_h, a_h))V_{h+1}^k + b_h^{k,pv}(s, a) \\
\leq \|p_h(\cdot | s, a) - \hat{p}_h^{k-1}(\cdot | s, a)\|_1 \|V_{h+1}^k\|_\infty + b_h^{k,pv}(s, a) \\
\leq H \|p_h(\cdot | s, a) - \hat{p}_h^{k-1}(\cdot | s, a)\|_1 + b_h^{k,pv}(s, a)
\]

(B.4)

See that the second relation is by the Cauchy-Schwarz inequality. The third is by the fact that for any $k, h, s, 0 \leq V_h^k(s) \leq H_h$, the last relation holds conditioned on the good event.

Plugging (B.3), (B.4) into (B.2) and then back to (B.1) we get

\[
(B.1) \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[2b_h^{k,c}(s, a_h) + 2b_h^{k,pv}(s, a_h) \mid s_1 = s, \pi, \mathcal{M}\right] \\
= C\sqrt{\ln \frac{2S\Delta h T}{\delta'}} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\sqrt{\frac{1}{n_h^{k-1}(s, a) \lor 1}} + H \sqrt{\frac{S}{n_h^{k-1}(s, a) \lor 1}} \mid s_1 = s, \pi, \mathcal{M}\right] \\
\leq C\sqrt{SH} \ln \frac{2S\Delta h T}{\delta'} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\sqrt{\frac{1}{n_h^{k-1}(s, a) \lor 1}} \mid s_1 = s, \pi, \mathcal{M}\right] \\
= C\sqrt{S} \ln \frac{2S\Delta h T}{\delta'} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\sqrt{\frac{1}{n_h^{k-1}(s, a) \lor 1}} \mid \mathcal{F}_{k-1}\right],
\]

where in the fourth relation we used the fact that the expectations are equivalent, since at the $k$-th episode we follow the policy $\pi_k$ in the MDP $\mathcal{M}$.

Applying Lemma 19 we get

\[
\text{Term (i)} \leq O(\sqrt{S^2\Delta h K}).
\]

\[\square\]

**Lemma 4** (OMD Term of the Stochastic Case). *Conditioned on the good event, we have that for any $\pi$*

\[
\text{Term (ii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\langle Q_h^k(s_h, \cdot), \pi^k_h(\cdot | s_h) - \pi_h(\cdot | s_h)\rangle \mid s_1 = s, \pi, P\right] \leq \sqrt{2H^4K \log A}.
\]

*Proof.* This term accounts for the optimization error, bounded by the OMD analysis.
By standard analysis of OMD with the KL divergence used as the Bregman distance (see Lemma 17) we have that for any \( h \in [H], s \in S \) and for policy \( \pi \),

\[
\sum_{k=1}^{K} (Q^k_h(s, \cdot) - \pi^k_h(s, \cdot) - \pi_h(s, \cdot)) \leq \frac{\log A}{t_K} + \frac{t_K}{2} \sum_{k=1}^{K} \sum_{a} \pi^k_h(a \mid s)(Q^k_h(s, a))^2
\]

where \( t_K \) is a fixed step size.

By the fact \( 0 \leq Q^k_h(s, a) \leq H \) (see Remark B.1), we have

\[
\sum_{k=1}^{K} (Q^k_h(s, \cdot) - \pi^k_h(s, \cdot) - \pi_h(s, \cdot)) \leq \frac{\log A}{t_K} + \frac{t_K H^2 K}{2}.
\]

Thus, we can bound Term (ii) as follows

\[
\text{Term (ii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} E_1 \left[ \sum_{k=1}^{K} (Q^k_h(s_h, \cdot) - \pi^k_h(s_h, \cdot) - \pi_h(s_h, \cdot)) \mid s_1 = s, \pi, p \right]
\]

\[
= \sum_{h=1}^{H} E_1 \left[ \sum_{k=1}^{K} (Q^k_h(s_h, \cdot) - \pi^k_h(s_h, \cdot) - \pi_h(s_h, \cdot)) \mid s_1 = s, \pi, p \right]
\]

\[
\leq \sum_{h=1}^{H} E_1 \left[ \frac{\log A}{t_K} + t_K H^2 K \mid s_1 = s, \pi \right] = \frac{H \log A}{t_K} + \frac{H t_K H^2 K}{2}.
\]

See that the first relation holds as the expectation does not depend on \( k \). Thus, by linearity of expectation, we can switch the order of summation and expectation. The second relation holds since (B.5) holds for any \( s \).

Finally, by choosing \( t_K = \sqrt{2 \log A / (H^2 K)} \), we obtain

\[
\text{Term (ii)} \leq \frac{2 H^4 K \log A}{2}.
\]

**Lemma 5** (Optimism Term of the Stochastic Case). *Conditioned on the good event, we have that for any \( \pi \)

\[
\text{Term (iii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} E_1 \left[ Q^k_h(s_h, a_h) - c_h(s_h, a_h) - p_h(s_h, a_h) V^k_{h+1} \mid s_1 = s, \pi, p \right] \leq 0.
\]

**Proof.** We have that

\[
\text{Term (iii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} E_1 \left[ Q^k_h(s_h, a_h) - c_h(s_h, a_h) - p_h(s_h, a_h) V^k_{h+1} \mid s_1 = s, \pi, p \right].
\]

By definition,

\[
Q^k_h(s, a) = \max \left\{ 0, c^k_h(s, a) - b^k_v(s, a) \right\} + \max \left\{ p^{k-1}(\cdot \mid s, a) V^{k-1}_{h+1} - b^k_v(s, a) \right\}.
\]

Now, by the fact that for any \( a, b, \max \{a \cdot b, 0\} \leq \max \{a, 0\} \cdot \max \{b, 0\} \), we have that

\[
Q^k_h(s, a) \leq \max \left\{ 0, c^k_h(s, a) - b^k_v(s, a) \right\} + \max \left\{ 0, p^{k-1}(\cdot \mid s, a) V^{k-1}_{h+1} - b^k_v(s, a) \right\}.
\]

Therefore, for any \( k, h, s, a \),
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\[ Q_h^k(s, a) - c_h(s, a) - p_h(\cdot \mid s, a)V_{h+1}^k \]
\[ \leq \max \left\{ 0, \bar{c}_h^k(s, a) - b_{h,c}^k(s, a) \right\} + \max \left\{ 0, \bar{p}_h^{k-1}(\cdot \mid s, a)V_{h+1}^k - b_{h}^{k,pv}(s, a) \right\} - c_h(s, a) - p_h(\cdot \mid s, a)V_{h+1}^k \]
\[ = \max \left\{ -c_h(s, a), \bar{c}_h^k(s, a) - c_h(s, a) - b_{h,c}^k(s, a) \right\} \]
\[ + \max \left\{ -p_h(\cdot \mid s, a)V_{h+1}^k, (\bar{p}_h^{k-1}(\cdot \mid s, a) - p_h(\cdot \mid s, a)V_{h+1}^k - b_{h}^{k,pv}(s, a) \right\} \]
\[ \leq \max \left\{ 0, \bar{c}_h^k(s, a) - c_h(s, a) - b_{h,c}^k(s, a) \right\} \]
\[ + \max \left\{ 0, (\bar{p}_h^{k-1}(\cdot \mid s, a) - p_h(\cdot \mid s, a)V_{h+1}^k - b_{h}^{k,pv}(s, a) \right\} \] (B.7)

Conditioned on the good event, we have that for any \((k, h, s, a)\),
\[ \bar{c}_h(s_h, a_h) - c_h(s_h, a_h) - b_{h,c}^k(s, a) \leq 0. \] (B.8)

Furthermore,
\[ (\bar{p}_h^{k-1}(\cdot \mid s_h, a_h) - p_h(\cdot \mid s_h, a_h)V_{h+1}^k - b_{h}^{k,pv}(s_h, a_h) \]
\[ \leq \| \bar{p}_h^{k-1}(\cdot \mid s_h, a_h) - p_h(\cdot \mid s_h, a_h) \|_1 \| V_{h+1}^k \|_\infty - b_{h}^{k,pv}(s_h, a_h) \]
\[ \leq H \| \bar{p}_h^{k-1}(\cdot \mid s_h, a_h) - p_h(\cdot \mid s_h, a_h) \|_1 - b_{h}^{k,pv}(s_h, a_h) \leq 0. \] (B.9)

The first relation holds by Cauchy Schwartz inequality. The second relation holds by the updating rule, which keeps \(0 \leq V_{h+1}^{\pi,\hat{c}} \leq H\) (see Remark B.1). The third relation holds conditioning on the good event.

Plugging (B.8), (B.9) into (B.7) we get
\[ \text{Term (iii)} \leq 0. \]
C. Adversarial MDPs

First, we restate here Algorithm 3 for readability:

**Algorithm 3** Optimistic POMD for Adversarial MDPs

**Require:** $I_K$, $\gamma$, $\pi_1$ is the uniform policy.

for $k = 1, ..., K$ do
  Rollout a trajectory by acting $\pi_k$
  for all $h, s$ do
    Compute $u_h^k(s)$ by $\pi_k, \mathcal{P}^{k-1}$, Eq. (7.1)
  end for
  # Policy Evaluation
  $\forall s \in \mathcal{S}$, $V_h^k(s) = 0$
  for $\forall h = H, ..., 1$ do
    for $\forall s, a \in \mathcal{S} \times \mathcal{A}$ do
      $\hat{c}_h^k(s, a) = \frac{c_h^k(s, a) 1 \{s = s_h^k, a = a_h^k\}}{u_h^k(s) \pi_h(a | s) + \gamma}$
      $\hat{p}_h^k(\cdot | s, a) \in \arg \min_{\hat{p}_h(\cdot | s, a) \in \mathcal{P}_h^{k-1}(s, a)} \hat{p}_h(\cdot | s, a)V_h^{k}$
      $Q_h^k(s, a) = \hat{c}_h^k(s, a) + \hat{p}_h^k(\cdot | s, a)V_h^{k}$
    end for
    for $\forall s \in \mathcal{S}$ do
      $V_h^k(s) = \langle Q_h^k(s, \cdot), \pi_h(\cdot | s) \rangle$
    end for
  end for
  # Policy Improvement
  for $\forall h, s, a \in [H] \times \mathcal{S} \times \mathcal{A}$ do
    $\pi_h^{k+1}(a | s) = \frac{\pi_h(a | s) \exp(-t_k Q_h^k(s, a))}{\sum_{a'} \pi_h(a' | s) \exp(-t_k Q_h^k(s, a'))}$
  end for
  Update counters and model, $n_k, \hat{p}_h^k$
end for

We define the costs of the online MDP at the $k$-th episode, for each $h \in [H], s \in \mathcal{S}, a \in \mathcal{A}$, and for any $\pi_h^k$

$$c_h^k(s, a) := \frac{c_h^k(s, a) 1 \{s = s_h^k, a = a_h^k\}}{u_h^k(s) \pi_h(a | s) + \gamma} \quad \text{(C.1)}$$

We also define the following optimistic model, $\hat{p}_h^k(\cdot | s, a)$, which is the solution to the following optimization problem:

$$\hat{p}_h^k(\cdot | s, a) \in \arg \min_{\hat{p}_h(\cdot | s, a) \in \mathcal{P}_h^{k-1}(s, a)} \hat{p}_h(\cdot | s, a)V_h^{k},$$

where $\mathcal{P}_h^{k}(s, a)$ is defined in (C.2) Finally, as for the stochastic case, we denote the empirical estimator of the transition function as

$$\hat{p}_h^k(s' | s, a) = \sum_{k=1}^{K'} \frac{1 \{s_k' = s, a_k' = a, s_{k+1}' = s'\}}{\sum_{s''} \sum_{k=1}^{K'} 1 \{s_k' = s, a_k' = a, s_{k+1}' = s''\}}.$$

**Remark C.1** (Bounded Q and value estimators). For any $k, h, s, a, Q_h^k(s, a) \in [0, H/\gamma]$ and $V_h^k(s) \in [0, H/\gamma]$. To see that, first note that $c_h^k(s, a) \in [0, 1/\gamma]$. By the fact that the estimators for the Q-function and value function are always calculated w.r.t. to some transition model $\hat{p}$, we get that the estimators are bounded as suggested.

The following lemmas, Lemma 6 and Lemma 7, will be essential to establish regret bounds for Algorithm 3. In the main body of the paper, we refer to these lemmas as claim 1 and claim 2, respectively. Lemma 6 is a very close adaptation of (Jin et al., 2019)[Lemma 11], which in itself based on (Neu, 2015)[Lemma 1]. Lemma 7 relies upon applying Lemma 6.
Lemma 6 (Bias of the adversarial costs). Let $\alpha^1, \ldots, \alpha^K$ be a sequence of functions, such that $\alpha^k \in [0, 2\gamma]^{S \times A}$ is $\mathcal{F}_{k-1}$-measurable for all $k$. Let $u^k_h(s) > 0$ for any $k, h, s$. Then, with probability of at least $1 - \delta$, for any $h \in [H]$, 
\[
\sum_{k=1}^{K} \sum_{s,a} \alpha^k(s,a) \left( \gamma^k_h(s,a) - \frac{d^k_h(s)}{u^k_h(s)} \gamma^k_h(s,a) \right) \leq \ln \frac{1}{\delta}.
\]

The full proof of Lemma 6 is given in section E.

Lemma 7 (Bias of the adversarial value functions). Let $\alpha^1, \ldots, \alpha^K$ be a sequence of functions, such that $\alpha^k \in [0, 1]$ is $\mathcal{F}_{k-1}$-measurable for all $k$. Furthermore, assume that for all $k, h, s u^k_h(s) > d^k_h(s) \geq 0$. Then, with probability of at least $1 - \delta$, for any fixed $h \in [H]$ and $s \in S$, 
\[
2\gamma \sum_{k=1}^{K} \alpha^k \left( V^\pi_{k,p,c}^k(s) - V^\pi_k^k(s) \right) \leq H \ln \frac{H}{\delta},
\]
where $V^\pi_{k,p,c}$ is the value of following the policy $\pi_k$ at the $h$-th step, on the MDP defined by the transitions $p$ and costs $\hat{c}$ (as defined in Appendix A).

Proof. For any $(h,s)$ we have 
\[
\sum_{k=1}^{K} \alpha^k \left( V^\pi_{k,p,c}^k(s) - V^\pi_k^k(s) \right) 
\] 
\[
= \sum_{k=1}^{K} \sum_{h'=h}^{H} \alpha^k E \left[ \hat{c}^k_h(s_{h'},a_{h'}) - c^k_h(s_{h'},a_{h'}) \mid s_h = s, \pi_k, p \right] 
\] 
\[
\leq \sum_{k=1}^{K} \sum_{h'=h}^{H} \alpha^k E \left[ \hat{c}^k_h(s_{h'},a_{h'}) - \frac{d^k_h(s_{h'})}{u^k_h(s_{h'})} \gamma^k_h(s_{h'},a_{h'}) \mid s_h = s, \pi_k, p \right] 
\] 
\[
= \sum_{h'=h}^{H} \sum_{k=1}^{K} \sum_{s_{h'}} \sum_{a_{h'}} \alpha^k \Pr(s_{h'},a_{h'} \mid s_h = s, \pi_k, p) \left( \hat{c}^k_h(s_{h'},a_{h'}) - \frac{d^k_h(s_{h'})}{u^k_h(s_{h'})} \gamma^k_h(s_{h'},a_{h'}) \right). 
\]

The first relation holds by Corollary 1, as both value functions are measured w.r.t. the same dynamics and are defined over the same policy. The second relation holds by the fact $c^k_h(s,a) \geq 0$ and by the fact that by the assumptions of the lemma, for any $h, k, s, d^k_h(s) \leq u^k_h(s)$.

Now, observe that $\Pr(s_{h'},a_{h'} \mid s_h = s, \pi_k, p) \in [0,1]$ and are measurable functions w.r.t. $\mathcal{F}_{k-1}$. By applying Lemma 6 for $\alpha^k_h(s_{h'},a_{h'}) = 2\gamma \alpha^k \Pr(s_{h'},a_{h'} \mid s_h = s, \pi_k, p)$ for all $s_{h'} \in S, a_{h'} \in A, h' \in [H]$, we obtain that w.p. $1 - \delta$
\[
\sum_{k=1}^{K} \alpha^k \left( V^\pi_{k,p,c}^k(s) - V^\pi_k^k(s) \right) \leq \frac{H}{\gamma} \ln \frac{H}{\delta}.
\]

C.1. Failure Events

In this section we define the high probability bounds which are later in use in the proof of Theorem 2. We divide the failure event into two different kinds of failure event: basic failure events which are independent on each other, and conditioned failure event which holds conditioned on the basic failure event.

The next sections are ordered in the following way: we first define the basic failure event and the resulting basic good event. Then, we describe the consequences of this basic good event. Finally, we describe the conditioned failure events, which rely on the consequences of the basic good event. By combining all failure events, we define the global failure event. In the proof, we condition our analysis on the event the global failure event does not hold. We also refer to this event as the good event.
C.1.1. Basic Failure Events:

\[ F_k^p = \left\{ \exists s, a, s', h : \left| \hat{p}_h(s' \mid s, a) - \tilde{p}_h^k(s' \mid s, a) \right| \geq 2 \sqrt{ \frac{\tilde{p}_h^k(s' \mid s, a)(1 - \tilde{p}_h^k(s' \mid s, a)) \ln \left( \frac{HSAK}{4\delta} \right)}{(n_h^k(s, a) - 1) \vee 1} } + \frac{14 \ln \left( \frac{HSAK}{4\delta} \right)}{3((n_h^k(s, a) - 1) \vee 1) \vee 1} \right\} \]

\[ F_k^N = \left\{ \exists s, a, h : n_h^k(s, a) \leq \frac{1}{2} \sum_{j < k} w_j(s, a, h) - H \ln \frac{SAH}{\delta} \right\} \]

\[ F_k^c = \left\{ \exists s, a, h : \sum_{k = 1}^{k'} c_h^k(s, a) - \frac{d_h^k(s)}{u_h^k(s)} c_h^k(s, a) \geq \ln \frac{\delta}{2} \right\} \]

Furthermore, the following relations hold.

- Let \( F^p = \bigcup_{k=1}^{K} F_k^p \). Then, \( \Pr\{F^p\} \leq \delta' \), by (Maurer & Pontil, 2009, Theorem 4) and union bounds.

- Let \( F^N = \bigcup_{k=1}^{K} F_k^N \). Then, \( \Pr\{F^N\} \leq \delta' \). The proof is given in (Dann et al., 2017) Corollary E.4.

- Fix \( k' \in [K] \) and let \( n_h^k(s) > 0 \) for all \( k \in [k'] \). Fix \( s, a, h \), and let \( \delta'' > 0 \). For any \( k \in [k'] \), define \( \alpha_h^k(s', a') = 2\gamma \mathbb{1}\{s = s', a = a'\} \) (which is a constant function, and hence measurable). We have that

\[
2\gamma \sum_{k = 1}^{k'} \sum_{s, a} \hat{c}_h^k(s, a) - c_h^k(s, a) = \sum_{k = 1}^{k'} \sum_{s, a} \alpha_h^k(s, a)(\hat{c}_h^k(s, a) - c_h^k(s, a)) \leq \ln \frac{H}{\delta''},
\]

by Lemma 6 w.p. \( 1 - \delta'' \) for any \( h \). Taking union bound on \( s, a \) and setting \( \delta'' = \frac{\delta'}{SAH} \), we get that \( \Pr\{F^c_k\} \leq \frac{\delta'}{K} \).

Finally, let \( F^c = \bigcup_{k'=1}^{K} F^c_{k'} \). By union bound, \( \Pr\{F^c\} \leq \delta' \).

Finally, setting \( \delta' = \frac{\delta}{6} \), and denote \( F^{\text{basic}} := F^p \cup F^N \cup F^c \). Then, by union bound \( \Pr\{F^{\text{basic}}\} \leq \delta \).

**Lemma 8** (Basic good event of the adversarial case). Denote \( G^{\text{basic}} := -F^{\text{basic}} \), then \( \Pr\{G^{\text{basic}}\} \geq 1 - \frac{\delta}{2} \). When \( G^{\text{basic}} \) occurs, we say that the basic good event holds.

C.1.2. Consequences Conditioning on the Basic Good Event

First, for any \( k, h, s, a \), we define the set

\[ \mathcal{P}_h^k(s, a) = \left\{ \hat{p}_h(\cdot \mid s, a) : \forall s' | \hat{p}_h(s' \mid s, a) - \tilde{p}_h^k(s' \mid s, a) \mid \leq \epsilon_h^k(s' \mid s, a), p_h(s' \mid s, a) \geq 0, \sum_{s'} \hat{p}_h(s' \mid s, a) = 1 \right\} \]

where

\[
\epsilon_h^k(s' \mid s, a) := 2 \sqrt{ \frac{\tilde{p}_h^k(s' \mid s, a)(1 - \tilde{p}_h^k(s' \mid s, a)) \ln \left( \frac{HSAK}{4\delta} \right)}{(n_h^k(s, a) - 1) \vee 1} } + \frac{14 \ln \left( \frac{HSAK}{4\delta} \right)}{3((n_h^k(s, a) - 1) \vee 1) \vee 1} \]

By using this definition conditioned on the basic good event, we get the following lemma from (Jin et al., 2019)[Lemma 8].
Lemma 9. Conditioned on the basic good event, for all \( k, h, s, a, a' \) and for all \( \hat{p}_h^k(\cdot \mid s, a) \in \mathcal{P}_{h-1}^k(s, a) \), there exists constants \( C_1, C_2 > 0 \) for which we have that
\[
\left| \hat{p}_h^k(s' \mid s, a) - p_h(s' \mid s, a) \right| = C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln(HSAK)}{(n_h^k(s, a) - 1) \lor 1} } + C_2 \frac{\ln(HSAK)}{(n_h^k(s, a) - 1) \lor 1}.\]

Lemma 10. Conditioned on the basic good event, for any \((s, a) \in S \times A, h \in [H], k \in [K]\)
\[
V_h^k(s) \leq V_h^{\pi_k, p, \hat{c}}(s),
\]
where \( V_h^{\pi_k, p, \hat{c}} \) is defined in Appendix A.

Proof. By definition of the update rule,
\[
Q_h^k(s, a) = c_h^k(s, a) + \hat{p}_h^k(\cdot \mid s, a)V_{h+1}^k.
\]

By the description of the algorithm, for each value, we solve the following minimization problem, for any \( k, h, s, a \)
\[
\hat{p}_h^k(\cdot \mid s, a) \in \arg \min_{p_h(\cdot \mid s, a) \in \mathcal{P}_{h-1}^k(s, a)} p_h(\cdot \mid s, a)V_h^k.
\]

Therefore, by conditioning on the good event and by lemma 9, for any \( k, h, s, a \) the following holds
\[
\hat{p}_h^k(\cdot \mid s, a)V_{h+1}^k \leq p_h(\cdot \mid s, a)V_{h+1}^k.
\]

By plugging in (C.4) in (C.3), we get
\[
Q_h^k(s, a) \leq c_h^k(s, a) + p_h(\cdot \mid s, a)V_{h+1}^k.
\]

Now, note that for \( H = h \) using the fact that \( V_{h+1}^k = 0 \) for any \( k, s \), we obtain,
\[
Q_h^k(s, a) = c_h^k(s, a) + \hat{p}_h(\cdot \mid s, a)V_{h+1} = c_h^k(s, a) = Q_H^{\pi_k, p, \hat{c}}(s, a),
\]
and therefore, for any \( k, s \) and policy \( \pi_k \)
\[
V_h^k(s) \leq V_H^{\pi_k, p, \hat{c}}.
\]

Using the above inequality, by backward recursion on \( h = H, H - 1, \ldots, 1 \) on (C.5), we get for any \( k, h, s, a \)
\[
Q_h^k(s, a) \leq c_h^k(s, a) + p_h(\cdot \mid s, a)V_{h+1}^k \leq c_h^k(s, a) + p_h(\cdot \mid s, a)V_{h+1}^{\pi_k, p, \hat{c}} = Q_h^{\pi_k, p, \hat{c}}(s, a),
\]
where in the second inequality we used the fact that \( p_h, V_{h+1}^k \) and \( V_{h+1}^{\pi_k, p, \hat{c}} \) are all non-negative.

Furthermore, this follows immediately.

C.1.3. Conditioned failure events

\[
F^c = \left\{ h : \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{I} \left[ \sum_{s} \sum_{a} d_h^k(s) \pi_k(a \mid s)c_h^k(s, a) \mid F_{k-1} \right] - \sum_{s} \sum_{a} d_h^k(s) \pi_k(a \mid s)c_h^k(s, a) \geq H \sqrt{K \ln \frac{H}{2\delta'}} \right\},
\]

\[
F_{k'}^{V, MD} = \left\{ h, s : \sum_{k=1}^{K} V_h^k(s) - V_{\pi_k}(s) \geq \frac{H}{\gamma} \ln \frac{H^2SAK}{\delta'} \right\},
\]

\[
F_{k'}^{V,1} = \left\{ h, s, s', a : \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s, a, s', k} \mathbb{I} \left[ \frac{p_h(s' \mid s, a)}{(n_h^{k-1}(s, a) - 1) \lor 1} \operatorname{Pr}(s_h = s, a_h = a \mid s_1, \pi_k, p)(V_{h+1}^k(s') - V_{\pi_k}(s')) \geq \frac{H^2S^2A}{\gamma} \ln \frac{H^2S^2AK}{\delta'} \right] \right\},
\]

\[
F_{k'}^{V,2} = \left\{ h, s, s', a : \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s, a, s', k} \operatorname{Pr}(s_h = s, a_h = a \mid s_1, \pi_k, p)(V_{h+1}^k(s') - V_{\pi_k}(s')) \geq \frac{H^2S^2A}{\gamma} \ln \frac{H^2S^2AK}{\delta'} \right\}.
\]
• Fix $h$ and let $\delta' > 0$. Conditioning on the the basic good event $G^{\text{basic}}$ for any $k, h$,
\[
\sum_{s} \sum_{a} d_{h}^{k}(s) \pi^{k}(a | s) c_{h}^{k}(s, a) = \sum_{s} \sum_{a} d_{h}^{k}(s) \pi^{k}(a | s) \frac{c_{h}^{k}(s, a)}{u_{h}^{k}(s) \pi^{k}(a | s) + \gamma} \leq \sum_{s} \sum_{a} \mathbb{I}(s_{h}^{k} = s, a_{h}^{k} = a) = 1,
\]
where we conditioned on the event $G^{\text{basic}}$ in which $d_{h}^{k}(s) \leq u_{h}^{k}(s)$. Therefore, $\sum_{s} \sum_{a} d_{h}^{k}(s) \pi^{k}(a | s) c_{h}^{k}(s, a) \in [0, 1]$. Thus, by Azuma-Hoeffding and taking union bound for all $H$, we have that for any $K$, $\Pr\{F^{c} \mid \neg F^{\text{basic}}\} \leq \delta'' = \frac{\delta'}{K}$.

• Fix $k' \in [K], h, s$ and let $\delta'' > 0$. Now, set for any $k \in [k']$, $\alpha^{k} = 1 \in [0, 1]$ (constant and thus measurable). Furthermore, conditioned on the basic good event, we have that for any $k, h, s$, $u_{h}^{k}(s) > d_{h}^{k}(s) \geq 0$. Thus, by applying Lemma 7, we get that w.p. $1 - \delta''$
\[
\sum_{k=1}^{K'} \alpha^{k} \left( V^{\pi_{+}, p, \delta}(s) - V^{\pi_{h}}(s) \right) \leq \frac{H}{\gamma} \ln \frac{H}{\delta''}.
\]
Now, conditioned on the basic good event, by Lemma 10 we have
\[
\sum_{k=1}^{K'} \alpha^{k} \left( V_{h}^{k}(s) - V_{h}^{\pi_{h}}(s) \right) \leq \frac{H}{\gamma} \ln \frac{H}{\delta''}.
\]
Taking union bounds on $h, s$ and setting $\delta'' = \frac{\delta'}{S^{2}AK}$, we get that $\Pr\{F^{v, MD}_{k'}\} \leq \frac{\delta'}{K}$. Finally, let $F^{v, MD} = \bigcup_{k'=1}^{K} F^{v, MD}_{k'}$. By union bound, $\Pr\{F^{v, MD}\} \leq \delta'$.

• Fix $k' \in [K], s, a, s', h$ and let $\delta'' > 0$. Now, set for any $k \in [k']$, $\alpha^{k} = \sum_{s_h, a_h} \frac{\Pr(s_h = s, a_h = a | s_k, x_k, p)}{\sqrt{(n_h^{k} - (s, a) - 1)v_{1}}} \in [0, 1]$, and note that it is $\mathcal{F}_{k-1}$-measurable. Furthermore, conditioned on the basic good event, we have that for any $k, h, s$, $u_{h}^{k}(s) > d_{h}^{k}(s) \geq 0$. Thus, by applying Lemma 7, we get that w.p. $1 - \delta''$
\[
\sum_{k=1}^{K'} \alpha^{k} \left( V^{\pi_{+}, p, \delta}(s) - V^{\pi_{h}}(s) \right) \leq \frac{H}{\gamma} \ln \frac{H}{\delta''}.
\]
Now, conditioned on the basic good event, by Lemma 10 we have
\[
\sum_{k=1}^{K'} \alpha^{k} \left( V_{h}^{k}(s) - V_{h}^{\pi_{h}}(s) \right) \leq \frac{H}{\gamma} \ln \frac{H}{\delta''},
\]
w.p. $1 - \delta''$. Taking union bound on $s, a, s', h$ and setting $\delta'' = \frac{\delta'}{S^{2}AK}$, we get that w.p. $1 - \delta'$
\[
\sum_{h=1}^{H} \sum_{s' = 1}^{S} \sum_{k=1}^{K} \alpha^{k} \left( V^{k}_{h}(s) - V^{\pi_{h}}(s) \right) \leq \frac{H^2 S}{\gamma} \ln \frac{H^2 S^2 AK}{\delta'},
\]
or in other words, $\Pr\{F^{v,1}_{k'}\} \leq \frac{\delta'}{\gamma}$. Finally, let $F^{v,1} = \bigcup_{k'=1}^{K} F^{v,1}_{k'}$. By union bound, $\Pr\{F^{v,1}\} \leq \delta'$.

• By following the same proof of event $F^{v,1}$, but using $\alpha^{k} = \sum_{s_h, a_h} \frac{\Pr(s_h = s, a_h = a | s_k, x_k, p)}{\sqrt{(n_h^{k} - (s, a) - 1)v_{1}}} \in [0, 1]$, we get that $\Pr\{F^{v,2}\} \leq \delta'$.

Now, denote the conditioned event, $F^{\text{conditioned}} := F^{c} \cup F^{v, MD} \cup F^{v,1} \cup F^{v,2}$. Next, we set $\delta' = \frac{\delta}{2}$. Then, by union bound $\Pr\{F^{\text{conditioned}} \mid \neg F^{\text{basic}}\} \leq \delta$. 

Lemma 11 (Conditioned good event of the adversarial case). Denote $G^{\text{conditioned}}$ as conditioned on $G^{\text{basic}} \geq 1 - \frac{\delta}{2}$. 
C.1.4. Global Failure Events

In this section, we combine both the basic and conditioned failure events into a single global failure event. The global failure event accounts for all failure events which can occur in the adversarial MDP case. Specifically, in our analysis we will always assume that none of the failure events occurs, which happens with probability of at least $1 - \delta$ since

$$\Pr\{\neg F_{\text{conditioned}} \cap \neg F_{\text{basic}} \} = \Pr\{\neg F_{\text{conditioned}} | \neg F_{\text{basic}}\} \Pr\{\neg F_{\text{basic}}\} \geq \left(1 - \frac{\delta}{2}\right) \left(1 - \frac{\delta}{2}\right) \geq 1 - \delta,$$

where we used the facts that $\Pr\{\neg F_{\text{basic}}\} \geq 1 - \frac{\delta}{2}$ by Lemma 8, and $\Pr\{\neg F_{\text{conditioned}} | \neg F_{\text{basic}}\} \geq 1 - \frac{\delta}{2}$ by Lemma 11.

**Lemma 12** (Good event of the adversarial case). Denote $G := G_{\text{conditioned}} \cap G_{\text{basic}} = \neg F_{\text{conditioned}} \cap \neg F_{\text{basic}}$, then $\Pr\{G\} \geq 1 - \delta$. When $G$ occurs, we say the algorithm outside the failure event or inside the good event.

C.2. Regret Analysis - Proof of Theorem 2

By conditioning our analysis on the good event which was formalized in the previous sections (see Lemma 12), we are ready to prove the following theorem, which establishes the convergence of Algorithm 3.

**Theorem 2.** For any $K' \in [K]$ the regret of Algorithm 3 is bounded by

$$\text{Regret}(K') \leq O\left(\sqrt{H^4 S^2 AK^{2/3}}\right),$$

for $\gamma = \tilde{O}(A^{-1/2} K^{-1/3})$ and $t_K = \tilde{O}(H^{-1} K^{-2/3}).$

**Proof.** First, we decompose the regret in the following way

$$\begin{align*}
\sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^{\pi}(s_1) &= \sum_{k=1}^{K} V_1^{\pi_k,t_c}(s_1) - V_1^{\pi}(s_1) + V_1^{k}(s_1) - V_1^{\pi}(s_1) \\
&= \sum_{k=1}^{K} V_1^{\pi_k}(s_1) - V_1^{k}(s_1) \\
&\quad + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h(\cdot | s_h) \rangle | s_1 = s, \pi, p\right] \\
&\quad + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[Q_h^k(s_h, a_h) - c_h(s_h, a_h) - p_h(\cdot | s_h, a_h) V_{h+1}^k | s_1 = s, \pi, p\right],
\end{align*}$$

where the second relation holds by using the extended value difference lemma (Lemma 1).

By applying Lemmas 13, 14 and 15 to bound each of the above three terms, respectively, we get that conditioned on the good event (see Lemma 12), for any $K' \in [K]$,

$$\begin{align*}
\text{Regret}(K') &\leq \tilde{O}\left(\sqrt{S^2 A H^4 K} + \gamma H S A K + \sqrt{H^2 K} + \frac{H^2 S}{\gamma}\right) \\
&\quad + \tilde{O}\left(\frac{H \log A}{t_K} + \frac{2t_K H^3}{\gamma^2} + \frac{t_K H^3 K}{\gamma} + \frac{H}{\gamma}\right).
\end{align*}$$
By choosing \( t_K = \tilde{O}\left(\sqrt{\log A/(H^2)K^{-2/3}}\right) \) and \( \gamma = \tilde{O}(A^{-1/2}K^{-1/3}) \), we obtain

\[
\text{Regret}(K) \leq \tilde{O}\left(\sqrt{S^2AH^2K} + HS\sqrt{AK^{2/3}} + \sqrt{H^2K} + H^2S^3/2K^{1/3}\right) + \tilde{O}\left(\sqrt{H^4\log AK^{2/3}} + A^2\log AH^2 + A\log AH^4K^{2/3} + H\sqrt{AK^{1/3}}\right)
\]
\[
\leq O\left(\sqrt{H^2S^2AK^{2/3}}\right),
\]
which concludes the proof. \( \square \)

The decomposition in the proof of Theorem 2 is the same as in the stochastic case. The analysis is different here due to the different nature of the estimators for the costs and transition model. Again, term (i) is a bias term between the value of the current policy and the estimation of that value, which is bounded in Lemma 13. Term (ii) is the linear approximation term used in the OMD optimization problem. This term will be bounded by the OMD analysis (see Lemma 14). Term (iii) is an optimism term. It represents the error of our \( Q \)-function estimation w.r.t. to the \( Q \)-function obtained by having the real model, and thus, applying the true 1-step Bellman operator. By the optimistic nature of our estimators, this term is (almost) negative given the good event (see Lemma 15).

**Lemma 13 (Bias Term of the Adversarial Case).** *Conditioned on the good event,*

\[
\text{Term (i)} = \sum_{k=1}^{K} V^\pi_k(s_1) - V^\pi(s_1) \leq \tilde{O}\left(\sqrt{S^2AH^2K} + HS\log AK + \frac{H^2S}{\gamma}\right)
\]

**Proof.** First, by Lemma 1, the following relations hold,

\[
\sum_{k=1}^{K} V^\pi_k(s_1) - V^\pi(s_1) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[\hat{c}^k_h(s_h, a_h) - c^k_h(s_h, a_h) \mid s_1 = s, \pi_k, P\right] \tag{A}
\]
\[
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E}\left[p_h(\cdot \mid s_h, a_h)V^k_{h+1} - \hat{\rho}^k_h(\cdot \mid s_h, a_h)V^k_{h+1} \mid s_1 = s, \pi_k, P\right]. \tag{B}
\]

**Term (A).** For any \((k, h, s, a)\),

\[
c^k_h(s, a) - \hat{c}^k_h(s, a) = c^k_h(s, a) - [\hat{c}^k_h(s, a) \mid \mathcal{F}_{k-1}] + [\hat{c}^k_h(s, a) \mid \mathcal{F}_{k-1}] - \hat{c}^k_h(s, a)
\]
\[
= c^k_h(s, a)\left(1 - \frac{d^k_h(s)\pi^k_h(a \mid s)}{u^k_h(s)\pi^k_h(a \mid s) + \gamma}\right) + [\hat{c}^k_h(s, a) \mid \mathcal{F}_{k-1}] - \hat{c}^k_h(s, a)
\]
\[
= c^k_h(s, a)\left(\frac{u^k_h(s)\pi^k_h(a \mid s) - d^k_h(s)\pi^k(a \mid s) + \gamma}{u^k_h(s)\pi^k_h(a \mid s) + \gamma}\right) + [\hat{c}^k_h(s, a) \mid \mathcal{F}_{k-1}] - \hat{c}^k_h(s, a).
\]
By plugging back to the first term of \((C.6)\),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s_h, a_h) - \hat{c}_h^k(s_h, a_h) \mid s_1 = s, \pi_k, \mathcal{M} \right] \\
= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s, a) \left( \frac{u_h^k(s) \pi_h^k(a \mid s) - d_h^k(s) \pi_h^k(a \mid s) + \gamma}{u_h^k(s) \pi_h^k(a \mid s) + \gamma} \right) \mid s_1 = s, \pi_k, \mathcal{M} \right] \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s, a) \mid \mathcal{F}_{k-1} \right] - \hat{c}_h^k(s, a) \mid s_1 = s, \pi_k, \mathcal{M} \\
= \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \sum_{a} d_h^k(s) \pi_h^k(a \mid s) c_h^k(s, a) \left( \frac{u_h^k(s) \pi_h^k(a \mid s) - d_h^k(s) \pi_h^k(a \mid s) + \gamma}{u_h^k(s) \pi_h^k(a \mid s) + \gamma} \right) \\
+ \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \sum_{a} d_h^k(s) \pi_h^k(a \mid s) \left( \mathbb{E} \left[ c_h^k(s, a) \mid \mathcal{F}_{k-1} \right] - \hat{c}_h^k(s, a) \right).
\]

(C.7)

First, we deal with the first term in \((C.7)\),

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \sum_{a} d_h^k(s) \pi_h^k(a \mid s) c_h^k(s, a) \left( \frac{u_h^k(s) \pi_h^k(a \mid s) - d_h^k(s) \pi_h^k(a \mid s) + \gamma}{u_h^k(s) \pi_h^k(a \mid s) + \gamma} \right) \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \sum_{a} \left( u_h^k(s) \pi_h^k(a \mid s) - d_h^k(s) \pi_h^k(a \mid s) \right) + \gamma HSAK \\
= \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \left( u_h^k(s) - d_h^k(s) \right) + \gamma HSAK,
\]

(C.8)

where in the inequality with use the fact that by conditioning on the good event, for any \(k, h, s\), \(d_h^k(s) \leq u_h^k(s)\), and therefore for any \(k, h, a\), \(\frac{d_h^k(s) \pi_h^k(a \mid s)}{u_h^k(s) \pi_h^k(a \mid s) + \gamma} \leq 1\)

As for the second term in \((C.7)\), conditioning on the good event we have that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \sum_{a} d_h^k(s) \pi_h^k(a \mid s) \left( \mathbb{E} \left[ c_h^k(s, a) \mid \mathcal{F}_{k-1} \right] - \hat{c}_h^k(s, a) \right) \leq H \sqrt{2K \ln \frac{H}{\delta}}.
\]

(C.9)

By combining \((C.8)\) and \((C.9)\), we obtain

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s_h, a_h) - \hat{c}_h^k(s_h, a_h) \mid s_1 = s, \pi_k, \mathcal{M} \right] \\
\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s} \left( u_h^k(s) - d_h^k(s) \right) + \gamma HSAK + H \sqrt{2K \ln \frac{H}{\delta}} \\
\leq O \left( HS \sqrt{AT \ln \frac{SAHK}{\delta'}} \right) + \gamma HSAK + H \sqrt{2K \ln \frac{H}{\delta}},
\]

where the last relation follows from Lemma 20.

**Term (B).** Now, its left to address the second term of \((C.6)\). Consider the following,
In the first transition we used the fact that

\[ \pi \quad \text{in the MDP} \]

\[ = p_h(\cdot \mid s, a) - \hat{p}_h^k(\cdot \mid s, a) \]

\[ \leq \sum_{s'} \left( C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta} }{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \right) V_{h+1}^k(s') \]

\[ = \sum_{s'} \left( C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta} }{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \right) V_{h+1}^{\pi_k}(s') \]

\[ + \sum_{s'} \left( C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta} }{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \right) (V_{h+1}^k(s') - V_{h+1}^{\pi_k}(s')). \quad (C.10) \]

The second transition is by the fact \( V_h^k \) is positive and by the conditioning on the good event and applying Lemma 9. The third transition is by the fact for any \( k, h, s, a, n_{h-1}^{-1}(s, a) \leq n_{h-1}^{-1}(s, a) \).

First, we deal with the first term. Conditioning on the good event, we have for any \((k, s, a, h)\)

\[ \sum_{s'} \left( C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta} }{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \right) V_{h+1}^{\pi_k}(s') \]

\[ \leq H \sum_{s'} \left( C_1 \sqrt{ \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta} }{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \right) \]

\[ \leq C_1 HS \sum_{s'} \frac{p_h(s' \mid s, a) \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} + \frac{C_2 H S \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \]

\[ = C_1 HS \sqrt{ \frac{S \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 H S \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1}. \quad (C.11) \]

In the first transition we used the fact that \( V_h^{\pi_k} \) is positive and bounded by \( H \) for any \( k, h, s' \). The second transition is by Jensen’s inequality and the fact that the square root is concave.

By summing as done in of (C.6) we get

\[ (C.11) = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ C_1 HS \sqrt{ \frac{S \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } + \frac{C_2 H S \ln \frac{HSAK}{\delta}}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \mid s_1 = s, \pi_k, \mathcal{M} \right] \]

\[ = C_1 HS \sqrt{ \frac{2SAHK}{\delta} } \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \sqrt{ \frac{1}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} } \mid \mathcal{F}_{k-1} \right] \]

\[ + C_2 H S \ln \frac{2SAHK}{\delta} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{(n_{h-1}^{-1}(s, a) - 1) \vee 1} \mid \mathcal{F}_{k-1} \right] \]

\[ \leq C_1 H \sqrt{ \frac{2SAHK}{\delta} } \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \sqrt{ \frac{1}{n_{h-1}^{-1}(s, a) \vee 1} } \mid \mathcal{F}_{k-1} \right] \]

\[ + 2 C_2 H S \ln \frac{2SAHK}{\delta} \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{n_{h-1}^{-1}(s, a) \vee 1} \mid \mathcal{F}_{k-1} \right]. \quad (C.12) \]

Note that in the first relation we used the fact that the expectations are equivalent, since at the \( k \)-th episode we follow the policy \( \pi_k \) in the MDP \( \mathcal{M} \). The third relation holds by the fact that for any \( n \geq 0 \), it holds that \( \frac{1}{(n+1)^{\gamma} \vee 1} \leq \frac{2}{n^{\gamma} \vee 1} \).
Finally, applying Lemma 19 and Lemma 18 and excluding constant and logarithmic factors in $K$, we get

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ C_1 H \sqrt{\frac{S_{\text{HSAK}}}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} + C_2 H S \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mid s_1 = s, \pi_k, \mathcal{M} \right] \leq \tilde{O} \left( \sqrt{S^2 A H^4 K} \right).
$$

Now, consider the second term of (C.10).

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \sum_{s'} \left( C_1 \frac{p_h(s' \mid s_h, a_h) \ln H_{\text{SAK}}}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} + C_2 \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mid (V_{h+1}^k(s') - V_{h+1}^{\pi_k}(s')) \mid s_1 = s, \pi_k, \mathcal{M} \right]
$$

$$
= \sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s_h, a_h} \sum_{s'} \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mathbb{P}(s_h, a_h \mid s_1 = s, \pi_k, \mathcal{M}) \left( \sum_{s'} \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mathbb{P}(s_h, a_h \mid s_1 = s, \pi_k, \mathcal{M}) \right) \left( V_{h+1}^k(s') - V_{h+1}^{\pi_k}(s') \right)
$$

$$
= C_1 \ln \frac{H_{\text{SAK}}}{\delta} \sum_{h=1}^{H} \sum_{s_h, a_h} \sum_{s'} \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mathbb{P}(s_h, a_h \mid s_1 = s, \pi_k, \mathcal{M}) \left( V_{h+1}^k(s') - V_{h+1}^{\pi_k}(s') \right)
$$

Next, conditioned on the good event, and specifically on events $F^{v,1}, F^{v,2}$ we have that

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \sum_{s'} \left( C_1 \frac{p_h(s' \mid s_h, a_h) \ln H_{\text{SAK}}}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} + C_2 \ln \frac{H_{\text{SAK}}}{\delta} \frac{1}{(n_h^{k-1}(s_h, a_h) - 1)}} \frac{1}{\sqrt{1}} \mid (V_{h+1}^k(s') - V_{h+1}^{\pi_k}(s')) \mid s_1 = s, \pi_k, \mathcal{M} \right] \leq O \left( \frac{H^2 S}{\gamma} \ln \frac{H_{\text{SAK}}}{\delta} \right).
$$

Finally, by combining the bounds, we get that

$$
\text{Term B} \leq \tilde{O} \left( \sqrt{S^2 A H^3 T} + \frac{H^2 S}{\gamma} \right).
$$

The result holds by combining the two above terms.

\[\square\]

**Lemma 14 (OMD Term of the Adversarial Case).** *Conditioned on the good event, for any pi,*

$$
\text{Term (ii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ Q_h^k(s_h, \pi_h(s_h \mid s_h)) - \pi_h(s_h \mid s_h) \right] \leq \frac{H \log A}{t_K} + \frac{2t_K H^3}{\gamma^2} + \frac{2t_K H^3 K}{\gamma}.
$$
This term accounts for the optimization error, bounded by the OMD analysis when the KL-divergence is used as the Bregman divergence.

By Lemma 17, we have that for any \( h \in [H], s \in \mathcal{S} \) and for policy \( \pi \),

\[
\sum_{k=1}^{K} \langle Q_h^k(s, \cdot), \pi_h^k(\cdot | s) - \pi_h(\cdot | s) \rangle \leq \frac{\log A}{t_K} + \frac{Ht_K}{2} \sum_{k=1}^{K} \sum_{a} \pi_h^k(a | s) (Q_h^k(s, a))^2
\]

where \( t_K \) is a fixed step size.

Now, conditioning on the good event, the following holds,

\[
(Q_h^k(s, a))^2 = (\hat{c}_h^k(s, a) + \hat{p}_h^k(\cdot | s, a)V_{h+1}^k)^2 \\
\leq 2(\hat{c}_h^k(s, a))^2 + 2(\hat{p}_h^k(\cdot | s, a)V_{h+1}^k)^2 \\
\leq \frac{2H}{\gamma} (\hat{c}_h^k(s, a) + \hat{p}_h^k(\cdot | s, a)V_{h+1}^k) \\
= \frac{2H}{\gamma} Q_h^k(s, a).
\]

Note that the second relation holds by \((a + b)^2 \leq 2a^2 + 2b^2\). The third relation is by the fact that both terms are bounded by \( \frac{2H}{\gamma} \). The fourth relation is by the definition of the update rule.

Plugging this into (C.13) we get for any \( s \in \mathcal{S}, h \in [H] \)

\[
\sum_{k} \langle Q_h^k(s, \cdot), \pi_h^k(\cdot | s) - \pi_h(\cdot | s) \rangle \\
\leq \frac{\log A}{t_K} + \frac{Ht_K}{2} \sum_{k=1}^{K} \sum_{a} \pi_h^k(a | s) Q_h^k(s, a) \\
= \frac{\log A}{t_K} + \frac{Ht_K}{2} \sum_{k} V_h^k(s) \\
\leq \frac{\log A}{t_K} + \frac{Ht_K}{2} \ln \frac{H^2S}{\delta} + \frac{Ht_K}{\gamma} \sum_{k} V_h^\pi(s) \\
\leq \frac{\log A}{t_K} + \frac{Ht_K}{2} \ln \frac{H^2S}{\delta} + \frac{2H \gamma}{\gamma} t_K K.
\]

The second relation holds by definition. The third relation holds by conditioning on the good event, specifically, event \( F^{w, MD} \). The fourth relation holds since the value function of the true MDP is bounded by \( H \).

Thus,

\[
\text{Term (ii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h(\cdot | s_h) \rangle \mid s_1 = s, \pi, p \right] \\
= \sum_{h=1}^{H} \mathbb{E} \left[ \sum_{k=1}^{K} \langle Q_h^k(s_h, \cdot), \pi_h^k(\cdot | s_h) - \pi_h(\cdot | s_h) \rangle \mid s_1 = s, \pi, p \right] \\
\leq \frac{H \log A}{t_K} + \frac{2t_K H^3}{\gamma^2} + \frac{2t_K H^3 K}{\gamma}.
\]

**Lemma 15** (Optimism Term of the Adversarial Case). Conditioned on the good event, for any \( \pi \),

\[
\text{Term (iii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ Q_h^k(s_h, a_h) - c_h(s_h, a_h) - p_h(\cdot | s_h, a_h)V_{h+1}^k \mid s_1 = s, \pi, p \right] \leq \tilde{O} \left( \frac{H}{\gamma} \right).
\]
We shall prove that, conditioned on the good event,

\[ \text{Term (iii)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ Q_h^k(s_h, a_h) - c_h(s_h, a_h) - p_h(\cdot | s_h, a_h)V_{h+1}^k | s_1 = s, \pi, p \right] \]

\[ = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s_h, a_h) - c_h^k(s_h, a_h) | s_1 = s, \pi, p \right] \]

\[ = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left\{ p_h^k(\cdot | s_h, a_h)V_{h+1}^k - p_h(\cdot | s_h, a_h)V_{h+1}^k | s_1 = s, \pi, p \right\}. \]

We shall prove that, conditioned on the good event,

\[ \text{Term (iii)} \leq \frac{H}{\gamma} \ln \frac{SAH}{\delta'}. \]

**Term (A).** We have that for any \( s, a, h, \) conditioning on the good event

\[ \sum_k c_h^k(s, a) - c_h^k(s, a) \leq \sum_k c_h^k(s, a) - \frac{d_h^k(s)}{u_h^k(s)} c_h^k(s, a) \leq \frac{1}{2\gamma} \ln \frac{SAH}{\delta'}, \]

where we used that fact that conditioned on the good event, \( 0 \leq d_h^k(s) < u_h^k(s) \) for any \( k, h, s. \) Thus,

\[ \text{Term (A)} = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h^k(s_h, a_h) - c_h^k(s_h, a_h) | s_1 = s, \pi, p \right] \]

\[ = \sum_{h=1}^{H} \mathbb{E} \left[ \sum_{k=1}^{K} c_h^k(s_h, a_h) - c_h^k(s_h, a_h) | s_1 = s, \pi, p \right] \leq \frac{H}{2\gamma} \ln \frac{SAH}{\delta'}. \]

**Term (B).** For any \( k, h, s, a, \)

\[ \hat{p}_h^k(\cdot | s, a)V_{h+1}^k - p_h(\cdot | s, a)V_{h+1}^k = \min_{\hat{p}(\cdot | s, a) \in \mathcal{P}_{h-1}^k(s, a)} \hat{p}(\cdot | s, a)V_{h+1}^k - p_h(\cdot | s, a)V_{h+1}^k \leq 0, \]

since \( p_h(\cdot | s, a) \in \mathcal{P}_h^k(s, a) \) conditioning on the good event.

The result follows by combining the two above terms.

\[ \square \]

### D. Difference Lemmas

The following lemma is similar to the analysis of the first term, in (Cai et al., 2019)[Lemma 4.2].

**Lemma 1 (Extended Value Difference).** Let \( \pi, \pi' \) be two policies, and \( \mathcal{M} = (\mathcal{S}, \mathcal{A}, \{p_h\}_{h=1}^{H}, \{c_h\}_{h=1}^{H}) \) and \( \mathcal{M}' = (\mathcal{S}, \mathcal{A}, \{p'_h\}_{h=1}^{H}, \{c'_h\}_{h=1}^{H}) \) be two MDPs. Let \( \hat{Q}^{-\mathcal{M}}_h(s, a) \) be an approximation of the \( Q \)-function of policy \( \pi \) on the MDP.
\( \mathcal{M} \) for all \( h, s, a \), and let \( \hat{V}_h^{\pi, \mathcal{M}}(s) = \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) \rangle \). Then,

\[
\hat{V}_1^{\pi, \mathcal{M}}(s_1) - V_1^{\pi', \mathcal{M}'}(s_1) = \\
\sum_{h=1}^{H} \mathbb{E} \left[ \langle \hat{Q}_h^{\pi, \mathcal{M}}(s_h, \cdot), \pi_h^\prime(\cdot | s_h) - \pi_h(\cdot | s_h) \rangle \mid s_1, \pi', p' \right] + \\
\sum_{h=1}^{H} \mathbb{E} \left[ \hat{Q}_h^{\pi, \mathcal{M}}(s_h, a_h) - c_h^\prime(s_h, a_h) - p_h^\prime(|s_h, a_h) \hat{V}_{h+1}^{\pi, \mathcal{M}} \mid s_1, \pi', p' \right]
\]

where \( V_1^{\pi', \mathcal{M}'} \) is the value function of \( \pi' \) in the MDP \( \mathcal{M}' \).

**Proof.** For any two policies \( \pi, \pi' \), and for any \( h \) and \( s \), by the definition \( \hat{V}_h^{\pi, \mathcal{M}}(s) = \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) \rangle \) and by the definition of \( V_h^{\pi, \mathcal{M}}, V_h^{\pi', \mathcal{M}'} \),

\[
\hat{V}_h^{\pi, \mathcal{M}}(s) - V_h^{\pi', \mathcal{M}'}(s) = \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) \rangle - \langle \hat{Q}_h^{\pi', \mathcal{M}'}(s, \cdot), \pi_h^\prime(\cdot | s) \rangle \\
= \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) - \pi_h^\prime(\cdot | s) \rangle + \langle \hat{Q}_h^{\pi', \mathcal{M}'}(s, \cdot), \pi_h(\cdot | s) \rangle \\
= \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) - \pi_h^\prime(\cdot | s) \rangle \\
+ \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot) - c_h^\prime(s, a) - \sum_{s'} p_h^\prime(s' | s, a) \hat{V}_{h+1}^{\pi', \mathcal{M}'}(s') \rangle, \pi_h(\cdot | s) \rangle
\]

where in the last relation we used the fixed-policy Bellman equation on the MDP \( \mathcal{M}' \). I.e., for any \( s, a \), we have that

\[
Q_h^{\pi', \mathcal{M}'}(s, a) = c_h^\prime(s, a) + \sum_{s'} p_h^\prime(s' | s, a) \hat{V}_{h+1}^{\pi', \mathcal{M}'}(s').
\]

Now, by adding and subtracting \( \sum_{s'} p_h^\prime(s' | s, \cdot) \langle \hat{V}_{h+1}^{\pi, \mathcal{M}}(s'), \pi_h(\cdot | s) \rangle \), we get

\[
\hat{V}_h^{\pi, \mathcal{M}}(s) - V_h^{\pi', \mathcal{M}'}(s) = \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot), \pi_h(\cdot | s) - \pi_h^\prime(\cdot | s) \rangle \\
+ \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot) - c_h^\prime(s, a) - \sum_{s'} p_h^\prime(s' | s, a) \hat{V}_{h+1}^{\pi, \mathcal{M}}(s') \rangle, \pi_h(\cdot | s) \rangle \\
+ \langle \sum_{a} \pi_h(a | s) \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, a) - c_h^\prime(s, a) - \sum_{s'} p_h^\prime(s' | s, a) \hat{V}_{h+1}^{\pi, \mathcal{M}}(s') \rangle, \pi_h(\cdot | s) \rangle \\
+ \langle \sum_{s'} \sum_{a} p_h^\prime(s' | s, a) \pi_h(a | s) \langle \hat{V}_{h+1}^{\pi, \mathcal{M}}(s') - V_{h+1}^{\pi', \mathcal{M}'}(s') \rangle, \pi_h(\cdot | s) \rangle \\
+ \langle \hat{Q}_h^{\pi, \mathcal{M}}(s, \cdot) - \pi_h^\prime(\cdot | s) \rangle \rangle
\]

\[
+ \mathbb{E} \left[ \hat{Q}_h^{\pi, \mathcal{M}}(s, a) - c_h^\prime(s, a) - \sum_{s'} p_h^\prime(s' | s, a) \hat{V}_{h+1}^{\pi, \mathcal{M}}(s') \mid s_h = s, \pi', \mathcal{M}' \right] \\
+ \mathbb{E} \left[ \hat{V}_{h+1}^{\pi, \mathcal{M}}(s_{h+1}) \mid s_h = s, \pi', \mathcal{M}' \right] \]

\[
\]

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By using the above relation recursively, we obtain,

\[
\hat{V}_1^{\pi, \mathcal{M}}(s) - V_1^{\pi', \mathcal{M}'}(s) = E \sum_{h=1}^{H} \left[ \left\langle \hat{Q}_h^{\pi, \mathcal{M}}(s_h, \cdot), \pi_h(\cdot | s_h) - \pi_h'(\cdot | s_h) \right\rangle \mid s_1 = s, \pi', \mathcal{M}' \right] \\
+ E \sum_{h=1}^{H} \left[ \hat{Q}_h^{\pi, \mathcal{M}}(s_h, a_h) - c_h(s_h, a_h) - \sum_{s'} p_h'(s' | s_h, a_h) \hat{V}_h^{\pi, \mathcal{M}'}(s') \mid s_h = s, \pi', \mathcal{M}' \right] \\
+ E \left[ \hat{V}_{H+1}^{\pi, \mathcal{M}}(s_{H+1}) - V_{H+1}^{\pi', \mathcal{M}'}(s_{H+1}) \mid s_1 = s, \pi', \mathcal{M}' \right].
\]

which concludes the proof.

By replacing the approximation in the last lemma with the real expected value, we get the following well known result:

**Corollary 1** (Value difference). Let \( \mathcal{M}, \mathcal{M}' \) be any \( H \)-finite horizon MDP. Then, for any two policies \( \pi, \pi' \), the following holds

\[
V_1^{\pi, \mathcal{M}}(s) - V_1^{\pi', \mathcal{M}'}(s) = \\
E \sum_{h=1}^{H} \left[ \left\langle Q_h^{\pi, \mathcal{M}}(s_h, \cdot), \pi_h(\cdot | s_h) - \pi_h'(\cdot | s_h) \right\rangle \mid s_1 = s, \pi', \mathcal{M}' \right] \\
+ E \sum_{h=1}^{H} \left[ (c_h(s_h, a_h) - c_h'(s_h, a_h)) + (p_h(\cdot | s_h, a_h) - p_h'(\cdot | s_h, a_h)) \hat{V}_h^{\pi, \mathcal{M}'} \mid s_h = s, \pi', \mathcal{M}' \right].
\]

**E. Useful Lemmas**

**E.1. Online Mirror Descent**

In each iteration of Online Mirror Descent (OMD), the following problem is solved:

\[
x_{k+1} \in \arg \min_{x \in \Delta_d} t_k \langle g_k, x - x_k \rangle + B_\omega(x, x_k).
\]

The following lemma, (Orabona, 2019)[Theorem 10.4], provides a fundamental inequality which will be used in our analysis.

**Lemma 16** (Fundamental inequality of Online Mirror Descent). Assume for \( g_{k,i} \geq 0 \) for \( k = 1, ..., K \) and \( i = 1, ..., d \). Let \( C = \Delta_d \) and \( \eta > 0 \). Using OMD with the KL-divergence, learning rate \( t_k \), and with uniform initialization, \( x_1 = [1/d, ..., 1/d] \), the following holds for any \( u \in \Delta_d \).

\[
\sum_{k=1}^{K} (g_k, x_k - u) \leq \frac{\log d}{t_K} + \frac{t_K}{2} \sum_{k=1}^{K} \sum_{i=1}^{d} x_{k,i} g_{k,i}^2.
\]
In our analysis, we will be solving the OMD problem for each time-step \( h \) and state \( s \) separately,

\[
\pi_{k+1}^h(\cdot | s) \in \arg\min_{\pi \in \Delta_A} t_K \langle Q_k^h(s, \cdot), \pi - x_k^h(\cdot | s) \rangle + d_{KL}(\pi || \pi_k^h(\cdot | s)).
\] (E.2)

Therefore, by adapting the above lemma to our notation, we get the following lemma,

**Lemma 17** (Fundamental inequality of Online Mirror Descent for RL). Let \( t_K > 0 \). Let \( \pi_1^h(\cdot | s) \) be the uniform distribution for any \( h \in [H] \) and \( s \in S \). Then, by solving (E.2) separately for any \( k \in [K], h \in [H] \) and \( s \in S \), the following holds for any stationary policy \( \pi \),

\[
\sum_{k=1}^K \langle Q_k^h(\cdot | s), \pi_k^h(\cdot | s) - \pi_h(\cdot | s) \rangle \leq \frac{\log A}{t_K} + \frac{t_K}{2} \sum_{k=1}^K \sum_{a \pi_k^h(a | s)(Q_k^h(s, a))^2}
\]

**Proof.** First, observe that for any \( k, h, s \), we solve the optimization problem defined in (E.2) which is the same as (E.1). By the fact that the estimators used in our analysis are non-negative, we can apply Lemma 16 separately for each \( h, s \) with \( g_k = Q_k^h(s, \cdot) \) and \( x_k = \pi_k^h(s, \cdot) \).

**E.2. Bounds on the Visitation Counts**

**Lemma 18** (e.g. Zanette & Brunskill 2019, Lemma 13). Outside the failure event, it holds that

\[
\sum_{k=1}^K \sum_{h=1}^H \sum_{t=1}^T \mathbb{E}\left[ \frac{1}{n_{k-1}(s_k^h, \pi_k(s_k^h)) \vee 1} | \mathcal{F}_{k-1} \right] \leq \tilde{O}(SAH^2).
\]

**Lemma 19** (e.g. Efroni et al. 2019, Lemma 38). Outside the failure event, it holds that

\[
\sum_{k=1}^K \sum_{h=1}^H \sum_{t=1}^T \mathbb{E}\left[ \sqrt{\frac{1}{n_{k-1}(s_k^h, \pi_k(s_k^h)) \vee 1}} | \mathcal{F}_{k-1} \right] \leq \tilde{O}\left(\sqrt{SAH^2K} + SAH\right).
\]

In both Zanette & Brunskill 2019; Efroni et al. 2019 these results were derived for MDPs with stationary dynamics. Repeating their analysis, in our case, an additional \( H \) factor emerges as we consider MDPs with non-stationary dynamics.

**E.3. Bias Lemmas**

**Lemma 6** (Bias of the adversarial costs). Let \( \alpha^1, \ldots, \alpha^K \) be a sequence of functions, such that \( \alpha^k \in [0, 2\gamma]^{S \times A} \) is \( \mathcal{F}_{k-1} \)-measurable for all \( k \). Let \( u_k^h(s) > 0 \) for any \( k, h, s \). Then, With probability of at least \( 1 - \delta \), for any \( h \in [H] \),

\[
\sum_{k=1}^K \sum_{s,a} \alpha^k(s, a) \left( \frac{c_k^h(s, a) - d_k^h(s)}{u_k^h(s)} \right) \leq \ln \frac{1}{\delta}.
\]
Define $\hat{X}_h^k = \sum_{s,a} \alpha_h^k(s,a) c_h^k(s,a)$ and $X_h^k = \sum_{s,a} \alpha_h^k(s,a) d_h^k(s,a) c_h^k(s,a)$. Next, we prove that $\mathbb{E}[\exp(\hat{X}_h^k) \mid \mathcal{F}_{k-1}] \leq \exp(X_h^k)$.

$$
\mathbb{E}[\exp(\hat{X}_h^k) \mid \mathcal{F}_{k-1}] = \mathbb{E}\left[\exp\left(\sum_{s,a} \alpha_h^k(s,a) c_h^k(s,a)\right) \mid \mathcal{F}_{k-1}\right] \\
\leq \mathbb{E}\left[\exp\left(\sum_{s,a} \frac{\alpha_h^k(s,a)c_h^k(s,a)}{2\gamma} \ln\left(1 + \frac{2\gamma c_h^k(s,a)}{u_h^k(s)\pi_h^k(a \mid s)}\right)\right) \mid \mathcal{F}_{k-1}\right] \\
\leq \mathbb{E}\left[\exp\left(\sum_{s,a} \ln\left(1 + \frac{\alpha_h^k(s,a)c_h^k(s,a)}{u_h^k(s)\pi_h^k(a \mid s)}\right)\right) \mid \mathcal{F}_{k-1}\right] \\
= \mathbb{E}\left[\prod_{s,a} \left(1 + \frac{\alpha_h^k(s,a)c_h^k(s,a)}{u_h^k(s)\pi_h^k(a \mid s)}\right) \mid \mathcal{F}_{k-1}\right] \\
= \mathbb{E}\left[1 + \sum_{s,a} \frac{\alpha_h^k(s,a)c_h^k(s,a)}{u_h^k(s)\pi_h^k(a \mid s)} \mid \mathcal{F}_{k-1}\right] \\
= 1 + \sum_{s,a} \frac{\alpha_h^k(s,a)d_h^k(s,a)c_h^k(s,a)}{u_h^k(s)\pi_h^k(a \mid s)} = 1 + X_h^k \leq \exp(X_h^k).
$$

Now, we use the above relation and apply Markov inequality to obtain

$$
\Pr\left[\sum_{k=1}^K \hat{X}_h^k - X_h^k > \frac{H}{\delta}\right] = \Pr\left[\exp\left(\sum_{k=1}^K \hat{X}_h^k - X_h^k\right) > \frac{H}{\delta}\right] \\
\leq \frac{\delta}{H} \mathbb{E}\left[\exp\left(\sum_{k=1}^K \hat{X}_h^k - X_h^k\right)\right] \\
= \frac{\delta}{H} \mathbb{E}\left[\exp\left(\sum_{k=1}^{K-1} \hat{X}_h^k - X_h^k\right) \mathbb{E}\left[\exp\left(\hat{X}_h^K - X_h^K\right) \mid \mathcal{F}_K\right]\right] \\
\leq \frac{\delta}{H} \mathbb{E}\left[\exp\left(\sum_{k=1}^{K-1} \hat{X}_h^k - X_h^k\right)\right] \leq \ldots \leq \frac{\delta}{H}.
$$
where the last inequality follows because $\mathbb{E}\left[\exp(\hat{X}_h^k) \mid \mathcal{F}_{k-1}\right] \leq \exp(X_h^k)$.

**Lemma 20** (Jin et al. 2019, Lemma 4). For any $k$, let $\{\tilde{p}^{k,s}\}_{s \in S}$ be any collection of transition functions which are all $\mathcal{F}_{k-1}$-measurable and belong to $\mathcal{P}$. Define the visitation frequencies

$$d_k^h(s) = \mathbb{E}\left[\mathbb{I}(s_h^k = s) \mid \pi^k, p]\right],$$

$$\tilde{d}_h^{k,s}(s) = \mathbb{E}\left[\mathbb{I}(s_h^k = s) \mid \pi^k, \tilde{p}^{k,s}\right],$$

for every $(s, h, k) \in S \times [H] \times [K]$. With probability at least $1 - \delta'$,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s \in S} |d_k^h(s) - \tilde{d}_h^{k,s}(s)| \leq O\left(\frac{HS}{\sqrt{AT \ln \frac{SAHK}{\delta'}}}\right).$$

Notice that $u_k^h(s) = \tilde{d}_h^{k,s}(s)$ for some $\tilde{p}^{k,s}$ which maximizes the probability to reach $s$ in the $h$ step of episode $k$. Thus, with probability at least $1 - \delta'$,

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \sum_{s \in S} |u_k^h(s) - d_k^h(s)| \leq O\left(\frac{HS}{\sqrt{AT \ln \frac{SAHK}{\delta'}}}\right).$$