A Four-Reggeon Vertex for $\mathbb{Z}_3$ Twisted Fermionic Fields

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Abstract

Using operator sewing techniques we construct the Reggeon vertex involving four external $\mathbb{Z}_3$-twisted complex fermionic fields. Generalizing a procedure recently applied to the ordinary Ramond four-vertex, we deduce the closed form of the $\mathbb{Z}_3$ vertex by demanding it to reproduce the results obtained by sewing.
Although a considerable amount of work has been devoted to using operator sewing techniques for string and conformal field theory (CFT) calculations involving twisted fields, most of the attention has so far been focused on $Z_2$ twisted fermions [1-16], i.e. the Ramond sector of the NSR string, and $Z_2$ twisted scalars [17-19,6,20-25,12,26]. The applications of fields with this basic twist ($Z_2$) are manifold, some of the more important ones being besides string theory certain statistical models (see e.g. [27]), and even some of (so far) purely mathematical interest like the monster module [28, 24, 25]. Even for this case of the simplest twist technical difficulties arise already at the tree level and for such a fundamental Reggeon vertex as the one with four (external) twisted legs. For fermions these old problems [2-6] were only recently overcome [15, 16]. In this short note we address the generalization of these questions to CFT’s containing fields with higher twists $Z_N, N > 2$ [20-23]. It turns out that $Z_N$ scalars play a very important role when considering e.g. strings moving on orbifolds [20, 21]. However, generalizations of operator sewing methods to CFT’s with higher twist fields ($Z_N, N > 2$) have so far been (as far as we know) conspicuously absent. We will focus on $Z_3$ twisted fermions for the simple reason that the corresponding case for scalars is likely to possess additional complications which are not yet understood even in the $Z_2$ situation [23]. The goal here is therefore to demonstrate, using $Z_3$ fermions, that the methods that were recently successfully applied to the case of $Z_2$ fermions in ref. [13], and which led to that the full four-Reggeon vertex can now be rigorously derived [16], can be taken over to higher twists. Although it turns out to be slightly more complicated to calculate explicitly the quantities appearing in the $Z_3$ twisted sewed vertex, we are able to deduce the closed form of the four-vertex. The answer generalizes the previous results for the Ramond string [13, 15, 16] in a very natural way, and further generalizations of our results to fermions with other twists is probably straightforward.

The twisted four-vertex is constructed by sewing together two dual four-vertices[ 29, 10] emitting complex $Z_3$ twisted external fields with mode expansions

$$\hat{\psi}(z) = i \sum_{n \in \mathbb{Z}} \hat{\psi}_n z^{-n-\frac{1}{3}}, \quad \hat{\bar{\psi}}(z) = i \sum_{n \in \mathbb{Z}} \hat{\bar{\psi}}_n z^{-n-\frac{2}{3}}, \quad \{\hat{\psi}_n, \hat{\bar{\psi}}_m\} = \delta_{n+m,0}$$

\(^3\text{See equation (7) below.}\)
and hermiticity properties
\[ (\hat{\psi}_n)^\dagger = \hat{\psi}_n, \quad (\hat{\bar{\psi}}_n)^\dagger = \hat{\bar{\psi}}_n \]
(2)

The ket vacuum is defined by
\[ \hat{\psi}_n |0\rangle = 0, \quad \forall n > 0; \quad \hat{\bar{\psi}}_n |0\rangle = 0, \quad \forall n \geq 0 \]
(3)

and the bra vacuum follows from (2) and \[ \langle 0 | = (\langle 0 |)^\dagger \]. The fermions may therefore be divided into the following creation and annihilation parts
\[ \hat{\psi}^{(+)}(z) = i \sum_{n=1}^{\infty} \hat{\psi}_n z^{n-\frac{3}{2}}, \quad \hat{\psi}^{(-)}(z) = i \sum_{n=0}^{\infty} \hat{\psi}_{-n} z^{n-\frac{3}{2}} \]
(4)
\[ \hat{\bar{\psi}}^{(+)}(z) = i \sum_{n=0}^{\infty} \hat{\bar{\psi}}_n z^{n-\frac{3}{2}}, \quad \hat{\bar{\psi}}^{(-)}(z) = i \sum_{n=1}^{\infty} \hat{\bar{\psi}}_{-n} z^{n-\frac{3}{2}} \]
(5)

with non-zero commutation relations (\(|z| > |w|\))
\[ \{ \hat{\psi}^{(+)}(z), \hat{\psi}^{(-)}(w) \} = -\left( \frac{w}{z} \right)^{\frac{1}{3}} \frac{1}{z-w}, \quad \{ \hat{\psi}^{(+)}(z), \hat{\bar{\psi}}^{(-)}(w) \} = -\left( \frac{\bar{w}}{\bar{z}} \right)^{\frac{1}{3}} \frac{1}{\bar{z}-\bar{w}} \]
(6)

The dual four-Reggeon vertex for complex fields is given by (see [10, 15, 26])
\[ \hat{W}(V) = 1 \langle 0 | : e^{\int dz \left[ \hat{\psi}^{V}_{aux}(z)(\hat{\psi}(z) + i\hat{\psi}_1(z)) + \hat{\bar{\psi}}^{V}_{aux}(z)(\hat{\bar{\psi}}(z) + i\hat{\bar{\psi}}_1(z)) \right] } : |0\rangle \]
(7)

where \( \hat{\psi}_1 \) is the (untwisted) normal ordering field and the dots refer only to the untwisted aux field. As explained in [28] the normal ordering field must be untwisted to ensure that the vertex transforms correctly under projective transformations (compare to the discussion in [1] leading to the corresponding vertex in the NSR case). This means that when we use the Baker-Hausdorff (BH) formula to eliminate the normal ordering field, thereby explicitly normal ordering the external fields \( \hat{\psi}(z), \hat{\bar{\psi}}(z) \), the total propagator in the BH term will not vanish. Instead this will give rise to a term bilinear in the aux field. This is the origin of the technical problems that appear as soon as one tries to incorporate twisted fields. The dual vertex describes the emission or absorption from the world sheet of two external twisted states located at \( z_1 = V(\infty) \) and \( \hat{z}_2 = V(0) \). The aux field is taken to represent the world sheet itself.

\[ ^4 \text{From [10] it is clear that the derivation of the dual four-vertex is independent of the twist properties of the external field.} \]
The four-vertex is constructed by multiplying two dual vertices together and performing a vacuum correlation in the \( \text{aux} \) field

\[
\hat{W}_4 = \text{aux} \langle 0 | \hat{W}_1(V_1) \hat{W}_2(V_2) | 0 \rangle_{\text{aux}}
\]  

(8)

By eliminating the normal ordering field and inserting the coherent state completeness relation in the \( \text{aux} \) Hilbert space the correlation becomes an infinite dimensional integral

\[
\hat{W}_4 = \int d^2 \psi_r e^{-\frac{1}{2} \tilde{\psi}^T G \tilde{\psi}}
\]

(9)

where

\[
\tilde{\psi}^T = \left( \psi_r \quad \psi_{-r} \quad \tilde{\psi}_r \quad \tilde{\psi}_{-r} \right) ; \quad G = \begin{pmatrix}
0 & 0 & M^{(++)} & -1 \\
0 & 0 & 1 & M^{(--)} \\
-M^{(++)}T & -1 & 0 & 0 \\
1 & -M^{(--)}T & 0 & 0
\end{pmatrix}
\]

(10)

\[
u^T = \left( \bar{U}_r(+) (V_1) \quad \bar{U}_r(-) (V_2) \quad U_r(+) (V_1) \quad U_r(-) (V_2) \right)
\]

(11)

Here the matrices \( M^{(\pm\pm)} \) and vectors \( U^{(\pm)}, \bar{U}^{(\pm)} \) are given by

\[
M_{rs}^{(++)} = - \int_0^\infty dz \int_0^\infty dw z^{-\frac{1}{2}} w^{s-\frac{1}{2}} \left( \frac{V_r^{-1}(z)}{V_r^{-1}(w)} \right)^{\frac{1}{2}} - 1 \]

\[
M_{rs}^{(--)} = - \int_\infty^\infty dz \int_\infty^\infty dw z^{-\frac{1}{2}} w^{s-\frac{1}{2}} \left( \frac{V_r^{-1}(z)}{V_r^{-1}(w)} \right)^{\frac{1}{2}} - 1
\]

\[
\bar{U}_r(+) (V_1) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n \int_0^\infty dz \sqrt{V_1^{-1}(z)} z^{-\frac{1}{2}} (V_1^{-1}(z))^{-n-\frac{4}{3}}
\]

\[
\bar{U}_r(-) (V_2) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n \int_\infty^\infty dz \sqrt{V_2^{-1}(z)} z^{-\frac{1}{2}} (V_2^{-1}(z))^{-n-\frac{4}{3}}
\]

\[
U_r(+) (V_1) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n \int_0^\infty dz \sqrt{V_1^{-1}(z)} z^{-\frac{1}{2}} (V_1^{-1}(z))^{-n-\frac{1}{3}}
\]

\[
U_r(-) (V_2) = \sum_{n \in \mathbb{Z}} \tilde{\psi}_n \int_\infty^\infty dz \sqrt{V_2^{-1}(z)} z^{-\frac{1}{2}} (V_2^{-1}(z))^{-n-\frac{1}{3}}
\]

(12)

After performing the integral we find

\[
\hat{W}_4 = \text{det}(G) e^{-\frac{1}{2} \bar{u}^T G^{-1} \bar{u}}
\]

(13)
Due to the projective invariance we can make a choice of our projective transformations. This choice is arbitrary as long as the transformations contain only one moduli parameter $\lambda$. The standard choice is to let the emission points of the external states be located at $\infty, -\frac{1}{\lambda}, -\lambda$ and 0. This can be implemented in four different ways \cite{15} given in Table 1 where we have defined the matrix

$$M_{rs} = \frac{r - \frac{1}{6}}{r + s} \begin{pmatrix} -\frac{4}{3} & -\frac{2}{3} \\ r - \frac{1}{2} & s - \frac{1}{2} \end{pmatrix} \lambda^{r+s}$$

and the vectors

$$u^{(n)}_r = \lambda^{n+r+\frac{1}{6}} \begin{pmatrix} -n - \frac{2}{3} \\ r - \frac{1}{2} \end{pmatrix}, \quad v^{(n)}_r = \lambda^{n+r-\frac{1}{6}} \begin{pmatrix} -n - \frac{1}{3} \\ r - \frac{1}{2} \end{pmatrix}$$

For the choice denoted as $I$ we find that \cite{13} becomes

$$\hat{W}_4 = \det(1 - M^2) \exp \left( \sum_{n,m \in \mathbb{Z}} \left[ -i\tilde{\psi}_n^1 u^{(n)T} M(1 - M^2)^{-1} v^{(m)} \tilde{\psi}_m^1 + i\tilde{\psi}_n^1 u^{(n)T} (1 - M^2)^{-1} v^{(m)} \tilde{\psi}_m^2 - i\tilde{\psi}_n^2 u^{(n)T} (1 - M^2)^{-1} v^{(m)} \tilde{\psi}_m^1 - \tilde{\psi}_n^2 u^{(n)T} M(1 - M^2)^{-1} v^{(m)} \tilde{\psi}_m^2 \right] \right)$$

Table 1: Projective transformations with one modulus $\lambda$. 

| $z^{(1)}_1, z^{(1)}_2$ | $V_1^{-1}$ | $M^{(+)M}$ | $U^{(+)}_{nr}(V_1)$ | $U^{(+)}_{nr}(V_1)$ |
|-----------------------|------------|-------------|----------------------|----------------------|
| $I$                   | $-\frac{1}{\lambda}, \infty$ | $z + \frac{1}{\lambda}$ | $-MT$ | $\tilde{\psi}_n^1 u^{(n)}_r$ | $\tilde{\psi}_n^1 v^{(n)}_r$ |
|                       | $-\lambda, 0$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $MT$ | $-i\tilde{\psi}_n^2 u^{(n)}_r$ | $-i\tilde{\psi}_n^2 v^{(n)}_r$ |
| $II$                  | $\infty, -\frac{1}{\lambda}$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $M$ | $i\tilde{\psi}_n^1 v^{(-n)}_r$ | $i\tilde{\psi}_n^1 u^{(-n)}_r$ |
|                       | $-\lambda, 0$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $MT$ | $-i\tilde{\psi}_n^2 u^{(-n)}_r$ | $-i\tilde{\psi}_n^2 v^{(-n)}_r$ |
| $III$                 | $\infty, -\frac{1}{\lambda}$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $M$ | $i\tilde{\psi}_n^1 v^{(-n)}_r$ | $i\tilde{\psi}_n^1 u^{(-n)}_r$ |
|                       | $0, -\lambda$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $-M$ | $-i\tilde{\psi}_n^2 u^{(-n)}_r$ | $-i\tilde{\psi}_n^2 v^{(-n)}_r$ |
| $IV$                  | $-\frac{1}{\lambda}, \infty$ | $z + \frac{1}{\lambda}$ | $-MT$ | $\tilde{\psi}_n^1 v^{(n)}_r$ | $\tilde{\psi}_n^1 u^{(n)}_r$ |
|                       | $0, -\lambda$ | $\frac{1}{z + \frac{1}{\lambda}}$ | $-M$ | $-i\tilde{\psi}_n^2 u^{(n)}_r$ | $-i\tilde{\psi}_n^2 v^{(n)}_r$ |
By using the matrix identity

\[ A^{-1}(A - B)B^{-1} = B^{-1} - A^{-1} \]  \hspace{1cm} (17)

all quantities occurring in the exponent can be expressed in terms of the quantities

\[ u^{T(n)}\xi^{(m)}, \quad u^{T(n)}\eta^{(m)}, \quad n, m \in \mathbb{Z} \]  \hspace{1cm} (18)

where

\[ \xi^{(n)} = (1 + M)^{-1}v^{(n)}, \quad \eta^{(n)} = (1 - M)^{-1}v^{(n)} \]  \hspace{1cm} (19)

Similar quantities were introduced for the Ramond string in the 1970’s [4, 18] (for \( n = m = 0 \)) and generalized to arbitrary \( m, n \) in [15]. Following a procedure similar to that used for Ramond in reference [4] we will now derive an equation for \( \xi = \xi^{(0)} \). For this purpose we introduce in addition to \( M \) in (14) the matrices (note that in the Ramond case only one type of \( M \) matrix is needed)

\[
\tilde{M}_{rs} = \frac{s - \frac{1}{6}}{r + s} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ r - \frac{1}{2} & s + \frac{1}{2} \end{pmatrix} \lambda^{r+s} \\
\bar{M}_{rs} = \frac{r + \frac{1}{6}}{r + s} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ r - \frac{1}{2} & s + \frac{1}{2} \end{pmatrix} \lambda^{r+s} \\
\tilde{\bar{M}}_{rs} = \frac{s + \frac{1}{6}}{r + s} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ r - \frac{1}{2} & s + \frac{1}{2} \end{pmatrix} \lambda^{r+s} \\
\bar{\tilde{M}}_{rs} = \frac{r - \frac{1}{6}}{r + s} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} \\ r - \frac{1}{2} & s + \frac{1}{2} \end{pmatrix} \lambda^{r+s}
\]

\hspace{1cm} (20) \hspace{1cm} (21) \hspace{1cm} (22)

and \( R_{rs} = (r - \frac{1}{6})\delta_{r,s}, \quad \bar{R}_{rs} = (r + \frac{1}{6})\delta_{r,s} \) such that

\[ R^{-1}MR = \tilde{M} \quad \bar{R}^{-1}\bar{M}\bar{R} = \bar{\tilde{M}} \]  \hspace{1cm} (23)

Also

\[
\frac{\partial M_{rs}}{\partial \lambda} = \frac{Rvu^T}{\lambda} \quad \frac{\partial \tilde{M}_{rs}}{\partial \lambda} = \frac{\bar{R}vu^T}{\lambda}
\]

\hspace{1cm} (24)

and

\[ M_{rs} + \tilde{M}_{rs} = vu^T \quad \bar{M}_{rs} + \bar{\tilde{M}}_{rs} = vu^T \]  \hspace{1cm} (25)
We define vectors $\tilde{\xi}, \bar{\xi}$ etc. similarly as in (19). Using (17) with $A = 1 + \tilde{M}$ and $B = 1 - \bar{M}$ gives
\[
\tilde{\xi} = \frac{\tilde{\eta}}{1 + u^T \tilde{\eta}}
\]
and with $A = 1 - \bar{M}, B = 1 + M$ we find
\[
\bar{\eta} = \frac{\xi}{1 - u^T \xi}
\]
Then
\[
\frac{\partial \xi}{\partial \lambda} = R \tilde{\xi} \lambda \frac{1}{1 + u^T \tilde{\eta}} = R \bar{\eta} \lambda \frac{1}{1 - u^T \bar{\eta}}
\]
Rearranging the factors, multiplying with $\lambda^{-\frac{1}{2}}$ and taking another derivative with respect to $\lambda$ gives the equation
\[
\frac{\partial}{\partial \lambda} \left( \lambda^{-\frac{1}{2}} \frac{1 + u^T \bar{\eta}}{1 - u^T \xi} \frac{\partial \xi}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} \left( \frac{R \bar{\eta} \lambda^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}} (1 + u^T \bar{\eta}) \right) \xi
\]
To find $\xi_r$ we must make an ansatz for the unknown functions $u^T \xi$ and $u^T \bar{\eta}$. By examining their Taylor expansions we find that
\[
u^T \xi = 1 - \frac{(1 - \lambda)^{\frac{1}{2}}}{(1 + \lambda)^{\frac{1}{2}}}, \quad u^T \bar{\eta} = -1 + \frac{(1 + \lambda)^{\frac{1}{2}}}{(1 - \lambda)^{\frac{1}{2}}}
\]
Inserting these functions in (29) gives the solution for $\xi_r$
\[
\xi_r = \frac{\lambda^r - \frac{1}{2}}{(1 + \lambda)^{2r - \frac{1}{2}}} \left( \begin{array}{c} -\frac{1}{2} \\ r - \frac{1}{2} \end{array} \right) \frac{4 \lambda}{(1 + \lambda)^2} \text{ } _2F_1(r, r + 1, 2r + 1, \frac{4 \lambda}{(1 + \lambda)^2})
\]
The next logical step would be to use the solution (31) to calculate $u^T \xi$ and $u^T \bar{\eta}$ thus verifying the ansatz (30). Unfortunately, to perform the summation over $r$ is more complicated in this case than for the Ramond string. To see why this is so let us recall what happens for Ramond. For the Ramond string the solution for the corresponding quantity $\xi^R_r$ is [18]\[
\xi^R_r = \frac{\lambda^2}{(1 + \lambda)^{2r}} \left( \begin{array}{c} -\frac{1}{2} \\ r - \frac{1}{2} \end{array} \right) \frac{4 \lambda}{(1 + \lambda)^2} \text{ } _2F_1(r, r + 1, 2r + 1, \frac{4 \lambda}{(1 + \lambda)^2})
\]
By using the integral representation for the hypergeometric function $\xi^R_r$ becomes, through a miraculous cancellation of $\Gamma$-functions,
\[
\xi^R_r = \frac{1}{\sqrt{2}} \int_C \frac{dt}{2 \pi i t} \omega(\lambda, t)^r, \quad \omega(\lambda, t) = \frac{t - \frac{(1 + \lambda)^2}{4 \lambda}}{t(1 - t)}
\]
The function $\omega(\lambda, t)$ is such that it removes the square root cuts in the sum leaving only poles. The simplification that occurs in (33) does not have a direct analogue in the $\mathbb{Z}_3$ case. For $\mathbb{Z}_3$ it seems like we would need a function with the ability to remove cubic root cuts. This could hopefully be achieved by some change of variables in the integral representation of the hypergeometric function but this has so far eluded us. However, we have checked that the Taylor expansions agree to high order in $\lambda$ making an error unlikely.

The closed form of the Ramond vertex was first suggested in [7] for three-vertices and generalized to an arbitrary number of legs in [13] using path integral arguments. A direct sewing verification of these expressions was given in [15] for the four-Ramond vertex where it was shown how to calculate the quantities occurring in the exponent of the sewed vertex at any oscillator level. It was finally proved rigorously for the four-Ramond vertex in [16]. The complex four-Ramond vertex can be written

$$\hat{W}_{R_1, R_2} = \langle 0 | : S(z_2^{(1)})S(z_1^{(1)}) :: S(z_2^{(2)})S(z_1^{(2)}) : | 0 \rangle$$

$$: \exp \left( \oint_{C_z} dz \oint_{C_w} dw \hat{\psi}_R^{V^{-1}}(z) \langle 0 | \psi(z) \bar{\psi}(w) : S(z_2^{(1)})S(z_1^{(1)})S(z_2^{(2)})S(z_1^{(2)}) : | 0 \rangle \hat{\psi}_R^{V^{-1}}(w) \right) :$$

where $\psi$ and $S$ are the bosonized fermion and spin fields respectively. The $z_i^{(j)}$ are the insertion points of the external states corresponding to any of the choices listed in Table 1. The $\hat{\psi}_R^{V^{-1}}$’s are sums of the external twisted fields transformed with their respective projective transformations. The contours $C_z$ and $C_w$ enclose all branch cuts appearing in $\hat{\psi}_R^{V^{-1}}(z)$ and $\bar{\psi}_R^{V^{-1}}(w)$ respectively.

This expression of course suggests a natural closed form of the complex $\mathbb{Z}_3$ twisted vertex. We just replace the spin fields $S(z) = S_{\mathbb{Z}_3}(z)$ by their $\mathbb{Z}_3$ analogues, the twist fields $S_{\mathbb{Z}_3}(z) = e^{i \phi(z)}$. Thus we propose the following form of the $\mathbb{Z}_3$ four-Reggeon vertex:

$$\hat{W}_4 = \langle 0 | : \tilde{S}_{\mathbb{Z}_3}(z_2^{(1)})\tilde{S}_{\mathbb{Z}_3}(z_1^{(1)}) :: \tilde{S}_{\mathbb{Z}_3}(z_2^{(2)})\tilde{S}_{\mathbb{Z}_3}(z_1^{(2)}) : | 0 \rangle$$

$$: \exp \left( \oint_{C_z} dz \oint_{C_w} dw \hat{\psi}_R^{V^{-1}}(z) \langle 0 | \psi(z) \bar{\psi}(w) : \tilde{S}_{\mathbb{Z}_3}(z_2^{(1)})\tilde{S}_{\mathbb{Z}_3}(z_1^{(1)})\tilde{S}_{\mathbb{Z}_3}(z_2^{(2)})\tilde{S}_{\mathbb{Z}_3}(z_1^{(2)}) : | 0 \rangle \hat{\psi}_R^{V^{-1}}(w) \right) :$$

The normal ordering is w.r.t. the bosonic field in terms of which the spin fields $S, \tilde{S}$ and fermion fields $\psi, \bar{\psi}$ are bosonized. This normal ordering is thus completely independent of the one indicated by the double dots outside the exponential and which refers to the external Ramond fields $\psi_R, \bar{\psi}_R$. Note that the bars on the external fields have moved compared to reference [13]. This is because we have placed the zero modes differently.
\[
\begin{array}{|c|c|c|}
\hline
n, m & I & IV \\
\hline
0, 0 & i\lambda (1 - \lambda^2)^{-\frac{1}{3}} & -\lambda^\frac{2}{3} \\
1, 0 & i\lambda^2 (1 - \lambda^2)^{-1} & -\lambda^\frac{3}{2} (1 - \lambda^2)^{-\frac{1}{3}} \\
0, 1 & i\lambda^2 (1 - \lambda^2)^{-\frac{1}{3}} (1 + \frac{\lambda^2}{3} (1 - \lambda^2)^{-1}) & -\lambda^\frac{4}{3} (1 - \lambda^2)^{-\frac{1}{3}} \\
1, 1 & i\lambda^3 (1 - \lambda^2)^{-1} (1 + \frac{4 \lambda^2}{3} (1 - \lambda^2)^{-1}) & (1 - \lambda^2)^{-\frac{1}{3}} (\frac{4 \lambda^3}{3} - \lambda^\frac{2}{3}) \\
-1, 0 & i (1 - \lambda^2)^{-\frac{1}{3}} (1 - \frac{2 \lambda^2}{3}) & -\lambda^\frac{4}{3} (1 - \lambda^2)^{-\frac{1}{3}} \\
0, -1 & i & -\lambda^\frac{3}{2} (1 - \lambda^2)^{-\frac{1}{3}} \\
-1, -1 & i (\frac{1}{3} + \frac{\lambda^2}{3}) & (1 - \lambda^2)^{-\frac{1}{3}} (\frac{4 \lambda^3}{3} - \lambda^\frac{2}{3}) \\
-1, 1 & i\lambda^2 (1 - \lambda^2)^{-\frac{1}{3}} \{ (1 - \frac{2 \lambda^2}{3}) (\frac{1}{3} + \frac{\lambda^2}{3} (1 - \lambda^2)^{-1}) - \lambda \} & -\lambda^\frac{4}{3} - \lambda^\frac{5}{3} - \lambda^\frac{5}{9} \\
1, -1 & i\lambda (1 - \lambda^2)^\frac{1}{3} & -\lambda (1 - \lambda^2)^{-\frac{2}{3}} \\
\hline
\end{array}
\]

Table 2: Coefficients of $\hat{\psi}_n^1 \hat{\psi}_m^2$ for two different choices of projective transformations.

We have verified that this is indeed the correct form of the vertex by calculating the functions occurring in the exponent for level zero and one oscillators (see Table 2 for some examples) and comparing their Taylor expansions to those obtained directly from the sewing expression (13). The expansion of the determinant prefactor in (13) also agrees with the correlation in (35) indicating that

\[
\det(1 + M^T M) = \det(1 + MM^T) = (1 - \lambda^2)^{-\frac{1}{3}}
\]

(36) \[ \det(1 - M^2) = (1 - \lambda^2)^{\frac{1}{3}} \] (37)

If we rewrite (35) in terms of the projective transformations $V_1$ and $V_2$ we find the alternative closed form of the vertex

\[
\hat{W}_4 = \left( V_2^{-1}(z_1^{(1)}) \right)^\frac{1}{3} : e^{\int_c dz \int_c dw \bar{\psi}^{V^{-1}}(z) G(V_1, V_2, z, w) \psi^{V^{-1}}(w) } : 
\]

(38)

where the propagator $G(V_1, V_2, z, w)$ is given by

\[
G(V_1, V_2, z, w) = \frac{V_2^{-1}(z) V_2^{-1}(w)}{V_1^{-1}(w) V_2^{-1}(w)} \frac{1}{z - w} 
\]

(39)
Although we have not proved that this is the closed form of the vertex we feel that there could be little doubt as to its correctness. A rigorous proof along the lines of [16] should be possible. The technical obstacles encountered do not appear insurmountable and remain objects of further study. It should also be straightforward to perform the steps presented here for fermions with arbitrary $\mathbb{Z}_N$ twists.

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