TOPOLOGICAL HOCHSCHILD HOMOLOGY OF \( K/p \) AS A \( K_p^\wedge \) MODULE

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ABSTRACT. Let \( R \) be an \( E_\infty \)-ring spectrum. Given a map \( \zeta \) from a space \( X \) to \( BGL_1R \), one can construct a Thom spectrum, \( X^\zeta \), which generalises the classical notion of Thom spectrum for spherical fibrations in the case \( R = S^0 \), the sphere spectrum. If \( X \) is a loop space (\( \simeq \Omega Y \)) and \( \zeta \) is homotopy equivalent to \( \Omega f \) for a map \( f \) from \( Y \) to \( B^2GL_1R \), then the Thom spectrum has an \( A_\infty \)-ring structure. The Topological Hochschild Homology of these \( A_\infty \)-ring spectra is equivalent to the Thom spectrum of a map out of the free loop space of \( Y \).

This paper considers the case \( X = S^1 \), \( R = K_p^\wedge \), the \( p \)-adic \( K \)-theory spectrum, and \( \zeta = 1 - p \in \pi_1BGL_1K_p^\wedge \). The associated Thom spectrum \( (S^1)^\zeta \) is equivalent to the mod \( p \) \( K \)-theory spectrum \( K/p \). The map \( \zeta \) is homotopy equivalent to a loop map, so the Thom spectrum has an \( A_\infty \)-ring structure. I will compute \( \pi_*THH_{K_p^\wedge}(K/p) \) using its description as a Thom spectrum.

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1. Introduction

The goal of this paper is to use generalised Thom spectra to calculate the Topological Hochschild Homology of \( K/p \) in the category of modules over \( K_p^\wedge \).

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Let $R$ be a ring spectrum and $GL_1R$ its space of units. It is the $H$-space of homotopy automorphisms of $R$ as an $R$-module. An $R$-twisting of a space $X$ is a continuous map $\zeta$ from $X$ to $BGL_1R$. Associated to $\zeta$, one can define the Thom spectrum of $\zeta$, $X^\zeta$ (see [2]). This notion specialises for $R = S^0$ to the Thom spectrum of a spherical fibration. The homotopy groups of $X^\zeta$ is the group of twisted $R$ homology classes with respect to the twisting $\zeta$.

Suppose that $R$ is an $E_\infty$-ring spectrum. Then its space of units is an infinite loop space. Given a map $f : BG \to B^2GL_1R$, let $\zeta \simeq \Omega f : G \to BGL_1R$. Then the Thom spectrum $G^\zeta$ admits an $A_\infty$ $R$-algebra structure.

1.1. $K/p$ as a module over $K^\wedge_p$. Suppose that $R = K^\wedge_p$, the spectrum of $p$-adic K-theory. Let $G$ be the group $S^1$. A twisting on $S^1$ is a map $\zeta : S^1 \to BGL_1K^\wedge_p$. This is classified by the group $\pi_1(BGL_1K^\wedge_p) \cong \pi_0(GL_1K^\wedge_p) \cong Z^p$. If we choose $\zeta = 1 - p \in Z^p$, then the Thom spectrum $(S^1)^\zeta \simeq K/p$, the mod $p$ $K$-theory spectrum. Moreover, the twisting $\zeta$ can be realised as a loop map, and so, for every way of writing $\zeta \simeq \Omega f$ we get an $A_\infty$-ring structure on $K/p$ as an $K^\wedge_p$-module.

1.2. Topological Hochschild Homology of Thom spectra. Given a map $f$ from $X$ to $B^2GL_1R$, let $G \simeq \Omega X$ and $\zeta \simeq \Omega f : G \simeq \Omega X \to BGL_1R$. In this case, the Thom spectrum $G^\zeta$ has an $A_\infty$-ring structure. We write $\eta^*f$ for the composite

$$LX \to LB^2GL_1R \overset{\cong}{\longrightarrow} B^2GL_1R \times BGL_1R \quad \downarrow \eta \times id \quad \cong$$

$$BGL_1R \times BGL_1R \longrightarrow BGL_1R$$

where $\eta : \Sigma R \to R$ is induced from $S^1 \overset{\eta}{\longrightarrow} S^0$ via $S^1 \wedge R \to S^0 \wedge R \simeq R$. In the above situation, $THH^R(G^\zeta) \simeq LX^{\eta^*f}$. The case $R = S^0$ was proved in [4]. The same argument applies for general $R$.

Using this identification of $THH$ as a Thom spectrum, we compute the Topological Hochschild Homology of $K/p$. For odd primes $p$,

$$\pi_*(THH_{K^\wedge_p}(K/p)) = \left\{ \begin{array}{ll} (Z/(p^\infty))^i & \text{if } * = 2k \\ 0 & \text{if } * = 2k + 1 \end{array} \right.$$ 

where $i$ is an integer between 1 and $p - 1$ depending on the choice of $f$ with $\zeta \simeq \Omega f$.

Similar results were obtained before by Angeltveit in [1]. He used the Bökstedt spectral sequence (see [5], chapter IX).

We can also form mod $p$ $K$-theory as a Thom spectrum by starting with $X = S^3$, $R = K^\wedge_p$ and $\zeta = p \in \pi_3(BGL_1K^\wedge_p) = \pi_2(GL_1K^\wedge_p) = Z_p$. Again, this $\zeta$ can be realised as a loop map and we can compute $THH$ of these $A_\infty$-ring structures in an analogous way. This gives the same results.
2. The Thom spectrum

The notion of a generalised Thom spectrum used here is discussed in detail in [2]. The construction resembles a twisted version of the group ring. Given an extension of a group \( G \) by the units in a field \( k \),

\[
(\tau) : 1 \to k^* \to E \to G \to 1
\]

the algebra \( k^*[G] = \mathbb{Z}[E] \otimes_{\mathbb{Z}[k^*]} k \) is a twisted group ring. If the extension \( \tau \) is trivial, one gets the group ring \( k[G] \). Imitating this definition of a twisted group ring for spectra leads to the construction of the Thom spectrum. One replaces the field \( k \) by an \( E_\infty \)-ring spectrum \( R \), and the units \( k^* \) by the space of units \( GL_1 \) acting on \( R \).

2.1. The space of units and the Thom spectrum. The space of units of a ring spectrum is a generalisation of the group of units of a commutative ring, the set of invertible elements under multiplication. It is defined to be the components of \( \Omega^\infty R \) that lie over the units in \( \pi_0(R) \). Following [2], we make the definition

**Definition 2.1.** Let \( R \) be an \( E_\infty \)-ring spectrum. Its space of units \( GL_1 R \) is defined to be the pullback,

\[
\begin{array}{ccc}
GL_1 R & \longrightarrow & \Omega^\infty(R) \\
\downarrow & & \downarrow \\
\pi_0(R)^x & \longrightarrow & \pi_0(R)
\end{array}
\]

It follows from the definition that the homotopy classes of maps from a space \( X \) to \( GL_1 R \) are given by

\[ [X, GL_1 R] = R^0(X)^x \]

the units of the cohomology ring \( R^0(X) = [X, \Omega^\infty R] \).

From the pullback diagram one can read off the homotopy groups of \( GL_1 R \),

\[
\pi_n(GL_1 R) = \begin{cases} 
\pi_n(R) & \text{if } n > 0 \\
\pi_0(R)^x & \text{if } n = 0
\end{cases}
\]

We note that \( GL_1 R \) is an \( H \)-space for any ring spectrum \( R \). If \( R \) is \( E_\infty \), then \( GL_1 R \) is an infinite loop space: there is a connective spectrum \( gl_1 R \) with 0th-space is \( GL_1 R \) (Theorem 3.2 in [2]).

We can view \( \Omega^\infty R \) as the space of endomorphisms \( End_R(R, R) \), in the topological category of \( R \)-modules, and \( GL_1 R = Aut_R(R, R) \subset End_R(R, R) \) as the subset of weak equivalences. Therefore, the units \( GL_1 R \) is the space of homotopy automorphisms of \( R \) in the category of \( R \)-modules. In this way, the infinite loop space \( GL_1 R \) acts on the spectrum \( R \) by weak equivalences, and \( R \) is a module over the \( E_\infty \) ring spectrum \( \Sigma^\infty GL_1 R_+ \).

**Definition 2.2.** Given a map \( \zeta : X \to BGL_1 R \), let \( P \) be the \( GL_1 R \) bundle classified by \( \zeta \) described as the pullback,
and define the associated Thom spectrum to be

\[ X^\zeta = \Sigma^\infty P_+ \wedge_L \Sigma^\infty GL_1(R)_+ R \]

In the above \( \wedge_L \) denotes the derived smash product in the category of modules over the \( E_\infty \)-ring spectrum \( \Sigma^\infty GL_1R_+ \) as in [5]. We note from section 7 of [2], that the Thom spectrum functor commutes with homotopy colimits, and from section 8.6 of [2] that it generalises the classical Thom spectrum of a spherical fibration.

The Thom spectrum of the map \( * \to BGL_1R \) is weakly equivalent to \( R \), since the universal bundle associated to the inclusion of a point in \( BGL_1R \) is isomorphic to \( GL_1R \) and \( \Sigma^\infty GL_1R_+ \wedge_L \Sigma^\infty GL_1R_+ R \simeq R \).

Similarly, the Thom spectrum of a map \( X \to BGL_1R \) which is null homotopic is weakly equivalent to \( R \wedge X_+ \). Indeed, the universal bundle associated to the constant map is \( X \times GL_1R \). Then the Thom spectrum is \( \Sigma^\infty( X \times GL_1R)_+ \wedge_L \Sigma^\infty GL_1R_+ R \simeq (\Sigma^\infty X_+ \wedge \Sigma^\infty GL_1R_+) \wedge_L \Sigma^\infty GL_1R_+ R \simeq R \wedge X_+ \).

Suppose that the space \( X \simeq \Sigma Y \), the reduced suspension on \( Y \). Then, a map \( X \xrightarrow{\zeta} BGL_1R \) is described by a map \( Y \xrightarrow{\tilde{\zeta}} GL_1R \), via \([X, BGL_1R] \cong [\Sigma Y, BGL_1R] \cong [Y, GL_1R]\). Such a \( \tilde{\zeta} \) is a unit in \( R^0(Y) \) which induces \( u_\zeta : R \wedge Y_+ \to R \).

**Proposition 2.3.** Suppose that \( \zeta \) is a map from \( X \simeq \Sigma Y \) to \( BGL_1R \). Then, the Thom spectrum \( X^\zeta \) is equivalent to the homotopy colimit of \((R \leftarrow R \wedge Y_+ \to R)\) where one of the maps is the projection \( p_Y \) and the other is \( u_\zeta \).

**Proof.** The space \( X \) is the homotopy colimit of \( * \leftarrow Y \to * \), and this gives a homotopy pushout square of Thom spectra,

\[
\begin{array}{ccc}
Y^\zeta & \longrightarrow & \#^\zeta \\
\downarrow & & \downarrow \\
\#^\zeta & \longrightarrow & (\Sigma Y)^\zeta
\end{array}
\]

The Thom spectrum \( *^\zeta \) is weakly equivalent to \( R \) and \( Y^\zeta \simeq R \wedge Y_+ \), so the homotopy pushout can be written as
From this, one obtains a Mayer Vietoris sequence for calculating the homotopy groups

\[ \ldots \to \pi_*(R \wedge Y) \to \pi_*(R) \oplus \pi_*(R) \to \pi_*(\Sigma Y) \to \ldots \]

To compute the maps in this sequence, one must examine the $GL_1 R$-bundle over $X \simeq \Sigma Y$. This restricts to trivial bundles over the two copies of the cone of $Y$ inside $X$ and on their intersection $Y$, the bundles are identified via the map $\hat{\zeta} : Y \to GL_1 R$.

In the long exact sequence, there are two maps $R^*(Y) \to \pi_*(R)$. One of these maps is given by the map from $Y$ to a point($p_Y$) and the other is the map $u_\zeta$ defined in the preceding paragraph.

Remark 2.4. The proposition describes the homotopy groups of the Thom spectrum as twisted $R$-homology groups. An $R$-twisting on a space $X$ can be defined as a 1-cocycle in the sheaf (of groupoids) $\{\text{units in } R^0(X)\}$. The groupoid of units in $R^0$ is classified by the units $GL_1 R$, and therefore, 1-cocycles on $X$ are equivalent to $[X, BGL_1 R]$. Therefore, a twisting is given by a continuous map $\zeta$ from $X$ to $BGL_1 R$.

For $X = \bigcup U_i$ a 1-cocycle defines units over $U_i \cap U_j$ satisfying a cocycle condition on further intersections. A twisted $R$ homology class is an element in each $R^*(U_i)$, two of which are identified using the values of the 1-cocycle on the intersections. The abelian group of these classes is defined to be the twisted $R$-homology of $X$ with respect to the twisting $\zeta$. This is isomorphic to the homotopy groups of the Thom spectrum $X^\zeta$. The proposition above verifies this in the case $X = \Sigma Y$, where $X$ is the union of two contractible open sets.

2.2. Computations of some Thom spectra.

Proposition 2.5. Suppose that $\zeta : S^1 \to BGL_1 K_p^\wedge$ represents $1 - p \in \pi_1(BGL_1(K_p^\wedge)) = \pi_0(GL_1(K_p^\wedge)) = \mathbb{Z}_p^\times$. Then, $(S^1)^\zeta \simeq K/p$.

Proof. By Proposition 2.3 with $Y = S^0$, the Thom spectrum is a homotopy pushout

\[ \begin{array}{ccc}
K_p^\wedge \lor K_p^\wedge & \to & K_p^\wedge \\
\downarrow & & \downarrow \\
K_p^\wedge & \to & (S^1)^\zeta
\end{array} \]

Therefore, there is a cofibre sequence

\[ K_p^\wedge \lor K_p^\wedge \to K_p^\wedge \lor K_p^\wedge \to (S^1)^\zeta \]
Proposition 2.3 also identifies the left map in the sequence in suitable coordinates, to be given by the matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 1 - p
\end{pmatrix}
\]

Therefore, the cofibre sequence can be rewritten as

\[K_p^\wedge p \to K_p^\wedge \to (S^1)^\zeta\]

so that \((S^1)^\zeta \simeq K_p^\wedge / p \simeq K/p\). \qed

Remark 2.6. Consider the map \(\zeta : S^1 \to BGL_1((S^0)^\wedge_p)\) given by \((1 - p)\) as in the previous proposition. Then, \((S^1)^\zeta \simeq (S^0)^\wedge_p / p \simeq M_p\) is the mod \(p\) Moore spectrum. In fact, for any \(\zeta : S^1 \to BGL_1R\), \((S^1)^\zeta \simeq cofibre(1 - \zeta : R \to R)\). This follows from the argument above.

Proposition 2.7. Let \(\zeta : S^3 \to BGL_1K^\wedge_p\) represent the element \(p\) of

\[\pi_3(BGL_1(K^\wedge_p)) = \pi_2(GL_1(K^\wedge_p)) = \pi_2(K^\wedge_p) \cong \mathbb{Z}/p\]

Then \((S^3)^\zeta \simeq K/p\).

Proof. The space \(S^3\) is homotopy equivalent to the suspension of \(S^2\). Proposition 2.3 implies the homotopy pushout

\[
\begin{array}{c}
K_p^\wedge \wedge S^2_p \rightarrow K_p^\wedge \\
\downarrow \quad \downarrow \\
K_p^\wedge \rightarrow (S^3)^\zeta
\end{array}
\]

and the associated Mayer Vietoris cofibre sequence

\[K_p^\wedge \wedge (S^2) \vee K_p^\wedge \rightarrow K_p^\wedge \vee K_p^\wedge \rightarrow (S^3)^\zeta.\]

In suitable coordinates, the map in the Mayer Vietoris sequence is given by the matrix

\[
\begin{pmatrix}
1 & 0 \\
1 & p
\end{pmatrix}
\]

and the sequence can be rewritten as

\[\Sigma^2 K_p^\wedge p \to K_p^\wedge \to (S^3)^\zeta\]

By Bott periodicity \(\Sigma^2 K_p^\wedge \simeq K_p^\wedge\) so that \((S^3)^\zeta \simeq K_p^\wedge / p\), as claimed. \qed

2.3. Ring Structures. Suppose \(R\) is an \(E_\infty\)-ring spectrum so that \(GL_1R\) is an infinite loop space. Given \(f : X \to B^2GL_1R\), and \(\zeta : G \simeq \Omega X \xrightarrow{\Omega f} BGL_1R\), the Thom spectrum \(G^\zeta\) has an \(A_\infty\)-ring structure. This follows from [3] and [2]. This raises the question when a map

\[\zeta : G \to BGL_1R\]

from a monoid \(G\) is homotopy equivalent to a loop map, i.e. \(\zeta \simeq \Omega f\) for

\[f : BG \to B^2GL_1R.\]

We have the standard maps

\[\Sigma G \xrightarrow{\sigma} BG, \Sigma GL_1R \xrightarrow{\sigma} BGL_1R\]
so the question is if
\[ \sigma \circ \Sigma \zeta : \Sigma G \to B^2GL_1R \]
extends over \( BG \),

\[
\begin{array}{ccc}
\Sigma G & \xrightarrow{\Sigma \zeta} & \Sigma BGL_1(R) \\
\downarrow \sigma & & \downarrow \sigma \\
BG & \xrightarrow{f} & B^2GL_1(R)
\end{array}
\]

**Proposition 2.8.** Let \( G = S^1 \), \( R = K_p^\wedge \) and \( \zeta = 1 - p \) as in Proposition 2.5, then \((S^1)^\zeta \simeq K/p\) has an \( A_\infty \)-ring structure.

*Proof.* The classifying space of \( S^1 \) is \( CP^\infty \) so, in this case the diagram above is

\[
\begin{array}{ccc}
S^2 & \xrightarrow{\Sigma(1-p)} & \Sigma BGL_1(K_p^\wedge) \\
\downarrow \sigma & & \downarrow \sigma \\
CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge)
\end{array}
\]

The space \( CP^\infty \) has a CW structure made of even dimensional cells so that all the cells are attached along odd dimensional spheres. The spectrum \( K_p^\wedge \) has non trivial homotopy groups only in even dimensions and hence, so does \( B^2GL_1K_p^\wedge \). Thus, all the obstructions to extending the map \( \Sigma 1 - p \) must vanish, which implies that there is an \( A_\infty \)-ring structure on the Thom spectrum \( K/p \).

**Proposition 2.9.** Suppose that \( G = S^3 \), \( R = K_p^\wedge \), and \( \zeta = p \) as in Proposition 2.7, then the Thom spectrum has an \( A_\infty \)-ring structure.

*Proof.* The classifying space of \( S^3 \) is the infinite quaternionic projective space \( HP^\infty \), and \( \Sigma S^3 = S^1 \to BS^3 = HP^\infty \) is obtained by attaching even cells along maps of odd dimensional spheres. Therefore the extension problem can always be solved. \( \square \)

### 3. Topological Hochschild Homology of Thom Spectra

In the last section, we observed that the Thom spectrum of a loop map carries an induced \( A_\infty \) structure. In this setting, there is a convenient description of the Topological Hochschild Homology as a Thom spectrum in the ideas of [4] and [10]. In the following \( G \) will be a group, \( X \) a space, and \( G \) homotopy equivalent to \( \Omega X \) as \( A_\infty \)-spaces. \( R \) will be an \( E_\infty \) ring spectrum.

The Thom spectrum of a map \( G \to BGL_1R \) is a twisted \( R \)-module generated by \( G \). If this is a loop map, the construction is that of a twisted group ring. Recall that the Hochschild Homology of group rings over a field is given by

\[ HH_*(k[G]) \cong k \otimes H_*(G, G) \]

where \( G \) acts on itself by conjugation. This is the homology of the Borel construction \( G_{hG} \simeq EG \times_G G \simeq LBG \), the free loop space of \( BG \), and so, \( HH_*(k[G]) \cong k \otimes H_*(LBG) \).
The analogous statement for Topological Hochschild Homology is the classical result of Bökstedt and Waldhausen,

\[ \text{THH}(\Sigma^\infty \Omega X_+) \simeq \Sigma^\infty LX_+. \]

In the category of \( R \)-modules, the theorem is \( \text{THH}^R(R \wedge \Omega X_+) \simeq R \wedge LX_+ \), computing the Topological Hochschild Homology of the Thom spectrum of the constant map. More generally, let \( f : X \to BGL_1 R \) and \( \zeta \simeq \Omega f : G \to BGL_1 R \), the Thom spectrum has an \( A_\infty \)-ring structure, and the Topological Hochschild Homology is the Thom spectrum of a map from \( LX \) to \( BGL_1 R \).

In the second part of the section, we apply the theorem for \( R = K^\wedge_p \) and \( G = S^1 \), in the computation of the previous section. This implies that the Thom spectrum is homotopy equivalent to the cofibre of a certain map \( K^\wedge_p \wedge CP^\infty_+ \to K^\wedge_p \wedge CP^\infty_+ \).

3.1. **Identifying Topological Hochschild Homology as a Thom spectrum.** Recall that the free loop space \( LY \) fits into a fibration

\[ \Omega Y \to LY \to Y \]

If \( Y \) is an \( H \)-space, then the fibration splits as \( LY \simeq Y \times \Omega Y \). This is an equivalence of \( H \)-spaces if \( Y \) is homotopy commutative.

Let \( f \) be a map from \( X \) to \( B^2GL_1 R \) and \( \eta : B^2GL_1 R \to \Omega B^2GL_1 R \) be induced from the Hopf map by

\[ B^2GL_1 R \simeq \text{Maps}(S^2, B^4GL_1 R) \xrightarrow{\eta^*} \text{Maps}(S^2, B^4GL_1 R) \]
\[ \simeq \text{Maps}(S^1, \Omega^2 B^4GL_1 R) \]
\[ \simeq \text{Maps}(S^1, B^2GL_1 R) \]
\[ \simeq \Omega B^2GL_1 R. \]

Let \( L^nf \) be the map from \( LX \) to \( BGL_1 R \) defined by the diagram

\[
\begin{array}{ccc}
LX & \xrightarrow{L^nf} & LB^2GL_1(R) \\
\downarrow & & \downarrow \simeq \\
\Omega B^2GL_1(R) & & BGL_1(R)
\end{array}
\]

The map \( \eta \times id : B^2GL_1 R \times \Omega B^2GL_1 R \to \Omega B^2GL_1 R \) is the product of the maps \( \eta \) and \( id \) using the \( H \)-space structure of \( \Omega B^2GL_1 R \). Without proof, we state:

**Theorem 3.1.** There is a homotopy equivalence

\[ \text{THH}^R(G^\zeta) \simeq (LX)^{L^nf} \]
This was proved in the case of the sphere spectrum in [4], [10]. A similar argument applies for any $E_\infty$-ring spectrum $R$. This will be accomplished in a future publication.

3.2. The example of $G = S^1$ and $R = K_p^\wedge$. By Proposition 2.8, we have the commutative diagram,

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\Sigma 1 - p} & \Sigma BGL_1(K_p^\wedge) \\
\downarrow & & \downarrow \\
CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge)
\end{array}
$$

and write $THH_K^p(K/p, f)$ for the Topological Hochschild Homology corresponding to this $A_\infty$-ring structure.

**Proposition 3.2.**

$$THH_K^p(K/p, f) \simeq (LCP^\infty)^{\hat{f}}$$

where $\hat{f}$ is the composite,

$$LCP^\infty \xrightarrow{L_\hat{f}} LB^2GL_1K_p^\wedge \simeq B^2GL_1K_p^\wedge \times BGL_1K_p^\wedge \xrightarrow{\eta} BGL_1K_p^\wedge$$

**Proof.** By Theorem 3.1, $THH_K^p(K/p, f) \simeq (LCP^\infty)^{L\eta f}$. Since $\pi_1(K_p^\wedge) = 0$, $\eta = 0$ in this case. Hence, the proposition. \qed

The focus of the rest of the paper will be the calculation of $\pi_*(LCP^\infty)^{\hat{f}} \simeq THH_K^p(K/p, f)$. First of all we note that:

**Proposition 3.3.** There is a long exact sequence

$$K_p^\wedge CP^\infty \to K_p^\wedge CP^\infty \to \pi_*THH_K^p(K/p, f) \to K_p^\wedge \cap CP_{p+1}^\infty \ldots$$

**Proof.** Note that $CP^\infty$ is an infinite loop space, and hence homotopy commutative, which implies that $LCP^\infty \simeq \Omega CP^\infty \times CP^\infty \simeq S^1 \times CP^\infty$. The space $S^1$ is a union of two contractible open sets whose intersection is $S^0$, so, there is a homotopy pushout,

$$
\begin{array}{ccc}
CP^\infty \sqcup CP^\infty & \to & CP^\infty \\
\downarrow & & \downarrow \\
CP^\infty & \to & LCP^\infty
\end{array}
\tag{*}
$$

and hence, a homotopy pushout square of Thom spectra
The two maps $CP^\infty \to LCP^\infty$ in (*) are the inclusion of constant loops, so, the two compositions $CP^\infty \to LCP^\infty \to LB^2GL_1K^\wedge_p \to BGL_1K^\wedge_p$ are nullhomotopic and the Thom spectra are $\simeq K^\wedge_p \land CP^\infty_+$. The map from $CP^\infty \sqcup CP^\infty$ to $BGL_1K^\wedge_p$ factors through $CP^\infty \to BGL_1K^\wedge_p$ so, the Thom spectrum $(CP^\infty \sqcup CP^\infty)^f \simeq K^\wedge_p \land CP^\infty_+ \land K^\wedge_p \land CP^\infty_+$. Therefore, the pushout can be written as:

$$
\begin{array}{c}
K^\wedge_p \land CP^\infty_+ \land K^\wedge_p \land CP^\infty_+ \to K^\wedge_p \land CP^\infty_+ \\
K^\wedge_p \land CP^\infty_+ \to (LCP^\infty)^{Lf}
\end{array}
$$

This gives a Mayer Vietoris sequence on homotopy groups,

$$
\ldots \to K^\wedge_p(CP^\infty) \oplus K^\wedge_p(CP^\infty) \to K^\wedge_p(CP^\infty) \oplus K^\wedge_p(CP^\infty) \to \pi_*(LCP^\infty)^{Lf} \ldots
$$

To simplify, one needs to understand the left hand map i.e., how $K^\wedge_p \land CP^\infty_+ \land K^\wedge_p \land CP^\infty_+$ maps to the two different copies of $K^\wedge_p \land CP^\infty_+$ in the pushout square. For that one needs to examine the structure of $Pf$, the $GL_1K^\wedge_p$-bundle over $S^1 \times CP^\infty$ classified by $\tilde{f}$.

Following the pushout square (*), we see that $Pf$ is obtained by identifying two trivial bundles over $CP^\infty$ after restricting over $CP^\infty \sqcup CP^\infty$, via a map $u : CP^\infty \sqcup CP^\infty \to GL_1K^\wedge_p$. The adjoint of $u$ is the map $\tilde{u}$ in the diagram,

$$
\begin{array}{c}
CP^\infty \sqcup CP^\infty \to CP^\infty \land CP^\infty \to S^1 \times CP^\infty \to \Sigma CP^\infty_+ \land \Sigma CP^\infty_+ \\
\downarrow 0 \quad \downarrow \tilde{u} \\
BGL_1K^\wedge_p
\end{array}
$$

The top row is the cofibre sequence associated to the pushout (*). Since the map $S^1 \times CP^\infty \to BGL_1K^\wedge_p$ is nullhomotopic on $CP^\infty \land CP^\infty$, it factors through $\Sigma CP^\infty_+ \land \Sigma CP^\infty_+$ as $\tilde{u}$.

The map $u$ gives two units $u_1, u_2$ in the $K^\wedge_p(0)(CP^\infty)$. In the Mayer Vietoris sequence for the Thom spectrum, these describe the map $K^\wedge_p \land CP^\infty_+ \land K^\wedge_p \land CP^\infty_+ \to K^\wedge_p \land CP^\infty_+ \land K^\wedge_p \land CP^\infty_+$ as the matrix,

$$
\begin{pmatrix}
1 & u_2 \\
u_1 & 1
\end{pmatrix}
$$
In fact, \( u_1 \) and \( u_2 \) are equal because each summand in \( \Sigma CP^\infty_+ \) of \( \Sigma CP^\infty_+ \vee \Sigma CP^\infty_+ \) is the cofibre of the map \( CP^\infty \rightarrow LCP^\infty = S^1 \times CP^\infty \) given by the inclusion of the constant loops and both can be defined by the same diagram,

\[
\begin{array}{ccc}
CP^\infty & \xrightarrow{0} & S^1 \times CP^\infty \\
\downarrow & & \downarrow \tilde{f} \\
\Sigma CP^\infty & \xrightarrow{u} & BGL_1(K_p^\wedge)
\end{array}
\]

In terms of \( u \), we can rewrite the Mayer Vietoris sequence as the long exact sequence,

\[
\ldots \rightarrow K^\wedge_{p*}(CP^\infty) \xrightarrow{u - 1} K^\wedge_{p*}(CP^\infty) \rightarrow \pi_*(LCP^\infty)^L f) \rightarrow \ldots \quad (\alpha)
\]

To calculate \( \pi_*(\text{THH}^{K_p}(K/p, f)) \), it remains to understand the map \( u \). This is done as follows:

**Proposition 3.4.** The adjoint of the map \( u : \Sigma CP^\infty_+ \rightarrow BGL_1R \), is homotopy equivalent to the composite \( \Sigma^2 CP^\infty_+ \xrightarrow{\mu} CP^\infty \xrightarrow{f} B^2GL_1K_p^\wedge \), where \( \mu \) is the composition \( \Sigma^2 CP^\infty_+ \simeq S^2 \wedge CP^\infty_+ \xrightarrow{\alpha \wedge id} CP^\infty \wedge CP^\infty_+ \rightarrow CP^\infty \).

**Proof.** The following diagram commutes:

\[
\begin{array}{ccc}
S^1 \wedge (S^1 \times CP^\infty) & \xrightarrow{\simeq} & S^1 \wedge LCP^\infty \\
\downarrow ev & & \downarrow ev \\
CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge)
\end{array}
\]

Consider the inclusion of the based loops \( BGL_1K_p^\wedge \rightarrow LB^2GL_1K_p^\wedge \). Under the composite,

\[
S^1 \times BGL_1K_p^\wedge \rightarrow S^1 \times LB^2GL_1K_p^\wedge \xrightarrow{\sigma} B^2GL_1K_p^\wedge,
\]

the copies \( S^1 \times * \) and \( * \times BGL_1K_p^\wedge \) map trivially. Thus, it factors through \( S^1 \wedge BGL_1K_p^\wedge \) as \( \Sigma BGL_1K_p^\wedge \xrightarrow{\sigma} B^2GL_1K_p^\wedge \). We are trying to figure out the map

\[
S^1 \times LCP^\infty \rightarrow S^1 \times LB^2GL_1K_p^\wedge \rightarrow S^1 \times BGL_1K_p^\wedge \rightarrow B^2GL_1K_p^\wedge
\]

Then, this factors through

\[
S^1 \wedge LCP^\infty \rightarrow S^1 \wedge LB^2GL_1K_p^\wedge \rightarrow S^1 \wedge BGL_1K_p^\wedge \xrightarrow{\sigma} B^2GL_1K_p^\wedge.
\]
Also \( LC\Pi \rightarrow BGL_1 K_p^{\wedge} \) factors through \( S^1 \wedge CP_+^\infty \) as \( u \). Putting all the remarks together, we have a commutative diagram,

\[
\begin{array}{ccc}
S^2 \wedge CP_+^\infty & \xrightarrow{\Sigma u} & S^1 \wedge BGL_1 K_p^{\wedge} \\
\downarrow & & \downarrow \\
S^1 \wedge (S^1 \times CP_+^\infty) & \xrightarrow{\approx} & S^1 \wedge LCP_+^\infty \\
\downarrow \Sigma f & & \downarrow S^1 \wedge Lf \\
CP_+^\infty & \xrightarrow{f} & B^2 GL_1(K_p^{\wedge}) \\
\end{array}
\]

The left hand vertical map from \( S^2 \wedge CP_+^\infty \) to \( S^1 \wedge (S^1 \times CP_+^\infty) \) is the inclusion of a factor in the splitting of the suspension of \( S^1 \wedge (S^1 \times CP_+^\infty) \approx (S^2 \wedge CP_+^\infty) \vee (S^1 \wedge CP_+^\infty) \).

It follows that \( \overline{u} \simeq \sigma \circ \Sigma u \simeq f \circ g \), where,

\[
g : S^2 \times \Sigma CP_+^\infty \rightarrow S^1 \wedge (S^1 \times CP_+^\infty) \approx S^1 \wedge LCP_+^\infty \xrightarrow{ev} CP_+^\infty
\]

and the composition \( g \simeq \mu \).

4. The Structure of \( GL_1(K_p^{\wedge}) \)

In this section, we prove a splitting of \( GL_1 K_p^{\wedge} \) using the logarithm \( l_p : gl_1 K_p^{\wedge} \rightarrow K_p^{\wedge} \) defined by Rezk (see [9]). Throughout this section, we assume that \( p \) is an odd prime.

**Proposition 4.1.** (Rezk, [9]) Let \( R \) be an \( E_\infty \) ring spectrum. Then there is a logarithmic cohomology operation, \( l_{p,n} \), from \( gl_1(R) \) to \( LK(n)(R) \) for every \( n \), and prime \( p \). If \( R \) is \( K(n) \)-local, this is a map from \( gl_1(R) \) to \( R \). When \( n = 1 \), \( l_p : gl_1 R \rightarrow R \) is given by the formula:

\[
l_p(x) = -\frac{1}{p} \log \left( \frac{\psi(x)}{x^p} \right)
\]

[Recall that a \( \theta \)-algebra structure is described by operations \( \psi \) and \( \theta \) (\( \psi \) is a ring homomorphism) such that \( \psi(x) = x^p + p\theta(x) \).]
**Proposition 4.2.** Suppose that $R = K_p^\wedge$. The operation $l_p : gl_1 K_p^\wedge \to K_p^\wedge$ factors through $ku_p^\wedge$, the connective cover of $K_p^\wedge$. On homotopy groups, the map is an isomorphism on $\pi_n$ for $n > 2$. At $n = 2$, it is 0. And for $n = 0$, this is the map

$$Z_p^\times \cong Z/(p-1) \times Z \overset{p}{\rightarrow} Z_p$$

**Proof.** The spectrum $K_p^\wedge$ is $K(1)$-local, and the operation $\psi$ is the Adams operation $\psi_p$. Since $gl_1 K_p^\wedge$ is connective, the map $l_p$ factors through $ku_p^\wedge$. Recall, that the homotopy groups of $gl_1 K_p^\wedge$ are given by

$$\pi_n(gl_1 K_p^\wedge) = \begin{cases} (K_p^{0,0}(S^n))^\times = \pi_n(K_p^\wedge) & \text{if } n > 0 \\ (K_p^{0,0}(S^0))^\times = \pi_0(K_p^\wedge)^\times & \text{if } n = 0 \end{cases}$$

Since $\pi_n K_p^\wedge$ is nonzero only for even $n$, it suffices to restrict our attention to even dimensional spheres. The $K$-theory of $S^{2n}$ is generated by $\epsilon$ where $1 - \epsilon$ = the tangent bundle of $S^{2n}$. Hence,

$$\pi_{2n}(gl_1 K_p^\wedge) = \widetilde{gl_1 K_p^\wedge}^0(S^{2n}) = (\widetilde{K_p^{0,0}}(S^{2n}))^\times = 1 + \epsilon \pi_{2n}(K_p^\wedge)$$

To calculate $l_p$ on $\pi_{2n} gl_1 K_p^\wedge$, one needs to compute $l_p(1 + k\epsilon)$ for $1 + k\epsilon \in \pi_{2n}(gl_1 K_p^\wedge(S^{2n})) = \pi_0(gl_1(K_p^\wedge S^{2n}))$. To accomplish this, we need to calculate $\psi_p(\epsilon)$. The map $p : (S^2)^n \to S^{2n}$ which quotients out the lower cells, induces an injection in $K$-theory, and splits $\epsilon$ as the product

$$p^*(\epsilon) = \prod (1 - L_i)$$

where $L_i$ is the canonical line bundle over the $i^{th}$ copy of $S^2 = CP^1$. Since the Adams operation $\psi_p$ raises line bundles to the $p^n$ power,

$$\psi_p(L_i) = L_i^p$$

$$\Rightarrow \psi_p(1 - L_i) = 1 - L_i^p = 1 - (1 - (1 - L_i))^p$$

The element $1 - L_i$ lies in the $K$-theory of $S^2$, so it squares to 0. Therefore,

$$\psi_p(1 - L_i) = 1 - (1 - p(1 - L_i)) = p(1 - L_i)$$

$$\Rightarrow \psi_p(\epsilon) = p^n \epsilon.$$ 

$$\Rightarrow \psi_p(1 + \epsilon) = 1 + p^n \epsilon$$

Hence,

$$l_p(1 + k\epsilon) = \frac{1}{p} \log \left( \frac{\psi(1 + k\epsilon)}{(1 + k\epsilon)^p} \right)$$

$$= \frac{1}{p} \log \left( \frac{1 + p^n k\epsilon}{(1 + k\epsilon)^p} \right)$$

$$\equiv \frac{1}{p} \log(1 + (p^n - p)k\epsilon) \pmod{p}$$
which becomes multiplication by \(1 - p^{n-1}\) (mod \(p\)) if \(n > 0\). Since the homotopy group \(\pi_{2n}(gl_1(K_p^\wedge)) = Z_p\) for \(n > 0\), this is an isomorphism for \(n > 1\). For \(n = 1\), this map is 0. For \(n = 0\), the map \(l_p : Z_p^\times \cong \mu_{p-1} \times Z_p \to Z_p\) is given by

\[
-\frac{1}{p} \log(x^{1-p})
\]

This map has kernel \(\nu_{p-1}\), the group of \((p - 1)^{st}\) roots of unity, as it takes \(p\)-adic integers of the form \(1 + pk\) to

\[
l_p(1 + pk) = -\frac{1}{p} \log((1 + pk)^{1-p})
\]

\[
= -\frac{1}{p} \log(1 + p(1-p)k)
\]

\[
= -(1-p)k + O(p)
\]

\[
\equiv -k \text{ (mod } p)\]

Therefore, the map \(l_p\) on \(Z_p^\times = \nu_{p-1} \times Z_p\), has kernel \(\nu_{p-1}\) and is an isomorphism onto \(Z_p\).

Recall that the spectrum \(ku_p^\wedge\) splits into Adams summands,

\[
ku_p^\wedge \simeq B \vee \Sigma^2 B \ldots \Sigma^{2p-4} B
\]

where \(B\) is the \(p\)-adic Adams summand \((\pi_*(B) = Z_p[v_1])\). Using this, we identify the image of the logarithmic cohomology operation. We construct \(K_p(\hat{2})\) from the spectrum \(ku_p^\wedge\) by killing the \(2^{nd}\) homotopy group:

**Definition 4.3.** Let \(B_2\) be the 2-connective cover of \(B\). Define

\[
K_p(\hat{2}) = B \vee \Sigma^2 B_2 \ldots \vee \Sigma^{2p-4} B
\]

**Proposition 4.4.** There is a split cofibre sequence,

\[
H\nu_{p-1} \vee \Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\hat{2})
\]

**Proof.** From the definition above, note that \(gl_1(K_p^\wedge) \to ku_p^\wedge \to K_p(\hat{2})\) is surjective on homotopy groups. The fibre \(F\) has homotopy only in dimensions 0 and 2. The Postnikov tower of \(F\) then is a cofibre sequence,

\[
\Sigma^2 HZ_p \to F \to H\nu_{p-1} \to \Sigma^3 HZ_p
\]

Since the group \(H^3(H\nu_{p-1}; Z_p) = 0\), the sequence splits and one obtains

\[
F \simeq H\nu_{p-1} \vee \Sigma^2 HZ_p
\]

Therefore, there is a cofibre sequence

\[
H\nu_{p-1} \vee \Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\hat{2})
\]

The next term in this sequence is

\[
\Sigma(H\nu_{p-1} \vee \Sigma^2 HZ_p) \simeq \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p
\]
and the next map is $K_p(\hat{2}) \to \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p$. Since the spaces in the Adams summands are retracts of $bu^\wedge_p$, their homology concentrated in even dimensions. Therefore,

$$\Sigma^{2k}B, \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p \cong H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p) \cong 0$$

Since the spectrum $B_2$ is 3-connected,

$$\Sigma^2 B_2, \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p \cong H^{-1}(B_2; \nu_{p-1}) \oplus H^1(B_2; Z_p) \cong 0$$

$$\Rightarrow [K_p(\hat{2}), H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p)] = 0$$

Hence, the cofibre sequence splits and

$$gl_1(K_p^\wedge) \simeq K_p(\hat{2}) \vee H\nu_{p-1} \vee \Sigma^2 HZ_p.$$ 

□

We will use this decomposition later to calculate homotopy classes of extensions. For that, we also have to understand how the splitting looks like when we map a space $X$ to $GL_1(K_p^\wedge)$. Recall, $[X, GL_1(K_p^\wedge)] = K_p^{\wedge 0}(X)^\times$. The map $l_p$ gives the way to map this to $[X, K_p(\hat{2})]$. The map $K_p^{\wedge 0}(X)^\times \to H^0(X; \nu_{p-1})$ is the composite

$$X \to GL_1(K_p^\wedge) \to \pi_0 GL_1(K_p^\wedge) \cong Z^\times_p \cong \nu_{p-1} \times Z_p \to \nu_{p-1} \simeq K(\nu_{p-1}, 0)$$

The third factor is $\Sigma^2 HZ_p$, and we have to understand the map from $H^2(X; Z_p)$ to $K_p^{\wedge 0}(X)^\times$. Now, $H^2(X; Z_p) = [X, K(Z_p, 2)] = [X, CP^\infty_p]$. The space $CP^\infty$ classifies line bundles which are invertible elements in $K$-theory.

**Proposition 4.5.** The map $H^2(X; Z_p) \to K_p^{\wedge 0}(X)^\times$ is given by $f \in [X, CP^{\wedge p}_p] \to L^f$ where $L^f$ is the line bundle classified by $f$.

**Proof.** The formula in the statement of the proposition defines a map of infinite loop spaces $CP^{\wedge p}_p \to GL_1 K_p^\wedge$, and hence, a map of spectra $\Sigma^2 HZ_p \to gl_1 K_p^\wedge$. Composing it with $l_p$, we get

$$l_p(L^f) = -\frac{1}{p} \log(\psi_p(L^f)^p)$$

$$= -\frac{1}{p} \log(\frac{(L^f)^p}{(L^f)^p})$$

$$= -\frac{1}{p} \log(1)$$

$$= 0$$

The computation above shows that the composition $\Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\hat{2})$ equals 0. Therefore, it factors through $\nu_{p-1} \times \Sigma^2 HZ_p$ in the diagram,
To complete this proof, we need to show that the map \( \Sigma^2 HZ_p \rightarrow H\nu_{p-1} \lor \Sigma^2 HZ_p \rightarrow \Sigma^2 HZ_p \) is an equivalence. The only non-zero homotopy group of \( \Sigma^2 HZ_p \) is \( \pi_2 \), so it suffices to check that the map \([S^2, CP^\infty] \rightarrow H^2(S^2; \mathbb{Z}_p)\) as described by the statement is an isomorphism. The left group is isomorphic to \( \mathbb{Z}_p \), via \( k \mapsto L^k \), \( L = \) the tangent bundle of \( S^2 \). The right group is \( H^2(S^2; \mathbb{Z}_p) \cong \mathbb{Z}_p \) inside \( K_p^\wedge(S^2)^\times \) as elements \( 1 + k\epsilon, \epsilon = 1 - L \).

The map between the two is \( L^k \mapsto (1 - \epsilon)^k = 1 - k\epsilon \) because \( \epsilon^2 = 0 \), and is evidently an isomorphism.

5. Calculation of THH

In this section, we complete the computation of THH for odd primes \( p \). We first parameterise the homotopy classes of extensions \( f \),

\[
\begin{align*}
S^2 \xrightarrow{\Sigma(1-p)} & \Sigma GL_1(K_p^\wedge) \\
\downarrow \sigma & \downarrow \sigma \\
CP^\infty \xrightarrow{f} & B^2 GL_1(K_p^\wedge)
\end{align*}
\]

using the results of the previous section.

Recall that,

\[
GL_1(K_p^\wedge) = \nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K_p(\overset{\sim}{2})
\]

\[
\implies B^2 GL_1(K_p^\wedge) = B^2 \nu_{p-1} \times K(Z_p, 4) \times \Omega^\infty \Sigma^2 K_p(\overset{\sim}{2})
\]

The condition on the map \( f \) is that its restriction to \( S^2 \) is \( 1 - p \). The homotopy classes of maps from \( S^2 \) to \( B^2 GL_1(K_p^\wedge) \) is split into three factors,

1. \([S^2, B^2 \nu_{p-1}] = H^2(S^2; \nu_{p-1}) \cong \nu_{p-1}\)
2. \([S^2, K(Z_p, 4)] = H^4(S^2; \mathbb{Z}_p) = 0\)
3. \([S^2, \Omega^\infty \Sigma^2 K_p(\overset{\sim}{2})] = [S^2, \Omega^\infty (\Sigma^2 B \lor \Sigma^4 B_2 \lor \Sigma^6 B \ldots \lor \Sigma^{2p-4} B)] = [S^2, \Omega^\infty \Sigma^2 B] = B^2(S^2) \cong \mathbb{Z}_p\)

In the splitting,

\[
[S^2, B^2 GL_1(K_p^\wedge)] = \nu_{p-1} \oplus B^2(S^2) \oplus H^4(S^2; \mathbb{Z}_p) = \nu_{p-1} \oplus \mathbb{Z}_p \oplus 0,
\]
1 − p is in the factor \( Z_p \), where it equals \( t_p(1 − p) = \alpha_p \) and,

\[
\alpha_p = -\frac{1}{p} \log((1 - p)^{1-p}) \\
\cong -\frac{1}{p} \log(1 - (1 - p)p) \\
\cong -1 \mod p
\]

5.1. **Calculation at the prime 3.** Let us begin the calculation at the prime 3. The cofibre sequence for \( gl_1K_3^\wedge \) is

\[
HZ/2 \vee \Sigma^2HZ_3 \to gl_1(K_3^\wedge) \to K_3(\hat{2})
\]

and

\[
K_3(\hat{2}) = B \vee \Sigma^2B_2.
\]

Therefore,

\[
GL_1K_3^\wedge = \mathbb{Z}/2 \times K(Z_3, 2) \times \Omega^\infty B \times \Omega^\infty B_2.
\]

We will study the extension to \( CP^\infty \) of the map \( 1 - p \), to the four factors \( \mathbb{Z}/2 \), \( K(Z_3, 2) \), \( \Omega^\infty B \), \( \Omega^\infty B_2 \) one by one. Let us start with the factor \( B \). The Adams summands are the eigenspaces of the action of the \((p - 1)^{st}\) roots of unity by Adams operations. The spectrum \( B \) is fixed by all the Adams operations. The projection from \( K^\wedge_p(X) \) to \( B^*(X) \) is given by

\[
\pi = \frac{1}{p - 1} (1 + \psi \zeta + \psi \zeta^2 + \ldots + \psi \zeta^{p - 2}),
\]

where \( \zeta \in \nu_{p-1} \subset \mathbb{Z}_p^\wedge \).

For the prime 3, we can take \( \zeta = -1 \) and then the projection operator is

\[
\pi = \frac{1 + \psi^{-1}}{2}
\]

Let us start by working out an example.

**Example 5.1.** Consider the element \( \beta L \in K_3^\wedge \) \( CP^\infty \) where \( \beta \) is the Bott element. Applying the projection, we get

\[
\pi(\beta L) = \frac{\beta(L - L^{-1})}{2}
\]

Restricting to \( S^2 \), using \( L = 1 - \epsilon \) and \( \epsilon^2 = 0 \), we obtain

\[
\frac{\beta((1 - \epsilon) - (1 - \epsilon)^{-1})}{2} = \frac{\beta((1 - \epsilon) - (1 + \epsilon))}{2}
\]

\[
= -\beta \epsilon
\]

\[
= -1
\]

In order for it to be an extension of the kind required, this restriction must be \( \alpha_3 \), so we multiply by \(-\alpha_3\). This defines,

\[
f = -\alpha_3 \frac{\beta(L - L^{-1})}{2}.
\]

Recall that, \( THH^K_3(K/3, f) \) is the cofibre of

\[
K_3^\wedge \land CP^\infty \xrightarrow{\nu^{-1}} K_3^\wedge \land CP^\infty \quad (\alpha)
\]
where $u \in K_3^{\wedge 0}(CP^\infty)^\times = [CP_+, GL_1(K_3^\wedge)]$ is the adjoint of,

\[
\begin{array}{ccc}
S^2 \wedge CP_+ & \xrightarrow{\mu} & CP^\infty \\
\downarrow & & \downarrow f \\
CP^\infty & \xrightarrow{\tilde{u}} & B^2GL_1(K_3^\wedge)
\end{array}
\]

The group structure of $CP^\infty$ classifies tensor product of line bundles so, $\mu^* L = L \otimes L$. This implies,

\[
\mu^*(f) = -\alpha_3 \frac{\beta(L \otimes L - L^{-1} \otimes L^{-1})}{2}
\]

The $K$ theory of $S^2$ is generated by $\epsilon = 1 - L$ with $\epsilon^2 = 0$. We can rewrite the equation using the generator

\[
\mu^*(f) = -\alpha_3 \frac{\beta((1 - \epsilon) \otimes L - (1 + \epsilon) \otimes L^{-1})}{2} = -\alpha_3 \frac{\beta \epsilon \otimes (L + L^{-1})}{2}
\]

Using the suspension isomorphism (given by $\beta \epsilon = 1$) we get,

\[
\mu^*(f) = -\alpha_3 \frac{L + L^{-1}}{2}
\]

To get $u$ we need to invert the logarithmic cohomology operation. Suppose that $u = h(x) \in K_3^{\wedge 0}(CP^\infty)^\times$. Then, we have to solve,

\[
- \frac{1}{3} \log \left( \frac{\psi_3(h(x))}{h(x)^3} \right) = -\alpha_3 \frac{L + L^{-1}}{2}
\]

\[
\Rightarrow \frac{\psi_3(h(x))}{h(x)^3} = \exp \left( 3\alpha_3 \frac{L + L^{-1}}{2} \right) \quad (\ast)
\]

Note that $\psi_3(x) = 1 - (1 - x)^3$ and hence,

\[
\frac{h(1 - (1 - x)^3)}{h(x)^3} = \exp \left( 3\alpha_3 \frac{L + L^{-1}}{2} \right)
\]

Let us look at the equation ($\mod 3^2, x^3$). The right side of the equation can be written in terms of $x$ using $L = 1 - x$, and then, $L^{-1} = 1 + x + x^2 \ (mod \ 3^2, x^3)$. Therefore, the right side simplifies to

\[
\exp \left( 3\alpha_3 \frac{L + L^{-1}}{2} \right) = \exp \left( 3\alpha_3 \frac{2 + x^2}{2} \right) = 1 + 3\alpha_3 + 3\frac{\alpha_3 x^2}{2}
\]

Now we will simplify the left side of ($\ast$). Suppose that $h(x) = a + bx + cx^2$. In order to solve the equation, we have to invert $l_3$. We know that $l_3$ has a kernel $Z/2 \vee K(Z_3, 2)$, so the equation can be solved once we know the restriction to these.
In the part of $HZ/2$, $\sigma : S^2 \to CP^\infty$ induces an isomorphism in $H^2(-; Z/2)$. Therefore, the extension is 0 here. The map $K_3^\wedge (CP^\infty) \to H^0(CP^\infty; Z/2)$ sends $a \mapsto a (\mod 3)$ (identifying $Z/2$ with the group of units in $\mathbb{F}_3$). Therefore, since $\mu^*(0) = 0$, we get the equation

$$a \equiv 1 (\mod 3)$$

In the factor $K(Z_3, 2)$, there is no restriction on $f$. Assume that it is trivial, so $\mu^*(0) = 0$. This maps into $GL_1(K_3^\wedge)$ by taking a line bundle over $CP^\infty$ to the corresponding unit in $K$-theory. If we look at $k \in Z_p = H^2(CP^\infty; Z_3) = [CP^\infty, K(Z_3, 2)]$, this is the line bundle $L^k = (1 - x)^k = 1 - kx + \frac{k(k-1)}{2}x^2 (\mod x^3)$. This is the only factor that gives a non zero coefficient of $x$ so, we get that $b = 0$.

Therefore, $h(x) = a + cx^2 (\mod 3^2, x^3)$ and $a \equiv 1 (\mod 3)$. The left side of (*) is

$$\psi(h(1 - (1 - x)^3)) = \frac{h(3x - 3x^2 + x^3)}{h(x)^3} \equiv \frac{a}{a^3 + 3ca^2x^2} \equiv a^{-2}(1 - 3\frac{c}{a} x^2) (\mod 3^2, x^3)$$

Working $(\mod 3^2, x^3)$, we have

$$a^{-2}(1 - 3\frac{c}{a} x^2) = 1 + 3\alpha_3 + 3\frac{\alpha_3 x^2}{2}$$

$$\Rightarrow a \equiv 1 + 3\alpha_3 (\mod 3^2) \text{ and } c \equiv \alpha_3 (\mod 3)$$

Thus $a = 1 + 3(\text{unit})$ and $c$ is a unit (since $\alpha_3$ is a unit).

Therefore, $u - 1$ looks like $3(\text{unit}) + x^2(\text{unit})$. We can choose a different parameterisation for $K$-theory of $CP^\infty$ to assume that $u - 1 = 3 + x^2$.

Now $K_3^\wedge (CP^\infty) = K_3^\wedge \{\beta_0, \beta_1, \ldots\}$ where $\beta_i$ is dual to $x^i$. Therefore,

$$< (u - 1)(\beta_i), x^j > = < \beta_i, x^j(3 + x^2) > = \begin{cases} 3 & \text{if } j = i \\ 1 & \text{if } j = i - 2 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, the map $u - 1$ on $K_3^\wedge (CP^\infty)$ is given by

$$(u - 1)(\beta_i) = \begin{cases} 3\beta_i & \text{if } i = 0, 1 \\ 3\beta_i + \beta_{i-2} & \text{if } i > 1 \end{cases}$$

Following the cofibre $(\alpha)$, we understand that $u - 1$ is injective, and its cokernel has two copies of $Z/(3^\infty)$ in even dimensions. Thus,

$$\pi_k(THH K_3^\wedge (K/3), f) = \begin{cases} Z/(3^\infty) \oplus Z/(3^\infty) & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

completing the calculation in this example.

Now we perform the calculation at the prime 3 for all extensions that are non trivial only on the factor $\Omega^\infty B$ of $GL_1(K_3^\wedge)$. The extension in the example was of this kind. So, we are looking at elements in $B^2(CP^\infty)$ which restrict to $\alpha_3$ in $S^2$. 

An element in $K_3^\wedge 2(CP^\infty)$ is given by $\beta g(x)$. Therefore, an element in $B^2(CP^\infty)$ is

$$
\pi(\beta g(x)) = \frac{\beta(g(x) - g(1 - \frac{1}{1-x}))}{2}
$$

Suppose that $g(x) = a' + b'x + c'x^2 \pmod{3^2, x^3}$. Restricting to $S^2$ (using $x = \epsilon$ and $\epsilon^2 = 0$) we get $b'$. We need to get $\alpha_3$. Thus, to get an extension we must have $b' = \alpha_3$. This gives us all possible extensions $f$ on the factor $B$. Let us work as before (mod $3^2, x^3$). Then,

$$
f = \frac{\beta(g(x) - g(1 - \frac{1}{1-x}))}{2} = \frac{\beta(a' + b'x + c'x^2 - g(-x - x^2))}{2} = \frac{\beta(2b'x + b'x^2)}{2} = \beta b' x + \frac{\beta b'}{2} x^2
$$

We have to calculate $u$ using

$$
\begin{array}{ccc}
S^2 \times CP^\infty & \xrightarrow{u} & B^2GL_1(K_3^\wedge) \\
\downarrow \mu & & \\
CP^\infty & \xrightarrow{f} & B^2GL_1(K_3^\wedge)
\end{array}
$$

By definition, the multiplication map takes $x$ to the formal group, which for $K$-theory is the multiplicative group. Therefore,

$$
x \mapsto \epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x
$$

$$
\implies x^2 \mapsto (\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^2 = 1 \otimes x^2 + 2\epsilon \otimes x - 2\epsilon \otimes x^2
$$

To get $\mu^*$ we must project onto the factor $S^2 \wedge CP^\infty$. Thus, we obtain

$$
\mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x, \quad \mu^*(x^2) = 2\epsilon \otimes x - 2\epsilon \otimes x^2.
$$

Using these formulae and the suspension isomorphism $\beta \epsilon = 1$ we calculate $\mu^*(f)$.

$$
\mu^*(f) = \beta b'(\epsilon \otimes 1 - \epsilon \otimes x) + \frac{\beta b'}{2}(2\epsilon \otimes x - 2\epsilon \otimes x^2)
$$

$$
= b'(1 - x) + \frac{b'}{2}(2x - 2x^2)
$$

$$
= b' - b'x^2
$$
To get \( u \), we have to invert the logarithmic cohomology operation \( l_3 \), as in the example. Suppose that \( u = h(x) \). Then, we need to solve

\[
l_3(u) = \frac{\psi_3(h(x))}{h(x)^3} = \exp(-3b'(1 - x^2))
\]

We have the formula \( \psi_3(x) = 1 - (1 - x)^3 \). Similar to the example, we assume that in our extension the contribution from \( HZ/2 \) is 1 and \( HZ_3 \) is 0. In the same way, this implies that if \( h(x) = a + bx + cx^2 \),

\[
a \equiv 1(\text{mod } 3), \quad b = 0
\]

Then, the equation becomes,

\[
a^{-2}(1 - 3\frac{c}{a}x^2) = \exp(-3b'(1 - x^2)) = 1 - 3b' + 3b'x^2
\]

In the same way, we understand that the unit \( u = 1 + 3.\text{unit} + x^2.\text{unit} \), and so, we obtain the same computation

\[
\pi_k(\text{THH}^{K/3}(K/3), f) = \begin{cases} 
0 & \text{if } k \text{ is odd} \\
Z/(3^\infty) \oplus Z/(3^\infty) & \text{if } k \text{ is even}
\end{cases}
\]

Now we want to see what happens if we allow extensions with non trivial contributions from the other 3 factors of \( GL_1(K_3) = Z/2 \times K(Z_3, 2) \times \Omega^\infty B \times \Omega^\infty B_2 \). In the part \( Z/2 \), the restriction \( H^2(CP^\infty; Z/2) \to H^2(S^2; Z/2) \) is an isomorphism. So, this factor always contributes trivially.

For the factor \( K(Z_3, 2) \), the group \([S^2, B^2K(Z_3, 2)] = [S^2, K(Z_3, 4)] = H^4(S^2; Z_3) = 0\). Therefore, there is no condition on \( f \) here. The group \( H^4(CP^\infty; Z_3) \) is generated by \( x^2 \) and \( f \) is given by \( ax^2 \) for some \( a \in Z_p \). To compute \( u \), consider:

\[
S^2 \times CP^\infty \xrightarrow{\mu} CP^\infty \xrightarrow{f} K(Z_3, 4)
\]

Note that in this case, \( \mu^*(x) = \epsilon \otimes 1 + 1 \otimes x \), which implies

\[
\mu^*(x^2) = (\epsilon \otimes 1 + 1 \otimes x)^2 = 2\epsilon \otimes x + 1 \otimes x^2
\]

To get \( u \) we have to project to \( S^2 \wedge CP_+^\infty \) and apply the suspension isomorphism. Then, we get \( 2ax \in H^2(CP^\infty; Z_3) \). Recall from the previous section that, from this we get the unit by taking \( L^{2a} \), where \( L = (1 - x) \) is the canonical line bundle. Therefore, the contribution to \( u \) from this factor is \( (1 - x)^{2a} \).

Now if \( a \) is divisible by 3 then, we still get that our \( u = 1 + 3.\text{unit} + x^2.\text{unit} \) which results in the same calculation for \( \pi_*(\text{THH}^{K/3}(K/3, f)) \). If \( a \) is not divisible by 3 then, it is a unit,
so that \( u = 1 + 3.\text{unit} + x.\text{unit} \). Therefore, by reparameterising we can write \( u - 1 = 3 + x \).

\[
\Rightarrow < (u - 1)(\beta_i), x^j > = < \beta_i, x^j(3 + x) > = \begin{cases} 
3 & \text{if } j = i \\
1 & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}
\]

\[
\Rightarrow (u - 1)(\beta_i) = \begin{cases} 
3\beta_i & \text{if } i = 0 \\
3\beta_i + \beta_{i-1} & \text{if } i > 0
\end{cases}
\]

Therefore, in this case,

\[
\pi_k(THH^{K/3}(K/3), f) = \begin{cases} 
0 & \text{if } k \text{ odd} \\
Z/(3^\infty) & \text{if } k \text{ even}
\end{cases}
\]

Now consider the factor \( \Sigma^2B_2 \). We know that this is 5-connected. So, if we look at extensions we know that they always restrict to \( 0 \in K^\wedge_3(CP^2) \). Since we are working \( (mod\ x^3) \), this means that these extensions always give 0.

Therefore, we get that, depending on \( f \) either \( \pi_*(THH^{K/3}(K/3, f)) = (Z/(3^\infty))^i \) in even degrees where \( i = 1 \) or 2 depending on \( f \).

This finishes our calculation at the prime 3.

5.2. Calculation at primes \( \geq 5 \). Let us now look at the other odd primes and work \( (mod\ x^p, p^2) \). Recall that there is a splitting

\[
GL_1(K^\wedge_p) = \nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K^\wedge_p(2),
\]

\[
K^\wedge_p(2) = B \lor \Sigma^2B_2 \lor \ldots \Sigma^{2p-4}B
\]

We start by working in the factor \( B \) of \( K^\wedge_p(2) \). The projection operator from \( K^\wedge_p(X) \) to \( B^*(X) \) is given by

\[
\pi = \frac{1 + \psi_\xi + \psi_{\xi^2} + \ldots \psi_{\xi^{p-2}}}{p - 1}
\]

Define \( \kappa \) to be the composite,

\[
K^\wedge_2(CP^\infty) \xrightarrow{\mu^*} K^\wedge_*(S^2 \wedge CP^\infty) \xrightarrow{\zeta} K^\wedge_{-2}(CP^\infty)
\]

First observe that the following diagram commutes:

\[
\begin{array}{ccc}
K^\wedge_2(CP^\infty) & \xrightarrow{\psi_u} & K^\wedge_2(CP^\infty) \\
\downarrow{\kappa} & & \downarrow{\kappa} \\
K^\wedge_0(CP^\infty) & \xrightarrow{\psi_u} & K^\wedge_0(CP^\infty)
\end{array}
\]

This implies all Adams operations hence \( \pi \), commutes with \( \kappa \).

Write \( x = 1 - L \) for the generator in \( K^\wedge_2(CP^\infty) \) and \( \epsilon \) its restriction to \( S^2 \). We have to look for \( f \) as in the diagram,
Suppose that \( f \) is given by \( \pi(\beta g(x)) \) where \( g(x) = a_0 + a_1 x + \ldots + a_{p-1} x^{p-1} \pmod{x^p, p^2} \).

**Claim:**
\[
\kappa(\beta g(x)) = g'(x)(1 - x)
\]

**Proof.** It is enough to check this on the generators \( x^n \). The multiplication takes \( x \) to the formal group of \( K \)-theory, which is the multiplicative formal group.

\[
\mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x
\]

Therefore,
\[
\mu^*(\beta x^n) = \beta(\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^n
\]
\[
= \beta(1 \otimes x^n + n\epsilon \otimes x^{n-1} - n\epsilon \otimes x^n)
\]

\( \kappa \) is obtained by projecting this onto the factor \( S^2 \wedge CP^\infty \) of the product, and then applying the suspension isomorphism \( (\beta \epsilon = 1) \). Therefore, we obtain
\[
\kappa(\beta x^n) = nx^{n-1} - nx^n
\]
\[
= nx^{n-1}(1 - x)
\]
\[
= (x^n)'(1 - x)
\]

\(\square\)

If we restrict \( f \) to \( S^2 \), we get
\[
\pi(\beta g(\epsilon)) = \pi(\beta(a_0 + a_1 \epsilon))
\]
\[
= \left( \frac{1 + \psi_\zeta + \psi_\zeta^2 + \ldots + \psi_\zeta^{p-2}}{p - 1} \right)^p(\beta(a_0 + a_1 \epsilon))
\]

The action of the Adams operations on the Bott element and \( \epsilon \) are given by
\[
\psi_a(\beta) = \frac{\beta}{a}, \; \psi_a(\epsilon) = 1 - (1 - \epsilon)^a = a\epsilon
\]

Therefore,
\[
\pi(\beta g(\epsilon)) = \left( \frac{1 + \psi_\zeta + \psi_\zeta^2 + \ldots + \psi_\zeta^{p-2}}{p - 1} \right)^p(\beta(a_0 + a_1 \epsilon))
\]
\[
= \beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p}) + (p - 1)a_1 \epsilon)
\]
\[
= \frac{\beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p})) + a_1 \epsilon}{p - 1}
\]
Since $\zeta$ is a $(p - 1)^{st}$ root of unity, we get

$$1 + \zeta^{-1} + \zeta^{-2} + \ldots + \zeta^{2-p} = \zeta^{p-1} + \zeta^{p-2} + \ldots + \zeta = 0$$

This shows that $\pi(\beta g(x))$ restricts to $a_1 \in B^2(S^2)$. Thus, we have that $a_1 = l_p(1-p) = \alpha_p$.

We need to calculate $u$ from the extension $\pi(\beta g(x))$ by solving,

$$l_p(u) = \kappa \pi(\beta g(x)) = \pi \kappa (\beta g(x)) = \pi(g'(x)(1-x))$$

Suppose that $h(x) = g'(x)(1-x) = c_0 + c_1 x + \ldots + c_{p-1} x^{p-1} (mod\ x^p, p^2)$. Then,

$$\pi(h(x)) = \frac{1 + \psi\zeta + \psi\zeta^2 + \ldots + \psi\zeta^{p-2}}{p-1}(h(x)) = \sum_{i=0}^{p-2} \frac{h(1-(1-x)^{\zeta^i})}{p-1}$$

Let us look at the coefficient of $x^a$ in the above equation.

$$[\pi(x^n)]_a = \sum_{i=0}^{p-2} \frac{(1-(1-x)^{\zeta^i})^n}{p-1}[a]$$

Since $\zeta$ is a $(p - 1)^{st}$ root of unity,

$$(\zeta)^i + (\zeta^2)^i + \ldots + (\zeta^{p-1})^i = \begin{cases} 0 & \text{if } i = 1, 2, \ldots, p-2 \\ p-1 & \text{if } i = 0, p-1 \end{cases}$$

The binomial coefficient $\binom{y}{a}$ is a polynomial in $y$ of degree $a$ with the constant term 0 and the top coefficient $1/a!$. Therefore,

$$\binom{l(\zeta)}{a} + \binom{l(\zeta^2)}{a} + \ldots + \binom{l(\zeta^{p-1})}{a} = \begin{cases} 0 & \text{if } a = 1, 2, \ldots, p-2 \\ \frac{p-1}{p^a} & \text{if } a = p-1 \\ \frac{p-1}{p^a} & \text{if } a = 0 \end{cases}$$
Therefore, we get
\[
\pi(x^n)_a = \begin{cases} 
\frac{1}{(p-1)!} \sum \binom{n}{l} (-1)^{p-1} & \text{if } a = 1, 2, \ldots, p - 2 \\
\sum \binom{n}{l} (-1)^l & \text{if } a = 0 \\
\end{cases}
\]
\[
\Rightarrow [\pi(x^n)]_0 = \sum \binom{n}{l} (-1)^l = \begin{cases} 
(1 - 1)^n = 0 & \text{if } n > 0 \\
1 & \text{if } n = 0 \\
\end{cases}
\]
The other possible non zero coefficient is $[\pi(x^n)]_{p-1}$. If $n = 0$, this must be 0. If $n > 0$ this gives
\[
[\pi(x^n)]_{p-1} \equiv \frac{1}{(p-1)!} \sum \binom{n}{l} (-1)^{p-1} \\
\equiv \frac{1}{(p-1)!} \sum \binom{n}{l} (-1)^{p-1} \\
\equiv - \sum \binom{n}{l} (-1)^l \\
\equiv -(1 - 1)^n + 1 \\
\equiv 1 \ (\mod p)
\]
Summarising the calculation $(\mod p)$, we get
\[
[\pi(x^n)]_a = \begin{cases} 
1 & \text{if } a = 0, n = 0 \\
1 & \text{if } a = p - 1, n > 0 \\
0 & \text{otherwise} \\
\end{cases}
\]
Now we are in a position to calculate $\pi(h(x))$ $(\mod p)$
\[
\pi(h(x)) = \pi(c_0 + c_1 x + \ldots + c_{p-1} x^{p-1}) \\
= c_0 \pi(1) + c_1 \pi(x) + \ldots + c_{p-1} \pi(x^{p-1}) \\
= c_0 + c_1 x^{p-1} + \ldots + c_{p-1} x^{p-1} \\
= c_0 + bx^{p-1}
\]
where $c_0 = a_1$ and
\[
b = c_1 + \ldots + c_{p-1} \\
= a_1 - 2a_2 + 2a_2 - 3a_3 \ldots - (p-1)a_{p-1} + (p-1)a_{p-1} - pa_p \\
\equiv a_1 \ (\mod p)
\]
Thus the equation for $u$ $(\mod p)$ reduces to
\[
l_p(u) = a_1 + bx^{p-1} \ (\mod p) \\
\Rightarrow \frac{-1}{p} \log\left(\frac{\psi_p(u(x))}{u^p}\right) = a_1 + bx^{p-1} \ (\mod p) \\
\Rightarrow \frac{\psi_p(u(x))}{u^p} = e^{p(a_1 + bx^{p-1})} = 1 - pa_1 + pbx^{p-1} \ (\mod p^2)
\]
We are looking at extensions which are non trivial only on the factor $B$. This implies $u(x) \in B_0^0(CP^\infty)$ which implies $u$ is in the image of $\pi$. By the calculations above, this implies that $u(x) = d_0 + d_1 x^{p-1}$ $(\mod x^p)$. Then,
\[
\psi_p(u(x)) = \frac{d_0}{d_0^p + pd_0^{p-1}d_1x^{p-1}} = (d_0)^{1-p}(1 - p\frac{d_1}{d_0}x^{p-1})
\]

Therefore, we obtain

\[
d_0^{1-p} = 1 - pa_1
\]

\[
\Rightarrow d_0 = (1 - pa_1)^{1-p} = 1 - \frac{p}{1-p}a_1 = 1 - pa_1 \pmod{p^2}
\]

\[
\Rightarrow d_1 = -d_0^pb = -1 \pmod{p}
\]

Therefore, \(d_0 = 1 + p.\text{unit}\) and \(d_1 = \text{unit}\). Thus, \(u = 1 + p.\text{unit} + \text{unit}x^{p-1}\). We can reparameterise so that \(u = 1 + p + x^{p-1}\).

\[
< (u - 1)(\beta_i), x^j > = \begin{cases} p & \text{if } j = i \\ 1 & \text{if } j = i - (p - 1) \\ 0 & \text{otherwise} \end{cases}
\]

\[
\Rightarrow (u - 1)(\beta_i) = \begin{cases} p\beta_i & \text{if } i = 0 \\ p\beta_i + \beta_{i-(p-1)} & \text{if } i > 0 \end{cases}
\]

Inputting this in the long exact sequence (\(\alpha\)), we get

\[
\pi_k(THH_{K^P}(K/p, f)) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ (Z/(p^\infty))^{p-1} & \text{if } k \text{ is even} \end{cases}
\]

Now let us look at what happens if we allow non trivial extensions in the other factors. Under restriction to \(S^2\), \(H\nu_{p-1}^2(S^2) \cong H\nu_{p-1}^2(CP^\infty) = \nu_{p-1}\). The element \(1 - p\) gives \(1 \in \nu_{p-1}\). So, this part always contributes trivially.

The factor \(\Sigma^2B_2\) is \((2p - 1)\)-connected. So, \([CP^{p-1}, \Sigma^2B_2] = 0\). Thus \((mod\ x^p)\) this factor is always trivial.

Next lets look at the factor \(\Sigma^2HZ_p\). Since \([S^2, \Sigma^1HZ_p] = HZ_p^{2}(S^2) = 0\), we have no condition on \(f\) from this factor. The group \(HZ_p^{2}(CP^\infty)\) is generated by \(x^2\). Suppose that \(f\) is given by \(ax^2 \in HZ_p^{2}(CP^\infty)\). To compute the contribution to \(u\), we have the diagram,
Under $\mu$, $x$ pulls back to the formal group and thus,
\[
\mu^*(x^2) = (\epsilon \otimes 1 + 1 \otimes x)^2
= 2\epsilon \otimes x + 1 \otimes x^2
\]
Projecting this to the factor $S^2 \wedge CP_+^\infty$, and applying the suspension isomorphism we get $2ax \in HZ_{p}^2(CP^\infty)$. The map from $HZ_{p}^2(CP^\infty) \to [CP^\infty, GL_1(K_p^\wedge)] = K_{p}^{\wedge}0(CP^\infty)^\wedge$ is given by $ax \to (1 + x)^a$.

Therefore, if $a$ is divisible by $p$ then we still get that $u = 1 + p.\text{unit} + x^{p-1}.\text{unit}$. This does not change the calculation of $\text{THH}^{K_{p}^{\wedge}}(K/p, f)$. If $a$ is not divisible by $p$, then it is a unit. Then, $u = 1 + p.\text{unit} + x.\text{unit}$. This can be reparameterised to $u = 1 + p + x$. Then

\[
< (u - 1)(\beta_i), x^j > = p \text{ if } j = i
= 1 \text{ if } j = i - 1
= 0 \text{ otherwise}
\]

Therefore, we obtain

\[
\pi_k(\text{THH}^{K_{p}^{\wedge}}(K/p, f)) = \begin{cases} 
0 & \text{if } k \text{ is odd} \\
\mathbb{Z}/(p^\infty) & \text{if } k \text{ is even}
\end{cases}
\]

The other factors are $\Sigma^{2k}B$ for $k = 2, 3, \ldots, p - 2$. These correspond to the eigenspaces of the action of the Adams operations where $\psi_{\zeta^i}$ acts as $\zeta^{ki}$. The projection operator is given by

\[
\pi_k = \frac{1 + \zeta^{-k}\psi_{\zeta} + \zeta^{-2k}\psi_{\zeta^2} + \ldots + \zeta^{-(p-2)k}\psi_{\zeta^{p-2}}}{p - 1}
\]

The group $[S^2, \Omega^\infty \Sigma^2 \Sigma^{2k}B] = B^{2k+2}(S^2) = 0$, so, there is no condition on restriction to $S^2$. Then, we may choose any $\pi_k(\beta h(x))$ for $f$, and $u$ must satisfy,

\[
l_p(u) = \kappa(\pi_k(\beta h(x))) = \pi_k(\kappa(\beta h(x))) = \pi_k(f'(x)(1 - x))
\]

Now assume $g(x) = h'(x)(1 - x) = c_0 + c_1 x + \ldots + c_{p-1} x^{p-1}$. Then

\[
\pi_k(g(x)) = \frac{1 + \zeta^{-k}\psi_{\zeta} + \zeta^{-2k}\psi_{\zeta^2} + \ldots + \zeta^{-(p-2)k}\psi_{\zeta^{p-2}}}{p - 1}(g(x))
= \frac{1 + \zeta^{-k}g(1 - (1 - x)^\zeta) + \zeta^{-2k}g(1 - (1 - x)^{2\zeta}) + \ldots + \zeta^{-(p-2)k}g(1 - (1 - x)^{p-2})}{p - 1}
\]

The following proposition is useful to complete the calculation.

**Proposition 5.2.** There is a polynomial $f_k(x) = x^k + a_{k+1}x^{k+1} + \ldots$ such that, $\text{Im}(\pi_k)$ has polynomials that are multiples of $f_k (mod x^p)$.
Proof. These polynomials are in the $p$-adic $K$-theory of $CP^\infty$. By looking ($mod$ $x^p$), we are restricting to the $K$-theory of $CP^{p-1}$. It splits into eigenspaces

$$K^\wedge 0_p(CP^{p-1}) = \bigoplus_{k=0}^{p-2} \Lambda_k$$

where $\Lambda_k = [CP^{p-1}, \Omega^\infty \Sigma^{2k} B] = B^{2k}(CP^{p-1})$ is the eigenspace on which the Adams operations $\psi_\zeta$ act as multiplication by $\zeta^k$. $\pi_k$ is the projection on to the eigenspace $\Lambda_k$. In this decomposition, $dim(\Lambda_0) = 2$ and $dim(\Lambda_k) = 1$ for all $k \geq 1$. Therefore, $\Lambda_k = \text{span}(f_k)$ for some polynomial $f_k$. To see how the polynomial $f_k$ looks like we compute $\pi_k(x)$ (note $\pi_k(1) = 0$, so we don’t get any information out of it).

$$\pi_k(x) = \frac{1 + \zeta^{-k} \psi_\zeta + \zeta^{-2k} \psi^2 \zeta + \ldots + \zeta^{-(p-2)} \psi^{p-2} \zeta^k}{p-1} (x)$$

$$= \frac{1}{p-1} \sum_{i=0}^{p-2} \zeta^{-ik} (1 - (1 - x) \zeta^i)$$

$$= \frac{1}{p-1} \sum_{i=0}^{p-2} \zeta^{-ik} (-1)^{i-1} \sum_{n=1}^\infty \left( \binom{\zeta^i}{n} \right)^x$$

$$= \frac{1}{p-1} \sum_{i=0}^{p-2} \sum_{n=1}^\infty (-1)^{i-1} \zeta^{-ik} \left( \binom{\zeta^i}{n} \right) x^n$$

Let’s look at the coefficient of $x^n$ in the above formula. $\left( \begin{array}{c} p \\ n \end{array} \right)$ is a polynomial of degree $n$ in $y$, and therefore, $(-1)^{i-1} y^{-k} \left( \begin{array}{c} p \\ n \end{array} \right)$ has terms of degree $-k$ to $-k + n$. So, if we sum the series, it is 0 if $k < n$. Thus, the first possible non zero coefficient of $x$ is in degree $k$. The coefficient of $x^k$ in $\pi_k(x)$ is given by

$$[\pi_k(x)]_k = \frac{1}{p-1} \sum_{i=0}^{p-2} (-1)^{i-1} \zeta^{-ik} \left( \binom{\zeta^i}{k} \right)$$

$$= \frac{1}{p-1} \sum_{i=0}^{p-2} (-1)^{i-1} \frac{1}{k!}$$

$$= \frac{1}{(p-1)k!} \neq 0 (mod p)$$

So, this is a unit in $\mathbb{Z}_p$. Therefore, $\text{Im}(\pi_k) = \text{Span}(f_k)$ where $f_k$ looks like $x^k + O(x^{k+1})$.

Therefore, $\pi_k(g(x)) = cf_k(x)$ for some constant $c$. The equation for $u$ is

$$l_p(u) = \pi_k(g(x)) = cf_k(x)$$

$$\implies \frac{\psi_p(u)}{u^p} = \exp(-pcf_k(x))$$

If $c$ is divisible by $p$, then ($mod$ $p^2$) the above equation is 0. If $c$ is not divisible by $p$, then the coefficient of $x^k$ in the right side is $p$ times an unit. We can solve for $u$ as in the
cases before. From here, we get a contribution = \text{unit}.x^k. Therefore, the unit becomes \( u = 1 + p.\text{unit} + x^k.\text{unit} \). As before, we have the long exact sequence (\( \alpha \))

\[
K_p^\wedge(CP^\infty)^{u-1} \to K_p^\wedge(CP^\infty) \to \pi_\ast(THH_{K_p}(K/p,f))
\]

and,

\[
< (u - 1)(\beta_i), x^j > = \begin{cases} 
p & \text{if } j = i \\
1 & \text{if } j = i - k \\
0 & \text{otherwise}
\end{cases}
\]

\[
\Rightarrow (u - 1)(\beta_i) = \begin{cases} 
p\beta_i & \text{if } i = 0 \\
p\beta_i + \beta_{i-k} & \text{if } i > 0
\end{cases}
\]

Therefore, we obtain that

\[
\Rightarrow \pi_n(THH_{K_p}(K/p,f)) = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
(Z/(p^\infty))^k & \text{if } n \text{ is even}
\end{cases}
\]

This ends the calculation for all odd primes. The homotopy groups of \( THH_{K_p}(K/p) \) are 0 in odd degrees and \( (Z/(p^\infty))^k \) in even degrees, where \( k \) is a number between 1 and \( p - 1 \) depending on the \( A_\infty \) structure on \( K/p \). This result was proved before by Angeltveit (I). He used the Bökstedt spectral sequence to calculate Topological Hochschild Homology.

**Remark 5.3.** This is the calculation identifying \( K/p \) as the Thom spectrum of \( S^1 \). A similar calculation can be carried out for the Thom spectrum of \( S^3 \) to get the same results.

**References**

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