ADAPTIVE CONTROL FOR SYNCHRONIZATION OF IDENTICAL AND NON-IDENTICAL CHAOTIC SYSTEMS WITH UNKNOWN PARAMETERS

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Abstract

In this paper, adaptive control theory is utilized to derive nonlinear controllers for the synchronization of two identical and non-identical chaotic systems with unknown parameters. Based on the Lyapunov stability theory, the adaptive control laws for synchronization controllers associated with adaptive update laws of system parameters are developed to make the states of two identical and non-identical systems synchronized. The feasibility of the obtained results are validated with numerical simulation.

Keywords: Synchronization, adaptive control, Lyapunov stability

1. Introduction

Chaotic phenomena in numerous natural and social systems have attracted great interested since E. Lorenz discovered the first physical chaotic attractor [1]. Chaos is also an interesting phenomenon of nonlinear systems. A deterministic chaotic system has some remarkable dynamics characteristics [2], such as system evolution sensitive to the initial conditions and broad spectrum of Fourier transform. It can be treated as a carrier to modulate signals that have the random characteristics. It also has the overall stability. When we use a chaotic signal to drive two identical systems, the two systems or certain parts of them will have the synchronous behaviour, which does well for confidential communication.

Since synchronization of chaos has been put forward by Pecora and Carroll in 1990 [3], the phenomenon and its application in secure communications attracts much attention [4, 5, 6, 7, 8, 9, 10]. Until now, many types of synchronization of chaotic system have been proposed, such
as generalized synchronization [11, 12], phase synchronization [13, 14], lag synchronization [15], anti-synchronization [16] and so on. At the same time, a wide variety of approaches for the synchronization and control of chaotic system have also been put forward in recent years [17,18, 19, 20, 21], such as PC method, adaptive control method [22], observer control method [23], fuzzy control method, global synchronization method, etc.

Recently, Hyperchaotic systems have attracted much attention in nonlinear area. Hyperchaotic system has more than one positive Lyapunov exponent which generates more complex dynamics than the low-dimensional chaotic system. Therefore Hyperchaotic system has much wider application than the low-dimensional chaotic system. For example, the adoption of Hyperchaotic system has been proposed for secure communication and the presence of more than one positive Lyapunov exponent clearly improves the security of the communication scheme [24].

In this paper, we study chaos synchronization of two identical Pan, and two identical Chen systems and non-identical Pan and Chen chaotic systems by adaptive control method. It is assumed that the parameters of the systems are unknown.

2. Mathematical description

Consider the drive chaotic system in the form of

$$\dot{x} = f(x) + F(x)\alpha$$  \hspace{1cm} (1)

where $x \in \Omega_1 \subset \mathbb{R}^n$ is the state vector, $\alpha \in \mathbb{R}^m$ is the unknown parameter vector of the system, $f(x)$ is an $n \times 1$ matrix, $F(x)$ is an $n \times m$ matrix, the elements $F_{ij}(x)$ in matrix $F(x)$ satisfies $F_{ij}(x) \in L_{\infty}$ for $x \in \Omega_1 \subset \mathbb{R}^n$. On the other hand, the response system is assumed by

$$\dot{y} = g(y) + G(y)\beta + u$$  \hspace{1cm} (2)

where $y \in \Omega_2 \subset \mathbb{R}^n$ is the state vector, $\beta \in \mathbb{R}^q$ is the unknown parameter vector of the system, $g(y)$ is an $n \times 1$ matrix, $G(y)$ is an $n \times q$ matrix, $u \in \mathbb{R}^n$ in control input vector, the elements $G_{ij}(y)$ in matrix $G(y)$ satisfies $G_{ij}(y) \in L_{\infty}$ for $y \in \Omega_2 \subset \mathbb{R}^n$.

Let $e = y - x$ is the synchronization error vector. Our goal is to design controller $u$ such that the trajectory of the response system (2) with initial condition $y_0$ can asymptotically approaches the drive system (1) with initial condition $x_0$ and finally implement the synchronization, that is,

$$\lim_{t \to \infty} \|e\| = \lim_{t \to \infty} \|y(t, y_0) - x(t, x_0)\| = 0$$  \hspace{1cm} (3)

where $\|\|$ is the Euclidean norm.

2.1. Adaptive synchronization controller design

Theorem 2.1 If the nonlinear control is selected as
\[ u = -f(x) - F(x)\dot{\alpha} - g(y) - G(y)\dot{\beta} - ke, \quad (4) \]

And adaptive laws of parameters are taken as
\[
\dot{\alpha} = [F(x)]^T e, \\
\dot{\beta} = [G(y)]^T e, \quad (5)
\]

Then the response system (2) can synchronize the drive system (1), where \( k > 0 \) is a constant, \( \dot{\alpha} \) and \( \dot{\beta} \) are respectively, estimations of the unknown parameters \( \alpha \) and \( \beta \) where \( \alpha \) and \( \beta \) are constants.

**Proof.** From eqns. (1) & (2), we get the error dynamical system as follows
\[
\dot{e} = F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) - ke \quad (6)
\]
Let \( \tilde{\alpha} = \alpha - \tilde{\alpha}, \tilde{\beta} = \beta - \tilde{\beta} \). If the Lyapunov function is chosen as
\[
V(e, \tilde{\alpha}, \tilde{\beta}) = \frac{1}{2} [e^T e + (\alpha - \tilde{\alpha})^T (\alpha - \tilde{\alpha}) + (\beta - \tilde{\beta})^T (\beta - \tilde{\beta})] \quad (7)
\]

Then the derivative of \( V \) along the trajectory of the error dynamical system is as follows
\[
\dot{V}(e, \tilde{\alpha}, \tilde{\beta}) = \dot{e}^T e + (\alpha - \tilde{\alpha})^T \dot{\alpha} + (\beta - \tilde{\beta})^T \dot{\beta} \\
= [F(x)(\alpha - \tilde{\alpha}) + G(y)(\beta - \tilde{\beta}) - ke]^T e - (\alpha - \tilde{\alpha})^T [F(x)]^T e - (\beta - \tilde{\beta})^T [G(y)]^T e \\
= -ke^T e, e < 0 \quad (8)
\]

As long as \( e \neq 0 \), thus, \( \frac{dV}{dt} < 0 \) for \( V > 0 \), and the proof follows from the Theorem of Lyapunov stability.

**Remark 2.2** Most typical chaotic systems can be described by (1), such as the Pan system, the Chen system.

**Remark 2.3** If system (1) and system (2) satisfies \( f(.) = g(.) \) and \( F(.) = G(.) \). Then the structure of system (1) and system (2) are identical. Therefore, Theorem 2.1 is also applicable to the adaptive synchronization of two identical chaotic systems with unknown parameters.

The chaotic Pan system is given by
\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_1 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - bx_3
\end{align*} 
\quad (9)
\]

where \( x_1, x_2, x_3 \) are state variables and \( a, b, c \) are constant. If \( a = 10, b = 8/3 \) and \( c = 16 \) then the Pan system is chaotic.
Also, Chen system is given by

\[
\begin{align*}
\dot{x}_1 &= \alpha (x_2 - x_1) \\
\dot{x}_2 &= \gamma x_1 - \alpha x_1 - x_1 x_3 + \gamma x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 
\end{align*}
\] (10)

where \( x_1, x_2 \) and \( x_3 \) are state variables and \( \alpha, \beta \) and \( \gamma \) are constants. If \( \alpha = 35, \beta = 3 \) and \( \gamma = 28 \) then the Chen system is chaotic.
3. Adaptive synchronization of identical Pan systems

In this study, we use adaptive control method to derive the synchronization of identical uncertain chaotic Pan System. Thus, the master system is described as follows.

\[
\dot{x}_1 = a(x_2 - x_1) \\
\dot{x}_2 = cx_1 - x_1 x_3 \\
\dot{x}_3 = x_1 x_2 - bx_3
\]

(11)

where \( x_1 - x_3 \) are the states variables, \( a, b \) and \( c \) are unknown parameters of the system. The slave system is described by the controlled chaotic Pan dynamics.

\[
\dot{y}_1 = a(y_2 - y_1) + u_1 \\
\dot{y}_2 = cy_1 - y_1 y_3 + u_2 \\
\dot{y}_3 = y_1 y_2 - by_3 + u_3
\]

(12)

where \( y_1 - y_3 \) are states variables and \( u_1 - u_3 \) are nonlinear controller to be designed.

The synchronization error is defined as follows.

\[
e_i = y_i - x_i \quad (i = 1, 2, 3)
\]

(13)

Then we get error dynamics as

\[
\dot{e}_1 = a(e_2 - e_1) + u_1 \\
\dot{e}_2 = ce_1 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 = y_1 y_2 - be_2 - x_1 x_2 + u_3
\]

(14)

Let us now define the Adaptive functions \( u_i(t) - u_i(t) \) as.

\[
\begin{align*}
u_i(t) &= -\hat{a}(e_2 - e_1) - k_i e_1 \\
u_2(t) &= -\hat{c}e_1 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\
u_3(t) &= \hat{b}e_2 - x_1 x_2 - k_3 e_3
\end{align*}
\]

(15)

where \( \hat{a}, \hat{b}, \hat{c} \) are the estimates of \( a, b, c \), respectively, and \( k_i \) \( (i = 1, 2, 3) \) are positive constants.

From (14) and (15) we get,

\[
\begin{align*}
\dot{e}_1 &= (a - \hat{a})(e_2 - e_1) - k_i e_1 \\
\dot{e}_2 &= (c - \hat{c})e_1 - k_2 e_2 \\
\dot{e}_3 &= -(b - \hat{b})e_3 - k_3 e_3
\end{align*}
\]

(16)

Let us now define the parameter estimation error as,
\[ e_a = a - \hat{a} \]
\[ e_b = b - \hat{b} \]
\[ e_c = c - \hat{c} \]  

(17)

From (16) and (17),
\[ \dot{e}_1 = e_a (e_2 - e_1) - k_e e_1 \]
\[ \dot{e}_2 = e_c e_1 - k_e e_2 \]
\[ \dot{e}_3 = -e_c e - k_e e_3 \]  

(18)

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used, consider the quadratic Lyapunov function as,
\[ V = \frac{1}{2} (e_1^2 + e_2^2 + e_3^2) \]  

(19)

Which is a positive definite function on \( \mathbb{R}^6 \), we also have
\[ \dot{e}_1 = -\hat{a} \]
\[ \dot{e}_2 = -\hat{b} \]
\[ \dot{e}_3 = -\hat{c} \]  

(20)

Differentiating (19) along the trajectories of (18) and using (20), we get
\[ \dot{V} = e_a e_1 (e_2 - e_1) - k_e e_1^2 + e_c e_2 e_c - k_e e_2^2 - e_c^2 e_c e_c - k_e e_c^2 - k_e e_a - \hat{a} e_a - \hat{b} e_b - \hat{c} e_c \]  

(21)

From (21), the estimated parameters are updated by the following law.
\[ \dot{\hat{a}} = e_1 (e_2 - e_1) + k_e e_a \]
\[ \dot{\hat{b}} = -e_2^2 + k_e e_b \]
\[ \dot{\hat{c}} = e_3 e_2 + k_e e_c \]  

(22)

where \( k_e, k_2 \) and \( k_6 \) are positive constants. From (21) and (22), we get
\[ \dot{V} = -k_e e_1^2 - k_e e_2^2 - k_e e_3^2 - k_e e_4^2 - k_e e_5^2 - k_e e_6^2 \]  

(23)

which is a negative definite function on \( \mathbb{R}^6 \). Thus by Lyapunov stability theory [25], it is immediate that the synchronization error \( e_i \) (\( i = 1, 2, 3 \)) and the parameter estimation error \( e_a, e_b, e_c \) decay to zero with time.

**Result 3.1** The identical chaotic Pan systems are synchronized by adaptive control law (15), where the update law for the parameter estimates is given by (22) and \( k_i (i = 1, 2, 3, 4, 5, 6) \) are positive constants.
Numerical Result: To solve the (11) and (12) with the adaptive nonlinear controller (15), by using mathematica. We take $k_i = 2(i = 1, 2, 3, 4, 5, 6)$. The parameters of the chaotic Pan systems is chosen as $a = 10, b = 8/3, c = 16$. The initial values of parameter estimates are taken as $\hat{a}(0) = 10, \hat{b}(0) = 24, \hat{c}(0) = 20$ and the initial values of master and slave systems are chosen as $x_i(0) = 15, x_2(0) = 12, x_3(0) = 32, y_i(0) = 24, y_2(0) = 20$ and $y_3(0) = 16$, respectively. Figure 3. Shows the synchronization of the Pan system and Figure 4. Shows the estimated values of the parameters, $\hat{a}, \hat{b}$ and $\hat{c}$ converge to the system parameters $a = 10, b = 8/3$ and $c = 16$, respectively.

Figure 3. Synchronization of Pan system
4. Adaptive synchronization of two identical chaotic Chen systems

The master system is given by

$$\begin{align*}
\dot{x}_1 &= \alpha(x_2 - x_1) \\
\dot{x}_2 &= \gamma x_1 - \alpha x_1 - x_1 x_3 + \gamma x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 
\end{align*}$$

(24)

And slave system is

$$\begin{align*}
\dot{y}_1 &= \alpha(y_2 - y_1) + u_i \\
\dot{y}_2 &= \gamma y_1 - \alpha y_1 - y_1 y_3 + \gamma y_2 + u_2 \\
\dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 
\end{align*}$$

(25)

where $u_i - u_3$ are nonlinear controller to be designed.

The synchronization error is

$$e_i = y_i - x_i \ (i = 1, 2, 3)$$

(26)

Then the error dynamics is defined as follows

$$\begin{align*}
\dot{e}_1 &= \alpha(e_2 - e_1) + u_i \\
\dot{e}_2 &= \gamma(e_1 + e_2) - \alpha e_1 - y_1 y_3 + x_1 x_3 + u_2 \\
\dot{e}_3 &= y_1 y_2 - \beta e_3 - x_1 x_2 + u_3 
\end{align*}$$

(27)

Now defining the adaptive control functions $u_i(t) - u_3(t)$.

$$\begin{align*}
u_i(t) &= -\dot{\alpha}(e_2 - e_1) - k_1 e_i \\
u_2(t) &= -\gamma(e_1 + e_2) + \alpha e_1 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\
u_3(t) &= -y_1 y_2 + \beta e_3 + x_1 x_2 - k_3 e_3 
\end{align*}$$

(28)
where \( \tilde{\alpha} - \tilde{\gamma} \) are estimates of \( \alpha - \gamma \), respectively, and \( k_i \) \((i = 1, 2, 3)\) are positive constants.

Then the error dynamics becomes

\[
\begin{align*}
\dot{e}_1 &= (\alpha - \tilde{\alpha})(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= (\gamma - \tilde{\gamma})(e_1 + e_2) - (\alpha - \tilde{\alpha})e_1 - k_2 e_2 \\
\dot{e}_3 &= -(\beta - \tilde{\beta})e_3 - k_3 e_3
\end{align*}
\]

(29)

The parameter error is given by

\[
\begin{align*}
e_{\alpha} &= \alpha - \tilde{\alpha} \\
e_{\beta} &= \beta - \tilde{\beta} \\
e_{\gamma} &= \gamma - \tilde{\gamma}
\end{align*}
\]

(30)

From (29) and (30)

\[
\begin{align*}
\dot{e}_1 &= e_{\alpha}(e_2 - e_1) - k_1 e_1 \\
\dot{e}_2 &= e_{\gamma}(e_1 + e_2) - e_{\alpha} e_1 - k_2 e_2 \\
\dot{e}_3 &= -e_{\beta} e_3 - k_3 e_3
\end{align*}
\]

(31)

We consider the quadratic Lyapunov function is given by

\[
V = \frac{1}{2}(e_{\alpha}^2 + e_{\beta}^2 + e_{\gamma}^2 + e_{\alpha} e_{\beta} + e_{\alpha} e_{\gamma} + e_{\beta} e_{\gamma})
\]

(32)

which is a positive definite function on \( \mathbb{R}^6 \), we also noted that

\[
\begin{align*}
\dot{e}_{\alpha} &= -\dot{\alpha} \\
\dot{e}_{\beta} &= -\dot{\beta} \\
\dot{e}_{\gamma} &= -\dot{\gamma}
\end{align*}
\]

(33)

Differentiating (32) we get

\[
\begin{align*}
\dot{V} &= e_{\alpha} e_{\alpha}(e_2 - e_1) - k_1 e_1^2 + e_{\gamma} e_{\gamma}(e_1 + e_2) - e_{\alpha} e_1 e_2 \\
&\quad + e_2^2 - e_{\beta} e_2^2 - k_3 e_3^2 + e_{\alpha} [-e_2(e_2 - e_1) + e_1 e_2 - k_2 e_{\alpha}] + e_{\beta} (e_3^2 - k_3 e_{\beta}) \\
&\quad + e_1 [-e_2(e_1 + e_2) - k_3 e_{\gamma}]
\end{align*}
\]

(34)

The parameter update law is given by

\[
\begin{align*}
\dot{\alpha} &= e_1(e_2 - e_1) - e_2 e_{\alpha} + k_4 e_{\alpha} \\
\dot{\beta} &= -e_{\beta}^2 + k_5 e_{\beta} \\
\dot{\gamma} &= e_2(e_1 + e_2) + k_6 e_{\gamma}
\end{align*}
\]

(35)

where \( k_4, k_5, k_6 \) are positive constants.
Then (34) becomes

\[ \dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_\alpha^2 - k_5e_\beta^2 - k_6e_\gamma^2 \]  

which is a negative definite function on $R^6$. Thus by Lyapunov stability theory [25], it is immediate that the synchronization error $e_i (i=1,2,3)$ and the parameter estimation error $e_\alpha, e_\beta$ and $e_\gamma$ decay to zero with time.

**Result 4.1** The identical chaotic Chen systems are synchronized by adaptive control law (28), where the update law for the parameter estimates is given by (35) and $k_i(i=1,2,3,4,5,6)$ are positive constants.

**Numerical Result:** To solve the (24) and (25) with the adaptive nonlinear controller (28), by using the mathematica. We take $k_i = 2(i=1,2,3,4,5,6)$. The parameters of the chaotic Chen systems is chosen as $\alpha = 35, \beta = 3, \gamma = 28$. The initial values of parameter estimates are taken as $\hat{\alpha}(0) = 17, \hat{\beta}(0) = 4, \hat{\gamma}(0) = 20$ and the initial values of master and slave systems are chosen as $x_i(0) = 5, x_i(0) = 18, x_i(0) = 26, y_i(0) = 20, y_i(0) = 35$ and $y_i(0) = 10$, respectively. Figure 5. Shows the synchronization of the Chen system and Figure 6. Shows the estimated values of the parameters, $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$ converge to the system parameters $\alpha = 35, \beta = 3$ and $\gamma = 28$, respectively.

5. **Adaptive synchronization of two different Pan and Chen systems**

Here, we take Pan System as the master system and Chen system as the slave system is as follows.

\[
\begin{align*}
\dot{x}_1 &= a(x_2 - x_1) \\
\dot{x}_2 &= cx_1 - x_1x_3 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{y}_1 &= \alpha(y_2 - y_1) + u_i \\
\dot{y}_2 &= \gamma y_1 - \alpha y_1 - y_1y_3 + \gamma y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - \beta y_3 + u_3
\end{align*}
\]

The synchronization error is defined as

\[ e_i = y_i - x_i \ (i=1,2,3) \]  

Then the error dynamics from (37) and (38) is as follows

\[
\begin{align*}
\dot{e}_1 &= \alpha(y_2 - y_1) - a(x_2 - x_1) + u_i \\
\dot{e}_2 &= \gamma y_1 - \alpha y_1 - y_1y_3 + \gamma y_2 - cx_1 + x_1x_3 + u_2 \\
\dot{e}_3 &= y_1y_2 - \beta y_3 - x_1x_2 + bx_3 + u_3
\end{align*}
\]

Defining the adaptive control function as
\[ u_i(t) = -\tilde{\alpha}(y_2 - y_i) + \tilde{\alpha}(x_2 - x_i) - k_i e_i \]
\[ u_2(t) = -\tilde{\gamma} y_1 + \tilde{\gamma} y_3 - \tilde{\gamma} y_2 + \hat{\alpha} x_1 - x_i x_3 - k_2 e_2 \]
\[ u_3(t) = -y_1 y_2 - \beta y_3 - x_i x_2 + \hat{b} x_3 - k_3 e_3 \]

Then the error dynamics becomes
\[ \dot{e}_1 = (\alpha - \tilde{\alpha})(y_2 - y_1) - (a - \tilde{a})(x_2 - x_1) - k_1 e_1 \]
\[ \dot{e}_2 = (\gamma - \tilde{\gamma}) y_1 - (\alpha - \tilde{\alpha}) y_1 + (\gamma - \tilde{\gamma}) y_2 - (c - \tilde{c}) x_1 - k_2 e_2 \]
\[ \dot{e}_3 = -\beta \gamma y_3 + (b - \tilde{b}) x_3 - k_3 e_3 \]  

Figure 5. Synchronization of Chen system
Figure 6. Adaptive parameters estimation errors: $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$

The parameter estimation error is defined as

\[
\begin{align*}
e_a &= a - \hat{a} \\
e_b &= b - \hat{b} \\
e_c &= c - \hat{c} \\
e_\alpha &= \alpha - \hat{\alpha} \\
e_\beta &= \beta - \hat{\beta} \\
e_\gamma &= \gamma - \hat{\gamma}
\end{align*}
\] (43)

From (42) and (43)

\[
\begin{align*}
\dot{e}_1 &= e_\alpha (y_2 - y_1) - e_\alpha (x_2 - x_1) - k_1 e_1 \\
\dot{e}_2 &= e_\gamma y_1 - e_\alpha y_1 + e_\gamma y_2 - e_\alpha x_1 - k_2 e_2 \\
\dot{e}_3 &= -e_\beta y_3 + e_\gamma x_3 - k_3 e_3
\end{align*}
\] (44)

Chose the quadratic Lyapunov function as

\[
V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2 + e_\alpha^2 + e_\beta^2 + e_\gamma^2)
\] (45)

which is positive definite function on $\mathbb{R}^9$, we also have

\[
\begin{align*}
\dot{e}_\alpha &= -\hat{\alpha}, \quad \dot{e}_\beta = -\hat{\beta} \\
\dot{e}_\gamma &= -\hat{\gamma}, \quad \dot{e}_\alpha = -\hat{\alpha} \\
\dot{e}_b &= -\hat{b}, \quad \dot{e}_b = -\hat{c}
\end{align*}
\] (46)
The parameters estimated update law is defined as
\[
\begin{align*}
\dot{a} &= e_1(y_2 - y_1) - e_2y_1 + k_\alpha e_\alpha \\
\dot{b} &= -e_3y_3 + k_\beta e_\beta \\
\dot{c} &= y_3e_2 + y_2e_3 + k_\gamma e_\gamma \\
\dot{\alpha} &= -e_1(x_2 - x_1) + k_\gamma e_\alpha \\
\dot{b} &= x_3e_3 + k_\beta e_b \\
\dot{c} &= -e_2x_1 + k_\gamma e_c
\end{align*}
\]
(47)

Now differentiating (45) we get
\[
\dot{V} = -k_\alpha e_\alpha^2 - k_\beta e_\beta^2 - k_\gamma e_\gamma^2 - k_\alpha e_\alpha^2 - k_\beta e_\beta^2 - k_\gamma e_\gamma^2 - k_\alpha e_\alpha^2 - k_\beta e_\beta^2 - k_\gamma e_\gamma^2
\]
(48)

which is a negative definite function on \( R^9 \). Thus by Lyapunov stability theory [25], it is immediate that the synchronization error \( e_i \) \((i = 1, 2, 3)\) and the parameter estimation error \( e_\alpha, e_\beta, e_\gamma, e_a, e_b, e_c \) and \( e_i \) decay to zero with time.

**Result 5.1** The chaotic Pan and Chen systems are synchronized by adaptive control law (41), where the update law for the parameter estimates is given by (47) and \( k_i (i = 1, 2, 3, 4, 5, 6) \) are positive constants.

**Numerical Result:** To solve the (37) and (38) with the adaptive nonlinear controller (41), by using the mathematica. We take \( k_i = 2 (i = 1, 2, 3, 4, 5, 6) \). The parameters of the chaotic Pan and Chen systems are chosen as \( a = 10, b = 8 / 3, c = 16 \) and \( \alpha = 35, \beta = 3, \gamma = 28 \), respectively. The initial values of parameter estimates are taken as \( \tilde{a}(0) = 17, \tilde{b}(0) = 4, \tilde{c}(0) = 20 \), \( \tilde{a}(0) = 10, \tilde{b}(0) = 24, \tilde{c}(0) = 20 \) and the initial values of master and slave systems are chosen as \( x_1(0) = 15, x_2(0) = 12, x_3(0) = 32, y_1(0) = 5, y_2(0) = 18 \) and \( y_3(0) = 26 \), respectively. Figure 7. Shows the synchronization of the Pan and Chen system and Figure: 8. Shows the estimated values of the parameters \( \tilde{a}, \tilde{\beta}, \tilde{\gamma}, \tilde{a}, \tilde{b} \) and \( \tilde{c} \) converge to the system parameters \( \alpha = 35, \beta = 3, \gamma = 28, a = 10, b = 8 / 3 \) and \( c = 16 \) respectively.

6. Conclusion

In this paper, we apply adaptive nonlinear control method for chaos synchronization of identical Pan Systems, identical Chen systems and nonidentical Pan and Chen systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Numerical results are shown the effectiveness of the synchronization scheme.
Figure 7. Synchronization of Pan and Chen systems

Figure 8. Adaptive parameters estimation errors: $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$
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