Stark resonances in a quantum waveguide with analytic curvature

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Abstract
We investigate the influence of an electric field on trapped modes arising in a two-dimensional curved quantum waveguide $\Omega$ i.e. bound states of the corresponding Laplace operator $-\Delta_{\Omega}$. Here the curvature of the guide is supposed to have an analytic extension in some complex domain around the real axis satisfying $|\gamma(z)| = O(|z|^{-\varepsilon})$, $\varepsilon > 3$ as $|\text{Re} z| \to \infty$. We show that under conditions on the electric field $F$, $H(F) = -\Delta_{\Omega} + F \cdot x$ has resonances near the discrete eigenvalues of $-\Delta_{\Omega}$.

Keywords: resonance, operator theory, Schrödinger operators

1. Introduction

This paper is a continuation and extension of earlier work [2] on the study of spectral resonances arising in a bent quantum waveguide when it is submitted to an electric field having a direction which has a non-zero projection on the longitudinal axis.

It is now well known that geometrical defects of the waveguide induce bound states, see e.g. [13] for a remarkable review of this problem. Basically, we are interesting whether these effects can be cancelled when the field is switched on. In other words do these bound states give rise to resonances under the perturbation induced by the field?

This question has not been studied so far. But it is worth to mention a work of Exner [6], the field he considered is chosen to be orthogonal to the guide outside a bounded region. The problem is then of a quite different nature and in particular the system has no resonance in the sense developed in this work.

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There are only a few results in mathematical physics literature on the resonance phenomena occurring in waveguides. However, we mention [5] where the authors also consider this problem in a two-dimensional case, but in another context.

Let us now recall the mathematical formulation of the problem; for more details we refer to [2, 6]. The object of our interest is the Stark operator

$$\mathbf{H}(F) = -\Delta_{\Omega} + F \cdot \mathbf{x}, \ F \in \mathbb{R}^2, \ \mathbf{x} = (x, y) \in \Omega$$

on $L^2(\Omega)$ where $\Omega$ is a curved strip in $\mathbb{R}^2$ of constant width $d > 0$ defined around a smooth curve $\Gamma$. The operator $-\Delta_{\Omega}$ is defined in a standard way by means of Dirichlet Boundary Conditions (DBC) on the boundary of $\Omega$, $\partial \Omega$ [15].

We assume that $\Omega$ is not straight, let $\mathbb{R} g_{11}$ be the signed curvature of $\Gamma$. In [2] it is supposed that $g \in C^2(\mathbb{R})$ and $d||g||_{\infty} < 1$. Here we consider a more general situation, namely we assume that

(h1) $g \in C^2(\mathbb{R})$ and there exist $0 < a_0 < \pi/2$, $0 < r_0$ s.t. $g$ has an analytic extension in $O_{a_0,r_0}$ = $\{z \in \mathbb{C}, \ |\arg z| < a_0 \} \cup \{z \in \mathbb{C}, \ |\pi - \arg z| < a_0\} \cap \{z \in \mathbb{C}, \ |\text{Re} z| > r_0\}$. Moreover $g$ satisfies $d||\text{Re} g||_{\infty} < 1$.

(h2) There exists $\varepsilon > 3$ s.t. $|g(z)| = O(|z|^{-\varepsilon})$ as $|\text{Re} z| \to \infty$.

Then $\Omega$ is asymptotically straight. We choose the lower boundary of $\Omega$ near $s = -\infty$ as the reference curve. Introduce orthogonal coordinates $(s, u) \in \Omega := \mathbb{R} \times (0, d)$, related to $(x, y) \in \Omega$ via the relations [7],

$$x(s, u) = \int_0^s \cos(\alpha(t))dt - u \sin(\alpha(s)), \ y(s, u) = \int_0^s \sin(\alpha(t))dt + u \cos(\alpha(s)),$$

where $\alpha(s) = \int_{-\infty}^s g(t) \ dt$. Set $\alpha_0 = \int_{-\infty}^{+\infty} g(t) \ dt$.

Since we have supposed $d \sup \{|g(s)|, s \in \mathbb{R}\} < 1$, the operator $-\Delta_{\Omega}$ is unitarily equivalent to

$$H = H_0 + V_0, \ H_0 = T_s + T_u$$

acting in the space $L^2(\Omega)$, with DBC on $\partial \Omega$ [7], where

$$T_s := -\partial_s g \partial_s, \ T_u := -\partial_u^2, \ g(s, u) = (1 + u g(s))^{-2}$$

and

$$V_0(s, u) = -\frac{\gamma(s)^2}{4(1 + u g(s))^2} + \frac{u \gamma''(s)}{2(1 + u g(s))^3} - \frac{5}{4} \frac{u^2 \gamma'(s)^2}{(1 + u g(s))^4}.$$  

With our assumptions, the potential $V_0$ is bounded and then $H = H_0 + V_0$ is a self-adjoint operator with the domain [8, 14],

$$D(H) = D(H_0) = \mathcal{H}_0^2(\Omega) \cap \mathcal{H}^2(\Omega).$$

Here and hereinafter we use standard notation for Sobolev spaces. Moreover the essential spectrum, $\sigma_{es}(H) = [\lambda_0, +\infty)$, where $\{\lambda_0, \lambda_1, \ldots\}$ are the transverse modes of the system i.e. the eigenvalues of the operator $-\partial_u^2$ on $L^2(0, d)$ with DBC on the boundary $[0, d]$ [4].

Denote the exterior field as $F = F(\cos(\eta), \sin(\eta))$. With respect to the new coordinates, the field interaction is then
Here we study a field regime which was not considered so far, i.e. the intensity of the field is the free parameter in $0 < F < 1$ and the direction $\eta$ is fixed satisfying

\[ |\eta| < \frac{\pi}{2} \quad \text{and} \quad |\eta - \alpha_0| > \frac{\pi}{2}. \]  

(1.8)

As discussed in section 2, this implies that $W(F)(s, u) \to -\infty$ as $s \to \pm \infty$. Thus the non-trapping region for a given negative energy $E$ i.e. $\{ (s, u) \in \Omega, W(F)(s, u) < E \}$ contains both a neighbourhood of $s = -\infty$ and $s = \infty$.

We denote by $H_0(F) = T_s + T_u + W(F)$ and $H(F) = H_0(F) + V_0$. Then a straightforward extension of the theorem 2.1 of [2] shows that for $F > 0$, the Stark operator $H(F)$ is essentially self-adjoint and $\sigma(H(F)) = \mathbb{R}$.

We are interested in the study of the influence of the electric field on the discrete spectrum of $H$. We want to show that the eigenvalues of $H$ give rise to resonances for the Stark operator $H(F)$, $F > 0$. The resonances of $H(F)$ are understood in the standard way [1, 11, 15] i.e. there are defined as the complex poles in the second Riemann sheet of the meromorphic continuation of the resolvent associated to $H(F)$. Evidently, if $H$ has no discrete eigenvalue below $\lambda_0$ this result proves that $H(F)$ has neither resonance or embedded eigenvalue in $\{ z \in \mathbb{C}, \text{Re} z < \lambda_0 \}$. For a discussion about eigenvalues of $H$ we refer the reader to [4].

To study this problem we need an additional assumption,

(h3) $\text{Im} \gamma \geq 0$ in $\{ z \in \mathcal{O}_{a_0, r_0}, 0 \leq \text{arg} z \leq a_0 \} \cup \{ z \in \mathcal{O}_{a_0, r_0}, 0 \leq \text{arg} z - \pi \leq a_0 \}$.

Remark 1.1. 1- In fact it is only necessary to suppose that the product $u \text{Im} \gamma \geq 0$ in $\{ z \in \mathcal{O}_{a_0, r_0}, 0 \leq \text{arg} z \leq a_0 \} \cup \{ z \in \mathcal{O}_{a_0, r_0}, 0 \leq \text{arg} z - \pi \leq a_0 \}$. So the case $\text{Im} \gamma \leq 0$ is reducing to the present one by taking the other boundary as reference curve.

2- We can check that $\gamma(s) = -\frac{\pi^2}{n \sin^2 \alpha}$; $n \geq 2, \alpha < 0$ satisfies our assumptions with $r_0 > 1$ and $0 < a_0 \leq \frac{\pi}{2 \alpha}$.

The study of the Stark effect was considered by several authors, see e.g. [3, 9, 10] for a discussion concerning the case of Shrödinger operators on $\mathbb{R}^d$ and [2, 6] for the Laplace operator defined on curved strips. In particular in [2], under assumptions on the curvature above mentioned and if $|\eta| < \frac{\pi}{2}$ and $|\eta - \alpha_0| < \frac{\pi}{2}$, it is proved the existence of Stark resonances having a width exponentially small w.r.t. $F$ as $F$ tends to zero. In this paper, we would like to extend this result under weaker assumptions on $\gamma$ i.e. hypotheses (h1–3) and in the field regime (1.8).

More precisely we will show the following.

Theorem 1.2. Suppose (h1–3). Let $E_0$ be a discrete eigenvalue of $H$ of finite multiplicity $n \in \mathbb{N}$. There exist $F_0 > 0$ and a dense subset $\mathcal{A}$ of $L^2(\Omega)$ such that for $0 < F \leq F_0$

(i) $z \in \mathbb{C}, \text{Im} z > 0 \mapsto \mathcal{R}_z(\varphi) = ((H(F) - z)^{-1} \varphi, \varphi), \varphi \in \mathcal{A}$

has a meromorphic extension in a complex neighbourhood $\nu_{E_0}$ of $E_0$ through the cut due to the presence of the continuous spectrum of $H(F)$.

(ii) $\bigcup_{\varphi \in \mathcal{A}} \{ \text{poles of } \mathcal{R}_z(\varphi) \} \cap \nu_{E_0}$ contains $n$ poles $Z_0(F), \ldots Z_{n-1}(F)$ converging to $E_0$ when $F \to 0$. 

(iii) Suppose that $E_0$ is a simple eigenvalue of $H$. Let $Z_0(F)$ as in (ii), then there exist two constants $0 < c_1, c_2$ such that for $0 < F \leq F_0$,
$$\left|\text{Im} Z_0\right| \leq c_1 e^{-c_2 F}.$$ 

In this paper, we only give elements we need to extend the strategy of [2] to the situation we now consider. In particular theorem 1.2 (iii) is covered by [2, section 6], so we omit the proof here.

The plan of this work is as follows. In section 2 we introduce a local modification of the operator $H(F)$ we need to perform the meromorphic continuation of the resolvent of $H(F)$. Some elements of the complex distortion theory are given in section 3. In sections 4 and 5 we define the extension of the resolvent of $H(F)$, this allow to define the resonances of $H(F)$. The existence of resonances is studied in section 6. Section 7 is devoted to some concluding remarks.

2. The reference operator

To prove the theorem 1.2 we use the distortion theory which can be found in [2, 3]. The first step is to consider a local modification of the operator $H_0(F)$ called the reference operator. It is defined as follows. Note that if $s < 0$, $W(F, s, u) = F(s \cos(\eta) + u \sin(\eta) + A_-) + R_-(F, s, u)$,
$$W(F, s, u) = F\left(\int_{-\infty}^{t} \left(\cos(\eta - \alpha(t)) - \cos(\eta)\right)dt + u\left(\sin(\eta - \alpha(s)) - \sin(\eta)\right)\right)$$
if $s \geq 0$, $W(F, s, u) = F(s \cos(\eta - \alpha_0) + u \sin(\eta - \alpha_0) + A_+) + R_+(F, s, u)$,

where
$$R_+(F, s, u) = F\left(\int_{t}^{\infty} \left(\cos(\eta - \alpha_0) - \cos(\eta - \alpha(t))\right)dt + u\left(\sin(\eta - \alpha(s)) - \sin(\eta - \alpha_0)\right)\right).$$

The constants $A_-, A_+$ are
$$A_- = \int_{-\infty}^{0} \left(\cos(\eta) - \cos(\eta - \alpha(t))\right)dt, \quad A_+ = \int_{0}^{\infty} \left(\cos(\eta - \alpha(t)) - \cos(\eta - \alpha_0)\right)dt.$$

In view of (h2) and
$$\alpha(s) = O\left(\frac{1}{|s|^{\frac{1}{2}}-1}\right), \quad \text{as} \; s \rightarrow -\infty; \quad \alpha(s) = \alpha_0 + O\left(\frac{1}{|s|^{\frac{1}{2}}-1}\right) \quad \text{as} \; s \rightarrow \infty$$
$$A_-, A_+ \text{ are well defined. Moreover we have,}$$
$$R_-(F, s, u), R_+(F, s, u) = O\left(\frac{F}{|s|^{\frac{1}{2}}-1}\right), \quad \text{as} \; s \rightarrow \pm\infty.$$ 

uniformly w.r.t. $u \in (0; d)$. Set $R_-(F, s, u) = 0$ for $s \geq 0$ and $R_+(F, s, u) = 0$ for $s < 0$. 

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Hence it is quite natural to consider a modified interaction defined as,

\[ \tilde{W}(F)(s, u) = F(s \cos(\eta) + u \sin(\eta)) + A_- \] for \( s < 0 \)

and

\[ \tilde{W}(F)(s, u) = F(s \cos(\eta - \alpha_0) + u \sin(\eta - \alpha_0)) + A_+ \] for \( s \geq 0 \).

In particular let

\[ R(F) := W(F) - \tilde{W}(F) = R_+(F) + R_-(F) \quad (2.13) \]

We have, uniformly w.r.t. \( u \in (0; d) \),

\[ R'(F, s, u) = O\left(\frac{F}{|s|^{\alpha}}\right), \quad R''(F, s, u) = O\left(\frac{F}{|s|^{\beta}}\right) \] as \( s \to \pm \infty \).

Let \( \hat{H}_0(F) \) be the reference operator in \( L^2(\Omega) \),

\[ \hat{H}_0(F) = H_0 + \tilde{W}(F). \]

\( \hat{H}_0(F) \) differs from \( H_0(F) \) by an additional bounded operator, then it is essentially self-adjoint and \( \sigma(\hat{H}_0(F)) = \mathbb{R} \).

### 3. Complex distortion

In this section, we give necessary elements of the complex distortion theory we need to define the family of distorted operators \( \{H_\theta(F), F \leq F_0\} \), for some \( F_0 > 0 \) and complex values of \( \theta \) [1, 3, 11]. We denote by \( \beta = \text{Im} \theta \).

Let \( E \in \mathbb{R}, E < 0 \) be the reference energy and \( 0 < \delta E < \frac{1}{2} \min \{1, |E|\} \). Denote \( E_- = E - \delta E \) and \( E_+ = E + \delta E \). We choose a real function \( \phi \in C^\infty(\mathbb{R}) \) such that,

\[ \phi(t) = \begin{cases} 1, & \text{if } t < E, \\ 0, & \text{if } t > E_+ \end{cases} \quad (3.15) \]

and satisfying \( \|\phi^{(k)}\|_\infty = O\left(\left(\frac{1}{\delta E}\right)^k\right) \).

For \( \theta \in \mathbb{R} \), we introduce the distortion on \( \Omega \), \( s_\theta(s, u) = (s + \theta \phi(s), u) \) with

\[ f(s) = \begin{cases} -\frac{1}{F \cos(\eta)} \Phi_-(s), & \text{if } s \leq 0, \\ -\frac{1}{F \cos(\eta - \alpha_0)} \Phi_+(s), & \text{if } s > 0, \end{cases} \quad (3.16) \]

where \( \Phi_-(s) = \phi(F \cos(\eta)s) \) and \( \Phi_+(s) = \phi(F \cos(\eta - \alpha_0)s) \). Set \( \Phi(s) = \Phi_-(s) + \Phi_+(s) \).

So for \( k \geq 1 \), \( \|\phi^{(k)}\|_\infty \leq \left(\frac{F}{\delta E}\right)^k, \quad \|f^{(k)}\|_\infty \leq \frac{F^{k-1}}{(\delta E)^k} \). In view of the definition (3.16), for small \( F, s_\theta \) is an translation along the longitudinal axis in neighbourhood of \( s = \pm \infty \) since

\[ f(s) = \begin{cases} \frac{1}{F \cos(\eta)}, & \text{if } s < -\frac{E}{F \cos(\eta)}, \\ \frac{-1}{F \cos(\eta - \alpha_0)}, & \text{if } s > \frac{E}{F \cos(\eta - \alpha_0)} \end{cases} \quad (3.17) \]
Assume $|\theta| < \delta E$. Let $U_\theta$ be the operator defined on $L^2(\Omega)$ by

$$ U_\theta \psi(s, u) = (1 + \theta f')^2 \psi(s_\theta(s, u)). $$

(3.18)

The operators $U_\theta$ are unitary and generate the family,

$$ H_\theta(F) = U_\theta H(F) U_\theta^{-1} = H_{0,\theta}(F) + V_{0,\theta}, $$

(3.19)

where

$$ H_{0,\theta}(F) := T_{s,\theta} + T_u + W_\theta(F), $$

(3.20)

with

$$ T_{s,\theta} = -(1 + \theta f')^{-2} \partial_s g_0 (1 + \theta f')^{-1} \partial_x (1 + \theta f')^{-2}, $$

(3.21)

$$ g_0 = (1 + u g_\theta)^2, \ g_\theta = \gamma \circ s_\theta, W_\theta(F) = W(F) \circ s_\theta \text{ and } V_{0,\theta} = V_0 \circ s_\theta. $$

We also have

$$ T_{s,\theta} = -\partial_s (1 + \theta f')^{-2} g_0 \partial_x + S_0, $$

(3.22)

where

$$ S_0 = -\frac{5g_\theta}{4} \frac{\theta f''^2}{(1 + \theta f')^2} + \frac{g_\theta}{2} \frac{\theta f''}{(1 + \theta f')^3} + \frac{g_\theta'}{2} \frac{\theta f''}{(1 + \theta f')^3}. $$

(3.23)

For small $F$, the analytic extension of the family $\{H_{0,\theta}(F), \ \theta \in \mathbb{R}, |\theta| < \delta E\}$ to a complex disk, $|\theta| \leq \beta_0$ for some $\beta_0 > 0$, depends strongly on the analytic property of the dilated curvature $g_\theta$. But clearly (h1) and the definition (3.18) of $s_\theta$ imply that if $0 < F \leq F_0$ with $F_0$ is small enough, $s \in \mathbb{R}, \ \theta \rightarrow g_\theta(s)$ is analytic in $|\theta| \leq |E| \tan(\alpha_0)$.

Then the same arguments as in the proof of [2, proposition 3.1] lead to

**Proposition 3.3.** Suppose (h1–2). There exist $F_0 > 0$ and $0 < \beta_0 \leq \min\{\delta E, |E| \tan(\alpha_0)\}$ such that for all $0 < F \leq F_0$, $\{H_{0,\theta}(F); |\theta| \leq \beta_0\}$ is a self-adjoint analytic family of operators [12].

It should be noted that all the critical values $\beta_0$ and $F_0$ that appear in this article are independent from each other.

In a similar way let $\hat{H}_{0,\theta}(F) = H_{0,\theta} + \hat{W}_\theta(F)$, where $\hat{W}_\theta(F) = \hat{W}_\theta(F) \circ s_\theta$. Since

$$ R_\theta(F) = W_\theta(F) - \hat{W}_\theta(F) = R(F) + i \beta R'(F) + O \left( \frac{F^2 |\beta|^2}{|s|^3} \right), $$

(3.24)

then from (2.13) and (2.14) and (h1), the multipliers $W_\theta(F) - \hat{W}_\theta(F)$ and $V_{0,\theta}$ have an analytic extension as bounded operators in $|\beta| \leq \beta_0$. Hence we get

**Corollary 3.4.** For all $0 < F < F_0$ the family of operators $\{\hat{H}_{0,\theta}(F); |\theta| \leq \beta_0\}$ is a self-adjoint analytic family of operators.

For some technical points of the section 6 below, we need to introduce another family of operators on $L^2(\Omega)$, let $\theta \in \mathbb{R}, |\theta| < \delta E$ and

$$ H_{0,\theta} = U_\theta (T_{s,\theta} + T_u) U_\theta^{-1} = T_{s,\theta} + T_u. $$

(3.25)

Note that $H_{0,\theta} = H_{0,0}(F = 0)$. Indeed the distortion is supported in $|s| \geq c'/F$ for some $c' > 0$, thus at least formally $H_{0,\theta}(F = 0) = H_0$. 

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Assume (h1) and (h2). Following arguments of [2, proposition 3.1] evoked above, there exist $F_0 > 0$ and $0 < \beta_0 \leq \min \{\delta E, |E|\tan(\alpha_0)\}$ such that for all $0 < F < F_0$, $|\theta| \leq \beta_0$, $D(H_{0, \theta}) = D(H_0)$ is also a self-adjoint analytic family of operators. In fact the main property of $H_{0, \theta}$ we use in section 6 below is:

**Proposition 3.5.** Suppose (h1–3) hold. Then there exists $0 < \beta_0$ and $0 < F_0$ such that for $0 < F < F_0$, $\beta = 1 \theta > 0$, $|\theta| \leq \beta_0 \leq \min \{\delta E, |E|\tan(\alpha_0)\}$ is a family of sectorial operators with a sector contained in

$$S = \{z \in \mathbb{C}, -2c\beta \leq \arg(z - \lambda_0 + \zeta) \leq 0\},$$

for some strictly positive constant $c$. Here $\zeta$ is an error term with $\text{Re} \, \zeta, \text{Im} \, \zeta \geq 0$ and $|\zeta| = O(\beta F^2)$.

**Proof.** We may suppose $\theta = i\beta$, $0 \leq \beta \leq \beta_0$. Note first that for $\beta$ small enough then

$$\gamma_0(s) = \gamma(s + i\beta'(s)) = \gamma(s) + i\beta'(s)\gamma'(s) + O\left(\frac{\beta^2}{|s|^3}\right). \tag{3.26}$$

By using (3.22) we have for $\varphi \in D(H_0)$, $\|\varphi\| = 1$,

$$((H_{0, \theta} - \lambda_0)\varphi, \varphi) = (G_\theta \partial_\theta \varphi, \partial_\theta \varphi) + (S_\theta \varphi, \varphi) + ((T_\theta - \lambda_0)\varphi, \varphi), \tag{3.27}$$

where $G_\theta = (1 + \theta\gamma)^{-1}$ and $S_\theta = (G_\theta \partial_\theta \varphi, \partial_\theta \varphi) + ((T_\theta - \lambda_0)\varphi, \varphi)$. By using (h2), we have for $F$ and $\beta$ small,

$$\text{Re} \, G_\theta^{-1} = (1 + u \text{Re} \, \gamma_0)^2 - (u \text{Im} \, \gamma_0)^2)(1 - \beta^2 F^2) - 4\beta' u \text{Im} \, \gamma_0(1 + \text{Re} \, \gamma_0) \tag{3.28}$$

$$\geq (1 + u \text{Re} \, \gamma_0)^2(1 - O(\beta^2)) + O(\beta^2 F^3) \geq c_1, \tag{3.29}$$

for some constant $c_1 > 0$. Then in view of the definition of $G_\theta$, the assumption (h1), (3.26) and (3.28), it is straightforward to check for $F$ and $\beta$ small $\text{Re} \, G_\theta \geq c'_1$ for some another constant $c'_1 > 0$. So $\text{Re} \, q \geq c'_1 \|\partial_\theta \varphi\|^2$. In the other hand,

$$\text{Im} \, G_\theta^{-1} = 2(\beta'(1 + u \text{Re} \, \gamma_0)^2 - (u \text{Im} \, \gamma_0)^2) + u \text{Im} \, \gamma_0(1 + \text{Re} \, \gamma_0)(1 - \beta^2 F^2)) \tag{3.30}$$

Since $f' \geq 0$ and by (h3), $\text{Im} \, \gamma_0 \geq 0$ then for $\beta$ and $F$ small $\text{Im} \, G_\theta^{-1} \geq 0$, hence $\text{Im} \, q \leq 0$. In the other hand in view of (3.26), it is straightforward to check that for $F$ and $\beta$ small enough there exist $c_2 > 0$ such that $|\text{Im} \, q| \leq 2\beta_2 \|\partial_\theta \varphi\|^2$. Then by (3.29),

$$q \in \{z \in \mathbb{C}, -2c \beta \leq \arg(z - \lambda_0) \leq 0\} \tag{3.31}$$

for some constant $c > 0$. But we know that $(S_\theta \varphi, \varphi) = O(\beta F^2)$, then (3.27) together with (3.31) conclude the proof of the proposition 3.5.

**Remark 3.6.** The main point in the proof of the propositions 3.5 and 4.7 below, is the fact that $\text{Im} \, \gamma_0 \geq 0$ which is insured by (h3). In view of (3.26), then for $\beta$ small, a necessary condition is then $f' \geq 0$. This means that $\gamma' \leq 0$ in a neighbourhood of $s = -\infty$ and $\gamma' \geq 0$ in a neighbourhood of $s = +\infty$. Roughly speaking this inequality is a geometrical non-trapping estimate in the spirit of [3] for the case of electric perturbations.

4. Spectral estimates

The main result in this section is the following. Let $\theta = i\beta$, $f' = \Phi - 1$ where $\Phi$ defined above in section 3, clearly $f'' < f'$. Set $\mu_\theta = 1 + \theta f^2$ and
\[ \nu_0 = \left\{ z \in \mathbb{C}, \Im \mu_0^2(E_+ + \lambda_0 - z) < \beta \frac{\delta E}{4} \right\}. \] (4.32)

In the sequel we denote by \( \nu_0^c \) the complement set of \( \nu_0 \). Let \( \rho(H_0(F)) \) be the resolvent set of \( H_0(F) \). Consider first \( \theta = i \beta \) then

**Proposition 4.7.** Let \( E < 0 \) be an reference energy. Suppose (h1–3). Then there exists \( 0 < \beta_0 \leq \min \{ \delta E, |E|\tan(a_0) \} \) and \( F_0 > 0 \) such that for \( 0 < \beta \leq \beta_0 \), and \( 0 < F \leq F_0 \).

(i) \( \nu_0 \subset \rho(H_0(F)) \)

(ii) For \( z \in \nu_0 \), \(|(\tilde{H}_{0, \theta}(F) - z)^{-1}| \leq \text{dist}^{-1}(z, \nu_0^c) \).

**Proof.** Note that in view of (3.21), we have

\[
\mu_y T_{s, \theta} \mu_y = T_{1}(\theta) + i T_{2}(\theta) + \mu_y (T_{s, \theta} \mu_y),
\]

where \( T_{1}(\theta) = -\partial_s \Re \mu_0^2 (1 + \theta f')^{-2} \gamma_0 \partial_s \) and \( T_{2}(\theta) = -\partial_s \Im \mu_0^2 (1 + \theta f')^{-2} \gamma_0 \partial_s \). \( T_{1}(\theta), T_{2}(\theta) \) are symmetric operators, let us check that under our assumptions \( T_{2}(\theta) \) is actually negative.

It then sufficient to show that \( q' = \Im \mu_0^2 (1 - i \beta f')^2 (1 + u_0^c)^2 \leq 0 \). We have

\[
q' = 2\beta(f' - f')(1 - \beta^2 f'^2)(1 + u \Re \gamma_0^2 - u^2 \Im \gamma_0^2) - 2u \Im \gamma_0 (1 + u \Re \gamma_0^2)(1 - \beta^2 (f')^2)(1 - \beta^2 (f')^2) + 4\beta^2 f'^4.
\]

By using (3.26) together with (h1), for \( F \) and \( \beta \) sufficiently small we have

\[
(1 - \beta^2 f'^2)((1 + u(\Re \gamma_0^2 - u^2 \Im \gamma_0^2))^2 \geq 0.
\]

We know that \( f' < f' \), therefore, in view of (h3) we get our claim.

In the other hand we have

\[
\Im \mu_y^2(\tilde{W}_0(F) - E_-) = (1 - \beta^2 f'^2) \Im \tilde{W}_0(F) + 2\beta f' (\Re \tilde{W}_0(F) - E_-)
\]

\[
= -\beta ((1 - \beta^2 (f')^2)\Phi + 2(1 - \Phi)(\tilde{W}(F) - E_-)).
\]

To estimate the rhs of this expression, we note that from the definition of \( \tilde{W}(F) \), if \( s \in \text{supp}(1 - \Phi), \tilde{W}(F) - E_- > \delta E + O(1) \). Accordingly for \( F \) and \( \beta \) small,

\[
\Im \mu_y^2(\tilde{W}_0(F) - E_-) \leq -\beta \frac{\delta E}{2}.
\] (4.33)

Thus we get

\[
\Im \mu_y(\tilde{H}_{0, \theta}(F) - z) \mu_y = \Im \mu_y(T_{s, \theta} \mu_y) + T_{2}(\theta) + \Im \mu_y^2 T_{s} + \Im \mu_y^2(\tilde{W}_0(F) - E_-)
\]

\[+ \Im \mu_y^2(E_- - z) \leq \Im \mu_y(T_{s, \theta} \mu_y) - \beta \frac{\delta E}{2} + \Im \mu_y^2(E_- + \lambda_0 - z)
\]

and since

\[
\Im \mu_y(T_{s, \theta} \mu_y) = O(\beta^2 F),
\] (4.34)

then for \( z \in \nu_0 \), for \( F \) and \( \beta \) small enough

\[
\Im \mu_y(\tilde{H}_{0, \theta}(F) - z) \mu_y \leq -\beta \frac{\delta E}{4} + \Im \mu_y^2(E_- + \lambda_0 - z) < 0.
\]

Thus the proof of the proposition follows (see [3] or [8] for more details). \( \Box \)
By standard arguments the proposition 4.7 holds for $0 < |\theta| \leq \beta_0$, $0 < \operatorname{Im} \theta$ and $0 < F < F_0$.

5. Meromorphic continuation of the resolvent

Under conditions of the proposition 4.7. Set $V_{\theta} = V_{0,\theta} + \tilde{W}_{\theta}(F) - \tilde{W}_{\theta}(F)$. Introduce the following operator, let $\theta \in \mathbb{C}$, $|\theta| < \beta_0$, $0 < F \leq F_0$, $z \in \nu_0$ and

$$K_{\theta}(F, z) = V_{\theta}(\tilde{H}_{0,\theta}(F) - z)^{-1}.$$  \hspace{1cm} (5.35)

Then

**Proposition 5.8.** Suppose $(h1–3)$. Then there exists $0 < \beta_0 \leq \min \{|\delta E|, |E|\tan(\alpha_0)\}$ and $F_0 > 0$ such that for $0 < |\theta| \leq \beta_0$, $|\theta| > 0$, and $0 < F \leq F_0$.

(i) $z \in \nu_0 \rightarrow K_{\theta}(F, z)$ is an analytic compact operator valued function.

(ii) For $z \in \nu_0$, $\operatorname{Im} z > 0$ large enough, $\|K_{\theta}(F, z)\| < 1$.

**Proof.** Let us first show that under these conditions, $K_{\theta}(F, z)$, $\theta = i \beta$, $\operatorname{Im} z > 0$ are compact operators, this allows to prove the proposition 5.8(i). We know from (3.24) and (h1–2) that $V_{\theta} = O \left( \frac{f}{|x|^3} \right)$ as $s \to \pm \infty$. Denote by $\mathbb{I}_H$ the identity operator on the space $\mathcal{H}$.

Let $h_0 = -\partial_z^2 \otimes \mathbb{I}_{L^2(0,\alpha)} + \mathbb{I}_{L^2(0,\alpha)} \otimes T_a$. Then the decay property of $V_{\theta}$ imply that the operator $V_{\theta}(h_0 - z)^{-1}$ is compact for $\operatorname{Re} z < \lambda_0$ (see e.g. [2] or [9]).

We have

$$\tilde{H}_{0,\theta}(F) - h_0 = \partial_z (1 - G_0) \partial_z + S_0 + \tilde{W}_{\theta}(F),$$  \hspace{1cm} (5.36)

where $G_0, S_0$ are given respectively by (3.23) and (3.27) and $\tilde{W}_{\theta}(F)$ is defined in section 3. Hence,

$$K_{\theta}(F, z) = V_{\theta}(h_0 - z)^{-1} + V_{\theta}(h_0 - z)^{-1}(\tilde{W}_{\theta}(F) + \partial_z (1 - G_0) \partial_z + S_0)(\tilde{H}_{0,\theta}(F) - z)^{-1}.$$  \hspace{1cm} (5.37)

Let first show that the operator $I_1 = V_{\theta}(h_0 - z)^{-1}\tilde{W}_{\theta}(F)(\tilde{H}_{0,\theta}(F) - z)^{-1}$ is compact. This follows from the Herbst’s argument [9]. Indeed let $l(s) = (1 + |s|^2)^{1/2}$, then

$$I_1 = V_{\theta}(h_0 - z)^{-1}\tilde{W}_{\theta}(F)\left( \frac{l}{\tilde{H}_{0,\theta}(F) - z} \right)^{-1} \sum_{[A, B]}$$

$$+ V_{\theta}(h_0 - z)^{-1}[l, h_0](h_0 - z)^{-1}\tilde{W}_{\theta}(F)\left( \frac{l}{\tilde{H}_{0,\theta}(F) - z} \right)^{-1},$$  \hspace{1cm} (5.38)

where $[A, B]$ denotes the commutator of the operators $A$ and $B$. The operator $V_{\theta}[l(h_0 - z)^{-1}$ is compact since $V_{\theta}(s, u)l(s) = O \left( \frac{f}{|x|^3} \right)$ as $s \to \pm \infty$ and $\varepsilon > 3$. This holds true for the first operator of the rhs of (5.38) since $\tilde{W}_{\theta}(F)(\tilde{H}_{0,\theta}(F) - z)^{-1}$ is a bounded operator. In the other hand by the closed graph theorem, $[l, h_0](h_0 - z)^{-1}$ is a bounded operator. So it follows that the second operator of of the rhs of (5.38) and then $I_1$ is also compact.

Set $I_2 = V_{\theta}(h_0 - z)^{-1}\partial_z (1 - G_0) \partial_z (\tilde{H}_{0,\theta}(F) - z)^{-1}$. Let us show $\partial_z (1 - G_0) \partial_z (\tilde{H}_{0,\theta}(F) - z)^{-1}$ is bounded. In view of the corollary 3.4, we are left to show that $\partial_z (1 - G_0) \partial_z (\tilde{H}_{0,\theta}(F) - z)^{-1}$ is bounded. We have
\[ \partial_i(1 - G_0)\partial_i(\hat{H}_0(F) - z)^{-1} = \partial_i(1 - G_0)\partial_i(H_0 - z)^{-1} + \partial_i(1 - G_0)\partial_i(H_0 - z)^{-1}\hat{W}(F)(\hat{H}_0(F) - z)^{-1}. \]

By the closed graph theorem \( \partial_i(1 - G_0)\partial_i(H_0 - z)^{-1} \) is bounded. Now the second term of the rhs of this equality can be written as
\[
\partial_i(1 - G_0)\partial_i(H_0 - z)^{-1}\frac{\hat{W}(F)}{l}(\hat{H}_0(F) - z)^{-1} + \partial_i(1 - G_0)\partial_i(H_0 - z)^{-1}[H_0, l](H_0 - z)^{-1}\frac{\hat{W}(F)}{l}(\hat{H}_0(F) - z)^{-1}.
\]

Notice that under (h2) then \((1 - G_0)l \) as well as \((1 - G_0)l'\) are bounded functions. Hence following the same arguments as above we are done. Evidently \( I_3 = I \) implies (ii).

In the other hand the proposition 4.7 implies (ii).

We now prove the first part of the theorem 1.2.

Let \( 0 < F \leq F_0 \) and \( E < 0 \). For \( 0 < |\theta| < \beta_0 \), \( \text{Im} \theta > 0 \) by the proposition 4.7, the lemma 5.8 and the fredholm alternative theorem, the operator \( \mathbb{1}_{L^2(\Theta)} + K_\theta(F, z) \) is invertible for all \( z \in \nu_\theta \setminus \mathcal{R} \) where \( \mathcal{R} \) is a discrete set. In the bounded operator sense, we have
\[ (H_0(F) - z)^{-1} = (\hat{H}_{\theta,0}(F) - z)^{-1}[\mathbb{1}_{L^2(\Theta)} + K_\theta(F, z)]^{-1}. \]

This implies that \( \nu_\theta \setminus \mathcal{R} \subset \rho(H_0(F)) \).

Choose \( \beta_0 \) so small that there exists a dense subset of analytic vectors associated to the transformation \( U_0 \) in \( |\theta| < \beta_0 \) (see [2, remark 3.3]). We denote this set by \( \mathcal{A} \). Then standards arguments of the distortion theory, and (5.40) imply that for all \( \varphi \in \mathcal{A} \)
\[ \mathcal{R}_\varphi(z) = ((H(F) - z)^{-1}\varphi, \varphi), \text{Im} z > 0 \] has an meromorphic extension in \( \nu_\theta \) given by
\[ \mathcal{R}_\varphi(z) = ((\hat{H}_{\theta,0}(F) - z)^{-1}(\mathbb{1}_{L^2(\Theta)} + K_\theta(F, z))^{-1}\varphi_{\theta}, \varphi_{\theta}). \]

We define the resonances of the operator \( H_0(F) \) as the poles of \( \mathcal{R}_\varphi \). They are locally \( \theta \)-independent and in view of (5.40) they are the discrete eigenvalues of the operator \( H_0(F) \).

6. Resonances

We want to prove the following result, let \( \theta = i\beta, 0 < \beta \leq \beta_0 \) as in the proposition 5.8 and the proof of the theorem 1.2 (i). Then

**Proposition 6.9.** Let \( E_0 \) is a negative eigenvalue of \( H \) with multiplicity \( n \). There exists \( F_0 > 0 \) such that for \( 0 < F \leq F_0, H_0(F) \) has exactly \( n \) eigenvalues denoted by \( Z_0, \ldots, Z_{n-1} \), satisfying \( \lim_{F \to 0}|Z_j - E_0| = 0, \quad j = 0, \ldots, n - 1 \).

The proposition 6.9 implying the theorem 1.2 (ii).

**Proof.** Following [2, section 5], to prove the proposition 6.9 we only have to prove that under (h1–3), for \( E < 0, \theta = i\beta, 0 < \beta \leq \beta_0 \). Then
\[ \|K_\theta(F, z) - K(z)\| \to 0 \text{ as } F \to 0 \] (6.42)
uniformly in \( z \in \mathcal{K} \), where \( \mathcal{K} \) is a compact subset of \( i\theta \cap \rho(H_0) \). By a continuity argument it is sufficient to prove that for some \( z_0 \in \mathcal{K} \). We have

\[
K_\theta(F, z_0) - K(z_0) = (V_{\theta, \theta} - V_0)(H_0 - z_0)^{-1} + (W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1} + V_{\theta, \theta}(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1} - (H_0 - z_0)^{-1}.
\]

From (6.43), the proof needs several steps. First we consider the two first term of the rhs of (6.43). In view of (3.19) and (6.43), we can see that

\[
\|V_{\theta, \theta} - V_0\|_\infty \leq C(\|\gamma_0 - \gamma\|_\infty + \|\gamma_{\theta}' - \gamma'\|_\infty + \|\gamma_{\theta}'' - \gamma''\|_\infty)
\]

for some constant \( C > 0 \). Hence we can estimate each term of the rhs of this last inequality by using Taylor expansions w.r.t. \( \theta \) (see e.g. (3.26)). Then we find

\[
\|\gamma_0 - \gamma\|_\infty, \|\gamma_{\theta}' - \gamma'\|_\infty \text{ and } \|\gamma_{\theta}'' - \gamma''\|_\infty = O(\beta F^{-1})
\]

and then \( \|V_{\theta, \theta} - V_0(H_0 - z_0)^{-1}\| \to 0 \) as \( F \to 0 \). The proposition 4.7 and (3.24) imply that \( \|(W_\theta(F) - \tilde{W}_\theta(F))(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1}\| \to 0 \) as \( F \to 0 \). Hence we are left to show that this is true for the third term of the rhs of (6.43) i.e.

\[
\delta H(z_0) \doteq V_{\theta, \theta}(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1} - (H_0 - z_0)^{-1}.
\]

Let \( H_{\theta, \theta} \) be the operator introduced in section 3. We use the estimate

\[
\|\delta H(z_0)\| \leq \|V_{\theta, \theta}(H_0, \theta - z_0)^{-1} - (\tilde{H}_{\theta, \theta}(F) - z_0)^{-1})\|
\]

\[
+ \|V_{\theta, \theta}((H_0, \theta - z_0)^{-1} - H_0 - z_0)^{-1})\|. \tag{6.44}
\]

First we consider

\[
\delta_1 H(z_0) = V_{\theta, \theta}(H_0, \theta - z_0)^{-1}W_\theta(F)(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1}.
\]

Following arguments of the proof of the proposition (5.8) we have

\[
\delta_1 H(z_0) = V_{\theta, \theta}I(H_0, \theta - z_0)^{-1}W_\theta(F)(\tilde{H}_{\theta, \theta}(F) - z_0)^{-1}
\]

\[
+ V_{\theta, \theta}[((H_0, \theta - z_0)^{-1} - H_0 - z_0)^{-1}]. \tag{6.45}
\]

Clearly \( \|\tilde{W}_\theta(F)\| \to 0 \) as \( F \to 0 \). In the other hand we know that \( V_{\theta, \theta}, V_{\theta, \theta}I \) are bounded uniformly w.r.t. \( F \). In view of the propositions 3.5 and 4.7, this also holds for the resolvents \( (H_0, \theta - z_0)^{-1} \) and \( (H_0, \theta - z_0)^{-1} \). Let us show that this is again true for \( (H_0, \theta - z_0)^{-1}, I \) then this will imply that \( \|\delta_1 H(z_0)\| \to 0 \) as \( F \to 0 \).

We have,

\[
[(H_0, \theta - z_0)^{-1}, I] = -(H_0, \theta - z_0)^{-1}(G_\theta l')' + 2l'G_\theta \partial_\theta(H_0, \theta - z_0)^{-1}
\]

and \( \Re G_\theta \geq 0, \Im G_\theta \leq 0 \). It is then sufficient to show that \( \|(\Re G_\theta)^{1/2} \partial_\theta(H_0, \theta - z_0)^{-1}\| \) and \( \|(-\Im G_\theta)^{1/2} \partial_\theta(H_0, \theta - z_0)^{-1}\| \) are uniformly bounded w.r.t. \( F \) for \( F \) small. Recall that \( H_{\theta, \theta} = -\partial_\theta G_\theta \partial_\theta + T_\theta + S_\theta \). Then our last claim follows from the arguments evoked above and that for \( \varphi \in L^2(\Omega), ||\varphi|| = 1 \),
\[ \| (\text{Re} G_\theta)^{1/2} \partial_\lambda (H_{0,\theta} - z_0)^{-1} \phi \| \leq \text{Re} ((H_{0,\theta} - z_0)^{-1} \phi, \varphi) \]

and

\[ \| (-\text{Im} G_\theta)^{1/2} \partial_\lambda (H_{0,\theta} - z_0)^{-1} \phi \| = -\text{Im} ((H_{0,\theta} - z_0)^{-1} \phi, \varphi) \]

\[ + \text{Im} ((H_{0,\theta} - z_0)^{-1} \phi, (S_\theta + z_0)(H_{0,\theta} - z_0)^{-1} \varphi). \]

Evidently this imply that there exists a constant \( c' > 0 \) s.t.

\[ \| (\text{Re} G_\theta)^{1/2} \partial_\lambda (H_{0,\theta} - z_0)^{-1} \|, \| (-\text{Im} G_\theta)^{1/2} \partial_\lambda (H_{0,\theta} - z_0)^{-1} \| \leq \| (H_{0,\theta} - z_0)^{-1} \| + c' \|(H_{0,\theta} - z_0)^{-1}\|^2. \]

Now consider

\[ \delta^2 H(z_0) = V_{0,\theta}(H_{0,\theta} - z_0)^{-1}(T_x - T_x,\theta)(H_0 - z_0)^{-1}. \]

For \( F \) and \( \beta \) small, \( T_x - T_x,\theta = \partial_\lambda G_\theta + S_\theta \) with \( G = (\gamma - \gamma) G_1 + i\beta F G_2 \) where \( G_1, G_2 \) are uniformly bounded functions w.r.t. \( F \) (see e.g. (1.4) and (3.22)).

Moreover we know from (3.26) that \( \gamma - \gamma = O(\beta F^{-1}) \) and \( S_\theta = O(\beta F^2) \) then

\[ \|\| \| V_{0,\theta}(H_{0,\theta} - z_0)^{-1}(\partial_\lambda G_\theta + S_\theta)(H_0 - z_0)^{-1} \| \]

\[ \leq C_\beta \| (F) \| \| (H_{0,\theta} - z_0)^{-1} \| \| \partial_\lambda G_\theta \partial_\lambda (H_0 - z_0)^{-1} \|

\[ + F \| (H_{0,\theta} - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| \| (H_0 - z_0)^{-1} \| (6.46) \]

for some constant \( C > 0 \). Where \( G_3 \) is uniformly bounded w.r.t. \( F \). So this term vanishes as \( F \to 0 \).

To study the second term, we use a different strategy. We have

\[ V_{0,\theta}(H_{0,\theta} - z_0)^{-1}\delta f G_2 \partial_\lambda (H_0 - z_0)^{-1} \]

\[ = V_{0,\theta}(H_0 - z_0)^{-1} \partial_\lambda f G_2 \partial_\lambda (H_0 - z_0)^{-1} \]

\[ = V_{0,\theta}(H_0 - z_0)^{-1} \partial_\lambda f G_2 \partial_\lambda (H_0 - z_0)^{-1}. \quad (6.47) \]

We know that the operator \( V_{0,\theta}(H_0 - z_0)^{-1} \) is a compact operator (see e.g. the proof of the lemma 5.8) then to prove that the operator in the lhs of (6.47) converges in the norm sense to \( 0_{\Omega(L^2(\Omega))} \) as \( F \to 0 \), it is sufficient to show that \( (H_0 - z_0)(H_{0,\theta} - z_0)^{-1} \partial_\lambda f G_2 \partial_\lambda (H_0 - z_0)^{-1} \) converges strongly to \( 0_{\Omega(L^2(\Omega))} \) as \( F \to 0 \).

Recall that \( C = \{ \phi = \tilde{\phi}\} \), \( \phi \in C_0^\infty(\mathbb{R}^2), \phi|_{\Omega} = 0 \} \) is a core of \( H_0 \), thus for \( \zeta \in \rho(H_0), \zeta = (H_0 - z_0)\psi \) is dense in \( L^2(\Omega) \). Set \( \psi = (H_0 - z_0)\phi, \phi \in C \).

Since the field \( f \) is chosen s.t. \( f' \) has support contained in \( |s| > c'/F \) for some \( c' > 0 \) then \( \lim_{F \to 0} \| \partial_\lambda f' G_2 \partial_\lambda \phi \| = 0 \). By using standard arguments of the perturbation theory and the proposition 3.5, for \( F \) small, the operator \( (H_0 - z_0)(H_{0,\theta} - z_0)^{-1} \) has a norm which is uniformly bounded w.r.t. \( F \). This proves our claim on \( C' \).

In the other hand \( \partial_\lambda f' G_2 \partial_\lambda (H_0 - z_0)^{-1} \) is bounded operator with a norm uniformly bounded w.r.t. \( F \), for \( F \) small. Then the strong convergence follows.

\[ \square \]

7. Concluding remarks

In this last section we would like to give some remarks about the field regime chosen here. Let us recall that in [6] it was assumed \( \eta = \frac{\pi}{2} \) and \( \alpha_0 = 0 \).

This question was then addressed by us in [2] with \( |\eta| < \frac{\alpha_0}{2} \) and \( |\eta - \alpha_0| < \frac{\alpha_0}{2} \) (or the symmetric case, \( |\eta| > \frac{\alpha_0}{2} \) and \( |\eta - \alpha_0| > \frac{\alpha_0}{2} \). Roughly speaking, this corresponds to the usual picture of the Stark effect for one-dimensional Schrödinger operators with local potential i.e.
the field interaction $W(F) \rightarrow -\infty$ as $s \rightarrow -\infty$ and $W(F) \rightarrow \infty$ as $s \rightarrow \infty$. We have proved an analog of the theorem 1.2 in this case.

Suppose now that $|\eta| > \frac{s}{2}$ and $|\eta - \alpha_0| < \frac{s}{2}$. Then from (2.9) and (2.10), clearly $W(F) \rightarrow \infty$ as $s \rightarrow \pm \infty$ i.e. it is a confining potential. By using standard arguments (see e.g. [15]) it easy to see that $H(F)$ has a compact resolvent and then only discrete spectrum.

Indeed consider the following operator in $L^2(\Omega)$. Let $F > 0$,

$$h(F) = T_s + T_n + w(F),$$

where $w(F) = F \cos(\eta)s$ if $s \leq 0$ and $w(F) = F \cos(\eta - \alpha_0)s$ if $s > 0$. Then by (h1) and (h2), there exists a strictly positive constant $c$ such that in the form sense we have

$$h(F) \geq h_1(F) = (-c\partial_x^2 + w(F)) \otimes I_n + I_s \otimes -\partial_y^2.$$  

But the operator $-\partial_x^2 + w(F)$ has a compact resolvent [15]. Let $\{p_n, n \geq 0\}$ the eigen-projectors corresponding to the transverse modes $\{\lambda_n, n \geq 0\}$. Since

$$(h_1(F) + 1)^{-1} = \bigoplus_{n \geq 1} (-c\partial_x^2 + w(F) + \lambda_n + 1)^{-1}) \otimes p_n.$$  

Then $(h_1(F) + 1)^{-1}$ is a compact operator as a norm limit of compact operators. Hence $h_1(F)$ satisfies the Rellich criterion [15] and in view of (7.48) it is also true for $h(F)$ so $(h(F) + 1)^{-1}$ is compact. Since $H(F) = h(F)$ is bounded then $(H(F) + 1)^{-1}$ is also a compact operator.

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