Precompact groups and convergence.

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Abstract

We consider precompact sequential and Fréchet group topologies and show that some natural constructions of such topologies always result in metrizable groups answering a question of D. Dikranjan et al. We show that it is consistent that all sequential precompact topologies on countable groups are Fréchet (or even metrizable). For some classes of groups (for example boolean) extra set-theoretic assumptions may be omitted (although in this case such groups do not have to be metrizable).

We also build (using ♦) an example of a countably compact Fréchet group that is not $\alpha_3$ and obtain a counterexample to a conjecture of D. Shakhmatov as a corollary.

1 Introduction

Compact topological groups play a special role throughout mathematics, in areas ranging from probability theory to harmonic analysis. Locally compact sequential groups (or even countably tight, see below for the relevant definitions) are metrizable thus shifting the focus of study to the convergence properties in groups with weaker compactness conditions. D. Shakhmatov in [12] proposes a systematic study of sequential and Fréchet properties in countably compact, and, more generally, precompact topological groups (recall that a topological group $G$ is precompact if it can be densely embedded as a subgroup of a compact group).

The $\Sigma$-product of $\omega_1$ copies of the two element group provides a standard example of a countably compact Fréchet group that is not first countable. Note that all countable subgroups of this example are metrizable. Paper [8] began the study of precompact Fréchet topologies on Abelian groups. Among the results proved in [8] is the consistent existence of a nonmetrizable Fréchet precompact topology on every infinite countable Abelian group.

In this paper we consider precompact groups that are sequential. We show that in ZFC, some natural ‘minimal’ precompact topologies on countable groups are sequential if an only if they are metrizable (Corollary 1), answering a question of D. Dikranjan et. al in [5]. We use the effective topology techniques pioneered by S. Todorčević (see [18] and [19]) to study such groups.
In the case of general precompact topologies we show that in some classes of groups (such as boolean, i.e. satisfying the identity $a + a = 0$) every countable precompact sequential group is Fréchet (Corollary 2). We show that consistently, all countable precompact sequential groups may be Fréchet (Theorem 1, see also Remark 1).

In the final section we construct an example of a countably compact Fréchet boolean group that is in some sense 'barely Fréchet'. More precisely, the group does not have the $\alpha_3$ property of A. Arkhangel’iskii (see below for the definitions) which allows us to disprove a conjecture of D. Shakhmatov mentioned in [13].

We use standard notation and definitions for set-theoretical and topological concepts (see [9]). All the definitions and properties used in this paper related to topological groups can be found in [3]. Book [7] is a good reference on Peter-Weyl theorem for compact groups. We only use a corollary of Peter-Weyl theorem that states that a precompact group $G$ can be embedded as a subgroup in a product of $U(n)$’s. The only property of $U(n)$ we use is that it is a compact metric group.

All spaces are assumed to be (completely) regular. We use $\overline{A}$ to denote the closure of $A$ in the topology $\tau$ (the reference to $\tau$ will be omitted if the topology is clear from the context). We use $\cdot$ and 1 to denote the operation and the unit in an arbitrary group $G$. For an Abelian group these become the traditional $+$ and 0, respectively. If $G$ is a group and $D \subseteq G$ then $\langle D \rangle$ stands for the subgroup (algebraically) generated by $D$ (i.e. the smallest subgroup of $G$ that contains $D$).

A topological space $X$ is called Fréchet if for any $A \subseteq X$ and any $x \in \overline{A}$ there exists an $S \subseteq A$ such that $S \rightarrow x$ (here $S \rightarrow x$ means $S \subseteq^* U$ for every open $U \ni x$ if $S$ is infinite and $S = \{x\}$ otherwise). More generally, $X$ is called sequential if for any $A \subseteq X$ such that $\overline{A} \neq A$ there exists an $S \subseteq A$ such that $S \rightarrow x \in \overline{A} \setminus A$. Trivially, every Fréchet space is sequential. A free topological group over a one point compactification of a countable discrete space provides an easy example of a countable sequential topological group that is not Fréchet.

We use the following intrinsic characterization of precompactness (sometimes called total boundedness, see [3]): a topological group $G$ is precompact if for any nonempty open $U \subseteq$ there exists a finite $F \subseteq G$ such that $U \cdot F = G$ (or $F \cdot U = G$).

## 2 Small precompact groups

A topological group $G$ is called ss-characterized (see [5]) if there exists an $S \subseteq G$ (called the characterizing sequence) such that the topology of $G$ is the finest precompact topology in which $S \rightarrow 1$.

Recall that a topology $\tau$ on a countable set $X$ (considered naturally as a subset of $2^\omega$ in the product topology) is called analytic if $\tau$ is a continuous image of the space of the irrational numbers. Papers [18], [19], and [15] contain a number of results on sequential spaces with analytic topologies.
Lemma 1. Let $G$ be an ss-characterized countable topological group. Then the topology of $G$ is analytic.

Proof. Assume $G = \omega$ and let $S = \{m_n : n \in \omega\}$ be the characterizing sequence. Let $U(\infty) = \prod_{n \in \omega} U(n)$. Note that for each $n \in \omega$ the group $U(n)$ is naturally embedded in $U(\infty)$ as a subgroup. Consider the following set

$$C_n = \{p : G \to U(n) \subseteq U(\infty) : p \text{ is an algebraic homomorphism}\}$$

Note that for every $n \in \omega$ the set $C_n$ is a closed subset of $\prod_{g \in C} U(\infty)$ in the topology of pointwise convergence. Given $n, m, k \in \omega$ each set

$$C_n^{m,k} = \{p \in C_n : d(1, p(m)) < (k + 1)^{-1} \text{ for } i > m\}$$

is Borel as are the sets $C_n^k = \cup_{m \in \omega} C_n^{m,k}$, $C_n = \cap_{k \in \omega} C_n^k$ and $C = \cup_{n \in \omega} C_n$.

Suppose $p \in C$ and $\epsilon > 0$. Let $k \in \omega$ be such that $(k + 1)^{-1} < \epsilon$. Let $p \in C_n^{m,k}$ for some $n \in \omega$. Then $p \in C_n^{m,k}$ for some $m \in \omega$ so $d(1, p(m)) < (k + 1)^{-1} < \epsilon$ for $i > m$. Thus $S \subseteq p^{-1}(B_n^\epsilon)$ for every $\epsilon > 0$ where $B_n^\epsilon = \{x \in U(n) : d(1, x) < \epsilon\}$. Thus $S \rightarrow 1$ in the precompact topology $\tau$ generated on $G$ by the prebase $\{p^{-1}(B_n^\epsilon) : \epsilon > 0, p \in C_n\}$.

Let $\tau'$ be the maximal precompact topology on $G$ in which $S \rightarrow 1$. Let $U \subseteq G$ be an open neighborhood of $1$. By Peter-Weyl theorem there are $\epsilon_1, \ldots, \epsilon_t > 0$ and continuous homomorphisms $p_1 : G \to U(n_1), \ldots, p_t : G \to U(n_t)$ such that $\cap_{i \leq t} p_i^{-1}(B_n^{\epsilon_i}) \subseteq U$. Since $S \rightarrow 1$ in $\tau'$ and $p_i$ is continuous, for every $k \in \omega$ there is an $m \in \omega$ such that $d(1, p_i(m)) < (k + 1)^{-1}$ for all $j > m$. So $p_i \in C_n^{m,k} \subseteq C_n^k$ for every $k \in \omega$. Thus $p_i \in C_n^{m,\omega} \subseteq C$ and $\tau' = \tau$.

Let $k \in \omega$ and consider the map $\pi_{k,n} : U(n)^\omega \to 2^\omega$ given by $\pi_{k,n}(x_1, \ldots, x_i, \ldots) = (\sigma_1, \ldots, \sigma_i, \ldots)$ where $\sigma_i = 1$ if $d(x_i, 1) < (k + 1)^{-1}$ and $\sigma_i = 0$ otherwise. Now $\pi_{k,n}$ is a measurable map. Put $f_{k,n}(p) = f_k^p \in 2^\omega$ where $p \in C_n$ and $f_k^p(m) = 1$ if and only if $d(1, p(m)) < (k + 1)^{-1}$. Thus the set $B = \cup_{k,n \in \omega} \pi_{k,n}(C_n) \subseteq 2^\omega$ is an analytic set of characteristic functions of some prebase at $1 \in G$. Thus $\tau$ is analytic by [18], Proposition 3.2(iii).

The following definition is used in several constructions.

Definition 1. Let $(X, \tau)$ be a topological space. Then $(X, \tau)$ is called $k_\omega$ if there exists a countable family $K$ of subspaces of $X$ such that $U \in \tau$ if and only if $U \cap K$ is relatively open in $K$ for every $K \in K$. We say that $\tau$ is determined by $K$ and write $\tau = k_\omega(K)$.

The next simple lemma shows that precompactness and $k_\omega$ are in some sense orthogonal properties. For a class of countable abelian groups whose topology is determined by a $T$-sequence this result was proved in [11], Proposition 2.3.12.

Lemma 2. A $k_\omega$ topological group is precompact if and only if it is compact.

Proof. Let $K = \{K_n : n \in \omega\}$ be a countable family of compact subspaces of $G$ that determines its topology. We may assume $1 \in K_n \subseteq K_{n+1}$ for every $n \in \omega$. If $G$ is not compact there exists an infinite closed discrete subset $D = \{d_n : n \in \omega\} \subseteq G$ where $d_n \neq d_m$ for $n \neq m$.

By induction build a sequence $\{O_n : n \in \omega\}$ of subsets of $G$ such that
(1) \(1 \in O_n \subseteq K_n\), \(O_n\) is relatively open in \(K_n\).

(2) \(\overline{O}_n \subseteq O_{n+1}\) for every \(n \in \omega\);

(3) \((\overline{O}_n \cdot K_m) \cap D = (\overline{O}_m \cdot K_m) \cap D\) for every \(m \leq n\).

If \(O_n\) has been built, note that \(\overline{O}_n\) is compact so for every \(m \leq n\) there exists an open \(O_m\) such that \(\overline{O}_n \subseteq O_m\) and \((\overline{O}_n \cdot K_m) \cap D = (\overline{O}_m \cdot K_m) \cap D\). Properties (1)–(3) are easy to check.

It follows from the choice of \(K\) and properties (1) and (2) that \(O = \cup_{n \in \omega} O_n\) is an open neighborhood of 1 in \(G\). Now (3) and the choice of \(D\) imply that \((O \cdot K_n) \cap D = (O_n \cdot K_n) \cap D\) is finite for every \(n \in \omega\).

Now if \(F \subseteq G\) is finite there is an \(n \in \omega\) such that \(F \subseteq K_n\). Thus \(O \cdot F \neq G\) for any finite \(F \subseteq G\) contradicting the precompactness of \(G\).

The lemmas above can now be used to answer Question 4.6 of [5].

**Corollary 1.** Every sequential countable subgroup of a countable ss-characterized group is metrizable.

**Proof.** Being sequential and analytic by Lemma [1] \(G\) is either first countable or \(k\omega\) by [15], Theorem 1. Since an infinite \(k\omega\) countable group cannot be precompact by Lemma [2] \(G\) is metrizable.

We now turn to the general (i.e. not necessarily definable) precompact countable sequential groups. Recall that a base of open neighborhoods of 1 of a topological group \(G\) is called linear if every element of the base is a normal subgroup of \(G\).

**Lemma 3.** Every countable sequential group with a linear base is Fréchet.

**Proof.** Let \(S \rightarrow 1\) be an arbitrary convergent sequence. Let \(U \subseteq G\) be an open neighborhood of 1 and let \(\overline{V} \subseteq U\) for some open subgroup \(V \subseteq G\). Then \(\langle S \setminus F \rangle \subseteq \overline{V} \subseteq U\) for some finite \(F \subseteq S\). Since \(\langle S \setminus F \rangle\) does not contain any isolated points the set \(\langle S \setminus F \rangle\) cannot be countably compact and thus contains an infinite closed and discrete subset \(D_F \subseteq \langle S \setminus F \rangle\). Thus \(G\) contains a closed copy of the space \(L = \{\omega\} \cup \omega \times \omega\) in which the neighborhoods of the only nonisolated point \(\omega\) are given by \(U_n = \{\omega\} \cup \omega \times (\omega \setminus n)\).

If \(G\) is sequential and not Fréchet it contains a closed copy of the sequential fan \(S(\omega)\). This is impossible by Lemma 4 in [4].

Since every torsion (i.e. such that for every \(g \in G\), \(g^n = 1\) for some \(n \in \mathbb{Z}\)) compact group has a linear base of open neighborhoods of 1 (see, for example, [3]) we obtain the following corollary.

**Corollary 2.** Every countable sequential precompact group of finite exponent (i.e. such that there is an \(n \in \mathbb{Z}\) with the property that \(g^n = 1\) for every \(g \in G\), in particular, every boolean) is Fréchet.
We do not know if the finite exponent restriction above can be dropped in ZFC. To prove the consistency of the property above for the class of all precompact sequential groups we need the following lemma.

**Lemma 4.** Let $G$ be a countable sequential precompact group. Let $\mathcal{H}$ be a countable family of nowhere dense subsets of $G$. Then there exists an infinite $S \subseteq G$ such that $S \to 1$ and $S \cap H$ is finite for every $H \in \mathcal{H}$.

**Proof.** Let $\mathcal{H} = \{ H_i : i \in \omega \}$ be a family of nowhere dense subsets of $G$ with the property that for every convergent sequence $S \subseteq G$ such that $S \to 1$ there is an $H_i \in \mathcal{H}$ such that $H_i \cap S$ is infinite. By extending $\mathcal{H}$ if necessary we may assume that $1 \in H_0$, each $H_i$ is closed in $G$ and $H_1 \cdot F \in \mathcal{H}$ for every $H_i \in \mathcal{H}$ and every finite $F \subseteq G$. Let $\overline{G} \supseteq G$ be a compact group that contains $G$ as a dense subgroup. Then for each $H_i \in \mathcal{H}$ its closure $\overline{H_i}$ in $\overline{G}$ is nowhere dense and for any $S \subseteq G$ such that $S \to g \in G$ there exists an $i \in \omega$ such that $H_i \cap S$ is infinite.

Using recursion, build a countable set $D = \{ d_i : i \in \omega \} \subseteq G$ and a family $U = \{ U_i : i \in \omega \}$ of open subsets of $G$ such that $U_{i+1} \subseteq U_i$, $U_i \cap (\bigcup_{j \leq i} H_j) = \emptyset$, and $d_k \in U_i \setminus \{ d_j : j < k \}$ for every $i \in \omega$ and $k > i$.

Note that the set $D$ has the property that for every $i \in \omega$ there exists an open set $U \subseteq \overline{G}$ such that $U \cap D$ is finite and $U \subseteq \overline{H_j} \subseteq U$.

Let $\{ F_i : i \in \omega \}$ list all the finite subsets of $G$. Build, by induction, subsets $V_i \subseteq \bigcup_{j \leq i} H_j$ so that the following properties are satisfied.

1. $V_0 = \emptyset$.
2. $V_0 \subseteq \overline{G}$.
3. $\{ V_i : i \in \omega \}$ have been built. Note that $V_i \subseteq \overline{G}$ is compact and $(\bigcup_i \cdot \bigcup_{j \leq i} F_j) \subseteq \bigcup_{i < I} H_j$ for some $I \in \omega$ by [4] and the properties of $\mathcal{H}$. By the property of $D$, there exists a subset $U \supseteq \bigcup_i (\bigcup_i \cdot \bigcup_{j \leq i} F_j)$, open in $\overline{G}$ such that $U \cap D$ is finite. Using the compactness of $V_i$, find an open subset $U' \supseteq V_i$ of $\overline{G}$ such that $(U' \cdot \bigcup_{j \leq i} F_j) \subseteq U$ so $F_j = (U' \cdot F_j) \cap D$ is finite for each $j \leq i$.

Using a similar argument and [5] for each $j \leq i$ find an open $U_j \supseteq V_i$ such that $(\bigcup_i \cdot F_j) \cap F_j = (\bigcup_i \cdot F_j) \cap F_j = (\bigcup_i \cdot F_j) \cap D = (\bigcup_i \cdot F_j) \cap D$. Put $V_{i+1} = \bigcup_{j \leq i+1} \overline{U_j} \cap \bigcap_{j \leq i} U_j \cap U'$.

Define $V = \bigcup_{i \in \omega} V_i \cap G$. If $F \subseteq G$ is finite then $F = F_i$ for some $i \in \omega$ and by [4] and [5] the set $(V \cdot F) \cap D = (\bigcup_i \cdot F_j) \cap D$ is finite.

Suppose $G \setminus V$ is not closed. Since $G$ is sequential, there exists an $S \subseteq G \setminus V$ such that $S \to g \in V$ for some $g$. By the choice of $\mathcal{H}$ there exist $i,j \in \omega$ such that $g \in V_i$, $i \geq j$, and $H_j \cap S$ is infinite. Since $V_i$ is relatively open in $\bigcup_{k \leq j} H_k \supseteq H_j$, $V_j \cap S$ is infinite, contradicting $S \cap V = \emptyset$.

Thus $V$ is a non-empty open subset of $G$ such that $G \neq V \cdot F$ for any finite $F \subseteq G$ contradicting the precompactness of $G$. \hfill \Box
Lemma 4 can be used to show the consistency of the property in Lemma 2 in the class of general precompact countable sequential groups.

**Theorem 1.** In the model obtained by adding $\omega_2$ Cohen reals to a model of CH every countable precompact sequential group is Fréchet.

**Proof.** The proof is almost a word-for-word reproduction of the proof of Lemma 7 of [16] so we only give a short outline. Let $M$ be an $\omega$-closed $(|M|^\omega \subseteq M)$ elementary submodel of $H(2^\omega)$ of size $\omega_1$ such that $G, \tau \in M$ for some $\text{Fn}(\omega_1, 2)$-name $\dot{\tau}$ of a sequential precompact topology on $G$. Let the ground model $V$ satisfy CH and let $G$ be a $\text{Fn}(\omega_1, 2)$-generic set over $V$. Just as in the proof of Lemma 7 of [16], one shows that the group $(G, \tau_M) \in V[G \cap M]$ is sequential, where $\tau_M$ consists of $G \cap M$-interpretations of sets in $\dot{\tau}(\alpha)$ where $\alpha \in M$ (here $\tau$ is interpreted as an $\omega_2$-list of subsets of $G$). Using elementarity, one shows that $(G, \tau_M)$ is also precompact.

The rest of the proof is nearly identical to that of Lemma 7 in [16] and uses Lemma 4 instead of Lemma 6 of [16].

**Remark 1.** Lemma 4 may also be used to complement Lemma 16 of [17] in the proof of Theorem 2 of [17] to obtain a model of ZFC in which every countable precompact sequential group is metrizable thus generalizing Lemma 1 to the class of countable precompact sequential groups. The saturation argument that uses the elementary submodel $M$ in the theorem above is then replaced by the use of $\diamond(S^2_1)$ in the ground model and the $\sigma$-centeredness of the forcing. We omit the details.

### 3 A Fréchet-Urysohn group.

Recall the definition of $\alpha_i$ properties introduced by A. Arkhangel’skii under a different name. Several of these properties, including a number of variations, have been independently defined and studied by other authors under different names although Arkhangel’skii was likely the first to undertake a systematic study of spaces with these properties. We follow the established notation below.

**Definition 2 ([1], [2], see also [12]).** Let $X$ be a topological space. For $i = 1, 2, 3,$ and 4 we say that $X$ is an $\alpha_i$-space provided for every countable family $\{S_n : n \in \omega\}$ of sequences converging to some point $x \in X$ there exists a ‘diagonal’ sequence $S$ converging to $x$ such that:

- $(\alpha_1)$ $S_n \setminus S$ is finite for all $n \in \omega$,
- $(\alpha_2)$ $S_n \cap S$ is infinite for all $n \in \omega$,
- $(\alpha_3)$ $S_n \cap S$ is infinite for infinitely many $n \in \omega$,
- $(\alpha_4)$ $S_n \cap S \neq \emptyset$ for infinitely many $n \in \omega$.  


P. Nyikos (see [10]) noted that \( \alpha_i \) properties are of great utility in the study of convergence properties in groups and other algebraic objects. Thus he showed in [10] that every Fréchet group is \( \alpha_4 \). A number of authors have since uncovered various connections (or lack thereof) between \( \alpha_i \) properties in the presence of an algebraic structure and other restrictions. Survey [12] provides a fairly comprehensive overview of these results.

In this section we build an example of a countably compact Fréchet boolean group that is not \( \alpha_3 \). Such a group always has a base of open neighborhoods of 0 consisting of subgroups (this follows from Pontryagin’s duality, although in the example below this property can be verified directly) which allows us to answer a question asked by D. Shakhmatov in 1990.

We borrow some terminology and techniques from [14] for the analysis of topologies on the boolean group.

**Definition 3.** Call \((G, K)\) a \(k_\omega\)-pair (with respect to \(\tau\)) if \((G, \tau)\) is a boolean topological group with the \(k_\omega\) topology \(\tau\) and \(K\) is a countable family of compact subspaces of \(G\) closed under finite sums and intersections such that \(\tau = k_\omega(K)\) and \(\bigcup K = G\).

**Lemma 5 ([14]).** Let \((G, K)\) be a \(k_\omega\)-pair, \(D' \subseteq G\) be infinite, closed and discrete in \(k_\omega(K)\). Then there exists an infinite independent \(D \subseteq D'\) such that \(\langle D\rangle\) is closed and discrete in \(k_\omega(K)\).

The topology of the example will be a simultaneous limit of \(k_\omega\) and first countable precompact topologies. The next definition is a convenient shortcut and can be viewed as an approximation of the final topology.

**Definition 4.** Call \((G, K, \mathcal{U})\) a convenient triple if \(G\) is a boolean group, \(\mathcal{U}\) is a countable family of subgroups closed under finite intersections that forms an open base of neighborhoods of 0 in some Hausdorff precompact topology \(\tau(\mathcal{U})\) on \(G\), and \(K\) is a countable family of compact (in \(\tau(\mathcal{U})\)) subgroups of \(G\) closed under finite sums and intersections such that \(\bigcup K = G\).

The next lemma was proved in [14].

**Lemma 6 ([14]).** Let \((G, K, \mathcal{U})\) be a convenient triple, let \(H\) be a subgroup of \(G\) closed in \(k_\omega(K)\). Then there exists a countable family of open (in \(k_\omega(K)\)) subgroups of finite index \(\mathcal{U}_0 \supseteq \mathcal{U}\) such that \(\overline{H}^{\tau(\mathcal{U})} \cap G = \cap \{ U \in \mathcal{U}_0 : H \subseteq U \} = H\).

The construction of the example is by an \(\omega_1\) recursion. Each step consists of adding a new convergent sequence such that the following properties are satisfied.

**Definition 5.** Let \((G, K, \mathcal{U})\) and \((G', K', \mathcal{U}')\) be convenient triples and an independent \(D \subseteq G\) be such that \(\langle D\rangle\) is closed and discrete in \(k_\omega(K)\). Call \((G', K', \mathcal{U}')\) a primitive sequential extension (pse for short) of \((G, K, \mathcal{U})\) over \(D\) if the following conditions hold:
(6) $G \subseteq G'$, $\bigcup F^{\sigma(u')} \in \mathcal{U}'$ for every $U \in \mathcal{U}$, and $\mathcal{K}'$ is the closure of $\mathcal{K} \cup \{L\}$ under finite sums where $L = \bigcup F^{\sigma(u')}$ (thus $L$ is compact in $k\omega(\mathcal{K}')$);

(7) $\langle D \setminus F \rangle^{\sigma(u')} \cap G = (D \setminus F)$ for any $F$;

It is easy to show that in the notation above, $K' = K + L$ for every $K' \in \mathcal{K}'$ and some $K \in \mathcal{K}$. The proof of the next lemma can be extracted from the results of [14] but we present it here for the reader’s convenience.

Lemma 7. Let $(\mathcal{G}, \mathcal{K}, \mathcal{U})$ be a convenient triple and let $D \subseteq G$ be a countable infinite independent subset such that $\langle D \rangle$ is closed and discrete in $k\omega(\mathcal{K})$. Then there exists a pse $(\mathcal{G}', \mathcal{K}', \mathcal{U}')$ of $(\mathcal{G}, \mathcal{K}, \mathcal{U})$ over $D$. If $D \to 0$ in $\tau(\mathcal{U})$ then $(\mathcal{G}', \mathcal{K}', \mathcal{U}')$ can be chosen so that $D \to 0$ in $\tau(\mathcal{U}')$.

Proof. Let $F \subseteq D$ be finite. Use Lemma 6 to find a countable family $\mathcal{U}_F$ of open in $k\omega(\mathcal{K})$ subgroups of $G$ of finite index such that $\langle D \setminus F \rangle = \cap \{ U \in \mathcal{U}_F : \langle D \setminus F \rangle \subseteq U \}$. Let $\mathcal{U}_0' = \mathcal{U}_0 \cup \{ U \in \bigcup_{F \subseteq D} \mathcal{U}_F : D \subseteq U \}$ and let $\mathcal{U}_0''$ be the closure of $\mathcal{U}_0'$ under finite intersections. Since every element of $\mathcal{U}_0''$ is a subgroup of finite index of $G$ open in $k\omega(\mathcal{K})$ and $(\mathcal{G}, \mathcal{K}, \mathcal{U})$ is a convenient triple, $\tau(\mathcal{U}_0'')$ is precompact. If $D \to 0$ in $\tau(\mathcal{U})$ then $D \to 0$ in $\tau(\mathcal{U}_0'')$ by the choice of $\mathcal{U}_0''$.

Let $(\mathcal{G}', \mathcal{K}', \mathcal{U}')$ be the compact group such that $G \subseteq G'$ is a dense subgroup and $\tau|_G = \tau(\mathcal{U}_0'')$. Let $L = \langle D \rangle$ and let $\mathcal{K}'$ be the closure of $\mathcal{K} \cup \{ L \}$ under finite sums. Put $G' = G + L \subseteq G''$ and $\mathcal{U}' = \{ \bigcup F^{\sigma'} U \subseteq \mathcal{U}_0' : U \in \mathcal{U}_0'' \}$.

Let $F \subseteq D$ and $g \in G$ be such that $g \not\in \langle D \setminus F \rangle$. If $g \not\in \langle D \rangle$ let $U \in \mathcal{U}_0$ be such that $D \subseteq U$ and $g \not\in U$. Then $U \in \mathcal{U}_0''$ is clopen in $k\omega(\mathcal{K})$ so $g \not\in \bigcup F^{\sigma} U$.

If $g \in \langle D \rangle$ then $g \not\in \langle D \setminus \{ f \} \rangle$ for some $f \in D \cap F$. Let $U \in \mathcal{U}_0 \setminus \{ f \}$ be such that $D \setminus \{ f \} \subseteq U$ and $g \not\in U$. The rest of the argument is similar to the case when $g \not\in \langle D \rangle$. Thus [7] holds. The rest of the properties follow from the construction. 

We now turn to the limit stages of the construction. Let $\gamma$ be an ordinal. Suppose for every $\sigma < \gamma$ a convenient triple $(H^\sigma, \mathcal{K}^\sigma, \mathcal{U}^\sigma)$ is defined so that the following conditions hold:

(8) $H^\sigma' \subseteq H^\sigma$, $\mathcal{K}^\sigma' \subseteq \mathcal{K}^\sigma$, and $\mathcal{U}^\sigma' \subseteq \{ U \cap H^\sigma' : U \in \mathcal{U}^\sigma \}$ if $\sigma' \leq \sigma < \gamma$;

(9) $H^\sigma$ is dense in $H^\sigma$ in $k\omega(\mathcal{K}^\sigma)$ for every $\sigma' \leq \sigma$;

Define $(H^{<\gamma}, \mathcal{K}^{<\gamma}, \mathcal{U}^{<\gamma})$ by taking $H^{<\gamma} = \cup_{\sigma < \gamma} H^\sigma$, $\mathcal{K}^{<\gamma} = \cup_{\sigma < \gamma} \mathcal{K}^\sigma$, $\mathcal{U}^{<\gamma} = \{ U^\omega(\mathcal{K}^{<\gamma}) : U \in \mathcal{U}^\sigma, \sigma < \gamma \}$.

Note that in the case of a successor $\gamma$, $(H^{<\gamma}, \mathcal{K}^{<\gamma}, \mathcal{U}^{<\gamma}) = (H^\gamma, \mathcal{K}^\gamma, \mathcal{U}^\gamma)$ where $\gamma' + 1 = \gamma$.

Lemma 8 ([14]). The family $\mathcal{U}^{<\gamma}$ forms a base of clopen subgroups of finite index for a precompact group topology $\tau(\mathcal{U}^{<\gamma})$ on $H^{<\gamma}$ and each $H^\sigma$, $\sigma < \gamma$ is dense in $H^{<\gamma}$ in $k\omega(\mathcal{K}^{<\gamma})$. If $\gamma < \omega_1$ then $(H^{<\gamma}, \mathcal{K}^{<\gamma}, \mathcal{U}^{<\gamma})$ is a convenient triple.
If, in addition to properties \([8]\) and \([9]\) each \((H^\sigma, K^\sigma, U^\sigma)\) is a pse of \((H^{<\sigma}, K^{<\sigma}, U^{<\sigma})\) over some \(D \subseteq H^{<\sigma}\), we will call \(\{(H^\sigma, K^\sigma, U^\sigma) : \sigma < \gamma\}\) a pse-chain.

To ensure that the final group does not satisfy \(\alpha_3\) the following stronger property is introduced. Below \(K\) will be a element of \(K\) for some convenient triple \((G, K, U)\) and \(\{S_n : n \in \omega\}\) will be a specially chosen family of convergent sequences in \(G\).

\(\text{(10)}\) let \(K \subseteq G\), then there exists an \(n(K) \in \omega\) such that \(K \cap \langle S_n \rangle\) is finite for every \(n > n(K)\).

The preservation of \([10]\) after adding a new convergent sequence is the subject of the next lemma.

**Lemma 9.** Let \((G, K)\) be a \(k_\omega\)-pair such that each \(K \in \mathcal{K}\) satisfies \([10]\). Suppose also that each \(n\{S_n\} \subseteq K(n)\) for some \(K(n) \in \mathcal{K}\). Let \(D \subseteq G\) be an infinite set such that \(\langle D \rangle\) is closed and discrete in \(G\). Then for every \(K \in \mathcal{K}\) the set \(K + \langle D \rangle\) satisfies \([10]\).

**Proof.** Let \(K \in \mathcal{K}\) and \(n(K) \in \omega\) be such that \(K \cap \langle S_n \rangle\) is finite for every \(n > n(K)\). Note that \(K + \langle S_n \rangle\) is compact in \(k_\omega(K)\) so \(F = \langle D \rangle \cap (K + \langle S_n \rangle)\) is finite for every \(n \in \omega\). If \(n > n(K)\) the set \(F' = K \cap \langle S_n \rangle\) is finite. Suppose \(d \in \langle D \rangle, s \in \langle S_n \rangle\), and \(a \in K\) are such that \(a + d = s\). Then \(\exists d \in F = \{a^i + s^i : a_i \in K, s^i \in \langle S_n \rangle, i \in |F|\}\) so \(s = a + a^i + s^i\) for some \(i \in |F|\) and \(a + a^i \in F' = \{f^j : j \in |F'|\}\). Thus \(s = f^j + s^i\) for some \(i \in |F|, j \in |F'|\) and \(F' = F\). Hence \(K + \langle D \rangle \cap \langle S_n \rangle\) is finite for every \(n > n(K)\). \(\square\)

The first step of the construction is given in the following simple lemma.

**Lemma 10.** There exists a convenient triple \((G, K, U)\) such that there are infinite countable independent sets \(S_n \subseteq G\) where \(S_n \to 0\) for every \(n \in \omega\) and for any \(K \in \mathcal{K}\) there exists an \(n \in \omega\) such that \(K \cap \langle S_n \rangle = \emptyset\).

**Proof.** It is straightforward to construct a compact boolean group \(T = \langle S \rangle\) where \(S \to 0\) is an infinite independent subset. Let \(G\) be the inductive limit of \(T^n, n \in \omega\), and \(\tau(U)\) be the topology inherited from the product topology on \(T^\omega\). Put \(S_n = \{0\}^{n-1} \times S\). Note that the topology of \(G\) is determined by the family \(K = \{T^n : n \in \omega\}\). All the properties of \((G, K, U)\) are easy to check. \(\square\)

We now present the recursive construction of the main example. Since all the groups \(G_\alpha\) in the construction have cardinality \(2^n\) we will assume that every \(G_\alpha\) is a subgroup (algebraically) of \(2^n\). Let \(\{C_\alpha : \alpha < \omega_1\} \subseteq [2^n]^{\omega}\) and \(\{P_\alpha : \alpha < \omega_1\} \subseteq [2^n]^\omega\) be some families of subsets of \(2^n\).

Let \((G_0, K_0, U_0) = (G, K, U)\) where the convenient triple \((G, K, U)\) and the sets \(S_n \subseteq G\) have been constructed in Lemma \([10]\). Below we use the notation \(cC_\alpha\) for the closure of \(C_\alpha\) in \(k_\omega(K, \alpha)\).

**Lemma 11.** There exists a pse-chain \(\{(G_\alpha, K_\alpha, U_\alpha) : \alpha < \omega_1\}\) such that
(11) every $K \in \mathcal{K}_\alpha$ satisfies \([10]\).

(12) if $P_\alpha \subseteq G_{<\alpha}$ is an infinite subset that is closed and discrete in $k_\omega(\mathcal{K}_{<\alpha})$ then there exists an infinite $S_\alpha \subseteq P_\alpha$ such that $S_\alpha \rightarrow s_\alpha$ in $k_\omega(\mathcal{K}_\alpha)$;

(13) if $0 \in \overline{P_\alpha}^{k_\omega(\mathcal{K}_{<\alpha})}$ then there exists an infinite $S_\alpha \subseteq P_\alpha$ such that $S_\alpha \rightarrow 0$ in $k_\omega(\mathcal{K}_\alpha)$;

(14) if $0 \in cC_\alpha^{\tau(\mathcal{U}_{<\alpha})}$ then there exists an infinite $T_\alpha \subseteq C_\alpha$ such that $T_\alpha \rightarrow 0$

Proof. Let $D' \subseteq G_{<\alpha}$ be such that $0 \in \overline{D'}^{\tau(\mathcal{U}_{<\alpha})}$ and there is no $S \subseteq D'$ such that $S \rightarrow 0$ in $k_\omega(\mathcal{K}_{<\alpha})$. Pick an infinite $D \subseteq D'$ such that $D \rightarrow 0$ in $\tau(\mathcal{U}_{<\alpha})$. Then $D$ is closed and discrete in $k_\omega(\mathcal{K}_{<\alpha})$ so using Lemma \[3\] we may assume that $\langle D \rangle$ is closed and discrete in $k_\omega(\mathcal{K}_{<\alpha})$. Apply Lemma \[4\] to construct a pse $(G_\alpha, \mathcal{K}_\alpha, \mathcal{U}_\alpha)$ of $(G_{<\alpha}, \mathcal{K}_{<\alpha}, \mathcal{U}_{<\alpha})$ over $D$ such that $D \rightarrow 0$ in $k_\omega(\mathcal{K}_\alpha)$.

Using either $P_\alpha$ of $cC_\alpha$ in place of $D'$, the argument in the previous paragraph, and putting $S_\alpha = D$ one can show that \([13]\) and \([14]\) are satisfied. Property \([12]\) can be treated using a similar construction (note that the conditions for \([12]\) and \([13]\) are mutually exclusive) by omitting the condition that $D \rightarrow 0$ in $\tau(\mathcal{U}_{<\alpha})$.

Since $\langle D \rangle$ was chosen to be closed and discrete in $k_\omega(\mathcal{K}_{<\alpha})$, Lemma \[7\] implies that $K + \langle D \rangle$ satisfies \([10]\). If $K'' \in k_\omega(\mathcal{K}_\alpha)$ then $K'' = K + \langle D \rangle^{\tau(\mathcal{U}_\alpha)}$ for some $K \in k_\omega(\mathcal{K}_{<\alpha})$. Suppose $g \in K'' \cap S_n$ for some $n \in \omega$. Then $g = a + d$ for some $a \in K$ and $d \in \langle D \rangle^{\tau(\mathcal{U}_\alpha)}$. Since $g \in G_{<\alpha}$, $d \in G_{<\alpha}$ so by \([7]\) $d \in \langle D \rangle$. Thus $K''$ satisfies \([10]\) and \([11]\) holds.

We next show that when the families $C_\alpha$ and $P_\alpha$ are chosen to have some special properties, the construction of Lemma \[11\] results in a desired group.

Suppose $\diamondsuit$ holds and let $\{ C_\alpha : \alpha < \omega_1 \}$ be a $\diamondsuit$-sequence. Identifying $\omega_1$ and $2^{\omega_2}$ we may assume that each $C_\alpha \subseteq 2^{\omega_2}$. Let $\{ P_\alpha : \alpha < \omega_1 \}$ list all infinite countable subsets of $2^{\omega_2}$ so that each $P_\alpha$ is listed $\omega_1$ times.

**Lemma 12.** Let $(G_{\omega_1}, \mathcal{K}_{\omega_1}, \mathcal{U}_{\omega_1}) = (G_{<\omega_1}, \mathcal{K}_{<\omega_1}, \mathcal{U}_{<\omega_1})$ where $(G_\alpha, \mathcal{K}_\alpha, \mathcal{U}_\alpha)$, $\alpha < \omega_1$ have been constructed in Lemma \[11\] Then $k_\omega(\mathcal{K}_{\omega_1}) = \tau(\mathcal{U}_{\omega_1})$, $G_{\omega_1}$ is countably compact, Fréchet, and not $\alpha_3$.

Proof. Suppose $A \subseteq G_{\omega_1}$ is such that $0 \not\in A$, $A \cap K$ is closed for every $K \in \mathcal{K}_{\omega_1}$ and $A \cap U \neq \emptyset$ for every $U \in \mathcal{U}_{\omega_1}$.

Let $\theta$ be large enough. Consider the sets of the form $A \cap M$ where $M$ is a countable elementary submodel of $H(\theta)$ and $X \in M$ is a countable set containing the details of the construction of $G_{\omega_1}$. The set

$$\{ \gamma \in \omega_1 : \gamma = M \cap \omega_1, X \in M, M \leq H(\theta) \}$$

is a club in $\omega_1$. Thus $C_\gamma = A \cap M$ for some $\gamma < \omega_1$ where $M \cap \omega_1 = \gamma$. Note that $\gamma < \omega_1$ is a limit and $cC_\gamma = A \cap G_{<\gamma}$. 


Let $U \in \mathcal{U}_\alpha$ for some $\alpha < \gamma$. Then $\overline{U^{k_1}}(K_{\omega_1}) \in \tau(\mathcal{U}_{\omega_1})$ so there exists a $\beta \geq \alpha$, a $K \in K_\beta$ and a $g \in A \cap K \cap \overline{U^{k_1}}(K_{\omega_1})$ by the choice of $A$. By elementarity, we may assume that $\beta < \gamma$ and $g \in M$ so $g \in A \cap \overline{U^{k_1}}(K_{\omega_1})$. Thus $0 \in \text{cC}_\gamma^\tau(\mathcal{U}_{\omega_1})$ and by (14) there exists a $T_\gamma \subseteq \text{cC}_\gamma \subseteq A$ such that $T_\gamma \rightarrow 0$ in $k_\omega(K_{\gamma})$. This shows that $k_\omega(K_{\omega_1}) = \tau(\mathcal{U}_{\omega_1})$.

Now simple arguments show that (12) implies the countable compactness of $k_\omega(K_{\omega_1})$, (13) implies that $k_\omega(K_{\omega_1})$ is Fréchet, and (11) implies that $k_\omega(K_{\omega_1})$ is not $\alpha_3$. $\square$

Lemmas (11) and (12) now imply the following theorem.

**Theorem 2** (**♦**). There exists a countably compact boolean Fréchet group that is not $\alpha_3$.

The next corollary provides a counterexample to Conjecture 9.4 in [13] and a partial negative answer to Question 4.3 in [12].

**Corollary 3** (**♦**). There exists a precompact countable Fréchet boolean group $G$ with a base of neighborhoods of $0$ consisting of subgroups of finite index such that $G$ is not $\alpha_3$.

**Proof.** Let $G$ be the countable group algebraically generated by the sheaf that provides a counterexample to the $\alpha_3$ property in the group from Theorem 2. $\square$

### 4 Questions.

Corollary 2 and Theorem 1 (see also the remark following Theorem 1) leave open a number of natural questions. The question below seems to be open even for $\mathbb{Z}$.

**Question 1.** Is every countable precompact sequential group Fréchet in ZFC?

A negative answer to Question 1 would follow from a negative answer to Question 2 below (in ZFC). Such a negative answer would also allow the (consistent) extension of the conclusion of Theorem 1 to the class of all (not necessarily countable) precompact groups.

**Question 2.** Can a precompact sequential group contain a closed copy of $S(\omega)$?

The conclusion of Lemma 1 can be generalized to groups having the maximal precompact topology in which a countable family of sequences converge to $1$. A more careful proof shows that when ‘countable’ is replaced by ‘analytic’ the resulting topology can be shown to be coanalytic (i.e. a complement of some analytic set). Note also that it can be shown that the set of convergent sequences of a space with an analytic topology is coanalytic. This leads to the following natural question.
Question 3. Let $G$ be a countable group and $S \subseteq 2^G$ be a Borel (analytic, coanalytic) family of subsets. Let $\tau$ be the maximal precompact group topology on $G$ such that $S \rightarrow 1$ in $\tau$ for each $S \in S$. Must $\tau$ be analytic?

The result of Theorem 2 hints at the possible positive answers to the next two questions. Note that such groups do not exist in ZFC alone and cannot be $\alpha_3$. Thus a positive answer to one of the questions below would strengthen the conclusion of Theorem 2.

Question 4. Do there exist countably compact Fréchet groups $H$ and $G$ such that $H \times G$ is not Fréchet?

Question 5. Does there exist a countably compact Fréchet group whose square is not Fréchet?

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