Superposition, transition probabilities and primitive observables in infinite quantum systems

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Abstract

The concepts of superposition and of transition probability, familiar from pure states in quantum physics, are extended to locally normal states on funnels of type I$_\infty$ factors. Such funnels are used in the description of infinite systems, appearing for example in quantum field theory or in quantum statistical mechanics; their respective constituents are interpreted as algebras of observables localized in an increasing family of nested spacetime regions. Given a generic reference state (expectation functional) on a funnel, e.g. a ground state or a thermal equilibrium state, it is shown that irrespective of the global type of this state all of its excitations, generated by the adjoint action of elements of the funnel, can coherently be superimposed in a meaningful manner. Moreover, these states are the extreme points of their convex hull and as such are analogues of pure states. As further support of this analogy, transition probabilities are defined, complete families of orthogonal states are exhibited and a one–to–one correspondence between the states and families of minimal projections on a Hilbert space is established. The physical interpretation of these quantities relies on a concept of primitive observables. It extends the familiar framework of observable algebras and avoids some counter intuitive features of that setting. Primitive observables admit a consistent statistical interpretation of corresponding measurements and their impact on states is described by a variant of the von Neumann–Lüders projection postulate.
1 Introduction

Local quantum physics [10], as opposed to quantum mechanics, incorporates the idea that one can assign observables to bounded spacetime regions where corresponding measurements can be carried out. This point of view, relying on the Heisenberg picture, has proved to be fruitful in the analysis of states in systems with an infinite number of degrees of freedom, appearing for example in quantum field theory or in quantum statistical mechanics. The global properties of these states can be quite different from those in standard quantum mechanics. Their description often requires algebras which are distinct from the familiar algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on some separable Hilbert space $\mathcal{H}$, being the prototype of a factor of type $I_\infty$ according to the classification of von Neumann algebras. In fact, one also encounters algebras of type $II_\infty$ and III. For states on the latter algebras the celebrated superposition principle fails and there exists up to now no operationally meaningful definition of transition probabilities, such as for pure states.

The restrictions of these global states (expectation functionals) to observables in bounded spacetime regions, however, behave generically like states with a limited number of degrees of freedom [4]. So a description of these partial states in terms of type $I_\infty$ factors is meaningful and also instructive [9]. This insight triggered studies of funnels of type $I_\infty$ factors $\mathcal{N}_n \subset \mathcal{N}_{n+1}$ with common identity, $n \in \mathbb{N}$, which are interpreted as algebras of observables associated with an increasing family (net) of nested spacetime regions [12] [21]. Thinking of a net of strictly increasing regions it is natural to assume that the algebras $\mathcal{N}_n$ are also strictly increasing with increasing $n$. This feature can be expressed by the condition that the subalgebra of operators in $\mathcal{N}_{n+1}$ which commute with all operators in $\mathcal{N}_n$, in notation $\mathcal{N}_n' \cap \mathcal{N}_{n+1}$, has infinite dimension and hence is also a type $I_\infty$ factor, $n \in \mathbb{N}$. Such funnels were named “proper sequential type $I_\infty$ funnels” by Takesaki [21]; since we restrict attention here to this case we will use the shorter term funnel for them. We denote the algebra generated by a given funnel by $\mathcal{N} = \bigcup_n \mathcal{N}_n$; it can be interpreted as the algebra generated by all observables which are localized in bounded spacetime regions.

The physical states on a funnel are described by positive, linear and normalized expectation functionals $\omega : \mathcal{N} \to \mathbb{C}$ which are locally normal, viz. weak operator continuous on the unit ball of each subalgebra $\mathcal{N}_n$, $n \in \mathbb{N}$. Given such a state $\omega$ one obtains by the Gelfand–Naimark–Segal (GNS) construction a faithful representation of the funnel on some separable Hilbert space $\mathcal{H}$; hence one can identify $\mathcal{N}$ with a subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover, there is a unit vector $\Omega \in \mathcal{H}$ such that $\omega(A) = \langle \Omega, A\Omega \rangle$, $A \in \mathcal{N}$, and $\Omega$ is cyclic for $\mathcal{N}$, viz. the subspace $\mathcal{N}\Omega$ is dense in $\mathcal{H}$. Depending on the choice of state, the closure of $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ in the weak operator
topology, denoted by $\mathcal{M}$, can be of any infinite type. Yet this fact is of no relevance here.

We will restrict attention to states $\omega$ whose representing vector $\Omega \in \mathcal{H}$ is separating for $\mathcal{N}$, viz. the equality $A\Omega = 0$ for $A \in \mathcal{N}$ implies $A = 0$. Hence no observable in $\mathcal{N}$ has a sharp (non–fluctuating) value in the state, so one does not have any a priori information about local properties of the corresponding ensemble. This feature is implied by the more stringent condition that $\Omega$ is cyclic for each algebra $\mathcal{N}_n' \cap \mathcal{N}_{n+1}$, $n \in \mathbb{N}$. We shall say that $\omega$ is a generic state on $\mathcal{N}$ if its GNS–vector $\Omega$ has the latter property. Indeed, it follows from a result of Dixmier and Marechall [8] that almost all rays $\mathbb{C}\Omega \subset \mathcal{H}$ arise from generic states $\omega$ (they form a $G_\delta$ set which is dense).

Generic states appear also frequently in physics, prominent examples being states of finite energy [1, Lem. 5], thermal equilibrium states in quantum field theory [15] and states on curved spacetimes satisfying a microlocal spectrum condition [20]. They may be regarded as some ad hoc description of a global background ("state of the world") in which local operations and measurements are performed. Given a generic state $\omega$ on $\mathcal{N}$ we consider its local excitations $\omega_\mathcal{N} = \{\omega_A = \omega \circ \text{Ad} A : A \in \mathcal{N}, \omega_A(1) = 1\}$, where $\text{Ad} A$ denotes the adjoint action of $A$ on $\mathcal{N}$ given by $\text{Ad} A(B) = A^*BA$, $B \in \mathcal{N}$. The completion of the convex hull of $\omega_\mathcal{N}$ in the norm topology induced by $\mathcal{N}$, called the folium of $\omega$, coincides with the set of normal states on $\mathcal{M}$ [12]. But we will not deal with this completion here.

In the present investigation we analyze for given generic state $\omega$ on $\mathcal{N}$ the structure of its local excitations $\omega_\mathcal{N}$ and show that they have many properties in common with the set of pure states in finite quantum systems. In the subsequent section we prove that there exists a canonical (bijective) lift from $\omega_\mathcal{N}$ to "normalized" rays in $\mathcal{N}$ given by $\omega_A \mapsto \mathbb{T}A$, $\mathbb{T}$ being the group of phase factors. (We will use the term "ray" also for these normalized sections.) Since $\mathcal{N}$ is in a natural way a vector space equipped with a scalar product induced by $\omega$, this operation is analogous to lifting pure states to rays of Hilbert space vectors. Thus the states in $\omega_\mathcal{N}$ can coherently be superimposed also in those cases where the underlying state $\omega$ on $\mathcal{N}$ is not pure and the weak closure $\mathcal{M}$ of $\mathcal{N}$ is of any type. Moreover, the states $\omega_\mathcal{N}$ are the extreme points of their (algebraic) convex hull, similarly to the case of finite quantum systems, where pure states are by definition the extreme points of their convex hull.

Making use of these facts we introduce in Sect. 3 an intrinsic concept of transition probability between pairs of states $\omega_A, \omega_B \in \omega_\mathcal{N}$, putting $\omega_A \cdot \omega_B \doteq |\omega(A^*B)|^2$. This product is locally continuous in both entries and can be extended to the convex hull of $\omega_\mathcal{N}$. The states are said to be orthogonal if $\omega_A \cdot \omega_B = 0$ and there exist families of mutually orthogonal states $\omega_{A_n} \in \omega_\mathcal{N}$, $n \in \mathbb{N}$, satisfying the completeness relation $\sum_n \omega_{A_n} \cdot \omega_B = 1$ for all $\omega_B \in \omega_\mathcal{N}$. 

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In analogy to the relation between pure states and one-dimensional projections whose linear span forms a \(*\)-algebra of compact operators, we show in Sect. 4 that the linear span of \(\omega_N\), denoted by \(\text{Span}\omega_N\), carries a product \(\text{Span}\omega_N \times \text{Span}\omega_N \rightarrow \text{Span}\omega_N\) and a \(*\)-operation. They turn \(\text{Span}\omega_N\) into a \(*\)-algebra which is isomorphic to a subalgebra of the compact operators. There is also a spectral theorem for \(\text{Span}\omega_N\) which implies that all states in the convex hull of \(\omega_N\) can be decomposed into convex combinations of orthogonal states.

In Sect. 5 we discuss the effect of inner operations on the states \(\omega_A \in \omega_N\) and propose a concept of “primitive observables”. A primitive observable is fixed by specifying the adjoint action \(\text{Ad}U\) of some unitary \(U \in N\) which induces the map \(\omega_A \mapsto \omega_A \circ \text{Ad}U = \omega_{UA}\). It is conceptually important that the resulting transition probabilities \(\omega_A \cdot \omega_{UA} = |\omega_A(U)|^2\) between the initial and final states may be regarded as observable since the unitaries \(U\) are normal operators and can therefore be decomposed into two commuting observables whose mean values can in principle be determined experimentally. Hence the above formula for transition probabilities is a physically meaningful extension of the corresponding one for pure states, involving one-dimensional projections. We therefore assert that the result of a measurement of a primitive observable in a given ensemble corresponds to the transition probability fixed by the corresponding operation in the given state.

We will show that one can recover from primitive observables the familiar observables and their expectation values by tuning the underlying unitaries and proceeding to a suitable limit. But the primitive observables also provide tools for the analysis of states which are not available in the conventional framework of observable algebras. For example, they allow to evade certain counter intuitive features and apparent paradoxes in the interpretation of relativistic quantum field theories originating from the Reeh–Schlieder property of the vacuum \([5, 13]\). Moreover, the primitive observables lead naturally to a concept of commensurability which generalizes the condition of commutativity for commensurable observables. This generalization could be useful in the discussion of causality properties of theories which do not comply with the standard postulate of locality of observables \([10]\). The article concludes with a brief summary and outlook.

2 Superpositions and mixtures

We establish in this section the asserted extension of the superposition principle to the local excitations \(\omega_N\) of a given generic state \(\omega\) on the algebra \(N\) generated by a funnel \(N_n \subset N_{n+1}\), \(n \in \mathbb{N}\). As explained in the introduction, we may assume that \(N\) is concretely given on some
Hilbert space $\mathcal{H}$, $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$, and that $\omega$ is a vector state induced by some unit vector $\Omega \in \mathcal{H}$ which is cyclic for $\mathcal{N}_n' \cap \mathcal{N}_{n+1}$, $n \in \mathbb{N}$. The following basic lemma is an easy consequence of this cyclicity. It is used at various points in the subsequent analysis.

**Lemma 2.1.** Let $\omega_A \in \omega_\mathcal{N}$ and $c_m \in \mathbb{C}$, $m = 1, \ldots, M$, such that $\sum_{m=1}^{M} c_m \omega_A = 0$. Then $\sum_{m=1}^{M} c_m A_m^* CA_m = 0$ for every $C \in \mathcal{N}$.

**Proof.** There exists some $n \in \mathbb{N}$ such that $A_n \in \mathcal{N}_n$, $m = 1, \ldots, M$. Given any $C \in \mathcal{N}$ one may assume, choosing $n$ sufficiently large, that also $C \in \mathcal{N}_n$. Now let $X,Y \in \mathcal{N}_n' \cap \mathcal{N}_{n+1}$, then

$$
\sum_{m=1}^{M} c_m \langle X \omega, A_m^* CA_m Y \omega \rangle = \sum_{m=1}^{M} c_m \langle \Omega, A_m^* X^* CY A_m \Omega \rangle = \sum_{m=1}^{M} c_m \omega_A (X^* CY) = 0. 
$$

Since this relation holds for arbitrary $X,Y \in \mathcal{N}_n' \cap \mathcal{N}_{n+1}$ and $\Omega$ is cyclic for this algebra it follows that $\sum_{m=1}^{M} c_m A_m^* CA_m = 0$, as claimed. $\square$

The second technical ingredient in our analysis is the basic fact that for any given $n \in \mathbb{N}$ there exists a pure state $\omega_n$ on the type $I_\infty$ factor $\mathcal{N}_{n+1}$ which is an extension of $\omega \upharpoonright \mathcal{N}_n$, viz. $\omega_n (C) = \omega (C)$, $C \in \mathcal{N}_n$. This is an immediate consequence of the assumption that all algebras $\mathcal{N}_n$ are of type $I_\infty$, implying $\mathcal{N}_{n+1} = \mathcal{N}_n \otimes (\mathcal{N}_n' \cap \mathcal{N}_{n+1})$. Since the second tensor factor is infinite dimensional, any normal state (density matrix) on $\mathcal{N}_n$ can be extended to a pure state on $\mathcal{N}_{n+1}$. Having chosen an extension $\omega_n$ of $\omega \upharpoonright \mathcal{N}_n$, there is some non–trivial minimal (“one–dimensional”) projection $E_n \in \mathcal{N}_{n+1}$ satisfying $E_n CE_n = \omega_n (C) E_n$, $C \in \mathcal{N}_{n+1}$. We will make repeatedly use of this result. The following fundamental proposition is based on these technical ingredients.

**Proposition 2.2.** Let $\omega_A, \omega_B \in \omega_\mathcal{N}$ satisfy $\omega_A = \omega_B$. There is a phase factor $t \in \mathbb{T}$ such that $B = tA$. Conversely, if $B = tA$, $t \in \mathbb{T}$, then $\omega_A = \omega_B$. Consequently, there exists a bijective lift from the states in $\omega_\mathcal{N}$ to rays in $\mathcal{N}$, given by $\omega_A \mapsto \mathbb{T}A$.

**Proof.** Given $\omega_A, \omega_B \in \omega_\mathcal{N}$ there is some $n \in \mathbb{N}$ such that $A, B \in \mathcal{N}_n$ and, as explained above, there is a minimal projection $E_n \in \mathcal{N}_{n+1}$ such that $E_n CE_n = \omega (C) E_n$, $C \in \mathcal{N}_n$. Moreover, since $\omega_A = \omega_B$ one infers from the basic lemma that $A^* CA = B^* CB$, $C \in \mathcal{N}_n$. Inserting here $C = BE_n$ and multiplying the resulting equality from the left by $E_n$ yields $E_n \omega (A^* B) A = E_n \omega (B^* B) B$. This implies $\omega (A^* B) A = \omega (B^* B) B$ since $\Omega$ is separating for $\mathcal{N}_n$ and $\|E_n Z\|^2 = \|E_n ZZ^* E_n\| = \|Z^* \Omega\|^2$ for $Z \in \mathcal{N}_n$. It then follows from the normalization condition $\omega (A^* A) = \omega (B^* B) = 1$ that $B = \omega (A^* B) A$ and $\omega (A^* B) \in \mathbb{T}$. The second statement is obvious, completing the proof. $\square$
It is an immediate consequence of this result that there holds the following version of the superposition principle in the present framework.

**Definition:** Let \( A, B \in \mathcal{N} \) and \( c_A, c_B \in \mathbb{C} \) such that \( \omega((c_A A + c_B B)^*(c_A A + c_B B)) = 1 \). There is a unique state \( \omega_{c_A A + c_B B} \in \mathcal{W}_\mathcal{N} \) corresponding to the ray \( T(c_A A + c_B B) \). It defines a coherent superposition of the states \( \omega_A, \omega_B \in \mathcal{W}_\mathcal{N} \).

In the next step we establish continuity properties of the lifts from \( \mathcal{W}_\mathcal{N} \) to rays in \( \mathcal{N} \). To this end we equip \( \mathcal{W}_\mathcal{N} \) with the norm topology induced by \( \mathcal{N} \), i.e. the norm distance between states \( \omega_A, \omega_B \in \mathcal{W}_\mathcal{N} \) is defined by \( \| \omega_A - \omega_B \| = \sup_{C \in \mathcal{N}} |\omega_A(C) - \omega_B(C)|/\|C\| \).

**Proposition 2.3.** Let \( n \in \mathbb{N} \) be fixed and let \( A_m \in \mathcal{N}_n, m \in \mathbb{N} \), be a uniformly bounded sequence of operators such that \( \omega_{A_m}, m \in \mathbb{N} \), form a Cauchy sequence in \( \mathcal{W}_\mathcal{N} \). There exists a sequence \( t_m \in \mathbb{T}, m \in \mathbb{N} \), such that \( t_mA_m, m \in \mathbb{N} \), converges to some operator \( A \in \mathcal{N}_n \) in the strong operator topology, and the sequence \( \omega_{A_m}, m \in \mathbb{N} \), converges to \( \omega_A \in \mathcal{W}_\mathcal{N} \) in norm.

**Proof.** One makes use again of the basic fact that there exists a minimal projection \( E_n \in \mathcal{N}_{n+1} \) satisfying \( E_n CE_n = \omega(C)E_n, C \in \mathcal{N}_n \). Hence, picking \( X, Y \in \mathcal{N}_n \cap \mathcal{N}_{n+1} \), one obtains the estimate for any \( l, m \in \mathbb{N} \)

\[
|\langle X\Omega, A^*_m A_m E_n A^*_m A_m Y\Omega \rangle - \langle X\Omega, A^*_l A_l E_n A^*_l A_l Y\Omega \rangle| \\
= |\omega_{A_m}(X^* A_m E_n A^*_m Y) - \omega_{A_l}(X^* A_l E_n A^*_l Y)| \leq \sup_k \| A_k \|^2 \| X \| \| Y \| \| \omega_{A_m} - \omega_{A_l} \|.
\]

Since \( \Omega \) is cyclic for \( \mathcal{N}_n \cap \mathcal{N}_{n+1} \) it follows that \( (A^*_m A_m E_n A^*_m A_m - A^*_l A_l E_n A^*_l A_l) \to 0 \) in the weak operator topology as \( l, m \to \infty \). Multiplying this relation from the left and right by \( E_n \) one finds that \( (E_n - 1|\omega(A^*_m A_m)|^2 E_n) \to 0 \) and consequently \( |\omega(A^*_m A_m)| \to 1 \) in this limit. Moreover, since \( \|(A_l - \omega(A^*_m A_l))\Omega\|^2 = (1 - |\omega(A^*_m A_l)|^2) \) one obtains, taking scalar products between \( A_k \Omega \) and the vector under the norm, \( |\omega(A^*_m A_l) - \omega(A^*_m A_m)| \to 0 \), uniformly in \( k \in \mathbb{N} \). Since \( \mathbb{T} \) is compact it follows from these facts that there exist \( t_l, t_m \in \mathbb{T} \) such that \( |\omega(A^*_m A_l) - t_m t_l^{-1}| \to 0 \) and, making use of this information in the above vector norm, one arrives at \( \|(t_l A_l - t_m A_m)\Omega\| \to 0 \) as \( l, m \to \infty \). Since the sequence \( t_mA_m \in \mathcal{N}_n, m \in \mathbb{N} \), is uniformly bounded and \( \Omega \) is separating for \( \mathcal{N}_n \) it is then clear that it converges in the strong operator topology and has a limit \( A \in \mathcal{N}_n \) since \( \mathcal{N}_n \) is complete. The remaining part of the statement follows from the simple estimate \( \|\omega_{A_m} - \omega_A\| \leq \inf_{t \in \mathbb{T}} \|tA_m\Omega - A\Omega\|. \)

From the point of view of physics this result says that the family of states \( \mathcal{W}_\mathcal{N} \) is complete under the action of all possible operations which can be performed in the given state \( \omega \) within bounded spacetime regions of arbitrary size. We therefore say \( \mathcal{W}_\mathcal{N} \) is **locally complete.**
set is thus a natural framework for the description of those states which can realistically be prepared in a given background. Secondly, the above result shows that there exist pointwise continuous sections of the lift from $\omega_N$ to rays in $\mathcal{N}$. More explicitly, given any $\omega_A \in \omega_N$, let $\omega_{A_n}$ be any sequence of states, generated by uniformly bounded operators $A_n \in \mathcal{N}_n$, $m \in \mathbb{N}$, for some $n \in \mathbb{N}$, which converges in norm to $\omega_A$. Then there is a sequence $t_m \in \mathbb{T}$, $m \in \mathbb{N}$, such $t_mA_n \rightarrow A$ in the strong operator topology as $m \rightarrow \infty$. We will refer to this fact by saying the lift is \textit{locally continuous} and use this term similarly for other functions on $\omega_N$.

Next, we turn to the analysis of mixtures of states in $\omega_N$. For any given $M \in \mathbb{N}$, arbitrary states $\omega_{A_m} \in \omega_N$ and numbers $0 \leq p_m \leq 1$ summing up to 1, $m = 1, \ldots, M$, we consider the convex combinations $\sum_{m=1}^M p_m \omega_{A_m}$. They form the algebraic convex hull Conv $\omega_N$ of $\omega_N$. In the subsequent proposition it is shown that $\omega_N$ consists of the extreme points of Conv $\omega_N$.

\textbf{Proposition 2.4.} Let $\omega_A \in \omega_N$ and let $\sum_{m=1}^M p_m \omega_{A_m} = \omega_A$, where $\omega_{A_m} \in \omega_N$ and $p_m$ are positive numbers summing up to 1, $m = 1, \ldots, M$. Then $\omega_{A_1} = \cdots = \omega_{A_M} = \omega_A$.

\textbf{Proof.} Let $n \in \mathbb{N}$ be big enough such that (the rays of) $A, A_1, \ldots, A_M \in \mathcal{N}_n$ and let $E_n \in \mathcal{N}_{n+1}$ be a minimal projection satisfying $E_nCE_n = \omega(C)E_n$, $C \in \mathcal{N}_n$. Now the input of the statement implies, making use of Lemma 2.1, $A^*E_nA = \sum_{m=1}^M p_m A_m^*E_nA_m$. Furthermore, $A^*E_nACA^*E_nA = \omega_n(ACA^*)A^*E_nA$, $C \in \mathcal{N}_{n+1}$, where $\omega_n$ is the chosen extension of $\omega | \mathcal{N}_n$ to a pure state on $\mathcal{N}_{n+1}$. Inserting $C = 1$ and noticing that $\omega_n(AA^*) = \omega(\omega^*) = \|A^*\Omega\|^2 \neq 0$ since $\Omega$ is separating, it is clear that $A^*E_nA$ is a multiple of a minimal projection in $\mathcal{N}_{n+1}$. This applies equally to the operators $A_m^*E_nA_m$, $m = 1, \ldots, M$. Since $p_m > 0$, $m = 1, \ldots, M$, and a minimal projection cannot be decomposed into a sum of different positive operators, it follows that $\omega(\omega^*)A_m^*E_nA_m = \omega(A_mA_m^*)A^*E_nA$, $m = 1, \ldots, M$. Multiplying these equalities from the left by $E_nA$ and making use again of the fact that $\Omega$ is separating for $\mathcal{N}_n$ and that $\omega(AA^*) \neq 0$, one arrives at $\omega(\omega^*)A_m^*A_m = \omega(A_mA_m^*)A_m$, $m = 1, \ldots, M$. Taking also into account the normalization condition $\omega(A^*)A = \omega(A_1^*A_1) = \cdots = \omega(A_M^*A_M) = 1$, one concludes that $|\omega(\omega^*)| = \omega(A_mA_m^*) \neq 0$, so there exist $t_m \in \mathbb{T}$ such that $A_m = t_mA$, $m = 1, \ldots, M$. The statement then follows. \qed

\section{Transition probabilities}

We have seen in the previous section that the states in $\omega_N$ have many properties in common with pure states, irrespective of the type of the underlying state $\omega$ on $\mathcal{N}$. Further support for this interpretation is provided by the existence of a meaningful concept of transition probabilities on $\omega_N$. 
**Definition:** Let $\omega_A, \omega_B \in \omega_N$. The transition probability between these states is given by $\omega_A \cdot \omega_B = |\omega(A^* B)|^2$. The states are said to be orthogonal if $\omega_A \cdot \omega_B = 0$. (Note that the definition is meaningful since the lifts $\omega_A \mapsto TA$, $\omega_B \mapsto TB$ are injective.)

Another generalization of the concept of transition probabilities was proposed by Uhlmann in [22]. Our present definition of $\omega_A \cdot \omega_B$ differs from $\omega_A^u \cdot \omega_B$, the quantity given by Uhlmann. In fact, $\omega_A \cdot \omega_B \leq \omega_A^u \cdot \omega_B$, where one has equality for arbitrary states in $\omega_N$ only if $\omega$ is pure. Whereas the Uhlmann concept has proved to be useful in mathematics, we believe that our present definition is more adequate in physics. Some basic properties are compiled in the following proposition.

**Proposition 3.1.** Let $\omega_A, \omega_B \in \omega_N$. Then 

(i) $0 \leq \omega_A \cdot \omega_B \leq 1$ and $\omega_A \cdot \omega_B = \omega_B \cdot \omega_A$.

(ii) $\omega_A \cdot \omega_B \leq 1 - (1/4) ||\omega_A - \omega_B||^2$; equality holds for all $\omega_A, \omega_B$ iff $\omega$ is a pure state on $N$.

(iii) The map $\omega_A, \omega_B \mapsto \omega_A \cdot \omega_B$ is locally continuous.

**Proof.** Point (i) follows from the Schwarz inequality $0 \leq |\omega(A^* B)|^2 \leq \omega(A^* A) \omega(B^* B) = 1$ and the fact that $\omega(A^* B) = \omega(B^* A)$. For the proof of point (ii) one makes use of the fact that the states in $\omega_N$ can be extended to $\mathcal{B}(\mathcal{H})$ in the GNS representation induced by $\omega$ and defines $||\omega_A - \omega_B||_{\mathcal{B}(\mathcal{H})} \doteq \sup_{X \in \mathcal{B}(\mathcal{H})} |\langle A \Omega, X A \Omega \rangle - \langle B \Omega, X B \Omega \rangle||/||X||$. Since $N \subset \mathcal{B}(\mathcal{H})$ it is clear that $||\omega_A - \omega_B|| \leq ||\omega_A - \omega_B||_{\mathcal{B}(\mathcal{H})}$; moreover, $\omega_A \cdot \omega_B = |\langle A \Omega, B \Omega \rangle|^2 = 1 - (1/4) ||\omega_A - \omega_B||^2_{\mathcal{B}(\mathcal{H})}$, as has been shown in [19]. The inequality in (ii) then follows. If $\omega$ is a pure state on $N$, whereby the weak closure of $N$ coincides with $\mathcal{B}(\mathcal{H})$, one has $||\omega_A - \omega_B|| = ||\omega_A - \omega_B||_{\mathcal{B}(\mathcal{H})}$, so equality in (ii) obtains in this case. If $\omega$ is not pure, the commutant $N' \subset \mathcal{B}(\mathcal{H})$ contains some non-trivial unitary operator $V$. Since $\Omega$ is cyclic for $N$ (by the GNS–construction) it is separating for $N'$ and consequently $|\langle \Omega, V \Omega \rangle| < 1$. Now let $A_m \in N$, $m \in \mathbb{N}$, be any sequence such that $A_m \Omega \rightarrow V \Omega$ strongly as $m \rightarrow \infty$. Then $\lim_{m \rightarrow \infty} ||\omega_{A_m} - \omega|| = 0$ and $\lim_{m \rightarrow \infty} |\langle \Omega, A_m \Omega \rangle| < 1$, so one cannot have equality in (ii) for arbitrary states, as claimed. The remaining point (iii) is an immediate consequence of the fact that the lifts $\omega_A \mapsto TA$ and $\omega_B \mapsto TB$ are locally continuous, cf. the remark after Proposition 2.3 and this continuity property is passed on to the transition probabilities. □

Based on the concept of transition probability for the states in $\omega_N$ it is easy to exhibit complete families of orthogonal states. For, by the Gram–Schmidt algorithm, one finds orthonormal systems of vectors $A_m \Omega \in N \Omega$, $m \in \mathbb{N}$, which are complete in $\mathcal{H}$ since $N \Omega$ is dense. In particular, $\sum_m |\langle B \Omega, A_m \Omega \rangle|^2 = 1$ for every $B \in N$ satisfying $\|B \Omega\| = 1$. Hence the states $\omega_{A_m} \in \omega_N$, $m \in \mathbb{N}$, form a complete family of orthogonal states in the following sense.
Definition: A family of orthogonal states $\omega_{A_m} \in \omega_N$, $m \in \mathbb{N}$, is said to be complete if $\sum_m \omega_B \cdot \omega_{A_m} = 1$ for every $\omega_B \in \omega_N$.

The existence of such complete families implies that one has a consistent statistical interpretation of the transition probabilities for the states in $\omega_N$.

4 Algebra of states

We continue our analysis of the states $\omega_N$ by showing that their linear span $\text{Span} \omega_N$ can be equipped with an associative product and a star operation, turning it into a $*$-algebra. Making use of the polarization identity $\omega(A^*CB) = (1/4) \sum_{j=0}^3 i^j \omega((A + i^j B)^* (A + i^j B))$ for $A, B, C \in \mathbb{N}$, it is apparent that the functional $\omega(A^* \cdot B)$ on $\mathbb{N}$ is contained in $\text{Span} \omega_N$. Whereas this functional depends on the operators $A, B \in \mathbb{N}$, the functional in the subsequent definition does not depend on the specific choice of operators from the rays $TA$ and $TB$, respectively. It is therefore well defined on $\omega_N \times \omega_N$.

Definition: Let $\omega_A, \omega_B \in \omega_N$. The functional $\omega_A \times \omega_B \in \text{Span} \omega_N$ is defined by

$$\omega_A \times \omega_B(C) = \omega(A^* B) \omega(B^* CA), \quad C \in \mathbb{N}.$$

Note that $\omega_A \times \omega_B(1) = \omega_A \cdot \omega_B$.

In the subsequent proposition we show that the map $\omega_A, \omega_B \mapsto \omega_A \times \omega_B$ extends linearly in both entries to an associative product on $\text{Span} \omega_N$. We also define an antilinear involution $\dagger$ on this space.

Proposition 4.1. The map $\omega_A, \omega_B \mapsto \omega_A \times \omega_B$ from $\omega_N \times \omega_N$ to $\text{Span} \omega_N$ extends linearly in both entries to an associative product on $\text{Span} \omega_N$, given by

$$\left( \sum_{k=1}^K c_k \omega_{A_k} \right) \times \left( \sum_{l=1}^L d_l \omega_{B_l} \right) \doteq \sum_{k=1}^K \sum_{l=1}^L c_k d_l \omega_{A_k} \times \omega_{B_l}. \quad (4.1)$$

Moreover, the antilinear involution $\dagger$ on $\text{Span} \omega_N$, given by

$$\left( \sum_{k=1}^K c_k \omega_{A_k} \right) \dagger(C) \doteq \sum_{k=1}^K \bar{c}_k \omega_{A_k}(C), \quad C \in \mathbb{N}, \quad (4.2)$$

is algebraically compatible with this product. Equipped with these operations, $\text{Span} \omega_N$ becomes a $*$-algebra, denoted by $\mathcal{C}$. 
Proof. For the proof that the definition (4.1) is consistent we only need to show that the right hand side of this equation vanishes whenever either one of the sums on the left hand side vanishes. So let \( \sum_{k=1}^{K} c_k \omega_{A_k} = 0 \) and hence, by Lemma 2.1 \( \sum_{k=1}^{K} c_k A_k^* C A_k = 0 \) for every \( C \in N \). Given any functional \( \sum_{l=1}^{L} d_l \omega_{B_l} \) and operator \( C \in N \), there is some \( n \in N \) such that (the rays of) \( A_k, B_l, C \in N_n \) for \( k = 1, \ldots, K, \ l = 1, \ldots, L \). Let \( E_n \in N_{n+1} \) be a non–trivial minimal projection such that \( E_n C E_n = \omega(C) E_N, \ C \in N_n \), and let \( \tau_n \) be the standard semifinite trace on \( N_{n+1} \). Inserting \( C = \sum_{l=1}^{L} d_l B_l E_n B_l^* C \in N_{n+1} \) into the above relation one gets the relation \( \sum_{k=1}^{K} \sum_{l=1}^{L} c_k d_l A_k^* B_l E_n B_l^* C A_k = 0 \). Multiplying it from the left and right by \( E_n \) and evaluating its trace \( \tau_n \) one arrives at

\[
0 = \sum_{k=1}^{K} \sum_{l=1}^{L} c_k d_l \tau_n( E_n A_k^* B_l E_n B_l^* C A_k E_n ) = \sum_{k=1}^{K} \sum_{l=1}^{L} c_k d_l A_k^* B_l \omega(B_l^* C A_k) \tau_n(E_n) = \sum_{k=1}^{K} \sum_{l=1}^{L} c_k d_l \omega_{A_k} \times \omega_{B_l}(C).
\]

Since \( C \in N \) was arbitrary, this shows that the right hand side of relation (4.1) vanishes whenever the first sum on the left hand side vanishes, and an analogous argument leads to the same conclusion if the second sum vanishes. It is then obvious that the product is linear in both entries. The proof that the product is associative is a consequence of the relations for \( A, B, C, D \in N \)

\[
(\omega(A^* B) \omega(B^* \cdot A) \times \omega_C)(D) = \omega(A^* B) \omega(B^* C) \omega(C^* DA),
\]

\[
(\omega_A \times \omega(B^* C) \omega(C^* \cdot B))(C) = \omega(B^* C) \omega(A^* B) \omega(C^* DA),
\]

which follow from the definition of the product and the polarization identity for the functionals \( \omega(B^* \cdot A) \) and \( \omega(C^* \cdot B) \). Finally, relation (4.2) and the polarization identity imply for \( A, B, C \in N, \ c, d \in C \),

\[
(\epsilon \omega_A \times d \omega_B)^{\dagger}(C) = (1/4) \sum_{j=0}^{3} c d \omega(A^* B) \epsilon^j \omega((B + i^j A)^* C(B + i^j A))
\]

\[
= (\epsilon d /4) \sum_{j=0}^{3} \omega(B^* A) (-i)^j \omega((A + (-i)^j B)^* C(A + (-i)^j B)) = (\overline{d \omega_B} \times \overline{\epsilon \omega_A})(C),
\]

proving the algebraic compatibility of the antilinear involution \( \dagger \) with the product \( \times \) on \( \text{Span} \ \omega_N \). The conclusion then follows.

Since the states \( \omega_A \in \omega_N \) satisfy \( \omega_A \times \omega_A = \omega_A \) and \( \omega_A^{\dagger} = \omega_A \), they correspond to symmetric projections in \( \mathcal{E} \). Moreover, they satisfy \( \omega_A \times \omega_C \times \omega_A = (\omega_A \cdot \omega_C) \omega_A, \omega_C \in \omega_N \),
and hence are minimal projections. It is also clear that \( \omega_A \times \omega_B = 0 \) iff \( \omega_A, \omega_B \) are orthogonal states. We will show next that any symmetric element of \( \mathcal{C} \) can be decomposed into a sum of orthogonal minimal projections with real coefficients ("spectral theorem").

**Proposition 4.2.** Let \( \psi = \psi^\dagger \in \mathcal{C} \). There exist mutually orthogonal states \( \omega_{A_m} \in \mathcal{W} \) and coefficients \( r_m \in \mathbb{R}, m = 1, \ldots, M \), such that \( \psi = \sum_{m=1}^{M} r_m \omega_{A_m} \). If \( \psi \in \text{Conv} \, \mathcal{W} \subset \mathcal{C} \), then \( r_m \geq 0, m = 1, \ldots, M, \) and \( \sum_{m=1}^{M} r_m = 1 \); hence every "mixed state" in \( \text{Conv} \, \mathcal{W} \) can be decomposed into a convex combination of orthogonal "pure states" in \( \mathcal{W} \).

**Proof.** If \( \psi = 0 \) the statement holds trivially. So let \( \psi = \sum_{m=1}^{M} c_m \omega_{B_m} = \psi^\dagger \) where, without loss of generality, the projections \( \omega_{B_m} \in \mathcal{C} \) are linearly independent and the coefficients \( c_m \) are real and different from 0 for \( m = 1, \ldots, M \). A given \( \psi \) has in general different decompositions of this type. Fixing any one, one considers the one–dimensional projections \( B_m E_\Omega \Omega^\dagger \in \mathcal{B}(\mathcal{H}) \), \( m = 1, \ldots, M \), where \( E_\Omega \) is the projection onto the ray \( \mathbb{C} \Omega \in \mathcal{H} \). Denoting by \( \tau \) the trace on \( \mathcal{B}(\mathcal{H}) \) one obtains \( \tau((\sum_{m=1}^{M} d_m B_m E_\Omega \Omega^\dagger C)) = \sum_{m=1}^{M} d_m \omega_{B_m}(C) \) for \( C \in \mathcal{N} \) and \( d_m \in \mathbb{C} \), \( m = 1, \ldots, M \). Thus the projections \( B_m E_\Omega \Omega^\dagger \), \( m = 1, \ldots, M \), are linearly independent as well. Moreover, putting \( \Psi \doteq \sum_{m=1}^{M} c_m B_m E_\Omega \Omega^\dagger B_m \in \mathcal{B}(\mathcal{H}) \), one recovers the given functional, \( \tau(\Psi C) = \psi(C), C \in \mathcal{N} \).

The vectors \( B_m \Omega \in \mathcal{H}, m = 1, \ldots, M, \) span an \( M \)-dimensional subspace \( \mathcal{H}_M \subset \mathcal{H} \) which is stable under the action of the self–adjoint operator \( \Psi \). Hence there exists a non–singular matrix \( S_{m,l}, m, l = 1, \ldots, M, \) such that the vectors \( \sum_{l=1}^{M} S_{m,l} \Omega \in \mathcal{H}_M \), \( m = 1, \ldots, M, \) are orthogonal, normalized and diagonalize \( \Psi \mid \mathcal{H}_M, \) i.e. \( (\Psi - r_m \Omega)(\sum_{l=1}^{M} S_{m,l} \Omega) = 0, \) where \( r_m \in \mathbb{R}, m = 1, \ldots, M \). Hence \( \Psi \) can be represented in the form \( \Psi = \sum_{m=1}^{M} r_m A_m E_\Omega \Omega^\dagger A_m^\star, \) where the operators \( A_m = \sum_{l=1}^{M} S_{m,l} B_l, m = 1, \ldots, M, \) are elements of \( \mathcal{N} \). It follows that \( \tau(\Psi C) = \sum_{m=1}^{M} r_m \omega_{A_m}(C), C \in \mathcal{N} \). Since by construction \( \omega_m \cdot \omega_l = |\langle A_m \Omega, A_l \Omega \rangle|^2 = \delta_{m,l}, \) \( l, m = 1, \ldots, M, \) this establishes the desired decomposition of \( \psi \).

Finally, let \( \psi = \sum_{m=1}^{M} c_m \omega_{B_m}, \) where \( c_m \geq 0, m = 1, \ldots, M. \) Then

\[
(\psi \times \omega_{A_l})(1) = \sum_{m=1}^{M} c_m (\omega_{B_m} \times \omega_{A_l})(1) = \sum_{m=1}^{M} c_m \omega_{B_m} \cdot \omega_{A_l} \geq 0, \quad l = 1, \ldots, M.
\]

Making use of this information in the orthogonal decomposition \( \psi = \sum_{m=1}^{M} r_m \omega_{A_m} \), it follows that \( r_l = (\psi \times \omega_{A_l})(1) \geq 0, l = 1, \ldots, M. \) Since the states in \( \text{Conv} \, \mathcal{W} \) are normalized one also has \( 1 = \psi(1) = \sum_{m=1}^{M} r_m, \) completing the proof of the theorem.

We conclude this discussion of the algebraic properties of \( \mathcal{C} = \text{Span} \, \omega_{\mathcal{N}} \) by introducing a left and right action of \( \mathcal{N} \) on this space.
**Definition:** Let $\psi \in \mathcal{C}$. The left, respectively, right actions of $A \in \mathcal{N}$ on $\psi$ are given by 

$$(A \times \psi)(C) \doteq \psi(AC) \quad \text{and} \quad (\psi \times A)(C) \doteq \psi(CA), \quad C \in \mathcal{N}.\quad \text{(1)}$$

With this definition $\mathcal{C}$ becomes an $\mathcal{N}$–bimodule.

Since $\mathcal{C}$ does not contain an identity, the underlying algebra $\mathcal{N}$ does not correspond to a subalgebra of $\mathcal{C}$. Yet one can recover the operators in $\mathcal{N}$ as “weak limits” of operators in $\mathcal{C}$. The appropriate topology is induced by the states in $\omega_\mathcal{N}$ which determine elements of the dual space of $\mathcal{C}$, defined subsequently. It follows from Proposition 4.1 that this definition is consistent.

**Definition:** Let $\omega_A \in \omega_\mathcal{N}$. Its dual action $\omega_A : \mathcal{C} \to \mathbb{C}$ is defined by

$$\omega_A(\psi) \doteq (\omega_A \times \psi)(1), \quad \psi \in \mathcal{C}.\quad \text{(2)}$$

The dual action of the elements of Conv $\omega_\mathcal{N}$ on $\mathcal{C}$ is defined analogously.

Making use of Proposition 4.1 one obtains by a straightforward computation the basic equalities for any $\omega_A, \omega_B, \omega_C, \omega_D \in \omega_\mathcal{N}$

$$(\omega_A \times \omega_B \times \omega_C)(1) = \omega(A^*B)\omega(B^*C)\omega(C^*A),\quad \text{(3)}$$

$$(\omega_A \times \omega_B \times \omega_C \times \omega_D)(1) = \omega(A^*B)\omega(B^*C)\omega(C^*D)\omega(D^*A),\quad \text{(4)}$$

and similarly for higher products. Note that the numerical values of these expressions do not change under cyclic permutations of the operators $A, B, C, D$. These relations are a key ingredient in the proof of the following result.

**Proposition 4.3.** (i) Let $\omega_A \in \omega_\mathcal{N}$. The map $\omega_A : \mathcal{C} \to \mathbb{C}$ satisfies $\omega_A(\psi^\dagger \times \psi) \geq 0$, $\psi \in \mathcal{C}$, and hence is a positive linear functional on $\mathcal{C}$.

(ii) The GNS–representation of $\mathcal{C}$ induced by the underlying state $\omega$ is faithful.

(iii) There exists a spatial isomorphism between the, as in (ii) represented, algebra $\mathcal{C}$ and the algebra $\mathcal{C}_\Omega \subset \mathcal{B}(\mathcal{H})$ which is generated by the projections $\{AE_\Omega A^* : A \in \mathcal{N}, \|A\Omega\| = 1\}$, where $E_\Omega$ denotes the projection onto the ray $\mathbb{C}\Omega \subset \mathcal{H}$.

**Proof.** (i) Let $\psi = \sum_m c_m \omega_{B_m} \in \mathcal{C}$. Then, by the first basic equality given above,

$$\omega_A(\psi^\dagger \times \psi) = \sum_{k=1}^M \sum_{l=1}^M c_k c_l (\omega_A \times \omega_{B_k} \times \omega_{B_l})(1) = \sum_{k=1}^M \sum_{l=1}^M c_k c_l \omega(A^*B_k)\omega(B_k^*B_l)\omega(B_l^*A) \geq 0,$$

where the asserted positivity follows from the relation $\omega(B_k^*A) = \omega(A^*B_k)$ and the fact that $\omega(B_k^*B_l)$ is a non–negative matrix, $k, l = 1, \ldots, M$. 


For the proof of (ii), let $0 \neq \psi \in \mathfrak{C}$ such that $\omega(\omega_A \times \psi \times \omega_B) = 0$ for any choice of states $\omega_A, \omega_B \in \omega_N$. Taking the complex conjugate of this equality gives $\omega(\omega_B \times \psi^* \times \omega_A) = 0$ for $\omega_A, \omega_B \in \omega_N$. Hence it suffices to consider the case $\psi^* = \psi$. According to Proposition 4.2, the given $\psi$ can then be brought into the form $\psi = \sum_{m=1}^{M} r_m \omega_{A_m}$ where the states $\omega_{A_m}$ are mutually orthogonal and $r_m \neq 0$, $m = 1, \ldots, M$. Making use of the second basic equality given above, one gets

$$0 = \omega(\omega_A \times \psi \times \omega_A) = \sum_{m=1}^{M} r_m (\omega \times \omega_A \times \omega_{A_m} \times \omega_A)(1) = \sum_{m=1}^{M} r_m |\omega(A)|^2 |\omega(A_m^* A)|^2.$$ 

If $\omega(A_l) \neq 0$ for some $l \in \{1, \ldots, M\}$ one puts $A = A_l$ into this equality, giving $r_l = 0$, in conflict with the assumption that $\psi \neq 0$. If $\omega(A_m) = 0$, $m = 1, \ldots, M$, one puts $A = c1 + A_l$ for some $c \neq 0$ and arrives at the same conclusion. Thus there exists no non-zero operator $\psi \in \mathfrak{C}$ which vanishes in the GNS-representation induced by $\omega$, hence this representation is faithful. But note that $\omega$ is not a faithful state on $\mathfrak{C}$.

As to the proof of (iii), recall that the GNS representation of $\mathcal{N}$ induced by $\omega$ consists of two ingredients: (a) a vector space $\mathcal{H}_\omega$ spanned by the equivalence classes $|\psi\rangle$ of elements of $\mathfrak{C}$, modulo the left ideal which annihilates $\omega$, which is equipped with a scalar product $\langle \psi_1 | \psi_2 \rangle \doteq \omega(\psi_1^* \times \psi_2)$; (b) a homomorphism $\pi$ from $\mathcal{N}$ to bounded operators acting on $\mathcal{H}_\omega$, given by $\pi(\psi_1) | \psi_2 \rangle \doteq | \psi_1 \times \psi_2 \rangle$, $\psi, \psi_1, \psi_2 \in \mathcal{N}$. After these preparations it is straightforward to prove that the mapping $W : \mathcal{H}_\omega \rightarrow \mathcal{H}$ defined by

$$W | \sum_{m=1}^{M} c_m \omega_{A_m} \rangle \doteq \sum_{m=1}^{M} c_m \omega(A_m^* A_m \Omega), \quad \sum_{m=1}^{M} c_m \omega_{A_m} \in \mathfrak{C},$$

is an isomorphism with dense domain in $\mathcal{H}_\omega$ and dense range in $\mathcal{H}$. In fact,

$$\langle \sum_{l=1}^{M} c_l \omega_{A_l} | \sum_{m=1}^{M} c_m \omega_{A_m} \rangle = \sum_{l=1}^{M} \sum_{m=1}^{M} \bar{c}_l c_m \omega(\omega_{A_l} \times \omega_{A_m})$$

$$= \sum_{l=1}^{M} \sum_{m=1}^{M} \bar{c}_l c_m \omega(A_l)\omega(A_l^* A_m^* A_m) = \sum_{l=1}^{M} \sum_{m=1}^{M} \bar{c}_l c_m \omega(\omega_{A_l})\omega(A_m^*) \langle A_l \Omega, A_m \Omega \rangle.$$

By a similar computation one obtains for arbitrary $\omega_A, \omega_B \in \omega_N$

$$\langle \omega_A | \pi \left( \sum_{m=1}^{M} c_m \omega_{A_m} \right) | \omega_B \rangle$$

$$= \omega(\omega_A \times (\sum_{m=1}^{M} c_m \omega_{A_m}) \times \omega_B) = \sum_{m=1}^{M} c_m \omega(A)\omega(A_m^* A_m)\omega(B_m^*)$$

$$= \omega(A) \langle A \Omega, (\sum_{m=1}^{M} c_m A_m E_\Omega A_m^* B) \Omega \rangle \omega(B^*) = \langle \omega_A | W^* \left( \sum_{m=1}^{M} c_m A_m E_\Omega A_m^* \right) W | \omega_B \rangle.$$
By comparison of the left and right hand side of this equality one obtains

$$W \pi \left( \sum_{m=1}^{M} c_m \omega_{A_m} \right) = \left( \sum_{m=1}^{M} c_m A_m E_\Omega A_m^* \right) W, \quad \sum_{m=1}^{M} c_m \omega_{A_m} \in \mathcal{E}.$$  

Hence $W$ establishes a spatial isomorphism between $\pi(\mathcal{E})$ and $\mathcal{E}_\Omega$, completing the proof of the proposition.

The preceding proposition establishes a simple universal picture of the space $\mathcal{E} = \text{Span } \omega_N$, spanned by the local excitations of any generic state $\omega$ on the funnel $N$, which does not depend on the global type of $\omega$: the space may be identified with the bimodule obtained by left and right (product) action of $N$ on the projection $E_\Omega$, where $\Omega \in \mathcal{H}$ is the chosen vector representing $\omega$. Thus it corresponds to a specific subspace $\mathcal{E}_\Omega \subset \mathcal{B}(\mathcal{H})$ of finite rank (hence trace class) operators. Moreover, the transition probabilities of the states in $\omega_N$, defined above, can be expressed in terms of $\mathcal{E}_\Omega$ by the familiar dual action of trace class operators onto themselves under the trace of $\mathcal{B}(\mathcal{H})$. However, since $\mathcal{E}_\Omega \cap N = \{0\}$ (as $\Omega$ is separating for $N$), the elements of $\mathcal{E}_\Omega$ may in general not be regarded as genuine observables, in contrast to quantum mechanics, where trace class operators are part of the observable algebra. The physical interpretation of these quantities therefore requires some explanations which will be provided in the subsequent section.

Let us mention in conclusion that part of the preceding mathematical results could have been also established by making use of the well–known fact that there exist isomorphisms (universal localizing maps \cite{3}) $\phi_n : \mathcal{B}(\mathcal{H}) \to N_{n+1}$, which leave $N_n$ pointwise fixed, $n \in \mathbb{N}$. Yet our present approach reveals more closely the intrinsic nature of the proposed concepts.

## 5 Primitive observables

Having exhibited the mathematical similarities between the local excitations of generic states in infinite quantum systems and pure states in quantum mechanics, let us turn now to the discussion of the physical significance of this observation. What is missing so far is an argument that the transition probabilities, as defined above, are in principle accessible to observations. We will establish such a link in this section by making use of the concept of operations.

The effect of non–mixing operations on states are generally described by the formula \cite{11, 14} $\omega \mapsto \frac{1}{\omega(V^*V)} \omega \circ \text{Ad}_V$, i.e. the operations are identified with the adjoint action of arbitrary elements $V \in N$ on the states, followed by a normalization. In this generality, one incorporates operations where parts of the resulting ensemble are discarded by the observer
and the remainder is considered as a new ensemble, as in the von Neumann–Lüders projection postulate. Yet such “state reducing operations” require a re–normalization and induce highly non–linear mappings on the space of states. We therefore restrict attention here to operations induced by operators $V \in \mathcal{N}$ satisfying the condition $V^*V = 1$, i.e. to isometries.

Operations described by isometries appear naturally in the context of physics, prominent examples being the effects of temporary inner perturbations of the dynamics in the Heisenberg picture. The special case of unitary operators $U \in \mathcal{N}$ is of particular interest here since unitary operators, being normal, are linear combinations of commuting selfadjoint operators, viz. $U = (1/2)(U + U^*) + (i/2)i(U^* - U)$; they can therefore be regarded as observables. Moreover, any isometry and any projection in $\mathcal{N}$ can be obtained as an appropriate limit of unitaries, as is shown in the lemma below. We will therefore base our physical interpretation of the mathematical framework on such unitaries.

In the proof of the subsequent lemma we make use of the fact that every non–zero projection $E \in \mathcal{N}$ has infinite dimension (its commutant in $\mathcal{N}$ is infinite dimensional); moreover, there exist isometries $V \in \mathcal{N}$ with range projection $E$, viz. $VV^* = E$.

**Lemma 5.1.** Let $E \in \mathcal{N}$ be any non–zero projection and let $V \in \mathcal{N}$ be any isometry with range projection $E$. Then

(i) there exists a sequence of unitaries $U_m \in \mathcal{N}$, $m \in \mathbb{N}$, such that $U_m \rightarrow V$ in the strong operator topology.

(ii) there exists a sequence of isometries $V_m \in \mathcal{N}$, $m \in \mathbb{N}$, with common range projection $E$ such that $V_m \rightarrow E$ in the weak operator topology and $V_m^* \rightarrow E$ in the strong operator topology.

(iii) for any $\omega_A \in \mathcal{W}_\mathcal{N}$ one has $\sup_V |\omega_A(V)| = \omega_A(E)$, where the supremum is taken with respect to all isometries $V \in \mathcal{N}$ with range projection $E$.

**Proof.** (i) Let $V \in \mathcal{N}_n$ for some $n \in \mathbb{N}$. Picking any strictly increasing sequence of projections $E_m \in \mathcal{N}_n$ which converges to 1 in the strong operator topology, one puts $V_m \doteq V E_m$, $m \in \mathbb{N}$. Then $(1 - V_m V_m^*)$ and $(1 - V_m^* V_m) = (1 - E_m)$ are infinite projections in $\mathcal{N}_n$, so there exist partial isometries $W_m \in \mathcal{N}_n$ such that $W_m W_m^* = (1 - V_m V_m^*)$ and $W_m^* W_m = (1 - E_m)$, $m \in \mathbb{N}$. By construction $(V_m + W_m)^* (V_m + W_m) = (V_m + W_m)(V_m + W_m)^* = 1$, hence these operators are unitaries, $m \in \mathbb{N}$. Moreover, $V_m \rightarrow V$ and $W_m \rightarrow 0$ in the strong operator topology as $m \rightarrow \infty$, so statement (i) follows.

(ii) Let $n \in \mathbb{N}$ be sufficiently large such that there is a sequence of unitaries $U_m \in \mathcal{N}_n$, $m \in \mathbb{N}$, as in (i) which converges to $V$ in the strong operator topology. Then $V_m^* \doteq U_m V^* \rightarrow V V^* = E$ in the strong operator and $V_m = V U_m^* \rightarrow E$ in the weak operator topology. Since $V_m V_m^* = E$
and $V_m^*V_m = 1$, statement (ii) follows.

(iii) For the proof of the remaining statement one first considers the case where $\omega_A$ is a faithful state on $\mathcal{N}$. Putting $w_A := \sup V |\omega_A(V)|$, it follows from the preceding step, inserting the weakly convergent sequence of isometries $VU_m^*$, $m \in \mathbb{N}$, into $\omega_A$, that $w_A \geq \omega_A(E)$. Next, let $F$ be the weak operator limit of some sequence of isometries with range projection $E$ which saturates the bound $w_A$. Since $\langle A\Omega, FA\Omega \rangle = \langle EA\Omega, FA\Omega \rangle$ the bound is attained if $EA\Omega$ and $FA\Omega$ are parallel. As $A\Omega$ is separating this implies $F = cE$ and since $\|F\| \leq 1$ it follows that $|c| \leq 1$, whence $w_A \leq \omega_A(E)$. Since the faithful states are norm dense in the states $\omega_N$ on the funnel, this shows that $w_A = \omega_A(E)$ for all states $\omega_A \in \omega_N$, completing the proof of the statement.

The adjoint action of a given unitary $U \in \mathcal{N}$ on the states, $\omega_A \mapsto \omega_A \circ \text{Ad} U = \omega_U A$, describes the effect of a fixed physical operation on the corresponding ensembles. The respective transition probabilities between the initial and final states are given by $\omega_A \cdot \omega_U A = |\omega_A(U)|^2$. They can be determined by measurements of the commuting observables underlying $U$ in the ensemble described by $\omega_A$. Simple examples illustrating this fact are the unitaries $U_t = (E + t(1 - E))$, where $E \in \mathcal{N}$ is a projection and $t \in \mathbb{T}$. Then $\omega_A \cdot \omega_U A = \omega_A(E)^2 + \omega_A(1 - E)^2 + 2 \text{Re} (t) \omega_A(E) \omega_A(1 - E)$, so in this case it suffices to determine the expectation value of $E$ in state $\omega_A$ in order to determine the transition probability induced by the corresponding operation.

Let us mention as an aside that the customary term “transition probability” is slightly misleading in the present context since, given the initial state $\omega_A$, the resulting final state $\omega_U A$ is uniquely determined. Rather, the quantity $\omega_A \cdot \omega_U A$ represents the probability of finding in the final ensemble members of the original ensemble which “survived” the operation. A more suggestive notion for $\omega_A \cdot \omega_U A$ would therefore be the term “fidelity”, introduced by Jozsa [16]. It could be interpreted as a measure for the degree of compatibility of an operation with the properties of a given initial ensemble. Yet in order to avoid confusion we will continue to use the term “transition probability”.

In order to shed further light on the significance of the operations and the resulting transition probabilities, let us discuss next how one can derive from them some pertinent physical information. Quantities of primary interest are the probabilities $\omega_A(E)$ of meeting in an ensemble, described by a state $\omega_A \in \omega_N$, members with specific properties, described by a projection $E \in \mathcal{N}$. As we will see, these data can be derived from the transition probabilities for suitably tuned operations (reminiscent of tuned up detectors).

According to part (iii) of the preceding lemma one has for any given projection $E$ and
isometry \( V \in \mathfrak{N} \) with range \( E \) the \textit{a priori} bound \( |\omega_A(V)| \leq \omega_A(E) \). It then follows from (ii) that for any given finite number of states and \( \varepsilon > 0 \) there exist appropriate isometries \( V \) satisfying \( |\omega_A(V) - \omega_A(E)| < \varepsilon \), \textit{i.e.} they saturate the upper bound with arbitrary precision. Moreover, as an immediate consequence of part (i), there exist for any \( \varepsilon > 0 \) unitaries \( U \in \mathfrak{N} \) such that \( \|\omega_{U_A} - \omega_{V_A}\| < \varepsilon \) for given isometry \( V \) and states \( \omega_A \). Combining this information one finds that \( \omega_{U_A}(1 - E) < \varepsilon \) and \( |\omega_A \cdot \omega_{U_A} - \omega_A(E)| < 4\varepsilon \) for the given states. The corresponding operations \( \text{Ad} U \) thus (a) determine within the given margins the probabilities \( \omega_A(E) \approx (\omega_A \cdot \omega_{U_A})^{1/2} \) of finding the property \( E \) in the initial states and (b) create final states which have the property \( E \) with probability \( \omega_{U_A}(E) \approx 1 \) without relying on the standard process of (von Neumann–Lüders) state reduction. This construction can be extended to several commuting projections \( E_m, m = 1, \ldots, M \). One can then determine the mean values of observables \( O = \sum_{m=1}^{M} o_m E_m \) in the states \( \omega_A \) with arbitrary precision by means of operations \( \text{Ad} U_m, m = 1, \ldots, M \), making use of the relation \( \omega_A(O) \approx \sum_{m=1}^{M} o_m (\omega_A \cdot \omega_{U_m A})^{1/2} \).

Having seen how suitably tuned operations allow for the recovery of observables and their statistical interpretation, let us stress that there exist operations providing information which cannot be obtained in the conventional setting of observable algebras. To illustrate this fact, let us recall that for any non-trivial projection \( E \in \mathfrak{N} \) one has \( \omega(E) > 0 \) for generic states \( \omega \), the vacuum in quantum field theory being an example. This Reeh–Schlieder property of the vacuum is often regarded as a conceptual problem \footnote{[5][13]}. For it implies that there do not exist perfect detectors, described by projections \( E \), which give a non-zero signal exclusively in states which are different from the vacuum, \textit{i.e.} for which \( \omega_A(E) \neq 0 \) implies \( \omega_A \neq \omega \). Yet, relying on the notion of operations and transition probabilities, one can model such detectors: picking any unitary \( U \in \mathfrak{N} \) with the property that \( \omega(U) = 0 \), the operation \( \omega_A \mapsto \omega_A \circ \text{Ad} U = \omega_{U_A} \), \( \omega_A \in \mathfrak{N} \), has the desired property since \( \omega_A \cdot \omega_{U_A} = |\omega_A(U)|^2 \neq 0 \) implies \( \omega_A \neq \omega \). The quantity \( \omega_A \cdot \omega_{U_A} \) may therefore consistently be interpreted as the probability that the detector registers some deviation from \( \omega \) in the state \( \omega_A \in \mathfrak{N} \).

The upshot of this discussion is the insight that operations and the resulting transition probabilities provide a meaningful extension of the conventional framework of observable algebras and their statistical interpretation. We therefore propose to call these operations \textit{primitive observables}, (primitive in the sense of being basic). Fundamental concepts which are familiar from the conventional setting, such as the characterization of commensurable observables, can be transferred to the primitive observables in a meaningful manner. Two primitive observables, described by operations \( \text{Ad} U_1 \) and \( \text{Ad} U_2 \), are regarded as commensurable if the corresponding composed operations coincide, \( \text{Ad} U_1 U_2 = \text{Ad} U_2 U_1 \). They then give rise
to coinciding transition probabilities for all states, i.e. $|\omega_A(U_1U_2)|^2 = |\omega_A(U_2U_1)|^2$, $\omega_A \in \omega_N$. Note that this condition does not necessarily imply that the unitaries commute. Yet if one has two sequences of commuting primitive observables constructed from unitaries which are tuned as above so as to approximate projections $E_1, E_2$ with arbitrary precision, then one can show that these projections commute. So one recovers the familiar commutativity of commensurable observables in this case. However, the generalized condition of commensurability imposes weaker constraints on the primitive observables. It may therefore be useful in discussions of causality properties of theories, where the standard postulate of locality of observables [10] cannot be applied. In any case, the proposed generalized notion of commensurability fits naturally into the present setting and is physically meaningful.

Up to this point we have concentrated on properties of the states in $\omega_N$ which do not depend on the type of the underlying generic state $\omega$. Yet there is some physically relevant feature of these states which does depend on their type. It originates from the fact that the notion of transition probability acquires physical significance through the existence of tunable operations acting on ensembles. Within the theoretical framework this fact is expressed by the assertion that for any $U \in \mathcal{N}$ the mapping $\omega_A \mapsto \omega_{UA}$ has some operational meaning and the corresponding transition probability $\omega_A \cdot \omega_{UA}$ is in principle accessible to observations. One may therefore ask whether it is possible to determine in this way the transition probabilities $\omega_A \cdot \omega_B$ between arbitrary states $\omega_A, \omega_B \in \omega_N$.

An affirmative answer to this question requires that for any pair of states there exists some unitary operator $U \in \mathcal{N}$ such that $\omega_B = \omega_{UA}$ or, slightly less restrictive, that there exist suitable unitaries for which this equality can be established with arbitrary precision. In other words, the inner operations induced by unitary operators have to act (topologically) transitively on the states $\omega_N$. Otherwise, it would not be possible to determine by physical operations the transition probabilities for arbitrary pairs of states. As a matter of fact, there do exist generic states $\omega$ on $\mathcal{N}$ where this obstruction occurs. These are primary states of type $\text{III}_\lambda$, $0 \leq \lambda < 1$, for which there exist certain pairs $\omega_A, \omega_B \in \omega_N$ whose minimal distance $\inf_U \|\omega_B - \omega_{UA}\|$ with regard to all possible operations is strictly positive [6].

There are, however, two important classes of states $\omega$ which comply with the condition of (topologically) transitive action of operations on the respective states $\omega_N$. The first class consists of pure states $\omega$ on $\mathcal{N}$, which are known to satisfy this condition according to a well–known theorem by Kadison [17]. The second class consists of states $\omega$ on $\mathcal{N}$ which are of type $\text{III}_1$. They also comply with this condition, as has been shown in [7]. So these two classes of states have much more in common than one might infer from their mathematical...
definition based on modular theory. More remarkably, these two classes consist exactly of those states which abound in the context of infinite quantum systems \cite{2,10,23}. Thus the concepts introduced here fully cover these classes of primary physical interest.

6 Summary and outlook

In the present investigation we have established some universal properties of states in infinite quantum systems, which are described by funnels of type $I_\infty$ factors. These states may be regarded as excitations of some generic reference state, describing a global background in which local measurements and operations take place. The states are complete with regard to local operations and therefore provide a meaningful framework for the discussion of their physical properties. Even though the states can be of any infinite Connes–von Neumann type, they share many basic properties with pure states. In particular, they allow for an intrinsic definition of coherent superpositions and of transition probabilities. The physical interpretation of this novel framework is based on the concept of primitive observables which extends the familiar notion of observables in terms of operator algebras. Primitive observables admit a consistent statistical interpretation, related to actual observations, and they bypass certain counter intuitive features and apparent paradoxes of the conventional operator algebraic setting. Moreover, they comply with a generalized condition of commensurability which entails the standard commutativity property of commensurable observables but is less restrictive. It thereby allows to discuss causality properties of theories without relying on the usual condition of locality. The present framework thus provides a promising setting for further study of the properties and the interpretation of infinite quantum systems.

On the mathematical side, the present results offer a new look at the state spaces of hyperfinite factors. They are all completions of “skeleton spaces” of finite rank operators which are bi–modules relative to the funnel generating the respective factor. It is of interest in this context that, algebraically, the funnels are all isomorphic and that funnels generating a given factor are related by inner automorphisms \cite{21}. Thus this description of the state spaces in terms of finite rank operators is intrinsic. It seems an interesting problem to understand how the Connes–von Neumann type of the factors is encoded in the structure of their respective skeleton spaces.
References

[1] Hans-Jürgen Borchers: On the vacuum state in quantum field theory II, Commun. Math. Phys. 1, 57–79 (1965).

[2] Ola Bratteli and Derek W. Robinson: Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States. Models in Quantum Statistical Mechanics, Springer 1997

[3] Detlev Buchholz, Sergio Doplicher and Roberto Longo: On Noether’s theorem in quantum field theory, Annals of Physics 170, 1–17 (1986)

[4] Detlev Buchholz and Eyvind H. Wichmann: Causal Independence and the Energy Level Density of States in Local Quantum Field Theory, Commun. Math. Phys. 106, 321–344 (1986)

[5] Detlev Buchholz and Jakob Yngvason: There are no causality problems for Fermi’s two-atom system, Phys. Rev. Lett. 73, 613–616 (1994)

[6] Alain Connes, Uffe Haagerup and Erling Størmer: Diameters of state spaces of type III factors, pp. 91–116 in: Lecture Notes in Mathematics 1132, Springer Verlag 1985

[7] Alain Connes and Erling Størmer: Homogeneity of the state space of factors of type III$_1$, J. Funct. Analysis 28, 187—196 (1987)

[8] Jacques Dixmier and Odile Maréchal: Vecteurs totalisateurs d’une algebre de von Neumann Commun. Math. Phys. 22, 44—50 (1971)

[9] Sergio Doplicher and Roberto Longo: Local Aspects of Superselection Rules II, Commun. Math. Phys. 88, 399–409 (1983)

[10] Rudolf Haag: Local Quantum Physics: Fields, Particles, Algebras, Springer 1996

[11] Rudolf Haag and Daniel Kastler: An algebraic approach to quantum field theory, J. Math. Phys. 5, 848–861 (1964)

[12] Rudolf Haag, Richard Kadison and Daniel Kastler: Nets of C*-Algebras and Classification of States, Commun. Math. Phys. 16, 81—104 (1970)

[13] Hans Halvorson: Reeh–Schlieder defeats Newton-Wigner: On alternative localization schemes in relativistic quantum field theory, Philosophy of Science 68, 111–133 (2001)

[14] Klaus E. Hellwig and Karl Kraus: Pure operations and measurements, Commun. Math. Phys. 11, 214–220 (1969)
[15] Christian Jäkel: The Reeh—Schlieder property for thermal field theories, J. Math. Phys. 41, 1745–1754 (2000)

[16] R. Jozsa: Fidelity for mixed quantum states, Journal of Modern Optics 41, 2315—2323 (1994)

[17] Richard Kadison: Irreducible operator algebras, Proc. Nat. Acad. Sci. USA 43, 273–276 (1957)

[18] Roberto Longo: Notes on algebraic invariants for non–commutative dynamical systems, Commun. Math. Phys. 69, 195—207 (1979)

[19] Gert Roepstorff and John E. Roberts: Some basic concepts of algebraic quantum theory, Commun. Math. Phys. 11, 321–338 (1968)

[20] Alexander Strohmaier, Rainer Verch and Manfred Wollenberg: Microlocal analysis of quantum fields on curved space–times: Analytic wave front sets and Reeh—Schlieder theorems, J. Math. Phys. 43, 5514–5530 (2002)

[21] Masamichi Takesaki: Algebraic equivalence of locally normal representations, Pacific J. Math. 34, 807–816 (1970)

[22] Arnim Uhlmann: The “transition probability” in the state space of a *–algebra, Rep. Math. Phys. 9, 273–279 (1976)

[23] Jakob Yngvason: The Role of Type III Factors in Quantum Field Theory, Rep. Math. Phys. 11, (2004)