Chance Constrained Programming for Capacitated Open Vehicle Routing Problems

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Abstract. In open vehicle routing problems, the vehicles are not required to return to the depot after completing service. In this paper, the problem is extended by considering the reliability of the fulfillment of customer demands. Each customer has a demand and each customer is necessarily to be served by a single vehicle and no vehicle may serve a set of customers whose total demand exceeds the vehicle capacity. Each vehicle route must start at the depot and after serving the last customer it is not necessary for the vehicle to return to the depot. The objective is to determine the set of vehicle routes that minimizes the total costs, with chance constrained to guarantee that customers’ demand is satisfied. We solve the stochastic model using sample average approximation approach.

1. Introducing
In the classic version of Vehicle Routing Problems (VRPs), vehicles must be returned to the depot after service has been completed. (see for example, [1]). However, vehicles do not need to do so in open VRPs. The vehicle routes are therefore not closed but open paths, beginning at the depot and ending with one of the customers.

At first glance, having open routes rather than closed routes seems to be a minor change. In fact, if travel expenses are asymmetrical, there is essentially no difference between the open and closed variants: to convert the open version into the closed version, it is sufficient to set the cost to zero for any customer to travel to the depot. If travel costs are symmetrical, moreover, things are subtler. In fact, we show in the next segment that, surprisingly, the open version turns out to be more particular than the closed version, in the sense that any closed VRP for n customers can be transformed into an open VRP for n customers, but there is no reverse path.

In addition, there are several practical applications in which VRPs are open normally. For example, if a company does not own a fleet of vehicles and all its deliveries from a central depot are carried out by hired vehicles that are not required to return to the depot. In such cases, the distribution costs may be proportional to the distance travelled during loading. A practical case study of this type is portrayed in [2] and [3]. The same model can also be used for pick-ups, where vehicles start empty at any customer and have to meet the demands of each customer on their way and deliver them to the warehouse.

Open VRPs are easily seen as highly NP-hard by reducing the problem of the Hamilton path. Therefore, research on open VRPs has concentrated so far on developing effective heuristics to solve them. For the version with capacity constraints only, [17] addressed a two-phase heuristic with minimum spanning trees, [2] a population-based heuristic and [3] a threshold acceptance heuristic. For
a more general version featuring both capacity and distance constraints, [4] and [5] define tabu search heuristics, [6] record travel heuristics and [7] adaptive neighborhood heuristics. Heuristics was also designed for open VRPs with other types of constraints; see for example [8] and [9]. A feasible neighborhood search approach was proposed by Azis and Mawengkang (2017) to solve VRP applied to catering service problem.

The classic Vehicle Routing Problem (VRP) determines the optimal set of routes used by a fleet of vehicles to assist a certain number of customers in a predefined chart; the objective is to minimize total travel costs (proportional to travel times or distances) and operating costs (proportional to the number of vehicles used). Stochastic VRP (SVRP) occurs when certain VRP parameters are random (e.g. demand and journey time). For example, the uncertainty turns out in customer presence (see for example, [11] and [12]), uncertainty in customer’s demand (see, [13] and [18]. Interesting literature review on VRP with uncertainty can be found, for example, in [14] and [15].

In this paper we consider how to guarantee that customers’ demand is satisfied. In order to be able to grasp this situation in model we use chance constrained stochastic program. The result model is then solved using sample average approximation due to [16].

2. Methodology

2.1. Problem Description

In this paper we present the Capacitated Open Vehicle Routing Problem (COVRP), which is defined as follows. A complete undirected graph \( G = (V, E) \) is given, with \( V = \{0, ..., n\} \). Vertex 0 represents the depot, the other vertices represent customers. The cost of travel from vertex \( i \) to vertex \( j \) is denoted by \( c_{ij} \), and we assume costs are symmetric, so \( c_{ij} = c_{ji} \). A fleet of \( K \) identical vehicles, each of capacity \( Q > 0 \), is given. Each customer \( i \) has a demand \( q_i \), with \( 0 < q_i \leq Q \). Each customer must be serviced by a single vehicle and no vehicle may serve a set of customers whose total demand exceeds its capacity. Each vehicle route must start at the depot and end at the last customer it serves. The objective is to define the set of vehicle routes that minimizes the total costs, with a guarantee that the demand of each customer is satisfied.

Notations used are defined as follows.

Set:
- \( T \) The set of workdays in the planning horizon,
- \( K \) The set of vehicles,
- \( N \) The set of customers,
- \( N_0 \) The set of customers and depot \( N_0 = \{0, n + 1\} \cup N \),
- \( K_i \) The set of preferable vehicles for customer \( i \) \( \in N \),

Parameters
- \( Q_k \) The capacity of vehicle \( k \) \( \in K \),
- \( C_{ij} \) The travel cost from node \( i \in N_0 \) to node \( j \in N_0 \),
- \( [a_i, b_i] \) The earliest and the latest visit time at node \( i \in N_0 \),
- \( d'_t \) The service time of node \( i \in N_0 \) on day \( t \in T_i \),
- \( \beta_i \) Delivery quantity for customer I on day \( t \in T_i \),
- \( d'_t \) The service time of node \( i \in N_0 \) on day \( t \in T_i \),
- \( q'_i \) The amount of demand needed at node \( i \in N_0 \) on day \( t \in T_i \).

Variables.

Binary:
- \( X_{i,j,k} \) Flow variables, equal to 1 if \((i, j)\) is traversed by vehicle \( k \) and 0 otherwise.

Non-negative:
Arrival time for vehicle type $k \in K$ at customer $i \in V_c$

Duration of service of vehicle type $m \in K_m$ at customer $i \in V_c$

2.2. Mathematical Model

The objective function is to minimize the travelling cost

$$\text{Minimize } \sum_{i \in N_0} \sum_{j \in N} \sum_{k \in K} x_{ijk}$$

Constraints

Constraints (2)-(6) characterize the vehicle flows on the path and enforce the route feasibility.

$$\sum_{i \in D} \sum_{k \in K} X_{0, i, k} \leq n$$

$$\sum_{j \in D} X_{0, j, k} = 1 \quad (\forall k \in K)$$

$$\sum_{j \in N_0} \sum_{k \in K} X_{i, j, k} = 1 \quad (\forall i \in N)$$

$$\sum_{j \in N_0} \sum_{k \in K} X_{j, i, k} = 1 \quad (\forall i \in N)$$

$$\sum_{j \in N_0} X_{i, j, k} = \sum_{j \in N_0} X_{j, i, k} \quad (\forall i \in N, k \in K)$$

Constraint (2) states that the number of vehicles to service must not exceed the available number of vehicle ready at the center node at the beginning of the planning horizon. The number of vehicles to service is stated by the total number of vehicles flowing from the depot. Constraint (3) represents each vehicle flow from and back to the depot only once. Constraints (4) and (5) state that each demand node must be visited only once. Constraint (6) requires that all vehicles who flow into a demand point must flow out of it.

$$\sum_{j \in N} x'_{0, jk} \leq 1, \quad \forall k \in K, \forall t \in T$$

Constraint (7) states that only one vehicle is to be used once a day.

$$\sum_{k \in K} \sum_{i \in N} x'_{0, jk} = 0, \quad \forall t \in T$$

As this is an open VRP, Constraint (8) is to ensure that a vehicle is not allowed to return to the depot.

$$\sum_{(i, j) \in N} x_{ijk} \leq ND - 1, \quad \forall k \in K$$

Constraint (9) is about the sub-tour elimination process.

$$\sum_{i \in N_0} \sum_{k \in K} x'_{ijk} \leq Q_k, \quad \forall k \in K, \forall t \in T$$

Constraint (10) is imposed for the allowable capacity of each vehicle.

$$\sum_{k \in K} \sum_{i \in N_0} x'_{ijk} = 0, \quad \forall t \in T$$

Constraint (11) is to avoid loops.

$$l'_{ik} \leq a_i \sum_{j \in N} x'_{ijk}, \quad \forall i \in N_0, \forall k \in K, \forall t \in T$$

$$a_i \sum_{j \in N} x'_{ijk} \leq l'_{ik} + u'_{ik} \leq b_i \sum_{j \in N} x'_{ijk}, \quad \forall i \in N_0, \forall k \in K, \forall t \in T$$
Constraints (12) and (13) present the time window imposed for each customer’s delivery.

\[ x_{ijk} \in \{0,1\}, \quad \forall i, j \in N_0, \forall k \in K \]  

\[ l_{ik}^t, u_{ik}^t \geq 0, \quad \forall i \in N_0, \forall k \in K, \forall t \in T \]  

Constraints (14) to (15) state the nature of each variable.

In order to guarantee that the quantity of demand for each customer is satisfied we impose a chance constraint to replace Constraint (10).

\[ P_r \{ \beta^{ik}_k \mid \sum_{j=1}^{N} \beta^{ik}_j \sum_{j=1}^{N} x_{ijk}^{j} \leq Q_k \} \geq 1-\alpha_N, \quad \forall k \in K, \forall t \in T \]  

Now the problem has become a chance constrained integer program.

### 3. Results and Discussion

#### 3.1. Chance-Constrained Programming

For simplification, assume that the constraint function \( G : \mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R} \) in (22) is scalar valued. Therefore, there are some constraints \( G_i(x, \xi) \leq 0, i = 1, \ldots, m \), can be equivalently exchanged by one constraint \( G(x, \xi) := \max_{1 \leq i \leq m} G_i(x, \xi) \leq 0 \).

Eq. (17) can then be rewritten as

\[ \min_{x \in X} f(x) \text{ subject to } q(x) \leq \varepsilon \]  

where \( q(x) := \Pr\{G(x, \xi) > 0\} \)

Now let \( \xi^1, \ldots, \xi^N \) be an independent identically distributed (iid) sample of \( N \) realizations of random vector \( \xi \). Given \( x \in X \) let

\[ \hat{q}_N(x) := N^{-1} \sum_{j=1}^{N} 1_{(0, \infty)}(G(x, \xi^j)), \]

where \( 1_{(0, \infty)} : \mathbb{R} \rightarrow \mathbb{R} \) is the indicator function of \( (0, \infty) \). That is, \( \hat{q}_N(x) \) is equal to the proportion of realizations with \( G(x, \xi^j) > 0 \) in the sample. For some given \( \gamma \in (0, 1) \) consider the following optimization problem associated with a sample \( \xi^1, \ldots, \xi^N \),

\[ \min_{x \in X} f(x) \text{ subject to } \hat{q}_N(x) \leq \gamma \]  

Respectively, Eq. (18) and (19) are called true and sampled average approximation (SAA) problems at the respective risk levels \( \varepsilon \) and \( \gamma \).

#### 3.3. Solving Sample Approximations

Now, we create how to solve these problems.

If \( \gamma = 0 \) then the SAA problem can be expressed as

\[ \min_{x \in X} f(x) \text{ subject to } G(x, \xi^j) \leq 0, \quad j = 1, \ldots, N \]  

When the functions \( f(x) \) and \( G(x, \xi^j) \) for \( j = 1, \ldots, N \) are convex (linear) and the set \( X \) is convex (polyhedral) then (26) is a convex (linear) program.

The SAA problem (18) can be expressed as the following mixed-integer problem (MIP)
\[ \begin{align*}
\min & \quad f(x) \\
\text{subject to} & \quad G(x, \xi_j) \leq M_j z_j, \quad j = 1, \ldots, N \\
& \quad \sum_{j=1}^{N} z_j \leq \gamma N \\
& \quad z_j \in \{0, 1\} \\
& \quad x \in X
\end{align*} \]  

where \( z_j \) is a binary variable and \( M_j \) is a large positive number such that \( M_j \geq \max_{x \in X} G(x, \xi_j) \) for all \( j = 1, \ldots, N \). Note that if \( z_j = 0 \) then the constraint \( G(x, \xi_j) \leq 0 \) is implemented. On the other hand \( z_j = 1 \) does not pose any restriction on \( G(x, \xi_j) \). The cardinality constraint \( \sum_{j=1}^{N} z_j \leq \gamma N \) requires that at least \( N \) of the \( N \) constraints \( G(x, \xi_j) \leq 0 \) for \( j = 1, \ldots, N \) are enforced.

3.4. The Algorithm

Let \( x = [x] + f, \quad 0 \leq f \leq 1 \) be the (continuous) solution of the relaxed problem, \([x]\) is the integer component of non-integer variable \( x \) and \( f \) is the fractional component.

Step 1. Get row \( i^* \) the smallest integer infeasibility, such that
\[ \delta_i = \min \{ f_i, 1 - f_i \} \]

Step 2. Calculate
\[ v_i^T = \alpha_i^T B^{-1} \]
This is a pricing operation.

Step 3. Calculate \( \sigma_j = v_i^T a_j \)

Step 4. Calculate
\[ \alpha_j^* = B^{-1} \alpha_j \]
i.e. solve \( B \alpha_j^* = \alpha_j \) for \( \alpha_j^* \).

Step 5. Ratio test; there would be three possibilities for the basic variables in order to stay feasible due to the releasing of nonbasic \( j^* \) from its bounds.
Repeat from step 1.

4. Conclusion

The Stochastic VRP (SVRP) occurs when certain VRP parameters are random (e.g. demand and travel time). In this paper we propose the Capacitated Open Vehicle Routing Problem (COVRP), where the demand is uncertain. The problem model turns out to be a hassle-based stochastic program. We use the Sample Average Approximation to convert the model into a mixed integer programming model. Then we solve the integer model using a strategy to release non-basic variables from their limits in combination with the "active constraint".

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