Palindromic k-Factorization in Pure Linear Time

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Abstract

Given a string $s$ of length $n$ over a general alphabet and an integer $k$, the problem is to decide whether $s$ is a concatenation of $k$ nonempty palindromes. Two previously known solutions for this problem work in time $O(kn)$ and $O(n \log n)$ respectively. Here we settle the complexity of this problem in the word-RAM model, presenting an $O(n)$-time online deciding algorithm. The algorithm simultaneously finds the minimum odd number of factors and the minimum even number of factors in a factorization of a string into nonempty palindromes. We also demonstrate how to get an explicit factorization of $s$ into $k$ palindromes with an $O(n)$-time offline postprocessing.

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1 Introduction

Algorithmic and combinatorial problems involving palindromes attracted the attention of researchers since the first days of stringology. Recall that a string $s = a_0a_1 \cdots a_{n-1}$ is a palindrome if it is equal to its reversal $\bar{s} = a_{n-1} \cdots a_1a_0$; palindromic (k-)factorization is a representation of a string as a concatenation of $(k)$ nonempty palindromes. There is a bunch of results on palindromic factorization. One of them is a hardness result: deciding the existence of a factorization into distinct palindromes is NP-complete [2]; for all other natural types of palindromic factorization, sooner or later a linear-time algorithm in the word-RAM model was designed. In 1977, Knuth, Morris, and Pratt [10] presented such an algorithm deciding whether a string has a factorization into even-length palindromes. Soon, Galil and Seiferas [8] did the same for factorization into palindromes of length $>1$, and also for 2-, 3-, and 4-factorization. The existence of $k$-factorization was shown to be decidable in $O(kn)$ time [11]. Most of the recent results were related to palindromic length of a string, which is the minimum number of factors in its palindromic factorization. There are several combinatorics papers, see e.g. [7, 6, 14], studying the conjecture that every aperiodic infinite word has finite factors of arbitrarily big palindromic length. On algorithmic side, the palindromic length of a string of length $n$ was shown to be computable in $O(n \log n)$ time [3, 9, 13]. In [3], the optimal $O(n)$ bound was reached, using bit compression and range operations.

A linear-time algorithm for palindromic length gave a hope for better results on palindromic k-factorization, because finding the latter is equivalent to computing the minimum even and minimum odd number of factors in a palindromic factorization. An $O(n \log n)$ algorithm for palindromic length can be easily transformed into an $O(n \log n)$ algorithm for even/odd palindromic length [13]. However, the properties of palindromic length used in linear-time algorithm does not hold for even/odd palindromic length. In this paper we show how to overcome the technical difficulties and present the following optimal result.
Theorem 1. There exists an online algorithm deciding, in $O(n)$ time independent of $k$, whether a length-$n$ input string over a general alphabet admits a palindromic $k$-factorization.

In addition, we show how to find a palindromic $k$-factorization explicitly within the linear time (but offline).

The paper is organized as follows: after preliminaries on strings and palindromes, we give an $O(n \log n)$-algorithm for computing the even/odd palindromic length of a string in Section 2. The main section is Section 3 where a linear-time algorithm is described. We conclude with an algorithm finding an explicit $k$-factorization (Section 4).

1.1 Preliminaries

For any $i,j$, $[i..j]$ denotes the range $\{ k \in \mathbb{Z} : i \leq k \leq j \}$; we abbreviate $[i..i]$ as $[i]$. We use ranges, in particular, to index strings and arrays. For strings we write $s = s[0..n-1]$, where $n = |s|$ is the length of $s$. The empty string is denoted by $\varepsilon$. A string $u$ is a substring of $s$ if $u = s[i..j]$ for some $i, j$ ($j < i$ means $u = \varepsilon$). Such pair $(i,j)$ is not necessarily unique; $i$ specifies an occurrence of $u$ at position $i$. A substring $s[0..j]$ (resp., $s[i..n-1]$) is a prefix (resp. suffix) of $s$. An integer $p \in [1..n]$ is a period of $s$ if $s[0..n-p+1] = s[p..n-1]$. We write $s^k$ for the concatenation of $k$ copies of $s$ (thus $s^k$ has period $|s|$). We write $\overline{s}$ for a string or array obtained from $s$ by reversing the order of elements (thus the equality $s = \overline{s}$ defines a palindrome). A substring (resp. suffix, prefix) that is a palindrome is called a subpalindrome (resp. suffix-palindrome, prefix-palindrome). If $s[i..j]$ is a subpalindrome of $s$, the numbers $(j + i)/2$ and $(j - i + 1)/2$ are respectively the center and the radius of $s[i..j]$.

A representation $s = w_1 \cdots w_k$, where $w_1, \ldots, w_k$ are palindromes, is a palindromic $(k)$-factorization of $s$. Efficient algorithms for palindromic factorization [3, 4, 9, 11, 13] used the following (or very similar) combinatorial properties.

Lemma 2 ([11 Lemmas 2, 3]). For any palindrome $w$ and any $p \in [1..|w|]$, the following conditions are equivalent: (1) $p$ is a period of $w$, (2) there are palindromes $u, v$ such that $|uv| = p$ and $w = (uv)^k u$ for some $k \geq 1$, (3) $w[p..|w|-1]$ is a palindrome.

Lemma 3 ([11 Lemma 7]). Suppose that $w = (uv)^k u$, where $k \geq 1$, $u$ and $v$ are palindromes, and $|uv|$ is the minimal period of $w$; then, the center of any subpalindrome $x$ of $w$ such that $|x| \geq |uv| - 1$ coincides with the center of some $u$ or $v$ from the decomposition.

The minimum $k$ such that a palindromic $k$-factorization of a string $s$ exists is the palindromic length of $s$, denoted by $p_l(s)$. We define the even palindromic length $p^e_l(s)$ and odd palindromic length $p^o_l(s)$ as the minimum even (resp., odd) $k$ among all such factorizations of $s$. If no such $k$ exists, the corresponding length is set to $\infty$. For example, $p_l(abcba) = p^e_l(abcba) = 1$, $p^o_l(abcba) = \infty$; $p_l(acaba) = p^e_l(acaba) = 2$, $p^o_l(acaba) = 5$.

A palindromic factorization $s = w_1 \cdots w_k$ with $k < |s| - 2$ can be easily transformed into a palindromic $(k+2)$-factorization: either $|w_i| \geq 3$ for some $i$, so $w_i = uaa$ for some letter $a$ and palindrome $u$, or $|w_i| = |w_j| = 2$ for some $i, j$, so $w_i$ and $w_j$ can be replaced by four 1-letter factors. This leads to the following crucial observation (see [13 Sect. 4.1]).

Lemma 4. (1) A string $s$ has a palindromic $k$-factorization iff $p^{k \mod 2}_l(s) \leq k$.

(2) A palindromic $k$-factorization of $s$ can be obtained in $O(n)$ time from its palindromic $p^{k \mod 2}_l(s)$-factorization.

Remark 5. Due to Lemma 4 throughout the paper we study the palindromic $k$-factorization problem for a string $s$ as the problem of finding $p^e_l(s)$ and $p^o_l(s)$.
Recall that the method for computing $pl(s)$ is dynamic programming: we compute the array $pl[0..n−1]$ such that $pl[i] = pl(s[i..])$ using an artificial initial value $pl[−1] = 0$ and the rule $pl[k] = 1 + \min_{i ∈ S_k} pl[i−1]$, where $S_k$ is the set of positions of all suffix-palindromes of $s[0..k]$.

This rule can be easily adapted to compute $pl^0(s)$ and $pl^1(s)$: for $j ∈ \{0, 1\}$ we define the array $pl^j[0..n−1]$ and employ the scheme

$$pl^0[−1] = 0, \quad pl^1[−1] = ∞, \quad pl^j[k] = 1 + \min_{i ∈ S_k} pl^{j−1}[i−1], \quad \text{where} \quad S_k = \{i : s[i..] = s[i..k]\} \quad (1)$$

Henceforth, $s$ denotes the input string of length $n$ and $PL[i] = (pl^0[i], pl^1[i])$. All considered algorithms work in the unit-cost word-RAM model with $Θ(\log n)$-bit machine words and standard operations like in the C programming language. All algorithms except Algorithm 2 are online and work in iterations: $i$th iteration begins with reading $s[i]$ and ends before reading $s[i+1]$.

2 An $O(n \log n)$ Algorithm

2.1 Palindromic Iterator

We process the input string using a data structure called (palindromic) iterator [11], which stores a string $s$ and answers the following queries in $O(1)$ time:
- $rad(x) / len(x)$ returns the radius / length of the longest palindrome in $s$ with the center $x$;
- $maxPal$ returns the center of the longest suffix-palindrome of $s$;
- $nextPal(x)$ returns the center of the longest proper suffix-palindrome of the suffix-palindrome of $s$ with the center $x$.

The iterator stores an array of radii for all possible centers of palindromes and a list of centers of suffix-palindromes in increasing order. The update query $add(a)$ appends letter $a$ to $s$. Performing this query, the iterator emulates an iteration of Manacher’s algorithm [12] and updates the list of suffix-palindromes, all within $O(1 + maxPal_{new} − maxPal_{old})$ time, which is $O(n)$ for $n$ updates to the originally empty structure. Below we assume that $add(a)$ returns the list of deleted centers in the form $dead(x) = (x, answers to all queries about x)$.

With the iterator, dynamic program [1] can be implemented to work in time proportional to the number of subpalindromes in $s$ ($Ω(n^2)$ in the worst case); one iteration is as follows:
1. $add(s[n])$; $pl^0[n], pl^1[n] ← +∞$
2. for $(x ← maxPal; x ≠ n + \frac{1}{2}; x ← nextPal(x))$ do
3. for $(j ∈ \{0, 1\})$ do $pl^j[n] ← \min\{pl^j[n], 1 + pl^{j−1}[n − len(x)]\}$

2.2 Series of Palindromes

Let $u_1, \ldots, u_k$ be all non-empty suffix-palindromes of a string $s$ in the order of decreasing length. For any $i < j$, since $u_j$ is a suffix of $u_i$, any period of $u_i$ is a period of $u_j$. Hence the sequence of minimal periods of $u_1, \ldots, u_k$ is non-increasing. The groups of suffix-palindromes with the same minimal period are series of palindromes (of $s$):

$$u_{i_1}, \ldots, u_{i_1+p_1}, \ldots, u_{i_2}, \ldots, u_{i_2+p_2}, \ldots, u_{i_k}, \ldots, u_{i_k+p_k}.$$
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Lemma 6 ([4, 8, 11]). For any string $s[0..n]$, if $r$ is the length of a tail of a series, then the head of the next series has length less than $2r/3$. In particular, if $p_1, \ldots, p_t$ are periods of all series of $s[0..n]$, then $t = O(\log n)$ and $p_1 + \cdots + p_t = O(n)$.

Note that strings with $\Omega(\log n)$ series for $\Omega(n)$ prefixes do exist [4].

Below we present an $O(n \log n)$ algorithm finding even and odd palindromic lengths of a string. Due to symmetry, we always write $pl^j$ assuming $j \in \{0, 1\}$. The algorithm, which is a straightforward adaptation of the algorithm of [3] for palindromic length, serves as a base for the linear-time solution, so we provide necessary details. We rewrite rule [1] as

$$pl^j[n] = 1 + \min_{U \in \{0, 1\}} \min_{w \in U} pl^{1-j}[n-|u|],$$

where $U$ runs through the series of $s$, and compute the internal minimum in $O(1)$ time using precalculations based on the structure of series, described in the following lemma.

Lemma 7 ([3]). For a string $s$ and $p \geq 1$, let $U$ be a $p$-series of palindromes, $k = \#U$. Then there exist unique palindromes $u, v$ with $|uv| = p, v \neq \varepsilon$ such that one of three conditions hold:

1) $U = \{(uv)^{k+1}u, (uv)^ku, \ldots, (uv)^2u\}$ and the next series begins with $uvu$,
2) $U = \{(uv)^ku, (uv)^{k-1}u, \ldots, wu\}$ and the next series begins with $w$,
3) $U = \{v^k, v^{k-1}, \ldots, v\}, p = 1, |v| = 1, u = \varepsilon$, and $U$ is the last series for $s$.

Proof. The lemma appeared in [3] without a proof, so we provide it here. If $p = 1, 3)$ trivially holds, so let $p > 1$. By Lemma 2, the head $w$ of $U$ has the form $(uw)^q u$ for some $q \geq 1$ and some palindromes $u$ and $v$ such that $|uv| = p$ and the longest proper suffix-palindrome of $w$ is $(uv)^{q-1}u$. Applying Lemma 2 iteratively, we finally get $U = \{(uw)^q u, \ldots, (uw)^{q-k+1} u\}$. Note that the only suffix-palindrome of $(uw)^q u$ with length $\geq |uw|$ is $uwu$ by Lemma 3. Hence by Lemma 2 the minimal period of $(uw)^q u$ equals $p$. So either $(uw)^q u \in U$ and thus $U$ satisfies 1) or 2), or $U = \{uw\}$, also satisfying 2).

Our algorithm maintains an array $\text{left}[1..n]$; for $p \in [1..n]$, if there is a $p$-series, then $\text{left}[p]$ is such that $s[\text{left}[p]+1..n]$ is the longest suffix (which is not necessarily a palindrome) of $s[0..n]$ with period $p$; otherwise, $\text{left}[p]$ is undefined. For example, if $s[0..n] = \ldots aabaababa$ and $p = 3$, then the mentioned suffix is $s[n-6..n] = aabaab$ and $\text{left}[3] = n - 7$. If we extend $s$ by $b$, this will break period 3 and make $\text{left}[3]$ undefined. If we then append $ba$, the resulting string $s \cdot bba = \ldots aabaabaababa$ will have a suffix-palindrome of period 3 again, and $\text{left}[3]$ will get a new value $n - 2$.

Remark 8. We do not explicitly make $\text{left}[p]$ undefined if it was defined earlier. We compute it at the iterations where a $p$-series is present. If the new value differs from the old one, we conclude that since we saw the previous $p$-series, period $p$ broke.

Lemma 9. Given a number $p$ and the length of the head of the $p$-series of $s$, $\text{left}[p]$ can be computed in $O(1)$ time.

Proof. Let $w = (uv)^k u$ be the head of the $p$-series (see Lemma 2), $x$ be the center of $w$, and $z(uv)^k u$ be the longest suffix of $s[0..n]$ with period $p$ (for $s$ in the above example, $u = \varepsilon$, $v = aba$, $x = n - 5/2$, $z = a$). Then $z$ is a proper suffix of $uv$. Hence the center $x_1$ of the prefix-palindrome $u$ of $w$ satisfies $\text{len}(x_1) = 2z + |u|$ (in the example, $x_1 = n - 11/2$, $\text{len}(x_1) = |a| u$).

Note that $|z| = \text{len}(x) \mod p$ and $x_1 = 2x - \text{cnt}(|u|)$, where $\text{cnt}(d) = n - (d - 1)/2$ is the center of the length $d$ suffix-palindrome of $s[0..n]$. Thus, $|z|$ and $\text{left}[p] = n - \text{len}(x) - |z|$ are computed in $O(1)$ time.
Computations of internal minima in \( \mathcal{P} \) are based on the following idea. Let \( U \) be a \( p \)-series for \( s[0..n] \) with \( k > 1 \) palindromes (w.l.o.g., \( U = \{(uv)^k u, \ldots, uuv\} \)). Updating \( \text{pl}^p[n] \) using \( U \), we compute \( m = \min \{ \text{pl}^{1−j}[n−kp−|u|], \ldots, \text{pl}^{1−j}[n−p−|u|] \} \). Now note that \( s[0..n] \) ends with \( (uv)^k u \) but not with \( (uv)^{k+1} u \); otherwise, the latter string would belong to \( U \). Then \( s[0..n−p] \) ends with \( (uv)^{k−1} u \) but not with \( (uv)^k u \) and thus has the \( p \)-series \( U' = \{(uv)^{k−1} u, \ldots, uuv\} \).

Thus, at that iteration we computed \( m' = \min \{ \text{pl}^{1−j}[n−kp−|u|], \ldots, \text{pl}^{1−j}[n−2p−|u|] \} \) for updating \( \text{pl}^p[n−p] \). If we saved \( m' \) into an auxiliary array, then \( m = \min \{ m', \text{pl}^{1−j}[n−p−|u|] \} \) is computed in constant time, as required. We store all precomputed minima in two arrays \( \text{pre}^j[1..n] \), where \( j ∈ \{0, 1\} \) and each \( \text{pre}^j[p] \) is, in turn, an array \( \text{pre}^j[p][0..p−1] \) such that

\[
\text{pre}^j[p][i] = \min \{ \text{pl}^j[t] : t ≥ \text{left}[p] \text{ and } (t − \text{left}[p]) \text{ mod } p = i \}
\]

and \( s[t+1..n] \) has a prefix-palindrome of minimal period \( p \). (3)

(See Fig. 1). We also denote \( \text{PRE}[p][i] = (\text{pre}^0[p][i], \text{pre}^1[p][i]) \). If the minimum in (3) is taken over the empty set (i.e., the related prefix of \( s \) has no \( p \)-series), then \( \text{pre}^j[p][i] \) is undefined. If \( \{u_1, \ldots, u_k\} \) is a \( p \)-series for \( s[0..n] \), then (3) implies \( \text{pre}^j[p][n−u_1−\text{left}[p]] = \min \{ \text{pl}^j[n−|u_1|] \} \), which is exactly the value \( m \) mentioned above.

![Figure 1 Precomputed values. Palindromes with minimal period \( p = 4 \) are shown. By (3), \( \text{pre}^0[p][0..2] = [\text{pl}^0[n−10]; \min \{ \text{pl}^0[n−9], \text{pl}^1[n−5]\}; \text{pl}^2[n−8]; \text{pre}^0[p][3] \) is undefined.](image)

Overall, given a new letter \( s[n] \), we compute \( \text{PL}[n] \) as follows:

**Algorithm 1**: \( O(\log n) \) algorithm, \( n \)th iteration

1: \( \text{add}(s[n]); \text{PL}[n] ← (+\infty, +\infty); \)
2: \( \text{for } (x ← \max \text{Pal}; x ≠ n + \frac{1}{2}); x ← \text{nextPal}(\text{cntr}(d)) \) do \( \triangleright \) goes to next head each time
3: \( p ← \text{len}(x) − \text{len}(\text{nextPal}(x)); \triangleright \) min. period of the head centered at \( x \)
4: \( d ← p + (\text{len}(x) \text{ mod } p); \triangleright \) length of candidate tail in \( p \)-series
5: \( \text{if } \text{len}(\text{cntr}(d)) − \text{len}(\text{nextPal}(\text{cntr}(d))) ≠ p \text{ then } d ← d + p; \triangleright \) corrected length of tail
6: \( y ← \text{left}[p]; \text{compute left}[p]; i ← n − \text{len}(x) − \text{left}[p]; \triangleright \) \( O(1) \) time by Lemma 9
7: \( \text{if } \text{left}[p] > y \text{ then clear PRE}[p]; \triangleright \) delete obsolete values
8: \( \text{if } \text{len}(x) = d \text{ then PRE}[p][i] ← \text{PL}[n−d]; \)
9: \( \text{PRE}[p][i] ← (\min \{ \text{pre}^0[p][i], \text{pl}^0[n−d]\}, \min \{ \text{pre}^1[p][i], \text{pl}^1[n−d]\}); \)
10: \( \text{PL}[n] ← (\min \{ \text{pl}^0[n], 1 + \text{pre}^0[p][i]\}, \min \{ \text{pl}^1[n], 1 + \text{pre}^1[p][i]\}); \)

**Proposition 10.** Algorithm 1 correctly computes \( \text{PL}[0..n] \) in \( O(\log n) \) time.

**Proof.** All queries \( \text{add} \) require \( O(n) \) time in total, so we ignore them. Let \( x \) be the center of a processed suffix-palindrome \( w = (uv)^k u \) with period \( p \). By Lemma 2 \( p = |uv| = \text{len}(x) − \text{len}(\text{nextPal}(x)) \).

Let \( x' = \text{cntr}(p + (\text{len}(x) \text{ mod } p)) \), so that \( x' \) is the center of the suffix-palindrome \( uvu \). By Lemma 7 \( uvu \) is, depending on its period \( \text{len}(x') − \text{len}(\text{nextPal}(x')) \), either the tail of the \( p \)-series or the head of the next series. Thus in lines 3–5 Algorithm 1 computes, using \( O(1) \) queries to the iterator, the period and the length of the tail of the series containing \( w \). This means, in particular, that the \( \text{for} \) loop iterates over all heads of series in \( s \), for the total of \( O(\log n) \) runs.
Next note that when a symbol added to s breaks period p, all values in $\text{pre}^i[p]$ become obsolete and should be deleted. Algorithm 1 does this in lines 6–7: if the “new” and “old” values of $\text{left}[p]$ differ, then period $p$ broke since the last update of $\text{pre}^i[p]$.

Let $\{v_1, \ldots, v_k\}$ be a p-series, $i = n - |u_1| - \text{left}[p]$. If $k = 1$, there was no p-series $p$ iterations ago, so the undefined value $\text{pre}^i[p][\cdot]$ is set to $p!\text{[n-|u_k|]}$ in line 8. Otherwise one has $\text{pre}^i[p][\cdot] = \min(p!\text{[n-|u_1|]}, \ldots, p!\text{[n-|u_k-1|]})$ by (3), and this value is updated using $p!\text{[n-|u_k|]}$ in line 9 so $\text{pre}^i$ is correctly maintained. Finally, in line 10 the rule (2) is implemented. So the algorithm is correct and each run of the for loop takes $O(1)$ time. The result now follows.

If $s[0..n]$ has a suffix-palindrome $w$ centered at $x$, we say that $w$ survives the next iteration if $x$ is the center of a suffix-palindrome of $s[0..n+1]$. Otherwise, $w$ (or $x$) dies at that iteration. We refer to the number of iterations $x$ survives as its time-to-live, denoted by $\text{ttl}(x)$. The same notions apply to any p-series of $s[0..n]$ and to its period $p$; we write $\text{ttl}(p)$ for the time-to-live of $p$. If $p$ dies, then p-series also dies, but not vice versa. More precisely, while $p$ is live, p-series evolves as follows (cf. Fig. 1).

A p-series appears at $n$th iteration as a single palindrome $\text{head}(p) = \text{tail}(p) = uvu$ centered at $x$. Then at some iteration $n+i$, $0 < i < p$, $x$ reaches $\text{left}[p]$ on the left and dies together with the series. At some iteration $n+i+j$, $0 \leq j < p$, a palindrome centered at $x + \frac{j}{p}$ “resurrects” the series (if $j = 0$, we can say that the series has not died). Finally, at the $(n+p)$th iteration a palindrome centered at $x+p$ joins the series; every subsequent $p$ iterations follow the same pattern (but the death of the head no longer means the death of the series).

In Fig. 4 $p = 4, i = 2, j = 1$.

**Lemma 11.** Suppose that $s[0..n]$ has p-series and q-series such that $p > q$ and $\text{ttl}(p) \geq p$. Then $\text{ttl}(q) < p$.

**Proof.** Let $\text{head}(p) = (uv)^q u$. If $\text{ttl}(q) \geq p$, the string $s[n-q+1..n+p]$ of length $p + q$ has periods $p$ and $q$. Hence it has period $d = \gcd(p, q)$ by the Fine-Wilf theorem [5]. Thus $uvu$ is a $(p/d)$-power of a shorter word; so $(uv)^q u$ has period $d < p$, contradicting the definition of p-series.

**Lemma 12.** Let a string $s[0..n]$ have series with periods $P = p_1 > p_2 > \cdots > p_l$ and let $t > 0$. Then $\sum_{i=1}^{l} \text{ttl}(p_i) t = O(P + t)$.

**Proof.** Divide the periods in two groups: those with $\text{ttl}(p) < p$ and the rest. In the first group, the sum of ttl’s is majorized by $\sum_{i=1}^{l} p_i$, which is $O(P)$ by Lemma 6. For the second group, $\text{ttl}(p_i)$ is smaller than the previous period from this group by Lemma 11. So we can take $t + \sum_{i=1}^{l} p_i = O(P + t)$ as the upper bound. The result now follows.

Lemma 12 has an immediate corollary used in the linear algorithm.

**Corollary 13.** Suppose that Algorithm 3 processed $s[0..n]$ and $p < n$. During the next $t$ iterations, the number of times the for loop of Algorithm 7 processes series which survived since $s[0..n]$ and have periods $\leq p$ is $O(p + t)$.

## 3 Linear Algorithm

### 3.1 Resources for Speed-Up

In some cases, dynamic programming over arrays can be sped up by a log $n$ factor by a technique called four Russians’ trick [1]. The idea is to store the DP array(s) in a compressed...
form requiring \( O(1) \) bits per element and update \((\log n)\)-size chunks of the compressed array using \( O(1) \) operations on machine words. The key operations used are table operations: the total number of valid inputs for such an operation is \( o(n) \) and the results for all valid inputs can be computed in advance in \( o(n) \) time and stored in an auxiliary table; the typical time/space restriction is \( O^\ast(n^\alpha) \), where \( \alpha < 1 \) and \( O^\ast \) suppresses polylog factors. To apply this technique to Algorithm \([1]\) we should meet several necessary conditions:

1) the array PL can be compressed to \( O(1) \) bits per element;
2) each array \( \text{PRE}[p] \) can be compressed to \( O(p + \log n) \) bits;
3) each update of arrays PL, \( \text{PRE}[p] \) (lines 8–10 operations of Algorithm \([1]\)) can be performed without decompression, simultaneously for \( \Omega(\log n) \) successive iterations with a constant number of operations over machine words, including table operations;
4) all intermediate states of arrays PL and \( \text{PRE}[p] \) during the course of the algorithm are valid inputs for the compression scheme.

Let us recall how it works for palindromic length (see \([3]\)). Condition 1 is provided by

\[ \text{Lemma 14 (}[13, \text{Lemma 4.11}]. \] If \( w \) is a string and \( a \) is a letter, then \( |\text{pl}(wa) − \text{pl}(w)| \leq 1. \]

Due to Lemma \([14]\) it is possible to store any subarray \( \text{pl}[i..j] \) as one number \( \text{pl}[i] \) followed by \((j − i)\) 2-bit codes for the differences \( \text{pl}[k] − \text{pl}[k−1]. \) The situation is more subtle with the arrays \( \text{pre}[p], \) but it was proved in \([3]\) that each of these arrays can be efficiently split into a constant number of chunks, where successive elements of the same chunk differ by at most one. For each chunk, the same encoding as for \( \text{pl} \) works; so condition 2 is also met.

The following observation shows the resource for range updates (condition 3). Let \( s = \cdots a(uv)^k u \) have \( p \)-series \( U \) with the head \( (uv)^k u \) (cf. Lemma \([7]\)), where \( |uv| = p, a \) is the last letter of \( v, \) and an element \( \text{pre}[p][i], \) for some \( i, \) was updated at the current iteration. Thus \( \text{pl}[n] \) was just updated using \( \text{pre}[p][i]. \) Now if the next letter is \( a, \) the string \( s' = sa \) retains the \( p \)-series, and each palindrome in it extends one letter left and right, retaining the same center as its counterpart in \( s. \) This means that during this iteration we update \( \text{pre}[p][i−1], \) and then \( \text{pl}[n+1] \) using \( \text{pre}[p][i−1]. \) If this situation repeats, say, \( t \) times, one can assign \( \text{pl}[n..n+t] \leftarrow \min\{\text{pl}[n..n+t], 1 + \text{pre}[p][i−t..i]\} \) in line 10 of Algorithm \([1]\)

Let \( t = \lceil \log \frac{n}{2} \rceil. \) Formally, a chunk is an array \( A = A[0..i], \) where \( i < t, \) of \((\log n)\)-bit numbers, in which any two consecutive numbers differ by at most one; it is stored as a \((\log n)\)-bit number followed by \( i \) 2-bit codes encoding the differences between consecutive values. If the length of a chunk is less than \( t, \) then the unused 2-bit code is added to the end.

Note that in both \( \tilde{A} \) and \( \min\{A, B\} \) consecutive elements differ by at most 1, so these chunks can be compressed. Therefore, condition 4 is also met in view of the following lemma.

\[ \text{Lemma 15. The following operations can be performed in } O(1) \text{ time using table operations:} \]

1) increment chunk \( A; \)
2) extract an element: given chunk \( A \) and number \( l < |A|, \) return the number \( A[l]; \)
3) extract a chunk: given chunks \( A, B \) with \( |B[0] − A[|A|−1]| \leq 1 \) and numbers \( l, d, \) return chunk \( AB[l..r], \) where \( r = \min\{l+d−1,|AB|−1\} \) and \( AB \) is the concatenation of \( A \) and \( B; \)
4) reverse a chunk: given chunk \( A, \) return \( \tilde{A}; \)
5) concatenate two chunks: given chunks \( A \) and \( B, \) return a chunk \( C = AB \) or, if \( |AB| > t, \) two chunks \( C, D \) such that \( CD = AB \) and \( |C| = t; \)
6) extend a chunk: given a chunk \( A \) and numbers \( l, r \) such that \( |A| + l + r \leq t, \) return a chunk \( C \) of length \( |A| + i + r \) such that \( C[l..l+|A|−1] = A, C[i+1] = C[i]−1 \) for each \( i < l, \) and \( C[i+1] = C[i]+1 \) for each \( i \geq l + |A|; \)
7) take minimum of two chunks: given chunks \( A, B, |A| = |B|, \) return the chunk \( C \) of length \( |A| \) such that \( C[i] = \min\{A[i], B[i]\} \) for each \( i. \)
There is a factorization with three cases possible.

2) Create a table containing, for any 2-bit codes, the sequence of 2-bit codes for suffix-palindrome at the beginning of the phase. Since in the beginning of a phase, an assumption is made that the next iteration will not change the previous phase and continues until one of three conditions is fulfilled:

- The table has less than size of chunks with 2-bit codes. (Or $l+d \leq |A|$ and $B$ is not involved at all.)
- Add $l$ to $A[0]$, append $l$ codes of -1 from the left and $r$ codes of 1 from the right to the codes in $A$.
- If $|A[0] - B'[0]| \geq 2t$, then the answer is the chunk with the smaller number. Otherwise, the difference is a $\lfloor 1 + \log t \rfloor$-bit number. One has $C[0] = \min\{A[0], B[0]\}$. Further, construct a table with $O(n^{1/2})$ entries which contains, for the given difference and two sequences of 2-bit codes, the sequence of 2-bit codes for $C$. Since the size of $C$ is $O(n^{3/4})$ bits, this is a table operation.

Overview of linear-time algorithm for palindromic length. The algorithm [3] is based on grouping consecutive iterations into phases. Each phase begins immediately after the end of the previous phase and continues until one of three conditions is fulfilled: $t = \lfloor \frac{|A[0]|}{s_0} \rfloor$ iterations passes, the input string ends, or the next iteration will break the longest suffix-palindrome. Under this assumption, $O(1)$ queries to the iterator is sufficient to compute the number $t_p = \min\{t, t_{tl}(p)\}$. Then the corresponding updates to $\text{pre}[p]$ and $\text{pl}$ are applied simultaneously for the next $t_p$ iterations, using $O(1)$ operations described in Lemma [15].

After processing all series, actual $t$ letters are added one by one; each time the iterator is updated, constant number of new series (with freshly appeared centers) are added, and the arrays $\text{pl}$ and $\text{pre}[p]$ are updated using these new series. When the processing of the letter $s[i]$ is finished, $\text{pl}[i]$ gets its true value. If an input symbol violates the assumption (i.e., it changes $\text{maxPal}$), the phase is aborted, unfinished updates are deleted, and a new phase is started from the current symbol. The of operations performed is $O(t)$ for a phase of $t$ iterations and $O(P + i)$ for a phase of $i < t$ iterations, where $P$ is the period of the longest suffix-palindrome at the beginning of the phase. Since $P \leq 2 \cdot (\text{maxPal}_{\text{new}} - \text{maxPal}_{\text{old}})$, and $\text{maxPal}$ monotonely increases from 0 to at most $n$ during the course of the algorithm, all phases take $O(n)$ time in total.

3.2 Even/Odd Palindromic Length

The following analog of Lemma [14] allows one to store the array $\text{PL}$ in compressed form.

Lemma 16. If $w$ is a string, $a, b$ are letters, $j \in \{0, 1\}$, then $\text{pl}^j(wab) \in \{\text{pl}^{j-1}(wa) + 1, \text{pl}^{j-1}(wa) - 1, \text{pl}^{j-1}(wa) + 1\}$.

Proof. Let $k = \text{pl}^j(wab)$. Consider all palindromic factorizations of the form $wab = w_1 \cdots w_k$. Three cases are possible.

1. There is a factorization with $w_k = b$. Then $k = \text{pl}^{j-1}(wa) + 1$.
2. There is a factorization with $w_k = bub$, $u \neq \varepsilon$, and no factorization with $w_k = b$. Then $wa$
has no \((k-1)\)-factorization, but has a \((k+1)\)-factorization \(w_1 \cdots w_{k-1} bw\); so \(k = pl^{1-j}(wa) - 1\).

3. \(w_k = ab\) in each factorization (so \(a = b\)). Then \(k = pl^{1-j}(w) + 1\).

As above, we set \(t = \lfloor \frac{\log n}{2} \rfloor\). The chunks are segments \(A = A[0..i] = (A^0[0..i], A^1[0..i])\) of PL or PRE\([p]\). We encode them using \(O(t) = O(\log n)\) bits. The difference with the case of palindromic length is that we store four explicit values: \(A^0[0], A^1[0], A^0[1], A^1[1]\). Subsequent elements are encoded by 2-bit codes associated with the cases of Lemma 16; the unused 2-bit code indicates the end of a shorter chunk. Below we demonstrate a problem with this encoding and show the solution.

Let us consider the graphs of two functions \(f_1(i) = pl^{(i-1) \mod 2} [i]\) and \(f_2(i) = pl^{i \mod 2} [i]\) (\(f_2\) is undefined for small values of \(i\) until two consecutive equal letters occur in \(s\)). At most points, \(f_1\) and \(f_2\) have the first difference \pm 1, but in some points, corresponding to case 3 from Lemma 16, the upper graph drops down to the lower one; see Fig. 2.

\[ s = b \ c \ c \ b \ a \ a \ a \]

\[ pl^1 = 1 \ 3 \ 3 \ 1 \ 5 \ 3 \ 3 \]

\[ pl^0 = 2 \ 2 \ 1 \ 1 \ 2 \ 2 \ 2 \]

\( \text{Figure 2} \) even/odd palindromic length visualized by \(f_1(i) = pl^{(i-1) \mod 2} [i]\) and \(f_2(i) = pl^{i \mod 2} [i]\).

The drops cause the following problem: taking minimum of two chunks can result in a chunk having no valid encoding (condition 4 is violated). For example, taking the minimum of two valid chunks \(\cdots 3 \ 7 \ 3 \cdots\) and \(\cdots 1 \ 7 \ 3 \cdots\) we get the chunk \(\cdots 1 \ 7 \ 3 \cdots\). The marked element does not satisfy any alternative from Lemma 16. To remedy this, we consider the following smoothing operation on the computed array \(PL[0..n-1]\): for each \(i \in [0..n-2]\), \(j \in \{0,1\}\) replace \(pl^j[i]\) by min\(\{pl^j[i], pl^{(j+1) \mod 2}[i+1], \ldots, pl^{(j+n-1-i) \mod 2}[n-1]+n-1-i\}\). For example, the graphs of \(f_1\) and \(f_2\) from Fig. 2 after smoothing will look as follows (empty circles represent redefined values):

The idea of smoothing is clarified by the following lemma.

\(\textbf{Lemma 17.}\) Smoothing of the array \(PL[0..n-1]\) does not affect the value \(PL[n]\) computed by rule (\(\Box\)).

\textbf{Proof.} Assume that we applied smoothing and then computed \(k = pl^{k \mod 2} [n]\) by rule (\(\Box\)) getting the minimum \(k-1\) as the element \(pl^{(k-1) \mod 2} [r]\). Since the minimum cannot increase after smoothing, we just need to check that \(s[0..n]\) indeed has a palindromic \(k\)-factorization. This is obvious if \(pl^{(k-1) \mod 2} [r]\) was not changed by smoothing. Otherwise,
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$k-1 = pl^{(k-1+i-r) \mod 2[i] + i - r}$ for some $i > r$. Since $s[r+1..n]$ is a palindrome (see (1)), we factor it as $wu$, where $u = s[r+1..i]$. So, $s[0..i] = s[0..i]vu$ indeed has a $k$-factorization: $s[0..i]$ can be factored into $k-1-i+r$ palindromes and $u$ into $i-r$ palindromes.

We extend smoothing to subarrays of the form $PL[l..r]$; for each $i \in [l..r-1]$, $j \in \{0, 1\}$ replace $pl^{j}[i]$ by $\min\{pl^{0}[i], pl^{j+1}(i+1) \mod 2[i+1] + 1, \ldots, pl^{j+r-i}(r) \mod 2[r] + r - i\}$. The resulting values are always between the values from the original $PL$ and those from smoothed $PL$, so Lemma 17 stays true for this sort of operations. The following lemma is crucial.

**Lemma 18.** Smoothing a chunk can be done in $O(1)$ time.

**Remark 19.** Let $(j, l) = \{1, 2\}$ and let $f_j$ have a drop at position $i$ (i.e., $f_j(i) = f_i(i-2) + 1 \leq f_j(i-1) - 3$). Then neither $f_j$ nor $f_1$ has a drop at position $i+1$. Indeed, $f_j(i+1) \leq f_j(i-2) + 3 < f_j(i-1)$ and $f_1(i-1) = f_1(i-2) + 2 = f_j(i) + 3$ (so either $f_i(i-1) \geq f_j(i+1)$ or $f_j(i+1) > f_j(i) + 3$).

**Proof of Lemma 18.** Assume that a chunk $A$ is given. We enumerate its positions from 1 to $t$ and write $\hat{A}$ for the result of its smoothing. The functions $f_1, f_2$ are defined as above; let $f_1, f_2$ be their counterparts after smoothing. If we know $f_j$ up to an additive constant (i.e., know some function $f_j + c$), then it is easy to obtain $\hat{f}_j + c$. Hence if we know both $f_1 + c$ and $f_2 + c$, we can reconstruct $\hat{A}$ up to the same additive constant $c$. Note that if $f_j$ has a drop at position 3, then $f_j(1)$ can be replaced by $f_j(2) + 1$ without affecting $\hat{f}_j$. So if we know the difference $\delta = f_j(2) - f_j(2)$ and 2-bit codes for the second and all subsequent elements of the chunk, we can compute $f_1$ and $f_2$ up to the constant $c = -f_2(2)$. Indeed, if $f_j$ has no drop at position 3, then we compute $f_j(3) + c$ from $\delta$ and the corresponding 2-bit code; if $f_j$ has a drop at 3, i.e., $f_j(3) = f_3 - j(1)$, then by Remark 19 there is no drop at 2, so we compute $f_{3-j}(1) + c$ and increment it to get $f_j(3) + c$. After computing $f_1(3), f_2(3)$, the rest is easily reconstructed from 2-bit codes.

Now we observe that if $|\delta| > 2t$, only two cases are possible, and they can be easily distinguished: either there are no drops, so $\hat{f}_j = f_j$ for $j = 1, 2$, or the leftmost drop is in $f_j$ at position $i$, and then $\hat{f}_j(i-r) = f_j(i) + r$ for all $r < i$. So in this case only the sign of $\delta$ matters, and we assign $\delta = \pm \infty$. Thus we obtained a table operation which, given $(2t - 2)$ 2-bit codes and a number $\delta$ with $O(t)$ distinct values, returns a compressed chunk; adding $f_3(2)$ to the explicit values in this chunk, one gets $\hat{A}$.

### 3.3 Linear-Time Algorithm: An Overview

Below we describe one phase of Algorithm EOPL computing even and odd palindromic lengths of the input string. We avoid pseudocode since it is too long.

**Prerequisites.** P1. Arrays $PL$ and $PRE[p]$ are stored in compressed form as sequences of length-$t$ chunks (the last chunk in a sequence can be short). We maintain a work chunk $W$ to compute the reversal of a new chunk of $PL$ during the current phase; we move symbols from $W$ to $PL$ one by one and thus avoid the reversal operation. List $wait$ stores new palindromes and palindromes that changed periods, to perform $PRE$ updates at the end of the phase.

P2. The operations 1–3, 5 from Lemma 15 are based on extracting an element and trivially extend to our type of chunks. We smooth (in $O(1)$ time by Lemma 18) chunks from $PL$ before applying min operations; the correctness of this approach is justified by Lemma 17. Since the first difference of the functions $f_1, f_2$ in a smoothed chunk is $\pm 1$, a smoothed chunk can safely be extended by extending these functions as in Lemma 15; this approach was
justified in [11, 13]. Then computing min of smoothed chunks can be easily done in \(O(1)\) time as in Lemma [15, 17]. The arrays \(\text{PRE}[p]\) consist of smoothed chunks and \(W\) is also smoothed; drops may occur only between the last element of \(\text{PL}\) and the first element of \(W\).

P3. Big \(p\)-series \((p \geq t)\) and small \(p\)-series \((p < t)\) are processed separately.

**First iteration.** F1. Read the next symbol (say, \(s[n]\)), perform \(\text{add}(n)\); at this moment
a. the iterator stores \(s[0..n]\) and returns the list \(\text{dead}\);
b. the array \(\text{PL}[0..n-1]\) and the arrays \(\text{PRE}[p]\) for \(s[0..n-1]\) are correctly computed;
c. the chunk \(W\) is initialized by \(\infty\)'s, the list \(\text{wait}\) is empty;
d. the maximum number of iterations in the phase is set to \(t' = \min\{n - \text{len}(\text{maxPal}), t\}\) (if the first number is smaller, \(\text{maxPal}\) must change after at most \(t'\) iterations).

F2. Loop through the list of big \(p\)-series of \(s[0..n-1]\):
   a. compute \(\text{left}[p]\) and clear \(\text{PRE}[p]\) if necessary;
   b. compute \(\text{ttf}'[p] = \min\{\text{ttf}_{i-1}[p], t'\}\) using symmetry inside palindromes;
   c*. using \(\text{left}[p]\) and \(\text{ttf}'[p]\), update \(\text{PRE}[p]\), \(W\), and \(\text{wait}\) (details in a separate item below).

F3. Process new suffix palindromes with centers \(n - \frac{1}{2}\) (if \(s[n-1] = s[n]\)) and \(n\):
   a. update \(W\) directly using the chunks of \(\text{PL}\) ending at position \(n-1\) (and possibly \(n-2\));
   b. using symmetry again, check whether \(n\) and/or \(n - \frac{1}{2}\) remain centers of suffix-palindromes after the last iteration of the phase; add the center(s) with the answer “yes” to the list \(\text{wait}\).

F4. Small series and finalization:
   a. extract from \(W\) its last element \(m = (m_0, m_1)\) and delete it from \(W\);
   b. process small \(p\)-series of \(s[0..n-1]\) directly by Algorithm [1], substituting \(m_i\) for \(\text{plf}[n]\);
   c. assign \(\text{PL}[n] = (m_0, m_1)\);
   d. update arrays \(\text{PRE}[p]\) with all palindromes from the list \(\text{dead}\).

**Subsequent iterations.**

S1. Read \(s[i]\), compare it to \(s[i-1-\text{len}(\text{maxPal})]\) and then the operations from steps F3 and F4, replacing \(n\) by \(i\).

S2. If \(s[i]\) extends the longest suffix-palindrome, the phase continues: perform \(\text{add}(s[i])\) and

S3. If \(s[i]\) breaks the longest suffix-palindrome, the phase is aborted:
   a. update the lists \(\text{PRE}[p]\) using suffix-palindromes from the list \(\text{wait}\), clear \(\text{wait}\);
   b. start the next phase with \(\text{add}(s[i])\).

S4. If there was no abortion and all \(t'\) symbols were processed, the phase is terminated:
   a. update the lists \(\text{PRE}[p]\) using suffix-palindromes from \(\text{wait}\), clear \(\text{wait}\);
   b. start the next phase with \(\text{add}(s[n+t])\).

**F2c detailed.** Let \(x\) be the center of \(\text{tail}(p)\), \(\text{pttlf}(x) = \min\{\text{pttl}_{i-1}(x), t'\}\). \(d = \min\{\text{ttlf}'(p), \text{pttlf}(x)\}\). Update \(\text{PRE}[p]\) with a length-\(d\) chunk from \(\text{PL}\) and \(W\) with the updated length-\(d\) chunk from \(\text{PRE}[p]\). If \(\text{ttlf}'(p) < \text{pttlf}(x)\) (after \(d\) iterations period \(p\) dies, but \(\text{head}(p) = \text{tail}(p)\) survives, becoming a palindrome with a bigger period), additionally update \(W\) with a chunk of length \(\text{pttlf}(x)\); if \(\text{pttlf}(x) \geq t'\), add \(x\) to \(\text{wait}\). Note that if \(p\)-series gets new tail \(x + \frac{d}{2}\) during the phase, then at the start of the phase \(x + \frac{d}{2}\) was the center of the head of some \(p'\)-series, survived the death of that series, and thus was used for an additional update of \(W\). Necessary updates of \(\text{PRE}[p]\) are carried when the lists \(\text{dead}\) and \(\text{wait}\) are processed. All the same stays true for \(x+p\) which is the next potential tail of the \(p\)-series.

**Lemma 20.** Algorithm EOPL correctly computes the array \(\text{PL}\).

**Proof.** The computation of \(\text{PL}\) by Algorithm EOPL differs from the correct computation by Algorithm [1] in a few points. The use of smoothed chunks is justified by Lemmas [17, 18].
so below we take smoothing into account speaking about correctness of the arrays $\text{PRE}[p]$. We prove the lemma by induction, with an obvious base and $F1b$ as the hypothesis; more precisely, we assume that at the start of a new phase with $s[n]$, the array $\text{PL}[0..n-1]$ and all arrays $\text{PRE}[p]$ are correct, where $p$ runs through the set of live periods of $s[0..n-1]$ (i.e., $s[0..n-1]$ has a $p$-periodic suffix with at least one $p$-periodic palindrome in it).

First we note that in an aborted phase some predictions in $F2c$ can be made beyond the actual end of the phase (i.e., the phase processes $s[n..n+i-1]$, and updates are made for more than $i$ elements of $W$ and $\text{PRE}[p]$). For $W$, this does not matter, because $W$ is translated to $\text{PL}$ one element at a time; see $F4a$. For $\text{PRE}[p]$, the situation means that the actual symbol $s[n+i]$ breaks period $p$, because it differs from the predicted symbol which preserved the period; thus, $p$ is dead and $\text{PRE}[p]$ will be cleared if more $p$-series appear in the future.

Next, if $p$ is alive for $s[0..n+i-1]$, all necessary updates for $\text{PRE}[p]$ were made at steps $F2c$, $F4d$ (useful if a palindrome dies while its period survives), and $S3a/S4a$. Note that repeated updates using the same palindrome (say, first at step $F2c$ and then at $F4b$) cannot harm.

Finally, all palindromes ending in $s[n-1..n+i-2]$ were used to update $W$, so $\text{PL}[n..n+i-1]$ is computed according to rule 1. The result now follows.

Lemma 21. If Algorithm EOPL spends, apart from the $\text{add}$ queries, $O(\log P + i)$ time for each phase of $i$ iterations, where $P$ is the period of the longest suffix-palindrome at the start of the phase, then it works in $O(n)$ time.

Proof. For a phase of $t$ iterations, $\log P = O(t)$, giving us $O(1)$ time per iteration. A phase of less than $t$ iterations either was aborted or fell into the case $F1d$; in any case, it was followed by an increment of $\maxPal$ by at least $P/2$. Since $\maxPal$ never decreases and the iterator performs $n$ $\text{add}$’s in $O(n)$ time, we obtain the desired bound.

In the next subsection we show how to organize the details of Algorithm EOPL in order to satisfy the time bound of Lemma 21.

### 3.4 Details of Algorithm EOPL

#### 3.4.1 Maintaining $\text{PRE}[p]$ Arrays

There are two problems related to the crucial task of maintaining the arrays $\text{PRE}[p]$. First, a linear-time RAM algorithm should fit into $O(n)$ words of memory. Since Algorithm EOPL clears an array $\text{PRE}[p]$ only before using it next time, these arrays are stored for all periods of palindromes in the word. Note that a string may contain palindromes with $\Omega(n)$ distinct periods (e.g., $a_1 \cdots a_i a_{i+1} a_i \cdots a_1$ contains palindromes with all even periods up to $2i$). We cannot allocate $\Omega(n^2)$ bits of memory, so instead we store only the elements we computed at least once. The number of such elements is the number of series of all prefixes in the string, i.e., $O(n \log n)$; this upper bound is OK for storing $\text{PRE}$ in compressed form. We allow for $O(\log n)$ additional elements per array to ease range operations.

The second problem is that $\text{PRE}[p]$ may consist of several unrelated parts, which hardens the use of compression. An example given in Fig. 3.5 represents the situation in a quite general form: for the string $s[0..n] = \cdots babaaabaab$ the array $\text{PRE}[p]$ ($p = 4$) evolves as follows. At $(n-4)$th iteration, the suffix palindrome $baab$ appeared, $\text{PRE}[4]$ was cleared, and $\text{PRE}[p][1] = PL[n-9]$ was assigned. At the next iteration we assigned $\text{PRE}[p][0] = PL[n-10]$ (note that we moved left by 1 in the array $\text{PL}$); $\text{PRE}[p][0..1]$ is part 1 of the array. Then at
the \((n-2)\text{th}\) iteration the palindrome \(aba \cdot a \cdot aba\) with the center \(n-6\) died, reaching \(\text{left}[4]\); however, its suffix \(aba\) survived since period 4 survived as well; \(aba\) extended to \(aaba\), which has period 4 but also period 3. Thus, at this moment there was no \(p\)-series and \(\text{PRE}[p][3]\) remained undefined; it constitutes part 3 of the array. Part 3 always corresponds to biggest indices in the array due to the fact that \(\text{left}[p]\) was just reached; this corresponds to jumping \((p-1)\) symbols right in \(\text{PL}\). At the \((n-1)\text{th}\) iteration we assigned \(\text{PRE}[p][2] = \text{PL}[n-8]\) (again moving left by 1 in \(\text{PL}\)); \(\text{PRE}[p][2]\) constitutes part 2. Finally, at the \(n\)th iteration we returned to part 1, using \(\text{PL}[n-5]\) (a jump \((p-1)\) symbols right in \(\text{PL}\)) to update \(\text{PRE}[p][1]\). If the period survives further, then after updating part 1 we make another jump in \(\text{PL}\) and fill part 3, then update part 2, and so on.

![Figure 3](image)

**Figure 3** Filling and updating \(\text{PRE}[p]\) array. Currently, part 3 is being filled. Arrows below \(\text{PL}\) show the order in which the elements of \(\text{PL}\), gathered in chunks, are used to fill/update \(\text{PRE}[p]\).

The general pattern (see Fig. 3) can be restored looking at the order in which Algorithm 1 fills and updates \(\text{PRE}[p]\). Filling \(\text{PRE}[p]\) is started from the element \(i = n - \text{len}(x) - \text{left}[p]\), where \(x\) is the center of the head of \(p\)-series, and continues left to the 0’s element (part 1). At the next iteration \(x\) dies; the palindrome with the center \(y = \text{nextPal}(x)\) either has a period smaller than \(p\) and we enter part 3, or no such period, and we immediately enter part 2 (part 3 has length 0) to fill the \((p-1)\)th element, jumping in \(\text{PL}\) by \(p-1\) elements to the right. As the period survives, the filling and updating of \(\text{PRE}[p]\) continues as in the above example.

We store \(\text{PRE}[p]\) as a triple of arrays \((L, C, R)\), \(|L| + |C| + |R| \leq \text{ttl}[p]\), where \(i\) is the iteration in which \(\text{PRE}[p]\) was cleared last time. The arrays represent, respectively, part 1, part 2, and part 3 of \(\text{PRE}[p]\). We also store \(|L|, |C|, |R|\), and the rightmost positions \(\text{end}_L, \text{end}_C\) of the first two arrays in \(\text{PRE}[p]\). The arrays are stored as sequences of smoothed chunks. If we just cleared \(\text{PRE}[p]\), we define \(\text{end}_L\) and add a chunk from \(\text{PL}\) to \(L\). All subsequent updates can be partitioned into “early” \((|L| + |C| + |R| < p)\) and “late” \((|L| + |C| + |R| = p)\). We first assume \(p > t\). Since a chunk is shorter than \(p\), no jumps inside a chunk is possible. So during early updates we first completely fill \(L\), then set \(\text{end}_C\) and completely fill \(C\). Next we update the whole \(L\) (using min operations on smoothed chunks), and finally fill \(R\) (the stage of filling \(R\) is shown in Fig. 3). If some chunk should be applied partially to \(C\) and partially to \(R\), we split it in two. During late updates we just perform min operations with chunks from corresponding arrays. Overall we perform each update in \(O(1)\) time using table operations (see Lemmas 15 and 19). Each of three arrays consists of compressed smoothed chunks; so \(\text{PRE}[p]\) occupies \(O(|L| + |C| + |R|)\) bits of space and any chunk needed to update \(\text{PL}\) can be constructed in \(O(1)\) time. By construction, the next lemma is immediate.

**Lemma 22.** *Updating \(\text{PRE}[p]\) with one chunk requires \(O(1)\) time.*

In the special case of small series we split each input chunk into several smaller chunks which fit into the arrays \(L, C, R\). Then we can state the result about the “range” updates of the lists \(\text{PRE}[p]\).
Lemma 23. During a phase of length $i$, all range updates of $\text{PRE}[p]$ at steps F2c and S3a/S4a spend in total $O(\log P + i)$ time, where $P$ is the minimum period, at the end of the phase, of the longest suffix-palindrome of $s$.

Proof. For one run of the loop F2, a constant number of updates of $\text{PRE}[p]$ with one chunk is performed (the maximum is 3 updates in case 3), each in $O(1)$ time by Lemma 22. Since the longest suffix-palindrome belongs to $P$-series, the head of the next series has length $< 2P$ by Lemma 7. Then by Lemma 3, the number of series is $O(\log P)$.

Similarly, applying palindromes from the list wait requires $O(1)$ chunk updates per palindrome if its period is $> i$, and one could write to the list at most $O(\log P)$ times at step F2c and at most $O(i)$ times at step F3b, which is exactly the desired bound.

Finally, the palindrome of periods $< i$ are used to update $\text{PRE}$ in chunks only at steps S3a/S4a, and we have $O(i)$ such palindromes. The number of chunks needed for an update using a palindrome $w$ with period $p$ is $O(i/p)$. However, $w$ has $O(i/p)$ suffix-palindromes in its series, and all of them are separate elements of wait. So we have only $O(i)$ chunks to apply to $\text{PRE}$. The result now follows.

3.4.2 Other Details

We analyze the remaining steps of Algorithm EOPL one-by-one, omitting the add queries. The steps F1c, F1d require $O(1)$ time; the same is true for F2a by Lemma 9 and for F2b by Lemma 24 below.

Lemma 24. For a string $s$, suppose that $s[0..n-1]$ is processed, $t' = \min\{t, n-\text{len}(\text{maxPal})\}$ and the segment $s[n..n+t'-1]$ will be processed in one phase. Let $\text{ttl}'[p] = \min\{\text{ttl}_{n-1}[p], t'\}$ and let $p_1 > p_2 > \cdots > p_m$ be the periods of all big series of $s[0..n-1]$. Then all values $\text{ttl}'[p_1], \ldots, \text{ttl}'[p_m]$ can be computed in $O(m)$ time.

Proof. We loop through the heads of series as in Algorithm 1. Let $x$ be a center of the head $(uv)^k u$ of $p_i$-series. As in Lemma 9, we compute $|u| = \text{len}(x) \mod p_i$ and $y = \text{cntr}(|u|)$. Then $y' = 2\text{maxPal} - y$ is the center of the prefix-palindrome $u$ of the longest suffix-palindrome $s[i..n-1]$ of $s[0..n-1]$; note that the longest suffix-palindrome of $s[i..n+t'-1]$ is $z = s[i-t'..n+t'-1]$ by conditions of the lemma. Let $w$ be the longest palindrome with the center $y'$; using the query $\text{rad}(y')$ we determine whether $w$ is located inside $z$ or begins outside it. In the latter case, $z$ has a $p_i$-periodic prefix ending with $(uv)^k u$ and then a $p_i$-periodic suffix beginning with $(uv)^k u$. So $\text{ttl}_{n-1}[p_i] \geq t'$ and then $\text{ttl}'[p_i] = t'$. In the former case, $w = \hat{w}_1 \hat{w}_1$, where $\hat{w}_1$ is a proper prefix of $\hat{w}$ because $|\hat{w}| = p_i > t' \geq |\hat{w}_1|$. Hence the period $p_i$ breaks on the left of $w$, which means that $\text{rad}(y') = \text{rad}(y) = \text{ttl}_{n-1}[p_i]$. All computations above take $O(1)$ time. The result now follows.

Next, the radii of suffix-palindromes with the centers $n$, $n - \frac{1}{2}$ can be found by the use of symmetry inside the longest suffix-palindrome; it is sufficient to query $\text{rad}(2\text{maxPal} - n)$ and $\text{rad}(2\text{maxPal} - n + \frac{1}{2})$. After this, step F3a requires a constant number of chunk updates, $O(i)$ in total for a phase of $i$ iterations. By Lemma 15, this takes $O(i)$ time.

For step F4b, Lemma 15 allows one to simulate Algorithm 1 in $O(1)$ time per update as follows: extract an element of $W$, an element of $\text{PRE}[p]$ and an element of $\text{PL}$, then update the second with the third and the first with the second. After that, “return” the updated elements to $\text{PRE}[p]$ and $W$, extending 1-element chunk and using min operations. Now Corollary 15 gives us the time bound $O(t)$ for a phase of $t$ iterations and $O(P+1)$ for the phase aborted after $i$ iterations, where $P$ is the period of the longest suffix-palindrome.
Finally, for step F4d note that we can safely use $O(1)$ time to process one palindrome from the list, because the total length of all lists dead is upper bounded by $2n$. Since dead($x$) contains the answers to all queries about $x$, we compute the period $p$ and left[$p$] in $O(1)$ time.

By Lemma 22, we can update PRE[$p$] in $O(1)$ time if $p \geq t$. For $p < t$, we need a closer look into the purpose of step F4d. This update is useful only if the only palindrome in the $p$-series died but period $p$ survived (part 3 of PRE[$p$]; see Fig. 3). Being the only palindrome in a series means len($x$) < 3$p$. Therefore, if len($x$) ≥ 3$p$, we make no update (period $p$ died, so PRE[$p$] will be cleared before the next use), and if len($x$) < 3$p$, then the update requires $O(1)$

The above analysis, together with Lemma 23, covers all steps of Algorithm EOPL. Then in total we have $O(\log P + i)$ bound for one phase, plus the aggregate $O(n)$ time bound for all add queries and step F4d. Now the references to Lemma 21, Lemma 20 and Remark 9 finish the proof of Theorem 1.

4 Computing $k$-factorization

A run of linear-time Algorithm EOPL leaves the iterator containing the input string $s = s[0..n-1]$ and the arrays $pl[0..n-1]$, $pl'[0..n-1]$ for $s$. Let us use them to construct an explicit $k$-factorization in additional linear time. Let $k' = pl^k$ mod 2[$n-1$]. By Lemma 4, it suffices to build a $k'$-factorization of $s$ in $O(n)$ time, which can be done as follows.

\begin{algorithm}
\begin{verbatim}
   1: for (i ← 1; i < k'; i++) do build a list list[i] of all positions j : pl^mod 2[j] = i 
   2: end ← n - 1; factors ← {} 
   3: for (i ← k' - 1; i > 0; i--) do
   4:    for j ∈ list[i] do
   5:       if len($s[i+1..\text{end}]$) = end − j then break \texttt{\triangleright} $s[j+1..\text{end}]$ is a palindrome
   6:          add $s[j+1..\text{end}]$ to factors; end ← j
   7:    add $s[0..\text{end}]$ to factors
\end{verbatim}
\end{algorithm}

Algorithm 2 builds a $k'$-factorization of $s$ in reversed order, using a trivial observation that if a string $w$ has even (odd) palindromic length $i$ then $w = uv$, where $u$ has odd (resp., even) palindromic length $i-1$ and $v$ is a nonempty palindrome. All lists are build in parallel in linear time. In the main cycle, each position serves as $j$ at most once, so the algorithm performs $O(n)$ len queries to the iterator. Thus we proved

\textbf{Theorem 25.} There is a linear-time word-RAM algorithm which, given a string $s$ over a general alphabet and a number $k$, builds a palindromic $k$-factorization of $s$ or reports that no such factorization exists.

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