FAYERS’ CONJECTURE AND THE SOCLES OF CYCLOTOMIC WEYL MODULES

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Abstract. Gordon James proved that the socle of a Weyl module of a classical Schur algebra is a sum of simple modules labelled by p-restricted partitions. We prove an analogue of this result in the very general setting of “Schur pairs”. As an application we show that the socle of a Weyl module of a cyclotomic q-Schur algebra is a sum of simple modules labelled by Kleshchev multipartitions and we use this result to prove a conjecture of Fayers that leads to an efficient LLT algorithm for the higher level cyclotomic Hecke algebras of type A. Finally, we prove a cyclotomic analogue of the Carter-Lusztig theorem.

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1. Introduction

In their landmark paper [35], Lascoux, Leclerc and Thibon conjectured that the decomposition matrices of the Iwahori-Hecke algebras of type A could be computed using the canonical bases of the level one Fock spaces. Ariki [2] generalised, and proved, this conjecture for the cyclotomic Hecke algebras of arbitrary level in type A. Unlike in level one, the canonical bases of higher level Fock spaces are quite difficult to compute. Building on [35], Fayers [23] gave a more efficient algorithm for computing the canonical bases of the higher level Fock spaces. As a result of his calculations he made a natural conjecture for the maximal degrees of the LLT-polynomials, which are certain parabolic Kazhdan-Lusztig polynomials that are computed by Fayers’ algorithm. This conjecture, if true, further improves the efficiency of Fayers’ algorithm. This paper started as a project to prove Fayers’ conjecture but the machinery that we develop to do this has several other applications.

Fayers’ conjecture is a purely combinatorial statement that gives an upper bound on the degrees of certain polynomials. As we will show, the representation theoretic significance of his conjecture gives information about the socle of the Weyl modules. To prove Fayers’ conjecture, we take advantage of recent advances by Stroppel-Webster [49], Maksimau [38], Brundan-Kleshchev [10, 11] and the authors [29] in the graded representation theory of the cyclotomic Hecke and Schur algebras to connect the higher level LLT-polynomials with the graded decomposition numbers of these algebras. This machinery allows us to reduce Fayers’ conjecture to understanding the socle of a Weyl module.

This paper starts by analysing the endomorphism algebra $S = \text{End}_A(M)$ of a Schur pair $(A, M)$, where $A$ is a self-injective algebra and $M$ is a faithful $A$-module. In this very general setting, which includes the cyclotomic Hecke algebras $H_n$ and the Schur algebras $S_n$ as special cases, we classify the projective-injective $S$-modules and show that the modules that can appear in the socle of the projective
S-modules are labelled by the simple $A$-modules. We then proceed to apply this general theory to the cyclotomic Hecke algebras and Schur algebras. Our main results are:

- We prove that the simple modules appearing in the socle of a Weyl modules are indexed by Klebschel multipartitions (Theorem 5.2).
- We prove Fayers Conjecture (Theorem 6.8).
- We give a cyclotomic generalisation of the Carter-Lusztig theorem [13] (Theorem 7.12).

Along the way we prove a number of useful new results about the cyclotomic Hecke and Schur algebras, such as a new description of the simple $H_n$-modules (Lemma 3.12) and an interpretation of Ringel duality at the level of Hecke algebras (Theorem 4.7).

All of the results in this paper apply to what are traditionally called the degenerate and non-degenerate cyclotomic Hecke algebras and Schur algebras (but not to the Ariki-Koike algebras with $\xi^2 = 1$). This is because, following [30], we use a slightly different presentation (Definition 3.2) of the cyclotomic Hecke algebras that simultaneously captures the degenerate and non-degenerate cyclotomic Hecke algebras.

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2. Schur pairs and the socle of tensor space

Partly inspired by the extensive literature of Schur algebras acting on tensor space, such as [17, 25, 44], this section studies the endomorphism algebra of a finite dimensional faithful module of a finite dimensional self-injective algebra. Our main aim is to understand, in this general setting, what simple modules can appear in the socle of “tensor space”.

Throughout this paper we fix a field $K$. If $A$ is a $K$-algebra let $A$-mod and mod-$A$ be the categories of finite dimensional left and right $A$-modules, respectively.

Recall that a trace form on an algebra $A$ is a linear map $\tau : A \to K$ such that $\tau(ab) = \tau(ba)$, for all $a, b \in A$. The algebra $A$ is symmetric if it has a non-degenerate symmetric bilinear form $(, ) : A \times A \to K$ such that

$$\langle xa, y \rangle = \langle x, ay \rangle, \quad \text{for all } a, x, y \in A$$

The algebra $A$ is self-injective if the regular representation of $A$ is injective as an $A$-module. As is well-known [14, 86], symmetric algebras are self-injective.

2.1. Definition. A Schur pair is an ordered pair $(A, M)$, where $A$ is a self-injective finite dimensional $K$-algebra and $M$ is a faithful finite dimensional right $A$-module. If $(A, M)$ is a Schur pair let

$$S = S_A(M) = \text{End}_A(M)$$

be the algebra of $A$-module endomorphisms of $M$.

By extending $K$, if necessary, we will always assume that $A$ and $S$ are both split over $K$.

An explicit example to keep in mind is the case where $A$ is the Iwahori-Hecke algebra of the symmetric group and $M$ is tensor space [17, §1 and §2], so that $S$ is the $q$-Schur algebra. More generally, as we show in Proposition 4.1, we can take $A$ to be a cyclotomic Hecke algebra of type $\tilde{A}$, in which case there are two natural choices for $M$.

For the remainder of this section we assume that $(A, M)$ is a Schur pair and that $S = S_A(M)$. Our aim is to understand the interplay between the representation theory of the algebras $A$ and $S$. Our starting point is the following easy fact.

2.2. Proposition. Suppose that $(A, M)$ is a Schur pair.

a) There exists an integer $r \geq 1$ and an $A$-module $N$ such that $M^{\oplus r} \cong A \oplus N$ as $A$-modules.

b) If $r \geq 1$ then $(A, M^{\oplus r})$ is a Schur pair and the algebras $S_A(M)$ and $S_A(M^{\oplus r})$ are Morita equivalent.

Proof. First consider (a). Since $M$ is a faithful finite dimensional $A$-module we can find an integer $r > 0$ such that $A$ is isomorphic to an $A$-submodule of $M^{\oplus r}$ as a right $A$-module. Indeed, if $\{m_1, \ldots, m_d\}$ is a basis of $M$ then the map $A \to M^{\oplus d}$ given by $a \mapsto (am_1, \ldots, am_d)$ is injective. Hence, $A$ is isomorphic to a submodule of $M^{\oplus r}$ whenever $r \geq d = \dim_K M$. Fix such an integer $r$. Since $A$ is self-injective the map $A \to M^{\oplus r}$ splits, so $M^{\oplus r} \cong A \oplus N$ as an $A$-module. This completes the proof of part (a).
Part (b) follows directly from the definitions since $M$ and $M^\oplus r$ have the same indecomposable summands.

In terms of the representation theory of $S$, Proposition 2.2 says that there is no harm in assuming that $A$ is a direct summand of $M$ whenever $(A, M)$ is a Schur pair. This will be convenient in several of the arguments below.

The algebra $S$ acts from the left on $M$ as an algebra of endomorphisms, with $φ \cdot m = φ(m)$, for $φ ∈ S$ and $m ∈ M$. Moreover, since $A$ acts faithfully on $M$, there is an algebra embedding $A^{op} \hookrightarrow \text{End}_K(M)$, where we identify $a ∈ A$ with the endomorphism $ρ_a ∈ \text{End}_K(M)$ given by $ρ_a(m) = ma$, for $m ∈ M$. By construction,

$$(φ \cdot m)a = φ(m)a = φ(ma) = φ \cdot (ma), \quad \text{for all } φ ∈ S, m ∈ M \text{ and } a ∈ A,$$

Hence, the left and right actions of $S$ and $A$ on $M$ commute with each other and $A^{op}$ is a subalgebra of $\text{End}_S(M)$. The next result will let us show that $A^{op} ≅ \text{End}_S(M)$.

2.3. Lemma. Suppose that $(A, M)$ is a Schur pair and $r ≥ 1$. Then $\text{End}_{S(A)}(M) ≅ \text{End}_{S(A)^{op}}(M^{\oplus r})$.

Proof. The proof is elementary but we give the argument because we need this idea below. There is an embedding $Θ : \text{End}_{S(A)}(M) → \text{End}_{S(A)^{op}}(M^{\oplus r})$ given by $Θ(φ) = φ^{op}$, where $φ^{op}$ is the endomorphism of $M^{\oplus r}$ given by $φ^{op}(m) = (φ(m_1), \ldots, φ(m_r))$, for $m = (m_1, \ldots, m_r) ∈ M^{\oplus r}$. For $1 ≤ i, j ≤ r$ let $π_i$ be projection onto the $i$th component of $M^{\oplus r}$ and let $σ_{ij}$ be the map that swaps the $i$th and $j$th components. Then $π_i, σ_{ij} ∈ Θ_1(M^{\oplus r})$. Therefore, if $φ ∈ \text{End}_{S(A)^{op}}(M^{\oplus r})$ and $m ∈ M^{\oplus r}$ then $φ(π_i m) = π_i φ(m)$. Hence, $φ$ maps the $i$th component of $M^{\oplus r}$ into the $i$th component.

The next result shows that $A$ and $S$ enjoy a Double Centralizer Property in the sense that $A ≃ \text{End}_S(M)^{op}$ and $S ≃ \text{End}_A(M)$. This result is well-known and appears as [14, Theorem 59.6]. We give a self-contained proof of this result both because it is central to all of the results in this section and because the proof is quite short.

2.4. Theorem (Double centraliser property). Let $(A, M)$ be a Schur pair. Then $A^{op} ≅ \text{End}_S(M)$.

Proof. By Proposition 2.2 and Lemma 2.3, we can assume that $M ≃ A ⊕ N$, for some $A$-module $N$. This reduces the proof of the theorem to the claim that $\text{End}_S(A ⊕ N) ≃ A^{op}$, where $S = \text{End}_A(A ⊕ N)$. Recall that $A^{op} ≃ \text{End}_A(A^A)$, where $A^A$ is the left regular representation of $A$. With this observation the claim now follows easily using a similar argument to the proof of Lemma 2.3: see [14, Lemma 59.4] for more details.

2.5. Corollary. Let $(A, M)$ be a Schur pair such that $A$ is a direct summand of $M$ and let $e : M → A$ be the natural projection map. Then $e ∈ S$, $A \cong eS$ as an algebra and $M \cong eS$ as an $(S, A)$-bimodule.

Proof. By definition, $e$ is an endomorphism of $M$ that commutes with the action of $A$ so we can consider $e$ to be an element of $S$. Then $Se ≃ M$ as a left $S$-module. Explicitly, the isomorphism is given by $s ↦ s \cdot 1_A$, where $1_A$ is the identity element of $A$. Hence, by Theorem 2.4, $A ≃ (\text{End}_S(M))^{op} ≃ (\text{End}_S(eS))^{op} ≃ eS$ as an algebra. Finally, if we identify $A$ with $eS$ then $Se ≃ M$ as an $(S, A)$-bimodule.

Suppose that $(A, M)$ is a Schur pair. Define a functor $F : \text{mod}-S → \text{mod}-A$ by $F(X) = X ⊗_S M$, for any $X ∈ \text{mod}-S$ and $F(g) = g ⊗_S 1_M$ for any $g ∈ \text{Hom}_S(X, Y)$. Note that $F(X)$ is a right $A$-module with $A$-action given by $(x \otimes m)a = x \otimes ma$, for all $x, m ∈ M, a ∈ A$. Moreover, if $A$ is a direct summand of $M$ with $e : M → A$ be the natural projection map, then $F(X) = X ⊗_S M ≃ X ⊗_S eS ≃ Xe$.

2.6. Corollary. The module $M$ is projective as a left $S$-module.

Proof. In view of Proposition 2.2, it is sufficient to consider the case when $A$ is a direct summand of $S$. Then $M ≃ eS$ as an $S$-module by Corollary 2.5, so $M$ is a projective $S$-module.

2.7. Corollary. The functor $F$ is exact and fully faithful on projective $S$-modules.
Proof. The functor $F$ is exact because $M$ is projective as an $S$-module by Corollary 2.6. To prove that $F$ is fully faithful on projectives it is enough to consider the case when $A$ is a direct summand of $M$ by Proposition 2.2. Then $A \cong eSe$ and $F(X) \cong Xe$ by Corollary 2.5 and the discussion above Corollary 2.6, for any $S$-module $X$. Hence, $F$ is fully faithful on projectives by Theorem 2.4. □

By Corollary 2.5, if $A$ is a direct summand of $M$ then $A \cong eSe$ and $F(X) = Xe$, for some idempotent $e \in S$. Therefore, in order to understand the connection between $S$-modules and $A$-modules we can apply Auslander’s theory of quotient functors, or Schur functors, as in [8, §3.1] or [25]. Let $\{L^\lambda | \lambda \in \mathcal{P}\}$ be a complete set of pairwise non-isomorphic simple $S$-modules in mod-$S$, where $\mathcal{P}$ is an indexing set. For each $\lambda \in \mathcal{P}$, define the $A$-module

$$D^\lambda = F(L^\lambda)$$

and set $\mathcal{K} = \{\lambda \in \mathcal{P} | D^\lambda \neq 0\}$. By the general theory of quotient functors, such as in [25, (6.2g)], $\{D^\lambda | \lambda \in \mathcal{K}\}$ is a complete set of pairwise non-isomorphic simple right $A$-modules. By Proposition 2.2 this is true whenever $S = S_A(M)$ and $(A, M)$ is a Schur pair (that is, we do not need to assume that $A$ is a direct summand of $M$).

2.8. Definition. Let $\lambda \in \mathcal{P}$. Let $P^\lambda$ be the projective cover of $L^\lambda$ and define the Young module $Y^\lambda = F(P^\lambda)$.

Thus, $P^\lambda$ and $L^\lambda$ are $S$-modules and $D^\lambda$ and $Y^\lambda$ are $A$-modules. In the special case when $S$ is the Schur algebra and $A$ is the group algebra of the symmetric group, $Y^\lambda$ is the usual Young module.

If $E$ is a right $B$-module for an algebra $B$ then the socle of $E$, soc$_B(E)$, is the maximal semisimple (right) submodule of $E$. Dually, the head of $E$, hd$_B(E)$, is its maximal semisimple (right) quotient module. When the algebra $B$ is clear then we simply write soc $E$ and hd $E$.

2.9. Proposition. Suppose that $\lambda, \mu \in \mathcal{P}$. Then the following hold.

a) The Young module $Y^\lambda$ is an indecomposable $A$-module.

b) There is an isomorphism $Y^\lambda \cong Y^\mu$ if and only if $\lambda \equiv \mu$.

c) The Young module $Y^\lambda$ is projective if and only if $\lambda \in \mathcal{K}$.

d) If $\lambda \in \mathcal{K}$ then $Y^\lambda$ is the projective cover of $D^\lambda$.

Proof. By Corollary 2.7, the functor $F$ is fully faithful on projective $S$-modules. Therefore, End$_A(Y^\lambda) \cong \text{End}_S(P^\lambda)$ is a local $K$-algebra since $P^\lambda$ is indecomposable. Therefore, $Y^\lambda$ is an indecomposable $A$-module. Further, Hom$_A(Y^\lambda, Y^\mu) \cong \text{Hom}_S(P^\lambda, P^\mu)$, so $Y^\lambda \cong Y^\mu$ if and only if $\lambda \equiv \mu$. We have established parts (a) and (b), so it remains to prove (c) and (d).

By Corollary 2.5 we may assume that $A$ is a direct summand of $M$, as an $A$-module, so that $A \cong eSe$ for some idempotent $e \in S$. Therefore, $F(S) = Se \cong eSe \oplus (1 - e)Se$ as right $A$-modules. Therefore, every indecomposable projective right $A$-module is isomorphic to $Y^\nu$, for some $\nu \in \mathcal{P}$.

Fix $\mu \in \mathcal{K}$ and let $Y(\mu)$ be the projective cover of the simple $A$-module $D^\mu$. By the last paragraph, $Y(\mu) \cong Y^\lambda$ for some $\lambda \in \mathcal{P}$. Suppose, by way of contradiction, that $\lambda \neq \mu$. As $P^\mu$ is the projective cover of $L^\mu$, by applying $F$ there is a surjective map $Y^\mu \twoheadrightarrow D^\mu$ since $F(L^\mu) = D^\mu$. Let $s \geq 1$ be the multiplicity of $D^\mu$ in the head of $Y^\mu$. As $Y^\mu \not\cong Y^\lambda \cong Y(\mu)$ is indecomposable we can find a non-zero map $\theta : (Y^\lambda)^{\oplus s} \rightarrow Y^\mu$ such that $D^\mu$ does not appear in the head of $Y^\mu/\text{im} \theta$. Hence, by Corollary 2.7, there exists a non-zero map $\tilde{\theta} : (P^\lambda)^{\oplus s} \rightarrow P^\mu$ such that $L^\mu$ does not appear in the head of $P^\mu/\text{im} \tilde{\theta}$. Note that $\text{im}(\tilde{\theta}) \neq P^\mu$ as $\text{hd}(P^\mu) \not\subseteq \text{hd}(P^\lambda)$. However, this is a contradiction because $P^\mu$ has simple head $L^\mu$. Hence, $\lambda = \mu$ and $Y(\mu) \cong Y^\mu$ as we wanted to show. This completes the proofs of parts (c) and (d). □

2.10. Theorem. Suppose that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map.

a) As a right $S$-module, $eS \cong \bigoplus_{\mu \in \mathcal{K}} D^\mu \oplus P^\mu$.

b) As a right $A$-module, $M \cong eSe \cong \bigoplus_{\lambda \in \mathcal{P}} L^\lambda \otimes Y^\lambda$.

Proof. By Corollary 2.5, $A \cong eSe$. First consider (a). There exist non-negative integers $d_\mu$ such that

$$eS \cong \bigoplus_{\mu \in \mathcal{K}} (P^\mu)^{\oplus d_\mu}.$$
Therefore, using Corollary 2.5 for the second equality,
\[ A \cong eSe = F(eS) = \bigoplus_{\mu \in \mathcal{P}} F(P_\mu) \oplus d_{\mu} = \bigoplus_{\mu \in \mathcal{P}} (Y_\mu) \oplus d_{\mu}. \]

The direct summands of \( A \) are necessarily projective \( A \)-modules, so \( d_{\mu} \neq 0 \) if and only if \( \mu \in \mathcal{X} \) and \( Y_\mu \) is the projective cover of \( D_\mu \) by Proposition 2.9(d). Moreover, since \( A \) is \( K \)-split by assumption, \( d_{\mu} = \dim D_\mu \) for all \( \mu \in \mathcal{P} \). This completes the proof of (a).

Now consider (b). As a right \( S \)-module, \( S \cong \bigoplus_{\lambda \in \mathcal{P}} L_\lambda \otimes P_\lambda \) since \( S \) is split over the field \( K \). By definition, \( Y_\lambda = F(P_\lambda) \). In view of Corollary 2.5, \( M \cong Se \cong F(S) \cong \bigoplus_{\lambda \in \mathcal{P}} L_\lambda \otimes Y_\lambda \) as a right \( A \)-module. \( \square \)

By Proposition 2.9, \( \{Y_\mu \mid \mu \in \mathcal{X}\} \) is a complete set of pairwise non-isomorphic indecomposable projective right \( A \)-modules. As \( A \) is self-injective, \( \{Y_\mu \mid \mu \in \mathcal{X}\} \) is also a complete set of pairwise non-isomorphic indecomposable injective right \( A \)-modules.

The next few results assume that \( A \cong eSe \) as in Corollary 2.5. For these results we identify the algebras \( A \) and \( eSe \) using this isomorphism. For the next result, say that a right \( S \)-submodule \( K \) of \( eS \) is generated by \( A \) if \( K = (K \cap eSe)S \).

2.11. Lemma. Suppose that \( A \) is a direct summand of \( M \) and let \( e : M \rightarrow A \) be the natural projection map. Then:

a) The map \( I \mapsto IS \) defines an inclusion preserving bijection between the set of right \( A \)-submodules of \( eSe \) and the set of right \( S \)-submodules of \( eS \) generated by \( A \).

b) If \( I \) and \( J \) are right \( A \)-submodules of \( eSe \) then \( \text{Hom}_A(I,J) \cong \text{Hom}_S(IS,JS) \).

c) If \( I \) and \( J \) are right \( A \)-submodules of \( eSe \) then \( I \cong J \) as \( A \)-modules if and only if \( IS \cong JS \) as \( S \)-modules.

Proof. Let \( I \) be a right \( A \)-submodule of \( eSe \). We claim that \( I = IS \cap eSe \). Certainly, \( I \subseteq IS \cap eSe \). Conversely, if \( x \in IS \cap eSe \) then \( x = xe \) and we can write \( x = \sum_{s \in S} a_s se \), for some \( a_s \in I \). Therefore,
\[ x = xe = \sum_{s \in S} a_s se = \sum_{s \in S} a_s(se) \in I, \]
since \( ese \in eSe \) and \( I \) is a right \( A \)-module. Therefore, \( I = IS \cap eSe \) as claimed. Hence, the ideal \( IS \) is generated by \( A \). Moreover, if \( I \) and \( J \) are right \( A \)-submodules of \( eSe \) then \( IS = JS \) if and only if \( I = J \).

We have now proved part (a).

As (c) follows immediately from (b), it remains to prove (b). Let \( I \) and \( J \) be \( A \)-submodules of \( eSe \). By the last paragraph, any homomorphism \( \psi : IS \rightarrow JS \) restricts to a well-defined \( A \)-module homomorphism \( \hat{\psi}_e : I \rightarrow J \) since \( A \cong eSe \). Conversely, since \( A \) is self-injective any homomorphism between two right ideals of \( A \) is given by left multiplication by \([14, \text{Theorem 61.2}]. That is, if \( \hat{\psi}_e : I \rightarrow J \) is an \( A \)-module homomorphism then there exists an \( a \in eSe \) such that \( \psi(x) = ax \), for all \( x \in I \). Therefore, there is a well-defined \( S \)-module homomorphism \( \psi : IS \rightarrow JS \) given by \( \psi(y) = ay \), for all \( y \in IS \). By construction, \( \psi \) is uniquely determined by \( \hat{\psi}_e \), and it restricts to \( \hat{\psi}_e \), so (b) follows. \( \square \)

Suppose that the algebra \( A \) comes equipped with an anti-involution \( * \). If \( X \) is any right \( A \)-module then the dual module \( X^* = \text{Hom}_K(X,K) \) becomes a right \( A \)-module with action \((f x)(a) = f(xa^*) \) for \( a \in A \), \( f \in X^* \) and \( x \in X \).

2.12. Definition. A self-dual Schur pair \((A,M)\) is a Schur pair \((A,M)\) where \( A \) is equipped with an anti-involution \(* \) such that \( M \cong M^* \) as \( A \)-modules.

Except for the last two results in this section, we now assume that \((A,M)\) is a self-dual Schur pair and we fix an \( A \)-module isomorphism \( \theta : M \rightarrow M^* \). Following [17, (1.5)], define an anti-isomorphism
\[ \tau : \text{End}_A(M) \rightarrow \text{End}_A(M^*) : s \mapsto s^\tau, \]
where \( s^\tau(f) = f \circ s \), for \( s \in S = \text{End}_A(M) \) and \( f \in M^* \). Then the anti-isomorphism \(* \) on \( A \) induces an anti-isomorphism \( \tau \) on \( S = \text{End}_A(M) \) that is given by \( \tau(s) := s^* := \theta^{-1} \circ s^* \circ \theta \), for all \( s \in S \). By definition, if \( s \in S \) and \( m \in M \) then \( s^*m = s^{-1}(\theta(m) \circ s) \) so that \( \theta(s^*m) = \theta(m) \circ s \) or, equivalently, \( \theta(sm) = \theta(m) \circ s^* \). Hence, \( \theta \) is also an \( S \)-module isomorphism.

2.13. Lemma. Suppose that \((A,M)\) is a self-dual Schur pair. Then \( M \) is self-dual and projective-injective as an \( S \)-module.
Proof. In the last paragraph we observed that $\theta$ is an $S$-module homomorphism, so $M \cong M^*$ as an $(S,A)$-bimodule. In particular, $M$ is self-dual as an $S$-module. By Corollary 2.6, $M$ is projective as an $S$-module so $M^* \cong M$ is injective giving the remaining claim. □

2.14. Definition. Suppose that $(A,M)$ is a self-dual Schur pair such that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map. Assume that $e^* = e$. For any $(S,A)$-bimodule $X$ let $X_R$ be the $(A,S)$-module such that $X_R = X$ as a $K$-vector space and $a x s = s^* x a^*$, for $a \in A, s \in S$ and $x \in M$.

Suppose that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map. By definition and Corollary 2.5, $S_A(M) = \text{End}_A(M) \cong \text{End}_A(Se)$. The next lemma, which is part of the motivation for introducing self-dual Schur pairs, allows us to replace $Se$ with $eS$. Note that $eS$ is a left $A$-module since $A \cong eSe$.

2.15. Lemma. Suppose that $(A,M)$ is a self-dual Schur pair such that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map. Assume that $e^* = e$. Then $*$ induces an isomorphism

$$(S_A(M))^* \cong S_A(M),$$

and right multiplication induces an algebra isomorphism $S_A(M)^* \cong \text{End}_A(eS_A(M))$.

Proof. We identify $eS_A(M)e$ with $S_A(M)$ and let $\theta$ be the natural projection map. Assume that $e^* = e$. Then $*$ induces an isomorphism $S_A(M)^* \cong \text{End}_A(eS_A(M))$. On the other hand,

$$\dim \text{End}_A(eS_A(M)) = \dim \text{End}_A(eS_A(M)e) = \dim S_A(M).$$

It follows that the injection above is an isomorphism. □

2.16. Lemma. Suppose that $(A,M)$ is a self-dual Schur pair such that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map. Assume that $e^* = e$. Let $X$ be a non-zero right $S$-submodule of $eS$. Then $X \cap eSe$ is a non-zero right $A$-submodule of $eSe$.

Proof. It is easy to see that $X \cap eSe$ is a right $A$-submodule of $eS$. We need to prove that $X \cap eSe \neq \{0\}$.

Before we start the proof observe that we can regard $eS$ as a left $A$-module because $A \cong eSe$. Let $D = \text{soc}_A(eS)$ and let $Q = QD$ be the injective hull of $D$. As $A$ is self-injective $Q$ is also projective as a left $A$-module. Therefore, we can find an integer $t \geq 0$ such that $Q$ embeds into $A^t$ as a left $A$-module. Recalling that $Q$ is the injective hull of $D = \text{soc}_A(eS)$, this implies that there exists a left $A$-module homomorphism $\vartheta : eS \rightarrow A^t$ such that $\vartheta$ restricts to the identity map on $D$.

Now are now ready to prove the lemma. Recall that $\vartheta(s) = s^*$ for each $s \in S$. By assumption $\vartheta(e) = e^* = e$. Since $A_A$ is a direct summand of $Se$ as a right $A$-module. It follows that the left regular $A$-module $(\cong \vartheta(A_A))$ is a direct summand of $eS = \vartheta(Se)$ as a left $A$-module. Fix a non-zero element $x \in X$. Then $x = ex \in eS$ so we can find a non-zero element $a = eae \in eS \cong A$ such that $ax \in D = \text{soc}_A(eS)$ because $0 \neq \text{soc}_A(eSx) \subseteq \text{soc}_A(eS) = D$. Therefore, $\vartheta(ax) \neq 0$. By composing $\vartheta$ with a suitable projection of $A^t$ onto $A$, and then using the inclusion $A \hookrightarrow eS$, we obtain a left $A$-module homomorphism $\vartheta : eS \rightarrow eS$ such that $\vartheta(A) \subset A$ and $\vartheta(ax) \neq 0$. Consequently, $\vartheta(x) \neq 0$ since $\vartheta(ax) = a\vartheta(x)$. Write $\vartheta(x) = ye + z(1 - e)$, for some $y, z \in eS$. Then $aye + az(1 - e) = \vartheta(ax) \in eS$, so $aye \neq 0 = az(1 - e)$. In particular, $ye = ye e$ is a non-zero element of $A$. To complete the proof recall that $S^*_A \cong \text{End}_A(eS)$ by Lemma 2.15. Therefore, we may assume that $\vartheta \in S$ and consider the endomorphism $e \circ \vartheta$ of $Se$. Since $X$ is an $S$-module, $ye = \vartheta(x)e = x(\vartheta^* \circ e) \in X \cap eSe$, where $x\vartheta^* = \vartheta(x)$ follows by definition. That is, $ye$ is a non-zero element of $X \cap eSe$, so $X \cap eSe \neq 0$ as we wanted to show. □

We can now prove the key result of this section. Recall that $\{L^\lambda | \lambda \in \mathcal{P}\}$ is a complete set of irreducible $S$-modules and that $\{D^\mu | \mu \in \mathcal{X}\}$ is a complete set of irreducible $A$-modules.

2.17. Theorem. Suppose that $(A,M)$ is a self-dual Schur pair such that $A$ is a direct summand of $M$ and let $e : M \rightarrow A$ be the natural projection map. Assume that $e^* = e$. Let $\mu \in \mathcal{P}$. The simple $S$-module $L^\mu$ is isomorphic to a submodule of $eS$ if and only if $\mu \in \mathcal{X}$. 

Lemma 2.16, we assume that \( (\lambda, \mu) \) says that Theorem 2.17 is a self-dual Schur pair. Theorem 2.10
\( (\lambda, \mu) \)
Corollary 2.23
\( \mu \)
\( \mu \)
\( \mu \)
\( \mu \)

2.20. Lemma. Suppose that \( (A, M) \) is a symmetric algebra and \( (D^\mu)^* \cong D^\mu \), for all \( \mu \in \mathcal{X} \). Then \( (Y^\mu)^* \cong Y^\mu \) as \( A \)-modules for all \( \mu \in \mathcal{X} \).

2.19. Corollary. Suppose that \((A, M)\) is a self-dual Schur pair. The functor \( F \) induces a one-to-one correspondence between the blocks of \( S \) and the blocks of \( A \).

2.22. Theorem. Suppose that \((A, M)\) is a self-dual Schur pair such that \( A \) is a direct summand of \( M \) and that \( A \) is a symmetric algebra such that \((D^\mu)^* \cong D^\mu \), for all \( \mu \in \mathcal{X} \). Let \( e : M \to A \) be the natural projection map and assume that \( e^* = e \).

\( \text{If } N \in \text{mod-}S \text{ then } F(N) = Ne \text{ by definition. It is easy to check that the map } f \mapsto \psi(f) : x \mapsto f(xe), \forall f \in N^*, x \in N \text{ defines a right } A \text{-module isomorphism } \psi : N^*e \cong (Ne)^*. \text{ In other words, the Schur functor } F \text{ commutes with the duality functor. As a consequence, it follows easily that}

\[ (L^\mu)^* \cong L^\mu, \text{ for all } \mu \in \mathcal{X}. \]

The next result shows that the projective-injective \( S \)-modules are indexed by the simple \( A \)-modules. The assumption that the simple \( A \)-modules are self-dual is used to ensure that the head and socle of \( P^\lambda \) are isomorphic whenever \( P^\lambda \) is self-dual. In applications \( A \) is usually a cellular algebra, in the sense of \[24\], in which case this assumption is automatic.

2.23. Lemma. Suppose that \((A, M)\) is a self-dual Schur pair such that \( A \) is a direct summand of \( M \) and let \( e : M \to A \) be the natural projection map. Assume that \( e^* = e \) and \((D^\mu)^* \cong D^\mu \), for all \( \mu \in \mathcal{X} \). Let \( \lambda \in \mathcal{P} \). Then the following are equivalent:

- \( \lambda \in \mathcal{X} \),
- \( D^\lambda \neq 0 \),
- \( L^\lambda \) is a right \( S \)-submodule of \( M_R \),
- \( P^\lambda \) is a direct summand of \( M_R \) as a right \( S \)-module,
- \( P^\lambda \) is a projective-injective right \( S \)-module,
\[
P^\lambda \text{ is self-dual.}
\]

**Proof.** By construction, \( M_R \cong eS \) and, by definition, \( \lambda \in \mathcal{K} \) if and only if \( D^\lambda \neq 0 \). Further, (a) and (c) are equivalent by Theorem 2.17 and (a) and (d) are equivalent by Theorem 2.10. Hence, (a), (b), (c) and (d) are equivalent. To complete the proof we show that (a) \( \implies (f) \implies (e) \implies (d) \).

First suppose that (a) holds so that \( \lambda \in \mathcal{K} \). By definition, \( P^\lambda \) has a simple head \( L^\lambda \). Then \( P^\lambda \) is an indecomposable direct summand of \( M_R \) by Theorem 2.10. By Lemma 2.13, \( M^* \cong M \) as a left \( S \)-module. It follows from Definition 2.14 that \((M_R)^* \cong M_R \) as a right \( S \)-module. Therefore, \((P^\lambda)^* \) is an indecomposable summand of \( M_R \), so \((P^\lambda)^* \cong P^\lambda \) for some \( \mu \in \mathcal{K} \). Hence, by (2.21), \( P^\mu \) has simple socle \( L^\lambda \) which implies that there exists a non-zero homomorphism \( f \) from \( P^\lambda \) to \( P^\mu \) such that \( \text{im}(f) \cong L^\lambda \). As the functor \( F \) is fully faithful there exists a non-zero homomorphism \( f_Y \) from \( Y^\lambda \) to \( Y^\mu \) such that \( \text{im}(f_Y) \cong D^\lambda \neq 0 \). By Lemma 2.20, \( Y^\mu \) is self-dual so it has simple socle \( D^\mu \). Therefore, \( \lambda = \mu \) so that \((P^\lambda)^* \cong P^\lambda \) is self-dual. Hence, (a) \( \implies (f) \).

Now suppose that (f) holds so that \( P^\lambda \) is self-dual. By definition, \( P^\lambda \cong (P^\lambda)^* \) is projective so it is also injective and (e) holds.

Finally, suppose that (e) holds so that \( P^\lambda \) is a projective-injective \( S \)-module. Since \( M_R \) is a faithful right \( S \)-module, \( S \) embeds into \( M_R^{\otimes t} \), for some \( t \geq 0 \), by the argument of Proposition 2.2. Fix such a \( t \). Since \((P^\lambda)^* \) is projective it embeds into \( M_R^{\otimes t} \). Taking duals there is a surjection \( M_R^{\otimes t} \cong (M_R)^{\otimes t} \rightarrow P^\lambda \). As \( P^\lambda \) is projective this map splits, so \( P^\lambda \) is isomorphic to an indecomposable submodule of \( M_R^{\otimes t} \). Hence, \( P^\lambda \) is isomorphic to an indecomposable submodule of \( M_R \). Hence, (e) \( \implies (d) \) as we wanted to show. \( \square \)

If \( S \) is a quasi-hereditary algebra then Theorem 2.22(e) says that for any \( \mu \in \mathcal{K} \), the projective indecomposable module \( P^\mu \) is also the injective hull of \( L^\mu \) and the indecomposable (partial) tilting module corresponding to \( \mu \); see Chapter 4.

**2.23. Corollary.** Suppose that \( X \) is a right \( S \)-module such that \( \text{soc} X : L^\lambda \neq 0 \) only if \( \lambda \in \mathcal{K} \). Then
\[
\text{soc} X : L^\lambda = \text{soc} F(X) : D^\lambda \quad \text{for all } \lambda \in \mathcal{K}.
\]
In particular, if \( P \) is a projective \( S \)-module then \( \text{soc} P : L^\lambda = \text{soc} F(P) : D^\lambda \), for all \( \lambda \in \mathcal{K} \).

**Proof.** If \( X = P \) is projective then \( \text{soc} P : L^\lambda \neq 0 \) only if \( \lambda \in \mathcal{K} \) by Corollary 2.18. Suppose, then, that \( X \) is a right \( S \)-module such that \( \text{soc} X : L^\lambda \neq 0 \) only if \( \lambda \in \mathcal{K} \). By exactness, \( \text{soc}(X) \subseteq \text{soc}(F(X)) \). We need to show that the reverse inclusion \( \text{soc}(F(X)) \subseteq \text{soc}(X) \) holds. By assumption,
\[
\text{soc} X = \bigoplus_{\mu \in \mathcal{K}} (L^\mu)^{\oplus x_\mu}
\]
for some non-negative integers \( x_\mu \). Let \( I_X \) be the injective hull of \( X \). Then 
\[
I_X = \bigoplus_{\mu \in \mathcal{K}} (P^\mu)^{\oplus x_\mu}
\]
since \( P^\mu \) is self-dual for all \( \mu \in \mathcal{K} \) by Theorem 2.22. Therefore we have injections
\[
\bigoplus_{\mu \in \mathcal{K}} (L^\mu)^{\oplus x_\mu} \hookrightarrow \text{soc} X \hookrightarrow I_X = \bigoplus_{\mu \in \mathcal{K}} (P^\mu)^{\oplus x_\mu}.
\]
Applying the functor \( F \) gives injections
\[
\bigoplus_{\mu \in \mathcal{K}} (D^\mu)^{\oplus x_\mu} \hookrightarrow F(\text{soc} X) \hookrightarrow F(I_X) = \bigoplus_{\mu \in \mathcal{K}} (Y^\mu)^{\oplus x_\mu}.
\]
In particular, the socle of \( F(X) \) is contained in \( \text{soc} F(I_X) = \bigoplus_{\mu \in \mathcal{K}} (D^\mu)^{\oplus x_\mu} = F(\text{soc} X) \). Hence, \( \text{soc} F(X) = F(\text{soc} X) \) and the corollary follows. \( \square \)

We conclude this section by relating hom-spaces of certain \( A \)-modules and \( S \)-modules. These results do not assume that \( (A,M) \) is a self-dual Schur pair. If \( B \) is an algebra and \( X \) is a subset of a left or right \( B \)-module, respectively, define the left and right **annihilators** of \( X \) to be
\[
L_B(X) = \{ b \in B \mid bx = 0 \text{ for all } x \in X \}
\]
\[
R_B(X) = \{ b \in B \mid xb = 0 \text{ for all } x \in X \}.
\]
Set \( LR_B(X) = L_B(R_B(X)) \) and \( RL_B(X) = R_B(L_B(X)) \). If \( X = \{ x \} \) write \( L_B(x) = L_B(X) \) and similarly for \( R_B(x) \), \( LR_B(x) \) and \( RL_B(x) \). In particular, \( LR_B(x) = \{ b \in B \mid bx = 0 \text{ whenever } xa = 0, \text{ for } a \in B \} \).

The key property of the double annihilators \( LR_B(x) \) and \( RL_B(x) \) that we need is the following well-known fact, which characterises self-injective algebras.
2.24. Lemma ([14, Theorem 61.2]). Suppose that $B$ is a self-injective algebra and that $x \in B$. Then $LR_B(x) = Bx$ and $RL_B(x) = xB$.

2.25. Lemma. Let $M$ be an $(S, A)$-bimodule and $m \in M$.

(a) If $X$ is a left $S$-submodule of $S$ then $\text{Hom}_S(Sm, X) \cong RL_S(m) \cap X$ as vector spaces.

(b) If $Y$ is a right $A$-submodule of $A$ then $\text{Hom}_A(mA, Y) \cong LR_A(m) \cap Y$ as vector spaces.

Proof. (a) If $x \in RL_S(m) \cap X$ then the map $sm \mapsto sx$ is an $S$-module homomorphism. Conversely, if $f \in \text{Hom}_S(Sm, X)$ then $f(m) \in RL_S(m) \cap X$ since $sf(m) = f(sm) = 0$ whenever $s \in L_S(m)$. It is straightforward to check that these maps are mutually inverse isomorphisms.

(b) By Lemma 2.24, it is enough to show that $\text{Hom}_A(mA, Y) \cong LR_A(m) \cap Y$. The argument to prove this is almost identical to part (a) in that if $x \in LR_A(m) \cap Y$ then the map $ma \mapsto xa$, for $a \in A$, is an $A$-module homomorphism. Conversely, let $f \in \text{Hom}_A(mA, Y)$ and $x = f(m)$. We need to show that $xa = 0$ whenever $a \in R_A(m)$. That is, $xa = 0$ whenever $ma = 0$. In fact, $xa = f(m)a = f(ma) = 0$. □

If $A$ is self-injective and $m \in A$ in Lemma 2.25(b) then $LR_A(m) = Am$ by Lemma 2.24, so $\text{Hom}_A(mA, Y) \cong Am \cap Y$.

3. Cyclotomic Hecke algebras

The main results of this paper apply the results of Chapter 2 to the cyclotomic Schur algebras $S_n$, which were introduced in [18]. In order to do this we first need to produce a Schur pair $(A, M)$ that defines the cyclotomic Schur algebras. Using the notation of Definition 2.1, the algebra $A$ will be a cyclotomic Hecke algebra $H_n$ and $M$ will be a direct sum of permutation modules for $H_n$. This section recalls the representation theory of the cyclotomic Hecke algebras of type $A$ and proves some new results about the simple modules of these algebras that will be needed later.

Fix a non-negative integer $n$. Let $S_n$ be the symmetric group on $n$ letters and let $\{s_1, \ldots, s_{n-1}\}$ be the standard set of Coxeter generators for $S_n$, where $s_r = (r, r + 1)$ for $1 \leq r < n$. If $w \in S_n$ then the length of $w$ is $\ell(w) = \min \{k \geq 0 \mid w = s_{r_1} \cdots s_{r_k}\}$.

Fix a field $K$ and $\xi \in K^\times$. If $k \in \mathbb{Z}$ then the quantum integer $[k] = [k]_\xi$ is

$$\begin{align*}
[k] &= [k]_\xi = \begin{cases} 
\xi + \xi^3 + \cdots + \xi^{2k-1}, & \text{if } k \geq 0, \\
-\xi^{-1} - \xi^{-3} - \cdots - \xi^{-2k+1}, & \text{if } k < 0.
\end{cases}
\end{align*}$$

So, $[k]_\xi = -[-k]_\xi^{-1}$ and if $\xi^2 \neq 1$ then $[k] = (\xi^{2k} - 1)/(\xi - \xi^{-1})$.

The quantum characteristic of $\xi$ is the smallest positive integer $e$ such that $[e]_\xi = 0$, where we set $e = \infty$ if $[k]_\xi \neq 0$ for all $k \in \mathbb{N}$. Notice that $\xi$ and $\xi^{-1}$ have the same quantum characteristic and if $\xi = 1$ then the quantum characteristic of $\xi$ is the characteristic of $K$.

Finally fix an integer $\ell > 0$ and a multicharge $\kappa = (\kappa_1, \ldots, \kappa_\ell) \in \mathbb{Z}^\ell$.

3.2. Definition (Cyclotomic Hecke algebras of type $A$ [5, 30]). The cyclotomic Hecke algebra of type $A$ with Hecke parameter $\xi$ and multicharge $\kappa$ is the unital associative $K$-algebra $H_n = H_n^\kappa(\xi)$ with generators $T_1, \ldots, T_n$, $L_1, \ldots, L_n$ and relations

$$\prod_{t=1}^{\ell} (L_t - [\kappa_t]) = 0,$$

$$(T_r - \xi)(T_r + \xi^{-1}) = 0,$$

$L_tL_r = L_rL_t$, \quad $T_tT_s = T_sT_t$ if $|r-s| > 1$,

$T_tT_{s+1}T_s = T_{s+1}T_sT_{s+1}$, \quad $T_tL_r = L_tT_r$ if $t \neq r, r + 1$,

$L_{r+1} = T_rL_tT_r + T_t$, \quad \text{where } 1 \leq r < n, 1 \leq s < n - 1 \text{ and } 1 \leq t \leq n.$

These algebras are almost the same as the Arik-Koike algebras introduced in [5] except that the presentation of Definition 3.2 changes the algebras when $\xi^2 = 1$. This allows the so-called degenerate ($\xi^2 = 1$) and non-degenerate ($\xi^2 \neq 1$) cases, to be treated simultaneously. See [30, §2] for more details.

3.3. Remark. Definition 3.2 is a renormalisation of the presentation of the cyclotomic Hecke algebras given in [30]. Explicitly, if $T_r$ and $L_s$ are the generators of the algebra $H_n(\xi^2, \kappa)$ given in [30, Definition 2.2] then $T_r = \xi^{-1}T_r$ and $L_s = \xi^{-1}L_s$, for $1 \leq r < n$ and $1 \leq s \leq n$. (In the notation of [30], the cyclotomic parameters of $H_n(\xi^2, \kappa)$ are $Q_r = \xi^{2n r}$, for $1 \leq r \leq \ell$.)

If $w \in S_n$ has reduced expression $w = s_{r_1} \cdots s_{r_k}$, so that $k = \ell(w)$, then set $T_w = T_{r_1} \cdots T_{r_k}$. Then $T_w$ depends only on $w$ and $\{L_{a_1} \cdots L_{a_{\ell}} T_w \mid 0 \leq a_\ell < \ell \text{ and } w \in S_n\}$ is a $K$-basis of $H_n$ by
the argument of [5]. Let $*$ be the unique anti-isomorphism of $H_n$ that fixes each of the generators. Then $T^*_w = T_{w^{-1}}$ and $L^*_m = L_m$, for all $w \in S_n$ and $1 \leq m \leq n$.

3.4. **Theorem** (Malle-Mathas [39] and Brundan [7]). The algebra $H_n$ is a symmetric algebra with non-degenerate trace form $\tau$. In particular, $H_n$ is a self-injective algebra.

The paper [39] proves this result when $\xi^2 \neq 1$ whereas [7] treats the case when $\xi^2 = 1$. In both cases the trace form $\tau$ is described explicitly, however, dual bases are known only when $\ell = 1, 2$.

A partition of an integer $m$ is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ of non-negative integers such that $|\lambda| = \sum \lambda_i = m$. A multipartition, or $\ell$-partition, of $n$ is an ordered $\ell$-tuple $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of partitions such that $|\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| = n$. The diagram of $\lambda$ is the set $|\lambda| = \{(k, r, c) \mid r \geq 1, 1 \leq c \leq \lambda^{(k)}_l \text{ and } 1 \leq k \leq \ell\}$.

A multipartition is uniquely determined by its diagram. A node is any element of the diagram of some multipartition.

Let $\mathcal{P}_n$ be the set of multipartitions of $n$. If $\lambda, \mu \in \mathcal{P}_n$ then $\lambda$ dominates $\mu$, written $\lambda \trianglerighteq \mu$, if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{i=1}^{k} \lambda^{(s)}_i \geq \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{i=1}^{k} \mu^{(s)}_i, \quad \text{for all } 1 \leq s \leq \ell \text{ and } k \geq 1.$$  

Dominance defines a partial order on $\mathcal{P}_n$.

If $X$ is a set then an $X$-valued $\lambda$-tableau is a function $T: [\lambda] \rightarrow X$. If $T$ is a $\lambda$-tableau write $\text{Shape}(T) = \lambda$. For convenience we identify $T = (T^{(1)}, \ldots, T^{(\ell)})$ with a labeling of the diagram $|\lambda|$ by elements of $X$ in the obvious way. Thus, we can talk of the components, rows and columns of $T$.

A standard $\lambda$-tableau is a map $t: |\lambda| \rightarrow \{1, 2, \ldots, n\}$ such that for $s = 1, \ldots, \ell$ the entries in each row of $t^{(s)}$ increase from left to right and the entries in each column of $t^{(s)}$ increase from top to bottom. Let $\text{Std}(\lambda)$ be the set of standard $\lambda$-tableaux and set

$$\text{Std}(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda) \quad \text{and} \quad \text{Std}^2(\mathcal{P}_n) = \bigcup_{\lambda \in \mathcal{P}_n} \text{Std}(\lambda) \times \text{Std}(\lambda).$$

If $t \in \text{Std}(\mathcal{P}_n)$ and $1 \leq m \leq n$ let $\text{comp}_m(t) = k$ if $m$ appears in the $k$-th component of $t$.

The conjugate of $\lambda$ is the multipartition $\lambda'$ whose diagram is obtained from $|\lambda|$ by reversing the order of the components and then swapping rows and columns. Thus, $|\lambda'| = \{(k, r, c) \mid (\ell - k + 1, c, r) \in |\lambda|\}$. The conjugate of $t$ is the $\lambda'$-tableau $t'$ obtained by reversing the order of the components in $t$ and then transposing the tableau in each component (that is, swapping rows and columns).

Let $\lambda^\Delta$ be the standard $\lambda$-tableau such that the numbers $1, 2, \ldots, n$ are entered in order from left to right along the rows of $t^{(1)}$, and then $t^{(2)}$, $\ldots$, $t^{(\ell)}$. Similarly, let $t^\Delta$ be the standard $\lambda$-tableau with the numbers $1, 2, \ldots, n$ entered in order down the columns of $t^{(1), \ldots , t^{(\ell)}}$. Then $(t^{(1)})' = t^\Delta$, for all $\lambda \in \mathcal{P}_n$.

If $t$ is a standard $\lambda$-tableau let $d(t), d'(t) \in S_n$ be the unique permutations such that $t = t^\Delta d(t)$ and $t = t^\Delta d'(t)$. Let $w_\lambda = d(t\lambda)$. It is easy to see that $d'(t) = d(t)$ and $w_\lambda = d(t) d(t)^{-1}$, for all $t \in \text{Std}(\lambda)$.

If $t$ is a tableau with entries in $\{1, 2, \ldots, n\}$ let $\text{Row}(t)$ and $\text{Col}(t)$ be the subgroups of $S_n$ that stabilise the rows and columns of $t$, respectively. If $\lambda \in \mathcal{P}_n$ let $S_\lambda = S_{\lambda^{(1)}} \times \cdots \times S_{\lambda^{(\ell)}}$ be the corresponding Young subgroup, or parabolic subgroup, of $S_n$. In particular, $\text{Row}(\lambda^\Delta) = S_\lambda$ and $\text{Col}(\lambda^\Delta) = S_{\lambda'}$, where $S_\lambda$ and $S_{\lambda'}$ are the natural subgroups of $S_n$ associated with the multipartitions $\lambda$ and $\lambda'$.

For $\lambda \in \mathcal{P}_n$ define $m_\lambda = u^\lambda x^\lambda$, where

$$u^\lambda = \prod_{m=1}^{n} \prod_{i=\text{comp}_m(t^{(i)})+1}^{\ell} (L_m - [n_i]) \quad \text{and} \quad x^\lambda = \sum_{w \in \text{Row}(\lambda^\Delta)} \xi^{(w)} T_w.$$  

Let $M^{\lambda} = m_\lambda H_n$ and set $M = \bigoplus_{\lambda \in \mathcal{P}_n} M^{\lambda}$. Below we define the cyclotomic Schur algebra to be the endomorphism algebra of $M$.

Fix $\lambda \in \mathcal{P}_n$ and define $m_{st} = T^*_d(s) m_\lambda T_d(t)$, for $s, t \in \text{Std}(\lambda)$. By [18, Theorem 3.26],

$$\{m_{st} \mid s, t \in \text{Std}(\mu) \text{ and } \lambda \in \mathcal{P}_n\}$$

is a cellular basis of $H_n$, where $\mathcal{P}_n$ is ordered by dominance. Consequently, if $H_2^{\lambda, \mu}$ is the $K$-subspace of $H_n$ spanned by $\{m_{st} \mid s, t \in \text{Std}(\mu) \text{ for some } \mu \in \mathcal{P}_n \text{ with } \mu \trianglerighteq \lambda\}$, then $H_2^{\lambda, \mu}$ is a two-sided ideal of $H_n$.

Invoking the theory of cellular algebras [24, 40], for $\lambda \in \mathcal{P}_n$ there is a right cell module $S^\lambda$, called a Specht module. By definition, $S^\lambda \cong m_\lambda H_n / (m_\lambda H_n \cap H_2^{\lambda, \mu})$. Set $m_t = m_{t^\Delta} + H_2^{\lambda, \mu}$, for
t ∈ Std(λ). Then \( \{ m_t | t ∈ \text{Std}(\lambda) \} \) is a \( K \)-basis for \( S^\lambda \). The Specht module comes equipped with an associative bilinear form \( \langle , \rangle : S^\lambda × S^\lambda → K \) that is determined by
\[
\langle m_s m_t u, m_s m_t u \rangle = \langle m_s, m_t \rangle m_u,
\]
for all \( s, t, u ∈ \text{Std}(\lambda) \).

Let \( \text{rad} S^\lambda = \{ x ∈ S^\lambda | \langle x, y \rangle = 0 \text{ for all } y ∈ S^\lambda \} \). Then \( \text{rad} S^\lambda \) is an \( H_\nu \)-submodule of \( S^\lambda \). We define \( D^\lambda = S^\lambda / \text{rad} S^\lambda \).

We need a parallel construction for the dual Specht modules. For \( \lambda ∈ \mathcal{P}_n \) set \( n_{\lambda, λ} = u_{\lambda, λ} \), where
\[
u_{\lambda, λ} = \prod_{m=1}^{n} \prod_{t=1}^{\text{comp}_u(t_x)-1} (L_m - [κ_t]) \text{ and } \lambda_{\lambda, λ} = \sum_{w ∈ \text{Col}(t_x)} (-ξ)^{-\ell(w)} T_w.
\]

For \( s, t ∈ \text{Std}(\lambda) \) set \( n_{st} = T_{d(\lambda)} n_{\lambda, \lambda} T_{d(t_x)} \). By \([21, (2.7)]\), \( \{ n_{st} | s, t ∈ \text{Std}(\lambda) \text{ and } \lambda ∈ \mathcal{P}_n \} \) is a second cellular basis of \( H_\nu \), where \( \mathcal{P}_n \) is ordered by reverse dominance.

For each multipartition \( \lambda ∈ \mathcal{P}_n \), the theory of cellular algebras gives us a dual Specht module \( S_\lambda = n_{\lambda, \lambda} H_\nu / (n_{\lambda, \lambda} H_\nu ∩ H_\nu^n) \). Here, \( H_\nu^n \) is the two-sided ideal of \( H_\nu \) with basis \( \{ n_{st} | s, t ∈ \text{Std}(\mu) \text{ for some } \mu ∈ \mathcal{P}_n \text{ with } \lambda ⊳ \mu \} \).

Much as before, set \( D_\lambda = S_\lambda / \text{rad} S_\lambda \).

3.6. Remark. The labelling that we are using for the \( n \)-basis is conjugate to that used in \([21, 41, 42]\) so \( n_{s, s} \) should be replaced with \( n_\lambda \), and \( n_{st} \) with \( n_{s, t} \), when comparing with these papers. Our notation reflects that fact that the elements \( m_\lambda \) and \( n_\lambda \) come from looking at the row and column stabilizers of \( \lambda \) and \( t_\lambda \), respectively. Similarly, our labelling of the dual Specht modules follows the same convention, which is in agreement with the papers \([29, 34]\). In \([41]\) the module \( S_\lambda \) is written as \( S_\lambda \).

Implicitly, the definitions of the elements \( m_\lambda, n_\lambda, m_{st} \) and \( n_{st} \) all depend upon the choice of Hecke parameter \( ξ \) and the multicharge \( κ \). This remark will be important below when we vary these parameters. For future use we summarise the properties of these modules that follow directly from the general theory of cellular algebras.

3.7. Theorem (\([19, 24, 41]\)). Suppose that \( K \) is a field and \( n ≥ 0 \).

a) The \( \{ D_\mu | \mu ∈ \mathcal{P}_n \text{ and } D_\mu ≠ 0 \} \) is a complete set of pairwise non-isomorphic irreducible \( H_\nu \)-modules. Moreover, \( (D_\mu)^* ≅ D_\mu \).

b) The \( \{ D_\mu | \mu ∈ \mathcal{P}_n \text{ and } D_\mu ≠ 0 \} \) is a complete set of pairwise non-isomorphic irreducible \( H_\nu \)-modules. Moreover, \( (D_\mu)^* ≅ D_\mu \).

For the main results in this paper we need to describe the Specht modules and dual Specht modules as submodules of \( H_\nu \), which is already known, and we need to determine the isomorphisms between the two sets of simple modules given by Theorem 3.7.

We extend the dominance ordering to the set of all standard tableaux by defining \( s ⊃ t \)
\[
\text{Shape}(s_{1,m}) ⊃ \text{Shape}(t_{1,m}),
\]
for \( 1 ≤ m ≤ n \).

As remarked above, if \( ℓ > 2 \) then no pairs of dual bases for \( H_\nu \) are known. The following fundamental result implies that the two bases \( \{ n_{st} \} \) and \( \{ n_{st} \} \) are dual bases of \( H_\nu \) "modulo higher terms".

3.8. Proposition (Hu and Mathas [28, Corollary 2.10], Mathas [41, Theorem 5.5]). Suppose that \( (s, t, (u, v) ∈ \text{Std}^2(\mathcal{P}_n) \) are pairs of tableaux of the same shape. Then \( n_{st} n_{tv} ≠ 0 \) and \( n_{st} n_{uv} ≠ 0 \) only if \( u ≥ t \). Similarly, \( n_{uv} n_{st} ≠ 0 \) and \( n_{uv} n_{st} ≠ 0 \) only if \( v ≥ s \).

For \( \lambda ∈ \mathcal{P}_n \) set \( z_\lambda = n_{\lambda, \lambda} T_{w_\lambda} m_\lambda \) and \( z_\lambda = m_\lambda T_{w_\lambda} n_\lambda \). Observe that \( z_\lambda = z_\lambda^* \) since \( w^{-1} = w_\lambda \). Moreover, \( z_\lambda = n_{\lambda, \lambda} n_{\lambda, \lambda} = n_{\lambda, \lambda} m_{\lambda, \lambda} ≠ 0 \) and \( z_\lambda = m_{\lambda, \lambda} n_{\lambda, \lambda} = m_{\lambda, \lambda} n_{\lambda, \lambda} ≠ 0 \) by Proposition 3.8.

3.9. Lemma (Du and Rui [21, Theorem 2.9]). Suppose that \( \lambda ∈ \mathcal{P}_n \). Then \( S_\lambda ≅ z_\lambda H_\nu \) and \( S_\lambda ≅ z_\lambda H_\nu \) as right \( H_\nu \)-modules.

To prove this observe that, because \( z_\lambda = n_{\lambda, \lambda} T_{w_\lambda} m_{\lambda, \lambda} \), there is a well-defined \( H_\nu \)-module homomorphism \( α : S^\lambda → z_\lambda H_\nu \) given by \( α(m_\lambda h) = z_\lambda h \), for \( h ∈ H_\nu \). To complete the proof it remains to check that \( \{ z_\lambda T_{d(t_x)} | t ∈ \text{Std}(\lambda) \} \) is a basis of \( z_\lambda H_\nu \). See \([21, \text{Theorem 2.9]} \) or \([42, \text{Proposition 3.13]} \) for details. The proof that \( z_\lambda H_\nu ≅ S_\lambda \) is similar.

As their names suggest, the Specht modules and dual Specht modules are dual to each other. The proof of this requires the trace form \( τ \) from Theorem 3.4 and a strengthening of Proposition 3.8.
3.10. **Theorem** (Mathas [42, Theorem 5.9]). Suppose that $\lambda \in \mathcal{P}_n$. Then $\tau(z^\lambda T_{w_\lambda}) = \tau(z^\lambda T_{w_\lambda^*}) \neq 0$.

We can now prove that $S^\lambda$ and $S_\lambda$ are dual to each other.

3.11. **Corollary** (Mathas [41, Corollary 5.7]). Let $\lambda \in \mathcal{P}_n$. Then $S^\lambda \cong (S_\lambda)^*$ and $S_\lambda \cong (S^\lambda)^*$.

**Proof.** This result is proved in [41] but we give a proof of the isomorphism $S^\lambda \cong (S_\lambda)^*$ here. We need the details below. For $s \in \text{Std}(\lambda)$ let $\theta_s \in (S^\lambda)^*$ be the linear map determined by $\theta_s(m_\eta) = \tau(n_{1\lambda} m_{1\lambda} T_{w_\lambda})$, for all $t \in \text{Std}(\lambda)$. Now define a linear map $\theta : S_\lambda \to (S^\lambda)^*$ by $\theta(n_\eta) = \theta_s$, for all $s \in \text{Std}(\lambda)$. Then $\theta$ is a vector space isomorphism by Proposition 3.8 and Theorem 3.10. For $h \in \mathcal{H}_n$ write $n_{\eta} h = \sum r_{\eta} n_{v_\eta}$, for $r_{\eta} \in K$. Fix $t \in \text{Std}(\lambda)$. Using Proposition 3.8,

$$\theta(n_\eta h)(m_\eta) = \sum_{v \in \text{Std}(\lambda)} r_{\eta} \theta_v(m_\eta) = \sum_{v \in \text{Std}(\lambda)} r_{\eta} \tau(n_{1\lambda} m_{1\lambda} T_{w_\lambda})$$

$$= \tau(n_{1\lambda} m_{1\lambda} T_{w_\lambda}) = \theta_s(m_\eta h^*) = (\theta(n_\eta h)(m_\eta)).$$

Hence, $\theta$ is an $H_\lambda$-module homomorphism, completing the proof. \qed

In Chapter 5 below we need an analogue of Corollary 3.11 relating the simple modules $D^\mu$ and $D_\nu$, for $\mu, \nu \in \mathcal{P}_n$. To establish this we use the following characterisation of the simple $H_n$-modules together with the “dual” algebras $H_\lambda$ that are introduced below. This description of the simple $H_n$-modules as submodules of $H_\lambda$ generalises a remark made by James [31, p. 41] for the symmetric groups.

3.12. **Lemma.** Suppose that $\mu \in \mathcal{P}_n$.

a) The simple module $D^\mu \neq 0$ if and only if $z^\mu H_n m_\mu \neq 0$.

b) The simple module $D_\mu \neq 0$ if and only if $z_\mu H_n m_\mu \neq 0$.

Moreover, if $D^\mu \neq 0$ then $D_\mu \cong z_\mu T_{w_\mu} m_\mu H_n$ and if $D_\mu \neq 0$ then $D_\mu = z_\mu T_{w_\mu} m_\mu H_n$.

**Proof.** We consider only the claims for $D^\mu$. For those for $D_\mu$ can be proved in the same way. As remarked after Lemma 3.9, the map $S^\mu \to z^\mu H_n$ determined by $m_\mu \mapsto T_{d(\mu)}$, for $t \in \text{Std}(\mu)$, is an isomorphism. If $s, t \in \text{Std}(\mu)$ then using the definitions

$$\langle m_\eta, m_\zeta \rangle = \langle m_\eta, m_\zeta \rangle_{w_\mu} = m_\mu w_{\mu} m_{w_\mu} w_{\mu} = z^\mu T_{d(\mu)} T_{d(\mu)} m_{w_\mu},$$

where the second equality follows from (3.5) because $n_{w_\mu} H_n^\mu = 0$ by Proposition 3.8. Hence, $D^\mu \neq 0$ if and only if $z^\mu H_n m_\mu \neq 0$, proving (a).

For the second claim, suppose that $D^\mu \neq 0$. We need to prove that $D^\mu \cong z_\mu T_{w_\mu} m_\mu H_n$. Combining Lemma 3.9 and Corollary 3.11, there are $H_n$-module homomorphisms

$$S^\mu \xrightarrow{\alpha} z^\mu H_n \xrightarrow{\beta} z_\mu H_n \xrightarrow{\theta} (S^\mu)^*,$$

where $\alpha(n_\eta) = z^\mu T_{d(\mu)}$, $\beta(x) = m_\mu T_{w_\mu} x$ and $\theta(z_\mu T_{d(\mu)}) = \theta_s$, for $s \in \text{Std}(\mu)$ and $x \in z_\mu H_n$, where $\theta_s$ is defined in the proof of Corollary 3.11. Moreover, $\alpha$ and $\theta$ are both isomorphisms. Let $\eta = \theta \circ \beta \circ \alpha$. We claim that, up to a non-zero scalar, $\eta$ is the map $S^\mu \to (S^\mu)^*$ induced by the inner product $\langle , \rangle$ on $S^\mu$. To see this fix $s \in \text{Std}(\mu)$ and write

$$n_{\mu} T_{w_\mu} m_\mu T_{d(\mu)} = n_{\mu} m_{\mu} = \sum_{u, \mu} c_{u, \mu} m_{\mu}, \quad \text{for } c_{u, \mu} \in K.$$

As $\{m_{\mu}\}$ is a cellular basis, with $\mathcal{P}_n$ ordered by reverse dominance, it follows that $c_{u, \mu} \neq 0$ only if $\mu \cong \text{Shape}(u) = \text{Shape}(v)$ with equality only if $u = t_\mu$. If $v \in \text{Std}(\mu)$ set $c_{v, \mu} = c_{u, v}$. Using Proposition 3.8 for the third equality,

$$\vartheta(m_\eta) = \theta(z_\mu T_{w_\mu} m_\mu T_{d(\mu)}) = \theta \left( \sum_{u, \mu} c_{u, \mu} m_{w_\mu} m_{t_\mu} n_{\mu} \right) = \theta \left( \sum_{v \in \text{Std}(\mu)} c_{v, \mu} m_{t_\mu} m_{u_\mu} n_{\mu} \right) = \sum_{v \in \text{Std}(\mu)} c_{v, \mu} \vartheta_v.$$

Therefore, if $t \in \text{Std}(\mu)$ then, using Proposition 3.8 and (3.5) again,

$$\vartheta(m_\eta)(m_\mu) = \sum_{v \in \text{Std}(\mu)} c_{v, \mu} \vartheta_v(m_\mu) = \sum_{v \in \text{Std}(\mu)} c_{v, \mu} \tau(m_{w_\mu} m_{t_\mu} T_{w_\mu}) = \tau \left( \sum_{v \in \text{Std}(\mu)} c_{v, \mu} m_{w_\mu} m_{t_\mu} m_{w_\mu} \right)$$

$$= \tau \left( n_{\mu} T_{w_\mu} m_{w_\mu} m_{w_\mu} T_{w_\mu} \right) = \langle m_\eta, m_\zeta \rangle = \tau \left( n_{\mu} T_{w_\mu} m_{w_\mu} T_{w_\mu} \right).$$

By Theorem 3.10, $\tau(z^\mu T_{w_\mu}) = \tau(z_\mu T_{w_\mu})$ is a non-zero element of $K$. Hence, up to an invertible scalar, the map $\vartheta$ coincides with the natural map induced by the bilinear form on $S^\mu$. In particular, if $D^\mu \neq 0$ then $D^\mu \cong \text{im} \vartheta \cong \text{im}(\beta \circ \alpha)$. By definition, $\text{im}(\beta \circ \alpha) = z_\mu T_{w_\mu} m_\mu H_n$ so this completes the proof. \qed
To identify the isomorphic simple modules $D^\mu$ and $D_\nu$ we need to shift to a “dual” Hecke algebra.

As we make precise in Theorem 4.7 below, this is a shadow of Ringel duality in the Hecke algebra world.

Let $\mathcal{H}_n' = \mathcal{H}_n'^{\xi^{-1}}$ be the cyclotomic Hecke algebra with Hecke parameter $\xi^{-1}$ and multicharge $\kappa' = (-\kappa_1, \ldots, -\kappa_t)$. To help distinguish between the elements of the algebras $\mathcal{H}_n$ and $\mathcal{H}_n'$ let $T_1', \ldots, T_n', L_1', \ldots, L_n'$ be the generators of $\mathcal{H}_n'$. More generally, we decorate all elements of $\mathcal{H}_n'$ with an appropriately placed ‘. For example, $m'_X = u^*_\lambda x^*_\lambda$ and $n'_X = u^*_\lambda x^*_\lambda$ both belong to $\mathcal{H}_n'$. Similarly, there are cellular bases $\{m'_i\}$ and $\{n'_i\}$ that give rise to Specht modules $S^\lambda$ and dual Specht modules $S'_\lambda$, for $\lambda \in \mathcal{P}_n$. Let $D^\lambda = S^\lambda / \text{rad } S^\lambda$ and $D'_\lambda = S'_\lambda / \text{rad } S'_\lambda$.

The next result, which does not appear to be in the literature, generalises the sign automorphism on the group algebra of a symmetric group. We leave the proof as an exercise to the reader because it follows by simply inspecting the relations in the two algebras $\mathcal{H}_n$ and $\mathcal{H}_n'$. (Recall from after (3.1) that $[k]_q = [-k]_{q^{-1}}$, for all $k \in \mathbb{Z}$.)

3.13. Lemma. There is a unique algebra isomorphism $\#: \mathcal{H}_n' \to \mathcal{H}_n$ such that

$$(T'^r)'^# = -T_r \quad \text{and} \quad (L'^s)'^# = -L_s, \quad \text{for } 1 \leq r < n \text{ and } 1 \leq s \leq n.$$  

Moreover, $m'_\lambda = \pm (n'_\lambda)^#$, $n'_\lambda = \pm (m'_\lambda)^#$, $m_{st} = \pm (n'_\lambda)^#$, $n_{st} = \pm (m'_\lambda)^#$, for all $\lambda \in \mathcal{P}_n$ and all $s, t \in \text{Std}(\lambda)$.

The isomorphism $\#: \mathcal{H}_n' \to \mathcal{H}_n$ induces an equivalence of categories $E_n^\# : \text{mod-}\mathcal{H}_n' \to \text{mod-}\mathcal{H}_n$. To describe the effect of $E_n^\#$ on the simple $\mathcal{H}_n'$-modules define

$$K_n = \{ \mu \in \mathcal{P}_n | D^\mu \neq 0 \} \quad \text{and} \quad K_n' = \{ \mu \in \mathcal{P}_n | D'^\mu \neq 0 \}.$$  

Recall from after (3.1) that $\xi$ and $\xi^{-1}$ both have quantum characteristic $e$. By the results in [3, 11], $\lambda \in K_n$ if and only if $\lambda$ is a Kleshchev, or restricted, multipartition with respect to $(e, \kappa)$ and $\lambda \in K_n'$ if and only if $\lambda$ is Kleshchev with respect to $(e', \kappa')$. The Kleshchev multipartitions index the crystal graphs of certain integrable highest weight modules, so their definition is explicit but recursive [3]. The crystal graph combinatorics show that there is a crystal isomorphism, known as the Mullineux map,

$$m : K_n \rightarrow K_n'; \mu \mapsto m(\mu).$$

(Using the set $I$ defined in Chapter 6, if $i \in I^n$ labels a path from $0 = (0) \ldots (0)$ to $\mu$ in the crystal graph $\bigcup_{m \geq 0} K_m$ then $-i$ labels a path from $0$ to $m(\mu)$ in the crystal graph $\bigcup_{m \geq 0} K_m$. This property determines the map $m$ uniquely. See [11, Theorem 4.12] or [4] for more details.) The next result connects the Mullineux map with the representation theory of $\mathcal{H}_n$.

3.14. Proposition (cf. [11, 34, 43]). Suppose that $\lambda \in \mathcal{P}_n$ and $\mu \in K_n$. Then

$$S^\lambda_n \cong E_n^\#(S^\lambda), \quad S'_n \cong E_n^\#(S^\lambda) \quad \text{and} \quad D^m(\mu) \cong E_n^\#(D^\mu)$$

as right $\mathcal{H}_n'$-modules.

Proof. For graded Specht modules the first two isomorphisms are proved in [34, Theorem 8.5], although it is not completely clear that the isomorphism in Lemma 3.13 agrees with the homogeneous sign isomorphism considered in [34]. Fortunately, these two isomorphisms follow directly from the definitions because $m'_{\lambda^T} = \pm (n_{\lambda^T})^#$ and $n'_{\lambda^T} = \pm (m_{\lambda^T})^#$ by Lemma 3.13. The isomorphism $S^\lambda_n \cong E_n^\#(S^\lambda)$, together with the modular branching rules [4, 26] and a standard argument due to Kleshchev, now imply that $D^m(\mu) \cong E_n^\#(D^\mu)$. For the corresponding result for the graded simple modules see [43, Theorem 3.6.6] and [11, (3.53)].

We can now prove the promised comparison results for the simple $\mathcal{H}_n$-modules.

3.15. Corollary. Suppose that $\mu \in \mathcal{P}_n$. Then $D^\mu \neq 0$ if and only if $\mu' \in K_n'$. Moreover, if $\mu \in K_n$ then $D^\mu \cong D_{m(\mu)}$ as $\mathcal{H}_n$-modules.

Proof. By Lemma 3.12(b) and Lemma 3.13, $D^\mu \neq 0$ if and only if $D'^\mu \neq 0$, so that $D^\mu \neq 0$ if and only if $\mu' \in K_n'$. Hence, using Proposition 3.14 twice, if $\mu \in K_n$ then

$$E_n^\#(D_{m(\mu)}) \cong E_n^\#(\text{hd } S_{m(\mu)}) \cong \text{hd } E_n^\#(S_{m(\mu)}) \cong \text{hd } E_n^\#(S_{m(\mu)}) \cong D_{m(\mu)} \cong E_n^\#(D^\mu).$$

Hence, $D_{m(\mu)} \cong D^\mu$ as required.

3.16. Corollary. Suppose that $\mu \in \mathcal{P}_n$. Then $D'_\mu \neq 0$ if and only if $\mu' \in K_n$. Moreover, if $\mu' \in K_n$ then $D'_\mu \cong D'^{m(\mu)}$ as $\mathcal{H}_n'$-modules.
Proof. By Lemma 3.12(a) and Lemma 3.13, $\text{D}_\mu' \neq 0$ if and only if $\text{D}_\mu'' \neq 0$, so $\text{D}_\mu' \neq 0$ if and only if $\mu' \in \mathcal{K}_n$. Now suppose that $\mu' \in \mathcal{K}_n$. Then

$$\text{D}_\mu' = \text{hd} S_\mu' \cong \text{hd} E_n^\#(S\mu') = E_n^\#(\text{hd}S\mu') \cong E_n^\#(\text{D}_\mu' \cong \text{D}_m(\mu'),$$

where the second isomorphism follows by Proposition 3.14.

3.17. Corollary. Suppose that $\mu \in \mathcal{K}_n$. Then $\text{soc} S_\mu \cong \text{D}_\mu \cong \text{soc} S_m(\mu')$.

Proof. If $\mu \in \mathcal{K}_n$ then $\text{D}_\mu \cong S\mu$. Moreover, $\text{hd} S_{m(\mu')} \cong D_{m(\mu')} \cong D_\mu$ by Corollary 3.15. Now take duals using Corollary 3.11 (and Theorem 3.7).

4. Cyclotomic Schur algebras

We are now ready to introduce the cyclotomic Schur algebras, which are one of the main objects of study in this paper. Our first goal is to describe which simple modules appear in the socles of the Weyl modules, tilting modules and the projective indecomposable modules and, as an application describe the socles of the Young modules. In the last section we saw that the Specht modules and dual Specht modules came from two different cellular bases for $\mathcal{H}_n$, and we needed to introduce the algebra $\mathcal{H}_n$ to understand the difference in the labelling of the simple modules coming from these two bases. Similarly, in this section we introduce two variations of the cyclotomic Schur algebras which we will use to translate information about the socle of a tilting module to the socle of a projective module.

As a prelude to defining the Schur algebras, for $\lambda \in \mathcal{P}_n$ let

$$M^\lambda = m_\lambda \mathcal{H}_n, \quad N^\lambda = n_\lambda \mathcal{H}_n, \quad M^\lambda_n = m_\lambda \mathcal{H}_n', \quad N^\lambda_n = n_\lambda \mathcal{H}_n',$$

and set

$$M = \bigoplus_{\lambda \in \mathcal{P}_n} M^\lambda, \quad N = \bigoplus_{\lambda \in \mathcal{P}_n} N^\lambda, \quad M_i = \bigoplus_{\lambda \in \mathcal{P}_n} M^\lambda_i, \quad N_i = \bigoplus_{\lambda \in \mathcal{P}_n} N^\lambda_i.$$

Then $M^\lambda, N^\lambda, M$ and $N$ are $\mathcal{H}_n$-modules and $M^\lambda_n, N^\lambda_n, M_i$ and $N_i$ are $\mathcal{H}_n'$-modules.

4.1. Proposition. Suppose that $n \geq 0$. Then $(\mathcal{H}_n, M)$, $(\mathcal{H}_n, N)$, $(\mathcal{H}_n', M_i)$ and $(\mathcal{H}_n', N_i)$ are self-dual Schur pairs.

Proof. By Theorem 3.4, $\mathcal{H}_n$ and $\mathcal{H}'_n$ are both symmetric algebras and so, in particular, they are self-injective. Moreover, since $\mathcal{H}_n \cong M^{[0][0]...[0]}(\cong N^{[0][0]...[0]})$ as right $\mathcal{H}_n$-modules, $M$ and $N$ are both faithful $\mathcal{H}_n$-modules. Therefore, $(\mathcal{H}_n, M)$ and $(\mathcal{H}_n, N)$ are both Schur pairs. Similarly, $(\mathcal{H}_n', M')$ and $(\mathcal{H}_n', N')$ are both Schur pairs. It remains to show that these Schur pairs satisfy the conditions of Definition 2.12.

As algebras $\mathcal{H}_n$ and $\mathcal{H}'_n$ are both cellular. They have an anti-isomorphism, which is the unique anti-isomorphism that fixes each of the generators of $\mathcal{H}_n$ and of $\mathcal{H}_n'$, respectively. Moreover, by Theorem 3.7, the simple $\mathcal{H}_n$-modules and the simple $\mathcal{H}_n'$-modules are self-dual with respect to these involutions. To show that $(\mathcal{H}_n, M)$, $(\mathcal{H}_n, N)$, $(\mathcal{H}_n', M_i)$ and $(\mathcal{H}_n', N_i)$ are self-dual Schur pairs it is enough to show that each of the modules $M^\lambda$, $N^\lambda$, $M^\lambda_n$ and $N^\lambda_n$ is self-dual, for $\lambda \in \mathcal{P}_n$. This is proved in [41, Proposition 5.13]. (The careful reader will notice that [41, Proposition 5.13] assumes that the cyclotomic parameters $Q_1, \ldots, Q_\ell$ are invertible. As explained in [43, §1.1], because of Definition 3.2 this condition translates into the vacuous requirement that $|\kappa_r|(|\xi - \xi^{-1}) + 1 = \xi^{2\kappa_r}$ in $K$, for $1 \leq r \leq \ell$. Hence, the requirement from [41] that $Q_1, \ldots, Q_\ell$ be invertible is satisfied.)

Consequently, all of the results in Chapter 2 apply to the endomorphism algebras $S_\lambda(M)$, where $(A, M)$ is one of the Schur pairs given in Proposition 4.1. Notice that, by definition, $\mathcal{H}_n$ is a direct summand of $M$ and $N$ since $M^{[0][0]...[0]}(\cong N^{[0][0]...[0]})$ as right $\mathcal{H}_n$-modules. Similarly, $\mathcal{H}_n'$ is a direct summand of $M_i$ and $N_i$.

4.2. Definition (Cyclotomic Schur algebras [9, 18, 41]). The **cyclotomic Schur algebra** and the **twisted cyclotomic Schur algebra** of $\mathcal{H}_n$ are the endomorphism algebras

$$S_n(M) = \text{End}_{\mathcal{H}_n}(M) \quad \text{and} \quad S'_n(M_i) = \text{End}_{\mathcal{H}_n}(M_i).$$

The cyclotomic Schur algebras include, as special cases, the classical Schur algebras studied by Green [25] and the $q$-Schur algebras introduced by Dipper and James [16]. Just as our definition of $\mathcal{H}_n$ differs slightly from the definition given by Ariki and Koike [5], the algebra $S_n(M)$ is isomorphic to a cyclotomic $\xi$-Schur algebra of [18] when $\xi^2 \neq 1$ and it is isomorphic a **degenerate** cyclotomic Schur algebra [9] when $\xi^2 = 1$. 


We are mainly interested in the cyclotomic Schur algebra $S_n = S_n(M)$, however, we need the algebras $S'_n = S'_n(M)$ to prove some of our results about $S_n$-modules. Given Proposition 4.1 it is also natural to consider the algebras $S_n(N) = \text{End}_{H_n}(N)$ and $S'_n(N) = \text{End}_{H'_n}(N)$. In fact, these algebras give no additional information because Proposition 3.14 readily implies the following.

4.3. Lemma. If $\lambda \in \mathcal{P}_n$ then $N^X_\lambda \cong E^\#_{(M^\lambda)}$ and $M^X_\lambda \cong E^\#_{(N^\lambda)}$ as right $H_n$-modules. Therefore, $S_n(M) \cong S'_n(N)$ and $S'_n(M) \cong S_n(N)$.

We leave the proof of Lemma 4.3 to the reader. In fact, as we recall below, $S_n = S_n(M)$ and $S'_n = S'_n(M)$ are quasi-hereditary algebras with weight posets $\mathcal{P}_n$ and $\mathcal{P}^{op}_n$, respectively. Similarly, $S_n(N)$ and $S'_n(N)$ are quasi-hereditary with weight posets $\mathcal{P}^{op}_n$ and $\mathcal{P}_n$, respectively. The isomorphisms $S_n(M) \cong S'_n(N)$ and $S_n(N) \cong S'_n(M)$ of Lemma 4.3 are both isomorphisms of quasi-hereditary algebras.

The algebras $S_n$ and $S'_n$ are finite dimensional and quasi-hereditary K-algebras with simple modules labelled by $\mathcal{P}_n$. To describe cellular bases of $S_n$ and $S'_n$ we need some more combinatorics.

Fix $\mu \in \mathcal{P}_n$ and $\lambda \in \mathcal{P}_n$. A $\lambda$-tableau of type $\mu$ is a map $T : [\lambda] \to \{(k, r) | 1 \leq k \leq \ell \text{ and } r \geq 1\}$ such that $\mu^{(k)} = \# \{\alpha \in [\lambda] | T(\alpha) = (k, r)\}$ for all $1 \leq k \leq \ell$ and $r \geq 1$. Order the pairs $(k, r)$ lexicographically. Following [18, Definition 4.4], a $\lambda$-tableau $T$ of type $\mu$ is row semistandard if

a) the entries in $T$ are weakly increasing along rows,

b) the entries in $T$ are strictly increasing down columns,

c) if $a = (k, r, c) \in [\lambda]$ then $T(\alpha) \geq (k, c)$.

Let $T_{row}(\lambda, \mu)$ be the set of row semistandard $\lambda$-tableaux of type $\mu$. There is a bijection $\text{Std}(\lambda) \to T_{row}(\lambda, \omega)$ given by replacing each entry $m$ in a tableau with $(\ell, m)$, where $\omega = (0 \ldots | 0 | 1 \ldots | 1)$.

Let $T_{col}(\lambda, \mu) = \{S' | S \in T_{row}(\lambda', \mu')\}$ be the set of column semistandard $\lambda$-tableaux of type $\mu$, where conjugation reverses the order of components, swaps rows and columns and replaces each entry $(k, r)$ with $(\ell - k + 1, r)$. (We need these tableaux in Chapter 7.) If $T \in T_{col}(\lambda, \mu)$ then the entries in $T$ weakly increase down columns, strictly increase along rows and if $(k, r, c) \in [\lambda]$ then $T(\alpha) \leq (k, c)$.

By construction, if $T \in T_{col}(\lambda, \mu)$ then $T$ is a tableau of type $\mu$.

Observe that $T_{row}(\lambda, \mu) \neq \emptyset$ only if $\lambda \trianglerighteq \mu$. Similarly, $T_{col}(\lambda, \mu) \neq \emptyset$ only if $\mu \trianglerighteq \lambda$. Let $T_{row}(\lambda) = \bigcup_{\mu} T_{row}(\lambda, \mu)$ and $T_{col}(\lambda) = \bigcup_{\mu} T_{col}(\lambda, \mu)$.

For $t \in \text{Std}(\lambda)$ let $row_t(t)$ be the tableau of type $\mu$ obtained from $t$ by replacing each entry $m$ in $t$ with $(k, r)$ if $m$ appears in row $r$ of the $k$th component of $t^\mu$ and define $col_t(t)$ similarly except that we use the column index of $m$ in $t^\mu$ instead. It is not hard to see that every semistandard tableau of type $\mu$ is equal to $row_t(t)$ and $col_t(t)$, for some $t \in \text{Std}(\lambda)$, although the converse is not true in general. Let $T^\lambda_{row} = \text{row}_{\lambda}(\lambda^\lambda)$ and $T^\lambda_{col} = \text{col}_{\lambda}(\lambda^\lambda)$, respectively, be the unique row and column semistandard $\lambda$-tableaux of type $\lambda$. As a quick exercise in these definitions, $T^{\lambda}_{row} = T^\lambda_{row}$. By [18, Theorem 4.14], $M^\mu = m_{\mu}H_n$ has $K$-basis \{m_{\mu} | S \in T_{row}(\lambda, \mu), t \in \text{Std}(\lambda), \lambda \in \mathcal{P}_n\}$, where

$$m_{\mu} = \sum_{t \in \text{Std}(\lambda)} \zeta^{(d(t))}m_{\mu}.\tag{4.4}$$

Extending this idea, following [18], if $S \in T_{row}(\lambda, \mu)$ and $T \in T_{row}(\lambda, \nu)$ define

$$m_{ST} = \sum_{t \in \text{Std}(\lambda)} \zeta^{(d(t))}m_{ST}.$$

Then $m_{ST} \in m_{\mu}H_n \cap H_n m_{\nu}$ by the remarks above. Therefore, the map $\varphi_{ST} : M^\nu \to M^\mu$ is defined by $\varphi_{ST}(m_{\nu}h) = m_{ST}h$, for all $h \in H_n$, is an $H_n$-module homomorphism. We consider $\varphi_{ST}$ as an element of $S_n$. By [18, Theorem 6.6], $(\varphi_{ST} | S \in T_{row}(\lambda, \mu)$ and $T \in T_{row}(\lambda, \nu)$ for $\lambda, \mu, \nu \in \mathcal{P}_n$ is a cellular basis of $S_n$.

For each $\lambda \in \mathcal{P}_n$ the algebra $S_n$ has a Weyl module $W^\lambda$, which is the corresponding right cell module of $S_n$. To make this more explicit, let $S'^{S\lambda}_n$ be the two-sided ideal of $S_n$ with basis the $\varphi_{ST}$ where $S$ and $T$ are semistandard $\mu$-tableaux with $\mu \triangleright \lambda$. Then $W^\lambda \cong \varphi_{ST}S_n/(\varphi_{ST}S_n \cap S'^{S\lambda}_n)$. If $S \in T_{row}(\lambda)$ then set $\varphi_S = \varphi_{ST} + S'^{S\lambda}_n \in W^\lambda$. Then $\varphi_{ST}S \cong S_{\varphi_{ST}}$ and $S_{\varphi_{ST}} \in W^\lambda$. Then $\langle \varphi_{ST}, S \rangle$ is a $K$-basis of $W^\lambda$. Exactly as for a Specht module, the Weyl module $W^\lambda$ comes equipped with a bilinear form, also written as $\langle \cdot, \cdot \rangle$, determined by

$$\langle \varphi_S, \varphi_T \rangle = \langle \varphi_S, \varphi_T \rangle, \text{ for all } S, U, V \in T_{row}(\lambda).$$

Let $L^\lambda = W^\lambda/\text{rad } W^\lambda$ where $\text{rad } W^\lambda$ is the radical of the form $\langle \cdot, \cdot \rangle$. By definition, $\varphi_{ST+\lambda}$ is the identity map on $M^\lambda$, so $\langle \varphi_{ST+\lambda}, \varphi_{ST} \rangle = 1$ and, consequently, $L^\lambda \neq 0$. FAYERS’ CONJECTURE AND THE SOCLES OF CYCLOTOMIC WEYL MODULES 15
In a similar way we can define elements \( m'_\psi \in M^\lambda \) and homomorphisms \( \varphi'_\psi \in \text{Hom}_H(L^\lambda, M^\mu) \), for semisimple tableaux \( S, T \in \text{Tr}_\lambda(\lambda) \). Hence, we can define Weyl modules \( W^\lambda \) and simple modules \( L^\lambda = W^\lambda / \text{rad} W^\lambda \) for \( S^\prime \).

The cellular algebra involutions on \( H_n \) and \( H'_n \) induce involutions \( * \) on \( S_n \) and \( S'_n \), with the property that \( \varphi_\psi \mapsto \varphi'_\psi \) and \( \psi'_\gamma \mapsto \varphi'_\psi \). For \( \lambda \in \mathcal{P} \) let \( V^\lambda = (W^\lambda)^* \) and \( \lambda^* \) be costandard modules for \( S_n \) and \( S'_n \), respectively.

4.5. Theorem (18, 41). Suppose that \( n \geq 0 \).

a) The algebra \( S_n \) is a quasi-hereditary cellular algebra with weight poset \( \mathcal{P}_n \), cellular basis

\[
\{ \varphi_\psi | S \in \text{Tr}_\lambda(\lambda, \sigma) \text{ and } T \in \text{Tr}_\tau(\lambda, \tau) \text{ for } \lambda, \sigma, \tau \in \mathcal{P}_n \},
\]

standard modules \( \{ W^\lambda | \lambda \in \mathcal{P}_n \} \), costandard modules \( \{ V^\lambda | \lambda \in \mathcal{P}_n \} \) and pairwise non-isomorphic self-dual simple modules \( \{ L^\lambda | \lambda \in \mathcal{P}_n \} \).

b) The algebra \( S'_n \) is a quasi-hereditary cellular algebra with weight poset \( \mathcal{P}^\text{op}_n \), cellular basis

\[
\{ \psi'_\psi | S \in \text{Tr}_\lambda(\lambda, \sigma) \text{ and } T \in \text{Tr}_\tau(\lambda, \tau) \text{ for } \lambda, \sigma, \tau \in \mathcal{P}_n \},
\]

standard modules \( \{ W^\lambda | \lambda \in \mathcal{P}_n \} \), costandard modules \( \{ V^\lambda | \lambda \in \mathcal{P}_n \} \) and pairwise non-isomorphic self-dual simple modules \( \{ L^\lambda | \lambda \in \mathcal{P}_n \} \).

Although we will not need them, there are analogous definitions of elements \( n_{\psi \theta} \in N^\lambda \) and \( n'_{\psi \theta} \in N'_{\lambda} \) and maps \( \psi_{\psi \theta} \in \text{Hom}_H(N^\lambda, N'^\lambda) \) and \( \psi'_{\psi \theta} \in \text{Hom}_H(N'^\lambda, N'^\lambda) \), where \( S \) and \( T \) are column semisimple tableaux. Then \( \text{Lemma 4.3 andTheorem 4.5 imply that } \{ \psi_{\psi \theta} \} \text{ and } \{ \psi'_{\psi \theta} \} \text{ are cellular bases for the quasi-hereditary algebras } S_n(N) \text{ and } S'_n(N), \text{ respectively.} \)

Let \( \pi \) be the natural projection from \( M \) onto \( H_n \). Then it is easy to check that \( \pi^* = \pi \). Recall that \( (H_n, M) \) is a Schur pair. The results in Chapter 2, we have that

\[
S_n \pi \cong M = \bigoplus_{\lambda \in \mathcal{P}_n} m_\lambda H_n, \quad \pi S_n \cong \bigoplus_{\lambda \in \mathcal{P}_n} m_\lambda H_n m_\lambda \text{ and } S'_n \cong \text{End}_H \left( \bigoplus_{\lambda \in \mathcal{P}_n} H_n m_\lambda \right).
\]

In particular, we can regard \( \oplus_{\lambda \in \mathcal{P}_n} m_\lambda H_n \) as a right \( S_n \)-module. Similar results hold for the Schur pair \( (H'_n, M') \). In what follows we consider \( z^\lambda \) to be an element of \( H_n m_\lambda \) and \( z^\lambda \) as an element of \( H'_n m_\lambda \).

The next result shows how the classical definitions of Weyl modules extend to the cyclotomic case. In the semisimple case this follows directly from \text{Lemma 2.11 and Lemma 3.9} but in general we need to work harder. In the special case when \( \ell = 1 \) this is due to Dipper and James [17].

4.6. Proposition. Suppose that \( \lambda \in \mathcal{P}_n \).

a) As \( S_n \)-modules, \( W^\lambda \cong z^\lambda S_n \). In particular, \( W^\lambda \) is (isomorphic to) a submodule of \( \oplus_{\lambda \in \mathcal{P}_n} H_n m_\lambda \).

b) As \( S'_n \)-modules, \( W^\lambda \cong z^\lambda S'_n \). In particular, \( W^\lambda \) is (isomorphic to) a submodule of \( \oplus_{\lambda \in \mathcal{P}_n} H'_n m_\lambda \).

Proof. We prove (a) and leave (b) for the reader. Recall that \( W^\lambda \cong \varphi_{\phi_{\gamma \mu}} S_n / (\varphi_{\phi_{\gamma \mu}} S_n \cap S'_{\phi \lambda}) \). Define an \( S_n \)-module homomorphism by

\[
\theta : \varphi_{\phi_{\gamma \mu}} S_n \longrightarrow \bigoplus_{\lambda \in \mathcal{P}_n} H_n m_\lambda; \quad \phi \mapsto z^\lambda \phi, \quad \text{for all } \phi \in \varphi_{\phi_{\gamma \mu}} S_n.
\]

Fix semisimple tableaux \( S \in \text{Tr}_\lambda(\mu, \lambda), T \in \text{Tr}_\mu(\mu), \) for some \( \mu \in \mathcal{P}_n \). Then

\[
\theta(\varphi_{\psi \theta}) = z^\lambda : \varphi_{\psi \theta} = (n_{\lambda T_{w_{\lambda}} m_\lambda})_{\psi \theta} = n_{\lambda T_{w_{\lambda}} m_{\psi \theta}}.
\]

By \text{Proposition 3.8}, if \( s, t \in \text{Std}(\mu) \) then \( n_{\lambda m_{st}} \neq 0 \) only if \( \lambda \geq \mu \). Consequently, \( \theta(\varphi_{\psi \theta}) = 0 \) if \( \lambda \not\geq \mu \).

In particular, \( S^\lambda_n \cap \varphi_{\phi_{\gamma \mu}} S_n \leq \ker \theta \) since \( \varphi_{\psi \theta} | S, T \in \text{Tr}_\lambda(\mu, \lambda) \) where \( \mu \not\geq \lambda \) is a basis of \( S'_n \lambda \). On the other hand, if \( S \in \text{Tr}_\lambda(\lambda) \) then \( \theta(\varphi_{\psi \theta}) = n_{\lambda T_{w_{\lambda}} m_{\psi \theta}} = n_{\lambda T_{w_{\lambda}} m_{\lambda}} \). In view of \text{Lemma 3.9},

\[
\{ \theta(\varphi_{\psi \theta}) | T \in \text{Tr}_\lambda(\lambda) \} = \{ n_{\lambda T_{w_{\lambda}} m_{\lambda}} | S \in \text{Tr}_\lambda(\lambda) \}
\]

is linearly independent (see also \text{[21, (2.10)]} or \text{[41, Proposition 5.9]}). Therefore, \( W^\lambda = \varphi_{\phi_{\gamma \mu}} S_n / \ker \theta \cong \text{im} \theta = z^\lambda S_n \) as required.

The last result in this section explains the significance of the algebras \( S'_n \cong S_n(N) \) in the representation theory of \( S_n \) (and hence why we need them in this paper). To do this we first recall the definition of the Ringel dual of a quasi-hereditary algebra.

Let \( S \) be a quasi-hereditary algebra with standard modules \( W^i \) and costandard modules \( V^i \), where \( i \) runs over a poset \( (I, \geq) \). Let \( \text{mod-} F_w(S) \) be the full subcategory of \( \text{mod-} S \) consisting of \( \Delta \)-filtered \( S \)-modules. Thus, \( X \in \text{mod-} F_w(S) \) if and only if \( X \) has a filtration with each subquotient isomorphic to a Weyl module \( W^i \), for \( i \in I \). Similarly, let \( \text{mod-} F_V(S) \) be the full subcategory of \( \text{mod-} S \) consisting of
\(\nabla\)-filtered \(S\)-modules. If \(X \in \text{mod-}F_W(S)\) let \((X : W^i)\) be the number of subquotients of \(X\) isomorphic to \(W^i\). Define \((Y : V^i)\) in the same way when \(Y \in \text{mod-}F_V(S)\). Since \(S\) is quasi-hereditary the multiplicities \((X : W^i)\) and \((Y : V^i)\) are independent of the choices of \(\Delta\) and \(\nabla\) filtrations.

An \(S\)-tilting module is any \(S\)-module in \(\text{mod-}F_W(S) \cap \text{mod-}F_V(S)\). As \(S\) is quasi-hereditary, by [20, A4] for each \(i \in I\) there is a unique indecomposable tilting module \(T^i\) for \(S\) such that \((T^i : W^i) = 1\) and \((T^i : W^j) \neq 0\) only if \(i \geq j\). Moreover, up to isomorphism \(\{T^i \mid i \in I\}\) is a complete set of pairwise non-isomorphic indecomposable tilting modules.

A full tilting module is a tilting module that contains \(\bigoplus_{i \in I} T^i\) as a summand. A Ringel dual of the algebra \(S\) is any algebra \(S^{RD} = \text{End}_S(T)\), where \(T\) is any full tilting module. Then \(S^{RD}\) is quasi-hereditary with respect to the opposite poset \(I^{op}\). By construction, the Ringel dual is unique up to Morita equivalence. There is an exact functor \(R_n : \text{mod-}F_V(S) \to \text{mod-}F_W(S^{RD})\) given by \(X \mapsto \text{Hom}_S(T, X)\).

Returning now to the cyclotomic Schur algebras, let \(T^\lambda\) and \(T^\lambda_n\) be the tilting modules for \(S_n\) and \(S'_n\), for \(\lambda \in \mathcal{P}_n\). Let \(P^\lambda\) and \(P^\lambda_n\) be the projective covers of \(L^\lambda\) and \(L^\lambda_n\), respectively.

Let \(F_n : \text{mod-}S_n \to \text{mod-}H_n\) and \(F'_n : \text{mod-}S'_n \to \text{mod-}H'_n\) be the Schur functors, defined in Chapter 2.

4.7. **Theorem** (Ringel duality for cyclotomic Schur algebras). The twisted cyclotomic Schur algebra \(S'_n \cong S_n^{RD}\) is Ringel dual to \(S_n\). Moreover, there is an exact functor \(R_n : \text{mod-}F_V(S_n) \to \text{mod-}F_W(S'_n)\) such that the following diagram of functors commutes:

\[
\begin{array}{ccc}
\text{mod-}F_V(S_n) & \xrightarrow{R_n} & \text{mod-}F_W(S'_n) \\
F_n \downarrow & & \downarrow F'_n \\
\text{mod-}H_n & \xrightarrow{E^\#_n} & \text{mod-}H'_n
\end{array}
\]

The functor \(R_n\) is determined by \(R_n(V^\lambda) \cong W^\lambda\), for all \(\lambda \in \mathcal{P}_n\). Moreover, \(R_n(T^\lambda) \cong P^\lambda\) as \(S_n\)-modules.

**Proof.** By [41, Corollary 7.3], there is a full tilting module \(E\) for \(S_n\) such that \(S_n(N) \cong \text{End}_{S_n}(E)\). By Lemma 4.3, \(S'_n \cong S_n(N)\) so \(S'_n\) is a Ringel dual of \(S_n\). The isomorphism \(S'_n \cong S_n(N)\) induces an equivalence of categories \(\text{mod-}S_n(N) \xrightarrow{\sim} \text{mod-}S'_n\), so Ringel duality gives an exact functor \(R_n : \text{mod-}F_V(S_n) \to \text{mod-}F_W(S'_n)\) that sends indecomposable tilting modules to indecomposable projective modules. By induction on the dominance order it follows that \(R_n(V^\lambda) \cong W^\lambda\), for all \(\lambda \in \mathcal{P}_n\). Since \(R_n\) is exact on \(\text{mod-}F_V(S'_n)\), it is uniquely determined by its action on the costandard modules. In particular, \(R_n(T^\lambda) \cong P^\lambda\) since \(V^\lambda\) is a quotient of \(T^\lambda\).

It remains to check that the diagram above commutes. By exactness it is sufficient to check commutativity on the costandard \(S_n\)-modules. By Corollary 2.5, Lemma 3.9 and Proposition 4.6, if \(\lambda \in \mathcal{P}_n\) then \(F_n(V^\lambda) \cong S^\lambda\) and \(F'_n(W^\lambda) \cong S'_n\). Therefore, as \(H_n\)-modules,

\[
(F'_n \circ R_n)(V^\lambda) \cong F'_n(W^\lambda) \cong S'_n \cong E^\#_n(S^\lambda) \cong (E^\#_n \circ F_n)(V^\lambda),
\]

where the third isomorphism comes from Proposition 3.14.

It is worth mentioning that the Ringel duality functor \(R_n\), when considered a functor from \(\text{mod-}S_n\) to \(\text{mod-}S'_n\), is only left exact, and not right exact. We restrict to the subcategory \(\text{mod-}F_V(S_n)\) in Theorem 4.7 only because this ensures that \(R_n\) is exact.

Following [41, §3-4], and Definition 2.8, we make the following definition.

4.8. **Definition** ([41, §3, §4, §7]). The **Young modules** and the **twisted Young modules** are the \(H_n\)-modules \(Y^\lambda = F_n(P^\lambda)\) and \(Y'_n = F'_n(T^\lambda)\), respectively, for \(\lambda \in \mathcal{P}_n\).

Similarly, define Young modules for \(H'_n\) by setting \(Y^\lambda_n = F'_n(P^\lambda_n)\) and \(Y'_n = F'_n(T^\lambda_n)\). We need Ringel duality for the next corollary, which we will use in Chapter 5 to prove results about the socles of Young modules for \(H_n\) and projective modules for \(S_n\).

4.9. **Corollary.** Suppose that \(\lambda \in \mathcal{P}_n\). Then \(E^\#_n(Y^\lambda) \cong Y^\lambda\) as \(H'_n\)-modules.

**Proof.** Using the definitions and Theorem 4.7, as \(H'_n\)-modules,

\[
E^\#_n(Y^\lambda) \cong (E^\#_n \circ F_n)(T^\lambda) \cong (F'_n \circ R_n)(T^\lambda) \cong F'_n(P^\lambda_n) \cong Y^\lambda_n,
\]

as required.
Let $I^\lambda$ be the injective hull of $L^\lambda$, for $\lambda \in \mathcal{P}_n$. Then $I^\lambda \in \text{mod-} \mathcal{F}_V(S_n)$.

4.10. Corollary. Suppose that $\lambda \in \mathcal{P}_n$. Then $E_n^\#(Y^\lambda) \cong Y_n^\lambda$.

Proof. The Schur functor commutes on the dualities on $\text{mod-} \mathcal{S}_n$ and $\text{mod-} \mathcal{H}_n$ and $(Y^\mu)^* \cong Y^\mu$ by [41, Corollary 5.14] (when $\lambda \in \mathcal{K}_n$, this follows from Lemma 2.20). Moreover, Ringel duality sends injective modules to tilting modules. Therefore,

$$E_n^\#(Y^\lambda) \cong E_n^\#((Y^\lambda)^*) \cong (E_n^\# \circ F_n)((P^\lambda)^*) \cong (E_n^\# \circ F_n)(I^\lambda) \cong F_n(I^\lambda) \cong Y_n^\lambda.$$ 

The final result in this section is part of the folklore for $\mathcal{H}_n$, but as far as we are aware the result is not in the literature.

4.11. Corollary. Suppose that $\mu \in \mathcal{P}_n$. Then:

a) If $\mu \in \mathcal{K}_n$ then $Y^\mu$ is the projective cover of $D^\mu$.

b) If $\mu' \in \mathcal{K}_n$ then $Y_{\mu'}$ is the projective cover of $D_{\mu'}$.

c) If $\mu \in \mathcal{K}_n$ then $Y^\mu \cong Y_{\mu(\mu')}$ as $\mathcal{H}_n$-modules.

Proof. By Proposition 2.9, $Y^\mu$ is projective if and only if $\mu \in \mathcal{K}_n$, in which case it is the projective cover of $D^\mu$. This proves (a). Similarly, $Y_{\mu'}$ is projective if and only if $\mu' \in \mathcal{K}_n$, in which case it is the projective cover of $D_{\mu'}$. Hence, (b) follows by Corollary 4.9 and Corollary 3.15. Part (c) is now automatic from parts (a) and (b). 

5. Socles of Weyl modules, tilting modules and projective modules

We now come to the first main result of this paper, which motivated much of the development of Chapter 2. In more detail, we describe the simple $\mathcal{S}_n$-modules that can appear in the socles of the Weyl modules, tilting modules and projective indecomposable modules and give similar results for the socles of the (twisted) Young modules.

Before we begin we note the following immediate consequence of Proposition 4.1 and Theorem 2.22.

5.1. Theorem. Suppose that $\lambda \in \mathcal{P}_n$. Then the following are equivalent:

a) $\lambda \in \mathcal{K}_n$,

b) $D^\lambda \neq 0$,

c) $L^\lambda$ is a right submodule of $M_R$,

d) $P^\lambda$ is a direct summand of $M_R$,

e) $P^\lambda$ is a projective-injective right $\mathcal{S}_n$-module,

f) $P^\lambda$ is an indecomposable tilting module,

g) $P^\lambda$ is self-dual.

There are analogous results for $\mathcal{S}_n$-modules.

The first of these results generalises a classical result of James [32, Theorem 2.8] in level one (when $\xi = 1$ and $\ell = 1$). This result will be used to prove Fayers’ conjecture in Theorem 6.8 below.

5.2. Theorem. Suppose that $\lambda, \mu \in \mathcal{P}_n$.

a) The simple module $L^\mu$ is a submodule of $W^\lambda$ only if $\mu \in \mathcal{K}_n$.

b) The simple module $L^\mu$ is a submodule of $W^\lambda$ only if $\mu \in \mathcal{K}_n$.

Proof. Both parts can be proved in the same way so we consider only (a). By Proposition 4.1, $(\mathcal{H}_n, M)$ is a Schur pair. Therefore, by Theorem 2.17, $L^\mu$ is an $\mathcal{S}_n$-submodule of $M_R$ if and only if $D^\mu \neq 0$, which is if and only if $\mu \in \mathcal{K}_n$. On the other hand, $W^\lambda$ is isomorphic to an $\mathcal{S}_n$-submodule of $M_R$ by Proposition 4.6, so $\text{soc} W^\lambda \cong \text{soc} M_R$. Hence, $L^\mu$ is a submodule of $W^\lambda$ only if $\mu \in \mathcal{K}_n$, as claimed.

5.3. Corollary. Suppose that $X \in \text{mod-} \mathcal{F}_W(S_n)$ and $\lambda, \mu \in \mathcal{P}_n$. Then $L^\mu$ is a submodule of $X$ only if $\mu \in \mathcal{K}_n$. In particular, $L^\mu$ is a submodule of $D^\lambda \oplus T^\lambda$ only if $\mu \in \mathcal{K}_n$.

By Proposition 4.6 and Lemma 2.11, or working directly with the definitions, $S^\lambda \cong F_n(W^\lambda)$ for all $\lambda \in \mathcal{P}_n$. Hence, applying the Schur functor to Theorem 5.2 gives the following.

5.4. Corollary. Suppose that $\lambda, \mu \in \mathcal{P}_n$. Then

$$[\text{soc} W^\lambda : L^\mu] = [\text{soc} S^\lambda : D^\mu] \quad \text{and} \quad [\text{soc} T^\lambda : L^\mu] = [\text{soc} Y^\lambda : D^\mu].$$

In particular, these two multiplicities are non-zero only if $\mu \in \mathcal{K}_n$. 

Proof. Recall from Proposition 4.1 that \((\mathcal{H}_n, M)\) is a self-dual Schur pair. Therefore, the result follows by Corollary 5.3 and Corollary 2.23. □

5.5. Corollary. Assume \(\mu \in \mathcal{K}_n\). Then \(\text{soc } W^{m(\mu)} \cong L^\mu\). Equivalently, \(\text{hd } V^{m(\mu)} \cong L^\mu\).

Proof. Combine Corollary 5.4 and Corollary 3.17 for the first isomorphism and take duals for the second. □

6. Proof of Fayers’ conjecture

In this section, as our second application of the results in Chapter 2, we prove Fayers Conjecture [23]. This is one of the main result of this paper. We first give a precise statement of Fayers’ conjecture and then recall some recent results that we need from the graded representation theory of \(\mathcal{H}_n\) and \(\mathcal{S}_n\).

Throughout this section we assume that \(K = \mathbb{C}\) and we fix \(\xi \in \mathbb{C}\), an element of quantum characteristic \(e\) (that is, a primitive 2\(e\)th root of unity in \(\mathbb{C}\)), and a multicharge \(\kappa \in \mathbb{Z}^L\). In fact, we will work with \(\kappa' = (-\kappa_1, \ldots, -\kappa_1)\) because our argument uses results of Stroppel and Webster [49] who worked with the twisted cyclotomic Schur algebra \(\mathcal{S}'_n \cong \mathcal{S}_n(N)\), by Lemma 4.3.

Set \(I = \mathbb{Z}/e\mathbb{Z}\), where we adopt the convention that \(e\mathbb{Z} = e\mathbb{Z} \cap \mathbb{Z} = \{0\}\) when \(e = \infty\) (so that \(I = \mathbb{Z}\) when \(e = \infty\)). Let \(q\) be an indeterminate over \(\mathbb{Q}\). Let \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\) be the quantised enveloping algebra of the Kac-Moody algebra \(\widehat{\mathfrak{sl}}_L\) (in particular, we consider \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_\infty)\) when \(e = \infty\)). The algebra \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\) is a Hopf algebra with coproduct \(\Delta\) determined by

\[
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \text{and} \quad \Delta(K_i) = K_i \otimes K_i,
\]

for \(i \in I\). Let \(\{\Lambda_i \mid i \in I\}\) be the set of fundamental weights and \(\{\alpha_i \mid i \in I\}\) the simple roots for \(\widehat{\mathfrak{sl}}_L\) and set \(P^+ = \bigoplus_{i \in I} \mathbb{N}_\Lambda\) and \(Q^+ = \bigoplus_{i \in I} \mathbb{N}_\alpha\). Then \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\) is the \(\mathbb{Q}(q)\)-algebra generated by elements \(\{E_i, F_i, K_i^\pm \mid i \in I\}\) subject to the well-known quantised relations [36]. The bar involution \(\overline{\cdot}\) is the \(\mathbb{Q}(q)\)-linear automorphism of \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\) determined by

\[
\overline{E_i} = E_i, \quad \overline{F_i} = F_i, \quad \overline{K_i} = K_i^{-1} \quad \text{and} \quad \overline{q} = q^{-1},
\]

for \(i \in I\).

The (combinatorial) Fock space \(\mathcal{F}(\kappa')\) is the \(\mathbb{Q}(q)\)-vector space

\[
\mathcal{F}(\kappa') = \bigoplus_{n \geq 0} \bigoplus \mathbb{Q}(q)s_{\lambda}.
\]

The residue of a node \(A = (k, r, c) \in [\lambda]\) is \(\text{res } A = -\kappa_r+1-k+c-r+e\mathbb{Z}\in I\). If \(\text{res } A = i \in I\) then \(A\) is an \(i\)-node. Following Hayashi [27] and Misra and Miwa [45], the action of \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\) on \(\mathcal{F}(\kappa')\) can be described explicitly using the combinatorics of admissible and removable \(i\)-nodes. At the categorical level the \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\)-action corresponds to graded \(i\)-induction and \(i\)-restriction for the cyclotomic Hecke algebras [11, 12, 28, 43]. As we do not need the precise details we refer interested reader to [11, §3.6] or [43, §3.5].

Let \(\Lambda' = \sum_{i \in I} l_i \Lambda_i,\) where \(l_i = \# \{1 \leq l \leq \ell \mid i = -\kappa_i + e\mathbb{Z}\}\), and set \(s_{\Lambda'} = s_{(0)\cdots(0)} \in \mathcal{F}(\Lambda')\). Then \(\Lambda' \in P^+\) and \(L(\Lambda') = \mathcal{U}_q(\widehat{\mathfrak{sl}}_L)s_{(0)\cdots(0)}\) is isomorphic to the integrable highest weight \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\)-module of highest weight \(\Lambda'\). The bar involution \(\overline{\cdot}\) induces a unique \(\mathbb{Q}(q)\)-linear bar involution \(\overline{\cdot}\) on \(L(\Lambda')\) such that \(s_{\lambda'} = s_{\lambda'}\overline{\cdot}\) and \(\overline{\sum_{\lambda \in \mathcal{P}_n}} s_{\lambda'} = \sum_{\lambda \in \mathcal{P}_n} s_{\lambda'}\), for all \(u \in \mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\). Brundan and Kleshchev [11, Theorem 3.26] show that the bar involution on \(L(\Lambda')\) extends to an involution \(\overline{\cdot}\) on \(\mathcal{F}(\kappa')\) and hence that the following holds.

6.1. Theorem. Suppose that \(\mu \in \mathcal{P}_n\). Then there is a unique bar-invariant element \(G^\mu \in \mathcal{F}(\kappa')\) such that

\[
G^\mu = \sum_{\lambda \in \mathcal{P}_n} d_{\lambda\mu}(q)s_{\lambda},
\]

for some polynomials \(d_{\lambda\mu}(q) \in \delta_{\lambda\mu} + q\mathbb{Z}[q]\).

Using the coproduct \(\Delta\) on \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\), it is straightforward to show that \(\mathcal{F}(\kappa') \cong \mathcal{F}(-\kappa_1) \otimes \cdots \otimes \mathcal{F}(-\kappa_1)\) as \(\mathcal{U}_q(\widehat{\mathfrak{sl}}_L)\)-modules. Uglov [50] has proved a more general version of these results where the Fock space does not necessarily satisfy this tensor product decomposition.

Fayers [23] used the tensor product decomposition of \(\mathcal{F}(\kappa')\) to give an algorithm for computing the elements \(G^\mu\) whenever \(\mu = (\mu^{(1)} \cdots \mu^{(r)})\) where the partition \(\mu^{(r)}\) is \(e\)-restricted, for \(1 \leq r \leq \ell\). This set of multipartitions contains \(\mathcal{K})\). (A partition \(\mu\) is \(e\)-restricted if \(\mu_k - \mu_{k+1} < e\), for \(k \geq 1\).)
Given $\lambda \in \mathcal{P}_n$ define $\beta_\lambda = \sum_{A \in [\lambda]} \alpha_{res} A \in \mathbb{Q}^+$. The combinatorial classification of the blocks of $\mathcal{H}'_n$ and $\mathcal{S}'_n$ given in [7, 37] is equivalent to the statement that the Specht modules, or Weyl modules, indexed by $\lambda$ and $\mu$ are in the same block if and only if $\beta_\lambda = \beta_\mu$. Following [10] define the defect of $\lambda$ by

$$\text{def } \lambda = (\lambda', \beta_\lambda) - \frac{1}{2}(\beta_\lambda, \beta_\lambda) \in \mathbb{N}. \quad (6.2)$$

By the remarks above, the defect is a block invariant. The defect is easily seen to be equivalent to the combinatorial definition of the weight $w(\lambda)$ of $\lambda$ given by [22, §2.1]. We can now state Fayers’ conjecture [23].

6.3. Conjecture (Fayers [23]). Let $\lambda, \mu \in \mathcal{P}_n$ Then $\text{deg } d_{\lambda \mu}(q) \leq \text{def } \mu$ and, moreover, $\text{deg } d_{\lambda \mu}(q) = \text{def } \mu$ only if $\mu \in \mathcal{K}_n$.

To prove this conjecture, we work in a graded setting where we can interpret $d_{\lambda \mu}(v)$ as a graded decomposition number. Building on work of Khovanov and Lauda [33] and Rouquier [48], Brundan and Kleshchev [10] showed that (each block of) $\mathcal{H}_n$ is a $\mathbb{Z}$-graded algebra. Extending this result, Stroppel and Webster [49] and the authors (when $e = \infty$) [29] showed that the cyclotomic Schur algebras admit a $\mathbb{Z}$-grading. Following Stroppel and Webster [49], we will work with the twisted cyclotomic Schur algebra $\mathcal{S}'_n$.

Let $\mathcal{S}'_n$ and $\mathcal{H}'_n$ be the basic graded algebras of $\mathcal{S}'_n$ and $\mathcal{H}'_n$, respectively, and let $\text{grmod-}\mathcal{S}'_n$ and $\text{grmod-}\mathcal{H}'_n$ be the corresponding categories of finite dimensional graded modules with homogeneous maps of degree zero. (Unlike in the ungraded setting, graded basic algebras are not uniquely determined up to isomorphism, as graded algebras. There is, however, a unique grading on $\mathcal{S}'_n$ such that Theorem 6.6.5 below holds and by applying the graded Schur functor this fixes the grading on $\mathcal{H}'_n$. More explicitly, $\mathcal{S}'_n$ is the graded endomorphism algebra of $\bigoplus_{\lambda \in \mathcal{P}_n} \mathbb{P}_\lambda^\mathcal{L}$, where $\mathbb{P}_\lambda^\mathcal{L}$ is the graded projective cover of the corresponding self-dual graded simple module.)

The algebra $\mathcal{S}'_n$ comes equipped with graded standard modules $\mathbb{W}_\lambda^\mathcal{L}$ and graded simple modules $\mathbb{L}_\lambda^\mathcal{L}$, where we fix the grading on these modules by requiring that $\mathbb{L}_\lambda^\mathcal{L}$ is graded self-dual and that there is a homogeneous surjection $\mathbb{W}_\lambda^\mathcal{L} \twoheadrightarrow \mathbb{L}_\lambda^\mathcal{L}$, for $\lambda \in \mathcal{P}_n$. Similarly, the graded algebra $\mathcal{H}'_n$ has graded analogues of the Specht modules $\mathbb{S}_\lambda^\mathcal{L}$ and graded simple modules $\mathbb{D}_\lambda^\mathcal{L}$ where we again fix the gradings requiring that $\mathbb{D}_\lambda^\mathcal{L}$ is graded self-dual and that $\mathbb{S}_\lambda^\mathcal{L}$ surjects onto $\mathbb{D}_\lambda^\mathcal{L}$, for $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{K}_n$. In particular, this implies that $\mathbb{L}_\lambda^\mathcal{L}$ and $\mathbb{D}_\lambda^\mathcal{L}$ are both one dimensional modules concentrated in degree zero.

If $M = \bigoplus_{d \in \mathbb{Z}} M_d$ is a $\mathbb{Z}$-graded module let $\langle \cdot \rangle$ be the shift functor so that $M(s)_d = M_{d-s}$. Then the graded decomposition numbers of $\mathcal{S}'_n$ and $\mathcal{H}'_n$ are the Laurent polynomials

$$[\mathbb{W}_\lambda^\mathcal{L} : \mathbb{L}_\lambda^\mathcal{L}]_q = \sum_{d \in \mathbb{Z}} [\mathbb{W}_\lambda^\mathcal{L} : \mathbb{L}_\lambda^\mathcal{L}(d)] q^d$$

and

$$[\mathbb{S}_\lambda^\mathcal{L} : \mathbb{D}_\lambda^\mathcal{L}]_q = \sum_{d \in \mathbb{Z}} [\mathbb{S}_\lambda^\mathcal{L} : \mathbb{D}_\lambda^\mathcal{L}(d)] q^d,$$

for $\lambda, \mu \in \mathcal{P}_n$. Abusing notation slightly, as in Chapter 2 there is a graded Schur functor

$$F'_n : \text{grmod-}\mathcal{S}'_n \rightarrow \text{grmod-}\mathcal{H}'_n$$

such that $F'_n(\mathbb{W}_\lambda^\mathcal{L}) \cong \mathbb{S}_\lambda^\mathcal{L}$ and $F'_n(\mathbb{D}_\lambda^\mathcal{L}) \cong \mathbb{D}_\lambda^\mathcal{L}$ which, of course, is zero if $\mu \notin \mathcal{K}'_n$.

6.4. Theorem (Stroppel-Webster [49, Theorem 7.11]). Suppose that $K = \mathbb{C}$ and let $\lambda, \mu \in \mathcal{P}_n$. Then

$$[\mathbb{W}_\lambda^\mathcal{L} : \mathbb{L}_\lambda^\mathcal{L}]_q = d_{\lambda \mu}(q).$$

Moreover, if $\mu \in \mathcal{K}_n$ then $[\mathbb{S}_\lambda^\mathcal{L} : \mathbb{D}_\lambda^\mathcal{L}]_q = d_{\lambda \mu}(q)$. In particular, $d_{\lambda \mu}(q) = \delta_{\lambda \mu} + q \mathbb{N}[q]$, for all $\lambda, \mu \in \mathcal{P}_n$.

The graded decomposition numbers of $\mathcal{H}'_n$ were first computed by Brundan and Kleshchev [11]. We need one property of these polynomials, which was part of Fayers’ motivation for Conjecture 6.3.

6.5. Lemma ([11, Remark 3.19] and [43, Corollary 3.6.7]). Suppose that $K = \mathbb{C}$, $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{K}'_n$. Then $0 \leq \text{deg } d_{\lambda \mu}(q) \leq \text{def } \mu$. Moreover, the following are equivalent:

a) $\text{deg } d_{\lambda \mu}(q) = \text{def } \mu$,

b) $d_{\lambda \mu}(q) = q^{\text{def } \mu}$,

c) $\lambda = \mathfrak{m}^{-1}(\mu')$.

Using properties of the polynomials $d_{\lambda \mu}(q)$ from Theorem 6.1, Fayers [22, Corollary 2.4] proved that if $\mu \in \mathcal{K}'_n$ then there is a unique multipartition $\lambda \in \mathcal{P}_n$ such that $d_{\lambda \mu}(q) = q^{w(\mu)}$, where $w(\mu)$ is his weight function. Combining Fayers’ result with Lemma 6.5 gives a representation theoretic proof that $w(\mu) = \text{def } \mu$. 

We need one more (deep) result from the graded representation theory of \( \mathcal{S}_n \). We refer the reader to [6, §2] for the definition of a Koszul algebra.

### 6.6. Theorem (Hu-Mathas [29] (\( e = \infty \)) and Maksimau [38]).
Suppose that \( K = \mathbb{C} \). Then the basic twisted cyclotomic Schur algebra \( \mathcal{S}_n \) is a Koszul algebra.

The papers [29, 38] actually prove this result for the blocks of \( \mathcal{S}_n \), however, this implies Theorem 6.6 because the direct sum of two Koszul algebras is again Koszul.

Notice that Theorem 6.4 implies that \( \mathcal{S}_n \) and \( \mathcal{H}_n \) are both positively graded algebras (strictly, non-negatively graded algebras). Moreover, since \( \text{Gr}_d^{\lambda} \) and \( \text{Gr}_d^{\Lambda} \) are both concentrated in degree zero, this implies that each of the modules \( W^{\lambda} \), \( L^{\lambda} \), \( \mathcal{S}^{\lambda} \) and \( \text{Gr}^{\lambda} \) is positively graded since \( d^{\lambda \mu}(q) \in \delta \lambda \mu + q\mathbb{N}[q] \).

If \( M = \bigoplus_{d \geq 0} M_d \) is a positively graded \( \mathcal{S}_n \)-module then its grading filtration is

\[
M = \text{Gr}_0 M \supseteq \text{Gr}_1 M \supseteq \ldots
\]

is given by \( \text{Gr}_r M = \bigoplus_{d \geq r} M_d \). Since \( \mathcal{S}_n \) is a positively graded algebra, \( \text{Gr}_r \) is an \( \mathcal{S}_n \)-module for \( r \geq 0 \).

In particular, \( W^{\lambda} \) has a grading filtration. For \( \lambda, \mu \in \mathcal{P}_n \) write

\[
d^{\lambda \mu}(q) = \sum_{r \geq 0} d^{(r)}_{\lambda \mu} q^r,
\]

where \( d^{(r)}_{\lambda \mu} \in \mathbb{N} \).

Let \( \text{rad}^0 W^{\lambda} \supseteq \text{rad}^1 W^{\lambda} \supseteq \ldots \) be the radical filtration of \( W^{\lambda} \).

### 6.7. Corollary.
Suppose that \( K = \mathbb{C} \) and let \( \lambda \in \mathcal{P}_n \). Then \( \text{rad}^r W^{\lambda} = \text{Gr}_r W^{\lambda} \). Consequently,

\[
[\text{rad}^r W^{\lambda} / \text{rad}^{r+1} W^{\lambda}] : \text{rad}^r W^{\lambda} = d^{(r)}_{\lambda \mu} q^r,
\]

for all \( r \geq 0 \).

**Proof.** By [6, Corollary 2.3.3], any Koszul algebra is quadratic. Therefore, since \( W^{\lambda} / \text{rad} W^{\lambda} \cong L^{\lambda} \) is simple and concentrated in degree zero, the radical filtration of \( W^{\lambda} \) coincides with the grading filtration of \( W^{\lambda} \) by [6, Proposition 2.4.1].

We can now prove a stronger version of Conjecture 6.3 (and of Lemma 6.5).

### 6.8. Theorem (Fayers’ Conjecture).
Suppose \( K = \mathbb{C} \) and that \( \lambda, \mu \in \mathcal{P}_n \). Then \( \deg d^{\lambda \mu}(q) \leq \text{def} \mu \) with equality only if \( \mu \in K'_n \). Moreover, the following are equivalent:

a) \( \deg d_{\lambda \mu}(q) = \text{def} \mu \),

b) \( d_{\lambda \mu}(q) = q^{\text{def} \mu} \),

c) \( \lambda = m^{-1}(\mu)' \),

d) \( \text{soc} W^{\lambda} = L^{\mu}_{\text{def} \mu} \),

e) \( \text{soc} L^{\lambda} = D^{\mu}_{\text{def} \mu} \).

**Proof.** First observe that by the comments before (6.2), if \( d_{\lambda \mu}(q) \neq 0 \) then \( \lambda = \text{def} \mu \). Fix \( \mu \in \mathcal{P}_n \) such that \( d_{\lambda \mu}(q) \neq 0 \) and \( d = \deg d_{\lambda \mu}(q) \) is maximal in the sense that \( \deg d_{\nu \mu}(q) \leq d \) for all \( \nu \in \mathcal{P}_n \).

By the maximality of \( d \), \( \text{Gr}_{d+1} W^{\lambda} = 0 \) and \( L^{\mu}_{d}(q) \) is a summand of \( \text{Gr}_{d} W^{\lambda} \), where the last equality comes from Corollary 6.7. Consequently, \( L^{\mu}_{d}(q) \) is a summand of the socle of \( W^{\lambda} \). Forgetting the gradings, this implies that \( L^{\mu}_{d} \) is contained in socle of \( W^{\lambda} \). Therefore, \( \mu \in K'_n \) by Theorem 5.2. As \( \mu \in K'_n \) and \( L^{\mu}_{d}(q) \) appears in soc \( W^{\lambda} \) whenever \( d = \deg d_{\lambda \mu}(q) \) is maximal the theorem now follows by applying Lemma 6.5.

Notice that by interchanging the roles of \( \kappa \) and \( \kappa' \), Theorem 6.8 becomes a result about \( \mathcal{S}_n \)-modules. By Theorem 6.8 if \( \lambda' \notin K_n \), then the socle of \( W^{\lambda} \) will not have a component in degree \( \text{def} \lambda \) and, a priori, \( \text{soc} W^{\lambda} \) is not necessarily homogeneous. On the other hand, by Theorem 5.2 and Theorem 6.8, if \( d \in \mathbb{Z} \) and \( L^{\mu}_{d}(q) \) appears in soc \( W^{\lambda} \) then \( 0 \leq d \leq \text{def} \lambda \) and \( \mu \in K'_n \).

### 7. Homomorphisms between Weyl modules and Specht modules

In our final section we return to ungraded representation theory of the cyclotomic Schur algebras \( \mathcal{S}_n \) and we prove some results relating the hom-spaces between Specht modules and Weyl modules. The main result in this section is a cyclotomic analogue of the classical Carter-Lusztig Theorem [13, Theorem 3.7]. The \( q \)-analogue of this result in level one was proved by Dipper and James [17].

### 7.1. Lemma.
Suppose that \( X, Y \in \text{mod-}F_{\kappa}(\mathcal{S}_n) \). Then the Schur functor \( F_{\kappa} \) induces an injection

\[
\text{Hom}_{\mathcal{S}_n}(X, Y) \rightarrow \text{Hom}_{\mathcal{H}_n}(F_{\kappa}(X), F_{\kappa}(Y)); \phi \mapsto F_{\kappa}(\phi).
\]
Proof. Since $F_n$ is exact it suffices to consider the case when $X = W^\lambda$ and $Y = W^\mu$, for some $\lambda, \mu \in \mathcal{P}_n$. Suppose that $f \in \text{Hom}_{S_n}(W^\lambda, W^\mu)$ and that $F_n(f) = 0$. It follows that

$$F_n(\text{im}(f)) = \text{im}(F_n(f)) = 0.$$ 

Now, if $f \neq 0$ then $\text{im } f \neq 0$ so there exists $\mu \in S_n$ such that $[\text{im } f : L^\mu] \neq 0$ by Theorem 5.2. Therefore, $F_n(\text{im } f) \neq 0$, showing that $F_n(f) = 0$ if and only if $f = 0$. 

The main result of this section gives sufficient conditions for the map of Lemma 7.1 to be an isomorphism of vector spaces. In the language of Rouquier [47, Definition 4.37], our result gives sufficient conditions for $S_n$ to be a 0-faithful cover of $H_n$.

By Lemma 7.1 there is an injection $\text{Hom}_{S_n}(W^\lambda, W^\mu) \rightarrow \text{Hom}_{H_n}(S^\lambda, S^\mu)$, for all $\lambda, \mu \in \mathcal{P}_n$. We want to recast this in the framework developed in Chapter 2 to prove Lemma 2.25. To this end, for $\lambda \in \mathcal{P}_n$ let $1_\lambda = \varphi_{n^\lambda}$ be the identity map on $M^\lambda$ and define $E^\lambda = S_n 1_\omega$, where $\omega = (0|\ldots|0|1^n) \in \mathcal{P}_n$.

Using Corollary 3.11 to take duals gives an isomorphism $\text{Hom}_{H_n}(S^\lambda, S^\mu) \cong \text{Hom}_{H_n}(S^\mu, S^\lambda)$. We work with homomorphisms between dual Specht modules because this better fits the framework developed in Chapter 2.

Fix $\mu \in \mathcal{P}_n$ and recall from Chapter 3 that $z_\mu = m_\mu T_{w_0} n_\mu$. Let $1_\mu$ be the identity map on $M^\mu = m_\mu H_n$ and let $\pi_\mu : H_n \rightarrow M^\mu$ be the natural surjection given by $\pi_\mu(h) = m_\mu h$ for $h \in H_n$. In fact, $1_\mu = \varphi_{n^\mu}$ and $\pi_\mu = \varphi_{n^\mu}$ are both elements of $S_n$. The isomorphism $H_n \cong \text{End}_{H_n}(H_n)$, which maps $h \in H_n$ to left multiplication by $h$, identifies $H_n$ with the subalgebra $1_\omega S_n 1_\omega$, where $1_\omega = \varphi_{n^\omega}$ is the identity map on $H_n$. Identify $H_n$ with its image under this map. Let $\zeta_\mu = \pi_\mu T_{w_0} n_\mu$. Then $\zeta_\mu \in 1_\mu S_n 1_\mu$. Define $\Delta^\mu = S_n \zeta_\mu$.

7.2. Lemma. Suppose that $\zeta \in S_n 1_\omega$. Then $L_{R_{S_n}}(\zeta) = S_n 1_\omega \cap L_{S_n}(R_{H_n}(\zeta))$.

Proof. First observe that $\zeta(1-1_\omega) = 0$ because $\zeta \in S_n 1_\omega$, so if $s \in L_{R_{S_n}}(\zeta)$ then $s(1-1_\omega) = 0$. Hence, $L_{R_{S_n}}(\zeta) \subseteq S_n 1_\omega$ and, consequently, $L_{R_{S_n}}(\zeta) \subseteq S_n 1_\omega \cap L_{S_n}(R_{H_n}(\zeta))$. To prove the reverse inclusion, let $x \in S_n 1_\omega \cap L_{S_n}(R_{H_n}(\zeta))$ and fix $s \in R_{S_n}(\zeta)$. Then $0 = \zeta s - \zeta 1_\omega s = \zeta 1_\omega s \pi_\lambda$, for all $\lambda \in \mathcal{P}_n$ (of course, $1_\omega s \pi_\lambda$ could be zero). By definition, $1_\omega s \pi_\lambda \in H_n$, so $x 1_\omega s \pi_\lambda = 0$ since $x \in L_{S_n}(R_{H_n}(\zeta))$. Hence, $x 1_\omega s 1_\lambda = 0$ since $\pi_\lambda = \iota_\lambda \pi_\lambda$ is surjective, for $\lambda \in \mathcal{P}_n$, so

$$xs = x 1_\omega s = \sum_{\lambda \notin \mathcal{P}_n} x 1_\omega s 1_\lambda = 0.$$ 

Hence, $x \in L_{R_{S_n}}(\zeta)$ as we needed to show. 

As in Chapter 2, the $*$-isomorphism of $H_n$ induces an anti-isomorphism of $S_n$, which we also call $*$ (see after Definition 2.14), if $X$ is a right $S_n$-module let $X_L$ be the left $S_n$-module that is equal to $X$ as a vector space and with left action given by $s \cdot x = x^*$, for $s \in S_n$ and $x \in X$. Similarly, if $X$ is a right $H_n$-module then let $X_L$ be the corresponding left $H_n$-module.

7.3. Lemma. Suppose that $\lambda, \mu \in \mathcal{P}_n$. Then $S^\lambda \cong \zeta_\lambda H_n$, as right $H_n$-modules, $\Delta^\lambda \cong W^\lambda$, as left $S_n$-modules, and there are vector space isomorphisms,

$$\text{Hom}_{S_n}(W^\lambda, W^\mu) \cong \text{Hom}_{S_n}(\Delta^\lambda, \Delta^\mu) \cong \Delta^\mu \cap R_{L_{S_n}}(\zeta_\lambda)$$

and

$$\text{Hom}_{H_n}(S^\lambda, S^\mu) \cong \text{Hom}_{H_n}(S^\mu, S^\lambda) \cong R_{L_{S_n}}(\zeta_\mu) \cap S^\lambda.$$ 

Proof. The map $m_\lambda h \mapsto \pi_\lambda h$, for $h \in H_n$, defines a $H_n$-module isomorphism $M^\lambda \cong \varphi_{S_n} \varphi_{\omega}$ that sends $\pi_\lambda$ to $\zeta_\lambda$. Therefore, $S^\lambda \cong \zeta_\lambda H_n$ by Lemma 3.9. Similarly, $\Delta^\lambda \cong (W^\lambda)^*$ as left $S_n$-modules by Proposition 4.6. Now consider the two hom-spaces. First, since $\Delta^\mu \cong W^\mu_L$ for $\nu \in \mathcal{P}_n$,

$$\text{Hom}_{S_n}(W^\lambda, W^\mu) \cong \text{Hom}_{S_n}(\Delta^\lambda, \Delta^\mu) \cong \text{Hom}_{S_n}(S^\lambda \zeta_\lambda, \Delta^\mu) \cong R_{L_{S_n}}(\zeta_\lambda) \cap \Delta^\mu,$$

using Lemma 2.25(a). Finally, if $\nu \in \mathcal{P}_n$ then $(S^\nu)^* \cong S^\nu$ so taking duals,

$$\text{Hom}_{S_n}(S^\lambda, S^\mu) \cong \text{Hom}_{S_n}(S^\mu, S^\lambda) \cong \text{Hom}_{S_n}(\zeta_\mu H_n, \zeta_\lambda H_n).$$

To complete the proof it is enough to show that $\text{Hom}_{H_n}(\zeta_\mu H_n, \zeta_\lambda H_n) \cong R_{L_{S_n}}(\zeta_\mu) \cap S^\lambda$. By Lemma 7.2, if $f \in \text{Hom}_{H_n}(\zeta_\mu H_n, \zeta_\lambda H_n)$ then $f(\zeta_\mu) \in L_{R_{S_n}}(\zeta_\mu)$. Conversely, if $z \in L_{R_{S_n}}(\zeta_\mu)$ then Lemma 7.2 implies that there is a well-defined $H_n$-module homomorphism $f : \zeta_\mu H_n \rightarrow \zeta_\lambda H_n$ given by $f(\zeta_\mu h) = zh$, for $h \in H_n$. Hence, $\text{Hom}_{H_n}(\zeta_\mu H_n, \zeta_\lambda H_n) \cong R_{L_{S_n}}(\zeta_\mu) \cap S^\lambda$ via the map $f \mapsto f(\zeta_\mu)$. 

\qed
So, to prove that $\text{Hom}_{S_n}(W^\lambda, W^\mu) \cong \text{Hom}_{S_n}(S^\lambda, S^\mu)$ it is enough to show that
\[
\Delta^\mu \cap \text{RL}_{S_n}(\langle \zeta \rangle) = \text{LR}_{H_n}(\langle \zeta \rangle) \cap \zeta H_n.
\]
To compare these hom-spaces we need to describe $\text{RL}_{S_n}(\langle \zeta \rangle)$ and $\text{LR}_{S_n}(\langle \zeta \rangle)$. To compute $\text{LR}_{S_n}(\langle \zeta \rangle)$ we prove a series of results about the “left Weyl modules” $\Delta^\lambda$, for $\lambda \in \mathcal{P}_n$. By Lemma 7.3, $W^\lambda = \Delta^\lambda$ so all of these results translate into statements about Weyl modules, which we leave as an exercise for the reader.

First, we need a fact that is of the folklore for $H_n$ but it does not seem to be in the literature. In level one, this is a result of Dipper and James [15, Lemma 4.1].

7.4. Lemma. Suppose that $m_\nu H_n n_\mu \neq 0$, for $\sigma, \nu \in \mathcal{P}_n$. Then $\nu \geq \sigma$.

Proof. By [18, Theorem 4.14], and as discussed in Chapter 4, $M^\sigma = m_\sigma H_n$, has basis $\{m_\sigma|S \in T_{row}(\tau, \sigma), \tau \in \text{Std}(\tau) \text{ for } \tau \in \mathcal{P}_n\}$. Therefore, $m_\sigma H_n n_\mu \neq 0$ only if there exist tableaux $S \in T_{row}(\tau, \nu)$ and $t \in \text{Std}(\tau)$, for some $\tau \in \mathcal{P}_n$, such that $m_\nu H_n \neq 0$. Hence, $\nu \geq \sigma$ by Proposition 3.8. Consequently, $\nu \geq t$ by Proposition 3.8, as required. \[\square\]

Recall the element $n_\mu \in H_n$ from Chapter 3. Let $\theta_\mu$ be the natural embedding of $N^\mu = m_\mu H_n$ into $H_n = M^\mu$ and set $E^\mu = S_\mu \theta_\mu$. Then $E^\mu$ is a left $S_n$-submodule of $\text{Hom}_{S_n}(N^\mu, M)$. (The right $S_n$-module $\bigoplus_\mu E^\mu_\mu$ is the full tilting module underpinning the proof of Theorem 4.7; see [41, Theorem 6.18].)

The remaining results in this section depend on [41], which only considers the non-degenerate Hecke algebras, so henceforth we assume that $\xi^2 \neq 1$. In view of Definition 3.2 and [28, Corollary 2.10], the arguments of [41] should extend to the degenerate case, however, these results do not appear in the literature so we cannot use them.

For the proof of the next result recall from Chapter 4 that $T_{row}(\nu, \mu)$ and $T_{col}(\nu, \mu)$, respectively, are the sets of row and column standard $\nu$-tableau of type $\mu$. If $T \in T_{col}(\nu, \mu)$ let $T$ be the unique standard $\nu$-tableau such that $\text{col}_\nu(T) = T$ and $\ell(d(t)) \leq \ell(d(t))$ whenever $t$ is standard and $\text{col}_\mu(t) = T$. Mirroring the definition of the $\varphi$-basis of $S_n$, for $U \in T_{col}(\varphi, \mu)$ and $V \in T_{col}(\varphi, \nu)$, where $\varphi \in \mathcal{P}_n$, define $\psi_{\nu \varphi} \in \text{Hom}_{S_n}(N^\mu, N^\nu)$ by $\psi_{\nu \varphi}(n_\mu h) = \nu m_\nu h$, for $h \in H_n$ and $n_\nu = \sum_{\nu \in \text{Std}(\nu)} (-\xi)^{-r(d(\nu)) - r(d(\nu))} n_{\nu \varphi}$, where the sum is over all standard tableaux such that $\text{col}_\mu(u) = U$ and $\text{col}_\nu(v) = \nu$. Then Theorem 4.5 and Proposition 3.14 imply that $\{\psi_{\nu \varphi}\}$ is a cellular basis of $S_n(N)$.

The next result describes the left Weyl module $\Delta^\mu$ as a kernel intersection.

7.5. Lemma. Suppose that $\xi^2 \neq 1$ and that $\mu \in \mathcal{P}_n$ and for $\nu \in \mathcal{P}_n$ let $\delta_{\mu \nu} = \text{Hom}_{S_n}(E^\mu, E^\nu)$. Then
\[
\Delta^\mu = \bigcap_{\nu \geq \mu, \mu \in \delta_{\mu \nu}} \ker \Psi.
\]

Proof. In view of [41, Proposition 6.4], a basis of $E^\mu$ is given by the maps
\[
\{\theta_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)} = T_{col}(\sigma, \mu) \text{ for } \alpha, \sigma \in \mathcal{P}_n\}
\]
where $\theta_\sigma(n_\mu h) = m_\sigma n_\mu h$, for $h \in H_n$. As noted in [41, Theorem 6.5], this implies that $E^\mu$ has a filtration by left Weyl modules $\Delta^\sigma$ such that $(E^\mu : \Delta^\sigma) = \#T_{col}(\sigma, \mu)$. In particular, $(E^\mu : \Delta^\sigma) \neq 0$ only if $\mu \geq \sigma$. By [41, Proposition 7.1], the hom-space $\delta_{\mu \nu}$ has basis
\[
\{\psi_{\nu \varphi} \in \text{Hom}_{S_n}(\tau, \mu) \text{ and } \varphi \in \text{Std}(\nu, \varphi) \text{ for } \tau \in \mathcal{P}_n\},
\]
where $\psi_{\nu \varphi}(\theta) = \mu \circ \psi_{\nu \varphi}$ for $\theta \in E^\mu$. (As in Remark 3.6, the notation used in [41] is slightly different with what we use here in that we are working with left $H_n$-modules here, so some care must be taken when comparing our results with those of [41].) Armed with these facts we can prove the lemma.

Fix $S \in T_{row}(\sigma, \alpha), T \in T_{col}(\sigma, \mu), U \in T_{col}(\tau, \mu)$ and $V \in T_{col}(\tau, \nu)$. Then the $H_n$-module homomorphism $\psi_{\nu \varphi} \in E^\nu$ is completely determined by
\[
\psi_{\nu \varphi}(\theta_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)}) = (\theta_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)})(U)(V) = \theta_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)}(U)(V) = m_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)} = m_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)},
\]
where $n_\mu = n_\mu h_{\varphi}$. In particular, $\psi_{\nu \varphi}(\theta_{T_{row}(\nu, \mu), T_{col}(\nu, \mu)}) \neq 0$ if $\nu \geq \sigma$ by Lemma 7.4.

By construction, the module $\Delta^\mu$ has basis $\{\theta_{T_{row}(\mu, \alpha), T_{col}(\mu, \mu)} = T_{col}(\sigma, \mu) \text{ for } \alpha \in \mathcal{P}_n\}$, so $\Delta^\mu$ is a submodule of $E^\mu$. Indeed, $\mu \geq T_{row}(\sigma, \mu)$ so the map $\phi \mapsto \phi_{T_{row}(\sigma, \mu)}$ is an injection. Hence, taking $\sigma = \mu$ and $T = T_{row}(\mu, \mu)$ in (7.6), $\Delta^\mu \subseteq \bigcap_{\nu \geq \mu, \mu \in \delta_{\mu \nu}} \ker \Psi$.

Conversely, if $\sigma \neq \mu$ and there exist tableaux $S \in T_{row}(\sigma, \alpha)$ and $T \in T_{col}(\sigma, \mu)$ then $\theta_{T_{row}(\alpha, \sigma), T_{col}(\sigma, \mu)} \in E^\mu$ and $\mu \geq \sigma$ since $T_{col}(\sigma, \mu) \neq 0$. Moreover, $\psi_{\nu \varphi}(\theta_{T_{row}(\alpha, \sigma), T_{col}(\sigma, \mu)}) \neq 0$ by (7.6) and Proposition 3.8. That is, $\theta_{T_{row}(\sigma, \mu)} \in \ker \psi_{\nu \varphi}$. This calculation is independent of $S$ and, in fact, it shows that $\psi_{\nu \varphi}(\sum c_\sigma \theta_{T_{row}(\sigma, \mu)}) \neq 0$ for any scalars $c_\sigma \in K$, which are not all zero. More generally, suppose that $\sigma = \sum_{\nu \in \delta_{\mu \nu}} \psi_{\nu \varphi} \in E^\mu$, for
some scalars $c_{pq} \in K$ such that $c_{pq} \neq 0$ only if $\text{Shape}(V) \triangleright \mu$. Fix $(S, T)$ such that $c_{ST} \neq 0$ and where $T$ is minimal in the sense that $T \ntriangleright V$ whenever $c_{pq} \neq 0$ for some $(u, V)$. Combining Proposition 3.8 with what we have just shown, it follows that $\Psi_{\tau \pi}(\theta) = \sum_{\sigma \in T_{row}(\sigma)} c_{ST} \Psi_{\tau \pi}(\theta_{ST}) \neq 0$.

Combining the last two paragraphs completes the proof. □

7.7. Corollary. Suppose that $\xi_1 \neq 1$ and that $\mu \in \mathcal{P}_n$. Then $\Delta^\mu = E^\mu \cap LR_{S_n}(\zeta_{\mu})$.

Proof. We have $\Delta^\mu = S_n \zeta_{\mu} \subseteq E^\mu \cap LR_{S_n}(\zeta_{\mu})$ since $\zeta_{\mu} \in E^\mu$. On the other hand, if $\nu \ntriangleleft \mu$ then every homomorphism $\Psi \in \mathcal{P}_{\mu}$ is given by right multiplication by some element $\psi$ by (7.6). Moreover, $\Delta^\mu \subseteq \ker \Psi$ Lemma 7.5, so if $\psi \in S_n$ then $\zeta_{\mu} \psi = \Psi(\zeta_{\mu}) = 0$. Therefore, if $x \in E^\mu \cap LR_{S_n}(\zeta_{\mu})$ then $\Psi(x) = x\psi = 0$. Therefore, $E^\mu \cap LR_{S_n}(\zeta_{\mu}) \subseteq \ker \Psi$, for all $\Psi \in \mathcal{P}_{\mu}$. The corollary now follows by Lemma 7.5. □

7.8. Lemma. Suppose that $\xi_1 \neq 1$ and $\kappa_r \neq \kappa_s \mod (c, Z)$, for $1 \leq r < s \leq \ell$. Then $LR_{S_n}(n_{\mu}) = E^\mu$.

Proof. On the one hand, we have that $E^\mu \subseteq LR_{S_n}(\theta_{\mu}) = LR_{S_n}(n_{\mu})$ by definition. On the other hand, under exactly these assumptions, [41, Corollary 6.11] says that

\[
E^\mu = \text{Hom}_{S_n}(M, N^\mu) \cong \text{Hom}_{S_n}(N^\mu_1, M_L) \cong \text{Hom}_{S_n}(1_\omega(S_n\theta_{\mu}), 1_\omega S_n) \\
\cong \text{Hom}_{S_n}(1_\omega S_n \ominus_{\mu} S_n \lambda_{\mu}, 1_\omega S_n) \cong \text{Hom}_{S_n}(S_n \lambda_{\mu}, \text{Hom}_{S_n}(1_\omega S_n, 1_\omega S_n)) \cong \text{Hom}_{S_n}(S_n \lambda_{\mu}, S_n).
\]

Therefore, $E^\mu = LR_{S_n}(\theta_{\mu}) \cap S_n = LR_{S_n}(n_{\mu}) \cap S_n = LR_{S_n}(n_{\mu})$ by Lemma 2.25(a). □

7.9. Lemma. Suppose that $\xi_1 \neq 1$ and $\mu' \in K'_n$. Then $LR_{S_n}(\zeta_{\mu}) \subseteq E^\mu$.

Proof. By Corollary 3.15, $D^\mu \neq 0$ and so $n_{\mu} H_n n_{\mu} \notin H_n^{\mu'\mu}$. In particular, $\zeta_{\mu} H_n n_{\mu} \neq 0$, so $\zeta_{\mu} \lambda_{n_{\mu}} \neq 0$. So, there exists $h \in H_n$ and $0 \neq c \in K$ such that $\zeta_{\mu} = c n_{\mu}$, or equivalently, $\zeta_{\mu} = c n_{\mu} = 0$. Suppose that $\phi \in LR_{S_n}(\zeta_{\mu})$. Then $\phi(c - h n_{\mu}) = 0$. Therefore, $c \phi = c \phi h n_{\mu} = 0$, so that $\phi \in S_n \mu_{\mu}$ since $c \neq 0$. Hence, by Lemma 7.8, $LR_{S_n}(\zeta_{\mu}) \subseteq S_n \mu_{\mu} \subseteq LR_{S_n}(n_{\mu}) = E^\mu$ as required. □

Before we can prove the main result of this section we need some analogous results for $RL_{S_n}(\zeta_{\lambda})$. We start with an analogue of Lemma 7.5 for $S_n$.

7.10. Lemma. Suppose that $\xi_1 \neq 1$ and that $\lambda \in \mathcal{P}_n$. Then

\[
\zeta_{\lambda} H_n = \{ m \in \pi \lambda H_n | \phi m = 0 \text{ for all } \phi \in 1_\omega S_n 1_\lambda \text{ for } \nu \in \mathcal{P}_n \text{ with } \lambda \ntriangleleft \nu \}.
\]

Proof. The argument is similar to the proof of Lemma 7.5 so we just sketch the proof. Let $X_{\lambda} = \{ m \in \pi \lambda H_n | \phi m = 0 \text{ for all } \phi \in 1_\omega S_n 1_\lambda \text{ for } \nu \in \mathcal{P}_n \text{ with } \lambda \ntriangleleft \nu \}$, so we need to show that $\zeta_{\lambda} H_n = X_{\lambda}$. In view of [41, Proposition 5.9], $\pi \lambda H_n$ has basis

\[
\{ \pi n_{\tau} | S \in T_{row}(\mu, \lambda) \text{ and } t \in \text{Std}(\mu) \text{ for some } \mu' \in \mathcal{P}_n \}
\]

and $\zeta_{\lambda} H_n$ is the $H_n$-submodule of $\pi \lambda H_n$ with basis $\{ \zeta_{\lambda} T_{d(t)}(1) | t \in \text{Std}(\lambda) \}$, where $S^0$ is the minimal standard $\lambda$-tableau (under the dominance order) such that $\text{row}_\mu(S^0) = \mu$. Observe that $\zeta_{\lambda} T_{d(t)}(1) = \pi n_{\tau} x_{\lambda}$, for $t \in \text{Std}(\lambda)$. If $\nu \in \mathcal{P}_n$ and $\nu \in T_{row}(\mu, \lambda)$ then $\varphi_{\mu}(\zeta_{\lambda} T_{d(t)}(1)) = m \nu H_n n_{\lambda} T_{d(t)}$. Consequently, if $\lambda \ntriangleleft \nu$ then $\phi_{\nu}(\zeta_{\lambda} T_{d(t)}(1)) = 0$. Therefore, $\zeta_{\lambda} H_n \subseteq X_{\lambda}$. To prove the reverse inclusion, if $\mu \ntriangleleft \lambda$ and $S \in T_{row}(\mu, \lambda)$ then $\mu \triangleright \lambda$ and $\varphi_{\nu S} \pi \lambda T_{d(t)}(1) \neq 0$ by Proposition 3.8. Arguing as in the last paragraph of Lemma 7.5 now completes the proof. □

7.11. Proposition. Let $\lambda \in \mathcal{P}_n$ and suppose that $\xi_1 \neq 1$. Then $\zeta_{\lambda} H_n = RL_{S_n}(\zeta_{\lambda}) \cap S_n 1_\lambda$.

Proof. Certainly, $\zeta_{\lambda} H_n = \zeta_{\lambda} S_n \cap S_n 1_\omega \subseteq RL_{S_n}(\zeta_{\lambda}) \cap S_n 1_\omega$. Conversely, suppose that $x \in RL_{S_n}(\zeta_{\lambda}) \cap S_n 1_\omega$. Then $x \in 1_\lambda S_n 1_\omega = 1_\lambda H_n$ since $(1 - 1_\lambda) \zeta_{\lambda} = 0$. If $\lambda \ntriangleleft \nu$ and $S \in T_{row}(\mu, \nu)$, $T \in T_{row}(\nu, \lambda)$ then $\varphi_{\nu T}(m \mu H_n n_{\lambda}) = 0$ since $m \mu H_n n_{\lambda} = 0$ by Lemma 7.4. Therefore, $\varphi_{\nu T} x = 0$ so that $x \in \zeta_{\lambda} H_n$ by Lemma 7.10. The result follows. □

Finally we can prove our cyclotomic generalisation of the Carter-Lusztig Theorem [13].

7.12. Theorem. Let $\mu, \lambda \in \mathcal{P}_n$ and assume that $\xi_1 \neq 1$ and that either

\footnote{Note that the element $n_{\nu'}$ in this paper corresponds to the element $n_{\nu'}$ in the notation of [41]. This accounts for the difference between our description of this basis and [41, Proposition 5.9].}
a) \( \mu' \in K_n' \), or
b) \( \xi^2 \neq -1 \) and \( \kappa_r \not\equiv \kappa_s \pmod{\ell} \), for \( 1 \leq r < s \leq \ell \).

Then \( \text{Hom}_{S_n}(W^\lambda, W^\mu) \cong \Delta^\mu \cap \chi^\lambda H_n \cong \text{Hom}_{H_n}(S^\lambda, S^\mu) \) as vector spaces.

Proof. By Lemma 7.3 and Proposition 7.11, \( \text{Hom}_{S_n}(W^\lambda, W^\mu) \cong \Delta^\mu \cap \chi^\lambda H_n \) and \( \text{Hom}_{H_n}(S^\lambda, S^\mu) \cong L_{S_n}(R_{S_n}(\chi^\mu)) \cap \chi^\lambda H_n \), so it is enough to show that \( L_{S_n}(\chi^\mu) = \Delta^\mu \), under the assumptions of the theorem. By Corollary 7.7, \( \Delta^\mu = E^\mu \cap L_{S_n}(\chi^\mu) \), so it is enough to show that \( L_{S_n}(\chi^\mu) \subseteq E^\mu \) by Lemma 7.2. If \( \mu' \in K_n' \) then this is immediate from Lemma 7.9. On the other hand, if the conditions in (b) hold then \( E^\mu = L_{S_n}(\eta_\mu) \) by Lemma 7.8, but \( L_{S_n}(\eta_\mu) \subseteq L_{S_n}(n_\mu) = E^\mu \), since \( \eta_\mu = n_\mu T_{n_\mu} n_\mu \), so again the result follows.

By Lemma 7.1, the Schur functor induces an injective map \( \text{Hom}_{S_n}(W^\lambda, W^\mu) \hookrightarrow \text{Hom}_{H_n}(S^\lambda, S^\mu) \). Under the assumptions of Theorem 7.12 this map is an isomorphism. If neither of the conditions in (a) and (b) in Theorem 7.12 hold then it is not difficult to find examples where these two hom-spaces are not isomorphic. Examples that show that the conclusions of Theorem 7.12 do not hold in general are easy to construct starting from easy observations that if \( \xi^2 = -1 \) then \( S^{(2)} \cong S^\xi \) and if \( \kappa_r = \kappa_s \) then \( S^{|r-s|} \cong S^{|r-s|} \), where \( 1 \leq r < s \leq \ell \) and \( \eta_r = (\eta_r^{(1)}, \ldots, \eta_r^{(t)}) \) is the multipartition with \( \eta_r^{(k)} = 1 \) if \( t = k \) and \( \eta_r^{(k)} = 0 \) otherwise. Compare with [41, Remark 6.10].

The assumption that \( \xi^2 \neq 1 \) in Theorem 7.12 is almost certainly unnecessary. As stated above, we include it because [41] does not consider the degenerate Hecke algebras, although as far as we have checked the arguments from [41] also apply when \( \xi^2 = 1 \). Under the assumptions in part (b) the isomorphism of Theorem 7.12 is implied by [47, Theorem 6.6]. As here, Rouquier’s argument relies on results from [41] and, in fact, the conditions in Theorem 7.12(b) come from [41, Theorem 6.9]. We state the full result here because almost no extra effort is required to prove Theorem 7.12 under the assumptions in parts (a) and (b).

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