A Theory Of Turbulent Particle Pair Diffusion: Locality Versus Non-locality

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Richardson’s 1926 theory of particle pair diffusion in a field of homogeneous turbulence is based upon the hypothesis of locality in space, and implies that in turbulence with generalised energy spectra \( E(k) \sim k^{-p} \), for \( 1 < p \leq 3 \), the turbulent pair diffusivity scales like \( D_{p}(\Delta) \sim \sigma^{(1+p)/2}_\Delta \), where \( \sigma_\Delta = \sqrt{\langle \Delta^2 \rangle} \), \( \Delta = |\Delta(t)| \) is the pair separation, and the angled brackets \( \langle \rangle \) denotes the ensemble average over particle pairs.

Here, we show through a novel mathematical analysis based upon the statistical theory of homogeneous turbulence that the turbulent pair diffusion in the inertial subrange is governed by both local and non-local diffusive processes, and that the pair diffusivity scales like,

\[
D_{p}(\Delta) \sim \sigma^{\gamma_p}_\Delta, \quad 1 < p \leq 3
\]

where the scaling exponent \( \gamma_p \) is such that \( (1 + p)/2 < \gamma_p < 2 \). The mean square separation scales like \( \langle \Delta^2 \rangle_p \sim t^{\chi_p} \) where \( t \) is the time and \( \chi_p = 1/(1 - \gamma_p/2) \).

Our analysis shows that there is some value of \( p = p_* < 5/3 \), where \( \gamma_{p_*} = 4/3 \), and where we obtain a non-Richardson 4/3-power law, \( D_{p_*} \sim \sigma^{4/3}_\Delta \), and a corresponding \( t^3 \)-regime, \( \langle \Delta^2 \rangle_{p_*} \sim t^3 \). For Kolmogorov turbulence we expect that \( \gamma_{5/3} > 4/3 \).

Key words: Turbulence, particle pair, relative, diffusion, pair diffusivity, locality, non-locality, correlations, non-Richardson.

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1. Introduction

Turbulent particle pair (relative) diffusion is important because it impacts upon many other turbulence related problems, such as the spread of pollution, inertial particle diffusion, bubbles in turbulence, and heat transport in turbulent flows. Whereas the mean distribution of a scalar quantity is related to single particle diffusion, fluctuations of the same quantity from its mean is related to the statistics of particle pair motion and it is often the fluctuations that are the most important quantities to model, (Durbin 1980, Thomson 1990, Malik et al 1992).

The most widely accepted theory for the diffusion of fluid particle pairs in a field of homogeneous turbulence is based upon Richardson’s locality in space hypothesis, in which for fluid particles located at $x_1(t)$ and $x_2(t)$ at time $t$, initially close together $\Delta_0 = |x_1(0) - x_2(0)| \ll L$, and such that their distance apart at later times $\Delta = |x_1(t) - x_2(t)|$ is within the inertial subrange of turbulent motions, $Max(\Delta_0, \eta) \ll \Delta \ll L$, the further increase in the pair separation is governed only by the energy spectrum at that scale $E(1/\Delta)$. This is equivalent to saying that only local quantities whose correlation extend over distances of the same scale as $\Delta$ are effective in this process. $L$ is the scale of energy containing eddies, $\eta$ is the Kolmogorov scale.

For Kolmogorov turbulence, since $E(1/\Delta)$ depends upon $\Delta$ and $\varepsilon$, Richardson locality implies that turbulence quantities such as the pair diffusivity $D = d\langle \Delta^2 \rangle / dt$, depend upon the local scale $\Delta$, and upon $\varepsilon$ the rate of kinetic energy dissipation per unit mass.

From geophysical data, Richardson assumed an approximate fit to the data for the turbulent pair diffusivity $D \sim \varepsilon^{2/3} \Delta^{4/3}$; this is equivalent to $\langle \Delta^2 \rangle \sim \varepsilon t^3$ (Obukov 1941) which is the best known form of Richardson’s theory, often referred to as the R-O $t^3$-regime. The angled brackets represents the ensemble average. However, a
review of Richardson’s fit clearly shows that other powers close to 4/3 are equally plausible.

And yet for a theory that has been so widely accepted and unchallenged by the scientific community since it was first proposed, it is surprising that it has never actually been verified with complete satisfaction because a very large inertial subrange is required in order to observe Richardson scaling, which presents formidable experimental and numerical challenges. As such locality still remains an unproven hypothesis.

It is believed that only large scale geophysical turbulent flows, such as in the atmosphere and seas, can produce inertial subranges large enough for this regime to be observable. Observations in geophysical flows suggesting the existence of a $t^3$-regime have been made by Tatarski (1960), Wilkins (1958), and Sullivan (1971). More recent observations include Julian et al (1977) in the atmosphere, and LaCasce and Ohlmann (2003), and Ollitrault et al. (2005) in the oceans.

Particle Tracking Velocimetry laboratory experiments (PTV, Malik and Dracos (1991), Malik et al (1993)) have been providing pair diffusion statistics at low to moderate Reynolds numbers. Virant and Dracos (1997) were the first obtain pair diffusion measurements from PTV. More recently, Berg et al. (2006) obtained measurements in a water tank at $R_\lambda = 172$, and Bourgoin et al. (2006) and Ouellette et al. (2006) tracked hundreds of particles at high temporal resolution at $R_\lambda = 815$.

DNS of particle pair studies at low Reynols numbers have been carried out by a number of researchers. The first was by Yeung (1994) at $R_\lambda = 90$. More recent DNS results are due to Boffetta and Sokolov (2002) at $R_\lambda = 200$, Ishihara and Kaneda (2002) at $R_\lambda = 283$, Yeung and Borgas (2004) at $R_\lambda = 230$, Sawford et al. (2008) at $R_\lambda = 650$, Scatamacchia et al. (2012) at $R_\lambda = 300$. The reader is referred to Salazar and Collins (2009) for more details.

A Lagrangian method, Kinematic Simulations (KS), has produced an approximate $t^3$-regime, Fung et al (1992). Since then, many studies using KS have...
re-produced similar results under different conditions, including in flows with
generalised power-law energy spectra of the form $E(k) \sim k^{-p}$ for $p > 1$; Fung
and Vassilicos (1996), Malik (1996), Thomson and Devenish (2005), Nicolleau
and Nowakowski (2011).

Although the general consensus among scientists in this field at the current time
is that the collection of observational, experimental and DNS data suggests a
convergence towards a Richard-Obukov scaling, the error levels, low Reynolds
and other assumptions made in analysing the data means that this is by no
means a forgone conclusion, as noted by Salazar and Collins (2009), ".. there
has not been an experiment that has unequivocally confirmed R-O scaling over a
broad-enough range of time and with sufficient accuracy."

DNS is even further behind than experiments because current DNS has produced
turbulence with very short and approximate inertial subranges. For example,
Ishihara et al (2009) perform a DNS with $4096^3$ with $Re_\lambda \sim 1200$, with an
inertial range of just $\sim 40$ (the flat range in Fig. 3 in their paper). It is believed
that we need an inertial range of around $10^5$ in order to observe an unequivocal
inertial range scaling in the pair diffusion.

The purpose of this paper is to re-examine turbulent pair diffusion in the inertial
subrange and to propose a new theory that accounts for both local and non-local
diffusion processes. Our main focus will be to obtain scaling laws for the pair
diffusivity through a new formal derivation based upon the statistical theory of
homogeneous turbulence and to estimate the relative magnitude of the contribu-
tions to the pair diffusivity from local and non-local processes in the inertial
subrange.

In section 2, we review Richardson’s locality theory of particle pair diffusion.
In section 3, we propose a new mathematical scheme for deriving an expression
for the turbulent pair diffusivity $D_p$ in turbulence with generalised power-law
turbulent energy spectra of the form $E(k) \sim k^{-p}$ for $1 < p \leq 3$. In section 4,
we investigate the implication of locality and non-locality, and obtain the general
2. Richardson’s 1926 locality

In 1926, Richardson published a paper entitled 'Atmospheric diffusion shown on a distance-neighbor graph' in which he set out a theory of how pairs of particles initially close together move apart due to the effects of atmospheric winds and turbulence. The problem is to determine the pair diffusivity $D = \langle \Delta \cdot v \rangle$ and the related mean square separation $\langle \Delta^2(t|\Delta_0, t_0) \rangle$ of a pair of fluid elements at $x_1(t)$ and $x_2(t)$ at time $t$, where $\Delta(t) = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2} = |x_2(t) - x_1(t)|$. The initial separation is denoted by $\Delta_0 = |x_2(t_0) - x_1(t_0)|$ at some earlier time $t_0$. $v(t) = u_2(t) - u_1(t)$ is the pair relative velocity at time $t$.

It is well known that there exists an initial ballistic regime for very short times due to the high correlation in time with the initial conditions, which leads to $\langle \Delta^2 \rangle = \Delta_0^2 + \langle v^2(\Delta_0) \rangle (t - t_0)^2$, for $|t - t_0| \ll T_{min}$ where $\langle v^2(\Delta_0) \rangle$ is the ensemble mean square relative velocity in the flow across a distance $\Delta_0$. The ballistic regime is true in any flow field and is valid for a very short time here designated by $T_{min}$.

After this time, if $\Delta$ is still smaller than the Kolmogorov scale, then the pair separation will increase due to the pure strain set up by the larger scales of motion leading to exponential growth, Fung et al. (1992), until the separation enters the inertial subrange.

At much larger times, when the individual particles in a pair are separated by distances of the order of the integral length scale or greater, their motions become independent and the pair diffusion collapses to twice the one-particle Taylor diffusion, $\langle \Delta^2 \rangle \to (12T_L(u')^2)t$, where $T_L$ is the turbulent Lagrangian time scale, and $u'$ is the rms turbulent velocity fluctuations (in one direction).

We will not consider these regimes any further, as our interest here is in the inertial subrange scaling.
In turbulent pair diffusion studies, it is commonly assumed that the pair initial separation is smaller than the Kolmogorov scale, \( \Delta_0 < \eta \), although this is not a formal requirement of the theory. The particles will diffuse apart and eventually decorrelate with the initial conditions – they will 'forget' their initial conditions, \((\Delta_0, t_0)\) – after some travel time \(t_{\Delta_0}\) (Batchelor 1952, Fung et al. 1992) before they enter inside the inertial subrange of turbulent motions.

Without loss of generality, henceforth we will assume that \(t_0 = 0\). If the pair loses dependency on the initial separation after an (approximate) travel time \(t_{\Delta_0}\) when the separation is \(\delta = \sqrt{\langle \Delta^2 \rangle(t_{\Delta_0})} \), then it is from \(t = t_{\Delta_0}\) that Richardson’s hypothesis should be applied.

Richardson’s locality hypothesis states that for pairs of particles whose separation is within the inertial subrange of turbulent motions, \(\eta \ll \Delta \ll L\), the pair diffusion process is governed by the energy spectrum at the local scale, \(E(1/\Delta)\); and since the energy spectrum is a function of \(1/\Delta\) and \(\varepsilon\), the rate of kinetic energy dissipation per unit mass, this means that the pair diffusion process depends only upon \(\Delta\) and \(\varepsilon\).

This implies the following scaling for the pair separation, Obukov (1941),

\[
\langle \Delta^2(t) \rangle = \delta^2 + g\varepsilon(t - t_{\Delta_0})^3 \quad \text{for } t > t_{\Delta_0}
\]  

(2.1)

\(g\) is assumed to be a universal constant of order \(O(1)\).

The appearance of the adjusted travel time \(\tau = t - t_{\Delta_0}\) is important. As \(\tau \to \infty\) then we expect that \(\tau \to t\) and the above expression approaches \(\langle \Delta^2(t) \rangle \to g\varepsilon t^3\), so long as the mean square separation remains inside the inertial subrange. However, for finite times the use of the adjusted travel time is useful in observing longer ranges over which the relation (2.1) can be observed, particularly in turbulence with short inertial subranges where using the raw time in plotting (2.1) can be misleading.

\(t_{\Delta_0}\) need not be precisely determined, so long as it is close to when initial conditions have been 'forgotten'. But it has been found (Fung et al. 1992) that taking
$t_{\Delta_0}$ to be the time when the pair ensemble distance is equal to the Kolmogorov scale, $\delta(t_{\Delta_0}) = \sqrt{\langle \Delta^2 \rangle} = \eta$ gives good results. (But when considering generalised power law spectra, there will be occasions when we need to start with initial separation that is greater than the Kolmogorov scale, so that $\eta < \delta \ll L$. In this case, $\delta$ is not an ensemble average but each pair is equal to this initial separation.)

Richardson’s original formulation was in terms of the turbulent pair diffusivity, which is obtained from equation (2.1) by differentiating with respect to time,

$$D = \frac{d\langle \Delta^2 \rangle}{dt} \sim \varepsilon (t - t_{\Delta_0})^2 \sim \varepsilon^{1/3} (\langle \Delta^2 \rangle - \delta^2)^{2/3} \quad (2.2)$$

or

$$D \sim \varepsilon^{1/3} \langle \Delta^2 \rangle^{2/3} \quad \text{when} \quad \delta \ll \sqrt{\langle \Delta^2 \rangle} \ll L \quad (2.3)$$

Richardson presented this as $D \sim \Delta^{4/3}$ based upon observations in geophysical flow, and is often referred to as the 4/3-law for the turbulent diffusivity.

An advantage of working with the relative diffusivity is that it eliminates the initial conditions and an adjusted travel time from the expression, so there is less ambiguity in interpreting the results.

### 3. A theory based upon local and non-local correlations

In this section, we derive an expression for the pair diffusivity through a formal analysis based upon the statistical theory of turbulence which sheds new light upon closure assumptions and physical reasoning hidden behind simple scaling arguments.

#### 3.1. Mathematical framework

The pair relative velocity is defined as $\mathbf{v} = d\Delta/dt$, and the pair diffusivity is defined formally by the ensemble average of the scalar product of $\mathbf{v}$ with $\Delta$,

$$D = \langle \Delta \cdot \mathbf{v} \rangle = \frac{1}{2} \frac{d\langle \Delta^2 \rangle}{dt} \quad (3.1)$$
We are interested in the scaling laws, so we will follow a policy in the rest of the paper of suppressing constants wherever possible.

For homogeneous, isotropic, incompressible, reflectional and statistically stationary turbulence in an infinite domain, the statistical theory of homogeneous turbulence, Batchelor (1953), gives the well known expression for the velocity $u(x)$ at position $x(t)$,

$$u(x) = \int A(k) \exp (i k \cdot x) \, d^3k$$  \hspace{1cm} (3.2)

$A$ is the Fourier transform of the flow field, $k$ is the associated wavenumber.

The quantity of interest is the relative velocity $v$ across a finite displacement $\Delta(t)$, viz

$$v(\Delta) = u(x_2) - u(x_1)$$

where $\Delta(t) = x_2(t) - x_1(t)$ and $x_1(t)$ and $x_2(t)$ are the locations of particles (fluid elements) 1 and 2 respectively at time $t$. The relative velocity is then given formally by,

$$v = \int [A \exp (i k \cdot (x_1 + \Delta)) - A \exp (i k \cdot x_1)] \, d^3k$$  \hspace{1cm} (3.3)

which simplifies to,

$$v = \int A [\exp (i k \cdot \Delta) - 1] \exp (i k \cdot x_1) \, d^3k.$$  \hspace{1cm} (3.4)

Taking the scalar product with $\Delta$, we obtain

$$\Delta \cdot v = \int (\Delta \cdot A)[\exp (i k \cdot \Delta) - 1] \exp (i k \cdot x_1) \, d^3k$$  \hspace{1cm} (3.5)

We designate by $\langle \cdot \rangle$ the ensemble average over particle pairs in many flow fields, to obtain and expression for the pair diffusivity,

$$D(\Delta) = \langle \Delta \cdot v \rangle = \int ((\Delta \cdot A)[\exp (i k \cdot \Delta) - 1] \exp (i k \cdot x_1)) \, d^3k$$  \hspace{1cm} (3.6)

Because of homogeneity, the ensemble average in (3.6) averages out the factor $\exp (i k \cdot x_1)$. We also assume that the scalings of pair statistics based upon $\Delta$
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will be unaffected by this averaging. (Note that although \( \langle \exp (i \mathbf{k} \cdot \mathbf{x}_1) \rangle = 0 \), the right hand side contains other factors that ensures a non-zero ensemble average.) We thus obtain an expression for the turbulent pair diffusivity scaling,

\[
D(\Delta) \approx \int \langle (\mathbf{A} \cdot \mathbf{k}) \exp (i \mathbf{k} \cdot \Delta) \rangle \, d^3 \mathbf{k}.
\]

(3.7)

Let the wavemode \( k \) be the magnitude of the wavenumber \( \mathbf{k} \), i.e. \( k = |\mathbf{k}| \). And let \( k_\Delta(t) = 1/\Delta(t) \) be the wavemode associated with the instantaneous pair separation at time \( t \).

It is tempting to approximate the integrand in (3.7) by expanding the exponential term such that \( \exp (i \mathbf{k} \cdot \Delta) - 1 \approx i \mathbf{k} \cdot \Delta \). However, the accuracy of this expansion depends on the range of \( |\mathbf{k}| \). For a given separation this expansion will be accurate for small \( |\mathbf{k}| \ll k_\Delta \) because \( |\mathbf{k} \cdot \Delta| \ll 1 \); but for \( |\mathbf{k}| \approx k_\Delta \), we have \( |\mathbf{k} \cdot \Delta| \approx 1 \) and the expansion is fair but implies some corrections. For \( |\mathbf{k}| \gg k_\Delta \), we have \( |\mathbf{k} \cdot \Delta| \gg 1 \) and the expansion to leading order is not accurate.

3.2. Three diffusion processes and uncertainty

Here, we make the main assumption about the nature of the physical processes occurring in the system, namely that there are three distinct and broadly independent physical processes that contribute to the pair diffusivity \( D(\Delta) \), each process acting from its own range of wavenumbers with the associated frequencies.

For a given pair separation \( \Delta \), the three physical processes operate, respectively, over the scales of motion that are smaller than \( \Delta \), over the scales that are local to \( \Delta \), and over the scales that are non-local to \( \Delta \). The associated frequencies are given by, \( \omega(k) \propto \sqrt{k^3 E(k)} \), assuming that the frequencies are close to the inverse turnover time of the eddy at wavenumber \( k \) according to usual thinking. At the local scales, the average frequency is \( \omega_\Delta \sim \sqrt{k_\Delta^3 E(k_\Delta)} \), and the turnover time is \( T_\Delta \sim 1/\omega_\Delta \).

On this physical basis, we partition the integral in equation (3.7) in to three wavenumber ranges, defined by:
s : small scales such that $k \gg k_\Delta$, and $|k \cdot \Delta| \gg 1$, and the associated frequencies are much larger than $\omega_\Delta$, $\omega(k) \gg \omega_\Delta$.

l : local scales such that $k \approx k_\Delta$, and $|k \cdot \Delta| \approx 1$, and the associated frequencies are of the same order as $\omega_\Delta$, $\omega(k) \approx \omega_\Delta$.

nl : non-local scales such that $k \ll k_\Delta$, and $|k \cdot \Delta| \ll 1$, and the associated frequencies are much smaller than $\omega_\Delta$, $\omega(k) \ll \omega_\Delta$.

With this partitioning, (3.7) becomes

$$D(\Delta) \approx \left( \int_{nl} + \int_l + \int_s \right) \langle (\Delta \cdot A) (\exp (ik \cdot \Delta) - 1) \rangle d^3k. \quad (3.8)$$

We rephrase this as,

$$D(\Delta) \approx D_{nl} + D_l + D_s. \quad (3.9)$$

There is uncertainty in defining the 'size' of a turbulent eddy, a problem which is inherent to the nature of turbulence (Tennekes and Lumley 1972). Consequently, it is not possible to define the precise cutoffs between the local and non-local scales and between the local and small scales of motion. At first glance, this may appear to be an intractable problem. But we remark that such uncertainty is a defining characteristic of turbulence and is implicit even in Richardson’s locality hypothesis where a lack of precise definition of the size of a local eddy does not prevent scaling laws from being obtained. Likewise, we will show in the ensuing analysis that some asymptotic results and the general scaling for the relative diffusivity can also be obtained in the more general case considered here.

In view of such uncertainties, (3.9) will be viewed as much as a conceptual paradigm as an equation. We will treat each physical process independently of the others.
A simplification can be made with respect to the small scales whose contribution to the diffusion process is more or less universal. We have,

\[ D^s(\Delta) \approx \int \langle (\Delta \cdot A)(\exp (i\mathbf{k} \cdot \Delta) - 1) \rangle \, d^3\mathbf{k}. \]  

(3.10)

Fortunately, we do not need to evaluate this integral directly, because the net ensemble effect can be assessed on physical grounds alone. \( D^s \) is the integral over large wavemodes and represents the contribution from scales of turbulent motion which are smaller than the instantaneous pair separation \( \Delta \), i.e for \( k \gg k_\Delta \).

The energy contained in these small scales is very small if the energy spectrum is decreasing as \( k \) increases such as an inverse power law of the type \( E(k) \sim k^{-p} \), with \( p > 1 \).

Furthermore, the small scales are associated with unsteadiness of high frequency, \( \omega^s(k) \gg \omega_\Delta \). Statistically, these high frequency motions induce random and rapid changes in the direction and magnitude of \( \Delta \).

Thus overall we expect that the changes in the statistics of \( \Delta \) induced by the action of the random small scale low energy turbulent velocity fluctuations to be extremely small.

We therefore assume that the net ensemble effect of the high wavenumbers \( k \gg k_\Delta \) in the relative diffusion process is negligible, i.e. \( D^s \ll \text{Max}(D^l, D^{nl}) \), and henceforth neglect this term in (3.9).

3.4. The physics of the local and non-local scales of motion

With the effect of small scale contributions eliminated, we are left with the simplified expression for the pair diffusivity,

\[ D(\Delta) \approx \int_{nl} \langle (\Delta \cdot A)(\exp (i\mathbf{k} \cdot \Delta) - 1) \rangle \, d^3\mathbf{k} + \int_l \langle (\Delta \cdot A)(\exp (i\mathbf{k} \cdot \Delta) - 1) \rangle \, d^3\mathbf{k} \]

\[ \approx D^{nl} + D^l \]  

(3.11)
As mentioned earlier, the expansion of the exponential in the integrand to leading order is accurate in the non-local range where $|k \cdot \Delta| \ll 1$, but in the local range where $|k \cdot \Delta| \approx 1$ such an expansion is only moderately accurate and some assessment of the ensemble effect of the corrections must be made.

Physically, the frequencies $\omega^l(k)$ associated with the local wavemodes is of the same order as $\omega_\Delta$. Such frequencies represent a moderate level of unsteadiness which will likewise moderately affect the local diffusion process without killing it entirely (as with the small scales).

We assume therefore that the ensemble effect of the unsteadiness of the local wavemodes is to quantitatively reduce the magnitude of the local diffusivity, but without altering its overall scalings. Then we can then still use the expansion of the exponential to leading order as $\exp(i k \cdot \Delta) - 1 \approx i k \cdot \Delta$ in $D^l$ but with the understanding that the net contribution in the magnitude of $D^l$ is smaller than expected.

With this assumption, expanding the exponential in (3.11), we obtain

$$D(\Delta) \approx \int_{nl} \langle (\Delta \cdot A)(i k \cdot \Delta) \rangle d^3k + F_l \int_l \langle (\Delta \cdot A)(i k \cdot \Delta) \rangle d^3k$$  \hspace{1cm} (3.12)

where $F_l = F_l(p, K, C) < 1$ is a correction factor accounting for the effect of the moderate local unsteadiness. We do not expect $F_l$ to be a universal constant because it will depend upon $p$, $C$ and $K$, where $K = k_\Delta/k_1 = L/\Delta$, and $C$ is a measure of the 'size' of local eddy in wavenumber space (more on this in section 4). After absorbing constants, we can re-write the integrand as,

$$\langle \Delta^2 |A||k| \cos(\alpha) \cos(\beta) \rangle$$ \hspace{1cm} (3.13)

where $\alpha$ is the angle between $\Delta$ and $A$, and $\beta$ is the angle between $\Delta$ and $k$. For isotropic random fields averaging the above expression over all directions, again, does not affect the scaling. (The distributions of $\alpha$ and $\beta$ are not uniform in direction; it is well known that $\Delta$ aligns preferentially in the positive strain directions, which ensures that the ensemble average above is non-zero.)
Retaining the angled brackets $\langle \cdot \rangle$ to include averaging over all directions, (3.12) with (3.13) simplifies to,

$$D(\Delta) \approx \int \int_{nl} \langle \Delta^2 ak \rangle \ dA(k) \ dk + F_l \int \int \langle \Delta^2 ak \rangle \ dA(k) \ dk$$

(3.14)

where $a = |A|$ and $k = |k|$ are the magnitudes of their respective vectors. $dA(k)$ is the element of surface area at radius $k$ is wavenumber space.

If we assume the closure $\langle \Delta^2 ak \rangle \sim \langle \Delta^2 \rangle \langle ak \rangle$, this integral scales like,

$$D(\Delta) \sim \langle \Delta^2 \rangle \int \int_{nl} \langle ak \rangle \ dA(k) \ dk + F_l \langle \Delta^2 \rangle \int \int \langle ak \rangle \ dA(k) \ dk$$

(3.15)

(Note that because $k$ and $a$ are magnitudes, the average $\langle ka \rangle \neq 0$ even though the vectors $k$ and $A$ are orthogonal.)

From the statistical theory of turbulence (Batchelor 1953), $\int \langle a^2 \rangle \ dA(k)$ represents the energy density per unit wavemode averaged over all direction, which scales like $\sim E(k)/k$. If we assume the closure, $\int \langle ak \rangle \ dA(k) \sim k \sqrt{\int \langle a^2 \rangle \ dA(k)}$, then (3.15) becomes,

$$D(\Delta) \sim \langle \Delta^2 \rangle \int \int_{nl} \sqrt{kE(k)} \ dk + F_l \langle \Delta^2 \rangle \int \int \sqrt{kE(k)} \ dk$$

(3.16)

As a further check, this can also be derived as follows. The velocity variance due to scales $k$ to $k + dk$ is $E(k)dk$, and the variance of velocity gradient is $k^2 E(k)dk$. Particle pair velocity difference variance $\sim \langle \Delta^2 \rangle k^2 E(k)dk$. Time scale of eddies of wave number $k$ is $1/\sqrt{k^3 E(k)}$. So the incremental contribution to the diffusivity due to these scales is velocity difference variance times time scale, $dD \sim \langle \Delta^2 \rangle \sqrt{kE(k)}dk$, which leads to (3.16).

To make further progress we need to specify the actual form of the turbulence spectrum. The turbulence will have an upper scale and a small scale between which the turbulence actually exists. Furthermore, in this work our focus is upon high Reynolds number turbulence which contains a large inertial subrange. For pair diffusion statisticstics, the form of the energy spectrum at the very highest length scales is immaterial. Thus, in the following, we will assume a generalised inverse power-law inertial energy spectrum in a large but finite range of wavenum-
where \( C_p \) is a constant. A lengthscale \( L \) is necessary for dimensional consistency. \( L \) scales with any lengthscale that is characteristic of the large scale forcing that is assumed to be driving the turbulence cascade – typically this is the integral length scale. Such generalised power law spectra are commonly used in turbulent pair studies, see the references after equation (4.3).

As we have now introduced the generalised power spectrum \( \sim k^{-p} \), henceforth we will add a suffix ‘\( p \)’ to important quantities to denote their dependence on the energy spectrum. Let \( \sigma_\Delta^2 = \langle \Delta^2 \rangle \), then equation (3.16) with (3.17) becomes,

\[
D_p(\Delta) \sim \varepsilon^{1/3} L^{(5/3-p)/2} \sigma_\Delta^2 \langle \Delta^2 \rangle \int_{nl} k^{(1-p)/2} \, dk + \int_{l} F l \varepsilon^{1/3} L^{(5/3-p)/2} \sigma_\Delta^2 \langle \Delta^2 \rangle \int_{l} k^{(1-p)/2} \, dk
\]

(3.18)

4. Locality and non-locality

Expression (3.18) represents a general mathematical formulation for the turbulent particle pair diffusivity and it is valid for both local and non-local theories up to this point.

In order to check the effectiveness of the closures and scalings assumed in (3.18), the locality limit must be derived from it, for which we have a known theory (Richardson), and see if it gives the correct result.

4.1. Validation: locality and the influence of local scales

Richardson’s locality hypothesis corresponds to the assumption that the non-local term in (3.18) is zero, thus retaining just the second term which is the contribution from wavemodes local to \( k_\Delta \). In order to evaluate this integral, we assume some cutoff \( k_* \) such that locality is valid in the range \([k_*, k_\Delta]\). Upon integrating over
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In this range, we obtain,

\[ D_l \sim F_l \varepsilon^{1/3} L^{(5/3-p)/2} \sigma_\Delta^2 \frac{2k^{(3-p)/2}_\Delta}{3-p} \left( 1 - \left( k_*/k_\Delta \right)^{(3-p)/2} \right). \]  \hfill (4.1)

\( k_* \) is unspecified, but locality implies that it must scale with \( k_\Delta \) in the local range of scales. Accordingly, we let \( k_* = k_\Delta / C \) for some finite \( C, 1 < C < \infty \), to obtain,

\[ D_l \sim F_l \varepsilon^{1/3} L^{(5/3-p)/2} \sigma_\Delta^2 k_\Delta^{(3-p)/2} \frac{2}{3-p} \left( 1 - \frac{1}{C^{(3-p)/2}} \right). \]  \hfill (4.2)

\( C = C(p, K) \) is again not expected to be a universal constant. \( C \) represents the ‘size’ of a local eddy in wavenumber space. It is a function of the energy spectrum \( \sim k^{-p} \) which becomes shallower as \( p \to 1 \), so there is relatively more energy in the local scales as \( p \) approaches 1. This indicates that the \( C \) probably increases in this limit. Conversely, \( C \) decreases closer towards 1 as \( p \to 3 \). \( C \) may also be a function of \( K \).

If we take \( k_\Delta \sim 1/\sigma_\Delta \), we obtain from (4.2) the turbulent pair diffusivity scaling based upon locality to be,

\[ D_l \sim F_l \varepsilon^{1/3} L^{(4/3-\gamma_p)/2} \sigma_\Delta^{\gamma_p} k_\Delta^{\gamma_p} \]  with \( \gamma_p = (1 + p)/2 \) \hfill (4.3)

where \( \sigma_\Delta = \sqrt{\langle \Delta^2 \rangle} \). For Kolmogorov turbulence \( p = 5/3 \), we obtain \( D_l^{5/3} \sim \varepsilon^{1/3} \sigma_\Delta^{4/3} \) which recovers the Richardson’s 4/3-scaling (assuming similarity with \( \sim \Delta^{4/3} \)).

The scaling (4.3) has been proposed on the basis of dimensional arguments alone in 2D by Morel and Larchaveque (1974), and in both 2D and 3D by Malik (1996); see also Fung and Vassilicos (1996), Nicolleau and Nowakowski (2011).

Equation (4.3) reproduces the generalised Richardson locality scaling, \( \gamma_p = (1 + p)/2 \), and as such provides the necessary validation for the formal method proposed here, and justifies the various closures and scaling assumptions assumed in arriving at (4.3).
4.2. Non-locality and the influence of non-local scales

The non-local contribution is given by the first term in (3.18),

$$D_p^{nl} \sim \varepsilon^{1/3} L^{(5/3-p)/2} \sigma_\Delta^2 \int_{k_{nl}} \frac{k^{(1-p)/2}}{\omega_{nl}(k)} \, dk$$  (4.4)

Physically, scales of turbulence that are non-local to the pair separation, $k < k_\ast$, will set up local straining fields at the scale $\Delta$ which will incrementally increase the pair separation (on average). Previous theories have always assumed that such non-local effects are negligible. However, there are three factors that suggest that this may be an oversimplification.

Firstly, the larger scales at $k$ possess much larger energies than at $k_\Delta$, when $k \ll k_\Delta$, and this will increase their relative influence in the pair diffusion process.

Secondly, the time scale of the non-local scales, $T_{nl}(k) \sim 1/\omega_{nl}(k)$, are much larger than the local turnover time scale, $T_{nl}(k) \gg T_\Delta \sim 1/\omega_\Delta$. This means that the straining fields set up by non-local wavemodes will persist (on average) for times that are longer than the time scale $T_\Delta$ of the local wavemodes at the separation scale. This will also serve to enhance their effectiveness in the pair diffusion process.

(Note that these two factors are opposite to the effects of the small scales mentioned in section 3.1 where high frequencies and low energies nullified their contributions.)

Thirdly, although an individual non-local wavemode may indeed have a weak effect on the diffusion process, the integral is over a much larger part of the energy spectrum than in the local process. The cumulative effect over the entire range of non-local wavemodes will again enhance the effectiveness of the non-local scales in the diffusion process.

There is therefore at least a chance that the non-local contributions are significant in the turbulent pair diffusion process.

It is well known that non-local scales of motion that are sufficiently bigger than
the pair separation, $k \ll k_\Delta$, set up straining fields at the local scale $\Delta$ leading to diffusivities that are proportional to $\langle \Delta^2 \rangle$, (and to exponential growth in time in $\langle \Delta^2 \rangle(t)$). The expression in (4.4) already contains this information because an incremental contribution to the diffusivity from a strain-like field at wavemode $k$ is, $dD_{nl}(k) \sim \sigma^2 \Delta$.

We therefore define the range of non-local scales to be those which produce such straining motion at the separation scale $\Delta$, namely the range $k_1 \leq k < k^*_\Delta$. Physically, when considering straining motion in this range, non-locality implies that we simply treat $k^*_\Delta$ as a large number.

In (4.4) we change variables in the integral to $s = k/k_1$ to obtain,

$$D_{nl} \sim \epsilon^{1/3} L^{(5/3-p)/2} k_1^{(3-p)/2} \sigma^2 \Delta \int_{nl} s^{(1-p)/2} ds$$

The integral in the brackets is a definite integral and yields a non-dimensional number, call it $S_{nl} = S_{nl}(p, C, K)$. Again we do not expect $S_{nl}$ to be a universal constant. This gives,

$$D_{nl} \sim S_{nl} \epsilon^{1/3} L^{(5/3-p)/2} k_1^{(3-p)/2} \sigma^2 \Delta.$$  \hspace{1cm} (4.5)

If we further assume that the upper end of the inertial range scales with the integral scale, $k_1 \sim 1/L$, then this simplifies to,

$$D_{nl} \sim S_{nl} \epsilon^{1/3} L^{-2/3} \sigma^2 \Delta, \quad \gamma_{nl}^p = 2.$$  \hspace{1cm} (4.6)

Note that $\gamma_{nl}^p$ is the non-locality scaling, and it is always equal to 2, independent of $p$. $D_{nl}^p$ is strain dominated, being proportional to $\sigma^2 \Delta$.

4.3. A general expression for the pair diffusivity $D_p$

4.3.1. The formal scaling for $D_p$

The scaling for the turbulent pair diffusivity is to leading order the sum of the local and non-local contributions, (4.3) and (4.7)

$$D_p \sim O \left( F \epsilon^{1/3} L^{(4/3-\gamma_p^l)} \sigma^2 \Delta \right) + O \left( S_{nl} \epsilon^{1/3} L^{-2/3} \sigma^2 \Delta \right).$$  \hspace{1cm} (4.8)
As a conceptual picture, this expression illustrates that there are broadly two physical processes contributing to the pair diffusion process, namely the local and non-local processes which are essentially decoupled from each other, and with their own scalings and unknown factors, $F_l$ and $S_{nl}$.

In deriving this expression the pair diffusivity is expressed naturally in terms of $\sigma_\Delta$ (or equivalently $\sigma_\Delta^2 = \langle \Delta^2 \rangle$). Other scalings have been proposed in the past, such as $\langle |\Delta| \rangle^{4/3}$, and $\langle |\Delta|^{4/3} \rangle$, but additional closure assumptions must be made in these cases.

Locality has been adopted in previous pair diffusion theories as the most natural assumption. But it is remarkable that from the mathematical formulation developed here, locality emerges as an unwarranted assumption in the sense that the second term in (4.8) is simply neglected.

The current locality/non-locality theory that leads to (4.8) requires no such assumption and is therefore a more natural framework for turbulent pair diffusion studies.

4.3.2. Asymptotics and trends in the balance of local and non-local contribution

From (4.8), it is apparent that the local and non-local contributions are always present for $1 < p \leqslant 3$. However, there are many uncertainties surrounding the factors that define their actual strengths. Uncertainty comes from a number of sources, most importantly from the intrinsic uncertainty regarding the size of an eddy, $C$, which is a known problem in turbulence, and also from the unknown factors $F_l$ and $S_{nl}$ which account for the various closures and scalings assumed in the analysis.

For these reasons, although (4.8) contains the correct scalings, the quantitative balance between the local and non-local terms cannot be obtained by comparing the two terms in equation (4.8) directly.

But it will turn out that we do not need the exact balance of the terms – it will
be sufficient to obtain the asymptotic balances at $p = 1$ and $p = 3$, and its general trend in the range $1 < p < 3$. These are obtained as follows.

Starting from (4.2) and (4.4), the relative contributions to the pair diffusivity $M_D = D_{nl} / D_p$ as a function of $p$, $K$ and $C$ is estimated to be,

$$M_D(p, C, K) \approx 1 - \frac{(C/K)^{(3-p)/2}}{F_l((C^{(3-p)/2}) - 1)}$$ \hspace{1cm} (4.9)

As $p \to 1$, then $M_D \to (1 - C/K)/F_l(C - 1)$. For large inertial subrange $K \gg 1$, and if say $F_l \approx 0.5$ and $4 < C \ll K$ then $M_D < 1$, i.e. locality dominates.

As $p \to 3$, then $M_D \to (\ln(K)/\ln(C) - 1)/F_l$. If $F_l \approx 0.5$ and $C \ll K^{2/3}$ then $M_D > 1$, i.e. non-locality dominates.

We have argued on physical grounds that we expect that $C$ increases (but remains finite) as $p \to 1$; but approaches closer to 1 as $p$ approaches 3. So the conditions $4 < C \ll K$ as $p \to 1$ and $K \gg 1$, and $C \ll K^{2/3}$ as $p \to 3$ are likely to be met in reality.

We cannot examine the range $1 < p < 3$ without knowing the form of $C$ as a function of $p$ and $K$ and the form of $F_l$ as a function of $C$, $p$ and $K$. But it is still instructive to test with some reasonable guesses and to see if this captures the essential trends.

For this purpose, we simplify by assuming that $C$ depends only upon $p$, and by taking a constant value for $F_l = 0.5$. Let us examine the case when the pair separation is deep inside the inertial subrange, say $K \sim L/\sigma_\Delta = 10^4$.

So let us choose an exponential function, $C = 1.1 + (C_1 - 1.1) \exp(-5(p-1))$, which approximately meets the requirements mentioned above: $C(1) = C_1$, $C(3) \approx 1.1$. Putting this in to (4.9) and computing $M_D$ for $1 < p < 3$, we obtain Fig. 1(a), which shows five cases for $C_1 = 4, 10, 20, 50$ and 100.

A second set of calculations for a linear function, $C = C_1 - (C_1 - 1.1)(p - 1)/2$, for $C_1 = 4, 10, 20, 50$ and 100, is shown in Fig. 1(b). Here $C(1) = C_1$, $C(3) = 1.1$
And as a benchmark, a third set of calculations for constant \( C = 4, 10, 20, 50 \) and 100, is shown in Fig. 1(c).

In all the cases considered, \( M_D \gg 1 \) as \( p \to 3 \), showing that non-locality is always dominant in this limit. And in all the cases considered, \( M_D \ll 1 \) as \( p \to 1 \) where locality must be dominant.

Imporantly, in all the three forms of \( C \) considered, \( M_D \) increases monotonically and approaches an asymptotic high value as \( p \to 3 \). Figs. 1(b) and 1(c) follow almost identical trends, showing that there is not much difference between the constant and linear cases.

We emphasise that there are many possible choices for \( C_1 \) and \( F_1 \) and the functional choices for \( C \). On the other hand we have chosen such values for these parameters and functions that are at least consistent with our physical reasoning, and as such we expect the behaviour of \( M_D \) to be broadly similar to that shown in Fig. 1. Hence, it is reasonable to draw the following general conclusions.

First, as \( p \to 3 \) then \( M_D \gg 1 \), and therefore \( D^{nl}_p \gg D^l_p \), the non-locality limit, yielding

\[
D_p \to D^{nl}_3 \sim \sigma^2_\Delta \quad \text{as} \quad p \to 3
\]  

(4.10)

This gives exponential growth, \( \langle \Delta^2 \rangle \sim \Delta^2_0 \exp(st) \) as \( p \to 3 \) for some strain-rate \( s \).

Second, as \( p \to 1 \) then \( M_D \ll 1 \), and therefore \( D^{nl}_p \ll D^l_p \), the locality limit, yielding

\[
D_p \to D^l_1 \sim \sigma_\Delta \quad \text{as} \quad p \to 1.
\]  

(4.11)

Third, in going smoothly from \( p = 1 \) to 3, \( M_D \) increases smoothly and monotonically indicating that non-local scales exert increasingly stronger strained motion at the scale \( \Delta \) until they are completely dominant in the pair diffusion process.

But the exact balance at a given \( p \) cannot be obtained for the reasons given above.
Figure 1. $M_D$ against $p$, from equation (4.9). (a) Top: exponential, $C = 1.1 + (C_1 - 1.1) \exp(-5(p - 1))$; (b) Middle: linear, $C = C_1 - (C_1 - 1.1)(p - 1)/2$; (c) Bottom: constant, $C = C_1$. Five cases shown in each case, $C_1 = 4, 10, 20, 50$ and 100. $F_I = 0.5$ in all cases.
The actual form of $D_p$ as a function of $\sigma_\Delta$ and $p$, must be found from additional considerations.

We remark that we could have started from (4.8) from the beginning, posing the pair diffusivity as the sum of two well known scalings, $\sim \sigma_\Delta^{1/3}$ from the locality scaling and $\sim \sigma_\Delta^2$ from the strain dominated non-local scaling. However, part of this work is also concerned with unearthing the closures and assumptions that are hidden behind simple scaling arguments, whether considering locality or non-locality. The current mathematical analysis allows us to do this, and additionally provides us with the asymptotics and the general trend in the relative balance of the non-local and local terms $M_D$.

4.3.3. The form of $D_p$ and related properties

The main purpose of this work is to obtain the general scaling for the turbulent pair diffusivity $D_p$. The mathematical analysis developed here has served its purpose in producing equation (4.8) as the sum of two power law scalings. We have also deduced the asymptotic balance of the non-local and local contributions at $p = 1$ and 3, and found a plausible trend in $M_D$ in the range $1 < p < 3$. But we cannot go any further with this analysis because of the uncertainty in defining $C(p,K)$ and $F_l(p,K,C)$.

To progress, we need an other principle. We note that $D_p$ is a continuous function of $p$ (and also of $\sigma_\Delta$), and it is reasonable to expect that $D_p$ will display a smooth and uniform transition between the two asymptotic states as $p$ passes smoothly from 1 to 3.

It is possible that one or the other power law dominates; but that would imply a discontinuous jump between one power law and the other at some value of $p$ in order to satisfy the asymptotic limiting cases. This is unlikely.

It is also possible that for every $p$ both local and non-local power laws exist in different parts of the inertial subrange, with a crossover between them in the middle of the inertial subrange. In this case, a new separation scale $\Delta_p$ where the
crossover occurs has to be introduced in to the theory. It also implies that two different inertial ranges existing simultaneously for any given $p$, locality dominant in one and non-locality in the other. This is also unlikely.

A third possibility is that both local and non-local correlations are effective in the pair diffusion process at all separations $\Delta$ in the range of validity of inertial range scaling for $1 < p \leq 3$. $D_p$ will then appear as a smooth transition as a function of $p$ along the entire range of $\Delta$ for which it is valid, which will therefore manifest as an intermediary power law scaling,

\[ D_p \sim \varepsilon^{1/3} L (4/3 - \gamma_p) \sigma_\Delta^{\gamma_p} \quad \text{for} \quad 1 < p \leq 3, \tag{4.12} \]

with $\gamma_p$ such that $\gamma_l < \gamma_p < \gamma_{nl}$, and as $p \to 1$ then $\gamma_p \to 1$; and as $p \to 3$ then $\gamma_p \to 2$. Globally, we must have $1 < \gamma_p < 2$ and $\gamma_p$ must change smoothly as $p$ goes from 1 to 3.

The ratio of the power law scalings $M_\gamma(p) = \gamma_p/\gamma_l$ is equal to 1 at both $p = 1$ and $p = 3$, and since $M_\gamma > 1$ in $1 < p < 3$, there must be a maximum in $M_\gamma$ at some $p_m$ in this range; this may be an indicator of the energy spectrum, $\sim k^{-p_m}$, that produces local and non-local correlations that are equally influential in the pair diffusion process.

Equation (4.12) is equivalent to the pair separation scaling,

\[ \langle \Delta^2 \rangle_p \sim \delta^2 + \varepsilon^{\chi_p/3} L^{2(1-\chi_p/3)} \tau_p^{\chi_p} \tag{4.13} \]

where $\tau_p$ is a suitably adjusted travel time, and $\chi_p = 1/(1 - \gamma_p/2)$.

Since $M_\gamma > 1$, then for Kolmogorov turbulence $p = 5/3$, we expect $\gamma_{5/3} > 4/3$, and $\chi_{5/3} > 3$.

We remark that from (4.12), $D_p$ is independent of $L$ for some value of $p = p_*$ where $\gamma_{p_*} = 4/3$, so that $\chi_{p_*} = 3$. $p_*$ is not Kolmogorov turbulence, we in fact expect that $p_* < 5/3$. For this power spectrum, $E \sim k^{-p_*}$, we obtain a non-Richardson 4/3-power law pair diffusivity,

\[ D_{p_*} \sim \varepsilon^{1/3} \sigma_\Delta^{4/3} \tag{4.14} \]
which corresponds to a non-Richardson-Obukov $t^3$-regime for the mean square separation,

$$\langle \Delta^2 \rangle_p = \delta^2 + g_p \varepsilon \tau_p^3$$

where $g_p$ is a universal constant.

The actual form of $\gamma_p$ as a function of $p$ and the values of $p_m$ and $p_*$ cannot be determined theoretically, but must be obtained from numerical simulations or from experiment.

5. Discussion and Conclusions

In a field of homogeneous turbulence, with generalised power-law energy spectra of the form $E(k) \sim k^{-p}$ for $1 < p \leqslant 3$, the power law scaling for the turbulent particle pair diffusivity $D_p$ given in equation (4.12) is the main result of this work. It is based upon the concept that both local and non-local correlations are active and govern the turbulent pair diffusion process inside the inertial subrange. Together with (4.13)–(4.15), the current theory provides a new framework for turbulent particle pair diffusion studies and its applications.

Whereas previous theories of turbulent pair diffusion have been based on the idea of locality and have been derived on the basis of dimensionality arguments alone, here an expression for the turbulent pair diffusivity $D_p$ has been obtained by using a novel method in which the Fourier representation for the relative velocity $v(\Delta)$, from the statistical theory of turbulence, is put directly into the definition $D_p = \langle \Delta \cdot v \rangle$. The principle that $D_p(\Delta)$ is a continuous function of $p$ within the inertial subrange is then used to obtain the final scaling.

The main physical idea in this analysis is the separation of the diffusion process into local and non-local processes which operate in different wavenumber ranges. One of the most interesting aspects of this analysis is that locality emerges as an unwarranted assumption. The inclusion of both local and non-local correlations is therefore the more natural framework in which to study turbulent pair diffusion.
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The method also sheds light on the fact that there are many closure and scaling assumptions that are common to both the local and non-local theories, but are hidden when simple scaling arguments alone are adopted. The fact that the correct locality limit (Richardson) is obtained from this method provides justification for these assumptions.

Importantly, the relative balance of the non-local and local contributions to the pair diffusivity $M_D$ is estimated in equation (4.9), which highlights the sensitivity of this balance to the 'size' $C$ of a local eddy in wavenumber space — a problem inherent to the very nature of turbulence. The trend in this balance is such that non-local straining fields exert increasing influence inside the inertial subrange as $p$ passed from 1 to 3 until they become totally dominant, as highlighted in Fig. 1. The case of Kolmogorov turbulence $p = 5/3$ is a case 'in between' where both the local and non-local processes exert a significant influence.

The ratio of the scalings, $M(p) = \gamma_p/\gamma_p^l$, peaks at some $p_m$ in the range $1 < p_m < 3$, where it is possible that the turbulence with this energy spectrum $E \sim k^{-p_m}$ the local and non-local correlations induce comparable influence on the pair diffusion process.

For some particular spectrum $E \sim k^{-p_*}$, where $p_* < 5/3$ and $\gamma_{p_*} = 4/3$, the pair diffusivity $D_{p_*}$ is independent of $L$, and gives a non-Richardson 4/3-power law $D_{p_*} \sim \sigma_{\Delta}^{4/3}$, and $\langle \Delta^2 \rangle_{p_*} \sim t^3$.

But for Kolmogorov turbulence $p = 5/3$, we expect that $\gamma_{5/3} > 4/3$, and $\chi_{5/3} > 3$.

Although $p > 3$ is outside the scope of the current study, in view of the above results, we remark that we expect the non-local straininging motions to continue to dominate.

Finally, the fact that turbulent pair diffusion is influenced by both local and non-local correlations is important may have important implications for our understanding of turbulence and for turbulent diffusion modeling strategies.

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