TWISTED SUMS OF $c_0(I)$

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Abstract. We study in this paper a few remarkable properties of twisted sums $Z(\kappa, X)$ of $c_0(\kappa)$ and a Banach space $X$. We first prove a representation theorem for such twisted sums from which we will obtain, among others, the following: (a) twisted sums of $c_0(\kappa)$ and $c_0(I)$ are either subspaces of $\ell_\infty(\kappa)$ or contain a complemented copy of $c_0(\kappa^+)$; (b) under the hypothesis $[p = c]$, when $K$ is either a suitable Corson compact, a separable Rosenthal compact or a scattered compact of finite height, there is a twisted sum of $c_0$ and $C(K)$ that is not isomorphic to a space of continuous functions; (c) all twisted sums $Z(\kappa, X)$ are isomorphically Lindenstrauss spaces when $X$ is a Lindenstrauss space; (d) all twisted sums $Z(\kappa, X)$ are isomorphically polyhedral when $X$ is a polyhedral space with a $\sigma$-discrete boundary, which solves a problem of Castillo and Papini.

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1. Introduction and preliminaries. See below for all unexplained terminology. Twisted sums of the form $0 \rightarrow c_0 \rightarrow Z \rightarrow c_0(I) \rightarrow 0$ are simultaneously simple and very complex mathematical objects. The examples of $Z$ known so far are:

- Spaces of continuous functions $C(K_A)$ over the Stone compactum $K_A$ of a Boolean algebra generated by an uncountable almost disjoint family $A$ of subsets of $\mathbb{N}$.

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• When \(|I| = \mathfrak{c}\), the spaces constructed in [32] (under some form of Martin’s axiom), whose main feature is that they are not isomorphic to any \(C(K)\)-space.

• Products and \(c_0\)-sums of both.

The nature and properties of such twisted sums depend on set-theoretic assumptions: in [27] it is shown that, under \([\text{CH}]\), there exist \(2^{\aleph_1}\) non-isomorphic \(C(K_A)\) spaces generated by families of size \(\aleph_1\); while in [5] it is shown that, under \([\text{MA} + \aleph_1 < \mathfrak{c}]\) all such spaces are isomorphic. And in [2] it is shown that \(\operatorname{Ext}(C(K_A), c_0) \neq 0\) under \([\text{CH}]\) while [27] proves that \(\operatorname{Ext}(C(K_A), c_0) = 0\) provided \(A\) is of size \(\aleph_1\) under \([\text{MA} + \aleph_1 < \mathfrak{c}]\). All this made us think that time is ripe to undertake a classification of properties such twisted sum spaces.

1.1. Background on exact sequences. An exact sequence of Banach spaces is a diagram of Banach spaces and (linear, continuous) operators

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Y & \xrightarrow{i} & Z & \xrightarrow{Q} & X & \rightarrow & 0
\end{array}
\] (1.a)

so that the kernel of every operator coincides with the image of the preceding one. The middle space \(Z\) is usually called a twisted sum of \(Y\) and \(X\). The open mapping theorem asserts that in such a situation \(Y\) is isomorphic to a closed subspace of \(Z\) and \(X\) is isomorphic to \(Z/Y\). Even if, in general, the middle space \(Z\) in (1.a) where \(Y\) and \(X\) are both Banach spaces is only a quasi-Banach space and does not need to be a Banach space, we will not have the need for such considerations, since a deep result of Kalton and Roberts [24] guarantees that \(Z\) is a Banach space whenever \(X\) is an \(L_\infty\)-space.

Two exact sequences \(0 \rightarrow Y \rightarrow Z_i \rightarrow X \rightarrow 0, i = 1, 2\) are said to be equivalent if there is an operator \(T : Z_1 \rightarrow Z_2\) making commutative the diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & Y & \xrightarrow{i_1} & Z_1 & \xrightarrow{T} & X & \rightarrow & 0
\end{array}
\]

The 3-lemma and the open mapping theorem assure that \(u\) is an isomorphism, and so this defines a true equivalence relation. We say that the exact sequence (1.a) is trivial, or that it splits, when it is equivalent to the exact sequence \(0 \rightarrow Y \rightarrow Y \oplus X \rightarrow X \rightarrow 0\). This happens if and only if \(i[Y]\) is a complemented subspace of \(Z\), and if and only if there is a linear continuous right-inverse for the quotient map \(Q\). We shall write \(\operatorname{Ext}(X, Y)\) for the set of equivalence classes of exact sequences (1.a). This set can be endowed with a vector space structure so that the zero element is the class of trivial sequences. Therefore, \(\operatorname{Ext}(X, Y) = 0\) means that every twisted sum of \(Y\) and \(X\) is trivial.

Two basic constructions regarding exact sequences are the pullback and the pushout. Consider an exact sequence (1.a) and operators \(T : X' \rightarrow X\) and \(S : Y \rightarrow X\).
Y'. Then there are new exact sequences forming commutative diagrams (1.b)

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & \text{PB} & \longrightarrow & X' & \longrightarrow & 0
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & \text{PO} & \longrightarrow & X & \longrightarrow & 0
\end{array}
\]

In the left diagram, the pullback space is the subspace \( \text{PB} = \{(z, x') \in Z \times X' : Qz = T x'\} \). The maps \( T \) and \( Q \) are the restrictions of the canonical projections from \( Z \times X' \) into \( Z \) and \( X' \), respectively, and \( \iota \) is simply the natural inclusion \( \iota(y) = (y, 0) \). In the right diagram, the pushout space \( \text{PO} \) is the quotient of the direct sum \( Y' \oplus Z \) by the closure of \( \{(-S(y), \iota(y)) : y \in Y\} \). The maps \( S \) and \( \iota \) are just the composition of the natural inclusions of \( Y' \) and \( Z \) into \( Y' \oplus Z \), respectively, with the quotient map, and \( Q \) arises from the factorization of the operator \( Y' \oplus Z \to X \), \( (y', z) \mapsto q(z) \). We shall call the lower rows of the left and right diagrams in (1.b) the pullback and pushout sequences, respectively. A pullback sequence splits if and only if there is a linear continuous lifting for \( T \); that is, an operator \( T' : X' \to Z \) such that \( qT' = T \). On the other hand, a pushout sequence splits if and only if \( S \) extends to \( Z \), which means there is an operator \( S' : Z \to Y' \) so that \( S'\iota = S \).

1.2. \( C \)-spaces. We will say that a Banach space is a \( C \)-space when it is isomorphic to some space \( C(K) \) of continuous functions on some compact space \( K \). As usual, we identify the dual of \( C(K) \) with the space \( M(K) \) of Radon measures on \( K \), with the variation norm. \( M_1(K) \) denotes the closed unit ball of \( M(K) \) endowed with the weak* topology. From now on, given a topological subspace \( S \subset K \) of a compact space, \( r : C(K) \to C(S) \) will always denote the natural restriction map, and \( e : X \to C(B_{X^*}) \) denotes the canonical evaluation map \( e(f)(\mu) = \langle \mu, f \rangle \).

2. Everything old is new again. Let us begin by stating the properties that every twisted sum space \( 0 \to c_0 \to Z \to c_0(I) \to 0 \) must have.

**Proposition 2.1.** The twisted sum space \( Z \):

1. Is \( c_0 \)-saturated and \( c_0 \)-uppersaturated.

2. Is an Asplund space. Consequently, it has weak*-sequentially compact dual ball, it is weak*-extensible and has the Gelfand-Phillips property.

3. Is isomorphic to a Lindenstrauss space; consequently, it has Pełczyński’s property \((V)\).

4. Is WCG if and only if is isomorphic to \( c_0(I) \).

5. \( Z \cong Z \oplus c_0 \).

6. Is separably injective and not universally separably injective.
Proof. Recall that a space is $c_0$-saturated if every closed subspace contains a copy of $c_0$; the space is $c_0$-uppersaturated [1, Definition 2.25] if every separable subspace is contained in a copy of $c_0$ contained in the space. Both are 3-space properties: the proof for $c_0$-saturation is in [6] while the proof for $c_0$-uppersaturation is in [1, Proposition 6.2]. Concerning (2), the Asplund character is a consequence of (1). Now, that Asplund spaces have weak*-sequentially compact dual ball appears in [18, Corollary 2], and the weak*-extensibility character follows from [39]. See also [7, Proposition 6]. This also shows that [36, Example 0.5.6] does not require any special choice of the almost disjoint family. The Gelfand-Phillips property comes as a consequence of the weak*-sequential compactness of the dual ball, see [6, Proposition 6.8.c]. Let us remark that, according to [28, Theorem 1.1], the space $Z$ has the additional property of having weak*-sequential dual ball (a property not shared by all Asplund spaces). We are grateful to the referee for this remark. As for (3), the fact that every twisted sum of $c_0$ and a Lindenstrauss space is isomorphic to a Lindenstrauss space is hinted in [32, Theorem 5.1]. See also Theorem 4.1 below.

**Lemma 2.2.** $c_0$ is isomorphic to an $M$-ideal in every superspace.

**Proof.** Indeed, $c_0$ is an $M$-ideal in $\ell_\infty$, hence in every $\ell_\infty(I)$, in its natural position. Since $\ell_\infty(I)$ is separably automorphic [1, Section 2.6], it means that $c_0$ is an $M$-ideal in any $\ell_\infty(I)$ in every position. Finally, if $c_0 \subset Y \subset X$ is an $M$-ideal in $X$ then it is also an $M$-ideal in $Y$. See also [21].

Assertion (3) follows immediately from that since being an $M$-ideal means that $Z^* = (c_0)^* \oplus_1 (Z/c_0)^*$ and thus, if $Z/c_0$ is a Lindenstrauss space, then $Z^* = \ell_1 \oplus_1 L_1(\mu)$ and $Z$ is a Lindenstrauss space. Lindenstrauss spaces have property (V) and are Lindenstrauss-Pelczyński spaces in the sense of [8]. In particular, $Z$ is an $\mathcal{L}_\infty$-space, but recall that not all $\mathcal{L}_\infty$-spaces enjoy property (V) or have to be a Lindenstrauss-Pelczyński space [8]. A proof for the more general fact that all twisted sums of $c_0(J)$ and a space with property (V) have property (V) can be seen in [11]. It is well-known that the space $c_0(I)$ is WCG and also that every copy of $c_0$ in a WCG space is complemented [22], which is enough to prove (4). Now, to prove (5), note that the quotient map $Q$ is necessarily an isomorphism when restricted to some subspace $Z_0$ of $Z$ isomorphic to $c_0$. Since $Q[Z_0]$ must be complemented in $c_0(I)$ by the result of Granero [16], $Z_0$ must be complemented in $Z$, and we conclude: $Z \simeq Y \oplus Z_0 \simeq Y \oplus Z_0 \oplus Z_0 \simeq Z \oplus Z_0$. Finally, to dispose of (6), recall that “to be separably injective” is a 3-space property [1, Proposition 2.11] in the category of Banach spaces and that $c_0(I)$ is separably injective, but no twisted sum of $c_0$ and $c_0(I)$ can be universally separably injective since they are never Grothendieck spaces [1, Proposition 2.8].

The space $c_0(I)$ is the simplest twisted sum of the type we are considering. One has:

**Lemma 2.3.** $c_0(I)$ has the following properties:
(1) Every copy of \( c_0(J) \) inside \( c_0(I) \) is complemented.

(2) Every nonseparable subspace of density \( \kappa \) contains a complemented copy of \( c_0(\kappa) \).

\textbf{Proof.} The first assertion is in [16]. The second is somehow folklore and can be partially found in different places [13, 31, 35]. Probably the deepest argument is given in [29, Lemma 2.7]: for any subspace \( X \) of \( c_0(I) \) there is a decomposition

\[ I = \bigcup_{i \in J} I_i \]

with \( |J| = |I| \) and such that if \( X_i = \{ x \in X : \text{supp } x \subseteq I_i \} \) then \( X = c_0(J, X_i) \) and the following sequences are isomorphic.

\[
\begin{array}{cccccc}
0 & \longrightarrow & X & \longrightarrow & c_0(I) & \longrightarrow & c_0(I)/X & \longrightarrow & 0 \\
0 & \longrightarrow & c_0(J, X_i) & \longrightarrow & c_0(J, c_0(I)) & \longrightarrow & c_0(J, c_0(I)/X_i) & \longrightarrow & 0 \\
\end{array}
\]

If \( \kappa \) is the density character of \( X \), there are exactly \( \kappa \)-many spaces \( X_i \neq \{0\} \) and thus it is clear that \( X \) contains a complemented copy of \( c_0(\kappa) \). \( \square \)

2.1. Twisted sums of \( c_0 \) and \( C(K) \) that are not \( C \)-spaces. The purpose of [32] was to construct, for every Eberlein compactum \( K \) of weight \( \mathfrak{c} \), a twisted sum of \( c_0 \) and \( C(K) \) which is not a \( C \)-space. Let us recall the key points of the construction. The cardinal \( p \) is defined in [15, 11D].

- First, Theorem 3.4 in [32] produces, under \( p = \mathfrak{c} \), a very special almost disjoint family \( A \) of subsets of natural numbers and size \( \mathfrak{c} \). For convenience, let us write \( A = \{ A_0^\xi, A_1^\xi, A_2^\xi : \xi < \mathfrak{c} \} \). The existence of such an \( A \) relies on the fact that, since \( K \) is Eberlein, \( M_1(K) \) is a sequentially compact space and \( |M_1(K)| = \mathfrak{c} \).

- Next, one needs to find a subspace of the form \( \Sigma = \{ \sigma_\xi, -\sigma_\xi : \xi < \mathfrak{c} \} \cup \{0\} \) inside \( M_1(K) \) which is homeomorphic to \( \mathbb{A}(\mathfrak{c}) \), the one point compactification of a discrete space of cardinality \( \mathfrak{c} \). This is done in [32, Lemma 4.1]. Actually, it suffices to find any copy of \( \mathbb{A}(\mathfrak{c}) \) inside \( M_1(K) \), for if \( L = \{ \mu_\xi : \xi < \mathfrak{c} \} \cup \{ \mu \} \subseteq M_1(K) \) is homeomorphic to \( \mathbb{A}(\mathfrak{c}) \), and we call \( \sigma_\xi = \frac{1}{2}(\mu_\xi - \mu) \) for \( \xi < \mathfrak{c} \), then \( \Sigma = \{ \sigma_\xi, -\sigma_\xi : \xi < \mathfrak{c} \} \cup \{0\} \) lies inside \( M_1(K) \) and it is also a copy of \( \mathbb{A}(\mathfrak{c}) \).

- Then, consider a compactum \( L \) defined as the adjunction space of \( K_A \) and \( M_1(K) \) under the map \( \psi : K'_A \to \Sigma \subseteq M_1(K) \) defined by

\[
\psi(A_0^\xi) = 0 \quad , \quad \psi(A_1^\xi) = -\sigma_\xi \quad , \quad \psi(A_2^\xi) = \sigma_\xi
\]

Note that \( L \) is a \textit{countable discrete extension} of \( M_1(K) \); namely, it contains \( M_1(K) \) in such a way that \( L \setminus M_1(K) \) is a copy of a discrete countable space.
The desired twisted sum space will be denoted $Z(\omega, K, \Sigma)$ to stress the role of $\Sigma$ and appears in the pullback sequence of the diagram:

![Diagram](attachment:image.png)

Having in mind the previous requisites, one can substantially enlarge the list of compacta $K$ for which there exists a twisted sum of $c_0$ and $C(K)$ that is not a $C$-space.

**Theorem 2.4.** $|p = c|$ There is a twisted sum $0 \to c_0 \to Z(\omega, K, \Sigma) \to C(K) \to 0$ in which $Z(\omega, K, \Sigma)$ is not a $C$-space provided $K$ has weight $c$ and belongs to any of the following classes:

1. Corson compacta with property (M).
2. Separable Rosenthal compacta.
3. Scattered compacta of finite height.

**Proof.** Recall that a compactum $K$ has property (M) if every regular (Radon) measure on $K$ has separable support. A good place to look for the standard facts about Corson compacta is [30, Chapter 6]. In particular, we mention here that separable Corson compacta are metrizable, and that Corson compacta are sequentially compact. Moreover, for a Corson compactum $K$, having property (M) is equivalent to $M_1(K)$ being Corson. Under $[\mathsf{MA} + \aleph_1 < c]$, every Corson compactum has property (M), but this is not true under $[\mathsf{CH}]$. As a consequence, it is readily seen that $M_1(K)$ is sequentially compact, and also that $|M_1(K)| = c$ because $K$ has $c$ many metrizable subspaces and each of them can only support $c$ many measures.

Now, in order to find a copy of $A(c)$ inside $M_1(K)$ when $K$ is Corson, we note that [32, Lemma 4.1] is also valid for Corson compacta.

Now we assume that $K$ is a separable Rosenthal compactum. The properties of Rosenthal compacta needed to carry out the proof are contained in the survey paper by Marciszewski [26]. Especially, we infer that $M_1(K)$ is sequentially compact and $|M_1(K)| = c$ from the fact that $M_1(K)$ is separable Rosenthal provided $K$ is. To conclude, observe that $M_1(K)$ contains a copy of $A(c)$ by virtue of a theorem of Todorčević [37, Theorem 9]. Indeed, Todorčević’s theorem contains the additional requirement that the separable Rosenthal compact has a non-$G_\delta$-point; but if $0 \in M_1(K)$ were a weak* $G_\delta$-point, then $K$ would be metrizable: write $\{0\} = \bigcap_{n=1}^\infty V_n$ as the intersection of weak* basic neighbourhoods of $0$, which are of the form

$$V_n = \{\mu \in M_1(K) : |\langle \mu, f_{(i,n)} \rangle| < \varepsilon_n\}, \quad 1 \leq i \leq k_n$$

Clearly, if some $\mu$ vanishes on every $f_{(i,n)}$ then $\mu \in \bigcap_{n=1}^\infty V_n = \{0\}$ and thus the span of $\bigcup_{n=1}^\infty \{f_{(1,n)}, \ldots, f_{(k_n,n)}\}$ would be dense in $C(K)$, which makes $C(K)$ separable.
To conclude, assume that $K$ is a scattered compacta of finite height and weight $c$. Then $C(K)$ is Asplund, and so $M_1(K)$ is sequentially compact. Also, $M(K)$ is isomorphic to $\ell_1(c)$, so $|M_1(K)| = c$. Finally, the assumption of finite height enables us to find a copy of $\mathbb{A}(c)$ inside $K$, and therefore inside $M_1(K)$, thanks to [2, Lemma 6.4]. We point out that $c$ is a regular cardinal under $p = c$ because $p$ is regular [15, 21K]. 

Compacts of the form $K_A$ are scattered and of finite height, therefore part (3) yields the existence of exact sequences $0 \rightarrow c_0 \rightarrow Z \rightarrow C(K_A) \rightarrow 0$ in which $Z$ is not a $C$-space. It is an open problem whether $\beta\mathbb{N}$ can be added to the list of Theorem 2.4 (see [23, Remark p. 67] or else [4]).

Another question that stems from [32] is whether the spaces $Z(\omega, K, \Sigma)$ can be chosen subspaces of $\ell_{\infty}$. In general, observe that the existence of an exact sequence $0 \rightarrow c_0 \rightarrow Z \rightarrow C(K) \rightarrow 0$ in which $Z$ is a subspace of $\ell_{\infty}$ implies that $C(K)$ is a subspace of $\ell_{\infty}/c_0$: indeed, $C(K) \simeq Z/c_0$ must be a subspace of some quotient $\ell_{\infty}/X$ with $X \simeq c_0$, but all such quotients [1] are isomorphic to $\ell_{\infty}/c_0$. It is true that if $\eta : C(K) \rightarrow \ell_{\infty}/c_0$ is an into isomorphism then the pullback sequence

$$
\begin{array}{ccccccc}
0 & \rightarrow & c_0 & \rightarrow & \ell_{\infty} & \rightarrow & \ell_{\infty}/c_0 & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & \omega & & \\
0 & \rightarrow & c_0 & \rightarrow & \text{PB} & \rightarrow & C(K) & \rightarrow & 0
\end{array}
$$

produces a twisted sum of $c_0$ and $C(K)$ that is a subspace of $\ell_{\infty}$; but this is not enough to guarantee that PB is not a $C$-space, since the pullback sequence could split. The simplest example of $c_0(c)$ sets the red lines of what is and what is not possible. On one hand, choosing $\Sigma = \{e_i^*, -e_i^*: i \in I\} \cup \{0\}$ in the unit ball of $\ell_1(c)$ produces a twisted sum $Z(\omega, K, \Sigma)$ of $c_0$ and $c_0(c)$ that is a subspace of $\ell_{\infty}$ and is not a $C$-space. Such twisted sum space will be denoted as PS in the sequel. On the other hand, $\text{PS} \oplus c_0(c)$ cannot be a subspace of $\ell_{\infty}$ and is still a twisted sum of $c_0$ and $c_0(c)$. Moreover, $\text{PS} \oplus c_0(c)$ can be regarded as a $Z(\omega, K, \Sigma)$ space, hence not a $C$-space either: consider $J \subset I$ such that $|J| = |I \setminus J| = c$; it is easy to see that when $\Sigma = \{e_j^*, -e_j^*: j \in J\} \cup \{0\}$ the space $Z(\omega, K, \Sigma)$ is isomorphic to $\text{PS} \oplus c_0(c)$.

The best result we can offer is the following one.

**Proposition 2.5.** $[p = c]$. Let $K$ be a compactum under the assumptions of Theorem 2.4. Suppose that:

- $C(K)$ admits a Markušević basis $\{(f_\xi, \mu_\xi) \in C(K) \times M(K) : \xi < c\}$; that is, a fundamental biorthogonal system so that $\{f_\xi : \xi < c\}$ separates the points in $M(K)$ and $\{\mu_\xi : \xi < c\}$ separates the points in $C(K)$.

- The basis is norming for $C(K)$; namely, there exists $C > 0$ such that $\|g\| \leq C\sup\{|\langle \mu_\xi, g \rangle| : \xi < c\}$ for each $g \in C(K)$.  


Then there is an exact sequence \( 0 \rightarrow c_0 \rightarrow Z \rightarrow C(K) \rightarrow 0 \) in which
\( Z \) is isomorphic to a subspace of \( \ell_\infty \) and it is not a \( C \)-space.

**Proof.** The set \( \Sigma = \{ \mu_\xi, -\mu_\xi : \xi < \varsigma \} \cup \{ 0 \} \) is a copy of \( A(c) \) inside \( M_1(K) \): indeed, given \( \mu_\xi \) no other \( \mu_\alpha \) is in the weak*-neighborhood \( \{ \mu \in M_1(K) : |\langle \mu, f_\xi \rangle| > 1/2 \} \). By a theorem of Plichko [34] we can assume the basis is bounded [20, Theorem 5.13] and thus we will assume without loss of generality that \( \|\mu_\xi\| \leq 1 \). Therefore, when we produce the countable discrete extension \( L \) of \( M_1(K) \) that generates the space \( Z(\omega, K, \Sigma) \), we obtain a set \( \{ \mu_\xi, -\mu_\xi : \xi < \varsigma \} \cup \{ 0 \} \) contained in the closure of the countable discrete set \( L \setminus M_1(K) \), which we can identify with \( \mathbb{N} \). The sequence of functionals \( (\delta_n)_{n=1}^\infty \), where \( \delta_n(f, g) = f(n) \) separates points and thus it defines an injective operator \( \omega : Z(\omega, K, \Sigma) \rightarrow \ell_\infty \) given by \( \omega(f) = (\delta_n(f))_n \). To show that it is an into isomorphism, pick \( (f, g) \in Z(\omega, K, \Sigma) \) and observe that \( \|(f, g)\| = \|f\| \) since, with the notations of diagram (2.a), \( \|g\| = \|e(g)\| = \|r(f)\| \leq \|f\| \). Now, \( f \) must attain its norm at some point in \( L \). If such point belongs to \( L \setminus M_1(K) \subseteq \mathbb{N} \), then clearly \( \|f\| = \|(\delta_n(f))\|_\infty \). Otherwise, there is some \( \mu \in M_1(K) \) such that

\[
\|f\| = |f(\mu)| = |\langle \mu, g \rangle| \leq C \cdot \sup_{\xi < \varsigma} |\langle \mu_\xi, g \rangle| = C \|(\delta_n(f))\|_\infty
\]

and so the proof concludes. \( \square \)

The problem with this proposition is that we do not know if it can be applied to a large class of compacta. Corson compacta with property (\( M \)) admit a Markušević basis \( \{ (f_\xi, \mu_\xi) \in C(K) \times M(K) : \xi < \varsigma \} \) but they are not necessarily norming. In fact, the existence of a norming Markušević basis for \( C(K) \) is not assured even if \( K \) is Eberlein [19]. On the other hand, it is a classical result [12, Theorem 11.23] that \( K \) is scattered Eberlein if and only if \( C(K) \) admits a shrinking Markušević basis, meaning that \( \overline{\text{span}}\{ \mu_\xi : \xi < \varsigma \} = M(K) \). Finally, we do not know if a separable Rosenthal compactum admits a norming Markušević basis. However, every twisted sum of \( c_0 \) and \( C(K) \) where \( K \) is a separable compactum is automatically a subspace of \( \ell_\infty \), since in such a case \( C(K) \) is a subspace of \( \ell_\infty \) and “to be a subspace of \( \ell_\infty \)” is a 3-space property in the category of Banach spaces (even if it is not in the category of quasi-Banach spaces).

### 3. New skin for the old ceremony.
Let us fix an infinite cardinal number \( \kappa \) and a compactum \( K \). A \( \kappa \)-discrete extension of \( K \), or \( \kappa \)-DE for short, is another compactum \( K \cup \kappa \) which contains a copy of \( K \) and a disjoint copy of a discrete set of size \( \kappa \). It is clear that if \( L = K \cup \kappa \) is a \( \kappa \)-DE, then there is an exact sequence

\[
0 \rightarrow c_0(\kappa) \rightarrow C(L) \rightarrow C(K) \rightarrow 0.
\]

**Theorem 3.1.** For every exact sequence \( 0 \rightarrow c_0(\kappa) \xrightarrow{r} Z \xrightarrow{q} X \rightarrow 0 \)
there is a $\kappa$-discrete extension $L$ of $B_{X^*}$ and a commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & c_0(\kappa) & \rightarrow & C(L) & \rightarrow & C(B_{X^*}) & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & c_0(\kappa) & \rightarrow & Z & \rightarrow & X & \rightarrow & 0
\end{array}
\]

Proof. The dual $0 \rightarrow X^* \rightarrow Z^* \rightarrow \ell_1(\kappa) \rightarrow 0$ of the original sequence obviously splits. Therefore, for every $\alpha < \kappa$ there is $z_\alpha^* \in Z^*$ such that $\iota^* z_\alpha^* = e_\alpha^*$, where $\{e_\alpha^* : \alpha < \kappa\}$ denotes the canonical basis of $\ell_1(\kappa)$. This implies that any weak*-cluster point of $\{z_\alpha^*\}_{\alpha<\kappa}$ belongs to $i[c_0(\kappa)]^{\perp} = Q^* [X^*]$, since $\langle z_\alpha^*, u \rangle = \langle e_\alpha^*, x \rangle \rightarrow 0$ for every $x \in c_0(\kappa)$. On the other hand, it is clear that no $z_\alpha^*$ belongs to $i[c_0(\kappa)]^{\perp}$. Consider $M > 0$ so that $\|z_\alpha^*\| \leq M \|Q^*\|$ for all $\alpha \in \kappa$ and set $L = Q^* [MB_{X^*}] \cup \{z_\alpha^* : \alpha \in \kappa\}$ endowed with the weak* topology of $Z^*$. Since $Q^* [MB_{X^*}]$ is homeomorphic to $B_{X^*}$, $L$ can be readily identified with a $\kappa$-discrete extension of $B_{X^*}$, which we call again $L$. The map $u : Z \rightarrow C(L)$ defined by $u(z)(x^*) = \langle Q^* x^*, z \rangle$ and $u(z)(z_\alpha^*) = \langle z_\alpha^*, z \rangle$ makes diagram (3.a) commutative. \qed

It follows from diagram (3.a) that $Z \cong \{(f, x) \in C(L) \oplus_\infty X : f |_{B_{X^*}} = ex\}$ and thus we can identify $Z$ with the subspace of functions $f \in C(L)$ satisfying $f |_{B_{X^*}} \in X$, endowed with the norm inherited from $C(L)$. Formally: $u(f, x) = f$ is an into isometry.

**Definition 3.1.** Let $X$ be a Banach space. A discrete extension $L$ of $B_{X^*}$ is realizable inside $X^*$ if the canonical embedding $B_{X^*} \hookrightarrow X^*$ can be extended to an embedding $L \hookrightarrow X^*$.

The standard splitting criterion can be reformulated now in the language of discrete extensions:

**Proposition 3.2.** The lower sequence in (3.a) is trivial if and only if $L$ is realizable inside $X^*$.

Proof. Let Comp denote the category of compact spaces and continuous functions (in particular, $\text{cont}(K, S)$ will denote the set of continuous functions between the compact spaces $K, S$) and by Ban$_1$ the category of Banach spaces and contractive operators (i.e., operators having norm at most 1) so that $\Sigma_1(X, Y)$ denotes the set of contractive operators between $X$ and $Y$. Consider the contravariant functor $\bigcirc^* : \text{Ban}_1 \rightarrow \text{Comp}$ that assigns to each Banach space $X$ the compact set $(B_{X^*}, \text{weak}^*)$ and to each contractive operator $\tau : X \rightarrow Y$ the continuous function $\tau^* : B_{Y^*} \rightarrow B_{X^*}$; and consider then the contravariant functor $C(\cdot) : \text{Comp} \rightarrow \text{Ban}_1$ that assigns to each compact space $K$ the Banach space $C(B_{X^*})$ and to each continuous function $f : K \rightarrow S$ the contractive operator $C(S) \rightarrow C(K)$ of composition with $f$. The proof is a direct consequence from the fact that these two functors are adjoints on the right; which means that there is a natural identification $\text{cont}(L, B_{X^*}) = \Sigma_1(X, C(L))$. \qed
4. Both sides now. The representation in Theorem 3.1 can be used to solve several problems in the area:

4.1. Twisted sums of $c_0(\kappa)$ and Lindenstrauss spaces are isomorphically Lindenstrauss. A Banach space will be called isomorphically Lindenstrauss when it can be renormed to be a Lindenstrauss space.

**Theorem 4.1.** Every twisted sum of $c_0(\kappa)$ and a Lindenstrauss space is isomorphically Lindenstrauss.

We present three proofs for this result, each of them containing some extra features worth attention.

*First proof.* Observe that the proof for $\kappa = \aleph_0$ has been already obtained as in Proposition 2.1 (3). We just need to extend the arguments used there to uncountable cardinals $\kappa$. And this just requires to check that $\ell_\infty(\kappa)$ is $c_0(\kappa)$-automorphic, a consequence of [1, Lemma 5.9] plus [1, Theorem 5.30].

*Second proof.* Let $Z$ be a twisted sum of $c_0(\kappa)$ and $X$ and observe that no specific norm has been assigned yet: we only know that some norm can be assigned. By the previous results, let $L$ be a $\kappa$-DE extension of $B_X^*$ generating the diagram (3.a), which, once completed, yields

\[
\begin{array}{c}
0 & \rightarrow & c_0(\kappa) & \rightarrow & C(L) & \rightarrow & r & C(B_X^*) & \rightarrow & 0 \\
\uparrow & & \downarrow u & & \uparrow r & & \uparrow e \downarrow & & \uparrow e \\
0 & \rightarrow & c_0(\kappa) & \rightarrow & Z & \rightarrow & Q & \rightarrow & X & \rightarrow & 0
\end{array}
\]

Thus $u : Z \rightarrow C(L)$ is an into isomorphism. Renorm $Z$ to make $u$ an into isometry, so that $Z/c_0(\kappa)$ is still $X$ isometrically. Every functional on $Z$ is of the form $u^*\mu$ for some $\mu \in M(L)$ acting in the form

\[
\langle u^*\mu, (f, x) \rangle = \langle \mu, f \rangle = \int f \, d\mu|_\kappa + \int_{B_X^*} e(x) \, d\mu|_{B_X^*} = \langle \mu|_\kappa, f \rangle + \langle e^*(\mu|_{B_X^*}), x \rangle
\]

Recall that the dual sequence of the middle column is

\[
\begin{array}{c}
0 & \rightarrow & Z^\perp & \rightarrow & M(L) & \rightarrow & Z^* & \rightarrow & 0;
\end{array}
\]
hence
\[ \| u^* \mu \| = \inf_{\nu \in Z_+} \| \mu - \nu \| = \| \mu \| + \inf_{\nu \in Z_+} \| (\mu - \nu) \|_{B_{X^*}} \]
because \( \nu \|_{\kappa} = 0 \) for every \( \nu \in Z_+ \). Moreover \( \inf_{\nu \in Z_+} \| (\mu - \nu) \|_{B_{X^*}} = \| e^*(\mu) \|_{B_{X^*}} \), the inequality \( \| e^*(\mu) \| \leq \| (\mu - \nu) \|_{B_{X^*}} \) follows from the fact that \( e^* \) is a norm-one operator and \( e^*(\nu)_{B_{X^*}} = 0 \) whenever \( \nu \in Z(L)_+ \). To see the converse inequality, assume \( \| \mu \| \leq 1 \) and set \( \nu = \mu - r^* e^*(\mu) \) so that \( \| (\mu - \nu) \|_{B_{X^*}} = \| r^* e^*(\mu) \| \leq \| e^*(\mu) \|_{B_{X^*}} \). Therefore, \( Z^* \) is isometrically isomorphic to \( \ell_1(\kappa) \oplus_1 X^* \), and \( Z \) is a Lindenstrauss space provided \( X \) is.

**Third proof.** We need to summon here (see either [1, Definition A.9] or [4, Chapter 5]) the notion of local complementation. A subspace \( Y \) of a Banach space \( X \) is said to be locally \( \lambda^+ \)-complemented (also, an isometric exact sequence \( 0 \to Y \to Z \to X \to 0 \) is said to locally \( \lambda^+ \)-split) if the bidual exact sequence
\[ 0 \to Y^{**} \to Z^{**} \to X^{**} \to 0 \]
is \( \lambda \)-splits. When the number \( \lambda \) is not necessary one just speaks of local complementation (an exact sequence locally splits if and only if the bidual sequence splits). We bring now the reader’s attention to the fact that a locally \( 1^+ \)-complemented subspace of a Lindenstrauss space is a Lindenstrauss space: in general, a locally complemented subspace of an \( \mathcal{L}_{\infty,\lambda} \) space is an \( \mathcal{L}_{\infty,\mu} \) space [1, Lemma A.12] but the parameters \( \lambda, \mu \) can be different. However, they coincide for the 1 case (see [4, Comment after Proposition 5.1.6]): a locally \( 1^+ \)-complemented subspace of an \( \mathcal{L}_{\infty,1^+} \) space is an \( \mathcal{L}_{\infty,1^+} \) space. The last piece we need is the observation that every exact sequence
\[ 0 \to c_0(\kappa) \to Z \to X \to 0 \]
locally \( 1^+ \)-splits for obvious reasons. Now, from the representation Theorem 3.1

\[ \begin{array}{ccc}
0 & \to & c_0(\kappa) \\
\| & & \downarrow u \\
0 & \to & c_0(\kappa) \\
\| & & \downarrow Q \\
0 & \to & Z \\
\| & & \downarrow e \\
0 & \to & C(L) \\
\| & & \downarrow r \\
0 & \to & C(B_{X^*}) \\
\end{array} \]

we obtain the diagonal pullback sequence (see [4, Chapter 2] for details)
\[ 0 \to Z \to C(L) \oplus_\infty X \to C(B_{X^*}) \to 0 \]

with embedding \( j(z) = (uz, Qz) \) and quotient map \( \pi(f, x) = rf - ex \). Since this sequence fits into a commutative diagram
\[ \begin{array}{ccc}
0 & \to & c_0(\kappa) \\
\| & & \downarrow i \\
0 & \to & Z \\
\| & & \downarrow j \\
0 & \to & C(B_{X^*}) \oplus_\infty X \\
\| & & \downarrow \pi \\
0 & \to & C(B_{X^*}) \\
\end{array} \]

the upper sequence locally \( 1^+ \)-splits and \( j \) is an into isometry then the lower sequence also locally \( 1^+ \)-splits. Hence \( Z \) is locally \( 1^+ \)-complemented in \( C(B_{X^*}) \oplus_\infty X \),
which is a Lindenstrauss space when $X$ is Lindenstrauss space, and the proof is done.

Each of these three proofs contains some additional feature. The first proof extends Lemma 2.2 to $c_0(\kappa)$ is isomorphic to an $M$-ideal in every superspace. The second proof has the virtue of making explicit the equivalent norm that makes $Z$ a Lindenstrauss space and provides us with the duality formula, something that will be used in the proof of Theorem 4.2. The third proof goes deeper showing that the representation Theorem 3.1 is optimal and somehow expresses a unique feature of $c_0(\kappa)$ spaces: indeed, even if one considers the “simplest” compact space $\omega^\omega$ after $\omega$ the result fails since an exact sequence $0 \to C(\omega^\omega) \to \Omega \to X \to 0$ does not necessarily fit into a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & C(\omega^\omega) & \to & C(L) & \to & C(B_{X^*}) & \to & 0 \\
\downarrow & & \uparrow & & \uparrow & & e \\
0 & \to & C(\omega^\omega) & \to & \Omega & \to & X & \to & 0
\end{array}
$$

To prove this, we recall from [4, Chapter 8 and 9] the existence of an exact sequence

$$
0 \to C(\omega^\omega) \to \Omega \to c_0 \to 0
$$
in which $\Omega$ is not isomorphic to a Lindenstrauss space (even if its dual can be renormed to be $\ell_1$). If the representation Theorem 3.1 remained valid for $\omega^\omega$ and, informally speaking, we could speak about “$\omega^\omega$-discrete extensions” $L$ of $C(B_{X^*})$, whatever this could mean, then the diagram argument would apply, $\Omega$ would be a locally $1^+$-complemented subspace of $C(L) \oplus \infty X$ and therefore it would be an isomorphically Lindenstrauss space.

### 4.2. Twisted sums that are $G$-spaces.

There are other types of Lindenstrauss spaces in the literature; among them the $M$-spaces and $G$-spaces. The former are the sublattices of $C(K)$-spaces while the latter are the subspaces $X$ of $C(K)$-spaces so that for a certain set of triples $A = \{(k_1^\alpha, k_2^\alpha, \lambda_\alpha) \in K \times K \times \mathbb{R}\}$ one has

$$
X = \{f \in C(K) : f(k_1^\alpha) = \lambda_\alpha f(k_2^\alpha)\} \text{ for all } \alpha \in A.
$$

We do not know whether every twisted sum of $c_0(\kappa)$ and $c_0(I)$ is a $G$-space. The examples known so far were $C$-spaces, so they certainly are $G$-spaces. Let us show that the example of Plebanek-Salguero presented in [32] is also a $G$-space. The construction was explained before and we just need to observe that the one point compactification $\{\sigma_\alpha, -\sigma_\alpha : \alpha < c\} \cup \{0\}$ can be just set $\{e_\alpha^*, -e_\alpha^* : \alpha < c\} \cup \{0\}$. Thus, once the almost-disjoint family $A$ has been obtained, the countable discrete extension $L = B_{\ell_1(c)} \cup \omega$ is obtained identifying $A_1^\alpha$ with $e_\alpha^*$, $A_2^\alpha$ with $-e_\alpha^*$ and $A_0^\alpha$ with 0. This means that

$$
\mathcal{PS} = \{f \in C(K_A) : \forall \alpha < c \quad f(A_1^\alpha) = -f(A_2^\alpha) \quad \text{and} \quad f(A_0^\alpha) = 0\}
$$

and thus $\mathcal{PS}$ is a $G$-space.
4.3. Twisted sums of $c_0(I)$ are isomorphically polyhedral. A Banach space is polyhedral if every finite dimensional (equivalently, 2-dimensional) subspace is isometric to a subspace of some finite-dimensional $\ell^n_\infty$. A Banach space is isomorphically polyhedral if it is isomorphic to a polyhedral space. For instance, $c_0$ is polyhedral and $c$ is isomorphically polyhedral but not polyhedral (see [17] for a short and beautiful proof); in fact, no $C(K)$-space can be polyhedral in its natural sup norm. Many times and in many places the first author asked whether “being isomorphically polyhedral” is a 3-space property, a problem that is yet open, to the best of our knowledge. We provide a partial affirmative answer to such question, and in particular, we conclude that any twisted sum of two $c_0(I)$-spaces must be isomorphically polyhedral, which is another related question repeatedly posed by Castillo and Papini [9, 10].

Some of the most useful criteria for recognizing isomorphically polyhedral spaces involve the behaviour of boundaries. Given $X$ a Banach space, a subset $B$ of $S_{X^*}$ is a boundary if for every $x \in X$ there is $x^* \in B$ so that $\langle x^*, x \rangle = \|x\|$. A boundary is said to be $\sigma$-discrete if it can be written as a countable union of relatively discrete sets in the weak* topology. Any Banach space possessing a $\sigma$-discrete boundary is polyhedral [14, Theorem 11].

**Theorem 4.2.** Let $X$ be a Banach space with a $\sigma$-discrete boundary. Then every twisted sum of $c_0(\kappa)$ and $X$ can be renormed so that it has a $\sigma$-discrete boundary. In particular, it is isomorphically polyhedral.

**Proof.** Let $Z$ be a twisted sum of $c_0(\kappa)$ and $X$ renormed, as in Theorem 3.1, as a subspace of $C(B_{X^*} \cup \kappa)$. Assume $B$ is a $\sigma$-discrete boundary for $X$ and consider

$$\hat{B} = \{\delta_\alpha, -\delta_\alpha : \alpha < \kappa\} \cup \{\delta_b : b \in B\}.$$

We claim that $\hat{B}$ constitutes a $\sigma$-discrete boundary for $Z$. It is a boundary, since every $f \in Z$ attains its norm at some point $t \in B_{X^*} \cup \kappa$. If $t \in \kappa$, then $|\langle \delta_t, f \rangle| = \|f\|$; otherwise, $\|f\| = \|f|_{B_{X^*}}\|$; and so there is $b \in B$ satisfying $\langle \delta_b, f \rangle = \langle b, f|_{B_{X^*}} \rangle = \|f|_{B_{X^*}}\| = \|f\|$. To prove that $\hat{B}$ is $\sigma$-discrete, write $B = \bigcup_{n=1}^{\infty} D_n$ as a union of relatively weak*-discrete subsets of $X^*$. This, of course, means that given $d$ in some $D_n$, there exist $x_1, \ldots, x_k \in X$ and $\varepsilon > 0$ so that $V \cap D_n = \{d\}$ for $V = \{x^* \in X^* : |\langle x^* - d, x_j \rangle| < \varepsilon, \forall 1 \leq j \leq k\}$. Letting $f_1, \ldots, f_k \in Z$ be such that $f_j|_{B_{X^*}} = x_j$, the following weak*-open subset of $Z^*$

$$\hat{V} = \{z^* \in Z^* : |\langle z^* - \delta_d, f_j \rangle| < \varepsilon, \forall 1 \leq j \leq k\}$$

clearly satisfies $\hat{V} \cap \{\delta_{x^*} : x^* \in D_n\} = \{\delta_d\}$. Finally, since $\{\delta_\alpha, -\delta_\alpha : \alpha < \varepsilon\}$ is relatively weak* discrete in $Z^*$, we conclude that $\hat{B}$ is $\sigma$-discrete. 

**Corollary 4.3.** Every twisted sum of $c_0(\kappa)$ and $c_0(I)$ is isomorphically polyhedral.
4.4. A dichotomy for twisted sums of $c_0(I)$. With the same notations as
before, let $L = B_{\ell^1(I)} \cup \kappa$. We set $\kappa_\perp = \{ f \in Z : f(\alpha) = 0 \ \forall \alpha < \kappa \}$ and
$\beta = \text{dens}(\kappa_\perp)$. One has:

**Theorem 4.4.** Every twisted sum of $c_0(\kappa)$ and $c_0(I)$ is either a subspace of $\ell_\infty(\kappa)$
or it is trivial on a copy of $c_0(\kappa^+)$. 

*Proof.* Let $(f_b)_{b < \beta}$ be a norming family of functionals on $\kappa_\perp$. The set
$\{ f_b, \delta_\alpha \}_{b < \beta, \alpha < \kappa}$ is norming in $Z$, which implies $Z$ is a subspace of $\ell_\infty(\max\{\beta, \kappa\})$. If $\beta \leq \kappa$ then $Z$ is a subspace of $\ell_\infty(\kappa)$. If $\beta > \kappa$ then let us look at the restriction map $r : \kappa_\perp \rightarrow C(B_{\ell^1(I)})$, which is an isomorphism. Therefore, $r[\kappa_\perp]$ is a subspace of $c_0(I)$ of density $\beta$ and so it contains a copy of $c_0(\kappa^+)$. Since that copy is complemented in $c_0(I)$ by the result of Granero [16], it is complemented in the twisted sum space too. $\square$

5. Yesterday, once more. The paper [33] contains a consistent negative so-

tion to the longstanding open problem about complemented subspaces of C-

spaces. If the space $PS$ were complemented in some $C$-space it would also be a

solution. The paper [33] contains an additional example: an exact sequence

$0 \rightarrow c_0 \rightarrow C(K_M) \rightarrow X \rightarrow 0$ in which $X$ is not a $C$-space. This is rather

interesting when combined with other results in this paper: Observe that if $J$ is

an ideal of $C(S)$ then $C(S)/J$ is a $C$-space [21], and thus, Lemma 2.2 is no longer valid for

isomorphic copies. On the other hand, the specific copy of $c_0$ mentioned

above is isomorphic to an $M$-ideal of $C(K_M)$ but it is not itself an $M$-ideal. Now

all together: if one forms the pullback diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & c_0 & \rightarrow & C(K_M) & \rightarrow & c_0(I) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & c_0 & \rightarrow & PB & \rightarrow & PS & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
c_0 & = & \rightarrow & & \rightarrow & & \rightarrow & & c_0 \\
\end{array}
\]

then either PB is a $C$-space or it is not:

- If PB is a $C$-space then the middle horizontal sequence provides another

  example of $c_0$ inside some $C(K)$-space such that the quotient $C(K)/c_0 = PS$

  is not a $C$-space

- If PB is not a $C$-space then the middle vertical sequence provides a twisted

  sum of $c_0$ and $C(K_M)$ that is not a $C$-space. Moreover, the diagonal pullback

  sequence $0 \rightarrow c_0 \rightarrow PB \rightarrow c_0(I) \rightarrow 0$ would be another example of

  twisted sum of $c_0$ and $c_0(I)$ that is not a $C$-space.

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Twisted sums of $c_0(I)$

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