Can two or more gauge bosons propagate in the light-front?

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Gauge fields in the light-front are usually fixed via the $\mathbf{n} \cdot \mathbf{A} = 0$ condition yielding the non-local singularities of the type $(k \cdot \mathbf{n})^{-\alpha} = 0$ and $\alpha = 1, 2, ..$ in the gauge boson propagator which must be addressed conveniently. In calculating this propagator for $n$ noncovariant gauge bosons those non-local terms demand the use of a prescription to ensure causality. We show that from 2 gauge bosons onward the implementation of such a prescription does not remove certain pathologies such as the non existence of two or more free propagating gauge bosons in the light-front form.

1 Light-front

In 1949 Dirac [1], showed that it is possible to construct dynamical forms from the description of a initial state of a given relativistic system in any space-time surface whose lengths between points have no causal connection. The dynamical evolution corresponds to the system following a trajectory through the hyper-surfaces. For example, the hyper-surface $t = 0$ is our three-dimensional space. It is invariant under translations and rotations. However, in any transformation of inertial reference frames which involves “boosts”, the temporal coordinate is modified, and therefore the hyper-surface in $t = 0$. Other hyper-surfaces can be invariant under some type of “boost”. It is the case of the hyperplane called null plane, defined by $x^+ = t + z$, which in analogy to the usual coordinate system, is commonly referred to as the “time” coordinate for the front form (light front). For example, a “boost” in the $z$ direction does not modify the null plane.

A point in the usual four-dimensional space-time is defined through the set of coordinates $(x^0, x^1, x^2, x^3)$, where $x^0$ is the time coordinate, that is, $x^0 = t$, with the usual convention of taking the speed of light equal to unity, $c = 1$. The other coordinates are the three-dimensional Euclidean space coordinates $x^1 = x$, $x^2 = y$ and $x^3 = z$.

The light-front coordinates are defined in terms of these by the following relations:

$$
\begin{align*}
    x^+ &= x^0 + x^3, \\
    x^- &= x^0 - x^3, \\
    \vec{x}_\perp &= x_1 \hat{i} + x_2 \hat{j},
\end{align*}
$$

(1)

where $\hat{i}$ and $\hat{j}$ are the unit vectors in the direction of the coordinates $x$ and $y$. The null plane is defined by $x^+ = 0$, that is, this condition defines the hyper-surface which is tangent to the light cone, the reason why some authors call those light-cone coordinates.

Note that for the usual four-dimensional Minkowski space-time whose metric $g^{\mu\nu}$ is defined such that its signature is $(1, -1, -1, -1)$ we have

$$
\begin{align*}
    x^+ &= x^0 + x^3 = x_0 - x_3 \equiv x_-, \\
    x^- &= x^0 - x^3 = x_0 + x_3 \equiv x_+, \\
    \vec{x}_\perp &= x_1 \hat{i} + x_2 \hat{j} = -x_1 \hat{i} - x_2 \hat{j} \equiv -x_\perp,
\end{align*}
$$

(2)
The initial boundary conditions for the dynamics in the light front are defined in this hyper-plane. Note that the axis $x^+$ is orthogonal to the plane $x^+ = 0$. Therefore, a displacement of this hyper-surface for $x^+ > 0$ is analogous to the displacement of the plane $t = 0$ for $t > 0$ of the usual four-dimensional space-time. With this analogy we identify $x^+$ as the "time" coordinate for the null plane. Of course, since there is a conspicuous discrete symmetry between $x^+ \leftrightarrow x^-$, one could choose $x^-$ as his "time" coordinate. However, once chosen, one has to stick to the convention adopted. We shall adhere to the former one.

The canonically conjugate momenta for the coordinates $x^+, x^-$ and $x^\perp$ are defined respectively by:

\[
\begin{align*}
  k^+ &= k^0 + k^3, \\
  k^- &= k^0 - k^3, \\
  k^\perp &= (k^1, k^2).
\end{align*}
\]  
(3)

The scalar product in the light front coordinates becomes therefore

\[
\begin{align*}
  a^\mu b_\mu &= \frac{1}{2} (a^+ b^- + a^- b^+) - \vec{a}^\perp \cdot \vec{b}^\perp,
\end{align*}
\]  
(4)

where $\vec{a}^\perp$ and $\vec{b}^\perp$ are the transverse components of the four vectors. All four vectors, tensors and other entities bearing space-time indices such as Dirac matrices $\gamma^\mu$ can be expressed in this new way, using components $(+, -, \perp)$.

From (4) we can get the scalar product $x^\mu k_\mu$ in the light front coordinates as:

\[
\begin{align*}
  x^\mu k_\mu &= \frac{1}{2} (x^+ k^- + x^- k^+) - \vec{x}^\perp \cdot \vec{k}^\perp.
\end{align*}
\]  
(5)

Here again, in analogy to the usual four-dimensional Minkowski space-time where such a scalar product is

\[
\begin{align*}
  x^\mu k_\mu &= x^0 k^0 - \mathbf{x} \cdot \mathbf{k}
\end{align*}
\]  
(6)

where $\mathbf{x}$ is the three-dimensional vector, with the energy $k^0$ associated to the time coordinate $x^0$, we have the light-front "energy" $k^-$ associated to the light-front "time" $x^+$. Note, however, that there is a crucial difference between the two formulations: while the usual four-dimensional space-time is Minkowskian, the light-front coordinates projects this onto two sectorized Euclidean spaces, namely $(+, -)$, and $(\perp, \perp)$.

In the Minkowski space described by the usual space-time coordinates we have the relation between the rest mass and the energy for the free particle given by $k^\mu k_\mu = m^2$. Using (4), we have

\[
\begin{align*}
  k^\mu k_\mu &= \frac{1}{2} (k^+ k^- + k^- k^+) - \vec{k}^\perp \cdot \vec{k}^\perp,
\end{align*}
\]  
so that

\[
\begin{align*}
  k^- &= \frac{-\vec{k}^2 + m^2}{k^+}.
\end{align*}
\]  
(7)

Note that the energy of a free particle is given by $k^0 = \pm \sqrt{m^2 + \mathbf{k}^2}$, which shows us a quadratic dependence of $k^0$ with respect to $\mathbf{k}$. These positive/negative energy possibilities for such a relation were the source of much difficulty in the interpretation of the negative energy particle states in the beginning of the quantum field theory description for particles, finally solved by the antiparticle interpretation given by Feynman. In contrast to this, we have a linear dependence between $(k^+)/^{-1}$ and $k^-$ (see Eq. (4)), which immediately reminds us of the non-relativistic quantum mechanical type of relationship for one particle state systems.
2 Classical Propagator

In a recent work of ours [2], we showed that a single Lagrange multiplier defined by \((n \cdot A)(\partial \cdot A)\) with \(n \cdot A = \partial \cdot A = 0\) at the classical level leads to a propagator in the light-front gauge that has no residual gauge freedom left.

Thus, for the relevant gauge fixing term that enters in the Lagrangian density which we define as

\[
(n \cdot A)(\partial \cdot A) = 0,
\]

gives for the Abelian gauge field Lagrangian density:

\[
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(2n_\mu A^\mu \partial_\nu A^\nu) = \mathcal{L}_E + \mathcal{L}_{GF}
\]

where the gauge fixing term is conveniently written so as to symmetrize the indices \(\mu\) and \(\nu\), and the gauge parameter can assume complex values. By partial integration and considering that terms which bear a total derivative don’t contribute and that surface terms vanish since \(\lim_{x \to \infty} A^\mu(x) = 0\), we have

\[
\mathcal{L}_E = \frac{1}{2}A^\mu(\Box g_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu
\]

and

\[
\mathcal{L}_{GF} = -\frac{1}{\alpha}(n \cdot A)(\partial \cdot A) = -\frac{1}{2\alpha}A^\mu(n_\mu \partial_\nu + n_\nu \partial_\mu) A^\nu
\]

so that

\[
\mathcal{L} = \frac{1}{2}A^\mu \left( \Box g_{\mu\nu} - \partial_\mu \partial_\nu - \frac{1}{\alpha}(n_\mu \partial_\nu + n_\nu \partial_\mu) \right) A^\nu
\]

To find the gauge field propagator we need to find the inverse of the operator between parenthesis in (12). That differential operator in momentum space is given by \(O_{\mu\nu}(k) = -k^2 g_{\mu\nu} + k_\mu k_\nu + \frac{1}{\alpha} (n_\mu k_\nu + n_\nu k_\mu)\), so that the propagator of the field, which we call \(S^{\mu\nu}(k)\), must satisfy the following equation \(O_{\mu\nu}S^{\nu\lambda}(k) = \delta^\lambda_\mu\), where \(S^{\nu\lambda}(k)\) can now be constructed from the most general tensor structure that can be defined, i.e., all the possible linear combinations of the tensor elements that composes it (the most general form includes the light-like vector \(m_\mu\) dual to the \(n_\mu\) [3] – but for our present purpose it is in fact indifferent):

\[
G^{\mu\nu}(k) = g^{\mu\nu} A + k^\mu k^\nu B + k^\mu n^\nu C + n^\mu k^\nu D + k^\nu m^\mu E + m^\mu k^\nu F + n^\mu n^\nu G + m^\mu m^\nu H + n^\mu m^\nu I + m^\mu n^\nu J
\]

Then, it is a matter of straightforward algebraic manipulation to get the relevant propagator in the light-front gauge, namely,

\[
S^{\mu\nu}(k) = -\frac{1}{k^2} \left\{ g^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k^+} + \frac{n^\mu n^\nu}{(k^+)^2} k^2 \right\}
\]

3 Quantum gauge boson propagator

The Feynman quantum propagator for the gauge boson can be derived integrating over all the momenta in (13). Projecting out this propagator on to the light-front we get a gauge boson particle propagating at equal light-front times. We are going to restrict our calculation to the total momentum \(P^+\) positive and corresponding forward light-front time propagation. In this case the propagator from \(x^+ = 0\) to \(x^+ > 0\) is given by:

\[
\tilde{S}^{(1)\mu\nu}(x_1^i) = i \int \frac{d^4k_1}{(2\pi)^4} \frac{N^{\mu\nu\rho\sigma} e^{-ik_1^\mu x_1^\rho}}{k_1^+ + i\varepsilon}.
\]
where
\[ N^{\mu\nu} = \frac{-g^{\mu\nu} k_1^+ k_1^- + (k_1^+ n^\nu + n^\mu k_1^-) k_1^+ - n^\mu n^\nu k_1^2}{k_1^2}. \] (16)

Note that because of the structure of the light-front propagator only three of the component projections are non-vanishing, namely,
\[ N^{\perp\perp} = -g^{\perp\perp}, \quad N^{\perp-} = \frac{n^- k_1^+}{k_1^+}, \quad N^{-\perp} = \frac{n^- n^\perp}{k_1^+}. \] (17)

At equal light-front times, we have:
\[ \tilde{S}^{(1)\mu\nu}(x^+) = \frac{i}{4} \int \frac{d^2 k_1^-}{(2\pi)^2} k_1^+ \frac{N^{\mu\nu} e^{-\frac{i}{2} k_1^- x^+}}{k_1^- (k_1^- - \frac{(k_1^-)^2}{k_1^+} + \frac{i\varepsilon}{k_1^+})}. \] (18)

so that, in terms of the component projections we have immediately
\[ \tilde{S}^{++} = \tilde{S}^{+-} = \tilde{S}^{\perp\perp} = 0, \quad \tilde{S}^{\perp+} \neq \tilde{S}^{\perp-} \neq \tilde{S}^{--} \neq 0. \] (19)

The \((\perp, \perp)\) component presents no particular difficulty in evaluation nor does it present any overwhelming troublesome feature. However, the components \((\perp, -)\) and \((-,-)\), while having similar technical difficulties in the performing of the relevant computation, do come with an overwhelming troublesome result which will become clearer as we proceed further on in our analysis. This feature is nothing more nothing less than the troublesome zero mode problem in the light front. We shall therefore restrict ourselves to the analysis of the \((\perp, -)\) component.

In terms of Fourier transform we have:
\[ S^{(1)\mu\nu}(p^-) = \int dx^+ e^{i p^- x^+} \tilde{S}^{(1)\mu\nu}(x^+), \] (20)

so that the component \(S^{\perp-}\) is:
\[ S^{(1)\perp-}(p^-) = i \int d^2 k_1^- k_1^+ n^- \delta \left(p^- - k_1^-\right) \left[ \frac{1}{(k_1^+)^2} \right] \delta \left(p^- - k_1^-\right) \left[ \frac{1}{(k_1^+)^2} \right]_{\text{ML}} \]
\[ = i \int d^2 k_1^- k_1^+ n^- \delta \left(p^- - k_1^-\right) \left[ \frac{1}{(k_1^+)^2} \right] \delta \left(p^- - k_1^-\right) \left[ \frac{1}{(k_1^+)^2} \right]_{\text{ML}} \]
\[ \times \left[ \frac{k_1^-}{k_1^+ (k_1^- + \frac{i\varepsilon}{k_1^+})} \right]^2 \] (21)

where
\[ \delta \left(p^- - k_1^-\right) = \frac{1}{2\pi} \int dx^+ e^{i \frac{x^-}{2} (p^- - k_1^-) x^+} \]

and the index ML stands for the Mandelstam-Leibbrandt prescription for the treatment of the \((k^+)^{-1}\) poles, namely,
\[ \left[ \frac{1}{k_1^+} \right]_{\text{ML}} = \lim_{\varepsilon \to 0} \left[ \frac{k^-}{k^+ k^- + i\varepsilon} \right]_{\text{ML}} \] (22)
The result is:

\[ S^{(1)\perp -}(p^-) = \frac{\theta(p^+) p^\perp n^-}{p^2} \frac{i}{(p^- - K_0^{(1)\perp} + i\varepsilon)}, \]  

(23)

where we have introduced the definition

\[ K_0^{(1)\perp} = \frac{p_\perp^2}{p^2}, \]  

(24)

as the light-front Hamiltonian of the free one-particle system. Note that for \( x^+ < 0 \), the \( S^{(1)}(x^+) = 0 \) because \( p^+ > 0 \). Moreover, observe that \( S^{(1)\perp -}(p^-) \) is in an operator form with respect to \( p^+ \) and \( p_\perp \). Consequently we have a clear manifestation of the zero mode problem in the factor \((p^+)^{-2} = 0\).

4 Two gauge boson propagator

The two-boson gauge propagator can be derived from the covariant propagator for two particles propagating at equal light-front times. Without losing generality, we are going to restrict our calculation to the total momentum \( P^+ \) positive and corresponding forward light-front time propagation. In this case the propagator from \( x^+ = 0 \) to \( x^+ > 0 \) is given by:

\[ \tilde{S}^{(2)\mu\nu;\alpha\beta}(x^\mu, x^\nu) = \int \frac{dk_1^-}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i N^{\mu\nu} e^{-ik_1^\lambda (x_1^\lambda - x_2^\lambda)}}{k_1^2 + i\varepsilon} \frac{i N^{\alpha\beta} e^{-ik_2^\lambda (x_2^\lambda - x_2^\lambda)}}{k_2^2 + i\varepsilon}. \]

(25)

At equal light-front times \( x_1^+ = x_2^+ = 0 \) and \( x_1^+ = x_2^+ = x^+ \), the propagator is written as:

\[ \tilde{S}^{(2)}(x^+) = \tilde{S}^{(1)}_1(x^+) \tilde{S}^{(1)}_2(x^+), \]

where the one-body propagators, \( \tilde{S}^{(1)}_i \), corresponding to the light-front propagators of particles \( i = 1 \) or \( 2 \), are defined by Eq.(15). We have explicitly:

\[ \tilde{S}^{(2)\mu\nu;\alpha\beta}(x^+) = -\frac{2}{4} \int \frac{dk_1^- dk_2^-}{(2\pi)^4} \frac{N^{\mu\nu} e^{\frac{i}{2} k_1^+ x^+}}{k_1^+ - \frac{k_2^+ - i\varepsilon}{k_2^-}} \frac{N^{\alpha\beta} e^{\frac{i}{2} k_2^+ x^+}}{k_2^+} \frac{1}{k_2^- - \frac{k_1^+ - i\varepsilon}{k_2^-}}. \]

(26)

The Fourier transform to the total light-front energy \( P^- \) is given by

\[ S^{(2)\mu\nu;\alpha\beta}(P^-) = \frac{1}{2} \int dx^+ e^{i P^- x^+} \tilde{S}^{(2)\mu\nu;\alpha\beta}(x^+). \]

(27)

As before, we can recognize immediately that (we omit the (2) index as well as the \( P^- \) dependence for shortness)

\[ S^{++++,} = S^{+-+,+} :=:: S^{+++,+} :=:: S^{++,++} :=:: S^{++,+,+} :=:: S^{++,++} :=:: 0 \]

\[ S^{++,---} \neq S^{+-,---} \neq:: S^{--+,+} \neq:: S^{----} \neq:: S^{---,---} \neq:: S^{[+++,---} \neq:: S^{+++,---} \neq:: 0 \]

(28)
which result in

\[
S^{(2)}_{-,-,-}(P^-) = -\frac{1}{(2\pi)} \int \frac{dk^-}{k_1^+ (P^+ - k_1^+)} \left[ \frac{k^- - k^-_{1\text{on}}}{k_1^+ (k^- - k^-_{1\text{on}} + \frac{i\varepsilon}{k_1^+})} \right]^2 ML \left( k^- - \frac{k^2_{1\text{on}}}{k_1^+} + \frac{i\varepsilon}{k_1^+} \right) \\
\times \left[ \frac{P^- - k^-_1 - k^-_{2\text{on}}}{(P^+ - k_1^+) (P^- - k^-_1 - k^-_{2\text{on}} + \frac{i\varepsilon}{P^+ - k_1^+})} \right]^2 ML \\
\times \frac{N_{-,-,-}^{(2)}}{(P^- - k^-_1 - \frac{(P^- - k^-_1)^2}{k_1^+} + \frac{i\varepsilon}{P^+ - k_1^+})},
\]

where \( P^-_{-+,-} = k^-_{1,+,-} + k^-_{2,+,-} \).

We perform the analytical integration in the \( k^-_1 \) momentum by evaluating the residue at the pole

\[
k^-_1 = k^-_{1\text{on}} - \frac{i\varepsilon}{k_1^+};
\]

\[
k^-_1 = P^- - k^-_{2\text{on}} + \frac{i\varepsilon}{P^+ - k_1^+}.
\]

It implies that only \( k_1^+ \) in the interval \( 0 < k_1^+ < K^+ \) gives a nonvanishing contribution to the integration. It implies that only \( k_1^+ \) in the interval \( 0 < k_1^+ < P^+ \) gives a nonvanishing contribution to the integration. The result is

\[
S^{(2)}_{-,-,-}(P^-) = \frac{\theta(k_1^+) \theta(P^+ - k_1^+) (k^-_1 n^-) (P^+ - k^-_1) n^-}{k_1^+ (P^+ - k_1^+)^2} \frac{i}{(P^- - K_{0}^{(2)-} + i\varepsilon)},
\]

and for the \((\perp,\perp,\perp)\) we have

\[
S^{(2)}_{\perp,\perp,\perp}(P^-) = \frac{\theta(k_1^+) \theta(P^+ - k_1^+) i (-g^{\perp,\perp}) (-g^{\perp,\perp})}{k_1^+ (P^+ - k_1^+)} \frac{i}{(P^- - K_{0}^{(2)-} + i\varepsilon)},
\]

where

\[
K_{0}^{(2)-} = \frac{k_{1\text{on}}^2}{k_1^+} + \frac{k_{2\text{on}}^2}{P^+ - k_1^+}.
\]

\( K_{0}^{(2)-} \) is the light-front Hamiltonian of the free two-particle system. For \( x^+ < 0 \), \( S^{(2)}(x^+) = 0 \) due to our choice of \( P^+ > 0 \). Observe that \( S^{(2)}(P^-) \) is written in Eq.(30) and Eq.(31) in operator form with respect to \( k^+ \) and \( k^\perp \). Again we have problems of a divergent factor for \( k_1^+ = 0 \) in [29].

5 Conclusion

Projecting the Feynman covariant space propagator in light-front coordinates and using the Mandelstam-Leibbrandt prescription to treat \( k^- = 0 \) singularities we get propagation of one or two bodies in the light-front for some components such as \((\perp,\perp,-)\) and \((\perp,\perp,\perp)\). Now, the \((\perp,-,\perp)\) component presents the zero mode problem for \( k^+ = 0 \). The same happens with other nonvanishing components, except for the \((\perp,\perp,\perp)\) component where there is no singularity of this type.
We observe that even with the use of Leibbrandt-Mandelstam prescription, it was not possible to remove the built-in singularity in $k^+ = 0$.

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