The QCD $\beta$-function at $O(1/N_f)$

J.A. Gracey,
Department of Applied Mathematics and Theoretical Physics,
University of Liverpool,
P.O. Box 147,
Liverpool,
L69 3BX,
United Kingdom.

Abstract. The leading order coefficients of the $\beta$-function of QCD are computed in a large $N_f$ expansion. They are in agreement with the three loop $\overline{\text{MS}}$ calculation. The method involves computing the anomalous dimension of the operator $(G_{\mu\nu}^a)^2$ at the $d$-dimensional fixed point in the non-abelian Thirring model to which QCD is equivalent in this limit. The effect the $O(1/N_f)$ corrections have on the location of the infrared stable fixed point for a range of $N_f$ is also examined.
The strong force of the standard model is described by quantum chromodynamics, (QCD), which is an $SU(3)$ gauge theory which is asymptotically free. At large energies the fundamental quarks behave as though they were non-interacting. In terms of the field theory itself this property is a consequence of the leading coefficient of the $\beta$-function being negative. The initial one loop calculation was carried out in [1]. Higher order corrections have also been determined. The remaining scheme independent term, ie the two loop contribution, was calculated in [2]. The three loop term was computed in the Feynman gauge in [3] using dimensional regularization in the $\overline{MS}$ scheme. More recently this result was checked in [4] in an arbitrary covariant gauge where the remaining renormalization group functions, such as the gluon anomalous dimension, were also deduced in arbitrary gauge, [4]. The three loop quark mass anomalous dimension, which is gauge independent, is available too, [5, 6]. One reason for such precise information is that, for example, it allows one to obtain a more accurate insight into the variation of quantities with energy scale. Fundamental in this respect is the $\beta$-function as it always appears in the appropriate renormalization group equation, (rge).

These higher order analytic calculations of the rge functions are exceedingly tedious to compute, however, due to the huge number of Feynman diagrams that arise. For such results to be credible it is important to have independent checks on the expressions obtained aside from the obvious one of performing another complete evaluation which may be a waste of resources. In this letter we provide the results of such a procedure for the QCD $\beta$-function. This is the large $N_f$ technique of determining exact all orders results of the rge functions of gauge theories at successive orders in powers of $1/N_f$, where $N_f$ is the number of fundamental fields. The technique was initially developed for low dimensional models in a series of impressive papers, [7, 8, 9]. Briefly the method involves computing appropriate critical exponents at the $d$-dimensional fixed point of the $\beta$-function as $N_f \to \infty$. Through the critical rge these $d$-dimensional exponents encode all orders information on the coefficients of the corresponding rge function. Clearly the values will overlap with the lowest known orders, providing the partial check we have indicated. From a technical point of view one benefit of this approach is the exploitation of the conformal symmetry at the fixed point which simplifies the resummation of the minimal set of Feynman diagrams comprising the relevant Schwinger Dyson equation. The calculation of the $O(1/N_f)$ QCD $\beta$-function here completes the leading order analysis as the quark, gluon and ghost dimensions were deduced in [11] in the Landau gauge and agreed with the three loop results of [4]. Another motivation arises from a comment in [4] in regard to future calculations. It is indicated that the four loop QCD $\beta$-function is attainable. The main obstacle, though, would appear to be correctly generating and treating the vast numbers of Feynman diagrams. Therefore the new coefficients we will deduce from our results will be important in this respect.

We recall that the $O(1/N_f)$ computation of the QED $\beta$-function is available, [12]. That calculation was carried out by inserting the implicit bubble sum of the photon propagator in the 2 and 3 point functions and then deducing the $\overline{MS}$ coefficients of the renormalization constants using dimensional regularization. Those results have been reproduced in the critical point approach, [13]. One interesting aspect of [12] was the search for other fixed points in the strictly four dimensional QED $\beta$-function, for a range of values of the coupling. Although none were observed it would be a worthwhile exercise to repeat that analysis in the non-abelian case especially as at two loops such a point exists, [14], for a range of $N_f$.

We recall the fundamental ingredients for treating QCD in large $N_f$ in our approach. The lagrangian is

$$L = i\bar{\psi}^i \gamma^\mu D^i_{\mu} \psi^i - \frac{(G^a_{\mu\nu})^2}{4e^2}$$

(1)

where $\psi^i$ is the quark field, $A^a_{\mu}$ is the gluon field, the covariant derivative is $D_\mu = \partial_\mu + iT^a A^a_\mu$.
with $T^a$ the generators of the colour gauge group and $G_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c / \epsilon$ is the field strength with $f^{abc}$ the structure constants. The ranges of the indices are $1 \leq i \leq N_f$, $1 \leq a \leq (N_c^2 - 1)$ and $1 \leq I \leq N_c$ and the Casimirs are $\text{tr}(T^a T^b) = T(R)\delta^{ab}$, $T^a T^a = C_2(R)$ and $f^{ac} f^{bd} = C_2(G)\delta^{ab}$. To three loops, in $d$-dimensions, \[ \beta(g) = (d - 4)g + \left[ \frac{2}{3} T(R)N_f - \frac{11}{6} C_2(G) \right] g^2 + \left[ \frac{1}{2} C_2(R)T(R)N_f + \frac{5}{6} C_2(G)T(R)N_f - \frac{17}{12} C_2^2(G) \right] g^3 - \left[ \frac{11}{72} C_2(R)T^2(R)N_f^2 + \frac{79}{432} C_2(G)T^2(R)N_f^2 + \frac{1}{16} C_2^2(R)T(R)N_f \right. \\
- \left. \frac{205}{288} C_2(R)C_2(G)T(R)N_f - \frac{1415}{864} C_2^2(G)T(R)N_f + \frac{2857}{1728} C_2^3(G) \right] g^4 + O(g^5) \] where our coupling $g$ is $g = (e / 2\pi)^2$. The presence of the $O(g)$ term of (2), corresponding to the dimension of the coupling in $d$-dimensions, gives rise to our non-trivial fixed point, $g_c$. Explicitly \[ g_c = \frac{3\epsilon}{T(R)N_f} + \frac{1}{4T^2(R)N_f} \left[ 33C_2(G)\epsilon - (27C_2(R) + 45C_2(G)) \right] \] \[ + \left( \frac{99}{4} C_2(R) + \frac{237}{8} C_2(G) \right) \epsilon^3 + O(\epsilon^4) \] \[ + O \left( \frac{1}{N_f^3} \right) \] where $d = 4 - 2\epsilon$. In the neighbourhood of this point the quark and gluon anomalous dimension were deduced in the Landau gauge as, at leading order in $1 / N_f$, respectively, \[ \eta = \frac{C_2(R)\eta_0}{T(R)N_f} \] \[ \eta + \chi = -\frac{C_2(G)\eta_0}{2(\mu - 2)T(R)N_f} \] where $\eta_0 = (2\mu - 1)(\mu - 2)\Gamma(2\mu) / [4\Gamma^2(\mu)\Gamma(\mu + 1)\Gamma(2 - \mu)]$ and $d = 2\mu$. Moreover, the asymptotic scaling forms of the respective propagators are, as $k^2 \to \infty$, \[ \psi(k) \sim \frac{Ak}{(k^2)^{\mu - \alpha}} , \quad A_{\mu \nu}(k) \sim \frac{B}{(k^2)^{\mu - \beta}} \left[ \eta_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} \right] \] where $\alpha = \mu + \frac{3}{2}\eta$, $\beta = 1 - \eta - \chi$ and $A$ and $B$ are amplitudes though only the combination $z = A^2B$ appears in calculations. Specifically $z = \Gamma(\mu + 1)\eta_0 / [2(2\mu - 1)(\mu - 2)T(R)N_f]$.

One feature which simplifies the fixed point analysis both for computing the $O(1/N_f)$ corrections to $\beta(g)$ and (4,5) arises from the universality class in which QCD belongs. For instance, it is widely accepted that the $O(N)$ bosonic $\sigma$ model and the $O(N)$ $\phi^4$ model are equivalent, or in the same universality class, at the $d$-dimensional fixed point analogous to (3). In other words critical exponents calculated in either field theory at this fixed point are the same. So if one wished to examine the critical behaviour of the three dimensional $O(3)$ Heisenberg ferromagnet to which both are equivalent then either model can be used. Another widely studied equivalence is the Yukawa interaction and the Gross Neveu model with the same chiral properties, \[ \text{(15)} \]. Likewise the Thirring model and QED are equivalent and have been the subject of recent interest, \[ \text{(16, 17)} \]. For QCD the relevant model is the non-abelian Thirring model, (NATM), whose lagrangian is \[ L = i\bar\psi^I \gamma^\mu \partial_\mu \psi^I + \bar\psi^I \gamma^\mu T^a_{IJ} \psi^J A^a_\mu - \frac{(A^a_\mu)^2}{2\lambda} \] \[ \text{(7)} \]
where $A^{a}_{\mu}$ is an auxiliary field, which if eliminated leads to a 4-fermi interaction, and $\lambda$ is the coupling whose dimension is $(d-2)$ when compared to the $(d-4)$ of the QCD coupling. It has been demonstrated in [18] that it is equivalent to QCD in the large $N_{f}$ limit. One feature of that work was the correct reproduction of the three and four gluon vertices of non-abelian theories, which are absent in (7), by integrating out quark loops in the gluon 3- and 4-point functions. For our computation the major simplification is the fact that by calculating with (7) at $g$, the resulting exponents, which are universal, will be equivalent to those computed in (1). Then by decoding the exponent using (3), we can deduce information on the perturbative structure of the rge functions. A test of this argument will be the correct reproduction of the $\overline{\text{MS}}$ coefficients. Importantly though we need only consider graphs which are built out of the single interaction of (7).

We now turn to the details of the calculation. Ordinarily one computes $\omega = -\frac{1}{2} \beta'(g)$ by considering corrections to (6) in the Dyson equations, [7, 13]. Equivalently one can identify the composite operator in the lagrangian whose coupling relates to the ordinary coupling constant, [8]. Then the anomalous dimension of that operator is related via a scaling law deduced from the lagrangian to $\beta'(g_{c})$. In QCD the appropriate operator is $(G^{a}_{\mu\nu})^{2}$ as we use a formulation, (1), where the coupling is defined in such a way that the 3-point interaction is $\bar{\psi} T^{a} \psi A^{a}_{\mu}$. Thus the canonical dimensions for the fields essentially satisfy the condition for conformal integration or uniqueness, [14]. From the second term of (1), therefore, we have the scaling relation

$$\omega = \eta + \chi + \chi_{G}$$

(8)

The quantity $\chi_{G}$ is the critical exponent corresponding to the renormalization of the pure (composite) operator $(G^{a}_{\mu\nu})^{2}$, whilst the gluon dimension arises because of the wave function renormalization of the constituent fields of the composite. Thus computing $\chi_{G}$ gives $\omega$ from (8).

To evaluate $\chi_{G}$ one substitutes the critical propagators (6) into the relevant $O(1/N_{f})$ set of Feynman diagrams and determines the residue of the simple pole in $\Delta$, [8]. This regularizing parameter is introduced by the shift $\beta \rightarrow \beta - \Delta$. The contributing graphs are illustrated in fig. 1 and each is computed in the Landau gauge to avoid mixing with (7), [9]. The first two graphs occur in QED and have been computed in [13]. Here we note their respective colour group factors are $C_{2}(R)$ and $[C_{2}(R) - C_{2}(G)/2]$. The third graph arises from the cubic term of the operator and it and the final graph are purely non-abelian, each having group factors $C_{2}(G)$. The computation was carried out by the application of standard techniques for massless integrals including integration by parts and conformal methods. Nevertheless several difficult subintegrals lurk within the final graph which were tedious to determine. To verify that we had obtained the correct values for them we calculated several graphs of [4] which contained the same subintegrals and checked that the total expression we computed agreed with the values given in [8]. Useful in this and other respects were the packages REDUCE [20] and FORM [21]. We note that the values are respectively,

$$\left( \begin{array}{c} \frac{\mu(\mu-1)(2\mu-1)\eta_{1}^{0}}{(\mu+1)}, \\
\frac{(4\mu^{2}+\mu-9)\eta_{1}^{0}}{(\mu+1)}, \\
\frac{(4\mu^{3}-2\mu^{2}-4\mu+1)\eta_{1}^{0}}{2(\mu+1)(\mu-1)(\mu-2)}, \\
\frac{(4\mu^{6}-6\mu^{5}+18\mu^{4}-67\mu^{3}+85\mu^{2}-38\mu+6)\eta_{1}^{0}}{4(2\mu-1)(\mu+1)(\mu-1)(\mu-2)} \end{array} \right)$$

(9)

Further at leading order there are no ghost contributions. One can see this by attempting to include them in the formalism we will describe later and then observing that the first appearance of any contribution is at $O(1/N_{f}^{2})$. This feature was also observed in [4] where the non-abelian generalization of the $CP(N)$ $\sigma$ model was studied. Indeed we make use of some of the observations of [4] here.
The final result is
\[
\omega = (\mu - 2) - \left[ (2\mu - 3)(\mu - 3)C_2(R) - \frac{(4\mu^4 - 18\mu^3 + 44\mu^2 - 45\mu + 14)C_2(G)}{4(2\mu - 1)(\mu - 1)} \right] \frac{\eta_1^0}{T(R)N_f} \tag{10}
\]

A final check on (10) is that it correctly reproduces the $O(1/N_f)$ terms of the three loop $\beta$-function of (2), [1-4]. This amounts to the terms with $C_2(G)$ as those with $C_2(R)$ have been verified for QED in [23, 24]. In three dimensions
\[
\omega = - \frac{1}{2} - \frac{10C_2(G)}{3\pi^2T(R)N_f} \tag{11}
\]

From (10) we can now deduce higher order coefficients which will appear in the $\overline{\text{MS}}$ $\beta$-function, by carrying out the $\epsilon$-expansion of (10) and using (3). Thus defining the leading order large $N_f$ coefficients by $a_n$,
\[
\beta(g) = \beta_0g^2 + \sum_{n=1}^{\infty} a_{n+1}[T(R)N_f]^n g^{n+2} \tag{12}
\]
with $\beta_0 = [2T(R)N_f/3 - 11C_2(G)/6]$, then
\[
a_4 = - \frac{[154C_2(R) + 53C_2(G)]}{3888}
\]
\[
a_5 = \frac{[(288\zeta(3) + 214)C_2(R) + (480\zeta(3) - 229)C_2(G)]}{31104}
\]
\[
a_6 = \frac{1}{233280}[(864\zeta(4) - 1056\zeta(3) + 502)C_2(R) + (1440\zeta(4) - 1264\zeta(3) - 453)C_2(G)]
\]
\[
a_7 = \frac{1}{1679616}[(3456\zeta(5) - 3168\zeta(4) - 2464\zeta(3) + 1206)C_2(R)
\]  
\[
+ (5760\zeta(5) - 3792\zeta(4) - 848\zeta(3) - 885)C_2(G)] \tag{13}
\]

Having obtained the set \{a_n\} for non-abelian theories we can examine the purely four dimensional $\beta$-function and search for fixed points other than the well known infrared stable point of Banks and Zaks, $g_{c,BZ}^4$, [4]. Its existence is important for recent developments in supersymmetric theories in relation to electric-magnetic duality, [23, 24]. Those infrared fixed points are determined by using exact non-perturbative arguments in the limit $N_c, N_f \to \infty$ with $N_f/3N_c$ held fixed, [24]. (The one loop coefficient of the $\beta$-function for that model is $(N_f - 3N_c)$ for $SU(N_c).$) Further in the context of $1/N_f$ expansions a $(16\pi - N_f)$ expansion from $g_{c,BZ}^4$ has been used to obtain an estimate for $\alpha_S$ for low $N_f$, [25]. Before studying the effect $O(1/N_f)$ corrections have on $\beta(g)$, we first recall properties of $g_{c,BZ}^4$. In [4] it was observed that for a range of $N_f$ the two terms of the two loop $\beta$-function have a different sign which therefore gives rise to a non-zero critical coupling, $g_{c,BZ}^4$. For $SU_c(3)$ this range is $8 < N_f < 17$, [4], and we have recorded the explicit values of $g_{c,BZ}^4$ in this case in our notation in table 1. Subsequently one can also study the effect that the inclusion the three loop term of the $\beta$-function has on the location of $g_{c,BZ}^4$. We have analysed (2) numerically and determined the three loop values of $g_{c,BZ}^4$ which are given in the second column of table 1. Several features are apparent. First, the range of $N_f$ for the existence of such an infrared fixed point is extended to $5 < N_f < 17$. Second the effect the three loop correction has is to move the location of $g_{c,BZ}^4$ towards the origin. In other words to a region where perturbation theory would be valid. These observations, however, ought to be qualified. It is not clear whether this picture is meaningful because as $N_f$ decreases $g_{c,BZ}^4$ clearly increases away from the region where perturbation theory is useful. In other words for low values of $N_f$ we can not make a reliable statement on even, say, the three loop range of $N_f$ for which $g_{c,BZ}^4$ occurs. One indication of where the perturbative picture may not be valid can be
deduced from the values of the critical exponent $\beta'(g^BZ_c)$ which is a physically meaningful and calculable quantity. In table 2 we have given the corresponding values for $\beta'(g^BZ_c)$ as deduced from the two and three loop values of $g^BZ_c$ respectively. Clearly the three loop corrections do not affect the two loop values appreciably for $N_f = 14, 15$ and $16$ suggesting that higher loop corrections are small. For lower values the divergence is evident indicating that the four and higher loop contributions would be needed to make an accurate estimate of the exponent. In light of the critical coupling being smaller for a larger range of $N_f$ it would better of either, though, to take the three loop values of $\beta'(g^BZ_c)$ as being the more reliable.

Now we consider the effect that the $O(1/N_f)$ corrections of (10) have. We have studied the case $N_c = 3$ in various ways. First, as in [12] we examined the $\beta$-function given by just taking all the leading order coefficients $a_n$ which was improved by non-abelianization, [24]. This entails replacing $N_f$ by the one loop $\beta$-function coefficient through the shift $N_f \rightarrow (N_f − 11C_2(G)/[4T(R)])$. It turns out that in searching for zeroes of the four dimensional $\beta$-function that the contributions from these $O(1/\beta_0)$ coefficients on their own are not sufficient for even obtaining a fixed point $g^BZ_c$. This is the same as was found in the QED case. [12], in the range of couplings where the series was convergent. Instead, to improve this situation we took the two and three loop $\beta$-functions of (2) and then included all subsequent information included in (10). The point of view being that one can at least study the effect the $O(1/N_f)$ corrections have on the fixed point $g^BZ_c$ which is known to exist. It turns out that in this approach we did not observe any non-trivial fixed points other than $g^BZ_c$ in $g > 0$ which was independent of the number of terms included. From a practical point of view in our analysis we truncated the series for $\beta(g)$ at around 14 terms. The effect of including more terms is negligible on the results we give in both tables until $N_f \lesssim 8$ when perturbation theory can not be regarded as reliable anyway. The remaining columns of our tables are the results of this analysis. Clearly the effect the $O(1/\beta_0)$ corrections have is not to move $g^BZ_c$ significantly from the three loop value for a large range of $N_f$. Also for $N_f = 14, 15$ and $16$ the values of $\beta'(g^BZ_c)$ are not that different from the perturbative estimates of $\beta'(g^BZ_c)$.

In conclusion we have produced the leading order corrections to the QCD $\beta$-function in a $1/N_f$ expansion which extends the calculation of [12]. Consequently we examined the effect they had on the known infrared fixed point in four dimensional $\beta$-function. It transpires that perturbation theory is valid for analysing the fixed point when the value of $N_f$ is near the upper bound for the existence of $g^BZ_c$ and estimates for a critical exponent were obtained then. It would be interesting, though, to compare the values of these exponents with results from other techniques such as the lattice, which would be expected to reliably cover the lower part of the range. In this case a resummation would be necessary to try and improve the lack of convergence which is apparent when the three loop values are compared to the two loop ones. Further, we believe it would be useful to repeat our analysis for the supersymmetric extension of QCD in large $N_f$ in relation to [23, 24]. Once the analogous expression to (10) is available it would be possible to study the effect the $O(1/N_f)$ corrections have on the infrared fixed point when $N_c$ is large as well as for orthogonal and symplectic gauge groups. As a first step one would need to determine which field theory supersymmetric QCD is equivalent to at the $d$-dimensional fixed point and verify, for example, that the correct triple and quartic interactions are obtained in the large $N_f$ limit similar to [8]. It would be hoped that there is a small set of interactions, as in the non-abelian Thirring model, to reduce the amount of calculation that would occur.

Acknowledgements. This work was carried out with the support of PPARC through an Advanced Fellowship. The author thanks Drs D.J. Broadhurst, D.R.T. Jones and H. Osborn for useful conversations and Dr T.J. Morris for drawing his attention to [18]. The figures were designed using the package FEYNDIAGRAM version 1.21.
References.

[1] D.J. Gross & F.J. Wilczek, Phys. Rev. Lett. 30 (1973), 1343; H.D. Politzer, Phys. Rev. Lett. 30 (1973), 1346.

[2] W.E. Caswell, Phys. Rev. Lett. 33 (1974), 244; D.R.T. Jones, Nucl. Phys. B75 (1974), 531.

[3] O.V. Tarasov, A.A. Vladimirov & A.Yu. Zharkov, Phys. Lett. 93B (1980), 429.

[4] S.A. Larin & J.A.M. Vermaseren, Phys. Lett. B303 (1993), 334.

[5] D.V. Nanopoulos & D.A. Ross, Nucl. Phys. B157 (1979), 273; R. Tarrach, Nucl. Phys. B183 (1981), 384.

[6] O.V. Tarasov, JINR preprint P2-82-900 (in Russian).

[7] A.N. Vasil’ev, Yu.M. Pis’mak & J.R. Honkonen, Theor. Math. Phys. 46 (1981), 157; ibid. 47 (1981), 291.

[8] A.N. Vasil’ev & M.Yu. Nalimov, Theor. Math. Phys. 55 (1982), 423; ibid. 56 (1982), 643.

[9] A.N. Vasil’ev, M.Yu. Nalimov & J.R. Honkonen, Theor. Math. Phys. 58 (1984), 111.

[10] J.A. Gracey, Phys. Lett. B318 (1993), 177.

[11] E.S. Egorian & O.V. Tarasov, Theor. Math. Phys. 41 (1979), 26.

[12] A. Palanques-Mestre & P. Pascual, Commun. Math. Phys. 95 (1984), 277.

[13] J.A. Gracey, Int. J. Mod. Phys. A8 (1993), 2465.

[14] T. Banks & A. Zaks, Nucl. Phys. B196 (1982), 189.

[15] J. Zinn-Justin, Nucl. Phys. B367 (1991), 105.

[16] S.J. Hands, Phys. Rev. D51 (1995), 5816.

[17] K.-I. Kondo, Nucl. Phys. B450 (1995), 251.

[18] A. Hasenfratz & P. Hasenfratz, Phys. Lett. B297 (1992), 166.

[19] M. d’Eramo, L. Peliti & G. Parisi, Lett. Nuovo Cim. 2 (1971), 878.

[20] A.C. Hearn, “REDUCE Users Manual” version 3.4, Rand publication CP78, (1991).

[21] J.A.M. Vermaseren, “FORM” version 1.1, CAN publication, (1992).

[22] S.G. Gorishny, A.L. Kataev, S.A. Larin & L.R. Surguladze, Phys. Lett. B256 (1991), 81.

[23] N. Seiberg & E. Witten, Nucl. Phys. B426 (1994), 19.

[24] N. Seiberg, Nucl. Phys. B435 (1995), 129.

[25] P.M. Stevenson, Phys. Lett. B331 (1994), 187.

[26] D.J. Broadhurst & A.G. Grozin, Phys. Rev. D52 (1995), 4082.
### Table 1. Location of infrared fixed point $g_{c}^{BZ}$ for $SU_c(3)$.

| $N_f$ | Two loop | Three loop | Two loop | Three loop |
|-------|----------|------------|----------|------------|
|       |          |            | + $O(1/\beta_0)$ | $+ O(1/\beta_0)$ |
| 6     | -        | 4.050746   | (0.803122)   | 0.517376   |
| 7     | -        | 0.782073   | (0.829257)   | 0.463261   |
| 8     | -        | 0.466021   | (0.858331)   | 0.388537   |
| 9     | 1.666667 | 0.327211   | 0.878128    | 0.305364   |
| 10    | 0.702703 | 0.243297   | 0.858362    | 0.238810   |
| 11    | 0.392857 | 0.184156   | 0.666176    | 0.185119   |
| 12    | 0.240000 | 0.138426   | 0.283504    | 0.138208   |
| 13    | 0.148936 | 0.100763   | 0.155791    | 0.100726   |
| 14    | 0.088496 | 0.068282   | 0.089480    | 0.068278   |
| 15    | 0.045455 | 0.039271   | 0.045534    | 0.039270   |
| 16    | 0.013245 | 0.012647   | 0.013246    | 0.012647   |

### Table 2. Values of $\beta'(g_{c}^{BZ})$ for $SU_c(3)$.

| $N_f$ | Two loop | Three loop | Two loop | Three loop |
|-------|----------|------------|----------|------------|
|       |          |            | + $O(1/\beta_0)$ | $+ O(1/\beta_0)$ |
| 6     | -        | 81.682972  | (15.663662)  | 8.095948   |
| 7     | -        | 5.972522   | (17.792600)  | 4.929522   |
| 8     | -        | 2.658882   | (16.790094)  | 2.717457   |
| 9     | 4.166667 | 1.475455   | 13.456718   | 1.510817   |
| 10    | 1.522523 | 0.871775   | 7.901656    | 0.879995   |
| 11    | 0.720238 | 0.516977   | 1.671025    | 0.518561   |
| 12    | 0.360000 | 0.295517   | 0.360750    | 0.295784   |
| 13    | 0.173759 | 0.155581   | 0.173789    | 0.155616   |
| 14    | 0.073746 | 0.069899   | 0.073751    | 0.069903   |
| 15    | 0.022727 | 0.022307   | 0.022727    | 0.022306   |
| 16    | 0.002208 | 0.002203   | 0.002207    | 0.002203   |
Fig. 1. Graphs for anomalous dimension of \((G^a_{\mu \nu})^2\).