The deceiving simplicity of problems with infinite charge distributions in electrostatics

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Abstract

We show that for an infinite, uniformly charged plate no well defined electric field exists in the framework of electrostatics, because it cannot be defined as a mathematically consistent limit of a solution for a finite plate. We discuss an infinite wire and an infinite stripe as examples of infinite charge distributions for which the electric field can be determined as a limit in a formal, mathematical way. We also propose a didactic framework that can help students understand subtleties related to the problems of limits in electrostatics. The framework consists of heuristic tools (claims) that help to align intuitions in the spirit of a rigorous definition of an integral. We thoroughly discuss to what degree the solution for a finite plate agrees with the traditional but unfortunately ill-defined solution for an infinite plate. Physics is a science of approximations. One can ask why the use of mathematically ill-defined formulae and objects should be forbidden if they make life simpler. In our opinion, approximations should have solid physical and mathematical foundations.

1 Introduction

In this paper, we discuss conceptual problems related to teaching electrostatics to college students. Many exercises involve sophisticated integrating over bounded or unbounded domains. However, the problem of the existence of integrals over unbounded domains is rarely discussed. Generally, the teaching process focuses on the application of "symmetry" as a leading heuristic rule, but a mathematical perspective on validity and the drawbacks of such an approach are not presented, even in standard textbooks (e.g. [1], [2], [3], [4], [5], [6], [7], [8]). The absence of such a discussion is permanent and hard to accept. Students who attend lectures have completed at least a basic calculus course and should be capable of understanding explanations related to the existence of limits, the Riemann integral over an unbounded domain and the integral in the Cauchy principal value sense. More than fifty years ago R. Shaw [9] expressed his frustration in the following words:
Presumably not unconnected with this uncritical acceptance of arguments based on symmetry is the fact that false, or at best incomplete, arguments of this type are quite common in elementary textbooks on electricity.

We will show that the "symmetry heuristics" in electrostatics do more harm than good and do not agree with the formal mathematical definition of limit. Even the Cauchy principal value, sometimes presented as a mathematical representation of a "symmetry heuristics", does not work in the long run as it clashes with invariance under translations. We understand that a heuristic is necessary to frame student intuition and give a general feeling of the subject \[10\]. Attempts to "associate meaning with certain structures" in case of definite integral in the context of electrostatics are presented in \[11,12\]. However, we did not find any discussion about a "concept image" related to integration over an unbounded domain. Therefore, we propose a new leading concept for the case of charge distributions over unbounded domains.

Unbounded distributions are problematic in various aspects. Here we focus on the existence of electric field integrals. However, other approaches are present in the literature. For example, the authors of \[13\] discuss asymptotic conditions of an unbounded charge distribution necessary to obtain the assumed asymptotics of the potential. We show our ideas in action discussing a few examples of unbounded charge distributions: the infinite wire, the infinite stripe, a quarter of the infinite plate, and the infinite plate.

We disagree with the popular opinion that calculating the electric field of the infinite plate is the simplest and correct way to obtain the approximation of the field of a big but finite plate. Let us assume that we somehow convince a student that for a large plate, far from its edges the field should be nearly uniform and nearly perpendicular to the plate. The student uses textbook procedures and receives the result. This approach has three significant flaws. First, the student has no idea how precise is the result. What is the error of the result? Is it 10% or $10^{-6}$%? (for a detailed discussion see chapter 5) Second, this approach strengthens the conviction that the field of the infinite plate exists as – intuitively but not mathematically – the limit of the enlarging procedure. Third, from the beginning the student is exposed to dirty tricks dressed up as fundamental principles.

2 Integrals over unbounded domains in electrostatics

2.1 The didactic challenge

The electric field of uniformly charged infinite objects such as an infinite wire and a plate is one of the standard topics present in introductory courses in electrostatics. Given some specific volumetric distribution of charges $\rho(\vec{r})$ confined in some finite volume (domain) $V \in \mathbb{R}^3$, the electric field at point $\vec{r}$ is given by the formula:

$$\vec{E}(\vec{r}) = k \int_V \frac{\rho(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$

where $k = \frac{1}{4\pi \varepsilon_0}$. In the case of infinite volume, the integral of the electric field over a non-compact domain should be computed as a limit:

$$\vec{E}_\infty(\vec{r}) = \lim_{{V \to \mathbb{R}^3}} \int_V \frac{\rho(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} dV'$$
The existence of the limit (2) is treated as a default in textbooks. Authors of textbooks (e.g. [1], [2], [3], [4], [5], [6], [7], [8]) implicitly assume that integrals over unbounded domains are computable and they focus on presenting the most effective ways to calculate the limit (2) (often using Gauss’s law), so the discussion has a technical and not an existential nature – for more details see Appendix C. Unfortunately, a discussion about the existence of limit (2) is unavoidable, even in the case of such a standard problem of electrostatics as a charged infinite plate. The absence of such a discussion is difficult to understand. One of the pessimistic explanations can be found in [14]. However, we optimistically believe that the authors could not find a satisfying way to explain all the subtleties to students. Indeed, comments like [15] (p. 181) do not help:

A double integral \( \int f(x, y) \, dx \, dy \) over an infinite region \( R \) can be defined by taking a sequence of regions \( \{ R_n \} \) such that, for any part of \( R \), this part is included in all \( R_n \) for \( n \) greater than some \( m \). If the double integral over \( R_n \) has a unique limit for all such sequences, this limit can be taken as the definition of the integral over \( R \). Improper double integrals may be defined similarly. It appears, however, that unless the same process gives a unique value when \( |f(x, y)| \) is substituted for \( f(x, y) \) the value of the limit will depend on the shapes of the regions \( R_n \), and consequently a non-absolutely convergent double integral has no meaning unless these are specified.

However true, these thoughts are convoluted enough to present a didactic challenge. Unfortunately, the over-abundant "symmetry heuristics" presented as obvious in textbooks makes detailed discussion about the existence of limit (2) more difficult. The didactic challenge is solvable but to do this the "symmetry" argument should not be used as a leading idea in electrostatics. A concise presentation of problems related to limit (2) could involve the following steps:

1. Downgrade "symmetry" intuitions as they do not help with the nuances of calculations over unbounded domains.
2. Find intuitions/heuristics that help to understand the mathematical subtleties of limit (2).
3. Check which classical problems of electrostatics can be computed directly from definition (2).
4. Accept the fact that some problems become ill-posed when extended to an unbounded domain.
5. Discuss the finite domain solutions for non-extendable problems.

### 2.2 Drawbacks of symmetry intuitions

We present simple examples of how intuition built on the "symmetry" argument conflicts with strict mathematical definitions. We believe that the typical second year student is capable of understanding the examples that follow.

#### 2.2.1 Limits

To show that the limit

\[
\lim_{x \to +\infty} \cos(x)
\]
does not exist (see also: [16], p. 66), it is enough to show a counterexample – for two different sequences: 
\[ x_n = 2\pi n, \quad y_n = \pi + 2\pi n, \quad n \in \mathbb{N}, \]
the limit (3) gives two different results:
\[ \lim_{n \to +\infty} \cos(2\pi n) = 1, \quad \lim_{n \to +\infty} \cos(\pi + 2\pi n) = -1. \] (4)
One would get into serious trouble during a calculus exam arguing that
\[ \lim_{x \to +\infty} \cos(x) = 0, \] (5)
using the "let's take the average" or "symmetry with respect to the x-axis" argument, even if
\[ \lim_{n \to \infty} \cos\left(\frac{\pi}{2} + \pi n\right) = 0. \] (6)
The truth is that not every sequence has a limit.

2.2.2 Integrals

Imagine one has to compute the integral of a real function \( f(x) \) over \( \mathbb{R} \). The existence of such an integral, by definition, is related to the existence of two independent limits:
\[ \int_{-\infty}^{+\infty} f(x) \, dx := \lim_{A \to -\infty} \lim_{B \to +\infty} \int_{A}^{B} f(x) \, dx \] (7)
In this spirit the integral:
\[ \int_{-\infty}^{+\infty} \sin(x) \, dx = \lim_{A \to -\infty} \lim_{B \to +\infty} \int_{A}^{B} \sin(x) \, dx = \lim_{B \to +\infty} (-\cos(B)) - \lim_{A \to -\infty} (-\cos(A)) \] (8)
does not exist because each limit for \( \cos(x) \) does not exist as we have shown in (4).
Why cannot one use the argument of symmetry and claim that the integral (5) is equal to zero because the \( \sin(x) \) is an odd function? The symmetry argument applied here essentially means that we treat variables \( A \) and \( B \) as not independent – now we impose an additional constraint \( A = -B \) and we want to solve problem (5) by computing:
\[ \lim_{B \to +\infty} \int_{-B}^{B} \sin(x) \, dx = \lim_{B \to +\infty} (-\cos(B)) - (-\cos(-B)) = 0, \] (9)
However from a mathematical point of view, the value of integral (5) is equal to (4) only when (5) exists in the sense of (7)! The nuance lies in the implication: if the integral defined in (7) exists then the result does not depend on the way we link the \( A \) and \( B \) values, say \( A = -B^2 \) or \( A = -2B \), etc. But the converse is not true.

To save the "symmetry" argument one could abandon formal definitions and say that every integral in electrostatics should be understood in the "symmetric" sense:
\[ \text{v. p.} \int_{-\infty}^{+\infty} f(x) \, dx := \lim_{A \to +\infty} \int_{-A}^{A} f(x) \, dx \] (10)
where v. p. means the Cauchy principal value (see also: [17], p. 45, example 11). Unfortunately such an approach also clashes with the "symmetry" heuristic when one tries to apply it to symmetric functions such as \( \cos(x) \):
\[ \lim_{A \to +\infty} \int_{-A}^{A} \cos(x) \, dx = \lim_{A \to +\infty} \sin(A) - \sin(-A) = \lim_{A \to +\infty} 2\sin(A) \] (11)
It is easy to show, as we did for (3), that the last limit in (11) does not exist. The conflict also manifests itself at the level of intuitions. Physicists like the idea of translational invariance as much as symmetry. On the computational level this means that the integral (in the principal value sense as well) over an unbounded domain should not change if we shift the graph of the function by $\frac{\pi}{2}$, so the result for $\sin(x)$ should be the same as for $\cos(x)$.

3 A conceptual framework for understanding electrostatics

We would like our students possess the ability to first think about whether a problem has a solution before going into the technical nuances of finding the best shortcut for solving it. The first step should not involve a discussion about the possible symmetries of the problem for it treats the existence of solutions by default. We need a leading idea that focuses on the nuances of the existence of limit (2) and at the same time could be accepted on the heuristic level as it relates to physical objects. Therefore we propose two equivalent claims:

Claim 1 The property of a system should not depend on the method of dividing the system into subsystems.

Claim 2 The property of the system should not depend on the method of constructing the system from subsystems.

The above claims consider two important facts related to limit (2): 1) The existence of limit for an unbounded region means that all possible ways to fill–up that region must lead to the same result. 2) If the result of (2) is not independent of the choice of division into smaller parts, the limit does not exist. Claim 1 represents a static approach to the system while Claim 2 focuses on its dynamical aspect. Both should appeal to different mathematical and physical intuitions of our students.

In light of the above claims students would be less surprised to see that the integral (2) in the case of an infinite charged plate gives different values depending on the particular prescription of extending volume $V$ (in a two-dimensional case) to infinity. Students can check that such a field, understood as a unique solution of (2), does not exist and has the same meaningless status as limit (3). In the next sections we will revisit standard problems of electrostatics and use Claims 1 and 2 as the leading ideas.

4 Classical problems of electrostatics revisited

4.1 The didactic challenge, part II

We aim to show that the application of Claims 1 and 2 can lead to interesting results or can at least provoke refreshing discussions with students. We examine the existence of the electric field for: a uniformly charged infinite wire, an infinite stripe and an infinite plate by computing appropriate limits of solutions for a finite wire and a rectangle. For linear and surface charge distributions, we use the following variants of formula (1)
\[
E(\vec{r}) = k \int_L \frac{\lambda(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dL'
\]

(12)

where \( \lambda(\vec{r}) \) is a linear charge distribution along some finite length curve \( L \), and

\[
\vec{E}(\vec{r}) = k \int_S \frac{\sigma(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \, dS'
\]

(13)

where \( \sigma(\vec{r}) \) is a surface charge distribution on some finite area surface \( S \). We do not discuss how to derive (12) and (13) from (1) by treating charge density in the rigorous, distributive sense. Such an approach, however preferable, would pose another didactic challenge as first year students are not familiar with the theory of distributions.

### 4.2 From finite to infinite straight wire

First we consider a one-dimensional, straight, uniformly charged wire with linear charge density \( \lambda \). We start with a wire \( L \) of finite length extending from point \( a \) to \( b \) on the \( X \) axis. We determine the electric field at point \( \vec{r} = [0, y, z] \), assuming \( y \neq 0 \) or \( z \neq 0 \). Using Coulomb’s law and superposing contributions from infinitesimal charge elements \( \lambda \, dx' \) at point \( \vec{r}' = [x', 0, 0] \) one obtains:

\[
\vec{E}(\vec{r}) = k \lambda \int_L \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, dx'
\]

where

\[
\int_L \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, dx' = \hat{e}_x \int_L \frac{-x'}{|\vec{r} - \vec{r}'|^3} \, dx' + \hat{e}_y \int_L \frac{y}{|\vec{r} - \vec{r}'|^3} \, dx' + \hat{e}_z \int_L \frac{z}{|\vec{r} - \vec{r}'|^3} \, dx'
\]

and

\[
\vec{r} - \vec{r}' = [-x', y, z] \quad |\vec{r} - \vec{r}'| = \sqrt{x'^2 + y^2 + z^2}
\]

As the \( y \) component of the electric field is analogous to the \( z \) component, for simplicity we continue calculation of the field at point \( \vec{r} = [0, 0, z] \), on the \( Z \) axis (assuming \( z \neq 0 \)). In this case \( E_y = 0 \). We calculate \( x \) and \( z \) components of \( \vec{E} \):

\[
E_x = k \lambda \int_a^b \frac{-x'}{\sqrt{x'^2 + z^2}^3} \, dx' = k \lambda \left( \frac{1}{\sqrt{b^2 + z^2}} - \frac{1}{\sqrt{a^2 + z^2}} \right)
\]

(14)

\[
E_z = k \lambda \int_a^b \frac{z}{\sqrt{x'^2 + z^2}^3} \, dx' = k \lambda \frac{1}{z} \left( \frac{b}{\sqrt{b^2 + z^2}} - \frac{a}{\sqrt{a^2 + z^2}} \right)
\]
Discussion

Our goal is to obtain the formula for the electric field of an infinite wire. First, we cannot assume that a solution in the sense of (2) exists. Therefore, we cannot set \( a = -b \) and calculate the limit \( b \to +\infty \) for \( E_x \) and \( E_z \) in (14). We cannot assume only from symmetry that the field component parallel to the wire, \( E_x \), is zero as is usually done in approaches using Gauss’ law. Only after we prove that a solution exists – which means that we have to compute limits \( a \to -\infty \) and \( b \to +\infty \) independently – then any symmetry-inspired methods or other shortcuts can be used and would give the same result. These considerations may seem superfluous, but such nuances play a crucial role in the case of the infinite plate.

The results for \( E_x \) and \( E_z \) are independent of any order in which limits \( a \to -\infty \) and \( b \to +\infty \) are calculated and in agreement with textbooks:

\[
\begin{align*}
\lim_{b \to +\infty} \lim_{a \to -\infty} E_x &= 0 \\
\lim_{b \to +\infty} \lim_{a \to -\infty} E_z &= 2k\lambda \frac{1}{z}
\end{align*}
\]

4.3 From the rectangle to the infinite plate

In an analogy to the case of the finite wire, we start with a finite rectangle and analyse what happens if sides of the rectangle are independently extended to infinity. It will be shown that in some cases the integral (2) does not exist.

4.3.1 The rectangle

Let us consider a two dimensional, uniformly charged rectangle \( P = [a, b] \times [c, d] \) on the \( XY \)-plane. The choice of coordinates is shown in Fig. 1, where \( \sigma \) denotes a constant surface charge density.

We determine the components of the electric field at point \( \vec{r} = [0, 0, z] \) on the \( Z \) axis, assuming \( z \neq 0 \). Details are presented in Appendix A. The \( x \)-component of the electric field is equal to

\[
E_x = k\sigma \ln \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) \quad (15)
\]

As the result for \( E_y \) can be easily obtained after a change of variables in equation (15),

\[
E_y = k\sigma \ln \left( \frac{b + \sqrt{d^2 + b^2 + z^2}}{a + \sqrt{d^2 + a^2 + z^2}} \right) \left( \frac{a + \sqrt{c^2 + a^2 + z^2}}{b + \sqrt{c^2 + b^2 + z^2}} \right) \quad (16)
\]

we limit our considerations to \( E_x \) only. The \( z \)-component of the electric field is equal to

\[
E_z = k\sigma \left\{ \arctan \left[ \frac{bd}{z\sqrt{b^2 + d^2 + z^2}} \right] - \arctan \left[ \frac{bc}{z\sqrt{b^2 + c^2 + z^2}} \right] \right\} \quad (17)
- \arctan \left[ \frac{ad}{z\sqrt{a^2 + d^2 + z^2}} \right] + \arctan \left[ \frac{ac}{z\sqrt{a^2 + c^2 + z^2}} \right]
\]
These results will be used in the following sections to calculate the electric field of infinite charge distributions.

4.3.2 From the rectangle to the infinite stripe

We extend the rectangle to the infinite stripe by setting \( d \to +\infty \) and \( c \to -\infty \). A discussion about the limits would be identical to the one from section 4.2. After computing limits independently for \( d \) and \( c \) one obtains (see Appendix B) well-defined components of the field

\[
E_x \text{ stripe} = k\sigma \ln \frac{a^2 + z^2}{b^2 + z^2}
\]

\[
E_y \text{ stripe} = 0
\]

\[
E_z \text{ stripe} = 2k\sigma \left\{ \arctan \left[ \frac{b}{z} \right] - \arctan \left[ \frac{a}{z} \right] \right\}
\]

(18)

4.3.3 From the stripe to the infinite plate

This procedure breaks down if we “extend” the infinite straight stripe to the infinite plate, calculating

\[
E_x \text{ plane} = \lim_{b \to +\infty} \lim_{a \to -\infty} E_x \text{ stripe} = \lim_{b \to +\infty} \lim_{a \to -\infty} k\sigma \ln \frac{a^2 + z^2}{b^2 + z^2}
\]
We aim to show that such a limit does not exist using a method similar to the case of limit (3). To prove that various procedures lead to different results, let us assume that

\[ a = -\xi b \]

where \( \xi \) is an arbitrary constant, \( \xi > 0 \). Then:

\[ E_{x, \text{plane}} = \lim_{b \to +\infty} E_{x, \text{stripe}} = k\sigma \ln\xi^2 \quad (19) \]

It is clear that any result is obtainable. For example, if we set \( \xi = 1 \) then \( E_{x, \text{plane}} = 0 \). But for \( \xi = e \) one obtains \( E_{x, \text{plane}} = 2k\sigma \). Similar reasoning shows that the \( y\)-component also does not exist. To help students, we can use our claims and explain the mathematical fact of non-existence of a limit on the level of intuition: the electric field of the infinite plate depends on the way the plate is built because different methods for extending the stripe to infinity give different results. This means that the electric field for the infinite plate does not exist.

Problems with \( E_{x, \text{plane}} \) and \( E_{y, \text{plane}} \) do not influence the existence of the third limit for \( E_{z, \text{plane}} \)

\[ E_{z, \text{plane}} = \frac{z}{|z|} \frac{\sigma}{2\varepsilon_0} \]

The last result is presented in standard textbooks as the \( z \) component of the electric field of the infinite plate, the remaining components are set to be zero as a result of "symmetry". However, with the help of formula (19) we see that the \( E_{x, \text{plane}} \) and \( E_{y, \text{plane}} \) can be arbitrary so we cannot talk about the vector quantity \( \vec{E}_{\text{plane}} \) in a meaningful way as two of its components are undefined.

### 4.3.4 A quarter of \( \mathbb{R}^2 \)

Another aspect of asymptotics of the electric field of the rectangle from section 4.3.1 will be revealed if, instead of extending opposite sides, one extends the rectangle to the first quarter of the \( XY \)-plane by extending the adjacent sides. We set \( a = 0 \), \( c = 0 \), \( b \to +\infty \) and \( d \to +\infty \). Then the limit of the argument of the logarithm in equation (15) for \( E_x \) equals zero:

\[
\lim_{b \to +\infty} \lim_{d \to +\infty} \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{\sqrt{b^2 + z^2}} \frac{\sqrt{z^2}}{d + \sqrt{d^2 + z^2}} \right) = 0
\]

Thus one has

\[ E_{x, \mathbb{R}_+^2} = \lim_{b \to +\infty} \lim_{d \to +\infty} E_x = -\infty \]

One obtains the same result for \( E_{y, \mathbb{R}_+^2} \) in equation (16) by calculating the same limit. Once more two components of the electric field are undefined. The \( z \)-component of the electric field is equal to

\[ E_{z, \mathbb{R}_+^2} = \frac{z}{|z|} \frac{\sigma}{8\varepsilon_0} \]
which is a quarter of the standard solution for the $z$-component of the field from an infinite plate. One could try to build the solution for an infinite plate of four such quarters. Unfortunately, the vector $\vec{E}_{Rz}$ is undefined and the existence of a well defined system made from four undefined subsystems cannot be accepted in a mathematical and intuitive sense.

We showed that the solution for the infinite wire exists, but there is no solution for the infinite plate. We did not find such a discussion in any textbook. For example, in [5] (problem 33, p. 1014) students are encouraged only to calculate the field of a half of an infinite wire. The result could be used to verify the existence of a solution for the infinite wire. The next, natural step would be to calculate the field from a half of an infinite plate. That would necessarily lead to a discussion on the existence of a solution.

5 How important are finite size and asymmetry

Although the problem of the existence of a solution for an infinite plate is fundamental, it may be treated as the next academic curio. A more practical question is: How much finite size and asymmetry in an infinite plate. That would necessarily lead to a discussion on the existence of a solution.

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We demonstrated the behaviour of these ratios in two simple cases:

(a) An extending stripe. The field is calculated in point $(0, 0, z)$ where $z > 0$ (Fig. 2). To be in a reasonable distance from the edges, we set the width of the rectangle to be 20 times larger than the distance $z$. Thus, we set three sides at $d = b = 10z$ and $c = -10z$. The length of the rectangle, and the position of the fourth side, we relate to the asymmetry parameter $\xi > 0$ by setting $a = -10z\xi$. For example, if $\xi = 1$, the square is obtained. The dependencies of $E_x/\sigma_{\infty}$ and $E_x/E_z$ on $\xi$ in this case are shown in Fig. 3 note that $E_y = 0$. It is clear that $E_x$ cannot be neglected, it is a significant component of the electric field: $|E_x/E_z| \geq 20\%$ for $\xi$ smaller than 0.5 or greater than 3.

It is worthwhile to comment about asymptotic behaviour: there are finite, non-zero limits for $E_z$ and $E_x$ as $\xi \to \infty$ (this describes a stripe that is infinitely long on one side, here – on the negative part of the $X$ axis).

(b) An extending square. The field is calculated at point $(0, 0, z)$ where $z > 0$ (Fig. 4). To be in a reasonable distance from the edges we set the distance to the top and right edges, of the rectangle to be 10 times the distance $z$ by setting $d = b = 10z$. Both, the length and the width of the rectangle we relate to the asymmetry parameter $\xi > 0$ by setting $a = c = -10z\xi$. In this case, two sides of the resulting square “move away” as the asymmetry parameter $\xi$ increases. The dependencies of $E_x/\sigma_{\infty}$ and $E_x/E_z$ on $\xi$ in this case are shown in Fig. 5. It should be noted that $E_y = E_z$. For $\xi > 2$ or $\xi < 0.5$ the $E_x$ component of the field is greater than around 20% of $E_z$. For $\xi > 200$ the $E_x$ component of the field is greater than $E_z$. However, for $\xi > 30$ the field component parallel to the plate, $\sqrt{E_x^2 + E_y^2} = \sqrt{2}|E_x|$, is greater than $E_z$. 

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Figure 2: The extending stripe. The rectangle $[a, b] \times [c, d]$ is charged with the constant surface density $\sigma$. The $Z$ axis is perpendicular to the $XY$-plane. The electric field is determined at point $\vec{r} = [0, 0, z]$. Three sides are set at $d = b = 10z$ and $c = -10z$. The length of the rectangle, and the position of the fourth side, is related to the asymmetry parameter $\xi > 0$ by setting $a = -10z\xi$. If $\xi = 1$, the square is obtained. As an example, the rectangle for the asymmetry parameter $\xi = 3$ is shown.

Figure 3: The dependence of $E_z/\frac{\sigma}{2\varepsilon_0}$ and $E_x/E_z$ on the asymmetry parameter $\xi$ for the extending stripe case. The electric field is determined at point $\vec{r} = [0, 0, z]$. Three sides of the uniformly charged rectangle placed on $XY$-plane are set at $d = b = 10z$ and $c = -10z$ (Fig. 2). The length of the rectangle, and the position of the fourth side, is related to the asymmetry parameter $\xi > 0$ by setting $a = -10z\xi$. For example, if $\xi = 1$, the square is obtained (in this case $E_x$ is equal to 0).
Figure 4: The extending square. The rectangle $[a, b] \times [c, d]$ is charged with the constant surface density $\sigma$. The Z axis is perpendicular to the XY-plane. The electric field is determined at point $\vec{r} = [0, 0, z]$. Two sides are set at $d = b = 10z$. Both, the length and the width of the rectangle are related to the asymmetry parameter $\xi > 0$ by setting $a = c = -10z\xi$. In this case, two sides of the resulting square “move away” as the asymmetry parameter $\xi$ increases. As an example, the square for the asymmetry parameter $\xi = 3$ is shown.

Figure 5: The dependence of $E_z/\sigma/2\varepsilon_0$ and $E_x/E_z$ on $\xi$ for the extending square case. The electric field is determined at point $\vec{r} = [0, 0, z]$. Two sides of the uniformly charged rectangle placed on XY-plane are set at $d = b = 10z$ (Fig. 4). Both, the length and the width of the rectangle are related to the asymmetry parameter $\xi > 0$ by setting $a = c = -10z\xi$. In this case, two sides of the resulting square “move away” as the asymmetry parameter $\xi$ increases. For $\xi = 1$ a center of the square is at point $(0, 0, 0)$, and $E_x = 0$ as expected.
The asymptotic behaviour is different than in the case of the extending stripe. Only the perpendicular component, $E_z$, is bounded. The parallel component is unbounded, $\lim_{\xi \to \infty} E_x = \infty$ and $\lim_{\xi \to \infty} E_y = \infty$, as in the case discussed in section 4.3.4.

For completeness we show how the field $E_z$ above the centre of the extending square varies. The field is calculated at point $(0, 0, z)$ where $z > 0$. To be above the centre of the square we set $b = d = \eta z$ and $a = c = -\eta z$ where $\eta > 0$. Thus, the length of a side of the square is equal to $2\eta z$. The dependency of $E_z/\sigma/\varepsilon_0$ on the ratio $\eta$ in this case, is shown in Fig. 6. It should be noted that here $E_y = E_x = 0$. If $\eta = 1$, which means that the length of a side of the square is equal to $2z$, the $z$-component of the field is only around $35\%$ of $\sigma/\varepsilon_0$. The field magnitude reaches $95\%$ of $\sigma/\varepsilon_0$ for $\eta = 20$ (the length of a side of the square is equal to $40z$). The field at the distance of $5$ cm above the centre of a square with a side of length $60$ cm ($\eta = 6$) would be equal to around $85\%$ of $\sigma/\varepsilon_0$.

![Figure 6: The dependence of $E_z/\sigma/\varepsilon_0$ above the centre of the extending square on the ratio $\eta$. The electric field is determined at point $\vec{r} = [0, 0, z]$. Four sides of the uniformly charged rectangle placed on the $XY$-plane are set at $b = d = \eta z$ and $a = c = -\eta z$ where $\eta > 0$. Thus, the length of a side of the square is equal to $2\eta z$. For example, the field at the distance of $5$ cm above the centre of a square with a side of length $60$ cm ($\eta = 6$) would be equal to around $85\%$ of $\sigma/\varepsilon_0$.]

6 Conclusions

We showed that for an infinite, uniformly charged plate no well defined electric field exists in the framework of electrostatics. We propose heuristic tools (the claims) that would help to align intuitions in the spirit of the rigorous definition of an integral. We want students to first consider the existence of the solution. We demonstrated that unfortunately some classical problems present in textbooks cannot be defined in a meaningful way – it is hard to talk about an electric field when only one component of the vector quantity is not ill-defined. Such problems seem to be very simple but their
simplicity is deceptive.

The good news is that a discussion about the applicability of solutions for a finite plate to an "infinite plate" problem is relatively simple. The transition from a rectangle to an infinite plate can lead through an infinite stripe or a quarter of $\mathbb{R}^2$ and help to understand where the solution ceases to exist. As we showed, a more rigorous discussion during classes is possible. Moreover, it may be interesting for students as a working example of the advantages of taking a closer look at definitions of mathematical objects. The didactic challenge can be overcome.

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A Uniformly charged rectangle

Let us consider a two dimensional, uniformly charged – with constant surface charge density $\sigma$ – rectangle $P = [a, b] \times [c, d]$ on the $XY$-plane. The choice of coordinates is shown in Fig. 1. We determine the electric field at point $\vec{r} = [0, 0, z]$ on the $Z$ axis, assuming $z \neq 0$. Using Coulomb's law and superposing contributions from infinitesimal charge elements $\sigma dS'$ at point $\vec{r}' = [x', y', 0]$ one obtains:

$$\vec{E}(\vec{r}) = k\sigma \int_{P} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} dS'$$

where

$$\int_{P} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} dS' = \hat{e}_x \int_{P} \frac{-x'}{|\vec{r} - \vec{r}'|^3} dS' + \hat{e}_y \int_{P} \frac{-y'}{|\vec{r} - \vec{r}'|^3} dS' + \hat{e}_z \int_{P} \frac{z}{|\vec{r} - \vec{r}'|^3} dS'$$

and

$$\vec{r} - \vec{r}' = [-x', -y', z]$$

$$|\vec{r} - \vec{r}'| = \sqrt{x'^2 + y'^2 + z'^2}$$

$x' \in [a, b]$ and $y' \in [c, d]$ at $z' = 0$. Let us focus on the $x$-component of $\vec{E}$:

$$E_x = k\sigma \int_{c}^{d} \int_{a}^{b} \frac{-x'}{\sqrt{x'^2 + y'^2 + z'^2}} dx' dy'$$

After the first integration one obtains

$$\int_{a}^{b} \frac{-x'}{\sqrt{(x'^2 + y'^2 + z'^2)^3}} dx' = \left. \frac{1}{\sqrt{x'^2 + y'^2 + z'^2}} \right|_{a}^{b} = \frac{1}{\sqrt{b^2 + y'^2 + z'^2}} - \frac{1}{\sqrt{a^2 + y'^2 + z'^2}}$$
The second integration leads to
\[
\int_c^d \frac{1}{\sqrt{b^2 + y'^2 + z^2}} dy' = \ln |y' + \sqrt{b^2 + y'^2 + z^2}|_c^d
\]
\[
= \ln \left| \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right|
\]
Finally, the \(x\)-component of the electric field is equal to:
\[
E_x = k\sigma \ln \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right)
\] (20)
The result for \(E_y\) can be easily obtained after a change of variables in equation (20).
To fully describe the electric field of the uniformly charged rectangle we calculate the \(z\) component of \(\vec{E}\):
\[
E_z = k\sigma \int_c^d \int_a^b \frac{z}{\sqrt{x'^2 + y'^2 + z^2}} dx' dy'
\]
The first integration:
\[
\int_a^b \frac{1}{\sqrt{x'^2 + y'^2 + z^2}} dx' = \left. \frac{x'}{(y'^2 + z^2)\sqrt{x'^2 + y'^2 + z^2}} \right|_a^b
\]
\[
= \left. \frac{b}{(y'^2 + z^2)\sqrt{b^2 + y'^2 + z^2}} \right|_a^b - \left. \frac{a}{(y'^2 + z^2)\sqrt{a^2 + y'^2 + z^2}} \right|
\]
The next integral is more complicated:
\[
\int_c^d \frac{b}{(y'^2 + z^2)\sqrt{b^2 + y'^2 + z^2}} dy' = \frac{1}{z} \arctan \left[ \frac{by'}{z\sqrt{b^2 + y'^2 + z^2}} \right]_c^d
\]
\[
= \frac{1}{z} \left\{ \arctan \left[ \frac{bd}{z\sqrt{b^2 + d^2 + z^2}} \right] - \arctan \left[ \frac{bc}{z\sqrt{b^2 + c^2 + z^2}} \right] \right. \}
\]
Finally, we obtain
\[
E_z = k\sigma \left\{ \arctan \left[ \frac{bd}{z\sqrt{b^2 + d^2 + z^2}} \right] - \arctan \left[ \frac{bc}{z\sqrt{b^2 + c^2 + z^2}} \right] \right. \}
\]
\[
- \arctan \left[ \frac{ad}{z\sqrt{a^2 + d^2 + z^2}} \right] + \arctan \left[ \frac{ac}{z\sqrt{a^2 + c^2 + z^2}} \right] \}
\]
It is simple to show that
\[
\lim_{b\to\infty} \lim_{a\to\infty} \lim_{d\to\infty} \lim_{c\to\infty} E_z = \frac{z}{|z|} k\sigma \left\{ \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} \right\} = \frac{z}{|z|} \frac{\sigma}{2\varepsilon_0}
\]
This limit for \(E_z\) is equal to the result well known from textbooks.
B  From a rectangle to an infinite stripe

We “extend” the rectangle to the infinite stripe by setting \( d \to +\infty \) and \( c \to -\infty \):

\[
\lim_{c \to -\infty} \lim_{d \to +\infty} \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) = \lim_{c \to -\infty} \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) = \frac{a^2 + z^2}{b^2 + z^2}
\]

One obtains a well defined \( x \)-component of the field:

\[
E_{x \text{ stripe}} = k\sigma \ln \frac{a^2 + z^2}{b^2 + z^2}
\]

C  Examples of inconsistencies

We are aware that it is a risky task to pinpoint the inconsistencies in well established textbooks. However, as university teachers that have to explain the issue to confused students every year, we would be more than satisfied to be able to recommend a textbook in which the authors present a consistent approach to problems with infinite charge distributions. Unfortunately, we did not find a mathematically correct treatment of such cases. To show that the problem is widespread, we present an arbitrary list of a few introductory courses in electrostatics in which the existence of the electric field or the force due to an uniformly charged infinite object is taken for granted.

- In [4] (Cancelling Components, pp. 639-640) the authors explain that in the case of a uniformly charged ring the components perpendicular to the ring axis are cancelled. This result is used as well in the case of a uniformly charged disk (pp. 643-644). However, at the end of this section the authors obtain the electric field for an infinite plate by extending the radius of the disk to infinity. There is no discussion of the existence of the presented integral if the radius of the disk is infinite. So, the components perpendicular to the axis of the disk are obtained on the same basis as in result [10]. In the following (p. 673) or in [1] (p. 13), the field from an infinite sheet is calculated using Gauss’ law, with the same assumption that the field parallel to the plate is zero. As we show in section [13.3] or [13.4] this field does not exist in the framework of electrostatics.

- In [8] (p. 51) the infinite sheet is built from infinite wires. The author observes that the integrand is an odd function, thus the result must be zero. Once more, students may think about \( \sin(x) \) as the integrand (see Eq. [8]) and wonder why physics lectures are not compatible with mathematical ones.

- We find in [2] (section 13-4, pp. from 13-13 to 13-14) that in the case of an infinite plate only the perpendicular component of the gravitational or the electric field is considered.

- In [5] (problem 33, p. 1052) and in [7] (section 4.8, p. 31) we have examples of the standard superposition of the fields from infinite plates. It is similar to superposing undefined quantities – such are the field components parallel to an infinite sheet.
In [5] (problem 37, p. 1053) the authors instruct students on how they should think: "THINK To calculate the electric field at a point very close to the center of a large, uniformly charged conducting plate, we replace the finite plate with an infinite plate having the same charge density. Planar symmetry then allows us to apply Gauss' law to calculate the electric field."

In [3] (p. 53) we read "In some textbook problems the charge itself extends to infinity (we speak, for instance, of the electric field of an infinite plane, or the magnetic field of an infinite wire). In such cases the normal boundary conditions do not apply, and one must invoke symmetry arguments to determine the fields uniquely." This suggests that the authors do not doubt that electrostatics is able to describe the case. The only problem is how to change the game rules to prove the result we believe in.

In [6] (pp. 45-46) the field components parallel to an infinite sheet are calculated and zero values are obtained. The author integrates first over an azimuthal angle, as the result is zero, the next integration over a radius is not necessary. However, students who usually already know Fubini's theorem can try to integrate first over the radius that leads them to infinity!

In all these cases the existence of the solution is assumed, and the authors' main goal is to obtain a mathematical formula, usually via some technical shortcut. A discussion of the existence of the solution would be beneficial for the didactic process, and is likely to lead to the correct result.