PP-waves and logarithmic conformal field theories

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Abstract

We provide a world-sheet interpretation to the plane wave limit of a large class of exact supergravity backgrounds in terms of logarithmic conformal field theories. As an illustrative example, we consider the two-dimensional conformal field theory of the coset model \(SU(2)_N/U(1)\) times a free time-like boson \(U(1)_\text{free}\), which admits a space-time interpretation as a three-dimensional plane wave solution by taking a correlated limit à la Penrose. We show that upon a contraction of Saletan type, in which the parafermions of the compact coset model are combined with the free time-like boson, one obtains a novel logarithmic conformal field theory with central charge \(c = 3\). Our results are motivated at the classical level using Poisson brackets of the fields, but they are also explicitly demonstrated at the quantum level using exact operator product expansions. We perform several computations in this theory including the evaluation of the four-point functions involving primary fields and their logarithmic partners, which are identified. We also employ the extended conformal symmetries of the model to construct an infinite number of logarithmic operators. This analysis can be easily generalized to other exact conformal field theory backgrounds with a plane wave limit in the target space.
1 Introduction

Searching for exact conformal field theories with interesting space-time interpretation in string theory, as in black hole physics and cosmology, has been a very active area of research in recent years (see, for instance, [1]-[5]). The advantages of having an exact description are rather obvious, since one can in principle perform a quantum mechanical stringy investigation of gravitational effects and deal with singularities and other important issues that appear in classical gravitational theories. Nevertheless, the exact description of a gravitational background in string theory does not necessarily guarantee that the corresponding conformal field theory will be easy to solve. The reason is that the resulting conformal field theories can be quite complicated, as they are based on non-compact groups that are difficult to analyze even at the Lie algebra level [6]-[9]. Moreover, there are many other backgrounds of great physical interest that are still awaiting an exact conformal field theory description. While many technical questions concerning the applications of string theory to gravitational problems are still remaining open to this day, any new progress is particularly welcome.

Plane waves arise as classical solutions to theories of gravity and share the attractive feature that they depart from the trivial flat space solution in the most controllable possible manner. This is essentially due to the existence of a covariantly constant null Killing vector, which, as it turns out, guarantees that curvature effects are kept to a minimum. Plane waves, being relatively simple solutions, are particularly easy to treat in various computations, which are usually performed on gravitational backgrounds at the semi-classical level. Therefore, the plane wave solutions, although might not be as interesting as, for instance, black hole and other solutions, they have the advantage of being simple enough to analyze in depth, yet without being completely trivial as flat space-time. Combining with the previous remarks, it is then natural to expect that the conformal field theories (CFT) corresponding to plane wave backgrounds will be relatively easy to study and in fact prove that they can be completely solvable.

Two distinct categories of exact CFT with the space-time interpretation of plane waves were constructed in recent years. The first category comprises of current algebra theories corresponding to WZW models based on non-semi-simple groups. The prototype example is the four-dimensional plane wave solution of [10], which was further analyzed using CFT techniques in [11]. This model can also be obtained using a limiting procedure starting
from the action for the WZW model for $SU(2)_N \times U(1)_{-N}$, where the last factor represents a time-like free boson playing the rôle of time [12]. This procedure is actually a limit of Penrose type that was first introduced in general relativity [13], but here it also involves a NS–NS two-form. The limit of WZW models involves taking the level $N$ of the current algebra very large, but in such a correlated way with a rescaling of the variables, that the resulting geometry is a plane wave, instead of flat space. This particular limiting procedure works out as a contraction of Saletan type for the underlying current algebra. Using these methods, a large class of models have been constructed in [14, 15] and various aspects of current algebras based on non-semisimple groups were subsequently investigated [17]-[19]. Note for completeness that the Lie algebra analog of this procedure is fairly known in the mathematical literature (see, for instance, [14]). We only emphasize here that the Saletan contraction is different from the Inonu–Wigner contraction, which leads to flat space-time, instead of a plane wave solution, and hence to a trivial CFT.

The second category of plane wave backgrounds with an exact CFT interpretation in string theory, comprises of coset theories corresponding to gauged WZW models based on non-semisimple groups. These can also be obtained, as it turns out in all known cases, by performing the same limiting procedure of Penrose type; illustrative examples of such models can be found in [12, 20, 21]. However, a detailed analysis of such theories is still missing at the CFT level and it is the purpose of this paper to initiate such investigation. We will find that the resulting theories belong to the class of so-called Logarithmic Conformal Field Theories (LCFT) [22, 23] which have been the subject of intense research over the past few years for various diverse reasons [24]-[27], starting with condensed matter physics [28, 29]; for a recent review of the subject see [30] and references therein. Curiously, our present work is the first in the literature to provide an example of a LCFT with clear space-time interpretation.

We will illustrate the main idea of our construction by focusing on a three-dimensional plane wave background, which is constructed from the metric and dilaton fields corresponding to the direct product theory $(SU(2)_N/U(1)) \times U(1)_{-N}$. We have, in particular,

$$\frac{1}{N} ds^2 = -dt^2 + d\theta^2 + \cot^2 \theta d\phi^2,$$

$$e^{-2\Phi} = e^{-2\Phi_0} \sin^2 \theta. \quad (1.1)$$

Consider next the change of variables from $(t, \theta, \phi) \rightarrow (u, v, x)$, given by

$$\theta = \epsilon v + u, \quad t = u, \quad \phi = \sqrt{\epsilon} x, \quad N = \frac{\alpha}{\epsilon}, \quad (1.2)$$

Note for completeness that the Lie algebra analog of this procedure is fairly known in the mathematical literature (see, for instance, [14]). We only emphasize here that the Saletan contraction is different from the Inonu–Wigner contraction, which leads to flat space-time, instead of a plane wave solution, and hence to a trivial CFT.

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where $\alpha, \epsilon$ are two parameters, followed by the limit $\epsilon \to 0$; $\alpha$ remains finite and fixed through the limiting process. Then, the resulting gravitational background is given by

$$\frac{1}{\alpha} ds^2 = 2dudv + \cot^2 u dx^2 ,$$

$$e^{-2\Phi} = e^{-2\Phi_0} \sin^2 u .$$

(1.3)

It represents the three-dimensional plane wave solution of interest, which has also been constructed in [21] in the same way.

The problem we would like to address in this paper is to find the nature of the exact CFT corresponding to the gravitational background (1.3). Along the way, we will also be interested in what happens to the symmetries of the original model (1.1) under the plane wave limit (1.2). The relevant discussion will naturally involve the time-like boson corresponding to the factor $U(1)^{-N}$, but also the compact parafermions of the coset model $SU(2)_N/U(1)$ [31], which are the natural chirally conserved objects for this exact CFT. In section 2, for the benefit of many readers, we will therefore review the basic elements of the parafermionic operator algebra, following [31], which can nevertheless be skipped by the experts. In section 3, we consider the classical counterpart of the parafermion currents that were introduced systematically in [32]. As we will see later, they admit a representation in terms of the metric variables appearing in (1.1) and they obey Poisson bracket relations as classical counterpart of the parafermionic operator algebra. The limiting procedure (1.2) will be naturally manifested at the Poisson algebra level and it will provide a guide on how to perform a similar limit quantum mechanically. Hence, the emergence of an exact LCFT will also be evident already at the classical level. In section 4, we perform the limiting procedure at the quantum level, using operator product expansions among the simplest operators of the model, and we explicitly show the emergence of a LCFT, as has been advertized. We also compute 2- and 4-point functions of operators that are logarithmic partners of each other and include a brief review of the relevant aspects of logarithmic conformal field theories. In section 5, we obtain as byproduct a free field realization of the elementary logarithmic partner fields. We also employ the extended conformal symmetries of the model to construct an infinite number of logarithmic operators. However, our analysis is not complete in the general case as we have not yet studied the representation theory of the limiting model in great detail. More results in this direction will be reported elsewhere. Finally, in section 6, we present the conclusions and briefly discuss some future directions.
It is also worth mentioning that although our present results are limited to a very simple background with an exact conformal field theory description, more general models can be tackled in a similar fashion. The two most important aspects of our work are first the world-sheet interpretation of the plane wave limit in supergravity theory and second the construction of a novel new class of logarithmic conformal field theories that comes as a bonus in this interpretation.

2 Parafermion operator algebra

In the following, we turn to the parafermion algebra that arises in the quantum mechanical treatment of the conformal field theory $SU(2)_N/U(1)$ for any integer value of the level $N$ and prepare to discuss carefully its large $N$ limits that are appropriate for the extended coset model $(SU(2)_N/U(1)) \times U(1)_{-N}$. We review the basic elements of the parafermionic operator algebra, following [31], and provide a brief summary of their correlation functions that will be used in the sequel. In the next section we will also consider the classical analogue of the parafermion currents, together with their commutation relations with respect to Poisson brackets, in order to motivate (among other things) appropriate field redefinitions and the new algebraic structures that emerge by taking $N$ to infinity. This will make it possible to extract the basic information on the world-sheet in terms of the target space coordinates and examine the different contractions of the resulting gravitational background, like its plane wave (Penrose) limit.

Recall that in ordinary conformal field theories with $Z_N$ symmetry, as in the coset model $SU(2)_N/U(1)$, there exists a set of $2N - 1$ parafermion fields $\psi_l(z)$ and $\bar{\psi}(\bar{z})$ with $l = 0, 1, \ldots, N - 1$ and $\psi_0 = 1 = \bar{\psi}_0$ corresponding to the identity operator, which are chirally conserved, i.e., $\bar{\partial}\psi_l = 0$ and $\partial\bar{\psi}_l = 0$. The parafermion fields are semi-local generalizing the usual fermion fields of the simplest $Z_2$ model to any other integer value $N$ and have fractional conformal dimension $d_l$ depending on $N$. First note that the fields $\psi_l(z)$ and $\bar{\psi}_l(\bar{z})$ have $Z_N \times \tilde{Z}_N$ charges $(l, l)$ and $(l, -l)$ respectively and as a result one picks up a phase by encircling one parafermion field around another inside a correlation function. Since any two fields with charges $(p, q)$ and $(p', q')$ yield a phase factor $\Omega = \exp(2\pi i \theta)$, where $\theta = -(pq' + p'q)/N$, the chiral fields $\psi_l$ and $\psi_{l'}$ have a mutual locality exponent $\theta = -2ll'/N$. For $N = 2$, we have the usual fermions in two dimensions with $l = l' = 1$ and $\Omega = 1$, whereas $\theta$ is non-integer for $N > 2$; local fields with $\Omega = 1$ are recovered
again in the large $N$ limit, as the parafermions turn into bosons. Moreover, the operator product expansion of two parafermions assumes the general form

\[
\psi_l(z)\psi_{l'}(w) = \frac{C_{l,l'}}{(z-w)^{d_l+d_{l'}-d_{l+l'}}} \sum_{n=0}^{\infty} (z-w)^n \Psi_{l+l'}^{(n)}(w), \tag{2.1}
\]

where $\Psi_{l+l'}^{(n)}$ are appropriate fields with charges $(l+l', l+l')$ and $C_{l,l'}$ are suitable structure constants. The leading term in the power series expansion, $\Psi_{l+l'}^{(0)}$, coincides with the parafermion field $\psi_{l+l'}$, whereas more complicated terms follow to higher orders in the operator product expansion. The anti-holomorphic sector of the theory that describes the operator product expansion of the remaining parafermion fields $\bar{\psi}_l(z)$ among themselves is identical to (2.1) using the anti-holomorphic coordinates $\bar{z}$ and $\bar{w}$ instead of $z$ and $w$, and it will be omitted from now on.

Taking into account the mutual locality exponent of the fields $\psi_l$ and $\psi_{l'}$, one easily obtains the following relation for the conformal dimension of the parafermion fields:

\[
d_l + d_{l'} - d_{l+l'} = 2\frac{ll'}{N} \mod Z . \tag{2.2}
\]

The conformal dimension of the $N$ chiral fields $\psi_l(z)$ that solves this equation and defines the parafermion operator algebra for $Z_N$ symmetric theories is

\[
d_l = d_{N-l} = \frac{l(N-l)}{N} ; \quad l = 0, 1, 2, \ldots, N-1 , \tag{2.3}
\]

where we also impose the hermiticity relation $\psi_l(z) = \psi_{N-l}(z)$ among its generators. Then, the operator product expansion of the parafermion fields assumes the following complete form:

\[
\begin{align*}
\psi_l(z)\psi_{l'}(w) & = \frac{C_{l,l'}}{(z-w)^{2ll'/N}} (\psi_{l+l'}(w) + \mathcal{O}(z-w)) ; \quad \text{for } l+l' < N , \\
\psi_l(z)\psi_{l'}^\dagger(w) & = \frac{C_{l,N-l'}}{(z-w)^{2(l-N-l')/N}} (\psi_{l-l'}(w) + \mathcal{O}(z-w)) ; \quad \text{for } l' < l , \tag{2.4} \\
\psi_l(z)\psi_{l'}^\dagger(w) & = \frac{1}{(z-w)^{2(N-l)/N}} \left( 1 + \frac{2d_l}{c} (z-w)^2 T(w) + \mathcal{O}(z-w)^3 \right).
\end{align*}
\]

The parameter $c$ is the central charge of the Virasoro algebra generated by the stress-energy tensor of the model, which is given by

\[
c = 2\frac{N-1}{N+2} , \tag{2.5}
\]
for all integer $N$. Moreover, the structure constants $C_{l,l'}$ are determined by the associativity property of the operator product expansion and they turn out to be

$$C_{l,l'}^2 = \frac{\Gamma(N-l+1)\Gamma(N-l'+1)\Gamma(l+l'+1)}{\Gamma(l+1)\Gamma(l'+1)\Gamma(N+1)\Gamma(N-l-l'+1)}. \tag{2.6}$$

One clearly has $C_{l,N-l} = 1$.

We add for completeness the operator product expansion of the stress-energy tensor $T(z)$ of the model with the parafermion currents,

$$T(z)\psi_l(w) = \frac{d_l}{(z-w)^2}\psi_l(w) + \frac{1}{z-w}\partial\psi_l(w) + \mathcal{O}(1), \tag{2.7}$$

which state that they are primary fields with conformal dimension $d_l$, and the operator product expansion with the stress-energy tensor itself,

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + \mathcal{O}(1), \tag{2.8}$$

which yields the Virasoro algebra with central charge $c$, as it is required.

The correlation functions among parafermion fields can be computed using the operator product expansions above. For instance, the 2-point function

$$\langle \psi_l(z)\psi_l^\dagger(w) \rangle = \frac{\delta_{l,l'}}{(z-w)^{2d_l}}, \tag{2.9}$$

follows immediately and provides the standard normalization of the parafermions. Higher order correlation functions also exhibit singularities with power law behaviour, which can be determined recursively using the structure of the operator algebra. Thus, a $2n$-point parafermion correlation function is related to the $(2n-2)$-point function as in

$$\langle \psi_1(z_1)\ldots\psi_1(z_n)\psi_1^\dagger(w_1)\ldots\psi_1^\dagger(w_n) \rangle = \prod_{i=2}^{n} \frac{1}{(z_1-z_i)^{2/N}} \prod_{j=1}^{n} \frac{1}{(z_1-w_j)^{2/N}} \times$$

$$\sum_{k=1}^{n} \left( \frac{1}{(z_1-w_k)^2} + \frac{2}{N(z_1-w_k)} \left( \sum_{i=2}^{n} \frac{1}{w_k-z_i} - \sum_{m \neq k} \frac{1}{w_k-w_m} \right) \right) \prod_{q=2}^{n} \frac{1}{(z_q-w_k)^{2/N}} \times$$

$$\prod_{p=1}^{k-1} \frac{1}{(w_p-w_k)^{2/N}} \prod_{r=k+1}^{n} \frac{1}{(w_k-w_r)^{2/N}} \langle \psi_1(z_2)\ldots\psi_1(z_n)\psi_1^\dagger(w_1)\ldots\psi_1^\dagger(w_k)\ldots\psi_1^\dagger(w_n) \rangle, \tag{2.10}$$

where $\hat{\psi}_1^\dagger(w_k)$ means that this factor has been removed by taking the operator product expansion with $\psi_1(z_1)$. Although general correlation functions can be obtained in closed
form by straightforward but cumbersome iteration of the formula, it will be sufficient for our purposes to consider only the exact form of the 4-point parafermion function, which turns out to be

\[
\langle \psi_1(z_1)\psi_1^\dagger(z_2)\psi_1(z_3)\psi_1^\dagger(z_4) \rangle = \left( \frac{z_{12}z_{14}z_{34}z_{23}}{z_{13}z_{24}} \right)^{2/N} \left( 1 + \frac{2}{N} \frac{z_{12}z_{34}}{z_{23}z_{24}} \right) + (z_2 \leftrightarrow z_4)
\]

(2.11)

where \(z_{ij} = z_i - z_j\) stands for simplicity.

We conclude this section with a result that has been known for some time [33, 34]. Recall that the parafermion currents become bosonic in the limit \(N \to \infty\), and as a result the various operator product expansions of the parafermion fields as well as their correlation functions become identical to those of two real free bosons (or equivalently a complex free boson) in two dimensions. In particular, one obtains an infinite selection of fields \(\psi_l(z)\) and \(\psi_l^\dagger(z)\) with \(l = 0, 1, 2, \cdots\), having integer conformal dimension \(d_l = l\), which can be represented in terms of a complex scalar field \(\phi(z)\) as

\[
\psi_l(z) =: (i\partial \phi(z))^l : , \quad \psi_l^\dagger(z) =: (-i\bar{\partial} \bar{\phi}(z))^l :
\]

(2.12)

and can be used to obtain a bosonic field realization of all subleading terms in the operator product expansion of the \(Z_\infty\) parafermion currents. In other words, \(\psi_1(z)\) and \(\psi_1^\dagger(z)\) are the two basic bosonic currents, whereas the remaining higher \(Z_\infty\) parafermions belong in their enveloping algebra. Then, the central charge of the Virasoro algebra of the \(SU(2)_N/U(1)\) coset model, which is given by \(c = 2(N - 1)/(N + 2)\), assumes its classical value \(c = 2\) as \(N \to \infty\). On the other hand, the free boson model \(U(1)_{-N}\) has a current algebra generated by the chiral fields \(J_0(z)\) (and \(\bar{J}_0(z)\) for the anti-holomorphic sector) with conformal dimensions \((1, 0)\) and \((0, 1)\) respectively. The operator product expansion of the holomorphic currents is

\[
J_0(z)J_0(w) = -\frac{N}{2(z - w)^2} + O(1)
\]

(2.13)

and also has a well defined large \(N\) limit obtained by rescaling the components of the \(U(1)\) current by \(\sqrt{N}\). The central charge of the Virasoro algebra is \(c = 1\) in this case, and therefore the total central charge of the product model \((SU(2)_N/U(1)) \times U(1)_{-N}\) becomes 3 in the large \(N\) limit, thus describing three free bosons in total. In accordance with this limit the corresponding background metric in (1.1) becomes flat and the dilaton constant.

It is important to realize that there is an alternative large \(N\) limit of the same product model that will be taken in a correlated way between the two factors, which yields an
inequivalent theory having logarithmic structure in its correlation functions. This logarithmic CFT will be discussed later in all detail, but the central charge will remain \( c = 3 \) irrespective of the limiting procedure.

### 3 Classical parafermion currents

In this section we consider the classical analogue of the parafermion currents, together with their commutation relations with respect to Poisson brackets, in order to motivate the appropriate field redefinitions and the new algebraic structures that emerge by taking \( N \to \infty \). In terms of the target space variables they are represented as

\[
\psi_1 = \sqrt{N} (\partial_+ \theta + i \cot \theta \partial_+ \phi) e^{-i(\phi - f \cot^2 \theta \partial_+ \phi)} ,
\]

\[
\psi_1^\dagger = \sqrt{N} (\partial_+ \theta - i \cot \theta \partial_+ \phi) e^{+i(\phi - f \cot^2 \theta \partial_+ \phi)} .
\]

We chose appropriate overall normalization factors so that they are the same for both classical and quantum parafermion currents.

It can be easily shown, using the equations of motion that follow from the two-dimensional \( \sigma \)-model action corresponding to the metric (1.1), that they are chirally conserved, i.e., \( \partial_- \psi_1 = \partial_- \psi_1^\dagger = 0 \). It is also well known that they obey the equal “time” (with respect to \( \sigma_- \), say) Poisson bracket algebra

\[
\{ \psi_1, \psi_1^\dagger \} = -\frac{2}{\pi} \delta'(\sigma_+ - \sigma_+') - \frac{\pi}{2N} \epsilon(\sigma_+ - \sigma_+') \psi_1(\sigma_+)^\dagger \psi_1(\sigma_+') ,
\]

\[
\{ \psi_1^\dagger, \psi_1^\dagger \} = \frac{\pi}{2N} \epsilon(\sigma_+ - \sigma_+') \psi_1(\sigma_+^\dagger) \psi_1(\sigma_+') ,
\]

\[
\{ \psi_1, \psi_1^\dagger \} = -\frac{\pi}{2N} \epsilon(\sigma_+ - \sigma_+') \psi_1(\sigma_+^\dagger) \psi_1(\sigma_+') ,
\]

where one assumes in writing Poisson brackets that the first parafermion current is evaluated at \( \sigma_+ \) and the second one at \( \sigma_+^\dagger \), whereas \( \sigma_- \) is common to both of them. Also we have used the antisymmetric step function defined as \( \epsilon(\sigma_+ - \sigma_+') = 1(-1) \) if \( \sigma_+ > \sigma_+' \) (\( \sigma_+ < \sigma_+' \)) and \( \delta' \) is the derivative of the usual \( \delta \)-function. In addition to (3.2), we have the Poisson bracket corresponding to the time-like boson,

\[
\{ J_0, J_0 \} = \frac{N}{\pi} \delta'(\sigma_+ - \sigma_+') ,
\]

where \( J_0 = -N \partial_+ t \).
Taking the limit \((1.2)\), we find the following interesting behaviour for the parafermion current:

$$
\sqrt{N}\psi_1 = \frac{1}{2}\Psi + \frac{1}{2\epsilon}\Phi + \frac{i}{\sqrt{\epsilon}}P + \mathcal{O}(\sqrt{\epsilon})
$$

(3.4)

whereas for \(\psi_1^\dagger\) we get its complex conjugate. Then, the real fields \(\Psi, \Phi\) and \(P\) can be represented in terms of the variables of the background \((1.3)\) as

\[
\begin{align*}
\Psi &= \alpha(2\partial_+ v + 2\cot u \partial_+ x - A^2 \partial_+ u), \\
\Phi &= 2\alpha \partial_+ u, \\
P &= \alpha(\cot u \partial_+ x - A \partial_+ u),
\end{align*}
\]

(3.5)

where \(A = x - \int \cot^2 u \partial_+ x\). These fields are chirally conserved on-shell by construction, but one may nevertheless check this explicitly. We note for consistency that the same limiting procedure of the extra \(U(1)\) generator \(J_0\) yields an expression for the field \(\Phi\) which is identical to the one appearing in \((3.3)\); thus, our limiting procedure is sensible and we may proceed further with the calculations. The Poisson brackets for \(\Phi, \Psi\) and \(P\) are easily found using \((3.2), (3.3)\) and \((3.4)\). The result reads

\[
\begin{align*}
\{\Phi, \Phi\} &= 0, \\
\{\Phi, \Psi\} &= -\frac{2\alpha}{\pi} \delta'(\sigma_+ - \sigma'_+), \\
\{\Psi, \Psi\} &= -\frac{\pi}{2\alpha} \epsilon(\sigma_+ - \sigma'_+)P(\sigma_+)P(\sigma'_+), \\
\{P, P\} &= -\frac{\alpha}{\pi} \delta'(\sigma_+ - \sigma'_+) - \frac{\pi}{8\alpha} \epsilon(\sigma_+ - \sigma'_+)\Phi(\sigma_+)\Phi(\sigma'_+), \\
\{\Phi, P\} &= 0, \\
\{\Psi, P\} &= \frac{\pi}{4\alpha} \epsilon(\sigma_+ - \sigma'_+)P(\sigma_+)\Phi(\sigma'_+).
\end{align*}
\]

(3.6)

Note that unlike the Poisson bracket algebra of the classical parafermions \((3.2)\), which is extended non-trivial at the quantum level in \((2.4)\) by receiving \(1/N\)-corrections to all orders, the algebra \((3.6)\) is actually exact to all orders in \(\alpha\) in the world-sheet perturbation theory. There are several ways to see this as, for instance, the solution \((1.3)\) does not receive any quantum corrections in string perturbation theory on the world-sheet due to its plane wave nature \([35, 36]\). This is easily argued using the fact that the constant \(\alpha\) appearing in \((1.3)\), which sets the scale in perturbation theory, can be set equal to any value by an appropriate rescaling of the variables \(v\) and \(x\). Hence, it is natural to expect
that the classical symmetries of this model will also be exact symmetries at the quantum level. This observation is also in agreement with the fact that the constant $\alpha$ appearing in the algebra (3.6) can be absorbed by obvious simultaneous rescaling of the currents $\Phi$ and $P$ in an obvious way. Keeping these remarks in mind, we are setting $\alpha = 1$ from now in the rest of the paper.

4 Correlated large $N$ limit and LCFT

We return now to the operator product expansions of the quantum operators of the model $(SU(2)_N/U(1)) \times U(1)_{-N}$ and consider the large $N$ limit that is obtained by forming linear combinations of the parafermion currents of the $SU(2)_N/U(1)$ factor with the $U(1)_{-N}$ currents. We will find this appropriate for taking the plane wave limit of the extended coset model as it has already been indicated by our previous classical analysis in terms of Poisson brackets. In this context, we will arrive at a rather surprising result stating that the correlated large $N$ limit defines a logarithmic conformal field theory for the model $(SU(2)_{\infty}/U(1)) \times U(1)_{-\infty}$ with central charge $c = 3$, as for the ordinary large $N$ limit. The new feature here is the appearance of logarithmic divergences in certain correlation functions that are defined in this limit, and hence one can interpret the plane wave limit of ordinary conformal field theories as providing new examples of logarithmic conformal field theories (LCFT), which have not been realized so far to the best of our knowledge. Although our analysis is presently confined to the coset model $(SU(2)_N/U(1)) \times U(1)_{-N}$, generalizations can also be considered to other conformal field theories as $N \to \infty$.

Following the lines of the classical analysis presented earlier, it is natural to define the three Hermitian chiral operators

$$\Phi(z) = \epsilon \left( \frac{\sqrt{N}}{2} (\psi_1 + \psi_1^\dagger) - J_0 \right), \quad \Psi(z) = \frac{\sqrt{N}}{2} (\psi_1 + \psi_1^\dagger) + J_0,$$

$$P(z) = \sqrt{\epsilon} \frac{\sqrt{N}}{2i} (\psi_1 - \psi_1^\dagger)$$

(4.1)

instead of $\psi_1(z)$, $\psi_1^\dagger(z)$ and $J_0(z)$ whose $U(1)$ central charge is $-N$; the non-vanishing 2-point functions $\langle \psi_1(z)\psi_1^\dagger(w) \rangle$ and $\langle J_0(z)J_0(w) \rangle$ are then normalized accordingly to 1 and $-N/2$ respectively. The parameter $\epsilon$ is defined as before, i.e., $\epsilon = \alpha/N = 1/N$. Note that the first two linear combinations do not have well-defined conformal dimension for finite values of $N$, as the weight is $1 - 1/N$ for each one of the parafermion currents and 1 for
the $U(1)$ current $J_0(z)$; however, the dimensions match in the large $N$ limit that will be taken shortly. Also, in the limit $N \to \infty$, the operator $\Phi(z)$ appears to scale to zero as $1/\sqrt{N}$, the operator $\Psi(z)$ scales to infinity as $\sqrt{N}$, whereas $P(z)$ remains finite. Finally, the stress-energy tensor of the extended model is defined as usual,

$$ T(z) = T_{SU(2)/U(1)} + T_{U(1)} $$

and will be used to compute the operator product expansion with each one of the fields $\Phi(z)$, $\Psi(z)$ and $P(z)$. It is here that we will first encounter the characteristic behaviour of a LCFT upon taking the large $N$ limit.

We first compute the operator product expansion

$$ T(z)\Psi(w) = \frac{1}{2(z-w)^2} \left( \left( 1 - \frac{1}{N} \right) \left( \Psi(w) + \frac{1}{\epsilon} \Phi(w) \right) + \Psi(w) - \frac{1}{\epsilon} \Phi(w) \right) $$

$$ + \frac{\partial \Psi(w)}{z-w} + O(1) , $$

using the definition of the field $\Psi$ and rewriting the right-hand side in terms of $\Phi$ and $\Psi$. Although there is nothing particular happening for finite values of $N$, we observe that in the large $N$ limit ($\epsilon \to 0$) the term which appears to order $1/(z-w)^2$ becomes

$$ \frac{1}{2} \lim_{\epsilon \to 0} \left( \left( 1 - \frac{1}{N} \right) \left( \Psi + \frac{1}{\epsilon} \Phi \right) + \Psi - \frac{1}{\epsilon} \Phi \right) = \Psi - \frac{\Phi}{2} . $$

Thus, we obtain in this limit the result

$$ T(z)\Psi(w) = \frac{\Psi(w) - \Phi(w)/2}{(z-w)^2} + \frac{\partial \Psi(w)}{z-w} + O(1) , $$

which is a characteristic operator product expansion in LCFT. Before making this connection more precise, we also compute the operator product expansion of the stress-energy tensor with the remaining two fields under study. We then obtain

$$ T(z)\Phi(w) = \frac{\epsilon}{2(z-w)^2} \left( \left( 1 - \frac{1}{N} \right) \left( \Psi(w) + \frac{1}{\epsilon} \Phi(w) \right) - \left( \Psi(w) - \frac{1}{\epsilon} \Phi(w) \right) \right) $$

$$ + \frac{\partial F(w)}{z-w} + O(1) , $$

which yields in the large $N$ limit the result

$$ T(z)\Phi(w) = \frac{\Phi(w)}{(z-w)^2} + \frac{\partial \Phi(w)}{z-w} + O(1) . $$
As for the field \( P \), which has definite conformal dimension for all \( N \), we obtain in the large \( N \) limit

\[
T(z)P(w) = \frac{P(w)}{(z-w)^2} + \frac{\partial P(w)}{z-w} + \mathcal{O}(1) ,
\]

(4.8)

stating that it is an ordinary primary field of weight 1.

Next, we compute the 2-point functions among the fields \( \Psi \) and \( \Phi \). Using the 2-point functions of the parafermions and the \( U(1) \) current \( J_0 \), we first obtain

\[
\langle \Psi(z)\Psi(w) \rangle = \frac{N}{2(z-w)^2} \left( (z-w)^{2/N} - 1 \right) ,
\]

(4.9)

which is valid for all values of \( N \). In taking the large \( N \) limit, we may expand

\[
(z-w)^{2/N} = 1 + \frac{2}{N} \ln(z-w) + \mathcal{O} \left( \frac{1}{N^2} \right) ,
\]

(4.10)

thus arriving at the result for the 2-point function

\[
\langle \Psi(z)\Psi(w) \rangle = \frac{\ln(z-w)}{(z-w)^2} ,
\]

(4.11)

which exhibits a logarithmic dependence on \( z-w \) on top of the usual power law behaviour of ordinary two-dimensional conformal field theories. Proceeding along similar lines we compute the 2-point function between the fields \( \Phi \) and \( \Psi \). In this case the logarithmic dependence cancels as \( N \to \infty \) and the final result in the large \( N \) limit reads as

\[
\langle \Phi(z)\Psi(w) \rangle = \frac{1}{(z-w)^2} .
\]

(4.12)

Finally, we obtain that

\[
\langle \Phi(z)\Phi(w) \rangle = 0 ,
\]

(4.13)

which is immediate and obvious in the large \( N \) limit as the field \( \Phi \) scales like \( 1/\sqrt{N} \). As for the 2-point function \( \langle P(z)P(w) \rangle \), the resulting expression is as in ordinary conformal field theory for a primary field of weight 1, i.e.,

\[
\langle P(z)P(w) \rangle = \frac{1}{2(z-w)^2} .
\]

(4.14)

The remaining 2-point functions are zero.

We also write down for completeness the operator product expansions of the various fields obtained from the parafermion and the free boson operator algebras after the limit
is taken. We obtain, in particular,
\[
\Psi(z)\Phi(w) = \frac{1}{(z-w)^2} + \mathcal{O}(1) ,
\]
\[
\Psi(z)\Psi(w) = \ln\left(\frac{z-w}{z-w}\right) + 2\ln(z-w) : P^2(w) : + \frac{1}{2}\ln^2(z-w) : \Phi^2(w) : + \mathcal{O}(1) ,
\]
\[
\Psi(z)P(w) = -\ln(z-w) : (P\Phi)(w) : + \mathcal{O}(1) ,
\]
\[
P(z)P(w) = \frac{1}{2(z-w)^2} + \frac{1}{2}\ln(z-w) : \Phi^2(w) : + \mathcal{O}(1) .
\]

Summarizing the results we have obtained so far, it is rather amusing to note the logarithmic dependence of the correlation functions that arises in this large \(N\) limit, which signals that a LCFT is at work in the plane wave (Penrose) limit of our exact conformal field theory background \((SU(2)_N/U(1)) \times U(1)_{-N}\). As such, it provides us with a worldsheet interpretation of the plane wave limit in supergravity, which can also be generalized to more complicated string backgrounds with an exact conformal field theory description. This observation can also be used to provide us with a new class of LCFT beyond the class of examples that have been studied so far. In this context, the field \(\Phi\) is a primary field of weight 1, whereas the field \(\Psi\) constitutes its logarithmic partner; as for \(P\), which is also a primary field of weight 1, it has no logarithmic partner.

To be more precise, we recall the bare facts of LCFT in two dimensions, which involve a selection of primary fields \(\{\Phi_h\}\) of conformal weight \(h\) and their logarithmic partners \(\{\Psi_h\}\). It is well known that in a LCFT any two logarithmic partners transform under a chiral conformal transformation \(z \rightarrow \tilde{z}(z)\) as doublet, which can be formally written as
\[
\left( \begin{array}{c} \Phi(z) \\ \Psi(z) \end{array} \right) = \left( \begin{array}{cc} \partial \tilde{z} \\ \partial z \end{array} \right) \left( \begin{array}{cc} h & 0 \\ 1 & h \end{array} \right) \left( \begin{array}{c} \Phi(\tilde{z}) \\ \Psi(\tilde{z}) \end{array} \right)
\]
and similarly for the anti-holomorphic sector. The infinitesimal form of these transformations give rise to the particular operator product expansions of the stress-energy tensor with the fields \(\Phi\) and \(\Psi\) that have been encountered above, provided that \(\Phi\) is rescaled by \(-1/2\); in our case, we also have \(h = 1\). Put it differently, the states \(|\Phi\rangle = \Phi(z=0)|0\rangle\) and \(|\Psi\rangle = \Psi(z=0)|0\rangle\), which correspond to the two fields in question, satisfy the general relations
\[
L_0|\Phi\rangle = h|\Phi\rangle ,
\]
\[
L_0|\Psi\rangle = h|\Psi\rangle + |\Phi\rangle
\]
(4.17)
and therefore the zero mode of the Virasoro algebra $L_0$ can no longer be diagonalized in LCFT, as it is in ordinary conformal field theories. Moreover, the 2-point function of any two chiral primary fields vanishes,

$$\langle \Phi_h(z)\Phi_{h'}(w) \rangle = 0 \tag{4.18}$$

while for correlators involving their logarithmic partners we have the following general structure:

$$\langle \Phi_h(z)\Psi_{h'}(w) \rangle = A \frac{A}{(z-w)^{h+h'}}$$

$$\langle \Psi_h(z)\Psi_{h'}(w) \rangle = \delta_{h,h'} \left( \frac{B - 2A \ln(z-w)}{z-w} \right)^{h+h'} \tag{4.19}$$

with $A$ and $B$ free constants; note that the constant $B$ can be consistently set to zero by a field redefinition. This is precisely the behaviour that we have also encountered in the large $N$ limit of the 2-point functions of the fields $\Psi$ and $\Phi$ with $h = h' = 1$, provided that the free constants are set equal to $A = -1/2$ and $B = 0$, while $\Phi$ is also rescaled by $-1/2$.

Having established the logarithmic structure of the resulting large $N$ theory by computing the relevant 2-point functions, we may now proceed to the calculation of higher point correlation functions. Recall that the form of the 4-point functions in LCFT can be easily obtained on general grounds; we summarize below the general expressions for the non-vanishing 4-point functions that involve at least one primary field $\Phi$ and its logarithmic partner:

$$\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Psi(z_4) \rangle = \prod_{i<j} z_{ij}^{i_{ij}} F^{(0)}(x),$$

$$\langle \Phi(z_1)\Phi(z_2)\Psi(z_3)\Psi(z_4) \rangle = \prod_{i<j} z_{ij}^{i_{ij}} \left( F^{(1)}_{34}(x) - 2F^{(0)}(x) \ln(z_{34}) \right),$$

$$\langle \Phi(z_1)\Psi(z_2)\Psi(z_3)\Psi(z_4) \rangle = \prod_{i<j} z_{ij}^{i_{ij}} \left[ F^{(2)}_{234}(x) - \sum_{2 \leq i<j \leq 4} \tilde{F}^{(1)}_{ij}(x) \ln(z_{ij}) \right] + 2F^{(0)}(x) (\ln(z_{23}) \ln(z_{24}) + \ln(z_{23}) \ln(z_{34}) + \ln(z_{24}) \ln(z_{34})) - F^{(0)}(x) \left( \ln^2(z_{23}) + \ln^2(z_{24}) + \ln^2(z_{34}) \right) \tag{4.20}$$

1 We immediately conclude that all odd-point functions involving the fields $\Phi, \Psi$ and $P$ of our model are zero. This is due to the vanishing of correlators involving an odd number of $\psi_1$ and $\psi_1^\dagger$ due to charge conservation and similarly for correlators involving an odd number of $J_0$'s. Hence, we will not mention the general form of the 3-point functions. Also, all $2n$-point functions involving more that $n$ insertions of the field $\Phi$ are zero in our case.
and omit the general expression for the correlation function \( \langle \Psi(z_1)\Psi(z_2)\Psi(z_3)\Psi(z_4) \rangle \) that is rather lengthy. Note that we used the abbreviations

\[
z_{ij} = z_i - z_j, \quad \mu_{ij} = \frac{1}{3} \sum_{k=1}^{4} h_k - h_i - h_j, \quad (4.21)
\]
in order to simplify the expressions, where \( h_i \) denotes the conformal dimension of the field inserted at any point \( z_i \). The coefficients \( F^{(0)}(x), F^{(1)}_{ij}(x) \) and \( F^{(2)}_{ijk}(x) \) are functions of the anharmonic ratio of the four points, \( x = (z_{12}z_{34})/(z_{14}z_{23}) \), as it is required by conformal invariance, but whose explicit form depend on the particular theory. According to their definition, the functions \( F^{(1)}_{ij}(x) \) are all related to a single function, say \( F^{(1)}(x) \), by the transformation

\[
F^{(1)}_{34}(x) = F^{(1)}(x), \quad F^{(1)}_{23}(x) = F^{(1)} \left( \frac{1}{1 - x} \right), \quad F^{(1)}_{24}(x) = F^{(1)}(1 - x).
\]

Finally, we used the abbreviation

\[
\tilde{F}^{(1)}_{ij}(x) = F^{(1)}_{ik}(x) + F^{(1)}_{jk}(x) - F^{(1)}_{ij}(x), \quad (4.23)
\]
with \( k \) corresponding to the index of the remaining third logarithmic field in the correlation functions above.

We have calculated the corresponding 4-point functions of the operators \( \Psi \) and \( \Phi \) in the \( (SU(2)_N/U(1)) \times U(1)_{-N} \) model, using the 4-point function of the parafermion operators \( \langle \psi_1(z_1)\psi^\dagger_1(z_2)\psi_1(z_3)\psi^\dagger_1(z_4) \rangle \) that has been given earlier. Expanding the factor in \( (2.11) \),

\[
\left( \frac{z_{12}z_{14}z_{34}z_{23}}{z_{13}z_{24}} \right)^{2/N} = 1 + \frac{2}{N} \ln \left( \frac{z_{12}z_{14}z_{34}z_{23}}{z_{13}z_{24}} \right) + \mathcal{O}\left( \frac{1}{N^2} \right) \quad (4.24)
\]
as a power series in \( 1/N \), we may subsequently take the large \( N \) limit of the correlation functions, which is appropriate for describing the plane wave limit of the supergravity background corresponding to our extended coset model. As for the 2-point functions, we also find here a remnant logarithmic dependence of the 4-point functions as \( N \to \infty \), which precisely fits the general formulae of LCFT according to our expectations. We spare the details of the calculation and only give the result

\[
\langle \Phi(z_1)\Phi(z_2)\Phi(z_3)\Psi(z_4) \rangle = 0,
\]
\[
\langle \Phi(z_1)\Phi(z_2)\Psi(z_3)\Psi(z_4) \rangle = \frac{1}{z_{12}^2 z_{23}^2} + \frac{1}{z_{14}^2 z_{23}^2}, \quad (4.25)
\]
\[
\langle \Phi(z_1)\Psi(z_2)\Psi(z_3)\Psi(z_4) \rangle = \frac{\ln z_{23}}{z_{12}^2 z_{34}^2} + \frac{\ln z_{24}}{z_{13}^2 z_{34}^2} + \frac{\ln z_{23}}{z_{14}^2 z_{23}^2}.
\]
These correlators can be cast into the general form (4.20) using that \( \mu_{ij} = -2/3, \forall i, j = 1, 2, 3, 4 \). We also have for the three key functions that enter into the correlators

\[
F^{(0)} = F^{(2)}_{234} = 0 ,
\]

\[
F^{(1)} = (1 + x^2) \left( \frac{x}{x - 1} \right)^{2/3} .
\]  

(4.26)

Of course, in order to match the overall normalization, our field \( \Phi \) has to be rescaled by \(-1/2\), as it was mentioned before. From these expressions we also compute the abbreviated functions in (4.23),

\[
\tilde{F}^{(1)}_{34} = 2 \prod_{i<j=1}^{4} z_{ij}^{2/3} \frac{1}{z_{12}^{2} z_{34}^{2}},
\]

\[
\tilde{F}^{(1)}_{23} = 2 \prod_{i<j=1}^{4} z_{ij}^{2/3} \frac{1}{z_{12}^{2} z_{23}^{2}},
\]  

(4.27)

\[
\tilde{F}^{(1)}_{24} = 2 \prod_{i<j=1}^{4} z_{ij}^{2/3} \frac{1}{z_{13}^{2} z_{24}^{2}}.
\]

Other correlation functions can also be computed in the large \( N \) limit, if it is necessary, using the same procedure. In particular, we can also show that the 4-point function having only logarithmic fields assumes the form given in the literature [30].

## 5 Free field representation and further results

We first consider the free field realization of the parafermion operators, which follows from the standard free field representation of the \( SU(2)_N \) current algebra. Introducing two real free bosons,

\[
\langle \phi_i(z) \phi_j(w) \rangle = -\delta_{ij} \ln(z - w) ; \quad i, j = 1, 2 ,
\]  

(5.1)

one may represent the elementary parafermion currents as

\[
\psi_1 = \frac{1}{\sqrt{2}} \left( -\sqrt{1 + 2/N} \partial \phi_1 + i \partial \phi_2 \right) e^{+\sqrt{2/N} \phi_2} ,
\]

\[
\psi_1^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{1 + 2/N} \partial \phi_1 + i \partial \phi_2 \right) e^{-\sqrt{2/N} \phi_2} .
\]  

(5.2)

We will denote the time-like free boson for the \( U(1)_{-N} \) factor by \( \phi_0 \). It obeys

\[
\langle \phi_0(z) \phi_0(w) \rangle = \ln(z - w) .
\]  

(5.3)
The stress-energy tensor of the entire theory is
\[ T(z) = \frac{1}{2} (\partial \phi_0)^2 - \frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 + \frac{i}{\sqrt{2(N + 2)}} \partial^2 \phi_1. \] (5.4)

Let us now consider the scalar field redefinition
\[ \phi_+ = \sqrt{\frac{1}{2N}} (\phi_0 + \phi_1), \quad \phi_- = \sqrt{\frac{N}{2}} (\phi_0 - \phi_1), \quad \phi_2 = \phi, \quad (5.5) \]
followed by the limit \( N \to \infty \). Then, the new set of scalars obey
\[ \langle \phi_+(z) \phi_-(w) \rangle = -\langle \phi(z) \phi(w) \rangle = \ln(z - w) \quad (5.6) \]
and have zero correlators otherwise. Following a procedure similar to section 3, we find that
\[ \Psi = -\frac{i}{2} (\phi^2 + 1) \partial \phi_+ - \phi \partial \phi + i \partial \phi_-, \quad \Phi = -i \partial \phi_+, \]
\[ P = \frac{1}{\sqrt{2}} (i \partial \phi - \phi \partial \phi_+) \quad (5.7) \]
Also, the stress-energy tensor of the theory becomes
\[ T(z) = -\frac{1}{2} (\partial \phi)^2 + \partial \phi_+ \partial \phi_- - \frac{i}{2} \partial^2 \phi_+ \quad (5.8) \]
and we thus complete the free field realization of the basic operators in our logarithmic conformal field theory.

We conclude this section by presenting some ideas towards the construction of more logarithmic partner fields in the model. We may take advantage of the extended conformal symmetries of the time-like free boson model \( U(1)_{-N} \) in order to define more fields of higher dimension, which become logarithmic partners in the large \( N \) limit. For this purpose, we use the higher parafermion currents \( \psi_l \) and \( \psi_l^\dagger \) to define the fields
\[ \Phi_l(z) = \epsilon \left( \frac{\sqrt{N}}{2} (\psi_l + \psi_l^\dagger) - \sqrt{NW_l} \right), \quad \Psi_l(z) = \frac{\sqrt{N}}{2} (\psi_l + \psi_l^\dagger) + \sqrt{NW_l}, \]
\[ P_l(z) = \sqrt{\epsilon} \frac{\sqrt{N}}{2i} (\psi_l - \psi_l^\dagger) \quad (5.9) \]
where \( W_l \) are the chiral \( W \)-currents of the extra time-like boson. They naturally generalize the fields that were defined earlier for \( l = 1 \) to operators of integer conformal dimension \( l \geq 2 \) using the higher parafermions, as one further takes the limit \( N \to \infty \).
Recall at this point that the conformal field theory of a free boson admits a $W_{1+\infty}$ symmetry\footnote{Actually, there is a realization of the $W_{1+\infty}$ algebra in terms of a complex fermion, which can be bosonized to yield a realization in terms of a real free scalar field.} with central charge $c = 1$; it is generated by the $U(1)$ current $J_0/\sqrt{N} = V_1$, the stress energy tensor $T_{U(1)} = V_2$, and an infinite collection of higher spin fields $V_l$ of integer dimension $l = 3, 4, 5, \cdots$. This algebra can be twisted by the $U(1)$ current of the model to yield a $W_\infty$ algebra generated by higher spin operators $W_l$ for all $l \geq 2$. $W_2$ is the improved stress-energy tensor, whereas the remaining $W_l$ generators are also obtained by twisting the corresponding $W_{1+\infty}$ generators $V_l$. The details of the algebra are not important for our present purposes apart from the following points: first, the twisting allows for the $U(1)$ current algebra to decouple from the remaining generators $W_l$; it also amounts to changing the central charge of the Virasoro algebra from $c = 1$ to $c = -2$; finally, there is a quasi-primary basis for the $W_\infty$ generators in which the algebra linearizes and the 2-point function of all generating fields becomes diagonal, i.e.,

$$\langle W_l(z)W_{l'}(w) \rangle = \frac{c}{4(z-w)^{l+l'}} \delta_{ll'}; \quad \text{with } c = -2 ,$$

(5.10)

where any spin dependent normalization factors have been absorbed into the definition of the $W_l$ currents. Further technical details on the subject of $W_\infty$ algebras can be found in \cite{33, 34}, for instance, and references therein.

It is then straightforward to verify, in analogy with the calculations that were performed earlier, that the fields $\Phi_l$ and $\Psi_l$ have 2-point correlation functions of the form

$$\langle \Psi_l(z)\Psi_{l'}(w) \rangle = \frac{l^2 \ln(z-w)}{(z-w)^{l+l'}} \delta_{ll'} , \quad \langle \Phi_l(z)\Psi_{l'}(w) \rangle = \frac{1}{(z-w)^{l+l'}} \delta_{ll'} ,$$

$$\langle \Phi_l(z)\Phi_{l'}(w) \rangle = 0$$

(5.11)

as $N \to \infty$, for all values $l = 1, 2, 3, \cdots$. Therefore, they qualify to be logarithmic partners in the resulting LCFT. However, it is fair to say that the fields $\Psi_l$ defined in this fashion do not always have well-defined operator product expansion with the ordinary stress-energy tensor of the theory, as central terms can appear with a $\sqrt{N}$ dependence.

6 Conclusions

Plane wave solutions have been studied extensively in the theory of general relativity, but also in string theory, as they exhibit many interesting physical properties. Their simplicity
has also proven very useful to address problems of gravitational physics within strings. In
fact, there have been some exactly solvable models based on non-semi-simple groups that
exhibit plane waves as supergravity backgrounds.

The interest in plane wave solutions arising in string and M-theory has been recently
revived following the realization that the original AdS/CFT correspondence can be ex-
tended in a non-trivial way to include the effect of highly massive string states using such
backgrounds [37]. The prototype example in these cases is the maximally supersymmetric
plane wave solution of type-IIB supergravity [38, 39]. This solution, in turn, can be ob-
tained as a Penrose limit of the maximally supersymmetric vacuum solution, $AdS_5 \times S^5$
[40]. Note that, although the Penrose limit was originally considered in general relativ-
ity, it can be straightforwardly extended to supergravities that correspond to low energy
string theories [41]. Then, in accordance with experience and general expectations, the
superalgebra for the $AdS_5 \times S^5$ background contracts to the corresponding plane wave
superalgebra, as it has been explicitly demonstrated in [42]. Here, we have derived an
unexpected result relating plane wave solutions to an exact logarithmic conformal field
theory. Although there is no direct evidence that the logarithmic structure we have en-
countered in the conformal field theory living on the world-sheet has any relation to the
logarithmic behaviour of correlators in gauge theories (see, for instance, [43]), we think
that such occurrence might not be accidental and worth exploring in the future.

The primary purpose of the present work was to understand the world-sheet interpre-
tation of the plane wave limit in supergravity, as we believe it will sharpen our present
understanding of all current issues that are involved. We have been able to report progress
in this direction by considering a simple, yet illustrative, example of an exact conformal
field theory whose Saletan limit (in the language of the world-sheet current algebra) gives
rise to plane waves in target space. From the technical point of view, we made extensive
use of the parafermion algebra, which is present in this representative model, to define
operators that are correlated to $U(1)$ currents and yield logarithmic correlation functions
in a suitable large $N$ limit. Actually, our findings could have been reported some years ago
in the context of two-dimensional conformal field theories, but there was no good reason to
expect at that time that there was a correlated large $N$ limit, which made sense and could
also provide us with a totally different algebraic structure from the ordinary situation.
The emergence of a logarithmic conformal field theory is quite interesting, as it can be
easily generalized to many more backgrounds having exact CFT description. Moreover,
our solution may be used to provide new examples of logarithmic conformal field theories, which have been studied so far only occasionally and in a different context. Thus, it may put both subjects in a different perspective and catalyze further the exchange of ideas among them.

There are certainly a number of questions that require further study. The representation theory of the chiral algebras that arise in these logarithmic conformal field theories is one of them and it has to be developed in view of their applications in the gravitational plane wave backgrounds. It might also be possible to obtain new results for the generalized AdS/CFT correspondence, as well as explain several of the new findings, by relying on the logarithmic structure that seems to exist on the string world-sheet. Finally, more examples of logarithmic fields, as well as a more systematic study of other coset models with plane wave limit should be considered in detail. We hope to report on all these issues in future publications.

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