AN INTRODUCTION TO POD-GREEDY-GALERKIN REDUCED BASIS METHOD

Pierfrancesco Siena  
MathLab, Mathematics Area,  
SISSA International School for Advanced Studies,  
Via Bonomea, 265, 34136, Trieste, Italy,  
psiena@sissa.it

Michele Girfoglio  
MathLab, Mathematics Area,  
SISSA International School for Advanced Studies,  
Via Bonomea, 265, 34136, Trieste, Italy,  
mgifogl@sissa.it

Gianluigi Rozza  
MathLab, Mathematics Area,  
SISSA International School for Advanced Studies,  
Via Bonomea, 265, 34136, Trieste, Italy,  
grozza@sissa.it

ABBREVIATIONS

| Abbreviation | Definition                  |
|--------------|-----------------------------|
| FOM          | Full order model            |
| PDE          | Partial differential equation|
| POD          | Proper orthogonal decomposition|
| RB           | Reduced basis               |
| ROM          | Reduced order method        |

1 INTRODUCTION AND MOTIVATION

PDEs can be used to model many problems in several fields of application including, e.g., fluid mechanics, heat and mass transfer, and electromagnetism. Accurate discretization methods (e.g., finite element or finite volume methods, the so-called FOM) are widely used to numerically solve these problems. However, when many physical and/or geometrical parameters are involved, the computational cost required by FOMs becomes prohibitively expensive and this is not acceptable for real-time computations that are becoming more and more popular for rapid prototyping. Therefore, there is the need to introduce ROMs (also referred to as RB methods) able to provide, as the input parameters change, fast and reliable solutions at a reduced computational cost.

The basic idea of ROM is related to the fact that often the parametric dependence of the problem at hand has an intrinsic dimension much lower than the number of degrees of freedom associated to the FOM. So the core of a ROM framework consists of computing a basis to be used to reconstruct adequately the solution of the problem. Different techniques have been explored during the last decades to generate the reduced space: see, e.g., Porsching, 1985; Ito and Ravindran, 1998b; Lassila et al., 2013; Bui-Thanh et al., 2003; Christensen et al., 1999; Gunzburger et al., 2007; Huynh et al., 2012; Rozza et al., 2008. Of course the POD and the greedy algorithm are the most used. Typically, in a one-dimensional parameter domain, a POD procedure is preferred, whilst for multi-dimensional spaces, strategies of greedy nature are favorite. This is due to the fact that the singular value decomposition of big snapshots matrices could lead to an high computational cost. Further details about advantages and disadvantages of greedy and POD strategies can be found in Huynh et al. (2012); Rozza et al. (2008).

Preliminary studies showed a limited computational improvement provided by ROM, due to the lack of a full decoupling between ROM and FOM (Noor, 1981; Porsching and Lee, 1987; Porsching, 1985). On the other hand, more recent works achieved a complete decoupling framework by means of an offline-online paradigm. The offline stage is related to the collection of a database of several high-fidelity solutions by solving the FOM for different values...
of physical and/or geometrical parameters. Then all the solutions are combined and compressed to extract a set of basis functions (computed by using the POD, the greedy algorithm or other techniques) that approximates the low-dimensional manifold on which the solution lies. On the other hand, in the online stage the information obtained in the offline stage is used to efficiently compute the solutions for new parameters instances. An high level of accuracy for the reduced system is then reached with the employment of a posterior error estimation technique (Ito and Ravindran, 1998b; Gunzburger, 2012; Ito and Ravindran, 1998a, 2001; Peterson, 1989; Girfoglio et al., 2021a,b,c; Stabile et al., 2017; Stabile and Rozza, 2018; Papapicco et al., 2022; Hijazi et al., 2020). Thanks to the significant research work carried out in the last years, ROMs are able to provide a very general framework that has successfully been applied to a wide range of problems from different physical contexts, e.g. Navier-Stokes equations for fluid dynamics applications within both biomedical and industrial framework including multiphysics scenarios such as turbulence, multiphase flows and fluid-structure interaction (Ito and Ravindran, 1998b; Gunzburger, 2012; Ito and Ravindran, 1998a, 2001; Peterson, 1989; Girfoglio et al., 2021a,b,c; Prud’Homme et al., 2002; Huyhn et al., 2007). In addition, the recent introduction of the empirical interpolation method allowed to reach an efficient offline-online decomposition also for complex problems involving non linear models (Barrault et al., 2004; Grepl et al., 2007).

This chapter is structured as follows. A brief description about affine linear elliptic coercive problems is reported in Section 2.1 Parametric weak formulation. Section 2.2 Parametric coercive coercive problems provides some notes about the posterior error bounds for the ROM offline decomposition also for complex physical phenomena, for instance the Burgers equation (Veroy et al., 2003b,c; Deparis and Rozza, 2009). Moreover, although most ROM works have been developed in a finite element environment, we highlight that also other discretization methods have been investigated, e.g. finite volume (Girfoglio et al., 2021a,b,c; Stabile et al., 2017; Stabile and Rozza, 2018; Papapicco et al., 2022; Ballarin and Rozza, 2016; Nonino et al., 2019; Rozza et al., 2018; Cuong et al., 2005; Ballarin et al., 2017, 2016; Siena et al., 2022; Zambelli et al., 2020; Hijazi et al., 2020; Løvgren et al., 2006a,b,c), heat transfer and thermo-mechanical problems (Shah et al., 2021; Guérin et al., 2021; Hernandez-Becerro et al., 2021; Benner et al., 2019; Grepl, 2005; Grepl and Patera, 2005; Rozza et al., 2009), Maxwell equations for the electromagnetism (Chen et al., 2009, 2010, 2012; Jabbar and Azeman, 2004) and some toy models for the basic representation of complex physical phenomena, for instance the Burgers equation (Veroy et al., 2003b,c; Deparis and Rozza, 2009). This chapter is structured as follows. A brief description about affine linear elliptic coercive problems is reported in Section 2. The ROM methodology is explained in Section 3. In particular the POD and the greedy algorithms to build the reduced space are introduced. Section 4 provides some notes about the posterior error bounds for the ROM approach. Finally, conclusions are provided in Section 5.

## 2 PARAMETRIZED DIFFERENTIAL EQUATIONS

Let us consider the domain $\Omega \subset \mathbb{R}^d$ where $d = 1, 2$ or 3, and $\partial \Omega$ is its boundary. Both vector fields $(d_v = d)$ and scalar fields $(d_v = 1)$ may be considered: $w : \Omega \mapsto \mathbb{R}^{d_v}$. The portions of $\partial \Omega$ where Dirichlet boundary conditions are imposed are identified by $\Gamma^D_{\Omega}$, with $1 \leq i \leq d_v$.

Let $V_i(\Omega)$ be the scalar space:

$$V_i(\Omega) = \{ v \in H^1(\Omega) : v|_{\Gamma^D_{\Omega}} = 0 \}, \quad 1 \leq i \leq d_v. \quad (1)$$

We have $H^1_0(\Omega) \subset V_i(\Omega) \subset H^1(\Omega)$ and for $\Gamma^D_\Omega = \partial \Omega$, $V_i(\Omega) = H^1_0(\Omega)$. From now on, we refer to $V_i(\Omega)$ as $V_i$, for an easier exposition.

Let us define the space $V = V_1 \times \cdots \times V_{d_v}$, whose general element is given by $w = (w_1, \ldots, w_{d_v})$. The space $V$ is equipped with an inner product denoted as $(w, v)_V, \forall w, v \in V$ and the induced norm is $\|w\|_V = (w, w)^{1/2}_V, \forall w \in V$. We observe that $V$ is an Hilber space.

Let us introduce a closed parameter domain $P \subset \mathbb{R}^p$. An element of $P$ is given by $\mu = (\mu_1, \ldots, \mu_p)$. So we define the parametric field variable for a parameter value, $u(\mu) = (u_1(\mu), \ldots, u_{d_v}(\mu)) : P \mapsto V$.

### 2.1 Parametric weak formulation

The abstract formulation of a stationary problem reads:
Given $\mu \in \mathcal{P} \subset \mathbb{R}^p$, find $u(\mu) \in \mathcal{V}$ such that
\begin{equation}
 a(u(\mu), v; \mu) = f(v; \mu) \quad \forall v \in \mathcal{V}
\end{equation}
and evaluate
\begin{equation}
 s(\mu) = l(u(\mu); \mu).
\end{equation}

The form $a : \mathcal{V} \times \mathcal{V} \times \mathcal{P} \mapsto \mathbb{R}$ is bilinear with respect to $\mathcal{V} \times \mathcal{V}$, $f : \mathcal{V} \times \mathcal{P} \mapsto \mathbb{R}$ and $l : \mathcal{V} \times \mathcal{P} \mapsto \mathbb{R}$ are linear with respect to $\mathcal{V}$ and $s : \mathcal{P} \mapsto \mathbb{R}$ is an output of interest of the model.

Hereinafter we assume that the problems of interest are compliant, i.e.:
\begin{itemize}
  \item $f(\cdot; \mu) = l(\cdot; \mu) \quad \forall \mu \in \mathcal{P},$
  \item $a(\cdot, \cdot; \mu)$ is symmetric $\forall \mu \in \mathcal{P}$.
\end{itemize}

The compliant hypothesis is verified for a wide range of case studies by allowing to significantly simplify the mathematical framework.

### 2.2 Well-posedness of the problem

The well-posedness of the abstract problem formulation \[\text{Eq. 2-3}\] could be established by using the Lax-Milgram theorem (Quarteroni and Valli, 2008) based on the following two assumptions:

1. The bilinear form $a(\cdot, \cdot; \mu)$ is coercive and continuous on $\mathcal{V} \times \mathcal{V}$.
   The coercivity requires that $\forall \mu \in \mathcal{P}$, $\exists \alpha(\mu) : 0 < \alpha \leq \alpha(\mu)$ such that:
   \begin{equation}
   a(v, v; \mu) \geq \alpha(\mu)\|v\|^2_{\mathcal{V}}, \quad \forall v \in \mathcal{V}.
   \end{equation}
   The coercivity constant $\alpha(\mu)$ is defined as:
   \begin{equation}
   \alpha(\mu) = \inf_{v \in \mathcal{V}} \frac{a(v, v; \mu)}{\|v\|^2_{\mathcal{V}}}, \quad \forall \mu \in \mathcal{P}.
   \end{equation}
   On the other hand, the continuity requires that $\forall \mu \in \mathcal{P}$, $\exists \gamma(\mu) : \gamma(\mu) \leq \gamma < \infty$ such that:
   \begin{equation}
   a(w, v; \mu) \leq \gamma(\mu)\|w\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \quad \forall w, v \in \mathcal{V},
   \end{equation}
   where the continuity constant $\gamma(\mu)$ is defined as:
   \begin{equation}
   \gamma(\mu) = \sup_{v \in \mathcal{V}} \sup_{w \in \mathcal{V}} \frac{a(w, v; \mu)}{\|w\|_{\mathcal{V}}\|v\|_{\mathcal{V}}}, \quad \forall \mu \in \mathcal{P}.
   \end{equation}

2. The linear form $f(\cdot; \mu)$ is continuous on $\mathcal{V}$, i.e. $\exists \delta(\mu) : \delta(\mu) \leq \delta < \infty$ such that:
   \begin{equation}
   f(v; \mu) \leq \delta(\mu)\|v\|_{\mathcal{V}}, \quad \forall v \in \mathcal{V},
   \end{equation}

Note that the norm of $\mathcal{V}$ can be taken equal, or equivalent to:
\begin{equation}
\|v\|_{\mathcal{V}} = a(w, w; \mu)^{1/2}, \quad \forall w \in \mathcal{V},
\end{equation}
which is the norm induced by $a(\cdot, \cdot; \mu)$ for a fixed parameter $\mu \in \mathcal{P}$.

### 2.3 Discretization of the weak formulation

Let us consider a discrete and finite dimensional subset $\mathcal{V}_\delta \subset \mathcal{V}$ with $\text{dim}(\mathcal{V}_\delta) = N_\delta$. As an example, it can be built as a standard finite element method with a triangulation and defining suitable linear basis functions on each element (Bathel, 2007).

The discretized form of Eqs. [2-3] is:

Given $\mu \in \mathcal{P}$, find $u_\delta(\mu) \in \mathcal{V}_\delta$ such that
\begin{equation}
 a(u_\delta(\mu), v_\delta; \mu) = f(v_\delta; \mu) \quad \forall v_\delta \in \mathcal{V}_\delta
\end{equation}
and evaluate
\begin{equation}
 s_\delta(\mu) = l(u_\delta(\mu); \mu).
\end{equation}
where

\[ \{ \} \]

As disclosed in Sec. 1, ROM is developed to deal with repeated model evaluation over a wide range of parameters despite the fact that the solution could be achieved with as high accuracy as desired.

Thanks to the continuity and the coercivity of \( a(\cdot, \cdot; \mu) \), the Galerkin orthogonality holds:

\[
   a(u(\mu) - u_\delta(\mu), v_\delta; \mu) = 0, \quad \forall v_\delta \in V_\delta. \tag{12}
\]

In addition, based on the triangle inequality, we obtain:

\[
   \|u(\mu) - u_\delta(\mu)\|_V \leq \|u(\mu) - v_\delta\|_V + \|v_\delta - u_\delta(\mu)\|_V, \quad \forall v_\delta \in V_\delta. \tag{13}
\]

Moreover, it holds that

\[
   \alpha(\mu)\|v_\delta - u_\delta(\mu)\|_V^2 \leq a(v_\delta - u_\delta(\mu), v_\delta - u_\delta(\mu); \mu) = a(v_\delta - u(\mu), v_\delta - u_\delta(\mu); \mu) \leq \gamma(\mu)\|v_\delta - u(\mu)\|_V\|v_\delta - u_\delta(\mu)\|_V, \quad \forall v_\delta \in V_\delta, \tag{14}
\]

and finally by applying the coercivity and continuity assumptions, and the Galerkin orthogonality, the Cea’s lemma \cite{Monk2003} is recovered:

\[
   \|u(\mu) - u_\delta(\mu)\|_V \leq \left( 1 + \frac{\gamma(\mu)}{\alpha(\mu)} \right) \inf_{v_\delta \in V_\delta} \|u(\mu) - v_\delta\|_V, \quad \forall v_\delta \in V_\delta. \tag{15}
\]

Thus the approximation error \( \|u(\mu) - u_\delta(\mu)\|_V \) is closely related to the best approximation error of \( u(\mu) \) in the approximation space \( V_\delta \) through the constants \( \alpha(\mu) \) and \( \gamma(\mu) \).

Concerning the implementation of the truth solver, the stiffness matrix \( A_\delta^{\mu} \) and the vector \( f_\delta^{\mu} \) can be assembled as:

\[
   (A_\delta^{\mu})_{i,j} = a(\phi_j, \phi_i; \mu) \quad \text{and} \quad (f_\delta^{\mu})_i = f(\phi_i; \mu), \quad 1 \leq i, j \leq N_\delta, \tag{16}
\]

where \( \{ \phi_i \}_{i=1}^{N_\delta} \) is a basis of \( V_\delta \). Therefore, the FOM problem in matrix form is:

\[
   \text{Given } \mu \in P, \text{ find } u_\delta^{\mu} \in R^{N_\delta} \text{ such that: } \quad \quad A_\delta^{\mu}u_\delta^{\mu} = f_\delta^{\mu}. \tag{17}
\]

Then evaluate

\[
   s_\delta(\mu) = (u_\delta^{\mu})^T f_\delta^{\mu}. \tag{18}
\]

The field approximation \( u_\delta(\mu) \) is obtained by

\[
   u_\delta(\mu) = \sum_{i=1}^{N_\delta} (u_\delta^{\mu})_i \phi_i \text{ where } (u_\delta^{\mu})_i \text{ denotes the } i\text{-th coefficient of the vector } u_\delta^{\mu}. \]

3 REDUCED ORDER MODEL

As disclosed in Sec. 1, ROM is developed to deal with repeated model evaluation over a wide range of parameters values by cutting down the large computational cost due to solving the FOM many times. In particular, the aim is to extract from a collection of solutions of the parameterized problem related to determined values of the parameters (named solution manifold) a small number of basis functions to be used in order to obtain an accurate reconstruction of the solution for a new instance of the parameters at a reduced computational cost. Such a goal is conducd by introducing an offline-online paradigm. The offline stage requires the resolution of \( N \) truth problems, therefore its computational cost is elevated due to the high value of \( N_\delta \). Then a reduced basis of size \( N \) is identified. On the other hand, during the online stage, a Galerkin projection onto the space spanned by the reduced basis is carried out, enabling to investigate the parameter space at a considerably reduced cost.

3.1 The reduced approximation

Let us define the solution manifold of Eq. 2 the set of all the exact solutions \( u(\mu) \in V \) varying the parameter \( \mu \):

\[ \mathcal{M} = \{ u(\mu) : \mu \in P \} \subset V. \tag{19} \]

At a discrete level, we can define similarly the solution manifold of Eq. 10

\[ \mathcal{M}_\delta = \{ u_\delta(\mu) : \mu \in P \} \subset V_\delta. \tag{20} \]

Of course, the cost to find the truth solutions can be very large because it depends on \( N_\delta \). ROM aims to find a small number of basis functions whose linear combination represents accurately the numerical solution \( u_\delta(\mu) \). Let \( \{ \xi_i \}_{i=1}^{N} \)
be the $N$-dimensional set of the reduced basis (the main techniques to compute it are explained in Section 3.2). Then the space generated is:

$$V_{rb} = \text{span}\{\xi_1, \ldots, \xi_N\} \subset V_\delta,$$

(21)

where it is assumed $N \ll N_\delta$. Therefore, the reduced form of the Eqs. 10-11 is:

$$\|u(\mu) - u_{rb}(\mu)\|_V \leq \|u(\mu) - u_\delta(\mu)\|_V + \|u_\delta(\mu) - u_{rb}(\mu)\|_V,$$

(24)

under the assumption that there exists a numerical method which allows to have $\|u(\mu) - u_\delta(\mu)\|_V$ small enough (see Section 2.3).

Let us introduce the Kolmogorov $N$-width which measures the distance between the truth solution and the reduced solution:

$$d_N(M_\delta) = \inf_{V_{rb}} E(M_\delta, V_{rb}) = \inf_{V_{rb}} \sup_{u_\delta \in M_\delta} \inf_{v_{rb} \in V_{rb}} \|u_\delta - v_{rb}\|_V.$$

(25)

We observe that when $d_N(M_\delta)$ decreases quickly as $N$ increases, a low number of basis functions could approximate properly the solution manifold $M_\delta$.

In addition, the Cea’s lemma applied to $V_{rb}$ states:

$$\|u(\mu) - u_{rb}(\mu)\|_V \leq \left(1 + \frac{\|f(\mu)\|_V}{\|A_{rb}(\mu)\|_V}\right) \inf_{v_{rb} \in V_{rb}} \|u(\mu) - v_{rb}\|_V,$$

(26)

where the coercivity and the continuity constants occur. Thus the ROM accuracy depends on the problem at hand.

The reduced solution matrix $A_{rb}$ and the vector $f_{rb}$ are:

$$(A_{rb})_{m,n} = a(\xi_m, \xi_n; \mu) \quad \text{and} \quad (f_{rb})_m = f(\xi_m; \mu), \quad 1 \leq m, n \leq N.$$

(27)

So the ROM in matrix form is:

$$\text{Given } \mu \in \mathbb{P}, \text{ find } u^\mu_{rb} \in \mathbb{R}^N \text{ such that:}$$

$$A_{rb}u^\mu_{rb} = f^\mu_{rb},$$

(28)

and evaluate

$$s_{rb}(\mu) = (u^\mu_{rb})^T f^\mu_{rb},$$

(29)

### 3.2 Reduced basis generation

In this section we analyse the two main strategies to generate the reduced basis space: the POD and the greedy algorithm. We highlight that, in literature, many other techniques can be found, also involving machine learning approaches: see, e.g., [Fu et al. (2021); Lee and Carlberg (2020); Gonzalez and Balajewicz (2018); Kashima (2016)]. However the description of these other methods is out of the scope of this chapter.

Let us introduce a discrete and finite-dimensional set of parameters $\mathbb{P}_h$. It can be obtained from an equispaced or a random sampling of $\mathbb{P}$. Therefore, the following solution manifold of dimension $M = |\mathbb{P}_h|$ can be considered:

$$M_\delta(\mathbb{P}_h) = \{u_\delta(\mu) : \mu \in \mathbb{P}_h\}.$$

(30)

By definition, it holds:

$$M_\delta(\mathbb{P}_h) \subset M_\delta \quad \text{and} \quad \mathbb{P}_h \subset \mathbb{P}.$$

(31)

Consequently, if $\mathbb{P}_h$ is sufficiently rich, $M_\delta(\mathbb{P}_h)$ is able to represent properly $M_\delta$. 


3.2.1 Proper orthogonal decomposition

POD is a technique whose aim consists of squeezing data. Once the parameter space is discretized and the high fidelity solutions for each element of $P_h$ are computed, POD extracts and retains only the essential information with a resulting compact form of the problem. The POD-space of dimension $N$ is the solution of the following minimization problem:

$$
\min_{V_{rb}} \|v_{rb}\|_N = N \left( \int_{\mu \in P_h} \inf_{v_{rb} \in V_{rb}} \|u_\delta(\mu) - v_{rb}\|_V^2 d\mu \right)^{1/2},
$$

(32)

whose discrete version corresponds to:

$$
\min_{V_{rb}} \|v_{rb}\|_N = N \left( \frac{1}{M} \sum_{\mu \in P_h} \inf_{v_{rb} \in V_{rb}} \|u_\delta(\mu) - v_{rb}\|_V^2 \right)^{1/2}.
$$

(33)

Let us sort the elements of $P_h$

$$
\{\mu_1, \ldots, \mu_M\},
$$

(34)

and denote with

$$
\{\psi_1, \ldots, \psi_M\}
$$

(35)

the elements of $M_\delta(P_h)$, $\psi_m = u_\delta(\mu_m)$ for $m = 1, \ldots, M$ (the so-called snapshots). If $V_M = \text{span}\{u_\delta(\mu) : \mu \in P_h\}$, we can introduce the symmetric and linear operator $C : V_M \rightarrow V_M$ defined as:

$$
C(v_\delta) = \frac{1}{M} \sum_{m=1}^M (v_\delta, \psi_m) \psi_m, \quad v_\delta \in V_M,
$$

(36)

whose eigenvalues and normalized eigenvectors are denoted as $(\lambda_i, \xi_i) \in \mathbb{R} \times V_M$ and satisfy:

$$(C(\xi_i), \psi_m) = \lambda_i (\xi_i, \psi_m), \quad 1 \leq m \leq M.
$$

(37)

Here we assume that the eigenvalues are sorted in descending order, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M$. The basis functions for the space $V_M$ are given by the eigenfunctions $\{\xi_1, \ldots, \xi_M\}$.

The first $N \ll M$ eigenfunctions $\{\xi_1, \ldots, \xi_N\}$ generate the $N$-dimensional reduced space $V_{POD} = \text{span}\{\xi_1, \ldots, \xi_N\}$ which fulfills Eq. (33) By projecting the elements of $M_\delta(P_h)$ onto $V_{POD}$, it can be proven that the error is related to the neglected eigenvalues:

$$
\frac{1}{M} \sum_{m=1}^M \|\psi_m - P_N[\psi_m]\|_V^2 = \sum_{m=N+1}^M \lambda_m,
$$

(38)

where $P_N[\psi_m] = \sum_{i=1}^N (\psi_m, \xi_i) \xi_i$ is the projection of $\psi_m$ onto $V_{POD}$. Due to the orthonormality of the eigenvectors in $||\cdot||_{L^2(\mathbb{R}^M)}$, we can derive the orthogonality property for the eigenfunctions:

$$(\xi_m, \xi_q) = M \lambda_i \delta_{mq}, \quad 1 \leq m, q \leq M,
$$

(39)

where $\delta_{mq}$ is the Kronecker delta.

In matrix form, the correlation matrix $C \in \mathbb{R}^{M \times M}$ corresponding to the linear operator $C$ is:

$$
C_{mq} = \frac{1}{M} (\psi_m, \psi_q) \psi, \quad 1 \leq m, q \leq M.
$$

(40)

The eigenvalues problem equivalent to Eq. (37) is:

$$
C v_i = \lambda_i v_i, \quad 1 \leq i \leq N.
$$

(41)

Then, the orthogonal basis functions are given by:

$$
\xi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{m=1}^M (v_i)_m \psi_m, \quad 1 \leq i \leq N,
$$

(42)

where $(v_i)_m$ denotes the $m$-th coefficient of the eigenvector $v_i \in \mathbb{R}^M$.

The computational cost to perform the POD can be very high. In fact, it is not possible to know the number of high fidelity solutions needed to ensure a good solution a priori and it depends on the problem at hand. Therefore, a lot of FOMs, $M \gg N$, have often to be solved, leading to an expensive offline phase. In addition, when $M$ and $N_\delta$ are large, also the cost to extrapolate the eigenfunctions, scaling as $O(N_\delta^2)$, increases.
3.2.2 Greedy algorithm

The greedy algorithm, schematized in Algorithm 1, is an iterative strategy where at each iteration one new basis function is added and the overall precision of the basis set is improved. It requires one truth solution to be computed per iteration and a total of \( N \) truth solutions to generate the \( N \)-dimensional reduced basis space. Let us suppose that an estimation \( \eta(\mu) \) of the error due to the replacement of \( V_\delta \) with \( V_{rb} \) is available (see Section 4 for further details), i.e.

\[
\| u_\delta(\mu) - u_{rb}(\mu) \| \leq \eta(\mu), \quad \forall \mu \in P.
\] (43)

At the \( n \)-th step of the iterative process, a new parameter is selected:

\[
\mu_{n+1} = \arg\max_{\mu \in P} [\eta(\mu)],
\] (44)

and the corresponding full order solution \( u_\delta(\mu_{n+1}) \) is computed. Then it is included in the reduced basis space \( V_{rb} = \text{span}\{u_\delta(\mu_1), \ldots, u_\delta(\mu_{n+1})\} \). In other words, at each iteration, if the dimension of \( V_{rb} \) is \( n \), the \( n + 1 \) basis function maximizes the estimation of the model order reduction error over \( P \). This is repeated until the maximal estimated error is under a fixed tolerance. So while the POD procedure finds a basis which is optimal for the \( L^2 \)-norm, the greedy approach is based on the maximum norm over \( P \).

From an operational point of view, as for the POD approach, we introduce the discrete parameter space \( P_h \). Since the greedy approach needs only the evaluation of the error estimator and not the resolution of Eq. (10) for each point in \( P_h \), the parameter space can be denser than the one used with the POD procedure, simply because the computational cost is considerably smaller. So the main advantages of this approach are two: the resolution of a big eigenvalue problem is avoided and only \( N \) truth solutions are computed. As a general rule, if \( F = \{ f(\mu) : \Omega \mapsto \mathbb{R} : \mu \in P \} \) is a set of parametrized functions, the \( n + 1 \)-th basis function is chosen as

\[
f_{n+1} = \arg\max_{\mu \in P} \| f(\mu) - P_n f(\mu) \|_V,
\] (45)

where \( P_n f \) is the orthogonal projection onto \( F_n = \text{span}\{f_1, \ldots, f_n\} \). Some results about the convergence of the greedy algorithm are shown in the following [Binev et al., 2011; DeVore et al., 2013].

**Theorem 3.1** Let us assume that \( F \) has an exponentially small Kolmogorov \( N \)-width, \( d_N(F) \leq ce^{-\alpha N} \), with \( \alpha > \log(2) \). Then, \( \exists \beta > 0 \) such that the set \( F_N = \text{span}\{f_1, \ldots, f_N\} \) resulting from the greedy algorithm is exponentially accurate in the sense that:

\[
\| F - P_N f \|_V \leq Ce^{-\beta N}
\]

Hence, if a problem allows an efficient and compact reduced basis, the greedy algorithm will provide an exponentially convergent approximation to it.

**Theorem 3.2** Let us assume that all the solutions \( M \) have an exponentially small Kolmogorov \( N \)-width, \( d_N(M) \leq ce^{-\alpha N} \), with \( \alpha > \log(1 + \sqrt{\gamma}) \) (where \( \gamma \) and \( \alpha \) are the continuity and coercivity constants introduced in Section 2.2). Then, the reduced basis approximation converges exponentially fast in the sense that \( \exists \beta > 0 \) such that:

\[
\forall \mu \in P : \| u_\delta(\mu) - u_{rb}(\mu) \|_V \leq Ce^{-\beta N}
\]

Therefore the reduced basis approximation \( u_{rb}(\mu) \) goes exponentially to the truth solution \( u_\delta(\mu) \). Note that it is not sufficient in order to conclude that the reduced method works properly, in fact it is necessary to suppose that truth solution is a good approximation of the exact solution \( u(\mu) \).

It is important to observe that, using a greedy approach, the snapshots for each parameter in \( P_h \), \( u_\delta(\mu_1), \ldots, u_\delta(\mu_N) \), may be linearly dependent, by leading to a large condition number of the associated solution matrix. Therefore, in order to overcome such an issue, it is recommended to orthonormalize \( u_\delta(\mu_1), \ldots, u_\delta(\mu_N) \) in order to find the reduced basis \( \xi_1, \ldots, \xi_N \).

3.3 Notes on the affine decomposition

The computation of the reduced solution \( u_{rb}(\mu) \) for each new parameter \( \mu \in P \) is the main task of the online phase. The associated computational cost should be independent on the complexity of the truth problem and depend only on the reduced basis space \( N \). Therefore, in order to ensure the efficiency of the reduced order framework, the affine decomposition strategy is considered. The following forms are introduced:

\[
a_q : V \times V \mapsto \mathbb{R}, \quad f_q : V \mapsto \mathbb{R}, \quad l_q : V \mapsto \mathbb{R},
\] (46)
AN INTRODUCTION TO POD-GREEDY-GALERKIN REDUCED BASIS METHOD

Algorithm 1 The greedy algorithm

\textbf{Input:} $\mu_1$, tolerance

\textbf{Output:} $V_{rb}$

1: Compute $u_\delta(\mu_n)$
2: $V_{rb} = \text{span}\{u_\delta(\mu_1), \ldots, u_\delta(\mu_n)\}$
3: \textbf{for} $\mu \in P_h$ \textbf{do}
4: \hspace{1em} compute $u_{rb}(\mu)$
5: \hspace{1em} evaluate $\eta(\mu)$
6: \textbf{end for}
7: $\mu_{n+1} = \text{argmax}_{\mu \in P} \left[\eta(\mu)\right]$
8: \hspace{1em} if $\eta(\mu_{n+1}) > \text{tolerance}$ \textbf{then}
9: \hspace{2em} $n = n + 1$
10: \hspace{1em} \textbf{else}
11: \hspace{2em} break.
12: \hspace{1em} \textbf{end if}

which are independent on the parameter $\mu \in P$ so that

\begin{equation}
\alpha(u, v; \mu) = \sum_{q=1}^{Q_a} \theta_a^q(\mu) a_q(w, v),
\end{equation}

\begin{equation}
f(v; \mu) = \sum_{q=1}^{Q_f} \theta_f^q(\mu) f_q(v),
\end{equation}

\begin{equation}
l(v; \mu) = \sum_{q=1}^{Q_l} \theta_l^q(\mu) l_q(v).
\end{equation}

The quantities $\theta_a^q, \theta_f^q, \theta_l^q$ are the scalar coefficients depending only on the parameter values $\mu \in P$.

As result of this decomposition, if $A_{rb}^q$ is the matrix associated to $a_q(\cdot, \cdot)$ and $f_{rb}^q$, $l_{rb}^q$ are the vectors associated to $f_q(\cdot)$ and $l_q(\cdot)$, respectively, they can be precomputed during the offline phase (because independent on the parameter $\mu$). Then, during the online phase, one obtains:

\begin{equation}
A_{rb}^\mu = \sum_{q=1}^{Q_a} \theta_a^q(\mu) A_{rb}^q, \quad f_{rb}^\mu = \sum_{q=1}^{Q_f} \theta_f^q(\mu) f_{rb}^q, \quad l_{rb}^\mu = \sum_{q=1}^{Q_l} \theta_l^q(\mu) l_{rb}^q.
\end{equation}

If the problem does not allow an affine representation, it can be approximated using other techniques, as the empirical interpolation method (see, e.g., [Hesthaven et al., 2016]).

4 ERROR ANALYSIS

An accurate posterior error estimation is desirable to have a robust and powerful reduced order model. Reliable approximation of the error is a significant aspect during both the offline and online phases. In fact, ROMs are problem dependent and can not be directly related to specific spatial or temporal scales, so problem intuition is of limited value and may even be wrong. In addition, sometimes they are used for real-time predictions when no time is available for an offline check.

When the parameter space is very large, some regions of $P$ can be unexplored due to the sampling strategies chosen to obtain $P_h$. Consequently, only the error in $P_h$ and not also in $P$ can be bounded. However, it is possible to control the output error during the online phase with several posteriori error estimations, for every $\mu \in P$, as we are going to show in this section. Some properties have to be satisfied: the error bounds has to be rigorous and valid for all $N$ and for every $\mu \in P$; furthermore, the bounds must be reasonably sharp since overly conservative errors will yield inefficient approximations; finally, the bounds have to be computable at low cost, independent on $N_h$.  

8
4.1 Error representation

Let us introduce the discrete version of the coercivity and the continuity constants (Eqs. 5 and 7):

\[
\alpha_{\delta}(\mu) = \inf_{v_\delta \in V_\delta} \frac{a(v_\delta, v_\delta; \mu)}{\|v_\delta\|^2_{V}}, \quad \gamma_{\delta}(\mu) = \sup_{v_\delta \in V_\delta} \sup_{w_\delta \in V_\delta} \frac{a(w_\delta, v_\delta; \mu)}{\|w_\delta\| V \cdot \|v_\delta\|_V}.
\]

By considering that \(V_\delta \subset V\), one obtains:

\[
\alpha(\mu) \leq \alpha_{\delta}(\mu), \quad \gamma_{\delta}(\mu) \leq \gamma(\mu).
\]

If the error is \(e(\mu) = u_{\delta}(\mu) - u_{rb}(\mu) \in V_\delta\) from the bilinearity of \(a(\cdot, \cdot; \mu)\) it holds:

\[
a(e(\mu), v_\delta; \mu) = r(v_\delta; \mu), \quad \forall v_\delta \in V_\delta,
\]

where the residual \(r(\cdot, \mu)\) in the dual space \(V_\delta^*\) is:

\[
r(v_\delta; \mu) = f(v_\delta; \mu) - a(u_{rb}(\mu), v_\delta; \mu), \quad \forall v_\delta \in V_\delta.
\]

Eq. 53 can be written as:

\[
a(e(\mu), v_\delta; \mu) = (\tilde{r}_{\delta}(\mu), v_\delta), \quad \forall v_\delta \in V_\delta,
\]

where \(\tilde{r}_{\delta}(\mu) \in V_\delta\) is the Riesz representation of \(r(\cdot, \mu) \in V_\delta^*\):

\[
(\tilde{r}_{\delta}(\mu), v_\delta) = r(v_\delta; \mu), \quad \forall v_\delta \in V_\delta.
\]

Therefore it holds:

\[
\|\tilde{r}_{\delta}(\mu)\|_V = \|r(\cdot; \mu)\|_{V_\delta^*} = \sup_{v_\delta \in V_\delta} \frac{r(v_\delta; \mu)}{\|v_\delta\|_V}, \quad \forall v_\delta \in V_\delta.
\]

Due to the hypothesis of compliant problem (see Sec. 2.1), it can be introduced the Prop. 4.1 about some error relations, useful for future proofs.

**Proposition 4.1** If the problem is compliant (see Sec. 2.1), then \(\forall \mu \in P:\)

\[
s_{\delta}(\mu) - s_{rb}(\mu) = \|u_{\delta}(\mu) - u_{rb}(\mu)\|_\mu^2, \quad \implies \quad s_{\delta}(\mu) \geq s_{rb}(\mu).
\]

**Proof:** Let us consider \(u_{\delta}(\mu) \in V_\delta\) and \(u_{rb}(\mu) \in V_{rb}\) and recall the Galerkin orthogonality:

\[
a(u_{\delta}(\mu) - u_{rb}(\mu), v_\delta; \mu) = 0, \quad \forall v_\delta \in V_{rb}.
\]

For the linearity of \(f(\cdot; \mu)\), the hypothesis of compliant problem and the Galerkin orthogonality, we get:

\[
s_{\delta}(\mu) - s_{rb}(\mu) = f(e(\mu); \mu) = a(u_{\delta}(\mu), e(\mu); \mu) = a(e(\mu), e(\mu); \mu) = \|e(\mu)\|_\mu^2.
\]

\(\square\)

4.2 Error bounds

Let us assume to have a lower bound \(\alpha_{LB}(\mu)\) for the discrete coercivity constant \(\alpha_{\delta}(\mu)\) (Eq. 51), independent on \(N_\delta\). Let us introduce the following error estimators:

\[
\eta_{en}(\mu) = \frac{\|\tilde{r}_{\delta}(\mu)\|_V}{\alpha_{LB}^{1/2}(\mu)}, \quad \text{for the energy norm},
\]

\[
\eta_{u}(\mu) = \frac{\|\tilde{r}_{\delta}(\mu)\|_V^2}{\alpha_{LB}(\mu)} = \eta_{en}(\mu)^2, \quad \text{for the output},
\]

\[
\eta_{s,rel}(\mu) = \frac{\|\tilde{r}_{\delta}(\mu)\|_V^2}{\alpha_{LB}(\mu) s_{rb}(\mu)} = \frac{\eta_{u}(\mu)}{s_{rb}(\mu)}, \quad \text{for the relative output}.
\]

It holds the Prop. 4.2 for the error of the reduced solution.

**Proposition 4.2** For all \(\mu \in P:\)

\[
\|u_{\delta}(\mu) - u_{rb}(\mu)\|_\mu \leq \eta_{en}(\mu),
\]

\[
s_{\delta}(\mu) - s_{rb}(\mu) \leq \eta_{u}(\mu),
\]

\[
\frac{s_{\delta}(\mu) - s_{rb}(\mu)}{s_{\delta}(\mu)} \leq \eta_{s,rel}(\mu).
\]
AN INTRODUCTION TO POD-GREEDY-GALERKIN REDUCED BASIS METHOD

Proof: Using Eq. 55 with \( v_\delta = e(\mu) \) and the Cauchy-Schwarz inequality, it holds:
\[
\|e(\mu)\|_\mu^2 = a(e(\mu), e(\mu); \mu) \leq \|\hat{r}_\delta(\mu)\|_V \|e(\mu)\|_V.
\] (66)

In addition, for the coercivity of \( a(\cdot, \cdot; \mu) \) and by considering that \( \alpha_{LB}(\mu) \leq \alpha_\delta(\mu) \) by definition, one obtains:
\[
\alpha_{LB}(\mu) \|e(\mu)\|_\mu^2 \leq a(e(\mu), e(\mu); \mu) = \|e(\mu)\|_\mu^2.
\] (67)

Combining thus Eqs. 66 and 67 with Prop. 4.1 yields Eqs. 63, 64 and 65.

Let us define the following effectivity indexes:
\[
eff_{en}(\mu) = \frac{\eta_{en}(\mu)}{\|u_\delta(\mu) - u_{rb}(\mu)\|_\mu},
\]
(68)
\[
eff_f(\mu) = \frac{\eta_f(\mu)}{s_\delta(\mu) - s_{rb}(\mu)},
\]
(69)
\[
eff_{s,rel}(\mu) = \frac{\eta_{s,rel}(\mu)}{s_\delta(\mu) - s_{rb}(\mu)}.
\]
(70)

These quantities are \( \geq 1 \) as ensured by Prop. 4.2 and the quality of the estimators increases as the effectivity indexes are closer to one. If the problem is coercive and compliant, they are bounded as specified in the next proposition.

Proposition 4.3 For all \( \mu \in P \), the effectivity indexes are controlled by an upper bound as follows:
\[
eff_{en}(\mu) \leq \sqrt{\gamma_\delta(\mu)/\alpha_{LB}(\mu)},
\] (71)
\[
eff_f(\mu) \leq \gamma_\delta(\mu)/\alpha_{LB}(\mu),
\] (72)
\[
eff_{s,rel}(\mu) \leq (1 + \eta_{s,rel})\gamma_\delta(\mu)/\alpha_{LB}(\mu).
\] (73)

Proof: Using Eq. 55 with \( v_\delta = \hat{r}_\delta(\mu) \) and the Cauchy-Schwarz inequality, it holds:
\[
\|\hat{r}_\delta(\mu)\|_V^2 = a(e(\mu), \hat{r}_\delta(\mu); \mu) \leq \|\hat{r}_\delta(\mu)\|_\mu \|e(\mu)\|_\mu.
\] (74)

In addition, the continuity of the bilinear form \( a(\cdot, \cdot; \mu) \) yields:
\[
\|\hat{r}_\delta(\mu)\|_\mu^2 = a(\hat{r}_\delta(\mu), \hat{r}_\delta(\mu); \mu) \leq \gamma_\delta(\mu)\|\hat{r}_\delta(\mu)\|_V^2 \leq \gamma_\delta(\mu)\|\hat{r}_\delta(\mu)\|_\mu \|e(\mu)\|_\mu.
\] (75)

Combining Eqs. 74 and 75 we obtain:
\[
\eta_{en}(\mu) = \frac{\|\hat{r}_\delta(\mu)\|_V^2}{\alpha_{LB}(\mu)} \leq \frac{\|\hat{r}_\delta(\mu)\|_\mu \|e(\mu)\|_\mu}{\alpha_{LB}(\mu)} \leq \frac{\gamma_\delta(\mu)}{\alpha_{LB}(\mu)} \|e(\mu)\|_\mu^2,
\] (76)

from which it follows Eq. 71.

Using Prop. 4.1 yields:
\[
eff_f(\mu) = \eff_{en}(\mu),
\] (77)

which proves Eq. 72.

Finally, by definition of \( \eta_{s,rel} \), it holds:
\[
eff_{s,rel}(\mu) = \frac{s_\delta(\mu)}{s_{rb}(\mu)} \eff_f(\mu).
\] (78)

Then, from Prop. 4.1 and Eq. 64 one obtains:
\[
\frac{s_\delta(\mu)}{s_{rb}(\mu)} = 1 + \frac{s_\delta(\mu) - s_{rb}(\mu)}{s_{rb}(\mu)} \leq 1 + \frac{\eta_f(\mu)}{s_\delta(\mu)} = 1 + \eta_{s,rel}(\mu),
\] (79)

which proves Eq. 73.

So far, the error of the reduced solution \( u_\delta(\mu) - u_{rb}(\mu) \) is controlled with the parameter dependent norm \( \|\cdot\|_\mu \). However, it also proves useful to find an error estimation in a norm which is independent on \( \mu \). So, for the fixed parameter \( \tilde{\mu} \), we refer to the norm over \( V, \|\cdot\|_V \).

Let us define the error estimators:
\[
\eta_V(\mu) = \frac{\|\hat{r}_\delta(\mu)\|_V}{\alpha_{LB}(\mu)},
\] (80)
\[
\eta_{V,rel}(\mu) = \frac{2\|\hat{r}_\delta(\mu)\|_V}{\alpha_{LB}(\mu)\|u_{rb}(\mu)\|_V}.
\] (81)
Proposition 4.4 For all $\mu \in P$
\[ \|u_\delta(\mu) - u_{rb}(\mu)\|_V \leq \eta_V(\mu). \] (82)
Moreover, if $\eta_{V,rel}(\mu) \leq 1$:
\[ \frac{\|u_\delta(\mu) - u_{rb}(\mu)\|_V}{\|u_\delta(\mu)\|_V} \leq \eta_{V,rel}(\mu), \] (83)
for some $\mu \in P$.

Proof: The first inequality can be proven similarly to what done for Eq. [63]
For the second one, due to the hypothesis $\eta_{V,rel}(\mu) \leq 1$, it holds:
\[ \|u_\delta(\mu)\|_V = \|u_{rb}(\mu)\|_V + \|u_\delta(\mu)\|_V - \|u_{rb}(\mu)\|_V \geq \|u_{rb}(\mu)\|_V \geq \|u_\delta(\mu) - u_{rb}(\mu)\|_V \]
\[ \geq \|u_{rb}(\mu)\|_V - \eta_V(\mu) = (1 - \frac{1}{2}\eta_{V,rel}(\mu))\|u_{rb}(\mu)\|_V \geq \frac{1}{2}\|u_{rb}(\mu)\|_V. \] (84)

Finally, it yields:
\[ \eta_{V,rel}(\mu) = 2\frac{\|\hat{u}_\delta(\mu)\|_V}{\alpha_{LB}(\mu)\|u_{rb}(\mu)\|_V} = 2\frac{\|u_\delta(\mu)\|_V \eta_V(\mu)}{\|u_\delta(\mu)\|_V} \geq \frac{\eta_V(\mu)}{\|u_\delta(\mu)\|_V} \geq \frac{\|u_\delta(\mu) - u_{rb}(\mu)\|_V}{\|u_\delta(\mu)\|_V}, \]
which concludes the proof. □
The associated effectivities indexes can be defined as:
\[ eff_V(\mu) = \frac{\eta_V(\mu)}{\|u_\delta(\mu) - u_{rb}(\mu)\|_V} \] (85)
\[ eff_{V,rel}(\mu) = \frac{\eta_{V,rel}(\mu)\|u_\delta(\mu)\|_V}{\|u_\delta(\mu) - u_{rb}(\mu)\|_V} \] (86)

Proposition 4.5 For all $\mu \in P$
\[ eff_V(\mu) \leq \frac{\gamma_\delta(\mu)}{\alpha_{LB}(\mu)}. \] (87)
Moreover, if $\eta_{V,rel}(\mu) \leq 1$:
\[ eff_{V,rel}(\mu) \leq 3\frac{\gamma_\delta(\mu)}{\alpha_{LB}(\mu)}. \] (88)
for some $\mu \in P$.

Proof: Using the inequality $\|e(\mu)\|_\mu \leq \gamma_\delta(\mu)\|e(\mu)\|_V$ and Eq. [71] it holds the first statement:
\[ eff_V(\mu) = \frac{eff_{en}(\mu)}{\alpha_{LB}(\mu)} \leq \frac{\gamma_\delta^{1/2}(\mu)}{\alpha_{LB}(\mu)} \frac{\|e(\mu)\|_\mu}{\|e(\mu)\|_V} \leq \frac{\gamma_\delta(\mu)}{\alpha_{LB}(\mu)}. \] (89)
As for the second inequality, combining the hypothesis $\eta_{V,rel}(\mu) \leq 1$ and Eq. [82] one obtains:
\[ \|u_\delta(\mu)\|_V - \|u_{rb}(\mu)\|_V \leq \|u_\delta(\mu) - u_{rb}(\mu)\|_V \]
\[ \leq \|\hat{u}_\delta(\mu)\|_V \frac{\|u_{rb}(\mu)\|_V}{\alpha_{LB}(\mu)} = \frac{1}{2}\|u_{rb}(\mu)\|_V \eta_{V,rel}(\mu) \leq \frac{1}{2}\|u_{rb}(\mu)\|_V. \] (90)
Finally, we get
\[ eff_{V,rel}(\mu) = 2\frac{\|u_\delta(\mu)\|_V}{\|u_{rb}(\mu)\|_V} eff_V(\mu) = 2 \left(1 + \frac{\|u_\delta(\mu)\|_V - \|u_{rb}(\mu)\|_V}{\|u_{rb}(\mu)\|_V}\right) eff_V(\mu) \leq 3\frac{\gamma_\delta(\mu)}{\alpha_{LB}(\mu)}. \] □
5 CONCLUSIONS

The aim of this chapter is to provide to the reader a brief overview about the ROM framework. However, the main ingredients here presented can be useful to study many applications and sometimes they can be easily extended to more complex cases, as reported in others chapters of the book.

In this context, we considered an affine linear coercive formulation in a finite element environment for a simple explanation. We investigated the POD and the greedy algorithm in order to understand how to extract the reduced space. On the other hand, the Galerkin projection is applied to compute the coefficients of the reduced solution. Moreover, we gave some insights related to the posterior error estimation.

To conclude, it is clear that the offline-online paradigm enables to improve the computational speed-up in many scenarios because the dimension of the reduced space is typically much smaller than the full order one.

ACKNOWLEDGMENTS

We acknowledge the support provided by the European Research Council Executive Agency by the Consolidator Grant project AROMA-CFD "Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics" - GA 681447, H2020-ERC CoG 2015 AROMA-CFD, PI G. Rozza, and INdAM-GNCS 2019-2020 projects.

References

Ballarin, F., Faggiano, E., Ippolito, S., Manzoni, A., Quarteroni, A., Rozza, G., Scrofani, R., 2016. Fast simulations of patient-specific haemodynamics of coronary artery bypass grafts based on a POD–Galerkin method and a vascular shape parametrization. Journal of computational physics 315, 609–628.

Ballarin, F., Faggiano, E., Manzoni, A., Quarteroni, A., Rozza, G., Ippolito, S., Antona, C., Scrofani, R., 2017. Numerical modeling of hemodynamics scenarios of patient-specific coronary artery bypass grafts. Biomechanics and modeling in mechanobiology 16, 1373–1399.

Ballarin, F., Rozza, G., 2016. POD–Galerkin monolithic reduced order models for parametrized fluid-structure interaction problems. International journal for numerical methods in fluids 82, 1010–1034.

Balmès, E., 1996. Parametric families of reduced finite element models. Theory and applications. Mechanical systems and signal processing 10, 381–394.

Barrault, M., Maday, Y., Nguyen, N.C., Patera, A.T., 2004. An ‘empirical interpolation’ method: application to efficient reduced-basis discretization of partial differential equations. Comptes rendus mathematique 339, 667–672.

Bathe, K.J., 2007. Finite element method. Wiley encyclopedia of computer science and engineering , 1–12.

Benner, P., Herzog, R., Lang, N., Riedel, I., Saak, J., 2019. Comparison of model order reduction methods for optimal sensor placement for thermo-elastic models. Engineering optimization 51, 465–483.

Benner, P., Schilders, W., Grivet-Talocia, S., Quarteroni, A., Rozza, G., Miguel Silveira, L., 2020. Model Order Reduction: Volume 3 Applications. volume 3. De Gruyter.

Binev, P., Cohen, A., Dahmen, W., DeVore, R., Petrova, G., Wojtaszczyk, P., 2011. Convergence rates for greedy algorithms in reduced basis methods. Journal on mathematical analysis 43, 1457–1472.

Bui-Thanh, T., Damodaran, M., Willcox, K., 2003. Proper orthogonal decomposition extensions for parametric applications in compressible aerodynamics, in: 21st applied aerodynamics conference, AIAA. p. 4213.

Chen, Y., Hesthaven, J.S., Maday, Y., Rodríguez, J., 2009. Improved successive constraint method based a posteriori error estimate for reduced basis approximation of 2D Maxwell’s problem. Mathematical modelling and numerical analysis 43, 1099–1116.

Chen, Y., Hesthaven, J.S., Maday, Y., Rodríguez, J., 2010. Certified reduced basis methods and output bounds for the harmonic Maxwell’s equations. Journal on scientific computing 32, 970–996.

Chen, Y., Hesthaven, J.S., Maday, Y., Rodríguez, J., Zhu, X., 2012. Certified reduced basis method for electromagnetic scattering and radar cross section estimation. Computer methods in applied mechanics and engineering 233, 92–108.

Christensen, E.A., Brøns, M., Sørensen, J.N., 1999. Evaluation of proper orthogonal decomposition–based decomposition techniques applied to parameter-dependent nonturbulent flows. Journal on scientific computing 21, 1419–1434.

Cuong, N.N., Veroy, K., Patera, A.T., 2005. Certified real-time solution of parametrized partial differential equations, in: Handbook of materials modeling. Springer, pp. 1529–1564.
AN INTRODUCTION TO POD-GREEDY-GALERKIN REDUCED BASIS METHOD

Deparis, S., Rozza, G., 2009. Reduced basis method for multi-parameter-dependent steady Navier–Stokes equations: applications to natural convection in a cavity. Journal of computational physics 228, 4359–4378.

DeVore, R., Petrova, G., Wojtaszczyk, P., 2013. Greedy algorithms for reduced bases in Banach spaces. Constructive approximation 37, 455–466.

Fu, R., Xiao, D., Navon, I., Wang, C., 2021. A data driven reduced order model of fluid flow by auto-encoder and self-attention deep learning methods. arXiv preprint arXiv:2109.02126.

Girfoglio, M., Quaini, A., Rozza, G., 2021a. A POD-Galerkin reduced order model for a LES filtering approach. Journal of computational physics 436, 110260.

Girfoglio, M., Quaini, A., Rozza, G., 2021b. Pressure stabilization strategies for a LES filtering reduced order model. Fluids 6, 302.

Girfoglio, M., Scandurra, L., Ballarin, F., Infantino, G., Nicolò, F., Montalto, A., Rozza, G., Scrofani, R., Comisso, M., Musumeci, F., 2021c. A non-intrusive data-driven ROM framework for hemodynamics problems. Acta mechanica sinica 37, 1183–1191.

Gonzalez, F.J., Balajewicz, M., 2018. Deep convolutional recurrent autoencoders for learning low-dimensional feature dynamics of fluid systems. arXiv preprint arXiv:1808.01346.

Grepl, M.A., 2005. Reduced-basis approximation a posteriori error estimation for parabolic partial differential equations. Ph.D. thesis. Massachusetts Institute of Technology.

Grepl, M.A., Maday, Y., Nguyen, N.C., Patera, A.T., 2007. Efficient reduced-basis treatment of nonaffine and nonlinear partial differential equations. Mathematical modelling and numerical analysis 41, 575–605.

Grepl, M.A., Patera, A.T., 2005. A posteriori error bounds for reduced-basis approximations of parametrized parabolic partial differential equations. Mathematical modelling and numerical analysis 39, 157–181.

Guérin, N., Thorin, A., Thouverez, F., Legrand, M., Almeida, P., 2019. Thermomechanical model reduction for efficient simulations of rotor-stator contact interaction. Journal of engineering for gas turbines and power 141, 1–10.

Gunzburger, M.D., 2012. Finite element methods for viscous incompressible flows: a guide to theory, practice, and algorithms. Elsevier.

Gunzburger, M.D., Peterson, J.S., Shadid, J.N., 2007. Reduced-order modeling of time-dependent PDEs with multiple parameters in the boundary data. Computer methods in applied mechanics and engineering 196, 1030–1047.

Hernández-Becerro, P., Spescha, D., Wegener, K., 2021. Model order reduction of thermo-mechanical models with parametric convective boundary conditions: focus on machine tools. Computational mechanics 67, 167–184.

Hess, M.W., Quaini, A., Rozza, G., et al., 2018. A spectral element reduced basis method for navier–stokes equations with geometric variations. Lecture notes in computational science and engineering 234, 561–571.

Hess, M.W., Rozza, G., 2017. A spectral element reduced basis method in parametric CFD, in: European conference on numerical mathematics and advanced applications, Springer. pp. 693–701.

Hesthaven, J.S., Rozza, G., Stamm, B., et al., 2016. Certified reduced basis methods for parametrized partial differential equations. volume 590. Springer.

Hijazi, S., Stabile, G., Mola, A., Rozza, G., 2020. Data-driven POD-Galerkin reduced order model for turbulent flows. Journal of computational physics 416, 109513.

Huynh, D., Knezevic, D., Patera, A., 2012. Certified reduced basis model validation: A frequentistic uncertainty framework. Computer methods in applied mechanics and engineering 201, 13–24.

Huynh, D.B.P., Rozza, G., Sen, S., Patera, A.T., 2007. A successive constraint linear optimization method for lower bounds of parametric coercivity and inf–sup stability constants. Comptes Rendus Mathematique 345, 473–478.

Ito, K., Ravindran, S., 1998a. A reduced basis method for control problems governed by PDEs, in: Control and estimation of distributed parameter systems. Springer. pp. 153–168.

Ito, K., Ravindran, S.S., 1998b. A reduced-order method for simulation and control of fluid flows. Journal of computational physics 143, 403–425.

Ito, K., Ravindran, S.S., 2001. Reduced basis method for optimal control of unsteady viscous flows. International journal of computational fluid dynamics 15, 97–113.

Jabbar, M., Azeman, A.B., 2004. Fast optimization of electromagnetic-problems: the reduced-basis finite element approach. Transactions on magnetics 40, 2161–2163.
Kashima, K., 2016. Nonlinear model reduction by deep autoencoder of noise response data, in: 55th conference on decision and control, IEEE. pp. 5750–5755.

Lassila, T., Manzoni, A., Quarteroni, A., Rozza, G., 2013. Generalized reduced basis methods and n-width estimates for the approximation of the solution manifold of parametric PDEs, in: Analysis and numerics of partial differential equations. Springer, pp. 307–329.

Lee, K., Carlberg, K.T., 2020. Model reduction of dynamical systems on nonlinear manifolds using deep convolutional autoencoders. Journal of computational physics 404, 108973.

Løvgren, A., Maday, Y., Ronquist, E., 2006a. A reduced basis element method for complex flow systems, in: Proceedings of the european conference on computational fluid dynamics, Citeseer. pp. 1–17.

Løvgren, A.E., Maday, Y., Rønquist, E.M., 2006b. A reduced basis element method for the steady Stokes problem. Mathematical modelling and numerical analysis 40, 529–552.

Løvgren, A.E., Maday, Y., Rønquist, E.M., 2006c. The reduced basis element method for fluid flows, in: Analysis and simulation of fluid dynamics. Springer, pp. 129–154.

Manzoni, A., Quarteroni, A., Rozza, G., 2012. Computational reduction for parametrized PDEs: strategies and applications. Milan journal of mathematics 80, 283–309.

Monk, P., et al., 2003. Finite element methods for Maxwell’s equations. Oxford University Press.

Nonino, M., Ballarin, F., Rozza, G., Maday, Y., 2019. Overcoming slowly decaying Kolmogorov n-width by transport maps: application to model order reduction of fluid dynamics and fluid–structure interaction problems. arXiv preprint arXiv:1911.06598.

Noor, A.K., 1981. Recent advances in reduction methods for nonlinear problems. Computational methods in nonlinear structural and solid mechanics, 31–44.

Papapicco, D., Demo, N., Girfoglio, M., Stabile, G., Rozza, G., 2022. The Neural Network shifted-proper orthogonal decomposition: A machine learning approach for non-linear reduction of hyperbolic equations. Computer methods in applied mechanics and engineering 392, 114687.

Peterson, J.S., 1989. The reduced basis method for incompressible viscous flow calculations. Journal on scientific and statistical computing 10, 777–786.

Pintore, M., Pichi, F., Hess, M., Rozza, G., Canuto, C., 2021. Efficient computation of bifurcation diagrams with a deflated approach to reduced basis spectral element method. Advances in Computational Mathematics 47, 1–39.

Porsching, T., Lee, M.L., 1987. The reduced basis method for initial value problems. Journal on numerical analysis 24, 1277–1287.

Porsching, T.A., 1985. Estimation of the error in the reduced basis method solution of nonlinear equations. Mathematics of computation 45, 487–496.

Prud’Homme, C., Rovas, D.V., Veroy, K., Machiels, L., Maday, Y., Patera, A.T., Turinici, G., 2002. Reliable real-time solution of parametrized partial differential equations: Reduced-basis output bound methods. Journal of fluids engineering 124, 70–80.

Quarteroni, A., Rozza, G., 2007. Numerical solution of parametrized Navier–Stokes equations by reduced basis methods. Numerical methods for partial differential equations 23, 923–948.

Quarteroni, A., Valli, A., 2008. Numerical approximation of partial differential equations. volume 23. Springer Science & Business Media.

Rozza, G., 2005. Reduced-basis methods for elliptic equations in sub-domains with a posteriori error bounds and adaptivity. Applied numerical mathematics 55, 403–424.

Rozza, G., Huynh, D.B.P., Patera, A.T., 2008. Reduced basis approximation and a posteriori error estimation for affinely parametrized elliptic coercive partial differential equations. Archives of computational methods in engineering 15, 229–275.

Rozza, G., Malik, H., Demo, N., Tezzele, M., Girfoglio, M., Stabile, G., Mola, A., 2018. Advances in reduced order methods for parametric industrial problems in computational fluid dynamics. ECCOMAS , 1–18.

Rozza, G., Nguyen, C., Patera, A.T., Deparis, S., 2009. Reduced basis methods and a posteriori error estimators for heat transfer problems, in: Heat transfer summer conference, ASME. pp. 753–762.

Shah, N.V., Girfoglio, M., Quintela, P., Rozza, G., Lengomin, A., Ballarin, F., Barral, P., 2021. Finite element based model order reduction for parametrized one-way coupled steady state linear thermomechanical problems. arXiv preprint arXiv:2111.08534.
Siena, P., Girfoglio, M., Ballarin, F., Rozza, G., 2022. Data-driven reduced order modelling for patient-specific hemodynamics of coronary artery bypass grafts with physical and geometrical parameters. submitted.

Stabile, G., Hijazi, S., Mola, A., Lorenzi, S., Rozza, G., 2017. POD-Galerkin reduced order methods for CFD using Finite Volume Discretisation: vortex shedding around a circular cylinder. Communications in applied and industrial mathematics 8, 210–236.

Stabile, G., Rozza, G., 2018. Finite volume POD-Galerkin stabilised reduced order methods for the parametrised incompressible Navier–Stokes equations. Computers & Fluids 173, 273–284.

Veroy, K., Prud’Homme, C., Patera, A.T., 2003a. Reduced-basis approximation of the viscous Burgers equation: rigorous a posteriori error bounds. Comptes rendus mathematique 337, 619–624.

Veroy, K., Prud’Homme, C., Rovas, D., Patera, A., 2003b. A posteriori error bounds for reduced-basis approximation of parametrized noncoercive and nonlinear elliptic partial differential equations, in: 16th computational fluid dynamics conference, AIAA. p. 3847.

Zainib, Z., Ballarin, F., Fremes, S., Triverio, P., Jiménez-Juan, L., Rozza, G., 2020. Reduced order methods for parametric optimal flow control in coronary bypass grafts, toward patient-specific data assimilation. International journal for numerical methods in biomedical engineering 37, e3367.