A $q$-deformed Version of the Heavenly Equations

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Abstract

Using a $q$-deformed Moyal algebra associated with the group of area preserving diffeomorphisms of the two-dimensional torus $T^2$, $\text{sdiff}_q(T^2)$, a $q$-deformed version for the Heavenly equations is given. Finally, the two-dimensional chiral version of Self-dual gravity in this $q$-deformed context is briefly discussed.
1 Introduction

New directions in mathematical physics seem to converge to Self-dual gravity in different ways. A most interesting formulation is due to Plebański [1]. In this Ref. it is found that the Self-dual gravity is described by the non-linear second order partial differential equation for a holomorphic function (the Heavenly equation).

At present time there exists great interest to find a deep relation between Self-dual Yang-Mills gauge theory and Self-dual gravity in dimension four [2-5]. In these papers Self-dual gravity is obtained from a dimensional reduction of the Self-dual Yang-Mills equations on a 4-dimensional flat space-time $M$ with infinite dimensional gauge group $SU(\infty)$. This group was taken to be the group of area preserving diffeomorphisms of the two-surface $\Sigma$, $SDiff(\Sigma)$.

Some symmetries involved here are infinite dimensional. In particular the large-$N$ limit of the algebra of Zamolodchikov $W_N$, i.e. $W_\infty$, plays a crucial role. The connection between $W_\infty$ algebras and Self-dual gravity was pointed out for the first time by Bakas in the first paper of Ref [6]. In the present paper we use this fact.

In a similar spirit, in the paper [7], we show that working with the Self-dual Yang-Mills fields on the 4-dimensional flat manifold $M$ with the Lie algebra $sdiff(\Sigma)$-valued connection 1-form, the first and the second Heavenly equations emerge in a natural manner. Here $sdiff(\Sigma)$ is the Lie algebra of $SDiff(\Sigma)$.
On the other hand, there was a considerable interest recently in the application of quantum groups and noncommutative geometry in gauge theory [8,9,10]. In Refs. [11,12] the Yang-Mills gauge theory on the classical space-time was defined using the quantum group SU\(_q\)(2) as the “symmetry group”. In Ref. [13] it is shown how General Relativity can be put in the context of noncommutative geometry. Until now a general theory for Einstein’s theory of gravity including quantum groups as well as quantum spaces has not yet been considered. This work forms a part of our research for the case of Self-dual gravity. We think that this generalization may include some new insights in a realistic theory of quantum gravity.

In this paper we use a \(q\)-deformed version of the Moyal algebra associated with the Lie algebra of the group of area preserving diffeomorphisms of the torus \(T^2\) (\(\Sigma = T^2\)), \text{sdiff}_q(T^2)\ [14, 15] in order to show new insights in four-dimensional Self-dual gravity. Specifically, we obtain a \(q\)-deformed version of the first and the second heavenly equations of Self-dual gravity [1]. This forms part of our attempts to construct the general theory of \(\mathcal{H}\) and \(\mathcal{HH}\) quantum spaces with quantum group as the symmetry group in a close philosophy with [16].

The paper is organized as follow, in section 2 we describe the necessary tools and the quantum algebras used in section 3. The section 3 is devoted to obtain a \(q\)-deformed version for the Heavenly equations. In section 4 we briefly discuss the recent two-dimensional chiral formulation of Self-dual
gravity given by Husain in Ref. [17] in the context of $q$-deformed algebras. Finally, in section 5, we give our conclusions.

2 Basic Tools and Quantum Algebras

The large $N$-limit of the algebra $W_N$ generated by the primary conformal fields with spins $1, 2, ..., N$ is known as a $W_\infty$ algebra. In Ref. [18] Bakas found that this algebra provides the representation of an infinite dimensional (sub)algebra of the area preserving diffeomorphisms of the plane.

Thus, taking the generators $W_{(k,m)}$ (of spin $k, k \geq 0, m \in \mathbb{Z}$) of the $W_\infty$ algebra, they satisfy the relation

$$[W_{(k,m)}, W_{(l,n)}] = \{(l + 1)(n + 1) - (k + 1)(m + 1)\}W_{(k+l,m+n)}.$$  \hspace{1cm} (1)

As is well known, if one takes the large $N$-limit of the Lie algebra $\text{su}(N)$ it can be identified in a natural manner with $\text{sdiff}(T^2)$, i.e. $\text{su}(\infty) \simeq \text{sdiff}(T^2)$ [19].

The algebra $\text{sdiff}(T^2)$ come given by the Poisson algebra

$$[L_f, L_g] = L_{\{f, g\}}$$  \hspace{1cm} (2)

where $\{f, g\}$ is the usual Poisson bracket and $L_f = \frac{\partial f}{\partial x} \frac{\partial}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial}{\partial x}$.

Taking a basis for the torus
\[ f \to f_m = \exp(\im \cdot x), \quad g \to f_n = \exp(\im \cdot x), \quad (3) \]

where \( m \) and \( n \) are 2-vectors with integer entries, \( m = (m_1, m_2) \) and \( x = (x_1, x_2) \). In this basis the Poisson algebras becomes the algebra \( \text{sdiff}(T^2) \)

\[ [L_m, L_n] = (m \times n)L_{m+n} \quad (4) \]

where \( m \times n = m_1n_2 - m_2n_1 \).

A deformation of the Poisson algebra (2) is the Moyal algebra. For the torus, the Moyal algebra reads

\[ [L_m, L_n] = \frac{1}{\kappa} \sin(\kappa m \times n)L_{m+n} \quad (5) \]

\( \kappa \) is here the parameter of the deformation.

A \( q \)-deformed version of this \( W_\infty \) algebra is

\[ [W^q_{(k,m)}, W^q_{(l,n)}] = W^q_{(k,m)}W^q_{(l,n)} - q \cdot W^q_{(l,n)}W^q_{(k,m)} \]

\[ = \{(l + 1)(n + 1) - (k + 1)(m + 1)\}W^q_{(k+l,m+n)} \quad (6) \]

where \( q \) is a complex parameter, in particular a root of unity, \( q = \exp(\im h) \), where \( h \) is a real number.

This algebra corresponds to a \( q \)-deformed version of \( \text{sdiff}(\Sigma) \), namely \( \text{sdiff}_q\Sigma \). However it is not very general because it is limited to anti-commute
The Moyal algebra (5) is also a Lie algebra and therefore can be $q$-deformed. There exist a $q$-deformation of the Moyal bracket (‘quantum Moyal’) proposed for the first time by Devchand, Fairlie, Fletcher and Sudbery [15].

On the other hand, there is other possible $q$-deformation of $\text{sdiff}(\Sigma)$. In Ref. [14], Fairlie found such a $q$-deformation from the construction of algebras of $q$-symmetrized polynomials in the $q$-Heisenberg operators $P$ and $Q$. That is, operators which satisfy the $q$-deformed Heisenberg algebra

$$PQ - qQP = i\lambda,$$

where $\lambda$ is a real parameter. Fairlie shown [14] that the $q$-deformed Heisenberg algebra leads to a $q$-deformation of the Moyal algebra

$$q^{n \times m} \cdot W_m W_n - q^{m \times n} \cdot W_n W_m = (\omega^{m \times n/2} - \omega^{n \times m/2})W_{m+n} + a \cdot m \delta_{m+n,0} \quad (7)$$

where $a$ is a constant 2-vector which characterizes the central extension (see also Ref. [20]). The classical limit of (7) gives precisely the Moyal algebra (5).

In this algebra (7), appears two parameters $q$ and $\omega$. If it is took $\omega = \exp (i\kappa)$ and $q = \exp (ih) \to 1$ (or $h \to 0$) we recover of course the Moyal algebra (5).

\footnote{We thank Prof. C. Zachos for pointing out this consequence.}
In order to apply the above $q$-deformed Moyal algebra is convenient to take $\mathbf{m} = (m_1, 0), \mathbf{n} = (0, n_2)$ in Eq. (7). Thus we have

$$q^x \cdot W_m W_n - q^{-x} \cdot W_n W_m = (\omega^{x/2} - \omega^{-x/2}) W_{m+n} + a_1 m_1 \delta_{(m_1, n_2), 0}$$

(8)

where $x \equiv m_1 n_2$.

We will apply the $q$-deformed Moyal algebra to the compatibility conditions for the usual Self-dual Yang-Mills equations on $\mathcal{M}$ (“classical” space) with local coordinates $\{t, x, y, z\}$ and after this we compare the corresponding results.

Redefining the local coordinates on $\mathcal{M}$ to be, $\alpha = t + iz$, $\bar{\alpha} = t - iz$, $\beta = x + iy$ and $\bar{\beta} = x - iy$ the Self-dual Yang-Mills equations are

$$F_{\alpha\beta} = 0, \quad F_{\bar{\alpha}\bar{\beta}} = 0, \quad F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0$$

(9)

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

(10)

and $i, j \in \{\alpha, \bar{\alpha}, \beta, \bar{\beta}\}$. The compatibility condition reads

$$[\partial_\alpha + \lambda \partial_\beta, A_\beta - \lambda A_\bar{\alpha}] - [\partial_\beta - \lambda \partial_\alpha, A_\alpha + \lambda A_\bar{\beta}]$$

$$= [A_\alpha + \lambda A_\bar{\beta}, A_\beta - \lambda A_\bar{\alpha}].$$

(11)
Now, we will work with the bundle

\[ \text{SDiff}_q(\Sigma) \to P \to \mathcal{M} \]  \hspace{1cm} (12)

with the connection 1-form \( A_i \) on \( \mathcal{M} \) taking values precisely on \( \text{sdiff}_q(T^2) \), i.e.

\[ A_i = \Phi_{i,s} \frac{\partial}{\partial r} - \Phi_{i,r} \frac{\partial}{\partial s} \in \text{sdiff}_q(T^2) \]  \hspace{1cm} (13)

where \( \Phi = \Phi(\alpha, \bar{\alpha}, \beta, \bar{\beta}; q) \) and \( r, s \) are local coordinates on the two-dimensional torus \( T^2 \). They are the generators of \( \text{sdiff}_q(T^2) \) and satisfy the algebra (7) or (8).

The compatibility condition in the \( q \)-deformed version is

\[ [\partial_\alpha + \lambda \partial_\beta, A_\beta - \lambda A_\alpha] - [\partial_\beta - \lambda \partial_\alpha, A_\alpha + \lambda A_\beta] = [A_\alpha + \lambda A_\beta, A_\beta - \lambda A_\alpha]_q. \]  \hspace{1cm} (14)

This equation will provide us the way of obtaining the \( q \)-deformed version of Heavenly equations.

3 The \( q \)-deformed Heavenly Equations

We start by substituting Eq. (13) into (14) and use the \( q \)-deformed commutation relations (8). Comparing powers of \( \lambda \) at order zero, two and one
respectively, we obtain the set of equations

\[ \Phi_{\alpha,\beta s} - \Phi_{\beta,\alpha s} + (q^x \cdot \Phi_{\alpha,r} \Phi_{\beta,ss} + q^{-x} \cdot \Phi_{\beta,s} \Phi_{\alpha,rs}) - (q^x \cdot \Phi_{\alpha,s} \Phi_{\beta,rs} + q^{-x} \cdot \Phi_{\beta,r} \Phi_{\alpha,ss}) \]

\[ + \mathcal{F}(s, \alpha, \bar{\alpha}, \beta, \bar{\beta}, q) = 0, \]  \hspace{1cm} (15)

\[ \Phi_{\bar{\alpha},\bar{\beta} s} - \Phi_{\bar{\beta},\bar{\alpha} s} + (q^x \cdot \Phi_{\bar{\alpha},r} \Phi_{\bar{\beta},ss} + q^{-x} \cdot \Phi_{\bar{\beta},s} \Phi_{\bar{\alpha},rs}) - (q^x \cdot \Phi_{\bar{\alpha},s} \Phi_{\bar{\beta},rs} + q^{-x} \cdot \Phi_{\bar{\beta},r} \Phi_{\bar{\alpha},ss}) \]

\[ + \bar{\mathcal{F}}(s, \alpha, \bar{\alpha}, \beta, \bar{\beta}, q) = 0, \]  \hspace{1cm} (16)

\[ \Phi_{\alpha,\bar{s}} - \Phi_{\bar{\alpha},\alpha \bar{s}} + \Phi_{\beta,\bar{s}} - \Phi_{\bar{\beta},\beta \bar{s}} + \left[ (q^{-x} \cdot \Phi_{\bar{\alpha},s} \Phi_{\alpha,rs} + q^x \cdot \Phi_{\alpha,r} \Phi_{\bar{\alpha},ss}) - (q^{-x} \cdot \Phi_{\bar{\alpha},r} \Phi_{\alpha,ss} + q^x \cdot \Phi_{\alpha,s} \Phi_{\bar{\alpha},rs}) \right] \]

\[ + \mathcal{G}(s, \alpha, \bar{\alpha}, \beta, \bar{\beta}, q) = 0, \]  \hspace{1cm} (17)

here \( \Phi_{\alpha,rs} = \frac{\partial^2 \Phi_{\alpha}}{\partial r \partial s} \), etc.

A. The q-deformed First Heavenly Equation

Now, assuming

\[ \Phi_\alpha = \Omega_{\alpha}, \quad \Phi_\beta = \Omega_{\beta}, \quad \Phi_{\bar{\alpha}} = \Phi_{\bar{\beta}} = 0, \]
\[
\mathcal{F} = 0 = \mathcal{G}, \quad \mathcal{F} = \mathcal{F}(s, \alpha, \beta, q), \quad \mathcal{F} \neq \overline{\mathcal{F}}
\]  

being \( \Omega = \Omega(\alpha, \beta, r, s; q) \) some holomorphic function of its arguments. Thus, Eqs. (16) and (17) are satisfied trivially and Eq. (15) gives

\[
q^x \cdot \Omega_{,ar} \Omega_{,\beta ss} + q^{-x} \cdot \Omega_{,\beta s} \Omega_{,ars} - (q^x \cdot \Omega_{,as} \Omega_{,\beta rs} + q^{-x} \cdot \Omega_{,\beta r} \Omega_{,ass}) + \mathcal{F}(s, \alpha, \beta, q) = 0
\]  

(19)

where \( \Omega_{,ars} = \frac{\partial^3 \Omega}{\partial \alpha \partial r \partial s} \) etc. After the change of coordinates \( r\mathcal{F} \to r \), this equation leads directly to

\[
q^x \cdot \Omega_{,ar} \Omega_{,\beta ss} + q^{-x} \cdot \Omega_{,\beta s} \Omega_{,ars} - (q^x \cdot \Omega_{,as} \Omega_{,\beta rs} + q^{-x} \cdot \Omega_{,\beta r} \Omega_{,ass}) = 1.
\]  

(20)

Notice that the \( q \)-deformed first Heavenly equation is a third order non-linear partial differential equation (TONLPDE).

When the parameter \( q \to 1 \) (or equivalently \( h \to 0 \)), we recover the first heavenly equation in its usual form [1]

\[
\Omega_{,ar} \Omega_{,\beta s} - \Omega_{,as} \Omega_{,\beta r} = 1.
\]

B. The \( q \)-deformed Second Heavenly Equation

Taking
\[\Phi_\alpha = \theta_s, \quad \Phi_\beta = -\theta_r, \quad \Phi_\alpha = \Phi_\beta = 0,\]

\[\bar{\Phi} = 0 = \mathcal{G}, \quad \mathcal{F} = \mathcal{F}(s, \alpha, \beta, q), \quad \theta = \theta(\alpha, \beta, r, s; q), \quad \mathcal{F} \neq \bar{\mathcal{F}}. \quad (21)\]

Thus, Eqs. (16) and (17) hold trivially and Eq. (15) yields

\[q^x \cdot \theta,ss \theta,rrs + q^{-x} \cdot \theta,rr \theta,sss - (q^x + q^{-x})\theta,rs \theta,rss + \theta,r\alpha s + \theta,s\beta s + \mathcal{F}(s, \alpha, \beta, q) = 0. \quad (22)\]

Now, making the substitution

\[\Theta = \theta + rsf, \quad f = f(\alpha, \beta, q), \quad f,\alpha = \mathcal{F} \quad (23)\]

(with \(\Theta = \Theta(\alpha, \beta, r, s; q)\)) into Eq. (22) we finally obtain

\[(q^x \cdot \Theta,ss \Theta,rrs + q^{-x} \cdot \Theta,rr \Theta,sss) - (q^x + q^{-x})\Theta,rs \Theta,rss + \Theta,r\alpha s + \Theta,s\beta s = 0. \quad (24)\]

This equation is again a TONLPDE. Taking the limit \(q \to 1\) we recover of course the second Heavenly equation [1]

\[\Theta,rr \Theta,ss - \Theta^2 + \Theta,r\alpha + \Theta,s\beta = 0.\]
4 Comments on the q-deformed Chiral Model of the Self-dual Gravity

In this last section we make some comments about the possible generalization of the Husein work [17]. In this paper it is shown that Self-dual Einstein equations are equivalent to the two-dimensional Chiral Model with gauge group, precisely SDiff(Σ).

Since the gauge group here is SDiff(Σ) the connection 1-form are sdiff(Σ)-valued. Thus, we will generalize some results in [17] using the $q$-deformed Moyal algebra (8) associated with $\text{sdiff}_q(T^2)$ instead of the Moyal algebra (5) associated with $\text{sdiff}(T^2)$.

The Ashtekar-Jacobson-Smolin formulation of Self-dual gravity [21] leads to a set of equations on the four manifold $\mathcal{M}^4 = \mathcal{K}^3 \times \mathbb{R}$ with local complex coordinates $\{x_0, x_1, x_2, x_3\}$. The equations are (see also [22])

\[
\text{Div} V_i^a = 0 \quad (25)
\]

\[
\frac{\partial V_i^a}{\partial t} = \frac{1}{2} \varepsilon_{ijk} [V_i, V_k]^a \quad (26)
\]

where $V_i^a$ are three spatial vector fields on $\mathcal{K}^3$ and $[,]$ is the Lie bracket. The solutions of these equations lead to the self-dual metric

\[
g^{ab} = (detV)^{-1} [V_i^a V_j^b \delta^{ij} + V_0^a V_0^b] \quad (27)
\]
where \(i, j = 1, 2, 3\) and \(V_0^j\) is a vector field used in the decomposition \(\mathcal{M}^4 = \mathcal{K}^3 \times \mathcal{R}\). The Eq. (26) is equivalent to the equations

\[
[T, X] = [U, V] = 0,
\]

\[
[T, U] + [X, V] = 0,
\]

where \(T = V_0 + iV_1, X = V_2 - iV_3, U = V_0 - iV_1\) and \(V = V_2 + iV_3\). Using the gauge freedom we can put in terms of the new coordinates \(\beta = x_0 + ix_1, \alpha = x_2 - ix_3, u = x_0 - ix_1\) and \(v = x_2 + ix_3\)

\[
T^a = \left(\frac{\partial}{\partial \beta}\right)^a, \quad X^a = \left(\frac{\partial}{\partial \alpha}\right)^a.
\]

As Husein showed [17], the Eqs. (28) and (29) are equivalent to the two-dimensional Chiral Model on the plane with coordinates \((\alpha, \beta)\).

The relation with the first Heavenly equation arises in this context when we take

\[
U = -\Omega_{\alpha s} \frac{\partial}{\partial r} + \Omega_{\alpha r} \frac{\partial}{\partial s},
\]

\[
V = \Omega_{\beta s} \frac{\partial}{\partial r} - \Omega_{\beta r} \frac{\partial}{\partial s}.
\]

These are precisely the generators of the Lie algebra \(\text{sdiff}(T^2)\) \(i.e. U, V \in \text{sdiff}(T^2)\) just as Eq. (13). As we saw before these generators satisfy the
Poisson algebra given by (2). The Eq. (28) leads directly to the first Heavenly equation in the form

$$\Omega_{,\alpha r} \Omega_{,\beta s} - \Omega_{,\alpha s} \Omega_{,\beta r} = 1.$$  

Thus, the natural question arises: what is the modification of the first Heavenly equation when $\mathcal{U}, \mathcal{V}$ are $\text{sdiff}_q(\Sigma)$-valued? To see this, notice that the only modification in the field equations is

$$[\mathcal{U}, \mathcal{V}]_q = q^x \cdot \mathcal{U} \mathcal{V} - q^{-x} \cdot \mathcal{V} \mathcal{U}$$  \hfill (33)

where now, $\mathcal{U}, \mathcal{V}$ depends on the parameter $q$ and $\Omega = \Omega(\alpha, \beta, r, s; q)$. Substituting (31) and (32) into (33) we obtain after a few steps

$$q^x \cdot \Omega_{,\alpha r} \Omega_{,\beta s} + q^{-x} \cdot \Omega_{,\beta s} \Omega_{,\alpha r} - (q^x \cdot \Omega_{,\alpha s} \Omega_{,\beta r} + q^{-x} \cdot \Omega_{,\beta r} \Omega_{,\alpha s}) = \mathcal{O}(s, \alpha, \beta; q)$$  \hfill (34)

where $\mathcal{O}$ is some holomorphic function its argument. Taking $\mathcal{O}(s, \alpha, \beta; q) = 1$ we obtain the $q$-deformed first Heavenly equation from the $q$-deformed two dimensional Chiral Model. It is

$$q^x \cdot \Omega_{,\alpha r} \Omega_{,\beta s} + q^{-x} \cdot \Omega_{,\beta s} \Omega_{,\alpha r} - (q^x \cdot \Omega_{,\alpha s} \Omega_{,\beta r} + q^{-x} \cdot \Omega_{,\beta r} \Omega_{,\alpha s}) = 1$$  \hfill (35)

and corresponds exactly to the result given in Eq. (20) for the Self-dual Yang-Mills case. This equation is the $q$-deformed version of the Eq. (29)
of the first Ref. [17]. Finally, taking the $q \to 1$ we obtain again the first heavenly equation in its usual form.

5 Conclusions

In this paper we have considered some applications of the $q$-deformed Moyal algebra (8) associated with the group of area preserving diffeomorphisms of the torus $T^2$. Since the Heavenly equations are very closely related with this symmetry [23,24], a $q$-deformed version of these equations is a generalization for they. In particular, we showed that the modification prevents the terms in Eqs. (20,24,34) to be complete derivatives which are integrated out in Heavenly equations. In such a way the $q$-deformed version of the Heavenly equations is of the third order. As usual, when we take the limit $q \to 1$ we recover the usual first and second Heavenly equations. A similar situation occurs for the two-dimensional Chiral Model representation of the Self-dual gravity.

Though the understanding of the $q$-deformation of infinite dimensional Lie algebras is still incomplete, the impact in physics is direct. Any advance in this direction will give new insights for Self-dual gravity and perhaps in its quantized version.
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References

[1] J.F. Plebański, J. Math. Phys. 16, 2395, (1975).

[2] L.J. Mason and E.T. Newmann, Commun. Math. Phys. 121, 659, (1989).

[3] I. Bakas, in Nonlinear Fields: Classical, Random, Semiclassical, edited by P. Garbaczewski and Z. Popowicz (World Scientific, Singapore, 1991).

[4] Q-Han Park, Int. J. Mod. Phys. A7, 1415, (1992).

[5] C. Castro, J. Math. Phys. 34, 681, (1993).

[6] I. Bakas, “Area Preserving Diffeomorphisms and Higher Spin Fields in Two Dimensions”, Proceedings of the Trieste Conference on Superme-
membranes and Physics in 2 + 1 Dimensions, Eds. M. Duff, C. Pope and E. Sezgin, World Scientific (1990) 352-362; Commun. Math. Phys. 134, 487, (1990); Int. J. Mod. Phys. A6, 2071, (1991).

[7] J.F. Plebański, M. Przanowski and H. García-Compeán, “From Self-dual Yang-Mills Fields to Self-dual Gravity” Acta Phys. Pol. B 25, 1079, (1994).

[8] S. Woronowicz, Publ. RIMS. Kyoto Univ. 23, 117, (1987); Commun. Math. Phys. 111, 613, (1987); 122, 125, (1989); 130, 381, (1990); Invent. Math. 93, 35, (1988).

[9] Y. Manin Quantum Groups and Non-Commutative Geometry, Cent. R. Math. 1561, Univ. Montréal, 1988.

[10] A. Connes, Publ. Math. IHES, 62, 44, (1983).

[11] I.Ya. Aref’eva and I.V. Volovich, Mod. Phys. Lett. A6, 893, (1991).

[12] T. Brzeziński and S. Majid, Phys. Lett. B298, 339, (1993).

[13] A.H. Chamseddine, G. Felder and J. Fröhlich, Commun. Math. Phys. 155, 205, (1993).

[14] D. Fairlie, in the Proceedings of the Argonne Workshop on Quantum Groups, Eds. T. Curtright, D. Fairlie and C. Zachos, World Scientific, 1991.
[15] P. Fletcher, in the *Proceedings of the Argonne Workshop on Quantum Groups*, Eds. T. Curtright, D. Fairlie and C. Zachos, World Scientific, 1991.

[16] T. Brzeziński and S. Majid, Commun. Math. Phys. **157**, 591, (1993).

[17] V. Husain, Phys. Rev. Lett. **72**, 800, (1994); Class. Quantum Grav. **11**, 927, (1994).

[18] I. Bakas, Phys. Lett. **B228**, 57, (1989).

[19] D.B. Fairlie, P. Fletcher and C. K. Zachos, Phys. Lett. **B218**, 203 (1989); D.B. Fairlie and C.K. Zachos, Phys. Lett. **B224**, 101, (1989); D.B. Fairlie, P. Fletcher and C.K. Zachos, J. Math. Phys. **31**, 1088, (1990).

[20] C. Zachos, “Paradigms of Quantum Algebras”, Preprint ANL-HEP-PR-90-61, Updated January 1992; Preprint ANL-HEP-CP-89-55.

[21] A. Ashtekar, T. Jacobson and L. Smolin, Commun. Math. Phys. **115**, 631, (1988).

[22] J.D.E. Grant, Phys. Rev. D **48**, 2606, (1993).

[23] C.P. Boyer and P. Winternitz, J. Math. Phys. **30**, 1081, (1989).

[24] M. Przanowski, J. Math. Phys. **31**, 300, (1990).