Stability results for a reaction-diffusion system with a single measurement

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Abstract. For a two by two reaction-diffusion system on a bounded domain we give a simultaneous stability result for one coefficient and for the initial conditions. The key ingredient is a global Carleman-type estimate with a single observation acting on a subdomain.

This paper is a review of a recent paper of the authors [7] and it is devoted to the simultaneous identification of one coefficient and the initial conditions in a reaction-diffusion system using the least number of observations as possible.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain of \( \mathbb{R}^n \) with \( n \leq 3 \), (the assumption \( n \leq 3 \) is necessary in order to obtain the appropriate regularity for the solution using classical Sobolev embedding, see [3]). We denote by \( \nu \) the outward unit normal to \( \Omega \) on \( \Gamma = \partial \Omega \) assumed to be of class \( C^1 \).

Let \( T > 0 \) and \( t_0 \in (0, T) \). We shall use the following notations \( Q_0 = \Omega \times (0, T) \), \( Q = \Omega \times (t_0, T) \), \( \Sigma = \Gamma \times (t_0, T) \) and \( \Sigma_0 = \Gamma \times (0, T) \). We consider the following reaction-diffusion system:

\[
\begin{align*}
\partial_t u &= \Delta u + a(x)u + b(x)v & \quad & \text{in } Q_0, \\
\partial_t v &= \Delta v + c(x)u + d(x)v & \quad & \text{in } Q_0, \\
u(t, x) &= g(t, x), & & \text{on } \Sigma_0, \\
u(0, x) &= u_0 & & \text{in } \Omega. \\
\end{align*}
\]

A number of models used in biology, ecology and biochemistry comprise reaction of “species” in the presence of diffusion: hence reaction-diffusion systems arise. Turing (see [19]) proposed that, under certain conditions, a chemical reaction in the presence of diffusion could produce spatial pattern of the chemical concentration. In biology and biomedicine reaction-diffusion systems are frequently used to model the emergence of pattern formation, wound healing, cancer and angiogenesis.

Our problem can be stated as follows:

Is it possible to determine the coefficient \( b(x) \) and the initial conditions \( u_0(x) \), \( v_0(x) \) for \( x \in \Omega \) from the following measurements:

\[
\partial_t v_{|((t_0, T) \times \omega)} \quad \text{and} \quad \Delta u(T', \cdot), \ u(T', \cdot), \ v(T', \cdot) \ \text{in } \Omega \ \text{for } T' = \frac{t_0 + T}{2}.
\]
Throughout this paper, let us consider the following set

$$\Lambda(R) = \{ \Phi \in L^\infty(\Omega); \|\Phi\|_{L^\infty(\Omega)} \leq R \},$$

where $R$ is a given positive constant.

If we assume that $(u_0, v_0)$ belongs to $(H^2(\Omega))^2$ and $g$, $h$ are sufficiently regular (e.g. $\exists \varepsilon > 0$ such that $g, h \in H^{1}(t_0, T, H^{2+\varepsilon}(\partial \Omega)) \cap H^{2}(t_0, T, H^{2}(\partial \Omega))$, then (1) admits a solution in $H^{1}(t_0, T, H^{2}(\Omega))$ (see [16]). We will later use this regularity result.

Let $\omega$ be a subdomain of $\Omega$. Let $(u, v)$ (resp. $(\tilde{u}, \tilde{v})$) be solution of (1) associated to $(a, b, c, d, u_0, v_0)$ (resp. $(\tilde{a}, \tilde{b}, c, d, \tilde{u}_0, \tilde{v}_0)$) satisfying some regularity and “positivity” properties:

$$\begin{align*}
&\begin{cases}
a, b, c, d, \tilde{b} \in \Lambda(R), \\
\text{There exist two constants } r > 0, \ c_0 > 0 \text{ such that } \\
\tilde{u}_0 \geq 0, \ \tilde{v}_0 \geq r, \ c \geq c_0, \ \tilde{b} > 0, \ c + dr \geq 0, \\
g \geq 0 \text{ and } h \geq r.
\end{cases}
\end{align*}$$

Such assumptions allows us to state that the function $\tilde{v}$ satisfies $|\tilde{v}(x, T')| \geq r > 0$ in $\Omega$ (see [18] thm 14.7 p. 200).

If we set $U = u - \tilde{u}$, $V = v - \tilde{v}$, $y = \partial_t (u - \tilde{u})$, $z = \partial_t (v - \tilde{v})$, $\gamma = b - \tilde{b}$, then $(y, z)$ is solution of

$$\begin{align*}
&\begin{cases}
\partial_t y = \Delta y + ay + bz + \gamma \partial_t \tilde{v} & \text{in } Q_0, \\
\partial_t z = \Delta z + cy + dz & \text{in } Q_0, \\
y(t, x) = z(t, x) = 0 & \text{on } \Sigma_0, \\
y(0, x) = \Delta U(0, x) + aU(0, x) + bV(0, x) + \gamma \tilde{v}(0, x), & \text{in } \Omega, \\
z(0, x) = \Delta V(0, x) + cU(0, x) + dV(0, x) & \text{in } \Omega.
\end{cases}
\end{align*}$$

The key ingredient to these stability results is a global Carleman estimate for a two by two system with one observation. Controllability for such parabolic systems has been studied in [1]. The Carleman estimate obtained in [1] cannot be used to solve the inverse problem of identification of one coefficient and initial conditions because of the weight functions which are different in the left and right hand side of their estimate. We establish a new Carleman estimate with one observation involving the same weight function in the left and right hand side. We prove a stability result for the initial conditions following the method of [20]. Concerning the stability of the initial conditions we use an extension of the logarithmic convexity method (see [9]). We can also cite [14]; he provides an estimate for the initial condition for a general parabolic operator, while the logarithmic convexity method works only for self-adjoint operators (see Theorem 3 in [14]).

The simultaneous reconstruction of one coefficient and initial conditions from the measurement of one solution $v$ over $(t_0, T) \times \omega$ and some measurement at fixed time $T'$ is an essential aspect of our result. In the perspective of numerical reconstruction, such problems are ill-posed. Stability results are thus of importance.

For the first time, the method of Carleman estimates was introduced in the field of inverse problems in the work of Bukhgeim and Klibanov [5]; also see, e.g., [4], [12] and [13] for some follow up publications of these authors. So far, the method of [5] is the only one enabling to prove uniqueness and stability results for inverse problems with single measurement data in the $n$-dimensional case with $n \geq 2$, which has generated many publications, including this one. While this method can provide Hölder stability results, the topic of the Lipschitz stability is a more delicate one. The first Lipschitz stability result for a multidimensional inverse problem (for a hyperbolic equation) was obtained by Puel and Yamamoto [17], using a modification of the idea of [5]. In [10] this result was extended for a parabolic equation. The main difference between our work and [10] is that we consider a coupled system of parabolic equations, and
the additional data are given only for one component of this system, the function \( \partial_t v(x, t) \) for \((x, t) \in \omega \times (t_0, T)\). Inverse problems for parabolic equations are well studied (see [6], [10], [20]). A recent book of Klibanov and Timonov [15] is devoted to the Carleman estimates applied to inverse coefficient problems. In our knowledge, there is no work about inverse problems for coupled parabolic systems.

The used method allows us to give a stability result for the coefficient \( a(x) \) adapting assumption (2). On the other hand, we only measure \( \partial_t v \) on \( \omega \), we cannot obtain such stability results for the coefficients \( c(x) \) or \( d(x) \) of the second equation of (1). For the reconstruction of two coefficients the problem is more complicated. We obtain partial results with restrictive assumptions on the coefficients \( a(x), b(x), c(x) \) and \( d(x) \). In order to avoid such assumptions, we think it is necessary to use other methods such as those used in [11].

In a first step, we derive a global Carleman estimate for system (3) with a single measurement i.e. the measurement of one solution \( v \) over \((t_0, T) \times \omega\). Let us introduce the following notations: let \( \omega' \Subset \omega \) and let \( \tilde{\beta} \) be a \( C^2(\Omega) \) function such that

\[
\tilde{\beta} > 0, \text{ in } \Omega, \quad \tilde{\beta} = 0 \text{ on } \partial \Omega, \quad \min \{|\nabla \tilde{\beta}(x)|, \ x \in \Omega \setminus \omega'\} > 0 \quad \text{and} \quad \partial_t \tilde{\beta} < 0 \text{ on } \partial \Omega.
\]

Then, we define \( \beta = \tilde{\beta} + K \) with \( K = m\|\tilde{\beta}\|_\infty \) and \( m > 1 \). For \( \lambda > 0 \) and \( t \in (t_0, T) \), we define the following weight functions

\[
\varphi(x, t) = \frac{e^{\lambda \beta(x)}}{(t - t_0)(T - t)}, \quad \eta(x, t) = \frac{e^{2\lambda K} - e^{\lambda \beta(x)}}{(t - t_0)(T - t)}.
\]

**Theorem 1 (Carleman Estimate)** We assume \( a, b, c, d \in \Lambda(R) \) and that exists \( c_0 > 0 \) such that \( c \geq c_0 \) in \( \omega \). Then there exist \( \lambda_1 = \lambda_1(\Omega, \omega) \geq 1 \), \( s_1 = s_1(\lambda_1, T) > 1 \) and a positive constant \( C_1 = C_1(\Omega, \omega, c_0, R, T) \) such that, for any \( \lambda \geq \lambda_1 \) and any \( s \geq s_1 \), the following inequality holds:

\[
I(y) + I(z) \leq C_1 \left[s^7 \lambda^8 \int_{t_0}^{T} \int_{\omega} e^{-2\eta \gamma \varphi^7 |\varphi|^2} \, dx \, dt + \int_{\Omega} e^{-2\eta \gamma |\partial_t \tilde{\varphi}|^2} \, dx \, dt\right],
\]

for any solution \((y, z)\) of (3).

In a second step, we establish a stability inequality and deduce a uniqueness result for the coefficient \( b \). This inequality (7) estimates the difference between the coefficients \( b \) and \( \tilde{b} \) with an upper bound given by some Sobolev norms of the difference between the solutions \( v \) associated to \((a, b, c, d, u_0, v_0)\) and \( \tilde{v} \) associated to \((a, b, c, d, \tilde{u}_0, \tilde{v}_0)\).

The Carleman estimate (4) will be the key ingredient in the proof of stability result.

Let \( T' = \frac{1}{2}(T + t_0) \) the point for which \( \Phi(t) = \frac{1}{(t - t_0)(T - t)} \) has its minimum value. The assumption (2) allows us to state that the solution \( \tilde{v} \) is such that \(|\tilde{v}(x, T')| \geq r > 0 \) in \( \Omega \). Furthermore if we assume that \( \tilde{u}_0, \tilde{v}_0 \) in \( H^2(\Omega) \), the solutions \( \tilde{u} \) and \( \tilde{v} \) belong to \( H^1(t_0, T, H^2(\Omega)) \).

Then using classical Sobolev embedding (see [3]), we can write for \( n \leq 3 \), that \( \partial_t \tilde{v} \) belongs to \( L^2(t_0, T, L^\infty(\Omega)) \) and we assume that \( |\partial_t \tilde{v}|^2 \in L^2(t_0, T) \in \Lambda(R) \).

We set \( \psi = e^{-\eta y} \) and using the following operator

\[
M_2 \psi = \partial_t \psi + 2s\lambda \varphi \nabla \beta \nabla \psi + 2s\lambda^2 \varphi |\nabla \beta|^2 \psi,
\]

we introduce, following [2],

\[
\mathcal{I} = \Re \int_{t_0}^{T'} \int_{\Omega} M_2 \psi \, \psi \, dx \, dt
\]

We have the following estimates.
Lemma 2 Let $\lambda \geq \lambda_1$ and $s \geq s_1$ and let $a, b, c, d \in \Lambda(R)$. We assume that assumption (2) is satisfied then there exists a constant $C = C(\Omega, \omega, T)$ such that

$$|I| \leq C s^{-3/2} \lambda^{-2} \left( s^7 \lambda^8 \int_{t_0}^{T} \int_{\omega} e^{-2s\eta} |\varphi|^2 dx \, dt + \int_{t_0}^{T} \int_{\Omega} e^{-2s\eta} |\gamma|^2 |\partial_t \widetilde{v}|^2 dx \, dt \right).$$

Lemma 3 Let $\lambda \geq \lambda_1$, $s \geq s_1$ and let $a, b, c, d \in \Lambda(R)$. Furthermore, we assume that $\tilde{u}_0, \tilde{v}_0$ in $H^2(\Omega)$ and the assumption (2) is satisfied. Then there exists a constant $C = C(\Omega, \omega, T)$ such that

$$\int_{\Omega} e^{-2s\eta(T',x)} |\Delta \, U(T', x) + aU(T', x) + bV(T', x)|^2 dx$$

$$\leq C s^{-3/2} \lambda^{-2} \left( s^7 \lambda^8 \int_{t_0}^{T} \int_{\omega} e^{-2s\eta} |\varphi|^2 dx \, dt + \int_{t_0}^{T} \int_{\Omega} e^{-2s\eta} |\gamma|^2 |\partial_t \widetilde{v}|^2 dx \, dt \right)$$

$$+ C \int_{\Omega} e^{-2s\eta(T',x)} |\Delta \, U(T', x) + aU(T', x) + bV(T', x)|^2 dx.$$

for the proof see [7].

Using the two previous lemmata we establish our main stability result.

Theorem 4 (First Stability Result) Let $\omega$ be a subdomain of an open set $\Omega$ of $\mathbb{R}^n$, let $a, b, c, d, u_0, v_0$ in $H^2(\Omega)$ and the assumption (2) is satisfied. Let $(u, v)$ associated to $(a, b, c, d, u_0, v_0)$ and $(\tilde{u}, \tilde{v})$ associated to $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{u}_0, \tilde{v}_0)$. Then there exists a constant $C$

$$C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0$$

such that

$$|b - \tilde{b}|^2_{L^2(\Omega)} \leq C |\partial_t v - \partial_t \tilde{v}|^2_{L^2((t_0, T) \times \omega)} + C |\Delta u(T', \cdot) - \Delta \tilde{u}(T', \cdot)|^2_{L^2(\Omega)}$$

$$+ C |u(T', \cdot) - \tilde{u}(T', \cdot)|^2_{L^2(\Omega)} + C |v(T', \cdot) - \tilde{v}(T', \cdot)|^2_{L^2(\Omega)}.$$

Remark 5 If we assume that $u(T', \cdot) = \tilde{u}(T', \cdot)$ and $v(T', \cdot) = \tilde{v}(T', \cdot)$ (such an additional assumption is sometimes made, e.g. in [10]), then the stability estimate becomes

$$|b - \tilde{b}|^2_{L^2(\Omega)} \leq C |\partial_t v - \partial_t \tilde{v}|^2_{L^2((t_0, T) \times \omega)}.$$

With Theorem 4 we have the following uniqueness result

Corollary 6 Under the same assumptions as in theorem 4 and if

$$(\partial_t v - \partial_t \tilde{v})(t, x) = 0 \quad \text{in} \quad (t_0, T) \times \omega,$$

$$\Delta u(T', x) - \Delta \tilde{u}(T', x) = 0 \quad \text{in} \quad \Omega,$$

$$v(T', x) - \tilde{v}(T', x) = 0 \quad \text{in} \quad \Omega.$$

Then $b = \tilde{b}$. 
Finally, we prove a stability result for initial conditions. We use the same method as in [20] to state a stability estimate for the initial conditions \( u_0, v_0 \). The idea is to prove logarithmic-convexity inequality. The following method has been used to obtain continuous dependence inequalities in initial value problems. If \((y, z)\) is solution of (3), we introduce \((y_1, z_1)\) and \((y_2, z_2)\) that satisfy

\[
\begin{align*}
\partial_t y_1 &= \Delta y_1 + ay_1 + bz_1 + \gamma \partial_t \tilde{v} \quad \text{in} \ 0, \\
\partial_t z_1 &= \Delta z_1 + cy_1 + dz_1 \quad \text{in} \ 0, \\
y_1(t, x) &= z_1(t, x) = 0 \quad \text{on} \ \Sigma, \\
y_1(0, x) &= 0, \quad \text{in} \ \Omega, \\
z_1(0, x) &= 0 \quad \text{in} \ \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t y_2 &= \Delta y_2 + ay_2 + bz_2 \quad \text{in} \ 0, \\
\partial_t z_2 &= \Delta z_2 + cy_2 + dz_2 \quad \text{in} \ 0, \\
y_2(t, x) &= z_2(t, x) = 0 \quad \text{on} \ \Sigma, \\
y_2(0, x) &= \Delta U(0, x) + aU(0, x) + bV(0, x) + \gamma \tilde{v}(0, x) \quad \text{in} \ \Omega, \\
z_2(0, x) &= \Delta V(0, x) + cU(0, x) + dV(0, x) \quad \text{in} \ \Omega.
\end{align*}
\]

Then, we have

\[ y = y_1 + y_2 \quad \text{and} \quad z = z_1 + z_2. \tag{10} \]

We give an \( L^2 \) estimate for \((y_1, z_1)\):

**Lemma 7** Let \( a, b, c, d \) \( |\partial_t \tilde{v}|_{L^2(t_0, T)} \in \Lambda(R) \). Then there exists a constant

\[ C = C(t_0, T', R) > 0, \]

such that

\[ |y_1(t)|^2_{L^2(\Omega)} + |z_1(t)|^2_{L^2(\Omega)} \leq C|\gamma|^2_{L^2(\Omega)}, \quad t_0 \leq t \leq T'. \tag{11} \]

Then we use a logarithmic-convexity inequality for \((y_2, z_2)\):

**Lemma 8** Let \( a, b, c, d \in \Lambda(R) \) and \( u_0, v_0, \tilde{u}_0, \tilde{v}_0 \) in \( H^1(\Omega) \). Then there exist constants \( M > 0, C = C(R) > 0 \) and \( C_1 = C_1(t_0, T', R) > 0 \) such that

\[ |y_2(t)|^2_{L^2(\Omega)} + |z_2(t)|^2_{L^2(\Omega)} \leq C_1 M^{1-\mu(t)}(|y_2(T')|^2_{L^2(\Omega)} + |z_2(T')|^2_{L^2(\Omega)})^{\mu(t)}, \tag{12} \]

for \( t_0 \leq t \leq T' \), where \( \mu(t) = \left(\frac{e^{-Ct_0} - e^{-Ct}}{e^{-Ct_0} - e^{-Ct'}}\right) \).

For the proofs see [7].

The two previous lemmata allow us to prove the following theorem:

**Theorem 9 (Second Stability Result)** Let \( \omega \) be a subdomain of an open set \( \Omega \) of \( \mathbb{R}^n \), let \( a, b, c, d \in \Lambda(R) \). Furthermore, we assume that \( u_0, v_0, \tilde{u}_0, \tilde{v}_0 \) in \( H^4(\Omega) \) and assumption (2) is satisfied. Let \((u, v)\) associated to \((a, b, c, d, u_0, v_0)\) and \((\tilde{u}, \tilde{v})\) associated to \((a, \tilde{b}, c, d, \tilde{u}_0, \tilde{v}_0)\). We set

\[ E = |\partial_t v - \partial_t \tilde{v}|^2_{L^2(t_0, T) \times \omega} + |u(T', \cdot) - \tilde{u}(T', \cdot)|^2_{H^2(\Omega)} + |v(T', \cdot) - \tilde{v}(T', \cdot)|^2_{H^2(\Omega)}. \]

Then there exists a constant \( C = C(\Omega, \omega, c_0, t_0, T, r, R) > 0 \) such that

\[ |u_0 - \tilde{u}_0|^2_{L^2(\Omega)} + |v_0 - \tilde{v}_0|^2_{L^2(\Omega)} \leq \frac{C}{|\log E|}, \quad \text{for} \quad 0 < E < 1. \]

For the proof see [7].
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