EXISTENCE OF A STABLE BLOW-UP PROFILE FOR THE NONLINEAR HEAT EQUATION WITH A CRITICAL POWER NONLINEAR GRADIENT TERM

SLIM TAYACHI AND HATEM ZAAG\footnote{This author is supported by the ERC Advanced Grant no. 291214, BLOWDISOL and by the ANR project ANAÉ ref. ANR-13-BS01-0010-03.}

ABSTRACT. We consider the nonlinear heat equation with a nonlinear gradient term: $\partial_t u = \Delta u + \mu |\nabla u|^q + |u|^{p-1}u$, $\mu > 0$, $q = 2p/(p+1)$, $p > 3$, $t \in (0,T)$, $x \in \mathbb{R}^N$. We construct a solution which blows up in finite time $T > 0$. We also give a sharp description of its blow-up profile and show that it is stable with respect to perturbations in initial data. The proof relies on the reduction of the problem to a finite dimensional one, and uses the index theory to conclude. The blow-up profile does not scale as $(T - t)^{1/2} |\log(T - t)|^{1/2}$, like in the standard nonlinear heat equation, i.e. $\mu = 0$, but as $(T - t)^{1/2} |\log(T - t)|^\beta$ with $\beta = (p + 1)/(2(p - 1)) > 1/2$. We also show that $u$ and $\nabla u$ blow up simultaneously and at a single point, and give the final profile. In particular, the final profile is more singular than the case of the standard nonlinear heat equation.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

We consider the problem
\begin{equation}
\partial_t u = \Delta u + \mu |\nabla u|^q + |u|^{p-1}u, \\
(1.1)
\end{equation}
where $u = u(x,t) \in \mathbb{R}$, $t \in [0,T)$, $x \in \mathbb{R}^N$, and the parameters $\mu$, $p$ and $q$ are such that
\begin{equation}
\mu > 0, p > 3, q = \frac{2p}{p+1}. \\
(1.2)
\end{equation}

Equation (1.1) is wellposed in $W^{1,\infty}(\mathbb{R}^N)$. See [2] Proposition 4.1, p. 18, [35] Corollary 3, p. 67] and [36] for $\mu < 0$. The proof for the case $\mu > 0$ follows similarly, using a fixed point argument. Precisely, there exists a unique maximal solution on $[0,T)$ of (1.1) with $T \leq \infty$. Moreover, according to [2], since $p > 1$ and $1 < q < 2$, nonglobal solutions, i.e. solutions with $T < \infty$, blow up in the $L^\infty$-norm.

The value $q = 2p/(p+1)$ is a critical exponent for the equation (1.1) for different reasons. One reason is that, when $q = 2p/(p+1)$, equation (1.1) is invariant under the transformation: $u_\lambda(t,x) = \lambda^{2/(p-1)}u(\lambda^2t, \lambda x)$, as for the equation without the gradient term, that is $\mu = 0$. Another reason is related to the behavior of global or blowing up solutions. Indeed, it is proved in [30] and [29] that when $\mu \neq 0$, the large time behavior of global solutions of (1.1) depends on the position of $q$ with respect to $2p/(p+1)$. On the other hand, it is pointed out in some previous works that the position of $q$ with respect to $2p/(p+1)$ has an influence on the behavior of blowing up solutions for $\mu < 0$. See [5] [6] [31] [32] [33] [34] [35] and references therein.

Many works has been devoted to the blow-up profiles for the equation without the gradient term, i.e. (1.1) with $\mu = 0$. See [4] [22] [3] [17] and references therein. But, few results are known for the equation with $\mu \neq 0$. For self-similar blow-up profiles, see [34] when $q = 2p/(p+1)$, $\mu < 0$ and [12] [13] when $q = 2$, $\mu > 0$. A blow-up profile is derived in [8] for the equation (1.1) when $q < 2p/(p+1)$. This
Theorem 1 (Blow-up profile for Equation (1.1)). Let $\mu > 0$ and $p, q$ be two real numbers such that

$$q = \frac{2p}{p+1} \quad \text{and} \quad p > 3. \quad (1.3)$$

Then, for any $\varepsilon > 0$, Equation (1.1) has a solution $u(x,t)$ such that $u$ and $\nabla u$ blow up in finite time $T > 0$ simultaneously. Moreover:

(i) For all $t \in [0,T)$,

$$\left\| (T-t)^{\frac{1}{p-1}} u(y\sqrt{T-t}, t) - \varphi^0 \left( \frac{y}{\log(T-t)^{\frac{1}{2}}} \right) \right\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C \frac{1}{1 + |\log(T-t)|^{\min\left(\frac{2}{p-1}, \frac{p-3}{2(p-1)}\right) - \varepsilon}}, \quad (1.4)$$

where

$$\varphi^0(z) = (p - 1 + b|z|^2)^{-\frac{1}{p-1}}, \quad z \in \mathbb{R}^N, \quad (1.5)$$

and $C$ is a positive constant.

(ii) The functions $u$ and $\nabla u$ blow up at the origin and only there.

(iii) For all $x \neq 0$, $u(x,t) \to u^*(x)$ as $t \to T$ in $C^1 \left( \frac{1}{R} < |x| < R \right)$ for any $R > 0$, with

$$u^*(x) \sim \left( \frac{b|x|^2}{2|\log|x||^{\frac{p+1}{p-1}}} \right)^{\frac{1}{p-1}}, \quad \text{as} \quad x \to 0, \quad (1.7)$$

and for $|x|$ small,

$$|\nabla u^*(x)| \leq C \frac{|x|^{\frac{p+1}{p-1}}}{|\log{|x||}^{\frac{1-3p}{(p-1)^2} - \varepsilon}}, \quad \text{if} \quad 3 < p \leq 7, \quad (1.8)$$

$$|\nabla u^*(x)| \leq C \frac{|x|^{\frac{p+1}{p-1}}}{|\log{|x||}^{\frac{-p^2+2p-5}{2(p-1)^2} - \varepsilon}}, \quad \text{if} \quad p > 7, \quad (1.9)$$

where $C$ is a positive constant.

Remark 1.1. From the previous Theorem, we have for all $t \in [0,T)$,

$$\left\| (T-t)^{1/(p-1)} u(x,t) - \left( p - 1 + \frac{b|x|^2}{(T-t)|\log(T-t)|^{(p+1)/(p-1)}} \right)^{-1/(p-1)} \right\|_{L^\infty(\mathbb{R}^N)} \leq C \frac{1}{1 + |\log(T-t)|^{\min\left(\frac{2}{p-1}, \frac{p-3}{2(p-1)}\right) - \varepsilon}},$$

where $C$ is a positive constant.
Remark 1.2. To have a flavor of the appearance of the particular shape for our profile $\phi^0$ in (1.5) together with the scaling factor $\beta$ and the parameter $b$ in (1.6), see the formal approach in Section 2 below. However, we would like to emphasize the fact that in the actual proof, those particular values are crucially needed in various algebraic identities. See the Remark 1.11 following Lemma 1.10 below and Proposition 1.18 below.

Remark 1.3. The initial data giving rise to the constructed solution is given in Proposition 1.2 below.

Remark 1.4. Note that the solution constructed in the above theorem does not exist in the case of the standard nonlinear heat equation, i.e. when $\mu = 0$ in (1.1). Indeed, our solution has a profile depending on the reduced variable

$$z = \frac{x}{\sqrt{T-t} |\log(T-t)|^q}$$

whereas, we know from the results in [37, 38] that the blow-up profiles in the case $\mu = 0$ depend on the reduced variables

$$z = \frac{x}{\sqrt{T-t} |\log(T-t)|^q} \text{ or } z = \frac{x}{(T-t)^{\frac{1}{2m}}}, \text{ where } m \geq 2 \text{ is an integer.}$$

Remark 1.5. We conjecture that identity (1.7) holds also after differentiation. Unfortunately, we have been able to derive only the weaker results given in (1.8) and (1.9).

Remark 1.6. In the case $\mu = 0$, the final profile of the standard nonlinear heat equation is given by

$$u^*_0(x) \sim C \left(\frac{|x|^2}{|\log |x||}\right)^{-\frac{1}{p-1}}, \text{ as } x \to 0,$$

(1.10)

where $C$ is a positive constant (see [30]). In our case, ($\mu > 0$), the final profile is given by (1.7). Let us denote it by $u^*_\mu$. Since $1 < q < 2$, $u^*_\mu$ is more singular than $u^*_0$ for $x$ close to 0, in the sense that

$$u^*_0(x) \ll u^*_\mu(x), \text{ as } x \to 0.$$ 

This shows the effect of the forcing gradient term in equation (1.1) with $\mu > 0$ in the equation. On the other hand, $u^*_\mu$ and $u^*_0$ have the same singularity near $x = 0$ for the limiting case $q = 2$ (that is when $p \to \infty$).

Let us note that in [31] and with $\mu < 0$ in the equation (1.1), that is with a damping gradient term in the equation, a less singular finale profile is obtained: $v^*_\mu(x) \sim |x|^{-\frac{2}{p-1}}$ as $x \to 0$.

Remark 1.7. We strongly believe that our strategy breaks down when $1 < p \leq 3$. See the remark following Lemma 1.10 below.

As a consequence of our techniques, we show the stability of the constructed solution, with respect to perturbations in initial data. More precisely, we have obtained the following result.

Theorem 2 (Stability of the blow-up profile (1.1)). Let $\mu > 0$ and $p$, $q$ be two real numbers such that

$$p > 3 \text{ and } q = \frac{2p}{p+1}.$$ 

(1.11)

Let $\hat{u}$ be the solution of (1.1) given by Theorem 1 with initial data $\hat{u}_0$ and which blows up at time $\hat{T}$. Then, there exists a neighborhood $V_0$ of $\hat{u}_0$ in $W^{1,\infty}(\mathbb{R}^N)$ such that for any $u_0 \in V_0$, Equation (1.1) has a unique solution $u$ with initial data $u_0$, $u$ blows up in finite time $T(u_0)$ and at a single point $a(u_0)$. Moreover, items (i)-(iii) of Theorem 1 are satisfied by $u(x-a,t)$ and

$$T(u_0) \to \hat{T}, a(u_0) \to 0, \text{ as } u_0 \to \hat{u}_0 \text{ in } W^{1,\infty}(\mathbb{R}^N).$$

Remark 1.8. In fact, we have a stronger version of the stability theorem, valid for single-point blow-up solutions of equation (1.1), enjoying the profile (1.5) only for a sequence of time. See Theorem 2 in page 58 below.
Let us give an idea of the methods used to prove the results. We construct the blow-up solution with the profile in Theorem 1 by following the methods of [4] and [22], though we are far from a simple adaptation, since the gradient term needs genuine new ideas as we explain shortly below. This kind of methods has been applied for various nonlinear evolution equations. For hyperbolic equations, it has been successfully used for the construction of multi-solitons for the semilinear wave equation in one space dimension (see [7]). For parabolic equations, it has been used in [19] and [40] for the complex Ginzburg-Landau equation with no gradient structure. See also the cases of the wave maps in [25], the Schrödinger maps in [21], the critical harmonic heat follow in [26], the two-dimensional Keller-Segel equation in [27] and the nonlinear heat equation involving a subcritical nonlinear gradient term in [8]. Recently, this method has been applied for a non variational parabolic system in [24] and for a logarithmically perturbed nonlinear heat equation in [23].

Unlike in the subcritical case in [8], the gradient term in the critical case induces substantial changes in the blow-up profile as we pointed-out in the comments following Theorem 1. Accordingly, its control requires special arguments. So, working in the framework of [22], some crucial modifications are needed. In particular, we have to overcome the following challenges:

- The prescribed profile is not known and not obvious to find. See Section 2 for a formal approach to justify such a profile, and the introduction of the parameter \( \beta \) given by (2.22) below.
- The profile is different from the profile in [22], hence also from all the previous studies in the parabolic case ([22, 8, 23, 24]). Therefore, brand new estimates are needed. See Section 4 below.
- In order to handle the new parameter \( \beta \) in the profile, we introduce a new shrinking set to trap the solution. See Definition 4.2 below. Finding such a set is not trivial, in particular the limitation \( p > 3 \) in related to the choice of such a set.
- A good understanding of the dynamics of the linearized operator of equation (2.2) below around the new profile is needed, taking into account the new shrinking set.
- Some crucial global and pointwise estimates of the gradient of the solution as well as fine parabolic regularity results are needed (see Section 4.2.3 below).

Then, following [22], the proof is divided in two steps. First, we reduce the problem to a finite dimensional one. Second, we solve the finite dimensional problem and conclude by contradiction, using index theory.

To prove the single point blow-up result for the constructed solution, we establish a new “no blow-up under some threshold” criterion for a parabolic inequality with a nonlinear gradient term. See Proposition 5.1 below. The final blow-up profile is determined then using the method of [40], Proposition 5.1 and [20].

The stability result, Theorem 2, is proved similarly as in [22] by interpreting the finite dimensional problem in terms of the blow-up time and the blow-up point.

Let us remark that if \( u \) is a solution of equation (1.1), then

\[
\overline{u}(x,t) = u\left(t, \frac{\mu^{\frac{1}{2-q}}}{t^{\frac{2-q}{2}}} x \right), t \in \left[0, \mu^{\frac{1}{2-q}} T\right], x \in \mathbb{R}^N,
\]

is a solution of the equation

\[
\partial_t \overline{u} = \Delta \overline{u} + |\nabla \overline{u}|^q + \mu^{-\frac{2}{2-q}} |\overline{u}|^{p-1} \overline{u}.
\]

Also,

\[
\tilde{u}(x,t) = u\left(\mu^{-1} t, x \right), t \in [0, \mu T], x \in \mathbb{R}^N,
\]

is solution of the equation

\[
\partial_t \tilde{u} = \mu^{-1} \Delta \tilde{u} + |\nabla \tilde{u}|^q + \mu^{-1} |\tilde{u}|^{p-1} \tilde{u}.
\]

And for \( \delta \neq 1 \),

\[
\underline{u}(x,t) = \lambda^{2/(p-1)} u(\lambda^{\delta} x, \lambda^2 t), \lambda = \mu^{\frac{1}{\delta}} \mu^{-\frac{1}{p-1}}, t \in \left[0, \mu^{-\frac{2}{p-1-q}} T\right], x \in \mathbb{R}^N,
\]
is solution of the equation
\[ \partial_t u = \mu^{-2/q} \Delta u + |\nabla u|^q + |u|^{p-1} u. \]

Then, since Theorem 1 is valid for all \( \mu > 0 \), we obtain the blow-up profile for a perturbed Viscous Hamilton-Jacobi (VHJ) equation, as well as for a perturbed Hamilton-Jacobi (HJ) equation, and nonlinear Hamilton-Jacobi (NHJ) equation. More precisely, this is our statement:

**Corollary 3** (Blow-up in the Hamilton-Jacobi style). Theorems 1 and 2 yield stable blow-up solutions:

1. **For the perturbed VHJ equation:**
   \[ \partial_t u = \Delta u + |\nabla u|^q + \nu |u|^{p-1} u, \quad \text{with } \nu > 0, \ 3/2 < q < 2, \ p = \frac{q}{2 - q}. \]

2. **For the perturbed HJ equation:**
   \[ \partial_t u = |\nabla u|^q + \nu' \Delta u + \nu'|u|^{p-1} u, \quad \text{with } \nu' > 0, \ 3/2 < q < 2, \ p = \frac{q}{2 - q}. \]

3. **For the perturbed NHJ equation:**
   \[ \partial_t u = |\nabla u|^q + |u|^{p-1} u + \nu'' \Delta u, \quad \text{with } \nu'' > 0, \ 3/2 < q < 2, \ p = \frac{q}{2 - q}. \]

In the three cases, the solutions and their gradients blow up simultaneously and only at one point. The blow-up profile is given by (1.4) with appropriate scaling.

The organization of the rest of this paper is as follows. In Section 2, we explain formally how we obtain the the profile and the exponent \( \beta \). In Section 3, we give a formulation of the problem in order to justify the formal argument. Section 4 is divided into two subsections: In subsection 4.1 we give the proof of the existence of the profile assuming the technical results. In particular, we construct a shrinking set and give an example of initial data giving rise to the prescribed blow-up profile. Subsection 4.2 is devoted to the proof of the technical results which are needed in the proof of the existence. Section 5 is devoted to the proof of the single point blow-up and the determination of the final profile. In particular, a new “no blow-up under some threshold” is established for parabolic equations (or inequalities) with nonlinear gradient terms. See Proposition 5.4 below. Finally, in Section 6, we prove the stability result, that is Theorem 2 and give a more general stability statement (see Theorem 2 page 58). In all the paper the notation \( A \ll B \) for positive real numbers \( A \) and \( B \) means that \( A \) is very smaller with respect to \( B \).

2. A Formal Approach

The aim of this section is to explain formally how we derive the behavior given in Theorem 1. In particular, how we obtain the profile \( \varphi^0 \) in (1.5), the parameter \( b \) and the exponent \( \beta = 2(p + 1)/(p - 1) \) in (1.6). Consider an arbitrary \( T > 0 \) and the self-similar transformation of (1.1)

\[ w(y, s) = (T - t)^{-1 - \frac{1}{p-1}} u(x, t), \quad y = \frac{x}{\sqrt{T - t}}, \quad s = - \log (T - t). \quad (2.1) \]

It follows that if \( u(x, t) \) satisfies (1.1) for all \( (x, t) \in \mathbb{R}^N \times [0, T) \), then \( w(y, s) \) satisfies the following equation:

\[ \partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p - 1} w + \mu |\nabla w|^q + |w|^{p-1} w, \quad (2.2) \]

for all \( (y, s) \in \mathbb{R}^N \times [- \log T, \infty) \). Thus, constructing a solution \( u(x, t) \) for the equation (1.1) that blows up at \( T < \infty \) like \( (T - t)^{-\frac{1}{p-1}} \) reduces to constructing a global solution \( w(y, s) \) for equation (2.2) such that

\[ 0 < \varepsilon \leq \limsup_{s \to \infty} \|w(s)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{1}{\varepsilon}. \quad (2.3) \]
A first idea to construct a blow-up solution for (1.1), would be to find a stationary solution for (2.2), yielding a self-similar solution for (1.1). It happens that when \( \mu < 0 \) and \( p \) is close to 1, the first author together with Souplet and Weissler were able in [34] to construct such a solution. Now, if \( \mu > 0 \), we know, still from [34] that it is not possible to construct such a solution in some restrictive class of solutions (see [34, Remark 2.1, p. 666]), of course, apart from the trivial constant solution \( w \equiv \kappa \) of (2.2), where

\[
\kappa = \left( \frac{1}{p-1} \right)^{\frac{1}{p-1}}.
\]  

### 2.1. Inner expansion.

Following the approach of Bricmont and Kupiainen in [4], we may look for a solution \( w \) such that \( w \to \kappa \) as \( s \to \infty \). Writing \( w = \kappa + \overline{w} \), we see that \( \overline{w} \to 0 \) as \( s \to \infty \) and satisfies the equation:

\[
\partial_s \overline{w} = \mathcal{L} \overline{w} + B(\overline{w}) + \mu |\nabla \overline{w}|^q,
\]

where

\[
\mathcal{L} = \Delta - \frac{1}{2} y \cdot \nabla + 1,
\]

and

\[
B(\overline{w}) = |\overline{w} + \kappa|^{p-1}(\overline{w} + \kappa) - \kappa^p - p\kappa^{p-1}w.
\]

Note that

\[
|B(\overline{w}) - \frac{p}{2\kappa}w^2| \leq C|w^3|,
\]

where \( C \) is a positive constant.

Let us recall some properties of \( \mathcal{L} \). The operator \( \mathcal{L} \) is self-adjoint in \( D(\mathcal{L}) \subset L^2_\rho(\mathbb{R}^N) \) where

\[
L^2_\rho(\mathbb{R}^N) = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} (f(y))^2 \rho(y)dy < \infty \right\}
\]

and

\[
\rho(y) = \frac{e^{-|y|^2}}{(4\pi)^N/2}, \quad y \in \mathbb{R}^N.
\]

The spectrum of \( \mathcal{L} \) is explicitly given by

\[
\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2} \mid m \in \mathbb{N} \right\}.
\]

It consists only in eigenvalues. For \( N = 1 \), all the eigenvalues are simple, and the eigenfunctions are dilations of Hermite polynomials: the eigenvalue \( 1 - \frac{m}{2} \) corresponds to the following eigenfunction:

\[
h_m(y) = \sum_{n=0}^{[m/2]} \frac{m!}{n!(m-2n)!} (-1)^n y^{m-2n}.
\]  

In particular \( h_0(y) = 1, h_1(y) = y \) and \( h_2(y) = y^2 - 2 \). Notice that \( h_m \) satisfies:

\[
\int_{\mathbb{R}} h_n h_m \rho dx = 2^n n! \delta_{nm} \quad \text{and} \quad \mathcal{L} h_m = \left( 1 - \frac{m}{2} \right) h_m.
\]

We also introduce

\[
k_m = \frac{h_m}{\|h_m\|^2_{L^2_\rho(\mathbb{R})}}.
\]

For \( N \geq 2 \), the eigenspace corresponding to \( \lambda = 1 - \frac{m}{2} \) is given by

\[
\{h_{m_1}(y_1) \ldots h_{m_N}(y_N) \mid m_1 + \cdots + m_N = m \}.
\]
In particular, for $\lambda = 1$, the eigenspace is $\{1\}$, for $\lambda = \frac{1}{2}$, it is $\{y_i \mid i = 1, \ldots, N\}$, for $\lambda = 0$, it is given by $\{h_2(y_i), y_i y_j \mid i = 1 \ldots N, i \neq j\}$.

In compliance with the spectral properties of $\mathcal{L}$, we may look for a solution expanded as follows:

$$\bar{w}(y, s) = \sum_{(m_1, \ldots, m_N) \in \mathbb{N}^N} \bar{w}_{(m_1, \ldots, m_N)}(s) h_{m_1}(y_1) \ldots h_{m_N}(y_N).$$

Since the eigenfunctions for $m_1 + \cdots + m_N \geq 3$ correspond to negative eigenvalues of $\mathcal{L}$, assuming $\bar{w}$ radial, we may consider that

$$\bar{w}(y, s) = \bar{w}_0(s) + \bar{w}_2(s) \sum_{i=1}^{N} h_2(y_i) = \bar{w}_0(s) + \bar{w}_2(s)(|y|^2 - 2N), \quad (2.10)$$

with $\bar{w}_0, \bar{w}_2 \to 0$ as $s \to \infty$.

Projecting Equation (2.5), and writing $\mu |\nabla \bar{w}|^q = \mu 2^q |y|^q |\bar{w}_2|^q$, we derive the following ODE system for $\bar{w}_0$ and $\bar{w}_2$:

$$\begin{align*}
\bar{w}_0' &= \bar{w}_0 + \frac{p}{2\kappa} (\bar{w}_0^2 + 8N \bar{w}_2^2) + \tilde{c}_0 \bar{w}_2^q + O \left( |\bar{w}_0|^3 + |\bar{w}_2|^3 \right), \\
\bar{w}_2' &= 0 + \frac{p}{\kappa} (\bar{w}_0 \bar{w}_2 + 4 \bar{w}_2^2) + \tilde{c}_2 \bar{w}_2^q + O \left( |\bar{w}_0|^3 + |\bar{w}_2|^3 \right),
\end{align*}$$

where

$$\tilde{c}_0 = \mu 2^q \int_{\mathbb{R}^N} |y|^q \rho \quad \text{and} \quad \tilde{c}_2 = \frac{\mu 2^q}{8N} \int_{\mathbb{R}^N} |y|^q (|y|^2 - 2N) \rho.$$

Note that for this calculation, we need to know the values of

$$\begin{align*}
\int_{\mathbb{R}^N} (|y|^2 - 2N)^2 \rho(y) dy &= N \int_{\mathbb{R}} (|\xi|^2 - 2)^2 \rho(\xi) d\xi = 8N, \\
\int_{\mathbb{R}^N} (|y|^2 - 2N^2) \rho(y) dy &= N \int_{\mathbb{R}} (|\xi|^2 - 2)^3 \rho(\xi) d\xi = 64N.
\end{align*}$$

Note also that the sign of $\tilde{c}_0$ and $\tilde{c}_2$ is the same as for $\mu$. Indeed, obviously $\int_{\mathbb{R}^N} |y|^q \rho(y) dy > 0$, and for $\int_{\mathbb{R}^N} |y|^q (|y|^2 - 2N) \rho(y) dy$, using the radial coordinate $r = |y|$ and an integration by parts, we write

$$\frac{8N \tilde{c}_2}{2^q \mu} = \int_{\mathbb{R}^N} |y|^q (|y|^2 - 2N) \rho(y) dy = \int_{\mathbb{R}^N} |y|^{q+2} \rho(y) dy - 2N \int_{\mathbb{R}^N} |y|^q \rho(y) dy$$

$$= 2(q + N) \int_{\mathbb{R}^N} |y|^q \rho(y) dy - 2N \int_{\mathbb{R}^N} |y|^q \rho(y) dy = 2q \int_{\mathbb{R}^N} |y|^q \rho(y) dy > 0. \quad (2.11)$$

From the equation on $\bar{w}_2$, we write

$$\bar{w}_2' = \tilde{c}_2 |\bar{w}_2|^q \left( 1 + O \left( |\bar{w}_2|^{2-q} \right) \right) + \frac{p}{\kappa} \bar{w}_0 |\bar{w}_2| + O \left( |\bar{w}_0|^3 \right),$$

and assuming that

$$|\bar{w}_0| \ll |\bar{w}_2|^q, \quad |\bar{w}_0|^3 \ll |\bar{w}_2|^q, \quad (2.12)$$

we get that

$$\bar{w}_2' \sim \text{sign}(\mu) |\tilde{c}_2| |\bar{w}_2|^q,$$

with $\text{sign}(\mu) = 1$ if $\mu > 0$ and $-1$ if $\mu < 0$.

In particular, if $\mu > 0$, then $\bar{w}_2$ is increasing tending to $0$ as $s \to \infty$ hence $\bar{w}_2 < 0$, while if $\mu < 0$, $\bar{w}_2$ is decreasing tending to $0$ as $s \to \infty$, hence $\bar{w}_2 > 0$. Then, since $1 < q < 2$, we get

$$\bar{w}_2 \sim -\text{sign}(\mu) \frac{B}{s^{q-1}},$$

with

$$B = [(q-1)|\tilde{c}_2|]^{\frac{1}{q-1}} = \left[ \frac{2^q-2}{N} q(q-1)|\mu| \int_{\mathbb{R}} |y|^q \rho \right]^{-\frac{1}{q-1}}. \quad (2.13)$$
From (2.11), we write
\[ w' = w_0 (1 + O(w_0)) + \tilde{c}_0|w_2|^q \left( 1 + O\left(|w_2|^{2-q}\right) \right), \]
and assuming that
\[ |w'| \ll w_0, \quad |w'| \ll |w_2|^q, \tag{2.14} \]
we derive that
\[ w_0 \sim -\tilde{c}_0|w_2|^q \sim -\tilde{c}_0 B_{\frac{q}{\nu}} \ll |w_2|. \]
Such \( w_0 \) and \( w_2 \) are compatible with the hypotheses (2.12) and (2.14).

Therefore, since \( w = \kappa + w_2 \), it follows from (2.10) that
\[ w(y,s) = \kappa + w_2(s) \left(|y|^2 - 2N\right) + O(w_2) \]
\[ = \kappa - \frac{\text{sign}(\mu)}{s^{\frac{q}{\nu}}} B(|y|^2 - 2N) + O\left(\frac{1}{s^{\frac{q}{\nu}}}\right) \]
\[ = \kappa - \text{sign}(\mu)B \frac{|y|^2}{s^{\frac{q}{\nu}}} + 2N \frac{s^\mu}{s^{\frac{q}{\nu}}} B + O\left(\frac{1}{s^{\frac{q}{\nu}}}\right), \tag{2.15} \]
in \( L^2(\mathbb{R}^N) \), and also uniformly on compact sets by standard parabolic regularity.

2.2. Outer expansion. From (2.15), we see that the variable
\[ z = \frac{y}{s^{\nu}}, \quad \text{with } \beta = \frac{1}{2} \left(\frac{q}{\nu} - 1\right) = \frac{p}{2} + \frac{1}{2(p-1)}, \]
as given in (1.6), is perhaps the relevant variable for blow-up. Unfortunately, (2.15) provides no shape, since it is valid only on compact sets (note that \( z \to 0 \) as \( s \to \infty \) in this case). In order to see some shape, we may need to go further in space, to the “outer region”, namely when \( z \neq 0 \).

In view of (2.15), we may try to find an expression of \( w \) of the form
\[ w(y,s) = \varphi^0(z) + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right), \tag{2.16} \]
for some \( \nu > 2\beta \). Plugging this ansatz in equation (2.2), keeping only the main order, we end-up with the following equation on \( \varphi^0(z) \):
\[ -\frac{1}{2} z \cdot \nabla \varphi^0(z) - \frac{1}{p-1} \varphi^0(z_0) + [\varphi^0(z)]^p = 0, \quad z = \frac{y}{s^{\beta}}. \tag{2.17} \]
Recalling that our aim is to find \( w \) a solution of (2.2) such that \( w \to \kappa \) as \( s \to \infty \) (in \( L_\rho^2(\mathbb{R}^N) \), hence uniformly on every compact set), we derive from (2.16) (with \( y = z = 0 \)) the natural condition
\[ \varphi^0(0) = \kappa. \]
Recalling also that we already adopted radial symmetry for the inner equation, we do the same here. Therefore, integrating equation (2.17), we see that
\[ \varphi^0(z) = \left(p - 1 + b|z|^2\right)^{-\frac{1}{p-1}}, \tag{2.18} \]
for some \( b \in \mathbb{R} \). Recalling also that we want a solution \( w \in L^\infty(\mathbb{R}^N) \), (see (2.3)), we see that \( b \geq 0 \) and for a nontrivial solution, we should have
\[ b > 0. \tag{2.19} \]

Thus, we have just obtained from (2.16) that
\[ w(y,s) = \left(p - 1 + b|z|^2\right)^{-\frac{1}{p-1}} + \frac{a}{s^{2\beta}} + O\left(\frac{1}{s^\nu}\right), \quad \text{with } z = \frac{y}{s^{\beta}} \text{ and } \nu > 2\beta. \tag{2.20} \]
We should understand this expansion to be valid at least on compact sets in $z$, that is for $|y| < Rs^\beta$, for any $R > 0$.

2.3. Matching asymptotics. Since (2.20) holds for $|y| < Rs^\beta$, for any $R > 0$, it holds also uniformly on compact sets, leading to the following expansion for $y$ bounded:

$$w(y, s) = \kappa - \frac{\kappa b}{(p-1)^2} \frac{|y|^2}{s^{2\beta}} + \frac{a}{s^{2\beta}} + O \left( \frac{1}{s^\nu} \right).$$

Comparing with (2.15), we find the following values for $b$ and $a$:

$$b = \text{sign}(\mu) \frac{B(p-1)^2}{\kappa} \quad \text{and} \quad a = 2N \text{sign}(\mu) B.$$

In particular, from (2.19) we see that $\mu > 0$.

In conclusion, using (2.13), we see that we have just derived the following profile for $w(y, s)$:

$$w(y, s) \sim \varphi(y, s)$$

with

$$\varphi(y, s) = \varphi^0 \left( \frac{y}{s^\beta} \right) + \frac{a}{s^{2\beta}} := \left( p - 1 + \frac{b}{s^{2\beta}} \right)^{\frac{p-1}{p-1}} + \frac{a}{s^{2\beta}}. \quad (2.21)$$

$$\beta = \frac{p+1}{2(p-1)}, \quad (2.22)$$
$$a = \frac{2Nbk}{(p-1)^2}, \quad (2.23)$$

$$b = \frac{1}{2}(p-1)^{\frac{p-2}{p-1}} \left( \frac{(4\pi)^{\frac{N}{2}}}{p \int_{\mathbb{R}^N} |y|^{q} e^{-|y|^2/4} dy} \right)^{\frac{p+1}{p-1}} \mu^{-(p+1)/(p-1)}, \quad (2.24)$$

3. Formulation of the problem

In this section we formulate the problem in order to justify the formal approach given in the previous section. Very soon, actually starting from (3.12) given below, we will only focus on the case $N = 1$ for simplicity. The proof in higher dimensions is no more difficult.

Let $w$, $y$ and $s$ be as in (2.1). Let us introduce $v(y, s)$ such that

$$w(y, s) = \varphi(y, s) + v(y, s), \quad (3.1)$$

where $\varphi$ is given by (2.21). If $w$ satisfies the equation (2.2), then $v$ satisfies the following equation:

$$\partial_s v = (\mathcal{L} + V)v + B(v) + G(v) + R(y, s), \quad (3.2)$$

where $\mathcal{L}$ is defined by (2.6) and

$$V(y, s) = p \varphi(y, s)^{p-1} - \frac{p}{p-1}, \quad (3.3)$$
$$B(v) = |\varphi + v|^{p-1}(\varphi + v) - \varphi^p - p\varphi^{p-1}v, \quad (3.4)$$
$$R(y, s) = \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{\varphi}{p-1} + \varphi^p - \frac{\partial \varphi}{\partial s} + \mu |\nabla \varphi|^q \quad (3.5)$$

and

$$G(v) = \mu |\nabla \varphi + \nabla v|^q - \mu |\nabla \varphi|^q. \quad (3.6)$$
Our aim is to construct initial data \( v(y,s) \) such that the equation (3.2) has a solution \( v(y,s) \) defined for all \( (y,s) \in \mathbb{R}^N \times [-\log T, \infty) \), and satisfies:
\[
\lim_{s \to \infty} \| v(s) \|_{W^{1,\infty}(\mathbb{R}^N)} = 0. \tag{3.7}
\]
From Equation (2.21), one sees that the variable \( z = \frac{y}{s} \) plays a fundamental role. Thus we will consider the dynamics for \( |z| > K \) and \( |z| < 2K \) separately for some \( K > 0 \) to be fixed large. Since
\[
|B(v)| \leq C|v|^2, \quad \| R(\cdot, s) \|_{L^\infty} \leq \frac{C}{s}, \quad \| G(v) \|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{s}} \| v \|_{L^\infty(\mathbb{R})}, \tag{3.8}
\]
for \( s \) large enough, (see (4.23), (4.25) and (4.38) below), it is then reasonable to think that the dynamics of equation (3.2) are influenced by the linear part, namely \( L + V \).

The properties of the operator \( L \) were given in Section 2. In particular, \( L \) is predominant on all the modes, except on the null modes where the terms \( V \) and \( G(v) \) play a crucial role (see item (ii) in Proposition 4.18 below).

As for the potential \( V \), it has two fundamental properties which will strongly influence our strategy:

(i) we have \( V(\cdot, s) \to 0 \) in \( L^2(\mathbb{R}) \) when \( s \to \infty \). In practice, the effect of \( V \) in the blow-up area \(|y| \leq Cs^\beta\) is regarded as a perturbation of the effect of \( L \) (except on the null mode).

(ii) outside of the blow-up area, we have the following property: for all \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) and \( s_\epsilon \) such that
\[
\sup_{s \geq s_\epsilon, |y| \geq C_\epsilon} \left| V(y,s) - \left( -\frac{p}{p-1} \right) \right| \leq \epsilon,
\]
with \(-\frac{p}{p-1} < -1\). As 1 is the largest eigenvalue of the operator \( L \), outside the blow-up area we can consider that the operator \( L + V \) is an operator with negative eigenvalues, hence, easily controlled.

Considering the fact that the behavior of \( V \) is not the same inside and outside the blow-up area, we decompose \( v \) as follows. Let us consider a non-increasing cut-off function \( \chi_0 \in C^\infty([0,\infty),[0,1]) \) such that \( \text{supp}(\chi_0) \subset [0,2] \) and \( \chi_0 \equiv 1 \) in \([0,1] \), and introduce
\[
\chi(y,s) = \chi_0 \left( \frac{|y|}{Ks^\beta} \right) \tag{3.9}
\]
with \( K = \max(6, K_5) \) and \( K_5 = K_5(N,p,\mu) \) is some large enough constant introduced below in Lemma 1.20.

Then, we write
\[
v(y,s) = v_b(y,s) + v_e(y,s), \tag{3.10}
\]
with
\[
v_b(y,s) = v(y,s)\chi(y,s) \quad \text{and} \quad v_e(y,s) = v(y,s)(1 - \chi(y,s)). \tag{3.11}
\]
We remark that
\[
\text{supp } v_b(s) \subset B(0,2Ks^\beta), \quad \text{supp } v_e(s) \subset \mathbb{R}^N \setminus B(0,Ks^\beta).
\]
As for \( v_b \), we will decompose it according to the sign of the eigenvalues of \( L \), by writing
\[
v_b(y,s) = \sum_{m=0}^{2} v_m(s)h_m(y) + v_-(y,s), \tag{3.12}
\]
where for \( 0 \leq m \leq 2 \), \( v_m = P_m(v_b) \) and \( v_-(y,s) = P_-(v_b) \), with \( P_m \) the \( L^2_\rho \) projector on \( h_m \), the eigenfunction corresponding to \( \lambda = 1 - \frac{m}{N} \geq 0 \), and \( P_- \) the projector on \( \{h_i, \quad i \geq 3\} \), the negative subspace of the operator \( L \) (as announced in the beginning of the section, hereafter, we assume that
4.1. Proof of the existence assuming technical results.

In the second subsection we give the proofs of the technical details. We proceed in two subsections. In the first subsection, we give the proof assuming the technical details.

Hereafter, we denote by \( (3.9) \), itself depending on \( p \), the function \( C \) here.

Definition 4.1 (Choice of the initial data) Let us define, for \( A \geq 1 \), \( s_0 = -\log T > 1 \) and \( d_0, d_1 \in \mathbb{R} \), the function

\[
\psi_{s_0,d_0,d_1}(y) = \frac{A}{s_0^{2\beta+1}} (d_0h_0(y) + d_1h_1(y)) \chi(2y,s_0),
\]

where \( h_i, i = 0, 1 \) are defined by \( (2.28) \) and \( \chi \) is defined by \( (3.9) \).

The solution of equation \( (3.2) \) will be denoted by \( v_{s_0,d_0,d_1} \) or \( v \) when there is no ambiguity. We will show that if \( A \) is fixed large enough, then, \( s_0 \) is fixed large enough depending on \( A \), we can fix the parameters \( (d_0, d_1) \in [-2,2]^2 \), so that the solution \( v_{s_0,d_0,d_1}(s) \to 0 \) as \( s \to \infty \) in \( W^{1,\infty}(\mathbb{R}) \), that is, \( (4.1) \) holds. Owing to the decomposition given in \( (4.2) \), it is enough to control the solution in a shrinking set defined as follows:

Definition 4.2 (A set shrinking to zero). Let \( \gamma \) be any real number such that

\[
3\beta < \gamma < \min(5\beta - 1, 2\beta + 1).
\]

For all \( A \geq 1 \) and \( s \geq 1 \), we define \( \mathcal{A}_A(s) \) as the set of all functions \( r \in L^{\infty}(\mathbb{R}) \) such that

\[
||r_{-}||_{L^{\infty}(\mathbb{R})} \leq \frac{A}{s^{2\beta+1}}, \quad \|r_{+}(y)/1 + |y|^3\|_{L^{\infty}(\mathbb{R})} \leq \frac{A}{s^{\gamma}},
\]

\[
|r_0|, |r_1| \leq \frac{A}{s^{2\beta+1}}, \quad |r_2| \leq \frac{\sqrt{A}}{s^{4\beta+1}},
\]

where \( r_{-}, r_{e} \) and \( r_{m} \) are defined in \( (4.12) \).

Remark 4.3. Since \( p > 3 \), it follows that \( \frac{1}{2} < \beta < 1 \), in particular the range for \( \gamma \) in \( (4.3) \) is not empty. Of course, the set \( \mathcal{A}_A(s) \) depends also on the choice of \( \gamma \) satisfying \( (4.3) \). However, while \( A \) will be chosen large enough so that various estimates hold, \( \gamma \) will be fixed once for all throughout the proof.
Since $A \geq 1$, then the sets $\vartheta_A(s)$ are increasing (for fixed $s$) with respect to $A$ in the sense of inclusion. We also show the following property of elements of $\vartheta_A(s)$:

For all $A \geq 1$, there exists $s_0(A) \geq 1$ such that, for all $s \geq s_0$ and $r \in \vartheta_A(s)$, we have

$$
\|r\|_{L^\infty(\mathbb{R})} \leq C \frac{A^2}{s^\gamma - 3\beta},
$$

where $C$ is a positive constant (see Proposition 4.7 below for the proof).

By (4.4), if a solution $v$ stays in $\vartheta_A(s)$ for $s \geq s_0$, then it converges to 0 in $L^\infty(\mathbb{R})$ (the convergence of the gradient will follow from parabolic regularity). Reasonably, our aim is then reduced to prove the following proposition:

**Proposition 4.4 (Existence of solutions trapped in $\vartheta_A(s)$).** There exists $A_2 \geq 1$ such that for $A \geq A_2$ there exists $s_{02}(A)$ such that for all $s_0 \geq s_{02}(A)$, there exists $(d_0, d_1)$ such that if $v$ is the solution of (3.13) with initial data at $s_0$, given by (4.2), then $v(s) \in \vartheta_A(s)$, for all $s \geq s_0$.

This proposition gives the stronger convergence to 0 in $L^\infty(\mathbb{R})$ thanks to (4.4), and the convergence in $W^{1,\infty}(\mathbb{R})$ will follow from parabolic regularity as we explain below.

Let us first be sure that we can choose the initial data such that it starts in $\vartheta_A(s_0)$. In other words, we will define a set where we will at the end select the good parameter $(d_0, d_1)$ that will give the conclusion of Proposition 4.4. More precisely, we have the following result:

**Proposition 4.5 (Properties of initial data).** For each $A \geq 1$, there exists $s_{03}(A) > 1$ such that for all $s_0 \geq s_{03}$:

(i) There exists a rectangle $\mathcal{D}_{s_0} \subset [-2, 2]^2$ (4.5)

such that the mapping

$$
\Phi : \mathbb{R}^2 \to \mathbb{R}^2,
$$

$$(d_0, d_1) \mapsto (\psi_0, \psi_1),$$

(where $\psi$ stands for $\psi_{s_0, d_0, d_1}$) is linear, one to one from $\mathcal{D}_{s_0}$ onto $[-A/s_0^{\beta+1}, A/s_0^{\beta+1}]^2$ and maps $\partial \mathcal{D}_{s_0}$ into $\mathcal{D}([-A/s_0^{\beta+1}, A/s_0^{\beta+1}]^2)$. Moreover, it has degree one on the boundary.

(ii) For all $(d_0, d_1) \in \mathcal{D}_{s_0}$, $\psi := \psi_{s_0, d_0, d_1} \in \vartheta_A(s_0)$ with strict inequalities except for $(\psi_0, \psi_1)$, in the sense that

$$
\psi_0 \equiv 0, \quad |\psi(y)| \leq \frac{1}{s_0^\gamma} (1 + |y|^\beta), \quad \forall \ y \in \mathbb{R}, \quad (4.6)
$$

$$
|\psi_0| \leq \frac{A}{s_0^{2\beta+1}}, \quad |\psi_1| \leq \frac{A}{s_0^{2\beta+1}}, \quad |\psi_2| \leq \frac{1}{s_0^{4\beta-1}}. \quad (4.7)
$$

(iii) Moreover, for all $(d_0, d_1) \in \mathcal{D}_{s_0}$, we have

$$
\|\nabla \psi\|_{L^\infty(\mathbb{R})} \leq C \frac{A}{s_0^{2\beta+1}} \leq \frac{1}{s_0^{\gamma-3\beta}}, \quad (4.8)
$$

$$
|\nabla \psi(y)| \leq \frac{1}{s_0^\gamma} (1 + |y|^\beta), \quad \forall \ y \in \mathbb{R}. \quad (4.9)
$$

The proof of the previous proposition is postponed to Subsection 4.2. Let us now give the proof of Proposition 4.4.
Proof of Proposition 4.4. Let us consider $A \geq 1$, $s_0 \geq s_{03}$, $(d_0, d_1) \in \mathcal{D}_{s_0}$, where $s_{03}$ is given by Proposition 4.5. From the existence theory (which follows from the Cauchy problem for equation (1.1) in $W^{1,\infty}(\mathbb{R})$) mentioned in the introduction, starting in $\vartheta_A(s_0)$ which is in $\vartheta_{A+1}(s_0)$, the solution stays in $\vartheta_A(s)$ until some maximal time $s_* = s_*(d_0, d_1)$. If $s_*(d_0, d_1) = \infty$ for some $(d_0, d_1) \in \mathcal{D}_{s_0}$, then the proof is complete. Otherwise, we argue by contradiction and suppose that $(d_0, d_1) \in \mathcal{D}_{s_0}$. By continuity and the definition of $s_*$, the solution at the point $s_*$ is on the boundary of $\vartheta_A(s_*).$ Then, by definition of $\vartheta_A(s_*)$, one at least of the inequalities in that definition is an equality. Owing to the following proposition, this can happen only for the first two components. Precisely, we have the following result:

**Proposition 4.6** (Control of $v(s)$ by $(v_0(s), v_1(s))$ in $\vartheta_A(s)$). There exists $A_4 \geq 1$ such that for each $A \geq A_4$, there exists $s_{04}(A) \in \mathbb{R}$ such that for all $s_0 \geq s_{04}(A)$, the following holds:

If $v$ is a solution of (3.2) with initial data at $s = s_0$ given by (4.2) with $(d_0, d_1) \in \mathcal{D}_{s_0}$, and $v(s) \in \vartheta_A(s)$ for all $s \in [s_0, s_1]$, with $v(s_1) \in \partial \vartheta_A(s_1)$ for some $s_1 \geq s_0$, then:

(i) (Reduction to a finite dimensional problem) We have:

$$ (v_0(s_1), v_1(s_1)) \in \partial \left( \begin{bmatrix} -A & A \\ s_1^{2\beta+1} & s_1^{2\beta+1} \end{bmatrix} \right)^2. $$

(ii) (Transverse crossing) There exist $m \in \{0, 1\}$ and $\omega \in \{-1, 1\}$ such that

$$ \omega v_m(s_1) = \frac{A}{s_1^{2\beta+1}} \quad \text{and} \quad \omega v_m'(s_1) > 0. $$

Assume the result of the previous proposition, for which the proof is given in Subsection 4.2 below, and continue the proof of Proposition 4.4. Let $A \geq A_4$ and $s_0 \geq s_{04}(A)$. It follows from Proposition 4.6 part (i), that $(v_0(s_*), v_1(s_*)) \in \partial \left( \begin{bmatrix} -A & A \\ s_*^{2\beta+1} & s_*^{2\beta+1} \end{bmatrix} \right)^2$, and the following function

$$ \Phi : \mathcal{D}_{s_0} \to \partial \left( [-1, 1]^2 \right) $$

$$(d_0, d_1) \mapsto \frac{s_*^{2\beta+1}}{A} (v_0, v_1)(d_0, d_1)(s_*), \quad \text{with} \ s_* = s_*(d_0, d_1),$$

is well defined. Then, it follows from Proposition 4.6 part (ii) that $\Phi$ is continuous. On the other hand, using Proposition 4.5 parts (i) and (ii) together with the fact that $v(s_0) = \psi_{s_0, d_0, d_1}$, we see that when $(d_0, d_1)$ is on the boundary of the rectangle $\mathcal{D}_{s_0}$, we have strict inequalities for the other components. Applying the transverse crossing property given in Proposition 4.6 part (ii), we see that $v(s)$ leaves $\vartheta_A(s)$ at $s = s_0$, hence $s_*(d_0, d_1) = s_0$. Using Proposition 4.5 part (i), we see that the restriction of $\Phi$ to the boundary is of degree 1. A contradiction then follows from the index theory. Thus, there exists a value $(d_0, d_1) \in \mathcal{D}_{s_0}$ such that for all $s \geq s_0$, $v(s_0, d_0, d_1)(s) \in \vartheta_A(s)$. This concludes the proof of Proposition 4.4.

Completion of the proof of (4.1). By Proposition 4.4 and 4.2, it remain only to show that $\|\nabla v(s)\|_{L^\infty(\mathbb{R})} \to 0$ as $s \to \infty$. We will prove the following parabolic regularity for equation (3.2):

For all $A \geq 1$, there exists $s_{05}(A) \geq s_{04}(A)$ such that for all $s_0 \geq s_{05}(A)$ the following holds: If $v(s)$ is a solution of equation (3.2) on $[s_0, s_1]$ where $s_1 \geq s_0$ with initial data at $s = s_0$, $v(s_0) = \psi_{s_0, d_0, d_1}$, $(d_0, d_1) \in \mathcal{D}_{s_0}$, $v(s) \in \vartheta_A(s)$ for all $s \in [s_0, s_1]$, then, for all $s \in [s_0, s_1]$, we have

$$ \|\nabla v(s)\|_{L^\infty(\mathbb{R})} \leq \frac{C_1 A^2}{s_0^{5-3\beta}}, \tag{4.10} $$

where $C_1$ is a positive constant.
This will be proved in Subsection 4.2 below. Then, from (4.3), Proposition 4.4 and (4.10), we have
\[\|v(s)\|_{W^{1,\infty}(\mathbb{R})} \leq \frac{C(A)}{s^{\gamma-3\beta}},\]
hence, by (4.3), (4.1) follows by taking \(s_0 \geq \max (s_{01}, s_{03}, s_{04}, s_{05})\).

4.2. Proof of the technical results. In this section, we prove all the technical results used without proof in the previous one, thus, finishing the argument for the proof of the existence of a solution of (3.2) satisfying (4.1). More precisely, we proceed in 4 steps, each given in a separate section.

- We first establish the needed properties on initial data and stated in Proposition 4.5. In particular, we show that initial data is trapped in \(\vartheta_A(s_0)\), provided that \(s_0\) is large enough, and the parameters \((d_0, d_1)\) are in a suitable set.
- Then, we show that the rest and the nonlinear terms of equation (3.2) are trapped in \(\vartheta_C(s)\) for some positive \(C\), assuming \(v \in \vartheta_A(s)\) if necessary. For the potential term, we show that it is in \(\vartheta_C(s)\), assuming \(v \in \vartheta_A(s)\).
- In the third step, we give parabolic regularity estimates, proving in particular estimate (4.10).
- Finally, we prove Proposition 4.6 concerning the reduction of the problem to a two-dimensional one.

4.2.1. Preparation of the initial data. In this subsection, we give some properties of the set \(\vartheta_A(s)\) introduced in Definition 4.2 and prove Proposition 4.5 concerning initial data. We first claim the following:

**Proposition 4.7 (Properties of elements of \(\vartheta_A(s)\)).** For all \(A \geq 1\), there exists \(s_{10} \geq 1\) such that, for all \(s \geq s_{10}\) and \(r \in \vartheta_A(s)\), we have

(i) \(\|r\|_{L^\infty(|y| \leq 2Ks^\beta)} \leq \frac{A}{s^{\gamma-3\beta}}\) and \(\|r\|_{L^\infty(\mathbb{R})} \leq \frac{A^2}{s^{\gamma-4\beta}}\).

(ii) \(|r_b(y)| \leq \frac{A}{s^{4\beta-1}}(1 + |y|^{3})\), \(|r_e(y)| \leq \frac{A^2}{s^{\gamma}}(1 + |y|^{3})\), and \(|r(y)| \leq \frac{A^2}{s^{4\beta-1}}(1 + |y|^3)\), \(\forall y \in \mathbb{R}\).

(iii) \(|r(y)| \leq \frac{A}{s^{2\beta+1}}(1 + |y|) + \frac{A^2}{s^{\gamma}}(1 + |y|^2)\), \(\forall y \in \mathbb{R}\),

where \(C\) is a positive constant.

**Proof.** Take \(A \geq 1\), \(s \geq 1\), \(r \in \vartheta_A(s)\) and \(y \in \mathbb{R}\). Recall that \(r(y) = r_b(y) + r_e(y)\) where
\[r_b(y) = \sum_{m=0}^{2} r_m h_m(y) + r_-(y),\]
with \(\chi\) defined by (3.3). In particular, \(\text{supp } r_b \subset \{|y| \leq 2Ks^\beta\}\) and \(\text{supp } r_e \subset \{|y| \geq Ks^\beta\}\).

(i) If \(|y| \leq 2Ks^\beta\), using the definition (2.8) of \(h_m\) and that of \(\vartheta_A(s)\) we get:
\[|r_b(y)| \leq (1 + |y|) \frac{A}{s^{2\beta+1}} + C(1 + |y|^2) \frac{\sqrt{A}}{s^{4\beta-1}} + (1 + |y|^{3}) \frac{A}{s^{\gamma}}.\] (4.11)

It follows, for \(s\) sufficiently large, that
\[
|r_b(y)| \leq (1 + 2Ks^\beta) \frac{A}{s^{2\beta+1}} + C(1 + (2Ks^\beta)^2) \frac{\sqrt{A}}{s^{4\beta-1}} + (1 + (2Ks^\beta)^3) \frac{A}{s^{\gamma}} \\
\leq C \frac{A}{s^{\beta+1}} + C \frac{\sqrt{A}}{s^{4\beta-1}} + C \frac{A}{s^{\gamma-3\beta}} \\
\leq C \frac{A}{s^{\gamma-3\beta}},
\]

□
since $\gamma < 5\beta - 1$ and $A \geq 1$. Moreover,
\[
\|r\|_{L^\infty(\mathbb{R})} \leq \|r_b\|_{L^\infty(\mathbb{R})} + \|r_e\|_{L^\infty(\mathbb{R})} \leq C \frac{A}{s^{\gamma-3\beta}} + \frac{A^2}{s^{\gamma-3\beta}} \leq C \frac{A^2}{s^{\gamma-3\beta}},
\]
which gives (i).

(ii) Since $A \geq 1$, $s \geq 1$ and $4\beta - 1 < \min(\gamma, 2\beta + 1)$ (by definition (4.3) of $\gamma$), if $|y| \leq 2Ks^\beta$, we write from (4.11):
\[
|r_b(y)| \leq C \frac{A}{s^{4\beta-1}}(1 + |y|^3).
\]
Since $r_b(y, s) \equiv 0$ when $|y| \leq 2Ks^\beta$, the last inequality is obviously true also.

If $|y| \geq Ks^\beta$, we have the definition of $\partial_A(s)$,
\[
|r_e(y)| \leq \|r_e\|_{L^\infty(\mathbb{R})} \leq \frac{A^2}{s^{\gamma-3\beta}} \leq \frac{A^2}{s^{\gamma-3\beta}} \left(\frac{|y|^3}{Ks^\beta}\right) \leq \frac{CA^2}{s^\gamma}(1 + |y|^3). \quad (4.12)
\]
The inequality is verified for all $y \in \mathbb{R}$, since $r_e(y, s) \equiv 0$ when $|y| \leq Ks^\beta$.

On the other hand, by the conditions on $A$ and $\beta$, we have
\[
|r(y)| \leq |r_b(y)| + |r_e(y)| \leq C \frac{A^2}{s^{4\beta-1}}(1 + |y|^3), \quad \forall y \in \mathbb{R},
\]
which gives (ii).

(iii) Using (4.11) and (4.12), we get
\[
|r(y)| \leq |r_b(y)| + |r_e(y)| \leq (1 + |y|) \frac{A}{s^{2\beta+1}} + C(1 + |y|^2) \frac{A}{s^{2\beta+1}} + (1 + |y|^3) \frac{A}{s^\gamma} + C \frac{A^2}{s^\gamma}(1 + |y|^3)
\]
\[
\leq (1 + |y|) \frac{A}{s^{2\beta+1}} + C(1 + |y|^2) \frac{A}{s^{2\beta+1}} + C \frac{A^2}{s^\gamma}(1 + |y|^3).
\]
This finishes the proof of Proposition 4.7. \qed

We need a second technical estimate before proving Proposition 4.8.

**Lemma 4.8.** There exists $s_{10}'$ such that for all $s_0 \geq s_{10}'$, if $g$ is given by $\chi(2y, s_0)$ or $y\chi(2y, s_0)$, then
\[
ge_e(y) \equiv 0, \quad \left\| \frac{g_e(y)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq C \frac{2^{\beta}}{s^3},
\]
and all $g_m$ are less then $Ce^{-s_0^{2\beta}}$, except:

(i) $|g_0 - 1| \leq Ce^{-s_0^{2\beta}}$ when $g(y) = \chi(2y, s_0)$,

(ii) $|g_1 - 1| \leq Ce^{-s_0^{2\beta}}$ when $g(y) = y\chi(2y, s_0)$.

**Remark 4.9.** Here in this lemma, we need the fact that $K \geq 6$.

**Proof.** Since $(1 - \chi(y, s_0))\chi(2y, s_0) \equiv 0$ and $\chi(y, s_0)\chi(2y, s_0) \equiv 1$, it follows that $g_e(y) \equiv 0$ and $g_0(y) = \chi(y, s_0)g(y) = g(y)$. In particular, $g_m = P_m(g)$ and $g_\perp = P_\perp(g)$, where $P_m$ and $P_\perp$ are the $L^2$ projectors on $h_m$ and $\{h_i \mid i \geq 3\}$, respectively. Writing $g(y) = \tilde{g}(y) + r(y)$ where $\tilde{g}(y) = 1$ or $y$ and $r(y) = \tilde{g}(y)(\chi(2y, s_0) - 1)$, the result will follow by linearity.

Starting first with $\tilde{g}$, we see that $P_\perp(\tilde{g}) \equiv 0$ and all $P_m(\tilde{g})$ are zero except
\[
P_0(\tilde{g}) = 1, \quad \text{when } \tilde{g}(y) = 1, \quad \text{and } P_1(\tilde{g}) = 1, \quad \text{when } \tilde{g}(y) = y.
It remains then to handle \( r \). Since \( 1 - \chi(y, s_0) = 0 \) for \( |y| \leq K_s^0 \), we see that
\[
0 \leq 1 - \chi(y, s_0) \leq \left( \frac{|y|}{K_s^0} \right)^i, \quad 0 \leq i \leq 3, \text{ hence } 0 \leq 1 - \chi(2y, s_0) \leq \left( \frac{2|y|}{K_s^0} \right)^2, \tag{4.13}
\]
and since \( K \geq 6 \) (see \( 3.9 \)),
\[
\rho(y)(1 - \chi(y, s_0)) \leq \sqrt{\rho(y)} \left( \frac{2|y|}{K_s^0} \right)^2 \leq Ce^{-\frac{K^2 s^3}{2} \sqrt{\rho(y)}} \leq Ce^{-\frac{K^2 s^3}{2} \sqrt{\rho(y)}}, \tag{4.14}
\]
and similarly,
\[
\rho(y)(1 - \chi(y, s_0)) \leq Ce^{-\frac{K^2 s^3}{2} \sqrt{\rho(y)}}. \tag{4.14}
\]
Therefore,
\[
|r(y)| \leq C(1 + |y|) \left( \frac{2|y|}{K_s^0} \right)^2 \leq \frac{C}{s_0^2} (1 + |y|^3) \quad \text{and} \quad |r_m| \leq Ce^{-\frac{K^2 s^3}{2}}, \quad m = 0, 1, 2.
\]
Hence, using the fact that \( |h_m(y)| \leq C(1 + |y|), \quad m = 0, 1, \) we get also
\[
|r_-(y)| \leq \frac{C}{s_0^2} (1 + |y|^3).
\]
This concludes the proof of Lemma 4.8.

With Lemma 4.8 at hand, we are ready to give the proof of Proposition 4.5.

**Proof of Proposition 4.5.** For simplicity we write \( \psi \) instead of \( \psi_{s_0, d_0, d_1} \).

(i) Using Lemma 4.8 we see that
\[
\psi_0 = d_0 \left( \frac{A}{s_0^{2\beta+1}} + O \left( e^{-s_0^{2\beta}} \right) \right) \quad \text{and} \quad \psi_1 = d_1 \left( \frac{A}{s_0^{2\beta+1}} + O \left( e^{-s_0^{2\beta}} \right) \right), \tag{4.15}
\]
and the conclusion of item (i) follows directly.

(ii) The fact that \( |\psi_m| \leq \frac{d_0}{s_0} \) for \( m = 0, 1 \) follows from item (i). Then, using Lemma 4.8 and linearity, we see that
\[
\psi_\infty(y) \equiv 0, \quad \left\| \psi_\infty(y) \right\|_{L^\infty(\mathbb{R})} \leq \frac{CA}{s_0^{4\beta+1}} (|d_0| + |d_1|), \quad |\psi_2| \leq C(|d_0| + |d_1|) e^{-s_0^{2\beta}}. \tag{4.16}
\]
Since
\[
|d_m| \leq 2 \quad \text{for} \quad m = 0, 1 \tag{4.17}
\]
from item (i), recalling that \( \gamma < 4\beta + 1 \) from \( 4.3 \), we get the conclusion of item (ii).

(iii) From \( 4.2 \), we have that
\[
\partial_y \psi(y) = d_1 \left( \frac{A}{s_0^{2\beta+1}} \chi(2y, s_0) + \frac{A}{s_0^{2\beta+1}} (d_0 + d_1 y) \chi_0 \left( \frac{2y}{K_s^0} \right) \right), \tag{4.18}
\]
where \( \chi_0 \) is defined by \( 3.9 \). Since \( \|z \chi_0(z)\|_{L^\infty(\mathbb{R})} \) and \( \frac{2z}{K_s^0} \) are bounded, then for \( s_0 \) sufficiently large we have, from \( 4.17 \) and the definition \( 4.3 \) of \( \gamma \),
\[
\|\partial_y \psi\|_{L^\infty(\mathbb{R})} \leq C \frac{A}{s_0^{2\beta+1}} \leq \frac{1}{s_0^{\gamma - 3\beta}}.
\]
As for the estimate on \( \partial_y \psi_- \), since \( \psi_\infty \equiv 0 \), we write from \( 3.13 \)
\[
\psi_-(y) = \psi(y) - (\psi_0 + \psi_1 y + \psi_2 (y^2 - 1)),
\]
and hence
\[
\partial_y \psi_-(y) = \partial_y \psi(y) - (\psi_1 + 2\psi_2 y).
\]
Using (4.18), (4.15) and (4.16), we see that
\[
\partial_y \psi_-(y) = d_1 \frac{A}{s_0^{2+\beta}} \left( \chi(2y, s_0) - 1 \right) + O \left( e^{-s_0^{2\beta}} \right) |y| + \frac{A}{s_0^{2+\beta}} (d_0 + d_1 y) \chi' \frac{2y}{K s_0^\beta}.
\] (4.19)

We remark now that
\[
|\chi_0'(z)| \leq C |z|^i, \quad i = 0, 1, 2, 3
\] (4.20)
in fact, \(\chi_0'(z) = 0\) for \(|z| \leq 1\) and it is bounded for \(z \in \mathbb{R}\). Using (4.17), (4.13), (4.20) and (4.19), we obtain
\[
|\partial_y \psi_-(y)| \leq C \left[ \frac{A}{s_0^{2+\beta}} + \frac{A}{s_0^{6+\gamma}} + e^{-s_0^{2\beta}} \right] (1 + |y|^3) \leq \frac{CA}{s_0^{2+\beta}} (1 + |y|^3) \leq \frac{1}{s_0^\beta} (1 + |y|^3),
\]
since \(0 < \gamma < 5\beta + 1\). Thus, the last inequality in item (iii) follows. This concludes the proof of Proposition 4.5.

4.2. Preliminary estimates on various terms of equation (3.2). In this step, we show that the rest term is trapped in \(\partial_C(s)\) for some \(C > 0\), provided that \(s\) is large. Then, assuming in addition that \(v(s) \in \partial_A(s)\), we show that the nonlinear term in also trapped in \(\partial_C(s)\), and the potential term \(Vv\), in \(\partial_C(s)\).

This is our first statement.

**Lemma 4.10** (Estimates on the rest term and the potential). There exists \(s_{11}\) sufficiently large such that for \(s \geq s_{11}\), we have the following

(i) \(R \in \partial_C(s)\) and \(|R_2(s)| \leq \frac{C}{s^{17}}\),

(ii) \(|V(s)|_{L^\infty(\mathbb{R})} \leq C, |V(y, s)| \leq C \frac{(1+|y|^2)}{s^{17}}, \forall y \in \mathbb{R}\).

where \(C\) is a positive constant, \(V\) and \(R\) are given by (3.3) and (3.3).

**Remark 4.11.** As we stated in a remark following Theorem 1, the particular value of \(b\) we fixed in (1.6) is natural from the formal approach in Section 2 above. In fact, it is crucial in the algebraic identity leading from (4.34) to (4.35). Indeed, with a different \(b\), we would have a larger \(R_2 \sim \frac{C}{s^{17}} \gg \frac{1}{s^{17}}\), making the convergence of \(v\) to zero in (3.7) more difficult (and probably impossible) to obtain.

**Remark 4.12.** With some more work, we may show that:
\[
\left\| \frac{R_-(y, s)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \sim \frac{C}{s^{2\beta + 1}}, \quad \text{as} \quad s \to \infty.
\]

This implies, in particular, that any attempt to adapt the powers of \(1/s\) in the definition of \(\partial_A(s)\) should respect
\[
\gamma \leq 2\beta + 1
\] (4.21)
where \(\gamma\) is such that
\[
\left\| \frac{v_-(y, s)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq C \frac{1}{s^\gamma},
\]
on the one hand. On the other hand, bearing in mind that we need to evaluate \(|v_-(y, s)|\) on the support of \(v_b\), that is, when \(|y| \leq 2Ks^\beta\), we see that
\[
\text{for } |y| \leq 2Ks^\beta, \quad \text{we have } |v_-(y, s)| \leq \frac{SAK^3}{s^{\gamma - 3\beta}},
\]
and the right hand side of the last inequality goes to zero if and only if
\[
\gamma > 3\beta.
\] (4.22)
From (4.24) and (4.22), we see that $3 \beta < \gamma \leq 2 \beta + 1$, which yields the natural condition

$$\beta < 1, \text{ i.e. } p > 3.$$  

Presumably, our strategy based on the shrinking set $\partial A(s)$, in the same style as [4] and [22] breaks down when $\beta \geq 1$. However, we are not saying that Theorem 4 is not true for $1 < p \leq 3$. Perhaps a substantial adaptation of the method of [4] and [22], or some other strategy, may give the result. The question remain open when $1 < p \leq 3$.

Before proving Lemma 4.10, let us state and prove the following lemma, where we make an expansion of $R(y,s)$:

**Lemma 4.13 (Bound and expansion of the rest term).** For $s$ large enough, we have

$$\|R(s)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{s}, \quad (4.23)$$

and

$$R(y,s) = \frac{a}{s^{2\beta}} - \frac{2b\kappa v_{s}^2}{(p-1)^2} \left( \frac{y}{s^\beta} \right)^p + \frac{2\beta a}{(p-1)^4} \left( \frac{1}{s^2\beta} \right)^2 \left( \frac{y}{s^\beta} \right)^p - \frac{2b\kappa v_{s}^2}{(p-1)^2} \left( \frac{y}{s^\beta} \right)^p + \frac{2\beta a}{(p-1)^4} \left( \frac{1}{s^2\beta} \right)^2 \left( \frac{y}{s^\beta} \right)^p$$

$$- \frac{2\beta b}{(p-1)^2} \left( \frac{y}{s^\beta} \right)^p + \mu \left( \frac{2b\kappa v_{s}^2}{(p-1)^2} \right) \left( \frac{1}{s^2\beta} \right)^2 \left( \frac{y}{s^\beta} \right)^p + O\left( \frac{z^4}{s} \right) + O\left( \frac{1}{s^4\beta} \right) + O\left( \frac{|z|^{q+2}}{s^{q\beta}} \right), \quad (4.24)$$

where $z = \frac{y}{s^\beta} \in \mathbb{R}$.

**Proof.** By the definitions of $\varphi$ and $\varphi^0$, we have

$$\partial_y \varphi(y,s) = -\frac{2b}{(p-1)s^\beta} \left( \frac{y}{s^\beta} \right)^p, \text{ hence } \|\partial_y \varphi(s)\|_{\infty} \leq \frac{C}{s^\beta}, \quad (4.25)$$

$$\partial_s \varphi(y,s) = \frac{2\beta b}{(p-1)s^\beta} \left( \frac{y}{s^\beta} \right)^2, \text{ hence } \|\partial_s \varphi(s)\|_{\infty} \leq \frac{C}{s^\beta}, \quad (4.26)$$

$$\partial_y^2 \varphi(y,s) = \frac{2b}{(p-1)s^\beta} \left[ - (\varphi^0(z))^p + \frac{2bp}{p-1} z^2 \varphi^0(z)^{p-1} \right], \text{ hence } \|\partial_y^2 \varphi(s)\|_{\infty} \leq \frac{C}{s^2\beta}. \quad (4.27)$$

On the other hand, since we have from (2.17),

$$-\frac{1}{2} \frac{y}{s^\beta} \varphi^0(y/s^\beta) + \left( \frac{\varphi^0(y/s^\beta)}{s^\beta} \right)^p = \frac{1}{p-1} \varphi^0(y/s^\beta) = 0,$$

we write

$$-\frac{1}{2} y \partial_y \varphi - \frac{1}{p-1} \varphi + \varphi^p = -\frac{1}{2} \frac{1}{s^\beta} \left[ \varphi^0 \right] + \frac{1}{p-1} \varphi^0 \left( \frac{y}{s^\beta} \right) - \frac{a}{(p-1)s^2\beta}$$

$$+ \varphi^p - \left[ \varphi^0 \right] \left( \frac{y}{s^\beta} \right)^p + \left[ \varphi^0 \right] \left( \frac{y}{s^\beta} \right)^p$$

$$= \varphi^p - \left[ \varphi^0 \right] \left( \frac{y}{s^\beta} \right)^p - \frac{a}{(p-1)s^2\beta}. \quad (4.28)$$

By Lipschitz property, we have that

$$|\varphi^p - (\varphi^0)^p| \leq \frac{C}{s^2\beta}.$$  

Hence

$$\| -\frac{1}{2} y \partial_y \varphi - \frac{1}{p-1} \varphi + \varphi^p \|_{L^\infty(\mathbb{R})} \leq \frac{C}{s^2\beta}. \quad (4.29)$$

Since $1 - \frac{p}{p-1} = q \beta < 2 \beta$, we see that by definition (3.5) of $R$ that (4.23) holds.
Now, as for the expansion \((4.24)\), it simply follows from Taylor expansions, derived from \((4.25), (4.26), (4.27)\) and \((4.28)\), for all \(s \geq 1\) and \(y \in \mathbb{R}^2\):

\[
\begin{aligned}
\partial_y^2 \varphi(y, s) &= \frac{2b}{(p-1)s^{2\beta}} \left[ -\frac{\kappa}{p-1} \left( 1 - \frac{bp}{(p-1)^2} z^2 \right) + \frac{2b\kappa p}{(p-1)^3} z^2 + O(z^4) \right], \\
\frac{1}{2} y \partial_y \varphi - \frac{1}{p-1} \varphi + \varphi^p &= \frac{pa}{(p-1)s^{2\beta}} \left[ 1 - \frac{b}{p-1} z^2 + O(z^4) \right] - \frac{a}{(p-1)s^{2\beta}} + O \left( \frac{1}{s^{4\beta}} \right), \\
\partial_s \varphi(y, s) &= \frac{2\beta b}{(p-1)s} z^2 \left( \frac{\kappa}{p-1} + O(z^2) \right) - \frac{2\beta a}{s^{2\beta+1}}, \\
|\partial_y \varphi|^q &= \left( \frac{2b}{(p-1)s^{\beta}} \right)^q \left( \frac{\kappa}{p-1} \right)^q |z|^q \left( 1 + O(z^2) \right).
\end{aligned}
\]

This concludes the proof of Lemma 4.13.

With Lemma 4.13, we are ready to prove Lemma 4.10.

**Proof of Lemma 4.10.**

(i) Following the decomposition \((3.13)\), we write \(R\) as

\[
R = R\chi + R(1 - \chi) = \left( \sum_{m=0}^{2} R_m h_m + R_\omega \right) + R_e.
\]

Since \(R\) is symmetric with respect to \(y\), we have \(R_1 = 0\).

Furthermore, inequality \((4.24)\) in the previous lemma implies in particular that

\[
\|R_e(s)\|_{L^\infty(\mathbb{R})} \leq \frac{C}{s} \leq \frac{C}{s^{2\beta+1}},
\]

by definition of \(\gamma\) given by \((4.3)\).

As for \(R_0\), using \((4.23)\) and the fact that \(2(q+1)\beta = 4\beta + 1\), we write

\[
R_0 = \int_{\mathbb{R}} R\chi \rho = \int_{\mathbb{R}} \rho \left( \frac{1}{s^{2\beta}} + O \left( \frac{1}{s^{2\beta+1}} \right) \right) = O \left( \frac{1}{s^{2\beta+1}} \right),
\]

from the choice of \(a\) made in \((2.23)\), therefore

\[
|R_0| \leq \frac{C}{s^{2\beta+1}}. \tag{4.33}
\]

Now, considering \(R_2\) and using the fact that

\[
\int h_2 \rho = 0, \quad \int k_2 y^2 \rho = 1, \quad 2(q+1)\beta = 4\beta + 1,
\]

we write from \((4.24)\) together with \((4.14)\),

\[
R_2 = \int_{\mathbb{R}} R\chi k_2 \rho = \int_{\mathbb{R}} R k_2 \rho + \int_{\mathbb{R}} R(1-\chi) k_2 \rho
\]

\[
= \int_{\mathbb{R}} R k_2 \rho + O \left( e^{-s^{2\beta}} \right)
\]

\[
= \left[ -\frac{2b\beta}{(p-1)^2} + \mu \left( \frac{2b\kappa}{(p-1)^2} \right)^q \int |y|^q k_2 \rho \right] \frac{1}{s^{2\beta+1}} + O \left( \frac{1}{s^{4\beta}} \right). \tag{4.34}
\]

Since

\[
\frac{2b\beta}{(p-1)^2} \|h_2\|_{L^\beta}^2 = \mu \left( \frac{2b\kappa}{(p-1)^2} \right)^q \int |y|^q h_2 \rho dy,
\]

\[\square\]
from the choice of $b$ in (2.24), we have that

$$|R_2| \leq C \frac{s^{4\beta}}{s^{4\beta-1}} .$$  \hspace{1cm} (4.35)$$

It remains now to bound $R_-(y, s)$. From (4.21) in Lemma 4.13 and the choice of $a$ in (2.23) and $b$ in (2.24), we see that

$$R(y, s) = -\frac{2\beta b \kappa}{(p - 1)^2} \frac{h_2(y)}{s^{2\beta+1}} + \mu \left( \frac{2b\kappa}{(p - 1)^2} \right)^q \frac{|y|^q}{s^{2\beta+1}} + \frac{6b^2 p \kappa}{(p - 1)^4} \frac{y^2}{s^{4\beta}} + O\left( \frac{y^4}{s^{4\beta+1}} \right) + O\left( \frac{1}{s^{4\beta}} \right).$$

Using that $|y| \leq 2Ks^\beta$ on the support of $\chi$, we see that

$$|R^\chi|(y, s) \leq C \frac{1}{s^{2\beta+1}} (1 + |y|^3).$$

Using (4.33) and (4.35), we see that

$$|R^-| = |R^\chi - (R_0 + R_2h_2)| \leq |\chi R| + |R_0| + C|R_2|(1 + |y|^2) \leq \frac{C}{s^{2\beta+1}} (1 + |y|^3) \leq \frac{C}{s^\gamma} (1 + |y|^3),$$

by the hypotheses (4.3) on $\gamma$. This concludes the proof of item (i) of the lemma.

(ii) Since $b > 0$, the first statement follows. By (3.3) and (2.21), for the second statement, we write, by definition (3.3) of $V$ and a Taylor expansion, (remark that $p > 3$):

$$V(y, s) = p\left( \varphi^0 \left( \frac{y}{s^\beta} \right) + \frac{a}{s^{2\beta}} \right)^{p-1} - \frac{p}{p-1} = p\left( \varphi^0 \left( \frac{y}{s^\beta} \right) \right)^{p-1} - \frac{p}{p-1} + O\left( \frac{1}{s^{2\beta}} \right).$$

Since $[\varphi^0(0)]^{p-1} = 1/(p - 1)$, the second statement follows. This concludes the proof of Lemma 4.10.

\[ \square \]

Lemma 4.14. Let $V$, $B$ and $G$ be given by (3.3), (3.4), (3.5) and (3.6). Then, for all $A \geq 1$, there exits $s_{12}$, sufficiently large, such that for all $s \geq s_{12}$, if $v \in \partial\varrho_A(s)$, then we have the following:

(i) $\|Vv(s)\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{4\beta+1}}, \|Vv(y, s)\| \leq \frac{CA^2}{s^{4\beta}} (1 + |y|^2), \forall y \in \mathbb{R},$ 

$$|(Vv)_m| \leq \frac{1}{s^{2\beta+1}}, \text{ for } m = 0, 1, |(Vv)_2| \leq \frac{1}{s^{4\beta}},$$

$$|(Vv)_-(y, s)| \leq \frac{CA}{s^\gamma} (1 + |y|^3) \text{ and } \|(Vv)_c\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{\gamma-3\beta}}.$$ In particular, $Vv \in \partial\varrho_G(s)$.

(ii) $Bv \in \partial\varrho_C(s)$ and $|(Bv)_2(s)| \leq \frac{C}{s^{4\beta}}.

(iii) Furthermore, if $\|\varrho v(s)\|_\infty \leq \frac{CA^2}{s^{4\beta+5}}$ and

$$|\varrho v(y, s)| \leq \frac{CA}{s^{2\beta+1}} (1 + |y|) + \frac{C\sqrt{A}}{s^{4\beta-1}} (1 + |y|^2) + \frac{CA^2}{s^\gamma} (1 + |y|^3), \forall y \in \mathbb{R},$$

then $G \in \partial\varrho_C(s)$, for some positive constant $C$.\[ \square \]
Proof. (i) Since \( v \in \vartheta_A(s) \), we have from Proposition 4.7

\[
\|v(s)\|_{L^\infty(\mathbb{R})} \leq C \frac{A^2}{s^{\gamma-3\beta}}.
\]

Then, using item (ii) of Lemma 4.10 we get,

\[
\|Vv(s)\|_{L^\infty(\mathbb{R})} \leq \|V(s)\|_{L^\infty(\mathbb{R})}\|v(s)\|_{L^\infty(\mathbb{R})} \leq C \frac{A^2}{s^{\gamma-3\beta}},
\]

and

\[
|Vv|(y,s) \leq \|v(s)\|_{L^\infty(\mathbb{R})}|V|(y,s) \leq C \frac{A^2}{s^{\gamma-\beta}}(1+|y|^2).
\]

Furthermore, using item (iii) of Proposition 4.7 and Definition 4.2 of \( \vartheta_A(s) \), we see that

\[
|(Vv)_b(y,s)| = |Vv_b(y,s)| \leq \frac{C(1+|y|^2)}{s^{2\beta}} \left( \sum_{m=0}^{2} |v_m(s)|(1+|y|)^m + \frac{v_v(y,s)}{1+|y|^3}(1+|y|^3) \right)
\]

\[
\leq \frac{CA}{s^{4\beta+1}}(1+|y|^3) + \frac{C\sqrt{A}}{s^{6\beta-1}}(1+|y|^4) + \frac{CA}{s^{3\gamma+2\beta}}(1+|y|^5).
\]

By definition of \((Vv)_m\) we see that

\[
|(Vv)_m(s)| = \left| \int_R k_m(y)(Vv)_b\rho(y)dy \right|
\]

\[
\leq \frac{CA}{s^{4\beta+1}} + \frac{C\sqrt{A}}{s^{6\beta-1}} + \frac{CA}{s^{\gamma+2\beta}}
\]

\[
\leq \frac{1}{s^{4\beta}},
\]

since \( \frac{1}{2} < \beta < 1 \) and \( \gamma \) satisfies \( (4.3) \).

As for \((Vv)_-\), noting that \( |y| \leq 2Ks^\beta \) on the support of \((Vv)_b\), we write from \( (4.37) \):

\[
\frac{|(Vv)_b|(y,s)}{1+|y|^3} \leq \frac{CA}{s^{4\beta+1}} + \frac{C\sqrt{A}}{s^{5\beta-1}} + \frac{CA}{s^\gamma} \leq \frac{C_1A}{s^\gamma},
\]

since \( \gamma < \min(5\beta-1,2\beta+1) \). Therefore, \( \frac{|(Vv)_-|(y,s)}{1+|y|^3} \leq \frac{CA}{s^\gamma} \), which is the desired estimate for \((Vv)_-\). As for the estimate on \( \|(Vv)_e\|_{L^\infty(\mathbb{R})} \), it follows from \( (4.36) \).

(ii) From a Taylor expansion, we have

\[
|B(v)| \leq C|v|^2.
\]

Since \( v \in \vartheta_A(s) \), from item (i) in Proposition 4.7, we have

\[
\|B(v)_e\|_{L^\infty(\mathbb{R})} \leq \|B(v)\|_{L^\infty(\mathbb{R})} \leq C\|v\|_{L^\infty(\mathbb{R})}^2 \leq C \frac{A^4}{s^{2(\gamma-3\beta)}} \leq \frac{1}{s^{\gamma-3\beta}},
\]
for sufficiently large $s$. Moreover, we have that,
\[ |(B(v))_b(y, s)| = |\chi(y, s)B(v)(y, s)| \leq C|v(y, s)|^2 \]
\[ \leq C\left( \sum_{m=0}^{2} |v_m|^2 h_m^2 + |v_-(y, s)|^2 + |v_e(y, s)|^2 \right) \]
\[ \leq C\left( \frac{A^2}{s^{4\beta+2}}(1 + |y|^2) + \frac{A}{s^{4\beta-2}}(1 + |y|^4) + \frac{A^4}{s^{2\gamma}}(1 + |y|^6)\right) 1_{|y| \leq 2Ks^\beta} + 1_{|y| > Ks^\beta} \frac{A^4}{s^{2\gamma-6\beta}} \] (4.39)

where $1_X$ is the characteristic function of a set $X$.
Hence, using (4.39), we write by definition of $(B(v))_m$,
\[ |B(v)_m| \leq C\left( \frac{A^2}{s^{4\beta+2}} + \frac{A}{s^{4\beta-2}} + \frac{A^4}{s^{2\gamma}} \right), \]
Therefore, by the conditions on $\gamma$ given by (4.3) and since $\beta > 1/2$, we see that
\[ |B(v)_m| \leq \frac{1}{s^{2\beta+1}}, \quad m = 0, 1 \quad \text{and} \quad |B(v)_2| \leq \frac{C}{s^{4\beta}} \leq \frac{1}{s^{4\beta-1}}. \]
Furthermore, by the expression of $\gamma$ given by (4.3), and since $\beta > 1/2$, we have
\[ |B(v)(y, s)| = |B(v)_b(y, s) - \sum_{m=0}^{2} (Bv)_m h_m| \]
\[ \leq C\left( \frac{A^2}{s^{4\beta+2}} + \frac{A}{s^{4\beta-2}} + \frac{A^4}{s^{2\gamma-3\beta}} \right)(1 + |y|^3) \]
\[ \leq \frac{1}{s^\gamma(1 + |y|^3)}, \]
for $s$ sufficiently large. This finishes the proof of item (ii).

(iii) Using the inequality
\[ ||x + x'||^q - |x|^q \leq C \left( |x|^{q-1}|x'| + |x'|^q \right), \quad \forall x \in \mathbb{R}, \quad x' \in \mathbb{R}, \]
we deduce that
\[ |G(y, s)| \leq C \left( |\partial_y \varphi|^{q-1}|\partial_y v| + |\partial_y v|^q \right). \] (4.40)
Since
\[ \|\partial_y v(s)\|_\infty \leq \frac{CA^2}{s^{\gamma-3\beta}} \quad \text{and} \quad \|\partial_y \varphi(s)\|_\infty \leq \frac{C}{s^{\beta}}, \]
it follows that
\[ \|G(s)\|_\infty \leq \frac{1}{s^{(q-1)}} \frac{CA^2}{s^{\gamma-3\beta}} + \frac{CA^q}{s^{q(\gamma-3\beta)}} \]
\[ \leq \frac{1}{s^{\gamma-3\beta}}, \]
because $q > 1$. Hence
\[ \|G_e(s)\|_\infty \leq \frac{1}{s^{\gamma-3\beta}}. \] (4.41)
On the other hand, from the fact that
\[ |\partial_y \varphi(y, s)| \leq C \frac{|y|}{s^{2\beta}}, \quad \forall y \in \mathbb{R}, \]
and
\[ |\partial_y v(y, s)| \leq \frac{CA}{s^{2\beta+1}}(1+|y|) + \frac{C\sqrt{A}}{s^{4\beta-1}}(1+|y|^2) + \frac{CA^2}{s^{\gamma}}(1+|y|^3), \quad \forall \ y \in \mathbb{R}, \]

using (4.40) and the identity
\[ 2\beta(q-1) = 1, \]
we deduce that
\[ |G(y, s)| \leq C\left(\frac{|y|^{q-1}}{s} |\partial_y v| + |\partial_y v|^q\right), \]
hence,
\[ |G(y, s)| \leq \frac{CA}{s^{2\beta+2}}|y|^{q-1}(1+|y|) + \frac{C\sqrt{A}}{s^{4\beta}}|y|^{q-1}(1+|y|^2) + \frac{CA^2}{s^{\gamma+1}}|y|^{q-1}(1+|y|^3) + \frac{CA q}{s^{q(2\beta+1)}}|y|^q + \frac{CA q/2}{s^{q(4\beta-1)}}(1+|y|^{2q}) + \frac{CA^{2q}}{s^{q\gamma}}(1+|y|^{3q}). \]

Therefore, from the conditions on \( \gamma \) given by (4.3), we have
\[ |G_m(s)| \leq \frac{1}{s^{2\beta+1}} \quad \text{for } m = 0, 1. \]
Since \( \beta < 3/2 \) and \( \gamma > 4\beta - 1 \), we get
\[ |G_2(s)| \leq \frac{1}{s^{4\beta-1}}. \]

We now turn to the estimate on \( G_- \).

Using (4.3) and the fact that \( |y| \leq 2K s^\beta \) on the support of \( G_- \), for \( G_b \) we write from (4.42) and the fact that \( q > 3/2 \) (which follows from the fact that \( p > 3 \) and \( q = 2p/(p+1) \)):
\[ \frac{|G_b(y, s)|}{1+|y|^3} \leq \frac{CA}{s^{2\beta+2}} + \frac{C\sqrt{A}}{s^{4\beta}} + \frac{CA^2}{s^{\gamma+1}} + \frac{CA q}{s^{q(2\beta+1)}} + \frac{CA q/2}{s^{q(4\beta-1)-(2q-3)\beta}} + \frac{CA^{2q}}{s^{q\gamma-3(q-1)\beta}}. \]

Noting from the conditions (4.3) on \( \gamma \) and (4.42) that
\[ 2\beta + 2 > 2\beta + 1 > \gamma; \quad 4\beta > 5\beta - 1 > \gamma; \quad q(2\beta + 1) > 2\beta + 1 > \gamma; \quad q(4\beta - 1) - (2q - 3)\beta = 2q\beta - q + 3\beta = 5\beta + 1 - q > 5\beta - 1 > \gamma; \]
\[ q\gamma - 3(q-1)\beta = q\gamma - \frac{3}{2} = (q-1)\gamma - \frac{3}{2} + \gamma = \gamma - \frac{3\gamma}{2\beta} + \gamma > \gamma, \]
we see that
\[ \frac{|G_b(y, s)|}{1+|y|^3} \leq \frac{1}{s^\beta}, \]
hence,
\[ |G_m(s)| \leq \frac{C}{s^{\gamma}}, \quad m = 0, 1, 2, \]
and
\[ |G_-(y, s)| \leq \frac{C}{s^{\gamma}(1+|y|^3)}, \quad \forall \ y \in \mathbb{R}. \]
This finishes the proof of Lemma 4.14. 

4.2.3. Parabolic regularity. In this subsection, we prove the parabolic regularity results and prove (4.10). To do so:

- We first give some linear parabolic regularity estimates on the linear operator $\mathcal{L}$ defined in (2.6). See Lemma 4.15 below.
- Then, since we aim at obtaining some fine estimates on $\partial_y v_-$ if $v(s) \in \mathcal{V}_A(s)$, we will write down the equation satisfied by $v_-$ and bound its source term. See Lemma 4.16 below.
- Finally, using the linear estimate along with the preliminary estimates given in the previous subsection, we derive the intended regularity estimates for the full equation (3.2).

Let us first give some linear regularity estimates:

**Lemma 4.15** (Properties of the semigroups $e^{\theta \mathcal{L}}$). The Kernel $e^{\theta \mathcal{L}}(y, x)$ of the semi-group $e^{\theta \mathcal{L}}$ is given by:

$$e^{\theta \mathcal{L}}(y, x) = \frac{e^{\theta}}{\sqrt{4\pi(1-e^{-\theta})}} \exp \left[-\frac{(ye^{-\theta/2} - x)^2}{4(1-e^{-\theta})}\right].$$

(4.45)

for all $\theta > 0$, and $e^{\theta \mathcal{L}}$ is defined by

$$e^{\theta \mathcal{L}}r(y) = \int_\mathbb{R} e^{\theta \mathcal{L}}(y, x)r(x)dx.$$  

(4.46)

We also have the following:

(i) If $r_1 \leq r_2$ then $e^{\theta \mathcal{L}}r_1 \leq e^{\theta \mathcal{L}}r_2$.

(ii) $\|\nabla(e^{\theta \mathcal{L}}r)\|_{L^\infty(\mathbb{R})} \leq C e^{\theta} \|\nabla r\|_{L^\infty(\mathbb{R})}$, $r \in W^{1,\infty}(\mathbb{R})$.

(iii) $\|\nabla(e^{\theta \mathcal{L}}r)\|_{L^\infty(\mathbb{R})} \leq \frac{Ce^\theta}{\sqrt{1-e^{-\theta}}} |r|_{L^\infty(\mathbb{R})}$, $r \in L^\infty(\mathbb{R})$.

(iv) If $|r(x)| \leq \eta(1 + |x|^m)$, $\forall x \in \mathbb{R}$, then $|e^{\theta \mathcal{L}}r(y)| \leq C\eta e^{\theta}(1 + |y|^m)$, $\forall y \in \mathbb{R}$.

(v) If $|\nabla r(x)| \leq \eta(1 + |x|^m)$, $\forall x \in \mathbb{R}$, then $|\nabla(e^{\theta \mathcal{L}}r)(y)| \leq C\eta e^{\theta}(1 + |y|^m)$, $\forall y \in \mathbb{R}$.

(vi) If $|r(x)| \leq \eta(1 + |x|^m)$, $\forall x \in \mathbb{R}$, then $|\nabla(e^{\theta \mathcal{L}}r)(y)| \leq C\eta e^{\theta}(1 + |y|^m)$, $\forall y \in \mathbb{R}$, where $C$ is a positive constant and $m \geq 0$.

Proof. The expressions of $e^{\theta \mathcal{L}}(y, x)$ and $e^{\theta \mathcal{L}}$ are given in [1] Formula (44), p. 554. See also [28].

(i) Follows by the positivity of the kernel. (ii) and (iii) Follow by simple calculations using (4.45) and (4.46) so we omit the proof. (iv) Follows from (4.45) and (4.46). See also [4] Lemma 4, p.555. (v)-(vi) follow also by simple calculations.

Now, we write down in the following, the equation satisfied by $v_-$ and estimate its source term:

**Lemma 4.16** (Equation satisfied by $v_-$). For all $A \geq 1$, there exists $s_{13}(A)$ sufficiently large such that for $s \geq s_{13}$, if $v \in \mathcal{V}_A(s)$,

$$\|\partial_y v(s)\|_{\infty} \leq \frac{CA^2}{s^{\gamma - \frac{1}{2}}}$$

(4.47)

and

$$|\partial_y v(y, s)| \leq \frac{CA}{s^{\gamma + \frac{1}{2}}(1 + |y|) + \frac{C\sqrt{A}}{s^{\gamma - \frac{1}{2}}}(1 + |y|^2) + \frac{CA^2}{s^{\gamma}}(1 + |y|^3)}, \forall y \in \mathbb{R},$$

(4.48)

then we have

$$\partial_s v_- = \mathcal{L}v_- + \mathcal{F}(y, s),$$

(4.49)

with

$$|\mathcal{F}(y, s)| \leq \frac{CA^2}{s^{\gamma}}(1 + |y|^3), \forall y \in \mathbb{R}.$$  

(4.50)
Proof. From equation (3.2), we have
\[(\partial_s v)_- = (L v)_- + (V v)_- + (B(v))_- + (G(v))_- + (R(y,s))_- .\]

Using the hypotheses of the Lemma, the results obtained in Lemmas 4.10 and 4.14, we deduce that, for \(s\) sufficiently large,
\[|(\partial_s v)_- - (L v)_-| \leq \frac{CA}{s^\gamma} (1 + |y|^3), \quad \forall y \in \mathbb{R} .\]  

(4.51)

Furthermore, we have from (3.11) and (3.12)
\[v_b = \chi v = \sum_{m=0}^{2} v_m h_m + v_- , \]  

(4.52)

hence
\[(\partial_s \chi)v + \chi \partial_s v = \sum_{m=0}^{2} v'_m h_m + \partial_s v_- , \]
on the one hand. On the other hand, applying (3.12) to \(\partial_s v\), we have
\[\chi \partial_s v = \sum_{m=0}^{2} (\partial_s v)_m h_m + (\partial_s v)_- . \]

Then, by taking the difference of the last two identities, we get
\[|\partial_s v_\alpha - (\partial_s v)_-| \leq |v||\partial_s \chi| + C \left( \sum_{m=0}^{2} |v'_m - (\partial_s v)_m| \right) (1 + |y|^2) . \]  

(4.53)

But we have by definition
\[v'_m = \frac{d}{ds} \left( \int_{\mathbb{R}} \chi k_m v \rho \right) = \int_{\mathbb{R}} \chi k_m \partial_s v \rho + \int_{\mathbb{R}} \partial_s \chi k_m v \rho\]
and
\[(\partial_s v)_m = \int_{\mathbb{R}} \chi k_m \partial_s v \rho . \]

Then, since \(\|v(s)\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{4\beta - 3\gamma}}\) by item (i) of Proposition 4.7 and
\[|\partial_s \chi| \leq \frac{C}{s} 1_{\{Ks^\beta < |y| < 2Ks^\beta\}} , \]  

(4.54)

where \(1_{\{Ks^\beta < |y| < 2Ks^\beta\}}\) is the characteristic function of the set \(\{Ks^\beta < |y| < 2Ks^\beta\}\), we deduce that
\[|v'_m - (\partial_s v)_m| = \left| \int_{\mathbb{R}} \partial_s \chi h_m v \rho \right| \leq \frac{CA^2}{s^{\gamma - 3\beta + 1}} e^{-s^{2\beta}} \leq \frac{1}{s^{4\beta}} \leq \frac{1}{s^{2\beta + 1}} \leq \frac{1}{s^\gamma} . \]

(4.55)

for \(s\) large enough. Now, using (4.54), we see that
\[|\partial_s \chi| \leq C \frac{|y|^i}{s^{1+i\beta}}, \quad i = 0, 1, 2 . \]

Therefore, since \(v \in \vartheta_A(s)\), by Proposition 4.7 part (iii) and by using the conditions on \(\gamma\) given by (4.3), we get
\[|v||\partial_s \chi| \leq C \left( \frac{A}{s^{4\beta + 2}} + \frac{\sqrt{A}}{s^{5\beta}} + \frac{A^2}{s^{7\gamma + 1}} \right) (1 + |y|^3) \]
\[\leq C \frac{A^2}{s^{7\gamma + 1}} (1 + |y|^3) . \]  

(4.56)
and we deduce, from (4.52), (4.55) and (4.56) that
\[ |\partial_s v - (\partial_s v)_-| \leq C \frac{A^2}{s^{\gamma+1}} (1 + |y|^3), \quad \forall \ y \in \mathbb{R}. \]  
(4.57)

Since \( L(h_m) = (1 - \frac{m}{2}) h_m \), we write from (4.52) and (4.12) applied to \( L \),
\[ L(\chi v) = \sum_{m=0}^{2} v_m \left( 1 - \frac{m}{2} \right) h_m + L v_-, \]
\[ \chi L v = \sum_{m=0}^{2} (L v)_m h_m + (L v)_-. \]

Therefore, we deduce
\[ |L v_ - (L v)_-| \leq |L(\chi v) - \chi L v| + \left( \sum_{m=0}^{2} \left| v_m \left( 1 - \frac{m}{2} \right) - (L v)_m \right| \right) \left( 1 + |y|^2 \right). \]  
(4.58)

Since \( L \) is self-adjoint, we have
\[ (L v)_m = \int_\mathbb{R} \chi k_m L v \rho = \int_\mathbb{R} L(\chi k_m) v \rho, \]
then
\[ \left| v_m \left( 1 - \frac{m}{2} \right) - (L v)_m \right| = \left| \int_\mathbb{R} \chi (L k_m) v \rho - \int_\mathbb{R} L(\chi k_m) v \rho \right| \]
\[ = \left| \int_\mathbb{R} (\chi (L k_m) - L(\chi k_m)) v \rho \right|. \]

Now, using the expression of \( L \) given by (2.11), we have for any regular function \( r \),
\[ |L(\chi r) - \chi L r| \leq |r| \left( \frac{1}{2} |y||\partial_y \chi| + |\partial_y^2 \chi| \right) + 2|\partial_y \chi| |\partial_y r|. \]  
(4.59)

Since
\[ s^\beta |\partial_y \chi| + s^{2\beta} |\partial_y^2 \chi| \leq 1_{\{K_s^\beta < |y| < 2K_s^\beta\}}, \]  
(4.60)

using item (i) of Proposition 4.7 and (4.47) with \( r = \chi k_m \), we get,
\[ \left| v_m \left( 1 - \frac{m}{2} \right) - (L v)_m \right| \leq \frac{C A^2}{s^{\gamma-3\beta}} e^{-s^{2\beta}} \leq \frac{1}{s^{4\beta}} \leq \frac{1}{s^{2\beta+1}} \leq \frac{1}{s^\gamma}, \]  
(4.61)

for \( s \) sufficiently large. On the other hand, since
\[ |y| |\partial_y \chi| + |\partial_y \chi| + |\partial_y^2 \chi| \leq C \frac{|y|^i}{s^\beta}, \quad i = 0, 1, 2, \]
from (4.60), using the hypotheses on \( v \) and \( \partial_y v \), (namely item (iii) of Proposition 4.7 and (4.48)), then by applying the inequality (4.59) with \( r = v \), we get
\[ |L(\chi v) - \chi L v| \leq \left( \frac{C A}{s^{4\beta+1}} + \frac{C \sqrt{A}}{s^{5\beta-1}} + \frac{C A^2}{s^\gamma} \right) (1 + |y|^3) \leq C \frac{A^2}{s^\gamma} (1 + |y|^3), \]  
(4.62)

since \( \gamma \leq \min(5\beta - 1, 4\beta + 1) \). Then, we deduce from (4.58), (4.61) and (4.62),
\[ |L v_- - (L v)_-| \leq C \frac{A^2}{s^\gamma} (1 + |y|^3). \]  
(4.63)

Now using (4.51), (4.57) and (4.63) we conclude the proof of the lemma. \( \square \)

Now, we are in a position to give our parabolic regularity statement:
Proposition 4.17 (Parabolic regularity for equation (3.2)). For all \( A \geq 1 \), there exists \( s_{14}(A) \) such that for all \( s_0 \geq s_{14}(A) \) the following holds:

Consider \( v(s) \) a solution of equation (3.2) on \([s_0, s_1]\) where \( s_1 \geq s_0 \) with initial data at \( s = s_0 \)

\[ v(y, s_0) = \psi_{s_0, d_0, d_1}(y) \]

defined in (4.2) with \((d_0, d_1) \in D_{s_0}\), and

\[ v(s) \in \mathcal{V}_A(s) \text{ for all } s \in [s_0, s_1]. \quad (4.64) \]

Then, for all \( s \in [s_0, s_1] \), we have

(i) \( \|\nabla v(s)\|_{L^\infty(\mathbb{R})} \leq C \frac{A^2}{s^{3-\beta}} \).

(ii) \( |\nabla v(y, s)| \leq C \frac{A^2}{s^{3-\beta}} (1 + |y|) + C \frac{\sqrt{A}}{s^{3\beta-1}} (1 + |y|^2) + C \frac{A^2}{s^\gamma} (1 + |y|^3), \forall y \in \mathbb{R}. \)

(iii) \( |\nabla v(y, s)| \leq C \frac{A^2}{s^\gamma} (1 + |y|^3), \forall y \in \mathbb{R}, \)

where \( C \) is a positive constant.

Proof. We consider \( A \geq 1, s_0 \geq 1 \) and \( v(s) \) a solution of equation (3.2) defined on \([s_0, s_1]\) where \( s_1 \geq s_0 \geq 1 \) and \( v(s_0) \) given by (4.2) with \((d_0, d_1) \in D_{s_0}\). We also assume that \( v(s) \in \mathcal{V}_A(s) \) for all \( s \in [s_0, s_1] \). For each of the items (i), (ii) and (iii), we consider two cases in the proof: \( s \leq s_0 + 1 \) and \( s > s_0 + 1 \).

Proof of Part (i)

Case 1: \( s \leq s_0 + 1 \). Let \( s'_1 = \min(s_0 + 1, s_1) \) and take \( s \in [s_0, s'_1] \). Then, since \( s_0 \geq 1 \), we have for any \( t \in [s_0, s] \),

\[ s_0 \leq t \leq s \leq s_0 + 1 \leq 2s_0, \text{ hence } \frac{1}{2} \leq \frac{t}{s} \leq \frac{1}{s_0} \leq \frac{1}{2}. \quad (4.65) \]

From equation (3.2), we write for any \( s \in [s_0, s'_1] \),

\[ v(s) = e^{(s-s_0)\mathcal{L}}v(s_0) + \int_{s_0}^s e^{(s-t)\mathcal{L}}F(t)dt, \quad (4.66) \]

where

\[ F(x, t) = Vv(x, t) + G(x, t) + B(v) + R(x, t). \quad (4.67) \]

Hence

\[ \nabla v(s) = \nabla e^{(s-s_0)\mathcal{L}}v(s_0) + \int_{s_0}^s \nabla e^{(s-t)\mathcal{L}}F(t)dt, \quad (4.68) \]

and

\[ |\nabla v(y, s)| \leq |\nabla e^{(s-s_0)\mathcal{L}}v(s_0)| + \int_{s_0}^s |\nabla e^{(s-t)\mathcal{L}}F(t)|dt. \]

Then, we write from (4.68) and Lemma 4.15 for all \( s \in [s_0, s'_1] \),

\[ \|\nabla v(s)\|_{L^\infty(\mathbb{R})} \leq \|\nabla e^{(s-s_0)\mathcal{L}}v(s_0)\|_{L^\infty} + \int_{s_0}^s \|\nabla e^{(s-t)\mathcal{L}}F(t)\|_{L^\infty}dt \]

\[ \leq C\|\nabla v(s_0)\|_{L^\infty} + C \int_{s_0}^s \frac{\|F(t)\|_{L^\infty}}{\sqrt{1 - e^{-(s-t)}}}dt. \quad (4.69) \]

Using (4.8) and (4.65), we write

\[ \|\nabla v(s_0)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{s_0^{\gamma-3\beta}} \leq \frac{C}{s_0^{\gamma-3\beta}}. \quad (4.70) \]
Furthermore, using \((4.25), (4.40)\) and \((4.42)\), we write
\[
|G(x,t)| \leq C|\nabla \varphi|^{q-1}|\nabla v| + C|\nabla v|^q
\leq \frac{C}{s^{2(q-1)}} \|\nabla v\|_\infty + C\|\nabla v\|^q_
\leq \frac{C}{s^{2}} \|\nabla v\|_\infty + C\|\nabla v\|^q.
\]

Using \((4.64), (ii) of Proposition 4.7\) Lemmas 4.10 and 4.14, together with \((4.65)\), we write for all \(t \in [s_0, s]\) and \(x \in \mathbb{R},\)
\[
|F(x,t)| \leq \frac{CA^2}{s^{\gamma-3\beta}} + \frac{C}{s^{\gamma-3\beta}} \|\nabla v(t)\|_{L_\infty(\mathbb{R})} + C\|\nabla v(t)\|^q_{L_\infty(\mathbb{R})}.
\]
Therefore, from \((4.70)\) and \((4.71)\) we write with \(g(s) = \|\nabla v(s)\|_{L_\infty(\mathbb{R})},\)
\[
g(s) \leq \frac{C_1A^2}{s^{\gamma-3\beta}} + C \int_{s_0}^s \frac{t^{-\frac{2}{q}}(s-t)}{\sqrt{1-e^{-(s-t)}}} dt,
\]
for some universal constant \(C_1 > 0\). Using a Gronwall’s argument, we claim that
\[
\forall s \in [s_0, s_1], \ g(s) \leq \frac{2C_1A^2}{s^{\gamma-3\beta}},
\]
for \(s_0\) large enough. Indeed, let
\[
s_* = \sup \left\{ s \in [s_0, s_1] \mid \forall s' \in [s_0, s], \ g(s') \leq \frac{2C_1}{s^{\gamma-3\beta}} \right\}.
\]
From \((4.72), s_*\) is well defined and \(s_* > s_0\). Suppose, by contradiction, that \(s_* < s_1\). In this case, by continuity, we have:
\[
g(s_*) = \frac{2C_1}{s_*^{\gamma-3\beta}}.
\]
By \((4.72), (4.65)\) and the definition of \(s_*\), we have, for all \(s \in [s_0, s_*],\)
\[
g(s) \leq \frac{C_1A^2}{s^{\gamma-3\beta}} + C \int_{s_0}^s \left(\frac{2C_1A^2}{s^{\gamma-3\beta}}\right) dq + 2C_1A^2\beta^{\beta-\frac{1}{2}}
\leq \frac{C_1A^2}{s^{\gamma-3\beta}} + C \left[\frac{2C_1A^2}{s^{\gamma-3\beta}} \int_{s_0}^s \frac{1}{\sqrt{1-e^{-(s-t)}}} dt\right]
\leq \frac{C_1A^2}{s^{\gamma-3\beta}} + C \left[\frac{2C_1A^2}{s^{\gamma-3\beta} + 1}\right]
\leq \frac{3C_1}{2s^{\gamma-3\beta}},
\]
for \(s_0\) sufficiently large, since \(q > 1\). This is a contradiction by \((4.73)\). Hence \(s_1' = s_*\), and \((4.73)\) holds.
This concludes the proof of Part (i) when \(s \leq s_0 + 1\).

**Case 2:** \(s > s_0 + 1\). (Note that this case does not occur when \(s_1 \leq s_0 + 1\)). Take \(s \in (s_0 + 1, s_1]\). Then, we have for any \(s' \in (s-1, s]\) and \(t \in [s-1, s']\), \(s \geq s_0 + 1 \geq 2\), hence \(s = s-1 + 1 \leq 2(s-1)\). Therefore,
\[
s-1 \leq t \leq s' \leq 2(s-1) \text{ hence } \frac{1}{s} \leq \frac{1}{s'} \leq \frac{1}{t} \leq \frac{1}{s-1} \leq \frac{2}{s}.
\]

From equation (3.2), we write for any \( s' \in [s-1,s] \),

\[
v(s') = e^{(s'-s+1)t} v(s-1) + \int_{s-1}^{s'} e^{(s'-t)t} F(t) dt,
\]

where \( F(x,t) \) and \( e^tL \) are given in (4.67) and (4.45). Using Lemma 4.15 we see that for all \( s' \in [s-1,s] \)

\[
\| \nabla v(s') \|_{L^\infty(\mathbb{R})} \leq \| \nabla e^{(s'-s+1)t} v(s-1) \|_{L^\infty(\mathbb{R})} + \int_{s-1}^{s'} \| \nabla e^{(s'-t)t} F(t) \|_{L^\infty(\mathbb{R})} dt \\
\leq \frac{C}{\sqrt{1-e^{-(s'-s+1)}}} \| v(s-1) \|_{L^\infty(\mathbb{R})} + C \int_{s-1}^{s'} \| F(t) \|_{L^\infty(\mathbb{R})} dt.
\] (4.77)

Recall from (4.64), Proposition 4.7 and (4.75) that

\[
\| v(s-1) \|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{(s-1)^{\gamma-3\beta}} \leq \frac{CA^2}{s^{\gamma-3\beta}}
\]

for \( s_0 \) sufficiently large. Therefore, using (4.71), (4.75), we write with \( g(s') = \| \nabla v(s') \|_{L^\infty(\mathbb{R})} \),

\[
g(s') \leq \frac{C_A^2}{s^{\gamma-3\beta}} \sqrt{1-e^{-(s'-s+1)}} + C \int_{s-1}^{s'} \frac{t^{-1/2}g(t) + g(t)^q}{\sqrt{1-e^{-(s'-t)}}} dt.
\]

Using a Gronwall’s argument, as for the previous case, we see that for \( s \) large enough,

\[
\forall s' \in [s-1,s], \ g(s') \leq \frac{2CA^2}{s^{\gamma-3\beta}} \sqrt{1-e^{-(s'-s+1)}}.
\]

Taking \( s' = s \) concludes the proof of Proposition 4.17 part (i), when \( s > s_0 + 1 \).

**Proof of Part (ii)** Let us define the function

\[
H(y,s) = \frac{A}{s^{2\beta+1}}(1+|y|) + \frac{\sqrt{A}}{s^{\beta-1}}(1+|y|^2) + \frac{A^2}{s^{\gamma}}(1+|y|^3), \quad \forall \ s > 0, \ \forall \ y \in \mathbb{R}.
\]

We consider \( s \in [s_0,s_1] \). Since \( v \in \partial A(s) \), using Proposition 4.7 part (iii), we see that

\[
|v(y,s)| \leq CH(y,s), \quad \text{for all } y \in \mathbb{R},
\]

provided that \( s_0 \) is sufficiently large. Similarly, since \( Vv(s) \in \partial CA(s) \), \( B(v) \in \partial C(s) \) and \( R \in \partial C(s) \) by Lemma 4.10 and 4.14 it follows that

\[
|Vv(y,s)| + |B(v)| + |R(y,s)| \leq CH(y,s).
\]

Using (4.25) and (4.40), we see by definition (4.67) of \( F \) that

\[
|F(y,s)| \leq CH(y,s) + \frac{C}{\sqrt{s}} |\nabla v(y,s)| + |\nabla v(y,s)|^q.
\]

Using the fact that \( v \in \partial A(s) \), it follows by the previous Part (i) that

\[
|\nabla v(s)|^{q-1}_{L^\infty(\mathbb{R})} \leq \frac{CA^{2(q-1)}}{s^{(q-1)(\gamma-3\beta)}},
\]

which is sufficiently small for \( s_0 \) large. Therefore, noting from (4.42) and (4.44) that

\[
(q - 1)(\gamma - 3\beta) = \frac{\gamma - 3\beta}{2\beta} < \frac{1}{2},
\]

we see that

\[
|F(y,s)| \leq CH(y,s) + \frac{CA^{2(q-1)}}{s^{(q-1)(\gamma-3\beta)}|\nabla v(y,s)|}.
\] (4.78)
Furthermore, we have by Proposition 4.15 Part (iii) that
\[ |\nabla v(y, s_0)| \leq C_0 H(y, s_0), \]
for some \( C_0 > 0 \). If we introduce the norm (depending on \( s \))
\[ \mathcal{N}(r) = \left\| \frac{r}{H(\cdot, s)} \right\|_{L^\infty(\mathbb{R})}, \]
then we see that
\[ \mathcal{N}(\nabla v(s_0)) \leq C_0. \]  \hspace{1cm} (4.79)
We should show that
\[ \sup_{s \in [s_0, s_*]} \mathcal{N}(\nabla v(s)) \leq M, \]  \hspace{1cm} (4.80)
where \( M \) is a fixed constant to be determined later, in particular \( M > 2C_0 \). We argue by contradiction. From (4.79), we consider \( s_* \in (s_0, s_1) \) the largest one such that the previous inequality is satisfied. Using item (i) of Proposition 4.17 (which is already proved), we can restrict \( y \) to some compact interval depending on \( s \). Then, using the uniform continuity, we deduce that
\[ \mathcal{N}(\nabla v(s_*)) = M, \]  and \( \mathcal{N}(\nabla v(s)) \leq M, \]  for all \( s \in [s_0, s_*]. \)  \hspace{1cm} (4.81)
We handle two cases:

**Case 1:** \( s_* \leq s_0 + 1 \). Recalling the integral equation (4.68), we have that
\[ \forall s \in [s_0, s_*], \nabla v(s) = \nabla e^{(s-s_0)\mathcal{L}} v(s_0) + \int_{s_0}^{s} \nabla \left( e^{(s-t)\mathcal{L}} F(t) \right) dt. \]  \hspace{1cm} (4.82)
Then, by Lemma 4.15 Part (ii) we have that
\[ \mathcal{N}(\nabla e^{(s-s_0)\mathcal{L}} v(s_0)) \leq C_1 C_0, \]  \hspace{1cm} (4.83)
where \( C_0 \) is used in (4.79), and \( C_1 > 0 \) is a constant. On the other hand, using (4.79) and (4.81), we see that
\[ |F(x, t)| \leq CH(x, t) + \frac{CA^{2(q-1)}}{t^{(\gamma-3\beta)}} |\nabla v(x, t)| \]
\[ \leq C \left( 1 + \frac{A^{2(q-1)}}{x^{(\gamma-3\beta)}} M \right) H(x, t), \]  \( \forall t \in [s_0, s_*]. \)
Using Lemma 4.15 Part (vi) and (4.65) we deduce that for \( s_0 \) sufficiently large,
\[ \left| \int_{s_0}^{s} \nabla e^{(s-t)\mathcal{L}} F(t) dt \right| \leq \int_{s_0}^{s} \left| \nabla e^{(s-t)\mathcal{L}} F(t) \right| dt \]
\[ \leq C \left( 1 + \frac{A^{2(q-1)}}{s_0^{(\gamma-3\beta)}} M \right) \int_{s_0}^{s} \frac{H(y, t)}{\sqrt{1 - e^{-(s-t)}}} dt \]
\[ \leq C \left( 1 + \frac{A^{2(q-1)}}{s_0^{(\gamma-3\beta)}} M \right) \left( \int_{s_0}^{s} \frac{1}{\sqrt{1 - e^{-(s-t)}}} dt \right) H(y, s) \]
\[ \leq C_2 \left( 1 + \frac{A^{2(q-1)}}{s_0^{(\gamma-3\beta)}} M \right) H(y, s), \]  \hspace{1cm} (4.84)
for some \( C_2 > 0 \). Hence, using also (4.83), we get from (4.82),
\[ |\nabla v(y, s)| \leq \left( C_1 C_0 + C_2 \left( 1 + \frac{A^{2(q-1)}}{s_0^{(\gamma-3\beta)}} M \right) \right) H(y, s), \]  \( \forall s \in [s_0, s_*]. \)  \hspace{1cm} (4.85)
Assuming that
\[ M \geq \max \left( 2C_0, 2C_1C_0 + 4C_2 \right), \]
then taking \( s_0 \) large so that
\[ \frac{A^{2(q-1)}}{s_0^{(q-1)(\gamma-3\beta)}} M \leq 1, \]
we see from (4.85) that
\[ \forall s \in [s_0, s_*], \forall y \in \mathbb{R}, |\nabla v(y, s)| \leq (C_0C_1 + 2C_2)H(y, s) \leq \frac{M}{2} H(y, s), \]
which is a contradiction by (4.81).

**Case 2:** \( s_* > s_0 + 1 \). From (4.76), we write for any \( s' \in [s_* - 1, s_*] \),
\[ \nabla v(s') = \nabla e^{(s' - s_*)L} v(s_* - 1) + \int_{s_* - 1}^{s'} \nabla e^{(t-s_*)L} F(t)dt. \] (4.87)
Since \( v(s_* - 1) \in \partial A(s_* - 1) \), by Proposition 4.7 Part (iii) we have \( |v(s_* - 1)| \leq CH(y, s_*) \), for \( s_0 \) sufficiently large. By Lemma 4.15 Part (vi), we have for all \( s' \in (s_* - 1, s_*) \)
\[ |\nabla e^{(s' - s_*)L} v(s_* - 1)| \leq C \frac{e^{(s' - s_*)L}}{\sqrt{1 - e^{-(s' - s_*)L}}} H(y, s_*). \]
Hence at \( s' = s_* \), we have
\[ |\nabla e^L v(s_* - 1)| \leq C_3 H(y, s_*), \]
for some constant \( C_3 > 0 \). Proceeding as for (4.81), we write
\[ \left| \int_{s_* - 1}^{s_*} \nabla e^{(t-s_*)L} F(t)dt \right| \leq C_2 \left( 1 + \frac{A^{2(q-1)}}{s_0^{(q-1)(\gamma-3\beta)}} M \right) H(y, s_*). \]
Then, from (4.87), we derive that
\[ |\nabla v(y, s_*)| \leq |\nabla e^L v(s_* - 1)| + \left| \int_{s_* - 1}^{s_*} \nabla e^{(t-s_*)L} F(t)dt \right|, \]
\[ \leq \left[ C_3 + C_2 \left( 1 + \frac{A^{2(q-1)}}{s_0^{(q-1)(\gamma-3\beta)}} M \right) \right] H(y, s_*). \]
Fixing
\[ M = 2 \max \left( C_0, C_1C_0 + 2C_2, C_3 + 2C_2 \right), \]
and taking \( s_0 \) large enough so that (4.86) holds, we see that
\[ |\nabla v(y, s_*)| \leq (C_3 + 2C_2)H(y, s_*) \leq \frac{M}{2} H(y, s_*), \]
and we get a contradiction with (4.81).

Since the value we have just fixed for \( M \) also leads to a contradiction in Case 1, we have just proved the validity of (4.80). This finishes the proof of Part (ii).

**Proof of Part (iii)** By Lemma 4.16 we have that
\[ \partial_s v_\ast = L \nu_\ast + \overline{F}(y, s), \] (4.88)
with
\[ |\overline{F}(y, s)| \leq \frac{CA^2}{s^7} (1 + |y|^3), \forall y \in \mathbb{R}. \] (4.89)
By Proposition 4.5, Part (iii), we have that
\[
|\nabla v_-(y, s_0)| \leq \frac{1}{s_0^\gamma}(1 + |y|^3), \quad \forall \ y \in \mathbb{R}.
\] (4.90)

We should show that
\[
|\nabla v_-(y, s)| \leq \frac{M}{s_0^\gamma}(1 + |y|^3), \quad \forall \ y \in \mathbb{R}, \quad \forall \ s \in [s_0, s_1],
\] (4.91)

for some $M \geq 2$ that will be fixed later, provided that $s_0$ is large enough. We then distinguish two cases.

**Case 1:** $s_\ast \leq s_0 + 1$. From (4.88), we write
\[
v_-(s) = e^{(s-s_0)} L v_-(s_0) + \int_{s_0}^{s} e^{(s-t)} L F(t) dt,
\]
and
\[
|\nabla v_-(s)| \leq |\nabla e^{(s-s_0)} L v_-(s_0)| + \int_{s_0}^{s} |\nabla e^{(s-t)} L F(t)| dt.
\]

Then, by (4.89), Lemma 4.15, Parts (v) and (vi) together with (4.90) and (4.89), we see that
\[
\forall \ y \in \mathbb{R}, \ |\nabla v_-(y, s)| \leq \frac{C}{s_0^\gamma} (1 + |y|^3) + \frac{C A^2}{s_0^\gamma} \left( \int_{s_0}^{s} \frac{1}{\sqrt{1 - e^{-(s-t)}}} dt \right) (1 + |y|^3)
\]
\[
\leq \frac{C_4 A^2}{s_0^\gamma} (1 + |y|^3),
\]
for $s_0$ sufficiently large and some constant $C_4 > 0$.

**Case 2:** $s_\ast > s_0 + 1$. Using (4.88), we write
\[
\nabla v_-(s) = \nabla e^{(s-s_\ast+1)} L v_-(s_\ast - 1) + \int_{s_\ast-1}^{s} \nabla e^{(s-t)} L F(t) dt, \quad s \in (s_\ast - 1, s_\ast).
\]

Since
\[
|v_-(y, s_\ast - 1)| \leq \frac{A}{s_\ast} (1 + |y|^3),
\]
from the fact that $v \in \partial A(s_\ast - 1)$, using Lemma 4.15 Part (vi), together with (4.89) we get
\[
|\nabla v_-(y, s_\ast)| \leq \frac{CA}{(s_\ast - 1)^\gamma \sqrt{1 - e^{-1}}} (1 + |y|^3) + \frac{CA^2}{s_\ast^\gamma} (1 + |y|^3) \left( \int_{s_\ast-1}^{s_\ast} \frac{1}{\sqrt{1 - e^{-(s-t)}}} dt \right)
\]
\[
\leq \frac{C_5 A^2}{s_\ast^\gamma} (1 + |y|^3).
\]

Fixing
\[
M = 2 \max(2, C_4 A^2, C_5 A^2),
\]
we see that (4.91) follows. This finishes the proof of Proposition 4.17.

4.2.4. Reduction to a finite dimensional problem. In the following, we reduce the problem to a finite-dimensional one. Namely, we prove Proposition 4.6. To do so, we project Equation (3.2) on the different components of the decomposition (3.13). Let us first give those projections in the following proposition, then use it to derive Proposition 4.6. This is the statement of the following proposition.
Proposition 4.18 (Dynamics of the different components). There exists $A_6 \geq 1$ such that for all $A \geq A_6$ there exists $s_0(A)$ large enough such that the following holds for all $s_0 \geq s_0(A)$:

Assume that for some $s_1 \geq \tau \geq s_0$, we have

$$v(s) \in \partial A(s), \text{ for all } s \in [\tau, s_1],$$

and that $\nabla v(s)$ satisfies the estimates stated in Parts (i)-(ii)-(iii) of Proposition 4.17. Then, the following holds for all $s \in [\tau, s_1]$:

(i) (ODE satisfied by the positive modes) For $m = 0$ and $m = 1$, we have

$$\left| v'_m(s) - \left(1 - \frac{m}{2}\right)v_m(s) \right| \leq \frac{C}{s^{2\beta + 1}}.$$

(ii) (ODE satisfied by the null mode) For $m = 2$, we have

$$\left| v'_2(s) + \frac{2\beta + 1}{s}v_2(s) \right| \leq \frac{C}{s^{4\beta}}.$$

(iii) (Control of the negative and outer modes) We have

$$\left\| \frac{v_-(y, s)}{1 + |y|^3} \right\|_{L^\infty} \leq Ce^{-(\frac{s-\tau}{2})} \left\| \frac{v_-(y, \tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C(1 + s - \tau)}{s^\gamma},$$

$$\left\| v_-(s) \right\|_{L^\infty} \leq Ce^{-(\frac{s-\tau}{p})} \left\| v_-(\tau) \right\|_{L^\infty} + Ce^{s-\tau} s^{3\beta} \left\| \frac{v_-(y, \tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C(1 + (s-\tau)e^{s-\tau})}{s^{\gamma-3\beta}}.$$

Remark 4.19. In item (ii), the value of the coefficient $2\beta + 1$ in front of $\frac{v_2}{s}$ crucially comes from an algebraic identity at the end of the proof of this item, involving the parameter $b$ defined in (1.6).

Proof. The proof will be carried out in 3 steps:

- In the first step, we write equations satisfied by $v_0$, $v_1$. Then, we prove (i) of Proposition 4.18.
- In the second step, we write an equation satisfied by $v_2$. Then, we prove (ii) of Proposition 4.18.
- In the third step, we write integral equations satisfied by $v_-$ and $v_c$. Then, we prove Part (iii) of Proposition 4.18.

Step 1: The positive modes. As in the proof of Lemma 4.10, we project equation (3.2), on the modes $m = 0, 1$ defined in (3.13):

$$v(t) = (L_0 v)_m + V(v)_m + (B(v))_m + (G(v))_m + R_m, \text{ with } m = 0, 1, 2 \text{ in (4.92)}.$$

Recall from (4.55) and (4.61) that

$$\left| v'_m - (\partial_s v)_m \right| + \left| \left(1 - \frac{m}{2}\right)v_m - (L_0 v)_m \right| \leq \frac{2}{s^{2\beta + 1}}, \text{ with } m = 0, 1, 2 \text{ in (4.93)}.$$

On the other hand, for $s_0$ sufficiently large, by Lemmas 4.10 and 4.14 together with the hypotheses, we have

$$\left| (V(v)_m + (B(v))_m + (G(v))_m + R_m \right| \leq \frac{C}{s^{2\beta + 1}}, \text{ with } m = 0, 1.$$

This proves (i) of the proposition.

Step 2: The null mode. From (4.92), we have

$$(\partial_s v)_2 = (L_0 v)_2 + (V_v)_2 + (B(v))_2 + R_2 + (G(v))_2.$$

Note that the first terms are estimated in (4.93). Using Lemmas 4.10 and 4.14 we write

$$\left| (V(v)_2 + (B(v))_2 + R_2 \right| \leq \frac{C}{s^{4\beta}}.$$
Thus, it remains to prove that
\[ |(G(v))_2 + \frac{2\beta + 1}{s}v_2(s)| \leq \frac{C}{s^{4\beta}} \] (4.94)
in order to conclude. Let us prove that. Write
\[ v = v_1h_1 + v_2h_2 + \tilde{v}. \] (4.95)
Then
\[ \partial_y v = v_1 + 2yw_2 + \partial_y \tilde{v}. \]
Write also
\[
G(v) = \mu \left[ |\partial_y \varphi + \partial_y v|^q - |\partial_y \varphi|^q \right] \\
= \mu \left[ |\partial_y \varphi + \partial_y v|^q - |\partial_y \varphi + v_1 + 2yv_2|^q \right] \\
+ \mu \left[ |\partial_y \varphi + v_1 + 2yv_2|^q - |\partial_y \varphi|^q \right] \\
:= G^1(v) + G^2(v). \] (4.96)

We begin by estimating \( G^1(v) \). If \( \chi \) is defined in (3.9), we have
\[
|\chi G^1(v)| = \mu \chi \left[ |\partial_y \varphi + \partial_y v|^q - |\partial_y \varphi + v_1 + 2yv_2|^q \right] \\
= \mu \chi \left[ |\partial_y \varphi + v_1 + 2yv_2 + \partial_y \tilde{v}|^q - |\partial_y \varphi + v_1 + 2yv_2|^q \right] \\
\leq C|\partial_y \varphi + v_1 + 2yv_2 + \theta \partial_y \tilde{v}|^{q-1} |\partial_y \tilde{v}|^\chi,
\]
for some \( \theta \in [0,1] \). Note from (4.95) and (4.13) that \( \tilde{v} = v_0 + v_\gamma + v_\nu \). Then, \( \partial_y \tilde{v} = \partial_y v_\gamma + \partial_y v_\nu = \partial_y v_\nu + (1-\gamma)\partial_y v - \partial_y \chi v \). Since we assumed that \( v(s) \in \partial_A(s) \) and \( \partial_y v \) satisfies the identities stated in Parts (i), (ii) and (iii) of Proposition 4.17, it follows that
\[
\forall \ |y| \leq 2Ks^\beta, \ |\partial_y \tilde{v}| \leq \frac{CA^2}{s^\gamma} (1 + |y|^{3}), \text{ for } s_0 \text{ large enough.}
\]

Then, we write from (4.25), (4.42) and the fact that \( v(s) \in \partial_A(s) \), for all \( |y| \leq 2Ks^\beta \),
\[
|\partial_y \varphi + v_1 + 2yv_2 + \theta \partial_y \tilde{v}|^{q-1} \leq C \left[ |\partial_y \varphi|^{q-1} + |v_1|^{q-1} + |y|^{q-1} |v_2|^{q-1} + |\partial_y \tilde{v}|^{q-1} \right] \\
\leq C |y|^{q-1} s + \frac{CA^{q-1}}{s^{(2\beta+1)(q-1)}} + \frac{CA^{q-1}}{s^{(\beta-1)(q-1)}} |y|^{q-1} \\
+ \frac{CA^{2(q-1)}}{s^{(q-1)}} \left( 1 + |y|^{3(q-1)} \right) \\
\leq \frac{C}{s} (1 + |y|^{q-1}) + \frac{CA^{2(q-1)}}{s^{(q-1)}} (1 + |y|^{3(q-1)}),
\]
for \( s_0 \) large enough, where we need the fact, that \( \gamma < 2\beta + 1 \), and \( (4\beta - 1)(q-1) = (4\beta - 1)/(2\beta) = 2 - 1/(2\beta) > 1 \), because \( \beta > 1/2 \). Hence, since \( \gamma q > \gamma + 1 \) from (4.3) and (4.12), we get
\[
|\chi G^1(v)| \leq \frac{CA^2}{s^{\gamma+1}}(1 + |y|^{q+2}) + \frac{A^{2q}}{s^q} (1 + |y|^{3q}) \leq \frac{CA^2}{s^{\gamma+1}} (1 + |y|^{3q}).
\]
Then, since \( \gamma > 4\beta - 1 \), we get for \( s_0 \) sufficiently large,
\[
\left| \int \chi G^1(v)k_2 \right| \leq \frac{CA^{2q}}{s^{\gamma+1}} < \frac{1}{s^{4\beta}}, \quad (4.97)
\]
Let us now consider \( G^2(v) \). Using (4.25) we write
\[
\partial_y \varphi = -\frac{2\kappa}{(p-1)^2} s^{2\beta} \left[ 1 + O \left( \frac{|y|^2}{s^{2\beta}} \right) \right]
\]
hence

\[ |\partial_y \varphi|^q = \left( \frac{2bk}{(p-1)^2} \right)^q \frac{|y|^q}{s^{2\beta+1}} + O \left( \frac{|y|^{q+2}}{s^{4\beta+1}} \right). \]  

(4.98)

On the other hand, since

\[ |v_2s^{2\beta}| \leq \frac{\sqrt{A}}{s^{2\beta-1}} \to 0 \quad \text{as} \quad s \to \infty \quad \text{and} \quad |v_1s^{2\beta}| \leq \frac{A}{s} \to 0 \quad \text{as} \quad s \to \infty, \]  

(4.99)

we write

\[ \partial_y \varphi + v_1 + 2yv_2 = -\frac{2bk}{(p-1)^2} \frac{y}{s^{2\beta}} \left[ 1 - v_1 \frac{s^{2\beta}1}{y} - 2v_2s^{2\beta}1 \right] + O \left( \frac{|y|^{q+2}}{s^{4\beta+1}} \right), \]

for \( |y| \leq 2Ks^\beta, \ y \neq 0 \) and where \( a \) is given in (2.23). Let us consider two subsets:

\[ I = \left\{ y \mid |y| \geq \frac{2}{a}|v_1|s^{2\beta} \right\}, \quad J = \left\{ y \mid |y| < \frac{2}{a}|v_1|s^{2\beta} \right\}. \]  

(4.100)

Let \( \varepsilon_0 > 0 \). If \( |y| < \varepsilon_0 s^\beta \) and \( y \in I \), since \( 2\beta q = 2\beta + 1 \), we have

\[ |\partial_y \varphi + v_1 + 2yv_2|^q = \left( \frac{2bk}{(p-1)^2} \right)^q \frac{|y|^q}{s^{2\beta+1}} \left[ 1 - qv_1 \frac{s^{2\beta}1}{y} - 2qv_2s^{2\beta}1 \right] + O \left( \frac{|y|^{q+2}}{s^{4\beta+1}} \right) + O(v_2^2s^{2\beta-1}|y|^q) + O \left( v_1^2s^{2\beta-1}|y|^{q-2} \right), \]

hence, from (4.96) and (4.98),

\[ G^2(v) = -qmuq^{-1}v_1 \frac{|y|q}{y} - 2qmuq^{-1}v_2 \frac{|y|^q}{s} + O \left( \frac{|y|^{q+2}}{s^{4\beta+1}} \right) + O(v_2^2s^{2\beta-1}|y|^q) + O \left( v_1^2s^{2\beta-1}|y|^{q-2} \right). \]

Then, since \( 1 < q < 2 \), using the fact that function \( \frac{|y|^q}{y} \) is odd, we get from (4.99), for \( s_0 \) large enough,

\[ \int_{I \cap \{|y| < \varepsilon_0 s^\beta\}} \chi G^2(v)k_2\rho = \int_{\mathbb{R}} |y|^q k_2\rho - \int_{\mathbb{R}} |y|^q k_2\rho \left( \frac{1}{s^{4\beta+1}} + O \left( \frac{A}{s^{6\beta-1}} \right) + O \left( \frac{A^2}{s^{2\beta+3}} \right) \right). \]

Then we write

\[ \int_{\mathbb{R}} |y|^q k_2\rho = \int_{\mathbb{R}} |y|^q k_2\rho - I_{ext} - I_{int} \]

with

\[ I_{ext} = \int_{|y| > \varepsilon_0 s^\beta} |y|^q k_2\rho \quad \text{and} \quad I_{int} = \int_{|y| < \frac{2}{a}|v_1|s^{2\beta}} |y|^q k_2\rho. \]

Using (4.99), we write

\[ I_{int} \leq \int_{|y| < \frac{2A}{a}} |y|^q k_2\rho \leq \frac{CA^{q+1}}{s^{q+1}} \]

and

\[ I_{ext} \leq Ce^{-s^\beta}. \]
Therefore, using again (4.99) we see that
\[
\int_{|y|<\varepsilon_0 s^{\beta}} \chi G^2(v) k_2 \rho = -2q\mu a^{q-1} v_2(s) \int_{\mathbb{R}} |y|^q k_2 \rho + O\left(\frac{A^{q+1}}{s^{4\beta+q-1}}\right)
\]
\[
+ O\left(\frac{1}{s^{4\beta+1}}\right) + O\left(\frac{A}{s^{6\beta}}\right) + O\left(\frac{A^2}{s^{2\beta+3}}\right)
\]
\[
= -2q\mu a^{q-1} v_2(s) \int_{\mathbb{R}} |y|^q k_2 \rho + O\left(\frac{1}{s^{4\beta}}\right),
\]
(4.101)
since \(p > 3\), \(\beta < \frac{3}{2}\).

Since \((4\beta - 1)q > 2\beta q = 2\beta + 1\), using (4.25) and (4.99), we see by definition (4.96) that
\[
|G^2(v)| \leq CA^q|\partial_y \varphi|^q + C|v_1|^q + C|v_2|^q |y|^q
\]
\[
\leq \frac{CA^q}{s^{q(2\beta + 1)}} + \frac{CA^{q/2} |y|^q}{s^{2\beta + 1}}.
\]
In particular,
\[
\left| \int_{|y|>\varepsilon_0 s^{\beta}} \chi G^2(v) k_2 \rho \right| \leq \frac{CA^q e^{-2\beta}}{s^{2\beta + 1}} \leq \frac{1}{s^{4\beta}},
\]
(4.102)
and, using (4.99) again, we see that \(J \subset \{|y| < \frac{2a}{\varepsilon_0}\}\), hence
\[
\left| \int_{J} \chi G^2(v) k_2 \rho \right| \leq \frac{CA^q}{s^{q(2\beta + 1)}} \int_{|y|<\frac{2a}{\varepsilon_0}} \rho + \frac{CA^{q/2}}{s^{2\beta + 1}} \int_{|y|<\frac{2a}{\varepsilon_0}} |y|^q \rho
\]
\[
\leq \frac{CA^{q+1}}{s^{q(2\beta + 1)+1}} + \frac{CA^{q/2+q+1}}{s^{q(2\beta + 1)+q+1}} \leq \frac{1}{s^{4\beta}},
\]
(4.103)
from the fact that \(q(2\beta + 1) + q = 1 > q(2\beta + 1) + 1 = 2\beta + q + 2 > 4\beta\), by (4.101) and the fact that \(q > 1 > \beta\). Finally, by definition (4.100) of the sets \(I\) and \(J\), and estimates (4.101), (4.102) and (4.103), we see that
\[
\int \chi G^2(v) k_2 \rho = -2q\mu a^{q-1} v_2(s) \int_{\mathbb{R}} |y|^q k_2 \rho + O\left(\frac{1}{s^{4\beta}}\right).
\]
(4.104)
In conclusion, from (4.97) and (4.104), we have
\[
(G(v))_2 = -\widetilde{c} v_2 + O\left(\frac{1}{s^{4\beta}}\right),
\]
with
\[
\widetilde{c} = 2q\mu \left(\frac{2b\kappa}{(p-1)^2}\right)^{q-1} \int_{\mathbb{R}} |y|^q k_2 \rho.
\]
From the particular choice of \(b\) given in (1.6), together with the definition (2.9) of \(k_2\) and identity (2.11), it appears that
\[
\widetilde{c} = 2\beta + 1
\]
and (4.94) follows. Since (4.94) was the last identity to check for item (ii) of Proposition 4.18, we are done.

**Step 3: The infinite-dimensional part \(v_-\) and \(v_e\).** Let us write the equation (3.12) on \(v\) in the following integral form:
\[
v(s) = S(s, \tau) v(\tau) + \int_\tau^s S(s, \sigma) B(v(\sigma)) d\sigma + \int_\tau^s S(s, \sigma) R(\sigma) d\sigma + \int_\tau^s S(s, \sigma) G(\sigma) d\sigma,
\]
(4.105)
where \(S\) is the fundamental solution of the operator \(\mathcal{L} + V\). We write
\[
v = A + B + C + D
\]
where
\[
A(s) = S(s, \tau)v(\tau), \quad B(s) = \int_\tau^s S(s, \sigma)B(v(\sigma))d\sigma, \quad \tag{4.106}
\]
\[
C(s) = \int_\tau^s S(s, \sigma)R(\sigma)d\sigma, \quad D(s) = \int_\tau^s S(s, \sigma)G(\sigma)d\sigma. \quad \tag{4.107}
\]

We assume that \(v(s)\) is in \(\vartheta_A(s)\) for each \(s \in [\tau, \tau + \rho]\), where \(\rho > 0\). Clearly, from the choice we made for \(K\) right after (3.9), the proof of Proposition 4.18 Part (iii) follows from the following:

**Lemma 4.20** (Estimates of the different terms of the Duhamel formulation (4.105)). There exists some \(K_5 = K_5(N, p, \mu) > 0\) such that for whenever \(K \geq K_5\), there exists \(A_5 > 0\) such that for all \(A \geq A_5\), and \(\rho > 0\) there exists \(s_0(A, \rho)\), such that for all \(s_0 \geq s_0(A, \rho)\), if we assume that for all \(s \in [\tau, \tau + \rho]\), \(v(s)\) satisfies (3.2), \(v(s) \in \vartheta_A(s)\) and \(\partial_y v\) satisfies the estimates given in Parts (i)-(ii)-(iii) of Proposition 4.17 with \(\tau \geq s_0\), then, we have the following results for all \(s \in [\tau, \tau + \rho]\):

(i) **(Linear term)**
\[
\|A_e(s)\|_{L^\infty} \leq Ce^{-\frac{1}{2}(s-\tau)}\|v_e(\tau)\|_{L^\infty} + Ce^{-s-\tau}\|B_e(\tau)\|_{L^\infty} + \frac{C}{s^\gamma},
\]
\[
\|A_e(s)\|_{L^\infty} \leq Ce^{-(s-\tau)}\|v_e(\tau)\|_{L^\infty} + Ce^{-s-\tau}\|B_e(\tau)\|_{L^\infty} + \frac{C}{s^\gamma},
\]
\[
\|A_e(s)\|_{L^\infty} \leq Ce^{-s-\tau}\|v_e(\tau)\|_{L^\infty} + Ce^{-s-\tau}\|B_e(\tau)\|_{L^\infty} + \frac{C}{s^\gamma}.
\]

(ii) **(Nonlinear source term)**
\[
|\mathcal{B}_e(s)| \leq \frac{C}{s^\gamma}(s-\tau)(1 + |y|^3), \quad \|\mathcal{B}_e(s)\|_{L^\infty} \leq \frac{C}{s^\gamma}(s-\tau)e^{s-\tau},
\]
\[
|\mathcal{C}_e(s)| \leq \frac{C}{s^\gamma}(s-\tau)(1 + |y|^3), \quad \|\mathcal{C}_e(s)\|_{L^\infty} \leq \frac{C}{s^\gamma}(s-\tau)e^{s-\tau}.
\]

(iii) **(Corrective term)**
\[
|\mathcal{C}_e(s)| \leq \frac{C}{s^\gamma}(s-\tau)(1 + |y|^3), \quad \|\mathcal{C}_e(s)\|_{L^\infty} \leq \frac{C}{s^\gamma}(s-\tau)e^{s-\tau}.
\]

(iv) **(Nonlinear gradient term)**
\[
|\mathcal{D}_e(s)| \leq \frac{C}{s^\gamma}(s-\tau)(1 + |y|^3), \quad \|\mathcal{D}_e(s)\|_{L^\infty} \leq \frac{C}{s^\gamma}(s-\tau)e^{s-\tau}.
\]

**Proof of Lemma 4.20**: We consider \(A \geq 1, \rho > 0\) and \(s_0 \geq \rho\). We then consider \(v(s)\) a solution of (3.2) satisfying \(v(s) \in \vartheta_A(s)\) such that \(\partial_y v\) satisfies the estimates given in Parts (i)-(ii)-(iii) of Proposition 4.17 for all \(s \in [\tau, \tau + \rho]\), for some \(\tau \geq s_0\).

(i) The proof of (i) follows by simple modifications of the argument for [22, Lemma 3.5], see also [4]; in fact, making the change of variable in the potential
\[
V(y, \tau) = V(y, s), \quad \text{with} \quad \sqrt{s} = s^\beta,
\]
we reduce to the case of the standard nonlinear heat equation treated in [22]. For that reason, we omit the proof.

(ii)-(iv) Consider \(s_0 \geq \rho\). If we take \(\tau \geq s_0\), then \(\tau + \rho \leq 2\tau\) and if \(\tau \leq \sigma \leq s \leq \tau + \rho\) then
\[
\frac{1}{2\tau} \leq \frac{1}{s} \leq \frac{1}{2\rho} \leq \frac{1}{\tau} \leq \frac{1}{\sigma} \leq \frac{1}{\tau}.
\]

Let us recall from Bricmont and Kupiainen [4] that for all \(y, x \in \mathbb{R}\)
\[
|K(s, \sigma, y, x)| \leq Ce^{(s-\sigma)\sqrt{x}}(y, x), \quad 1 \leq \sigma \leq s \leq 2\sigma, \tag{4.109}
\]
\[
\int |K(s, \sigma, y, x)|(1 + |x|^m)dx \leq Ce^{(s-\sigma)(1 + |y|^m)}, \quad m \geq 0, \quad 1 \leq \sigma \leq s \leq 2\sigma. \tag{4.110}
\]
where $e^{\theta L}$ is given in (4.15) (see [2] p. 545)). The proof of (ii)-(iv) follows by using the fact that $R$, $B$, $G$ are in $\partial C(s)$, the linear part estimates (i), and by integration. This concludes the proof of Lemma 4.20 and Proposition 4.18 too.

Now, with Proposition 4.18 which gives the projection of the equation (3.2) on the different components of the decomposition (3.13), we are ready to prove Proposition 4.6.

**Proof of Proposition 4.6.** Let us consider $A \geq \max(A_6, 1)$, and $s_0 \geq \max(s_{14}(A), s_{06}(A))$. Later in the proof we will fix $A$ larger, and $s_0$ larger. From our conditions on $A$ and $s_0$, we already see that the conclusion of Proposition 4.17 holds, and so does the conclusion of Proposition 4.18. We then consider $v$ a solution of (3.2) with initial data $\psi_{s_0,d_0,d_1}$ given by (4.2) with $(d_0,d_1) \in D_T$, such that $v(s) \in \partial A(s)$, for all $s \in [s_0,s_1]$ with $v(s_1) \in \partial A(s_1)$.

By definition 4.2 of $\partial (s_1)$, it is enough to prove that

$$|v_2(s_1)| < \frac{\sqrt{A}}{s_1^{4\beta+1}}, \quad \left\| \frac{v_-(y,s_1)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R})} < \frac{A}{s_1^1}, \quad \text{and} \quad \left\| v_e(s_1) \right\|_{L^\infty(\mathbb{R})} < \frac{A^2}{s_1^{4\beta-3\beta}}. \quad (4.111)$$

We begin with $v_2$. Assume by contradiction that

$$v_2(s_1) = \frac{\sqrt{A}}{s_1^{4\beta+1}}.$$

Since for all $s \in [s_0,s_1]$, $|v_2(s)| \leq \frac{\sqrt{A}}{s_1^{4\beta+1}}$, it follows by differentiation that

$$v'_2(s_1) \geq -(4\beta-1)\frac{\sqrt{A}}{s_1^{4\beta}},$$

on the one hand. On the other hand, since we have from Proposition 4.18

$$v'_2(s) = -\frac{2\beta+1}{s} v_2 + O\left(\frac{1}{s^{4\beta}}\right),$$

we see that

$$v'_2(s_1) \leq \frac{2\beta+1}{s_1} v_2(s_1) + \frac{C}{s_1^{4\beta}} = \frac{-A(2\beta+1)}{s_1^{4\beta}}.$$

This leads to a contradiction with the previous inequality, for $A$ sufficiently large, since $\beta < 1$, hence $4\beta-1 < 2\beta+1$. Similarly, we get a contradiction if $v_2(s_1) = -\frac{\sqrt{A}}{s_1^{4\beta}}$. Thus, $|v_2(s_1)| < \frac{\sqrt{A}}{s_1^{4\beta+1}}$.

To prove that (4.111) holds for $v_-$ and $v_e$, it is enough to prove that, for all $s \in [s_0,s_1]$,

$$\left\| v_e(s) \right\|_{L^\infty(\mathbb{R})} \leq \frac{A^2}{2s^{4\beta-3\beta}}, \quad \left\| \frac{v_-(y,s)}{1+|y|^3} \right\|_{L^\infty(\mathbb{R})} \leq \frac{A}{2s^1}, \quad (4.112)$$

provided that $A$ is large enough, and $s_0$ is large enough. Define $\sigma = \log(A/C_0)$, with $C_0 > 0$ an appropriate constant to be chosen later, and take $s_0 \geq \sigma$, so that for all $\tau \geq s_0$ and $s \in [\tau,\tau+\sigma]$, we have

$$\tau \leq s \leq \tau + \sigma \leq \tau + s_0 \leq 2\tau, \quad \text{hence} \quad \frac{1}{2\tau} \leq \frac{1}{s} \leq \frac{1}{\tau} \leq \frac{2}{s}. \quad (4.113)$$

We consider two cases in the proof of (4.112).
Case 1: $s \leq s_0 + \sigma$. From our estimates on initial data stated in item (ii) of Proposition 4.15, it is clear that $(4.112)$ is satisfied with $\tau = s_0$, provided that $A \geq 2$. Using Proposition 4.18 Part (iii), with $\tau = s_0$, as well as $(4.113)$ and estimate $(4.16)$, we get
\[
\left\| \frac{v_-(y,s)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq C e^{-\frac{(s-s_0)}{2}} \left\| \frac{v_-(y,s_0)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} + \frac{C}{s^{3\beta}} e^{-(s-s_0)^2} \left\| v_e(s_0) \right\|_{L^\infty} + \frac{C}{s^\gamma} (1 + s - s_0)
\]
\[
\leq \frac{C}{s^{\gamma-3\beta}} e^{s-s_0} + \frac{C}{s^{\gamma-3\beta}} (1 + (s - s_0)e^{s-s_0})
\]
\[
\leq \frac{C}{s^{\gamma-3\beta}} (e^\sigma + 1 + \sigma e^\sigma).
\]
Since $\sigma = \log \frac{A}{s_0}$, taking $A$ sufficiently large, we get $(4.112)$.

Case 2: $s > s_0 + \sigma$. Let $\tau = s - \sigma > s_0$. Applying Part (iii) of Proposition 4.18 and using the fact that $v(\tau) \in \partial \vartheta_A(\tau)$, we have
\[
\left\| \frac{v_-(y,s)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq C e^{-\frac{(s-\tau)}{2}} \left\| \frac{v_-(y,\tau)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} + \frac{C}{s^{3\beta}} e^{-(s-\tau)^2} \left\| v_e(\tau) \right\|_{L^\infty} + \frac{C}{s^\gamma} (1 + s - \tau),
\]
\[
\leq \frac{C}{s^\gamma} \left( e^{-\frac{\sigma}{2}} A + e^{-\sigma^2} A^2 + 1 + \sigma \right)
\]
and
\[
\left\| v_e(s) \right\|_{L^\infty} \leq C e^{-\frac{(s-\tau)}{2}} \left\| v_e(\tau) \right\|_{L^\infty} + C e^{s-\tau} s^{3\beta} \left\| \frac{v_-(\tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C}{s^{\gamma-3\beta}} (1 + (s - \tau)e^{s-\tau})
\]
\[
\leq \frac{C_0'}{s^{\gamma-3\beta}} \left( e^{-\frac{\sigma}{2}} A^2 + e^\sigma A + 1 + \sigma e^\sigma \right),
\]
for some $C_0' > 0$. Fixing $C_0 = 10C_0'$ and $\sigma = \log \frac{A}{s_0}$, then taking $A$ sufficiently large we see that $(4.112)$ follows.

In conclusion, we have just proved that $(4.112)$ follows in both cases, hence $(4.111)$ follows too. Since $v(s_1) \in \partial \vartheta_A(s_1)$, we see that $(v_0(s_1), v_1(s_1)) \in \partial \left( \left[ -\frac{A}{s_1^{2\beta+1}}, \frac{A}{s_1^{2\beta+1}} \right] \right)^2$, by definition 4.7 of $\vartheta_A(s)$. This concludes the proof of Part (i) of Proposition 4.6

(ii) From Part (i), there exists $m = 0, 1$ and $\omega = \pm 1$ such that
\[
v_m(s_1) = \omega \frac{A}{s_1^{2\beta+1}}.
\]
Using Part (i) of Proposition 4.18 we have that
\[
\omega v_m'(s_1) \geq \left( 1 - \frac{m}{2} \right) \omega v_m(s_1) - \frac{C}{s_1^{2\beta+1}} \left[ \left( 1 - \frac{m}{2} \right) A - C \right] \frac{1}{s_1^{2\beta+1}}.
\]
Hence, for $A$ large enough, $\omega v_m'(s_1) > 0$. This concludes the proof of Proposition 4.6.
5. SINGLE POINT BLOW-UP AND FINAL PROFILE

In this section we prove Theorem 1. To do so, we first establish a result of no blow-up under some threshold. This is done in a separate subsection. In the second subsection, we give the proof of Theorem 1.

5.1. No blow-up under some threshold. In this section, we prove the following result which is similar in the spirit to the result of Giga and Kohn in [16, Theorem 2.1, p. 850], though we need genuine new parabolic regularity estimates to control the nonlinear gradient term, and this makes the originality of our contribution. This is our statement:

Proposition 5.1 (No blow-up under some threshold). For all $C_0 > 0$, there exists $\varepsilon_0 > 0$ such that if $u(\xi, \tau)$ satisfies for all $|\xi| < 1$ and $\tau \in [0, 1)$,

$$|\partial_\tau u - \Delta u| \leq C_0 (1 + |u|^p + |\nabla u|^q),$$

$$|\partial_\tau \nabla u - \Delta (\nabla u) - \mu \nabla |\nabla u|^q| \leq C_0 (1 + |\nabla u||u|^{p-1}),$$

with $q = \frac{2p}{p+1}$, $p > 3$ and $\mu \in \mathbb{R}$, together with the bound

$$|u(\xi, \tau)| + \sqrt{1 - \tau |\nabla u(\xi, \tau)|} \leq \varepsilon_0 (1 - \tau)^{-\frac{1}{p-1}},$$

then

$$\forall |\xi| \leq 1/16, \forall \tau \in [0, 1), |u(\xi, \tau)| + |\partial_\tau u(\xi, \tau)| \leq M\varepsilon_0,$$

for some $M = M(C_0, p, \mu) > 0$. In particular, $u$ and $|\nabla u|$ do not blow up at $\xi = 0$ and $\tau = 1$.

We proceed in 2 parts in order to prove this proposition:

- In Part 1, we write Duhamel formulations satisfied by truncations of the solution and its gradient. We also recall from [16] an integral computation table.

- In Part 2, using the previous estimates and an iteration process, we give the proof of Proposition 5.1.

Part 1: A toolbox for the proof

In this part, we give a Duhamel formulation for the equations satisfied by $u$ and its gradient.

Denoting by $B_r$ the set $\{x \in \mathbb{R}^N \mid |x| < r\}$, where $r > 0$, we establish the following Lemma:

Lemma 5.2 (Truncation and Duhamel Formulations of the equations). Let $r \in (0, 1]$ and $\phi_r$ be a smooth function supported on $B_r$ such that $\phi_r \equiv 1$ on $B_{r/2}$ and $0 \leq \phi_r \leq 1$. Let $w = \phi_r u$, $w_1 = \phi_r v$, $v = \nabla u$, where $u$ is as in Proposition 5.1. Then we have the following for all $\xi \in \mathbb{R}$ and $0 \leq \tau < 1$,

(i) $|\partial_\tau w + \Delta w + u \Delta \phi_r - 2\nabla \cdot (u \nabla \phi_r)| \leq C_0 (1 + |w||u|^{p-1} + |\nabla u|^q)$.

(ii) $|\partial_\tau w_1 + \Delta w_1 + v \Delta \phi_r - 2\nabla \cdot (v \nabla \phi_r) + \mu \nabla \cdot (|v|^{q-2} v w_1) - \mu |v|^{q} \nabla \phi_r| \leq C_0 (1 + |w_1||u|^{p-1})$.

(iii) $\|w(\tau)\|_{L^\infty} \leq \|u(0)\|_{L^\infty(B_{r/2})} + C \int_0^\tau (\tau - s)^{-1/2}\|u(s)\|_{L^\infty(B_r)} ds$

$$+ C \int_0^\tau \|u(s)\|_{L^\infty(B_r)}^{p-1}\|w(s)\|_{L^\infty} ds + C \int_0^\tau \|\nabla u(s)\|_{L^\infty(B_r)}^q ds.$$

(iv) $\|w_1(\tau)\|_{L^\infty} \leq \|\nabla u(0)\|_{L^\infty(B_r)} + C \int_0^\tau (\tau - s)^{-1/2}\|\nabla u(s)\|_{L^\infty(B_r)} ds + C \int_0^\tau \|u(s)\|_{L^\infty(B_r)}^{p-1}\|w_1(s)\|_{L^\infty} ds$

$$+ C \int_0^\tau (\tau - s)^{-1/2}\|w_1(s)\|_{L^\infty} \|\nabla u(s)\|_{L^\infty(B_r)}^{q-1} ds + C \int_0^\tau \|\nabla u(s)\|_{L^\infty(B_r)}^q ds.$$

Remark 5.3. The truncated functions $w$ and $w_1$ do depend on $r$, but we omit that dependence in order to avoid complicated notations.
Lemma 5.4

(iii) follows. This concludes the proof of Lemma 5.2. □

Lemma 5.2 together with the bound (5.1):

The proof is trivial.

Proof.

(i)-(ii) The proof is trivial.

(iii)-(iv) We only prove item (iii) since item (iv) follows similarly.

Writing the equation given in item (i) in its Duhamel formulation, we see that

$$ \left| w(\tau) - e^{\tau \Delta} w(0) - \int_0^\tau e^{(\tau-s)\Delta} \left[ u\Delta \phi_r - 2\nabla(u\nabla \phi_r) \right] ds \right| $$

$$ \leq C \int_0^\tau e^{(\tau-s)\Delta} \left[ |u|^{p-1} |w| \right] ds + C \int_0^\tau e^{(\tau-s)\Delta} |\nabla u|^q ds, $$

where $e^{\tau \Delta}$ is the heat semigroup. Since the truncation $\phi_r$ is supported in the ball $B_r$, only the $L^\infty(B_r)$ of $u$ and $\nabla u$ are involved, and item (iii) follows. This concludes the proof of Lemma 5.2. □

Now, we recall the following integral computation table from Giga and Kohn [16]:

Lemma 5.4 (An integral computation table; see Lemma 2.2 page 851 in Giga and Kohn [16]). For $0 < \alpha < 1$, $\theta > 0$ and $0 \leq \tau < 1$, the integral

$$ I(\tau) = \int_0^\tau (\tau - s)^{-\alpha} (1 - s)^{-\theta} ds $$

satisfies

(i) $I(\tau) \leq \left( (1-\alpha)^{-1} + (\alpha + \theta - 1)^{-1} \right) (1 - \tau)^{1-\alpha-\theta}$ if $\alpha + \theta > 1$,

(ii) $I(\tau) \leq (1-\alpha)^{-1} + |\log(1-\tau)|$ if $\alpha + \theta = 1$,

(iii) $I(\tau) \leq (1 - \alpha - \theta)^{-1}$ if $\alpha + \theta < 1$.

Part 2: The proof of Proposition 5.1

Using the various estimates of Part 1, we are ready to proceed by iteration to prove Proposition 5.1.

Proof of Proposition 5.1. We follow the iteration method of Giga and Kohn [16] Theorem 2.1, p. 850, based on the Duhamel formulations given in Lemma 5.2, though we need here new ideas coming from sharp parabolic estimates. We give the proof in 4 steps, improving [5.1] step by step, in order to prove the boundedness of $u$ and $\nabla u$ at the end of the 4th step.

Step 1: We apply Lemma 5.2 with $r = 1$. If $w_1 = \phi_1 v$ with $v = \nabla u$, then we have from item (iv) in Lemma 5.2 together with the bound [5.1]:

$$ \|w_1(\tau)\|_{L^\infty} \leq \|\nabla u(0)\|_{L^\infty(B_1)} + C \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|v(s)\|_{L^\infty(B_1)} ds + C \int_0^\tau \|u(s)\|_{L^\infty(B_1)}^{p-1} \|w_1(s)\|_{L^\infty} ds $$

$$ + C \int_0^\tau (\tau - s)^{-\frac{1}{2}} \|w_1(s)\|_{L^\infty} \|\nabla u(s)\|_{L^\infty(B_1)}^{q-1} ds + C \int_0^\tau \|\nabla u(s)\|_{L^\infty(B_1)}^q ds $$

$$ \leq \varepsilon_0 + C\varepsilon_0 \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1-s)^{-\frac{1}{p-1}} ds + C\varepsilon_0^{p-1} \int_0^\tau (1-s)^{-1} \|w_1(s)\|_{L^\infty} ds $$

$$ + C\varepsilon_0^{q-1} \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1-s)^{-\frac{1}{p-1}} \|w_1(s)\|_{L^\infty} ds + C\varepsilon_0^q \int_0^\tau (1-s)^{-\frac{1}{p-1}} ds. $$

Since $q < p$, using Lemma 5.4 we see that for $\varepsilon_0$ small enough, we have

$$ \|w_1(\tau)\|_{L^\infty} \leq C\varepsilon_0 (1-\tau)^{-\frac{1}{p-1}} + C\varepsilon_0^{q-1} \int_0^\tau (\tau - s)^{-\frac{1}{2}} (1-s)^{-\frac{1}{p-1}} \|w_1(s)\|_{L^\infty} ds. $$
Using a Gronwall’s argument together with Lemma 5.4, we see that
\[ \forall \tau \in [0, 1), \|w_1\|_{L^\infty} \leq 2C\varepsilon_0(1 - \tau)^{-\frac{1}{p-1}}, \]
for \( \varepsilon_0 \) sufficiently small. In particular, since \( w_1 = \nabla u \) when \( |\xi| \leq \frac{1}{4} \), it follows that
\[ \forall \tau \in [0, 1), \forall |\xi| \leq \frac{1}{4}, |\nabla u(\xi, \tau)| \leq 2C\varepsilon_0(1 - \tau)^{-\frac{1}{p-1}}, \tag{5.2} \]
which improves the bound on \( \nabla u \) in (5.1), when \( |\xi| \leq \frac{1}{4} \).

**Step 2:** Now, we take \( r = \frac{1}{4} \) and focus on \( w = \phi_{1/2}u \). Using item (iii) in Lemma 5.2 together with (5.1), (5.2) and Lemma 5.4, we write
\[
\|w(\tau)\|_{L^\infty} \leq \|u(0)\|_{L^\infty(B_{1/2})} + C \int_0^\tau (\tau - s)^{-1/2}\|u(s)\|_{L^\infty(B_{1/2})} ds + C \int_0^\tau \|w(s)\|_{L^\infty} ds + C \int_0^\tau \|\nabla u(s)\|_{L^\infty}^q ds
\]
\[ \leq \varepsilon_0 + C\varepsilon_0 \int_0^\tau (\tau - s)^{-1/2}(1 - s)^{-\frac{1}{p-1}} ds + C\varepsilon_0^p \int_0^\tau (1 - s)^{-q/(p-1)} ds \]
\[ \leq C\varepsilon_0 + C\varepsilon_0^p \int_0^\tau (1 - s)^{-1}\|w(s)\|_{L^\infty} ds, \]
(remember that \( p > 3 \)). By Gronwall’s inequality, we deduce that
\[ \|w(\tau)\|_{L^\infty} \leq C(1 - \tau)^{-c\varepsilon_0^{p-1}}, \forall \tau \in [0, 1). \]
Since \( w(\xi, \tau) = u(\xi, \tau) \), for all \( |\xi| \leq \frac{1}{4} \), it follows that
\[ \forall \tau \in [0, 1), \forall |\xi| \leq \frac{1}{4}, |u(\xi, \tau)| \leq C(1 - \tau)^{-c\varepsilon_0^{p-1}}. \tag{5.3} \]

**Step 3:** For the step 3 we take \( r = 1/4 \). We now consider \( \bar{w}_1 = \phi_{1/4}u := \phi_{1/4}\nabla u \). Applying item (iv) in Lemma 5.2 and using the bounds (5.2) and (5.3), together with Lemma 5.4, we see that
\[
\|\bar{w}_1(\tau)\|_{L^\infty} \leq \|\nabla u(0)\|_{L^\infty(B_{1/4})} + C \int_0^\tau (\tau - s)^{-\frac{1}{2}}\|\nabla u(s)\|_{L^\infty(B_{1/4})} ds + C \int_0^\tau (\tau - s)^{-\frac{1}{2}}\|\bar{w}_1(s)\|_{L^\infty} \|\nabla u(s)\|_{L^\infty(B_{1/4})}^q ds
\]
\[ + C \int_0^\tau \|\nabla u(s)\|_{L^\infty(B_{1/4})}^q ds \]
\[ \leq \varepsilon_0 + C\varepsilon_0 \int_0^\tau (\tau - s)^{-1/2}(1 - s)^{-1/(p-1)} ds + C \int_0^\tau (1 - s)^{-c(p-1)c_0^{p-1}} ds \]
\[ + C\varepsilon_0^{p-1} \int_0^\tau (\tau - s)^{-1/2}(1 - s)^{-1/(p-1)} ds \]
\[ \leq C\varepsilon_0 + C\varepsilon_0^{p-1} \int_0^\tau (\tau - s)^{-1/2}(1 - s)^{-1/(p+1)} ds \]
for \( \varepsilon_0 \) small enough (remember that \( p > 3 \)). Using again Gronwall’s technique, together with Lemma 5.4, we see that
\[ \|\bar{w}_1(\tau)\|_{L^\infty} \leq C\varepsilon_0, \forall \tau \in [0, 1). \]
Since \( \ddot{w}(\xi, \tau) = \nabla u(\xi, \tau) \) when \( |\xi| \leq \frac{1}{8} \), it follows that
\[
\forall \tau \in [0, 1], \quad \forall |\xi| \leq \frac{1}{8}, \quad |\nabla u(\xi, \tau)| \leq C \varepsilon_0. \tag{5.4}
\]

**Step 4:** For the step 4 we take \( r = 1/8 \). We now consider \( \dddot{w} = \phi_{1/8} u \) and use item (iii) in Lemma 3.2, the bounds in (3.1), (5.2) and (5.4), together with Lemma 5.2 to derive that
\[
\|\dddot{w}(\tau)\|_{L^\infty} \leq \|u(0)\|_{L^\infty(B_{1/8})} + C \int_0^\tau (\tau - s)^{-1/2} \|u(s)\|_{L^\infty(B_{1/8})} ds
+ C \int_0^\tau (1 - s)^{-c(p-1)^{p-1}} \|\dddot{w}(s)\|_{L^\infty} ds
\leq C \varepsilon_0 + C \int_0^\tau (1 - s)^{-c(p-1)\varepsilon_0^{p-1}} \|\dddot{w}(s)\|_{L^\infty} ds
\]
(use the fact that \( p > 3 \)). Again, by Gronwall’s argument, we see that
\[
\|\dddot{w}(\tau)\|_{L^\infty} \leq M \varepsilon_0, \quad \forall \tau \in [0, 1).
\]
Since \( \dddot{w}(\xi, \tau) = u(\xi, \tau) \), for all \( |\xi| \leq \frac{1}{16} \) and \( \tau \in [0, 1) \), it follows that
\[
\forall \tau \in [0, 1), \quad \forall |\xi| \leq \frac{1}{16}, \quad |u(\xi, \tau)| \leq C \varepsilon_0. \tag{5.5}
\]
Using (5.4) and (5.5), we get the conclusion of the proof of Proposition 5.1. \( \square \)

### 5.2. Proof of Theorem 1

This section is dedicated to the proof of Theorem 1. We will present the proofs of items (i), (ii) and (iii) separately.

**Proof of Theorem 1 Part (i).** If we introduce for \( \epsilon > 0 \)
\[
\gamma = \min(5\beta - 1, 2\beta + 1) - \epsilon, \tag{5.6}
\]
and recall that \( p > 3 \), then we see that (4.23) holds, provided \( \epsilon \) is small enough. Therefore, our strategy in Section 4 works, and we get from Propositions 4.4 and 4.7 the existence of a solution \( v \) to equation (3.2), defined for all \( y \in \mathbb{R} \) and \( s \geq s_0 \), for some \( s_0 \geq 1 \), such that
\[
\forall s \geq s_0, \quad \|v(s)\|_{W^{1, \infty}(\mathbb{R})} \leq \frac{C}{s^{\gamma - 3\beta}}.
\]
Then, using (3.1), this yields \( w(y, s) \), a solution to equation (2.2), defined for all \( y \in \mathbb{R} \) and \( s \geq s_0 \), such that
\[
\|w(y, s) - \varphi(y, s)\|_{W^{1, \infty}(\mathbb{R})} \leq \frac{C}{s^{\gamma - 3\beta}}.
\]
where the profile \( \varphi \) is introduced in (2.2).

Introducing
\[
T = e^{-s_0}
\]
and the function \( u(x, t) \) defined from \( w(y, s) \) by (2.1), we see that \( u \) is a solution to equation (1.1) defined for all \( x \in \mathbb{R} \) and \( t \in [0, T) \), which satisfies
\[
\forall t \in [0, T), \quad \left\| (T - t)^{p-1} u(y \sqrt{T - t}, t) - \varphi^0 \left( \frac{y}{\log(T - t)} \right) \right\|_{W^{1, \infty}(\mathbb{R})} \leq \frac{C}{\log(T - t)^{\gamma - 3\beta}} \tag{5.7}
\]
(see here the fact that \( \gamma > 3\beta < 2\beta \) which comes from (4.3)).

In particular, \( u(t) \in W^{1, \infty}(\mathbb{R}) \), for all \( t \in [0, T) \), and \( \lim_{t \to T} (T - t)^{p-1} u(0, t) = (p - 1)^{p-1} \), hence \( \lim_{t \to T} u(0, t) = \infty \), which means that \( u \) blows up at time \( t = T \), (at least) at the origin. Moreover, we have
\[
\|u(t)\|_{L^\infty(\mathbb{R})} \leq C(T - t)^{-1/p-1}, \quad \forall t \in [0, T).
\]
We now turn to proving that $|\nabla u(x, t)|$ blows up at the origin. By Proposition 4.17 part (ii), and (5.1), we have for all $s \geq -\log T$ and $y \in \mathbb{R}$,

$$|\nabla w(y, s) - \nabla \varphi(y, s)| = |\nabla v(y, s)| \leq C \frac{A}{s^{2\beta-1}} (1 + |y|) + C \frac{\sqrt{A}}{s^{2\beta}} (1 + |y|^2) + C \frac{A^2}{s^\gamma} (1 + |y|^3).$$

Put $y := y(s) = s^\alpha$, with $0 < \alpha < \min(2\beta - 1, \frac{\gamma - 2\beta}{2})$. Then, from (1.25), we see that

$$\partial_y \varphi(y(s), s) = -\frac{2b}{(p-1)} s^{2\beta} \left( \varphi^0 \left( \frac{y(s)}{s^\beta} \right) \right)^p \sim C \frac{s^{2\beta-\alpha}}{s^{2\beta-\alpha}} \text{ as } s \to \infty.$$

The conditions on $\alpha$ imply that $|\nabla w(y(s), s)| \sim C s^{2\beta}$, as $s \to \infty$. By the relation between $w, u, y, s$ and $\alpha$ given in (2.1), we get

$$|w(u_{\beta})| \sim C(0^\alpha) \sim C(T - t)^{-\frac{1}{p-1}} |\log(T - t)|^{\alpha - 2\beta} \to \infty \text{ as } t \to T.$$

In particular,

$$\|\nabla u(t)\|_{L^\infty(\mathbb{R})} \geq C(T - t)^{-\frac{1}{p-1}} |\log(T - t)|^{\alpha - 2\beta}.$$

Since $\sqrt{T - t} |\log(T - t)|^{\alpha} \to 0$ as $t \to T$, $\nabla u$ blows up at time $T$, (at least) at the origin. Since (1.4) follows from (5.7) and (5.6), this concludes the proof of item (i) of Theorem 1.

\[\square\]

**Proof of Theorem 1 Part (ii).** Consider $x_0 \neq 0$. By Part (i), we have that

$$\|w(\cdot, s) - \varphi(\cdot, s)\|_{W^1, \infty(\mathbb{R})} \leq C s^{\gamma - 2\beta},$$

for all $s \geq s_0 = -\log T$. Using the relation between $w, u, y, s$ and $x, t$ given by (2.1), and by the definition of $\varphi$ given by (2.21), we get that

$$\sup_{x \in \mathbb{R}} (T - t)^{-\frac{1}{p-1} + \frac{1}{2}} |\nabla u(x, t)| \leq \frac{\|\nabla \varphi^0\|_{L^\infty(\mathbb{R})}}{|\log(T - t)|^{\beta}} + \frac{C}{|\log(T - t)|^{\gamma - 3\beta}} \to 0, \text{ as } t \to T,$$

and

$$\sup_{|x - x_0| < \frac{|x_0|}{2}} (T - t)^{-\frac{1}{p-1}} |u(x, t)| \leq \varphi^0 \left( \frac{\frac{|x_0|}{2}}{\sqrt{T - t} |\log(T - t)|^{\beta}} \right) \to C, \text{ as } t \to T.$$

Consider $\delta > 0$ to be chosen small enough so various estimates hold. Then, for $K_0 > 0$ to be fixed later, we define $t_0(x_0)$ by:

$$|x_0| = K_0 \sqrt{T - t_0(x_0) |\log(T - t_0(x_0))|^{\beta}}, \quad \text{if } |x_0| \leq \delta,$$

$$t_0(x_0) = t_0(\delta), \quad \text{if } |x_0| > \delta.$$ 

Note that $t_0(x_0)$ is unique if $|x_0|$ is sufficiently small. Note also that $t_0(x_0) \to T$ as $x_0 \to 0$. Let us introduce the rescaled functions

$$U(x_0, \xi, \tau) = (T - t_0(x_0))^{-\frac{1}{p-1}} u(x, t),$$

$$V(x_0, \xi, \tau) := \nabla_\xi U(x_0, \xi, \tau),$$

where

$$x = x_0 + \xi \sqrt{T - t_0(x_0)}, \quad t = t_0(x_0) + \tau(T - t_0(x_0)), \quad \xi \in \mathbb{R}, \quad \tau \in [-\frac{t_0(x_0)}{T - t_0(x_0)}, 1).$$

It is easy to see from (5.8) and (5.9) that Proposition 5.1 applies to $U(x_0, \cdot, \cdot)$, hence that $x_0$ is not a blow-up point of $u$ and $\nabla u$. Let us insist on the fact that our argument works for any $x_0 \neq 0$ without any smallness assumptions, thanks to the adapted definition of $t_0(x_0)$ in (5.10). This proves the single point blow-up result. Thus we deduce that $u$ and $\nabla u$ blow up simultaneously at time $T$ and only at $x = 0$. \[\square\]
Proof of Theorem 1. Part (iii). We divide the proof into two parts. In the first part we show the existence of the final profile $u^*$. In the second part, we find an equivalent of $u^*$, and bound $|\partial_x u^*|$.

Part 1: Existence of the final profile. In this part, we show the existence of a blow-up final profile $u^* \in C^1 (\mathbb{R} \setminus \{0\})$ such that $u(t,x) \to u^*$ as $t \to T$, in $C^1$ of every compact of $\mathbb{R} \setminus \{0\}$. The case $\mu = 0$ is treated in [40]. In comparison, with the case $\mu = 0$, we need to use advanced parabolic regularity as in Proposition 5.1, which is the extended Giga-Kohn no-blow-up result for parabolic equations with a nonlinear gradient term.

If $v = \nabla u$, then, we derive from equation (1.1) the following system satisfied by $(\partial_t u, \partial_t v)$:

\[
\begin{align*}
\partial_t u &= \Delta u + \mu |v|^q + |u|^{p-1} u, \\
\partial_t v &= \Delta v + \mu \nabla (|v|^q) + p |u|^{p-1} v.
\end{align*}
\] (5.14) (5.15)

Since we know from Part (ii) that $u$ and $v$ are bounded uniformly on $K \subset \mathbb{R} \setminus \{0\}$, using parabolic regularity techniques, similar to Proposition 5.1, we can show that $\partial_t u$ and $\partial_t v$ are also bounded on $K' \times [T/2, T]$ for any $K' \subset \mathbb{R} \setminus \{0\}$. Therefore as in [20], there exists $u^*$ in $C^1 (\mathbb{R} \setminus \{0\})$ such that $u(t,x) \to u^*(x)$ and $\partial_x u(x,t) \to \partial_x u^*(x)$ as $t \to T$, uniformly on compact sets of $\mathbb{R} \setminus \{0\}$.

Part 2: Equivalent of the final profile. Let us now find an equivalent of $u^*$ as $x \to 0$. Consider $x_0 \neq 0$. Since $u$ is a solution of the equation (1.1) and $q = 2p/(p + 1)$, it follows that $U$ and $V$ defined in (5.11)-(5.12) satisfy the equations:

\[
\begin{align*}
\partial_x U &= \Delta_x U + \mu |\nabla_x U|^q + |U|^{p-1} U, \\
\partial_x V &= \Delta_x V + \mu \nabla_x (|V|^q) + p |U|^{p-1} V.
\end{align*}
\] (5.16) (5.17)

By Part (i) of Theorem 1, we have:

\[
\sup_{|\xi| \leq 6 \log(T-t_0(x_0))^{\beta/2}} |U(x_0,\xi,0) - \phi^0(K_0)| \leq \frac{C}{|\log(T-t_0(x_0))|^{\gamma}},
\] (5.18)

with

\[
\gamma' = \min(\beta/2, \gamma - 3\beta),
\] (5.19)

and

\[
\sup_{|\xi| \leq 6 \log(T-t_0(x_0))^{\beta/2}} |V(x_0,\xi,0)| \leq \frac{C}{|\log(T-t_0(x_0))|^{\gamma - 3\beta}}.
\] (5.20)

Using (5.9) and (5.9'), we see that for all $\tau \in [0,1]$ and $|\xi| \leq 6 |\log(T-t_0(x_0))|^{\beta/2}$, we have

\[
(1 - \tau)^{1-p} (|\phi^0(K_0/2)| + \sqrt{1-\tau} |V(x_0,\xi,\tau)|) \leq \phi^0 \left(\frac{K_0}{2}\right) + \frac{C}{|\log(T-t_0(x_0))|^{\gamma - 3\beta}} \equiv \tilde{\epsilon}_0(K_0,x_0),
\] with

\[
\tilde{\epsilon}_0 \to 0 \text{ as } |x_0| \to 0 \text{ and } K_0 \to \infty.
\] (5.21)

Fixing $K_0$ large enough and $|x_0|$ small enough, we can make $\tilde{\epsilon}_0(K_0,x_0) \leq \epsilon_0$, where $\epsilon_0$ is defined in Proposition 5.1. Applying Proposition 5.1 we deduce that for all $\tau \in [0,1]$,

\[
\sup_{|\xi| \leq 5 \log(T-t_0(x_0))^{\beta/2}} |U(x_0,\xi,\tau)| + |V(x_0,\xi,\tau)| \leq M_0 \equiv M_{\tilde{\epsilon}_0},
\] (5.22)

for some $M > 0$. With these estimates, we proceed in three steps to finish the proof of item (iii) in Theorem 1, using a truncation argument, as for Proposition 5.1 and recalling the definition of $\gamma'$ given in (5.19):

- In Step 1, we show that

\[
\forall \tau \in [0,1], \forall |\xi| \leq 2 |\log(T-t_0(x_0))|^{\beta/2}, |V(x_0,\xi,\tau)| \leq \frac{C}{|\log(T-t_0(x_0))|^{\gamma'}}.
\] (5.23)
- In Step 2, we show that
\[
\forall \tau \in [0,1), \forall |\xi| \leq |\log(T-t_0(x_0))|^{\beta/2}, |U(x_0,\xi,\tau) - \overline{U}_{K_0}| \leq \frac{C}{|\log(T-t_0(x_0))|^{\gamma}}, \quad (5.24)
\]
where \( \overline{U}_{K_0} \) is the solution of the ordinary differential equation
\[
\overline{U}_{K_0}'(\tau) = \overline{U}_{K_0}^p(\tau),
\]
with initial data
\[
\overline{U}_{K_0}(0) = \varphi^0(K_0),
\]
given by
\[
\overline{U}_{K_0}(\tau) = ((p-1)(1-\tau) + bK^2_0)^{-\frac{1}{p-1}}. \quad (5.25)
\]

- In Step 3, we justify that
\[
\forall, \tau \in [\frac{1}{2},1), \forall |\xi| \leq \frac{1}{2} |\log(T-t_0(x_0))|^{\beta/2}, |\partial_\tau U(x_0,\xi,\tau)| + |\partial_\tau V(x_0,\xi,\tau)| \leq C, \quad (5.26)
\]
which yields limits for \( U \) and \( V \) as \( \tau \to 1 \), hence for \( u \) and \( \partial_\tau u \) as \( t \to T \), by definitions \((5.11)\) and \((5.12)\) of \( U \) and \( V \).

Let \( \phi \in C^\infty(\mathbb{R}) \) be a cut-off function, with \( \text{supp } \phi \subset B(0,1), 0 \leq \phi \leq 1 \) and \( \phi \equiv 1 \) on \( B(0,1/2) \).

Introducing
\[
\phi_r(\xi) = \phi \left( \frac{\xi}{r|\log(T-t_0(x_0))|^{\beta/2}} \right),
\]
we see that
\[
\|\nabla \phi_r\|_{L^\infty(\mathbb{R})} \leq \frac{C}{|\log(T-t_0(x_0))|^{\beta}}, \quad \text{and} \quad \|\Delta \phi_r\|_{L^\infty(\mathbb{R})} \leq \frac{C}{|\log(T-t_0(x_0))|^{\beta}}. \quad (5.27)
\]

\textbf{Step 1: Proof of (5.23).} Let us consider \( r = 2 \) and introduce
\[
V_2 = \phi_2 V. \quad (5.28)
\]
Then, arguing as for Lemma \((5.2)\) we have for all \( \xi \in \mathbb{R} \) and \( \tau \in [0,1), \)
\[
\partial_\tau V_2 = \Delta_\xi V_2 + V \Delta_\xi \phi_2 - 2\nabla_\xi (V \nabla_\xi \phi_2) + p|V|^{p-1}V_2 + \mu \nabla_\xi (|V|^{q-2}VV_2) - \mu |V|^{q} \nabla_\xi \phi_2.
\]
Taking the \( L^\infty \)-norm on the Duhamel equation satisfied by \( V_2 \), then using \((5.27), (5.22)\) and \((5.20)\), we get for all \( \tau \in [0,1), \)
\[
\|V_2(\tau)\|_{L^\infty(\mathbb{R})} \leq \|V_2(0)\|_{L^\infty(\mathbb{R})} + \frac{CM_0}{|\log(T-t_0(x_0))|^{\beta}} + \frac{CM_0}{|\log(T-t_0(x_0))|^{\beta/2}} + M_0^{p-1} \int_0^\tau \|V_2(s)\|_{L^\infty(\mathbb{R})} ds + C \int_0^\tau \frac{\|V_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\tau-s}} ds + \frac{CM_0^q}{|\log(T-t_0(x_0))|^{\beta/2}} \leq \frac{C}{|\log(T-t_0(x_0))|^{\gamma}} + C \eta_0 \int_0^\tau \frac{\|V_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\tau-s}} ds,
\]
where \( \eta_0 = \max(M_0^{p-1}, M_0^{q-1}, |\log(T-t_0(x_0))|^{-\gamma}) \). Since \( \eta_0 \) can be made sufficiently small by taking \( |x_0| \) small enough and \( K_0 \) large enough (see \((5.21)\) and \((5.22)\)), using Gronwall’s inequality, we deduce that
\[
\|V_2(\tau)\|_{L^\infty(\mathbb{R})} \leq \frac{2C}{|\log(T-t_0(x_0))|^{\gamma}}, \quad \forall \tau \in [0,1),
\]
and \((5.23)\) follows.
Step 2: Proof of (5.24). Let us consider \( r = 1 \) and introduce
\[
U = U - U_{K_0}, \quad U_1 = \phi_1 U.
\]
Then \( U_1 \) satisfies the equation
\[
\partial_\tau U_1 = \Delta U_1 + U_1 \Delta \phi_1 - 2\nabla(U \nabla \phi_1) + a U_1 + \mu \nabla U \nabla \phi_1,
\]
where
\[
a(x_0, \xi, \tau) = \frac{|U(x_0, \xi, \tau)|^{p-1} U(x_0, \xi, \tau) - \overline{U}_{K_0}^p(\tau)}{U(x_0, \xi, \tau) - \overline{U}_{K_0}}, \quad \text{if} \quad U(x_0, \xi, \tau) \neq \overline{U}_{K_0},
\]
and satisfies
\[
|a(x_0, \xi, \tau)| \leq C \eta_1
\]
where \( \eta_1 = \max(K_0^{-2}, M_0^{p-1}) \to 0 \) as \( |x_0| \to 0 \) and \( K_0 \to \infty \), by (5.25) and (5.22). Using (5.18), (5.23), (5.27) and the Duhamel equation on \( U_1 \), we get
\[
\|U_1(\tau)\|_{L^\infty(\mathbb{R})} \leq \|U_1(0)\|_{L^\infty(\mathbb{R})} + \frac{CM_0}{\log(T - t_0(x_0))} + \frac{CM_0}{|\log(T - t_0(x_0))|^{\beta/2}}
\]
\[
+C_\eta_1 \int_0^\tau \|U_1(s)\|_{L^\infty(\mathbb{R})} ds + \frac{C}{|\log(T - t_0(x_0))|^{\gamma'}} + C_\eta_2 \int_0^\tau \|U_1(s)\|_{L^\infty(\mathbb{R})} ds,
\]
where \( \eta_2 = \min(\eta_1, 1/|\log(T - t_0(x_0))|^{\beta/2}) \). Since \( \eta_2 \to 0 \) as \( |x_0| \to 0 \) and \( K_0 \to \infty \), using Gronwall’s inequality, we deduce that
\[
\|U_1(\tau)\|_{\infty} \leq \frac{C}{|\log(T - t_0(x_0))|^{\gamma'}}, \quad \forall \tau \in [0, 1),
\]
and (5.24) follows.

Step 3: Proof of (5.26) and conclusion of the Proof of Theorem 7. From (5.16) and (5.17), we write the following system verified by \( (\partial_\tau U, \partial_\tau V) \), for all \( \tau \in [0, 1) \) for all \( \xi \in \mathbb{R} \),
\[
\partial_\tau (\partial_\tau U) = \Delta \xi (\partial_\tau U) + q \mu (\partial_\tau V) |V|^{q-2} V + p |U|^{p-1} (\partial_\tau U)
\]
\[
\partial_\tau (\partial_\tau V) = \Delta \xi (\partial_\tau V) + p q \nabla \xi \left( (\partial_\tau V) |V|^{q-2} V + p \nabla (|U|^{p-1} \partial_\tau U) \right).
\]
Using (5.23) and (5.24), and classical parabolic regularity, as in Steps 1 and 2, we see that (5.26) follows. Then, as in [20], we have that \( \lim_{\tau \to 1} U(x_0, 0, \tau) \) and \( \lim_{\tau \to 1} V(x_0, 0, \tau) \) exist for \( x_0 \) sufficiently small. Moreover, using (5.23) and (5.24), we see that
\[
\lim_{\tau \to 1} U(x_0, 0, \tau) \sim \overline{U}_{K_0}(1) = (b K_0^2)^{-\frac{1}{p-1}}, \quad \text{as} \quad x_0 \to 0,
\]
and
\[
|\lim_{\tau \to 1} V(x_0, 0, \tau)| \leq \frac{C}{|\log(T - t_0(x_0))|^{\gamma'}},
\]
for \( |x_0| \) small enough, where \( \gamma' \) is defined in (5.19). Recall from the beginning of the proof Part (iii) of Theorem 1 (see Part 1 page 45), that \( \lim_{\tau \to T} U(x_0, t) := u^*(x_0) \) and \( \lim_{\tau \to T} \nabla u(x_0, t) := \partial_\tau u^*(x_0) \). Therefore, from the definitions (5.11) and (5.12) of \( U \) and \( V \), we write
\[
u^*(x_0) = \lim_{t \to T} u(x_0, t) = \lim_{\tau \to 1} \frac{U(x_0, 0, \tau)}{(T - t_0(x_0))^{\frac{1}{p-1}}} \sim (b K_0^2)^{-\frac{1}{p-1}} \frac{1}{(T - t_0(x_0))^{\frac{1}{p-1}}}, \quad \text{as} \quad x_0 \to 0,
\]
and
\[|\partial_x u^*(x_0)| = \lim_{\tau \to 1} |\partial_x u(x_0, t)| = \lim_{\tau \to 1} \left| V(x_0, 0, \tau) \right| = \frac{C}{(T - t_0(x_0))^\frac{1}{p-\gamma} \frac{1}{\log(T - t_0(x_0))} \gamma'},\]

Using (5.10), we have
\[\log |x_0| \sim \frac{1}{2} \log(T - t_0(x_0)) \text{ and } T - t_0(x_0) \sim \frac{|x_0|^2}{2^{2\beta K_0^2} \log |x_0|^2}, \text{ as } x_0 \to 0.\]

Hence,
\[u^*(x_0) \sim \left(\frac{b|x_0|^2}{2 \log |x_0|^2}ight)^{-\frac{1}{p-1}}, \text{ as } x_0 \to 0,\]

and
\[|\partial_x u^*(x_0)| \leq \frac{C|x_0|^{-\frac{p+1}{p-1}}}{2^{2\beta K_0^2} \log |x_0|^2}.\]

Since \(\gamma = \min(2\beta + 1, 5\beta - 1) - \varepsilon \) with \(\varepsilon > 0\) by (5.6) and \(\gamma' = \min(\gamma - 3\beta, \frac{\beta}{7})\) by (5.19), we easily see that
\[\gamma' - 2\beta^2 = \begin{cases} \frac{1-3p}{p-1} - \varepsilon, & \text{if } 3 < p \leq 7 \\ \frac{-p^2 + 2p - 5}{2(p-1)^2} - \varepsilon, & \text{if } p > 7. \end{cases}\]

This concludes the proof of Theorem 1. \(\Box\)

6. Stability

In this section we will prove Theorem 2. In fact, the stability is a natural by-product of the existence proof, thanks to a geometrical interpretation of the parameters of the finite-dimensional problem (i.e. \((d_0, d_1)\) in (1.2)) in terms of the blow-up time and the blow-up point.

Let us first explain the strategy of the proof, and leave the technical details for the following subsection. Finally, in the third section, we briefly conclude the proof of Theorem 2, then state a stronger version, valid for blow-up solutions having the profile (6.35) only for a subsequence (see Theorem 2 page 58 below).

6.1. Strategy of the proof. Let us consider \(\hat{u}\) the constructed solution of equation (1.1) in Theorem 1 and call \(\hat{u}_0\) its initial data in \(W^{1,\infty}(\mathbb{R})\), and \(\hat{T}\) its blow-up time. From the construction method in Section 4 (see Proposition 4.4), consider \(\hat{A} \geq 1\) such that
\[\forall s \geq - \log \hat{T}, \hat{v}(s) \in \partial_A(s), \tag{6.31}\]

where
\[\hat{v}(y, s) = \hat{w}(y, s) - \varphi(y, s), \hat{w}(y, s) = e^{-\frac{s}{p-1}} \hat{u} \left(y e^{-\frac{s}{2}}, \hat{T} - e^{-s} \right) \tag{6.32}\]

and \(\varphi\) is defined in (2.21) (here and throughout this section, we consider the constant \(K\) defining the truncation in (3.3) as fixed). Now, we consider \(u_0 \in W^{1,\infty}(\mathbb{R})\) such that \(\|\varepsilon_0\|_{W^{1,\infty}(\mathbb{R})}\) is small, where
\[\varepsilon_0 = u_0 - \hat{u}_0. \tag{6.33}\]

We denote by \(u_{\varepsilon_0}\) the solution of equation (1.1) with initial data \(u_0\) and \(T(u_0) \leq +\infty\) its maximal time of existence, from the Cauchy theory in \(W^{1,\infty}(\mathbb{R})\).

Our aim is to show that, for some \(A_0 > 0\) and \(\sigma_0 \geq - \log(\hat{T})\), large enough, if \(\varepsilon_0\) is small enough, then \(T(u_0)\) is finite, and \(u_{\varepsilon_0}\) blows up at time \(T(u_0)\) only at one blow-up point \(a(u_0)\), with
\[T(u_0) \to \hat{T}, a(u_0) \to 0 \text{ as } \varepsilon_0 = u_0 - \hat{u}_0 \to 0 \text{ in } W^{1,\infty}(\mathbb{R}), \tag{6.34}\]
and

\[ v_{T(u_0),a(u_0),u_0}(s) \in \vartheta_{A_0}(s) \]

for \( s \geq s_0 \) large enough, where, for any \( (T,a) \in \mathbb{R}^2 \), we introduce

\[ v_{T,a,u_0}(y,s) = w_{T,a,u_0}(y,s) - \varphi(y,s), \]  \hspace{1cm} (6.35)

and \( w_{T,a,u_0} \) is the similarity variable version centered at \( (T,a) \) of \( u_{u_0}(x,t) \), the solution of equation \( (1.1) \) with initial data \( u_0 \). More precisely, we have

\[ w_{T,a,u_0}(y,s) = (T-t)^{\frac{1}{p-1}}u(x,t), \quad y = (x-a)/\sqrt{T-t}, \quad s = -\log(T-t). \]  \hspace{1cm} (6.36)

Indeed, with the estimates of Section 5, we deduce that \( u_{u_0} \) satisfies the same estimates as \( \dot{u} \), given in Theorem 1, which is the desired conclusion of Theorem 2. Note that for the moment, we don’t even know that \( T(u_0) \) is finite, hence asking the question of the existence of \( a(u_0) \) is non relevant and talking about \( v_{T(u_0),a(u_0),u_0} \) is meaningless at this stage.

Anticipating our aim in (6.34), we will study \( v_{T,a,u_0} \), where \( (T,a) \) is arbitrary in a small neighborhood of \( (\hat{T},0) \), hoping to have some hint that some particular value \( (\hat{T}(u_0), \hat{a}(u_0)) \) close to \( (\hat{T},0) \), will correspond to the aimed \( (T(u_0), a(u_0)) \). Note that all the \( v_{T,a,u_0} \) satisfy the same equation, namely (6.32), for all \( (y,s) \in \mathbb{R} \times [-\log T, - \log(T-T(u_0))_+] \) (by convention, we note \(-\log(0_+)=\infty\)). Introducing

\[ \epsilon(x,t) = u(x,t) - \dot{u}(x,t), \quad \text{for all} \quad x \in \mathbb{R} \quad \text{and} \quad 0 \leq t < \min(T(u_0), \hat{T}), \]

we see from (6.35), (6.36) and (6.33) that for any \( \sigma_0 \in [-\log \hat{T}, -\log(\hat{T}-T(u_0))_+] \), we have

\[ v_{T,a,u_0}(y,s_0) = \hat{v}(\sigma_0, u_0, T,a,y) \equiv (1+\tau)^{\frac{1}{p-1}} \left[ \hat{v}(z, \sigma_0) + \varphi(z, \sigma_0) + e^{-\sigma_0} \left( e^{\frac{\sigma_0}{2}} z, \hat{T} - e^{-\sigma_0} \right) \right] - \varphi(y,s_0) \]

with

\[ \tau = (T-\hat{T})e^{\sigma_0}, \quad \alpha = ae^{\sigma_0}, \quad s_0 = s_0(\sigma_0, \tau) = \sigma_0 - \log(1+\tau) \quad \text{and} \quad z = y\sqrt{1+\tau} + \alpha \]  \hspace{1cm} (6.37)

(note that we used the fact that

\[ \hat{v} = v_{T,0,\hat{a}_0} = w_{T,0,\hat{a}_0} - \varphi = \hat{w} - \varphi, \]

which follows from (6.32) and (6.33)).

In this context, \( \hat{v}(\sigma_0, u_0, T,a,y) \) appears as initial data for equation (3.2) at initial time \( s = s_0 \). Though the initial time \( s_0 = s_0(\sigma_0, \tau) \) is changing with \( T \), this reminds us of an analogous situation: in the constructing procedure in Section 4, we had initial data \( \psi_{s_0,d_0,d_1} \) at \( s = -\log \hat{T} \), for the same equation (3.2), depending on two parameters \( (d_0,d_1) \).

What if by chance, the application \( (T,a) \mapsto \hat{v}(\sigma_0, u_0, T,a,y) \) satisfies the same initialization estimates as \( (d_0,d_1) \mapsto \psi_{s_0,d_0,d_1} \) (See Proposition 4.3) ?

In that case, the construction procedure would work, starting from \( \hat{v}(\sigma_0, u_0, T,a,y) \) at time \( s = s_0 \), including the reduction to a finite dimensional problem, and the topological argument involving the two parameters \( T \) and \( a \), resulting in the existence of \( (\hat{T}(u_0), \hat{a}(u_0)) \), such that equation (3.2) with initial data at time \( s = s_0 \), \( \hat{v}(\sigma_0, u_0, \hat{T}(u_0), \hat{a}(u_0)) \), has a solution \( \hat{v}_{\sigma_0,u_0} \) such that

\[ \forall \ s \geq s_0, \ \hat{v}_{\sigma_0,u_0}(s) \in \vartheta_{A_0}(s). \]  \hspace{1cm} (6.39)

But then, remember that by definition, \( \hat{v}_{T(u_0),\hat{a}(u_0),u_0} \) is the initial data also at time \( s = s_0 \) defined in (6.38), for \( v_{T(u_0),\hat{a}(u_0),u_0}(y,s) \), another solution of the same equation (3.2). From uniqueness in the Cauchy problem, both solutions are equal, and have the same domain of definition, and the same trapping property in \( \vartheta_{A_0}(s) \). In particular, recalling that \( v_{T(u_0),\hat{a}(u_0),u_0}(y,s) \) is defined for all \( (y,s) \in \mathbb{R} \times [-\log T(u_0), -\log ((\hat{T}(u_0) - T(u_0))^+)) \), this implies that

\[ \hat{T}(u_0) = T(u_0), \]
and

$$\forall s \geq s_0, \ v_{T(u_0), \tilde{a}(u_0), u_0}(y, s) = v_{\sigma_0, u_0}(y, s)$$

and from (6.39), we have

$$\forall s \geq s_0, \ v_{T(u_0), \tilde{a}(u_0), u_0}(s) \in \partial A_0(s).$$  \hspace{1cm} (6.40)

Using our technique in Section 5, we see that our original function blows up in finite time $T(u_0)$ only at one blow-up point, $\tilde{a}(u_0)$, and that $u(x - \tilde{a}(u_0))$ satisfies the profile estimates given in Theorem 4.4.

Of course, all this holds, provided that we check that the application $(T, a) \mapsto \psi(\sigma_0, u_0, T, a)$ satisfies a statement analogous to Proposition 4.5, and that we show that

$$T(u_0) \rightarrow \hat{T}, \ \tilde{a}(u_0) \rightarrow 0 \ \text{as} \ \varepsilon_0 = u_0 - \hat{u}_0 \rightarrow 0 \ \text{in} \ W^{1, \infty}(\mathbb{R}).$$

Let us do that in the following subsections.

6.2. Behavior of “initial data” $\tilde{\psi}$ for $(T, a)$ near $(\hat{T}, 0)$. As explained in the previous subsection, here, we are left with the proof of an analogous statement to Proposition 4.5 for initial data $\tilde{\psi}(\sigma_0, u_0, T, a)$ (6.37) as a function of $(T, a)$. This is the new statement:

**Proposition 4.5 (Properties of initial data) $\tilde{\psi}(\sigma_0, u_0, T, a)$**

There exists $\tilde{C} > 0$ such that for any $A_0 \geq CA$, there exists $\sigma_0(A_0) > 0$ large enough, such that for any $\sigma_0 \geq \sigma_0(A_0)$, there exists $\hat{c}_0(\sigma_0) > 0$ small enough such that for all $u_0 \in B_{W^{1, \infty}}(\hat{u}_0, \hat{c}_0(\sigma_0)) := \{u \in W^{1, \infty} | \|u - \hat{u}_0\|_{W^{1, \infty}} \leq \hat{c}_0(\sigma_0)\}$, the following holds:

(i) There exists a set

$$\mathcal{D}_{A_0, \sigma_0, u_0} \subset \{(T, a) \mid |T - \hat{T}| \leq \frac{2e^{-\sigma_0}A_0(p - 1)}{\kappa \sigma_0^{2\beta + 1}}, |a| \leq \frac{e^{-\sigma_0}A_0(p - 1)}{b \kappa \sigma_0}\}$$

whose boundary is a Jordan curve such that the mapping

$$(T, a) \mapsto s_0^{2\beta + 1}(\tilde{\psi}_0, \tilde{\psi}_1)(\sigma_0, u_0, T, a)$$

where $s_0 = s_0(\sigma_0, \tau) = \sigma_0 - \log(1 + \tau)$

is one to one from $\mathcal{D}_{A_0, \sigma_0, u_0}$ onto $[A_0, A_0]^2$. Moreover, it is of degree $-1$ on the boundary.

(ii) For all $(T, a) \in \mathcal{D}_{A_0, \sigma_0, u_0}$, $\tilde{\psi}(\sigma_0, u_0, T, a) \in \partial A_0(s_0)$ with strict inequalities except for the two first, namely $\tilde{\psi}(\sigma_0, u_0, T, a)\), in the sense that

$$|\tilde{\psi}_m| \leq \frac{A_0}{s_0^{2\beta + 1}}, \ m = 0, 1, \ |\tilde{\psi}_2| < \frac{C \sqrt{A}}{s_0^{4\beta + 1}}, \ |\tilde{\psi}_-(y)| < \frac{3A}{s_0^3}(1 + |y|^3), \ \forall y \in \mathbb{R}, \ |\tilde{\psi}_c|_{L^\infty} < \frac{C A^2}{s_0^{3\gamma - 3}}.$$

(iii) Moreover, for all $(T, a) \in \mathcal{D}_{u_0}$, we have

$$\|\nabla \tilde{\psi}\|_{L^\infty}(\mathbb{R}) \leq \frac{C A^2}{s_0^{3\gamma - 3}} \text{ and } |\nabla \tilde{\psi}_-(y)| \leq \frac{C A^2}{s_0^3}(1 + |y|^3), \ \forall y \in \mathbb{R}.$$

**Remark 6.1.** Since $\tilde{\psi}(\sigma_0, u_0, T, a)$ is the considered initial data for equation (3.2) at time $s = s_0$, we naturally decompose it according to (3.10)-(3.13) with $s = s_0$ defined in (6.38). In fact, this statement follows directly from the following expansion of $\tilde{\psi}(\sigma_0, u_0, T, a)$ (6.37) for $(T, a)$ close to $(\hat{T}, 0)$:

**Lemma 6.2 (Expansion of modes for $(T, a)$ close to $(\hat{T}, \hat{a})$).** For $\sigma_0 > 0$ large enough, there exists $C_0(\sigma_0) > 0$ such that for any

$$\|\varepsilon_0\|_{W^{1, \infty}(\mathbb{R})} \leq \frac{1}{C_0}, \ |\tau| \leq \frac{1}{2}, \ |\alpha| \leq 1,$$

we have the following expansions:
Remark 6.3. As already stated in Remark 6.7, we decompose \( \tilde{\psi} \) according to (3.10) - (3.13) with \( s = s_0 \) defined in (6.38).

Indeed, let us first use this lemma to derive Proposition 6.5, then, we will give its proof.

Proof of Proposition 6.5 assuming that Lemma 6.2 holds.
(i) Introducing the following change of functions and variables:

\[
\tilde{\psi}_m(\sigma_0, \varepsilon_0, \tilde{\tau}, \tilde{\alpha}) = s_0^{2\beta + 1} \tilde{\psi}_m(\sigma_0, u_0, T, a) \quad \text{for} \quad m = 0, 1, \quad \varepsilon_0 = u_0 - \hat{u}_0, \quad \tilde{\tau} = s_0^{2\beta + 1} \tau \quad \text{and} \quad \tilde{\alpha} = \sigma_0 \alpha, \quad (6.43)
\]

we readily see from the previous lemma that whenever

\[
\|\varepsilon_0\|_{W^{1, \infty}(\mathbb{R})} \leq \frac{1}{C_0(\sigma_0)}, \quad |\tilde{\tau}| \leq \frac{s_0^{2\beta + 1}}{2}, \quad |\tilde{\alpha}| \leq \sigma_0,
\]

and \( \sigma_0 \) is large enough, it follows that

\[
|\tilde{\psi}(\sigma_0, u_0, T, a) - \tilde{\psi}_0(\sigma_0) - \kappa \tilde{\tau}| \leq C\left( \frac{1}{\sigma_0^{2\beta + 1}} + \frac{|\tilde{\alpha}|}{\sigma_0^{\gamma - 3\beta}} + \frac{|\tilde{\alpha}|}{\sigma_0} + \frac{\tilde{\tau}^2}{\sigma_0^{2\beta + 1}} + \frac{|\tilde{\tau}|}{\sigma_0} \right) + C_0 \varepsilon_0 \|\varepsilon_0\|_{W^{1, \infty}(\mathbb{R})}, \quad (6.44)
\]

\[
|\tilde{\psi}_1(\sigma_0, u_0, T, a) - \tilde{\psi}_0(\sigma_0) - \tilde{\psi}_1(\sigma_0)| \leq C\left( \frac{s_0^{2\beta + 1}}{0^{\gamma - 3\beta}} + \frac{2b\kappa}{(p - 1)^2} \tilde{\alpha} \right) \leq C\left( \frac{s_0^{2\beta + 1} e^{-\sigma_0^\beta}}{\sigma_0^{\gamma - 3\beta}} + \frac{|\tilde{\alpha}|}{\sigma_0^{2\beta - 1}} + \frac{\tilde{\tau}^2}{\sigma_0^{2\beta + 1}} + \frac{|\tilde{\tau}|}{\sigma_0} \right) + C_0 \varepsilon_0 \|\varepsilon_0\|_{W^{1, \infty}(\mathbb{R})}, \quad (6.45)
\]
and
\[ \text{Jac}_{\tilde{\tau}, \tilde{\alpha}}(\sigma_0, \varepsilon_0, \tilde{\psi}_0, \tilde{\psi}_1)(\tilde{\tau}, \tilde{\alpha}) = \begin{vmatrix} \frac{\kappa}{p-1} & 0 \\ \frac{2b}{p-1} & -\frac{2b}{p-1} \end{vmatrix} + C(\frac{\sqrt{\hat{A}}}{\sigma_0^{2\beta-1}} + \frac{|\tilde{\tau}|}{\sigma_0} + \frac{|\tilde{\alpha}|}{\sigma_0} + \frac{\alpha^2}{\sigma_0^{2(1-\beta)}}) + C_0|\varepsilon_0||\varepsilon_0|_{W^{1,\infty}(\mathbb{R})}. \]

Consider now \( A_0 \geq 2\hat{A}, \sigma_0 \) large and \( \varepsilon_0 \) such that
\[ ||\varepsilon_0||_{W^{1,\infty}(\mathbb{R})} \leq \varepsilon_0(\sigma_0) \equiv \frac{1}{\sigma_0^{2\beta+1} + C_0(\sigma_0)}. \] (6.46)

From the above-mentioned expansions, we see that for \( \sigma_0 \) large enough, the function
\[ (\tilde{\tau}, \tilde{\alpha}) \mapsto (\tilde{\psi}_0, \tilde{\psi}_1)(\sigma_0, \varepsilon_0, \tilde{\tau}, \tilde{\alpha}) \]
is a \( C^1 \) diffeomorphism from the rectangle
\[ \mathcal{R}_{A_0} \equiv \left[-\frac{2(p-1)A_0}{\kappa}, \frac{2(p-1)A_0}{\kappa}\right] \times \left[-\frac{(p-1)^2A_0}{b\kappa}, \frac{(p-1)^2A_0}{b\kappa}\right] \]
onto a set \( \mathcal{E}_{A_0, \sigma_0, u_0} \) which approaches (in some appropriate sense) from the rectangle \([\sigma_0^{2\beta+1}v_0(\sigma_0) - 2A_0, \sigma_0^{2\beta+1}v_0(\sigma_0) + 2A_0] \times [\sigma_0^{2\beta+1}v_0(\sigma_1) - 2A_0, \sigma_0^{2\beta+1}v_1(\sigma_0) + 2A_0] \) as \( \sigma_0 \to \infty \). Since \( \hat{v}(\sigma_0) \in \theta_A(\sigma_0) \) by (6.31), hence \( |\sigma_0^{2\beta+1}v_m(\sigma_0)| \leq \hat{A} \) for \( m = 0, 1 \), we clearly see that
\[ |\varepsilon_0(\sigma_0)| \leq C_0(\sigma_0) \] for \( \sigma_0 \) large enough, hence, there exists a set \( \hat{D}_{A_0, \sigma_0, u_0} \subset \mathcal{R}_{A_0} \) such that
\[ (\tilde{\psi}_0, \tilde{\psi}_1)(\sigma_0, \varepsilon_0, \tilde{D}_{A_0, \sigma_0, u_0}) = [-A_0, A_0]^2. \]
Moreover, from (6.44)-(6.45), the function \( (\tilde{\psi}_0, \tilde{\psi}_1) \) has degree \(-1\) on the boundary of \( \hat{D}_{A_0, \sigma_0, u_0} \). Using back the transformation (6.44) gives the conclusion of item (i).

(ii) Take \((T, a) \in \hat{D}_{A_0, \sigma_0, u_0} \) and let us check that \( \tilde{\psi}(\sigma_0, u_0, T, a) \in \theta_{\hat{A}}(s_0) \).

First, from item (i), we know by construction that \( |\tilde{\psi}_m(\sigma_0, u_0, T, a)| \leq A_0\sigma_0^{-2\beta-1} \) for \( m = 0, 1 \). For the other estimates to be checked, note first from (6.44) and the definition (6.38) of \((\tau, \alpha)\) that we have
\[ |\tau| \leq \frac{2A_0(p-1)}{\kappa\sigma_0} \]
and \[ |\alpha| \leq \frac{A_0(p-1)^2}{b\kappa\sigma_0}. \]
Therefore, using (6.46) then items (iii), (viii) and (ix) of Lemma 6.2 we see that for \( \sigma_0 \) large, we have
\[ |\tilde{\psi}_2(\sigma_0, u_0, T, a)| \leq C\sqrt{\hat{A}} \]
and \[ |\tilde{\psi}_e(\sigma_0, u_0, T, a)| \leq C\hat{A}. \]

If \( A_0 \geq \hat{C}\hat{A} \), for some large \( \hat{C} > 0 \), then we see that the conclusion follows.

(iii) The conclusion follows from items (x) and (xi) in Lemma 6.2 proceeding similarly to what we did for item (ii).

This concludes the proof of Proposition 4.5, assuming that Lemma 6.2 holds. \( \square \)

Now, we are left with the proof of Lemma 6.2.

**Proof of Lemma 6.2.** The proof is similar in the spirit to the case of the unperturbed semilinear heat equation (with \( \mu = 0 \) in (1.1)) treated in [22] (see Lemma B.2 page 186 in that paper). However, due to the difference of the scaling in the profile \( \varphi \) (we had \( \beta = \frac{1}{2} \) in [22] whereas we have \( \beta = \frac{p+1}{2(p-1)} \) here), we need to carefully give details for all the computations in the proof.

From (6.37), we write
\[ \tilde{\psi}(\sigma_0, u_0, T, a, y) = \tilde{\psi}^1(y, s_0) + \tilde{\psi}^2(y, s_0) + \tilde{\psi}^3(y, s_0) + \tilde{\psi}^4(y, s_0) \]
where

\[
\tilde{\psi}^1(y, s_0) = (1 + \tau) \frac{1}{\tau} e^{-\frac{2s_0}{\tau}} \left( e^{-\frac{\tau}{2} z, \hat{T} - e^{-\sigma_0}} \right), \quad \tilde{\psi}^2(y, s_0) = (1 + \tau) \frac{1}{\tau} \tilde{\psi}(z, \sigma_0),
\]

\[
\tilde{\psi}^3(y, s_0) = (1 + \tau) \frac{1}{\tau} \varphi(z, \sigma_0), \quad \tilde{\psi}^4(y, s_0) = -\varphi(y, s_0), \quad (6.47)
\]

where \( \tau, \alpha \) and \( s_0 \) are given in (6.38). In the following, we proceed in 4 steps, to prove analogous statements to Lemma [6.2] with \( \tilde{\psi} \) replaced by \( \tilde{\psi}^i \) for \( i = 1, \ldots, 4 \). Of course, Lemma [6.2] then follows by addition. Note that as for \( \psi \), we decompose \( \tilde{\psi}_i \) according to (3.10), (3.13) with \( s = s_0 \) (6.38). This also justifies the lighter notation \( \tilde{\psi}_i(y, s_0) \), where we insist on the dependence on the space variable \( y \) and the time variable \( s_0 \) corresponding to the initial time where \( \tilde{\psi} \) is considered (see Remark [6.1]).

**Step 1: Expansions of \( \tilde{\psi}^1 \)**

Let us first note that since equation (1.1) is wellposed in \( W^{1,\infty}(\mathbb{R}) \) through a simple fixed-point argument (see page [1]), it follows that

\[
\| \varepsilon(\hat{T} - e^{-\sigma_0}) \|_{W^{1,\infty}(\mathbb{R})} \leq C_0 \| \varepsilon_0 \|_{W^{1,\infty}(\mathbb{R})},
\]

whenever \( \| \varepsilon_0 \|_{W^{1,\infty}(\mathbb{R})} \leq \frac{1}{C_0} \), for some \( C_0 = C_0(\sigma_0) \).

Therefore, using our techniques throughout this paper, we clearly see that for \( \sigma_0 \) large enough and \( (\tau, \alpha) \) satisfying (6.42), we have

\[
|\tilde{\psi}_0^1(s_0) + |\tilde{\psi}_1^1(s_0)| + |\tilde{\psi}_2^1(s_0)| + |\partial_{\tau}(\frac{s_0}{\sigma_0})^{2\beta+1} \tilde{\psi}_0^1(s_0)| + |\partial_{\tau}(\frac{s_0}{\sigma_0})^{2\beta+1} \tilde{\psi}_1^1(s_0)| + |\partial_{\alpha} \tilde{\psi}_0^1(s_0)| + |\partial_{\alpha} \tilde{\psi}_1^1(s_0)|
\]

\[
+ \left\| \tilde{\psi}_0^1(y, s_0) \right\|_{L^\infty(\mathbb{R})} + \left\| \tilde{\psi}_1^1(s_0) \right\|_{L^\infty(\mathbb{R})} + \left\| \nabla \tilde{\psi}_0^1(s_0) \right\|_{L^\infty(\mathbb{R})} + \left\| \nabla \tilde{\psi}_1^1(y, s_0) \right\|_{L^\infty(\mathbb{R})} \leq C \| \varepsilon(\hat{T} - e^{-\sigma_0}) \|_{W^{1,\infty}(\mathbb{R})} \leq C_0 \| \varepsilon_0 \|_{W^{1,\infty}(\mathbb{R})}.
\]

**Step 2: Expansions of \( \tilde{\psi}^2 \)**

Anticipating Step 3, where we handle \( \tilde{\psi}^3 \), we note that both functions \( \tilde{\psi}^2 \) and \( \tilde{\psi}^3 \) are of the form \( (1 + \tau)^{\frac{1}{\tau-1}} g(z, \sigma_0) \), where \( g(\sigma_0) \in W^{1,\infty}(\mathbb{R}) \). For that reason, we can handle both functions at the same time, by first expanding in the style of Lemma [6.2] a function \( \bar{g} \) defined by

\[
\bar{g}(y, s_0) = (1 + \tau)^{\frac{1}{\tau-1}} g(z, \sigma_0) \quad \text{where} \quad z = y \sqrt{1 + \tau} + \alpha \quad \text{and} \quad s_0 = \sigma_0 - \log(1 + \tau), \quad (6.48)
\]

where \( g(\sigma_0) \in W^{1,\infty}(\mathbb{R}) \). The following statement allows us to conclude for \( \tilde{\psi}^2 \):
Lemma 6.4 (Expansions of $\tilde{g}(y, \sigma_0)$). If $g(\sigma_0) \in W^{1,\infty}(\mathbb{R})$ and $\tilde{g}$ is defined by (6.48), then, the following expansions hold for $\sigma_0$ large enough and $(\tau, \alpha)$ satisfying (6.42):

(i) \[
\left(\frac{s_0}{\sigma_0}\right)^{2\beta_1+1} \tilde{g}_0(s_0) = g_0(\sigma_0) \left(1 + \frac{\tau}{p-1}\right) + O(\alpha g_1(\sigma_0)) + O((|\tau| + \alpha^2)g_2(\sigma_0)) \]
\[
+ O((e^{-\sigma_0^{2\beta_1}} + \tau^2 + \frac{|\tau|}{\sigma_0} + |\tau\alpha| + |\alpha|^3)|g(\sigma_0)||L^{\infty}), \]

(ii) \[
\left(\frac{s_0}{\sigma_0}\right)^{2\beta_1+1} \tilde{g}_1(s_0) = g_1(\sigma_0) \left(1 + \frac{(p+1)\tau}{2(p-1)}\right) + 2\alpha g_2(\sigma_0) + O(g_3(\sigma_0)(|\tau| + \alpha^2)) \]
\[
+ O((e^{-\sigma_0^{2\beta_1}} + \tau^2 + \frac{|\tau|}{\sigma_0} + |\tau\alpha| + |\alpha|^3)|g(\sigma_0)||L^{\infty}), \]

(iii) $\tilde{g}_2(s_0) = g_2(\sigma_0) + O(g_3(\sigma_0)|\alpha| + O(g_4(\sigma_0)\alpha^2) + O((e^{-\sigma_0^{2\beta_1}} + |\tau| + |\alpha|^3)|g(\sigma_0)||L^{\infty}),$

(iv) $s_0^{-2\beta_1-1} \partial_{\tau} \left[\frac{s_0}{\sigma_0}^{2\beta_1+1} \tilde{g}_0(s_0)\right] = \frac{g_0(\sigma_0)}{p-1} + O(g_2(\sigma_0)) + O((\sigma_0^{-1} + |\tau| + |\alpha|)|g(\sigma_0)||L^{\infty}),$

(v) $\partial_{\tau} \left[\frac{s_0}{\sigma_0}^{2\beta_1+1} \tilde{g}_1(s_0)\right] = O(g_1(\sigma_0)) + O(g_3(\sigma_0)) + O((\sigma_0^{-1} + |\tau| + |\alpha|)|g(\sigma_0)||L^{\infty}),$

(vi) $\partial_{\tau} \tilde{g}_0(s_0) = O(g_1(\sigma_0)) + O(\alpha g_2(\sigma_0)) + O((e^{-\sigma_0^{2\beta_1}} + |\tau| + \alpha^2)|g(\sigma_0)||L^{\infty}),$

(vii) $\partial_{\tau} \tilde{g}_1(s_0) = O(g_2(\sigma_0)) + O(\alpha g_3(\sigma_0)) + O((e^{-\sigma_0^{2\beta_1}} + |\tau| + \alpha^2)|g(\sigma_0)||L^{\infty}),$

(viii) \[
\left\| \frac{\tilde{g}(y, s_0)}{1 + |y|^3} \right\|_{L^{\infty}(\mathbb{R})} \leq C \left\| \frac{g(z, \sigma_0)}{1 + |z|^3} \right\|_{L^{\infty}(\mathbb{R})} + C \sigma_0^{-4\beta_1} + |\tau| + |\alpha|^3 \right\| |g(\sigma_0)||L^{\infty}(\mathbb{R}) \]
\[
+ C|\alpha| (|g_1(\sigma_0)| + |g_2(\sigma_0)|), \]

(ix) $\|\tilde{g}(y, s_0)\|_{L^{\infty}(\mathbb{R})} \leq \|g(z, \sigma_0)\|_{L^{\infty}(\mathbb{R})} + C(|\tau| + |\alpha|)|g(\sigma_0)||L^{\infty}(\mathbb{R})$, \]

(x) $\|\nabla \tilde{g}(s_0)\|_{L^{\infty}(\mathbb{R})} \leq C \|\nabla g(\sigma_0)\|_{L^{\infty}(\mathbb{R})}$,

(xi) \[
\left\| \frac{\nabla \tilde{g}(y, s_0)}{1 + |y|^3} \right\|_{L^{\infty}(\mathbb{R})} \leq C \left\| \frac{\nabla g(z, \sigma_0)}{1 + |z|^3} \right\|_{L^{\infty}(\mathbb{R})} + C \|\nabla g(\sigma_0)\|_{L^{\infty}(\mathbb{R})} \left( |\tau| + \frac{|\alpha|^3}{\sigma_0^{5\beta_1}} \right) + O(\alpha g_2(\sigma_0)) \]
\[
+ O(\alpha g_3(\sigma_0)) + O(\alpha^2 g_4(\sigma_0)) + C|g(\sigma_0)||L^{\infty}(\mathbb{R}) \left( |\sigma_0^{-5\beta_1} + |\tau| + |\alpha|^3 \right). \]

Remark 6.5. We would like to insist on the fact that the notation $O(g)$ stands here for a function bounded by $Cg$, where $C$ is a universal constant, depending only on $p$ and $\mu$, the function $\chi$ and the constant $K > 0$ defining the truncation in (3.39).

Proof. The proof is straightforward from the definition of the decomposition in (3.13). As it is only technical, we leave it to Appendix A. \qed

Indeed, using (6.31), item (i) of Proposition 4.6 and parabolic regularity stated in Proposition 4.17 (using the part of the proof with $s \geq s_0 + 1$ only), we see that for $\sigma_0$ large enough, we have

$\hat{\nu}(\sigma_0) \in \vartheta \hat{A}(\sigma_0), \; |\hat{\nu}_j(\sigma_0)| \leq \frac{C \hat{A}}{\sigma_0^4}$ for $j = 3$ or 4,

$\|\hat{\nu}(\sigma_0)\|_{L^{\infty}(\mathbb{R})} + \|\nabla \hat{\nu}(\sigma_0)\|_{L^{\infty}(\mathbb{R})} \leq \frac{C \hat{A}^2}{\sigma_0^{5\beta_1}}$ and $\left\| \frac{\nabla \hat{\nu}(z, \sigma_0)}{1 + |z|^3} \right\|_{L^{\infty}(\mathbb{R})} \leq \frac{C \hat{A}^2}{\sigma_0^5}.$
Using the definition of $\tilde{v}_3$ given in Definition (4.2), then applying Lemma 6.4 with $g = \tilde{v}$, we see from (4.3) that for $\sigma_0$ large enough, we have

$$
\left| \left( \frac{s_0}{\sigma_0} \right)^{2\beta} \psi^2_0(s_0) - \tilde{v}_0(\sigma_0) \right| \leq \frac{C\hat{A}^2 e^{-\sigma_0^2\beta}}{\sigma_0^{4-3\beta}} + \frac{C\hat{A}^2}{\sigma_0^{4-3\beta}} |\tau| + |\alpha|^3 + \frac{C\hat{A}|\alpha|}{\sigma_0^{2\beta}} + \frac{C\sqrt{A}}{\sigma_0^{2\beta-1}} \alpha^2,
$$

$$
|s_0^{-2\beta-1} \partial_\tau \left[ \frac{s_0^{2\beta+1}}{\sigma_0} \psi_0^2(s_0) \right]| \leq \frac{C\hat{A}^2}{\sigma_0^{-3+1}} + \frac{C\hat{A}^2}{\sigma_0^{-3\beta}} (|\tau| + |\alpha|),
$$

$$
|s_0^{-2\beta-1} \partial_\tau \left[ s_0^{2\beta+1} \psi_0^2(s_0) \right]| \leq \frac{C\hat{A}^2}{\sigma_0^{-3+1}} + \frac{C\hat{A}^2}{\sigma_0^{-3\beta}} (|\tau| + |\alpha|),
$$

$$
|\partial_\alpha \psi^2_0(s_0)| \leq \frac{C\sqrt{A}}{\sigma_0^{2\beta+1}} + \frac{C\hat{A}}{\sigma_0^{4\beta}} |\alpha| + \frac{C\hat{A}^2}{\sigma_0^{4\beta-1}} (|\tau| + \alpha^2),
$$

$$
|\partial_\alpha \psi^2_1(s_0)| \leq \frac{C\sqrt{A}}{\sigma_0^{2\beta}} + \frac{C\hat{A}}{\sigma_0^{4\beta}} |\alpha| + \frac{C\hat{A}^2}{\sigma_0^{4\beta-1}} (|\tau| + \alpha^2),
$$

$$
\left\| \frac{\psi^2_0(y, s_0)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq \frac{C\hat{A}}{\sigma_0^{4\beta}} + \frac{C\sqrt{A}}{\sigma_0^{4\beta-1}} |\alpha| + \frac{C\hat{A}^2}{\sigma_0^{4\beta}} (|\tau| + |\alpha|^3),
$$

$$
|\nabla \psi^2_0(s_0)|_{L^\infty(\mathbb{R})} \leq \frac{C\hat{A}^2}{\sigma_0^{4\beta}},
$$

$$
\left\| \frac{\nabla \psi^2_0(y, s_0)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq \frac{C\hat{A}^2}{\sigma_0^{4\beta}} + \frac{C\sqrt{A}}{\sigma_0^{4\beta-1}} |\alpha| + \frac{C\hat{A}^2}{\sigma_0^{4\beta}} (|\tau| + |\alpha|^3).
$$

**Step 3: Expansions of $\tilde{v}^3$**

The first part of the estimates follows from Lemma 6.4 as for $\tilde{v}^2$. The second part needs more refinements, which are eased by the fact that $\tilde{v}^3$ is explicit.

In order to apply Lemma 6.4 with $g = \varphi$, we first introduce the following estimates on $\varphi$ which follow from straightforward computations:

For $s$ large enough, we have

$$
\varphi_0(s) = \kappa + O(s^{-4\beta}), \quad \varphi_2(s) = -\frac{a}{2} s^{-2\beta} + O(s^{-4\beta}), \quad \varphi_1(s) = \varphi_3(s) = 0, \quad \varphi_4(s) = O(s^{-4\beta}),
$$

$$
\|\varphi(s)\|_{L^\infty(\mathbb{R})} \leq C.
$$
where $a$ is introduced in (2.23). Applying Lemma 6.4 with $g = \varphi$ and using the value of $a$ in (2.23), we see that

$$
\left| \frac{s_0}{\sigma_0} \right|^{2\beta+1} \psi_0^3(s_0) - \kappa - \frac{\kappa \tau}{p - 1} \leq C \left( \frac{1}{\sigma_0^4} + \frac{\alpha}{\sigma_0^2} + \tau^2 + \left| \tau \right| + \left| \alpha \right| + \left| \alpha^3 \right|, 
\right.
$$

and

$$
\left| \frac{s_0}{\sigma_0} \right|^{2\beta+1} \psi_1(s_0) + \frac{2b \kappa \alpha}{(p - 1)^2 \sigma_0^2} \leq C \left( \frac{\alpha}{\sigma_0^4} + e^{-\alpha^3} + \tau^2 + \left| \tau \right| + \left| \alpha \right| + \left| \alpha^3 \right|, 
\right.
$$

Therefore, by definition of the decomposition (3.13), we see that

$$
\tilde{\psi}_2^3(s_0) = -\frac{a}{2\sigma_0^2} + O \left( \frac{1}{\sigma_0^4} \right) + O(\left| \tau \right| + \left| \alpha^3 \right|),
$$

$$
\sigma_0^{-2\beta-1} \partial_r \left[ \sigma_0^{-2\beta+1} \psi_0^3(s_0) \right] = \frac{\kappa}{p - 1} + O \left( \frac{1}{\sigma_0^2} \right) + O(\left| \tau \right| + \left| \alpha \right|),
$$

$$
|\sigma_0^{-2\beta-1} \partial_r \left[ \sigma_0^{-2\beta+1} \psi_1^3(s_0) \right]| \leq C(\sigma_0^{-1} + \left| \tau \right| + \left| \alpha \right|),
$$

$$
|\partial_\alpha \tilde{\psi}_0^3(s_0)| \leq C \left( \frac{\alpha}{\sigma_0^4} + e^{-\alpha^3} + \left| \tau \right| + \left| \alpha^2 \right| \right), \quad \partial_\alpha \tilde{\psi}_1^3(s_0) = -\frac{2b \kappa}{(p - 1)^2 \sigma_0^2} + O \left( \frac{1}{\sigma_0^4} \right) + O(\left| \tau \right| + \left| \alpha^2 \right|).
$$

Now, for the remaining components, we need more refined estimates, based on the explicit formula of $\varphi$ defined in (2.21), in particular the following, for $\sigma_0$ large enough and for all $y \in \mathbb{R}$,

$$
|\tilde{\psi}_0^3(y, s_0) - \varphi(y, \sigma_0)| \leq C \left( \left| \tau \right| + \frac{\left| \alpha \right|}{\sigma_0^2} \right)(1 + \left| y \right|^2), \quad |\tilde{\psi}_0^3(y, s_0) - \varphi(y, \sigma_0)| \leq C \left( \left| \tau \right| + \frac{\left| \alpha \right|}{\sigma_0^2} \right),
$$

$$
|\nabla \tilde{\psi}_0^3(y, s_0) - \nabla \varphi(y, \sigma_0)| \leq C \left( \frac{\left| \tau \right|}{\sigma_0^3} + \frac{\left| \alpha \right|}{\sigma_0^2} \right), \quad \nabla \tilde{\psi}_1^3(y, s_0) - \nabla \varphi(y, \sigma_0)| \leq C \left( \frac{\left| \tau \right|}{\sigma_0^3} + \frac{\left| \alpha \right|}{\sigma_0^2} \right),
$$

(6.49)

(for the second and third estimates, use a first order Taylor expansion, and evaluate the error according to the position of $|y|$ with respect to 1). In fact, in Step 4, we will obtain analogous estimates for $\tilde{\psi}_4(y, s_0)$. Therefore, as far as the remaining components are concerned, we wait for Step 4, where we will directly obtain the contribution of $\tilde{\psi}_3(y, s_0) + \tilde{\psi}_4(y, s_0)$ to those components.

**Step 4: Expansions of $\tilde{\psi}_4$**

Here, we need to refine the estimates we gave for $\varphi$ in Step 3. From further refinements, we write for $s$ large enough and for all $y \in \mathbb{R}$,

$$
|\varphi(y, s) - \left[ \kappa - \frac{a}{2\sigma_0^2} h_2(y) \right]| \leq C \frac{|y|^4}{s^{4\beta}}, \quad |\partial_s \varphi(y, s)| \leq C \min \left( \frac{1}{s}, \frac{1 + y^2}{s^{2\beta+1}} \right), \quad |\partial_s \nabla \varphi(y, s)| \leq C \frac{s^{-1}}{s^{2\beta+1}}.
$$

Since $\tilde{\psi}_4(y) = -\varphi(y, s_0)$ from (6.47) with $s_0 = \sigma_0 - \log(1 + \tau)$, it follows that

$$
\left| \tilde{\psi}_4(y, s_0) + \left[ \kappa - \frac{a}{2\sigma_0^2} h_2(y) \right] \right| \leq C \frac{|y|^4}{s^{4\beta}} + C \frac{|\tau|(1 + y^2)}{s^{2\beta+1}},
$$

(6.50)

$$
\left| \tilde{\psi}_4(y, s_0) + \varphi(y, \sigma_0) \right| \leq C |\tau| \min \left( \frac{1}{\sigma_0}, \frac{(1 + \tau^2)}{\sigma_0^{2\beta+1}} \right), \quad |\nabla \tilde{\psi}_4(y, s_0) + \nabla \varphi(y, \sigma_0)| \leq C |\tau| \sigma_0^{-1-\beta}.
$$

Therefore, by definition of the decomposition (3.13), we see that

$$
\left( \frac{s_0}{\sigma_0} \right)^{2\beta+1} \tilde{\psi}_4(s_0) = -\kappa + O(\sigma_0^{-4\beta}) + O(\tau \sigma_0^{-1}), \quad \tilde{\psi}_4^1(s_0) = 0, \quad \tilde{\psi}_4^2(s_0) = \frac{a}{2\sigma_0^2} + O(\sigma_0^{-4\beta}) + O(\tau \sigma_0^{-2\beta-1}),
$$

$$
|\sigma_0^{-2\beta-1} \partial_r \left[ \sigma_0^{-2\beta+1} \psi_0^4(s_0) \right]| \leq C \frac{1}{\sigma_0}, \quad \partial_r \left[ \sigma_0^{-2\beta+1} \psi_0^4 \right](s_0) = 0, \quad \partial_\alpha \tilde{\psi}_0^4(s_0) = \partial_\alpha \tilde{\psi}_1^4(s_0) = 0.
As for the remaining components, we will directly estimate the contribution of \( \tilde{\psi}^3 (y, s_0) \) and \( \tilde{\psi}^4 (y, s_0) \). Using (6.49) and (6.50), we see that

\[
| \tilde{\psi}^3 (y, s_0) + \tilde{\psi}^4 (y, s_0) | \leq C \min \left\{ \left( | \tau | + \frac{\alpha}{\sigma_0^3} \right) (1 + |y|^3), \left( | \tau | + \frac{\sigma_0}{\beta} \right) \right\},
\]

\[
| \nabla \tilde{\psi}^3 (y, s_0) + \nabla \tilde{\psi}^4 (y, s_0) | \leq C \left( \frac{| \tau |}{\sigma_0^3} + \frac{\alpha}{\sigma_0^3} \right) (1 + |y|^3).
\]

Using these estimates, we see that for all \( y \in \mathbb{R} \),

\[
| \tilde{\psi}^3 (y, s_0) + \tilde{\psi}^4 (y, s_0) | \leq C \left( | \tau | + \frac{\alpha}{\sigma_0^3} \right) (1 + |y|^3), \quad | \tilde{\psi}^3 (y, s_0) + \tilde{\psi}^4 (y, s_0) | \leq C \left( | \tau | + \frac{\alpha}{\sigma_0^3} \right),
\]

\[
| \nabla \tilde{\psi}^3 (y, s_0) + \nabla \tilde{\psi}^4 (y, s_0) | \leq C \left( \frac{| \tau |}{\sigma_0^3} + \frac{\alpha}{\sigma_0^3} \right) (1 + |y|^3)
\]

(use the fact that

\[
\left\| \frac{v_- (y, s_0)}{1 + |y|^3} \right\|_{L^\infty (\mathbb{R})} \leq C \left\| \frac{v (y, s_0)}{1 + |y|^3} \right\|_{L^\infty (\mathbb{R})},
\]

\[
| \nabla v_- (y, s_0) | \leq C \left( \left\| \nabla v (s_0) \right\|_{L^\infty (\mathbb{R})} + \frac{\left\| v (s_0) \right\|_{L^\infty (\mathbb{R})}}{\sigma_0^{4 \beta}} \right) (1 + |y|^3),
\]

which follow from the definition of the decomposition (5.13).

**Conclusion of the proof of Lemma 6.2** In fact, the conclusion follows by adding the various estimates obtained in Steps 1 through 4. This concludes the proof of Lemma 6.2.

Since we have already showed that Proposition 4.5 follows from Lemma 6.2, this also concludes the proof of Proposition 4.5. \( \square \)

### 6.3. Conclusion of the proof of Theorem 2 and generalization.

In this subsection, we briefly explain how to conclude the proof of Theorem 2 from the initialization given in Proposition 4.5 and our techniques developed for the existence proof. Then, we give a stronger version, valid for blowing-up solutions having the profile (1.5) for blowing-up solutions.

**Conclusion of the proof of Theorem 2** As we will see here, thanks to Proposition 4.5, the stability proof reduces to an existence proof. Note that having an initial time \( s_0 \) for equation (2.2) depending on the parameter \( T \) changes nothing to the situation. Note also that thanks to Proposition 4.5, we may write a statement Proposition 4.5' analogous to Proposition 4.5, valid for \( A \geq \bar{C}' \bar{A} \), for some large enough constant \( \bar{C}' > 0 \). Let us now briefly explain the proof of Theorem 2.

Fix some \( A_0 = \max (\bar{C}, \bar{C}') \) where \( \bar{C} \geq 0 \) is introduced in Proposition 4.5, and \( \bar{C}' > 0 \) introduced above. Consider then an arbitrary \( \eta > 0 \), and fix \( \sigma_0 \) large enough, so that

\[
\frac{2e^{-\sigma_0} A_0 (p - 1)}{\kappa \sigma_0^{2p + 1}} + \frac{e^{-\sigma_0}}{b \kappa \sigma_0} A_0 (p - 1)^2 \leq \eta.
\]

Then, take initial data \( u_0 \in B_{W^{1,1} (\mathbb{R})} (\hat{u}_0, \hat{\sigma}_0 (\sigma_0)) \), where \( \hat{\sigma}_0 (\sigma_0) \) is defined in Proposition 4.5. As explained above in the strategy of the proof, we find some parameters \( (\bar{T}(u_0), \bar{\sigma}(u_0)) \) satisfying

\[
| \bar{T}(u_0) - \bar{T}| + | \bar{\sigma}(u_0) | \leq \eta,
\]

such that

\[
\forall s \geq \bar{s}_0 \equiv \sigma_0 - \log (1 + (\bar{T}(u_0) - \bar{T}) e^{\sigma_0}), \quad v_{\bar{T}(u_0), \bar{\sigma}(u_0), u_0} (s) \in \partial A_0 (s).
\]
In particular, \( u(x, t) \) blows up at time \( T(u_0) = \hat{T}(u_0) \), only at one blow-up point \( a(u_0) = \hat{a}(u_0) \), with the profile (1.5). Since \( \eta \) is arbitrary in (6.51), this means that
\[
(T(u_0), a(u_0)) \to (\hat{T}, \hat{a}) \text{ as } u_0 \to \hat{u}_0.
\]
This concludes the proof of Theorem 2. \( \Box \)

Validity of the stability result: As a concluding remark, we would like to mention that our stability proof works not only around the constructed solution in Theorem 1 but also for blow-up solutions having the profile (1.5) only for a subsequence. More precisely, this is our “twin” statement for the stability result stated in Theorem 2.

Theorem 2 (Stability of blow-up solutions of equation (1.1) having the profile (1.5) only for a subsequence). Consider \( \hat{u}(x, t) \) a solution to equation (1.1) with initial data \( \hat{u}_0 \), which blows up at time \( \hat{T} \) only at one blow-up point \( \hat{a} \), such that
\[
v_{\hat{T}, \hat{a}, \hat{u}_0}(s_n) \in \partial \hat{A}(s_n), \quad \| \nabla v_{\hat{T}, \hat{a}, \hat{u}_0}(s_n) \|_{L^\infty(\mathbb{R})} \leq C \hat{A}^2 s_n^{-3\beta} \quad \text{and} \quad \left\| \frac{\nabla v_{\hat{T}, \hat{a}, \hat{u}_0}(y,s_n)}{1 + |y|^3} \right\|_{L^\infty(\mathbb{R})} \leq \frac{C \hat{A}^2}{s_n}, \quad (6.52)
\]
for some \( \hat{A} > 0 \) and for some sequence \( s_n \to \infty \), where \( v_{\hat{T}, \hat{a}, \hat{u}_0} \) is defined in (6.33). Then, there exists a neighborhood \( V_0 \) of \( \hat{u}_0 \) in \( W^{1,\infty}(\mathbb{R}^N) \) such that for any \( \hat{u}_0 \in V_0 \), Equation (1.1) has a unique solution \( u \) with initial data \( u_0 \), \( u \) blows up in finite time \( T(u_0) \) and at a single point \( a(u_0) \). Moreover, (1.4) is satisfied by \( u(x - a(u_0), t) \) (with \( T \) replaced by \( T(u_0) \) and for all \( t \in [T(u_0) - e^{-s_n}, T(u_0)] \) for some \( n \in \mathbb{N} \), not just for a sequence). We also have
\[
T(u_0) \to \hat{T}, \quad a(u_0) \to \hat{a}, \quad \text{as} \quad u_0 \to \hat{u}_0 \quad \text{in} \quad W^{1,\infty}(\mathbb{R}^N).
\]

Remark 6.6. As a consequence of this theorem, we see that \( \hat{u} \) has the profile (1.5) not only for a sequence, but for the whole time range \( [T(u_0) - e^{-s_n}, T(u_0)] \) (in particular, (6.52) holds also for all \( s \) large, and not just for a sequence). In other words, having the profile (1.5) only for a sequence (together with some estimates on the gradient) is equivalent to having that profile for the whole range of times.

Appendix A. A technical result related to the projection (3.13)

In this section, we prove Lemma 6.4.

Proof of Lemma 6.4. Consider \( \sigma_0 \) to be taken large enough. We assume that \( (\tau, \alpha) \) satisfies (6.42).\( (i)-(ii) \) Consider \( m = 0, 1, 2 \). From (3.13), we see that
\[
\bar{g}_m(s_0) = \int_{\mathbb{R}} \bar{g}(y, s_0) k_m(y) \chi(y, s_0) \rho(y) dy,
\]
where \( k_m(y) \) and \( \chi(y, s_0) \) are given in (2.9) and (3.9). Using (6.48) and making the change of variables \( y \to z \), we write
\[
\bar{g}_m(s_0) = (1 + \tau)^{-\frac{1}{p-1} - \frac{1}{2}} \int_{\mathbb{R}} g(z, \sigma_0) k_m(y) \chi(y, s_0) \rho(y) dz. \quad (A.53)
\]
First, note that one easily checks that
\[
|\sigma_0^{2\beta + 1} - \sigma_0^{2\beta + 1}| \leq C |\tau| \sigma_0^{2\beta},
\]
\[
|\chi(y, s_0) - \chi(z, \sigma_0)| \leq C (|\tau z| + |\alpha|)^{1} \{ |z| \geq \frac{M}{\sigma_0} \}, \quad (A.54)
\]
\[
\rho(y) - \rho(z) \left[ 1 + \frac{\alpha^2}{4} + \frac{\alpha}{2} \frac{2\tau + 2}{\theta} \frac{2\tau^2 + \alpha^2}{z^2} \right] \leq C (\tau^2 + |\tau \alpha| + |\alpha^3|) (\rho(z))^4, \quad (A.55)
\]
whenever \((\tau, \alpha)\) satisfies \((6.42)\). Then, we give the following conversion table for the polynomials involved in the expression of \(\bar{g}_m\):

\[
(1 + \tau)\frac{1}{p-1} - \frac{1}{2}k_m(y) \left[ 1 - \frac{\alpha^2}{4} + \frac{\alpha}{2} z + \frac{2\tau + \alpha^2}{8} z^2 \right]
\]

\[
= \left( 1 + \frac{\tau}{p-1} \right) k_0(z) + \alpha k_1(z) + (2\tau + \alpha^2) k_2(z) + O((\tau^2 + |\tau\alpha|)(1 + |z|^2)) \text{ if } m = 0,
\]

\[
= \left( 1 + \frac{(p+1)\tau}{2(p-1)} \right) k_1(z) + 2\alpha k_2(z) + 3(2\tau + \alpha^2) k_3(z) + O((\tau^2 + |\tau\alpha| + |\alpha|^3)(1 + |z|^3)) \text{ if } m = 1.
\]

Recalling that

\[
g_j(s) = \int_{\mathbb{R}} g(z) k_j(z) \chi(z, s) \rho \, dz,
\]

the result follows for items (i) and (ii).

(iii) Now, we take \(m = 2\). The proof follows the same pattern as for the previous items, though we need less accuracy in \(\tau\). Indeed, we need this less precise version of \((A.55)\):

\[
\left| \rho(y) - \rho(z) \left[ 1 - \frac{\alpha^2}{4} + \frac{\alpha}{2} z + \frac{\alpha^2}{8} z^2 \right] \right| \leq C (|\tau| + |\alpha|^3) (\rho(z))^{\frac{1}{2}}.
\]

As before, we need the following expansion:

\[
(1 + \tau)\frac{1}{p-1} - \frac{1}{2}k_2(y) \left[ 1 - \frac{\alpha^2}{4} + \frac{\alpha}{2} z + \frac{\alpha^2}{8} z^2 \right] = k_2(z) + 3\alpha k_3(z) + 6\alpha^2 k_4(z) + O(|\tau| + |\alpha|^3)(1 + z^4)).
\]

Using \((A.54)\) and \((A.56)\) yields item (ii).

(iv)-(v) Take \(m = 0\) or \(1\). From \((6.48)\) and \((A.53)\), we write

\[
\begin{align*}
\pi^{-1} \partial r \{ s_0^{2\beta + 1} g_m \} (s_0) &= \left( \frac{1}{p-1} - \frac{1}{2} \right) \bar{g}_m(s_0) - (2\beta + 1) \frac{s_0}{s_0(1 + \tau)} \bar{g}_m(s_0) \\
&+ (1 + \tau)\frac{1}{p-1} - \frac{1}{2} \int_{\mathbb{R}} g(z, s_0) \partial_r y \left( \frac{1}{2} k_{m-1}^r(y) \chi(y, s_0) + k_m(y) \partial y \chi(y, s_0) - \frac{y}{2} k_m(y) \chi(y, s_0) \right) \rho(y) \, dz \\
&+ (1 + \tau)\frac{1}{p-1} - \frac{1}{2} \int_{\mathbb{R}} g(z, s_0) k_m(y) \partial_r s_0 \partial_s \chi(y, s_0) \rho(y) \, dz,
\end{align*}
\]

with the convention that \(k_{-1}(y) = 0\). Noting from \((6.48)\) and the definition \((3.3)\) of \(\chi\) that

\[
y = \frac{z - \alpha}{\sqrt{1 + \tau}}, \quad \partial_r y = -\frac{z - \alpha}{2(1 + \tau)^2}, \quad |\partial_y \chi(y, s_0)| \leq \frac{C}{\sigma_0^2} 1\{|z| \geq k_3 \sigma_0^2\},
\]

\[
\partial_r s_0 = -\frac{1}{1 + \tau}, \quad |\partial_s \chi(y, s_0)| \leq \frac{C}{\sigma_0} 1\{|z| \geq k_3 \sigma_0^2\},
\]

then, using the various estimates presented for the proof of items (i) to (iii), we directly get the conclusions for items (iv) and (v).

(vi)-(vii) Take \(m = 0\) or \(1\). From \((A.53)\), we write

\[
\partial_0 \bar{g}_m(s_0) = (1 + \tau)\frac{1}{p-1} - \frac{1}{2} \int_{\mathbb{R}} g(z, s_0) \partial_0 y \left( \frac{1}{2} k_{m-1}^r(y) \chi(y, s_0) + k_m(y) \partial y \chi(y, s_0) - \frac{y}{2} k_m(y) \chi(y, s_0) \right) \rho(y) \, dz.
\]

Note that

\[
\partial_0 y = -\frac{1}{\sqrt{1 + \tau}}.
\]
Using this even less precise version of (A.55):
\[
\left| \rho(y) - \rho(z) \left[ 1 + \frac{\alpha}{2} z \right] \right| \leq C \left( |\tau| + \alpha^2 \right) (\rho(z))^{\frac{3}{2}},
\]

with (A.54) and (A.57), we see that
\[
\partial_\alpha \bar{g}_0(\sigma_0) = \int_{\mathbb{R}} g(z, \sigma_0) \frac{z - \alpha}{2} \rho(z) dz + O((e^{-\sigma_0^2} + |\tau| + \alpha^2)\|g(\sigma_0)\|_{L^\infty}),
\]
\[
\partial_\alpha \bar{g}_1(\sigma_0) = \int_{\mathbb{R}} g(z, \sigma_0) \left( \frac{(z - \alpha)^2}{4} - \frac{1}{2} \right) \rho(z) dz + O((e^{-\sigma_0^2} + |\tau| + \alpha^2)\|g(\sigma_0)\|_{L^\infty}).
\]

Since
\[
\frac{z - \alpha}{2} \left( 1 + \frac{\alpha}{2} z \right) = k_1(z) + 2\alpha k_2(z) + O(\alpha^2(1 + z^2)),
\]
\[
\left( \frac{(z - \alpha)^2}{4} - \frac{1}{2} \right) \left( 1 + \frac{\alpha}{2} z \right) = 2k_2(z) + 6\alpha k_3(z) + O(\alpha^2(1 + |z|^2)),
\]
arguing as for the previous items gives the result.

(viii) From (3.13), we write
\[
\bar{g}_-(y, s_0) = \chi(y, s_0) \bar{g}(y, s_0) - \sum_{i=0}^{2} \bar{g}_i(s_0) h_i(y),
\]
\[
g_-(z, \sigma_0) = \chi(z, \sigma_0) g(z, \sigma_0) - \sum_{i=0}^{2} g_i(\sigma_0) h_i(z).
\]

Making the difference and using the definition (0.48) of \( \bar{g} \), we see that
\[
\bar{g}_-(y, s_0) - g_-(z, \sigma_0) = g(z, \sigma_0) \left( (1 + \tau)^{\frac{1}{2r+1}} - \frac{1}{2} \chi(y, s_0) - \chi(z, \sigma_0) \right)
\]
\[
- \sum_{i=0}^{2} (\bar{g}_i(s_0) - g_i(\sigma_0)) h_i(y) + \sum_{i=0}^{2} g_i(\sigma_0) (h_i(z) - h_i(y)).
\]

Since we have from (A.54),
\[
\left| (1 + \tau)^{\frac{1}{2r+1}} - \frac{1}{2} \chi(y, s_0) - \chi(z, \sigma_0) \right| \leq C|\tau| + \frac{C}{\sigma_0^3} (|\tau y| + |\alpha|) 1_{\{|y| \leq \frac{4}{\sigma_0^2}\}} \leq C \left( |\tau| + \frac{|\alpha|}{\sigma_0^3} \right) (1 + |y|^3),
\]
\[
(A.59)
\]

and from the definition (2.8) of \( h_i \),
\[
h_0(y) - h_0(z) = 0, \ |h_1(y) - h_1(z)| \leq C(|\alpha| + |\tau y|), \ |h_2(y) - h_2(z)| \leq C(\alpha^2 + |\alpha y| + |\tau y^2|),
\]
and from (6.48)
\[
1 + |z|^3 \leq 2(1 + |y|^3), \quad (A.60)
\]

we see that the conclusion follows from items (i)-(iii) (use the fact that \( |g_j(\sigma_0)| \leq C \|\frac{g(z, \sigma_0)}{1 + |z|^3}\|_{L^\infty(\mathbb{R})} \) for \( j = 3, 4 \), which follows from (3.121)).

(ix) From (3.11), we write
\[
\bar{g}_e(y, s_0) = (1 - \chi(y, s_0)) \bar{g}(y, s_0) \quad \text{and} \quad g_e(z, \sigma_0) = (1 - \chi(z, \sigma_0)) g(z, \sigma_0),
\]
therefore, by definition (6.48) of \( \bar{g} \), we have
\[
\bar{g}_e(y, s_0) = g_e(z, \sigma_0) + \left( \left[ (1 + \tau)^{\frac{1}{2r+1}} - \frac{1}{2} \right] (1 - \chi(y, s_0)) - \chi(y, \sigma_0) - \chi(z, s_0) \right) g(z, \sigma_0).
\]
Since we have from the definition (3.9) of \( \chi \) and the expression (6.48) of \( y \),
\[
|\chi(y, s_0) - \chi(z, \sigma_0)| \leq \frac{C}{\sigma_0^\beta}(|y| + |\alpha|)^{1}(\frac{K \sqrt{\sigma_0^\beta}}{\sigma_0^\gamma} \leq \frac{3K \sqrt{\sigma_0^\beta}}{\sigma_0^\gamma}) \leq C(|\tau| + |\alpha|),
\]
the result follows.

(x) The result follows from the differentiation of (6.48):
\[
\nabla \tilde{g}(y, s_0) = (1 + \tau)^{\frac{1}{p-1}} \frac{1}{2} \nabla g(z, \sigma_0) \text{ with } y = z \sqrt{1 + \tau + \alpha}.
\]

(xi) Differentiating (A.58), we see that
\[
\nabla \tilde{g}^-(y, s_0) - \sqrt{1 + \tau} \nabla \tilde{g}^-(z, \sigma_0) = \sqrt{1 + \tau} \nabla \chi(z, \sigma_0) \left((1 + \tau)^{\frac{1}{p-1}} \frac{1}{2} \chi(y, s_0) - \chi(z, \sigma_0)\right)
\]
\[
+ g(z, \sigma_0) \left((1 + \tau)^{\frac{1}{p-1}} \frac{1}{2} \nabla \chi(y, s_0) - \sqrt{1 + \tau} \nabla \chi(z, \sigma_0)\right)
\]
\[
- \sum_{i=0}^2 \frac{\tilde{g}_i(s_0) - g_i(\sigma_0))}{\tau h_{i-1}(y)} + \sum_{i=0}^2 \frac{g_i(\sigma_0)}{\tau h_{i-1}(y)} \left(1 + \tau h_{i-1}(z) - h_{i-1}(y)\right).
\]
Since
\[
\left|\left(1 + \tau\right)^{\frac{1}{p-1}} \frac{1}{2} \nabla \chi(y, s_0) - \sqrt{1 + \tau} \nabla \chi(z, \sigma_0)\right| \leq C \frac{|\tau|}{\sigma_0^\beta} + C \frac{|\tau y| + |\alpha|}{\sigma_0^\gamma} \left(\frac{K \sqrt{\sigma_0^\beta}}{\sigma_0^\gamma} \leq \frac{3K \sqrt{\sigma_0^\beta}}{\sigma_0^\gamma}\right)
\]
\[
\leq C \frac{|\alpha|}{\sigma_0^\gamma} (1 + |y|^3)
\]
by definition (3.9) of \( \chi \), using the fact that
\[
|\sqrt{1 + \tau h_0(\cdot) - h_0(\cdot)}| \leq C|\tau| \text{ and } |\sqrt{1 + \tau h_1(\cdot) - h_1(\cdot)}| \leq C(|\tau y| + |\alpha|),
\]
we derive the conclusion from (A.60), (A.59) together with items (i)-(iii).

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Université de Tunis El Manar, Faculté des Sciences de Tunis, Département de Mathématiques, Laboratoire Équations aux Dérivées Partielles LR03ES04, 2092 Tunis, Tunisie. e-mail: slim.tayachi@fst.rnu.tn

Université Paris 13, Sorbonne Paris cité, Institut Galilée, CNRS UMR 7539 LAGA, 99 Avenue Jean-Baptiste Clément 93430 Villetaneuse, France. e-mail: Hatem.Zaag@univ-paris13.fr