Absence of ghost in a new bimetric-matter coupling

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Abstract: Interactions in bimetric theory, which can describe gravity in the presence of an extra spin-2 field, are severely constrained by the requirement of the absence of the Boulware-Deser ghost instability. Recently an interesting new matter coupling was proposed in terms of a composite metric but it was claimed to reintroduce the ghost. In this paper we carry out a nonlinear Hamiltonian analysis of this new matter coupling and show that it is indeed ghost-free. The analysis involves using a new set of variables that naturally appear in the relation between the metric and vielbein formulations of bimetric theory. In terms of these variables we show that the new matter coupling does not reduce the number of constraints in bimetric theory and hence does not reintroduce the Boulware-Deser ghost.

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1 Introduction

Attempts to generalize linear Fierz-Pauli theory [1] and construct nonlinear interactions for massive spin-2 fields remained unsuccessful for decades due to the notorious appearance of the Boulware-Deser ghost instability [2, 3]. Only a few years ago the development of new approximation methods in [4, 5] lead to the proposal of a candidate action for ghost-free nonlinear massive gravity [6, 7]. The formulation of this type of theory requires the presence of a second metric tensor (a “background metric”) which, in this first approach, was taken to be of Minkowski form. In [8] it was shown that this model indeed avoids the Boulware-Deser ghost and therefore gives the first consistent description of nonlinear spin-2 self-interactions. A reformulation and generalization to arbitrary background metrics appeared in [9], for which the absence of ghost has also been demonstrated [10–12]. Moreover, it turned out that the second metric can have its own dynamics without spoiling the consistency of the theory [11, 13]. This ghost-free bimetric theory, whose spectrum around Einstein solutions has been analyzed in [14], describes nonlinear interactions of a massive and a massless spin-2 field. It can be regarded as a theory of gravitational interactions in the presence of an extra spin-2 field.

The only known consistent matter couplings in bimetric theory so far have been of the same form as in general relativity, but now there are two metrics that could possibly interact with two different types of matter. From the ghost proof in [13], it is clear that both of the metrics can couple independently to different matter sources without introducing inconsistencies. Recent work has shown that a problem occurs in general when both metrics couple to the same matter [15, 16]. In this case, the constraint that removes the Boulware-Deser ghost in bimetric theory is destroyed and the fatal instability reappears. These findings are in agreement with earlier attempts to couple matter to a certain combination of the two metrics, which possesses massless fluctuations around maximally symmetric solutions [14].

Recently, the authors of [16] identified another particular combination of the metrics that leads to interesting consequences when coupled to matter. As for the ghost issue, they showed that the new couplings were ghost-free in some simplifying limits, but from a more detailed
Hamiltonian analysis in a perturbative setup, they concluded that the ghost instability was indeed present sourced by the new matter couplings.\footnote{Shortly after, an independent group proposed the same kind of coupling \cite{17}, based on a simple, necessary but not sufficient condition for the absence of ghost.} If true, the proposed matter coupling would reintroduce fatal instabilities and thus would be of little use for phenomenological applications. Although \cite{16} argued that the theory could still be treated as a valid effective theory with a cut-off below the ghost mass, it is not obvious that such an interpretation is valid. Since ghost modes can create energy from the vacuum, they are produced in interactions with healthy fields even in the absence of external energy \cite{18, 19}. Spontaneous vacuum decay into heavy ghost modes can release energy above the cut-off scale, rendering an effective field theory description invalid.

In this paper we carry out a detailed non-linear Hamiltonian analysis of the matter couplings proposed in \cite{16} and show that they are free of the Boulware-Deser ghost. The analysis involves a modification of the construction that was used to prove the absence of ghost is massive gravity and bimetric theory \cite{8, 10, 13}, and naturally appears in the context of the connection between the metric \cite{13} and vielbein \cite{20} formulations of bimetric theory.

This paper is organized as follows. In section 2 we review the bimetric action and its ghost analysis, explaining why a main step in the the analysis is guaranteed to work. In section 3 we introduce a new set of bimetric variables that simplifies the expressions and makes the analysis feasible. The absence of ghost in the proposed matter coupling is demonstrated in section 4. For completeness, in section 5 we show that the new variables introduced possess an interesting interpretation in the vielbein formulation. Finally, our results are discussed in section 6 and some mathematical details are relegated to the appendices.

### 2 Review of bimetric theory

In this section we first review bimetric theory and its matter couplings and then provide a simplified overview of the ghost analysis in the Hamiltonian framework.

#### 2.1 The ghost-free bimetric action and matter couplings

The ghost-free action for the two spin-2 fields $g_{\mu\nu}$ and $f_{\mu\nu}$ is given by \cite{11, 13},

$$S = \int d^4x \left[ m_g^2 \sqrt{g} R(g) + m_f^2 \sqrt{f} R(f) - 2 m^4 \sqrt{g} \sum_{n=0}^4 \beta_n e_n \left( \sqrt{g^{-1}} f \right) \right].$$

(2.1)

Here, $m_g$ and $m_f$ are the Planck masses for the two metrics, $m$ is a mass scale and $\beta_n$ are interaction parameters. $e_n(X)$ denote the elementary symmetric polynomials of the eigenvalues of the matrix $X$. They appear in the expansion of the determinant,

$$\det(\mathbb{1} + X) = \sum_{n=0}^4 e_n(X).$$

(2.2)
The interaction potential originated in the massive gravity context [7], as reformulated in [9]. The dependence on the square-root matrix \( X = \sqrt{g^{-1}f} \), defined through \((\sqrt{g^{-1}f})^2 = g^{-1}f\), is crucial for the absence of the Boulware-Deser ghost [3]. We will briefly review the ghost analysis of (2.1) in the next subsection.

The bimetric action has a well-defined mass spectrum around proportional backgrounds \( \bar{f}_{\mu\nu} = c^2 \bar{g}_{\mu\nu} \). Then the perturbations of the two metrics diagonalize into a massless and a massive fluctuation, \( \delta G \propto \delta g / m_g^2 \) and \( \delta M \propto \delta f - c^2 \delta \bar{g} \) [14]. Hence the metrics \( g_{\mu\nu} \) and \( f_{\mu\nu} \) are combinations of massless and massive modes. The metric with a larger Planck mass has a larger massless component.

The known ghost-free matter couplings that can be added to (2.1) are of the form,

\[
S_{\text{matter}} = \int d^4 x \sqrt{g} \mathcal{L}_g(g, \phi_g) + \int d^4 x \sqrt{f} \mathcal{L}_f(f, \phi_f),
\]

where \( \phi_g \) and \( \phi_f \) denote different types of matter fields that minimally couple to the respective metrics in the standard way. In the limit \( m_g >> m_f \), the metric \( g_{\mu\nu} \) is mostly massless and can be regarded as the gravitational metric with \( M_p = m_g \) [13, 14], while \( f_{\mu\nu} \) is an extra spin-2 field that modifies gravity. In this framework, the observed high scale \( M_p \), or the weak strength, of gravity is correlated with the effective masslessness of the interaction.

Coupling both metrics to the same type of matter was considered in [21] but it can be shown that such couplings reintroduce the BD ghost [15, 16]. It is also known that coupling the most obvious nonlinear extension of the massless mode to matter also reintroduces the ghost [14].

### 2.2 A new proposed matter coupling

Recently, [16] considered coupling matter to an effective metric built out of \( g \) and \( f \) as,

\[
G_{\mu\nu} = a^2 g_{\mu\nu} + 2ab g_{\mu\rho} S^{\rho\nu} + b^2 f_{\mu\nu},
\]

where \( a \) and \( b \) are arbitrary parameters.\(^2\) Note that for \( a, b \neq 0 \), these are degenerate with the bimetric parameters and can always be set to one by rescalings \( g \rightarrow g/a^2 \), \( f \rightarrow f/b^2 \) and absorbing the factors that are generated in the bimetric action into \( m_g, m_f \) and \( \beta_n \). Some interesting features of the new matter couplings were discussed in [16]. Some other features will be discussed below.

The coupling of \( G \) to matter passes some simple necessary checks for being ghost-free. [16] showed that in two simplifying limits, the mini-superspace approximation and the decoupling limit, the new couplings were ghost-free. But a more general Hamiltonian analysis of perturbations around flat space showed a ghost at the sixth order. Hence, [16] concludes that beyond the approximations considered, the theory is not ghost-free.

Since the mass of the ghost seemed to be above the energy scale of the theory, one may try to argue, as in [16], that the new matter couplings could still be considered in an effective

\(^2\)This combination can serve as a metric since \( gS \) is symmetric, which can be easily seen by formally expanding \( S = \sqrt{1 + (g^{-1}f - 1)} \).
theory below a cutoff. Such an argument is obviously valid for non-ghost excitations. But its validity for a ghost mode is in doubt since the production of ghost modes do not require external energy, hence vacuum can spontaneously decay into ghost modes, releasing energy above the cutoff scale. This would make an effective theory which contains a ghost above the cutoff inherently inconsistent. For a discussion, see, for example, \[18, 19\]. Hence the presence of a ghost would be fatal for the new matter couplings irrespective of its mass scale.

The aim of this work is to perform a complete Hamiltonian analysis of the new matter couplings and show that they do not reintroduce the ghost into the bimetric theory.

2.3 Review of ghost analysis in bimetric theory

Here we briefly review the main aspects of the ghost analysis of bimetric theory \[8, 10, 11, 13\]. This also provides the framework for addressing the ghost issue in new matter couplings. In particular, we emphasize a crucial aspect of the analysis not sufficiently clearly stated earlier.

To carry out a Hamiltonian analysis, we start with the usual ADM decomposition \[22\] of the two metrics, which in matrix notation reads,

\[
g = \begin{pmatrix} -N^2 + \nu^T \gamma \nu & \nu^T \gamma \\ \gamma \nu & \gamma \end{pmatrix}, \quad f = \begin{pmatrix} -L^2 + \lambda^T \phi \lambda & \lambda^T \phi \\ \phi \lambda & \phi \end{pmatrix}.
\]

(2.5)

Here, \(N\) and \(L\) are the lapses, \(\nu^i\) and \(\lambda^i\) are the shift vectors and \(\gamma_{ij}\) and \(\phi_{ij}\) are the spatial 3-metrics. In these variables, the bimetric Lagrangian takes the form (in phase space variables and up to surface terms),

\[
\mathcal{L} = \pi^{ij} \partial_t \gamma_{ij} + p^{ij} \partial_t \phi_{ij} + NR^0(g) + \nu^i R_i^g + LR^0(f) + \lambda^i R_i^f - 2m^4 \tilde{V}(N, L, \nu, \lambda, \gamma, \phi).
\]

(2.6)

Here \(\pi^{ij}\) and \(p^{ij}\) are the momenta canonically conjugate to \(\gamma_{ij}\) and \(\phi_{ij}\). The 8 lapses and shifts appear without time derivatives and are non-dynamical, while, a priori, \(\gamma_{ij}\) and \(\phi_{ij}\) contain 12 dynamical fields (24 phase space degrees of freedom). These include ghost modes that must be removable by gauge fixing and by constraints arising from the equations of motion for \(N, L, \nu^i, \lambda^i\). But since \(\tilde{V}\) is highly nonlinear in the \(N, L, \nu^i, \lambda^i\), the corresponding 8 equations of motion could potentially depend on all the non-dynamical variables and determine them in terms of \(\gamma_{ij}\) and \(\phi_{ij}\), rather than becoming constraints on \(\gamma_{ij}\) and \(\phi_{ij}\). This is the basis of the Boulware-Deser argument for presence of a ghost in such theories \[3\]. However, it is also possible that these equations do not determine all lapses and shifts in which case some of them will instead impose constraints on \(\gamma_{ij}\) and \(\phi_{ij}\).\(^3\) We outline the analysis below.

If it turns out that the 8 equations of motion arising from \(N, L, \nu^i, \lambda^i\) can determine only 3 combinations of these variables, say, \(n^i(N, L, \nu, \lambda, \gamma, \phi)\), in terms of \(\gamma_{ij}\) and \(\phi_{ij}\), then the remaining 5 equations will not depend on the non-dynamical variables. They become constraints on \(\gamma_{ij}\) and \(\phi_{ij}\) and eliminate some of the unwanted modes.\(^4\)

\(^3\)In the massive gravity context this possibility was discussed in \[6, 7\]

\(^4\)This counting is for ghost-free bimetric theory \[13\]. Other counting possibilities have been considered in \[23, 24\] in the massive gravity context.
In the action let us now trade 3 of the non-dynamical variables, say, the $\nu^i$, for the combinations $n^i$ (the possibilities are restricted by 3-dimensional general covariance). For the above picture to hold, the remaining 5 variables $\lambda^i$, $N$ and $L$ must now appear linearly as Lagrange multipliers that enforce the 5 constraint equations. Since the action \((2.6)\) already contains a term $\nu^i R^{(g)}_i$, this in turn implies that the expression for $\nu^i$ in terms of $n^i$ must be linear in the remaining lapses and shifts. For the bimetric action \((2.1)\), a combination that works is \([10, 13]\),

$$\nu^i = \lambda^i + L n^i + N D^i_j n^j . \tag{2.7}$$

The metric $D^i_j$ will be specified below. On eliminating $\nu^i$ in favor of $n^i$, the action takes the form \([13]\),

$$S = \int d^4x \left( \pi^{ij} \partial_t \gamma_{ij} + p^{ij} \partial_t \phi_{ij} + \lambda^i C^{(\lambda)}_i + L C_L + N C_N \right) , \tag{2.8}$$

where $C^{(\lambda)}_i = R^{(g)}_i + R^{(f)}_i$ is independent of $n^i$. The explicit forms of $C_N$ and $C_L$ are given below. The important point is that in terms of $n^i$, the theory is linear in the $\lambda^i$, $L$ and $N$.

Since the $n^i$ equations of motion are linear in $L$ and $N$, naively it seems that $n^i$ depend on the lapses, contrary to the assumption. However, since $n^i$ enter only through $\nu^i$, we have,

$$\frac{\delta S}{\delta n^i} = \left( \frac{\delta S}{\delta \nu^j} \right)_{\nu=\nu(n)} \frac{\delta \nu^j}{\delta n^i} = 0 , \tag{2.9}$$

or explicitly,

$$N \frac{\partial}{\partial n^i} C_N + L \frac{\partial}{\partial n^i} C_L = C_j^{(\nu)} \left( L \delta^i_j + N \frac{\partial}{\partial n^j} (Dn)^i \right) = 0 . \tag{2.10}$$

Note that both sides of the first equality are linear in $N$ and $L$, hence it is obvious that $C_j^{(\nu)}$, when expressed as a function of $n^i$, must be independent of the lapses. Since the Jacobian factor $\delta \nu^j/\delta n^i$ is invertible, it follows that $n^i$ are determined by the equations

$$C_j^{(\nu)} (n, \gamma, \phi, \pi, p) = 0 , \tag{2.11}$$

and are independent of the lapses and shifts as desired. This property has been explicitly verified for the action \((2.1)\) \([8, 10, 11, 13]\), but the above discussion shows that this is always the case whenever the action can be made linear in the lapses and shifts through field redefinitions of the type \((2.7)\). The Lagrange multipliers in \((2.8)\) now lead to 5 constraints on the a priori dynamical variables,

$$C^{(\lambda)}_i = 0 , \quad C_L = 0 , \quad C_N = 0 . \tag{2.12}$$

The last of these is accompanied by a secondary constraint $C_N^{(2)} \equiv \partial_t C_N = 0$ and the pair remove the Boulware-Deser ghost and its conjugate momentum \([11]\). The other 4 constraints
are associated with general covariance and along with gauge fixing, eliminate another 8 phases pace degrees of freedom, reducing to phase space degrees of freedom to 24-8=14, or 7 dynamical fields. Some of the original dynamical equations now reduce to further constraints that determine $N, L$ and $\lambda^i$. As an aside, (2.10) implies that when the $n^i$ equations are satisfied, then $\partial C_N/\partial n^i = 0$ and $\partial C_L/\partial n^i = 0$, hence all constraints become $n^i$-independent.

The feasibility of proving the absence of ghost in this way depends on the possibility of converting an action that is non-linear in the lapses and shifts (2.1), to a partially linear form (2.8) through a redefinition of the form (2.7). In the bimetric case, a further complication is the appearance of the square-root matrix in the action. The redefinition (2.7) resolves both these problems provided the $3\times3$ matrix $D_{ij}$ in (2.7) satisfies a condition that solves to [10],

$$D = \sqrt{\gamma^{-1} \phi(x\mathbb{1} + n n^T \phi)} (x\mathbb{1} + n n^T \phi)^{-1}, \quad x \equiv 1 - n^T \phi n.$$  \hspace{1cm} (2.13)

Here we use matrix notation and $\mathbb{1}$ stands for the matrix with components $\delta_{ij}$. In terms of the $n^i$, the action takes the form (2.8) where $C_L$ and $C_N$ are given by [13],

$$C_L = R_0^{(f)} + R_i^{(g)} n^i + 2m^4 \sqrt{\det \gamma} U,$$

$$C_N = R_0^{(g)} + R_i^{(g)} D_{ij} n^j + 2m^4 \sqrt{\det \gamma} V.$$  \hspace{1cm} (2.14)

$U$ and $V$ are the contributions from the interaction potential and read, in matrix notation,

$$U = \sqrt{x} \left( \sum_{n=0}^{2} \beta_{n+1} e_n (\sqrt{x} D) + \beta_3 (e_1(D) n^T \phi Dn - (Dn)^T \phi Dn) \right) + \beta_2 n^T \phi Dn + \beta_4 \sqrt{\det \gamma} \sqrt{\det \phi},$$

$$V = \sum_{n=0}^{3} \beta_n e_n (\sqrt{x} D).$$  \hspace{1cm} (2.15)

### 3 A new redefinition of shift variables

The present form of the redefinition (2.7) is asymmetric in the Hamiltonian variables for the two metrics. In this section we introduce a new parametrization of the shift vectors $\nu^i$ which appears more symmetric. The new variables simplify the expressions and facilitate the ghost analysis for the new matter coupling. Later, in section 5, we show that the new shift variables are related to the Lorentz boost that symmetrizes a combination of the vielbeins for $g$ and $f$ such that the square root $\sqrt{g^{-1}f}$ can be evaluated.

#### 3.1 The new redefinition

We begin by decomposing the spatial metrics into vielbeins,

$$\gamma_{ij} = e^a_i \delta_{ab} e^b_j, \quad \phi_{ij} = \phi^a_i \delta_{ab} \phi^b_j.$$  \hspace{1cm} (3.1)

The vielbeins are defined up to two independent local Lorentz rotations. Let us consider the following further redefinition of the shift vector $n^i$ in terms of new variables $v_a$,

$$n^i = (\tilde{\phi}^{-1})_a \delta_{ab} v_b.$$  \hspace{1cm} (3.2)
Here, $\tilde{\varphi} = R\varphi$ and $R$ is a specific Lorentz rotation obtained below. Next, we express the matrix $D$ in (2.13) in terms of the new variables,

$$D = \sqrt{e^{-1} \tilde{T}^{-1}(e^{-1})^\top \tilde{\varphi}^T R^T (x \tilde{I} + vv^\top) R \varphi \left( x \mathbb{1} + \varphi^{-1} \tilde{T}^{-1} vv^\top R \varphi \right)^{-1}},$$

where now $x = 1 - v^\top \tilde{T}^{-1} v$ and for the matrices that raise and lower local Lorentz indices we have introduced the following symbols,

$$\tilde{I}_{ab} = \delta_{ab}, \quad (\tilde{T}^{-1})_{ab} = \delta^{ab}. \quad (3.4)$$

Observe that we can write,

$$x \tilde{I} + vv^\top = x (\mathbb{1} + \frac{1}{x + \sqrt{x}} vv^\top) \tilde{T}^{-1} (\mathbb{1} + \frac{1}{x + \sqrt{x}} vv^\top). \quad (3.5)$$

The square root of the $3 \times 3$ matrix in (3.3) can be evaluated if we demand the following symmetry property,

$$\left( \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) R \varphi e^{-1} \right)^\top = \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) R \varphi e^{-1}. \quad (3.6)$$

This requirement can always be satisfied by choosing an appropriate local Lorentz rotation $R$. The quantity considered is of the form $ARB$ with $R^T \tilde{T} R = \tilde{I}$ and can be symmetrized by,

$$R = \sqrt{(A^T B^{-1} \tilde{T}^{-1})(A^T B^{-1} \tilde{T}^{-1})^\top (A^T B^{-1} \tilde{T}^{-1})^\top}. \quad (3.7)$$

Here $A = A^\top = (\mathbb{1} + \frac{1}{x + \sqrt{x}} vv^\top)$ and $B = \varphi e^{-1}$. This is a sandwiched version of the standard polar decomposition, the matrix under the square-root is now positive and $R$ always exists, exactly as in polar decomposition (see appendix A for a derivation).

Now, using the notation $\tilde{\varphi} = R\varphi$, with $R$ fixed as above, the matrix $D$ becomes,

$$D = \sqrt{x} e^{-1} \tilde{I}^{-1} (e^{-1})^\top \tilde{\varphi}^T \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) \tilde{T}^{-1} \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) \tilde{\varphi} (x \mathbb{1} + \varphi^{-1} \tilde{T}^{-1} vv^\top \tilde{\varphi})^{-1}$$

$$= \sqrt{x} e^{-1} \tilde{I}^{-1} \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) \tilde{\varphi} e^{-1} \tilde{T}^{-1} \left( \tilde{I} + \frac{1}{x + \sqrt{x}} vv^\top \right) \tilde{\varphi} (x \mathbb{1} + \varphi^{-1} \tilde{T}^{-1} vv^\top \tilde{\varphi})^{-1}$$

$$= \sqrt{x} e^{-1} \tilde{I}^{-1} \left( \mathbb{1} + \frac{1}{x + \sqrt{x}} vv^\top \right) \tilde{\varphi} (x \mathbb{1} + \varphi^{-1} \tilde{T}^{-1} vv^\top \tilde{\varphi})^{-1}. \quad (3.8)$$

Finally, using,

$$(x \mathbb{1} + \varphi^{-1} \tilde{T}^{-1} vv^\top \tilde{\varphi})^{-1} = x^{-1} (\mathbb{1} - \varphi^{-1} \tilde{T}^{-1} vv^\top \tilde{\varphi}),$$

we arrive at the simple expression,

$$D = \frac{1}{\sqrt{x}} e^{-1} \left( \mathbb{1} - \frac{1}{x + \sqrt{x}} \tilde{T}^{-1} vv^\top \tilde{\varphi} \right), \quad (3.10)$$

where $\mathbb{1}^a_b = \delta^a_b$. When acting on $n^i = (\tilde{\varphi}^{-1})^i_a \delta^{ab} v_b$, this matrix gives $D^i_j n^j = (e^{-1})^i_a \delta^{ab} v_b$. Putting everything together, we find that the redefinition (2.7) of the original shift vector now reads,

$$n^i = \lambda^i + \left( L(\tilde{\varphi}^{-1})^i_a + N (e^{-1})^i_a \right) \delta^{ab} v_b, \quad (3.11)$$
which has a simpler symmetric form. But note that the gauge fixed $\tilde{\phi}_i$ now also depends on $\phi_i$, $e_i^a$ and $v_a$ through $R^a_b$. This complication however does not affect the ghost argument.

The new variables also answer another question: The expression (2.13) for $D$, which appeared in the massive gravity and bimetric ghost analysis, involves a $3 \times 3$ square-root matrix the existence of which is not evident. The analysis here shows that in the Hamiltonian framework employed, the matrix $D$ always exists.

The action in terms of the new variables can easily be obtained from (2.8) where one replaces the spatial metrics in terms of vielbeins as well as $n^i$ and $D^{ij}$ in terms of $v_a$ using (3.2) and (3.10). Clearly, with the new redefinition, the ghost proof of [10, 11, 13] which we reviewed in section 2.3 goes through in the same way as before, with $n^i$ replaced by $v_a$.

4 Absence of ghost in the new matter coupling

We now turn to the ghost analysis of the matter coupling for a composite metric proposed in [16], where it was also concluded that the new couplings reintroduced the Boulware-Deser ghost instability into the theory. This would render the new couplings unusable even in an effective theory sense, as discussed in section 2.2. However, using the new bimetric variables $v^a$, we show that the new matter couplings are linear in the lapses $N$, $L$ and the shift $\lambda^i$. Moreover, the equations for $v_a$ are independent of $N$, $L$ and $\lambda^i$. Thus, the theory, including the new matter coupling, contains the same number of constraints as pure bimetric theory and hence should be free of the Boulware-Deser ghost mode.

4.1 Matter coupling of the effective metric

Recently, [16] proposed coupling the ghost-free bimetric theory to matter through an “effective” metric,

$$G_{\mu\nu} = a^2 g_{\mu\nu} + 2ab g_{\mu\rho} \left( \sqrt{g^{-1}f} \right)^\rho_\nu + b^2 f_{\mu\nu}.$$  (4.1)

As argued in section 2.2, we can set $a = b = 1$ without loss of generality. Then, $G = g(1+S)^2$.

Let us denote the ADM variables of $G$ by $N_{\text{eff}}$ (lapse), $\nu_{\text{eff}}^i$ (shift) and $(\gamma_{\text{eff}})_{ij}$ (spatial metric), such that,

$$G = \begin{pmatrix}
-N_{\text{eff}}^2 + \nu_{\text{eff}}^k (\gamma_{\text{eff}})_{kli} \nu_{\text{eff}}^l
&\nu_{\text{eff}}^k (\gamma_{\text{eff}})_{kj}

(\gamma_{\text{eff}})_{ik} \nu_{\text{eff}}^k
&(\gamma_{\text{eff}})_{ij}
\end{pmatrix}.  \quad (4.2)
$$

From general relativity it is known that, for standard minimal matter couplings, the matter Lagrangian expressed in terms of phase space variables, takes the form,\(^5\)

$$\mathcal{L}_{\text{matter}} = \mathcal{L}_0 + N_{\text{eff}} \Theta + \nu_{\text{eff}}^i \Theta_i.  \quad (4.3)$$

\(^5\)For coupling to fermions, one needs to invoke the vierbein (5.19) for $G$. Also in this case, it follows from standard results in general relativity that the vierbein-matter interactions are linear in $N_{\text{eff}}$ and $\nu_{\text{eff}}^i$.
This is linear in \( N_{\text{eff}} \) and \( \nu_{\text{eff}} \), which, however, are highly nonlinear in the ADM variables of \( g \) and \( f \). In order not to spoil the consistency of the bimetric potential, \( L_{\text{matter}} \) must become linear in \( N, L, \) and \( \lambda^i \) after the redefinition (3.11) has been performed. We thus need to show that \( N_{\text{eff}} \) and \( \nu_{\text{eff}} \) are linear functions of \( N, L, \) and \( \lambda^i \) and that the latter do not appear in the \( v_a \) equations of motion.

### 4.2 Linearity in the lapses

It is straightforward to work out the expressions for \( \nu_{\text{eff}} \) and \( \gamma_{\text{eff}} \). First, we note that after the redefinition (3.11) the ADM decomposition of the term \( g \sqrt{g^{-1}f} \) in \( G \) reads,

\[
g \sqrt{g^{-1}f} = \left( -\sqrt{x}NL + (\lambda + L\bar{\phi}^{-1}\hat{I}^{-1}v)^T \chi \bar{\phi}(\lambda + L\bar{\phi}^{-1}\hat{I}^{-1}v) \right) \left( \lambda + L\bar{\phi}^{-1}\hat{I}^{-1}v \right)^T \chi \bar{\phi}.
\]

To keep the expression shorter, we have used matrix notation and defined,

\[
\chi_{bc} = \delta_{bc} - \frac{1}{1 + \sqrt{x}v b v c}.
\]

Plugging this together with the ADM decompositions for \( g \) and \( f \) into the effective metric \( G = g + 2gS + f \) and comparing the result to (4.2), we can easily read off the spatial metric and the shift of \( G \),

\[
(\gamma_{\text{eff}})_{ij} = \gamma_{ij} + \phi_{ij} + e^a_i \chi_{ab} \bar{\phi}_b + e^a_j \chi_{ab} \bar{\phi}_b,
\]

\[
\nu_{\text{eff}}^i = \lambda^i + L(\bar{\phi}^{-1})_a \delta_{ab} \nu_b + (\gamma_{\text{eff}})^{-1} (Ne^a_j - L\bar{\phi}_j^a) v_a.
\]

In principle, the lapse \( N_{\text{eff}} \) can be derived in the same manner, but this requires a tedious computation (which we have performed, verifying that the result agrees with the expressions derived below). For the sake of transparency, we provide a simpler derivation here. A third approach is presented in appendix B. Consider,

\[
\sqrt{\det g_{\text{eff}}} = N_{\text{eff}} \sqrt{\det \gamma_{\text{eff}}}.
\]

Using (2.2), this is also equal to,

\[
\sqrt{\det g_{\text{eff}}} = \sqrt{\det g} \det \left( 1 + \sqrt{g^{-1}f} \right) = N \sqrt{\det \gamma} \sum_{n=0}^{4} e_n \left( \sqrt{g^{-1}f} \right) .
\]

Thus we have,

\[
N_{\text{eff}} = \frac{1}{\sqrt{\det \gamma_{\text{eff}}}} \left( N \sqrt{\det \gamma} \sum_{n=0}^{4} e_n \left( \sqrt{g^{-1}f} \right) \right) .
\]

\(^6\)Note that the symmetry of the matrix \( e^T \chi \bar{\phi} \) is equivalent to the symmetry of \((\hat{1} + \frac{1}{x+\sqrt{x}v^T})\bar{\phi}^{-1}v\) which is imposed by our choice of rotational gauge in (3.6).
Note that the right-hand side of (4.9) is the bimetric potential with all $\beta_n$ set to one. After the redefinition of the shift $\nu^i$, this expression is linear in the $N$ and $L$ and does not contain the shifts $\lambda^i$, as discussed in section 2.3. More precisely, we have,

$$N\sqrt{\det \gamma} \sum_{n=0}^{4} e_n(S) = N\sqrt{\det \gamma} V + L\sqrt{\det \gamma} U,$$

(4.11)

where the scalar functions $U$ and $V$ are defined as in (2.15) but with $\beta_k = 1$ for all $k$. In terms our new shift variables $v_a$ (3.11) they read,

$$U = \sqrt{x} \left( \sum_{n=0}^{2} e_n(\sqrt{x} D) + e_1(D) v^T \tilde{\varphi} e^{-1} \tilde{I}^{-1} v - (\tilde{\varphi} e^{-1} \tilde{I}^{-1} v)^T \tilde{I} \tilde{\varphi} e^{-1} \tilde{I}^{-1} v \right) + v^T \tilde{\varphi} e^{-1} \tilde{I}^{-1} v + \det(\tilde{\varphi} e^{-1}),$$

$$V = \sum_{n=0}^{3} e_n(\sqrt{x} D),$$

(4.12)

where,

$$D = \frac{1}{\sqrt{x}} e^{-1} \left( \hat{I} - \frac{1}{1+\sqrt{x}} \hat{I}^{-1} v v^T \right) \tilde{\varphi}.$$

(4.13)

This shows that the effective lapse $N_{\text{eff}}$ and shift $\nu_{\text{eff}}^i$ are linear functions of $N$, $L$ and $\lambda^i$. Hence, the matter coupling will not introduce nonlinearities for these variables.

### 4.3 Absence of the lapses in the shift equations

Let us recapitulate what we have established so far. After the redefinition that renders the bimetric potential linear in $N$, $L$ and $\lambda^i$, the same holds for the matter coupling. Thus the complete bimetric plus matter Lagrangian takes the form,

$$\mathcal{L} = \mathcal{L}'_{\text{dyn}} + \lambda^i C'_i - N C'_N - L C'_L,$$

(4.14)

where $\mathcal{L}'_{\text{dyn}}$ contains the kinetic terms in (2.8) plus any new dynamics from the matter sector, i.e. the terms in (4.3) that do not depend on $N$, $L$ and $\lambda^i$. The constraints $C'_i$, $C'_N$ and $C'_L$ consist of the original terms from the bimetric potential as well as the new contributions from the matter coupling (4.3). Variation of the Lagrangian with respect to $N$, $L$ and $\lambda^i$ produces equations that are independent of the variables themselves and hence constrain the remaining phase space degrees of freedom. We note further that $C'_i(\lambda^i)$ does not depend on the components of $v_a$ and consequently $\lambda^i$ does not show up in their equations.

However, $C'_N$ and $C'_L$ depend on the $v_a$ and the only thing left to check is that $N$ and $L$ do not appear in the equations of motion for $v_a$. This is crucial because if the $v_a$ equations did depend on $N$ or $L$, then they would not determine $v_a$ in terms of the phase space variables alone. Consequently there would not exist enough constraints on the latter to eliminate the
ghost.\(^7\) In this case, the Boulware-Deser ghost would propagate and destroy the consistency of the theory.

We thus need to demonstrate that the \(v_a\) equations do not determine \(N\) nor \(L\). The argument for this is the same as that outline for the pure bimetric theory. Naively, one finds that \(N\) and \(L\) multiply nonlinear functions of \(v_a\) and are thus expected to appear in its equations. However, the new shift vector \(v_a\) enters the action only through the original variable \(\nu^i\) (3.11). Hence its equations of motion can be written as,

\[
0 = \frac{\delta S}{\delta v_a} = \frac{\delta S}{\delta \nu^j} \frac{\delta \nu^j}{\delta v_a} = \frac{\delta S}{\delta \nu^j} \left( N(e^{-1})^j a + L \frac{\delta}{\delta v_a} ((\tilde{\varphi}^{-1})^j b \delta^{bc} v_c) \right). \tag{4.15}
\]

Now, since the action is linear in the lapses, so is the variation with respect to \(v^a\) on the left-hand-side. Since the Jacobian factor \(\frac{\delta \nu^j}{\delta v_a}\) is already linear in the lapses, it follows that \(\frac{\delta S}{\delta \nu^j}\), when expressed in terms of the \(v_a\), cannot depend on \(L\) and \(N\) (otherwise there would be nonlinear terms). Furthermore, in order for the redefinition to be well defined, we need the Jacobian to be invertible and hence, the \(v_a\) equations of motion are equivalent to \(\frac{\delta S}{\delta \nu^j} = 0\), which does not involve the lapses. We emphasize the generality of this statement: For a redefinition that renders the Lagrangian linear in the lapses and that is linear in the lapses itself, the equation for the redefined shift vector are always independent the the lapses.

This completes the proof that the number of constraints in bimetric theory is not altered when the composite metric (4.1) is coupled to matter and hence the Boulware-Deser ghost is not reintroduced by the novel matter coupling.

Our result is at variance with the conclusion in \([16]\) that in the presence of the new matter couplings, the theory is no longer linear in the lapses and hence is not ghost-free. Since \([16]\) provides the result of a perturbative analysis without the calculational details, the discrepancy remains unexplained.

5 Relation to vielbein formulation

This section is devoted to the interesting interpretation of the new bimetric variables introduced in section 3 in the context of the vierbein formulation of bimetric theory. The consistency proof of the matter coupling in the previous section does not rely on this background material.

5.1 Symmetrization condition

A reformulation of bimetric theory in terms of vierbeins which avoids the square-root matrix has been proposed in \([20]\). In order to express the interaction potential of the metric formulation in terms of vierbeins, we decompose the two metrics, \(g_{\mu\nu} = e(g)^{a}_{\mu} \eta_{ab} e(g)^{b}_{\nu}\) and

---

\(^7\)One can also see the absence of the lapse constraint in the case where the shift equations depend on the lapse as follows: If the solution for the shift depends on the lapse, then nonlinear functions of the lapse will appear in the action after the shift has been integrated out. Hence, the lapse equation of motion will now determine the lapse itself instead of imposing a constraint on other variables.
\[ f_{\mu \nu} = e(\theta)_{\mu}^{a} \eta_{ab} e(\theta)_{b}^{\nu}. \]  

The square root \( \sqrt{g^{-1}f} \) can be evaluated in terms of the vierbeins provided that the following symmetry condition is satisfied,

\[ \eta_{ac} e(\theta)_{a}^{c} (e(g)^{-1})_{b}^{\mu} = \eta_{bc} e(\theta)_{c}^{e} (e(g)^{-1})_{a}^{\mu}. \]  \hspace{1cm} (5.1)

If this is the case one obtains, in matrix notation,

\[ e(g)^{-1} e(\theta) e(g)^{-1} = e(g)^{-1} (e(g)^{-1})^T e(\theta)^T e(g)^{-1} = g^{-1} f, \]  \hspace{1cm} (5.2)

and hence,

\[ \sqrt{g^{-1}f} = e(\theta)^{-1} e(\theta). \]  \hspace{1cm} (5.3)

The question whether the symmetry condition (5.1) follows from the vierbein equations or can be implemented in the dynamics by extending the theory is beyond the scope of this paper. For related work, see [25–27].

Here we are interested in exploring the relation between the symmetry condition and our new redefinition of the shift vector. A general vierbein can be parametrized by a Lorentz boost acting on a triangular vierbein. We therefore start with partially gauge fixed vierbeins in triangular form,

\[ e_t(g) = \begin{pmatrix} N & 0 \\ e_a^k \nu_k & e_i^a \end{pmatrix}, \quad e_t(\theta) = \begin{pmatrix} L & 0 \\ \varphi_a^k \lambda_k & \varphi_i^a \end{pmatrix}, \]  \hspace{1cm} (5.4)

whose components translate into the ADM variables (2.5) of the metrics with \( \gamma_{ij} = e^a_i \delta_{ab} e^b_j \) and \( \phi_{ij} = \varphi^a_i \delta_{ab} \varphi^b_j \). In the above Lorentz gauge, the combined matrix \( \hat{S} \equiv e_t(\theta) e_t(g)^{-1} \) reads,

\[ \hat{S} \equiv \begin{pmatrix} \Sigma & 0 \\ \sigma^a & s^a_b \end{pmatrix} = \begin{pmatrix} L & 0 \\ \frac{1}{N} \varphi^a_k \lambda_k - \nu^k & \varphi^a_k (e^{-1})^b_k \end{pmatrix}. \]  \hspace{1cm} (5.5)

Note that for the triangular vierbeins the symmetry condition (5.1) is not satisfied by \( \hat{S} \). However, the general form for \( \hat{S} \) is given by a Lorentz boost acting on (5.5). In order to make the connection to the metric formulation, we thus need to find a Lorentz boost \( \Lambda \) that symmetrizes \( \hat{S} \),

\[ S_s = \eta \Lambda \hat{S}, \quad S_s^T = S_s. \]  \hspace{1cm} (5.6)

The rotational subgroup of the Lorentz transformations is not fixed by the triangular form. To account for the rotations we could write \( \varphi = R \varphi' \), where \( \varphi' \) is the gauge fixed spatial metric. For notational simplicity we refrain from doing so here but keep in mind that \( \varphi \) still contains the rotational degrees of freedom.

We now derive the expression for the Lorentz boost that achieves the symmetrization of \( \hat{S} \). A general boost can be parametrized in terms of a boost velocity vector \( v_a \) and the corresponding boost factor \( \Gamma = (\sqrt{1 - v_a \delta_{ab} v_b})^{-1} \),

\[ \Lambda = \begin{pmatrix} \Gamma & \Gamma v_b \\ \frac{v^a}{\Gamma} & \frac{v^a}{\Gamma} + \frac{v^2}{1 + \Gamma} v_a v_b \end{pmatrix}. \]  \hspace{1cm} (5.7)
Here and in what follows, $v^a$ with upper indices denotes $\delta^{ab} v_b$. For the general form of $S_s$ we now have,

$$ S_s = \eta \Lambda \hat{S} = \begin{pmatrix} -\Gamma \Sigma - \Gamma v^c \sigma_c & -\Gamma v_c s^c_b \\ \Gamma \Sigma v_a + \left( \delta_{ac} + \frac{\Gamma^2}{1+\Gamma} v_a v_c \right) \sigma^c & \left( \delta_{ac} + \frac{\Gamma^2}{1+\Gamma} v_a v_c \right) s^c_b \end{pmatrix}. \quad (5.8) $$

Demanding its symmetry gives the following conditions,

$$ -\Gamma v_b v_c s^c_a = \Gamma \Sigma v_a + \left( \delta_{ac} + \frac{\Gamma^2}{1+\Gamma} v_a v_c \right) \sigma^c, \quad (5.9a) $$

$$ \left( \delta_{ac} + \frac{\Gamma^2}{1+\Gamma} v_a v_c \right) s^c_b = \left( \delta_{bc} + \frac{\Gamma^2}{1+\Gamma} v_b v_c \right) s^c_a. \quad (5.9b) $$

The second condition can be met by fixing the rotations in $\varphi = R \varphi'$. As we argued before, this is always possible thanks to the polar decomposition theorem. We therefore focus on the first condition that we try to satisfy by fixing the velocity vector $v_a$ of the Lorentz boost.

Multiplying the first condition (5.9a) with $v_b$ results in the following matrix equation,

$$ -\Gamma v_b v_c s^c_a = \Gamma \Sigma v_a + v_b \left( \delta_{ac} + \frac{\Gamma^2}{1+\Gamma} v_a v_c \right) \sigma^c. \quad (5.10) $$

Subtracting its transpose from the equation gives,

$$ \Gamma (\delta_{ac} s^c_b - \delta_{bc} s^c_a) = v_b \sigma^c \delta_{cb} - v_a \sigma^c \delta_{ca}, \quad (5.11) $$

which we insert into the second condition (5.9b) to get,

$$ \delta_{ac} s^c_b - \delta_{bc} s^c_a = \frac{\Gamma}{1+\Gamma} (v_a \sigma^c \delta_{cb} - v_b \sigma^c \delta_{ca}). \quad (5.12) $$

Contraction with $v^b$ leads to,

$$ (\delta_{ac} s^c_b - \delta_{bc} s^c_a) v^b = \frac{\Gamma}{1+\Gamma} (v_a \sigma^c \delta_{cb} - v_b \sigma^c \delta_{ca}) v^b $$

$$ = \frac{\Gamma}{1+\Gamma} (\Gamma^{-2} - 1) \sigma^c \delta_{ca} + \frac{\Gamma}{1+\Gamma} v_a \sigma^b v_b, \quad (5.13) $$

which implies,

$$ \left( \delta_{ab} + \frac{\Gamma^2}{1+\Gamma} v_a v_b \right) \sigma^b = (\delta_{ac} s^c_b - \delta_{bc} s^c_a) v^b. \quad (5.14) $$

Plugging this back into the first condition (5.9a), we arrive at,

$$ (s^a_{\ b} + \Sigma \delta^a_b) v^b = -\sigma^a, \quad (5.15) $$

which finally yields the solution for the velocity vector of the Lorentz boost that symmetrizes the matrix $\hat{S} = e(f) e(g)^{-1}$,

$$ v^a = -(s + \Sigma \hat{1})^{-1} a_{\ b} \sigma^b. \quad (5.16) $$

In terms of the components of $e(g)$ and $e(f)$ this expression reads,

$$ v^a = \left( [Ne^{-1} + L \varphi^{-1}]^{-1} \right)^a_{\ i} (v^i - \lambda^i). \quad (5.17) $$
Remarkably, this equation is exactly the same as (3.11) which means that the redefined shift vector of our new ADM variables can be identified with the velocity vector of the Lorentz boost that symmetrizes the matrix $\hat{S}$. Note also that the scalar $x$ in (2.13) is identified with $\Gamma^{-2}$. Moreover, the symmetry condition for the 3x3 matrix, equation (5.9b), is the same as the gauge condition (3.6) that we had to impose on the spatial vielbeins in order to compute the matrix square root in the solution for $D$.

The effective metric $G = a^2g + 2abg\sqrt{g^{-1}f} + b^2f$ in terms of the vierbeins simply becomes,

$$G_{\mu\nu} = a^2e_\nu ^{a}\eta _{ab}e_\mu ^{b} + b^2e_\nu ^{a}\eta _{ab}e_\mu ^{b} + ab\left[e_\nu ^{a}\eta _{abc}e_\mu ^{c} + e_\nu ^{a}\eta _{abc}e_\mu ^{c}\right] = e(G)^a_\mu \eta _{ab}e(G)^b_\nu,$$

(5.18)

where the vierbein for $G$ is (modulo an overall Lorentz transformation),

$$e(G)^a_\mu = ae_\nu ^{a}\mu + b\Lambda ^{ab}_c e_\nu ^{b}\mu ,$$

(5.19)

in agreement with [17].

5.2 Bound on the variables

Since $v_a$ is a Lorentz velocity vector, we must have $v_a\delta ^{ab}_b < 1$ such that $\Gamma$ is finite. The symmetrization conditions are therefore only solvable for the velocity vector if the vierbein variables satisfy the bound,

$$\delta ^{ab}_a\left[(Ne^{-1} + L\varphi ^{-1})^{-1}\right]^a_i (\nu ^i - \lambda ^i)\left[(Ne^{-1} + L\varphi ^{-1})^{-1}\right]_b^j (\nu ^j - \lambda ^j) < 1.$$

(5.20)

It is easy to see that the condition $v_a\delta ^{ab}_b < 1$ translates into $x > 0$ which is the requirement for the existence of $\sqrt{x}$ and hence the square root-matrix $\sqrt{g^{-1}f}$. This means that it is possible to symmetrize the matrix $\hat{S}$ if and only if $\sqrt{g^{-1}f}$ exists, as has already been pointed out in [26] following a different approach.

If the above bound is not satisfied, then bimetric theory does not possess a formulation in terms of vielbeins. Whether the vierbein formulation in this case is free of the Boulware-Deser ghost or not is still an open question.

6 Discussion

We have proven the absence of ghost at the classical level for a recently proposed matter coupling in bimetric theory. The effective metric that can consistently couple to matter is a combination of the bimetric variables $g$ and $f$. Some relevant issues are discussed below.

Our ghost proof is important for the theoretical consistency of the theory. As mentioned in the introduction, [16] reported a perturbative analysis showing that the theory had a ghost at the classical level. It was further argued that this ghost was harmless for the low-energy theory because its mass was found to lie above a certain cut-off scale. However, unlike healthy fields, ghosts can create energy from the vacuum and thus they appear in
interactions with other particles, or in the process of vacuum decay, irrespective of their mass or the available energy \([18, 19]\). A ghost mass above the cut-off scale of the theory would imply that vacuum decay can spontaneously release energy above the cut-off scale and render the effective description invalid. Therefore the classical consistency of a theory requires the absence of ghosts on all energy scales.

Quantum corrections may destabilize the specific structure of the potential and/or the matter coupling and the Boulware-Deser ghost may reappear \([16, 28]\). This is not surprising since already the formulation of a consistent quantum theory for massive spin-1 fields requires the introduction of additional fields and a Higgs mechanism. It is expected that a similar extension of bimetric theory needs to be developed in order to achieve unitarity also at the quantum level. The search for an analogue of the Higgs mechanism for spin-2 fields is still ongoing and, until it is found, bimetric theory should be regarded as valid only at the classical level.

Suppose that one of the parameters \(a\) and \(b\) in the effective metric \(G\) is fixed such that \(b/a = m_f/m_g \equiv \alpha\). Then \(G = a^2\alpha(\alpha^{-1}g + g\sqrt{g^{-1}f} + \alpha f)\) is invariant under the interchange of \(\alpha^{-1}g\) and \(\alpha f\). This interchange symmetry becomes an invariance of the full theory if in addition the \(\beta_n\) parameters in the bimetric interaction potential satisfy \(\alpha^{4-n}\beta_n = \alpha^n\beta_{4-n}\). Models with this specific symmetry have been discussed in \([29]\), where it was shown that they possess solutions without a well-defined massive gravity limit. The presence of the interchange symmetry at the classical level may have implications for the quantum version of the theory because it is expected that the symmetry is preserved at the quantum level provided that it is compatible with the quantization procedure.

The fluctuations of the effective metric that respects the interchange symmetry are massless around proportional backgrounds of bimetric theory in vacuum \([14]\). While the fluctuations of the original metrics around maximally symmetric backgrounds are never mass eigenstates, the parameters \(a\) and \(b\) that appear in the effective metric (4.1) can be tuned to make its fluctuation massless or massive. The coupling of a different effective metric with massless fluctuations to matter has been studied before but turned out to reintroduce the Boulware-Deser ghost \([14]\). Coupling a massless spin-2 field to matter is interesting because it avoids the linear vDVZ discontinuity \([30, 31]\). On the other hand, the phenomenology of the theory could still differ from general relativity because the effective metric does not possess a standard kinetic term of Einstein-Hilbert form. Moreover, the interaction with the massive spin-2 field may alter predictions for observations. It is therefore interesting to study, for instance, cosmological solutions and their perturbations in this new version of bimetric theory including matter.

For most phenomenological applications, it is necessary to find classical solutions to the equations of motion. The new ghost-free matter coupling complicates the derivation of the equations for the metrics \(g\) and \(f\): In order to compute the variation of the matter coupling with respect to one of the metrics, it is necessary to know the variation of the square root matrix \(\sqrt{g^{-1}f}\). In principle, this can be read off from the results in \([32]\) where the second variation of the bimetric potential was computed. Another option is to switch variables from
$g$ and $f$ to, for instance, $g$ and $G$, and compute the equations for the new fields.\footnote{In \cite{14}, the bimetric action has been rewritten in terms of $g$ and $S = \sqrt{g^{-1}f}$. From there, one can obtain the action in terms of $g$ and $G$ by replacing $S = \sqrt{g^{-1}G - 1}$.} In this case, the variation of the matter coupling will be as simple as in general relativity. The variation of the bimetric action becomes more complicated but can still be computed in a straightforward way.

Acknowledgments: We thank C. Deffayet, J. Enander, E. Mörtsell, B. Sundborg and M. von Strauss for helpful discussions.

A The sandwiched polar decomposition

For matrices $A$ and $B$ and an orthogonal transformation $R$, such that $R^\dagger \hat{I} = \hat{I}$, consider the matrix $ARB$.\footnote{The matrix $\hat{I}$ is defined in (3.4) and we use this form of orthogonal rotations consistent with the conventions in this paper.} We determine $R$ such that

$$ARB = (ARB)^\dagger$$ \hspace{1cm} (A.1)

On inverting, using $R^{-1} = \hat{I}^{-1} R^\dagger\hat{I}$ and manipulating the outcome, this can be recast as,

$$\left( A^\dagger B^{-1} \hat{I}^{-1} R \right)^2 = (A^\dagger B^{-1} \hat{I}^{-1})(A^\dagger B^{-1} \hat{I}^{-1})^\dagger \geq 0$$ \hspace{1cm} (A.2)

Hence the square-root of the right-hand-side exists. On taking the square root and solving the right hand side for $R$ one gets the desired result,

$$R = \sqrt{(A^\dagger B^{-1} \hat{I}^{-1})(A^\dagger B^{-1} \hat{I}^{-1})^\dagger (A^\dagger B^{-1} \hat{I}^{-1})^{-1}^\dagger}$$ \hspace{1cm} (A.3)

B Alternative derivation of the effective lapse

Another straightforward way to compute $N_{\text{eff}}$ is directly using (5.19),

$$e(G) = \begin{pmatrix} N & 0 \\ \epsilon^a_j \nu^j & e^a_i \end{pmatrix} + \begin{pmatrix} \Gamma & \Gamma v_b \\ \Gamma v^a & \delta^{ac} \chi_{eb} \end{pmatrix} \begin{pmatrix} L & 0 \\ \varphi^j \lambda^j & \varphi^b_i \end{pmatrix}$$

$$= \begin{pmatrix} N + L \Gamma + \Gamma v_b \varphi^b_i \lambda^j + \delta^{ab} \chi_{bc} \varphi^c_j \lambda^j + \epsilon^a_j \nu^j \delta^{ab} \chi_{bc} \varphi^b_i + \epsilon^a_i \\ L \Gamma v^a + \delta^{ab} \chi_{bc} \varphi^c_j \lambda^j + \epsilon^a_j \nu^j \delta^{ab} \chi_{bc} \varphi^b_i + \epsilon^a_i \end{pmatrix}.$$ \hspace{1cm} (B.1)

The matrix $\chi$ has been defined in (4.5) and again we raise indices on $v_a$ with $\delta^{ab}$. We parametrize the vierbein $e(G)$ in terms of its block structure,

$$e(G) = \begin{pmatrix} e_{00} & e_{01}^\dagger \\ e_{10} & e_{11} \end{pmatrix},$$ \hspace{1cm} (B.2)
where \( e_{00} \) is a scalar, \( e_{01} \) and \( e_{10} \) are vectors and \( e_{11} \) is a spatial matrix. Then we can identify the lapse of the corresponding metric, \( G = e(G)^T n e(G) \), as,

\[
N_{\text{eff}}^2 = (e_{10}' e_{11} - e_{00} e_{01}') (e_{11}' e_{11} - e_{01}' e_{01})^{-1} (e_{11}' e_{10} - e_{00} e_{01}) + e_{00}^2 - e_{10}' e_{10}.
\]  

(B.3)

After invoking the matrix inversion lemma and performing some algebra, this simplifies to,

\[
N_{\text{eff}} = \frac{e_{00} - e_{01}' e_{11} e_{10}}{\sqrt{1 - |(e_{11})^{-1} e_{01}|^2}},
\]

(B.4)

where we use the notation \(|u|\) for the norm of the vector \( u \) with respect to \( \delta_{ab} \). Identifying the components of (B.1) with (B.2) and substituting \( \nu' = \lambda^i + (N e^{-1} + L e^{-1})^{-i} a \delta_{ab} v_b \), we obtain,

\[
e_{00} = N + L \Gamma + \Gamma v^T \varphi \lambda,
\]

(B.5)

\[
e_{01}' = \Gamma v^T \varphi,
\]

(B.6)

\[
e_{10} = L (\tilde{\Gamma}^{-1} \chi + e \varphi^{-1}) \tilde{\Gamma}^{-1} v + N \tilde{\Gamma}^{-1} v + (\tilde{\Gamma}^{-1} \chi + e \varphi^{-1}) \varphi \lambda,
\]

(B.7)

\[
e_{11} = (\tilde{\Gamma}^{-1} \chi + e \varphi^{-1}) \varphi,
\]

(B.8)

where, as before, the matrix \( \tilde{\Gamma}^{-1} \) has components \( \delta_{ab} \). From this we can evaluate the numerator and denominator of \( N_{\text{eff}} \),

\[
e_{00} - e_{01}' e_{11} e_{10} = L \Gamma^{-1} + N \left( 1 - v^T (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right),
\]

(B.9)

\[
\sqrt{1 - |(e_{11})^{-1} e_{01}|^2} = \sqrt{1 - \left| (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right|^2}.
\]

(B.10)

After assembling, we obtain,

\[
N_{\text{eff}} = c_1 N + c_2 L,
\]

(B.11)

\[
c_1 = \frac{1 - v^T (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v}{\sqrt{1 - \left| (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right|^2}},
\]

(B.12)

\[
c_2 = \frac{\Gamma^{-1}}{\sqrt{1 - \left| (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right|^2}}.
\]

(B.13)

where it can be verified that \( \left| (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right|^2 < 1 \). The expression for \( c_1 \) can be further simplified to,

\[
c_1 = \frac{\Gamma^{-1}}{\sqrt{1 - \left| (\tilde{\Gamma} + e \varphi^{-1} \chi e^{-1} \tilde{\Gamma})^{-1} v \right|^2}},
\]

(B.14)

which, when compared to \( c_2 \), reflects the symmetry of the equations with respect to \( e(g) \) and \( e(f) \). Finally, we have verified that these expressions are in agreement with our result for \( N_{\text{eff}} \) in (4.10).
References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173 (1939) 211.
[2] D. G. Boulware and S. Deser, Phys. Lett. B 40 (1972) 227.
[3] D. G. Boulware and S. Deser, Phys. Rev. D 6 (1972) 3368.
[4] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305 (2003) 96 [hep-th/0210184].
[5] P. Creminelli, A. Nicolis, M. Papucci and E. Trincherini, JHEP 0509 (2005) 003 [hep-th/0505147].
[6] C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020 [arXiv:1007.0443 [hep-th]].
[7] C. de Rham, G. Gabadadze and A. J. Tolley, Phys. Rev. Lett. 106 (2011) 231101 [arXiv:1011.1232 [hep-th]].
[8] S. F. Hassan and R. A. Rosen, Phys. Rev. Lett. 108 (2012) 041101 [arXiv:1106.3344 [hep-th]].
[9] S. F. Hassan, R. A. Rosen, JHEP 1107 (2011) 009. [arXiv:1103.6055 [hep-th]].
[10] S. F. Hassan, R. A. Rosen and A. Schmidt-May, JHEP 1202 (2012) 026 [arXiv:1109.3230 [hep-th]].
[11] S. F. Hassan and R. A. Rosen, JHEP 1204 (2012) 123 [arXiv:1111.2070 [hep-th]].
[12] S. F. Hassan, A. Schmidt-May and M. von Strauss, Phys. Lett. B 715 (2012) 335 [arXiv:1203.5283 [hep-th]].
[13] S. F. Hassan and R. A. Rosen, JHEP 1202 (2012) 126 [arXiv:1109.3515 [hep-th]].
[14] S. F. Hassan, A. Schmidt-May and M. von Strauss, JHEP 1305 (2013) 086 [arXiv:1208.1515 [hep-th]].
[15] Y. Yamashita, A. De Felice and T. Tanaka, arXiv:1408.0487 [hep-th].
[16] C. de Rham, L. Heisenberg and R. H. Ribeiro, arXiv:1408.1678 [hep-th].
[17] J. Noller and S. Melville, arXiv:1408.5131 [hep-th].
[18] R. P. Woodard, Lect. Notes Phys. 720 (2007) 403 [astro-ph/0601672].
[19] F. Sbis, arXiv:1406.4550 [hep-th].
[20] K. Hinterbichler and R. A. Rosen, JHEP 1207 (2012) 047 [arXiv:1203.5783 [hep-th]].
[21] Y. Akrami, T. S. Koivisto, D. F. Mota and M. Sandstad, JCAP 1310 (2013) 046 [arXiv:1306.0004 [hep-th]].
[22] R. L. Arnowitt, S. Deser and C. W. Misner, Gen. Rel. Grav. 40 (2008) 1997 [gr-qc/0405109].
[23] D. Comelli, M. Crisostomi, F. Nesti and L. Pilo, Phys. Rev. D 86 (2012) 101502 [arXiv:1204.1027 [hep-th]].
[24] D. Comelli, F. Nesti and L. Pilo, arXiv:1407.4991 [hep-th].
[25] S. F. Hassan, A. Schmidt-May and M. von Strauss, arXiv:1204.5202 [hep-th].
[26] C. Deffayet, J. Mourad and G. Zahariade, JHEP 1303 (2013) 086 [arXiv:1208.4493 [gr-qc]].
[27] M. Baados, C. Deffayet and M. Pino, Phys. Rev. D 88 (2013) 12, 124016 [arXiv:1310.3249 [hep-th]].

[28] C. de Rham, L. Heisenberg and R. H. Ribeiro, Phys. Rev. D 88 (2013) 084058 [arXiv:1307.7169 [hep-th]].

[29] S. F. Hassan, A. Schmidt-May and M. von Strauss, arXiv:1407.2772 [hep-th].

[30] H. van Dam and M. J. G. Veltman, Nucl. Phys. B 22 (1970) 397.

[31] V. I. Zakharov, JETP Lett. 12 (1970) 312 [Pisma Zh. Eksp. Teor. Fiz. 12 (1970) 447].

[32] P. Guarato and R. Durrer, Phys. Rev. D 89 (2014) 084016 [arXiv:1309.2245 [gr-qc]].