SPECTRAL GAP ON RIEMANNIAN PATH SPACE OVER STATIC AND EVOLVING MANIFOLDS

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ABSTRACT. In this article, we continue the discussion of Fang-Wu (2015) to estimate the spectral gap of the Ornstein-Uhlenbeck operator on path space over a Riemannian manifold of pinched Ricci curvature. Along with explicit estimates we study the short-time asymptotics of the spectral gap. The results are then extended to the path space of Riemannian manifolds evolving under a geometric flow. Our paper is strongly motivated by Naber’s recent work (2015) on characterizing bounded Ricci curvature through stochastic analysis on path space.

1. INTRODUCTION

Suppose $(M, g)$ is a $d$-dimensional complete smooth manifold. Let $\nabla$ be the Levi-Civita connection and $\Delta$ the Laplacian on $M$. Given a $C^1$ vector field $Z$ on $M$, we consider the Bakry-Emery curvature $\text{Ric}^Z := \text{Ric} - \nabla Z$ for the so-called Witten Laplacian $L = \frac{1}{2}(\Delta + Z)$ where $\text{Ric}$ is the Ricci curvature tensor with respect to $g$.

It is well known that the spectral gap of $L$ can be estimated in terms of a lower curvature bound $K$, i.e.,

$$\text{Ric}^Z \geq K$$

for some constant $K$, see e.g. [4, 5, 10]. These results reveal the close relationship between spectral gap, convergence to equilibrium and hypercontractivity of the corresponding semigroup. For example, Poincaré inequalities and log-Sobolev inequalities which can be used to characterize the convergence for the semigroup, imply certain lower bound for the spectral gap.

In this article, we extend this circle of ideas to the Riemannian path space over $M$ and revisit the problem of estimating the spectral gap of the Ornstein-Uhlenbeck operator under the following general curvature condition: there exist constants $k_1$ and $k_2$ such that

$$k_1 \leq \text{Ric}^Z \leq k_2.$$
the log-Sobolev inequality [1]; the constant in this inequality has been estimated in [19] in terms of curvature bounds.

Very recently, A. Naber [24] proved that certain log-Sobolev inequalities and $L^p$-inequalities on path space are equivalent to a constant upper bound for the norm of Ricci curvature on the base manifold $M$; R. Haslhofer and A. Naber [17] extended these results to characterize solutions of the Ricci flow, see also [18]. Inspired by this work, S. Fang and B. Wu [16] gave an estimate of the spectral gap under the curvature condition that there exist two constants $k_1$ and $k_2$ with $k_1 + k_2 \geq 0$ such that

$$k_1 \leq \text{Ric}^Z \leq k_2.$$ 

Dealing with the case "$k_1 + k_2 < 0"$ however, the same argument may lead to a loss of information concerning $k_2$. We revisit this topic in this article. Our aim is to remove the restriction $k_1 + k_2 \geq 0$ in the curvature condition and to establish sharper short-time asymptotics for the spectral gap.

Our methods rely strongly on suitable extensions and generalizations of recent estimates on Riemannian path space, due to Naber [24], resp. Haslhofer and Naber [17] [18]. This work is crucial for our arguments, as it allows to characterize bounded Ricci curvature in terms of stochastic analysis on path space.

We start by briefly introducing the context. Let $X_t^x$ be a diffusion process with generator $L$ starting from $X_0^x = x$. We call $X_t^x$ an $L$-diffusion process. We assume that $X_t^x$ is non-explosive. Let $B_t = (B_t^1, ..., B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. It is well known that the $L$-diffusion process $X_t^x$ starting from $x$ solves the equation

$$dX_t^x = u_t^i \circ dB_t + \frac{1}{2} Z(X_t^x) \, dt, \quad X_0^x = x,$$

where $u_t^i$ is the horizontal process of $X_t^x$ taking values in the orthonormal frame bundle $O(M)$ over $M$ such that $\pi(u_0^i) = x$. Furthermore

$$//_{s,t} := u_t^i \circ (u_s^j)^{-1} : T_{X_s^x} \to T_{X_t^x}, \quad s \leq t,$$

denotes parallel transport along the paths $r \mapsto X_r^x$. As usual, orthonormal frames $u \in O(M)$ are identified with isometries $u: \mathbb{R}^d \to T_x M$ where $\pi(u) = x$.

For fixed $T > 0$ define $W^T := C([0,T]; M)$ and let

$$\mathcal{F} C_{0,T}^\infty = \{ W^T \ni \gamma \mapsto f(\gamma_1, \ldots, \gamma_n) : \ n \geq 1, \ 0 < t_1 < \ldots < t_n \leq T, \ f \in C_0^\infty(M^n) \}$$

be the class of smooth cylindrical functions on $W^T$. Let $X_{[0,T]} = \{ X_t : 0 \leq t \leq T \}$ for fixed $T > 0$. Then, for $F \in \mathcal{F} C_{0,T}^\infty$ with $F(\gamma) = f(\gamma_1, \ldots, \gamma_n)$, we define the intrinsic gradient as

$$D_t F(X_t^x) = \sum_{i=1}^n \mathbf{1}_{t < t_i} //_{t_i, t}^{-1} \nabla_i f(X_t^x, \ldots, X_n^x), \quad t \in [0,T],$$

where $\nabla_i$ denotes the gradient with respect to the $i$-th component. The generator $\mathcal{L}$ associated to the Dirichlet form

$$\mathcal{E}(F,F) = \mathbb{E} \left[ \int_0^T |D_t F|^2(X_{[0,T]}) \, dt \right]$$

is called Ornstein-Uhlenbeck operator. Let $\text{gap}(\mathcal{L})$ be the spectral gap of the Ornstein-Uhlenbeck operator $\mathcal{L}$.

In this article, we continue the topic of estimating $\text{gap}(\mathcal{L})$ under general lower and upper bounds of the Ricci curvature. For the sake of conciseness, let us first introduce some notation: for constants
$K_1$ and $K_2$, define

$$C(T, K_1, K_2) = \begin{cases} 
1 + \frac{k_2 T}{2} + \frac{k_2^2 T^2}{8}, & K_1 = 0; \\
(1 + \beta)^2 - \beta \sqrt{(2 + \beta) \left(2 + 2\beta - \beta e^{-k_1 T/2}\right) e^{-k_1 T/4}}, & K_1 > 0; \\
\frac{1}{2} + \frac{1}{2} \left(1 + \beta (1 - e^{-k_1 T/2})\right)^2, & K_1 < 0,
\end{cases} \quad (1.2)$$

where $\beta = k_2 / k_1$.

**Theorem 1.1.** Let $(M, g)$ be a complete manifold. Assume that

$$k_1 \leq \text{Ric}^Z \leq k_2. \quad (1.3)$$

The following estimate holds:

$$\text{gap}(\mathcal{L})^{-1} \leq C(T, k_1, [k_1] \lor [k_2]) \land \left[ C \left( T, k_1, \frac{k_2 - k_1}{2} \right) \times C \left( T, k_1 + k_2, \frac{|k_1 + k_2|}{2} \right) \right]. \quad (1.4)$$

Let us mention that the first bound in inequality $(1.4)$, i.e.,

$$\text{gap}(\mathcal{L})^{-1} \leq C(T, k_1, [k_1] \lor [k_2]),$$

is due to Fang and Wu [16].

**Remark 1.2.** In explicit terms we may expand the upper bound as follows:

$$C(T, k_1, [k_1] \lor [k_2])$$

$$= \begin{cases} 
1 + \frac{k_2 T}{2} + \frac{k_2^2 T^2}{8}, & k_1 = 0; \\
(\gamma + 1)^2 - \gamma \sqrt{(2 + \gamma) \left(2 \gamma + 2 - \gamma e^{-k_1 T/2}\right) e^{-k_1 T/4}}, & k_1 > 0; \\
\frac{1}{2} + \frac{1}{2} \left(1 + \gamma - \gamma e^{-k_1 T/2}\right)^2, & k_1 + k_2 \geq 0 \text{ and } k_1 < 0; \\
\frac{1}{2} \left(1 + e^{-k_1 T}\right), & k_1 + k_2 < 0,
\end{cases}$$

and

$$C \left( T, k_1, \frac{k_2 - k_1}{2} \right) \times C \left( T, k_1 + k_2, \frac{|k_1 + k_2|}{2} \right)$$

$$= \begin{cases} 
\left(1 + \frac{k_2 T}{4} + \frac{k_2^2 T^2}{32}\right) \left(4 - (12 - 3e^{-k_2 T/4})^{1/2} e^{-k_2 T/4}\right), & k_1 = 0; \\
\frac{1}{4} \left\{ (\gamma + 1)^2 - (\gamma - 1) (\gamma + 3)^{1/2} \left(2 \gamma + 2 - (\gamma - 1) e^{-k_1 T/2}\right) \right\} e^{-k_1 T/4}, & k_1 > 0; \\
\frac{1}{2} \left\{ 1 + \frac{1}{4} (\gamma + 1 - (\gamma - 1) e^{-k_1 T/2})^2 \right\} \times \left(4 - (12 - 3e^{-k_1 T/4})^{1/2} e^{-k_1 T/4}\right), & k_1 + k_2 \geq 0 \text{ and } k_1 < 0; \\
\frac{1}{4} \left\{ 1 + \frac{1}{4} (\gamma + 1 - (\gamma - 1) e^{-k_1 T/2})^2 \right\} \left(1 + e^{-k_1 T/2}\right), & k_1 + k_2 < 0,
\end{cases}$$

where $\gamma := k_2 / k_1$.

By means of Theorem 1.1 we are now in position to determine the asymptotic behavior of $\text{gap}(\mathcal{L})$ as $T$ tends to 0.
Theorem 1.3. Assume $k_1 \leq \text{Ric}^Z \leq k_2$. Then, as $T \to 0$, the following asymptotics hold:

(i) for $k_1 \geq 0$,
$$\text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{1}{8} \left( k_2^2 - \frac{(7k_1 + k_2)(k_1 + k_2)}{6(3k_1 + k_2)} \right) T^2 + o(T^2);$$

(ii) for $k_1 + k_2 \geq 0$ and $k_1 < 0$,
$$\text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{1}{8} \left( k_2^2 + \frac{2k_1^2 - k_2^2 - 5k_1k_2}{6} \right) T^2 + o(T^2);$$

(iii) for $k_1 + k_2 < 0$,
$$\text{gap}(\mathcal{L})^{-1} \leq 1 - \frac{k_1}{2} T + \frac{1}{8} \left( k_1^2 + \frac{3k_1^2 + k_2^2}{4} \right) T^2 + o(T^2).$$

Remark 1.4. Note that as $T \to 0$, up to the first order, the two upper bounds in Theorem 1.1 have the same short-time behaviour, however when considered up to second order, our estimates provide sharper asymptotics (see the proof of Theorem 1.3). For instance, from [16, Proposition 3.6] we know that if $k_1 \to 0$, then
$$\text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{1}{8} k_2^2 T^2 + o(T^2).$$
In this case, from Theorem 1.3 we deduce that
$$\text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{5}{48} k_2^2 T^2 + o(T^2)$$
with a smaller coefficient of $T^2$ when compared to estimate (1.5).

In Section 3 below we shall extend these results to the path space of an evolving manifold $(M, g_t)$. Stochastic analysis on evolving manifolds began with an appropriate notion of Brownian motion on $(M, g_t)$ (called $g_t$-Brownian motion), see [2]. Since then there has been a lot of subsequent work, see for instance, [22, 23, 21, 24, 6, 7, 8]. Here, we deal with diffusions $X_t$ generated by $L_t = \frac{1}{2}(\Delta_t + Z_t)$ which are assumed to be non-explosive. The first author [7] developed a Malliavin calculus on the path space of $X_t$ by means of an appropriate derivative formula and an integration by parts formula. Recently, Naber [24] characterized solutions to the Ricci flow in terms of semigroup gradient estimates. Inspired by this work, we consider in Section 3 one-parameter families of Ornstein-Uhlenbeck type operators on the path space and adapt a known method to obtain a family of log-Sobolev inequalities and Poincaré inequalities on the path space of the $L_t$-diffusion under a modified pinched curvature condition. This curvature contains information about the time derivative of the metric as well. In the particular case of the Ricci flow this modified curvature tensor equals to zero.

The rest of the paper is organized as follows. In the next section we establish first a log-Sobolev inequality and a Poincaré inequality on Riemannian path space; these inequalities are the tools to establish our main results of Section 4. As already indicated, Section 3 is then devoted to the extension of the results to evolving manifolds under a geometric flow.

2. PROOFS OF MAIN RESULTS

To prove the main results, we introduce a two-parameter family $\{Q_{r,s}\}_{0 \leq r < s}$ of multiplicative functionals as follows: the $Q_{r,s}$ are a random variable taking values in the linear automorphisms of $T_{X_t}M$ satisfying for fixed $r \geq 0$ the pathwise equation:
$$\frac{dQ_{r,s}}{dr} = -\frac{1}{2} Q_{r,s} \text{Ric}^Z, \quad Q_{r,r} = \text{id}, \quad (2.1)$$
where $\text{Ric}^Z = \lim_{r \to 0} \text{Ric}^Z_{t,r} \circ \lim_{r \to 0} \circ [r, t]$, see [20] and [25, Theorem 4.1.1]. As usual, $\text{Ric}^Z$ operates as linear homomorphism on $T_{X_t}M$ via $\text{Ric}^Z_{t,v} = \text{Ric}^Z(\cdot, v)^2, v \in T_{X_t}M$. 

It is easy to see that if $\text{Ric}^Z \geq K$ for some constant $K$, then for any $0 \leq r \leq t < T$,  
\[
\|Q_{r,t}\| \leq e^{-K(t-r)/2}, \quad \text{a.s.,}
\]
where $\| \cdot \|$ denotes the operator norm. The functionals $Q_{r,t}$ (or the “damped parallel transport” defined as $//_{r,t} o Q_{r,t}$) are well-known ingredients in the stochastic representation of the heat flow on one-forms and for Bismut-type derivative formulas for the diffusion semigroup \{$P_t$\}$_{t \geq 0}$, see [3][14].

On path space a canonical gradient operator is given in terms of $Q_{r,t}$. For any $F \in \mathcal{P}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_1, \ldots, \gamma_n)$, the damped gradient $\hat{D}_t F(X^t_{[0,T]})$ is defined as  
\[
\hat{D}_t F(X^t_{[0,T]}) = \sum_{i=1}^n \mathbf{1}_{\{t < \tau_i\}} Q_{t,\tau_i} ///_{t,\tau_i}^{-1} \nabla_i f(X^\gamma_{\tau_1}, \ldots, X^\gamma_{\tau_n}), \quad t \in [0,T].
\]
By estimating the damped gradient, a log-Sobolev inequality and a Poincaré inequality on path space can be obtained. Let us first introduce the following function: for any constants $K_1, K_2$ and $c$,  
\[
\Lambda^c(t, T, K_1, K_2) := \beta(t) = \int_0^t \frac{K_2}{\beta} e^{-\frac{1}{2}K_1 + c} (t-s) \, ds,
\]
where $\beta(t) = 1 + \frac{K_2}{t} e^{-\frac{1}{2}K_1 - c} (s-t) \, ds$. Define  
\[
S(T, K_1, K_2) = \inf_{s \in \mathbb{R}} \sup_{t \in [0,T]} \Lambda^c(t, T, K_1, K_2).
\]

**Theorem 2.1.** Assume $k_1 \leq \text{Ric}^Z \leq k_2$. Let  
\[
H(T, k_1, k_2) := S(T, k_1, |k_1|, |k_2|) \wedge \left[ S \left( T, k_1, \frac{k_2 - k_1}{2} \right) S \left( T, \frac{k_2 + k_1}{2}, \frac{k_2 + k_1}{2} \right) \right],
\]
Then for $F \in \mathcal{P}C^\infty_{0,T}$, we have  
(i) $\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2H(T, k_1, k_2) \mathbb{E} \int_0^T |D_tF|^2 \, dr$;
(ii) $\mathbb{E}[F - \mathbb{E}[F]]^2 \leq H(T, k_1, k_2) \mathbb{E} \int_0^T |D_tF|^2 \, dr$.

First, let us introduce some functional inequalities on path space under pinched curvature condition, which extend the estimates in [24]. For $F \in \mathcal{P}C^\infty_{0,T}$ with $F(\gamma) = f(\gamma_1, \ldots, \gamma_n)$, we define a modified gradient as  
\[
\hat{D}_t F(X^t_{[0,T]}) = \sum_{i=1}^n \mathbf{1}_{\{t \leq \tau_i\}} e^{-\frac{k_2 - k_1}{4} (t - \tau_i)} ///_{t,\tau_i}^{-1} \nabla_i f(X^\gamma_{\tau_1}, \ldots, X^\gamma_{\tau_n}), \quad t \in [0,T].
\]
In what follows, if there no ambiguity, we write briefly $D_tF$, $\hat{D}_t F$ and $\tilde{D}_t F$ instead of $D_tF(X^t_{[0,T]})$, $\hat{D}_t F(X^t_{[0,T]})$ and $\tilde{D}_t F(X^t_{[0,T]})$.

**Proposition 2.2.** Let $(M, g)$ be a complete Riemannian manifold. Let $k_1, k_2$ be two real constants such that $k_1 \leq k_2$. The following conditions are equivalent:

(i) $k_1 \leq \text{Ric}^Z \leq k_2$;
(ii) for any $F \in \mathcal{P}C^\infty_{0,T}$,

\[
|\nabla_x \mathbb{E}[F(X^t_{[0,T]})]| \leq \mathbb{E}|\hat{D}_0 F| + \frac{k_2 - k_1}{4} \int_0^T e^{-k_1 s/2} \mathbb{E}|\hat{D}_s F| \, ds;
\]

(iii) for any $F \in \mathcal{P}C^\infty_{0,T}$ and constant $c$,  
\[
|\nabla_x \mathbb{E}[F(X^t_{[0,T]})]|^2 \leq \left( 1 + \frac{k_2 - k_1}{4} \int_0^T e^{-\frac{k_2 - k_1}{4} s} \, ds \right) \left( \mathbb{E}|\hat{D}_0 F|^2 + \frac{k_2 - k_1}{4} \int_0^T e^{-(k_2 + c) s} \mathbb{E}|\hat{D}_s F|^2 \, ds \right);
\]
(iv) for any $F \in \mathcal{F}C^0_{0,T}$, constant $c$ and $t_1 < t_2$ in $[0,T]$,
\[
\mathbb{E}\left[\mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \log \mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \right] - \mathbb{E}\left[\mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \log \mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \right] \\
\leq 2 \int_{t_1}^{t_2} \left( 1 + \frac{k_2 - k_1}{4} \right) t e^{-(\frac{1}{4} + c)(s-t)} ds \left( \mathbb{E}[\tilde{D}_1 F]^2 + \frac{k_2 - k_1}{4} \int_t^T e^{-(\frac{1}{4} + c)(s-t)} \mathbb{E}[\tilde{D}_1 F]^2 ds \right) dt;
\]

(v) for any $F \in \mathcal{F}C^0_{0,T}$, constant $c$ and $t_1 < t_2$ in $[0,T]$,
\[
\mathbb{E}\left[\mathbb{E}[F(X_{t_1})|\mathcal{F}_{t_1}]^2 \right] - \mathbb{E}\left[\mathbb{E}[F(X_{t_1})|\mathcal{F}_{t_1}]^2 \right] \\
\leq 2 \int_{t_1}^{t_2} \left( 1 + \frac{k_2 - k_1}{4} \right) t e^{-(\frac{1}{4} + c)(s-t)} ds \left( \mathbb{E}[\tilde{D}_1 F]^2 + \frac{k_2 - k_1}{4} \int_t^T e^{-(\frac{1}{4} + c)(s-t)} \mathbb{E}[\tilde{D}_1 F]^2 ds \right) dt.
\]

Proof. (a) The following inequalities are well known (see [13] and [25] Chapter 4). For convenience of the reader we include them with precise statements.

1) for $F \in \mathcal{F}C^0_{0,T}$, one has
\[
\nabla_x \mathbb{E}[F(X_{t_1})] = \mathbb{E}[\tilde{D}_0 F(X_{t_1})];
\]

2) for $F \in \mathcal{F}C^0_{0,T}$, one has
\[
\mathbb{E}\left[\mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \log \mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \right] - \mathbb{E}\left[\mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \log \mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}] \right] \\
\leq 2 \mathbb{E}\int_{t_1}^{t_2} |\tilde{D}_1 F(X_{t_1})|^2 dt;
\]

3) for $F \in \mathcal{F}C^0_{0,T}$, one has
\[
\mathbb{E}\left[\mathbb{E}[F^2(X_{t_1})|\mathcal{F}_{t_1}]^2 \right] - \mathbb{E}\left[\mathbb{E}[F(X_{t_1})|\mathcal{F}_{t_1}]^2 \right] \leq \mathbb{E}\int_{t_1}^{t_2} |\tilde{D}_1 F(X_{t_1})|^2 dt.
\]

Hence it suffices to estimate $|\tilde{D}_1 F(X_{t_1})|$. For the sake of brevity, let $k = \frac{k_1 + k_2}{2}$ and $\tilde{k} = \frac{k_1 - k_2}{2}$. It is easy to see that
\[
\tilde{D}_1 F = \tilde{D}_1 F + \sum_{i=1}^{N} \mathbf{1}_{\{t_i \leq s\}} \left( e^{k_i(s-t)/2} Q_{t_i} - \text{id} \right) e^{-k_i(s-t)/2}^{-1} \nabla_i F \\
= \tilde{D}_1 F + \int_t^T e^{-k(s-t)/2} \frac{d(e^{k(s-t)/2} Q_{s})}{ds}^{-1} \tilde{D}_1 F ds.
\]
As
\[
\frac{d(e^{k(s-t)/2} Q_{s})}{ds} = -\frac{1}{2} \left( e^{k(s-t)/2} Q_{s} \right) \left( \text{Ric}_{/s}^Z - k \text{id} \right),
\]
we get
\[
|\tilde{D}_1 F| \leq |\tilde{D}_1 F| + \frac{1}{2} \int_t^T \|Q_{s}\| \cdot \|\left( \text{Ric}_{/s}^Z \right)^2 - k \text{id} \| \cdot |\tilde{D}_1 F| ds \\
\leq |\tilde{D}_1 F| + \frac{1}{2} \tilde{k} \int_t^T e^{-\frac{1}{2} \tilde{k}(s-t)} |\tilde{D}_1 F| ds.
\]
It follows that
\[
|\tilde{D}_1 F|^2 \leq e^{2\alpha t} \left( e^{-\alpha t} |\tilde{D}_1 F| + \frac{1}{2} \tilde{k} \int_t^T e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} e^{-\alpha s} |\tilde{D}_1 F| ds \right)^2.
\]
Thus, by Cauchy’s inequality, we obtain

\[
\left| D_t F \right|^2 \leq e^{2c t} \left( 1 + \frac{1}{2} \int_0^T k e^{-\left(\frac{4}{2k_1-c}\right)(s-t)} ds \right) \left( e^{-2c t} |\hat{D}_t F|^2 + \int_t^T \frac{1}{2} k e^{-\left(\frac{4}{2k_1-c}\right)(s-t)} - 2c \left| \hat{D}_s F \right|^2 ds \right).
\]

This allows to complete the proof of (i) implies (ii)–(v).

(b) Conversely, to prove (ii)–(v) \(\Rightarrow\) (i), by a similar argument as in \([24, 26]\), it suffices to prove that (iii) implies (i). Following \([24]\), we first take \(F(X_{[0,T]}^x) = f(X_t^x)\) as test functional. In this case, (iii) reduces to

\[
\left| \nabla P_t f \right|^2(x) \leq \left[ \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) \right] \left| \nabla f \right|^2(x).
\]

By means of the formula from \([25, \text{Theorem 2.2.4}]\):

\[
\text{Ric}^Z(\nabla f, \nabla f)(x) = \lim_{t \to 0} \frac{P_t |\nabla f|^2(x) - |\nabla P_t f|^2(x)}{t} = \text{C}_0^\infty(M),
\]

we obtain the inequality \(\text{Ric}^Z \geq k_1\). Taking however \(F(X_{[0,T]}^x) = f(x) - \frac{1}{2} f(X_t^x)\) as test functional, then (iii) reduces to the inequality:

\[
\left| \nabla f(x) - \frac{1}{2} \nabla P_t f(x) \right|^2 \leq \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) \times \left( \left\langle \nabla f(x) - \frac{1}{2} \nabla P_t f(x) \right\rangle + \frac{k_2 - k_1}{16} \left( \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) e^{-k t/2} P_t |\nabla f|^2(x) \right).
\]

Expanding the last inequality, we arrive at

\[
\left| \nabla P_t f(x) \right|^2 - \left[ \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) \right] P_t |\nabla f|^2(x)
\]

\[
\leq (k_2 - k_1) \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds |\nabla f(x)|^2 + 4 \left\langle \nabla f(x), \nabla P_t f(x) \right\rangle
\]

\[
- 4 \left( 1 + \frac{k_2 - k_1}{4} \int_0^t e^{-\left(\frac{4}{2k_1-c}\right)s} ds \right) e^{-k t/2} \left\langle \nabla f(x), \text{C}_0^\infty(X_{[0,T]}^x) \right\rangle.
\]

Then by \([9, \text{Lemma 2.5}]\) it is straightforward to derive the upper bound \(\text{Ric}^Z \leq k_2\).

\[\square\]

\textbf{Remark 2.3.} In our forthcoming paper \([9]\) we use a direct method which does not need to use the advanced theory on path space, to prove the result that the pinched curvature condition is equivalent to the coupled conditions \((2.3)\) and \((2.4)\) when \(c = (k_1 + k_2)/4\).

\textbf{Proof of Theorem 2.7} The following inequalities are well known (see \([13]\) and \([25, \text{Chapter 4}]\)). For convenience of the reader we include them here, as we have done in the proof of Proposition 2.2

1) for \(F \in F_{[0,T]}^\infty\), one has

\[
\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2 \mathbb{E} \int_0^T \left| \hat{D}_t F(X_{[0,T]}^x) \right|^2 dt;
\]

2) for \(F \in F_{[0,T]}^\infty\), one has

\[
\mathbb{E} \left[ F - \mathbb{E}[F] \right]^2 \leq \mathbb{E} \int_0^T \left| \hat{D}_t F(X_{[0,T]}^x) \right|^2 dt.
\]

Hence, it suffices to estimate \(\mathbb{E} \int_0^T \left| \hat{D}_t F \right|^2 dt\) where \(\hat{D}_t F = \hat{D}_t F(X_{[0,T]}^x)\). By \([24]\), we know that

\[
|\hat{D}_t F| \leq |D_t F| + \frac{|k_1| + |k_2|}{2} \int_t^T e^{-\frac{4}{2k_1-c}(s-t)} |D_s F| ds.
\]
It follows that for any constant \( c \), we have
\[
|\hat{D}_t F|^2 \leq e^{2c t} \left( e^{-ct} |D_t F| + \frac{|k_1| \vee |k_2|}{2} \int_t^T e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} e^{-cs} |D_s F| \, ds \right)^2.
\]
Thus, by Cauchy’s inequality, we obtain
\[
|\hat{D}_t F|^2 \leq \left( 1 + \frac{|k_1| \vee |k_2|}{2} \int_t^T e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} \, ds \right) \left( |D_t F|^2 + \frac{|k_1| \vee |k_2|}{2} \int_t^T e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |D_s F|^2 \, ds \right).
\]

Let
\[
\alpha_1(t) = 1 + \frac{|k_1| \vee |k_2|}{2} \int_t^T e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} \, ds.
\]
Then, integrating both sides of Eq. (2.5) from 0 to \( T \) yields
\[
\int_0^T |\hat{D}_t F|^2 \, dt \leq \int_0^T \alpha_1(t) \left( |D_t F|^2 + \frac{|k_1| \vee |k_2|}{2} \int_t^T e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |D_s F|^2 \, ds \right) \, dt
\]
\[
= \int_0^T \left( \alpha_1(t) + \frac{|k_1| \vee |k_2|}{2} \int_0^t \alpha_1(s) e^{-\left(\frac{1}{2} k_1 + c\right)(t-s)} \, ds \right) |D_t F|^2 \, dt
\]
\[
\leq S(T, k_1, |k_1| \vee |k_2|) \int_0^T |D_t F|^2 \, dt.
\]

We are now going to prove
\[
\mathbb{E} \int_0^T |\hat{D}_t F(X_{[0,T]}))|^2 \, dt \leq S \left(T, k_1, \frac{k_2 - k_1}{2} \right) S \left(T, \frac{k_1 + k_2}{2}, \frac{|k_2 + k_1|}{2} \right) \int_0^T \mathbb{E} |D_t F(X_{[0,T]}))|^2 \, dt.
\]

Our first step is to show that
\[
\mathbb{E} \int_0^T |\hat{D}_t F(X_{[0,T]}))|^2 \, dt \leq S \left(T, k_1, \frac{k_2 - k_1}{2} \right) \int_0^T \mathbb{E} |D_t F(X_{[0,T]}))|^2 \, dt.
\]

Recall the notations introduced above
\[
\hat{D}_t F(X_{[0,T]})) := \sum_{i=1}^n \mathbf{1}_{\{t \leq t_i\}} e^{-\frac{1}{2} k \left( t-t_i \right) / \tau_{X_i}} \nabla f(X_1, \ldots, X_n)
\]
and \( k := \frac{k_1 + k_2}{2}, \tilde{k} := \frac{k_2 - k_1}{2} \). By Proposition 2.2, for any constant \( c \), we have
\[
|\hat{D}_t F|^2 \leq \left( 1 + \frac{1}{2} \int_t^T \tilde{k} e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} \, ds \right) \left( |\hat{D}_t F|^2 + \frac{1}{2} \int_t^T \tilde{k} e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |\hat{D}_s F|^2 \, ds \right).
\]

Integrating both sides from 0 to \( T \) yields
\[
\int_0^T |\hat{D}_t F|^2 \, dt \leq \int_0^T \left( 1 + \frac{1}{2} \int_t^T \tilde{k} e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} \, ds \right) \left( |\hat{D}_t F|^2 + \frac{1}{2} \tilde{k} e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |\hat{D}_s F|^2 \, ds \right) \, dt.
\]

Let \( \alpha_2(t) = 1 + \frac{1}{2} \int_t^T \tilde{k} e^{-\left(\frac{1}{2} k_1 - c\right)(s-t)} \, ds \). Then
\[
\int_0^T |\hat{D}_t F|^2 \, dt \leq \int_0^T \alpha_2(t) \left( |\hat{D}_t F|^2 + \frac{1}{2} \tilde{k} e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |\hat{D}_s F|^2 \, ds \right) \, dt
\]
\[
= \int_0^T \alpha_2(t) |\hat{D}_t F|^2 \, dt + \int_0^T \alpha_2(t) \int_t^T \frac{1}{2} \tilde{k} e^{-\left(\frac{1}{2} k_1 + c\right)(s-t)} |\hat{D}_s F|^2 \, ds \, dt
\]
\[
= \int_0^T \left( \alpha_2(t) + \frac{1}{2} \int_0^t \alpha_2(s) e^{-\left(\frac{1}{2} k_1 + c\right)(t-s)} \, ds \right) |\hat{D}_t F|^2 \, dt
\]
\[
= \int_0^T \mathcal{N}(t, T, k_1, \tilde{k}) |\hat{D}_t F|^2 \, dt.
\]
Therefore, we have
\[ \int_0^T |\hat{D}_t F|^2 \, dt \leq \inf_{c \in \mathbb{R}} \sup_{t \in [0,T]} \Lambda^c(t, T, k_1, \tilde{k}) \int_0^T |\hat{D}_t F|^2 \, dt. \]

Our second step is to prove
\[ \int_0^T \mathbb{E} |\hat{D}_t F|^2 \, dt \leq S(T, k, |k|) \int_0^T \mathbb{E} |D_t F|^2 \, dt. \]

To this end, we first observe that
\[ |\hat{D}_t F| = \left| \sum_{i=1}^N \mathbb{1}_{t \leq t_i} e^{-\frac{1}{2}(t_i-t)} / i t \nabla_i F \right| \leq |D_t F| + \frac{1}{2} |k| \int_t^T e^{-\frac{1}{2}(s-t)} |D_s F| \, ds. \]

Let \( \alpha_3(t) = 1 + \frac{1}{2} |k| \int_t^T e^{-\frac{1}{2}(s-t)} \, ds \) for some constant \( c \). We have
\[
\int_0^T |\hat{D}_t F|^2 \, dt \leq \int_0^T \left( |D_t F| + \frac{1}{2} |k| \int_t^T e^{-\frac{1}{2}(s-t)} |D_s F| \, ds \right)^2 \, dt \\
\leq \int_0^T \left( \alpha_3(t) + \frac{1}{2} |k| \int_0^t \alpha_3(s) e^{-(\frac{1}{2} c)(t-s)} \, ds \right) |D_t F|^2 \, ds.
\]

It is easy to see that
\[ \Lambda^c(t, T, k, |k|) = \alpha_3(t) + \frac{1}{2} |k| \int_0^t \alpha_3(s) e^{-(\frac{1}{2} c)(t-s)} \, ds. \]

Hence, we arrive at
\[ \int_0^T \mathbb{E} |\hat{D}_t F|^2 \, dt \leq S(T, k_1, \tilde{k}) S(T, k, |k|) \int_0^T \mathbb{E} |D_t F|^2 \, dt, \]
which completes the proof of Theorem 2.1.

In the proof of Theorem 2.1 the function \( \Lambda := \Lambda^0 \) will play an important role. More precisely, for constants \( K_1 \) and \( K_2 \), we have
\[
\Lambda(t, T, K_1, K_2) = \begin{cases} 
(1 + \beta)^2 - (\beta + \beta^2) e^{-K_1 t/2} - \frac{2\beta + \beta^2}{2} e^{-K_1 (T-t)/2} + \frac{\beta^2}{2} e^{-K_1 (T+t)/2}, & \text{if } K_1 \neq 0, \\
1 + \frac{K_2 T}{2} + \frac{K_2^2}{8} (2 T t - t^2), & \text{if } K_1 = 0
\end{cases}
\]
where \( \beta = K_2 / K_1 \). We choose here the value \( c = 0 \), since it seems to give the best asymptotics as \( T \to 0 \).

**Proposition 2.4.** Let \( K_1 \) and \( K_2 \) be two constants such that \( K_2 \geq 0 \). Then
\[ C(T, K_1, K_2) = \sup_{t \in [0,T]} \Lambda(t, T, K_1, K_2), \]
where \( C(T, K_1, K_2) \) is defined as in (1.2).

**Proof.** For the case \( K_1 + K_2 \geq 0 \), the reader is referred to [16 Proposition 3.3]. It suffices to deal with the remaining case \( K_1 + K_2 < 0 \). The idea is similar to the proof of [16 Proposition 3.3].

When \( K_1 + K_2 < 0 \) and \( K_2 \geq 0 \), we must have \( K_1 < 0 \). Taking derivative of \( \Lambda \) with respect to \( t \), we obtain
\[
\Lambda'(t, T, K_1, K_2) = \frac{K_1}{4} e^{-K_1 t/2} \left[ 2(\beta + \beta^2) - \beta^2 e^{-K_1 T/2} - (2\beta + \beta^2) e^{-K_1 T/2} e^{K_1 t} \right],
\]
where \( \beta = K_2 / K_1 \). From this it is easy to see that there exists at most one point \( t \) such that \( \Lambda'(t, T, K_1, K_2) = 0 \).
In addition, for the boundary values \( t = 0, T \), we have
\[
\Lambda'(0, T, K_1, K_2) = \frac{1}{2} \beta (K_1 + K_2) \left( 1 - e^{-K_1 T/2} \right) < 0;
\]
\[
\Lambda'(T, T, K_1, K_2) = -\frac{K_2}{2} \left( 1 - e^{-K_1 T/2} \right) - \frac{K_2^2}{4K_1} \left( 1 - e^{-K_1 T/2} \right)^2 > 0.
\]
Thus, we obtain that the maximal value of \( \Lambda \) over the interval \([0, T]\) is reached either at \( t = 0 \) or at \( t = T \). Moreover, by inspection it is easy to see that \( \Lambda(0, T, K_1, K_2) \leq \frac{1}{4} + \frac{1}{2} \Lambda(0, T, K_1, K_2) = \Lambda(T, T, K_1, K_2) \). All this taken together, we may conclude that
\[
\sup_{t \in [0, T]} \Lambda(t, T, K_1, K_2) = \Lambda(T, T, K_1, K_2).
\]

**Proof of Theorem 7.4** From Theorem 2.1 we conclude that
\[
\text{gap} (\mathcal{L})^{-1} \leq H(T, k_1, k_2).
\]
Moreover, it is easy to be observed that
\[
S(T, K_1, K_2) \leq \sup_{t \in [0, T]} \Lambda(t, T, K_1, K_2) = C(T, K_1, K_2),
\]
which allows to complete the proof of Theorem 7.1. □

**Proof of Theorem 7.3** We check the short-time behavior of \( C(T, K_1, K_2) \) for \( K_2 \geq 0 \) first. If \( K_1 > 0 \), then
\[
C(T, K_1, K_2) = (1 + \beta)^2 - \beta \sqrt{(2 + \beta) (2 + 2 \beta - \beta e^{-K_1 T/2}) e^{-K_1 T/4}}
\]
\[
= (1 + \beta)^2 - \beta (2 + \beta) e^{-K_1 T/4} \sqrt{1 + \frac{\beta}{2 + \beta} (1 - e^{-K_1 T/2})}.
\]

Note that
\[
\sqrt{1 + \frac{\beta}{2 + \beta} (1 - e^{-K_1 T/2})} = 1 + \frac{\beta}{2(2 + \beta)} (1 - e^{-K_1 T/2}) - \frac{\beta^2}{8} (1 - e^{-K_1 T/2})^2 - o(T^2)
\]
\[
= 1 + \frac{\beta}{2(2 + \beta)} \left( \frac{1}{2} K_1 T - \frac{1}{8} (K_1 T)^2 \right) - \frac{\beta^2}{32(2 + \beta)^2} (K_1 T)^2 + o(T^2)
\]
\[
= 1 + \frac{\beta}{4(2 + \beta)} K_1 T - \frac{\beta^2 (4 + 3 \beta)}{32(2 + \beta)^2} (K_1 T)^2 + o(T^2).
\]

Thus,
\[
C(T, K_1, K_2) = (1 + \beta)^2 - \beta (2 + \beta) \left( 1 - \frac{1}{4} K_1 T + \frac{1}{32} (K_1 T)^2 + o(T^2) \right)
\times \left( 1 + \frac{\beta}{4(2 + \beta)} K_1 T - \frac{\beta^2 (4 + 3 \beta)}{32(2 + \beta)^2} (K_1 T)^2 + o(T^2) \right)
\]
\[
= 1 + \frac{K_2 T}{2} + \left( 1 - \frac{(K_1 + K_2) K_1}{(2K_1 + K_2) K_2} \right) \frac{K_2^2 T^2}{8} + o(T^2).
\]

If \( K_1 < 0 \), then
\[
C(T, K_1, K_2) = \frac{1}{2} + \frac{1}{2} \left( 1 + \beta (1 - e^{-K_1 T/2}) \right)^2
\]
\[
= \frac{1}{2} + \frac{1}{2} \left( 1 + \beta \frac{K_1 T}{2} - \beta \frac{(K_1 T)^2}{8} + o(T^2) \right)^2
\]
\[
= 1 + \frac{1}{2} K_2 T + \left( 1 - \frac{K_1}{K_2} \right) \frac{(K_2 T)^2}{8} + o(T^2).
\]
Hence, for \( C(T, k_1, |k_1| \vee |k_2|) \), we obtain

\[
C(T, k_1, |k_1| \vee |k_2|) = \begin{cases} 
1 + \frac{k_2}{2} T + \frac{k_2^2}{8} T^2 - \frac{k_1 k_2 (k_1 + k_2)}{8(2k_1 + k_2)} T^2 + o(T^2), & k_1 \geq 0, \\
1 + \frac{k_2}{2} T + \frac{k_2^2}{8} T^2 - \frac{k_1 k_2}{8} T^2 + o(T^2), & k_1 + k_2 \geq 0 \text{ and } k_1 < 0, \\
1 - \frac{k_1}{2} T + \frac{k_2^2}{8} T^2 + \frac{(k_1 T)^2}{8} + o(T^2), & k_1 + k_2 < 0.
\end{cases}
\]

We now turn to estimate \( C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_1 + k_2}{2}, \frac{|k_2 + k_1|}{2}) \).

(i) When \( k_1 + k_2 < 0 \), we have

\[
C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_2 + k_1}{2}, \frac{k_2 + k_1}{2}) = 1 + \frac{k_2}{2} T + \frac{k_2^2}{8} T^2 + \frac{3k_1^2 + k_2^2}{32} T^2 + o(T^2);
\]

(ii) when \( k_1 + k_2 \geq 0 \) and \( k_1 \leq 0 \),

\[
C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_2 + k_1}{2}, \frac{k_2 + k_1}{2}) = 1 + \frac{k_2}{2} T + \frac{k_2^2}{8} T^2 + \frac{2k_1^2 - k_2^2 - 5k_1 k_2}{48} T^2 + o(T^2);
\]

(iii) when \( k_1 > 0 \),

\[
C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_2 + k_1}{2}, \frac{k_2 + k_1}{2}) = 1 - \frac{k_1}{2} T + \frac{k_2^2}{8} T^2 - \frac{(7k_1 k_2 + k_2^2)(k_1 + k_2)}{48(3k_1 + k_2)} T^2 + o(T^2).
\]

Summarizing the estimates above, we conclude that as \( T \to 0 \),

\[
C(T, k_1, |k_1| \vee |k_2|) \text{ and } C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_1 + k_2}{2}, \frac{|k_2 + k_1|}{2})
\]

have the same first order term, i.e. coefficient of \( T \), and we only need to compare the coefficients of \( T^2 \).

(i) If \( k_1 \geq 0 \), then

\[
-\frac{(7k_1 k_2 + k_2^2)(k_1 + k_2)}{48(3k_1 + k_2)} + \frac{k_1 k_2 (k_1 + k_2)}{8(2k_1 + k_2)} = -\frac{(k_2^2 - k_1^2)(4k_1 + k_2)k_2}{48(3k_1 + k_2)(2k_1 + k_2)} \leq 0.
\]

(ii) If \( k_1 + k_2 \geq 0 \) and \( k_1 < 0 \), then

\[
\frac{2k_1^2 - k_2^2 - 5k_1 k_2}{48} + \frac{k_1 k_2}{8} = \frac{(2k_1 - k_2)(k_1 + k_2)}{48} \leq 0.
\]

(iii) If \( k_1 + k_2 < 0 \), then

\[
\frac{3k_1^2 + k_2^2}{32} - \frac{k_2^2}{8} = \frac{k_2^2 - k_1^2}{32} < 0.
\]

From this we conclude that

\[
C(T, k_1, \frac{k_2 - k_1}{2}) C(T, \frac{k_1 + k_2}{2}, \frac{|k_2 + k_1|}{2})
\]

has a smaller coefficient in \( T^2 \). The proof is then completed by using Theorem 1.1.
3. EXTENSION TO THE PATH SPACE OF EVOLVING MANIFOLDS

In this section, our base space is a differentiable manifold carrying a geometric flow of complete Riemannian metrics, more precisely, a $d$-dimensional differential manifold $M$ equipped with a family of complete Riemannian metrics $(g_t)_{t \in [0,T_c)}$ for some $T_c \in (0,\infty]$, which is $C^1$ in $t$.

Let $\nabla$ and $\Delta_t$ be the Levi-Civita connection and the Laplace-Beltrami operator associated with the metric $g_t$, respectively. Let $(Z_t)_{t \in [0,T_c)}$ be a $C^1,\infty$-family of vector fields. Consider the diffusion process $X_t^x$ generated by $L_t = \frac{1}{2}(\Delta_t + Z_t)$ (called $L_t$-diffusion process) starting from $x$ at time 0, which is assumed to be non-explosive before $T_c$ (see [22] for sufficient conditions).

It is well-known (e.g. [2,12]) that by Itô’s formula, for any $\langle \cdot, \cdot \rangle_0$ is the inner product on $T_xM$ given by the initial metric $g_0$. In other words, $X_t^x$ is a diffusion generated by $L_t$.

For the sake of brevity, we introduce the following notation: for $X,Y \in TM$ such that $\pi(X) = \pi(Y)$ let

$$\mathcal{R}_t^X(X,Y) := \text{Ric}_t(X,Y) - \langle \nabla^X X, Y \rangle_t - \partial_t g_t(X,Y)$$

where $\text{Ric}_t$ is the Ricci curvature tensor with respect to the metric $g_t$ and $\langle \cdot, \cdot \rangle_t = g_t(\cdot, \cdot)$. In what follows, given functions $\phi, \psi$ on $[0,T_c) \times M$, we write $\psi \leq \mathcal{R}_t^X \leq \phi$ if

$$|\psi|_t^2 \leq \mathcal{R}_t^X(X,X) \leq |\phi|_t^2$$

holds for all $X \in TM$, where $|X|_t := \sqrt{g_t(X,X)}$.

Similarly to Eq. (2.1) we define a two-parameter family of multiplicative functionals $\{Q_{r,t}\}_{r \leq t}$ as solution to the following equation: for $0 \leq r \leq t < T_c$ let

$$\frac{dQ_{r,t}}{dr} = -\frac{1}{2}Q_{r,t} \mathcal{R}_t^Z(X_t^x) , \quad Q_{r,r} = \text{id}, \quad \mathcal{R}_t^Z(X_t^x)$$

where by definition

$$\mathcal{R}_t^Z(X_t^x) := \mathcal{R}_t^Z(\cdot)_t \circ \mathcal{R}_t(X_t^x) \circ \mathcal{R}_t.$$ 

Let $W(M)$ be the path space of $M$. Fixing $T \in (0,T_c)$ we have the space of smooth cylindrical functions on $W(M)$ defined as

$$\mathcal{FC}_{0,T}^n := \{ F : F = f(X_{t_1}, \ldots, X_{t_n}), \quad 0 < t_1 < \ldots < t_n \leq T, \quad f \in C^n_0(M^n) \}.$$ 

For $F \in \mathcal{FC}_{0,T}^n$ we consider again different types of gradients:

(i) intrinsic gradient:

$$D_t F(X_{0,T}) = \sum_{i=1}^n \mathbf{1}_{\{t \leq t_i\}} \langle |t_h^{-1} \nabla^h f(X_{t_1}, \ldots, X_{t_n}) , t \in [0,T];$$
(ii) damped gradient:
\[ \tilde{D}_t F(X_{[0,T]}) = \sum_{i=1}^{n} I_{[t \leq t_i]} \frac{\partial}{\partial \tilde{t}_i} \nabla_{\tilde{t}_i}^2 f(X_{[t_i, \ldots, t_n]}), \quad t \in [0, T]; \]

(iii) modified gradient:
\[ \check{D}_t F(X_{[0,T]}) = \sum_{i=1}^{n} I_{[t \leq t_i]} \exp^{-\frac{1}{2} \int_{t}^{\tilde{t}_i} (k_1 + k_2)(r) dr} / \int_{t}^{\tilde{t}_i} \nabla_{\tilde{t}_i}^2 f(X_{[t_i, \ldots, t_n]}). \]

We again write briefly \( D_t F, \check{D}_t F \) and \( \tilde{D}_t F \) instead of \( D_t F(X_{[0,T]}), \check{D}_t F(X_{[0,T]}) \) and \( \tilde{D}_t F(X_{[0,T]}) \) if there is no ambiguity. In terms of the intrinsic gradient \( D_0 \), we consider the one-parameter family of Ornstein-Uhlenbeck type operators given as
\[ \mathcal{L} = -D_0^* D_0. \]

Our aim is to give an estimate for the spectral gap of \( \mathcal{L} \), denoted by \( \text{gap}(\mathcal{L}) \). To this end, we use the Poincaré inequality and log-Sobolev inequality of the next theorem. For the precise statement some notation is required. Given three functions \( K_1, K_2 \) and \( c \) in \( C([0, T]) \), we define
\[ \check{\Lambda}^c(t, T, K_1, K_2) = \alpha(t) + \frac{1}{2} K_2(t) \int_{0}^{t} \alpha(s) \exp^{-\frac{1}{2} \int_{s}^{t} (k_1 + 2c)(r) dr} ds \]
where
\[ \alpha(t) = 1 + \int_{t}^{T} K_2(s) \exp^{-\frac{1}{2} \int_{s}^{t} (k_2 - 2c)(r) dr} ds. \]
Furthermore let
\[ \check{S}(T, K_1, K_2) = \inf_{c \in C([0, T])} \sup_{t \in [0, T]} \check{\Lambda}(t, T, K_1, K_2). \]

Note that if \( K_1, K_2, c \) are constants then
\[ \check{\Lambda}^c(t, T, K_1, K_2) = \Lambda^c(t, T, K_1, K_2). \]

Analogously to Theorem 2.1, we recall the following two inequalities.

**Theorem 3.1.** Assume that there exist continuous functions \( k_1, k_2 \) such that for every vector field \( X \),
\[ k_1(t) |X|^2 \leq \check{\mathcal{R}}^2(X, X) \leq k_2(t) |X|^2, \quad t \in [0, T]. \]

Then,
(i) for every cylindrical function \( F \in \mathcal{F} C_{0, T}^\omega \),
\[ \mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2 \check{H}(T, k_1, k_2) \int_{0}^{T} \mathbb{E}[D_t F]^2 ds, \]

where
\[ \check{H}(T, k_1, k_2) = \check{S}(T, k_1, |k_1| \lor |k_2|) \wedge \left[ \check{S} \left( T, k_1, \frac{k_2 - k_1}{2} \right) + \check{S} \left( T, k_1 + k_2, \frac{|k_1 + k_2|}{2} \right) \right]; \]

(ii) for every cylindrical function \( F \in \mathcal{F} C_{0, T}^\omega \),
\[ \mathbb{E} \left[ F - \mathbb{E}[F] \right]^2 \leq \check{H}(T, k_1, k_2) \int_{0}^{T} \mathbb{E}[D_t F]^2 ds. \]

Similarly to Section 2, we need the characterizations of modified pinched curvature condition on path space to prove Theorem 3.1. In the following, we will use the notation:
\[ \mathbb{E}^{(x)}[\cdot] := \mathbb{E}[\cdot | \mathcal{F}_t, X_t = x]. \]

**Proposition 3.2.** Let \( (M, g_t)_{t \in [0,T]} \) be a smooth manifold carrying a family of complete metrics \( g_t \). Let \( k_1, k_2 \) be two continuous functions in \( C([0, T]) \) such that \( k_1 \leq k_2 \). For any \( T \in (0, T_0) \), the following conditions are equivalent:
(i) for any $t \in [0, T]$, 
$$k_1(t) \leq \mathcal{L}_t^2 \leq k_2(t);$$
(ii) for any $F \in \mathcal{F}C_{0,T}^\infty$, 
$$|\nabla_x E^{(x,t)}(F(X_{[0,T]}))|_t \leq E^{(x,t)}(|\tilde{\nabla}F|) + \frac{1}{2} \int_t^T \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} E^{(x,t)}(|\tilde{\nabla}F|) ds$$
where $\tilde{k} = (k_2 - k_1)/2$;
(iii) for any $F \in \mathcal{F}C_{0,T}^\infty$ and any continuous function $c$ on $[0,T]$, 
$$|\nabla_x E^{(x,t)}(F(X_{[0,T]}))|^2_t \leq \left(1 + \frac{1}{2} \int_t^T \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} ds \right)$$
$$\times \left(E^{(x,t)}(|\tilde{\nabla}F|)^2_t + \frac{1}{2} \int_t^T \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} E^{(x,t)}(|\tilde{\nabla}F|)^2 ds \right);$$
(iv) for any $F \in \mathcal{F}C_{0,T}^\infty$, any continuous function $c$ on $[0,T]$, and any $t_1 < t_2$ in $[0,T]$, 
$$\mathbb{E} \left[ E[F^2(X_{[0,T]}), \mathcal{F}_{t_2}] \log E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \right] - \mathbb{E} \left[ E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \log E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \right] \leq 2 \int_{t_1}^{t_2} \left(1 + \frac{1}{2} \int_r^{t_2} \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} ds \right)$$
$$\times \left(E[|\tilde{\nabla}F|)^2_t + \frac{1}{2} \int_t^{t_2} \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} E[|\tilde{\nabla}F|)^2 ds \right) dr;$$
(v) for any $F \in \mathcal{F}C_{0,T}^\infty$, any continuous function $c$ on $[0,T]$, and any $t_1 < t_2$ in $[0,T]$, 
$$\mathbb{E} \left[ E[F(X_{[0,T]}), \mathcal{F}_{t_2}]^2 \right] - \mathbb{E} \left[ E[F(X_{[0,T]}), \mathcal{F}_{t_1}]^2 \right] \leq \int_{t_1}^{t_2} \left(1 + \frac{1}{2} \int_r^{t_2} \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} ds \right)$$
$$\times \left(E[|\tilde{\nabla}F|)^2_t + \frac{1}{2} \int_t^{t_2} \tilde{k}(s) e^{-\frac{1}{2} \int_r^s \frac{k_1(r)}{2} dr} E[|\tilde{\nabla}F|)^2 ds \right) dr.$$

Remark 3.3. When $k_1 = k_2 = 0$, $c = 0$ and $L = \frac{1}{2} \Delta$, the inequalities in Proposition 3.2 are proved in [24] to characterize solutions of the Ricci flow.

**Proof of Proposition 3.2** By [7 Theorem 4.3] we know that for $F \in \mathcal{F}C_{0,T}^\infty$, one has 
$$|\nabla_x E^{(x,t)}(F(X_{[0,T]}))|_t \leq E^{(x,t)}|\tilde{\nabla}F|,$$
and 
$$\mathbb{E} \left[ E[F^2(X_{[0,T]}), \mathcal{F}_{t_2}] \log E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \right] - \mathbb{E} \left[ E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \log E[F^2(X_{[0,T]}), \mathcal{F}_{t_1}] \right] \leq 2 \int_{t_1}^{t_2} E[|\tilde{\nabla}F|)^2_t ds.$$
Analogously, by a similar discussion as in the proof of [7 Theorem 4.3], we have 
$$\mathbb{E} \left[ E[F(X_{[0,T]}), \mathcal{F}_{t_2}]^2 \right] - \mathbb{E} \left[ E[F(X_{[0,T]}), \mathcal{F}_{t_1}]^2 \right] \leq \int_{t_1}^{t_2} E[|\tilde{\nabla}F|)^2_t ds.$$
Hence it suffices again to estimate $|\tilde{\nabla}F|.$
Defining 
$$\bar{k} = \frac{k_1 + k_2}{2} \quad \text{and} \quad \tilde{k} = \frac{k_2 - k_1}{2},$$
recall 
$$\tilde{\nabla}_t F(X_{[0,T]}) = \sum_{i=1}^n 1_{(t \leq t_i)} e^{-\frac{1}{2} \int_{t_i}^{t} \bar{k}(r) dr} \|\tilde{\nabla}_t f(X_{t_1}, \ldots, X_{t_n}).$$
Then, we have
\[
\bar{D}_t F = D_t F + \sum_{i=1}^{n} \mathbf{1}_{[t \leq t_i]} (\bar{Q}_{t, t_i} - \text{id}) e^{-\frac{1}{2} \int_{t_i}^{t} \bar{k}(r) \, dr} \|_{t_i, t}^{-1} \nabla_{t_i} f(X_{t_1}, \ldots, X_{t_n}) \\
= D_t F - \frac{1}{2} \int_{t}^{T} Q_{t, s} \left( \mathcal{R}_{||t, s} - \bar{k}(s) \text{id} \right) \|_{t, s}^{-1} \bar{D}_s F \, ds
\]
(3.3)
where \( \bar{Q}_{t, s} = e^{\frac{1}{2} \int_{t}^{s} \bar{k}(r) \, dr} Q_{t, s} \). Using similar arguments as in the proof of Proposition 2.2, we obtain \((i) \Rightarrow (ii)-(v)\).

Conversely, to prove \((ii)-(v) \Rightarrow (i)\), the essential part is to prove \((iii) \Rightarrow (i)\). The trick is again to use the test functionals \( F(X_{[0, T]} = f(X_t) \) and \( F(X_{[0, T]} = f(X_t - \frac{1}{2} f(X_t)) \). We refer the reader to [24][9] for detailed calculations.

**Proof of Theorem 3.1**

For convenience of the reader, we first recall that for \( F \in \mathcal{F}_{C^2_{0, T}} \), one has
\[
\mathbb{E}[F^2 \log F^2] - \mathbb{E}[F^2] \log \mathbb{E}[F^2] \leq 2 \int_{0}^{T} \mathbb{E} |\bar{D}_s F|_c^2 \, ds
\]
and
\[
\mathbb{E} [F - \mathbb{E}[F]]^2 \leq \int_{0}^{T} \mathbb{E} |\bar{D}_s F|_c^2 \, ds.
\]

Hence it suffices to estimate \( \int_{0}^{T} \mathbb{E} |\bar{D}_s F|_c^2 \, ds \). Under condition (3.2), we obtain the bounds
\[
|\mathcal{R}_1^T (X, X)| \leq (|k_1| \lor |k_2|)(t) |X|^2
\]
and
\[
\mathcal{R}_2^T (X, X) \geq k_1(t)|X|^2
\]
for all \( X \in TM \). Then
\[
\bar{D}_t F = D_t F + \sum_{i=1}^{n} \left( \int_{t_i}^{t} \frac{dQ_{t, s}}{ds} \right) \|_{t_i, t}^{-1} \nabla_{t_i} f(X_{t_1}, \ldots, X_{t_n})
= D_t F - \frac{1}{2} \int_{t}^{T} Q_{t, s} \mathcal{R}_{||t, s} \|_{t, s}^{-1} \bar{D}_s F \, ds
\]
which implies that
\[
|\bar{D}_t F| \leq |D_t F| + \frac{1}{2} \int_{t}^{T} (|k_1| \lor |k_2|)(s) e^{-\frac{1}{2} \int_{t}^{s} k_1(r) \, dr} |D_s F|_c \, ds.
\]

Using a similar argument as in the proof of Theorem 2.1, we arrive at
\[
\int_{0}^{T} |\bar{D}_t F|_c^2 \, dt \leq \int_{0}^{T} \bar{K}^c(t, T, k_1, |k_1| \lor |k_2|) |D_t F|_c^2 \, dt.
\]
(3.4)

On the other hand, by Proposition 3.2, we have
\[
|\bar{D}_t F|_c^2 \leq \left( 1 + \frac{1}{2} \int_{t}^{T} \bar{k}(s) e^{-\frac{1}{2} \int_{t}^{s} (\frac{k_1}{2} + k_2) \, dr} \, ds \right) \mathbb{E} |\bar{D}_t F|_c^2 + \frac{1}{2} \int_{t}^{T} \bar{k}(s) e^{-\frac{1}{2} \int_{t}^{s} (k_1 + k_2) \, dr} \mathbb{E} |\bar{D}_t F|_c^2 \, ds
\]
(3.5)

Moreover, for \( |\bar{D}_t F|_c \), it is easy to see that
\[
|\bar{D}_t F|_c = |D_t F| + \frac{1}{2} \int_{t}^{T} \bar{k}(s) e^{-\frac{1}{2} \int_{t}^{s} k_2 \, dr} |D_s F|_c \, ds.
\]
(3.6)

Combining this with Eq. (3.5), and using similar arguments as in the proof of Theorem 2.1, we obtain
\[
\int_{0}^{T} |\bar{D}_t F|_c^2 \, dt \leq \bar{S} \left( t, T, k_1, \frac{k_2 - k_1}{2} \right) \int_{0}^{T} \bar{K}^c \left( t, T, \frac{k_2 + k_1}{2}, \frac{k_2 + k_1}{2} \right) |D_t F|_c^2 \, dt
\].
From this and by means of Eq. (3.4), the proof is directly completed. □

The following result is a direct consequence of Theorem 3.1.

**Theorem 3.4.** Assume that there exist two continuous functions $k_1$ and $k_2$ such that
\[
    k_1(t) |X|^2_t \leq R^2_t(X, X) \leq k_2(t) |X|^2_t, \quad t \in [0, T]
\]
for any vector field $X$ on $M$. Then
\[
    \text{gap}(\mathcal{L})^{-1} \leq \tilde{H}(T, k_1, k_2).
\]
For the special case that $k_1$ and $k_2$ are constants, the following asymptotics hold as $T \to 0$:

(i) for $k_1 \geq 0$,
\[
    \text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{1}{8} \left( k_2^2 - \frac{(7k_1 + k_2)(k_1 + k_2)}{6(3k_1 + k_2)} \right) T^2 + o(T^2);
\]
(ii) for $k_1 + k_2 \geq 0$ and $k_1 < 0$,
\[
    \text{gap}(\mathcal{L})^{-1} \leq 1 + \frac{k_2}{2} T + \frac{1}{8} \left( k_2^2 + \frac{2k_2^2 - k_2^3 - 5k_1k_2}{6} \right) T^2 + o(T^2);
\]
(iii) for $k_1 + k_2 < 0$,
\[
    \text{gap}(\mathcal{L})^{-1} \leq 1 - \frac{k_1}{2} T + \frac{1}{8} \left( k_1^2 + \frac{3k_1^2 + k_2^2}{4} \right) T^2 + o(T^2).
\]

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