CARLSON-GRIFFITHS’ THEORY VIA BROWNIAN MOTION

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Abstract. Early in 1970s, Carlson-Griffiths made a significant progress in the study of Nevanlinna theory, who devised equi-distribution theory for holomorphic mappings from $\mathbb{C}^m$ into a projective algebraic manifold intersecting divisors. In this paper, we develop Carlson-Griffiths’ theory by generalizing the source manifold $\mathbb{C}^m$ to complete Kähler manifolds.

1. Introduction

Nevanlinna theory, devised by R. Nevanlinna in 1925, is part of the theory of meromorphic functions which generalizes the Picard’s little theorem. This theory was later generalized to parabolic manifolds by Stoll [25, 26]. Early in 1970s, Carlson and Griffiths [7, 14] made a significant progress in the study of Nevanlinna theory, who devised the equi-distribution theory of holomorphic mappings from $\mathbb{C}^m$ into complex projective algebraic manifolds intersecting divisors. Later, Griffiths and King [13, 14] proceeded to generalize the theory to affine algebraic manifolds. More generalizations were done by Sakai [23] in terms of Kodaira dimension, the singular divisor was considered by Shiffman [24]. Now let’s first review Carlson-Griffiths’ work briefly.

Let $V$ be a complex projective algebraic manifold satisfying $\dim \mathbb{C} V \leq m$. In general, we set for two holomorphic line bundles $L_1, L_2$ over $V$ that

\[
\frac{c_1(L_2)}{c_1(L_1)} = \inf \left\{ s \in \mathbb{R} : \omega_2 < s \omega_1 ; \exists \omega_1 \in c_1(L_1), \exists \omega_2 \in c_1(L_2) \right\},
\]

\[
\frac{c_1(L_2)}{c_1(L_1)} = \sup \left\{ s \in \mathbb{R} : \omega_2 > s \omega_1 ; \exists \omega_1 \in c_1(L_1), \exists \omega_2 \in c_1(L_2) \right\}.
\]

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Let $f : \mathbb{C}^m \to V$ be a holomorphic mapping. We use $\delta_f(D)$ to denote the defect of $f$ with respect to $D$, defined by

$$\delta_f(D) = 1 - \limsup_{r \to \infty} \frac{N_f(r, D)}{T_f(r, L)},$$

where $N_f(r, D), T_f(r, L)$ are defined in Remark 3.3. Carlson-Griffiths proved

**Theorem A.** Let $f : \mathbb{C}^m \to V$ be a differentiably non-degenerate holomorphic mapping. Let $L \to V$ be a positive line bundle and let a divisor $D \in |L|$ be of simple normal crossing type. Then

$$\delta_f(D) \leq \left[ \frac{c_1(K^*_V)}{c_1(L)} \right].$$

The purpose of this paper is to generalize Theorem A to complete Kähler manifolds. Our method is to combine Logarithmic Derivative Lemma (LDL) with stochastic technique developed by Carne and Atsuji. So, the first task here is to establish LDL for meromorphic functions on complete Kähler manifolds (see Theorem 1.1 below), which may be of its own interest. Recall that the first probabilistic proof of Nevanlinna’s Second Main Theorem of meromorphic functions on $\mathbb{C}$ is due to Carne [8], who re-formulated Nevanlinna’s functions in terms of Brownian motion. Later, Atsuji [1, 2, 3, 4] obtained a Second Main Theorem of meromorphic functions defined on complete Kähler manifolds. Recently, Dong-He-Ru [10] re-visited this technique and provided a probabilistic proof of Cartan’s Second Main Theorem.

We introduce the main results in this paper, the notations will be provided in the later sections. Note by Remark 3.3 that the definitions of Nevanlinna’s functions in the Kähler manifold case are natural extensions of the classical ones in the $\mathbb{C}^m$ case. Indeed, for technical reasons, all the Kähler manifolds (as domains) considered in this paper are assumed to be open.

Let $M$ be a complete Kähler manifold with $\dim_{\mathbb{C}} M \geq \dim_{\mathbb{C}} V$. Let $L \to V$ be an arbitrary holomorphic line bundle and fix a Hermitian metric $\omega$ on $V$.

We first establish the following Logarithmic Derivative Lemma:

**Theorem 1.1.** Let $\psi$ be a nonconstant meromorphic function on $M$. Then for any $\delta > 0$, there exists a function $C(o, r, \delta) > 0$ independent of $\psi$ and a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure such that

$$m\left(r, \frac{\|\nabla_M \psi\|}{|\psi|}\right) \leq \left(1 + \frac{(1 + \delta)^2}{2}\right) \log T(r, \psi) + \log C(o, r, \delta)$$

holds for $r > 1$ outside $E_\delta$, where $o$ is a fixed reference point in $M$.

The estimate of term $C(o, r, \delta)$ will be provided when $M$ is non-positively curved (see (19)). Let $\text{Ric}_M$ and $\mathcal{R}_M$ be the Ricci curvature tensor and Ricci
curvature form of $M$ respectively. Set

$$\kappa(t) = \frac{1}{2 \dim C - 1} \min_{x \in B_0(t)} R_M(x),$$

where $R_M(x)$ is the pointwise lower bound of Ricci curvature defined by

$$R_M(x) = \inf_{\xi \in TM} \frac{\text{Ric}_M(\xi, \bar{\xi})}{\|\xi\|^2}.$$  

We have the following Second Main Theorem (SMT) for complete Kähler manifolds:

**Theorem 1.2.** Let a divisor $D \in |L|$ be of simple normal crossing type. Let $f : M \to V$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta > 0$, there exists a function $C(o, r, \delta) > 0$ independent of $f$ and a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure such that

$$T_f(r, L) + T_f(r, K_V) + T(r, R_M) \leq \overline{N}_f(r, D) + O(\log T_f(r, \omega) + \log C(o, r, \delta))$$

holds for $r > 1$ outside $E_\delta$.

When $M$ is non-positively curved, by estimating $C(o, r, \delta)$ and $T(r, R_M)$, we obtain

**Theorem 1.3.** Let a divisor $D \in |L|$ be of simple normal crossing type. Let $f : M \to V$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta > 0$

$$T_f(r, L) + T_f(r, K_V) \leq \overline{N}_f(r, D) + O(\log T_f(r, \omega) - \kappa(r)r^2 + \delta \log r)$$

holds for $r > 1$ outside a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure.

We denote by $\Theta_f(D)$ another defect (without counting multiplicities) of $f$ with respect to $D$, defined by

$$\Theta_f(D) = 1 - \limsup_{r \to \infty} \frac{\overline{N}_f(r, D)}{T_f(r, L)}.$$

**Corollary 1.4 (Defect relation).** Assume the same conditions as in Theorem 1.3. If $f$ satisfies the growth condition

$$\liminf_{r \to \infty} \frac{r^2 \kappa(r)}{T_f(r, \omega)} = 0,$$

then

$$\Theta_f(D) \left[ \frac{c_1(L)}{\omega} \right] \leq \left[ \frac{c_1(K^*_V)}{\omega} \right].$$

In particular, if $M = \mathbb{C}^n$, then we have $\kappa(r) \equiv 0$. So, Corollary 1.4 implies Theorem A. More general, we have SMT for singular divisors:
Theorem 1.5. Let $D$ be a hypersurface of $V$. Let $f : M \to V$ be a differentiably non-degenerate meromorphic mapping. Then for any $\delta > 0$

\[
T_f(r,L_D) + T_f(r,K_V) - N_f(r,D) \leq m_f(r,\text{Sing}(D)) + O\left(\log T_f(r,\omega) - \kappa(r)r^2 + \delta \log r\right)
\]

holds for $r > 1$ outside a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure.

2. Preliminaries

We introduce some basics referred to [5, 6, 9, 13, 16, 17, 18, 21].

2.1. Poincaré-Lelong formula.

Let $M$ be a $m$-dimensional complex manifold. A divisor $D$ on $M$ is said to be of normal crossings if $D$ is locally defined by an equation $z_1 \cdots z_k = 0$ for a local holomorphic coordinate system $z_1, \ldots, z_m$. Additionally, if every irreducible component of $D$ is smooth, one says that $D$ is of simple normal crossings. A holomorphic line bundle $L \to M$ is said to be Hermitian if $L$ is equipped with a Hermitian metric $h = (\{h_\alpha\}, \{U_\alpha\})$, where $h_\alpha : U_\alpha \to \mathbb{R}^+$ are positive smooth functions such that $h_\beta = |g_{\alpha\beta}|^2 h_\alpha$ on $U_\alpha \cap U_\beta$, and $\{g_{\alpha\beta}\}$ is a transition function system of $L$. Let $\{e_\alpha\}$ be a local holomorphic frame of $L$, we have $\|e_\alpha\|^2_h = h_\alpha$. A Hermitian metric $h$ of $L$ defines a global, closed and smooth $(1,1)$-form $-dd^c \log h$ on $M$, where

\[
d = \partial + \bar{\partial}, \quad d^c = \frac{\sqrt{-1}}{4\pi} (\bar{\partial} - \partial), \quad dd^c = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial}.
\]

We call $-dd^c \log h$ the Chern form denoted by $c_1(L, h)$ associated with metric $h$, which determines a Chern class $c_1(L) \in H^2_{DR}(M, \mathbb{R})$, $c_1(L, h)$ is also called the curvature form of $L$. If $c_1(L) > 0$, namely, there exists a Hermitian metric $h$ such that $-dd^c \log h > 0$, then we say that $L$ is positive, written as $L > 0$.

Let $TM$ denote the holomorphic tangent bundle of $M$. The canonical line bundle of $M$ is defined by

\[
K_M = \bigwedge^m T^*M
\]

with transition functions $g_{\alpha\beta} = \det(\partial z_j^\beta / \partial z_i^\alpha)$ on $U_\alpha \cap U_\beta$. Given a Hermitian metric $h$ on $K_M$, it well defines a global, positive and smooth $(m,m)$-form

\[
\Omega = \frac{1}{i\pi} \bigwedge_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j
\]

on $M$, which is therefore a volume form of $M$. The Ricci form of $\Omega$ is defined by $\text{Ric}\Omega = dd^c \log h$. Clearly, $c_1(K_M, h) = -\text{Ric}\Omega$. Conversely, if let $\Omega$ be a
volume form on $M$ which is compact, there exists a unique Hermitian metric $h$ on $K_M$ such that $dd^c \log h = \text{Ric} \Omega$.

Let $H^0(M, L)$ denote the vector space of holomorphic global sections of $L$ over $M$. For any $s \in H^0(M, L)$, the divisor $D_s$ is well defined by $D_s \cap U_\alpha = (s)|_{U_\alpha}$. Denote by $|L|$ the complete linear system of all effective divisors $D_s$ with $s \in H^0(M, L)$. Let $D$ be a divisor on $M$, then $D$ defines a holomorphic line bundle $L_D$ over $M$ in such manner: let $\{g_\alpha\}, \{U_\alpha\}$ be the local defining function system of $D$, then the transition system is given by $g_{\alpha \beta} = g_\alpha / g_\beta$.

Note that $\{g_\alpha\}$ defines a global meromorphic section on $M$ written as $s_D$ of $L_D$ over $M$, called the canonical section associated with $D$.

We introduce the famous Poincaré-Lelong formula:

**Lemma 2.1** (Poincaré-Lelong formula, [7]). Let $L \to M$ be a holomorphic line bundle equipped with a Hermitian metric $h$, and let $s$ be a holomorphic section of $L$ over $M$ with zero divisor $D_s$. Then $\log \|s\|_h$ is locally integrable on $M$ and it defines a current satisfying the current equation

$$dd^c \log \|s\|_h^2 = D_s - c_1(L, h).$$

**2.2. Brownian motions.**

Let $(M, g)$ be a Riemannian manifold with the Laplace-Beltrami operator $\Delta_M$ associated with metric $g$. A Brownian motion $X_t$ in $M$ is a heat diffusion process generated by $\Delta_M/2$ with transition density function $p(t, x, y)$ being the minimal positive fundamental solution of heat equation

$$\frac{\partial}{\partial t} u(t, x) - \frac{1}{2} \Delta_M u(t, x) = 0.$$ 

In particular, when $M = \mathbb{R}^m$

$$p(t, x, y) = \frac{1}{(2\pi t)^{m/2}} e^{-\|x-y\|^2/2t}.$$ 

Let $X_t$ be the Brownian motion in $M$ with generator $\Delta_M/2$. We denote by $\mathbb{P}_x$ the law of $X_t$ starting from $x \in M$, and denote by $\mathbb{E}_x$ the expectation with respect to $\mathbb{P}_x$.

**A. Co-area formula**

Let $D$ be a bounded domain with the smooth boundary $\partial D$ in $M$. Denote by $d\pi_x^{\partial D}(y)$ the harmonic measure on $\partial D$ with respect to $x$, and by $g_D(x, y)$ the Green function of $\Delta_M/2$ for $D$ with Dirichlet boundary condition and a pole at $x$, i.e.,

$$-\frac{1}{2} \Delta_M g_D(x, y) = \delta_x(y), \quad y \in D; \quad g_D(x, y) = 0, \quad y \in \partial D.$$
For each \( \phi \in \mathcal{C}_b(D) \) (space of bounded and continuous functions on \( D \)), the co-area formula \(^5\) says that

\[
E_x \left[ \int_0^{\tau_D} \phi(X_t) dt \right] = \int_D g_D(x,y) \phi(y) dV(y).
\]

From Proposition 2.8 in \(^5\), we note the relation of harmonic measures and hitting times as follows

\[
E_x [\psi(X_{\tau_D})] = \int_{\partial D} \psi(y) d\pi_{\partial D}(y)
\]

for \( \psi \in \mathcal{C}(\overline{D}) \). Since "\( E_x \)”, the co-area formula and \(^3\) still work when \( \phi, \psi \) are of a pluripolar set of singularities.

**B. Itô formula**

The following identity is called the Itô formula (see \([1, 17, 18]\))

\[
u(X_t) - u(x) = B \left( \int_0^t \| \nabla_M u \|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad \mathbb{P}_x \text{ - a.s.}
\]

for \( u \in \mathcal{C}^2(M) \) (space of bounded \( \mathcal{C}^2 \)-class functions on \( M \)), where \( B_t \) is the standard Brownian motion in \( \mathbb{R} \), and \( \nabla_M \) is the gradient operator on \( M \). It follows the Dynkin formula

\[
E_x[u(X_T)] - u(x) = \frac{1}{2} E_x \left[ \int_0^T \Delta_M u(X_t) dt \right]
\]

for a stopping time \( T \) such that each term in the above formula makes sense. Note that Dynkin formula still holds for \( u \in \mathcal{C}^2(M) \) if \( T = \tau_D \). In further, it also works when \( u \) is of a pluripolar set of singularities, particularly for a plurisubharmonic function \( u \).

**2.3. Curvatures.**

Let \((M, g)\) be a Kähler manifold of complex dimension \( m \). We can express the Ricci curvature of \( M \) as

\[
\text{Ric}_M = \sum_{i,j} R_{ij} dz_i \otimes d\bar{z}_j,
\]

where

\[
R_{ij} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{st}).
\]

A well-known theorem by S. S. Chern asserts that the Ricci curvature form

\[
\mathcal{R}_M := -dd^c \log \det(g_{st}) = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^m R_{ij} dz_i \wedge d\bar{z}_j
\]

is a real and closed \((1,1)\)-form which represents a cohomology class of the de Rham cohomology group \( H^2_{\text{DR}}(M, \mathbb{R}) \). This cohomology class depends only
on the complex structure of \( M \), is called the first Chern class of \( M \). Let \( s_M \) denote the Ricci scalar curvature of \( M \) defined by

\[
s_M = \sum_{i,j=1}^{m} g^{i\bar{j}} R_{i\bar{j}},
\]
where \((g^{i\bar{j}})\) is the inverse of \((g_{i\bar{j}})\). Invoking (11), we obtain

\[
s_M = -\frac{1}{4} \Delta_M \log \det(g_{st}).
\]

**Lemma 2.2.** Let \( R_M \) be the pointwise lower bound of Ricci curvature of \( M \). Then we have

\[
s_M \geq m R_M.
\]

**Proof.** Fix any point \( x \in M \), we take a local holomorphic coordinate system \( z \) around \( x \) such that \( g_{i\bar{j}}(x) = \delta^i_j \). We get

\[
s_M(x) = \sum_{j=1}^{m} R_{j\bar{j}}(x) \geq \sum_{j=1}^{m} \text{Ric}_M \left( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j} \right)_x \geq m R_M(x)
\]
which proves the lemma. \( \square \)

### 3. First Main Theorem

We generalize the notions of Nevanlinna’s functions to the general Kähler manifolds and show a First Main Theorem of meromorphic mapping defined on Kähler manifolds. Let \((M, g)\) be a Kähler manifold of complex dimension \( m \), the associated Kähler form is defined by

\[
\alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{m} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.
\]

Fix \( o \in M \) as a reference point. Denote by \( B_o(r) \) the geodesic ball centered at \( o \) with radius \( r \), and by \( S_o(r) \) the geodesic sphere centered at \( o \) with radius \( r \). By Sard’s theorem, \( S_o(r) \) is a submanifold of \( M \) for almost all \( r > 0 \). Also, one denotes by \( g_{r}(o, x) \) the Green function of \( \Delta_M/2 \) for \( B_o(r) \) with Dirichlet boundary condition and a pole at \( o \), and by \( d\pi^r_o(x) \) the harmonic measure on \( S_o(r) \) with respect to \( o \).

#### 3.1. Nevanlinna’s functions.

Let

\[
f : M \to N
\]
be a meromorphic mapping to a compact complex manifold \( N \), which means that \( f \) is defined by such a holomorphic mapping \( f_0 : M \setminus I \to N \), where \( I \) is some analytic subset of \( M \) with \( \dim_{\mathbb{C}} I \leq m - 2 \), called the indeterminacy
set of $f$ such that the closure $\overline{G(f_0)}$ of the graph of $f_0$ is an analytic subset of $M \times N$ and the natural projection $\overline{G(f_0)} \to M$ is proper.

Let a $(1,1)$-form $\eta$ on $M$, we use the following convenient notation

$$e_\eta(x) = 2m \eta \wedge \alpha^{m-1} / \alpha^m.$$

For an arbitrary $(1,1)$-form $\omega$ on $N$, we define the characteristic function of $f$ with respect to $\omega$ by

$$T_f(r, \omega) = \frac{1}{\pi^m} \int_{B_o(r)} g_r(o, x) e^f_\omega(x) \, dV(x)$$

where $dV$ is the Riemannian volume measure on $M$. Let a holomorphic line bundle $L \to N$. Equip $L$ with a Hermitian metric $h$. Since $N$ is compact, we well define

$$T_f(r, L) := T_f(r, c_1(L, h))$$

up to a bounded term.

In what follows, we define the proximity function and counting function.

**Lemma 3.1.** $\Delta_M \log(h \circ f)$ is well defined on $M \setminus I$ and

$$\Delta_M \log(h \circ f) = -4m f^* c_1(L, h) \wedge \alpha^{m-1}. $$

Hence, we have

$$e_f^* c_1(L, h) = -\frac{1}{2} \Delta_M \log(h \circ f).$$

**Proof.** Let $(\{U_\alpha\}, \{e_\alpha\})$ be a local trivialization covering of $(L, h)$ with transition function system $\{g_{\alpha\beta}\}$. On $U_\alpha \cap U_\beta$,

$$e_\beta = g_{\alpha\beta} e_\alpha, \ h_\alpha = h|_{U_\alpha} = \|e_\alpha\|^2, \ h_\beta = h|_{U_\beta} = \|e_\beta\|^2.$$

Thus, we get

$$\Delta_M \log(h_\beta \circ f) = \Delta_M \log(h_\alpha \circ f) + \Delta_M \log |g_{\alpha\beta} \circ f|^2$$

on $f^{-1}(U_\alpha \cap U_\beta) \setminus I$. Notice that $g_{\alpha\beta}$ is holomorphic and nowhere vanishing on $U_\alpha \cap U_\beta$, we see that $\log |g_{\alpha\beta} \circ f|^2$ is harmonic on $f^{-1}(U_\alpha \cap U_\beta) \setminus I$. So, $\Delta_M \log(|g_{\alpha\beta} \circ f|) = \Delta_M \log(h_\alpha \circ f)$ on $f^{-1}(U_\alpha \cap U_\beta) \setminus I$. Thus $\Delta_M \log(h \circ f)$ is well defined on $M \setminus I$. Fix $x \in M$, we choose a normal holomorphic coordinate system $z$ near $x$ in the sense that $g_{\bar{z}j}(x) = \delta_j^i$ and all the first-order derivative of $g_{\bar{z}j}$ vanish at $x$. Then at $x$, we have

$$\Delta_M = 4 \sum_j \frac{\partial^2}{\partial z_j \partial \bar{z}_j}, \ \alpha^m = m! \prod_{j=1}^{m} \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j.$$
and
\[ f^*c_1(L, h) \wedge \alpha^{m-1} = -(m-1)! \text{tr} \left( \frac{\partial^2 \log(h \circ f)}{\partial z_i \partial \bar{z}_j} \right) \wedge \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j, \]
where “tr” stands for the trace of a square matrix. Indeed, by (5)
\[ \Delta_M \log(h \circ f) = 4 \text{tr} \left( \frac{\partial^2 \log(h \circ f)}{\partial z_i \partial \bar{z}_j} \right) \]
at \(x\). This proves the lemma. \(\square\)

Let \(s \in H^0(N, L)\) which is not equal to 0. Locally, we write \(s = \tilde{s}e\), where \(e\) is a local holomorphic frame of \(L\). Then
\[ \Delta_M \log \|s \circ f\|^2 = \Delta_M \log(h \circ f) + \Delta_M \log |\tilde{s} \circ f|^2. \]
By the similar arguments as in the proof of Lemma 3.1, we get
\[ \Delta_M \log |\tilde{s} \circ f|^2 = 4m \frac{dd^c \log |\tilde{s} \circ f|^2}{\alpha^m}. \]

**Lemma 3.2.** Let \(s \in H^0(N, L)\) with \((s) = D\). If \((L, h) \geq 0\), then

(i) \(\log \|s \circ f\|^2\) is locally the difference of two plurisubharmonic functions, and hence \(\log \|s \circ f\|^2 \in \mathcal{L}_{\text{loc}}(M)\).

(ii) \(dd^c \log \|s \circ f\|^2 = f^*D - f^*c_1(L, h)\) in the sense of currents.

**Proof.** Locally, we can write \(s = \tilde{s}e\), where \(e\) is a local holomorphic frame of \(L\) with \(h = \|e\|^2\). Then
\[ \log \|s \circ f\|^2 = \log |\tilde{s} \circ f|^2 + \log(h \circ f). \]
Since \(c_1(L, h) \geq 0\), one obtains \(-dd^c \log(h \circ f) \geq 0\). Indeed, \(\tilde{s}\) is holomorphic, hence \(dd^c \log |\tilde{s} \circ f|^2 \geq 0\). This follows (i). Poincaré-Lelong formula implies that \(dd^c \log |\tilde{s} \circ f|^2 = f^*D\) in the sense of currents, hence (ii) holds. \(\square\)

Assume that \(L \geq 0\). The **proximity function** of \(f\) with respect to \(D \in |L|\) is defined by
\[ m_f(r, D) = \int_{S_0(r)} \log \frac{1}{\|s_D \circ f(x)\|} d\pi^r(x). \]
Write
\[ \log \|s_D \circ f\|^{-2} = \log(h \circ f)^{-1} - \log |\tilde{s}_D \circ f|^2 \]
as the difference of two plurisubharmonic functions. Then it defines a Riesz charge \(d\mu = d\mu_1 - d\mu_2\), here \(d\mu_2\) gives a Riesz measure for \(f^*D\). The counting
function of $f$ with respect to $D$ is defined by

$$N_f(r, D) = \frac{1}{4} \int_{B_o(r)} g_r(o, x) \Delta_M \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

$$= \pi^m \int_{B_o(r)} g_r(o, x) \Delta_M \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

Similarly, one can define $N(r, \text{Supp}\, f^* D)$. We write $\overline{N}_f(r, D) = N(r, \text{Supp}\, f^* D)$ in short.

3.2. Probabilistic expressions of Nevanlinna’s functions.

We reformulate Nevanlinna’s functions in terms of Brownian motion $X_t$. Let $I$ be the indeterminacy set of $f$. Set the stopping time

$$\tau_r = \inf \{ t > 0 : X_t \notin B_o(r) \}.$$

Put $\omega := -dd^c \log h$. By co-area formula, we have

$$T_f(r, L) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} e_{f^* \omega}(X_t) dt \right].$$

The relation between harmonic measures and hitting times gives that

$$m_f(r, D) = \mathbb{E}_o \left[ \log \frac{1}{\| s_D \circ f(X_{\tau_r}) \|} \right].$$

To counting function $N_f(r, D)$, we use an alternative probabilistic expression (see [1, 4, 8]) as follows

$$(6) \quad N_f(r, D) = \lim_{\lambda \to \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq t \leq \tau_r} \log \frac{1}{\| s_D \circ f(X_t) \|} > \lambda \right).$$

Following the arguments in [12] related to the local martingales, we see that the above limit exists. By Dynkin formula and co-area formula, it equals

$$\lim_{\lambda \to \infty} \lambda \mathbb{P}_o \left( \sup_{0 \leq t \leq \tau_r} \log \frac{1}{\| s_D \circ f(X_t) \|} > \lambda \right)$$

$$= \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_M \log \frac{1}{\| s_D \circ f(X_t) \|} \, dt \right]$$

$$= \frac{1}{4} \int_{B_o(r)} g_r(o, x) \Delta_M \log |\tilde{s}_D \circ f(x)|^2 dV(x)$$

$$= N_f(r, D).$$
Remark 3.3. The definitions of Nevanlinna’s functions in above are natural extensions of the classical ones. To see that, we recall the $\mathbb{C}^m$-case:

\[
T_f(r, L) = \int_0^r \frac{dt}{t^{2m-1}} \int_{B_0(t)} f^* c_1(L, h) \wedge \alpha^{m-1},
\]

\[
m_f(r, D) = \int_{S_o(r)} \log \frac{1}{\|s_D \circ f\|} \gamma,
\]

\[
N_f(r, D) = \int_0^r \frac{dt}{t^{2m-1}} \int_{B_0(t)} dd^c \log |s_D \circ f|^2 \wedge \alpha^{m-1},
\]

where $o = (0, \cdots, 0)$ and

\[
\alpha = dd^c \|z\|^2, \quad \gamma = dd^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1}.
\]

Notice the facts that

\[
\gamma = d\pi_r(z), \quad g_r(o, z) = \begin{cases} \frac{\|z\|^{2-2m-r^2-2m}}{(m-1)\omega_{2m-1}}, & m \geq 2; \\ \frac{1}{\omega} \log \frac{1}{|z|}, & m = 1. \end{cases}
\]

where $\omega_{2m-1}$ is the volume of unit sphere in $\mathbb{R}^{2m}$. Apply integration by part, we see it coincides with ours.

3.3. First Main Theorem.

Let $N$ be a complex projective algebraic manifold. There is a very ample holomorphic line bundle $L' \to V$. Equip $L'$ with a Hermitian metric $h'$ such that $\omega := -dd^c \log h' > 0$. For an arbitrary holomorphic line bundle $L \to N$ equipped with a Hermitian metric $h$, whose Chern form says $\omega := -dd^c \log h$, we can pick $k \in \mathbb{N}$ large enough so that $\omega + k\omega' > 0$. Take the natural product Hermitian metric $\| \cdot \|$ on $L \otimes L'^k$, then the Chern form is $\omega + k\omega'$. Choose $\sigma \in H^0(M, L')$ such that $f(M) \not\subset \text{Supp}(\sigma)$. Due to $\omega + k\omega' > 0$ and $\omega' > 0$, we see that $\log \|s_D \otimes \sigma^k \circ f\|^2$ and $\log \|\sigma \circ f\|^2$ are locally the difference of two plurisubharmonic functions, where $D \in |L|$. Thus,

\[
\log \|s_D \circ f\|^2 = \log \|(s_D \otimes \sigma^k) \circ f\|^2 - k \log \|\sigma \circ f\|^2
\]

is locally the difference of two plurisubharmonic functions. Namely, $m_f(r, D)$ can be defined.

We have the First Main Theorem (FMT):

Theorem 3.4 (FMT). We have

\[
T_f(r, L) = m_f(r, D) + N_f(r, D) + O(1).
\]

Proof. Set

\[
T_{\lambda,r} = \inf \left\{ t > 0 : \sup_{s \in [0, t]} \log \frac{1}{\|s_D \circ f(X_s)\|} > \lambda \right\},
\]
where \( T_{I,r} = \{0 \leq t \leq \tau_r : X_t \in I\} \) and \( I \) is the indeterminacy set of \( f \). Due to the definition of \( T_{\lambda,r} \), \( X_t \) does not pass through \( f^*D \) as well as those points in \( I \) near which \( \log \|s_D \circ f(X_t)\|^{-1} \) is unbounded when \( 0 \leq t \leq \tau_r \land T_{\lambda,r} \). By Dynkin formula, it follows that

\[
\mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{\tau_r \land T_{\lambda,r}})\|} \right] = \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r \land T_{\lambda,r}} \Delta_M \log \frac{1}{\|s_D \circ f(X_t)\|} dt \right] + \log \frac{1}{\|s_D \circ f(o)\|},
\]

where \( \tau_r \land T_{\lambda,r} = \min\{\tau_r, T_{\lambda,r}\} \). Note that \( \Delta_M \log |\tilde{s}_D \circ f| = 0 \) outside \( I \cup f^*D \), we see that

\[
\Delta_M \log \frac{1}{\|s_D \circ f(X_t)\|} = -\frac{1}{2} \Delta_M \log h \circ f(X_t)
\]

for \( t \in [0, T_{\lambda,r}] \). Thus, (7) turns to

\[
\mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{\tau_r \land T_{\lambda,r}})\|} \right] = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r \land T_{\lambda,r}} \Delta_M \log h \circ f(X_t) dt \right] + O(1).
\]

The monotone convergence theorem leads to

\[
\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r \land T_{\lambda,r}} \Delta_M \log h \circ f(X_t) dt \right] \to \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r^*} e_{f^*\omega}(X_t) dt \right] = T_f(r,L)
\]

as \( \lambda \to \infty \), due to \( T_{\lambda,r} \to \infty \) a.s. as \( \lambda \to \infty \). We handle the first term in (7), write it as two parts:

\[
I + II = \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{\tau_r})\|} : \tau_r < T_{\lambda,r} \right] + \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{T_{\lambda,r}})\|} : T_{\lambda,r} \leq \tau_r \right].
\]

By the monotone convergence theorem again,

\[
I \to \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{\tau_r})\|} \right] = m_f(r,D)
\]

as \( \lambda \to \infty \). Finally, we deal with II. By the definition of \( T_{\lambda,r} \), we see that

\[
II = \lambda \mathbb{P}_o \left( \sup_{t \in [0,\tau_r] \setminus T_{I,r}} \log \frac{1}{\|s_D \circ f(X_t)\|} > \lambda \right) \to N_f(r,D)
\]

as \( \lambda \to \infty \). Put the above together, we show the theorem. \( \square \)

**Corollary 3.5 (Nevanlinna inequality).** We have

\[
N_f(r,D) \leq T_f(r,L) + O(1).
\]
4. Logarithmic Derivative Lemma

The goals of this section are to prove the Logarithmic Derivative Lemma for Kähler manifolds (i.e., Theorem 1.1) and provide an estimate of $C(o, r, \delta)$ when the Kähler manifolds are non-positively curved. The Logarithmic Derivative Lemma plays an useful role in derivation of the Second Main Theorem in Section 5.

4.1. Logarithmic Derivative Lemma.

Let $(M, g)$ be a $m$-dimensional complete Kähler manifold, and $\nabla_M$ be the gradient operator on $M$ associated with $g$. Let $X_t$ be the Brownian motion in $M$ with generator $\Delta_M/2$.

We first prepare some lemmas:

**Lemma 4.1** (Calculus Lemma, [1]). Let $k \geq 0$ be a locally integrable function on $M$ such that it is locally bounded at $o \in M$. Then for any $\delta > 0$, there exists a function $C(o, r, \delta) > 0$ independent of $k$ and a set $E_\delta \subset [0, \infty)$ of finite Lebesgue measure such that

\[ E_o [k(X_{r})] \leq C(o, r, \delta) \left( E_o \left[ \int_0^r k(X_t)dt \right] \right)^{(1+\delta)^2} \]

holds for $r > 1$ outside $E_\delta$.

Let $\psi$ be a meromorphic function on $M$. The norm of the gradient of $\psi$ is defined by

\[ \|\nabla_M \psi\|^2 = \sum_{i,j} g^{i\overline{j}} \frac{\partial \psi}{\partial z_i} \frac{\overline{\partial \psi}}{\partial \overline{z_j}}, \]

where $(g^{i\overline{j}})$ is the inverse of $(g_{\overline{i}j})$. Locally, we write $\psi = \psi_1/\psi_0$, where $\psi_0, \psi_1$ are holomorphic functions so that $\text{codim}_C(\psi_0 = \psi_1 = 0) \geq 2$ if $\dim_C M \geq 2$. Identify $\psi$ with a meromorphic mapping into $\mathbb{P}^1(\mathbb{C})$ by $x \mapsto [\psi_0(x) : \psi_1(x)]$. The characteristic function of $\psi$ with respect to the Fubini-Study form $\omega_{FS}$ on $\mathbb{P}^1(\mathbb{C})$ is defined by

\[ T_\psi(r, \omega_{FS}) = \frac{1}{4} \int_{B_o(r)} g_r(o, x) \Delta_M \log(|\psi_0(x)|^2 + |\psi_1(x)|^2) dV(x). \]

Let $i : \mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ be an inclusion defined by $z \mapsto [1 : z]$. Via the pull-back by $i$, we have a $(1,1)$-form $i^*\omega_{FS} = dd^c \log(1 + |\zeta|^2)$ on $\mathbb{C}$, where $\zeta := w_1/w_0$ and $[w_0 : w_1]$ is the homogeneous coordinate system of $\mathbb{P}^1(\mathbb{C})$. The characteristic function of $\psi$ with respect to $i^*\omega_{FS}$ is defined by

\[ \hat{T}_\psi(r, \omega_{FS}) = \frac{1}{4} \int_{B_o(r)} g_r(o, x) \Delta_M \log(1 + |\psi(x)|^2) dV(x). \]
Clearly,
\[ \hat{T}_\psi(r,\omega_{FS}) \leq T_\psi(r,\omega_{FS}). \]
We adopt the spherical distance \( \| \cdot, \cdot \| \) on \( \mathbb{P}^1(\mathbb{C}) \), the proximity function of \( \psi \) with respect to \( a \in \mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{ \infty \} \) is defined by
\[ \hat{m}_\psi(r,a) = \int_{S_\omega(r)} \log_+ \frac{1}{\| \psi(x), a \|} d\pi_o^r(x). \]
Again, set
\[ \hat{N}_\psi(r,a) = \pi \hat{m}(m-1)! \int_{\psi^{-1}(a) \cap B_\omega(r)} g_r(o,x)\alpha^{m-1}. \]
Using the similar arguments as in the proof of Theorem \[3.4\], we easily show that
\[ \hat{T}_\psi(r,\omega_{FS}) = \hat{m}_\psi(r,a) + \hat{N}_\psi(r,a) + O(1). \]
We also define the Nevanlinna’s characteristic function
\[ T(r,\psi) := m(r,\psi) + N(r,\psi), \]
where
\[ m(r,\psi) = \int_{S_\omega(r)} \log_+ |\psi(x)| d\pi_o^r(x), \]
\[ N(r,\psi) = \pi \hat{m}(m-1)! \int_{\psi^{-1}(\infty) \cap B_\omega(r)} g_r(o,x)\alpha^{m-1}. \]
Clearly, \( N(r,\psi) = \hat{N}_\psi(r,\infty) \) and \( m(r,\psi) = \hat{m}_\psi(r,\infty) + O(1) \). Thus,
\[ (9) \quad T(r,\psi) = \hat{T}_\psi(r,\omega_{FS}) + O(1), \quad T\left(r, \frac{1}{\psi - a}\right) = T(r,\psi) + O(1). \]
On \( \mathbb{P}^1(\mathbb{C}) \), we take a singular metric
\[ \Phi = \frac{1}{|\zeta|^2(1 + \log^2 |\zeta|)} \sqrt{-1} \frac{1}{4\pi^2} d\zeta \wedge d\bar{\zeta}. \]
A direct computation shows that
\[ (10) \quad \int_{\mathbb{P}^1(\mathbb{C})} \Phi = 1, \quad 2m\pi \frac{\psi^* \Phi \wedge \alpha^{m-1}}{\alpha^m} = \frac{\| \nabla_M \psi \|^2}{\psi^2(1 + \log^2 |\psi|)}. \]
Set
\[ T_\psi(r,\Phi) = \frac{1}{2} \int_{B_\omega(r)} g_r(o,x)e_{\psi^* \Phi}(x)dV(x). \]
Invoking \[10\], we obtain
\[ (11) \quad T_\psi(r,\Phi) = \frac{1}{2\pi} \int_{B_\omega(r)} g_r(o,x)\frac{\| \nabla_M \psi \|^2}{\psi^2(1 + \log^2 |\psi|)}(x)dV(x). \]
**Lemma 4.2.** We have
\[ T_\psi(r,\Phi) \leq T(r,\psi) + O(1). \]
Proof. By Fubini’s theorem and Corollary 3.5

\[ T_\psi(r, \Phi) = m \int_{B_o(r)} g_r(o, x) \psi^* \Phi \wedge \alpha^{m-1} \alpha_m \, dV(x) \]

\[ = \frac{\pi^m}{(m-1)!} \int_{\zeta \in \mathbb{P}^1(C)} \Phi(\zeta) \int_{\psi^{-1}(\zeta) \cap B_o(r)} g_r(o, x) \alpha^{m-1} \, dV(x) \]

\[ = \int_{\zeta \in \mathbb{P}^1(C)} N_\psi(r, \zeta) \Phi(\zeta) \]

\[ \leq \int_{\zeta \in \mathbb{P}^1(C)} (T(r, \psi) + O(1)) \Phi(\zeta) \]

\[ = T(r, \psi) + O(1). \]

The proof is completed. □

Lemma 4.3. Assume that \( \psi(x) \not\equiv 0 \). For any \( \delta > 0 \), there are \( C(o, r, \delta) > 0 \) independent of \( \psi \) and \( E_\delta (1, \infty) \) of finite Lebesgue measure such that

\[ \mathbb{E}_o \left[ \log^+ \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_{\tau_r}) \right] \leq (1 + \delta)^2 \log T(r, \psi) + \log C(o, r, \delta) \]

holds for \( r > 1 \) outside \( E_\delta \).

Proof. By Jensen inequality, it is clear that

\[ \mathbb{E}_o \left[ \log^+ \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_{\tau_r}) \right] \leq \mathbb{E}_o \left[ \log \left( 1 + \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_{\tau_r}) \right) \right] \]

\[ \leq \log^+ \mathbb{E}_o \left[ \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_{\tau_r}) \right] + O(1). \]

By Lemma 4.1 and co-area formula, there is \( C(o, r, \delta) > 0 \) such that

\[ \log^+ \mathbb{E}_o \left[ \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_{\tau_r}) \right] \]

\[ \leq (1 + \delta)^2 \log^+ \mathbb{E}_o \left[ \int_0^{\tau_r} \frac{\| \nabla_M \psi \|^2}{|\psi|^2(1 + \log^2 |\psi|)}(X_t) \, dt \right] + \log C(o, r, \delta) \]

\[ \leq (1 + \delta)^2 \log T(r, \psi) + \log C(o, r, \delta) + O(1), \]

where Lemma 4.2 and (11) are applied. Modify \( C(o, r, \delta) \) such that the term \( O(1) \) is removed, then we get the desired inequality. □

Define

\[ m \left( r, \frac{\| \nabla_M \psi \|}{|\psi|} \right) = \int_{S_o(r)} \log^+ \frac{\| \nabla_M \psi \|}{|\psi|} \, d\pi_o^r(x). \]

We now prove Theorem 1.1.
Proof. On the one hand,
\[ m\left( r, \frac{\|\nabla_M \psi\|}{|\psi|} \right) \leq \frac{1}{2} \int_{\mathcal{S}_o(r)} \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} \right) (x) d\pi_o^r(x) \]
\[ + \frac{1}{2} \int_{\mathcal{S}_o(r)} \log^+ \left( 1 + \log^2 |\psi(x)| \right) d\pi_o^r(x) \]
\[ = \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} \right) (x_{\tau r}) \right] \]
\[ + \frac{1}{2} \int_{\mathcal{S}_o(r)} \log \left( 1 + \log^2 |\psi(x)| \right) d\pi_o^r(x) \]
\[ \leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} \right) (x_{\tau r}) \right] \]
\[ + \frac{1}{2} \int_{\mathcal{S}_o(r)} \log \left( 1 + \log^2 |\psi(x)| + \log^+ \left( \frac{1}{|\psi(x)|} \right)^2 \right) d\pi_o^r(x) \]
\[ \leq \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} \right) (x_{\tau r}) \right] \]
\[ + \int_{\mathcal{S}_o(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x). \]

Lemma 4.3 implies that
\[ \frac{1}{2} \mathbb{E}_o \left[ \log^+ \left( \frac{\|\nabla_M \psi\|^2}{|\psi|^2 (1 + \log^2 |\psi|)} \right) (x_{\tau r}) \right] \]
\[ \leq \frac{(1 + \delta)^2}{2} \log T(r, \psi) + \frac{1}{2} \log C(o, r, \delta) + O(1). \]

On the other hand, by Jensen inequality and (9)
\[ \int_{\mathcal{S}_o(r)} \log \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x) \]
\[ \leq \log \int_{\mathcal{S}_o(r)} \left( 1 + \log^+ |\psi(x)| + \log^+ \frac{1}{|\psi(x)|} \right) d\pi_o^r(x) \]
\[ \leq \log \left( m(r, \psi) + m(r, 1/\psi) \right) + O(1) \]
\[ \leq \log T(r, \psi) + O(1). \]

Replacing \( C(o, r, \delta) \) by \( C^2(o, r, \delta) \) and combining the above, then the theorem can be proved. \( \square \)

4.2. Estimate of \( C(o, r, \delta) \).

Let \( M \) be a complete Kähler manifold of non-positive sectional curvature. Indeed, we let \( \kappa \) be defined by (1). Clearly, \( \kappa \) is a non-positive, non-increasing and continuous function on \([0, \infty)\). Treat the differential equation
\[ G''(t) + \kappa(t)G(t) = 0; \quad G(0) = 0, \quad G'(0) = 1 \]
on $[0, \infty)$. Now compare (12) with $y''(t) + \kappa(0)y(t) = 0$ under the same initial conditions, we see that $G$ can be estimated simply as

$$G(t) = t \quad \text{for} \quad \kappa \equiv 0; \quad G(t) \geq t \quad \text{for} \quad \kappa \neq 0.$$  

This follows that

$$G(r) \geq r \quad \text{for} \quad r \geq 0; \quad \int_{1}^{r} \frac{dt}{G(t)} \leq \log r \quad \text{for} \quad r \geq 1. \tag{13}$$  

On the other hand, we rewrite (12) as the form

$$\log' G(t) \cdot \log' G'(t) = -\kappa(t).$$

Since $G(t) \geq t$ is increasing, then the decrease and non-positivity of $\kappa$ imply that for each fixed $t$, $G$ must be satisfied one of the following two inequalities

$$\log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for} \quad t > 0; \quad \log' G'(t) \leq \sqrt{-\kappa(t)} \quad \text{for} \quad t \geq 0.$$  

By virtue of $G(t) \to 0$ as $t \to 0$, by integration, $G$ is bounded from above by

$$G(r) \leq r \exp (r \sqrt{-\kappa(r)}) \quad \text{for} \quad r \geq 0. \tag{14}$$

In what follows, one assumes that $M$ is simply connected. The purpose of this section is to show the following Logarithmic Derivative Lemma (LDL) by estimating $C(o,r,\delta)$.

**Theorem 4.4 (LDL).** Let $\psi$ be a nonconstant meromorphic function on $M$. Then

$$m \left( r, \left\| \nabla_{M} \psi \right\| / |\psi| \right) \leq \left( 1 + \frac{(1 + \delta)^2}{2} \right) \log T(r, \psi) + O \left( r \sqrt{-\kappa(r)} + \delta \log r \right),$$

where $\kappa$ is defined by (1).

**Remark.** The LDL still holds when $M$ is multi-connected, one just needs to lift $M$ to the universal covering, see the arguments in Section 5.3.

We first introduce some lemmas.

**Lemma 4.5 ([4]).** Let $\eta > 0$ be a number. Then there is a constant $C > 0$ such that

$$g_{r}(o,x) \int_{\gamma}^{r} G^{1-2m}(t)dt \geq C \int_{r(x)}^{r} G^{1-2m}(t)dt$$

holds for $r > \eta$ and $x \in B_{o}(r) \setminus \overline{B_{o}(\eta)}$, where $G$ is defined by (12).

**Lemma 4.6 ([11, 16]).** Let $M$ be a simply-connected, non-positively curved and complete Hermitian manifold of complex dimension $m$. Then

(i) $g_{r}(o,x) \leq \begin{cases} \pi \log r^{2}, & m = 1 \\ (\frac{1}{m-1})^{2m-1} \log^{2-2m}(x) - r^{2-2m}, & m \geq 2 \end{cases}$;  

(ii) $d\pi^{r}_{o}(x) \leq \frac{1}{\omega_{2m-1}r^{2m-1}}d\sigma_{r}(x).$
where \( g_r(o, x) \) denotes the Green function of \( \Delta_M/2 \) for \( B_o(r) \) with Dirichlet boundary condition and a pole at \( o \), and \( d\sigma_r^\gamma(x) \) is the harmonic measure on \( S_o(r) \) with respect to \( o \), and \( \omega_{2m-1} \) is the Euclidean volume of unit sphere in \( \mathbb{R}^{2m} \), and \( d\sigma_r(x) \) is the induced Riemannian volume measure on \( S_o(r) \).

**Lemma 4.7** (Borel Lemma, [22]). Let \( T \) be a strictly positive nondecreasing function of \( \phi^1 \)-class on \((0, \infty)\). Let \( \gamma > 0 \) be a number such that \( T(\gamma) \geq e \), and \( \phi \) be a strictly positive nondecreasing function such that

\[
c_\phi = \int_e^\infty \frac{1}{t\phi(t)} dt < \infty.
\]

Then, the inequality

\[
T'(r) \leq T(r)\phi(T(r))
\]

holds for \( r \geq \gamma \) outside a set of Lebesgue measure not exceeding \( c_\phi \). Particularly, take \( \phi(T) = T^\delta \) for a number \( \delta > 0 \), we have \( T'(r) \leq T^{1+\delta}(r) \) holds for \( r > 0 \) outside a set \( E_\delta \subset (0, \infty) \) of finite Lebesgue measure.

We now prove the following so-called Calculus Lemma (see also [1]) which gives an estimate of \( C(o, r, \delta) \).

**Lemma 4.8** (Calculus Lemma). Let \( k \geq 0 \) be a locally integrable function on \( M \) such that it is locally bounded at \( o \in M \). Then for any \( \delta > 0 \), there is a constant \( C > 0 \) independent of \( k, \delta \), and a set \( E_\delta \subset (1, \infty) \) of finite Lebesgue measure such that

\[
\mathbb{E}_o[k(X_{r\gamma})] \leq \frac{C(1+\delta)^2 \log(1+\delta)^2 r^{(1-2m)\delta}}{r^{(1-2m)\delta}e^{(1-2m)(1+\delta)r\sqrt{-\kappa(r)}}} \left( \mathbb{E}_o \left[ \int_0^{r\gamma} k(X_t)dt \right] \right)^{(1+\delta)^2}
\]

holds for \( r > 1 \) outside \( E_\delta \), where \( \kappa \) is defined by [1].

**Proof.** By Lemma 4.5 and Lemma 4.6 with (13), we get

\[
\mathbb{E}_o \left[ \int_0^{r\gamma} k(X_t)dt \right] = \int_{B_o(r)} g_r(o, x)k(x)dV(x)
\]

\[
= \int_0^r dt \int_{S_o(t)} g_r(o, x)k(x)d\sigma_t(x)
\]

\[
\geq C_0 \int_0^r \int_t^{r\gamma} G^{1-2m}(s)ds dt \int_{S_o(t)} k(x)d\sigma_t(x)
\]

\[
= \frac{C_0}{\log r} \int_0^r dt \int_t^{r\gamma} G^{1-2m}(s)ds \int_{S_o(t)} k(x)d\sigma_t(x)
\]

and

\[
\mathbb{E}_o[k(X_{r\gamma})] = \int_{S_o(r)} k(x)d\pi_o^\gamma(x) \leq \frac{1}{\omega_{2m-1}r^{2m-1}} \int_{S_o(r)} k(x)d\sigma_r(x),
\]
where \( \omega_{2m-1} \) denotes the Euclidean volume of unit sphere in \( \mathbb{R}^{2m} \), \( d\sigma_r \) is the induced volume measure on \( S_o(r) \). Hence,

\[
E_o \left[ \int_0^r k(X_t) dt \right] \geq \frac{C_0}{\log r} \int_0^r dt \int_t^r G^{1-2m}(s) ds \int_{S_o(t)} k(x) d\sigma_t(x)
\]

and

\[
(15) \quad E_o[k(X_{\tau_r})] \leq \frac{1}{\omega_{2m-1} r^{2m-1}} \int_{S_o(r)} k(x) d\sigma_r(x).
\]

Put

\[
\Gamma(r) = \int_0^r dt \int_t^r G^{1-2m}(s) ds \int_{S_o(t)} k(x) d\sigma_t(x).
\]

Then

\[
(16) \quad \Gamma'(r) = G^{1-2m}(r) \int_0^r dt \int_{S_o(t)} k(x) d\sigma_t(x).
\]

A simple computation shows that

\[
\Gamma'(r) = G^{1-2m}(r) \int_0^r dt \int_{S_o(t)} k(x) d\sigma_t(x).
\]

By this with (15)

\[
(17) \quad E_o[k(X_{\tau_r})] \leq \frac{1}{\omega_{2m-1} r^{2m-1}} \frac{d}{dr} \left( \frac{\Gamma'(r)}{G^{1-2m}(r)} \right).
\]

Using Lemma 4.7 twice, for any \( \delta > 0 \) we have

\[
(18) \quad \frac{d}{dr} \left( \frac{\Gamma'(r)}{G^{1-2m}(r)} \right) \leq \frac{\Gamma(1+\delta)^2(r)}{G^{(1-2m)(1+\delta)}(r)}
\]

holds outside a set \( E_\delta \subset (1, \infty) \) of finite Lebesgue measure. Using (16)--(18) and (14), it is not hard to conclude that

\[
E_o[k(X_{\tau_r})] \leq \frac{C(1+\delta)^2 \log(1+\delta)^2 r}{r(1-2m)\delta e(1-2m)(1+\delta) r^{1-\kappa(r)}} \left( E_o \left[ \int_0^\tau k(X_t) dt \right] \right)^{(1+\delta)^2}
\]

with \( C = 1/C_0 > 0 \) being a constant independent of \( k, \delta \).

Lemma 4.8 implies an estimate

\[
C(\sigma, r, \delta) \leq \frac{C(1+\delta)^2 \log(1+\delta)^2 r}{r(1-2m)\delta e(1-2m)(1+\delta) r^{1-\kappa(r)}}
\]

Thus, we get

\[
(19) \quad \log C(\sigma, r, \delta) \leq O \left( r^{1-\kappa(r)} + \delta \log r \right).
\]

We prove Theorem 4.4:

**Proof.** Combining Theorem 1.1 with (19), we show the theorem. \( \square \)
5. Second Main Theorem

5.1. Meromorphic mappings into \( \mathbb{P}^n(\mathbb{C}) \).

In this subsection, \( M \) is a general Kähler manifold. Let \( \psi : M \to \mathbb{P}^n(\mathbb{C}) \) be a meromorphic mapping, i.e., there exists an open covering \( \{U_\alpha\} \) of \( M \) such that \( \psi \) has a local representation \( [\psi_0^\alpha : \cdots : \psi_n^\alpha] \) on each \( U_\alpha \), where \( \psi_0^\alpha, \cdots, \psi_n^\alpha \) are holomorphic functions on \( U_\alpha \) satisfying

\[
\text{codim}_\mathbb{C}(\psi_0^\alpha = \cdots = \psi_n^\alpha = 0) \geq 2.
\]

Let \( [w_0 : \cdots : w_n] \) denote the homogeneous coordinate of \( \mathbb{P}^n(\mathbb{C}) \). Assuming that \( w_0 \circ \psi \not\equiv 0 \). Let \( i : \mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C}) \) be an inclusion given by \( (z_1, \cdots, z_n) \mapsto [1 : z_1 : \cdots : z_n] \). Clearly, \( \omega_{FS} \) induces a \((1,1)\)-form \( i^* \omega_{FS} = dd^c \log(|\zeta_0|^2 + |\zeta_1|^2 + \cdots + |\zeta_n|^2) \) on \( \mathbb{C}^n \), where \( \zeta_j := w_j/w_0 \) for \( 0 \leq j \leq n \). The characteristic function of \( \psi \) with respect to \( i^* \omega_{FS} \) is well defined by

\[
\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{4} \int_{B_o(r)} g_r(o,x) \Delta_M \log \left( \sum_{j=0}^{n} |\zeta_j \circ \psi(x)|^2 \right) dV(x).
\]

Clearly,

\[
\hat{T}_\psi(r, \omega_{FS}) \leq \frac{1}{4} \int_{B_o(r)} g_r(o,x) \Delta_M \log \|\psi(x)\|^2 dV(x) = T_\psi(r, \omega_{FS}).
\]

The co-area formula leads to

\[
\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{4} \mathbb{E}_o \left[ \int_0^r \Delta_M \log \left( \sum_{j=0}^{n} |\zeta_j \circ \psi(X_t)|^2 \right) dt \right].
\]

Note that the pole divisor of \( \zeta_j \circ \psi \) is pluripolar. By Dynkin formula

\[
\hat{T}_\psi(r, \omega_{FS}) = \frac{1}{2} \int_{S_o(r)} \log \left( \sum_{j=0}^{n} |\zeta_j \circ \psi(x)|^2 \right) d\pi_o^r(x) - \frac{1}{2} \log \left( \sum_{j=0}^{n} |\zeta_j \circ \psi(o)|^2 \right),
\]

\[
\hat{T}_{\zeta_j \circ \psi}(r, \omega_{FS}) = \frac{1}{2} \int_{S_o(r)} \log \left( 1 + |\zeta_j \circ \psi(x)|^2 \right) d\pi_o^r(x) - \frac{1}{2} \log \left( 1 + |\zeta_j \circ \psi(o)|^2 \right).
\]

**Theorem 5.1.** We have

\[
\max_{1 \leq j \leq n} T(r, \zeta_j \circ \psi) + O(1) \leq \hat{T}_\psi(r, \omega_{FS}) \leq \sum_{j=1}^{n} T(r, \zeta_j \circ \psi) + O(1).
\]
Proof. On the one hand,
\[ \hat{T}_\psi(r, \omega_{FS}) \]
\[ \leq \frac{1}{2} \sum_{j=1}^{n} \left( \int_{S_{0}(r)} \log \left( 1 + |\zeta_j \circ \psi(x)|^2 \right) d\pi_x(\omega) - \log \left( 1 + |\zeta_j \circ \psi(o)|^2 \right) \right) + O(1) \]
\[ = \sum_{j=1}^{n} T(r, \zeta_j \circ \psi) + O(1). \]

On the other hand,
\[ T(r, \zeta_j \circ \psi) = \hat{T}_{\zeta_j \circ \psi}(r, \omega_{FS}) + O(1) \]
\[ \leq \frac{1}{4} \int_{B_0(r)} g_{r}(o, x) \Delta_M \log \left( \sum_{j=0}^{n} |\zeta_j \circ \psi(x)|^2 \right) dV(x) + O(1) \]
\[ = \hat{T}_\psi(r, \omega_{FS}) + O(1). \]

We conclude the proof. \(\square\)

Corollary 5.2. We have
\[ \max_{1 \leq j \leq n} T(r, \zeta_j \circ \psi) \leq T_\psi(r, \omega_{FS}) + O(1). \]

Let \( V \) be a complex projective algebraic variety and \( \mathbb{C}(V) \) be the field of rational functions defined on \( V \) over \( \mathbb{C} \). Let \( V \hookrightarrow \mathbb{P}^{N}(\mathbb{C}) \) be a holomorphic embedding, and let \( H_{V} \) be the restriction of hyperplane line bundle \( H \) over \( \mathbb{P}^{N}(\mathbb{C}) \) to \( V \). Denote by \([w_0 : \cdots : w_N]\) the homogeneous coordinate system of \( \mathbb{P}^{N}(\mathbb{C}) \) and assume that \( w_0 \neq 0 \) without loss of generality. Notice that the restriction \( \{\zeta_j := w_j/w_0\} \) to \( V \) gives a transcendental base of \( \mathbb{C}(V) \). Thereby, any \( \phi \in \mathbb{C}(V) \) can be represented by a rational function in \( \zeta_1, \cdots, \zeta_N \)
\[ \phi = Q(\zeta_1, \cdots, \zeta_N). \]

Theorem 5.3. Let \( f : M \to V \) be an algebraically non-degenerate meromorphic mapping. Then for \( \phi \in \mathbb{C}(V) \), there is a constant \( C > 0 \) depending on \( \phi \) such that
\[ T(r, \phi \circ f) \leq C T_f(r, H_V) + O(1). \]

Proof. Assume that \( w_0 \circ f \neq 0 \) without loss of generality. Since \( Q_j \) is rational, there is constant \( C' > 0 \) such that \( T(r, \phi \circ f) \leq C' \sum_{j=1}^{N} T(r, \zeta_j \circ f) + O(1) \). By Corollary 5.2, \( T(r, \zeta_j \circ f) \leq T_f(r, H_V) + O(1) \). This proves the theorem. \(\square\)

Corollary 5.4. Let \( f : M \to V \) be an algebraically non-degenerate meromorphic mapping. Fix a positive \((1, 1)\)-form \( \omega \) on \( V \). Then for any \( \phi \in \mathbb{C}(V) \), there is a constant \( C > 0 \) depending on \( \phi \) such that
\[ T(r, \phi \circ f) \leq C T_f(r, \omega) + O(1). \]
Proof. The compactness of $V$ and Theorem 5.3 implies the assertion. 

5.2. Estimate of $E_o[\tau_r]$.

We let $M$ be a simply-connected complete Kähler manifold of non-positive sectional curvature, and let $X_t$ be the Brownian motion in $M$ with generator $\Delta_M/2$ started at $o$. Recall that $\dim_C M = m$, $\tau_r = \inf\{t > 0 : X_t \not\in B_o(r)\}$.

Lemma 5.5. We have

$$E_o[\tau_r] \leq \frac{2r^2}{2m - 1}.$$  

Proof. The argument follows essentially from Atsuji [4], but here we provide a simpler proof though a rougher estimate. We refer the reader to [4] for a better estimate that $E_o[\tau_r] \leq r^2/2m$. Let $M$ be a simply-connected complete Kähler manifold of non-positive sectional curvature, and let $X_t$ be the Brownian motion in $M$ with generator $\Delta_M/2$ started at $o$. Let $r_1(x)$ be the distance function of $x$ from $o$. Apply Itô formula to $r_1(x)$

$$r_1(X_t) - r_1(X_0) = B_t - L_t + \frac{1}{2} \int_0^t \Delta_M r_1(X_s) ds,$$

here $B_t$ is the standard Brownian motion in $\mathbb{R}$, and $L_t$ is a local time on cut locus of $o$, an increasing process which increases only at cut loci of $o$. Since $M$ is simply connected and non-positively curved, then

$$\Delta_M r_1(x) \geq \frac{2m - 1}{r_1(x)}, \quad L_t \equiv 0.$$

By (20), we arrive at

$$r_1(X_t) \geq B_t + \frac{2m - 1}{2} \int_0^t \frac{ds}{r_1(X_s)}.$$

Let $t = \tau_r$ and take expectation on both sides of the above inequality, then it yields that

$$\max_{x \in S_o(r)} r_1(x) \geq \frac{(2m - 1)E_o[\tau_r]}{2 \max_{x \in S_o(r)} r_1(x)}.$$

Let $o' \to o$, we are led to the conclusion. 

5.3. Second Main Theorem.

Let $M$ be a complete Kähler manifold of non-positive sectional curvature. Consider the (analytic) universal covering

$$\pi : \tilde{M} \to M.$$  

Via the pull-back by $\pi$, $\tilde{M}$ can be equipped with the induced metric from the metric of $M$. So, under this metric, $\tilde{M}$ becomes a simply-connected complete Kähler manifold of non-positive sectional curvature. Take a diffusion process $\tilde{X}_t$ in $\tilde{M}$ such that $X_t = \pi(\tilde{X}_t)$, here $X_t$ is the Brownian motion started at
o ∈ M, then ˜X_t is a Brownian motion generated by ∆M/2 induced from the pull-back metric. Let ˜X_t start at ˜o ∈ ˜M with o = π(˜o). Then

$E_o[\phi(X_t)] = E_{\tilde{o}}[\phi \circ \pi(\tilde{X}_t)]$

for $\phi \in \mathcal{C}_b(M)$. Set

$\tilde{\tau}_r = \inf \{ t > 0 : \tilde{X}_t \notin B_{\tilde{o}}(r) \}$,

where $B_{\tilde{o}}(r)$ is a geodesic ball centered at $\tilde{o}$ with radius $r$ in $\tilde{M}$. If necessary, one can extend the filtration in probability space where $(X_t, \mathbb{P}_o)$ are defined so that $\tilde{\tau}_r$ is a stopping time with respect to a filtration where the stochastic calculus of $X_t$ works. By the above arguments, we may assume $M$ is simply connected by lifting $f$ to the universal covering.

Let $V$ be a complex projective algebraic manifold with complex dimension $n \leq m = \dim_{\mathbb{C}} M$, and let $L \to V$ be a holomorphic line bundle. Let a divisor $D \in |L|$ be of simple normal crossing type, one can express $D = \sum_{j=1}^q D_j$ as the union of irreducible components. Equipping $L_{D_j}$ with Hermitian metric which then induces a natural Hermitian metric $h$ on $L = \otimes_{j=1}^q L_{D_j}$. Fixing a Hermitian metric form $\omega$ on $V$, which gives a (smooth) volume form $\Omega := \omega^n$ on $V$. Pick $s_j \in H^0(V, L_{D_j})$ with $(s_j) = D_j$ and $\|s_j\| < 1$. On $V$, one defines a singular volume form

$$\Phi = \Omega \prod_{j=1}^q \|s_j\|^2.$$

Set

$$\xi^{\alpha} = f^* \Phi \wedge \alpha^{m-n}.$$

Note that

$$\alpha^m = m! \det(g_{ij}) \prod_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j.$$

A direct computation leads to

$$dd^c \log \xi \geq f^* c_1(L, h) - f^* \text{Ric} \Omega + \mathcal{R}_M - \text{Supp} f^* D$$

in the sense of currents, where $\mathcal{R}_M = -dd^c \log \det(g_{ij})$. This follows that

$$\frac{1}{4} \int_{B_o(r)} g_{r}(o, x) \Delta_M \log \xi(x) dV(x) \geq T_f(r, L) + T_f(r, K_V) + T(r, \mathcal{R}_M) - \mathcal{N}_f(r, D) + O(1).$$

We now prove Theorem 1.2

**Proof.** By Ru-Wong’s arguments (see [22], Page 231-233), the simple normal crossing type of $D$ implies that there exists a finite open covering $\{U_\lambda\}$ of $V$
together with rational functions \( w_{\lambda 1}, \ldots, w_{\lambda n} \) on \( V \) for \( \lambda \) such that \( w_{\lambda 1}, \ldots \) are holomorphic on \( U_\lambda \) as well as

\[
dw_{\lambda 1} \wedge \cdots \wedge dw_{\lambda n}(y) \neq 0, \quad \forall y \in U_\lambda,
\]

\[
D \cap U_\lambda = \{ w_{\lambda 1} \cdots w_{\lambda h_{\lambda}} = 0 \}, \quad \exists h_{\lambda} \leq n.
\]

In addition, we can require \( L_{D_j}|_{U_\lambda} \cong U_\lambda \times \mathbb{C} \) for \( \lambda, j \). On \( U_\lambda \), we get

\[
\Phi = e_{\lambda} |w_{\lambda 1}| \cdots |w_{\lambda h_{\lambda}}|^2 \wedge \frac{1}{2\pi} \sum_{k=1}^{n} \sqrt{-1} \, dw_{\lambda k} \wedge d\bar{w}_{\lambda k},
\]

where \( \Phi \) is given by (21) and \( e_{\lambda} \) is a smooth positive function. Let \( \{ \phi_{\lambda} \} \) be a partition of unity subordinate to \( \{ U_\lambda \} \), then \( \phi_{\lambda} e_{\lambda} \) is bounded on \( V \).

Set \( \Phi_{\lambda} = \phi_{\lambda} e_{\lambda} |w_{\lambda 1}| \cdots |w_{\lambda h_{\lambda}}|^2 \wedge \frac{1}{2\pi} \sum_{k=1}^{n} \sqrt{-1} \, dw_{\lambda k} \wedge d\bar{w}_{\lambda k} \).

Put \( f_{\lambda k} = w_{\lambda k} \circ f \), then on \( f^{-1}(U_\lambda) \) we obtain

\[
(23) \quad f^*\Phi_{\lambda} = \frac{\phi_{\lambda} \circ f \cdot e_{\lambda} \circ f}{|f_{\lambda 1}|^2 \cdots |f_{\lambda h_{\lambda}}|^2} \wedge \frac{1}{2\pi} \sum_{k=1}^{n} \sqrt{-1} \, df_{\lambda k} \wedge d\bar{f}_{\lambda k}.
\]

Set

\[
f^*\Phi \wedge \alpha^{m-n} = \xi \alpha^m, \quad f^*\Phi_{\lambda} \wedge \alpha^{m-n} = \xi_{\lambda} \alpha^m
\]

which arrives at (22). Clearly, we have \( \xi = \sum_{\lambda} \xi_{\lambda} \). Again, set

\[
(24) \quad f^*\omega \wedge \alpha^{m-1} = \varrho \alpha^m
\]

which follows that

\[
(25) \quad \varrho = \frac{1}{2m} e f^*\omega.
\]

For each \( \lambda \) and any \( x \in f^{-1}(U_\lambda) \), take a local holomorphic coordinate system \( z \) around \( x \). Since \( \phi_{\lambda} \circ f \cdot e_{\lambda} \circ f \) is bounded, it is not very hard to see from (23) and (24) that \( \xi_{\lambda} \) is bounded from above by \( P_{\lambda} \), where \( P_{\lambda} \) is a polynomial in

\[
\varrho, \quad g^{ij} \frac{\partial f_{\lambda k}}{\partial z_i} \frac{\overline{\partial f_{\lambda k}}}{\partial z_j} / |f_{\lambda k}|^2, \quad 1 \leq i, j \leq m, \quad 1 \leq k \leq n.
\]

This yields that

\[
(26) \quad \log^+ \xi_{\lambda} \leq O \left( \log^+ \varrho + \sum_k \log^+ \| \nabla M f_{\lambda k} \| / |f_{\lambda k}| \right) + O(1).
\]

Thus, we conclude that

\[
(27) \quad \log^+ \xi \leq O \left( \log^+ \varrho + \sum_{k,\lambda} \log^+ \| \nabla M f_{\lambda k} \| / |f_{\lambda k}| \right) + O(1)
\]
on $M$. On the one hand,

$$\frac{1}{4} \int_{B_o(r)} g_t(o, x) \Delta_M \log \xi(x) dV(x) = \frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_t}) \right] + O(1)$$

due to co-area formula and Dynkin formula. Hence, by (22) we have

$$(28) \quad \frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_t}) \right] \geq T_f(r, L) + T_f(r, K_V) + T(r, \mathcal{R}_M) - \nabla f(r, D) + O(1).$$

On the other hand, since $f_{\lambda k}$ is the pull-back of rational function $w_{\lambda k}$ on $V$ by $f$, Corollary 5.4 implies that

$$(29) \quad T(r, f_{\lambda k}) \leq O(T_f(r, \omega)) + O(1).$$

Using (26) and (29) with Theorem 1.1

$$\frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_t}) \right] \leq \sum_{k, \lambda} m(r, \|\mathcal{N}_M f_{\lambda k}\|/|f_{\lambda k}|) + \log \mathbb{E}_o [\log^+ \varrho(X_{r_t})] + O(1)$$

In the meanwhile, Lemma 4.1 and (25) imply

$$\log^+ \mathbb{E}_o [\varrho(X_{r_t})] \leq (1 + \delta)^2 \log^+ \mathbb{E}_o \left[ \int_0^{r_t} g(X_t) dt \right] + \log C(o, r, \delta)$$

By this with (28), we prove the theorem. \hfill \Box

We proceed to prove Theorem 1.3.

**Lemma 5.6.** Let $\kappa$ be defined by (11). If $M$ is non-positively curved, then

$$T(r, \mathcal{R}_M) \geq 2 \kappa r^2.$$
Proof. Lemma 2.2 implies that \( 0 \geq s_M \geq mR_M \). By co-area formula
\[
T(r, \mathcal{R}_M) = -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_M \log \det(g_{i\bar{j}}(X_t)) dt \right]
\]
\[
= \mathbb{E}_o \left[ \int_0^{\tau_r} s_M(X_t) dt \right] \geq m \mathbb{E}_o \left[ \int_0^{\tau_r} R_M(X_t) dt \right] \geq m(2m - 1) \kappa(r) \mathbb{E}_o[\tau_r].
\]
We have \( \mathbb{E}_o[\tau_r] \leq 2r^2/(2m - 1) \) by Lemma 5.5. The proof is completed. □

Proof. With the estimate of \( C(o,r,\delta) \) given by (19) and estimate of \( T(r, \mathcal{R}_M) \) given by Lemma 5.6, Theorem 1.3 follows from Theorem 1.2. □

If \( M = \mathbb{C}^m \), then \( \kappa \equiv 0 \). Theorem 1.3 implies that

**Corollary 5.7** (Carlson-Griffiths, [7]; Noguchi, [19]). *Let a divisor \( D \in |L| \) be of simple normal crossing type. Let \( f : \mathbb{C}^m \to V \) be a differentiably non-degenerate meromorphic mapping. Then*

\[
T_f(r, L) + T_f(r, K_V) \leq N_f(r, D) + O(\log T_f(r, \omega) + \delta \log r) \|.
\]

6. **Second Main Theorem for singular divisors**

We extend the Second Main Theorem for divisors of simply normal crossing type to general divisors. Given a hypersurface \( D \) of a complex projective algebraic manifold \( V \). Let \( S \) denote the set of the points of \( D \) at which \( D \) has a non-normal-crossing singularity. By Hironaka’s resolution of singularities (see [15]), there exists a proper modification

\[
\tau : \tilde{V} \to V
\]
such that \( \tilde{V} \setminus \tilde{S} \) is biholomorphic to \( V \setminus S \) under \( \tau \), and \( \tilde{D} \) is only of normal crossing singularities, where \( \tilde{S} = \tau^{-1}(S) \) and \( \tilde{D} = \tau^{-1}(D) \). Let \( \bar{D} = \bar{D} \setminus \tilde{S} \) be the closure of \( \tilde{D} \setminus \tilde{S} \), and \( \tilde{S}_j \) be the irreducible components of \( \tilde{S} \). Put

\[
\tau^*D = \tilde{D} + \sum p_j\tilde{S}_j = \tilde{D} + \sum (p_j - 1)\tilde{S}_j, \quad R_\tau = \sum q_j\tilde{S}_j,
\]
where \( R_\tau \) is ramification divisor of \( \tau \), and \( p_j, q_j > 0 \) are integers. Again, set

\[
S^* = \sum \varsigma_j\tilde{S}_j, \quad \varsigma_j = \max \{p_j - q_j - 1, 0\}.
\]
We endow \( L_{S^*} \) with a Hermitian metric \( \| \cdot \| \) and take a holomorphic section \( \sigma \) of \( L_{S^*} \) with \( \text{Div}\sigma = (\sigma) = S^* \) and \( \|\sigma\| < 1 \). Let

\[
f : M \to V
\]
be a meromorphic mapping from a complete Kähler manifold \( M \) such that \( f(M) \not\subset \mathcal{D} \). The proximity function of \( f \) with respect to the singularities of \( \mathcal{D} \) is defined by

\[
m_f(r, \operatorname{Sing}(\mathcal{D})) = \int_{S_o(r)} \frac{1}{\log ||\sigma \circ \tau^{-1} \circ f(x)||} d\pi_o^r(x).
\]

Let \( \tilde{f} : M \to \tilde{V} \) be the lift of \( f \) given by \( \tau \circ \tilde{f} = f \). Then, we verify that

\[
m_f(r, \operatorname{Sing}(\mathcal{D})) = m_f(r, S^*) = \sum s_j m_{\tilde{f}}(r, S_j).
\]

We now prove Theorem 1.5:

**Proof.** We first suppose that \( \mathcal{D} \) is the union of smooth hypersurfaces, namely, no irreducible component of \( \tilde{D} \) crosses itself. Let \( E \) be the union of generic hyperplane sections of \( V \) so that the set \( A = \tilde{D} \cup E \) has only normal-crossing singularities. By (30) with \( K_{\tilde{V}} = \tau^* K_V \otimes L_{R^*} \), we have

\[
K_{\tilde{V}} \otimes L_{\tilde{D}} = \tau^* K_V \otimes \tau^* L_D \otimes \bigotimes_{S_j} L_{\tilde{S}_j}^{(1-p_j+q_j)}.
\]

Applying Theorem 1.3 to \( \tilde{f} \) for divisor \( A \),

\[
T_{\tilde{f}}(r, L_A) + T_{\tilde{f}}(r, K_{\tilde{V}}) \leq N_{\tilde{f}}(r, A) + O(\log T_{\tilde{f}}(r, \tau^* \omega) - r^2 \kappa(r) + \delta \log r).
\]

The First Main Theorem implies that

\[
T_{\tilde{f}}(r, L_A) = m_{\tilde{f}}(r, A) + N_{\tilde{f}}(r, A) + O(1)
\]

\[
= m_{\tilde{f}}(r, \tilde{D}) + m_{\tilde{f}}(r, E) + N_{\tilde{f}}(r, A) + O(1)
\]

\[
\geq m_{\tilde{f}}(r, \tilde{D}) + N_{\tilde{f}}(r, A) + O(1)
\]

\[
= T_{\tilde{f}}(r, L_{\tilde{D}}) - N_{\tilde{f}}(r, \tilde{D}) + N_{\tilde{f}}(r, A) + O(1),
\]

which leads to

\[
T_{\tilde{f}}(r, L_A) - N_{\tilde{f}}(r, A) \geq T_{\tilde{f}}(r, L_{\tilde{D}}) - N_{\tilde{f}}(r, \tilde{D}) + O(1).
\]

Combining \( T_{\tilde{f}}(r, \tau^* \omega) = T_{\tilde{f}}(r, \omega) \) and \( N_{\tilde{f}}(r, \tilde{D}) = N_{\tilde{f}}(r, D) \) with the above,

\[
T_{\tilde{f}}(r, L_{\tilde{D}}) + T_{\tilde{f}}(r, K_{\tilde{V}}) \leq N_{\tilde{f}}(r, \tilde{D}) + O(\log T_{\tilde{f}}(r, \omega) - r^2 \kappa(r) + \delta \log r).
\]

It yields from (33) that

\[
T_{\tilde{f}}(r, L_{\tilde{D}}) + T_{\tilde{f}}(r, K_{\tilde{V}}) = T_{\tilde{f}}(r, \tau^* L_D) + T_{\tilde{f}}(r, \tau^* K_V) + \sum (1 - p_j + q_j) T_{\tilde{f}}(r, L_{\tilde{S}_j})
\]

\[
= T_f(r, L_D) + T_f(r, K_V) + \sum (1 - p_j + q_j) T_f(r, L_{\tilde{S}_j}).
\]
Since $N_f(r, \tilde{S}) = 0$, it follows from (31) and (32) that
\[
\sum (1-p_j + q_j)T_f(r, L_{\tilde{S}_j}) = \sum (1-p_j + q_j)m_f(r, \tilde{S}_j) + O(1) \\
\leq \sum \varsigma_j m_f(r, \tilde{S}_j) + O(1) \\
= m_f(r, \text{Sing}(D)) + O(1).
\]

Combining (34)-(36), we show the theorem.

To prove the general case, according to the above proved, one only needs to verify this claim for an arbitrary hypersurface $D$ of normal-crossing type. Note by the arguments in [24], Page 175 that there is a proper modification $\tau: \tilde{V} \to V$ such that $\tilde{D} = \tau^{-1}(D)$ is only the union of a collection of smooth hypersurfaces of normal crossings. Thus, $m_f(r, \text{Sing}(D)) = 0$. By the special case of this theorem proved, the claim holds for $D$ by using Theorem 1.3.

\[\square\]

**Corollary 6.1** (Shiffman, [24]). Let $D \subset V$ be an ample hypersurface. Let $f: \mathbb{C}^n \to V$ be a differentiably non-degenerate meromorphic mapping. Then
\[
T_f(r, L_D) + T_f(r, K_V) \\
\leq \mathcal{N}_f(r, D) + m_f(r, \text{Sing}(D)) + O(\log T_f(r, L_D) + \delta \log r)
\]

\[\leq \Theta_f(D) \left[ \frac{c_1(L)}{\omega} \right] + \limsup_{r \to \infty} \frac{m_f(r, \text{Sing}(D))}{T_f(r, \omega)}.
\]

For further consideration of defect relations, we introduce some additional notations. Let $A$ be a hypersurface of $V$ such that $A \supset S$, where $S$ is a set of non-normal-crossing singularities of $D$ given before. We write
\[
\tau^* A = \hat{A} + \sum t_j \tilde{S}_j, \quad \hat{A} = \tau^{-1}(A) \setminus \bar{S}.
\]
Set
\[
\gamma_{A,D} = \max \frac{\varsigma_j}{t_j}
\]
where $\varsigma_j$ are given by (31). Clearly, $0 \leq \gamma_{A,D} < 1$. Note from (37) that
\[
m_f(r, A) = m_f(r, \tau^* A) \geq \sum t_j m_f(r, \tilde{S}_j) + O(1).
\]

By (32), we see that
\[
m_f(r, \text{Sing}(D)) \leq \gamma_{A,D} \sum t_j m_f(r, \tilde{S}_j) \leq \gamma_{A,D} m_f(r, A) + O(1).
\]

\[\square\]
Theorem 6.3. Let $L \to V$ be a holomorphic line bundle, and let $D_1, \cdots, D_q \in |L|$ be hypersurfaces such that any two among them have no common components. Let $A \subset V$ be a hypersurface containing the non-normal-crossing singularities of $\sum_{j=1}^q D_j$. Let $f : M \to V$ be a differentiably non-degenerate meromorphic mapping. If $f$ satisfies the growth condition
\[
\liminf_{r \to \infty} \frac{r^2 \kappa(r)}{T_f(r, \omega)} = 0,
\]
where $\kappa$ is defined by (1), then
\[
\sum_{j=1}^q \Theta_f(D_j) \left( \frac{c_1(L)}{c_1(K_V)} \right) \leq \frac{1}{q} \left( \frac{c_1(K_V)}{c_1(L)} \right) + \frac{\gamma_{A,D}}{q} \left( \frac{c_1(L_A)}{c_1(L)} \right).
\]

Proof. By (39), we get
\[
\sum_{j=1}^q \limsup_{r \to \infty} \frac{m_f(r, \text{Sing}(D_j))}{T_f(r, \omega)} \leq \gamma_{A,D} \left( \frac{c_1(L_A)}{c_1(L)} \right).
\]
Note that $L_{D_1 + \cdots + D_q} = L^q$. By Theorem 6.2 we show the theorem. \hfill \Box

Corollary 6.4 (Shiffman, [24]). Let $L \to V$ be a positive line bundle, and let $D_1, \cdots, D_q \in |L|$ be hypersurfaces such that any two among them have no common components. Let $A \subset V$ be a hypersurface containing the non-normal-crossing singularities of $\sum_{j=1}^q D_j$. Let $f : \mathbb{C}^m \to V$ be a differentiably non-degenerate meromorphic mapping. Then
\[
\sum_{j=1}^q \Theta_f(D_j) \leq \frac{1}{q} \left( \frac{c_1(K_V)}{c_1(L)} \right) + \frac{\gamma_{A,D}}{q} \left( \frac{c_1(L_A)}{c_1(L)} \right).
\]

Proof. Replace $\omega$ by $c_1(L, h)$ in Theorem 6.3. \hfill \Box

Corollary 6.5. Let $L \to V$ be a positive line bundle, and let $D \in |L|$ be a hypersurface. If there is a hypersurface $A \subset V$ containing the non-normal-crossing singularities of $D$ such that
\[
\left( \frac{c_1(K_V)}{c_1(L)} \right) + \gamma_{A,D} \left( \frac{c_1(L_A)}{c_1(L)} \right) < 1,
\]
then every meromorphic mapping $f : M \to V \setminus D$ satisfying
\[
\liminf_{r \to \infty} \frac{r^2 \kappa(r)}{T_f(r, L)} = 0
\]
is differentiably degenerate, where $\kappa$ is defined by (1).
Corollary 6.6. Let $D \subset \mathbb{P}^n(\mathbb{C})$ be a hypersurface of degree $d_D$. If there is a hypersurface $A \subset \mathbb{P}^n(\mathbb{C})$ of degree $d_A$ containing the non-normal-crossing singularities of $D$ such that $d_A \gamma_{A,D} + n + 1 < d_D$. Then every meromorphic mapping $f : M \rightarrow \mathbb{P}^n(\mathbb{C}) \setminus D$ satisfying
\[ \liminf_{r \to \infty} \frac{r^2 \kappa(r)}{T_f(r, L_D)} = 0 \]
is differentiably degenerate, where $\kappa$ is defined by (1).

Proof. The conditions imply that
\[ \left[ c_1(K_{\mathbb{P}^n(\mathbb{C})}^*)/c_1([D]) \right] + \gamma_{A,D} \left[ c_1([A])/c_1([D]) \right] = \frac{n + 1}{d_D} + \gamma_{A,D} \frac{d_A}{d_D} < 1. \]
By Corollary 6.5 we see that the corollary holds. □

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