Wake Effects on Drift in Two-Dimensional Inviscid Incompressible Flows

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This investigation analyzes the effect of vortex wakes on the Lagrangian displacement of particles induced by the passage of an obstacle in a two-dimensional incompressible and inviscid fluid. In addition to the trajectories of individual particles, we also study their drift and the corresponding total drift areas in the Föppl and Kirchhoff potential flow models. Our findings, which are obtained numerically and in some regimes are also supported by asymptotic analysis, are compared to the wakeless potential flow which serves as a reference. We show that in the presence of the Föppl vortex wake some of the particles follow more complicated trajectories featuring a second loop. The appearance of an additional stagnation point in the Föppl flow is identified as a source of this effect. It is also demonstrated that, while the total drift area increases with the size of the wake for large vortex strengths, it is actually decreased for small circulation values. On the other hand, the Kirchhoff flow model is shown to have an unbounded total drift area. By providing a systematic account of the wake effects on the drift, the results of this study will allow for more accurate modeling of hydrodynamic stirring.

Keywords: drift; wakes; Föppl flow; Kirchhoff flow

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I. INTRODUCTION

When a body passes through an unbounded fluid, it induces a net displacement of fluid particles. The difference between the initial and final positions of a fluid particle is defined as the particle's "drift" \[4\], and plays an important role in characterization of the stirring occurring in multiphase flows \[9\] and due to swimming bodies \[33\]. Hereafter we will exclusively focus on flows with velocity fields stationary in a suitable steadily translating frame of reference, and will consider flows symmetric with respect to the flow centerline. Analysis of drift in time-dependent flows is more involved and some efforts in this direction have been made using methods of chaotic dynamics \[17, 36\].

Following the seminal study by Munk \[24\], the phenomenon of drift has recently received a lot of attention in the context of mixing in the oceans caused by swimming organisms \[18\]. However, most of the theoretical descriptions of stirring rely on irrotational flow models used to compute or estimate the drift (an exception to this is a recent study \[29\] focused on the Stokesian approximation). The goal of the present contribution is to understand the effect of vortex wakes on the drift in inviscid flows. This will be accomplished in the two-dimensional (2D) setting using a combination of careful numerical computations and mathematical analysis. The set-up of the problem is illustrated in Figure 1 with \((r, \theta)\) and \((r', \theta')\) representing, respectively, the polar coordinates in the fixed and moving frame of reference.

Drift has been investigated for over a century with the earliest work belonging to Maxwell \[5\] who showed that, when the passing object is a circular cylinder inducing a simple potential flow, then surrounding fluid particles follow trajectories in the form of "elastica" curves (a more modern account of this problem can be found in monograph \[23\]). For a given fluid particle, an elastica-shaped trajectory approaches a straight line parallel to the path of the moving cylinder for points far upstream and downstream, and exhibits a loop with a fore-and-aft symmetry (when the particle travels along this loop, the cylinder is underneath it). The historical origins and some other applications of elasticas are surveyed in \[21\]. Another major contribution to this area is due to Darwin who, in addition to particle trajectories, studied the problem of drift area and drift volume which are global quantities characterizing the particle drift in a given flow. Darwin's proposition \[8\], also referred to as a "theorem", is a key result relating the drift area or volume of a moving body to its added mass. Its utility consists in the fact that the latter quantity tends to be easier to evaluate for flows past objects with complex shape. There has been some debate \[1, 10, 34, 35\] concerning a rigorous proof of this result in its full generality which was centered on the evaluation method for conditionally convergent integrals. Connections between the Darwinian drift and the Stokesian drift, related to the wave motion, were explored in \[12\].

The concept of drift was recently generalized for the case of flows induced by propagating vortices (vortex rings) in \[7\]. Motivated by biofluid applications, a recent study \[29\] investigated the effects of vortex wakes on the drift induced by simple swimmers moving in the Stokes fluid. In the present study we provide a thorough account of the effects of different vortex wakes on the drift in inviscid flows. We will focus on 2D flows, because they offer simple solutions amenable to straightforward analysis, so that closed-form results can be obtained.

In our paper we begin in Section II by precisely defining the drift and the total drift area, and explaining how these quantities can be evaluated in a given flow. Then, in Section III, we introduce the different vortex flows considered in our study and identify their key parameters. Computational results are presented in Section IV together with a validation of the numerical approaches, whereas their discussion and a posteriori justification via asymptotic analysis are offered in Section V. Conclusions and outlook are deferred to Section VI.

II. DRIFT: DEFINITION AND CALCULATION

We will consider a circular cylinder of unit radius \((a = 1)\) passing through an incompressible inviscid fluid in a 2D unbounded domain \(\Omega\). The flow satisfies the continuity and momentum equations

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \quad \text{in } \Omega, \tag{1a}
\]

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \tag{1b}
\]
in which the fluid density is \( \rho = 1 \). The cylinder passes with its center along the \( x \)-axis from \( x = -\infty \) to \( x = \infty \) with constant unit speed. Hence, in the cylinder’s frame of reference there is a uniform stream at infinity such that \( \mathbf{u} \to U \hat{\mathbf{x}} \) as \( |x| \to \infty \), where \( U = -1 \) and \( \hat{\mathbf{x}} \) is the unit vector associated with the \( x \)-axis. In this frame of reference, the flow is steady, i.e., \( \mathbf{u} = \mathbf{u}(x) \), and satisfies the no through-flow boundary condition \( \mathbf{u} \cdot \mathbf{n} = 0 \) on the cylinder boundary \( \partial \Omega \), where \( \mathbf{n} \) is the unit normal vector. Euler system (1) is known to admit nonunique solutions and different such solutions will be discussed below.

Hereafter, without the risk of confusion, we will interchangeably use the vector and complex notation for various vector quantities. A point \((x, y) \in \Omega\) will be represented as \( \mathbf{x} = [x, y]^T \) or \( z = x + iy \), where \( i := \sqrt{-1} \) (the symbol “:=” defines the quantity on the left-hand side with the quantity on the right-hand side). The fluid velocity will be denoted \( \mathbf{u}(\mathbf{x}) = [u_x, u_y]^T \) or \( V(z) = (u_x - iu_y)(z) \), where \( u_x \) and \( u_y \) are the \( x \) and \( y \) components. Assuming that the velocity field is incompressible and irrotational, it will be expressed in terms of the complex potential \( W(z) = (\phi + i\psi)(z) \) as \( V(z) = dW/dz \), where \( \phi \) and \( \psi \) are, respectively, the scalar potential and the streamfunction.

In much the same way that Maxwell [5] and Darwin [8] studied the problem of drift, we will consider the trajectories and drifts of individual particles in the fluid as the cylinder passes. Let the initial position of the particle at \( t = 0 \) be \( \mathbf{x}_0 \) and \([x(t; \mathbf{x}_0), y(t; \mathbf{x}_0)]^T\) denote the corresponding particle trajectory. Then, the displacement, or drift, of the particle initially at \( \mathbf{x}_0 \) is defined as

\[
\xi(\mathbf{x}_0) := \int_{-\infty}^{\infty} u_x(x(t; \mathbf{x}_0), y(t; \mathbf{x}_0)) \, dt, \tag{2}
\]

where the horizontal velocity component \( u_x \) is given in the absolute frame of reference. Integral (2) is improper and the question of its convergence will be addressed further below. By changing the integration variable from time \( t \) to the polar angle \( \theta' \) in the moving frame of reference, cf. Figure 1, it can be transformed to an integral (still improper) defined over a finite interval \( \theta' \in [0, \pi] \) with the bounds corresponding to the position of the particle in front and behind the obstacle. Rewriting the velocity in the polar coordinate system in the moving frame of reference as \( \mathbf{u} = u_r \hat{\mathbf{r}}' + u_{\theta'} \hat{\mathbf{\theta}}' \), where

![Figure 1: Schematic of the problem indicating representative particle trajectories and the coordinate systems used.](image)
\{ \hat{r}', \hat{\theta}' \} are the two unit vectors, we have for the azimuthal component

\[ u_\theta(r', \theta') = r' \frac{d\theta'}{dt} \implies dt = \frac{r' d\theta'}{u_\theta(r', \theta')} \]

and (2) becomes

\[ \xi(x_0) := \int_0^\pi r' u_x(r', \theta') \frac{d\theta'}{u_\theta(r', \theta')} d\theta'. \]  

Form (4) is more convenient for some of the manipulations we will need to perform when deriving the drift in wakeless flow (Section III A).

The key quantity of interest in practical applications is the total drift area \( D \) representing the integral displacement of particles initially located on a line perpendicular to the path of the obstacle at an infinite upstream distance (Figure 1)

\[ D := 2 \int_0^\infty \xi(y_\infty) dy_\infty = \int_{-\infty}^{+\infty} \xi(\psi) d\psi, \]  

where \( y_\infty \) is the transverse coordinate of the particle’s position when \( t \to -\infty \) (with a slight abuse of notation, \( \xi \) may be equivalently considered a function of \( x_0, y_\infty \) or \( \psi \)). The two integrals in (5) are equal, because \( \psi \to y_\infty \) as \( x \to \infty \). The total drift area \( D \) involves two nested improper integrals (in expressions (2) and (5)). Whether this quantity is actually well-defined has been the subject of a debate [1, 10, 34, 35] with the conclusion that this is indeed the case, provided the order of integration is as used here, i.e., first with respect to the streamwise coordinate and then with respect to the transverse coordinate. On the other hand, reversing the order of integration will result in a conditionally convergent expression.

There are two ways to evaluate the total drift area \( D \). First, we can use a suitably transformed definition of formula (5) combined with the particle displacement given in (2). From the practical point of view, the most convenient way to evaluate the improper integral (2) is to set the particle positions \( x_0 \) at \( t = 0 \) and then obtain the trajectories by integrating the system

\[ \frac{dx(t)}{dt} = u(x(t)), \quad x(0) = x_0 \]  

forward and backward in time, i.e., for \( t \to \pm \infty \), for different \( x_0 \). Since the initial particle positions in formula (5) are given at infinity, they need to be transformed to positions with finite streamwise locations, e.g., \( x_0 = [0, y_0]^T \). Since for a particle on a given streamline, \( \psi \) is constant and equal to some \( C \), we have

\[ C = \psi(0, y_0) = \lim_{x \to \infty} \psi(x, y_\infty) = y_\infty. \]

Defining \( g(y_0) := \psi(0, y_0) = y_\infty \) as the map between the \( y \)-coordinates of the particle at \( x = 0 \) and at \( x = \infty \), we obtain

\[ \frac{dy_\infty}{dy_0} = g'(y_0), \]

so that (5) becomes

\[ D = 2 \int_1^\infty \xi(g(y_0)) g'(y_0) dy_0, \]

where the lower bound is now set to unity, because the particle on the streamline with \( \psi = 0 \) has the coordinate \( y_0 = 1 \) at \( x_0 = 0 \). We note that function \( g(y_0) \) will be different for different solutions of Euler system (1).
The second method to evaluate the total drift area is to use Darwin’s theorem [8] which stipulates that \( D = M \), where \( M \) is the added mass, and the fluid density is assumed equal to the unity. For our problem, the added mass is given by a line integral over the contour \( C \) which is the boundary of the largest region with closed streamlines

\[
M = \oint_C \phi n_x \, ds, \tag{10}
\]

where \( n_x \) is the \( x \)-component of the unit normal vector. In addition to the boundary of the obstacle, contour \( C \) also comprises the boundary of the recirculation region, if it is present in the flow. The reason is that, in obtaining relation (10), the divergence theorem cannot be applied on regions where singularities (point vortices) are present.

Alternatively, one can bypass evaluation of integral (10) by the application of Taylor’s added mass theorem [32]. If we consider the union of our cylinder and the recirculation region as a single “body” \( B \) in motion, this theorem allows us to compute the added mass in terms of the singularities within this region. Suppose that our composite body contains \( N \) sources and sinks with locations \( z_i \) and strength \( m_i \). In addition, it contains \( M \) doublets (or dipoles) with strength \( \mu_j \) and continuously distributed sources and sinks with area density defined by \( \sigma \). Then for irrotational flows, the generalized form of the added mass given in [19] is

\[
A_{\alpha 1} + B_{\alpha 1} + i(A_{\alpha 2} + B_{\alpha 2}) = 2\pi \rho \left[ \int_S \sigma_\alpha z \, dA + \sum_{i=1}^{N} m_{i\alpha} z_i + \sum_{j=1}^{M} \mu_{j\alpha} \right], \quad \alpha = 1, 2, \tag{11}
\]

where \( A \) is the added mass tensor and \( B \) a tensor representing the mass of the displaced fluid per unit area of the body with entries given by

\[
B_{\alpha\beta} = \rho \oint_C x_\beta n_\alpha \, ds, \quad \alpha, \beta = 1, 2 \tag{12}
\]

in which \( x_1 = x, \, x_2 = y, \, u_1 = u_x, \, u_2 = u_y \). For our problem, formula (11) simplifies quite significantly. In particular, since we are considering rectilinear motion in the \( x \)-direction of a body symmetric with the OX axis, we need only consider the element of the added mass tensor with \( \alpha, \beta = 1 \) so we may take the real part of (11) and drop these indices. Further, as there are no continuous sources or sinks and \( \rho = 1 \), we get for the added mass (now writing \( A = M \))

\[
M = 2\pi \Re \left[ \sum_{i=1}^{N} m_{i} z_i + \sum_{j=1}^{M} \mu_{j} \right] - B. \tag{13}
\]

In addition, since \( ds \) is an infinitesimal distance along the body, we have \( n_x \, ds = dy \). Thus, \( B \) can be simplified and interpreted as the area of the cylinder augmented by the area of the wake

\[
B = \oint_C x_n x \, ds = \oint_C x(y) \, dy. \tag{14}
\]

We remark that relation (13) can be interpreted as consisting of two parts: a “universal” part represented by the first term involving only the far-field expansion of the velocity field induced by the obstacle together with its vortex system and a second part characterizing the specific flow and represented by \( B \). An analogous decomposition of the total drift area was obtained in [29] for a swimmer in the Stokes flow. While all three approaches, involving definition formula (5), added-mass relation (10) and Taylor’s theorem (13)–(14), are equivalent as far as the evaluation of the total drift area is concerned, the first one offers additional insights in the form of the particle trajectories responsible for the observed drift.
III. MODEL PROBLEMS

In this Section we describe the three model flows we will consider in our study. In addition to the wakeless potential flow for which the questions of drift are well understood and which will serve as a reference, we will also investigate the Föppl and Kirchhoff flows which will be shown to have quite different properties. These two flows are often invoked as the possible inviscid limits of steady viscous Navier-Stokes flows [30]. For simplicity, in all three cases the cylinder radius and the free stream at infinity have unit values, $a = 1$ and $U = -1$.

A. Wakeless Potential Flow

In the frame of reference attached to the obstacle, this flow in defined by the complex potential

$$W(z) = - \left( z + \frac{1}{z} \right)$$

which does not involve any parameters. The flow field exhibits no separation and is characterized by symmetry with respect to both OX and OY axes. The streamline pattern and the velocity fields are illustrated in Figures 2a,b.

B. Föppl Flow

The Föppl vortex system [15] is a one-parameter family of solutions constructed by superimposing a pair of opposite-sign vortices with circulations $\Gamma > 0$ and $-\Gamma$ located symmetrically at $z_1 = x_1 + iy_1$, $y_1 > 0$, and $\overline{z}_1$, where the overbar denotes complex conjugation, on the flow with potential (15). The resulting potential of the Föppl flow is thus

$$W(z) = - \left( z + \frac{1}{z} \right) + \frac{\Gamma}{2\pi i} \log \left( \frac{z - z_1}{z - \overline{z}_1} \right) - \frac{\Gamma}{2\pi i} \log \left( \frac{z - \overline{z}_1}{z - z_1} \right).$$

The locus of equilibrium vortex locations, the so-called Föppl curve, is described by the algebraic relation

$$r_1^2 - 1 = 2r_1 y_1,$$

where $r_1 := \sqrt{x_1^2 + y_1^2}$. The circulation of the vortices is related to their position through

$$\Gamma = 2\pi \frac{(r_1^2 - 1)(r_1^4 - 1)}{r_1^2}.$$  

For a given circulation $\Gamma > 0$, the Föppl system is a limiting solution (as the vortex area goes to zero) of a family of Euler flows with finite-area vortex patches discovered by Elcrat et al. [14] (see also [26]). The Föppl system features a closed recirculation region with size growing with $\Gamma$. As is evident from (16), in the limit $\Gamma \to 0$ the wakeless potential flow from Section III A is recovered. The streamline pattern and the velocity field of a representative Föppl flow are illustrated in Figures 2c,d. The Föppl system has been successfully employed as a model in a number of studies concerning the stability and control of separated wake flows [22, 25, 27, 28, 31].

C. Kirchhoff Flow

The Kirchhoff flow is a manifestation of the free-streamline theory of the 2D ideal flows [20]. It features an object with two free streamlines in the upper and lower half-planes that separate the external fluid
from the region behind the object, called the cavity region, where the velocity is zero and the pressure is constant. Commonly, the object used for these types of flows is a flat plate, however, for consistency with the wakeless potential and the Föppl flows, we will consider here a 1st-order approximation of a circular cylinder presented in [2]. The Kirchhoff flow is interesting as an inviscid model, because it features an infinite wake and a finite drag.

We will first clarify the notation: variable $z$ denotes the physical plane we are interested in, where the circular cylinder is of unit radius centered at $(0,0)$ and the flow is moving from right to left, whereas variable $Z$ refers to the physical plane as used in [2], where the cylinder instead has a radius of approximately 1.77 and is centered at approximately $(1.38,0)$ with flow going in the opposite direction. We can define a map to switch between the two spaces

$$Z(z) := -1.770434824562303z + 1.377445608362303.$$  

(19)

The complex potential is defined as a modified Levi-Civita transformation [2]

$$W(\tau) = -\frac{(\tau - \frac{1}{\tau})^2}{4}$$  

(20)

where $\tau = \rho e^{i\sigma}$ and $0 \leq \rho \leq 1$, $-\pi/2 \leq \sigma \leq \pi/2$. Unlike the models described in Sections III A and III B, potential (20) is not given in terms of the variable in the physical space and additional transformations are needed, so that it can be evaluated at $z$ or $Z$. An intermediate map $\zeta(\tau)$ may be used to connect the $\tau$ and $Z$ planes

$$\zeta = \frac{dZ}{dW}$$  

(21)

and for a 1st-order approximation of a circular cylinder we have

$$\zeta(\tau) = \frac{1 + \tau}{1 - \tau} e^{-0.9426\tau + 0.0191\tau^3}.$$  

(22)

Then, using the chain rule, we may write

$$\frac{dZ}{d\tau} = \frac{dZ}{dW} \frac{dW}{d\tau}^{-1},$$  

(23)

where the first derivative factor is (22) and the second can be derived from (20). Thus, $Z(\tau)$ can be determined up to a constant through the integration

$$Z(\tau) = \int_{\tau_0}^{\tau} \frac{dZ}{dW} \frac{dW}{d\tau^2} d\tau' = \frac{1}{2} \int_{\tau_0}^{\tau} \left(1 + \tau'ight) \left(1 + \frac{1}{\tau'^2}ight) \left(\tau' - \frac{1}{\tau'}\right) e^{-0.9426\tau' + 0.0191\tau'^3} d\tau'$$  

(24)

where $\tau_0$ is an arbitrary constant. Integral (24) does not lend itself to analytical treatment, however, a generalized series expansion for the integrand was found up to $O(\tau^2)$ around $\tau = 0$, so that, after integration, we obtain

$$Z(\tau) = -\frac{1}{2} \left(c_1\tau^{-2} + c_2\tau^{-1} + c_3\log \tau + c_4\tau + c_5\tau^2\right) + Z_0,$$  

(25)

where $c_1 = 0.5$, $c_2 = 1.0574$, $c_3 = -0.55904738$, $c_4 = -0.8828122332$, $c_5 = 0.1113906656$ and $Z_0$ is some constant.

From (21) and (22), we can now compute the velocities in the $Z$-plane in terms of the $\tau$ variable

$$u_x(\tau) = \Re\left(\frac{1}{\zeta(\tau)}\right),$$  

(26a)

$$u_y(\tau) = -\Im\left(\frac{1}{\zeta(\tau)}\right).$$  

(26b)
Since we are interested in the flow in the direction opposite to the one in the $Z$-plane \[2\], we set $V(z) = (u_x - iu_y)(z) = (-u_x - iu_y)(\tau)$. In order to be able to evaluate velocities (26) at a given location $Z$ in the physical space, we need to invert map (25), i.e., find $\tau = Z^{-1}(z)$. This is done by applying Newton’s method to

$$F(\tau) = -\frac{1}{2} \left( c_1 \tau^{-2} + c_2 \tau^{-1} + c_3 \log \tau + c_4 \tau + c_5 \tau^2 \right) - Z = 0. \quad (27)$$

Once $\tau$ is found, the velocity at the required location can be computed using (26). The streamlines and velocity field of the Kirchhoff flow can be seen in Figures 2e and 2f.

**IV. RESULTS**

In this Section we compare the trajectories of individual particles, their drift and the corresponding total drift areas in the three flows introduced in the previous section. While, as reviewed below in Section IV A, these quantities can be determined analytically in the wakeless potential flow, they have to be computed numerically in the case of the Föppl and Kirchhoff flows, and the computational techniques are described and validated in Section IV B. Finally, the main results are presented in Section IV C.

**A. Lagrangian Trajectories and Drift in the Wakeless Potential Flow**

These classical results, recalled here for completeness, were derived by Maxwell \[5\] and were also surveyed in \[23\]. A key relation which makes this problem analytically tractable allows one to express the radial coordinate of the particle in the cylinder’s frame of reference $r'$ in terms of its azimuthal angle $\theta'$ with the streamfunction $\psi$ used as a parameter

$$r'(\theta') = \psi + \sqrt{\psi^2 + 4a^2 \sin^2 \theta'}. \quad (28)$$

Then, when expressed using the angle $\eta$ made by the tangent to the particle trajectory at a given point and the OX axis as the dependent variable and the arc-length $s$ as the independent variable, the equation governing the particle trajectories is of the form

$$\frac{d\eta}{ds} = 4 \frac{a^2}{\sqrt{1 - k^2 \sin^2 \theta}} \left( y - \frac{1}{2} \psi \right) \quad (29)$$

implying that the trajectories are examples of “elasticae”, a family of curves with a long history in mathematics \[21\]. The quantity $d\eta/ds$ represents the curvature of the trajectory and the connection with elasticae was first recognized by Milne-Thompson \[23\]. However, for our purposes, it is more convenient to start from equation (6) in which the independent variable is changed from $t$ to $\theta'$ as shown in (3). Then, combining the resulting equation with relation (28) and integrating we obtain

$$x(u) = \frac{a}{k} \left[ \left( 1 - \frac{1}{2} k^2 \right) u - E(u) \right], \quad (30a)$$

$$y(u) = \frac{a}{k} \left[ \frac{dk}{d\psi} + d\psi(u) \right]. \quad (30b)$$

where $k := 2a/\sqrt{\psi^2 + 4a^2}$ (in our case $a = 1$),

$$E(u) := \int_{\theta' - \pi/2}^{\theta' + \pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta,$$

$$d\psi(u) := \sqrt{1 - k^2 \sin^2 (\theta' - \pi/2)}$$
Figure 2: Streamlines (left column) and velocity field (right column) of the wakeless potential flow (top row), Föppl flow with $\Gamma = 8.84$ (middle row) and in the Kirchhoff flow (bottom row).
which are, respectively, an incomplete elliptic integral of the second type and a Jacobi elliptic function. The variable $u$ parameterizing trajectories (30) is defined as an incomplete elliptic integral involving the polar angle $\theta'$ (Figure 1)

$$u := \int_0^{\theta' - \pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta. \quad (31)$$

We note that the initial position of the particle is encoded in the value of the streamfunction $\psi$ appearing in the expression for $k$. The drift corresponding to $t \in (-\infty, \infty)$, cf. (2) is then obtained by taking the limit $\theta' \to \pi/2$ in (30a) which yields, after setting $a = 1$ and noting (7),

$$\xi_1(y_0) = \frac{2}{k} \left( 1 - \frac{1}{2} k^2 \right) K - E, \quad (32)$$

where

$$K = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta'}} d\theta', \quad E = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta'} d\theta'$$

are the complete elliptic integrals of the first and second type. Using Darwin’s theorem, the total drift volume can then be shown to be

$$D_1 = \pi. \quad (33)$$

These results will be illustrated in Section IV C.

B. Numerical Computation of Particle Trajectories, Drift and Total Drift Area in the Föppl and Kirchhoff Flows

Since explicit relations of the type (28) are not available for the Föppl and Kirchhoff flows, we need to resort to numerical computations in order to determine the particle trajectories, drift and the total drift area. The particle trajectories are computed as described in Section II by solving system (6) with the initial data $x_0 = [0, y_0]^T$, where $y_0 > 1$ is a parameter (we note that, when $y_0 = 1$ in Föppl flow, the particle is on the streamline connected to the stagnation point and the drift $\xi(1)$ is infinite). In the case of the Föppl flow the particle trajectories are additionally parameterized by the vortex circulation $\Gamma$. The velocity on the right-hand side of (6) is obtained, respectively, by complex-differentiating potential (16) and using expressions (26) in the two cases. System (6) is integrated for different values of $y_0$ and, in the case of the Föppl flow, $\Gamma$ using MATLAB routines ode23 and ode45 with adaptive adjustment of the time step. Numerical evaluation of the drift, given by an improper integral (2), is a subtle issue requiring judicious choice of the truncation $[-T, T]$ of the original unbounded interval $(-\infty, \infty)$. Questions concerning the partial drift, evaluated for finite times $t \in [-T, T]$, were investigated in [3, 10] where it was shown that such truncation of the integration domain leads to nontrivial corrections to the drift as defined in (2). In order to exclude these finite-time effects from the numerical integration, one has to make sure that $T$ is chosen sufficiently large. For Föppl flow, this is achieved by setting $T$ close to realmax, the largest positive floating-point number in the IEEE double-precision standard [16], which is of the order $O(10^{308})$ and then balancing the accuracy with the computational time by adjusting the relative and absolute tolerances, RelTol and AbsTol, in the routines ode23 and ode45. Owing to the adaptive adjustment of the time step employed in these routines, the total computational time required for a single particle trajectory does not typically exceed one minute on a state-of-the-art workstation even for the finest tolerances. This approach is validated by computing the particle trajectories $x(t; y_0)$ and the associated drift $\xi(y_0)$ numerically for the wakeless potential flow (obtained setting $\Gamma = 0$ in (16)) and then comparing them to the analytical expressions (30) and (32) (since these formulas involve special functions, care must be taken to enforce a required level of precision in the evaluation of these functions as well). The results obtained for a single trajectory with $y_0 = 2$ are
presented in Figure 3a, where we show a segment of the particle trajectory computed numerically and given by expression (30), and in Figure 3b in which we show the difference between the exact drift value \( \xi_1(2) = 2.011398641052742 \times 10^{-1} \) and its numerical approximation \( \hat{\xi}_1(2) \) for different fixed \( \text{RelTol} \) and \( \text{AbsTol} \). As is evident from Figure 3b, the error in the evaluation of the drift is rather small and decreases algebraically with the refinement of both \( \text{RelTol} \) and \( \text{AbsTol} \), Thus, in all subsequent calculations we will use routine \texttt{ode45} with \( \text{RelTol} = \text{AbsTol} = 10^{-13} \).

Unlike in the case of the wakeless and Föppl flow where \( T \) was allowed to extend close to \( \text{realmax} \), in Kirchhoff flow we have to restrict the truncation of the time axis to \( T = 10^3 \) which is due to the failure of Newton’s method applied to (27) to converge for such large values of \( t \). However, since the structure of the flow advecting the particles does not change much when \( |t| > T \), we will compensate for this by extrapolating the velocity for large times. Since, as will be shown below, the velocity field following the particle trajectory is for sufficiently large \( t \) a power-law function of time, this extrapolation will be performed using the formula

\[
h(t) = ct^{\beta},
\]

where \( c \in \mathbb{R} \) and \( \beta < 0 \), using 10 data points corresponding to the largest available times.

As regards evaluation of the total drift area \( D \), three different approaches can be used: definition formula (5), or more conveniently (9), added-mass formula (10) and Taylor’s theorem (13)–(14). In the first approach the parameter space \( y_0 \) is discretized in such a way that the relative variation of \( \xi(y_0) \) between two adjacent discrete values of \( y_0 \) would not exceed 1%. The function \( g(y_0) \) and its derivative needed in (9) are identified for the Föppl flow as follows

\[
g(y_0) = - \left( y_0 - \frac{1}{y_0} \right) + \frac{\Gamma}{2\pi} \left[ \log \left( \frac{\sqrt{x_1^2 + (y_0 + y_1)^2}}{\sqrt{x_1^2 + (y_0 - y_1)^2}} \right) \right]
+ \log \left( \frac{\left( \frac{x_1}{x_1 + y_0} \right)^2 + \left( y_0 - \frac{y_1}{x_1 + y_0} \right)^2}{\left( \frac{x_1}{x_1 + y_1} \right)^2 + \left( y_0 + \frac{y_1}{x_1 + y_1} \right)^2} \right),
\]

\[
g'(y_0) = - \left( 1 + \frac{1}{y_0^2} \right) + \frac{\Gamma}{2\pi} \left[ \frac{y_0 + y_1}{x_1^2 + (y_0 + y_1)^2} - \frac{y_0 - y_1}{x_1^2 + (y_0 - y_1)^2} \right]
+ \frac{y_0 - \frac{y_1}{x_1 + y_0} \right)^2}{\left( \frac{x_1}{x_1 + y_0} \right)^2 + \left( y_0 - \frac{y_1}{x_1 + y_0} \right)^2} \right) \right].
\]

Concerning the computation of the total drift area via Taylor’s theorem (13)–(14), the two Föppl vortices and their images inside the cylinder make the contributions \( m = \pm \frac{\Gamma}{2\pi} \) each, whereas the dipole at the origin contributes \( \mu = Ua^2 \). Therefore, after setting \( a = 1 \) and \( U = -1 \), equation (13) becomes

\[
M = 2\pi\Re \left( \frac{\Gamma}{2\pi t} z_1 - \frac{\Gamma}{2\pi t z_1} - \frac{\Gamma}{2\pi t} z_2 + \frac{\Gamma}{2\pi t z_2} - 1 \right) - B
= -2\pi + 2\Gamma \left( y_1 - \frac{y_1}{x_1^2 + y_1^2} \right) - B.
\]

Since it does not appear possible to find an analytic expression for \( B \) representing the area of the recirculation bubble, it has to be evaluated numerically using (14).

C. Comparison of Particle Trajectories, Drift and Total Drift Area in Flows with Different Wake Models

The particle trajectories corresponding to several different initial positions \( x_0 = [0, y_0]^T \) are shown in Figure 4 for the wakeless potential flow, the Föppl flow with different circulations \( \Gamma \) and for the
Kirchhoff flow (the wakeless potential flow is obtained as the Föppl flow with $\Gamma = 0$). The initial positions corresponding to the indicated cylinder locations are marked with circles, whereas crosses indicate the particle positions at the same instances of time in the different cases. First, we see that in the wakeless potential flow (Figure 4a) all the particle trajectories have the form of elastica curves symmetric with respect to the OY axis. The presence of the vortex wakes in the Föppl flows breaks the fore-and-aft symmetry of the trajectories and, for small values of $y_0$, spawns a second loop on the trajectory which becomes larger for increasing vortex circulations $\Gamma$. The presence of this secondary loop results from the fact that, for sufficiently small $y_0$, the transverse component $u_y$ of the particle velocity must change sign when the particle is flowing around the recirculation region. Multimedia versions of Figure 4 available on-line contain animations of particle trajectories in the different cases.

Next, in Figure 5a, we show the drift $\xi$ of the individual particles as a function of the circulation $\Gamma$ and, in Figure 5b, as a function of the initial distance $y_0$ from the horizontal axis. Since this is how it is often presented, the latter data is replotted in Figure 5c using the linear scaling with the drift $\xi$ marked on the horizontal axis and the vertical axis representing $y_\infty$, cf. (7). In Figure 5a we see that the dependence of the drift $\xi$ on the vortex circulation $\Gamma$ is not monotonous regardless of the initial position of the particle. Moreover, in a certain range of $\Gamma$ there are two initial positions $y_0$ such that the corresponding drift $\xi(y_0)$ is equal to the drift $\xi_1(y_0)$ in the wakeless flow. While for sufficiently large circulations the drift ultimately increases as compared to the wakeless flow (corresponding to $\Gamma = 0$), for small values of $\Gamma$ the drift is actually reduced. In other words, for every $y_0 > 1$ there exists a “critical” circulation $\Gamma_0 > 0$ such that the Föppl flow has the same drift $\xi$ as the wakeless flow. This critical circulation is a nonmonotonous function of the distance $y_0$ from the flow centerline. In addition to confirming these observations, Figure 5b shows that drift $\xi(y_0)$ is a decreasing function of $y_0$ which exhibits two distinct asymptotic regimes (see Section V for more details on this).

It turns out that, regardless of the initial position $x_0 = [0, y_0]^T$, in the Kirchhoff flow drift (2) is unbounded. This is evident from Figure 6 showing an extrapolation using formula (34) of the velocity component $u_x(t)$ following the particle trajectory for large positive and negative times. We observe that, while for positive times the asymptotic behavior is characterized by the exponent $\beta = -1.1172$, for negative times the exponent is $\beta = -0.5092$ implying that $u_x(t)$ is not in fact integrable. Although
Figure 4: Particle trajectories for different initial conditions \(x_0 = [0, y_0]^T\) in the wakeless potential flow (a), the Föppl flow with different circulations (b,c,d) and the Kirchhoff flow (e). The x’s represent the particle positions at unit time intervals, whereas the o’s correspond to the particle positions at \(t = 0\), at which the cylinder, recirculation bubble for Föppl flow and the cavity for Kirchhoff flow are also indicated. The total drift areas produced by the fluid displacements shown in figures (a) and (c) are approximately equal, cf. (9), even though the individual particles with the same initial locations have quite different trajectories.

For brevity in Figure 6 the data was shown for one trajectory only (corresponding to \(y_0 = 5\)), analogous results we also obtained for other trajectories. Thus, the drift data is not shown for the Kirchhoff flow in Figure 5.

Finally, in Figure 7, we show the dependence of the total drift area \(D\) on the vortex circulation \(\Gamma\) computed using the three methods discussed in Section IV B, all of which show excellent agreement. We
see that the total drift area exhibits a well-defined minimum which is a manifestation of the competing effects observed in Figure 5a. The smallest drift area $D = 2.93$ is achieved for $\Gamma = 1.97$, whereas for $\Gamma = 3.6$ drift area is approximately $D = \pi$, the same as in the wakeless potential flow, cf. (33). The particle trajectories corresponding to these two cases are shown in Figures 4b and 4c.

V. ASYMPTOTIC ANALYSIS

As was discussed in Section IV C, the drift of a particle in Föppl flow depends on two parameters, namely, the vortex circulation $\Gamma$ and the initial distance $y_0$ between the particle and the flow centerline.
In this section we derive expressions characterizing the drift when the parameters take some limiting values. The asymptotic study of the drift in the wakeless potential flow as \( y_0 \to 1 \) and \( y_0 \to \infty \) is presented in [4], and our approach will build on this analysis.

A first, trivial, observation is that in the limit \( \Gamma \to 0 \) the drift of the wakeless potential flow is obtained uniformly in \( y_0 \). Here we will consider the limit \( y_0 \to \infty \). Since the required transformations are rather complicated, requiring the use of symbolic algebra tools (Maple), for brevity below we will only highlight the key steps.

We start by taking the Taylor expansion of the velocity component \( u_x \) in equation (6) about the initial position of the particle \( x_0 = [0, y_0]^T \) and truncate it at the order \( \mathcal{O}(\|x - x_0\|^2) \). This is justified by the observation, cf. Figure 4, that for large \( y_0 \) the particle trajectories are close to being circular and in the proximity of \( x_0 \). Our goal will be to integrate this expansion with respect to time, cf. (2), but first we have to substitute for \( x(t) \) and \( y(t) \) and in the proximity of \( y_0 \) by the observation, cf. Figure 4, that for large initial position of the particle only . In the limit \( y_0 \to \infty \) the trajectories \( x(t) \) can be approximated with the solutions \( \dot{x}(t) = [\tilde{x}(t), \tilde{y}(t)]^T \) of system (6) in which the right-hand side is evaluated at \( x_0 \), i.e., \( d\tilde{x}(t)/dt = u([0, y_0]^T, t) \), which is written out as

\[
\begin{align*}
\frac{d\tilde{x}}{dt} &= \frac{t^2 - y_0^2}{(t^2 + y_0^2)^2} + \frac{\Gamma}{2\pi} \left[ -\frac{y_0 - y_1}{(-t - x_1)^2 + (y_0 - y_1)^2} + \frac{y_0 - y_1}{(-t - x_1)^2 + (y_0 - y_1)^2} \right. \\
&\quad \left. + \frac{y_0 + y_1}{(-t - x_1)^2 + (y_0 + y_1)^2} - \frac{y_0 + y_1}{(-t + x_1)^2 + (y_0 + y_1)^2} \right] - \frac{t - x_1}{t^2 + y_0^2} \\
\frac{d\tilde{y}}{dt} &= -\frac{2ty_0}{(t^2 + y_0^2)^2} + \frac{\Gamma}{2\pi} \left[ \frac{t - x_1}{(-t - x_1)^2 + (y_0 - y_1)^2} - \frac{t - x_1}{(-t + x_1)^2 + (y_0 - y_1)^2} \right. \\
&\quad \left. - \frac{1}{(-t - x_1)^2 + (y_0 + y_1)^2} + \frac{1}{(-t + x_1)^2 + (y_0 + y_1)^2} \right].
\end{align*}
\]

Relations (38a)–(38b) are integrated analytically for \( \tilde{x}(t) \) and \( \tilde{y}(t) \) and, before the resulting expressions are substituted in the series expansion of \( u_x \), they are expanded in a Taylor series with respect to \( \Gamma \) which is assumed small. Noting (18) and relations \( y_1 = (r_1^2 - 1)/(2r_1) \) and \( r_1 = \)
Figure 5: Dependence of drift $\xi$ on (a) the vortex circulation $\Gamma$ for initial particle positions $y_0 \in \{1.001, 1.002, \ldots, 1.01, 1.02, \ldots, 1.1, 1.2, \ldots, 2.0\}$ (larger $y_0$ corresponding to lower curves), (b) the initial distance $y_0$ and (c) the distance $y_\infty$ from the flow centerline measured at infinity, cf. (7), for the circulation values indicated in the legend.

$\sqrt{2\sqrt{x_1^2 - x_1^4} + 1 + 2x_1^2 - 1/\sqrt{3}}$, this expansion can be re-expressed only in terms of $x_1$, which is the downstream coordinate of the Föppl vortex. Finally, integrating the resulting expression from $t = -\infty$ to $t = \infty$ and keeping only the leading-order term in $y_0$, we obtain the following approximation to the drift

$$\xi = \frac{\pi}{2y_0^3} \left[1 + 64(x_1 + 1)^4 + 192((x_1 + 1)^5 + (x_1 + 1)^6)\right] + O((x_1 + 1)^7)$$

(39)

valid for $y_0 \to \infty$ and $x_1 \to -1$ (equivalently, $\Gamma \to 0$). As is evident from this relation, the presence of the Föppl vortices introduces a correction to the expression $\pi/(2y_0^3)$ characterizing the drift in the wakeless potential flow in the limit $y_0 \to \infty$ [4]. Asymptotic relation (39) is compared to the actual data for $\Gamma \to 0$ in Figure 8a and for $y_0 \to \infty$ in Figure 8b showing a very good agreement in both cases.
(a) $t < 0$, $\beta = -0.5092$.

(b) $t > 0$, $\beta = -1.1172$.

Figure 6: Behavior of the velocity component $u_x$ following the trajectory of the particle located at $y_0 = 5$ at $t = 0$ in Kirchhoff flow for large (a) negative and (b) positive times (solid line); the dashed line represents the power-law fit (34) with the exponent values indicated in the captions.

Figure 7: Total drift area $D$ in the Föppl flows as a function of the vortex circulation $\Gamma$ evaluated based on definition formula (9) (empty circles), added-mass formula (10) (crosses) and Taylor’s theorem (13)–(14) (dashed line).

Analysis of the drift in the presence of the Föppl vortices in the limit $y_0 \to 0$ is more complicated and is beyond the scope of the present study.
VI. DISCUSSION, CONCLUSIONS AND OUTLOOK

In this study we presented a comprehensive analysis, based on careful numerical computations supported in some regimes by asymptotic analysis, of the effects of vortex wakes on the Darwinian drift induced by steadily translating obstacles. We focused on the Föppl and Kirchhoff flows featuring, respectively, a closed and open wake, which were compared to the wakeless potential flow used as a reference. We also discussed three different approaches to the computation of the total drift area, with the method based on Taylor’s theorem leading to a decomposition of $D$ into a “universal” part and a “flow-specific” part, in analogy with the decomposition established in [29] for the Stokes flow.

The particle trajectories in Föppl and Kirchhoff flows are quite different (cf. Figures 4b-d and 4e). In Föppl flow for certain values of $\Gamma$ and $y_0$ the particle trajectories exhibit a secondary loop corresponding to the instant of time when the particle change direction to circumnavigate the recirculation bubble. An interesting, and perhaps somewhat unexpected, finding is that while for large values of circulation $\Gamma$ the presence of the recirculation region in Föppl flow increases the total drift area, an opposite effect occurs for smaller values of $\Gamma$ (Figure 5a). The increase of the total drift area for large $\Gamma$ can be understood by analyzing the particle trajectories in the context of changes to the flow topology. Inspection of Figure 4a, corresponding to the wakeless potential flow, reveals that the largest displacement occurs when the particle is close to one of the stagnation points (front or rear). The presence of the wake vortices in Föppl flow introduces another stagnation point (see Figure 2c) in the neighborhood of which particles can be trapped and dragged for a long time. This effect is illustrated in Figure 9 where we show several particle trajectories in the neighborhood of the separation point where the boundary of the recirculation zone meets the obstacle. Symbols on the trajectories mark positions at equal time intervals, indicating that the particles closer to the separation point are trapped there for a longer time. Passage near this separation point corresponds to the climb on the second loop in the trajectories shown in Figures 4b-d. There are some interesting similarities and differences with respect to the wake effects on the drift in Stokes flows reported in [29]. In both cases the drift of an individual particle decays as $y_0^{-3}$ when the particle’s position becomes large, cf. (39). This is a consequence of the fact that both the Föppl flow considered here and the Stokesian swimmer flow studied in [29] have a dipolar far-field representation (even though the spatial dimensions are different). On the other hand, in contrast to the behavior observed here, in the Stokes case a significant reflux (negative particle displacements) was observed resulting in a negative total drift area corresponding to large wake sizes.

In regard to Kirchhoff flow we demonstrated that drift $\xi$ of individual particles is in fact not bounded

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Figure 8: Dependence of drift $\xi$ on (a) the circulation $\Gamma$ for $y_0 = 1000$ and (b) the initial particle position $y_0$ for $\Gamma = 0.38023$; solid lines represent the actual data whereas the dashed lines correspond to asymptotic formula (39).
and, consequently, the total drift area is not defined either. This finding should not be surprising, given that Kirchhoff flow has an infinite open wake (and hence can be “seen” by the particles as a moving body of an infinite extent). We note that another instance in which an unbounded total drift volume was found was the Stokes flow past a spherical droplet [11]. Since like Kirchhoff flow and in contrast to the Stokesian swimmers analyzed in [29], this flow is characterized by a finite drag, we may by analogy conjecture that unbounded total drift area is a feature of steady flows in unbounded domains which exhibit a nonzero drag.

We expect that the results reported here may help improve the accuracy of modeling efforts concerning biogenic mixing, such as those reported in [18]. There is a number of open questions which may deserve further study concerning, for example, the drift induced by pairs or larger groups of moving obstacles (in the context of the potential flow theory, such flows can be studied using the formalism based on the Schottky-Klein function [6]), or obstacles with asymmetric wakes as were recently reported in [13]. The problem of identifying the shape of the obstacle which will produce a prescribed drift will lead to some interesting shape-optimization problems.

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