RANK-ONE PERTURBATION
OF TOEPLITZ OPERATORS AND REFLEXIVITY

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Abstract. It was shown that rank-one perturbation of the space of Toeplitz operators preserves 2-hyperreflexivity.

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1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators on $\mathcal{H}$.

It is well known that the space of trace class operators $\tau c$ is a predual to $\mathcal{B}(\mathcal{H})$ with the dual action $\langle A, f \rangle = tr(Af)$, for $A \in \mathcal{B}(\mathcal{H})$ and $f \in \tau c$. The trace norm in $\tau c$ will be denoted by $\| \cdot \|_1$. Denote by $F_k$ the set of operators of rank at most $k$. Every rank-one operator may be written as $x \otimes y$, for $x, y \in \mathcal{H}$, and $(x \otimes y)z = \langle z, y \rangle x$ for $z \in \mathcal{H}$. Moreover, $tr(T(x \otimes y)) = \langle Tx, y \rangle$.

Let $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a subspace (when we write subspace we mean a norm closed linear manifold). By $d(T, \mathcal{M})$ we will denote the standard distance from an operator $T$ to a subspace $\mathcal{M}$, i.e., $d(T, \mathcal{M}) = \inf\{\|T - M\| : M \in \mathcal{M}\}$. It is known that when $\mathcal{M}$ is weak* closed $d(T, \mathcal{M}) = sup\{|tr(Tf)| : f \in \mathcal{M}_\perp, \|f\|_1 \leq 1\}$, where $\mathcal{M}_\perp$ denotes the preannihilator of $\mathcal{M}$.

Recall that the reflexive closure of a subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is given by

$$\text{ref } \mathcal{M} = \{ T \in \mathcal{B}(\mathcal{H}) : Tx \in [\mathcal{M}x] \text{ for all } x \in \mathcal{H} \},$$

where $[\cdot]$ denotes the norm-closure. A subspace $\mathcal{M}$ is called reflexive if $\mathcal{M} = \text{ref } \mathcal{M}$. Due to Longstaff [14] we know that when $\mathcal{M}$ is a weak* closed subspace of $\mathcal{B}(\mathcal{H})$, then $\mathcal{M}$ is reflexive if and only if $\mathcal{M}_\perp$ is a closed linear span of the set of all operators of rank one contained in $\mathcal{M}$ (i.e., $\mathcal{M}_\perp = [\mathcal{M}_\perp \cap F_1]$). A subspace $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called $k$-reflexive if $\mathcal{M}^{(k)} = \{ M^{(k)} : M \in \mathcal{M} \}$ is reflexive in $\mathcal{B}(\mathcal{H}^{(k)})$, where
proved that a weak* closed subspace \( M \subset H \) is \( k \)-reflexive if and only if \( M \) is a closed linear span of rank-\( k \) operators contained in \( M \) (i.e., \( M = [M \cap F_h] \)).

In [2] Arveson defines an algebra \( A \) as hyperreflexive if there is a constant \( a \) such that \( d(T,A) \leq a \sup\{||P^{-1}TP|| : P \in \text{Lat} A \} \) for all \( T \in B(H) \). In [11] this definition was generalized to subspaces of operators. A subspace \( M \subset B(H) \) is called hyperreflexive if there is a constant \( a \) such that

\[
d(T,M) \leq a \sup\{||Q^{-1}TP|| : P, Q \text{ are projections and } Q^{1/2}MP = 0\}
\]

for all \( T \in B(H) \). As it was shown in [12] the supremum on the right hand side is equal to \( \sup\{||T,g \otimes h|| : g \otimes h \in M \cap F_k, ||g \otimes h||_1 \leq 1\} \).

Recall after [10] the definition of \( k \)-hyperreflexivity. Let \( M \subset B(H) \) be a subspace. For any \( T \in B(H) \) denote

\[
\alpha_k(T,M) = \sup\{|\text{tr}(Tf)| : f \in M \cap F_k, ||f||_1 \leq 1\}.
\]

A subspace \( M \) is called \( k \)-hyperreflexive if there is a \( a > 0 \) such that for any \( T \in B(H) \) the following inequality holds:

\[
d(T,M) \leq a \alpha_k(T,M).
\]

Let \( \kappa_k(M) \) be the infimum of the collection of all constants \( a \) such that inequality (1.1) holds, then \( \kappa_k(M) \) is a constant of \( k \)-hyperreflexivity. Operator \( T \) is \( k \)-hyperreflexive if the WOT closed algebra generated by \( T \) and identity is \( k \)-hyperreflexive.

When \( k = 1 \) the definition above coincides with the definition of hyperreflexivity and the letter \( k \) will be omitted.

2. Reflexivity of Perturbated Toeplitz operators

Let \( T \) be the unit circle on the complex plane \( \mathbb{C} \). Denote \( L^2 = L^2(\mathbb{T}, m) \) and \( L^\infty = L^\infty(\mathbb{T}, m) \), where \( m \) is the normalized Lebesgue measure on \( \mathbb{T} \). Let \( H^2 \) be the Hardy space corresponding to \( L^2 \) and \( P_{H^2} \) be a projection from \( L^2 \) onto \( H^2 \). For each \( \phi \in L^\infty \) we define \( T_\phi : H^2 \rightarrow H^2 \) by \( T_\phi f = P_{H^2}(\phi f) \) for \( f \in H^2 \). Operator \( T_\phi \) is called a Toeplitz operator and \( T \) will denote the space of all Toeplitz operators.

The unilateral shift \( S \) can be realized as the multiplication operator by independent variable \( T_z \). Moreover, \( T = \{T_\phi : \phi \in L^\infty\} = \{A : T^*_z AT_z = A\} \) ([9, Corollary 1 to Problem 194]). Hence \( T \) is weak* closed.

Let \( \{e_j\} \in \mathbb{N} \) be the usual basis in \( H^2 \). Denote by \( M_{lm} \) the subspace \( T + \mathbb{C}(e_l \otimes e_m) \). In [4, Theorem 3.1] the authors proved that the space of all Toeplitz operators is not reflexive but it is 2-reflexive. We will show that the subspace \( M_{lm} \) has the same properties.

**Proposition 2.1.** The subspace \( M_{lm} \) is not reflexive but it is 2-reflexive.

**Proof.** Notice that \( (M_{lm})_\perp = T_\perp \cap (e_l \otimes e_m)_\perp \). Since \( T_\perp \) contains no nonzero rank-one operators, then \( M_{lm} \) is not reflexive.
Proposition 2.5. Subspace we have that by Arveson in [1, Proposition 5.2], which has the property that for any \( T \) operators is hyperreflexive. Moreover, the space of all Toeplitz operators

\[
\mathcal{T}_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \ldots \},
\]

where \( S \) is the unilateral shift. Therefore,

\[
(\mathcal{M}_tm)_\perp = \text{span}\{e_i \otimes e_j - Se_i \otimes Se_j : i, j = 1, 2, \ldots, (i, j) \neq (l, m) \text{ and } (i + 1, j + 1) \neq (l, m)\}.
\]

Hence \( \mathcal{M}_tm \) is 2-reflexive. \( \square \)

Recall after [5] the following definition.

**Definition 2.2.** Subspace \( \mathcal{M} \subset B(\mathcal{H}) \) has property \( \mathcal{A}_{1/k} \) if \( \mathcal{M} \) is weak* closed and for any weak* continuous functional \( \phi \) on \( \mathcal{M} \) there is \( g \in F_k \) such that \( \phi(M) = tr(Mg) \) for \( M \in \mathcal{M} \).

**Proposition 2.3.** The subspace \( \mathcal{M}_tm = \mathcal{T} + \mathbb{C}(e_l \otimes e_m) \) has property \( \mathcal{A}_{1/4} \).

**Proof.** Let \( t \in \tau \). Since \( \mathcal{T} \) has property \( \mathcal{A}_{1/2} \) ([10, Proposition 4.1]), there is \( f \in F_2 \) such that \( (t - f) \in \mathcal{T}_\perp \). If \( (t - f) \in (\mathbb{C}e_l \otimes e_m)_\perp \), then \( (t - f) \in (\mathcal{M}_tm)_\perp \). If

\[
(t - f) \notin (\mathbb{C}e_l \otimes e_m)_\perp, \text{ then } (t - f - \lambda e_l \otimes e_m + \lambda e_{l+1} \otimes e_{m+1}) \in (\mathcal{M}_tm)_\perp,
\]

where \( \lambda = P_{\mathbb{C}e_l}(t - f)P_{\mathbb{C}e_m} \) and \( P_{\mathbb{C}e} \) denotes the orthogonal projection on \( \mathbb{C}e \). So \( \mathcal{M}_tm \) has property \( \mathcal{A}_{1/4} \). \( \square \)

In [13] Larson proved that if \( \mathcal{M} \) is \( k \)-reflexive, then any weak* closed subspace \( \mathcal{L} \subset \mathcal{M} \) is \( k \)-reflexive if and only if \( \mathcal{M} \) has property \( \mathcal{A}_{1/k} \). It follows immediately from Proposition 2.1 and Proposition 2.3 that:

**Corollary 2.4.** Every weak*-closed subspace of \( \mathcal{M}_tm = \mathcal{T} + \mathbb{C}(e_l \otimes e_m) \) is 4-reflexive.

On the other hand, due to [8] we know that the algebra of analytic Toeplitz operators is hyperreflexive. Moreover, the space of all Toeplitz operators \( \mathcal{T} \) is \( 2 \)-hyperreflexive and \( \kappa_2(\mathcal{T}) \leq 2 \) (see [10,15]). We will show that the subspace \( \mathcal{M}_tm \) is \( 2 \)-hyperreflexive. In the proof we will use the projection \( \pi : B(H^2) \to \mathcal{T} \) constructed by Arveson in [1, Proposition 5.2], which has the property that for any \( A \in B(H^2) \) the operator \( \pi(A) \) belongs to the weak* closed convex hull of the set \( \{ T_n^* A T_m : n \in \mathbb{N} \} \).

**Proposition 2.5.** Subspace \( \mathcal{M}_tm = \mathcal{T} + \mathbb{C}(e_l \otimes e_m) \) is 2-hyperreflexive with constant \( \kappa_2(\mathcal{M}_tm) \leq 2 \).

**Proof.** Let \( A \in B(H^2) \). For \( \lambda \in \mathbb{C} \) define \( A_\lambda = A - \lambda e_l \otimes e_m \). Notice that for any \( \lambda \in \mathbb{C} \)

\[
d(A, \mathcal{M}_tm) \leq \| A - \pi(A) - \lambda e_l \otimes e_m \| = \| A_\lambda - \pi(A_\lambda) \|.
\]

Since the space of Toeplitz operators \( \mathcal{T} \) is 2-hyperreflexive with constant at most 2, we have that

\[
d(A_\lambda, \mathcal{T}) \leq \| A_\lambda - \pi(A_\lambda) \| \leq 2\kappa_2(A_\lambda, \mathcal{T}) \quad \text{for details see [10]}.\]
To complete the proof it is enough to show that for any \( A \in \mathcal{B}(H^2) \) there is \( \lambda \in \mathbb{C} \) such that
\[
\alpha_2(A_\lambda, T) = \alpha_2(A, \mathcal{M}_{lm}).
\] (2.1)

Note that
\[
\alpha_2(A_\lambda, T) = \sup \{|tr(A_\lambda t)| : 2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k}, k \geq 1, i, j = 0, 1, 2, \ldots \}.
\]
If this supremum is realized by \( 2t = e_i \otimes e_j - e_{i+k} \otimes e_{j+k} \) for \( (i, j) \neq (l, m) \) and \( (i+k, j+k) \neq (l, m) \), then equality (2.1) holds. So, it is enough to consider the case when
\[
\alpha_2(A_\lambda, T) = \sup \{|tr(A_\lambda t)| : 2t = e_l \otimes e_m - e_{l+k} \otimes e_{m+k}, k \geq \min\{-l, -m\}\} = \sup \{|a_{lm} - a_{l+k,m+k}| : k \geq \min\{-l, -m\}\}.
\]
Suppose that \( \alpha_2(A, \mathcal{M}_{lm}) = \beta > 0 \). Note that for any \( \lambda \) we have \( \beta \leq \alpha_2(A_\lambda, T) \).
If we choose \( \lambda = a_{lm} - a_{l+1,m+1} \), then
\[
\alpha_2(A_\lambda, T) = \sup \{|a_{l+1,m+1} - a_{l+k,m+k}| : k \geq \min\{-l, -m\}\} \leq \beta.
\]
Hence \( \alpha_2(A_\lambda, T) = \alpha_2(A, \mathcal{M}_{lm}) \), which completes the proof. \( \square \)

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