Wave functions of SU(3) pure gauge glueballs on the lattice

Jian Liang,1,* Ying Chen,1† Wei-Feng Chiu,1 Long-Cheng Gui,2 Ming Gong,1 and Zhaofeng Liu 1
1 Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
2 Department of Physics, Hunan Normal University, Changsha 410081, China

The Bethe-Salpeter wave functions of SU(3) pure gauge glueballs are revisited in this study. The ground and the first excited states of scalar and tensor glueballs are identified unambiguously by using the variational method on the basis of large operator sets. We calculate their wave functions in the Coulomb gauge and use two lattices with different lattice spacings to check the discretization artifacts. For ground states, the radial wave functions are approximately Gaussian and the size of the tensor is twice as large as that of the scalar. For the first excited states, the radial nodes are clearly observed for both the scalar and the tensor glueballs, such that they can be interpreted as the first radial excitations. These observations may shed light on the theoretical understanding of the inner structure of glueballs.

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I. INTRODUCTION

In quantum chromodynamics, gluons have strong interactions with each other and can make a sort of hadron states, called glueballs, which are different from the conventional mesons that are made up of quarks and described by the quark model. The conventional $q\bar{q}$-mesons can be viewed as bound states of valence quarks and antiquarks. However for glueballs, there is not yet a reliable phenomenological description of their intrinsic degrees of freedom. The simplest low-energy color singlet systems made up of gluons are always conjectured as two-gluon glueballs, whose $J^{PC}$ quantum numbers are expected to be $0^{++}$, $2^{++}$, $0^{+-}$, etc. (If one considers the fact that the gluons are massless gauge bosons, $1^{++}$ quantum number will not appear for the two-gluon glueball. All quantum numbers are permitted for multi-gluon systems). These arguments seem in qualitative agreement with the observation of lattice QCD calculation of the glueball spectrum [1–4], where the lowest-lying states are $0^{++}$, $2^{++}$, $0^{+-}$ glueballs in the order of their mass levels from below.

As for the gluon dynamics inside glueballs, there are various effective approaches such as the bag model [5], QCD in Coulomb gauge [6, 7], and the potential models [8–10]. The potential models depict a glueball as a bound state of two or more constituent gluons through a confining interacting potential. Even though still controversial, the potential models with proper theoretical assumptions can reproduce the glueball spectrum from lattice QCD.

Apart from the spectrum, some other static properties of glueballs, such as the Bethe-Salpeter (BS) wave functions, can also be investigated through lattice QCD studies, from which one can infer some qualitative information on the size and the spatial profile of a glueball. For a glueball dominated by a two-gluon component, the BS wave function is defined through the matrix element of a two-gluon operator $A_\mu(x)A_\nu(y)$ between the glueball state and the vacuum, which can be derived from the lattice QCD calculation of relevant two-point functions. This kind of wave function may reflect the spatial structure of a hadron to some extent. Since the two-gluon operator $A_\mu(x)A_\nu(y)$ is obviously gauge variant, the calculation should be performed in a fixed gauge, for example, the Coulomb gauge. The pioneering study in this section was carried out for the SU(2) pure gauge glueballs [11]. The similar study in the SU(3) case can be found in Ref. [12].

There are actually several phenomenological studies [13, 14] devoted to understand the structure and the gluonic dynamics within glueballs using the BS wave functions obtained from lattice QCD calculations, where the C-parity states are taken as mainly two-gluon states and the interacting potential between the constituent gluons is extracted. However, the previous works focus only on ground states and the results are not precise enough, a more complete and careful analysis on the BS wave functions of both the ground and the excited states is in demand and is actually the aim of this work. It is known that the key point in the calculation of the above matrix elements is to identify the glueball state unambiguously. In order for this, we adopt the sophisticated technique used in the glueball spectrum study [1–3]. This technique implements the variational method based on a large operator set for each quantum number of glueballs, through which the ground and the first excited states can be well determined. Finally, the BS wave functions of both the ground and the first excited glueball states are derived to a high precision. We hope our results can provide more information to the understanding of the nature of glueballs.

This article is organized as follows: Sec. II gives a detailed definition of two-gluon operators on the lattice and Sec. III are the numerical details of the calculation of BS wave functions, where the parameterizations of these wave functions and their physical implications are discussed. A brief summary is given in Sec. IV.
II. TWO-GLUON OPERATORS

It is known that the Bethe-Salpeter equation [13] provides a relativistic description of a two-body system where the relativistic Bethe-Salpeter (BS) wave function of a bound state can then be defined. For a two-gluon glueball state $|G\rangle$ in its rest frame, the BS wave function is expressed as [11]:

$$\phi_G(r) = \langle 0 | \sum_{\mu, \nu} \alpha(\mu, \nu) \sum_{|\vec{r}|=r} Y_{lm}(\hat{r}) A_\mu(\vec{x}) A_\nu(\vec{x} + \vec{r}) |G\rangle,$$

(1)

where $\hat{r}$ stands for the spatial orientation of $\vec{r}$, the summations on $\alpha(\mu, \nu)$ and $|\vec{r}|$ guarantee the right quantum number of the state $|G\rangle$, and the gauge field $A_\mu$ acts as the creation operator for a gluon. However, the fundamental gluonic variables are the gauge links $U_\mu(x)$ instead of $A_\mu(x)$ on a lattice, such that one has to build the lattice counter part of the two-gluon operator $O_{A,\mu\nu}(x, \vec{r}) = A_\mu(\vec{x}) A_\nu(\vec{x} + \vec{r})$. The most straightforward way to do this is to make use of the relation

$$U_\mu(x) = e^{iag A_\mu(x)}, \quad U_\mu^\dagger(x) = e^{-iag A_\mu(x)},$$

(2)

so that the gauge field $A_\mu(x)$ is obtained as

$$A_\mu(x) \sim [U_\mu(x) - U_\mu(x)^\dagger] (1 + O(a^2)).$$

(3)

through the classical small-$a$ expansion of $U_\mu(x)$. We denote this kind of derivation of the two-gluon operator as $O_{A,\mu\nu}$. In Ref. [11] they construct the operator $A_\mu^\dagger A_\nu$ from gauge links directly rather and argue that in this way the possible mixing with the flux states can be minimized. Although flux states do not appear in this work (see below), we also follow their formula to build our second kind of operator $O_A^2$ under $SU(3)$ gauge to to check the possible difference,

$$O_{A,\mu\nu}(x, \vec{r})$$

$$= ReTr[U_\mu^\dagger(x) U_\nu(x + \vec{r})] - \frac{1}{3} ReTr[U_\mu(x)] \cdot ReTr[U_\nu(x + \vec{r})]$$

$$= (ag)^2 A_\mu(x) A_\nu(x + \vec{r}) + O(a^4).$$

(4)

However, the classical expansion of $U_\mu(x)$ is not justified due to the tadpole diagrams if the quantum effects are considered, and thereby the high order lattice artifacts is not well under control. In view of this fact, we propose an alternative way to define $A_\mu(x)$ from $U_\mu(x)$ through a non-linear derivation,

$$A_\mu(x) \sim \text{Ln}[U_\mu(x)].$$

(5)

For the $SU(3)$ element $U_\mu(x)$, there exists an unitary matrix $R$ satisfying

$$R^\dagger U R = \text{diag}(\lambda_1, \lambda_2, \lambda_3),$$

(6)

with $\lambda_i, i = 1 \sim 3$ being the 3 eigenvalues of matrix $U$. So for each $U_\mu(x)$ on each configuration, we first diagonalize it and obtain the correspond $R$ and the eigenvalues $\lambda_i, i = 1, 2, 3$, and then define the gauge field $A_\mu(x)$ as

$$A_\mu(x) \sim R \cdot \text{diag}(\log(\lambda_1), \log(\lambda_2), \log(\lambda_3)) \cdot R^\dagger,$$

(7)

so $O_A^3$ is achieved.

Within the lattice QCD framework, the continuum Lorentz group becomes the discrete octahedral point group $O$. The irreducible representations (irreps) of this group are known as $A_1$, $A_2$, $E$, $T_1$ and $T_2$, whose lowest continuum spin correspondences are $J = 0, 3, 2, 1, 2$, respectively and higher spin components in each of these irreps will disappear in the continuum limit. To be specific, the quantum number of the scalar glueball is given actually by the $A_1^{++}$ irreps on the lattice, while that of the tensor glueball is split into $E^{++}$ and $T_2^{++}$. In order to assign a proper quantum number to the two-gluon operators mentioned above, we sum over all the symmetrical spatial displacements $\vec{r}$ and properly combine spatial orientations $\mu$. Explicit forms are as follows:

$$O_{A_1}^{A_1^{++}}(r) = \sum_{|\vec{r}|, x, \mu} A_\mu(\vec{x} + \vec{r}) A_\mu(\vec{x}), \quad \mu = 1 \sim 3$$

$$O_{A_1}^{E_+^{++}}(r) = \sum_{|\vec{r}|, x} \frac{1}{\sqrt{2}} (A_1(\vec{x} + \vec{r}) A_1(\vec{x}) - A_2(\vec{x} + \vec{r}) A_2(\vec{x}))$$

$$O_{A_1}^{E_+^{++}+}(r) = \sum_{|\vec{r}|, x} \frac{1}{\sqrt{6}} (2A_3(\vec{x}+\vec{r}) A_3(\vec{x}) - A_1(\vec{x} + \vec{r}) A_1(\vec{x})$$

$$- A_2(\vec{x} + \vec{r}) A_2(\vec{x}))$$

$$O_{A_1}^{T_2^{++}}(r) = \sum_{|\vec{r}|, x} \frac{1}{\sqrt{2}} (A_3(\vec{x} + \vec{r}) A_3(\vec{x}) + A_3(\vec{x} + \vec{r}) A_3(\vec{x}))$$

$$O_{A_1}^{T_2^{++}}(r) = \sum_{|\vec{r}|, x} \frac{1}{\sqrt{2}} (A_1(\vec{x} + \vec{r}) A_3(\vec{x}) + A_2(\vec{x} + \vec{r}) A_3(\vec{x}))$$

$$O_{A_1}^{T_2^{++}}(r) = \sum_{|\vec{r}|, x} \frac{1}{\sqrt{2}} (A_1(\vec{x} + \vec{r}) A_3(\vec{x}) + A_2(\vec{x} + \vec{r}) A_3(\vec{x}))$$

(8)

Note that the dimensions of the irreps $A_1$, $E$, and $T_2$ are 1, 2, and 3, respecitively and the time coordinate in the above 5 equations are omitted for convenience.

III. NUMERICAL DETAILS

We generate the gauge configurations on two anisotropic lattices using the tadpole-improved gauge action [1]. The lattice sizes are $L^3 \times T = 8^3 \times 96$ and $L^3 \times T = 12^3 \times 144$, respectively, and the temporal lattice spacing $a_t$ is made much smaller than the spatial one $a_s$ with the anisotropy $\xi \equiv a_s/a_t = 5$. Thus we can obtain a higher resolution of hadron correlation functions in the temporal direction, which helps us to handle glueball states whose signal-to-noise ratios damp rapidly in time. The relevant input parameters are listed in Tab. [11] where the $a_s$ values are determined through the static potential
with the scale parameter $r_0^{-1} = 410(20) \text{MeV}$. There are 5000 gauge configurations generated for each lattice. The volumes of the two lattices are both about $1.7 fm$, which has been tested to be large enough for glueballs [3]. The BS wave function in the rest frame of a glueball state can be derived by calculating the following two-point functions of the two-gluon operator $O_A(t, r)$ and an operator $O_B$ which creates glueball states with a specific quantum number,

$$C_2(t, r) = \langle 0|O_A(t, r)O_B(0)|0 \rangle$$

$$= \sum_n \frac{1}{2m_n} \langle 0|O_A(r)\langle n|O_B(0)e^{-m_n t}$$

$$= \sum_n \phi_n(r)e^{-m_n t} \tag{9}$$

where $m_n$ is the mass of the $n$-th state, and $\phi_n(r)$ is its BS wave function by the definition in Eq. (1) (up to an irrelevant normalization factor). Since the two-gluon operators $O_A(t, r)$ are gauge variant objects, we need to calculate the two-point functions $C_2(t, r)$ in a fixed gauge. So in the practical calculation, we first fix all the gauge configurations to the Coulomb gauge, in which the BS wave functions are usually assumed to have connection with the wave functions of potential models. In this work we intend to obtain the wave functions of both the ground and the first excited glueball states, therefore the key question is to identify a specific state unambiguously enough. For this purpose, we adopt the sophisticated techniques applied in the calculation of the glueball spectrum to construct the optimal glueball operators $O_{\text{opt}}$ which couple to specific states dominantly. The procedure is outlined as follows.

As is mentioned above that the quantum numbers of glueballs are realized through the irreps $A_1, A_2, E, T_1$, and $T_2$ of the lattice point group $O$. Therefore, we first build the same 10 prototypes of Wilson loops as those in Refs. [2, 3], based on which several steps of smearing schemes are applied to obtain more interpolation operators for glueballs. We perform the 24 spatial operations of the $O$ group to these Wilson loops and linearly combine them to realize the irreps of $A_1, E$ and $T_2$. As such we obtain 24 different operators for each of the quantum numbers $R = A_{1}^{++}, E^{++}$, or $T_{2}^{++}$. Finally, we implement the variational method to get the optimal operators by solving the generalized eigenvalue problem. To be specific, for a given quantum number $R$, we first calculate the $24 \times 24$ correlation matrix $C_{\alpha \beta}^{R}(t)$ through the operator set $\{\phi_{\alpha}^{R}, \alpha = 1 \rightarrow 24\}$,

$$C_{\alpha \beta}^{R}(t) = \langle 0|\phi_{\alpha}^{R}(t)\phi_{\beta}^{R}(0)|0 \rangle, \tag{10}$$

and then solve the generalized eigenvalue problem

$$C^{R}(t_0)V_{i}^{R} = e^{-m_{i}^{R}t}C(0)V_{i}^{R}, \tag{11}$$

to obtain the eigenvector $V_{i}^{R}$ for the $i$-th eigenvalue $e^{-m_{i}^{R}t}$. In this work we choose $t_0 = 1$. The eigenvector $V_{i}^{R}$ yields the combination coefficients for the optimal operator,

$$O_{\text{opt},i}^{R} = \sum_{\alpha=1}^{24} V_{i}^{R}(\alpha) \cdot \phi_{\alpha}^{R}. \tag{12}$$

It has been verified in the calculation of the spectrum that $O_{\text{opt},i}^{R}$ is highly optimized and its correlation function is saturated by the $i$-th state even at the beginning time slice. In other words, the correlation function of $O_{\text{opt}}^{R}$ can be expressed as

$$C_{2, \text{opt}}(t) = \langle O_{\text{opt}}^{R}(t)O_{\text{opt}}^{R}(0) \rangle \approx e^{-m_{R}t}, \tag{13}$$

with the normalization $C_{\text{opt},2}(0) = 1$. The effective mass plateaus of the ground and the first excited states of the scalar and tensor glueballs are all plotted in Fig. [1] where one can see that effective mass plateau of each channel almost starts from the first time slice. The masses of the ground and the first excited glueball states in the $A_1^{++}$, $E^{++}$, and $T_2^{++}$ channels are given in Tab. [1] in the physical units through the scale parameter $r_0^{-1} = 410(20) \text{MeV}$. The mass values are consistent with previous works [2, 3]. It is also seen that the masses of the $A_1^{++}$ glueballs have obvious lattice spacing dependence, while the masses of the $E^{++}$ and $T_2^{++}$ glueballs depend little on the lattice spacing. The coincidence of the masses of the $E^{++}$ and $T_2^{++}$ glueballs at the same lattice spacing implies that the rotational symmetry breaking effects are not important on the lattices we are using.

By using this kind of optimal glueball operators, the two-point function $C_2(t, r)$ can be simplified as

$$C_{2}^{R}(t, r) = \langle 0|O_{A}^{R}(t, r)O_{\text{opt}}^{R}(0)|0 \rangle \approx N \phi_{R}(r)e^{-m_{R}t}, \tag{14}$$

where $m_{R}$ is the mass of the specific glueball state, $N$ is an irrelevant normalization factor. With this prescription, we can fit $C_{2}^{R}(t, r)$ through a single exponential function form to extract the wave function $\phi_{R}(r)$. Since the two-point functions $C_2^{R}(t, r)$ with different $r$’s have the same mass parameter $m_{R}$ as that of $C_{\text{opt},2}(t, r)$, we perform a joint data analysis on all the $C_{2}^{R}(t, r)$ along with $C_{2, \text{opt}}(t, r)$. We divide the 50000 measurements of the two-point functions into 100 bins and take the average of the 50 measurements in each bin as an independent measurement. After that we construct the jackknife covariance matrix of the two-point functions, through which

\begin{table}
\centering
\caption{The input parameters for the calculation. Values of the coupling $\beta$, anisotropy $\xi$, the lattice size, and the number of measurements are listed. $a_s/r_0$ is determined by the static potential, and the first error of $a_s$ is the statistical one and the second comes from the uncertainty of the scale parameter $a_0^{-1} = 410(20) \text{MeV}$.}
\begin{tabular}{cccccc}
$\beta$ & $\xi$ & $a_s/r_0$ & $a_s(\text{fm})$ & $L_a(\text{fm})$ & $T$ & $N_{\text{conf}}$
\hline
2.4 & 5 & 0.461(4) & 0.222(2)(11) & $\sim 1.78$ & $8^3 \times 96$ & 5000
2.8 & 5 & 0.288(2) & 0.138(1)(7) & $\sim 1.66$ & $12^3 \times 144$ & 5000
\end{tabular}
\end{table}
By the use of the following function form,

\[ \phi_G(r) = e^{-\left( \frac{r}{r_0} \right)^\alpha}, \]

where \( r_0 \) and \( \alpha \) are fit parameters, whose best fit results are listed in Tab. III for the three channels on the finer

The wave functions \( \phi_R(r) \) with different \( r \) can be obtained simultaneously. For all the channels we are interested in, the qualities of the correlated fits are very good.

We first test the three definitions of the two-gluon operators on the two lattices we are using. The derived wave functions in the \( A_1^{++} \), \( E^{++} \), and \( T_2^{++} \) channels are plotted in Fig. 1 where the horizontal axis are in the units of \( fm \) converted from the lattice spacings listed Tab. II for comparison. It is seen that for each channel, the wave functions from the three definitions lie on each other within errors and signal the smallness of the lattice artifacts.

We also calculate the root-of-mean-square radii (RMS) \( r_{\text{RMS}} \) for the ground state glueballs with the definition

\[ r_{\text{RMS}}^2 = \frac{\int dr r^2 \phi^*(r) \phi(r)}{\int dr \phi^*(r) \phi(r)}, \]

where \( \phi(r) \) is the wave function with the best-fit parameters. The results are also listed in Tab. III. The \( r_{\text{RMS}} \)'s of \( E^{++} \) and \( T_2^{++} \) glueballs are almost two times larger than that of \( A_1^{++} \) glueball.

By the benefit of the variational method discussed above, we can also identify clearly the first excited states

\[ \beta \quad A_1^{++} \quad (GeV) \quad A_1^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \\
2.4 \quad 1.36(1)(7) \quad 2.58(4)(13) \quad 2.40(4)(12) \quad 3.29(13)(16) \quad 2.36(4)(12) \quad 3.34(13)(16) \\
2.8 \quad 1.52(2)(8) \quad 2.92(20)(15) \quad 2.31(4)(11) \quad 3.42(15)(17) \quad 2.31(3)(11) \quad 3.49(21)(17) \]

FIG. 1: Effective mass plateaus of the ground and the first excited states in \( A_1^{++}, E^{++} \), and \( T_2^{++} \) channels. The left plot is from \( 8^3 \times 96 \) lattice while the right one is from \( 12^3 \times 144 \). The time and the mass axis are both in lattice units.

FIG. 2: This figure shows the comparison of the ground state wave functions \( \Phi(r) \) (normalized to 1 at \( r = 0 \)) extracted through the different definitions of the two-gluon operators in \( A_1^{++}, E^{++} \), and \( T_2^{++} \) channels. The finite lattice spacing effects are also checked by the comparison of the wave functions calculated on \( 8^3 \times 96 \) and \( 12^3 \times 144 \) lattices. It is seen that, for each channel, the wave functions lie on each other within errors and signal the smallness of the lattice artifacts.

\[ \alpha \quad A_1^{++} \quad (GeV) \quad A_1^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \\
0.1 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 0.1 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \]

\[ \beta \quad A_1^{++} \quad (GeV) \quad A_1^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \\
2.8 \quad 1.52(2)(8) \quad 2.92(20)(15) \quad 2.31(4)(11) \quad 3.42(15)(17) \quad 2.31(3)(11) \quad 3.49(21)(17) \]

\[ \beta \quad A_1^{++} \quad (GeV) \quad A_1^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad E^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \quad T_2^{++} \quad (GeV) \\
2.8 \quad 1.52(2)(8) \quad 2.92(20)(15) \quad 2.31(4)(11) \quad 3.42(15)(17) \quad 2.31(3)(11) \quad 3.49(21)(17) \]

TABLE II: The masses are given in physical units for the ground and the first excited states of glueballs in the \( A_1^{++}, E^{++} \), and \( T_2^{++} \) channels (the first excited states are labeled by an asterisk on the quantum numbers). The first error of the glueball mass is the statistical one and the second comes from the uncertainty of the scale parameter \( r_0^{-1} = 410(20) \) MeV.
TABLE III: The best-fit results of the ground state wave functions in the three channels using the parameterization of Eq. (15). The RMS radii $r_{\text{RMS}}$ of different ground states glueballs of ground state glueball wave functions are also listed. The size of the tensor glueball is roughly two times larger than that of the scalar glueball.

| $r_0 (\text{fm})$ | $A_1^{++}$ | $E^{++}$ | $T_2^{++}$ |
|------------------|-------------|----------|------------|
| $\alpha$         | 0.154(8)    | 0.258(5) | 0.296(3)   |
| $r_{\text{RMS}} (\text{fm})$ | 2.1(4)    | 2.1(1)  | 2.07(7)   |

![FIG. 3](image)

FIG. 3: The ground state wave functions derived on $12^3 \times 144$ lattice are plotted by data points. The filled curves are the fit using the parameterization in Eq. (15).

of glueballs in the three channels. This enables us to derive their wave functions as well and the procedure is similar to that for the ground states. Fig. 3 (the points) shows the wave functions of the $A_1^{++}$, $E^{++}$ and $T_2^{++}$ excited glueballs. We do observe the radial nodes for these wave functions, which can be viewed as a strong support that the excited state is the first radial excited state of the glueball in each channel. Inspired by the function form of the two-body non-relativistic Schrödinger equation, we parameterize the wave function of the first excited state as follows,

$$\phi_G(r) = (1 - \beta \cdot r^\alpha) e^{-\left(\frac{r}{r_0}\right)^\alpha}, \quad (17)$$

where the parameter $\alpha$ depends on the potential, for example, $\alpha = 1$ for the Coulomb potential and $\alpha = 2$ for the harmonious oscillator potential. Since $\phi_G(r)$s with different $r$ are derived from the same gauge ensemble, we perform a correlated minimal-$\chi^2$ fit to the wave functions with a bootstrapped covariance matrix. The fit results for the $12^3$ ensemble are listed in Tab. IV. It is interesting that the parameters $\alpha$ and $r_0$ in the tensor channel ($T_2$ irreps) are close to that of the ground state, and especially, $\alpha$ is still close to 2, which is exactly the case for the harmonious oscillator potential model. For the first excited scalar glueball, the fitted parameter $\alpha$ deviates from 2 a little. The reason for this is not clear yet. Recalling that size of the ground scalar glueball is as small as 0.127 fm, while the lattice spacings we are using are comparable to it or even larger, a possibility is that the radial behavior of the first excited scalar glueball is not accurately extracted.

Based on the observation of the BS wave functions of glueballs in this study, one can infer some helpful information on the inner structure of glueballs. For the ground states, the BS wave functions of the scalar and the tensor glueball show the similar radial behaviors which can be well described by Gaussian-like function form. Assuming that the glueballs in these two channels are dominated by the two-gluon components, since the scalar is the lowest state, the possible two constituent gluons of the scalar must be in $S$-wave. The similarity of the wave function of the tensor glueball implies that the two constituent gluons are also in the $S$-wave. With the parameterizations in Eq. (15) and Eq. (17), the wave functions of the ground and the excited tensor glueballs are compatible with the $1S$ and $2S$ wave function of a non-relativistic harmonious oscillator, this may provide useful information for understanding the interaction between gluons within glueballs in the potential model picture.

TABLE IV: The best-fit results of wave functions of the first excited states in the three channels using the parameterization of Eq. (17). The errors are only statistical.

| $r_0 (\text{fm})$ | $A_1^{++}$ | $E^{++}$ | $T_2^{++}$ |
|------------------|-------------|----------|------------|
| $\beta$          | 21(4)       | 11(1)    | 11(1)      |
| $\alpha$         | 1.7(1)      | 2.0(1)   | 2.0(1)     |
| $r_{\text{RMS}} (\text{fm})$ | 0.21(1)    | 0.30(1)  | 0.31(1)    |

![FIG. 4](image)

FIG. 4: Excited glueball wave functions for both quantum numbers $0^{++}$ and $2^{++}$. Points are the lattice data and the lines are the fit results. Symmetric components for the tensor glueball are all averaged.
IV. SUMMARY AND DISCUSSIONS

The Bethe-Salpeter wave functions of the pure gauge scalar and tensor glueballs are revisited in this work. The feature of this study is the precise identification of the ground and the first excited glueball states in these two channels through the variational method based on large operator sets. We test the different definitions of two-gluon operators and find no sizeable difference, which implies the finite lattice spacing artifacts are small due to the implementation of the improved gauge action. This is also reinforced by checking the results from two lattices with different lattice spacings.

With a large statistics, the BS wave functions of both the ground and the first excited states in the scalar and tensor channels are extracted precisely. Instead of the exponential fall-off of the wave functions observed in previous works, we find that the wave functions of the ground states are Gaussian-like. The size of the ground tensor glueball is roughly two times larger than that of the scalar glueball. For the first time, we observe the radial nodes of the wave functions of the first excited states, which support them as the first radial excitations. We use the function forms inspired by potential models to parameterize the wave functions, and the fitted parameters show that the wave functions of the ground and the first excited state in the tensor channel are compatible with the $1S$ and $2S$ wave functions of a harmonious oscillator. These observations are helpful for a qualitative understanding of the inner structure of glueballs. Furthermore, there have been quite a few phenomenological studies on glueballs in the picture of potential models [13, 14, 16, 17], which use the scalar glueball wave function as input to derive the interacting potentials between constituent gluons. The results in this work can provide much finer and more precise information in this sector.

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