A new class of Laguerre-based Apostol type polynomials

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Abstract: In this paper, we introduce a generating function for a new generalization of Laguerre-based Apostol-Bernoulli polynomials, Apostol-Euler and Apostol-Genocchi polynomials. By making use of the generating function method and some functional equations mentioned in the paper, we conduct a further investigation in order to obtain symmetric identities of these polynomials.

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1. Introduction

Throughout the paper, we make use of the following notations:

\[ \mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\} \]

and

\[ \mathbb{Z}^- = \{-1, -2, -3, \ldots\} = \mathbb{Z}_0 \setminus \{0\}. \]

ABOUT THE AUTHOR

Serkan Araci was born in Hatay, Turkey, on 1 October 1988. He received his BS and MS degrees in mathematics from the University of Gaziantep, Gaziantep, Turkey, in 2010 and 2013, respectively. Additionally, the title of his MS thesis is “Bernstein polynomials and their reflections in analytic number theory” and, for this thesis, he received the Best Thesis Award of 2013 from the University of Gaziantep. He has published more than 90 papers in reputed international journals. His research interests include p-adic analysis, analytic theory of numbers, q-series and q-polynomials, and theory of umbral calculus. Araci is an editor and a referee for some international journals.

PUBLIC INTEREST STATEMENT

In the paper, we have established the generating functions for the Laguerre-based Apostol-type polynomials and Laguerre-based Apostol-type Hermite polynomials by making use of Tricomi function of the generating function for Laguerre polynomials. The equivalent forms of these generating functions can be derived by using Equations. (1.1), (1.6), and (2.1). They can be viewed as the equivalent forms of the generating functions (2.3), (2.6), and (2.8), respectively. In the previous sections, we have used the concepts and the formalism associated with Laguerre polynomials to introduce the Laguerre-based Apostol-type polynomials and Laguerre-based Apostol-type Hermite polynomials and establish their properties. The approach presented here is general and we have established the summation rules, which can be used to derive the results for Laguerre-based Apostol-type polynomials from the results of the corresponding Appell polynomials.
Here, as convention, \( \mathbb{Z} \) denotes the set of integers, \( \mathbb{R} \) denotes the set of real numbers and \( \mathbb{C} \) denotes the set of complex numbers.

The generating function of Laguerre polynomials are defined by means of the generating function (Srivastava & Manocha, 1984):

\[
\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n \quad (|t| < 1) \tag{1.1}
\]

or equivalently by

\[
\exp(yt) J_0\left(2 \sqrt{x}\right) = \sum_{n=0}^{\infty} y^n L_n\left(\frac{x}{y}\right)t^n \tag{1.2}
\]

where \( J_0\left(2 \sqrt{x}\right) \) are called 0th order Bessel function, and \( n \)th order Bessel function \( J_n(x) \) are given by the series:

\[
x^n J_n\left(2 \sqrt{x}\right) = \sum_{r=0}^{\infty} \frac{(-1)^n x^n}{r!(n+r)!} \quad (n \in \mathbb{N}_0). \tag{1.3}
\]

We recall that the Gould–Hopper generalized Hermite polynomials are defined as

\[
g_m^n(x, y) = n! \sum_{r=0}^{\infty} \frac{x^{n-ry}}{r!(n - mr)!}.
\]

where \( m \) is positive integer (see Srivastava & Manocha, 1984). These polynomials are specified by the generating function

\[
\exp(xt + yt^m) = \sum g_m^n(x, y) \frac{t^n}{n!}
\]

(see Srivastava & Manocha, 1984).

In particular, we note that

\[
g_0^n(x, y) = H_n(x, y)
\]

where \( H_n(x, y) \) are called 2-variable Hermite–Kampé de Fériet polynomials (Srivastava & Manocha, 1984) that can be defined by the generating function:

\[
\exp\left(xt + yt^2\right) = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!} \tag{1.4}
\]

and it reduces to the ordinary Hermite polynomials \( H_n(x) \) (see Srivastava & Manocha, 1984) when we take the values \( y = -1 \) and \( 2x \) instead of \( x \) in the Equation (1.4). Furthermore, we recall that the 3-variable Laguerre–Hermite polynomials (3VLHP) \( L_n(x, y, z) \) are defined by the series (Kurt, 2010)

\[
L_n(x, y, z) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{z^n y^{n-2k} L_{n-2k}(\frac{z}{y})}{k!(n-2k)!}.
\]

The generating function of the Equation (1.5) is that

\[
\frac{1}{1-zt} \exp\left(\frac{-xt}{1-zt} + \frac{yt^2}{1-zt^2}\right) = \sum_{n=0}^{\infty} L_n(x, y, z)t^n
\]
and it also equals to
\[
\exp(yt + zt^2)L_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} H_n(x, y, z) \frac{t^n}{n!}.
\]

(1.7)

At the value \(Z = -\frac{1}{2}\) in the Equation (1.7), we have
\[
L_n(x, y, -\frac{1}{2}) = L_n(x, y)
\]

and
\[
L_n(x, 1, -1) = L_n(x)
\]

where \(L_n(x, y)\) denotes 2-variable Laguerre–Hermite polynomials (2VLP) (Magnus, Oberhettinger, & Soni, 1966) and \(L_n(x)\) denotes the Laguerre–Hermite polynomials (LHP) (Ozarslan, 2013).

The literature contains a large number of interesting properties and relationships involving these polynomials (Araci, Bagdasaryan, & Srivastava, 2014; Araci, Şen, Acikgoz, & Orucoglu, 2015; Comtet, 1974; Khan, Al Saad, & Khan, 2010; Kurt, 2010; Luke, 1969; Luo et al., 2006, 2011; Luo & Srivastava, 2005, 2006, 2011a, 2011b).

\[
\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^{\alpha} = 1)
\]

(1.8)

\[
\left(\frac{2}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^{\alpha} = 1)
\]

(1.9)

\[
\left(\frac{2t}{e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^{\alpha} = 1)
\]

(1.10)

The literature contains a large number of interesting properties and relationships involving these polynomials (Araci, Bagdasaryan, & Srivastava, 2014; Araci, Şen, Acikgoz, & Orucoglu, 2015; Comtet, 1974; Khan, Al Saad, & Khan, 2010; Kurt, 2010; Luke, 1969; Luo et al., 2006, 2011a, 2011b; Magnus, Oberhettinger, & Soni, 1966; Ozarslan, 2013, 2011; Ozden, 2010, 2011; Ozden, Simsek, & Srivastava, 2010; Pathan, 2012; Pathan & Khan, 2015; Kilbas, Srivastava, & Trujillo, 2006; Srivastava & Manocha, 1984; Srivastava et al., 2014; Srivastava, Kurt, & Simsek, 2012; Srivastava, Garg, & Choudhary, 2011; Tuenter, 2001). Luo and Srivastava (2005, 2006, 2011b) introduced the generalized Apostol–Bernoulli polynomials \(B_n^{(\alpha)}(x)\) of order \(\alpha\). Luo (2006) also investigated the generalized Apostol–Euler polynomials \(E_n^{(\alpha)}(x)\) and the generalized Apostol–Genocchi polynomials \(G_n^{(\alpha)}(x)\) of (non-negative integer) order \(\alpha\) (see also Luo, 2006, 2011a; Luo & Srivastava, 2011).

Let \(\alpha\) be a non-negative integer. The generalized Apostol–Bernoulli polynomials \(B_n^{(\alpha)}(x; \lambda)\) of order \(\alpha\), the generalized Apostol–Euler polynomials \(E_n^{(\alpha)}(x; \lambda)\) of order \(\alpha\), and the generalized Apostol–Genocchi polynomials \(G_n^{(\alpha)}(x; \lambda)\) of order \(\alpha\) are defined, respectively, by the following generating functions (see Luo & Srivastava, 2011b)

\[
\left(\frac{t}{\lambda e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < 2\pi \text{ when } \lambda = 1; |t| < ||\log \lambda|| \text{ when } \lambda \neq 1)
\]

(1.11)

\[
\left(\frac{2}{\lambda e^t + 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < ||\log (-\lambda)||)
\]

(1.12)

and
\[
\left( \frac{2t}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}, \quad (|t| < |\log (-\lambda)|). \tag{1.13}
\]

It can be easily noted that
\[
B_n^{(\alpha)}(x) = B_n^{(\alpha)}(x; 1), \quad E_n^{(\alpha)}(x) = E_n^{(\alpha)}(x; 1) \quad \text{and} \quad G_n^{(\alpha)}(x) = G_n^{(\alpha)}(x; 1).
\]

Recently, Kurt (2010) gave the following generalization of the Bernoulli polynomials of order \( \alpha \), which is recalled in Definition 1.

**Definition 1** For arbitrary real or complex parameter \( \alpha \), the generalized Bernoulli polynomials \( B_n^{(\alpha-m-1)}(x; m \in \mathbb{N}) \) are defined in centered at \( t = 0 \) by means of the generating function:

\[
\left( \frac{t^m}{e^t - \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha+m-1)}(x; \lambda) \frac{t^n}{n!}.	ag{1.14}
\]

Clearly, if we take \( m = 1 \) in (1.14), then the definition (1.14) becomes the definition (1.13).

More recently, Tremblay, Gaboury, and Fugère (2011) further gave the following generalization of Kurt’s definition (1.14) in the following form.

**Definition 2** For arbitrary real or complex parameter \( \lambda \) and \( \alpha \) and the natural numbers \( m \in \mathbb{N} \), the generalized Bernoulli polynomials \( B_n^{(\alpha-m-1)}(x; \lambda) \) are defined in centered at \( t = 0 \), with \( |t| < |\log \lambda| \), by means of the generating function:

\[
\left( \frac{t^m}{\lambda e^t - \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha+m-1)}(x; \lambda) \frac{t^n}{n!}.	ag{1.15}
\]

Clearly, if we take \( m = 1 \) in (1.15), then the definition (1.15) becomes the definition (1.11).

We now give the following definition for the generalized Euler polynomials \( E_n^{(\alpha)}(x) \).

**Definition 3** For arbitrary real or complex parameter \( \alpha \) and natural number \( m \in \mathbb{N} \), the generalized Euler polynomials \( E_n^{(\alpha-m-1)}(x) \) are defined in centered at \( t = 0 \), with \( |t| < \pi \), by means of the generating function:

\[
\left( \frac{2^m}{e^t + \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha+m-1)}(x) \frac{t^n}{n!}.	ag{1.16}
\]

Obviously, setting \( m = 1 \) in (1.16), we have \( E_n^{(\alpha, 0)}(x; 1) = E_n^{(\alpha)}(x) \).

**Definition 4** For arbitrary real or complex parameter \( \lambda \) and \( \alpha \) and the natural number \( m \), the generalized Euler polynomials \( E_n^{(\alpha-m-1)}(x) \) are defined in centered at \( t = 0 \), with \( |t| < |\log (-\lambda)| \), by means of the generating function:
\[
\left( \frac{2^m}{\lambda e^t + \sum_{n=0}^{\infty} \frac{t^n}{n!}} \right)^u e^{xt} = \sum_{n=0}^{\infty} E_n^{(x,m-1)}(x;\lambda) \frac{t^n}{n!}, \tag{1.17}
\]

It is easy to see that setting \( m = 1 \) in (1.17), we have \( E_n^{(x,0)}(x;\lambda) = E_n^{(x)}(x;\lambda) \). From (1.17) we have

\[
E_n^{(x,m-1)}(x;\lambda) = \left( \frac{2^m}{\lambda + 1} \right)^u . \tag{1.18}
\]

**Definition 5** For arbitrary real or complex parameter \( a \) and natural number \( m \in \mathbb{N} \), the generalized Genocchi polynomials \( G_n^{(a,m-1)}(x) \) are defined in centered at \( t = 0 \), with \( |t| < \pi \), by means of the generating function:

\[
\left( \frac{2^m t^m}{e^t + \sum_{n=0}^{\infty} \frac{t^n}{n!}} \right)^u e^{xt} = \sum_{n=0}^{\infty} G_n^{(a,m-1)}(x) \frac{t^n}{n!} . \tag{1.19}
\]

Obviously, setting \( m = 1 \) in (1.19), we have \( G_n^{(a,0)}(x;1) = G_n^{(a)}(x) \).

**Definition 6** For arbitrary real or complex parameter \( \lambda \) and \( a \), and the natural number \( m \), the generalized Genocchi polynomials \( G_n^{(a,m-1)}(x) \) are defined in centered at \( t = 0 \), with \( |t| < |\log (-\lambda)| \), by means of the generating function:

\[
\left( \frac{2^m t^m}{\lambda e^t + \sum_{n=0}^{\infty} \frac{t^n}{n!}} \right)^u e^{xt} = \sum_{n=0}^{\infty} G_n^{(a,m-1)}(x;\lambda) \frac{t^n}{n!} . \tag{1.20}
\]

It is easy to see that setting \( m = 1 \) in (1.20), we have \( G_n^{(a,0)}(x;\lambda) = G_n^{(a)}(x;\lambda) \).

In this paper, we introduce a new class of generalized Apostol-type polynomials, a countable set of polynomials \( Y_{n,h}^{(a)}(x,y;k,a,b) \) generalizing Apostol-type Laguerre-Bernoulli, Apostol-type Laguerre-Euler and Apostol-type Laguerre-Genocchi polynomials and Laguerre polynomials of 2-variables \( L_n(x,y) \) specified by the generating relation (1.2) and Mittag-Leffler function.

In this paper, we develop some elementary properties and derive the implicit summation formulæ for these generalized polynomials by using different analytical means on their respective generating functions.

**2. A new class of Laguerre-based Apostol-type polynomials**

Recently, Ozden (2010, 2011), Ozden, Simsek, and Srivastava (2010) and Ozarslan (2011, 2013) introduced the unification of the Apostol-type polynomials including Bernoulli, Euler and Genocchi polynomials \( Y_n^{(a)}(x;k,a,b) \) of higher order \( a \) which are defined by

\[
\left( \frac{2^{1-k} t^k}{\beta e^t - a} \right)^u e^{xt} = \sum_{n=0}^{\infty} Y_n^{(a)}(x;k,a,b) \frac{t^n}{n!} . \tag{2.1}
\]

\[
\left| t + b \log \left( \frac{\beta}{a} \right) \right| < 2\pi, x \in \mathbb{R}; 1^+: = 1; k \in \mathbb{N}_0; a, b \in \mathbb{R} \setminus \{0\}; a, \beta \in \mathbb{C} .
\]

Ozarslan (2011) gave the following precise conditions of convergence of the series involved in (2.1):
\( \text{(i) if } a^0 > 0 \text{ and } k \in \mathbb{N}, \text{ then } \left| t + b \log \left( \frac{\beta}{a} \right) \right| < 2\pi, x \in \mathbb{R}; 1^* : = 1; \alpha, \beta \in \mathbb{C} \)

\( \text{(ii) if } a^0 > 0 \text{ and } k = 0, \text{ then } 0 < \Im \left( t + b \log \left( \frac{\beta}{a} \right) \right) < 2\pi, x \in \mathbb{R}; 1^* : = 1; \alpha, \beta \in \mathbb{C} \)

\( \text{(iii) if } a^0 < 0 \text{ and } k \in \mathbb{N}_0, \text{ then } \left| t + b \log \left( \frac{\beta}{a} \right) \right| < 2\pi, x \in \mathbb{R}; 1^* : = 1; \alpha, \beta \in \mathbb{C} \)

**Definition 7** The generalized Laguerre-based Apostol-type Bernoulli, Laguerre-based Apostol-type Euler and Laguerre-based Apostol-type Genocchi polynomials \( Y_{\alpha,\beta}^{m}(x, y; k, a, b) \), \( m \geq 1 \) for a real or complex parameter \( a \) defined in a suitable neighborhood of \( t = 0 \) by means of the following generating function

\[
\left( \frac{21^{-k}t^k}{\beta^\alpha e^t - a^\alpha \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha \exp(\gamma t) J_0 \left( 2 \sqrt{\gamma t} \right) = \sum_{n=0}^{m} Y_{\alpha,\beta}^{m}(x, y; k, a, b) \frac{t^n}{n!}
\]

so that

\[
Y_{\alpha,\beta}^{m}(x, y; k, a, b) = \sum_{n=0}^{m} \binom{n}{r} \left( \frac{\gamma}{m-r} \right) B_{\alpha,\beta}^{m}(x, y; k, a, b) \frac{t^n}{n!}.
\]

For \( x = 0 \) in Equation (2.2), the result reduces to known result of Ozden (2010); 2011 and Ozden et al. (2010).

For \( k = a = b = 1 \) and \( \beta = \lambda \) in (2.2), we state the following definition.

**Definition 8** Let \( \alpha \) and \( \beta \) be arbitrary real or complex parameters. The generalized Laguerre Apostol-type Bernoulli polynomials are defined by

\[
\left( \frac{t}{\lambda e^t - \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha \exp(y t) J_0 \left( 2 \sqrt{\gamma t} \right) = \sum_{n=0}^{m} B_{\alpha}^{n}(x, y; \lambda) \frac{t^n}{n!}.
\]

At the value \( m = 1 \) in the Equation (2.3), the result reduces to the known result of Khan et al. (2010):

\[
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha \exp(y t) J_0 \left( 2 \sqrt{\gamma t} \right) = \sum_{n=0}^{\infty} B_{\alpha}^{n}(x, y; \lambda) \frac{t^n}{n!}.
\]

\( (|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log(-\lambda)| \text{ when } \lambda \neq 1. \)

Setting \( k + 1 = -\alpha = b = 1 \) and \( \beta = \lambda \) in (2.2), we define the following.

**Definition 9** Let \( \alpha \) and \( \beta \) be arbitrary real or complex parameters. The generalized Laguerre Apostol-type Euler polynomials are defined by

\[
\left( \frac{2}{\lambda e^t + \sum_{n=0}^{m-1} \frac{t^n}{n!}} \right)^\alpha \exp(y t) J_0 \left( 2 \sqrt{\gamma t} \right) = \sum_{n=0}^{m} E_{\alpha}^{n}(x, y; \lambda) \frac{t^n}{n!}.
\]

For \( x = 0 \) in the Equation (2.6), further taking \( m = 1 \), the result reduces to the known result of Khan et al. (2010):
\[
\left(\frac{2}{\lambda e^t + 1}\right)^a \exp(yt)J_0\left(2 \sqrt{xt}\right) = \sum_{m=0}^{\infty} E^\omega_m(x,y;\lambda) \frac{t^n}{n!}.
\]

\((|t| < \kappa, \text{ when } \lambda = 1; |t| < |\log(-\lambda)|, \text{ when } \lambda \neq 1).\)

Setting \(k + 1 = -2a = b = 1\) and \(2\beta = \lambda\) in (2.2), we define the following.

**Definition 10** \ Let \(a\) and \(\lambda\) be arbitrary real or complex parameters. The generalized Laguerre Apostol-type Genocchi polynomials are introduced by

\[
\left(\frac{2t}{\lambda e^t + 1}\right)^a \exp(yt)J_0\left(2 \sqrt{xt}\right) = \sum_{m=0}^{\infty} G^\omega_m(x,y;\lambda) \frac{t^n}{n!}.
\]

For \(x = 0\) in Equation (2.6), Further taking \(a,m, \lambda = 1\), the result reduces to the known result of Khan et al. (2010):

\[
\left(\frac{2t}{\lambda e^t + 1}\right)^a \exp(yt)J_0\left(2 \sqrt{xt}\right) = \sum_{m=0}^{\infty} G_m(x,y;\lambda) \frac{t^n}{n!}
\]

\((|t| < \kappa, \text{ when } \lambda = 1; |t| < |\log(-\lambda)|, \text{ when } \lambda \neq 1).\)

**Definition 11** \ The generalized Laguerre-based Apostol-type Hermite-Bernoulli, Laguerre-based Apostol-type Hermite-Euler and Laguerre-based Apostol-type Hermite-Genocchi polynomials \(\ell \phi_n(x,y,z)\), for a real or complex parameter \(a\) defined in a suitable neighborhood of \(t = 0\) by means of the following generating function:

\[
\left(\frac{2^{1-m} e^t}{\beta^a e^t - a^b \sum_{n=0}^{\infty} h^n}\right)^a \exp(yt)J_0\left(2 \sqrt{xt}\right) = \sum_{n=0}^{\infty} \psi_n(x,y,z;\lambda) \frac{t^n}{n!}
\]

where \(\psi_n(x,y,z) = \psi_{m,n}(x,y,z;\lambda,a,b)\) contain as its special cases both generalized Apostol-type polynomials (2.1), \(y^a_{m,n}(x;\lambda,a,b)\), (1.15) to (1.20) and Kampé de Fériet generalization of the Hermite polynomials \(H_n(x,y)\) (cf. Equation (1.4)).

By substituting \(x = y = z = 0\) in (2.9), we obtain the corresponding unification of the generalized Apostol-type Bernoulli, Apostol-type Euler and Apostol-type Genocchi numbers \(y_{n,d}^{m}(k,a,b)(m \geq 1)\) are defined for a real or complex parameter \(a\) by means of the generating function

\[
\left(\frac{2^{1-m} e^t}{\beta^a e^t - a^b \sum_{n=0}^{\infty} h^n}\right)^a \psi_{m,n}(k,a,b) \frac{t^n}{n!}
\]

Then by (2.9) and (1.7), we have the representation

\[
\ell \phi_n(x,y,z;\lambda,a,b) = \sum_{n=0}^{\infty} \left(\begin{array}{c} n \\ r \end{array}\right) y_{n-r,d}^{m}(k,a,b) \psi_n(x,y,z).
\]

For \(a = 0\), in Equation (2.9), the result reduces to Equation (1.7).

Setting \(x = 0, m = 1\) and replacing \(y\) by \(x\) and \(z\) by \(y\), respectively, in (2.9), we get a recent result of Pathan and Khan (2015). For \(k = \beta = a = b = 1, x = 0\) and replacing \(y\) by \(x\) and \(z\) by \(y\), respectively,
in (2.9), the result reduces to the known result of Pathan and Khan (2015). Further if $\alpha = 1$ the result reduces to known result of Pathan (2012):

$$\left( \frac{t}{e^t - 1} \right) e^{y + yt} = \sum_{n=0}^{\infty} B_n(x, y) \frac{t^n}{n!} \quad (2.11)$$

Besides by (2.10), we can also obtain the generalized Hermite-Euler polynomials $E_{\alpha}^{(x, y)}(x, y)$ and the generalized Hermite-Genocchi polynomials $G_{\alpha}^{(x, y)}(x, y)$ each of order $\alpha$ and degree $n$, respectively, defined by the following generating functions

$$\left( \frac{2}{e^t + 1} \right) e^{xt-yt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x, y) \frac{t^n}{n!} \quad (|t| < \pi; 1^\alpha = 1)$$

(2.12)

and

$$\left( \frac{2t}{e^t + 1} \right) e^{xt-yt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x, y) \frac{t^n}{n!} \quad (|t| < \pi; 1^\alpha = 1).$$

(2.13)

It may be seen that for $y = 0$, (2.11) to (2.13) are, respectively, the generalizations of (1.8) to (1.10).

We continue with another basic example of (2.9) by taking $m, k = 2$ and $\alpha = 1$. Thus we have

$$2^{-1} \left( \frac{t^2}{\beta \rho e^t - \alpha^b(1 + t)} \right) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{H_n(x, y, z) t^n}{n!} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Phi_n(x, y, z) t^n}{n!}$$

where $\Phi_n(x, y) = \sum_{n=0}^{\infty} Y_{n, \alpha}^{(2, 1)}(x, y, z; 2, \alpha, b)$. We have

$$2^{-1} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{H_n(x, y, z) t^n}{n!}$$

$$= \beta^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Phi_{n-p-1}(x, y, z) t^{n-2}}{(n-2)!} - \alpha^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Phi_{n-q-1}(x, y, z) t^{n-2}}{(n-2)!} - \alpha^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Phi_{n-r-1}(x, y, z) t^{n-2}}{(n-2)!}.$$

Replace $n$ by $n - p + 2$, $p \leq n - 2$

$$\frac{1}{2} \frac{H_n(x, y, z)}{n!} = \beta^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Phi_{n-p-1}(x, y, z) t^{n-2}}{(n-2)!} - \alpha^b \left( \frac{\Phi_{n+2}(x, y, z)}{(n+2)!} - \frac{\Phi_{n+1}(x, y)}{(n+1)!} \right).$$

This formula gives a representation of $H_n(x, y, z)$ in terms of sums of $\Phi$. This is the key to the next conclusion for finding another representation of $H_n(x, y, z)$ in terms of sums of $\Psi$ where $\Psi_n(x, y, z) = \sum_{n=0}^{\infty} Y_{n, \beta}^{(1, 1)}(x, y, z; 1, \alpha, b)$. For this taking $\alpha = m = k = 1$ in (2.9), we have

$$\left( \frac{t}{\beta \rho e^t - \alpha^b} \right) e^{x+y+zt} J_0 \left( 2 \sqrt{\alpha t} \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{\Psi_n(x, y, z) t^n}{n!}$$

$$= \beta^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Psi_{n-p-1}(x, y, z) t^{n-2}}{(n-2)!} - \alpha^b \left( \frac{\Psi_{n+2}(x, y, z)}{(n+2)!} - \frac{\Psi_{n+1}(x, y)}{(n+1)!} \right).$$

Comparing the coefficients of $t^n$, we have

$$\frac{H_n(x, y, z)}{n!} = \beta^b \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \frac{\Psi_{n-p-1}(x, y, z) t^{n-2}}{(n-2)!} - \alpha^b \left( \frac{\Psi_{n+2}(x, y, z)}{(n+2)!} - \frac{\Psi_{n+1}(x, y)}{(n+1)!} \right).$$

When investigating the connection between Hermite polynomials $H_n(x, y, z)$ and generalized Apostol-type polynomials $\phi_n(x, y, z)$, the following theorem is of great importance.
Theorem 1 The following holds true

\[ lH_n(x, y) = \sum_{k=0}^{n!} \frac{\Gamma(n+1)}{(n+1)!} [\phi_n^b(x, y, z; k, a, b) - \phi_n^d(x, y, z; k, a, b)] \]

where \( \phi_n^b(x, y, z) = \psi_{n+1}^{1,1}(x, y, z; k, a, b) \).

Proof We begin with the definition 11 and write

\[ e^{x+y+z} J_0(2 \sqrt{x+y}) = \frac{1}{2^{1/4}} \left( \frac{2^{1/4} e^{x+y+z}}{\beta^2 e^{x+y+z}} \right) J_0(2 \sqrt{x+y}) = \frac{1}{2^{1/4}} \left[ \frac{2^{1/4} e^{x+y+z}}{\beta^2 e^{x+y+z}} \right] J_0(2 \sqrt{x+y}) - \frac{1}{2^{1/4}} \left( \frac{2^{1/4} e^{x+y+z}}{\beta^2 e^{x+y+z}} \right) J_0(2 \sqrt{x+y}). \]

Then using the definition of Kampé de Fériet generalization of the Laguerre-Hermite polynomials

\[ \theta H_n(x, y) \phi_n^d(x, y, z; k, a, b) \]

holds true:

\[ \sum_{n=0}^{\infty} H_n(x, y, z) n^n! = \sum_{n=0}^{\infty} \frac{n!}{(n+1)!} [\phi_n^b(x, y, z; k, a, b) - \phi_n^d(x, y, z; k, a, b)] n^n. \]

Finally, comparing the coefficients of \( t^n \), we complete the proof of the theorem. 

Corollary 1 The following formula holds:

\[ lH_n(x, y, z) = \frac{1}{(n+1)!} \left[ \phi_n^b(x, y, z; k, a, b) - \phi_n^d(x, y, z; k, a, b) \right] \]

where \( \psi_n^b(x, y, z) = \psi_{n+1}^{1,1}(x, y, z; k, a, b) \).

Theorem 2 The following formula involving Laguerre-Apostol-type polynomials \( \psi_n^m(x, y; k, a, b) \) holds true:

\[ \psi_n^m(x, y; z; k, a, b) = \sum_{r=0}^{n} \binom{n}{r} \psi_n^{m-r}(x, y; k, a, b) \]

Proof By Definition 7, we easily get the proof of the theorem. So we omit it.

Theorem 3 The following formula involving Laguerre-Apostol-type polynomials \( \psi_n^m(x, y; k, a, b) \) holds true:

\[ \psi_n^m(x, y; k, a, b) = \sum_{r=0}^{n} \psi_n^{m-r}(k, a, b) \]

Proof The proof of this theorem follows from Definition 7. So we omit the proof.

3. Implicit formulae involving Laguerre-based Apostol-type polynomials

This section is devoted to employing the definition of the Laguerre-based Apostol-type polynomials \( \psi_n^m(x, y; k, a, b) \). First we prove the following results involving Laguerre-based Apostol-type polynomials \( \psi_n^m(x, y; k, a, b) \).

Theorem 4 The following implicit summation formulae for Laguerre-based Apostol-type polynomials \( \psi_n^m(x, y; k, a, b) \) holds true:
\[ l Y_{q,+i}^{(m)}(x, z; k, a, b) = \sum_{n=0}^{q} \sum_{p=0}^{l} \binom{q}{n} \binom{l}{p} (z - y)^{n+p} Y_{q,-p,n}^{(m)}(x, y; k, a, b). \]

**Proof** \ We replace \( t \) by \( t + u \) and rewrite the generating function (2.2) as

\[
\left( \frac{2^{1-k}(t + u)^k}{e^{t-u} - q^k \sum_{m=0}^{\infty} \frac{t^m}{m!}} \right)^n J_0(2 \sqrt{(t + u)}) = e^{-y(t+u)} \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!}.
\]

Replacing \( y \) by \( z \) in the above equation and equating the resulting equation to the above equation, we get

\[
e^{2 - y(t+u)} \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!} = \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!}.
\]

On expanding the exponential function in the above gives

\[
\sum_{N=0}^{\infty} \frac{(z - y)(t + u)^N}{N!} = \sum_{n,m=0}^{\infty} \frac{f(n + m)}{n! m!} \frac{y^n}{n!} \frac{z^m}{m!}
\]

in the left-hand side, becomes

\[
\sum_{n,p=0}^{\infty} \frac{(z - y)^{n+p} t^n u^p}{n! p!} \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!} = \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!}.
\]

Now replacing \( q \) by \( q - n \), \( l \) by \( l - p \) and using the lemma (Srivastava & Manocha, 1984) in the left-hand side of (3.1), we get

\[
\sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!} = \sum_{q,j=0}^{\infty} \frac{t^q}{q!} \frac{u^j}{j!}.
\]

Finally on equating the coefficients of the like powers of \( t^q \) and \( u^j \) in the above equation, we get the required result. \( \square \)

For \( k = a = b = 1 \) and \( \beta = \lambda \) in Theorem 4, we get the following corollary.

**COROLLARY 2** \ The following implicit summation formulae for Laguerre-based Apostol-type Bernoulli polynomials \( l B_{q,+i}^{(m-1)}(x, y; z, k, a, b) \) holds true:

\[
l B_{q,+i}^{(m-1)}(x, z; k, a, b) = \sum_{n=0}^{q} \sum_{p=0}^{l} \binom{q}{n} \binom{l}{p} (z - y)^{n+p} B_{q,-p,n}^{(m-1)}(x, y; k, a, b).
\]

For \( k = a = b = 1 \) and \( \beta = \lambda \) in Theorem 4, we get the corollary.
**Corollary 3**  The following implicit summation formulae for Laguerre-based Apostol-type Euler polynomials \( L^{\alpha,m}_{n}(x,y;\lambda) \) holds true:

\[
L^{\alpha,m}_{q+l}(x,z;\lambda) = \sum_{n=0}^{q} \sum_{p=0}^{l} \left( \begin{array}{c} q \\ n \\ \end{array} \right) \left( \begin{array}{c} l \\ p \\ \end{array} \right) (z - y)^{n+p} L^{\alpha,m}_{q+l-p-n}(x,y;\lambda).
\]

Letting \( k = -2a = b = 1 \) and \( 2\beta = \lambda \) in Theorem 4, we get the corollary.

**Corollary 4**  The following implicit summation formulae for Laguerre-based Apostol-type Bernoulli polynomials \( B^{\alpha,m}_{n}(x,y;\lambda) \) holds true:

\[
B^{\alpha,m}_{n}(x,y;z;\lambda) = \sum_{s=0}^{n} \binom{n}{s} B^{\alpha,m}_{n-s}(z;\lambda) y^{s} L_{s}\left( \frac{x}{y} \right).
\]

**Theorem 5**  The following implicit summation formula involving Laguerre-based Apostol-type polynomials \( Y^{\alpha,m}_{n}(x,y;k,a,b) \) holds true:

\[
Y^{\alpha,m}_{n}(x,y+z;k,a,b) = \sum_{s=0}^{n} \binom{n}{s} Y^{\alpha,m}_{n-s}(z;k,a,b) y^{s} L_{s}\left( \frac{x}{y} \right).
\]

**Proof**  When we replace \( y \) by \( y + z \) in (2.2), use (1.2) and rewrite the generating function, we conclude the proof of this theorem.

For \( k = a = b = 1 \) and \( \beta = \lambda \) in Theorem 5, we get the following corollary.

**Corollary 5**  The following implicit summation formulae for Laguerre-based Apostol-type Bernoulli polynomials \( B^{\alpha,m}_{n}(x,y;\lambda) \) holds true:

\[
B^{\alpha,m}_{n}(x,y;z;\lambda) = \sum_{s=0}^{n} \binom{n}{s} B^{\alpha,m}_{n-s}(z;\lambda) y^{s} L_{s}\left( \frac{x}{y} \right).
\]

For \( k + 1 = -a = b = 1 \) and \( \beta = \lambda \) in Theorem 5, we get the corollary.

**Corollary 6**  The following implicit summation formulae for Laguerre-based Apostol-type Euler polynomials \( E^{\alpha,m}_{n}(x,y;\lambda) \) holds true:

\[
E^{\alpha,m}_{n}(x,y+z;\lambda) = \sum_{s=0}^{n} \binom{n}{s} E^{\alpha,m}_{n-s}(z;\lambda) y^{s} L_{s}\left( \frac{x}{y} \right).
\]

Letting \( k = -2a = b = 1 \) and \( 2\beta = \lambda \) in Theorem 5, we get the corollary.

**Corollary 7**  The following implicit summation formulae for Laguerre-based Apostol-type Genocchi polynomials \( G^{\alpha,m}_{n}(x,y;\lambda) \) holds true:

\[
G^{\alpha,m}_{n}(x,y+z;\lambda) = \sum_{s=0}^{n} \binom{n}{s} G^{\alpha,m}_{n-s}(z;\lambda) y^{s} L_{s}\left( \frac{x}{y} \right).
\]

**Theorem 6**  The following implicit summation formulae for Laguerre-based Apostol-type polynomials \( Y^{\alpha,m}_{n}(x,y;k,a,b) \) holds true:

\[
Y^{\alpha,m}_{n}(x,y;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} Y^{\alpha,m}_{n-r}(y - u; x, a, b) x^{r} L_{s}\left( \frac{x}{y} \right).
\]
Khan et al., Cogent Mathematics (2016), 3: 1243839
http://dx.doi.org/10.1080/23311835.2016.1243839

Proof By exploiting the generating function (1.2), we can write Equation (2.2) as

$$
\left( \frac{2^{1-k} t}{\beta^2 e^t - a^k \sum_{n=0}^{\infty} \frac{t^n}{n!}} \right)^n e^{y-u(\beta t)} J_0 (2 \sqrt{\beta} x t) = \sum_{n=0}^{\infty} y_{n,\beta}^{(m)} (y-u(\beta t,a,b)) t^n \frac{n!}{m!} \sum_{r=0}^{\infty} x'^r L_r \left( \frac{y}{u(\beta t)} \right). \tag{3.2}
$$

Now replacing $n$ by $n-r$ in the right-hand side and using the lemma (Srivastava & Manocha, 1984) in the right-hand side of Equation (3.2), we complete the proof of the theorem. $\square$

For $k = a = b = 1$ and $\beta = \lambda$ in Theorem 6, we get the following corollary.

COROLLARY 8 The following implicit summation formulae for Laguerre-based Apostol-type Bernoulli polynomials $L_n^{(a-1)}(x,y;\lambda)$ holds true:

$$
L_n^{(a-1)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(a-1)}(y-u,\lambda) x^r L_r \left( \frac{x}{u} \right).
$$

For $k+1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 6, we get the corollary.

COROLLARY 9 The following implicit summation formulae for Laguerre-based Apostol-type Euler polynomials $E_n^{(a-1)}(x,y;\lambda)$ holds true:

$$
E_n^{(a-1)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} E_{n-r}^{(a-1)}(y-u,\lambda) x^r L_r \left( \frac{x}{u} \right).
$$

Letting $k = -2a = b = 1$ and $2\beta = \lambda$ in Theorem 6, we get the corollary.

COROLLARY 10 The following implicit summation formulae for Laguerre-based Apostol-type Genocchi polynomials $G_n^{(a-1)}(x,y;\lambda)$ holds true:

$$
G_n^{(a-1)}(x,y;\lambda) = \sum_{r=0}^{n} \binom{n}{r} G_{n-r}^{(a-1)}(y-u,\lambda) x^r L_r \left( \frac{x}{u} \right).
$$

THEOREM 7 The following implicit summation formulae for Laguerre-based Apostol-type polynomials $L_n^{(a,m)}(x,y;k,a,b)$ holds true:

$$
L_n^{(a,m)}(x,y+1;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} L_r^{(a,m)}(x,y;k,a,b).
$$

Proof By using the generating function (2.2), it is easy to prove this theorem. $\square$

For $k = a = b = 1$ and $\beta = \lambda$ in Theorem 7, we get the following corollary.

COROLLARY 11 The following implicit summation formulae for Laguerre-based Apostol-type Bernoulli polynomials $B_n^{(a,m)}(x,y;\lambda)$ holds true:

$$
B_n^{(a,m)}(x,y+1;\lambda) = \sum_{r=0}^{n} \binom{n}{r} B_{n-r}^{(a,m)}(x,y;\lambda).
$$

For $k+1 = -a = b = 1$ and $\beta = \lambda$ in Theorem 7, we get the corollary.
Corollary 12. The following implicit summation formulae for Laguerre-based Apostol-type Euler polynomials \( L_n^{(a,m-1)}(x,y;\lambda) \) holds true:

\[
L_n^{(a,m-1)}(x,y+1;\lambda) = \sum_{r=0}^{n} \binom{n}{r} L_{n-r}^{(a,m-1)}(x,y;\lambda).
\]

Letting \( k = -2a = b = 1 \) and \( 2\beta = \lambda \) in Theorem 6, we get the corollary.

Corollary 13. The following implicit summation formulae for Laguerre-based Apostol-type Genocchi polynomials \( G_n^{(a,m-1)}(x,y;\lambda) \) holds true:

\[
L_n^{(a,m-1)}(x,y+1;\lambda) = \sum_{r=0}^{n} \binom{n}{r} G_{n-r}^{(a,m-1)}(x,y;\lambda).
\]

4. Symmetry identities for the Laguerre-based Apostol-type polynomials

In this section, we give general symmetry identities for the generalized Laguerre-based Apostol-type polynomials \( L_n^{(a,m)}(x,y;k,a,b) \) by applying the generating function (2.1) and (2.2).

Theorem 8. The following identity holds true:

\[
\sum_{r=0}^{n} \binom{n}{r} d^n L_n^{(r,a,m)}(dx,dy;k,a,b) L_r^{(a,m)}(cw,cz;k,a,b) = \sum_{r=0}^{n} \binom{n}{r} c^n L_n^{(r,a,m)}(cx,dy;k,a,b) L_r^{(a,m)}(dw,dz;k,a,b).
\]

Proof. We start to prove by the following expression:

\[
g(t) = \frac{c^k d^k \alpha^x \beta^y}{\alpha^x \beta^y} \left( \sum_{r=0}^{m} \frac{t^r}{r!} L_r^{(a,m)}(cx,dy;k,a,b) L_r^{(a,m)}(dw,dz;k,a,b) \right).
\]

Then the expression for \( g(t) \) is symmetric in \( a \) and \( b \) and we can expand \( g(t) \) into series in two ways to obtain

\[
g(t) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(ct)^r}{n!} \frac{(dt)^r}{r!} L_n^{(a,m)}(cx,dy;k,a,b) L_r^{(a,m)}(dw,dz;k,a,b).
\]

On similar lines we can show that

\[
g(t) = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(ct)^r}{n!} \frac{(dt)^r}{r!} L_n^{(a,m)}(cx,dy;k,a,b) L_r^{(a,m)}(dw,dz;k,a,b).
\]

by comparing the coefficients of \( t^n \) on the right-hand sides of the last two equations we arrive at the desired result.

For \( k = a = b = 1 \) and \( \beta = \lambda \) in Theorem 8, we get the following corollary.
Corollary 14. We have the following symmetry identity for the Laguerre-based generalized Apostol-Bernoulli polynomials
\[
\sum_{r=0}^{n} \binom{n}{r} d^r c^{n-r} L_{n-r}^{[a,m]}(dx, dy; \lambda) L_{r}^{[a,m]}(cw, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} c^r d^{n-r} L_{n-r}^{[a,m]}(cx, cy; \lambda) L_{r}^{[a,m]}(dw, dz; \lambda).
\]
For \(k + 1 = -a = b = 1\) and \(\beta = \lambda\) in Theorem 8, we get the corollary.

Corollary 15. We have for each pair of positive even integers \(c\) and \(d\) or for each pair of positive odd integers \(c\) and \(d\).
\[
\sum_{r=0}^{n} \binom{n}{r} d^r c^{n-r} L_{n-r}^{[a,m]}(dx, dy; \lambda) L_{r}^{[a,m]}(cw, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} c^r d^{n-r} L_{n-r}^{[a,m]}(cx, cy; \lambda) L_{r}^{[a,m]}(dw, dz; \lambda).
\]
Letting \(k = -2a = b = 1\) and \(2\beta = \lambda\) in Theorem 8, we get the corollary.

Corollary 16. We have for each pair of positive even integers \(c\) and \(d\) or for each pair of positive odd integers \(c\) and \(d\).
\[
\sum_{r=0}^{n} \binom{n}{r} c^r d^{n-r} G_{n-r}^{[a,m]}(dx, dy; \lambda) G_{r}^{[a,m]}(cw, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} d^r c^{n-r} G_{n-r}^{[a,m]}(cx, cy; \lambda) G_{r}^{[a,m]}(dw, dz; \lambda).
\]

Theorem 9. The following identity holds true:
\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^r d^{n-r} L_{n-r}^{[a,m]}(dx + \frac{d}{c} i + j, dy; k, a, b) L_{r}^{[a,m]}(cw, cz; k, a, b)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} d^r c^{n-r} L_{n-r}^{[a,m]}(cx + \frac{c}{d} i + j, cy; k, a, b) L_{r}^{[a,m]}(dw, dz; k, a, b).
\]

Proof. Let
\[
g(t) = \left( \frac{2^{2k-m}c^k t^{2k}}{\left( \beta^k e^{2z} - \alpha^k \sum_{h=0}^{m-1} \frac{e^{zh}}{h!} \right) \left( \beta^k e^{2\lambda} - \alpha^k \sum_{h=0}^{m-1} \frac{e^{\lambda h}}{h!} \right)} \right)^x \exp \left( (e^{cdt} - 1)^2 \right) / \left( (e^{dt} - 1)(e^{\lambda t} - 1) \right) \cdot \frac{\exp(\text{cd}(x + z)t)}{J_0(2 \sqrt{\text{cd}yt}) J_0(2 \sqrt{\text{cd}wt})}.
\]
From this formula and using same technique as in Theorem 9, we arrive at the desired result. \(\square\)

For \(k = a = b = 1\) and \(\beta = \lambda\) in Theorem 8, we get the following corollary.

Corollary 17. We have the following symmetry identity for the Laguerre-based generalized Apostol-Bernoulli polynomials
\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-r} d^r L B^{(u,m-1)}_{n-r}(dx + \frac{d}{c}j + dy; \lambda) L B^{(u,m-1)}_{r}(cw, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^{n-r} d^r L E^{(u,m-1)}_{n-r}(cx + \frac{c}{d}i + jy; \lambda) L E^{(u,m-1)}_{r}(cw, dz; \lambda).
\]

For \( k + 1 = -\alpha = b = 1 \) and \( \beta = \lambda \) in Theorem 9, we get the corollary.

**Corollary 18** We have for each pair of positive even integers and \( d \) or for each pair of positive odd integers \( c \) and \( d \)

\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-r} d^r L C^{(u,m-1)}_{n-r}(dx + \frac{d}{c}j + dy; \lambda) L C^{(u,m-1)}_{r}(cw, dz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^{n-r} d^r L E^{(u,m-1)}_{n-r}(cx + \frac{c}{d}i + jy; \lambda) L E^{(u,m-1)}_{r}(cw, dz; \lambda).
\]

Letting \( k = -2\alpha = b = 1 \) and \( 2\beta = \lambda \) in Theorem 9, we get the corollary.

**Corollary 19** We have for each pair of positive even integers \( c \) and \( d \) or for each pair of positive odd integers \( c \) and \( d \).

\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-r} d^r G^{(u,m-1)}_{n-r}(dx + \frac{d}{c}j + dy; \lambda) L G^{(u,m-1)}_{r}(cw, dz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^{n-r} d^r L G^{(u,m-1)}_{n-r}(cx + \frac{c}{d}i + jy; \lambda) L G^{(u,m-1)}_{r}(cw, dz; \lambda).
\]

**Theorem 10** The following identity holds true:

\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-r} d^r P^{(u,m)}_{n-r,\alpha}(dx + \frac{d}{c}j + dy; k, a, b) L P^{(u,m)}_{r,\beta}(cw + \frac{c}{d}i; cz, k, a, b)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^{n-r} d^r L P^{(u,m)}_{n-r,\alpha}(cx + \frac{c}{d}i + jy; k, a, b) L P^{(u,m)}_{r,\beta}(dw + \frac{d}{c}j; dz, k, a, b).
\]

**Proof** The proof is analogous to Theorem 9. So we omit the proof of this theorem. \( \square \)

For \( k = \alpha = b = 1 \) and \( \beta = \lambda \) in Theorem 10, we get the following corollary.

**Corollary 20** We have the following symmetry identity for the Laguerre-based generalized Apostol-Bernoulli polynomials

\[
\sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{c-1} \sum_{j=0}^{d-1} c^{n-r} d^r L B^{(u,m-1)}_{n-r}(dx + \frac{d}{c}j + dy; \lambda) L B^{(u,m-1)}_{r}(cw + \frac{c}{d}i; cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \binom{n}{r} \sum_{i=0}^{d-1} \sum_{j=0}^{c-1} c^{n-r} d^r L B^{(u,m-1)}_{n-r}(cx + \frac{c}{d}i + jy; \lambda) L B^{(u,m-1)}_{r}(dw + \frac{d}{c}j; dz; \lambda).
\]

For \( k + 1 = -\alpha = b = 1 \) and \( \beta = \lambda \) in Theorem 10, we state the following corollary.

**Corollary 21** We have for each pair of positive even integers \( c \) and \( d \) or for each pair of positive odd integers \( c \) and \( d \).
\[
\sum_{m=0}^{n} \left( \begin{array}{l} n \\ r \end{array} \right) \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} c^{n-r} d^{r} E_{n-r}^{(a,m-1)} \left( dx + \frac{d}{c} i, dy; \lambda \right) E_{r}^{(a,m-1)}(cw + \frac{c}{d} j, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \left( \begin{array}{l} n \\ r \end{array} \right) \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} c^{n-r} d^{r} E_{n-r}^{(a,m-1)} \left( cx + \frac{c}{d} j, cy; \lambda \right) G_{r}^{(a,m-1)}(dw + \frac{d}{c} j, dz; \lambda).
\]

Letting \( k = -2a = b = 1 \) and \( 2\beta = \lambda \) in Theorem 10, we get the corollary.

**Corollary 22** We have for each pair of positive even integers \( c \) and \( d \) or for each pair of positive odd integers \( c \) and \( d \)
\[
\sum_{r=0}^{n} \left( \begin{array}{l} n \\ r \end{array} \right) \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} c^{n-r} d^{r} E_{n-r}^{(a,m-1)} \left( dx + \frac{d}{c} i, dy; \lambda \right) E_{r}^{(a,m-1)}(cw + \frac{c}{d} j, cz; \lambda)
\]
\[
= \sum_{r=0}^{n} \left( \begin{array}{l} n \\ r \end{array} \right) \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} c^{n-r} d^{r} E_{n-r}^{(a,m-1)} \left( cx + \frac{c}{d} j, cy; \lambda \right) G_{r}^{(a,m-1)}(dw + \frac{d}{c} j, dz; \lambda).
\]

**5. Conclusion**

In Section 2, we have established the generating functions for the Laguerre-based Apostol-type polynomials and Laguerre-based Apostol-type Hermite polynomials by making use of Tricomi function of the generating function for Laguerre polynomials. The equivalent forms of these generating functions can be derived by using Equations (1.1), (1.6) and (2.1). They can be viewed as the equivalent forms of the generating functions (2.3), (2.6) and (2.8), respectively. In the previous sections, we have used the concepts and the formalism associated with Laguerre polynomials to introduce the Laguerre-based Apostol-type polynomials and Laguerre-based Apostol-type Hermite polynomials and establish their properties. The approach presented here is general and we have established the summation rules, which can be used to derive the results for Laguerre-based Apostol-type polynomials from the corresponding Appell polynomials.

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