Chiral heat wave in cold Fermi liquid and modified zero sound

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We discuss kinetic equations involving the anomalous terms responsible for the chiral anomaly. The general chiral heat wave in cold Fermi liquid is described and the modification of the anomalous zero sound at small temperature and vorticity is found.

\section{Introduction}

The anomalous contributions in the kinetic equations and hydrodynamics have been overlooked for many years however now the corresponding modifications are well established \cite{1,2} in both settings. The anomalous terms follow from the nontrivial connection in the momentum space emerging due to the level crossing phenomena. For the relativistic particle the Berry connection involves the monopole connection where helicity yields the corresponding charge of the particle. Both chiral anomaly in the external magnetic field and the mixed anomaly modify the kinetic equations \cite{2,3}. Recently the second order approximation to the kinetic and hydrodynamical equations in the chiral matter has been considered \cite{6}.

The new terms in the kinetics imply the possibility of the new types of the collective excitations and modification of the old ones. The first example of the new modes was developed in \cite{4} for the chiral magnetic wave and later the similar chiral vortical wave has been found \cite{5}. The mixing of these new modes as well as the modes emerging from the temperature gradients have been discussed in \cite{7,8,14}. The most general pattern of the mixing considered so far has been analyzed in \cite{10}.

The analysis takes some care due to the subtleties concerning the Lorentz invariance \cite{13,15,16}. In particular it was argued that condition of the Lorentz invariance imposes nontrivial restriction on the dynamics of the collective modes and for instance the careful analysis shows the way how the total contribution for the chiral magnetic effect comes from the several independent contributions \cite{13}.

In this Letter we focus on the collective wave in the cold Fermi liquid \cite{12} assuming that temperature is small enough. We investigate the mixing of the different modes in terms of the kinetic equations in two opposite limits in the relaxation time. In the first limit we consider the hydro-
dynamical approximation and reproduce the dispersion law of the collective modes derived in [10] in the different approach. In the opposite limit we consider the zero sound corresponding to the fluctuations of the Fermi surface. The modification of the anomalous zero sound discussed in [9, 11] due to the vorticity and temperature will be derived.

The note is organized as follows. First, we remind the kinetic equations involving the anomalous terms. In Section 3 we consider the hydrodynamic limit of the kinetic equations while in Section 4 we derive the modifications of the anomalous zero sound in the cold Fermi liquid.

II. KINETIC DESCRIPTION

A natural way to describe Fermi liquid is through right and left quasiparticles. We denote their distribution functions as \( n_R(t, \mathbf{x}, \mathbf{p}) \) and \( n_L(t, \mathbf{x}, \mathbf{p}) \), respectively. Energy of the quasiparticles is given by the energy functionals \( \epsilon_R(\mathbf{p})[n_R, n_L] \) and \( \epsilon_L(\mathbf{p})[n_R, n_L] \). Later we always refer to the quasiparticles as particles, because we will not deal with any actual particles. We begin with usual kinetic equations for both kinds of particles:

\[
\frac{\partial n_{R/L}}{\partial t} + \dot{x}_{R/L} \cdot \frac{\partial n_{R/L}}{\partial \mathbf{x}} + \dot{\mathbf{p}}_{R/L} \cdot \frac{\partial n_{R/L}}{\partial \mathbf{p}} = C_{R/L}[n_R, n_L],
\]

where \( C_{R/L} \) are collision integrals. They are supplemented with the equations of motion which take into account rotation of the liquid (treated as a perturbation to the metric) and anomalous Berry curvature contribution. The gravitomagnetic analogy tells us that for small angular velocities quasiparticles in rotating background behave as if they were in an external energy-dependent magnetic field \( \mathbf{B} \sim 2\epsilon\omega \), where \( \epsilon \) is a quasiparticle energy \([17]\). The equations of motion for quasiparticles with Berry curvature in an external magnetic field are known \([12]\), so in our case they are:

\[
\sqrt{G_R} \dot{\mathbf{x}}_R = \mathbf{v}_R + 2\epsilon_R \omega (\mathbf{v}_R \cdot \mathbf{b}_R) + \mathbf{E}_R \times \mathbf{b}_R,
\]

\[
\sqrt{G_L} \dot{\mathbf{x}}_L = \mathbf{v}_L + 2\epsilon_L \omega (\mathbf{v}_L \cdot \mathbf{b}_L) + \mathbf{E}_L \times \mathbf{b}_L,
\]

\[
\sqrt{G_R} \dot{\mathbf{p}}_R = \mathbf{E}_R + 2\epsilon_R \mathbf{v}_R \times \omega + (\mathbf{E}_R \cdot \omega)2\epsilon_R \mathbf{b}_R,
\]

\[
\sqrt{G_L} \dot{\mathbf{p}}_L = \mathbf{E}_L + 2\epsilon_L \mathbf{v}_L \times \omega + (\mathbf{E}_L \cdot \omega)2\epsilon_L \mathbf{b}_L,
\]

where \( \mathbf{v}_{R/L} = \frac{\partial \epsilon_{R/L}}{\partial \mathbf{p}} \) are particle velocities, \( p = |\mathbf{p}| \), \( v_F \) is Fermi speed (we assume that it is the same for left and right particles), \( \mathbf{E}_{R/L} = -\frac{\partial \epsilon_{R/L}}{\partial \mathbf{x}} \), \( \mathbf{b}_{R/L} = \pm \frac{\dot{\mathbf{p}}}{2p^3} \) are Berry connections in momentum space and factors \( \sqrt{G_{R/L}} = 1 + 2\epsilon_{R/L}(\mathbf{b}_{R/L} \cdot \omega) \) modify phase space volume due to the interplay between Berry phase and vorticity.
We are interested in waves propagating in the liquid so we parametrize the distribution functions as planar wave fluctuations above the equilibrium configuration (which we assume to be the same for left and right particles):

\[ n_{R/L} = n^0 + \frac{\partial n^0}{\partial \mu} h_{R/L}(p) e^{i(\nu t - kx)}, \]

(6)

where \( n^0 = \frac{1}{e^{\beta(e^0 - \mu)} + 1} \) is usual Fermi-Dirac distribution (note that \( e^0 \) are energies of particles having this particular distribution function). Throughout this paper we consider \( \mu \gg T \) which corresponds to cold regime.

Such parametrization of the distribution functions leads to parametrization of energy fluctuations and collision integral as

\[ \delta \epsilon_{R/L} = \epsilon_{R/L} - e^0 = F_{R/L}[h_R, h_L] e^{i(\nu t - kx)} + O(h^2), \]

(7)

and, since in equilibrium the collision integral is zero,

\[ C_{R/L} = \frac{\partial n^0}{\partial \mu} I_{R/L}[h_R, h_L] e^{i(\nu t - kx)} + O(h^2), \]

(8)

where \( F \)'s and \( I \)'s are linear functionals. Plugging all these (2)-(8) back into the kinetic equations after some rearrangement we obtain:

\[ -i\nu h_R + \dot{x}_R^0 \cdot (i k + 2\frac{e^0}{\sqrt{G_R}} \omega \times \frac{\partial}{\partial p}) \{ h_R + F_{RR}[h_R] + F_{RL}[h_L] \} = I_R[h_R, h_L], \]

(9)

\[ -i\nu h_L + \dot{x}_L^0 \cdot (i k + 2\frac{e^0}{\sqrt{G_L}} \omega \times \frac{\partial}{\partial p}) \{ h_L + F_{LR}[h_R] + F_{LL}[h_L] \} = I_L[h_R, h_L], \]

(10)

where we have split the linearized energy functionals into left and right parts: \( F_{R}[h_R, h_L] = F_{RR}[h_R] + F_{RL}[h_L] \) and \( F_{L}[h_R, h_L] = F_{LR}[h_R] + F_{LL}[h_L] \). Here \( \dot{x}_{R/L}^0 = \frac{1}{\sqrt{G_{R/L}}} [\dot{v}^0 + 2\epsilon_{R/L} \omega (\dot{v}^0 \cdot b_{R/L})] \).

Further we are going to study two opposite regimes: hydrodynamic regime of low frequency and small wave vector and zero sound regime of high frequency and large wave vector. If \( \tau \) is some characteristic relaxation time entering the collision integral, these regimes are given, respectively, by \( \nu \tau \ll 1 \) and \( \nu \tau \gg 1 \).

### III. HYDRODYNAMIC REGIME

In hydrodynamic regime \( \nu, k \to 0 \). Note that there exist solutions to the kinetic equations given by infinitesimal constant shift of chemical potentials and temperature, accompanied by the
corresponding shifts in the functions of particle energy in equilibrium (the last is because for Fermi liquid energy of particles is, generally speaking, a functional of the distribution function). So we parametrize the fluctuations as (note that we allow the chemical potentials of right and left particles to fluctuate independently):

\[ h_{R/L} = \delta \mu_{R/L} - \delta \epsilon_{R/L} - \frac{\epsilon^0 - \mu}{T} \delta T \]  

(11)

Since this is a solution for small constant \( \delta \mu_{R/L} \) and \( \delta T \), we expect that to the lowest order in \( \nu \) and \( k \) the solution we are looking for will have the same form. From here on let us assume the simplest possible form of the linearized energy functionals: \( F_{RR}[h_R] = F_S \langle h_R \rangle_R, F_{RL}[h_L] = F_A \langle h_L \rangle_L, F_{LR}[h_R] = F_A \langle h_R \rangle_R \) and \( F_{LL}[h_L] = F_S \langle h_L \rangle_L \), where we have introduced averaging over momentum space:

\[ \langle ... \rangle_R = \frac{1}{\chi} \int \sqrt{G_R} \frac{\partial n^0}{\partial \mu} (...) \]  

(12)

\[ \langle ... \rangle_L = \frac{1}{\chi} \int \sqrt{G_L} \frac{\partial n^0}{\partial \mu} (...) \]  

(13)

Here \( \int_p = \int \frac{d^3p}{(2\pi)^3}, \chi = \int_p \sqrt{G_R} \frac{\partial n^0}{\partial \mu} = \int_p \sqrt{G_L} \frac{\partial n^0}{\partial \mu} \) since the equilibrium distribution is isotropic and the differences between \( \sqrt{G_R} \) and \( \sqrt{G_L} \) do not show up. Note that the normalization is so that \( \langle 1 \rangle_R = \langle 1 \rangle_L = 1 \). So energy fluctuations have the form

\[ \delta \epsilon^0_R = F_S \langle h_R \rangle_R + F_A \langle h_L \rangle_L, \]  

(14)

\[ \delta \epsilon^0_L = F_A \langle h_R \rangle_R + F_S \langle h_L \rangle_L. \]  

(15)

Let us for convenience further denote \( \delta \mu_R \) as \( h_1 \), \( \delta \mu_L \) as \( h_2 \) and \( (\delta T) \) as \( h_3 \).

To get rid of the collision integrals let us make use of conservation of the number of right and left particles and energy. It implies for collision integrals

\[ \int_p \sqrt{G_R} C_R[n_R, n_L] = 0, \]  

(16)

\[ \int_p \sqrt{G_L} C_L[n_R, n_L] = 0, \]  

(17)

\[ \int_p (\sqrt{G_R} \epsilon_R[n_R, n_L] C_R[n_R, n_L] + \sqrt{G_L} \epsilon_L[n_R, n_L] C_L[n_R, n_L]) = 0, \]  

(18)

for any \( n_R, n_L \) and corresponding \( \epsilon_R[n_R, n_L], \epsilon_L[n_R, n_L] \) which implies for any \( h_R, h_L \):
\[ \langle I_R[h_R, h_L] \rangle_R = 0, \tag{19} \]
\[ \langle I_L[h_R, h_L] \rangle_L = 0, \tag{20} \]
\[ \langle \epsilon_R I_R[h_R, h_L] \rangle_R + \langle \epsilon_L I_L[h_R, h_L] \rangle_L = 0. \tag{21} \]

So we act on the equations (9), (10) with the averaging operation to obtain

\[ -i\nu \langle h_R \rangle + ik \cdot \langle \dot{x}_R^0(h_R + F_S\langle h_R \rangle + F_A\langle h_L \rangle) \rangle_R = 0, \tag{22} \]
\[ -i\nu \langle h_L \rangle + ik \cdot \langle \dot{x}_L^0(h_L + F_A\langle h_R \rangle + F_S\langle h_L \rangle) \rangle_L = 0, \tag{23} \]
\[ -i\nu(\langle h_R^0 \rangle + \langle h_L^0 \rangle) + ik \cdot (\langle \dot{x}_R^0(h_R + F_S\langle h_R \rangle + F_A\langle h_L \rangle)^0 \rangle_R + \langle \dot{x}_L^0(h_L + F_A\langle h_R \rangle + F_S\langle h_L \rangle)^0 \rangle_L) = 0. \tag{24} \]

Here we used the fact that since the quantities such as \( h_R, h_L, h_R^0, h_L^0 \) are isotropic we may forget about the \( \sqrt{G} \)'s factors and average them just as

\[ \langle ... \rangle = \frac{1}{\chi} \int_p \frac{\partial n^0}{\partial \mu}. \tag{25} \]

Also this isotropy leads to vanishing of the terms, proportional to \( \omega \times \frac{\partial}{\partial p} \). Note that since \( h \)'s are small and \( \epsilon_{R/L} = \epsilon^0 + \text{"terms linear in } h \" \) in the equation (21) we only keep \( \epsilon^0 \) since we are solely interested in the terms, linear in \( h \). Let us for future convenience change the notation here. From here on we will work with the variables \( h_V = h_1 + h_2 \) and \( h_A = h_1 - h_2 \) having the meaning of fluctuations of vector and axial chemical potentials correspondingly. Also for convenience in future we will not work with the two first equations of the system (22) separately but instead we will sum them up and subtract the second from the first one.

Then,

\[ \langle h_R \rangle + \langle h_L \rangle = h_V - (F_S + F_A)(\langle h_R \rangle + \langle h_L \rangle) + 2\langle \frac{\epsilon^0 - \mu}{T} \rangle h_3, \tag{26} \]
\[ \langle h_R \rangle - \langle h_L \rangle = h_A + (F_A - F_S)(\langle h_R \rangle - \langle h_L \rangle), \tag{27} \]

so

\[ \langle h_R + h_L \rangle = \frac{h_V + 2\langle \frac{\epsilon^0 - \mu}{T} \rangle h_3}{1 + F_S + F_A}, \tag{28} \]
\[ \langle h_R - h_L \rangle = \frac{h_A}{1 + F_S - F_A}. \tag{29} \]
Similarly
\[
(h_R + h_L)\epsilon^0 = [h_V - (F_S + F_A)(h_R + h_L)]\langle \epsilon^0 \rangle + 2\langle \frac{\epsilon^0 - \mu}{T} \epsilon^0 \rangle h_3.
\] (30)

Now we calculate \( \langle \epsilon^0 \rangle \) and \( \langle (\epsilon^0)^2 \rangle \). Since the temperature is low, \( T \ll \mu \), we will only keep terms up to quadratic in temperature. At low temperatures all the excitations are localized near the Fermi sphere, so we will expand the dispersion relation of the particles near it:

\[
\epsilon^0 = \mu + v_F(p - p_F),
\] (31)

Here \( p_F \) is Fermi momentum and we assume \( p_F v_F \sim \mu \). In fact, the next terms with the second and the third powers of \( p - p_F \) would also contribute to some quantities we are going to calculate in the order, interesting to us. However, it doesn’t seem that including such terms will give rise to any conceptual difference. So we will omit them to reduce the mess in the calculations and restrict ourselves with the linear dispersion relation for the particles not far from the Fermi surface. Reversing this relation we obtain

\[
p = p_F + \frac{\epsilon^0 - \mu}{v_F}.
\] (32)

Then,

\[
\langle \epsilon^0 \rangle = \frac{1}{\chi} \int_p \frac{\partial n^0}{\partial \mu} \epsilon^0 = \frac{1}{2\pi^2 \chi} \int_0^\infty \frac{\beta \epsilon_0 e^{\beta(\epsilon_0 - \mu)}}{(e^{\beta(\epsilon_0 - \mu)} + 1)^2} \beta^2 d\mu \approx \frac{T}{2\pi^2 \chi v_F} \int_{-\infty}^\infty \frac{x + \beta \mu}{(e^x + 1)^2} \left( p_F^2 + \frac{2p_F T x}{v_F} + \frac{x^2 T^2}{v_F^2} \right) dx
\]
\[
\approx \frac{1}{2\pi^2 \chi v_F} \left( \mu p_F^2 + \frac{T^2 \pi^2}{3v_F^2} \left( 2p_F v_F + \mu \right) \right).
\] (33)

Note that when integrating over \( x = \beta(\epsilon^0 - \mu) \) we have set the lower integration limit to minus infinity, since it is actually \( -\beta \mu \) where the function is exponentially small and does not contribute to the integral. Similarly,

\[
\langle (\epsilon^0)^2 \rangle \approx \frac{1}{2\pi^2 \chi v_F} \left[ p_F^2 \mu^2 + \frac{T^2 \pi^2}{3} \left( p_F^2 + \frac{4 \mu}{v_F} p_F + \frac{\mu^2}{v_F^2} \right) + \frac{7 \pi^4 T^4}{15} v_F^2 \right].
\] (34)

Let us also present the expression for \( \chi \):

\[
\chi = \int_p \frac{\partial n^0}{\partial \mu} = \frac{1}{2\pi^2 v_F} \left( p_F^2 + \frac{T^2 \pi^2}{3v_F^2} \right). \] (35)

Then,
\[ \frac{\langle e^0 - \mu \rangle}{T} = \frac{p_F T}{3\chi v_F^2}, \]  
\[ \langle e^0 - \mu \rangle = \frac{T}{6\chi v_F} \left( p_F^2 + 2p_F \frac{\mu}{v_F} + \frac{7\pi^2 T^2}{5 v_F^2} \right). \]  

So, finally

\[ \langle h_R + h_L \rangle = \frac{h_V + h_3 \frac{2\beta T}{3\chi v_F}}{1 + F_S + F_A}, \]  
\[ \langle h_R - h_L \rangle = \frac{h_A}{1 + F_S - F_A}, \]  
\[ \langle (h_R + h_L) e^0 \rangle = \frac{h_V}{1 + F_S + F_A} \left( \mu + \frac{2T^2 \pi^2}{3p_F v_F} \right) + \frac{h_3 T}{3\chi v_F} \left( \frac{p_F^2}{v_F} + \frac{2p_F \mu}{v_F(1 + F_S + F_A)} + \frac{\pi^2 T^2(1 + F_S + F_A)}{15 v_F^2(1 + F_S + F_A)} \right). \]  

Now let's proceed to calculating terms like \( \langle \dot{\mathbf{x}}_R^0 h_R \rangle \) and \( \langle \dot{\mathbf{x}}_R^0 e^0 h_R \rangle \). Since the only term in \( h \)'s depending on \( p \) is \( \frac{e^0 - \mu}{T} \), it will suffice to calculate \( \langle \dot{\mathbf{x}}_R^0 \rangle \) and \( \langle \dot{\mathbf{x}}_R^0 (e^0 - \mu) \rangle \). Let us remind that

\[ \dot{\mathbf{x}}_{R/L} = \frac{1}{\sqrt{G_{R/L}}} [\mathbf{v}^0 + 2\mathbf{b}_{R/L} + \mathbf{b}_{R/L} \cdot \mathbf{b}_{R/L}] . \]  
Note that the first term will not give a contribution after averaging since the rest of the expression is isotropic.

\[ \langle \dot{\mathbf{x}}_R^0 \rangle = \frac{2\omega}{2\pi^2 \chi} \int_0^\infty e^0 v^0 \frac{1}{2p^2} p^2 dp \frac{\partial n^0}{\partial \mu} = \frac{\omega}{2\pi^2 \chi} \int_{-\infty}^{\infty} T(x + \beta \mu) v_F \beta e^x \frac{T}{(e^x + 1)^2} dx = \frac{\omega \mu}{2\pi^2 \chi}. \]  

Similarly,

\[ \langle \dot{\mathbf{x}}_R^0 (e^0 - \mu) \rangle = \frac{\omega T^2}{6\chi}. \]  

For the left particles all that changes compared to the right ones is \( \mathbf{b}_R \rightarrow \mathbf{b}_L = -\mathbf{b}_R \), which effectively just causes terms with \( \omega \) to change sign. So, we obtain

\[ \langle \dot{\mathbf{x}}_R^0 (h_R + F_S (h_R) + F_A (h_R)) \rangle + \langle \dot{\mathbf{x}}_L^0 (h_L + F_A (h_L)) \rangle = \frac{\mu \omega}{2\pi^2 \chi} h_A, \]  
\[ \langle \dot{\mathbf{x}}_R^0 (h_R + F_S (h_R) + F_A (h_R)) \rangle - \langle \dot{\mathbf{x}}_L^0 (h_L + F_A (h_R) + F_S (h_L)) \rangle = \frac{\mu \omega}{2\pi^2 \chi} h_V + \frac{\omega T}{3\chi} h_3, \]  
\[ \langle \dot{\mathbf{x}}_R^0 (h_R + F_S (h_R) + F_A (h_L)) e^0 \rangle + \langle \dot{\mathbf{x}}_L^0 (h_L + F_A (h_R) + F_S (h_L)) e^0 \rangle = \frac{\omega}{\chi} \left( \frac{\mu^2}{2\pi^2} + \frac{T^2}{6} \right) h_A. \]  

So, the system (22) - (24) transforms into
\[ \nu \left( \frac{h_V}{F_1} + h_3 \frac{2p_F T}{3v_F^2 F_1} \right) - \frac{(k \cdot \omega)}{\chi} h_A \frac{\mu}{2\pi^2} = 0, \quad (46) \]

\[ \nu \frac{h_A}{F_2} - \frac{(k \cdot \omega)}{\chi} \left( \frac{h_V \mu}{2\pi^2} + \frac{h_3 T}{3} \right) = 0, \quad (47) \]

\[ \nu \left[ \frac{h_V \mu}{F_1} \left( 1 + \frac{2\pi^2 T^2}{3\mu_F v_F} \right) + \frac{h_3 T}{3\chi v_F} \left( \frac{p_F^2 + \frac{2p_F \mu}{v_F F_1} + \frac{\pi^2 T^2 (20 + F_1)}{15 v_F^2 F_1}}{\chi} \right) \right] - \frac{h_A (k \cdot \omega)}{\chi} \left( \frac{\mu^2}{2\pi^2} + \frac{T^2}{6} \right) = 0. \quad (48) \]

Here we have introduced \( F_1 = 1 + F_S + F_A, \) \( F_2 = 1 + F_S - F_A. \) For this system to be consistent the following dispersion relation should be true (so that it’s determinant is zero):

\[ \nu = \pm \frac{\mu (k \cdot \omega)}{2\pi^2 \chi} \sqrt{F_1 F_2} \left[ 1 + \frac{T^2 \pi^2}{6\mu^2 F_1} \left( 1 - 4A + 4A^2 \right) \right] \quad (49) \]

Here we have introduced dimensionless parameter \( A = \frac{\mu}{v_F p_F}. \)

At the first sight, at zero temperature this dispersion relation differs from the known expression for the Chiral Vortical Wave (first obtained in [5]):

\[ \nu = \pm \frac{\mu (k \cdot \omega)}{2\pi^2 \chi}. \quad (50) \]

However, we should take into account that, firstly, our definition of \( \chi \) differs from the one in [5], and secondly, that in [5] it was supposed, that vector and axial susceptibilities are the same, which is not true in our case (if \( F_A \neq 0 \)). Namely, our definition of \( \chi \) (which we will later refer to \( \chi_{our} \)) is given by

\[ \chi_{our} = \int_p \sqrt{G_R} \frac{\partial n^0}{\partial \mu} = \int_p \frac{\partial n^0}{\partial \mu}, \]

which is not exactly susceptibility \( \chi = \frac{\delta N^0}{\delta \mu} \) as in [5] (where \( N^0 \) is density of particles) due to the interactions. As to the second point, if one does not suppose \( \chi_V = \chi_A \), where \( \chi_V = \frac{\delta N_V}{\delta \mu_V} \) and \( \chi_A = \frac{\delta N_A}{\delta \mu_A} \) (here \( N_{A/V} \) are vector and axial charge densities respectively and \( \mu_{V/A} \) are vector and axial chemical potentials respectively), then in the expression [50] \( \chi \) (denoting both \( \chi_V \) and \( \chi_A \) in case they coincide) should be replaced with \( \sqrt{\chi_V \chi_A} \). In order to show that (49) at zero temperature indeed coincides with (50) we will derive the expressions for \( \chi_V \) and \( \chi_A \) via \( \chi_{our}, F_1 \) and \( F_2 \):

\[ \chi = \frac{\delta N_V}{\delta \mu_V} = \frac{\delta N_R + \delta N_L}{\delta \mu_V}, \quad (51) \]

Since

\[ \delta N_R = \int_p \frac{\partial n_R}{\partial \mu} \delta h_R, \quad \delta N_L = \int_p \frac{\partial n_L}{\partial \mu} \delta h_L, \quad (52) \]

where

\[ \delta h_R = \delta \mu_R - \delta \epsilon_R, \quad \delta h_L = \delta \mu_L - \delta \epsilon_L, \quad (53) \]
and

\[ \delta \epsilon_R = F_S \delta h_R + F_A \delta h_L, \quad \delta \epsilon_L = F_A \delta h_R + F_S \delta h_L, \]  

(54)

we get

\[ \delta N_R = \int_{p} \frac{\partial n^0 \delta \mu_R}{\partial \mu} F_1, \quad \delta N_L = \int_{p} \frac{\partial n^0 \delta \mu_L}{\partial \mu} F_1, \]  

(55)

and finally

\[ \delta N_V = \int_{p} \frac{\partial n^0 \delta \mu_R + \delta \mu_L}{\partial \mu} F_1 = \int_{p} \frac{\partial n^0 \delta \mu_V}{\partial \mu} F_1, \]  

(56)

which gives

\[ \chi_V = \frac{\chi_{\text{our}}}{F_1}. \]  

(57)

Similar calculation gives

\[ \chi_A = \frac{\chi_{\text{our}}}{F_2}, \]  

(58)

which confirms that our answer (49) coincides with (50) at zero temperature.

At non-zero temperatures the corrections quadratic in temperature represent the mixing of Chiral Heat and Vortical Waves ([7, 10]).

IV. ZERO SOUND

A. Zero temperature

The regime opposite to the hydrodynamic one is given by \( \nu \tau \gg 1 \), so we may neglect collision terms. Then kinetic equations (9) and (10) look like

\[ -i \nu h_R + x_R^0 \cdot \left( \frac{ik + 2\epsilon^0 \omega \times \partial}{\partial p} \right) \{ h_R + \mathcal{F}_{RR}[h_R] + \mathcal{F}_{RL}[h_L] \} = 0, \]  

(59)

\[ -i \nu h_L + \dot{x}_L^0 \cdot \left( \frac{ik + 2\epsilon^0 \omega \times \partial}{\partial p} \right) \{ h_L + \mathcal{F}_{LR}[h_R] + \mathcal{F}_{LL}[h_L] \} = 0, \]  

(60)

At first let us work at zero temperature and later we will find the temperature corrections perturbatively. For simplicity we are going to analyze excitations propagating along the vorticity \( \mathbf{k}||\omega \). Let us look for the axially-symmetric solutions, so that \( h_R = h_R(\theta), h_L = h_L(\theta) \), where \( \theta \) is the angle
between \( p \) and \( k \). Under these assumptions the terms of the form \( \omega \times \frac{\partial}{\partial p} \) will vanish in the above equations. We will also assume, like in the previous subsection, that the linearized energy functionals have the simplest possible form \( F_{RR}[h_R] = F_S(h_R)_R, F_{RL}[h_L] = F_A(h_L)_L, F_{LR}[h_R] = F_A(h_R)_R \) and \( F_{LL}[h_L] = F_S(h_L)_L \). Performing the averaging in the axially-symmetric case we obtain

\[
F_{RR}[h_R] = \frac{F_S}{\chi} \int_p \frac{\partial n^0}{\partial \mu} \sqrt{G_R h_R(\theta)} = \frac{F_S}{4\pi^2 \chi} \int_0^\pi \sin \theta h_R(\theta) d\theta \int_0^\infty p^2 dp e^x \left( 1 + \frac{2e^0 \omega \cos \theta}{2p^2} \right)
\]

\[
= F_S \left( \int_0^\pi \frac{\sin \theta h_R(\theta) d\theta}{2} + \frac{\mu \omega}{4\pi^2 v_F} \int_0^\pi \sin \theta \cos \theta h_R(\theta) d\theta \right). \tag{61}
\]

Let us denote

\[
B^0_{R/L} = \int_0^\pi \frac{\sin \theta h_{R/L}(\theta) d\theta}{2}, \quad D^0_{R/L} = \int_0^\pi \frac{\sin \theta \cos \theta h_{R/L}(\theta) d\theta}{2},
\]

\[
a_0 = \frac{\mu \omega}{2\pi^2 v_F} \bigg|_{T=0} = \frac{\mu \omega}{p_F} \tag{63}
\]

Then

\[
F_{RR}[h_R(\theta)] = F_S(B^0_R + a_0 D^0_R), \quad F_{RL}[h_L(\theta)] = F_A(B^0_L - a_0 D^0_L), \tag{64}
\]

\[
F_{RL}[h_R(\theta)] = F_A(B^0_R + a_0 D^0_R), \quad F_{LL}[h_L(\theta)] = F_S(B^0_L - a_0 D^0_L). \tag{65}
\]

Taking all this into account and plugging back into (59)-(60) we obtain

\[
-s h_R(\theta) (1 + a_0 \cos \theta) + (\cos \theta + a_0) [h_R(\theta) + F_S(B^0_R + a_0 D^0_R) + F_A(B^0_L - a_0 D^0_L)] = 0, \tag{66}
\]

\[
-s h_L(\theta) (1 - a_0 \cos \theta) + (\cos \theta - a_0) [h_R(\theta) + F_A(B^0_R + a_0 D^0_R) + F_S(B^0_L - a_0 D^0_L)] = 0. \tag{67}
\]

Here we denoted \( s = \frac{\nu}{v_F k} \). Expressing \( h^0_{R/L} \) in terms of \( B^0_{R/L}, D^0_{R/L} \) and plugging it back into the definitions of \( B^0_{R/L}, D^0_{R/L} \) we get the closed system:

\[
A^0_R = I(a_0)(F_S A^0_R + F_A A^0_L), \tag{68}
\]

\[
A^0_L = I(-a_0)(F_A A^0_R + F_S A^0_L), \tag{69}
\]

where we denoted \( A^0_{R/L} = B^0_{R/L} \pm a_0 D^0_{R/L} \) and introduced the following integral

\[
I(a_0) = \frac{1}{2} \int_{-1}^{1} \frac{dx}{s - x + a_0(sx - 1)}. \tag{70}
\]
The determinant of the system \([\text{68}]-[\text{69}]\) should be zero for it to be consistent, which gives us an equation upon \(I(a_0), I(-a_0)\) and, therefore, \(s\):

\[I(a_0)I(-a_0)(F_S^2 - F_A^2) - F_S[I(a_0) + I(-a_0)] + 1 = 0.\] (71)

Note that we assume that the vorticity is small, which implies \(a_0 \ll 1\) and we may calculate \(I(a_0)\) only up to the lowest orders in \(a_0\). It follows from (71) that terms linear in \(a_0\) will not contribute, so we keep terms up to the second order. Parametrizing the integral as

\[I(a_0) = L_0(s_0) + a_0L_1(s_0) + a_0^2L_2(s_0),\] (72)

we easily find that

\[L_0(s_0) = s_0 \text{arcotanh } s_0 - 1,\] (73)

\[L_1(s_0) = 3s_0(s_0 \text{arcotanh } s_0 - 1),\] (74)

\[L_2(s_0) = 2s_0[-3s_0 + (3s_0^2 - 1) \text{arcotanh } s_0].\] (75)

Here we changed the notation from \(s\) to \(s_0\) to emphasize that it is the solution at zero temperature. The equation (71) is now transformed into

\[[L_0^2 - (L_1^2 - 2L_0L_2)a_0^2](F_S^2 - F_A^2) - 2F_S(L_0 + a_0^2L_2) + 1 = 0\] (76)

By resolving this equation in the zero order in \(a_0\) we find that \(s_0\), which we will call in this order \(s_0^0\) has to satisfy the irrational equation

\[\text{arcotanh } s_0^0 = \frac{1}{s_0^0} \left( \frac{1}{F_S \pm F_A} + 1 \right),\] (77)

which is unmodified zero sound dispersion relation in case of two fermion species. For the quadratic in \(a_0\) correction which actually contains the information about the vortical modification we have

\[\delta s_0 = s_0 - s_0^0 = -\pm a_0^2 \frac{F_SL_2(s_0^0) + [L_1(s_0^0)^2 - 2L_0(s_0^0)L_2(s_0^0)](F_S^2 - F_A^2)}{F_A[\text{arcotanh } s_0^0 - \frac{s_0^0}{2(s_0^0)^2 - 1}]}\] (78)

These last two equations completely define the modified zero sound dispersion relation at zero temperature in the lowest order in vorticity (remember that \(a_0\) is linear in \(\omega\), so the corrections are quadratic in \(\omega\)).
B. Non-zero temperature

Now let us find thermal corrections to the modified zero sound dispersion relation. To do that, we will introduce a new term in the fluctuation of distribution function: it will be still axially-symmetric but there will be some dependence on the absolute value of momentum now:

\[
h_{R/L} = h_{R/L}(\theta) + \delta h_{R/L}(p, \theta)
\]

(79)

Here we assume that the second term is small compared to the first one (since it is related to the temperature it is of order of some power of \( T \)). Since both temperature and vorticity are small we will neglect terms of the form \( \delta h_{R/L} \omega \). In this case

\[
F_{RR}[h_R] = F_S(\langle h_R(\theta) \rangle + \delta h_R) = F_S\left( A^0_R - \frac{\omega \mu \pi^2 T^2}{3p_F^4 v_F^2} D^0_R + \delta A_R \right) = F_S(A^0_R + \delta a D^0_R + \delta A_R),
\]

(80)

\[
F_{RL}[h_L] = F_A(A^0_L - \delta a D^0_L + \delta A_L),
\]

(81)

\[
F_{LR}[h_R] = F_A(A^0_R + \delta a D^0_R + \delta A_R),
\]

(82)

\[
F_{LL}[h_L] = F_S(A^0_L - \delta a D^0_L + \delta A_L).
\]

(83)

Here we denoted \(-\omega \mu \pi^2 T^2 / 3p_F^4 v_F^2\) as \( \delta a \), \( \langle \delta h_R \rangle_R \) as \( \delta A_R \) and \( \langle \delta h_L \rangle_L \) as \( \delta A_L \). When we plug these into (59) - (60) and note, that terms arising from \( \omega \times \partial / \partial p \) either vanish as before or are negligible in our approximation (we do not try to keep track of terms linear in \( \delta h_{R/L} \) multiplied by some power of \( \omega \)), we obtain:

\[
-s(h_R(\theta) + \delta h_R) \left( 1 + \frac{\epsilon_0 \omega}{p^2} \cos \theta \right) + \left( \cos \theta + \frac{\epsilon_0 \omega}{p^2} \right) \cdot [h_R(\theta) + \delta h_R + F_S(A^0_R + \delta a D^0_R + \delta A_R) + F_A(A^0_L - \delta a D^0_L + \delta A_L)] = 0,
\]

(84)

\[
-s(h_L(\theta) + \delta h_L) \left( 1 - \frac{\epsilon_0 \omega}{p^2} \cos \theta \right) + \left( \cos \theta - \frac{\epsilon_0 \omega}{p^2} \right) \cdot [h_L(\theta) + \delta h_L + F_A(A^0_R + \delta a D^0_R + \delta A_R) + F_S(A^0_L - \delta a D^0_L + \delta A_L)] = 0.
\]

(85)

Using these equations we express \( \delta h_{R/L} \) via everything else (note that we know the expressions for \( h_{R/L}(\theta) \)) and then averaging the corresponding expressions we find that the resulting system is consistent provided that \( a_0 \) in the discussion of zero-temperature case above is shifted to \( a_0 + \delta a \) where \( \delta a \) was defined above as \(-\omega \mu \pi^2 T^2 / 3p_F^4 v_F^2\). Doing that exploits the fact that

\[
\langle \frac{\epsilon_0 \omega}{p^2} \rangle_R = \langle \frac{\epsilon_0 \omega}{p^2} \rangle_L = a_0 + \delta a.
\]

(86)
Finally, summing it all up, the expression for the dispersion relation in question is

\[ s = s_0^0 + \delta s, \]  

where \( s_0^0 \) is the solution of one of the two equations

\[ \text{arcotanh} \, s_0^0 = \frac{1}{s_0^0} \left( \frac{1}{2(F_S \pm F_A)} + 1 \right), \]  

and \( \delta s \) is given by

\[ \delta s = \pm (a_0 + \delta a)^2 F_S L_2(s_0^0) + \left( L_1(s_0^0)^2 - 2L_0(s_0^0)L_2(s_0^0) \right) (F_S^2 - F_A^2) \]

\[ s_0^0 \]

\[ F_A \left( \text{arcotanh} \, s_0^0 - \frac{s_0^0}{2((s_0^0)^2-1)} \right) \]

\[ \pm \frac{\omega^2 \mu^2}{P_T^4} \left( 1 - \frac{2\pi^2 T^2}{3v_T^2 P_T^2} \right) \]

\[ F_A \left[ \text{arcotanh} \, s_0^0 - \frac{s_0^0}{2((s_0^0)^2-1)} \right] \]

\[ \approx \pm \frac{\omega^2 \mu^2}{P_T^4} \left( 1 - \frac{2\pi^2 T^2}{3v_T^2 P_T^2} \right) \]

So, we see that both at zero and non-zero temperature there are two modes of zero sound. In both cases the lowest order correction to the velocity of zero sound is proportional to the square of angular velocity. For the non-zero temperature case there is an extra correction, proportional to squares of both angular velocity and temperature.

V. CONCLUSION

In this note we have discussed the mixing of the collective modes in the anomalous kinetic equations in the cold Fermi liquid. We have reproduced the previous results in the hydrodynamical limit and find a new modification of the anomalous zero sound when the vorticity and temperature are taken into account.

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[1] D. T. Son and P. Surowka, Hydrodynamics with Triangle Anomalies , Phys. Rev. Lett.103 , 191601 (2009) [arXiv:0906.5044].

[2] M. A. Stephanov, Y. Yin, Chiral Kinetic Theory Phys. Rev. Lett. 109,162001 (2012) [arXiv:1207.0747].

[3] K. Landsteiner, E. Megias and F. Pena-Benitez, Gravitational Anomaly and Transport , Phys. Rev. Lett. 107, 021601 (2011) [arXiv:1103.5006].
K. Landsteiner, E. Megias, L. Melgar and F. Pena-Benitez, Holographic Gravitational Anomaly and Chiral Vortical Effect, JHEP 1109, 121 (2011) [arXiv:1107.0368].

[4] D. E. Kharzeev and H. U. Yee, Chiral Magnetic Wave, Phys. Rev. D 83, 085007 (2011) [arXiv:1012.6026].

[5] Y. Jiang, X. G. Huang and J. Liao, Chiral vortical wave and induced flavor charge transport in a rotating quark-gluon plasma, [arXiv:1504.03201 [hep-ph]].

[6] E. V. Gorbar, D. O. Rybalka and I. A. Shovkovy, “Second-order dissipative hydrodynamics for plasma with chiral asymmetry and vorticity,” [arXiv:1702.07791 [hep-th]].

E. V. Gorbar, V. A. Miransky, I. A. Shovkovy and P. O. Sukhachov, “Second-order chiral kinetic theory: chiral magnetic and pseudomagnetic waves,” [arXiv:1702.02950 [cond-mat.mes-hall]].

[7] M. N. Chernodub, Chiral Heat Wave and wave mixing in chiral media, [arXiv:1509.01245 [hep-th]].

[8] T. Kalaydzhyan and E. Murchikova, “Thermal chiral vortical and magnetic waves: new excitation modes in chiral fluids,” [arXiv:1609.00024 [hep-th]].

[9] M. A. Stephanov, H. U. Yee and Y. Yin, Collective modes of chiral kinetic theory in a magnetic field, Phys. Rev. D91, 125014 (2015) [arXiv:1501.00222].

[10] D. Frenklakh, “Chiral heat wave and mixed waves in kinetic theory,” Phys. Rev. D 94, no. 11, 116010 (2016) doi:10.1103/PhysRevD.94.116010 [arXiv:1603.08971 [hep-th]].

[11] A. Gorsky and A. V. Zayakin, “Anomalous Zero Sound,” JHEP 1302, 124 (2013) [arXiv:1206.4725 [hep-th]].

[12] D. T. Son and N. Yamamoto, Berry Curvature, Triangle Anomalies, and Chiral Magnetic Effect in Fermi Liquids,” [arXiv:1203.2697 [cond-mat.mes-hall]]

[13] D. E. Kharzeev, M. A. Stephanov and H. U. Yee, “Anatomy of chiral magnetic effect in and out of equilibrium,” [arXiv:1612.01674 [hep-ph]].

[14] J. Y. Chen, D. T. Son and M. A. Stephanov, “Collisions in Chiral Kinetic Theory,” Phys. Rev. Lett. 115, no. 2, 021601 (2015) doi:10.1103/PhysRevLett.115.021601 [arXiv:1502.06966 [hep-th]].

[15] Y. Hidaka, S. Pu and D. L. Yang, “Relativistic Chiral Kinetic Theory from Quantum Field Theories,” [arXiv:1612.04630 [hep-th]].

[16] J. Y. Chen, D. T. Son, M. A. Stephanov, H. U. Yee and Y. Yin, “Lorentz Invariance in Chiral Kinetic Theory,” Phys. Rev. Lett. 113, no. 18, 182302 (2014) doi:10.1103/PhysRevLett.113.182302 [arXiv:1404.5963 [hep-th]].

[17] G. Basar, D. E. Kharzeev, I. Zahed ”Chiral and Gravitational Anomalies on Fermi Surfaces” Phys. Rev. Lett. 111, 161601 (2013) doi:10.1103/PhysRevLett.111.161601 [arXiv:1307.2234]