A revised sequential quadratic semidefinite programming method for nonlinear semidefinite optimization

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Abstract

In 2020, Yamakawa and Okuno proposed a stabilized sequential quadratic semidefinite programming (SQSDP) method for solving, in particular, degenerate nonlinear semidefinite optimization problems. The algorithm is shown to converge globally without a constraint qualification, and it has some nice properties, including the feasible subproblems, and their possible inexact computations. In particular, the convergence was established for approximate-Karush-Kuhn-Tucker (AKKT) and trace-AKKT conditions, which are two sequential optimality conditions for the nonlinear conic contexts. However, recently, complementarity-AKKT (CAKKT) conditions were also consider, as an alternative to the previous mentioned ones, that is more practical. Since few methods are shown to converge to CAKKT points, at least in conic optimization, and to complete the study associated to the SQSDP, here we propose a revised version of the method, maintaining the good properties. We modify the previous algorithm, prove the global convergence in the sense of CAKKT, and show some preliminary numerical experiments.

Keywords: approximate Karush-Kuhn-Tucker conditions, sequential quadratic programming, sequential optimality conditions, nonlinear semidefinite programming.

1 Introduction

The following nonlinear semidefinite programming (NSDP) problem is considered:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) = 0, \quad X(x) \in S^d_+,
\end{align*}
\]  

(NSDP)

where \( f: \mathbb{R}^n \to \mathbb{R}, \) \( g: \mathbb{R}^n \to \mathbb{R}^m, \) and \( X: \mathbb{R}^n \to S^d \) are twice continuously differentiable functions, \( S^d \) is the linear space of all real symmetric matrices of dimension \( d \times d, \) and \( S^d_+ \)

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is the cone of all positive semidefinite matrices in \( S^d \). The above problem extends some well-known optimization problems, including the nonlinear programming and the linear semidefinite programming (linear SDP).

The research associated to NSDP is currently increasing, with many possible applications, for instance in control theory [6], structural optimization [14,17] and finance [16]. In particular, methods like the interior point [13,23,25], the augmented Lagrangian-type [6,8,10,18], and the sequential quadratic programming (SQP) type [7,9,22] had been proposed in the last two decades (see [24] and references therein). Here, we focus in SQP-type methods for NSDP, which basically consist in solving simpler quadratic SDP problems in each iteration, extending the classical SQP method for nonlinear programming. These methods are also called \textit{sequential quadratic semidefinite programming} (SQSDP). One of the initial SQP-type methods for NSDP were proposed by Fares, Noll and Apkarian [9] with a local convergent algorithm, and by Correa and Ramirez [7], with a global convergent algorithm. Afterwards, some variants had been also proposed in the literature (see, for instance [15,20,27]).

In particular, Yamakawa and Okuno [22] recently proposed a \textit{stabilized SQSDP} method for NSDP, which is an extension of the stabilized SQP method for nonlinear programming proposed by Gill and Robinson [11]. The term “stabilized” is used because stabilized subproblems are solved in each iteration in order to handle degenerate problems. They proved that all accumulation points of the sequence generated by the algorithm is either infeasible but stationary for a feasibility problem, or satisfy some necessary optimality conditions. The conditions used in their work were actually proposed by Andreani, Haeser, and Viana [4] for NSDP problems, and is called \textit{approximate-Karush-Kuhn-Tucker} (AKKT) and \textit{trace-AKKT} (TAKKT) conditions.

These conditions are included in a class called \textit{sequential optimality conditions} [3,5]. They are shown to be necessary for optimality without any constraint qualification, which is a fact that differs from the classical KKT conditions. The sequential optimality conditions are essentially KKT variants, designed for building convergence theory of iterative algorithms. The research related to these conditions in general conic programming context, in particular NSDP, are still ongoing. Some kinds of sequential optimality conditions were proposed for NSDP so far, for example AKKT, TAKKT, and \textit{complementarity-AKKT} (CAKKT). In particular, CAKKT is a newer sequential optimality and was shown that CAKKT implies AKKT (TAKKT), and that CAKKT (or AKKT, TAKKT) with Robinson (or Mangasarian-Fromovitz) constraint qualification imply KKT [1,2].

Moreover, some algorithms were shown to generate points satisfying these sequential optimality conditions. However, in the conic context, the few algorithms that were proved to generate, in particular, CAKKT points are the augmented Lagrangian method [2], and the primal-dual interior point method [1]. To the best of authors’ knowledge, there is no SQP-type method producing CAKKT points. Although Yamakawa and Okuno’s [22] stabilized SQSDP method can generate AKKT or TAKKT points, it is preferable that the method obtains CAKKT ones, because as we mentioned, CAKKT is a sufficient condition under which AKKT (TAKKT) holds. Therefore, we refine the existing stabilized SQSDP
method [22] so that it can find CAKKT points. Although the modification in the algorithm is small, the proofs change considerably, and it becomes necessary to assume the so-called generalized Lojasiewicz inequality to prove the global convergence.

This paper is organized as follows. In Section 2 we define some basic notations and review the sequential optimality conditions for NSDP. In Section 3 we describe our proposed method, including the subproblem structure, the used merit function, and ways to update the Lagrange multipliers and the penalty parameters. We also prove the global convergence of the method in Section 4 and present some numerical experiments in Section 5. Finally, in Section 6 we show some final remarks.

2 Preliminaries

Let us start with some notations that will be used throughout the paper. The trace of a matrix $Z \in \mathbb{S}^d$ is denoted by $\text{tr}(Z) := \sum_{i=1}^{d} Z_{ii}$. In addition, if $Y \in \mathbb{S}^d$, then the inner product of $Y$ and $Z$ is written as $\langle Y, Z \rangle := \text{tr}(YZ)$, and the Frobenius norm of $Z$ is given by $\|Z\|_F := \langle Z, Z \rangle^{1/2}$. We use the notation $Y \perp Z$ to denote $\langle Y, Z \rangle = 0$. Moreover, the eigenvalues of $Z \in \mathbb{S}^d$ are written as $\lambda_1(Z) \leq \cdots \leq \lambda_d(Z)$, in ascending order. A positive definite (positive semidefinite) matrix $Z \in \mathbb{S}^d$ is written as $Z \succ O$ ($Z \succeq O$). We also define the following operator for all $Z \in \mathbb{S}^d$:

$$\text{svec}(Z) := (Z_{11}, \sqrt{2}Z_{21}, \ldots, \sqrt{2}Z_{d1}, Z_{22}, \sqrt{2}Z_{32}, \ldots, \sqrt{2}Z_{d2}, Z_{33}, \ldots, Z_{dd})^\top,$$

where $\top$ means transpose. The inner product of vectors $x, y \in \mathbb{R}^n$ is written as $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$ where $x_i$ and $y_i$ represent the $i$-th entry of $x$ and $y$ respectively, and the Euclidean norm of $x$ is denoted by $\|x\| := \langle x, x \rangle^{1/2}$. For all $Y, Z \in \mathbb{S}^d$, we note that

$$\langle Y, Z \rangle = \text{svec}(Y)^\top \text{svec}(Z), \quad \text{and} \quad \|Z\|_F = \|\text{svec}(Z)\|$$

(2.1)

hold. For a given $\rho : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$, the gradient and the Hessian of $\rho$ at $x$ is written as $\nabla \rho(x)$ and $\nabla^2 \rho(x)$, respectively. If $\tilde{\rho} : \mathbb{R}^n \times \mathbb{S}^d \to \mathbb{R}$, then its gradient at $(x, Z) \in \mathbb{R}^n \times \mathbb{S}^d$ with respect to $x$ is denoted by $\nabla_x \tilde{\rho}(x, Z)$.

With the dimensions given by each context, we denote by $I$ the identity matrix, and we use $e$ as the vector with all entries being equal to one. Moreover, a diagonal matrix with diagonal entries $r_1, \ldots, r_n \in \mathbb{R}$ is given by

$$\text{diag} [r_1, \ldots, r_n] := \begin{pmatrix} r_1 & & O \\ & \ddots & \\ O & & r_n \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Now, assuming that $Z \in \mathbb{S}^d$ can be diagonalized, we write its decomposition as $Z = PDP^\top$, where $P$ is an orthogonal matrix and $D$ is defined by $D = \text{diag} [\lambda_1^P(Z), \ldots, \lambda_d^P(Z)]$. Also, the projection of $Z$ onto $\mathbb{S}_+^d$ is given as follows:

$$[Z]_+ := P \text{diag} \left[ [\lambda_1^P(Z)]_+ , \ldots, [\lambda_d^P(U)]_+ \right] P^\top,$$
where \([ \cdot ]_+ : \mathbb{R} \rightarrow \mathbb{R}\) means \([r]_+ := \max\{0, r\}\). Furthermore, the cardinality of a set \(T\) is written as \(\text{card}(T)\).

Finally, let us define some notations related to problem (NSDP). Let \(g := (g_1, \ldots, g_m)\) with \(g_i : \mathbb{R}^n \rightarrow \mathbb{R}\) for all \(i = 1, \ldots, m\). The transposed Jacobian matrix of \(g\) at \(x\) is denoted by \(\nabla g(x) \in \mathbb{R}^{n \times m}\), i.e., \(\nabla g(x) := [\nabla g_1(x), \ldots, \nabla g_m(x)]\). The matrix \(A_j(x) \in \mathbb{S}^d\) is defined as the partial derivative \(A_j(x) := \partial X(x)/\partial x_j\) for all \(j = 1, \ldots, n\). Also, the operator \(A(x) : \mathbb{R}^n \rightarrow \mathbb{S}^d\) and the adjoint operator \(A^*(x) : \mathbb{S}^d \rightarrow \mathbb{R}^n\) are given respectively as:

\[
\begin{align*}
A(x)u &:= u_1A_1(x) + \cdots + u_nA_n(x), \\
A^*(x)U &:= [\langle A_1(x), U \rangle, \ldots, \langle A_n(x), U \rangle]^\top.
\end{align*}
\]

### 2.1 Optimality conditions

Here, we will review the concept of sequential optimality conditions for NSDP, that was developed in \cite{4}. Let \(L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^d_+ \rightarrow \mathbb{R}\) be the Lagrangian function of (NSDP):

\[
L(x, y, Z) := f(x) - \langle g(x), y \rangle - \langle X(x), Z \rangle,
\]

where \(y \in \mathbb{R}^m\) and \(Z \in \mathbb{S}^d_+\) are the Lagrange multipliers associated to the equality and the conic constraints, respectively. From the definition (2.2), we observe that the gradient of \(L\) with respect to \(x\) is given by

\[
\nabla_x L(x, y, Z) = \nabla f(x) - \nabla g(x)y - A^*(x)Z.
\]

Thus, we can define the KKT conditions of (NSDP) as follows.

**Definition 2.1** (KKT conditions). We say that \((x, y, Z) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{S}^d_+\) satisfies the KKT conditions for problem (NSDP) if

\[
\begin{align*}
\nabla_x L(x, y, Z) &= 0, \\
g(x) &= 0, \\
\langle X(x), Z \rangle &= 0, \\
X(x) &\succeq O, \\
Z &\succeq O.
\end{align*}
\]

If \((x, y, Z)\) satisfies the above KKT conditions, then we call \(x\) a KKT point, and \((y, Z)\) are the corresponding Lagrange multipliers. Moreover, as it is well known, a local optimal point needs to satisfy some constraint qualification in order to be a KKT point. As alternative optimality conditions, the sequential optimality have been studied in the last decade. They were first introduced for NLP and later developed for more general conic programming, which includes NSDP \cite{1,3,4}. Also, these conditions are known to be necessary for optimality without requiring a constraint qualification. In the following, we remark two sequential optimality conditions for NSDP problems, called AKKT and TAKKT.

**Definition 2.2.** \cite{4} Definition 4] We say that \(x \in \mathbb{R}^n\) satisfies the AKKT conditions for problem (NSDP) if \(g(x) = 0, X(x) \succeq O\) and there exist sequences \(\{x_k\} \subset \mathbb{R}^n, \{y_k\} \subset \mathbb{R}^m\) and \(\{Z_k\} \subset \mathbb{S}^d_+\) such that

- \(\lim_{k \to \infty} x_k = x\),
• \( \lim_{k \to \infty} \left( \nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)Z_k \right) = 0, \)

• \( \lambda^U_j(X(x)) > 0 \implies \) there exists \( k_j \in \mathbb{N} \) such that \( \lambda^U_k(Z_k) = 0 \) for all \( k \geq k_j \), where \( U \) and \( U_k \) are orthogonal matrices that satisfy \( U_k \to U \) when \( k \to \infty \), and

\[
X(x) = U \text{ diag} \left[ \lambda_1^U(X(x)), \ldots, \lambda_d^U(X(x)) \right] U^T,
\]

\[
Z_k = U_k \text{ diag} \left[ \lambda_1^U(Z_k), \ldots, \lambda_d^U(Z_k) \right] U_k^T
\]

for all \( j = 1, \ldots, d \).

Definition 2.3. [4, Definition 5] We say that \( x \in \mathbb{R}^n \) satisfies the TAKKT (trace-AKKT) conditions for problem \((\text{NSDP})\) if \( g(x) = 0, X(x) \succeq O \) and there exist sequences \( \{x_k\} \subset \mathbb{R}^n, \{y_k\} \subset \mathbb{R}^m \) and \( \{Z_k\} \subset S^d_+ \) such that

• \( \lim_{k \to \infty} x_k = x, \)

• \( \lim_{k \to \infty} \left( \nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)Z_k \right) = 0, \)

• \( \lim_{k \to \infty} (X(x_k), Z_k) = 0. \)

It was proved in [4] Theorems 2 and 5 that the local minimizers of \((\text{NSDP})\) always satisfy the AKKT and the TAKKT. Note also that, differently from the AKKT, the TAKKT avoids computation of eigenvalues. However, concerning the relation between these conditions, there are examples showing that the AKKT and the TAKKT conditions do not imply each other (see [4] Example 3] and [1] Example 3.1). The following CAKKT was proposed as another sequential optimality condition, free of eigenvalue computations, more suitable for the conic context, but having also a clear relationship with both AKKT and TAKKT.

Definition 2.4. [2, Section 2] We say that \( x \in \mathbb{R}^n \) satisfies the CAKKT (complementarity-AKKT) conditions for problem \((\text{NSDP})\) if \( g(x) = 0, X(x) \succeq O \) and there exist sequences \( \{x_k\} \subset \mathbb{R}^n, \{y_k\} \subset \mathbb{R}^m \) and \( \{Z_k\} \subset S^d_+ \) such that

• \( \lim_{k \to \infty} x_k = x, \)

• \( \lim_{k \to \infty} \left( \nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)Z_k \right) = 0, \)

• \( \lim_{k \to \infty} X(x_k) \circ Z_k = O, \)

where \( \circ \) denotes the Jordan product, i.e., \( A \circ B := (AB + BA)/2 \) for all \( A, B \in S^d \).

In [1] Section 3] and [2] Theorem 2.3], it was shown that the CAKKT implies both AKKT and TAKKT conditions. Furthermore, the CAKKT is equivalent to the KKT conditions when the Mangasarian-Fromovitz constraint qualification holds [1, Theorem 3.3]. In the next section, we will propose a method that generates these CAKKT sequences, by modifying the SQP-type method proposed in [22].
3 The proposed SQSDP method

We describe a brief outline of our SQSDP method. The proposed SQSDP method is based on the SQP-type method developed in [22], and mainly consists of three steps: solving a subproblem, updating the current point, and updating Lagrange multipliers and parameters. In the following, we provide explanation of each step.

3.1 The subproblem of the proposed SQSDP method

Let \( k \in \mathbb{N} \) be the current iteration. For a given point \((x_k, y_k, Z_k) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{S}^d\), the proposed SQSDP method solves the following subproblem:

\[
\begin{align*}
\text{minimize} & \quad \langle \nabla f(x_k) - \nabla g(x_k) s_k, \xi \rangle + \frac{1}{2} (M_k \xi, \xi) + \frac{\sigma_k}{2} \|\Sigma\|^2_F \\
\text{subject to} & \quad A(x_k) \xi + \sigma_k (\Sigma - T_k) \geq O,
\end{align*}
\]

where \( \sigma_k > 0 \) is a penalty parameter and \( s_k, T_k, \) and \( M_k \) are defined as follows:

\[
s_k := y_k - \frac{1}{\sigma_k} g(x_k), \quad T_k := Z_k - \frac{1}{\sigma_k} X(x_k), \quad M_k := H_k + \frac{1}{\sigma_k} \nabla g(x_k) \nabla g(x_k)^\top,
\]

and \( H_k \in \mathbb{R}^{n \times n} \) is the Hessian \( \nabla^2_{xx} L(x_k, y_k, Z_k) \) of the Lagrangian function (2.2) or its approximation. Problem (3.1) is derived from the following one, which is an extension of the existing stabilized subproblem proposed in [21]:

\[
\begin{align*}
\text{minimize} & \quad \langle \nabla f(x_k), \xi \rangle + \frac{1}{2} (H_k \xi, \xi) + \frac{\sigma_k}{2} \|\Sigma\|^2_F + \frac{\sigma_k}{2} \|\xi\|^2_F \\
\text{subject to} & \quad g(x_k) + \nabla g(x_k)^\top \xi + \sigma_k (\xi - y_k) = 0, \\
& \quad X(x) + A(x_k) \xi + \sigma_k (\Sigma - Z_k) \geq O.
\end{align*}
\]

Problem (3.1) is obtained by eliminating the variable \( \zeta \) in (3.2) via \( \zeta = y_k - \frac{1}{\sigma_k} (g(x_k) + \nabla g(x_k)^\top \xi_k) \) and has various useful properties below [22]:

(i) It always has a strictly feasible point \( (\xi, \Sigma) = (0, I + T_k) \);

(ii) if \( M_k \succ O \), then it has a unique global optimum.

Item (i) implies that (3.1) is solvable even if the current point \((x_k, y_k, Z_k) \) is not sufficiently close to the KKT point of (NSDP) and satisfies Slater’s constraint qualification. Moreover, these facts and item (ii) ensure that if the approximate Hessian \( H_k \) is designed so that \( M_k \succ O \) at each iteration, then the proposed SQSDP can obtain the KKT point of (3.1). Therefore, we suppose that \( M_k \succ O \) in the subsequent discussion. After obtaining the unique optimum \( (\xi_k, \Sigma_k) \) of (3.1), we set a search direction \( p_k \) and trial Lagrange multipliers \( \bar{y}_{k+1} \) and \( \bar{Z}_{k+1} \) as

\[
p_k := \xi_k, \quad \bar{y}_{k+1} := y_k - \frac{1}{\sigma_k} \left( g(x_k) + \nabla g(x_k)^\top \xi_k \right), \quad \bar{Z}_{k+1} := \Sigma_k,
\]

respectively.
3.2 Updating the current iterate

After computing the search direction $p_k$, we consider updating $x_k$ along $\xi_k$. To decide the step size which indicates how far we update $x_k$ along $\xi_k$, we introduce the following merit function $F: \mathbb{R}^n \to \mathbb{R}$:

$$F(x; \sigma, y, Z) := f(x) + \frac{1}{2\sigma} \| \sigma y - g(x) \|^2 + \frac{1}{2\sigma} \| \sigma Z - X(x) \|_F^2,$$

(3.4)

where only $x$ is the variable. From [4, Lemma 5], the merit function $F$ is differentiable on $\mathbb{R}^n$ and its gradient at $x$ is given by

$$\nabla F(x; \sigma, y, Z) = \nabla f(x) - \nabla g(x) \left( y - \frac{1}{\sigma} g(x) \right) - A^*(x) \left[ Z - \frac{1}{\sigma} X(x) \right]_+.$$

(3.5)

Furthermore, the merit function $F$ has the following nice property.

**Proposition 3.1.** [22] Proposition 2] Assume that $M_k \succ 0$. Then, the unique optimum $(\xi_k, \Sigma_k)$ of (3.1) satisfies $\langle \nabla F(x_k; \sigma_k, y_k, Z_k), \xi_k \rangle \leq -\langle M_k \xi_k, \xi_k \rangle - \sigma_k \| \Sigma_k - Z_k \|_F^2$. Moreover, $\nabla F(x_k; \sigma_k, y_k, Z_k) = 0$ if and only if $(\xi_k, \Sigma_k) = (0, [T_k]_+)$. Proposition 3.1 guarantees that $p_k = \xi_k$ is a descent direction of $F$. By applying a backtracking line search with $F$, we determine the step size $\alpha_k$, namely, we set $\alpha_k := \beta^{\ell_k}$, where $\beta \in (0, 1)$ is a positive constant and $\ell_k \geq 0$ is the smallest nonnegative integer such that

$$F(x_k + \beta^{\ell_k} p_k; \sigma_k, y_k, Z_k) \leq F(x_k; \sigma_k, y_k, Z_k) + \tau \beta^{\ell_k} \Delta_k,$$

(3.6)

with $\Delta_k := \max \{ \langle \nabla F(x_k; \sigma_k, y_k, Z_k), p_k \rangle, -\omega \| p_k \|^2 \}, \omega \in (0, 1)$, and $\tau \in (0, 1)$. Using the step size $\alpha_k$, the current point $x_k$ is updated by $x_{k+1} := x_k + \alpha_k p_k$.

3.3 Updating Lagrange multipliers

This section describes a procedure to update Lagrange multipliers. Although the ordinary SQP-type methods would immediately set the new Lagrange multiplier pair $(y_{k+1}, Z_{k+1})$ as $(\bar{y}_{k+1}, \bar{Z}_{k+1})$, the proposed SQSDP method does not update them like the ordinary methods and first check whether the triplet $(x_{k+1}, \bar{y}_k, \bar{Z}_{k+1})$ is approaching the KKT or CAKKT point. To this end, we define two functions to measure the distance between a given point $(x, y, Z)$ and the KKT point:

$$\Phi(x, y, Z) := r_V(x) + \kappa r_O(x, y, Z), \quad \Psi(x, y, Z) := \kappa r_V(x) + r_O(x, y, Z),$$

(3.7)

where $\kappa \in (0, 1)$ and

$$r_V(x) := \| g(x) \| + [\lambda_{\max}(-X(x))]_+, \quad r_O(x, y, Z) := \| \nabla_x L(x, y, Z) \| + \| X(x) Z \|_F.$$  

(3.8)

Note that $r_V(x) = r_O(x, y, Z) = 0$ if and only if $(x, y, Z)$ is the KKT point of (NSDP). Moreover, we utilize the function $\| \nabla F(\cdot; \sigma_k, y_k, Z_k) \|$ to measure the distance between a
given point $x$ and the CAKKT point. By combining these concepts, we provide the following procedure to update the Lagrange multipliers. This procedure was first proposed by Gill and Robinson [11] and extended for NSDP by Yamakawa and Okuno [22]. We point out that the main difference between this method and our proposal is in the definition of the function $r_O$ (the second definition in (3.8)), and consequently, in the functions $\Phi$ and $\Psi$ given in (3.7).

**Procedure 3.2.**

**Step 1.** *(V-iterate)* If $\Phi(x_{k+1}, y_{k+1}, Z_{k+1}) \leq \frac{1}{2} \phi_k$, then set

$$\phi_{k+1} := \frac{1}{2} \phi_k, \quad \psi_{k+1} := \psi_k, \quad \gamma_{k+1} := \gamma_k, \quad y_{k+1} := y_k, \quad Z_{k+1} := Z_{k+1},$$

and end the procedure. Otherwise, go to Step 2.

**Step 2.** *(O-iterate)* If $\Psi(x_{k+1}, y_{k+1}, Z_{k+1}) \leq \frac{1}{2} \psi_k$, set

$$\phi_{k+1} := \phi_k, \quad \psi_{k+1} := \frac{1}{2} \psi_k, \quad \gamma_{k+1} := \gamma_k, \quad y_{k+1} := y_k, \quad Z_{k+1} := Z_{k+1},$$

and end the procedure. Otherwise, go to Step 3.

**Step 3.** *(M-iterate)* If $\|\nabla F(x_{k+1}; \sigma_k, y_k, Z_k)\| \leq \gamma_k$, set

$$\phi_{k+1} := \phi_k, \quad \psi_{k+1} := \psi_k, \quad \gamma_{k+1} := \frac{1}{2} \gamma_k,$$

$$y_{k+1} := \Pi_C(y_k - \frac{1}{\gamma_k} g(x_{k+1})), \quad Z_{k+1} := \Pi_D([Z_k - \frac{1}{\sigma_k} X(x_{k+1})]_+),$$

and end the procedure. Here, $\Pi_C$ and $\Pi_D$ are projections onto the sets

$$C := \{y \in \mathbb{R}^m | -y_{\text{max}} e \leq y \leq y_{\text{max}} e\} \quad \text{and} \quad D := \left\{Z \in \mathbb{S}^d | O \preceq Z \preceq z_{\text{max}} I\right\},$$

respectively, and $y_{\text{max}} > 0$, $z_{\text{max}} > 0$ are constants. Otherwise, go to Step 4.

**Step 4.** *(F-iterate)* Set $\phi_{k+1} := \phi_k, \quad \psi_{k+1} := \psi_k, \quad \gamma_{k+1} := \gamma_k, \quad y_{k+1} := y_k, \quad Z_{k+1} := Z_k$.

If the conditions $\Phi(x_{k+1}, y_{k+1}, Z_{k+1}) \leq \frac{1}{2} \phi_k$ and $\Psi(x_{k+1}, y_{k+1}, Z_{k+1}) \leq \frac{1}{2} \psi_k$ in Steps 1 and 2 are satisfied, then we consider that $(x_{k+1}, y_k, Z_k)$ is approaching the KKT point. Therefore, we adopt the trial Lagrange multiplier pair $(y_{k+1}, Z_{k+1})$ as the new Lagrange multiplier pair $(y_k, Z_k)$ because the pair has a nice tendency. If $\|\nabla F(x_{k+1}; \sigma_k, y_k, Z_k)\| \leq \gamma_k$ in Step 3 is satisfied, then we consider that $x_{k+1}$ is approaching the CAKKT point. In this case, we adopt the updating rule used in the augmented Lagrangian method [4] because the function $F$, which is regarded as the augmented Lagrange function, is minimizing. If all the conditions in Steps 1, 2, and 3 are not satisfied, we do not update the Lagrange multipliers in this iteration.

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1The V, O, M and F-iterations mean violation, optimality, merit (function), and failure, respectively.
3.4 Updating the penalty parameter

This paper considers that (NSDP) does not satisfy any constraint qualification, and hence it is possible that there is no Lagrange multiplier pair satisfying the KKT conditions. Therefore, we need to design the proposed method so that it can find a CAKKT point. This is done by minimizing the merit function $F$, which is called the augmented Lagrange function, as shown in the next section. Since the case where $\|\nabla F(x_{k+1}; \sigma_k, y_k, Z_k)\| \leq \gamma_k$, seen in Step 3 in Procedure 3.2, corresponds to the step which minimizes the augmented Lagrangian function, it is reasonable to update the penalty parameter $\sigma_k$ based on the manner used in the augmented Lagrangian method as follows:

$$
\sigma_{k+1} := \begin{cases} 
\min \left\{ \frac{1}{2} \sigma_k, r(x_{k+1}, y_{k+1}, Z_{k+1})^{\frac{3}{2}} \right\}, & \text{if } \|\nabla F(x_{k+1}; \sigma_k, y_k, Z_k)\| \leq \gamma_k, \\
\sigma_k, & \text{otherwise,}
\end{cases}
$$

(3.9)

where

$$
r(x, y, Z) := r_V(x) + r_O(x, y, Z)
$$

and $r_V, r_O$ are defined in (3.8). Note that the term $r(x_{k+1}, y_{k+1}, Z_{k+1})^{\frac{3}{2}}$ has an effect to achieve fast local convergence and is also utilized in [11].

3.5 Proposed algorithm

Summarizing the above discussion, we propose the following method for solving (NSDP).

Algorithm 3.3.

Step 0. Set constants $\tau \in (0, 1)$, $\omega \in (0, 1)$, $\beta \in (0, 1)$, $\kappa \in (0, 1)$, $y_{\text{max}} > 0$, $z_{\text{max}} > 0$, $k_{\text{max}} \in \mathbb{N}$, and $\varepsilon > 0$. Choose an initial point $(x_0, y_0, Z_0)$ and parameters $\phi_0 > 0$, $\psi_0 > 0$, $\sigma_0 > 0$, $k := 0$, $y_0 := y_0$, and $Z_0 := Z_0$. Go to Step 1.

Step 1. If $r(x_k, y_k, Z_k) \leq \varepsilon$, $\gamma_k \leq \varepsilon$, or $k = k_{\text{max}}$, then stop. Otherwise, go to Step 2.

Step 2. If $\|\nabla F(x_k; \sigma_k, y_k, Z_k)\| = 0$, set

$$
x_{k+1} := x_k, \quad y_{k+1} := y_k - \frac{1}{\sigma_k} g(x_{k+1}), \quad Z_{k+1} := \left[Z_k - \frac{1}{\sigma_k} X(x_{k+1})\right]_+
$$

and go to Step 5. Otherwise, go to Step 3.

Step 3. Choose $H_k \succ O$ and find the global optimum $(\xi_k, \Sigma_k)$ by solving (3.1). Set

$$
p_k := \xi_k, \quad y_{k+1} := y_k - \frac{1}{\sigma_k} \left(g(x_k) + \nabla g(x_k)^\top \xi_k\right), \quad Z_{k+1} := \Sigma_k
$$

and go to Step 4.

Step 4. Compute the smallest nonnegative integer $\ell_k$ with (3.6). Set $x_{k+1} := x_k + \beta^\ell_k p_k$ and go to Step 5.
Step 5. Compute \( y_{k+1}, Z_{k+1}, \phi_{k+1}, \psi_{k+1} \) and \( \gamma_{k+1} \) with Procedure 3.2 and go to Step 6.

Step 6. Update \( \sigma_k \) using (3.9) and go to Step 7.

Step 7. Set \( k := k + 1 \) and go to Step 1.

Note that if \( \|\nabla F(x_k; \sigma_k, y_k, Z_k)\| = 0 \) holds in Step 2, then Proposition 3.1 indicates that the unique global minimizer of (3.1) is \((\xi_k, \Sigma_k) = (0, [T_k]_+)\). Thus, in this case, we do not need to solve the subproblem (3.1) and we go to Step 5 immediately.

4 Global convergence

In this section, we discuss convergence properties of Algorithm 3.3, assuming that an infinite sequence of points are generated. For that, we suppose that the following assumption holds.

Assumption 4.1. Let \( \{x_k\} \) be a sequence generated by Algorithm 3.3 for problem (NSDP). Then, we assume the following assertions:

(a) The functions \( f, g \) and \( X \) are twice continuously differentiable.

(b) There exists a compact set containing \( \{x_k\} \).

(c) For all \( k \), there exist constants \( \nu_1 \) and \( \nu_2 \) satisfying

\[
\nu_1 \leq \lambda_{\min} \left( H_k + \frac{1}{\sigma_k} \nabla g(x_k) \nabla g(x_k)^\top \right) \quad \text{and} \quad \lambda_{\max}(H_k) \leq \nu_2,
\]

where \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) denote, respectively, the minimum and the maximum eigenvalues of \( A \in \mathbb{S}^d \).

(d) Let \( x^* \) be an accumulation point of \( \{x_k\} \). Then, there exist \( \delta > 0 \) and a continuous function \( \varphi: B(x^*, \delta) \to \mathbb{R} \) satisfying \( \lim_{x \to x^*} \varphi(x) = 0 \) and

\[
|P(x) - P(x^*)| \leq \varphi(x) \|\nabla P(x)\|,
\]

where \( B(x^*, \delta) \) is the Euclidean ball with radius \( \delta \) around \( x^* \), and \( P \) is a feasibility measure defined by

\[
P(x) := \frac{1}{2} \|g(x)\|^2 + \frac{1}{2} \|[-X(x)]_+\|^2. \tag{4.1}
\]

In the above assumption, we note that (a) and (b) are standard, and also used in [22]. The assumption (c) is also used in the same work, and it holds, for instance, when \( H_k \) is positive definite and bounded [22]. Moreover, (d) is a weak assumption on the smoothness of \( g(\cdot) \) and \( X(\cdot) \), which is defined in [5] and known as a generalized Lojasiewicz inequality.
Before showing the main convergence result, let us consider the following partitions for the algorithm’s iterations:

\[ \mathcal{K}_{VO} := \{ k \mid \text{V-iterate or O-iterate is executed in } k\text{-th iteration of Procedure 3.2} \}, \]
\[ \mathcal{K}_M := \{ k \mid \text{M-iterate is executed in } k\text{-th iteration of Procedure 3.2} \}, \]
\[ \mathcal{K}_F := \{ k \mid \text{F-iterate is executed in } k\text{-th iteration of Procedure 3.2} \}. \]

We now give two lemmas, associated with the above set of iterations.

**Lemma 4.2.** Suppose that Assumption 4.1 holds. Then, we have:

(i) If \( \text{card}(\mathcal{K}_{VO}) = \infty \), then \( \phi_k \to 0 \) or \( \psi_k \to 0 \) when \( k \to \infty \).

(ii) If \( \text{card}(\mathcal{K}_{VO}) < \infty \), then \( \{Z_k\} \) is bounded.

(iii) If \( \text{card}(\mathcal{K}_{VO}) < \infty \) and \( \text{card}(\mathcal{K}_M) = \infty \), then \( \sigma_k \to 0 \) and \( \gamma_k \to 0 \) when \( k \to \infty \).

(iv) The situation \( \text{card}(\mathcal{K}_{VO}) < \infty \), \( \text{card}(\mathcal{K}_M) < \infty \), and \( \text{card}(\mathcal{K}_F) = \infty \) never occurs.

**Proof.** See [22, Lemma 4 and Theorem 3].

**Lemma 4.3.** Let \( \{x_k\} \) be a sequence generated by Algorithm 3.3, and suppose that Assumption 4.1 holds. Assume that \( \text{card}(\mathcal{K}_{VO}) < \infty \) and define \( \mathcal{K}'_M := \{ k \in \mathbb{N} \mid k-1 \in \mathcal{K}_M \} \). Moreover, let \( M \subset \mathcal{K}'_M \) such that \( x_k \to x^* \) when \( M \ni k \to \infty \), where \( x^* \) is a feasible point of \( (\text{NSDP}) \). Then the following statements hold:

(i) \( (1/\sigma_{k-1})\lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \left[ \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \right]_+ \to 0 \) when \( M \ni k \to \infty \) for all \( i = 1, \ldots, m \).

(ii) \( \|\nabla P(x_k)/\sigma_{k-1}\| \) is bounded when \( k \in M \).

**Proof.** See Appendix A.

Under Assumption 4.1 and supposing also that an infinite sequence of points are generated by Algorithm 3.3, we obtain the following convergence result. Basically, any accumulation point of the sequence generated by the method is either a CAKKKT point, or it is infeasible but stationary for the feasibility measure \( P \).

**Theorem 4.4.** Let \( \{x_k\} \) be a sequence generated by Algorithm 3.3, and suppose that Assumption 4.1 holds. Then, any accumulation point of \( \{x_k\} \) satisfies one of the following:

(i) It is a CAKKKT point of \( (\text{NSDP}) \);

(ii) It is an infeasible point of \( (\text{NSDP}) \), but a stationary point of the feasibility measure \( P \), defined in (4.1).

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Proof. We consider two cases: (a) \( \text{card}(\mathcal{K}_{V^O}) = \infty \), and (b) \( \text{card}(\mathcal{K}_{V^O}) < \infty \).

**Case (a)** Assume that \( \text{card}(\mathcal{K}_{V^O}) = \infty \). We show that statement (i) holds in this case. Let \( \mathcal{K}'_{V^O} := \{ k \in \mathbb{N} \mid k - 1 \in \mathcal{K}_{V^O} \} \). It is clear that \( \text{card}(\mathcal{K}'_{V^O}) = \infty \) because \( \text{card}(\mathcal{K}_{V^O}) = \infty \). Using this fact and Assumption 4.1 (b), \( \{ x_k \}_{k \in \mathcal{K}'_{K^O}} \) has an accumulation point, say \( x^* \). Then, there exists \( J \subset \mathcal{K}_{K^O} \) such that \( x_k \rightarrow x^* \) when \( J \ni k \rightarrow \infty \). From (3.7), Lemma 4.2 (i), and the steps in Procedure 3.2, it follows that \( r_V(x_k) \rightarrow 0 \) and \( r_O(x_k, y_k, Z_k) \rightarrow 0 \) when \( J \ni k \rightarrow \infty \). Therefore, from the definition of \( r_V \) and \( r_O \) given in (3.8), we conclude that

\[
\lim_{J \ni k \rightarrow \infty} \left( \nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)Z_k \right) = 0,
\]

\[
\lim_{J \ni k \rightarrow \infty} \|X(x_k)Z_k\|_F = \lim_{J \ni k \rightarrow \infty} \|Z_kX(x_k)\|_F = 0,
\]

\[
\lim_{J \ni k \rightarrow \infty} g(x_k) = g(x^*) = 0,
\]

\[
\lim_{J \ni k \rightarrow \infty} [\lambda_{\max}(-X(x_k))]_+ = [\lambda_{\max}(-X(x^*))]_+ = 0,
\]

which show that \( x^* \) is a CAKKT point of (NSDP).

**Case (b)** Assume that \( \text{card}(\mathcal{K}_{V^O}) < \infty \). To begin with, we show that the gradient of the Lagrange function converge to 0 in some subset of \( \mathcal{K}_M \). Recalling that an infinite sequence of iterates are generated, it follows from Lemma 4.2 (iv) that \( \text{card}(\mathcal{K}'_M) = \infty \). This shows that \( \text{card}(\mathcal{K}'_M) = \infty \), where \( \mathcal{K}'_M := \{ k \in \mathbb{N} \mid k - 1 \in \mathcal{K}_M \} \). Once again by Assumption 4.1 (b), \( \{ x_k \}_{k \in \mathcal{K}'_M} \) has at least one accumulation point. Let \( x^* \) be such a point. Thus, there exists \( J \subset \mathcal{K}'_M \) such that \( x_k \rightarrow x^* \) when \( J \supset k \rightarrow \infty \). Moreover, from Lemma 4.2 (ii)–(iii), we have the boundedness of \( \{ Z_k \} \), and that \( \sigma_k \rightarrow 0 \) and \( \gamma_k \rightarrow 0 \) when \( k \rightarrow \infty \). Defining for simplicity the terms below,

\[
y_k := y_{k-1} - \frac{1}{\sigma_{k-1}}g(x_k), \quad \tilde{Z}_k := \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right]_+,
\]

and recalling (3.3), we obtain

\[
\|\nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)\tilde{Z}_k\| = \|\nabla F(x_k; \sigma_{k-1}, y_{k-1}, Z_{k-1})\| \leq \gamma_{k-1},
\]

when \( k \in J \). These facts and Lemma 4.2 (iii) yield

\[
\lim_{J \ni k \rightarrow \infty} \left( \nabla f(x_k) - \nabla g(x_k)y_k - A^*(x_k)\tilde{Z}_k \right) = 0. \tag{4.2}
\]

Now, we consider two cases: (b1) when \( x^* \) is feasible for (NSDP), and (b2) when \( x^* \) is an infeasible point of (NSDP). We will show that (i) holds in the first case, and (ii) is satisfied in the latter case.

**Case (b1)** Let us show that situation (i) holds when \( x^* \) is feasible for (NSDP). From the definition of CAKKT, as well as (4.2), it is sufficient to show that \( X(x_k) \circ \tilde{Z}_k \rightarrow O \).
when \( k \to \infty \). Moreover, it is sufficient to prove that \( X(x_k)\tilde{Z}_k \to O \) when \( k \to \infty \) since \( \|X(x_k)\tilde{Z}_k\|_F = \|\tilde{Z}_kX(x_k)\|_F \).

Consider the diagonalization \( \sigma_{k-1}Z_{k-1} - X(x_k) = S_kD_kS_k^T \), where \( S_k \) is an orthogonal matrix. Hence, simple calculations show that

\[
X(x_k)\tilde{Z}_k = (\sigma_{k-1}Z_{k-1} - S_kD_kS_k^T) \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right] +
\]

\[
= \sigma_{k-1}Z_{k-1} \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right] + S_kD_kS_k^T \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right] +
\]

\[
= \sigma_{k-1}Z_{k-1} \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right] + S_kD_kS_k^T \left[ \frac{1}{\sigma_{k-1}}S_kD_kS_k^T \right] +
\]

\[
= \sigma_{k-1}Z_{k-1} \left[ Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k) \right] + \frac{1}{\sigma_{k-1}}S_kD_k [D_k]_+ S_k^T.
\]

We now prove that \( \sigma_{k-1}Z_{k-1} [Z_{k-1} - \frac{1}{\sigma_{k-1}}X(x_k)]_+ \) and \( (1/\sigma_{k-1})S_kD_k [D_k]_+ S_k^T \) converge to \( O \) when \( k \to \infty \). The first term clearly converges to \( O \) because \( x^* \) is a feasible point of (NSDP), namely, \( X(x^*) \succeq O \) and Lemma 4.2 (ii)–(iii) hold. To show that the same can be said for the second term, observe that

\[
\frac{1}{\sigma_{k-1}}S_kD_k [D_k]_+ S_k^T = \frac{1}{\sigma_{k-1}} \sum_{i=1}^m \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \left[ \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \right]_+ s_i^k \left[ s_i^k \right]^T,
\]

where \( s_i^k \) denotes the \( i \)-th column of \( S_k \). From Lemma 4.3 (i), we have

\[
\frac{1}{\sigma_{k-1}}\lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \left[ \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \right]_+ \to 0, \quad i = 1, \ldots, m,
\]

which proves our claim. Therefore, we conclude that \( x^* \) is a CAKKT point.

**Case (b2)** Finally, we show that condition (ii) holds if \( x^* \) is an infeasible point of (NSDP). Since \( x_k \to x^* \) when \( k \to \infty \), there exists \( \delta > 0 \) and \( \kappa \in \mathbb{R} \) such that \( x_k \in B(x^*, \delta) \) for all \( k \geq \kappa \). In the following, let \( k \geq \kappa \). From the definition of \( P \) given in (4.1), we have

\[
\nabla P(x) = \nabla g(x)g(x) - A^x(x) [A^x(x)]_+.
\]

From Lemma 4.3 (ii), there exists a positive constant \( M \) such that \( \|\nabla P(x_k)/\sigma_{k-1}\| \leq M \). Also, by Lemma 4.2 (iii), we have \( \sigma_k \to 0 \) and \( \|\nabla P(x_k)\| \to \|\nabla P(x^*)\| = 0 \) when \( k \to \infty \). Therefore, condition (ii) holds if \( x^* \) is an infeasible point of (NSDP).

**5 Numerical experiments**

In this section, we present some simple numerical experiments to check the validity of Algorithm 5.3. We consider two NSDP problems here: one with no KKT solution and
another where degeneracy occurs. The program was implemented in MATLAB R2020b, and we ran Algorithm 3.3 on a machine with Intel Core i9-9900K, with 3.60GHz of CPU and 128GB of RAM. The subproblem used in the algorithm is computed using SDPT3, version 4.0 [19, 20].

Let us first describe the setting of Algorithm 3.3. The initial point was set as \((x_0, y_0, Z_0) = (0, 0, O)\). For the stopping criteria, we use the following three conditions:

\[
    r(x_k, y_k, Z_k) := r_V(x_k) + r_O(x_k, y_k, Z_k) \leq 10^{-4}, \quad \gamma_k \leq 10^{-4}, \quad \text{or} \quad k_{max} = 200.
\]

As usual, the condition \(\|\nabla F(x_k; \sigma_k, y_k, Z_k)\| = 0\) in Step 2 of Algorithm 3.3 is relaxed to \(\|\nabla F(x_k; \sigma_k, y_k, Z_k)\| \leq 10^{-4}\). Moreover, the constants and parameters were set as follows: \(\tau := 10^{-4}, \quad \omega := 10^{-4}, \quad \beta := 0.5, \quad \kappa := 10^{-5}, \quad y_{max} := 10^{6}, \quad z_{max} := 10^{6}, \quad \epsilon := 10^{-4}, \quad \phi_0 := 10^3, \quad \psi_0 := 10^3, \quad \gamma_0 := 10^{-1}, \quad \sigma_0 := 10^{-1}\). We also set the stopping condition’s parameter for the subproblem (“gaptol” in SDPT3) as \(10^{-10}\).

### 5.1 Problem with no KKT solution

We consider the following one-dimensional problem:

\[
    \text{minimize} \quad 2x \\
    \text{subject to} \quad \begin{pmatrix} 0 & -x \\ -x & 1 \end{pmatrix} \succeq O.
\]

The unique feasible point, which is also optimal for this problem is \(x = 0\). However, it is not a KKT point, as proved in [4, Example 1]. For this problem, Algorithm 3.3 was able to obtain the solution at the 35-th iteration and the value of the measure \(r\) was equal to \(9.98 \times 10^{-5}\). Although this is a toy problem, we can see that the absence of KKT does not affect the proposed SQSDP method, at least in this example.

### 5.2 Degenerated problem

Next, we solve the following degenerated problem:

\[
    \text{minimize} \quad \langle C, X \rangle \\
    \text{such that} \quad X_{ii} = 1, \quad i = 1, \ldots, n, \\
    \langle J, X \rangle = 0, \quad X \succeq O,
\]

where \(J\) is an \(n\)-dimensional square matrix with all elements equal to 1, and \(C\) is an \(n\)-dimensional symmetric matrix whose elements satisfy \(C_{ij} \in [-1, 1]\) for all \(i, j = 1, \ldots, n\). Since \(e^\top X e = \langle J, X \rangle = 0\), there is no strictly feasible point for this problem, which means that Slater’s constraint qualification does not hold.

Here, we generated 10 instances of problems with \(n = 5\) and \(n = 10\), and solved them by using Algorithm 3.3. Table I shows its computational results, where “Average iterations” means the average iterations spent by Algorithm 3.3. In addition, “Average \(r\)”, “Maximum
“Average $r$” and “Minimum $r$” represent, respectively, the average, maximum, and minimum values of the measure $r$ at the final iteration. As it is possible to observe in the table, in most problems, the proposed algorithm was able to find a solution.

Table 1: Performance for the degenerated problem

|               | $n = 5$          | $n = 10$          |
|---------------|------------------|------------------|
| Average $r$   | $2.45 \times 10^{-3}$ | $7.01 \times 10^{-3}$ |
| Maximum $r$   | $1.50 \times 10^{-2}$ | $6.67 \times 10^{-2}$ |
| Minimum $r$   | $6.09 \times 10^{-9}$ | $8.95 \times 10^{-9}$ |

6 Conclusion

In this paper, we proposed a revised sequential quadratic semidefinite programming method for NSDP. The algorithm is based on a recent work of Yamakawa and Okuno [22], and the main difference is that now we are able to generate CAKKT points. By adding a not strict assumption, called generalized Lojasiewicz inequality, we also proved the global convergence of the method. Simple numerical experiments were also done, showing the validity of the method. One future work will be to analyze the convergence rate of the method.

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Appendix A

Here, we give a proof for Lemma 4.3. Before that, we state the following useful result concerning eigenvalues [12]. If $A, B \in S^m$, then for all $i = 1, \ldots, m$, we have

$$\lambda_1(A) + \lambda_i(B) \leq \lambda_i(A + B) \leq \lambda_m(A) + \lambda_i(B).$$  \hspace{1cm} (6.1)

Proof of Lemma 4.3 (i):

For simplicity, let us define

$$\tilde{\lambda}_i^k := \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)).$$

We will prove that $(1/\sigma_{k-1})\tilde{\lambda}_i^k [\tilde{\lambda}_i^k]_+ \to 0$ by considering two cases, when $i \in \{1, \ldots, m\}$ satisfies (a) $\lambda_i(-X(x^*)) < 0$ or (b) $\lambda_i(-X(x^*)) = 0$.

Case (a) Let $i \in \{1, \ldots, m\}$ be such that $\lambda_i(-X(x^*)) < 0$. 


First, recalling that $\sigma_{k-1} > 0$, it follows from the second inequality of (6.1) that
\[
\frac{1}{\sigma_{k-1}} \tilde{\lambda}_i^k \leq \lambda_m(Z_{k-1}) + \lambda_i \left( -\frac{1}{\sigma_{k-1}} X(x_k) \right).
\] (6.2)

From the assumption that $\lambda_i(-X(x^*)) < 0$, there exists a positive integer $\overline{k}$ such that $\lambda_i(-X(x_k)) < 0$ for all $k \geq \overline{k}$. In the following, we suppose that $k \geq \overline{k}$. We obtain from Lemma 4.2 (ii)–(iii) that $\{Z_{k-1}\}$ is bounded and $\sigma_{k-1} \to 0$ when $k \to \infty$. Thus, we have
\[
\lambda_m(Z_{k-1}) + \lambda_i \left( -\frac{1}{\sigma_{k-1}} X(x_k) \right) \to -\infty.
\]

This, together with (6.2) shows that $(1/\sigma_{k-1})\tilde{\lambda}_i^k \to -\infty$, which yields $(1/\sigma_{k-1})\tilde{\lambda}_i^k+ = 0$ for sufficiently large $k$. Moreover, $\tilde{\lambda}_i^k$ is bounded from Assumption 4.1 (b) and Lemma 4.2 (ii), and therefore $(1/\sigma_{k-1})\tilde{\lambda}_i^k+ \to 0$ in this case.

**Case (b)** Let $i \in \{1, \ldots, m\}$ be such that $\lambda_i(-X(x^*)) = 0$.

Here, we consider once again two cases: (b1) there exists a positive integer $\hat{k}$ such that $\lambda_i(-X(x_k)) \leq 0$ for all $k \geq \hat{k}$, and (b2) there exists a subset $\mathcal{J} \subset \mathcal{M}$ such that $\text{card}(\mathcal{J}) = \infty$ and $\lambda_i(-X(x_k)) > 0$ for all $k \in \mathcal{J}$.

**Case (b1)** Assume that there exists a positive integer $\hat{k}$ such that $\lambda_i(-X(x_k)) \leq 0$ for all $k \geq \hat{k}$. To prove our claim, here we will show that $(1/\sigma_{k-1})\tilde{\lambda}_i^k+$ is bounded and $\tilde{\lambda}_i^k \to 0$ when $k \to \infty$. In the following, let $k \geq \hat{k}$.

Let us first show that $(1/\sigma_{k-1})\tilde{\lambda}_i^k+$ is bounded. Since $\sigma_{k-1} > 0$, we have
\[
0 \leq (1/\sigma_{k-1})\tilde{\lambda}_i^k+ = \max \left\{ 0, \lambda_i \left( Z_{k-1} - \frac{1}{\sigma_{k-1}} X(x_k) \right) \right\} \]
\[
\leq \max \left\{ 0, \lambda_m(Z_{k-1}) + \lambda_i \left( -\frac{1}{\sigma_{k-1}} X(x_k) \right) \right\} \]
\[
\leq \max \left\{ 0, \lambda_m(Z_{k-1}) \right\} + \max \left\{ 0, \lambda_i \left( -\frac{1}{\sigma_{k-1}} X(x_k) \right) \right\},
\]
where the second inequality holds from the second inequality of (6.1), and the last one follows from a property of the $[\cdot]+$ function. From Lemma 4.2 (ii), the first term of the last expression is bounded. Moreover, $\lambda_i(-X(x_k))/\sigma_{k-1} \leq 0$ from assumption and because $\sigma_{k-1} > 0$, which means that the second term of the last inequality is equal to 0. Therefore, $(1/\sigma_{k-1})\tilde{\lambda}_i^k+$ is bounded as we claimed. Now, from Lemma 4.2 (ii)–(iii), we get
\[
\tilde{\lambda}_i^k = \lambda_i(\sigma_{k-1}Z_{k-1} - X(x_k)) \to \lambda_i(-X(x^*)) = 0,
\]
and the proof is complete for this case.
Case (b2) Assume that there exists a subset $J \subset M$ such that $\text{card}(J) = \infty$ and $\lambda_i(-X(x_k)) > 0$ for all $k \in J$. Now, take $k \in J$. From the first inequality of (6.1), we obtain

$$\tilde{\lambda}_i^k = \lambda_i^k(\sigma_{k-1}Z_{k-1} - X(x_k)) \geq \sigma_{k-1}\lambda_1(Z_{k-1}) + \lambda_i(-X(x_k)).$$

By assumption, we know $\lambda_i(-X(x_k)) > 0$. Moreover, from Step 3 of Procedure 3.2, $Z_{k-1}$ is a positive semidefinite matrix and thus $\lambda_1(Z_{k-1}) \geq 0$. Therefore, the above inequality shows that $\tilde{\lambda}_i^k > 0$. Now, recalling that $\sigma_{k-1} > 0$ and from (6.1),

$$\lambda_1(Z_{k-1}) + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k)) \leq \frac{1}{\sigma_{k-1}}\tilde{\lambda}_i^k \leq \lambda_m(Z_{k-1}) + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k)) \quad (6.3)$$

holds. Multiplying it by the positive term $\tilde{\lambda}_i^k$, and observing that $[\tilde{\lambda}_i^k]_+ = \tilde{\lambda}_i^k$, we have

$$\lambda_1(Z_{k-1})\tilde{\lambda}_i^k + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))\tilde{\lambda}_i^k \leq \frac{1}{\sigma_{k-1}}\tilde{\lambda}_i^k[\tilde{\lambda}_i^k]_+$$

$$\leq \lambda_m(Z_{k-1})\tilde{\lambda}_i^k + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))\tilde{\lambda}_i^k. \quad (6.4)$$

In addition, by Lemma 4.2 (ii)–(iii) and the fact that $\lambda_i(-X(x_k)) \to \lambda_i(-X(x^*)) = 0$ when $J \ni k \to \infty$, we obtain $\tilde{\lambda}_i^k \to 0$. Moreover, $\lambda_1(Z_{k-1})\tilde{\lambda}_i^k \to 0$ and $\lambda_m(Z_{k-1})\tilde{\lambda}_i^k \to 0$ hold. These limits, together with inequalities (6.4), show that $(1/\sigma_{k-1})\tilde{\lambda}_i^k[\tilde{\lambda}_i^k]_+ \to 0$ holds when

$$\frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))\tilde{\lambda}_i^k \to 0. \quad (6.5)$$

To prove this limit, note that $\lambda_i(-X(x_k)) > 0$ holds by assumption, and so we can multiply (6.3) by it to obtain

$$\lambda_1(Z_{k-1})\lambda_i(-X(x_k)) + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))^2 \leq \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))\tilde{\lambda}_i^k$$

$$\leq \lambda_m(Z_{k-1})\lambda_i(-X(x_k)) + \frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))^2.$$

Once again from Lemma 4.2 (ii) and the fact that $\lambda_i(-X(x_k)) \to 0$ when $J \ni k \to \infty$, show that $\lambda_1(Z_{k-1})\lambda_i(-X(x_k)) \to 0$ and $\lambda_m(Z_{k-1})\lambda_i(-X(x_k)) \to 0$. These limits, together with the above inequalities show that in order to prove (6.6), it becomes sufficient to prove

$$\frac{1}{\sigma_{k-1}}\lambda_i(-X(x_k))^2 \to 0. \quad (6.6)$$

Let us now prove (6.6). Since $x_k \to x^*$ when $k \to \infty$, there exists $\delta > 0$ and a positive integer $\bar{k}$ such that $x_k \in B(x^*, \delta)$ for all $k \geq \bar{k}$. In the following, we suppose $J \ni k \geq \bar{k}$. Regarding $P$ defined in 4.11, we have $P(x^*) = 0$ because $x^*$ is a feasible point of [NSDP]. Thus, from Assumption 4.1 (d), $|P(x^*)| = |P(x^*)| \leq \varphi(x^*)\|\nabla P(x^*)\| |P(x_k)| \leq \frac{\varphi(x_k)}{\sigma_{k-1}}\|\nabla P(x_k)\|$ holds, and therefore

$$0 \leq \frac{1}{\sigma_{k-1}}|P(x_k)| = \frac{1}{\sigma_{k-1}}|P(x_k)| \leq \frac{\varphi(x_k)}{\sigma_{k-1}}\|\nabla P(x_k)\|.$$
From Lemma 4.3 (ii), $\nabla P(x_k)/\sigma_{k-1}$ is bounded. Moreover, since $\varphi(x_k) \to \varphi(x^*) = 0$, the above inequality shows that

$$
\frac{1}{\sigma_{k-1}}|P(x_k)| = \frac{1}{2\sigma_{k-1}} \left( \|g(x_k)\|^2 + \|[-X(x_k)]_+\|_F^2 \right) \to 0.
$$

In addition, because $\|g(x_k)\|$ and $\|[-X(x_k)]_+\|_F$ are both nonnegative, we get the limit $(1/\sigma_{k-1})\|[-X(x_k)]_+\|^2 \to 0$. Since

$$
\frac{1}{\sigma_{k-1}} \sum_{j=1}^{m} \lambda_j ([{-X(x_k)}]_+)^2 = \frac{1}{\sigma_{k-1}} \|{-X(x_k)}\|_F^2 \to 0
$$

holds, we conclude that $\lambda_j ([{-X(x_k)}]_+)^2/\sigma_{k-1} \to 0$ for $j = 1, \ldots, m$. Therefore, because $\lambda_i([-X(x_k)]) > 0$ from assumption, (6.6) holds. We then conclude that (6.5) is also true, which in turn shows the lemma’s claim. 

\[\square\]

**Proof of Lemma 4.3 (ii):**

Let us prove that $\|\nabla P(x_k)/\sigma_{k-1}\|$ is bounded when $k \in \mathcal{M}$. Fixing $k \in \mathcal{M}$, clearly,

$$
\nabla P(x_k) = \nabla g(x_k)g(x_k) - A^*(x_k)[-X(x_k)]_+
$$

by the definition of $P$ in (6.1). Then, simple calculations show that

$$
\left\| \frac{\nabla P(x_k)}{\sigma_{k-1}} \right\| = \frac{1}{\sigma_{k-1}} \left\| \nabla g(x_k)g(x_k) - A^*(x_k) \left( \sigma_{k-1}Z_{k-1} - X(x_k) \right) \right\|
$$

$$
+ \frac{1}{\sigma_{k-1}} \left\| A^*(x_k) \left( [-X(x_k)]_+ - \sigma_{k-1}Z_{k-1} + X(x_k) \right) \right\|
$$

Moreover, the first term of the right-hand side of the above expression can be written as follows:

$$
\left\| \frac{1}{\sigma_{k-1}} \left( \nabla g(x_k)g(x_k) - A^*(x_k) \sigma_{k-1}Z_{k-1} - X(x_k) \right) \right\|
$$

$$
= \left\| \nabla g(x_k)g(x_k) - A^*(x_k) \sigma_{k-1}Z_{k-1} - X(x_k) \right\|
$$

$$
+ \frac{1}{\sigma_{k-1}} \nabla f(x_k) - \nabla g(x_k)y_{k-1}
$$

$$
\leq \|\nabla F(x_k; x_{k-1}, y_{k-1}, Z_{k-1})\| + \|\nabla f(x_k) - \nabla g(x_k)y_{k-1}\|
$$

$$
\leq \gamma_{k-1} + \|\nabla f(x_k) - \nabla g(x_k)y_{k-1}\|,
$$

20
where the first inequality holds from (3.5), and the second one follows because the iterate \( k \in \mathcal{M} \) means \((k - 1) \in \mathcal{K}_M\), so the condition in Step 3 of Procedure 3.2 is true. Once again from the Step 3 of Procedure 3.2, we know that \( y_{k-1} \) is bounded. With Assumption 4.1 (b), this implies the boundedness of \( \|\nabla f(x_k) - \nabla g(x_k)y_{k-1}\| \). Thus, the term in (6.8) is also bounded.

Let us now analyze the last term of the expression (6.7). Recall that

\[
\mathcal{A}^*(x)U := \begin{bmatrix}
\langle A_1(x), U \rangle \\
\vdots \\
\langle A_n(x), U \rangle
\end{bmatrix} = \begin{bmatrix}
svec(A_1(x))\top svec(U) \\
\vdots \\
svec(A_n(x))\top svec(U)
\end{bmatrix} = \begin{bmatrix}
svec(A_1(x))\top \\
\vdots \\
svec(A_n(x))\top
\end{bmatrix} svec(U)
\]

for any \( x \in \mathbb{R}^n \) and \( U \in \mathbb{S}^d \), where the second equality holds from (2.1). Using the above formulation, we have:

\[
\frac{1}{\sigma_{k-1}} \|\mathcal{A}^*(x_k)([-X(x_k)]_+ - [\sigma_{k-1}Z_{k-1} - X(x_k)]_+)\| = \frac{1}{\sigma_{k-1}} \left\| \begin{bmatrix}
svec(A_1(x_k))\top \\
\vdots \\
svec(A_n(x_k))\top
\end{bmatrix} svec([-X(x_k)]_+ - [\sigma_{k-1}Z_{k-1} - X(x_k)]_+) \right\| 
\leq \frac{1}{\sigma_{k-1}} \left\| \begin{bmatrix}
svec(A_1(x_k))\top \\
\vdots \\
svec(A_n(x_k))\top
\end{bmatrix} \right\| \|svec([-X(x_k)]_+ - [\sigma_{k-1}Z_{k-1} - X(x_k)]_+)|. \tag{6.9}
\]

Now, from (2.1), the last norm is bounded as follows:

\[
\|svec([-X(x_k)]_+ - [\sigma_{k-1}Z_{k-1} - X(x_k)]_+)\| = \| [-X(x_k)]_+ - [\sigma_{k-1}Z_{k-1} - X(x_k)]_+ \|_F 
\leq \|\sigma_{k-1}Z_{k-1}\|_F = \sigma_{k-1}\|Z_{k-1}\|_F,
\]

where the inequality follows from the expansion property of projections. The above inequality, together with Assumption 4.1 (b) and Lemma 4.2 (ii), shows that (6.9) is also bounded. Therefore, from (6.7) the claim holds.