A Study of Quantum Many-Body Systems With Nearest And Next-to-Nearest Neighbour Long-Range Interactions

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Abstract

The scattering and bound states of the many-body systems, related to the short-range Dyson model, are studied. First, we show that the scattering states can be realized as coherent states and the scattering Hamiltonian can be connected to a free system. Unlike the closely related Calogero-Moser model, only a part of the partial waves acquire energy independent phase shifts, after scattering. The cause of the same is traced to the reduction in the degeneracies. The bound state Hamiltonian for the full-line problem is also studied and the relationship of its Hilbert space with that of the decoupled oscillators elucidated. Finally, we analyze the related models on circle and construct a part of the excitation spectrum through symmetry arguments.

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I. INTRODUCTION

Exactly solvable and quantum integrable many-body systems, with long-range interactions, are one of the most active fields of current research. The Calogero-Sutherland model (CSM)\textsuperscript{1-3}, the Sutherland model (SM)\textsuperscript{4} and their variants are the most prominent examples among such systems\textsuperscript{5}. These models have found application in various branches of physics\textsuperscript{6-10}, ranging from quantum Hall effect\textsuperscript{11-13}, gauge theories\textsuperscript{14,15}, chaos\textsuperscript{6,16-17}, fractional statistics\textsuperscript{8,18-23} etc. The CSM and SM type long-range interactions have also manifested in models dealing with pairing interactions\textsuperscript{24} and phase transitions\textsuperscript{25}. It is known that, the CSM is related to random matrix theory\textsuperscript{2,6,16,26-28} and enables one to capture the universal aspects of various physical phenomena\textsuperscript{6}. The Brownian motion model of Dyson connects the random matrix theory (RMT) with exactly solvable models\textsuperscript{29}. The role of RMT in the description of the level statistics of chaotic systems is well-known\textsuperscript{30}.

Some time back, a short-range Dyson model was introduced to understand the spectral statistics of systems, which are “non-universal with a universal trend”\textsuperscript{31}. It is known that, there are dynamical systems which are neither chaotic nor integrable, the so called pseudo-integrable systems which exhibit the above mentioned level statistics\textsuperscript{32}. Aharanov-Bohm billiards\textsuperscript{33}, three dimensional Anderson model at the metal-insulator transition point\textsuperscript{34} and some polygonal billiards\textsuperscript{35} fit into the above description. Quite recently, a new class of one dimensional, exactly solvable many-body quantum mechanical models on the line, with nearest and next-to-nearest neighbour interactions, have been introduced\textsuperscript{36,37}, which are related to this short-range Dyson model. Further, using the symmetrized version of this model, it has been shown that there exists an off-diagonal long-range order in the system which indicates the presence of different quantum phases\textsuperscript{38}. Apart from possessing a good thermodynamic limit \textit{i.e.}, $\lim_{N \to \infty} \frac{E_0}{N}$ is finite, these models are more physical in the sense that, unlike the CSM and SM, where all particles experience pairwise interaction of identical strength, irrespective of their distances, here the interactions are only nearest neighbour and next-to-nearest neighbour. These models are exactly solvable, but not integrable. Hence, it
is interesting to enquire as to how many features of the integrable CSM type systems are retained in the present case. For example, the scattering phenomena is quite interesting in the CSM case, since the outgoing waves can be shown to be of the incoming type, with momenta $k'_i = k_{N+1-i}(i = 1, 2, \cdots, N)$, where $k_i$’s are the incoming momenta. Remarkably, the phase shifts are energy independent, a result ascribable to the scale invariance of the inverse square interaction. Since, in the present case also, scale invariance holds and the interaction goes to zero as the particle separation increases, it is of deep interest to study the scattering phenomena. Similarly, for the bound state problem, CSM can be exactly made equivalent to a set decoupled oscillators, via a similarity transformation\textsuperscript{39}. Hence, it is also of interest to check the same here to understand the precise differences in the Hilbert space structure between integrable and non-integrable Hamiltonians, possessing identical spectra. It should be mentioned that, in the present case, the degeneracy is less, since the lack of quantum integrability i.e., a desired set of mutually commuting operators having common eigenfunctions with the Hamiltonian, reduces the degeneracy. Also, these models, being of recent origin, need to be analyzed thoroughly, in order to unravel their properties, as has been done for the CSM, SM and their generalizations.

The present paper is devoted to an investigation of these models, and deals with, both the scattering and the bound state problems. It is organized as follows: In Sec.II, we study the scattering problem and show that the scattering state is a coherent state. The connection between the scattering Hamiltonian and that of the free particles in this model is then demonstrated. We then study the scattering phase shift and point out its similarities and differences with the Calogero case. In Sec.III, we analyze the relationship of the bound state problem with the decoupled oscillators and find some wavefunctions explicitly in the Cartesian basis. This analysis reveals explicitly that the degeneracy of this model is less as compared to the CSM. Finally, in Sec.IV, the nearest neighbour and next-to-nearest neighbour $A_{N-1}$ and $BC_N$ models on the circle are studied; a part of their excitation spectra is obtained through symmetry arguments.
II. THE SCATTERING PROBLEM

A. Realization of the scattering states as coherent states

The scattering Hamiltonian, with nearest and next-to-nearest neighbour, inverse square interactions, in the units $\hbar = m = 1$, is given by,

$$H_{\text{sca}} = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \beta(\beta - 1) \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})^2} - \beta^2 \sum_{i=2}^{N} \frac{1}{(x_{i-1} - x_i)(x_i - x_{i+1})},$$

(1)

where $x_{N+i} = x_i$. The corresponding bound state Hamiltonian, contains an additional oscillator potential:

$$H = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \beta(\beta - 1) \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})^2} - \beta^2 \sum_{i=1}^{N} \frac{1}{(x_{i-1} - x_i)(x_i - x_{i+1})}.$$  (2)

It is interesting to note that, the scattering eigenstates can be constructed as coherent states of the bound state eigenfunctions like that of the Calogero case. To be precise, we show that the polynomial part of the bound-state wavefunctions enter into the construction of the scattering states. For this purpose, we first identify an $SU(1, 1)$ algebra containing the bound and scattering Hamiltonians as its elements, after suitable similarity transformations:

$$Z^{-1}(-H_{\text{sca}})Z = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})} (\partial_i - \partial_{i+1}) \equiv T_+,$$

$$\hat{S}^{-1}(-H/2)\hat{S} = -\frac{1}{2} \left( \sum_i x_i \partial_i + E_0 \right) \equiv T_0.$$  (3)

Here, $Z \equiv \prod_{i=1}^{N} \left| x_i - x_{i+1} \right|^\beta$ and $\hat{S} \equiv \exp\left\{-\frac{1}{2} \sum_i x_i^2\right\} Z \exp\left\{-\frac{1}{2} T_+\right\}.$

Defining,

$$\frac{1}{2} \sum_i x_i^2 \equiv T_-,$$  (4)

one can easily check that $T_\pm$ and $T_0$ satisfy the usual $SU(1, 1)$ algebra:

$$[T_+, T_-] = -2T_0, \quad [T_0, T_\pm] = \pm T_\pm.$$  

The quadratic Casimir for the above algebra is given by,
\[ \hat{C} = T_- T_+ - T_0 (T_0 + 1) = T_+ T_- - T_0 (T_0 - 1) \quad . \] (5)

By finding a canonical conjugate of \( T_+ \) \[ [T_+, \tilde{T}_-] = 1 \quad , \] (6)
one can construct the coherent state \( < x | m, k > \), the eigenstate of \( T_+ \) which are nothing but the scattering states,

\[ < x | m, k > = U^{-1} P_m(x) = e^{-\frac{1}{2} k^2 \tilde{T}} P_m(x) \quad , \] (7)

with

\[ T_+ P_m(x) \equiv \left[ \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})} (\partial_i - \partial_{i+1}) \right] P_m(x) = 0 \quad . \] (8)

It is known that this equation admits homogeneous solutions i.e, \( T_0 P_m(x) = -[(m + E_0)/2] P_m(x) \). Here \( m \) refers to the degree of homogeneity of \( P_m(x) \). It is easy to see that, \( U^{-1} P_m(x) \) is the eigenstate of \( T_+ \).Starting from \( T_+ P_m(x) = 0 \), one gets,

\[ U^{-1} T_+ U U^{-1} P_m(x) = 0 \quad , \] (9)
i.e,

\[ T_+ U^{-1} P_m(x) = -\frac{1}{2} k^2 U^{-1} P_m(x) \quad . \] (10)

The scattering state is given by, \( \psi_{sca} = Z U^{-1} P_m(x) \) , since \( H_{sca} = -Z T_+ Z^{-1} \), therefore, \( H_{sca} \psi_{sca} = \frac{k^2}{2} \psi_{sca} \). To find \( < x | m, k > \) explicitly, we have to determine \( \tilde{T}_- \). By choosing \( \tilde{T}_- = T_- F(T_0) \), Eq. (8) becomes

\[ [T_+, T_- F(T_0)] = F(T_0) T_+ T_- - F(T_0 + 1) T_- T_+ = 1 \quad , \]

\[ F(T_0) \{ \hat{C} + T_0 (T_0 - 1) \} - F(T_0 + 1) \{ \hat{C} + T_0 (T_0 + 1) \} = 1 \quad , \] (11)
yielding,

\[ F(T_0) = \frac{-T_0 + a}{\hat{C} + T_0 (T_0 - 1)} \quad . \] (12)
Here, $a$ is a parameter to be fixed along with the value of the quadratic Casimir $\hat{C}$, by demanding that the above commutator is valid in the eigenspace of $T_0$. Eq. (6) when used on $P_m(x)$, yields, $a = 1 - (E_0 + m)/2$. Similarly,

$$\hat{C}P_m(x) = (T_+T - T_0(T_0 + 1))P_m(x) = CP_m(x) \ ,$$

(13)

where, $C = \frac{1}{2}(m + E_0)(1 - (m + E_0)/2)$. One then finds,

$$F(T_0)P_m(x) = \frac{-T_0 + a}{C + T_0(T_0 - 1)}P_m(x)$$

$$= -\frac{1}{T_0 - (m + E_0)/2}P_m(x) \ .$$

(14)

Explicitly, we have,

$$<x | m, k> = e^{-\frac{1}{2}k^2\tilde{T}_-}e^{-T_+}P_m(x)$$

$$= e^{-\frac{1}{2}k^2}e^{-T_+}e^{-\frac{1}{2}k^2\tilde{T}_-}P_m(x)$$

$$= e^{-\frac{1}{2}k^2}\sum_{n=0}^{\infty} \frac{(k^2/2)^n}{(E_0 + m + n)!}L_{n}^{(E_0 - 1 + m)}(r^2/2)P_m(x)$$

$$= e^{-k^2/4}(\frac{k}{2})^{-(E_0 - 1 + m)}(r)^-(E_0 - 1 + m)J_{E_0 - 1 + m}(kr)P_m(x) \ ,$$

(15)

where, $r^2 = \sum_i x_i^2$ and $J_{E_0 - 1 + m}(kr)$ is the Bessel function. Note that, there is an additional factor of $e^{-T_+}$ in the above equation, this has been introduced for calculational convenience and does not alter our results, since $e^{-T_+}P_m = P_m$. In order to arrive at the above result, we have made use of the following:

$$T_+(r^{2n}P_m(x)) = 2n(E_0 - 1 + m + n)r^{2(n-1)}P_m(x) \ ,$$

and also the identity\textsuperscript{45},

$$J_\alpha(2\sqrt{xz})e^z(xz)^{-\alpha/2} = \sum_{n=0}^{\infty} \frac{z^n}{(n + \alpha + 1)!}L_n^\alpha(x) \ .$$

Note that, the above wavefunction can also be obtained by solving the scattering Hamiltonian explicitly. However, we have chosen this algebraic method, since it will be of subsequent use.
B. Connection to Free Particles

The fact that as in the Calogero case, the spectrum of the scattering Hamiltonian matches with that of the free particles and that, the phase shift, as will be shown later, are energy independent, suggests a possible connection of this system with free particles. We now show the same by making use of the algebraic structures already introduced.

The following generators,

\[ K_+ = H_{sca}, \quad \tilde{K}_+ = H_{sca}(\beta = 0), \]

\[ K_- = \frac{1}{2} \sum_{i=1}^{N} x_i^2 = \tilde{K}_-, \]

and

\[ K_0 = -\frac{i}{4} \sum_{i=1}^{N} (2x_i \partial_i + 1) = \tilde{K}_0, \quad (16) \]

satisfy \([K_0, K_\pm(\tilde{K}_\pm)] = \pm iK_\pm(\tilde{K}_\pm), [K_-(\tilde{K}_-), K_+(\tilde{K}_+)] = 2iK_0(\tilde{K}_0).\] The normalization of the generators have been chosen differently for convenience. It can be verified that, the following operator,

\[ U = e^{i\frac{\pi}{2}(\tilde{K}_++\tilde{K}_-)} e^{-i\frac{\pi}{2}(K_++K_-)}, \]

maps the \(H_{sca},\) with interactions, to a interaction-free system, \(i.e.,\)

\[ U \ H_{sca} \ U^\dagger = H_{sca}(\beta = 0) = -\frac{1}{2} \sum_{i} \frac{\partial^2}{\partial x_i^2}. \quad (17) \]

This result motivates one to analyze, explicitly, the precise correspondence of the respective Hilbert spaces of the interacting and non-interacting systems. Care has to be taken to ensure that, the mapped states are members of the Hilbert space. This analysis, for the present case, as well as for the Calogero model has not been carried out so far. We hope to address this problem in the near future.
C. Analysis of the Scattering Phase Shift

As is well known, in order to obtain the scattering phase shifts, one has to analyze the asymptotic behaviour of the eigenfunctions of the Hamiltonian, when all particles are far apart from each other. In the present case, the most general stationary eigenfunction can be written as a superposition of all the \( m \) dependent states as,

\[
\psi = Z \sum_{m=0}^{\infty} \sum_{q} C_{m,q} \left( \frac{r}{2} \right)^{-(E_0-1+m)} J_{E_0-1+m}(kr) P_{m,q}(x),
\]  

where, \( C_{m,q} \)'s are some coefficients and the index \( q \) refers to the number of independent \( P_m \)'s at the level \( m \).

The asymptotic limit of Eq. (18) can be obtained from the asymptotic behaviour of the Bessel function \( J_{E_0-1+m}(kr) \) and from the fact that \( r^{-m} P_{m,q}(x) \) does not change in this limit, since \( P_{m,q} \)'s are homogeneous functions of degree \( m \). One obtains,

\[
\psi \sim \psi_{\text{in}} + \psi_{\text{out}},
\]

where,

\[
\psi_{\text{in}} \equiv e^{i(E_0-1)\pi/2}(2\pi kr)^{-\frac{1}{2}} \left( Z r^{-(E_0-1)} \sum_{m=0}^{\infty} \sum_{q} C_{m,q} \left( \frac{r}{2} \right)^{-m} e^{-im\pi/2} P_{m,q}(x) \right) e^{-ikr},
\]

and

\[
\psi_{\text{out}} \equiv e^{-i(E_0-1)\pi/2}(2\pi kr)^{-\frac{1}{2}} \left( Z r^{-(E_0-1)} \sum_{m=0}^{\infty} \sum_{q} C_{m,q} \left( \frac{r}{2} \right)^{-m} e^{-im\pi/2} P_{m,q}(x) \right) e^{ikr}.
\]

The crucial differences between the Calogero and the present model arise from this point. Denoting the equivalent of \( P_{m,q}(x) \)'s in the Calogero case, as \( \tilde{P}_{m,q}(x) \)'s, we first describe the Calogero scattering problem, and then compare the results of this model with the same. Though the \( \tilde{P}_{m,q}(x) \)'s, which are the solutions of the generalized Laplace equation, have not been found explicitly so far, for arbitrary \( m \), Calogero had shown the existence and completeness of these homogeneous symmetric polynomials. Hence, by choosing suitable values for the coefficients \( C_{m,q} \)'s, one could characterize the incoming wave, for the Calogero
case as,

$$\psi_{\text{in}} \equiv c \exp \left\{ i \sum_i k_i x_i \right\},$$

with \( k_i \leq k_{i+1}, i = 1, 2, \ldots, N-1 \), as the stationary eigenfunction in the center-of-mass frame, \( i.e., \sum_i k_i = 0 \). Further, since, the \( \tilde{P}_{m,q} \)'s are symmetric under cyclic permutations and are homogeneous, one obtains, \( \tilde{P}_{m,q}(-T x) = e^{-i m \pi} \tilde{P}_{m,q}(x) \). where, \( T \) denotes a cyclic permutation of the particle coordinates. Hence, \( \psi_{\text{out}} \) for the Calogero model can be written as,

$$\psi_{\text{out}} \equiv e^{-i(E_0-1)\pi} (2\pi \bar{k}_r)^{-\frac{1}{2}} \left( Zr^{-(E_0-1)} \sum_{m=0}^{\infty} \sum_q C_{m,q} (r/2)^{-m} e^{i m \pi/2} \tilde{P}_{m,q}(-T x) \right) e^{-i \bar{k}_r} \quad (22)$$

where, \( \bar{k} = -k \), and \( -Tx_i = -x_{N+1-i} \). The action of \(( -T)\) takes a given particle ordering \( x_i \geq x_{i+1} \) to \( -x_{N+1-i} \geq -x_{N-i} \), and hence preserves the order \( i.e., \), the sector of the configuration space. This is the reason, invariance of \( \tilde{P}_{m,q}(x) \)'s under cyclic permutation is enough to compute the phase shifts. Comparison with \( \psi_{\text{in}} \), yields,

$$\psi_{\text{out}} = ce^{-i(E_0-1)\pi} \exp \left\{ -i \sum_i \bar{k}_i T x_i \right\}$$

$$= ce^{-i(E_0-1)\pi} \exp \left\{ i \sum_i k_{N+1-i} x_i \right\}, \quad (23)$$

where, we have used \( \bar{k}_i = -k_i \) and the cyclic permutation has been carried on the momentum variables. From above, it is clear that, the initial scattering situation characterized by the initial momenta, \( k_i(i = 1, 2, \ldots, N) \), goes over to the final configuration, characterized by the final momenta, \( k_i' = k_{N+1-i} \) and the phase shifts are energy independent. In the present model, \( P_{m,q}(x) \)'s are the solutions of Eq. (B), which is symmetric only under cyclic permutations. Hence, one needs to check if the same steps as narrated above for Calogero’s case also applies here. First of all, we should find out if the number of \( P_{m,q} \)'s are same here. For the sake of clarity, we first consider the four particle case and concentrate on the homogeneous solutions of degree four. Since the monomial symmetric functions, provide a linearly independent basis set, we can expand \( P_4(x_i) \) as,

$$P_4 = a \sum_{i=1}^{4} x_i^4 + b \sum_{i \neq j}^{4} x_i^2 x_j + c \sum_{i<j}^{4} x_i^2 x_j^2 + d \sum_{i \neq j \neq l}^{4} x_i^2 x_j x_l + e \sum_{i<j<l<p}^{4} x_i x_j x_l x_p \quad . \quad (24)$$

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The operation of $T_+$ on $P_4$ gives the following three sets of conditions,

$$\begin{align*}
\left[2(3 + 4\beta)a - 2\beta b + (3 + 4\beta)c - 2\beta d\right] \sum_{i=1}^{4} x_i^2 &= 0 , \\
\left[4\beta a + 2(3 + 4\beta)b - 2\beta c + 2(1 - \beta)d - \beta e\right] \sum_{i=1}^{4} x_i x_{i+1} &= 0 , \\
\left[6(1 + 2\beta)b + 2(1 - 2\beta)d\right] \sum_{i=1}^{4} x_i x_{i+2} &= 0 , \\
\end{align*}$$

(25)

where $a$, $b$, $c$, $d$ and $e$ are unknown constants to be determined from the above equations.

For the purpose of comparison, in the Calogero case,

$$\tilde{T}_+ \tilde{P}_4(x) \equiv \left[\frac{1}{2} \sum_{i=1}^{4} \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i<j}^{4} \frac{1}{(x_i - x_j)}(\partial_i - \partial_j)\right] \tilde{P}_4(x) = 0 ,$$

(26)

does not hold in the present model as compared to the Calogero case. This reduction in the number of solutions takes place because, the interaction term contained in the $T_+$ operator, leads to the split of the monomial symmetric functions of degree $(m-2)$, giving additional conditions unlike the Calogero case. Explicit calculations for a number of few body examples yields similar results. Hence, it is clear that, $P_{m,q}$’s do not form a complete set. All the solutions obtained so far are symmetric and belongs to a subset of the Calogero case. The reason of the symmetric nature of the polynomials lies in the interaction term in $T_+$. In order that the action of the interaction term, $\beta \sum_{i=1}^{N} \frac{1}{x_i - x_{i+1}}(\partial_i - \partial_{i+1})$, on the polynomial $P_m(x)$ results in a polynomial of degree $m-2$, the denominator needs to be cancelled. This results in the symmetrization of the polynomial, since the nearest neighbour couplings arising from $\frac{1}{x_i - x_{i+1}}$ connects each particle with every other member of the set. The reduction in the number of symmetric polynomials can also be understood from the point of view of integrability. Since Calogero model is fully integrable, there are desired
number of operators, commuting with the Hamiltonian, which can be used for connecting
the members of a given set of $P_{m,q}$’s akin to the angular momentum raising and lowering
operators in the central force problem. The fact that, the number of $P_{m,q}$’s are less here
indicates that, the corresponding commuting constants of motion are less here. This point
will be further elaborated in the next section.

As has been mentioned earlier, the completeness of the solutions of the generalized
Laplace equation in the Calogero model enables one to relate all the partial waves of the
incoming state with those of the outgoing state, with constant energy independent phase
shifts. As is clear from Eqs. (20), (21) and (22), in the present case, the outgoing wavefront
can be made to look like the incoming wavefront. However, only those partial waves, which
are solutions of Eq. (8) will acquire energy independent phase shifts $e^{-i(E_0-1)\pi}$, the rest
will be unaffected by the interaction. This difference between the Calogero and the present
model arises, because of the reduction of the number of $P_{m,q}$’s here.

**III. THE BOUND STATE PROBLEM**

**A. Mapping of the model to decoupled oscillators**

The bound state Hamiltonian is given by,

$$H = -\frac{1}{2} \sum_{i=1}^{N} \partial_{i}^2 + \frac{1}{2} \sum_{i=1}^{N} x_{i}^2 + \beta(\beta - 1) \sum_{i=1}^{N} \frac{1}{(x_{i} - x_{i+1})^2} - \beta^2 \sum_{i=1}^{N} \frac{1}{(x_{i-1} - x_{i})(x_{i} - x_{i+1})},$$

(28)

where, $\partial_{i} \equiv \frac{\partial}{\partial x_{i}}$ and $x_{N+i} \equiv x_{i}$. It is worth pointing out that, for three particles, this model
is equivalent to the CSM, since the three body term vanishes in this case. The ground-state
wavefunction and the energy of this system are respectively given by, $\psi_0 = G Z$ and
$E_0 = (N/2 + N\beta)$, where $G$ and $Z$ have been defined earlier. The first term of $E_0$ is the
ground state energy of $N$ oscillators and the second one comes from the interaction.

In the following, we make use of a method developed in Ref. (39) to show the equivalence
of this model to a set of decoupled oscillators. For that purpose, we perform a similarity
transformation on the Hamiltonian, by its ground-state wavefunction, to yield
\[ H' \equiv \psi_0^{-1} H \psi_0 = \sum_i x_i \partial_i + E_0 - T_+ \quad , \]  

where, \( T_+ \equiv \frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \beta \sum_{i=1}^{N} \frac{1}{(x_i - x_{i+1})} (\partial_i - \partial_{i+1}) \). Here, we confine ourselves to a sector of the configuration space given by \( x_1 \geq x_2 \geq \cdots \geq x_{N-1} \geq x_N \). Using the identity,  

\[ [\sum_i x_i \partial_i , e^{-T_+/2}] = T_+ e^{-T_+/2} \quad , \]

it is easy to see that  

\[ \bar{H} \equiv e^{T_+/2} H' e^{-T_+/2} = \sum_i x_i \partial_i + E_0 \quad . \]  

From the above diagonalized form, it is evident that, the spectrum of \( H \) is like that of \( N \) uncoupled oscillators and is linear in the coupling parameter \( \beta \). Explicitly, \( \bar{H} \) can be made equivalent to the decoupled oscillators:

\[ G e^{-T_+,(\beta=0)/2} \bar{H} e^{T_+,(\beta=0)/2} G^{-1} = -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + E_0 - N/2 \quad . \]  

Making inverse similarity transformations, one can write down the raising and lowering operators for \( H \), akin to the CSM. However, the eigenfunctions of \( H \) can be constructed straightforwardly by making use of Eq. (30); since the eigenfunctions of \( \sum_i x_i \partial_i \) are homogeneous polynomials of degree \( n \) in the particle coordinates, \( n \) being any integer. Although, the similarity transformation formally maps the Hilbert space of the interacting problem to that of the free oscillators, it needs a careful study. The singular terms present in \( T_+ \) may yield states, which are not members of the Hilbert space. Below, we clarify this point.

**B. Eigenfunctions in the Cartesian Basis**

In the following, we present some eigenfunctions for the \( N \)-particle case, computed using the power-sum basis, \( P_l(x) = \sum_{i=1}^{N} x_i^l \), i.e., \( \psi_l = \psi_0 S_l \); here, \( S_l \equiv e^{-T_+/2} P_l \), and the corresponding energy eigenvalue is \( E_l = (l + E_0) \), \( l \) being an integer.

The wavefunction (unnormalized) corresponding to the center-of-mass degree of freedom, \( R = \frac{1}{N} \sum_{i=1}^{N} x_i \), is found to be (we use the notation \( \psi_{n_1,n_2,\ldots,n_N} = \psi_0 \exp\{-\frac{1}{2}T_+\} \prod_l P_{l}^{n_l} \),
\[
\psi_{n_1,0,0,\ldots} = \psi_0 \exp\left\{-\frac{1}{2}T_+\right\} R^{n_1} = \psi_0 \exp\left\{-\frac{1}{4} \sum_{i=1}^{N} \partial_i^2\right\} R^{n_1} \tag{32}
\]

This can be cast in the form

\[
\psi_{n_1,0,0,\ldots} = c \psi_0 \sum_{m_i = n_1}^{N} \prod_{i=1}^{N} \frac{H_{m_i}(x_i)}{m_i!},
\]

where, \(H_{m_i}(x_i)\)'s are the Hermite polynomials, and \(c\) is a constant. Similarly, the eigenfunction for the radial degree of freedom, \(r^2 = \sum_i x_i^2\), can be obtained from,

\[
\psi_{0,n_2,0,\ldots} = \psi_0 \exp\left\{-\frac{1}{2}T_+\right\} (r^2)^{n_2} = \psi_0 e^{-\frac{1}{2}T_+} P_2^{n_2} \tag{34}
\]

For the sake of clarity, we give below the explicit derivation for \(\psi_{0,1,0,0}\) for the four particle case and then generalize the result for arbitrary number of particles and levels. For four particles,

\[
\psi_{0,1,0,0} = \psi_0 e^{-\frac{1}{2}T_+} P_2 = \psi_0 e^{-\frac{1}{2}T_+} (x_1^2 + x_2^2 + x_3^2 + x_4^2) \tag{35}
\]

We note that, \(T_+ P_2 = 4(2\beta + 1)\), \(T_+^2 P_2 = 0\), and hence, \(\psi_{0,1,0,0} = \psi_0 (P_2 - 2(2\beta + 1))\). For \(N\) particles, it can be verified that, \(T_+ r^{2n} = 2n(E_0 - 1 + n)r^{2(n-1)}\) and this gives \(\psi_{0,n_2,0,\ldots}\) as

\[
\psi_{0,n_2,0,\ldots} = \psi_0 \sum_{m=0}^{n_2} \frac{(-1)^m}{m!(n_2 - m)!} \frac{(E_0 - 1 + n_2)!}{(E_0 - 1 + m)!} (r^2)^m = c \psi_0 L_{n_2}^{E_0-1}(r^2),
\]

where, \(L_{n_2}^{E_0-1}(r^2)\) is the Lagurre polynomial.

Further, for four particles, \(S_3 = P_3 - \frac{1}{3}(2\beta + 1)P_1\). However, it can be checked that, \(S_4 = e^{-T_+/2} P_4\), does not terminate as a polynomial and results in a function with negative powers of the particle coordinates, which is not normalizable with respect to the ground-state wavefunction as a measure. This indicates that, there is less degeneracy in the present model, as compared to the symmetrized states of the decoupled oscillators. Finding the other wavefunctions explicitly, for an arbitrary number of particles, and also the exact degeneracy structure of this model remains an open problem. In the above, we have concentrated in
finding the wavefunctions in the Cartesian basis; the interested readers are referred to Ref.(36) for some wavefunctions in the angular basis.

Below, we list a few eigenfunctions constructed by using the elementary symmetric functions (we follow the notations of Ref.(49) for the symmetric polynomials).

\[
e_1 = \sum_{1 \leq i \leq N} x_i, \quad e_2 = \sum_{1 \leq i < j \leq N} x_i x_j, \quad e_3 = \sum_{1 \leq i < j < k \leq N} x_i x_j x_k, \ldots, e_N = \prod_{i=1}^N x_i.
\] (37)

In this case, \( \psi_{\{m_i\}} = \psi_0 B_{\{m_i\}}, \) \( B_{\{m_i\}} = e^{-T_s/2} \prod_i (e_i)^{m_i} \), and the corresponding eigenvalues are, \( E_{\{m_i\}} = \sum_i i m_i + E_0 \). Some of the \( B_{\{m_i\}} \)'s for the four particle case are listed below:

\[
B_{2,0,0,0} = e_1^2 - 2, \quad B_{1,1,0,0} = e_1 e_2 - \frac{1}{2} (3 - 4 \beta) e_1, \quad B_{3,0,0,0} = e_1^3 - 6 e_1,
\]

\[
B_{2,1,0,0} = e_1^2 e_2 - (3 - 2 \beta) e_1^2 - 2 e_2 + (3 - 4 \beta), \quad B_{4,0,0,0} = e_1^4 - 12 e_1^2 + 12,
\]

\[
B_{3,1,0,0} = e_1^3 e_2 - \frac{1}{2} (9 - 4 \beta) e_1^3 - 6 e_1 e_2 + 6 (3 - 2 \beta) e_1, \quad B_{5,0,0,0} = e_1^5 - 20 e_1^3 + 60 e_1,
\]

\[
B_{4,1,0,0} = e_1^4 e_2 - 2 (3 - \beta) e_1^4 - 12 e_1^2 e_2 + 12 e_2 + 6 (9 - 4 \beta) e_1^2 - 12 (3 - 2 \beta).
\] (38)

At this point, it is worth recollecting the Stanley-Macdonald conjecture\(^{49}\), which states that, the coefficients of the interaction parameter \( \beta \) are positive integers, when the Jack polynomials\(^{49}\) are expressed in terms of the monomial symmetric functions with a suitable normalization. This conjecture was later proved by Sah\(^{50}\). Similar feature appears in the case of the Hi-Jack polynomials\(^{51}\), which are the polynomial part of the wavefunctions of the CSM, but with an exception that the coefficient \( \beta \) can also be negative. Remarkably, from the above explicit computations of the polynomials, we also find that the coefficients of the interacting parameter \( \beta \) are integers (both positive and negative), though we have used elementary symmetric functions. It will be interesting to check whether the modified Stanley-Macdonald conjecture also holds in the present case for \( N \) particles.

**IV. MODEL ON A CIRCLE**
A. $A_{N-1}$ Model

Recently, Jain et al. [36,37] have studied a model with nearest and next to nearest neighbour interactions and with periodic boundary conditions as given by

$$H = -\frac{1}{2} \sum_j \partial_j^2 + \beta(\beta - 1) \frac{\pi^2}{L^2} \sum_j \frac{1}{\sin^2[\pi(x_j - x_{j+1})]}$$

$$- \beta^2 \frac{\pi^2}{L^2} \sum_j \cot[\frac{\pi}{L}(x_{j-1} - x_j)] \cot[\frac{\pi}{L}(x_j - x_{j+1})],$$

(39)

with $x_{i+N} = x_i$. They have shown that the ground state energy eigenvalue and the eigenfunction for this model are given by

$$\psi_0 = \prod_j [\sin \frac{\pi}{L}(x_j - x_{j+1})]^\beta, \quad E_0 = N\beta^2 \frac{\pi^2}{L^2}.$$  

(40)

The purpose of this section is to obtain a part of the excitation spectrum of this model. To that end, we substitute

$$\psi = \psi_0 \phi,$$

(41)

in the eigenvalue equation for the Hamiltonian. It is then easily shown that $\phi$ satisfies the equation

$$\left( -\frac{1}{2} \sum_{j=1}^N \partial_j^2 - \beta \frac{\pi}{L} \sum_{j=1}^N \left[ \cot \frac{\pi}{L}(x_j - x_{j+1}) - \cot \frac{\pi}{L}(x_{j-1} - x_j) \right] \partial_j + E_0 - E \right) \phi = 0.$$  

(42)

Introducing, $z_j = \exp(2i\pi x_j/L)$, Eq. (42) reduces to

$$H_1 \phi = (\epsilon - \epsilon_0) \phi$$  

(43)

where

$$H_1 = \sum_{j=1}^N D_j^2 + \beta \sum_{j=1}^N \left[ \frac{z_j + z_{j+1}}{z_j - z_{j+1}} \right] (D_j - D_{j+1}),$$  

(44)

with $D_j \equiv \frac{\partial}{\partial z_j}$ and $\epsilon - \epsilon_0 = (E - E_0) \frac{L^2}{2\pi^2}$. It is worth pointing out that the Eqs. (43) and (44) are structurally similar to those in the SM except $z_k$ is replaced by $z_{j+1}$ in our case.

It may be noted that $H_1$ commutes with the momentum operator $P = \frac{2\pi}{L} \sum_{i=1}^N \frac{\partial}{\partial z_i}$. Hence $\phi$ is also an eigenstate of the momentum operator, i.e.,
Further, if $\phi$ is an eigenstate of $H_1$ and $P$ then

$$\phi' = G^q\phi, \quad G = \prod_{i=1}^{N} z_i,$$

is also an eigenstate of $H'$ and $P$ with eigenvalues $\epsilon - \epsilon_0 + Nq^2 + 2q\kappa$ and $\kappa + Nq$ respectively. Here $q$ is any integer (both positive and negative). Note that the multiplication by $G$ implements Galilei boost.

It may be noted that the Hamiltonian and hence the $\phi$ equation is invariant under $z_j \rightarrow z_j^{-1}$. Since, $z_j = e^{2i\pi x_j/L}$, hence $z_j^{-1} = e^{-2i\pi x_j/L}$ thereby indicating the presence of left and right moving modes with momentum $\kappa$ and $-\kappa$. Hence it follows that, if one obtains a solution with momentum $\kappa$, then by changing $z_j \rightarrow z_j^{-1}$, one can get another solution with the same energy but with the opposite momentum ($-\kappa$). Thus all the excited states with nonzero momentum are (at least) doubly degenerate.

Finally, let us discuss the solutions to the $\phi$ equation. So far we have been able to obtain the following four solutions.

(i) $\phi = e_1, \quad \epsilon - \epsilon_0 = 1 + 2\beta,$

(ii) $\phi = e_{N-1}, \quad \epsilon - \epsilon_1 = N - 1 + 2\beta,$

(iii) $\phi = e_1e_{N-1} - \frac{N}{1 + 2\beta}e_N, \quad \epsilon - \epsilon_0 = N + 2 + 4\beta,$

(iv) $\phi = e_N, \quad \epsilon - \epsilon_0 = N.$

Here $e_j$ (j=1,2,...,N) denotes the elementary symmetric functions as defined by Eq. (37) (defined in terms of $z_j$). For example, $e_2 = z_1z_2 + \ldots + z_{N-1}z_N$ and it has $N(N-1)/2$ number of terms.

As mentioned above, each of these solution is doubly degenerate. For example solutions $e_1$ and $e_{N-1}/e_N$ are degenerate. By taking the linear combinations of these two complex solutions, it is easily seen that the two degenerate real solutions are

$$\phi = \sum_{i=1}^{N} \cos u_i, \quad \phi = \sum_{i=1}^{N} \sin u_i,$$
where \( u_i = \frac{2\pi x_i}{L} \). Similarly, all other doubly excited state solutions can be rewritten as two independent real solutions. It would appear from this discussion that all the excited states are doubly degenerate. However, this is not so. In particular, consider

\[
\phi = \frac{e_1 e_{N-1}}{e_N} - \frac{N}{1 + 2\beta}.
\]

It is easily shown that it is an exact solution to Eq. (43) with \( \epsilon - \epsilon_0 = 2 + 4\beta \) but with momentum eigenvalue \( \kappa = 0 \). This is a nondegenerate solution as it remains invariant under \( z_i \to z_i^{-1} \). In terms of the trigonometric functions it can be rewritten as

\[
\phi = \sum_{i<j}^N \cos(u_i - u_j) + \frac{N\beta}{1 + 2\beta}.
\]

At first sight it appears somewhat surprising that whereas in the Sutherland model, there are so many excited state solutions, in our case one is able to obtain so few solutions. In this context it may be noted that whereas the Hamiltonian in the Sutherland case is invariant under the full permutation group \( S_N \), in our case for \( N > 3 \) the Hamiltonian \( H \) as given by Eq. (44) has only cyclic symmetry. If one looks at the solutions to Eq. (44), then one finds that the condition of no pole in the \( \beta \)-dependent term almost forces \( \phi \) to be invariant under \( S_N \). Now out of the various \( e_i \) (i=1,2,...,N), the only ones in which the demand of cyclic invariance necessarily ensures invariance under full permutation group are precisely \( e_1, e_{N-1}, e_N \), in terms of which we have obtained the four solutions.

**B. BC\(_N\) Model**

Recently Auberson et al.\(^{37}\) have studied a \( BC_N \) model with nearest and next-to-nearest neighbour interactions and with periodic boundary conditions as given by

\[
H = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \beta(\beta - 1) \frac{\pi^2}{L^2} \sum_{i=1}^N \left[ \frac{1}{\sin^2 L} (x_i - x_{i+1}) + \frac{1}{\sin^2 L} (x_i + x_{i+1}) \right]
- \beta^2 \frac{\pi^2}{L^2} \sum_{i=1}^N \left[ \cot \frac{\pi}{L} (x_{i-1} - x_i) - \cot \frac{\pi}{L} (x_{i-1} + x_i) \right]
\left[ \cot \frac{\pi}{L} (x_i - x_{i+1}) + \cot \frac{\pi}{L} (x_i + x_{i+1}) \right] + g_1 \frac{\pi^2}{L^2} \sum_i \frac{\pi}{L} x_i + g_2 \frac{\pi^2}{L^2} \sum_i \frac{1}{\sin^2 L} x_i.
\]

\[ (51) \]
Following them, we restrict the coordinates \( x_i \) to the sector \( L \geq x_1 \geq x_2 \geq \ldots \geq x_N \geq 0 \). As shown by Auberson et al.\(^3\), the ground state eigenfunction is given by

\[
\psi_0 = \prod_{i=1}^{N} \sin^{\gamma} \theta_i \prod_{i=1}^{N} \left( \sin^2 2\theta_i \right)^{\gamma_1/2} \prod_{i=1}^{N} \left[ \sin^2 (\theta_i - \theta_{i+1}) \right]^{\beta/2} \prod_{i=1}^{N} \left[ \sin^2 (\theta_i + \theta_{i+1}) \right]^{\beta/2},
\]

(52)

where \( g_1, g_2 \) are related to \( \gamma, \gamma_1 \) by

\[
g_1 = \gamma \left( \gamma + 2\gamma_1 - 1 \right), \quad g_2 = 2\gamma_1 (\gamma_1 - 1).
\]

(53)

The corresponding ground state energy turns out to be

\[
E_0 = \frac{N\pi^2}{2L^2} (\gamma + 2\gamma_1 + 2\beta)^2.
\]

(54)

By setting one or both of the coupling constants \( \gamma, \gamma_1 \) to zero we get the other root systems i.e.

\[
B_N : \gamma_1 = 0, \quad C_N : \gamma = 0, \quad D_N : \gamma = \gamma_1 = 0.
\]

(55)

The purpose of this subsection is to obtain a part of the excitation spectrum of the \( BC_N, B_N, C_N, D_N \) models. To that end, we substitute

\[
\psi = \psi_0 \phi,
\]

(56)

in the eigenvalue equation for the above Hamiltonian where \( \psi_0 \) is as given by Eq. (52). It is easy to show that in that case \( \phi \) satisfies the equation

\[
\begin{align*}
&\left[ \sum_{j=1}^{N} \frac{\partial^2}{\partial \theta_j^2} + 2\beta \sum_{j=1}^{N} \cot(\theta_j - \theta_{j+1}) \left( \frac{\partial}{\partial \theta_j} - \frac{\partial}{\partial \theta_{j+1}} \right) \\
&+ 2\beta \sum_{j=1}^{N} \cot(\theta_j + \theta_{j+1}) \left( \frac{\partial}{\partial \theta_j} + \frac{\partial}{\partial \theta_{j+1}} \right) + 2\gamma \sum_{j=1}^{N} \cot \theta_j \frac{\partial}{\partial \theta_j} \\
&+ 4\gamma_1 \sum_{j=1}^{N} \cot 2\theta_j \frac{\partial}{\partial \theta_j} + (E - E_0) \frac{2L^2}{\pi^2} \right] \phi = 0,
\end{align*}
\]

(57)

where \( \theta_j = \frac{\pi x_j}{L} \). Introducing, \( z_j = \exp(2i\theta_j) \), Eq. (57) reduces to

\[
H_1 \phi = (\epsilon - \epsilon_0) \phi
\]

(58)
where
\[
H_1 = \sum_{j=1}^{N} D_j^2 + \gamma \sum_{j=1}^{N} \frac{z_j + 1}{z_j - 1} D_j + 2\gamma_1 \sum_{j=1}^{N} \frac{z_j^2 + 1}{z_j - 1} D_j \\
+ \beta \sum_{j=1}^{N} \left[ \frac{z_j + z_{j+1}}{z_j - z_{j+1}} \right] (D_j - D_{j+1}) + \beta \sum_{j=1}^{N} \left[ \frac{z_j z_{j+1} + 1}{z_j z_{j+1} - 1} \right] (D_j + D_{j+1}).
\] (59)

Here \(D_j \equiv z_j \frac{\partial}{\partial z_j}\) while \(\epsilon - \epsilon_0 = (E - E_0) \frac{L^2}{2\pi^2}\). It is worth pointing out that the above equation is structurally similar to that of the \(BC_N\) of SM, except \(z_{j+1}\) is replaced by \(z_k\) in our case.

Note that apart from the cyclic symmetry, the Hamiltonian and hence the \(\phi\) equation is also invariant under \(z_j \rightarrow z_j^{-1}\). As a consequence, as in the \(BC_N\) Sutherland model, it turns out that even in our case the polynomial eigenfunctions of \(H_1\) with \(BC_N\) symmetry as given by Eq. (59) are symmetric polynomials in \((z_j + \frac{1}{z_j})\) i.e. in \(\cos(2\pi x_j/L)\).

Finally, let us discuss the solutions to the \(\phi\) equation. So far we have been able to obtain only one solution in the \(BC_N\) case but are able to obtain several solutions in the \(D_N\) case and also a few in the \(B_N\) and \(C_N\) cases.

C. Exact Solution for the \(BC_N\) Model

It is easily checked that the exact solution is
\[
\phi \equiv \phi_{BC_N} = \phi_1 + \alpha, \quad \epsilon - \epsilon_0 = 1 + \gamma + 2\gamma_1 + 4\beta,
\] (60)

where
\[
\phi_1 = \sum_{j=1}^{N} (z_j + \frac{1}{z_j}), \quad \alpha = \frac{2N\gamma}{1 + \gamma + 2\gamma_1 + 4\beta}.
\] (61)

We will see that this solution (and in fact other solutions, if any, in the \(BC_N\) case) will play important roles in our construction of solutions for other systems like \(B_N, C_N\) and \(D_N\).

D. Exact Solutions For the \(B_N\) Model \((\gamma_1 = 0)\)

Apart from the obvious solution \(\phi(z_j; \beta, \gamma, \gamma_1 = 0)\) as given by Eq. (60) it turns out that there are other solutions corresponding to the spinorial representation for the \(B_N\) model. These are
\[ \phi \equiv \phi^+ = \Pi_{j=1}^{N} (\sqrt{z_j} + \frac{1}{\sqrt{z_j}}), \quad \epsilon^+ - \epsilon_0 = \frac{N}{4} [1 + 2\gamma + 4\beta]. \] (62)

In order to obtain the other solution, we start with the ansatz

\[ \phi = \phi^+ \psi^+ \] (63)

where \( \phi^+ \) is as given by Eq. (62) and consider the equation \( H_1 \phi = (\epsilon - \epsilon_0) \phi \). It is easily seen that in that case \( \psi^+ \) satisfies

\[ H_1^+ (z_j; \beta, \gamma, \gamma_1 = 0) \psi^+ = (\epsilon - \epsilon^+) \psi^+, \] (64)

where \( \epsilon^+ \) is as given by Eq. (62) while

\[ H_1^+ = H_1 (z_j; \beta, \gamma, \gamma_1 = 0) + \sum_{j=1}^{N} \frac{z_j - 1}{z_j + 1} D_j \]
\[ = H_1 (z_j; \beta, \gamma - 1, \gamma_1 = 1). \] (65)

We thus see that \( \psi^+ \) essentially satisfies the same equation as satisfied by the \( BC_N \) Hamiltonian but with the value of \( \gamma \) and \( \gamma_1 \) shifted to \( \gamma - 1 \) and 1 respectively. Thus it follows that once we obtain solutions of the \( BC_N \) problem, all these will give us new solutions of the spinorial type for the \( B_N \) model as given by Eqs. (63) to (65). Unfortunately, so far we have been able to obtain only one solution in the \( BC_N \) case as given by Eqs. (60) and (61). Using that solution, it then follows that the new spinorial solution for the \( B_N \) model is as given by Eq. (63) with energy

\[ \epsilon - \epsilon_0 = 2 + \gamma + 4\beta + \frac{N}{4} [1 + 2\gamma + 4\beta], \] (66)

while \( \psi^+ \equiv \phi_{BC_N} (z_j; \beta, \gamma - 1, \gamma_1 = 1) \) with \( \phi_{BC_N} \) being given by Eqs. (60) and (61).

**E. Exact Solutions For the \( D_N \) Model (\( \gamma = \gamma_1 = 0 \))**

Apart from the above solutions (60), (62) and (63) (with \( \gamma = \gamma_1 = 0 \)), we have found several other solutions in the \( D_N \) case. This is related to the fact that unlike \( B_N \), there are
two distinct classes of spinor representations for $D_N$. Besides, there are also some additional solutions in this case. It may be noted that in this case the Hamiltonian $H_1$ acting on $\phi$ is

$$H_1(z_j; \beta) = H_1(z_j; \beta, \gamma = 0, \gamma_1 = 0).$$ (67)

The first new solution that we have is given by

$$\phi \equiv \phi^- = \Pi_{j=1}^N (\sqrt{z_j} - \frac{1}{\sqrt{z_j}}), \quad \epsilon^- - \epsilon_0 = \frac{N}{4}[1 + 4\beta].$$ (68)

Note that the two solutions (62) (with $\gamma = 0$) and (68) which correspond to the two different spinorial representations are degenerate in energy.

In order to obtain the other solution, we start with the ansatz

$$\phi = \phi^- \psi^-$$ (69)

where $\phi^-$ is as given by Eq. (68) and consider the equation $H_1 \phi = (\epsilon - \epsilon_0)\phi$. It is easily seen that in that case $\psi^-$ satisfies

$$H_1^- (z_j; \beta, \gamma = 0, \gamma_1 = 0) \psi^- = (\epsilon^- - \epsilon^-) \psi^- ,$$ (70)

where $\epsilon^-$ is as given by Eq. (68) while

$$H_1^- = H_1(z_j; \beta, \gamma = 0, \gamma_1 = 0) + \sum_{j=1}^N \frac{z_j + 1}{z_j - 1} D_j$$

$$= H_1(z_j; \beta, \gamma = 1, \gamma_1 = 0).$$ (71)

We thus see that $\psi^-$ essentially satisfies the same equation as satisfied by the $B_N$ Hamiltonian but with the value of $\gamma$ being fixed at 1. Using the solution for the $B_N$ case as given by Eqs. (60) and (61) (with $\gamma = 1, \gamma_1 = 0$) it then follows that the new spinorial solution for the $D_N$ model is as given by Eq. (69) with energy

$$\epsilon - \epsilon_0 = 2 + 4\beta + \frac{N}{4}[1 + 4\beta],$$ (72)

while $\psi^- \equiv \phi_{BC_N}(z_j; \beta, \gamma = 1, \gamma_1 = 0)$ with $\phi_{BC_N}$ being given by Eqs. (60) and (61). Notice that this solution is degenerate in energy with the solution (68) (with $\gamma = 0$).
In addition, we find that the product of the two “spinorial solutions” is also a solution of the $D_N$ model, i.e.

$$\phi \equiv \phi^+ \phi^- = \Pi_{j=1}^{N}(z_j - \frac{1}{z_j}), \quad \epsilon^{+-} - \epsilon_0 = N[1 + 2\beta]. \quad (73)$$

In order to obtain another solution, as above we start with the ansatz

$$\phi = \phi^+ \phi^- \psi^{+-} \quad (74)$$

where $\phi^+, \phi^-$ are as given by Eqs. (72) and (77) respectively and consider the equation $H_1 \phi = (\epsilon - \epsilon_0) \phi$. It is easily seen that in that case $\psi^{+-}$ satisfies

$$H_1^{+-}(z_j; \beta, \gamma = 0, \gamma_1 = 0) \psi^{+-} = (\epsilon - \epsilon^{+-}) \psi^{+-}, \quad (75)$$

where $\epsilon^{+-}$ is as given by Eq. (73) while

$$H_1^{+-} = H_1(z_j; \beta, \gamma = 0, \gamma_1 = 0) + 2 \sum_{j=1}^{N} \frac{z_j^2 + 1}{z_j^2 - 1} D_j$$

$$= H_1(z_j; \beta, \gamma = 0, \gamma_1 = 1). \quad (76)$$

We thus see that $\psi^{+-}$ essentially satisfies the same equation as satisfied by the $C_N$ Hamiltonian but with the value of $\gamma_1$ being fixed at 1. Using the solution for the $BC_N$ case as given by Eqs. (60) and (61) (with $\gamma = 0, \gamma_1 = 1$), it then follows that the new solution for the $D_N$ model is as given by Eq. (74) with energy

$$\epsilon - \epsilon_0 = 3 + 4\beta + N[1 + 2\beta], \quad (77)$$

while $\psi^{+-} \equiv \phi_{BC_N}(z_j; \beta, \gamma = 0, \gamma_1 = 1)$ with $\phi_{BC_N}$ being given by eqs. (60) and (61).

F. Exact Solution For the $C_{N=4}$ Model ($\gamma = 0$)

So far we have discussed all the solutions which are valid for any $N$. In addition, in the special case of $N = 4$, we have been able to obtain a solution in the $C_{N=4}$ (and two solutions in the $D_{N=4}$) case. The solution for the $C_{N=4}$ case is given by
\[ \phi \equiv \phi^{31} = A \left[ (z_1 + \frac{1}{z_1})(z_2 + \frac{1}{z_2})(z_3 + \frac{1}{z_3}) + C.P. \right] + B \phi_1, \quad (78) \]

where

\[ \epsilon - \epsilon_0 = 3 + 6\gamma_1 + 8\beta, \quad B = \frac{8A\beta}{1 + 2\gamma_1 + 2\beta}. \quad (79) \]

Here, by C.P. one means cyclic permutations and \( \phi_1 \) is as given by Eq. (60).

\textbf{G. Exact Solutions For the } D_{N=4} \textbf{ Model (} \gamma = \gamma_1 = 0 \textbf{)}

Clearly, one obvious solution in the \( D_{N=4} \) case is obtained from the solution (78) by putting \( \gamma_1 = 0 \). The other solution is obtained by making use of the ansatz as given by Eq. (74), i.e. let

\[ \phi = \phi^{31} \phi^{+-}, \quad (80) \]

with \( \phi^{31} \) being given by Eq. (78). Using Eqs. (74) to (76) it then follows that this is a solution for the \( D_{N=4} \) model with

\[ \epsilon - \epsilon_0 = 13 + 16\beta, \quad B = \frac{8A\beta}{3 + 2\beta}. \quad (81) \]

\textbf{V. CONCLUSIONS}

In conclusion, we have carried out a systematic study of the many-body Hamiltonian, related to the short range Dyson model. The scattering state of this model is obtained and is shown to be a coherent state. Akin to the CSM, the connection of the scattering Hamiltonian to a free system is established. Unlike the Calogero model, analysis of the scattering process for the present model reveals that, only a part of the partial waves acquire energy independent phase shifts. We then showed the mapping of the bound system to decoupled oscillators and by explicitly computing some of the bound-state eigenfunctions, we find that, the present model has less degeneracy as compared to the Calogero-Sutherland...
model. Finally, we have studied the $A_{N-1}$, $BC_N$, $B_N$, $C_N$ and $D_N$ models on a circle and obtained a part of their excitation spectrum. A number of open problems, like finding the $P_{m,q}(x)$’s explicitly for the N-body problem, characterizing the degeneracy structure, finding the complete eigenspectra and conserved quantities for these type of models still remains to be tackled. We hope to come back to some of these issues in future.

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