Complete minimal submanifolds with nullity in the hyperbolic space

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Abstract

We investigate complete minimal submanifolds $f : M^3 \to H^n$ in hyperbolic space with index of relative nullity at least one at any point. The case when the ambient space is either the Euclidean space or the round sphere was already studied in [6] and [7], respectively. If the scalar curvature is bounded from below we conclude that the submanifold has to be either totally geodesic or a generalized cone over a complete minimal surface lying in an equidistant submanifold of $H^n$.

In this paper, we continue our study of a class of minimal isometric immersions of complete Riemannian manifolds in space forms $f : M^m \to Q^n_c$ initiated in [6] for sectional curvature $c = 0$ and continued in [7] for $c > 0$. The basic hypothesis is that the index of relative nullity satisfies $\nu \geq m - 2$ everywhere. The goal is to conclude that under some reasonable assumption the submanifold has to be of a simple geometric type other than totally geodesic. For instance, under the hypothesis that the Omori-Yau maximum principle holds on the manifold, we showed in [6] that the Euclidean submanifold has to be a $(m - 2)$-cylinder.

In any of the two cases already studied, the proofs reduced to analyzing the situation of the three dimensional submanifolds. In fact, for submanifolds in spheres only this case turned out to be possible. This paper is devoted to complete minimal submanifolds in hyperbolic space $f : M^m \to H^n$ under the assumption that $\nu \geq m - 2$. In this ambient space this condition turns out to be quite less restrictive than in the previously studied cases. Nevertheless, we have reasons to believe that the manifold being three-dimensional is still quite special and this is why this case allows a characterization of a class of submanifolds that is contained in the following description.

Let $i : Q^{n-\ell}_c \to H^n$, $1 \leq \ell \leq n - 3$ and $c \neq -1$, denote an inclusion as a complete simply connected umbilical submanifold. Thus $Q^{n-\ell}_c$ is either a totally geodesic submanifold

This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Science and Technology Agency of the Región de Murcia. This work was partially supported by MINECO/FEDER project reference MTM2015-65430-P and Fundación Séneca project reference 19901/GERM/15, Spain.
of a geodesic sphere or an equidistant hypersurface or a horosphere if \( c > 0, \ c < 0 \) or \( c = 0 \), respectively. Let \( g: \mathbb{L}^2 \to \mathbb{Q}_n^{n-\ell} \) be an isometric immersion of a two-dimensional Riemannian manifold. The normal bundle of \( h = i \circ g: \mathbb{L}^2 \to \mathbb{H}^n \) splits orthogonally as

\[
N_h L = i_* N_g L \oplus N_i \mathbb{Q}_n^{n-\ell}
\]

where \( N_i \mathbb{Q}_n^{n-\ell} \) is regarded as a subbundle of \( N_h L \). Let \( G: N_i \mathbb{Q}_n^{n-\ell} \to \mathbb{H}^n \) be defined by

\[
G(x, w) = \exp_{g(x)} w
\]

where \( \exp \) denotes the exponential map of \( \mathbb{H}^n \). Then we denote by \( M_{m}, m = 2 + \ell \), the open subset of \( N_i \mathbb{Q}_n^{n-\ell} \) where \( G \) is an immersion endowed with the metric induced by the map \( G \).

**Definition:** The *generalized cone* in hyperbolic space over a surface \( g: \mathbb{L}^2 \to \mathbb{Q}_n^{n-\ell} \) is the isometric immersion \( F_g: M_{m} \to \mathbb{H}^n, m = 2 + \ell \), defined as \( F_g = G|_{M_{m}} \).

**Theorem 1.** Let \( f: M^3 \to \mathbb{H}^n \) be a minimal isometric immersion with index of relative nullity at least \( \nu \geq 1 \) at any point. Assume that \( M^3 \) is complete with scalar curvature bounded from below. Then \( f \) is either totally geodesic or a generalized cone over a complete minimal surface with bounded Gauss curvature lying in an equidistant submanifold of \( \mathbb{H}^n \).

Notice that generalized cones over minimal surfaces contained in the other two types of umbilical submanifolds are not part of the theorem. In fact, if the surface lies inside a geodesic sphere then the generalized cone is never complete whereas if it lies in a horosphere then the scalar curvature of the cone is unbounded.

Like it happens for \( c \geq 0 \), in the present case where \( c < 0 \) there are plenty of local examples other than generalized cones. As a matter of fact, a local parametrization of all minimal submanifolds \( f: M^m \to \mathbb{H}^n \) with index of relative nullity \( \nu = m - 2 \) was given in [12] in terms of certain elliptic space-like surfaces in either the Lorentzian sphere or the Lorentzian flat space according to \( n - m \) being even or odd, respectively. Moreover, from the results in [3], [9] and [11] it is clear that this parametrization can be used to construct complete examples of any dimension other than generalized cones.

1 Preliminaries

Let \( f: M^m \to \mathbb{H}^n \) denote an isometric immersion of an \( m \)-dimensional Riemannian manifold \( M^m \) into hyperbolic space. The *relative nullity subspace* of \( f \) at \( x \in M^m \) is defined as

\[
\mathcal{D}(x) = \{ X \in T_x M: \alpha(X, Y) = 0 \ \text{for all} \ Y \in T_x M \}
\]

where \( \alpha: TM \times TM \to N_f M \) denotes the second fundamental form with values in the normal bundle.
The dimension \( \nu(x) \) of \( D(x) \) is called the *index of relative nullity* of \( f \) at \( x \in M^m \). If \( f \) has constant index of relative nullity on an open subset \( U \subset M^m \), then \( D \) is integrable along \( U \), its leaves are totally geodesic submanifolds of \( M^m \) and their images under \( f \) are totally geodesics submanifolds of \( \mathbb{H}^n \).

Let \( U \subset M^m \) be an open subset where the index of relative nullity \( \nu = s > 0 \) is constant. The following is a fundamental result in the theory of isometric immersions into space forms; cf. [4].

**Proposition 2.** Let \( \gamma: [0, b] \to M^m \) be a geodesic such that \( \gamma([0, b]) \subset U \) is contained in a leaf of relative nullity foliation. Then \( \nu(\gamma(b)) = s \).

We decompose any \( X \in TM \) as

\[
X = X^v + X^h
\]

according to the orthogonal decomposition \( TM = D \oplus D^\perp \), and denote by \( \nabla^v \) and \( \nabla^h \) the components of the Levi-Civita connection with respect to that decomposition.

The *splitting tensor* \( C: D \times D^\perp \to D^\perp \) is defined as

\[
C(T, X) = -\nabla^h_X T = -(\nabla_X T)^h
\]

for any \( T \in D \) and \( X \in D^\perp \). If \( x \in M^m \) and \( T \in D(x) \), then the tensor gives rise to an endomorphism \( C_T = C(T, \cdot): D^\perp(x) \to D^\perp(x) \). Accordingly, we regard \( C \) as a map

\[
C: \Gamma(D) \to \Gamma(\text{End}(D^\perp)).
\]

The following differential equations are a well known consequence of the Codazzi equation:

\[
\nabla_S C_T = C_T \circ C_S + C_{\nabla^h_S T} - \langle T, S \rangle I
\]

(1)

where \( I \) is the identity map,

\[
(\nabla^h_X C_T)Y - (\nabla^h_Y C_T)X = C_{\nabla^h_X T}Y - C_{\nabla^h_Y T}X
\]

(2)

and

\[

abla_T A_\xi|_{D^\perp} = A_\xi|_{D^\perp} \circ C_T + A_{\nabla^h_T \xi}|_{D^\perp}
\]

(3)

for any \( S, T \in \Gamma(D) \), \( X, Y \in \Gamma(D^\perp) \) and with

\[
\langle A_\eta X, Y \rangle = \langle \alpha(X, Y), \eta \rangle
\]

for any \( \eta \in N_f M \). See [4] or [5].
A main ingredient in the proof of our result is the Omori-Yau maximum principle. Recall that the Omori-Yau maximum principle is said to hold on $M^m$ if for any function $\varphi \in C^2(M)$ bounded from above there exists a sequence of points $\{x_j\}_{j \in \mathbb{N}}$ such that

$$\varphi(x_j) > \sup \varphi - 1/j, \quad \|\nabla \varphi\|(x_j) \leq 1/j \quad \text{and} \quad \Delta \varphi(x_j) \leq 1/j$$

for any $j \in \mathbb{N}$. It is well-known that the principle is valid if the manifold is complete and the Ricci curvature does not decay fast to $-\infty$. It also holds for properly immersed submanifolds in space forms with bounded length of the mean curvature vector field; cf. [1].

The following two results that will be used for the proofs are consequence of the Omori-Yau maximum principle.

**Proposition 3.** Let $M^m$ be a Riemannian manifold for which the Omori-Yau maximum principle holds. If $\varphi \in C^\infty(M)$ satisfies that $\Delta \varphi \geq 2 \varphi^2$ then $\sup \varphi = 0$.

**Proof:** See [1] or [10] Lemma 4.1. □

**Proposition 4.** Let $M^m$ be a Riemannian manifold which Ricci curvature bounded from below by $-K$ for some constant $K \geq 0$. If $f \in C^\infty(M)$ is a harmonic function which is bounded from below on $M^m$, then

$$\|\nabla f\| \leq \sqrt{(m - 1)K} (f - \inf f).$$

**Proof:** See [16] Theorem 3′′. □

## 2 Generalized cones

In this section, we discuss several basic properties of generalized cones in hyperbolic space.

That a smooth tangent distribution $D$ is umbilical means that there exists a smooth section $\delta$ of $D^\perp$ such that

$$\langle \nabla_X Y, T \rangle = \langle X, Y \rangle \langle \delta, T \rangle$$

for all $X, Y \in D$ and $T \in D^\perp$.

**Proposition 5.** Let $f : M^m \to \mathbb{H}^n$ be an isometric immersion with constant index of relative nullity $1 \leq \nu \leq m - 2$. Assume that the conullity distribution $D^\perp$ is umbilical. Then $f$ is locally a generalized cone over an isometric immersion $g : \Sigma^{m-\nu} \to \mathbb{Q}^{n-\nu}$ into an umbilical submanifold of $\mathbb{H}^n$. Moreover, the submanifold is globally a generalized cone if the relative nullity leaves are complete.
Proof: Let \( j: \Sigma^{m-\nu} \to M^n \) denote the inclusion of a leaf \( \Sigma^{n-\nu} \) of \( \mathcal{D} \) into \( M^n \). Then set \( h = f \circ j: \Sigma^{m-\nu} \to \mathbb{H}^n \). The normal bundle of \( h \) splits as
\[
N_h \Sigma = f_* N_j \Sigma \oplus N_f M = f_* \mathcal{D}|_\Sigma \oplus N_f M.
\]
By assumption, there exists \( \delta \in \Gamma(\mathcal{D}) \) such that \( C_T = \langle T, \delta \rangle I \) for all \( T \in \Gamma(\mathcal{D}) \). Thus,
\[
\tilde{\nabla}_X f_* T = f_* \nabla_X T = -f_* C_T X + f_* \nabla^n_X T = -\langle T, \delta \rangle f_* X + f_* \nabla^n_X T
\]
for all \( X \in T \Sigma \) and \( T \in \Gamma(\mathcal{D}) \), where \( \tilde{\nabla} \) is the induced connection on \( f^* \mathbb{H}^n \). Therefore the subbundle \( \mathcal{S} = f_* \mathcal{D}|_\Sigma \) of \( N_h \Sigma \) is parallel with respect to the normal connection and the shape operator of \( h \) with respect to any section \( \eta = f_* T \) of \( \mathcal{S} \), with \( T \in \Gamma(\mathcal{D}) \), is given by
\[
A^h_\eta = \langle T, \delta \rangle I.
\]
It follows easily that there exist an umbilical inclusion \( i: \mathcal{Q}_c^{n-\nu} \to \mathbb{H}^n \) and an isometric immersion \( g: \Sigma^{m-\nu} \to \mathcal{Q}_c^{n-\nu} \) such that \( h = i \circ g \). Moreover, at any \( x \in \Sigma^{m-\nu} \) the fiber \( \mathcal{S}(x) = f_* \mathcal{D}(x) \) coincides with the normal space of \( i \) at \( i(x) \). Thus the generalized cone over \( g \) coincides locally with \( f \). The global statement is immediate. □

Proposition 6. Let \( g: L^2 \to \mathcal{Q}_c^{n-\nu} \) be a minimal surface. With the notation given above we have that the following facts hold:

(i) The generalized cone \( F_g: M^n \to \mathbb{H}^n \), \( m = 2 + \nu \), over \( g \) is a minimal immersion with index of relative nullity at least \( \nu \) at any point.

(ii) The map \( G \) is an immersion if and only if \( \mathcal{Q}_c^{n-\nu} \) is a totally geodesic submanifold of either an equidistant hypersurface or a horosphere in \( \mathbb{H}^n \). In that case \( M^n \) is complete if and only if \( L^2 \) is complete. Moreover, if \( \mathcal{Q}_c^{n-\nu} \) is contained in an equidistant (respectively, horosphere) hypersurface then the scalar curvature of \( M^n \) is bounded (respectively, unbounded) along each fiber of the normal bundle of the umbilical inclusion \( i: \mathcal{Q}_c^{n-\nu} \to \mathbb{H}^n \).

Proof: Let \( i: \mathcal{Q}_c^{n-\nu} \to \mathbb{H}^n \) be a complete simply connected umbilical submanifold. Then let \( \eta_1, \eta_2, \ldots, \eta_\nu \) be a global orthonormal frame of the normal bundle of \( i \) such that \( \eta_1 \) points in the direction of the mean curvature vector field \( H \).

Since the normal bundle \( N_i \mathcal{Q}_c^{n-\nu} \) is a trivial vector bundle we have that the map \( G: L^2 \times \mathbb{R}^\nu \to \mathbb{H}^n \) is given parametrically by
\[
G(x, t_1, t_2, \ldots, t_\nu) = \cosh t_\nu f_{\nu-1}(x) + \sinh t_\nu \eta_\nu(x),
\]
where \( f_j \) are defined inductively by \( f_0 = g \) and
\[
f_j = \cosh t_j f_{j-1} + \sinh t_j \eta_j, \quad 1 \leq j \leq \nu.
\]
Set
\[ h_j = \Pi_{k=j+1}^{\nu} \cosh t_k, \quad 1 \leq j \leq \nu - 1 \]
and
\[ r = h_1 (\cosh t_1 - \|H\| \sinh t_1). \]
A straightforward computation gives
\[ G^*(X) = r g^*(X), \quad X \in TL, \]
\[ G^*(\partial_{t_j}) = h_j (\sinh t_j f_{j-1} + \cosh t_j \eta_j), \quad 1 \leq j \leq \nu - 1, \]
\[ G^*(\partial_{t_{\nu}}) = \sinh t_{\nu} f_{\nu-1} + \cosh t_{\nu} \eta_{\nu}. \]

It is clear that the map \( G \) is an immersion if and only if \( \|H\| \leq 1 \), which in turn is equivalent to \( Q^{m-\nu}_c \) being a totally geodesic submanifold of either an equidistant hypersurface or a horosphere in \( \mathbb{H}^n \). Moreover, its second fundamental form is given by
\[ \alpha_G(X,Y) = r \alpha_g(X,Y) \]
if \( X,Y \in TL \), and the fact that the vectors \( \partial_{t_1}, \ldots, \partial_{t_{\nu}} \) belong to the relative nullity subspace. This proves part (i).

The induced metric on \( L^2 \times \mathbb{R}^\nu \) is given by
\[ \langle \cdot, \cdot \rangle_G = r^2 \langle \cdot, \cdot \rangle_g + \langle \cdot, \cdot \rangle_0, \]
where the Euclidean space \( \mathbb{R}^\nu \) is equipped with the complete Riemannian metric
\[ \langle \cdot, \cdot \rangle_0 = h_1^2 dt_1^2 + \cdots + h_{\nu-1}^2 dt_{\nu-1}^2 + dt_{\nu}^2. \]
It follows from Lemma 7.2 in [2] that the manifold \( M^m \) is complete if and only if \( L^2 \) is complete.

Finally, the Gauss equation yields that the scalar curvature \( s \) of \( M^m \) is given by
\[ s = -m(m-1) - \frac{1}{r^2} \| \alpha_g \|^2. \]
This clearly implies that the scalar curvature of \( M^m \) is bounded (respectively, unbounded) along each fiber of the normal bundle of the umbilical inclusion \( i: Q^{m-\nu}_c \to \mathbb{H}^n \) if \( Q^{m-\nu}_c \) is a totally geodesic submanifold of an equidistant hypersurface (respectively, horosphere).

3 The proof of Theorem

We start the proof with a result that holds for submanifolds on any dimension.
Lemma 7. Let \( f : M^m \to \mathbb{H}^n \) be a minimal isometric immersion with constant index of relative nullity \( \nu = m - 2 \). Then \( C_T \in \text{span}\{I,J\} \) for any \( T \in \Gamma(D) \) where \( I \) is the identity endomorphism and \( J \) denotes the almost complex structure of \( D^\perp \).

Proof: We have from (3) that
\[
A_\xi|_{D^\perp} \circ C_T = C^I_T \circ A_\xi|_{D^\perp} \tag{4}
\]
for any \( T \in \Gamma(D) \). On the other hand, the minimality condition is equivalent to
\[
A_\xi|_{D^\perp} \circ J = J^I \circ A_\xi|_{D^\perp}. \tag{5}
\]

We first treat the hypersurface case \( n = m + 1 \). Let \( \xi \) denote a local unit normal vector field along \( f \) and let \( e_1, e_2 \) be an orthonormal tangent frame that diagonalizes \( A_\xi|_{D^\perp} \) such that \( Je_1 = e_2 \). Set
\[
u = \langle \nabla_{e_2} e_1, T \rangle \quad \text{and} \quad v = \langle \nabla_{e_1} e_1, T \rangle.
\]
From the Codazzi equations
\[
(\nabla_{e_1} A_\xi)e_2 = (\nabla_{e_2} A_\xi)e_1 \quad \text{and} \quad (\nabla_{e_i} A_\xi)T = (\nabla_T A_\xi)e_i \quad \text{for} \quad i = 1, 2,
\]
we obtain that
\[
u = -\langle \nabla_{e_1} e_2, T \rangle \quad \text{and} \quad v = \langle \nabla_{e_2} e_2, T \rangle.
\]
It follows that \( C_T = vI - uJ \).

Assume now that \( f \) does not reduce codimension to one. The normal subspaces
\[
N^f_1(x) = \text{span}\{\alpha(X,Y) : \text{for all} \ X,Y \in T_x M\}
\]
have dimension at most two due to minimality. If there is an open subset \( V \subset M^m \) where \( \dim N^f_1 = 1 \), then a simple argument using the Codazzi equation gives that the normal subbundle \( N^f_1 \) is parallel in the normal connection along \( V \). Hence \( f|_V \) reduces codimension to one. But then due to real analyticity of the immersion, the same would hold globally, and that is a contradiction. Therefore, there is an open dense subset of \( M^m \) where \( \dim N^f_1 = 2 \). From (4) and (5) it now follows easily that \( C_T \in \text{span}\{I,J\} \) as we wished.

We now point our attention to the three-dimensional case in which the harmonicity established by the following result will turn out to be fundamental.

Lemma 8. Assume that \( m = 3 \). Let \( e_1, e_2, e_3 \) be a local orthonormal tangent frame such that \( e_3 \) spans \( D^\perp \) and let \( v, u \) are smooth functions such that \( C_{e_3} = vI - uJ \). Then
\[
v = -\langle \nabla_{e_1} e_3, e_1 \rangle = -\langle \nabla_{e_2} e_3, e_2 \rangle \quad \text{and} \quad u = \langle \nabla_{e_1} e_3, e_2 \rangle = -\langle \nabla_{e_2} e_3, e_1 \rangle. \tag{6}
\]
Moreover,
\[
e_3(v) = v^2 - u^2 - 1, \quad e_3(u) = 2uv, \quad e_1(u) = e_2(v) \quad \text{and} \quad e_2(u) = -e_1(v). \tag{7}
\]
Furthermore, the functions \( v \) and \( u \) are harmonic.
The proof of (7) is a direct consequence of (1) and (2). The harmonicity of the functions \( v \) and \( u \) follows by a straightforward computation using (6) and (7) similar to the one given in [6] or [7].

Next we make use of the real analytic structure of a minimal submanifold in order to extend smoothly the relative nullity distribution to the totally geodesic points.

Let \( \mathcal{A} \) denote the set of totally geodesic points of \( f \). By Proposition 2, the relative nullity distribution \( \mathcal{D} \) is a line bundle on \( M^3 \setminus \mathcal{A} \). Since \( f \) is real analytic we have that \( \mathcal{A} \) is a real analytic set. According to Lojasiewicz’s structure theorem [13, Theorem 6.3.3], it follows that \( \mathcal{A} \) locally decomposes as

\[
\mathcal{A} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3
\]

where each set \( \mathcal{V}^k \), \( 0 \leq k \leq 3 \), is either empty or a disjoint finite union of \( k \)-dimensional real analytic subvarieties. A point \( x_0 \in \mathcal{A} \) is called a regular point of dimension \( k \) if there is a neighborhood \( \Omega \) of \( x_0 \) such that \( \Omega \cap \mathcal{A} \) is a \( k \)-dimensional real analytic submanifold of \( \Omega \). Otherwise \( x_0 \) is said to be a singular point. Then the set of singular points is locally a finite union of submanifolds.

We can assume that \( \mathcal{V}^3 \) is empty since, otherwise, we already have by real analyticity that \( f \) is a totally geodesic submanifold.

**Lemma 9.** The set \( \mathcal{V}^0 \) is empty.

**Proof:** The proof goes as in [6] and [7].

**Lemma 10.** The set \( \mathcal{V}^2 \) is empty.

**Proof:** The proof is similar with those given in [6] and [7]. All we have to show is that \( \mathcal{V}^2 \) does not contain regular points. Suppose to the contrary and let \( \Omega \subset M^3 \) be an open neighborhood of a regular point \( x_0 \in \mathcal{V}^2 \) such that \( L^2 = \Omega \cap \mathcal{A} \) is an embedded surface. Let \( e_1, e_2, e_3, \xi_1, \ldots, \xi_{n-3} \) be an orthonormal frame adapted to \( M^3 \) along \( \Omega \) near \( x_0 \).

The Gauss map \( \gamma : M^3 \to Gr(4, n+1) \) takes values into the Grassmannian of oriented space-like 4-dimensional subspaces in the Lorentzian space \( \mathbb{L}^{n+1} \). Regarding \( Gr(4, n+1) \) as a submanifold in \( \wedge^4 \mathbb{L}^{n+1} \) via the map for the Plücker embedding, we have that

\[
\gamma = f \wedge e_1 \wedge e_2 \wedge e_3.
\]

The coefficients of the second fundamental form are

\[
h^a_{ij} = \langle \alpha(e_i, e_j), \xi_a \rangle
\]

where from now on \( 1 \leq i, j, k \leq 3 \) and \( 1 \leq a, b \leq n - 3 \). It is easy to see that

\[
\gamma_* e_i = \sum_{j, a} h^a_{ij} f \wedge e_{ja}
\]

(8)
where $e_{ja}$ is obtained by replacing $e_j$ with $\xi_a$ in $f \wedge e_1 \wedge e_2 \wedge e_3$. Then
\[
\sum_i \langle \gamma_s e_i, \gamma_s e_i \rangle = \sum_{i,j,a} (h_{ij}^a)^2 \langle f \wedge e_{ja}, f \wedge e_{ja} \rangle = -\|\alpha\|^2
\]
where the inner product of two simple 4-vectors in $\wedge^4 \mathbb{L}^{n+1}$ is defined by
\[
\langle a_1 \wedge a_2 \wedge a_3 \wedge a_4, b_1 \wedge b_2 \wedge b_3 \wedge b_4 \rangle = \det \left( \langle a_i, b_j \rangle \right).
\]

A long but straightforward computation using the Codazzi equation yields
\[
\Delta \gamma = -\|\alpha\|^2 \gamma + \sum_{i,a \neq b,j \neq k} h_{ij}^a h_{ik}^b \langle f \wedge e_{ja,kb}, f \wedge e_{ja,kb} \rangle \tag{9}
\]
where $e_{ja,kb}$ is obtained by replacing $e_j$ with $\xi_a$ and $e_k$ with $\xi_b$ in $e_1 \wedge e_2 \wedge e_3$.

We identify $\wedge^4 \mathbb{L}^{n+1}$ with $\mathbb{L}_S^N$ where \((n+1)_4\) and $S = \binom{n}{3}$ and regard $\gamma$ as a map from $M^3$ into $\mathbb{L}_S^N$. Denoting by $\{A_J\}_{J \in \{1,\ldots,N\}}$ the corresponding base in $\mathbb{L}_S^N$, where $A_1, \ldots, A_S$ are timelike and the remaining vectors spacelike, we have that
\[
\gamma = \sum_{J=1}^N w_J A_J
\]
where $w_J = -\langle \gamma, A_J \rangle$ for $1 \leq J \leq S$ and $w_J = \langle \gamma, A_J \rangle$ for $S + 1 \leq J \leq N$.

We obtain from (9) that
\[
\Delta w_J = -\|\alpha\|^2 w_J - \epsilon_J \sum_{i,a \neq b,j \neq k} h_{ij}^a h_{ik}^b \langle f \wedge e_{ja,kb}, A_J \rangle \tag{10}
\]
where
\[
\epsilon_J = \begin{cases} +1, & 1 \leq J \leq S \\ -1, & S + 1 \leq J \leq N. \end{cases}
\]

Take a local chart $\phi: U \rightarrow \mathbb{R}^3$ of coordinates $x = (x_1, x_2, x_3)$ on an open subset $U$ of $\Omega$ and set
\[
e_i = \sum_j \mu_{ij} \partial_{x_j}. \tag{11}
\]
Setting $\theta_J = w_J \circ \phi^{-1}$, we obtain the map $\theta: \phi(U) \subset \mathbb{R}^3 \rightarrow \mathbb{L}_S^N$ given by
\[
\theta = \sum_{J=1}^N \theta_J A_J = (\theta_1, \ldots, \theta_N).
\]

Thus $\theta = \gamma \circ \phi^{-1}$ is the representation of the Gauss map with respect to the above mentioned charts. From (11) and
\[
h_{ij}^a = \sum_J \langle f \wedge e_{ja}, A_J \rangle e_i(w_J)
\]
we derive that
\[ h_{ij} = \sum_{k,J} \mu_{ik} \langle f \wedge e_{ja}, A_J \rangle (\theta_J)_{x_k}. \]  
(12)

Thus
\[ \|\alpha\|^2 = \sum_{i,j,a} \left( \sum_{k,I} \mu_{ik} \langle f \wedge e_{ja}, A_I \rangle (\theta_I)_{x_k} \right)^2. \]  
(13)

The Laplacian of \( M^3 \) is given by
\[ \Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \partial_{x_i} \left( \sqrt{g} g^{ij} \partial_{x_j} \right) \]
where \( g_{ij} \) are the components of the metric of \( M^3 \) and \( g = \det(g_{ij}) \). Using (12) and (13) we see that (10) is of the form
\[ \sum_{i,j} g_{ij} (\theta_J)_{x_i x_j} + C_J (x, \theta, \theta x_1, \theta x_2, \theta x_3) = 0, \]
where \( C_J : \phi(U) \times \mathbb{R}^{4N} \to \mathbb{R} \) is given by
\[ C_J(x, y, z_1, z_2, z_3) = \frac{1}{\sqrt{g}} \sum_{i,j} (\sqrt{g} g^{ij})_{x_i z_j} J + y_J \sum_{i,j,a} \left( \sum_{k,I} \mu_{ik} \langle f \wedge e_{ja}, A_I \rangle z_{kI} \right)^2 \]
and
\[ + \epsilon_J \sum_{I,K} \sum_{i,j,m} \frac{\mu_{il} \mu_{im}}{a \neq k, l \neq k} \langle f \wedge e_{ja, kb}, A_J \rangle \langle f \wedge e_{ja}, A_K \rangle \langle f \wedge e_{kb}, A_I \rangle z_{mI} z_{lK} \]
with \( y = (y_1, \ldots, y_N), z_i = (z_{i1}, \ldots, z_{iN}), i, m, l \in \{1, 2, 3\} \) and \( I, J, K \in \{1, \ldots, N\} \). Let \( A_{ij} = g^{ij} I_N, I_N \) being the identity \( N \times N \) matrix, \( C = (C_1, \ldots, C_N) \) and \( n \) the unit normal field to the surface \( \phi(L^2) \) in \( \mathbb{R}^3 \). Then, the vector valued map \( \theta = (\theta_1, \ldots, \theta_N) \) satisfies the elliptic equation
\[ L \theta = \sum_{i,j} A_{ij}(x) \theta_{x_i x_j} + C(x, \theta, \theta x_1, \theta x_2, \theta x_3) = 0 \]
with initial conditions: \( \theta \) is constant on \( \phi(L^2) \) and \( n_\alpha(\theta) = 0 \) on \( \phi(L^2) \).

According to the Cauchy-Kowalewsky theorem (cf. [15]) the above system has a unique solution if the surface \( \phi(L^2) \) is noncharacteristic. This latter condition is satisfied if \( Q(n) \neq 0 \), where \( Q \) is the characteristic form given by
\[ Q(\zeta) = \det(\Lambda(\zeta)) \]
with \( \zeta = (\zeta_1, \zeta_2, \zeta_3) \) and
\[ \Lambda(\zeta) = \sum_{i,j} g^{ij} \zeta_i \zeta_j I_N \]
is the symbol of the differential operator $\mathcal{L}$. That the surface $\phi(L^2)$ is noncharacteristic follows from 

$$Q(\zeta) = \left( \sum_{i,j} g^{ij} \zeta_i \zeta_j \right)^N.$$ 

Since $C(x,y,0,0,0) = 0$ the constant maps satisfy the system. Due to uniqueness of solutions to the Cauchy problem, we deduce that the Gauss map $\gamma$ is constant on an open subset of $M^3$ and that is not possible. □

**Lemma 11.** The relative nullity distribution can be extended analytically over the regular points of the set $A$.

**Proof:** Clearly $D$ extends continuously over the regular points of $A$. Let $e_1, e_2, e_3$ be a local orthonormal tangent frame on an open subset $U$ of $M^3 \setminus A$ as in Lemma 8. We view $e_3$ as a map $F: U \to T^1M$ into the unit tangent bundle of $M^3$ endowed with the Riemannian metric inherited from the Sasaki metric on $TM$. We argue that the map $F = e_3$ is harmonic. In fact, from (6) and (7) we obtain that

$$\Delta e_3 = \sum_{i=1}^{3} (\nabla_{e_i} \nabla_{e_i} e_3 - \nabla_{\nabla_{e_i} e_i} e_3)$$

$$= -2(u^2 + v^2)e_3$$

$$= -2\|\nabla e_1 e_3\|^2 + \|\nabla e_2 e_3\|^2) e_3.$$ 

Hence the map $F$ satisfies the differential equation

$$\Delta F + \|\nabla F\|^2 F = 0,$$

which is precisely the Euler-Lagrange equation for the energy functional of $F$ (cf. [17, Proposition 1.1]). Thus $F: U \to T^1M$ is harmonic. Since the singular set $A$ has Hausdorff dimension one, it follows that $\text{cap}_2(A) = 0$. From a result of Meier [14, Theorem 1] it follows that $F$ is of class $C^2$ on $U$. But then $F$ is real analytic by a result due to Eells and Sampson [8, Proposition p. 117]. □

**Lemma 12.** The set $A$ has no singular points.

**Proof:** It now follows immediately using Proposition 2. □

**Proof of Theorem 1:** We have seen that the relative nullity distribution $D$ extends to a global line bundle, also denoted by $D$. By passing to the 2-fold covering, if necessary, we have that this line bundle is trivial. Thus it is spanned by a globally defined unit section $e$. Hence, there is a unique, up to sign, orthogonal almost complex structure $J: D^\perp \to D^\perp$. By Lemmas 7 and 8 there are harmonic functions $u, v \in C^\infty(M)$ such that

$$C_e = vI - uJ.$$
To obtain the proof of the theorem all we have to show is that \( u \) vanishes. In fact, if that is the case then the result will follow from Propositions 5 and 6.

Making use of the first two equations in (7) and that the functions \( u, v \) are harmonic, we obtain that

\[
\Delta(u^2 + v^2 - 1) = 2\|\nabla u\|^2 + 2\|\nabla v\|^2 \\
\geq 2(e(u))^2 + 2(e(v))^2 \\
= 8u^2v^2 + 2(v^2 - u^2 - 1)^2 \\
\geq 2(u^2 + v^2 - 1)^2.
\]

Since the Ricci curvature of \( M^3 \) is bounded from below, then Proposition 3 applies and gives that \( u^2 + v^2 \leq 1 \). Hence \( u \) and \( v \) are bounded functions.

We claim that \( v^2 < 1 \). Suppose to the contrary that there is \( x_0 \in M^3 \) such that \( |v(x_0)| = 1 \). The maximum principle for harmonic functions yields that \( v = 1 \) or \( v = -1 \) everywhere. Hence \( C_e = \pm I \). We have using (3) that

\[
e(\|\alpha\|^2) = e\left(\sum_{j=1}^{n-3} \text{tr}(A_{\xi_j}^2)\right) = \sum_{j=1}^{n-3} \text{tr}(\nabla e A_{\xi_j}^2) = 2 \sum_{j=1}^{n-3} \text{tr}(A_{\xi_j} \circ C_e \circ A_{\xi_j}) = \pm 2\|\alpha\|^2
\]

where \( \xi_1, \ldots, \xi_{n-3} \) is an orthonormal normal frame parallel along \( \gamma \). Thus

\[
\|\alpha(\gamma(t))\|^2 = ce^{\pm t}
\]

where \( c > 0 \) is a constant. Therefore \( \|\alpha\| \) is unbounded along \( \gamma \). This clearly contradicts the assumption on the scalar curvature and proves the claim.

Let \( \gamma: \mathbb{R} \to M^3 \) be a unit speed geodesic contained in a leaf of the relative nullity foliation. Since \( v^2 < 1 \), we have from the first equation in (7) that

\[
(v \circ \gamma)' = (v \circ \gamma)^2 - (u \circ \gamma)^2 - 1 \leq (v \circ \gamma)^2 - 1.
\]

Hence the function \( v \circ \gamma: \mathbb{R} \to (-1, 1) \) is strictly decreasing and satisfies \( \sup v \circ \gamma = 1 \) and \( \inf v \circ \gamma = -1 \). Thus the function \( v \) changes sign only once along each leaf of the relative nullity foliation. From the first equation in (7) and \( v^2 < 1 \) it follows that

\[
e(v) = v^2 - u^2 - 1 < 0.
\]

Since 0 is a regular value of \( v \), the level set \( L^2 = v^{-1}(0) \) is a 2-dimensional connected submanifold of \( M^3 \) and the map \( \rho: L^2 \times \mathbb{R} \to M^3 \) defined by

\[
\rho(x, t) = \exp_x te(x)
\]

is a diffeomorphism. Consider the smooth function \( \phi: L^2 \times \mathbb{R} \to \mathbb{R} \) given by

\[
\phi \circ \rho^{-1} = \frac{-2v}{1 + u^2 + v^2 + \sqrt{(1 + u^2 + v^2)^2 - 4v^2}}.
\]
Setting $\psi = \phi \circ \rho^{-1}$, we have that
\[
\frac{\psi}{1 + \psi^2} = \frac{-v}{1 + u^2 + v^2}.
\tag{14}
\]
A straightforward computation using (7) yields
\[
e(\psi) = 1 - \psi^2.
\tag{15}
\]
Since $\phi$ vanishes on $L^2$ we obtain that $\phi(x, t) = \tanh t$. Thus $\psi$ is bounded on $M^3$. Hence $\theta \in C^\infty(M)$ given by
\[
\theta = u^2 + (v + \psi)^2
\]
is also bounded. Using (7) and (15) we readily see that
\[
e(\theta) = 2ue(u) + 2(\psi - u^2 - v^2)
= 2(\psi - \theta).
\tag{16}
\]
Since $u$ and $v$ are harmonic functions, we obtain that
\[
\Delta \theta = 2\|\nabla u\|^2 + 2(\psi - u^2 - v^2)\Delta \psi + 2\|\nabla (v + \psi)\|^2
\geq 8u^2v^2 + 2(\psi - u^2 - v^2)\Delta \psi + 2(\psi - \theta)^2.
\tag{17}
\]
On the other hand, it follows from (14) that
\[
\frac{(1 - \psi^2)(1 + u^2 + v^2)^2}{(1 + \psi^2)^2}\nabla \psi = 2uv\nabla u - (1 + u^2 - v^2)\nabla v.
\tag{18}
\]
Using the harmonicity of $u$ and $v$ again, a straightforward computation gives
\[
\frac{1 - \psi^2}{2(1 + \psi^2)^2}\Delta \psi = \frac{v(1 - 3u^2 + v^2)}{(1 + u^2 + v^2)^3}\|\nabla u\|^2 + \frac{2u(1 + u^2 - 3v^2)}{(1 + u^2 + v^2)^3}(\nabla u, \nabla v)
+ \frac{v(3 + 3u^2 - v^2)}{(1 + u^2 + v^2)^3}\|\nabla v\|^2 + \frac{3\psi - \psi^3}{(1 + \psi^3)^3}\|\nabla \psi\|^2.
\tag{19}
\]
Since $\theta$ is bounded, by the Omori-Yau maximum principle there is a sequence $\{x_j\}_{j \in \mathbb{N}}$ of points in $M^3$ such that
\[
(i) \lim \theta(x_j) = \sup \theta, \quad (ii) \|\nabla \theta(x_j)\| \leq 1/j \text{ and } (iii) \Delta \theta(x_j) \leq 1/j.
\tag{20}
\]
Taking a subsequence, we have that $\lim u(x_j) = u_0$, $\lim v(x_j) = v_0$ and $\lim \psi(x_j) = \psi_0$. Estimating at $x_j$ and letting $j \to \infty$, we obtain from (i) and (ii) of (20) and (16) that
\[
(v_0 - \psi_0)\sup \theta = 0.
\]

We conclude that $u$ has to vanish unless $v_0 = \psi_0$.

Suppose now that $v_0 = \psi_0$. We have from (14) that $v_0 = \psi_0 = 0$. On the other hand, since the Ricci curvature of $M^3$ is bounded from below it follows from Proposition 4 that $\|\nabla u\|$ and $\|\nabla v\|$ are bounded. Hence, from (18), (19) and since $\psi_0 = 0$, we have that $\Delta \psi(x_j)$ is bounded. Passing to the limit and using part (iii) of (20), we obtain from (17) that $u_0 = 0$. It follows using part (i) of (20) that $\sup \theta = 0$. Thus the function $u$ vanishes, and this concludes the proof. ■

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