Modelling Road Accident Blackspots Data with the Discrete Generalized Pareto distribution

Faustino Prieto\textsuperscript{a}\textsuperscript{1}, Emilio Gómez-Déniz\textsuperscript{b}, José María Sarabia\textsuperscript{a}

\textsuperscript{a}Department of Economics, University of Cantabria, Avenida de los Castros s/n, 39005 Santander, Spain
\textsuperscript{b}Department of Quantitative Methods in Economics, University of Las Palmas de Gran Canaria, 35017 Las Palmas de G.C., Spain.

Abstract

This study shows how road traffic networks events, in particular road accidents on blackspots, can be modelled with simple probabilistic distributions. We considered the number of accidents and the number of deaths on Spanish blackspots in the period 2003-2007, from Spanish General Directorate of Traffic (DGT). We modelled those datasets, respectively, with the discrete generalized Pareto distribution (a discrete parametric model with three parameters) and with the discrete Lomax distribution (a discrete parametric model with two parameters, and particular case of the previous model). For that, we analyzed the basic properties of both parametric models: cumulative distribution, survival, probability mass, quantile and hazard functions, genesis and r-th-order moments; applied two estimation methods of their parameters: the $\mu$ and $(\mu + 1)$ frequency method and the maximum likelihood method; and used two goodness-of-fit tests: Chi-square test and discrete Kolmogorov-Smirnov test based on bootstrap resampling. We found that those probabilistic models can be useful to describe the road accident blackspots datasets analyzed.

Key Words: Accident blackspots; Road traffic networks; Complex systems; Discrete Lomax distribution; Discrete generalized Pareto distributions.

\textsuperscript{1}Corresponding author. Tel.: +34 942 206758. Fax: +34 942 201603. E-mail address: faustino.prieto@unican.es (F. Prieto).
1 Introduction

Road traffic networks are key economic drivers in today’s world. They provide a quick, reliable and flexible transportation system, for people, goods and services. Unfortunately, traffic accidents happen everyday - road injury was one of the top 10 causes of death in the world in 2011, according to the World Health Organization (WHO, 2013). For that reason, many research has been devoted to the analysis of road accidents from different points of views (see, for example, Aguero-Valverde, 2013; Alemany et al., 2013; Brijs et al., 2007; Cafiso and Di Silvestro, 2011; Deublein et al., 2013; Kim et al., 2007; Li, 2012; Matínez et al., 2013, and references therein).

In particular, the analysis of blackspots (accident-prone road sections, Chen et al., 2011) has been considered one of the basic steps to reduce road accident rates. In this direction, several methods to identify blackspots have been proposed (accident frequency method; accident rate methods; quality control method; empirical Bayesian method; and many more; Hauer et al., 2002; Cheng and Washington, 2005; Jurenoks et al., 2008; Sabel et al., 2005; Pei and Ding, 2005; Gregoriades and Mouskos, 2013); Geographical Information Systems (GIS) have been incorporated in the analysis of blackspots (Chen et al., 2011; Mandloi and Gupta, 2003); the effectiveness of blackspot programs has been evaluated in different countries (Meuleners et al., 2008; Hoque et al., 2007); and it still remains as an active field of research.

The aim of this study was to show that the road traffic networks events, specifically the Spanish blackspots events, can be described by simple probabilistic models. Although the number of accidents and deaths on the road have decreased considerably in Spain in the last years, due to the advertising campaign promote by the authorities and the establishment of the force of the points system on driving licences, which there has been an improvement in responsible behaviour on the part of drivers, the high rate of mortality in the Spanish roads continues being very high. This fact motivates the study dealt here, where we concentrate our attention in modelling: (1) the number of accidents on Spanish blackspots, in the period 2003-2007, with the discrete generalized Pareto distribution (a discrete parametric model with three parameters; Asadi et al., 2001; Ekheden and Hössjer, 2012); and (2) the number of deaths on Spanish blackspots, in the same period 2003-2007, with the discrete Lomax distribution (a discrete parametric model with two parameters, and particular case of the previous model; Krishna and Singh, 2009). This paper also shows the basic properties of both models; proposes
two estimation methods of their parameters: the $\mu$ and $(\mu + 1)$ frequency method and the maximum likelihood method, leading the first one to initial estimators for the second one; and recommends two goodness-of-fit tests for both of them: the Chi-square test and the discrete Kolmogorov-Smirnov test.

The rest of this paper is organized as follows: in Section 2 we introduce the dataset considered, the two probabilistic models used and the estimation methods and goodness of fit tests applied; finally, the results are presented and discussed in Section 3.

2 Methods

In this section, we describe the road accident Spanish blackspots dataset used (section 2.1), we show the basic properties of the discrete generalized Pareto distribution (section 2.2) and the discrete Lomax distribution (section 2.3), we propose two estimation methods of the parameters for both distributions considered (section 2.4), and finally, we recommend two goodness-of-fit tests for those probabilistic distributions (section 2.5).

2.1 Road accident blackspot data

We considered road accidents on Spanish blackspots data from Spanish General Directorate of Traffic (D.G.T., 2003-2007) (DGT, 2013a), where blackspots are identified by an accident frequency method as follows: a blackspot is a road section of 100 meters with three or more traffic accidents in a one-year period.

The dataset considered in this work consists of 16552 accidents, occurred at Spanish road blackspots, from 2003 to 2007. In those accidents, unfortunately, 895 people died (within 30 days after accident) - which means, the 5.4 % of the deaths in Spanish road accidents in that period (DGT, 2013b).

Table 1 and Fig.1 show the total number of Spanish blackspots, and the total number of accidents and deaths happened on those blackspots, in each year of the period considered.

Note that we found four blackspots (one in the year 2003, two in the year 2005, and one more in the year 2006) with only two traffic accidents in the dataset published by DGT. Due to the blackspot definition (3 or more accidents), we didn’t include them in this study. However, we confirmed separately that the main conclusions reached in this work are the same including them as blackspots with three accidents instead of two.
We analyzed two discrete variables. The first one was the number of road accidents on Spanish blackspots for the year considered - the corresponding data can be seen in Table 2 (for example, in 2003, there were 525 blackspots in Spain where 3 accidents occurred). The second discrete variable was the number of deaths on Spanish blackspots for the year considered - see Table 3 (for example, in 2003, there were 797 blackspots where nobody died). Note that the first discrete variable selected (number of accidents) has the minimum value of 3 due to the blackspot definition, and the second discrete variable analyzed (number of deaths) has the minimum value of 0 corresponding to accidents without deaths.

Table 1: Number of Spanish road accident blackspots, and number of accidents and deaths on those blackspots in the period: 2003-2007.

|                | 2003 | 2004 | 2005 | 2006 | 2007 | Total |
|----------------|------|------|------|------|------|-------|
| Number of blackspots in Spain | 958  | 780  | 737  | 748  | 802  |       |
| Number of accidents on Spanish blackspots | 3941 | 3200 | 3051 | 3071 | 3289 | 16552 |
| Number of deaths on Spanish blackspots | 220  | 191  | 179  | 171  | 134  | 895   |

Figure 1: Number of Spanish road accident blackspots, and number of accidents and deaths on those blackspots in the period: 2003-2007.
Table 2: Number of road accident blackspots in Spain according to the number of road accidents. Period: 2003-2007.

| Number of accidents | 2003 | 2004 | 2005 | 2006 | 2007 |
|---------------------|------|------|------|------|------|
| 3                   | 525  | 438  | 390  | 404  | 445  |
| 4                   | 209  | 173  | 177  | 164  | 172  |
| 5                   | 94   | 71   | 68   | 89   | 77   |
| 6                   | 41   | 38   | 35   | 45   | 48   |
| 7                   | 34   | 23   | 22   | 20   | 19   |
| 8                   | 15   | 9    | 11   | 8    | 11   |
| 9                   | 22   | 8    | 4    | 4    | 11   |
| 10                  | 5    | 6    | 4    | 1    | 2    |
| 11                  | 2    | 1    | 3    | 3    | 4    |
| 12                  | 4    | 3    | 3    | 2    | 1    |
| 13                  | 1    | 2    | 3    |    | 5    |
| 14                  | 1    | 2    |    | 1    | 2    |
| 15                  |    | 2    | 1    |    | 2    |
| 16                  | 1    |    | 2    | 2    |    |
| 17                  | 1    |    |    |    | 1    |
| 18                  |    |    | 1    |    |    |
| 19                  | 1    | 1    |    |    |    |
| 20                  | 1    | 1    |    |    |    |
| 21                  |    |    |    | 1    |    |
| 22                  |    |    |    | 1    |    |
| 24                  |    |    |    | 1    |    |
| 25                  |    |    |    | 1    |    |
| 27                  |    | 1    |    |    |    |
| 29                  |    |    |    | 1    |    |
| 32                  |    |    |    | 1    |    |
| 33                  |    |    | 2    |    |    |
| 36                  |    |    | 1    |    |    |
| 39                  | 1    |    |    | 1    |    |
| 49                  |    | 1    |    |    |    |
| Total               | 958  | 780  | 737  | 748  | 802  |

Table 3: Number of road accident blackspots in Spain according to the number of deaths. Period: 2003-2007.

| Number of deaths | 2003 | 2004 | 2005 | 2006 | 2007 |
|------------------|------|------|------|------|------|
| 0                | 797  | 636  | 611  | 632  | 693  |
| 1                | 126  | 108  | 94   | 84   | 92   |
| 2                | 19   | 27   | 23   | 19   | 12   |
| 3                | 12   | 7    | 3    | 9    | 4    |
| 4                | 2    | 2    | 3    | 1    |    |
| 5                |    |    | 1    | 1    |    |
| 6                | 2    |    | 1    | 1    |    |
| 7                |    |    | 1    | 1    |    |
| Total            | 958  | 780  | 737  | 748  | 802  |
2.2 The discrete generalized Pareto distribution

The discrete generalized Pareto distribution is defined in terms of the cumulative distribution function (cdf). If \( F(x) = \Pr(X \leq x) \) we define:

\[
F(x) = 1 - [1 + \lambda(x - \mu + 1)]^{-\alpha}, \quad x = \mu, \mu + 1, \ldots
\]  
(1)

and \( F(x) = 0 \) if \( x < \mu \), where \( \alpha, \lambda, \mu > 0 \) are respectively the shape, scale and location parameters.

A random variable with cdf (1) will be denoted by \( X \sim \mathcal{DGP}(\alpha, \lambda, \mu) \).

The survival function, \( \bar{F}(x) = \Pr(X \geq x) = 1 - F(x-1) \) (Xekalaki, 1983), of the \( \mathcal{DGP} \) distribution can be written as

\[
\bar{F}(x) = [1 + \lambda(x - \mu)]^{-\alpha}, \quad x = \mu, \mu + 1, \ldots
\]  
(2)

The probability mass function (pmf) of the \( \mathcal{DGP} \) distribution is given by

\[
\Pr(X = x) = [1 + \lambda(x - \mu)]^{-\alpha} - [1 + \lambda(x - \mu + 1)]^{-\alpha}, \quad x = \mu, \mu + 1, \ldots
\]  
(3)

The \( \mathcal{DGP} \) distribution is unimodal with a modal value at \( x = \mu \). Letting Eq.(3), define \( \Pr(X = x) \) also for non-integer values of \( x \), then we obtain that for \( x \geq \mu \)

\[
\frac{d\Pr(X = x)}{dx} = \alpha\lambda[(1 + \lambda(x - \mu + 1))^{-(\alpha+1)} - (1 + \lambda(x - \mu))^{-(\alpha+1)}],
\]

which clearly is negative, and hence the probability mass function is a decreasing function on \( X \).

Fig.2 shows the different forms of pmf given by Eq.(3), according to different values of the parameters \( \alpha \) and \( \lambda \).

The \( \mathcal{DGP} \) distribution can be obtained by discretizing the continuous generalized Pareto distribution, by applying the following expression (Kemp, 2008; Nakagawa and Osaki, 1975; Gómez-Déniz, 2011; Gómez-Déniz et al., 2011; Gómez-Déniz and Calderín-Ojeda, 2011; Roy, 2004)

\[
\Pr(X = x) = \bar{F}(x) - \bar{F}(x + 1),
\]  
(4)

where \( \bar{F}(x) \) is the survival function of that continuous model and \( \Pr(X = x) \) is the probability mass function of the discrete model associated. Using Eq.(4) with the survival function of the continuous generalized Pareto distribution
(Arnold, 1983), given by \( \bar{F}(x) = [1 + \lambda(x - \mu)]^{-\alpha} \), we obtain the pmf of the \( \mathcal{DGP} \) distribution given in the Eq. (3).

The quantile function can be obtained by inverting Eq. (1),

\[
x_\gamma = \left\lceil \frac{(1 - \gamma)^{-1/\alpha} - 1}{\lambda} - 1 + \mu \right\rceil,
\]

where \( \lceil \cdot \rceil \) denotes the ceiling function (the smallest integer greater than or equal). In particular, the median is given by

\[
x_{0.5} = \left\lceil \frac{2^{1/\alpha} - 1}{\lambda} - 1 + \mu \right\rceil.
\]

The hazard function (failure rate), using Eqs. (2) and (3), can be written as

\[
r(x) = \frac{Pr(X = x)}{\bar{F}(x)} = 1 - \left[ \frac{1 + \lambda(x - \mu)}{1 + \lambda(x - \mu + 1)} \right]^\alpha, \quad x = \mu, \mu + 1, \ldots
\]

which is a decreasing function on \( X \).

The rth-order moment of \( \mathcal{DGP} \) distribution becomes

\[
E(X^r) = \sum_{x=\mu}^{\infty} x^r Pr(X = x) = \sum_{x=\mu+1}^{\infty} [x^r - (x-1)^r] \bar{F}(x)
= \sum_{x=\mu+1}^{\infty} \frac{x^r - (x-1)^r}{[1 + \lambda(x - \mu)]^\alpha}.
\]

In particular, the mean and the 2nd moment are given by

\[
E(X) = \sum_{x=\mu+1}^{\infty} \frac{1}{[1 + \lambda(x - \mu)]^\alpha},
E(X^2) = \sum_{x=\mu+1}^{\infty} \frac{2x - 1}{[1 + \lambda(x - \mu)]^\alpha},
\]

which are finite, respectively, if \( \alpha > 1 \) and \( \alpha > 2 \). Note that \( \partial E(X)/\partial \alpha < 0 \) and \( \partial E(X)/\partial \lambda < 0 \), and therefore the mean decreases with both parameter: \( \alpha \) and \( \lambda \).

Table 3 shows the index of dispersion \( D = [E(X^2) - (E(X))^2]/E(X) \), for different values of the parameters \( \alpha \) and \( \lambda \). It can be seen that this variance-to-mean ratio seems always be larger than 1, and therefore the \( \mathcal{DGP} \) distribution seems overdispersed.

7
Figure 2: Probability mass function of the DGP distribution for selected values of $\mu, \alpha, \lambda$.

Table 4: Index of dispersion $D = [E(X^2) - (E(X))^2]/E(X)$, for different values of $\alpha, \lambda$.

| $\lambda$ | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0.1       | 16.47| 7.70| 5.04| 3.79| 3.08| 2.63| 2.31| 2.08|
| 0.2       | 9.05 | 4.40| 2.99| 2.33| 1.96| 1.72| 1.56| 1.44|
| 0.3       | 6.58 | 3.31| 2.32| 1.86| 1.69| 1.44| 1.33| 1.25|
| 0.4       | 5.36 | 2.77| 1.99| 1.63| 1.43| 1.30| 1.22| 1.16|
| 0.5       | 4.63 | 2.45| 1.80| 1.49| 1.33| 1.23| 1.16| 1.11|
| 0.6       | 4.14 | 2.24| 1.67| 1.41| 1.26| 1.18| 1.12| 1.08|
| 0.7       | 3.79 | 2.09| 1.58| 1.34| 1.22| 1.14| 1.09| 1.06|
| 0.8       | 3.54 | 1.98| 1.51| 1.30| 1.19| 1.12| 1.08| 1.05|
| 0.9       | 3.34 | 1.89| 1.46| 1.26| 1.16| 1.10| 1.06| 1.04|
| 1.0       | 3.18 | 1.82| 1.42| 1.24| 1.14| 1.09| 1.05| 1.03|
| 2.0       | 2.45 | 1.51| 1.24| 1.12| 1.06| 1.03| 1.02| 1.01|
| 3.0       | 2.21 | 1.41| 1.18| 1.09| 1.04| 1.02| 1.01| 1.00|
| 4.0       | 2.09 | 1.36| 1.15| 1.07| 1.03| 1.02| 1.01| 1.00|
| 5.0       | 2.02 | 1.33| 1.14| 1.06| 1.03| 1.01| 1.00| 1.00|
| 6.0       | 1.97 | 1.31| 1.13| 1.06| 1.03| 1.01| 1.00| 1.00|
| 7.0       | 1.94 | 1.29| 1.12| 1.05| 1.02| 1.01| 1.00| 1.00|
| 8.0       | 1.91 | 1.28| 1.12| 1.05| 1.02| 1.01| 1.00| 1.00|
| 9.0       | 1.89 | 1.28| 1.11| 1.05| 1.02| 1.01| 1.00| 1.00|
| 10.0      | 1.87 | 1.27| 1.11| 1.05| 1.02| 1.01| 1.00| 1.00|
2.3 The discrete Lomax distribution

The class defined in Eq. (1) includes an important 2-parameter particular case: the discrete Lomax distribution \((\mu = 0)\).

The discrete Lomax distribution is also defined in terms of the cumulative distribution function (cdf). If \(F(x) = \Pr(X \leq x)\) we define:

\[
F(x) = 1 - [1 + \lambda(x + 1)]^{-\alpha}, \quad x = 0, 1, \ldots
\]

and \(F(x) = 0\) if \(x < 0\), where \(\alpha, \lambda > 0\) are respectively the shape and scale parameters.

A random variable with cdf \((8)\) will be denoted by \(X \sim DL_{\alpha,\lambda}\). In this case \(DG_{\alpha,\lambda} \equiv DL_{\alpha,\lambda}\).

The survival function is given by

\[
\bar{F}(x) = \Pr(X \geq x) = [1 + \lambda x]^{-\alpha}, \quad x = 0, 1, \ldots
\]

The pmf of the \(DL_\alpha\) distribution is given by

\[
\Pr(X = x) = [1 + \lambda x]^{-\alpha} - [1 + \lambda(x + 1)]^{-\alpha}, \quad x = 0, 1, \ldots
\]

The \(DL_\alpha\) distribution can be obtained also by discretizing the continuous Lomax distribution (also known as Pareto II distribution,\cite{arnold1983pareto}), by applying the Eq.(4).

Of course that because the \(DL_\alpha\) is a particular case of the \(DG_{\alpha,\lambda}\) distribution, again the distribution with unimodal with a zero vertex and the quantile, hazard function and the moments are obtained by putting \(\mu = 0\) in Eqs.(5), \(6\) and \(7\), respectively.

Furthermore, a more simple distribution can be obtained directly from Eq.(9) by assuming that \(\alpha = 1\) and obtained therefore a one-parameter distribution which has good properties. Although the mean of this distribution does not exist we will see some interesting properties of this distribution can be obtained easily. To the best of our knowledge, the discrete distribution presented here has not been previously studied in detail in statistical literature.

In this case, after straightforward computation we get the probability mass function of this simple distribution given by

\[
\Pr(X = x) = \frac{\lambda}{(1 + \lambda x)(1 + \lambda(x + 1))}, \quad x = 0, 1, \ldots; \lambda > 0.
\]
The quantiles, hazard function and the median are obtained from Eqs. (5), (6) and (7), respectively, after putting $\alpha = 1$ and $\mu = 0$. In this particular case, some algebra provides the probability generating function, which is given by 

$$G_X(z) = \frac{1}{z} [1 - (1 - z)\Phi(z, 1, 1/\lambda)], \quad |z| < 1,$$

where $\Phi(z, s, a)$ is the Hurwitz–Lerch transcendent function given by $\Phi(z, s, a) = \sum_{k=0}^{\infty} z^k (k + a)^{-s}$.

Because non-closed form exists for the moments of the distribution, alternatively we can use the following inverse moment

$$E\left(\frac{1}{X + 1}\right) = \frac{\lambda (\lambda + \psi(1/\lambda) + \gamma - 1)}{1 - \lambda},$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function and $\gamma$ is Euler’s constant, with approximate numerical value 0.577216. It must be reported that the Hurwitz-Lerch transcendent function and the digamma function are available in the Mathematica package. Additionally, since $\Pr(X = 0) \neq 0$, $\Pr(X = 1) \neq 0$ and $\Pr(X = j)/\Pr(X = j - 1)$, $j = 1, 2, \ldots$ forms a monotone increasing sequence, we have that the distribution is infinitely divisible (log-convex). See [Warde and Katti, 1971] for details. Therefore, $\Pr(X = x)$ is a decreasing sequence (see [Johnson and Kotz, 1982], p.75), which is congruent with the zero vertex of the new distribution. Moreover, as any infinitely divisible distribution defined on nonnegative integers is a compound Poisson distribution (see Proposition 9 in [Karlis and Xekalaki, 2005]), we conclude that the new probability mass function (pmf) given in Eq.(10) is a compound Poisson distribution.

Furthermore, the infinitely divisible distribution plays an important role in many areas of statistics, for example, in stochastic processes and in actuarial statistics. When a distribution $G$ is infinitely divisible then for any integer $x \geq 2$, there exists a distribution $G_x$ such that $G$ is the $x$-fold convolution of $G_x$, namely, $G = G_x^x$.

Finally, since the new distribution is infinitely divisible, a lower bound for the variance can be obtained when $\alpha = 1$ (see [Johnson and Kotz, 1982], p.75), which is given by $\text{var}(X) \geq \frac{\Pr(X=1)}{\Pr(X=0)} = \frac{1}{1+\lambda}$. 

10
2.4 Estimation of parameters

Let \(x_1, \ldots, x_n\) be a sample of size \(n\) drawn from a \(\mathcal{DP}\) distribution and \(x_{\text{min}}\) the smallest observation (\(x_i \geq x_{\text{min}}, \forall i\)). We assume that \(\mu\)-parameter is given and can be estimated using the sample minimum \(\hat{\mu} = x_{\text{min}}\) (in the case of \(\mathcal{DL}\) distribution, \(\hat{\mu} = x_{\text{min}} = 0\)). In consequence, we estimate both parameters \(\alpha\) and \(\lambda\). For that, we propose two estimation methods: the \(\mu\)-frequency and \((\mu + 1)\)-frequency method; and the maximum likelihood method. The first method leads to simple estimators, which can be used as initial estimators, the seed value \((\alpha_0, \lambda_0)\), in the maximum likelihood method.

To apply the \(\mu\)-frequency and \((\mu + 1)\)-frequency method for estimating the parameters \(\alpha\) and \(\lambda\), we have to calculate the relative frequencies of \(X = \mu\) and of \(X = (\mu + 1)\) in the sample, which we denote as \(\hat{p}_\mu\) and \(\hat{p}_{\mu+1}\) respectively, equate them to the corresponding pmf values from Eq.(3), and solve the two equations obtained, simultaneously for \(\alpha\) and \(\lambda\). From Eq.(3):

\[
\hat{p}_\mu = 1 - [1 + \lambda]^{-\alpha}, \tag{11}
\]

\[
\hat{p}_{\mu+1} = [1 + \lambda]^{-\alpha} - [1 + 2\lambda]^{-\alpha}. \tag{12}
\]

If we eliminate \(\alpha\) in Eqs.(11) and (12), we obtain the equation in \(\lambda\),

\[
\frac{\log(1 + 2\lambda)}{\log(1 + \lambda)} = \frac{1 - \hat{p}_\mu - \hat{p}_{\mu+1}}{1 - \hat{p}_\mu}. \tag{13}
\]

The previous equation can be solved by a simple computer algorithm, taking in account that the left hand of the equation\(\text{[13]}\) is a monotone function in \(\lambda\). Finally, the \(\alpha\) estimator is (using Eq.(11)):

\[
\hat{\alpha} = \frac{\log(1 - \hat{p}_\mu)}{\log(1 + \lambda)}. \tag{14}
\]

Now, we consider the second method proposed: the maximum likelihood method. The log-likelihood function is given by,

\[
\log \ell(\lambda, \nu) = \sum_{i=1}^{n} \log \Pr(X = x_i) = \sum_{i=1}^{n} \log \left[ (1 + \lambda(x_i - \mu))^{-\alpha} - (1 + \lambda(x_i - \mu + 1))^{-\alpha} \right].
\]

where \(\Pr(X = x_i)\) is the pmf defined in Eq.(3).
Taking partial derivatives with respect to $\alpha$ and $\lambda$, and equating them to zero, we obtain the normal equations:

$$
\frac{\partial \log \ell}{\partial \alpha} = \sum_{i=1}^{n} \frac{\log [1 + \lambda(x_i - \mu + 1)]}{[1 + \lambda(x_i - \mu + 1)]^\alpha [1 + \lambda(x_i - \mu)]^{-\alpha} - 1} - \sum_{i=1}^{n} \frac{\log [1 + \lambda(x_i - \mu)]}{1 - [1 + \lambda(x_i - \mu)]^\alpha [1 + \lambda(x_i - \mu + 1)]^{-\alpha}} = 0,
$$

$$
\frac{\partial \log \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{\alpha(x_i - \mu + 1)}{[1 + \lambda(x_i - \mu + 1)]^\alpha + 1 [1 + \lambda(x_i - \mu)]^{-\alpha} - 1 + \lambda(x_i - \mu + 1]} - \sum_{i=1}^{n} \frac{\alpha(x_i - \mu)}{1 + \lambda(x_i - \mu) - [1 + \lambda(x_i - \mu)]^{\alpha + 1} [1 + \lambda(x_i - \mu + 1)]^{-\alpha}} = 0,
$$

which can be solved to obtain the maximum likelihood estimators.

In this study, maximum likelihood estimates of $\alpha$ and $\lambda$ were computed by numerical methods, using the (Mathematica, version 8.0) software function `FindMaximum`; and taking, as the initial value $(\alpha_0, \lambda_0)$, the seed value obtained from the Eqs. (13) and (14) by the $\mu$-frequency and $(\mu + 1)$-frequency method.

### 2.5 Goodness-of fit test for DGPF distribution

Let $x_1, \ldots, x_n$ be a sample of size $n$ of the discrete random variable $X$. In the previous section, we have assumed that the data come from the discrete generalized Pareto distribution (or from the discrete Lomax distribution) and we have shown how to estimate the parameters $\alpha, \lambda, \mu$ of that distribution. Therefore, a crucial task is to test the goodness of fit (GOF) of the sample with the DGPF (or DLo) model. For that, in this study we have used two different GOF tests: the Chi-Square test and the discrete Kolmogorov-Smirnov test.

The Chi-Square GOF test was first proposed by Karl Pearson in 1990 [Pearson, 1900]. It gives us a measure of how close the observed values are to the expected values given by the fitted discrete model. The chi-square GOF test statistics is as follows

$$
\chi^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i},
$$
where $O_i$ and $E_i$ are, respectively, the observed and the expected frequency for bin $i$ (with bins combined if the expected frequency is less than 5, and $k$ is the resulting number of bins). Note that the expected frequency was calculated, in this study, with the maximum likelihood estimator obtained in the previous section.

The null hypothesis can be expressed as

$$H_0 : \text{the data follow the discrete } \mathcal{DG} \mathcal{P} \text{ (or } \mathcal{DL} \mathcal{O}) \text{ model},$$

where null hypothesis can be rejected with the selected level of significance $\alpha = 0.05$ if

$$\chi^2 > \chi^2_{0.95,k-r-1},$$

where $r$ is the number of parameter of the model fitted ($r = 3$ for $\mathcal{DG} \mathcal{P}$ distribution and $r = 2$ for $\mathcal{DL} \mathcal{O}$ distribution), and $\chi^2_{0.95,k-r-1}$ is the chi-square critical value with $k - r - 1$ degrees of freedom.

Additionally, the corresponding $p$-value can be obtained, and null hypothesis can be rejected with the selected level of significance if $p$-value < 0.05.

The discrete Kolmogorov-Smirnov ($KS$) test was first presented, as far as we know, by Norbert Henze in 1996 (Henze, 1996). It is an empirical distribution function (EDF) goodness-of-fit (GOF) test for discrete data, based on the use of a parametric bootstrap.

The discrete $KS$ GOF test statistics for $\mathcal{DG} \mathcal{P}$ (or $\mathcal{DL} \mathcal{O}$) distribution is given by

$$K_n = \sqrt{n} \max_{\mu \leq k \leq M} |F_n(k) - F_n(k, \hat{\vartheta}_n)|,$$  \hspace{1cm} (15)

where

$$F_n(k) = n^{-1} \sum_{i=1}^{n} 1 \{x_i \leq k\}$$  \hspace{1cm} (16)

is the EDF of the sample $x_1, \ldots, x_n$, of size $n$, in a sample value $k$; $\hat{\vartheta}_n = \hat{\vartheta}_n(x_1, \ldots, x_n)$ is the suitable estimator (in this study, the maximum likelihood estimator obtained previously, as described in section 2.4) of the unknown parameter vector $\vartheta = (\alpha, \lambda, \mu)$; and $M$ is the sample maximum

$$M = x_{max} = \max_{1 \leq i \leq n} x_i.$$

The null hypothesis can be expressed again as

$$H_0 : \text{the data follow the discrete } \mathcal{DG} \mathcal{P} \text{ (or } \mathcal{DL} \mathcal{O}) \text{ model},$$
Then, the procedure is as follows (Prieto et al., 2014):

1. calculate the empirical KS statistic for the observed data using Eq. (15);
2. generate by simulation, using Eq. (3), enough synthetic data sets with the same sample size \( n \) as the observed data (in this study, we generated 10000 datasets);
3. fit each synthetic dataset by maximum likelihood (see section 2.4), and obtained its theoretical cumulative distribution function cdf using Eq. (16);
4. calculate the KS statistic for each synthetic dataset, using its own theoretical cdf (instead of the EDF) and the synthetic dataset (instead of the observed data) in Eq. (15);
5. calculate the \( p \)-value as the fraction of synthetic datasets with a KS statistic greater than the empirical KS statistic;
6. null hypothesis can be rejected at the 0.05 level of significance if the \( p \)-value calculated is less than 0.05.

### 3 Results

Table 5 shows the parameter estimates \( \hat{\alpha} \) and \( \hat{\lambda} \), and their standard errors: (1) from the \( DGP \) distribution, fitted to the first variable data analyzed (number of accidents on blackspots), with \( \hat{\mu} = x_{\min} = 3 \); and (2) from the \( DL_0 \) distribution, fitted to the second variable data, (number of deaths on blackspots); both models fitted by maximum likelihood, in the period 2003-2007. It can be seen that most of the estimates are statistically significant at a 0.05 level of significance, assuming the asymptotic normality of the maximum likelihood estimates.

Table 6 shows the values of the chi-square statistic, the degree of freedom and the corresponding chi-square critical values, from \( DGP \) and \( DL_0 \) distributions fitted, respectively to variable 1 and variable 2 data, in 2003-2007. It can be seen that the values of chi-square statistic are less than the corresponding chi-square critical values, except in the case of variable 1 in the year 2003. In addition, table 6 shows the one tailed (right-tail) probability value (\( p \)-value) for the chi-square test, corresponding to both variables analyzed in each year from 2003 to 2007, confirming the results obtained: all the \( p \)-values are greater than 0.05, except in the case of number of accidents (variable 1) in the year 2003. It means that \( DGP \) model can not be ruled out with the 0.05 level of significance in four of the five years considered and can be ruled out with that level of significance in 2003. It also means that \( DL_0 \) model can not
be ruled out with the 0.05 level of significance in all the years considered. Table 7 shows the values of the empirical $KS$ statistic and the $p$-values obtained by bootstrap resampling. It can be seen that all the $p$-values obtained are greater than 0.05, therefore the discrete $KS$ GOF test indicates that our null hypothesis ($H_0$: the data follow the discrete $DGP$, or $DL_o$, model) can not be rejected at a 0.05 level of significance, in both variables (number of accidents and number of deaths), in the period 2003-2007.

In summary, we have checked the maximum likelihood fit of the discrete generalized Pareto model ($DGP$) to the number of road accidents on Spanish blackspots for each year of the period 2003-2007. Also, we have checked the maximum likelihood fit of discrete Lomax model ($DL_o$) to the number of deaths on Spanish blackspots in the same period. In both cases, using the Chi-Square goodness-of-fit test and the discrete Kolmogorov-Smirnov goodness-of-fit test. The results indicate that those probabilistic models can be useful to describe the road accident blackspots datasets analyzed.

4 Conclusions

We found a three parameter probability distribution that we can use to model the number of road accidents on blackspots: the discrete generalized Pareto distribution.

We found a two parameter probability distribution that we can also use to model road traffic networks events, in particular the number of deaths on road accident blackspots: the discrete Lomax distribution.

We considered road accidents on Spanish blackspots data, from Spanish General Directorate of Traffic (DGT), where blackspots are identified as road sections of 100 meters with three or more traffic accidents in a one-year period, and where a death is included if occurred within 30 days after the accident. We modelled 16552 accidents, occurred on Spanish road blackspots in the period 2003-2007; analyzed two discrete variables: the number of accidents and the number of deaths; fitted the discrete generalized Pareto and the discrete Lomax distributions, respectively to the data, by maximum likelihood; and tested the goodness-of-fit of those models by a Chi-square test and by a Kolmogorov-Smirnov test method based on bootstrap resampling.

In this study, we found that road traffic networks events, specifically road accident blackspots events, can be described by simple probabilistic models: the discrete generalized Pareto and the discrete Lomax distributions.
Table 5: Parameter estimates from DGP and DL0 models to the number of accidents (variable 1, $\mu = 3$) and deaths (variable 2), respectively, on Spanish blackpots dataset, by maximum likelihood (standard error in parenthesis).

|                | 2003       | 2004       | 2005       | 2006       | 2007       |
|----------------|------------|------------|------------|------------|------------|
| DGP model      |            |            |            |            |            |
| (no. accidents)| $\hat{\alpha}$ | 3.8227    | 3.2601    | 3.3883    | 4.0439    | 3.5710    |
|                |             | (0.6398)   | (0.5140)  | (0.5443)  | (0.7178)  | (0.6093)  |
| (no. deaths)   | $\hat{\lambda}$ | 0.2295    | 0.2933    | 0.2719    | 0.2182    | 0.2547    |
|                |             | (0.0482)   | (0.0599)  | (0.0559)  | (0.0479)  | (0.0552)  |

Table 6: Chi-square statistic values and the corresponding $p$-values, from DGP and DL0 models, to the number of accidents and deaths on Spanish blackpots respectively, from 2003 to 2007. $\chi^2 > \chi^2_{0.95,k-r-1}$ or $p$-value $< 0.05$ indicates than the model can be ruled out with the 0.05 level of significance.

|                | 2003       | 2004       | 2005       | 2006       | 2007       |
|----------------|------------|------------|------------|------------|------------|
| DGP model      | $\chi^2$  | 17.930     | 2.608      | 5.537      | 10.397     | 4.903      |
| (no. accidents)| df = $k - r - 1$ | 6          | 6          | 5          | 5          | 6          |
|                | $\hat{\chi}^{2}_{0.95,k-r-1}$ | 12.592     | 12.592     | 11.071     | 11.071     | 12.592     |
|                | p-value    | 0.0864     | 0.8561     | 0.3539     | 0.0647     | 0.5563     |
| DL0 model      | $\chi^2$  | 3.639      | 0.590      | 0.556      | 0.203      | 0.918      |
| (no. deaths)   | df = $k - r - 1$ | 1          | 1          | 1          | 1          | 1          |
|                | $\hat{\chi}^{2}_{0.95,k-r-1}$ | 3.841      | 3.841      | 3.841      | 3.841      | 3.841      |
|                | p-value    | 0.0564     | 0.4425     | 0.4560     | 0.6527     | 0.3380     |

Table 7: Empirical KS statistics and bootstrap $p$-values corresponding to both variables analyzed in each year from 2003 to 2007. $p$-value $< 0.05$ indicates than the model can be ruled out with the 0.05 level of significance.

|                | 2003       | 2004       | 2005       | 2006       | 2007       |
|----------------|------------|------------|------------|------------|------------|
| DGP model      | KS         | 0.3088     | 0.1712     | 0.3950     | 0.4810     | 0.1867     |
| (no. accidents)| p-value    | 0.3322     | 0.8087     | 0.1351     | 0.0518     | 0.7640     |
| DL0 model      | KS         | 0.1361     | 0.1152     | 0.0824     | 0.0475     | 0.0978     |
| (no. deaths)   | p-value    | 0.2606     | 0.3987     | 0.6226     | 0.9047     | 0.2962     |
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