DISCRETE-TIME REALIZATION OF FRACTIONAL-ORDER PROPORTIONAL INTEGRAL CONTROLLER FOR A CLASS OF FRACTIONAL-ORDER SYSTEM

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ABSTRACT. The approximation of the fractional-order controller (FOC) has already been recognized as a distinguished field of research in the literature of system and control. In this paper, a two-step design approach is presented to realize a fractional-order proportional integral controller (FOPI) for a class of fractional-order plant model. The design goals are based on some frequency domain specifications. The first stage of the work is focused on developing the pure continuous-time FOC, while the second stage actually realizes the FOPI controller in discrete-time representation. The presented approach is fundamentally dissimilar with respect to the conventional approaches of z-domain. In the process of realizing the FOC, the delta operator has been involved as a generating function due to its exclusive competency to unify the discrete-time system and its continuous-time counterpart at low sampling time limit. The well-known continued fraction expansion (CFE) method has been employed to approximate the FOPI controller in delta-domain. Simulation outcomes exhibit that the discrete-time FOPI controller merges to its continuous-time counterpart at the low sampling time limit. The robustness of the overall system is also investigated in delta-domain.

1. Introduction. Fractional calculus can be designated as a generalized form of calculus for expressing derivatives and integrals having both the integer orders and non-integer orders [21]. The study of the fractional-order system (FOS) has drawn significant attention among the researchers and engineers of various arenas in recent time. The natural systems are typically of fractional-order. But, such systems are still modeled using integer order calculus disregarding their small fractionality [1]. However, the precise modeling of many such systems demands the involvement of fractional calculus theory. The development of the fractional-order modeling was inadequate due to the absenteeism of solution till the nineties. However, during the last three decades, many methods have come up for approximating the fractional differ-integral and as a result, a new branch of research in system theory has been evolved, which is so called Fractional-Order System (FOS) [23].

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The application of the fractional calculus is a contemporary research topic in control science. Superior performance of the fractional-order controller (FOC) over the classical PID controller was first established by Oustaloup through the invention of CRONE controller [20]. After few years, Podlubny proposed a fractional-order PID (FOPID) controller with two extra tuning parameters and also established its superiority over the integer order PID controller in terms of system performance and robustness [22]. The use of the FOC is prevalent to control the integer order plant [30],[18],[2],[24]. However, the FOC has also been employed to control the fractional-order plant [33],[9],[11],[19]. In this paper, a fractional-order proportional integral controller has been realized for a class of fractional-order plant [12]. The design method comprises of two stages. In the first stage, the continuous-time FOPI controller is developed using an analytical approach for achieving some design goals like gain cross over frequency, phase margin etc. In the second stage, the discrete-time approximation of the FOPI controller is performed. The reason behind choosing the FOC instead of the integer order controller is that the above mentioned frequency domain specifications cannot be satisfied even by employing the classical PID controller for the concerned plant model [10].

The FOCs are approximated either indirectly or directly for their discrete-time realizations. Indirect methods undergo through the frequency domain fitting of the FOC in \( s \)-domain. Then, the discretization is performed on the fitted transfer function [3],[32]. Direct approximation of FOC does not require the stage of frequency domain fitting. It involves one generating function which is expanded for attaining the discrete-time FOC directly from its continuous-time representation [15],[4],[8],[25],[7],[27]. In this paper, a direct approximation method is chosen accordingly to realize the FOC after finding its continuous-time model analytically. The main contribution of the paper is to find a dissimilar approach for approximating the FOPI controller in discrete-time domain. The prevailing approaches are predominantly developed in \( z \)-domain. On contrary to that, the presented methodology finds approximation of the FOPI controller through casting delta operator as a generating function. The idea behind involving the delta operator as a generating function has been evolved due to its unification property which facilitates to amalgamate the discrete-time system and its continuous-time counterpart at low sampling time [16],[13],[31],[26]. The expansion of the generating function is prevalent to realize the FOC. In this work, CFE method has been chosen for accomplishing this task as it converges faster than the power series expansion (PSE) method [4]. Therefore, it produces comparatively lower order approximation as compared to PSE method.

The paper is organized as follows: Section 1 presents the introduction on FOS and FOC. In Section 2, the design of FOPI controller is discussed in two steps. In the first step, the tuning parameters of the ideal FOPI controller are obtained for the specified design objectives. In the second step, the discrete-time approximation of the FOPI controller is derived in delta-domain. Simulation results are analyzed in Section 3. Final conclusions are drawn in Section 4.

2. Design of fractional-order proportional integral controller. The fractional-order plant is specified as below:

\[
P(s) = \frac{K}{(\tau s^\mu + 1)}
\]  

(1)

A fractional-order proportional integral controller is chosen to achieve the following design specifications:
• Gain cross-over frequency = $\omega_{gc}$ rad/sec.
• Phase margin = $\phi^0_m$.
• A flat phase characteristics around $\omega_{gc}$ when plant gain K varies within a range.

The FOPI controller is represented by the following transfer function:

$$C(s) = K_p \left( 1 + \frac{K_i}{s^\nu} \right)$$  \hspace{1cm} (2)

The open loop system is expressed as follows:

$$G_{OL}(s) = P(s)C(s) = \frac{KK_p (1 + K_i)}{\tau s^\mu + 1}$$  \hspace{1cm} (3)

The proposed design objectives are formulated accordingly as shown from equations 4 to 6:

$$|G_{OL}(j\omega_{gc})| = |[P(j\omega)C(j\omega)]| = 1$$  \hspace{1cm} (4)

$$\text{Arg}\left\{G_{OL}(j\omega_{gc})\right\} = \text{Arg}\left\{P(j\omega)C(j\omega_{gc})\right\} = -\pi + \phi_m$$  \hspace{1cm} (5)

$$\left[ \frac{d\text{Arg}\{G_{OL}(j\omega)\}}{d\omega} \right]_{\omega=\omega_{gc}} = \left[ \frac{d\text{Arg}\{P(j\omega)C(j\omega)\}}{d\omega} \right]_{\omega=\omega_{gc}} = 0$$  \hspace{1cm} (6)

Equations 4, 5 and 6 have been solved to derive the controller parameters $K_p$, $K_i$ and $\nu$ using an analytical approach [28],[29]. Both the equations 5 and 6 are expressed as $K_i = g_1(\nu)$ and $K_i = g_2(\nu)$ . The intersection of $g_1(\nu)$ and $g_2(\nu)$ gives the desired values of $K_i$ and $\nu$. Utilizing these values, the remaining parameter $K_p$ is derived from the equation 4. The continuous-time FOPI controller thus obtained from the above design method has been realized in delta-domain. The philosophy of the delta operator theory is based on the amalgamation of the discrete-time system with its continuous-time counterpart at the lower sampling time limit. The time domain and frequency domain representations of the delta operator are given by the equations 7 and 8 as below:

$$\delta = \frac{q - 1}{\Delta}$$  \hspace{1cm} (7)

$$s = \frac{1}{\Delta} \ln(1 + \Delta \gamma)$$  \hspace{1cm} (8)

where $\Delta$ implies the sampling time and $q$ symbolizes the forward shift operator. Now, the following relationships can be developed from the equations 7 and 8 as given below:

$$\lim_{\Delta \to 0} \delta \Phi(t) = \lim_{\Delta \to 0} \frac{\Phi(t + \Delta) - \Phi(t)}{\Delta} = \frac{d}{dt} \Phi(t)$$  \hspace{1cm} (9)

$$\lim_{\Delta \to 0} \gamma = \lim_{\Delta \to 0} \frac{e^{\Delta} - 1}{\Delta} = s$$  \hspace{1cm} (10)

From the above equations, it is noticed that the delta operator effectively mingles with the corresponding continuous-time operators both in the time domain and in the frequency domain. This remarkable property of the delta operator remains the key motivation behind choosing it for FOC realization. In this work, the FOC has
been realized in delta-domain through renovating equation 8 using trapezoidal rule as given below:

\[
1 + \Delta \gamma = \left( e^{\frac{1}{2} s \Delta} \right) \times \left( e^{-\frac{1}{2} s \Delta} \right)^{-1} \approx \frac{1 + \frac{1}{2} s \Delta}{1 - \frac{1}{2} s \Delta}
\]  

(11)

\[
s \approx L \times \frac{\gamma}{\gamma + L}; \left( L = \frac{2}{\Delta} \right)
\]  

(12)

Using equation 12, the discrete-time FOPI controller is represented as below:

\[
C_\delta(\gamma) = K_p \left\{ 1 + \frac{K_i}{\left( L \times \frac{\gamma}{\gamma + L} \right)^\nu} \right\}
\]  

(13)

The FOPI controller contains the irrational term \( s^\nu \approx \left( L \times \frac{\gamma}{\gamma + L} \right)^\nu \). The fifth order approximation of \( s^\nu \approx \left( L \times \frac{\gamma}{\gamma + L} \right)^\nu \) is obtained using CFE method. The CFE of \((d+1)^\nu\) is given as below [5], [14]:

\[
(d+1)^\nu = \left[ 1; \frac{d\nu}{1}; \frac{d(1 - \nu)}{2}; \frac{d(1 + \nu)}{3}; \frac{d(2 - \nu)}{2}; \frac{d(2 + \nu)}{5}; \frac{d(3 - \nu)}{2}; \frac{d(3 + \nu)}{7}; \ldots \right]
\]  

(14)

Replacing \( d = \left( L \times \frac{\gamma}{\gamma + L} - 1 \right) \) in the equation (14), the fifth order approximation of \( s^\nu \) is obtained as below:

\[
s^\nu \approx \left( L \times \frac{\gamma}{\gamma + L} \right)^\nu \approx \frac{\sum_{i=0}^{5} J_i \gamma^{5-i}}{\sum_{i=0}^{5} K_i \gamma^{5-i}}
\]  

(15)

where, \( J_i \) and \( K_i (i = 0, 1, \ldots 5) \) of equation 15 are specified in Table 1.

The plant and the open loop system are represented in delta-domain as follows:

\[
P_\delta(\gamma) = \frac{K}{\left\{ \tau \left( L \times \frac{\gamma}{\gamma + L} \right)^\mu + 1 \right\}}
\]  

(16)

\[
G_{OL\delta}(\gamma) = P_\delta(\gamma)C_\delta(\gamma) = \frac{KK_p \left\{ 1 + \frac{K_i}{\left( L \times \frac{\gamma}{\gamma + L} \right)^\nu} \right\}}{\left\{ \tau \left( L \times \frac{\gamma}{\gamma + L} \right)^\mu + 1 \right\}}
\]  

(17)

The approximations of \( C_\delta(\gamma) \) and \( G_{OL\delta}(\gamma) \) are computed using equation 15 and Table 1. The frequency responses of the approximated FOPI controller and the approximated open loop system are studied in comparison to the respective underlying systems taking different values of \( \Delta \). The step responses of the approximated closed loop system are also compared with the ideal closed loop fractional-order system.
Table 1. Numerator and denominator coefficients obtained from fifth-order CFE approximation of $s^\nu \approx \left( L \times \frac{\gamma}{\gamma + L} \right)^\nu$.

| $J_0$ | $J_1$ | $J_2$ | $J_3$ | $J_4$ | $J_5$ |
|-------|-------|-------|-------|-------|-------|
| $\alpha_5 + L\alpha_4 + L^2\alpha_3 + L^3\alpha_2 + L^4\alpha_1 + L^5\alpha_0$ | $5L\alpha_5 + 4L^2\alpha_4 + 3L^3\alpha_3 + 2L^4\alpha_2 + L^5\alpha_1$ | $10L^2\alpha_5 + 6L^3\alpha_4 + 3L^4\alpha_3 + L^5\alpha_2$ | $10L^3\alpha_5 + 4L^4\alpha_4 + L^5\alpha_3$ | $5L^4\alpha_5 + L^5\alpha_4$ | $L^5\alpha_5$ |
| $K_0$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ | $K_5$ |
| $\alpha_0 = (120 + 274\nu + 225\nu^2 + 85\nu^3 + 15\nu^4 + \nu^5)$ | $\alpha_1 = (3000 + 3250\nu + 1005\nu^2 - 5\nu^3 - 45\nu^4 - 5\nu^5)$ | $\alpha_2 = (12000 + 4000\nu - 1230\nu^2 - 410\nu^3 + 30\nu^4 + 10\nu^5)$ | $\alpha_3 = (12000 - 4000\nu - 1230\nu^2 + 410\nu^3 + 305\nu^4 - 10\nu^5)$ | $\alpha_4 = (3000 - 3250\nu + 100\nu^2 + 5\nu^3 - 45\nu^4 + 5\nu^5)$ | $\alpha_5 = -(120 - 274\nu + 225\nu^2 - 85\nu^3 + 15\nu^4 - \nu^5)$ |

3. Simulation and Results. Assuming $K=1$ , $\tau=0.5$ and $\mu=0.1$ , the fractional-order plant model is given as below [12]:

$$P(s) = \frac{1}{(0.5s^{0.1} + 1)}$$  (18)

The design specifications for the overall controlled system is specified as follows [19], [15]:

- $\omega_{gc} = 7$ rad/sec.
- $\phi_m = 75^\circ$.
- A flat phase characteristics around $\omega_{gc}$.

As per the controller design method described in the Section 2, we obtain $K_i = 284.722$ and $\nu = 1.1497$ from Figure 1 as resulted by solving equations 5 and 6 for the specified design specifications. Now, using these values of $K_i$ and $\nu$, we obtain $K_p = 0.0531$ from equation 4. Replacing $K_i = 284.722$, $\nu = 1.1497$ and $K_p = 0.0531$
in the equation 2, the continuous time FOPI controller is obtained as below:

\[ C(s) = 0.0531 \left( 1 + \frac{284.722}{s^{1.1497}} \right) \]  (19)

Accordingly, using equation 3, the open loop system is obtained as below:

\[ G_{OL}(s) = P(s)C(s) = \frac{0.0531 \left( 1 + \frac{284.722}{s^{1.1497}} \right)}{0.5s^{0.1} + 1} \]  (20)

Using equations 13 and 17, the discrete-time FOPI controller and the discrete-time open loop system are derived as shown below:

\[ C_{\delta}(\gamma) = 0.0531 \left\{ 1 + \frac{284.722}{\left( L \times \frac{\gamma}{\gamma + L} \right) \left( L \times \frac{\gamma}{\gamma + L} \right)^{0.1497}} \right\}, \left( L = \frac{2}{\Delta} \right) \]  (21)

\[ G_{OL\delta}(\gamma) = \frac{0.0531 \left\{ 1 + \frac{284.722}{\left( L \times \frac{\gamma}{\gamma + L} \right) \left( L \times \frac{\gamma}{\gamma + L} \right)^{0.1497}} \right\}}{0.5 \left( L \times \frac{\gamma}{\gamma + L} \right)^{0.1} + 1}, \left( L = \frac{2}{\Delta} \right) \]  (22)

The expression of \( C_{\delta}(\gamma) \) and \( G_{OL\delta}(\gamma) \) include the irrational fractional term \( \left( L \times \frac{1}{\gamma + L} \right)^{0.1497} \) and \( \left( L \times \frac{1}{\gamma + L} \right)^{0.1} \). The approximations of these two irrational terms are calculated using Table 1 and equation 15. Accordingly, the approximation of \( C_{\delta}(\gamma) \) and \( G_{OL\delta}(\gamma) \) are listed in Tables 2 and 3 taking \( \Delta = 0.01 \) sec [27], [13], \( \Delta = 0.1 \) sec [15], [4] and \( \Delta = 1 \) sec [17].

The frequency response graphs of \( C(s) \) and \( C_{\delta}(\gamma) \) are shown in Figure 2. The frequency response graphs indicate that the discrete-time FOPI controller approaches towards the original continuous-time FOPI controller as the sample time goes low. This also validates the theory of the delta operator and justifies its efficacy for realization of the discrete-time FOPI controller. The reasonable approximation of the FOPI controller influences to obtain the worthwhile approximation of the open loop system as well. The frequency response graphs of \( G_{OL}(s) \) and \( G_{OL\delta}(\gamma) \) are shown.
Table 2. Approximation of FOPI controller at different sampling instants.

| Sampling time (sec) | Approximation of $G_{OL}(\gamma)$ |
|---------------------|----------------------------------|
| $\triangle = 1$     | $\frac{2.921 \times 10^9 \gamma^6 + 1.805 \times 10^4 \gamma^5 + 4.172 \times 10^4 \gamma^4 + 4.428 \times 10^3 \gamma^3 + 2.126 \times 10^7 \gamma^2 + 3.878 \times 10^6 \gamma + 1.61 \times 10^5}{\gamma(4.254 \times 10^5 \gamma^5 + 1.658 \times 10^5 \gamma^4 + 2.147 \times 10^5 \gamma^3 + 1.066 \times 10^7 \gamma^2 + 1.757 \times 10^6 \gamma + 5360)}$ |
| $\triangle = 0.1$   | $\frac{1.291 \times 10^9 \gamma^6 + 4.316 \times 10^5 \gamma^5 + 4.34 \times 10^5 \gamma^4 + 1.356 \times 10^5 \gamma^3 + 1.303 \times 10^7 \gamma^2 + 3.444 \times 10^6 \gamma + 1.61 \times 10^5}{\gamma(2.398 \times 10^5 \gamma^5 + 3.106 \times 10^5 \gamma^4 + 9.161 \times 10^5 \gamma^3 + 7.606 \times 10^7 \gamma^2 + 1.636 \times 10^6 \gamma + 5360)}$ |
| $\triangle = 0.01$  | $\frac{1.094 \gamma^6 + 1.188 \times 10^3 \gamma^5 + 2.664 \times 10^3 \gamma^4 + 1.129 \times 10^3 \gamma^3 + 1.226 \times 10^7 \gamma^2 + 3.4 \times 10^6 \gamma + 1.61 \times 10^5}{\gamma(1.179 \times 10^5 \gamma^5 + 2.327 \times 10^5 \gamma^4 + 8.154 \times 10^5 \gamma^3 + 7.312 \times 10^7 \gamma^2 + 1.624 \times 10^6 \gamma + 5360)}$ |

It is observed that the continuous-time open loop system satisfies the desired gain cross over frequency $\omega_{gc} = 7$ rad/sec and the phase margin $\varphi_m = -105^\circ + 180^\circ = 75^\circ$. The phase response graphs seem to be considerably flat around the gain cross over frequency too. The frequency response graphs of $G_{OL}(\gamma)$ gradually becomes closer to the $G_{OL}(s)$ as the sampling time $\triangle$ goes low. The frequency responses of $G_{OL}(\gamma)$ and $G_{OL}(s)$ resemble maximally at $\triangle = 0.01$ sec which once again validates the fundamental theory of the delta operator and highlights its implication in designing the discrete-time FOPI controller.

Figure 2. Frequency responses of $C(s)$ and $C_{\delta}(\gamma)$ taking different sampling instants.

It seems that the approximated open loop model complies all the specifications as met by its continuous-time counterpart. The step responses of the approximated
Table 3. Approximation of open loop system at different sampling instants.

| Sampling time (sec) | Approximation of $C_0(\gamma)$ |
|---------------------|--------------------------------|
| $\triangle = 1$     | $5.698 \times 10^{11}\gamma^{11} + 5.884 \times 10^{12}\gamma^{10} + 2.604 \times 10^{13}\gamma^9 + 6.462 \times 10^8\gamma^8 + 9.874 \times 10^{13}\gamma^7 + 9.615 \times 10^{13}\gamma^6 + 5.98 \times 10^{13}\gamma^5 + 2.32 \times 10^{13}\gamma^4 + 5.33 \times 10^{12}\gamma^3 + 6.624 \times 10^{11}\gamma^2 + 3.769 \times 10^{10}\gamma + 7.713 \times 10^8\gamma(1.274 \times 10^{11}\gamma^{10} + 1.016 \times 10^{12}\gamma^9 + 3.38 \times 10^{12}\gamma^8 + 6.098 \times 10^{12}\gamma^7 + 6.506 \times 10^{12}\gamma^6 + 4.203 \times 10^{12}\gamma^5 + 1.622 \times 10^{12}\gamma^4 + 3.565 \times 10^{11}\gamma^3 + 4.06 \times 10^{10}\gamma^2 + 1.995 \times 10^9\gamma + 3.381 \times 10^7)$ |
| $\triangle = 0.1$   | $1.072 \times 10^9\gamma^{11} + 5.213 \times 10^8\gamma^{10} + 9.603 \times 10^8\gamma^9 + 8.515 \times 10^{11}\gamma^8 + 3.874 \times 10^{12}\gamma^7 + 9.389 \times 10^{12}\gamma^6 + 1.22 \times 10^{13}\gamma^5 + 8.435 \times 10^{12}\gamma^4 + 2.979 \times 10^{12}\gamma^3 + 5.013 \times 10^{11}\gamma^2 + 3.387 \times 10^{10}\gamma + 7.713 \times 10^8\gamma(3.33 \times 10^9\gamma^{10} + 9.124 \times 10^8\gamma^9 + 9.082 \times 10^8\gamma^8 + 4.135 \times 10^{11}\gamma^7 + 9.518 \times 10^{11}\gamma^6 + 1.141 \times 10^{12}\gamma^5 + 7.148 \times 10^{11}\gamma^4 + 2.245 \times 10^{11}\gamma^3 + 3.283 \times 10^{10}\gamma^2 + 1.843 \times 10^9\gamma + 3.381 \times 10^7)$ |
| $\triangle = 0.01$  | $3.798 \times 10^8\gamma^{11} + 5.103 \times 10^8\gamma^{10} + 2.031 \times 10^8\gamma^9 + 3.232 \times 10^{11}\gamma^8 + 2.106 \times 10^{12}\gamma^7 + 6.439 \times 10^{12}\gamma^6 + 9.708 \times 10^{12}\gamma^5 + 7.399 \times 10^{12}\gamma^4 + 2.779 \times 10^{12}\gamma^3 + 4.862 \times 10^{11}\gamma^2 + 3.349 \times 10^{10}\gamma + 7.713 \times 10^8\gamma(7.238 \times 10^7\gamma^{10} + 3.138 \times 10^9\gamma^9 + 4.556 \times 10^9\gamma^8 + 2.641 \times 10^{11}\gamma^7 + 7.16 \times 10^{11}\gamma^6 + 9.579 \times 10^{11}\gamma^5 + 6.459 \times 10^{11}\gamma^4 + 2.128 \times 10^{11}\gamma^3 + 3.209 \times 10^{10}\gamma^2 + 1.828 \times 10^9\gamma + 3.381 \times 10^7)$ |

Figure 3. Frequency responses of $G_{OL}(s)$ and $G_{OL\delta}(\gamma)$ taking different sampling instants.
closed loop system along with the ideal closed loop system are shown in Figure 4. Clearly, the step response of the approximated closed loop system resembles the ideal closed loop system while choosing the lower sampling time $\Delta=0.01$ sec.

The variations in the plant gain are evident from the prevailing literature to analyze the robustness of the overall controlled system with FOC [6], [17], [24]. Accordingly, the actual plant gain $K=1$ has been deviated up to the level of 80% to evaluate the robustness of the overall controlled system in delta-domain. The phase margin of the closed loop system still remains constant in spite of these wide variations of the plant gain as seen from Figure 5. The maximum overshoot of the closed loop system remains unaltered under the similar variations of the
plant gain which indicates the iso damping property as noticed from Figure 6. Therefore, the robustness of the overall system has been ensured as the phase margin and the maximum overshoot have not been affected by the uncertain plant gain. This emphasizes on the fact that the designed FOPI controller of delta-domain is capable enough to handle the sudden variation of the plant gain and thus keeping the performance characteristics of the overall controlled systems unchanged.

4. Conclusions. Discrete-time approximations of FOCs are commonly accomplished in z-domain. Instead of this common practice, this work is focused to implement the FOC using a diverse approach through capitalizing the unification property of the delta operator. In this paper, a fractional-order proportional integral controller has been designed to control a class of fractional-order plant for achieving some design specifications like phase margin, gain cross-over frequency and robustness in terms of achieving flat phase response around the gain cross-over frequency. The design methodology involves two stages. In the first stage, the continuous-time FOPI controller has been obtained for the specified fractional-order plant model. Open loop frequency response characteristics ensures that all the design specifications are satisfied. In the later stage, the FOPI controller is approximated in delta-domain. The approximations of the FOPI controller and the overall controlled system tend to the corresponding original systems in terms of frequency response characteristics and the time response characteristics as the sampling time tends to zero. This justifies the efficacy of the presented methodology as a substitute of the conventional approaches as far as the implementation of the FOC is concerned. Finally, the robustness of the overall system is also examined in delta-domain by deviating the plant gain up to the level of 80%. The flat phase characteristics and the iso-damped step response graph ensure the robustness of the overall system.

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