The evolution of the law of random processes in the analysis of dynamic systems

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Abstract The paper presents a method for determining the evolution of the cumulative distribution function of random processes which are encountered in the study of dynamic systems with some uncertainties in the characterizing parameters. It is proved that these distribution functions are the solution of a partial differential equation, whose coefficients can be determined once the dynamic system has been solved, and whose numerical solution can be obtained with the finite difference method. Two simple problems are solved here both explicitly and numerically, then the obtained results are compared with each other.

Keywords Dynamic system · Probability · Random variable · Stochastic process

1 Introduction

The analysis of structures subjected to dynamic loads is now generally conducted using refined mechanical models and adequate numerical techniques. However, even when the problem is well-posed so that both the existence and the uniqueness of the solution are guaranteed, some parameters used to describe the geometric and mechanical characteristics of the structure and the external actions are generally affected by uncertainties that must be taken into account in the analysis. These input parameters are thus represented by random variables, which are defined on an appropriate probability space and whose law is supposed to be known. A stochastic process is thus obtained which is parameterized over time and has the Euclidean space as state space.

In many applications, it is necessary to determine the probability distribution of some output quantities. It is a question of determining the evolution over time of the law of some random process which is defined in terms of the solution of the dynamic system. This objective is generally achieved using the Monte Carlo method which, at least in principle, gives the possibility to consider complicated models and geometries without having to resort to unrealistic simplifying hypotheses. On the other hand, this method can require long computation times which may not be compatible with the required precision [1].

A different way to deal with the problem is to use the generalized density evolution equation, which is a consequence of the "principle of preservation of probability" [2]. This method leads to writing a linear PDE whose solution gives the probability density function for the quantities of interest. The coefficients of this equation at each instant depend on the value of the state variables and then can be obtained from the (deterministic) solution of the dynamic system. The method of the generalized density evolution equation has been implemented into the MADY code, which has already
the routines for dynamic analysis of plane, three-dimen-
sional, or beam and shell-based structures [3], and has
been applied to the study of some masonry construc-
tions [4, 5]. For these structures, the uncertainties have
particular relevance, due to the constitutive charac-
teristics of the material and the geometry.

In this paper, we deduce an equation similar to the
one proposed in [2] but which allows calculating the
cumulative distribution function directly, instead of
the density function, of the chosen random variable.
Indeed, while the former is always a locally integrable
function, it may happen that the latter is defined only
in a distributitional sense. The deduction is made using
classical theorems of the measure theory; although
most notions discussed below can be presented in their
intuitive sense, we prefer to give explicit definitions
and proofs to avoid misunderstandings. Thus, both the
approach proposed in this paper and the one proposed
in [2] are rigorously justified. Moreover, with the help
of the differential forms and the coarea formula, relations
are deduced for the explicit computation of both
the probability density function and the cumulative
distribution function; these are especially useful if the
number of random input parameters is greater than one.

Finally, two examples are presented in which the
evolution of the probability distribution of a chosen
output parameter is explicitly calculated. The solution
is then compared with the numerical one obtained with
the MASY code, which has meanwhile been updated
with the new numerical procedures presented in this
paper.

2 Background and notations

Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space. Here \(\Omega\) is the set
of the outcomes, \(\mathcal{A}\) is the \(\sigma\)-algebra on \(\Omega\) made of all
the events and \(\mathbb{P} : \mathcal{A} \to [0, 1]\) is a probability measure,
i.e. a positive measure on \((\Omega, \mathcal{A})\) such that \(\mathbb{P}(\Omega) = 1\).
Moreover, let \(\mathbb{R}^n\) and \(\mathcal{B}(\mathbb{R}^n)\) be the \(n\)-dimensional
Euclidean space and its corresponding Borel \(\sigma\)-algebra,
respectively.

A map \(\mathbf{X} : \Omega \to \mathbb{R}^n\) is said to be a (vector) random variable
if it is \(\mathcal{B}(\mathbb{R}^n)\)-measurable, i.e. if
\[
\{\mathbf{X} \in B\} \in \mathcal{A}, \text{ for each } B \in \mathcal{B}(\mathbb{R}^n),
\]
where, as usual, \(\{\mathbf{X} \in B\}\) denotes \(\mathbf{X}^{-1}(B)\). The measure
\(\mu_X\) denotes the law of \(\mathbf{X}\) or the image of measure
\(\mathbb{P}\) under \(\mathbf{X}\) [6], i.e. the measure defined on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\)
by
\[
\mu_X(B) = \mathbb{P}(\{\mathbf{X} \in B\}), \text{ for each } B \in \mathcal{B}(\mathbb{R}^n)
\]
and the function \(F_X : \mathbb{R}^n \to [0, 1]\), defined by
\[
F_X(a_1, a_2, \ldots, a_n) = \mu_X(\{\mathbf{X} \in \mathbb{R}^n : -\infty < x_i \leq a_i,
\text{ } i = 1, 2, \ldots, n\}),
\]
is called cumulative distribution function of \(\mathbf{X}\).

Let \(\mathcal{L}^n\) be the Lebesgue measure on \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\).
Measure \(\mu_X\) is said to be absolutely continuous
(with respect to \(\mathcal{L}^n\)) if \(\mu_X(B) = 0\) for every set
\(B \in \mathcal{B}(\mathbb{R}^n)\) with \(\mathcal{L}^n(B) = 0\). In this case, the
Radon-Nikodym theorem [6] guarantees the existence of a
positive integrable function \(p_X\) such that
\[
\mu_X(B) = \int_B p_X(x)dx \quad (1)
\]
for every \(B \in \mathcal{B}(\mathbb{R}^n)\). Function \(p_X\) is called the (joint)
probability density function of the vector random
variable \(\mathbf{X}\) and it holds
\[
F_X(x_1, x_2, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p_X(\xi_1, \xi_2, \ldots, \xi_n)d\xi_1, d\xi_2, \ldots, d\xi_n
\]
or
\[
p_X(x_1, x_2, \ldots, x_n) = \frac{dF_X(x_1, x_2, \ldots, x_n)}{d\xi_1 d\xi_2 \cdots d\xi_n}. \quad (2)
\]
If \(\mu_X\) is not absolutely continuous, then its probabil-
ity density function does not exist as an integrable
function. However, it can be defined as a generalized
function by interpreting (2) as a distributional deriva-
tive [7].

Let \(\mathbf{X}\) be a vector random variable and
\(f : \mathbb{R}^n \to \mathbb{R}^m\) a Borel function; then \(\mathbf{Y} = f(\mathbf{X})\)
is again a vector random variable and, for each
\(B \in \mathcal{B}(\mathbb{R}^n),
\]
\[
\mu_Y(B) = \mu_X(f^{-1}(B)). \quad (3)
\]
If \(\mathbf{X}\) has a probability density function \(p_X\), then
\[
\mu_Y(B) = \mu_X(f^{-1}(B)) = \int_{f^{-1}(B)} p_X(x)dx \quad (4)
\]
by (1) and (3). Moreover, if \(f\) is such that [8]
\( \mathcal{L}^m(B) = 0 \Rightarrow \mathcal{L}^n(f^{-1}(B)) = 0 \)

for every Borel subset \( B \) of \( \mathbb{R}^n \), then, in view of (4), \( \mu_Y \) is (finite and) absolutely continuous with respect to \( \mathcal{L}^m \) and thus it admits a probability density function \( p_Y \). In the particular case when \( n = m \) and \( f \) is a diffeomorphism, i.e. a bijective application such that both \( f \) and \( f^{-1} \) are continuously differentiable, then

\[
p_Y(y) = p_X(f^{-1}(y)) \left| Jf^{-1}(y) \right|
\]

where \( Jf^{-1} \) denotes the Jacobian of \( f^{-1} \).

More generally, suppose that there exists a countable family \( (B_i)_{i \in I} \) of pairwise disjoint open sets of \( \mathbb{R}^n \), such that (i) the event

\[
B = \bigcup_{i \in I} \{ Y \in B_i \}
\]

has probability \( \mathbb{P}(B) \) equal to 1; (ii) for each \( i \in I \), the restriction \( f_i \) of \( f \) to \( B_i \) is a diffeomorphism of \( B_i \) onto an open set \( C_i \subset \mathbb{R}^n \). Under this assumption, define

\[
(p_Y)_i(y) = p_X(f_i^{-1}(y)) \left| Jf_i^{-1}(y) \right| \chi_{C_i}(y)
\]

where

\[
\chi_{C_i}(y) = \begin{cases} 1 & \text{if } y \in C_i \\ 0 & \text{otherwise} \end{cases}
\]

is the indicator function of \( C_i \). Then

\[
p_Y = \sum_{i \in I} (p_Y)_i.
\]

A stochastic process \( \{ X_t, t \in D \} \), with \( D = [0, T] \) a real interval, is a family of vector random variables indexed by a parameter \( t \) defined on a common probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) and all with their values in \((\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))\), which is called the state space. By definition, for each \( t \in D \), \( X_t \) is an \( \mathcal{A} \)-measurable function and, for each \( \omega \in \Omega \), \( \{ X_t(\omega), t \in D \} \) is a function defined in \( D \) that is called sample function, realization or trajectory of the process.

Let be \( z \in \mathbb{R} \) and \( f : \mathbb{R}^m \to \mathbb{R} \) a smooth function. Then, the set

\[
\{ z = f \} = \{ x \in \mathbb{R}^m : z - f(x) = 0 \text{ and } |\nabla f(x)| > 0 \},
\]

if it is not empty, is a regular hypersurface of \( \mathbb{R}^m \) whose orientation is determined by the unit normal vector \( n = \nabla f |\nabla f| \). Let us put

\[
f_y = \frac{df}{dx_j}
\]

and denote by \( \mu_S \) the volume form on \( \{ z = f \} \) [9], i.e. the differential form

\[
\mu_S = \sum_{j=1}^n \frac{(-1)^{j-1}}{|\nabla f|} f_j dx_1 \wedge \ldots \wedge \hat{dx_j} \wedge \ldots \wedge dx_n,
\]

(where \( \hat{dx_j} \) means “omit the factor \( dx_j \)”). Recalling that on the hypersurface \( \{ z = f \} \) we have

\[
df = \sum_{j=1}^n f_j dx_j = 0,
\]

it is easy to verify that, for each \( j = 1, \ldots, n \) it turns out that [7]

\[
\mu_S = \frac{(-1)^{j-1}}{|\nabla f|} f_j dx_1 \wedge \ldots \wedge \hat{dx_j} \wedge \ldots \wedge dx_n,
\]

everywhere \( f_j \neq 0 \) holds.

For each \( \alpha \leq n \), let \( \mathcal{H}^\alpha \) be the Hausdorff measure in \( \mathbb{R}^n \) [10], defined in such a way that \( \mathcal{H}^n = \mathcal{L}^n \). Then for every function \( g \) which is integrable on \( \{ z = f \} \) we have

\[
\int_{\{ z = f \}} g d\mathcal{H}^{n-1} = \int_{\{ z = f \}} g \mu_S.
\]

The following useful result is a consequence of the coarea formula. Let \( f : \mathbb{R}^m \to \mathbb{R} \) be a smooth function such that \( |\nabla f(x)| > 0 \), and \( g : \mathbb{R}^m \to \mathbb{R} \) be an integrable function. Moreover, let us put

\[
\{ z > f \} = \{ x \in \mathbb{R}^m : z - f(x) > 0 \}.
\]

Then

\[
\int_{\{ z > f \}} g dx = \int_{-\infty}^z d\zeta \int_{\{ z = f \}} \frac{g}{|\nabla f|} d\mathcal{H}^{m-1}
\]

and, in particular,

\[
\frac{d}{dz} \int_{\{ z > f \}} g dx = \int_{\{ z = f \}} \frac{g}{|\nabla f|} d\mathcal{H}^{m-1}.
\]
3 Stochastic dynamic system

The equation of motion of a body, discretized by the finite element method with respect to the space variable is reduced to a system of ODEs

\[ \mathbf{M} \ddot{\mathbf{y}} + \mathbf{f}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{B}(\mathbf{y}, t)\xi(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \quad \dot{\mathbf{y}}(t_0) = \mathbf{\dot{y}}_0, \]

(9)

where \( t \in D \) is the time and \( \mathbf{y}, \dot{\mathbf{y}} \) and \( \ddot{\mathbf{y}} \) are the displacement, velocity and acceleration vector, respectively, which are defined on \( D \) and take their values in \( \mathbb{R}^n \). \( \mathbf{M} \) is the mass matrix, \( \mathbf{f} \) is the internal force vector (including damping and restoring force), \( \mathbf{B} \) is the input force influence matrix, \( \xi \) is the external excitation vector and \( \mathbf{y}_0 \) and \( \mathbf{\dot{y}}_0 \) are the initial displacement and velocity vector, respectively. By introducing the state vector

\[ \mathbf{x} = \left\{ \begin{array}{c} \mathbf{\dot{y}} \\ \mathbf{y} \end{array} \right\}, \]

Equation (9) can be rewritten as

\[ \dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}, t) + \mathbf{B}(\mathbf{x}, t)\xi(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{10} \]

where

\[ \mathbf{A}(\mathbf{x}, t) = \begin{pmatrix} -\mathbf{M}^{-1}\mathbf{f}(\mathbf{x}) \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{B}(\mathbf{x}, t) = \begin{pmatrix} \mathbf{M}^{-1}\mathbf{B}(\mathbf{x}, t) \\ \mathbf{0} \end{pmatrix}. \]

If randomness is present, coming from the initial conditions, the excitation, or the properties of the system, a random state equation can be written as [2]

\[ \dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}, t_0, t) \tag{11} \]

where \( \mathbf{G}(\mathbf{x}, t_0, t) \in \Lambda \subset \mathbb{R}^m \) is the vector of all random parameters, i.e., the values assumed by the vector random variable \( \mathbf{\Theta} : \Omega \to \mathbb{R}^m \), that is supposed to be time-independent and to have a (joint) probability density function \( p_{\mathbf{\Theta}} \).

If the (deterministic) problem (10) is well posed, then for each choice of \( \Theta \) and \( \mathbf{x}_0 \), Eq. (11) has one and only one solution

\[ \mathbf{x} = \mathbf{H}(\mathbf{x}_0, t_0, t), \quad \mathbf{H}(\mathbf{x}_0, 0, t) = \mathbf{x}_0 \]

with \( \mathbf{H} : \Lambda \times \mathbb{R}^{2n} \times D \to \mathbb{R}^{2n} \) a suitable smooth function.

Below, for sake of simplicity, we omit to explicitly indicate the dependence on \( \mathbf{x}_0 \) and write \( \mathbf{x}_t = \mathbf{H}(\mathbf{x}_0, t) \) to denote the solution of (11).

In applications, we are interested in considering stochastic processes of the type

\[ Z_t(\mathbf{\Theta}) = Z(\mathbf{\Theta}, t) = \psi \circ \mathbf{H}(\mathbf{\Theta}, t), \]

with

\[ Z(\mathbf{\Theta}, 0) = \psi(\mathbf{x}_0) = \mathbf{z}_0, \tag{12} \]

and in determining the evolution of the law of \( Z_t \). Here \( \psi : \mathbb{R}^{2n} \to \mathbb{R} \) is a deterministic smooth function. To this aim, for each \( t \in D \), let us consider the family of measures \( \{ \mu_t^\Theta : \Theta \in \Lambda \} \) which are defined on \( \mathbb{R} \) by

\[ \mu_t^\Theta(B) = \chi_B \circ Z_t \tag{13} \]

for every Borel subset \( B \) of \( \mathbb{R} \), with \( \chi_B \) the indicator function of \( B \). Measure \( \mu_t^\Theta \) is the law of the conditional probability of \( Z_t \) given \( \Theta \), and sometimes it is denoted by \( p_{Z_t}(z|\Theta = \Theta, t) \). Indeed, once \( \Theta \) and \( t \) are fixed, \( \mu_t^\Theta \) is the Dirac measure on \( \mathbb{R} \), concentrated in the point \( Z_t(\Theta) \). The cumulative distribution function of \( \mu_t^\Theta \) is the real function

\[ F_t^\Theta(z) = \mu_t^\Theta((-\infty, z]) = \begin{cases} 1 & \text{if } z \geq Z_t(\Theta), \\ 0 & \text{otherwise}. \end{cases} \tag{14} \]

Let \( C_0 \) be the space of the continuous real functions that are compactly supported in \( \mathbb{R} \), with the maximum norm \( |\cdot|_{C_0} \). For every function \( f \in C_0 \), we have

\[ \int_{\mathbb{R}} f(z) \mu_t^\Theta(dz) = f(Z_t(\Theta)) \tag{15} \]

(this is true for the indicator functions of measurable sets by (13) and then for simple functions so that the general case follows from an approximation argument). Therefore, the map \( \Theta \to \int_{\mathbb{R}} f(z) \mu_t^\Theta(dz) \) is measurable on \( \Lambda \).

Proposition (i) For fixed \( t \in D \), there is one and one measure \( \mu_t \) on \( \mathbb{R} \) such that

\[ \int_{\mathbb{R}} f(z) \mu_t(dz) = \int_{\Lambda} p_{\mathbf{\Theta}}(\mathbf{\Theta})d\mathbf{\Theta} \int_{\mathbb{R}} f(z) \mu_t^\Theta(dz) \tag{16} \]

for each \( f \in C_0 \).
(ii) $\mu$ is the law of $Z_t$, i.e., for every Borel subset $B$ of $\mathbb{R}$,

$$\mu_t(B) = \mathbb{P}(Z_t \in B).$$

**Proof** (i) Let be

$$\Phi(f) = \int_{\mathbb{R}} p_\Theta(\theta)d\theta \int_{\mathbb{R}} f(z)\mu^\theta_t(dz).$$

Then, noting that

$$\int_{\mathbb{R}} \mu^\theta_t(dz) = \mu^\theta_t(\mathbb{R}) = 1 \quad \text{and} \quad \int_{\mathbb{R}} p_\Theta(\theta)d\theta = \mathbb{P}(\Omega) = 1$$

we obtain

$$|\Phi(f)| \leq \int_{\mathbb{R}} p_\Theta(\theta)d\theta \int_{\mathbb{R}} |f(z)|\mu^\theta_t(dz)
\leq |f|_{c_1} \int_{\mathbb{R}} p_\Theta(\theta)d\theta \int_{\mathbb{R}} \mu^\theta_t(dz) = |f|_{c_1}.$$

Thus, $\Phi$ is a linear and continuous functional on $C_0$ and (i) follows from the Riesz representation theorem [11].

(ii) For every Borel subset $B$ of $\mathbb{R}$ we have

$$\mu_t(B) = \int_{\mathbb{R}} \chi_B d\mu_t(dz) = \int_{\mathbb{R}} p_\Theta(\theta)d\theta \int_{\mathbb{R}} \chi_B(z)\mu^\theta_t(dz)
= \int_{\mathbb{R}} \chi_B(Z_t(\theta))p_\Theta(\theta)d\theta
= \int_{\mathbb{R}} p_\Theta(\theta)\chi_{Z_t^{-1}(B)}(\theta)d\theta
= \int_{Z_t^{-1}(B)} p_\Theta(\theta)d\theta = \mathbb{P}(Z_t^{-1}(B)),
$$

where the second identity follows from (16), the third follows from (15) and the fourth follows from the relation $\chi_B \circ Z_t = \chi_{Z_t^{-1}(B)}$. \qed

If we denote by $F(t, z, t)$ the cumulative distribution function of $\mu$, with the help of (16) and (14) we can write

$$F_t(z, t) = \mu_t((-\infty, z]) = \int_{-\infty}^z f^{\theta}_t(z)dz = \int_{\mathbb{R}} p_\Theta(\theta)d\theta \int_{-\infty}^z \chi_{(-\infty, z]}(\theta)d\theta = \int_{\mathbb{R}} p_\Theta(\theta)d\theta \mathbb{P}(Z_t \leq z),$$

where

$$\{Z_t \leq z\} = \{\theta \in \Lambda : z - Z_t(\theta) \geq 0\}.$$
if on the hypersurface \( \{ Z_t = z \} \) the inequality \( \frac{\partial Z}{\partial \theta_j} \neq 0 \) holds.

As will be shown later in the examples, Eqs. (18), (21) and (22), (23) can be used for the explicit calculation of the cumulative distribution function and the probability density function, respectively, even if the dimension of \( \Lambda \) is greater than one.

4 The evolution of the law of \( Z_t \)

In [2] the authors obtain a linear PDE in the unknown

\[
p_{Z \theta}(z, \theta, t) = p_{Z \theta}(z|\theta = \theta, t)p_{\theta}(\theta)
\]

which once solved allows to determine the probability density function \( p_{Z \theta}(z, t) \) of \( Z_t \), by integrating \( p_{Z \theta} \) over \( \Lambda \), with respect to \( \theta \). Below we propose a similar method to directly determine the cumulative distribution function \( F_Z(z, t) \) of \( Z_t \). Let’s start with some notation.

Let be \( N = \mathbb{R} \times \Lambda \times D \subset \mathbb{R}^{m+2} \), \( G(z, \theta, t) = z - Z(\theta, t) \) and \( \chi_K \) be the indicator function of the region

\[
K = \{ (z, \theta, t) \in N : G \geq 0 \}
\]

that is

\[
\chi_K(z, \theta, t) = \begin{cases} 1 & \text{if } z - Z(\theta, t) \geq 0 \\ 0 & \text{otherwise} \end{cases}
\] (24)

(note that, for each pair \( z \) and \( t \), region \( \{ Z_t \leq z \} \) that has been defined in (19) is a subset of \( K \)). Let us denote by \( D \) the space of infinitely differentiable real functions with compact support in \( N \).

Consider the two distributions [7] which, for every \( \phi \in D \), are defined in \( N \) by

\[
< \chi_K \cdot \phi > = \int_K \chi_K(x)\phi(x)dx = \int_{\{ z - Z(\theta, t) \geq 0 \}} \phi(x)dx,
\]

with \( x=(z, \theta, t) \), and

\[
< \delta(G) \cdot \phi > = \int_N \delta(G(x))\phi(x)dx = \int_{\{ G(z, \theta, t) = 0 \}} \phi(x)dx,
\]

where

\( \mathcal{I} = \{ (z, \theta, t) \in N : G(z, \theta, t) = 0 \} \), is a regular hypersurface, because in \( N \) we have \( |\nabla G| = \sqrt{1 + |\nabla Z|^2} \geq 1 \). Note that while the first distribution is regular, the second is concentrated on a set that is negligible with respect to the Lebesgue measure.

It can be proved [7] that \( \delta(G) \) is the distributional derivative of \( \chi_K \), i.e. the distribution such that

\[
\frac{\partial \chi_K}{\partial z}(G) = \delta(G) \frac{\partial \chi_K}{\partial \theta_j}(G) = -\frac{\partial \chi_K}{\partial \theta_j}(G) = -\frac{\partial \chi_K}{\partial t}(G)
\]

or also

\[
\nabla \chi_K = \delta(G)\nabla G.
\]

From (25) to (25) we deduce the PDE

\[
\frac{\partial \chi_K}{\partial t}(z, \theta, t) + \frac{\partial Z(\theta, t)}{\partial t} \frac{\partial \chi_K}{\partial z}(z, \theta, t) = 0
\] (26)

with the initial conditions

\[
\chi_K(z, \theta, 0) = \begin{cases} 1 & \text{if } z \geq z_0 \\ 0 & \text{otherwise} \end{cases}
\]

which follows from (24) to (12).

Once \( \chi_K \) has been calculated, \( F_Z \) is obtained from the relation

\[
F_Z(z, t) = \int_{\mathcal{I}} \chi_K(z, \theta, t)p_{\theta}(\theta)d\theta.
\] (27)

In a similar way, [7] it is defined the distributional derivative \( \delta' \) of \( \delta \) and it is shown that it results

\[
\frac{\partial \delta(G)}{\partial z} = \delta'(G), \quad \frac{\partial \delta(G)}{\partial \theta_j} = -\frac{\partial G}{\partial \theta_j} \delta'(G), \quad \frac{\partial \delta(G)}{\partial t} = -\frac{\partial G}{\partial t} \delta'(G)
\]

from which the PDE that has been proposed in [2] can be obtained, i.e.

\[
\frac{\partial p_{Z \theta}(z, \theta, t)}{\partial t} + \frac{\partial Z(\theta, t)}{\partial t} \frac{\partial p_{Z \theta}(z, \theta, t)}{\partial z} = 0
\] (28)

The corresponding initial condition is

\[
p_{Z \theta}(z, \theta, 0) = \delta(z - z_0)p_{\theta}(\theta)
\]

or

\[
p_{Z \theta}(z, \theta, 0) = p_{Z_0}(z)p_{\theta}(\theta),
\]
depending on whether \( z_0 \) is deterministic or not. Once \( p_{Z\theta} \) has been calculated, \( p_Z \) is obtained from the relation

\[
p_Z(z, t) = \int_\Lambda p_{Z\theta}(z, \theta, t) d\theta.
\]  
(29)

Thus \( \chi_K \) and \( p_{Z\theta} \) satisfy the same distributional PDE but with different initial conditions.

In applications, system (11) and Eqs. (26) [or (28)] are both solved numerically, after which \( F_Z(z, t) \) is determined by the relation (27) [or \( p_Z(z, t) \) is determined by (29)]. Two examples are presented below where the whole procedure can be followed explicitly. Let’s briefly mention the solution technique for Eq. (26). The analogue for Eq. (28) is dealt with in detail in [2].

Using the method of characteristics [12] for this purpose, we can write the following system of ODEs, with the corresponding initial conditions

\[
\begin{align*}
\frac{dz}{ds} &= \frac{\partial Z(\theta, t)}{\partial t} \\
\frac{d\theta}{ds} &= t(0) = 0, \\
\frac{d\chi_k}{ds} &= 0 \\
\chi_k(\tau, \theta, 0) &= \begin{cases} 1 & \text{if } \tau \geq z_0, \\
0 & \text{otherwise}. \end{cases}
\end{align*}
\]  
(30)

From (30), we obtain \( t = s \), which in view of (30), provides the following two implications

\[
z(s, \tau) = Z(\theta, s) + \tau - Z(\theta, 0) \Rightarrow \\
\tau = z - Z(\theta, t) + Z(\theta, 0).
\]  
(31)

From (30), taking into account that \( Z(\theta, 0) = z_0 \) by (12), we deduce

\[
\chi_k(z, \theta, t) = \chi_k(\tau, \theta, 0) = \begin{cases} 1 & \text{if } \tau \geq Z(\theta, 0), \\
0 & \text{otherwise}. \end{cases}
\]

Finally, from (27) we obtain

\[
F_Z(z, t) = \int_\Lambda \chi_k(z, \theta, t) p_\theta(\theta) d\theta = \int_{\{z \leq \xi\}} p_\theta(\theta) d\theta
\]  
(32)

which coincides with (18).

5 Examples

5.1 The motion of a material point

Let us consider the rectilinear motion of a material point whose velocity is proportional to the traveled distance. The proportionality coefficient and the initial position are expressed by the two independent positive-valued random variables \( \Theta_1 \) and \( \Theta_2 \), respectively. Distance \( x \) is measured in meters and time \( t \) in seconds, and a superimposed dot indicates derivation with respect to time. We can write

\[
\dot{x} = \theta_1 x, \quad x(0) = \theta_2
\]

and

\[
\Theta = (\theta_1, \theta_2) \in \Lambda = \mathbb{R}^+ \times \mathbb{R}^+, \quad t \in D = [0, \infty).
\]

By separating the variables we obtain

\[
\ln \left( \frac{x}{\theta_2} \right) = \theta_1 t,
\]

from which, assuming that \( \psi \) is the identity function,

\[
Z(\theta, t) = x(\theta, t) = \theta_2 e^{\theta_1 t} \quad \text{and} \quad \frac{\partial Z}{\partial t} = \theta_1 \theta_2 e^{\theta_1 t}
\]

follow. As

\[
\nabla Z(\theta, t) = \left( \frac{\partial Z}{\partial \theta_1}, \frac{\partial Z}{\partial \theta_2} \right) = e^{\theta_1 t}(\theta_2 t, 1).
\]

from (21) we obtain the cumulative distribution function of \( Z \)

\[
F_Z(z, t) = \int_{-\infty}^z d\xi \int_{\{z = \xi\}} \frac{p_\theta}{e^{\theta_1 t} \sqrt{1 + \theta_2^2 t^2}} d\mu^1(\theta),
\]

where \( p_\theta \) is the joint probability density function of the random variables \( \Theta_1 \) and \( \Theta_2 \).

Alternatively, relation (18) can be used,

\[
F_Z(z, t) = \int_{\{z \leq \xi\}} p_\theta(\theta) d\theta.
\]

Particularly, we assume that \( \Theta_1 \) and \( \Theta_2 \) are two independent variables, each uniformly distributed on the interval \([0, 1]\), so that \( p_\theta \) is the indicator function of the square \( Q = [0, 1] \times [0, 1] \). Then, we obtain (Fig. 1)
\[ F_Z(z, t) = \int_{Q} p_{\Theta}(\theta) d\theta \]
\[
\begin{cases} 
0 & \text{if } 0 < z \leq 1, \\
\ln z/t + \int_{z}^{1} ze^{-\Theta_1} d\Theta_1 = \frac{1}{t} (\ln z + 1 - ze^{-t}) & \text{if } 1 < z \leq e', \\
1 & \text{if } z > e',
\end{cases}
\]

from which, deriving with respect to \( z \), we deduce the probability density function
\[
p_Z(z, t) = \begin{cases} 
\frac{1}{t} (1 - e^{-t}) & \text{if } 0 < z \leq 1, \\
\frac{1}{t} (1 - e^{-t}) & \text{if } 1 < z \leq e', \\
0 & \text{if } z > e'.
\end{cases}
\]  

Equation (26) and the corresponding initial conditions take the form
\[
\frac{\partial X_K(z, \Theta, t)}{\partial t} + \Theta_z e^{\Theta_1} \frac{\partial X_K(z, \Theta, t)}{\partial z} = 0
\]
and
\[
X_K(z, \Theta, 0) = \begin{cases} 
1 & \text{if } z \geq \Theta_z, \\
0 & \text{otherwise},
\end{cases}
\]
respectively.

The numerical solution can be accomplished by means of the following steps.

**Step 1** Construct a partition of \( \Lambda = [0, 1] \times [0, 1] \), in a finite number \( N \) of squares \( \Lambda_i \) with center \( \Theta' = (\Theta'_1, \Theta'_2) \) (\( i = 1, \ldots, N \)), and assign to each \( \Theta' \) the corresponding value \( p_{\Theta}(\Theta') = \frac{1}{N} \).

Fig. 1 Graph of \( z = Z_t \), for \( a \ 0 \leq z \leq 1, \ b \ 1 \leq z \leq e' \) and \( c \ z = e' \) (Example 1)
Step 2 Discretize the first quadrant of the $z$, $t$ plane by choosing a mesh width $\Delta z$ and a time step $\Delta t$, and defining the discrete mesh points $(z_j, t_n)$ by

$$z_j = j\Delta z, \quad t_n = n\Delta t.$$ 

Then, for each representative point $\theta^i$, solve the equation

$$\frac{\partial \chi_k(z, \theta^i, t)}{\partial t} + \theta_i^\alpha e^{\theta_i^\beta} \frac{\partial \chi_k(z, \theta^i, t)}{\partial z} = 0,$$

with the help of the initial condition

$$\chi_k(z, \theta^i, 0) = \begin{cases} 
1 & \text{if } z \geq \theta^i_2, \\
0 & \text{otherwise}, 
\end{cases}$$

by using the total variation diminishing (TVD) method [2, 13].

Step 3 Finally, determine the cumulative distribution function by means of the discretized version of the relation (32), i.e. for each mesh point $(z_j, t_n)$ compute

$$F_z(z_j, t_n) = \sum_{i=1}^{N} \chi_k(z_j, \theta^i, t_n).$$

Figs. 2 and 3 show the numerical results that have been obtained with $\Delta z = 6.25 \cdot 10^{-3} m$, $\Delta t = 6.25 \cdot 10^{-4} s$. Each side of $\Lambda$ has been divided into 40 equal sub-intervals, so that $N = 1600$.

In addition to the cumulative function, the probability density function $p(z, t)$ was also numerically calculated by means of Eqs. (28) and (29), at the same four instants previously considered. The calculations have been made for $N = 40 \times 40$, $N = 80 \times 80$ and $N = 160 \times 160$. We used $\Delta z = 6.25 \cdot 10^{-3} m$ and $\Delta t = 6.25 \cdot 10^{-4} s$ in the first two cases, and $\Delta z = 3.125 \cdot 10^{-3} m$ and $\Delta t = 3.125 \cdot 10^{-4} s$ in the third case. The obtained results are shown in Fig. 4 for $t = t_1$ and $t = t_4$; the other cases are similar.

At each instant $t$ a possible estimate of the relative error is given by the formula

$$\delta(t) = \sqrt{\frac{\sum_{i=1}^{N} (f(z_i, t) - g(z_i, t))^2}{\sum_{i=1}^{N} f(z_i, t)^2}},$$

where the values of $f(z_i, t)$ and $g(z_i, t)$ are obtained explicitly and numerically, respectively.

The error was calculated, for $0 \leq z \leq 4$, and in all the examined cases its order of magnitude turned out to be practically constant over time. In the case of the cumulative function, the maximum error was less than $1.7 \cdot 10^{-3}$. In the case of the density function, it was less than $1.9 \cdot 10^{-1}$ for $N = 40 \times 40$, less than $1.4 \cdot 10^{-1}$ for $N = 80 \times 80$ and less than $9 \cdot 10^{-1}$ for $N = 160 \times 160$.

**Fig. 2** Cumulative distribution functions at $t_1 = 0.25 s$, $t_2 = 0.75 s$, $t_3 = 1.25 s$ and $t_4 = 1.75 s$; comparison between the explicit and numerical solutions. (Example 1)
5.2 The Klein–Gordon equation

Let \( \mathcal{U} = \{x \in \mathbb{R} : 0 \leq x \leq l\} \) be the reference configuration of a string which is made of elastic material having a mass density \( \rho \) per unit length. The string is fixed at its ends and is subjected to a constant tension \( \tau \) and an additional force proportional to the displacement \( u \). The equation of motion for oscillations of small amplitude is

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \alpha^2 u = 0, \tag{35}
\]

where

\[
c = \sqrt{\frac{\tau}{\rho}} \tag{36}
\]

and \( \alpha^2 u \) is the additional force per unit mass. This is also the Klein–Gordon equation of quantum field theory [12, 14].

The function

\[
u(x, t) = A \sin \left( \frac{\pi x}{l} \right) \cos(\omega t) \tag{37}\]

solves Eq. (35) for \( \omega = \sqrt{\left( \frac{\pi c}{l} \right)^2 + \alpha^2} \),

\[
\tag{38}
\]

The boundary conditions

\[ u(0, t) = u(l, t) = 0 \]

and the initial conditions

\[
u(x, 0) = A \sin \left( \frac{\pi x}{l} \right) \quad \text{and} \quad \frac{\partial u(x, 0)}{\partial t} = 0.
\]

We assume that \( \pi c/l \) and \( \alpha \) are independent random variables, which will be indicated with \( \Theta_1 \) and \( \Theta_2 \) respectively, and that they are uniformly distributed over \( \Lambda = [\kappa_i, \kappa_f] \times [\alpha_i, \alpha_f] \).

We choose \( Z \) as the (dimensionless) displacement \( u \left( \frac{l}{2}, t \right) / A \) of the middle point of the string, so that it results

\[ Z(\Theta, t) = \cos \left( t \sqrt{\Theta_1^2 + \Theta_2^2} \right). \]

Equation (26) becomes

\[
\frac{\partial \chi_{K}(z, \Theta, t)}{\partial t} - \left( \sqrt{\Theta_1^2 + \Theta_2^2} \sin \left( t \sqrt{\Theta_1^2 + \Theta_2^2} \right) \right) \frac{\partial \chi_{K}(z, \Theta, t)}{\partial z} = 0. \tag{39}
\]
with the initial condition

\[ \chi_K(z, \theta, 0) = \begin{cases} 
1 & \text{if } z \geq 1, \\
0 & \text{otherwise}. 
\end{cases} \]

The calculations were made for \( t = 0.025 \text{s}, \) \( t = 0.05 \text{s}, \) and \( t = 0.075 \text{s}; \) we limit ourselves to describe the first case because the other two are similar. Let us denote by \( \gamma = \alpha_f (\kappa_f - \kappa_i) \) the area of \( \Lambda, \) and by

\[ z_1 = \cos \left( t \sqrt{\kappa_f^2 + \alpha_f^2} \right), \quad z_2 = \cos \left( t \kappa_f^2 \right), \]

\[ z_3 = \cos \left( t \sqrt{\kappa_f^2 + \alpha_f^2} \right) \quad \text{and} \quad z_4 = \cos(t\kappa_i) \]

The values of \( z \) for which the circumference with center at the origin and radius \( \cos^{-1}(z)/t \) passes through the corresponding vertices of \( \Lambda \) (Fig. 5). Then, with the help of (18), we obtain

(i) For \( z \leq z_1, \)

\[ F(z, t) = 0; \]

(ii) For \( z_1 \leq z \leq z_2, \)

\[ F(z, t) = \frac{1}{\zeta} \left( \alpha_f (\kappa_f - \kappa_a) - \int_{\kappa_a}^{\kappa_f} \frac{1}{t^2} \cos^{-1}(z)^2 - \theta_1^2 \right) d\theta_1 \]

where \( \kappa_a = \sqrt{\frac{1}{t^2} \cos^{-1}(z)^2 - \alpha_f^2}; \)

(iii) For \( z_2 \leq z \leq z_3, \)

\[ F(z, t) = \frac{1}{\zeta} \left( \alpha_f (\kappa_f - \kappa_a) - \int_{\kappa_a}^{\kappa_f} \sqrt{\frac{1}{t^2} \cos^{-1}(z)^2 - \theta_1^2} \right) d\theta_1 \]

where \( \kappa_b = \frac{1}{t} \cos^{-1}(z) \) and \( \kappa_c \) has the same expression as \( \kappa_a; \)

(iv) For \( z_3 \leq z \leq z_4, \)

\[ F(z, t) = 1 - \frac{1}{\zeta} \int_{\kappa_a}^{\kappa_f} \sqrt{\frac{1}{t^2} \cos^{-1}(z)^2 - \theta_1^2} d\theta_1; \]

(v) For \( z \geq z_4, \)

\[ F(z, t) = 1. \]
The following parameter values have been used: \( \kappa_i = 80, \kappa_f = 100, \alpha_i = 0 \) and \( \alpha_f = 10 s^{-1} \), and then \( \zeta = 200 s^{-2} \). The intervals \( [\kappa_i, \kappa_f] \) and \( [\alpha_i, \alpha_f] \) have been divided respectively into 40 and 20 equal sub-intervals, so that the partition of \( \Lambda \) is made up of \( N = 800 \) rectangles. Furthermore, \( \Delta z = 1.25 \cdot 10^{-2} \) and \( \Delta t = 7.81 \cdot 10^{-5} \) s were used. The relative error calculated with the formula (34) is of the order of \( 4 \cdot 10^{-2} \). Fig. 6 compares the numerical results with those obtained explicitly, at the three instants considered.

6 Conclusions

From the results obtained in the cases examined it appears that the accuracy of the cumulative distribution function, calculated with the proposed method, is good. The relative numerical procedure can be implemented in any code that allows the solution of the dynamic system and can be easily extended to vector-valued random processes.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Fig. 6 Cumulative distribution function at \( t_1 = 0.025 \) s, \( t_2 = 0.05 \) s and \( t_3 = 0.075 \) s; comparison between the explicit and numerical solutions (Example 2)
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