A perturbation theory approach to the ground state exciton energy in the limit of a weak magnetic field in anomalous exciton Hall effect.

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Abstract.

The anomalous exciton Hall effect is a phenomenon that occurs in a quantum well in the presence of an external magnetic field applied perpendicular to the surface due to the interaction of the exciton dipole moment with an electric field, formed by the charged impurities. The effect was fully described in [1] for different magnetic field regimes. In this paper, we focus on the way the perturbation method was used for finding the ground state energy of an exciton in the limit of a weak magnetic field.

1. Introduction

Excitons, propagating in the presence of an external magnetic field orthogonal to their velocity, acquire an electric dipole moment perpendicular to both the magnetic field and their propagation direction [2]. The existence of a dipole moment causes the excitons to have a nonzero asymmetric scattering rate while interacting with charged impurities. The $T$-matrix formalism was exploited to describe the symmetric and asymmetric contributions to the scattering rates. The problem then boiled down to the calculation of $T$-matrix elements, which satisfy the Lippmann–Schwinger equation:

$$T_{k,k'} = \hat{V}_{k,k'} + \int \frac{d^2g}{(2\pi)^2} \frac{\hat{V}_{k,g} T_{g,k'}}{E - \epsilon(g) + i\nu}. \tag{1}$$

By virtue of the smallness of the magnetic field, one may desire to solve the Lippmann–Schwinger equation perturbatively:

$$T_{k,k'} \approx \hat{V}_{k,k'} + \delta T_{k,k'}, \text{ where } \delta T_{k,k'} = P \int \frac{d^2g}{(2\pi)^2} \frac{\hat{V}_{k,g} \hat{V}_{g,k'}}{E - \epsilon(g)} - \frac{i\nu}{2} \int_0^{2\pi} \int \hat{V}_{k,g} \hat{V}_{g,k'} d\phi_g. \tag{2}$$
The goal of this paper is to find the ground state energy $\epsilon(\mathbf{g})$ of an exciton with a wave vector $\mathbf{g}$ in the presence of a magnetic field: $\nu = |\partial\epsilon(\mathbf{k})/\partial k|^{-1}k/(2\pi)$; $\hat{V}$ is the exciton–impurity scattering potential.

However, the perturbation theory approach turned out to be invalid in this case, since the calculated first-order correction was of the same order as the main contribution. Nevertheless, being inapplicable for solving the Lippmann–Schwinger equation, the method can be used for the calculation of the ground state energy $\epsilon(\mathbf{g})$ and the matrix elements of the scattering potential. In the following sections, the steps are presented for finding an analytical expression of the ground state energy of an exciton in the limit of a weak magnetic field using the perturbation method. During the calculations, several infinite sums appear, which are computed using an elegant method from [3].

2. Formulation of the problem

(i) We start by writing the unperturbed Hamiltonian of the relative electron–hole motion:

$$\hat{H}^{(0)} = -\frac{\hbar^2}{2\mu} \nabla_r^2 - \frac{e^2}{4\pi \varepsilon_0 r} + \frac{\hbar^2 k^2}{2M}$$

(3)

Its wave functions $\Phi_{m_1,m_2}(\mathbf{r})$ ( $\Phi_{m}(\mathbf{r}) = |m\rangle$) consist of a coordinate ($R_{m_1}(\mathbf{r})$) and an angular ($\Theta_{m_2}(\theta)$) part and can be found in [4]. The ground state wave function and energy read ($a_B = 4\pi \varepsilon_0 \hbar^2/(\mu e^2)$ is the Bohr radius):

$$\Phi_{m_0}(\mathbf{r}) = \left(\frac{8}{\pi}\right)^{1/2} \frac{1}{a_B} \exp \left(-\frac{2r}{a_B}\right); \quad \langle m_0 \rangle(k) = -\frac{\mu e^4}{2}\frac{1}{\hbar^2} \frac{1}{8\pi^2 \varepsilon_0^2} + \frac{\hbar^2 k^2}{2M} = \frac{\hbar^2 k^2}{2M} - \epsilon_0$$

(4)

(ii) The magnetic field affects the system through the perturbation potential $\hat{V}$, which assumed to be small:

$$\hat{V} = -\frac{i\hbar eB}{2} \left[ \frac{1}{m_e} - \frac{1}{m_h} \right] [\mathbf{r} \times \nabla_r]_z + \frac{e^2 B^2}{8\mu r^2} + \frac{e\hbar}{M} \hat{B} \cdot [\mathbf{r} \times \hat{k}]$$

(5)

(iii) The goal of this paper is to find the ground state energy $\epsilon(\mathbf{k})$ using a perturbation theory approach.

3. Calculation of the ground state energy

(i) Let us expand the ground energy up to the second order in the magnetic field strength $B$:

$$\epsilon(k) = \langle m_0 |\hat{V}| m_0 \rangle + \langle m_0 |\hat{V}| m_0 \rangle + \sum_{m \neq m_0} \frac{|\langle m_0 |\hat{V}| m_0 \rangle|^2}{\epsilon_{m_0}(k) - \epsilon_{m_0}(k)}$$

(6)

(ii) One may notice, that: $\langle m_0 |\hat{X}| m_0 \rangle = 0$ and $\langle m_0 |\hat{L}_z| m_0 \rangle = 0$. Thus (we omit dependence on $k$ in the energy for brevity; $B \parallel Oz$):

$$\langle m_0 |\hat{V}| m_0 \rangle = \frac{e^2 B^2}{8\mu} \langle m_0 |\hat{X}^2| m_0 \rangle = \frac{e^2 B^2}{8\mu} \cdot \frac{3a_B^2}{8} = \frac{3e^2 a_B^2 B^2}{64\mu}$$

(7)

$$\sum_{m \neq m_0} \frac{|\langle m_0 |\hat{V}| m_0 \rangle|^2}{\epsilon_{m_0}(k) - \epsilon_{m_0}(k)} = \sum_{m \neq m_0} \frac{|\langle m_0 |\hat{X}^2| m_0 \rangle|^2}{8\mu} = \frac{(e^2 B^2)^2}{8\mu} \sum_{m \neq m_0} \frac{1}{\epsilon_{m_0}(k) - \epsilon_{m_0}(k)}$$

(8)

One can see that the analytical solution requires a calculation of several infinite sums. The smart way to perform that is presented in next subsections.
3.1. First auxiliary task
First, we want to concentrate on the following problem: how a hermitian operator \( \hat{b}_1 \), satisfying the relation \( i\mu[\hat{H}_0, \hat{b}_1] = \hat{x} \), acts on \( \Phi_{m_0}(r) \).

(i) Let us introduce \( b_1(r) \) as \( \hat{b}_1 \Phi_{m_0}(r) = b_1(r)\Phi_{m_0}(r) \). Then:

\[
i\mu[\hat{H}_0, \hat{b}_1]\Phi_{m_0}(r) = \hat{x}\Phi_{m_0}(r)
\]

\[
\Rightarrow -\frac{i\hbar^2}{2} \left\{ \left[ \nabla^2 b_1(r) \right] \Phi_{m_0}(r) + 2 \left[ \nabla_r b_1(r) \right] \cdot \left[ \nabla_r \Phi_{m_0}(r) \right] \right\} = x\Phi_{m_0}(r)
\]

(ii) The solution for the equation above is:

\[
b_1(r) = -i\frac{a^2}{\hbar^2} \left[ \left( \frac{r^2}{4a^2} + \frac{3r}{16a} \right) \cos \phi + C \right]
\]

with \( C \) being an arbitrary constant.

3.2. Second auxiliary task
Now let us focus on the similar problem: how a hermitian operator \( \hat{b}_2 \), satisfying the relation \( i\mu[\hat{H}_0, \hat{b}_2] = \hat{r}^2 \), acts on \( \Phi_{m_0}(r) \).

(i) We introduce \( b_2(r) \) as \( \hat{b}_2 \Phi_{m_0}(r) = b_2(r)\Phi_{m_0}(r) \). Then, following exactly the same steps, we obtain the equation on \( b_2(r) \):

\[
-\frac{i\hbar^2}{2} \left\{ \left[ \nabla^2 b_2(r) \right] \Phi_{m_0}(r) + 2 \left[ \nabla_r b_2(r) \right] \cdot \left[ \nabla_r \Phi_{m_0}(r) \right] \right\} = \hat{r}^2\Phi_{m_0}(r)
\]

(ii) The solution reads:

\[
b_2(r) = \frac{i\hbar^4}{2\hbar^2} \left[ \left( \frac{r}{a_B} \right)^3 + \frac{3}{8} \left( \frac{r}{a_B} \right)^2 + \frac{3}{8} \frac{r}{a_B} + \frac{3}{32} \ln \left( \frac{r}{a_B} \right) + C \right]
\]

with \( C \) being an arbitrary constant.

3.3. Calculation of the infinite sums
Using the results of the previous section, we can find analytical expressions for several sums, which appear during energy calculation.

(i) From the relations operators \( \hat{b}_1 \) and \( \hat{b}_2 \) satisfy, one obtains:

\[
i\mu \left( \epsilon^{(0)}_{m_0} - \epsilon^{(0)}_m \right) \langle m_0 | \hat{b}_1 | m \rangle = \langle m_0 | \hat{x} | m \rangle; \quad i\mu \left( \epsilon^{(0)}_{m_0} - \epsilon^{(0)}_m \right) \langle m_0 | \hat{b}_2 | m \rangle = \langle m_0 | \hat{r}^2 | m \rangle
\]

(ii)

\[
I_1 = \sum_{m \neq m_0} \frac{|\langle m_0 | \hat{x} | m \rangle|^2}{(\epsilon^{(0)}_{m_0} - \epsilon^{(0)}_m)^2} = -\sum_m \frac{|\langle m_0 | \hat{b}_1 | m \rangle \langle m | \hat{b}_1 | m_0 \rangle|^2}{(\epsilon^{(0)}_{m_0} - \epsilon^{(0)}_m)^2} \cdot (i\mu)^2 \left( \epsilon^{(0)}_{m_0} - \epsilon^{(0)}_m \right)^2
\]

\[
= \mu^2 \langle m_0 | \hat{b}_2^2 | m_0 \rangle = \frac{159\alpha^6 \mu^2}{4096\hbar^4}
\]
\[ I_2 = \sum_{m \neq m_0} \frac{|\langle m_0 | \hat{x} | m \rangle|^2}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} = \sum_m \frac{\langle m_0 | \hat{x} | m \rangle \langle m | \hat{b}_1 | m_0 \rangle}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} \cdot (-i\mu) \left( \epsilon^{(0)}_m - \epsilon^{(0)}_m \right) \]
\[ = -i\mu \langle m_0 | \hat{x} \hat{b}_1 | m_0 \rangle = -\frac{21a_B^4\mu}{256\hbar^2} \]
\[ I_3 = \sum_{m \neq m_0} \frac{|\langle m_0 | \hat{p}^2 | m \rangle|^2}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} = \sum_{m \neq m_0} \frac{\langle m_0 | \hat{p}^2 | m \rangle \langle m | \hat{b}_2 | m_0 \rangle}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} \cdot (-i\mu) \left( \epsilon^{(0)}_m - \epsilon^{(0)}_m \right) \]
\[ = -i\mu \left( \langle m_0 | \hat{p}^2 \hat{b}_2 | m_0 \rangle - \langle m_0 | \hat{p}^2 | m_0 \rangle \langle m_0 | \hat{b}_2 | m_0 \rangle \right) = -\frac{105a_B^6\mu}{512\hbar^2} \]

3.4. Completion of the calculations

(i) Using the calculation of the sums from the previous section, 8 can be simplified:

\[
\begin{align*}
(a) & \quad \left( \frac{e^2B^2}{8\mu} \right)^2 \sum_{m \neq m_0} \left| \frac{\langle m | \hat{p}^2 | m_0 \rangle}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} \right|^2 = -\left( \frac{e^2B^2}{8\mu} \right)^2 \cdot \frac{105a_B^6\mu}{512\hbar^2} \sim B^4 \Rightarrow \text{we neglect this term} \\
(b) & \quad \left( \frac{eB}{M} \right)^2 \sum_{m \neq m_0} \left| \frac{\langle m | xk_y - yk_x | m_0 \rangle}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} \right|^2 = \left( \frac{eB}{M} \right)^2 \sum_{m \neq m_0} \left| \frac{\langle m | x | m_0 \rangle}{\epsilon^{(0)}_m - \epsilon^{(0)}_m} \right|^2 \\
& \quad = \left( \frac{eB}{M} \right)^2 \cdot \left( -\frac{21a_B^4\mu}{256\hbar^2} \right) 
\end{align*}
\]

(ii) Having all the necessary calculations done, the analytical expression for the ground state energy in a weak magnetic field limit can be found, using the perturbation approach:

\[ \epsilon(k) = \epsilon^{(0)}_{m_0}(k) + \frac{3e^2a_B^2B^2}{64\mu} - \left( \frac{eBk}{M} \right)^2 \frac{21a_B^4\mu}{256\hbar^2} \quad (12) \]

Introducing the magnetic length \( l_B = \sqrt{\hbar/(eB)} \) and \( l_2 = (3/128)^{1/4}a_B \), the energy can be represented in the form:

\[ \epsilon(k) = \frac{\hbar^2k^2}{2M} \left[ 1 - 2|\kappa| \left( \frac{a_B}{l_B} \right)^4 \right] - \epsilon_0 \left[ 1 - \left( \frac{l_2}{l_B} \right)^4 \right] \quad (13) \]

4. Conclusion

A perturbation theory approach is not applicable for solving the Lippmann–Schwinger equation, but it allows us to find the ground state energy and the scattering matrix elements analytically. They in turn are substituted into the Lippmann–Schwinger equation, which was solved numerically in the main work.

It is interesting to note that the obtained expression for \( \epsilon(k) \), having the same form as in [3], actually differs from the latter in the constant \( l_2 \).

References

[1] Kozin V K, Shabashov V A, Kavokin A V and Shelykh I A 2021 Phys. Rev. Lett. 126 036801
[2] Thomas D G and Hopfield J J 1961 Phys. Rev. 124 657
[3] Arseev P I, Dzyubenko A B 1998 J. Exp. Theor. Phys. 87 200–9
[4] Yang X L, Guo S H, Chan F T, Wong K W and Ching W Y 1991 Phys. Rev. A 43 1186