Elementary Darboux transformations and factorization

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Abstract

A general theorem on factorization of matrices with polynomial entries is proven and it is used to reduce polynomial Darboux matrices to linear ones. Some new examples of linear Darboux matrices are discussed.

1 Introduction

In the continuous case, for spectral problems with \( m \times m \) matrix potentials \( U \) polynomial in \( \lambda \)

\[
\frac{d}{dx} \Psi = U(x, \lambda) \Psi
\]

the construction of Darboux transformation

\[
\Psi \rightarrow \hat{\Psi} = Q(x, \lambda) \Psi, \quad \frac{d}{dx} Q = \hat{U} Q - QU \Rightarrow \frac{d}{dx} \hat{\Psi} = \hat{U}(x, \lambda) \hat{\Psi}
\]

can be based on the elementary Darboux transformations, for which

\[
Q(x, \lambda) = (\lambda - \alpha)P(x) + (\lambda - \beta)(E - P(x)), \quad P^2 = P.
\]

Here, \( E \) denotes the identity matrix and the projector \( P \) is uniquely defined by the nullspaces of \( Q \) :

\[
\text{Ker } Q|_{\lambda=\alpha} = \text{Im } P = M, \quad \text{Ker } Q|_{\lambda=\beta} = \text{Ker } P = N
\]
If \( v(x) \in M \) and \( w(x) \in N \), then the following equations have to be satisfied:

\[
\frac{d}{dx} v(x) = U(x, \alpha) v(x), \quad \frac{d}{dx} w(x) = U(x, \beta) w(x).
\] (4)

The zeroes of the determinant define the eigenvalues \( \lambda = \alpha, \beta \) introduced by the Darboux transformation (3). In particular, for the Zakharov-Shabat spectral problems (see, e.g., [1]),

\[
\hat{U} = \lambda \sigma + q(x), \quad q(x) = [\sigma, \tilde{q}(x)],
\] (5)

where \( \sigma \) is a constant matrix with simple eigenvalues, the function \( \hat{\Psi} \) satisfies the equation (1) with the potential

\[
U = \hat{U} + [Q, \sigma].
\]

The case \( \beta = \bar{\alpha}, \quad P^* = P \) where \( * \) means hermitian conjugation, corresponds to the self-adjoint potentials

\[
U^*(x, \bar{\lambda}) = U(\lambda, x), \quad q^* = q.
\]

Main Theorem in Section 2 \(^3\) formulates the necessary and sufficient conditions on the \( N \)th degree matrix polynomial \( A(\lambda) \) to be represented as the product of \( N \) first order polynomials (3), (6).

In application to Darboux transformations this factorization \( A = Q_N \cdots Q_1 \) gives rise the sequence of spectral problems (1)

\[
\frac{d}{dx} \Psi_j = U_j(x, \lambda) \Psi_j, \quad \Psi_{j+1} = Q_j(x, \lambda) \Psi_j,
\] (7)

related with each other by the transformations (2),(3). Observe that the composition of Darboux transformations \( \Psi_{j+1} = B_j \Psi_j \) is also a Darboux transformation since (Cf. (2))

\[
B_{j,x} = U_{j+1} B_j - B_j U_j, \quad B \overset{def}{=} B_n \cdots B_1 \Rightarrow B_x = U_{n+1} B - B U_1.
\] (8)

For a scalar differential operator \( A \), the elementary Darboux transformation \( \Psi \to \hat{\Psi} \) is defined by the first order differential operator:

\[
A \Psi = \lambda \Psi, \quad \hat{\Psi} = (D - f) \Psi, \quad \hat{A} \hat{\Psi} = \lambda \hat{\Psi} \quad \hat{A} = (D - f) A (D - f)^{-1}.
\]

For the operator \( A \) of order \( N \), an analog of Main Theorem yields the relation

\[
A \Psi = \frac{\langle \varphi_1, \ldots, \varphi_N, \Psi \rangle}{\langle \varphi_1, \ldots, \varphi_N \rangle} = (D - f_N) \cdots (D - f_2)(D - f_1) \Psi,
\] (9)

where \( \varphi_j \in \text{Ker} \ A \) and

\[
\langle \varphi_1, \ldots, \varphi_l \rangle \overset{def}{=} \det(D^{k-1}(\varphi_j)), \quad j, k = 1, \ldots, l, \quad D = \frac{d}{dx}.
\]

\(^3\)An analog of this theorem was recently used by [7] in order to construct an approximate solution of Riemann-Hilbert problem. See also [6].
The proof of (9) is obtained by induction (Cf. Main Theorem) based upon the identity valid for any smooth functions $\varphi_j$:

$$<\varphi_1,\ldots,\varphi_m> = \varphi_1 <\hat{\varphi}_2,\ldots,\hat{\varphi}_m>, \quad \hat{\varphi}_j = (D - f_1)\varphi_j, \quad f_1 = D\log \varphi_1.$$ 

In eq. (9) the factors $D - f_j$ are irreducible (they are first order differential operators). Such property appears to be crucial in the theory of dressing chains [2], [3], [4], where the functions $f_j$ are used as basic dynamical variables. On the other hand, the Darboux transformation (3) can be factorized into the composition of two more elementary ones with null spaces Ker $P$ and Im $P$, respectively. This factorization reformulated in terms of the corresponding matrices $B_1$ and $B_2$ gives rise to the following equations:

$$\Psi(m + 1,n) = B_1(m,n)\Psi(m,n), \quad \Psi(m,n + 1) = B_2(m,n)\Psi(m,n),$$

$$\Psi(m + 1,n + 1) = Q(m,n)\Psi(m,n), \quad Q = B_1 \circ B_2,$$

where by definition

$$Q = B_1 \circ B_2 \Leftrightarrow Q(m,n) = B_1(m,n + 1)B_2(m,n) = B_2(m + 1,n)B_1(m,n). \quad (10)$$

Equations (10) show that, unlike the continuous case (8), the Darboux transformations and the original discrete spectral problem have the same nature and it is possible to replace one with the other.

Thus, the problem of factorization of (3), considered in Section 3, brings up the discrete spectral problems with potentials linear in the spectral parameter $\lambda$. In the examples, we will restrict ourself to the simplest $2 \times 2$ case but without the assumption that $Q$ is factorized through orthogonal projectors (see (3)). In our considerations, the continuous independent variable $x$ will play the role of a parameter.

In the last subsection we consider solutions of the Yang-Baxter equation linear in the spectral parameter $\lambda$.

2 Main Theorem

In order to prove that the Darboux transformations (3) are generic ones, it sufficies to prove that the product of $N$ matrices (3) yields an $N$-th degree polynomial in $\lambda$ of a general form. We will use the characteristic property of the polynomials (3) as follows

$$Q_{\alpha\beta} \overset{\text{def}}{=} (\lambda - \alpha)P + (\lambda - \beta)(E - P) \Rightarrow Q_{\alpha\beta}Q_{\beta\alpha} = (\lambda - \alpha)(\lambda - \beta)E. \quad (11)$$

In case (6), we have

$$Q(\bar{\lambda})^*Q(\lambda) = (\lambda - \alpha)(\lambda - \bar{\alpha})E$$

, and such property holds, obviously, for the product of matrices (3) with the orthogonal projectors $P^* = P$. 

3
Theorem. Let $A(\lambda)$ be an $m \times m$ matrix with polynomial entries in $\lambda$ satisfying the conditions:

$$A(\bar{\lambda})^* A(\lambda) = \prod_{j=1}^{N} (\lambda - \alpha_j)(\lambda - \bar{\alpha}_j)E,$$

$$\lim_{\lambda \to \infty} \frac{A(\lambda)}{\lambda^N} = E. \quad (13)$$

Then $A(\lambda)$ can be factorized as

$$A(\lambda) = \prod_{j=1}^{N} \left( (\lambda - \alpha_j)P_{\alpha_j} + (\lambda - \bar{\alpha}_j)(E - P_{\alpha_j}) \right), \quad (14)$$

where the $P_{\alpha_j}$ are Hermitian projectors.

Proof: We proceed by induction on $N$. If $N = 0$, then from (13) and the Liouville theorem it follows that $A(\lambda) = E$.

Now we show that if the statement of our theorem holds for $N - 1$, it holds for $N$ as well. From (12) and (13) it follows:

$$\det(A(\lambda)) = \prod_{j=1}^{N} (\lambda - \alpha_j)^{k_j}(\lambda - \bar{\alpha}_j)^{m - k_j}. \quad (15)$$

We select a zero $\alpha_j$ of $\det(A(\lambda))$; from (15) it follows:

$$\dim(\text{Ker}(A(\alpha_j))) = k_j,$$

$$\dim(\text{Ker}(A(\bar{\alpha}_j))) = m - k_j. \quad (16)$$

Let $|v_i\rangle_{i=1}^{k_j}$ and $|w_i\rangle_{i=1}^{N-k_j}$ be an orthonormal basis for $\text{Ker}(A(\alpha_j))$ and $\text{Ker}(A(\bar{\alpha}_j))$, respectively. We denote by $P_{\alpha_j}$ and $P_{\bar{\alpha}_j}$ the Hermitian projectors

$$P_{\alpha_j} \equiv \sum_{i=1}^{k_j} |v_i\rangle \langle v_i|, \quad (17)$$

$$P_{\bar{\alpha}_j} \equiv \sum_{i=1}^{m-k_j} |w_i\rangle \langle w_i|. \quad (18)$$

Let us prove that

$$P_{\bar{\alpha}_j} = (E - P_{\alpha_j}). \quad (19)$$

To prove (19) is equivalent to proving that $\mathbb{C}^m$ admits an orthogonal decomposition

$$\mathbb{C}^m = \text{Ker}(A(\alpha_j)) \oplus \text{Ker}(A(\bar{\alpha}_j)). \quad (20)$$

From (12) it follows:

$$A(\alpha_j)A(\bar{\alpha}_j)^* = 0, \quad (21)$$

that is

$$\text{Im}(A(\bar{\alpha}_j)^*) \subset \text{Ker}(A(\alpha_j)). \quad (22)$$
but from (16) we have:
\[
\text{dim}(\text{Im}(A(\bar{\alpha}_j)^*)) = m - \text{dim}(\text{Ker}(A(\bar{\alpha}_j))) = k_j,
\]
\[
\text{dim}(\text{Ker}(A(\alpha_j))) = k_j,
\]
whence it follows:
\[
\text{Im}(A(\bar{\alpha}_j)^*) = \text{Ker}(A(\alpha_j)).
\] (23)

Equation (23) is equivalent to
\[
\text{Ker}(A(\bar{\alpha}_j)) = \text{Im}(A(\alpha_j)^*)
\] (24)
whence it follows
\[
\mathbb{C}^m = \text{Ker}(A(\alpha_j)) \oplus \text{Im}(A(\alpha_j)^*) = \text{Ker}(A(\alpha_j)) \oplus \text{Ker}(A(\bar{\alpha}_j)).
\]

Now the matrix
\[
A(\lambda) \left[ (\lambda - \bar{\alpha}_j)P_{\alpha_j} + (\lambda - \alpha_j)(E - P_{\alpha_j}) \right]
\] (25)
vanishes at \( \lambda = \alpha_j \) and \( \lambda = \bar{\alpha}_j \). Consequently, we have:
\[
A(\lambda) \left[ (\lambda - \bar{\alpha}_j)P_{\alpha_j} + (\lambda - \alpha_j)(E - P_{\alpha_j}) \right] = (\lambda - \alpha_j)(\lambda - \bar{\alpha}_j)\hat{A}(\lambda)
\] (26)
for a certain matrix \( \hat{A}(\lambda) \). Then
\[
A(\lambda) = \hat{A}(\lambda) \left[ (\lambda - \alpha_j)P_{\alpha_j} + (\lambda - \bar{\alpha}_j)(E - P_{\alpha_j}) \right]
\] (27)

Since \( P_{\alpha_j} \) is Hermitian, it follows that \( \hat{A}(\lambda) \) satisfies
\[
\hat{A}(\lambda)^*\hat{A}(\lambda) = \prod_{k \neq j}^N (\lambda - \alpha_k)(\lambda - \bar{\alpha}_k)E.
\] (28)

Moreover,
\[
\lim_{\lambda \to \infty} \frac{A(\lambda)}{\lambda^N} = \lim_{\lambda \to \infty} \frac{\hat{A}(\lambda)}{\lambda^{N-1}} = E.
\] (29)

In the case of distinct zeroes \( \alpha_j \neq \alpha_k \), the orthogonal projectors \( P_{\alpha_j} \) in (14) are uniquely defined by the subspaces
\[
M_j \overset{\text{def}}{=} \text{Ker} A(\lambda) |_{\lambda = \alpha_j}
\] (30)
and an ordering of zeroes \( \alpha_1, \ldots, \alpha_N \). Indeed, for instance, for \( N = 2 \), we have
\[
A(\lambda) = Q_1(\lambda)Q_2(\lambda), \quad Q_2(\lambda) = (\lambda - \alpha_j)P_{\alpha_j} + (\lambda - \bar{\alpha}_j)(E - P_{\alpha_j}), \quad j = 1, 2;
\]
with \( P_{\alpha_j} = P_{\alpha_j} \) and
\[
\text{Im} P_{\alpha_2} = \text{Ker}(E - P_{\alpha_2}) = \text{Ker} Q_2(\alpha_2) = \text{Ker} Q_1(\alpha_2)Q_2(\alpha_2) = M_2
\]
, since \( \det Q_1(\alpha_2) \neq 0 \). Similarly
\[
\text{Im} P_{\alpha_1} = Q_2(\alpha_1)M_1, \quad \text{since} \quad M_1 = \text{Ker}(E - P_{\alpha_1})Q_2(\alpha_1).
\]

In order to construct a dressing chain, we have to factorize (see the next section) the basic Darboux transformation (11) further:
\[
Q_{\alpha\beta}(\lambda) = (\lambda - \alpha)P_{\alpha\beta} + (\lambda - \beta)(E - P_{\alpha\beta}) = B_2(\lambda)B_1(\lambda)
\]
where these two factors \( B_j(\lambda) = \sigma_j \lambda + b_j \) are linear in \( \lambda \) and
\[
\det Q_{\alpha\beta}(\lambda) = (\lambda - \alpha)^k(\lambda - \beta)^{m-k} \Rightarrow \det B_1 = (\lambda - \alpha)^k, \quad \det B_2 = (\lambda - \beta)^{m-k}.
\]
Evidently,
\[
\text{Ker} B_1(\beta) = \text{Im} B_2(\alpha) = \text{Ker} P_{\alpha\beta},
\]

Since, due to (32) and the fact that \( \det B_1(\alpha) \neq 0 \), we have
\[
Q_{\alpha\beta}(\alpha) = (\alpha - \beta)(E - P_{\alpha\beta}) = B_2(\alpha)B_1(\alpha) \Rightarrow \text{Im} B_2(\alpha) = \text{Im}(E - P_{\alpha\beta}) = \text{Ker} P_{\alpha\beta}.
\]

### 3 \( 2 \times 2 \) case

In this section we consider some interesting, from our point of view, classes of \( 2 \times 2 \) matrices linear in \( \lambda \) related with (3). First, let us consider the chain of matrices \( B_j, j \in \mathbb{Z} \):
\[
B_j = \begin{pmatrix}
\lambda + h_j & \frac{1}{2} u_j \\
\frac{1}{2} v_{j+1} & 1
\end{pmatrix}, \quad h_j = \frac{1}{4} u_j v_{j+1} - \beta_j \quad \text{for} \quad j \quad \text{odd},
\]
\[
B_j = \begin{pmatrix}
1 & -\frac{1}{2} u_j \\
-\frac{1}{2} v_j & \lambda + h_j
\end{pmatrix}, \quad h_j = \frac{1}{4} v_j u_{j+1} - \beta_j \quad \text{for} \quad j \quad \text{even}.
\]

Thus
\[
\det B_j = \lambda - \beta_j \quad \text{for all} \quad j
\]
and (Cf (33)) it easy to see that
\[
\text{Ker} B_1|_{\lambda=\beta_1} = \text{Im} B_2|_{\lambda=\beta_2}, \quad \text{Ker} B_2|_{\lambda=\beta_2} = \text{Im} B_3|_{\lambda=\beta_3}, \ldots
\]

Moreover, these matrices satisfy the consistency conditions (10) (see Lemma below) and
\[
B_2B_1 = \lambda - \beta_1 + \beta_{12}P_{12}, \quad B_3B_2 = \lambda - \beta_2 + \beta_{23}P_{23}, \ldots, \quad \beta_{ij} \overset{\text{def}}{=} \beta_i - \beta_j,
\]
where \( P_{ij}^2 = P_{ij} \) are projectors. In particular,
\[
\beta_{12}P_{12} = \begin{pmatrix}
\frac{1}{2} u_1 v_{13} & \frac{1}{2} u_1 u_{13} \\
\frac{1}{2} v_2 (\beta_{12} - \frac{1}{4} u_2 u_{13}) & \beta_{12} - \frac{1}{4} u_2 u_{13}
\end{pmatrix}, \quad u_{13} \overset{\text{def}}{=} u_1 - u_3,
\]
\[
\beta_{23}P_{23} = \begin{pmatrix}
\beta_{23} + \frac{1}{4} u_2 v_{42} & \frac{1}{2} u_3 (\beta_{23} + \frac{1}{4} u_2 v_{24}) \\
\frac{1}{2} v_3 & \frac{1}{4} u_3 v_{24}
\end{pmatrix}, \quad v_{ij} \overset{\text{def}}{=} v_i - v_j.
\]
Thus these matrices provide us with a factorization of elementary Darboux transformations (3). Observe that, for odd \( i \), the product \( B_j B_i \), where \( j = i + 1 \), depends only on \( v_j, u_{ij} = u_i - u_j \) for \( i \) odd and on the difference \( v_{i+2} - v_{i+1} \) and \( u_{i+1} \) for \( i \) even.

In comparison with the factorization defined by main Theorem, the entries of matrices (34) and (35) are expressed directly in terms of the potentials

\[
U_j = \begin{pmatrix} \lambda & u_j \\ v_j & -\lambda \end{pmatrix} = \lambda \sigma + q_j
\]

of the 2 \times 2 Zakharov-Shabat spectral problem

\[
\psi^1_x - \lambda \psi^1 = u \psi^2, \quad \psi^2_x + \lambda \psi^2 = v \psi^1, \quad \text{or} \quad \psi_x = U \psi. \tag{36}
\]

Thus, the substitution of matrices (34) and (35) into the equations (8) yields the dressing chain for the spectral problem (36):

\[
(D + 2 \beta_j) v_{j+1} + 2v_j = \frac{1}{2} u_j v_{j+1}^2, \quad (D - 2 \beta_j) u_j - 2u_{j+1} = -\frac{1}{2} u_j^2 v_{j+1}, \tag{37}
\]

\[
(D - 2 \beta_j) u_{j+1} - 2u_j = -\frac{1}{2} u_{j+1}^2 v_j, \quad (D + 2 \beta_j) v_j + 2v_{j+1} = \frac{1}{2} u_{j+1} v_j^2. \tag{38}
\]

The first case corresponds to (34) and the second one to (35).

**Remark.** In the case of the general Zakharov-Shabat spectral problem (5) one could consider the following Darboux transformations (8)

\[
B_j = \lambda \sigma_j + b_j, \quad U_j = \lambda \sigma + q_j.
\]

Collecting quadratic and linear terms in \( \lambda \) in (8) one finds

\[
[\sigma, \sigma_j] = 0, \quad \sigma_{j,x} = [\sigma, A_j] + q_{j+1} \sigma_j - \sigma_j q_j.
\]

That implies that \( \sigma_{j,x} = 0 \) since the potentials \( q_j \) are off-diagonal in the basis where the matrix \( \sigma \) is diagonal. Thus, the matrices \( \sigma_j \) should be \( x \)-independent and could be any diagonal matrix in that basis.

Coming back to the spectral problem (36) we will show how, starting with \( (u_1, v_1) \), one construct a chain of Darboux matrices \( B_j \), transforming the equation (36) with the potential \( U_j \) into one with the potential \( U_{j+1} \), using the solution to the corresponding spectral problem Riccati equation:

\[
\rho_x = 2\lambda \rho - v_j \rho^2 + u_j, \quad \rho \overset{\text{def}}{=} \frac{\psi^1}{\psi^2}. \tag{39}
\]

This should highlight a similarity with the dressing chain considered in [2].
Proposition. There is a one-to-one correspondence between the matrices (34), (35) and solutions of the Riccati equation (39) with $\lambda = \beta_j$.

Proof: At the first step we have to find the potentials $u_2$ and $v_2$. The function $v_2$ is defined by (37) as solution of ODE

$$(D + 2\beta_1)v_2 + 2v_1 = \frac{1}{2}u_1v_2^2$$

which is reduced by rescaling $v_2 = 2\rho$ to (39) with $\lambda = \beta_1$. For the function $u_2$, we obtain now the exact formula

$$2u_2 = u_{1,x} - 2\beta_1u_1 + \frac{1}{2}u_1^2v_2.$$

At the next step, we find $u_3 = 2\rho^{-1}$ by solving (39) with $\lambda = \beta_2$. It satisfies the corresponding equation (38):

$$(D - 2\beta_2)u_3 - 2u_2 = \frac{1}{2}u_2^2v_2$$

while $v_3$ is defined by the second equation in (38). Thus in both cases the system of equations (37) and (38) are reduced to the Riccati equation (39).

Consistency problem. We will ignore now the dependence on the continuous independent variable $x$ and consider more general (as compared with (34), (35)) matrices linear in $\lambda$ and satisfying conditions analogous to (10). Having applications in mind, we will denote one of the matrices in (10) as $A(n, \lambda)$ and consider the second one as a Darboux transformation for the discrete spectral problem defined by $A$:

$$\Psi_{n+1} = A(n, \lambda)\Psi_n, \quad A = \left( \begin{array}{cc} \lambda + c & a \\ b & \alpha \end{array} \right), \quad \hat{\Psi} = B\Psi, \quad \hat{\Psi}_{n+1} = \hat{A}(n, \lambda)\hat{\Psi}. \quad (40)$$

If the spectral problem in (40) is a “good” one, then there should exist nontrivial Darboux transformations with

$$B = \left( \begin{array}{cc} \varepsilon & f \\ g & h - \lambda \end{array} \right) \quad (41)$$

and the consistency equations $T_1 \circ T_2 = T_2 \circ T_1$, where $T_1 \Psi = A\Psi$, $T_2 \Psi = B\Psi$ (Cf. (10)) yield

$$(\lambda + \hat{c})\varepsilon + g\hat{a} = \varepsilon'(\lambda + c) + f'\hat{b}, \quad (\lambda + \hat{c})f + \hat{a}(h - \lambda) = \varepsilon'a + f'\alpha$$

$$\varepsilon\hat{b} + \hat{a}g = (\lambda + c)g' + b(h' - \lambda), \quad \hat{a}(h - \lambda)\varepsilon + f\hat{b} = \alpha(h' - \lambda) + g'a$$

where $B' \overset{\text{def}}{=} T_1(B)$ and $\hat{A} \overset{\text{def}}{=} T_2(A)$. We see that

$$g' = T_1(g) = b, \quad \hat{a} = T_2(a) = f, \quad \varepsilon' = \varepsilon, \quad \hat{a} = \alpha$$

and thus the consistency equations are reduced to

$$\left\{ \begin{array}{ll} b(h' + c) = \varepsilon\hat{b} + \alpha g, & \varepsilon(\hat{c} - c) = f'g' - fg \\ \hat{a}(h + \hat{c}) = \varepsilon a + \alpha f', & \alpha(h' - h) = \hat{a}b - ab \end{array} \right. \quad (42)$$

It is easy to verify now

4The particular case of $A$ with $\alpha = 0$ is related with Toda chain.
Lemma. The matrices $B_j$ and $B_{j+1}$ defined by (34), (35) satisfy the consistency equations (42) for any $j$.

Generally we find that (Cf. (10))

$$A \circ B = \begin{pmatrix} \varepsilon (\lambda - \delta + h^\dagger) & \hat{a}h^\dagger \\ bh^\dagger & \alpha (h^\dagger - \lambda + \beta) \end{pmatrix}$$

where

$$h^\dagger = \hat{c} + h = c + h', \quad ab = \alpha (c + \beta), \quad fg = \varepsilon (h - \delta), \det A = \alpha (\lambda - \beta_1), \quad \det B = \varepsilon (\beta_2 - \lambda)$$

and thus our Lemma corresponds to the case $\varepsilon + \alpha = 0$.

The Darboux matrix $B$ in (40) is reciprocal to $A$ and (unlike the continuous case) one can replace it with $A \circ B$. This property allows us to use the matrix $A$ as a starting point in the spectral problem (40) with $\alpha = 0$ [3].

An $R$-matrix interlude. While working out examples for the above factorization theorem, we noticed that some of them were also solutions of the quantum Yang-Baxter equation

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu). \quad (43)$$

Here $R_{12} = R \otimes I\!d$, $R_{23} = I\!d \otimes R$ and $R_{13} = (I\!d \otimes P)R_{12}(I\!d \otimes P)$ acts trivially in the second of the three spaces, $P$ is the usual permutation operator:

$$P(u \otimes v) \overset{\text{def}}{=} v \otimes u.$$  

Thus in the simplest $2 \times 2$ case, (43) is an equation for a $4 \times 4$ matrix $R$ and a well known solution is given by

$$R(\lambda) = \begin{pmatrix} \lambda + i\eta & 0 & 0 & 0 \\ 0 & \lambda & i\eta & 0 \\ 0 & i\eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda + i\eta \end{pmatrix} = \frac{1}{2} [(\lambda + i\eta)P + (\lambda - i\eta)(1 - P)]. \quad (44)$$

and corresponds to the $XXX$–spin chain. Since $P$ is a projector, we can express it in the form:

$$P = \frac{|a\rangle\langle b|}{\langle a|b\rangle}, \quad \text{where} \quad 2|a\rangle = \langle b| = (0, 1, -1, 0) \quad (45)$$

We found that there exist two other possible choices of $|a\rangle$ and $\langle b|$ in (45) such that an $R$-matrix of the form

$$R(\lambda) = (\lambda + i\eta)P + (\lambda - i\eta)(1 - P)$$

satisfies the Yang-Baxter equation, namely:

$$|a\rangle = (\alpha^{-1}, 1, 1, \alpha), \quad \langle b| = (\beta^{-1}, 1, 1, \beta), \quad |a\rangle = (-1, \alpha, \beta, -\beta\alpha), \quad \langle b| = (\beta\alpha, \alpha, \beta, 1),$$

with $\alpha, \beta$ being two real parameters.
Having two new solutions of the Yang-Baxter equation, we can construct the associated Lax matrices and, finally, two monodromy matrices out of them. Then, taking the traces of the monodromy matrices we obtain two sets of commuting quantum observables. Unfortunately the elements of each set are functionally dependent. Hence, these two new solutions do not give rise to new quantum integrable systems.

Further investigations on Quantum Yang–Baxter equation showed that, if we generalize the $R$–matrix form (46) dropping the normalization condition (13), then other solutions, linear in the spectral parameter $\lambda$, can be found. We notice that $R$ can then be expressed in the form $R = \lambda A + B$, where $A$ and $B$ must be solutions to the two-dimensional constant quantum Yang-Baxter equation; all such solutions have been found by J. Hietarinta [5].

An example of solution that we believe could be of some interest is the following one:

$$R(\lambda) = \begin{pmatrix}
\eta & 0 & 0 & \lambda \\
0 & 0 & \lambda + \eta & 0 \\
0 & \lambda + \eta & 0 & 0 \\
\lambda & 0 & 0 & \eta
\end{pmatrix}, \quad R^{-1} = \frac{1}{\lambda^2 - \eta^2} \begin{pmatrix}
-\eta & 0 & 0 & \lambda \\
0 & 0 & \lambda - \eta & 0 \\
0 & \lambda - \eta & 0 & 0 \\
\lambda & 0 & 0 & -\eta
\end{pmatrix}$$

Work is in progress on this topic.

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