Strong convergence of tensor products of independent G.U.E. matrices

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Abstract

Given tuples of properly normalized independent \( N \times N \) G.U.E. matrices \( (X_N^{(1)}, \ldots, X_N^{(r_1)}) \) and \( (Y_N^{(1)}, \ldots, Y_N^{(r_2)}) \), we show that the tuple \( (X_N^{(1)} \otimes I_N, \ldots, X_N^{(r_1)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r_2)}) \) of \( N^2 \times N^2 \) random matrices converges strongly as \( N \) tends to infinity. It was shown by Ben Hayes [8] that this result implies that the Peterson-Thom conjecture is true.

1 Introduction

Our investigation is motivated by a result of Ben Hayes [8], showing that a conjecture about the structure of certain finite von Neumann algebras is implied by a strong convergence result for tuples of random matrices. Specifically, the conjecture (see [8, Conjecture 1]) is the following. Assume \( r \in \mathbb{N}, r > 1 \), is given. Denote by \( \mathbb{F}_r \) the free group with \( r \) free generators, and by \( L(\mathbb{F}_r) \) the free group von Neumann algebra, that is, the von Neumann algebra generated by the left regular representation of \( \mathbb{F}_r \) in the space \( B(\ell^2(\mathbb{F}_r)) \) of bounded linear operators on the Hilbert space \( \ell^2(\mathbb{F}_r) \). Assume that \( Q \) is a von Neumann subalgebra of \( L(\mathbb{F}_r) \) which is diffuse (meaning that it contains no minimal projections) and amenable (meaning that there exists a conditional expectation \( E: B(\ell^2(\mathbb{F}_r)) \rightarrow Q \)). Then there exists a unique maximal amenable von Neumann subalgebra \( P \) of \( L(\mathbb{F}_r) \) such that \( Q \subseteq P \). (For the terminology related to the structure of von Neumann algebras, we refer to [1] - specifically in our case to [1, Definition 2.4.12] and [1, Definition 10.2.1 and Proposition 10.2.2].) This conjecture, cited in [8], is known...
as the Peterson-Thom conjecture \cite{13}. Ben Hayes proved in \cite{8} Theorem 1.1] that the Peterson-Thom conjecture is implied by the following conjecture. For any \( N \in \mathbb{N} \), assume that \( X_N^{(1)}, \ldots, X_N^{(r)}, Y_N^{(1)}, \ldots, Y_N^{(r)} \) are \( N \times N \) independent random standard normalized G.U.E.-distributed matrices (see Definition \( 2 \) below). Denote by \( I_N \) the \( N \times N \) identity matrix. Assume also that \( s_1, \ldots, s_r \) are free standard semicircular random variables, and \( P \) is a polynomial in \( 2r \) selfadjoint noncommuting indeterminates. Let \( 1_s \) be the unit of \( C^*(s_1, \ldots, s_r) \), the unital \( C^* \)-algebra generated by the free semicirculars \( s_1, \ldots, s_r \). The conjecture states that

\[
\lim_{N \to \infty} \| P(X_N^{(1)} \otimes I_N, \ldots, X_N^{(r)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r)}) \| = \| P(s_1 \otimes 1_s, \ldots, s_r \otimes 1_s, 1_s \otimes s_1, \ldots, 1_s \otimes s_r) \|_{\text{min}}, \tag{1}
\]

in probability. (One views \( P(s_1 \otimes 1_s, \ldots, s_r \otimes 1_s, 1_s \otimes s_1, \ldots, 1_s \otimes s_r) \in C^*(s_1, \ldots, s_r) \otimes_{\text{min}} C^*(s_1, \ldots, s_r) \), the completion of the algebraic tensor product \( C^*(s_1, \ldots, s_r) \otimes C^*(s_1, \ldots, s_r) \) with respect to \( \| \cdot \|_{\text{min}} \), the minimal, or spatial, norm on \( C^*(s_1, \ldots, s_r) \otimes C^*(s_1, \ldots, s_r) \) - see \cite{8} Section 2.1 and \cite{11} Section II.9.1; since in this paper we consider only the spatial, minimal tensor product norm on our tensor products of \( C^* \)-algebras - or von Neumann algebras - from now on we suppress the “\( \text{min} \)” subscript in our norm notations.)

This phenomenon is called “strong convergence” of \( X_N^{(1)} \otimes I_N, \ldots, X_N^{(r)} \otimes I_N, I_N \otimes Y_N^{(1)}, \ldots, I_N \otimes Y_N^{(r)} \) to \( s_1 \otimes 1_s, \ldots, s_r \otimes 1_s, 1_s \otimes s_1, \ldots, 1_s \otimes s_r \). It is shown in \cite{8} Theorem 1.1] that if this strong convergence holds, then the Peterson-Thom conjecture is true as well. Recent works established results of this nature, dealing with matrices \( X_N^{(1)} \otimes I_M, \ldots, X_N^{(r)} \otimes I_M, I_N \otimes Y_M^{(1)}, \ldots, I_N \otimes Y_M^{(r)} \), where the dimension of the G.U.E. matrices \( Y_i \)’s is \( M \) and \( M = O(N^{1/4}) \) in \cite{14}, \( M = O(N^{1/3}) \) in \cite{15}, and \( M = o(N/(\log N)^3) \) in \cite{2}. As mentioned for instance in \cite{2}, this does not suffice for the purpose of \cite{8}, which requires \( M = N \) (see \cite{2} Proposition 9.3)).

In this paper, we prove the above strong convergence \cite{11}. More precisely, let \( W_N = \{ W_N^{(i)} \mid i = 1, \ldots r_1 \} \) and \( W_N = \{ W_N^{(j)} \mid i = 1, \ldots r_2 \} \) be independent standard G.U.E. \( N \times N \) matrices. Let \( (\mathcal{A}, \tau) \) be a \( C^* \)-probability space equipped with a faithful tracial state and \( \mathbf{s} = \{ s_i, i = 1, \ldots, r_1 \} \) and \( \tilde{\mathbf{s}} = \{ \tilde{s}_i, i = 1, \ldots, r_2 \} \) be (possibly different) free semi-circular systems, that is tuples of free standard semi-circular variables, in \( (\mathcal{A}, \tau) \). It is straightforward to deduce from \cite{19} Corollary 3.9] that for any noncommutative poly-
nomial \( P \) in \( r_1 + r_2 \) variables, one has

\[
(\text{tr}_N \otimes \text{tr}_N) \left[ P \left( \frac{W_N}{\sqrt{N}} \otimes I_N, I_N \otimes \frac{\tilde{W}_N}{\sqrt{N}} \right) \right] \to_{N \to +\infty} (\tau \otimes \tau)[P(s \otimes 1_A, 1_A \otimes \tilde{s})] \quad \text{a.s.}
\]

(\( \text{Tr}_p \) denotes the trace and \( \text{tr}_p = \frac{1}{p} \text{Tr}_p \) the normalized trace on \( M_p(\mathbb{C}) \): \( \text{Tr}_p(I_p) = p, \text{tr}_p(I_p) = 1 \).) We prove the following:

**Theorem 1.** Almost surely, for any noncommutative polynomial \( P \) in \( r_1 + r_2 \) variables,

\[
\left\| P \left( \frac{W_N}{\sqrt{N}} \otimes I_N, I_N \otimes \frac{\tilde{W}_N}{\sqrt{N}} \right) \right\| \to_{N \to +\infty} \left\| P(s \otimes 1_A, 1_A \otimes \tilde{s}) \right\| .
\]

We end the introduction with a brief outline of the rest of the paper. In Section 2 we introduce the main objects of interest for our study and re-phrase Theorem 1 in terms of the Cayley transforms of the selfadjoint variables involved (see Theorem 3). In Section 3 we introduce the necessary tools from noncommutative analysis and probability, and prove a number of auxiliary results about them. This section follows largely [10, 12, 22, 23]. Section 4 is dedicated to studying the first couple of terms in the \( 1/N^2 \) expansion of expectations of normalized traces of polynomials in Cayley transforms of tuples of independent G.U.E. matrices. This is essentially done by an application of Parraud’s work [12], with some adaptations. In Section 5 we explain in detail how the methods introduced by Haagerup, Thorbjørnsen, and Schultz for proving strong convergence of random matrices [7, 16] apply to prove Theorem 3. Section 6 contains the proof of the main step indicated in Section 5. Appendix 7 is dedicated to a technical result, namely the proof of a variance estimate.

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## 2 From Wigner to unitary matrices

Let us introduce the (very simple and classical) random matrix model our paper is concerned with.
Definition 2. A Gaussian Unitary Ensemble (G.U.E.) random matrix $W_N$ of size $N$ is a selfadjoint matrix such that the entries $(W_N)_{i,j}$ of $W_N$ are centered Gaussian (normal) random variables satisfying:

- $\{(W_N)_{i,i} : 1 \leq i \leq N\} \cup \{\Re(W_N)_{i,j}, \Im(W_N)_{i,j} : 1 \leq j < i \leq N\}$ are independent;
- $(W_N)_{i,i}, 1 \leq i \leq N$, and $\sqrt{2}\Re(W_N)_{i,j}, \sqrt{2}\Im(W_N)_{i,j}, 1 \leq j < i \leq N$ have all variance equal to 1.

We call $\frac{W_N}{\sqrt{N}}$ a normalized G.U.E.

A noncommutative random variable $x$ in a $C^*$-probability space $(\mathcal{A}, \tau)$ is a standard semicircular variable if $x = x^*$ and for any $k \in \mathbb{N}$, $\tau(x^k) = \int t^k \, d\mu_{sc}(t)$ where $d\mu_{sc}(t) = \frac{1}{2\pi} \sqrt{4-t^2} 1_{[-2,2]}(t) \, dt$. A standard semicircular system $\{s_i, i = 1, \ldots, r\}$ is a tuple of free standard semicircular variables (see Section 3.1.5 for the definition of freeness).

With the notation from the previous section, set

$$X = (X_1, \ldots, X_{r_1}) = W_N/\sqrt{N},$$

$$Y = (Y_1, \ldots, Y_{r_2}) = \tilde{W}_N/\sqrt{N},$$

to be tuples of independent normalized G.U.E. matrices (from now on, unless otherwise specified, all our G.U.E. random matrices will be so normalized). Consider the Cayley transform $\Psi : \mathbb{C} \to \mathbb{C}$ given by $\Psi(z) = \frac{z+1}{z-1}$ (as usual, we denote $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ the one-point compactification of the complex plane $\mathbb{C}$). By elementary functional calculus, one evaluates $\Psi$ on bounded linear operators on Hilbert spaces whose spectrum does not contain the complex number $i$. It is known (and easy to verify) that $\Psi(H) = \frac{H+i}{H-i}$ is a unitary operator whenever $H$ is a Hermitian (selfadjoint) operator, bounded or not. Conversely, if $U$ is a unitary operator whose spectrum does not contain 1, then $\Psi^{-1}(U)$ is a bounded selfadjoint operator (if 1 belongs to the spectrum of $U$, but it is not an eigenvalue of $U$, one may still define the unbounded symmetric operator $\Psi^{-1}(U)$). As usual, $\Psi^{-1}(w) = i\frac{w+1}{w-1}$ is the inverse with respect to composition of $\Psi$. More generally, the Cayley transform $\Psi$ can be viewed as an analytic free noncommutative (nc) map in the sense of [10], both on finite (matrix) and infinite dimensional spaces. In this paper we will need to consider restrictions of the free nc map $\Psi$ to the operator lower
half-plane (i.e. the set of operators whose imaginary part is strictly negative) and to the operator unit ball (i.e. the set of operators whose operator norm is strictly less than one). It is important to note that $\Psi$ sends the operator lower half-plane bijectively onto the operator unit ball.

Set
\[
(\Psi(X_1), \ldots, \Psi(X_{r_1})) = (U_1, \ldots, U_{r_1}) = U_{r_1}, \\
(\Psi(s_1), \ldots, \Psi(s_{r_1})) = (u_1, \ldots, u_{r_1}) = u_{r_1}, \\
(\Psi(Y_1), \ldots, \Psi(Y_{r_2})) = (V_1, \ldots, V_{r_2}) = V_{r_2}, \\
(\Psi(\tilde{s}_1), \ldots, \Psi(\tilde{s}_{r_2})) = (v_1, \ldots, v_{r_2}) = v_{r_2},
\]
(due to a lack of space, we find it convenient sometimes to use an underline in order to denote a vector of objects, with the index denoting the length of the vector). In order to prove Theorem 1 it is sufficient to prove the following

**Theorem 3.** For any selfadjoint polynomial $P$ in $r_1 + r_2$ variables and their adjoints, $\|P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}, U^* \otimes I_{N_{r_1}}, I_N \otimes V^*_{r_2})\|$ converges to $\|P(u \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}, u^* \otimes 1_{A_{r_1}}, 1_A \otimes v^*_{r_2})\|$.

Indeed, as we have seen above, $\Psi^{-1}(U) = \frac{I + U}{I + 1}$ and $\frac{I + U}{I + 1} \otimes I = I \otimes I + 2(U \otimes I - I \otimes I)^{-1}$. A noncommutative monomial $M$ of degree $n$ in $r_1 + r_2$ indeterminates is written as $M = \alpha x_{i_1} x_{i_2} \cdots x_{i_n}$ for some $i_1, \ldots, i_n \in \{1, \ldots, r_1 + r_2\}$. Then

\[
M(X_1 \otimes I_N, \ldots, X_{r_1} \otimes I_N, I_N \otimes Y_1, \ldots, I_N \otimes Y_{r_2}) \\
= \alpha \left( \prod_{j: i_j \in \{1, \ldots, r_1\}} X_{i_j} \otimes 1 \right) \left( \prod_{k: i_k \in \{r_1+1, \ldots, r_1+r_2\}} 1 \otimes Y_{i_k-r_1} \right) \\
= \alpha \left( \prod_{j: i_j \in \{1, \ldots, r_1\}} i [I \otimes I + 2(U_{i_j} \otimes I - I \otimes I)^{-1}] \right) \\
\times \left( \prod_{k: i_k \in \{r_1+1, \ldots, r_1+r_2\}} i [I \otimes I + 2(I \otimes V_{i_k-r_1} - I \otimes I)^{-1}] \right).
\]

Thus $M(X_1 \otimes I_N, \ldots, X_{r_1} \otimes I_N, I_N \otimes Y_1, \ldots, I_N \otimes Y_{r_2})$ is a rational function in $U_1 \otimes I_N, \ldots, U_{r_1} \otimes I_N, I_N \otimes V_1, \ldots, I_N \otimes V_{r_2}$. Since a polynomial is a sum of monomials, this implies that any noncommutative polynomial
in $X_1 \otimes I_N, \ldots, X_{r_1} \otimes I_N, I_N \otimes Y_1, \ldots, I_N \otimes Y_{r_2}$ is a rational function of $U_1 \otimes I_N, \ldots, U_{r_1} \otimes I_N, I_N \otimes V_1, \ldots, I_N \otimes V_{r_2}$. Therefore, if we are able to prove the strong convergence of $U_1 \otimes I_N, \ldots, U_{r_1} \otimes I_N, I_N \otimes V_1, \ldots, I_N \otimes V_{r_2}$ (that is Theorem 3) then we will deduce the strong convergence of $X_1 \otimes I_N, \ldots, X_{r_1} \otimes I_N, I_N \otimes Y_1, \ldots, I_N \otimes Y_{r_2}$ (that is Theorem 1) by Theorem 3 of [25]. Therefore, in the following, we will focus on the proof of Theorem 3.

For technical reasons, it will be necessary for us to represent the Cayley transform of a selfadjoint in terms of a Fourier-type integral. A straightforward verification shows that

$$\frac{x - i}{x + i} = 1 - 2 \int_{-\infty}^{0} e^{(-ix+1)y} \, dy$$
and

$$\frac{x + i}{x - i} = 1 - 2 \int_{-\infty}^{0} e^{(ix+1)y} \, dy.$$

Both relations hold not only for all $x \in \mathbb{R}$, but for all complex numbers $x$ whose imaginary part is strictly less than one in absolute value. In particular, $z \mapsto 1 - 2 \int_{-\infty}^{0} e^{(\pm iz+1)y} \, dy$ are analytic functions on $\{z \in \mathbb{C} : |\Im Z| < 1\}$. We wish, of course, to apply this function to various operators. We observe the following:

1. Analyticity of the correspondence $z \mapsto 1 - 2 \int_{-\infty}^{0} e^{(\pm iz+1)y} \, dy$ allows us to define $1 - 2 \int_{-\infty}^{0} e^{(\pm iz+1)y} \, dy$ as a bounded linear operator via analytic functional calculus for any bounded linear operator $Z$ on a Hilbert space whose spectrum is contained in $\{z \in \mathbb{C} : |\Im Z| < 1\}$. (For direct estimates, note that $\|e^{(\pm iZ+1)y}\| \leq \|e^{(\pm i\Re Z)y}\|\|e^{(\mp i\Im Z+1)y}\| = \|e^{(\mp i\Im Z+1)y}\|.$)

2. One has

$$1 - 2 \int_{-\infty}^{0} e^{(\pm iz+1)y} \, dy = \frac{Z \pm i}{Z \mp i},$$

for any $Z$ as in the previous item.

3. If in addition $Z = Z^*$, then the integral in the left hand side of Equality (4) converges in the norm topology if one understands $e^{(\pm iz+1)y}$ in the sense of continuous functional calculus.

4. If $Z = Z^*$ is unbounded, then the above statement still holds, but with norm convergence replaced by pointwise-norm convergence on a dense subspace (see [15, Chapter IX] for details).

5. Equality (4) may be viewed as an equality of noncommutative analytic functions in the sense of [10]. In particular, Taylor-Taylor power
series expansions around zero of the two sides of the equation coincide (specifically, if $A$ is a von Neumann algebra, then the two sides of (4) represent the same noncommutative analytic function on the noncommutative disk $\bigcup_{n \geq 1} \{Z \in M_n(A) : \|Z\| < 1\}$ and have the same convergent power series expansions on it). This will be important when we require an application of Voiculescu’s free difference quotient to (4).

We will come back to these facts in the following section.

The tools for modelling the expansion in $1/N$ of an ensemble of tuples of Gaussian random matrices, as elaborated by Parraud [12], are based on noncommutative analysis and noncommutative probability. In the following section we introduce these tools and prove the needed auxiliary results from these fields.

3 Noncommutative analysis and probability tools

From this point on, we fix a separable Hilbert space $H$ and a tracial von Neumann algebra $(A, \tau)$ on $H$, that is, a star-subalgebra $A$ of the space of all bounded linear operators on $H$ which contains the identity operator and is closed in the weak (hence strong) operator topology (see [4, Section I.9]), together with a normal, faithful tracial state $\tau: A \to \mathbb{C}$ (briefly, this means that $\tau$ is a state such that if $(x_i)_i$ is a bounded increasing net of positive elements of $A$, then $x = \sup x_i \implies \tau(x) = \lim \tau(x_i)$ — normality —, $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$ — traciality — and $\tau(xx^*) = 0 \implies x = 0$ — faithfulness; for an introduction to von Neumann algebras we refer the reader to [4,17,18]). We shall call such a pair $(A, \tau)$ a $W^*$-probability space, which is a particular case of a $C^*$-probability space (see [24]).

For any $N \in \mathbb{N}$, we define the von Neumann algebra of bounded $N \times N$ random matrices $M_N(\mathbb{C}) \otimes L^\infty([0,1], dx) \simeq M_N(L^\infty([0,1], dx))$, where $L^\infty([0,1], dx)$ is simply the space of essentially bounded measurable functions on the interval $[0,1]$ endowed with the Lebesgue measure $dx$ and the sigma-algebra of the Borel subsets. It is clear that $M_N(\mathbb{C}) \otimes L^\infty([0,1], dx)$ is a $W^*$-probability space when endowed with the tracial linear functional $\text{tr}_N \otimes \mathbb{E}$, where $\mathbb{E}$ is the expectation $\mathbb{E}(\cdot) = \int_{[0,1]} \cdot \ dx$ on $L^\infty([0,1], dx)$ and $\text{tr}_N$ is the normalized trace on $N \times N$ matrices: $\text{tr}_N((a_{i,j})_{1 \leq i,j \leq N}) = \frac{1}{N} (a_{1,1} + a_{2,2} + \cdots + a_{N,N})$. 


We shall work in the von Neumann algebra \( A_N = A \star M_N(L^\infty([0,1],dx)) \), where \( \star \) denotes the tracial von Neumann algebra free product of the two algebras: it is known from Voiculescu’s work that there exists a unique normal faithful tracial state \( \tau_N \) on \( A_N \) such that \( \tau_N|_A = \tau \) and \( \tau_N|_{M_N(C) \otimes L^\infty([0,1],dx)} = \text{tr}_N \otimes \mathbb{E} \). For an introduction to free products of von Neumann algebras, and free probability in general, we refer to [24].

We make in addition a technical assumption, namely that there exist in \( A \) countably many free semicircular random variables and that \( A \) is Connes embeddable (for instance, the free group factor \( L(F_n) \) for some \( n \geq 2 \) satisfies this condition). It is known that there exist countably many independent classical random variables in \( L^\infty([0,1],dx) \).

We denote by \( \widetilde{A}_N \supset A_N \) the star-algebra of possibly unbounded operators affiliated to \( A_N \) (see [3, Section 2] and references therein). The algebra \( L^\infty([0,1],dx) \) of random variables \( f: [0,1] \rightarrow \mathbb{C} \) which have all moments, that is, \( \int_{[0,1]} |f(x)|^k \, dx < \infty \) for all \( k \in \mathbb{N} \), is an algebra of operators affiliated to \( L^\infty([0,1],dx) \). This way one can find a countable family of independent G.U.E. matrices in \( M_N(C) \otimes L^\infty([0,1],dx) \subset \widetilde{A}_N \).

3.1 Noncommutative polynomials, functions, and distributions

3.1.1 Noncommutative polynomials

Consider the free semigroup \( \mathbb{F}_r^+ \) with \( r \) free generators \( \mathfrak{X}_1, \ldots, \mathfrak{X}_r \). For simplicity we denote \( \mathfrak{X}_r = (\mathfrak{X}_1, \ldots, \mathfrak{X}_r) \). Let \( C(\mathfrak{X}_r) \) be the star-algebra of the free semigroup with \( r \) free generators, where the star operation is given by the real-linear extension of the operation \((\alpha \mathfrak{X}_{i_1} \mathfrak{X}_{i_2} \cdots \mathfrak{X}_{i_n})^* = \pi \mathfrak{X}_{i_n} \cdots \mathfrak{X}_{i_2} \mathfrak{X}_{i_1}\) for all \( \alpha \in \mathbb{C}, n \in \mathbb{N}, i_1, i_2, \ldots, i_n \in \{1, \ldots, r\} \). We will refer to \( C(\mathfrak{X}_r) \) as the algebra of polynomials in \( r \) noncommuting selfadjoint indeterminates.

Remark 4. There are algebras of operators (bounded or not) which are star-isomorphic to \( C(\mathfrak{X}_r) \). For example, the unital star-algebra \( ^*\text{Alg}\{1, s_1, \ldots, s_r\} \) generated by an \( r \)-tuple of standard free semicircular random variables \( s_1, \ldots, s_r \in A \) and the unit 1 of \( A \) is star-isomorphic to \( C(\mathfrak{X}_r) \) via the linear extension of the map \( \alpha s_{i_1} s_{i_2} \cdots s_{i_n} \mapsto \alpha \mathfrak{X}_{i_1} \mathfrak{X}_{i_2} \cdots \mathfrak{X}_{i_n} \). This is in fact true for any free \( r \)-tuple of diffuse selfadjoint random variables, but not only. We call algebraically free any \( r \) tuple of selfadjoint random variables which generate a unital star-subalgebra of \( \widetilde{A}_N \) which is star-isomorphic to \( C(\mathfrak{X}_r) \).
We will need to work also with algebras of noncommutative polynomials in non-selfadjoint indeterminates. Consider $\mathcal{U} = (\mathcal{U}_1, \ldots, \mathcal{U}_r)$ to be free semi-group generators as above, so that $\mathcal{C}(\mathcal{U}) \simeq \mathcal{C}(\mathcal{X})$ as unital algebras, but not as star-algebras. We assume that there exists an $r$-tuple $\mathcal{U}_r^* = (\mathcal{U}_1^*, \ldots, \mathcal{U}_r^*)$ where $\mathcal{U}_j$ and $\mathcal{U}_k^*$ satisfy no algebraic relation. Let $\mathcal{C}(\mathcal{U}, \mathcal{U}_r^*)$ be the unital star-algebra generated by $\mathcal{U}$ and $\mathcal{U}_r^*$, where the star operation is the real linear extension of $(\alpha \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n)^* = \overline{\mathcal{U}_1} \mathcal{U}_2^* \cdots \mathcal{U}_n^* \mathcal{U}_1^*.$

It is obvious that a given polynomial $P \in \mathcal{C}(\mathcal{X})$ can be evaluated on any $r$-tuple of selfadjoint operators, and such an evaluation induces a surjective morphism onto the unital star-algebra generated by the $r$-tuple in question. Similarly, $Q \in \mathcal{C}(\mathcal{U}, \mathcal{U}_r^*)$ can be evaluated on any $r$-tuple of possibly non-selfadjoint operators, and such an evaluation induces a surjective morphism onto the unital star-algebra generated by the $r$-tuple in question. Thus, noncommutative polynomials can be viewed as functions of operators.

One may introduce on $\mathcal{C}(\mathcal{X})$ the operation of free difference quotient $[22, 23]:$

$$\partial_j: \mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X}) \otimes \mathcal{C}(\mathcal{X}), \quad \partial_j 1 = 0, \quad \partial_j \mathcal{X}_k = \delta_{j,k} 1 \otimes 1,$$

extended by linearity and the Leibniz rule: $\partial_j (PQ) = (\partial_j P)(1 \otimes Q) + (P \otimes 1)(\partial_j Q)$. A direct verification shows that the free difference quotient thus defined obeys the usual chain rule $\partial_j (\partial_j P(Q_1(\mathcal{X}), \ldots, Q_r(\mathcal{X}))) = (\partial_j P)(Q_1(\mathcal{X}), \ldots, Q_r(\mathcal{X})) \circ \partial_j Q_1(\mathcal{X}) + \cdots + (\partial_j P)(Q_1(\mathcal{X}), \ldots, Q_r(\mathcal{X})) \circ \partial_j Q_r(\mathcal{X}).$

The above free difference quotient can be extended to polynomial algebras with more general coefficients. For our purposes, we consider the case when the coefficients are in some arbitrary von Neumann algebra $\mathcal{T}$. That means, we consider the algebraic tensor product $\mathcal{T} \otimes \mathcal{C}(\mathcal{X})$ and view this new algebra as a $\mathcal{T}$-bimodule with the actions $t(\theta \otimes P) = (t\theta) \otimes P$ and $(\theta \otimes P)t = (\theta t) \otimes P$, extended by linearity. We take the algebraic tensor product $(\mathcal{T} \otimes \mathcal{C}(\mathcal{X})) \otimes_\mathcal{T} (\mathcal{T} \otimes \mathcal{C}(\mathcal{X}))$ obtained from the usual tensor product space over $\mathbb{C}$ factored by the relations $\langle (\theta_1 t \otimes P_1) \otimes (\theta_2 \otimes P_2) - (\theta_1 \otimes P_1) \otimes (t \theta_2 \otimes P_2) \rangle: t, \theta_1, \theta_2 \in \mathcal{T}, P_1, P_2 \in \mathcal{C}(\mathcal{X}) \}$. Note that for any elementary tensor one has $(\theta_1 \otimes P_1) \otimes_\mathcal{T} (\theta_2 \otimes P_2) = (\theta_1 \theta_2 \otimes P_1) \otimes_\mathcal{T} (1 \otimes P_2) = (1 \otimes P_1) \otimes_\mathcal{T} (\theta_1 \theta_2 \otimes P_2).$ This allows for the identification

$$(\mathcal{T} \otimes \mathcal{C}(\mathcal{X})) \otimes_\mathcal{T} (\mathcal{T} \otimes \mathcal{C}(\mathcal{X})) \simeq \mathcal{T} \otimes \mathcal{C}(\mathcal{X}) \otimes \mathcal{C}(\mathcal{X})$$

via $(t_j \otimes P_j) \otimes_\mathcal{T} (\theta_j \otimes Q_j) \mapsto (t_j \theta_j) \otimes P_j \otimes Q_j$, extended by linearity. We let the free difference quotient act only on the second tensor as $(\text{id}_\mathcal{T} \otimes \partial_j)$; via
the above identification, this comes down to defining

\[ \partial_j : \mathcal{T} \otimes \mathbb{C} \langle X_r \rangle \to (\mathcal{T} \otimes \mathbb{C} \langle X_r \rangle) \otimes_T (\mathcal{T} \otimes \mathbb{C} \langle X_r \rangle) \]  

(6)
as

\[ \partial_j (\theta \otimes X_k) = \delta_{j,k} (\theta \otimes 1) \otimes_T (1 \otimes 1), \quad \partial_j (\theta \otimes 1) = 0, \]

and

\[ \partial_j [(\theta_1 \otimes P_1) (\theta_2 \otimes P_2)] = [\partial_j (\theta_1 \otimes P_1)] (1 \otimes 1) \otimes_T (\theta_2 \otimes P_2) + (\theta_1 \otimes P_1) \otimes_T (1 \otimes 1) [\partial_j (\theta_2 \otimes P_2)]. \]

This view of the free difference quotient has some minor advantages for us in the formulation of our results, but is not essential. The reader is welcome to keep in mind the “amplification” \((\text{id}_T \otimes \partial_j)\) exclusively.

3.1.2 Extensions of the free difference quotient to non-polynomial functions

In this section we use one of the observations from Section 3.1.1 in order to extend the free difference quotient to noncommutative functions other than polynomials. One can easily conceive a term-by-term extension of the free difference quotient to formal power series, however, one should prove that such an extension makes sense. We will avoid this difficulty by recalling that \(\mathbb{C} \langle X_r \rangle \cong ^*\text{Alg}\{1, s_1, \ldots, s_r\}\) for any \(r\)-tuple of bounded selfadjoint operators from \(\mathcal{A}_N\) which are algebraically free. We need to apply \(\partial_j\) to only two simple types of non-polynomial functions, and will limit our analysis to those: exponentials and inverses.

For the exponential, we follow \([12]\). Let \(P \in \mathbb{C} \langle X_r \rangle\) and algebraically free selfadjoints \(s_1, \ldots, s_r\) be given. We formally let

\[ e^{P(s_1, \ldots, s_r)} = \sum_{n=0}^{\infty} \frac{P(s_1, \ldots, s_r)^n}{n!} \]

and

\[ \partial_j e^{P(s_1, \ldots, s_r)} = \sum_{n=0}^{\infty} \frac{\partial_j (P(s_1, \ldots, s_r)^n)}{n!} = \sum_{n=1}^{\infty} \frac{\partial_j (P(s_1, \ldots, s_r)^n)}{n!} \]

If \(s_1, \ldots, s_r\) are bounded, then it turns out (quite easily) that both the above series converge in the norm topology of \(\mathcal{A}_N\) and \(\mathcal{A}_N \otimes \mathcal{A}_N\) (von Neumann algebra tensor product), respectively. If they are not, then one usually cannot hope for anything better than pointwise convergence on a dense subspace. Fortunately, in our paper we will deal with a special case. But first, note that

\[ \partial_j (P(s_1, \ldots, s_r)^n) \]
\[ \sum_{k=0}^{n-1} (P(s_1, \ldots, s_r)^k \otimes 1)(\partial_j P(s_1, \ldots, s_r))(1 \otimes P(s_1, \ldots, s_r)^{n-k-1}). \]

To save space, in the following array we eliminate the variables \(s_k\) from notation.

\[
\partial_j e^P = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(P^k \otimes 1)(\partial_j P)(1 \otimes P^{n-k-1})}{n!} \\
= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(P^k \otimes 1)(\partial_j P)(1 \otimes P^{m-k})}{(m+1)!} \\
= \sum_{m=0}^{\infty} \int_0^1 a^m \frac{m!}{m^k} \sum_{k=0}^{m} (P^k \otimes 1)(\partial_j P)(1 \otimes P^{m-k}) \\
= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \int_0^1 \frac{a^k (1-a)^{m-k}}{k!(m-k)!} \, da \, (P^k \otimes 1)(\partial_j P)(1 \otimes P^{m-k}) \\
= \int_0^1 \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{(aP \otimes 1)^k (\partial_j P)(1 \otimes (1-a)P)^{m-k}}{k!(m-k)!} \, da \\
= \int_0^1 \sum_{p=0}^{\infty} \frac{(aP \otimes 1)^p}{p!} (\partial_j P) \sum_{q=0}^{\infty} \frac{(1 \otimes (1-a)P)^q}{q!} \, da \\
= \int_0^1 e^{aP \otimes 1}(\partial_j P)e^{1 \otimes (1-a)P} \, da.
\]

We have used \(\int_0^1 a^m \, da = \frac{1}{m+1}, \int_0^1 \frac{a^k (1-a)^{m-k}}{k!(m-k)!} \, da = \frac{1}{(m+1)!},\) and the obvious properties of series multiplication. This result is due to Parraud (see [12]), and will be useful to us in its embodiment for the very simple polynomials \((\pm is_j + 1)y, y \in \mathbb{R}\) a constant, although sometimes with \(s_j\) being an unbounded selfadjoint operator.

When all operators \(s_k\) are bounded, one observes that the whole formal power series argument showing that

\[
\partial_j e^P = \int_0^1 e^{aP \otimes 1}(\partial_j P)e^{1 \otimes (1-a)P} \, da, \quad P \in \mathbb{C}(\mathcal{F}),
\]

involves only absolutely convergent series (with convergence being uniform on any set \(\{(s_1, \ldots, s_r) \in \mathcal{A}_N^r : \|s_j\| \leq M\}, \ M > 0 \) fixed). Multiplication
of formal power series guarantees thus that, at least on bounded operators, the rules of calculation for the free difference quotient still hold on the algebra $\ast\text{Alg}\{P(s_1, \ldots, s_r), e^{P(s_1, \ldots, s_r)} : P \in \mathbb{C}(X)\}$ generated by all polynomials and exponentials of polynomials in algebraically free selfadjoint variables $s_1, \ldots, s_r$.

The other non-polynomial expression we are concerned with is the inverse $(z - P(s_1, \ldots, s_r))^{-1}$. This was entirely covered by Voiculescu’s work [23] (see, in particular, Sections 1 and 3), but we wish to take here the more elementary view via convergent power series. If the operators $s_k$ are bounded, then $(z - P(s_1, \ldots, s_r))^{-1} = \sum_{n=0}^{\infty} \frac{P(s_1, \ldots, s_r)^n}{z^{n+1}}$ for all $z \in \mathbb{C}$ with $|z| > \|P(s_1, \ldots, s_r)\|$. Then

$$\partial_j(z - P(s_1, \ldots, s_r))^{-1} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{1}{z^{n+1}} \left( P^k \otimes 1 \right)(\partial_j P)(1 \otimes P^{n-k-1})$$

$$= \sum_{m=0}^{\infty} \sum_{p+q=m} \frac{1}{z^{m+2}} (P \otimes 1)^p(\partial_j P)(1 \otimes P)^q$$

$$= \sum_{m=0}^{\infty} \sum_{p+q=m} \left( \frac{P^p}{z^{p+1}} \otimes 1 \right)(\partial_j P) \left( 1 \otimes \frac{P^q}{z^{q+1}} \right)$$

$$= \left( \sum_{p=0}^{\infty} \frac{P^p}{z^{p+1}} \otimes 1 \right)(\partial_j P) \left( \sum_{q=0}^{\infty} 1 \otimes \frac{P^q}{z^{q+1}} \right)$$

$$= \left( (z - 1)^{-1} \otimes 1 \right)(\partial_j P) \left( 1 \otimes (z - 1)^{-1} \right).$$

We have shown that if $|z| > \|P(s_1, \ldots, s_r)\|$, then

$$\partial_j(z - P(s_1, \ldots, s_r))^{-1} = \left( (z - P(s_1, \ldots, s_r))^{-1} \otimes 1 \right)(\partial_j P) \left( 1 \otimes (z - P(s_1, \ldots, s_r))^{-1} \right), \quad (8)$$

with the proof again involving exclusively term-by-term manipulations in convergent power series. We note that for any $\varepsilon > 0$, the convergences of all series involved above is uniform and absolute on a set $\{ (z; s_1, \ldots, s_r) \in \mathbb{C} \times \mathcal{A}_N^r : \|P(s_1, \ldots, s_r)\|/|z| < 1 - \varepsilon \}$. It follows now that, when all power series involved are convergent on (convenient) open sets, any sum, product, and composition of functions of the form $P(s_1, \ldots, s_r), e^{P(s_1, \ldots, s_r)}, (z - P(s_1, \ldots, s_r))^{-1}$ accepts an application of the free difference quotient, and the radius of convergence for the newly obtained series in $\mathcal{A}_N \otimes \mathcal{A}_N$ does not decrease.
Remark 5. Once the expressions of the series involved in both (7) and (8) are established, the necessity of \( s_1, \ldots, s_r \) being algebraically free disappears, and the statements regarding convergence apply for any operators \( s_1, \ldots, s_r \), not necessarily selfadjoint, which satisfy the required majorization conditions for each series. We will return to this matter in the following section.

Remark 6. It is worth noting that the application of the free difference quotient preserves in a certain sense the “class” of functions: polynomial, rational, and exponential. In the case of polynomial and rational functions, this is evident thanks to the definition of the free difference quotient and (8), respectively. In the case of exponential functions, (7) tells us that the statement should be understood “in the limit”: the free difference quotient applied to the exponential of a polynomial is a norm limit (when operators are bounded) of sums of terms of the form (exponential of polynomial \( \times \) polynomial) \( \otimes \) (polynomial \( \times \) exponential of polynomial). These statements are quite obvious, but we shall only need a particular case of the one corresponding to rational functions, namely resolvents, which shall be discussed and proven in detail.

3.1.3 Noncommutative functions

In this subsection we follow very closely [10], to which we refer for a thorough and detailed introduction to the subject. We do not introduce noncommutative functions in maximum generality, but in a context only as general as needed for our purposes. Thus, given a \( C^\ast \)-algebra \( B \), a noncommutative set over \( B^d \) is a family \( \Omega = \coprod_{n \in \mathbb{N}} \Omega_n \) such that

1. \( \Omega_n \subseteq M_n(B)^d \) for all \( n \in \mathbb{N} \);

2. If \( X_d \in \Omega_n, Y_d \in \Omega_m \), then \( X_d \oplus Y_d = \left( \begin{bmatrix} X_{d1} & 0 \\ 0 & Y_{d1} \end{bmatrix}, \ldots, \begin{bmatrix} X_{dn} & 0 \\ 0 & Y_{dn} \end{bmatrix} \right) \in \Omega_{n+m}, m, n \in \mathbb{N} \);

3. If \( X_d \in \Omega_n \) and \( U \in M_n(\mathbb{C}) \) is a unitary, then \( UX_dU^* = (UX_1U^*, \ldots, UX_dU^*) \in \Omega_n \).

A noncommutative function is a family \( f = \{ f_n \}_{n \in \mathbb{N}} \) such that

1. \( f_n : \Omega_n \to M_n(B) \) for all \( n \in \mathbb{N} \);

2. If \( X_d \in \Omega_n, Y_d \in \Omega_m \), then \( f_{n+m}(X_d \oplus Y_d) = f_n(X_d) \oplus f_m(Y_d), m, n \in \mathbb{N} \);
3. If \( X_d \in \Omega_n \) and \( S \in M_n(\mathbb{C}) \) is an invertible matrix such that \( SX_d S^{-1} \in \Omega_n \), then \( f_n(SX_d S^{-1}) = SF_n(X_d)S^{-1} \).

(Clearly \( f_n \) needs not take values in \( M_n(\mathcal{B}) \) for the same \( \mathcal{B} \), but we will not encounter any other case in this paper.) We call \( n \) the level of the noncommutative function/set. The following example will be useful.

**Example 7.** Consider a simply connected open set \( G \subseteq \mathbb{C} \). Then any analytic function \( f: G \to \mathbb{C} \) is the first level of a noncommutative function taking values in \( \mathbb{C}_{nc} := \bigoplus_n M_n(\mathbb{C}) \). Its natural domain of definition is \( \bigoplus_n \{ Z \in M_n(\mathbb{C}) : sp(Z) \subseteq G \} \) and it is defined via the analytic functional calculus:

\[
f_n(Z) = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta I_n - Z)^{-1} d\gamma(\zeta),
\]

for some simple closed curve \( \gamma \) in \( G \) and surrounding exactly once the spectrum \( sp(Z) \) of \( Z \). This is easily seen to be an extension of the polynomial evaluation.

An important property of noncommutative functions is that the derivative at a given level \( n \) is often recoverable from the simple evaluation at level \( 2n \). Specifically: if \( X_d \in \Omega_n, Y_d \in \Omega_m, \) and \( B_d \in M_{n \times m}(\mathcal{B}) \) is such that \( \begin{bmatrix} X_d & B_d \\ 0 & Y_d \end{bmatrix} \in \Omega_{n+m} \) and \( f \) is locally bounded on slices, then

\[
f_{n+m} \left( \begin{bmatrix} X_d & B_d \\ 0 & Y_d \end{bmatrix} \right) = \begin{bmatrix} f_n(X_d) & \Delta f_{n,m}(X_d,Y_d)(B_d) \\ 0 & f_m(Y_d) \end{bmatrix}.
\]

If there is an open set around zero such that \( \begin{bmatrix} X_d & B_d \\ 0 & Y_d \end{bmatrix} \in \Omega_{n+m} \) for all \( B_d \) in that set, then \( M_{n \times m}(\mathcal{B}) \) \( \supseteq \bigoplus_m B_d \mapsto \Delta f_{n,m}(X_d,Y_d)(B_d) \in M_{n \times m}(\mathcal{B}) \) is a \( \mathbb{C} \)-linear map. If \( m = n \), then \( \Delta f_{n,n}(X_d,Y_d)(X_d - Y_d) = f_n(X_d) - f_n(Y_d) \), and if in addition \( X_d = Y_d \), then \( \Delta f_{n,n}(X_d,X_d)(B_d) = f'_n(X_d)(B_d) \), the usual Fréchet derivative of the Banach space-valued map \( f_n \) (the dependence in \( X_d \) and \( Y_d \) of \( \Delta f_{n,m}(X_d,Y_d)(B_d) \) is of a nature that is very similar to the dependence of \( f_n \) on \( X_d \), or of \( f_m \) on \( Y_d \)). A rather spectacular fact that follows from this is that locally bounded noncommutative functions defined on sets that are “fat” enough (open sets, for instance) are automatically Fréchet analytic (see [10, Corollary 7.6]). We call \( \Delta f_{n,m}(X_d,Y_d) \) the (first) difference-differential operator.
This is important for us for several reasons. First, one easily notes that any $\mathbf{B}_d = (B_1, B_2, \ldots, B_d) \in M_{n \times m}(\mathcal{B})^d$ is written as $\mathbf{B}_d = B_1 e_1 + B_2 e_2 + \cdots + B_d e_d$, with $e_j = (0, \ldots, 1 \otimes I_m, \ldots, 0)$, where the identity is on position $j$. Thus, it makes sense to define

$$\Delta_j f_{n,m}(X_d, Y_d)(B) = \Delta f_{n,m}(X_d, Y_d)(B e_j) = \Delta f_{n,m}(X_d, Y_d)(0, \ldots, B, \ldots, 0),$$

the $j$th partial difference-differential operator $\Delta_j f_{n,m}(X_d, Y_d): M_{n \times m}(\mathcal{B}) \to M_{n \times m}(\mathcal{B})$. If $m = n$ and $X_d = Y_d$, then $\Delta_j f_{n,n}(X_d, X_d)$ is just the classical partial derivative in the $j$th coordinate. Second, one may repeat the above for larger matrices:

$$f_{n+m+p} \left( \begin{bmatrix} X_d & B_d & 0 \\ 0 & Y_d & C_d \\ 0 & 0 & Z_d \end{bmatrix} \right) = \begin{bmatrix} f_n(X_d) & \Delta f_{n,m}(X_d, Y_d)(B_d) & \Delta^2 f_{n,m,p}(X_d, Y_d, Z_d)(B_d, C_d) \\ 0 & f_m(Y_d) & \Delta f_{m,p}(Y_d, Z_d)(C_d) \\ 0 & 0 & f_p(Z_d) \end{bmatrix},$$

where $(B_d, C_d) \mapsto \Delta^2 f_{n,m,p}(X_d, Y_d, Z_d)(B_d, C_d)$ is a bilinear correspondence from $M_{n \times m}(\mathcal{B})^d \times M_{m \times p}(\mathcal{B})^d$ to $M_{m \times p}(\mathcal{B})$, and so on, as one considers larger and larger matrices (again $\Delta^2 f_{n,n,n}(X_d, X_d, X_d)$ is the classical second derivative of $f_n$). Generally,

$$f_{m_1+\ldots+m_{n+1}} \left( \begin{bmatrix} X_d^{(1)} & B_d^{(1)} & \cdots & 0 & 0 \\ 0 & X_d^{(2)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X_d^{(n)} & B_d^{(n)} \\ 0 & 0 & \cdots & 0 & X_d^{(n+1)} \end{bmatrix} \right) = \begin{bmatrix} f_{m_1}(X_d^{(1)}) & \Delta^n f_{m_1,\ldots,m_{n+1}}(X_d^{(1)}, \ldots, X_d^{(n+1)})(B_d^{(1)}, \ldots, B_d^{(n)}) \\ \vdots & \vdots & \vdots \\ 0 & \cdots & f_p(X_d^{(n+1)}) \end{bmatrix},$$

(9)

This process allows one to write power series expansions for noncommutative functions: according to [10] Theorems 7.2 and 7.4, if $f$ is locally bounded (which will always be the case for the functions we deal with), then

$$\frac{1}{K!} \frac{d^K}{dt^K} f_n(Y_d + t Z_d)_{|t=0} = \Delta^K f_{n,\ldots,n}(Y_d, \ldots, Y_d)(Z_d, \ldots, Z_d),$$

(10)

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\[ f_n(X_d) = \sum_{l=0}^{\infty} \Delta^l f_{n,...,n}(Y_d, \ldots, Y_d)(X_d - Y_d, \ldots, X_d - Y_d), \tag{11} \]

where (10) happens for all \( K, n \in \mathbb{N}, Y_d \in \Omega_n, Z_d \in M_n(B)^d \) (recall that each \( \Omega_n \) is assumed now to be open in the usual, norm topology of \( M_n(B)^d \)), and for any open ball \( \mathcal{Y} \subseteq \Omega_n \) centered at \( Y_d \) and \( \varepsilon > 0 \), (11) converges absolutely and uniformly on the set

\[ \mathcal{Y}_\varepsilon := \{ X_d \in \mathcal{Y} : Y_d + (1 + \varepsilon)(X_d - Y_d) \in \mathcal{Y} \}. \]

One has
\[
\sum_{l=0}^{\infty} \sup_{X_d \in \mathcal{Y}_\varepsilon} \left\| \Delta^l f_{n,...,n}(Y_d, \ldots, Y_d)(X_d - Y_d, \ldots, X_d - Y_d) \right\|_{M_n(B)^d} < \infty. \tag{12}
\]

Following [10], we refer to (11) as the \textit{Taylor-Taylor series expansion of \( f_n \) around \( Y_d \)}. The essential point for us about this series development is that its terms \( \Delta^l f_{n,...,n}(Y_d, \ldots, Y_d)(X_d - Y_d, \ldots, X_d - Y_d) \), \( n, l \in \mathbb{N} \), determine uniquely the function \( f \).

In our case, \( B \) will be either \( \mathbb{C} \) or, sometimes, a finite von Neumann algebra like \( A \) or \( A_N \) or \( M_N(L^\infty([0,1], dx)) \).

### 3.1.4 Relations between the difference-differential operator and the free difference quotient

We have seen in Sections 3.1.1 and 3.1.3 two perspectives on noncommutative functions and on derivatives. In this section, we intend to unify them to some extent. Let us begin by considering noncommutative functions on open subsets of \( \mathbb{C}_nc^r = \bigsqcup_n M_n(\mathbb{C})^r \). Given a monomial \( M \in \mathbb{C}(X_1, \ldots, X_r) \), one may view it as a noncommutative function simply by performing evaluations on \( r \)-tuples of \( n \times n \) complex matrices for all \( n \). Thus, for an \( M = X_{i_1}X_{i_2} \cdots X_{i_m} \in \{1, \ldots, r\} \), one has \( \partial_j M = \sum_{k : i_k = j} X_{i_1} \cdots X_{i_{k-1}} \otimes X_{i_k+1} \cdots X_{i_m} \), with the convention that the empty word is the unit 1 (i.e. if, say, \( i_m = j \), then the last summand is \( X_{i_1} \cdots X_{i_m} \otimes 1 \)).

Evaluation on an \( r \)-tuple of matrices \((X_1, \ldots, X_r)\) yields \( \partial_j M(X_1, \ldots, X_r) = \sum_{k : i_k = j} X_{i_1} \cdots X_{i_{k-1}} \otimes X_{i_k+1} \cdots X_{i_m} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \). However, \( M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \) identifies naturally with the space of linear maps from \( M_n(\mathbb{C}) \) to itself via

\[
M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \ni \sum_j A_j \otimes B_j \mapsto \left[ C \mapsto \sum_j A_j C B_j \right] \in \mathcal{L}(M_n(\mathbb{C}), M_n(\mathbb{C})).
\]
The identification is bijective, and if one considers the opposite algebra structure on the second tensor, it is also an algebraic isomorphism \( M_n(\mathbb{C}) \otimes M_n(\mathbb{C})^{op} \simeq \mathcal{L}(M_n(\mathbb{C}), M_n(\mathbb{C})) \). This allows one to easily identify the free difference quotient and the difference-differential operator: simply observe that

\[
M \left( \begin{bmatrix} X_1 & 0 \\ 0 & X_1 \end{bmatrix}, \ldots, \begin{bmatrix} X_j & C \\ 0 & X_j \end{bmatrix}, \ldots, \begin{bmatrix} X_r & 0 \\ 0 & X_r \end{bmatrix} \right)
= \begin{bmatrix} X_{i_1} \delta_{i_1,j}C \\ 0 & X_{i_1} \end{bmatrix} \begin{bmatrix} X_{i_2} \delta_{i_2,j}C \\ 0 & X_{i_2} \end{bmatrix} \cdots \begin{bmatrix} X_{i_m} \delta_{i_m,j}C \\ 0 & X_{i_m} \end{bmatrix}
= \begin{bmatrix} X_{i_1}X_{i_2} \cdots X_{i_m} \\ 0 & \sum_{k:i_k=j} X_{i_1} \cdots X_{i_{k-1}}CX_{i_{k+1}} \cdots X_{i_m} \end{bmatrix},
\]

which shows that on monomial functions defined on \( \mathbb{C}_r^{nc} \) the above isomorphism identifies the difference-differential operator and the free difference quotient when evaluated on \( r \)-tuples of complex matrices of all sizes. The extension to convergent power series is performed the obvious way.

### 3.1.5 Noncommutative distributions

The main object in noncommutative probability is the noncommutative distribution. In maximum generality, it is defined as being simply a linear functional \( \mu: \mathbb{C}(X_r) \to \mathbb{C} \) such that \( \mu(1) = 1 \). Such a distribution is called positive if \( \mu(P^*P) \geq 0 \) for all \( P \in \mathbb{C}(X_r) \), tracial if \( \mu(PQ) = \mu(QP) \) for all \( P, Q \in \mathbb{C}(X_r) \), and faithful if \( \mu(P^*P) = 0 \implies P = 0 \) in \( \mathbb{C}(X_r) \). The distribution of an \( r \)-tuple of selfadjoint elements \((s_1, \ldots, s_r) \) in \((\mathcal{A}, \tau)\) with respect to \( \tau \) is simply the linear functional \( \mu_{(s_1, \ldots, s_r)}: \mathbb{C}(X_r) \to \mathbb{C} \) given by \( \mu_{(s_1, \ldots, s_r)}(P) = \tau(P(s_1, \ldots, s_r)). \) Under our hypotheses on \( \tau \), \( \mu_{(s_1, \ldots, s_r)} \) is automatically positive, tracial, and, if \( s_1, \ldots, s_r \) are algebraically free, faithful. A sequence \( \{(s_1^{(N)}, \ldots, s_r^{(N)})\}_{N \in \mathbb{N}} \subseteq (\mathcal{A}_N, \tau_N) \) converges to \((s_1, \ldots, s_r) \) in distribution if \( \lim_{N \to \infty} \mu_{(s_1^{(N)}, \ldots, s_r^{(N)})}(P) = \mu_{(s_1, \ldots, s_r)}(P) \) for all \( P \in \mathbb{C}(X_r) \).

We shall be concerned in this paper with a notion of noncommutative independence introduced by Voiculescu, namely free independence. Elements \((s_1, \ldots, s_r) \in \mathcal{A} \) are said to be freely independent with respect to \( \tau \) (or just free) if for any \( n \in \mathbb{N} \) and any polynomials \( P_1, \ldots, P_n \in \mathbb{C}[X] \) (one indeterminate!), one has

\[
\tau((P_1(s_{i_1}) - \tau(P_1(s_{i_1}))) (P_2(s_{i_2}) - \tau(P_2(s_{i_2}))) \cdots (P_n(s_{i_n}) - \tau(P_n(s_{i_n})))) = 0
\]
whenever $i_1, \ldots, i_n \in \{1, \ldots, r\}$ are such that no two consecutive indices are equal: $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$. Of course, sets $S_1, \ldots, S_r$ are free if any elements $s_1 \in \text{Alg} S_1, \ldots, s_r \in \text{Alg} S_r$ picked from them are so. Variables $(s_1, \ldots, s_r)$ are star-free (or *-free) if $\text{Alg}\{1, s_1, s_1^*\}, \ldots, \text{Alg}\{1, s_r, s_r^*\}$ are free. The property of being free may extend to closures of sets/algebras in various topologies, depending on continuity properties of $\tau$.

Of course, one notices that $\mu((s_1, \ldots, s_r))$ extends from the algebra of noncommutative polynomials in $r$ selfadjoint indeterminates to a much larger algebra: in fact, if $\lim_i P_i(s_1, \ldots, s_r) = F \in \mathcal{A}$ in the weak operator topology, then $\mu((s_1, \ldots, s_r))(P_i) = \tau(P_i(s_1, \ldots, s_r)) \to \tau(F)$. Thus, $\mu((s_1, \ldots, s_r))$ extends to the von Neumann algebra generated by $(s_1, \ldots, s_r)$ in $\mathcal{A}$, which, in this precise sense, can be understood as the space of $L^\infty$ noncommutative functions of $(s_1, \ldots, s_r)$.

We shall deal in our paper with linear functionals on $\mathcal{C}(\mathcal{X}_r)$ which are not distributions, and extend only to much smaller spaces of noncommutative functions containing $\mathcal{C}(\mathcal{X}_r)$. However, all our functionals will share the property of being real: we call $\nu: \mathcal{C}(\mathcal{X}_r) \to \mathbb{C}$ real if $\nu(P^*) = \overline{\nu(P)}$ for all $P \in \mathcal{C}(\mathcal{X}_r)$.

We conclude this subsection with the definition of a star-distribution: the star-distribution of the possibly non-selfadjoint $r$-tuple $u_1, \ldots, u_r \in (\mathcal{A}, \tau)$ is the linear functional $\mu(u_1, \ldots, u_r): \mathcal{C}(\mathcal{U}_r, \mathcal{U}_r^*) \to \mathbb{C}$ given by $\mu(u_1, \ldots, u_r)(P) = \tau(P(u_1, \ldots, u_r, u_1^*, \ldots, u_r^*))$ for all $P \in \mathcal{C}(\mathcal{U}_r, \mathcal{U}_r^*)$. Definitions for positivity, faithfulness and traciality are unchanged, and for $\tau$ as in our context, $\mu(u_1, \ldots, u_r)$ satisfies them all.

### 3.2 Auxiliary results

As it has already been mentioned in Section 2, one of the main tools in our proofs is the Cayley transform viewed as a noncommutative function, and its “embodiment” as a Fourier-like integral:

$$\Psi(x)^\epsilon = \frac{x + \epsilon i}{x - \epsilon i} = 1 - 2 \int_{-\infty}^{0} e^{(ie+1)y} \, dy, \quad \epsilon \in \{\pm 1\}. \quad (13)$$

The first equality holds for all $x \in \mathbb{C}$, and the second for all $\{z \in \mathbb{C}: |\Im z| < 1\}$, as analytic functions (to be precise, when $\epsilon = 1$, the second equality holds on $\{z \in \mathbb{C}: \Im z < 1\}$, and when $\epsilon = -1$, on $\{z \in \mathbb{C}: \Im z > -1\}$). This has been shown in Section 2 to extend via analytic functional calculus to
bounded linear operators whose spectrum is included in the corresponding domain, and, for selfadjoint unbounded operators affiliated to \((A_N, \tau_N)\), via continuous functional calculus; the notation is obvious:

\[ \Psi(Z) = \frac{Z + \epsilon i Z - \epsilon i}{Z - \epsilon i} = 1 - 2 \int_{-\infty}^{0} e^{(i\epsilon Z + 1)} y \, dy, \quad \epsilon \in \{\pm 1\}. \]  

(14)

One usually writes \((Z + \epsilon i)(Z - \epsilon i)^{-1}\) instead of \(\frac{Z + \epsilon i}{Z - \epsilon i}\), but since \((Z + \epsilon i)(Z - \epsilon i)^{-1} = (Z - \epsilon i)^{-1}(Z + \epsilon i)\), we trust that the notation \(Z + \epsilon i Z - \epsilon i\) is not confusing and we will continue to use it.

Analyticity (see Example 7) guarantees that the Taylor-Taylor power series expansions around zero of the left and right hand sides in (14) coincide. However, a direct verification is possible:

\[ \frac{Z + \epsilon i}{Z - \epsilon i} = 1 - 2 \sum_{n=0}^{\infty} (-i\epsilon Z)^n, \quad \|Z\| < 1, \]

\[ 1 - 2 \int_{-\infty}^{0} e^{(i\epsilon Z + 1)y} \, dy = 1 - 2 \int_{-\infty}^{0} e^{y} e^{i\epsilon Z y} \, dy = 1 - 2 \int_{-\infty}^{0} e^{y} \sum_{n=0}^{\infty} \frac{(i\epsilon Z y)^n}{n!} \, dy \]

\[ = 1 - 2 \sum_{n=0}^{\infty} \left( \int_{-\infty}^{0} y^n e^{y} \, dy \right) (i\epsilon Z)^n \]

\[ = 1 - 2 \sum_{n=0}^{\infty} (-1)^n (i\epsilon Z)^n, \quad \|Z\| < 1. \]

(We have used the commutativity of the identity with \(Z\) to write \(e^{(i\epsilon Z + 1)y} = e^{y} e^{i\epsilon Z y}\), and the elementary equality \(\mathcal{I}_n = -n\mathcal{I}_{n-1}\) for \(\mathcal{I}_n = \int_{-\infty}^{0} y^n e^{y} \, dy\).)

This fact, together with Sections 3.1.4, 3.1.2, and the analyticity result of Section 3.1.3 imply the following essential facts:

**Lemma 8.** Let \(X = X^* \in \tilde{A}_N\) be an operator such that the free difference quotient with respect to \(X\) is well-defined, and let \(U^\epsilon = \frac{X + \epsilon i X}{X - \epsilon i}\). Then

\[ \partial U^\epsilon = \epsilon \frac{i}{2} (U^\epsilon - 1) \otimes (U^\epsilon - 1). \]  

(15)

**Proof.** This is an elementary computation, thanks to Voiculescu’s [23 Sections 1 and 3]. Using power series and analyticity, one can argue thus: we
have shown in Section 3.1.2 that \( \partial U^\epsilon = \partial (1 + 2i\epsilon(X - \epsilon i)^{-1}) = -2i\epsilon(X - \epsilon i)^{-1} \otimes (X - \epsilon i)^{-1} = \frac{d}{2}(U^\epsilon - 1) \otimes (U^\epsilon - 1) \). This holds via the power series argument for all \( X, \|X\| < 1 \). However, the identification of free difference quotient and difference-differential operators from Section 3.1.4 and the fact that the difference-differential operator is defined for all \( X \) with \( -1 < \Im X < 1 \), guarantee via the classical analytic continuation principle that Equality (1.5) holds for all selfadjoints \( X \).

As noted before, the free difference quotient is well-defined on all non-commutative functions introduced and discussed in Subsection 3.1.2 as well as on all compositions, sums, and products of such functions. Moreover, Remark 3 indicates that in a certain sense, these classes of functions are stable under \( \partial_j, 1 \leq j \leq r \). Thus, one can iterate the application of the free difference quotient by applying one of \( \partial_j \otimes \partial_j, \partial_j \otimes \partial_k, \partial_j \otimes \id \) to \( \partial_j F \) for a given \( F \) in this class of functions, then \( \partial_j \otimes \partial_j \otimes \partial_j \) etc. In each case, the result is either a tensor product of polynomials, or of formal power series which are convergent when evaluated on operators on which the power series representing the initial functions converges, and the analytic extensions thereof. Thus, for \( F \) picked from among these functions, one may define the following operations:

1. \( \text{ev}_n \) on \( \partial_j F(s_1, \ldots, s_r) \) via the linear extension or the extension to power series of \( \text{ev}_n : \mathcal{A}_N \otimes \mathcal{A}_N \rightarrow \mathcal{A}_N \), \( \text{ev}_n(a \otimes b) = asb, s \in \mathcal{A}_N \) (there is an obvious \( n \)-linear extension of this operation: \( \text{ev}_{s_1, s_2, \ldots, s_n} : \mathcal{A}_N^\otimes n+1 \rightarrow \mathcal{A}_N \), \( \text{ev}_{s_1, s_2, \ldots, s_n}(a_1 \otimes \cdots \otimes a_{n+1}) = a_1s_1a_2s_2 \cdots a_n s_n a_{n+1} \));

2. \( \text{flip} \) on \( \partial_j F(s_1, \ldots, s_r) \) via the linear extension or the extension to power series of the map \( \text{flip} : \mathcal{A}_N \otimes \mathcal{A}_N \rightarrow \mathcal{A}_N \otimes \mathcal{A}_N \), \( \text{flip}(a \otimes b) = b \otimes a \).

**Lemma 9.** Assume that \( \Psi_{\epsilon_1}, \Psi_{\epsilon_2}, \ldots, \Psi_{\epsilon_n} \in \mathbb{C} \langle \mathcal{U}, \mathcal{U}^\dagger \rangle \) is given. Consider operations \( A_1, \ldots, A_k \in \{ \text{ev}_1, \text{flip}, \text{flip} \otimes \text{flip} \} \cup \{ \partial_j, \partial_j \otimes \partial_j : 1 \leq j \leq r \} \), and assume that the composition \( A_1 \cdots A_k \) is well-defined. Then

\[
(A_1 \cdots A_k)(\Psi(s_{\epsilon_1})^{\epsilon_1}\Psi(s_{\epsilon_2})^{\epsilon_2} \cdots \Psi(s_{\epsilon_n})^{\epsilon_n}) = \\
\int_{(-\infty,0]^n} (A_1 \cdots A_k)(e^{i\epsilon_1 s_{\epsilon_1} + 1} y_1 e^{i\epsilon_2 s_{\epsilon_2} + 1} y_2 \cdots e^{i\epsilon_n s_{\epsilon_n} + 1} y_n) \prod_{j=1}^n (d\delta_0(y_j) - 2dy_j).
\]

The lemma essentially states that all of the operations in question “pass under” the Fourier-type integral representation of the Cayley transform.
Proof. We start as usual with an arbitrary \( r \)-tuple of algebraically free self-adjoint variables \((s_1, \ldots, s_r) \in A_N^r\) (so that the star-algebra generated by them is isomorphic to \(\mathbb{C}(\mathbb{A}_N)\)), which we assume in addition to be of norm strictly less than one. Our strategy is obvious: we argue that the formal power series in the left and right-hand side obtained after each application of one of the operations \(A_1, \ldots, A_k\) must still coincide term-by-term.

We write the power series expansions around zero for the two sides: first,

\[
e^{(i\epsilon_1 s_1 + 1)y_1} e^{(i\epsilon_2 s_2 + 1)y_2} \cdots e^{(i\epsilon_n s_n + 1)y_n} = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} \frac{ey_1^{m_1}ey_2^{m_2} \cdots ey_n^{m_n}}{m_1!m_2! \cdots m_n!} (i\epsilon_1 s_1)^{m_1} (i\epsilon_2 s_2)^{m_2} \cdots (i\epsilon_n s_n)^{m_n}.
\]

Our reason for only separating the \( y \) variables is made obvious by the power-series justification of (14). For convenience, we rewrite the power series expansion \(e^{\frac{z}{1+z}} = 1 - 2 \sum_{n=0}^{\infty} (-i\epsilon Z)^n = -1 + \sum_{n=1}^{\infty} 2(-1)^{n+1}(i\epsilon Z)^n = \sum_{n=0}^{\infty} c_n (i\epsilon Z)^n\), where \(c_n = (-1)^{n+1} \min\{n+1, 2\}\) \((c_0 = -1, c_1 = 2, c_2 = -2, c_3 = 2, \ldots)\). Then

\[
\Psi(s_1)^{e_1} \Psi(s_2)^{e_2} \cdots \Psi(s_n)^{e_n} = \sum_{m_1, m_2, \ldots, m_n=0}^{\infty} c_{m_1} c_{m_2} \cdots c_{m_n} (i\epsilon_1 s_1)^{m_1} (i\epsilon_2 s_2)^{m_2} \cdots (i\epsilon_n s_n)^{m_n}.
\]

For any fixed \( n \)-tuple \((m_1, m_2, \ldots, m_n) \in \mathbb{N}^n\), one may apply \((A_1 \cdots A_k)\) to \((i\epsilon_1 s_1)^{m_1} (i\epsilon_2 s_2)^{m_2} \cdots (i\epsilon_n s_n)^{m_n}\), under the infinite sum, because none of the operations in the set \(\{e^v, \text{flip}, \text{flip} \otimes \text{flip}\} \cup \{\partial_j, \partial_j \otimes \partial_j \mid 1 \leq j \leq r\}\) changes the radius of convergence. Thus, one needs only verify that

\[
c_{m_1} c_{m_2} \cdots c_{m_n} = \int_{(-\infty,0]^n} \frac{ey_1^{m_1}ey_2^{m_2} \cdots ey_n^{m_n}}{m_1!m_2! \cdots m_n!} \prod_{j=1}^{n} (d\delta_0(y_j) - 2dy_j).
\]

Clearly if all of \( m_1, m_2, \ldots, m_n \neq 0 \), then the integral with respect to the Dirac measure is zero, so that the equality to be proved becomes

\[
c_{m_1} c_{m_2} \cdots c_{m_n} = (-2)^n \int_{(-\infty,0]^n} \frac{ey_1^{m_1}ey_2^{m_2} \cdots ey_n^{m_n}}{m_1!m_2! \cdots m_n!} dy_1 dy_2 \cdots dy_n,
\]

which is obviously true. If some of the \( m_j \)'s are equal to zero, then we separate them form the rest. Since the scalar coefficients do commute, we may assume...
without loss of generality that \( m_1 = m_2 = \cdots = m_k = 0, m_{k+1}, \ldots, m_n > 0 \).

Recalling that \( c_0 = -1 \), the equality to be proved becomes

\[
(-1)^k c_{m_{k+1}} \cdots c_{m_n} =
\]

\[
(-2)^{n-k} \int_{(-\infty,0]^n} e^{y_1} \cdots e^{y_k} e^{y_{k+1}} \cdot \cdots \cdot e^{y_n} \frac{y_{k+1}^{m_{k+1}} \cdots y_n^{m_n}}{m_{k+1}! \cdots m_n!} \prod_{j=1}^k (d\delta_0(y_j) - 2dy_j) dy_{k+1} \cdots dy_n.
\]

This is again obvious: since the integral factors, one needs only notice that

\[
\int_{(-\infty,0]^k} e^{y_1} \cdots e^{y_k} \prod_{j=1}^k (d\delta_0(y_j) - 2dy_j) = (1 - 2)^k = (-1)^k.
\]

This proves our lemma for \( \|s_j\| < 1 \). However, thanks to Section \( \ref{sec:extension} \), it extends to the whole domain of analyticity of \( \Psi(\cdot) \) and \( \int_{(-\infty,0]} e^{(i\epsilon+1)y} (d\delta_0(y) - 2dy) \), respectively, that is, to the set of operators \( \{(s_1, \ldots, s_r) : -1 < \Re s_j < 1, 1 \leq j \leq r\} \). \( \square \)

### 3.2.1 Spaces of noncommutative functions

With \( s_r = (s_1, \ldots, s_r) \) being an \( r \)-tuple of algebraically free selfadjoint random variables from \((\mathcal{A}_N, \tau_N)\) as above, we let \( \mathcal{U}_r, \mathcal{U}_r^{-1} \) be given by \( \mathcal{U}_r = \Psi(s_j)^{\frac{1}{s_j + \epsilon_j}} \). Define

\[
\mathcal{S}_1 = \text{Alg} \left( \mathcal{T} \otimes \mathbb{C} \langle \mathcal{U}_1, \mathcal{U}_1^* \rangle \cup \{(z - P)^{-1} : P = P^* \in \mathcal{T} \otimes \mathbb{C} \langle \mathcal{U}_1, \mathcal{U}_1^* \rangle, z \not\in \text{sp}(P(\mathcal{U}_1, \mathcal{U}_1^*))) \right).
\]

(Again, here we consider unitary \( \mathcal{U}_1 \)'s, so it does not matter whether we write \( P(\mathcal{U}_1, \mathcal{U}_1^*) \) or \( P(\mathcal{U}_1, \mathcal{U}_1^{-1}) \), or \( \mathcal{T} \otimes \mathbb{C} \langle \mathcal{U}_1, \mathcal{U}_1^* \rangle \) or \( \mathcal{T} \otimes \mathbb{C} \langle \mathcal{U}_1, \mathcal{U}_1^{-1} \rangle \).) Normally one should write \((z(1_\mathcal{T} \otimes 1) - P)^{-1}\) instead of \((z - P)^{-1}\), but we choose to suppress the unit from the notation whenever the risk of confusion is small.) We define recursively \( \mathcal{S}_n \) out of \( \mathcal{S}_{n-1} \) the following way:

\[
\mathcal{S}_n = \text{Alg} \left( \mathcal{S}_{n-1} \cup \{(z - F_z)^{-1} : F_z \in \mathcal{S}_{n-1}, z \in \mathbb{C} \setminus \text{sp}(F_z(\mathcal{U}_r, \mathcal{U}_r^*)), \right.
\]

\[
\left. F_z = F_z^*, \text{ or } \Re z \Im F_z(\mathcal{U}_r, \mathcal{U}_r^*) \leq 0 \text{ and } F_z^*(\mathcal{U}_r, \mathcal{U}_r^*) = F_z(\mathcal{U}_r, \mathcal{U}_r^*)^* \right\}.
\]

In words: according to the induction hypothesis, any element of \( \mathcal{S}_{n-1} \) is evaluated in the \( r \)-tuple \( \langle \mathcal{U}_r, \mathcal{U}_r^* \rangle \) in order to yield a bounded operator which

\footnote{It is important to keep in mind that \( \mathcal{U}_j = \Psi(s_j) \) is unitary whenever \( s_j = s_j^* \), so that \( \mathcal{U}_j^* = \mathcal{U}_j^{-1} \), allowing us to alternate between the two notations. However, when considering evaluations on non-selfadjoint \( s_j \), the choice matters, and, unless specifically stated otherwise, we choose \( ^{-1} \) over \( ^* \).}
belongs to $\mathcal{T} \otimes \mathcal{A}_N$. An element belonging to $\mathcal{J}_{n-1}$ may depend on $z$ or not (it may just be a polynomial), so we denote it by $F_z$. In $\mathcal{J}_n$ we allow elements $F_z \in \mathcal{J}_{n-1}$ and resolvents of elements $F_z$, evaluated also in $z$, but only if either $F_z = F_z^*$ in $\mathcal{J}_{n-1}$ or the imaginary part of the operator $F_z(U, U^*)$ has the opposite sign compared to the imaginary part of $z$ when $z \not\in \mathbb{R}$ and is selfadjoint when $z \in \mathbb{R}$. In order to obtain a bounded operator when evaluating such a resolvent in $U, U^*$, we naturally require that $z \in \mathbb{C} \setminus \sigma(F_z(U, U^*))$. One should probably best view elements in any $\mathcal{J}_n$ as a subclass of rational functions of $(U, U^*) = (U_1, \ldots, U_r, U_{r+1}, \ldots, U_n)$ together with the “notational rule” on how to compute the evaluation of the rational function in question.

For clarity, an element in $\mathcal{J}_2$ could look like $(P - i)(z - P)^{-1} + (z - Q(z - R)^{-1}Q^*)^{-1}RQ(z - R)^{-1}$ for some $P = P^*, R = R^*, Q \in \mathcal{T} \otimes \mathbb{C}(U_r, U_r^*)$, or like $(z + iQ(z - R)^{-1}(z - R)^{-1}Q^*)^{-1}$. There is a major difference between the first and the second example: the first one, when evaluated in $U_r, U_r^*$, yields an analytic map defined on a subset of $\mathbb{C}$ which is the complement of a compact set, taking values in a von Neumann algebra, while the second does not. To separate these two cases, we define the subalgebras

$$\mathcal{J}_n^a = \{ F_z \in \mathcal{J}_n : z \mapsto F_z(U, U^*) \text{ is analytic on its domain} \}, \quad n \in \mathbb{N}.$$ 

These subalgebras do not contain $(z - P)^{-1}$, for instance. We define

$$\mathcal{J}_n^a = \bigcup_{n=1}^{\infty} \mathcal{J}_n^a,$$  \hspace{1cm} (16)

and, for any given state $\phi : \mathcal{T} \to \mathbb{C},$

$$\phi.\mathcal{J}_n^a = \{(\phi \otimes \text{id}_{\mathcal{A}_N})(F_z) : F_z \in \mathcal{J}_n^a \} \subset \mathcal{A}_N, \quad n = 0, 1, 2, \ldots, \infty.$$  \hspace{1cm} (17)

A polynomial $P \in \mathbb{C}(U_r, U_r^*)$ can be evaluated in $U_1 = \Psi(s_1), \ldots, U_r = \Psi(s_r), U_1^{-1} = \Psi(s_1)^{-1}, \ldots, U_r^{-1} = \Psi(s_r)^{-1}$ in the obvious way, by replacing $U_j$ with $U_j = \Psi(s_j)$ and $U_j^*$ with $U_j^{-1} = \Psi(s_j)^{-1}$. Motivated by (13) and (14), for any $P \in \mathcal{T} \otimes \mathbb{C}(U_r, U_r^*)$, $\partial_j P(U_r, U_r^*)$ is defined by requiring that it takes values in $(\mathcal{T} \otimes \mathbb{C}(U_r, U_r^*)) \otimes (\mathcal{T} \otimes \mathbb{C}(U_r, U_r^*))$, it respects the Leibniz rule, it is defined on monomials by

$$\partial_j (t \otimes U_{i_1}^{e_1}U_{i_2}^{e_2} \cdots U_{i_m}^{e_m})$$  \hspace{1cm} (18)
\[
\begin{align*}
\frac{i}{2} \sum_{1 \leq k \leq n} & \left[ t \otimes \mathcal{U}_{i_1}^1 \cdots \mathcal{U}_{i_{k-1}}^{c_k-1} (\mathcal{U}_{i_k} - 1) \right] \otimes_T \left[ 1_T \otimes (\mathcal{U}_{i_k} - 1) \mathcal{U}_{i_{k+1}}^{c_{k+1}} \cdots \mathcal{U}_{i_n}^{c_n} \right] \\
- \frac{i}{2} \sum_{1 \leq k \leq n} & \left[ t \otimes \mathcal{U}_{i_1}^1 \cdots \mathcal{U}_{i_{k-1}}^{c_k-1} (\mathcal{U}_{i_k} - 1) \right] \otimes_T \left[ 1_T \otimes (\mathcal{U}_{i_k}^* - 1) \mathcal{U}_{i_{k+1}}^{c_{k+1}} \cdots \mathcal{U}_{i_n}^{c_n} \right],
\end{align*}
\]

and is extended by linearity to polynomials. The reader may verify that the above is consistent with \( \partial(t \otimes \mathcal{U}^t) = \partial(t \otimes \mathcal{U}^t) = 0 \). The existence of such an object needs not be proven, since it follows from Voiculescu’s extension of the free difference quotient to the algebra generated by the Cayley transforms of the algebraically free selfadjoint variables \( s_1, \ldots, s_r \). The only reason we mention it is for the sake of simplifying notations in the rest of the paper.

Lemma 10. With the above notations, and under the above hypotheses, one has \( \partial_i : \mathcal{I}_n^a \to \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty^a \) for each \( 1 \leq i \leq r \), where \( \partial_i \) acts on \( \mathcal{U}, \mathcal{U}^* \) as described in [18].

Proof. It follows from Voiculescu’s result, see [8] that \( \partial_i((z - P)^{-1}) = ((z - P)^{-1} \otimes_T (1_T \otimes 1))(\partial_i P)((1_T \otimes 1) \otimes_T (z - P)^{-1}) ) \in \mathcal{I}_1^a \otimes_T \mathcal{I}_1^a \). Linearity and product rule together with [18] show then that \( \partial_i(\mathcal{I}_n^a) \subseteq \mathcal{I}_1^a \otimes_T \mathcal{I}_1^a \subseteq \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty^a \). Assume \( F_z \in \mathcal{I}_n^a \setminus \mathcal{I}_{n-1}^a \). Clearly (by linearity), \( \partial_i F_z \) is in \( \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty^a \) whenever all summands of \( F_z \) are mapped by \( \partial_i \) inside \( \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty^a \) (we do not imply here that \( F_z(\mathcal{U}, \mathcal{U}^*) \) has a unique expression as a sum - that would be false: \( z(z - P)^{-1} - 1 = (z - P)^{-1}P \) - our statement refers to the fact that, by definition, there exists at least one expression whose terms are all in \( \mathcal{I}_n^a \)). All summands which are in \( \mathcal{I}_{n-1}^a \) are mapped by \( \partial_i \) inside \( \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty^a \) by the induction hypothesis. Thus, we may assume without loss of generality that \( F_z \) has only one term, which is a product of elements of the form \( (z - E_z)^{-1} \) with \( E_z \in \mathcal{I}_{n-1}^a \), \( z \exists \exists E_z(\mathcal{U}, \mathcal{U}^*) \leq 0, E_z(\mathcal{U}, \mathcal{U}^*) = E_z(\mathcal{U}, \mathcal{U}^*)^* \), and \( P(\mathcal{U}, \mathcal{U}^*), P \in T \otimes \mathbb{C}(\mathcal{U}, \mathcal{U}^*) \). Since elements \( (z - E_z)^{-1} \in \mathcal{I}_n^a \) whenever \( E_z \in \mathcal{I}_j^a \), it follows that all factors \( (z - E_z)^{-1} \) for which \( E_z \in \mathcal{I}_j^a, j < n - 1 \), are themselves elements in \( \mathcal{I}_{n-1}^a \). Thus, one may write

\[ F_z = D_z^1(z - E_z^1)^{-1}D_z^2(z - E_z^2)^{-1}D_z^3 \cdots (z - E_z^{k-1})^{-1}D_z^{k+1}, \]

where \( D_z^1, \ldots, D_z^{k+1} \in \mathcal{I}_{n-1}^a, E_z^1, \ldots, E_z^k \in \mathcal{I}_{n-1}^a \setminus \mathcal{I}_{n-2}^a \), \( z \exists \exists E_z^j(\mathcal{U}, \mathcal{U}^*) \leq 0, E_z^j(\mathcal{U}, \mathcal{U}^*) = E_z^j(\mathcal{U}, \mathcal{U}^*)^* \), \( 1 \leq j \leq k \), \( \partial_i F_z \) is a sum of \( 2k + 1 \) terms: corresponding to each \( D_z^j \), one has the (possibly zero) term \( [D_z^j(z - E_z^1)^{-1}D_z^2 \cdots (z - E_z^{k-1})^{-1} \otimes_T (1_T \otimes 1)](\partial_i D_z^j) [(1_T \otimes 1) \otimes_T (z - E_z^1)^{-1}D_z^2 \cdots (z - E_z^{k-1})^{-1}D_z^{k+1}] \). By
the induction hypothesis, $\partial_i D_{i}^j \in \mathcal{I}^a_\infty \otimes \mathcal{I}_\infty$, and all other factors are in $\mathcal{I}_\infty$ by the definition of $F_z$. For each $E_{i}^j$, one has the (again possibly zero) term

$$[D_{i}^1 (z - E_{i}^1)^{-1} \cdots D_{i}^j (z - E_{i}^j)^{-1} \otimes_T (1_T \otimes 1)](\partial_i E_{i}^j)[(1_T \otimes 1) \otimes_T (z - E_{i}^j)^{-1} \times D_{i}^j \cdots (z - E_{i}^k)^{-1} D_{i}^{k+1}].$$

Again by induction hypothesis $\partial_i E_{i}^j \in \mathcal{I}^a_\infty \otimes \mathcal{I}_\infty$, while by definition both $D_{i}^1 (z - E_{i}^1)^{-1} \cdots D_{i}^j (z - E_{i}^j)^{-1}$ and $(z - E_{i}^j)^{-1} D_{i}^j \cdots (z - E_{i}^k)^{-1} D_{i}^{k+1}$ belong to $\mathcal{I}_\infty^a$.

Lemma 10 guarantees that $\mathcal{I}_\infty^a$ is stable under $\partial_i$, and hence under iterated applications of $\partial_i$ (that means, $(\text{id}_{\otimes_T^p} \otimes T \partial_i \otimes T \text{id}_{\otimes_T^{p-1}}) (\mathcal{I}_\infty^a) \subseteq (\mathcal{I}_\infty^a)^{\otimes_T^{p+1}}$ for all $p \in \mathbb{N}$, $0 \leq l \leq p - 1$).

Clearly,

flip: $\mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty \ni (t \otimes A) \otimes_T (\theta \otimes B) \mapsto (t \otimes B) \otimes_T (\theta \otimes A) \in \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty$, (19)

(the first tensor coordinate is obviously not flipped)

$$\text{ev}_1: \mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty \ni (t \otimes A) \otimes_T (\theta \otimes B) \mapsto (t \theta) \otimes (AB) \in \mathcal{I}_\infty^a,$$ (20)

so that $\mathcal{I}_\infty^a \otimes_T \mathcal{I}_\infty$ is mapped by $\text{ev}_1 \circ \text{flip}$ into $\mathcal{I}_\infty^a$.

**Lemma 11.** With the above notations, let $F_z \in \mathcal{I}_\infty^a$ be given. Assume that $K \subset \mathbb{R}$ is a compact set such that $z \mapsto F_z(U, U^*)$ is analytic on $\mathbb{C} \setminus K$. Then

$$z \mapsto (\partial_j F_z)(U, U^*; V, V^*) \in (T \otimes A_N) \otimes_T (T \otimes A_N)$$

is analytic on $\mathbb{C} \setminus K$ for each $1 \leq i \leq r$.

**Proof.** The fact is clearly true for $\mathcal{I}_1^a$: first, recalling Voiculescu’s formula

$$(\partial_j(z - P)^{-1})(U, U^*; V, V^*) = [(z - P(U, U^*))^{-1} \otimes_T (1_T \otimes 1)](\partial_j P)(U, U^*; V, V^*)[(1_T \otimes 1) \otimes_T (z - P(V, V^*))^{-1}],$$

a clearly analytic function defined on the resolvent of $P$ evaluated in an $r$-tuple of algebraically free unitaries, and with values in $(T \otimes A_N) \otimes_T (T \otimes A_N)$; one concludes the case $n = 1$ by applying the product rule and linearity. The induction hypothesis tells us that $z \mapsto (\partial_j F_z)(U, U^*; V, V^*)$ is analytic on the domain of $z \mapsto F_z(U, U^*)$ whenever $F_z \in \mathcal{I}_n^a$. As in the argument for Lemma 10 one only needs to prove this fact for a single product of the form $F_z = D_{i}^1 (z - E_{i}^1)^{-1} D_{i}^{2}(z -...
This page contains a complex mathematical text discussing the analytic properties of certain algebraic structures and operator-valued functions. The text is concerned with the behavior of sums of products of resolvents of bounded variables and the analyticity of these expressions in the neighborhood of infinity. Additionally, it discusses the extension of such properties to unitary elements and the implications for analytic continuation. The text also touches upon the use of free difference quotients and the convolution of operators in the context of analyticity.

The main results are presented in terms of lemmas, with proofs that involve induction processes and the use of power series expansions. The lemmas are numbered, with Lemma 12 being particularly noteworthy for its application of analyticity in the complex domain and the consideration of compact sets in the real line.

The text is structured in a way that allows for a progressive understanding of the concepts, starting from basic assumptions and building up to more complex statements. It is a typical example of advanced mathematical discourse, likely found in a specialized research paper or a graduate-level textbook on advanced operator theory.
Lemma 13. For any $i \in \{1, \ldots, r\}$, one has
\[ \partial_i \circ (\phi \otimes \text{id}_{A_N}) = \phi \circ \partial_i, \] (21)
where the $\partial_i$ in the right-hand side takes values in $(T \otimes A_N) \otimes_T (T \otimes A_N)$ (see Section 3.1.1), the $\partial_i$ in the left-hand side takes values in $A_N \otimes A_N$, and the $\phi$ in the right-hand side is the extension by linearity and continuity of the map $(t_1 \otimes a_1) \otimes_T (t_2 \otimes a_2) \mapsto \phi(t_1 t_2)(a_1 \otimes a_2)$.

(We should first note that the linear map $\phi$ in the right hand side of (21) is indeed well-defined: for any simple tensor $(t_1 \otimes a_1) \otimes_T (t_2 \otimes a_2) \in \mathcal{S}_\infty \otimes_T \mathcal{S}_\infty$, one can apply $\phi$ by imposing $(t_1 \otimes a_1) \otimes_T (t_2 \otimes a_2) \mapsto \phi(t_1 t_2)(a_1 \otimes a_2)$, and extend by linearity. This is well-defined because, as the reader will recall from Subsection 3.1.1, $(t_1 \otimes a_1) \otimes_T (t_2 \otimes a_2) = (t_1 t_2 \otimes a_1) \otimes_T (1 \otimes a_2) = (1 \otimes a_1) \otimes_T (t_1 t_2 \otimes a_2)$. Since $\phi$ is a state, the linear extension of this map further extends to the closure of this tensor product in any $C^*$-norm, which strictly includes the tensor product $\mathcal{S}_\infty \otimes_T \mathcal{S}_\infty$.)

Proof of Lemma 13. We first verify (21) on polynomials in algebraically free selfadjoint variables $s_1, \ldots, s_r$, with coefficients in $T$: if $P = \sum_w c_w \otimes s^w$, then $(\phi \otimes \text{id}_{A_N})(P) = \sum_w \phi(c_w)s^w$ and $\partial_i((\phi \otimes \text{id}_{A_N})(P)) = \sum_w \phi(c_w)\partial_i(s^w) = \sum_w \phi(c_w)\sum_{w=uiu} s^u \otimes s^v$, while $\partial_i P = \sum_w \partial_i(c_w \otimes s^w) = \sum_w \sum_{w=uiu}(c_w \otimes s^u) \otimes_T (1_T \otimes s^v)$ and $\phi(\sum_w \sum_{w=uiu}(c_w \otimes s^u) \otimes_T (1_T \otimes s^v)) = \sum_w \phi(c_w)\sum_{w=uiu}s^u \otimes s^v$ (again, here $\phi$ denoting also the extension $\phi: (T \otimes A_N) \otimes_T (T \otimes A_N) \to A_N \otimes A_N, \phi((t_1 \otimes a_1) \otimes_T (t_2 \otimes a_2)) = \phi(t_1 t_2)(a_1 \otimes a_2)$). This shows the
equality \( \partial_i \circ (\phi \otimes \text{id}_{\mathcal{A}_N}) = \phi \circ \partial_i \) on \( \mathcal{T} \otimes \mathbb{C}(s_1, \ldots, s_d) \). Next, we verify this on polynomials in the Cayley transform of \( s_1, \ldots, s_r \). If before the sums were indexed by elements from the free semigroup with \( r \) generators, now they must be indexed by the free group with \( r \) generators: if \( P = \sum_w c_w \otimes \mathcal{U}_w^w \), then \( (\phi \otimes \text{id}_{\mathcal{A}_N})(P) = \sum_w \phi(c_w)\mathcal{U}_w^w \) and \( \partial_i((\phi \otimes \text{id}_{\mathcal{A}_N})(P)) = \sum_w \phi(c_w)\partial_i(\mathcal{U}_w^w) = \sum_w \phi(c_w)\frac{i}{2}(\sum_{u=uiw} \mathcal{U}_u^u(U_i-1) \otimes (U_i-1)\mathcal{U}_w^w - \sum_{w=ui(-i)} \mathcal{U}_w^w(U_i^* - 1) \otimes (U_i^* - 1)\mathcal{U}_w^w) \).

When we apply first the free difference quotient, we obtain, according to \([18]\), \( \partial_i P = \sum_w \partial_i (c_w \otimes \mathcal{U}_w^w) = \frac{i}{2} \sum_w (\sum_{w=uiw} (c_w \otimes \mathcal{U}_w^w(U_i - 1)) \otimes \mathcal{T} (1_{\mathcal{T}} \otimes (U_i - 1)\mathcal{U}_w^w)) - \sum_{w=ui(-i)} (c_w \otimes \mathcal{U}_w^w(U_i^* - 1)) \otimes \mathcal{T} (1_{\mathcal{T}} \otimes (U_i^* - 1)\mathcal{U}_w^w)) \). Applying \( \phi \) yields the same \( \sum_w \phi(c_w)\frac{i}{2}(\sum_{w=uiw} \mathcal{U}_u^u(U_i - 1) \otimes (U_i - 1)\mathcal{U}_w^w - \sum_{w=ui(-i)} \mathcal{U}_w^w(U_i^* - 1) \otimes (U_i^* - 1)\mathcal{U}_w^w) \).

Next, thanks to the continuity of \( \phi \), it follows that this same equality extends on convergent power series as in the previous lemma. Indeed, one needs only recall that the series remains convergent under the application of \( \partial_i \). Thus, in particular, \( \partial_i (\phi \otimes \text{id}_{\mathcal{A}_N}) = \phi \circ \partial_i \) on the space of power series

\[
\sum_{n=0}^{\infty} \frac{(P_n(U_i^*, \mathcal{U}_i^*))^n}{z^{n+1}}, \quad P_n \in \mathcal{T} \otimes \mathbb{C}(\mathcal{U}_i^*, \mathcal{U}_i^*),
\]

applied term-wise, whenever \( \sum_{n=0}^{\infty} \frac{||P_n(U_i^*, \mathcal{U}_i^*)||^n}{z^{n+1}} < \infty \) for all \( z, |z| \) sufficiently large. An application of Lemmata \([11]\) and \([12]\) allows us to conclude. \( \square \)

### 3.2.2 The Schur complement

We only use in this paper a very simple version of this famous formula. Assuming that an operator \( X \) has a block decomposition \( X = \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix} \), suppose that \( x_{2,2} \) is invertible in the algebra reduced by the projection onto its domain. Then \( X \) is invertible if and only if \( x_{1,1} - x_{1,2}x_{2,2}^{-1}x_{2,1} \) is invertible in the algebra reduced by the projection onto the domain of \( x_{1,1} \), and the entries of the block decomposition of \( X^{-1} \) are given by

\[
X^{-1} = \begin{bmatrix}
(x_{1,1} - x_{1,2}x_{2,2}^{-1}x_{2,1})^{-1} & -(x_{1,1} - x_{1,2}x_{2,2}^{-1}x_{2,1})^{-1}x_{1,2}x_{2,2}^{-1} \\
-x_{2,2}^{-1}x_{2,1}(x_{1,1} - x_{1,2}x_{2,2}^{-1}x_{2,1})^{-1}x_{2,2}^{-1} + x_{2,2}^{-1}x_{2,1}(x_{1,1} - x_{1,2}x_{2,2}^{-1}x_{2,1})^{-1}x_{1,2}x_{2,2}^{-1} & x_{2,2}^{-1}
\end{bmatrix}
\]

(23)
For the specific case of a resolvent, the Schur complement formula is written as

\[
\begin{bmatrix}
    z - f & \iota \\
    i & z - j
\end{bmatrix}^{-1} = \begin{bmatrix}
    (z - f - \iota(z - j)^{-1}i)^{-1} & (z - f - \iota(z - j)^{-1}i)^{-1}i[j - z]^{-1}i(z - f - \iota(z - j)^{-1}i)^{-1}i[j - z]^{-1}i\\
    [j - z]^{-1}i(z - f - \iota(z - j)^{-1}i)^{-1} & [j - z]^{-1}i(z - f - \iota(z - j)^{-1}i)^{-1}i(z - f - \iota(z - j)^{-1}i)^{-1}i[j - z]^{-1}i
\end{bmatrix}.
\]

(All of \(f, \iota, i, j\) should be viewed as blocks of matrices/operators, and \(z\) as the multiplication of the identity corresponding to the block in question with the complex number \(z\).) We will mostly deal with selfadjoint matrices, so that \(f = f^*\), \(\iota^* = i\), and \(j = j^*\). In that case, it is known that the spectrum of the operator is included in \(\mathbb{R}\), so the above makes sense at least for \(z \in \mathbb{C}^\pm\).

The main object of our study in this paper is the resolvent of an affine matrix-valued polynomial. Specifically, we consider selfadjoint operators of the form

\[
S = \xi \otimes 1 \otimes 1 + 2 \sum_{j=1}^{r_1} \Re(\gamma_j \otimes u_j \otimes 1) + 2 \sum_{j=1}^{r_2} \Re(\beta_j \otimes 1 \otimes v_j), \tag{24}
\]

where \(\xi, \gamma_1, \ldots, \gamma_{r_1}, \beta_1, \ldots, \beta_{r_2} \in M_m(\mathbb{C})\) for some arbitrary \(m \in \mathbb{N}, m \geq 1\), and \(u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}\) are Cayley transforms of arbitrary (possibly unbounded) selfadjoint random variables in \(\tilde{A}_N\). (Recall that \(\Re X = \frac{X + X^*}{2}\) denotes the real part of the operator \(X\) and \(\Im X = \frac{X - X^*}{2i}\) its imaginary part.) We shall usually write \((z - S)^{-1}\) for the resolvent of \(S\), instead of \((zI_m \otimes 1 \otimes 1 - S)^{-1}\). The Schur complement formula applied to \((z - S)^{-1}\) makes clear the usefulness of the spaces \(\mathcal{S}_\infty^a\). For the following results, we do not need to assume that both the \(u\)’s and the \(v\)’s are Cayley transforms of selfadjoints, so we will drop this assumption for the rest of this section and simply assume that

\[
S = \xi \otimes 1 \otimes 1 + 2 \sum_{j=1}^{r_1} \Re(\gamma_j \otimes t_j \otimes 1) + 2 \sum_{j=1}^{r_2} \Re(\beta_j \otimes 1 \otimes v_j), \tag{25}
\]

where the \(\xi, \gamma_j, \beta_j, v_j\) are precisely as above, but \(t_1, \ldots, t_{r_1}\) are arbitrary bounded operators belonging to some von Neumann algebra \(\mathcal{T}\) endowed with a state \(\phi\).

**Lemma 14.** The resolvent \((z - S)^{-1}\) of the operator \(S \in M_m(\mathcal{T} \otimes \mathcal{A}_N)\) introduced in \((25)\) belongs to \(M_m(\mathcal{S}_\infty^a)\), that is, each of its entries is an element from \(\mathcal{S}_\infty^a\).
Proof. The lemma is intuitively obvious once one considers the Schur complement formula, but we were unable to find a formal complete argument that is short. The proof is again by induction, this time after the matrix size $m$, and by using the (explicit) Schur complement formula.

The element $\left( (zI_m - \xi) \otimes 1 \otimes 1 - \sum_{i=1}^{r_1} 2\Re(\gamma_i \otimes t_i \otimes 1) - \sum_{i=1}^{r_2} 2\Re(\beta_i \otimes 1 \otimes v_i) \right)^{-1} \in M_m(\mathcal{T} \otimes \mathcal{A}_N)$ is a matrix of the form

$$
\begin{bmatrix}
z - \ell_{1,1}(t_i \otimes 1, 1 \otimes v_j) & \ell_{1,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{1,m}(t_i \otimes 1, 1 \otimes v_j) \\
\ell_{1,2}(t_i \otimes 1, 1 \otimes v_j)^* & z - \ell_{2,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j) \\
\vdots & \vdots & \ddots & \vdots \\
\ell_{1,m}(t_i \otimes 1, 1 \otimes v_j)^* & \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j)^* & \cdots & z - \ell_{m,m}(t_i \otimes 1, 1 \otimes v_j)
\end{bmatrix}^{-1},
$$

where $\ell_{i,j} \in \mathbb{C}^{(\mathcal{U}_{r_1}, \mathcal{U}_{r_1}^*) \otimes \mathbb{C}^{(\mathcal{U}_{r_2}, \mathcal{U}_{r_2}^*)} \mathcal{A}_N}$ are affine polynomials. This matrix is selfadjoint whenever $z \in \mathbb{R}$; otherwise, we are looking at the resolvent of a selfadjoint operator, it is a normal operator with strictly negative imaginary part. If $m = 1$, then $((z - \xi)1 \otimes 1 - 2\Re \sum_{i=1}^{r_1} \gamma_i(t_i \otimes 1) - 2\Re \sum_{i=1}^{r_2} \beta_i (1 \otimes v_i))^{-1}$

$\in \mathcal{T} \otimes \mathcal{A}_N$, with $\xi \in \mathbb{R}, \gamma_1, \ldots, \gamma_{r_1}, \beta_1, \ldots, \beta_{r_2} \in \mathbb{C}$. We are looking at the resolvent of the affine selfadjoint polynomial $\xi 1 \otimes 1 + 2\Re \sum_{i=1}^{r_1} \gamma_i(t_i \otimes 1) + 2\Re \sum_{i=1}^{r_2} \beta_i (1 \otimes U_i) \in \mathcal{T} \otimes \mathbb{C}^{(\mathcal{U}_{r_2}, \mathcal{U}_{r_2}^*)}$. By definition, its resolvent, evaluated in $\mathcal{U}_{r_2}, \mathcal{U}_{r_2}^*$, belongs to $\mathcal{J}^{a}$. We could now go directly to the induction step, but it might help intuition to analyze the case $m = 2$ too. The above formula becomes (due to lack of space, we cannot write it in matrix form, we only specify which formulae correspond to which entry):

$(1,1) : (z - \ell_{1,1} - \ell_{1,2}[z - \ell_{2,2}]^{-1}\ell_{1,2})^{-1}$

$= [z - \ell_{1,1} ]^{-1} + [z - \ell_{1,1}]^{-1}\ell_{1,2}(z - \ell_{2,2} - \ell_{1,2}[z - \ell_{1,1}]^{-1}\ell_{1,2})^{-1}\ell_{1,2}[z - \ell_{1,1}]^{-1}$

$(1,2) : (z - \ell_{1,1} - \ell_{1,2}[z - \ell_{2,2}]^{-1}\ell_{1,2})^{-1}\ell_{1,2}[z - \ell_{2,2}]^{-1}$

$= [\ell_{1,1} - z]^{-1}\ell_{1,2}(z - \ell_{2,2} - \ell_{1,2}[z - \ell_{1,1}]^{-1}\ell_{1,2})^{-1}$

$(2,1) : [\ell_{2,2} - z]^{-1}\ell_{1,2}[z - \ell_{1,1} - \ell_{2,2}[z - \ell_{2,2}]^{-1}\ell_{1,2}]^{-1}$

$= (z - \ell_{2,2} - \ell_{1,2}[z - \ell_{1,1}]^{-1}\ell_{1,2})^{-1}\ell_{1,2}[z - \ell_{1,1}]^{-1}$

$(2,2) : [z - \ell_{2,2}]^{-1} + [z - \ell_{2,2}]^{-1}\ell_{1,2}(z - \ell_{1,1} - \ell_{1,2}[z - \ell_{2,2}]^{-1}\ell_{1,2})^{-1}\ell_{1,2}[z - \ell_{2,2}]^{-1}$

$= (z - \ell_{2,2} - \ell_{1,2}[z - \ell_{1,1}]^{-1}\ell_{1,2})^{-1}$.  

We recall that both $\ell_{1,1}, \ell_{2,2} \in \mathcal{T} \otimes \mathcal{A}_N$ are selfadjoint. A brief visual examination of the above formulae makes it clear that this fact places all of these entries in $\mathcal{J}_2^a$.  

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Now our statement has been proved for \( m = 1, 2 \). Assume it true for \( m - 1 \) (that is, the resolvent of any operator of the form \( \xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2\Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \), with \( \xi = \xi^*, \gamma_i, \beta_i \in M_{m-1}(\mathbb{C}) \), has all entries in \( \mathcal{F}_\infty^a \)). Denote

\[
Q_z = \begin{bmatrix} z - \ell_{2,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j) \\ \vdots & \ddots & \vdots \\ \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j)^* & \cdots & z - \ell_{m,m}(t_i \otimes 1, 1 \otimes v_j) \end{bmatrix},
\]

the lower right \((m - 1) \times (m - 1)\) corner of our matrix,

\[
u = \begin{bmatrix} \ell_{1,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{1,m}(t_i \otimes 1, 1 \otimes v_j) \end{bmatrix},
\]

the upper \(1 \times (m - 1)\) right corner of our matrix, and

\[
u^* = \begin{bmatrix} \ell_{1,2}(t_i \otimes 1, 1 \otimes v_j)^* \\ \vdots \\ \ell_{1,m}(t_i \otimes 1, 1 \otimes v_j)^* \end{bmatrix}
\]

its adjoint, which is the lower left \((m - 1) \times 1\) corner (the upper left \(1 \times 1\) corner is \(z - \ell_{1,1}(t_i \otimes 1, 1 \otimes v_j)\)). Schur’s formula tells us that

\[
\begin{bmatrix} z - \ell_{1,1}(t_i \otimes 1, 1 \otimes v_j) & \ell_{1,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{1,m}(t_i \otimes 1, 1 \otimes v_j) \\ \ell_{1,2}(t_i \otimes 1, 1 \otimes v_j)^* & z - \ell_{2,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j) \\ \vdots & \ddots & \vdots \\ \ell_{1,m}(t_i \otimes 1, 1 \otimes v_j)^* & \ell_{2,m}(t_i \otimes 1, 1 \otimes v_j)^* & \cdots & z - \ell_{m,m}(t_i \otimes 1, 1 \otimes v_j) \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} (z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1} & -(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1}uQ_z^{-1} \\ -Q_z^{-1}u^*(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1} & Q_z^{-1} + Q_z^{-1}u^*(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1}uQ_z^{-1} \end{bmatrix}
\]

\[
= \begin{bmatrix} (z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1} & -(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1}uQ_z^{-1} \\ -Q_z^{-1}u^*(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1} & Q_z^{-1} - u^*(z - \ell_{1,1} - uQ_z^{-1}u^*)^{-1}uQ_z^{-1} \end{bmatrix},
\]

where we have again suppressed the variables \(s, t\) in our notation. According to the induction hypothesis, all entries of \(Q_z^{-1}\) belong to \(\mathcal{F}_\infty^a\). The correspondence \(z \mapsto Q_z^{-1}\) is analytic on the resolvent set of the selfadjoint operator

\[
\begin{bmatrix} \ell_{2,2}(t_i \otimes 1, 1 \otimes v_j) & \cdots & -\ell_{2,m}(t_i \otimes 1, 1 \otimes v_j) \\ \vdots & \ddots & \vdots \\ -\ell_{2,m}(t_i \otimes 1, 1 \otimes v_j)^* & \cdots & \ell_{m,m}(t_i \otimes 1, 1 \otimes v_j) \end{bmatrix},
\]

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If Lemma 15. Let \( Q_z = (Q_z^{-1})^* \), and \( \Im z Q_z^{-1} \leq 0 \). This guarantees that \( \Im z (\ell_{1,1} + uQ_z^{-1}u^*) \leq 0 \) and \( \ell_{1,1} + uQ_z^{-1}u^* = (\ell_{1,1} + uQ_z^{-1}u^*)^* \). Since all entries of \( Q_z^{-1} \) are in \( S_\infty \), one has \( \ell_{1,1} + uQ_z^{-1}u^* \in S_\infty \). It follows that \( (z - \ell_{1,1} + uQ_z^{-1}u^*)^{-1} \in S_\infty \).

This, together with the formulae for the other entries, shows that all entries of

\[
\begin{bmatrix}
z - \ell_{1,1}(t_1 \otimes 1, 1 \otimes v_j) & \ell_{1,2}(t_1 \otimes 1, 1 \otimes v_j) & \cdots & \ell_{1,m}(t_1 \otimes 1, 1 \otimes v_j) \\
\ell_{1,2}(t_1 \otimes 1, 1 \otimes v_j)^* & z - \ell_{2,2}(t_1 \otimes 1, 1 \otimes v_j) & \cdots & \ell_{2,m}(t_1 \otimes 1, 1 \otimes v_j) \\
\vdots & \ddots & \ddots & \vdots \\
\ell_{1,m}(t_1 \otimes 1, 1 \otimes v_j)^* & \ell_{2,m}(t_1 \otimes 1, 1 \otimes v_j)^* & \cdots & z - \ell_{m,m}(t_1 \otimes 1, 1 \otimes v_j)
\end{bmatrix}^{-1}
\]

belong to \( S_\infty \), as claimed.

Finally, we record a statement relating our spaces \( S_\infty \) to Schwartz distributions - and thus justifying the choice of the notation \( S \):

**Lemma 15.** If \( F_z \in S_\infty \), then there exists a \( \kappa \in \mathbb{N} \) such that

\[
\lim_{\Im z \to 0} |\Im z|^\kappa \|F_z\| = 0,
\]

uniformly in \( \Re z \in \mathbb{R} \). In particular, for any state \( \phi: \mathcal{T} \to \mathbb{C} \), the map \( \mathbb{C}^+ \ni z \mapsto (\phi \otimes \tau_N)(F_z) \in \mathbb{C} \) is the Cauchy transform of a Schwartz distribution with compact support in \( \mathbb{R} \).

**Proof.** The proof is quite straightforward. Recall that by its definition, \( F_z \) is a finite sum of finite products of the form \( D^1(z - E_1^1)D^2(z - E_2^2)D^3 \cdots (z - E_k^k)D^k+1 \), where \( D^1, \ldots, D^{k+1} \) are polynomials with coefficients in \( \mathcal{T} \) (possibly equal to \( 1 \otimes 1 \)), and \( E_1^1, \ldots, E_k^k \) are either selfadjoint polynomials or elements from \( S_\infty \) whose imaginary parts have opposite sign to the imaginary part of \( z \). Thus, since if each term of a finite sum satisfies the conclusion of the lemma, then so does the sum (with a possibly larger \( \kappa \)), we may assume without loss of generality that

\[
F_z = D^1(z - E_1^1)^{-1}D^2(z - E_2^2)^{-1}D^3 \cdots (z - E_k^k)^{-1}D^{k+1}.
\]

For any given evaluation of \( F_z \) on unitaries \( U, U^* \), one has \( \|F_z\| \leq \|D^1\|\|D^2\|\|D^3\| \cdots \|D^{k+1}\| \). For each \( j \in \{1, \ldots, k\} \) and \( z \in \mathbb{C}^+ \), one has (by the definition of \( S_\infty \)) \( 0 < \Im z \leq \Im z - \Im E_j^j \implies \frac{1}{\Im z} \geq (\Im z - \Im E_j^j)^{-1} \geq 0 \), so

\[
\|(z - E_j^j)^{-1}\| = \|(\Re (z - E_j^j) + i\Im (z - E_j^j))^{-1}\|
\]

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Thus, \(|F_z| \leq \frac{\|D_1\| \|D_2\| \ldots \|D_{k+1}\|}{|z|^k}\). Choosing \(\kappa = k + 1\) yields the desired result.

The statement regarding \((\phi \otimes \tau_N)(F_z)\) follows from this estimate, Lemma \[12\] and the characterization of Cauchy transforms of Schwartz distributions with compact support in \(\mathbb{R}\) that can be found in [20].

\[\square\]

4 Fundamental preliminary result

One of the most important and fundamental results in free probability is Voiculescu’s asymptotic freeness result [21] for independent G.U.E. matrices: an \(r\)-tuple of independent G.U.E. random matrices \((X_1, \ldots, X_r) \in (M_N(L^\infty([0, 1], dx)), \text{tr}_N \otimes \mathbb{E})\) converges in distribution to an \(r\)-tuple \((s_1, \ldots, s_r) \in (\mathcal{A}, \tau)\) of standard free semicircular random variables. This result has been strengthened numerous times by many authors (including Voiculescu).

For our purposes it is important that the “rate of convergence” is now known. It is shown in [19], [12, Theorem 1.1] that the following linear functional exists:

\[\nu_1 : \mathcal{C}(\mathbb{F}_r) \to \mathbb{C}, \text{ given by }\]

\[\nu_1(P) := \lim_{N \to \infty} N^2 (\mathbb{E}[\text{tr}_N(P(X_1, \ldots, X_r))] - \tau(P(s_1, \ldots, s_r))), \quad (26)\]

for any \(P \in \mathcal{C}(\mathbb{F}_r)\), where \(s_1, \ldots, s_r\) are free standard semicirculars and \(X_1, \ldots, X_r\) are independent selfadjoint GUE matrices. Define also the linear functionals \(\nu_1^{(N)}, \nu_2^{(N)} : \mathcal{C}(\mathbb{F}_r) \to \mathbb{C}\) by setting

\[\nu_1^{(N)}(P) := N^2 (\mathbb{E}[\text{tr}_N(P(X_1, \ldots, X_r))] - \tau(P(s_1, \ldots, s_r))), \quad (27)\]

\[\nu_2^{(N)}(P) := N^2 \left[ N^2 (\mathbb{E}[\text{tr}_N(P(X_1, \ldots, X_r))] - \tau(P(s_1, \ldots, s_r))) - \nu_1(P(s_1, \ldots, s_r)) \right]. \quad (28)\]

(Observe that \(\nu_1 = \lim_{N \to \infty} \nu_1^{(N)}, \text{ pointwise.}\)
4.1 Some results due to Parraud

Parraud [12] showed that $\nu_1, \nu_2^{(N)}$ extend to significantly larger classes of noncommutative functions defined on certain operators (this comes as a consequence of some bounds on $\nu_j(P)$ in terms of $j$ and the polynomial $P$). For our purposes, it is important that they extend to exponential functions such as the ones from [14]. This comes from a remarkable set of explicit formulae for $\nu_1, \nu_j^{(N)}, j = 1, 2$, in terms of free difference quotients, flips, and evaluations, identified by Parraud that we are going to present now.

Consider free $r$-tuples of standard free semicircular variables $\tilde{\mathbf{w}}, \tilde{\mathbf{z}}^1, \tilde{\mathbf{z}}^2; \tilde{\mathbf{w}}, \tilde{\mathbf{z}}^1, \tilde{\mathbf{z}}^2$, and $\mathbf{x}$, and let $X_N$ be an $r$-tuple of independent G.U.E., free from the others. Define for $0 \leq t_1 \leq t_2$

\[
\begin{align*}
\tilde{z}^1_r(N) &= (1 - e^{-t_1})^{1/2} \tilde{z}^1_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_N, \\
\tilde{z}^2_r(N) &= (1 - e^{-t_1})^{1/2} \tilde{z}^2_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_N, \\
\tilde{z}^1_r(N) &= (1 - e^{-t_1})^{1/2} \tilde{z}^1_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_N, \\
\tilde{z}^2_r(N) &= (1 - e^{-t_1})^{1/2} \tilde{z}^2_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_N, \\
\end{align*}
\]

and

\[
\begin{align*}
\tilde{z}^1_r &= (1 - e^{-t_1})^{1/2} \tilde{z}^1_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_r, \\
\tilde{z}^2_r &= (1 - e^{-t_1})^{1/2} \tilde{z}^2_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_r, \\
\tilde{z}^1_r &= (1 - e^{-t_1})^{1/2} \tilde{z}^1_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_r, \\
\tilde{z}^2_r &= (1 - e^{-t_1})^{1/2} \tilde{z}^2_r + (e^{-t_1} - e^{-t_2})^{1/2} \tilde{w}_r + e^{-t_2/2} X_r.
\end{align*}
\]

The first four are $r$-tuples of unbounded selfadjoint random variables, all identically distributed, and the last four are standard semicircular variables whose distributions do not depend on $t_1, t_2$ (an easy verification). Define

\[\mathcal{F}_r = \mathbb{C}\langle e^{tR}, R \in \mathbb{C}\langle \mathbf{x} \rangle, R = R^\ast, \mathbb{C}\langle \mathbf{x} \rangle \rangle.\]

The following proposition is a corollary of [12, Proposition 3.7].

**Proposition 16.** [12, Proposition 3.7] For any $Q \in \mathcal{F}_r$, one has

\[
(\text{tr}_N \otimes \mathbb{E})(Q(X_1, \ldots, X_r)) = \tau(Q(s_1, \ldots, s_r)) + \frac{\nu_2^{(N)}(Q)}{N^2}.
\]

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where

\[
\nu_1^{(N)}(Q) = \int_0^{+\infty} \int_{t_2}^{+\infty} \tau_N(L^{t_1,t_2}(Q)(\tilde{z}_1^1(N), \tilde{z}_1^2(N), \tilde{z}_2^2(N), \tilde{z}_2^1(N))) \, dt_1 \, dt_2,
\]

with, for any four tuples \((y_1^1, \tilde{y}_1^1, \tilde{y}_2^2, y_2^2)\) of \(r\)-tuples,

\[
L^{t_1,t_2}(Q)(y_1^1, \tilde{y}_1^1, \tilde{y}_2^2, y_2^2) = \frac{1}{2} e^{-t_2-t_1} \sum_{1 \leq i,j \leq r} \Theta_{y_1^1, \tilde{y}_1^1, \tilde{y}_2^2, y_2^2}^{i,j} (\partial_j \otimes \partial_j (\partial_i (D_i Q))),
\]

\(D_i = ev_1 \circ \text{flip} \circ \partial_i\) and

\[
\Theta_{x_1, x_2, x_3, x_4} : A \otimes B \otimes C \otimes D \mapsto B(X_1)A(X_2)D(X_3)C(X_4).
\]

Moreover, \(\nu_1\) defined in (26) is explicitly given by

\[
\nu_1(Q) = \int_0^{+\infty} \int_{t_2}^{+\infty} \tau_N(L^{t_1,t_2}(Q)(\tilde{z}_1^1, \tilde{z}_1^2, \tilde{z}_2^2, \tilde{z}_2^1)) \, dt_1 \, dt_2.
\]

Now define

\[
J_1 = \{\{2, 1\}, \{3, 1\}, \{5, 4\}, \{6, 4\}\} = \{E_1, E_2, \tilde{E}_1, \tilde{E}_2\}
\]

and set

\[
X_{i,E_1} := y_1^1, \text{ the } i^{\text{th}} \text{ coordinate of the vector } y_1^1,
\]

\[
X_{i,\tilde{E}_1} := \tilde{y}_1^1, \text{ the } i^{\text{th}} \text{ coordinate of the vector } \tilde{y}_1^1,
\]

\[
X_{i,\tilde{E}_2} := \tilde{y}_2^2, \text{ the } i^{\text{th}} \text{ coordinate of the vector } \tilde{y}_2^2,
\]

\[
X_{i,E_2} := y_2^2, \text{ the } i^{\text{th}} \text{ coordinate of the vector } y_2^2.
\]

This makes \(L^{t_1,t_2}(Q)\) into a function in \(4r\) variables, indexed by the sets of \(J_1\), of \(r\)-tuples \((X_{i,E_1})_{1 \leq i \leq r}, (X_{i,\tilde{E}_1})_{1 \leq i \leq r}, (X_{i,\tilde{E}_2})_{1 \leq i \leq r}\) and \((X_{i,E_2})_{1 \leq i \leq r}\). Similarly, for \(I, K \in J_1\), and \(1 \leq i, j \leq r\), one has according to Voiculescu’s definition

\[
\partial_{I,K} X_{j,K} = \delta_{i,j} \delta_{I,K} 1 \otimes 1,
\]

and we define

\[
\partial_{i} X_{j,K} = \delta_{i,j} 1 \otimes 1.
\]
Let us introduce the following sets of sets. The reader might question whether this last version of the free difference and tuple of r-tuples. Let $J$ be free. Since there are four elements in each set $X$ is by definition the union of all these sets, that is,

$$J_2^{1,1} = \{\{8, 2, 1, 19\}, \{9, 3, 1, 19\}, \{11, 5, 4, 19\}, \{12, 6, 4, 19\}\},$$

$$J_2^{2,1} = \{\{8, 7, 1, 19\}, \{9, 7, 1, 19\}, \{11, 10, 4, 19\}, \{12, 10, 4, 19\}\},$$

$$J_2^{1,2} = \{\{14, 2, 1, 19\}, \{15, 3, 1, 19\}, \{17, 5, 4, 19\}, \{18, 6, 4, 19\}\},$$

$$J_2^{2,2} = \{\{14, 13, 1, 19\}, \{15, 13, 1, 19\}, \{17, 16, 4, 19\}, \{18, 16, 4, 19\}\},$$

$$J_2^{3,1} = \{\{8, 7, 20, 19\}, \{9, 7, 20, 19\}, \{11, 10, 20, 19\}, \{12, 10, 20, 19\}\},$$

$$J_2^{3,2} = \{\{14, 13, 21, 19\}, \{15, 13, 21, 19\}, \{17, 16, 21, 19\}, \{18, 16, 21, 19\}\}$$

and

$$J_2^{1,1} = \{\{29, 23, 22, 40\}, \{30, 24, 22, 40\}, \{32, 26, 25, 40\}, \{33, 27, 25, 40\}\},$$

$$J_2^{2,1} = \{\{29, 28, 22, 40\}, \{30, 28, 22, 40\}, \{32, 31, 25, 40\}, \{33, 31, 25, 40\}\},$$

$$J_2^{1,2} = \{\{35, 23, 22, 40\}, \{36, 24, 22, 40\}, \{38, 26, 25, 40\}, \{39, 27, 25, 40\}\},$$

$$J_2^{2,2} = \{\{35, 34, 22, 40\}, \{36, 34, 22, 40\}, \{38, 37, 25, 40\}, \{39, 37, 25, 40\}\},$$

$$J_2^{3,1} = \{\{29, 28, 41, 40\}, \{30, 28, 41, 40\}, \{32, 31, 41, 40\}, \{33, 31, 41, 40\}\},$$

$$J_2^{3,2} = \{\{35, 34, 42, 40\}, \{36, 34, 42, 40\}, \{38, 37, 42, 40\}, \{39, 37, 42, 40\}\}.$$

$J_2$ is by definition the union of all these sets, that is, $J_2 = \left(\bigcup_{l,p=1}^{2} J_2^{l,p}\right) \cup \left(\bigcup_{l,p=1}^{2} \tilde{J}_2^{l,p}\right) \cup J_2^{3,1} \cup J_2^{3,2} \cup \tilde{J}_2^{3,1} \cup \tilde{J}_2^{3,2}$ contains 48 sets of integer numbers smaller than 42. Define now

$$X_{1,1} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{1,1}}, \quad X_{2,1} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{2,1}}, \quad X_{1,2} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{1,2}}, \quad X_{2,2} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{2,2}}, \quad X_{3,1} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{3,1}}, \quad X_{3,2} = (X_{i,l})_{1 \leq i \leq r, l \in J_2^{3,2}}, \quad \tilde{X}_{1,1} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{1,1}}, \quad \tilde{X}_{2,1} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{2,1}}, \quad \tilde{X}_{1,2} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{1,2}}, \quad \tilde{X}_{2,2} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{2,2}}, \quad \tilde{X}_{3,1} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{3,1}}, \quad \tilde{X}_{3,2} = (X_{i,l})_{1 \leq i \leq r, l \in \tilde{J}_2^{3,2}}.$$

Since there are four elements in each set $J_2^{l,p}$, $\tilde{J}_2^{l,p}$, each $X_{l,p}$ and $\tilde{X}_{l,p}$ is a 4-tuple of r-tuples. Let $x_1^1, \ldots, x_4^{42}$ be free r-tuples of standard free semicircular
systems and $X_{N,r}$ be as before an $r$-tuple of independent $N \times N$ G.U.E. matrices. Define now for $T_2 = (t_1, t_2, t_3, t_4)$ with $t_4 \geq t_2 \geq t_1 \geq 0$ and $t_3 \geq t_4$, for any $1 \leq i \leq r$, for any $I \in J_2$, $I = \{I_1, I_2, I_3, I_4\}$ where the $I_p$'s are integer numbers smaller or equal to 42,

$$X_{i,I}^{N,T_2} = \sum_{l=1}^{4} \left( e^{-\tilde{t}_1} - e^{-\tilde{t}_l} \right)^{1/2} X_i^T_{l} + e^{-t_4/2} X_N^{(i)},$$

where $\tilde{t}_1 \leq \cdots \leq \tilde{t}_4$ are the $t_i$'s in increasing order, and $\tilde{t}_0 = 0$.

The following proposition is once more a corollary of [12, Proposition 3.7].

**Proposition 17.** [12, Proposition 3.7] For any $Q \in F_r$, one has

$$(\text{tr}_N \otimes \mathbb{E})(Q(X_1, \ldots, X_r)) = \tau(Q(s_1, \ldots, s_r)) + \frac{\nu_1(Q)}{N^2} + \frac{\nu_2^{(N)}(Q)}{N^2},$$

where

$$\nu_2^{(N)}(Q) = \int_{A_2} \tau_N \left( L_{T_2} \circ L^{t_1,t_2}(Q)(X_{N,T_2}) \right) dt_1 \cdots dt_4, \quad (31)$$

$A_2 = \{T_2 = (t_1, t_2, t_3, t_4), 0 \leq t_1 \leq t_2 \leq t_3 \leq t_4\},$

$$L_{T_2} = e^{-t_4-t_3} \left\{ 1_{[t_2,t_4]}(t_3)L_3 + 1_{[0,t_1]}(t_3)L_1 + 1_{[t_1,t_2]}(t_3)L_2 \right\},$$

with for $l = 1, 2$,

$$L_l = \frac{1}{2} \sum_{l \leq i,j \leq r} \sum_{I,J \in J_l} \Theta^{X_{i,1,1},X_{i,1,2},X_{i,2}} \left[ \partial_{j} \otimes \partial_{j}(\partial_{i}D_{i}) \right]$$

and

$$L_3 = \frac{1}{2} \sum_{l \leq i,j \leq r} \Theta^{X_{3,1,1},X_{3,1,2},X_{3,2}} \left[ \partial_{j} \otimes \partial_{j}(\partial_{i}D_{i}) \right].$$

It will be convenient to use Parraud’s formula (7), and replace in the above $\partial_j$ with $\partial_{a,j}$, $\partial_{j,I}$ with $\partial_{d,j,I}$ etc. In that case, (31) becomes

$$\nu_2^{(N)}(Q) = \int_{A_2} \int_{[0,1]^s} \tau_N \left( L_{T_2} \circ L^{t_1,t_2}(Q_{y_1,\ldots,y_n})(X_{N,T_2}) \right) da_1 db_1 dc_1 dd_1 da_2 db_2 dc_2 dd_2 dt_1 \cdots dt_4, \quad (32)$$
where \( L_{a_2,b_2,c_2,d_2} \), \( L_{a_1,b_1,c_1,d_1} \) coincide with \( L_{t_2}, L_{t_1}, t_2 \), except that \( D_i \)'s and \( \partial_i \)'s, \( \partial_{j,I} \)'s are replaced by their versions defined with the notation

\[
\partial_{a,j} e_P = e^a P \otimes 1 (\partial_j P) e^{1 \otimes (1-a)P},
\]

(33)

\[
L_{a_2,b_2,c_2,d_2} =
\]

\[
e^{-t_4-t_3} \left\{ 1_{[t_2,t_4]}(t_3)L_{3}^{a_2,b_2,c_2,d_2} + 1_{[0,t_1]}(t_3)L_{1}^{a_2,b_2,c_2,d_2} + 1_{[t_1,t_2]}(t_3)L_{2}^{a_2,b_2,c_2,d_2} \right\},
\]

where for \( l = 1, 2, \)

\[
L_{l}^{a_2,b_2,c_2,d_2} = \frac{1}{2} \sum_{1 \leq i,j \leq r} \sum_{I,J \in J_l} \Theta_{X_{l,1},X_{l,2},X_{l,2}} \left[ \partial_{d_2,j} \otimes \partial_{c_2,j} \left( \partial_{b_2,i} D_{a_2,i} \right) \right]
\]

and

\[
L_{3}^{a_2,b_2,c_2,d_2} = \frac{1}{2} \sum_{1 \leq i,j \leq r} \Theta_{X_{3,1},X_{3,2},X_{3,2}} \left[ \partial_{d_2,j} \otimes \partial_{c_2,j} \left( \partial_{b_2,i} D_{a_2,i} \right) \right].
\]

4.2 Extension of Parraud’s result to polynomials in Cayley transforms

We will need to extend Parraud’s result to polynomials in Cayley transforms of selfadjoint variables and prove the following proposition:

**Proposition 18.** With the notations from Section 3, for any selfadjoint polynomial \( P(U_r, U_*) \in \mathbb{C} \langle U_r, U_* \rangle \), one has

\[
(\operatorname{tr}_N \otimes \mathbb{E})(P(U_r, U_*)) = \tau(P(u_1, \ldots, u_r, u_1^*, \ldots, u_r^*)) + \frac{\nu_1^{(N)}(P(U_r, U_*))}{N^2} + \frac{\nu_2^{(N)}(P(U_r, U_*))}{N^4},
\]

where \( \nu_1^{(N)} \), \( \nu_1 \), \( \nu_2^{(N)} \) are real linear functionals on \( \mathbb{C} \langle U_r, U_* \rangle \),

\[
\nu_1^{(N)}(P(U_r, U_*)) = \]

(34)
\[
\int_0^{+\infty} \int_0^{t_2} \tau_N(L^{t_1,t_2}(P)(\Psi(z_1^1(N)), \Psi(z_1^2(N)), \Psi(z_1^3(N)), \Psi(z_1^4(N)))) \, dt_1 \, dt_2,
\]

where the Cayley transform \(\Psi\) is applied entrywise, and the free difference quotients in the expression of the \(L\)'s act according to (18) (following from (15)).

To clarify the nature of the linear functionals \(\nu_1^{(N)}\), \(\nu_1\), \(\nu_2^{(N)}\) in the proposition above: they have been defined in (26)–(28) on polynomials in self-adjoint indeterminates \(X^r\). Using Parraud’s work, we will show below that they extend to polynomials in Cayley transforms of selfadjoints, and any \(\nu_j\) evaluated in \(P(U^r, U^*)\) is given by replacing \(X^r\)’s and \(s^r\)’s in (26)–(28) with \(\Psi(X^r)^{\tau_j}\)'s and \(\Psi(s^r)^{\tau_j}\)'s, respectively:

\[
\nu_1(P(U^r, U^*)) = \lim_{N \to \infty} N^2 \left( \text{tr}_N(P(\Psi(X_1), \ldots, \Psi(X_r), \Psi(X_1)^*, \ldots, \Psi(X_r)^*)) \right)
- \tau(P(\Psi(s_1), \ldots, \Psi(s_r), \Psi(s_1)^*, \ldots, \Psi(s_r)^*))
= \lim_{N \to \infty} N^2 \left( \text{tr}_N(P(U_1, \ldots, U_r, U_1^*, \ldots, U_r^*)) \right) - \tau(P(u_1, \ldots, u_r, u_1^*, \ldots, u_r^*))
\]

and so on (recall the notations introduced just before Theorem 3). Thus, from now on, when we refer to either of \(\nu_1, \nu_j^{(N)}, j = 1, 2\), evaluated in \(P(U^r, U^*)\), this is what we mean, and when we refer to them evaluated in \(P(X^r)\), or simply in \(P\), we mean (26)–(28).

The statement of the proposition is essentially that (the part relevant to our purposes of) Parraud’s [12, Theorem 1.1] still holds for polynomials in Cayley transforms of semicirculars/G.U.E. matrices. We shall split most of the proof of this proposition in several lemmas. Recall that \(X_1, \ldots, X_r\) are \(N \times N\) G.U.E. matrices, and \(U_j = \Psi(X_j), 1 \leq j \leq r\), are their Cayley transforms. Since any polynomial is a finite sum of monomials, we assume
without loss of generality that \( P(U_1, U_r^*) = U_1^{\epsilon_1} \cdots U_r^{\epsilon_r} \) for some \( \epsilon_1, \ldots, \epsilon_r \in \{1, \ldots, r\} \) and \( \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\} \) (at this time we do not assume that \( P \) is selfadjoint). Let

\[ Q_{y_1, \ldots, y_n} : (s_1, \ldots, s_r) \mapsto e^{(i\epsilon_1 s_1 + 1)y_1} \cdots e^{(i\epsilon_n s_n + 1)y_n}, \quad y_1, \ldots, y_n \in (-\infty, 0], \]

where \((s_1, \ldots, s_r)\) is any \( r \)-tuple of (possibly unbounded) operators whose imaginary part is strictly between \(-1\) and 1. We evaluate:

\[
P(U_1, U_r^*) = U_1^{\epsilon_1} \cdots U_r^{\epsilon_r} = \frac{(X_i + \epsilon_1 i)}{(X_i - \epsilon_1 i)} \cdots \frac{(X_n + \epsilon_n i)}{(X_n - \epsilon_n i)},
\]

and

\[
P(u_1, u_r^*) = u_1^{\epsilon_1} \cdots u_r^{\epsilon_r} = \frac{(s_1 + \epsilon_1 i)}{(s_1 - \epsilon_1 i)} \cdots \frac{(s_n + \epsilon_n i)}{(s_n - \epsilon_n i)}.
\]

Therefore, by (14) and the considerations from Sections 3.1.3 and 3.1.4

\[
\tau_N \left( U_1^{\epsilon_1} \cdots U_r^{\epsilon_n} \right) = \int_{(-\infty, 0)^n} \tau_N \left( e^{(i\epsilon_1 X_1 + 1)y_1} \cdots e^{(i\epsilon_n X_n + 1)y_n} \right) (d\delta_0(y_1) - 2dy_1) \cdots (d\delta_0(y_n) - 2dy_n).
\]

(Recall that on G.U.E. matrices, \( \tau_N = tr_N \otimes \mathbb{E} \) by definition/construction.)

Thanks to the normality (hence so-continuity) of \( \tau_N \), one may indeed permute it with the integral, in order to obtain the above formula. We apply Proposition 16 to \( Q_{y_1, \ldots, y_n}(X_1, \ldots, X_r) = e^{(i\epsilon_1 X_1 + 1)y_1} \cdots e^{(i\epsilon_n X_n + 1)y_n} \) and obtain that

\[
\nu_1(P(U_1, U_r^*)) = \int_{(-\infty, 0)^n} \int_0^{\infty} \tau_N \left( L_{t_1, t_2}(Q_{y_1, \ldots, y_n})(z_{t_1 r}, z_{t_1 r}^*, z_{t_2 r}, z_{t_2 r}^*) \right) dt_1 dt_2 (d\delta_0(y_1) - 2dy_1) \cdots (d\delta_0(y_n) - 2dy_n),
\]

The formula for \( \nu_1^{(N)}(P(U_1, U_r^*)) \) is obtained by replacing \( z_{t_1 r}, z_{t_1 r}^*, z_{t_2 r}, z_{t_2 r}^* \) with \( z_{t_1 r}^1(N), z_{t_1 r}^2(N), z_{t_2 r}^1(N), z_{t_2 r}^2(N) \) in (37).

**Lemma 19.** There exist \( C, D > 0 \) such that for any \( n, N \in \mathbb{N} \setminus \{0\} \), for any \( \epsilon_1, \ldots, \epsilon_n \in \{\pm 1\} \)

\[
\left| \nu_1^{(N)}(U_1^{\epsilon_1} \cdots U_r^{\epsilon_r}) \right| \leq C(n + 2)^4 D^n,
\]

\[
\left| \nu_1(U_1^{\epsilon_1} \cdots U_r^{\epsilon_r}) \right| \leq C(n + 2)^4 D^n.
\]
Proof. In order to obtain the claimed estimates, it is convenient to express the free difference quotients of exponentials using Parraud’s formula (7): with the notation (33) one writes $\partial_j e^P = \int_0^1 \partial_{a,j} e^P \, da$ in (37); for simplicity, $D_{a,i} := e^{j_1 \cdot \diamond \cdot \diamond \cdot a_i}$. In the formula for $Q$, we may isolate the scalar exponentials: $Q_{y_1, \ldots, y_n}(s_1, \ldots, s_r) = e^{y_1 + \cdots + y_n} e^{s_1 y_1} \cdots e^{s_n y_n}$. When one applies $D_{a,i}$ to $Q$, one obtains a sum of at most $n$ terms corresponding to the choice of the exponential $e^{s_j y_j}$ with $i_j = i$ that we derive, each term of the sum being of the form $i e^{s_j y_j} e^{y_1 + \cdots + y_n}$ times a product of $n$ exponentials of similar kind to $e^{s_j y_j}$ (the difference being that in two instances $y_j$ is replaced by either $ay_j$ or $(1-a)y_j$, $0 \leq a \leq 1$). Applying the next free difference quotient in the formula, again under its form $\partial_{b,i}$ from (7), to such a term leads to at most $n+1$ terms (corresponding to the choice of the exponential, among the $n+1$ ones of the product that we derive), each term being of the form $e^{t_1 e^{s_j y_j}}$ (the difference being that in two instances $y_j$ is replaced by either $ay_j$ or $(1-a)y_j$, $0 \leq a \leq 1$). Applying the next free difference quotient in the formula, again under its form $\partial_{a,j}$ from (7), to each term of the resulting sum, one obtains a sum of at most $(n+1)(n+1)$ terms (corresponding to the choice of the exponential, among the $n+1$ ones of the product on the left and respectively on the right of the tensor product, that we apply the free difference quotient to); each term is of the form

$$e^{t_1 e^{s_j y_j}} e^{y_1 + \cdots + y_n} a^{p_1} (1-a)^{q_1} b^{p_2} (1-b)^{q_2} M_1 \otimes M_2 \otimes M_3 \otimes M_4,$$

where $p_1, q_1, p_2, q_2 \in \{0, 1, 2\}$ and each of $M_1, M_2, M_3, M_4$ is a product of at most $(n+1)$ exponentials. Therefore,

$$|\nu_1^{(N)}(\mathbf{U}_1^\alpha \cdots \mathbf{U}_n^\alpha)| \leq n(n+1)^3 \max_{p=0,1,2,3,4} \left( \int_{(-\infty,0]} |y|^p e^y dy \right)^n \int_0^t \frac{1}{2} e^{t_2-t_1} dt_1 dt_2 \times \max_{p_1, q_1, p_2, q_2 \in \{0,1,2\}} \int_{[0,1]^2} a^{p_1} (1-a)^{q_1} b^{p_2} (1-b)^{q_2} dadb,$$

with an identical estimate for $\nu_1$, which yields the result. (We have of course used the facts that $\tau$ is a state, hence of norm one, that it is normal, hence continuous in the strong operator topology and thus commuting with the integrals, and that $\|e^{s_j y_j}\| = 1$ for any — not necessarily bounded — selfadjoint $s_{ij,}$.)
Remark 20.

1. The proof of Lemma 19 does not use the fact that the evaluations are performed on elements of the specific form \(z_j^i(N)\) or \(z_j^i\) — hence our use of the notation \(s_1, \ldots, s_r\) for arbitrary algebraically free selfadjoint variables —, but only the fact that these elements are selfadjoint. By recalling the estimate \(\|e^{(\pm i\pi)s_1)}\| \leq \|e^{(\pi \pm i)s_1)}\| = \|e^{(\pi \pm i))}\|\) (see [9, Corollary 6.5.22]), one finds that the conclusion of Lemma 19 holds for non-selfadjoint bounded \(s_j\)’s as well, if their imaginary part is strictly smaller in norm than one.

2. The explicit formula \(37\) will be important later (in conjunction with Lemma 19, Lemma 11, Lemma 12, and Lemma 15) in analyzing the support of a certain compactly supported Schwartz distribution associated to \(\nu_1\). We shall not need such a precise analysis for \(\nu_2\), but only an analogue of Lemma 19, so we will focus less in the following on explaining the formula for \(\nu_2\). However, the estimates in Lemma 21 below will be seen equally easily to hold for variables with small imaginary part.

Now, for \(P(\mathcal{U}_r, \mathcal{U}_r)\) being the monomial \(\mathcal{U}_1, \ldots, \mathcal{U}_r ightarrow \mathcal{U}_{i_1} \cdots \mathcal{U}_{i_n}\), Proposition 17 also yields that

\[
\nu_2^{(N)}(P(\mathcal{U}_r, \mathcal{U}_r)) = \int_{-\infty,0}^{\infty} \int_{0,1}^{1/\|L\|} \rho_N (L_{t_2} \circ L_{t_1} (Q_{y_1, \ldots, y_n})(X_{N,T_2}))
\]

\[
d a_1 d b_1 d c_1 d d_1 d a_2 d b_2 d c_2 d d_2 d t_1 \cdots d t_4 (d \delta_0(y_1) - 2d y_1) \cdots (d \delta_0(y_n) - 2d y_n).
\]

Lemma 21. There exist \(C, D > 0\) such that for any \(n, N \in \mathbb{N} \setminus \{0\}\), for any \(\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}\),

\[
|\nu_2^{(N)}(\mathcal{U}_{i_1} \cdots \mathcal{U}_{i_n})| \leq C(n + 5)^8 D^n,
\]

Proof. \(L_{t_1, t_2}(Q)\) is a function in four \(r\)-tuples of variables, indexed by the sets of \(J_1, (X_{i,E_1})_{1 \leq i \leq r}, (X_{i,E_1})_{1 \leq i \leq r}, (X_{i,E_2})_{1 \leq i \leq r}\) and \((X_{i,E_2})_{1 \leq i \leq r}\). Moreover, we can deduce from the proof of Lemma 19 that it is equal to the sum of at most \(n(n + 1)^3\) terms of the kind

\[
\frac{1}{2} e^{-t_2 - t_1} \epsilon_{i_1} \epsilon_{i_2} \epsilon_{i_3} y_{i_1} y_{i_2} y_{i_3} y_{i_4} e^{y_{i_1} + \cdots + y_n} a^{\nu_1} (1 - a)^{\nu_1} b^{\nu_2} (1 - b)^{\nu_2} M,
\]
where \( p_1, q_1, p_2, q_2 \in \{0, 1, 2\} \) and \( M \) is a product of at most \( 4(n+1) \) exponentials. Repeating the arguments of Lemma 19 when we apply now \( L_{a,b,c,d}^{T_2}(Q) \), we obtain that there exist \( C_1 > 0, C_2 > 0 \) such that
\[
\nu_2(\Sigma_{i=1}^n \cdot \Sigma_{n}^\infty) \leq C_1 n(n+1)^3(4(n+1))(4(n+1)+1)^3 C_2^n,
\]
which easily yields the desired result. \( \square \)

**Proof of Proposition 18**  The proof proceeds by arguing that, thanks to Lemma 9, Lemma 19, and Lemma 21, Paraud’s formula for polynomials and exponentials in G.U.E. matrices and semicirculars extends to polynomials in Cayley transforms of G.U.E. matrices and semicirculars.

Up to this point we have argued that the functionals \( \nu_j, \nu_j^{(N)}, 1 \leq j \leq 2 \) as defined in (26)–(28), extend to products of exponentials \( Q = e^{(i\tau_1 s_1 + 1)y_1} \ldots e^{(i\tau_n s_n + 1)y_n} \), with explicit formulae (an application of [12, Proposition 3.7]); that integrating the product \( Q \) of exponentials with respect to the product measure \((d\theta_0(y_1) - 2dy_1)\cdots(d\theta_0(y_n) - 2dy_n)\) on \((-\infty, 0]^n\) commutes with the application of \( \tau_N \), yielding, by (14), \( \tau_N(U_{i_1}^{r_1} \cdots U_{i_n}^{r_n}) \) (normality of \( \tau_N \)); that linear functionals \( \nu_j(\Sigma_{i=1}^n \cdot \Sigma_{n}^\infty), \nu_j^{(N)}(\Sigma_{i=1}^n \cdot \Sigma_{n}^\infty), j = 1, 2 \), are well-defined (equalities (37), (38)) via integrals \( \int_{(-\infty, 0]^n} \nu_1(Q), \int_{(-\infty, 0]^n} \nu_2(Q) \) of the product of exponentials \( Q \) with respect to the same product measure; that \( \nu_1, \nu_2^{(N)}, 1 \leq j \leq 2 \), given via these formulae, are bounded on such monomials by expressions of the type \( p(n)D^n \) for some polynomial \( p \) and positive number \( D \), independently of the — bounded or unbounded — selfadjoints \( s_1, \ldots, s_r \), on which \( Q \) is evaluated (Lemma 19 and Lemma 21).

In order to complete the proof of our proposition, we need to still argue that the expressions for \( \nu_1(Q), \nu_2^{(N)}(Q) \) in terms of free difference quotients, flips, and evaluations, “go through” the integrals on \((-\infty, 0]^n\) with respect to \((d\theta_0(y_1) - 2dy_1)\cdots(d\theta_0(y_n) - 2dy_n)\). However, this is a direct application of Lemma 9 and of the fact that, by Lemmas 19 and 21 combined with the observations in Remark 20, the expressions \((s_{ij}^t)_{t=1}^{3,1} \in \mathbb{N}^4 \mapsto \tau_N(L_t^{1,2}((s_{ij}^1, s_{ij}^2, s_{ij}^3, s_{ij}^4)) \) and \((s_{ij}^{1,1}, s_{ij}^{2,1}, s_{ij}^{2,2}, s_{ij}^{2,2})_{t=1,2} \times (s_{ij}^{3,1}, s_{ij}^{3,2}, s_{ij}^{3,2}, s_{ij}^{3,2}) \mapsto \tau_N(L_t^{L_2 \circ L_2^T}(s_{ij}^t)) \) are noncommutative analytic functions on a small enough neighborhood of zero. Indeed, it has been shown in Lemma 9 via a power series argument combined with the unique extension property of noncommutative analytic functions, that any successive application of the operations involved in \( L_t^2 \) and \( L_t^{L_2} \) commutes with taking the integral on \((-\infty, 0]^n\) with respect to \((d\theta_0(y_1) - 2dy_1)\cdots(d\theta_0(y_n) - 2dy_n)\).
2dφn. Since the application of τN and the integration with respect to the variables \( t_i, 1 \leq i \leq 4 \), preserves both analyticity and convergence around zero, the analytic continuation argument extends to proving that \( ν_1(P(U, U^*)) , ν_j^{(N)}(P(U, U^*)) , j = 1, 2 \) defined via (35), (34) and (36) is the same object as \( ν_1(P(U, U^*)) , ν_j^{(N)}(P(U, U^*)) , j = 1, 2 \), defined via (37) and (38).

5 Haagerup, Thorbjørnsen and Schultz’s approach

We present here the arguments of [7] and [16] which still hold in our framework of Theorem 3 and point out the result which requires new ideas and techniques.

First, it is straightforward to deduce from Proposition 18 that for any noncommutative polynomial \( P \) in \( r_1 + r_2 \) variables and their adjoints,

\[
\text{etr}_N \otimes \text{tr}_N \left[ P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}, U^* \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*) \right]
\to_{N \to +\infty} \tau \otimes \tau \left[ P \left( u \otimes 1_{A_{r_1}}, u^* \otimes 1_{A_{r_1}}, v^* \otimes 1_{A_{r_2}}, v \otimes 1_{A_{r_2}} \right) \right]
\]

(recall the notations introduced before Theorem 3). Now, since \( Ψ(H) = I + 2i(H - i)^{-1} , Ψ(H)^{-1} = I - 2i(H + i)^{-1} \), it follows that \( \text{etr}_N \otimes \text{tr}_N \left[ P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}, U^* \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*) \right] \) is a linear combination of terms of the form

\[
\text{tr}_N \left( \prod_{j=1}^q (iε_j - X_j)^{-1} \right) \text{tr}_N \left( \prod_{k=1}^q (iε_k - Y_k)^{-1} \right)
\]

where, as before, \( ε_j, ε_k \in \{±1\} \) and the \( i_j \)'s and the \( l_k \)'s belong respectively to \( \{1, \ldots, r_1\} \) and \( \{1, \ldots, r_2\} \). It is then straightforward to see that there exists a constant \( C_P \) depending only on \( P \), such that the map on the real vector space \( (M_N(\mathbb{C})_{sa})^{r_1+r_2} \),

\[
(A_1, \ldots, A_{r_1}, A_{r_1+1}, \ldots, A_{r_1+r_2}) \mapsto \text{etr}_N \otimes \text{tr}_N \left[ P(Ψ(A_1) \otimes I_N, \ldots, Ψ(A_{r_1}) \otimes I_N, I_N \otimes Ψ(A_{r_1+1}), \ldots, I_N \otimes Ψ(A_{r_1+r_2})) \right]
\]

is \( \frac{C_P}{N} \)-Lipschitz with respect to the norm \( \| \cdot \|_e \) defined by

\[
\| (A_1, \ldots, A_{r_1+r_2}) \|_e^2 = \sum_{j=1}^{r_1+r_2} \text{Tr}_N(A_j^2).
\]
The norm \( \| \cdot \|_e \) corresponds to the usual Euclidean norm on \( \mathbb{R}^{(r_1+r_2)N^2} \). Once one identifies \( M_N(\mathbb{C})_{sa} \) and \( \mathbb{R}^{N^2} \) via the isomorphism \( \Phi_0 : M_N(\mathbb{C})_{sa} \to \mathbb{R}^{N^2} \) given by

\[
\Phi_0((a_{kl})_{1 \leq k, l \leq N}) = \left( (a_{kk})_{1 \leq k \leq N}, (\sqrt{2} Re a_{kl})_{1 \leq k < l \leq N}, (\sqrt{2} Im a_{kl})_{1 \leq k < l \leq N} \right).
\]

Therefore the classical Gaussian concentration phenomenon and (39) lead to the almost sure convergence of \( \text{tr}_N \otimes \text{tr}_N \left[ P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*, U^* \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*) \right] \) to \( \tau \otimes \tau \left[ P \left( u \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}, u^* \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}^* \right) \right] \) when \( N \) goes to infinity. It turns out that in proving Theorem 3, the minoration almost surely, for any \( P \),

\[
\lim \inf_{N \to +\infty} \left\| P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}, U^* \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*) \right\| \geq \left\| P \left( u \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}, u^* \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}^* \right) \right\|
\]

should follow rather easily (sticking to the proof of Lemma 7.2 in [7]). So, the main difficulty is the proof of the reverse inequality: almost surely for any \( P \),

\[
\lim \sup_{N \to +\infty} \left\| P(U \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}, U^* \otimes I_{N_{r_1}}, I_N \otimes V_{r_2}^*) \right\| \leq \left\| P \left( u \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}, u^* \otimes 1_{A_{r_1}}, 1_A \otimes v_{r_2}^* \right) \right\|.
\]

(40)

Thanks to a linearization trick (following [7], Section 2 and the proof of Proposition 7.3), in order to prove (40), it suffices to prove:

**Proposition 22.** For all \( m \in \mathbb{N} \), all matrices \( \xi = \xi^*, \gamma_1, \ldots, \gamma_{r_1}, \beta_1, \ldots, \beta_{r_2} \in M_m(\mathbb{C}) \) and all \( \varepsilon > 0 \), almost surely, for all large \( N \), we have

\[
\text{sp} (\xi \otimes I_N \otimes I_N + S_U + S_V) \subset \text{sp} (\xi \otimes 1_A \otimes 1_A + S_u + S_v) + (-\varepsilon, \varepsilon),
\]

where

\[
S_U = \sum_{i=1}^{r_1} (\gamma_i \otimes U_i \otimes I_N + \gamma_i^* \otimes U_i^* \otimes I_N) = 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N,
\]

\[
S_V = \sum_{i=1}^{r_2} (\beta_i \otimes I_N \otimes V_i + \beta_i^* \otimes I_N \otimes V_i^*) = 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes V_i,
\]

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Here, \( \text{sp}(T) \) denotes the spectrum of the operator \( T \), \( I_N \) the identity matrix and \( 1_A \) denotes the unit of \( A \).

Set

\[
S_u = \sum_{i=1}^{r_1} (\gamma_i \otimes u_i \otimes 1_A + \gamma_i^* \otimes u_i^* \otimes 1_A) = 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A,
\]

\[
S_v = \sum_{i=1}^{r_2} (\beta_i \otimes 1_A \otimes v_i + \beta_i^* \otimes 1_A \otimes v_i^*) = 2\Re \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i.
\]

The proof of Proposition 22 requires sharp estimate of \( g_N(z) - g(z) \) where for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
g_N(z) = \mathbb{E}(\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N)[(zI_m \otimes I_N \otimes I_N - S_N)^{-1}] \tag{43}
\]

and

\[
g(z) = (\text{tr}_m \otimes \tau \otimes \tau)[(zI_m \otimes 1_A \otimes 1_A - S)^{-1}]. \tag{44}
\]

This estimate is detailed in Section 6. More precisely we are going to establish that there exists a polynomial \( Q \) with nonnegative coefficients such that, for \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
|g_N(z) - g(z) - E(z)| \leq Q(|\Im z|^{-1}) \tag{45}
\]

where \( E(z) \) is the Stietjes transform of a distribution \( \hat{\Lambda} \) whose support is included in the spectrum of \( S \) and \( \hat{\Lambda}(1) = 0 \).

Now, we explain, for the reader’s convenience because of differences arising from the dimension of our tensor matrices, how (45) can be deduced from
by the approaches introduced in [7, 16]. Using the Stieltjes inversion formula for measures and compactly supported distributions, one obtains that, for any $\varphi$ in $C^{\infty}(\mathbb{R}, \mathbb{R})$ with compact support,

$$
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\varphi(S_N))] - \text{tr}_m \otimes \tau \otimes \tau(\varphi(S)) - \frac{\hat{\Lambda}(\varphi)}{N^2} = \frac{1}{\pi} \lim_{y \to 0^+} \Im \int_{\mathbb{R}} \varphi(x) \epsilon_N(x + iy) \, dx,
$$

where $\epsilon_N(z) = g_N(z) - g(z) - \frac{E(z)}{N^2}$ satisfies, according to (15), for any $z \in \mathbb{C \setminus \mathbb{R}},$

$$
|\epsilon_n(z)| \leq \frac{Q(|\Im z|^{-1})}{N^4}.
$$

We refer the reader to the Appendix of [6], where it is proved using the ideas of [7] that if $h$ is an analytic function on $\mathbb{C \setminus \mathbb{R}}$ which satisfies

$$
|h(z)| \leq Q(|\Im z|^{-1})(|z| + K)^\alpha
$$

for some polynomial $Q$ with nonnegative coefficients and degree $k$, then there exists a polynomial $\tilde{Q}$ such that

$$
\limsup_{y \to 0^+} \left| \int_{\mathbb{R}} \varphi(x) h(x + iy) \, dx \right| \leq \frac{C}{N^4},
$$

and then

$$
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\varphi(S_N))] - \text{tr}_m \otimes \tau \otimes \tau(\varphi(S)) - \frac{\hat{\Lambda}(\varphi)}{N^2} = O\left(\frac{1}{N^4}\right). \tag{46}
$$

Let $\rho \in C^{\infty}(\mathbb{R}, \mathbb{R})$ be such that $\rho \geq 0$, its support is included in $[-1, 1]$ and $\int_{\mathbb{R}} \rho(x) \, dx = 1$. Let $0 < \delta < 1$. Define for any $x \in \mathbb{R},$

$$
\rho_{\frac{x}{\delta}}(x) = \frac{2}{\delta} \rho\left(\frac{2x}{\delta}\right).
$$
Set
\[ K(\delta) = \{ x, \text{dist}(x, \text{sp}(S)) \leq \delta \} \]
and define for any \( x \in \mathbb{R} \),
\[ f_\delta(x) = \int_{\mathbb{R}} 1_{K(\delta)}(y) \rho_{\frac{\delta}{2}}(x - y) \, dy. \]
The function \( f_\delta \) is in \( C^\infty(\mathbb{R}, \mathbb{R}) \), \( f_\delta \equiv 1 \) on \( K(\frac{\delta}{2}) \); its support is included in \( K(2\delta) \). Since there exists \( C \) such that the spectrum of \( S \) is included in \( [-C; C] \), the support of \( f_\delta \) is included in \( [-C - 2; C + 2] \). Thus, according to (46),
\[
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(f_\delta(S_n))] - \text{tr}_m \otimes \tau \otimes \tau(f_\delta(S)) - \frac{\hat{\Lambda}(f_\delta)}{N^2} = O_\delta \left( \frac{1}{N^4} \right) \quad (47)
\]
and, since \( f_\delta' \equiv 0 \) on \( \text{sp}(S) \),
\[
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N((f_\delta')^2(S_n))] - \text{tr}_m \otimes \tau \otimes \tau((f_\delta')^2(S)) = O_\delta \left( \frac{1}{N^4} \right). \quad (48)
\]
Since \( \hat{\Lambda}(1) = 0 \), the function \( \psi_\delta \equiv 1 - f_\delta \) also satisfies
\[
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\psi_\delta(S_n))] - \text{tr}_m \otimes \tau \otimes \tau(\psi_\delta(S)) - \frac{\hat{\Lambda}(\psi_\delta)}{N^2} = O_\delta \left( \frac{1}{N^4} \right). \quad (49)
\]
Moreover, since \( \psi_\delta' = -f_\delta' \), it follows readily from (48) that
\[
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N((\psi_\delta')^2(S_n))] - \text{tr}_m \otimes \tau \otimes \tau((\psi_\delta')^2(S)) = O_\delta \left( \frac{1}{N^4} \right). \quad (50)
\]
Now, since \( \psi_\delta \equiv 0 \) on the spectrum of \( S \), we deduce that
\[
\mathbb{E}[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\psi_\delta(S_n))] = O_\delta \left( \frac{1}{N^4} \right) \quad (50)
\]
and
\[
\mathbb{E} \left[ \text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N \left( (\psi_\delta')^2(S_N) \right) \right] = O_\delta \left( \frac{1}{N^4} \right). \quad (51)
\]
Denoting by \( V(Z) \) the variance of a random variable \( Z \), according to Lemma 26 we have
\[
V[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\psi_\delta(S_n))] \leq \frac{C}{N^2} \mathbb{E} \left[ \text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N \{ (\psi_\delta'(S_N))^2 \} \right].
\]
Hence, using (51), one can deduce that
\[
V[\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\psi_\delta(S_N))] = O_\delta \left( \frac{1}{N^\alpha} \right).
\] (52)

Set
\[
Z_{N,\delta} := \text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N(\psi_\delta(S_N))
\]
and
\[
\Omega_{N,\delta} = \{|Z_{N,\delta} - \mathbb{E}(Z_{N,\delta})| > N^{-\alpha}\},
\]
where \(\alpha\) will be fixed later. Using (52), we have
\[
\mathbb{P}(\Omega_{N,\delta}) \leq N^{2\alpha}V\{Z_{N,\delta}\} = O_\delta \left( \frac{1}{N^{6-2\alpha}} \right).
\]

For any \(0 < \alpha < 5/2\), by Borel-Cantelli lemma, we deduce that, almost surely for all large \(N\),
\[
|Z_{N,\delta} - \mathbb{E}(Z_{N,\delta})| \leq N^{-\alpha}.
\] (53)

From (50) and (53), we deduce that there exists some constant \(C_\delta\) such that, almost surely for all large \(N\),
\[
|Z_{N,\delta}| \leq C_\delta N^{-4} + N^{-\alpha}.
\]

Choose \(2 < \alpha < 5/2\) and set \(\theta = \alpha - 2 > 0\). Thus,
\[
|Z_{N,\delta}| \leq C_\delta N^{-4} + N^{-\alpha} \leq N^{-2}(C_\delta N^{-2} + N^{-\theta}).
\]

Since \(\psi_\delta \geq 1_{\mathbb{R} \setminus K(2\delta)}\), it follows that, almost surely for all large \(N\), the number of eigenvalues of \(S_N\) which are in \(\mathbb{R} \setminus K(2\delta)\) is lower than \(m \left( N^{-\theta} + C_\delta N^{-2} \right) \) and thus obviously, almost surely for all large \(N\), the number of eigenvalues of \(S_N\) which are in \(\mathbb{R} \setminus K(2\delta)\) has to be equal to zero. Thus almost surely for all large \(N\), the spectrum of \(S_N\) is included in \(K(2\delta) = \{x, \text{dist}(x, \text{sp}(S)) \leq 2\delta\}\). Since this result holds for any \(m \times m\) matrices \(\xi, \gamma, \beta\), the proof of Proposition 22 is complete.

6 Proof of (45)

6.1 Extension of \(\nu_1, \nu_1^{(N)}, \nu_2^{(N)}\) to entries of resolvents of affine matrix polynomials

Recall that \(\nu_1, \nu_1^{(N)}, \nu_2^{(N)}\) have been extended in Proposition 18 to polynomials in Cayley transforms of selfadjoints. In this section, we extend them
to entries of resolvents of $m \times m$ matrices of affine polynomials. We do this in several steps.

### 6.1.1 On a neighborhood of infinity

As seen in Proposition [22], we are concerned with the behavior of the resolvent of the operator $S$ from [23] and evaluations of $\nu_1, \nu_j^{(N)}$, $j = 1, 2$ on the second or the third tensor (or on both). We first consider the case when the $\nu$’s are applied on the last tensor, and hence we investigate the more general form of $S$ from [25]. We agree to temporarily view the notation $\nu_i, 1 \leq i \leq r_2$ as identifying the Cayley transform of some arbitrary selfadjoint. It is clear that if $|z| > \| \xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes \nu_i \|$, then

\[
(zI_m - \xi) \otimes 1 \otimes 1 - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes \nu_i)
\]

\[
= \sum_{n=0}^{\infty} (\xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2\Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes \nu_i)^n,
\]

where the right hand side series converges in norm. It can be easily seen that $(T_0 + T_1 + \cdots + T_{r_2})^n = \sum_{0 \leq i_1, \ldots, i_n \leq r_2} T_{i_1} T_{i_2} \cdots T_{i_n}$, a sum of $(r_2 + 1)^n$ terms. For each $k \in \{0, 1, \ldots, n\}$, there are $r_2^{n-k} \binom{n!}{k!(n-k)!}$ terms that contain exactly $k$ instances of $T_0$ in this sum (i.e. for which exactly $k$ of the $i_1, \ldots, i_n$ are equal to zero). We isolate the Cayley transforms $\nu_j$ in the above-displayed numerator by applying this formula with $T_0 = \xi \otimes 1 \otimes 1 + \sum_{i=1}^{r_1} 2\Re (\gamma_i \otimes t_i) \otimes 1$ and $T_j = 2\Re (\beta_j \otimes 1 \otimes \nu_j), 1 \leq j \leq r_2$. Pick $i_1, \ldots, i_n$ with $k$ of the $i$’s being equal to 0, and the other $n-k$ being $i_{n-k}, \ldots, i_{n-k}$. Then

\[
T_{i_1} \cdots T_{i_n} = \sum_{\epsilon_{i_1}, \ldots, \epsilon_{i_{n-k}} \in \{\pm 1\}} \tilde{T}_{i_1} \cdots \tilde{T}_{i_n} \otimes \nu_{i_{1}}^{\epsilon_{i_1}} \cdots \nu_{i_{n-k}}^{\epsilon_{n-k}},
\]

where $\tilde{T}_{ij} \in \{\beta_{ij} \otimes 1, \beta_{ij}^* \otimes 1\}$ if $i_j \neq 0$ and $\tilde{T}_{ij} = \xi \otimes 1 + 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i$ if $i_j = 0$. Assuming in addition that the $\nu_j$’s are Cayley transforms of algebraically free variables and applying the estimates of Lemma [19] to the above, one obtains

\[
\|(id_m \otimes \id_T \otimes \nu_1)(T_1 \cdots T_n)\| \leq \sum_{\epsilon_{i_1}, \ldots, \epsilon_{i_{n-k}} \in \{\pm 1\}} \nu_1(\nu_{i_{1}}^{\epsilon_{i_1}} \cdots \nu_{i_{n-k}}^{\epsilon_{n-k}}) \| \tilde{T}_{i_1} \cdots \tilde{T}_{i_n} \|
\]

(55)
< C(2D)^{n-k}(n + 2)^4 \| \beta_{i_1} \| \cdots \| \beta_{i_{n-k}} \|
\times \left\| \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 \right\|^k.

The brutal estimate \( \| \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 \|^k \| \beta_{i_1} \| \cdots \| \beta_{i_{n-k}} \| \leq (\max \{ \| \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 \|, \| \beta_1 \|, \ldots, \| \beta_{r_2} \| \})^n =: \mathcal{M}^n \) allows us to write
\[
\| (\text{id}_m \otimes \text{id}_T \otimes \nu_1)(T_{i_1} \cdots T_{i_n}) \| < C(2D)^{n-k}\mathcal{M}^n(n + 2)^4.
\]

As already mentioned, there are \( r_2^{n-k} \frac{n!}{k!(n-k)!} \) such terms, so that
\[
\left\| (\text{id}_m \otimes \text{id}_T \otimes \nu_1) \left( \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2 \Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \right)^n \right\|
< C\mathcal{M}^n(n + 2)^4 \sum_{k=0}^{n} r_2^{n-k} (2D)^{n-k} \frac{n!}{k!(n-k)!} = C(n + 2)^4 \mathcal{M}^n (2Dr_2 + 1)^n.
\]

It follows that
\[
(\text{id}_m \otimes \text{id}_T \otimes \nu_1) \left[ \left( (zI_m - \xi) \otimes 1 \otimes 1 - 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2 \Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \right)^{-1} \right]
= \sum_{n=0}^{\infty} (\text{id}_m \otimes \text{id}_T \otimes \nu_1) \left[ \left( \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2 \Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \right)^n \right],
\]
where the series in the right-hand side converges in the norm topology on \( M_m(T) \) whenever \( |z| > \mathcal{M}(2Dr_2 + 1) \). Based on the identical estimate for \( \nu_1^{(N)} \) present in Lemma 19 all of the above holds with \( \nu_1 \) replaced by \( \nu_1^{(N)} \), \( N \in \mathbb{N} \).

An application of Lemma 21 allows us to conclude, with the constants \( C, D \) from that lemma, that
\[
(\text{id}_m \otimes \text{id}_T \otimes \nu_2^{(N)}) \left[ \left( (zI_m - \xi) \otimes 1 \otimes 1 - 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2 \Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \right)^{-1} \right]
= \sum_{n=0}^{\infty} (\text{id}_m \otimes \text{id}_T \otimes \nu_2^{(N)}) \left[ \left( \xi \otimes 1 \otimes 1 + 2 \Re \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2 \Re \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \right)^n \right],
\]
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the series converging in the same topology on $M_m(\mathcal{T})$ whenever $|z| > 2\Re(2Dr_2 + 1)$.

In particular, since $\text{tr}_m$ and $\phi$ are states, both (56) and (57) hold with $\text{id}_m$ replaced by $\text{tr}_m$ and/or $\text{id}_\mathcal{T}$ by $\phi$, with the same radii of convergence.

We have thus shown that the linear functionals $\nu_1, \nu_j^{(N)}, 1 \leq j \leq 2$, further extend from polynomials (possibly with operator coefficients) in Cayley transforms of algebraically free selfadjoints to convergent power series of such polynomials, possibly at the cost of reducing the radius of convergence of the series in question. To be precise, we have shown that (56) and (57) hold uniformly in $z$ and $t_1, \ldots, t_{r_1}$ on a set of the form $\{|z| > M\} \times \{(t_1, \ldots, t_{r_1}): |t_j| < M'\}$ for some $M, M' > 0$. The right-hand side of each of these expressions is a convergent power series on these sets. One defines $(\text{id}_m \otimes \phi \otimes \text{id}_A)(\nu_j)$ applied to the resolvents in the left-hand sides as being the convergent infinite sums in the right-hand sides.$^2$

Let us establish next that formulae (35), (34) and (36) extend (when applied entrywise) to the above power series expansion of the resolvent of (25).

We first consider the entrywise application of $L^{t_1,t_2}$ and of $L^{T_2} \circ L^{t_1,t_2}$ from (35), (34) and (36), to $(\text{id}_m \otimes \phi \otimes \text{id}_A_N)((z - S)^{-1})$. By using Lemma 13 (see (22)) together with Lemma 11 and Lemma 12, it follows that the application of a succession of free difference quotients, flips, and evaluations preserves the radius of convergence around infinity (recall that the variables $v_j$ are assumed to be Cayley transforms of selfadjoints; at this time, we do not even need to particularize the evaluations to the variables $\tilde{z}_j, \tilde{z}_j^{\mathcal{T}}, X^{N,T_2}$ - the argument remains valid with any given selfadjoint algebraically free variables). Applying $\tau_N$ and integrating preserves analyticity. Thus, if $|z| > \|\xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^{t_1} \gamma_i \otimes t_i \otimes 1 + 2\Re \sum_{i=1}^{t_2} \beta_i \otimes 1 \otimes v_i\|$, then an application of either of $L^{t_1,t_2}$ or $L^{T_2} \circ L^{t_1,t_2}$ to the right hand side of (34) (on the third tensor) preserves convergence (in norm).

$^2$While not needed for our argument, we note that a rather strong and precise continuity statement for the linear maps $\nu_j, \nu_j^{(N)}$ can be derived from Parraud’s work. We give a brief outline of that argument in Aside 23.

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6.1.2 Away from infinity

Next, we change the point around which we perform the power series expansion in order to extend the domain on which we can apply the above reasoning. Just in this paragraph, denote \( X := \xi \otimes 1 \otimes 1 + 2R \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2R \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i \). Assume \( |z| \leq \|X\| = \|\xi \otimes 1 \otimes 1 + 2R \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 + 2R \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i\|, \, z \in \mathbb{C}^+ \), as the case of the reverse inequality has just been dealt with. We write \((z-X)^{-1} = (z-\zeta-(X-\zeta))^{-1} = (z-\zeta)^{-1} \left(1 - \frac{X-\zeta}{z-\zeta}\right)^{-1}\).

Choose \( \zeta \in \mathbb{C}^- \) such that \( \Re \zeta \) is half-way between the upper and lower bound of the spectrum of \( X \): \( \Re \zeta = \frac{\max \sigma(X) + \min \sigma(X)}{2} \). The operator \( X - \zeta \) being normal, \( \|X - \zeta\|^2 = \|X - \Re \zeta\|^2 + |\Im \zeta|^2 \leq \|X\|^2 + |\Im \zeta|^2 \). Since \( z \in \mathbb{C}^+ \), \( \zeta \in \mathbb{C}^- \), one has \( |z - \zeta|^2 = (\Re z - \Re \zeta)^2 + (\Im z)^2 + (3\zeta)^2 + 2\Im z|\Im \zeta| \), so that in order to ensure that \( \|X - \zeta\| < |z - \zeta| \), one needs only make sure that \( \|X\|^2 + |\Im \zeta|^2 < (\Re z - \Re \zeta)^2 + (\Im z)^2 + (3\zeta)^2 + 2\Im z|\Im \zeta| \), i.e. \( 2\Im z|\Im \zeta| > \|X\|^2 - (\Re z - \Re \zeta)^2 - (\Im z)^2 \). By fixing a \( 3\zeta \ll 0 \), one finds an open set of \( z \)'s around our initial point on which \( \|X - \zeta\| \leq |z - \zeta| \) holds. It has been shown in Lemma 13 (see (22)) that \( \partial_t \circ (\phi \otimes \id_A) = \phi \circ \partial_t \) on the (entries of the) power series

\[
\sum_{n=0}^{\infty} \frac{(X - \zeta)^n}{(z - \zeta)^{n+1}} \in M_m(\mathcal{A}_\infty) \subset M_m(\mathcal{T} \otimes \mathcal{A}_N),
\]

again applied term-wise, for all \( z \) in the open set \( \mathbb{C}^+ \cap \{ z : (\Re z - \Re \zeta)^2 + (\Im z + |\Im \zeta|)^2 > \|X\|^2 + |\Im \zeta|^2 \} = \mathbb{C}^+ \cap \left[ \mathbb{C} \setminus \sqrt{\|X\|^2 + |\Im \zeta|^2}(\overline{\mathbb{C}} + \zeta) \right] \), the intersection of the upper half-plane with the complement of the closed disk centered at \( \zeta \) and having radius equal to \( \sqrt{\|X\|^2 + |\Im \zeta|^2} \) (recall that \( X - \zeta \) is an affine polynomial). In words, the set of points \( z \) on which the above holds is the subset of the upper half-plane obtained by taking out of it the points belonging to the closed disk whose center is \( \zeta \in \mathbb{C}^- \) and whose boundary passes through the point of the spectrum of \( X \) which is farthest away from the origin. It is obvious that any fixed \( z \in \mathbb{C}^+ \) belongs to this set if \( \zeta \in \mathbb{C}^- \) is properly chosen. Thus, as claimed, \( \partial_t \circ (\phi \otimes \id_A) = \phi \circ \partial_t \) on each entry of \( \left((zI_m - \xi) \otimes 1 \otimes 1 - 2R \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2R \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i\right)^{-1} \) for all \( z \in \mathbb{C}^+ \cup \mathbb{C}^- \). Thus, we have proven that \( \partial_t \circ (\phi \otimes \id_A) = \phi \circ \partial_t \) on all entries of \( \left((zI_m - \xi) \otimes 1 \otimes 1 - 2R \sum_{i=1}^{r_1} \gamma_i \otimes t_i \otimes 1 - 2R \sum_{i=1}^{r_2} \beta_i \otimes 1 \otimes v_i\right)^{-1} \) for all \( z \in \mathbb{C} \setminus [-\|X\|, \|X\|] \). Thanks to this, it follows that (35), (36) and (37) extend entrywise to \( (\id_m \otimes \phi \otimes \id_{\mathcal{A}_N})((z - \mathcal{S})^{-1}) \) for all \( z \in \mathbb{C} \setminus [-\|X\|, \|X\|] \).
(As an aside, note that the above argument is easily seen to apply to any element of $S^\infty_a$, meaning in particular that $\partial_i$, as defined in $A$, is well-defined on $\phi S^\infty_a$ and Lemmas 10 and 11 apply to $\partial_i$, as it is defined on $\phi S^\infty_a$.)

Recall that by Lemma 14 all entries of (25) belong to $S^\infty_a$. An element $F_z$ in $S^\infty_a$ necessarily satisfies the following property: there exists $k \in \mathbb{N}$ depending on the specific element $F_z$ such that

$$\lim_{y \to 0} |y|^k \| F_{\eta+iy}(U_r, U^*_r) \| = 0,$$

for all $\eta \in \mathbb{R}$. (59)

This has been shown in Lemma 15. Observing that free difference quotients, flips and evaluations on selfadjoints preserve selfadjointness, it follows that both $z \mapsto (\text{tr}_m \otimes \phi \otimes \nu_j^{(N)})( (z - S)^{-1} )$, $z \mapsto (\text{tr}_m \otimes \phi \otimes \nu_j)( (z - S)^{-1} )$ are Cauchy transforms of real Schwartz distributions with compact support (included in $[-\|X\|, \|X\|]$).

We have proved that:

1. $(\text{id}_m \otimes \text{id}_T \otimes \nu_j), (\text{id}_m \otimes \text{id}_T \otimes \nu_j^{(N)}), j = 1, 2$, can be applied term-wise to the power series expansion of the resolvent $(z - S)^{-1}$ on a neighborhood of infinity to yield a still convergent power series;

2. the formulae (35), (34) and (36) for $\nu_1, \nu_j^{(N)}$, $j = 1, 2$, can be applied to each entry of $(\text{id}_m \otimes \phi \otimes \text{id}_A_N)((z - S)^{-1})$ for all $z$ in the complement of a compact interval, and that this extends $z \mapsto (\text{id}_m \otimes \phi \otimes \nu_j)((z - S)^{-1}), (\text{id}_m \otimes \phi \otimes \nu_j^{(N)})((z - S)^{-1}), j = 1, 2$, from a neighborhood of infinity;

3. this extension is analytic (see Aside 23 below for an intuitive justification of these facts).

4. $z \mapsto (\text{tr}_m \otimes \phi \otimes \nu_j^{(N)})( (z - S)^{-1} )$, $z \mapsto (\text{tr}_m \otimes \phi \otimes \nu_j)( (z - S)^{-1} ), j = 1, 2$, are Cauchy transforms of real Schwartz distributions with compact support (included in $[-\|X\|, \|X\|]$).

This proves the needed extension of the $\nu_j$’s. Of course, the results above will eventually be applied with $T = A_N$, $\phi = \tau$ and $t_1, \ldots, t_r$ being a tuple of either independent G.U.E.s or free semicirculars (or sums of such).

**Aside 23.** Formulae (35), (34) and (36) for $\nu_1, \nu_j^{(N)}$ indicate the existence of a space of “smooth” functions on which these linear functionals are continuous. Since it is not needed for the purposes of our paper, we only
sketch the construction of such a space. Since it is the context of our paper, we start from \(\mathbb{C}(\mathcal{U}, \mathcal{U}^*)\) (in the context of \([12]\), one would have started from \(\mathbb{C}(\mathcal{U}, \mathcal{U}^*)\)—as the reader can easily verify, it makes no difference). On this space we consider a family of (semi)norms slightly similar to the ones introduced, for instance, in \([11]\): given \(P \in \mathbb{C}(\mathcal{U}, \mathcal{U}^*)\), we define

\[
\|P(\mathcal{U}, \mathcal{U}^*)\|_0 = \sup \left\{ \|P(\Psi(x_1), \ldots, \Psi(x_r), \Psi(x_1)^*, \ldots, \Psi(x_r)^*)\| : x_j = x_j^* \in \hat{A}_N, N \in \mathbb{N} \right\}.
\]

It is an easy verification that \(\| \cdot \|_0\) is a norm on the complex vector space \(\mathbb{C}(\mathcal{U}, \mathcal{U}^*)\). It is also obviously finite, being majorized by the sum of the absolute values of all the coefficients of the monomials of \(P\).

Next, \(\|P(\mathcal{U}, \mathcal{U}^*)\|_{D, j}^1 = \sup \{ \| (D_j P)(\Psi(x_1), \ldots, \Psi(x_r), \Psi(x_1)^*, \ldots, \Psi(x_r)^*)\| : x_j = x_j^{*} \in \hat{A}_N, N \in \mathbb{N} \}\). This is equally clearly a seminorm. We continue this way:

\[
\|P(\mathcal{U}, \mathcal{U}^*)\|_{D, j, k}^1 = \sup \{ \| \partial_k(D_j P)(\Psi(x_1), \ldots, \Psi(x_r), \Psi(x_1)^*, \ldots, \Psi(x_r)^*)\| : x_j = x_j^*, y_j = y_j^* \in \hat{A}_N, N \in \mathbb{N} \},
\]

and finally \(\|P(\mathcal{U}, \mathcal{U}^*)\|_{4, j, k, l}^1 = \sup \{ \| ev_{1, 1, 1}(\text{flip} \otimes \text{flip})(\partial_1 \otimes \partial_1)(\partial_k(D_j P))(\Psi(x_1), \ldots, \Psi(x_r), \Psi(x_1)^*, \ldots, \Psi(x_r)^*)\| : x_j = x_j^*, y_j = y_j^*, z_j = z_j^* \in \hat{A}_N, N \in \mathbb{N} \}\). We define the norm \(\|P(\mathcal{U}, \mathcal{U}^*)\|_{n_1} = \|P(\mathcal{U}, \mathcal{U}^*)\|_0 + \|P(\mathcal{U}, \mathcal{U}^*)\|_{D, 1}^1 + \|P(\mathcal{U}, \mathcal{U}^*)\|_{2, j, k}^1 + \|P(\mathcal{U}, \mathcal{U}^*)\|_{4, j, k, l}^1\). As a sum of a norm and four seminorms, \(\| \cdot \|_{n_1}\) is obviously a norm. If \(P_n, n \in \mathbb{N}\), is a sequence of polynomials such that \(\lim_{n \to \infty} \|P_n(\mathcal{U}, \mathcal{U}^*)\|_{n_1} = 0\), then the dominated convergence theorem together with formulae \((34)\) and \((35)\) guarantee that \(\nu_1(P_n(\mathcal{U}, \mathcal{U}^*)) \to 0\) and \(\nu_2^{(N)}(P_n(\mathcal{U}, \mathcal{U}^*)) \to 0\) as \(n \to \infty\). Lemma \((14)\) can be adapted to show that \(\phi \mathcal{S}^a_{\infty}\) is included in the closure of \(\mathbb{C}(\mathcal{U}, \mathcal{U}^*)\) with respect to this norm. Based on \((36)\), one can construct the obvious way a norm \(\| \cdot \|_{n_2}\) such that \(\nu_2^{(N)}\) are continuous in the topology it induces, and the closure of polynomials with respect to it includes \(\phi \mathcal{S}^a_{\infty}\).

### 6.2 Series expansion for large \(|z|\)

Recall the notations \(S_U, S_V\) from Proposition \((22)\) and denote

\[
G_{U+V}(b) = (\text{id}_m \otimes \text{tr}_N \otimes \text{tr}_N) \left[ (b \otimes I_N \otimes I_N - S_U - S_V)^{-1} \right].
\]
Thus, for $\|S_{U} + S_{V}\|$ may be a random matrix, $G_{U+V}$ may be a random noncommutative function. It is defined on the operator upper and lower half-planes and on the open set of elements $b$ such that $\|((3b)^{-1})\| < \|S_{U} + S_{V}\|$. Similarly, $G_{U}(b) = (id_{m} \otimes \text{tr}_{N} \otimes \text{tr}_{N}) \left[ (b \otimes I_{N} \otimes I_{N} - S_{U})^{-1} \right]$ and $G_{V}(b) = (id_{m} \otimes \text{tr}_{N} \otimes \text{tr}_{N}) \left[ (b \otimes I_{N} \otimes I_{N} - S_{V})^{-1} \right]$. To simplify the notation, let us rewrite $S_{V} = \sum_{i=1}^{2r_{2}} a_{i} \otimes I_{N} \otimes V_{i}$ where for $i \leq r_{2}$, $a_{i} = \beta_{i}$, $V_{i} = V_{i}$, $v_{i} = v_{i}$, and for $i > r_{2}$, $a_{i} = \beta_{i}^{*}$, $V_{i} = V_{i}^{*}$, $v_{i} = v_{i}^{*}$. With $\Delta^{n}$ denoting the the difference-differential operator as in (9),

$$
\|\Delta^{n}G_{U}(b, b, \ldots, b)(a_{1}, a_{2}, \ldots, a_{n})\| \leq \|(3b)^{-1}\|(3b)^{-1}\| \max_{i=1, \ldots, r_{2}} \{\|\beta_{i}\|\}^{n}.
$$

Thus, for $\|(3b)^{-1}\|$ sufficiently small, using Proposition 18, we obtain that

$$
\mathbb{E}(\text{tr}_{m} \otimes \text{tr}_{N} \otimes \text{tr}_{N}) \left[ (b \otimes I_{N} \otimes I_{N} - S_{U} - S_{V})^{-1} \right]
$$

$$
= \sum_{n=0}^{\infty} \sum_{1 \leq t_{1}, \ldots, t_{n} \leq 2r_{2}} \mathbb{E} \text{tr}_{m} [\Delta^{n}G_{U}(b, b, \ldots, b)(a_{1}, a_{2}, \ldots, a_{n})] \mathbb{E} \text{tr}_{N} \left( \prod_{j=1}^{n} V_{i_{j}} \right)
$$

$$
= \sum_{n=0}^{\infty} \sum_{1 \leq t_{1}, \ldots, t_{n} \leq 2r_{2}} \mathbb{E} \text{tr}_{m} [\Delta^{n}G_{U}(b, b, \ldots, b)(a_{1}, a_{2}, \ldots, a_{n})] \times \left\{ \tau \left( \prod_{j=1}^{n} V_{i_{j}} \right) + \frac{\nu_{1}(\prod_{j=1}^{n} V_{i_{j}})}{N^{2}} + \frac{\nu_{2}(\prod_{j=1}^{n} V_{i_{j}})}{N^{4}} \right\}
$$

$$
= \mathbb{E}(\text{tr}_{m} \otimes \text{tr}_{N} \otimes \tau) \left[ (b \otimes I_{N} \otimes 1_{A} - 2\mathfrak{R} \sum_{i=1}^{r_{1}} \gamma_{i} \otimes U_{i} \otimes 1_{A} - 2\mathfrak{R} \sum_{i=1}^{r_{2}} \beta_{i} \otimes I_{N} \otimes v_{i})^{-1} \right]
$$

$$
+ \frac{1}{N^{2}} \mathbb{E}(\text{tr}_{m} \otimes \text{tr}_{N} \otimes v_{1}) \left[ (b \otimes I_{N} \otimes 1_{A} - 2\mathfrak{R} \sum_{i=1}^{r_{1}} \gamma_{i} \otimes U_{i} \otimes 1_{A} - 2\mathfrak{R} \sum_{i=1}^{r_{2}} \beta_{i} \otimes I_{N} \otimes v_{i})^{-1} \right]
$$

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\[+ \frac{1}{N^4} \mathbb{E}(\text{tr}_m \otimes \text{tr}_N \otimes \nu_2^{(N)})(b \otimes I_N \otimes 1_A - 2\Re \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_A - 2\Re \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i)^{-1},\]

where the last equality has been proven to hold with \(b = zI_m - t;\) from here on, we make this assumption on \(b.\) It is clear from both the previously stated estimate and from Lemma 19 and Lemma 21 that if \(|\Im z|\) is sufficiently large, then each of the above terms has a convergent power series expansion. The last term in the above sum is \(O(1/N^4),\) thanks to the bounds independent of \(N\) obtained in Lemma 21 and applied in Section 6.1. But for the first two terms of the sum above, we have to iterate in order to replace the \(U_i\)'s by the \(u_i\)'s. For the first term, we will get similarly:

\[\mathbb{E}(\text{tr}_m \otimes \text{tr}_N \otimes \tau) \left[ b \otimes I_N \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes I_N \otimes v_i^* \right]^{-1}\]

\[= (\text{tr}_m \otimes \tau \otimes \tau) \left[ b \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right]^{-1}\]

\[+ \frac{1}{N^2} (\text{tr}_m \otimes \nu_1 \otimes \tau) \left[ b \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right]^{-1}\]

\[+ \frac{1}{N^4} (\text{tr}_m \otimes \nu_2^{(N)} \otimes \tau) \left[ b \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right]^{-1}.

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For the second term,
\[
\frac{1}{N^2} \mathbb{E} (\text{tr}_m \otimes \text{tr}_N \otimes \nu_1) \left[ \left( b \otimes I_N \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes 1_A \\
- \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes I_N \otimes v_i^* \right)^{-1} \right]
\]
\[
= \frac{1}{N^2} (\text{tr}_m \otimes \tau \otimes \nu_1) \left[ \left( b \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \\
- \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right]
\]+ \frac{1}{N^4} (\text{tr}_m \otimes \nu_1^{(N)} \otimes \nu_1) \left[ \left( b \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \\
- \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right]
\]

Each of the above is just the definition of \( \nu_j, \nu_j^{(N)} \), applied to the entries of resolvents of affine matrix polynomials; it was shown in the previous section that all three of these linear functionals extend to such elements. Finally, by regrouping, we obtain that for all \( z \) with \(|\Im z|\) large enough, for all \( N \),
\[
\mathbb{E} (\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N) \left[ \left( (zI_m - \xi) \otimes I_N \otimes I_N - S_U - S_V \right)^{-1} \right] =
\]
\[
(\text{tr}_m \otimes \tau \otimes \tau) \left[ \left( (zI_m - \xi) \otimes 1_A \otimes 1_A - S_u - S_v \right)^{-1} \right] + \frac{E(z)}{N^2} + \frac{\Delta_N(z)}{N^4},
\]
where
\[
\Delta_N(z) = \mathbb{E} (\text{tr}_m \otimes \text{tr}_N \otimes \nu_2^{(N)}) \left[ \left( (zI_m - \xi) \otimes I_N \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes 1_A \\
- \sum_{i=1}^{r_1} \gamma_i^* \otimes U_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes I_N \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes I_N \otimes v_i^* \right)^{-1} \right]
\]+ \mathbb{E} (\text{tr}_m \otimes \nu_2^{(N)} \otimes \tau) \left[ \left( (zI_m - \xi) \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \\
- \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right]
\]
\[-\sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \] 

+ \left( \text{tr}_m \otimes \gamma_1^{(N)} \otimes \nu_1 \right) \left[ \left( zI_m - \xi \right) \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \right. 

\left. - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right],

and

\[ E(z) = \left( \text{tr}_m \otimes \tau \otimes \nu_1 \right) \left[ \left( zI_m - \xi \right) \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \right. 

\left. - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right] 

+ \left( \text{tr}_m \otimes \nu_1 \otimes \tau \right) \left[ \left( zI_m - \xi \right) \otimes 1_A \otimes 1_A - \sum_{i=1}^{r_1} \gamma_i \otimes u_i \otimes 1_A \right. 

\left. - \sum_{i=1}^{r_1} \gamma_i^* \otimes u_i^* \otimes 1_A - \sum_{i=1}^{r_2} \beta_i \otimes 1_A \otimes v_i - \sum_{i=1}^{r_2} \beta_i^* \otimes 1_A \otimes v_i^* \right)^{-1} \right].

### 6.3 Extension of equality (60) to any \( z \in \mathbb{C}^+ \)

We remind the reader that identity (60) was established for all \( z \) with \(|\Im z|\) sufficiently large\(^3\) and both the left-hand side and the first term of the right-hand side of (60) are analytic in \( z \) on the complement of a compact interval in \( \mathbb{R} \). Moreover, in Subsections 6.1.1 and 6.1.2, we have proved that all summands of \( \Delta_N \) and of \( E \) extend analytically to the complement in \( \mathbb{C} \) of a compact interval (apriori possibly larger) in \( \mathbb{R} \) (using the estimates from Lemma 19 and Lemma 21, one showed analyticity around infinity, and then using formulae (34) and (36), one proved that all summands of \( \Delta_N \) and of \( E \) extend to \( \mathbb{C}^\pm \), when viewing \( \nu_1, \nu_j^{(N)}, j = 1, 2 \) as applied term-wise to power series of polynomials in the variables \( v_i \)). Since all involved objects have been

\(^3\)Strictly greater than \( \max\{\|\beta_1\|, \ldots, \|\beta_{r_2}\|\} \), to be precise.
shown to be analytic on the complement in $\mathbb{C}$ of a compact subinterval of $\mathbb{R}$ and the left and right hand side coincide on a neighborhood of infinity, the identity principle for analytic functions allows us to conclude.

6.4 Study of $E(z)$

We need one more result: we have shown that both $\Delta_N$ and $E$ are the Cauchy transforms of compactly supported real Schwartz distributions. Indeed, in Lemma 15 we have shown that states send elements from $S_a^\infty$ (to which entries of our resolvents are shown to belong in Lemma 14) into Cauchy transforms of compactly supported real Schwartz distributions and in Proposition 18 we have shown that all of $\nu_1, \nu_j^{(N)}, j = 1, 2$ are real linear functionals. Since

$$\lim_{z \to \infty} zE(z) = \lim_{z \to \infty} z\Delta_N(z) = 0,$$

the compactly supported distributions in question map the constant function 1 to zero. What remains to be shown is that $z \mapsto E(z)$ is the Cauchy transform of a compactly supported real distribution whose support is included in the spectrum of $\xi \otimes 1_A \otimes 1_A + S_u + S_v$:

**Proposition 24.** $E$ is the Stietjes transform of a distribution whose support is included in the spectrum of $\xi \otimes 1_A \otimes 1_A + S_u + S_v$, and which maps the constant function 1 to zero.

**Proof.** As mentioned just above, all it remains to do is show that the support of $z \mapsto E(z)$ is included in the spectrum of $\xi \otimes 1_A \otimes 1_A + S_u + S_v$. We show this by proving a slightly stronger result: with the (more general) $\mathcal{S}$ from (25), we prove that the support of the Schwartz distribution whose Cauchy transform is $z \mapsto (\text{tr}_m \otimes \phi \otimes \nu_1)((z - \mathcal{S})^{-1})$ is equal to the spectrum sp($\mathcal{S}$). Recall the expression under the sum and the integral in Parraud’s formula (37): $\Theta_{\frac{1}{z_1}\frac{1}{z_2}}\frac{1}{z_1}\frac{1}{z_2}((\partial_{\tilde{a}_1} \otimes \partial_{\tilde{a}_2}) \circ \partial_i(\text{ev}_1 \circ \text{flip} \circ \partial_i)\cdot\cdot\cdot))$. Of course, in our case it has to be modified to some extent in order to account for the coefficients living in $T$. The fact that $\partial_i \circ (\phi \otimes \text{id}_{A^N}) = \phi \circ \partial_i$ (see Lemma 13) allows us to perform coherently these modifications. Specifically, we write $\Theta_{\frac{1}{z_1}\frac{1}{z_2}}\frac{1}{z_1}\frac{1}{z_2}((\partial_{\tilde{a}_1} \otimes \partial_{\tilde{a}_2}) \circ \partial_i(\text{ev}_1 \circ \text{flip} \circ \partial_i)((\phi \otimes \text{id}_{A_N})(F_{\tilde{a}_1})))$ for $F_\tilde{a}_1$ being any of the entries of the resolvent of $\mathcal{S}$, which, we claim, makes sense. Indeed, as already seen, $\partial_i((\phi \otimes \text{id}_{A_N})(F_{\tilde{a}_1})) = \phi(\partial_i F_{\tilde{a}_1})$, and the right hand side makes sense thanks to Lemma 10. Moreover, thanks to Lemma 11, if $z \mapsto F_{\tilde{a}_1}(\mu_{a_1})$ is analytic on a given set for some tuple of variables $\mu_{a_1}$, then so is $z \mapsto \phi(\partial_i F_{\tilde{a}_1})(\mu_{a_1})$ for any $\mu_{a_1}$ having the same distribution as $\mu_{a_1}$. Next, one applies $\text{ev}_1 \circ \text{flip}$ to the result: $(\text{ev}_1 \circ \text{flip})(\partial_i((\phi \otimes \text{id}_{A_N})(F_{\tilde{a}_1}))\cdot\cdot\cdot))$. On each individual tensor in a given sum, this yields a product of two elements,
both belonging to $\phi S^\infty$. As we have proved that $\partial_i$ is well-defined on $\phi S^\infty$, applying it on the resulting object is simply done via the product rule. Since multiplication does not reduce the domain of analyticity, any evaluation on elements with a given distribution will either preserve or increase the set of analyticity in question. Relations (22) and (58) tell us that $\partial_j \otimes \partial_j$ is still well-defined on $\partial_i (ev_1 \circ flip \circ \partial_i)((\phi \otimes id_{A_N})(F_z))$ for all $z$ outside a compact subset of $\mathbb{R}$, and yet another application of Lemma 11 guarantees the preservation (or increase) of the domain of analyticity. Trivially, $\Theta^1 \hat{z}_1, \hat{z}_2, z_2^2$ is well-defined on the result. The analyticity of the correspondence in the complex variable $z$ follows from the (multi)linearity of the application. Since, as noted after their definition in Section 4, the distributions of the free tuples of variables $\hat{z}_1, \hat{z}_2, z_2^2$ do not depend on $t_1, t_2$, but are all free standard semicirculars (and our trace $\tau$ is assumed to be faithful), it follows that the evaluation on these variables is possible precisely when the evaluation on free tuples of standard free semicirculars is.

With this, we have shown that applying $\tau(\Theta^1 \hat{z}_1, \hat{z}_2, z_2^2)((\partial_j \otimes \partial_j) \circ \partial_i (ev_1 \circ flip \circ \partial_i)((\phi \otimes id_{A_N})(\cdot))))$ to any entry of the $m \times m$ matrix \((zI_m - \xi) \otimes 1 - 2\Re \sum_{i=1}^q \gamma_i \otimes t_i \otimes 1 - 2\Re \sum_{i=1}^r \beta_i \otimes 1 \otimes v_i \)^{-1} yields a function which is analytic on the complement of the spectrum of $\xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^q \gamma_i \otimes t_i \otimes 1 + 2\Re \sum_{i=1}^r \beta_i \otimes 1 \otimes s_i$. This guarantees that each such function extends analytically to the whole complement of the spectrum of $\xi \otimes 1 \otimes 1 + 2\Re \sum_{i=1}^q \gamma_i \otimes t_i \otimes 1 + 2\Re \sum_{i=1}^r \beta_i \otimes 1 \otimes v_i$. We conclude by simply applying this proof twice to the two summands of $E$, with $\phi$ replaced by $\tau$ and the $t_i$'s by our $u_i$'s. \(\square\)

6.5 Bound for $\Delta_N(z)$

**Proposition 25.** There exists $k$ in $\mathbb{N}$ and a nonnegative constant $C$ such that, for any $z \in \mathbb{C}^+$, for any $N$,

$$|\Delta_N(z)| \leq \frac{C}{|\Im z|^k}.$$

The proposition has in fact been proved in the several steps culminating in Section 6.3, one needs only observe that $C$ can be chosen independently from $N$. This follows easily from the estimate $\|F_z\| \leq \frac{\|D^1\| \|D^2\| \cdots \|D^{k+1}\|}{|\Im z|^k}$ from the proof of Lemma 11 and the fact that our variables are Cayley transforms of selfadjoints, hence unitaries.
Lemma 26. Let $W_N = \{W_N^{(i)}, i = 1, \ldots, r_1\}$ and $\tilde{W}_N = \{\tilde{W}_N^{(j)}, i = 1, \ldots, r_2\}$ be independent standard G.U.E. $N \times N$ matrices. Set $U_i = \Psi \left( \frac{W_N^{(i)}}{\sqrt{N}} \right)$, $V_j = \Psi \left( \frac{\tilde{W}_N^{(j)}}{\sqrt{N}} \right)$, and

$$S_N = \xi \otimes I_N \otimes I_N + 2\Re \left( \sum_{i=1}^{r_1} \gamma_i \otimes U_i \otimes I_N + \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes V_j \right).$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that $\varphi$ and $\varphi'$ are bounded. We have

$$V[(tr_m \otimes tr_N \otimes tr_N)(\varphi(S_N))] \leq \frac{16}{N^2} \left( \sum_{i=1}^{r_1} \|\gamma_i\|^2 + \sum_{j=1}^{r_2} \|\beta_j\|^2 \right) \mathbb{E} \left[ (tr_m \otimes tr_N \otimes tr_N)((\varphi')^2(S_N)) \right].$$

Proof. We follow the proof of [7, Proposition 4.7]. Set $E_{r_1+r_2,N} = (M_N(\mathbb{C})_{sa})^{r_1+r_2}$ and consider the mappings $g : E_{r_1+r_2,N} \to (M_m(\mathbb{C}) \otimes (M_N(\mathbb{C}) \otimes (M_N(\mathbb{C})))_{sa}$ and $f : E_{r_1+r_2,N} \to \mathbb{C}$ defined by

$$g(X_1, \ldots, X_{r_1}, Y_1, \ldots, Y_{r_2}) = \xi \otimes I_N \otimes I_N + \sum_{i=1}^{r_1} \gamma_i \otimes \Psi(X_i) \otimes I_N + \sum_{i=1}^{r_1} \gamma_i^* \otimes \Psi(-X_i) \otimes I_N$$

$$+ \sum_{j=1}^{r_2} \beta_j \otimes I_N \otimes \Psi(Y_j) + \sum_{j=1}^{r_2} \beta_j^* \otimes I_N \otimes \Psi(-Y_j),$$

$$f(X_1, \ldots, X_{r_1}, Y_1, \ldots, Y_{r_2}) = (tr_m \otimes tr_N \otimes tr_N) \{ \varphi[g(X_1, \ldots, X_{r_1}, Y_1, \ldots, Y_{r_2})] \}.$$
where \( \langle \cdot, \cdot \rangle_e \) denotes the trace-induced (Hilbert-Schmidt) scalar product. Thus
\[
\|\nabla f(v)\|_e^2 = \max_{w \in S_1(\mathcal{E}_{r_1+r_2,N})} |\langle \nabla f(v), w \rangle_e|^2
\]
\[
= \max_{w \in S_1(\mathcal{E}_{r_1+r_2,N})} \left| \frac{d}{dt} \left| f(v + tw) \right| \right|_{t=0}^2,
\]
with \( S_1(\mathcal{E}_{r_1+r_2,N}) \) denoting the unit sphere in the Euclidean metric on \( \mathcal{E}_{r_1+r_2,N} \) induced by \( \langle \cdot, \cdot \rangle_e \). Let \( v = (v_1, \ldots, v_{r_1}, \tilde{v}_1, \ldots, \tilde{v}_{r_2}) \in \mathcal{E}_{r_1+r_2,N} \) and \( w = (w_1, \ldots, w_{r_1}, \tilde{w}_1, \ldots, \tilde{w}_{r_2}) \in S_1(\mathcal{E}_{r_1+r_2,N}) \). We have
\[
\frac{d}{dt} \bigg|_{t=0} f(v + tw) = \frac{d}{dt} \bigg|_{t=0} (\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N)(\varphi(g(v + tw)))
\]
\[
= (\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N)[\varphi'(g(v)) \frac{d}{dt} \bigg|_{t=0} g(v + tw)]
\]
\[
= (\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N) \left[ \varphi'(g(v)) \right.
\]
\[
\times \left\{ -2i \sum_{i=1}^{r_1} \gamma_i \otimes (v_i - i)^{-1} w_i (v_i - i)^{-1} \otimes I_N
\]
\[
+ 2i \sum_{i=1}^{r_1} \gamma_i^* \otimes (v_i + i)^{-1} w_i (v_i + i)^{-1} \otimes I_N
\]
\[
- 2i \sum_{j=1}^{r_2} \beta_j \otimes (\tilde{v}_j - i)^{-1} \tilde{w}_j (\tilde{v}_j - i)^{-1} \otimes I_N
\]
\[
+ 2i \sum_{j=1}^{r_2} \beta_j^* \otimes (\tilde{v}_j + i)^{-1} \tilde{w}_j (\tilde{v}_j + i)^{-1} \otimes I_N \right\}.
\]
Using then Cauchy-Schwartz inequality for \( \text{Tr}_m \otimes \text{Tr}_N \otimes \text{Tr}_N \), we obtain
\[
\left| \frac{d}{dt} \bigg|_{t=0} f(v + tw) \right|^2 \leq \frac{1}{m^2 N^4} \text{Tr}_m \otimes \text{Tr}_N \otimes \text{Tr}_N[\varphi'(g(v))^2]
\]
\[
\times (\text{Tr}_m \otimes \text{Tr}_N \otimes \text{Tr}_N) \left[ \left\{ -2i \sum_{i=1}^{r_1} \gamma_i \otimes (v_i - i)^{-1} w_i (v_i - i)^{-1} \otimes I_N
\right.
\]
\[
+ 2i \sum_{i=1}^{r_1} \gamma_i^* \otimes (v_i + i)^{-1} w_i (v_i + i)^{-1} \otimes I_N \right].
\]
\[-2i \sum_{j=1}^{r_2} \beta_j \otimes (\tilde{v}_j - i)^{-1} \tilde{w}_j (\tilde{v}_j - i)^{-1} \otimes I_N \]
\[+ 2i \sum_{j=1}^{r_2} \beta_j^* \otimes (\tilde{v}_j + i)^{-1} \tilde{w}_j (\tilde{v}_j + i)^{-1} \otimes I_N \]}
\[= 2 \sum_{i=1}^{r_1} \gamma_i \otimes (v_i - i)^{-1} w_i (v_i - i)^{-1} \otimes I_N \]
\[+ 2 \sum_{i=1}^{r_1} \gamma_i^* \otimes (v_i + i)^{-1} w_i (v_i + i)^{-1} \otimes I_N \]
\[+ 2 \sum_{j=1}^{r_2} \beta_j \otimes (\tilde{v}_j - i)^{-1} \tilde{w}_j (\tilde{v}_j - i)^{-1} \otimes I_N \]
\[+ 2 \sum_{j=1}^{r_2} \beta_j^* \otimes (\tilde{v}_j + i)^{-1} \tilde{w}_j (\tilde{v}_j + i)^{-1} \otimes I_N \]}
\[\| \mathbf{H} \|_{HS} \leq 4 \sqrt{mN} \left( \sum_{i=1}^{r_1} \| \gamma_i \|^2 + \sum_{j=1}^{r_2} \| \beta_j \|^2 \right)^{1/2} \left( \sum_{i=1}^{r_1} \text{Tr}_N w_i^2 + \sum_{j=1}^{r_2} \text{Tr}_N \bar{w}_j^2 \right)^{1/2} \]
\[= 4 \sqrt{mN} \left( \sum_{i=1}^{r_1} \| \gamma_i \|^2 + \sum_{j=1}^{r_2} \| \beta_j \|^2 \right)^{1/2} \]
where we use the fact that \( w = (w_1, \ldots, w_{r_1}, \bar{w}_1, \ldots, \bar{w}_{r_2}) \in S_1(\mathcal{E}_{r_1+r_2,N}) \) in
the last line. Thus, for any \( v \in \mathcal{E}_{r_1+r_2,N} \) and \( w \in S_1(\mathcal{E}_{r_1+r_2,N}) \),

\[
\left| \frac{d}{dt} f(v + tw) \right|_{t=0}^2 \leq \frac{16}{N} \left( \sum_{i=1}^{r_1} \| \gamma_i \|^2 + \sum_{j=1}^{r_2} \| \beta_j \|^2 \right) (\text{tr}_m \otimes \text{tr}_N \otimes \text{tr}_N)[(\varphi'(g(v))^2],
\]

and the result follows. \( \square \)

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