Turning Mathematics Problems into Games: Reinforcement Learning and Gröbner bases together solve Integer Feasibility Problems

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Abstract
Can agents be trained to answer difficult mathematical questions by playing a game? We consider the integer feasibility problem, a challenge of deciding whether a system of linear equations and inequalities has a solution with integer values. This is a famous NP-complete problem with applications in many areas of Mathematics and Computer Science. Our paper describes a novel algebraic reinforcement learning framework that allows an agent to play a game equivalent to the integer feasibility problem. We explain how to transform the integer feasibility problem into a game over a set of arrays with fixed margin sums. The game starts with an initial state (an array), and by applying a legal move that leaves the margins unchanged, we aim to eventually reach a winning state with zeros in specific positions. To win the game the player must find a path between the initial state and a final terminal winning state, if one exists. Finding such a winning state is equivalent to solving the integer feasibility problem. The key algebraic ingredient is a Gröbner basis of the toric ideal for the underlying axial transportation polyhedron. The Gröbner basis can be seen as a set of connecting moves (actions) of the game. We then propose a novel RL approach that trains an agent to predict moves in continuous space to cope with the large size of action space. The continuous move is then projected onto the set of legal moves so that the path always leads to valid states. As a proof of concept we demonstrate in experiments that our agent can play well the simplest version of our game for 2-way tables. Our work highlights the potential to train agents to solve non-trivial mathematical queries through contemporary machine learning methods used to train agents to play games.

Introduction
Reinforcement learning has seen tremendous success in recent years, especially in playing games at levels that achieve superhuman performances (Mnih et al. 2015, Silver et al. 2017, Guo et al. 2016, Vinyals et al. 2019). The philosophical principle we introduce in this paper is to try to reformulate non-trivial mathematical problems as games and then try to adapt reinforcement learning techniques to play those games. By winning the game we solve the original mathematical problem. Of course this first requires (at least for now) a human creating the right game for the given mathematical problem. As a proof of concept, in this paper we use the integer feasibility problem (IFP). In its simplest form, the IFP is the decision question of whether a polyhedral system \( \{ x : Ax = b, x \geq 0 \} \) has an integer vector solution \( x \) (note that any system of equations and inequalities can be reduced to this standard form). This famous NP-complete problem is very important in mathematical optimization, discrete mathematics, algebra, and other areas of mathematics. What we propose here is to turn it into a game on arrays.

As we explain below we transform every instance of the integer IFP into a positional game over arrays or tables with fixed margin sums belonging to the axial transportation we use an old polynomial-time algorithm from (De Loera and Om 2004a) that canonically rewrites an instance of the IFP as the face of a \( l \times m \times n \) axial transportation polytope. Axial transportation polytopes are convex polytopes whose points are arrays or tables with fixed axial sums of entries (Yemelichev, Kovalév, and Kravtsov 1984). The second idea is that we know well the toric (polynomial ring) ideals of axial transportation polytopes have an explicit North-West Gröbner basis that connects all lattice points. The game starts with an initial state (an initial array), and by applying a legal move that leaves the margins unchanged, we aim to reach a winning state with zeros in specific positions. To win the game the player must find a path between the initial state and a final terminal winning state. Finding such a winning state is equivalent to solving the integer feasibility problem. As we see later the winning position, if one exists, is essentially a table with very specific zero entries. The Gröbner basis of the toric ideal for the underlying transportation polyhedron as a set generating all connecting moves that the agent can be trained to select the moves. Of course our games sometimes have no solution, then our method, at least for now, may have to exhaust all positions before concluding this. Reinforcement learning tools are then used to search the game space.

Reinforcement learning on games has achieved great success and a key point of our paper is that the success can be extended to train artificial agents to play these games with a mathematical origin. We make a successful practical demonstration of these ideas with experiments in the simplest form of our games, the situation for 2D tables where we trained an agent to predict the moves. Our game can be viewed as a variant of the stochastic shortest path problem (Bertsekas).
The sizes as a face $F$. Theorem 1. Any rational convex polyhedron can be written as a face of some polytope. We begin with stating the first algorithmic approach to table that satisfy same margins. In other words, we get an optimal solution for arbitrary demand and supply vectors. In general the algorithm always constructs an optimal solution for arbitrary demand and supply vectors $a^1$ and $a^2$ and cost vector $c$. But for some cost vectors and special sequences the solution will be optimal. Sequences and cost matrices which fulfill the property above are called Monge sequences.
For this paper, we need to find an initial integer table efficiently for 3-way axial transportation polytopes. The good news is that, once again, a very similar greedy algorithm for the classical 2-way problem explained above can be applied to obtain a feasible solution for this more general case. In pseudocode this algorithm will read as follows in the case of a 3-way \( l \times m \times n \) axial transportation polytope:

Take an arbitrary order of the variables, say the sequence \( S := \{[i_1, j_1, k_1], [i_2, j_2, k_2], \ldots, [i_{lmn}, j_{lmn}, k_{lmn}]\} \), and perform the subsequent greedy algorithm:

For \( s := 1 \) to \( lmn \) do
1. Set \( x_{i_s,j_s,k_s} := \min(a_1^s, a_2^s, a_3^s) \)
2. \( a_1^s := a_1^s - x_{i_s,j_s,k_s} \)
   \( a_2^s := a_2^s - x_{i_s,j_s,k_s} \)
   \( a_3^s := a_3^s - x_{i_s,j_s,k_s} \)

Again, given a sequence \( S \) of triples of indices, this greedy algorithm maximizes each variable of \( S \) in turn. When the algorithm ends it will give always a feasible solution, in fact an integer solution when the margins are integer. In (Rudolf 1998) we have a necessary and sufficient condition on \( S \) and the cost matrix \( c \) which guarantees that the solution is in fact an optimal LP solution for costs \( c_{i,j,k} \) associated to each entry:

**Lemma 1.** The generalized north-west rule algorithm finds a feasible solution for the three-dimensional axial transportation problem for all right-hand-side vectors \( a_1, a_2, a_3 \) whose sum of entries are equal. The solution is integer if the vectors \( a_1, a_2, a_3 \) are integer. Moreover if there is a cost matrix cost matrix \( c_{i,j,k} \) which is a three-dimensional Monge sequence in the sense of (Rudolf 1998), then the solution found is an optimal linear programming solution for the minimization problem.

In the rest of this section, we will briefly introduce the notion of Gröbner bases relevant for our project problems. Recall a polynomial ideal \( I \) is a set of polynomials in \( R = \mathbb{Q}[x_1, \ldots, x_n] \) that satisfies two properties: (1) If \( f, g \) are in \( I \) then \( f + g \in I \) (2) If \( f \in I \) and \( h \in R \) then \( fh \) is in \( I \). A Gröbner basis of an ideal \( I \) is a special finite generating set for \( I \) with special computational properties. Their computational powers include the ability to answer membership questions for the ideal, computing intersections of ideals, computing projections of ideals, etc. Gröbner bases in general can be computed with the well-known Buchberger algorithm. We are only interested in special kinds of ideals, called toric ideals whose Gröbner bases are better behaved: Given a matrix \( A \) with integer entries, the toric ideal \( I_A \) is the ideal generated by the binomials of the form \( x^e - x^f \) such that \( A^e - A^f = 0 \). Gröbner bases of toric ideals have been explored in the literature (see (Sturmfels 1996, De Loera, Hemmecke, and Köppe 2013, Cox, Little, and O Shea 2005) and references therein). If we find a Gröbner basis \( G_A = \{x^{v_1} - x^{v_2}, x^{v_3} - x^{v_4}, \ldots, x^{v_{n-1}} - x^{v_n}\} \) for \( I_A \), it is well known that the vectors \( \Gamma = \{v_1 - v_2, v_3 - v_4, \ldots, v_{n-1} - v_n\} \) will have the following properties. Let \( P = \{x : Ax = b, x \geq 0\} \) be any polytope that could be defined by the matrix \( A \) and by a choice of an integral right-hand-side vector \( b \). If we form a graph whose vertices are the lattice points of \( P \) and we connect any pair \( x_1, x_2 \) of them by an edge if there is a vector \( u - v \) in \( \Gamma \) such that \( x_1 - u + v = x_2 \) with \( x_1 - u \geq 0 \), then the resulting graph is connected (Sturmfels 1996, Chapter 4). Moreover if we orient the edges of this graph according to a term order we used to compute the Gröbner basis above (where the tail of an edge is bigger than its head) this directed graph will have a unique sink. Thus from any lattice point of \( P \) there is an “augmenting” path to this sink. We will call the process of traversing such an augmenting path a reduction. Moreover, we will refer to the elements in \( \Gamma \) as moves. It has been shown in (Hösten and Sullivant 2002) that while toric ideals of general transportation polytopes are as nasty as those for general polytopes, axial transportation polytopes enjoy a rich decomposable structure that essentially allow us to build Gröbner bases from the Gröbner bases of their slices. For instance, for 2-way tables we know everything about the Gröbner bases of their toric ideals

**Theorem 2.** Let \( A \) be the 0/1-matrix which is the matrix of the linear transformation that computes the row and column sums of a given 2-way table. Then \( A \) is (totally) unimodular and hence its minimal universal Gröbner basis consists of its circuits. These circuits are 2-way tables whose row and column sums are zero and with entries in \( \{0, +1, -1\} \) of minimal support.

For axial transportation polytope of size \( m \times n \times k \), can we find a similarly nice Gröbner basis for some term order. We explain several ways next.

**Integer Feasibility Testing is a Game on Tables**

We have seen that from Theorem 1 the (integer) tables with specified margins in the construction represent all the lattice solutions of the original IP. Those will be the states of the game. In order to check the integer feasibility of \( P \) we need to have a Gröbner basis (test set) of the axial transportation polyhedron \( Q \) such that the normal form of the North-West corner rule integer initial solution \( v \) is a feasible solution of \( P \) (if such a solution exists). Of course, such a Gröbner basis exists right away: a Gröbner basis of the axial transportation arrays with respect to an elimination term order where all variables corresponding to the “forbidden” entries of the arrays are bigger than the “enabled” entries will do the job. In principle, we are done. However in the rest of the section we explain how to find a more efficient solution avoiding to use Buchberger algorithm.

We construct such a Gröbner basis building from the Gröbner bases of 2-way transportation problems slices of 3-way tables (see Theorem 2 and discussion there). Moreover, we will prove that we do not need to explicitly compute and store this Gröbner basis in advance (which is a very large set of vectors). It is enough to compute an element of the Gröbner basis “on the fly” that will improve the current feasible solution. For our construction we will follow the ideas presented in (Hösten and Sullivant 2002). There a similar construction was given for any decomposable statistical problem. Fortunately, the 3-way axial problems are decomposable, so we can use those techniques. But we will
Lemma 2. Let \( \succ^1 \) and \( \succ^2 \) be term orders for \( l \times n \) and \( m \times n \) planar tables and let \( \succ_1, \ldots, \succ_n \) be \( n \) term orders for the \( l \times m \) planar tables. If \( \succ^i \) is the term order for \( l \times m \times n \) 3-way tables described in Theorem 3 then the relation \( \succ^i \) on such 3-way tables given by \( X \succ^i X' \) if

\[
\text{Proj}_{X,z}(X) \succ^1 \text{Proj}_{X,z}(X') \quad \text{or} \quad 
\text{Proj}_{X,z}(X) = \text{Proj}_{X,z}(X') \quad \text{and} \quad \text{Proj}_{y,z}(X) \succ^2 \text{Proj}_{y,z}(X') \quad \text{or} \quad 
\text{Proj}_{X,z}(X) = \text{Proj}_{X,z}(X') \quad \text{and} \quad \text{Proj}_{y,z}(X) = \text{Proj}_{y,z}(X') \quad \text{and} \quad 
X \succ^i X'
\]

is a term order.

The proof of Lemma 2 is in the Appendix.

Theorem 3. Let \( G_{r_1}, \ldots, G_{r_n} \) be \( n \) Gröbner bases of the 2-way \( l \times m \) transportation problem. Let \( \succ' \) be the term order on the entries of the \( l \times m \times n \) axial transportation problem where \( \{ Y[i,j,1] \} \succ' \{ Y[i,j,2] \} \succ' \cdots \succ' \{ Y[i,j,n] \} \) and the entries in the \( k \)th horizontal slice \( \{ Y[i,j,k] \} \) are ordered with respect to the term order \( \succ_k \). Then \( \mathcal{F}(G_{r_1}, \ldots, G_{r_n}) \) is a Gröbner basis with respect to \( \succ' \).

There is a second set of moves for the original transportation problem \( Q \) obtained from 2-way transportation problems. Now we will describe these moves. This time let \( T_{x,z} \) be the 2-way planar transportation problem. Let \( G_{x,z} \) be a Gröbner basis for this problem. Suppose \( X_1 - X_2 \) is an element in \( G_{x,z} \) where \( X_1 \) and \( X_2 \) are nonnegative tables with entries \( X_1[i,k] \) and \( X_2[i,k] \). We can “lift” such an element to a move for the original problem \( Q \) as follows. First note that \( X_1 - X_2 \) is homogeneous, i.e., \( \sum_{k} X_1[i,k] = \sum_{k} X_2[i,k] = t \). Second because we are in the setting of a 2-way transportation problem, for each \( k \) we have \( \sum_{i} X_1[i,k] = \sum_{i} X_2[i,k] \). Hence we can represent \( X_1 - X_2 \) as

\[
([i_1,k_1], [i_2,k_2], \ldots, [i_t,k_t]) - ([i'_1,k_1], [i'_2,k_2], \ldots, [i'_t,k_t]).
\]

Here we allow that some indices \( [i_t,k_t] \) repeated if the corresponding entry \( X_1[i_1,\cdots] \) (or \( X_2[i_1,\cdots] \)) in the table is bigger than one. Now given a sequence of indices (again repetitions are allowed) \( j_1, \ldots, j_t \) we get a move \( Y_1 - Y_2 \) for the transportation problem \( Q \) as

\[
([i_1,j_1,k_1], [i_2,j_2,k_2], \ldots, [i_t,j_t,k_t]) - ([i'_1,j_1,k_1], [i'_2,j_2,k_2], \ldots, [i'_t,j_t,k_t]).
\]

We let \( \mathcal{L}(G_{x,z}) \) to be the set of all moves obtained from all Gröbner basis elements in \( G_{x,z} \) using all possible liftings. Similarly we can define \( \mathcal{L}(G_{y,z}) \). Now we claim we can put together \( \mathcal{F}(G_{r_1}, \ldots, G_{r_n}), \mathcal{L}(G_{x,z}), \) and \( \mathcal{L}(G_{y,z}) \) to get a Gröbner basis for the toric ideal of \( l \times m \times n \) 3-way axial transportation problem. However, we first need to describe the appropriate term order.

Given a 3-way table \( X \) we can compute “projection” tables (marginals) in any axial direction. For instance \( \text{Proj}_{X,z}(X) \) would be the 2-way table whose \( (i,k) \) entry is \( \sum_{j} X[i,j,k] \).

Lemma 2. Let \( \succ^1 \) and \( \succ^2 \) be term orders for \( l \times n \) and \( m \times n \) planar tables and let \( \succ_1, \ldots, \succ_n \) be \( n \) term orders with respect to the term orders \( \succ^1 \) and \( \succ^2 \), respectively. Also let \( G_{r_1}, \ldots, G_{r_n} \) be \( n \) Gröbner bases for \( l \times m \) planar transportation problems with respect to the term orders \( \succ_1, \ldots, \succ_n \). Let \( \succ' \) be the term order for the \( l \times m \times n \) tables given in Theorem 3. Then the set

\[
G = \mathcal{L}(G_{x,z}) \cup \mathcal{L}(G_{y,z}) \cup \mathcal{F}(G_{r_1}, \ldots, G_{r_n})
\]

is a Gröbner basis for the 3-way axial transportation problems with respect to the term order \( \succ' \) of Lemma 2.

Now we are ready to construct a Gröbner basis for 3-way axial transportation problems with which we will solve the integer feasibility problem for any polytope \( P \) after it has been encoded as a 3-way axial transportation polytope \( Q \). In order to do this we will describe term orders \( \succ^1, \succ^2 \) and \( \succ_1, \ldots, \succ_n \) and then use Theorem 3. Recall that by the construction of \( Q \), the tables corresponding to the feasible solutions of \( P \) will be a face \( F \) given by forbidden entries which need to be set to zero. If \( X \) is such a table where all the enabled entries are positive, then we get forbidden entries for tables \( \text{Proj}_{x,z}(X), \text{Proj}_{y,z}(X) \) which are forced to be equal to zero. Also each horizontal slice \( X_1, \ldots, X_n \) will have its own forbidden entries. Let \( F_{x,z}, F_{y,z} \) and \( F_1, \ldots, F_n \) be these forbidden entries, and let \( E_{x,z}, E_{y,z} \) and \( E_1, \ldots, E_n \) be their complements, namely, the enabled entries. We let \( w_{x,z} \) be the weight vector where \( w_{x,z}[(i,k)] = 1 \) if \( (i,k) \in F_{x,z} \) and \( w_{x,z}[(i,k)] = 0 \) if \( (i,k) \in E_{x,z} \). We define \( w_{y,z} \) and \( w_1, \ldots, w_n \) in a similar way. We define the term order \( \succ^1 \) to be any elimination term order where \( F_{x,z} \succ_1 E_{x,z} \) (and the entries \( F_{x,z} \) and \( E_{x,z} \) within themselves are ordered in an arbitrary but fixed order) which refines the ordering giving by \( w_{x,z} \). In other words, given two \( l \times m \times n \) tables \( X \) and \( Y \), if the weight of \( \text{Proj}_{x,z}(X) \) is bigger than the weight of \( \text{Proj}_{x,z}(Y) \) then we declare \( \text{Proj}_{x,z}(X) \succ^1 \text{Proj}_{x,z}(Y) \). In the case of equality, we resort to the elimination term order that breaks the tie. Note that if the support of \( \text{Proj}_{x,z}(Y) \) is contained in \( E_{x,z} \) and that of \( F_{x,z}(X) \) is not, then we immediately declare \( \text{Proj}_{x,z}(X) \succ^1 \text{Proj}_{x,z}(Y) \). We define \( \succ^2 \) and \( \succ_1, \ldots, \succ_n \) similarly in which the forbidden entries are eliminated.

Theorem 5. Algorithm 2 solves the integer feasibility of any rational polytope \( P \) by first transforming it into an 3-way axial transportation polytope \( Q \) with specific 1-margins and finding a sequence of Gröbner basis moves from an initial
Algorithm 1: Integer Feasibility Testing Game

Input: A rational polytope $P \subset \mathbb{R}^d$ presented in its representation $P = \{ x : Ax = b, x \geq 0 \}$.
Output: YES or NO depending on whether $P$ contains an integer lattice point.

1. Compute the encoding of $P$ as a face of a 3-way axial transportation polytope $Q$.
2. Use the North-West-corner rule to find an initial table $\mathcal{V}$ that is a feasible integer solution in $Q$.
3. Use Gröbner basis elements with respect to $\succ_s$ constructed above to get the unique sink $W$. If the weight of $\text{Proj}_{x,z}(W)$ or of $\text{Proj}_{y,z}(W)$ is not zero, output NO.
4. Otherwise, using the Gröbner basis elements in $G_{x,z}$ and $G_{y,z}$ of weight zero (and their liftings) generate a set $\mathcal{S}$ of tables such that $\{\text{Proj}_{x,z}(T) : T \in \mathcal{S}\}$ and $\{\text{Proj}_{y,z}(T) : T \in \mathcal{S}\}$ are the set of 2-way $l \times n$ and $m \times n$ tables with the same row and column sums as $\text{Proj}_{x,z}(W)$ and $\text{Proj}_{y,z}(W)$, and with their support in $E_{x,z}$ and $E_{y,z}$ respectively.
5. For each $T \in \mathcal{S}$ reduce $T$ just using the Gröbner basis elements in $\mathcal{F}(G_{x_1}, \ldots, G_{x_n})$ to obtain $X$. If the weight of $X$ with respect to $w_k$ is zero for all $k = 1, \ldots, n$, output YES. If no $T$ in $\mathcal{S}$ gives rise to such an $X$, output NO.

The action space $A$ is generated as integer combination of the Gröbner basis of Theorem 2.

Learning to Play Games on 2-way Tables

As a proof of concept, we now describe our method of playing table games using reinforcement learning on the polyhedral Gröbner bases systems we presented. Here we only consider the simpler case of 2-way tables because we have an explicit list of all Gröbner basis moves already (see Theorem 5). Similar ideas can be extended to 3-way axial transportation polytopes. In this family, the state space $\mathcal{S}$ consists of $m \times n$ tables of a 2-way transportation polytope with fixed 1-margins,

$$\mathcal{S} = \{ s \in \mathbb{Z}^{m \times n} | \sum_i s_{i,j} = x_i, \sum_j s_{i,j} = y_j, s \geq 0 \}.$$  

The action space is the Gröbner basis generators that connect any two $m \times n$ tables in $\mathcal{S}$. They are computed using Theorem 5. It is noteworthy that the action space can be defined with more flexibility because one can consider the minimum set of actions that connects all tables, or with more complicated moves that are linear combinations of the minimum moves. There is a trade-off between the action space and the efficiency of solving the game. We refine our action space $\mathcal{A}$ as follows.

$$\mathcal{A} = \{ a \in \{-1, 0, 1\}^{m \times n} | \sum_i a_{i,j} = 0, \sum_j a_{i,j} = 0 \}.$$  

Note that any element of the action space $\mathcal{A}$ is generated as integer combination of the Gröbner basis of Theorem 2.

### Structured Action Prediction

Due to the fact that our action space is discrete, one natural way is to predict the action as a classification problem at each time step. However,
its feasibility is hindered by the large number of actions to compute and store in advance. We instead make the actor network predict a continuous action \( \mathbf{a} = \pi_\phi(s) \). Then, we obtain the integral solution \( \tilde{\mathbf{a}} = P_g(s)(\mathbf{a}) \) by projecting \( \mathbf{a} \) to the feasible set of discrete actions encoded by the following integer program.

\[
\begin{align*}
\min_{\mathbf{a}^+, \mathbf{a}^-} & \quad \sum_{i, j} ||\mathbf{a}^+ - \mathbf{a}^- - \mathbf{a}||_d \\
\text{s.t.} & \quad \sum_i \mathbf{a}_{i, j}^+ - \mathbf{a}_{i, j}^- = 0, \sum_j \mathbf{a}_{i, j}^+ - \mathbf{a}_{i, j}^- = 0 \\
& \quad \mathbf{a}^+ + \mathbf{a}^- \leq 1, \sum_{i, j} \mathbf{a}_{i, j}^+ \geq 1, \\
& \quad \mathbf{a}^+, \mathbf{a}^- \in \{0, 1\}^{m \times n},
\end{align*}
\]

and we obtain the projected discrete action by \( \hat{\mathbf{a}} = \mathbf{a}^+ - \mathbf{a}^- \). We add constraint (2) to exclude the action with all zeros. An illustration of the process is shown in Figure [1].

Splitting \( \hat{\mathbf{a}} \) into two binary variables \( \mathbf{a}^+ \) and \( \mathbf{a}^- \) is not only speeds up the projection but also provides more flexibility on the constraints of the actions. For instance, we can bound the number of non-zero elements in the action by adding \( c_1 \leq \sum_{i, j} \mathbf{a}_{i, j}^+ \leq c_2 \) to the system.

To train the critic network, we consider the projection operator a deterministic part of the environment and directly learn \( Q(s, \mathbf{a}) \). Similar strategies can be found in Dulac-Arnold et al. (2015). It is worthwhile to mention that one can also learn \( q(s, \hat{\mathbf{a}}) \). The non-differentiable projection layer can be tackled by straight-through estimators (Bengio, Léonard, and Courville 2013; van den Oord, Vinyals, and Kavukcuoglu 2017). However, for every mini-batch update, calculating a target in the temporal difference learning requires the projection operations to obtain \( \hat{\mathbf{a}} \). The projection will become a bottleneck and significantly slows down the whole training process. We also observed in experiments that learning \( Q(s, \mathbf{a}) \) has no obvious performance gain. The actor network is updated for each mini-batch \( B \) by gradient ascent

\[
\nabla J(\phi) = \frac{1}{|B|} \sum_{(s, \mathbf{a}, s', r)} \nabla_\theta Q_\theta(s, \mathbf{a}) \nabla_\phi \pi_\phi(s).
\]

**Learning from Demonstrations**

Learning from demonstrations is an effective strategy to improve sample efficiency. Due the large action space of our problem, we generate a set of demonstrations \( \mathcal{B}_D \) using a greedy algorithm to speed up the learning. We make sure that a fixed portion of samples in the mini-batch are drawn from the demonstration during each training step. Since the demonstrations generated by the greedy algorithm are not perfect, we adopt the Q-filter strategy (Nair et al. 2017) to only enforce a supervised learning loss when the demonstrated action \( \mathbf{a} \) has a higher \( Q \) value than the actor’s action. Thus, the demonstration loss on mini-batch \( B_D \) using demonstrated action \( \mathbf{a}_D \) can be summarized as:

\[
L_D = \frac{1}{|B_D|} \sum_{(s, \mathbf{a}_D)} \left| \left| (\mathbf{a}_D - \pi_\phi(s)) \right| \right|^2 \cdot 1_{Q_\theta(s, \mathbf{a}_D) > Q_\phi(s)}.
\]
can solve the problem whereas the BC model has a very low success rate and it takes longer steps to reach the goal state.

**Generalization Test**

In this section we wonder if the model can solve games that are unseen in the training data. We set up the experiment by training the model on the fixed upper 1-margin bound 20 with grid size 5 and 10, and test its success rates on larger bounds with the same corresponding grid size. The results in Table 1 demonstrate that the trained model can solve games that it never experiences during the policy training step. This further suggests that the model can successfully recognize patterns in the table and produces corresponding moves regardless of their magnitudes.

### Conclusions and Future Work

We proposed an algorithm that converts the integer feasibility problem into a game on tables. We formulated the table game as a reinforcement learning problem and developed novel techniques to tackle the algebraic structure of the action space. The experimental results on 2-way tables show the potential of solving integer feasibility problems using the Gröbner bases approach. Training on 3-way tables is the next stage of our work. Since IFP is an NP-complete problem, in some cases our game algorithm may have to visit all (exponentially many) tables before concluding there is no solution. It is an open problem to explore ways to cut the search as our games have no solution sometimes.
Appendix

In this appendix we include technical proofs of some of the main theorems. Including an explanation of how any rational polyhedron can be rewritten as a face of a 3-way axial transportation polytopes with fixed 1 marginals.

Proofs of Lemma [2] and Theorem [5]

Proof (Lemma [2]). If \( X \neq X' \) it is clear that either \( X \succ X' \) or \( X' \succ X \), and the relation is compatible with adding the same table \( Y \) to \( X \) and \( X' \). So we just need to show that \( \succ^* \) is transitive. So let \( X \succ X' \) and \( X' \succ X'' \). There is a total of nine possibilities to be checked depending on how the tables are aligned, so we give one of these for illustration. Suppose \( X \succ X' \) because \( \text{Proj}_{x,z}(X) = \text{Proj}_{x,z}(X') \) but \( \text{Proj}_{y,z}(X) \succ^* \text{Proj}_{y,z}(X') \), and also suppose that \( X' \succ X'' \) because \( \text{Proj}_{x,z}(X') \succ X' \) and \( \text{Proj}_{x,z}(X'') \succ X'' \). Then \( \text{Proj}_{x,z}(X) = \text{Proj}_{x,z}(X') \succ^* \text{Proj}_{x,z}(X'') \). Hence \( X \succ X'' \). \( \square \)

Proof (of Theorem [5]). First we show that if \( P \) is feasible then the \( w_{x,z} \)-weight of \( \text{Proj}_{x,z}(W) \) and the \( w_{y,z} \)-weight of \( \text{Proj}_{y,z}(W) \) must both be zero. Suppose \( U \) is a table corresponding to a feasible solution of \( P \). Then clearly the \( w_{x,z} \)-weight of \( \text{Proj}_{x,z}(U) \) and the \( w_{y,z} \)-weight of \( \text{Proj}_{y,z}(U) \) are zero. But then if one of these weights for \( W \) were bigger than zero we would get a contradiction to the assumption that \( W \) is the unique sink obtained by reduction of \( V \) with respect to \( \succ^* \). Now because the \( w_{x,z} \)-weight of both \( \text{Proj}_{x,z}(W) \) and \( \text{Proj}_{y,z}(W) \) are zero, and since \( \text{Proj}_{x,z}(W) \) is the unique sink of 2-way \( x \times y \) tables with row and column sums equal to those of \( \text{Proj}_{x,z}(V) \), there is a sequence of Gröbner basis elements in \( G_{x,z} \) which reduce \( \text{Proj}_{x,z}(U) \) to \( \text{Proj}_{x,z}(W) \). These elements must have \( w_{x,z} \)-weight zero. This means that we can reverse these moves and use their liftings to obtain a table \( W' \) such that \( \text{Proj}_{x,z}(W') = \text{Proj}_{x,z}(U) \). Note that \( \text{Proj}_{y,z}(W') = \text{Proj}_{y,z}(W) \), and we can use the same argument above to conclude that one can reach to a table \( T \) using lifted elements of \( G_{y,z} \) with \( w_{x,z} \)-weight zero such that \( \text{Proj}_{y,z}(T) = \text{Proj}_{y,z}(U) \). Of course, we also have \( \text{Proj}_{x,z}(T) = \text{Proj}_{x,z}(U) \). The table \( T \) will be an element of \( S \) in Step 4 above, if we use the (reversed) elements of \( G_{x,z} \) and \( G_{y,z} \) of respective weights zero to generate all possible \( 2 \times 2 \) \( x \times y \) tables with the same row and column sums as \( \text{Proj}_{x,z}(W) \) and \( \text{Proj}_{y,z}(W) \). By this construction, for each \( 1 \leq k \leq n \) the horizontal slices \( T_k \) and \( U_k \) have identical row and column sums. Since the \( w_{y,z} \)-weight of \( U_k \) is zero, if the same weight of \( T_k \) is not zero, one can find a sequence of Gröbner basis elements in \( G_{x,z} \) (and hence in \( F(G_{x,z},\ldots,G_{y,z}) \)) to obtain \( X \) where the \( w_{y,z} \)-weight of \( X_k \) is zero. Because \( \text{Proj}_{x,z}(X) = \text{Proj}_{x,z}(T) = \text{Proj}_{x,z}(U) \) and \( \text{Proj}_{y,z}(X) = \text{Proj}_{y,z}(T) = \text{Proj}_{y,z}(U) \), the table \( X \) corresponds to a feasible solution of \( P \). \( \square \)

Preprocessing: Coefficient Reduction

Let \( P = \{ y \geq 0 : Ay = b \} \) where \( A = (a_{i,j}) \) is an integer matrix and \( b \) is an integer vector. We represent it as a polytope \( Q = \{ x \geq 0 : Cx = d \} \), in polynomial-time, with \( \{ -1, 0, 1, 2 \} \)-valued matrix \( C = (c_{i,j}) \) of coefficients, as follows. Consider any variable \( y_j \) and let \( k_j := \max\left\{ \lfloor \log_2 |a_{i,j}| \rfloor : i = 1, \ldots, m \right\} \) be the maximum number of bits in the binary representation of the absolute value of any \( a_{i,j} \). We introduce variables \( x_{j,0}, \ldots, x_{j,k_j} \), and relate them by the equations \( 2x_{j,i} - x_{j,i+1} = 0 \). The representing injection \( \sigma \) is defined by \( \sigma(j) := (j, 0) \), embedding \( y_j \) as \( x_{j,0} \). Consider any term \( a_{i,j} y_j \) of the original system. Using the binary expansion \( |a_{i,j}| = \sum_{s=0}^{k_j} t_s 2^s \) with all \( t_s \in \{0, 1\} \), we rewrite this term as \( \pm \sum_{s=0}^{k_j} t_s x_{j,s} \). It is easy to see that this procedure provides a new representation, and we get the following.

Lemma 3. Any rational polytope \( P = \{ y \geq 0 : Ay = b \} \) is polynomial-time representable as a polytope \( Q = \{ x \geq 0 : Cx = d \} \) with \( \{ -1, 0, 1, 2 \} \)-valued defining matrix \( C \).

Representing Polytopes as 3-way Transportation Polytopes with 1 marginals and forbidden entries

Let \( P = \{ y \geq 0 : Ay = b \} \) where \( A = (a_{i,j}) \) is an \( m \times n \) integer matrix and \( b \) is an integer vector: we assume that \( P \) is bounded and hence a polytope, with an integer upper bound \( U \) (which can be derived from the Cramer’s rule bound) on the value of any coordinate \( y_i \) of any \( y \in P \).

For each variable \( y_i \), let \( r_j \) be the largest between the sum of the positive coefficients of \( y_j \) over all equations and the sum of absolute values of the negative coefficients of \( y_j \) over all equations,

\[
r_j := \max \left( \sum_k \{ a_{k,j} : a_{k,j} > 0 \}, \sum_k \{ a_{k,j} : a_{k,j} < 0 \} \right).
\]

Let \( r := \sum_{i=1}^{n} r_j \), \( R := \{ 1, \ldots, r \} \), \( m := m + 1 \) and \( H := \{ 1, \ldots, h \} \). We now describe how to construct vectors \( u, v, w \in \mathbb{Z}^r \), \( w \in \mathbb{Z}^h \), and a set \( E \subset R \times R \times H \) of triples - the “enabled”, non-“forbidden” entries - such that the polytope \( P \) is represented as the corresponding transportation polytope of \( r \times r \times h \) arrays with plane-sums \( u, v, w \) and only entries indexed by \( E \) enabled.

\[
T = \{ x \in \mathbb{R}^r_{\geq 0} : x_{i,j,k} = 0 \text{ for all } (i, j, k) \notin E, \text{ and } \sum_{k} x_{i,j,k} = w_{i}, \sum_{i} x_{i,j,k} = v_{j}, \sum_{j} x_{i,j,k} = u_{k} \}.
\]

We also indicate the injection \( \sigma : \{ 1, \ldots, n \} \to R \times R \times H \) giving the desired embedding of the coordinates \( y_i \) as the coordinates \( x_{i,j,k} \) and the representation of \( P \) as \( T \) (see paragraph following Theorem [1]).
Basically, each equation $k = 1, \ldots, m$ will be encoded in a “horizontal plane” $R \times R \times \{k\}$ (the last plane $R \times R \times \{h\}$ is included for consistency and its entries can be regarded as “slack”); and each variable $y_j$, $j = 1, \ldots, n$ will be encoded in a “vertical box” $R_j \times R_j \times H$, where $R = \bigcup_{j=1}^{n} R_j$ is the natural partition of $R$ with $|R_j| = r_j$, namely with $R_j := \{1 + \sum_{l<j} r_l, \ldots, \sum_{l\leq j} r_l\}$.

Now, all “vertical” plane-sums are set to the same value $U$, that is, $u_j := v_j := U$ for $j = 1, \ldots, r$. All entries not in the union $\bigcup_{j=1}^{n} R_j \times R_j \times H$ of the variable boxes will be forbidden. We now describe the enabled entries in the boxes; for simplicity we discuss the box $R_1 \times R_1 \times H$, the others being similar. We distinguish between the two cases $r_1 = 1$ and $r_1 \geq 2$.

In the first case, $R_1 = \{1\}$; the box, which is just the single line $\{1\} \times \{1\} \times H$, will have exactly two enabled entries $(1,1,1^+), (1,1,1^-)$ for suitable $1^+, 1^-$ to be defined later. We set $\sigma(1) := (1,1,1^+)$, namely embed $y_1 = x_{1,1,1^+}$.

We define the complement of the variable $y_1$ to be $\bar{y}_1 := U - y_1$ (and likewise for the other variables). The vertical sums $u, v$ then force $\bar{y}_1 = U - y_1 = U - x_{1,1,1^+} = x_{1,1,1^-}$, so the complement of $y_1$ is also embedded. Next, consider the case $r_1 \geq 2$. For each $s = 1, \ldots, r_1$, the line $\{s\} \times \{1\} \times H$ (respectively, $\{s\} \times \{1 + (s \bmod r_1)\} \times H$) will contain one enabled entry $(s,s,k^+(s))$ (respectively $(s,1 + (s \bmod r_1), k^-(s))$). All other entries of $R_1 \times R_1 \times H$ will be forbidden. Again, we set $\sigma(1) := (1,1,k^+(1))$, namely embed $y_1 = x_{1,1,k^+(1)}$; it is then not hard to see that, again, the vertical sums $u, v$ force $x_{s,s,k^+(s)} = x_{1,1,k^+(1)} = y_1$ and $x_{s,1+s \bmod r_1,k^-(s)} = U - x_{1,1,k^+(1)} = \bar{y}_1$ for each $s = 1, \ldots, r_1$. Therefore, both $y_1$ and $\bar{y}_1$ are each embedded in $r_1$ distinct entries.

To clarify the above description it is helpful to visualize the $R \times R$ matrix $(x_{i,j,+})$ whose entries are the vertical line-sums $x_{i,j,+} := \sum_{k=1}^{h} x_{i,j,k}$.

Next we encode the equations by defining the horizontal plane-sums $w$ and the indices $k^+(s), k^-(s)$ above as follows. For $k = 1, \ldots, m$, consider the $k$th equation $\sum_j a_{k,j} y_j = b_k$. Define the index sets $J^+ := \{j : a_{k,j} > 0\}$ and $J^- := \{j : a_{k,j} < 0\}$, and set $w_k := b_k + U \cdot \sum_{j \in J^-} |a_{k,j}|$. The last coordinate of $w$ is set for consistency with $u, v$ to be $w_m = w_{m+1} := r \cdot U - \sum_{k=1}^{m} w_k$. Now, with $\bar{y}_j := U - y_j$ the complement of variable $y_j$, as above, the $k$-th equation can be rewritten as

$$\sum_{j \in J^+} a_{k,j} y_j + \sum_{j \in J^-} |a_{k,j}| \bar{y}_j = \sum_{j=1}^{n} a_{k,j} y_j + U \cdot \sum_{j \in J^-} |a_{k,j}| = b_k + U \cdot \sum_{j \in J^-} |a_{k,j}| = w_k.$$

To encode this equation, we simply “pull down” to the corresponding $k$th horizontal plane as many copies of each variable $y_j$ or $\bar{y}_j$ by suitably setting $k^+(s) := k$ or $k^-(s) := k$. By the choice of $r_j$ there are sufficiently many, possibly with a few redundant copies which are absorbed in the last hyperplane by setting $k^+(s) := m + 1$ or $k^-(s) := m + 1$. For instance, if $m = 8$, the first variable $y_1$ has $r_1 = 3$ as above, its coefficient $a_{4,1} = 3$ in the fourth equation is positive, its coefficient $a_{7,1} = -2$ in the seventh equation is negative, and $a_{k,1} = 0$ for $k \neq 4, 7$, then we set $k^+(1) = k^+(2) = k^+(3) := 4$ (so $\sigma(1) := (1,1,4)$ embedding $y_1$ as $x_{1,1,4}$), $k^-(1) = k^-(2) := 7$, and $k^-(3) = h = 9$.

This way, all equations are suitably encoded, and Theorem 3 follows from the construction outlined above and Lemma 3.

**Theorem 1 (Restated).** Any rational polytope $P = \{y \in \mathbb{R}^n_{\geq 0} : Ay = b\}$ is polynomial-time representable as a plane-sum entry-forbidden 3-way transportation polytope

$$T = \{ x \in \mathbb{R}^{r \times r \times h}_{\geq 0} \mid x_{i,j,k} = 0 \text{ for all } (i,j,k) \notin E, \quad \text{and} \quad \sum_{i,j} x_{i,j,k} = w_k, \sum_{i,j} x_{i,j,k} = v_j, \sum_{j,k} x_{i,j,k} = u_i \}.$$