Instability by Chern-Simons and/or Transgressions

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ABSTRACT

It was demonstrated recently that there is an upper bound of the Chern-Simons coupling of the five-dimensional Einstein-Maxwell theory, beyond which the electrically charged AdS$_2 \times S^3$ vacuum solution becomes unstable. We generalize the result to a general class of gravity theories involving Chern-Simons and/or transgression terms and find their upper bounds for stability. We show that supergravities with AdS$\times$Sphere vacua satisfy the bounds.
1 Introduction

Chern-Simons and transgression terms associated with form fields are common occurrences in supergravities. Typically supergravities allow all possible such terms but with the coupling strengths dictated by the supersymmetry. For example, the $\frac{1}{6}$ factor of the Chern-Simons term in eleven-dimensional supergravity is indeed fixed by the supersymmetry [1]. It turns out this term plays an important role in quantizing the supermembrane tension [2]. It was demonstrated that the U-duality groups $E_{n(+n)}$ of maximum supergravities coming from the $n$-torus reduction would be broken to only the $GL(n, \mathbb{R})$, had this coefficient not been $\frac{1}{6}$ [3]. This enhancement of global symmetry from $GL(n, \mathbb{R})$ to $E_{n(+n)}$ is crucial [4] for the consistent $S^7$ [5, 6] or $S^4$ [7, 8] Kaluza-Klein reductions of eleven-dimensional supergravity. The consistency requires a delicate balance [9] between the properties of the Killing vectors in the spheres and the properties of eleven-dimensional supergravity, including the $\frac{1}{6}$ factor.

On the other hand, in most of the isotropic $p$-brane constructions [2, 10] in string and M-theory, there is no contribution from the Chern-Simons or the transgression terms. It is intriguing to question whether such terms have any effects on the $p$-brane physics. In particular, we are interested in the non-dilatonic $p$-branes whose decoupling limits give rise to AdS×Sphere backgrounds. These solutions are expected, and in some cases proven to be stable due to the supersymmetry they preserve. Nevertheless, it was recently observed [11] that the Chern-Simons term could in principle provide a source of instability. This was known in $D = 3$ where topologically massive gauge theory can developed a tachyon mode when the topological Chern-Simons term is introduced [12]. The example considered in [11] was Einstein-Maxwell theory in five dimensions with a generic Chern-Simons term. It was demonstrated that for the electrically-charged $\text{AdS}_2 \times S^3$ background, there is an upper bound of the Chern-Simons coupling, beyond which tachyon modes emerge. The coupling in supergravity satisfies this bound. For the magnetically-charged $\text{AdS}_3 \times S^2$ solution, there is no such instability.

In this paper, we examine a large class of Chern-Simons and transgression structures that could arise in supergravities. We relax the couplings to be arbitrary constants and discuss the instability that could arise due to these terms. In section 2, we examine theories involving form fields with Chern-Simons and/or transgression terms in flat spacetime backgrounds. In general, Chern-Simons terms always produce instability in backgrounds with electric charges whilst the transgression terms produce instability in magnetic backgrounds. In section 3, we couple the system to the Einstein-Hilbert action with a cosmological con-
stant. By making use of the Breitenlohner-Freedman bound of AdS backgrounds, we derive the the maximum coupling of the Chern-Simons and/or transgression terms, beyond which instability will arise. We apply the results in various supergravities in section 4, and demonstrate that for AdS×Sphere backgrounds in supergravities, the bounds are always satisfied; they would have been saturated had the momentum in the internal direction be continuous. We conclude our paper in section 5. In appendix A, we present a detailed linear analysis of eleven-dimensional supergravity in AdS$_4 \times S^7$ and AdS$_7 \times S^4$ backgrounds. We use this example to show that in general the linear perturbation of the form fields that depends on the Chern-Simons/transgression coupling decouples from the rest of the perturbation modes including the graviton modes and hence can be analyzed easily. We give the condition for which these modes are no longer decoupled from certain graviton modes.

2 A general case in flat background

2.1 Either Chern-Simons or transgression term

Let us consider a general case in flat spacetime background, involving $(n, p, q)$-form field strengths. The Lagrangian contains only the kinetic terms and one Chern-Simons term, namely

$$L_0 = -\frac{1}{2} * H_n \wedge H_n - \frac{1}{2} * F_p \wedge F_p - \frac{1}{2} * G_q \wedge G_q + \alpha C(n-1) \wedge F_p \wedge G_q. \quad (1)$$

where $H_n = dC_{(n-1)}$, $F_p = dA_{(p-1)}$, $G_q = dB_{(q-1)}$. It is clear that the spacetime dimension is $D = n + p + q - 1$. The constant $\alpha$ measures the strength of the Chern-Simons coupling. It is well-known that the Chern-Simons and transgression terms are sometimes related by the Hodge dualization. This can certainly be done for $(1)$. To be specific, if we perform the Hodge dual on the $H_n$ to become $(D-n)$-form $\tilde{H}_{(D-n)}$, the Lagrangian becomes

$$L_0' = -\frac{1}{2} * \tilde{H}_{(D-n)} \wedge \tilde{H}_{(D-n)} - \frac{1}{2} * F_p \wedge F_p - \frac{1}{2} * G_q \wedge G_q. \quad (2)$$

In this Lagrangian there is no longer any Chern-Simons term; however, the $(D-n)$-form field strength is modified by a transgression term, namely

$$\tilde{H}_{(D-n)} = dC_{(D-n-1)} + \alpha A_{(p-1)} \wedge G_q, \quad dH_{(D-n)} = \alpha F_p \wedge G_q. \quad (3)$$

Thus there is no need for us to discuss the case with purely the transgression term in detail, since it can be dualized to become a Chern-Simons term.
The equations of motion for (11) are given by

\[ d*H_{(n)} = (-1)^n d\alpha F_{(p)} \wedge G_{(q)}, \]
\[ d*F_{(p)} = (-1)^{pq} d\alpha H_{(n)} \wedge G_{(q)}, \]
\[ d*G_{(q)} = d\alpha H_{(n)} \wedge F_{(p)}. \]  

(4)

In terms of index notation, the equations are given by

\[ \partial_I H^{IJ_1\ldots J_{n-1}} = \frac{\alpha}{p!q!} \varepsilon^{J_1\ldots J_{n-1}K_1\ldots K_p L_1\ldots L_q} F_{K_1\ldots K_p} G_{L_1\ldots L_q}, \]
\[ \partial_I F^{IJ_1\ldots J_{p-1}} = \frac{(-1)^{(D-q)p} \alpha}{n!q!} \varepsilon^{J_1\ldots J_{p-1}K_1\ldots K_q L_1\ldots L_q} H_{K_1\ldots K_q} G_{L_1\ldots L_q}, \]
\[ \partial_I G^{IJ_1\ldots J_{q-1}} = \frac{(-1)^{qD} \alpha}{n!p!} \varepsilon^{J_1\ldots J_{q-1}K_1\ldots K_p L_1\ldots L_q} H_{K_1\ldots K_p} F_{L_1\ldots L_p}. \]  

(5)

Here the tensor \( \varepsilon \) is a pure number in flat background and we adopt the convention that \( \varepsilon_{012\ldots n} = 1 \).

Let us consider a background with vanishing form fields \( F_{(p)} \) and \( F_{(q)} \) but non-zero \( H_{(n-1)} \). The equations for the linear fluctuation \((U, V)\) for \((F, G)\) are then given by

\[ \partial_I U^{IJ_1\ldots J_{p-1}} = \frac{(-1)^{(D-q)p} \alpha}{n!q!} \varepsilon^{J_1\ldots J_{p-1}K_1\ldots K_q L_1\ldots L_q} H_{K_1\ldots K_q} F_{L_1\ldots L_q}, \]
\[ \partial_I V^{IJ_1\ldots J_{q-1}} = \frac{(-1)^{qD} \alpha}{n!p!} \varepsilon^{J_1\ldots J_{q-1}K_1\ldots K_p L_1\ldots L_q} H_{K_1\ldots K_p} G_{L_1\ldots L_p}. \]  

(6)

In this section, we examine the possible instability due to the Chern-Simons term for constant electric or magnetic \( H_{(n)} \), or dyonic in \( D = 2n \) dimensions. In the last subsection, we shall consider a system with both Chern-Simons and transgression terms. It should be emphasized that for our discussion it is equivalent to turn on each one of the \((n, p, q)\) forms. In special cases where two or all three field strengths are the same, some combinatoric factors can be altered without changing the essential conclusion.

2.2 Electric \( H_{(n)} \)

We first consider the case with \( H_{(n)} \) being electric, namely

\[ H_{(n)} = (-1)^n E dt \wedge dx^1 \wedge \ldots \wedge dx^{n-1}, \]  

(7)

where \( E \) is a constant. The minus factor in the above plays no essential role and it is merely to make the intermediate formulae better looking. The whole spacetime is split into \( n \)-dimensional sub-spacetime \( T \) with coordinates \( x^\mu \) and \((d = D - n = p + q - 1)\) dimensional space \( S \) with coordinates \( y^j \). The equations of motion (6) become

\[ \partial_I U^{i_1\ldots i_{p-1}} = -\frac{\alpha E}{q!} \varepsilon^{i_1\ldots i_{p-1}j_1\ldots j_q} V_{j_1\ldots j_q}, \]  

4
\[ \partial_t V^{i_1 \cdots i_{q-1}} = -\frac{(1)^{p q} \alpha E}{p!} \varepsilon^{i_1 \cdots i_{q-1} j_1 \cdots j_p} U^{j_1 \cdots j_p} , \]
\[ \partial_t U^{i_1 \cdots i_{p-2} \mu} = 0 , \quad \partial_t V^{i_1 \cdots i_{q-2} \mu} = 0 . \] 

Note that \( U \) and \( V \) satisfy the following Bianchi identity

\[ \partial_t U^{j_1 \cdots j_q} = 0 = \partial_t V^{j_1 \cdots j_q} . \] 

Let us define

\[ \tilde{U}^{i_1 \cdots i_q-1} = \frac{1}{p!} \varepsilon^{i_1 \cdots i_q-1 j_1 \cdots j_p} U^{j_1 \cdots j_p} , \quad \tilde{V}^{i_1 \cdots i_{p-1}} = \frac{1}{q!} \varepsilon^{i_1 \cdots i_{p-1} j_1 \cdots j_q} V^{j_1 \cdots j_q} . \]

This implies that

\[ U^{i_1 \cdots i_p} = \frac{(1)^{p(q-1)}}{(q-1)!} \varepsilon^{i_1 \cdots i_p j_1 \cdots j_{q-1}} \tilde{U}^{j_1 \cdots j_{q-1}} , \quad V^{i_1 \cdots i_q} = \frac{(1)^{q(p-1)}}{(p-1)!} \varepsilon^{i_1 \cdots i_q j_1 \cdots j_{p-1}} \tilde{V}^{j_1 \cdots j_{p-1}} . \]

Acting on the first and second equations in by \( \varepsilon_{\ell i_1 \cdots i_{p-1} k_1 \cdots k_{q-1}} \partial^\ell \) and \( \varepsilon_{\ell i_1 \cdots i_{q-1} k_1 \cdots k_{p-1}} \partial^\ell \) respectively, we have

\[ \Box \tilde{U}^{i_1 \cdots i_q-1} + \frac{\alpha E}{(p-1)!} \varepsilon^{i_1 \cdots i_q-1 j_1 \cdots j_{q-1} \ell} \partial_\ell \tilde{V}^{i_1 \cdots i_{p-1}} , \]
\[ \Box \tilde{V}^{j_1 \cdots j_{p-1}} + \frac{(1)^{p q} \alpha E}{(q-1)!} \varepsilon^{j_1 \cdots j_{p-1} \ell i_1 \cdots i_{q-1} \partial_\ell \tilde{U}^{i_1 \cdots i_{q-1}} , \] 

where

\[ \Box = \partial^\mu \partial_\mu + \partial^\ell \partial_\ell . \] 

The above two equations can be expressed in terms of form language, namely

\[ \Box s^* U^s = 0 , \quad \Box d^* V^d = 0 \] 

The subscript “\( s \)” denotes that the forms and the Hodge dual are defined in the \( S \)-space. 

In the momentum basis, \( e^{ip_{\mu} x^\mu + ik_i u^i} \), we have

\[ \begin{pmatrix} (M^2 - k^2) I_{N_p} & \alpha E J_{p,q} \\ \alpha E J^i_{p,q} & (M^2 - k^2) I_{N_q} \end{pmatrix} \begin{pmatrix} \tilde{U}^s \\ \tilde{V}^d \end{pmatrix} = 0 \] 

where \( M^2 = -p^\mu p_\mu , \quad k^2 = k^i k_i , \quad N_p = C_{d}^{p-1} \) and \( N_q = C_{d}^{q-1} \). Also \( J \) is the \( N_p \times N_q \) matrix of momenta \( k_i \) and \( I_N \) is the \( N \times N \) identity matrix.

Let us look at some specific examples of \( J \). For convenience we may arrange \((\tilde{U}, \tilde{V})\) in the lexical order. For \( q = 1 \), \( J_{p,1} \) is a row vector of dimension \( p \) and the components are given by

\[ (J_{p,1})_i = i (-1)^{i+1} k_{p+1-i} . \]
The next simplest example is \( p = 2 \) and \( q = 2 \), for which we have

\[
J_{2,2} = i \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}
\] (17)

The matrix in (15) is hermitian and hence guaranteed to have real eigenvalues. The mass of possible tachyon modes can be determined by the vanishing of the determinant of the matrix, which leads to the condition

\[
M^2 - k^2 \pm \alpha E k = 0.
\] (18)

(The \( M = 0 \) solution with non-vanishing \( k \) is incompatible with the Bianchi identity.) Thus there are tachyon modes for \( 0 < k < \frac{1}{2} |\alpha E| \). In section 3, we shall analyze the system coupled to gravity, in which case the constant electric \( F_n \) can support an AdS_\( n \times S^{D-n} \) background.

2.3 Magnetic \( H_n \)

We now examine the case with \( H_n \) being magnetic, namely

\[
H_n = (-1)^{D+1} B dy^1 \wedge \ldots \wedge dy^n.
\] (19)

We split the whole spacetime into two parts: the \( (d = p + q - 1) \) dimensional spacetime \( T \), with coordinates \( x^\mu \) and the \( n \)-dimensional space \( S \) with coordinates \( y^i \). The equations for the linear perturbations (6) become

\[
\partial_I U^{\mu_1 \ldots \mu_{p-1}} + \frac{\alpha B}{q!} \varepsilon^{\mu_1 \ldots \mu_{p-1} \nu_1 \ldots \nu_q} U^{\nu_1 \ldots \nu_q} = 0,
\]

\[
\partial_I V^{\mu_1 \ldots \mu_{q-1}} + \frac{(-1)^{pq} \alpha B}{p!} \varepsilon^{\mu_1 \ldots \mu_{q-1} \nu_1 \ldots \nu_p} U^{\nu_1 \ldots \nu_p} = 0,
\]

\[
\partial_I U^{IJ_{1 \ldots J_{p-2}i}} = 0, \quad \partial_I V^{IJ_{1 \ldots J_{q-2}i}} = 0.
\] (20)

We define

\[
\tilde{U}^{\mu_1 \ldots \mu_{q-1}} = \frac{1}{p!} \varepsilon^{\mu_1 \ldots \mu_{q-1} \nu_1 \ldots \nu_p} U^{\nu_1 \ldots \nu_p}, \quad \tilde{V}^{\mu_1 \ldots \mu_{p-1}} = \frac{1}{q!} \varepsilon^{\mu_1 \ldots \mu_{p-1} \nu_1 \ldots \nu_q} U^{\nu_1 \ldots \nu_q}.
\] (21)

This implies that

\[
U^{\mu_1 \ldots \mu_p} = -\frac{(-1)^{p(q-1)}}{(q-1)!} \varepsilon^{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_{q-1}} \tilde{U}^{\nu_1 \ldots \nu_{q-1}},
\]

\[
V^{\mu_1 \ldots \mu_q} = -\frac{(-1)^{q(p-1)}}{(p-1)!} \varepsilon^{\mu_1 \ldots \mu_q \nu_1 \ldots \nu_{p-1}} \tilde{V}^{\nu_1 \ldots \nu_{p-1}}.
\] (22)
The equations for $\tilde{U}$ and $\tilde{V}$ can be cast into the same form as (14), except that now Hodge dual and the forms are defined within the $T$-spacetime, namely

$$\square \tilde{U}_t + \alpha E_{*t}d\tilde{V}_t = 0, \quad \square \tilde{V}_t + (-1)^{pq}\alpha E_{*t}d\tilde{U}_t = 0.$$  \hspace{1cm}(23)$$

Here the subscript $t$ labels the $T$-spacetime. In this case, the characteristic equation for the mass $M$ and momentum modular $k$ is given by

$$M^2 - k^2 + \alpha BM = 0.$$  \hspace{1cm}(24)$$

It is thus clear that there is no tachyon mode.

2.4 Dyonic $H^{(n)}$

When $D = 2n$, the field strength $H^{(n)}$ can be both electric and magnetic, namely

$$H^{(n)} = (-1)^nE\, dt \wedge dx^1 \wedge \ldots \wedge dx^{n-1} - B dy^1 \wedge \ldots \wedge dy^n.$$  \hspace{1cm}(25)$$

The $D$-dimensional spacetime is split into the $n$-dimensional spacetime $T$ with coordinates $x^\mu$ and the $n$-dimensional space $S$ with coordinates $y^i$. It is straightforward to derive the linearized equations of motion, which contain the following

$$\square \tilde{U}_s + \alpha E_{*s}d\tilde{V}_s = 0, \quad \square \tilde{V}_s + (-1)^{pq}\alpha E_{*s}d\tilde{U}_s = 0,$$

$$\square \tilde{U}_t + \alpha B_{*t}d\tilde{V}_t = 0, \quad \square \tilde{V}_t + (-1)^{pq}\alpha B_{*t}d\tilde{U}_t = 0.$$  \hspace{1cm}(26)$$

Thus there are tachyon modes associated with $\tilde{U}_s$ and $\tilde{V}_s$.

2.5 Both Chern-Simons and transgression terms

There can be both Chern-Simons and transgression terms associated with the same field strength $H^{(n)}$. The corresponding equations of motion and Bianchi identity are characterized by

$$d*H^{(n)} = (-1)^nD\alpha F_{(p)} \wedge G_{(q)}, \quad dh^{(n)} = \beta \tilde{F}_{(\tilde{p})} \wedge \tilde{G}_{(\tilde{q})}.$$  \hspace{1cm}(27)$$

Let $\tilde{X}$ and $\tilde{Y}$ be associated with $\tilde{F}$ and $\tilde{G}$ in the same way as $\tilde{U}$ and $\tilde{V}$ associated with $F$ and $G$. When the $H^{(n)}$ is electric, we have

$$\square \tilde{U}_s + \alpha E_{*s}d\tilde{V}_s = 0, \quad \square \tilde{V}_s + (-1)^{pq}\alpha E_{*s}d\tilde{U}_s = 0,$$

$$\square \tilde{X}_t + \beta E_{*t}d\tilde{Y}_t = 0, \quad \square \tilde{Y}_t + (-1)^{pq}\beta E_{*t}d\tilde{X}_t = 0.$$  \hspace{1cm}(28)$$

When the $H^{(n)}$ is magnetic, we have

$$\square \tilde{U}_t + \alpha B_{*t}d\tilde{V}_t = 0, \quad \square \tilde{V}_t + (-1)^{pq}\alpha B_{*t}d\tilde{U}_t = 0,$$

$$\square \tilde{X}_s + \beta B_{*s}d\tilde{Y}_s = 0, \quad \square \tilde{Y}_s + (-1)^{pq}\beta B_{*s}d\tilde{X}_s = 0.$$  \hspace{1cm}(29)$$
\[ \Box \tilde{X}_s + \beta B \ast_s d\tilde{Y}_s = 0, \quad \Box \tilde{Y}_s + (-1)^{pq} \beta B \ast_s d\tilde{X}_s = 0. \]  

(29)

Thus in this case, there are tachyon modes regardless whether \( H_{(n)} \) is electric or magnetic.

### 3 Coupled to Gravity

In the previous section we consider the instability arising from the Chern-Simons or transgression terms of form fields in the flat Minkowskian background. We now examine the effect of gravity coupled to this system. We first consider the Lagrangian

\[ \mathcal{L} = (R - 2\Lambda) \ast \mathbb{I} + \mathcal{L}_0. \]  

(30)

where \( \mathcal{L}_0 \) takes the same form as \([1]\). The first term in the above is the Einstein-Hilbert term with a cosmological constant \( \Lambda \). Let us first consider the \( \text{AdS} \times S^{D-n} \) vacuum solution supported by the electric \( H_{(n)} \); it is given by

\[
\begin{align*}
    ds^2_D &= a^2 ds^2_n + b^2 d\Omega^2_{D-n}, \quad H_{(n)} = E a^n \epsilon_{(n)},
    \\
    E^2 &= \frac{2(n-1)}{a^2} + \frac{2(D-n-1)}{b^2}, \quad -2\Lambda = \frac{(n-1)^2}{a^2} - \frac{(D-n-1)^2}{b^2},
\end{align*}
\]

(31)

where \( ds^2_n \) and \( d\Omega^2_{D-n} \) are the unit \( \text{AdS}_n \) and \( S^{D-n} \) metrics, satisfying \( R_{\mu\nu} = -(n-1)g_{\mu\nu} \) and \( R_{ij} = (D-n-1)g_{ij} \) respectively.

As discussed in the end of appendix A, in the special case for \( n = 2 \) with \( H_{(2)} = F_{(2)} \) and/or \( H_{(2)} = G_{(2)} \), the gravitational fluctuation couples with that of the 2-form field strengths through the Chern-Simons coupling \( \alpha \). This case was studied in \([11]\) for five-dimensional Einstein-Maxwell theory. In more generic cases, as we show in appendix A, the gravitational perturbation is independent of \( \alpha \). Since the purpose of the paper is to examine the effect of \( \alpha \) on the stability of the \( \text{AdS} \times \text{Sphere} \) solutions, there is no need for us to present the perturbation of the metric here. The relevant modes are the same as the one discussed in the flat background, namely \( \tilde{U}_s \) and \( \tilde{V}_s \) defined by \([10]\). (Here the subscript \( s \) denotes that the quantities carry only the indices in the \( S^{D-n} \) directions.) They now satisfy

\[
\Delta \tilde{U}_s + \alpha E \ast_s d\tilde{V}_s = 0, \quad \Delta \tilde{V}_s + (-1)^{pq} \alpha E \ast_s d\tilde{U}_s = 0,
\]

(32)

where \( \Delta = -(dd^\dagger + d^\dagger d) \) is the Laplace operator with respect to the \( \text{AdS} \times \text{Sphere} \) background. This implies that the mass of the possible tachyon modes is again determined by

\[ M^2 - k^2 \pm \alpha Ek = 0. \]

(33)
Thus the minimum value of the mass for the tachyon modes is given by

\[ M_{\text{min}} = -\frac{1}{4} \alpha^2 E^2. \]  

(34)

For this to satisfy the Breitenlohner-Freedman (BF) bound of the AdS\(_n\) spacetime, namely

\[ M_{\text{min}}^2 \geq M_{\text{BF}}^2 = -\frac{(n-1)^2}{4a^2}, \]  

(35)

we have

\[ \alpha^2 \left(1 + \frac{2a^2 \Lambda}{(n-1)(D-2)} \right) \leq \frac{(n-1)(D-n-1)}{2(D-2)}. \]  

(36)

There are two cases arising. The first case is when \( \Lambda \leq 0 \), for which the \( \alpha \) has a maximum value, namely

\[ \alpha^2 \leq \alpha_{\text{max}}^2 = \frac{(n-1)(D-n-1)}{2(D-2)}. \]  

(37)

Once this condition is satisfied, there is no instability due to the Chern-Simons term for all the allowed parameter regions of the AdS\( \times \)Sphere solutions in (31). The second case is when \( \Lambda > 0 \). In addition to the condition that \( \alpha \) has to be smaller than \( \alpha_{\text{max}} \), there is a further requirement that the AdS radius has to be sufficiently small. For a given \( \alpha < \alpha_{\text{max}} \), the maximum radius for the AdS\(_n\) is given by

\[ a_{\text{max}}^2 = \frac{(n-1)(D-2)}{2\Lambda} \left( \frac{\alpha_{\text{max}}^2}{\alpha^2} - 1 \right). \]  

(38)

Solutions with \( a > a_{\text{max}} \) suffers from the instability due to the Chern-Simons term.

For the magnetic AdS\(_{D-n}\times S^n\) solution, it is straightforward to show that the mass formula is then given by

\[ M^2 - k^2 \pm \alpha BM = 0, \quad \rightarrow \quad M^2 = \left( \sqrt{k^2 + \frac{1}{4} \alpha^2 B^2} \pm \frac{1}{2} \alpha B \right)^2. \]  

(39)

Thus there is no instability due to the Chern-Simons term.

It is clear that if the \( H(n) \) has only the transgression term instead of the Chern-Simons term, the electric solution will always be stable whilst the magnetic solution will be stable only if the analogous condition discussed above with \( E \) replaced by magnetic flux parameter \( B \) is satisfied. When \( H(n) \) has both Chern-Simons and transgression terms, the above conditions have to be satisfied in order to avoid instability regardless whether the \( H(n) \) is electric or magnetic.

If \( H(n) \) is self-dual, then the corresponding AdS\(_n\) \( \times S^n \) is given by

\begin{align*}
    ds_{2n}^2 &= a^2 ds_n^2 + b^2 d\Omega_n^2, \\
    E^2 &= (n-1) \left( \frac{1}{a^2} - \frac{1}{b^2} \right), \\
    F(n) &= E(a^n e(n) + b^n \Omega(n)), \\
    -2 \Lambda &= (n-1)^2 \left( \frac{1}{a^2} - \frac{1}{b^2} \right). 
\end{align*}  

(40)
Thus, for Λ ≤ 0, there is no instability due to the Chern-Simons and/or transgression terms as long as we have α ≤ α_{max} with

\[ \alpha_{\text{max}}^2 = \frac{1}{2} (n - 1), \]

Note that this expression for α_{max} has a factor 2 difference compared to that in (37) specializing in D = 2n. For Λ > 0, when this condition is satisfied, there can still be unstable AdSₙ × Sⁿ solutions as long as the AdS radius is larger than a_{max}, where

\[ a_{\text{max}}^2 = \frac{(n - 1)^2}{\Lambda} \left( \frac{\alpha_{\text{max}}^2}{\alpha^2} - 1 \right). \]

4 Applications in supergravities

We now apply the results obtained in the previous section in supergravities. Let us first examine eleven-dimensional supergravity, which has AdS₄ × S⁷ and AdS₇ × S⁴ vacuum solutions. The detailed analysis of linearized perturbation in these backgrounds were presented in appendix A. Eleven-dimensional supergravity has the Chern-Simons term, given by

\[ \mathcal{L}_{FFA} = \alpha A_3 \wedge F_4 \wedge F_4, \]

where |α| = 1/6. The situation is slightly different from the examples discussed in sections 2 and 3, where the (n, p, q) forms are all different. The equation of motion for F₄ now produces a factor 3, namely

\[ d*F_4 = 3 \alpha F_4 \wedge F_4. \]

Furthermore, if we consider \( F_4 = \tilde{F}_4 + f_4 \), where \( \tilde{F}_4 \) is the background and \( f_4 \) is a small perturbation, the above equation picks another factor 2, i.e.

\[ d\delta(*\tilde{F}_4) + d\delta f_4 = 6 \alpha \tilde{F}_4 \wedge f_4. \]

As shown in appendix A, the first term plays no role, and it follows that the stability condition (37) is modified by a factor 6 and becomes

\[ |\alpha| \leq |\alpha_{\text{max}}| = \frac{1}{6}. \]

Thus the bound of Chern-Simons coupling for the stable AdS₄ × S⁷ is saturated naively by eleven-dimensional supergravity. (See Appendix for further discussion.) For the magnetic AdS₇ × S⁴, there is no such a bound.
Type IIB supergravity has an AdS$_5 \times S^5$ vacuum solution supported by the self-dual 5-form field strength $H^{(5)}$. It couples to the R-R and NS-NS 3-form field strengths by both Chern-Simons and transgression terms, leading to the equations of motion

$$
\begin{align*}
    d\ast H^{(5)} &= -F_1^{(3)} \wedge F_2^{(3)}, \\
    d\ast F_1^{(3)} &= H^{(5)} \wedge F_2^{(3)}, \\
    d\ast F_2^{(3)} &= -H^{(5)} \wedge F_1^{(3)},
\end{align*}
$$

(47)

where $F_1^{(3)} = dA_1^{(2)}$, $F_2^{(3)} = dA_2^{(2)}$ are the R-R and NS-NS 3-forms respectively. There is a subtlety with the 5-form normalization; it enters the energy-momentum tensor with a $1/\sqrt{2}$ factor, leading to $1/2$ of the contribution of the usual convention of the tensor. This implies that the condition of stability (41) is modified by a factor 2, leading to

$$
|\alpha| \leq |\alpha_{\max}| = 1.
$$

(48)

Thus the bound is also satisfied by type IIB supergravity.

Another example is the AdS$_3 \times S^3$ vacuum solution of six-dimensional supergravities supported by a self-dual 3-form. In maximum supergravity, there can be such Chern-Simons terms in the form of $A^{(2)} \wedge F_1^{(2)} \wedge F_2^{(2)}$ and/or analogous transgression terms. In less than maximum supergravities, they become $\frac{1}{2} A^{(2)} \wedge F_1^{(2)} \wedge F_2^{(2)}$. At the level of equations of motion for the 2-forms, the coupling constant for Chern-Simons and/or transgression terms can be viewed as $\alpha = 1$ in both cases. Since the stability bound for the AdS$_3 \times S^3$ supported by the self-dual 3-form is given by (41), it follows that the bound is also satisfied by six-dimensional supergravities.

The stability condition for the electric AdS$_2 \times S^3$ was studied in [11]. If one would naively apply the condition (47) taking into account that the Chern-Simons term involves the same $U(1)$ vector field, one would obtain the bound $\alpha_{\max} = 1/(6\sqrt{3})$, which is half of the supergravity value. However, it turns out that in this special case, as explained in appendix A, the linear perturbation of the the Maxwell field cannot be decoupled from all the linear perturbation modes of the metric. The mass formula is thus modified by the inclusion of certain relevant graviton modes. The consequence is that the bound is also satisfied by five-dimensional supergravity [11].

5 Conclusions

An important issue in the AdS/CFT correspondence is the stability of the AdS$\times$Sphere vacuum solutions in supergravities. These solutions are expected to be stable by the argument of supersymmetry. However the abundant Chern-Simons and/or transgression terms in supergravities could in principle produce a source of instability, even though they play

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no role in the construction of these solutions. In this paper we relax the couplings of the Chern-Simons/transgression terms to be arbitrary constants and show that there indeed are upper bounds for these couplings beyond which instability occurs.

We show that in general the tachyon modes arise from the linear perturbation of the form fields and decouple from the gravitational modes. However, in some special cases tachyon modes can also involve the gravitational modes and the analysis can be more involved. The conclusion is that the couplings of Chern-Simons/transgression terms in supergravities examined all satisfy their stability bounds. It would have saturated the bounds had the momenta in the internal direction be continuous. The result clearly indicates the special feature of supergravities and suggests that the AdS/CFT correspondence may only be valid within a sound theory such as supergravities.

Our focus of analysis has been the non-dilatonic $p$-branes whose decoupling limits give rise to AdS×Sphere backgrounds. It is of interest to investigate the analogous stability condition for dilatonic $p$-branes whose decoupling limit give rise to a product of a domain wall spacetime and a sphere, which may shed light on the domain wall/QFT correspondence.

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**A Instability analysis for $D=11$ supergravity**

The Lagrangian for the bosonic sector of eleven-dimensional supergravity is given by

$$\mathcal{L} = R \ast 1 - \frac{1}{2} \star F_{(4)} \wedge F_{(4)} + \frac{1}{6} A_{(3)} \wedge F_{(4)} \wedge F_{(4)},$$  \hspace{1cm}(49)

where $F_{(4)} = dA_{(3)}$. We shall consider a more general Lagrangian by adding a cosmological constant $\Lambda$ and replacing the Chern-Simons coupling $\frac{1}{6} \star$ by an arbitrary constant $\alpha$. The resulting equations of motion are modified, given by

$$R_{IJ} - \frac{2}{9} \Lambda g_{IJ} = \frac{1}{12} F_{IJ}^2 - \frac{1}{144} F_{IJ}^2 g_{IJ},$$

$$\partial_I (\sqrt{-g} F^{I J_1 J_2 J_3}) = \sqrt{-g} \nabla_I F^{I J_1 J_2 J_3} = \frac{3 \alpha}{(4!)^2} \epsilon^{J_1 J_2 J_3 K_1 \ldots K_8} F_{K_1 \ldots K_4} F_{K_5 \ldots K_8}.$$  \hspace{1cm}(50)

where $\epsilon^{0,1,\ldots,10} = -1$. The system admits AdS$_4 \times S^7$ and AdS$_7 \times S^4$ vacuum solutions. We shall analyze the stability conditions for both vacua.
A.1 AdS$_4 \times S^7$

The AdS$_4 \times S^7$ solution is given by

\[ ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \frac{R_4^2}{r^2} ds_4^2 + R_7^2 d\Omega_7^2, \quad F_{(4)} = E R_4^4 \epsilon_{(4)}, \]

\[ E = \sqrt{\frac{6}{R_4^2} + \frac{12}{R_7^2}}, \quad \Lambda = -\frac{9}{2R_4^2} + \frac{18}{R_7^2}, \]

(51)

where $\epsilon_{(4)}$ is the volume form for the $ds_4^2$. The metrics $ds_4^2$ and $d\Omega_7^2$ describe the unit AdS$_4$ and $S^7$ respectively. In other words, the curvatures of the background are given by

\[ \bar{R}_{\mu\nu\rho\sigma} = -\frac{1}{R_4^2} (g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}), \quad \bar{R}_{ijkl} = \frac{1}{R_7^2} (g_{ik} g_{jl} - g_{jk} g_{il}), \]

\[ \bar{R}_{\mu\nu} = -\frac{3}{R_4^2} g_{\mu\nu}, \quad \bar{R}_{ij} = \frac{6}{R_7^2} g_{ij}, \quad \bar{R} = -\frac{12}{R_4^2} + \frac{42}{R_7^2}. \]

(52)

It is clear that we have split the whole spacetime index $I, J = 0, 1, \ldots, 10$ into $\mu, \nu, \ldots = 0, 1, 2, 3$ to label the indices in AdS$_4$ and $i, j, \ldots = 4, 5, \ldots, 10$ to label the indices in the $S^7$ directions. The fluctuations of the metric and the 4-form are denoted by

\[ g_{IJ} = \bar{g}_{IJ} + h_{IJ}, \quad F_{IJKL} = \bar{F}_{IJKL} + f_{IJKL}. \]

(53)

We find

\[ R_{IJ} = \bar{R}_{IJ} + \frac{1}{2} \left( \bar{\nabla}_K \bar{\nabla}_J h^K_I + \bar{\nabla}_K \bar{\nabla}_I h^K_J - \bar{\nabla}_K \bar{\nabla}_J h^K_I - \bar{\nabla}_I \bar{\nabla}_J h^K_K \right), \]

\[ \delta(\sqrt{-g}) = \frac{1}{2} \sqrt{-g} h^I_I, \]

(54)

Our purpose is to discuss the instability due to the Chern-Simons $F F A$ term, hence we first consider the linearized equation of motion for the gauge field. It will become apparent presently that it is advantageous to adopt the traceless gauge

\[ h \equiv h^I_I = 0. \]

(55)

It follows that $\delta(\sqrt{-g}) = 0$. The variation of the 4-form field strength around the background is given by

\[ \delta F_{IJKL} = f_{IJKL} - \bar{F}_{MJKL} h^{M I} - \bar{F}^{I}_{MJ} h^{MJ} - \bar{F}^{I}_{MK} h^{MK} - \bar{F}^{IJK} M h^{ML}. \]

(56)

Specifically, we have

\[ \delta F^{\mu\nu\rho\sigma} = f^{\mu\nu\rho\sigma} - E \bar{\epsilon}^{\mu\nu\rho\sigma} h_{\lambda}^\lambda, \quad \delta F^{\mu\nu\rho i} = f^{\rho abc i} - E \bar{\epsilon}^{\mu\nu\rho\sigma} h_{\sigma}^i, \quad \delta F^{I\!J \!\!\!\, I\!J} = f^{I\!J \!\!\!\, I\!J}. \]

(57)

The linearized equations for the gauge field perturbations are

\[ \partial_{\sigma} \left( \sqrt{-\bar{g}} (f^{\sigma\mu\nu\rho} - E \bar{\epsilon}^{\sigma\mu\nu\rho} h_{\lambda}^\lambda) \right) + \partial_{i} \left( \sqrt{-\bar{g}} (f^{i\mu\nu\rho} - E \bar{\epsilon}^{\sigma\mu\nu\rho} h_{\sigma}^i) \right) = 0, \]
\[ \partial_\mu \left( \sqrt{-g} (f^{\rho\mu\nu} - E \tilde{\varepsilon}^{\rho\mu\nu\sigma} h_\sigma) \right) + \partial_\nu \left( \sqrt{-g} f^{\nu\mu} \right) = 0, \]
\[ \partial_\mu \left( \sqrt{-g} f^{\nu\muij} \right) + \partial_k \left( \sqrt{-g} f^{k\muij} \right) = 0, \]
\[ \partial_\mu \left( \sqrt{-g} f^{\nu\mu} \right) + \partial_i \left( \sqrt{-g} f^{ijk} \right) = -\frac{\alpha}{4} E \frac{R^4}{r^4} \epsilon^{ijkl} f_{ijkl}, \] (58)

Note that the parameter \( \alpha \) appears only in the last equation in (58), which can also be expressed as
\[ \nabla^\mu f_{\mu ij} + \nabla^l f_{lij} + \frac{\alpha}{4} E \tilde{\varepsilon}^{ijkl} f_{ijkl} = 0. \] (59)

Acting \( \tilde{\varepsilon}_{m_1 m_2 m_3}^{\mu
\nu m} \) on (59), we find
\[ 0 = \frac{1}{3!} \tilde{\varepsilon}_{m_1 m_2 m_3}^{\mu
\nu m} \nabla_m (\nabla^\mu f_{\mu ij} + \nabla^l f_{lij} + \frac{\alpha}{4} E \tilde{\varepsilon}^{ijkl} f_{ijkl}) \]
\[ = \frac{1}{3!} \tilde{\varepsilon}_{m_1 m_2 m_3}^{\mu
\nu m} \left[ \nabla^\mu \nabla_m f_{\mu ij} + \nabla^l \nabla_m f_{lij} + R^l_{m} n f_{mnj} + R^l_{m} n f_{nij} \right. \]
\[ + \left. R^l_{m} n f_{lmj} + R^l_{m} n f_{lij} + \frac{\alpha}{4} E \nabla_m (\tilde{\varepsilon}^{ijkl} f_{ijkl}) \right] \]
\[ = \left( \nabla^\mu f_{\mu ij} + \nabla^l f_{lij} + \frac{\alpha}{4} E \nabla_m (\tilde{\varepsilon}^{ijkl} f_{ijkl}) \right) \]
\[ = (\nabla^\mu f_{\mu ij} + \nabla^l f_{lij} + \frac{\alpha}{4} E \nabla_m (\tilde{\varepsilon}^{ijkl} f_{ijkl}) \right) \]
\[ = (\Box + \Delta_7) f_{m_1 m_2 m_3} + \alpha E \tilde{\varepsilon}_{m_1 m_2 m_3}^{\mu
\nu m} \nabla_m f_{\mu ij}, \] (60)

where we have defined
\[ f_{m_1 m_2 m_3} = \frac{1}{3!} \tilde{\varepsilon}_{m_1 m_2 m_3}^{\mu
\nu m} f_{\mu ij}. \] (61)

We also have used the Bianchi identity as well as the explicit form of the Laplace operator \( \Delta_7 = -(d_7 d^7 + d^7 d_7) \) acting on a tensor:
\[ \nabla_{[l} f_{ijk]} = \partial_{[l} f_{ijk]} + \nabla^m f_{mijk} - \tilde{\varepsilon}^{mijk} f_{mnk} + 2 \tilde{\varepsilon}^{mijk} f_{mkn} \]
\[ + 2 \tilde{\varepsilon}^{mijk} f_{mkn} = \nabla^m f_{mijk} - \frac{12}{R_7} f_{ijk}. \] (62)

The terms appearing in linear perturbation of the energy-momentum tensor are
\[ \delta F^2_{\mu\nu} = \frac{1}{2} \tilde{\varepsilon}_{\mu\nu\rho\sigma} f_{\rho_1 \rho_2 \rho_3 \rho_4} + 6 E^2 (\tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\rho_1 \rho_2 \rho_3 \rho_4} + \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\rho_1 \rho_2 \rho_3 \rho_4} + \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\rho_1 \rho_2 \rho_3 \rho_4}), \]
\[ \delta F^2_{\mu i} = E \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\nu \rho \sigma} = 0, \]
\[ \delta F^2_{ij} = E \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\nu \rho \sigma} + E^2 (\tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\nu \rho \sigma} + \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\nu \rho \sigma} + \tilde{\varepsilon}^{\mu\nu\rho\sigma} f_{\nu \rho \sigma}). \] (63)

Note that only the perturbations \( f_{\rho_1 \rho_2 \rho_3 \rho_4} \) and \( f_{\nu \rho \sigma} \) contribute to the tensor and the parameter \( \alpha \) does not appear in the Einstein equations. Combining with (58), we conclude that
$f_{ijk}$ are decoupled from other fluctuations in the linear order, with only the $f_{ijk}$ affected by the parameter $\alpha$. Therefore, the mass of the possible tachyon modes is determined by

$$M^2 - k^2 \pm 6\alpha Ek = 0 \quad (64)$$

It is clear that there are tachyon modes for $0 < k < |3\alpha E|$. The most negative tachyon mode happens at $k = |3\alpha E|$, where $M^2 = -(3\alpha E)^2$. The Breitenlohner-Freedman bound of AdS$_{d+1}$ is

$$m_{BF}^2 = -\frac{d^2}{4R_d^2}. \quad (65)$$

To avoid the physical instability, we need $(3\alpha E)^2 = 9\alpha^2 \left( \frac{6}{R_4^2} + \frac{12}{R_7^2} \right) \leq \frac{9}{4R_4^2}$. For eleven-dimensional supergravity we have $\alpha = \frac{1}{6}$, and hence the bound would be saturated if the internal momentum $k$ had been continuous. However, to saturate the bound requires that $k = |3\alpha E|$ be a possible mode in the spectrum of $f_{ijk}$ on the $S^7$. The spectrum of $f_{ijk}$ on $S^7$ is given by $k = \pm (l + 6)/R_7$, where $l = 0, 1, 2, \ldots$. On the other hand, given $\alpha = \frac{1}{6}$ and $\Lambda = 0$, we find that $3\alpha E = 3/R_7$. Therefore, although the Breitenlohner-Freedman bound is saturated by the naive minimum of the mass formula (64) for eleven-dimensional supergravity, it could not be saturated by any modes of $f_{ijk}$ on $S^7$. The lowest mode for eleven-dimensional supergravity corresponds to $M = 0$ which occurs when $k = \pm 6/R_7$. In fact, to make sure the lowest mode from the spectrum of $f_{ijk}$ on the $S^7$ satisfying the Breitenlohner-Freedman bound, we only need $\alpha \leq 5/24$.

It is worth emphasizing that although the traceless gauge is normally a “wasteful” choice, it serves our purpose in that the potential tachyon modes due to the Chern-Simons term decouple manifestly. The disadvantage is that the discussion for the linearized Einstein equations become more complicated, for which it is convenient to take the De Donder gauge. Since the graviton modes play no role in our conclusion of the stability, we shall not present them here.

A.2 AdS$_7 \times S^4$

The AdS$_7 \times S^4$ solution supported by the magnetic $F_{(4)}$ is given by

$$ds^2 = \bar{g}_{IJ} dx^I dx^J = R_7^2 ds_7^2 + R_4^2 d\Omega_4^2, \quad \bar{F} = B R_4^4 \Omega_{(4)},$$

$$B = \sqrt{\frac{6}{R_4^2} + \frac{12}{R_7^2}}, \quad \Lambda = \frac{9}{2R_4^2} - \frac{18}{R_7^2}. \quad (66)$$

Correspondingly we have

$$\bar{R}_{\mu\nu\rho\sigma} = -\frac{1}{R_7^2} (\bar{g}_{\mu\rho} \bar{g}_{\nu\sigma} - \bar{g}_{\nu\rho} \bar{g}_{\mu\sigma}), \quad \bar{R}_{ijkl} = \frac{1}{R_4^2} (\bar{g}_{ik} \bar{g}_{jl} - \bar{g}_{ij} \bar{g}_{kl}).$$
\[ \bar{R}_{\mu\nu} = -\frac{6}{R_7^2} g_{\mu\nu}, \quad \bar{R}_{ij} = \frac{3}{R_7^2} g_{ij}, \quad \bar{R} = \frac{42}{R_7^2} - \frac{12}{R_4^2}, \]  

(67)

where we have split the whole spacetime index \( I, J = 0, \ldots, 10 \) into \( \mu, \nu, \cdots = 0, 1, \ldots, 6 \) to label the indices in AdS\(_7\) and \( i, j, \cdots = 7, \ldots, 10 \) to label the indices in the S\(_4\) directions. The fluctuations of the metric and the 4-form are again given by (53). Under the traceless gauge we have \( \delta(\sqrt{-g}) = 0 \). The variation involving the 4-form is given by

\[
\begin{align*}
\delta F^{IJ} & = f^{IJ}, \\
\delta F^{ijkl} & = f^{ijkl} - B \bar{\varepsilon}^{ij} h^I, \\
\delta F^{ijkl}_m h^m & = f^{ijkl} - B \bar{\varepsilon}^{ij} h^I, \\
\delta F_{\mu
u} & = 0, \\
\delta F_{\mu
u} & = B \bar{\varepsilon}^{i} f_{\mu
u}, \\
\delta F_{\mu
u} & = B \bar{\varepsilon}^{ijkl} f_{ijkl} - 6B^2 (\bar{g}_{ij} h^k - h_{ij}), \\
\delta (\frac{1}{4!} F^2 g_{IJ}) & = \frac{2B}{4!} \bar{g}_{IJ} \bar{\varepsilon}^{ijkl} f_{ijkl} - B^2 (\bar{g}_{IJ} h^i - h_{IJ}).
\end{align*}
\]

(68)

The linearized equations for the 4-form perturbation are

\[
\begin{align*}
\partial_\lambda \left( \sqrt{-g} f^{\lambda\mu\rho} \right) + \partial_\mu \left( \sqrt{-g} f^{i\mu\rho} \right) & = \frac{\alpha}{4} B \sqrt{g} \epsilon^{\lambda\mu\rho\lambda_2 \lambda_3 \lambda_4} f_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}, \\
\partial_\rho \left( \sqrt{-g} f^{i\mu\rho} \right) + \partial_i \left( \sqrt{-g} f^{ij\mu} \right) & = 0, \\
\partial_\nu \left( \sqrt{-g} f^{ij\mu} \right) + \partial_k \left( \sqrt{-g} f^{ij\nu} + B \bar{\varepsilon}^{ijkl} h^I \right) & = 0, \\
\partial_\mu \left( \sqrt{-g} (f^{ijkl} - B \bar{\varepsilon}^{ij} h^I) \right) + \partial_i \left( \sqrt{-g} (f^{ij\mu} - B \bar{\varepsilon}^{ij} h^I) \right) & = 0.
\end{align*}
\]

(69)

The first equation in (69) is the only one that depends on the parameter \( \alpha \); it can be expressed as

\[
\nabla^i f_{i\mu\rho} + \nabla^\sigma f_{\sigma\mu\rho} - \frac{\alpha}{4} B \bar{\varepsilon}_{-\mu\rho}^{\sigma_1 \sigma_2 \sigma_3} f^{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = 0.
\]

(70)

Acting \( \bar{\varepsilon}_{\sigma_1 \sigma_2 \sigma_3}^{\sigma_4 \mu \rho} \nabla_d \) on (70), we find

\[
(\Box + \Delta_7) f_{\sigma_1 \sigma_2 \sigma_3} - \alpha B \bar{\varepsilon}_{\sigma_1 \sigma_2 \sigma_3}^{\sigma_4 \mu \rho} \nabla_\sigma f_{\mu \rho} = 0,
\]

(71)

where we have defined

\[
f^{\sigma_1 \sigma_2 \sigma_3} = \frac{1}{4!} \bar{\varepsilon}_{\sigma_1 \sigma_2 \sigma_3}^{\sigma_4 \mu \rho} f_{\sigma_1 \sigma_2 \sigma_3 \sigma_4}
\]

(72)

and we have also used the Bianchi identity as well as the explicit form of the Laplace operator \( \Delta_7 = -(d_7 d_7^I + d_7^I d_7) \) acting on a tensor

\[
\Delta_7 f_{\mu \rho} = \nabla^\sigma \nabla_\sigma f_{\mu \rho} + \frac{12}{R_7^2} f_{\mu \rho}.
\]

(73)

Analog with the case for electric flux, only the perturbation \( f_{i1i2i3i4} \) and \( f_{ijkl} \) contribute to the energy-momentum tensor and the parameter \( \alpha \) does not appear in the Einstein
equations. Therefore, $f_{\mu \nu \rho}$ are decoupled with other fluctuations in linear order. The mass of the these modes is decided by

$$M^2 - k^2 \pm 6\alpha B M = 0 \Rightarrow M^2 = \left( \sqrt{k^2 + (3\alpha B)^2} \pm 3\alpha B \right)^2.$$  \hspace{1cm} (74)

There is no tachyon instability.

An important lesson we learn from the above linear analysis is that the potential tachyon modes $f_{ijk}$ due to the Chern-Simons term decouple at the linear order from the rest of the perturbations. This is obvious for the generic $(n, p, q)$-system discussed in sections 2 and 3, where only $H_{(n)}$ is non-vanishing and $F_{(p)}$ and $G_{(q)}$ differ from $H_{(n)}$. In this example, however, the $(n, p, q)$-forms are all the same as one, namely the $F_{(4)}$. The background $\bar{F}_{(4)}$ is non-vanishing and the 4-form perturbation does couple with the gravitational perturbation. Nevertheless, as can be seen from (58) and (69), for an $n$-form, its equation of motion has $(n - 1)$ free indices, but only the modes involving at least $(n - 2)$ free indices in the parallel directions of the background flux couple with the graviton modes. These modes are independent of the parameter $\alpha$ for $n > 2$. Thus the potential tachyon modes decouple from the rest modes and satisfy a simple equation (32) for $n > 2$. It is clear that the above argument breaks down for $n = 2$ and the tachyon modes can no longer decouple from some gravitational modes. This situation happens in the Einstein-Maxwell Chern-Simons theory in five dimensions. The equations of motion for the tachyon modes are more complicated and the result was announced in [11]. It turns out that there is again an upper bound of the Chern-Simons coupling for stability and the naive minimum of the mass formula in five-dimensional supergravity would have saturated the bound, if the momentum in the internal direction had been continuous. However, to saturate the bound requires that $k = \left| 1/r_3 \right|$ while the corresponding spectrum on the $S^3$ is given by $k = \left| (l + 2)/r_3 \right|$ with $l = 0, 1, 2, \ldots$. Therefore, the bound could not be saturated by any modes on the sphere and the real lowest mode corresponds to $M = 0$. In fact, it is a common feature in all the supergravities we have examined.

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