Re-study on the contribution of scalar potential and spectra of $c\bar{c}$, $b\bar{b}$ and $b\bar{c}(bc)$ families

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We indicated in our previous work that for QED the role of the scalar potential which appears at the loop level is much smaller than that of the vector potential and in fact negligible. But the situation is different for QCD, one reason is that the loop effects are more significant because $\alpha_s$ is much larger than $\alpha$, and secondly the non-perturbative QCD effects may induce a sizable scalar potential. In this work, we phenomenologically study the contribution of the scalar potential to the spectra of charmonia, bottomonia and $b\bar{c}(bc)$ family. Taking into account both vector and scalar potentials, by fitting the well measured charmonia and bottomonia spectra, we re-fix the relevant parameters and test them by calculating other states of not only the charmonia, bottomonia, but also further the $b\bar{c}$ family. We also consider the Lamb shift of the spectra.

I. INTRODUCTION

The potential model has been proposed to evaluate the spectra of the quantum systems composing of heavy flavors, such as charmonia and bottomonia for many years [1]. The subject on heavy quarkonia was thoroughly discussed in an enlightening paper [2]. Even though one calculates the binding energy in terms of the non-relativistic Schrödinger equation, it is indeed a reasonable framework for such heavy resonant states. The main physics, no matter it is induced by the standard model (SM) or new physics beyond the SM, is included in the potential. In phenomenology, the potential contains two pieces, the Coulomb potential which is induced by the gluon exchange and the confinement which may come from the non-perturbative QCD effects or other sources. It was suggested that due to the symmetry consideration [3], for the QED case, if the Dirac equation has a higher symmetry degree than the corresponding Schrödinger equation, the Coulomb potential should have both scalar and vector parts with equal fractions. However, for such a combination the hydrogen potential would not possess the coupling of orbital angular momentum and spin ($L \cdot S$) and it definitely contradicts to the reality.

In our previous work [4], we analyzed the source of the scalar and vector potentials for the QED, namely the vector potential is induced by one-photon exchange for the vector-like gauge theory QED, while the scalar part must be coming from the loop effects. Therefore, for the QED case, the fraction of the scalar potential is very suppressed because the coupling $\alpha \sim 1/137$ is small and does not have substantial contribution to the spectrum of hydrogen. This conclusion is consistent with the precise measurement on the spectra of hydrogen-like atoms. However, for the QCD case, the situation is very different. First because the strong coupling $\alpha_s$ is much larger than $\alpha$, the loop suppression is not as much as for the QED and sometimes the NLO effect of QCD can even exceed the leading order [5, 6]. Secondly the non-perturbative effects may also change the whole scenario. Thus for QCD, the scalar potential may play an important role and its contribution may especially manifest at values of the spectrum line splitting due to the coupling of orbital angular momentum and spin. That was also suggested by some authors [7].

Because the non-perturbative QCD effects cannot be reliably derived from any underlying theory so far, we prefer to introduce phenomenological parameters to manifest their roles and the parameters are fixed by fitting the well-measured data. Indeed the parameters not only contain the contribution of the non-perturbative effects, but also that from the higher orders of the perturbative effects. In our last work [8], we introduced two more parameters to account for the scalar piece in the potential and then re-fitted the spectra of charmonia. The fitting is indeed improved comparing with that including only the vector piece. Thus for further investigating the involvement of scalar potential, we use the same strategy adopted in [8] to evaluate the spectra of the bottomonia ($bb$) and then the various resonant states of the $b\bar{c}(bc)$ system.

Namely, we write $U(r) = V(r) + S(r)$ and the vector potential is $V(r) = -c \chi_F \alpha_s/r + d\kappa^2 r$, while the scalar one is $S(r) = -(a - c)\chi_F \alpha_s/r + (b - d)\kappa^2 r$ with $a, b, c, d$ are four parameters to be determined. The induced terms, such as $L \cdot S$ coupling, $S_1 \cdot S_2$ coupling and the spin-independent corrections etc., are given in the very enlightening work by Lucha et al. [9-11]. Substituting all the expressions into the Schrödinger equation, we may obtain the spectra of heavy quark-antiquark systems.

Besides, the QED theory predicts the Lamb shift which is due to the vacuum effects. In QM, it only shifts the $S$-wave spectra because in the non-relativistic limit, it is proportional to $\delta(r)$, but by the quantum field theory, the $L \neq 0$ states are also affected. In other words, by considering the Lamb shift, the positions of the spectra would deviate from that obtained without the Lamb shift. For the hadron case, the governing theory is QCD which also induces the Lamb shift [12], and in this work, we include its contribution. It is noticed that, as the phenomenological parameters which are determined by fitting data are introduced, all higher order effects should also be automatically involved, so it seems that there is no need to consider the Lamb shifts which are induced by the loop effects. In fact, it is. However, as calculating the form factors of the hadronic transition matrix elements or obtaining...
the parton distribution functions, we always wish to squeeze the uncontrollable parts which are not calculable, such as the non-perturbative contributions, as small as possible. Similarly, here we include the NLO or even NNLO corrections i.e. the Lamb shifts and re-fit the parameters which are indeed not derivable.

Considering the Lamb shift, there is a byproduct, which may be very enlightening for understanding the theory. Obviously, only the products $\alpha_a$ and $\delta_s$ in the potential matter, but not $c$, $d$, $\alpha_a$, and $\kappa$ separately. However, for explicitly showing the roles of the scalar and vector pieces, we adopt a special strategy.

When the Lamb shift is taken into account, the situation may change slightly. In the expression of the Lamb shift, there is an ultraviolet divergent term which includes the renormalization scale $\mu$. Meanwhile the running coupling $\alpha_s$ also depends on the scale. The authors of [13] suggest an effective method to deal with the divergence and meanwhile fix the value of $\alpha_s$ (see the text, where we introduce the method in some details for the readers’ convenience). It is noted that $\mu$ is a complicated function of $\alpha_s$, quark mass $(m_b$ or $m_c)$, and the principal quantum number $n$. Taking a special way to determine $\mu$ which is corresponding to adopting a special renormalization scheme, and considering the dependence of $\alpha_s$ on $\mu$, one can eventually find the value of $\alpha_s$ for a certain flavor ($b$ or $c$) and a given principal quantum number. For example, for $\Upsilon (1S)$, we have $\alpha_s = 0.218$ for $m_b = 4.8$ GeV and the scale-parameter $\Lambda = 0.2$ GeV. Amazingly, this value is quite close to that adopted in literature by fitting the $\Upsilon$ spectra.

By contrast, the confinement term $\kappa r$ is fully coming from the non-perturbative QCD effects and cannot be theoretically derived so far. Thus we adopt the value given in [13].

With all the inputs, we calculate the spectra of the $c\bar{c}$, $b\bar{b}$ and $bc$($b\bar{c}$) systems.

This paper is organized as follows. In Sec. II and III we introduce the generalized Breit-Fermi Hamiltonian and the Schrödinger equation for the $bc$ bound states: $T(1S), \chi_{b0}(1P), \chi_{b1}(1P), \chi_{b2}(1P)$ and $T(2S)$. Then we numerically solve the eigen-equations for these bound states and fix the parameters as we did for dealing with the charmonia family in our previous work. In Sec. IV the Lamb shift is taken into account and another set of the parameters is given to improve our predictions. In Sec. V we give the spectra of the $bc$($b\bar{c}$) mesons. The last section is devoted to our conclusion and discussion.

II. THE GENERALIZED BREIT-FERMI Hamiltonian and SCHÖRDINGER EQUATION

The generalized Breit-Fermi Hamiltonian was given in Refs. [9–11] as

\[ H = H_0 + H_1 + \ldots, \]  
\[ H_0 = \frac{p^2}{m} + 2m + S(r) + V(r), \]  
\[ H_1 = H_{sd} + H_{si}, \]  
\[ H_{sd} = H_{ls} + H_{ss} + H_t \]

\[ = \frac{1}{2m^2r} (3V'' - S') \mathbf{L} \cdot (S_1 + S_2) + \frac{2}{3m^2} S_1 \cdot S_2 \nabla^2 V(r) \]

\[ + \frac{1}{12m^2} \left( \frac{1}{r} V' - V'' \right) S_{12}, \]  
\[ H_{si} = -\frac{p^4}{4m^3} + \frac{1}{4m^2} \left\{ \frac{2}{r} V'(r) \cdot \mathbf{L}^2 + [p^2, V - rV'] \right\} \]

\[ + 2 V - rV')p^2 + \frac{1}{2} \left[ \frac{8}{r} V'(r) + V'' - rV''' \right] \} \]  

where, $V$ and $S$ stand as the vector and scalar potentials and $H_{si}$ and $H_{sd}$ represent the spin-independent and spin-dependent pieces respectively. For the linear confinement piece we adopt the Cornell potential [16]. Thus the total potential at the lowest order reads

\[ U(r) = V(r) + S(r) = -a C_F \frac{\alpha_s}{r} + b \kappa^2 r \]  

where,

\[ \begin{align*}
V(r) &= -c C_F \alpha_s/r + d \kappa^2 r \\
S(r) &= -(a-c) C_F \alpha_s/r + (b-d) \kappa^2 r
\end{align*} \]

With the hamiltonian (1) and the potential (2), one can solve the Schrödinger equation

\[ H \Psi(r) = (H_0 + H_1) \Psi(r) = (E + 2m) \Psi(r). \]  

We can transform the radial wave function into $R(x)$ with a dimensionless variable: $x = \kappa r$, then the reduced radial equation is written as\(^1\)

\[ \frac{d^2}{dx^2} u(x) = A(x) u(x), \]

where,

\[ A(x) = -\tilde{m} \left( \tilde{E} - \tilde{U}(x) - \tilde{H}_1 \right) + \frac{1(1+1)}{x^2} \]

\[ -\frac{1}{4} \left( \tilde{E} - \tilde{U}(x) \right)^2 \]  

with

\[ \begin{align*}
\tilde{m} &= m/\kappa, \quad \tilde{E} = E/\kappa, \\
\tilde{H}_1 &= H_1/\kappa, \quad \tilde{U}(x) = U(x)/\kappa.
\end{align*} \]

In the simplified potential form [3], the approximation

\[ p^4 \sim \left[ m \left( E - U(r) \right) \right]^2 \]

is used. For the legitimacy of applying this approximation in the calculation and the error degree brought up in the numerical values are briefly discussed in the appendix.

\[^1\] The standard form of the radial equation can be easily found in Ref. [17], and the method to make it dimensionless is borrowed from Ref. [18].
III. THE ENERGY GAP FUNCTION FOR THE \( b \bar{b} \) BOTTOMONIA AND THE NUMERICAL RESULTS WITHOUT TAKING INTO ACCOUNT THE LAMB SHIFT

The radial equation (4) can be solved in terms of the method so-called “The iterative numerical process” which is introduced in literatures [17, 18]. We have improved this method, and then fix the parameters \( a, b, c, d \) by fitting the well measured spectra of bottomonia. In our previous work [8], we explain the reason for the choice of the input for charmonia. However, for bottomonia, the situation is slightly different and the masses of \( \Upsilon(1S) \), \( \chi_{b0}(1P) \), \( \chi_b(1P) \), \( \Upsilon(2S) \) are chosen for the obtaining the values of \( a, b, c \) and \( d \). Similar to the procedure we took in our previous work [8], instead of directly fitting the masses, we construct a series of relations which should be fitted:

\[
\begin{align*}
    m_{\Upsilon(2S)} - m_{\Upsilon(1S)} &= E_{[2^3S_1]} - E_{[1^3S_1]}; \\
    m_{\Upsilon(2S)} - m_{\chi_{b0}(1P)} &= E_{[2^3S_1]} - E_{[1^3P_0]}; \\
    m_{\Upsilon(2S)} - m_{\chi_b(1P)} &= E_{[2^3S_1]} - E_{[1^3P_1]}; \\
    m_{\Upsilon(2S)} - m_{\chi_b(2P)} &= E_{[2^3S_1]} - E_{[1^3P_2]},
\end{align*}
\]

where, \( E_{[n_s^{2s+1}l_j]} \) represents the eigen-values of the radial equations (4) with various quantum numbers \( n_s, j, l, \) and \( s \) for the bottomonia. Because the parameters \( a, b, c \) and \( d \) are involved in the potential (4), \( E_{[n_s^{2s+1}l_j]} \) must be functions of these parameters. \( m_{[\text{meson}]} \) are the masses of the individual states which are shown in the following Tab. II.

Sequentially, the parameters \( a, b, c \) and \( d \) are obtained by solving Eqs. (5). By employing the Newton’s iterative method (The details about the numerical method can be found in Ref. [20]), we have achieved:

\[ a = 1.2165, \quad b = 1.2988, \quad c = 0.8686, \quad d = 0.5886 \]

Here we set \( \alpha_s = 0.284 \) and \( \kappa = 0.42 \) GeV which seem somehow different from the values given in literatures [13, 21, 22]. But as noticed, the deviation may be included in the phenomenological parameters \( a, b, c \) and \( d \). The choice of \( \alpha_s \) has another reason which is associated with our treatment of the contribution of the Lamb shift (see next section), anyhow this value is not very far apart from that given in literature.

Given \( a, b, c \) and \( d \) in [6], the masses of the bottomonia states are determined as:

\[ M(n_s^{2s+1}l_j) = E_{[n_s^{2s+1}l_j]} + E_0 \]

where, \( E_0 \) is the zero-point energy:

\[ E_0 = m_{[\Upsilon(1S)]} - E_{[1^3S_1]} \]

and the final results are shown in the Tab. II below.

Explicitly, in the process, the masses of the mesons with superscript “fit” are taken as inputs to obtain the parameters and then the masses of other states in the table are predicted.

For readers’ convenience and a clear comparison, we also list the results for the charmonia which were obtained in our previous work [8].

| meson | \( M_{\text{EXP}} \) | Prediction | \( M_{\text{EXP}} \) |
|-------|-----------------|-------------|-----------------|
| \( \eta(1S_0) \) | 9.3620 | 9.4124 | 9.9932 |
| \( \Upsilon(1S_0) \) | 9.4603 | 9.4603 | 10.0233 |
| \( \chi_{b0}(1P_0) \) | 8.9854 | 8.9854 | 10.1611 |
| \( \chi_b(1P_0) \) | 9.8928 | 9.8928 | 10.1719 |
| \( h_b(1P_1) \) | 9.8897 | 9.8897 | 10.1827 |
| \( \chi_b(2P_2) \) | 9.9212 | 9.9212 | 10.3552 |

| meson | \( M_{\text{EXP}} \) | Prediction | \( M_{\text{EXP}} \) |
|-------|-----------------|-------------|-----------------|
| \( \eta(1S_0) \) | 2.9803 | 3.0189 | 3.5562 |
| \( J/\psi(1S_1) \) | 3.0969 | 3.0969 | 3.6370 |
| \( \chi_{b0}(1P_0) \) | 3.4148 | 3.4148 | 3.6861 |
| \( \chi_b(1P_0) \) | 3.5107 | 3.5107 | 4.1164 |
| \( h_b(1P_1) \) | 3.5259 | 3.5100 | |
The authors of Ref.\textsuperscript{12,14} gave the theoretical expressions for the binding energies which involve contributions of the Lamb shift. It is well known that the induced Hamiltonian contributing to the Lamb shift is due to the vacuum fluctuation and can be obtained by calculating the loop diagrams order by order. Thus the Lamb Shift starts at \(O(\alpha_s^3)\)\textsuperscript{13}.\footnote{We indicate that the Lamb shift effects start from \(O(\alpha_s^3)\), and it seems that this allegation conflicts with Eq.\textsuperscript{(11)}. As noted, in the expression Eq.\textsuperscript{(11)} which is given in Ref.\textsuperscript{13}, \(\Delta E_{LM}\) is not the potential derived from the loop, but the expectation value of the potential with the wavefunctions which are solutions of the Schrödinger equation containing only the Coulomb piece. Since such solutions possess an exponential factor \(\exp -\alpha_s \mu r\), the expectation value of any function of \(r\) as \(f(x)\) should be proportional to \(\alpha_s^2\) (\(n \geq 1\)), thus the \(\Delta E_{LM}\) in Eq.(11) start from \(\alpha_s^3\). But indeed the potential pieces corresponding to the Lamb shifts directly come from the loop diagrams which start from \(O(\alpha_s^2)\).} The Lamb shift is:

\[
\Delta E_{LM} = \langle \Psi | V_{\text{Lamb}} | \Psi \rangle,
\]

where \(\Psi\) is the solution of the Schrödinger equation containing only the Coulomb piece, can be written as:

\[
\Delta E_{LM}[n,j,l,s] = m \left[ \Delta E(\alpha_s^3) + \Delta E(\alpha_s^4) + \Delta E(\alpha_s^5) + \ldots \right].
\]

For illustrating the contribution of the Lamb shift to the spectra, let us directly copy Titard’s formulas \textsuperscript{13} below, where we dropped the tree-level terms and the relativistic corrections, and we have:

\[
\Delta E(\alpha_s^3) = -\frac{\alpha_s^3}{8\pi n^2} \left( 2\beta_0 \gamma_E + 4a_1 \right); \quad (11b)
\]

\[
\Delta E(\alpha_s^4) = -\frac{\alpha_s^4}{4\pi^2} \left( a_1 + \frac{\gamma_E \beta_0}{2} \right)^2 + 2 \left( \gamma_E a_1 \beta_0 + a_2 \right) + \left( \frac{\pi^2}{12} + \frac{\gamma_E}{2} \right) \beta_0^2 + b_1 \right) \right], \quad (11c)
\]

where, the \(n\) in Eq.\textsuperscript{(11)} stands for the principal quantum number as \(n = n + 1\), where, \(n\), and \(l\) are defined in Sec. \textsuperscript{III}. All the constants as \(a_1, a_2, b_1, b_2\) (\(i = 1, 2, 3\)) are given in Ref.\textsuperscript{24} (also see \textsuperscript{13,25,28}). Further more, Hoang et al. estimated the contribution of higher orders up to \(O(\alpha_s^5)\) and \(O(\alpha_s^6)\) to the binding energies (See \textsuperscript{12}).

When we calculate the QCD Lamb shift effects for the hadron spectra, the potential not only contains the Coulomb piece, but also the confinement, so that the expression would be more complicated than that shown in Eq.(11b) and (11c). In fact, there cannot be analytical expressions for the expectation values.

The Lamb Shift \(\Delta E_{LM}[n,j,l,s]\) depends on the coupling constant \(\alpha_s\) in Eq.\textsuperscript{(11)} as \textsuperscript{13}:

\[
\alpha_s(\mu^2) = \frac{2}{\beta_0} \ln \frac{\mu/\Lambda}{\beta_0^2} \left[ 1 - \frac{\beta_1}{\beta_0} \ln \frac{\mu^2/\Lambda^2}{\beta_0^2} \right]
\]

\[+ \frac{1}{\beta_0} \ln \frac{\mu^2/\Lambda^2}{\beta_0^2} \left[ \frac{\beta_2^2}{\beta_0^2} \ln \frac{\mu^2/\Lambda^2}{\beta_0^2} \right]
\]

\[- \frac{\beta_2^2}{\beta_0^2} \ln \frac{\mu^2/\Lambda^2}{\beta_0^2} - \frac{\beta_1 \beta_2}{\beta_0^2} \right]. \quad (12)
\]

Using the formulas given above, one can evaluate the Lamb shift of the charmonia states. The choice of the renormalization point \(\mu\) is suggested by Pineda et al., and “a natural value for this parameter” is \textsuperscript{13,24}:

\[
\mu = \frac{2}{na_B} \quad (13a)
\]

where,

\[
a_B = \frac{2}{mC_F \alpha_s} \quad (13b)
\]

\[
\tilde{\alpha}_s(\mu^2) = \alpha_s \left[ \frac{1 + (a_1 + \frac{\gamma_E \beta_0}{2}) \pi}{\gamma_E} \left( a_1 \beta_0 + \frac{\beta_1}{8} \right) + \frac{\pi^2}{12} + \frac{\gamma_E}{2} \right] + \frac{\beta_0^2}{4} + b_1 \right) \quad (13c)
\]

In the expression of the newly derived Hamiltonian there is a term \(\ln 2\alpha_s \mu r/\gamma\) (after a Fourier transformation from the momentum space to the configuration space), which is UV divergent. To deal with the divergence, it is suggested to take an effective method. For smaller range of \(r\) the Coulomb piece \(1/r\) obviously dominates, so that in \(\langle \Psi | H_{\text{Lamb}} | \Psi \rangle\) one can use the wavefunction \(\Psi\) which is the solution of the Schrödinger equation containing only the Coulomb potential, namely we can have an analytical solution for this asymptotic situation. Thus \(\langle \Psi | H_{\text{Lamb}} | \Psi \rangle \approx \ln(\mu/2).\) To make the UV divergence vanish, the suggested renormalization scheme is to set \(\mu = 2/na_B\). Indeed, in \textsuperscript{13}, other three alternative schemes were also suggested, here we just take this one and find the value of \(\alpha_s\) determined with this scheme is closer to that adopted in early literature for calculating the spectra of bottomonia.

The value of the parameter \(\Lambda\) is chosen as 0.2 GeV for bottomonia\textsuperscript{24}, and at this point,

\[
\alpha_s^n = 0.284 \quad (14)
\]

which is the value of \(\alpha_s\) we used in Sec.\textsuperscript{III}. It is noted that \(\alpha_s\) is different for different quantum number \(n\):

\[
\alpha_s^n = 0.24, \quad \alpha_s^n = 0.284, \quad \alpha_s^n = 0.316. \quad (15)
\]

We will use the \(n\)-related \(\alpha_s\) value for evaluating the spectra of the radially excited states of bottomonia.

Simply adding the Lamb shift to the total binding energy is like that we change the zero-point energy for each state. We still select masses of \(\Upsilon(1S), \chi_b(1P), \chi_b(1P), \chi_b(1P), \Upsilon(2S)\) as inputs, and solve the equation \textsuperscript{13} again as we did in last section. But the value of \(\alpha_s\) in \textsuperscript{13} is taken as that given in
Eq. (15) which depends on \( n \). The new solutions of \( a, b, c \) and \( d \) are:

\[
\begin{align*}
& a^{(LM)} = 1.4256, \quad b^{(LM)} = 1.3553, \\
& c^{(LM)} = 0.8077, \quad d^{(LM)} = 0.6849, 
\end{align*}
\]

where the superscript \( LM \) refers to that all the corresponding parameters are obtained as the Lamb shift being taken into account.

With these solutions, our predictions on the whole family spectra of bottomonia are presented in Tab. III.

| meson | \( \Delta E_{LM} \) | \( M \) | \( M' \) | \( M_{Exp} \) |
|-------|------------------|------|------|------------------|
| \( \Upsilon(1^3S_0) \) | -0.1064 | 9.5274 | 9.4210 | 9.3020 |
| \( \Upsilon(1^3S_1)\text{st} \) | -0.1114 | 9.5717 | 9.4603 | 9.6003 |
| \( \chi_{b0}(1^3P_0)\text{fit} \) | -0.0618 | 9.9212 | 9.8594 | 9.8594 |
| \( \chi_{b1}(1^3P_1)\text{fit} \) | -0.0620 | 9.9546 | 9.8928 | 9.8928 |
| \( h_b(1^1P_1) \) | -0.0621 | 9.9507 | 9.8887 |
| \( \chi_{b2}(1^3P_2)\text{fit} \) | -0.0622 | 9.9744 | 9.9122 | 9.9122 |
| \( \eta_b(2^1D_0) \) | -0.0549 | 10.0451 | 9.9902 |
| \( \Upsilon(2^1S_1)\text{st} \) | -0.0561 | 10.0794 | 10.0233 | 10.0233 |
| \( \Upsilon(1^3D_1) \) | -0.0412 | 10.2139 | 10.1727 | 10.1611 |
| \( \Upsilon(1^3D_2) \) | -0.0412 | 10.2272 | 10.1860 |
| \( \Upsilon(1^3D_3) \) | -0.0412 | 10.2399 | 10.1987 |
| \( \Upsilon(3^3S_1) \) | -0.0379 | 10.4380 | 10.4001 | 10.3552 |

**TABLE III: The mass spectra with the Lamb Shift (in GeV), where, the LM stands for the contribution of the Lamb Shift, \( M' \) is the predicted mass when the parameters are set as in Eq. (10) and \( M' \) stands for \( M' = M + \Delta E_{LM} \).**

\( \eta_b(1^3S_0) \) and \( \chi_{b0}(1^3P_0) \) are the states of the bottomonia, we may interpolate those parameters for the \( \Upsilon \) family lies between charmonia and the bottomonia. The difference of the quark masses \( \mu = m_b m_c / (m_b + m_c) \)

V. THE SPECTRA OF THE \( \bar{b}c(\bar{b}c) \) MESONS

In this section, we further study the spectra of the \( \bar{b}c(\bar{b}c) \) mesons. Except the ground state \( \bar{b}c \), the other states of the \( \bar{b}c \) mesons have not been well measured yet [19], we cannot directly fit the parameters from data as what we do for the charmonia and the bottomonia. It is noted that the parameters for charmonia and bottomonia are not drastically apart and since the \( \bar{b}c(\bar{b}c) \) family lies between charmonia and the bottomonia, we may interpolate those parameters for the \( \bar{b}c(\bar{b}c) \) family, namely average the values for charmonia and bottomonia to be that for the \( \bar{b}c(\bar{b}c) \) mesons (See the Tab. IV).

\[
H_{\text{sal}} = H_{1s} + H_{\infty} + H_{1} = \frac{1}{2r} [V'(r) - S'(r)] \left( \frac{L \cdot S_b}{m_b^2} + \frac{L \cdot S_c}{m_c^2} \right) + V'(r) L \cdot S + \frac{2}{3m_b m_c} \nabla^2 V(r) S_b \cdot S_c + \frac{1}{m_b m_c} \left\{ \frac{V'(r)}{r} - V''(r) \right\} \left( \frac{S_b \cdot r}{r^2} \frac{(S_c \cdot r)}{r} - \frac{1}{3} S_b \cdot S_c \right) \]

\[
H_{\text{si}} = - \frac{1}{8} \left\{ \frac{1}{m_b^2} + \frac{1}{m_c^2} - \frac{2}{m_b m_c} \right\} \nabla^2 V(r) + \frac{1}{4m_b m_c} \left\{ \frac{2}{r} V'(r) \cdot L^2 + \frac{p^2}{r} [V(r) - rV'(r)] \right\}
\]

So the Schrödinger equation we need is:

\[
H \Psi(r) = (H_0 + H_1) \Psi(r) = (E + m_b + m_c) \Psi(r) \quad (19f)
\]

where,

\[
\mu = \frac{m_b m_c}{m_b + m_c} \quad (20)
\]
With this equation and the concerned parameters, we can predict the spectra of the members of the whole \(bc(\bar{bc})\) family shown in Tab. V.

| Quantity | our predict\(^a\) | our predict\(^b\) | EFG\([29, 30]\) | KWLC\([31]\) |
|----------|------------------|------------------|-----------------|-----------|
| \(1^1S_1 - 1^3S_0\) | 0.0526 | 0.0448 | 0.0620 | 0.0348 |
| \(2^1S_0 - 1^3S_0\) | 0.5923 | 0.5843 | 0.5650 | 0.5863 |
| \(2^3S_1 - 1^5S_1\) | 0.5723 | 0.5744 | 0.5430 | 0.5795 |
| \(3^3S_1 - 2^5S_1\) | 0.4021 | 0.4067 | 0.3540 | 0.3652 |
| \(1^3P_0 - 1^5P_0\) | 0.3709 | 0.3885 | 0.3670 |
| \(1^3P_1 - 1^5P_1\) | 0.0492 | 0.0392 | 0.0350 |
| \(1^3P_2 - 1^5P_1\) | 0.0495 | 0.0411 |
| \(1^3D_1 - 2^5S_1\) | 0.1341 | 0.1509 | 0.1910 |
| \(1^3D_2 - 2^5S_1\) | 0.0176 | 0.0253 | 0.0050 |
| \(1^3D_3 - 2^5S_1\) | 0.0170 | 0.0261 | 0.0040 |

\(^a\) with the parameter in Tab. IV.

\(^b\) Considering the effect of the Lamb shift and the parameters are taken as \([13]\).

**VI. CONCLUSION AND DISCUSSION**

In this work, we study the role of scalar potential to the spectra of charmonia, bottomonia and the \(bc(\bar{bc})\) family. Our strategy is that the scalar and vector potentials have different fractions which manifest in their coefficients (in the text, they are \(a, b, c\) and \(d\) for the Coulomb and confinement pieces respectively). By fitting some members of charmonia and bottomonia which are more accurately measured, we fix them. Since except \(B_c\), the ground state of the \(bc(\bar{bc})\) family, other states have not been well measured yet, then we interpolate the parameters for charmonia and bottomonia to determine the concerned ones for the \(bc(\bar{bc})\) mesons. With those parameters, we further predict the mass spectra of the rest resonances of charmonia, bottomonia and the whole \(bc(\bar{bc})\) family.

It is shown that unlike the QED case where the fraction of the scalar potential is very small and negligible, for the quarkonia where QCD dominates, the fraction of scalar potential is of the same order of magnitude as the vector potential. This is consistent with the conclusion of Ref. [32] and is not surprising. As we indicated that for the vector-like coupling theories QED and QCD, the scalar potential can only appear at loop level or is induced by non-perturbative effect (QCD only). Thus it should be loop-suppressed. However, for QCD, the coupling is sizable and the higher order contributions and the non-perturbative effects somehow are significant, so that one can expect the fraction of the scalar potential is large.

Moreover, the Lamb shift is induced by the vacuum fluctuation and only appears at loop level, indeed its leading contribution is at \(O(\alpha_s^2)\). Therefore for the QED case, it is hard to observe the Lamb shift (observation of the Lamb shift is a great success for theory and experiment indeed), however, for QCD the effects are not ignorable. It is shown [5, 6] that the NLO QCD effects may exceed the LO contributions at some processes. By taking into account of the Lamb shift, we refit the model parameters and find they are obviously distinct from that without considering the Lamb shift.

The results help us to get better understanding of QCD, higher order effects and especially the non-perturbative effects. Even though it is only half-quantitative, it is an insight to the whole picture.

In this work, we adopt the renormalization scheme as \(\mu = 2/|a_B|\) [13], which determines the effective coupling \(\alpha_s\). It is worth emphasizing that \(\alpha_s\) depends on the principal quantum number \(n\) and this is different from that usually used in literature. But for the ground state of charmonia and bottomonia, the values of \(\alpha_s\) are quite close to that appearing in the literature.

The predictions on the \(bc(\bar{bc})\) family will be tested at LHCb experiments where a great amount of the excite states of \(bc(\bar{bc})\) will be produced. Comparing with the data, we will learn more about the QCD and structures of the “final” meson family.

Actually, in this work, we only consider the Cornell potential which is supported by the area theorem and commonly adopted in phenomenological studies where the spectra and wavefunctions of heavy mesons are involved. Indeed, there are some other proposals. For example, the authors of Ref. [33] use the harmonic oscillator model to deal with the confinement and further consider the effects of open charm loop for higher excited charmonia states which may induce energy shifts and change decay widths. Moreover, some authors introduce a phenomenological form for the spin-spin interaction [34] which may also result in energy level shift. In this work, we just restrict ourselves at the quark level QCD motivated potential whose form is given in Ref. [12, 10], but we may further our studies on the coupled-channel scenario in our coming work.

The contribution of the scalar potential to the hadron spectra was noticed by some authors [35] long time ago, and its importance was confirmed. In our present work, we re-emphasize its role and discuss the origin in comparison with the QED case. In terms of the newly achieved data on charmonia and bottomonia, we analyze the hadron spectra and gain all the concerned parameters. We also investigate the significance of the Lamb shift phenomenologically. Then we go on discussing the spectra of \(bc(\bar{bc})\) family within the same framework, the results will be tested in the future experiments.

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Appendix: Check the legitimacy of the approximation

\[ p^4 \sim \left[ 2\mu \left( E_0 - V(r) \right) \right]^2 \]

We investigate and elucidate the legitimacy of the approximation adopted in the text:

\[ p^4 \sim \left[ 2\mu E_0 - V(r) \right]^2 \]

through a few examples.

In fact, such problems have been thoroughly discussed in literature and even written in textbooks, for example in Ref. [35], and the relativistic corrections to the Cornell potential can be found in Ref. [36]. Here we just re-do the numerical computation to convince our readers and ourselves about the legitimacy of the approximation adopted in the text because it is very important for getting the spectra.

The 0-th order Schrödinger equation is:

\[ \left[ \frac{p^2}{2\mu} + V(r) \right] \Psi = E_0 \Psi \] (21)

With the relativistic correction, it will be:

\[ \left[ \frac{p^2}{2\mu} + V(r) - \frac{p^4}{4m^2} \right] \Psi = E_1 \Psi \] (22)

Note: here we ignore other irrelevant correction terms such as the L-S coupling etc. because we only concern the cause there exists an analytical solution.

We have the exact solution (the eigen-energy and the wave function):

\[ E_0^n = -\frac{a^2}{4m^2} \] (24a)
\[ R_{10}(r) = 2K^2 e^{-Kr} \] (24b)
\[ R_{20}(r) = \left( \frac{1}{2}K \right)^\frac{3}{2} (2 - Kr) e^{-\frac{3}{2}Kr} \] (24c)
\[ R_{30}(r) = \left( \frac{1}{3}K \right)^\frac{5}{2} \left[ 2 - \frac{4}{3}K + \frac{4}{27}K^2 \right] e^{-\frac{5}{2}Kr} \] (24d)

where,

\[ \mu = \frac{m^2}{2m} = \frac{m}{2}; \quad K = \frac{1}{2}ma \]

If \( m = 1.84 \text{ GeV} \), and \( a = 0.5 \) (these numbers are just taken for an illustration, but not for real physics), then:

\[ \begin{cases} E_0^{n=1} = -0.115 \\ E_0^{n=2} = -0.02875 \\ E_0^{n=3} = -0.012778 \end{cases} \] (25)

and in the perturbative method, the relativistic correction is:

\[ \Delta E^n = \int dr r^2 \frac{1}{4m^3} \left[ p^2 R_{n0}(r) \right]^2 \]
\[ \Rightarrow \begin{cases} \Delta E_1^{n=1} = 0.008984; \\ \Delta E_1^{n=2} = 0.001460; \\ \Delta E_1^{n=3} = 0.000558. \end{cases} \] (26)

On the other hand, as we use the approximation: \( p^4 \sim \left[ 2\mu \left( E_0 - V(r) \right) \right]^2 \) in Eq. (22), and then find the numerical solution of Eq. (22):

\[ \begin{cases} E_1^{n=1} = -0.1223; \\ E_1^{n=2} = -0.0300; \\ E_1^{n=3} = -0.0130. \end{cases} \] (27)

If we define: \( \Delta E_1 = E_0^n - E_1^n \), then we have:

\[ \begin{cases} \Delta E_1^{n=1} = 0.007265; \\ \Delta E_1^{n=2} = 0.001249; \\ \Delta E_1^{n=3} = 0.000228. \end{cases} \] (28)

From the Eqs. (26) and (28), we can find that the error is near 0.001 GeV. The relative error is:

\[ x^n = \frac{E_0^n + \Delta E^n - E_1^n}{E_0^n + \Delta E^n + E_1^n} \]
\[ \Rightarrow \begin{cases} x_1^{n=1} = 0.68\%; \\ x_1^{n=2} = 0.35\%; \\ x_1^{n=3} = 1.28\%. \end{cases} \] (29)

With the Coulomb potential the Schrödinger equation possesses an analytical solution, so it is easy to see the error. However, for the Cornell potential, there is no analytical solution available, so that we need to use the numerical solution for our analysis.

The Cornell potential is:

\[ V(r) = -\frac{a}{r} + br \] (30)

where, we set \( a = 0.5, b = 0.2 \) and consider the second radially excited state with \( n = 2 \) for an example. The numerical solution of Eq. (27) is:

\[ E_0^{n=2} = 0.856872 \] (31)

and the numerical solution of Eq. (22) is:

\[ E_1^{n=2} = 0.918242 \] (32)

So we have:

\[ \Delta E_1^{n=2} = E_1^{n=2} - E_0^{n=2} = 0.06137 \] (33)

\[ \text{here, we may use } E_0 \text{ as well, but for a clear comparison we use } E_1 \text{ instead. The error is not great.} \]
In the numerical solution of Eq. (21) we have the wave function $R_{n=2}(r)$, and with the perturbative method, the contribution of the relativistic correction is:

$$
\Delta E^{n=2} = \int \frac{1}{4m^3} \left[ p^2 R_{20}(r) \right]^2 r^2 dr
$$

$$
= 0.0669. \quad (34)
$$

Finally, from Eqs. (33) and (34), we have the relative error:

$$
x^{n=2} = \frac{E^{n=2} + \Delta E^{n=2} - E^{n=2}}{E^{n=2} + \Delta E^{n=2} - E^{n=2}} = 0.30\%. \quad (35)
$$

Even though the relativistic error seems large, in fact the error is only at order of a few of MeV. The QCD Lamb shift generally results in a few of tens of MeV, thus the error brought up by the approximation seems not too serious.

Here one important point should be clarified. Directly calculating Eq. (34), one would have an unrealistically large result. The reason is obvious that unlike the analytical solution near the zero point ($r \to 0$), the behavior of the numerical solution near the zero point is only at order of a few of MeV. The QCD Lamb shift generally results in a few of tens of MeV, thus the error brought up by the approximation seems not too serious.

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