CONSTRUCTION OF SEPARATELY CONTINUOUS FUNCTIONS
WITH GIVEN RESTRICTION

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Abstract. It is solved the problem on constructed of separately continuous
functions on product of two topological spaces with given restriction. In partic-
ular, it is shown that for every topological space \( X \) and first Baire class function
\( g : X \to \mathbb{R} \) there exists separately continuous function \( f : X \times X \to \mathbb{R} \) such
that \( f(x, x) = g(x) \) for every \( x \in X \).

1. Introduction

It ia well-known (see [1]) that the diagonals of separately continuous functions
of two real variables are exactly the first Baire functions. It is shown in [2] that for
every topological space \( X \) with the normal square \( X^2 \) and \( G_\delta \)-diagonal and every
function \( g : X \to \mathbb{R} \) of the first Baire class there exists a separately continuous
function \( f : X \times X \to \mathbb{R} \), for which \( f(x, x) = g(x) \), i.e. every the first Baire class
function on the diagonal can be extended to a separately continuous function on
all product. Analogous question for functions of \( n \) variables was considered in [3].

On other hand, in the investigations of separately continuous functions \( f : X \times Y \to \mathbb{R} \) defined on the product of topological spaces \( X \) and \( Y \) the fol-
lowing two topologies naturally arise (see [4]): the separately continuous topol-
ygy \( \sigma \) (the weakest topology with respect to which all functions \( f \) are continu-
ous) and the cross-topology \( \gamma \) (it consists of all sets \( G \) for which all \( x \)-sections
\( G_x = \{ y \in Y : (x, y) \in G \} \) and \( y \)-sections \( G^y = \{ x \in X : (x, y) \in G \} \) are open in \( Y \)
and \( X \) respectively). Since the diagonal \( \Delta = \{(x, x) : x \in \mathbb{R} \} \) is a closed discrete
set in \((\mathbb{R}^2, \sigma)\) or in \((\mathbb{R}^2, \gamma)\) and not every function defined on \( \Delta \) can be extended
to a separately continuous function on \( \mathbb{R}^2 \), even for \( X = Y = \mathbb{R} \) the topologies \( \sigma \) and \( \gamma \) are not normal (moreover, \( \gamma \) is not regular [4,5]). Besides, every separately
continuous function \( f : X \times Y \to \mathbb{R} \) is a Baire class function for a wide class of
the products \( X \times Y \), in particular, if at least one of the multipliers is matrizable
[6]. Thus, the following question naturally arises: for which sets \( E \subseteq X \times Y \) and
\( \sigma \)-continuous (\( \gamma \)-continuous) function \( g : E \to \mathbb{R} \) of the first Baire class there exists
a separately continuous function \( f : X \times Y \to \mathbb{R} \) for which the restriction \( f|_E \)
coincides with \( g \)?

In this paper we generalize an approach proposed in [2] and solve the problem
formulated above for sets \( E \) of some type in the product of topological spaces.

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mapping.
2. Notions and auxiliary statements

A set $A \subseteq X$ has the extension property in a topological space $X$, if every continuous function $g : A \to [0, 1]$ can be extended to a continuous function $f : X \to [0, 1]$. According to Tietze-Uryson theorem [7, p.116], every closed set in a normal space has the extension property.

**Lemma 2.1.** Let sets $X_1$ and $Y_1$ have the extension property in a topological spaces $X$ and $Y$ respectively, $e : X_1 \to Y_1$ be a homeomorphism, $E = \{(x, e(x)) : x \in X_1\}$ and $g : E \to [-1, 1]$ be a continuous function. Then there exist continuous functions $f : X \times Y \to [-1, 1]$ and $h : X \times Y \to [-1, 1]$, which satisfy the following conditions:

(i) $f|E = g$;
(ii) $E \subseteq h^{-1}(0)$;
(iii) for every $x', x'' \in X$ and $y', y'' \in Y$ if $x' = x''$ or $y' = y''$ then $|f(x', y') - f(x'', y'')| = |h(x', y') - h(x'', y'')|$. 

**Proof.** Consider the continuous function $\varphi : X_1 \to [-1, 1]$ and $\psi : Y_1 \to [-1, 1]$, which defined by: $\varphi(x) = g(e(x), e(x))$, $\psi(y) = g(e^{-1}(y), y)$. Since $X_1$ and $Y_1$ have the extension property in $X$ and $Y$ respectively, there exist continuous functions $\hat{\varphi} : X \to [-1, 1]$ and $\hat{\psi} : Y \to [-1, 1]$ such that $\hat{\varphi}|_{X_1} = \varphi$ and $\hat{\psi}|_{Y_1} = \psi$. Put $h(x, y) = \frac{\hat{\varphi}(x) + \hat{\psi}(y)}{2}$ and $h(x, y) = \frac{\hat{\varphi}(x) - \hat{\psi}(y)}{2}$. Clearly that $f$ and $h$ are continuous on $X \times Y$ and valued in $[-1, 1]$. Moreover, for every point $p = (x, y) \in E$ we have $\hat{\varphi}(x) = \varphi(x) = g(y) = \psi(y) = \hat{\psi}(y)$. Therefore $f|E = g$ and $h|E = 0$, i.e. the conditions (i) and (ii) are hold.

Let $x', x'' \in X$ and $y, y'' \in Y$. Then

$$f(x', y) - f(x'', y) = \frac{\hat{\varphi}(x') - \hat{\varphi}(x'')}{2} = h(x, y') - h(x, y'').$$

If $x \in X$ and $y', y'' \in Y$, then

$$f(x, y') - f(x, y'') = \frac{\hat{\psi}(y') - \hat{\psi}(y'')}{2} = h(x, y') - h(x, y'').$$

Thus, the condition (iii) is holds and lemma is proved. \qed

In the case if the set $E$ satisfies a compactness-type condition we will use the following proposition.

**Lemma 2.2.** Let $X$ be a topological space, $E$ be a pseudocompact set in $X$, $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions $f_n : X \to \mathbb{R}$ which pointwise converges on the set $E$. Then there exists a functionally closed set $F \subseteq X$ such that $E \subseteq F$ and the sequence $(f_n)_{n=1}^{\infty}$ pointwise converges on the set $F$.

**Proof.** Consider the diagonal mapping

$$f = \Delta_{n \in \mathbb{N}} f_n : X \to \mathbb{R}^\mathbb{N}, \ f(x) = (f_n(x))_{n \in \mathbb{N}}.$$ 

Since the set $E$ is pseudocompact and $f$ is continuous, the set $f(E)$ is a pseudocompact set in the metrizable space $\mathbb{R}^\mathbb{N}$. Therefore $f(E)$ is closed and the set $F = f^{-1}(f(E))$ is functionally closed. It remains to verify that the sequence $(f_n)_{n=1}^{\infty}$ pointwise converges on $F$. Let $x \in F$. Then there exists an $x_1 \in E$ such that $f(x) = f(x_1)$, i.e. $f_n(x) = f_n(x_1)$ for every $n \in \mathbb{N}$. Since the sequence $(f_n(x_1))_{n=1}^{\infty}$ is convergent, the sequence $(f_n(x))_{n=1}^{\infty}$ is convergent too. \qed
The following proposition we will use in a final stage of the construction of separately continuous functions with the given restriction.

**Lemma 2.3.** Let $X$ be a topological space, $F$ be a functionally closed set in $X$, $(h_n)_{n=1}^\infty$ be a sequence of continuous functions $h_n : X \to \mathbb{R}$ such that $F \subseteq h_n^{-1}(0)$ for every $n \in \mathbb{N}$. Then there exists a locally finite partition of the unit $(\varphi_n)_{n=0}^\infty$ on $G$ such that the supports $G_n = \text{supp}\varphi_n = \{ x \in G : \varphi_n(x) > 0 \}$ of functions $\varphi_n$ satisfy the conditions:

(a) $G_n \cap F = \emptyset$ for every $n = 0, 1, 2, \ldots$;
(b) $G_n \subseteq h_n^{-1}((-\frac{1}{n}, \frac{1}{n}))$ for every $n = 1, 2, \ldots$.

**Proof.** Let $h_0 : X \to [0, 1]$ be a continuous function such that $F = h_0^{-1}(0)$. For every $n \in \mathbb{N}$ we put

$$A_n = \bigcap_{k=0}^n h_k^{-1} \left( (-\frac{1}{n}, \frac{1}{n}) \right), \quad B_n = \bigcap_{k=0}^n h_k^{-1} \left( [-\frac{1}{n}, \frac{1}{n}] \right),$$

$$G_n = A_n \setminus B_{n+2} \text{ and, moreover, } G_0 = G \setminus B_2.\text{ Clearly that all sets } G_n \text{ are functionally open and } G_n \subseteq h_n^{-1}((-\frac{1}{n}, \frac{1}{n})) \text{ for every } n \in \mathbb{N}, \text{ i.e. the condition (b) holds. Note that}\text{ }$$

$$\bigcap_{n=1}^\infty A_n = \bigcap_{n=1}^\infty B_n = \bigcap_{n=0}^\infty h_n^{-1}(0) = F.$$ 

Since $A_{n+1} \subseteq B_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$,

$$A_n \setminus A_{n+1} \subseteq A_n \setminus B_{n+2} \subseteq A_n \setminus A_{n+2} = (A_n \setminus A_{n+1}) \cup (A_{n+1} \setminus A_{n+2}).$$

Therefore

$$\bigcup_{n=1}^\infty G_n = \bigcup_{n=1}^\infty (A_n \setminus B_{n+2}) = \bigcup_{n=1}^\infty (A_n \setminus A_{n+1}) = A_1 \setminus \left( \bigcap_{n=1}^\infty A_n \right) = A_1 \setminus F.$$ 

Thus, $\bigcup_{n=0}^\infty G_n = (G \setminus B_2) \cup (A_1 \setminus F) = G.$

We show that the family $(G_n : n = 0, 1, \ldots)$ is locally finite on $G$. Let $x \in G$, i.e. $h_0(x) \neq 0$. We choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < |h_0(x)|$. Then $x \notin B_{n_0}$ and the set $G \setminus B_{n_0}$ is a neighborhood of $x$. On other hand, $G_n \subseteq A_n \subseteq B_{n_0}$ for every $n \geq n_0$. Therefore $G_n \cap (G \setminus B_{n_0}) = \emptyset$ for every $n \geq n_0$. Thus, the family $(G_n : n = 0, 1, \ldots)$ is locally finite at the point $x$.

Since the sets $G_n$ are functionally open, there exist continuous functions $\psi_n : X \to [0, 1]$ such that $G_n = \psi_n^{-1}([0, 1])$. The function $\psi : G \to [0, +\infty)$ which defined by $\psi(x) = \sum_{n=0}^\infty \psi_n(x)$ is continuous, moreover, $\text{supp}\psi = G$. For every $x \in G$ and $n = 0, 1, \ldots$ we put $\varphi_n(x) = \frac{\psi_n(x)}{\psi(x)}$. The functions $\varphi_n$ are continuous and formed a locally finite partition of the unit on $G$, moreover $G_n = \text{supp}\varphi_n$.

It remains to verify the condition (a). Since $G_n \subseteq X \setminus B_{n+2} \subseteq X \setminus A_{n+2}$ and the set $X \setminus A_{n+2}$ is closed, $G_n \subseteq X \setminus A_{n+2}$, i.e. $G_n \cap A_{n+2} = \emptyset$. Moreover, $F \subseteq A_{n+2}$, therefore $\overline{G_n \cap F} = \emptyset$ for every $n = 0, 1, \ldots$. □
3. Main results

**Theorem 3.1.** Let sets $X_1$ and $Y_1$ have the extension property in topological spaces $X$ and $Y$ respectively, $e : X_1 \to Y_1$ be a homeomorphism, $E = \{(x, e(x)) : x \in X_1\}$, $g : E \to R$ be the first Baire class function and at least one of the following conditions: $E$ is pseudocompact, $E$ is functionally closed in $X \times Y$, $X_1$ is functionally closed in $X$, $Y_1$ is functionally closed in $Y$ holds. Then there exists a separately continuous function $f : X \times Y \to R$ such that $f|_E = g$.

**Proof.** We take a sequence of continuous functions $g_n : E \to [-n, n]$ which pointwise converges to the function $g$ and use Lemma 2.1. We obtain a sequence of continuous functions $f_n : X \times Y \to [-n, n]$ and $h_n : X \times Y \to [-n, n]$ which satisfy the following conditions (i)-(iii).

We show that the set $E$ is contained in some functionally closed set $F_1$ on which the sequence $(f_n)_{n=1}^{\infty}$ pointwise converges. If $E$ is functionally closed, then $F_1 = E$. It follows from Lemma 2.2 the existence of such set $F_1$ for pseudocompact set $E$. It remains to verify this in the case when $X_1$ or $Y_1$ is functionally closed in $X$ or $Y$ respectively. Let $X_1$ is functionally closed in $X$. Now we put

$$F_1 = (X_1 \times Y) \cap \left( \bigcap_{n=1}^{\infty} h_n^{-1}(0) \right).$$

It follows from the property (ii) that $E$ is contained in a functionally closed set $F_1$. We take a point $(x, y)$ from the set $F_1$. Using the condition (iii) of Lemma 2.1 we obtain $|f_n(x, y) - f_n(x, e(x))| = |h_n(x, y) - h_n(x, e(x))| = 0$. Hence, $f_n(x, y) = f_n(x, e(x))$. Since the sequence $(f_n)_{n=1}^{\infty}$ pointwise converges on $E$, the sequence $(f_n(x, y))_{n=1}^{\infty}$ converges, because $(x, e(x)) \in E$.

Now we use Lemma 2.3 to the functionally closed set

$$F = F_1 \cap \left( \bigcap_{n=1}^{\infty} h_n^{-1}(0) \right)$$

in the space $X \times Y$ and to the sequence of continuous functions $h_n$ and obtain a locally finite partition of the unit $(\varphi_n)_{n=0}^{\infty}$ on $G = (X \times Y) \setminus F$, which satisfies the conditions (a) and (b).

Let $f_0 \equiv 0$ on $X \times Y$. We consider the function

$$f(x, y) = \begin{cases} \sum_{n=0}^{\infty} \varphi_n(x, y)f_n(x, y), & \text{if } (x, y) \in G, \\ \lim_{n \to \infty} f_n(x, y), & \text{if } (x, y) \in F. \end{cases}$$

Since $(\varphi_n)_{n=0}^{\infty}$ is a locally finite partition of the unit on the set $G$ and all functions $f_n$ are continuous, the function $f$ is correctly defined and continuous on the set $G$. Note that $F \subseteq F_1$. Therefore the sequence $(f_n)_{n=0}^{\infty}$ pointwise converges on $F$ and the function $f$ correctly defined on $F$. Moreover, since $E \subseteq h_n^{-1}(0)$ for every $n$ and $E \subseteq F_1$, $E \subseteq F$ and $f|_E = \lim_{n \to \infty} f_n|_E = \lim_{n \to \infty} g_n = g$.

It remains to verify that the function $f$ is separately continuous at points of the set $F$. Let $p_0 = (x_0, y_0) \in F \setminus \varepsilon > 0$. We choose $n_0 \in N$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$ and $|f_n(p_0) - f(p_0)| < \frac{\varepsilon}{2}$ for every $n \geq n_0$. It follows from the condition (a) that the set

$$W = X \times Y \setminus \left( \bigcup_{n=0}^{n_0} G_n \right),$$

...
where \( G_n = \text{supp} \varphi_n \), is an open neighborhood of \( p_0 \) in \( X \times Y \). We take a neighborhood \( U \) of \( x_0 \) in \( X \) such that \( U \times \{y_0\} \subseteq W \). Let \( x \in U \). If \( p = (x, y_0) \in F \), then \( h_n(p) = 0 \) for every \( n \in \mathbb{N} \). Then according to the condition (iii), we obtain 
\[
|f_n(p) - f_n(p)| = |h_n(p_0) - h_n(p)| = 0, \text{ i.e. } f_n(p_0) = f_n(p).
\]
Therefore \( f(p_0) = f(p) \).

If \( p \notin F \), then
\[
f(p) = \sum_{n=0}^{\infty} \varphi_n(p)f_n(p) = \sum_{n=n_0}^{\infty} \varphi_n(p)f_n(p),
\]
because \( p \in W \). Then
\[
|f(p_0) - f(p)| = \left| \sum_{n=n_0}^{\infty} \varphi_n(p)(f(p_0) - f_n(p)) + \sum_{n=n_0}^{\infty} \varphi_n(p)f_n(p) - \sum_{n=n_0}^{\infty} \varphi_n(p)f_n(p) \right|
\]
\[
\leq \sum_{n=n_0}^{\infty} \varphi_n(p)f_n(p) \leq \sum_{n=n_0}^{\infty} \varphi_n(p)|f(p_0) - f_n(p)| +
\]
\[
+ \sum_{n=n_0}^{\infty} \varphi_n(p)|f_n(p) - f_n(p)| < \sum_{n=n_0}^{\infty} \varphi_n(p) \cdot \frac{\varepsilon}{2} +
\]
\[
+ \sum_{n=n_0}^{\infty} \varphi_n(p)|h_n(p_0) - h_n(p)| = \frac{\varepsilon}{2} + \sum_{n=n_0}^{\infty} \varphi_n(p)|h_n(p)|.
\]

It follows from the property (b) of sets \( G_n \) that if \( \varphi_n(p) \neq 0 \), then \( |h_n(p)| < \frac{1}{n} \).

Thus,
\[
\sum_{n=n_0}^{\infty} \varphi_n(p)|h_n(p)| \leq \sum_{n=n_0}^{\infty} \varphi_n(p) \cdot \frac{1}{n} \leq \frac{1}{n_0} \sum_{n=n_0}^{\infty} \varphi_n(p) = \frac{1}{n_0} < \frac{\varepsilon}{2}.
\]

Hence, \( |f(p_0) - f(p)| < \varepsilon \). Thus, \( f \) is continuous at \( p_0 \) with respect to \( x \).

The continuity of \( f \) at \( p_0 \) with respect to \( y \) can be proved analogously. Thus, \( f \) is separately continuous and the theorem is proved.

In the case of \( X = Y = X_1 = Y_1 \) we obtain the following theorem which generalized the result from \([2]\).

**Theorem 3.2.** Let \( X \) be a topological space and \( g : X \to \mathbb{R} \) be a function of the first Baire class. Then there exists a separately continuous function \( f : X \times X \to \mathbb{R} \) such that \( f(x, x) = g(x) \) for every \( x \in X \).

4. Functions on the Product of Compacts

Now we consider the case when \( X \) and \( Y \) satisfy compactness type conditions. A set \( E \) in a product \( X \times Y \) is called horizontally and vertically one-pointed if for every \( x \in X \) and \( y \in Y \) the sets \( E \cap \{x\} \times Y \) and \( E \cap (X \times \{y\}) \) are at most countable and horizontally and vertically \( n \)-pointed if corresponding sets contain at most \( n \) elements.

**Theorem 4.1.** Let \( X \) and \( Y \) be compacts, \( E \) be a closed horizontally and vertically one-pointed set in \( X \times Y \) and \( g : E \to \mathbb{R} \) be a function of the first Baire class. Then there exists a separately continuous function \( f : X \times Y \to \mathbb{R} \), for which \( f|_E = g \).
Proof. Since the set $E$ is horizontally and vertically one-pointed, the projections the compact set $E$ to the axis $X$ and $Y$ are continuous injective mappings. Hence, $E$ is the graph of a homeomorphism $e : X_1 \to Y_1$, where $X_1$ and $Y_1$ are the projections of $E$ on $X$ and $Y$ respectively. Now the existence of desired function $f$ follows from Theorem 3.1. \hfill \Box

**Theorem 4.2.** Let $X$ and $Y$ be a locally compact spaces such that $X \times Y$ be a paracompact, $E$ be a closed horizontally and vertically one-pointed set and $g : E \to \mathbb{R}$ be the first Baire class function. Then there exists a separately continuous function $f : X \times Y \to \mathbb{R}$ for which $f|_E = g$.

**Proof.** For every $p = (x, y) \in X \times Y$ we choose open neighborhoods $U_p$ and $V_p$ of $x$ and $y$ in $X$ and $Y$ respectively such that the closure $X_p = \overline{U_p}$ and $Y_p = \overline{V_p}$ are compacts and the set $E_p = E \cap (X_p \times Y_p)$ is horizontally and vertically one-pointed. According to Theorem 1.2 there exists a separately continuous function $f_p : X_p \times Y_p \to \mathbb{R}$ for which $f_p|_{E_p} = g|_{E_p}$. Since the space $X \times Y$ is a paracompact, there exists a partition of the unit $(\varphi_i : i \in I)$ on $X \times Y$ which is subordinated to the open cover $(W_p = U_p \times V_p : p \in X \times Y)$ of $X \times Y$ [7, p.447]. For every $i \in I$ we choose $p_i \in X \times Y$ such that $\text{supp} \varphi_i \subseteq W_{p_i}$ and put

$$g_i(x, y) = \begin{cases} f_p(x, y), & \text{if } (x, y) \in W_{p_i}, \\ 0, & \text{if } (x, y) \notin W_{p_i}. \end{cases}$$

Note that the functions $\varphi_i g_i$ are separately continuous on $X \times Y$ and $(\varphi_i g_i)|_E = (\varphi_i|_E)g$. Then the function $f = \sum_{i \in I} \varphi_i g_i$ is the required. \hfill \Box

5. Example

Finally we give an example which show the essentiality of conditions under the set $E$ in Theorem 1.1.

Let $X = Y = [0, 1]$, $E_1 = \{((2k-1)/2^n, (2k-1)/2^n + 1/2^{n+1}) : k = 1, \ldots, 2^n-1, n \in \mathbb{N}\}$, $E_2 = \{(x, x) : x \in X\}$, $E = E_1 \cup E_2$, $g : E \to \mathbb{R}$, $g(x) = \begin{cases} 1, & x \in E_1, \\ 0, & x \in E_2. \end{cases}$ Clearly that $E$ is a closed horizontally and vertically 2-pointed set in $X \times Y$ and $g$ is a function of the first Baire class. Since the set $E_1$ is dense in $E$, the set $D(g)$ of discontinuity points set of the function $g$ coincides with $E$. Therefore the projections of $D(g)$ on the axis $X$ and $Y$ coincide with $X$ and $Y$ respectively. On the other hand, it is well-known that for every separately continuous function $f : X \times Y \to \mathbb{R}$ the set $D(f)$ of points of discontinuity of $f$ is contained in the product $A \times B$ of meagre sets $A \subseteq X$ and $B \subseteq Y$ respectively. Thus, $D(g) \not\subseteq D(f)$ and the function $f$ can not be extension of the function $g$.

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