Euclidean LQG dynamics: an electric shift in perspective

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Abstract

Loop quantum gravity (LQG) is a non-perturbative attempt at quantization of a classical phase space description of gravity in terms of $SU(2)$ connections and electric fields. As emphasized recently Ashtekar and Varadarajan (2020 arXiv:2012.12094[gr-qc]), on this phase space, classical gravitational evolution in time can be understood in terms of certain gauge covariant generalizations of Lie derivatives with respect to a spatial $SU(2)$ Lie algebra valued vector field called the electric shift. We present a derivation of a quantum dynamics for Euclidean LQG which is informed by this understanding. In addition to the physically motivated nature of the action of the Euclidean Hamiltonian constraint so derived, the derivation implies that the spin labels of regulating holonomies are determined by corresponding labels of the spin network state being acted upon thus eliminating the ‘spin $j$-ambiguity’ pointed out by Perez. By virtue of Thiemann’s seminal work, the Euclidean quantum dynamics plays a crucial role in the construction of the Lorentzian quantum dynamics so that our considerations also have application to Lorentzian LQG.

Keywords: loop quantum gravity, Hamiltonian constraint, canonical gravity

1. Introduction

Canonical loop quantum gravity [2] is an attempt at constructing a non-perturbative canonical quantization of a classical Hamiltonian description of gravity in terms of an $SU(2)$ electric field and its conjugate connection. While the electric field bears the interpretation of a spatial triad field and hence can be used to construct the spatial geometry, the connection contains information of the extrinsic curvature of the slice as embedded in the dynamically emergent spacetime geometry. The quantum kinematics of loop quantum gravity (LQG) is constructed without recourse to any fixed background geometry and is by now very well understood. The basic kinematic operators are holonomies of the connection along spatial edges and electric

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fluxes through spatial surfaces and their representation hints at a picture of discrete quantum spatial geometry. On the other hand, despite significant progress, a satisfactory treatment of the quantum dynamics of LQG remains an open problem. This dynamics is driven by the quantum Hamiltonian constraint operator. While early pioneering efforts [3, 4] culminating in the seminal work of Thiemann [5] demonstrate the existence of such an operator, the construction method yields an operator which is far from unique. This operator construction method [5] first replaces the classical Hamiltonian constraint with a classical approximant built out of classical precursors of the basic operators used in LQG, replaces these classical precursors by their quantum correspondents thereby defining a regulated quantum operator, and in a final step defines the Hamiltonian constraint operator as the limit of these regulated operators when the regulating parameter is removed. In the last decade, these methods have been applied to, and further developed in, the context of progressively complicated toy models which capture more and more aspects of gravity [9–15]. In these models it turns out that the classical dynamics in time can be understood in terms of geometrical deformations of field variables in space. Progress in the quantum theory is then greatly facilitated by the incorporation of this property of classical evolution into the quantum dynamics.

Recent work [1] shows that classical time evolution in Euclidean gravity can also be understood in terms of spatial Lie derivatives, suitably generalised, with respect to Lie algebra valued vector field called the electric shift which is obtained by multiplying the electric field by the lapse function (which smears the Hamiltonian constraint) and scaling the result by an appropriate power of the determinant of the spatial metric. In this work we present a construction of a Hamiltonian constraint operator for Euclidean gravity which attempts to incorporate this central feature of the evolution equations. As a result, the ensuing quantum dynamics can be understood in terms of quantum state transformations generated by a corresponding electric shift operator. The central role of the electric shift (and its operator correspondent) in classical (and quantum) dynamics was recognized in earlier work on a toy model system obtained by replacing the SU(2) group in canonical Euclidean gravity equations by the abelian group U(1) both in 2 + 1 [11] as well as in 3 + 1 dimensions [12–15]. In the 3 + 1 dimensional case, the classical toy model is exactly that derived by Smolin [16] as a novel weak coupling limit of Euclidean gravity.

In this toy model [11–13, 15], while the starting point was a definition of the quantum dynamics mediated by the electric shift, the primary focus was on the construction of a constraint action which yielded a non-trivial anomaly free constraint algebra. As analyzed in detail in [18], the anomaly free property can be seen as an implementation of the physical property of spacetime covariance. Hence, while our main focus here is the derivation of Hamiltonian constraint actions for Euclidean gravity in which the electric shift plays a central role, we shall also try to incorporate features which are necessary for a putative demonstration of consistency of these actions with a non-trivial anomaly free quantum constraint algebra. In addition we shall also attempt to incorporate features which seem to facilitate a second physical property beyond spacetime covariance, namely the property of propagation (see references [19–21] and, particularly, reference [22]).

1 It is pertinent to note here that there exist various implementations of the Hamiltonian constraint in LQG, which differ in several aspects from Thiemann’s original proposal in [5], for example the constructions of references [6–8].
2 We note that given the complicated dynamical nature of classical general relativity and the absence of any preferred fixed background geometry to regulate operators, the existence of a well defined gravitational Hamiltonian constraint operator in LQG is already a remarkable achievement.
3 For recent work on the classical theory of this model with a view towards the quantum theory, see [17].
The layout of the paper is as follows. In section 2 we briefly review the classical Hamiltonian formulation of Euclidean gravity in terms of electric fields and connections and define the electric shift. Next we review the quantum kinematics underlying LQG. In doing so we find it useful to define holonomies in spin \( j \) representations for arbitrary spin \( j \). This ‘quantum’ part of the material in section 2 is standard and its main purpose is the establishment of notation rather than pedagogy; indeed, in the rest of the paper, we shall assume familiarity with standard LQG structures. In section 3 we construct the electric shift operator and provide a heuristic derivation of the action of the Hamiltonian constraint in which spin network deformations generated by the electric shift operator play a central role. The derivation relies on a key classical identity connected with the gauge covariant Lie derivative first introduced by Jackiw [23] to the best of our knowledge. The final constraint action can also be viewed in the more conventional terms of holonomy approximants to the curvature rather than deformations generated by the electric shift operator. When viewed in this way, it turns out that the holonomy approximants are defined with respect to representations of \( SU(2) \) which are tailored to the spin labels of the edges of the spin network being acted upon. This removes the ‘spin \( j \)-ambiguity’ in the representation choice of regulating holonomies raised by Perez [24]. The end result of section 3 is an action which is reminiscent of the standard ‘QSD’ action derived by Thiemann [5] in that it acts only at vertices roughly by adding a single extraordinary edge for each pair of edges emanating from the vertex. The edges are added ‘one at a time’ between each such pair and the result summed over. The main difference here is that, first, the extraordinary edge spin labels are tailored to those of the edges emanating from the vertex rather than being fixed at \( j = \frac{1}{2} \) and second, the way they are added is slightly different. The resulting constraint action shares the property with its QSD counterpart that a second such constraint action does not act on deformations created by the first.

As first noted in [25] in order to obtain a non-trivial constraint algebra on a ‘habitat’ [26], it is necessary for a second constraint action to act on deformations created by the first. Such actions were obtained in the \( 3 + 1 U(1)^3 \) toy model setting in [12, 13, 15] wherein state deformations created a new vertex at which the edges formed a certain conical structure on which the second action could (generically) act non-trivially. Hence, in section 4 we develop the action of section 3 further so as to obtain the exact analog of these conical deformations but now in the \( SU(2) \) context of Euclidean gravity. In the \( U(1)^3 \) case such deformations were shown to be consistent with anomaly free constraint commutators [15]. Preliminary calculations in progress indicate the existence of significant technical complications when one tries to generalise the considerations of [15] to the \( SU(2) \) case. Hence in section 5 we further develop the action of section 4 to obtain, roughly speaking, a mix of actions of the type presented in sections 3 and 4. The action seems simple enough that progress on the ‘anomaly free’ front may be within reach. It is also likely that an analysis similar to that of [22] reveals consistency with propagation. Section 6 is devoted to a discussion of our results and plans for future work.

Unless otherwise specified we shall use units in which \( G = h = c = 1 \). We also choose the Barbero–Immirzi [27, 28] parameter to be unity. For a first stab at the problem considered in this work, please see [29].

2. Review of classical theory and its quantum kinematics

2.1. Classical Hamiltonian formulation

The phase space variables \((A^a_i, E^i)\) are an \( su(2) \) connection and conjugate densitized electric field on the Cauchy slice \( \Sigma \) with \( a \) denoting a tangent space index on \( \Sigma \) and \( i \) an ‘internal’ index.
valued in \(su(2)\). The \(su(2)\) structure constant is denoted by \(\epsilon_{ijk}\). Internal indices are raised and lowered by the \(su(2)\) Cartan–Killing metric, denoted by the Kronecker delta symbol \(\delta_{ij}\). The density weight 2 contravariant metric on \(\Sigma\) is defined as \(q_{ab} := E_a^i E_b^j\), \(q\) being the determinant of the corresponding covariant metric \(q^{ab}\). The action of the gauge covariant derivative \(D_a\) associated with \(A_i^a\) on a Lie algebra valued scalar \(\lambda^i\) is:

\[
D_a \lambda^i = \partial_a \lambda^i + \epsilon^i_{jk} A_j^a \lambda^k
\]

where \(\partial_a\) is a flat (coordinate) derivative operator with respect to which the connection is specified. The curvature of \(D_a\) is then \(F_{ab} := \partial_a A_b^i - \partial_b A_a^i + \epsilon^{ik}_{
}

\[
G[\Lambda] = \int d^3x \Lambda' D_a E^a_i
\]

\[
D[\vec{N}] = \int d^3x N^a \left( E^i_a F^i_{ab} - A^i_j D_a E^b_j \right)
\]

\[
H[N] = \frac{1}{2} \int d^3x N q^{-1/3} \epsilon^{ijk} E^i_a E^j_b F_{ab}
\]

are the Gauss law, diffeomorphism, and Hamiltonian constraints of the theory. The Hamiltonian constraint density, \(\frac{2}{3} q^{-1/3} \epsilon^{ijk} E^i_a E^j_b F_{ab}\), in (2.4) is chosen to be of density weight \(\frac{4}{3}\) with the lapse \(N\) carrying a density weight of \(-\frac{1}{3}\). As discussed in [12, 13, 15] this choice of density weight is made in anticipation of the putative construction of a non-trivial anomaly free constraint algebra. However, our considerations in this paper are independent of this choice of density weight and (as we shall see, apart from an overall factor of regulating parameter) would go through for a Hamiltonian constraint density of weight 1 as well.

The Poisson brackets between the constraints are:

\[
\{G[\Lambda], G[\Lambda']\} = \{G[\Lambda], H[N]\} = 0
\]

\[
\{D[\vec{N}], G[\Lambda]\} = G[\mathcal{L}_{\vec{N}} \Lambda]
\]

\[
\{D[\vec{N}], D[\vec{M}]\} = D[\mathcal{L}_{\vec{N}} \vec{M}]
\]

\[
\{D[\vec{N}], H[N]\} = H[\mathcal{L}_{\vec{N}} N]
\]

\[
\{H[N], H[M]\} = D[\mathcal{L}_N + G[\Lambda \cdot \vec{L}], L^a := q^{-2/3} E^a_i E_i^b (M \partial_b N - N \partial_b M).
\]

Here \(\mathcal{L}_\nu\) denotes the Lie derivative with respect to the vector field \(N^\nu\). Note that the last Poisson bracket (between the Hamiltonian constraints) exhibits structure functions just as in Lorentzian gravity albeit with a different overall sign.

We define the electric shift \(N^a_i\) by

\[
N^a_i = NE^a_i q^{-1/3}
\]

As can be checked \(N^a_i\) transforms like a Lie algebra valued vector field.

We are concerned exclusively with the Hamiltonian constraint in this paper. Segregating an ‘electric shift’ part in brackets, we write (2.4) as:

\[
H(N) = \frac{1}{2} \int \epsilon^{ijk} \left( \frac{NE^i_j}{q^{1/3}} \right) F_{ab} E^b_j.
\]
As remarked in section 1 and shown in reference [1], the electric shift plays a key role in a geometrical interpretation of the evolution generated by the Hamiltonian constraint. More in detail, reference [1] generalises, in a natural gauge covariant manner, the notion of Lie derivative with respect to a vector field to one with respect to a Lie algebra valued vector field. It then shows that the evolution of the connection and electric field generated by the Hamiltonian constraint can be written as simple combinations of their gauge covariant Lie derivatives with respect to the (Lie algebra valued) electric shift (2.10). In this paper we try to incorporate this structural aspect of the evolution equations as best as we can in quantum theory. As we shall see, the quantum Hamiltonian constraint that we construct generates deformations of the graph underlying the quantum spin network state being acted upon, the nature of these deformations being tied to the action of the quantum electric shift operator.

2.2. Holonomies

Let $e : [0, s] \to \Sigma$ be an edge embedded in $\Sigma$. The classical holonomy function along the edge $e$ is defined as the path ordered exponential of the connection along the edge $e$. The path ordered exponential, $P \exp - \int e A$, is a $2j + 1 \times 2j + 1$ matrix in the spin $j$ representation of the group $SU(2)$ with matrix components defined as follows:

$$
\left( P \exp - \int e A \right)^B_C := \delta^B_C + \sum_m (-1)^m \int_0^s ds_1 \times \int_0^{s_1} \cdots \int_0^{s_{m-1}} ds_m (A(s_1)A(s_2) \cdots A(s_m))^B_C
$$

(2.12)

where the matrix $A(t)$ is defined through:

$$
A(t)^E_F := \dot{e}^E(t)A'_E(e(t)) (\tau_1^{(j)})^E_F
$$

(2.13)

with $A'_E(e(t))$ denoting the evaluation of the connection $A'_E$ at the point on the edge $e$ at parameter value $t \in [0, s]$, $\dot{e}^E(t)$ denoting the edge tangent at that point and $(\tau_1^{(j)})^E_F$ being the $e_F$ component of the spin $j$ matrix representative of the $i$th generator of the Lie algebra of $SU(2)$. The holonomy (2.12) is then a unitary $2j + 1 \times 2j + 1$ matrix with $B, C \in (-j, -j + 1, \ldots, j - 1, j)$ and we denote it by $h_{ij}(s, 0)$. In what follows we shall often omit the $(j)$ superscript to avoid notational clutter.

2.3. Quantum kinematics

Standard $SU(2)$ representation theory implies that all holonomies for $j > \frac{1}{2}$ can be obtained through suitable linear combinations of products of $j = \frac{1}{2}$ holonomies. Hence in LQG, while the quantum kinematics supports the action of holonomy operators for all $j$, it is the standard practice to treat operator correspondents of $j = \frac{1}{2}$ holonomies as fundamental. These operators act by multiplication on an orthonormal basis of spin network states [31, 32]. For our purposes a spin network state can be thought of as a wave function of the connection labeled by a closed oriented graph $\gamma$ consisting of a set of oriented edges $e_\alpha, \alpha = 1, \ldots, m$ each colored with a spin label $j_\alpha$, and a set of $SU(2)$ invariant tensors, $\{C_v\}$, one at every vertex $v$ which maps the tensor product of representations along incoming edges at the vertex to the tensor product of representations along outgoing edges at that vertex. These tensors are called intertwiners.

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4We shall work with the semianalytic category of edges, surfaces and Cauchy slices [30] although this will not be explicitly needed.
because they intertwine incoming representations with outgoing ones. We denote the entire set of labels by $S$. The label set $S$ specifies a corresponding wave function of the connection $S(A)$ as follows.

Let $e(t)$ denote the point on the edge $e$ at parameter value $t$. By convention we use edge parameterizations such that the beginning point of the edge is at $t = 0$ and the final point at $t = 1$, the orientation of the edge being in the direction of increasing parameter value. Evaluate the holonomy of the connection $A_i$ along an oriented edge $e_i$ of the graph in the representation $j_i$ to obtain the matrix $h_{j_i}^{A_i}(1, 0)_{e_i}^{e_i}$ so that the index $E_{e_i}$ is associated with the beginning point of $e_i$ and $F_{e_i}$ with its endpoint. The intertwiner at vertex $v$ is a tensor $C_v$ with the following index structure. For the upper index of every incoming edge at $v$, $C_v$ has a corresponding lower index, and for the lower index of every outgoing edge at $v$, it has a corresponding upper index. By virtue of this index structure, the upper and lower indices of $C_v$ can be contracted with lower and upper indices of outgoing and incoming edges at $v$ respectively. It follows that the upper and lower indices associated with the tensor product of intertwiners overall vertices can then be contracted with corresponding lower and upper indices of the tensor product of all edge holonomies for all edges of the graph. This contraction then yields a function of the connection $A_i$ which is the wave function $S(A)$. This wave function is gauge invariant by virtue of the group invariance of the intertwiners. This group invariance property is as follows. Consider a spin network $S$ with an $N$ valent vertex at the point $v$. Let the edges at $v$ be outgoing at $v$ so that the intertwiner $C$ at $v$ has $N$ upper indices,

$$C \equiv C^{A_1 \cdots A_N}. \quad (2.14)$$

Our notation implicitly assumes that $A_i$ is valued in the spin $j_i$ representation which colours the $i$th outgoing edge $e_i$ at $v$. Let $g_i$ be the matrix representative of the $SU(2)$ element $g$ in the $j_i$ representation. Then the group invariance property of the intertwiner is:

$$C^{A_1 \cdots A_N} g_1^{B_1} A_1 g_2^{B_2} A_2 \cdots g_N^{B_N} A_N = C^{B_1 \cdots B_N}. \quad (2.15)$$

We shall use this group invariance property repeatedly in sections 3–5. In order to reduce notational clutter when doing so we abuse notation slightly, suppress the $i$ index on $g_i$ and denote the matrix representative of $g$ also by $g$ (the spin representation of this matrix representative will be clear from context) and write the group invariance property as:

$$C^{A_1 \cdots A_N} g^{B_1} A_1 g^{B_2} A_2 \cdots g^{B_N} A_N = C^{B_1 \cdots B_N} \quad (2.15)$$

By choosing a suitable basis of intertwiners at each vertex, one obtains an orthonormal basis with respect to the $SU(2)$ Haar measure based Hilbert space measure of LQG (see for example [32]).

The electric field operator can be thought of as acting by functional differentiation on this wave function and its action on any $S(A)$ can be readily inferred from its action on a spin $j$ edge holonomy which in turn can be evaluated from (2.12). A straightforward computation yields:

$$\hat{E}_a(x) h_{e}^{A} b(1, 0) = -i \frac{\delta}{\delta A_a(x)} h_{e}^{A}(1, 0)$$

$$= i \int_0^1 d\tau \delta^3(x, e(\tau)) (h_{e}(1, \tau) \tau \delta^3(x, e(\tau)) h_{e}(0, \tau))^A A, \quad (2.16)$$

where $h_{e}^{A}(1, 0)$ as a functional of the connection is defined through equation (2.12) (with $s$ set to 1 and the indices $B_C$ in that equation replaced by $A_B$).
This expression is not a finite linear combination of edge holonomies because of the integral of the 3 dimensional Dirac delta function along the 1 dimensional edge. Integrating the expression over a 2 dimensional surface (if necessary, with an intermediate regularization of the delta function [32]) yields a well defined electric flux operator action which maps edge holonomies to finite linear combinations of edge holonomies. Nevertheless, we shall find it useful to employ the expression (2.16) in our heuristic argumentation of section 3 as well as in our derivation of the electric shift operator to which we turn next.

3. Derivation of an electric shift mediated Hamiltonian constraint action

In section 3.1 we construct the electric shift operator. In section 3.2 we display a key identity for our considerations in section 3.3. In section 3.3 we derive the action of the Hamiltonian constraint operator. Our arguments in sections 3.1 and 3.3 follow the established practice in LQG of defining operators through the following steps: (i) first by regulating them, (ii) next by defining their regulated action on spin network wave functions which are assumed to be functions of smooth connections, (iii) then by neglecting terms which are next to leading order in the regulating parameter and (iv) finally taking away the regulating parameter to zero. The key step is (iii) and the ambiguities in the final operator action in (iv) arise because the neglected terms which are small for any fixed smooth connection argument of the wave function, can be $O(1)$ in the norm of the Hilbert space wherein the wave functions reside. The procedure is well tested and its heuristic justification stems from the fact that on any fixed graph, the evaluation of a wave function on any element of the quantum configuration space is identical to that on some smooth connection. While in section 3.1 we shall be able to implement step (iv) in a simple way and obtain an operator on the kinematic Hilbert space, in section 3.3 we shall refrain from implementing step (iv) because the operator approximant does not have a limit on the kinematic Hilbert space. Instead, the ‘continuum limit’ of step (iv) must be sought in a suitable space of off shell states [15] and the construction of such a space represents work in progress. Additionally in section 3.3 we shall regulate a part of the quantum shift operator action of section 3.2 in a manner which, while not strictly following (i)–(iii), is motivated by the smoothness of classical phase space fields. We shall expand on this cryptic statement in footnote 9 of section 3.3 when we encounter the regulation in question.

3.1. Electric shift operator

Consider a point $v$ and a coordinate patch $\{x\}$ around $v$. Fix a coordinate ball $B_{\tau}(v)$ of radius $\tau$ centred at $v$, and restrict attention to small enough $\tau$ in the following manipulations so that all constructions happen within the domain of $\{x\}$. At the point $v$, define a classical approximant $N^{a}_{(\tau)}$ to the electric shift $N^{a}$ which agrees with $N^{a}$ as $\tau \to 0$:

$$N^{a}_{(\tau)} := N(x(v)) q^{-1/3} \frac{1}{4\pi^{2}} \int_{B_{\tau}(v)} \mathrm{d}^{3}x \hat{E}_{a}(x).$$

Here $q^{-1/3}$ denotes an approximant to $q^{-1/3}$ at $v$ so that it agrees with $q^{-1/3}$ as $\tau \to 0$ and $N(x(v))$ denotes the evaluation of the density weighted lapse at $v$ in the coordinate system $\{x\}$.

The corresponding operator, $\hat{N}^{a}_{(\tau)}$, is:

$$\hat{N}^{a}_{(\tau)} = N(x(v)) \frac{1}{4\pi^{2}} \left( \int_{B_{\tau}(v)} \mathrm{d}^{3}x \hat{E}_{a}(x) \right) q^{-1/3}$$

(3.2)
where we have ordered $q^{-1/3}$ to the right. We shall assume that $q^{-1/3}$ has the following properties in its action on a spin network:

(a) For small enough $\tau$, $\tau^{-2}q^{-1/3}$ is independent of $\tau$ and has trivial action if $v$ is a vertex of the spin network of valence less than 3.

(b) When $\tau^{-2}q^{-1/3}$ has non-trivial action at $v$, it only changes the intertwiner at $v$ while leaving the edges at $v$ untouched.

We note that any $q^{-1/3}$ deriving either through Thiemann like identities [5, 37] involving the volume operator or through the Tikhonov regularization\(^5\) involving spectral analysis of the volume operator [33] satisfies properties (a) and (b) above. Consider the action of $\tau^{-2}q^{-1/3}$ on a spin network state $S(A)$ with intertwiner $C$ at its $N$ valent vertex $v$. This action changes the intertwiner to a new intertwiner at $v$ which we denote as $C_\lambda$, thereby mapping the spin network to the new spin network $S_\lambda(A)$ where $S_\lambda$ is obtained from $S$ by replacing $C$ at $v$ by $C_\lambda$ at $v$. It follows that the regulated quantum shift action on $S$ for small enough $\tau$ is:

\[
\hat{N}_{(\tau_j)}aS(A) = N(x(v)) \left[ \frac{1}{3} \int_{B_\epsilon(v)} \text{d}^3x \hat{E}_i(x) \right] S_\lambda
\]

\[
= iN(x(v)) \left[ \frac{1}{3} \int_{B_\epsilon(v)} \text{d}^3x \sum_{i=1}^N \int_0^1 \text{d}t_i \hat{e}_i(t_i) \delta^3(x, e_i(t)) \right]
\]

\[
\times (h_{e_i}(1, t_i)\tau_i^{(b)}h_{e_i}(t_i, 0)) \frac{\partial S_\lambda}{\partial h_{e_i}(1, 0) A_j} e_i
\]

\[
= iN(x(v)) \left[ \frac{1}{3} \sum_{i=1}^N \int_0^1 \text{d}t_i \hat{e}_i(t_i) \tau_i^{(b)}h_{e_i}(t_i, 0) \frac{\partial S_\lambda}{\partial h_{e_i}(1, 0) A_j} e_i \right]
\]

\[
\times \frac{\partial S_\lambda}{\partial h_{e_i}(1, 0) A_j}. \tag{3.3}
\]

In the second line we have used (2.16) to evaluate the action of the triad operator on $S_\lambda$. In the third line the $\delta$-function has been integrated over $B_\epsilon(v)$ with the consequence that the $t_i$ integral is over the part of $i$th edge within $B_\epsilon(v)$ so that $t_{i,j}$ is the parameter value at which $e_i$ intersects the boundary of $B_\epsilon(v)$. Denoting the unit coordinate tangent vector to the edge $e_i$ at $v$ by $\hat{e}_i$, it follows that to leading order in $\tau$ we have:

\[
\hat{N}_{(\tau_j)}aS(A) = iN(x(v)) \left[ \frac{1}{3} \sum_{i=1}^N \tau_i^{(b)}h_{e_i}(1, 0) \frac{\partial S_\lambda}{\partial h_{e_i}(1, 0) A_j} e_i \right]. \tag{3.4}
\]

Note that the combination $(h_{e_i}(1, 0)\tau_i^{(b)}h_{e_i}(1, 0) A_j) \frac{\partial S_\lambda}{\partial h_{e_i}(1, 0) A_j}$ is exactly the result of the action of the left invariant vector field $\hat{X}_{i,j}$ of the $i$th copy of $SU(2)$ associated with the edge holonomy $h_{e_i}(1, 0)(A)$ on $S(A)$. Hence, taking the $\tau \to 0$ limit of right-hand side of (3.4),

\(^5\)The regularization [33] defines the action of the inverse of a positive operator $\hat{O}$ on a state $\psi$ as $\lim_{\epsilon \to 0^+} (\hat{O} + \epsilon)^{-1} \hat{O} \psi$ where $(\hat{O} + \epsilon)^{-1}$ is defined through spectral analysis. The result is an operator which annihilates the zero eigenvalue states of $\hat{O}$.
we define the action of the quantum shift operator at a vertex \( v \) of the spin network \( S \) to be

\[
\hat{N}^i_j(v)S(A) = \frac{3i}{4\pi}N(x(v)) \sum_{j=1}^N \hat{c}^\dagger_j \hat{X}_j jS(A). \tag{3.5}
\]

From the discussion above, the action of the quantum shift is non-trivial only when the ‘inverse metric determinant operator’ \( \lim_{\tau \to 0} \tau^{-2}q^{-1/3} \) has a non-trivial action. Vertices of \( S \) where the inverse metric determinant operator acts non-trivially will be referred to as non-degenerate vertices (from (b) above, such vertices have a minimum valence of 3).

### 3.2. An important identity

The following identity, and subsequent considerations below will be of use in the following subsection. For any vector field \( V^a \), it is straightforward to check that the following identity holds:

\[
V^a F^i_{ab} = L^i_v A^i_a - D_b (V \cdot A)^i
\tag{3.6}
\]

where \( L^i_v A^i_a \) is the Lie derivative of \( A^i_a \) with respect to \( V^a \):

\[
L^i_v A^i_a = V^b \partial_b A^i_a + A^i_b \partial_b V^a
\tag{3.7}
\]

Next, define the one parameter family of \( SU(2) \) group elements \( g(p, t), p \in \Sigma \) as:

\[
g(p, t) = \exp - \int_{c(p, t)} A
\tag{3.8}
\]

The path ordered exponential is along the curve \( c(p, t) \). The curve \( c(p, t) \) is the integral curve of the vector field \( V^a \) which starts at the point \( p \in \Sigma \) and ends at the point \( \phi(\vec{V}, t) \circ p \). Here \( \phi(\vec{V}, t) \) is the one parameter family of diffeomorphisms generated by \( V^a \) and parameterised by its affine parameter \( t \) (so that \( V^a = (\frac{d}{d \tau}) \vec{V}^a \)). It is understood that the path ordered exponential is evaluated in a spin \( j \) representation (we have suppressed the spin \( j \) label to avoid notational clutter). Next, we define:

\[
A^i_a(p, t)\tau_i = g^{-1}(p, t)(\phi(\vec{V}, t), A^i_a(p))\tau_i g(p, t) + g^{-1}(p, t)\partial_a g(p, t)
\tag{3.9}
\]

where \( \phi(\vec{V}, t) \) is the pull back action of the diffeomorphism \( \phi(\vec{V}, t) \), \( \tau_i \) is the \( i \)th generator of the Lie algebra of \( SU(2) \) in the spin \( j \) representation and the summation convention over the repeated index \( i \) is implicit.

It is then straightforward to check (as asserted by Jackiw [23]) that for small enough \( \epsilon \):

\[
V^a(p) F^i_{ab}(p)\tau_i = \frac{A^i_a(p, \epsilon)\tau_i - A^i_a(p)\tau_i}{\epsilon} + O(\epsilon)
\tag{3.10}
\]

As indicated in reference [23], the check consists of (a) expanding \( g(p, t), (\phi(\vec{V}, t), A^i_a(p)) \) with \( t = \epsilon \) to order \( \epsilon \) and neglecting higher order terms, (b) substituting the resulting expressions with higher order terms so neglected into (3.9), and (c) substituting \( A^i_a(p, \epsilon)\tau_i \) in (3.10) by the expression obtained in (b). While it is desirable to also carefully bound the contribution of the neglected terms to (3.10), we refrain from attempting to do so here due to the heuristic nature of the arguments which employ (3.10) in our analysis of the constraint action in section 3.3 (see remark (a), section 3.3).
3.3. Hamiltonian constraint operator action

The classical constraint (2.11) has three constituents: the right most triad field, the curvature term and the electric shift. Following earlier work on toy models [15] we aim for a constraint action constructed through the following 3 step schematic:

(a) The combination of the electric shift and the curvature result in a deformation of the connection $\delta A$ similar to (3.10)

(b) The remaining triad field acts as a functional derivative so that we have, schematically, the combination $\delta A \frac{\delta}{\delta \Psi(A)} \Psi(A)$

(c) The combination $\delta A \frac{\delta}{\delta \Psi(A)} \Psi(A)$ can be approximated in terms of the difference of wave functions: $\Psi(A + \delta A) - \Psi(A)$.

With this ‘guiding heuristic’ in mind, we proceed as follows. We order the quantum shift term to the right and obtain the following regulated constraint action on a spin network $S(A)$:

$$H(N)S(A) = \frac{1}{2} \int d^3 x e_i \# \hat{F}_{ab}^\mu \hat{E}_k^\mu (NE_j^\nu \bar{q}^{-1/3}) S(A)$$

$$= \frac{1}{2} \int d^3 x e_i \# \hat{F}_{ab}^\mu \hat{E}_k^\mu \hat{N}_j^m S(A)$$

(3.11)

where in the second line we have used the definition of the quantum shift operator. For simplicity we assume that the lapse $N$ is of compact support around a nondegenerate vertex $v$ of $S(A)$, with no other vertex of $S$ in this support. The action of the right most electric shift operator is then supported only at $v$. In order to obtain a non-trivial action of the constraint operator it is necessary to regulate the quantum shift action so as to endow it with a support of non-zero measure. We do this by replacing each edge tangent $\hat{e}_j^i$ in (3.5) by a vector field $\hat{e}_j^i(m)$ which vanishes outside an open neighborhood $U_{i,j}(v)$ of $v$. We choose $\hat{e}_j^i, U_{i,j}(v)$ such that, for small enough $\epsilon$, the following properties hold.

(a) $U_{i,j}(v)$ is a coordinate ball around $v$ of radius $\epsilon + \epsilon^m$ for some $m \gg 1$

(b) $e_j \cap U_{i,j}(v) = e_j(t_{i,j}, 0)$ where $e_j(t_{i,j}, 0)$ is a single segment of $e_j$ running from the vertex $v$ at $t_j = 0$ to the point $e_j(t_j = t_{i,j})$.

(c) Consider the point on $e_j$ which is located at a coordinate distance $\epsilon$ from $v$ along $e_j$. Let the parameter value $t_I$ for this point be $\epsilon_I$. On $e_j$, $\hat{e}_j^i(t_I) = \lambda(t_I)\hat{e}_j^i(t_I)$ for some $\lambda(t_I) \geq 0$ with $\lambda(t_I) = 1$ for $0 \leq t_I \leq \epsilon_I$. Thus the regulated edge tangent $\hat{e}_j^i(t_I)$ is the same as the unit coordinate edge tangent on $e_j(t_{i,j}, 0)$ except for a small neighborhood of the point $e_j(t_{i,j}, 0)$, this neighborhood being of coordinate size $O(\epsilon^3)$.

The action of the regulated quantum shift operator at $v$ is defined to be:

$$\hat{N}_j^m(v)S(A) = \frac{3i}{4\pi} N(x(v)) \sum_{j=1}^N \hat{e}_j^i \hat{X}_j \mu S(A).$$

(3.12)

7 If $e_I$ was a coordinate straight line, $v$ would be exactly at a distance $\epsilon$ from $e_I(e_I)$. Since $e_I$ may not be such a straight line, the coordinate distance between the points $v, e_I(e_I)$ is $\epsilon + O(\epsilon^2)$.

8 If $e_I$ was a coordinate straight line, from (a) this neighborhood would be of $O(\epsilon^3)$. 


This in turn defines the regulated Hamiltonian constraint operator \( \hat{H}_c(N) \): \(^9\)

\[
\hat{H}_c(N)S(A) = \frac{1}{2} \int d^3x \epsilon_i \hat{F}_i^\dagger \hat{E}_k^\dagger \hat{E}_k \hat{N}_i S(A)
\]
\[
= \frac{3i}{8\pi} N(x(v)) \sum_{I} \int_{U_{i,j}(v)} d^3x \epsilon_i \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i S(A)
\]
\[
= \frac{3i}{8\pi} N(x(v)) \sum_{I} \int_{U_{i,j}(v)} d^3x \epsilon_i \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i S(A).
\]

(3.13)

Next we act with \( \hat{E}_k^\dagger \) through (2.16), and integrate the Dirac delta function in that action over \( U_{i,j}(v) \). The result is, similar to the integral in (3.3), a sum over integrals one for each edge with the \( J \)th integral being over the part of \( J \)th edge within \( U_{i,j}(v) \). Recalling that \( t_{i,j} \) is the parameter value at which \( \epsilon_i \) intersects the boundary of \( U_{i,j}(v) \), we have:

\[
\hat{H}_c(N)S(A) = \frac{3i}{8\pi} N(x(v)) \sum_{I} \int_{U_{i,j}(v)} d^3x \epsilon_i \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i S(A)
\]
\[
= -\frac{3}{8\pi} N(x(v)) \sum_{I,J} \int_0^{t_{i,j}} dt_j \epsilon_j \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i \epsilon_i
\]
\[
\times [h_{\epsilon j}(1, t_j) \tau_k h_{\epsilon j}(t_j, 0)] \partial \hat{X}_{i,j} S(A) \frac{\partial \hat{N}_i}{\partial \hat{X}_{i,j} B_j}
\]
\[
= -\frac{3}{8\pi} N(x(v)) \sum_{I,J} \int_0^{t_{i,j}} dt_j \epsilon_j \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i \epsilon_i
\]
\[
\times (\tau_k + O(\epsilon)) \frac{\partial \hat{X}_{i,j} S(A)}{\partial \hat{X}_{i,j} B_j}
\]
\[
= -\frac{3}{8\pi} N(x(v)) \sum_{I,J} \sum_{j \neq I} \int_0^{t_{i,j}} dt_j \epsilon_j \hat{F}_i^\dagger \hat{E}_k \hat{E}_k \hat{N}_i \epsilon_i
\]
\[
\times (h_{\epsilon j}(1, 0) \tau_k) \partial \hat{X}_{i,j} S(A) \frac{\partial \hat{N}_i}{\partial \hat{X}_{i,j} B_j}
\]  

(3.14)

where in the last line we have only retained the leading order contribution in \( \epsilon \) (see (a) in the preamble of section 3). Further, since \( \hat{F}_{i,j} \) is a connection dependent operator, it acts

\(^9\) Note that this sort of regulation is not of the type (a)–(c) described in the preamble of section 3 as it does not directly involve connection dependent terms. However the necessity of that regulation as well as this one both arise due to the distributional nature of the quantum fields. More in detail the regulation (a)–(c) is necessary because the distributional nature of elements of the quantum configuration space of connections only allow holonomies to be defined as opposed to the local connection fields of the classical theory. Similarly the regulation of the quantum shift is necessitated by virtue of the distributional nature of the action of the quantum electric field operator (2.16) and that of the inverse determinant of the metric operator (see (a) and (b) of section 3.1) which is in contrast to the smoothness of the corresponding classical fields. Note that this behaviour of these electric field dependent operators is also, ultimately, a consequence of the distributional nature of elements of the quantum configuration space of connections. Finally, we note that the regulation has effectively the same consequence as that of conventional regulations of the Hamiltonian constraint which, despite the non-triviality of the action of the inverse metric determinant exclusively at isolated points, yield contributions to the regulated operator action from small neighbourhoods of these points [5].
by multiplication. Hence we have removed the ‘hat’ on it in the last line. Finally, note that if \( I = J \) we get no contribution by virtue of the antisymmetry of \( F^i_{ab} \), together with the fact (see (c) above of this section) that \( \hat{e}^j \) is parallel to \( \hat{e}^i \) along the \( i \)th edge. Hence, in the last line the second sum is over \( J \neq I \). Note that \( \tau_k \) is in the representation \( j_k \) which labels the edge \( e_j \).

Next, we replace \( e^b \tau_i \) by the commutator between \( \tau_i \) and \( \tau_j \) to create the combination \((\hat{e}^a_i, F^i_{ab}) \tau_i \) so as to facilitate the application of (3.10).10 We obtain:

\[
\hat{H}_e(N)S(A) = -\frac{3}{8\pi}N(x(v))\sum_j (h_{j,1}(1,0) \int_0^{t_{i,j}} dt \hat{e}^b_j \left[ (\hat{e}^a_i, F^i_{ab}) \tau_i \right]^A_{B_j}) \times \frac{\partial X_{i,B}}{\partial H_{A_j}}.
\]

Applying (3.10) with \( V^a = \hat{e}^a_i \) to the part of (3.15) which depends non-trivially on the variable of integration \( t_j \), and suppressing matrix indices, we have to leading order in \( \epsilon \) that:

\[
\int_0^{t_{i,j}} dt \hat{e}^b_j \hat{e}^i_{ab} = \epsilon \int_0^{t_{i,j}} dt \hat{e}^b_j \left( \frac{A^j_i(e_j(t_j), \epsilon) \tau_i - A^i_j(e_j(t_j)) \tau_j}{\epsilon} + O(\epsilon) \right),
\]

where the point on the edge \( e_j \) at parameter value \( t_j \) is denoted by \( e_j(t_j) \) Since the integral is over a small initial part of the \( j \)th edge of coordinate size of \( O(\epsilon) \), we replace the difference between the integrated connections by their holonomies. Accordingly, suppressing matrix indices, it is straightforward to check that:

\[
- \left( \int_0^{t_{i,j}} dt \hat{e}^b_j A^j_i(e_j(t_j), \epsilon) \tau_i - \int_0^{t_{i,j}} dt \hat{e}^b_j A^i_j(e_j(t_j)) \tau_j \right)
\]

\[
= h_{e_j(t_{i,j}, 0)}(A) - h_{e_j(t_{i,j}, 0)}(A) + O(\epsilon^3)
\]

\[
= \left( (h_{e_j(t_{i,j}, 0)}(A) - h_{e_j(t_{i,j}, 0)}(A)) \right) + O(\epsilon^3).
\]

Here \( A_i \) denotes the connection given by setting \( V^a = \hat{e}^a_i \) and \( t = \epsilon \) in (3.9).

\[
h_{e_j(t_{i,j}, 0)}(A) = (g(e_j(t_{i,j}, \epsilon), \epsilon)^{-1} h_{e_j(t_{i,j}, 0)}((\phi_{t, \epsilon})(A)) g(v, \epsilon)
\]

where we have denoted \( \phi(v, \epsilon) \) in (3.9) by \( \phi_{t, \epsilon} \). Recall from (3.8) that \( g(p, \epsilon) \) in the above equation is the holonomy along the integral curve of the vector field \( V^a = \hat{e}^a_i \) from \( p \) to the point \( \phi_{t, \epsilon}(p) \). Since this vector field vanishes outside \( U_{i,j}(v) \) and since \( e_j(t_{i,j}, 0) \) is the point at which \( e_j \) intersects the boundary of \( U_{i,j}(v) \), it follows that \( g(e_j(t_{i,j}, \epsilon), \epsilon) = 1 \). Next, note that by virtue of property (c) of \( \hat{e}^a_i \), above, the integral curve of \( \hat{e}^a_i \) starting at \( v \) runs along \( e_j \). It follows from (c) that:

\[
g(v, \epsilon) = h_{e_j(\epsilon)}(0)(A).
\]

where the parameter value \( t_1 = \epsilon_1 \) is such that the coordinate distance from \( v \) to \( e_j(\epsilon) \) is \( \epsilon \) to \( O(\epsilon^3) \). Finally note that:

\[
h_{e_j(t_{i,j}, 0)}(\phi_{t, \epsilon} A) = h_{\phi_{t, \epsilon}(A)}(t_{i,j}, 0)(A).
\]

10 Note that \( \tau_i, \tau_j \) in this commutator are matrices in the spin \( j_j \) representation which labels \( e_j \) because \( \tau_k \) is in this representation.
Here $\phi_{j,E}(e_j)$ denotes the image of the edge $e_j$ by $\phi_{j,E}$ and we have used the natural parameterization of $\phi_{j,E}(e_j)$ induced by $\phi_{j,E}$ from $e_j$. Explicitly, in this parameterization the point $(\phi_{j,E}(e_j))(t_j)$ on $\phi_{j,E}(e_j)$ with parameter $t_j$ is the image $\phi_{j,E}(e_j(t_j))$ of the point on $e_j$ with parameter $t_j$.

Putting all this together in (3.17), we get:

$$
- \left( \int_0^{l_{iJ}} dt_j \epsilon_j \phi_{iJ}(t_j, \epsilon) \tau_i \right) = \left[ (h_{e_j}(\epsilon, 0))^{-1} \left( h_{\phi_{j,E}(e_j)}(t, 0) h_{e_j}(\epsilon, 0) - h_{e_j}(t, 0) \right) \right] + O(\epsilon^3)
$$

$$
= (h_{e_j}(t, 0))^{-1} h_{\phi_{j,E}(e_j)}(t, 0) h_{e_j}(\epsilon, 0) - 1 + O(\epsilon^3)
$$

$$
= h_{ij, \epsilon} - 1 + O(\epsilon^3)
$$

(3.21)

where all holonomies are evaluated with respect to the connection $A$ and where we have defined the loop $l_{iJ, \epsilon}$ as:

$$
l_{iJ, \epsilon} = (e_{ij}(1, 0))^{-1} \circ \phi_{j,E}(e_{iJ, \epsilon, 0}) \circ e_{i(\epsilon, 0)}
$$

(3.22)

where the subscripts indicate the beginning and end values of the parameter values for each of the edges in the equation.

As shown in figure 1(c) this loop starts at $v$, goes along $e_j$ till it reaches the displaced image $\phi_{j,E}(v)$ of $v$, then runs along the displaced $j$th edge $\phi_{j,E}(e_j)$ till it hits $e_j$, and then runs back along $e_j$ in the incoming direction to $v$. From footnote 10 it follows that $h_{ij, \epsilon}$ in (3.21) is a holonomy in the spin $j_j$ representation and that $1$ in (3.21) is the identity in the spin $j_j$ representation. Using (3.21) and (3.16) in (3.15) and dropping the next to leading order term deriving from the $O(\epsilon^3)$ term in (3.21), we obtain our final result:

$$
\tilde{H}_j(N)S(A) = \frac{3}{8\pi} N(x(v)) \int \sum_i \sum_{l \neq j} \left( h_{e_j}(1, 0)[(h_{ij, \epsilon} - 1), \tau^I] \right)^1_{B_j}
$$

$$
\times \frac{\partial \hat{X}_{ij} S(A)}{\partial H_{ij, \epsilon}}
$$

$$
= \frac{3}{8\pi} N(x(v)) \int \sum_i \sum_{l \neq j} \left( h_{e_j}(1, 0)[h_{ij, \epsilon}, \tau^I] \right)^1_{B_j} \frac{\partial \hat{X}_{ij} S(A)}{\partial H_{ij, \epsilon}}
$$

(3.23)

(3.24)

The deformed spin networks arising from the first and second terms in the commutator in (3.24) are depicted in figure 2.

We make the following remarks:

(a) Equation (3.10) involves the approximation of a Lie derivative by a small finite diffeomorphism minus the identity. This is valid for vector fields which are independent of the smallness parameter $\epsilon$. However, in our considerations the relevant vector fields $\{ \hat{e}_i, i = 1, \ldots, N \}$ have $\epsilon$ dependent supports so that our application of (3.10) is not, strictly speaking, valid. We feel that this sloppiness is excusable at the level of heuristics employed in our arguments. Indeed, since a similar logical jump was made in our reasonably successful treatment of toy models [12], we are inclined to treat the finite deformation maps $\phi_{j,E}$ as primary with the vector fields $\{ \hat{e}_i \}$ serving as crutches to be discarded.
Figure 1. Figure (a) shows the undeformed vertex structure at $v$. $\phi_j$ is visualised to act on this vertex structure by deforming it along its $I$th edge as shown in figure (b) wherein the displaced vertex $v_I$, and intersection point $\tilde{v}_I$ between the $J$th edge and its deformed image are labeled. Figure (c) shows the loop $l_{IJ}$ which starts from $v$, runs along the $J$th edge to $v_I$, moves along the $J$th displaced edge to $\tilde{v}_J$ and then back to $v$ along (and in the opposite direction to) $e_J$.

Figure 2. Figures (a) and (b) depict the deformed vertex structure of the states created by the action (3.24) of the Hamiltonian constraint and correspond to the first and second terms in the commutator of (3.24) for fixed $I, J \neq I$. The box represents the intertwiner $C_\lambda$ located at the position of the undisplaced vertex $v$ of $S$ (see figure 1(a)). Insertions of $\tau_j$ are represented by circles. The touching of an object with the intertwiner box indicates an index contraction between an index of the object and that of the intertwiner so that whereas both the $\tau_j$’s have index contractions with $C_\lambda$ in figure (a), in figure (b) only the one on the $I$th edge has such an index contraction.

(b) While our argumentation was based on the finite deformations generated by the ‘vector field’ part of the quantum shift, the final result (3.23) can be naturally interpreted in terms of electric field operator actions and curvature approximants (as is usually done) as follows. Recall our starting point:

$$\hat{H}(N)S(A) = \frac{1}{2} \int d^3x \epsilon_{jk}^{\hat{q}} \hat{X}^{ij}(N E_j^a \hat{g}^{-1/3}) S(A).$$

(3.25)

Up to factors of $\epsilon$ the successive action of each term in this expression can be seen to result in (3.23). The operator $\hat{g}^{-1/3}$ maps $S$ to $S_1$ in (3.23). The lapse $N$ is evaluated at $v$ and so appears as an overall factor in (3.23). The $E_j^a$ inserts a $\tau_j$ on each edge $e_I$ at $v$ in the spin $j_I$ representation which colors the edge $e_I$, together with a factor of $\hat{q}^a$. The $\tau_j$ insertion corresponds to the $X_{j,I}$ operator in (3.23). The $\hat{E}_j^a$ operator likewise inserts $\tau_k$ in the spin
(d) Despite these similarities, the constraint action (3.24) differs from the QSD constraint in two ways. First, the extraordinary edge (and the loop $I_{j,J,e}$) in (3.24) is colored by the spin label of the $J$th edge in contrast to the QSD case in which the color is fixed at $j = \frac{1}{2}$. Since the coloring is fixed by our argumentation and not put in by hand, and since the small loop holonomy term serves as a curvature approximant by virtue of our discussion in (b) above, it follows that our derivation completely fixes the spin representation ambiguity pointed out by Perez [24]. The second way in which our action differs from that of the QSD constraint is in its implementation of the curvature term in terms of holonomy composition. Here the term combines with matrix representatives of the Lie algebra so as to obtain a result in which holonomies are composed. In contrast, standard treatments, including that of the QSD constraint approximate the curvature term in terms of small loop holonomy traces with respect to matrix representatives of the Lie algebra and the result is then a multiplication by such traces instead of the holonomy composition seen in (3.24). In our view,
(e) It is straightforward to check that the expression (3.24) is gauge invariant with respect to gauge transformations of the connection. To do so note first that the expression is explicitly gauge invariant with respect to gauge transformations away from $v$ by virtue of the gauge invariance of $S_\lambda$ together with the fact that all the $\tau$ matrix insertions are only at $v$. Next consider gauge transformations which are non-trivial at $v$. Let $g \in SU(2)$ be a gauge transformation at $v$. We shall slightly abuse notation and denote its matrix representative in the spin $j_I$ representation also as $g$. It will be clear from the context as to which spin representation $g$ is valued in. Suppressing matrix indices, the holonomy $h_e$ for an incoming edge at $v$ transforms as

$$h_e \rightarrow g^{-1} h_e g,$$

for an outgoing edge $e$ at $v$ as $h_e \rightarrow h_e g$ and for a loop $l$ starting in an outgoing direction at $v$ and ending in an incoming direction at $v$,

$$h_l \rightarrow g^{-1} h_l g.$$  

We write $S_\lambda(A)$ as:

$$S_\lambda(A) = C_\lambda^{A_1 \ldots A_N} \left( \prod_{J=1}^{N} h_{eJ}(1,0)^{B_J}_{A_J} \right) S_{rest}^{B_I \ldots B_N}(A) \tag{3.26}$$

where $S_{rest}$ depends on edge holonomies of edges which do not emanate from $v$. As mentioned earlier we have assumed that all edges at $v$ in $S$ are outgoing. It is immediate to check that the gauge invariance of $S_\lambda(A)$ is then a consequence of the group invariance property of the intertwiner $C_\lambda$ (see (2.15)). In the notation of (3.26) we have that:

$$\left( h_{eJ}(1,0)[h_{IJ}, \tau^j] \right)^{B_J}_{A_J} \frac{\partial X_{I,J} S(A)}{\partial h_{eJ}^{B_J}} = \left( h_{eJ}(1,0)[h_{IJ}, \tau^j] \right)^{B_J}_{A_J} \times \left( \prod_{K \neq I \neq J} h_{eK}(1,0)^{B_K}_{A_K} \right) S_{rest}^{B_I \ldots B_N}. \tag{3.27}$$

If follows from the behaviour of holonomies and the interliner under gauge transformations that under the action of $g$ we have that

$$\left( h_{eJ}(1,0)[h_{IJ}, \tau^j] \right)^{B_J}_{A_J} \frac{\partial X_{I,J} S(A)}{\partial h_{eJ}^{B_J}} \rightarrow \left( h_{eJ}(1,0)[h_{IJ}, g \tau^j g^{-1}] \right)^{B_J}_{A_J} \times \left( \prod_{K \neq I \neq J} h_{eK}(1,0)^{B_K}_{A_K} \right) S_{rest}^{B_I \ldots B_N}. \tag{3.28}$$

Next, note that since the $j$ index of $\tau_j$ rotates like a vector index under $SU(2)$ transformations, we have that

$$g \tau_j g^{-1} = R^j_k \tau_k \tag{3.29}$$

where $R^j_k$ is an orthogonal matrix so that $R^k_h R^j_g = \delta^j_h$. Using the orthogonality property of $R$ together with (3.29) in (3.28) then implies that the action of the Hamiltonian constraint (3.24) is gauge invariant.
(f) Returning to (a) in the light of (b), we note that the precise nature of the loop $l_{IJ}$ depends on how we interpret the action of $\phi_{I\epsilon}$ on the vertex structure at $v$. In what follows we shall interpret this transformation, as in [15], as an ‘abrupt pulling’ of the vertex structure around $v$. Hereon till section 6 we restrict attention to nondegenerate vertices $v$ at which no triple of edge tangents are linearly dependent. We call such vertices as Grot–Rovelli or ‘GR’ vertices [15]. For such vertices it is straightforward to implement $\phi_{I\epsilon}$ in such a way that no spurious intersections manifest between the set of undeformed and deformed edges (for details see [15]). This restriction to GR vertices is only for simplicity and we comment on the general case in section 6.

Let $v$ be a GR vertex (see figure 1(a)). Since $\phi_{I\epsilon}$ is the identity outside a coordinate distance of $O(\epsilon)$ from $v$, the deformation of the vertex structure$^{11}$ by the action of $\phi_{I\epsilon}$ is confined to a small neighborhood of $v$. From (c) (see the discussion before (3.12)), the vertex $v$ is displaced by a coordinate distance $\epsilon$ along the edge $e_I$ to its new position $v_{I\epsilon}$ at parameter value $t_I = \epsilon$. The part of each edge $e_J$, $J \neq I$ between $v$ (at parameter $t_J = 0$) and $\tilde{v}_J$ (at parameter value $t_{J, I \neq I}$) is deformed to an edge which connects $v_{I\epsilon}$ to $\tilde{v}_J$ so that at the vertex $v_{I\epsilon}$, the $N - 1$ edges $\{e_J, J \neq I\}$ form a ‘downward pointing cone’ with ‘upward’ axis along the $I$th edge, all edges assumed to be outgoing. Due to this abrupt pulling, a kink is created at each $\tilde{v}_J$. The resulting ‘conical deformation’ is depicted in figure 1(b). Since this deformation generates these kinks it cannot be a diffeomorphism.

This brings us further away from the interpretation of this deformation as a finite transformation generated by the smooth vector field $\hat{e}_a I$. It may be possible to generate such a deformation through an appropriately non-smooth vector field but rather than get into these technical fine points, we feel that at the level of heuristics employed in our argumentation, we are justified in directly defining the action of $\phi_{I\epsilon}$ without worrying about whether it can be generated by a vector field. We note that this sort of an action seems to be exactly the sort of action contemplated for the ‘extended diffeomorphisms’ of [34]. Note that from the perspective of curvature approximants this choice of deformation leads to a very natural choice of loop $l_{IJ\epsilon}$. We shall return to this discussion in section 6.

(g) From (c), it follows that a second action of the Hamiltonian constraint would not see the new vertices created by the action of the first. As discussed in [25] this behaviour trivialises the constraint algebra. Since we are interested in a non-trivial implementation of the constraint algebra we would like the second action to be non-trivial on deformations created by the first. Such an implementation was successfully employed in our demonstration of a non-trivial anomaly free constraint algebra in the toy model context of [15]. In the next section, we improve on the action (3.24) so as to generate such deformations through the addition of higher order terms to (3.24).

4. Quantum dynamics through conical deformations

The reason that the action (3.23) results in new vertices which, by virtue of their low valence and planar nature, are expected to be degenerate is that the action yields a sum over terms each corresponding to the deformation of a single edge at a time. The deformation of the single edge $e_J$ corresponds to its ‘abrupt pulling’ along the edge $e_I$ so that there are $N - 1$ terms in the sum for fixed $I$ in (3.23). In contrast the desired conical deformation of [15]

$^{11}$Note that our discussion pertains to the visualization of the action of $\phi_{I\epsilon}$ on the entire vertex structure i.e. on all the edges emanating from $v$. In contrast, in the constraint action (3.24), each term involves the restriction of the action of $\phi_{I\epsilon}$ so visualized to one edge at a time and depicted in figure 2 (see also the first sentence of section 4 in this regard).
involves a deformation of all the $N - 1$ edges together along the remaining edge. To achieve this, the idea is to convert an expression which looks schematically like $\sum_j \delta x_j \frac{\partial F(x_1, \ldots, x_N)}{\partial x_j}$ into $F(x_1 + \delta x_1, \ldots, x_N + \delta x_N) - F(x_1, \ldots, x_N)$. In our case $x_j$ corresponds to $h_{ij}(1,0)$, $\delta x_j$ to $(h_{ij} - 1)$ and $F$ to $S_\lambda$. Since $F \sim S_\lambda$ is a product (3.26) over the $h_{ij}(1,0)$, the desired form $F(x_1 + \delta x_1, \ldots, x_N + \delta x_N)$ which accommodates all the deformed holonomies together is also a product over (deformed) holonomies.

Hence, in this section we obtain the desired conical deformation by converting the sum over deformations to a difference of products over deformations (qualitatively similar to the sum to product modification in [10]) by the addition of higher order terms to (3.23). In accordance with the schematics discussed in the previous paragraph, these terms will be higher order in the (integrated) curvature approximants $h_{ij} - 1$ and hence higher order in the area of the small loops $l_{ij}$. Recall that the group invariance property of the intertwiner (2.15) involves a product of group elements. This suggests that we attempt to replace the matrix generators of $su(2)$, namely the $\tau_j$ matrices, with group elements and use the group invariance property (2.15) to manipulate the sum over deformed holonomies in (3.23) with the $\tau_j$ so replaced, into the desired difference of products which agrees with the sum to leading order. This is the strategy we will follow below. For simplicity we restrict attention to GR vertices i.e. vertices at which no triple of edge tangents is linearly dependent.

Define $g_{j,\delta}$ as the matrix

$$g_{j,\delta} = \exp \delta \tau_j$$

(4.1)

where the spin representation label of $\tau_j$ has been suppressed and will be clear from the context below. Note that for $\tau_j$ in the $f_j$ representation and $\delta$ small enough (and independent of $\epsilon$), we have that:

$$\frac{g_{j,\delta}^{-1}h_{ij,\delta}g_{j,\delta} - h_{ij,\delta}}{\delta} = [h_{ij,\delta}, \tau_j] + O(\delta).$$

(4.2)

Using (4.2) we rewrite (3.24) as:

$$\hat{H}_s(N)SA = \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} \sum_j \sum_{j \neq i} (h_{ij}(1,0)[h_{ij,\delta}, \tau_j])_{A_j} \frac{\partial X_j S(A)}{h_{ij,\delta} B_j}$$

$$= \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} \sum_j \sum_{j \neq i} \lim_{\delta \to 0} \frac{(h_{ij}(1,0)g_{j,\delta}^{-1}h_{ij,\delta}g_{j,\delta} - h_{ij}(1,0)h_{ij,\delta})_{A_j}}{\delta} \times \frac{\partial X_j S(A)}{h_{ij,\delta} B_j}.$$  

(4.3)

It is useful to define $\hat{H}_{c,\delta}(N)SA$ as

$$\hat{H}_{c,\delta}(N)SA = \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon\theta} \sum_j \sum_{j \neq i} (h_{ij}(1,0)g_{j,\delta}^{-1}h_{ij,\delta}g_{j,\delta}$$

$$- h_{ij}(1,0)h_{ij,\delta})_{A_j} \frac{\partial X_j S(A)}{h_{ij,\delta} B_j}.$$  

(4.4)
Let the area of the small loop $l_{IJ}$ be $\alpha_{IJ}$, and let the largest of these areas for all $I, J$ be $\alpha$. We shall proceed through the addition of terms higher order in $\alpha \epsilon$ to (4.4) to get our final expression (4.9) whose $\delta \to 0$ limit will be seen to correspond to a modification of (3.24) by terms of higher order in $\alpha$.

First note that from (3.26) it follows that:

\[
\hat{X}_{IJ} S_{\lambda}(A) = C_{X}^{A_{1} \cdots A_{N}} \left( \prod_{J \neq I} h_{ij}(1,0)^{B_{j}} A_{j} \right) (\hat{X}_{IJ} h_{iJ}(1,0))^{B_{I}} A_{I} S_{\text{rest}_{B_{I} \cdots B_{N}}}(A). \tag{4.5}
\]

Next, note that

\[
h_{ij}(1,0)g_{ij}^{-1} h_{ij}, g_{ij} - h_{ij}(1,0) h_{ij} = (h_{ij}(1,0)(g_{ij}^{-1} h_{ij}, g_{ij} - 1)) - (h_{ij}(1,0)(h_{ij} - 1)) \tag{4.6}
\]

where each of the two bracketed terms on the right-hand side are of the order of the area $\alpha$.

It is then straightforward to see that the discussion in the first paragraph of this section together with the fact that $\hat{X}_{IJ} S_{\lambda}$ is a product over edge holonomies (4.5), and the $O(\alpha \epsilon)$ behaviour of the terms on the right-hand side of (4.6) implies that:

\[
\sum_{J \neq I} (h_{ij}(1,0)(g_{ij}^{-1} h_{ij}, g_{ij} - 1))^{B_{J}} A_{J} \frac{\partial \hat{X}_{IJ} S_{\lambda}(A)}{\partial h_{ij}^{B_{J} B_{J}}} = \left( C_{X}^{A_{1} \cdots A_{N}} \left( \prod_{J \neq I} (h_{ij}(1,0)g_{ij}^{-1} h_{ij}, g_{ij})^{B_{J}} A_{j} \right) (\hat{X}_{IJ} h_{iJ}(1,0))^{B_{I}} A_{I} \right.
\]

\[
\times S_{\text{rest}_{B_{I} \cdots B_{N}}}(A) - \hat{X}_{IJ} S_{\lambda}(A) + O(\alpha^{2}), \tag{4.7}
\]

\[
\sum_{J \neq I} (h_{ij}(1,0)(h_{ij} - 1))^{B_{J}} A_{J} \frac{\partial \hat{X}_{IJ} S_{\lambda}(A)}{\partial h_{ij}^{B_{J} B_{J}}} = \left( C_{X}^{A_{1} \cdots A_{N}} \left( \prod_{J \neq I} (h_{ij}(1,0)h_{ij})^{B_{J}} A_{j} \right) (\hat{X}_{IJ} h_{iJ}(1,0))^{B_{I}} A_{I} \right.
\]

\[
\times S_{\text{rest}_{B_{I} \cdots B_{N}}}(A) - \hat{X}_{IJ} S_{\lambda}(A) + O(\alpha^{2}). \tag{4.8}
\]

Using (4.7) and (4.8) in (4.4) and dropping the $O(\alpha^{2})$ terms, we redefine the action of $H_{\epsilon, \delta}(N)$ to be:

\[\text{While in this work } \alpha_{IJ}, \alpha \sim O(\epsilon^{2}), \text{ we prefer to designate the small loop area parameter as } \alpha_{\epsilon} \text{ rather than } \epsilon^{2}, \text{ in anticipation of its role in future work. For a comment in this regard, please see remark (d) of section 6.}\]
\[ \hat{H}_{c,\delta}(N)S(A) := \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon\delta} \left\{ \sum_{I=1}^{N} C_{\lambda}^{A_{1}\cdots A_{N}} \right\} \]

\[ \times \left( \prod_{j \neq I} (h_{j,I}(1,0) g_{j,I}^{-1} h_{I,I,I} g_{I,I} g_{I,0})^{B_{j}}_{A_{j}} \right) \left( \hat{X}_{j,I} h_{j,I}(1,0) \right)^{B_{j}}_{A_{j}} \]

\[ \times S_{\text{rest}B_{j}B_{k}A}(A) \]

\[ - \sum_{I=1}^{N} C_{\lambda}^{A_{1}\cdots A_{N}} \left( \prod_{j \neq I=1}^{N} (h_{j,I}(1,0) h_{I,J,I}^{c})^{B_{j}}_{A_{j}} \right) \]

\[ \times (\hat{X}_{j,I} h_{j,I}(1,0))^{B_{j}}_{A_{j}} S_{\text{rest}B_{j}B_{k}A}(A) \} . \]  

(4.9)

Expanding out the \( g_{j,I}^{-1} \) factors in orders of \( \delta \), it is straightforward to see that

(a) The term in the second line is the 0th order in \( \delta \) contribution to the term in the first line

(b) The first order in \( \delta \) term in the first line can be expanded in powers of the difference between small loop holonomies and the identity i.e. in orders of \( \alpha \epsilon \). It is straightforward to see that the leading order in \( \alpha \epsilon \) term in the latter expansion yields (3.24), and that there are also higher order terms in \( \alpha \epsilon \).

It follows from (a) and (b) that the \( \delta \to 0 \) limit of (4.9) yields a higher order in \( \alpha \epsilon \) modification of (3.24). As discussed at the beginning of this section, this modification constitutes an acceptable definition of the action of \( \hat{H}_{c}(N) \). Accordingly, we define the action of \( \hat{H}_{c}(N) \) on \( S(A) \) as:

\[ \hat{H}_{c}(N)S(A) = \lim_{\delta \to 0} \hat{H}_{c,\delta}(N)S(A) \]  

(4.10)

with \( \hat{H}_{c,\delta}(N)S(A) \) defined through (4.9). Before taking the \( \delta \to 0 \) limit it is useful to simplify (4.9) by using the gauge invariance properties of the group invariant intertwiner \( C_{\lambda} \) (see (2.15)). Note that the group invariance property of \( C_{\lambda} \) can be rewritten in the form

\[ C_{\lambda}^{A_{1}\cdots A_{N}} \left( \prod_{j \neq I} g_{j,I}^{B_{j}}_{A_{j}} \right) = C_{\lambda}^{B_{1}\cdots B_{N}} g^{-1} A_{I} \] 

(4.11)

Using (3.22) and setting \( g \equiv h_{j,I}(\epsilon_{I},0) g_{j,I} \) in (4.11) to simplify the first part of the second line of (4.9), we get:

\[ C_{\lambda}^{A_{1}\cdots A_{N}} \left( \prod_{j \neq I} (h_{j,I}(1,0) g_{j,I}^{-1} h_{I,I,I} g_{I,I} g_{I,0})^{B_{j}}_{A_{j}} \right) = C_{\lambda}^{B_{1}\cdots B_{N}} \left( \prod_{j \neq I} (h_{j,I}(1,0) g_{j,I}^{-1} h_{I,I,I} g_{I,I})^{B_{j}}_{A_{j}} \right) \]

\[ \times \left( (h_{j,I}(\epsilon_{I},0) g_{j,I})^{-1} \right)^{B_{j}}_{A_{j}} . \] 

(4.12)

\(^{13}\) Since (3.23) implies that (3.24) is of \( O(\alpha^{2}) \), (4.9) is also \( O(\alpha^{2}) \) to leading order in \( \epsilon \) with corrections of \( O(\alpha^{2}) \).
Using (3.22) and setting $g \equiv h_{e_I}(\epsilon_I, 0)$ in (4.11), to simplify the first part of the third line of (4.9), we get:

$$C_{A_1 \cdots A_N}^{A_1 \cdots A_N} \left( \prod_{j \neq I} (h_{e_I}(1, 0) h_{e_J}(1, 0))^{B_j} \right)$$

$$= C_{A_1 \cdots A_N}^{A_1 \cdots A_N} \left( \prod_{j \neq I} (h_{e_I}(1, 0) h_{e_J}(t_{e_I J}, 0))^{-1} h_{\tau_{e_I J}(e_I)}(t_{e_I J}, 0))^{B_j} \right)$$

$$\times ((h_{e_I}(\epsilon_I, 0))^{-1} A_j)_{D_j}^{-1}$$

$$= C_{A_1 \cdots A_N}^{A_1 \cdots A_N} \left( \prod_{j \neq I} (h_{e_I}(1, t_{e_I J}, 0) h_{\tau_{e_I J}(e_I)}(t_{e_I J}, 0))^{B_j} \right)$$

$$\times ((h_{e_I}(\epsilon_I, 0))^{-1} A_j)_{D_j}^{-1}.$$  \hspace{1cm} (4.13)

Using these in (4.9), we get:

$$\hat{H}_{e_I}(N) S(A) := \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} S_{\text{rest} B_j \cdots B_N(A)} C_{A_1 \cdots A_N}^{A_1 \cdots A_N}$$

$$\times \left( \sum_{I=1}^{N} \left( \prod_{j \neq I} (h_{e_I}(1, 0) g_{e_I}(h_{e_I}(t_{e_I J}, 0))^{-1} h_{\tau_{e_I J}(e_I)}(t_{e_I J}, 0))^{B_j} \right) \right)$$

$$\times ((h_{e_I}(\epsilon_I, 0))^{-1} A_j)_{D_j}^{-1} (\hat{X}_{e_I} h_{e_I}(1, 0))^{B_j}.$$ \hspace{1cm} (4.14)

Taking the $\delta \to 0$ limit of (4.14), we obtain:

$$\hat{H}_{e_I}(N) S(A) := -\frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} S_{\text{rest} B_j \cdots B_N(A)} C_{A_1 \cdots A_N}^{A_1 \cdots A_N}$$

$$\times \left( \sum_{I=1}^{N} \left\{ \sum_{k \neq I} \left( \prod_{X \neq I, J} h_{e_X}(1, t_{e_X K}, 0) h_{\tau_{e_X K}(e_X)}(t_{e_X K}, 0))^{B_k} \right) \right) \right)$$

$$\times ((h_{e_I}(\epsilon_I, 0))^{-1} A_j)_{D_j}^{-1} (\hat{X}_{e_I} h_{e_I}(1, 0))^{B_j}.$$  \hspace{1cm} (4.14)
\[ + \sum_{l=1}^{N} \left\{ \left( \prod_{j \neq l} (h_{\xi_j}(1, t_{j,l}, t_{i,j,l}, 0) h_{0_l, (e_j)(t_{i,j,l}, 0)})^{B_{ij}} h_{\xi_l}(1, 0))^{B_{lij}} \right) \times (\tau_j h_{\xi_l}(\epsilon_l, 0))^{-1} h_{\xi_l}(1, 0))^{B_{lij}} \right\} \). \tag{4.15} \]

Note that in the last line of (4.15):

\[(\tau_j h_{\xi_l}(\epsilon_l, 0))^{-1} h_{\xi_l}(1, 0))^{B_{lij}} = (\tau_j h_{\xi_l}(\epsilon_l, 0))^{-1} h_{\xi_l}(1, 0))^{B_{lij}} = (h_{\xi_l}(1, 0) \tau_j(\epsilon_l, 0))^{-1} h_{\xi_l}(1, 0))^{B_{lij}} = - f_j(j_l + 1) h_{\xi_l}(1, 0))^{B_{lij}} \tag{4.16} \]

where in the last line we have used the fact that \( \tau_j \) is the spin \( j_l \) representative of the \( j_l \)th generator of \( SU(2) \).

Using (4.16) we have the result:

\[ \hat{H}_s(N)S(A) := \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} \sum_{B_l=1}^{B_0}(A) c_{\lambda}^{D_1 ... D_N} \times \left( \sum_{l=1}^{N} \left\{ (j_l)(j_l + 1) \left( \prod_{j \neq l} (h_{\xi_j}(1, t_{j,l}, t_{i,j,l}, 0) h_{0_l, (e_j)(t_{i,j,l}, 0)})^{B_{ij}} h_{\xi_l}(1, 0))^{B_{lij}} \right) \right\} \times (t_{j,l-1}, 0))^{B_{lij}} \right\} \]

\[ - \sum_{l=1}^{N} \left\{ \sum_{j \neq l} \left( \prod_{k \neq j,l} (h_{\xi_k}(1, t_{k,l}, t_{i,k,l}, 0) h_{0_l, (e_j)(t_{i,k,l}, 0)})^{B_{ij}} h_{\xi_l}(1, 0))^{B_{lij}} \right) \times (h_{\xi_l}(1, 0) \tau_j(\epsilon_l, 0))^{-1} h_{\xi_l}(1, 0))^{B_{lij}} \right\} \]. \tag{4.17} \]

We write the action (4.17) in the concise form:

\[ \hat{H}_s(N)S(A) := \frac{3}{8\pi} N(x(v)) \frac{1}{\epsilon} \sum_{l=1}^{N} \left( j_l(j_l + 1) S_{\lambda, j,l} - \sum_{l=1}^{N} \sum_{j \neq l} S_{\lambda, j,l} \right) \tag{4.18} \]

where we have defined the deformed states \( S_{\lambda, j,l}, S_{\lambda, j,l} \) to be:

\[ S_{\lambda, j,l} := c_{\lambda}^{D_1 ... D_N} \left( \prod_{j \neq l} (h_{\xi_j}(1, t_{j,l}, t_{i,j,l}, 0) h_{0_l, (e_j)(t_{i,j,l}, 0)})^{B_{ij}} h_{\xi_l}(1, 0))^{B_{lij}} \right) \times S_{\text{rest } B_1 ... B_N}, \tag{4.19} \]
Figure 3. The structures of the conical deformed states $S_{\lambda,I,\varepsilon}$, $S_{\lambda,I,\varepsilon}$ near the vertex $v$ of $S$ are depicted above. Figure (a) is identical to figure 1(b) and depicts the vertex structure of $S_{\lambda,I,\varepsilon}$. The intertwiner $C_\lambda$ sits at the displaced vertex $\tilde{v}_I$. The circles in figure (b) depict insertions of $\tau_j$ at the original vertex $v$. The $I$th edge at $\tilde{v}_I$ starts out oriented opposite to its counterpart in the undeformed state $S$ (see figure 1(a)) till $v$ where there is an insertion of $\tau_j$. It then traces out the same trajectory as its counterpart in $S$. The insertion structure on the undeformed part of $J$th edge between $\tilde{v}_J$ and $v$ is similarly depicted: the segment starts from $\tilde{v}_J$ moves opposite to its counterpart in $S$ to $v$ where there is an insertion and then runs back.

$$S_{\lambda,I,\varepsilon} := C_{\lambda}^{D_1 \ldots D_N} \left( \prod_{K \neq I,J} h_{ek}(1, t_{e,K}, (t_{e,K}, 0))^{b_k} D_k \right) \times (h_{\tau_j}(1, 0) \tau_j (h_{\tau_j}(t_{\tau_j,J,0}, 0))^{-1} D_j^{-1} S_{\text{rest} \ b_1 \ldots b_N}. \right)$$

Equation (4.20) is our final result. We make the following remarks.

1. Conically deformed states: $S_{\lambda,I,\varepsilon}, S_{\lambda,I,\varepsilon}$ correspond to states in which the vertex deformations in the vicinity of the vertex $v$ which are defined through (4.19) and (4.20). As shown in figure 3, both (4.19) and (4.20) describe deformations which create a new conically deformed vertex of the type encountered in the $U(1)^3$ toy model in [15]. Figure 3(a) depicts the ‘electric diffeomorphism’ type deformation [15] described by (4.19) wherein the original vertex at $v$ is displaced to its new position along the $I$th edge and the edge tangents at $v$ are conically deformed to yield the vertex structure at the displaced vertex, resulting in the state $S_{\lambda,I,\varepsilon}$. The vertex is actually $N+1$ valent but using the terminology of [15] we shall refer to it as an $N$ valent singly CGR vertex. A CGR vertex is one in which a pair of edge tangents are collinear and which is GR with respect to the set of edge tangents obtained by dropping one of this collinear pair. The state $S_{\lambda,I,\varepsilon}$ is not itself a spin network state because the initial parts of the $I,J$ edges are retraced after insertions of $\tau_i$.

2. Gauge invariance: it is straightforward to see, using an analysis similar to that of remark (e) at the end of section 3, that (4.19) and (4.20) are gauge invariant expressions. The gauge invariance of $S_{\lambda,I,\varepsilon}$ may also be inferred directly from figure 3(a): the figure shows that $S_{\lambda,I,\varepsilon}$
is obtained from \( S_\lambda \) by a displacement of the vertex structure near \( v \) in \( S_\lambda \) by \( \phi_{lj} \), thus implying that the gauge invariance of \( S_{\lambda, l, j} \) follows immediately from that of \( S_\lambda \). For \( S_{\lambda, l, j} \) gauge invariance under gauge transformations supported away from \( \phi_{lj} \), \( v \) is immediate. For gauge transformations at \( \phi_{lj} \), gauge invariance follows from group invariance of \( C_\lambda \). For gauge transformations at \( v \) gauge invariance follows from an argument identical to that in remark (e) of section 3 based on (3.29).

(3) **Constraint action and deformed states in orders of \( \alpha \):** from footnote 13, the combination of the two terms in the curly brackets of (4.17) is of order \( \alpha \). Consequently the right-hand side of (4.18) is also of order \( \alpha \). It is useful for our considerations in section 5 to display the constraint action in a form which explicitly segregates the order \( \alpha \) contributions to \( S_{\lambda, l, j} \). The relevant computations are straightforward and we relegate them to appendix A. The desired form of (4.18) turns out to be:

\[
\hat{H}_c(N)S(A) := \frac{3}{8\pi} N(x(v)) \left( \sum_{i=1}^{N} \frac{j_i(j_i + 1)(S_{\lambda, l, j} - S_\lambda)}{\epsilon} \right) - \sum_{i=1}^{N} \left\{ \left( \sum_{j \neq l} S_{\lambda, j, j} \right) - j_i(j_i + 1)S_\lambda \right\} \tag{4.21}
\]

As shown in the appendix, each of the curly brackets in (4.21) is of \( O(\alpha) \). Note that equation (4.21) is exactly the same as (4.17) because we have only added and subtracted the \( S_\lambda(A) \) term. The form of the constraint action (4.21) will be of use in section 5 wherein we further modify the constraint action by terms which are higher order in \( \alpha \).

(4) **Comments on anomaly free-ness and propagation:** the form of the constraint action (4.21) is reminiscent of the ‘\( U - 1 \)’ form in the \( U(1)^3 \) toy model [15]. This ‘\( U - 1 \)’ form played a crucial role in the demonstration of the anomaly free property of the constraint action in [15]. Therefore one may try to repeat the analysis of [15] and attempt to show that the constraint action (4.21) is consistent with an anomaly free constraint algebra. Before doing so, it is of interest to modify (4.21) through the addition of higher order terms so as to bring the constraint action into an even more optimal form to which the general methodology of [15] could be applied to investigate the issue of anomaly freeness. We do this in the appendix C. Since the modified deformations have no significant qualitative difference with those generated by the action (4.21), we do not display the modified constraint action here; it is of potential interest only for attempts at tackling the anomaly free issue along the lines of [15].

As mentioned in section 1, ongoing preliminary efforts to apply the considerations of [15], developed in the \( U(1)^3 \) setting, to the \( SU(2) \) case of interest here meet with significant technical complications. A brief description of these complications follows and may be skipped on a first reading. As mentioned in (1) above, the action of the constraint on the GR vertex \( v \) creates the states \( S_{\lambda, l, j} \) with a CGR vertex. The action of the constraint can be generalised in a straightforward manner so as to act on this CGR vertex, it then turns out that (a) among the set of deformed states generated by repeated actions of the constraint, there is a subset of states with vertices of increasing valence and (b) these states complicate attempts to demonstrate the existence of an anomaly free constraint algebra. In [15], this phenomenon of increasing valence was negated by a more involved choice of constraint action on CGR vertices through the use of technical tools called *interventions*. A generalisation of the application of interventions to the \( SU(2) \) case seems to lead to a very complicated and baroque action for the constraint. We also note that the action discussed in [15], while anomaly free in the specific sense defined in that work, is not consistent with 3D propagation [21]. It is then likely that any intervention based generalisation of the considerations of [15] to the \( SU(2) \) case would also suffer from this lack...
of adequate propagation. Hence in section 5 we further modify the action (4.21) through higher order terms so as to obtain a constraint action which, we believe, stands a better chance of displaying consistency with the properties of an anomaly free constraint algebra and propagation together.

Since the requirement of consistency with propagation forms an important part of our motivation to define the constraint action in the next section, we end this section with a discussion of the notion of propagation. Our discussion will be intuitive and succinct; for details the reader is urged to consult references [20, 21] and, especially, [22]. Our discussion is based primarily on the work [22] in progress with Thomas Thiemann. Due to limitations of space and the scope of this paper, our argumentation maybe slightly opaque but we hope to give a flavour of the key ideas.

Given a spin network state, the action of a constraint at a vertex deforms the state in the immediate vicinity of this vertex to yield a ‘perturbed’ spin net state. If the quantum dynamics is such that this perturbation moves to distant (in terms of graph connectivity) vertices of the original spin net state, the quantum dynamics will be said to encode long range propagation and the initially perturbed state and the final state can be said to be related by long range propagation. If the movement of the perturbation is confined to a set of nearby vertices, then the propagation will be said to be short range. Propagation is a property of physical states as follows. Physical states lie in the joint kernel of the quantum constraints and are constructed as sums of infinitely many spin net states. For mathematical precision the sum is over spin network bra states so that physical states lie in the (algebraic) dual space. A physical state \( \Psi \) is then written as a sum over spin net summands:

\[
\Psi = \sum_s \alpha_s \langle \bar{s} | .
\]

(4.22)

Given one such spin net summand, if there exists another summand which is related to the first by long range propagation then we shall say that the physical state encodes long range propagation. If there exist a sequence of summands, each element of which is related to the next by propagation, we shall call this sequence a propagation sequence. The larger the number of distinct propagation sequences, the more vigorous is said to be the propagation encoded by the physical state. As emphasized by Smolin [19], propagation sequences cannot be generated by repeated actions of the constraints on a given spin network state due to the ultralocal action of the Hamiltonian constraint i.e. the (bra correspondents of) the spin networks generated by such actions have independent webs of deformed structure around each vertex with no inter-vertex correlation of these deformed structures. Nevertheless, physical states can encode propagation by virtue of the following. A physical state \( \Psi \) must satisfy the condition

\[
\lim_{\epsilon \to 0} \Psi(\hat{H}_\epsilon(N)|s\rangle) = \lim_{\epsilon \to 0} \sum a_s \langle \bar{s} | \hat{H}_\epsilon(N) | s \rangle = 0
\]

(4.23)

for all spin nets \( |s\rangle \) and all \( N \). Typical solutions to the constraints are linear combinations of ‘children’ states which are (the bra correspondents of states) obtained from the action of the constraints on a restricted set of ‘parent states’.

Consider a physical state which has, as a summand, (the bra correspondent of) a child state \( s' \) obtained by the action of the constraint at vertex \( A \) of a parent state \( s_1 \). Now, the constraint action may be such that the same child \( s' \) can also be generated by the action of the constraint at a neighbouring vertex \( B \) of a distinct parent \( s_2 \) i.e. one whose colorings and vertex

\[14\] By ‘same’, we mean identical up to the action of diffeomorphisms; elaboration of this point is beyond the scope of this paper.
structure differs from that of the first parent (only) near $A$, $B$. Now the constraint action on $s_2$ at $B$ will also generate children $\{s''\}$ other than $s'$ and some of these children will typically not be generated by the constraint action on $s_1$ at $A$. The coefficients of the (bra counterparts of the) children $s'$, $\{s''\}$ in the linear combination (4.22) which defines the putative physical state are constrained by equation (4.23) evaluated with lapses with support only in a small vicinity of $A$ (with $s = s_1$ in (4.23)) and of $B$ (with $s = s_2$). These equations at $A$ and $B$ can, in principle, present themselves as non-trivially coupled by virtue of the same child $s'$ being created at both vertices. In this manner the existence of children created at $A$ as summands for $\Psi$ can determine the existence of children created at $B$ as summands for $\Psi$ and these children together can form parts of a propagation sequence. The argument can be repeated for the vertex $B$ and its neighboring vertex $C \neq A$ as long as there is a child created both from the constraint action at $B$ and at $C$. Thus, propagation is entirely dependent on the existence of sufficiently many children of non-unique parentage.

In this regard the constraint action of this section (putatively modified through interventions as indicated above) is expected to typically create $N$ valent (in the sense of [15]) child vertices from an $N$ valent parental vertex. Hence in the generic case of children of parent spin networks with vertices of distinct valence, there is a unique parental vertex for a given child and the lack of sufficient non-uniqueness in parentage is expected to inhibit propagation. The constraint action of the next section, however, does generate children some of which have child vertices of valence 2 or 3 independent of the parental vertex valence. It is then highly plausible that some of these children can be typically of non-unique parentage. This is why we expect the dynamics of the next section to be consistent with propagation. We leave an explicit demonstration of this consistency (along the lines of [22]) for further work.

5. Mixed action quantum dynamics

In section 5.1 we define a new constraint action by modifying the second set of contributions in (4.21) through the addition of higher order terms in $\alpha$, to $S_{\lambda,j,j,\epsilon}$. The resulting set of deformed spin nets are expected to facilitate propagation for reasons which we discuss in section 6. In section 5.2 we compute the commutator of two such Hamiltonian constraint actions. We also construct a particular operator correspondent of the Poisson bracket between the corresponding pair of classical Hamiltonian constraints. We find that the deformed spin nets generated by this operator correspondent of the constraint Poisson bracket are closely related to those generated by the constraint operator commutator. We discuss this property in relation to that of an unconventional view of the anomaly free property in section 5.3. As in section 4 we restrict attention to the constraint action on GR vertices i.e. vertices at which no triplet of edge tangents is linearly dependent. Additionally, in sections 5.2 and 5.3 we shall use density weight one constraints.

5.1. The Hamiltonian constraint action

Starting from (4.20), we have that:

$$S_{\lambda,j,j,\epsilon} := C_{\lambda}^{D_1, \ldots , D_N} \left( \prod_{k \neq ij} h_{\epsilon k}(1, 0) h_{\epsilon k}^{-1} (h_{\epsilon k}(\epsilon, 0))^{-1} D_{\epsilon k}^{-1} \right) \times (h_{\epsilon k}(1, 0) h_{\epsilon k}^{-1} (h_{\epsilon k}(\epsilon, 0))^{-1} B_{\epsilon k}^{-1} D_{\epsilon k}^{-1} (h_{\epsilon k}(1, 0) h_{\epsilon k}^{-1} B_{\epsilon k})^{-1} \times ((h_{\epsilon k}(\epsilon, 0))^{-1} B_{\epsilon k}^{-1} D_{\epsilon k}^{-1} (h_{\epsilon k}(1, 0) h_{\epsilon k}^{-1} B_{\epsilon k})^{-1}$$
where we have used (3.22) in the first equation and (2.15) in the second. Next we set $h_{ij\epsilon\tau} = (h_{ij\epsilon\tau} - 1) + 1$ and use the fact that $h_{ij\epsilon\tau} - 1 = O(\alpha)$ to expand (5.1) in $\alpha$. We get:

$$S_{\lambda, I, J, \epsilon} := \left\{ \begin{array}{c} S_{\text{rest} B_1 \ldots B_N} C^{D_1 \ldots D_N}_\lambda \\
\times \left( \prod_{K \neq I, J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
+ \sum_{K \neq I, J} \left( \prod_{M \neq K \neq I \neq J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
\times (h_{ij\epsilon\tau} - 1)^{\delta_{ij}}_{DK} \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
+ \left( \prod_{K \neq I, J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \right\} + O(\alpha^2). \quad (5.2)$$

We shall define a new Hamiltonian constraint action from (4.21) by neglecting the $O(\alpha^2)$ contribution to $S_{\lambda, I, J, \epsilon}$ in (5.2). It is convenient to abuse notation slightly and continue to denote the right-hand side of (5.2) without the $O(\alpha^2)$ term by $S_{\lambda, I, J, \epsilon}$. Using this notation, it is straightforward to see that (5.2) (without the $O(\alpha^2)$ term) can be expanded to give:

$$S_{\lambda, I, J, \epsilon} := S_{\text{rest} B_1 \ldots B_N} C^{D_1 \ldots D_N}_\lambda \left\{ -(N - 2) \left( \prod_{K \neq I, J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
\times (h_{ij\epsilon\tau} - 1)^{\delta_{ij}}_{DK} \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
+ \sum_{K \neq I, J} \left( \prod_{M \neq K \neq I \neq J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
\times (h_{ij\epsilon\tau} - 1)^{\delta_{ij}}_{DK} \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \\
+ \left( \prod_{K \neq I, J} h_{ek}(1, 0)^{\delta_{ek}}_{DK} \right) \left( h_{e}(1, 0) \tau J (h_{e}(1, 0) \tau J)^{\delta_{J}}_{D_l} \right) \right\}. \quad (5.3)$$
Figure 4. Figures 4(a) and (b) depict the deformed vertex structure of $S_{\lambda IJL}$, $S_{\lambda IJL}$, respectively. The box represents the intertwiner $C_\lambda$ located at the position of the undisplaced vertex $v$ of $S$ (see figure 1(a)). Similar to figure 3 insertions of $\tau_j$ are represented by circles. The touching of an object with the intertwiner box indicates an index contraction between an index of the object and that of the intertwiner so that whereas both the $\tau_j$'s have index contractions with $C_\lambda$ in figure (a), in figure (b) only the one on the Ith edge has such an index contraction.

The new constraint action is then defined to be:

$$\hat{H}_\epsilon(N)S(A) = \frac{3}{8\pi}N(x(v)} \left( \sum_{I=1}^{N} \frac{j(j+1)(S_{\lambda IJL} - S_{\lambda})}{\epsilon} \right)
- \sum_{I=1}^{N} \left\{ \sum_{I \neq J} S_{\lambda IJL} - j(j+1)S_{\lambda} \right\}.$$

(5.4)

with $S_{\lambda IJL}$ given by (5.3) rather than (4.20).

Next, note that the infinitesimal form of the group invariance property (2.15) implies that:

$$\sum_{I=1}^{N} C_{\lambda} B_{1\ldots N} \sum_{I \neq J}^{N} \left( \prod_{K \neq I,J} \right) \tau \left( \begin{array}{c} \alpha \\ \beta \\ \end{array} \right) = 0.$$

(5.5)

Using (5.5) to simplify the sum over $I \neq J$ of the first line of (5.3), we obtain:

$$= -\left( N - 2 \right) \sum_{I \neq I}^{N} S_{\text{rest}} B_{1\ldots N} C_{\lambda}^{D_1\ldots D_N} \left( \prod_{K \neq I,J} \right) h_{\epsilon} (1,0)^{\beta_K D_K}$$

$$\times \left( h_{\epsilon} (1,0)^{\beta_I D_I} \right)$$

$$= -\left( N - 2 \right) \sum_{I \neq I}^{N} S_{\text{rest}} B_{1\ldots N} C_{\lambda}^{D_1\ldots D_N} \left( \prod_{K \neq I,J} \right) h_{\epsilon} (1,0)^{\beta_K D_K}$$

$$\times \left( h_{\epsilon} (1,0)^{\beta_I D_I} \right)$$

$$= -\left( N - 2 \right) j(j+1)S_{\lambda}.$$

(5.6)
Defining:

\[ S_{\lambda,(1)IJ,K,\epsilon} = S_{\text{test}} R_1 R_2 C_{D_1 D_2} \left( \prod_{M \neq K \neq J \neq I} h_{ij} (1, 0) D_{M} \right) \]
\[ \times (h_{ij}(1, 0) h_{ij}(1, 0) D_{I}) (h_{ij}(1, 0) D_{J}) \]
\[ \times (h_{ij}(1, 0) \tau_{D_{I}}) \]
\[ S_{\lambda,(2)IJ,\epsilon} = S_{\text{test}} R_1 R_2 C_{D_1 D_2} \left( \prod_{k \neq I J} h_{ij}(1, 0) D_{K} \right) \]
\[ \times (h_{ij}(1, 0) \tau_{D_{I}}) \]

we have, finally:

\[ \mathcal{H}_N(S(A)) := \frac{3}{8 \pi N(x(t))} \left( \sum_{j=1}^{N} \frac{j(j+1)(S_{\lambda,J,\epsilon} - S_{\lambda})}{\epsilon} \right) \]
\[ = \frac{N}{\epsilon} \left( \sum_{j=1}^{N} \frac{\sum_{k \neq IJ} S_{\lambda,(1)IJ,K,\epsilon} + (\sum_{j \neq IJ} S_{\lambda,(2)IJ,\epsilon} - j(j+1)(N-1)S_{\lambda})}{\epsilon} \right) \]

We make the following remarks:

(a) We depict the vertex deformations in \( S_{\lambda,(1)IJ,K,\epsilon}, S_{\lambda,(2)IJ,\epsilon} \) in figure 4. The displaced vertex in each of the states \( S_{\lambda,(1)IJ,K,\epsilon}, S_{\lambda,(2)IJ,\epsilon} \) is planar with valence at most 3. The kinks which are created by the deformation are also planar and at most trivalent. The states \( S_{\lambda,(1)IJ,K,\epsilon}, S_{\lambda,(2)IJ,\epsilon} \) can be expanded in terms of spin networks. It follows that the displaced vertex and the kinks created by the deformation in each of these spin networks are also planar, at most trivalent and, from (b) below, gauge invariant. Hence these vertices have vanishing volume. The inverse metric determinant whether defined through a Thiemann like trick [5] with the Ashtekar–Lewandowski volume operator [35] or a Tikhonov regularization (see footnote 5), vanishes and these kinks and the displaced vertex are degenerate so that a second constraint action does not see them.

(b) Gauge invariance of the states \( S_{\lambda,(1)IJ,K,\epsilon}, S_{\lambda,(2)IJ,\epsilon} \) follows immediately from an argumentation along the lines of that employed in remark (e), section 3.3.

(c) The lemma in the appendix B may be applied to the retraced part of the Jth edge in \( S_{\lambda,(2)IJ,\epsilon} \) with the implication that every spin network in the spin network decomposition of \( S_{\lambda,(2)IJ,\epsilon} \) has this part of the Jth edge labeled by spin 1. The initial part of the Jth edge of a spin network in the spin network decomposition of \( S_{\lambda,(2)IJ,\epsilon} \) can carry a spin label \( j \) in the range \( |j_f - j_i| \) to \( j_f + j_i \) in accordance with the Clebsch–Gordan decomposition. If \( j = 0 \) is permissible, the corresponding spin network has an \( N - 1 \) valence vertex at \( v \). Spin nets with \( j \neq 0 \) have an \( N \) valent vertex at \( v \). For \( S_{\lambda,(1)IJ,K,\epsilon} \) similar conclusions hold except that \( J \) is replaced by \( K, N \) by \( N - 1, \) and \( N - 1 \) by \( N - 2 \). If the vertex \( v \) is non-degenerate it can support a second action of the Hamiltonian constraint but this action does not contribute to the commutator as it acts at the same vertex as the first constraint.
5.2. A curious property of the commutator

In this section and the next, we shall consider unit density weight constraints. As indicated in remark (c) of section 3.3, the considerations of that section go through for any density weight of the Hamiltonian constraint. It is straightforward to check that this holds for our argumentation in sections 4 and 5.1 as well, the only change being in the overall power of $\epsilon$ in the expression for the constraint action and a possible change in the intertwiner $C_\lambda$ which is obtained from the intertwiner $C$ of the state $S$ through the action of the power of the inverse metric determinant operator which governs the density weight of the constraint. In particular, for the case of density weight 1, all explicit factors of $\epsilon$ cancel so that there is not such factor at all, consistent with the observations of [5]. Moreover, the lapses are scalars and no coordinate patch is required for their evaluation.

We first compute the commutator between two constraint actions, each given by (5.9) i.e. we compute, with $\epsilon_1 \ll \epsilon$, the object:

$$[\hat{H}(M), \hat{H}(N)]_{\epsilon_1} S(A) := (\hat{H}_{\epsilon_1}(M)\hat{H}_\epsilon(N) - \hat{H}_\epsilon(N)\hat{H}_{\epsilon_1}(M)) S(A).$$

(5.10)

For simplicity we assume that $N, M$ are such that the only nondegenerate vertex in their support is $v$. Terms which involve the evaluation of both lapses at $v$ vanish by antisymmetry. It follows that the contributions from the commutator come from the action of a second constraint on deformed states of the type $S_{\lambda, I, \epsilon}$ created by the first:

$$[\hat{H}(M), \hat{H}(N)]_{\epsilon_1} S = \left( \frac{3}{8\pi} \right)^2 \sum_j j_1 (j_1 + 1)(N(v)M(v) - M(v)N(v))$$

$$\times \left( \sum_{I_1} j_1 (j_1 + 1)(S_{(M, I, \epsilon_1)(\lambda, I, \epsilon)} - S_{\lambda, I, \epsilon}) - \sum_{I_1} \sum_{I_1 \neq I_1, k_1 \neq I_1} S_{(M, I, \epsilon_1)(M, I_1, k_1, I_1)} \right)$$

$$+ \sum_{I_1} j_1 (j_1 + 1)(N - 1) S_{\lambda, I, \epsilon}.$$  

(5.11)

Our (more or less obvious) notation is as follows. We have slightly abused notation and used the subscript $\lambda$ to denote the state $S_{\lambda, I, \epsilon}$ with intertwiner $C_\lambda$ despite the fact that the intertwiner change is due to the action of $\hat{q}^{\frac{1}{2}}$ here as opposed to $\hat{q}^{\frac{1}{2}}$ in (5.9). As we have seen, the only type of first transformation to contribute is $(\lambda, I, \epsilon)$ which sends $S$ to $S_{\lambda, I, \epsilon}$. Recall that $S_{\lambda, I, \epsilon}$ is obtained by a displacing the vertex structure of $S$ at $v$ to the new vertex, here denoted by $v_{\epsilon_1}$, the intertwiner at $v_{\epsilon_1}$ now being $C_\lambda$. This intertwiner changes due to the action of $\hat{q}^{\frac{1}{2}}$ coming from the second constraint. Since the change in the intertwiner depends on the conically deformed vertex structure at $v_{\epsilon_1}$, we denote the resulting intertwiner by $C_{\lambda, I, \epsilon}$. The notation in square brackets indicate that the transformation specified by the second round bracket is followed by that specified by the first bracket. The second transformation can be any of:

(a) $(\lambda', I_1, \epsilon_1)$ which displaces the vertex structure around $v_{\epsilon_1}$ of $S_{\lambda, I, \epsilon}$ along its $I_1$th edge by the amount $\epsilon_1$ and changes the intertwiner $C_\lambda$ at this displaced vertex to $C_{\lambda', I, \epsilon}$. 

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(b) \((\lambda^i, (1), I_1, J_1, K_1, \epsilon_i)\) which changes the intertwiner \(C_\lambda\) at \(v_{I\epsilon}\) of \(S_{\lambda,J} \) to \(C_{\lambda', \lambda}\) and then alters the vertex structure around \(v_{I\epsilon}\) in accordance with figure 4(a), or

c) \((\lambda^i(2), I_1, J_1, \epsilon_i)\) which changes the intertwiner \(C_\lambda\) at \(v_{I\epsilon}\) of \(S_{\lambda,J} \) to \(C_{\lambda', \lambda}\) and then alters the vertex structure around \(v_{I\epsilon}\) in accordance with figure 4(b).

From the appendix D (see equation (D.32)), the operator correspondent \(\hat{O}(M, N)\) of the Poisson bracket between \(H(M)\) and \(H(N)\) can be regulated so as to act on \(S\) as:

\[
i\hat{O}_\lambda(M, N)S(A) = \left(\frac{3}{4\pi}\right)^2 \omega_0 \sum_{j=1}^{N} \sum_{i,j=1}^{N} \frac{j(j+1)(S_{\eta,\lambda,J_i} - S_{\eta,\lambda})}{\kappa} - \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \sum_{k=1}^{N} S_{\eta,\lambda}(i,j,k) - S_{\eta,\lambda}(i,j,k)\right) + \left(\sum_{i=1}^{N} S_{\eta,\lambda}(i,m,n,h,\lambda) - j(j+1)(N-1) S_{\eta,\lambda}\right).
\]

(5.12)

Here \(\omega_0\) is a quantity that depends solely on the lapses and \(\kappa\) is a constant which characterises the small loops \(I_{[K,r,\kappa]}\) whose value we are free to choose (see the appendix for details). From the appendix it turns out that each of the states, \(S_{\eta,\lambda,J_i}, S_{\eta,\lambda}(1,1), S_{\eta,\lambda}(1,2), S_{\eta,\lambda}(2,1), S_{\eta,\lambda}(2,2)\), are, respectively, closely related to the states \(S_{\eta,\lambda,J_i}(1,1), S_{\eta,\lambda}(1,2), S_{\eta,\lambda}(2,1), S_{\eta,\lambda}(2,2)\) in (5.11). In particular if we repeat the considerations of this work with \(\phi_{\lambda,\epsilon}\) chosen to be a diffeomorphism, each of the former states turns out, respectively, to be a diffeomorphic image of each of the latter.

5.3. An unconventional view of the quantum constraint algebra

Note that the lapse prefactor \(N(v)M(v_{I\epsilon}) - M(v)N(v_{I\epsilon})\), in (5.11) vanishes as \(\epsilon \to 0\). This vanishing is tied to the continuity of the lapse functions. Under the same assumption, it turns out that \(\omega_0\) also vanishes so that assumption 1 of the appendix D is satisfied with \(\omega_0 = 0\).

Let us now assume that the lapses can be discontinuous at isolated points. Let us also assume for the purposes of this section that the transformations \(\phi_{\lambda,\epsilon}\) are diffeomorphisms instead of conical transformations. Under these assumptions and setting \(\kappa^{-1} = \frac{\sum_{j=1}^{N} j(j+1)}{\omega_0}\), we find that (a) the \(\epsilon \to 0\) limit of the action of any diffeomorphism invariant distribution on the commutator (5.11) is the same as its action on (5.12) and (b) this limit is non-trivial whenever the vertex \(v\) coincides with an isolated point of discontinuity of at least one of the lapses.

This may possibly be viewed as an illustration of a non-trivial anomaly free action. Since this is an unconventional point of view, we do not wish to unduly embitter it and restrict ourselves to the following remarks:

(a) Note that the commutator is non-trivial only if the displaced vertex in the state \(S_{\lambda,J_i}\) is non-degenerate. The fact that the deformed vertex structure is diffeomorphic to the undeformed one implies that the non-degeneracy of the former guarantees that of the latter if the inverse determinant is defined in a diffeomorphism covariant way, as is usually done.

(b) The equality can be viewed as the equality of continuum limits in an operator topology defined by a family of semi-norms, each semi-norm defined by a pair of elements, the first being a diffeomorphism invariant distribution and the second a spin network state of the type \(S\) (see [12] for this viewpoint). Alternatively one could try to weaken the uniform Rovelli Smolin operator topology [5, 37] to allow a non-uniformity with respect to the

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15 This is expected on general grounds [15] for density weight 1 constraints and is the reason that we have considered the higher density constraint in earlier sections.
lapse evaluation. It would be of interest to make the notion of this ‘weakened’ topology precise. We also feel that a habitat based demonstration should be possible and we leave this for future work.

(c) If we implement $\phi_{I,\epsilon}$ as a conical deformation, the non-degeneracy condition is preserved for the displaced vertex $S_{I,\epsilon}$ if we use the Rovelli–Smolin volume in conjunction with the Tikhonov regularization (see footnote 5 and [33]). The conical deformation is an example of an ‘extended diffeomorphism’ [34]. Hence it seems likely that the non-trivial equality, $\lim_{\epsilon \to 0} \lim_{\epsilon_1 \to 0} \Phi([\hat{H}(M), H(N)]_{I,\epsilon}, S) = i \lim_{\epsilon \to 0} \Phi(\hat{O}(M, N)S)$, holds for spin network states $S$ of the type considered in section 5.1 and distributions $\Phi$ which are invariant with respect to these singular diffeomorphisms if we use the Rovelli–Smolin volume operator.

(d) Note that the extended diffeomorphisms of [34] cannot be generated by smooth vector fields; some amount of non-smoothness is required of any putative generator. From this point of view, the use of discontinuous lapse functions does not seem strange; if non-smooth shifts can be contemplated why not discontinuous lapses? On the other hand, also note that the GR condition on $S$ is not preserved by extended diffeomorphisms.

To summarise: the use of unit density constraints avoids many of the technical complications which arise in the consideration of higher density constraints. However for smooth lapses the commutator between two Hamiltonian constraints as well as the operator correspondent of their Poisson bracket trivialise. Trivialization can be avoided by the unconventional assumption of lapses with discontinuities. Then, (a) if these discontinuities are at isolated points, (b) if we define a valid but unconventional regulation of the operator correspondent of the constraint Poisson bracket, and (c) we implement the $\{\phi_{I,\epsilon}\}$ as diffeomorphisms, it is possible to define a non-trivial continuum limit of this regularization so that it reproduces the $(-i)$ times the continuum limit of the commutator between the constraint operators.

6. Discussion

In this work we have attempted to combine the standard construction methods of LQG [5, 37] together with geometric insights into the classical dynamics of Euclidean general relativity [1] in order to construct an action of the Hamiltonian constraint operator. The geometric interpretation of the classical dynamics derives from a rewriting of the Einstein equations in which time evolution of phase space fields can be understood as Lie derivatives, suitably generalised, with respect to a Lie algebra valued, phase space dependent, spatial vector field called the electric shift [1]. Our derivation accords a central role to the corresponding electric shift operator. Folding the action of the electric shift operator into that of the constraint operator results in the action (3.24) of section 3. The derivation of this action is our main result16. Its simplicity is tied to a crucial step in our derivation in which we combine the action of the curvature term with that of the electric field operators in a delicate manner without recourse to the relatively brutal substitution of curvature components by traces of holonomies with respect to $\tau_I$. In this regard our treatment of the crucial curvature term is closer to that of Thiemann’s implementation of the ‘right-hand side’ of the constraint commutator, $\hat{O}(M, N)$, in [37, 39] than to standard treatments of this term in the constraint operator itself.

16 Note that while we derived (3.24) under the assumption of lapse support around a single vertex, the derivation trivially extends to the case of arbitrary lapse support. The constraint action is then a sum over contributions of the type (3.24), one for every non-degenerate vertex of $S$. Similarly the results of sections 4 and 5 also trivially extend to arbitrary lapses by summing over all vertex contributions.
The combined action of the curvature and electric field dependent operators naturally brings the deformations \( \{ \phi_I, \epsilon \} \) to the fore, each \( \phi_I \), defining a deformation of the vertex structure at \( v \) along the edge \( e_I \) emanating therefrom. However, the resulting action (3.24) can also be viewed in terms of a specific choice of holonomy approximants to the curvature of the connection. In this specific choice, these approximants are labeled by spin quantum numbers which are tailored to the spin labels of the edges of the spin network state being acted upon. The qualitative reason for this ‘dynamical’ choice of spin labels is as follows. The transformations \( \phi_I, \epsilon \) deform the edges at the vertex of interest without changing their spin labels. In order to implement such a deformation through the action of holonomy operators, it would be necessary to eliminate the original edges by (inverse) holonomy multiplication and re-introduce their (deformed) images also by holonomy multiplication. Clearly this would need these regulating holonomies to be labeled by the spin labels of the edges. Therefore, while in our derivation this feature can be traced to our specific visualization of \( \phi_I, \epsilon \), the feature itself (and the consequent elimination of Perez’s spin ambiguity [24]) seems to be quite robust and valid for any reasonable implementation of \( \phi_I, \epsilon \).

In the standard construction of the Hamiltonian constraint [5, 37] most of the remaining ambiguity is tied to the detailed choice of loops (for e.g. their topology, placement and routing) for the regulating holonomies. In our derivation this is all encapsulated in our visualization of \( \phi_I \). Based on our intuition and experience with toy models, we have visualized each \( \phi_I \) to act as a conical deformation so that its action falls into the class of extended diffeomorphisms introduced by Fairbairn and Rovelli [34]. Another natural visualization would be that of a diffeomorphism as in our brief speculative detour of sections 5.2 and 5.3. It is remarkable that these natural possibilities give rise to the constraint action of section 3, which, purely in terms of edge placements, resembles the choices made in [5, 37]. In accordance with our view that these choices be confined through physical requirements, we further developed the constraint action of section 3 into the form derived in section 4 so as to confront the requirement of a non-trivial anomaly free constraint algebra\(^{17} \). This form closely resembles that in the \( U(1)^3 \) toy model [15]. However, preliminary calculations seem to indicate technical complications in any straightforward effort to generalise the considerations of [15] to the \( SU(2) \) case. Hence, we further developed the constraint action in section 5. The final action (5.9) results in two distinct types of deformations. The first is a conical deformation (and hence and extended diffeomorphism) and displacement of the original vertex structure as a whole and the second deforms the vertex structure around the original vertex ‘one edge at a time’. As a result, roughly speaking, the first piece is expected to play a crucial role in any putative demonstration of an anomaly free constraint algebra and the second piece is expected to seed propagation along the lines of [22]. Ongoing work consists of an investigation into these matters.

We close with the following remarks which expand on our brief summary above and point to open issues:

(a) **Use of the GR property** (this remark may be skipped on a first reading as it concerns technicalities detailed in [15]): the GR property played an important role in the treatment of the constraint algebra in the \( U(1)^3 \) toy model context [15] and we anticipate that (some

\(^{17}\) A necessary condition for non-triviality of the commutator between a pair of Hamiltonian constraints is that the second constraint act on deformations generated by the first [26]. This ensures that the second lapse is evaluated at a different vertex than the first and thereby yields a result which is not symmetric under interchange of lapses thus preventing the antisymmetry of this combination from trivialising the commutator. In contrast to the constraint action of section 3, the action of section 4 constructs deformed and displaced vertex structures on which a second action is not necessarily trivial.
weakened version of) the GR property may play an important role in analysing the constraint algebra with regard to the anomaly free property in the context of the $SU(2)$ case as well. While we restricted attention to nondegenerate vertices satisfying the GR property in sections 4 and 5 there seems to be no obstruction to repeating the considerations of those sections for general (nondegenerate) vertex structures. The completion of such an exercise hinges on a specification of the transformation $\phi_{I,\epsilon}$ on such vertices. In the GR case [15] it is straightforward to specify the deformation in such a way that no unwanted intersections between the set of deformed and undeformed edges are created. For a non-GR vertex with no colinear edge tangents, all our considerations would go through unchanged except that the specification of the action of $\phi_{I,\epsilon}$ on such vertices would have to confront issues similar to those by Thiemann in [5] with regard to routing of the deformed edges so that no unwanted intersections arise. This non-colinearity property is expected to be preserved by the mixed action of section 5. Hence for the purposes of probing the constraint algebra one may restrict attention to such vertices and view, similar to the GR condition [15], such a restriction as the analog of a nondegeneracy property in the quantum setting [15]. If sets of edges with colinear or anticolinear tangents are present new issues such as the increasing valence problem described in section 4 may arise and need to be resolved, with a satisfactory generalisation of the mixed action offering a likely resolution.

(b) The choice of volume operator: the volume operator plays a key role in the construction of the inverse metric determinant operator. Preservation of vertex non-degeneracy with respect to the inverse metric determinant operator under the deformations $\{\phi_{I,\epsilon}\}$ is likely to be a key ingredient in any proof of the anomaly free property (see the discussion of ‘eternal non-degeneracy’ in [15]). If these deformations were diffeomorphisms, the diffeomorphism covariance of any acceptable definition of the inverse metric determinant would ensure preservation of such non-degeneracy. Since we have chosen the $\{\phi_{I,\epsilon}\}$ to be conical deformations and since conical deformations fall into the class of extended diffeomorphisms defined by Fairbairn and Rovelli in [34], preservation of non-degeneracy under such deformations would require a construction of the inverse determinant metric which interacted well with such extended diffeomorphisms. For example, if one used the Rovelli–Smolin volume operator [36] in conjunction with the Tikhonov regularization (see footnote 5 and [33]), non-degeneracy would be preserved under conical deformations. As mentioned in section 5.3, the proposed use of lapses which are discontinuous at isolated points to probe the constraint algebra of unit density constraints seems to be in coherence with the Winston–Fairbairn proposal to extend the group of diffeomorphisms to maps which are diffeomorphisms almost everywhere except at isolated points [34, 38].

(c) The properties of propagation and anomaly free action: from [22], a key ingredient for a quantum dynamics to display propagation is the existence of ‘children of non-unique parentage’. Roughly speaking, by a child of non-unique parentage we mean a spin network which can be generated by the action of the constraint on two or more diffeomorphically distinct ‘parent’ spin networks i.e. a given deformed spin network does not arise from the constraint action on a unique ‘parent’ state. It is then very likely that arguments similar to those in [22] and applied to children of type $S_{M2,IJK,\epsilon}$ will establish vigorous propagation for the mixed action dynamics of section 5. On the other hand, notwithstanding the discussion in section 5.3, a comprehensive and satisfactory demonstration of a non-trivial anomaly free constraint algebra constitutes an open problem. In particular, relative to section 5.3, we would like to provide such a demonstration without recourse to discontinuous lapses and also without recourse to an unconventional regularization of $O(M,N)$.  

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Thus we would like such a demonstration to be based on higher density constraints using and generalising the techniques developed in [15], and, we would like the operator correspondent of the constraint Poisson bracket, \( \hat{O}(M,N) \), to act as a (perhaps, extended) diffeomorphism. Investigations into this issue constitute work in progress.

(d) A comment on the \( \alpha \epsilon \) expansion: in this work we have assumed that the small loop area parameter \( \alpha \epsilon \) is of \( O(\epsilon^2) \). However, the action of \( \phi_{L_s} \) in [15] corresponds to small loops for which \( \alpha \epsilon \) is of a higher order than \( \epsilon^2 \) and we plan to combine the results of this work together with the methods of [15] in an effort to demonstrate the existence of an anomaly free, propagating constraint action for Euclidean LQG. While a detailed discussion of this and related tensions between the structures underlying [15] and the considerations in this work is beyond the scope of this paper, it is pertinent to reiterate our general viewpoint: our viewpoint (motivated by the central role of the electric shift in the classical theory) is to treat all heuristic argumentation as a crutch whose purpose is to obtain a geometrically compelling constraint action in which the deformations \( \phi_{L_s} \) play a key role and whose justification would only be \textit{a posteriori} through a putative demonstration of its consistency with an anomaly free constraint algebra and the property of propagation.

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**Data availability statement**

No new data were created or analysed in this study.

**Appendix A. Explicit segregation of \( O(\alpha \epsilon) \) terms in (4.17)**

Note that from footnote 13, the combination of the two terms in the curly brackets of (4.17) is of order \( \alpha \epsilon \). We further manipulate (4.17) so as to explicitly segregate the order \( \alpha \epsilon \) contributions in each of the two terms as follows. To do so we note that:

\[
C_{\lambda_1...\lambda_N}^{D_1...D_N} \left( \prod_{j \neq I} (h_{\epsilon_j}(1, t_{\epsilon_j, j}) h_{\epsilon_j}(-t_{\epsilon_j, j}, 0))^{D_j} D_j \right) h_{\epsilon_I}(1, \epsilon_I)^{D_I} \epsilon_I = C_{\lambda_1...\lambda_N}^{D_1...D_N} \left( \prod_{j \neq I} (h_{\epsilon_j}(1, 0) h_{\epsilon_j}(-t_{\epsilon_j, 0}, 0))^{D_j} D_j \right) h_{\epsilon_I}(1, \epsilon_I)^{D_I} \epsilon_I
\]

\[
(A.1)
\]

\[
(A.2)
\]
\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon I}(0,0)^B_I E_I = C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon J}(1,0)^B_J E_J \]  
(A.3)

\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon I}(1,0)^B_I E_I = C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon I}(1,0)^B_I E_I \]  
(A.4)

\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon I}(1,0)^B_I E_I = C_{\lambda_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J E_J \right) h_{\varepsilon I}(1,0)^B_I E_I + O(\alpha) \]  
(A.5)

where we have used (3.22) in the second line, (4.11) with \( g \equiv (h_{\varepsilon J}(\varepsilon, 0))^{-1} \) in the third, and used the fact that \( h_{I J} = 1 + O(\alpha) \) in the last line. Using (3.22) and (4.11) in an identical manner and again noting that \( h_{I J} = 1 + O(\alpha) \), we have that:

\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{K \neq I, J} (h_{\varepsilon K}(1,0) h_{I K})^B_K D_K \right) (h_{\varepsilon J}(1,0) h_{I J})^B_J D_J \]  
(A.6)

\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{K \neq I, J} (h_{\varepsilon K}(1,0) h_{I K})^B_K D_K \right) (h_{\varepsilon J}(1,0) h_{I J})^B_J D_J \]  
(A.7)

\[ C_{\lambda_1\ldots\lambda_n} \left( \prod_{K \neq I, J} (h_{\varepsilon K}(1,0) h_{I K})^B_K D_K \right) (h_{\varepsilon J}(1,0) h_{I J})^B_J D_J + O(\alpha). \]  
(A.8)

Equation (5.5) implies that

\[ \sum_{J \neq I} C_{\lambda_1\ldots\lambda_n} \left( \prod_{K \neq I, J} (h_{\varepsilon K}(1,0) h_{I K})^B_K D_K \right) (h_{\varepsilon J}(1,0) h_{I J})^B_J D_J (h_{\varepsilon I}(1,0) h_{I J})^B_J D_J = -C_{\lambda_1\ldots\lambda_n} C_{I_1D_1\ldots\lambda_n} \left( \prod_{J \neq I} (h_{\varepsilon J}(1,0) h_{I J})^B_J D_J \right) (h_{\varepsilon I}(1,0) h_{I J})^B_J D_J \]  
(A.9)

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\[ \times (h_{\epsilon I}(1,0)\tau_j)^{B_j}_{D_j} \]
\[ = j_i(j_i + 1)C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \left( \prod_j h_{\epsilon I}(1,0)^{B_j}_{D_j} \right). \quad (A.9) \]

Equations (A.8) and (A.9) imply that:
\[ \sum_{j \neq I} C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \left( \prod_{k \neq I,J} h_{\epsilon K}(1, t_{\epsilon K I}) h_{\epsilon J}(t_{\epsilon J I}, 0)^{B_k}_{D_k} \right) \]
\[ \times (h_{\epsilon I}(1,0)\tau_j)^{B_j}_{D_j} ) h_{\epsilon I}(1, t_{\epsilon I J}) h_{\epsilon J}(t_{\epsilon J I}, 0)^{B_j}_{D_j} \]
\[ \times (h_{\epsilon I}(1,0)\tau_j)^{B_j}_{D_j} (h_{\epsilon I}(\epsilon I, 0))^{-1} h_{\epsilon J}(\epsilon J, 0)^{B_j}_{D_j} \]
\[ = j_i(j_i + 1)C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \left( \prod_j h_{\epsilon I}(1,0)^{B_j}_{D_j} \right) + O(\alpha). \quad (A.10) \]

Equations (A.5) and (A.10) imply that the right-hand side of (A.10) corresponds to the \( O(1) \) part of each of the curly brackets. Hence we may subtract out the \( O(1) \) contributions to each of these lines and write (4.17) in the form:
\[ \hat{H}_I(N)S(A) := \frac{3}{8\pi N(x(v))} \frac{1}{\epsilon} \sum_{l=1}^{N} S_{\text{rest}_{B_l \ldots B_N}}(A) \]
\[ \times \left( \left\{ j_i(j_i + 1)C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \left( \prod_{j \neq I} (h_{\epsilon I}(1, t_{\epsilon I J}) h_{\epsilon J}(t_{\epsilon J I}, 0)^{B_j}_{D_j} \right) \right\} \right) \]
\[ \times \left( \left\{ \prod_{j=1}^{N} (h_{\epsilon I}(1,0)^{B_j}_{D_j}) \right\} - \sum_{j \neq I} C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \right) \]
\[ \times \left( \left\{ \prod_{k \neq I,J} (h_{\epsilon K}(1, t_{\epsilon K I}) h_{\epsilon J}(t_{\epsilon J I}, 0)^{B_k}_{D_k} \right) \right) \]
\[ \times (h_{\epsilon I}(1,0)\tau_j)^{B_j}_{D_j} (h_{\epsilon I}(\epsilon I, 0))^{-1} h_{\epsilon J}(\epsilon J, 0)^{B_j}_{D_j} \]
\[ \times (h_{\epsilon I}(1,0)\tau_j)^{B_j}_{D_j} (h_{\epsilon I}(\epsilon I, 0))^{-1} h_{\epsilon J}(\epsilon J, 0)^{B_j}_{D_j} \]
\[ = j_i(j_i + 1)C_{\lambda D_1 \ldots D_N}^{D_1 \ldots D_N} \left( \prod_{j=1}^{N} (h_{\epsilon I}(1,0)^{B_j}_{D_j} \right) \}} \right) \quad (A.11) \]

where the terms in curly brackets are now each of \( O(\alpha). \) Note that the subtraction term in each curly bracket, when combined with \( S_{\text{rest}_{B_l \ldots B_N}}(A) \) yields exactly \( S_{\lambda}(A). \) Hence we may write
where we reproduce the definitions (4.19) and (4.20) of section 4 as (A.13) and (A.14) below for the convenience of the reader:

\[
S_{\lambda, I, \epsilon} := C_{\lambda}^{D_1 \cdots D_N} \left( \prod_{j \neq I} (h_{e_j}(1, t_{e_j,I}) h_{\delta_{e_j}(t_{e_j,I},0)})^{B_j}_{D_j} \right) h_{e_j}(1, \epsilon_j)^{B_j}_{D_j} \\
\times S_{\text{rest}}^{B_1 \cdots B_N}, \tag{A.13}
\]

\[
S_{\lambda, I, J, \epsilon} := C_{\lambda}^{D_1 \cdots D_N} \left( \prod_{K \neq I, J} (h_{e_K}(1, t_{e_K,I}) h_{\delta_{e_K}(t_{e_K,I},0)})^{B_K}_{D_K} \right) h_{e_J}(1, \epsilon_J)^{B_J}_{D_J} \\
\times (h_{e_J}(1, 0) \tau_I(h_{e_J}(t_{e_J,I}, 0))^{-1} h_{\delta_{e_J}(t_{e_J,I},0)})^{B_J}_{D_J} \\
\times (h_{e_I}(1, 0) \tau_I((h_{e_I}(\epsilon_I, 0))^{-1} h_{\delta_{e_I}(\epsilon_I,0)})^{B_I}_{D_I} S_{\text{rest}}^{B_1 \cdots B_N}. \tag{A.14}
\]

**Appendix B. A useful lemma**

**Lemma.** Consider the matrix expression \( f = (h^{(j)}_e(A))^{-1} \tau_I h^{(j)}_e(A) \) where \( h_e(A) \) is an edge holonomy of the connection \( A \) along the edge \( e \) and \( h^{(j)}_e(A), \tau_I \) are in a spin \( j \) representation. Then the Clebsch–Gordan decomposition of \( f \) has contributions only from spin 1.

**Proof.** We have that \( h^{(j)}_e(A) = :g^{(j)}_e(\cdot) : \) where \( g^{(j)}_e(\cdot) \) is the spin \( j \) matrix representative of the element \( g \in SU(2) \). Since \( \tau_I \) is the spin \( j \) representative of the \( j \)th generator of \( su(2) \) its index \( j \) behaves as vector index under the action of \( SU(2) \) transformations so that

\[
(g^{(j)}_e)^{-1} \tau_I g^{(j)}_e = R^I_J(g) \tau_K, \tag{B.1}
\]

where the rotation matrix \( R^I_J(g) \) bears the interpretation of the spin 1 representative of \( g \). It follows that \( R(g) = h^{(1)}_e(A) \) where we have suppressed the matrix indices and where the superscript (1) indicates that the holonomy is in the spin 1 representation. This completes the proof. \( \square \)

**Appendix C. A more general form of Hamiltonian constraint action through conical type deformations**

In this section we change each of the two terms in (4.21) through the addition of higher order terms in \( \alpha \) and thereby obtain a general form of the Hamiltonian constraint which encompasses (4.17) and (4.21) and which may, conceivably, be of use in future attempts at showing consistency with an anomaly free constraint algebra.
We define the loop $l_{cI}^{I,J,\epsilon}$ as follows:

(a) Similar to $l_{I,J}^{I,J,\epsilon}$, the loop $l_{cI}^{I,J,\epsilon}$ is a ‘triangle’ with two sides along $e_I$ and $e_J$ emanating from its vertex $v$ and a third side connecting the vertices on $e_I$, $e_J$ with each other.

(b) We adjust the placement of the vertices on the sides $e_J$, $e_I$ by a coordinate distance of $O(\epsilon)$ so that the area of the loop $l_{cI}^{I,J,\epsilon}$ is now $c_I\alpha_{IJ,\epsilon}$.

It follows from standard holonomy expansion that:

$$h_{cI}^{I,J,\epsilon}(A) - 1 = -\left(\epsilon^I_F\epsilon^J_F F^{ab}_v(v)\tau_c\alpha_{IJ,\epsilon} + O(\alpha_{IJ,\epsilon}^2)\right)$$

Equations (C.1), (A.4) and (A.5) imply that:

$$a_I \left( C^{E_1...E_N}_{\lambda} \left( \prod_{j \neq I} (h_{cI}(1,0)h_{IJ,\epsilon})^{B_j}_{E_j} \right) h_{cI}(1,0)^{B_I}_{E_I} - C^{E_1...E_N}_{\lambda} \right)$$

$$= C^{E_1...E_N}_{\lambda} \left( \prod_{j \neq I} (h_{cI}(1,0)h_{IJ,\epsilon})^{B_j}_{E_j} \right) h_{cI}(1,0)^{B_I}_{E_I} - C^{E_1...E_N}_{\lambda}$$

Similarly, equations (C.1), (A.7) and (A.8) imply that:

$$b_I \left( C^{D_1...D_N}_{\lambda} \left( \prod_{K \neq I,J} (h_{cI}(1,0)h_{IJ,\epsilon})^{B_K}_{D_K} \right) (h_{cI}(1,0)\tau_jh_{IJ,\epsilon})^{B_I}_{D_I} \right)$$

Next note that the right-hand side of (4.19) (modulo $S_{\text{rest}}$) can be replaced by (A.4). Replacing the loops $l_{I,J}^{I,J,\epsilon}$ in the resulting expression for $A.19$ by the loops $l_{cI}^{I,J,\epsilon}$, we obtain an expression which we denote by $S_{\lambda,I,\alpha,\epsilon}$. Similarly the right-hand side of (4.20) can be
replaced by (A.7). Replacing the loops \( l_{ij,\epsilon} \) in the resulting expression for (4.20) by the loops \( \hat{l}_{ij,\epsilon} \), we obtain an expression which we denote by \( \hat{S}_{\lambda,l,ij,\epsilon} \). Equations (C.2) and (C.3) then imply the action

\[
\hat{H}_\epsilon(N)S(A) := \frac{3}{8\pi} N(\epsilon(v)) \left( \sum_{i=1}^{N} \frac{A_i(j_i + 1)(S_{\lambda,i,ij,\epsilon} - S_{\lambda})}{a_i \epsilon} ight)
\]

\[
+ \sum_{i=1}^{N} \left( \frac{\sum_{j \neq i} S_{\lambda,ij,\epsilon}}{b_j \epsilon} - j_i^2 (j_i + 1) S_{\lambda} \right)
\]

(C.4)

agrees with the (4.21) to leading order i.e. up to terms of \( O(\alpha^2) \). Hence (C.4) constitutes a valid approximant to the Hamiltonian constraint. Equation (C.4) is our final result.

Appendix D. The operator correspondent of \( \{H(M),H(N)\} \)

On the Gauss law constraint surface (2.9) for density 1 constraints we have that:

\[
\{H[M],H[N]\} = \int d^3x \omega_{ad}(v) \epsilon_{abc} F_{bc}(v) := O(M,N)
\]

(D.1)

where

\[
\omega_{ad} := N \partial_a M - M \partial_a N.
\]

(D.2)

In this section we construct the action of a regulated operator correspondent of (D.1) on the gauge invariant spin network \( S(A) \) at its non-degenerate vertex \( v \) with \( v \) assumed to be GR. We shall use the notation of the main text, assume that the supports of the lapses \( N,M \) include only a single nondegenerate vertex \( v \) of \( S \) and choose an operator ordering in which the regulated inverse metric determinant acts first. Similar to the case of the regulated Hamiltonian constraint, the only nontrivial contribution to the regulated operator action then comes from a ball \( B_\epsilon(v) \) of coordinate size \( \epsilon \) around \( v \). Accordingly we define:

\[
\hat{O}_\epsilon(M,N)S(A) := \frac{4\pi}{3} \epsilon^3 (\omega_{a}(v))_a((F^d_{bc}(v))_a((\hat{E}^a_{bc})(\hat{E}^a_{bc})(\hat{E}^a_{bc})(q^{-1/2}).
\]

\[
\times (q^{-1/2})_a S(A)
\]

(D.3)

where we have omitted the ‘hat’ on the regulated curvature term as it acts by multiplication. The regulated inverse metric determinant and electric field operators above are defined as follows. \( \hat{q}^{-1/2} \) is obtained from the volume operator and we expect \( \epsilon^{-3} \hat{q}^{-1/2} \) to be a well defined operator which only changes the intertwiner. We denote:

\[
\epsilon^{-3} \hat{q}^{-1/2} S := S_{\lambda},
\]

(D.4)

\[
\epsilon^{-3} \hat{q}^{-1/2} S_{\lambda} := S_{\eta,\lambda}
\]

(D.5)

where \( S_{\eta,\lambda} \) differs from \( S_{\lambda} \) by a change in the intertwiner \( C_{\lambda} \) of the former to the new intertwiner \( C_{\eta,\lambda} \). Each of the electric field operators is regulated in a manner similar to the main text by averaging its action over the ball \( B_\epsilon \). Thus, if \( \hat{S} \) is a gauge invariant spin network and \( B_\epsilon \) contains
only a single nondegenerate vertex \( v \) of \( \bar{S} \), it is straightforward to repeat a derivation similar to that of (3.5) and obtain:

\[
(E_m^f)\bar{S} := \frac{3}{4\pi\epsilon} \int_{B_m(v)} \delta^3 x E_m^f(x) \bar{S}
\]

\[
= i \frac{3}{4\pi\epsilon} \sum_l \hat{e}_l^f \tilde{X}_{l,m} \bar{S}
\]

where \( \hat{e}_l^f \) is the unit coordinate tangent to the 1th edge at the vertex \( v \). It follows that:

\[
i\hat{O}_L(M,N)S(A) := \left( \frac{3}{4\pi} \right)^2 \sum_{IJK} (\hat{e}_J^f \omega_{\nu}) (\epsilon^2 F_{IJK}^\nu(v) \hat{e}_I^f \hat{e}_K^f \hat{X}_{IJK} \hat{X}_{IJK} S_{\eta,\lambda})
\]

\[
= - \left( \frac{3}{4\pi} \right)^2 \left( \sum_I \sum_{K \neq I} O_{IKJ} + \sum_I \sum_{J \neq I} O_{IJ} \right)
\]

\[
+ \sum_I \sum_{J \neq I} \sum_K \sum_{K \neq I} O_{IKJ}
\]

\[(D.6)\]

\[(D.9)\]

where

\[
O_{IKJ} := (\hat{e}_J^f \omega_{\nu}) (\epsilon^2 F_{IJK}^\nu(v) \hat{e}_I^f \hat{e}_K^f \hat{X}_{IJK} \hat{X}_{IJK} S_{\eta,\lambda}).
\]

\[(D.10)\]

and we have used the antisymmetry of \( F_{IJK}^\nu \) to drop the \( K = I \) terms in the sum in (D.8). Next we define regulated versions of \( (\hat{e}_J^f \omega_{\nu}) \), \( (\epsilon^2 F_{IJK}^\nu(v) \hat{e}_I^f \hat{e}_K^f \hat{X}_{IJK} \hat{X}_{IJK} S_{\eta,\lambda}) \) through:

\[
(\hat{e}_J^f \omega_{\nu}) := N(v)M(v^a + \hat{e}_J^f) - M(v)N(v^a + \hat{e}_J^f),
\]

\[(D.11)\]

where \( v^a + \hat{e}_J^f \) denotes the coordinates of the point obtained by traversing a coordinate distance \( \epsilon \) along the direction \( \hat{e}_J^f \) from \( v \), and,

\[
(-\epsilon^2 F_{IJK}^\nu(v) \hat{e}_I^f \hat{e}_K^f \hat{X}_{IJK} \hat{X}_{IJK} S_{\eta,\lambda}) := \frac{l_{IK,\nu}}{\kappa} \frac{1}{\kappa} + O(\alpha^2)
\]

\[(D.12)\]

where \( l_{IK,\nu} \) is a triangular loop similar to \( l_{IK,\nu} \) in the main text except that it has area \( \kappa \epsilon^2 \) for some constant \( \kappa \) to be specified later. Thus \( l_{IK,\nu} \) traverses \( e_1 \) from \( v \) at \( t_I = 0 \) to some \( v_{I,K,\nu} \) which lies at parameter value \( t_I = \epsilon_{I,K,\nu} \), then along an edge from \( v_{I,K,\nu} \) to a point on the edge \( e_K \) at some parameter value \( t_{I,K,J,\nu} \) and then from this point back to \( v \) along \( e_K \) in the inward direction. In addition we assume\(^{18}\) that the loops \( \{l_{IK,\nu}\} \) can be chosen in accordance with choice 1 below:

**Choice 1.** The loops \( \{l_{IK,\nu}, K \neq I\} \) are chosen to be diffeomorphic images of their counterparts \( \{l_{IK,\nu}, K \neq I\} \) by some diffeomorphism \( \psi_{I,K,\nu} \) such that

\[
l_{IK,\nu} = \psi_K(t_{I,K,I,\nu}, 0)^{-1} \circ \phi_{I,K,J,\nu}(e_K)(t_{I,K,J,\nu}, 0) \circ e_I(t_{I,K,\nu}, 0)
\]

\[(D.13)\]

\(^{18}\)While this assumption seems reasonable to us, an explicit construction of such a diffeomorphism is desirable. We leave this for future work.
where we have defined

$$\phi_{I,K} := \psi_{I,K} \circ \phi_{I},$$  \hspace{1cm} (D.14)

and where we have used the natural parameterization induced on the edge $\phi_{I,K}(e_K(t_{r,K,I,K}, 0))$ by $\phi_{I,K}$, from the edge $e_K$. Equation (D.13) implies that:

$$h_{I,K} = \left(h_{I,K}(t_{r,K,I,K}, 0)\right)^{-1} h_{\phi_{I,K}(e_K)(t_{r,K,I,K}, 0)} h_{I,1}(e_{I,1}, 0)$$  \hspace{1cm} (D.15)

We evaluate each of the terms in (D.9) in turn using (D.11), (D.12), (D.14) and (D.15). When using (D.12) we shall drop the $O(\alpha^2)$ term. We shall see that the first term can be modified by higher curvature terms in a manner similar to the ‘sum to product’ transformation of section 3 so as to obtain a contribution similar to that of the first set of terms in the constraint commutator, namely the $S_{(I,J_1,J_2,I_1,J_3,I_2)}$ terms in the second line of (5.11). The remaining two sets of terms in (D.10) will be seen to be analogs of the $S_{(I,J_1,J_2,J_3,I_1,I_2)}$, $S_{(I,J_2,J_1,J_3,I_1,I_2)}$ in the constraint commutator (5.11). We have that:

$$O_{IK} := (e^{\hat{e}_I}_{J} \omega_{J})_{k} (-e^{2} F_{k}^{I}) (v) e^{\hat{e}_I}_{K} \hat{X}_{I,J} \hat{X}_{K} S_{I,J},$$  \hspace{1cm} (D.16)

$$= (e^{\hat{e}_I}_{J} \omega_{J})_{k} (-e^{2} F_{k}^{I}) (v) e^{\hat{e}_I}_{K} \hat{X}_{I,J} \hat{X}_{K} S_{I,J},$$  \hspace{1cm} (D.17)

$$= -(j)(j+1)(e^{\hat{e}_J}_{I} \omega_{J})_{k} (-e^{2} F_{k}^{I}) (v) e^{\hat{e}_I}_{K} \hat{X}_{I,K} S_{I,J},$$  \hspace{1cm} (D.18)

$$= -(j)(j+1)(e^{\hat{e}_J}_{I} \omega_{J})_{k} (-e^{2} F_{k}^{I}) (v) e^{\hat{e}_I}_{K} \hat{X}_{I,K} S_{I,J},$$  \hspace{1cm} (D.19)

$$= -(j)(j+1)(e^{\hat{e}_J}_{I} \omega_{J})_{k} \left(h_{I,K}(1,0) \left(h_{I,K} - 1 \right) \right)_{A_K}$$  \hspace{1cm} (D.20)

Above, we have used $K \neq I$ to move $\hat{X}_I$ past $\hat{X}_K$ in (D.17) and used (D.12) in (D.20). Next we sum over $K$, and add higher order terms in small loop area to move from a sum to a product exactly as in section 3.3. It is then straightforward to obtain:

$$\sum_{K \neq I} O_{IK} = -\kappa^{-1}(j)(j+1)(e^{\hat{e}_J}_{I} \omega_{J})_{k} \{S_{I,J,K} - S_{I,K} \} + O(\alpha^2)$$  \hspace{1cm} (D.21)

where

$$S_{I,J,K} := C_{I,J,K} \left( \prod_{K \neq I} (h_{I,K}(1,0)) h_{\phi_{I,K}(e_K)(t_{r,K,I,K}, 0))^{B_{K}}) \right)$$  \hspace{1cm} (D.22)

Note that $S_{I,J,K}$ is given exactly by substituting $C_{I,J,K}$ and $\phi_{I,K}$ by $C_{I,K}$ and $\phi_{I,K}$ in (4.19).
Next consider
\[
O_{IJ} := (\epsilon \epsilon_j^\beta \omega_k)(-2 \epsilon^k_{\beta k})(v) \epsilon_j^\beta \epsilon^k_{\beta k} \hat{X}_l \hat{X}_l S_{\eta, \lambda} \tag{D.23}
\]
\[
= (\epsilon \epsilon_j^\beta \omega_k)(-2 \epsilon^k_{\beta k})(v) \epsilon_j^\beta \epsilon^k_{\beta k} (h_{e_k}(1,0) \tau_i \tau_i)_{\lambda_j} \frac{\partial \hat{X}_l}{\partial h_{e_k}(1,0)}_{\lambda_j}^{\beta j} \tag{D.24}
\]
\[
= \kappa^{-1} (\epsilon \epsilon_j^\beta \omega_k)(h_{e_k}(1,0) \tau_i \tau_i (h_{e_k}(1,0) - 1))_{\lambda_j} \frac{\partial \hat{X}_l}{\partial h_{e_k}(1,0)}_{\lambda_j}^{\beta j} \tag{D.25}
\]

Next, we sum (D.25) over \( J \). Let us assume that

**Assumption 1.** The \( O(1) \) contribution to \( (\epsilon \epsilon_j^\beta \omega_k) \) is independent of \( J \) for small enough \( \epsilon \) i.e.

\[
(\epsilon \epsilon_j^\beta \omega_k)_0 = \omega_0 + O(\epsilon) \tag{D.26}
\]

where \( \omega_0 \) is independent of \( \epsilon \).

It is then straightforward to see, using (5.5), that the sum over \( 1 \) contributions yields, to leading order in \( \epsilon \), the result \( (\epsilon \epsilon_j^\beta \omega_k)_0(j_i)(j_i + 1)S_{\eta, \lambda} \). It follows that the final result can be written to leading order in \( \epsilon \) as:

\[
O_{IJ} = \kappa^{-1} \omega_0 \sum_{J \neq j} (S_{\eta, \lambda}(I,J,I,J) - (j_i)(j_i + 1))S_{\eta, \lambda} \tag{D.27}
\]

where \( S_{\eta, \lambda}(I,J,I,J) \) is given exactly by substituting, in (5.8), \( C_\lambda \) and \( I_{IJ, J} \) by \( C_{\eta, \lambda} \) and \( I_{IJ, J, \kappa} \) respectively.

Finally, proceeding in a similar manner, we have that:

\[
O_{IK} \mid_{I \neq J \neq K} = (\epsilon \epsilon_j^\beta \omega_k)(-2 \epsilon^k_{\beta k})(v) \epsilon_j^\beta \epsilon^k_{\beta k} \hat{X}_l \hat{X}_l S_{\eta, \lambda} \tag{D.28}
\]
\[
= (\epsilon \epsilon_j^\beta \omega_k)(-2 \epsilon^k_{\beta k})(v) \epsilon_j^\beta \epsilon^k_{\beta k} (h_{e_k}(1,0) \tau_j \tau_j)_{\lambda_k} \frac{\partial \hat{X}_l}{\partial h_{e_k}(1,0)}_{\lambda_k}^{\beta k} \tag{D.29}
\]
\[
= \kappa^{-1} (\epsilon \epsilon_j^\beta \omega_k)(h_{e_k}(1,0)(h_{e_k}(1,0) - 1))_{\lambda_k} \frac{\partial \hat{X}_l}{\partial h_{e_k}(1,0)}_{\lambda_k}^{\beta k} \tag{D.30}
\]

Summing the \( I \) term over \( K \) yields \( N - 2 \) identical terms. Under Assumption 1 above, we may then sum over \( J \) and use (5.5). The result is \( (N - 2)(j_i)(j_i + 1)S_{\eta, \lambda} \). It follows that the final result can be written to leading order in \( \epsilon \) as:

\[
\sum_{J \neq K} O_{IK} = \kappa^{-1} \omega_0 (\sum_{J \neq K} (S_{\eta, \lambda}(I,J,I,K) - (N - 2)(j_i)(j_i + 1))S_{\eta, \lambda}) \tag{D.31}
\]

where \( S_{\eta, \lambda}(I,J,I,K) \) is given exactly by substituting, in (5.7), \( S_\lambda \) and \( I_{K,I} \) by \( S_{\eta, \lambda} \) and \( I_{K,I, \kappa} \) respectively.
Putting all this together we obtain our final result that to leading order in $\epsilon$ we have that:

$$i\hat{O}_c(N, M)S(A) = \left(\frac{3}{4\pi}\right)^2 \omega_0 \left( \sum_{I=1}^{N} j_I(j_I + 1)(S_{H,\lambda,c,e} - S_{H,\lambda}) \right)$$

$$- \sum_{I=1}^{N} \left( \sum_{j \neq j_I} \sum_{\kappa} S_{H,\lambda}(1)(j_I, J, K, \kappa) - \sum_{j \neq j_I} S_{H,\lambda}(2)(j_I, J, K, \kappa) - j_I(j_I + 1)(N - 1)S_{H,\lambda} \right)$$

(D.32)

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