THE DAUGAVET PROPERTY IN THE MUSIELAK-ORLICZ SPACES

ANNA KAMIŃSKA AND DAMIAN KUBIAK

Abstract. We show that among all Musielak-Orlicz function spaces on a \(\sigma\)-finite non-atomic complete measure space equipped with either the Luxemburg norm or the Orlicz norm the only spaces with the Daugavet property are \(L^1\), \(L^\infty\), \(L^1 \oplus L^\infty\) and \(L^1 \oplus^\infty L^\infty\). In particular, we obtain complete characterizations of the Daugavet property in the weighted interpolation spaces, the variable exponent Lebesgue spaces (Nakano spaces) and the Orlicz spaces.

1. Introduction

Let \((X, \| \cdot \|)\) be a Banach space and \(T : X \to X\) be a bounded linear operator. The equation
\[
\|T + I\| = 1 + \|T\|
\]
where \(I\) is the identity operator on \(X\), is called the Daugavet equation. If the Daugavet equation is satisfied by every rank one operator \(T\) then \(X\) is said to have the Daugavet property. It is known that if \(X\) has the Daugavet property then Eq. (1) is satisfied by every weakly compact operator.

Eq. (1) was first studied by I.K. Daugavet in the space \(C(0, 1)\) [10]. Examples of spaces which have the Daugavet property are \(L^1\) and \(L^\infty\) over a non-atomic measure space as well as \(C(K)\), where \(K\) is a compact Hausdorff space with no isolated points. Moreover, finite direct sums \(\oplus_1\) and \(\oplus_\infty\) of spaces with the Daugavet property possess that property as well [28]. It is known that Banach spaces with the Daugavet property fail the Radon-Nikodym property and do not embed into a space with an unconditional basis. For a historical overview on the Daugavet property we refer to [4]. An introductory exposition on the Daugavet equation and the Daugavet property can be found in [2].

It has been recently showed that among all rearrangement invariant function spaces over a non-atomic finite measure spaces only \(L^1\) and \(L^\infty\) have the Daugavet property [3, 17]. Inspired by that result we study the Daugavet property in the class of Musielak-Orlicz function spaces on a \(\sigma\)-finite non-atomic complete measure space. These spaces are not rearrangement invariant in general. The variable exponent Lebesgue spaces (Nakano spaces) and the Orlicz spaces appear as special cases of the Musielak-Orlicz spaces. It should be mentioned that the class of Musielak-Orlicz spaces we consider here is the most general and includes also the interpolation spaces \(L^1 \cap L^\infty\) and \(L^1 + L^\infty\), as well as their weighted versions which are studied in Section 3. In section 4, using an observation that the unit sphere of a Banach space with the Daugavet property does not contain a uniformly non-\(\ell_1^2\) point, we prove that the only Musielak-Orlicz spaces (equipped with either the Luxemburg norm or the Orlicz norm) with the Daugavet property are \(L^1\), \(L^\infty\), \(L^1 \oplus L^\infty\) or \(L^1 \oplus^\infty L^\infty\) with weights. This generalizes several results obtained earlier in [4] and solves the problem of the Daugavet property in the Musielak-Orlicz spaces completely. In the appendix we give a proof of Köthe duality in the most general case of the Musielak-Orlicz spaces. That result is of course well known but it seems that a direct proof in such generality has never been published.

Date: October 13, 2014; Revised March 03, 2015.
2010 Mathematics Subject Classification. 46B20, 46E30, 47B38.
Key words and phrases. Daugavet property, Musielak-Orlicz space, Orlicz space, variable exponent space, Nakano space, uniformly non-square point, diameter 2 property, slice.

The second author was supported by the Tennessee Technological University internal Faculty Research Grant during 2013-14.

Please cite this article in press as: A. Kamińska, D. Kubiak, The Daugavet property in the Musielak-Orlicz spaces, J. Math. Anal. Appl. 427 (2) (2015), 873-898, MR3323013, http://dx.doi.org/10.1016/j.jmaa.2015.02.035
2. Preliminaries

For a Banach space $X$, by $S(X)$ and $B(X)$ we denote the unit sphere and the unit ball, respectively. The space of all bounded linear functionals on $X$ is denoted by $X^*$. Let $(X, \| \cdot \|)$ be a (real) Banach space. For any $x^* \in S(X^*)$ and $\epsilon > 0$ the set

$$S(x^*; \epsilon) = \{ x \in B(X) : |x^*x| > 1 - \epsilon \}$$

is called a slice determined by $x^*$ and $\epsilon$. We say that a Banach space $(X, \| \cdot \|)$ has the slice (or local) diameter 2 property if every slice of $B(X)$ has diameter 2. It is known that every space with the Daugavet property has the slice diameter 2 property \( \Pi \).

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, respectively. By $X \oplus_1 Y$ we denote the Banach space consisting of all ordered pairs $(x, y)$ where $x \in X$ and $y \in Y$ with the norm $\| (x, y) \| = \| x \|_X + \| y \|_Y$. Similarly, by $X \oplus \infty Y$ we denote the Banach space consisting of all ordered pairs $(x, y)$ where $x \in X$ and $y \in Y$ with the norm $\| (x, y) \| = \max\{ \| x \|_X, \| y \|_Y \}$. It is clear that $(X \oplus_1 Y)^* = X^* \oplus_1 Y^*$ and $(X \oplus \infty Y)^* = X^* \oplus \infty Y^*$ with equality of norms. The fact that two Banach spaces $X$ and $Y$ are isometrically isomorphic is denoted by $X \simeq Y$ or by $X = Y$ if an isometric isomorphism between $X$ and $Y$ is obvious (for example the identity mapping).

In the sequel we assume that $(\Omega, \Sigma, \mu)$ is a non-atomic $\sigma$-finite complete measure space. By $L_0 = L_0(\Omega)$ we denote the set of all (equivalence classes with respect to the equality $\mu$-a.e.) of measurable extended-real valued functions on $\Omega$.

Let $(X, \| \cdot \|_X)$ be a Banach function lattice on $(\Omega, \Sigma, \mu)$, that is $X \subset L_0$ and if $|x| \leq |y|$ $\mu$-a.e. on $\Omega$, $x, y \in X$ then $x \in X$ and $\|x\|_X \leq \|y\|_X$. For any function $x \in L_0$, the support of $x$ is defined by $\text{supp}(x) = \{ t \in \Omega : x(t) \neq 0 \}$. Recall that $\text{supp}(X)$ is a measurable subset of $\Omega$ such that every element of $X$ vanishes $\mu$-a.e. on $\Omega \setminus \text{supp}(X)$ and for every measurable subset $E \subset \text{supp}(X)$ with positive measure there is a measurable set $F \subset E$ with finite and positive measure such that $\chi_F \in X$ [9 p. 14]. An element $x \in X$ is called order continuous if for every $0 \leq x_n \leq |x|$ such that $x_n \downarrow 0$ $\mu$-a.e. it holds $\| x_n \|_X \to 0$. By $X_0$ we denote the set of all order continuous elements of $X$. A Banach function lattice $X$ is said to have the Fatou property whenever for any sequence $(x_n)$ in $X$ and $x \in L_0$ such that $x_n \to x$ $\mu$-a.e. on $\Omega$ and $\sup \| x_n \|_X < \infty$, we have that $x \in X$ and $\| x \|_X = \liminf \| x_n \|_X$. Given a measurable set $\Gamma \subset \Omega$ we denote $X(\Gamma) = \{ x \in X : \mu(\text{supp}(x) \setminus \Gamma) = 0 \}$ and the norm $\| \cdot \|_{X(\Gamma)}$ on $X(\Gamma)$ is defined by $\| x \|_{X(\Gamma)} = \| x \chi_{\Gamma} \|_X$. It is clear that $X(\Gamma)$ is continuously embedded in $L_0$.

Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be Banach function lattices on $(\Omega, \Sigma, \mu)$ with the Fatou property and let $\Gamma_1, \Gamma_2$ be measurable sets such that $\Gamma_2 \subset \Gamma_1 \subset \Omega$. We define

$$X(\Gamma_1) \cap Y(\Gamma_2) = Y(\Gamma_2) \cap X(\Gamma_1) = \{ x \in L_0 : x \in X(\Gamma_1) \text{ and } x \chi_{\Gamma_2} \in Y(\Gamma_2) \},$$

and equip it with the norm

$$\| x \| = \| x \|_{X(\Gamma_1) \cap Y(\Gamma_2)} = \max\{ \| x \|_{X(\Gamma_1)}, \| x \chi_{\Gamma_2} \|_{Y(\Gamma_2)} \}.$$

It follows that $(X(\Gamma_1) \cap Y(\Gamma_2), \| \cdot \|)$ is a Banach function lattice on $(\Omega, \Sigma, \mu)$ with the Fatou property and $\text{supp}(X(\Gamma_1) \cap Y(\Gamma_2)) = \Gamma_1$. Moreover, we set

$$X(\Gamma_1) + Y(\Gamma_2) = Y(\Gamma_2) + X(\Gamma_1) = \{ z \in L_0 : z = x + y \text{ for some } x \in X(\Gamma_1) \text{ and } y \in Y(\Gamma_2) \},$$

and equip it with the norm

$$\| z \| = \| z \|_{X(\Gamma_1) + Y(\Gamma_2)} = \inf\{ \| x \|_{X(\Gamma_1)} + \| y \|_{Y(\Gamma_2)} : z = x + y, x \in X(\Gamma_1), y \in Y(\Gamma_2) \}.$$

Again, $(X(\Gamma_1) + Y(\Gamma_2), \| \cdot \|)$ is a Banach function lattice on $(\Omega, \Sigma, \mu)$ with the Fatou property and $\text{supp}(X(\Gamma_1) + Y(\Gamma_2)) = \Gamma_1$. The proof of completeness of $X(\Gamma_1) \cap Y(\Gamma_2)$ and $X(\Gamma_1) + Y(\Gamma_2)$ is essentially the same as the proof of Theorem 1.3 [6 p. 97].

It should be noted that given two Banach function lattices $X$ and $Y$ on $(\Omega, \Sigma, \mu)$ the space $X \cap Y$ is usually defined (explicitly or implicitly) as the set of functions belonging to both, $X$ and $Y$ (see for example [6 p. 97], [23 p. 9] or [2] p. 16). However, as we will see in Theorem 3.3 we need to consider spaces $X \cap Y$ as defined in the previous paragraph.

Recall [23 p. 44] that the Köthe dual $X'$ of $X$ is the collection of those $y \in L_0$ such that $\text{supp}(y) \subset \text{supp}(X)$ and

$$\| y \|_{X'} = \sup \left\{ \int_\Omega |xy|d\mu : \| x \|_X \leq 1 \right\} < \infty.$$
The space \((X', \| \cdot \|_{X'})\) is a Banach function lattice on \((\Omega, \Sigma, \mu)\) with the Fatou property and \(\text{supp}(X') = \text{supp}(X)\). A functional \(F \in X^*\) is said to be order continuous if \(F(x_n) \to 0\) whenever \(x_n, x \in X, x_n \to 0\) and \(|x_n| \leq \mu\text{-a.e. on } \Omega\). It is known that \(F \in X'_\epsilon\), the set of all order continuous functionals on \(X\), if and only if there exists a unique \(y \in X'\) such that \(F(x) = \int_\Omega xy \, d\mu\) and \(\|F\| = \|y\|_{X'}\). Hence \(X'\) is isometrically isomorphic to \(X^*_\epsilon\). Moreover \(X'' = X\) with equality of norms if and only if \(X\) has the Fatou property. For details on Banach lattices see \([6,9,23,24]\).

Recall also the following result on Köthe duality of spaces \(X \cap Y\) and \(X + Y\) (cf. Lemma 1.12 [9, p. 18]).

**Theorem 2.1.** Let \((X, \| \cdot \|_X)\) and \((Y, \| \cdot \|_Y)\) be Banach function lattices on \((\Omega, \Sigma, \mu)\) with the Fatou property and \(\Gamma_2 \subseteq \Gamma_1 \subseteq \Omega\), where \(\Gamma_1, \Gamma_2\) are measurable sets of positive measure. The following Köthe dualities hold true,

\[ (X(\Gamma_1) + Y(\Gamma_2))' = X(\Gamma_1)' \cap Y(\Gamma_2)' \]

and

\[ (X(\Gamma_1) \cap Y(\Gamma_2))' = X(\Gamma_1)' + Y(\Gamma_2)' \]

with equality of norms.

A function \(\varphi: [0, \infty) \to [0, \infty]\) is called an Orlicz function, if \(\varphi\) is not identically 0, \(\lim_{u \to 0^+} \varphi(u) = \varphi(0) = 0\), and \(\varphi\) is left continuous and convex on \((0, b_\varphi]\), where \(b_\varphi = \inf\{u > 0 : \varphi(u) < \infty\}\). It follows that \(\varphi\) is continuous on \((0, b_\varphi]\). For an Orlicz function \(\varphi\) we define \(a_\varphi = \inf\{u > 0 : \varphi(u) = 0\}\) and \(d_\varphi = \sup\{u \in [0, b_\varphi) : \varphi(u/2) = \varphi(u/2)\}\). Clearly \(0 < a_\varphi < d_\varphi < b_\varphi < \infty\), \(a_\varphi < \infty\), and \(b_\varphi > 0\). Moreover, if \(a_\varphi = 0\) and \(d_\varphi > 0\) then \(\varphi(u) = cu^\lambda\) for all \(u \in [0, d_\varphi]\) and some constant \(c > 0\). Clearly, if \(a_\varphi > 0\) then \(d_\varphi = a_\varphi\). For convenience we denote \(\varphi(\infty) = \infty\).

A function \(M: \Omega \times [0, \infty) \to [0, \infty]\) is called a Musielak-Orlicz function if for \(\mu\text{-a.e. } t \in \Omega\), \(M(t, \cdot)\) is an Orlicz function and for all \(u > 0\), \(M(\cdot, u)\) is measurable.

For a Musielak-Orlicz function \(M\) we define the functions \(a_M(t) = \inf\{u > 0 : M(t, u) = 0\}\), \(b_M(t) = \inf\{u > 0 : M(t, u) < \infty\}\) and \(d_M(t) = \sup\{u \in [0, b_M(t)) : M(t, u/2) = M(t, u/2)\}\). The functions \(a_M, b_M\) and \(d_M\) are measurable \([7,27]\). The basic pointwise properties of the functions \(a_M, b_M\) and \(d_M\) follow from the above discussion on Orlicz functions.

For a Musielak-Orlicz function \(M\) the complementary function \(N: \Omega \times [0, \infty) \to [0, \infty]\) is defined by

\[ N(t, u) = \sup\{uv - M(t, v) : v > 0\}, \quad t \in \Omega \text{ and } u \geq 0. \]

It is known, that the function \(N\) complementary to \(M\) is a Musielak-Orlicz function.

Let \(M\) and \(N\) be complementary Musielak-Orlicz functions. The semimodular \(\rho_M: L_0 \to [0, \infty]\) given by

\[ \rho_M(x) = \int_\Omega M(t, |x(t)|) \, d\mu \]

is convex and defines the Musielak-Orlicz function space

\[ L_M = \{x \in L_0 : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0\} \]

with the Luxemburg norm

\[ \|x\|_M = \inf\{\lambda > 0 : \rho_M(x/\lambda) \leq 1\}. \]

We also consider the space \(E_M\) of all finite elements in \(L_M\),

\[ E_M = \{x \in L_0 : \rho_M(\lambda x) < \infty \text{ for all } \lambda > 0\}. \]

For details on modular spaces and (semi)modulars we refer to \([26]\).

Recall that the Köthe dual \((L_M, \| \cdot \|_M)' = (L_N, \| \cdot \|_N)'\), where

\[ \|x\|_N = \sup\left\{ \int_\Omega xy \, d\mu : \rho_M(y) \leq 1 \right\} \]

is the Orlicz norm on \(L_N\) (see Theorem 5.4). The Orlicz norm is equal to the Amemiya norm \([12,16]\), that is

\[ \|x\|_N = \inf_{k>0} \frac{1}{k} [1 + \rho_N(kx)]. \]
Moreover \((L_M, \| \cdot \|_M) = (L_N, \| \cdot \|_N)\) (see Theorem 5.3). It is known that \(\|x\|_M \leq \|x\|_M^N \leq 2\|x\|_M\) for all \(x \in L_M\) [26, p. 9]. In the sequel, we denote \(L_M = (L_M, \| \cdot \|_M)\) and \(L_N^N = (L_N, \| \cdot \|_N^N)\). It follows from the general theory of Banach lattices that \(L_M^*\) is isomorphic to \((L_M)^* \oplus S\), where \(S\) is the set of all singular functionals on \(L_M\). Moreover, every singular functional evaluates to 0 at order continuous elements of \(L_M\) [25]. Since \((L_M)^* = L_N^N\), \(L_M^*\) is isometrically isomorphic to a subspace of \(L_N^N\).

If a Musielak-Orlicz function \(M(t, u) = \varphi(u)\) for all \(t \in \Omega\), where \(\varphi\) is an Orlicz function, then \(L_M = L_\varphi\) is called an Orlicz space. In this case we denote \(\psi(u) = N(t, u)\), where \(N\) is the complementary function of \(M\) [7].

Let \(p \in L_0\) be such that \(1 \leq p(t) \leq \infty\). If, for \(\mu\)-a.e. \(t \in \Omega\),
\[
M(t, u) = \begin{cases} \frac{u(t)}{p(t)} & \text{if } t \in \Omega \setminus \Omega_\infty, \\ \alpha(u) & \text{if } t \in \Omega_\infty,
\end{cases}
\]
where \(\Omega_\infty = \{t \in \Omega : p(t) = \infty\}\) and \(\alpha(u) = 0\) for \(0 \leq u \leq 1\) and \(\alpha(u) = \infty\) for \(u > 1\) then the space \(L_M\) is called a variable exponent Lebesgue space (or Nakano space) and is denoted by \(L_{p(t)}\).

We use the standard convention that \(a/0 = \infty\), \(a/\infty = 0\), \(\infty/a = \infty\) for \(a \in (0, \infty)\), \(0 \cdot \infty = 0\), \(\infty \leq \infty\), \(\inf \emptyset = \infty\).

3. Weighted Interpolation Spaces \(L_{1,v} + L_{\infty,w}\) and \(L_{1,w} \cap L_{\infty,v}\)

In this section we study the Daugavet property in weighted interpolation spaces \(L_{1,v} + L_{\infty,w}\) and \(L_{1,w} \cap L_{\infty,v}\), which are in fact, as we will see in the next section, the Musielak-Orlicz spaces generated by certain Musielak-Orlicz functions. We prove the criteria of the Daugavet property in both spaces which will be applied in the proofs of main results in section 4.

For a measurable set \(\Gamma \subset \Omega\), a function \(u \in L_0(\Omega)\) is called a weight function on \(\Gamma\) if \(0 < u < \infty\) \(\mu\)-a.e. on \(\Gamma\). Given an arbitrary weight function \(u\) on \(\Omega\), we denote
\[
L_{\infty,u}(\Omega) = \{x \in L_0 : x u \in L_{\infty}(\Omega)\}
\]
and equip it with the standard norm
\[
\|x\|_{\infty,u} = \|x u\|_{\infty} = \text{ess sup}_{t \in \Omega} |x(t) u(t)|.
\]
Similarly
\[
L_{1,u} = L_{1,u}(\Omega) = \{x \in L_0 : x u \in L_1(\Omega)\}
\]
and
\[
\|x\|_{1,u} = \|x u\|_1 = \int_{\Omega} |x u| \, d\mu.
\]

Let \(\Gamma \subset \Omega\) be a measurable set with \(\mu(\Gamma) > 0\) and \(u\) be a weight function on \(\Gamma\). It follows that
\[
L_{\infty,u}(\Gamma) = \{x \in L_0 : \mu(\text{supp}(x) \setminus \Gamma) = 0 \text{ and } x u \in L_{\infty}(\Gamma)\}
\]
and for \(x \in L_{\infty,u}(\Gamma)\), \(\|x\|_{\infty,u} = \text{ess sup}_{t \in \Gamma} |x(t) u(t)|\). Similarly
\[
L_{1,u}(\Gamma) = \{x \in L_0 : \mu(\text{supp}(x) \setminus \Gamma) = 0 \text{ and } x u \in L_1(\Gamma)\}
\]
and for \(x \in L_{1,u}(\Gamma)\), \(\|x\|_{1,u} = \int_{\Gamma} |x u| \, d\mu\). Both spaces \((L_{\infty,u}(\Gamma), \| \cdot \|_{\infty,u})\) and \((L_{1,u}(\Gamma), \| \cdot \|_{1,u})\) are Banach function lattices on \((\Omega, \Sigma, \mu)\) with the Fatou property continuously embedded in \(L_0\) and with \(\text{supp}(L_{\infty,u}(\Gamma)) = \text{supp}(L_{1,u}(\Gamma)) = \Gamma\).

It is clear that \(L_{1,u}(\Gamma) \simeq L_1(\Gamma)\) by the isometric isomorphism from \(L_{1,u}(\Gamma)\) to \(L_1(\Gamma)\) mapping \(x \mapsto x u\). Similarly, \(L_{\infty,u}(\Gamma) \simeq L_{\infty}(\Gamma)\). It follows that \((L_{1,u}(\Gamma))' = L_{\infty,1/u}(\Gamma)\) and \((L_{\infty,u}(\Gamma))' = L_{1,1/u}(\Gamma)\) with equality of norms.

Let \(v, w \in L_0\) be weight functions on \(\Gamma\) and \(\Omega\), respectively, where \(\mu(\Gamma) > 0\). We consider the following spaces (see introduction),
\[
L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma) = \{x \in L_0 : x \in L_{1,w}(\Omega) \text{ and } x \chi_{\Gamma} \in L_{\infty,v}(\Gamma)\}
\]
equipped with the norm
\[
\|x\|_{w,v} = \|x\|_{L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma)} = \max\{\|x\|_{1,w}, \|x \chi_{\Gamma}\|_{\infty,v}\},
\]
and
\[
L_{\infty,w}(\Omega) + L_{1,v}(\Gamma) = \{x \in L_0 : x = y + z \text{ for some } y \in L_{\infty,w}(\Omega) \text{ and } z \in L_{1,v}(\Gamma)\}
\]
equipped with the norm
\[ \|x\|_{w,v}^\Sigma = \|x\|_{L^{\infty,w}(\Omega) + L^1,v(\Gamma)}^\Sigma = \inf \{ \|y\|_{\infty,w} + \|z\|_{1,v} : x = y + z, y \in L^{\infty,w}(\Omega), z \in L^1,v(\Gamma) \} . \]

Both \( L^{1,v}(\Omega) \cap L^{\infty,w}(\Gamma) \) and \( L^{\infty,w}(\Omega) + L^1,v(\Gamma) \) with their respective norms are Banach function lattices on \( (\Omega, \Sigma, \mu) \) with the Fatou property and with \( \text{supp}(L^{1,v}(\Omega) \cap L^{\infty,w}(\Gamma)) = \text{supp}(L^{\infty,w}(\Omega) + L^1,v(\Gamma)) = \Omega \). Moreover, the following Köthe duality holds true (see Theorem 2.1),
\[ (L^{1,v}(\Omega) \cap L^{\infty,w}(\Gamma))^\prime = L^{\infty,1/w}(\Omega) + L^1,1/v(\Gamma) , \]
and
\[ (L^{\infty,1/w}(\Omega) + L^{1,1/v}(\Gamma))^\prime = L^{1,w}(\Omega) \cap L^{\infty,v}(\Gamma) . \]

The following lemma gives two useful conditions equivalent to the Daugavet property.

**Lemma 3.1** (See [18] Lemma 2.2). The following are equivalent.

(i) A Banach space \( (X, \| \cdot \|) \) has the Daugavet property.

(ii) For every \( x \in S(X) \) and \( y^* \in S(X^*) \) and every \( \epsilon > 0 \) there is \( x^* \in S(X^*) \) such that \( x^*(x) > 1 - \epsilon \) and \( \|x^* + y^*\| > 2 - \epsilon \).

(iii) For every \( x \in S(X) \) and \( x^* \in S(X^*) \) and every \( \epsilon > 0 \) there is \( y \in S(X) \) such that \( x^*(y) > 1 - \epsilon \) and \( \|x + y\| > 2 - \epsilon \).

We need two technical lemmas.

**Lemma 3.2.** Let \( \Gamma \) be a measurable subset of \( \Omega \) with \( \mu(\Gamma) > 0 \) and \( v, w \in \ell_0 \) be weight functions on \( \Gamma \) and \( \Omega \), respectively. Let the space \( X = L^1,v(\Gamma) + L^{\infty,w}(\Omega) \) be equipped with the norm
\[ \|x\| = \|x\|_{w,v}^\Sigma \]
Then
\[ X^* \simeq (L^{1,1/v}(\Gamma) \cap L^{\infty,1/w}(\Omega)) \oplus S \]
and the norm on \( X^* \) is given by
\[ \|F\| = \max\{ \|f\chi_\Gamma\|_{\infty,1/v}, \|f\|_{1,1/w} + \|T\| \} , \]
where \( F = F_1 + T \), \( F_1(y) = \int\int f(y) d\mu \) for some \( f \in L^{\infty,1/v}(\Gamma) \cap L^{1,1/w}(\Omega) \) and \( T \) is a singular functional on \( X \).

**Proof.** The method of proof is similar to the proof of Theorem 2.12 [4]. By Theorem 2.1 and from the general theory of Banach lattices we have that \( X^* \simeq (L^{\infty,1/v}(\Gamma) \cap L^{1,1/w}(\Omega)) \oplus S \). Let \( x \in X \), that is \( x = y + z \) where \( y \in L^{1,v}(\Gamma) \) and \( z \in L^{\infty,w}(\Omega) \). Let \( F, F_1, T \) and \( f \) be as in the statement of the theorem. Since \( y \) is an order continuous element of \( X \) we have that \( T(y) = 0 \). Hence
\[ |F(x)| \leq |F_1(y)| + |F_1(z)| + |T(z)| \leq \int \|f\chi_\Gamma\| d\mu + \int \|f\|_{1,1/w} + \|T\| \| z \|_{X} \]
\[ \leq \|f\chi_\Gamma\|_{\infty,1/v} + \|f\|_{1,1/w} + \|T\| \| z \|_{\infty,w} \]
\[ \leq \max\{ \|f\chi_\Gamma\|_{\infty,1/v} + \|f\|_{1,1/w} + \|T\| \} (\|y\|_{1,v} + \|z\|_{\infty,w}) . \]

Therefore
\[ |F| \leq \max\{ \|f\chi_\Gamma\|_{\infty,1/v} + \|f\|_{1,1/w} + \|T\| \} . \]

For every \( 0 \leq x = y + z \in X \) with \( y \in L^{1,v}(\Gamma) \) and \( z \in L^{\infty,w}(\Omega) \), since \( y \) is an order continuous element of \( X \), \( |T|(y) = 0 \). Hence, for every \( \epsilon > 0 \), there are \( 0 \leq y_0 \in L^{1,v}(\Gamma) \) and \( 0 \leq z_0 \in L^{\infty,w}(\Omega) \) such that
\[ \|y_0\|_{1,v} + \|z_0\|_{\infty,w} < 1 + \epsilon \text{ and } |T|(z_0) > \|T\| - \epsilon/2 . \]
Since \( f \in L^{1,1/w}(\Omega) \) there exists \( 0 \leq z_1 \in B(L^{\infty,w}(\Omega)) \) such that
\[ |F_1|(z_1) = \int \|f\|_{1,1/w} d\mu > \|f\|_{1,1/w} - \epsilon/2 . \]
For \( z_2 = \max\{ z_0, z_1 \} \) we have that \( 0 \leq z_2 \in X \) and
\[ |F(z_2)| = |F_1|(z_2) + |T|(z_2) = \int \|f\|_{1,1/w} d\mu + |T|(z_2) \geq \int \|f\|_{1,1/w} d\mu + |T|(z_0) > \|f\|_{1,1/w} + \|T\| - \epsilon . \]
Since \(\|z_2\|_{w,v} < 1 + \varepsilon\), we get that
\[
\|F\| \geq (\|f\|_{1,w} + \|T\| - \varepsilon)/(1 + \varepsilon).
\]
It follows that \(\|F\| \geq \|f\|_{1,w} + \|T\|\). We also have that
\[
\|F\| = \sup_{\|x\|_{w,v,\varepsilon} \leq 1} |F(x)| = \sup_{\|x\|_{w,v} \leq 1} |F(x)| = \sup_{\|x\|_{w,v} \leq 1} \left| \int_{\Omega} f \chi_{T}x \, d\mu \right| = \|f\chi_{T}\|_{\infty,l/v}.
\]
Hence \(\|F\| \geq \max\{\|f\chi_{T}\|_{\infty,l/v}, \|f\|_{1,w} + \|T\|\}\) and the claim follows. \(\square\)

**Lemma 3.3.** Let \(\Gamma\) be a measurable subset of \(\Omega\) with \(\mu(\Gamma) > 0\). Let \(v, w \in L_0\) be weight functions on \(\Gamma\) and \(\Omega\), respectively. The space \(X = L_{1,v}(\Gamma) + L_{\infty,w}(\Omega)\) equipped with the norm \(\|x\|_{w,v} = \|x\|_{w,v}\) is order continuous if and only if \(\mu(\Omega \setminus \Gamma) = 0\) and \(\int_{\Omega} v/w \, d\mu < \infty\).

**Proof.** Assume first that \(\mu(\Omega \setminus \Gamma) = 0\) and \(\int_{\Omega} v/w \, d\mu < \infty\). Let \(x \in X\) be arbitrary, that is \(x = y + z\) where \(y \in L_{1,v}\) and \(z \in L_{\infty,w}\). Since \(\|z\|_{w,v} = \int_{\Omega} |z|w/v \, d\mu \leq \|z\|_{\infty,w} \int_{\Omega} v \, d\mu < \infty\) we get \(\|x\|_{1,v} < \infty\), that is \(x \in L_{1,v}\). It follows that \(X = L_{1,v}\) as sets and \(\|x\| \leq \|x\|_{1,v}\) for every \(x \in X\), and consequently \((X, \|\cdot\|)\) is order continuous.

If \(\mu(\Omega \setminus \Gamma) > 0\) then in order to see that \(X\) is not order continuous it is enough to take \(x = (1/w)\chi_{\Omega \setminus \Gamma}\) and \(x_n = (1/w)\chi_{\Omega \setminus \Gamma}, n \in \mathbb{N}\), where \((A_n) \subset \Omega \setminus \Gamma\) is a sequence of measurable sets such that \(A_{n+1} \subset A_n\), \(n \in \mathbb{N}\) and \(\mu(A_n) \to 0\) as \(n \to \infty\).

Assume now that \(\mu(\Omega \setminus \Gamma) = 0\) and \(\int_{\Omega} v/w \, d\mu = \infty\). We have two cases.

Case 1. There is a measurable set \(A \subset \Omega\) of finite measure such that \(\int_A v \, d\mu = \infty\). In this case there exists a sequence \((A_n)\) of measurable subsets of \(A\) such that \(A_{n+1} \subset A_n\) for \(n \in \mathbb{N}\), \(\mu(A_n) \to 0\) as \(n \to \infty\) and \(\int_{A_n} v \, d\mu = \infty, n \in \mathbb{N}\). Let \(x = (1/w)\chi_A\) and \(x_n = (1/w)\chi_{A_n}, n \in \mathbb{N}\). We have that \(0 \leq x_n \leq x\) on \(\Omega\) and \(x_n \downarrow 0 \mu\text{-a.e. on } \Omega\). But \(\|x\| = \|x_n\| = 1\) for every \(n \in \mathbb{N}\). Indeed, clearly \(\|x_n\| \leq 1\). Let \(x_n = y_n + z_n\) where \(y_n \in L_{1,v}(\Gamma)\), \(z_n \in L_{\infty,w}\) and for \(c \in (0,1)\) denote \(B_c = \{t \in A : |y_n(t)| > c/w(t)\}\). Since \(y_n \in L_{1,v}(\Gamma)\) it must be that \(\int_{B_c} v \, d\mu < \infty\). It follows that \(\mu(A_n \setminus B_c) > 0, |z_n| \geq (1-c)/w\) on \(A_n \setminus B_c\) and \(\|y_n\|_{1,v} + \|z_n\|_{\infty,w} \geq \|z_n\|_{\infty,w} \geq 1 - c\). Since \(c \in (0,1)\) was arbitrary, we get that \(\|x_n\| = 1\), \(n \in \mathbb{N}\). Similarly \(\|x\| = 1\). Hence \(X\) is not order continuous.

Case 2. For every measurable set \(A \subset \Omega\) of finite measure \(\int_A v \, d\mu < \infty\). In this case \(\mu(\Omega) = \infty\). By \(\sigma\)-finiteness of \(\mu\) there exists a sequence \((A_n)\) of measurable sets such that \(\Omega = \bigcup_{n=1}^{\infty} A_n, A_n \subset A_{n+1}\), \(\mu(A_n) < \infty, n \in \mathbb{N}\). Clearly \(\int_{\Omega \setminus A_n} v \, d\mu = \infty, n \in \mathbb{N}\). Let \(x = (1/w)\chi_{\Omega \setminus A_n}\) and \(x_n = (1/w)\chi_{\Omega \setminus A_n}, n \in \mathbb{N}\). We have that \(0 \leq x_n \leq x\) on \(\Omega\) and \(x_n \downarrow 0 \mu\text{-a.e. on } \Omega\). But \(\|x\| = \|x_n\| = 1\) for every \(n \in \mathbb{N}\). Indeed, clearly \(\|x_n\| \leq 1\). Let \(x_n = y_n + z_n\) where \(y_n \in L_{1,v}(\Gamma)\), \(z_n \in L_{\infty,w}\) and for \(c \in (0,1)\) denote \(B_c = \{t \in \Omega \setminus A_n : |y_n(t)| > c/w(t)\}\). Since \(y_n \in L_{1,v}(\Gamma)\) it must be that \(\int_{B_c} v \, d\mu < \infty\). It follows that \(\mu(\Omega \setminus A_n) \setminus B_c) > 0, |z_n| \geq (1-c)/w\) on \(\Omega \setminus A_n \setminus B_c\) and \(\|y_n\|_{1,v} + \|z_n\|_{\infty,w} \geq \|z_n\|_{\infty,w} \geq 1 - c\). Thus \(\|x_n\| = 1\), \(n \in \mathbb{N}\). Analogously we show that \(\|x\| = 1\). Hence \(X\) is not order continuous. \(\square\)

The following lemma gives conditions on weights \(v\) and \(w\) for which the space \(L_{1,v}(\Gamma) + L_{\infty,w}(\Omega)\) fails the Daugavet property.

**Lemma 3.4.** Let \(\Gamma\) be a measurable subset of \(\Omega\) with \(\mu(\Gamma) > 0\) and \(v, w \in L_0\) be weight functions on \(\Gamma\) and \(\Omega\), respectively. The space \(X = L_{1,v}(\Gamma) + L_{\infty,w}(\Omega)\) equipped with the norm \(\|x\|_{w,v} = \|x\|_{w,v}\) does not have the Daugavet property whenever \(\mu(\Omega \setminus \Gamma) > 0\) or \(\int_\Gamma v/w \, d\mu > 1\).

**Proof.** We consider two cases.

Case 1. Suppose first that \(\mu(\Omega \setminus \Gamma) > 0\) or \(\int_\Gamma v/w \, d\mu = \infty\). By Lemma 3.3 the space \(X\) is not order continuous.

Clearly, there exist constants \(\alpha, \beta > 0\) and a measurable set \(\Omega_0 \subset \Gamma\) of positive and finite measure such that \(\alpha \leq w(t), v(t) \leq \beta\) for all \(t \in \Omega_0\). Let \(A \subset \Omega_0\) be a measurable set such that \(0 < \mu(A) \leq \beta/\alpha\). It follows that
\[
(2) \quad \frac{1}{\mu(A)} \frac{w(t)}{v(t)} \geq 1 \text{ for all } t \in A.
\]
Define
\[ x = \frac{1}{\mu(A)} \frac{1}{v} \chi_A. \]

Since \( \|x\|_{1,v} = 1 \) we have that \( \|x\| \leq 1 \). Let \( F_0 \in X^* \) be defined by the function \( f_0 = v \chi_A \), that is \( F_0(z) = \int_A v z \, d\mu \), \( z \in X \). Since \( f_0 \in X' = L_{1,1/v}^\infty(\Gamma) \cap L_{1,1/w}(\Omega) \), \( \|f_0\|_{\infty,1/v} = 1 \) and by (2)
\[ \|f_0\|_{1,1/w} = \int_A \frac{v}{w} \, d\mu \leq 1, \]
we get that \( \|F_0\| \leq 1 \). Moreover \( F_0(x) = 1 \). It follows that \( \|x\| = 1 \).

Fix \( b \in [1/2,1) \) and let \( 0 < \epsilon < 1 \) and \( c > 1 \) be such that
\[
0 < \frac{c \epsilon}{1 - c \epsilon} \leq \mu(A) \alpha/\beta \quad \text{and} \\
(4) \quad c \epsilon < 1 - b.
\]

Define \( g = -bv \chi_A \).

Clearly \( \|g\|_{\infty,1/v} = b \) and by (2), \( \|g\|_{1,1/w} \leq b \). Since the space \( X \) is not order continuous, there are non-trivial singular functionals on \( X \). Let \( S_1 \in B(X^*) \) be a singular functional on \( X \) with the norm \( \|S_1\| = 1 - \|g\|_{1,1/w} \). Define \( G \in X^* \) by
\[
G(y) = \int_{\Omega} gy \, d\mu + S_1(y), \quad y \in X.
\]

By Lemma 3.2, \( \|G\| = \max\{\|g\|_{\infty,1/v}, \|g\|_{1,1/w} + \|S_1\|\} = 1 \).

Let \( F \in S(X^*) \) be such that \( F(x) > 1 - \epsilon \). Since \( x \in L_{1,1}(\Gamma) \) and \( L_{1,v}(\Gamma) \) is order continuous and \( \|z\| \leq \|z\|_{1,v} \) for all \( z \in L_{1,1}(\Gamma) \) it is clear that \( x \) is an order continuous element of \( X \). In view of \( F = H + S_2 \) where \( H \in B(X^*) \) is an integral functional defined by a function \( h \in B(L_{\infty,1/v}(\Gamma) \cap L_{1,1/w}(\Omega)) \) and \( S_2 \in B(X^*) \) is a singular functional on \( X \) and \( x \) is an order continuous element of \( X \), we get that \( S_2(x) = 0 \). Hence
\[ 1 - \epsilon < F(x) = H(x) = \frac{1}{\mu(A)} \int_A \frac{h}{v} \, d\mu. \]

It is not difficult to see that for every \( d > 1 \) there exists a measurable subset \( B \subset A \) with \( \mu(B) > d^{-1} \mu(A) \) and
\[ h > (1 - de)v \mu \text{-a.e. on } B. \]

Taking \( d = c \) and the corresponding \( B \subset A \) from the above statement, in view of \( v \geq \alpha \) and \( w \leq \beta \) on \( A \), we get that
\[ \|h \chi_B\|_{1,1/w} \geq (1 - c \epsilon) \mu(B) \alpha/\beta. \]

Now, by Lemma 3.2 we get \( \|h\|_{1,1/w} + \|S_2\| \leq \|F\| = 1 \), and so
\[ \|h \chi_{\Omega \cap B}\|_{1,1/w} + \|S_2\| = \|h\|_{1,1/w} - \|h \chi_B\|_{1,1/w} + \|S_2\| \leq 1 - (1 - c \epsilon) \mu(B) \alpha/\beta. \]

Now we show that \( \|F + G\|_{X^*} \leq 2 - \epsilon \). Since \( B \subset \Gamma \), \( \|h \chi_{\Gamma}\|_{\infty,1/v} = \|(h/v) \chi_{\Gamma}\|_\infty \leq \|F\| = 1 \), \( h/v > 1 - \epsilon \geq b \mu \text{-a.e. on } B \) by (1), and \( 1 - b \leq b \) we get that
\[
\|h + g\|_{1,1/w} + \|S_1 + S_2\| \leq \|(h + g) \chi_B\|_{1,1/w} + \|(h + g) \chi_{\Omega \setminus B}\|_{1,1/w} + \|S_1\| + \|S_2\|
\leq \int_B \frac{h}{v} \, d\mu + \|h \chi_{\Omega \cap B}\|_{1,1/w} + \|S_2\| + \|g \chi_B\|_{1,1/w} + \|S_1\|
\leq (1 - b) \|v \chi_B\|_{1,1/w} + 1 - (1 - c \epsilon) \mu(B) \alpha/\beta + b \|v \chi_B\|_{1,1/w} + \|S_1\|
\leq 1 - (1 - c \epsilon) \mu(B) \alpha/\beta + \|g \chi_B\|_{1,1/w} + \|S_1\|
\leq 2 - (1 - c \epsilon) \mu(B) \alpha/\beta \leq 2 - \epsilon,
\]
where the last inequality follows from \( \mu(B) > c^{-1} \mu(A) \) and (3). Moreover, by (4) we get that
\[
\|(h + g) \chi_{\Gamma}\|_{\infty,1/v} \leq \|h \chi_{\Gamma}\|_{\infty,1/v} + \|g\|_{\infty,1/v} \leq 1 + b \leq 2 - \epsilon.
\]

It follows that
\[
\|F + G\|_{X^*} = \max\{\|(h + g) \chi_{\Gamma}\|_{\infty,1/v}, \|h + g\|_{1,1/w} + \|S_1 + S_2\|\} \leq 2 - \epsilon,
\]
hence $X$ fails the Daugavet property by Lemma 3.1(ii).

Case 2. Suppose now that $\mu(\Omega \setminus \Gamma) = 0$ and $1 < \int_{\Omega} v/w \, d\mu < \infty$. By Lemma 3.3 the space $X$ is order continuous. In this case the whole proof above can be repeated with the following modifications. If $\int_{\Omega} v/w \, d\mu \leq 2$ then fix $b = (\int_{\Omega} v/w \, d\mu)^{1/2}$, $g = -bv\chi_\Omega$, and $S_1 = S_2 = 0$. Otherwise, find a measurable subset $C \subset \Omega$ containing $A$ such that $\int_{C} v/w \, d\mu = 2$, fix $b = 1/2$, $g = -bv\chi_C$ and $S_1 = S_2 = 0$. In both cases we have $\|g\|_{1/w} = 1$, $\|g\|_{\infty} = b < 1$. Hence $\|G\| = \max\{\|g\|_{1/v} \|g\|_{1/w} \} = 1$.

Now we can characterize $L_{1,v}(\Gamma) + L_{\infty,w}(\Omega)$ spaces with the Daugavet property.

**Theorem 3.5.** Let $\Gamma$ be a measurable subset of $\Omega$ such that $\mu(\Gamma) > 0$ and $v, w \in L_0$ be weight functions on $\Gamma$ and $\Omega$, respectively. Let the space $X = L_{1,v}(\Gamma) + L_{\infty,w}(\Omega)$ be equipped with the norm $\|x\| = \|x\|_{\Sigma_{w,v}}$. The following conditions are equivalent.

(i) $X$ has the Daugavet property.

(ii) $X = L_{1,v}$.

(iii) $\mu(\Omega \setminus \Gamma) = 0$ and $\int_{\Omega} v/w \, d\mu \leq 1$.

**Proof.** If $\int_{\Omega} v/w \, d\mu \leq 1$ and $\Gamma = \Omega$ up to a set of measure zero then $\|x\|_{1,v} = \int_{\Omega} |x|v \, d\mu = \int_{\Omega} |x| \|w/v\| \, d\mu \leq \|x\|_{\infty,w} \int_{\Omega} v/w \, d\mu \leq \|x\|_{\infty,w}$ for every $x \in X$. It follows that, for every $x \in X$, if $x = y + z$ where $y \in L_{1,v}$ and $z \in L_{\infty,w}$ we have that $\|y\|_{1,v} + \|z\|_{\infty,w} \geq \|y\|_{1,v} + \|z\|_{1,v} \geq \|y + z\|_{1,v} = \|x\|_{1,v}$. Hence $\|x\| \geq \|x\|_{1,v}$. The opposite inequality is obvious. Therefore $\|x\| = \|x\|_{1,v}$ for every $x \in X$. Since $L_{1,v} \subset X$ we have that $X = L_{1,v}$. Hence (iii) implies (ii) which in turn clearly implies (i). By Lemma 3.1(iii) (iii) follows from (i). □

Similarly as above we describe $L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma)$ spaces with the Daugavet property.

**Lemma 3.6.** Let $\Gamma$ be a measurable subset of $\Omega$ such that $\mu(\Gamma) > 0$ and $v, w \in L_0$ be weight functions on $\Gamma$ and $\Omega$, respectively. The space $X = L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma)$, with the norm $\|x\| = \|x\|_{w,v}$ does not have the Daugavet property whenever $\mu(\Omega \setminus \Gamma) > 0$ or $\int_{\Gamma} w/v \, d\mu > 1$.

**Proof.** We show that the condition (iii) of Lemma 3.1 fails. We consider two cases.

(i) Assume that $\mu(\Omega \setminus \Gamma) > 0$. Let $c \in (0,1)$ and $A \subset \Gamma$ be a measurable set of positive and finite measure such that

$$c \int_A w/v \, d\mu = : \gamma \in (0,c).$$

Let $c_2 > 0$ and $A_2 \subset \Omega \setminus \Gamma$ be a measurable set such that $c_2 \int_{A_2} w \, d\mu = 1 - \gamma$. Define

$$x = c_2^{-1} \chi_A + c_2^2 \chi_{A_2}.$$  

Since $\|x\|_{1,w} = c < 1$ we have that $\|x\|_{1,w} = 1 = \|x\|_{1,w}$.

Let $F \in X^*$ be induced by $f = -(c/\gamma)w \chi_A$. Since $X' = L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma)$ and $\|f\|_{1,v} = 1$ we get that $\|F\| \leq 1$. Let $z = (1/v)\chi_{A}$. Clearly $\|z\|_{\infty,v} = 1$ and $\|z\|_{1,w} = \gamma/c < 1$. Whence $\|z\| = 1$. Moreover $F(z) = -1$. It follows that $\|F\| = 1$.

We finish the proof using methods similar to those in Theorem 2.9 [2] or Proposition 4.3 [5].

Let $0 < \epsilon < 2\gamma(1-c)/(1-c+2\gamma)$ and $y \in S(X)$ be such that $F(y) > 1 - \epsilon$. Observe that $\epsilon < 1 - c$.

Let

$$D = \{t \in A : -y(t) \geq 0\}, \ E = \left\{ t \in D : -y(t) \leq \frac{c}{v(t)} \right\}, \ B = \left\{ t \in D : -y(t) > \frac{c}{v(t)} \right\}.$$  

Since $|yv| \leq 1$ $\mu$-a.e. on $\Gamma$, $D = E \cup B$ and $E \cap B = \emptyset$, we get that

$$1 - \epsilon < F(y) = \int_{\Gamma} c \gamma \frac{w(-y)}{v} \, d\mu \leq \frac{c}{\gamma} \int_D w(-y) \, d\mu = \frac{c}{\gamma} \int_E w(-y) \, d\mu + \frac{c}{\gamma} \int_B w(-y) \, d\mu$$

$$\leq \frac{c}{\gamma} \int_E w \, d\mu + \frac{c}{\gamma} \int_B w \, d\mu \leq c \left( \frac{c}{\gamma} \int_E w \, d\mu - \frac{c}{\gamma} \int_B v \, d\mu \right) + \frac{c}{\gamma} \int_B v \, d\mu$$

$$= \frac{c}{\gamma} \left( \gamma - c \frac{c}{\gamma} \int_B \frac{w}{v} \, d\mu \right) + \frac{c}{\gamma} \int_B v \, d\mu = c + \frac{c}{\gamma}(1-c) \int_B \frac{w}{v} \, d\mu.$$
It follows that
\begin{equation}
\tag{5}
c \int_B \frac{w}{v} \, d\mu \geq \frac{\gamma}{1 - c}(1 - c - \epsilon).
\end{equation}
We also have that
\begin{equation}
\tag{6}
\| (x + y) \chi_{\Gamma} \|_{\infty, v} \leq \| x \chi_{\Gamma} \|_{\infty, v} + \| y \chi_{\Gamma} \|_{\infty, v} \leq c + 1 < 2 - \epsilon,
\end{equation}
and
\begin{align*}
\|(x + y) \chi_D\|_{1,w} &= \|(x + y) \chi_E\|_{1,w} + \|(x + y) \chi_B\|_{1,w} \\
&= \int_E \left| \frac{c}{v} - (-y) \right| w \, d\mu + \int_B \left| \frac{c}{v} - (-y) \right| w \, d\mu \\
&= \int_E \left( \frac{c}{v} - (-y) \right) w \, d\mu + \int_B \left( - \frac{c}{v} - y \right) w \, d\mu \\
&= c \left( \int_E \frac{w}{v} \, d\mu + \int_B \frac{w}{v} \, d\mu \right) + \|(x + y) \chi_B\|_{1,w}.
\end{align*}
Moreover
\begin{align*}
\| x \chi_D \|_{1,w} + \| y \chi_D \|_{1,w} &= \| x \chi_E \|_{1,w} + \| y \chi_B \|_{1,w} + \| y \chi_E \|_{1,w} + \| y \chi_B \|_{1,w} \\
&= c \left( \int_E \frac{w}{v} \, d\mu + \int_B \frac{w}{v} \, d\mu \right) + \| y \chi_E \|_{1,w} + \| y \chi_B \|_{1,w}.
\end{align*}
It follows that
\begin{equation}
\tag{7}
\| (x + y) \chi_D \|_{1,w} \leq \| x \chi_D \|_{1,w} + \| y \chi_D \|_{1,w} - 2c \int_B \frac{w}{v} \, d\mu.
\end{equation}
Finally by inequality (5) and definition of $\epsilon$, we get that
\begin{align*}
\| x + y \|_{1,w} &= \| (x + y) \chi_D \|_{1,w} + \|(x + y) \chi_\Omega \setminus D\|_{1,w} \\
&\leq \| x \chi_D \|_{1,w} + \| y \chi_D \|_{1,w} - 2c \int_B \frac{w}{v} \, d\mu + \| x \chi_\Omega \setminus D\|_{1,w} + \| y \chi_\Omega \setminus D\|_{1,w} \\
&= 2 - 2c \int_B \frac{w}{v} \, d\mu \leq 2 - \frac{2\gamma}{1 - c}(1 - c - \epsilon) < 2 - \epsilon.
\end{align*}
Whence by (6)
\begin{align*}
\| x + y \| &= \max \{ \| x + y \|_{1,w}, \| (x + y) \chi_{\Gamma} \|_{\infty, v} \} \leq 2 - \epsilon,
\end{align*}
which finishes the proof in this case by Lemma 3.1(ii).

(ii) Assume now that $\mu(\Omega \setminus \Gamma) = 0$ and $\int_{\Omega} w/v \, d\mu > 1$. There are a set $A \subset \Omega$ with finite and positive measure and a constant $c \in (0, 1)$ such that
\begin{equation}
\tag{8}
c \int_A \frac{w}{v} \, d\mu = 1.
\end{equation}
Moreover, there is a measurable set $A_1 \subset A$ such that
\begin{equation}
\tag{9}
\int_{A_1} \frac{w}{v} \, d\mu = 1.
\end{equation}
Let
\begin{equation}
\tag{10}
x = c \frac{1}{v} \chi_A,
\end{equation}
and $F \in X^*$ be induced by $f = -w \chi_{A_1}$. Since $X' = L_{\infty,1/w} + L_{1,1/v}$ and $\| f \|_{\infty,1/w} = 1$ we get that $\| F \| \leq 1$. Let $z = (1/v) \chi_{A_1}$. Clearly $\| z \|_{\infty,v} = \| z \|_{1,w} = 1 = \| z \|$. Moreover $F(z) = -1$. It follows that $\| F \| = 1$.

We finish the proof similarly as in the previous case.
Let $0 < \varepsilon < \min\{2c(1-c)/(1+c), 1-c\}$ and $y \in S(X)$ be such that $F(y) > 1 - \varepsilon$. Let

$$D = \{ t \in A_1 : -y(t) \geq 0 \}, \quad E = \left\{ t \in D : -y(t) \leq \frac{c}{v(t)} \right\}, \quad B = \left\{ t \in D : -y(t) > \frac{c}{v(t)} \right\}.$$ 

Since $|yv| \leq 1$ $\mu$-a.e. on $\Omega$, $D = E \cup B$ and $E \cap B = \emptyset$ we get that

$$1 - \varepsilon < F(y) = \int_{A_1} w(-y) \, d\mu \leq \int_{D} w(-y) \, d\mu = \int_{E} w(-y) \, d\mu + \int_{B} w(-y) \, d\mu$$

$$\leq \int_{E} c \frac{w}{v} \, d\mu + \int_{B} \frac{w}{v} \, d\mu \leq c \left( \int_{A_1} \frac{w}{v} \, d\mu - \int_{B} \frac{w}{v} \, d\mu \right) + \int_{B} \frac{w}{v} \, d\mu$$

$$\leq c \left( 1 - \int_{B} \frac{w}{v} \, d\mu \right) + \int_{B} \frac{w}{v} \, d\mu = c + (1 - c) \int_{B} \frac{w}{v} \, d\mu.$$ 

It follows that

$$(8) \quad c \int_{B} \frac{w}{v} \, d\mu \geq \frac{c}{1 - c} (1 - c - \varepsilon).$$

In the same way as in case (i) we obtain inequalities (6) and (7). Finally by inequalities (7), (8) and definition of $\varepsilon$, we get that

$$\|x + y\|_{1,w} = \|(x + y)\chi_D\|_{1,w} + \|(x + y)\chi_{\Omega \setminus D}\|_{1,w}$$

$$\leq \|x\chi_D\|_{1,w} + \|y\chi_D\|_{1,w} - 2c \int_{B} \frac{w}{v} \, d\mu + \|x\chi_{\Omega \setminus D}\|_{1,w} + \|y\chi_D\|_{1,w}$$

$$= 2 - 2c \int_{B} \frac{w}{v} \, d\mu \leq 2 - \frac{2c}{1 - c} (1 - c - \varepsilon) < 2 - \varepsilon.$$ 

Thus by (6)

$$\|x + y\| = \max\{\|x + y\|_{1,w}, \|x + y\|_{\infty,v}\} \leq 2 - \varepsilon,$$

which finishes the proof in this case by Lemma 3.1(iii). \hfill \Box

**Theorem 3.7.** Let $\Gamma$ be a measurable subset of $\Omega$ such that $\mu(\Gamma) > 0$ and $v, w \in L_0$ be weight functions on $\Gamma$ and $\Omega$, respectively. Let $X = L_{1,w}(\Omega) \cap L_{\infty,v}(\Gamma)$ be equipped with the norm $\|x\| = \|x\|_{w,v}$. The following conditions are equivalent.

(i) $X$ has the Daugavet property.

(ii) $X = L_{\infty,v}$.

(iii) $\mu(\Omega \setminus \Gamma) = 0$ and $\int_{\Omega} w/v \, d\mu \leq 1$.

**Proof.** Assuming (iii) we have that $\|x\|_{1,w} = \int_{\Omega} |x| w \, d\mu = \int_{\Omega} |x| v w/v \, d\mu \leq \|x\|_{\infty,v} \int_{\Omega} w/v \, d\mu \leq \|x\|_{\infty,v}$. It follows that $\|x\| = \|x\|_{\infty,v}$ and therefore $X = L_{\infty,v}$. Hence (iii) implies (ii) which in turn clearly implies (i). By Lemma 3.6 we have that (iii) follows from (i). \hfill \Box

4. THE DAUGAVET PROPERTY IN THE MUSIELAK-ORLICZ SPACES

We begin this section with a basic observation regarding Orlicz functions.

**Lemma 4.1.** Let $\varphi$ be an Orlicz function. For every closed and bounded interval $I \subset (d_{\varphi}, b_{\varphi})$ there is a constant $\sigma \in (0, 1)$ such that $2\varphi(u/2)/\varphi(u) \leq \sigma$ for $u \in I$. Moreover, if $\varphi(b_{\varphi}) < \infty$ then the same statement holds true for closed intervals $I \subset (d_{\varphi}, b_{\varphi})$.

**Proof.** It is not difficult to see that if $\varphi(u/2) = \varphi(u)/2$ for some $u \geq 0$ then $\varphi(v/2) = \varphi(v)/2$ for all $v \in [0, u]$. Indeed, suppose that there are $u > 0$ and $v \in (0, u)$ such that $\varphi(u/2) = \varphi(u)/2$ and $\varphi(v/2) < \varphi(v)/2$. Then $[\varphi(u/2) - \varphi(v/2)]/[u/2 - v/2] > [\varphi(u) - \varphi(v)]/[u - v]$ which contradicts the convexity of $\varphi$. It follows that for all $u \in (d_{\varphi}, b_{\varphi}), \varphi(u/2) < \varphi(u)/2$. Since the ratio $2\varphi(u/2)/\varphi(u)$ is a continuous function on $(d_{\varphi}, b_{\varphi})$, for every closed and bounded interval $I \subset (d_{\varphi}, b_{\varphi})$ there is a positive constant $\sigma < 1$ such that $2\varphi(u/2)/\varphi(u) \leq \sigma$ for $u \in I$. In the case when $\varphi(b_{\varphi}) < \infty$ the interval $I$ can include the point $b_{\varphi}$. \hfill \Box

The following lemma generalizes this fact to the case of a Musielak-Orlicz function.
Lemma 4.2. Let $M$ be a Musielak-Orlicz function. If $\mu\{t \in \Omega : d_M(t) < b_M(t)\} > 0$ then there exist a set $A \in \Sigma$ with positive and finite measure and numbers $a, b$ and $\sigma_1$ such that $a < b$, $0 < \sigma_1 < 1$, $[a, b] \subset (d_M(t), b_M(t))$, $2M(t, u/2)/M(t, u) \leq \sigma_1$ for $t \in A$ and $u \in [a, b]$, and $M$ is bounded on $A \times [a, b]$.

Proof. Let $D = \{t \in \Omega : d_M(t) < b_M(t)\}$ and $\{r_1, r_2, \ldots\} \subset [0, \infty)$ be a countable dense set. Since $D = \bigcup_n \{t \in D : d_M(t) < r_n < b_M(t)\}$, there are numbers $a < b$ and a set $D' \subset D$ such that $\mu(D') > 0$ and $[a, b] \subset (d_M(t), b_M(t))$ for $t \in D'$.

Let $\sigma(t) = \sup\{2M(t, u/2)/M(t, u) : u \in [a, b]\}$, $t \in D'$. Clearly $\sigma$ is a measurable function and $0 < \sigma(t) < 1$ for all $t \in D'$. Denote $D_n = \{t \in D' : 1 - 1/n \leq \sigma(t) < 1 - 1/(n + 1)\}$. Since $D' = \bigcup_n D_n$, $\mu(D_N) > 0$ for some $N \in \mathbb{N}$. Define $\sigma_1 = 1 - 1/(N + 1)$. Let $C_n = \{t \in D_N : n - 1 \leq M(t, b) < n\}$. Since $D_N = \bigcup_n C_n$ there is $N' \in \mathbb{N}$ such that $\mu(C_{N'}) > 0$. Now, by taking as $A \in \Sigma$ any subset of $C_{N'}$ with positive and finite measure the claim follows. \hfill \Box

Now we state a useful decomposition theorem for the Musielak-Orlicz spaces. The following notation will be used in the sequel. For a Musielak-Orlicz function $M$, we define

$$\Omega_\infty = \{t \in \Omega : a_M(t) = b_M(t)\}, \quad \nu : \Omega \to [0, \infty), \quad \nu = 1/b_M,$$

$$\Omega_1 = \{t \in \Omega \setminus \Omega_\infty : 0 = a_M(t) < d_M(t) = b_M(t) = \infty\},$$

$$\Omega_{1, \infty} = \{t \in \Omega \setminus \Omega_\infty : 0 = a_M(t) < d_M(t) < b_M(t) = \infty\},$$

$$w : \Omega \to [0, \infty), \quad w(t) = \begin{cases} M(t, u)/u & \text{where } u \in (0, b_M(t)) \text{ if } t \in \Omega_1 \cup \Omega_{1, \infty}, \\ 0 & \text{if } t \in \Omega \setminus (\Omega_1 \cup \Omega_{1, \infty}). \end{cases}$$

For $t \in \Omega_\infty$, $b_M(t) \in (0, \infty)$ and so $w(t) \in (0, \infty)$. Observe also that $w$ is well defined by the definitions of $\Omega_1$ and $\Omega_{1, \infty}$, and $w(t) \in (0, \infty)$ for $t \in \Omega_1 \cup \Omega_{1, \infty}$. In fact $w = a_N \mu$-a.e. on $\Omega$, where $N$ is the complementary function of $M$. Moreover, if $t \in \Omega_1 \cup \Omega_{1, \infty}$ then $M(t, \nu) = w(t)\nu$ for all $u \in [0, b_M(t))$. Clearly $\mu(\Omega_\infty \cap (\Omega_1 \cup \Omega_{1, \infty})) = 0$.

It is easy to see that if $\Omega_\infty = \Omega$ up to a set of measure zero, then $L_M = L_{\infty, v}(\Omega_{\infty}) = L_{\infty, 1/b_M}$ with $\|x\|_{L_M} = \|x\|_{L_{\infty, v}}$. If $d_M(t) = \infty$ for $\mu$-a.e. $t \in \Omega$ then $L_M = L_{1, w} = L_{1, a_N}$ with $\|x\|_{L_M} = \|x\|_{L_{1, w}}$.

Moreover, if $\mu(\Omega_\infty) = 0$ and $d_M(t) = b_M(t)$ $\mu$-a.e. on $\Omega$ then

$$L_M = L_{\infty, v}(\Omega_{1, \infty}) \cap L_{1, w}(\Omega) = L_{\infty, 1/b_M}(\Omega_{1, \infty}) \cap L_{1, a_N}(\Omega)$$

and $\|x\|_{L_M} = \|x\|_{L_{\infty, v}} = \max\{\|x\|_{L_{1, w}}, \|x\chi_{\Omega_{1, \infty}}\|_{L_{\infty, v}}\}$.

Theorem 4.3. Let $M$ be a Musielak-Orlicz function. Then for any $x \in L_M$,

$$\|x\|_{L_M} = \max\{\|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}, \|x\chi_{\Omega_{1, \infty}}\|_{L_{1, w}}\},$$

and thus

$$L_M = L_{\infty, v}(\Omega_{\infty}) \oplus L_{\infty, 1/b_M}(\Omega_{1, \infty}) \cap L_{1, a_N}(\Omega).$$

Moreover, if $d_M(t) = b_M(t)$ $\mu$-a.e. on $\Omega \setminus \Omega_\infty$ then

$$\|x\|_{L_M} = \max\{\|x\chi_{\Omega_{\infty} \cup \Omega_{1, \infty}}\|_{L_{\infty, v}}, \|x\chi_{\Omega_{1, \infty}}\|_{L_{1, w}}\},$$

and thus

$$L_M = L_{\infty, v}(\Omega_{\infty}) \oplus L_{\infty, 1/b_M}(\Omega_{1, \infty} \cap L_{1, w}(\Omega \setminus \Omega_\infty)).$$

Proof. It is easy to observe that, if $\lambda > 0$ is such that $\|x(t)/\lambda\ > b_M(t)$ on a subset of $\Omega_\infty$ of positive measure, then $\lambda < \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}$, then $\rho_M(\lambda) = \infty$. Moreover, if $\lambda > \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}$ then $\rho_M(\lambda) = 0$. Hence

$$\|x\|_{L_M} = \inf\{\lambda > \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}} : \rho_M(\lambda) \leq 1\} = \max\{\|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}, \|x\chi_{\Omega_{1, \infty}}\|_{L_{1, w}}\}. $$

Assume now that $d_M(t) = b_M(t)$ $\mu$-a.e. on $\Omega \setminus \Omega_\infty$. Let $x \in L_M$ be such that $\mu(\Omega_{\infty} \cap \text{supp}(x)) = 0$. Similarly as above, if $\lambda < \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}$ then $\rho_M(\lambda) \geq \rho(\lambda) = \infty$. If $\lambda > \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}$ then $\rho_M(\lambda) = \rho_M(\lambda) + \rho(\lambda) = \int_{\Omega_{\infty}} \frac{w(x)}{\lambda} \mu(dx)$. Hence

$$\|x\|_{L_M} = \inf\left\{\lambda > \|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}} : \int_{\Omega \setminus \Omega_\infty} \frac{w(x)}{\lambda} \mu(dx) \leq 1\right\} = \max\{\|x\chi_{\Omega_{\infty}}\|_{L_{\infty, v}}, \|x\chi_{\Omega_{1, \infty}}\|_{L_{1, w}}\}. $$
For an arbitrary $x \in L_M^+$, applying the first part, we get that
\[
\|x\|_M = \max\{\|x\chi_{\Omega_\infty}\|_{\infty,v}, \|x\chi_{1,\infty}\|_{\infty,v}, \|x\chi_{\Omega_\infty}\|_{1,w}\} = \max\{\|x\chi_{\Omega_\infty}\|_{\infty,v}, \|x\chi_{\Omega_\infty}\|_{1,w}\}.
\]

\[\square\]

To characterize Musielak-Orlicz spaces with the Daugavet property we will use the following simple observation. A point $x \in S(X)$ is called uniformly non-$\ell_1^2$ (or uniformly non-square), if there exists $\delta > 0$ such that $\min\{\|x+y\|, \|x-y\|\} < 2-\delta$ for all $y \in S(X)$. It is worth to mention that non-square points and non-squareness properties of the above type have been considered in context of many spaces [8][13][14][21][22].

**Proposition 4.4.** If $(X, \| \cdot \|)$ has the Daugavet property then there are no uniformly non-$\ell_1^2$ points either in $X$ or in $X^*$.

**Proof.** Let $x \in S(X)$ be arbitrary, $x^* \in S(X^*)$ be such that $x^*(x) = -1$ and $\epsilon > 0$. By Lemma 3.1(iii), there is $y \in S(X)$ such that $x^*(y) > 1-\epsilon$ and $\|x+y\| > 2-\epsilon$. Also $\|x-y\| = \|y-x\| \geq x^*(y-x) > 2-\epsilon$. Hence $x$ is not uniformly non-$\ell_1^2$.

Similarly, let $y^* \in S(X^*)$ and $\epsilon > 0$ be arbitrary. There is $x \in S(X)$ such that $y^*(x) < -1+\epsilon/2$. By Lemma 3.1(ii), there is $x^* \in S(X^*)$ such that $x^*(x) > 1-\epsilon/2$ and $\|x^*+y^*\| > 2-\epsilon$. Also $\|x^*-y^*\| \geq (x^*-y^*)(x) > 2-\epsilon$. Hence $y^*$ is not uniformly non-$\ell_1^2$.

\[\square\]

The next proposition is known (Theorem 4 [20]), but we present its proof for completeness.

**Proposition 4.5.** Let $M$ be a Musielak-Orlicz function. If $x \in L_M^+$ and $\rho_M(b_M\chi_{\supp(x)}) \leq 1$ then $\|x\|_M = \|x\|_{\infty,v}$. Moreover, $L_M = L_{\infty,v}$ with equality of norms if and only if $\rho_M(b_M) \leq 1$.

**Proof.** It is clear that the condition $\rho_M(b_M\chi_{\supp(x)}) \leq 1$ implies $b_M < \infty \mu$-a.e. on $\supp(x)$.

Suppose that $\|x\|_{\infty,v} = 1$, that is $\|x/b_M\|_{\infty} = 1$. Then for every $c > 0$ there is a set of positive measure $D \subset \supp(x)$ such that $\|x(t)| > (1-c)b_M(t), t \in D$. It follows that $\rho_M(x/(1-c)) = \infty$. Hence $\|x\|_M \geq 1$. Since $x \leq b_M \mu$-a.e. on $\supp(x)$, $\rho_M(x) \leq \rho_M(b_M\chi_{\supp(x)}) \leq 1$. It follows that $\|x\|_M = 1$.

Suppose now that $\|x\|_M = 1$. It follows that $\rho_M(x) \leq 1, \rho_M(x/(1-c)) > 1$ for every $c > 0$ and so $\|x\|_{\infty,v} \leq 1$. If $\|x\|_{\infty,v} = 1-c$ for some $c > 0$, that is $|x| \leq (1-c)b_M \mu$-a.e. on $\Omega$, then $\rho_M(x/(1-c)) \leq \rho_M(b_M\chi_{\supp(x)}) \leq 1$, which gives a contradiction with $\|x\|_M = 1$. Hence $\|x\|_{\infty,v} = 1$.

We have showed that for all $x \in L_M^+$ with $\rho_M(b_M\chi_{\supp(x)}) \leq 1$, $\|x\|_M = 1$ if and only if $\|x\|_{\infty,v} = 1$.

The first claim follows.

If $\rho_M(b_M) \leq 1$ then by the first part, it is clear that $L_M = L_{\infty,v}$ with equality of norms. Let $L_M = L_{\infty,v}$ with equality of norms. Clearly $\|b_M\|_M = \|b_M\|_{\infty,v} = 1$. It follows that $\rho_M(b_M) \leq 1$.

\[\square\]

**Proposition 4.6.** Let $M$ and $N$ be complementary Musielak-Orlicz functions. If $x \in L_M^+$ and $\rho_N(b_N\chi_{\supp(x)}) \leq 1$ then $\|x\|_M = \|x\|_{1,b_N}$. Moreover, $L_M^+ = L_{1,b_N}$ with equality of norms if and only if $\rho_N(b_N) \leq 1$.

**Proof.** By the assumption $\rho_N(b_N\chi_{\supp(x)}) \leq 1$ we have that $b_N < \infty \mu$-a.e. on $\supp(x)$. Since for $\mu$-a.a. $t \in \Omega$, $\lim_{u \to \infty} M(t,u)/u = b_N(t)$ and $M(t,u)/u$ is increasing on $[0,\infty)$ [16], we have that $M(t,u)/u \leq b_N(t)$ for $\mu$-a.a. $t \in \Omega$. Hence $\|x\|_M^2 = \inf_{k>0} k^{-1}\|1+\rho_M(kx)\| \leq \inf_{k>0} k^{-1}\|1+\int \Omega \chi_{b_N(x)} |x| \, \mu\| = \|x\|_{1,b_N}$.

On the other hand $\|x\|_M^2 = \sup_{\Omega} \int \Omega x \, h \, \mu : \rho_N(h) \leq 1 \|x\|_{1,b_N}$. The second statement follows trivially.

\[\square\]

We also need the following result [19], p. 64.

**Lemma 4.7.** Let $M$ be a Musielak-Orlicz function with $b_M = \infty \mu$-a.e. on $\Omega$. There exists an ascending sequence $(T_i)_{i=1}^\infty$ of measurable sets such that $\mu(T_i) < \infty$ for all $i \in \mathbb{N}$, $\mu(\Omega \setminus \bigcup_{i=1}^\infty T_i) = 0$ and $\sup_{t \in T_i} M(t,u) < \infty$ for all $u \geq 0$ and all $i \in \mathbb{N}$.

Before we proceed to the main theorem we need one more technical result.

**Lemma 4.8.** Let $M$ be a Musielak-Orlicz function. Then $E_M \neq \emptyset$ if and only if $\mu\{t \in \Omega : b_M(t) = \infty\} > 0$. 

Proof. It is clear that if \( b_M < \infty \) \( \mu \)-a.e. on \( \Omega \) then \( E_M = \{0\} \). Now let \( S = \{ t \in \Omega : b_M(t) = \infty \} \). If \( \mu(S) > 0 \), since \( \mu \) is \( \sigma \)-finite, by Lemma 4.7, \( S = \bigcup_{i=1}^{\infty} S_i \), where \( 0 < \mu(S_i) < \infty \), \( i \in \mathbb{N} \) and for every \( u \geq 0 \), the function \( M(\cdot, u) \) is essentially bounded on \( S_i \). It follows that \( \chi_{S_i} \in E_M, i \in \mathbb{N} \). □

The following theorem implies that a very wide class of Musielak-Orlicz spaces does not have the Daugavet property.

**Theorem 4.9.** Let \( M \) be a Musielak-Orlicz function. If \( \mu\{t \in \Omega : d_M(t) < b_M(t)\} > 0 \) and \( \rho_M(b_M) > 1 \) then there is a uniformly non-\( \ell^1 \) point in \( L_M \).

**Proof.** By Lemma 4.2 there exist a measurable set \( C \) with \( 0 < \mu C < \infty \) and numbers \( a, b \) and \( \sigma_1 \) such that \( a < b, 0 < \sigma_1 < 1, [a, b] \subset (d_M(t), b_M(t)) \),

\[
M(t, u/2) \leq \sigma_1 M(t, u)/2 \quad \text{for} \quad t \in C \quad \text{and} \quad u \in [a, b],
\]

and \( M \) is strictly positive and bounded on \( C \times [a, b] \).

Without loss of generality we assume that \( \rho_M(a\chi_C) \leq 1 \). Denote \( S = \{ t \in \Omega : b_M(t) = \infty \} \).

We consider two cases. First, suppose that \( \mu(S) > 0 \). Then, let \( A \subset C \) be such that \( \mu(S \setminus A) > 0 \) and \( \rho_M(a\chi_A) \leq 1 \). By Lemma 4.8 \( E_M(S \setminus A) \neq \{0\} \). By Lemma 4.7 there is a measurable set \( T \) of positive and finite measure such that \( \chi_T \in E_M(S \setminus A) \). Define \( x = a\chi_A + x_0 \), where \( x_0 = d_0\chi_T \) and \( d_0 \geq 0 \) is such that \( \rho_M(a\chi_x + x_0) = 1 \).

Now, suppose that \( \mu(S) = 0 \), that is \( b_M < \infty \) \( \mu \)-a.e. on \( \Omega \). Let \( A \subset C \) be such that \( \mu(A) > 0 \), \( \rho_M(a\chi_A) \leq 1 \) and \( \rho_M(b_M\chi_{\Omega \setminus A}) > 1 \). We can find a measurable set \( G \subset \Omega \setminus A \) with positive and finite measure such that \( \rho_M(b_M\chi_G) > 1 \). By the left continuity of the modular, we get that for some positive constant \( c_1 < 1 \), \( \rho_M(c_1b_M\chi_G) > 1 \). Let \( G_n = \{ t \in G : n - 1 \leq M(t, c_1b_M(t)) < n \}, n \in \mathbb{N} \). Since \( \cup_{n=1}^{\infty} G_n = G \), for \( r \in \mathbb{N} \) large enough, \( \infty > \rho_M(c_1b_M\chi_{\Omega \setminus G_n}) > 1 \). Let \( c_2 > 0 \) be such that

\[
\rho_M(x_0) = 1 - \rho_M(a\chi_A), \quad \text{where} \quad x_0 = \frac{c_1}{1 + c_2}b_M\chi_{\cup_{n=1}^{\infty} G_n}.
\]

Then \( \rho_M(a\chi_x + x_0) = 1 \) and define again \( x = a\chi_A + x_0 \).

In both cases \( \rho_M(x) = 1 \) and so \( \|x\|_M = 1 \). By the construction we have \( \rho_M((1 + \epsilon)\chi_x) < \infty \) and \( \rho_M((1 + \epsilon)x_0) < \infty \) for some \( \epsilon > 0 \). Since \( a\chi_A \) and \( x_0 \) have disjoint supports,

\[
\rho_M((1 + \epsilon)x) < \infty \quad \text{for some} \quad \epsilon > 0.
\]

Let \( y \in S(L_M) \) be arbitrary. We split \( A \) into disjoint union \( A = A_1 \cup A_2 \cup A_3 \), where \( A_1 = \{ t \in A : b_M(t) = \infty \} \), \( A_2 = \{ t \in A : b_M(t) < \infty \) and \( M(t, b_M(t)) = \infty \} \) and \( A_3 = \{ t \in A : M(t, b_M(t)) < \infty \} \). Since for every \( \lambda > 1 \), \( \rho_M(y/\lambda) \leq 1 \), there exist constants \( c \in (0, 1) \) and \( d > 0 \) so large that \( \mu(A \cap B) > 0 \), where

\[
B = \{ t \in \Omega : |y(t)| \leq d\chi_{A_1}(t) + cb_M(t)\chi_{A_2}(t) + b_M(t)\chi_{A_3}(t) \}.
\]

Let \( \sigma : \Omega \to [0, 1) \) be defined by

\[
\sigma(t) = \begin{cases}
\sup \{2M(t, u/2)/M(t, u) : u \in [a, d]\}, & \text{if} \ t \in A_1 \\
\sup \{2M(t, u/2)/M(t, u) : u \in [a, cb_M(t)]\}, & \text{if} \ t \in A_2 \\
\sup \{2M(t, u/2)/M(t, u) : u \in [a, b_M(t)]\}, & \text{if} \ t \in A_3 \\
0, & \text{otherwise}.
\end{cases}
\]

It is not difficult to see that \( \sigma \) is a finite measurable function and \( \sigma > 0 \) on \( A \). It follows that for every \( t \in A \),

\[
M(t, u/2) \leq \sigma(t)M(t, u)/2,
\]

for all \( u \in [a, d] \) if \( t \in A_1 \), for all \( u \in [a, cb_M(t)] \) if \( t \in A_2 \), and for all \( u \in [a, b_M(t)] \) if \( t \in A_3 \). Since \( A \cap B = \cup_{n=1}^{\infty}\{ t \in A \cap B : 1 - 1/n < \sigma(t) \leq 1 - 1/(n + 1) \} \), there is a subset \( H \subset A \cap B \), with \( \mu(H) > 0 \) and a constant \( \sigma_2 \in (0, 1) \) such that

\[
M(t, u/2) \leq \sigma_2 M(t, u)/2,
\]

for all \( u \in [a, d] \) if \( t \in H \cap A_1 \), for all \( u \in [a, cb_M(t)] \) if \( t \in H \cap A_2 \), and for all \( u \in [a, b_M(t)] \) if \( t \in H \cap A_3 \).
Let $\sigma_0 = \max\{\sigma_1, \sigma_2\}$, $\eta = \rho_M(a\chi_A)$ and $\gamma = \rho_M(a\chi_{A\setminus H})$. Clearly $\sigma_0 \in (0,1)$, $\eta \in (0,1]$ and $\gamma \in [0,\eta)$. It follows that
\begin{equation}
(12) \quad \rho_M(a\chi_H) = \rho_M(a\chi_A) - \rho_M(a\chi_{A\setminus H}) = \eta - \gamma > 0. \tag{12}
\end{equation}
Let $\delta \in (0, (1 - \sigma_0)(\eta - \gamma)/2)$, that is
\begin{equation}
(13) \quad (1 - \sigma_0)(\eta - \gamma)/2 - \delta > 0. \tag{13}
\end{equation}

Since $a < b$, by (10) we find $\epsilon > 0$ such that $(1 + \epsilon)a \leq b$, $\rho_M((1 + \epsilon)x) < \infty$, and
\begin{equation}
(14) \quad \rho_M((1 + \epsilon)x) < \rho_M(x) + \delta. \tag{14}
\end{equation}
Define $z = (1 + \epsilon)x$.

We finish the proof in a similar way as in the proof of Theorem 3.27 [7, p. 133].

Define $D = \{t \in H : x(t)y(t) \geq 0\}$, $E = H \setminus D$. Since $D \subset H \subset A \cap B$, by (11) and by definition of the set $B$, we have that
\[ \rho_M((y/2)\chi_D) \leq (\sigma_2/2)\rho_M(y\chi_D). \]
Moreover, since $D \subset A$, $z = (1 + \epsilon)(a\chi_A + x_0)$ and supp$(x_0) \cap A = \emptyset$ we have that $|z|\chi_D = (1 + \epsilon)a\chi_A$ and hence by (9), taking into account that $A \subset C$, we get
\[ \rho_M((z/2)\chi_D) \leq (\sigma_1/2)\rho_M(z\chi_D). \]

It follows that
\[ \rho_M\left(\frac{z - y}{2}\chi_D\right) \leq \rho_M\left(\frac{\max\{|z|, |y|\}}{2}\chi_D\right) \leq \frac{\sigma_0}{2}\rho_M(\max\{|z|, |y|\}\chi_D) \leq \frac{\sigma_0}{2}(\rho_M(z\chi_D) + \rho_M(y\chi_D)). \]

Similarly
\[ \rho_M\left(\frac{z + y}{2}\chi_E\right) \leq \frac{\sigma_0}{2}(\rho_M(z\chi_E) + \rho_M(y\chi_E)). \]

Hence
\begin{equation}
(15) \quad \rho_M\left(\frac{z + y}{2}\chi_H\right) + \rho_M\left(\frac{z - y}{2}\chi_H\right) = \rho_M\left(\frac{z + y}{2}\chi_D\right) + \rho_M\left(\frac{z + y}{2}\chi_E\right)
\end{equation}
\begin{equation}
+ \rho_M\left(\frac{z - y}{2}\chi_D\right) + \rho_M\left(\frac{z - y}{2}\chi_E\right) \leq \frac{1}{2}\rho_M(z\chi_D) + \frac{1}{2}\rho_M(y\chi_D)
\end{equation}
\begin{equation}
+ \frac{\sigma_0}{2}(\rho_M(z\chi_E) + \rho_M(y\chi_E)) + \frac{\sigma_0}{2}(\rho_M(z\chi_D) + \rho_M(y\chi_D))
\end{equation}
\begin{equation}
+ \frac{1}{2}\rho_M(z\chi_E) + \frac{1}{2}\rho_M(y\chi_E) = \frac{1}{2}\rho_M(z\chi_H) + \frac{1}{2}\rho_M(y\chi_H). \tag{15}
\end{equation}

By the inequality (14) and definition of $z$,
\[ 2 + \delta \geq \rho_M(y) + \rho_M(x) + \delta \geq \rho_M(z) + \rho_M(y). \]

By the above, (12), (15) and convexity of the functions $M(t, \cdot)$ for $\mu$-a.e. $t \in \Omega$, we conclude that
\[ 2 + \delta - \rho_M\left(\frac{z + y}{2}\right) - \rho_M\left(\frac{z - y}{2}\right) \]
\[ \geq \rho_M(z) + \rho_M(y) - \rho_M\left(\frac{z + y}{2}\right) - \rho_M\left(\frac{z - y}{2}\right) \]
\[ \geq \rho_M(z\chi_H) + \rho_M(y\chi_H) - \rho_M\left(\frac{z + y}{2}\chi_H\right) - \rho_M\left(\frac{z - y}{2}\chi_H\right) \]
\[ \geq \frac{1 - \sigma_0}{2}(\rho_M(z\chi_H) + \rho_M(y\chi_H)) \geq \frac{1 - \sigma_0}{2}\rho_M(z\chi_H) \geq \frac{1 - \sigma_0}{2}\rho_M(a\chi_H) = \frac{1 - \sigma_0}{2}(\eta - \gamma). \]

By (13) we get that
\[ 2 - \rho_M\left(\frac{z + y}{2}\right) - \rho_M\left(\frac{z - y}{2}\right) > 0, \]

thus
\[ \min \left\{ \rho_M\left(\frac{z + y}{2}\right), \rho_M\left(\frac{z - y}{2}\right) \right\} \leq 1. \]
If \( \rho_M \left( \frac{x+y}{2} \right) \leq 1 \) then \( \frac{\|x+y\|_M}{2} \leq 1 \) which gives \( \frac{\|x+y/(1+\epsilon)\|_M}{2} \leq \frac{1}{1+\epsilon} \). Hence
\[
\left\| \frac{x+y}{2} \right\|_M \leq \frac{\left\| x+y/(1+\epsilon) \right\|_M}{2} \leq \frac{1}{1+\epsilon}.
\]
This follows that
\[
\left\| \frac{y}{2(1+\epsilon)} \right\|_M = \frac{\epsilon}{2(1+\epsilon)}.
\]
Therefore
\[
\left\| \frac{x+y}{2} \right\|_M \leq \frac{1}{1+\epsilon} + \frac{\epsilon}{2(1+\epsilon)} = 1 - \frac{\epsilon}{2(1+\epsilon)}.
\]
If \( \rho_M \left( \frac{x-y}{2} \right) \leq 1 \) then we get similarly that
\[
\left\| \frac{x-y}{2} \right\|_M \leq 1 - \frac{\epsilon}{2(1+\epsilon)}.
\]
Finally, for all \( y \in S(L_M) \),
\[
\min \{ \|x+y\|_M, \|x-y\|_M \} \leq 2 - \frac{\epsilon}{1+\epsilon},
\]
that is \( x \) is a uniformly non-\( \ell^2 \) point.

From the above theorem and from Proposition 1.4 we have the following corollary.

**Corollary 4.10.** Let \( M \) be a Musielak-Orlicz function. If \( \mu \{ t \in \Omega : d_M(t) < b_M(t) \} > 0 \) and \( \rho_M(b_M) > 1 \) then \( L_M \) does not have the Daugavet property.

We need two more results before we state the main theorem. The following lemma is analogous to Lemma 4.2 [11] proved there for the maximum norm.

**Lemma 4.11.** Let \( X = X_1 \oplus X_2 \oplus \ldots \oplus X_n \) be a finite direct sum of Banach spaces \( (X_i, \| \cdot \|_i), i = 1, 2, \ldots, n \), equipped with the norm \( \|x\| = \|x_1\|_1 + \|x_2\|_2 + \ldots + \|x_n\|_n \), where \( x = (x_1, x_2, \ldots, x_n) \), \( x_i \in X_i, i = 1, 2, \ldots, n \). If \( X \) has the Daugavet property then it is inherited by each component \( X_i, i = 1, 2, \ldots, n \).

**Proof.** Without loss of generality we assume that \( n = 2 \). Suppose that \( X = X_1 \oplus X_2 \) has the Daugavet property. It is enough to show that \( X_1 \) has that property. Let \( T(x_1) = x_1^* (x_1) y_1, x_1 \in X_1 \), be an arbitrary rank 1 operator on \( X_1 \), where \( x_1^* \in X_1^* \), \( y_1 \in X_1 \) and \( \|x_1^*\| = \|y_1\|_1 = 1 \). Clearly \( \|T\| = 1 \) and \( \|I + T\|_{X_1 \rightarrow X_1} \leq 2 \). We will show the opposite inequality. For any \( x = (x_1, x_2) \in X \) define
\[
x^*(x) = x_1^* (x_1), y = (y_1, 0) \text{ and } \tilde{T}(x) = x^*(x)y = (x_1^*(x_1) y_1, 0).
\]
Since \( X^* \simeq (X_1^* \oplus_{\infty} X_2^*) \), \( \|x^*\| = \|x_1^*\| = 1 \). Moreover \( \|y\| = \|y_1\|_1 = 1 \), hence \( \|\tilde{T}\|_{X \rightarrow X} = 1 \). Since \( \tilde{T} \) is a rank one operator on \( X \), by the Daugavet property of \( X \),
\[
2 = \|I + \tilde{T}\|_{X \rightarrow X} = \sup_{\|x\| \leq 1} \|x + x^*(x)y\|
\]
\[
= \sup_{\|x_1\|_1 + \|x_2\|_2 \leq 1} \left\{ \|x_1 + x_1^*(x_1)y_1\|_1 + \|x_2\|_2 \right\}.
\]
Hence for every \( \epsilon > 0 \) there is \( x \in X, x = (x_1, x_2), \|x\| = \|x_1\|_1 + \|x_2\|_2 \leq 1 \) such that
\[
\|x_1 + x_1^*(x_1)y_1\|_1 > 2 - \epsilon - \|x_2\|_2 \geq 1 + \|x_1\|_1 - \epsilon.
\]
It follows that
\[
2 \geq \left\| \frac{x_1}{\|x_1\|_1} + x_1^* \left( \frac{x_1}{\|x_1\|_1} \right) y_1 \right\|_1 \geq \frac{1}{\|x_1\|_1} + 1 - \frac{\epsilon}{\|x_1\|_1}.
\]
Multiplying by \( \|x_1\|_1 \), we get that \( \|x_1\|_1 \geq 1 - \epsilon \). Hence by (16),
\[
\|I + T\| \geq \|x_1 + T(x_1)\|_1 = \|x_1 + x_1^*(x_1)y_1\|_1 > 2 - 2\epsilon.
\]
It follows that \( \|I + T\| = 2 \).
Proposition 4.12. Let $X$ be a Banach space. The following conditions are equivalent.

(i) The norm on $X$ is 2-rough, that is for all $x \in X$,
\[
\limsup_{\|h\| \to 0} \frac{\|x + h\| + \|x - h\| - 2\|x\|}{\|h\|} = 2.
\]

(ii) $X^*$ has the weak* slice (or local) diameter 2 property, that is every weak* slice of $B(X^*)$ has diameter 2.

(iii) The norm on $X$ is locally octahedral, that is no point of $S(X)$ is uniformly non-$l^2$.

Finally, we state theorem which characterizes Musielak-Orlicz spaces with the Daugavet property.

Theorem 4.13. Let $M$ and $N$ be complementary Musielak-Orlicz functions, $v = 1/b_M$, $w = a_N$, $\Omega_\infty = \{t \in \Omega : a_M(t) = b_M(t)\}$, $\Omega_1 = \{t \in \Omega \setminus \Omega_\infty : d_M(t) = b_M(t) = \infty\}$ and $\Omega_{1,\infty} = \{t \in \Omega \setminus \Omega_\infty : d_M(t) = b_M(t) < \infty\}$. The following conditions are equivalent.

(i) $L_M$ has the Daugavet property.

(ii) $L_M = L_{1,v}$ or $L_M = L_{\infty,v}$ or $L_M = L_{\infty,v}(\Omega_\infty) \oplus L_{1,v}(\Omega \setminus \Omega_\infty)$.

(iii) $L_N^o = L_{\infty,1/w}$ or $L_N^o = L_{1,1/v}$ or $L_N^o = L_{1,1/v}(\Omega_\infty) \oplus L_{\infty,1/w}(\Omega \setminus \Omega_\infty)$.

(iv) $L_N^o$ has the Daugavet property.

Proof. Let $L_M$ have the Daugavet property. By Corollary 4.10 we have that $\mu\{t \in \Omega : d_M(t) < b_M(t)\} = 0$ or $\rho_M(b_M) \leq 1$. The latter condition is equivalent to $L_M = L_{\infty,v}$ by Proposition 4.5. The former condition implies that for $\mu$-a.e. $t \in \Omega \setminus \Omega_\infty$ we have $d_M(t) = b_M(t)$. Hence in view of Theorem 4.3

\[
L_M = \begin{cases}
L_{1,v}, & \text{if } \mu(\Omega \setminus \Omega_1) = 0, \\
L_{\infty,v}, & \text{if } \mu(\Omega \setminus \Omega_\infty) = 0, \\
L_{\infty,v}(\Omega_\infty) \oplus L_{1,v}(\Omega \setminus \Omega_\infty), & \text{if } \mu(\Omega_\infty), \mu(\Omega \setminus \Omega_\infty) > 0 \text{ and } \mu(\Omega_{1,\infty}) = 0, \\
L_{\infty,v}(\Omega_\infty) \oplus (L_{1,v}(\Omega \setminus \Omega_\infty) \cap L_{\infty,v}(\Omega \setminus \Omega_\infty)), & \text{if } \mu(\Omega \setminus \Omega_\infty), \mu(\Omega_{1,\infty}) > 0.
\end{cases}
\]  

By Theorem 3.7 applied to $\Omega \setminus \Omega_\infty$ and $\Omega_{1,\infty}$ for $\Omega$ and $\Gamma$ respectively, the second component of the last space in (17) has the Daugavet property if and only if it is equal to $L_{\infty,v}(\Omega \setminus \Omega_\infty)$. Hence, in view of Lemma 4.2 [4] we see that (i) implies (ii). Since the Daugavet property is lifted from components of $\oplus$ sums to the whole space [28] we conclude that conditions (i) and (ii) are equivalent. Since the last statement is also true for $\oplus$ sums [28], we see that (iii) implies (iv).

Assume now that (iv) holds true. We will show that $\mu\{t \in \Omega : d_M(t) < b_M(t)\} = 0$ or $\rho_M(b_M) \leq 1$. Suppose that this condition is not satisfied. Then by Theorem 4.9 we have that $L_M$ is not locally octahedral. Hence, by Proposition 4.12 the dual space $(L_M)^* \simeq L_N^o \oplus S$ fails the weak* slice diameter 2 property. Therefore we can find a weak* slice $S(x, \epsilon) = \{f \in B((L_M)^*) : f(x) > 1 - \epsilon\}$ with the diameter less than 2, where $x \in S(L_M)$ and $\epsilon > 0$. Let $\kappa : L_M \to (L_M)^{**}$ be the canonical mapping defined by $(\kappa)(x^*)(x) = x^*(x)$, $x^* \in X^*$. Consider the sets

\[
S'(x, \epsilon) = \{f \in B((L_M)^*) : f \in (L_M)^*_c \text{ and } f(x) > 1 - \epsilon\}
\]

and

\[
S''(F, \epsilon) = \{y \in B(L_N^o) : F(y) > 1 - \epsilon\}
\]

where $F = \kappa(x)$. Since $(L_M)^*_c \simeq (L_M)' = L_N^o$, there is a bijective correspondence preserving norm between $S'(x, \epsilon)$ and $S''(F, \epsilon)$. Since $S'(x, \epsilon) \subset S(x, \epsilon)$ we see that the slice $S''(F, \epsilon)$ of $B(L_N^o)$ has the diameter less than 2. Hence $L_N^o$ fails the slice diameter 2 property. In particular $L_N^o$ does not have the Daugavet property, which contradicts (iv).

Hence, indeed it must be that $\mu\{t \in \Omega : d_M(t) < b_M(t)\} = 0$ or $\rho_M(b_M) \leq 1$. The latter condition is equivalent to $L_N^o = L_{1,1/v}$ by Proposition 4.6. Similarly as previously, the former condition together with Theorem 4.3 the Köthe duality $(L_M)' = L_N^o$ (Theorem 5.4) and (17) gives $L_N^o = L_{\infty,1/w}$, or $L_N^o = L_{1,1/v}(\Omega_\infty) \oplus L_{\infty,1/w}(\Omega \setminus \Omega_\infty)$, or $L_N^o = L_{1,1/v}(\Omega_\infty) \oplus L_{\infty,1/w}(\Omega \setminus \Omega_\infty)$.
Then the general case has never been published. Since the latter space has the Daugavet property, by Lemma 3.11 we infer that the second component of that space has the Daugavet property as well. Now we see that the condition (iii) follows from Theorem 3.5. Hence (iii) and (iv) are equivalent.

Conditions (ii) and (iii) are clearly equivalent by the Köthe duality $L^o_N = (L_M)'$ and $(L^o_N)' = L_M$ (see the appendix).

**Corollary 4.14.** Let $M$ be a Musielak-Orlicz function such that $0 < M(t, u) < \infty$ for $\mu$-a.a. $t \in \Omega$ and for all $u > 0$, that is $a_M = 0$ and $b_M = \infty$ $\mu$-a.e. on $\Omega$. Let $N$ be the function complementary to $M$. The following conditions are equivalent.

(i) $L_M$ has the Daugavet property.
(ii) $L_M = L_{1,a_N}$.
(iii) $L^o_N = L_{\infty,1/a_N}$.
(iv) $L^o_N$ has the Daugavet property.

As we noted in the introduction, if $M(t, u) = \varphi(u)$ for all $t \in \Omega$ and $u \geq 0$, where $\varphi$ is an Orlicz function then $L_M = L_\varphi$, the Orlicz space generated by $\varphi$. In this case $a_M = a_\varphi$ and $b_M = b_\varphi$ on $\Omega$, where $a_\varphi$ and $b_\varphi$ are constants defined in the introduction.

**Corollary 4.15.** Let $\varphi$ and $\psi$ be complementary Orlicz functions. The following statements are equivalent.

(i) $L_\varphi$ has the Daugavet property.
(ii) $L_\varphi = L_{1,a_\psi}$ or $L_\psi = L_{\infty,1/b_\varphi}$.
(iii) $L^o_\psi = L_{\infty,1/a_\psi}$ or $L^o_\psi = L_{1,b_\varphi}$.
(iv) $L^o_\psi$ has the Daugavet property.

Another corollary from Theorem 4.13 is the following generalization of Theorem 4.1 [4].

**Corollary 4.16.** Let $L_{p(t)}$ be a Nakano space, where $1 \leq p(t) \leq \infty$ and $1/p(t) + 1/q(t) = 1$ for $\mu$-a.a. $t \in \Omega$ with the usual convention that $q(t) = \infty$ if $p(t) = 1$. Denote $\Omega_\infty = \{t \in \Omega : p(t) = \infty\}$. The following statements are equivalent.

(i) $L_{p(t)}$ has the Daugavet property.
(ii) $L_{p(t)} = L_1$ or $L_{p(t)} = L_\infty$ or $L_{p(t)} = L_1(\Omega \setminus \Omega_\infty) \oplus L_\infty(\Omega_\infty)$.
(iii) $L^o_{q(t)} = L_\infty$ or $L^o_{q(t)} = L_1$ or $L^o_{q(t)} = L_\infty(\Omega \setminus \Omega_\infty) \oplus L_1(\Omega_\infty)$.
(iv) $L^o_{q(t)}$ has the Daugavet property.

From the proof of Theorem 4.13 we can also deduce the following result.

**Corollary 4.17.** If $L^o_N$ has the slice diameter 2 property then $L^o_N = L_{\infty,1/a_N}$, or $L^o_N = L_{1,b_M}$, or $L^o_N = L_{1,b_M}(\Omega_\infty) \oplus L_{\infty,1/a_N}(\Omega \setminus \Omega_\infty)$, or $L^o_N = L_{1,b_M}(\Omega_\infty) \oplus L_{1,b_M}(\Omega_\infty) + L_{\infty,1/a_N}(\Omega \setminus \Omega_\infty)$.

**Appendix: Köthe duality**

In this section we present a proof of Köthe duality of Musielak-Orlicz spaces. The result is well known. However, to the best of our knowledge, a direct self-contained proof of that result in the general case has never been published.

We need the following result on Orlicz functions characterizing equality in Young’s inequality [10].

**Proposition 5.1.** Let $\varphi$ and $\psi$ be a pair of complementary Orlicz functions and $\varphi^\prime_-, \varphi^\prime_+$ be the left and right derivative of $\varphi$, respectively. Let

$$\partial \varphi(u) = \{v \geq 0 : \varphi(u) + \psi(v) = uv\}, \ u \geq 0.$$

Then

(i) $\partial \varphi(0) = [0, a_\psi] = [0, \varphi^\prime_+(0)]$.
(ii) If $u \in (0, b_\varphi)$ then $\partial \varphi(u) = [\varphi^\prime_-(u), \varphi^\prime_+(u)]$.
(iii) If $\varphi^\prime_-(b_\varphi) < \infty$ then $\partial \varphi(b_\varphi) = [\varphi^\prime_-(b_\varphi), \infty]$.
(iv) If $\varphi(b_\varphi) = \infty$ then $\partial \varphi(b_\varphi) = 0$.
(v) If $u > b_\varphi$ then $\partial \varphi(u) = 0$. 
Lemma 5.2. Let $\varphi$ and $\psi$ be a pair of complementary Orlicz functions and $\psi_-$ be the left derivative of $\psi$. If $b_\varphi < \infty$ then $\varphi(\psi_-(u)) < \infty$ for all $u > 0$.

Theorem 5.3. Let $M$ be a Musielak-Orlicz function. The Köthe dual $(L_M^o)' = L_N$.

*Proof.* Recall that $(L_M^o)'$ is isometrically isomorphic to the space of all order continuous functionals $(L_M^o)_c$.

Let $g \in L_N$ and $F : L_M^o \to \mathbb{R}$ be defined by $F(f) = \int f \, g \, d\mu$. By definition of $\| \cdot \|_M^o$, $|F(f)| \leq \| f \|_M^o \| g \|_N$. Hence $\| F \| \leq \| g \|_N$. Thus $F$ is a bounded linear order continuous functional. Next we show the reverse inequality. Without loss of generality we assume that $\| g \|_N = 1$. It follows that $\rho_N(g) \leq 1$ and $\rho_N((1 + \epsilon)g) \geq 1$ for all $\epsilon > 0$. In the sequel, by $N'$ we denote the left-side derivative of $N$ with respect to $u$ (we define $N'(t, 0) = 0$, $t \in \Omega$).

We consider two cases.

Case 1. There is $\epsilon_0 > 0$ such that $(1 + \epsilon_0)|g| \leq b_N \, \mu$-a.e. on $\Omega$. Fix $\epsilon \in (0, \epsilon_0)$. In view of $(1 + \epsilon)|g| < b_N \, \mu$-a.e. on $\Omega$, by Lemma 5.2, the function $M(\cdot, N'(\cdot, (1 + \epsilon)|g|\cdot))$ is nonnegative and finite $\mu$-a.e. on $\Omega$. Since $\mu$ is $\sigma$-finite, there is an ascending sequence of measurable sets with finite and positive measure $(\Omega_n)_{n=1}^\infty$ such that $\Omega = \bigcup_{n=1}^\infty \Omega_n$. Let $T_n = \{ t \in \Omega : M(t, N'(t, (1 + \epsilon)|g|\cdot)) \leq n \}$, $n \in \mathbb{N}$. Clearly $(T_n)_{n=1}^\infty$ is an ascending sequence of measurable sets of finite measure satisfying
\begin{equation}
\sup_{t \in T_n} M(t, N'(t, (1 + \epsilon)|g|\cdot)) < \infty, \quad n \in \mathbb{N}.
\end{equation}
Moreover $\mu(\Omega \setminus \bigcup_n T_n) = 0$. Indeed, for $t \in \Omega \setminus \bigcup_n T_n$ we have that $t \in \Omega_n$ for all $n \geq n_0$, for some $n_0 \in \mathbb{N}$ and $t \notin \bigcup_{n=1}^\infty T_n$. This implies that for all $n \geq n_0$, $M(t, N'(t, (1 + \epsilon)|g|\cdot)) > n$. Since $M(\cdot, N'(\cdot, (1 + \epsilon)|g|\cdot))$ is finite $\mu$-a.e. on $\Omega$ we conclude that $\mu(\Omega \setminus \bigcup_n T_n) = 0$.

Let $\tilde{g}_n$ be a sequence of non-negative simple functions such that $\tilde{g}_n \uparrow |g|$ $\mu$-a.e. on $\Omega$ and $\mu(\text{supp} \tilde{g}_n) < \infty$, $n \in \mathbb{N}$. Define
\begin{equation}
g_n = \tilde{g}_n \chi_{T_n}, \quad n \in \mathbb{N}.
\end{equation}
Clearly $g_n \leq |g|$ and $g_n \uparrow |g|$ $\mu$-a.e. on $\Omega$. Therefore $\rho_N((1 + \epsilon)g) = \lim_{n \to \infty} \rho_N((1 + \epsilon)g_n)$ and $\rho_N((1 + \epsilon)g_n) \geq 1$ for all $n$ large enough. By Proposition 5.1 for all $n \in \mathbb{N}$,
\begin{equation}
\partial N(t, (1 + \epsilon)g_n(t)) = \emptyset \quad \text{for } \mu\text{-a.a. } t \in \Omega.
\end{equation}
Moreover, since $g_n \leq |g|$, in view of (18) we get that for $n \in \mathbb{N}$,
\begin{equation}
\int_{\Omega} M(t, N'(t, (1 + \epsilon)g_n(t))) \, d\mu < \infty.
\end{equation}
Define
\begin{equation}
y_n(t) = N'(t, (1 + \epsilon)g_n(t)) \text{sign}(g(t)), \quad t \in \Omega,
\end{equation}
and
\begin{equation}
f_n = \frac{y_n}{\rho_M(y_n) + 1}.
\end{equation}
The functions $f_n$ are well defined since $\rho_M(y_n) < \infty$ for every $n \in \mathbb{N}$ by (20). By Young’s inequality for every $h \in L_0$ and $\mu$-a.a. $t \in \Omega$,
\begin{equation}
|y_n(t)h(t)| = N'(t, (1 + \epsilon)g_n(t))|h(t)| \leq M(t, N'(t, (1 + \epsilon)g_n(t))) + N(t, |h(t)|).
\end{equation}
Hence
\begin{equation}
\| f_n \|_M^o = \sup \left\{ \int_{\Omega} \frac{|y_n|h|}{\rho_M(y_n) + 1} : \rho_N(h) \leq 1 \right\} \leq 1.
\end{equation}
By (19) the following equality in Young’s inequality holds true for $\mu$-a.a. $t \in \Omega$
\begin{equation}
\frac{1}{1 + \epsilon} N'(t, (1 + \epsilon)g_n(t))(1 + \epsilon)g_n(t) = \frac{1}{1 + \epsilon} \left[ M(N'(t, (1 + \epsilon)g_n(t))) + N(t, (1 + \epsilon)g_n(t)) \right].
\end{equation}
It follows that
\begin{equation}
\| F \| = \sup \left\{ \int_{\Omega} f \, g \, d\mu \mid \| f \|_M^o \leq 1 \right\} \geq \int_{\Omega} f_n g_n \text{sign}(g) \, d\mu = \int_{\Omega} N'(t, (1 + \epsilon)g_n(t)g_n(t) \frac{M(t, N'(t, (1 + \epsilon)g_n(t))) + N(t, (1 + \epsilon)g_n(t))}{\rho_M(y_n) + 1} \, d\mu = \frac{1}{1 + \epsilon} \frac{\rho_M(y_n) + \rho_N((1 + \epsilon)g_n)}{\rho_M(y_n) + 1}.
\end{equation}
Since $1 \leq \rho_N((1 + \epsilon)g) = \lim_{n \to \infty} \rho_N((1 + \epsilon)g_n)$, we conclude that $\|F\| \geq 1/(1 + \epsilon) = \|g\|_N/(1 + \epsilon)$.

But we can take $\epsilon$ arbitrarily close to 0, hence $\|F\| \geq \|g\|_N$.

Case 2. For every $\epsilon > 0$ there is a measurable set $E_\epsilon$ of positive measure such that $(1 + \epsilon)|g| > b_N \mu$-a.e. on $E_\epsilon$. Let
\begin{equation}
A_n = \{t \in \Omega : |g(t)| \geq (1 - 1/n)b_N(t)\}.
\end{equation}
Clearly $\mu(A_n) > 0$ and $b_N < \infty \mu$-a.e. on $A_n$ for every $n \in \mathbb{N}$. There are measurable sets $B_n \subset A_n$ of positive and finite measure such that $\int_{B_n} b_N \, d\mu < \infty$, $n \in \mathbb{N}$. Define
\[ f_n = \left(\int_{B_n} b_N \, d\mu\right)^{-1} \chi_{B_n}(g), \quad n \in \mathbb{N}. \]

Since for any $h \in L_0$ with $\rho_N(h) \leq 1$ we have $|h| \leq b_N \mu$-a.e. on $\Omega$, so
\[ \|f_n\|_M = \sup\left\{ \int_{\Omega} f_nh \, d\mu : \rho_N(h) \leq 1 \right\} \leq 1. \]

From (21) we get that
\[ \int_{\Omega} f_ng \, d\mu = \left(\int_{B_n} b_N \, d\mu\right)^{-1} \int_{B_n} g \, d\mu = \left(\int_{B_n} b_N \, d\mu\right)^{-1} \int_{B_n} |g| \, d\mu \geq 1 - 1/n. \]

It follows that $\|F\| \geq \|g\|_N$.

The fact that every order continuous functional on $L^0_M$ is of the integral form follows from the Radon-Nikodym Theorem. Indeed, let $F$ be an order continuous functional on $L^0_M$. From the Radon-Nikodym Theorem it follows that there is a measurable function $g$ such that $F(XE) = \int_X g \, d\mu$ for every measurable set $E$ with $\mu(E) < \infty$ such that $X \in L^0_M$. Let $f \in L^n_M$ be such that $f \geq 0 \mu$-a.e. on $\Omega$. There is a sequence $(f_n)$ of simple functions such that $0 \leq f_n \leq f$ and $f_n \uparrow f \mu$-a.e. on $\Omega$. Since $F$ is order continuous we have that $|F(f - f_n)| \to 0$ as $n \to \infty$. Hence $F(f) = \lim_n F(f_n) = \lim_n \int_{\Omega} f_n \, d\mu = \int_{\Omega} f \, d\mu$. Since an arbitrary $f \in L^0_M$ can be written as $f = f^+ - f^-$, where $f^+$ and $f^-$ are positive, we see that $F(f) = \int_{\Omega} f \, d\mu$ for every $f \in L^0_M$. Since $\|f\|_M^0 \leq 2\|f\|_M$ for every $f \in L^0_M$, we have that
\[ \|g\|_N^0 = \sup\left\{ \int_{\Omega} fg \, d\mu : \|f\|_M \leq 1 \right\} \leq \sup\left\{ \int_{\Omega} f \, d\mu : \|f\|_M^0 \leq 1 \right\} = 2\|F\| < \infty. \]

Hence $g \in L_N$. \hfill \Box

**Theorem 5.4.** Let $M$ be a Musielak-Orlicz function. The Kôthe dual $(L^0_M)^\prime = L^\infty_N$.

**Proof.** Let $g \in L^\infty_N$ and $F : L_M \to \mathbb{R}$ be defined by $F(f) = \int_{\Omega} fg \, d\mu$. Clearly $F$ is a bounded linear order continuous functional. Since for $f \in L_M$, $\|f\|_M \leq 1$ if and only if $\rho_M(f) \leq 1$ we get that
\[ \|F\| = \sup\{ \int_{\Omega} fg \, d\mu : \|f\|_M \leq 1 \} = \sup\{ \int_{\Omega} fg \, d\mu : \rho_M(f) \leq 1 \} = \|g\|_N^\infty. \]

The fact that every order continuous functional on $L_M$ is of the integral form follows similarly as in Theorem 5.3 \hfill \Box

**References**

1. T. Abrahamsen, V. Lima, and O. Nygaard, *Remarks on diameter 2 properties*, J. Convex Anal. 20 (2013), no. 2, 439–452. MR3098474
2. Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002. MR1921782 (2003b:46002)
3. M. D. Acosta and A. Kamińska, *The Daugavet property in rearrangement invariant spaces*, Trans. Amer. Math. Soc., to appear.
4. M. D. Acosta and A. Kamińska, *The Daugavet property and weak neighborhoods in Banach lattices*, J. Convex Anal. 19 (2012), no. 3, 875–912. MR3013764
5. M. D. Acosta and A. Kamińska, *Weak neighborhoods and the Daugavet property of the interpolation spaces $L_1 + L_\infty$ and $L_1 \cap L_\infty$*, Indiana Univ. Math. J. 57 (2008), no. 1, 77–96. MR2400252 (2009a:46019)
6. C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and Applied Mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988. MR928802 (89e:46001)
[7] S. Chen, *Geometry of Orlicz spaces*, Dissertationes Math. (Rozprawy Mat.) **356** (1996), 204. With a preface by Julian Musielak. MR1410390 (97i:46051)

[8] Y. Cui and R. Pluciennik, *Local uniform nonsquareness in Cesàro sequence spaces*, Comment. Math. Prace Mat. **37** (1997), 47–58. MR1608225 (99b:46025)

[9] M. Cwikel, P. G. Nilsson, and G. Schechtman, *Interpolation of weighted Banach lattices. A characterization of relatively decomposable Banach lattices*, Mem. Amer. Math. Soc. **165** (2003), no. 787, vi+127. MR1996919 (2006f:46024)

[10] I. K. Daugavet, *A property of completely continuous operators in the space C*, Uspehi Mat. Nauk **18** (1963), no. 5 (113), 157–158. MR0157225 (28 #461)

[11] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993. MR1211634 (94d:46012)

[12] X. L. Fan, *Amemiya norm equals Orlicz norm in Musielak-Orlicz spaces*, Acta Math. Sin. (Engl. Ser.) **23** (2007), no. 2, 281–288. MR2286921 (2008e:46037)

[13] P. Foralewski, H. Hudzik, and P. Kolwicz, *Non-squareness properties of Orlicz-Lorentz function spaces*, J. Inequal. Appl. (2013), 2013:32, 25. MR3022822

[14] Non-squareness properties of Orlicz-Lorentz sequence spaces, J. Funct. Anal. **264** (2013), no. 2, 605–629. MR2997393

[15] H. Hudzik and L. Maligranda, *Amemiya norm equals Orlicz norm in general*, Indag. Math. (N.S.) **11** (2000), no. 4, 573–585. MR1990821 (2003c:46039)

[16] V. Kadets, M. Martín, J. Merí, and D. Werner, *Lushness, numerical index 1 and the Daugavet property in rearrangement invariant spaces*, Canad. J. Math. **65** (2013), no. 2, 331–348. MR3028566

[17] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, *Banach spaces with the Daugavet property*, Trans. Amer. Math. Soc. **352** (2000), no. 2, 855–873. MR1621757 (2000c:46023)

[18] A. Kamińska, *Some convexity properties of Musielak-Orlicz spaces of Bochner type*, Proceedings of the 13th winter school on abstract analysis (Srní, 1985), 1985, pp. 63–73 (1986). MR894273 (88k:46039)

[19] On Musielak-Orlicz spaces isometric to $L_2$ or $L_\infty$, Collect. Math. **48** (1997), no. 4-6, 563–569. Fourth International Conference on Function Spaces (Zielona Góra, 1995). MR1602596 (99a:46049)

[20] A. Kamińska and D. Kubiak, *On isometric copies of $\ell_\infty$ and James constants in Cesàro-Orlicz sequence spaces*, J. Math. Anal. Appl. **372** (2010), no. 2, 574–584. MR2678885 (2012a:46103)

[21] P. Kolwicz and A. Panfil, *Non-square Lorentz spaces $\Gamma_{p,\omega}$*, Indag. Math. (N.S.) **24** (2013), no. 1, 254–263. MR2997764

[22] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semenov, *Interpolation of linear operators*, Translations of Mathematical Monographs, vol. 54, American Mathematical Society, Providence, R.I., 1982. Translated from the Russian by J. Szücs. MR649411 (84j:46010)

[23] W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces. Vol. I*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1971. North-Holland Mathematical Library. MR0511676 (58 #23483)

[24] Riesz spaces. Vol. II, North-Holland mathematical library, North-Holland Publishing Company, 1983.

[25] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. 1034, Springer-Verlag, Berlin, 1983. MR724434 (85m:46028)

[26] S. Shang, Y. Cui, and Y. Fu, *Nonsquareness in Musielak-Orlicz-Bochner function spaces*, Abstr. Appl. Anal. (2011), Art. ID 361525, 16. MR2784391 (2012d:46077)

[27] P. Wojtaszczyk, *Some remarks on the Daugavet equation*, Proc. Amer. Math. Soc. **115** (1992), no. 4, 1047–1052. MR1126202 (92k:47041)

Department of Mathematical Sciences, The University of Memphis, TN 38152-3240

E-mail address: kaminska@memphis.edu

Mathematics Department, Tennessee Technological University, 110 University Drive, Box 5054, Cookeville, TN 38505

E-mail address: dkubiak@tntech.edu