CTL* model checking for data-aware dynamic systems with arithmetic*

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Abstract. The analysis of complex dynamic systems is a core research topic in formal methods and AI, and combined modelling of systems with data has gained increasing importance in applications such as business process management. In addition, process mining techniques are nowadays used to automatically mine process models from event data, often without correctness guarantees. Thus verification techniques for linear and branching time properties are needed to ensure desired behavior. Here we consider data-aware dynamic systems with arithmetic (DDSAs), which constitute a concise but expressive formalism of transition systems with linear arithmetic guards. We present a CTL* model checking procedure for DDSAs that relies on a finite-state abstraction by means of a set of formulas that capture variable configurations. Linear-time verification was shown to be decidable in specific classes of DDSAs where the constraint language or the control flow are suitably confined. We investigate several of these restrictions for the case of CTL*, with both positive and negative results: CTL* verification is proven decidable for monotonicity and integer periodicity constraint systems, but undecidable for feedback free and bounded lookback systems. To demonstrate the feasibility of our approach, we implemented it in the SMT-based prototype ada, showing that many practical business process models can be effectively analyzed.

Keywords: verification · CTL* · counter systems · arithmetic constraints · SMT.

1 Introduction

The study of complex dynamic systems is a core research topic in AI, with a long tradition in formal methods. It finds application in a variety of domains, such as notably business process management (BPM), where studying the interplay between control-flow and data has gained momentum [46][9][10][25]. Processes are increasingly mined by automatic techniques [1][13] that lack any correctness guarantees, making verification even more important to ensure the desired behavior. However, the presence of data pushes verification to the verge of undecidability due to an infinite state space. This is aggravated by the use of arithmetic, in spite

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of its importance for practical applications. Indeed, model checking of transition systems operating on numeric data variables with arithmetic constraints is known to be undecidable, as it is easy to model a two-counter machine.

In this work, we focus on the concise but expressive framework of data-aware dynamic systems with arithmetic (DDSAs), also known as counter systems. Several classes of DDSAs have been isolated where specific verification tasks are decidable, notably reachability and linear-time model checking. Far fewer results are known about the case of branching time, with the exception of flat counter systems where loops may not be nested, and gap-order constraint systems where constraints are restricted to the form $x - y \geq 2$. However, many processes in BPM and beyond fall into neither of these two classes, as illustrated by the example below.

Example 1. The following DDSA $B$ models a management process for road fines by the Italian police. It maintains seven so-called case data variables (i.e., variables local to each process instance, called “case” in the BPM literature): $a$ (amount), $t$ (total amount), $d$ (dismissal code), $p$ (points deducted), $e$ (expenses), and time durations $ds$, $dp$, $dj$. The process starts by creating a case, upon which the offender is notified within 90 days, i.e., 2160h (send fine). If the offender pays a sufficient amount $t$, the process terminates via silent actions $\tau_1$, $\tau_2$, or $\tau_3$. For the less happy paths, the credit collection action is triggered if the payment was insufficient; while appeal to judge and appeal to prefecture reflect filed protests by the offender, which again need to respect certain time constraints.

This model was generated from real-life logs by automatic process mining techniques paired with domain knowledge, but without any correctness guarantee. For instance, data-aware soundness requires that the process can always reach a final state from any reachable configuration, expressed by the branching-time property $\text{AG}E \text{F} \text{end}$. This property is false here, as $B$ can get stuck in state $p_7$ if $d > 1$. In addition, process-specific linear-time properties are needed, e.g., that a send fine event is always followed by a sufficient payment (i.e., $\langle \text{send fine} \rangle \mathcal{T} \rightarrow F \langle \text{payment} \rangle (t \geq a)$, where $\langle \alpha \rangle$ is the next operator via action $\alpha$).

This example highlights how both linear-time and branching-time verification are needed. In this paper, we present a CTL* model checking algorithm for DDSAs, adopting a finite-trace semantics ($\text{CTL}^*_f$) to reflect the nature of
processes as in Ex. 1. More precisely, our approach can synthesize conditions on the initial variable assignment such that a given property holds. We then derive an abstract decidability criterion which is satisfied by two practical DDSA classes that restrict the constraint language to (a) monotonicity constraints [21,26], i.e., variable-to-variable or variable-to-constant comparisons over $\mathbb{Q}$ or $\mathbb{R}$, and (b) integer periodicity constraints [23,19], i.e., variable-to-constant and restricted variable-to-variable comparisons with modulo operators. On the other hand, the restrictions known as feedback-freedom [14] and the more general bounded lookback [28] restrict the control flow of DDSAs such that LTL$_f$ verification is decidable, but we show here that CTL$^*_f$ remains undecidable.

In summary, we make the following contributions:

1. We present a CTL$^*_f$ model checking algorithm for DDSAs;
2. As an abstract decidability criterion for our verification problem, we prove a termination condition for this algorithm (Cor. 1);
3. This result is used to show that CTL$^*_f$ verification is decidable for monotonicity constraint and integer periodicity constraint systems;
4. The cases of feedback-free and bounded-lookback systems are undecidable;
5. We implemented our approach in the prototype ada using SMT solvers as backends and tested it on a range of business processes from the literature.

The paper is structured as follows: The rest of this section compiles related work. In Sec. 2 we recall preliminaries about DDSAs and CTL$^*_f$. Sec. 3 is dedicated to LTL verification with configuration maps, which is used by our model checking procedure in Sec. 4. After giving an abstract termination criterion, Sec. 5 presents decidability results for concrete DDSA classes. We describe our implementation in Sec. 6. Complete proofs and experiments can be found in the appendix.

**Related work.** Verification of transition systems with arithmetic constraints, also called counter systems, has been studied in many areas including formal methods, database theory, and BPM. Reachability was proven decidable for a variety of classes, e.g., reversal-bounded counter machines [34], finite linear [29], flat [13], and gap-order constraint (GC) systems [6]. Considerable work has also been dedicated to linear-time verification: LTL model checking is decidable for monotonicity constraint (MC) systems [21], even comparing variables multiple steps apart. DDSAs with MCs are also considered in [20] from the perspective of LTL with a finite-run semantics (LTL$_f$ [16]), giving an explicit procedure to compute finite, faithful abstractions. Linear-time verification is also decidable for integer periodicity constraint systems, also with past time operators [19,23], and feedback-free systems, for an enriched constraint language that can refer to a read-only database [14]. Decidability of LTL$_f$ was also shown for systems with the abstract finite summary property [28], which includes MC, GC, and systems with $k$-bounded lookback, the latter being a generalization of feedback freedom.

Branching-time verification was less studied: Decidability of CTL$^*$ was proven for flat counter systems with Presburger-definable loop iteration [22], even in NP [20]. These results are orthogonal to ours: we do not demand flatness, but our approach does not cover their results. Moreover, it was shown that CTL$^*$
verification is decidable for pushdown systems, which can model counter systems with a single integer variable \[30\]. For integer relational automata (IRA), i.e., systems with constraints \(x \geq y\) or \(x > y\) and domain \(\mathbb{Z}\), CTL model checking is undecidable while the existential and universal fragments of CTL* remain decidable \[12\]. For GC systems, which extend IRAs to constraints of the form \(x - y \geq k\), the existential fragment of CTL* is decidable while the universal one is not \[8\]. A similar dichotomy holds for the EF and EG fragments of CTL \[42\]. A subclass of IRAs were considered in \[11,7\], allowing only periodicity and monotonicity constraints. While satisfiability of CTL* was proven decidable, model checking is not (as already shown in \[12\]), though it is decidable for properties in the fragment CEF+, an extension of the EF fragment \[7\]. In contrast, rather than restricting temporal operators, we show decidability of model checking under an abstract property of the DDSA and the verified property. This abstract property can be guaranteed by suitably constraining the constraint class in the system, or the control flow. More closely related is work by Gascon \[31\], who shows decidability of CTL* model checking for counter systems that admit a nice symbolic valuation abstraction, an abstract property which includes MC and integer periodicity constraint (IPC) systems. The relationship between our decidability criterion and the property defined by Gascon will need further investigation. Another difference is that we here adopt a finite-path semantics for CTL* as e.g. considered in \[47\], since for the analysis of real-world processes such as business processes it is sufficient to consider finite traces. On a high level, our method follows a common approach to CTL*: the verification property is processed bottom-up, and we compute solutions for each subproperty. These are then used to formulate an equivalent linear-time verification problem \[2\, p.429\]. For the latter, we can partially rely on earlier work \[28\].

2 Background

We start by defining the set of constraints over expressions of sort \(\text{int}, \text{rat},\) or \(\text{real}\), with associated domains \(\text{dom}(\text{int}) = \mathbb{Z}, \text{dom}(\text{rat}) = \mathbb{Q},\) and \(\text{dom}(\text{real}) = \mathbb{R}\).

Definition 1. For a given set of sorted variables \(V\), expressions \(e_s\) of sort \(s\) and atoms \(a\) are defined as follows:
\[
e_s := v_s \mid k_s \mid e_s + e_s \mid e_s - e_s \quad a := e_s = e_s \mid e_s < e_s \mid e_s \leq e_s \mid e_{\text{int}} \equiv_n e_{\text{int}}
\]
where \(k_s \in \text{dom}(s)\), \(v_s \in V\) has sort \(s\), and \(\equiv_n\) denotes equality modulo some \(n \in \mathbb{N}\). A constraint is then a quantifier-free boolean expression over atoms \(a\).

The set of all constraints built from atoms over variables \(V\) is denoted by \(C(V)\). For instance, \(x \neq 1\), \(x < y - z\), and \(x - y = 2 \land y \neq 1\) are valid constraints independent of the sort of \(\{x, y, z\}\), while \(u \equiv_y v + 1\) is a constraint for integer variables \(u\) and \(v\). We write \(\text{Var}(\varphi)\) for the set of variables in a formula \(\varphi\). For an assignment \(\alpha\) with domain \(V\) that maps variables to values in their domain, and a formula \(\varphi\) we write \(\alpha \models \varphi\) if \(\alpha\) satisfies \(\varphi\).

We are thus in the realm of SMT with linear arithmetic, which is decidable and admits quantifier elimination \[45\]: if \(\varphi\) is a formula in \(C(X \cup \{y\})\), thus
having free variables $X \cup \{y\}$, there is a quantifier-free $\varphi'$ with free variables $X$ that is equivalent to $\exists y. \varphi$, i.e., $\varphi' \equiv \exists y. \varphi$, where $\equiv$ denotes logical equivalence.

2.1 Data-aware Dynamic Systems with Arithmetic

From now on, $V$ will be a fixed, finite set of variables. We consider two disjoint, marked copies of $V$, $V^r = \{v^r | v \in V\}$ and $V^w = \{v^w | v \in V\}$, called the read and write variables. They will refer to the variable values before and after a transition, respectively. We also write $\overrightarrow{V}$ for a vector that contains the variables $V$ in an arbitrary but fixed order, and $\overrightarrow{V^r}$ and $\overrightarrow{V^w}$ for the vectors that order $V^r$ and $V^w$ in the same way.

**Definition 2.** A DDSA $B = \langle B, b_I, A, T, B_F, V, \alpha_I, \text{guard} \rangle$ is a labeled transition system where (i) $B$ is a finite set of control states, with $b_I \in B$ the initial one; (ii) $A$ is a set of actions; (iii) $T \subseteq B \times A \times B$ is a transition relation; (iv) $B_F \subseteq B$ are final states; (v) $V$ is the set of process variables; (vi) $\alpha_I$ the initial variable assignment; (vii) guard: $A \rightarrow \mathcal{C}(V^r \cup V^w)$ specifies the executability constraints.

**Example 2.** We consider the following DDSAs $B, B_{di},$ and $B_{ipc}$, where $x, y$ have domain $\mathbb{Q}$ and $a, v, s$ have domain $\mathbb{Z}$. Initial and final states have incoming arrows and double borders, respectively; $\alpha_I$ is not fixed for now.

![Diagram](image)

Also the system in Ex. 1 represents a DDSA. If state $b$ admits a transition to $b'$ via action $a$, namely $(b, a, b') \in \Delta$, this is denoted by $b \xrightarrow{a} b'$. A configuration of $B$ is a pair $(b, \alpha)$ where $b \in B$ and $\alpha$ is an assignment with domain $V$. A guard assignment is a function $\beta: V^r \cup V^w \rightarrow D$. For an action $a$, let $\text{write}(a) = \text{Var}(\text{guard}(a)) \cap V^w$. As defined next, an action $a$ transforms a configuration $(b, \alpha)$ into a new configuration $(b', \alpha')$ by updating the assignment $\alpha$ according to the action guard, which can at the same time evaluate conditions on the current values of variables and write new values:

**Definition 3.** A DDSA $B = \langle B, b_I, A, T, B_F, V, \alpha_I, \text{guard} \rangle$ admits a step from configuration $(b, \alpha)$ to $(b', \alpha')$ via action $a$, denoted $(b, \alpha) \xrightarrow{a} (b', \alpha')$, if $b \xrightarrow{a} b'$, $\alpha'(v) = \alpha(v)$ for all $v \in V \setminus \text{write}(a)$, and the guard assignment $\beta$ given by $\beta(v^r) = \alpha(v)$ and $\beta(v^w) = \alpha'(v)$ for all $v \in V$ satisfies $\beta \models \text{guard}(a)$.

For instance, for $B$ in Ex. 2 and initial assignment $\alpha_I(x) = \alpha_I(y) = 0$, the initial configuration admits a step $(b_1, \left[\begin{smallmatrix} x = 0 \\ y = 0 \end{smallmatrix}\right]) \xrightarrow{a_1} (b_2, \left[\begin{smallmatrix} x = 0 \\ y = -1 \end{smallmatrix}\right])$ with $\beta(x^r) = \beta(x^w) = \beta(y^r) = 0$ and $\beta(y^w) = 3$.

A run $\rho$ of a DDSA $B$ of length $n$ from configuration $(b, \alpha)$ is a sequence of steps $\rho: (b, \alpha) = (b_0, \alpha_0) \xrightarrow{a_1} (b_1, \alpha_1) \xrightarrow{a_2} \cdots \xrightarrow{a_n} (b_n, \alpha_n)$. We also associate with $\rho$ the symbolic run $\sigma$: $b_0 a_1 b_1 a_2 \cdots a_n b_n$ where state and action sequences are recorded without assignments, and say that $\sigma$ is the abstraction of $\rho$ (or, $\sigma$ abstracts $\rho$). For some $m < n$, $\sigma|_m$ denotes the prefix of $\sigma$ that has $m$ steps.
2.2 History Constraints

In this section, we fix a DDSA \( B = (B, b_1, A, T, B_F, V, \alpha_1, \text{guard}) \). We aim to build an abstraction of \( B \) that covers the (potentially infinite) set of configurations by finitely many states of the form \((b, \varphi)\), where \( b \in B \) is a control state and \( \varphi \) a formula that expresses conditions on the process variables \( V \). A state \((b, \varphi)\) will thus represent all configurations \((b, \alpha)\) s.t. \( \alpha \models \varphi \). To mimic steps on the abstract level, we define below the update function to express how such a formula \( \varphi \) is modified by executing an action. First, let the transition formula of action \( a \) be \( \Delta_a(V^e, V^w) = \text{guard}(a) \land \bigwedge_{v \in V \setminus \text{write}(a)} v^w = v^e \). Intuitively, this states conditions on variables before and after executing \( a \): \( \text{guard}(a) \) must be true and the values of all variables that are not written are propagated by inertia. As \( \Delta_a \) has free variables \( V^e \) and \( V^w \), we write \( \Delta_a(X, Y) \) for the formula obtained from \( \Delta_a \) by replacing \( V^e \) by \( X \) and \( V^w \) by \( Y \).

Definition 4. For a formula \( \varphi \) with free variables \( V \) and action \( a \), update\((\varphi, a) = \exists \bar{U} \varphi(\bar{U}) \land \Delta_a(\bar{U}, \bar{V}) \), where \( U \) is a set of variables that do not occur in \( \varphi \).

Our approach generates an abstraction using formulas of a special shape called history constraints \([28]\), obtained by iterated update operations in combination with a sequence of verification constraints \( \bar{U} \). The latter will later be taken from the transition labels of an automaton for the verified property. For now it is enough to consider \( \bar{U} \) an arbitrary sequence of constraints with free variables \( V \). Its prefix of length \( k \) is denoted by \( \bar{U}|_k \). We need a fixed set of placeholder variables \( V_0 \) that are disjoint from \( V \), and assume an injective variable renaming \( \nu : V \rightarrow V_0 \). Let \( \varphi_\nu \) be the formula \( \varphi_\nu = \bigwedge_{v \in V} \nu(v) \).

Definition 5. For a symbolic run \( \sigma : b_0 \xrightarrow{a_1} b_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} b_n \) and verification constraint sequence \( \bar{U} = \langle \vartheta_0, \ldots, \vartheta_n \rangle \), the history constraint \( h(\sigma, \bar{U}) \) is given by \( h(\sigma, \bar{U}) = \varphi_\nu \land \vartheta_0 \) if \( n = 0 \), and \( h(\sigma, \bar{U}) = \text{update}(h(\sigma|_{n-1}, \bar{U}|_{n-1}), a_n) \land \vartheta_n \) if \( n > 0 \).

Thus, history constraints are formulas with free variables \( V \cup V_0 \). Satisfying assignments for history constraints are closely related to assignments in runs

Lemma 1. For a symbolic run \( \sigma : b_0 \xrightarrow{a_1} b_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} b_n \) and \( \bar{U} = \langle \vartheta_0, \ldots, \vartheta_n \rangle \), \( h(\sigma, \bar{U}) \) is satisfied by assignment \( \alpha \) with domain \( V \cup V_0 \) iff \( \sigma \) abstracts a run \( \rho : (b_0, \alpha_0) \xrightarrow{a_1} \cdots \xrightarrow{a_n} (b_n, \alpha_n) \) such that (i) \( \alpha_0(\nu(v)) = \alpha(\nu(v)) \), and (ii) \( \alpha_i(v) = \alpha(v) \) for all \( v \in V_i \), and (iii) \( \alpha_i \models \vartheta_i \) for all \( i, 0 \leq i \leq n \).

2.3 CTL^*

For a DDSA \( B \) as above, we consider the following verification properties:

Definition 6. CTL^* state formulas \( \chi \) and path formulas \( \psi \) are defined by the following grammar, for constraints \( c \in \mathcal{C}(V) \) and control states \( b \in B \):

\[
\chi ::= \top | c | b | \chi \land \chi | \neg \chi | E \psi | X \psi | G \psi | \psi U \psi
\]

\(1\) Lem. 1 is a slight variation of [28], Lem. 3.5; Def. 5 differs from history constraints in [28] in that the initial assignment is not fixed. We provide a proof in App. A.
We use the usual abbreviations $\mathbf{F} \psi = T \cup \psi$, $\chi_1 \lor \chi_2 = \neg(\neg \chi_1 \land \neg \chi_2)$, and $A \psi = E \neg \psi$. To simplify the presentation, we do not explicitly treat next state operators $(a)$ via a specific action $a$, as used in Ex. 1 though this would be possible (cf. [28]). However, such an operator can be encoded by adding a fresh data variable $x$ to $V$, the conjunct $x^w = 1$ to $\text{guard}(a)$, and $x^w = 0$ to all other guards, and replacing $(a)\psi$ in the verification property by $X(\psi \land x = 1)$.

The maximal number of nested path quantifiers in a formula $\psi$ is called the quantifier depth of $\psi$, denoted by $qd(\psi)$. We adopt a finite path semantics for CTL*. For a control state $b \in B$ and a state assignment $\alpha$, let $FRuns(b, \alpha)$ be the set of final runs $\rho: (b, \alpha) = (b_0, \alpha_0) \xrightarrow{\alpha_0} \ldots \xrightarrow{\alpha_n} (b_n, \alpha_n)$ such that $b_n \in F$ is a final state. The $i$-th configuration $(b_i, \alpha_i)$ in $\rho$ is denoted by $\rho_i$.

**Definition 7.** The semantics of CTL$_l^*$ is inductively defined as follows. For a DDSA $B$ with configuration $(b, \alpha)$, state formulas $\chi, \chi'$, and path formulas $\psi, \psi'$:

$(b, \alpha) \models T$ iff $\alpha \models c$

$(b, \alpha) \models b'$ iff $b = b'$

$(b, \alpha) \models \chi \land \chi'$ iff $(b, \alpha) \models \chi$ and $(b, \alpha) \models \chi'$

$(b, \alpha) \models \neg \chi$ iff $(b, \alpha) \not\models \chi$

$(b, \alpha) \models E \psi$ iff $\exists \rho \in FRuns(b, \alpha)$ such that $\rho \models \psi$

where $\rho = \psi$ iff $\rho, 0 \models \psi$ holds, and for a run $\rho$ of length $n$ and all $i$, $0 \leq i \leq n$:

$\rho, i \models \chi$ iff $\rho_i \models \chi$

$\rho, i \models \neg \psi$ iff $\rho, i \not\models \psi$

$\rho, i \models \psi \land \psi'$ iff $\rho, i \models \psi$ and $\rho, i \models \psi'$

$\rho, i \models X \psi$ iff $i < n$ and $\rho, i + 1 \models \psi$

$\rho, i \models G \psi$ iff for all $j$, $i \leq j \leq n$, it holds that $\rho, j \models \psi$

$\rho, i \models \psi \lor \psi'$ iff $\exists k$ with $i + k \leq n$ such that $\rho, i + k \models \psi'$ and for all $j$, $0 \leq j < k$, it holds that $\rho, i + j \models \psi$.

Instead of simply checking whether the initial configuration of a DDSA $B$ satisfies a CTL$_l^*$ property $\chi$, we try to determine, for every state $b \in B$, which constraints on variables need to hold in order to satisfy $\chi$. As the number of configurations $(b, \alpha)$ of a DDSA $B$ is usually infinite, configuration sets cannot be enumerated explicitly. Instead, we represent a set of configurations as a configuration map $K: B \rightarrow C(V)$ that associates with every control state $b \in B$ a formula $K(b) \in C(V)$, representing all configurations $(b, \alpha)$ such that $\alpha \models K(b)$.

Our aim is thus to compute a solution $K$ to the following problem:

**Definition 8 (Verification problem).** For a DDSA $B$ and state formula $\chi$, is there a configuration map $K$ such that $(b, \alpha) \models \chi$ iff $\alpha \models K(b)$, for all $b \in B$?

We call the verification problem given by $B$ and $\chi$ solvable if a solution $K$ exists and can be effectively computed. For instance, for $B$ from Ex. 2 and $\chi_1 = AG(x \geq 2)$, a solution is given by $K = \{b_1 \rightarrow \bot, b_2 \rightarrow x \geq 2 \land y \geq 2, b_3 \rightarrow x \geq 2\}$. For $\chi_2 = EX(AG(x \geq 2))$, a solution is $K' = \{b_1 \rightarrow x \geq 2, b_2 \rightarrow y \geq 2, b_3 \rightarrow \bot\}$. As $b_1$ is the initial state, $B$ satisfies $\chi_2$ with every initial assignment that sets $\alpha_1(x) \geq 2$. Note that a solution $K$ to the verification
problem for \( B \) and \( \chi \) in particular allows to determine whether \((b_1, \alpha_1) \models \chi\) holds, by testing \( \alpha_1 \models K(b_1) \), so that \((b_1, \alpha_1) \models \chi\) is decidable for \( B \).

### 3 LTL with Configuration Maps

Following a common approach to CTL* verification, our technique processes the property \( \chi \) bottom-up, computing solutions for each subformula \( E \psi \), before solving a linear-time model checking problem \( \chi' \) in which the solutions to subformulas appear as atoms. Given our representation of sets of configurations, we use LTL formulas where atoms are configuration maps, and denote this specification language by \( \text{LTL}_f^B \). For a given DDSA \( B \), it is formally defined as follows:

\[
\psi := K \mid \psi \land \psi \mid \neg \psi \mid X \psi \mid G \psi \mid \psi \cup \psi
\]

where \( K \in K_B \), for \( K_B \) is the set of configuration maps for \( B \). We again use a finite-trace semantics [10].

**Definition 9.** A run \( \rho \) of length \( n \) satisfies an LTL\(_f^B \) formula \( \psi \), denoted \( \rho \models K \psi \), iff \( \rho, 0 \models K \psi \) holds, where for all \( i \), \( 0 \leq i \leq n \):

- \( \rho, i \models K \psi \land \psi' \) iff \( \rho, i \models K \psi \) and \( \rho, i \models K \psi' \);
- \( \rho, i \models K \neg \psi \) iff \( \rho, i \not\models K \psi \);
- \( \rho, i \models K X \psi \) iff \( \rho, i < n \) and \( \rho, i+1 \models K \psi \);
- \( \rho, i \models K G \psi \) iff \( \rho, i \models K \psi \) and \( (i = n \) or \( \rho, i+1 \models K G \psi \));
- \( \rho, i \models K \psi \cup \psi' \) iff \( \rho, i \models K \psi' \) or \( (i < n \) and \( \rho, i \models K \psi \) and \( \rho, i+1 \models K \psi \cup \psi' \)).

Our approach to LTL\(_f^B \) verification proceeds along the lines of the LTL\(_f \) procedure from [28], with the difference that simple constraint atoms are replaced by configuration maps. In order to express the requirements on a run of a DDSA \( B \) to satisfy an LTL\(_f^B \) formula \( \chi \), we use a nondeterministic automaton (NFA) \( N_\psi = (Q, \Sigma, \varphi, q_0, Q_F) \), where the states \( Q \) is a set of subformulas of \( \psi \), \( \Sigma = 2^{X^a} \) is the alphabet, \( \varphi \) is the transition relation, \( q_0 \in Q \) is the initial state, and \( Q_F \subseteq Q \) is the set of final states. The construction of \( N_\psi \) is standard [15,28], treating configuration maps for the time being as propositions; but for completeness it is described in App. [C] For instance, for a configuration map \( K, \psi = F K \) corresponds to the NFA \( \begin{array}{c} \circ \circ \circ \circ \end{array} \) and \( \psi' = XK \) to \( \begin{array}{c} \circ \circ \circ \circ \end{array} \). (For simplicity, edges labels \( \{K\} \) are shown as \( K \), and edge labels \( \emptyset \) are omitted.)

For \( w_i \in \Sigma \), i.e., \( w_i \) is a set of configuration maps, \( w_i(b) \) denotes the formula \( \bigwedge_{K \in w} K(b) \). Moreover, for \( w = w_0, \ldots, w_n \in \Sigma^* \) and a symbolic run \( \sigma: b_0 \xrightarrow{a_0} b_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} b_n \), let \( w \odot \sigma \) denote the sequence of formulas \( \langle w_0(b_0), \ldots, w_n(b_n) \rangle \), i.e., the component-wise application of \( w \) to the control states of \( \sigma \). A word \( w_0, \ldots, w_n \in \Sigma^* \) is consistent with a run \( (b_0, \alpha_0) \xrightarrow{a_0} (b_1, \alpha_1) \xrightarrow{a_2} \cdots \xrightarrow{a_n} (b_n, \alpha_n) \) if \( \alpha_i \models w_i(b_i) \) for all \( i \), \( 0 \leq i \leq n \). The key correctness property of \( N_\psi \) is the following (cf. [28] Lem. 4.4), and see App. [C] for the proof adapted to LTL\(_f^B \);

**Lemma 2.** \( N_\psi \) accepts a word that is consistent with a run \( \rho \) iff \( \rho \models K \psi \).
**Product Construction.** As a next step in our verification procedure, given a control state $b$ of $B$, we aim to find (a symbolic representation of) all configurations $(b, \alpha)$ that satisfy an LTL$_f^B$ formula $\psi$. To that end, we combine $N_{\psi}$ with $B$ to a cross-product automaton $N_{\psi}^B$. For technical reasons, when performing the product construction, the steps in $B$ need to be shifted by one with respect to the steps in $N_{\psi}$. Hence, given $b \in B$, let $B_b$ be the DDSA obtained from $B$ by adding a dummy initial state $\langle b \rangle$, so that $B_b$ has state set $B' = B \cup \{ \langle b \rangle \}$ and transition relation $T' = T \cup \{ (\langle b \rangle, a_0, b) \}$ for a fresh action $a_0$ with guard$(a_0) = \top$.

**Definition 10.** The product automaton $N_{\psi}^B$ is defined for an LTL$_f^B$ formula $\psi$, a DDSA $B$, and a control state $b \in B$. Let $B_b = (B', \langle b \rangle, A', T', B_F, V, \alpha_1, \text{guard})$ and $N_{\psi}$ as above. Then $N_{\psi}^B$ is as follows:

- $P \subseteq B' \times Q \times (V \cup V_0)$, i.e., states in $P$ are triples $(b, q, \phi)$ such that $b$ is in the set of final states $B'$, $q$ is in the set of final states $Q$, and $\phi$ is guarded by $a_0$ in $\text{guard}$(a).
- the initial state is $p_0 = (\langle b \rangle, q_0, \phi)$;
- if $b \xrightarrow{a} b'$ in $T'$, $q \xrightarrow{w} q'$ in $N_{\psi}$, and update$(\phi, a) \land w(b')$ is satisfiable, there is a transition $(b, q, \phi) \xrightarrow{a,w} (b', q', \phi')$ in $R$ such that $\phi' \equiv \text{update}(\phi, a) \land w(b')$;
- $(b', q', \phi')$ is in the set of final states $P_F \subseteq P$ iff $b' \in B_F$, and $q' \in Q_F$.

**Example 3.** Consider the DDSA $B$ from Ex. 2 and let $K = \{ b_1 \mapsto \bot, b_2 \mapsto x \geq 2 \land y \geq 2, b_3 \mapsto x = 3 \}$. The property $\psi = XK$ is captured by the NFA $\xrightarrow{\top} \xrightarrow{\top} \xrightarrow{\top} \xrightarrow{\top} \xrightarrow{\top}$). The product automata $N_{\psi}^B$ and $N_{\psi}^{b_2}$ are as follows:

- where the shaded nodes are final. The formulas in nodes were obtained by applying quantifier elimination to the formulas built using update according to Def. 10. $N_{\psi}^{b_2}$ consists only of the dummy transition and has no final states.

Def. 10 need not terminate if infinitely many non-equivalent formulas occur in the construction. In Sec. 4 we will identify a criterion that guarantees termination. Beforehand, we state the key correctness property, which lifts [28, Thm. 4.7] to LTL with configuration maps. Its proof is similar to the respective result in [28] but we provide it in the appendix for completeness.

**Theorem 1.** Let $\psi \in \text{LTL}_f^B$ and $b \in B$ such that there is a finite product automaton $N_{\psi}^B$. Then there is a final run $\rho: (b, \alpha_0) \rightarrow^* (b_F, \alpha_F)$ of $B$ such that $\rho \models \psi$, iff $N_{\psi}^B$ has a final state $(b_F, q_F, \phi)$ for some $q_F$ and $\phi$ such that $\phi$ is satisfied by assignment $\gamma$ with $\gamma(V_0) = \alpha_0(V)$ and $\gamma(V) = \alpha_F(V)$.

Thus, witnesses for $\psi$ correspond to paths to final states in the product automaton: e.g., in $N_{\psi}^{b_1}$ in Ex. 3 the formula in the left final node is satisfied by $\gamma(x_0) = \gamma(x) = \gamma(y) = 3$ and $\gamma(y_0) = 0$. For $\alpha_0$ and $\alpha_2$ such that $\alpha_0(V) = \gamma(V_0) = \{ x \mapsto 3, y \mapsto 0 \}$ and $\alpha_2(V) = \gamma(V) = \{ x \mapsto 3, y \mapsto 3 \}$ there is a witness run for $\psi$ from $(b_1, \alpha_0)$ to $(b_1, \alpha_2)$, e.g., $(b_1, \{ x_0 = 3, y_0 = 0 \}) \xrightarrow{2_1} (b_2, \{ x_0 = 3 \}) \xrightarrow{2_2} (b_3, \{ x_0 = 3, y_0 = 3 \})$.\]
4 Model Checking Procedure

We use the results of the previous section to define a model checking procedure for CTL formulas, shown in Fig. 4. First, we explain the tasks achieved by the three mutually recursive functions.

- **checkState(χ)** returns a configuration map representing the set of configurations that satisfy a state formula χ. In the base cases, it returns a function that checks the respective condition, for boolean operators we recurse on the arguments, and for a formula E ψ we proceed to the checkPath procedure.

- **checkPath(ψ)** returns a configuration map K that represents all configurations which admit a path that satisfies ψ. First, toLTLK is used to obtain an equivalent LTL formula ψ′ (which entails the computation of solutions for all subproperties E η). Then solution K is constructed as follows: For every control state b, we build the product automaton Nφ,b, and collect the set ΦF of formulas in final states. Every ϕ ∈ ΦF encodes runs from b to a final state of B that satisfy ψ′. The variables V′ and V in ϕ act as placeholders for the initial and the final values of the runs, respectively. By ϕ(V′, V) we rename variables to use instead V′ at the start and V at the end, we quantify existentially over U (as the final valuation is irrelevant), and take the disjunction over all ϕ ∈ ΦF. The resulting formula ϕ′ encodes all final runs from b that satisfy ψ′, so we set K(b) := ϕ′.

- **toLTLK(ψ)** computes an LTL formula equivalent to a path formula ψ. To this end, it performs two kinds of replacements in ψ: (a) ∨ b ∈ B, and constraints c are represented as configuration maps; and (b) subformulas E η are replaced by their solutions Kη, which are computed by a recursive call to checkPath.

To represent the base cases of formulas as configuration maps in Fig. 4 we define Kx := (λb ∨), Kb := (λb′, b′ ? ∨ : ⊥) for all b ∈ B, and Kc := (λc) for constraints c. We also write ¬K for (λb, ¬K(b)) and K ∧ K′ for (λb, K(b) ∧ K′(b)). The next example illustrates the approach.

**Example 4.** Consider χ = E X (A G (x ≥ 2)) and the DDSA B in Ex. 2. To get a solution K1 to checkState(χ) = checkPath(ψ1) for ψ1 = X (A G (x ≥ 2)), we first compute an equivalent LTL formula ψ′ 1 = XK2, where K2 is a solution to A G (x ≥ 2) ≡ ¬E F (x < 2). To this end, we run checkPath(ψ2) for ψ2 := F (x < 2), which is represented in LTL as ψ′ 2 := F (Kx < 2) with NFA . Next, checkPath builds Nφ,b for all states b. For instance, for b2 we get:

![Diagram showing the construction of configurations](image)

where dashed arrows indicate transitions to non-final sink states. For U = (Û, Ŷ), and the formulas ϕ1, ϕ2, and ϕ3 in final nodes, we compute
For every configuration 

\begin{equation}
\text{Procedure checkState}(\chi)
\end{equation}

1: \textbf{procedure} checkState(\chi) 
2: \hspace{1em} \textbf{switch} \chi \textbf{do} 
3: \hspace{2em} \textbf{case} \ T, b \in B, \text{ or } c \in \mathcal{C}: \textbf{return} K_\chi 
4: \hspace{2em} \textbf{case} \ \chi_1 \land \chi_2:\ \textbf{return} \ \text{checkState}(\chi_1) \land \text{checkState}(\chi_2) 
5: \hspace{2em} \textbf{case} \ \neg \chi:\ \textbf{return} \ \neg \text{checkState}(\chi) 
6: \hspace{2em} \textbf{case} \ \mathbf{E} \ \psi:\ \textbf{return} \ \text{checkPath}(\psi) 

1: \textbf{procedure} checkPath(\psi) 
2: \psi' := \text{toLTL}_K(\psi) 
3: \hspace{1em} \textbf{for} b \in B \textbf{ do} 
4: \hspace{2em} (P, R, p_0, P_\psi) := N_{\mathcal{G}_b}^{\psi'} \quad \triangleright \text{ product automaton for } \psi', \ B, \text{ and } b 
5: \hspace{2em} \Phi := \{\varphi \mid (b, q, \psi) \in P_\psi\} \quad \triangleright \text{ collect formulas in final states} 
6: \hspace{2em} K(b) := \bigvee_{\varphi \in \Phi} \varphi((\mathcal{V}, \mathcal{U})) 
7: \hspace{2em} \textbf{return} K 

1: \textbf{procedure} toLTL_K(\psi) 
2: \hspace{1em} \textbf{switch} \ \psi \textbf{ do} 
3: \hspace{2em} \textbf{case} \ T, b \in B, \text{ or } c \in \mathcal{C}: \textbf{return} K_{\psi} 
4: \hspace{2em} \textbf{case} \ \psi_1 \land \psi_2:\ \textbf{return} \ \text{toLTL}_K(\psi_1) \land \text{toLTL}_K(\psi_2) 
5: \hspace{2em} \textbf{case} \ \neg \psi:\ \textbf{return} \ \neg \text{toLTL}_K(\psi) 
6: \hspace{2em} \textbf{case} \ \mathbf{E} \ \psi:\ \textbf{return} \ \text{checkPath}(\psi) 
7: \hspace{2em} \textbf{case} \ \mathbf{X} \ \psi:\ \textbf{return} \ \mathbf{X} \ \text{toLTL}_K(\psi) 
8: \hspace{2em} \textbf{case} \ \mathbf{G} \ \psi:\ \textbf{return} \ \mathbf{G} \ \text{toLTL}_K(\psi) 
9: \hspace{2em} \textbf{case} \ \psi_1 \ \mathbf{U} \ \psi_2:\ \textbf{return} \ \text{toLTL}_K(\psi_1) \ \mathbf{U} \ \text{toLTL}_K(\psi_2) 

\begin{align*}
\exists x, \phi_1((\mathcal{V}, \mathcal{U})) & \equiv \exists x. \hat{y}. \hat{x} = x = \hat{y} \land x < 2 \equiv x < 2 \\
\exists x, \phi_2((\mathcal{V}, \mathcal{U})) & \equiv \exists x. \hat{y}. \hat{x} = \hat{y} \land y < 2 \equiv y < 2 \\
\exists x, \phi_3((\mathcal{V}, \mathcal{U})) & \equiv \exists x. \hat{y}. \hat{x} = \hat{y} \land x < 2 \equiv x < 2 
\end{align*}

so that \( K_3 := \text{checkPath}(\psi_2) \) sets \( K_3(b_2) = \bigvee_{i=1}^3 \exists x. \phi_i((\mathcal{V}, \mathcal{U})) \equiv x < 2 \lor y < 2 \). For reasons of space, the constructions for \( b_1 \) and \( b_3 \) are shown in Ex. \( \triangleright \) in App. \( \triangleright \). We obtain \( K_3(b_1) = \top \) and \( K_3(b_3) = x < 2 \). By negation, the solution \( K_2 \) to \( \mathbf{A} \ \mathbf{G} \ (x \geq 2) \) is \( K_2 = \neg K_3 = \{b_1 \mapsto \top, \ b_2 \mapsto x \geq 2 \land y \geq 2, \ b_3 \mapsto x \geq 2\} \). Now we can proceed with \( \text{checkPath}(\psi_1) \). The NFA and product automata for \( \psi_1 \equiv \mathbf{X} K_2 \) are as shown in Ex. \( \triangleright \) and in a similar way as above we obtain the solution \( K_1 \) for \( \mathbf{E} \mathbf{X} \mathbf{A} \mathbf{G} \ (x \geq 2) \) as \( K_1 = \{b_1 \mapsto x \geq 2, \ b_2 \mapsto y \geq 2, \ b_3 \mapsto \bot\} \). Thus, \( \mathcal{B} \) satisfies the property for any initial assignment \( \alpha_I \) with \( \alpha_I(x) \geq 2 \).

Next we prove correctness of \( \text{checkState}(\chi) \) under the condition that it is defined, i.e., all required product automata are finite. First we state our main result, but before giving its proof we show helpful properties of \( \text{toLTL}_K \) and \( \text{checkPath} \).

**Theorem 2.** For every configuration \( (b, \alpha) \) of the DDSA \( \mathcal{B} \) and every state property \( \chi \), if \( \text{checkState}(\chi) \) is defined then \( (b, \alpha) \models \chi \) iff \( \alpha \models \text{checkState}(\chi)(b) \).

**Lemma 3.** Let \( \psi \) be a path formula with \( \text{qd}(\psi) = k \). Suppose that for all configurations \( (b, \alpha) \) and path formulas \( \psi' \) with \( \text{qd}(\psi') < k \), there is a \( \rho' \in \text{FRuns}(b, \alpha) \) with \( \rho' \models \psi' \) iff \( \alpha \models \text{checkPath}(\psi')(b) \). Then \( \rho \models \psi \) iff \( \rho \models \text{toLTL}_K(\psi) \).
Proof (sketch). By induction on $\psi$. The base cases are by the definitions of $K_\top$, $K_b$, and $K_c$. In the inductive step, if $\psi = E \psi'$ then $\rho \models \psi$ iff $\exists \rho' \in \text{FRuns}(b_0, \alpha_0)$ with $\rho' \models \psi'$, for $\rho_0 = (b_0, \alpha_0)$. As $qd(\psi') < qd(\psi)$, this holds by assumption iff $\alpha_0 \models \text{checkPath}(\psi')(b_0)$. This is equivalent to $\rho \models_{K} \text{toLTL}_{K}(\psi) = \text{checkPath}(\psi')$.

All other cases are by the induction hypothesis and Defs. 7 and 9.

Lemma 4. If $\psi' = \text{toLTL}_{K}(\psi)$ such that for all runs $\rho$ it is $\rho \models \psi$ iff $\rho \models_{K} \psi'$, there is a run $\rho \in \text{FRuns}(b, \alpha)$ with $\rho \models \psi$ iff $\alpha \models \text{checkPath}(\psi)(b)$.

Proof. ($\Rightarrow$) Suppose there is a run $\rho \in \text{FRuns}(b, \alpha)$ with $\rho \models \psi$, so $\rho$ is of the form $(b, \alpha) \rightarrow^*(b_F, q_F, \varphi)$ for some $b_F \in B_F$. By assumption, this implies $\rho \models_{K} \psi'$, so that by Thm. 4 $N_{b, \alpha}$ has a final state $(b_F, q_F, \varphi)$ where $\varphi$ is satisfied by an assignment $\gamma$ with domain $V \cup V_0$ such that $\gamma(V_0) = \alpha(V)$ and $\gamma(V) = \alpha_F(V)$. By definition, $\text{checkPath}(\psi)(b)$ contains a disjunct $\exists \overline{U}, \varphi(\overline{V}, \overline{U})$. As $\gamma$ satisfies $\varphi$ and $\gamma(V_0) = \alpha(V)$, $\alpha \models \text{checkPath}(\psi)(b)$. ($\Leftarrow$) If $\alpha \models \text{checkPath}(\psi)(b)$, by definition of $\text{checkPath}$ there is a formula $\varphi$ such that $\alpha \models \exists \overline{U}, \varphi(\overline{V}, \overline{U})$ and $\varphi$ occurs in a final state $(b_F, q_F, \varphi)$ of $N_{b, \alpha}$. Hence there is an assignment $\gamma$ with domain $V \cup V_0$ and $\gamma(V_0) = \alpha(V)$ such that $\gamma \models \varphi$. By Thm. 4, there is a run $\rho: (b, \alpha) \rightarrow^*(b_F, \alpha_F)$ such that $\rho \models_{K} \psi'$. By the assumption, we have $\rho \models \psi$. \qed

At this point the main theorem can be proven:

Proof (of Thm. 3). We first show (⋆): for any path formula $\psi$, there is a run $\rho \in \text{FRuns}(b, \alpha)$ with $\rho \models \psi$ iff $\alpha \models \text{checkPath}(\psi)(b)$. The proof is by induction on $qd(\psi)$. If $\psi$ contains no path quantifiers, Lem. 3 implies that $\rho \models \psi$ iff $\rho \models_{K} \text{toLTL}_{K}(\psi)$ for all runs $\rho$, so (⋆) follows from Lem. 4. In the induction step, we conclude from Lem. 3 using the induction hypothesis of (⋆) as assumption, that $\rho \models \psi$ iff $\rho \models_{K} \text{toLTL}_{K}(\psi)$ for all runs $\rho$. Again, (⋆) follows from Lem. 4.

The theorem is then shown by induction on $\chi$: The base cases $\top, b' \in B$, $c \in C$ are easy to check, and for properties of the form $\neg \chi'$ and $\chi_1 \land \chi_2$ the claim follows from the induction hypothesis and the definitions. Finally, for $\chi = E \psi$, $(b, \alpha) \models \chi$ iff there is a run $\rho \in \text{FRuns}(b, \alpha)$ such that $\rho \models \psi$. By (⋆) this is the case iff $\alpha \models \text{checkPath}(\psi)(b) = \text{checkState}(\chi)(b)$. \qed

Termination We next show that the formulas generated in our procedure all have a particular shape, to obtain an abstract termination result. For a set of formulas $\Phi \subseteq C(V)$ and a symbolic run $\sigma$, let a history constraint $h(\sigma, \overline{\sigma})$ be over basis $\Phi$ if $\overline{\sigma} = (\delta_0, \ldots, \delta_n)$ and for all $i$, $1 \leq i \leq n$, there is a subset $T_i \subseteq \Phi$ s.t. $\delta_i = \bigwedge T_i$. Moreover, for a set of formulas $\Phi$, let $\Phi^\pm = \Phi \cup \{\neg \varphi \mid \varphi \in \Phi\}$.

Definition 11. For a DDSA $B$, a constraint set $C$ over free variables $V$, and $k \geq 0$, the formula sets $\Phi_k$ are inductively defined by $\Phi_0 = C \cup \{\top, \bot\}$ and $\Phi_{k+1} = \{\bigwedge_{\varphi \in H} \exists \overline{U}, \varphi(\overline{V}, \overline{U}) \mid H \subseteq H_k\}$, where $H_k$ is the set of all history constraints of $B$ with basis $\bigcup_{i \leq k} \Phi_i^\pm$. 
Note that formulas in $\Phi_k$ have free variables $V$, while those in $\mathcal{H}_k$ have free variables $V_0 \cup V$. We next show that these sets correspond to the formulas generated by our procedure, if all constraints in the verification property are in $C$.

**Lemma 5.** Let $\mathcal{E} \psi$ have quantifier depth $k$, $\psi' = \text{toLTL}_K(\psi)$, and $N_{B,b}^{\psi'}$ be a constraint graph constructed in $\text{checkPath}(\psi)$ for some $b \in B$. Then,

1. for all nodes $(b', q, \varphi)$ in $N_{B,b}^{\psi'}$ there is some $\varphi' \in \mathcal{H}_k$ such that $\varphi \equiv \varphi'$,
2. $\text{checkPath}(\psi)(b)$ is equivalent to a formula in $\Phi_{k+1}$.

The statements are proven by induction on $k$, using the results about the product construction (Lem. 6). From part (1) of this lemma and Thm. 2 we thus obtain an abstract criterion for decidability that will become useful in the next section:

**Corollary 1.** For a DDS $B$ as above and a state formula $\chi$, if $\mathcal{H}_j(b)$ is finite up to equivalence for all $j < \text{qd}(\chi)$ and $b \in B$, the verification problem is solvable.

**Proof.** By the assumption about the sets $\mathcal{H}_j(b)$ for $j < \text{qd}(\chi)$, all product automata constructions in recursive calls $\text{checkPath}(\psi)$ of $\text{checkState}(\chi)$ terminate if logical equivalence of formulas is checked eagerly. Thus $\text{checkState}(\chi)$ is defined, and by Thm. 2 it solves the verification problem. \qed

The property that all sets $\mathcal{H}_j(b)$, $j < \text{qd}(\chi)$, are finite might not be decidable itself. However, in the next section we will show means to guarantee this property. Moreover, we remark that finiteness of all $\mathcal{H}_j(b)$ implies a finite history set, a decidability criterion identified for the linear-time case [28, Def. 3.6]; but Ex. 5 below illustrates that the requirement on the $\mathcal{H}_j(b)$’s is strictly stronger.

## 5 Decidability of DDSA Classes

We here illustrate restrictions on DDSAs, either on the control flow or on the constraint language, that render our approach a decision procedure for $\text{CTL}^*$.  

**Monotonicity constraints** (MCs) restrict constraints (Def. 1) as follows: MCs over variables $V$ and domain $D$ have the form $p \circ q$ where $p, q \in D \cup V$ and $\circ$ is one of $\ldots, \neq, =, \leq, >, \geq, \lor$. The domain $D$ may be $\mathbb{R}$ or $\mathbb{Q}$. We call a boolean formula whose atoms are MCs an $MC$ formula, a DDSA where all atoms in guards are MCs an $MC$-$DDSA$, and a $\text{CTL}^*_f$ property whose constraint atoms are MCs an $MC$ property. For instance, $B$ in Ex. 2 is an $MC$-$DDSA$.

We exploit a useful quantifier elimination property: If $\varphi$ is an MC formula over a set of constants $L$ and variables $V \cup \{x\}$, there is some $\varphi' \equiv \exists x. \varphi$ such that $\varphi'$ is a quantifier-free MC formula over $V$ and $L$. Such a $\varphi'$ can be obtained by writing $\varphi$ in disjunctive normal form and applying a Fourier-Motzkin procedure [30, Sec. 5.4] to each disjunct, which guarantees that all constants in $\varphi'$ also occur in $\varphi$.

**Theorem 3.** The verification problem is solvable for all combinations of an $MC$-$DDSA$ $B$ and an $MC$ property $\chi$. 
Proof. Let \( \chi \) be an MC property, and \( L \) the finite set of constants in constraints in \( \chi \), \( \alpha_0 \), and guards of \( B \). Let moreover \( MC_L \) be the set of quantifier-free formulas whose atoms are MCs over \( V \cup V_0 \) and \( L \), so \( MC_L \) is finite up to equivalence.

We show the following property \((\ast)\): all history constraints \( h(\sigma, \overrightarrow{\nu}) \) over basis \( MC_L \) are equivalent to a formula in \( MC_L \). For a symbolic run \( \sigma : b_0 \rightarrow^n b_{n-1} \rightarrow b_n \) and a sequence \( \overrightarrow{\nu} = (\nu_0, \ldots, \nu_n) \) over \( MC_L \), the proof is by induction on \( n \).

In the base case, \( h(\sigma, \overrightarrow{\nu}) = \varphi_{\nu} \land \vartheta_0 \) is in \( MC_L \) because \( \varphi_{\nu} \) is a conjunction of equalities between \( V \cup V_0 \), and \( \vartheta_0 \in MC_L \) by assumption. By induction hypothesis, \( h(\sigma|_{n-1}, \overrightarrow{\nu}|_{n-1}) = \varphi \) for some \( \varphi \in MC_L \). Thus \( h(\sigma, \overrightarrow{\nu}) = \forall U. \varphi(U) \land \Delta_u(U, \overrightarrow{\nu}) \land \vartheta_n \).

As \( B \) is an MC-DDSA, \( \Delta_u(U, \overrightarrow{\nu}) \) is a conjunction of MCs over \( V \cup U \) and constants \( L \), and \( \vartheta_n \in MC_L \) by assumption. By the quantifier elimination property, there exists a quantifier-free MC-formula \( \varphi' \) over variables \( V_0 \cup V \) that is equivalent to \( \exists U. \varphi(U) \land \Delta_u(U, \overrightarrow{\nu}) \land \vartheta_n \), and mentions only constants in \( L \), so \( \varphi' \in MC_L \).

For \( \mathcal{C} \) the set of constraints in \( \chi \), we now show that \( \mathcal{H}_j \subseteq MC_L \) for all \( j \geq 0 \), by induction on \( j \). In the base case, \( (j = 0) \), the claim follows from \((\ast)\), as all constraints in \( \Phi_0 \), i.e., in \( \chi \), are in \( MC_L \). For \( j > 0 \), consider first a formula \( \varphi \in \Phi_j \) for some \( b \in B \). Then \( \varphi \) is of the form \( \varphi = \bigvee_{H \in H_j} \varphi(H, U) \land \vartheta_n \) for some \( H \subseteq H_{j-1} \). By the induction hypothesis, \( H \subseteq MC_L \), so by the quantifier elimination property of MC formulas, \( \varphi \) is equivalent to an MC-formula over \( V \) and \( L \) in \( MC_L \). As \( \mathcal{H}_j \) is built over basis \( \Phi_j \), the claim follows from \((\ast)\). \( \square \)

Notably, the above quantifier elimination property fails for MCs over integer variables; indeed, CTL model checking is undecidable in this case [12, Thm. 4.1].

**Integer periodicity constraint** systems confine the constraint language to variable-to-constant comparisons and restricted forms of variable-to-variable comparisons, and are for instance used in calendar formalisms [19, 23]. More precisely, **integer periodicity constraint** (IPC) atoms have the form \( x = y, x \odot d \) for \( \odot \in \{=, \neq, <, >\} \), \( x \equiv_k y + d \), or \( x \equiv_k d \), for variables \( x, y \) with domain \( Z \) and \( k, d \in N \). A boolean formula whose atoms are IPCs is an **IPC formula**, a DDSA whose guards are conjunctions of IPCs an **IPC-DDSA**, and a CTL_\text{f}^+ formula whose constraint atoms are IPCs an **IPC property**. For instance, \( B_{\text{ipc}} \) in Ex. 2 is an IPC-DDSA.

Using Cor. 1 and a known quantifier elimination property for IPCs [19, Thm. 2], one can show that the verification problem is also solvable for IPC-DDSAs, in a proof that resembles the one of Thm. 3 (see App. A).

**Theorem 4.** The verification problem is solvable for all combinations of an IPC-DDSA \( B \) and an IPC-property \( \chi \).

**Bounded lookback systems** [28] restrict the control flow of the DDSA rather than the constraint language, and is a generalization of the earlier criterion of **feedback-freedom** [14]. Intuitively, the property demands that the behavior of a DDSA at any point in time depends only on boundedly many events from the past. We refer to [28, Def. 5.9] for the formal definition. Systems that enjoy bounded lookback allow for decidable linear-time verification [28, Thm. 5.10]. However, we next show that this is not the case for branching time.
Example 5. We reduce control state reachability of two-counter machines (2CM) to decidability of CTL\textsuperscript{*} formulas for feedback-free (and hence bounded lookback) systems, inspired by \cite{42} Thm. 4.1. 2CMs have a finite control structure and two counters \(x_1, x_2\) that can be incremented, decremented, and tested for 0. It is undecidable whether a 2CM will ever reach a designated control state \(f\) \cite{43}.

For a 2CM \(M\), we build a feedback-free DDSA \(B = \langle B, b_I, A, T, B_F, V, \alpha_1, \text{guard} \rangle\) and a CTL\textsuperscript{*} property \(\chi\) such that \(B\) satisfies \(\chi\) iff \(f\) is reachable in \(M\). The set \(B\) consists of the control states of \(M\), together with an error state \(e\) and auxiliary states \(b_t\) for transitions \(t\) of \(M\), such that \(B_F = \{f, e\}\). The set \(V\) consists of \(x_1, x_2\) and auxiliary variables \(p_1, p_2, m_1, m_2\). Zero-test transitions of \(M\) are directly modeled in \(B\), whereas a step \(q \rightarrow q'\) that increments \(x_i\) by one is modeled as:

\[
\begin{align*}
0 & \xrightarrow{x_i' = x_i + 0 \land p_i^w = x_i} 0 \\
0 & \xrightarrow{x_i' \neq p_i + 1} 0
\end{align*}
\]

The step \(q \rightarrow b_t\) writes \(x_i\), storing its previous value in \(p_i\), but if the write was not an increment by exactly 1, a step to state \(e\) is enabled. Decrements are modeled similarly. For \(C = \emptyset\) and a symbolic run \(\sigma\) of \(B\), the only possible non-equality edge in \(G_{\sigma,C}\) is a final step to \(e\). Thus, there is no non-equality path between different instants of the same variable, so \(B\) is feedback-free. As increments are not exact, \(B\) overapproximates \(M\). However, \(\chi = E G(\neg EX e)\) asserts existence of a path that never allows for a step to \(e\) (i.e., it properly simulates \(M\)) but reaches the final state \(f\). Thus, \(B\) satisfies \(\chi\) iff \(f\) is reachable in \(M\).

6 Implementation

We implemented our approach in the prototype ada (arithmetic DDS analyzer) in Python; source code, benchmarks, and a web interface are available (https://ctlstar.adatool.dev). The tool takes a CTL\textsuperscript{*} property \(\chi\) together with either a DDSA in JSON format, or a (bounded) Petri net with data (DPN) in PNML format \cite{5} as input, in the latter case the system is transformed into a DDSA. The tool then applies the algorithm in Fig. 1. If successful, it outputs the configuration map returned by \(\text{checkState}(\chi)\), and it can visualize the product constructions. To perform SMT checks and quantifier elimination, ada interfaces CVC5 \cite{24} and Z3 \cite{18}. Besides numeric variables, ada also supports variables of type boolean and string. In addition to the operations in Def. 4, ada allows next operators \langle a \rangle via an action \(a\), which are useful for verification.

We tested ada on a set of business process models presented as Data Petri nets (DPNs) in the literature. As these nets are bounded, they can be transformed into DDSAs. The results are reported in the table below. We indicate whether the system belongs to a decidable class, the verified property and whether it is satisfied by the initial configuration, the verification time, the number of SMT checks, and the sizes of both the DDSA \(B\), and the sum of all product constructions, as numbers of nodes/transitions. We used CVC5 as SMT solver; times are without visualization, which tends to be time-consuming for large graphs. All tests were run on an Intel Core i7 with 4\times 2.60GHz and 19GB RAM.
We briefly comment on the benchmarks: For all examples we checked the property no deadlock that abbreviates $\text{AGF} \chi_f$, where $\chi_f$ is a disjunction of all final states. This is one of the two requirements of the crucial soundness property (cf. Ex. 1). Weak soundness that relaxes that allows dead transitions, but all firable transitions must lead to final states. We write weak sound($a$) for the property $\text{EF}(\langle a \rangle \top \rightarrow \text{AG} \langle a \rangle \top \rightarrow \text{F} \chi_f)$, stating the requirements for action $a$.

(a)-(c) are versions of the road fine process from Ex. 1. The DPNs for (a) [40, Fig. 12.7] and (b) [37, Fig. 13] were mined automatically from logs, while (c) is the normative version [41, Fig. 7] shown in Ex. 1. While (a) and (c) are unsound (no deadlock is violated), this issue was fixed in version (b). We can also check whether specific states are deadlock-free, as by $\psi_{a1} = \text{AG} (p_7 \rightarrow \text{EF} \text{end})$, which actually holds in (a)-(c) as $p_7$ is not the problematic state.

Other considered properties are $\psi_{a2} = \text{AG} (\text{end} \rightarrow \text{total} \leq \text{amount})$, $\psi_{c1} = \text{EF} (dS \geq 2160)$, $\psi_{c2} = \text{EF} (dP \geq 1440)$, and $\psi_{c3} = \text{EF} (dJ \geq 1440)$ check whether the time constraints can be violated.

(d) models a billing process in a hospital [40, Fig. 15.3]. The tool verifies that it is deadlock-free. Moreover, $\psi_{d1} = \text{EF} (p16 \land \neg \text{isClosed})$ checks whether there exists a run where in the final state $p16$ the isClosed flag is not set.

(e) is a normative model for a sepsis triage process in a hospital [40, Fig. 13.3], and (f) is a version of the same process that was mined purely automat-
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...from logs [40, Fig. 13.6]. Both versions are deadlock-free. According to [40, Sec. 13], it is assumed that triage happened before antibiotics are administered, i.e., $\psi_e = AG(sink \rightarrow timeTriage < timeAntibiotics)$, which is actually not satisfied by (e). However, the desired time limit $\psi_{e2} = AG(sink \rightarrow timeTriage + 60 \geq timeAntibiotics)$ holds. We can check that variable lacticAcid is not written until a certain activity happens, i.e., $\psi_f = AG(\neg lacticAcid U (diagnosticLacticAcid)\top)$ holds.

(g)–(i) reflect activities in patient logistics of a hospital, based on logs of real-life processes [40, Fig. 14.3]. While the no deadlock property is satisfied by all initial configurations, the output of ada reveals that in case of (h) this need not hold for other initial assignments. The tool also confirms that if the variable org1 has value 207 in state $p_2$ then this value will be maintained, $\psi_{i1} = AG(p_2 \land org_1 = 207 \rightarrow AG\! org_1 = 207)$. We also verify that in this process either the transfer or history activity happens, but not both, by $\psi_{i2} = A(\exists F\! (transfer)\top \land \exists F\! (history)\top)$ and $\psi_{i3} = \neg E\! (F\! (transfer)\top \land F\! (history)\top)$.

(j) is a credit approval process [17, Fig. 3]. It can be verified that a loan is only granted if the application passed the customer verification and the decision stages ($\psi_{j1} = AG(openLoan)\top \rightarrow ver \land dec$); though even if the verification and the decision variables are set, it is not guaranteed that a loan is granted ($\psi_{j2} = A(F\! (ver \land dec) \rightarrow F\! (openLoan)\top)$), but it is possible ($\psi_{j3} = A(F\! (ver \land dec) \rightarrow E\! (openLoan)\top)$)

(k) is a package handling routine [27, Fig. 5]. The properties $\psi_{k1} = E\! F\! (fetch)\top$ and $\psi_{k2} = E\! F\! (\sigma_6)\top$ are not satisfied, so the process has dead transitions.

(l) models an auction process [28, Ex. 1.1], for which ada reveals a deadlock. We also check the properties $\psi_{l1} = E\! F\! (sold \land d > 0 \land o \leq t)$ and $\psi_{l2} = E\! F\! (b = 1 \land o > t \land F\! (sold \land b > 1))$ considered in [28, Ex. 1.1].

7 Conclusion

This paper presents a CTL* verification technique for DDSAs that is a decision procedure for monotonicity and integer periodicity constraint systems. To the best of our knowledge, this is the first proof of decidability of CTL* for these classes. In contrast, the cases of feedback-free and bounded lookback systems are shown undecidable. We implemented our approach in the tool ada and showed its usefulness on a range of business processes from the literature.

We see various opportunities to extend this work. A richer verification language could support past time operators [19] and the possibility to compare variables multiple steps apart [21]. Further decidable fragments could be sought using covers [33], or aiming for compatibility with locally finite theories [32]. Moreover, a restricted version of the bounded lookback property could guarantee decidability of CTL*, similarly to the way feedback freedom was strengthened in [35]. We conjecture that many of the DPNs used in the experiments could be in such a class. The implementation could be improved to avoid the computation of many similar formulas, thus gaining efficiency. Finally, the complexity class that our approach implies for CTL* in the decidable classes is yet to be clarified.
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A Proofs

Lemma 1 For a symbolic run $\sigma : b_0 \xrightarrow{\alpha_0} b_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} b_n$ and $\vartheta = (\vartheta_0, \ldots, \vartheta_n)$, $h(\sigma, \vartheta)$ is satisfied by assignment $\alpha$ with domain $V \cup V_0$ iff $\sigma$ abstracts a run $\rho : (b_0, \alpha_0) \xrightarrow{\alpha_0} \ldots \xrightarrow{\alpha_n} (b_n, \alpha_n)$ such that (i) $\alpha_0(v) = \alpha(\nu(v))$, and (ii) $\alpha_n(v) = \alpha(v)$ for all $v \in V$, and (iii) $\alpha_i \models \vartheta_i$ for all $i$, $0 \leq i \leq n$.

Proof. ($\Leftarrow$) By induction on $n$. If $n = 0$, the assumptions imply $\alpha(v) = \alpha(\nu(v))$ for all $v \in V$, so $\alpha \models \varphi_\nu$. As $\alpha_0$ satisfies $\vartheta_0$, $\alpha$ also satisfies $\varphi_\nu \land \vartheta_0 = h(\sigma, \vartheta_0)$, so the claim holds. For the induction step, let $\sigma$ be a symbolic run $\sigma : b_0 \xrightarrow{\alpha_0} b_1 \xrightarrow{\alpha_1} \ldots b_{n+1}$ that abstracts a run $\rho : (b_0, \alpha_0) \xrightarrow{\alpha_0} \ldots \xrightarrow{\alpha_n} (b_n, \alpha_n)$, and let $\vartheta = (\vartheta_0, \ldots, \vartheta_n, \vartheta_{n+1})$. Then $\sigma|_n$ also abstracts $\rho|_n$, so by the induction hypothesis the assignment $\alpha'$ with domain $V \cup V_0$ given by $\alpha'(\nu(v)) = \alpha_0(v)$ and $\alpha'(v) = \alpha_n(v)$ for all $v \in V$ satisfies $h(\sigma|_n, \vartheta|_n)$. By definition of a step, the guard assignment $\beta$ given by $\beta(\nu^v) = \alpha_n(v)$ and $\beta(\nu^w) = \alpha_{n+1}(v)$ for all $v \in V$
satisfies $\text{guard}(a)$. For $\varphi := h(\sigma|_n, \overline{\nu}|_n)$ we thus have

$$h(\sigma, \overline{\nu}) = \text{update}(\varphi, a) \land \vartheta_{n+1}$$

$$= \exists \overline{U}, \varphi(\overline{U}) \land \Delta_a(\overline{U}, \overline{\nu}) \land \vartheta_{n+1}$$

$$= \exists \overline{U}, \varphi(\overline{U}) \land (\text{guard}(a) \land \bigwedge_{v \in V \cap \text{write}(a)} v^w = v^r)(\overline{U}, \overline{\nu}) \land \vartheta_{n+1}$$

As $\alpha' \models \varphi$, by the construction of $\beta$ above, it holds that $\alpha_{n+1}$ satisfies the first conjunct of this formula, using values $\alpha_n(\overline{\nu})$ as witnesses for the existentially quantified variables $\overline{U}$. Since moreover $\alpha_{n+1} \models \vartheta_{n+1}$ by assumption, it follows that $\alpha_{n+1}$ satisfies $h(\sigma, \overline{\nu})$.

($\implies$) By induction on $n$. For $n = 0$, suppose that $\alpha$ satisfies $h(\sigma, \overline{\nu}) = \varphi \land \vartheta_0$. By definition of $\varphi_\nu$, this implies $\alpha(v) = \alpha(\nu(v))$ for all $v \in V$. The empty run $(b_0, 0_0)$ thus satisfies the claim, with $\alpha_0(v) = \alpha(v)$ for all $v \in V$. For the inductive step, let $\sigma : b_0 \xrightarrow{a} b_n \xrightarrow{a} b_{n+1}$ and $\overline{\nu} = \langle \vartheta_0, \ldots, \vartheta_{n+1} \rangle$ satisfy $\alpha \models h(\sigma, \overline{\nu})$. Since

$$h(\sigma, \overline{\nu}) = \text{update}(h(\sigma|_n, \overline{\nu}|_n), a) \land \vartheta_{n+1}$$

$$= \exists \overline{U}, h(\sigma|_n, \overline{\nu}|_n)(\overline{U}) \land \Delta_a(\overline{U}, \overline{\nu}) \land \vartheta_{n+1}$$

it must hold that $\alpha \models \vartheta_{n+1}$ and there must be an assignment $\gamma$ with domain $U \cup V_0 \cup V$ such that $\gamma(v) = \alpha(v)$ for all $v \in V \cup V_0$, and $\gamma$ satisfies both $h(\sigma|_n, \overline{\nu}|_n)(\overline{U})$ and $\Delta_a(\overline{U}, \overline{\nu})$. We can write $\overline{V} = \langle v_1, \ldots, v_k \rangle$ and $\overline{U} = \langle u_1, \ldots, u_k \rangle$ for some $k$. Let $\alpha'$ be the assignment with domain $V \cup V_0$ such that $\alpha'(v_i) = \gamma(u_i)$ for all $1 \leq i \leq k$, and $\alpha'(v) = \gamma(v)$ for all $v \in V_0$. Then $\alpha'$ satisfies $h(\sigma|_n, \overline{\nu}|_n)$. Therefore, by the induction hypothesis $\sigma|_n$ abstracts a run $\rho : (b_0, \alpha_0) \xrightarrow{a_1} (b_1, \alpha_1) \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} (b_n, \alpha_n)$ such that $\alpha_i \models \vartheta_i$ for all $0 \leq i \leq n$. Let $\beta$ be the guard assignment such that $\beta(v') = \alpha'(v)$ and $\beta(v^w) = \alpha(v)$ for all $v \in V$. By definition of $\alpha'$, since $\gamma$ satisfies $\Delta_a(\overline{U}, \overline{\nu})$, $\beta$ satisfies $\Delta_a(\overline{\nu}, \overline{\nu}^w)$ and hence $\beta \models \text{guard}(a)$. Thus $\rho$ can be extended with a step $(b_n, \alpha_n) \xrightarrow{a} (b_{n+1}, \alpha_{n+1})$ such that $\alpha_{n+1}(v) = \alpha(v)$ for all $v \in V$. Moreover, as $\alpha$ satisfies $\vartheta_{n+1}$, $\alpha_i \models \vartheta_i$ for all $0 \leq i \leq n + 1$. This proves the claim.

Given a path $\pi$ in $\mathcal{N}^\psi_{\mathcal{B},b}$ of the form

$$\pi : (b, q_0, \varphi_\nu) \xrightarrow{a_0,w_0} (b_0, q_1, \varphi_1) \xrightarrow{a_1,w_1} (b_1, q_2, \varphi_2) \xrightarrow{a} (b_n, q_{n+1}, \varphi_{n+1})$$  \hspace{0.5cm} (1)

where the last node is final, we write $\sigma(\pi)$ for the symbolic run $\sigma : b = b_0 \xrightarrow{a_1} b_1 \xrightarrow{a} b_n$ (ignoring the initial dummy transition in $\pi$), and $w(\pi) = w_0, \ldots, w_n$.

**Lemma 6.** Let $\psi \in \text{LTL}_f^\mathcal{B}$ be over $\Phi$.

1. If a word $w$ is accepted by $\mathcal{N}_\psi$ and $\sigma$ is a symbolic run such that $h(\sigma, w \otimes \sigma)$ is satisfiable, there is a path $\pi$ of the form $[1]$ in $\mathcal{N}^\psi_{\mathcal{B},b}$ such that $\sigma = \sigma(\pi)$, $w = w(\pi)$, and $\varphi_{n+1} \equiv h(\sigma, w \otimes \sigma)$.

2. If $\pi$ is a path of the form $[1]$ in $\mathcal{N}^\psi_{\mathcal{B},b}$ then $w(\pi)$ is accepted by $\mathcal{N}_\psi$, $\varphi_{n+1}$ is satisfiable, and $\varphi_{n+1} \equiv h(\sigma(\pi), w(\pi) \otimes \sigma(\pi))$. 
Proof. (1) By induction on the length $n$ of $\sigma$. If $n = 0$ then $\sigma$ is empty and $w = \varsigma$ for some $\varsigma \in \Sigma$. By assumption, $h(\sigma, w \otimes \sigma) = \varphi_\nu \land \varphi_0(b)$ is satisfiable. Thus, $\text{update}(\varphi_\nu, a_0) \land \varphi_0(b) = \varphi_\nu \land \varphi_0(b)$ is satisfiable (using $\text{guard}(a_0) = \top$), so by Def. 10 there is a step $(b, q_0, \varphi_\nu) \xrightarrow{a_0} (b, q_1, \varphi_1)$ such that $\varphi_1 \equiv \text{update}(\varphi_\nu, a_0) \land \varphi_0(b)$.

In the inductive step, $\sigma$ has the form $b_0 \overset{\varsigma_1}{\rightarrow} b_n \overset{\varsigma_2}{\rightarrow} b_{n+1}$, and $w = \varsigma_0 \cdots \varsigma_n$ is accepted by $N_\psi$, such that $h(\sigma, w \otimes \sigma)$ is satisfiable. Let $\sigma' = \sigma|_n$ and $w' = w|_n$. By the induction hypothesis, $N_\psi$ has a node $p_{n+1} = (b_n, q_{n+1}, \varphi_{n+1})$ and a path $\pi: p_0 \rightarrow^* p_{n+1}$ such that $\varphi_{n+1} \equiv h(\sigma', w' \otimes \sigma')$. Therefore,

$$\text{update}(\varphi_{n+1}, a) \land \varphi_{n+1}(b_{n+1}) \equiv \text{update}(h(\sigma', w' \otimes \sigma'), a) \land \varphi_{n+1}(b_{n+1}) = h(\sigma, w \otimes \sigma)$$

is satisfiable. Therefore, $N_\psi$ must have a node $p' = (b_{n+1}, q_{n+2}, \varphi_{n+2})$ such that $\varphi_{n+2} \equiv \text{update}(\varphi_{n+1}, a) \land \varphi_{n+1}(b_{n+1})$ and an edge $p_n \overset{a, \varphi_{n+1}}{\rightarrow} p'$ can be appended to $\pi$.

(2) By induction on $n$. If $n = 0$ then $\pi$ consists of the single step $(b, q_0, C_{a_0}) \xrightarrow{a_0} (b, q_1, \varphi_1)$ and $\sigma$ consists only of state $b$. By Def. 10 this step exists because $\text{update}(\varphi_\nu, a_0) \land \varphi_0(b) = \varphi_\nu \land \varphi_0(b)$ is satisfiable, for some $q_0 \overset{a_0}{\rightarrow} q_1$, using the fact that $\text{guard}(a_0) = \top$. The formula $\varphi_1$ must satisfy $\varphi_1 \equiv \varphi_\nu \land \varphi_0(b)$. For $w(\pi) = \varsigma_0$ we indeed have $h(\sigma, w(\pi) \otimes \sigma(\pi)) = \varphi_\nu \land \varphi_0(b)$, so the claim holds.

In the inductive step, consider a path $\pi: p_0 \rightarrow^* p_{n+1} \xrightarrow{a_0} p_{n+2}$ for $p_0$ the initial node of $N_\psi$ and $p_i = (b_{i-1}, q_i, \varphi_i)$ for all $1 \leq i \leq n + 2$. Let $\sigma = \sigma(\pi)$ be the symbolic run $b_0 \overset{\varsigma_1}{\rightarrow} b_n \overset{\varsigma_2}{\rightarrow} b_{n+1}$, $\sigma' = \sigma|_n$, and $w = w(\pi)$. By the induction hypothesis, there is a run $q_0 \overset{\varsigma_1}{\rightarrow} q_1 \overset{\varsigma_2}{\rightarrow} \cdots \overset{\varsigma_n}{\rightarrow} q_{n+1}$ in $N_\psi$ such that for $w' = w(\sigma|_n) = \varsigma_0 \cdots \varsigma_n$, the history constraint $h(\sigma', w' \otimes \sigma')$ is satisfiable and equivalent to $\varphi_{n+1}$ (\ast). Since there is an edge $(b_n, q_{n+1}, \varphi_{n+2}) \xrightarrow{a_0} (b_{n+1}, q_{n+2}, \varphi_{n+2})$, by Def. 10 there must be a transition $q_{n+1} \overset{\varsigma_i}{\rightarrow} q_{n+2}$ in $N_\psi$, such that $\varphi_{n+2} \equiv \text{update}(\varphi_{n+1}, a) \land \varsigma_i(b_{n+1})$ is satisfiable. Using (\ast) and abbreviating $\vartheta = \varsigma_i(b_{n+1})$,

$$\text{update}(\varphi_{n+1}, a) \land \vartheta \equiv \text{update}(h(\sigma', w' \otimes \sigma'), a) \land \vartheta = h(\sigma, w \otimes \sigma)$$

holds and since $\varphi_{n+2}$ is satisfiable the claim holds. \qed

For instance, the path to the left final node in $N_\psi$ in Ex. 3 corresponds to the word $w = \langle b, \{ K \} \rangle$ accepted by $N_\psi$ and $\sigma: b_1 \overset{a_1}{\rightarrow} b_2 \overset{a_2}{\rightarrow} b_3$, and the formula is equivalent to $h(\sigma, \vartheta)$ for $\vartheta = (\top, K(b_2), \top)$. Now, the product construction serves to check whether there exists a run that satisfies an LTL formula:

**Theorem** 1 Let $\psi \in \text{LTL}_B$ and $b \in B$. There is a final run $\rho: (b, a_0) \rightarrow^* (b_F, a_F)$ of $B$ such that $\rho \models K \psi$, if $N_\psi$ has a final state $(b_F, q_F, \varphi)$ for some $q_F$ and $\varphi$ such that $\varphi$ is satisfied by assignment $\gamma$ with $\gamma(V_0) = a_0(V)$ and $\gamma(V) = a_F(V)$.

**Proof.** ($\Rightarrow$) Suppose $\rho \models K \psi$, and let $\sigma$ be the abstraction of $\rho$. By Lem. 2 $N_\psi$ accepts a word $w = w_0 \cdots w_n$ that is consistent with $\rho$, i.e., $\alpha_i \models w_i(b_i)$ for all $i$, $0 \leq i \leq n$. Thus assignment $\gamma$ satisfies $h(\sigma, w \otimes \alpha) \otimes b$ by Lem. 1. By Lem. 6 there is
Lemma 3 Let \( \psi \) be a path formula with \( qd(\psi) = k \). Suppose that for all configurations \((b, \alpha)\) and path formulas \( \psi' \) with \( qd(\psi') < k \), there is a \( \rho' \in \text{FRuns}(b, \alpha) \) with \( \rho', 0 \models \psi' \) iff \( \alpha \models \text{checkPath}(\psi')(b) \). Then \( \rho, 0 \models \psi \) iff \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) \).

Proof. We suppose that \( \rho_0 = (b_0, \alpha_0) \) and apply induction on \( \psi \). First, if \( \psi = \top \) then \( \rho, 0 \models_{\text{K}} \top \) and \( \text{toLTL}_{\text{K}}(\psi) = \top \), so the claim holds. Second, if \( \psi = \neg \psi \) then \( \rho, 0 \models \psi \) iff \( \alpha \models \text{checkPath}(\psi')(b_0) \), and moreover \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) = K_b \) iff \( \alpha_0 \models K_b(b_0) \), which holds if \( b = b_0 \). Third if \( \psi = c \), then \( \rho, 0 \models \psi \) iff \( \alpha \models c \), and \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) = K_c \) iff \( \alpha_0 \models K_c(b_0) \). For the induction step, we perform again a case distinction on \( \psi \). If \( \psi = \mathbf{E} \psi' \) then \( \rho, 0 \models \psi \) iff there is some \( \rho' \in \text{FRuns}(b_0, \alpha_0) \) with \( \rho', 0 \models \psi' \). As \( qd(\psi') < qd(\psi) \), this holds by assumption iff \( \alpha \models \text{checkPath}(\psi')(b_0) \). Moreover, \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) = \text{checkPath}(\psi')(b_0) \) by definition of \( \models_{\text{K}} \), which proves the claim.

Theorem 2 For every configuration \((b, \alpha)\) of \( B \) and state property \( \chi \), \((b, \alpha) \models \chi \) iff \( \alpha \models \text{checkState}(\chi)(b) \).

Proof. We first show property \((*): \): there is a run \( \rho \in \text{FRuns}(b, \alpha) \) with \( \rho, 0 \models \psi \) iff \( \alpha \models \text{checkPath}(\psi')(b) \). The proof is by induction on \( qd(\psi) \). If \( \psi \) contains no path quantifiers, Lem. 3 implies that \( \rho, 0 \models \psi \) iff \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) \) for all runs \( \rho \), so \((*)\) follows from Lem. 4. In the induction step, we conclude from Lem. 3 (a), using the induction hypothesis as assumption, that \( \rho, 0 \models \psi \) iff \( \rho, 0 \models_{\text{K}} \text{toLTL}_{\text{K}}(\psi) \) for all runs \( \rho \). Then \((*)\) follows from Lem. 4.

The claim of the lemma can be shown by induction on \( \chi \): There are three base cases: if \( \chi = \top \) or \( \chi = c \) then the claim is trivial as \( \text{checkState}(\chi)(b) = \chi \); and if \( \chi = b' \) for some \( b' \in B \), \((b, \alpha) \models \chi \) iff \( b = b' \), and the same condition applies to \( \alpha \models \text{checkState}(\chi)(b) \) by definition of \( K_b \). The inductive step also distinguishes three cases: First, if \( \chi = \neg \chi' \) then by the induction hypothesis \((b, \alpha) \models \chi' \) iff \( \alpha \models \text{checkState}(\chi')(b) \). By definition, \((b, \alpha) \models \chi \) iff \( \alpha \neq \text{checkState}(\chi')(b) \), which holds by definition of negation of configuration maps iff \( \alpha \models \text{checkState}(\chi)(b) \). Second, if \( \chi = \chi_1 \land \chi_2 \) then by the induction hypothesis \((b, \alpha) \models \chi_i \) iff \( \alpha \models \text{checkState}(\chi_i)(b) \) for both \( i \in \{1, 2\} \). So by definition, \((b, \alpha) \models \chi \) iff both \((b, \alpha) \models \chi_i \), which holds iff \( \alpha \models \text{checkState}(\chi_i)(b) \) for both \( i \in \{1, 2\} \). By definition of conjunction on configuration maps this is equivalent to \( \alpha \models \text{checkState}(\chi)(b) \). Finally, if \( \chi = \mathbf{E} \psi \) then \((b, \alpha) \models \chi \) iff there is a run \( \rho \in \text{FRuns}(b, \alpha) \) such that \( \rho, 0 \models \psi \). We use \((*)\) to conclude that this is the case iff \( \alpha \models \text{checkPath}(\psi)(b) = \text{checkState}(\chi)(b) \).
Lemma 5 Let ψ have quantifier depth k, ψ′ = toLTLK(ψ), and N′B,b be a constraint graph constructed in checkPath(ψ) for some b ∈ B. Then,
(1) for all nodes (b′, q, ϕ) in N′B,b there is some ϕ′ ∈ Hk such that ϕ ⊨ ϕ′,
(2) checkPath(ψ)(b) is equivalent to a formula in Φk+1.

Proof. We prove the statements by induction on k. In the base case, ψ contains no path quantifiers, so by definition of toLTLK, all atoms K′ occurring in ψ′ satisfy K′(b′) ∈ C ∪ {⊤, ⊥} = Φ0 for all b′ ∈ B, so ψ′ is an LTL formula over basis Φ0. Let π be a path to a node (b′, q, ϕ) in N′B,b, ϕ := σ(π) be the associated symbolic run σ: b0 ⊓1, ..., ⊓n b, and w := w(π) the associated word in Σ. By Lem. 3(1), ϕ ≡ h(σ, w ⊩ σ). The word w = w0, . . . , wn satisfies w0 ∈ 2Kσ(Φ0), so for w ⊩ σ = θ0, . . . , θn we have θi = ∩ Ti for some Ti ⊆ Φ0. Therefore h(σ, w ⊩ σ) ∈ H0, so that (1) holds. Furthermore, the symbolic configuration map K returned by checkPath satisfies K(b) = ϕ(ϕ, ϕ, ϕ), where every ϕ is equivalent to some formula in Φ0. Hence K(b) is equivalent to a formula in Φ1 by definition of Φ, so that (2) holds.

In the step case, we consider a formula ψ of quantifier depth k + 1, and the induction hypothesis is that (1) and (2) hold for a formula of depth k. The call to toLTLK(ψ) replaces all occurrences of subformulas Φη by checkPath(η), where η has quantifier depth at most k. By part (2) of the induction hypothesis, checkPath(η)(b) is equivalent to a formula in Φk+1 for all b ∈ B. If we abbreviate Θ := ∪k+1 Φk+1, we can thus assume that ψ′ = toLTLK(ψ) is a formula over Θ. Let π be a path to a node (b′, q, ϕ) in N′B,b, ϕ := σ(π) be the associated symbolic run σ: b0 ⊓1, ..., ⊓n b, and w := w(π) the associated word. By Lem. 3(1), ϕ ≡ h(σ, w ⊩ σ). The word w = w0, . . . , wn satisfies w0 ∈ 2Kσ(Θ), so for w ⊩ σ = θ0, . . . , θn we have θi = ∩ Ti for some Ti ⊆ Θ. Therefore h(σ, w ⊩ σ) ∈ Hk+1, so that (1) holds. Furthermore, the symbolic configuration map K returned by checkPath satisfies K(b) = ϕ(ϕ, ϕ, ϕ), where every ϕ is equivalent to some formula in Hk+1. Hence K(b) is equivalent to a formula in Φk+2, so that (2) holds.

In IPCs of the form x ⊩ d for ⊩ ∈ {=, =, <, >}, x ≡k y + d, and x ≡k d, we call d a constant and k a modulus.

Theorem 4 For any IPC-DDSA B and IPC-property χ the verification problem is decidable.

Proof. Let χ be an IPC-property, L the finite set of constants d in χ, o0, and guards of B, and K the least common multiple of all moduli k1, ..., km that occur in χ and guards of B. Let moreover ΦIPC be the set of quantifier-free boolean formulas whose atoms are IPCs over variables V ∪ V0, moduli k1, ..., km, K, and constants L, so ΦIPC is finite up to equivalence.

We show the following property (⋆): all history constraints h(σ, n) over ΦIPC are equivalent to a formula in ΦIPC. For a symbolic run σ: b0 ⊓1, b1 ⊓2, ..., ⊓n b, and a sequence n = (n0, . . . , n) over ΦIPC, the proof is by induction on n. In the base case n = 0, h(σ, ) = ϕv ∧ 0 is in ΦIPC because ϕv is a conjunction
of equalities between variables, and \( \vartheta_0 \in \Phi_{IPC} \) by assumption. In the induction step, \( h(\sigma, \vartheta) = \text{update}(h(\sigma|_{n-1}, \vartheta|_{n-1}), a_n) \wedge \vartheta_n \). By induction hypothesis, \( h(\sigma|_{n-1}, \vartheta|_{n-1}) \) is equivalent to a formula \( \varphi \) in \( \Phi_{IPC} \). Thus \( h(\sigma, \vartheta) \equiv \varphi \) for the formula \( \varphi = \exists U. \varphi(U) \land \Delta_\vartheta(U, V) \land \vartheta_n \). By assumption, \( \Delta_\vartheta(U, V) \) is a conjunction of IPCs over \( V \cup U \), moduli \( k_1, \ldots, k_m \), and \( L \), and \( \vartheta_n \in \Phi_{IPC} \) as well.

According to the quantifier elimination property proven in \cite{TCS19} Thm. 2, there exists a quantifier-free IPC-formula \( \varphi' \) over variables \( V_0 \cup V \), modulus \( K \), and \( L \) that is equivalent to \( \exists U. \varphi(U) \land \Delta_\vartheta(U, V) \land \vartheta_n \), so \( \varphi' \in \Phi_{IPC} \).

We now show that \( \mathcal{H}_j(b) \subseteq \Phi_{IPC} \) for all \( j \geq 0 \), by induction on \( j \). In the base case \( (j = 0) \) the claim follows from (\( \ast \)), since all constraints in \( \chi \) are in \( \Phi_{IPC} \). For \( j > 0 \), consider first a formula \( \hat{\varphi} \in \Phi_j \). Then \( \hat{\varphi} \) is of the form \( \bigvee_{\varphi \in H} \exists U. \varphi(U, V) \) for some \( H \subseteq \mathcal{H}_{j-1} \). By the induction hypothesis, \( H \subseteq \Phi_{IPC} \), so by the above quantifier elimination property, \( \hat{\varphi} \) is equivalent to a formula \( \varphi' \in \Phi_{IPC} \). As \( \mathcal{H}_j \) consists of all history constraints over \( \Phi_j \), the claim follows from (\( \ast \)).

\section*{B Examples}

\textbf{Example 6.} We show here the product automata that were omitted in Ex. 4 for lack of space. For the LTL\(_j^\mathcal{B}\) formula \( \psi_2' = \mathcal{F}(K_{x<2}) \) and state \( b_1 \), we get the following automaton:

For \( U = (\hat{x}, \hat{y}) \), and the formulas \( \varphi_1, \varphi_2, \) and \( \varphi_3 \) in final nodes, we compute

\( \exists U. \varphi_1(V, U) \equiv \exists \hat{x} \hat{y}. \hat{x} = x \land \hat{x} > 0 \land \hat{x} \leq 2 \equiv x > 0 \land x < 2 \)

\( \exists U. \varphi_2(V, U) \equiv \exists \hat{x} \hat{y}. \hat{x} = y \land \hat{y} < 0 \land \hat{y} < 2 \equiv \mathcal{F} \)

\( \exists U. \varphi_3(V, U) \equiv \exists \hat{x} \hat{y}. \hat{x} = y \land \hat{y} > 0 \land \hat{y} > x \equiv x < 2 \)

so that \( K_3 = \text{checkPath}(\psi_1) \) sets \( K_3(b_1) = \bigvee_{i=1}^3 \exists U. \varphi_i(V, U) \equiv \mathcal{T} \). For state \( b_3 \), we get the following simple automaton:

For the formula \( \varphi \) in the final state we have \( \exists U. \varphi_3(V, U) = \exists \hat{x} \hat{y}. \hat{x} = x \land \hat{y} = y \land x < 2 \equiv x < 2 \) so that \( K_j(b_3) = x < 2 \).

The next example illustrates our approach on some simple properties to illustrate how the branching time requirements are reflected.
Example 7. Let $B$ be the following simple DDSA:

- Consider $\psi_1 = \operatorname{EX} ((x = 1) \land \operatorname{EX} (x = 2))$. We first evaluate $\operatorname{EX} (x = 2)$ on all states. The NFA for the formula $\psi_0 = X K_{x=2}$ is as follows, for $q_1 = \psi_1$ and $q_2 = K_{x=2}$:

This leads to the product automata shown next:

For $b_1$, line 6 of `checkPath` thus yields $K'(b_1) = (\exists x. x = 2)(\top) = \top$, and $K'(b_2) = (\exists x. x = 2 \land x_0 = 2)(\top) = (x = 2)$. Overall, the evaluation of $\operatorname{EX} (x = 2)$ thus yields $K'$ such that $K' = \{ b_1 \mapsto \top, b_2 \mapsto (x = 2), b_3 \mapsto \bot, b_4 \mapsto \bot \}$. We hence construct the NFA for the formula $\psi'_1 = X(K_{x=1} \land K')$, which looks as follows, for $q_1 = \psi'_1$ and $q_2 = K_{x=1} \land K'$:

We focus now on the evaluation of $\psi_1$ in state $b_1$. The product construction for $B$, $b_1$, and $\psi'_1$ is started as follows:

However, at this point the next product transition would combine $b_1 \xrightarrow{x' \geq 0} b_2$ with $q_2 \{ (K_{x=1}, K') \} \top$: The labels $K_{x=1}$ and $K_1$ evaluated at the destination state $b_2$, yield the constraints $x = 1$ and $x = 2$, but since their conjunction is unsatisfiable, the product construction stops at this point. Hence there are no final states, so that the resulting configuration map $K := \text{checkPath}(\psi_1)$ sets $K(b_1) = \bot$, as expected.

- Consider $\psi_2 = (\operatorname{EX} (x = 1)) \land (\operatorname{EX} (x = 2))$. To recursively process the formula, we first evaluate $\operatorname{EX} (x = 1)$ on all states as above, which yields $K_1 = \{ b_1 \mapsto \top, b_2 \mapsto (x = 1), b_3 \mapsto \bot, b_4 \mapsto \bot \}$, and similarly for $\operatorname{EX} (x = 1)$ we obtain $K_2 = \{ b_1 \mapsto \top, b_2 \mapsto (x = 2), b_3 \mapsto \bot, b_4 \mapsto \bot \}$. When evaluating $\psi_2$, we hence return $K_1 \land K_2 = \{ b_1 \mapsto \top, b_2 \mapsto \bot, b_3 \mapsto \bot, b_4 \mapsto \bot \}$. 


Consider $\psi_3 = E X (E X (x = 1) \land E X (x = 2))$. As above, we first evaluate $K_1$ and $K_2$ as above. We then construct the NFA for the formula $\psi'_3 = X (K_1 \land K_2)$, which looks as follows, for $q_1 = \psi'_3$ and $q_2 = K_1 \land K_2$:

![NFA Diagram]

To evaluate $\psi_3$ in state $b_1$ the product construction for $B$, $b_1$, and $\psi'_3$ is again started as follows:

- $b_0 | x = x_0$ → $b_1 | x = x_0$)

However, at this point the next product transition would combine $b_1 \quad x' \geq 0 \Rightarrow b_2$ with $q_2 \{K_1, K_2\}, \top$: The labels $K_1$ and $K_2$ evaluated at the desination state $b_2$ yield the constraints $x = 1$ and $x = 2$, and since their conjunction is unsatisfiable, the product construction stops at this point, without producing a final state. Hence the resulting configuration map $K$ sets $K(b_1) = \bot$.

### C NFA Construction

For the following construction, we assume that $\psi \in \text{LTL}^B_f$ is in negation normal form. To this end we need to extend the grammar for $\text{LTL}^B_f$ formulas to allow disjunction $\psi_1 \lor \psi_2$ and a weak next operator $Y \psi$. The semantics Def. 9 is extended as $\rho, i \models \psi_1 \lor \psi_2$ iff $\rho, i \models \psi_1$ or $\rho, i \models \psi_2$, and $\rho, i \models Y \psi$ iff $i = n$ or $\rho, i + 1 \models \psi$. Then a formula $\neg X \psi$ can be written as $Y \neg \psi$, so that we can assume $\psi$ to be in negation normal form. We can assume that the only base case is $K \in \mathcal{K}_B(\Phi)$ because for every $K$ also $\neg K$ is in $\mathcal{K}_B(\Phi)$.

We build an NFA $N_\psi = (Q, \Sigma, \varrho, q_0, Q_F)$, where (i) the set $Q$ of states is a set of quoted $\text{LTL}^B_f$ formulas together with $\{\top, \bot\}$; (ii) $\Sigma = 2^{K_B(\Phi)}$ is the alphabet; (iii) $\varrho \subseteq Q \times \Sigma \times Q$ is the transition relation; (iv) $q_0 \in Q$ is the initial state; (v) $Q_F \subseteq Q$ is the set of final states.

Following [15], we define $\varrho$ using an auxiliary function $\delta$ and a new proposition $\lambda$ that marks the last element of the trace. The input of $\delta$ is a (quoted) formula $\psi \in \text{LTL}^B_f$, and its output a set of tuples $(\psi', \zeta)$ where $\psi'$ has the same type as $\psi$ and $\zeta \in 2^{\Sigma_L(\lambda, \neg \lambda)}$. For two sets of such tuples $R_1, R_2$, and $\cap$ either $\land$ or $\lor$, let $R_1 \cap R_2 = \{ (\psi_1 \cap \psi_2', \zeta_1 \cup \zeta_2) \mid (\psi_1', \zeta_1) \in R_1, (\psi_2', \zeta_2) \in R_2 \}$, where we simplify $\psi_1 \cap \psi_2$ if possible. The function $\delta$ is as follows:

1. $\delta(\top') = \{(\top', \emptyset)\}$ and $\delta(\bot') = \{(\bot', \emptyset)\}$
2. $\delta(\neg\lambda') = \{(\neg\lambda', \emptyset)\}$
3. $\delta(\lambda') = \{(\lambda', K)\}, (\bot', \emptyset)\}$ if $K \in \mathcal{K}_B(\Phi)$
4. $\delta(\psi_1 \lor \psi_2') = \delta(\psi_1') \lor \delta(\psi_2')$
5. $\delta(\psi_1 \land \psi_2') = \delta(\psi_1') \land \delta(\psi_2')$
6. $\delta(\neg\psi') = \{(\neg\psi', \neg\lambda)\}, (\bot', \emptyset)\}$
7. $\delta(\lambda') = \{(\lambda', K)\}, (\bot', \emptyset)\}$
8. $\delta(\neg\psi') = \{(\neg\psi', K)\}$
9. $\delta(\neg\lambda') = \{(\neg\lambda', \emptyset)\}$
10. $\delta(\lambda') = \{(\lambda', K)\}$
11. $\delta(\neg\psi') = \{(\neg\psi', \neg\lambda)\}$
12. $\delta(\neg\lambda') = \{(\neg\lambda', \emptyset)\}$
13. $\delta(\lambda') = \{(\lambda', \emptyset)\}$
14. $\delta(\neg\lambda') = \{(\neg\lambda', \emptyset)\}$

where $\delta_3$ abbreviates $\{ (\neg\psi', \neg\lambda), (\bot', \emptyset) \}$. While the symbol $\lambda$ is needed for the construction, we can omit it from the NFA, and define $N_\psi$ as follows:

**Definition 12.** For a formula $\psi \in \text{LTL}^B_f$, let the NFA $N_\psi = (Q, \Sigma, \varrho, q_0, \{q_f, q_c\})$ be given by $q_0 = \psi^\top$, $q_f = \top$ and $q_c$ is an additional final state, and $Q$, $\varrho$ are the...
with step easy to check for every base case of the definition of
if word \( \varsigma \) such that

By induction on the structure of \( \Sigma \) let \( \varsigma \in \Sigma \) is consistent with step \( i \) of a run
\[
\rho: (b_0, \alpha_0) \overset{a_1}{\rightarrow} (b_1, \alpha_1) \overset{a_2}{\rightarrow} \cdots \overset{a_n}{\rightarrow} (b_n, \alpha_n)
\]
if \( \alpha_i \models \varsigma(b_i) \). Moreover, \( \varsigma \in \Sigma \) is \( \lambda \)-consistent with step \( i \) of \( \rho \) if \( \varsigma \) is consistent with step \( i \) of \( \rho \), if \( i < n \) then \( \lambda \not\in \varsigma \), and if \( i = n \) then \( \lambda \not\in \varsigma \). By definition, a word \( q_0q_1\cdots q_n \in \Sigma^* \) is consistent with a run \( \rho \) if \( \varsigma_i \) is consistent with step \( i \) of \( \rho \) for all \( i, 0 \leq i \leq n \).

We first note that the function \( \delta \) is total in the sense that for every assignment \( \alpha \) and run \( \rho \), the returned set has an entry that is \( \lambda \)-consistent with \( \alpha \) and \( \rho \).

Lemma 7. For every run \( \rho \) of the form \( \square \), every \( i, 0 \leq i \leq n \), and \( \psi \in \text{LTL}_f^B \), there is some \( (\psi', \varsigma) \in \delta(\psi') \) such that \( \varsigma \) is \( \lambda \)-consistent with step \( i \) of \( \rho \).

Proof. By induction on the structure of \( \psi \) using the definition of \( \delta \). The claim is easy to check for every base case of the definition of \( \delta \), and in all other cases it follows from the induction hypothesis.

We next show a crucial feature of the function \( \delta \), namely that it preserves and reflects the property of a run satisfying a formula. Both directions are proven by tedious but straightforward induction proofs on the formula structure.

Lemma 8. Let \( \psi \in \text{LTL}_f^B \cup \{\top, \bot\} \), \( \rho \) a run of the form \( \square \), and \( 0 \leq i \leq n \). Then \( \rho, i \models_K \psi \) holds if and only if there is some \( (\psi', \varsigma) \in \delta(\psi') \) such that \( \varsigma \) is \( \lambda \)-consistent with step \( i \) of \( \rho \).

Proof. (\( \Rightarrow \)) We first note that if \( \psi' = \top \) then \( (\ast) \) holds for both \( i < n \) and \( i = n \). The proof is by induction on \( \psi \).
- If \( \psi = \top \), we can choose \( (\psi', \varsigma) = (\top, \emptyset) \). Then, \( \emptyset \) is \( \lambda \)-consistent with any step, and \( (\ast) \) follows from \( (\ast) \).
- If \( \rho, i \models_K K \) for some \( K \in K_B(\Phi) \), we may take \( (\top, \emptyset) \) in \( \delta(\top) \). As \( \rho, i \models_K K \), \( \alpha_i \) satisfies \( K(b_i) \), so consistency holds and we use \( (\ast) \) for \( (\ast) \).
- If \( \rho, i \models_K X \psi \) then \( i < n \) and \( \rho, i+1 \models_K \psi \). For \( (\psi', \neg \lambda) \) in \( \delta(X \psi') \), part \( (\ast) \) holds since \( \neg \lambda \in \varsigma \) and \( i < n \), and \( (\ast) \) because of \( \rho, i+1 \models_K \psi \).
- Suppose \( \rho, i \models_K Y \psi \). If \( i = n \) then \( (\ast) \) is by definition, and \( (\ast) \) by \( (\ast) \). If \( i < n \) then \( \rho, i+1 \models_K \psi \). For \( (\psi', \neg \lambda) \) in \( \delta(Y \psi') \), part \( (\ast) \) holds since \( \neg \lambda \in \varsigma \) and \( i < n \), and \( (\ast) \) because of \( \rho, i+1 \models_K \psi \).
- Suppose \( \psi = \psi_1 \land \psi_2 \). By assumption \( \rho, i \models_K \psi_1 \land \psi_2 \), and hence \( \rho, i \models_K \psi_1 \) and \( \rho, i \models_K \psi_2 \). By the induction hypothesis, there are \( (\psi_1', \varsigma_1) \) in \( \delta(\psi_1') \) and \( (\psi_2', \varsigma_2) \) in \( \delta(\psi_2') \) such that for both \( k \in \{1, 2\} \), \( (\ast') \) \( \varsigma_k \) is consistent with step \( i \) of \( \rho \) and \( (\ast') \) either \( \rho, i+1 \models_K \psi'_k \), or \( i = n \) and \( \psi'_k = \top \). By

\( \square \)
definition of $\delta$, we can choose $(\psi_1' \land \psi_2'', \varsigma_1 \cup \varsigma_2) \in \delta(\psi_1' \land \psi_2'')$. Then (a) follows from $(a')$ and $\varsigma = \varsigma_1 \cup \varsigma_2$, and (b) if $i = n$ then $(b')$ implies $\psi' = T$, and otherwise $\rho, i+1 \models K \psi_1' \land \psi_2''$.

- Suppose $\psi = \psi_1 \lor \psi_2$. By assumption $\rho, i \models_K \psi_1 \lor \psi_2$, and hence $\rho, i \models_K \psi_1$ or $\rho, i \models_K \psi_2$. We assume the former. By the induction hypothesis, there is some $(\psi_1'', \varsigma_1) \in \delta(\psi_1'')$ such that $(a') \varsigma_1 = \psi_1'$ is $\lambda$-consistent with step $i$ of $\rho$, and $(b') \rho, i+1 \models_K \psi_1'$, or $i = n$ and $\psi_1' = T$. As $\delta$ is total (Lem. 7), there must be some $(\psi_2'', \varsigma_2) \in \delta(\psi_2'')$ such that $\varsigma_2$ is $\lambda$-consistent with step $i$ of $\rho$. By definition of $\delta$, we can choose $(\psi''', \varsigma')$ as $(\psi_1'' \lor \psi_2''', \varsigma_1 \cup \varsigma_2) \in \delta(\psi_1' \lor \psi_2'')$. Then (a) follows from $(a')$ and $\varsigma_1$ being $\lambda$-consistent with step $i$ of $\rho$, and (b) if $i = n$ then $(b')$ implies $\psi''' = T$, hence $\psi' = T$; otherwise $\rho, i+1 \models_K \psi_1' \lor \psi_2''$.

- Suppose $\rho, i \models_K G \psi$, so $\rho, i \models_K \psi$ and either (1) $i = n$, or (2) $\rho, i+1 \models_K G \psi$.

We have $\delta(G \psi) = \delta(\psi') \land (\delta(\langle \cdot \rangle G \psi) \lor \delta_\chi) = (\delta(\psi') \land \delta(\langle \cdot \rangle G \psi)) \lor (\delta(\psi') \land \delta_\chi)$. In either case, by the induction hypothesis there is some $(\psi''', \varsigma') \in \delta(\psi')$ such that $(a') \varsigma' = \psi''$ is $\lambda$-consistent with step $i$ of $\rho$, and $(b') \rho, i+1 \models_K \psi''$, or $i = n$ and $\psi'' = T$.

(1) Let $(\psi_1'', \varsigma_1)$ be $(\psi'' \land \psi_1') \in \delta(\psi'') \land \delta_\chi$. We have $(a_1) \varsigma_1$ is $\lambda$-consistent with step $i$ of $\rho$ because of $(a')$ and $\varsigma = \{K\}$. As $\alpha_i \models K(b_i)$ by $\lambda$-consistency, $\rho, i \models_K K$.

Let $\psi = X \chi$. As $\psi'$ is $\perp$ or $\rho, i+1 \models_K \psi'$, by definition of $\delta$ the only possibility is $\psi' = \chi''$ and $\varsigma = \{\neg \lambda\}$. As $\varsigma$ is consistent with step $i$ of $\rho$ and $\neg \lambda \in \varsigma$, we must have $i < n$, so $\rho, i+1 \models_K \psi'$ and hence $\rho, i \models_K \psi$ by Def. 9.

- Suppose $\psi = Y \chi$. If $\psi' = T$ then $i = n$ and $\rho, i \models_K \psi$ holds by definition. Otherwise, we can reason as in the case above.

- If $\psi = \psi_1 \land \psi_2$ then by $(\psi'', \varsigma) \in \delta(\psi'')$ and the definition of $\delta$ there are $\psi_1'$ and $\psi_2''$ such that $(\psi_1'', \varsigma_1) \in \delta(\psi_1'')$ and $(\psi_2'', \varsigma_2) \in \delta(\psi_2'')$, and $\psi' = \psi_1' \land \psi_2''$ and $\varsigma = \varsigma_1 \cup \varsigma_2$. Therefore, either $i = n$ and $\psi' = \psi_1' = \psi_2'' = T$, or $i < n$ and $\rho, i+1 \models_K \psi_1'$ and $\rho, i+1 \models_K \psi_2''$. In either case, $\rho, i \models_K \psi_1$ and $\rho, i \models_K \psi_2$ hold by the induction hypothesis, so $\rho, i \models_K \psi_1 \land \psi_2$.

- Similarly, if $\psi = \psi_1 \lor \psi_2$ then there are $\psi_1'$ and $\psi_2''$ such that $(\psi_1'', \varsigma_1) \in \delta(\psi_1'')$ and $(\psi_2'', \varsigma_2) \in \delta(\psi_2'')$, $\psi' = \psi_1' \lor \psi_2''$ and $\varsigma = \varsigma_1 \cup \varsigma_2$. If $i = n$ and $\psi' = T$, then $\psi_1' = T$ or $\psi_2'' = T$. If otherwise $i < n$ then $\rho, i+1 \models_K \psi'$ implies $\rho, i+1 \models_K \psi_1'$ or $\rho, i+1 \models_K \psi_2''$. From the induction hypothesis we obtain in either case $\rho, i \models_K \psi_1$ or $\rho, i \models_K \psi_2$, so $\rho, i \models_K \psi_1 \lor \psi_2$. 

(\text{Note that the assumptions exclude } \psi' = \perp. \text{ We apply induction on } \psi, \text{ and use the definition of } \delta \text{ for each case.})

- If $\psi = T$ then $(\psi'', \varsigma) \in \delta(\psi'')$ implies $\psi' = T$, and $\rho, i \models_K T$ holds.

- If $\psi = K \in \mathcal{K}_G(\Phi)$, we have $\psi' = T$ and $\varsigma = \{K\}$. As $\alpha_i \models K(b_i)$ by $\lambda$-consistency, $\rho, i \models_K K$.

- Let $\psi = X \chi$. As $\psi'$ is $\perp$ or $\rho, i+1 \models_K \psi'$, by definition of $\delta$ the only possibility is $\psi' = \chi''$ and $\varsigma = \{\neg \chi\}$. As $\varsigma$ is consistent with step $i$ of $\rho$ and $\neg \chi \in \varsigma$, we must have $i < n$, so $\rho, i+1 \models_K \psi'$ and hence $\rho, i \models_K \psi$ by Def. 9.
If \( \psi = G \chi \) then we can distinguish two cases:

1. There are \( \psi_1 \) and \( \psi_2 \) such that \( (\psi_1', \zeta_1) \in \delta(\chi') \), \( (\psi_2', \zeta_2) \in \delta \chi \), \( \psi' = \psi_1' \lor \psi_2' \) and \( \zeta = \zeta_1 \cup \zeta_2 \). As \( (\psi_2', \zeta_2) \in \delta \chi \), we must have \( \psi_2' = T \) and \( \zeta_2 = \{ \lambda \} \) (otherwise, we would have \( \psi' = \bot \)). By consistency, \( \lambda \in \zeta \) implies \( i = n \), so \( \psi' = T \) by assumption and therefore we must have \( \psi_1' = T \). From the induction hypothesis and \( (\psi_1', \zeta_1) \in \delta(\chi') \) we conclude \( \rho, i \models K \chi \), so by Def. 8 Proof. 

2. There are \( \psi_1' \) and \( \psi_2' \) such that \( (\psi_1', \zeta_1) \in \delta(\chi') \), \( (\psi_2', \zeta_2) \in \delta(\{\} G \chi') \), \( \psi' = \psi_1' \lor \psi_2' \) and \( \zeta = \zeta_1 \cup \zeta_2 \). We have \( \neg \lambda \in \zeta_2 \), so by consistency \( i < n \). As \( \rho, i+1 \models K \psi' = \psi_1' \lor \psi_2' \), by Def. 8 \( \rho, i+1 \models K \psi_1' \) and \( \rho, i+1 \models K \psi_2' \). By the induction hypothesis, \( (\psi_1', \zeta_1) \in \delta(\chi') \) and \( \rho, i+1 \models K \psi_1' \) imply \( \rho, i \models K \chi \). Moreover, \( (\psi_2', \zeta_2) \in \delta(\{\} G \chi') \) and \( \rho, i+1 \models K \psi_2' \) imply \( \psi_2' = G \chi \) by Def. 8 so we have \( \rho, i \models K (\{\} G \chi) \). Thus \( \rho, i \models K G \chi \).

The case for \( U \) is similar. \( \square \)

Let a word \( \zeta_0 \zeta_1 \cdots \zeta_n \in \Sigma^* \) be well-formed if \( \lambda \notin \zeta_i \) for all \( 0 \leq i \leq n \), and \( \neg \lambda \notin \zeta_n \).

**Lemma 9.** A well-formed word \( w \in \Sigma^* \) that is consistent with a run \( \rho \) satisfies \( \top \in \delta(\psi', w) \) iff \( \rho, 0 \models K \psi \).

**Proof.** \( (\Longrightarrow) \) Let \( w = \zeta_0 \zeta_1 \cdots \zeta_n \) and \( \chi_0, \chi_1, \ldots, \chi_{n+1} \) be the sequence of formulas witnessing \( \top \in \delta(\psi', w) \), so that \( \chi_0 = \psi, \chi_{n+1} = T \), and \( (\chi_{i+1}, \zeta_i) \in \delta(\chi'_i) \) for all \( i, 0 \leq i \leq n \). As \( w \) is well-formed and consistent with \( \rho \), by definition \( \zeta_i \) is \( \lambda \)-consistent with \( \rho \) at \( i \) for all \( i, 0 \leq i \leq n \). In order to show that \( \rho, 0 \models K \psi \) holds, we verify that \( \rho, i \models K \chi_i \) for all \( i, 0 \leq i \leq n \), by induction on \( n - i \). In the base case \( i = n \). We have \( \chi_{n+1} = T \) and \( (\chi_n, \zeta_0, \zeta_n) \in \delta(\chi'_n) \), and from Lem. 9(\( \Longrightarrow \)) it follows that \( \rho, n \models K \chi_n \). If \( i < n \), we assume by the induction hypothesis that \( \rho, i+1 \models K \chi_{i+1} \). We have \( (\chi_{i+1}, \zeta_i) \in \delta(\chi'_i) \), so \( \rho, i \models K \chi_i \) follows again from Lem. 9(\( \Longrightarrow \)), which concludes the induction step. Finally, the claim follows for the case \( i = 0 \) because \( \chi_0 = \psi \).

\( (\Longleftarrow) \) Let \( \rho \) be of the form \( [2] \). We show that for all \( i, 0 \leq i \leq n \), and every formula \( \chi \), if \( \rho, i \models K \chi \), then there is a word \( w_i = \zeta_{i+1} \cdots \zeta_n \) of length \( n - i + 1 \) such that \( \top \in \delta(\chi', w_i) \), and \( \zeta_j \) is \( \lambda \)-consistent with step \( j \) of \( \rho \) for all \( j, i \leq j \leq n \). The proof of is by induction on \( n - i \).

In the base case where \( i = n \), we assume that \( \rho, n \models K \chi \). By Lem. 9(\( \Longleftarrow \)) there is some \( \zeta_n \) such that \( (\top, \zeta_n) \in \delta(\chi') \), and \( \zeta_n \) is \( \lambda \)-consistent with step \( n \) of \( \rho \). For the induction step, assume \( i < n \) and \( \rho, i \models K \chi \). By Lem. 9(\( \Longleftarrow \)) there is some \( (\chi', \zeta_i) \in \delta(\chi') \) such that \( \rho, i+1 \models K \chi' \), and moreover \( \zeta_i \) is \( \lambda \)-consistent with \( \rho \) at step \( i \). By the induction hypothesis, there is a word \( w_{i+1} = \zeta_{i+1} \cdots \zeta_n \) such that \( \top \in \delta(\chi', w_{i+1}) \), and \( \zeta_j \) is \( \lambda \)-consistent with \( \rho \) at instant \( j \) for all \( j, i \leq j \leq n \). We can define \( w_i = \zeta_i \zeta_{i+1} \cdots \zeta_n \), which satisfies \( \top \in \delta(\chi', w_i) \) and \( \zeta_j \) is \( \lambda \)-consistent with \( \rho \) at \( j \) for all \( j, i \leq j \leq n \), so the induction step works.

By assumption, \( \rho, 0 \models K \psi \) holds. From the case \( i = 0 \) of the above statement, we obtain a word \( w = w_0 \) such that \( \top \in \delta(\psi', w) \) and \( w \) is \( \lambda \)-consistent with all steps of \( \rho \), i.e., \( w \) is well-formed and consistent with \( \rho \). \( \square \)

We next show some simple properties that will be useful to show correctness of the automaton without \( \lambda \).
Lemma 10. Let \( \psi \in \text{LTL}_f^B \) and \((\chi, \varsigma) \in \delta(\psi)\). (1) If \( \lambda \in \varsigma \) and \( -\lambda \not\in \varsigma \) then \( \chi = \top \ or \ \chi = \bot \). (2) Suppose \( -\lambda \in \varsigma \), \( \lambda \not\in \varsigma \), and \( \chi = \top \), and \( \varsigma \) is consistent with step \( i \) of run \( \rho \). Then there is some \((\top, \varsigma') \in \delta(\psi)\) such that \( -\lambda \not\in \varsigma' \) and \( \varsigma' \) is consistent with step \( i \) of \( \rho \) as well. (3) If \( \chi \) is not \( \top \ or \ \bot \) then \( \varsigma \) has \( \lambda \) or \( -\lambda \).

Proof. All three statements are shown simultaneously by induction on \( \psi \).

- If \( \psi \) is \( \top \), \( \bot \), or an atom \( K \) then \( \chi = \top \ or \ \bot \), so (1) and (3) hold, and \( -\lambda \not\in \varsigma \), so also (2) is satisfied.

- If \( \psi = X \psi' \) then \( \delta(\psi) = \{ (\psi'', \{\lambda\}), (\bot'', \{\lambda\}) \} \). (1) is satisfied by \((\bot'', \{\lambda\})\), and (2) holds for any \( \psi' \) such that \( (1) \) and (3) hold, and \( -\lambda \not\in \varsigma \), so also (2) is satisfied.

- If \( \psi = \forall \psi' \) then \( \delta(\psi) = \{ (\forall \psi'', \{\lambda\}), (\exists \psi'', \{\lambda\}) \} \). (1) is satisfied by \((\exists \psi'', \{\lambda\})\), and (2) holds because \( -\lambda \not\in \varsigma \), and (3) is satisfied anyway.

- If \( \psi = \psi_1 \lor \psi_2 \) then we must have \( \chi = \chi_1 \lor \chi_2 \) such that \((\chi_i, \varsigma_i) \in \delta(\psi_i)\) for both \( i \in \{1, 2\} \), and \( \varsigma = \varsigma_1 \cup \varsigma_2 \). (1) Suppose \( \lambda \not\in \varsigma \) and \( -\lambda \not\in \varsigma \). First, assume \( \lambda \not\in \varsigma_1 \), \( -\lambda \not\in \varsigma_1 \), and \( -\lambda \not\in \varsigma_2 \). By the induction hypothesis (1), \( \chi_1 \) is either \( \top \) or \( \bot \). In the former case, \( \chi_1 = \top \), and as by assumption \( -\lambda \not\in \varsigma_2 \), by the induction hypothesis (3), \( \chi_2 \) must be \( \top \ or \ \bot \). Otherwise, \( \chi = \chi_2 \). Then, if \( \lambda \not\in \varsigma_2 \), we can again use the induction hypothesis to conclude that \( \chi = \chi_2 \) is \( \top \ or \ \bot \). Otherwise, we have \( \lambda \not\in \varsigma_2 \) and \( -\lambda \not\in \varsigma_2 \), so \( \chi_2 \) must be \( \top \ or \ \bot \) by the induction hypothesis (3).

- If \( \psi = \psi_1 \land \psi_2 \) then we must have \( \chi = \chi_1 \land \chi_2 \) such that \((\chi_i, \varsigma_i) \in \delta(\psi_i)\) for both \( i \in \{1, 2\} \). (1) Suppose \( \lambda \not\in \varsigma \) and \( -\lambda \not\in \varsigma \). W.l.o.g., we can assume \( \lambda \not\in \varsigma_1 \), \( -\lambda \not\in \varsigma_1 \), and \( -\lambda \not\in \varsigma_2 \). By the induction hypothesis (1), \( \chi_1 \) is either \( \top \) or \( \bot \). In the latter case, \( \chi = \bot \), so the claim holds. Otherwise, \( \chi = \chi_2 \), and as by assumption \( -\lambda \not\in \varsigma_2 \), by the induction hypothesis (3), \( \chi_2 \) must be \( \top \ or \ \bot \). (2) Suppose \( -\lambda \in \varsigma \), \( \lambda \not\in \varsigma \), and \( \chi = \top \), and \( \varsigma \) is consistent with step \( i \) of \( \rho \). We can assume \( \lambda \not\in \varsigma_1 \), \( \lambda \not\in \varsigma_2 \), and \( \chi_1 = \chi_2 = \top \). We must have \( \lambda \not\in \varsigma_1 \), \( \lambda \in \varsigma_2 \), or both. However, for each \( i \in \{1, 2\} \) such that \( \lambda \not\in \varsigma_i \), by the induction hypothesis (2) there is some \((\top, \varsigma_i') \in \delta(\psi_i)\) such that \( \lambda \not\in \varsigma_i' \) and \( \varsigma_i' \) is consistent with step \( i \) of \( \rho \). If \( \lambda \not\in \varsigma_i \), set \( \varsigma_i' = \varsigma_i \). Hence \((\top, \varsigma_i' \cup \varsigma_i') \in \delta(\psi_i)\) such that \( \lambda \not\in \varsigma_i' \cup \varsigma_i' \) is consistent with step \( i \) of \( \rho \) and \( -\lambda \not\in \varsigma_i' \cup \varsigma_i' \).

- For \( \psi = G \psi' \), note that \( \delta(\psi) = \{ (\top, \{\lambda\}), (\bot, \{\lambda\}) \} \) satisfies the properties. The result then follows from the cases for \( \lor \) and \( \land \).

- All other cases follow from the cases for \( \lor \) and \( \land \). \( \square \)
Lemma 2 \( N_\psi \) accepts a word that is consistent with a run \( \rho \) iff \( 0 \models_\mathcal{K} \psi \).

Proof. \((\Rightarrow)\) Let \( w = \varsigma_0\varsigma_1 \cdots \varsigma_n \) be accepted, and \( q_0 \xrightarrow{\varsigma_0} q_1 \xrightarrow{\varsigma_1} \cdots \xrightarrow{\varsigma_n} q_{n+1} \) be the respective accepting run of \( N_\psi \). By Def. 12 there are \( \varsigma'_i \), such that \( \varsigma_i = \varsigma'_i \setminus \{ \lambda, \neg\lambda \} \) and \( \{ \lambda, \neg\lambda \} \nsubseteq \varsigma'_i \) for all \( i, 0 \leq i \leq n \). Let \( w' = \varsigma'_0\varsigma'_1 \cdots \varsigma'_n \). Then \( w' \) is consistent with \( \rho \) because so is \( w \). Moreover, by Lem. 10 (2) we can choose \( \varsigma'_n \) such that \( \neg\lambda \notin \varsigma'_n \), and \( \varsigma'_n \) is consistent with \( \rho \) at \( n \). Then \( w' \) is well-formed: indeed, since edges to "\( \top \)" labeled \( \lambda \) are redirected to \( q_e \) and "\( \bot \)" cannot occur in the accepting sequence, by Lem. 10 (1) we have \( \lambda \notin \varsigma'_i \) for \( i < n \). Thus by Def. 12 we have "\( \top \)" \( \in \delta^*(\psi, w') \). According to Lem. 9 \( \rho \models_\mathcal{K} \psi \).

\((\Leftarrow)\) If \( \rho \models_\mathcal{K} \psi \) then by Lem. 9 there is a well-formed word \( w = \varsigma_0\varsigma_1 \cdots \varsigma_n \) that is consistent with \( \rho \) such that "\( \top \)" \( \in \delta^*(\psi, w) \). As \( w \) is well-formed, no \( \varsigma_i \) has both \( \lambda \) and \( \neg\lambda \). Hence all \( \delta \)-steps are reflected by transitions in \( N_\psi \). As "\( \psi \)" is the initial state, by Def. 12 there is an accepting run in \( N_\psi \) to "\( \top \)" or \( q_e \). \( \Box \)