Bimodules over Relative Rota-Baxter Algebras and Cohomologies

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Abstract
A relative Rota-Baxter algebra is a generalization of a Rota-Baxter algebra. Relative Rota-Baxter algebras are closely related to dendriform algebras. In this paper, we introduce bimodules over a relative Rota-Baxter algebra that fits with the representations of dendriform algebras. We define the cohomology of a relative Rota-Baxter algebra with coefficients in a bimodule and then study abelian extensions of relative Rota-Baxter algebras in terms of the second cohomology group. Finally, we consider homotopy relative Rota-Baxter algebras and classify skeletal homotopy relative Rota-Baxter algebras in terms of the above-defined cohomology.

Keywords Relative Rota-Baxter algebras · Bimodules · Cohomology · Abelian extensions · Homotopy algebras

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1 Introduction
Rota-Baxter operators were first appeared in a 1960’s paper of Baxter [5] as a tool to study fluctuation theory in probability. Subsequently, such operators were investigated by Rota [23], Cartier [6], Atkinson [2] among others. In the last twenty years, Rota-Baxter operators received very much attention due to their connection with many branches of mathematics...
and mathematical physics. For instance, Rota-Baxter operators play a key role in the combinatorial study of trees and shuffles [15], splitting of algebras [3], infinitesimal bialgebras [1], Hopf algebras [29], double algebras [11] and renormalizations in quantum field theory [7]. Let $A$ be an associative algebra. A linear map $R : A \rightarrow A$ is said to be a Rota-Baxter operator on $A$ if

$$R(a) \cdot R(a') = R\left(R(a) \cdot a' + a \cdot R(a')\right), \text{ for } a, a' \in A.$$ 

A pair $(A, R)$ consisting of an associative algebra $A$ and a Rota-Baxter operator $R$ is called a Rota-Baxter algebra. The notion of Rota-Baxter bimodules over a Rota-Baxter algebra was introduced by Guo and Lin in [16]. Recently, Wang and Zhou [28] defined the cohomology of a Rota-Baxter algebra with coefficients in a Rota-Baxter bimodule. Such cohomology is closely related to abelian extensions and deformations of Rota-Baxter algebras. See [14] for more details about Rota-Baxter operators.

In [27] Uchino introduced a notion of generalized Rota-Baxter operator as an operator analogue of Poisson structures in the noncommutative setup. Later, generalized Rota-Baxter operators are also named as relative Rota-Baxter operators or $O$-operators [4, 9]. They are mostly known as relative Rota-Baxter operators in the literature. Let $A$ be an associative algebra and $M$ be an $A$-bimodule. A linear map $R : M \rightarrow A$ is said to be a relative Rota-Baxter operator if

$$R(m) \cdot R(m') = R(R(m) \cdot_M m' + m \cdot_M R(m')), \text{ for } m, m' \in M.$$ 

Here $\cdot_M$ denotes the left and right $A$-actions on $M$. Recently, relative Rota-Baxter operators are cohomologically studied in [9]. A relative Rota-Baxter (associative) algebra is a triple $(A, M, R)$ in which $A$ is an associative algebra, $M$ is an $A$-bimodule, and $R : M \rightarrow A$ is a relative Rota-Baxter operator. In the present paper, we denote a relative Rota-Baxter algebra as above by $M \xrightarrow{R} A$. It follows that any Rota-Baxter algebra $(A, R)$ can be considered as a relative Rota-Baxter algebra $A \xrightarrow{R} A$, where $A$ is equipped with the adjoint $A$-bimodule structure. It has been observed in [27] that a relative Rota-Baxter algebra $M \xrightarrow{R} A$ induces a dendriform structure on $M$.

Rota-Baxter operators and relative Rota-Baxter operators in the context of Lie algebras first appeared in the study of classical $r$-matrices [19]. Cohomologies of relative Rota-Baxter operators on Lie algebras were initially studied in [26]. Then using higher derived brackets, the cohomology theory of relative Rota-Baxter Lie algebras were successfully established by Lazarev, Sheng and Tang in [20]. Recently, cohomologies of relative Rota-Baxter Lie algebras with coefficients in an arbitrary representation were given by Jiang and Sheng in [17]. Applications of these cohomologies were given to study infinitesimal deformations, abelian extensions and homotopies of relative Rota-Baxter Lie algebras (we refer to [12, 13] for deformations and cohomologies of algebraic structures). Some aspects of the above study are already considered for relative Rota-Baxter (associative) algebras by the present authors [10]. More precisely, we defined cohomologies of relative Rota-Baxter algebras, which can be used to characterize infinitesimal deformations. Homotopy relative Rota-Baxter algebras were also given in the same paper. However, cohomologies of relative Rota-Baxter algebras are not yet studied from the representation-theoretic points of view.

The main aim of this paper is to define bimodules over a relative Rota-Baxter algebra and introduce the cohomology of a relative Rota-Baxter algebra with coefficients in a bimodule. Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra. A bimodule over it consists of a 2-term chain complex $N \xrightarrow{S} B$ in which $B$ and $N$ are both $A$-bimodules, together with two bilinear maps
$l : M \otimes B \to N$ and $r : B \otimes M \to N$ satisfying some compatibility relations (see Definition 3.1). Note that our notion of bimodule over a relative Rota-Baxter algebra generalizes the Rota-Baxter bimodule over a Rota-Baxter algebra (see Example 3.2). Further, any relative Rota-Baxter algebra can be regarded as a bimodule over itself, called the adjoint bimodule. Given a bimodule over a relative Rota-Baxter algebra, we construct the corresponding dual bimodule.

Moreover, there is a semidirect product construction associated with a bimodule over a relative Rota-Baxter algebra (cf. Proposition 3.9). Our notion of bimodule over a relative Rota-Baxter algebra nicely fits with the representations of dendriform algebras. More precisely, we show that a bimodule over a relative Rota-Baxter algebra gives rise to a representation of the induced dendriform algebra (cf. Proposition 3.10).

Next, we introduce the cohomology of a relative Rota-Baxter algebra with coefficients in a bimodule. The cohomology of a relative Rota-Baxter algebra introduced in [10] turns out to be the cohomology of the relative Rota-Baxter algebra with coefficients in the adjoint bimodule. Then we consider abelian extensions of a relative Rota-Baxter algebra by a bimodule. We show that isomorphism classes of such abelian extensions are classified by the second cohomology group (cf. Theorem 5.3).

The notion of homotopy relative Rota-Baxter operators in the context of $A_\infty$-algebras (strongly homotopy associative algebras) was introduced in [10]. Like the classical case, a homotopy relative Rota-Baxter algebra consists of an $A_\infty$-algebra $A$, an $A$-bimodule $M$ and a homotopy relative Rota-Baxter operator. In this paper, we consider homotopy relative Rota-Baxter algebras whose underlying graded vector spaces $A, M$ are concentrated in degrees 0 and 1. In particular, we classify skeletal homotopy relative Rota-Baxter algebras by third cocycles of relative Rota-Baxter algebras introduced in the present paper (cf. Theorem 6.7).

The paper is organized as follows. In Section 2, we recall the Hochschild cohomology of associative algebra with coefficients in a bimodule. Then we recall relative Rota-Baxter algebras, which are the main objective in this paper, and their relation with dendriform algebras. Our main references are [9, 22, 27].

Let $(A, \mu)$ be an associative algebra. For any two elements $a, a' \in A$, we write the multiplication $\mu(a, a')$ by $a \cdot a'$. An associative algebra $(A, \mu)$ is often denoted by $A$.

Let $A$ be an associative algebra. An $A$-bimodule is a vector space $M$ together with two bilinear maps (called left and right $A$-actions) $l_M : A \otimes M \to M$, $(a, m) \mapsto a \cdot_M m$ and $r_M : M \otimes A \to M$, $(m, a) \mapsto m \cdot_M a$ satisfying

\begin{align*}
(a \cdot a') \cdot_M m &= a \cdot_M (a' \cdot_M m), \\
(a \cdot_M m) \cdot_M a' &= a \cdot_M (m \cdot_M a') \quad \text{and} \quad (m \cdot_M a) \cdot_M a' = m \cdot_M (a \cdot a'),
\end{align*}

(cf. Springer)
for $a, a' \in A$ and $m \in M$. An $A$-bimodule as above is simply denoted by $M$ when the left and right $A$-actions are clear from the context. Any associative algebra $A$ is obviously an $A$-bimodule with left and right $A$-actions are given by the algebra multiplication map. We call this the adjoint $A$-bimodule.

Given an $A$-bimodule $M$, the dual space $M^*$ also carries an $A$-bimodule structure with left and right $A$-actions given by

$$(a \cdot_M f)(m) = f(m \cdot_M a) \quad \text{and} \quad (f \cdot_M a)(m) = f(a \cdot_M m), \quad \text{for } a \in A, \ f \in M^*, \ m \in M.$$  

The above bimodule structure is called the dual $A$-bimodule structure on $M^*$. In particular, the dual of the adjoint $A$-bimodule is called the coadjoint $A$-bimodule.

Let $A$ be an associative algebra and $M$ be an $A$-bimodule. Then the direct sum $A \oplus M$ carries an associative algebra structure with the multiplication given by

$$(a, m) \cdot (a', m') = (a \cdot a', a \cdot_M m' + m \cdot_M a'), \quad \text{for } (a, m), (a', m') \in A \oplus M.$$  

The pair $(A \oplus M, \cdot)$ is called the semidirect product algebra. The semidirect product algebra will play a crucial role in our study.

We will now recall the Hochschild cohomology of an associative algebra with coefficients in a bimodule. Let $A$ be an associative algebra and $M$ be an $A$-bimodule. For each $k \geq 0$, define an abelian group $C^k_H(A, M) := \text{Hom}(A^k, M)$. Moreover, there is a differential $\delta_{A, M} : C^k_H(A, M) \to C^{k+1}_H(A, M)$, for $k \geq 0$, given by

$$(\delta_{A, M} f)(a_1, \ldots, a_{k+1}) = a_1 \cdot_M f(a_2, \ldots, a_{k+1}) + \sum_{i=1}^{k} (-1)^i f(a_1, \ldots, a_i \cdot a_{i+1}, \ldots, a_{k+1})$$

$$+ (-1)^{k+1} f(a_1, \ldots, a_k) \cdot_M a_{k+1},$$

for $f \in C^k_H(A, M)$ and $a_1, \ldots, a_{k+1} \in A$. The cohomology groups of the cochain complex $\{C^*_H(A, M), \delta_{A, M}\}$ are called the Hochschild cohomology groups of $A$ with coefficients in $M$.

**Definition 2.1** (i) Let $A$ be an associative algebra and $M$ be an $A$-bimodule. A linear map $R : M \to A$ is said to be a **relative Rota-Baxter operator** on $M$ over the algebra $A$ if $R$ satisfies

$$R(m) \cdot R(m') = R(R(m) \cdot_M m' + m \cdot_M R(m')), \quad \text{for } m, m' \in M. \quad (2)$$

(ii) A **relative Rota-Baxter algebra** is a triple $(A, M, R)$ consisting of an associative algebra $A$, an $A$-bimodule $M$ and a relative Rota-Baxter operator $R : M \to A$.

**Notation 2.2** For some conventional reason, we denote a relative Rota-Baxter algebra by $M \xrightarrow{R} A$ instead of the triple $(A, M, R)$. Note that both of these notations capture the same pieces of information.

**Example 2.3** Let $A = (A_1 \xrightarrow{d} A_0)$ be a 2-term chain complex. Define $\text{End}(A)$ to be the space of all chain maps on $A$, i.e.,

$$\text{End}(A) = \{(f_0, f_1) \mid f_0 \in \text{End}(A_0), \ f_1 \in \text{End}(A_1) \text{ and } f_0 \circ d = d \circ f_1\}.$$  

Note that $\text{End}(A)$ carries an associative algebra structure with multiplication given by

$$(f_0, g_0) \cdot (g_0, g_1) = (f_0 \circ g_0, f_1 \circ g_1), \quad \text{for } (f_0, g_0), (g_0, g_1) \in \text{End}(A).$$
Moreover, there is an \( \text{End}(A) \)-bimodule structure on the space \( M = \text{Hom}(A_0, \ker d) \) given by

\[
(f_0, f_1) \cdot_M \phi = f_1 \circ \phi \quad \text{and} \quad \phi \cdot_M (f_0, f_1) = \phi \circ f_0,
\]
for \((f_0, f_1) \in \text{End}(A), \phi \in M = \text{Hom}(A_0, \ker d)\).

Finally, it is easy to verify that the map \( R : M \to \text{End}(A), \phi \mapsto (0, \phi \circ d) \) is a relative Rota-Baxter operator. In other words, \( M \xrightarrow{R} \text{End}(A) \) is a relative Rota-Baxter algebra.

Some other examples of relative Rota-Baxter algebras can be found in [27].

**Definition 2.4** Let \( M \xrightarrow{R} A \) and \( N \xrightarrow{S} B \) be two relative Rota-Baxter algebras. A **morphism** of relative Rota-Baxter algebras from \( M \xrightarrow{R} A \) to \( N \xrightarrow{S} B \) is a pair \((\phi, \psi)\) consisting of an algebra morphism \( \phi : A \to B \) and a linear map \( \psi : M \to N \) satisfying

\[
\psi(a \cdot_M m) = \phi(a) \cdot_N \psi(m),
\]
\[
\psi(m \cdot_M a) = \psi(m) \cdot_N \phi(a)
\]
and \( \phi \circ R = S \circ \psi \), for \( a \in A, m \in M \).

It is called an isomorphism if \( \phi \) and \( \psi \) are linear isomorphisms.

Next, we recall dendriform algebras that are intimately connected with relative Rota-Baxter algebras [1, 21, 27].

**Definition 2.5** A **dendriform algebra** is a vector space \( D \) together with two bilinear operations \( \prec, \succ : D \otimes D \to D \) satisfying

\[
(x \prec y) \prec z = x \prec (y \prec z + y \succ z), \tag{3}
\]
\[
(x \succ y) \prec z = x \succ (y \prec z), \tag{4}
\]
\[
(x \prec y + x \succ y) \succ z = x \succ (y \succ z), \quad \text{for} \ x, y, z \in D. \tag{5}
\]

Let \( (D, \prec, \succ) \) be a dendriform algebra. It follows from (3)–(5) that the new operation \( x \ast y = x \prec y + x \succ y \), for \( x, y \in D \), makes \( D \) into an associative algebra. We call this the ‘total associative algebra’, denoted by \( D_{\text{Tot}} \).

**Definition 2.6** Let \( (D, \prec, \succ) \) be a dendriform algebra. A **representation** of it consists of a vector space \( E \) together with four bilinear maps (called action maps)

\[
\prec : D \otimes E \to E, \quad \succ : D \otimes E \to E, \quad \prec : E \otimes D \to E \quad \text{and} \quad \succ : E \otimes D \to E
\]
satisfying the 9 identities where each tuple of 3 identities correspond to (3)–(5) with exactly one of \( x, y, z \) comes from \( E \).

**Remark 2.7** (i) Any dendriform algebra \( D \) is a representation of itself, where action maps are dendriform operations on \( D \). It is called the adjoint representation.

(ii) Let \( D \) be a dendriform algebra and \( E \) be a representation of it. Then \( E \) can be given a \( D_{\text{Tot}} \)-bimodule structure with left and right \( D_{\text{Tot}} \)-actions

\[
x \cdot_E e = x \prec e + x \succ e \quad \text{and} \quad e \cdot_E x = e \prec x + e \succ x, \quad \text{for} \ x \in D_{\text{Tot}}, e \in E.
\]

The following result can be found in [9, 27].
Proposition 2.8  

(i) Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra. Then \( M \) carries a dendriform algebra structure given by 

\[
m \prec_R m' = m \cdot_M R(m') \quad \text{and} \quad m \succ_R m' = R(m) \cdot_M m', \quad \text{for} \ m, m' \in M.
\]

Hence \((M, \ast_R)\) is an associative algebra, where \( m \ast_R m' = R(m) \cdot_M m' + m \cdot_M R(m') \), for \( m, m' \in M \). (We denote this total associative algebra by \( M_{\text{Tot}} \)). With this associative structure, the map \( R : M \rightarrow A \) is a morphism of associative algebras.

(ii) If \((\phi, \psi)\) is a morphism of relative Rota-Baxter algebras from \( M \xrightarrow{R} A \) to \( N \xrightarrow{S} B \), then the map \( \psi : M \rightarrow N \) is a morphism of dendriform algebras.

3 Bimodules over Relative Rota-Baxter Algebras

In this section, we introduce bimodules over relative Rota-Baxter algebras and provide various examples. We construct the corresponding semidirect product in the context of relative Rota-Baxter algebras. Finally, we show that a bimodule over a relative Rota-Baxter algebra gives rise to a representation of the induced dendriform algebra and find out a relationship between the cohomology of a relative Rota-Baxter operator and the cohomology of the induced dendriform algebra.

Definition 3.1  

Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra. A bimodule over it consists of a tuple \((N \xrightarrow{S} B, l, r)\) in which \( N \xrightarrow{S} B \) is a 2-term chain complex with both \( B \) and \( N \) are \( A \)-bimodules, and there are two bilinear maps \( l : M \otimes B \rightarrow N \) and \( r : B \otimes M \rightarrow N \) satisfying

\[
l(a \cdot_M m, b) = a \cdot_N l(m, b), \quad l(m \cdot_M a, b) = l(m, a \cdot_B b), \quad l(m, b \cdot_B a) = l(m, b) \cdot_N a, \tag{6}
\]

\[
r(a \cdot_B b, m) = a \cdot_N r(b, m), \quad r(b \cdot_B a, m) = r(b, a \cdot_M m), \quad r(b, m \cdot_M a) = r(b, m) \cdot_N a, \tag{7}
\]

and

\[
R(m) \cdot_B S(n) = S(R(m) \cdot_N n + l(m, S(n))), \tag{8}
\]

\[
S(n) \cdot_B R(m) = S(r(S(n), m) + n \cdot_N R(m)). \tag{9}
\]

for \( a \in A, \ b \in B, \ m \in M, \ n \in N \).

A bimodule as above is often denoted by the complex \( N \xrightarrow{S} B \) when the bilinear maps \( l, r \) are clear from the context.

Example 3.2 (Rota-Baxter bimodule)  

Let \( (A, R) \) be a Rota-Baxter algebra. Then it can be considered as a relative Rota-Baxter algebra \( A \xrightarrow{R} A \), where \( A \) is equipped with the adjoint \( A \)-bimodule structure. A Rota-Baxter bimodule \([16]\) over the Rota-Baxter algebra \( (A, R) \) consists of a pair \((M, R_M)\) in which \( M \) is an \( A \)-bimodule and \( R_M : M \rightarrow M \) is a linear map satisfying

\[
R(a) \cdot_M R_M(m) = R_M(R(a) \cdot_M m + a \cdot_M R_M(m)),
\]

\[
R_M(m) \cdot_M R(a) = R_M(R_M(m) \cdot_M a + m \cdot_M R(a)).
\]
for \( a \in A, \; m \in M \). Then it follows that \( M \xrightarrow{RM} M \) is a bimodule over the relative Rota-Baxter algebra \( A \xrightarrow{R} A \), where the bilinear maps \( l \) and \( r \) are respectively left and right \( A \)-actions on \( M \).

Here we consider an example of a Rota-Baxter bimodule. Let \( A \) be an associative algebra. For an element \( r = r_1 \otimes r_2 \in A \otimes A \), we define the following three elements

\[
\begin{align*}
\text{r}_{13}\text{r}_{12} &= r_1 \cdot \tilde{r}_1(1) \otimes \tilde{r}_2(2) \otimes r_2, \\
\text{r}_{12}\text{r}_{23} &= r_1 \otimes r_2 \cdot \tilde{r}_1(1) \otimes \tilde{r}_2(2) \\
\text{r}_{23}\text{r}_{13} &= r_1 \otimes \tilde{r}_1(1) \otimes \tilde{r}_2(2) \cdot r_2
\end{align*}
\]

of the tensor product \( A \otimes A \otimes A \). Here \( \text{r} = \tilde{r}_1(1) \otimes \tilde{r}_2(2) \) is an another copy of \( r \). We consider the following equation, called the associative Yang-Baxter equation (in short AYBE)

\[
\text{r}_{13}\text{r}_{12} - \text{r}_{12}\text{r}_{23} + \text{r}_{23}\text{r}_{13} = 0,
\]

equivalently,

\[
r_1 \cdot \tilde{r}_1(1) \otimes \tilde{r}_2(2) \otimes r_2 - r_1 \otimes r_2 \cdot \tilde{r}_1(1) \otimes \tilde{r}_2(2) + r_1 \otimes \tilde{r}_1(1) \otimes \tilde{r}_2(2) \cdot r_2 = 0.
\]

A solution of the associative Yang-Baxter equation is called an associative \( r \)-matrix.

Let \( A \) be an associative algebra and \( r = r_1 \otimes r_2 \in A \otimes A \) be an associative \( r \)-matrix. We define a map \( R : A \rightarrow A \) by \( R(a) = r_1 \cdot a \cdot r_2 \), for \( a \in A \). Then it has been shown in [1] that \( R \) is a Rota-Baxter operator on \( A \). In other words, \((A, R)\) is a Rota-Baxter algebra. Further, if \( M \) is an \( A \)-bimodule, we define a map \( R_M : M \rightarrow M \) by \( R_M(m) = r_1 \cdot M \cdot m \cdot r_2 \), for \( m \in M \). Then it can be checked that \((M, R_M)\) is a Rota-Baxter bimodule over the Rota-Baxter algebra \((A, R)\). To see this, we observe that

\[
R(a) \cdot_M R_M(m) = (r_1 \cdot a \cdot r_2) \cdot_M (r_1 \cdot_M m \cdot_M r_2)
\]

\[
= r_1 \cdot \tilde{r}_1(1) \cdot a \cdot \tilde{r}_2(2) \cdot_M m \cdot_M r_2 + r_1 \cdot a \cdot \tilde{r}_1(1) \cdot_M m \cdot_M \tilde{r}_2(2) \cdot r_2
\]

\[
= R_M( R(a) \cdot_M m + a \cdot_R R_M(m) )
\]

and

\[
R_M(m) \cdot_M R(a) = (r_1 \cdot_M m \cdot_M r_2) \cdot_M (r_1 \cdot a \cdot r_2)
\]

\[
= r_1 \cdot \tilde{r}_1(1) \cdot m \cdot_M \tilde{r}_2(2) \cdot a \cdot r_2 + r_1 \cdot M \cdot r_1 \cdot a \cdot \tilde{r}_2(2) \cdot r_2
\]

\[
= R_M( R_M(m) \cdot_M a + m \cdot_M R(a) ).
\]

**Example 3.3 (Adjoint bimodule)** Any relative Rota-Baxter algebra \( M \xrightarrow{R} A \) is a bimodule over itself, where \( A \) is equipped with the adjoint \( A \)-bimodule, \( M \) is equipped with the given \( A \)-bimodule, and the bilinear maps \( l \) and \( r \) are respectively right and left \( A \)-actions on \( M \).

**Proposition 3.4 (Dual bimodule)** Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra and \( N \xrightarrow{S} B \) be a bimodule over it. Then \( B^* \xrightarrow{-S^*} N^* \) is also a bimodule, where the \( A \)-bimodule structures on \( N^* \) and \( B^* \) are given by the dual \( A \)-bimodules, and the bilinear maps \( l^* : M \otimes N^* \rightarrow B^* \) and \( r^* : N^* \otimes M \rightarrow B^* \) are given by

\[
l^*(m, f_N)(b) = f_N(r(b, m)) \quad \text{and} \quad r^*(f_N, m)(b) = f_N(l(m, b)),
\]

for \( m \in M, \; f_N \in N^* \) and \( b \in B \).
Proof We only need to check the identities of (6), (7), (8) and (9) for the above dual structures. For any $a \in A, m \in M, f_N \in N^*$ and $b \in B$, we have

\[
I^*(a \cdot_M m, f_N)(b) = f_N(r(b, a \cdot_M m))
\]

\[
\equiv f_N(r(b \cdot_B a, m)) = I^*(m, f_N)(b \cdot_B a) = (a \cdot_B I^*(m, f_N))(b),
\]

\[
l^*(m \cdot_M a, f_N)(b) = f_N(r(b, m \cdot_M a))
\]

\[
\equiv f_N(r(b, m \cdot_N a) = (a \cdot_N f_N)(r(b, m)) = l^*(m, a \cdot_N f_N)(b),
\]

\[
l^*(m, f_N \cdot_N a)(b) = f_N(a \cdot_N r(b, m))
\]

\[
\equiv f_N(r(a \cdot_B b, m)) = l^*(m, f_N)(a \cdot_B b) = (l^*(m, f_N) \cdot_B a)(b).
\]

This shows that the identities in (6) are hold. Similarly, one can verify the identities in (7).

Finally, for any $m \in M$, $f_B \in B^*$ and $n \in N$, we observe that

\[
(R(m) \cdot_N^* (-S^*)(f_B))(n) = (-S^*)(f_B)(n \cdot_N^* R(m))
\]

\[
= -f_B(S(n \cdot_N R(m))
\]

\[
\equiv -(f_B(S(n) \cdot_B R(m)) - S \circ r(S(n), m))
\]

\[
= -(R(m) \cdot_B f_B)(S(n)) + l^*(m, S^*(f_B))(S(n))
\]

\[
= (-S^*)(R(m) \cdot_B f_B + l^*(m, -S^*(f_B)))(n)
\]

and

\[
((-S^*)(f_B) \cdot_N R(m))(n) = (-S^*)(f_B)(R(m) \cdot_N^* n)
\]

\[
= -f_B(S(R(m) \cdot_N^* n))
\]

\[
\equiv f_B(S \circ l(m, S(n)) - R(m) \cdot_B S(n))
\]

\[
= S^*(f_B)(l(m, S(n))) - f_B(R(m) \cdot_B S(n))
\]

\[
= r^*(S^*(f_B), m)(S(n)) - (f_B \cdot_B R(m))(S(n))
\]

\[
= (-S^*)(r^*(-S^*(f_B), m) + f_B \cdot_B R(m))(n).
\]

This shows that (8) and (9) also holds for the above dual structures. Hence $(B^*, \overleftarrow{S^*}, I^*, r^*)$ is a bimodule over the relative Rota-Baxter algebra $M \overrightarrow{R} A$. \qed

Example 3.5 (Coadjoint bimodule) This example is dual to the adjoint bimodule given in Example 3.3. Let $M \overrightarrow{R} A$ be a relative Rota-Baxter algebra. Then $(A^*, \overrightarrow{R^*} M^*, I^*, r^*)$ is a bimodule, where the bilinear maps $l^* : M \otimes M^* \rightarrow A^*$ and $r^* : M^* \otimes M \rightarrow A^*$ are given by

\[
l^*(m, f_M)(a) = f_M(a \cdot_M m)
\]

\[
\text{ and } r^*(f_M, m)(a) = f_M(m \cdot_M a),
\]

for $m \in M, f_M \in M^*, a \in A$.\n
Example 3.6 (Bimodule induced by a morphism) Let $M \overrightarrow{R} A$ and $N \overrightarrow{S} B$ be two relative Rota-Baxter algebras and $(\phi, \psi)$ be a morphism between them (see Definition 2.4). Then $N \overrightarrow{S} B$ can be given a bimodule structure over the relative Rota-Baxter algebra $M \overrightarrow{R} A$, where the $A$-bimodule structures on $B$ and $N$ are respectively given by

\[
a \cdot_B b = \phi(a) \cdot b, \quad b \cdot_B a = b \cdot \phi(a)
\]

\[
\text{ and } a \cdot_N n = \phi(a) \cdot_B n, \quad n \cdot_N a = n \cdot_B \phi(a),
\]

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for \(a \in A, b \in B, n \in N\). Here \(B_N^N\) denote the left and right \(B\)-actions on \(N\). Finally, the bilinear maps \(l : M \otimes B \to N\) and \(r : B \otimes M \to N\) are given by

\[
l(m, b) = \psi(m) \cdot B_N^N b \quad \text{and} \quad r(b, m) = b \cdot B_N^N \psi(m), \quad \text{for } m \in M, b \in B.
\]

**Example 3.7** Let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. A linear map \(d : A \to M\) is said to be a derivation on \(A\) with values in \(M\) if

\[
d(a \cdot b) = a \cdot_M d(b) + d(a) \cdot_M b, \quad \text{for } a, b \in A.
\]

In this case, we call the triple \((A, M, d)\) a relative differential algebra. A bimodule over a relative differential algebra \((A, M, d)\) consists of a quintuple \((B, N, \delta, l, r)\) in which \(B\) and \(N\) are both \(A\)-bimodules, \(l : M \otimes B \to N\) and \(r : B \otimes M \to N\) are bilinear maps satisfying the identities in (6), (7), and \(\delta : B \to N\) is a linear map such that followings are hold:

\[
\delta(a \cdot_B b) = a \cdot_N \delta(b) + l(d(a), b) \quad \text{and} \quad \delta(b \cdot_B a) = r(b, d(a)) + \delta(b) \cdot_N a, \quad \text{for } a \in A, b \in B.
\]

Let \((A, M, d)\) be a relative differential algebra and \((B, N, \delta, l, r)\) be a bimodule over it. Suppose \(\dim M = \dim A\) and \(\dim N = \dim B\), and the maps \(d, \delta\) are invertible. Then it is easy to see that \(M \xrightarrow{d^{-1}} A\) is a relative Rota-Baxter algebra and \((N \xrightarrow{\delta^{-1}} B, l, r)\) is a bimodule over it.

In [27] Uchino observed that a relative Rota-Baxter operator lifts to a Rota-Baxter operator on the semidirect product algebra. More precisely, let \(A\) be an associative algebra and \(M\) be an \(A\)-bimodule. Consider the semidirect product algebra structure on \(A \oplus M\) with the product

\[
(a, m) \cdot (a', m') = (a \cdot a', a \cdot_M m' + m \cdot_M a'), \quad \text{for } (a, m), (a', m') \in A \oplus M.
\]

A linear map \(R : M \to A\) is a relative Rota-Baxter operator if and only if the map \(\widehat{R} : A \oplus M \to A \oplus M\) defined by \(\widehat{R}(a, m) = (R(m), 0)\) is a Rota-Baxter operator on the semidirect product algebra \(A \oplus M\). In the following, we show that a bimodule over a relative Rota-Baxter algebra can also be lifted.

**Proposition 3.8** Let \(M \xrightarrow{R} A\) be a relative Rota-Baxter algebra and \(N \xrightarrow{S} B\) be a 2-term chain complex in which \(B\) and \(N\) are both \(A\)-bimodules. Suppose there are maps \(l : M \otimes B \to N\) and \(r : B \otimes M \to N\) satisfying the conditions of (6), (7). Then \(B \oplus N\) carries an \((A \oplus M)\)-bimodule structure with left and right actions

\[
(a, m) \cdot_{B \oplus N} (b, n) = (a \cdot_B b, a \cdot_N n + l(m, b)),
\]

\[
(b, n) \cdot_{B \oplus N} (a, m) = (b \cdot_B a, r(b, m) + n \cdot_N a).
\]

Moreover, \((N \xrightarrow{S} B, l, r)\) is a bimodule over the relative Rota-Baxter algebra \(M \xrightarrow{R} A\) if and only if the map \(\widehat{S} : B \oplus N \to B \oplus N\) is the map \(\widehat{S}(b, n) = (S(n), 0)\), for \((b, n) \in B \oplus N\).

**Proof** The first part is a straightforward calculation. To prove the second part, we observe that

\[
\widehat{R}(a, m) \cdot_{B \oplus N} \widehat{S}(b, n) = (R(m), 0) \cdot_{B \oplus N} (S(n), 0) = (R(m) \cdot_B S(n), 0).
\]
and
\[
\widehat{S}(\hat{R}(a, m) \cdot_{B \oplus N} (b, n) + (a, m) \cdot_{B \oplus N} \hat{S}(b, n)) = \widehat{S}(\hat{R}(m, 0) \cdot_{B \oplus N} (b, n) + (a, m) \cdot_{B \oplus N} S(n, 0))
\]
\[
= \widehat{S}(R(m) \cdot_{B} b + a \cdot_{B} S(n), R(m) \cdot_{N} n + l(m, S(n)))
\]
\[
= (S(R(m) \cdot_{N} n + l(m, S(n))), 0).
\]
This shows that \( \hat{R}(a, m) \cdot_{B \oplus N} \hat{S}(b, n) = \widehat{S}(\hat{R}(a, m) \cdot_{B \oplus N} (b, n) + (a, m) \cdot_{B \oplus N} \hat{S}(b, n)) \) holds if and only if (8) holds. Similarly, one can verify that
\[
\widehat{S}(b, n) \cdot_{B \oplus N} \hat{R}(a, m) = \widehat{S}(b, n) \cdot_{B \oplus N} (a, m) + (b, n) \cdot_{B \oplus N} \hat{R}(a, m)
\]
holds if and only if (9) holds. This implies that \((B \oplus N, \widehat{S})\) is a Rota-Baxter bimodule over the Rota-Baxter algebra \((A \oplus M, \hat{R})\) if and only if \((N \xrightarrow{\widehat{S}} B, l, r)\) is a bimodule over the relative Rota-Baxter algebra \(M \xrightarrow{R} A\).

Let \(M \xrightarrow{R} A\) be a relative Rota-Baxter algebra and \((N \xrightarrow{S} B, l, r)\) be a bimodule over it. Consider the direct sum \(A \oplus B\) with the semidirect product algebra structure given by
\[
(a, b) \cdot_{\times} (a', b') = (a \cdot a', a \cdot_{B} b' + b \cdot_{B} a'), \quad \text{for } (a, b), (a', b') \in A \oplus B.
\]
We claim that the space \(M \oplus N\) carries a bimodule structure over the associative algebra \(A \oplus B\) with left and right actions
\[(a, b) \triangleright (m, n) = (a \cdot_{M} m, a \cdot_{N} n + r(b, m)) \quad \text{and} \quad (m, n) \triangleleft (a, b) = (m \cdot_{M} a, l(m, b) + n \cdot_{N} a),\]
for \((a, b) \in A \oplus B\) and \((m, n) \in M \oplus N\). To see this, we observe that
\[
((a, b) \cdot_{\times} (a', b')) \triangleright (m, n) - (a, b) \triangleright ((a', b') \triangleright (m, n))
\]
\[
= (a \cdot a', a \cdot_{B} b' + b \cdot_{B} a') \triangleright (m, n) - (a, b) \triangleright (a' \cdot_{M} m, a' \cdot_{N} n + r(b', m))
\]
\[
= (a \cdot a') \cdot_{M} m - a \cdot_{M} (a' \cdot_{M} m), (a \cdot a') \cdot_{N} n + r(a \cdot_{B} b', m) + r(b \cdot_{B} a', m)
\]
\[
- a \cdot_{N} (a' \cdot_{N} n) - a \cdot_{N} r(b', m) - r(b, a' \cdot_{M} m)) \quad (1),(7) \equiv 0.
\]
This proves that \(((a, b) \cdot_{\times} (a', b')) \triangleright (m, n) = (a, b) \triangleright ((a', b') \triangleright (m, n))\). Similarly, we have
\[
((a, b) \triangleright (m, n)) \triangleleft (a', b') - (a, b) \triangleright ((m, n) \triangleleft (a', b'))
\]
\[
= (a \cdot_{M} m, a \cdot_{N} n + r(b, m)) \triangleleft (a', b') - (a, b) \triangleright (m \cdot_{M} a', l(m, b') + n \cdot_{N} a')
\]
\[
= ((a \cdot_{M} m) \cdot_{M} a' - a \cdot_{M} (m \cdot_{M} a'), l(a \cdot_{M} m, b') + (a \cdot_{N} n) \cdot_{N} a' + r(b, m) \cdot_{N} a'
\]
\[
- a \cdot_{N} l(m, b') - a \cdot_{N} (n \cdot_{N} a') - r(b, m \cdot_{M} a')) \quad (1),(6),(7) \equiv 0.
\]
and
\[
((m, n) \triangleleft (a, b)) \triangleleft (a', b') - (m, n) \triangleleft ((a, b) \cdot_{\times} (a', b'))
\]
\[
= (m \cdot_{M} a, l(m, b) + n \cdot_{N} a) \triangleleft (a', b') - (m, n) \triangleleft (a \cdot a', a \cdot_{B} b' + b \cdot_{B} a')
\]
\[
= ((m \cdot_{M} a) \cdot_{M} a' - m \cdot_{M} (a \cdot a'), l(m \cdot_{M} a, b') + l(m, b) \cdot_{N} a' + (n \cdot_{N} a) \cdot_{N} a'
\]
\[
- l(m, a \cdot_{B} b') - l(m, b \cdot_{B} a') - n \cdot_{N} (a \cdot a')) \quad (1),(6) \equiv 0.
\]
Thus, our claim holds. With all these notations, we are now in a position to construct the semidirect product relative Rota-Baxter algebra.
Proposition 3.9 (Semidirect product) Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and $(N \xrightarrow{S} B, l, r)$ be a bimodule over it. Then $M \oplus N \xrightarrow{R \oplus S} A \oplus B$ is a relative Rota-Baxter algebra.

Proof For any $(m, n), (m', n') \in M \oplus N$, we have

\[
(R \oplus S)(m, n) \cdot (R \oplus S)(m', n') = (R(m), S(n)) \cdot (R(m'), S(n'))
\]

\[
= (R(m) \cdot R(m'), R(m) \cdot S(n') + S(n) \cdot R(m'))
\]

\[
(2),(8),(9) 
\]

\[
= (R(R(m) \cdot_M m' \cdot_M R(m')), S(R(m) \cdot_N n' + l(m, S(n'))) + S(r(S(n), m') + n \cdot_N R(m'))
\]

\[
= (R \oplus S)(R(m) \cdot_M m' + m \cdot_M R(m'), R(m) \cdot_N n' + r(S(n), m') + l(m, S(n')) + n \cdot_N R(m'))
\]

\[
= (R \oplus S)((R(m), S(n)) \cdot (m', n') + (m, n) \cdot (R(m'), S(n')))
\]

\[
= (R \oplus S)\big((R \oplus S)(m, n)) \cdot (m', n') + (m, n) \cdot (R \oplus S)(m', n')\big).
\]

This shows that $R \oplus S : M \oplus N \rightarrow A \oplus B$ is a relative Rota-Baxter operator. Hence the result follows.

Relations with Dendriform Algebras Here, we show that a bimodule over a relative Rota-Baxter algebra gives rise to a representation of the induced dendriform algebra. Furthermore, we find a converse of this result. In the end, we discuss the relationship between the cohomology (with coefficients) of a relative Rota-Baxter operator and the cohomology (with coefficients) of the induced dendriform algebra.

The proof of the following result is straightforward, hence we omit the details.

Proposition 3.10 Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and $(N \xrightarrow{S} B, l, r)$ be a bimodule over it. Then $N$ is a representation of the induced dendriform algebra $(M, \prec_R, \succ_R)$ with action maps given by

\[
m \prec n = l(m, S(n)), \quad m \succ n = R(m) \cdot_N n, \quad n \prec m = n \cdot_N R(m) \quad \text{and} \quad n \succ m = r(S(n), m).
\]

Remark 3.11 Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and consider the adjoint bimodule $M \xrightarrow{R} A$ given in Example 3.3. Then the representation of the dendriform algebra $(M, \prec_R, \succ_R)$ on the vector space $M$ given in the above proposition coincides with the adjoint representation.

In Proposition 3.10, we observed that a bimodule over a relative Rota-Baxter algebra gives rise to a representation of the induced dendriform algebra. In the following, we will discuss the converse of this result. Let $(D, \prec, \succ)$ be a dendriform algebra. Consider the total associative algebra $D_{\text{Tot}}$. Then there is a $D_{\text{Tot}}$-bimodule structure on $D$ with left and right actions

\[
x \cdot x := x \succ x \quad \text{and} \quad x \cdot x := x \prec x, \quad \text{for } x \in D_{\text{Tot}}, x \in D.
\]

With this bimodule, it is easy to verify that $D \xrightarrow{\text{id}_D} D_{\text{Tot}}$ is a relative Rota-Baxter algebra. Moreover, the induced dendriform structure on $D$ coincides with the given one.
Next, we take a representation $E$ of the dendriform algebra $(D, <, >)$. Then there are two $D_{\text{Tot}}$-bimodule structures on $E$. The first one is given by

$$x \cdot^1_E e := x < e + e > e \quad \text{and} \quad e \cdot^1_E x = e < x + e > x,$$

the second one is given by

$$x \cdot^2_E e = x > e \quad \text{and} \quad e \cdot^2_E x = e < x,$$

for $x \in D_{\text{Tot}}$ and $e \in E$. We denote the first $D_{\text{Tot}}$-bimodule by $E_{\text{Tot}}$ and the second one simply by $E$. Then it can be checked that $(E \xrightarrow{id_E} E_{\text{Tot}}, l, r)$ is a bimodule over the relative Rota-Baxter algebra $D \xrightarrow{id_D} D_{\text{Tot}}$, where the bilinear maps $l : D \otimes E_{\text{Tot}} \to E$ and $r : E_{\text{Tot}} \otimes D \to E$ are given by

$$l(x, e) = x < e \quad \text{and} \quad r(e, x) = e > x, \quad \text{for } x \in D, e \in E_{\text{Tot}}.$$

Moreover, the induced representation of the dendriform algebra $D$ on the vector space $E$ coincides with the given one.

**Remark 3.12** By lifting the relative Rota-Baxter algebra $D \xrightarrow{id_D} D_{\text{Tot}}$ and its bimodule $(E \xrightarrow{id_E} E_{\text{Tot}}, l, r)$, we obtain the Rota-Baxter algebra $(D_{\text{Tot}} \oplus D, \bar{id}_D)$ and its Rota-Baxter bimodule $(E_{\text{Tot}} \oplus E, \bar{id}_E)$. See Proposition 3.8 for details. It is easy to see that the inclusion $D \hookrightarrow D_{\text{Tot}} \oplus D$ is an embedding of the dendriform algebra $D$ into the Rota-Baxter algebra $(D_{\text{Tot}} \oplus D, \bar{id}_D)$. In other words, the inclusion map $D \hookrightarrow D_{\text{Tot}} \oplus D$ is a morphism of dendriform algebras, where $D_{\text{Tot}} \oplus D$ is equipped with the dendriform algebra induced from the Rota-Baxter operator $\bar{id}_D$. Similarly, the inclusion $E \hookrightarrow E_{\text{Tot}} \oplus E$ is an embedding of the representation $E$ (of the dendriform algebra $D$) into the Rota-Baxter bimodule $(E_{\text{Tot}} \oplus E, \bar{id}_E)$ of the Rota-Baxter algebra $(D_{\text{Tot}} \oplus D, \bar{id}_D)$.

In the following, we find some cohomological relations with dendriform algebras. Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra. Then we have seen in Proposition 2.8 that $M_{\text{Tot}} = (M, *_R)$ is an associative algebra, where

$$m *_R m' = R(m) \cdot_M m' + m \cdot_M R(m'), \quad \text{for } m, m' \in M.$$

Let $(N \xrightarrow{S} B, l, r)$ be a bimodule over the relative Rota-Baxter algebra $M \xrightarrow{R} A$. We define bilinear maps $>_{B} : M \otimes B \to B$ and $<_{B} : B \otimes M \to B$ by

$$m >_{B} b = R(m) \cdot_B b - S(l(m, b)) \quad \text{and} \quad b <_{B} m = b \cdot_B R(m) - S(r(b, m)), \quad \text{for } m \in M, b \in B.$$

Then we have the following.

**Proposition 3.13** With the above notations, $B$ is a bimodule over the associative algebra $M_{\text{Tot}}$ (We denote this bimodule by $B_{>,<}$).

**Proof** For any $m, m' \in M$ and $b \in B$, we have

$$\begin{align*}
(m *_R m') >_{B} b - m >_{B} (m' >_{B} b) &= R(m *_R m') \cdot_B b - S(l(m *_R m', b)) - R(m) \cdot_B (m' >_{B} b) + S(l(m, m' >_{B} b)) \\
&= (R(m) \cdot R(m')) \cdot_B b - S(l(R(m) \cdot_M m', b)) - S(l(m \cdot_M R(m'), b)) \\
&\quad - R(m) \cdot_B (R(m') \cdot_B b) + R(m) \cdot_B S(l(m', b)) + S(l(m, R(m') \cdot_B b)) - S(l(m, S \circ l(m', b))) \\
&= -S(l(R(m) \cdot_M m', b)) + S(R(m) \cdot_M l(m', b)) + l(m, S \circ l(m', b)) - S(l(m, S \circ l(m', b))) = 0.
\end{align*}$$
Similarly,
\[(m \rightharpoonup_B b) \triangleleft_B m' - m \rightharpoonup_R (b \triangleleft_B m')\]
\[= (m \rightharpoonup_B b) \cdot_B R(m') - S(r(m \rightharpoonup_B b, m')) - R(m) \cdot_B (b \triangleleft_B m') + S(l(m, b \triangleleft_B m'))\]
\[= (R(m) \cdot_B b) \cdot_B R(m') - S(l(m, b)) \cdot_B R(m') - S(r(R(m) \cdot_B b, m')) + S(r(S \circ l(m, b), m'))\]
\[\quad - R(m) \cdot_B (b \cdot_B R(m')) + R(m) \cdot_B S(r(b, m')) + S(l(m, b \cdot_B R(m'))) - S(l(m, S \circ r(b, m')))\]
\[\quad + S(R(m) \cdot_N r(b, m') + l(m, S \circ r(b, m'))) + S(l(m, b \cdot_B R(m'))) - S(l(m, S \circ r(b, m'))) = 0\]
and
\[\quad (b \triangleleft_B m) \triangleleft_B m' - b \triangleleft_B (m \star_R m')\]
\[= (b \triangleleft_B m) \cdot_B R(m') - S(r(b \triangleleft_B m, m')) - b \cdot_B R(m \star_R m') + S(r(b, m \star_R m'))\]
\[= (b \cdot_B R(m')) \cdot_B R(m') - S(r(b, m')) \cdot_B R(m') - S(r(b \cdot_B R(m), m')) + S(r(S \circ r(b, m), m'))\]
\[\quad - b \cdot_B (R(m) \cdot_B m') + S(r(b, R(m) \cdot_M m')) + S(r(b, m \cdot_M R(m')))\]
\[= -S(r(S \circ r(b, m), m') + r(b, m') \cdot_N R(m')) + S(r(S \circ r(b, m), m')) + S(r(b, m \cdot_M R(m'))) = 0.\]

This proves the result. \(\square\)

It follows from the above proposition that we may consider the Hochschild cohomology of the associative algebra \(M_{\text{Tot}}\) with coefficients in the bimodule \(B_{\triangleleft, \triangleleft}\). More precisely, we define the cochain complex \(\{C^*_{\text{H}}(M_{\text{Tot}}, B_{\triangleleft, \triangleleft}), \delta_{M, B}\}\), where \(C^k_{\text{H}}(M_{\text{Tot}}, B_{\triangleleft, \triangleleft}) = \text{Hom}(M^{\otimes k}, B)\), for \(k \geq 0\), and \(\delta_{M, B} : C^0_{\text{H}}(M_{\text{Tot}}, B_{\triangleleft, \triangleleft}) \to C^1_{\text{H}}(M_{\text{Tot}}, B_{\triangleleft, \triangleleft})\) given by
\[
(\delta_{M, B} f)(m_1, \ldots, m_{k+1})
= m_1 \rightharpoonup_B f(m_2, \ldots, m_{k+1}) + \sum_{i=1}^k (-1)^i f(m_1, \ldots, m_i \star_R m_{i+1}, \ldots, m_{k+1})
+ (-1)^{k+1} f(m_1, \ldots, m_k) \triangleleft_B m_{k+1}
= R(m_1) \cdot_B f(m_2, \ldots, m_{k+1}) - S(l(m_1, f(m_2, \ldots, m_{k+1})))
+ \sum_{i=1}^k (-1)^i f(m_1, \ldots, R(m_i) \cdot_M m_{i+1}, \ldots, m_{k+1})
+ \sum_{i=1}^k (-1)^i f(m_1, \ldots, m_i \cdot_M R(m_{i+1}), \ldots, m_{k+1})
+ (-1)^{k+1} f(m_1, \ldots, m_k) \cdot_B R(m_{k+1}) - S(r(f(m_1, \ldots, m_k), m_{k+1})).
\]

We denote the corresponding cohomology groups by \(H^*_{\text{H}}(M_{\text{Tot}}, B_{\triangleleft, \triangleleft})\).

Remark 3.14 Let \(M \xrightarrow{R} A\) be a relative Rota-Baxter algebra and consider the adjoint bimodule \(M \xrightarrow{R} A\) (see Example 3.3). Then it follows from the previous discussion that the vector space \(A\) carries a bimodule structure over the associative algebra \(M_{\text{Tot}}\). This bimodule has considered first by Uchino [27] and further studied in [9]. The corresponding Hochschild cohomology groups are called the cohomology of the relative Rota-Baxter operator \(R\).
Given a relative Rota-Baxter algebra $M \xrightarrow{R} A$, in [9] the author finds a connection between the cohomology of the relative Rota-Baxter operator $R$ and the cohomology of the induced dendriform algebra $(M, \prec_R, \succ_R)$ with coefficients in the adjoint representation. We will extend this result in the context of relative Rota-Baxter algebras equipped with bimodules.

We first recall the cohomology of a dendriform algebra with coefficients in a representation [8]. Let $(D, \prec, \succ)$ be a dendriform algebra. Consider the associative algebra $D_{\text{Tot}} \oplus D$ given in Remark 3.12. Note that the associative multiplication on $D_{\text{Tot}} \oplus D$ is given by

$$(x, x) \cdot (y, y) = (x \ast y, x \succ y + x \prec y), \text{ for } (x, x), (y, y) \in D_{\text{Tot}} \oplus D.$$  

If $E$ is a representation of the dendriform algebra $D$, then $E_{\text{Tot}} \oplus E$ is a bimodule over the associative algebra $D_{\text{Tot}} \oplus D$ with left and right actions given by

$$(x, x) \cdot (e, e) = (x \ast e + x \succ e, x \succ e + x \prec e) \text{ and } (e, e) \cdot (x, x) = (e \ast x + e \succ x, e \succ x + e \prec x),$$  

for $(x, x) \in D_{\text{Tot}} \oplus D$ and $(e, e) \in E_{\text{Tot}} \oplus E$. Therefore, we may define the Hochschild cochain complex $\{C^*_H(D_{\text{Tot}} \oplus D, E_{\text{Tot}} \oplus E), \delta_{D_{\text{Tot}} \oplus D, E_{\text{Tot}} \oplus E}\}$. Using this complex, we will now define the cochain complex of the dendriform algebra $D$ with coefficients in the representation $E$. We first need the following notations. For each $k \geq 1$, let $C_k$ be the set of first $k$ natural numbers. For convenience, we write $C_k = \{[1], [2], \ldots, [k]\}$. We define the $k$-th cochain group as

$$C^k_D(D, E) := \text{Hom}(k[C_k] \otimes D^\otimes k, E), \text{ for } k \geq 1.$$  

For any $f \in C^k_D(D, E)$, there is an element $\widehat{f} \in C^k_H(D_{\text{Tot}} \oplus D, E_{\text{Tot}} \oplus E)$ given by

$$\widehat{f}((x_1, 0), \ldots, (x_k, 0)) = \left( \sum_{i=1}^k f([i]; x_1, \ldots, x_k), 0 \right),$$  

$$\widehat{f}((x_1, 0), \ldots, (0, x_i), \ldots, (x_k, 0)) = (0, f([i]; x_1, \ldots, x_k)),$$

$$\widehat{f}((x_1, 0), \ldots, (0, x_i), \ldots, (0, x_j), \ldots, (x_k, 0)) = (0, 0).$$  

Note that $f$ can be obtained from $\widehat{f}$ by the following

$$f([i]; x_1, \ldots, x_k) = \text{pr}_2 \circ \widehat{f}((x_1, 0), \ldots, (0, x_i), \ldots, (x_k, 0)), \text{ for } [i] \in C_k,$$

where $\text{pr}_2 : E_{\text{Tot}} \oplus E \to E$ is the projection onto the second factor. We define a map $\delta_D : C^k_D(D, E) \to C^{k+1}_D(D, E)$ implicitly by the following formula

$$\overline{\delta_D(f)} = \delta_{D_{\text{Tot}} \oplus D, E_{\text{Tot}} \oplus E}(\widehat{f}), \text{ for } f \in C^k_D(D, E).$$

The explicit formula of the differential $\delta_D$ can be found in [8]. The cohomology groups of the cochain complex $\{C^*_H(D, E), \delta_D\}$ are called the cohomology of the dendriform algebra $D$ with coefficients in the representation $E$.

Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and $(N \xrightarrow{S} B, l, r)$ be a bimodule over it. We consider the cochain complex $\{C^*_H(M_{\text{Tot}}, B_{\otimes}), \delta_M, B\}$. On the other hand, we know from Proposition 3.10 that $N$ is a representation of the dendriform algebra $(M, \prec_R, \succ_R)$. Hence we may consider the dendriform cochain complex $\{C^*_D(M, N), \delta_D\}$.  

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For each \( k \geq 1 \), we define a map \( \psi_k : C^k_H(M_{\text{Tot}}, B_{\varphi, \circ}) \to C^{k+1}_D(M, N) \) by
\[
\psi_k(f)([r]; m_1, \ldots, m_{k+1}) = \begin{cases} 
( -1 )^{k+1} l( m_1, f( m_2, \ldots, m_{k+1} ) ) & \text{if } r = 1 \\
0 & \text{if } 2 \leq r \leq k \\
r( f( m_1, \ldots, m_k ), m_{k+1} ) & \text{if } r = k + 1.
\end{cases}
\]

**Proposition 3.15** With the above notations, we have
\[
\delta_D \circ \psi_k = \psi_{k+1} \circ \delta_{M, B}.
\]
Hence the collection \( \{ \psi_k \}_{k \geq 1} \) of maps induces a map \( \psi_* : H^*_{\text{H}}(M_{\text{Tot}}, B_{\varphi, \circ}) \to H^{*+1}_D(M, N) \) on cohomology.

**Proof** We first consider the semidirect product relative Rota-Baxter algebra \( M \oplus N \xrightarrow{R \oplus S} A \oplus B \) given in Proposition 3.9. Hence it follows from Proposition 2.8 that the space \( M \oplus N \) carries a dendriform algebra structure with products
\[
(m, n) \prec_{R \oplus S} (m', n') = (m \cdot_M R(m'), l(m, S(n')) + n \cdot_N R(m')), \\
(m, n) \succ_{R \oplus S} (m', n') = (R(m) \cdot_M m', R(m) \cdot_N n' + r(S(n), m')).
\]
The corresponding total associative multiplication is given by
\[
(m, n) \ast_{R \oplus S} (m', n') = (m \ast_R m', l(m, S(n')) + n \cdot_N R(m') + R(m) \cdot_N n' + r(S(n), m')).
\]
Moreover, \( A \oplus B \) is a bimodule over this associative algebra with left and right actions
\[
(m, n) \triangleright_{A \oplus B} (a, b) = (R(m) \cdot a - R(m \cdot M a), R(m) \cdot_B b + S(n) \cdot_B a - S(l(m, b) + n \cdot_N a)), \\
(a, b) \triangleleft_{A \oplus B} (m, n) = (a \cdot_R m - R(a \cdot_M m), a \cdot_B S(n) + b \cdot_B R(m) - S(a \cdot_N n + r(b, m))).
\]
It follows from [9, Proposition 3.5] that the collection \( \{ \Psi_k \}_{k \geq 1} \) of maps
\[
\Psi_k : C^k_H(M \oplus N, A \oplus B) \to C^{k+1}_D(M \oplus N, M \oplus N)
\]
\[
\Psi_k(F)([r]; (m_1, n_1), \ldots, (m_{k+1}, n_{k+1})) = \begin{cases} 
( -1 )^{k+1} ( m_1, n_1 ) \triangleright_F ( m_2, n_2, \ldots, (m_{k+1}, n_{k+1}) ) & \text{if } r = 1 \\
0 & \text{if } 2 \leq r \leq k \\
F((m_1, n_1), \ldots, (m_k, n_k)) \triangleleft_{(m_{k+1}, n_{k+1})} & \text{if } r = k + 1
\end{cases}
\]
defines a morphism of cochain complexes from \( \{ C^*_{\text{H}}(M \oplus N, A \oplus B), \delta_{M \oplus N, A \oplus B} \} \) to \( \{ C^{*+1}_D(M \oplus N, M \oplus N), \delta_D \} \). Note that \( \{ C^*_{\text{H}}(M_{\text{Tot}}, B_{\varphi, \circ}), \delta_{M, B} \} \) is a subcomplex of \( \{ C^*_{\text{H}}(M \oplus N, A \oplus B), \delta_{M \oplus N, A \oplus B} \} \), and \( \{ C^{*+1}_D(M, N), \delta_D \} \) is a subcomplex of \( \{ C^{*+1}_D(M \oplus N, M \oplus N), \delta_D \} \). Finally, the result follows as the restriction of \( \Psi_k \) to the subspace \( C^k_H(M_{\text{Tot}}, B_{\varphi, \circ}) \) is given by \( \psi_k \), for all \( k \geq 1 \). This proves the result. \( \square \)

### 4 Cohomology of Relative Rota-Baxter Algebras with Coefficients in Bimodule

In this section, we define the cohomology of a relative Rota-Baxter algebra with coefficients in a bimodule. This cohomology generalizes the cohomology of relative Rota-Baxter algebras (with coefficients in the adjoint bimodule) defined in [10].
We first recall the cochain complex of a relative Rota-Baxter algebra $M \xrightarrow{R} A$. Let $\mathcal{A}^{k,1}_{k-1,1}$ be the direct sum of all possible $k$ tensor powers of $A$ and $M$ in which $A$ appears $k - 1$ times (hence $M$ appears exactly once). For instance,

$\mathcal{A}^{0,1} = M, \quad \mathcal{A}^{1,1} = (A \otimes M) \oplus (M \otimes A), \quad \mathcal{A}^{2,1} = (A \otimes A \otimes M) \oplus (A \otimes M \otimes A) \oplus (M \otimes A \otimes A)$.

For each $k \geq 0$, the space of $k$-cochains $C^k(A, M, R)$ is given by

$C^k(A, M, R) = \begin{cases} 0 & \text{if } k = 0, \\ \text{Hom}(A, A) \oplus \text{Hom}(M, M) & \text{if } k = 1, \\ \text{Hom}(A \otimes^k, A) \oplus \text{Hom}(\mathcal{A}^{k-1,1}, M) \oplus \text{Hom}(M \otimes^{k-1}, A) & \text{if } k \geq 2. \end{cases}$

The coboundary operator $D : C^k(A, M, R) \rightarrow C^{k+1}(A, M, R)$ is given by

$D(\alpha, \beta) = (\delta_A(\alpha), \delta^A_{A,M}(\beta), h_R(\alpha, \beta))$, for $(\alpha, \beta) \in C^1(A, M, R) = \text{Hom}(A, A) \oplus \text{Hom}(M, M)$,

$D(\alpha, \beta, \gamma) = (\delta_A(\alpha), \delta^A_{A,M}(\beta), \delta_{M,A}(\gamma) + h_R(\alpha, \beta))$, for $(\alpha, \beta, \gamma) \in C^{k+2}(A, M, R)$.

Here $\delta_A$ is the Hochschild coboundary operator of the associative algebra $A$ with coefficients in the adjoint bimodule, and for any $\alpha \in \text{Hom}(A \otimes^k, A)$, the map $\delta^A_{A,M} : \text{Hom}(\mathcal{A}^{k-1,1}, M) \rightarrow \text{Hom}(\mathcal{A}^{k-1,1}, M)$ is given by

$\delta^A_{A,M}(\beta)(a_1, \ldots, a_{k+1}) = (\mu + l_M)(a_1, (\alpha + \beta)(a_2, \ldots, a_{k+1}))$

$+ \sum_{i=1}^k (-1)^i \beta(a_1, \ldots, (\mu + l_M + r_M)(a_i, a_{i+1}), \ldots, a_{k+1})$

$+ (-1)^{k+1} (\mu + r_M)((\alpha + \beta)(a_1, \ldots, a_k), a_{k+1}),$

for $\beta \in \text{Hom}(\mathcal{A}^{k-1,1}, M)$ and $a_1, \ldots, a_s, \ldots, a_{k+1} \in A$, $a_s \in M$ ($1 \leq s \leq k + 1$). The map $\delta_{M,A}$ is the Hochschild coboundary operator of the associative algebra $M_{\text{Tot}}$ with coefficients in the $M_{\text{Tot}}$-bimodule $A$ considered in Remark 3.14, and the map $h_R : \text{Hom}(A \otimes^k, A) \oplus \text{Hom}(\mathcal{A}^{k-1,1}, M) \rightarrow \text{Hom}(M \otimes^k, A)$ is given by

$h_R(\alpha, \beta)(m_1, \ldots, m_k) = (-1)^k \{\alpha(R(m_1), \ldots, R(m_k)) - \sum_{i=1}^k R \circ \beta(R(m_1), \ldots, m_i, \ldots, R(m_k))\}$.

It has been shown in [10] that $\{C^*(A, M, R), D\}$ is a cochain complex. The corresponding cohomology groups are denoted by $H^*(A, M, R)$, and called the cohomology of the relative Rota-Baxter algebra $M \xrightarrow{R} A$.

In the same reference, the authors also considered formal deformations of a relative Rota-Baxter algebra $M \xrightarrow{R} A$ and showed that the above cohomology govern such deformations.

**Cohomology of a Relative Rota-Baxter Algebra with Coefficients in a Bimodule** Here we define the cochain complex of a relative Rota-Baxter algebra with coefficients in a bimodule. Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and $(N \xrightarrow{S} B, l, r)$ be a bimodule over it. We fix the following notations.

1. Let $\delta_{A,B} : \text{Hom}(A \otimes^k, B) \rightarrow \text{Hom}(A \otimes^{k+1}, B)$, for $k \geq 1$, be the Hochschild coboundary operator of the algebra $A$ with coefficients in the $A$-bimodule $B$.  

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It can be easily verified that the map \( \delta \) is the restriction of the map \( \delta_{A,N}^k \) to the subcomplex of the cochain complex \( M^{\otimes k} \rightarrow A; N \rightarrow B \).

We are now in a position to define the cohomology of the relative Rota-Baxter algebra \( M \rightarrow A \) with coefficients in the bimodule \( (N \rightarrow B, l, r) \). For each \( k \geq 0 \), we define \( C_{krb}^k(M \rightarrow A; N \rightarrow B) \) by

\[
C_{krb}^k(M \rightarrow A; N \rightarrow B) = \begin{cases} 
0 & \text{if } k = 0, \\
\text{Hom}(A, B) \oplus \text{Hom}(M, N) & \text{if } k = 1, \\
\text{Hom}(A^{\otimes k}, B) \oplus \text{Hom}(A^{k-1,1}, N) \oplus \text{Hom}(M^{\otimes k-1}, B) & \text{if } k \geq 2.
\end{cases}
\]

Define a map \( \delta_{rb} : C_{krb}^k(M \rightarrow A; N \rightarrow B) \rightarrow C_{krb}^{k+1}(M \rightarrow A; N \rightarrow B) \) by

\[
\delta_{rb}(\alpha, \beta) = (\delta_{A,B}(\alpha), \delta_{A,N}^k(\beta), h_R(\alpha, \beta)), \quad \text{for } (\alpha, \beta) \in C_{krb}^k(M \rightarrow A; N \rightarrow B),
\]

\[
\delta_{rb}(\alpha, \beta, \gamma) = (\delta_{A,B}(\alpha), \delta_{A,N}^k(\beta), \delta_{M,B}(\gamma) + h_R(\alpha, \beta)), \quad \text{for } (\alpha, \beta, \gamma) \in C_{krb}^{k+2}(M \rightarrow A; N \rightarrow B).
\]

**Proposition 4.1** With all these notations, we have \( (\delta_{rb})^2 = 0 \).

**Proof** If \( M \rightarrow A \) is a relative Rota-Baxter algebra and \( (N \rightarrow B, l, r) \) is a bimodule over it, then by Proposition 3.9 we have the semidirect product relative Rota-Baxter algebra \( M \oplus N \rightarrow A \oplus B \). Let \( \mathcal{C}^*(A \oplus B, M \oplus N, R \oplus S, \mathcal{D}) \) be the cochain complex of this semidirect product relative Rota-Baxter algebra (with coefficients in the adjoint bimodule). Then there is an obvious inclusion

\[ C_{krb}^k(M \rightarrow A; N \rightarrow B) \hookrightarrow C^k(A \oplus B, M \oplus N, R \oplus S), \quad \text{for } k \geq 1. \]

It can be easily verified that the map \( \delta_{rb} \) is the restriction of the map \( \delta \) on the collection of subspaces \( C_{krb}^k(M \rightarrow A; N \rightarrow B) \). In other words, \( \{C_{krb}^k(M \rightarrow A; N \rightarrow B), \delta_{rb}\} \) is a subcomplex of the cochain complex \( \mathcal{C}^*(A \oplus B, M \oplus N, R \oplus S, \mathcal{D}) \), which implies that \( (\delta_{rb})^2 = 0 \). \( \square \)
It follows from the above proposition that \( \{ C^\bullet \text{RB}(M \xrightarrow{R} A; N \xrightarrow{S} B), \delta_{\text{RB}} \} \) is a cochain complex. Let \( Z^k_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B) \) and \( B^k_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B) \) be the space of \( k \)-cochains and \( k \)-coboundaries, respectively. The corresponding quotients

\[
H^k_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B) := \frac{Z^k_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B)}{B^k_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B)}, \text{ for } k \geq 0
\]

are called the cohomology of the relative Rota-Baxter algebra \( M \xrightarrow{R} A \) with coefficients in the bimodule \( (N \xrightarrow{S} B, l, r) \).

Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra and \( (N \xrightarrow{S} B, l, r) \) be a bimodule over it. A pair \((\alpha, \beta) \in \text{Hom}(A, B) \oplus \text{Hom}(M, N) = C^1_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B)\) is said to be a derivation on the relative Rota-Baxter algebra \( M \xrightarrow{R} A \) with values in the bimodule \( (N \xrightarrow{S} B, l, r) \) if the following identities are held:

\[
\begin{align*}
\alpha(a \cdot a') &= \alpha(a) \cdot_B a' + a \cdot_B \alpha(a'), \\
\beta(a \cdot_M m) &= r(\alpha(a), m) + a \cdot_N \beta(m), \\
\beta(m \cdot_M a) &= \beta(m) \cdot_N a + l(m, \alpha(a)), \\
\alpha(R(m)) &= S(\beta(m)),
\end{align*}
\]

for \( a, a' \in A \) and \( m \in M \). The set of all derivations are denoted by \( \text{Der}(M \xrightarrow{R} A; N \xrightarrow{S} B) \).

It follows from the definition that \( Z^1_{\text{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B) = \text{Der}(M \xrightarrow{R} A; N \xrightarrow{S} B) \).

**Remark 4.2** Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra and consider it as a adjoint bimodule over itself (Example 3.3). Then the cochain complex \( \{ C^\bullet(M \xrightarrow{R} A; M \xrightarrow{R} A), \delta_{\text{RB}} \} \) coincides with the complex \( \{ C^\bullet(A, M, R), D \} \) given in [10]. Hence the corresponding cohomology groups are the same.

**Cohomology of a Rota-Baxter algebra with coefficients in a Rota-Baxter bimodule**

Let \((A, R)\) be a Rota-Baxter algebra and \((M, R_M)\) be a Rota-Baxter bimodule. In Example 3.2, we observed that the Rota-Baxter algebra \((A, R)\) can be considered as a relative Rota-Baxter algebra \( A \xrightarrow{R} A \) and the Rota-Baxter bimodule \((M, R_M)\) can be considered as a bimodule \( M \xrightarrow{R_M} M \) over the relative Rota-Baxter algebra \( A \xrightarrow{R} A \). Consider the cochain complex \( \{ C^\bullet_{\text{RB}}(A \xrightarrow{R} A; M \xrightarrow{R_M} M), \delta_{\text{RB}} \} \) of the relative Rota-Baxter algebra \( A \xrightarrow{R} A \) with coefficients in the bimodule \( M \xrightarrow{R_M} M \).

For each \( k \geq 0 \), we define \( C^k_{\text{RB}}(A, R; M, R_M) \) by

\[
C^k_{\text{RB}}(A, R; M, R_M) = \begin{cases} 
0 & \text{if } k = 0, \\
\text{Hom}(A, M) & \text{if } k = 1, \\
\text{Hom}(A \otimes^k M) \oplus \text{Hom}(A \otimes^{k-1} M) & \text{if } k \geq 2.
\end{cases}
\]

Then there is an embedding \( i : C^k_{\text{RB}}(A, R; M, R_M) \to C^k_{\text{RB}}(A \xrightarrow{R} A; M \xrightarrow{R_M} M) \) given by

\[
i(\alpha) = (\alpha, \alpha) \quad \text{and} \quad i(\beta, \gamma) = (\beta, \beta, \gamma), \quad \text{for } \alpha \in \text{Hom}(A, M) \text{ and } (\beta, \gamma) \in C^{k+2}_{\text{RB}}(A, R; M, R_M).
\]

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It is easy to see that $\delta_{\text{RB}}(\text{im}(i)) \subset \text{im}(i)$. Hence the differential $\delta_{\text{RB}}$ restricts to a differential (which we denote by $\delta_{\text{RB}}$) on $C^\bullet_{\text{RB}}(A, R; M, R_M)$. In other words, the complex $[C^\bullet_{\text{RB}}(A, R; M, R_M), \delta_{\text{RB}}]$ is a subcomplex of $[C^\bullet_{\text{RB}}(A \to A; M \to M), \delta_{\text{RB}}]$. The cohomology groups of $[C^\bullet_{\text{RB}}(A, R; M, R_M), \delta_{\text{RB}}]$ are called the cohomology of the Rota-Baxter algebra $(A, R)$ with coefficients in the Rota-Baxter bimodule $(M, R_M)$. Note that the above cochain complex coincides with the one given in [28]. Hence the cohomologies are also same.

5 Abelian Extensions of Relative Rota-Baxter Algebras

In this section, we study abelian extensions of a relative Rota-Baxter algebra by a bimodule. Our main result classifies isomorphism classes of such abelian extensions by the second cohomology group of the relative Rota-Baxter algebra.

Let $M \to A$ be a relative Rota-Baxter algebra and $N \to B$ be a 2-term chain complex. Note that $N \to B$ can be regarded as a relative Rota-Baxter algebra with trivial associative multiplication on $B$ and its trivial bimodule structure on $N$.

**Definition 5.1** An abelian extension of a relative Rota-Baxter algebra $M \to A$ by a 2-term cochain complex $N \to B$ is a short exact sequence of relative Rota-Baxter algebras

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{\bar{p}} M \longrightarrow 0$$

Let (11) be an abelian extension of the relative Rota-Baxter algebra $M \to A$ by a 2-term cochain complex $N \to B$. A section of (11) is a pair $(s, \bar{s})$ of linear maps $s : A \to \hat{A}$ and $\bar{s} : M \to \hat{M}$ satisfying $p \circ s = \text{id}_A$ and $\bar{p} \circ \bar{s} = \text{id}_M$. Note that a section always exists.

Let $(s, \bar{s})$ be a section of (11). We define maps $l_B : A \otimes B \to B$, $(a, b) \mapsto a \cdot_B b$ and $r_B : B \otimes A \to B$, $(b, a) \mapsto b \cdot_B a$ by

$$a \cdot_B b = s(a) \cdot_{\hat{A}} i(b) \quad \text{and} \quad b \cdot_B a = i(b) \cdot_{\hat{A}} s(a).$$

Here $\cdot_{\hat{A}}$ denotes the multiplication on the associative algebra $\hat{A}$. It is easy to see that the maps $l_B, r_B$ defines an $A$-bimodule structure on $B$. Similarly, we define maps $l_N : A \otimes N \to N$, $(a, n) \mapsto a \cdot_N n$ and $r_N : N \otimes A \to N$, $(n, a) \mapsto n \cdot_N a$ by

$$a \cdot_N n = s(a) \cdot_{\hat{M}} \bar{i}(n) \quad \text{and} \quad n \cdot_N a = \bar{i}(n) \cdot_{\hat{M}} s(a).$$

Here $\cdot_{\hat{M}}$ denotes both left and right $\hat{A}$-actions on $\hat{M}$. The maps $l_N, r_N$ defines an $A$-bimodule structure on $N$. We also define maps $l : M \otimes B \to N$ and $r : B \otimes M \to N$ by

$$l(m, b) = \bar{s}(m) \cdot_{\hat{M}} i(b) \quad \text{and} \quad r(b, m) = i(b) \cdot_{\hat{M}} \bar{s}(m).$$
It simply follows that the maps $l$ and $r$ satisfy the conditions of (6) and (7). Finally, we observe that
\[
R(m) \cdot_B S(n) = sR(m) \cdot_{\hat{A}} iS(n) = \hat{R}(\hat{S}(m)) + M \cdot_{\hat{A}} \hat{R}(\hat{i}(n)) = \hat{R}(sR(m) \cdot_{\hat{M}} \hat{i}(n) + \hat{\sigma}(m) \cdot_{\hat{M}} \hat{R}(\hat{i}(n))) = \hat{R}(sR(m) \cdot_{\hat{M}} \hat{i}(n) + \hat{\sigma}(m) \cdot_{\hat{M}} iS(n)) = \hat{R}(sR(m) \cdot_{\hat{M}} n + i(m, S(n))).
\]

Thus, $(N \xrightarrow{S} B, l, r)$ defines a bimodule over the relative Rota-Baxter algebra.

Next, we claim that this bimodule is independent of the choice of the section $(s, \hat{\sigma})$ of (11). To prove this, take another section $(s', \hat{\sigma}')$ of (11). We first observe that $s(a) - s'(a) \in \ker p = \im i$ and $\hat{\sigma}(m) - \hat{\sigma}'(m) \in \ker p = \im i$, for $a \in A$, $m \in M$.

Hence we have
\[
a \cdot_B b - a' \cdot_B b = (s(a) - s'(a)) \cdot_{\hat{A}} i(b) = 0 \quad \text{and} \quad b \cdot_B a - b' \cdot_B a = i(b) \cdot_{\hat{A}} (s(a) - s'(a)) = 0.
\]

Here $'_{B}$ denotes the $A$-bimodule structure on $B$ induced by the section $(s', \hat{\sigma}')$. It follows that the $A$-bimodule structure on $B$ does not depend on the choice of section. Similarly,
\[
a \cdot_{\hat{N}} n - a' \cdot_{\hat{N}} n = (s(a) - s'(a)) \cdot_{\hat{M}} \hat{i}(n) = 0 \quad \text{and} \quad n \cdot_{\hat{N}} a - n' \cdot_{\hat{N}} a = \hat{i}(n) \cdot_{\hat{M}} (s(a) - s'(a)) = 0
\]

which shows that the $A$-bimodule structure on $N$ is independent of the choice of section.

Finally, if $l': M \otimes B \to N$ and $r': B \otimes M \to N$ denote the bilinear maps induced by the section $(s', \hat{\sigma}')$ then
\[
l(m, b) - l'(m, b) = (s(m) - s'(m)) \cdot_{\hat{M}} i(b) = 0 \quad \text{and} \quad r(b, m) - r'(b, m) = i(b) \cdot_{\hat{M}} (s(m) - s'(m)) = 0.
\]

Hence the bilinear maps $l : M \otimes B \to N$ and $r : B \otimes M \to N$ are also independent of the choice of section, which proves our claim.

**Definition 5.2** Let $\hat{M} \xrightarrow{R} \hat{A}$ and $\hat{M}' \xrightarrow{R} \hat{A}'$ be two abelian extensions of the relative Rota-Baxter algebra $M \xrightarrow{R} A$ by the 2-term chain complex $N \xrightarrow{S} B$. They are said to be **isomorphic** if there is an isomorphism $(\phi, \psi)$ of relative Rota-Baxter algebras from $\hat{M} \xrightarrow{R} \hat{A}$ to $\hat{M}' \xrightarrow{R} \hat{A}'$ making the following diagram commutative
\[
\begin{array}{ccccccccc}
0 & \to & N & \xrightarrow{\psi} & \hat{M} & \xrightarrow{\phi} & \hat{A} & \xrightarrow{0} \\
& & N & \xrightarrow{\psi} & \hat{M}' & \xrightarrow{\phi} & \hat{A}' & \xrightarrow{0} \\
0 & \to & B & \xrightarrow{0} & \hat{A} & \xrightarrow{0} & \hat{A}' & \xrightarrow{0} \\
\end{array}
\]

Let $M \xrightarrow{R} A$ be a relative Rota-Baxter algebra and $(N \xrightarrow{S} B, l, r)$ be a bimodule over it. We denote by $\text{Ext}(M \xrightarrow{R} A; N \xrightarrow{S} B)$ the set of isomorphism classes of abelian extensions.
of the relative Rota-Baxter algebra \( M \xrightarrow{R} A \) by the 2-term chain complex \( N \xrightarrow{S} B \) so that the induced bimodule structure on \( N \xrightarrow{S} B \) is the prescribed one.

With these notations, we have the following.

**Theorem 5.3** Let \( M \xrightarrow{R} A \) be a relative Rota-Baxter algebra and \( (N \xrightarrow{S} B, l, r) \) be a bimodule over it. Then there is a one-to-one correspondence between \( \text{Ext}(M \xrightarrow{R} A; N \xrightarrow{S} B) \) and the second cohomology group \( H^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \).

**Proof** Let \((11)\) be an abelian extension of \( M \xrightarrow{R} A \) by the 2-term chain complex \( N \xrightarrow{S} B \). For any section \((s, \overline{s})\), we define maps

\[
\alpha \in \text{Hom}(A^{\otimes 2}, B), \quad \beta \in \text{Hom}(A^1, N), \quad \gamma \in \text{Hom}(M, B),
\]

\[
\alpha(a, a') := s(a) \cdot \overline{s}(a') - s(a \cdot a'),
\]

\[
\beta(a, m) := s(a) \cdot \overline{M}s(m) - \overline{s}(a \cdot M m),
\]

\[
\gamma(m) := \overline{R}(\overline{s}(m)) - s(R(m)).
\]

By a straightforward calculation, it follows that \((\alpha, \beta, \gamma)\) is a 2-cocycle in \( Z^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \). Hence the abelian extension \((11)\) corresponds to a cohomology class in \( H^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \). Moreover, the cohomology class does not depend on the choice of section.

Next, let \( \widehat{M} \xrightarrow{\widehat{R}} \widehat{A} \) and \( \widehat{M}' \xrightarrow{\widehat{R}'} \widehat{A}' \) be two isomorphic abelian extensions and the isomorphism is given by \((\phi, \psi)\) (see Definition 5.2). Let \((s, \overline{s})\) be a section of the first abelian extension. Then we have

\[
p' \circ (\phi \circ s) = p \circ s = \text{id}_A \quad \text{and} \quad p' \circ (\psi \circ \overline{s}) = p \circ \overline{s} = \text{id}_M.
\]

Therefore, \((\phi \circ s, \psi \circ \overline{s})\) is a section of the second abelian extension. If \((\alpha', \beta', \gamma')\) denotes the 2-cocycle in \( Z^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \) corresponding to the second abelian extension and its section \((\phi \circ s, \psi \circ \overline{s})\), then

\[
\alpha'(a, a') = (\phi \circ s)(a) \cdot \overline{s}(a') - (\phi \circ s)(a \cdot a')
\]

\[
= \phi(s(a) \cdot \overline{s}(a') - s(a \cdot a'))
\]

\[
= \phi(\alpha(a, a')) = \alpha(a, a') \quad (\because \phi|_B = \text{id}_B).
\]

Similarly, one can show that \( \beta = \beta' \) and \( \gamma = \gamma' \). So, \((\alpha, \beta, \gamma)\) and \((\alpha', \beta', \gamma')\) corresponds to the same element in \( H^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \). Hence there is a well-defined map

\[
\Theta_1 : \text{Ext}(M \xrightarrow{R} A; N \xrightarrow{S} B) \rightarrow H^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B).
\]

To define the map in the other direction, we take a 2-cocycle \((\alpha, \beta, \gamma) \in Z^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \). In other words, we have

\[
\delta_{A,B}(\alpha) = 0, \quad \delta_{A,N}(\beta) = 0 \quad \text{and} \quad \delta_{M,B}(\gamma) + h_R(\alpha, \beta) = 0. \quad (12)
\]
Let $\hat{A} = A \oplus B$ and $\hat{M} = M \oplus N$. We define maps

$\cdot\hat{A} : \hat{A} \otimes \hat{A} \to \hat{A}$, \hspace{1em} $\cdot\hat{M} : \hat{A} \otimes \hat{M} \to \hat{M}$ \hspace{1em} and \hspace{1em} $\cdot\hat{M} : \hat{M} \otimes \hat{A} \to \hat{M}$

by

$$(a, b) \cdot\hat{A} (a', b') = (a \cdot a', a \cdot b b' + b \cdot b a' + \alpha(a, a')), \hspace{1em} (a, b) \cdot\hat{M} (m, n) = (a \cdot_M m, a \cdot_N n + r(b, m) + \beta(a, m)), \hspace{1em} (m, n) \cdot\hat{M} (a, b) = (m \cdot_M a, l(m, b) + n \cdot_N a + \beta(m, a)).$$

Since $\alpha$ and $\beta$ satisfies the first two conditions of (12), it follows that the above structure maps defines an associative algebra structure on $\hat{A}$ and an $\hat{A}$-bimodule structure on $\hat{M}$. Finally, we define a map $\hat{R} : \hat{M} \to \hat{A}$ by

$$\hat{R}(m, n) = (R(m), S(n) + \gamma(m)), \text{ for } (m, n) \in \hat{M}.$$  

It is also straightforward to see that $\hat{R} : \hat{M} \to \hat{A}$ is a relative Rota-Baxter operator (follows from the last condition of (12)). In other words, $\hat{M} \xrightarrow{\hat{R}} \hat{A}$ is a relative Rota-Baxter algebra. Moreover, it is an abelian extension of $M \xrightarrow{R} A$ by the 2-term chain complex $N \xrightarrow{S} B$ as of (11) with the structure maps

$$i(b) = (0, b), \hspace{1em} \bar{i}(n) = (0, n), \hspace{1em} p(a, b) = a \text{ and } \bar{p}(m, n) = m.$$  

Let $(\alpha', \beta', \gamma') \in Z^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B)$ be another 2-cocycle cohomologous to $(\alpha, \beta, \gamma)$, say

$$(\alpha, \beta, \gamma) - (\alpha', \beta', \gamma') = \delta_{rRB}(\theta, \vartheta),$$

for some $(\theta, \vartheta) \in C^1_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B)$. We define maps $\phi : A \oplus B \to A \oplus B$ and $\psi : M \oplus N \to M \oplus N$ by

$$\phi(a, b) = (a, b + \theta(a)), \hspace{1em} \psi(m, n) = (m, n + \vartheta(m)).$$

Then it is easy to check that $(\phi, \psi)$ defines an isomorphism of abelian extensions from $\hat{M} \xrightarrow{\hat{R}} \hat{A}$ to $\hat{M} \xrightarrow{\hat{R}} \hat{A}'$. Thus, there is a well-defined map

$$\Theta_2 : H^2_{rRB}(M \xrightarrow{R} A; N \xrightarrow{S} B) \to \text{Ext}(M \xrightarrow{R} A; N \xrightarrow{S} B).$$

Finally, the maps $\Theta_1$ and $\Theta_2$ are inverses to each other. This completes the proof. \hfill \Box

### 6 Classifications of Skeletal Homotopy Relative Rota-Baxter Algebras

The notion of homotopy relative Rota-Baxter Lie algebras was introduced by Lazarev, Sheng, and Tang [20]. Homotopy theory of relative Rota-Baxter associative algebras is studied in [10]. Roughly, a homotopy relative Rota-Baxter (associative) algebra is a triple consisting of an $A_\infty$-algebra, a bimodule over the $A_\infty$-algebra and a homotopy relative Rota-Baxter operator. In this section, we mainly focus on homotopy relative Rota-Baxter algebras whose underlying $A_\infty$-algebra and the bimodule are both concentrated in degrees 0 and 1. We call them 2-term homotopy relative Rota-Baxter algebras. We classify 'skeletal' homotopy relative Rota-Baxter algebras.

**Definition 6.1** [25] A 2-term $A_\infty$-algebra $\mathcal{A} = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3)$ consists of a chain complex $A_1 \xrightarrow{d} A_0$ together with bilinear maps $\mu_2 : A_i \otimes A_j \to A_{i+j}$ ($0 \leq i, j, i + j \leq 1$)
and a trilinear map $\mu_3 : A_0 \otimes A_0 \otimes A_0 \rightarrow A_1$ satisfying the following set of identities: for $a, b, c, e \in A_0$ and $p, q \in A_1$,

(i) $d\mu_2(a, p) = \mu_2(a, dp)$,
(ii) $\mu_2(dp, q) = \mu_2(dp, a)$,
(iii) $d\mu_3(a, b, c) = \mu_2(\mu_2(a, b), c) - \mu_2(a, \mu_2(b, c))$,
(iv) $\mu_3(\mu_2(a, b), c, e) - \mu_3(a, \mu_2(b, c), e) + \mu_3(a, b, \mu_2(c, e)) = \mu_2(\mu_3(a, b, c), e) + \mu_2(a, \mu_3(b, c), e)$.

The notion of bimodules over a 2-term $A_\infty$-algebra is a particular case of bimodules over an $A_\infty$-algebra [18].

**Definition 6.2** Let $\mathcal{A} = (A_1 \xrightarrow{\ i \ \ } A_0, \mu_2, \mu_3)$ be a 2-term $A_\infty$-algebra. A bimodule over it consists of a tuple $\mathcal{M} = (M_1 \xrightarrow{\ d \ \ } M_0, \mu_2^M, \mu_3^M)$ of a chain complex $M_1 \xrightarrow{\ d^M \ \ } M_0$ together with bilinear maps $\mu_2^M : A_i \otimes M_j \rightarrow M_{i+j}, \quad \mu_3^M : M_i \otimes A_j \rightarrow M_{i+j}$ $(0 \leq i, j, i + j \leq 1)$ and a trilinear map $\mu_3^M : A_0^{2,1} \rightarrow M_1$ (here $A_0^{2,1} = (A_0 \otimes A_0 \otimes M_0) \oplus (A_0 \otimes M_0 \otimes A_0) \oplus (M_0 \otimes A_0 \otimes A_0)$) satisfying the all possible identities corresponding to (i)-(iv) of Definition 6.1 where exactly one of the variables in each identity is replaced by an element of $\mathcal{M}$ and the corresponding operation $d, \mu_2$ or $\mu_3$ is replaced by $d^M, \mu_2^M$ or $\mu_3^M$.

**Example 6.3** Any 2-term $A_\infty$-algebra $\mathcal{A} = (A_1 \xrightarrow{\ i \ \ } A_0, \mu_2, \mu_3)$ is a bimodule over itself. This is called the adjoint bimodule.

**Definition 6.4** Let $\mathcal{A} = (A_1 \xrightarrow{\ i \ \ } A_0, \mu_2, \mu_3)$ be a 2-term $A_\infty$-algebra and $\mathcal{M} = (M_1 \xrightarrow{\ d^M \ \ } M_0, \mu_2^M, \mu_3^M)$ be a bimodule over it. A (2-term) homotopy relative Rota-Baxter operator (on $\mathcal{A}$ with respect to $\mathcal{M}$) is a triple $\mathcal{R} = (R_0, R_1, R_2)$ of maps $R_0 : M_0 \rightarrow A_0, \quad R_1 : M_1 \rightarrow A_1$ and $R_2 : M_0 \otimes M_0 \rightarrow A_1$ satisfying

(i) $d \circ R_1 = R_0 \circ d^M$,
(ii) $R_0(\mu_2^M(R_0(m), m')) + \mu_2^M(m, R_0(m')) - \mu_2(R_0(m), R_0(m')) = d(R_2(m, m'))$,
(iii) $R_1(\mu_2^M(R_0(m), n) + \mu_2^M(m, R_1(n))) - \mu_2(R_0(m), R_1(n)) = R_2(m, d^M(n))$,
(iv) $R_2(R_0(m), R_2(m', m'')) - R_1(\mu_2^M(m, R_2(m', m''))) - \mu_2^M(R_0(m), m') + \mu_2^M(m, R_0(m')) + \mu_2^M(R_0(m'), m'')$,
(v) $R_1(\mu_2^M(R_2(m, m'), m'')) + \mu_3^M(R_0(m), R_0(m''), R_0(m'')) + \mu_3^M(R_0(m), R_0(m)', R_0(m''))$,

for $m, m', m'' \in M_0$ and $n \in M_1$. 

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A (2-term) homotopy relative Rota-Baxter algebra is a triple \( (\mathcal{A}, \mathcal{M}, \mathcal{R}) \) in which \( \mathcal{A} \) is a 2-term \( A_\infty \)-algebra, \( \mathcal{M} \) is an \( \mathcal{A} \)-bimodule and \( \mathcal{R} \) is a homotopy relative Rota-Baxter operator. Like nonhomotopic case, we denote a homotopy relative Rota-Baxter algebra as above by \( \mathcal{M} \xrightarrow{\mathcal{R}} \mathcal{A} \).

**Remark 6.5** Let \( \mathcal{R} \) be a (2-term) homotopy relative Rota-Baxter operator on an associative 2-algebra \( \mathcal{A} \) with respect to a bimodule \( \mathcal{M} \). Then the standard skew-symmetrization process yields a relative Rota-Baxter operator (in the sense of [24]) on the associated Lie 2-algebra with respect to the representation. For more details, we refer to [10, Proposition 7.8].

Next, we consider skeletal homotopy relative Rota-Baxter algebras and classify them in terms of 3-cocycles in the cochain complex of relative Rota-Baxter algebras.

**Definition 6.6** A homotopy relative Rota-Baxter algebra \( \mathcal{M} \xrightarrow{\mathcal{R}} \mathcal{A} \) with

\[
\mathcal{A} = (A_1 \xrightarrow{d} A_0, \mu_2, \mu_3), \quad \mathcal{M} = (M_1 \xrightarrow{d^\mathcal{M}} M_0, \mu_2^\mathcal{M}, \mu_3^\mathcal{M}) \quad \text{and} \quad \mathcal{R} = (R_0, R_1, R_2)
\]

is said to be **skeletal** if \( d = 0 \) and \( d^\mathcal{M} = 0 \).

**Theorem 6.7** There is a one-to-one correspondence between skeletal homotopy relative Rota-Baxter algebras and triples \( (M \xrightarrow{R} A; N \xrightarrow{S} B; (\alpha, \beta, \gamma)) \) in which \( M \xrightarrow{R} A \) is a relative Rota-Baxter algebra, \( N \xrightarrow{S} B \) is a bimodule and \( (\alpha, \beta, \gamma) \in Z^3_{\text{RBA}}(M \xrightarrow{R} A; N \xrightarrow{S} B) \) is a 3-cocycle.

**Proof** Let \( \mathcal{M} \xrightarrow{\mathcal{R}} \mathcal{A} \) be a skeletal homotopy relative Rota-Baxter algebra, where

\[
\mathcal{A} = (A_1 \xrightarrow{0} A_0, \mu_2, \mu_3), \quad \mathcal{M} = (M_1 \xrightarrow{0} M_0, \mu_2^\mathcal{M}, \mu_3^\mathcal{M}) \quad \text{and} \quad \mathcal{R} = (R_0, R_1, R_2).
\]

Since \( \mathcal{A} = (A_1 \xrightarrow{0} A_0, \mu_2, \mu_3) \) is a 2-term \( A_\infty \)-algebra, it follows from Definition 6.1 that \( A_0 \) is an associative algebra with the multiplication \( \mu(a, b) := \mu_2(a, b) \), for \( a, b \in A_0 \), and \( A_1 \) is a bimodule over the associative algebra \( A_0 \) with left and right \( A_0 \)-actions \( l_{A_1} : A_0 \otimes A_1 \rightarrow A_1 \) and \( r_{A_1} : A_1 \otimes A_0 \rightarrow A_1 \) given by

\[
l_{A_1}(a, p) := \mu_2(a, p), \quad r_{A_1}(p, a) := \mu_2(p, a), \quad \text{for} \ a \in A_0, \ p \in A_1.
\]

Further, the map \( \mu_3 \in \text{Hom}(A_0^\otimes 3, A_1) \) is a 3-cocycle in the Hochschild cochain complex of \( A_0 \) with coefficients in the \( A_0 \)-bimodule \( A_1 \). Moreover, \( \mathcal{M} = (M_1 \xrightarrow{0} M_0, \mu_2^\mathcal{M}, \mu_3^\mathcal{M}) \) is a bimodule over \( \mathcal{A} \) implies that both \( M_0 \) and \( M_1 \) are bimodules over the associative algebra \( A_0 \) with left and right actions given by

\[
l_{M_0}(a, m) = \mu_2^\mathcal{M}(a, m), \quad r_{M_0}(m, a) = \mu_2^\mathcal{M}(m, a), \quad \text{for} \ a \in A_0, \ m \in M_0,
\]
\[
l_{M_1}(a, n) = \mu_2^\mathcal{M}(a, n), \quad r_{M_1}(n, a) = \mu_2^\mathcal{M}(n, a), \quad \text{for} \ a \in A_0, \ n \in M_1.
\]

Further, we define maps \( l : M_0 \otimes A_1 \rightarrow M_1 \) and \( r : A_1 \otimes M_0 \rightarrow M_1 \) by \( l(m, p) = \mu_2^\mathcal{M}(m, p) \) and \( r(p, m) = \mu_2^\mathcal{M}(p, m) \), for \( m \in M_0, \ p \in A_1 \). It is also easy to verify that the map \( \mu_3^\mathcal{M} \in \text{Hom}(A_0^\otimes 3, M_1) \) satisfies \( \delta_{\mu_3^\mathcal{M}}(\mu_3^\mathcal{M}) = 0 \).

Finally, since \( \mathcal{R} = (R_0, R_1, R_2) \) is a homotopy relative Rota-Baxter operator, it follows from the conditions (ii)-(iv) of Definition 6.4 that \( M_0 \xrightarrow{R_0} A_0 \) is a relative Rota-Baxter
algebra and \((M_1 \xrightarrow{R_1} A_1, l, r)\) is a bimodule over it. Finally, the condition (v) of Definition 6.4 implies that \(\delta_{M_0, A_1}(R_2) + h_{R_0}(\mu_3, \mu_3^{M}) = 0\). Hence, we have

\[
\delta_{\mathfrak{RB}}((\mu_3, \mu_3^{M}, R_2)) = \left(\delta_{A_0, A_1}^{R_3}(\mu_3), \delta_{R_{A_0, A_1}}^{R_3}(\mu_3^{M}), \delta_{M_0, A_1}(R_2) + h_{R_0}(\mu_3, \mu_3^{M})\right) = 0.
\]

This shows that \((M_0 \xrightarrow{R_0} A_0; M_1 \xrightarrow{R_1} A_1; (\mu_3, \mu_3^{M}, R_2))\) is a required triple.

Conversely, let \((M \xrightarrow{R} A; N \xrightarrow{S} B; (\alpha, \beta, \gamma))\) be a triple consisting of a relative Rota-Baxter algebra \(M \xrightarrow{R} A\), a bimodule \((N \xrightarrow{S} B, l, r)\) and a 3-cocycle \((\alpha, \beta, \gamma) \in Z^3_{\mathfrak{RB}}(M \xrightarrow{R} A; N \xrightarrow{S} B) \subset \text{Hom}(A \otimes B) \oplus \text{Hom}(A^{2,1}, N) \oplus \text{Hom}(M \otimes B, B)\). Then it is easy to verify that \(A = (B \xrightarrow{0} A, \mu_2, \mu_3)\) is a 2-term \(A_\infty\)-algebra, where \(\mu_2\) and \(\mu_3\) are given by

\[
\mu_2(a, a') = a \cdot a', \quad \mu_2(a, b) = a \cdot b, \quad \mu_2(b, a) = b \cdot a, \quad \text{for } a, a' \in A, b \in B, \mu_3(a, a', a'') = \alpha(a, a', a''), \quad \text{for } a, a', a'' \in A.
\]

Moreover, \(\mathcal{M} = (N \xrightarrow{0} M, \mu_2^{M}, \mu_3^{M})\) is a bimodule over \(A\), where

\[
\mu_2^{M}(a, m) = a \cdot_{M} m, \quad \mu_2^{M}(a, n) = a \cdot_{N} n, \quad \mu_2^{M}(b, m) = r(b, m), \quad \mu_2^{M}(m, a) = m \cdot_{A} a, \quad \mu_2^{M}(n, a) = n \cdot_{N} a, \quad \mu_2^{M}(m, b) = l(m, b),
\]

for \(a \in A, b \in B, m \in M, n \in N\), and \(\mu_3^{M} : A^{2,1} \to N\) is given by \(\mu_3^{M} = \beta\). Further, one can show that \(\mathcal{R} = (R, S, \gamma)\) is a homotopy relative Rota-baxter operator. In other words, \(\mathcal{M} \xrightarrow{R} A\) is a (skeletal) homotopy relative Rota-Baxter algebra.

The above two correspondences are inverses to each other, which completes the proof. 

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