New $AdS_3$ Branes and Boundary States

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Abstract

We examine D-branes on $AdS_3$, and find a three-brane wrapping the entire $AdS_3$, in addition to 1-branes and instantonic 2-branes previously discussed in the literature. The three-brane is found using a construction of Maldacena, Moore, and Seiberg. We show that all these branes satisfy Cardy’s condition and extract the open string spectrum on them.

1 Introduction

Perturbative string theory on $AdS_3$, which is isomorphic to the SL(2, $\mathbb{R}$) group manifold, is considerably more complicated than string theory on compact group manifolds. (For previous work on SL(2, $\mathbb{R}$), see [1, 2, 3, 4, 5, 6, 7, 8].) Some of the subtleties of the closed string spectrum were worked out in [9], where a proposal for the closed string spectrum was proposed, and checked by an explicit computation of the partition function.

There has also been work on D-branes and open strings in $AdS_3$. Semiclassical descriptions of D-branes were given in [10, 11, 12], and the direct quantization of the open strings on the branes was performed in [13]. The boundary state description of the D-branes was found in [14].

In this paper, we will extend the results of [14] to describe boundary states for other branes on $AdS_3$ and Euclidean $AdS_3$. (Semiclassical descriptions of some of these branes had been given in [10, 11, 12]). We verify that Cardy’s condition is satisfied and use this to extract the spectrum of open strings connecting various branes.

The interesting aspect in these calculations is that the infinite volume of $AdS_3$ makes quantities like the overlaps of branes infinite. As we shall see (and was already shown in [14]) all these divergences can be tamed by a careful regularization of the volume.

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We start by reviewing the closed string spectrum of [9]. Then we review the D-brane spectrum on Euclidean AdS$_3$, which is a straightforward extension of [10, 11, 12]. We find a whole zoo of objects, which fall into three classes. The boundary states for these branes can be written down, following the methods of [14], and the overlaps can be computed.

It turns out that one of these three classes actually fails to satisfy Cardy’s condition. The remaining two classes of objects (which we call 1-branes and 2-instantons) satisfy Cardy’s condition and hence are physical states in the theory.

We then investigate a new class of branes (which we call parafermionic branes) which were not discussed by [10, 11, 12]. These new branes are the extension to SL(2, $\mathbb{R}$) of branes found in SU(2) by [13]. This yields one new brane that satisfies Cardy’s condition and appears to be a three-brane wrapping the entire AdS$_3$.

We close with a summary of our results.

Recently, the paper [16] appeared with related results.

## 2 Review of closed strings on AdS$_3$

In cylindrical (global) coordinates, the metric of AdS$_3$ is

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2$$

All modes of particles propagating in AdS$_3$ should fall into representations of the isometry group, namely SL(2, $\mathbb{R}$)$_L \times$ SL(2, $\mathbb{R}$)$_R$. Explicit expressions for the generators of SL(2, $\mathbb{R}$)$_L \times$ SL(2, $\mathbb{R}$)$_R$ are [17]

\begin{align*}
J^3 &= \frac{i}{2} \partial_u \\
J^+ &= \frac{i}{2} e^{-2iu} \left[ \coth 2\rho \partial_u - \frac{1}{\sinh 2\rho} \partial_v + i\partial_\rho \right] \\
J^- &= \frac{i}{2} e^{2iu} \left[ \coth 2\rho \partial_u - \frac{1}{\sinh 2\rho} \partial_v - i\partial_\rho \right]
\end{align*}

where $u = \frac{1}{2}(t + \phi)$, $v = \frac{1}{2}(t - \phi)$. The generators of the other SL(2, $\mathbb{R}$) algebra are obtained by exchanging $u$ and $v$ in the above expressions.

The Laplacian on Lorentzian AdS$_3$ is given by

$$\Box = \partial_\rho^2 + 2\frac{\cosh 2\rho}{\sinh 2\rho} \partial_\rho + \frac{1}{\sinh^2 \rho} \partial_\phi^2 - \frac{1}{\cosh^2 \rho} \partial_t^2$$

The Laplacian on Euclidean AdS$_3$ is obtained by replacing $t = i\tau$.

We will consider (delta function) normalizable states in Euclidean AdS$_3$. This forces the eigenvalues of the Laplacian to be of the form $4j(j - 1)$ with $j = \frac{1}{2} + is$ (s real). These are the continuous representations of the covering group SL(2, $\mathbb{C}$).

These do not have highest or lowest weight states, and therefore the spectrum of $J^3$ in this representation is unbounded from above and from below. This leads to a divergence in the characters of these representations.
For each representation of the zero mode algebra \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \) one can construct a module of the Kac-Moody algebra by a repeated application of oscillator modes \( J_{-n}^a, n > 0, a = \pm, 3 \). We denote such representations by \( \hat{D}^\pm_j \) and \( \hat{C}^\alpha_j \). These representations were dubbed positive energy representations in \([9]\), as the spectrum of \( L_0 \) is bounded from below.

In \([18, 9]\), a new set of representations of the KM algebra was constructed. These are obtained from the above representations by an application of the spectral flow, defined as:

\[
J_n^3 \rightarrow J_n^3 - \frac{k}{2} \omega \delta_{n,0}, \quad J_n^+ \rightarrow J_{n+\omega}^+ \quad J_n^- \rightarrow J_{n-\omega}^-
\]

(4)

This preserves the KM algebra, and was conjectured in \([9]\) to be a symmetry of the closed string spectrum. The new representations have \( L_0 \) unbounded from below, but satisfy a no-ghost theorem, proven in \([9]\). The complete closed string spectrum then consists of the representations allowed by the no-ghost theorem above, together with all their images under the spectral flow—the so-called flowed representations.

3 The brane zoo

3.1 Classical descriptions

We now review the D-brane spectrum found in \([10, 11, 12]\). Following \([19]\), we shall look for D-branes wrapping regular and twined conjugacy classes.

We will use the following quantities, defined in terms of Euclidean cylindrical coordinates (see the Appendix)

\[
\sinh \psi \equiv \sinh \rho \sin \phi \\
\sinh \alpha \equiv \cosh \rho \sinh \tau \\
\cosh \lambda \equiv \cosh \rho \cosh \tau
\]

(5)

3.1.1 1-branes

One set of conjugacy classes in Lorentzian \( AdS_3 \) is described by the equation \([12]\)

\[
\sinh \psi = \text{constant} \equiv \sinh \psi_0
\]

(6)

A brane wrapped on this hypersurface is extended along the time direction and hence describes a D1-brane in Lorentzian \( AdS_3 \).

In Euclidean \( AdS_3 \), the conjugacy class is described by the same equation, and is a hypersurface in Euclidean \( AdS_3 \).

In Poincare coordinates on Euclidean \( AdS_3 \), the corresponding equation for the conjugacy class is

\[
\frac{x}{r} = \text{constant}
\]

(7)
3.1.2 2-instantons

A second set of conjugacy classes in Lorentzian $AdS_3$ is described by the equation $\cosh \rho \sin t = \text{constant}$. A brane wrapped on this hypersurface is not extended along the time direction and hence describes an instantonic D2-brane in Lorentzian $AdS_3$.

In Euclidean $AdS_3$, the conjugacy class is described by the equation

$$\sinh \alpha = \text{constant} \equiv \sinh \alpha_0$$

(8)

In Poincaré coordinates in Euclidean $AdS_3$, the corresponding equation for the conjugacy class is

$$\tilde{\tau} \equiv r = \text{constant}$$

(9)

This shows the similarity between the 2-instanton and the 1-brane above.

Equation (8) can be understood as being related to equation (8) by the replacement $\rho \rightarrow \rho + \frac{i\pi}{2}, (\tau + \phi) \rightarrow (\tau + \phi), (\tau - \phi) \rightarrow -(\tau - \phi)$, which leaves Euclidean $AdS_3$ invariant. This is an inner automorphism of the covering group $SL(2, C)$.

3.1.3 0-instantons

The Lorentzian $AdS_3$ conjugacy class $\cosh \rho \sin t = \text{constant}$ can be translated to the conjugacy class $\cosh \rho \cos t = \text{constant}$ by a shift $t \rightarrow t + \frac{\pi}{2}$, which is an inner automorphism of $SL(2, \mathbb{R})$. This new equation has a different continuation to Euclidean $AdS_3$, i.e.

$$\cosh \lambda = \text{constant} \equiv \cosh \lambda_0$$

(10)

A brane wrapped on this hypersurface is also instantonic. In particular, the case $\lambda_0 = 0$ describes a pointlike object at $\rho = 0, \tau = 0$. (This is, as we shall see, not the usual D-instanton.) The more general case $\lambda_0 > 0$ describes a hypersurface in Euclidean $AdS_3$ surrounding the point $\rho = 0, \tau = 0$.

As we shall see, these branes do not satisfy Cardy’s condition, and therefore do not correspond to physical objects.

These conjugacy classes are related to the ones in the previous subsection by the shift $\tau \rightarrow \tau - \frac{\pi}{2}$, which is an inner automorphism of the covering group $SL(2, C)$.

3.1.4 Circular branes

More generally, the Lorentzian $AdS_3$ conjugacy class

$$\cosh \rho \left( \sin t \cos t_0 + \cos t \sin t_0 \right) = \text{constant}$$

(11)

continues in Euclidean $AdS_3$ to $\cosh \rho (i \sinh \tau \cos t_0 + \cosh \tau \sin t_0) = \text{constant}$ which is actually two equations

$$\cosh \rho \sinh \tau = C_1 \quad \cosh \rho \cosh \tau = C_2$$

(12)

This describes a circle centered at $\rho = 0$, at some fixed value of $\tau$.

As we shall see, none of these conjugacy classes lead to boundary states satisfying Cardy’s condition unless $\sin t_0 = 0$, which reduces to the previous cases.
3.2 Boundary states

3.2.1 1-branes

The boundary state for the 1-brane was found in [14]. We review this construction here.

Firstly, the boundary state $|B^1D\rangle$ for the 1-brane satisfies $(J^a + \bar{J}^a)|B^1D\rangle = 0$. For example, the zero-mode equation $(J^3 + \bar{J}^3)|B^1D\rangle = 0$ translates to $\partial_t |B^1D\rangle = 0$ i.e. the brane is extended along the time direction. The other components work similarly, as shown in [14].

The solution for the Ishibashi states satisfying $(J^a_n + \bar{J}^a_{-n})|I\rangle = 0$ (13) was found in [20]. Suppose we are given a KM primary $|\Phi_j\rangle$ in the $\hat{C}^\alpha_j \times \hat{\bar{C}}^\alpha_j$ representation. By that we mean a state which is annihilated by all lowering operators $J^a_n, n > 0$, but not by the zero modes (since $C^\alpha_j$ has no highest weight states). Define [21, 22]

$$|I^j\rangle = \sum_{I,J} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |\Phi_j\rangle$$ (14)

Here $I, J$ are ordered strings of indices $(n_1, a_1) \cdots (n_r, a_r)$, and

$$J_I = J^a_{n_1} \cdots J^a_{n_r}. \quad (15)$$

For later convenience we choose an ordering such that all zero modes act from the left. This sums over all the descendants in the KM module, with the normalization defined as

$$M_{IJ} = \langle \Phi^j | J_I J_{-I} | \Phi^j \rangle$$ (16)

$M_{IJ}$ is invertible for any KM module. (For degenerate modules, one has to mod out by the null vectors). It is easy to see that $|I^j\rangle$ satisfies (13) by showing that $(J^a_n + \bar{J}^a_{-n})|I^j\rangle$ is orthogonal to all states in the module based on $|\Phi^j\rangle$.

It turns out that it is necessary to separate out the oscillator modes from the zero-modes (which we sometimes refer to as primaries). This is because there are divergences in the brane overlaps, which come from the infinite sum over primaries. We therefore want to separate out the finite oscillator contribution from this divergence.

Hence we define a modified Ishibashi state, which is a coherent state defined as

$$|I^j_{m\bar{m}}\rangle = \sum_{I,J} M^{-1}_{IJ} J_{-I} \bar{J}_{-J} |\Phi^j_{m\bar{m}}\rangle$$ (17)

Here $|\Phi^j_{m\bar{m}}\rangle$ is annihilated by all oscillator modes except for the zero modes (as above), and has magnetic quantum numbers $m$ and $\bar{m}$. The sum $\sum$ over descendants here is defined to exclude any action by the zero modes $J^a_0$. It is clear that formally

$$|I^j\rangle = \sum_m |I^j_{m,-m}\rangle$$ (18)
The Cardy state was found in the large $k$ limit by requiring the branes to be localized on the conjugacy class in this limit. The boundary state for the D1-brane was then found to be [14]

$$|B^{1D}\rangle = T \sum_s e^{-is\psi_0} |I^s\rangle \equiv T \int \frac{ds}{2\pi} e^{-is\psi_0} |I^s\rangle$$ (19)

where $|I^s\rangle$ is the coherent state defined in [17]

$$|I^s\rangle = \sum_{I,J} M^{-1}_{IJ} J_{-I} J_{-J} |\Phi^s\rangle$$ (20)

and $|\Phi^s\rangle$ is the primary satisfying $\langle x | \Phi^s \rangle = \frac{e^{i\alpha\psi}}{\cosh \psi}$. ($\psi$ is defined in [8].) $T$ is an overall normalization constant. In the flat space case it was determined by [23].

### 3.2.2 2-instantons

The conjugacy class $\cosh \rho \sinh \tau = \text{constant}$ was obtained from the conjugacy class (8) by the continuation $\rho \to \rho + \frac{i\pi}{2}, (\tau + \phi) \to (\tau + \phi), (\tau - \phi) \to -(\tau - \phi)$.

This continuation affects the currents as

$$J^a \to J^a, \quad \bar{J}^3 \to -\bar{J}^3, \quad \bar{J}^+ \to -\bar{J}^-, \quad \bar{J}^- \to -\bar{J}^+$$ (21)

Accordingly the Ishibashi state (17) is replaced by (cf. [24])

$$|\hat{I}_{mn}\rangle = \sum_{I,J} M^{-1}_{IJ} K_{-I} K_{-J} |\Phi^I_m\rangle$$ (22)

with

$$K^a_n = J^a_n, \quad \bar{K}^3_n = -\bar{J}^3_n, \quad \bar{K}^+_n = -\bar{J}^-_n, \quad \bar{K}^-_n = -\bar{J}^+_n$$ (23)

The boundary state for the D2-instanton can be written in analogy to (19)

$$|B^{2I}\rangle = T \sum_p e^{-ip\alpha} |\hat{I}_p\rangle \equiv T \int \frac{dp}{2\pi} e^{-ip\alpha} |\hat{I}_p\rangle$$ (24)

where $|\hat{I}_p\rangle$ is the modified Ishibashi state (22) built over the primary satisfying $\langle x | \Phi^p \rangle = \frac{e^{i\alpha\psi}}{\cosh \psi}$. ($\alpha$ is defined in [8].)

We now consider the flowed representations. It was argued in [14] that the 1-brane did not couple to flowed representations. Heuristically, the flowed representations represent strings winding near the boundary of $AdS_3$. The 1-brane does not wind near the boundary and hence cannot couple to these states. It might seem that the 2-instanton can couple to these flowed representations. However, the symmetry between the 1-brane and the 2-instanton in Poincare coordinates shows that in fact there is no such coupling.

This result is also a consequence of Cardy’s condition. The overlap of the 2-instantons should be the same as that of the 1-branes (again due to the symmetry in Poincare coordinates.) If the 2-instanton couples to flowed representations, this matching is destroyed. Hence we conclude that the 2-instanton does not couple to flowed representations.
3.2.3 0-instantons

The conjugacy class \( \cosh \rho \cosh \tau = \text{constant} \) was obtained from the conjugacy class \( \cosh \rho \sinh \tau = \text{constant} \) by the further shift \( \tau \to \tau - \frac{i\pi}{2} \). This shifts the currents as

\[
J^3 \to J^3, \quad J^+ \to iJ^+, \quad J^- \to -iJ^- \\
\bar{J}^3 \to \bar{J}^3, \quad \bar{J}^+ \to i\bar{J}^+, \quad \bar{J}^- \to -i\bar{J}^-.
\]

The Ishibashi state corresponding to a primary \( |\Phi_{jm\bar{m}}^j\rangle \) is therefore

\[
|\tilde{I}_{jm}^j\rangle = \sum_{I,J} M^{-1}_{IJ} K_{IJ} |\Phi_{jm\bar{m}}^j\rangle
\]

with

\[
K_n^3 = J_n^3, \quad K_n^+ = iJ_n^+, \quad K_n^- = -iJ_n^-
\]

\[
\bar{K}_n^3 = -\bar{J}_n^3, \quad \bar{K}_n^+ = i\bar{J}_n^+, \quad \bar{K}_n^- = -i\bar{J}_n^+-\frac{1}{2}
\]

The boundary state for the D0-instanton can then be written as

\[
|B^{0I}\rangle = T \sum_q e^{-iq\lambda_0} |\tilde{I}_q^I\rangle \equiv T \int dq e^{-iq\lambda_0} |\tilde{I}_q^I\rangle
\]

where \( |\tilde{I}_q^I\rangle \) is the modified Ishibashi state \((26)\) built over the primary satisfying \( \langle x|\Phi_q^0\rangle = \frac{e^{iq\lambda}}{\sinh \lambda} \). (\( \lambda \) is defined in \((5)\).)

3.2.4 Circular branes

It does not seem possible to write a boundary state for these branes using the method of \((14)\). They do not have a good flat space limit either. It is likely that these conjugacy classes do not correspond to physical branes.

4 Overlaps of branes

4.1 General remarks

The overlap of the branes \( |B\rangle \) and \( |\tilde{B}\rangle \) is defined as \( \langle \tilde{B}|q^{L_0+\bar{L}_0-\hat{P}}|B\rangle \).

We shall find it useful to separate the oscillator contribution from the contribution of the primaries. To do this we define new coherent states which only have overlap with the primaries.

\[
|C^{1I}\rangle = \sum_s e^{-is\psi_0} |\Phi^s\rangle
\]

\[
|C^{2I}\rangle = \sum_p e^{-ip\alpha_0} |\Phi^p\rangle
\]

\[
|C^{0I}\rangle = \sum_q e^{-iq\lambda_0} |\Phi^q\rangle
\]

(29)
Note that
\[ \langle x|C^{1D}\rangle = \delta (\sinh \psi - \sinh \psi_0) \]
\[ \langle x|C^{2I}\rangle = \delta (\sinh \alpha - \sinh \alpha_0) \]
\[ \langle x|C^{0I}\rangle = \delta (\cosh \lambda - \cosh \lambda_0) \]  
(30)

which shows that the branes are localized on conjugacy classes.

We will often use the relation
\[ \langle \tilde{C}|q^{L_0+L_0-\frac{c}{12}}|C\rangle = \int dx \langle \tilde{C}|x\rangle \langle x|q^{L_0+L_0-\frac{c}{12}}|C\rangle \]
(31)

We will also need the formula:
\[ \left(L_0 + \bar{L}_0 - \frac{c}{12}\right) |\Phi^s\rangle = \left(-2j(j-1) - \frac{k}{4(k-2)}\right) |\Phi^s\rangle = \left(-\frac{s^2}{2(k-2)} - \frac{1}{4}\right) |\Phi^s\rangle \]
(32)

### 4.2 Preliminary comments on 0-instantons

We now present our first evidence that the 0-instantons are not physical states.

We start with the case \( \lambda_0 = 0 \), which describes a pointlike object, which at first glance may appear to be the standard D-instanton. This is in fact not the case.

To see this, note that the usual D-instanton looks like a ten dimensional delta function. The brane wrapped on the conjugacy class instead satisfies
\[ \langle x|C^{0I}\rangle = \delta (\cosh \lambda - 1) \]
(33)

and hence
\[ \int dx \langle x|C^{0I}\rangle = \int d\lambda \sinh^2 \lambda \delta (\cosh \lambda - 1) = 0 \]
(34)

Hence the normalization is incorrect for this state to be a D-instanton. Similarly the other 0-instantons do not resemble known D-branes in the flat space limit.

As we shall now see, the 0-instantons do not satisfy Cardy’s condition either. To see this, we shall compute the overlap of a 2-instanton with a 0-instanton.

### 4.3 Overlaps of 2-instantons and 0-instantons

The 0-instanton will be labelled by \( \lambda_0 \), and the 2-instanton will be labelled by \( \alpha_0 \).

The overlap of primaries (using (31)) is
\[ \langle \tilde{C}^0|q^{L_0+L_0-\frac{c}{12}}|C^{2I}\rangle = \int_0^\infty d\lambda \int_{-\pi/2}^{\pi/2} d\sigma \int_0^{2\pi} d\phi \sinh^2 \lambda \cos \sigma \]

\[ \times \delta (\cosh \lambda - \cosh \lambda_0) \int \frac{ds}{2\pi} q^{-\frac{s}{4}} q^{rac{s^2}{2(k-2)}} e^{i(\alpha-\alpha_0)s} \]
(35)

For \( \lambda_0 = 0 \), the integral is easily seen to be zero.
More generally, changing variables from $\sigma$ to $\alpha$, the integral is seen to be

$$\langle \tilde{C}_0^{1I} | q^{L_0+L_0-\frac{2}{\pi}} | C^{2I} \rangle = 2\pi q^{-\frac{1}{4}} \int_{-\lambda_0}^{\lambda_0} d\alpha \sqrt{\frac{k-2}{2i\pi \tau}} \exp \left( \frac{(\alpha - \alpha_0)^2(k-2)}{2i\pi \tau} \right)$$

(36)

The oscillators contribute an extra factor $[\prod_{n=1}^{\infty} (1 - q^{2n})]^{-3}$.

Combining this with the contribution from the primaries, we find the overlap of the boundary states. This resulting expression does not yield a good open string partition function upon modular transformation.

This indicates that both the 2-instanton and the 0-instanton cannot be included consistently in the theory. Since the 2-instanton is related to the 1-brane by flipping $x, \tilde{\tau}$ in Poincare coordinates, we should have both these branes in the theory. On the other hand, we have seen that the 0-instanton does not have a good flat space limit. Hence we find that the 0-instantons should be discarded.

### 4.4 Overlaps of 2-instantons and 1-branes

We now turn to the overlap of a 2-instanton and a 1-brane. We need to regulate a divergence, because the intersection of the D1-brane and the 2-instanton is the common solution in $AdS_3$ of the equations $\cosh \rho \sinh \tau = \text{const}$ and $\sinh \rho \sin \phi = \text{const}$, which has an infinite volume. Indeed, the volume of the intersection region is

$$\int dx \, \delta(\sinh \alpha - \sinh \alpha_0) \, \delta(\sinh \psi - \sinh \psi_0)$$

$$= \int \frac{d\rho d\psi d\alpha}{\sqrt{\sinh^2 \rho - \sinh^2 \psi}} \frac{\sinh \rho \cosh \rho \cosh \psi \cosh \alpha}{\cos^2 \rho + \sinh^2 \alpha} \times \delta(\sinh \alpha - \sinh \alpha_0) \, \delta(\sinh \psi - \sinh \psi_0)$$

(37)

For large $\rho$ (at fixed $\alpha, \psi$), there is a volume divergence as the volume grows as

$$V \sim \int_{\rho_0}^{\rho_0} d\rho \sim \rho_0$$

(38)

This implies that the open string partition function will have a volume divergence. It can be written as

$$Z_{\text{open}} \sim \rho_0 \tilde{Z}$$

(39)

where $\tilde{Z}$ is finite.

Now since the open string partition function is divergent, for consistency the overlap of the branes on the closed string side should also be divergent with a leading divergence proportional to $\rho_0$, and as we shall see, this is true. To check Cardy’s condition, we can drop the common divergent factor $\rho_0$, and examine the modular transformation of the remaining piece.

The overlap of primaries (using (31)) is

$$\langle \tilde{C}^{2I} | q^{L_0+L_0-\frac{2}{\pi}} | C^{1D} \rangle = \int \frac{d\rho d\psi d\alpha}{\sqrt{\sinh^2 \rho - \sinh^2 \psi}} \frac{\sinh \rho \cosh \rho \cosh \psi \cosh \alpha}{\cos^2 \rho + \sinh^2 \alpha} \times \delta(\sinh \alpha - \sinh \alpha_0) \int \frac{ds}{2\pi} q^{-\frac{1}{4}} q^{2\tau \rho - \frac{1}{2}} \frac{e^{i(q - \psi_0)s}}{\cosh \psi}$$

(40)
For large $\rho$ at fixed $\alpha, \psi$, this behaves as

$$\int_{\rho_0}^{\rho} d\rho d\psi d\alpha \, \delta(\alpha - \alpha_0) \int \frac{d\sigma}{2\pi} q^{-\frac{1}{4}q(\frac{\sigma^2}{2n} - \frac{\sigma^2}{2(k-n)})} e^{i(\psi - \tilde{\psi}_0)s} \sim q^{-\frac{1}{4}\rho_0}$$

(41)

The oscillator contribution comes from states with $J^3 = \bar{J}^3 = 0$, since only they have overlap with both the 1-brane and the 2-instanton. They occur with a relative phase in the two Ishibashi states; for every action of $\bar{J}^3_n$ or $\bar{J}^2_n \,(n \geq 0)$, the relative phase is multiplied by $(-1)$. The oscillators therefore contribute a factor

$$\frac{1}{\prod_{n=1}^{\infty}(1 - q^{-2n})(1 + q^{2n})(1 + q^{2n})}$$

(42)

After modular transformation, we find that

$$\tilde{Z} \propto \frac{1}{\sqrt{\ln(q)} \prod_{n=1}^{\infty}(1 - q^{2n})(1 - q^{2n-1})(1 - q^{2n-1})}$$

(43)

which we interpret as being the partition function of three bosons, one with Neumann boundary conditions, and the other two satisfying Neumann-Dirichlet boundary conditions. (There is no residual current algebra, since the boundary conditions of the 1-brane and the 2-instanton are incompatible.)

5 Parafermionic branes

We start by reviewing the $SU(2)$ case. Here the A-branes were constructed by [25]. They can be written

$$|A, \hat{j}_C \rangle = \sum_j \frac{S_j^j}{\sqrt{S_0^j}} |A, j \rangle$$

(44)

Here the LHS is the Cardy state, while $|A, j \rangle$ is the $SU(2)$ Ishibashi state.

The idea of [15] is to use a different CFT realization of the $SU(2)$ theory using parafermions. The Hilbert space of the $SU(2)$ WZW model can be decomposed as $SU(2)_k = (U(1)_k \times (PF)_k) / \mathbb{Z}_k$, where $(PF)_k$ is the parafermion theory.

There is a corresponding decomposition of the Cardy states into Cardy states of the parafermion theory and the $U(1)$ theory [15]

$$|A, \hat{j}_C \rangle = \frac{1}{\sqrt{k}} \sum_{\hat{n}} |A\hat{n}_C \rangle^{PF} |A\hat{n}_C \rangle^{U(1)}$$

(45)

The detailed description of the Ishibashi states and Cardy states can be found in [15]. We will not need the detailed knowledge of the parafermionic theory.

We can now perform a T-duality on the $U(1)$ theory. This exchanges Dirichlet and Neumann conditions. The A-brane above is the one satisfying Dirichlet conditions

$$(\partial_\zeta - \partial_{\bar{\zeta}}) |A\hat{n}_C \rangle^{U(1)} = 0$$

(46)
(we are representing the \(U(1)\) theory by a free boson \(\zeta\).)

The branes with Neumann boundary conditions are the B-branes satisfying

\[
(\partial_\zeta + \partial_{\bar{\zeta}})|B, \eta\rangle^{U(1)}_C = 0 \tag{47}
\]

Thus, after T-duality, we find a new set of branes in the \(SU(2)\) theory \cite{15}

\[
|B, \hat{j}, \eta = \pm 1\rangle_C = \frac{1}{\sqrt{k}} |B, \eta\rangle^{U(1)}_C \sum_{\hat{n}=0}^{2k-1} |A\hat{j}\hat{n}\rangle^{PF}_C \tag{48}
\]

5.1 Shape of the branes

The shape of these branes can be found in the semiclassical limit.

The primaries of the \(SU(2)\) theory can be written

\[
\Phi^{j}_{m\bar{m}} = V^{j}_{m\bar{m}} e^{im\zeta + \bar{m}\bar{\zeta}} \tag{49}
\]

where \(V^{j}_{m\bar{m}}\) is an operator in the parafermion theory and \(\zeta\) is a free boson.

In coordinates where the metric of \(SU(2)\) is \(ds^2 = d\eta^2 + \cos^2 \eta d\gamma^2 + \sin^2 \eta d\theta^2\)

\(m + \bar{m}\) and \(m - \bar{m}\) are the momenta conjugate to translations in \(\gamma\) and \(\theta\) respectively.

The B-branes are Neumann with respect to \(\zeta\) i.e. they only couple to states with \(m = \bar{m} = 0\). The coupling to these states is the same for both A- and B-branes.

In other words, if the A-brane satisfies in the large \(k\) limit

\[
\langle x|A\rangle = \sum c^{j}_{m\bar{m}} \langle x|\Phi^{j}_{m\bar{m}}\rangle \tag{50}
\]

then the B-brane satisfies

\[
\langle x|B\rangle = \sum c^{j}_{00} \langle x|\Phi^{j}_{00}\rangle \tag{51}
\]

Hence (upto an overall constant)

\[
\langle x|B\rangle = \int \frac{d\theta}{2\pi} \frac{d\gamma}{2\pi} \langle x|A\rangle \tag{52}
\]

In coordinates where the metric of \(SU(2)\) is \(ds^2 = d\tilde{\psi}^2 + \sin^2 \tilde{\psi}(d\tilde{\chi}^2 + \sin^2 \tilde{\chi}d\tilde{\omega}^2)\),

the explicit expression for the shape of the A-brane is

\[
\langle x|A\rangle \sim T \delta(\cos \tilde{\psi} - \cos \tilde{\psi}_0) \tag{53}
\]

These coordinates are related to the other coordinates through the relation

\[
\cos \tilde{\psi} = \cos \eta \sin \gamma \quad \sin \tilde{\psi} \sin \tilde{\chi} = \sin \eta \quad \tilde{\omega} = \theta \tag{54}
\]

Hence

\[
\langle x|B\rangle = \int \frac{d\gamma}{2\pi} T \delta(\cos \tilde{\psi} - \cos \tilde{\psi}_0) = \frac{T}{\pi \sqrt{2}} \frac{\Theta(\cos \eta - \cos \tilde{\psi}_0)}{\sqrt{(\cos 2\eta - \cos 2\tilde{\psi}_0)}} \tag{55}
\]

in agreement with \cite{15}. 

11
5.2 Parafermionic branes in AdS$_3$

We expect a similar set of new branes in AdS$_3$. We shall use the intuition from the SU(2) model to find the shape of these new branes. This is not guaranteed to work, as the structure of the primaries in the SL(2, $\mathbb{R}$) model is so different. As we shall see, the naive calculation we do here does not give the correct answer, but provides a guide to it.

The shape of the A-branes in the large $k$ limit is

$$\langle x|A \rangle \sim T\delta(\sinh \psi - \sinh \psi_0)$$ (56)

$\psi$ is related to cylindrical coordinates through the relation $\sinh \psi = \sinh \rho \sin \phi$. If we use the formula (52), then we find the shape of the B-branes to be

$$\langle x|B \rangle \sim T \int d\phi \frac{\delta(\sinh \psi - \sinh \psi_0)}{2\pi} = \frac{T}{\pi} \Theta(\sinh \rho - \sinh \psi_0) \sqrt{\sinh^2 \rho - \sinh^2 \psi_0}$$ (57)

These branes can be visualized as covering the whole of AdS$_3$ except for a cylindrical hole in the middle.

5.3 Overlaps of the branes

To check these calculations, we now ask if the branes satisfy Cardy’s condition. First we will look at the overlap of the A-brane with the B-brane.

The overlaps of the A- and B-branes in AdS$_3$ is divergent and can be understood in the same way as in section (4.4).

The intersection region has a volume

$$\int d\rho d\psi d\tau \left( \frac{\cosh \rho \sinh \rho \cosh \psi}{\sqrt{\sinh^2 \rho - \sinh^2 \psi_0}} \right) (T\delta(\sinh \psi - \sinh \psi_0))$$

$$\times \left( \frac{T}{\pi} \Theta(\sinh \rho - \sinh \psi_0) \sqrt{\sinh^2 \rho - \sinh^2 \psi_0} \right)$$ (58)

which is divergent as $\rho$ tends to infinity at fixed $\psi, \tau$. In this region the volume is

$$V \sim \int_0^{\psi_0} d\rho d\psi d\tau \frac{T^2}{\pi} \delta(\psi - \psi_0) \sim \int d\tau \rho_0 \frac{T^2}{\pi}$$ (59)

The overlap of primaries is now

$$\int d\rho d\psi d\tau \left( \frac{\cosh \rho \sinh \rho \cosh \psi}{\sqrt{\sinh^2 \rho - \sinh^2 \psi_0}} \right) \left( \frac{T}{\pi} \Theta(\sinh \rho - \sinh \psi_0) \sqrt{\sinh^2 \rho - \sinh^2 \psi_0} \right)$$

$$\times T \int \frac{ds}{2\pi} q^{-\frac{1}{4}} q^{s^2 - \frac{1}{2}} e^{-i\tilde{\psi}_0 s} e^{i\tilde{\psi} s} \cosh \psi$$ (60)
As \( \rho \) tends to infinity at fixed \( \psi, \tau \), this tends to

\[
\int_{\rho_0}^{\rho} d\rho d\psi d\tau \int \frac{ds}{2\pi} \frac{T^2}{\pi} q^{-\frac{\rho}{2}q^{\frac{s^2}{4(k-2)}}} e^{-i\tilde{\psi}s} e^{i\tilde{\psi}s}
= \int d\tau \frac{T^2}{\pi} q^{-\frac{1}{4}\rho_0} \tag{61}
\]

The oscillator contribution comes from \( \zeta \), which has Neumann boundary conditions on the B-brane and Dirichlet boundary conditions on the A-brane, and from the parafermions, which have the same boundary conditions on both branes. Accordingly the oscillators contribute a factor

\[
\frac{1}{\prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n}) (1 + q^{2n})} \tag{62}
\]

After modular transformation, we find that the resulting open string partition function can be understood as the partition function of three bosons, one having Neumann-Dirichlet conditions, and the other two satisfying Neumann boundary conditions.

Hence the overlap so far is consistent with Cardy’s condition. However, this only tests the proposal \([57]\) in the limit \( \rho \sim \infty \). So we have established so far that

\[
\lim_{\rho \to \infty} \langle x | B \rangle = e^{-\rho} \tag{63}
\]

We now turn to the overlap of the B-brane with itself. As usual there is a divergence in the overlap coming from the region \( \rho \sim \infty \).

At \( \rho \sim \infty \), \( e^{-\rho} \) is an eigenvalue of the Laplacian

\[
\Box e^{-\rho} = \frac{1}{\cosh \rho \sinh \rho} \partial_\rho \left( \cosh \rho \sinh \rho \partial_\rho e^{-\rho} \right) \sim -e^{-\rho} \tag{64}
\]

with eigenvalue \(-1 - s^2 = -1\). Hence in this limit

\[
\langle x | q^{L_0 + \bar{L}_0 - \frac{k}{2}} | B \rangle \sim q^{-\frac{1}{4}} e^{-\rho} \tag{65}
\]

In other words, in this limit, only the primary with \( s = 0 \) contributes.

The overlap of primaries is then

\[
\int dp d\phi d\tau \cosh \rho \sinh \rho \langle B | x \rangle \langle x | q^{L_0 + \bar{L}_0 - \frac{k}{2}} | B \rangle \sim \int d\tau q^{-\frac{1}{4}} \rho_0 \tag{66}
\]

The oscillators contribute the usual factor \([\prod_{n=1}^{\infty} (1 - q^{2n})]^{-3}\).

The open string partition function is obtained after a modular transformation, and can easily be seen to have the interpretation of the partition function of three free bosons with Neumann boundary conditions.

This is so far only the leading divergent piece. If this is to be the exact final answer, then the overlap of the branes should only come from the \( s = 0 \) state. In the large \( k \) limit, this is the primary satisfying

\[
\Box \Phi = -\Phi \tag{67}
\]
which is solved by $\Phi = \frac{1}{\cosh \rho} \, 2F_1(\frac{1}{2}, \frac{1}{2}, 1, \tanh^2 \rho)$.

In this limit the shape of the B-brane is therefore

$$\langle x | B \rangle \propto \frac{1}{\cosh \rho} \, 2F_1(\frac{1}{2}, \frac{1}{2}, 1, \tanh^2 \rho)$$

(68)

which is consistent with the limit (63).

We can therefore write the boundary state for the parafermionic brane as

$$| B \rangle = T \, | I^0 \rangle$$

(69)

where $| I^0 \rangle$ is the coherent state defined in (17)

$$| I^0 \rangle = \sum_{I,J} M_{IJ}^{-1} K_{-i} K_{-j} | \Phi^0 \rangle$$

(70)

where $K^1, K^2$ are the parafermionic modes, and $K^3$ is the free boson $\zeta$. $| \Phi^0 \rangle$ is the primary satisfying $(L_0 + \bar{L}_0 - \frac{c}{12}) \, | \Phi^0 \rangle = (-\frac{1}{2}) \, | \Phi^0 \rangle$

This describes a 3-brane covering the entire $AdS_3$ space.

In the flat space limit, the brane couples only to modes with $s^2 \sim p^2 = 0$. Hence it appears locally like a flat space three-brane. The other branes which do not cover the whole $AdS_3$ do not satisfy Cardy’s condition. This is in agreement with the flat space limit, where we do not have boundary states for open branes.

The total integral of the three-form field strength through the brane is nonzero. However, the brane is noncompact, so there is no obvious problem.

6 Summary

We have discussed several different branes in $AdS_3$. The ones that satisfy Cardy’s condition are

- The 1-brane: This is a brane wrapping the conjugacy class $\sinh \psi = \text{constant}$.
  The boundary state for this brane is given in (19).

- The 2-instanton: This is a brane wrapping the conjugacy class $\cosh \rho \sinh \tau = \text{constant}$. The boundary state for this brane is given in (24).

- The parafermionic 3-brane: This is a brane wrapping the whole $AdS_3$ space.
  The boundary state for this brane is given in (69).

The overlaps of various combinations of these branes have been computed in [14], and in this paper, and from them the open string spectrum on these branes can be calculated.

The overlap of a 1-brane with a 1-brane was calculated in [14], and the open string spectrum was found. In the Lorentzian framework, the states on the brane were given by the discrete states $\hat{D}_j^{\pm \omega}$ and the continuous states $\hat{C}_j^{\alpha \omega}$. (This was originally found...
in [13]). In the Euclidean theory, the open strings are in the continuous representations of SL(2, C).

The open string spectrum between a 2-instanton and a 2-instanton in the Euclidean theory is the same as that between a 1-brane and a 1-brane i.e. the strings are in the continuous representations. There is no Lorentzian analogue for the 2-instantons, since these are not states in the Lorentzian theory.

The open string spectrum between a 2-instanton and a 1-brane does not preserve the current algebra, and is the theory of three free bosons, two with Neumann-Dirichlet conditions and one with Neumann boundary conditions.

The theory on the open strings connecting the three-brane to itself is the theory of three free bosons with Neumann boundary conditions.

The theory on the open strings connecting the three-brane to a 1-brane is the theory of three free bosons, two with Neumann boundary conditions and the third with Neumann-Dirichlet boundary conditions. The theory on the open strings connecting the three-brane to a 2-instanton is the same.

We have only found the spectrum of the various open strings. The interactions of these strings are still largely unknown. This is an important question that deserves further study.

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8 Appendix

$AdS_3$ is the universal cover of the group manifold $SL(2, \mathbb{R})$. A convenient parametrization of a $SL(2, \mathbb{R})$ element is provided by the Euler angles:

$$g = e^{i \sigma_2} e^{\rho \sigma_3} e^{i \sigma_2} =$$

$$= \begin{pmatrix}
    \cos t \cosh \rho + \cos \phi \sinh \rho & \sin t \cosh \rho - \sin \phi \sinh \rho \\
    -\sin t \cosh \rho - \sin \phi \sinh \rho & \cos t \cosh \rho - \cos \phi \sinh \rho
\end{pmatrix}$$

(71)

where $\sigma_i, i = 1, 2, 3$ are the Pauli matrices, and we define:

$$u = \frac{1}{2}(t + \phi) \quad v = \frac{1}{2}(t - \phi)$$

(72)

These are the cylindrical (global) coordinates of $AdS_3$. The metric is then:

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2$$

(73)

After a Wick rotation we obtain Euclidean $AdS_3$, which is the homogeneous space $SL(2, \mathbb{C})/SU(2)$. We can parametrize Euclidean $AdS_3$ in the following coordinate systems:
**Cylindrical coordinates.** In these coordinates

\[ ds^2 = d\rho^2 + \sinh^2 \rho d\phi^2 + \cosh^2 \rho d\tau^2 \]  

(74)

The coordinates satisfy \( 0 \leq \rho < \infty, 0 \leq \phi < 2\pi, -\infty < \tau < \infty \).

**Poincare coordinates.** In these coordinates

\[ ds^2 = \frac{1}{r^2} \left( dr^2 + dx^2 + d\tilde{\tau}^2 \right) \]  

(75)

The coordinates satisfy \( 0 \leq r < \infty, -\infty < x < \infty, -\infty < \tilde{\tau} < \infty \).

**1-brane coordinates** or **AdS\(_2\) coordinates** defined as

\[ \sinh \psi = \sinh \rho \sinh \phi, \quad \cosh \psi \sinh \chi = -\sinh \rho \cos \phi, \quad \tau = \tau \]  

(76)

The metric in these coordinates is

\[ ds^2 = d\psi^2 + \cosh^2 \psi d\chi^2 + \cosh^2 \psi \cosh^2 \chi d\tau^2 \]  

(77)

**2-instanton coordinates**, defined as

\[ \sinh \alpha = \cosh \rho \sinh \tau \quad \cosh \alpha \cosh \beta = \cosh \rho \cosh \tau, \quad \phi = \phi \]  

(78)

with \( -\infty < \alpha < \infty, 0 \leq \beta < \infty \). The metric in these coordinates is

\[ ds^2 = d\alpha^2 + \cosh^2 \alpha d\beta^2 + \cosh^2 \alpha \sinh^2 \beta d\phi^2 \]  

(79)

**0-instanton coordinates**, defined as

\[ \cosh \lambda = \cosh \rho \cosh \tau \quad \sinh \lambda \sin \sigma = \cosh \rho \sinh \tau, \quad \phi = \phi \]  

(80)

with \( 0 \leq \lambda < \infty, -\frac{\pi}{2} < \sigma \leq \frac{\pi}{2} \). The metric in these coordinates is

\[ ds^2 = d\lambda^2 + \sinh^2 \lambda d\sigma^2 + \sinh^2 \lambda \cos^2 \sigma d\phi^2 \]  

(81)

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