BILIMITS ARE BIFINAL OBJECTS

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Abstract. We prove that a (lax) bilimit of a 2-functor is characterized by the existence of a limiting contraction in the 2-category of (lax) cones over the diagram. We also investigate the notion of bifinal object and prove that a (lax) bilimit is a limiting bifinal object in the category of cones. Everything is developed in the context of marked 2-categories, so that the machinery can be applied to different levels of laxity, including pseudo-limits.

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Introduction

The theory of limits and colimits sits at the core of category theory and it has become an essential tool to express natural constructions of interest to many areas of mathematics. As the formalism of category theory matured, it became clear that some important phenomena are better understood when framed in a 2-categorical setting. It was therefore natural to pursue a generalization of the useful concept of (co)limits in such a context. The notions of 2-limit and 2-colimit were first formulated independently byAuderset in [5], where the Eilenberg–Moore and the Kleisli category of a monad are recovered as a 2-limit and 2-colimit, and by Borceux–Kelly [6], who introduced the notion of enriched limits and colimits and so in particular limits and colimits enriched in the cartesian closed category of

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small categories. These notions were further studied and developed by Street [24], Kelly [19, 18] and Lack [22], who also introduced and investigated the lax and weighted versions.

More recently two papers by clingman and Moser appeared in the literature, namely [9] and [10], where they investigate whether the well-known result that limits are terminal cones extends to the 2-dimensional framework. The first one proves that the answer is negative, i.e. terminal cones are no longer enough to capture the correct universal property, no matter what flavour of slice category one uses. In the second paper, the authors leverage on results from double-category theory on representability of Cat-valued functors to show that being terminal still captures the notion of limit, provided one is willing to work with an alternative 2-category than that of cones, there denoted by $\text{mor}(F)$ for a given diagram $F$.

**Motivations.** The main goal of this paper is to clarify, with its main result (Theorem 4.10), that a natural characterization of lax bilimits in terms of cones is still possible. The use of a marking on the domain of the 2-functor of which we want to study the bilimit addresses in a fundamental way all the possible levels of laxity of the bilimit (pseudo, lax or anything in between). We remark that this further level of generality is necessary from a technical viewpoint to coherently interpolate laxity from pseudo to lax in all the 2-categorical constructions. For example, a similar technique is also needed in the development of flat pseudo-functors, and therefore in the theory of 2-categorical filteredness as well as in the theory of 2-topoi, carried out by Descotte, Dubuc and Szyld in [13].

Another reason that drove us to write this paper was filling the gap in the literature as concerns final objects in 2-category theory. Although some results on final 2-functors already appeared in [1] by Abellán García and Stern (and in the $(\infty,2)$-categorical context in [14] by the authors), we consider establishing the connections between final objects, contractions and bilimits long overdue.

Finally, further motivation comes from the realm of homotopy theory and in particular of $(\infty,2)$-category theory. On the one hand, we wanted to build some low-dimensional intuition based on the $(\infty,2)$-categorical treatment of (lax, weighted) (co)limits we developed in [14]. On the other hand, we are convinced that classical 2-category theory and $(\infty,2)$-category theory can benefit significantly from each other techniques and results. Indeed, the results of this paper have been written with a fibrational point of view typical of weak higher categories. From this perspective, the use of marked edges is all the more natural. Conversely, 2-category theory encompasses many structures and results that would be useful to generalize to $(\infty,2)$-categories, as they are needed in derived algebraic geometry. Consider for instance the theory of 2-(co)filtered 2-categories and 2-Ind/2-Pro constructions. These are worked out by Descotte and Dubuc in [12] for 2-categories and their $(\infty,2)$-categorical counter-part is a useful tool in geometry, see for instance [23, §A.3] of Porta and Sala. Profiting to a greater extend of such an interactive relationship will be the subject of further investigations.

**Technical summary.** In this work, we insist on keeping the category of cones as the object of interest, but we claim that the notion of “terminality” is not the correct one to consider (hence the no-go theorem of [9]). Instead, we focus
our attention on bifinal objects, and we make use of the formalism of marked 2-categories and contractions to characterize (lax) bilimits as bifinal objects in the 2-category of cones. The reason why this is not needed in the 1-dimensional case is that in such context final objects coincide with terminal ones. That is to say, an object $c$ of a category $C$ is terminal if and only if the inclusion functor $\{c\} \to C$ is a final functor or equivalently if the projection $C_{/c} \to C$ has a section mapping $c$ to the identity $1_c$. It is therefore natural to investigate the meaning of finality of a 2-functor $\{c\} \to \mathcal{C}$, with $\mathcal{C}$ a 2-category with a set of marked 1-cells, and to inspect well-behaved sections of the projection 2-functor $\mathcal{C}_{/c} \to \mathcal{C}$ for a pseudo/lax version of the slice 2-category. By doing so, one is led to consider conditions of local terminality, by which we mean choices of terminal objects in the hom-categories $\mathcal{C}(a,c)$, for all objects $a$ of $\mathcal{C}$. Exploiting this point of view, we show that the appropriate 2-categorical generalizations of these characterizations of terminality still coincide and are equivalent to a “coherent” notion of local terminality known as contraction.

An archetypical example of terminal object in category theory is given by the identity arrow $1_x$, thought as an object of $C_{/x}$ for $C$ a category and $x$ an object of $C$. Armed with the understanding of 2-categorical terminal conditions as contractions or as suitable final 2-functor from a singleton, one can examine the (lax) 2-slice $\mathcal{C}_{/x}$ for a 2-category $\mathcal{C}$ and an object $x$ of $\mathcal{C}$. This is the 2-categorical equivalent of the slice, but where the triangles of $\mathcal{C}$ with tip $x$ defining the 1-cells $\mathcal{C}_{/x}$ are filled with a 2-cell of $\mathcal{C}$. A first interesting observation concerns the local terminal objects: for an arrow $f : a \to x$ of $\mathcal{C}$, the terminal objects of the hom-category $\mathcal{C}_{/x}(f, 1_x)$ are precisely the cartesian edges of the projections $\mathcal{C}_{/x} \to \mathcal{C}$. These cartesian edges are simply the triangles $\mathcal{C}$ with edges $f$ and $1_x$ such that the 2-cell filling it is invertible. Said otherwise, such a cartesian edge is the data of a 1-cell $g : a \to x$ of $\mathcal{C}$ together with an invertible 2-cell between $f$ and $g$. But there are potentially other cartesian edges of $\mathcal{C}_{/x}$ from $f : a \to x$ to another arrow $g : b \to x$ of $\mathcal{C}$, which are just 1-cells $h : a \to b$ together with an invertible 2-cell between $f$ and $hg$. Adopting a marking of these cartesian edges, which acts as a book-keeping device, we show that $1_x$ satisfies a finer finality property with respect to all the cartesian edges. In particular, we find that $1_x$ is biterminal in the sub-2-category where the objects are the same as in $\mathcal{C}_{/x}$ and the hom-categories are the full sub-categories spanned by the cartesian edges. So a more classical terminality property can be obtained, if we are willing to restrict to these triangles filled with invertible 2-cells. But this biterminal property does not characterise the stronger local terminality in the full (lax) 2-slice.

Once we turn to (lax) bilimits, keeping track of all the cartesian edges proves to be the right approach. Given 2-functor $F : \mathcal{J} \to \mathcal{C}$ and an object $\ell$ of $\mathcal{C}$, a (lax) 2-cone over $F$ with tip $\ell$ is a (lax) 2-natural transformation from the constant 2-functor on $\ell$ to $F$. This is the 2-categorical equivalent of a cone over $F$, where the triangles constituting the 2-cone are filled with 2-cells, which can be invertible or not according to the laxity prescribed. The (lax) 2-cones over $F$ can be canonically organised to form a 2-category of (lax) 2-cones $C_{/F}$. Again this is similar to the classical category of cones over a functor, but now every triangle is filled with a 2-cell of $\mathcal{C}$. A pair $(\ell, \lambda)$ is said to be a (lax) bilimit of the functor $F$ if the representable hom-category $\mathcal{C}(x, \ell)$ is equivalent to the category of (lax) 2-cones over $F$ with tip $x$; that is, the pair $(\ell, \lambda)$ 2-represents the (lax) 2-cones over $F$. The main result of the paper (Theorem 4.10, together with Proposition 4.6) states that the pair $(\ell, \lambda)$
is a (lax) bilimit of $F$ if and only if the pair $(\ell, \lambda)$, thought as an object of the 2-category $C_{/F}$ of (lax) 2-cones over $F$, is final with respect to the class of cartesian edges of the projection $C_{/F} \to \mathcal{C}$. The local terminality that we get is of the following kind: If $(\ell, \lambda)$ is a (lax) bilimit of the 2-functor $F: \mathcal{J} \to \mathcal{C}$ and $(x, \alpha)$ is another (lax) 2-cone over $F$, then every morphism of (lax) 2-cones $(f, \sigma)$ from $(x, \alpha)$ to $(\ell, \lambda)$, i.e., any object of the hom-category $C_{/x} \to (x, \alpha), (\ell, \lambda) /\emptyset$, with $\sigma$ an invertible “2-cell” is terminal; here $f: x \to \ell$ is a morphism of $\mathcal{C}$, $\alpha$ is a (lax) cone of $x$ over $F$ and $\sigma$ is a modification from $\lambda$ precomposed (component-wise) by $f$ to $\alpha$. Said otherwise, any such morphism of (lax) cones over $F$, morally a triangle over $F$, filled by an invertible 2-cell is terminal in the appropriate hom-category. These morphisms of (lax) 2-cones are precisely the cartesian edges of the projection $C_{/F} \to \mathcal{C}$ with target the (lax) 2-cone $(\ell, \lambda)$. In particular, if we restrict to the hom-categories spanned by the cartesian edges we get that $(\ell, \lambda)$ is bifinal.

**Structure of the article.** The paper is structured as follows. After a preliminary section where we fix the notation, we briefly recall the necessary background on 2-categories and relevant constructions, namely joins, slices and the Grothendieck construction for fibrations of 2-categories.

We then move on to Section 2, where we introduce lax marked bilimits and contractions (following Descotte, Dubuc and Szlyd [13]), and we prove in Proposition 2.2.11 that final objects can be characterized in several ways, one of which involves contractions.

Next, Section 3.1 is an investigation on representable fibrations and the properties of the corresponding representing objects. By looking at the cartesian morphisms of the projection $p: C_{/\ell} \to \mathcal{C}$ we are able to highlight the difference from the simpler case of ordinary 1-categories, drawing a parallelism in Proposition 3.2.4. Moreover, we carve out a full subcategory of the category of maps over a given cone on a diagram, that we prove in Proposition 3.3.8 to be equivalent to the slice over the tip of the given cone. Note that this is a low-tech version of Corollary 5.1.6 from [14].

Finally, in the last section we blend all together, culminating in the main result recorded as Theorem 4.10, which characterizes (lax, marked) bilimits as limiting bifinal cones. Furthermore, we prove that such bilimits are also terminal in the appropriate subcategory of cones obtained by restricting to cartesian morphisms between them.

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1. **Preliminaries**

1.1. **Basic notions and notations.**

1.1.1. Recall that a 2-category $\mathcal{C}$ is composed by a class $\text{Ob}(\mathcal{C})$ of objects or 0-cells, a class of 1-cells, a class of 2-cells, identities for objects and 1-cells, source and target functions for 1-cells and 2-cells, together with a “horizontal” composition $*_{0}$ for 1-cells and 2-cells and a “vertical” composition $*_{1}$ for 2-cells. The identity of an object or 1-cell $x$ will be denoted by $1_{x}$. We will denote the composition of 1-cells
just by juxtaposition. Given a 1-cell \( f: x \to y \) and a 2-cell \( \alpha: g \to g': y \to z \), we will denote by \( \alpha *_0 f \) the whiskering, by which we mean the composition \( \alpha *_0 1_f \). For any pair of objects \( x, y \), we will denote by \( C(x, y) \) the category whose objects are 1-cells of \( C \) from \( x \) to \( y \) and the morphisms are the 2-cells between these.

The category of small 2-categories and strict 2-functors will be denoted by \( 2\text{-}C \), while its full subcategory spanned by 1-categories will be denoted by \( C \). The 2-category of small 2-categories, pseudo-functors and pseudo-natural transformations will be denoted by \( 2\text{-}C_{ps} \). By pseudo-functor we will always mean normal pseudo-functors, i.e., unit preserving.

Example 1.1.2. The canonical example of a 2-category is given by \( C \), where natural transformations play the role of 2-cells.

1.1.3. We will denote by
\[
D_0 = \bullet, \quad D_1 = 0 \to 1, \quad D_2 = \begin{array}{c} 0 \\ \searrow \end{array} 1
\]
the 2-categories corepresenting, respectively, the free-living object, 1-cell and 2-cell in \( 2\text{-}C \).

1.1.4. Given a 2-category \( C \), one can consider the 2-category \( C^{\text{op}} \) obtained from \( C \) by reversing all the 1-cells, i.e., formally swapping source and target. By reversing the 2-cells of \( C \), we get another 2-category which we denote by \( C_{co} \). Thus, for every pair of objects \( x, y \), we have \( C(x, y)^{\text{op}} = C_{co}(x, y) \). Combining these two dualities, we get a 2-category \( C_{co}^{\text{op}} \), where both 1-cells and 2-cells of \( C \) have been reversed.

1.1.5. A 1-cell \( f: x \to y \) of a 2-category \( C \) is an equivalence if we can find a 1-cell \( g: y \to x \) together with invertible 2-cells \( gf \to 1_x \) and \( fg \to 1_y \). A biequivalence of 2-categories is a 2-functor \( F: C \to D \) such that
\[
C(x, y) \to D(Fx, Fy)
\]
is an equivalence of categories for every pair of objects \( x, y \) of \( C \), and for every object \( z \) of \( D \) we can find an object \( c \) of \( C \) equipped with an equivalence \( Fc \to z \). Biequivalences of 2-categories are the weak equivalences for a model category structure of 2-categories established by Lack [20, 21]. In particular, we shall use the fact that the class of biequivalences enjoys the 2-out-of-3 property. Furthermore, in order to check whether a 2-functor \( F: C \to D \) is a biequivalence it is sufficient to check that:

- the 2-functor \( F \) is surjective on objects,
- the 2-functor \( F \) is full on 1-cells, i.e., lifts 1-cells with prescribed lifts of the boundary 0-cells (i.e., objects),
- the 2-functor \( F \) is fully faithful on 2-cells, i.e., it uniquely lifts 2-cells with prescribed lifts of the boundary 1-cells.

In fact, such a 2-functor is a trivial fibration in the above-mentioned model category structure. In detail, the first condition corresponds to a lifting property of the form:
\[
\begin{array}{c}
\varnothing \\
\downarrow \\
D_0
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow \\
\varnothing 
\end{array}
\begin{array}{c}
C \\
\downarrow \\
F
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
D
\end{array}
\]
The second one is encoded by the following lifting property:

\[
\begin{array}{c}
D_0 \coprod D_0 \longrightarrow \mathcal{C} \\
\downarrow (s,t) \quad \downarrow F \\
D_1 \longrightarrow \mathcal{D}.
\end{array}
\]

Here we used the globular notation, so that \( s,t : D_0 \to D_1 \) are the functors mapping the unique object of \( D_0 \) to 0, resp. 1. Finally, the third condition merges together two lifting properties, depicted below:

\[
\begin{array}{cc}
\begin{array}{c}
D_1 \\
\downarrow (s,t)
\end{array} & \begin{array}{c}
D_0 \coprod D_0 \\
\downarrow F
\end{array} \\
\begin{array}{c}
D_2 \\
\longrightarrow \mathcal{D}
\end{array} & \begin{array}{c}
D_1 \coprod D_1 \\
\downarrow (\text{id}, \text{id}) \\
\longrightarrow \mathcal{D}
\end{array}
\end{array}
\]

An analogous characterisation holds if \( F : \mathcal{C} \to \mathcal{D} \) is instead a pseudo-functor (see [21]).

**Definition 1.1.6.** A marked 2-category is a pair \((\mathcal{C}, E)\) where \( \mathcal{C} \) is a 2-category and \( E \) is a class of 1-cells in \( \mathcal{C} \) containing the identities. A marked 2-functor between marked 2-categories \( F : (\mathcal{C}, E_\mathcal{C}) \to (\mathcal{D}, E_\mathcal{D}) \) is a 2-functor \( F : \mathcal{C} \to \mathcal{D} \) that maps every marked 1-cell in \( E_\mathcal{C} \) to a marked 1-cell in \( E_\mathcal{D} \), i.e., \( F(E_\mathcal{C}) \subseteq E_\mathcal{D} \). We denote by \( 2\text{-Cat}^\ast \) the category of marked 2-categories and marked 2-functors.

Whenever we will consider a marked 2-category, we will only mention the non-trivial marked 1-cells, i.e., those which are not identities.

1.1.7. We say that an object \( c \) of a 2-category \( \mathcal{C} \) is quasi-terminal, or that \( \mathcal{C} \) admits \( c \) as a quasi-terminal object, if for any object \( x \) of \( \mathcal{C} \) the category \( \mathcal{C}(x, c) \) has a terminal object. The notion of quasi-terminal object was introduced in [17] by Jay under the name of locally terminal object. The term quasi-terminal appears also in [3], where Ara and Maltsiniotis extend this to strict \( n \)-categories, for \( 1 \leq n \leq \infty \).

1.1.8. Let \( \mathfrak{J} = (\mathfrak{J}, E) \) be a marked 2-category, \( \mathcal{C} \) be a 2-category and \( F, G : \mathfrak{J} \to \mathcal{C} \) be 2-functors. A lax \( E \)-natural transformation \( \alpha : F \to G \) consists of the following data:

- a 1-cell \( \alpha_i : F_i \to G_i \) of \( \mathcal{C} \) for any \( i \) in \( \text{Ob}(\mathfrak{J}) \),
- a 2-cell \( \alpha_k : G(k) *_0 \alpha_i \to \alpha_j *_0 F(k) \) of \( \mathcal{C} \), that we depict as follows

\[
\begin{array}{ccc}
F_i \downarrow \alpha_i \downarrow & & \alpha_j \downarrow \\
\alpha_k \downarrow & \simeq & \downarrow \\
G_i \downarrow \alpha_k \downarrow & & G_j
\end{array}
\]

for every 1-cell \( k : i \to j \) of \( \mathfrak{J} \),

satisfying the following conditions:

**identity:** we have \( \alpha_1 = 1_{\alpha_i} \) for all objects \( i \) of \( \mathfrak{J} \);

**marking:** the 2-cell \( \alpha_k \) is invertible for all \( k \) in \( E \);

**compositions:** we have \( \alpha_k = (\alpha_i *_0 Fk) *_1 (Gl *_0 \alpha_k) \), for all \( k : i \to i' \) and \( l : i' \to i'' \) 1-cells in \( \mathfrak{J} \),

**compatibility:** we have \( \alpha_l *_1 (G\delta *_0 \alpha_i) = (\alpha_j *_0 F\delta) *_1 \alpha_k \), for every 2-cell \( \delta : k \to l: i \to j \) of \( \mathfrak{J} \).
We will denote by \([J, C]_E\) the 2-category of 2-functors from \(J\) to \(C\), lax \(E\)-natural transformations and modifications.

Remark 1.1.9. In the notation of the previous paragraph, if \(E\) consists only of the identity 1-cells of \(J\) (resp. all the 1-cells of \(J\)), we recover the notion of lax natural transformation (resp. pseudo natural transformation).

Remark 1.1.10. The notion of \(E\)-natural transformation was introduced in [13, Definition 2.1.1] by Descotte, Dubuc and Szyld, where they are called \(\sigma\)-natural transformation and the marking is denoted by \(\Sigma\).

1.2. Join and slices of 2-categories.

1.2.1. The notions of join and slice were generalized by Ara and Maltsiniotis [4] to the category of strict \(\infty\)-categories (also known as \(\omega\)-categories). By truncating, their notion also provides a definition for strict \(n\)-categories (see [4, Ch. 8]). In fact, there are at least two sensible notions of join that one can give for strict higher categories, due to the choice of the variance of the higher cells: the lax and the oplax join (cf. [4, Remark 6.37]). The two joins collapse to the classical 1-categorical join once we truncate with respect to the higher cells. This choice of variance is important in relation to the kind of slice one intends to consider. We shall mainly consider slices over a 2-functor, and in this case the lax variance enjoys better formal properties. We start by recalling the general formalism that allows us to get the generalised (op)lax slices.

1.2.2. Given two 2-categories \(A\) and \(B\), their lax join \(A \star B\) is the following 2-category:

- the objects are those of \(A \amalg B\), that we denote by \(a \star \emptyset\) and \(\emptyset \star b\), for \(a \in \text{Ob}(A)\) and \(b \in \text{Ob}(B)\);
- to the 1-cells of the coproduct 2-category \(A \amalg B\), that we denote by \(f \star \emptyset\) and \(\emptyset \star g\), for any \(f:a \to a'\) in \(A\) and any \(g:b \to b'\) in \(B\), we add a 1-cell \(a \star b:a \star \emptyset \to \emptyset \star b\) for every pair of objects \((a, b)\) in \(\text{Ob}(A) \times \text{Ob}(B)\), and then closing by composition in the obvious way;
- to the 2-cells of the coproduct 2-category \(A \amalg B\), that we denote by \(\alpha \star \emptyset\) and \(\emptyset \star \beta\), for any \(\alpha: f \Rightarrow f'\) in \(A\) and every \(\beta: g \Rightarrow g'\) in \(B\), we add 2-cells

\[
\begin{array}{ccc}
\emptyset \star b & \Rightarrow & \emptyset \star g \\
\alpha \star & \Rightarrow & \beta \\
\emptyset \star b' & \Rightarrow & \emptyset \star g'
\end{array}
\]

and

\[
\begin{array}{ccc}
a \star \emptyset & \Rightarrow & a \star b \\
a \star & \Rightarrow & a \star b \\
a' \star \emptyset & \Rightarrow & a' \star b
\end{array}
\]

for any \(f:a \to a'\) in \(A\) and \(g:b \to b'\) in \(B\). We close by composition in the obvious way, but imposing the following relations.
- For any \(a\) in \(\text{Ob}(A)\) and \(b\) in \(\text{Ob}(B)\), the 2-cells \(a \star 1_b\) and \(1_a \star b\) are identities;
- for any \(\alpha: f \Rightarrow f'\) in \(A\) and \(b\) in \(\text{Ob}(B)\), we impose

\[f' \star b \star 1 (a' \star b \star 0 \alpha \star \emptyset) = f \star b;\]

- for any \(a\) in \(\text{Ob}(A)\) and \(\beta: g \Rightarrow g'\) in \(B\), we impose

\[a \star g' \star 1 (\emptyset \star \beta \star 0 a \star b) = a \star g;\]
This gives a monoidal category structure on 2-Cat, the category of small 2-categories, whose unit is the empty 2-category. This monoidal structure is not closed, but it is locally closed in the sense that the functors

\[ 2\text{-Cat} \to 2\text{-Cat}^{A/}, \quad \mathcal{B} \mapsto (A \star \mathcal{B}, i_A: A \to A \star \mathcal{B}) \]

\[ 2\text{-Cat} \to 2\text{-Cat}^{B/}, \quad \mathcal{A} \mapsto (A \star \mathcal{B}, i_{\mathcal{B}}: \mathcal{B} \to A \star \mathcal{B}), \]

both have a right adjoint, where \( A \xrightarrow{i_A} A \star \mathcal{B} \xleftarrow{i_{\mathcal{B}}} \mathcal{B} \) are the canonical inclusions. These adjoint functors are denoted by

\[ 2\text{-Cat}^{A/} \to 2\text{-Cat}, \quad (\mathcal{C}, u: A \to \mathcal{C}) \mapsto \mathcal{C}_{u/} \]

\[ 2\text{-Cat}^{B/} \to 2\text{-Cat}, \quad (\mathcal{C}, v: \mathcal{B} \to \mathcal{C}) \mapsto \mathcal{C}_{/v} \]

and called the (generalized) lax slice functors. The 2-categories \( \mathcal{C}_{u/} \) and \( \mathcal{C}_{/v} \) correspond to the 2-category of lax cocones under \( u \) and the 2-category of lax cones over \( v \), respectively. If \( A \) and \( \mathcal{B} \) are both the terminal (2-)category \( D_0 \) and \( u \) and \( v \) correspond to the object \( c \) of \( C \), then the generalized lax slices over and under the object \( c \) will simply be denoted by \( \mathcal{C}_{c/} \) and \( \mathcal{C}_{/c} \).

1.2.3. We now provide an explicit description of the 2-categorical lax slice over a point. Let \( \mathcal{C} \) be a 2-category and \( c \) be an object of \( \mathcal{C} \).

- The objects of \( \mathcal{C}_{c/} \) are pairs \((x, \alpha)\), with \( x \) in \( \text{Ob} \mathcal{C} \) and \( \alpha: x \to c \) a 1-cell of \( \mathcal{C} \). They correspond to 2-functors \( D_0 \times D_0 \to \mathcal{C} \) and \( D_0 \times D_0 \cong D_1 \).
- A 1-cell of \( \mathcal{C}_{c/} \) from \((x, \alpha)\) to \((y, \beta)\) is given by a 2-functor \( D_1 \times D_0 \to \mathcal{C} \) such that its restriction to \( \{0\} \times D_0 \) is \((x, \alpha)\) (resp. to \( \{1\} \times D_0 \) is \((y, \beta)\)). Explicitly, this means that such a 1-cell is a pair \((f, \gamma)\), where \( f: x \to y \) is a 1-cell and \( \gamma: \beta f \to \alpha \) is a 2-cell of \( \mathcal{C} \), that we depict by

- A 2-cell of \( \mathcal{C}_{c/} \) from \((f, \gamma)\) to \((g, \delta)\) is given by a 2-functor \( D_2 \times D_0 \to \mathcal{C} \) such that the appropriate restrictions to \( D_1 \times D_0 \) are \((f, \gamma)\) to \((g, \delta)\). Explicitly, this means that such a 2-cell is given by a 2-cell \( \Xi: f \to g \) of \( \mathcal{C} \) satisfying

\[ \delta \star_1 (\beta \star_0 \Xi) = \gamma, \]
that we can depict by

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{\alpha} & \searrow{g} & \downarrow{\beta} \\
  c & \quad & c
\end{array}
\]

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{\gamma} & \searrow{\delta} & \downarrow{\xi} \\
  c & \quad & c
\end{array}
\]

Compositions of cells is inherited by \(\mathcal{C}\).

The 2-category \(\mathcal{C}_{\mathcal{J}}\) admits a similar description, and it is canonically isomorphic to the 2-category \((\mathcal{C}^{op})^{op}\).

1.2.4. Based on the description of the slice over an object of a 2-category, we provide the description of the lax slice of a 2-category \(\mathcal{C}\) over a 2-functor \(F: \mathcal{J} \to \mathcal{C}\).

- The objects of \(\mathcal{C}_{F}\) are pairs \((x, \alpha)\), where \(x\) is an object of \(\mathcal{C}\), and \(\alpha\) consists of a family of 1-cells \((\alpha_i: x \to F_i)_{i \in \text{Ob}(\mathcal{J})}\) of \(\mathcal{C}\), and a family of 2-cells indexed by 1-cells \(k: i \to j\) of \(\mathcal{J}\), such that for every 2-cell \(\Lambda: k \to k'\) in \(\mathcal{J}\) the following relation is satisfied

\[
\alpha_k = \alpha_{k'} \ast_1 (F \Lambda \ast_0 \alpha_i).
\]

Furthermore, this assignment has to respect identities and be functorial. More precisely, \(\alpha_1 = 1_{\alpha_i}\) for all objects \(i\) of \(\mathcal{J}\) and given another 1-cell \(h: i' \to i''\) of \(\mathcal{J}\), we have

\[
\alpha_{hk} = \alpha_h \ast_1 (F h \ast_0 \alpha_k).
\]

They correspond to 2-functors \(D_0 \ast \mathcal{J} \to \mathcal{C}\) such that the restriction to the 2-functor \(\emptyset \ast \mathcal{J} \to \mathcal{C}\) is \(F\), i.e., lax cones over \(F\).

- A 1-cell of \(\mathcal{C}_{F}\) from \((x, \alpha)\) to \((y, \beta)\) is given by a 2-functor \(D_1 \ast \mathcal{J} \to \mathcal{C}\) such that its restriction to \(\{0\} \ast D_0\) is \((x, \alpha)\) (resp. to \(\{1\} \ast D_0\) is \((y, \beta)\)). It is a lax morphism of lax cones over \(F\). Explicitly, this means that such a 1-cell is a pair \((f, \mu)\), where \(f: x \to y\) is a 1-cell of \(\mathcal{C}\) and \(\mu_i: \beta_i f \to \alpha_i\) is a 2-cell of \(\mathcal{C}\) for all \(i\) in \(\text{Ob}(\mathcal{J})\), that we depict by

\[
\begin{array}{ccc}
  x & \xrightarrow{f} & y \\
  \downarrow{\alpha} & \searrow{\beta} & \downarrow{\beta_i} \\
  F_i & \quad & F_i
\end{array}
\]

That is, for every object \(i\) of \(\mathcal{J}\) the pair \((f, \mu_i)\) is a 1-cell in \(\mathcal{C}_{F_i}\), which corresponds to the restriction \(D_1 \ast \{i\} \to \mathcal{C}\). Moreover, the set of 2-cells \(\mu\) must satisfy the following relations:

\[
\alpha_k \ast_1 (F k \ast_0 \mu_i) = \mu_j \ast_1 (\beta_k \ast_0 f)
\]
for every 1-cell \(k:i \to j\) of \(\mathcal{J}\), that we can depict as the following commutative diagram

\[
x \xrightarrow{\alpha_j} F j \\
\downarrow \alpha_k \downarrow \beta_i \downarrow \delta_i \\
y \xrightarrow{\alpha_i} F i
\]

in \(\mathcal{C}\) and corresponds to \(D_1 \star \{k:i \to j\} \to \mathcal{C}\).

- A 2-cell of \(\mathcal{C}_F\) from \((f, \mu)\) to \((g, \delta)\) is given by a 2-functor \(D_2 \star \mathcal{J} \to \mathcal{C}\) whose restrictions to \(D_1 \star \mathcal{J}\) are \((f, \mu)\) to \((g, \delta)\). Explicitly, this means that such a 2-cell is given by a 2-cell \(\Xi : f \to g\) of \(\mathcal{C}\) satisfying \(\delta_i \star (\beta_i \star \Xi) = \mu_i\) for all \(i\) in \(\text{Ob}(\mathcal{J})\), i.e., it provides a 2-cell from \((f, \mu_i)\) to \((g, \delta_i)\) in the slices of \(\mathcal{C}\) over the objects \(F i\).

**Remark 1.2.6.** For a 1-cell \((f, \mu):(x, \alpha) \to (y, \beta)\) of \(\mathcal{C}_F\) we have not imposed any equations to be satisfied for a 2-cell \(\Gamma : k \to k'\) of \(\mathcal{J}\). This is because they are implied by the relations already present. To be more precise, let us consider \(D_1 \star D_2\) as a strict \(\infty\)-category, in fact a strict 4-category, and let us describe its 2-categorical truncation in detail.

There are four “generating” 3-cells in \(D_1 \star D_2\) (generating here is meant in the sense of polygraphs or computads, see for instance [4, 1.4]). These are sent via \((f, \mu):D_1 \star D_2 \to \mathcal{C}\) to the identities witnessing:

- the relation (1) for \(x\) and \(\Lambda\), that we denote by \(x \star \Lambda\);
- the relation (1) for \(y\) and \(\Lambda\), that we denote by \(y \star \Lambda\);
- the relation (2) for \(f\) and \(k\), that we denote by \(f \star k\);
- the relation (2) for \(f\) and \(k'\), that we denote by \(f \star k'\).

These are all relations that we have by assumption on \((x, \alpha)\) and \((y, \beta)\) or by the requirement for 1-cells of \(\mathcal{J}\) that \((f, \mu)\) must satisfy. The unique 4-cell of \(D_1 \star D_2\) has as source (resp. target) an appropriate whiskered composition of the preimages of \(x \star \Lambda\) and \(f \star k\) (resp. of \(f \star k'\) and \(y \star \Lambda\)). Since truncation identifies source and target 2-cells of every 3-cell, we see that the condition for (what we can denote by) \(f \star \Lambda\) is already satisfied.

A similar argument explains why a 2-cell \(\Xi\) of \(\mathcal{C}_F\) needs no further coherence constraints than those corresponding to objects of \(\mathcal{J}\), the ones with respect to 1-cells and 2-cells of \(\mathcal{J}\) being automatically satisfied.

1.2.7. Let \(\mathcal{J} = (\mathcal{J}, E)\) be a marked 2-category and \(F:\mathcal{J} \to \mathcal{C}\) a 2-functor. As with the natural transformations, we can use the marking \(E\) in order to impose invertibility conditions on the slice \(\mathcal{C}_F\). We denote by \(\mathcal{C}^{/\mathcal{J}}_E\), or simply by \(\mathcal{C}^{/\mathcal{J}}\) when \(E\) is clear from the context, the full sub-2-category of \(\mathcal{C}_F\) spanned by the objects \((x, \alpha)\) for which the 2-cell \(\alpha_k : F k \star 0 \alpha_k \to \alpha_j\) is invertible for all \(k:i \to j\) in \(E\).

If \(E\) consists of just the identity 1-cells of \(\mathcal{J}\), the 2-categories \(\mathcal{C}^{/\mathcal{J}}_E\) and \(\mathcal{C}^{/\mathcal{J}}\) coincide. If \(E\) consists of all the 1-cells of \(\mathcal{J}\) then we get the pseudo slice of \(\mathcal{C}\) over \(F\).
1.2.8. For $\tilde{A} = (A, E_A)$ and $\tilde{B} = (B, E_B)$ two marked categories, it is possible to define their marked join $\tilde{A} * \tilde{B}$. This is the 2-category $\mathcal{A} * \mathcal{B}$ where we formally invert all the 2-cells of the kind

$$\begin{array}{ccc}
a \ast \emptyset & \xrightarrow{a + b} & \emptyset * b \\
\downarrow a \ast g & & \downarrow b \\
\emptyset & & \emptyset
d
d\end{array}$$

and

$$\begin{array}{ccc}
a \ast \emptyset & \xrightarrow{a + b} & b \\
\downarrow f \ast \emptyset & & \downarrow f \ast b \\
\emptyset & & \emptyset
d\end{array}$$

for $f$ an element of $E_A$ and $g$ an element of $E_B$. This operation defines a functor $\mathcal{C} \rightarrow \mathcal{C} * \mathcal{C}$, which agrees with the standard join on minimally marked 2-categories. The functors

$$\begin{array}{ccc}
\mathcal{C} & \rightarrow & \mathcal{C} * \mathcal{C} \\
\mathcal{C} & \rightarrow & \mathcal{C} * \mathcal{C}
d\end{array}$$

both have a right adjoint functor. These adjoint functors are denoted by

$$\begin{array}{ccc}
\mathcal{C} & \rightarrow & \mathcal{C} \\
\mathcal{C} & \rightarrow & \mathcal{C}
d\end{array}$$

We stress the fact that we consider the output of the marked join as a bare 2-category, without marking. Even though it inherits a natural marking from the two inputs, it will play no role in what follows.

1.2.9. Following the notation of the previous paragraph, let $\tilde{A}$ be the terminal marked 2-category $\mathcal{D}_0$, so that a marked 2-functor $\mathcal{D}_0 \rightarrow \mathcal{C}$ simply corresponds to an object $c$ of $\mathcal{C}$. The marked 2-category $\mathcal{C}^{\mathcal{C}^{\mathcal{C}}}$ (resp. $\mathcal{C}^{\mathcal{C}^{\mathcal{C}}}$) as underlying 2-category (cf. 1.2.3) and the marked edges are the triangles filled with an invertible 2-cell. These marked 2-categories appear in [1] where Abellán García and Stern denote them by $\mathcal{C}^{\mathcal{C}^{\mathcal{C}}}$.

1.3. BILIMITS ARE BIFINAL OBJECTS. In this section we recall the notions of $\mathcal{E}$-bilimit of a 2-functor with source a marked 2-category $(\mathcal{J}, \mathcal{E})$.

1.3.1. Let $\mathcal{J} = (\mathcal{J}, \mathcal{E})$ be a marked 2-category, $\mathcal{C}$ a 2-category and $F: \mathcal{J} \rightarrow \mathcal{C}$ a 2-functor. For a given object $x$ of $\mathcal{C}$, we denote by $\Delta x$ the constant 2-functor on $x$ from $\mathcal{J}$ to $\mathcal{C}$. Given a 2-functor $F: \mathcal{J} \rightarrow \mathcal{C}$, an $\mathcal{E}$-lax $F$-cone $\alpha$ with vertex $x$, that we shall often denote by $(x, \alpha)$, is an object of $[\mathcal{J}, \mathcal{E}]_x(\Delta x, F)$. This means that for every
object \( i \) of \( \mathcal{J} \) we have a 1-cell \( \alpha_i : x \to Fi \) of \( \mathcal{C} \), for every 1-cell \( k : i \to j \) of \( \mathcal{J} \) we have a 2-cell
\[
\begin{array}{ccc}
x & \xrightarrow{\alpha_k} & x \\
\downarrow{\alpha_i} & \cong & \downarrow{\alpha_j} \\
Fi & \xrightarrow{Fk} & Fj
\end{array}
\]
in invertible whenever \( k \) is in \( E \). Moreover, this assignment has to respect the identities and be functorial. Finally, for every 2-cell \( \Gamma : k \to k' \) of \( \mathcal{J} \) we have the relation
\[
\alpha_k = \alpha_{k'} * \alpha._1 (FT *_0 \alpha_i).
\]

Hence, an \( E \)-lax \( F \)-cone with vertex \( x \) corresponds to an object \((x, \alpha)\) of the 2-category \( \mathcal{C}^F \), that is the \( E \)-lax slice of \( \mathcal{C} \) over \( F \).

**Remark 1.3.2.** Notice that the assignment of a 2-functor \( F' : D_0 \triangleright \mathcal{J} \to \mathcal{C} \) corresponds to the data of the 2-functor \( F \) together with an \( E \)-lax cone \( \lambda : \Delta x \to F \) over \( F \), where \( x \) is the image via \( F' \) of the unique object of \( D_0 \).

The 2-category \( D_0 \triangleright \mathcal{J} \) will be denoted by \( \mathcal{J}^\triangleright \), and we will use \( \mathcal{J}^\triangleleft \) for the dual case of cocones.

1.3.3. Let \( F : \mathcal{J} \to \mathcal{C} \) be a 2-functor, with \( \mathcal{J} = (\mathcal{J}, E) \) a marked 2-category, and \( \alpha : \Delta x \to F \) an \( E \)-lax \( F \)-cone. For every object \( z \) of \( \mathcal{C} \) there is a canonical functor
\[
\alpha^* = \alpha \cdot \Delta(-) : \mathcal{C}(z, x) \to [\mathcal{J}, \mathcal{C}](\Delta z, F)
\]
given by post-composition with \( \alpha \). More explicitly, for \( f : z \to x \) a 1-cell of \( \mathcal{C} \) we define \( \alpha \cdot f \) to be \( \alpha_i f : z \to Fi \) for every \( i \) in \( \text{Ob}(\mathcal{J}) \) and \( \alpha_k *_0 f \) for every 1-cell \( k : i \to j \) of \( \mathcal{J} \). A 2-cell \( \zeta : f \to g : z \to x \) defines an evident modification \( \alpha \cdot \zeta : \alpha \cdot f \to \alpha \cdot g \).

**Definition 1.3.4.** For a 2-functor \( F : \mathcal{J} \to \mathcal{C} \), with \( \mathcal{J} = (\mathcal{J}, E) \) a marked 2-category, an \( E \)-lax \( F \)-cone \((\ell, \lambda)\) is said to be an \( E \)-bilimit cone if the canonical functor
\[
\lambda' = \lambda \cdot \Delta(-) : \mathcal{C}(x, \ell) \to [\mathcal{J}, \mathcal{C}](\Delta x, F)
\]
is an equivalence of categories for every \( x \) in \( \text{Ob}(\mathcal{C}) \). We also say that \((\ell, \lambda)\) is the \( E \)-bilimit of \( F \).

The dual definition gives the notion of \( E \)-bicolimit of \( F \), which is just the \( E^{\text{op}} \)-bilimit of the 2-functor \( F^{\text{op}} : \mathcal{J}^{\text{op}} \to \mathcal{C}^{\text{op}} \).

**Remark 1.3.5.** The notions of \( E \)-cones and \( E \)-bilimits were introduced in [13, Definition 2.4.3], and called \( \sigma \)-cones and \( \sigma \)-colimits by Descotte, Dubuc and Szyld where the marking \( E \) is there denoted by \( \Sigma \), and emerge naturally in the study of flat pseudo-functors.

**Remark 1.3.6.** The case of weighted \( E \)-bilimits can be recovered from that of \( E \)-bilimits, as proven in [13, Theorem 2.4.10]. In fact, given a 2-functor \( F : \mathcal{J} \to \mathcal{C} \) and a weight \( W : \mathcal{J} \to \text{Cat} \), if we denote by \( p : E_{\mathcal{W}} \to \mathcal{J} \) the 2-fibration associated with the 2-functor \( W \) (cf. Proposition 1.4.15), then the \( E \)-bilimit of \( F \) weighted by \( W \) is precisely the \( E_{\mathcal{W}} \)-bilimit of \( F \circ p : E_{\mathcal{W}} \to \mathcal{C} \), where the marking \( E_{\mathcal{W}} \) is described in [13, Definition 2.4.8].
1.4. Grothendieck construction for 2-categories. Fibrations of 2-categories were initially introduced by Hermida in his paper [16], but his definition is not powerful enough to obtain a Grothendieck construction for such fibrations. This notion was later perfected by Buckley, who gave the correct definition in [7] and proved the corresponding (un)straightening theorem. In what follows we present a concise summary of the main results which are relevant for our treatment. We start by recalling the standard notion of cartesian edge and cartesian fibration for the 1-categorical setting.

**Definition 1.4.1.** Let $p: E \rightarrow B$ be a functor between categories. An arrow $f: x \rightarrow y$ of $E$ is $p$-cartesian if the square

$$
\begin{array}{ccc}
E(a, x) & \xrightarrow{f_\circ a} & E(a, y) \\
\downarrow_{p_{a, x}} & & \downarrow_{p_{a, y}} \\
B(pa, px) & \xrightarrow{p(f)_\circ a} & B(pa, py)
\end{array}
$$

is a pullback square of sets for any $a$ in $\text{Ob}(E)$. The functor $p: E \rightarrow B$ is a cartesian, or Grothendieck, fibration if for any object $e \in E$ and any arrow $f: b \rightarrow p(e)$ in $B$ there exists a $p$-cartesian arrow $h: a \rightarrow e$ in $E$ with $p(h) = f$.

Now we recall the notion of cartesian 1-cell and 2-cell for the 2-categorical setting.

**Definition 1.4.2.** Let $p: E \rightarrow B$ be a 2-functor between 2-categories.

- A 1-cell $f: x \rightarrow y$ in $E$ is $p$-cartesian if the following square is a pullback of categories for every $a$ in $\text{Ob}(E)$:

$$
\begin{array}{ccc}
E(a, x) & \xrightarrow{f_\circ a} & E(a, y) \\
\downarrow_{p_{a, x}} & & \downarrow_{p_{a, y}} \\
B(pa, px) & \xrightarrow{p(f)_\circ a} & B(pa, py)
\end{array}
$$

- A 2-cell $\alpha: f \Rightarrow g: x \rightarrow y$ in $E$ is $p$-cartesian if it is a $p_{x, y}$-cartesian 1-cell, with $p_{x, y}: \mathcal{E}(x, y) \rightarrow \mathcal{B}(px, py)$.

1.4.3. We spell out explicitly the pullback defining a $p$-cartesian 1-cell $f: x \rightarrow y$, since we will need it in the following sections. The pullback gives existence and uniqueness property both at the level of 1-cells and of 2-cells.

**1-cells:** for every 1-cell $g: a \rightarrow y$ of $\mathcal{E}$ and every 1-cell $k: pa \rightarrow px$ in $\mathcal{B}$ such that $pf \circ_0 k = pg$, there is a unique 1-cell $\bar{k}: a \rightarrow x$ such that $f\bar{k} = g$ and $p\bar{k} = k$.

**2-cells:** for every 2-cell $\alpha: g \Rightarrow g': a \rightarrow y$ of $\mathcal{E}$ and every 2-cell $\tau: k \rightarrow h: pa \rightarrow px$ in $\mathcal{B}$ such that $pf \circ_0 \tau = \bar{h}$, there is a unique 2-cell $\beta: \bar{k} \rightarrow \bar{h}$ such that $f \circ_0 \beta = \alpha$ and $p\beta = \tau$.

1.4.4. The notion of cartesian fibration for 2-categories amounts to the existence of enough cartesian lifts, as in the 1-dimensional case, but it also requires an additional property: cartesian 2-cells must be closed under horizontal composition. Note that, by definition and by the obvious fact that cartesian 1-cells are closed under composition, cartesian 2-cells are automatically closed under vertical composition.

**Definition 1.4.5.** A 2-functor between 2-categories $p: \mathcal{E} \rightarrow \mathcal{B}$ is called a 2-fibration if it satisfies the following properties:
(1) for every object $e \in \mathcal{E}$ and every 1-cell $f : h \to p(e)$ in $\mathcal{B}$ there exists a $p$-cartesian 1-cell $h : a \to e$ in $\mathcal{E}$ with $p(h) = f$.

(2) for every pair of objects $x, y$ in $\mathcal{E}$, the map $p_{x,y} : \mathcal{E}(x,y) \to \mathcal{B}(px,py)$ is a Cartesian fibration of categories.

(3) cartesian 2-cells are closed under horizontal composition, i.e., for every triple of objects $(x, y, z)$ in $\mathcal{E}$, the functor $e_{x,y,z} : \mathcal{E}(y, z) \times \mathcal{E}(x, y) \to \mathcal{E}(x, z)$ sends $p_{y,z} \times p_{x,y}$-cartesian 1-cells to $p_{x,z}$-ones.

In this case we call $\mathcal{E}$ the total 2-category of the fibration $p$, and $B$ is said to be the base 2-category.

**Remark 1.4.6.** Observe that condition (3) can be rephrased by requiring that given 1-cells in $\mathcal{E}$ of the form $f : w \to x$ and $g : y \to z$, the whiskering functors $\circ f : \mathcal{E}(x, y) \to \mathcal{E}(w, y)$ and $g \circ - : \mathcal{E}(x, y) \to \mathcal{E}(x, z)$ preserve cartesian 1-cells. This follows from the fact that horizontal composition can be obtained from vertical composition and whiskerings.

**Definition 1.4.7.** Given a cartesian 2-fibration $p : \mathcal{E} \to \mathcal{B}$ we will denote by $\mathcal{E}_{\text{cart}}$ the sub-2-category of $\mathcal{E}$ spanned by all objects and the $p$-cartesian edges between them. More precisely, $\mathcal{E}_{\text{cart}}$ is the 2-category whose objects are the elements of $\text{Ob}(\mathcal{E})$, and such that, for every pair of objects $x$ and $y$, the hom-category $\mathcal{E}_{\text{cart}}(x, y)$ is the full subcategory of $\mathcal{E}(x, y)$ spanned by $p$-cartesian edges.

The motivation for introducing 2-fibrations is that they are a convenient way to encode functors into $\text{2-Cat}$ or $\text{2-Cat}_{ps}$. More precisely, we have the following result, which is a combination of Theorems 2.2.11 and 3.3.12 in [7].

**Theorem 1.4.8.** There exists a biequivalence of 2-categories between $\text{2Fib}_{ps}(\mathcal{B})$ and $[\text{B}_{\text{co}}^{op}, \text{2-Cat}_{ps}]_{ps}$, the former being the 2-category of 2-fibrations over $\mathcal{B}$ equipped with a choice of cartesian lifts compatible with composition, pseudo-functors preserving 1-cartesian and 2-cartesian cells and making the obvious triangle commute up-to-isomorphism and pseudo-natural transformations satisfying the obvious property, while the latter is the 2-category of (strict) 2-functors into $\text{2-Cat}_{ps}$, pseudo-natural transformations and modifications.

**Remark 1.4.9.** In the proof of Theorem 1.4.8 (specifically in the proof of [7, Theorem 2.2.11]), Buckley exhibits a “Grothendieck construction” 2-functor $[\text{B}_{\text{co}}^{op}, \text{2-Cat}] \to \text{2Fib}_{ps}(\mathcal{B})$ (in fact, a 3-functor, but we will not need this additional structure) which is surjective on objects up-to-isomorphism. In particular, such a biequivalence of 2-categories preserves and reflects equivalences (cf. 1.1.5); the equivalences of $\text{2Fib}_{ps}(\mathcal{B})$ are simply the cartesian preserving pseudo-functors that are biequivalences of 2-categories, while the equivalences of $[\text{B}_{\text{co}}^{op}, \text{2-Cat}]$ are the pseudo-natural transformations that are object-wise biequivalences of 2-categories.

The reason why we need to introduce pseudo-functoriality and pseudo-naturality is that the quasi-inverse of a strict 2-functor $F : A \to \mathcal{B}$ that is a biequivalence is not in general a strict 2-functor, but instead a pseudo-functor (see [20, Example 3.1]). In turn, via this biequivalence this also reflects the fact that given a 2-natural transformation between two 2-functors $F, G : A \to \text{2-Cat}$ that is object-wise a biequivalence, then the transformation built up using the object-wise quasi-inverses is not a strict 2-natural transformation in general but instead a pseudo-natural one, even if the quasi-inverses are all strict 2-functors. This phenomenon is already evident for 2-natural transformations between 2-functors with values in the 2-subcategory $\text{Cat}$.
of $2\text{-}\text{Cat}$ that are object-wise equivalences; in fact, this will be the case of interest for us.

**Notation 1.4.10.** By considering fibrations over $\mathcal{B}^{op}$, $\mathcal{B}_{co}$ or $\mathcal{B}_{co}^{op}$ we obtain four different variants of 2-fibrations, corresponding to the four possible types of variance for 2-functors $\mathcal{B} \to 2\text{-}\text{Cat}$. Instead of using four different names, we will adopt the name *fibration* for each of these cases, and specify the variance when needed.

1.4.11. In the same paper [7], Buckley proves several weakening of Theorem 1.4.8, by looking at fibrations without a choice of lifts (which correspond to pseudofunctors) and fibrations of bicategories. We content ourselves with the strict case as this is the level of generality needed for this work.

**Remark 1.4.12.** If every 2-cell in the total 2-category $\mathcal{E}$ of a fibration $p: \mathcal{E} \to \mathcal{B}$ is $p$-cartesian, then the fibers $\mathcal{E}_x$ are $(2,1)$-categories for every $x \in \mathcal{B}$, i.e., 2-categories where every 2-cell is invertible. Furthermore, there is at most a 2-cell between each pair of 1-cells $f, g$ in $\mathcal{E}_x$, so we can view these fibers as 1-categories, by quotienting out these invertible 2-cells. We shall call *1-fibrations* this class of fibrations.

**Example 1.4.13.** One of the canonical examples of a 1-fibration is given by slice projections. Given a 2-category $\mathcal{B}$ and an object $x \in \mathcal{B}$, it is easy to see that the projection $p: \mathcal{B}/x \to \mathcal{B}$ from the lax slice of $\mathcal{B}$ over $x$ to $\mathcal{B}$ is a 1-fibration. It corresponds to the hom-2-functor $\mathcal{B}(-, x): \mathcal{B}^{op} \to \text{Cat}$. In particular, every 2-cell in $\mathcal{B}/x$ is $p$-cartesian, which is the reason why the associated 2-functor factors through the inclusion $\text{Cat} \hookrightarrow 2\text{-}\text{Cat}$.

We will only be using the part of Theorem 1.4.8 that deals with 1-fibrations, that we now record.

**Corollary 1.4.14.** There exists a biequivalence of 2-categories between $1\text{Fib}_s(\mathcal{B})_{ps}$ and $[\mathcal{B}_{co}^{op}, \text{Cat}]_{ps}$, the former being the 2-category of 1-fibrations equipped with a choice of cartesian lifts compatible with composition, pseudo-functors preserving 1-cartesian cells and making the obvious triangle commute up-to-isomorphism and pseudo-natural transformations satisfying the obvious property, while the latter is the 2-category of (strict) 2-functors into $\text{Cat}$, pseudo-natural transformations and modifications.

Given the explicit description of the biequivalence in Theorem 1.4.8 provided by Buckley in his paper, we can detail the action on objects of the biequivalence of the previous corollary.

**Proposition 1.4.15.** The following facts hold true:

1. The fibration corresponding to the identity 2-functor $\text{Id}: \text{Cat} \to \text{Cat}$ is given by the forgetful functor

   $$U: \text{Cat}_{D_0}/ \to \text{Cat}$$

   from the lax slice of $\text{Cat}$ under the terminal category to $\text{Cat}$.

2. The fibration $p: \mathcal{E} \to \mathcal{B}$ associated with a 2-functor $F: \mathcal{B} \to \text{Cat}$ is obtained by forming the pullback displayed below.

   $$\begin{array}{ccc}
   \mathcal{E} & \to & \text{Cat}_{D_0}/ \\
   p \downarrow & & \downarrow U \\
   \mathcal{B} & \to & \text{Cat}.
   \end{array}$$
Example 1.4.16. The slice fibration \( \mathcal{C}^{/F} \to \mathcal{C} \) classifies the 2-functor
\[
[\mathcal{J}, \mathcal{C}](\Delta x, F) : \mathcal{C}^{\text{op}} \to \text{Cat}.
\]
Indeed, thanks to Proposition 1.4.15 we have to compute the pullback
\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{p} & (\text{Cat}_{D_0})^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{C} & \xrightarrow{[\mathcal{J}, \mathcal{C}](\Delta x, F)} & \text{Cat}^{\text{op}}.
\end{array}
\]

An object of \( \mathcal{E} \) is a pair \((x, \alpha)\), where \( x \) is an object of \( \mathcal{C} \) and \( \alpha \) an object of \( [\mathcal{J}, \mathcal{C}](\Delta x, F) \). That is, \((x, \alpha)\) is a \( \mathcal{E} \)-lax \( F \)-cone with vertex \( x \), which by what we observed in paragraph 1.3.1 corresponds precisely to an object \((x, \alpha)\) in \( \mathcal{C}^{/F} \).

A 1-cell in \( \mathcal{E} \) from \((x, \alpha)\) to \((y, \beta)\) is a pair \((f, \mu)\), where \( f : x \to y \) is a 1-cell in \( \mathcal{C} \) and \( \mu : \beta \cdot \Delta f \to \alpha \) is a modification. For every object \( i \) in \( \mathcal{J} \), this corresponds to a commutative diagram
\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
F_i & \xrightarrow{\gamma_i} & \beta_i
\end{array}
\]
in \( \mathcal{C} \), i.e., the pair \((f, \mu)\) is a 1-cell of \( \mathcal{C}^{/F} \).

A 2-cell in \( \mathcal{E} \) from \((f, \mu)\) to \((f', \mu')\) corresponds to a 2-cell \( \Xi \) of \( \mathcal{C} \) which is a modification from \( \mu \) to \( \mu' \). That is, for every \( i \) in \( \mathcal{J} \) we have a commutative diagram
\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow & & \downarrow \\
F_i & \xrightarrow{\alpha_i} & \beta_i
\end{array}
\]
\[
\begin{array}{ccc}
x & \xleftarrow{\Xi} & y \\
\downarrow & & \downarrow \\
F_i & \xleftarrow{\gamma_i} & \beta_i
\end{array}
\]
This is precisely a 2-cell \( \Xi : (f, \mu) \to (f', \mu') \) in \( \mathcal{C}^{/F} \).

2. Bifinality

In this section we motivate and provide the definition of contraction and bifinal object and we study some of their basic properties.

2.1. Contractions and local terminality.

2.1.1. Recall that a category \( C \) has a terminal object \( c \) if and only if the canonical projection functor \( C_{/c} \to C \) has a section mapping \( c \) to \( 1_c \). This is straightforward, since such a section provides for each object \( x \) of \( C \) a morphism \( \gamma_x : x \to c \) and moreover for every morphism \( f : x \to c \) the triangle
\[
\begin{array}{ccc}
x & \xrightarrow{f} & c \\
\downarrow & & \downarrow \\
\gamma_x & \xrightarrow{=} & c
\end{array}
\]
of \( C \) must commute.
**Definition 2.1.2.** Let $\mathcal{C}$ be a 2-category and $c$ an object of $\mathcal{C}$. A contraction on $\mathcal{C}$ with center $c$ is section $\gamma: \mathcal{C} \to \mathcal{C}_{/c}$ of the canonical projection $\mathcal{C}_{/c} \to \mathcal{C}$, i.e., a 2-functor $\mathcal{C} \to \mathcal{C}_{/c}$ making the triangle

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\gamma} & \mathcal{C}_{/c} \\
\downarrow & & \downarrow \\
\mathcal{C} & \end{array}
$$

of 2-categories commute, such that:

- the image of the center $c$ is equal to $1_c$; and
- for every $x$ in $\text{Ob}(\mathcal{C})$ the 2-cell $\gamma(\gamma(x))$ (that we will denote by $\gamma_x^2$) is the identity of $\gamma_x$.

We will also say that the object $c$ is *bifinal* if there is a contraction on $\mathcal{C}$ having $c$ as center.

2.1.3. Let us spell out the content of the definition of contraction. Suppose we have such a contraction $\gamma$ on $\mathcal{C}$ with center $c$.

- For every $x$ in $\text{Ob}(\mathcal{C})$ we have a 1-cell $\gamma_x: x \to c$, such that $\gamma_c = 1_c$.
- For every 1-cell $f: x \to y$ of $\mathcal{C}$ we have a 2-cell

$$
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\gamma_x & \downarrow & \gamma_y \\
\gamma f & \\
\end{array}
$$

such that $\gamma_{\gamma_x} = \text{Id}_{\gamma_x}$.
- For every composable pair of 1-cells $f, g$ the relation $\gamma_{g*0f} = \gamma_f *_1 (\gamma_g *_0 f)$ holds.
- For every 2-cell $\alpha: f \to g$ of $\mathcal{C}$ the relation

$$
\gamma_f = \gamma_g *_1 (\gamma_g *_0 \alpha)
$$

holds.

From this explicit description it is easy to see that a contraction on $\mathcal{C}$ with center $c$ also corresponds to the datum of an object $\gamma$ in $[\mathcal{C}, \mathcal{C}](1_c, \Delta_c)$ where $\gamma_c = 1_c$ and such that the collection of the 1-cells of the form $\gamma_x$ are mapped by $\gamma$ to identities. A contraction uniquely determines a set $E_c$ of 1-cells consisting of (the identities together with) the 1-cells of the form $\gamma_x$. With this at hand, it is clear that such a contraction determines an object of $[\mathcal{C}, \mathcal{C}]_{E_c}(1_c, \Delta_c)$.

2.1.4. The notion of contraction in the more general case of strict $\infty$-categories is due to Ara and Maltsiniotis [3] and it also appears under the name of “initial/terminal structure” in an unpublished text of Burroni [8]. Beware that what we call here contraction for simplicity is in fact the dual of the notion given in [3]: our notion of contraction corresponds to what they call dual contraction. Notice also that their definition is equivalent but stated differently. In fact, they use the notion of (lax) Gray tensor product in the definition of contraction. This is a closed monoidal structure that was first introduced by Gray on 2-categories [15] and was later generalized to strict higher categories by Al-Agl and Steiner [2] and alternatively by Crans in his thesis [11].
With the Gray tensor product $\otimes: 2\text{-}Cat \times 2\text{-}Cat \to 2\text{-}Cat$ at our disposal, we can define a contraction on $\mathcal{C}$ with center $c$ to be a 2-functor

$$\gamma: D_1 \otimes \mathcal{C} \to \mathcal{C}$$

such that

- the restriction to $\gamma_{\{0\} \times \mathcal{C}}$ is the identity on $\mathcal{C}$ and the restriction to $\gamma_{\{1\} \times \mathcal{C}}$ is constant on $c$,
- the image of $\{(0 \to 1), \{c\}\}$ is the identity of $c$, and
- for every $x$ in $\text{Ob}(\mathcal{C})$ the image of $\gamma_x = \gamma_{\{(0 \to 1), x\}}$ is the identity 2-cell.

**Remark 2.1.5.** A priori, one might want to weaken the notion of contraction, according to the homotopy coherent approach of (weak) higher categories. Indeed, for a contraction $\gamma$ on $\mathcal{C}$ with center $c$, we might request the 2-cell $\gamma_x$ to be an invertible 2-cell, while we require the stronger condition of being the identity of $\gamma_x$. However, in our 2-categorical framework this stronger condition follows from the relations we impose on 2-cells. More precisely, the 2-cell $\gamma_x = \gamma_{\{(0 \to 1), x\}}$.

As $\gamma_x$ is an invertible 2-cell, this implies that it must be the identity 2-cell of $\gamma_x$.

2.1.6. If $c$ is a bifinal object of a 2-category $\mathcal{C}$, then the contractions we can associate to it are essentially unique. Indeed, let $\alpha$ and $\beta$ be two contractions on $\mathcal{C}$ with center $c$. For every object $x$ of $\mathcal{C}$ we have two 1-cells $\alpha_x, \beta_x: x \to c$. Applying $\alpha$ to $\beta_x$ and $\beta$ to $\alpha_x$ we get two 2-cells

\[
\begin{array}{ccc}
  x & \xrightarrow{\alpha_x} & c \\
  \downarrow^\alpha & \searrow & \downarrow^\beta \\
  c & \swarrow & c
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
  x & \xrightarrow{\beta_x} & c \\
  \downarrow^\beta & \nearrow & \downarrow^\alpha \\
  c & \searrow & c
\end{array}
\]

of $\mathcal{C}$. Finally, applying once more $\alpha$ and $\beta$ to $\beta_{\alpha_x}$ and $\alpha_{\beta_x}$ respectively we get the relations

$$\alpha_{\beta_x} \ast 1_{\beta_x} = 1_{\alpha_x} \quad \text{and} \quad \beta_{\alpha_x} \ast 1_{\alpha_x} = 1_{\beta_x}.$$  

Hence, $\alpha_{\beta_x}$ and $\beta_{\alpha_x}$ are inverses of one another. Similarly, for every 1-cell $f: x \to y$ of $\mathcal{C}$ one gets

(4) \quad $\alpha_{\beta_x} \ast 1_f = \alpha_f \ast (\alpha_{\beta_x} \ast 0_f)$

so that $\alpha_f$ and $\beta_f$ can be obtained one from the other via compositions with invertible 2-cells. In particular, equation (4) provides a pseudo natural transformation $\rho: \alpha \to \beta: \mathcal{C} \to \mathcal{C}_{/c}$ defined by $\rho_x = (1_x, \alpha_{\beta_x})$ and for which the square

\[
\begin{array}{ccc}
  \alpha_x & \xrightarrow{\alpha_f} & \alpha_y \\
  \downarrow^{\rho_x} & \downarrow & \downarrow^{\rho_y} \\
  \beta_x & \xrightarrow{\beta_f} & \beta_y
\end{array}
\]

in $\mathcal{C}_{/c}$ commutes for every 1-cell $f: x \to y$ of $\mathcal{C}$.

The above paragraph shows that if a contraction with center $c$ exists then it is essentially unique. The following result concerns the existence of such a contraction, relating it to the existence of an appropriate supply of terminal arrows with target $c$. 

Proposition 2.1.7. Let \( \mathcal{C} \) be a 2-category and \( c \in \text{Ob}(\mathcal{C}) \) an object. Let \( f_a: a \to c \) be a choice of a morphism from each object \( a \in \text{Ob}(\mathcal{C}) \) to \( c \). Then the following are equivalent:

1. There exists a contraction \( \gamma \) with center \( c \) such that \( \gamma_a = f_a \) for every \( a \).
2. Every \( f_a \) is terminal in \( \mathcal{C}(a, c) \) and \( f_c = 1_c \).

Proof. Assume given a contraction \( \gamma \) with center \( c \) and \( \gamma_a = f_a \) for every \( a \), so that in particular \( f_c = 1_c \) by definition. Then for every \( g:a \to c \) we have a 2-cell \( \gamma_g: g \Rightarrow \gamma_a = f_a \). We claim that this is the unique 2-cell from \( g \) to \( \gamma_a \). Indeed, Given another 2-cell \( \beta: g \Rightarrow \gamma_a \), we can apply \( \gamma \) to it, resulting in the equality \( \gamma_g = \gamma_{\gamma_a} \circ \beta \).

Since \( \gamma_{\gamma_a} \) is an identity 2-cell, we get that \( \beta \) and \( \gamma_g \) must coincide.

On the other hand, suppose that \( f_c = 1_c \) and that each \( f_a \) is terminal in \( \mathcal{C}(a, c) \). We define a contraction \( \gamma \) as follows. On objects we define \( \gamma_a = f_a \) and on 1-cells \( g:a \to b \) set \( \gamma_g \) to be the unique 2-cell \( \gamma_b \circ_0 g \Rightarrow f_a \). We have to check that this assignment defines a contraction. For every pair \( x \xrightarrow{f} y \xrightarrow{g} z \) of composable 1-cells of \( \mathcal{C} \), we have two parallel 2-cells \( \gamma_{g \circ_0 f} \) and \( \gamma_f \circ_1 (\gamma_g \circ_0 f) \) from \( \gamma_z \circ_0 g \circ_0 f \) to \( \gamma_x \).

As the target \( \gamma_z \) is a terminal object in \( \mathcal{C}(x, c) \), it follows that these 2-cells must be equal. For the same reason, for every 2-cell \( \Xi: f \Rightarrow g \) of \( \mathcal{C} \) the relation

\[
\gamma_f = \gamma_g \circ_1 (\gamma_g \circ_0 \Xi)
\]

holds. Finally, \( \gamma_{\gamma_a}; \gamma_a \Rightarrow \gamma_a \) is a 2-cell from a terminal 1-cell to itself and is hence the identity. \( \Box \)

Remark 2.1.8. Proposition 2.1.7 implies in particular that if there exists a contraction with center \( c \) then \( c \) is quasi-terminal. This fact is proven in much greater generality (for strict \( \infty \)-categories) by Ara and Maltsiniotis in [3, Proposition B.13].

2.2. Marked bfinality and final 2-functors. In category theory, a functor \( w: \mathcal{J} \to \mathcal{C} \) is final if for every functor \( F: \mathcal{C} \to \mathcal{D} \) the colimit \( \text{lim} F \) exists if and only if the colimit \( \text{lim} Fu \) does and, whenever they exist, the canonical morphism \( \text{lim} Fu \to \text{lim} F \) is an isomorphism. We now give the appropriate 2-categorical generalization.

Definition 2.2.1. A 2-functor \( w: (\mathcal{C}, E_\mathcal{C}) \to (\mathcal{D}, E_\mathcal{D}) \) of marked 2-categories is said to be final if for every 2-functor \( F: \mathcal{D} \to \mathcal{E} \) with \( \mathcal{E} \) a 2-category, and for every \( x \in \mathcal{E} \), the induced map

\[
[\mathcal{D}, \mathcal{E}]_{E_\mathcal{D}}(F, \Delta x) \to [\mathcal{C}, \mathcal{E}]_{E_\mathcal{C}}(Fu, \Delta x)
\]

is an equivalence of categories.

Remark 2.2.2. It follows directly from the 2-categorical Yoneda lemma that if \( w: (\mathcal{C}, E_\mathcal{C}) \to (\mathcal{D}, E_\mathcal{D}) \) is a final 2-functor and \( F: \mathcal{D} \to \mathcal{E} \) is a 2-functor then \( F \) admits an \( E_\mathcal{D} \)-bicoinit if and only if \( Fu \) admits an \( E_\mathcal{C} \)-bicofinit, and when these two equivalent conditions hold, the canonical 1-cell \( \text{lim} Fu \to \text{lim} F \) is an equivalence.

Remark 2.2.3. The definition of final 2-functor given above is equivalent to the one studied in [1] by Abellán García and Stern, as can be verified by comparing the explicit description of marked cocones in [1, Definition 6.1.3] with the explicit description of \( E \)-lax cocones in Paragraph 1.3.1 above.

A terminal object \( c \) in a category \( C \) can be characterized by the fact that the functor \( c: D_0 \to C \) is final. Indeed, \( c \) is terminal if and only if the identity functor
1_C: C \to C \text{ admits } c \text{ as its colimit. Our goal in this section is discuss the analogous situation in the setting of final 2-functors } D_0 \to \mathcal{C}, \text{ for } \mathcal{C} \text{ a 2-category. For this, we will need to extend the notion of contraction to the framework of marked 2-categories.}

**Definition 2.2.4.** Let \((\mathcal{C}, M)\) be a marked 2-category and \(\gamma\) a contraction on \(\mathcal{C}\) with center \(c\). We say that \(\gamma\) is an \(M\)-contraction if the following two conditions hold:

1. For every \(x \in \text{Ob}(\mathcal{C})\) the edge \(\gamma_x\) of \(\mathcal{C}\) is marked.
2. For every marked edge \(f\) of \((\mathcal{C}, M)\), the 2-cell \(\gamma_f\) of \(\mathcal{C}\) is invertible.

If such an \(M\)-contraction exists, we will say that \(c\) is \(M\)-bifinal.

**Remark 2.2.5.** Any contraction \(\gamma\) with center \(c\) is automatically an \(M_\gamma\)-contraction, where \(M_\gamma\) denotes the collection of edges \(\gamma_a\) for \(a \in \text{Ob}(\mathcal{C})\) (together with the identities).

**Remark 2.2.6.** Let \((\mathcal{C}, M)\) be a marked 2-category, and \(\gamma, \rho\) two contractions with center \(c\). By Paragraph 2.1.6, we have that \(\gamma\) and \(\rho\) are isomorphic as contractions. It then follows that when the collection of 1-cells \(M\) is closed under isomorphism of 1-cells then \(\gamma\) is an \(M\)-contraction if and only if \(\rho\) is an \(M\)-contraction.

**Proposition 2.2.7.** Let \(c \in \mathcal{C}\) be an object and for each \(a \in \text{Ob}(\mathcal{C})\) fix a morphism \(f_a: a \to c\) in such a way that \(f_c = 1_c\). Then \(c\) is bifinal with a contraction \(\gamma\) satisfying \(\gamma_a = f_a\) if and only if the inclusion \(\{c\} \to (\mathcal{C}, M)\) is a final 2-functor, where \(M_c = \{f_a\}_{a \in \text{Ob}(\mathcal{C})}\).

The previous proposition is a particular case of a more general result characterizing the final 2-functors of the form \(\{c\} \to (\mathcal{C}, M)\), for a given marking \(M\) of \(\mathcal{C}\) containing \(M_c\).

**Definition 2.2.8.** Let \((\mathcal{C}, M)\) be a marked 2-category and \(c\) an object of \(\mathcal{C}\). We say that \(c\) is \(\text{pre-final}\) if

1. for every \(a\) in \(\text{Ob}(\mathcal{C})\) the category \(\mathcal{C}(a, c)\) has a terminal object.
2. the identity 1-cell of \(c\) is terminal in \(\mathcal{C}(c, c)\).

Moreover, we say that the pre-final object \(c\) is \(M\)-final if

3. for every \(a\) in \(\text{Ob}(\mathcal{C})\) there exists a marked edge \(a \to c\) in \((\mathcal{C}, M)\);
4. for every marked edge \(f: a \to b\) in \(M\), the induced functor \(f^*: \mathcal{C}(b, c) \to \mathcal{C}(a, c)\) preserves terminal objects.

**Remark 2.2.9.** If \(c\) is \(M\)-final in \((\mathcal{C}, M)\) then every marked edge \(a \to c\) with target \(c\) is terminal in \(\mathcal{C}(a, c)\). To see this, note that since \(1_c\) is terminal in \(\mathcal{C}(c, c)\) and pre-composition with marked edges preserves terminal objects it follows that every marked edges \(a \to c\) is terminal in \(\mathcal{C}(a, c)\). On the other hand, such a marked terminal edge exists in \(\mathcal{C}(a, c)\) by condition (3), and hence any other terminal edge in \(\mathcal{C}(a, c)\) is isomorphic to it. In particular, if the collection \(M\) is closed under isomorphisms of 1-cells then the marked edges in \(\mathcal{C}(a, c)\) are exactly the terminal ones.

**Remark 2.2.10.** In [13, §3.3] Descotte, Dubuc and Szyld work out a theory of \(M\)-final 2-functor in the case where the source is an \(M\)-filtered 2-category. For a marked 2-functor \(\{c\} \to (\mathcal{C}, M)\) the definition of loc. cit. is equivalent to \(c\) being \(M\)-final in the sense of Definition 2.2.8 above, at least when the collection of marked
edges $M$ is closed under composition of 1-cells (as it is assumed in loc. cit.). Indeed, their definition adapted to the case where the source is $D_0$ states that the following conditions are satisfied:

- **C0** for every $a$ in $\text{Ob}(\mathcal{C})$ there exists a 1-cell $a \to c$ in $M$;
- **C1** for every object $a$ of $\mathcal{C}$ and every pair of parallel 1-cells $f, g: a \to c$ with $g$ in $M$, there is a 2-cell $\alpha: f \to g$;
- **C2** for every object $a$ of $\mathcal{C}$, every pair of parallel 1-cells $f, g: a \to c$ with $g$ in $M$, the 2-cell $\alpha: f \to g$ is unique.

These conditions all hold when $c$ is $M$-final by Remark 2.2.9. On the other hand, the combination of the above conditions is clearly equivalent to the statement that for each $a \in \text{Ob}(\mathcal{C})$ there exist marked edges $a \to c$, and that these are all terminal in their respective hom categories. In particular, conditions (1) and (3) hold in this case. In addition, since all identities are marked (2) is implied, and when the collection of marked edges is closed under composition we also get that (4) holds.

The following result shows that the notions of finality for an object that we have introduced so far coincide.

**Proposition 2.2.11.** Let $(\mathcal{C}, M)$ be a marked 2-category, $c$ an object of $\mathcal{C}$, and $f_a: a \to c$ a choice of a marked 1-cell for every $a \in \text{Ob}(\mathcal{C})$ such that $f_c = 1_c$. Then the following statements are equivalent:

1. $c$ is $M$-final;
2. $c$ is $M$-bifinal with contraction $\gamma$ such that $\gamma_a = f_a$;
3. the inclusion $\{c\} \to (\mathcal{C}, M)$ is final.

**Warning 2.2.12.** In Proposition 2.2.11 we implicitly assume in advance that a marked edge $a \to c$ exists for every $a \in \text{Ob}(\mathcal{C})$. This assumption automatically holds if $c$ is $M$-final or $M$-bifinal, but not necessarily if one only assumes that the inclusion $\{c\} \to (\mathcal{C}, M)$ is final. In particular, while Proposition 2.2.11 shows that for a given object $c \in \text{Ob}(\mathcal{C})$ the properties of being $M$-final and $M$-bifinal are equivalent, these conditions are only shown to imply the finality of the inclusion $\{c\} \to (\mathcal{C}, M)$, and to be implied by it if $(\mathcal{C}, M)$ possesses sufficiently many marked edges.

**Proof of Proposition 2.2.11.** $(i) \Rightarrow (ii)$. Assume that $c$ is $M$-final. By Remark 2.2.9 we have that $f_a$ is terminal in $\mathcal{C}(a, c)$ for every $a \in \text{Ob}(\mathcal{C})$. By Proposition 2.1.7 there exists a contraction $\gamma$ such that $\gamma_a = f_a$ for every $a \in \text{Ob}(\mathcal{C})$. To see that $\gamma$ is an $M$-contraction we now note that $\gamma_a$ is marked by construction, and for $g: a \to b$ a marked 1-cell, condition (4) entails that $\gamma_g: \gamma_b \circ g \to \gamma_a$ is a 2-cell between two terminal 1-cells, and is hence invertible. This shows that $\gamma$ is an $M$-contraction.

$(ii) \Rightarrow (iii)$. Let $c$ be an $M$-bifinal object and $\gamma$ an $M$-contraction with center $c$ such that $\gamma_a = f_a$ for every $a \in \text{Ob}(\mathcal{C})$. We need to show that for any 2-functor $F: \mathcal{C} \to \mathcal{D}$, and any $d \in \mathcal{D}$, the evaluation at $c$ map

$$\bullet : (\mathcal{C}, \mathcal{D})_M(F, \Delta d) \to \mathcal{D}(Fc, d),$$

is an equivalence of categories. We construct an explicit inverse

$$\gamma^*: \mathcal{D}(Fc, d) \to (\mathcal{C}, \mathcal{D})_M(F, \Delta d)$$

to this functor by means of post-composition with $\gamma$. More precisely, given $f: Fc \to d$, we can consider the family of 1-cells

$$\{f F(\gamma_x): Fx \to d\}_{x \in \text{Ob}(\mathcal{C})}.$$
and that of 2-cells
\[
(fF(\gamma_h); fF(\gamma_x) \to fF(\gamma_y)F(h))_{h \in \epsilon(x,y)}.
\]
It is clear that these data all organize into an $M$-lax natural transformation $\gamma^*(f)$ from $F$ to $\Delta d$ and one can similarly obtain a modification $\gamma^*(\alpha) \cdot \gamma^*(f) \to \gamma^*(g)$ from a 2-cell $\alpha \cdot f \to g$. We now claim that $\gamma^*$ is inverse to $\bullet_c$. Indeed, on the one hand, the composition $\bullet_c \circ \gamma^*$ is the identity on $\mathcal{D}(Fc, d)$. In the other direction, we need to show that the $M$-lax transformations $\rho$ and $\gamma^*(\rho_c)$ are isomorphic objects in $[\mathcal{C}, \mathcal{D}]/M(F, \Delta d)$, naturally in $\rho$. For $x$ an object of $\mathcal{C}$, we get $\gamma^*(\rho_c)_x = \rho_c F \gamma_x$. This gives us a triangle
\[
\begin{array}{c}
Fx \\ \downarrow \rho_x \\
Fc \
\end{array} \xleftarrow{F \gamma_x} \begin{array}{c}
\downarrow \beta_x \\
d \\
\end{array} \begin{array}{c}
\downarrow \rho_c \\
\end{array}
\]
in $\mathcal{D}$, where the 2-cell $\beta_x$ is $\rho_{\gamma_x}$. As $\gamma_x$ is in $M$ the 2-cell $\rho_{\gamma_x}$ is invertible by assumption. We need to check that the collection of the 2-cells $\beta_x$, for all objects $x$ of $\mathcal{C}$, organizes into an invertible modification between $\rho$ and $\gamma^*(\rho_c)$. For this we must check that given any 1-cell $f: x \to y$ of $\mathcal{C}$, the equation
\[
\rho_f \star_1 (\rho_{\gamma_x} \star_0 Ff) = \rho_{\gamma_x} \star_1 (\rho_c \star_0 F\gamma f).
\]
holds. This is precisely the relation satisfied by the action of $\rho$ on the 2-cell
\[
\begin{array}{c}
x \\ \downarrow \gamma_x \\
\downarrow \gamma_f \\
y \\
\end{array} \xleftarrow{f} \begin{array}{c}
\downarrow \gamma_y \\
c \\
\end{array}
\]
of $\mathcal{C}$. Hence, the functor $\bullet_c$ is an equivalence.

(iii) $\Rightarrow$ (i). Suppose that $c: D_0 \to (\mathcal{C}, M)$ is a final 2-functor. The object $c$ with the cocone $1_c$ is clearly the bicolimit of the 2-functor $c$. In light of Remark 2.2.2 it then follows that the identity 2-functor $1_c: (\mathcal{C}, M) \to \mathcal{C}$ admits an $M$-bicolimit cocone $\gamma$ such that $\gamma_c = 1_c$. Now by assumption every object $a \in \text{Ob}(\mathcal{C})$ admits a marked edge $f_a: a \to c$. Since $\gamma$ is an $M$-lax cocone the 2-cell $\gamma_f: f_a \Rightarrow \gamma_a$ is invertible, and hence $f_a$ and $\gamma_a$ are isomorphic 1-cells in $\mathcal{C}(a, c)$. We now observe that for a general morphism $g: a \to c$ the condition that $\gamma_g$ is invertible is closed under isomorphism in $\mathcal{C}(a, c)$. It then follows that $\gamma_{\gamma_a}$ is invertible as well, so that $\gamma$ constitutes a contraction with center $c$. By Proposition 2.1.7 we now get that each $\gamma_a$ is terminal in $\mathcal{C}(a, c)$, and since $\gamma_c = 1_c$, this means that $c$ is pre-final. In addition, Condition (3) holds by assumption, and since $\gamma$ is an $M$-lax cocone we have that for every marked edge $g: a \to b$ the 2-cell $\gamma_g$ is invertible, and so $\gamma_a \cong g \circ_0 \gamma_b$. We thus have that pre-composition $\mathcal{C}(b, c) \to \mathcal{C}(a, c)$ with any marked edge $g: a \to b$ preserves terminal objects, which shows (4).

Proof of Proposition 2.2.7. This is a particular case of the implication (ii) $\iff$ (iii) of Proposition 2.2.11 applied to $M_c$.  

3. Slice fibrations

In this section we further study the slice fibrations $p: \mathcal{C}^F \to \mathcal{C}$ associated to a marked 2-category $\mathcal{J} = (\mathcal{J}, E)$ and a 2-functor $F: \mathcal{J} \to \mathcal{C}$ as in Paragraph 1.2.7. We begin in §3.1 by giving an explicit description of the $p$-cartesian edges in $\mathcal{C}^F$, and
show that they coincide with the marked edges with respect to the marking of Paragraph 1.2.8. In §3.2 we then focus on the particular case where $\mathcal{J} = D_0$, so that $\mathcal{C}^F = \mathcal{C}^F_{/\mathcal{J} \hookrightarrow \mathcal{C}}$ is a representable fibration, and show that in this case the object $1_{\mathcal{F}(\ast)}$ is the center of a contraction relative to the collection of $p$-cartesian edges. Finally, in §3.3 we construct a modified 2-category of cones which projects to $\mathcal{C}^F$, and show that this projection is a biequivalence if and only if $F$ admits an $E$-bilimit.

3.1. Cartesian edges in slice fibrations.

3.1.1. Let us fix a marked 2-category $\mathcal{J} = (\mathcal{J}, E)$, a 2-functor $F: \mathcal{J} \to \mathcal{C}$, and consider the associated projection 2-functor $p: \mathcal{C}^F \to \mathcal{C}$. We wish to describe the $p$-cartesian morphisms of $\mathcal{C}^F$. Consider an object $(x, \lambda): D_0 \ast \mathcal{J} \to \mathcal{C}^F$, that we can represent by

\[
\begin{array}{ccc}
\lambda_i & x & \lambda_j \\
\downarrow & \downarrow & \downarrow \\
F_i & \mu & F_j
\end{array}
\quad \text{for } h:i \to j \text{ in } \mathcal{J},
\]

with $\lambda_h$ invertible whenever $h$ is in $E$. We claim that every 1-cell $(f, \sigma): D_1 \ast \mathcal{J} \to \mathcal{C}$ from $(z, \alpha)$ to $(x, \lambda)$:

\[
\begin{array}{cc}
z & \overset{f}{\longrightarrow} & x \\
\downarrow & \downarrow & \downarrow \\
\alpha_j & \downarrow & \downarrow \\
F_j & \longrightarrow & F_j
\end{array}
\]

such that the 2-cell $\sigma_j$ of $\mathcal{C}$ is invertible for every object $j$ of $\mathcal{J}$, is a $p$-cartesian lift of $f: z \to x$. In the notations of Paragraph 1.2.8, we are claiming that $(f, \sigma)$ is a $p$-cartesian lift whenever it is represented by a 2-functor $D^1_0 \ast \mathcal{J} \to \mathcal{C}$. One possible such choice is given by the precomposition of $(x, \lambda)$ with $f$, i.e., where $\sigma_j$ is the identity of $\lambda_j f$ for all $j$ in $\text{Ob}(\mathcal{J})$.

Indeed, let $(f', \mu): (z', \alpha') \to (x, \lambda)$ be a 1-cell of $\mathcal{C}^F$ and $g: z' \to z$ a 1-cell of $\mathcal{C}$ such that $fg = f'$. We wish to find an edge $(g, \bar{\mu}): (z', \alpha') \to (z, \alpha)$ such that

\[(f, \sigma) \ast_0 (g, \bar{\mu}) = (f', \mu).\]

Now $(f, \sigma) \ast_0 (g, \bar{\mu}) = (fg, \bar{\mu} \bullet \sigma)$, where we set $(\bar{\mu} \bullet \sigma)_j = \bar{\mu}_j \ast_1 (\sigma_j \ast_0 g)$ so that necessarily $f \ast_0 g = f'$ and $\mu = \bar{\mu} \bullet \sigma$. As $p(g, \bar{\mu}) = g$, we must have $(g, \bar{\mu}) = (g, \mu \bullet \sigma^{-1})$, where we set

\[(\mu \bullet \sigma^{-1})_j = \mu_j \ast_1 (\sigma_j^{-1} \ast_0 g)\]

for all objects $j$ in $\mathcal{J}$. Said otherwise, if such a 1-cell $(g, \bar{\mu})$ of $\mathcal{C}^F$ exists, then for any object $j$ of $\mathcal{J}$ the 2-cell $\bar{\mu}$ of $\mathcal{C}$ must be equal to $\mu_j \ast_1 (\sigma_j^{-1} \ast_0 g)$. This shows that if $(g, \bar{\mu})$ satisfying (5) exists, then it is unique. As for its existence, it is easy to check that the following diagram

\[
\begin{array}{ccc}
z' & \overset{\alpha'}{\longrightarrow} & F_j \\
\downarrow & \downarrow & \downarrow \\
\bar{\mu} & \sigma_0 & \mu \bullet \sigma^{-1} \\
\downarrow & \downarrow & \downarrow \\
z & \overset{z}{\longrightarrow} & F_i
\end{array}
\]

in $\mathcal{C}$ commutes for every edge $h:i \to j$ in $\mathcal{J}$, so that $\bar{\mu} \cdot \alpha' \to \alpha$ defines a 1-cell in $\mathcal{C}^F$. This concludes the existence and uniqueness property for 1-cells.
Let \( \Xi: (f', \mu) \to (f'', \mu') \) be a 2-cell of \( \mathcal{C}^F \). In particular, for any object \( i \) of \( \mathcal{J} \) we have a commutative diagram

\[
\begin{array}{ccc}
z' & \xrightarrow{f'} & x \\
\downarrow{\alpha'_i} & \searrow{\mu'_i} & \downarrow{\lambda_i} \\
F i & = & \mathcal{M} & \rightarrow & F i
\end{array}
\]

in \( \mathcal{C} \). Fix \( \tau: g \to g' \) a 2-cell of \( \mathcal{C} \) such that \( p(\Xi) = \Xi = f \circ p(\tau) \). We know from the argument above that there are unique lifts

\[
(\mu, \tilde{\mu}): (z', \alpha') \to (z, \alpha) \quad \text{and} \quad (g', \overline{\mu}): (z', \alpha') \to (z, \alpha)
\]
such that \( (f, \sigma) \circ_0 (g, \overline{\mu}) = (f', \mu) \) and \( (f, \sigma) \circ_0 (g', \overline{\mu}) = (f'', \mu') \). We wish to find a unique 2-cell \( \Xi: (g, \overline{\mu}) \to (g', \overline{\mu'}) \) of \( \mathcal{C}^F \) such that \( p(\Xi) = \tau \) and

\[
(\nu, \tilde{\mu}) \circ_0 \tau = \Xi.
\]
in \( \mathcal{C}^F \). The condition \( p(\Xi) = \tau \) forces the uniqueness. Since \( 1_{(f, \sigma)} \circ_0 \tau = 1_f \circ_0 \tau = \Xi \), the existence part follows once we show that \( \tau \) is a 2-cell from \( (g, \overline{\mu}) \) to \( (g', \overline{\mu'}) \). For every \( i \) in \( \text{Ob}(\mathcal{J}) \) we have

\[
\begin{align*}
\overline{\mu} \circ_1 (\alpha \circ_0 \tau) &= \mu' \circ_1 (\sigma^{-1} \circ_0 g) \circ_1 (\alpha \circ_0 \tau) \\
&= \mu' \circ_1 (\lambda \circ_0 f \circ_0 \tau) \circ_1 (\sigma^{-1} \circ_0 g') \\
&= \mu' \circ_1 (\lambda \circ_0 \Xi \circ_1 (\sigma^{-1} \circ_0 g')) \\
&= \mu \circ_1 (\sigma^{-1} \circ_0 g') = \tilde{\mu}_i,
\end{align*}
\]
where the second equality is given by the interchange law, the third and fourth by the assumptions and the first and the last one are the definitions of \( \overline{\mu}' \) and \( \tilde{\mu} \), respectively.

We have thus shown that the 1-cell \( (f, \sigma) \) satisfies the properties of \( p \)-cartesian 1-cell detailed in paragraph 1.4.3.

**Proposition 3.1.2.** The \( p \)-cartesian edges \( (f, \sigma): (z, \beta) \to (x, \alpha) \) of \( \mathcal{C}^F \) are all of the form

\[
\begin{array}{ccc}
z & \xrightarrow{f} & x \\
\downarrow{\beta} & \searrow{\alpha} & \downarrow{\alpha_i} \\
\mathcal{M} & \rightarrow & F i
\end{array}
\]

with \( \sigma_i \) an invertible 2-cell of \( \mathcal{C} \) for all \( i \) in \( \text{Ob}(\mathcal{J}) \). In particular, the \( p \)-cartesian edges of \( \mathcal{C}^F \) are precisely its marked edges, as detailed in Paragraph 1.2.8.

**Proof.** We have shown in the paragraphs above that the edges of \( \mathcal{C}^F \) of this form are indeed \( p \)-cartesian. If \( (g, \mu): (z, \beta) \to (x, \alpha) \) is a \( p \)-cartesian edge, then there exists a unique 1-cell \( (1_z, \mu'): (z, \alpha \cdot g) \to (z, \beta) \) such that

\[
(\nu, \tilde{\mu}) \circ_0 (1_z, \mu') = (g, 1_{\alpha \cdot g}),
\]
which is equivalent to the condition \( \mu_i \circ_0 \mu_i = 1_{\alpha_i \cdot g} \) for all \( i \) in \( \text{Ob}(\mathcal{J}) \). But \( \mu_i \) is an invertible 2-cell of \( \mathcal{C} \), since by [7, Proposition 2.1.4(2)] the 1-cell \( (1_z, \mu') \) is an isomorphism of \( \mathcal{C}_F \) and so in \( \mathcal{C}^F \). Hence, \( \mu_i \) is an invertible 2-cell of \( \mathcal{C} \) for all objects \( i \) of \( \mathcal{J} \). \( \square \)
3.1.3. Proposition 3.1.2 yields an explicit description of the sub-2-category \((\mathcal{C}^2F)_{\text{cart}}\) of \(\mathcal{C}^2F\) spanned by all objects and the \(p\)-cartesian edges between them (see Definition 1.4.7). Indeed, the 1-cells of \((\mathcal{C}^2F)_{\text{cart}}\) are given by

\[
\begin{array}{ccc}
z & \xrightarrow{g} & z' \\
\downarrow & \alpha & \downarrow \\
F_i & \xleftarrow{\sigma} & F_i'
\end{array}
\]

with \(\sigma_i\) an invertible 2-cell of \(\mathcal{C}\) for every \(i\) in \(\text{Ob}(\mathcal{J})\).

3.2. **Representable fibrations.** In this section we focus on the particular case of slice fibrations where \(\mathcal{J} = D_0\), that is, a 2-functor \(F: \mathcal{J} \to \mathcal{C}\) simply corresponds to an object \(x \in \mathcal{C}\). In this case \(\mathcal{C}/x\) is equipped with a designated object \(1_x\), whose finality properties we wish to understand. We start by recalling the classical 1-categorical scenario.

**The 1-categorical case.** Let \(p: E \to B\) be a cartesian fibration of 1-categories and suppose that it is represented by an object \(x \in \mathcal{B}\). By this we mean that the functor

\[
B(-, x): B^{\text{op}} \to \text{Set}
\]

is classified by \(p: E \to B\), i.e., we have a pullback square

\[
\begin{array}{ccc}
E & \xrightarrow{p} & (\text{Set}(*))^{\text{op}} \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & \text{Set}^{\text{op}}
\end{array}
\]

of categories. In this case, \(E\) is isomorphic to \(B_{/x}\), and we may consider the object \(\bar{x} \in \text{Ob}(E)\) corresponding under this isomorphism to \(1_x \in B_{/x}\). The object \(\bar{x} \in \text{Ob}(E)\) can then be internally characterized inside \(E\) as being a terminal object.

**The 2-categorical case.** Let \(p: \mathcal{E} \to \mathcal{B}\) be a cartesian 2-fibration and suppose that it is represented by an object \(x \in \mathcal{B}\). That is, the 2-functor

\[
\mathcal{B}(-, x): \mathcal{B}^{\text{op}} \to \text{Cat}
\]

is classified by \(p: \mathcal{E} \to \mathcal{B}\), meaning that by Proposition 1.4.15 we have a pullback square

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{p} & (\text{Cat}_{D_0})^{\text{op}} \\
\downarrow & & \downarrow \\
\mathcal{B} & \xrightarrow{} & \text{Cat}^{\text{op}}
\end{array}
\]

of 2-categories. In this case, \(\mathcal{E}\) is isomorphic to \(\mathcal{B}_{/x}\), and we may consider the object \(\bar{x} \in \text{Ob}(\mathcal{E})\) corresponding under this isomorphism to \(1_x \in \mathcal{B}_{/x}\).

Unlike the 1-categorical case, the object \(\bar{x}\) of \(\mathcal{E}\) is not biterminal in general (\(\bar{x}\) biterminal means that for every object \(z\) in \(\mathcal{E}\) the category \(\mathcal{E}(z, \bar{x})\) is equivalent to \(D_0\)). Instead, we have the following:

**Proposition 3.2.1.** Let \(p: \mathcal{E} \to \mathcal{B}\) be a cartesian 2-fibration represented by \(x \in \text{Ob}(\mathcal{B})\) and let \(\bar{x} \in \mathcal{E}\) be the associated lift of \(x\). Let \(M_{\text{cart}}\) denote the collection of \(p\)-cartesian edges. Then \(\bar{x}\) is \(M_{\text{cart}}\)-final in \(\mathcal{E}\).
Proof. Without loss of generality we may assume that \( E = \mathcal{B}_{/x} \) and \( \bar{x} = 1_x \). The objects of the category \( \mathcal{B}_{/x} (\alpha, 1_x) \) are of the form \((\beta, \zeta)\)

\[
\begin{array}{ccc}
y & \xrightarrow{\beta} & x \\
\alpha & \swarrow \nearrow & x \\
\end{array}
\]

, that is \( y \xrightarrow{\beta} x \), and its morphisms \( \Xi: (\beta, \zeta) \to (\beta', \zeta') \) are of the form

\[
\begin{array}{ccc}
y & \xrightarrow{\beta} & x \\
\alpha & \swarrow \nearrow & x \\
\end{array}
\]

, that is \( \zeta' \ast 1_\Xi = \zeta \).

We claim that the object \((\alpha, 1_x)\) of \( \mathcal{B}_{/x} (\alpha, 1_x) \) is terminal. Indeed, for every other object \((\beta, \zeta)\) in \( \mathcal{B}_{/x} (\alpha, 1_x) \) we have that \( \zeta: (\beta, \zeta) \to (\alpha, 1_x) \) is a morphism in \( \mathcal{B}_{/x} (\alpha, 1_x) \). Moreover, it is clearly the unique going from \((\beta, \zeta)\) to \((\alpha, 1_x)\), as any such morphism \( \Xi \) must satisfy \( 1_x \ast 1_\Xi = \Xi = \zeta \). In addition, taking \( \alpha = 1_x \) we also get that the identity on \( 1_x \) is terminal in \( \mathcal{B}_{/x} (1_x, 1_x) \).

We have thus established that \( 1_x \) is a pre-final object of \( \mathcal{B}_{/x} \). To show that it is \( \mathcal{M}_{\mathrm{cart}} \)-final we now point out that by Proposition 3.1.2 each of the arrows \((\alpha, 1_x) : \alpha \to 1_x \) just contracted are \( p \)-cartesian, so that (3) holds, and that any other \( p \)-cartesian edge \( \alpha \to 1_x \) is of the form

\[
\begin{array}{ccc}
y & \xrightarrow{\beta} & x \\
\alpha & \swarrow \nearrow & x \\
\end{array}
\]

and hence isomorphic in \( \mathcal{B}_{/x} (\alpha, 1_x) \) to \((\alpha, 1_x)\). This means that all \( p \)-cartesian edges \( \alpha \to 1_x \) are in fact terminal, and since the collection of \( p \)-cartesian edges is also closed under isomorphism we have that these are exactly the terminal edges in \( \mathcal{B}_{/x} (\alpha, 1_x) \). Condition (4) is then a consequence of the fact that \( p \)-cartesian edges are closed under composition. \( \square \)

3.2.2. Combining Proposition 3.2.1 and Proposition 2.1.7 we obtain a contraction \( \gamma \) on \( \mathcal{B}_{/x} \) with center \( 1_x \). This contraction can be explicitly described as follows:

- To a given object \((y, \alpha)\) we assign the 1-cell \((\alpha, 1_x)\), that we can depict by

\[
\begin{array}{ccc}
y & \xrightarrow{\alpha} & x \\
\alpha & \swarrow \nearrow & x \\
\end{array}
\]

- For every 1-cell \((f, \zeta): (y, \alpha) \to (z, \beta) \) we have to provide a 2-cell going from \( \gamma(z, \beta) \ast_0 (f, \zeta) \) to \( \gamma(y, \alpha) \). By definition, \( \gamma(z, \beta) \ast_0 (f, \zeta) = (\beta f, \zeta) \) and
3.2.3. We give an explicit description of the 2-category $\mathcal{B}_{/x}$ (see Definition 1.4.7). The objects are the elements of $\text{Ob}(\mathcal{B}_{/x})$. For every pair of objects $\alpha: y \to x$ and $\alpha': y' \to x$, the hom-category $(\mathcal{B}_{/x})_{\text{cart}}((y, \alpha), (y', \alpha'))$ is the full subcategory of $\mathcal{B}_{/x}((y, \alpha), (y', \alpha'))$ spanned by $p$-cartesian edges. By Proposition 3.1.2, these are the 1-cells $(\beta, \sigma)$ of $\mathcal{B}_{/x}$ of the form

$$z \xrightarrow{\beta} y$$

with $\sigma$ an invertible 2-cell of $\mathcal{B}$.

**Proposition 3.2.4.** The object $(x, 1_x)$ is biterminal in $(\mathcal{B}_{/x})_{\text{cart}}$.

**Proof.** Consider any object $\alpha: y \to x$ of $(\mathcal{B}_{/x})_{\text{cart}}$. We wish to show that the category $(\mathcal{B}_{/x})_{\text{cart}}((y, \alpha), (x, 1_x))$ is equivalent to the terminal category $D_0$. First of all, this category is non-empty, since it has $\alpha, 1_x$ as an object. Given a 1-cell $(\beta, \zeta): (y, \alpha) \to (x, 1_x)$, the composition $1_x *_0 \beta = \beta$ must be the source of $\zeta$ and moreover $\zeta: \beta \to \alpha$ must be an invertible 2-cell of $\mathcal{B}$ by Proposition 3.1.2. For every pair of 1-cells $(\beta, \zeta), (\beta', \zeta') : (y, \alpha) \to (x, 1_x)$, a 2-cell $\tau: (\beta, \zeta) \to (\beta', \zeta')$ must satisfy the relation $\zeta' *_1 \tau = \zeta$. It follows that such a 2-cell is invertible and in fact it always exists and it is unique, namely it is given by $\tau = (\zeta')^{-1} *_1 \zeta$. This shows that for every object $(y, \alpha)$ in $(\mathcal{B}_{/x})_{\text{cart}}$ the category $(\mathcal{B}_{/x})_{\text{cart}}((y, \alpha), (x, 1_x))$ is the chaotic category on the objects of the form $(\beta, \zeta)$, with $\zeta: \beta \to \alpha$ invertible, and it is thus equivalent to the terminal category $D_0$, i.e., $(x, 1_x)$ is biterminal. \qed

**Remark 3.2.5.** In the general case of a 2-functor $F: \mathcal{J} \to \mathcal{C}$ and a marking $E$ on $\mathcal{J}$, the 2-category $\mathcal{C}^{/F}$ does not necessarily have a quasi-terminal object and similarly $(\mathcal{C}^{/F})_{\text{cart}}$ does not necessarily have a biterminal object.

3.3. **The modified 2-category of cones.** Let $\mathcal{J} = (\mathcal{J}, E)$ be a marked 2-category, $F: \mathcal{J} \to \mathcal{C}$ be a 2-functor, $\ell$ an object of $\mathcal{C}$, $\lambda: \Delta \ell \to F$ an $E$-cone over $F$ and $F^{\otimes}: \mathcal{J}^{\otimes} \to \mathcal{C}$ the corresponding 2-functor (cf. Remark 1.3.2). In this section we introduce an auxiliary 2-category of cones that is always biequivalent to $\mathcal{C}^{/\ell} = \mathcal{C}^{/\ell}$ and that is biequivalent to $\mathcal{C}^{/F}$ if and only if $(\ell, \lambda)$ is an $E$-bilimit of $F$.

**Definition 3.3.1.** We define $\mathcal{C}_{/F}^{\otimes}$ as the 2-full sub-2-category of $\mathcal{C}^{/F}$ whose objects are the 2-functors $D_0 * D_0 * (j) \to \mathcal{C}$ such that their restrictions to $D_0 * D_0 * \{j\} \to$
\( \mathcal{C} \), for any object \( j \) of \( \mathcal{J} \), determine diagrams

\[
\begin{array}{ccc}
(x, h) & \xrightarrow{h} & (\ell, h) \\
\downarrow_{\alpha_j} & \swarrow_{\sigma_j} & \downarrow_{\lambda_j} \\
F(j) & & F(j)
\end{array}
\]

in \( \mathcal{C} \) with \( \sigma_j: \lambda_j h \to \alpha_j \) an invertible 2-cell of \( \mathcal{C} \).

3.3.2. We denote an object of \( \mathcal{C}_{/F} \) by \( (x, h, \alpha, \sigma) \), where we mean that:

- \( h: x \to \ell \) is a 1-cell of \( \mathcal{C} \);
- the pair \( (x, \alpha) \) is an \( E \)-cone over \( F \), so that in particular for every 1-cell \( \kappa: i \to j \) in \( \mathcal{J} \) we have a triangle

\[
\begin{array}{ccc}
x & \xrightarrow{\alpha_i} & x \\
\downarrow_{\alpha_j} & \swarrow_{\alpha_j} & \downarrow_{\lambda_j} \\
Fi & \to & Fj
\end{array}
\]

with \( \alpha_k \) invertible whenever \( \kappa \in E \).
- for every \( i \) in \( \text{Ob}(\mathcal{J}) \), \( \sigma_i: \lambda_i h \to \alpha_i \) is an invertible 2-cell of \( \mathcal{C} \), that we can depict as

\[
\begin{array}{ccc}
x & \xrightarrow{h} & (\ell, h) \\
\downarrow_{\alpha_i} & \swarrow_{\sigma_i} & \downarrow_{\lambda_i} \\
F(i) & & F(i)
\end{array}
\]

Notice that by Remark 1.2.5 we have that \( \sigma: \lambda \cdot h \to \alpha \) is an invertible modification.

The objects of the form \( (x, h, \lambda \cdot h, \iota) \), with \( \iota_i \) the identity 2-cell of \( \lambda_i h \) for all \( i \) in \( \text{Ob}(\mathcal{J}) \), are particularly simple and we will make use of them in the following proofs in order to simplify the coherences appearing in explicit form of the cells of \( \mathcal{C}_{/F} \).

We shall also commit the notational abuse of denoting by \( \iota \) the appropriate trivial modification, regardless of the 1-cell of \( \mathcal{C} \) involved; for instance, for \( g: y \to \ell \) another 1-cell of \( \mathcal{C} \) we shall write \( (y, g, \lambda \cdot g, \iota) \) where we mean that here \( \iota_i \) is the identity of \( \lambda_i g \) for all \( i \in \text{Ob}(\mathcal{J}) \).

A 1-cell from \( (x, h, \alpha, \sigma) \) to \( (y, g, \beta, \tau) \) is given by a triple \( (f, \zeta, \mu) \), where \( f: x \to y \) is a 1-cell in \( \mathcal{C} \), \( \zeta: g \circ f \to h \) is a 2-cell of \( \mathcal{C} \) and \( \mu \) is the modification satisfying \( \mu \circ (\tau \circ f) = \sigma \circ \lambda \cdot \zeta \). Since \( \tau \) is an invertible modification, we actually have that \( \mu \) is defined as \( \sigma \circ \lambda \cdot \zeta \circ (\tau^{-1} \circ f) \). Notice that a 1-cell from \( (x, h, \lambda \cdot h, \iota) \) to \( (y, g, \lambda \cdot h, \iota) \) must be of the form \( (f, \zeta, \lambda \cdot \zeta) \).

In light of Lemma 3.3.3 below it will suffice to describe the 2-cells of \( \mathcal{C}_{/F} \) whose 0-dimensional source and target are of the form \( (x, h, \lambda \cdot h, \iota) \) and \( (y, g, \lambda \cdot h, \iota) \). Here and in what follows, we will always commit the abuse of denoting by \( \iota \) the appropriate trivial modification. In this case, if we are given 1-cells \( (f, \zeta, \lambda \cdot \zeta) \) and \( (f', \zeta', \lambda \cdot \zeta') \) from \( (x, h, \lambda \cdot h, \iota) \) to \( (y, g, \lambda \cdot h, \iota) \), then we have that a 2-cell of \( \mathcal{C}_{/F} \) is precisely a 2-cell of \( \mathcal{C}_{/\ell} \), that is a 2-cell \( \Xi: f \to f': x \to y \) of \( \mathcal{C} \) satisfying \( \zeta' \circ \lambda g \circ \Xi = \zeta \). Indeed, if for instance \( k: i \to j \) is a 1-cell of \( \mathcal{J} \), then by the coherences imposed by the 1-cells of \( \mathcal{C}_{/F} \) we have

\[
(\lambda_j \zeta' \circ \lambda g \circ \Xi) \circ \mu = \lambda_k h \circ (\lambda g \circ \Xi)
\]

\[
= \lambda_k h \circ (\lambda g \circ \Xi)
\]

where we mean that here \( \mu \) is the identity of \( \lambda_i g \) for all \( i \in \text{Ob}(\mathcal{J}) \).
and similarly for a 2-cell of $\beta$. Morally, the coherences of a 2-cell of $\mathcal{C}_{/F^0}$ are encoded by higher cells of $D_2 * D_0 * D_m$, with $m = 0, 1, 2$, but since we are truncating at the 2-dimensional level, many of these higher cells are trivial (cf. Remark 1.2.6) and moreover we are choosing specific objects (and 1-cells) of $\mathcal{C}_{/F^0}$ which further simplify some of the coherences involved with identities.

We shall now prove that the 2-functor $q: \mathcal{C}_{/F^0} \to \mathcal{C}_{/\ell}$ is a biequivalence, following the strategy outlined in 1.1.5. The proof is subdivided in few steps and a preliminary auxiliary lemma, that we shall use to simplify some of the coherences involved.

**Lemma 3.3.3.** Let $(x, h, \alpha, \sigma)$ be an object of $\mathcal{C}_{/F^0}$. Then it is isomorphic to the object $(x, h, \lambda \cdot h, i)$, where $i_\ell$ is the identity 2-cell of $\lambda \cdot h$ for all $i$ in $\text{Ob}(\beta)$.

**Proof.** The 1-cell $(1_x, \sigma)$ of $\mathcal{C}_{/F}$ is an isomorphism and it is such that $(h, i) \ast 0 (1_x, \sigma) = (h, \sigma)$. This is equivalent to say that $(1_x, 1_h, \sigma^{-1})$ is an isomorphism from $(x, h, h \cdot \lambda, i)$ to $(x, h, \alpha, \sigma)$ in $\mathcal{C}_{/F^0}$. $\square$

3.3.4 (Surjectivity on objects). Consider an object $h: x \to \ell$ of $\mathcal{C}_{/\ell}$. Then the object $(x, h, h \cdot \lambda, i)$ of $\mathcal{C}_{/F^0}$ maps to $(x, h)$.

3.3.5 (Fullness on 1-cells). Let $(f, \zeta): (x, h) \to (y, g)$ be a 1-cell in $\mathcal{C}_{/\ell}$, that we can depict as

$$
\begin{array}{c}
\xymatrix{
x \ar[r]^f & y \\
\downarrow^h & \downarrow^g \\
\ell & \zeta
}\end{array}
$$

Given two objects $(x, h, \alpha, \sigma)$ and $(y, g, \beta, \tau)$ of $\mathcal{C}_{/F^0}$ that lift the source and target (respectively) of $(f, \zeta)$, we want to find a 1-cell $D_1 * D_0 * \beta \to \mathcal{C}$ from $(x, h, \alpha, \sigma)$ to $(y, g, \beta, \tau)$ such that its projection via $q$ is the 1-cell $(f, \zeta)$ of $\mathcal{C}_{/\ell}$. According to paragraph 3.3.2, the triple $(f, \zeta, \mu)$ is a 1-cell of $\mathcal{C}_{/F^0}$ with the correct boundary, with $\mu = \sigma \ast 1 \lambda \cdot \zeta \ast_1 (\tau^{-1} \ast 0 f)$, and it is clearly mapped to $(f, \zeta)$. Notice that this lifting requires no additional data, but just commutativity conditions, so that such a lift must be unique. We also remark that if we choose $(x, h, \lambda \cdot h, i)$ and $(y, g, \lambda \cdot g, i)$ as objects lifting the source and target of $(f, \zeta)$, then the 1-cell $(f, \zeta, \lambda \cdot \zeta)$ between these two objects of $\mathcal{C}_{/F^0}$ maps to $(f, \zeta)$. A simple verification shows that the square

$$
\begin{array}{c}
\xymatrix{(x, h, \alpha, \sigma) \ar[r]^{(f, \zeta, \mu)} & (y, g, \beta, \tau) \\
(1_x, 1_h, \sigma) & (1_y, 1_g, \tau) \\
(x, h, \lambda \cdot h, i) \ar[r]_{(f, \zeta, \lambda \cdot \zeta)} & (y, g, \lambda \cdot \zeta, i)
}\end{array}
$$

of $\mathcal{C}_{/F^0}$ is commutative, where the vertical 1-cells are (the inverses of) those of Lemma 3.3.3. In particular, this means that it was actually enough to find the (unique) lift $(f, \zeta, \lambda \cdot \zeta)$ of $(f, \zeta)$ with $(x, h, \lambda \cdot h, i)$ and $(y, g, \lambda \cdot g, i)$ as source and target, since then the composition

$$(1_y, 1_g, \tau^{-1}) \ast_0 (f, \zeta, \lambda \cdot \zeta) \ast_0 (1_x, 1_h, \sigma)$$

is a lift of $(f, \zeta)$ with $(x, h, \alpha, \sigma)$ and $(y, g, \beta, \tau)$ as source and target.
3.3.6 (Fullness on 2-cells). Let $\Xi: (f, \zeta) \to (f', \zeta')$ be a 2-cell in $C_{/\ell}$, that we can depict as

$$
\begin{array}{c}
\xymatrix{
  x & \ar[d]^-h \ar[r]^-g & y \\
  \ell & \ar[u]^-f \ar[r]_-\zeta & \ell \\
}
\end{array}
\quad=
\begin{array}{c}
\xymatrix{
  x & \ar[d]^-h \ar[r]^-g & y \\
  \ell & \ar[u]^-f \ar[r]_-f' & \ell \\
}
\end{array}
$$

Given two objects $(x, h, \alpha, \sigma)$ and $(y, g, \beta, \tau)$ of $\om{C_{/\ell}}$, and lifts of $(f, \zeta, \mu)$ and $(f', \zeta', \mu')$ to $\om{C_{/\ell}}$, we wish to find a 2-cell $D_2 \ast D_0 \ast \jmath \to C_{/\ell}$ from $(f, \zeta, \mu)$ to $(f', \zeta', \mu')$ such that its projection via $q$ is the 2-cell $\Xi$ of $C_{/\ell}$. Thanks to Lemma 3.3.3 and the observation at the end of the previous point, we can again assume that $\alpha = \lambda \cdot h$, $\beta = \lambda \cdot g$ and $\sigma_i$ and $\tau_i$ are the identity 2-cells for all objects $i$ of $\beta$. Indeed, if we find a lift $\Xi'$ of $\Xi$ with $(x, h, \lambda \cdot h, \iota)$ and $(y, g, \lambda \cdot g, \iota)$ as source and target, then the 2-cell $(y, g, (\iota)^{-1}) \ast_0 \Xi' \ast_0 (1, 1, h, \iota)$ is a lift of $\Xi$ having $(x, h, \alpha, \sigma)$ and $(y, g, \beta, \tau)$ as source and target. By the discussion in paragraph 3.3.2, we know that $\Xi$ is a 2-cell of $\om{C_{/\ell}}$ with correct source and target and that it clearly lifts $\Xi$.

3.3.7 (Faithfulness on 2-cells). By the observation in paragraph 3.3.2, a 2-cell of $\om{C_{/\ell}}$ from $(f, \zeta, \lambda \cdot \zeta)$ to $(f', \zeta', \lambda \cdot \zeta')$ simply amounts to a 2-cell $\Xi: f \to f'$ of $C$ satisfying $\Xi \ast_1 (g \ast_0 \Xi) = \zeta$, which is the same thing as a 2-cell of $C_{/\ell}$ from $(f, \zeta)$ to $(f', \zeta')$. Therefore the map

$$
\om{C_{/\ell}}((f, \zeta, \lambda \cdot \zeta), (f', \zeta', \lambda \cdot \zeta')) \to C_{/\ell}((f, \zeta), (f', \zeta'))
$$

is a bijection. Consider now two generic parallel 1-cells $(f, \zeta, \mu)$ and $(f', \zeta', \mu')$ of $\om{C_{/\ell}}$, say from $(x, h, \alpha, \sigma)$ to $(y, g, \beta, \tau)$. The commutativity of diagram (6) implies that we have an isomorphism of categories

$$
\om{C_{/\ell}}((x, h, \alpha, \sigma), (y, g, \beta, \tau)) \cong \om{C_{/\ell}}((x, h, \lambda \cdot h, \iota), (y, g, \lambda \cdot g, \iota))
$$

mapping $(f, \zeta, \mu)$ to $(f, \zeta, \lambda \cdot \zeta)$, so that in particular we get a bijection

$$
\om{C_{/\ell}}((f, \zeta, \mu), (f', \zeta', \mu')) \cong \om{C_{/\ell}}((f, \zeta, \lambda \cdot \zeta), (f', \zeta', \lambda \cdot \zeta')).
$$

Hence, the 2-functor $q: \om{C_{/\ell}} \to C_{/\ell}$ is faithful on 2-cells.

Putting together the previous four points, we get the following result.

**Proposition 3.3.8.** The canonical projection $q: \om{C_{/\ell}} \to C_{/\ell}$ is a biequivalence.

**Corollary 3.3.9.** Let $(\jmath, E)$ be a marked 2-category, $F: \jmath \to C$ a 2-functor, and $(\ell, \lambda) \in C_{/\ell}$ an $E$-lax cone over $F$. Then $(\ell, \lambda)$ is an $E$-bilimit cone if and only if the projection $\om{C_{/\ell}} \to C^{/F}$ is a biequivalence.

**Proof.** By definition, $(\ell, \lambda)$ is an $E$-bilimit cone if and only if the canonical 2-natural transformation

$$
\lambda \cdot (-): C(-, \ell) \to [\jmath, C]_E(\Delta -, F)
$$

of functors $\om{C^{op}} \to \om{Cat}$, is an equivalence. Since the fibrations classifying these two 2-functors are $C_{/\ell}$ and $C^{/F}$, respectively (cf. Example 1.4.16), by virtue of Theorem 1.4.8 (see also Remark 1.4.9), this 2-natural transformation is an equivalence if
and only if the corresponding 2-functor $\mathcal{C}_{/\ell} \rightarrow \mathcal{C}^{/F}$ depicted below is a biequivalence.

\[
\begin{array}{ccc}
\mathcal{C}_{/\ell} & \longrightarrow & \mathcal{C}^{/F} \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathcal{E}^{/F}
\end{array}
\]

Now, the triangle

\[
\begin{array}{ccc}
\mathcal{C}^{/F} & \xrightarrow{\sim} & \mathcal{C}^{/F} \\
\uparrow & & \uparrow \\
\mathcal{C}_{/\ell} & \xrightarrow{\cong} & \mathcal{C}_{/\ell}
\end{array}
\]

of 2-categories commutes up-to an invertible 2-natural transformation given by Lemma 3.3.3 (see also diagram (6)). Hence, applying Proposition 3.3.8 together with the 2-out-of-3 property of biequivalences and the stability of biequivalences by 2-natural isomorphisms to the previous triangle we deduce that the 2-functor $\mathcal{C}_{/\ell} \rightarrow \mathcal{C}^{/F}$ is a biequivalence if and only if $\mathcal{C}^{/F \circ} \rightarrow \mathcal{C}^{/F}$ is one. \qed

4. Bilimits and bifinal cones

4.1. We fix a marked 2-category $\mathcal{J} = (\mathcal{J}, E)$ and a 2-functor $F: \mathcal{J} \rightarrow \mathcal{C}$, and we denote by $p: \mathcal{E}^{/F} \rightarrow \mathcal{E}$ the projection 2-functor. In this section we show that a cone $(\ell, \lambda)$ over $\mathcal{C}_{/\ell}$ is an $E$-bilimit cone if and only if it is the center of a limiting contraction $H$ on $\mathcal{E}^{/F}$, as defined below.

Notation 4.2. In what follows, we will use the letter $H$ to denote a contraction, in contrast with the use of $\gamma$ and other greek letters in previous sections. In fact, we will confine our use of greek letters to denote cones on diagrams.

Definition 4.3. Let $H$ be a contraction on $\mathcal{E}^{/F}$. We say that $H$ is a limiting contraction if it is an $M_{\text{cart}}$-contraction, where $M_{\text{cart}}$ is the collection of $p$-cartesian arrows in $\mathcal{C}^{/F}$. A cone $(\ell, \lambda)$ over $\mathcal{C}^{/F}$ will be called limiting bifinal if it is the center of a limiting contraction on $\mathcal{C}^{/F}$.

4.4. More explicitly, the condition of being an $M_{\text{cart}}$-contraction means that

1. For every cone $(x, \alpha)$ over $F$, the 1-cell $H(x, \alpha)$ is $p$-cartesian.
2. For every $p$-cartesian edge $(f, \mu):(x, \alpha) \rightarrow (y, \beta)$, the 2-cell $H(f, \mu)$ of $\mathcal{E}^{/F}$ is invertible.

The first condition then means that for all objects $i$ of $\mathcal{J}$ the 2-cell

\[
\begin{array}{ccc}
x & \xrightarrow{\ell} & \mathcal{E} \\
\alpha \downarrow & & \downarrow \lambda \\
F \mathcal{E} & \xrightarrow{\cong} & \mathcal{E}
\end{array}
\]
of $H(x, \alpha)$ is invertible. The second condition states that in the commutative diagram

\[ (7) \]

\[
\begin{array}{ccc}
  x & \xrightarrow{\alpha_i} & F_i \\
  f & \downarrow & \leftarrow \sigma(x, \alpha) \\
  y & \xrightarrow{h(x, \alpha)} & \ell \\
\end{array}
\]

\[
\begin{array}{ccc}
  x & \xrightarrow{\alpha_i} & F_i \\
  f & \downarrow & \leftarrow \sigma(y, \beta) \\
  y & \xrightarrow{h(y, \beta)} & \ell \\
\end{array}
\]

of $C$, where $H(x, \alpha) = (h(x, \alpha), \sigma(x, \alpha))$ and $H(y, \beta) = (h(y, \beta), \sigma(y, \beta))$. Whenever the 2-cells $\mu_i$ are invertible for all $i \in \text{Ob}(\mathcal{J})$, the 2-cell $H(f, \mu)$ is invertible. Equivalently, the 2-cell $H(f, \mu)$ is invertible whenever the 2-cells $\lambda_i *_0 H(f, \mu)$ are invertible for all $i \in \text{Ob}(\mathcal{J})$.

**Remark 4.5.** It follows from Remark 2.2.6 and the fact that $M_{\text{cart}}$ is closed under isomorphisms of 1-cells that if $H$ and $K$ are two contractions on $\mathcal{C}^{\mathcal{F}}$ with center $(\ell, \lambda)$ then $H$ is limiting if and only if $K$ is limiting.

Applying Proposition 2.2.11 to the present case yields:

**Proposition 4.6.** $\mathcal{C}^{\mathcal{F}}$ admits a limiting contraction with center $(\ell, \lambda)$ if and only if every cone $(\ell', \lambda')$ admits a $p$-cartesian edge $(\ell', \lambda') \rightarrow (\ell, \lambda)$ and the 2-functor $\{(\ell, \lambda)\} \rightarrow (\mathcal{C}^{\mathcal{F}}, M_{\text{cart}})$ is final.

Given a contraction $H$ on $\mathcal{C}^{\mathcal{F}}$ with center $(\ell, \lambda)$, we may consider the canonical projection $\overline{\mathcal{C}^{\mathcal{F}}} \rightarrow \mathcal{C}^{\mathcal{F}}$, where $\mathcal{F}^\downarrow : \mathcal{D} \rightarrow \mathcal{C}$ is the cone determined by $(\ell, \lambda)$.

**Proposition 4.7.** If $(\ell, \lambda)$ is limiting bifinal then the associated projection 2-functor $\overline{\mathcal{C}^{\mathcal{F}}} \rightarrow \mathcal{C}^{\mathcal{F}}$ is a biequivalence.

**Proof.** Call $H$ the $M_{\text{cart}}$-contraction. We will prove that the projection $\overline{\mathcal{C}^{\mathcal{F}}} \rightarrow \mathcal{C}^{\mathcal{F}}$ is a trivial fibration.

**Surjectivity on objects:** For a cone $(x, \alpha)$ over $F$, the 1-cell $H(x, \alpha)$ is by definition an object of $\overline{\mathcal{C}^{\mathcal{F}}} = \mathcal{C}^{\mathcal{F}}$ that lifts $(x, \alpha)$.

**Fullness on 1-cells:** Let $(x, h, \alpha, \sigma)$ and $(y, g, \beta, \tau)$ be objects of $\overline{\mathcal{C}^{\mathcal{F}}}$ and fix a 1-cell $(f, \mu) : (x, \alpha) \rightarrow (y, \beta)$ of $\mathcal{C}^{\mathcal{F}}$. We wish to lift $(f, \mu)$ to $\overline{\mathcal{C}^{\mathcal{F}}}$, e.g., by setting $\Gamma H(x, \alpha) \rightarrow H(y, \beta)$. Applying the contraction to $(f, \mu)$ gives a triangle

\[
\begin{array}{ccc}
  (x, \alpha) & \xrightarrow{(f, \mu)} & (y, \beta) \\
  H(x, \alpha) & \xrightarrow{H(f, \mu)} & H(y, \beta) \\
 (\ell, \lambda) & \xrightarrow{H(g, \tau)} & (\ell, \lambda) \\
\end{array}
\]

in $\mathcal{C}^{\mathcal{F}}$, where we have set $\Gamma = H(f, \mu)$. By assumption, the 1-cells $(h, \sigma) : (x, \alpha) \rightarrow (\ell, \lambda)$ and $(g, \tau) : (y, \beta) \rightarrow (\ell, \lambda)$ of $\mathcal{C}^{\mathcal{F}}$ are $p$-cartesian. Applying to these 1-cells the limiting contraction $H$, by point (2) we get two invertible 2-cells

$$H(h, \sigma) : (h, \sigma) \rightarrow H(x, \alpha) \quad \text{and} \quad H(g, \tau) : (g, \tau) \rightarrow H(y, \beta)$$
of $\mathcal{C}^{/F}$. The composite 2-cell $\zeta = H(h, \sigma)^{-1} \ast_1 H(f, \mu) \ast_1 (H(g, \tau) \ast_0 (f, \mu))$ of $\mathcal{C}^{/F}$ fills the triangle

$$
\begin{array}{c}
\begin{array}{c}
(x, \alpha) \\
\downarrow^\Xi
\end{array}
\end{array}
\xymatrix{
\ar[rrr]_{(f, \mu)} &&& (y, \beta) \\
\ar[rrr]_{(h, \sigma)} \ar[u]^\zeta &&& (g, \tau) \ar[u]_{(\ell, \lambda)}
}
$$

therefore providing a 1-cell $(f, \zeta, \mu, \nu)$: $(x, h, \alpha, \sigma) \to (y, g, \beta, \tau)$ of $\mathcal{C}^{/F}$ lifting the 1-cell $(f, \mu)$ of $\mathcal{C}^{/F}$.

**Fullness on 2-cells:** Let $(f, \zeta, \mu)$ and $(f', \zeta', \mu')$ be two parallel 1-cells of $\mathcal{C}^{/F}$ from $(x, h, \alpha, \sigma)$ to $(y, g, \beta, \tau)$. We wish to find a lift to $\mathcal{C}^{/F}$ for any 2-cell $\Xi: (f, \mu) \to (f', \mu')$ of $\mathcal{C}^{/F}$. Using the description of 3.3.2, we may view $\Xi$ itself as such a lift: for this, we have to check that the identity

$$
(8) \quad \zeta' \ast_1 (g \ast_0 \Xi) = \zeta
$$

is satisfied in $\mathcal{C}^{/F}$. Applying the constraint of the contraction $H$ to the 2-cell $\zeta: (g f, \tau, \mu) \to (h, \sigma)$: $(x, \alpha) \to (\ell, \lambda)$, where $(g f, \tau, \mu) = (g, \tau) \ast_0 (f, \mu)$, of $\mathcal{C}^{/F}$ we get the relation

$$
H(g f, \tau, \mu) = H(h, \sigma) \ast_1 \zeta.
$$

By the functoriality of the contraction, the left-hand side of this equation is equal to $H(f, \mu) \ast_1 (H(g, \tau) \ast_0 (f, \mu))$ and moreover the 2-cell $H(h, \sigma)$ is invertible, as $(h, \sigma)$ is $p$-cartesian by assumption. This implies that we can write

$$
\zeta = H(h, \sigma)^{-1} \ast_1 H(f, \mu) \ast_1 (H(g, \tau) \ast_0 (f, \mu))
$$

and similarly

$$
\zeta' = H(h, \sigma)^{-1} \ast_1 H(f', \mu') \ast_1 (H(g, \tau) \ast_0 (f', \mu')).
$$

Hence, equation (8) is satisfied if and only if we have

$$
H(f', \mu') \ast_1 (H(g, \tau) \ast_0 (f', \mu')) \ast_1 ((g, \tau) \ast_0 \Xi) = H(f, \mu) \ast_1 (H(g, \tau) \ast_0 (f, \mu)).
$$

Notice that by the interchange rule for

$$
\begin{array}{c}
\begin{array}{c}
(x, \alpha) \\
\downarrow^\Xi
\end{array}
\end{array}
\xymatrix{
\ar[rrr]_{(f, \mu)} &&& (y, \beta) \\
\ar[rrr]_{(f', \mu')} \ar[u]^H \ar[u]_H &&& (\ell, \lambda) \ar[u]_{H(g, \tau)}
}
$$

we actually have

$$
(H(g, \tau) \ast_0 (f', \mu')) \ast_1 ((g, \tau) \ast_0 \Xi) = (H(y, \beta) \ast_0 \Xi) \ast_1 (H(g, \tau) \ast_0 (f, \mu))
$$

and the 2-cell $H(y, \beta)$ is invertible, since $(g, \tau)$ is $p$-cartesian by assumption, so that equation (8) is satisfied if and only if the following equation is:

$$
(9) \quad H(f', \mu') \ast_1 (H(y, \beta) \ast_0 \Xi) = H(f, \mu).
$$

Now, using the coherence for 2-cells given by the contraction $H$ applied to the 2-cell on the left-hand side of the previous equation and using the constraints for which $H(H(x, \alpha)) = 1_{H(x, \alpha)}$ and $H(\ell, \lambda) = 1_{(\ell, \lambda)}$, we finally get that equation (9) is indeed satisfied.
Faithfulness on 2-cells: This is clear by the explicit description of the 2-cells of \( \mathcal{C} \), which are just 2-cells of \( \mathcal{C} \) satisfying some coherence conditions.

We now prove the converse of Proposition 4.7. For convenience, we first record the following observation about bilimits.

**Proposition 4.8.** If \((\ell, \lambda)\) is an \( E \)-bilimit of the 2-functor \( F : \mathcal{J} \to \mathcal{C} \), then \( \mathcal{C}^{/F} \) admits a limiting contraction with center \((\ell, \lambda)\).

**Proof.** Let \((\ell, \lambda)\) be an \( E \)-bilimit of \( F \). Our goal is to construct a limiting contraction \( H \) with center \((\ell, \lambda)\).

Since \((\ell, \lambda)\) is an \( E \)-bilimit the functor

\[
\lambda \cdot (-) : \mathcal{C}(x, \ell) \to [\mathcal{J}, \mathcal{C}](\Delta x, F)
\]

is an equivalence for every \( x \in \text{Ob}(\mathcal{C}) \). In particular, \( \lambda \cdot (-) \) is essentially surjective, so that we can find, for any \( \alpha \in [\mathcal{J}, \mathcal{C}](\Delta x, F) \), a 1-cell \( h : x \to \ell \) of \( \mathcal{C} \) and an isomorphism \( \sigma : \alpha \cong \lambda \cdot h \). The data of \( \alpha, h \) and \( \sigma \) then determine a \( p \)-cartesian 1-cell of \( \mathcal{C}^{/F} \) depicted by the triangle

\[
(x, \alpha) \xrightarrow{(1_x, \sigma)} (x, \lambda \cdot h) \xrightarrow{(\ell, \lambda) \cdot \sigma} (\ell, \lambda)
\]

We define our contraction on the level of objects by associating to \((x, \alpha)\) the \( p \)-cartesian edge \( H(x, \alpha) := (h, \sigma) \), where we may assume without loss of generality that we have picked \( H(\ell, \lambda) \) to be \( (1_\ell, 1_\lambda) \).

Now consider a 1-cell \((f, \mu) : (x, \alpha) \to (y, \beta)\) in \( \mathcal{C}^{/F} \), that can be seen as a 1-cell \( \mu \) of \( [\mathcal{J}, \mathcal{C}](\Delta x, F) \) from \( \alpha \) to \( \beta \cdot f \) (that is, a modification between the \( E \)- lax natural transformations \( \alpha \) and \( \beta \cdot f \)). Let \((h, \sigma) = H(x, \alpha)\) and \((g, \tau) = H(y, \beta)\) be the 1-cells constructed above. The composite \( \rho = \sigma \cdot \mu \cdot \tau \cdot (\tau \cdot \Delta f) \) then gives a morphism from \( \lambda \cdot (gf) \) to \( \lambda \cdot h \) in the category \([\mathcal{J}, \mathcal{C}](\Delta x, F)\). Since \( \lambda \cdot (-) \) is fully faithful this morphism lifts to a unique morphism \( \Gamma : gf \to h \) in \( \mathcal{C}(x, \ell) \), which we can write as a 2-cell

\[
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{h} & & \downarrow{g} \\
\ell & \xrightarrow{\Gamma} & \ell
\end{array}
\]

in \( \mathcal{C} \). One immediately checks, by applying \( \lambda \cdot (-) \), that we have

\[
\sigma_i \cdot \mu \cdot \tau \cdot (\tau \cdot \Delta f) = \mu_i \cdot \tau \cdot (\tau \cdot \Delta f)
\]

for all \( i \in \text{Ob}(\mathcal{J}) \). We may thus consider \( \Gamma \) as a 2-cell of \( \mathcal{C}^{/F} \) with source \( H(y, \beta) \cdot \mu \cdot \tau \cdot (\tau \cdot \Delta f) \) and target \( H(x, \alpha) \). We then extend our contraction to the level of 1-morphisms by setting \( H(f, \mu) = \Gamma \). Explicitly, the 2-cell \( H(f, \mu) \) is given by the pasting

\[
(x, \alpha) \xrightarrow{(1_x, \sigma)} (x, \lambda \cdot h) \xrightarrow{(f, \lambda \cdot \Gamma)} (y, \lambda \cdot g) \xrightarrow{(1_y, \tau \cdot \Gamma)} (y, \beta)
\]

(10)

\[
\begin{array}{ccc}
(x, \alpha) & \xrightarrow{(1_x, \sigma)} & (x, \lambda \cdot h) \\
\downarrow{(h, \sigma)} & & \downarrow{(h, 1_{\lambda \cdot h})} \\
(\ell, \lambda) & \xrightarrow{(g, \tau \cdot \Gamma)} & (y, \lambda \cdot g) \\
\end{array}
\]

\[
\begin{array}{ccc}
(y, \lambda \cdot g) & \xrightarrow{(1_y, \tau \cdot \Gamma)} & (y, \beta) \\
\end{array}
\]
in $\mathcal{C}^F$, where the left-most and the right-most triangles are commutative by definition. With this description at hand, and given that $\Gamma$ is uniquely determined by the fully faithfulness of $\lambda \cdot (\cdot)$, it is easy to verify that the assignment $H$ is compatible with composition of 1-cells, that is

$$H((f', \mu') \circ (f, \mu)) = H(f, \mu) \ast_1 (H(f', \mu') \circ_0 (f, \mu))$$

for any pair of composable 1-cells $(f, \mu), (f', \mu')$ of $\mathcal{C}^F$. In addition, since $\lambda \cdot \Gamma = \sigma^{-1} \ast_1 \mu \ast_1 (\tau \circ_0 \Delta f)$ with $\tau$ and $\sigma$ invertible and $\lambda \cdot (\cdot)$ is an equivalence we have that $\Gamma$ is invertible whenever $\mu$ is invertible. We conclude that $H$ sends $p$-cartesian 1-cells to invertible 2-cells, and consequently that $H(H(x, \alpha)) = 1_{H(x, \alpha)}$ by Remark 2.1.5.

To show that $H$ constitutes a limiting contraction it will now suffice to show that for any 2-cell $\Xi: (f, \mu) \to (f', \mu') : (x, \alpha) \to (g, \beta)$ of $\mathcal{C}^F$, the relation

$$(11) \quad H(f, \mu) = H(f', \mu') \ast_1 (H(y, \beta) \circ_0 \Xi)$$

is satisfied. We begin by noticing that since $\Xi$ is a 2-cell of $\mathcal{C}^F$, by definition we have

$$\mu = \sigma \ast_1 \lambda \cdot \Gamma \ast_1 (\tau^{-1} \circ_0 \Delta f), \quad \mu' = \sigma \ast_1 \lambda \cdot \Gamma' \ast_1 (\tau^{-1} \circ_0 \Delta f').$$

So the modification $\mu$ is equal to

$$\sigma \ast_1 \lambda \cdot \Gamma' \ast_1 (\tau^{-1} \circ_0 \Delta f') \ast_1 (\beta \circ_0 \Delta \Xi),$$

which using the interchange law can be rewritten as

$$\sigma \ast_1 \lambda \cdot \Gamma' \ast_1 (\lambda \cdot g \circ_0 \Delta \Xi) \ast_1 (\tau^{-1} \circ_0 \Delta f).$$

Since $\sigma$ and $\tau$ are invertible by construction, we obtain the relation

$$(12) \quad \lambda \cdot \Gamma = \lambda \cdot \Gamma' \ast_1 (\lambda \cdot g \circ_0 \Delta \Xi).$$

Observe that since $(1_x, \sigma^{-1}) : (x, \lambda \cdot h) \to (x, \alpha)$ and $(1_y, \tau) : (y, \beta) \to (y, \lambda \cdot g)$ are invertible 1-cells of $\mathcal{C}^F$, equation (11) holds if and only if it is whiskered by these two cells. Namely, we must show that

$$\begin{align*}
(x, \lambda \cdot h) & \xrightarrow{(1_x, \sigma^{-1})} (x, \alpha) & \xrightarrow{(f, \mu)} & \xrightarrow{(1_y, \tau)} & (y, \lambda \cdot g) \\
(h, 1_{\lambda \cdot h}) & \xrightarrow{(h, \sigma)} & \xrightarrow{(g, \lambda \cdot g)} & \xrightarrow{(g, 1_{\lambda \cdot g})} & (\ell, \lambda)
\end{align*}$$

is equal to

$$\begin{align*}
(x, \lambda \cdot h) & \xrightarrow{(1_x, \sigma^{-1})} (x, \alpha) & \xrightarrow{(f', \mu')} & \xrightarrow{(1_y, \tau')} & (y, \lambda \cdot g) \\
(h, 1_{\lambda \cdot h}) & \xrightarrow{(h, \sigma)} & \xrightarrow{(g, \lambda \cdot g)} & \xrightarrow{(g, 1_{\lambda \cdot g})} & (\ell, \lambda)
\end{align*}$$
We can rewrite this as

\[ (x, \lambda \cdot h) \xrightarrow{(f, \lambda) \Gamma} (y, \lambda \cdot g) \]

\[ (h, 1_{\lambda h}) \]

\[ (\ell, \lambda) \xleftarrow{(g, 1_{\lambda g})} \]

\[ (x, \lambda \cdot h) \xrightarrow{(f, \lambda) \Gamma} (y, \lambda \cdot g) \]

But the equality between these 2-cells of \( \mathcal{C}^{/ F} \) is expressed precisely by equation (12), which is satisfied by assumption. This complete the definition of a limiting contraction \( H \) with center \((\ell, \lambda)\), thereby finishing the proof.

We are now ready to prove the converse of Proposition 4.7.

**Corollary 4.9.** Let \((\ell, \lambda)\) be an \( E \)-lax cone over the 2-functor \( F: \mathcal{J} \to \mathcal{C} \). If the projection 2-functor \( \mathcal{C}^{/ F} \to \mathcal{C}^{/ F} \) is a biequivalence then \((\ell, \lambda)\) is limiting bifinal.

**Proof.** This follows from Corollary 3.3.9 and Proposition 4.8.

We have all the elements to finally state and prove our main theorem.

**Theorem 4.10.** Let \((\ell, \lambda)\) be an \( E \)-lax cone of the 2-functor \( F: \mathcal{J} \to \mathcal{C} \). The following statements are equivalent:

1. the \( E \)-lax \( F \)-cone \((\ell, \lambda)\) is an \( E \)-bilimit of \( F \);
2. the induced 2-functor \( \mathcal{C}^{/ F} \to \mathcal{C}^{/ F} \) is a biequivalence;
3. the object \((\ell, \lambda)\) of \( \mathcal{C}^{/ F} \) is limiting bifinal.

**Proof.** The equivalence between statements (1) and (2) is provided by Corollary 3.3.9. By virtue of Proposition 4.7 and its converse Corollary 4.9, \( \mathcal{C}^{/ F} \to \mathcal{C}^{/ F} \) is a biequivalence if and only if \((\ell, \lambda)\) is limiting bifinal in \( \mathcal{C}^{/ F} \). This proves the equivalence between statements (2) and (3), thereby concluding the proof of the theorem.

**Remark 4.11.** Given a 2-functor \( F: \mathcal{J} \to \mathcal{C} \), a marking \( E \) on \( \mathcal{J} \) and a weight \( W: \mathcal{J} \to \mathcal{C} \text{at} \), we observed in Remark 1.3.6 that the \( W \)-weighted \( E \)-bilimit can be expressed as the \( E_W \)-bilimit of the functor \( \mathcal{E}_F \xrightarrow{p} \mathcal{J} \xrightarrow{F} \mathcal{C} \), where \( p: \mathcal{E}_F \to \mathcal{J} \) is the fibration classifying the weight \( 2 \)-functor \( W \) and \( E_W \) is a canonical marking on \( \mathcal{E}_F \) detailed in [13, Definition 2.1.4] by Descotte, Dubuc and Szyld. In particular, a \( W \)-weighted \( E \)-bilimit of \( F \) is equivalent to a limiting bifinal object \((\ell, \lambda)\) in \( \mathcal{C}^{/ F} \).

Restricting to the cartesian edges of \( \mathcal{C}^{/ F} \) as a fibration over \( \mathcal{C} \) (see 3.1.3) and using the analysis on cartesian edges performed in §3.1, we can deduce the following statement, which already appears as Proposition 5.4 in [9] by Klingman and Moser.

**Corollary 4.12.** An \( E \)-bilimit \((\ell, \lambda)\) of the 2-functor \( F: \mathcal{J} \to \mathcal{C} \) is a biterminal object in \( (\mathcal{C}^{/ F})_{\text{cart}} \).

**Proof.** It follows from the previous theorem that the 2-functor \( \mathcal{C}_{/\ell} \to \mathcal{C}^{/ F} \) induced by \((\ell, \lambda)\) is a biequivalence, that we think of as a morphism of fibrations over \( \mathcal{C} \). This biequivalence of fibrations induces a 2-functor \( (\mathcal{C}_{/\ell})_{\text{cart}} \to (\mathcal{C}^{/ F})_{\text{cart}} \), since biequivalences preserve and create cartesian edges, and it is clear that such a 2-functor is still a biequivalence. By virtue of Proposition 3.2.4 the object \((\ell, 1_{\ell})\) is a biterminal object in \( \mathcal{C}_{/\ell} \) and therefore its image \((\ell, \lambda)\) is a biterminal object in \( (\mathcal{C}^{/ F})_{\text{cart}} \).
BILIMITS ARE BIFINAL OBJECTS

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