A UNITED-SET FORMULA

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Let a morphism \( f : X \to Y \) of algebraic varieties be given. A \textit{united set} or \textit{united} \( k \)-\textit{tuple} for \( f \) is a \( k \)-tuple \( x_1, \ldots, x_k \) of distinct points on (or “infinitely near”) \( X \), such that \( f(x_1) = \cdots = f(x_k) \). The purpose of this note is to announce an enumerative formula, valid under a restrictive hypothesis, for the united \( k \)-tuples of a map, i.e., a formula for the rational equivalence (or homology) class of a suitable cycle which parameterizes them. This yields as special cases formulas for the united \( k \)-tuples which contain a \( k_1 \)-tuple, a \( k_2 \)-tuple, etc., of mutually infinitely-near points. For our united-\( k \)-tuple cycle even to be defined, the morphism \( f \) has to admit a certain kind of “resolution” (essentially it must factor through a “generic” map into a variety fibred by smooth curves over \( Y \)). Our result is sufficient, however, to yield formulas for the lines having prescribed contacts with a given projective variety having “generic” singularities and arbitrary dimension and codimension; these in turn yield formulas for the Thom-Boardman-Roberts singularity schemes \([8]\) of a generic projection of such a variety. Classically such formulas were known for curves, for surfaces in \( \mathbb{P}^3 \), and in a few other cases, cf. \([1]\). Some recent results were obtained by Lascoux \([6]\), Roberts \([9]\) and LeBarz \([7]\). Our result yields new formulas even for surfaces in \( \mathbb{P}^4 \). For a modern account of these and related matters, see Kleiman’s surveys \([3, 5]\).

Admittedly, the hypothesis of existence of a “resolution” is a severe restriction on the morphism \( f \). I am hopeful, however, that by pursuing further the same principles as in this paper, I will eventually obtain a united-set formula valid without such a restriction, and which would moreover be completely “intrinsic”, in the sense of taking place on a suitable space associated solely to \( X \) (which is not the case with the present formula).

We shall work in the category of complete (usually nonsingular) varieties over a field. Everything goes through with no change, however, in the category of compact complex manifolds.

1. Set-up. Fix a morphism \( f : X \to Y \) of nonsingular varieties, and put \( m = \dim X, n = \dim Y \). A \textit{resolution} of \( f \) is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow f & & \\
\tilde{f} & & Z
\end{array}
\]

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\(^2\)This term is \textit{not} consistent with the classical one of united point of a correspondence.

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where \( \pi \) is a smooth morphism of relative dimension 1, and \( \bar{f} \) is an embedding.

More generally to define a \( k \)-\textit{quasi-resolution} assume about \( \bar{f} \) only that its \( k' \)-fold locus (cf. [4]), for all \( k' \leq k \), has codimension \( (k' - 1)(n - m) \) in \( X \) (this is weaker than assuming that the latter locus has its expected codimension, which is \( (k' - 1)(n - m + 1) \)). Now fix such a \( k \)-quasi-resolution.

We will define some auxiliary objects. \( Z^k \) is the \( k \)-fold fibre product of \( Z \) over \( Y \); \( Z^k_X \) is the fibre product \( Z^{k-1} \times_Y X \), with coordinate projections \( \pi^k : Z^k_X \to Z^{k-1}, \ p^k : Z^k_X \to X; j^k : Z^k_X \to Z^k \) is \( \pi^k \times \bar{f} ; \ \pi^k : Z^k_X \to Z \) is \( \pi^k \) followed by the \( i \)th coordinate projection; \( q^k_i : Z^k_X \to Z^{k-1}_X \) is "delete the \( i \)th coordinate".

For \( i < j < k \), the divisor \( D^k_{i,j} \subset Z^k_X \) is defined by \( \pi^k_i(\cdot) = \pi^k_j(\cdot) \); similarly, for \( i < k, D^k_{i,k} \subset Z^k_X \) is defined by \( \pi^k_i(\cdot) = p^k(\cdot) \); also put \( D^k = \sum_{i=1}^{k-1} D^k_{i,k} \).

\[ U_k(\bar{f}) = \left\{ z \in Z^k_X : \sigma(z) \subset \bar{f}(X) \text{ as schemes} \right\}. \]

This definition is justified by the fact that the image of \( U_k(\bar{f}) \) in the Hilbert scheme of \( X \) coincides, up to a lower-dimensional set, with the set of length-\( k \) subschemes of \( X \) which are mapped by \( f \) to a single reduced point. The image of \( U_k(\bar{f}) \) in \( X \) coincides with the \( k \)-fold locus of \( \bar{f} \) as defined by Kleiman [4].

Now the set \( U_k = U_k(\bar{f}) \) can (see §4) naturally be made into a cycle of "expected dimension" \( km - (k-1)n \) (i.e. expected codimension \( (k-1)(n-m+1) \)) in \( Z^k_X \), and we seek a formula for the class \([U_k]\) of \( U_k \) in the Chow group of \( Z^k_X \) (though we could instead work in \( Z^k \), working in \( Z^k_X \) yields finer results).

The result we get is an inductive one, and goes as follows.

**THEOREM.** Given a \( k \)-\textit{quasi-resolution} as above, assume that \( U_k \) and \( U_{k-1} \) have their expected codimensions. Then if \( k \geq 2 \) we have

\[
[U_k] = (\pi^k)^*([j^{k-1}]_*)([U_{k-1}]) - \left( \sum_{i=1}^{k-1} (q^k_i)^*[([U_{k-1}] \cdot [D^k_{i,k}] ) \left( (p^k)^*(c(\nu)) \right) \left( 1 + [D^k] \right) \right)_{n-m}
\]

in \( CH^{(k-1)(n-m+1)}(Z^k_X) \); where \( \nu \) denotes the virtual normal bundle of \( \bar{f} \), i.e., \( \bar{f}^*(TZ) - TX \), \( c(\nu) \) denotes its total Chern class, and \( \{ \} \) \( n-m \) denotes the part in degree \( n - m \). Also \( U_1 = 1 \).

**3. Applications.** By pushing the formula for \([U_k]\) down to \( X \), we obtain a multiple-point formula à la Kleiman [4]. However, our formula contains more information than that. In particular, note that \( U_k \cdot D^k_{i,j} \cdot D^k_{j,i} \cdots \) parametrizes those united \( k \)-tuples whose \( i \)th and \( j \)th, \( i \)th and \( j \)th, etc. points are infinitely near each other, so the theorem yields enumerative formulas for the united \( k \)-tuples which are the union of Thom-Boardman \( S^1(i_1) \)-singularities, \( i = 1, \ldots, d \), \( q_1 + \cdots + q_d = k \), cf. Roberts [8].
I know two main types of maps which admit quasi-resolutions.

(a) Let \( g: V \to \mathbb{P}^N \) be a map whose image has generic singularities (cf. [3]). Put \( X = \{(v,L) \in V \times G(1,\mathbb{P}^N) : g(v) \in L\} \), \( Y = G(1,\mathbb{P}^N) \), and let \( f: X \to Y \) be the projection, \( Z \to Y \) the tautological \( \mathbb{P}^1 \) bundle, and \( X \to Z \) the natural map. The united \( k \)-tuples of \( f \) correspond to the \( k \)-secant lines of \( g(V) \), and thus the theorem yields enumerative formulas for these. They include formulas for the \( k \)-secant lines having prescribed types of contact with \( V \), as well as for the "varieties of contact" of such lines. For instance, if \( V \) is a surface in \( \mathbb{P}^4 \), it will in general have a finite number of inflexional tangent lines meeting it elsewhere, say \( L_1, \ldots, L_r \); a formula for \( r \) was already given by LeBarz [7]. Put \( L_i \cap V = 3p_i + q_i \). By pushing down to \( V \) the formula for \([U_4] \cdot [D_{1,2}] \cdot [D_{1,3}]\), we get a formula for the rational equivalence class of \( p_1 + \cdots + p_r \) (resp. \( q_1 + \cdots + q_r \)) on \( V \).

(b) Let \( g: V \to \mathbb{P}^N \) be as above, and let \( f: V \to Y = \mathbb{P}^n \) be \( f \) followed by projection from a general center \( M = \mathbb{P}^{N-n-1} \subset \mathbb{P}^N \), where \( n \geq \dim V \). Then projection from a general codimension-1 subspace \( M' \subset M \) yields a quasi-resolution \( \tilde{f}: V \to \mathbb{P}^{n+1} \), so the theorem applies, yielding some formulas for the singularities of \( f \), including those of Thom-Boardman-Roberts as above. Actually this case is a special case of case (a), because united points of \( f \) correspond to \( k \)-secants of \( \tilde{f}(V) \) passing through a fixed point.

4. Proof. As in other recent work on similar questions (see [3, 4, 5]), a key ingredient in the proof is an application of a "residual-intersection formula", of which we only require a relatively simple case, due to Fulton and MacPherson [2]. Consider the following cartesian diagram:

\[
\begin{array}{ccc}
I & \to & r^{-1} j^{k-1} U_{k-1} \\
\downarrow & & \downarrow \\
Z^k_X & \overset{j^k}{\to} & Z^k
\end{array}
\]

where \( r: Z^k \to Z^{k-1} \) is projection onto the first \( k-1 \) coordinates. One can show that \( I \) consists of \( U_k \) plus a "residual" cycle, namely \((\pi^k)^{-1}(j^{k-1} U_{k-1}) \cdot D^k\). Now [2] tells us how to compute the contribution of this residual cycle to the intersection-cycle \([r^{-1}(j^{k-1} U_{k-1})] \cdot Z^k_X\), and this yields our formula.

ADDED IN PROOF. The hope expressed in the introduction is now a reality: a united-set formula taking place in a suitable "configuration space" \( X^{[k]} \) and valid "modulo \( \mathbb{S}_2(f) \)" has been obtained, as a consequence of a general formula for the rational-equivalence class of \( V^{[k]} \) on \( Z^{[k]} \), where \( V \subset Z \) are arbitrary manifolds. As an application, among others, I obtain a formula for the class, in the moduli space of curves, of the locus of curves carrying a \( g^r_d \), for given \( r, d \).

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REFERENCES

1. H. F. Baker, *Principles of geometry*, vols. V, VI, Cambridge Univ. Press, New York, 1933.
2. W. Fulton and R. MacPherson, *Intersecting cycles on an algebraic variety*, Real and Complex Singularities (P. Holm (ed.)), Sijthoff & Noordhoff, 1977, pp. 179–197.
3. S. L. Kleiman, *The enumerative theory of singularities*, ibid., pp. 297–396.
4. _____, *Multiple-point formulas. I: Iteration*, Acta Math. 147 (1981), 13–49.
5. _____, *Multiple-point formulas for maps*, Proc. Conf. Algebraic Geometry (Nice, 1981) (in press).
6. A. Lascoux, *Calcul de certains polynômes de Thom*, C. R. Acad. Sci. Paris Sér. A 278 (1974), 889–891.
7. P. LeBars, *Formules pour les multisécantes des surfaces*, C. R. Acad. Sci. Paris Sér. A 292 (1981), 797–800.
8. J. Roberts, *Singularity subschemes and generic projections*, Trans. Amer. Math. Soc. 212 (1975), 229–268.
9. _____, *Some properties of double-point schemes*, Compositio Math. 41 (1980), 61–94.

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