Radiation of two hard gluons in $e^+e^- \rightarrow q\bar{q}$

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Abstract

The process $e^+e^- \rightarrow q\bar{q}gg$ where all final particles are hard is treated in a semi-analytic approach. Cuts on the invariant masses of the quark and the gluon pair can be applied. The total cross section is obtained by three numerical integrations.

1 Introduction

The study of $W$ pair production is an important task of the present experiments at LEP [1]. The $W$ Bosons are only accessible through their decay products. Therefore, the complete process $e^+e^- \rightarrow f_1 \bar{f}_2 f_3 \bar{f}_4$ must be studied. It can be calculated by Monte Carlo or semi-analytic techniques [4]. A Monte Carlo description is absolutely necessary to describe a real detector and realistic experimental cuts. A semi-analytic description allows only for simple cuts in the phase space but it is very fast and still flexible enough for fits to data. This is proved by the semi-analytic code GENTLE/4fan [4], which is used by all four LEP collaborations to determine the $W$ mass [1].

A jet due to a light quark cannot be distinguished experimentally from a jet due to a gluon. Hence the reaction

$$e^+(k_1)e^-(k_2) \rightarrow q(p_1)\bar{q}(p_2)g(p_3)g(p_4)$$

(1)
could give a signature similar to that of the reaction

$$e^+e^- \rightarrow q_1\bar{q}_2 q_3\bar{q}_4.$$ (2)

Reaction (1) is therefore an important incoherent background to the process (2). The process (1) can be measured in more detail when the quark flavor $q$ can be tagged. For heavy quarks, it is also a background to Higgs production [4].

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The process (1) was investigated many years ago [5] because a jet due to a light quark cannot be distinguished from a jet due to the quark accompanied by soft and/or collinear gluons. Monte Carlo studies of the process (1) with four hard final particles can be found in references [6].

The semi-analytic calculation of 4-fermion final states neglecting quark masses leads to very compact formulae for double and triple differential cross sections [7]. Only one (two) numerical integrations are necessary to evaluate simple distributions (cross sections). This makes the semi-analytic codes fast compared to Monte Carlo programs.

The aim of this paper is the derivation of semi-analytic formulae describing the process (1), which will be added to the code GENTLE/4fan [3] in order to enhance the flexibility of this code. In the calculation of reaction (1), we must apply the same cuts as to the reaction (2). In particular, the invariant energies of the gluon and quark pairs in (1) are bounded from below. This is necessary to keep the cross section out of the range of hadronic resonances where perturbation theory breaks down. Such cuts also ensure that the gluons cannot become soft. Where all four final particles are hard, we can limit ourselves to a tree-level calculation.

The numerical largest corrections due to the radiation of photons from the initial state can be taken into account by a convolution,

\[ \sigma^{ISR}(s) = \int_{s'} ds' \frac{d\sigma^0(s')}{s'} \rho(s'/s). \quad (3) \]

See reference [3] for further details and references.

2 Calculation

We parameterize the phase space by 6 angles and 2 invariants,

\[ d\Gamma = \prod_{i=1}^{4} \frac{d^3p_i}{2p_i^0} \times \delta^4(k_1 + k_2 - \sum_{i=1}^{4} p_i) \]

\[ = \frac{\sqrt{\lambda(s, s_q, s_g)}}{8s} \frac{\sqrt{\lambda(s_q, m_1^2, m_2^2)}}{8s_q} \frac{\sqrt{\lambda(s_g, m_3^2, m_4^2)}}{8s_g} \]

\[ ds_q ds_g d\Omega_q d\Omega_g d\Omega_g, \quad (4) \]

with

\[ \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \quad \lambda \equiv \lambda(s, s_q, s_g), \]

\[ s = (k_1 + k_2)^2, \quad s_q = (p_1 + p_2)^2 \quad \text{and} \quad s_g = (p_3 + p_4)^2, \]

\[ p_i^2 = m_i^2, \quad m_1 = m_2 = m_q, \quad m_3 = m_4 = m_g = 0. \quad (5) \]

The solid angles \( \Omega_q \) and \( \Omega_g \) are defined in the rest frames of the quark and gluon pairs. \( \Omega_0 \) is the solid angle of \( \vec{p}_1 + \vec{p}_2 \) in the laboratory frame. We have \( d\Omega_i = d\cos \theta_i d\phi_i \) for \( i = 0, q, g \). The kinematical ranges of the integration variables are

\[ (m_1 + m_2)^2 \leq s_q \leq (\sqrt{s} - m_3 - m_4)^2, \quad (m_3 + m_4)^2 \leq s_g \leq (\sqrt{s} - \sqrt{s_g})^2, \]

\[ -1 \leq \cos \theta_i \leq 1, \quad 0 \leq \phi_i \leq 2\pi. \quad (6) \]

The process (1) is described by 16 Feynman diagrams, 8 with virtual photon exchange and 8 with virtual Z-Boson exchange. 6 of the 8 diagrams are obtained by attaching the two gluons in all possible combinations to the quark pair, the other two diagrams contain the triple-gluon vertex. The two types of diagrams are presented below in Figure 1.

We obtain distributions and cross sections by integrating over the matrix element squared, which is obtained by traditional trace techniques. Our semi-analytic approach does not allow
for angular cuts to separate gluons from quarks. Therefore, we keep a non–zero quark mass to avoid collinear singularities. The calculation is done with the help of the symbolic manipulation program \textsc{FORM} \cite{8}.

Without transversal beam polarization, the integration over the angle $\phi_0$ is trivial giving $2\pi$. While the first 2 integrations over $\phi_q$ and $\phi_g$ as azimuthal angles of the $qq$- and $gg$-subsystems in the chosen parameterization can be done relatively easily and the integration over the solid angle $\cos\theta_0$ is trivial because only terms proportional to $1$, $\cos\theta_0$ and $\cos^2\theta_0$ arise, the last 2 angular integrations prove to be more complicated \cite{9}. This is due to logarithms with quadratic or biquadratic polynomials in $\cos\theta_{q,g}$ in their arguments after one integration. In order to avoid singularities for the last 2 angles $\cos\theta_{q,g}$ – these reflect the collinear singularities of the original amplitude for $m_q = 0$ – the quark mass $m_q$ cannot be neglected. So from these last 2 integrations one can be done analytically leading to dilogarithms of the type $\text{Li}_2(s_q, s_g; s, m^2_q)$ and the above mentioned logarithmic integrals as dominant contributions, while the other integration is treated numerically. An analytical integration over $\cos\theta_g$ in the massless $gg$-subsystem with a numerical integration over $\cos\theta_q$ afterwards leads to shorter expressions of the integrand than vice versa and is favored.

The total cross section may then be written in the form

$$
\sigma(s) = \int_{s_q}^{s} ds_q \int_{s_g}^{(\sqrt{s}-\sqrt{s_q})^2} ds_g \int_{-1}^{1} d\cos\theta_q \frac{\sqrt{\lambda}}{\pi s} C_{qqgg}(e; q; s) G_{qqgg}(s_q, s_g, \cos\theta_q; s, m^2_q),
$$

with numerical integrations over $\cos\theta_q$, $s_q$ and $s_g$. In the integration \cite{4}, we allow for cuts $\bar{s}_q$ and $\bar{s}_g$ on the invariant masses. After neglecting quark masses, the function $C_{qqgg}$ containing couplings and gauge boson propagators, is

$$
C_{qqgg}(e; q; s) = \frac{2}{(6\pi^2)^2} \alpha_s^2(4\pi)^2 \Re e \sum_{V_i, V_j = \gamma, Z} \frac{1}{D_{V_i}(s)} \frac{1}{D_{V_j}(s)} \left[ L_{e}(V_i) L_{e}(V_j) + R_{e}(V_i) R_{e}(V_j) \right] \left[ L_{q}(V_i) L_{q}(V_j) + R_{q}(V_i) R_{q}(V_j) \right],
$$

where $L_f(V)$ and $R_f(V)$ are the left- and right-handed couplings of the fermion $f$ to the vector Boson $V$ with the mass $M_V$ and the width $\Gamma_V$. The preceding factor is kept for convenience with reference \cite{12}.

The analytic expression for the function $G_{qqgg}(s_q, s_g, \cos\theta_q; s, m^2_q)$ is much longer than those obtained for the four-fermion final states investigated in \cite{12}. This is partially due to the richer topology of the Feynman diagrams, partially due to the non-zero quark mass appearing as an
additional dimensional parameter and partially due to the fact that the integration over $\theta_q$ cannot be done analytically. As was recognized in \[7\], the analytic expressions always tend to be longer before the integration over the last angle is done.

Several checks are applied to ensure the validity of our calculation. The matrix element must satisfy Bose symmetry. In addition, it must vanish when the polarization vector of anyone of the gluons is substituted by the momentum vector of this gluon. All “artificial” poles appearing from partial fraction decomposition must be compensated by corresponding terms in the nominator. In particular, the result must be finite in the limit $\lambda \to 0$ and $\cos \theta_i \to \pm 1$. Taylor expansions are necessary in different regions of the phase space to obtain results, which are numerically stable. All analytical integrals entering our calculation are checked numerically. In addition, our analytic result is checked numerically against our amplitude squared with a 7-fold integration by the integration routine VEGAS \[10\]. Finally, we compared our result numerically with COMPHEP \[11\] for different c.m. energies, quark masses and cuts.

3 Results

The numerical input for our figures is

- $\alpha_s = 0.121$, $\alpha = 1/128$, $\sin^2 \theta_W = 0.232$, $M_Z = 91.186 \text{ GeV}$, and $\Gamma_Z = 2.495 \text{ GeV}$.

The total cross section for $q = b, c$ is shown in Figure 2. To illustrate the mass dependence, the cross section is given for different $b$-quark masses beyond the $Z$ peak. The considered process has a large cross section. For light quark masses and large energies $\sqrt{s} \geq 200 \text{ GeV}$, it becomes comparable to the cross section of $e^+ e^- \to q\bar{q}$. This is due to the radiation of two collinear gluons. For $\sqrt{s} = 200 \text{ GeV}$ and $m_q = 1.5 \text{ GeV}$, every collinear gluon generates a logarithm $\ln \frac{s}{m_q^2} \approx 9.8$, which numerically compensates the coupling constant $\alpha_s$. Other contributions to the cross section

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{The total cross section as a function of the c.m. energy for $m_b = 4.3 \text{ GeV}$ and $m_c = 1.5 \text{ GeV}$. For $\sqrt{s} > 120 \text{ GeV}$, the cross section is also shown for $m_b = 1.5 \text{ GeV}$ and $m_b = 10 \text{ GeV}$. The cuts $\sqrt{s_q} = \sqrt{s_g} = 20 \text{ GeV}$ are applied.
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{The differential distributions $d\sigma/d\sqrt{s_q}$ and $d\sigma/d\sqrt{s_g}$ for $\sqrt{s} = 200 \text{ GeV}$, $m_c = 1.5 \text{ GeV}$ and $m_b = 4.3 \text{ GeV}$. The cuts $\sqrt{s_q} = \sqrt{s_g} = 20 \text{ GeV}$ are applied.
\end{figure}
are of comparable importance for lower energies or larger quark masses. The logarithmic mass terms are the reason for the increase of the total cross section with smaller quark masses.

Figure 3 shows the differential distributions of the invariant energies of the quark and gluon pairs in the final state. The distributions are zero for $\sqrt{s_{i}} < 20 \text{GeV}$ and $\sqrt{s_{i}} > 180 \text{GeV}$ due to our kinematical cuts. The distributions for $\frac{d\sigma}{d\sqrt{s_{i}}}$ peak for small $s_{g}$ reflecting the infrared peak due to the radiation of soft gluons. The distributions for $\frac{d\sigma}{d\sqrt{s_{q}}}$ are largest for values of $s_{q}$ near the upper limit $(\sqrt{s} - \sqrt{s_{g}})^{2}$ again due to the radiation of soft and collinear gluons.

To summarize, we performed a semi-analytic calculation of the process $e^{+}e^{-} \rightarrow q\bar{q}gg$ where all final particles are assumed to be hard. The process has a large cross section. It is a large incoherent background to the four-fermion processes currently measured at LEP.

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