Exact non-equilibrium full counting statistics and proof of dynamical Jarzynski equality for Luttinger liquid tunnel junctions

Gleb A. Skorobagatko

1Institute for Condensed Matter Physics of National Academy of Sciences of Ukraine, Sivenitskii Str.1,79011 Lviv, Ukraine

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Full counting statistics for a wide class of Luttinger liquid tunnel junctions in a “weak link” regime is considered in all orders in tunnel coupling and out of the equilibrium in the time domain. Especially, two important mathematical statements: the FCS-S-theorem and the FCS-S-lemma about exact re-exponentiation of Keldysh-contour-ordered evolution operator are proven. These statements are the generalizations of S-theorem and S-lemma have been recently proven by the author in Ref.[G.A.Skorobagatko, Phys.Rev.B, 98, 045409 (2018)]. It is shown that FCS-S-theorem can be treated also as the proof of dynamical Jarzynski equality for tunnel electron transport out of the equilibrium (in time domain). As the result, exact time-dependent cumulant generating functional is derived being valid at arbitrary electron-electron repulsion in the Luttinger liquid leads of the junction, arbitrary temperature and arbitrary bias voltage. Respective general formula in its long-time asymptotics turns into a non-perturbative Luttinger liquid generalization of a well-known Levitov-Lesovik formula for cumulant generating function. Hence, demonstrated proof of FCS-S-theorem can be considered also as the proof of detailed balance theorem for the long-time limit of strongly correlated electron transport in arbitrary tunnel junctions. As the consequence of obtained results, a new measure of disequilibrium in Luttinger tunnel junction is introduced and discussed.

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I. INTRODUCTION

Extensive development of modern physics and quantum engineering \(^{1–6}\) demands rigorous knowledge about the details of quantum electron transport in low-dimensional nanowires \(^{7–10}\). Since the times of seminal Kane and Fisher papers \(^{11,12}\) on renormalization group analysis of electron tunneling through different types of Luttinger liquid junctions, the features of non-equilibrium electron ballistic transport in quasi-one dimensional electron systems has become more and more principal object of attention both in theory \(^{11–13,15–21}\) and in the experiment \(^{2–10,14,37}\). The common issue for such nanoelectronic devices is the one-dimensionality (1D) of chiral electron density excitations being responsible for respective ballistic electron transport in such structures at low enough temperatures \(^{11–13,15–21}\). Here a well-known bosonization technique \(^{12,43}\) with basic relations between fermionic and bosonic field operators \(^{11–13,15–21}\) paves the way to effective theoretical description of e.g. quantum Hall edge states (QHE-states) \(^{18–21}\) and 1D quantum wires \(^{11–13,15–21}\) within Tomonaga-Luttinger liquid (TLL) model in terms of charge- and current densities of plasmonic excitations in such systems.

From both fundamental and practical points of view generic bosonized QPC model is still far from its detailed and, at the same time, comprehensive understanding. However, due to their extreme practical importance, such QPC models stimulated a number of papers \(^{21,44–47}\) on each among two overlapped main approaches to their theoretical descriptions. Those approaches are: 1) various practical implementations \(^{18–20,30–33,36}\), the latter consider various particular modifications of most general bosonized QPC model, but such treatment remains perturbative only up to the lowest orders in small tunnel coupling (for non-equilibrium effects \(^{19,20,30–33}\)) or in small backscattering (e.g. for problems of electron injection into QH edge states \(^{18–21,36–47}\)) and 2) fundamental exact non-perturbative results \(^{15,16,22–25}\) on different aspects of general Luttinger liquid QPC models though having limited implementations either due to extreme mathematical difficulties in the derivations of any simple analytical consequences from general too abstract solutions \(^{15,16,22–25}\), or due to uncontrollability (i.e. variance) of assumptions, one needs to overcome \(^{16,25}\) in order to ”distill” any analytical non-perturbative answer from very general and very complicated expressions.

First type of studies is more common than second one and includes a variety of different non-equilibrium results on QHE- and single-electron transistor systems with 1D leads, which though remain valid only in the lowest orders in tunnel couplings \(^{36–39}\). Somewhere in between the studies of first (perturbative) and of the second (exact) type, the original method of functional bosonization \(^{21,45,46}\) is placed. Within latter framework one deals with scattering phases of relevant plasmonic excitations in the weakly constricted Luttinger liquid quantum wire or in fractional quantum Hall edge states \(^{21,44–47}\), such systems with weak electron backscattering one can treat exactly within the functional bosonization method as it was shown by Gutman, Gefen and Mirlin in the series of related papers \(^{21,44–45}\). However, the latter is applicable, strictly speaking, only for those quasi-1D quantum wires having weak
enough backscattering from the impurities rather than for quantum wires being interrupted by means of large enough tunnel barriers. This is because the Fredholm determinant factorization lying in the core of all the method implies a picture of multiple consecutive scatterings of quantum plasmonic excitations in the effective external time-dependent potential created by weak impurity (or weak constringtion of the wire) being "dressed" into long-wavelength "classical" component of plasmonic charge density field. However, the latter picture seems to have no direct implementations for the electron transport problems involving electron tunneling. This is just because in the latter case the tunneling of a bare electron through a junction is a strongly non-perturbative process, which involves a destruction/creation of entire "cloud" of plasmonic excitations (the electron in bosonized picture is just a 2π-kink of bosonic quantum field). Therefore, electron tunneling in the weak-link (or alternatively, weak tunneling) limit of a Luttinger liquid junction cannot be performed only just as forward scattering of chiral plasmons delocalised throughout a junction as it takes place within the functional bosonization method of Refs. [21,44,47]. In other words, functional bosonization is valid for 1D quantum wires which have quite "smooth" constrictions and/or inhomogeneities (as it takes places in FQH-edge states quantum dynamics) rather than for "sharp" boundary conditions which define common tunnel junction. Hence, the various functional bosonization results performed in Refs. [21,44,47] should be useful mostly in the cases of well-defined fractional-quantum-Hall effect (FQHE) edge states 18-21,31,36, where electrons with initially prepared distributions are injected into the edge states of FQH systems and these wave packets interacts only weakly with each other. At the same time, for Luttinger liquid tunnel junction of interest the functional bosonization method is problematic to use for the non-perturbative treatment because of superposition of two aspects: i) strong non-linearity of the tunnel Hamiltonian in its bosonic representation and ii) "sharp" boundary conditions in the vicinity of tunnel contact (i.e. due to strong backscattering from the tunnel barrier).

In the studies of the second type (exact treatment), especially for the regime of "weak tunneling" in 1D junctions, people make use of the exact integrability methods, such as thermodynamic Bethe ansatz (TBA) 15,16,24,25 being applied to a cumbersome exact solution 22,23 of a well-known boundary Sine-Gordon (BSG-)model, this case is reduced to. In more details, exactly solvable BSG-model 22,23 allows one to obtain an infinite chain of connected exact integral equations, which might be decoupled from each other within TBA-method and, thus, solved explicitly 16,24,25. However, the latter is possible only under some specific assumptions about the eigenvalues and eigenstates for respective exactly solvable model 22,25. This research direction is represented by several basic papers by Ludwig, Fendley and Saleur 15,23,24, Komnik and Saleur 16,25, where the latter authors especially pointed out on the importance of subtleties hidden in the implementations of TBA-method 22. In particular, they noticed that all the method is sensitive to the type of eigenstates (e.g. to different numbers of kink-antikink pairs, breathers, etc.) chosen for further analysis of TBA equations, because definite combinations of such eigenstates (e.g. kink-antikink pairs) allow for well-defined distribution functions only up to the calculation of real-time correlators of 3-d order in corresponding cumulant expansions, whereas already in 4-th order cumulant the kink-antikink pair becomes strongly fluctuating, which breaks the legitimacy of the decoupling of TBA chain of equations and, thus, all the TBA results for all higher order cumulants 22.

As one can see from the above, the complete description of real-time quantum dynamics out of the equilibrium for arbitrary Luttinger liquid QPC still remains a very challenging task. However, there are some theoretical hints 28,30-33 on the fact, that even in the case of interacting electrons that complicated, their charge transfer statistic 26,27,29 still represents a much more simple and much more general picture. Especially, from the times of well-known Levitov and Lesovik papers 30,31, where authors derived in the lowest orders in tunnel coupling a remarkable generic (Levitov-Lesovik) formula for cumulant generating function of non-interacting electrons, it has been understood 28 that above mentioned extremely complicated (and strictly speaking, unknown in its details) picture of electron real-time correlations in any Luttinger liquid QPC should not be relevant for electron counting statistics at least in the long-time limit of their tunneling through the junction 28. Interestingly, this point of view is indirectly supported by a number of parallel investigations on another aspects of electron transport in quasi-one dimensional electron systems, such as: 1) the validity of linked cluster theorem for orthogonality catastrophe calculation 22,13; 2) general relation between high order cumulants derived from the summation of selected linked-cluster diagrams in lowest order in electron tunneling 24; 3) simple expression for Fano factor from complicated exact BSG model solution reported in Ref. [24]; 4) Sukhorukov-Loss perturbative prediction 22 about the universality of Fano-factor for interacting and non-interacting electrons; 5) the possibility to incorporate electron-electron interactions into functional bosonization description of weakly constricted 1D quantum wires reported in Refs. [21,44,45] together with predicted in Refs. [35,36] consequence about electron charge fractionalization in certain full counting statistics measurements; 6) recent observation by Gutman and Heiblum 34 about universal noise formula for quantum Hall edge states, which follows from common statistical mechanics.

However, all above mentioned separate hints on the universality of full counting statistics for interacting electrons in their "steady flow" (or, simpler, steady) state, on one hand, make use of general statistical arguments, such as e.g. detailed balance or, more general, fluctuation-dissipation theorem, as hypothesis. In turn, the
detailed balance relation between "forward" and "backward" tunneling rates are just a "steady state version" of more general fluctuation-dissipation theorem (or Jarzynski equality) for non-equilibrium processes\textsuperscript{39,40}. On the other hand, importantly all results that general for the case of interacting electrons in tunnel junction has remained, in fact, only perturbative, e.g. one is unable to claim, whether the detailed balance relations, or Jarzynski equality out of the equilibrium - remain valid in arbitrary order of perturbation theory, or beyond the "steady flow" regime (i.e. on finite observation timescales). Moreover, in the derivation of classical Levitov-Lesovik formula\textsuperscript{26,28} for non-interacting electrons the detailed balance relation for electron tunneling rates has been indirectly used in order to obtain correct renormalization of tunneling amplitudes by bias voltage\textsuperscript{28}. However, to be precise, such type of renormalization has been justified only for the non-interacting electrons, whereas for the case of interacting electrons in the leads (Luttinger liquid situation) all the above complicated quantum soliton-like dynamics\textsuperscript{24,25} should come into play with unknown details of respective finite-time evolution\textsuperscript{37}.

Therefore, this paper is intended to bring certainty into existing phenomenological hints on the universality of counting statistics and fluctuation-dissipation theorem for interacting and non-interacting electrons in tunnel junctions, thus, filling the gap between existing fragmentary cases of exact treatment of the real-time non-equilibrium dynamics for arbitrary Luttinger liquid quantum-point contacts in the weak tunneling limit. Especially, here I calculate exactly all the infinite Keldysh contour-ordered expansion of evolution operator for biased Luttinger liquid QPC in the weak tunneling limit and in real time beyond long-time asymptote, for arbitrary temperature, bias voltage and electron-electron interaction in junction electrodes. In particular, I make use of so-called S-theorem and S-lemma, which both I had introduced and proved in Ref.\textsuperscript{[38]}. Naturally, performed rigorous proof of FCS-S-theorem at the same time is a rigorous mathematical proof of the detailed balance theorem in steady state and proof of Jarzynski equality - for non-stationary (or non-equilibrium) quantum many-body dynamics of interacting electrons in arbitrary Luttinger liquid tunnel junction in the weak tunneling regime.

II. MODEL

The QPC under consideration represents a tunnel contact in the regime of weak tunneling\textsuperscript{12,13} which connects right (R) and left (L) Luttinger liquids. The total Hamiltonian of our problem consists of two terms $H_S = H_{LL} + H_{int}$, where $H_{LL}$ represents the Hamiltonian of the left and right Luttinger liquids and $H_{int}$ stands for the tunnel interaction between those two. If we consider the QPC to be located at $x = 0$, then $H_{LL} = \frac{1}{2\pi} \sum_{j=L,R} v_j \int_{-\infty}^{\infty} \left\{ g \left( \partial_x \varphi_j \right)^2 + \frac{1}{\pi} \left( \partial_x \theta_j \right)^2 \right\} dx$(Here and everywhere in all the numbered formulas I put $\hbar = 1$ and $e = 1$ ) where $\theta_{L(R)}(x) = \pi \int_{-\infty}^{x} dx' P_{LL}(x')$ and $\varphi_{L(R)}(x) = \pi \int_{-\infty}^{x} dx' j_{L(R)}(x')$ are the usual charge- and phase- bosonic quantum fields ($\langle \theta_{L(R)}(x,t) \rangle = 0$ and $\langle \varphi_{L(R)}(x,t) \rangle = 0$) in the Luttinger liquid liquid of semi-infinite 1D quantum wire, those corresponding to fluctuating parts of charge- and current electron densities in the QPC lead\textsuperscript{11,13,17,25,38,43}. Here $g$ is a dimensionless correlation parameter which is defined as $g \approx (1 + U_s / 2E_F)^{-1/2}$ (for repulsive interactions it fulfills $0 < g \leq 1$) while $v_j$ is the group velocity of collective plasmonic excitations in the leads\textsuperscript{32,33}. Further, within the "weak tunneling" approach\textsuperscript{12,13} "charge"- bosonic fields: $\theta_{L,R}(x = 0, t)$ are pinned on the edges of respective QPC electrodes at point $x = 0$ by means of the condition\textsuperscript{12,13,17,38,43}: $\theta_{L}(x = 0, t) = \theta_R(x = 0, t) = 0$. Hence, if one defines following non-local bosonic charge- and phase-fields\textsuperscript{17,38} $\varphi_{\pm} = [\varphi_L \pm \varphi_R]$ it is possible to write tunnel Hamiltonian of our system in the following bosonized form $H_{int} = \lambda \cos (\varphi_- + eVt)|_{x=0}$ (1)

where $\lambda = t / (\pi a_0)$, with $t$ being a "bare" tunneling amplitude for given QPC. The parameter $a_0$ is the lattice constant of the model. This constant provides a natural high-energy cut-off $\Lambda_0 = v_j / a_0$, and goes to zero in the continuum limit ($a_0 \to 0$)\textsuperscript{33}. Quantity $eV$ refers to external bias voltage applied between left and right Luttinger liquid electrodes of QPC\textsuperscript{33}.

To detect the state of charge transport through our Luttinger liquid tunnel junction (or, QPC) we should examine the full counting statistics (FCS) of our system\textsuperscript{28,29}. Probability distribution (see e.g. Ref.\textsuperscript{[29]}): $\mathcal{P}(N,t)$ for the total electrical charge $q = eN$ transmitted through the QPC during the fixed interval of time $t$ from the so-called Keldysh partition function (KPF) $\tilde{\chi}(\xi, t) = \sum_{\mathcal{N}} \mathcal{P}(\mathcal{N}, t) e^{i\mathcal{N}} = e^{-\mathcal{F}(\xi,t)}$ (2)

where $\mathcal{F}(\xi,t)$ is often called cumulant generating function (CGF)\textsuperscript{28,29}. If one manages somehow to find exact expression for $\mathcal{F}(\xi,t)$, then the latter allows to obtain all "irreducible" moments (cumulants) of the time-dependent distribution: $\mathcal{P}(\mathcal{N}, t)$ as well as that distribution by itself\textsuperscript{28}. Hence, by definition one has for the cumulant of arbitrary order $k$

$$\langle N^k(t) \rangle = (-i)^k \frac{\partial^k \mathcal{F}(\xi, t)}{\partial \xi^k} |_{\xi=0, k=1,2...,}$$ (3)

On the other hand, as it is well-known in the context of FCS that the Keldysh partition function (KPF) can be expressed as the averaged Keldysh contour-ordered T-exponent (evolution operator) with respective tunnel Hamiltonian\textsuperscript{28,29}. In our case (see also Refs.\textsuperscript{[17,38]}) this expression reads $\tilde{\chi}(\xi, t) = \langle T_K \exp(-i\lambda \int_{-\infty}^{\xi} A_{\xi}(\tau) d\tau) \rangle$ (4)
where the thermal averaging is performed over the ground state of Luttinger liquid leads with the appropriate Gibbs’ factors. In Eq.(4) symbol $T_K$ marks the sum over all possible time-orderings on the complex-time Keldysh contour $\mathcal{C}_K \in (0 \pm i \vartheta; t)$. At the same time, for the time-dependent ”quantum potential” $A_{\xi(\tau)}(\tau)$ in ”bosonic language” one has

$$A_{\xi(\tau)}(\tau) = \cos \left[ \varphi_{\beta}(\tau) + f_{\xi(\tau)}(\tau) \right]$$  \hspace{1cm} (5)

with the time- and counting field-dependent function $f_{\xi(\tau)}(\tau)$, where $\xi(\tau) = \pm \xi$ is a counting field defined on the upper (lower) branch of the Keldysh contour $\mathcal{C}_K \in (0 \pm i \vartheta; t)$.

III. FCS CASE OF SUMMATION (S-)
THEOREM AND ITS CONSEQUENCES FOR
LUTTINGER LIQUID TUNNEL JUNCTION

Now to calculate KPF of Eq.(4) exactly one can do this with the help of the FCS-S-Theorem (and FCS-S-Lemma) proven in what follows(see Supplementary materials below and Ref.[38]).

FCS-S-Theorem (proven in the Appendix B below) states following exact relation

$$\chi(t) = \left\langle T_K \exp(-i\hat{\lambda} \int_{\mathcal{C}_K} A_{\xi(\tau)}(\tau) \, d\tau) \right\rangle$$

$$= \exp \left\{ -\frac{\hat{\lambda}^2}{2} \int_{\mathcal{C}_K} d\tau_1 d\tau_2 \left\langle T_K A_{\xi(\tau_1)}(\tau_1) A_{\xi(\tau_2)}(\tau_2) \right\rangle \right\}$$

$$= \exp \left\{ -\frac{\hat{\lambda}^2}{2} \int_{\mathcal{C}_K} d\tau_1 d\tau_2 \left\langle T_K A_{\xi(\tau_1)}(\tau_1) A_{\xi(\tau_2)}(\tau_2) \right\rangle \right\}$$

$$= \exp \left\{ -\left\langle T_K A_{\xi(\tau)}(\tau) \, d\tau \right\rangle \right\}$$

(6)

Remarkably, the exactness of Eq.(6) has been proven here by FCS-S-theorem can be treated also as the manifestation of famous Jarzynski equality\textsuperscript{29,30}, which holds for given quantum system with fluctuating quantum bosonic field $\varphi_{\beta}(t)$ in real time domain. Indeed, conventional Jarzynski equality\textsuperscript{25} (i.e. the one for imaginary time-domain, where $\tau \in (0; 0 - i \beta)$ and $\beta = 1/T$ -is inverse temperature) for abstract statistical system out of the equilibrium reads

$$\langle \exp^{-\beta W} \rangle = \exp^{-\beta \Delta F}$$ \hspace{1cm} (7)

here $W$ is certain fluctuating quantity (say, external "work" on the system along its stochastic trajectory); the averaging is performed over different stochastic trajectories of the system (i.e. over different "external measurement" realizations) and $\Delta F$ in the r.h.s. of Eq.(7) represents trajectory-independent change in system’s free energy. By comparing Eq.(6) and FCS-S-theorem with Eq.(7) one can conclude that in our case fluctuating quantity $W$ correspond to fluctuating action of interacting quantum field $\varphi_{\beta}(t)$, while the averaging over different stochastic trajectories in "conventional” Jarzynsky case\textsuperscript{29,30} corresponds to thermal averaging together with Keldysh contour integrations over different contour arrangements of quantum field operators $\varphi_{\beta}(t)$ of all orders in our case.

Equally, in Eq.(6) one can perform time-dependent cumulant generating functional $\mathcal{F}(\xi, t)$ in the following exact form

$$\mathcal{F}(\xi, t) = \left( \frac{\hat{\lambda}}{2} \right)^2 \times$$

$$\frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 \nu_\beta(\tau_1 - \tau_2) \times$$

$$\left\{ \cos \left[ \pi/\beta + eV(\tau_1 - \tau_2) \right] \left( e^{2\xi} \left[ \sigma \left( \frac{\pi}{\beta} + eV(\tau_1 - \tau_2) \right) \right] - 1 \right) \right\}$$

(8)

$$+ \left( \frac{\hat{\lambda}}{2} \right)^2 \times$$

$$\frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 \nu_\beta(\tau_1 - \tau_2) \times$$

$$\left\{ \cos \left[ \pi/\beta - eV(\tau_1 - \tau_2) \right] \left( e^{-2\xi} \left[ \sigma \left( \frac{\pi}{\beta} - eV(\tau_1 - \tau_2) \right) \right] - 1 \right) \right\}$$

with symmetric pair correlator

$$u_\beta(\tau - \tau') = \left( e^{i\xi} - e^{-i\xi} \right)$$

$$= \left[ \frac{\pi T/\Lambda_g}{\sin \left( \frac{\pi T}{\beta T} |\tau - \tau'| \right) } \right]^{2/g}$$

(9)

which should be regularized properly at $|\tau - \tau'| \to 0$ under corresponded time integrations of Eq.(8)(see Appendix C below). Formulas (8,9) -represent exact general expression for the time-dependent Luttinger liquid QPC cumulant generating functional.

Remarkably, in the limit: $t \to \infty$ one will have $\lim_{t \to \infty} \mathcal{F}(\xi, t) = W_B(\xi)t$, where, calculating analytically two corresponded integrals $J_{C,B} = \lim_{t \to \infty} J_C(t)$ and $J_{S,B} = \lim_{t \to \infty} J_S(t)$(see Appendix C below) one can obtain for $W_B(\xi)$ following exact Luttinger liquid generalization of well-known ”Levitov-Lesovik” formula

$$W_B(\xi) = \{ \Gamma_{g,\pm} \left[ e^{2\xi} - 1 \right] \} + \Gamma_{g,\pm} \left[ e^{-2\xi} - 1 \right] \}$$

(10)

with

$$\Gamma_{g,\pm} = \Gamma_{g,\pm}(eV, T) = \left( \frac{\hat{\lambda}^2}{4\Lambda_g} \right) F_g(eV, T) e^{\pm eV/\beta T}$$

(11)

where

$$F_g(eV, T) = \left[ \frac{\Gamma (1/g + i [eV/2\pi T])}{\Gamma (2/g)} \right] \left[ \frac{2\pi T}{\Lambda_g} \right]^{(2/g - 1)}$$

(12)

is our ”Luttinger liquid” - (or ”interaction”) factor. Equations (10-12) give rise to the exact long-time description of the Luttinger liquid tunnel junction in the framework of its full counting statistics (FCS).

Obviously, from the equations (11,12) it follows the fundamental property for the long-time limit of quantum ballistic transport of interacting electrons, being demonstrated previously only for the case of noninteracting electrons in tunnel junction\textsuperscript{28}

$$\frac{\Gamma_{g,\pm}(eV, T)}{\Gamma_{g,\pm}(eV, T)} = e^{\pm eV/T}$$

(13)
Equation (13) represents a "steady state version" of dynamical Jarzynski equality (see Eqs.(6,7)) proven here by means of FCS-S-theorem. Thus, an important exact result, which is proven here states that: In the "long-term" perspective, i.e. in the limit \( t \to \infty \) arbitrary repulsive electron-electron interactions in the leads of Luttinger liquid tunnel junction are unable to break the detailed statistical balance between physical "bare" electrons being transmitted through and reflected from such a tunnel contact.

Now substituting formulas (10,11) into Eq.(3) one easily obtains following exact relations

\[
\lim_{t \to \infty} \langle \langle N^k \rangle \rangle_B = \left\{ \begin{array}{ll}
- (\Gamma_{g,+} + \Gamma_{g,-})t & , k = \text{even} \\
- (\Gamma_{g,+} - \Gamma_{g,-})t & , k = \text{odd}
\end{array} \right. 
\]

which represent Luttinger liquid generalization of similar expressions have been derived perturbatively (only in the lowest order in tunnel coupling) by Levitov and Reznikov for the case noninteracting electrons. In particular, from general formulas (11,12,14) it follows for the average current (here we put \( e = |e| = 1 \) together with \( h = 1 \)) \( \langle \langle J_B eV, T \rangle \rangle_B = - \left( \hat{\gamma}^2 / \gamma \right) \tilde{g}(eV,T) \sinh(eV/2T) \) and for respective noise power \( \tilde{S}_B(eV,T) = - \left( \hat{\gamma}^2 / \gamma \right) \tilde{g}(eV,T) \cosh(eV/2T) \) with \( \tilde{g}(eV,T) \) from Eq.(13). Obviously, these expressions give correct "high-" and "low-temperature" asymptotes, which coincide with well-known Kane-Fisher scaling for arbitrary values of Luttinger liquid correlation parameter: \( 0 < g \leq 1 \). Therefore, for the ratio of two subsequent irreducible correlators for arbitrary Luttinger liquid tunnel junction in a "detailed balance" ("steady flow") regime of strongly correlated electron tunneling

\[
F_B = \frac{\langle \langle N^{2k} \rangle \rangle_B}{\langle \langle N^{2k-1} \rangle \rangle_B} = \frac{\tilde{S}_B}{I_B} = \coth(eV/2T). 
\]

(15)

(for \( k = 1, 2, 3, \ldots \)), where \( F(eV,T) \) represents a universal Fano-factor (i.e. noise power to current ratio), which remarkably coincides with the Fano-factor \( F_0(eV,T) \) derived by Sukhorukov and Loss for noninteracting electrons in the leads.

Obviously, Eqs.(3,8,9) enable one to introduce a non-equilibrium analog of \( F_B \) for any finite moment of time \( t \) after the beginning of the tunnel transport measurements. I called the respective quantity the "steady flow" rate, it reads

\[
R_{SF}(t) = \frac{1}{2} \left( t - \langle \langle N^{2k}(t) \rangle \rangle - \langle \langle N^{2k-1}(t) \rangle \rangle \right) = \frac{1}{2} \left( \frac{1 - \hat{\gamma}^2 J_C(t)}{1 - \hat{\gamma}^2 J_S(t)} \right), 
\]

(16)

where \( J_C(t) = \frac{\tilde{J}_C(t)}{2} \left( \frac{\pi T}{\lambda_g} \right)^{2/g} \) and \( J_S(t) = \frac{\tilde{J}_S(t)}{2} \left( \frac{\pi T}{\lambda_g} \right)^{2/g} \) with

\[
\tilde{J}_C(t) = \int_0^t ds \frac{\cos(eVs) \cos [\pi/g - 2\eta(s)/g]}{[\sin^2(\pi Ts) + (\pi T/\lambda_g)^2 \cosh^2(\pi Ts)]^{1/g}} 
\]

(17)

and

\[
\tilde{J}_S(t) = \int_0^t ds \frac{\sin(eVs) \sin [\pi/g - 2\eta(s)/g]}{[\sin^2(\pi Ts) + (\pi T/\lambda_g)^2 \cosh^2(\pi Ts)]^{1/g}}. 
\]

(18)

with \( \eta(t) = \arctan[(\pi T/\lambda_g) \coth(\pi T t)] \). One can see from Eqs.(17,18) that introduced "measure of disequilibrium" (or "steady flow" rate) \( R_{SF}(t) \) of Eq.(16) has following remarkable property

\[
\lim_{t \to \infty} R_{SF}(t) \approx \frac{F_B}{2} = \frac{1}{2} \coth(eV/2T) = \frac{1}{2}. 
\]

(19)

Naturally, the "steady flow" rate \( R_{SF}(t) \) defined by means of Eqs.(16-18) represents a clear measure of disequilibrium for electron transport in any 1D tunnel junction, if to understand by the equilibrium in such system its steady state with constant average transport characteristics of Eqs.(10-15,19).

IV. DISCUSSION

On Fig.1a,b quantity \( R_{SF}(t) \) is plotted as the function of time (in the units of \( \delta_g \simeq \Lambda^{-1}_g \)), according to obtained exact Eqs.(16-18) for several distinct values of Luttinger liquid correlation parameter \( g \) in the low-temperature regime of electron tunneling. Fig.1a depicts non-interacting and weakly interacting cases \( (g = 1 \) and \( g = 0.875 \)), while Fig.1b refers to strongly interacting case \( (g = 0.505 \) and \( g = 0.45 \), correspondingly). One can see from both figures (Figs.1a,b) that for any strength of electron-electron repulsive interactions in the electrodes (i.e. for any possible value of parameter \( 0 < g \leq 1 \)) the respective weak-tunneling equilibration represents a fast enough and, at the same time, a tiny process, which evolves to the equilibrium on timescales not bigger than \( \approx 10^2 \delta_g \). Its first and most "unequilibrated" stage takes relatively short time \( \sim 15 - 20 \delta_g \) - for weakly interacting electrons and even shorter time, of the order of \( \sim 3 - 5 \delta_g \) - for strongly interacting electrons in the QPC leads. Besides that, as one can see from both insets to Figs.1a,b on the later stages of system's quantum dynamics (of the order of \( \sim 10 - 50 \delta_g \)) the quantity \( R_{SF}(t) \) - slightly oscillates with very small amplitude around its low-temperature equilibrium value being equal
to 1/2 and the amplitude of such oscillations quickly decays to the values beyond the accuracy of equilibrium value (1/2) detection. - These tiny oscillations represent a reminiscence of two different effects: the one defined by bias voltage (i.e. due to $\cos(\epsilon V t)$ and $\sin(\epsilon V t)$ factors in Eqs(17,18)) - has the same nature as one of a non-stationary Josephson effect in the superconductivity,\footnote{since in both cases one has fluctuating phase-field $\varphi(t)$ which enters Hamiltonian via $\cos(\varphi(t) + \epsilon V t)$ term; whereas the second effect is due to the oscillating factors $\cos(\pi g/2 + 2\eta(t)/g)$ and $\sin(\pi g/2 + 2\eta(t)/g)$ in Eqs.(17,18), it is similar to “Friedel oscillations” phenomena though realized in the time domain and is compatible with similar effects in the weak backscattering limit of constricted 1D quantum wires.}

The most interesting effect of electron-electron interaction being evident from Fig.1 is that regime of strong electron correlations (i.e. the “steady flow” of strongly interacting electrons in the leads) establishes much faster in the system than ”steady flow” regime of the same kind for weakly interacting (and non-interacting) electrons in QPC. This feature is remarkable just because in the steady state of ballistic electron transport the ”interacting” Luttinger liquid tunnel junctions provide much slower electron tunneling than ”non-interacting” ones due to Kane-Fisher suppression of electron density of states on the edge of 1D quantum wire (Kane-Fisher effect\footnote{\cite{11,12,17,38}}). Here, as it can be seen from Fig.1, Kane-Fisher suppression of electron tunneling is manifested in much lower amplitudes of non-equilibrium oscillations of ”steady flow” rate $R_{SF}(t)$ for interacting electrons, however the latter does not contradict much faster decay of these non-equilibrium oscillations to the equilibrium value of $R_{SF}(t) = 1/2$ being revealed here for arbitrary interacting electrons (when $g < 1$) in the tunnel junction. Moreover, it seems to be quite natural when strong electron-electron correlations in the system ”enforce” its evolution to equilibrium state (or steady state) more effectively than ”weak” correlations in the non-interacting and weakly interacting cases.

Here one might ask, why revealed non-equilibrium effects on the early stages of ballistic electron transport in tunnel junctions are so small by magnitude if to consider respective ”steady flow” rate quantity? - The answer is in the statistical meaning of introduced $R_{SF}(t)$ rate with respect to weak tunneling regime under consideration. The point is, one may treat $R_{SF}(t)$ as the conditional probability to find the electron in its ”steady current-carrying” (or chiral, or ”steady-flow”) quantum state, ”living” in both electrodes of QPC simultaneously: one can see from all graphs on Fig.1 that for large enough timescales function $R_{SF}(t)$ tends to the value 1/2, which corresponds to the situation where mentioned ”steady-flow” state of the electron has been already formed with probability equal to one. Equally, one may think about the introduced ”steady flow” rate $R_{SF}(t)$ as about time-dependent conditional probability for bare electron to be in its ”steady flow” (or chiral) quantum state in any among two QPC
electrodes but, at the same time, not to be involved into the (both virtual and real) tunnelings through the tunnel contact of interest. Then from Fig.1 one can see that such the probability fluctuates only weakly due to assumed weak tunneling regime and (because of the same reason) - it saturates quickly to its equilibrium low-temperature value 1/2. The latter means, that in the "steady flow" regime of electron transport all tunnelings of the junction tend to form a collective chiral quantum state and, therefore, the probability to find any bare electron in one among two QPC leads tends to be equal to 1/2 in the described "steady flow" regime of electron tunneling.

V. CONCLUSIONS

To conclude, due to rigorous mathematical statements (FCS-S-theorem and FCS-S-lemma) have been proven here, it becomes possible to calculate exactly the time-dependent cumulant generating functional in all orders of tunnel coupling for arbitrary Luttinger liquid tunnel junctions in the regime of "weak link", for arbitrary temperatures and bias voltages. The resulting exact formulas describe explicitly all real-time correlations in the system at any instant of time and in the long-time limit obtained formulas take the generalized Levitov-Lesovik structure, similarly to the case of non-interacting electrons in the leads. The exactness of obtained results also proves the validity of two important theorems from statistical physics for arbitrary Luttinger liquid tunnel junction.

These are: 1) the non-equilibrium Jarzynski equality and 2) the detailed balance theorem for tunnel current of interacting electrons in the long-time limit (steady state). On the other hand, exact time-dependent correlators of all orders (obtained in the above from the exact cumulant generating functional) allow to introduce and to study a new measure of disequilibrium in the system: a "steady flow" rate, which describes the character of the Luttinger liquid QPC equilibrium to its current-carrying steady state (here I call latter as "steady flow" state). As the result, the equilibration of ballistic electron transport in Luttinger liquid tunnel junction has been studied explicitly. Especially, it was found that strong electron-electron interactions in the QPC electrodes brings QPC to its steady state much faster than the same occurs for the case of non-interacting and weakly interacting electrons in the tunnel junction. All the obtained results are believed to be important for a wide range of problems in modern low-dimensional quantum mesoscopics from fundamental, such as collective behavior of strongly interacting electrons in quasi-one dimensional junctions; exact charge transfer statistics of strongly correlated electrons - to applied ones, such as non-equilibrium effects in the ballistic transport in one-dimensional quantum wires, quantum Hall edge states manipulations, decoherence, as well as dynamics and engineering of various more general quantum circuits and devices.

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Appendix A: Basic real time pair correlator

Let us pay more attention to pair correlator of Eq.(9). The basic ingredient in its calculation is well-known integral over $k$ with nonzero high-energy cutoff $e^{-\alpha_0 k}$, which can be calculated exactly (see e.g. Refs.[17,38,43]) and the answer is

$$\langle [\varphi(t) - \varphi(0)]^2 \rangle = I(t)/g$$

$$= \frac{4}{g} P \int_0^\infty \frac{dk}{k} e^{-\alpha_0 k} [1 - \cos(v_g k t)] n_B(v_g k)$$

$$= \frac{2}{g} \log \left[ \frac{\Gamma(1 + \alpha_0 T/v_g)}{\Gamma(1 + \alpha_0 T/v_g - iT) \Gamma(1 + \alpha_0 T/v_g + iT)} \right]$$

(A1)

where $\Gamma$ is the Gamma function. However, one can convince numerically that much more simple and suitable for analytics expression

$$I(t) = 2 \log \left[ \frac{\sinh^2(\pi T t) + (\pi T_0/v_g)^2 \cosh^2(\pi T t)}{(\pi T_0/v_g)^2} \right].$$

(A2)

- perfectly (i.e. with great numerical accuracy) approximates the exact result of Eq.(A1) at all values of argument $t$ in the whole range of parameters. One can obtain the latter expression (A2) by means of complex contour integration neglecting the infinitesimally small contributions from certain parts of complex contour. From Eq.(A2) it is evident that basic correlator $\langle [\varphi(t) - \varphi(0)]^2 \rangle$ is properly regularized at $t = 0$ (i.e. at equal times $\tau = \tau'$) as it should be by the definition. Formulas (A1) and (A2) tell that concrete form of such regularization of $I(s)$ at $s = 0$ is a matter of taste to some extent, since the behavior of an electron in the leads on corresponding short-time scale $|\tau - \tau'| = |t| \lesssim \Lambda_g^{-1}$ is beyond the control in the "long wavelength" approach of Luttinger liquid model. Thus, for our purposes, it is enough to take only a "large" $t$ asymptotic of basic correlator (A2) (or (A1)). As the result, one obtains

$$\langle \Delta \varphi_+(t)^2 \rangle = \langle [\varphi_+(t) - \varphi_+ (0)]^2 \rangle$$

$$= \frac{I(t)}{g} \approx \frac{2}{g} \log \left( \frac{\sinh^2(\pi T t)}{(\pi T_0/v_g)^2} \right).$$

(A3)

at $|t| \gg \Lambda_g^{-1} = \frac{\pi \alpha_0}{v_g} (\Lambda_g^{-1} \ll 1)$. The latter simple expression for the time-correlator for bosonic fields has been used frequently in the framework of Luttinger liquid calculations (see e.g. Giamarchi’s book of Ref.[43]). However, writing similar real-time basic pair correlator $f_g(\tau - \tau') = (e^{i\varphi_-(\tau')} e^{-i\varphi_-(\tau')} )^2 = e^{i[\varphi_-(\tau') - \varphi_+(\tau)]} e^{i[\varphi_-(\tau) - \varphi_+(\tau')]} = u_g(\tau - \tau') [e^{i[\varphi_-(\tau') - \varphi_+(\tau)]}]$ but under the sequence of time-integrations in the expansion of r.h.s. of Eq.(4), one should keep a proper regularization for equal times $\tau = \tau'$ even on the level of long-time asymptote of $u_g(\tau - \tau') (|\tau - \tau'| > \delta_g)$ since such a regularization provides correct analytical continuation for the time integral in calculation of decoherence rate and FCS (see Appendix C below). Thus, one needs
to rewrite a basic Luttinger liquid real time pair correlator \( f_g(\tau - \tau') \) in the form

\[
f_g(\tau - \tau') = \langle e^{i\varphi_-(\tau)} e^{-i\varphi_-(\tau')} \rangle = e^{\langle [\varphi_-(\tau) - \varphi_-(\tau')]^2 \rangle} e^{i\varphi_-(\tau') - i\varphi_-(\tau)}
\]

where positive \( \delta \simeq \Lambda_g^{-1} \geq 0 \) supposed to be infinitesimally small (\(|\tau - \tau'| > \delta_g\)). Formula (A4) for Luttinger liquid real time pair correlator is well-known in the literature (see e.g. the appendix of the Giamarchi’s book of Ref.[43]) and is used here for derivation of basic time-dependent integrals in Appendix C.

Appendix B: The FCS generalizations of S-Theorem and S-lemma

Let us generalize S-Theorem and S-lemma from Ref.[38] on the case of time-dependent quantum potential \( A_\xi(t) \) with a counting field \( \xi(\tau) \) which is dependent on the branch of corresponding complex Keldysh contour \( C_K \in (0 \pm i0; t) \)

\( \triangleleft \) FCS-S-Theorem: “The case of exact exponentiation of Keldysh contour time-integration for bosonic quantum potential with counting field”

\( \blacktriangleleft \) For any exponential bosonic operator of the form:

\[
A_{\xi(\tau)}(\tau) = \cos [\varphi_-(\tau) + f_{\xi(\tau)}(\tau)]
\]

where

\[
f_{\xi(\tau)}(\tau) = \begin{cases} f_{+\xi}(\tau), & \text{Im}\{\tau\} > 0 \\ f_{-\xi}(\tau), & \text{Im}\{\tau\} < 0 \end{cases}
\]

with property

\[
|f_i(\tau_1) - f_j(\tau_2)| = |f_i(\tau_2) - f_j(\tau_1)|
\]

with \( i, j = \pm \xi \) - is fulfilled for \( f_{\pm\xi}(\tau) \), being certain functions of time defined as \( f_{+\xi}(\tau) \) (\( f_{-\xi}(\tau) \)) on the upper (lower) branch of Keldysh contour in the complex plane and \( \varphi_-(\tau) \) is time-dependent bosonic field with zero mean \( \langle \varphi_-(\tau) \rangle = 0 \), which fulfils commutation relation of the form:

\[
[\varphi_-(\tau_n), \varphi_-(\tau_{n'})] = -2i\vartheta_g \text{sgn}[\tau_n - \tau_{n'}]
\]

where \( \vartheta_g = \text{const} \) (notice, that in our particular case: \( \vartheta_g = \pi/g \).

\( \blacktriangleright \) it follows for time-dependent and Keldysh contour-ordered average from a quantum potential with Keldysh contour-dependent counting field \( \xi(\tau) \)

\[
\tilde{\chi}(t) = \left\langle T_K \exp(-i\tilde{\lambda} \int_{C_K} A_{\xi(\tau)}(\tau) d\tau) \right\rangle
\]

\[
= \exp\left\{-F(\xi, t)\right\}
\]

\( \text{where for the function } F(\xi, t) \text{ one has} \)

\[
F(\xi, t) = \left(\lambda\right)^2 \left\{ \frac{1}{2} \int_{C_K} d\tau_1 d\tau_2 \langle T_K A_{\xi(\tau_1)}(\tau_1) A_{\xi(\tau)}(\tau_2) \rangle \right\}
\]

or, alternatively

\[
F(\xi, t) = \left(\lambda\right)^2 \frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \times
\]

\[
\left\{ e^{i\vartheta_g \Delta \kappa_+(\tau_1 - \tau_2)} + e^{-i\vartheta_g \Delta \kappa_-(\tau_1 - \tau_2)} \right\}
\]

with

\[
\Delta \kappa_\pm(\tau_1 - \tau_2) = [\tilde{\kappa}_\pm(\tau_1 - \tau_2) - \kappa_\pm(\tau_1 - \tau_2)]
\]

\( \text{properly regularized at the point } \tau_1 = \tau_2. \)

\( \triangleright \) The proof of FCS-S-Theorem.

In what follows we will slightly modify the proof of S-theorem for Keldysh propagator \( \chi(t) \) with respect to new quantum potential \( A_{\pm\xi}(\tau) \) of Eq.(5) and(B1,B2). First, as well as for usual S-theorem and S-lemma of Ref.[38], let us consider only the pair-correlator as the building block for the rest of material. Especially, in our case, in accordance with Eqs.(A4,B1,B2) we will have

\[
\langle A_{\xi(\tau)}(\tau_n) A_{\xi'(\tau')}(\tau_{n'}) \rangle = \langle A_{\xi(\tau)}(\tau_n) A_{\xi'(\tau')}(\tau_{n'}) \rangle_R \times
\]

\[
\left\{ \begin{array}{l} e^{-i\vartheta_g}, \tau_n > \tau_{n'} \\ e^{i\vartheta_g}, \tau_n < \tau_{n'} \\ 1, \tau_n = \tau_{n'} \end{array} \right\}
\]

where the “real” (but not necessary symmetrized) pair-correlator: \( \langle A_{\xi(\tau)}(\tau_n) A_{\xi'(\tau')}(\tau_{n'}) \rangle_R \) reads

\[
\langle A_{\pm\xi}(\tau_1) A_{\pm\xi}(\tau_{n+1}) \rangle_R = u(\tau_n - \tau_{n+1}) \kappa_\pm(\tau_{n+1})
\]

\( \text{with} \)

\[
\kappa_\pm(\tau_{n+1}) = \cos [f_{\pm\xi}(\tau_n) - f_{\pm\xi}(\tau_{n+1})].
\]

Also, here one should consider a ”mixed” average

\[
\langle A_{\pm\xi}(\tau_1) A_{\mp\xi}(\tau_{n+1}) \rangle_R = u(\tau_n - \tau_{n+1}) \tilde{\kappa}_\pm(\tau_{n+1})
\]

\( \text{where} \)

\[
\tilde{\kappa}_\pm(\tau_{n+1}) = \cos [f_{\pm\xi}(\tau_n) - f_{\mp\xi}(\tau_{n+1})].
\]
where \( u(\tau_1 - \tau_{+1}) \) is the correlator of Eq.(9) being properly regularized in its denominator at \( \tau_1 = \tau_{+1} \) according to Eq.(A4). Functions \( \kappa_{\pm}(\tau_1 - \tau_{+1}) \) and \( \tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) \) -correspond to a different "cos" terms generated by \( f(\tau) \) dependence of quantum potential (B1). These factors take care about two possible arrangements of a pair of time arguments \( \tau_1 \) and \( \tau_{+1} \) on a complex Keldysh contour: \( C_K \in (0 \pm i0; t) \). Here \( \kappa_{\pm}(\tau_1 - \tau_{+1}) \) -correspond to the case where \( \tau_1 \) and \( \tau_{+1} \) are on the same (upper or lower) branch of the Keldysh contour, while \( \tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) \) -to the case when those time arguments are placed on different branches of the contour.

Analogously to the case of Ref.[38] the general expansion of Keldysh partition function of Eq.(4) will be straightforward

\[
\tilde{\chi}(t, \xi) = 1 + \sum_{n=2}^{\text{even}} \sum_{j,k=0}^n \frac{(-i\lambda)^n}{n!} C^n_{j,k} e^{i\phi_j \tau_j} e^{-i\phi_k \tau_k} \prod_{j=0}^{n/2-k} \int_0^t d\tau_j \int_0^{\tau_j} d\tau_{j+1} \ldots \int_0^{\tau_{n/2-k}} d\tau_{n/2-k+1} \int_0^{\tau_{n/2-k+1}} d\tau_{n/2-k+2} \ldots \int_0^{\tau_{n-k}} d\tau_{n-k} \int_0^{\tau_{n-k}} d\tau_{n-k+1} \ldots \int_0^{\tau_{n-1}} d\tau_{n-1} \int_0^{\tau_{n-1}} d\tau_n \times \langle A_{+\xi}(\tau_k) \ldots A_{+\xi}(\tau_l) A_{-\xi}(\tau_{n-k}) \ldots A_{-\xi}(\tau_{n-l}) \rangle S.
\]

(B16)

Here \( C^n_{j,k} = \frac{(-i\lambda)^n}{n!} \) are standard binomial coefficients and \( \langle A_{+\xi}(\tau_m) \ldots A_{-\xi}(\tau_n) \rangle S \) one should understand as the "symmetric" part of the correlator \( \langle A_{+\xi}(\tau_m) \ldots A_{-\xi}(\tau_n) \rangle \) with respect to exchange \( \tau_m \leftrightarrow \tau_n \) (or \( \tau_m \leftrightarrow \tau_n \), or \( \tau_m \leftrightarrow \tau_n \)) for any pair of time variables \( \tau_m \) and \( \tau_n \). In Eq.(B16) a following modification of quantum potential \( A_0(\tau) \) from Ref.[38] is performed

\[
A_{\pm\xi}(\tau) = \cos \ [\varphi(\tau) + f_{\pm\xi}(\tau)].
\]

(B17)

Importantly, to apply the FCS-modification of S-Theorem from Ref.[38] to present case, where functions \( f_{\pm\xi}(\tau) \) are defined by Eq.(B2) one needs to modify these functions somehow to make them fulfill the constraint (B3). Such a modification is straightforward: first, applying the same "plugging procedure" as in the proof of S-lemma in Ref.[38] for all "non-time-ordered" averages in the r.h.s. of Eq.(B16) one can reduce all \( n \) different time integrations to one sequence of \( n \) time-ordered time integrations in each \( n \)-th order in \( \lambda_{1/2} \). One might convince from the proof of S-Lemma in the Appendix B of Ref.[38] that such rearrangement can be made before the proof of "crossing" diagrams cancellation without any loss of generality of S-lemma. Second, since all time variables of integration \( \tau_1, \tau_{+1} \) (\( l \leq 1, n \)) are ordered now, the "signature" function \( \text{sgn}[\tau_1 - \tau_{+1}] \) will be always equal to 1 if \( \tau_1 > \tau_{+1} \) (\( l \leq l' \)). Therefore, nothing changes in time-integrations of Eq.(B16) if one replace the difference: \( eV(\tau_1 - \tau_{+1}) + 2\epsilon \) in Eq.(B17) by \( eV(\tau_1 - \tau_{+1}) + 2\epsilon \text{sgn}[\tau_1 - \tau_{+1}] \) in the arguments of the "cos" function (B17) in the r.h.s. of Eq.(B16). With respect to this replacement one can rewrite Eqs.(B13,B15) as follows:

\[
\kappa_{\pm}(\tau_1 - \tau_{+1}) = \cos [eV(\tau_1 - \tau_{+1})], \quad (B18)
\]

and

\[
\tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) = \cos [eV(\tau_1 - \tau_{+1}) + 2\epsilon \text{sgn}(\tau_1 - \tau_{+1})]. \quad (B19)
\]

Obviously, functions (B18,B19) fulfill the condition (B3) of the validity of FCS-versions of S-Theorem and S-lemma. After such redefinition of quantum potentials in expansion (B16), one can straightforwardly apply the proof of more simple S-Theorem and S-lemma of Ref.[38] to more general FCS (full counting statistics) case of interest.

Now, evidently, for each sequence of \( k \) and \( n-k \) connected integrations in each term of \( k \)-th (and \( n-k \)-th) order in the infinite sum of r.h.s. of Eq.(B16) one can repeat the same logic steps as in the proof of S-Lemma from the Appendix B of Ref.[38] just replacing everywhere \( f(\tau) \) by \( f_{\pm\xi}(\tau) \) in the sequence of \( k \)- and by \( f_{\mp\xi}(\tau) \) in the sequence of \( n-k \) integrations, correspondingly. As the result, one will obtain for any even natural \( l \) a following statement

\[
\langle A_{\pm\xi}(\tau_1) A_{\pm\xi}(\tau_{+1}) \rangle S = u(\tau_1 - \tau_{+1}) \kappa_{\pm}(\tau_1 - \tau_{+1}) \quad (B21)
\]

where

\[
\kappa_{\pm}(\tau_1 - \tau_{+1}) = \cos [f_{\pm\xi}(\tau_1) - f_{\pm\xi}(\tau_{+1})]. \quad (B22)
\]

Also, in the framework of Eqs.(B16,B20) one should consider a "mixed" average

\[
\langle A_{\pm\xi}(\tau_1) A_{\mp\xi}(\tau_{+1}) \rangle S = u(\tau_1 - \tau_{+1}) \tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) \quad (B23)
\]

where

\[
\tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) = \cos [f_{\pm\xi}(\tau_1) - f_{\mp\xi}(\tau_{+1})]. \quad (B24)
\]

As it was already mentioned in the above, in Eq.(B22,B24) functions \( \kappa_{\pm}(\tau_1 - \tau_{+1}) \) and \( \tilde{\kappa}_{\pm}(\tau_1 - \tau_{+1}) \) take care about two possible types of arrangement for a pair of time arguments \( \tau_1 \) and \( \tau_{+1} \) on a complex Keldysh
contour $C_K \in (0 \pm i0; t)$ ($\kappa_{\pm}(\tau_l - \tau_{l+1})$ correspond to the case where $\tau_l$ and $\tau_{l+1}$ are on the same (upper or lower) branch of the Keldysh contour and $\bar{\kappa}_{\pm}(\tau_l - \tau_{l+1})$ to the case when those time arguments are placed on different branches of the contour). Thus the equality (B20) represents a generalized Wick theorem for quantum potential with branch-dependent counting field $\xi(\tau)$. Eq. (B20) also means that all “crossing” diagrams from the expansion (B16) do not affect the result of corresponding time integration and the “linked cluster approximation” (in its modified form of Eq. (B20)) remains exact also for quantum potential $A_{\pm}(\tau)$ with contour branch-dependent counting field $\xi(\tau)$ in the framework of full counting statistics.

Now, in order to bring the statement (B20) to the form more similar to one of conventional S-lemma of Ref.[38], first let us note that, if

$$|f_i(\tau_l) - f_j(\tau_{l+1})| = |f_i(\tau_{l+1}) - f_j(\tau_l)|$$  \hspace{1cm} (B25)

with $i, j = \pm \xi$, is fulfilled for the functions $f_{\pm \xi}(\tau)$ then from Eqs.(B22, B24) it follows that

$$\kappa_{\pm}(\tau_l, \tau_{l+1}) = \kappa_{\pm}(\tau_{l+1}, \tau_l)$$  \hspace{1cm} (B26)

$$\bar{\kappa}_{\pm}(\tau_l, \tau_{l+1}) = \bar{\kappa}_{\pm}(\tau_{l+1}, \tau_l)$$

hence, from Eqs.(B21)

$$\langle A_{\pm \xi}(\tau_l) A_{\pm \xi}(\tau_{l+1}) \rangle_S = \langle A_{\pm \xi}(\tau_{l+1}) A_{\pm \xi}(\tau_l) \rangle_S$$  \hspace{1cm} (B27)

and as well from (B23)

$$\langle A_{\pm \xi}(\tau_l) A_{\mp \xi}(\tau_{l+1}) \rangle_S = \langle A_{\pm \xi}(\tau_{l+1}) A_{\mp \xi}(\tau_l) \rangle_S.$$  \hspace{1cm} (B28)

Obviously, in the latter case, $\langle A_{\pm \xi}(\tau_l) A_{\pm \xi}(\tau_{l+1}) \rangle_S$ and $\langle A_{\pm \xi}(\tau_l) A_{\mp \xi}(\tau_{l+1}) \rangle_S$ play the role of “symmetrized” pair-correlators $\langle A_0(\tau_{l+1}) A_0(\tau_l) \rangle_S$ from the proof of S-lemma in the Appendix B of Ref.[38]. Thus, having in hands Eqs.(B27, B28) one can apply to the integrals in the r.h.s. of Eq.(B16) the same “plugging” procedure as takes place in the proof of S-lemma in Ref.[38] but in the present (FCS-) case, such the procedure will give a slightly different result because of the presence of new ingredients: $\kappa_{\pm}(\tau)$ and $\bar{\kappa}_{\pm}(\tau)$ which mark four different arrangements of two time arguments $\tau_l$ and $\tau_{l+1}$ on two (time- and anti-time-) branches of complex Keldysh contour. In particular, applying to time integrals in the r.h.s. of Eq.(B16) such the procedure and making use of formulas (B20, B27, B28) for $2m$-fold integral from the symmetrized function, one can obtain

(below everywhere $n = 2m$ is an even natural number)

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{k-2}} d\tau_{k-1} \int_0^{\tau_{k-1}} d\tau_k \times$$

$$\int_0^t d\tau_1' \int_0^{\tau_1'} d\tau_2' \cdots \int_0^{\tau_{(n-k)-2}} d\tau_{n-k-1} \int_0^{\tau_{(n-k)-1}} d\tau_{n-k} \times$$

$$\times \langle A_{+\xi}(\tau_k) \cdots A_{+\xi}(\tau_1) A_{-\xi}(\tau_{(n-k)}) \cdots A_{-\xi}(\tau_1') \rangle_S$$

$$= \frac{1}{(n/2)!} \sum_{j=0}^k C^n_{j} \sum_{j'=0}^{(n/2-k)} C^{(n/2-k)}_{j'} \times$$

$$\left\{ \frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \kappa_{+}(\tau_1 - \tau_2) \right\}^{j}\times$$

$$\left\{ \frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \kappa_{-}(\tau_1 - \tau_2) \right\}^{j'}\times$$

$$\left\{ \frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \kappa_{-}(\tau_1 - \tau_2) \right\}^{(n/2-k-j')} \times$$

$$\left\{ \frac{1}{2} \int_0^t d\tau_1 \int_0^t d\tau_2 (\tau_1 - \tau_2) \kappa_{+}(\tau_1 - \tau_2) \right\}^{(n/2-k-j')}. \hspace{1cm} (B29)$$

Here the ”minus” sign in the first (and third) brackets in the r.h.s. of the latter equation is due to inversion of the limits in one among two time integrations when one ”plugs” the ”empty place” in the time-(anti-time) ordered sequence of integrations by one integral from anti-time- (and time-) ordered sequence of integrals originated from one ”mixed” term of Eq.(B28). In Eq. (B29) the binomial coefficients $C^k_j = \frac{k!}{j!(k-j)!}$ and $C^{(n/2-k)}_{j'} = \frac{(n/2-k)!}{j'!(n/2-k-j')!}$ count the number of ways one can compose $j$ and $j'$ ”mixed” pair correlators of the form (B28) from the product of $k$ and $(n/2-k)$ available terms, correspondingly. Now using twice a binomial formula: $(x+y)^n = \sum_{k=0}^{m} \binom{m}{k} x^k y^{n-k}$ one can obtain from Eq.(B29) following expression

$$\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{k-2}} d\tau_{k-1} \int_0^{\tau_{k-1}} d\tau_k \times$$

$$\int_0^t d\tau_1' \int_0^{\tau_1'} d\tau_2' \cdots \int_0^{\tau_{(n-k)-2}} d\tau_{n-k-1} \int_0^{\tau_{(n-k)-1}} d\tau_{n-k} \times$$

$$\times \langle A_{+\xi}(\tau_k) \cdots A_{+\xi}(\tau_1) A_{-\xi}(\tau_{(n-k)}) \cdots A_{-\xi}(\tau_1') \rangle_S$$

$$= \frac{(-1)^{n/2}}{(n/2)!} \left\{ \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \Delta \kappa_{+}(\tau_1 - \tau_2) \right\}^k \times$$

$$\times \left\{ \int_0^t d\tau_1 \int_0^t d\tau_2 u(\tau_1 - \tau_2) \Delta \kappa_{-}(\tau_1 - \tau_2) \right\}^{(n/2-k)} \hspace{1cm} (B30)$$

where

$$\Delta \kappa_{\pm}(\tau_1 - \tau_2) = [\bar{\kappa}_{\pm}(\tau_1 - \tau_2) - \kappa_{\pm}(\tau_1 - \tau_2)] \hspace{1cm} (B31)$$

Evidently, the equations (B29, B30) have been derived above state following
\( FCS-S\text{-Lemma:} \) "The case of the exact factorization for Wick’s theorem for full-counting statistics quantum potential."

- For any exponential bosonic operator of the form:
  \[ A_{\xi(\tau)}(\tau) = \cos[\varphi_-(\tau) + f_{\xi}(\tau)] \]
  where
  \[ f_{\xi}(\tau) = \begin{cases} f_{+\xi}(\tau), \text{Im}(\tau) > 0 \\ f_{-\xi}(\tau), \text{Im}(\tau) < 0 \end{cases} \]
  the property
  \[ |f_i(\tau_1) - f_j(\tau_2)| = |f_i(\tau_2) - f_j(\tau_1)| \quad (B32) \]
  with \( i, j = \pm \xi \) is fulfilled for \( f_{\pm \xi}(\tau) \) being certain functions of time and defined as \( f_{+\xi}(\tau) \) (\( f_{-\xi}(\tau) \)) on the upper (lower) branch of Keldysh contour in the complex plane and \( \varphi_-(\tau) \) is time-dependent bosonic field with zero mean \( \langle \varphi_-(\tau) \rangle = 0 \) , which fulfills commutation relation of the form:
  \[ [\varphi_-(\tau), \varphi_-(\tau')] = -2i\partial_{\tau}\text{sgn}[\tau - \tau'] \]

\( \nabla \) it follows for time-dependent and Keldysh contour-ordered average from a quantum potential with Keldysh contour-dependent counting field \( \xi(\tau) \)

\[
\begin{align*}
\int_0^t \int_0^{\tau_1} dt_1 \int_0^{\tau_2} dt_2 \ldots \int_0^{\tau_{k-1}} dt_{k-1} \int_0^{\tau_k} dt_k \\
\int_0^t \int_0^{\tau'_1} dt'_1 \int_0^{\tau'_2} dt'_2 \ldots \int_0^{\tau'_{n-k-1}} dt'_{n-k-1} \int_0^{\tau'_{n-k}} dt'_{n-k} \\
\times \langle A_{+\xi}(\tau_k) \ldots A_{+\xi}(\tau_1) A_{-\xi}(\tau_{n-k-1}) \ldots A_{-\xi}(\tau_1) \rangle \\
= (-1/2)^{n/2} \left\{ \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) \Delta \kappa_+(\tau_1 - \tau_2) \right\}^k \\
\times \left\{ \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) \Delta \kappa_- (\tau_1 - \tau_2) \right\}^{(n/2-k)}
\end{align*}
\]

where
\[
\Delta \kappa_{\pm}(\tau_1 - \tau_2) = [\kappa_{+}(\tau_1 - \tau_2) - \kappa_{-}(\tau_1 - \tau_2)]
\]
\[
\kappa_{\pm}(\tau_1 - \tau_2) = \cos[f_{+\xi}(\tau_1) - f_{-\xi}(\tau_2)]
\]

with function \( u(\tau_1 - \tau_2) \) being a pair correlator:

\[ u_g(\tau_1 - \tau_2) = e^{i\langle (\varphi_-(\tau_1) - \varphi_-(\tau_2))^2 \rangle} \]

properly regularized at the point \( \tau_1 = \tau_2 \).

Therefore, by means of Eqs.(B30,B31) the FCS-S-Lemma is proven. \( \blacksquare \)

Now using the FCS-S Lemma and substituting the r.h.s. of Eq.(B30) into the r.h.s. of Eq.(B16) one immediately obtains

\[
\hat{\chi}(t, \xi) = 1 + \sum_{n=2}^{\infty} (-i\lambda)^n \sum_{k, j=0}^{n-2} C^n_{k, j} e^{i\theta_k e^{-i\theta_j (n/2-k)}} \\
\times (-1/2)^{n/2} \left\{ \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) \Delta \kappa_+(\tau_1 - \tau_2) \right\}^k \\
\times \left\{ \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) \Delta \kappa_- (\tau_1 - \tau_2) \right\}^{(n/2-k)}
\]

(B33)

Expansion (B33) automatically takes care about all possible combinations of pair correlators (B27) and (B28) which can appear in formula (B16) and it remains valid in all orders of \( n \) giving rise to nonperturbative calculation of \( \hat{\chi}(t) \). Indeed, noting that in Eq.(B33) \( n = 2m \), \( (m = 1, 2, 3, \ldots) \) and using a standard binomial formula: \( (x + y)^m = \sum_{k=0}^{m} C_r^m x^k y^{m-k} \) one obtains from Eq.(B33) following power series for time-dependent (i.e. nonequilibrium) Keldysh partition function (KPF)

\[
\hat{\chi}(t, \xi) = 1 + \sum_{m=1}^{\infty} \left( \frac{-\lambda^2}{m!} \right)^m \left\{ \frac{1}{2} \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) F_C((\tau_1 - \tau_2), \xi) \right\}^m
\]

(B34)

where

\[
F_C((\tau_1 - \tau_2), \xi) = \{ e^{i\theta_g \Delta \kappa_+(\tau_1 - \tau_2)} + e^{-i\theta_g \Delta \kappa_- (\tau_1 - \tau_2)} \}
\]

(B35)

is a "kernel" function being responsible for the four different layouts of two time arguments \( \tau_1 \) and \( \tau_2 \) on the complex Keldysh contour: \( C_k \in (0 \pm i0; t) \). Obviously, infinite series (B33,B34) re-exponentiates exactly into the compact form

\[
\hat{\chi}(t, \xi) = \exp(-F(\xi, t))
\]

(B36)

where for the generating functional \( F(\xi, t) \) one has following exact formula

\[
F(\xi, t) = \left( \frac{\lambda}{2} \right)^2 \times \left\{ \frac{1}{2} \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) F_C((\tau_1 - \tau_2), \xi) \right\} \\
= \left( \frac{\lambda}{2} \right)^2 \times \left\{ \int_0^t \int_0^t dt_1 \int_0^t dt_2 u(\tau_1 - \tau_2) \times \{ e^{i\theta_g \Delta \kappa_+(\tau_1 - \tau_2)} + e^{-i\theta_g \Delta \kappa_- (\tau_1 - \tau_2)} \} \right\}
\]

(B37)

Remarkably, Eq.(B37) can be also rewritten in the form of a double integral from our "initial" (non-symmetrized) average \( \langle A_{\xi(\tau)}(\tau_1) A_{\xi(\tau)}(\tau_2) \rangle \) over complex Keldysh con-
tour $C_K \in (0 \pm i0; t)$

$$J_F(\xi, t) = \left( \hat{\lambda} \right)^2 \times$$

$$\left\{ \frac{1}{2} \int_{C_K} d\tau_1 d\tau_2 \langle T_K A_{\xi(\tau)}(\tau_1) A_{\xi(\tau)}(\tau_2) \rangle \right\}. \tag{B38}$$

Evidently, equations (B36-B38) complete the proof of the FCS-S-Theorem. Therefore, the FCS-S-Theorem is proven. ■

Appendix C: Exact calculation of a basic time integral in the long-time limit

To calculate real-time integrals $J_C$ and $J_S$ from Eqs.(17,18) in their "long-time" limit: $t \rightarrow \infty$ let us perform it as following sum

$$J_{C(S)} = \frac{(J_d^+ + J_d^-)}{4} \tag{C1}$$

where

$$J_d^\pm = \left[ \frac{\pi T}{\Lambda_g} \right]^{2/g} \lim_{\delta \rightarrow 0} \int_{-\infty}^{+\infty} ds \left\{ \frac{e^{\pm i(\epsilon V_k)\tau} e^{-i(\pi/g)\text{sgn}(s)}}{\sinh^{2/g}\left[ \pi T(s - i\delta_g\text{sgn}(s)) \right]} \right\}. \tag{C2}$$

In the latter formula one can replace infinitesimal $\delta_g \rightarrow 0$ introducing instead the principal value of corresponding integral "without" the infinitesimal vicinity of the point $s = 0$. The result is

$$J_d^\pm = \left[ \frac{2\pi T}{\Lambda_g} \right]^{2/g} \times \mathcal{P} \int_{-\infty}^{+\infty} ds \left( e^{\pi T s + i(\pi/2)\text{sgn}(s)} + e^{-\pi T s - i(\pi/2)\text{sgn}(s)} \right)^{2/g} \tag{C3}$$

Now changing variable of integration to $z = e^{\pi Ts + i\pi/2}$ and $ds = z^{-1} dz/\pi T$ one obtains

$$J_d^\pm = \frac{e^{\pm iV/2T}}{\pi T} \left[ \frac{2\pi T}{\Lambda_g} \right]^{2/g} \mathcal{P} \int_{+i0}^{+i\infty} \frac{z^{(2/g - 1)\pm iV/\pi T}}{[z^2 + 1]^{2/g}} dz. \tag{C4}$$

And, finally, changing $z$ to $\tilde{z} = z^2$ we arrive at the expression

$$J_d^\pm = J_d^\pm \frac{e^{\pm iV/2T}}{\Lambda_g} \left[ \frac{2\pi T}{\Lambda_g} \right]^{(2/g - 1)} \tag{C5}$$

where we define following complex integral

$$J_C^\pm = -\mathcal{P} \int_{-\infty}^{0} u_\pm(\tilde{z}) d\tilde{z} = -\mathcal{P} \int_{-\infty}^{0} \frac{\tilde{z}^{[1/g-1\pm iV/2\pi T]} \left( [\tilde{z}^2 + 1]^{2/g} \right)}{d\tilde{z}}. \tag{C6}$$

Notice, that symbol $\mathcal{P}$ in r.h.s. of Eq.(C6) means that we have excluded the point $\tilde{z} = -1$ with its infinitesimal vicinity from the integration along the real axis.

Let us consider the integral in the r.h.s. of Eq.(C6) on the complex plane. Function $u_\pm(\tilde{z})$ under the integral is analytical everywhere on the upper (lower) half of a complex plane, excepting the points: $\tilde{z} = \tilde{z}_k = \exp(i(2k+1))$, where $k = 0, 1, 2, 3, \ldots$ ($k = 0, -1, -2, -3, \ldots$ and, thus, function $u_+(\tilde{z}_k)$ (or $u_-(\tilde{z}_k)$) - diverges. Further, let us introduce for the integrals $J_C^\pm$ two closed complex contours $C_\pm$ consisting of five parts each. Let first part of both contours $C_\pm$ goes from $\tilde{z} = -\infty$ to $\tilde{z} = -1$ along the real axis. Then let the second part of the contour $C_+$ (or $C_-$) goes along the line $Re\tilde{z} = -1$ from $Im\tilde{z} = 0$ to $Im\tilde{z} = +\infty$ and back to $Im\tilde{z} = 0$ (from $Im\tilde{z} = 0$ to $Im\tilde{z} = -\infty$ and back to $Im\tilde{z} = 0$ for $C_-$) being "bent" around all points $\tilde{z} = \tilde{z}_k$ which are on this line. Then, let the third part of both contours goes from $\tilde{z} = -1$ to $\tilde{z} = 0$ and the fourth part - from $\tilde{z} = 0$ to $\tilde{z} = +\infty$ along the real axis. And, finally, let us "close" our contour $C_+$ (or $C_-$) by its fifth part i.e. by the semi-circle of infinite radius ($R_\pm \rightarrow \infty$) in the upper (lower) half of a complex plane.

Both introduced closed contours $C_\pm$ contain no special points for functions $u_\pm(\tilde{z})$ correspondingly. It means that, by Residue theorem, the integral from the function $u_\pm(\tilde{z})$ (or $u_\mp(\tilde{z})$) along closed complex contour $C_+$ (or $C_-$, correspondingly) - is equal to zero. Thus, one can write

$$\int_{C_\pm} u_\pm(\tilde{z}) d\tilde{z} = \int_{I_\pm} u_\pm(\tilde{z}) d\tilde{z} + \int_{II_\pm} u_\pm(\tilde{z}) d\tilde{z} + \int_{III_\pm} u_\pm(\tilde{z}) d\tilde{z} + \int_{IV_\pm} u_\pm(\tilde{z}) d\tilde{z} + \int_{V_\pm} u_\pm(\tilde{z}) d\tilde{z} = 0. \tag{C7}$$

Now, from the form of second and fifth parts of our contours as well as from the analytic properties of complex functions $u_\pm(\tilde{z})$ one can conclude that

$$\int_{I_{\pm}} u_\pm(\tilde{z}) d\tilde{z} = \int_{V_{\pm}} u_\pm(\tilde{z}) d\tilde{z} = 0. \tag{C8}$$

Evidently, from Eqs.(C7,C8) it follows

$$\int_{I_\pm} u_\pm(\tilde{z}) d\tilde{z} + \int_{III_\pm} u_\pm(\tilde{z}) d\tilde{z} = - \int_{IV_\pm} u_\pm(\tilde{z}) d\tilde{z} \tag{C9}$$

But, on the other hand, by the definition of chosen contours $C_\pm$ we have

$$\int_{I_{\pm}} u_\pm(\tilde{z}) d\tilde{z} + \int_{III_\pm} u_\pm(\tilde{z}) d\tilde{z} = \mathcal{P} \int_{-\infty}^{0} u_\pm(\tilde{z}) d\tilde{z} = -J_{C}^\pm. \tag{C10}$$
Equations (C9,C10) allow us to re-define our ”initial” complex integral (C6) as

$$J^\pm_C = \int_{IV(\pm)} u_\pm(\tilde{z}) d\tilde{z} = \int_0^{+\infty} u_\pm(\tilde{z}) d\tilde{z}$$  \hspace{1cm} \text{(C11)}$$

It means that

$$J^\pm_C = \int_0^{+\infty} \frac{\tilde{z}^{1/g - 1/2} eV/2\pi T}{[\tilde{z} + 1]^{2/g}} d\tilde{z}.$$  \hspace{1cm} \text{(C12)}$$

The latter integral, in turn, is well-tabulated and is nothing more than the integral representation of beta-function $B(p; q)$ of two complex arguments $p, q$

$$B(p; q) = \int_0^{+\infty} \frac{\tilde{z}^{p-1}}{[\tilde{z} + 1]^{p+q}} d\tilde{z}.$$ \hspace{1cm} \text{(C13)}$$

Comparing Eqs.(C12,C13) one can conclude that in our case $p = q^* = 1/g + i(eV/2\pi T)$ and, hence,

$$J^+_C = J^-_C = B \left( \frac{1}{g} + i \frac{eV}{2\pi T}; \frac{1}{g} - i \frac{eV}{2\pi T} \right).$$  \hspace{1cm} \text{(C14)}$$

Finally, using well-known identity

$$B(p; q) = B(q; p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \hspace{1cm} \text{(C15)}$$

with Euler’s gamma-function $\Gamma(z)$ of complex argument, one can obtain following explicit formula for the integral $J^\pm_C$, which is just a real number

$$J^+_C = J^-_C = \frac{|\Gamma(1/g + i [eV/2\pi T])|^2}{\Gamma(2/g)}.$$ \hspace{1cm} \text{(C16)}$$

Substituting formula (C16) into Eq.(C5) we will have

$$J^\pm_d = \frac{e^{\pm eV/2T}}{A_g} \left[ \frac{2\pi T}{A_g} \right]^{(2/g - 1)} \frac{|\Gamma(1/g + i [eV/2\pi T])|^2}{\Gamma(2/g)}.$$ \hspace{1cm} \text{(C17)}$$

In general, formulas being similar to Eq.(C17) are known in the literature on Luttinger liquid effects (see Ref.[43]), since they appear for the time integrals from Luttinger liquid real time correlators, e.g. when one calculates transition rates from Luttinger liquid lead to quantum dot using the Fermi golden rule.

At last, substituting formulas (C17) into Eq.(C1) we obtain the desired expressions for the ”long-time” asymptotics ($t \rightarrow \infty$) of our two basic real time integrals from Eqs.(11,12)

$$J_C = \frac{\cosh(eV/2T)}{2A_g} \left[ \frac{2\pi T}{A_g} \right]^{(2/g - 1)} \frac{|\Gamma(1/g + i [eV/2\pi T])|^2}{\Gamma(2/g)}.$$ \hspace{1cm} \text{(C18)}$$

and

$$J_S = \frac{\sinh(eV/2T)}{2A_g} \left[ \frac{2\pi T}{A_g} \right]^{(2/g - 1)} \frac{|\Gamma(1/g + i [eV/2\pi T])|^2}{\Gamma(2/g)}.$$ \hspace{1cm} \text{(C19)}$$

Obviously, formulas (C17-C19) are exact, they give the expressions (11,12) from the main text and represent main ”building blocks”, carrying voltage- and temperature-dependent ”steady-flow” (i.e. equilibrium) Luttinger liquid effects in our model.