Computing the local metric dimension of a graph from the local metric dimension of primary subgraphs

Juan A. Rodríguez-Velázquez, Carlos García Gómez and Gabriel A. Barragán-Ramírez
Departament d’Enginyeria Informàtica i Matemàtiques,
Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain.
juanalberto.rodriguez@urv.cat, gbrbcn@gmail.com

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Abstract
For an ordered subset \( W = \{w_1, w_2, \ldots, w_k\} \) of vertices and a vertex \( u \) in a connected graph \( G \), the representation of \( u \) with respect to \( W \) is the ordered \( k \)-tuple \( r(u|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)) \), where \( d(x, y) \) represents the distance between the vertices \( x \) and \( y \). The set \( W \) is a local metric generator for \( G \) if every two adjacent vertices of \( G \) have distinct representations. A minimum local metric generator is called a local metric basis for \( G \) and its cardinality the local metric dimension of \( G \). We show that the computation of the local metric dimension of a graph with cut vertices is reduced to the computation of the local metric dimension of the so-called primary subgraphs. The main results are applied to specific constructions including bouquets of graphs, rooted product graphs, corona product graphs, block graphs and chain of graphs.

1 Introduction

A generator of a metric space is a set \( S \) of points in the space with the property that every point of the space is uniquely determined by its distances from the elements of \( S \). Given a simple and connected graph \( G = (V, E) \), we consider the metric \( d_G : V \times V \to \mathbb{N} \), where \( d_G(x, y) \) is the length of a shortest path between \( x \) and \( y \). \((V, d_G)\) is clearly a metric space. A metric generator of a connected graph \( G \) is a subset of vertices, \( W \subset V(G) \), for which, given any pair of vertices \( u, v \in V(G) \) there is at least one element \( w \in W \) for which we have

\[
d_G(u, w) \neq d_G(v, w).
\]

We say then, that \( w \) is able to distinguish the pair of vertices \( u, v \). A metric generator with minimum cardinality is defined as a metric basis for \( G \). The cardinality of this set is denoted by \( \dim(G) \) and is referred as the metric dimension of \( G \).
We can see a metric basis $S$ of $G$ as an ordered set $S = \{s_1, s_2, \ldots, s_d\}$. In this sense, we refer to the vector
\[
r(u|S) = (d_G(u, s_1), d_G(u, s_2), \ldots, d_G(u, s_d))
\]
as the coordinate vector of $u$ with respect to the basis $S$. Note that since $S$ is a metric basis for $G$, for any pair of vertices $u$ and $v$ of $G$, it holds that $r(u|S) \neq r(v|S)$. Hence, each vertex is uniquely determined by its coordinate vector with respect to a basis.

These concepts were first introduced by Slater in [30], where the metric generators were called locating sets. The concept of metric dimension of a connected graph was introduced independently by Harary and Metler in [14], where metric generators received the name of resolving sets. After these papers were published several authors developed diverse theoretical works about this topic, for instance, we cite [1, 16, 2, 3, 4, 5, 6, 7, 9, 10, 11, 14, 15, 21, 20, 23, 27, 28, 29]. Slater described the usefulness of these ideas into long range aids to navigation [30]. Also, these concepts have some applications in chemistry for representing chemical compounds [19, 18] or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [22]. Other applications of this concept to navigation of robots in networks and other areas appear in [3, 17, 21].

In this paper we are interested in a local version of metric generators introduced by Okamoto et al. in [24]. Given a connected graph $G$, we define a local metric generator as a set of vertices that distinguishes any pair of adjacent vertices. This means that, given two adjacent vertices $u, v \in V(G)$ there is at least an element of this set, say $w$, for which we have $d_G(u, w) \neq d_G(v, w)$.

If a local metric generator has minimum cardinality among all local metric generators, then we call this set a local metric basis for $G$. The cardinality of the local metric basis is denoted by $\dim_l(G)$ and it is called the local metric dimension of $G$. Note that each metric generator is also a local metric generator because each metric generator distinguishes any pair of vertices, while a local metric generator only distinguishes pairs of neighbours. Then the following relation between the local metric dimension and the metric dimension of a graph is valid
\[
1 \leq \dim_l(G) \leq \dim(G) \leq n - 1.
\]

In this paper we show that the computation of the local metric dimension of a graph with cut vertices is reduced to the computation of the local metric dimension of the so-called primary subgraphs. The main results are applied to specific constructions including bouquets of graphs, rooted product graphs, corona product graphs, block graphs and chain of graphs.

The following basic results, established in [24], will be used in this paper.

**Theorem 1.** [24] Let $G$ be a nontrivial connected graph of order $n$. Then $\dim_l(G) = n - 1$ if and only if $G = K_n$ and $\dim_l(G) = 1$ if and only if $G$ is bipartite.

**Theorem 2.** [24]. A connected graph $G$ of order $n \geq 3$ has local metric dimension $\dim_l(G) = n - 2$ if and only if the clique number of $G$ is $\omega(G) = n - 1$.

In this work the remain definitions are given the first time that the concept is found in the text.
2 Main results

Let $G[\mathcal{H}]$ be a connected graph constructed from a family of pairwise disjoint (non-trivial) connected graphs $\mathcal{H} = \{G_1, ..., G_k\}$ as follows. Select a vertex of $G_1$, a vertex of $G_2$, and identify these two vertices. Then continue in this manner inductively. More precisely, suppose that we have already used $G_1, ..., G_i$ in the construction, where $2 \leq i \leq k-1$. Then select a vertex in the already constructed graph (which may in particular be one of the already selected vertices) and a vertex of $G_{i+1}$; we identify these two vertices. Note that any graph $G[\mathcal{H}]$ constructed in this way has a tree-like structure, the $G'_i$s being its building stones (see Figure 1).

We will briefly say that $G[\mathcal{H}]$ is obtained by point-attaching from $G_1, ..., G_k$ and that $G'_i$s are the primary subgraphs of $G[\mathcal{H}]$. We will also say that the vertices of $G[\mathcal{H}]$ obtained by identifying two vertices of different primary subgraphs are the attachment vertices of $G[\mathcal{H}]$. The above terminology was previously introduced in [8] where the authors obtained an expression that reduces the computation of the Hosoya polynomials of a graph with cut vertices to the Hosoya polynomial of the so-called primary subgraphs.

To begin with the study of the local metric dimension of $G[\mathcal{H}]$ we need some additional terminology. Given an attachment vertex $x$ of $G[\mathcal{H}]$ and a primary subgraph $G_j$ such that $x \in V(G_j)$, we define the subgraph $G_j(x^+)$ of $G[\mathcal{H}]$ as follows. We remove from $G[\mathcal{H}]$ all the edges connecting $x$ with vertices in $G_j$, then $G_j(x^+)$ is the connected component which has $x$ as a vertex. For instance, Figure 2 shows the subgraph $G_1(x^+)$ of the graph $G[\mathcal{H}]$ shown in Figure 1.

Let $J_H \subseteq [k]$ be the set of subscripts such that $j \in J_H$ whenever $G_j$ is a non-bipartite primary subgraph of $G[\mathcal{H}]$. Note that $J_H = \emptyset$ if and only if $G[\mathcal{H}]$ is bipartite, i.e., $J_H = \emptyset$ if and only if $\dim_l(G[\mathcal{H}]) = 1$. From now on we assume that $J_H \neq \emptyset$.

![Figure 1: A graph $G[\mathcal{H}]$ obtained by point-attaching from $\mathcal{H} = \{G_1, G_2, ..., G_7\}$](image-url)
Now, let \( C_j \) be the set composed by attachment vertices of \( G[H] \) belonging to \( V(G_j) \) such that \( x \in C_j \) whenever \( G_j(x^+) \) is not bipartite. For instance, if \( G_2, G_3 \) and \( G_7 \) are the non-bipartite primary subgraphs of the graph shown in Figure 1 then \( C_2 = \{x, w\} \).

For any \( j \in J_H \) we define

\[
\alpha_j = \max_{B \in \mathcal{B}(G_j)} \{|C_j \cap B|\},
\]

where \( \mathcal{B}(G_j) \) is the set of local metric bases of \( G_j \), i.e., \( \alpha_j \) is the maximum cardinality of a set \( \{x_{j_1}, x_{j_2}, ..., x_{j_{\alpha_j}}\} \subseteq V(G_j) \) composed by attachment vertices of \( G[H] \) belonging simultaneously to a local metric basis of \( G_j \) such that for every \( l \in \{1, ..., \alpha_j\} \) the subgraph \( G_j(x_{j_l}^+) \) is not bipartite.

**Theorem 3.** For any non-bipartite graph \( G[H] \) obtained by point-attaching from a family of connected graphs \( \mathcal{H} = \{G_1, ..., G_k\} \),

\[
\dim_l(G[H]) \leq \sum_{j \in J_H} (\dim_l(G_j) - \alpha_j).
\]

**Proof.** For any \( j \in J_H \) we take \( B_j \in \mathcal{B}(G_j) \) and \( M_j \subseteq B_j \cap C_j \) such that \(|M_j| = \alpha_j\). We claim that \( B = \bigcup_{j \in J_H} (B_j - M_j) \) is a local metric generator for \( G[H] \).

First of all, note that by the structure of \( G[H] \) we have that for any \( v \in M_j \) there exists a non-bipartite primary subgraph \( G_r \), which is a subgraph of \( G_j(v^+) \), such that \( B_r - M_r \neq \emptyset \). To see this we take a non-bipartite primary subgraph \( G_{j_1} \), which is a subgraph of \( G_j(v^+) \), next, if \( B_{j_1} = M_{j_1} \), then we take \( v_1 \in V(G_{j_1}) \) and, as above, we take a non-bipartite primary subgraph \( G_{j_2} \), which is a subgraph of \( G_j(v_1^+) \), and if \( B_{j_2} = M_{j_2} \) then we repeat this process until obtain a non-bipartite primary subgraph \( G_{j_t} \), which is a subgraph of \( G_j(v_1^+) \) such that \(|B_{j_t}| > \left|M_{j_1}\right| \).
worst, we will arrive to a subgraph $G_j(v_{l-1}^+)$ containing only one non-bipartite primary subgraph. With this fact in mind, we differentiate the following cases for two adjacent vertices $x, y \in V(G_i)$. 

Case 1. $i \in J_H$. If the pair $x, y$ is distinguished by some $u \in B_i - M_i$, then we are done. Now, if the pair $x, y$ is distinguished by $v \in M_i$, then we take $G_r$ as a non-bipartite primary subgraph of $G_i(v^+)$ such that $B_r - M_r \neq \emptyset$. Since the pair $x, y$ is distinguished by any vertex of $G_i(v^+)$, it is also distinguished by any $u \in B_r - M_r$.

Case 2. $i \in [k] - J_H$. In this case, we take $j \in J_H$ such that $B_j - M_j \neq \emptyset$ and, since $G_j$ is bipartite, the pair $x, y$ is distinguished by any $u \in B_j - M_j$.

Hence, $B$ is a local metric generator for $G[\mathcal{H}]$ and, as a consequence,

$$\dim_l(G[\mathcal{H}]) \leq |B| = \sum_{j \in J_H} (|B_j| - |M_j|) = \sum_{j \in J_H} (\dim_l(G_j) - \alpha_j).$$

Therefore, the result follows. \hfill \Box

**Theorem 4.** Let $G[\mathcal{H}]$ be a non-bipartite graph obtained by point-attaching from a family of connected graphs $\mathcal{H} = \{G_1, \ldots, G_k\}$. If for each $j \in [k]$ it holds that any minimal local metric generator for $G_j$ is minimum, then

$$\dim_l(G[\mathcal{H}]) = \sum_{j \in J_H} (\dim_l(G_j) - \alpha_j).$$

*Proof.* Since $G[\mathcal{H}]$ is a non-bipartite graph, any vertex belonging to a local metric basis of $G[\mathcal{H}]$ distinguishes every pair of adjacent vertices included in a bipartite primary subgraph of $G[\mathcal{H}]$. Hence, we take a local metric basis $A$ of $G[\mathcal{H}]$ which does not contain vertices belonging to the bipartite primary subgraphs of $G[\mathcal{H}]$. i.e., for any $i \in [k] - J_H$ it holds $A \cap V(G_i) = \emptyset$. Now, for each $j \in J_H$ we define $A_j = A \cap V(G_j)$.

We claim that $C_j \cup A_j$ is a local metric generator for $G_j$. Suppose that there exist two adjacent vertices $x, y \in V(G_j)$ which are not distinguished by the elements of $A_j$. In such a case, there exists $x_r \in A_r, r \in J_H - \{j\}$, which distinguishes $x, y$, and so there must exists $v \in C_j$ such that $G_r$ is a subgraph of $G_j(v^+)$ and, as a result, $v$ distinguishes the pair $x, y$. Hence, $C_j \cup A_j$ is a local metric generator for $G_j$.

Moreover, if $j \in J_H$, then for any attachment vertex $w \in C_j$ it holds that $|A \cap V(G_j(w^+))| > 0$, as $G_j(w^+)$ is not bipartite. Hence, given two adjacent vertices $x, y \in V(G_j)$, which are distinguished by $w$, there exists $w' \in A_r \cap V(G_j(w^+), r \in J_H - \{j\}$, which distinguishes $x, y$, and so the minimality of $A$ leads to $C_j \cap A_j = \emptyset$.

Now, if any minimal local metric generator for $G_j$ is minimum, then there exists a set $C'_j \subseteq C_j$ such that $C'_j \cup A_j$ is a local metric basis for $G_j$. Thus, $|C'_j| + |A_j| = |C'_j \cup A_j| = \dim_l(G_j)$. Therefore,

$$\dim_l(G[\mathcal{H}]) = |A| = \sum_{j \in J_H} |A_j| = \sum_{j \in J_H} (\dim_l(G_j) - |C'_j|) \geq \sum_{j \in J_H} (\dim_l(G_j) - \alpha_j).$$

We conclude the proof by Theorem 3. \hfill \Box
For any \( j \in J_{\mathcal{H}} \) we define \( \Gamma(G_j) \) as the family of local metric generators for \( G_j \), and
\[
\rho_j = \min_{S \subseteq V(G_j)} \{|S| : S \cup C_j \in \Gamma(G_j)\}.
\]
Also, any set for which the above minimum is attained will be denoted by \( R_j \). Notice that such a set is not necessarily unique.

With the above notation in mind we can state our next result.

**Theorem 5.** For any non-bipartite graph \( G[\mathcal{H}] \) obtained by point-attaching from a family of connected graphs \( \mathcal{H} = \{G_1, \ldots, G_k\} \),
\[
\dim_l(G[\mathcal{H}]) = \sum_{j \in J_{\mathcal{H}}} \rho_j.
\]

**Proof.** We will show that \( X = \bigcup_{j \in J_{\mathcal{H}}} R_j \) is a local metric generator for \( G[\mathcal{H}] \).

First of all, note that by the structure of \( G[\mathcal{H}] \) we have that for any \( v \in C_j, j \in J_{\mathcal{H}} \), there exists a non-bipartite primary subgraph \( G_i \), which is a subgraph of \( G_j(v^+) \), such that \( R_i \neq \emptyset \). To see this we take a non-bipartite primary subgraph \( G_{j_1} \), which is a subgraph of \( G_j(v^+) \), next, if \( R_{j_1} = \emptyset \), then we take \( v_1 \in V(G_{j_1}) - \{v\} \) and, as above, we take a non-bipartite primary subgraph \( G_{j_2} \), which is a subgraph of \( G_j(v_1^+) \), and if \( R_{j_2} = \emptyset \) then we repeat this process until obtain a non-bipartite primary subgraph \( G_j \), which is a subgraph of \( G_j(v_{t-1}^+) \) such that \( R_{j_t} \neq \emptyset \) (at worst, we will arrive to a subgraph \( G_j(v_{t-1}^+) \) containing only one non-bipartite primary subgraph). Hence, \( X \neq \emptyset \) and, as a result, if \( G_i \) is bipartite, then any pair of adjacent vertices \( x, y \in V(G_i) \) is distinguished by any vertex belonging to \( X \).

Now, if \( x, y \) are adjacent in a non-bipartite primary subgraph \( G_j \), then there exists \( v \in R_j \cup C_j \) which distinguishes them. In the case that \( v \in C_j \), we know that there exists a primary subgraph of \( G_j(v^+) \), such that \( R_i \neq \emptyset \) and any \( w \in R_i \) also distinguishes \( x, y \). As a result, \( X \) is a local metric generator for \( G[\mathcal{H}] \). Therefore,
\[
\dim_l(G[\mathcal{H}]) \leq |X| = \sum_{j \in J_{\mathcal{H}}} \rho_j.
\]

It remains to show that \( \dim_l(G[\mathcal{H}]) \geq |X| = \sum_{j \in J_{\mathcal{H}}} \rho_j \). Since \( G[\mathcal{H}] \) is a non-bipartite graph, any vertex belonging to a local metric basis of \( G[\mathcal{H}] \) distinguishes every pair of adjacent vertices included in a bipartite primary subgraph of \( G[\mathcal{H}] \). Hence, we take a local metric basis \( A \) of \( G[\mathcal{H}] \) which does not contain vertices belonging to the bipartite primary subgraphs of \( G[\mathcal{H}] \) i.e., for any \( i \in [k] - J_{\mathcal{H}} \) it holds \( A \cap V(G_i) = \emptyset \). For each \( j \in J_{\mathcal{H}} \) we define \( A_j = A \cap V(G_j) \). Note that \( A_j \cup C_j \) is a local metric generator for \( G_j \) and, by the minimality of \( A \), we have \( A_j \cap C_j = \emptyset \). Hence, \( |A_j| \geq |R_j| = \rho_j \). Therefore,
\[
\dim_l(G[\mathcal{H}]) = |A| = \sum_{j \in J_{\mathcal{H}}} |A_j| \geq \sum_{j \in J_{\mathcal{H}}} \rho_j.
\]

\( \square \)
If $G_j$ is the only non-bipartite primary subgraph of $G[\mathcal{H}]$, then $|J_\mathcal{H}| = 1$ and $\rho_j = \dim_l(G_j)$. Then we obtain the following particular case of Theorem 5.

**Corollary 6.** Let $G[\mathcal{H}]$ be a graph obtained by point-attaching from the family of connected graphs $\mathcal{H} = \{G_1, ..., G_k\}$. If $G_j$ is the only non-bipartite primary subgraph of $G[\mathcal{H}]$, then

$$\dim_l(G[\mathcal{H}]) = \dim_l(G_j).$$

It is well-known that that a unicyclic graph $G$ is bipartite if and only if its cycle has even length. For the case of non-bipartite unicyclic graphs we can apply Corollary 6 to deduce that for any non-bipartite unicyclic graph $G$ it holds that $\dim_l(G) = 2$.

There are other cases in which $\rho_j$ and $\alpha_j$ are very easy to obtain. For instance, if $C_j = \{v\}$, then $\rho_j = \dim_l(G_j) - \alpha_j$, where $\alpha_j = 1$ if $v$ belongs to a local metric basis for $G_i$ and $\alpha_j = 0$ in otherwise. Also, if $C_j = V(G_j)$, then $\rho_j = 0$ and $\alpha_j = \dim_l(G_j)$.

The remain sections of this article are devoted to derive some consequences of Theorem 5. We also give several families of graphs where the equality of Theorem 5 is achieved.

### 3 Rooted product graphs

Rooted product graphs can be constructed as follows. Let $\mathcal{H}$ be a sequence of $n$ graphs $H_1, H_2, \ldots, H_n$. In each of these graphs a particular vertex $v_i$ is selected. This vertex will be called the root of the graph $H_i$. The **rooted product graph** $G \circ \mathcal{H}$ is the graph obtained by identifying the root of the graph $H_i$ with the $i$-th vertex of $G$, as defined by Godsil and Mckay [13]. Clearly, any rooted product graph is obtained by point-attaching from $G, H_1, H_2, ..., H_n$. Therefore, as a consequence of Theorem 5 we obtain a formula for the local metric dimension of any rooted product graph. To begin with, we consider the case where every $H_i$ is a bipartite graph.

**Corollary 7.** Let $G$ be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of $n$ connected bipartite graphs $H_1, H_2, \ldots, H_n$. Then for any rooted product graph $G \circ \mathcal{H}$,

$$\dim_l(G \circ \mathcal{H}) = \dim_l(G).$$

If every $H_i$ is non-bipartite, the result can be expressed as follows.

**Corollary 8.** Let $G$ be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of $n$ connected non-bipartite graphs $H_1, H_2, \ldots, H_n$. Then for any rooted product graph $G \circ \mathcal{H}$,

$$\dim_l(G \circ \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(H_j) - \alpha_j).$$

Note that in this case $\alpha_j = 1$ if the root of $H_j$ belongs to a local metric basis of $H_j$ and $\alpha_j = 0$ in otherwise.

Now we will restrict ourselves to a particular case of rooted product graphs where the sequence $H_1, H_2, \ldots, H_n$ consists of $n$ isomorphic graphs of order $n'$, and will be using in each of them the same root vertex $v$. The resulting rooted product graph is denoted by the expression $G \circ_v H$. In this case Corollary 8 is simplified as follows.
Remark 9. Let $H$ be a connected non-bipartite graph and let $v$ be a vertex of $H$.

(i) If $v$ does not belong to any metric basis for $H$, then for any connected graph $G$ of order $n$,
\[
\dim_l(G \circ_v H) = n \cdot \dim_l(H)
\]

(ii) If $v$ belongs to a metric basis for $H$, then for any connected graph $G$ of order $n \geq 2$,
\[
\dim_l(G \circ_v H) = n \cdot (\dim_l(H) - 1)
\]

Lemma 10. If $H$ is a connected graph of order $n'$ with clique number $\omega(H) = n' - 1$, and $G$ is a connected graph of order $n \geq 2$, then for any $v \in V(H)$,
\[
\dim_l(G \circ_v H) = n(n' - 3).
\]

Proof. Since $H$ has clique number $\omega(H) = n' - 1$, by Theorem 2 we have $\dim_l(H) = n' - 2$. To conclude the proof by Remark 9 we need to prove that any vertex of $H$ belongs to a local metric basis. With this aim, we consider three vertices $v_i, v_j, v_k \in V(H)$ and a maximum clique $Q$ of $H$ such that $v_i \notin V(Q)$, $v_j \in N_H(v_i)$ and $v_k \notin N_H(v_i)$ (Here $N_H(x)$ denotes the set of neighbours that $x$ has in $H$). Then we have the following:

- Since $v_i$ distinguishes the pair of adjacent vertices $v_j, v_k$, the set $B_i = V(H) - \{v_j, v_k\}$ is a local metric basis of $H$.
- Since $v_i v_k \notin E(H)$, the set, $B_j = V(H) - \{v_i, v_k\}$ is a local metric basis of $H$.
- Since $v_k$ distinguishes the pair of adjacent vertices $v_i, v_j$, the set $B_k = V(H) - \{v_i, v_j\}$ is a local metric basis of $H$.

Therefore, any vertex of $H$ belongs to a local metric basis. \qed

The equality $\dim_l(G \circ_v H) = n(n' - 3)$ is not exclusive for connected graphs of order $n'$ with clique number $\omega(H) = n' - 1$. Consider for instance the graph $H = \langle v \rangle + (K_r \cup K_s)$, $r \geq 2$ and $s \geq 2$, i.e., $H$ is the graph $K_r \cup K_s$ together with all the edges joining an isolated vertex $v$ to every vertex of $K_r \cup K_s$. In this case the order of $H$ is $n' = r + s + 1$, while its local metric dimension is $\dim_l(H) = n' - 3$. Note however, that the vertex $v$ can not be in any local metric basis. Hence, in this particular case for any connected graph $G$ of order $n \geq 2$, the local metric dimension of the rooted product graph $G \circ_v H$ is calculated from Remark 9 giving
\[
\dim_l(G \circ_v H) = n \cdot \dim_l(H) = n(n' - 3).
\]

Proposition 11. Let $G$ be a connected graph of order $n \geq 2$. Let $H$ be a connected non-bipartite graph of order $n'$ and let $v \in V(H)$. Then the following assertions hold.

(i) $n \leq \dim_l(G \circ_v H) \leq n(n' - 2)$. 

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(ii) $\dim_l(G \circ_v H) = n$ if and only if $\dim_l(H) = 2$ and the root vertex $v$ belongs to any local metric basis of $H$.

(iii) $\dim_l(G \circ_v H) = n(n' - 2)$ if and only if $H \cong K_{n'}$.

(iv) If $H \not\cong K_{n'}$, then $\dim_l(G \circ_v H) \leq n(n' - 3)$.

Proof. Remark 9 directly leads to the lower bound. Note that $\dim_l(H) \geq 2$, as $H$ is not bipartite. Now, if $v$ belongs to a local metric basis of $H$ and $\dim_l(H) = 2$, then Remark 9 (ii) leads to $\dim_l(G \circ_v H) = n$. Otherwise, if $v$ does not belong to any local metric basis of $H$, then Remark 9 leads to $\dim_l(G \circ_v H) \geq 2n$. This proves (ii).

Now, if $H \cong K_{n'}$, then $\dim_l(H) = n' - 1$ and, since $v$ belongs to a local metric basis of $H$, Remark 9 (ii) leads to $\dim_l(G \circ_v H) = n(n' - 2)$. On the other hand, if $H$ is a connected non-complete graph of order $n'$, then we have $\dim_l(H) \leq n' - 2$. So, Remark 9 leads to the upper bound.

Note that if $\dim_l(H) = n' - 2$, then Theorem 2 and Lemma 10 lead to $\dim_l(G \circ_v H) \leq n(n' - 3)$. Thus, (iii) and (iv) follows.

4 Corona product graphs

Let $G$ be a graphs of order $n$ and let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be a family of graphs. Recall that the corona product $G \odot \mathcal{H}$ is defined as the graph obtained from $G$ and $\mathcal{H}$ by taking one copy of $G$ and joining by an edge each vertex from $H_i$ with the $i^{th}$-vertex of $G$, [12]. The join $G + H$ is defined as the graph obtained from disjoint graphs $G$ and $H$ by taking one copy of $G$ and one copy of $H$ and joining by an edge each vertex of $G$ with each vertex of $H$. Notice that the particular case of corona graph $K_1 \odot H$ is isomorphic to the join graph $K_1 + H$. We can obtain any corona graph $G \odot \mathcal{H}$ by point-attaching from $G$, $K_1 + H_1$, $K_1 + H_2$, ..., $K_1 + H_n$. Note that if $H_i$ is a non-trivial graph, then the primary subgraph $K_1 + H_i$ is not bipartite. In fact, we can see the corona graph as a particular case of rooted product graph.

Corollary 12. Let $G$ be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a sequence of $n$ non-empty graphs $H_1, H_2, \ldots, H_n$. Then for any corona product graph $G \odot \mathcal{H}$,

$$\dim_l(G \odot \mathcal{H}) = \sum_{j=1}^{n} (\dim_l(K_1 + H_j) - \alpha_j).$$

Note that in this case $\alpha_j = 1$ if the vertex of $K_1$ belongs to a local metric basis of $K_1 + H_j$ and $\alpha_j = 0$ in otherwise.

The particular case of corona product graphs where the sequence $H_1, H_2, \ldots, H_n$ consists of $n$ isomorphic graphs of order $n'$ was previously studied in [25, 26]. The resulting corona graph is denoted by the expression $G \odot H$. As a particular case of Corollary 12, we derive the next result which was previously obtained in [25].

Remark 13. [25] Let $H$ be a non-empty graph. The following assertions hold.
(i) If the vertex of $K_1$ does not belong to any local metric basis for $K_1 + H$, then for any connected graph $G$ of order $n$,
\[ \dim_l(G \circ H) = n \cdot \dim_l(K_1 + H). \]

(ii) If the vertex of $K_1$ belongs to a local metric basis for $K_1 + H$, then for any connected graph $G$ of order $n \geq 2$,
\[ \dim_l(G \circ H) = n(\dim_l(K_1 + H) - 1). \]

The reader is referred to [25, 26] for a more detailed study on the local metric dimension of corona product graphs.

5 Block graphs

A block graph is a graph whose blocks are cliques. Since any block graph is obtained by point-attaching from $G_1 = K_{t_1}$, $G_2 = K_{t_2}$, ..., $G_k = K_{t_k}$, as a consequence of Theorem 5 we obtain a formula for the local metric dimension of any block graph. Our next result shows how the formula is reduced when every block has order $t_i \geq 3$.

Corollary 14. Let $\mathcal{H} = \{G_1 = K_{t_1}, G_2 = K_{t_2}, ..., G_k = K_{t_k}\}$ be a finite sequence of pairwise disjoint complete graphs of order $t_i \geq 3$, $i = 1, ..., k$. Then for any block graph $G[\mathcal{H}]$,
\[ \dim_l(G[\mathcal{H}]) = \sum_{j=1}^{k} (t_j - 1 - \alpha_j). \]

In this case $\alpha_j$ becomes $t_j - 1$ if every vertex of $K_{t_j}$ is a cut vertex of $G[\mathcal{H}]$ and it becomes the number of cut vertices of $G[\mathcal{H}]$ belonging to the clique $K_{t_i}$ in otherwise.

6 Bouquet of graphs

Let $\mathcal{H} = \{G_1, G_2, ..., G_k\}$ be a finite sequence of pairwise disjoint connected graphs and let $x_i \in V(G_i)$. By definition, the bouquet $\mathcal{H}_x$ of the graphs in $\mathcal{H}$ with respect to the vertices $\{x_i\}_{i=1}^{k}$ is obtained by identifying the vertices $x_1, x_2, ..., x_k$ with a new vertex $x$. Clearly, the bouquet $\mathcal{H}_x$ is a graph obtained by point-attaching from $G_1, G_2, ..., G_k$. Therefore, as a consequence of Theorem 5 we obtain the following result.

Corollary 15. Let $\mathcal{H} = \{G_1, G_2, ..., G_k\}$ be a finite sequence of pairwise disjoint connected graphs and let $x_i \in V(G_i)$ such that $J_H \neq \emptyset$. If $\mathcal{H}_x$ is the bouquet obtained from $\mathcal{H}$ by identifying the vertices $x_1, x_2, ..., x_k$ with a new vertex $x$, then
\[ \dim_l(\mathcal{H}_x) = \sum_{j \in J_H} (\dim_l(G_j) - \alpha_j). \]

Note that in this case $\alpha_i = 1$ if $x_i$ belongs to a local metric basis of $G_i$ and $\alpha_i = 0$ in otherwise.
7 Chain of graphs

Let $\mathcal{H} = \{G_1, G_2, ..., G_k\}$ be a finite sequence of pairwise disjoint connected non-trivial graphs and let $x_i, y_i \in V(G_i)$. By definition, the chain $\mathcal{C}(\mathcal{H})$ of the graphs in $\mathcal{H}$ with respect to the set of vertices $\{y_1, x_k\} \cup (\bigcup_{i=2}^{k-1}\{x_i, y_i\})$ is the connected graph obtained by identifying the vertex $y_i$ with the vertex $x_{i+1}$ for $i \in [k-1]$. Clearly, the chain $\mathcal{C}(\mathcal{H})$ is a graph obtained by point-attaching from $G_1, G_2, ..., G_k$.

![Figure 3: A chain $\mathcal{C}(\mathcal{H})$ obtained by point-attaching from $\mathcal{H} = \{G_1, G_2, G_3, G_4\}$.](image)

For every $j \in J_{\mathcal{H}}$ we say that $x_j$ is replaceable in $\mathcal{C}(\mathcal{H})$ if and only if there exists a local metric basis $B_j$ of $G_j$ such that $x_j \in B_j$ and there exists $k < j$ such that $G_k$ is a non-bipartite primary graph. Analogously, we say that $y_j$ is replaceable in $\mathcal{C}(\mathcal{H})$ if and only if there exists a local metric basis $B'_j$ of $G_j$ such that $y_j \in B'_j$ and there exists $k > j$ such that $G_k$ is a non-bipartite primary subgraph. We say that $x_j$ and $y_j$ are simultaneously replaceable in $\mathcal{C}(\mathcal{H})$ if both are replaceable in $\mathcal{C}(\mathcal{H})$ and there exists a local metric basis of $G_j$ containing both $x_j$ and $y_j$.

The formula for $\dim_l(\mathcal{C}(\mathcal{H}))$ is directly obtained from Theorem 5. In this case we have the following possibilities for the value of $\rho_j$.

- If $1 \in J_{\mathcal{H}}$ and $y_1$ is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_1 = \dim_l(G_1) - 1$.
- If $1 \in J_{\mathcal{H}}$ and $y_1$ is not replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_1 = \dim_l(G_1)$.
- If $k \in J_{\mathcal{H}}$ and $x_k$ is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_k = \dim_l(G_1) - 1$.
- If $k \in J_{\mathcal{H}}$ and $x_k$ is not replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_k = \dim_l(G_1)$.

For $j \in J_{\mathcal{H}} \cap \{2, ..., k-1\}$ we have the following possibilities.

- If neither $x_j$ nor $y_j$ is replaceable in $\mathcal{C}(\mathcal{H})$, then either $\rho_j = \dim_l(G_j)$ or $\rho_j = \dim_l(G_j) - 1$.
- If $x_j$ and $y_j$ are simultaneously replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_j = \dim_l(G_j) - 2$.
- If $x_j$ and $y_j$ are not simultaneously replaceable in $\mathcal{C}(\mathcal{H})$ and $x_j$ (or $y_j$) is replaceable in $\mathcal{C}(\mathcal{H})$, then $\rho_j = \dim_l(G_j) - 1$.

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