Coarse rigidity of Euclidean buildings

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Abstract

We prove the following rigidity results. Coarse equivalences between metrically complete Euclidean buildings preserve thick spherical buildings at infinity. If all irreducible factors have rank at least two, then coarsely equivalent Euclidean buildings are isometric (up to scaling factors); if in addition none of the irreducible factors is a Euclidean cone, then the isometry is unique and has finite distance from the coarse equivalence. This generalizes work of Kleiner and Leeb.

We prove coarse (i.e. quasi-isometric) rigidity results for trees (simplicial trees and $\mathbb{R}$-trees) and, more generally, for discrete and nondiscrete Euclidean buildings. For trees, a key ingredient is a certain equivariance condition. Without this condition, quasi-isometric rigidity fails. The ‘infinite letters’ $X$ and $H$ are, for example, coarsely equivalent without being isometric. Note, too, that any two simplicial trees of finite constant valence are coarsely equivalent without being necessarily isometric [21]. Our main results are as follows.

**Theorem I** Let $G$ be a group acting isometrically on two metrically complete leafless trees $T_1, T_2$. Assume that there is a coarse equivalence $f : T_1 \to T_2$, that $T_2$ has at least 3 ends and that the induced map $\partial f : \partial T_1 \to \partial T_2$ between the ends of the trees is $G$-equivariant. If the $G$-action on $\partial T_1$ is 2-transitive, then (after rescaling the metric on $T_2$) there is a $G$-equivariant isometry $\bar{f} : T_1 \to T_2$ with $\partial f = \partial \bar{f}$. If $T_1$ has at least two branch points, then $\bar{f}$ is unique and has finite distance from $f$.

The precise result is [5,6] below, where we also consider products of trees and Euclidean spaces.

**Theorem II** Let $X_1$ and $X_2$ be metrically complete nondiscrete Euclidean buildings whose spherical buildings at infinity $\partial_{cpl} X_1$ and $\partial_{cpl} X_2$ are thick. Let $f : X_1 \times \mathbb{R}^{m_1} \to X_2 \times \mathbb{R}^{m_2}$ be a coarse equivalence. Then $m_1 = m_2$ and there is a combinatorial isomorphism $f_s : \partial_{cpl} X_1 \to \partial_{cpl} X_2$ between the spherical buildings at infinity which is characterized by the fact that the $f$-image of an affine apartment $A \subseteq X_1$ has finite Hausdorff distance from the $f_s$-image of $A$.

This is [5,7] below. We remark that the boundary map $f_s$ is constructed in a combinatorial way from $f$. In general, a coarse equivalence between CAT(0)-spaces will not induce a map between the respective Tits boundaries.

**Theorem III** Let $f : X_1 \times \mathbb{R}^{m_1} \to X_2 \times \mathbb{R}^{m_2}$ be as in Theorem II and assume in addition that $X_1$ has no tree factors. Then there is (after rescaling the metrics on the irreducible factors of $X_2$) an isometry $\bar{f} : X_1 \to X_2$ with boundary map $\bar{f}_s = f_s$. Put $f(x \times y) = f_1(x \times y) \times f_2(x \times y)$. If none of the de Rham factors of $X_1$ is a Euclidean cone over its boundary, then $\bar{f}$ is unique and $d(f_1(x \times y), f(x))$ is bounded as a function of $x \in X_1$.

For a more general statement see [12,3] below. Theorem II and Theorem III were proved by Kleiner and Leeb under the additional assumptions that the Euclidean buildings are thick (i.e. that the thick points are cobounded) and that the spherical buildings at infinity are Moufang [15, 1.1.3] or compact [19, 1.3]. (These results extended, in turn, Mostow-Prasad rigidity [20,]) By Tits’ extension theorem [20, 4.1.2] [31, 1.6], every thick irreducible spherical building of rank at least 3 is automatically Moufang. The spherical building at infinity of an irreducible 2-dimensional Euclidean building, on the other hand, need not be either Moufang or compact; see, for example, [21]. In contrast to [15], we construct the combinatorial boundary map $f_s$ of Theorem II first and then use it to obtain a simpler approach to Theorem III.

For the proof of Theorem III we use Tits’ rigidity result [20, Thm. 2] which says that a Euclidean building is determined by its spherical building at infinity plus a certain forest; see Section 11 below. We refer to the preservation of this forest as the ‘ecological’ condition. Theorem I allows us to show that the boundary map is indeed ecological (tree preserving) and leads to a proof of Theorem III. This approach through the forest is also taken in [19]. We remark that Tits’ result [20, Thm. 2] is fundamental both in [15, 4.10.1] and in [19, Sec. 5.5].
Rigidity of quasi-isometries for higher rank symmetric spaces was proved by Kleiner-Leeb [15] and Eskin-Farb [12]. In [17] another proof was indicated which uses a combination of Lie theory and model theory, and which generalizes to simple algebraic groups over real closed fields. These rigidity results are in a certain sense complimentary to the present paper.

1 Geodesic spaces and trees

We recall a few notions from metric geometry. The metrics which we consider here are always real-valued. Let \((X,d)\) be a metric space. For \(r > 0\) and \(x \in X\) we put \(B_r(x) = \{ y \in X \mid d(x,y) < r \}\). For \(Y \subseteq X\) we put \(B_r(Y) = \bigcup \{ B_r(y) \mid y \in Y \}\). A subset \(Y\) of a metric space \(X\) is called bounded if it is contained in some sufficiently large ball, \(Y \subseteq B_r(x)\). A map \(f : X \to Y\) between metric spaces which preserves distances is called an isometric embedding; if \(f\) is onto, it is called an isometry. A geodesic in a metric space \(X\) is an isometric embedding \(\gamma : J \to X\), where \(J \subseteq \mathbb{R}\) is a closed interval. The image \(\gamma(J)\) is called a geodesic segment. If any two points of \(X\) are contained in some geodesic segment, \(X\) is called a geodesic space. If every geodesic \(\gamma : J \to X\) admits a geodesic extension \(\tilde{\gamma} : \mathbb{R} \to X\), we say that \(X\) has extensible geodesics (the extension will in general not be unique).

1.1 Definition [4] p. 167 [10] p. 29 A metric space \(T\) is called a tree (or \(\mathbb{R}\)-tree) if it has the following two properties.

\((T1)\) For any two points \(x, y \in T\), there is a unique geodesic \(\gamma : [0, d(x,y)] \to T\) with \(\gamma(0) = x\) and \(\gamma(d(x,y)) = y\). We put \([x,y] = \gamma([0, d(x,y)])\).

\((T2)\) If \(0 < r < s\) and if \(\gamma : [0, s] \to T\) is an injection such that \(\gamma|[0,r]\) and \(\gamma|[r,s]\) are geodesics, then \(\gamma\) is a geodesic.

An \(\mathbb{R}\)-tree with extensible geodesics is called a leafless tree. An apartment in \(T\) is an isometric image of \(\mathbb{R}\). Basic references for trees are [1], [10] and [20].

If \(z\) is in \([x,y]\) but different from \(x\) and \(y\), we say that \(z\) is between \(x\) and \(y\). Given two geodesic segments \([x,y]\) and \([x',y']\), there is a unique point \(z\) with \([x,y] \cap [x',y'] = [x,z]\) [10] p. 30. A point \(z\) is called a branch point if there are three points \(u,v,w\) distinct from \(z\) such that \([u,v] \cap [v,w] \cap [w,u] = \{z\}\).

1.2 Ends A ray in a tree \(T\) is an isometric image of \([0, \infty)\). Two rays are equivalent if their intersection is again a ray; the resulting equivalence classes are called the ends of \(T\), and \(\partial T\) denotes the set of ends of \(T\). Given \(x \in T\) and \(u \in \partial T\), there is a unique geodesic \(\gamma : [0, \infty) \to T\) with \(\gamma(0) = x\) whose image is in the class of \(u\) [10] p. 60; we put \(\gamma([0, \infty)) = [x,u]\) and \((x,u) = [x,u) - \{x\}\). Every apartment in \(T\) determines two ends. Conversely, if \(u,v\) are distinct ends, then there is a unique apartment whose ends are \(u\) and \(v\) [10] p. 61] and which we denote \((u,v)\). If \(A \subseteq T\) is an apartment and \(z \in T\), then there exists a unique point \(\pi_A(z)\) in \(A\) which has minimal distance from \(z\) and every geodesic segment \([z,x]\) with \(x \in A\) contains \(\pi_A(z)\) [10] p. 61].

1.3 Isometric group actions The isometries of a metric space \(X\) onto itself form a group \(I(X)\), the isometry group of \(X\). An isometric action of a group \(G\) on \(X\) is an homomorphism \(G \to I(X)\). If \(G\) acts isometrically on a tree \(T\), then it also acts on the set \(\partial T\) of ends. Throughout this paper we let groups act from the left. If \(G\) acts on \(X\), then \(X^G\) denotes the fixed point set of the action, and for \(x \in X\), the \(G\)-stabilizer of \(x\) is denoted \(G_x\). A subgroup \(P \subseteq G\) is called bounded if it has a bounded orbit in \(X\). If \(P\) is bounded, then every \(P\)-orbit in \(X\) is bounded.

The following result is fundamental for us. It is a special case of the Bruhat-Tits fixed point theorem.

1.4 Theorem Let \(G\) be a group acting isometrically on a metrically complete tree \(T\). If \(P \subseteq G\) is bounded, then \(P\) has a fixed point, \(T^P \neq \emptyset\).

Proof. Trees are CAT(0)-spaces [4] p. 167, so the result follows from [9] 3.2.3, [4] II.2.8 or [18] 1.2.
2 Trees with 2-transitive actions on their ends

Let $G$ be a group acting isometrically on a metrically complete tree $T$. We call a point $x \in T$ $G$-isolated if $T^{G_x} = \{x\}$. For such a $G$-isolated point $x$, the stabilizer $G_x$ is by definition a maximal bounded subgroup.

Throughout this section, we assume that $G$ acts 2-transitively on the ends of a leafless metrically complete tree $T$: given ends $u \neq v$ and $u' \neq v'$, there is an element $g \in G$ with $g(u) = u'$ and $g(v) = v'$. We note that such a group acts transitively on the set of apartments of $T$. In this section we derive the structure of these trees.

2.1 Proposition Let $u, v, w$ be three distinct ends of $T$ and let $x$ be the branch point determined by these three ends, $(u, v) \cap (u, w) = (u, x)$. Then there exists an element $g \in G_x$ which fixes $u$ and maps $v$ to $w$.

Proof. Since $G$ is 2-transitive on $\partial T$, we find $h \in G_u$ such that $h(v) = w$. Since $h(x) \in h(u, v) = (u, w)$, one of the following three cases must hold:

1. If $h(x) = x$ we are done, with $g = h$.
2. If $h(x) \in (u, x)$, then $A = \bigcup \{h^{-n}(u, x) \mid n \in \mathbb{N}\}$ is an $h$-stable apartment, on which $h$ induces a translation of length $d(x, h(x))$. As all apartments are conjugate under $G$, we find an isometry $h'$ fixing $u$ and $w$ which maps $h(x)$ to $x$. Thus $g = h'h$ fixes $x$ and $u$ and maps $v$ to $w$.
3. If $x \in (u, h(x))$, we put $A = \bigcup \{h^n(u, x) \mid n \in \mathbb{N}\}$ and argue similarly as in (2). \qed

2.2 Corollary Branch points are $G$-isolated.

The 2-transitive action on $\partial T$ has strong consequences for the structure of $T$, as we prove now. We start with a topological dichotomy.

2.3 Proposition The set of branch points is either closed and discrete, or it is dense in every apartment.

Proof. Let $A = (u, v)$ be an apartment and assume that the set of branch points is not dense in $A$. If $A$ contains no branch point, then $A = T$, and if $A$ contains only one branch point, then every apartment contains only one branch point. In either case, the claim follows.

Assume that $x \in A$ is a branch point and that $A$ contains another branch point $y$. By 2.1 we find an element $g \in G_x$ which interchanges $(u, x)$ and $(v, x)$. Then $g(y)$ is also a branch point. In this way, we get infinitely many branch points in $A$ at uniform distance $d(x, y)$. Because the set of branch points in $A$ was assumed not to be dense, there has to be a minimal distance $t$ between these, and they are distributed uniformly at this distance in $A$.

Since all apartments are conjugate, the set of branch points is either dense or discrete with uniform distance $t$ in every apartment. In the discrete case, the minimal distance between two branch points is $t$, so the set is closed and discrete in $T$. \qed

2.4 Corollary Assume that the set of branch points is discrete. There are three possibilities for the structure of $T$.

Type (0). There are no branch points, $T \cong \mathbb{R}$.
Type (I). There is a single branch point. Then $T$ is the Euclidean cone over its set $\partial T$ of ends, i.e. $T$ is a quotient of $[0, \infty) \times \partial T$, where $0 \times \partial T$ is identified with $\partial T$, for all $u, v \in \partial T$.
Type (II). There is an infinite discrete set of branch points. Then $T$ is a simplicial metric tree, every vertex has valence at least 3, and all edges have the same length $t$. \qed

This classification can be refined in terms of the $G$-action. For type (0), $G$ induces a group $\{\pm 1\} \times R$ on $T$, where $R$ is a subgroup of $(\mathbb{R}, +)$. If $R = 0$, there are infinitely many $G$-isolated points. For type (I), there is a unique $G$-isolated point. For type (II), $G_x$ acts transitively on the set of edges containing $z$ for each branch point $z$. Hence there are two subcases: either $G$ acts transitively on the branch points (vertices), or $T$ is bipartite and $G$ has two orbits on the vertices. In the first case, the mid-points of the edges are $G$-isolated (and $G$ acts with inversion), in the second case, only the branch points are $G$-isolated. We note that for the types (I), (II), $T$ admits the structure of a simplicial metric tree with a simplicial $G$-action.

In the remaining non-discrete case (III), we have the following.
2.5 Proposition If the set of branch points is dense, then every point \( x \in T \) is \( G \)-isolated.

Proof. Let \( x \in T \). If \( G_x \) fixes another point \( y \neq x \), then \( G_x \) fixes the geodesic segment \([x, y]\). There is a branch point \( z \) between \( x \) and \( y \). By 2.1, \( y \) is not a fixed point of \( G_x \). But \( G_{x,y} = G_x \supseteq G_{x,z} \), a contradiction. □

Every tree with a 2-transitive action on its ends corresponds to one of the types (0)-(III). Now we clarify the role of the maximal bounded subgroups. We noted already that by the Bruhat-Tits fixed point theorem, the stabilizers of \( G \)-isolated points are maximal bounded subgroups.

2.6 Proposition If \( P \subseteq G \) is a maximal bounded subgroup, then \( P \) is the stabilizer of a \( G \)-isolated point.

Proof. Assume this is false, so \( T_P \) contains a geodesic segment \([x, y]\) with \( x \neq y \). If \([x, y]\) contains a branch point \( z \), then \( G_z = P \) (by maximality), whence \( T_P = T^{G_z} = \{ z \} \), a contradiction. So the set of branch points cannot be dense in \( T \). By 2.3, it is closed and discrete in \( T \). By 2.4, we can assume that \( T \) is a simplicial tree and \( P \) fixes some edge of \( T \) elementwise. Therefore it fixes also some branch point \( z \), which is again a contradiction. □

3 Recovering the tree from the \( G \)-action

We continue to assume that \( G \) acts 2-transitively on the ends of the leafless metrically complete tree \( T \). We let \( i_G(T) \) denote the set of \( G \)-isolated points of \( T \). By the results of the previous section, \( i_G(T) \) corresponds bijectively to the set of maximal bounded subgroups of \( G \). With respect to the conjugation action of \( G \) on subgroups, this correspondence is \( G \)-equivariant. Our aim is to show that \( T \) can be recovered from the \( G \)-actions on \( i_G(T) \) and \( \partial T \). We let \( b(T) \subseteq i_G(T) \) denote the set of branch points and consider the different types (0)-(III) of trees. By the combinatorial structure of a tree we mean the underlying set \( T \) together with the collection of all apartments in \( T \) (without any metric). First, we dispose of the two degenerate types (0) and (I).

3.1 Type (0) is characterized by \( \# \partial T = 2 \) (and \( b(T) = \emptyset \)). The tree \( T \cong \mathbb{R} \) is unique up to isometry and the group induced by \( G \) splits semidirectly as \( \mathbb{Z}/2 \rtimes \mathbb{R} \), where \( R \) is a subgroup of \((\mathbb{R},+)\). Since \( \mathbb{R} \) contains subgroups which are abstractly, but not topologically, isomorphic, the \( G \)-action on \( T \) cannot be recovered from the action on \( i_G(T) \) and \( \partial T \).

Type (I) is characterized by \( \# \partial T \geq 3 \) and \( \# i_G(T) = 1 \). This determines both the combinatorial structure of the tree \( T \) and the \( G \)-action, but not the metric.

In the remaining cases, both \( \partial T \) and \( i_G(T) \) are infinite. This situation is much more rigid.

3.2 Suppose that \( T \) is of type (II). Then \( x \in i_G(T) \) is a branch point if and only if \( G_x \) has no orbit of length 2 in \( i_G(T) \). So we can recover the set \( b(T) \) of branch points in \( i_G(T) \). By 2.4, two branch points \( x, y \) are adjacent if and only if the only branch points fixed by \( G_{x,y} \) are \( x \) and \( y \), so the simplicial structure of \( T \) can be recovered from the \( G \)-action on \( i_G(T) \). (We note that then \( G_{x,y} \) has at most 3 fixed points in \( i_G(T) \).) Since all edges of \( T \) have the same length, \( T \) is determined as a metric space up to a scaling factor.

The following result shows how for type (III) apartments can be described in terms of the \( G \)-action.

3.3 Lemma Assume that \( T \) is of type (III). Let \( A = (u, v) \) be an apartment. Then \( x \) is contained in \( A \) if and only if \( x \) is the unique fixed point of \( \langle G_{x,u} \cup G_{x,v} \rangle \).

Proof. Suppose that \( x \notin A \). Then both \( G_{x,u} \) and \( G_{x,v} \) fix \( \pi_A(x) \neq x \). Now assume that \( x \in A \) and that \( z \neq x \). If \( \pi_A(z) = x \) then \( x \) is a branch point and \( G_{x,u} \) moves \( z \) by 2.4. If \( \pi_A(z) \) is between \( x \) and \( u \), then there is a branch point between \( x \) and \( \pi_A(z) \) and therefore \( G_{x,u} \) moves \( z \) by 2.4. Similarly, \( G_{x,v} \) moves \( z \) if \( \pi_A(z) \) is between \( x \) and \( v \). □

3.4 If \( T \) is of type (III), then \( i_G(T) = T \). The stabilizer \( G_{x,y} \) of two distinct points \( x, y \) fixes the infinite set \([x, y] \), so type (III) can be distinguished from type (II). By 3.3 we can recover the apartments in \( T \) from the group action. Let \( A = (u, v) \) be an apartment and let \( z \in A \). Then \( (u, z) = A \cap T^{G_{u,z}} \), so we can also recover
the rays in $A$. Therefore $G$ determines the topology of $A$. Since the branch points are dense $A$, $G_{u,v}$ induces a group $H$ of translations on $A$ which has a dense orbit. Up to a scaling factor, there is just one $H$-invariant metric on $A$ which satisfies $d(h^n(x), y) = nd(x, y)$ for all $h \in H$ and all $n \in \mathbb{N}$. So the metric on $A$ is determined up to scaling. Since all apartments are conjugate, the metric on $T$ is unique up to scaling.

Summarizing these facts we have the following result.

3.5 Proposition Assume that $\#\partial T \geq 3$. Given the $G$-actions on $\partial T$ and $i_G(T)$, the combinatorial structure of the tree $T$ is uniquely determined. If $T$ is of type (II) or type (III), the metric is determined up to a scaling factor.

4 Metric spaces and coarse geometry

We recall some notions from coarse geometry [24]. A map $f : X \longrightarrow Y$ between metric spaces is called controlled if there is a monotonic real function $\rho : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$d_Y(f(x), f(y)) \leq \rho(d_X(x, y))$$

holds for all $x, y \in X$. If in addition the preimage of every bounded set is bounded, then $f$ is called a coarse map. Neither $f$ nor $\rho$ is required to be continuous. Note that the image of a bounded set under a controlled map is bounded. Two maps $g, f : X \longrightarrow Y$ between metric spaces have finite distance if the set \{ $d_Y(f(x), g(x))$ | $x \in X$ \} is bounded. This is an equivalence relation which leads to the coarse metric category whose objects are metric spaces and whose morphisms are equivalence classes of coarse maps. A coarse equivalence is an isomorphism in this category.

4.1 Lemma If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are controlled and if $g \circ f$ has bounded distance from the identity map on $X$, then $f$ is coarse. In particular, $f$ is a coarse equivalence if $f \circ g$ also has bounded distance from the identity.

Proof. Suppose that $f(Z)$ is bounded. Since $g$ is controlled, $g(f(Z))$ is also bounded, $g(f(Z)) \subseteq B_r(x)$ for some $x \in X$. Now $Z$ is contained in some $s$-neighborhood of $g(f(Z))$, so $Z \subseteq B_{r+s}(x)$ is bounded. \qed

A controlled map with control function $\rho(t) = t$ is called a 1-Lipschitz map. If $\rho(t) = ct + d$, with $c \geq 1$ and $d \geq 0$, then $f$ is called large-scale Lipschitz. A coarse equivalence is a quasi-isometry if both the map and its coarse inverse are large-scale Lipschitz. A rough isometry is a coarse equivalence where the control functions in both directions are of the form $\rho(t) = t + d$ (such maps are sometimes called coarse isometries). A map which has finite distance from an isometry is a rough isometry.

4.2 Lemma Let $f : X \longrightarrow Y$ be controlled. If $X$ is geodesic, then $f$ is large-scale Lipschitz.

Proof. For $x, y \in X$ let $d(x, y) = m + s$, with $m \in \mathbb{N}$ and $0 \leq s < 1$. Let $\gamma$ be a geodesic from $x$ to $y$. Then

$$d(f(\gamma(j)), f(\gamma(j+1))) \leq \rho(1)$$

for $j = 0, \ldots, m-1$, so $d(f(\gamma(0)), f(\gamma(m+s))) \leq m\rho(1) + \rho(s) \leq (m+s)\rho(1) + \rho(1)$. \qed

In particular, every coarse equivalence of geodesic spaces is a quasi-isometry.

4.3 Hausdorff distance Two subsets $U, V$ of a metric space $X$ have Hausdorff distance at most $r$ if $U \subseteq B_r(V)$ and $V \subseteq B_r(U)$. Then we write $Hd(U, V) < r$. For example, a nonempty subset is bounded if and only if it has finite Hausdorff distance from some point. More generally, we say that $V$ dominates $U$ if $U \subseteq B_r(V)$ for some $r > 0$, and we write $U \subsetneq_{Hd} V$. This defines a preorder on the subsets of $X$. We call $Y \subseteq X$ cobounded if $Y$ dominates $X$.

If $f : X \longrightarrow Y$ is controlled with control function $\rho$, and if $U \subseteq B_r(V)$, then $f(U) \subseteq B_{\rho(r)}(V)$. So if $f$ is a coarse equivalence, then $U$ and $V$ have finite Hausdorff distance if and only if $f(U)$ and $f(V)$ have finite Hausdorff distance, and $U$ dominates $V$ if and only if $f(U)$ dominates $f(V)$. 

5
5 Coarse equivariant rigidity for trees

Now we can prove our main rigidity result for trees. We use a result about coarse equivalences of hyperbolic spaces which goes back to M. Morse. It says that the coarse image of a geodesic is Hausdorff close to a geodesic \[ \text{III.H.1.7} \] \[ \text{1.3.2}. \] Since trees are hyperbolic, it follows that there is a positive constant \( r > 0 \) such that the coarse image \( f(A) \) of an apartment \( A \) has Hausdorff distance at most \( r \) from a unique apartment \( A' \). We put \( f_*A = A' \) (note that \( f_* \) just maps apartments to apartments, it is not a map between the trees).

However, for our application to Euclidean buildings we have to consider the more general situation of a coarse equivalence \( T_1 \times \mathbb{R}^m_1 \to T_2 \times \mathbb{R}^m_2 \). Then Morse’s result has to be replaced by the following theorem which is (up to a small modification) due to Kleiner and Leeb.

5.1 Theorem Let \( T_1, T_2 \) be metrically complete leafless trees. If \( f : T_1 \times \mathbb{R}^m_1 \to T_2 \times \mathbb{R}^m_2 \) is a coarse equivalence, then \( m_1 = m_2 \) and there exists a constant \( r_f > 0 \) such that the following holds. For every apartment \( A \) of \( T_1 \), there is a unique apartment \( A' \subseteq T_2 \) such that \( A' \times \mathbb{R}^m_1 \) has Hausdorff distance at most \( r_f \) from \( f(A) \times \mathbb{R}^m_1 \).

Proof. A leafless \( \mathbb{R} \)-tree is a Euclidean building, so \[ \text{8.2} \] applies. Since trees are 1-dimensional, \( m_1 = m_2 \). \( \square \)

We put \( f_*A = A' \); if \( g \) is a coarse inverse for \( f \), then \( g_* \) is an inverse for \( f_* \). We also need the following auxiliary result on trees.

5.2 Lemma Let \( \mathcal{F} \) be a collection of apartments in a tree \( T \) and let \( r > 0 \). If the subset \( \bigcap \{ B_r(A) \mid A \in \mathcal{F} \} \subseteq T \) is unbounded, then the apartments in \( \mathcal{F} \) have a common end.

Proof. The result is a special case of \[ \text{8.3} \] below, but we give a direct proof. Let \( (u, v) \in \mathcal{F} \) and choose a sequence \( x_n \in X \) such that \( \pi_{(u,v)}(x_n) \) converges to one end of \( (u, v) \), say \( u \). This is possible since \( X \subseteq B_r((u, v)) \) is unbounded. Let \( A \in \mathcal{F} \). Then \( d(\pi_A(x_n), \pi_{(u,v)}(x_n)) \leq 2r \) and the unbounded sequence \( \pi_A(x_n) \) subconverges to an end \( w \) of \( A \). If \( w \neq u \), then the values \( d(\pi_A(x_n), \pi_{(u,v)}(x_n)) \) would be unbounded. Thus \( w = u \). \( \square \)

In the previous lemma the end is unique unless \( \mathcal{F} \) consists of a single apartment.

5.3 Proposition If \( T_1 \) has at least one branch point and if \( f : T_1 \times \mathbb{R}^m \to T_2 \times \mathbb{R}^m \) is a coarse equivalence, then the map \( f_* \) between the sets of apartments of the trees induces a bijection \( f_* : \partial T_1 \to \partial T_2 \) between the ends of the trees, in such a way that \( f_*(u, v) = (f_*u, f_*v) \).

Proof. Let \( r > 0 \) be as in \[ \text{5.1} \] and let \( \mathcal{F} \) be a finite collection of apartments in \( T_1 \) in having an end \( u \) in common. Put \( (u, x) = \bigcap \{ A \mid A \in \mathcal{F} \} \) and \( Y = \bigcap \{ B_{r_1}(f_*A) \mid A \in \mathcal{F} \} \). Then \( f((u, x) \times \mathbb{R}^m) \subseteq Y \times \mathbb{R}^m \). If the set \( f_*\mathcal{F} \) has no common end, then \( Y \) is bounded by \[ \text{5.2} \] so \( Y \times \mathbb{R}^m \) is quasi-isometric to \( \mathbb{R}^m \).

We obtain then a quasi-isometric embedding of \( [0, \infty) \times \mathbb{R}^m \) into \( \mathbb{R}^m \), which induces a continuous injection between the asymptotic cones, \( [0, \infty) \times \mathbb{R}^m \cong \text{Cone}([0, \infty) \times \mathbb{R}^m) \to \text{Cone}(\mathbb{R}^m) \cong \mathbb{R}^m \); see \[ \text{15} \] \[ \text{17}. \] This is impossible by topological dimension invariance. Therefore \( Y \) is unbounded and \( f_*\mathcal{F} \) has a common end. Applying the same argument to a coarse inverse \( g \) of \( f \), we see that a finite set \( \mathcal{F} \) has an end in common if and only if \( f_*\mathcal{F} \) has an end in common.

Consider now the simplicial complex whose simplexes are the finite collections of apartments in \( T_1 \) having a common end. The ends of \( T_1 \) correspond to the maximal complete subcomplexes (a simplicial complex is complete if any two simplexes are contained in some simplex). Since \( f_* \) preserves the simplicial structure, it preserves ends. \( \square \)

The complex constructed in the previous proof is a special case of the apartment complex; see \[ \text{6.9} \] It is the nerve of the covering of \( \partial T_1 \) given by all two-element subsets. The last paragraph of the proof is actually a special case of \[ \text{6.10} \]

The next result is essentially a special case of \[ \text{15} \text{ 8.3.11}. \]

5.4 Proposition Let \( G \) be a group acting isometrically on two metrically complete leafless trees \( T_1, T_2 \). Assume that \( f : T_1 \times \mathbb{R}^m \to T_2 \times \mathbb{R}^m \) is a coarse equivalence and that the induced map \( f_* \) on the apartments is \( G \)-equivariant. If \( T_1 \) has at least three ends, then a subgroup \( P \subseteq G \) has a bounded orbit in \( T_1 \) if and only if it has a bounded orbit in \( T_2 \).
**Proof.** Suppose that \( P \subseteq G \) is bounded. Let \( x \) be a branch point in \( T_1 \) and consider the bounded orbit \( P(x) \). We put \( f(x \times 0) = y \times p \) and we show that \( y \) has a bounded orbit \( P(y) \) in \( T_2 \).

Let \( F \) denote the set of all apartments in \( T_1 \) which intersect the orbit \( P(x) \) nontrivially. This set \( F \) is obviously \( P \)-invariant and has no common end (because \( x \) is a branch point). Let \( F' = f \circ F \) denote the corresponding set of apartments in \( T_2 \). Since \( f_b \) is \( G \)-equivariant, \( F' \) is also \( P \)-invariant, and the apartments in \( F' \) have no common end by 5.3. For \( s > 0 \) consider the \( P \)-invariant set \( X_s = \{ B_s(A') \mid A' \in F' \} \subseteq T_2 \). By 5.2 \( X_s \) is bounded (or empty).

Since \( P(x) \) is bounded, there is a constant \( r' > 0 \) such that \( d(x, \pi_A(x)) < r' \) for all \( A \in F \). We put \( f(\pi_A(x) \times 0) = y' \times p' \), for some \( A \in F \). Then \( d(y', \pi_{f(A)}(y')) \leq r_f \) by 5.1. Let \( \rho \) be the control function for \( f \), then \( d(y, y') \leq \rho(r') \). This holds for all \( A \in F \), whence \( y \in X_{\rho(r') + r_f} \). As this set is bounded and \( P \)-invariant, \( P(y) \) is bounded.

If \( g \) is a coarse inverse of \( f \), then \( g_b \) is \( G \)-equivariant (because it is the inverse of the equivariant map \( f_b \)).

Now we get to our main result on trees, which implies Theorem I in the introduction. The map \( f_* : \partial T_1 \longrightarrow \partial T_2 \) is defined as in 5.3.

5.5 **Theorem** Let \( G \) be a group acting isometrically on two metrically complete leafless trees \( T_1, T_2 \), with \( \# \partial T_1 \geq 3 \) and assume that the action of \( G \) on \( \partial T_1 \) is 2-transitive. If there is a coarse equivalence \( f : T_1 \times \mathbb{R}^m \longrightarrow T_2 \times \mathbb{R}^m \) and if the induced map \( f_* : \partial T_1 \longrightarrow \partial T_2 \) is \( G \)-equivariant, then the following hold.

(i) After rescaling the metric on \( T_2 \) by a constant \( a > 0 \), there is a \( G \)-equivariant isometry \( \bar{f} : T_1 \longrightarrow T_2 \), with \( \bar{f}_b = f_b \). If \( T_1 \) has at least 2 branch points, then both \( \bar{f} \) and \( a \) are unique.

(ii) Put \( f(x \times p) = f_1(x \times p) \times f_2(x \times p) \). If \( T_1 \) has at least two branch points, then there is a constant \( s > 0 \) such that \( d(f_1(x \times p), f(x)) \leq s \) holds for all \( x \times p \in T_1 \times \mathbb{R}^m \). The constant \( s \) depends only on \( T_1 \), the control function \( \rho \), \( a \) and the constant \( r_f \) from 5.7. In particular, \( f \) and \( \bar{f} \) have finite distance if \( m = 0 \).

(iii) If \( f \) is a rough isometry, then we can put \( a = 1 \).

**Proof.** (i) By 5.4, both \( G \)-actions have the same set of maximal bounded subgroups. These subgroups correspond by 2.6 to the \( G \)-isolated points. Therefore we have equivariant bijections \( \bar{f} : i_G(T_1) \longrightarrow i_G(T_2) \) and \( f_* : \partial T_1 \longrightarrow \partial T_2 \). The trees of type (I), (II), and (III) can be distinguished by the \( G \)-action on \( i_G(T_1) \); see 3.2, 5.2, and 5.3 (our assumptions exclude trees of type (0)). The combinatorial structure is also encoded in the \( G \)-action, as we noted in 3.5. By the results in Section 3, we can rescale the metric on \( T_2 \) by a constant \( a > 0 \) in such a way that \( \bar{f} : i_G(T_1) \longrightarrow i_G(T_2) \). The trees of type (I) and (II) have unique \( \bar{f} \). From the construction it is clear that \( \bar{f}_b = f_b \). For trees of type (II) and (III), \( \bar{f} \) and \( a \) are unique by 5.3.

(ii) Let \( z \in b(T_1) \) and put \( r_1 = 1 + \inf \{d(x, y) \mid x, y \in b(T_1), x \neq y \} \). Then \( T_1 \) is covered by the \( G \)-translates of \( B_{r_1}(z) \). Let \( F \) be the collection of all apartments of \( T_1 \) containing \( z \). By 5.1 there is a constant \( r_f > 0 \) such that \( f_1(z \times p) = \bigcap \{ B_{r_f} \} = A \in F \} = \bigcap \{ B_{r_f}(f(A)) \mid A \in F \} = B_{r_f}(f(z)) \). It follows that \( d(f_1(x \times p), f(x)) \leq ar_1 + r_f + \rho(r_1) \) for general \( x \in T_1 \).

(iii) For trees of type (I) it is clear that no rescaling is necessary in order to find \( \bar{f} \). Suppose that \( f \) is a rough isometry, with control function \( \rho(t) = t + b \). For trees of type (II) and (III) we have by (iii) \( d(\bar{f}(x), \bar{f}(x')) \leq d(x, x') + b + 2(ar_1 + r_f + b) \). On the other hand, \( d(f(x), \bar{f}(x')) = ad(x, x') \). Since \( T_1 \) is unbounded, \( a \leq 1 \). Applying the same argument to \( \bar{f}^{-1} \), we see that \( a = 1 \).

For the special case of a thick locally finite simplicial tree, a similar result was proved by Leeb [19, 4.3.1].

6 **Buildings**

We record some basic notions and facts for buildings. Everything we need can be found in [5], [25], [29] and [31]. For our present purposes it is convenient to view buildings as simplicial complexes (or even geometric realizations of simplicial complexes). This is essentially Tits’ approach in [30]; see also [11].

6.1 **Simplicial complexes** Let \( V \) be a set and \( S \) a collection of finite subsets of \( V \). If \( \bigcup S = V \) and if \( S \) is closed under going down (i.e. \( a \subseteq b \in S \) implies \( a \in S \)), then the poset \( (S, \subseteq) \) is called a simplicial complex. More generally, any poset isomorphic to such a poset \( S \) will be called a simplicial complex. The join \( S \ast T \) of two simplicial complexes \( S, T \) is the product poset; it is again a simplicial complex. Homomorphisms between
simplicial complexes are defined in the obvious way [4, 7A.1]. A homomorphism which maps k-simplices to k-simplices is called nondegenerate.

The geometric realization $|S|$ of the simplicial complex $S$ is the set of all functions $p : V \longrightarrow [0, 1]$ with $p^{-1}(0, 1] \in S$ and $\sum_{v \in V} p(v) = 1$. The set $|S|$, endowed with the weak topology, is a CW complex. Moreover, $|S \ast T| \cong |S| \ast |T|$, where the right-hand side is the topological join. In the case of spherical buildings, $|S|$ is often endowed with a stronger, metric topology [4, II.10A]. By a result due to Dowker, the identity map is a homotopy equivalence between these two topologies [4, I.7].

### 6.2 Coxeter groups and buildings
Let $(W, I)$ be a Coxeter system. Thus $W$ is a group with a (finite) generating set $I$ consisting of involutions and a presentation of the form $W = \langle I \mid (ij)^{or(d(ij))} = 1 \rangle$ for all $i, j \in I$. For a subset $J \subseteq I$ we put $W_J = \langle J \rangle$. Then $(W_J, J)$ is again a Coxeter system. The poset $\Sigma = \bigcup \{W/W_J \mid J \subseteq I\}$, ordered by reversed inclusion, is a simplicial complex, the Coxeter complex $\Sigma = \Sigma(W, I)$. The type of a simplex $wW_J$ is $t(wW_J) = I - J$. The type function may be viewed as a non-degenerate simplicial epimorphism from $\Sigma$ to the power set $2^I$ of $I$ (viewed as a simplicial complex).

A building $B$ is a simplicial complex together with a collection $\text{Apt}(B)$ of subcomplexes, called apartments, which are isomorphic to a fixed Coxeter complex $\Sigma$. The apartments have to satisfy the following compatibility conditions.

(B1) For any two simplices $a, b \in B$, there is an apartment $A$ containing $a, b$.

(B2) If $A, A' \subseteq B$ are apartments containing the simplices $a, b$, then there is a (type preserving) isomorphism $A \longrightarrow A'$ fixing $a$ and $b$.

The type functions of the apartments are pairwise compatible and extend to a non-degenerate simplicial epimorphism $t : B \longrightarrow 2^I$. The cardinality of $I$ is the rank of the building ($\text{rank}(B) = \dim(B) + 1$). A building of rank 1 is just a set (of cardinality at least 2), the apartments are the two-element subsets.

The maximal simplices in a building are called chambers, and $\text{Cham}(B)$ is the set of all chambers. Every simplex of a building is contained in some chamber (so buildings are pure simplicial complexes). Recall that the dual graph of a pure complex is the graph whose vertices are the maximal simplices and whose edges are the simplices of codimension 1; this is the chamber graph of $\Delta$. A gallery is a simplicial path in the chamber graph, and a non-stammering gallery is a path where consecutive chambers are always distinct. The chamber graph of a building is always connected. A minimal gallery is a shortest path in the chamber graph.

A building is called thick if every non-maximal simplex is contained in at least 3 distinct chambers (it is always contained in at least 2 distinct chambers). We allow non-thick buildings (in [29], all buildings are assumed to be thick, but the results from [29] which we collect in this section hold for non-thick buildings as well).

### 6.3 Residues and panels
Let $a \in B$ be simplex of type $I - \{j\}$. The residue of $a$ is the poset $\text{Res}(a)$ consisting of all simplices containing $a$; this poset is again a building, whose Coxeter complex is modeled on $W_{I - j}$ [29, 3.12]. If $a$ is a simplex of codimension 1 and type $I - \{j\}$, then $\text{Res}(a)$ is called a $j$-panel.

There is an order-reversing poset isomorphism between the simplicial complex $B$ and the set of all residues in $B$. Residues can also be defined in terms of the chamber graph, viewed as edge-colored graphs. Basically, this is a dictionary which allows the passage from buildings, viewed as simplicial complexes (as in [29]) to buildings viewed as edge colored graphs (or chamber systems) (as in [31]). In view of this correspondence, we call a simplex of type $I - \{j\}$ also a $j$-panel.

The join of two buildings is again a building. Conversely, a building decomposes as a join if its Coxeter group is decomposable (i.e. if $I \subseteq W$ decomposes into two subsets which centralize each other).

### 6.4 Spherical buildings
A Coxeter complex $\Sigma$ is called spherical if it is finite. Then the geometric realization $|\Sigma|$ is a combinatorial sphere of dimension $#I - 1$. A building is called spherical if its apartments are finite.

### 6.5 Thick reductions
A non-thin spherical building $B$ can always be ‘reduced’ to a thick building as follows. There exists a thick spherical building $B_0$ such that $B$ is a simplicial refinement of $S^0 \ast \cdots \ast S^0 \ast B_0$ (we view $S^0$ as a thin spherical building of rank 1) [8, 9, 15, 3.7, 28]. For the geometric realization, we have then $|B| = S^k \ast |B_0|$, where $k$ is the number of $S^0$-factors in the join. So non-thin buildings are suspensions of thick buildings. The geometric realization of a thin spherical buildings is a sphere.
The following lemma will be used later.

**6.6 Lemma** Let $B$ be a spherical building and let $A \subseteq B$ be an apartment. If every panel $a \in A$ is contained in at least three different chambers of $B$, then $B$ is thick.

*Proof.* Let $a$ be an arbitrary panel in $B$, and let $c_0, \ldots, c_k$ be a shortest gallery with the property that the first chamber $c_0$ contains $a$ and the last chamber $c_k$ is in $A$. This gallery can be continued inside $A$ as a reduced gallery until it reaches a panel $b \in A$ which is opposite $a$. Then there is a bijection $[a:b]$ between $Res(a)$ and $Res(b)$; see [9.1] below. It follows that $a$ is contained in at least three different chambers. 

Euclidean buildings give rise to a family of building epimorhism. The following fact is useful; see [3, 2.8].

**6.7 Lemma** Let $B, B'$ be spherical buildings of the same type and let $\varphi : B \longrightarrow B'$ be a simplicial map over $I$. Then $\varphi$ is an epimorphism if and only if its restriction to every panel is surjective.

*Proof.* Since the chamber graph is connected, it is clear that the local surjectivity conditions implies global surjectivity. Conversely, suppose that $\varphi$ is an epimorphism. Let $a', b' \in B'$ be $i$-adjacent chambers and let $a$ be a preimage of $a'$. We have to find a preimage $b$ of $b'$ which is $i$-adjacent to $a$. Let $a'$ be opposite $a'$ and let $a' \rightarrow b' \rightarrow \cdots \rightarrow c'$ be a minimal gallery which we denote by $\gamma'$. Let $c$ be a preimage of $c'$. Since $\varphi$ does not increase distances in the chamber graph, $c$ is opposite $a$. Thus there is a gallery $a \rightarrow b \rightarrow \cdots \rightarrow c$ in $B$ of the same type as $\gamma'$, which we denote by $\gamma$. Since $\gamma'$ is the unique gallery of its type in $B'$ from $a'$ to $c'$, it follows that $\varphi(\gamma) = \gamma'$. In particular, $\varphi(b) = b'$.

**6.8 Corollary** Let $B, B'$ be spherical buildings of the same type and let $\varphi : B \longrightarrow B'$ be a simplicial epimorphism over $I$. If $B'$ is thick, then $B$ is also thick.

A thick spherical building is determined by its apartment complex which we introduce now. This complex appeared already in 5.3. We will see later that the apartment complex of the spherical building at infinity is a coarse invariant of a Euclidean building.

**6.9 The apartment complex** Let $B$ be a building and $Apt(B)$ its set of apartments. The apartment complex $AC(B)$ is the simplicial complex whose simplices are finite subsets $A_1, \ldots, A_k$ of apartments, with $A_1 \cap \cdots \cap A_k \neq \emptyset$ (in other words, $AC(B)$ is the nerve of the covering $Apt(B)$ of $B$).

If $B$ is thick, then every simplex $a \in B$ can be written as an intersection of finitely many apartments.

**6.10 Proposition** Let $B_1, B_2$ be thick spherical buildings and let $\varphi : AC(B_1) \longrightarrow AC(B_2)$ be a simplicial isomorphism. Then there is a unique isomorphism $\Phi : B_1 \longrightarrow B_2$ such that $\varphi(A) = \Phi(A)$ for all $A \in Apt(B_1)$.

*Proof.* The map $\varphi$ maps sets of apartments with the finite intersection property to sets with the finite intersection property. Because $B_1$ is thick and has finite apartments, the maximal subsets with the finite intersection property in $Apt(B_1)$ are precisely the sets $S_v = \{ A \in Apt(B_1) \mid v \in A \}$ for all vertices $v$ of $B$. Therefore $\varphi$ induces a bijection $\Phi$ between the vertices of the buildings. If $v \in A$, then $\Phi(v) \in \varphi(A)$. Since $B_1$ is thick, two vertices $u, v$ in $B_1$ are adjacent if and only if the following holds: no other vertex $w$ is in the intersection of all apartments containing $u$ and $v$. This can be expressed in $AC(B_1)$ as follows: if $S_w \supseteq S_u \cap S_v$, then $S_w = S_u$ or $S_w = S_v$. So $\Phi$ preserves the 1-skeleton of $B_1$. But every building is the flag complex of its 1-skeleton [29, 3.16], therefore $\Phi$ is simplicial. A simplicial isomorphism between buildings is a building isomorphism (which might not be type preserving).

In the previous proof, thickness is essential. It is clear that the thick reduction (see 6.5) of a spherical building has the same apartment complex as the building itself.
7 Euclidean buildings

The notion of a (nondiscrete) Euclidean building is due to Tits [30] (Prior to their axiomatization in [30], the nondiscrete Euclidean buildings that arise from reductive groups over valued fields were studied in [6, 15, 22, 30].) We rely on Parreau’s work [22] which contains many important structural results for Euclidean buildings. She showed in particular that the axioms given by Kleiner-Leeb [15] are equivalent to Tits’ original axioms plus metric completeness.

7.1 The affine Weyl group We fix a spherical Coxeter group \((W, I)\) in its standard representation on \(\mathbb{R}^n\) (where \(n = \#I\)). A Weyl chamber or sector in \(\mathbb{R}^n\) is the closure of a connected component in \(\mathbb{R}^n - (H_1 \cup H_2 \cup \cdots \cup H_t)\), where the \(H_k\) are the reflection hyperplanes of \(W\). The closure of a connected component of \(\mathbb{R}^n - H_i\) is called a half space. A wall is a reflection hyperplane. Note that we do not require that \(W\) is irreducible.

We also fix a \(W\)-invariant inner product on \(\mathbb{R}^n\). Up to scaling factors on the irreducible \(W\)-submodules of \(\mathbb{R}^n\), such an inner product is unique. The group \(W\) normalizes the translation group \((\mathbb{R}^n, +)\), and the semidirect product \(W \oplus \mathbb{R}^n\) acts isometrically on \(\mathbb{R}^n\). We call this group the affine Weyl group.

In [30, 15] or [22], the translation group may be some \(W\)-invariant subgroup of \((\mathbb{R}^n, +)\). Since we are in this paper only concerned with metric properties of Euclidean buildings, and since the affine Weyl group can always be enlarged to the full translation group without changing the underlying metric space and the set of affine apartments [22, 1.2], there is no loss in generality here. From this viewpoint, every \(p \in X\) is a special point [3, 1.3.7].

7.2 Euclidean buildings Let \(W\) be a spherical Coxeter group and \(W \oplus \mathbb{R}^n\) the corresponding affine Weyl group. Let \(X\) be a metric space. A chart is an isometric embedding \(\varphi : \mathbb{R}^n \to X\), and its image is called an affine apartment. We call two charts \(\varphi, \psi\) \(W\)-compatible if \(Y = \varphi^{-1} \psi(\mathbb{R}^n)\) is convex (in the Euclidean sense) and if there is an element \(w \in W \oplus \mathbb{R}^n\) such that \(\psi \circ w|_Y = \varphi|_Y\) (this condition is void if \(Y = \emptyset\)). We call a metric space \(X\) together with a collection \(\mathcal{A}\) of charts a Euclidean building if it has the following properties.

(EB1) For all \(\varphi \in \mathcal{A}\) and \(w \in W \oplus \mathbb{R}^n\), the composition \(\varphi \circ w\) is in \(\mathcal{A}\).

(EB2) Any two points \(x, y \in X\) are contained in some affine apartment.

(EB3) The charts are \(W\)-compatible.

The charts allow us to map Weyl chambers, walls and half spaces into \(X\); their images are also called Weyl chambers, walls and half spaces. The first three axioms guarantee that these notions are coordinate independent.

(EB4) If \(a, b \subseteq X\) are Weyl chambers, then there is an affine apartment \(A\) such that the intersections \(A \cap a\) and \(A \cap b\) contain Weyl chambers.

(EB5) If \(A_1, A_2, A_3\) are affine apartments which intersect pairwise in half spaces, then \(A_1 \cap A_2 \cap A_3 \neq \emptyset\).

7.3 Example: Euclidean cones over spherical buildings Let \(B\) be a spherical building and let \(EB(B)\) denote the quotient of \(|B| \times [0, \infty)\) where \(|B| \times 0\) is identified to a point. Let \(d_{|B|}\) denote the spherical metric on \(|B|\), and put \(d(x \times s, y \times t)^2 = s^2 + t^2 - 2st \cos(d_{|B|}(x, y))\); see [1] I.5.6. With this metric, \(EB(B)\) is the infinite Euclidean cone over \(|B|\). It is not difficult to see that \(EB(B)\) is a Euclidean building. The affine apartments in \(EB(B)\) correspond bijectively to the apartments of \(B\). These buildings are generalizations of the trees of type (I), and we call them Euclidean buildings of type (I). One can view \(EB(B)\) as the affine building with respect to the trivial valuation on the spherical building \(B\). We note that this construction if functorial: every automorphism of \(B\) extends to an isometry of \(EB(B)\). In this way, every spherical building can be viewed as a Euclidean building. This functor \(EB\) was considered by Rousseau in [26] where \(EB(B)\) is called inmeable vectoriel.

7.4 Example: leafless trees A leafless tree is a Euclidean building. In particular \(\mathbb{R}\) is a Euclidean building. The affine Weyl group is \(W \mathbb{R} = \{x \mapsto \pm x + c \mid c \in \mathbb{R}\}\).

7.5 Example: simplicial affine buildings The geometric realization of an affine simplicial building is a Euclidean building [5, 30].
A Weyl simplex in $\mathbb{R}^n$ is an intersection of Weyl chambers. The Weyl simplices in $\mathbb{R}^n$, ordered by inclusion, form a simplicial complex which is isomorphic to the Coxeter complex $\Sigma$ of $W$. The images of the Weyl simplices in $X$ under the charts are also called Weyl simplices; the image of the origin in $\mathbb{R}^n$ is called the base point of the Weyl simplex.

The Weyl simplices lead to spherical buildings in two different ways; the first captures the asymptotic geometry of $X$, while the second is an ‘infinitesimal’ version of the Euclidean building, similar to the tangent space of a Riemannian manifold. The first will be considered now and the second in Section 10.

7.6 The spherical building at infinity We call two Weyl simplices $a, a' \subseteq X$ equivalent if they have finite Hausdorff distance. The equivalence class of $a$ is denoted $\partial a$. The preorder $\subseteq_{Hd}$ defined in [4.3] induces a partial order on these equivalence classes. Let $\partial_A X$ denote the set of all equivalence classes of Weyl simplices, partially ordered by domination $\subseteq_{Hd}$. For every affine apartment $A$, the poset $\partial A$ consisting of the Weyl simplices contained in $A$ may be viewed as a sub-poset of $\partial_A X$.

7.7 Proposition The poset $\partial_A X$ is a spherical building. The map $A \mapsto \partial A$ is a one-to-one correspondence between the affine apartments in $X$ and the apartments of the spherical building $\partial_A X$.

Proof. See eg. Parreau [22, 1.5]. □

7.8 The maximal atlas The spherical building at infinity depends very much on the chosen set of charts $A$. Similarly as for differentiable manifolds, a Euclidean building admits a unique maximal atlas $\hat{A}$ which is characterized by the following property: every subspace $A \subseteq X$ which is isometric to $\mathbb{R}^n$ is an affine apartment in $\hat{A}$ [22, 2.6]. The set $\hat{A}$ is called the complete apartment system. We denote the spherical building at infinity which corresponds to the complete apartment system by $\partial_{cpl} X$.

7.9 Product decompositions and reductions If the Coxeter group $W$ is reducible, then $X$ decomposes as a metric product $X = X_1 \times X_2$ of Euclidean buildings, with $\partial_A X_1 \star \partial_A X_2 = \partial_A X$ [22, 2.1]. If $S^k \star B_0 = \partial_A X$ is the thick reduction of $\partial_A X$ and if $W_0$ is the Coxeter group of $B_0$, then there is a Euclidean building $X_0$ with affine Weyl group $W_0 \mathbb{R}^m$ and an isometry $X \cong \mathbb{R}^{n-m} \times X_0$, with $\partial_{A_n} X_0 = B_0$ and $k = n - m - 1$ [15, 4.9].

8 Coarse equivalences induce isomorphisms at infinity

In the present section we prove that a coarse equivalence of Euclidean buildings induces an isomorphism between the spherical buildings at infinity. In order to show this, we have to describe the spherical building at infinity by coarse data.

We will use the following fact. Euclidean buildings are CAT(0) spaces [6, 3.2] [22, 2.9]. If $Y$ is a closed convex subset of a metrically complete CAT(0)-space, then there is a retraction map $\pi_Y : X \longrightarrow Y$ which is 1-Lipschitz [4, II.2.4].

8.1 Proposition Let $A_1, \ldots, A_k$ be affine apartments in a Euclidean building $X$. The following are equivalent.

(i) $\partial A_1 \cap \cdots \cap \partial A_k \neq \emptyset$.

(ii) There is an unbounded set which is dominated by $A_1, \ldots, A_k$.

Proof. To see that (i) implies (ii), let $a \subseteq X$ be a Weyl simplex representing an element $\partial a \in \partial A_1 \cap \cdots \cap \partial A_k$. For each $j$ there is a Weyl simplex $a_j \subseteq A_j$ representing $\partial a$. Since $a$ and $a_j$ have finite Hausdorff distance, each $A_j$ dominates $a$.

Before we prove the converse implication, we note the following. If an affine apartment $A$ dominates $Z$, then $\pi_A Z$ has finite Hausdorff distance from $Z$. Now we assume that the unbounded set $Z$ is dominated by $A_1, \ldots, A_k$. Consider the unbounded set $Y = \pi_{A_1} Z$. Since $A_1$ is a finite union of Weyl simplices, there is a Weyl simplex $a \subseteq A_1$ of minimal dimension which dominates an unbounded subset $Y_1 \subseteq Y$. We claim that $\partial a \in \partial A_1 \cap \cdots \cap \partial A_k$.

Let $j > 1$. In $A_j$ we find a Weyl chamber $c_j$ which dominates an unbounded subset $Y_j$ of $\pi_{A_j} (Y_1)$. Let $A'_j$ be an affine apartment containing representatives $a'$ and $c'_j$ of $\partial a$ and $\partial c_j$. Then $Y'_j = \pi_{A'_j} (Y_j)$ has finite Hausdorff distance from $Y_j$. Since both $a'$ and $c'_j$ dominate the unbounded set $Y'_j$, and since $a'$ is also a Weyl simplex of minimal dimension which dominates an unbounded subset of $Y$, we have $\partial a' \subseteq \partial c_j \in \partial A_j$. □
The fact that a coarse equivalence preserves the apartment complex depends on the following fundamental result due to Kleiner-Leeb [15, 7.1.5]. It is the higher dimensional analog of Morse’s Lemma; see Section 5.

8.2 Theorem Let \( f : X_1 \longrightarrow X_2 \) be a coarse equivalence between metrically complete Euclidean buildings, both endowed with the maximal apartment systems. Then \( \dim X_1 = \dim X_2 \) and there is a constant \( r_f \) such that the following holds. For every affine apartment \( A \subseteq X \), the image \( f(A) \) has Hausdorff distance at most \( r_f \) from a unique affine apartment \( A' \subseteq X_2 \).

Suppose that \( X_1 \) and \( X_2 \) are Euclidean buildings and that \( f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2} \) is a coarse equivalence. We may view \( \mathbb{R}^{m_1} \) as an affine building. Then every affine apartment in \( X_1 \times \mathbb{R}^{m_1} \) is of the form \( A \times \mathbb{R}^{m_1} \), where \( A \subseteq X_1 \) is an affine apartment. By \( \text{8.2} \) there is a map \( f_* \) from the affine apartments of \( X_1 \) to the affine apartments of \( X_2 \), such that \( f(A \times \mathbb{R}^{m_1}) \) has finite Hausdorff distance from \( f_*A \times \mathbb{R}^{m_2} \). If \( g \) is a coarse inverse of \( f \), then \( g_* \) is an inverse of \( f_* \).

8.3 Proposition Suppose that \( X_1 \) and \( X_2 \) are metrically complete Euclidean buildings. Let \( f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^m \) be a coarse equivalence. Then \( f_* \) induces an isomorphism between the apartment complexes \( AC(\partial_{cpl}X_1) \) and \( AC(\partial_{cpl}X_2) \).

Proof. Let \( A_1, \ldots, A_k \subseteq X_1 \) be affine apartments with \( \partial A_1 \cap \cdots \cap \partial A_k \neq \emptyset \). Let \( a \) be a 1-dimensional Weyl simplex representing a vertex \( \partial a \in \partial A_1 \cap \cdots \cap \partial A_k \). Choose \( s > 0 \) such that \( a \subseteq B_s(A_1) \cap \cdots \cap B_s(A_k) \). Let \( r_f > 0 \) be as in \( \text{8.2} \) and let \( Y = B_{r_f + \rho(s)}(f_*A_1) \cap \cdots \cap B_{r_f + \rho(s)}(f_*A_k) \), where \( \rho \) is the control function for \( f \). Then \( f \) restricts to a quasi-isometric embedding of \( a \times \mathbb{R}^m \) into \( Y \times \mathbb{R}^m \). If \( Y \) is bounded, we get a quasi-isometric embedding of \( a \times \mathbb{R}^m \) into \( \mathbb{R}^m \), which is impossible (by topological dimension invariance, applied to the asymptotic cones). So \( Y \) has to be unbounded. Therefore \( \{ f_*A_1, \ldots, f_*A_k \} \) is a simplex in \( AC(\partial_{cpl}X_2) \) by \( \text{8.1} \). It follows that \( f_* \) is a simplicial map \( f_* : AC(\partial_{cpl}X_1) \longrightarrow AC(\partial_{cpl}X_2) \). If \( g \) is a coarse inverse of \( f \), then \( g_* \) is a simplicial inverse of \( f_* \).

The next result is proved in the appendix. [13.4]

8.4 Proposition Let \( X_1 \) and \( X_2 \) be metrically complete Euclidean buildings. Let \( f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2} \) be a coarse equivalence. If \( \partial_{cpl}X_1 \) and \( \partial_{cpl}X_2 \) are thick, then \( m_1 = m_2 \).

Combining these results, we have the following first main result about coarse equivalences between Euclidean buildings. This is Theorem II in the introduction.

8.5 Theorem Let \( X_1 \) and \( X_2 \) be metrically complete Euclidean buildings whose spherical buildings at infinity \( \partial_{cpl}X_1 \) and \( \partial_{cpl}X_2 \) are thick. Let \( f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2} \) be a coarse equivalence. Then \( m_1 = m_2 \) and the map \( f_* \) on the affine apartments extends uniquely to a simplicial isomorphism \( f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2 \).

Proof. By \( \text{8.4} \) we have \( m_1 = m_2 \). By \( \text{8.3} \) the map \( f_* \) induces a simplicial isomorphism between \( AC(\partial_{cpl}X_1) \) and \( AC(\partial_{cpl}X_2) \). By \( \text{6.10} \) \( f_* \) induces a simplicial isomorphism \( f_* : \partial_{cpl}X_1 \longrightarrow \partial_{cpl}X_2 \).

The thickness of the spherical buildings is essential for the argument. However, the Euclidean factors which are allowed in the theorem lead also to a result for the case that the spherical buildings at infinity are weak buildings.

8.6 Corollary Let \( f : X_1 \longrightarrow X_2 \) be a coarse equivalence between metrically complete Euclidean buildings. Then the induced map on the affine apartments induces an isomorphism between the thick building factors in the reductions of \( \partial_{cpl}X_1 \) and \( \partial_{cpl}X_2 \).

Proof. This follows from \( \text{7.49} \) and \( \text{8.5} \).

Note that we do not claim that \( f \) induces directly a map \( \partial f : \partial X_1 \longrightarrow \partial X_2 \) between the boundaries in the sense of CAT(0) geometry. This will in general not be the case; for example, \( f \) could be a bilipschitz homeomorphism of the Euclidean cone \( EB(B) \) over a thick spherical building \( B \). Such a self homeomorphism can be rather wild at infinity. Our construction of \( f_* \) applies nevertheless.
9 The projectivity groupoid of a spherical building

The proof of Theorem III relies on a combination of Theorem I and Theorem II. In order to obtain trees with large automorphism groups we consider projectivities in buildings. For projective planes, this is a classical technique for producing 2-transitive groups.

9.1 Projections in buildings Let \( c \) be a chamber and \( a \) a simplex in a building. Then there is a unique chamber \( d \) in \( \text{Res}(a) \) which has minimal distance from \( c \) (with respect to the distance in the chamber graph), and which is denoted \( d = \text{proj}_a c \). If \( b \) is a simplex then \( \text{proj}_a b \) is defined to be the simplicial intersection of the chambers \( \text{proj}_a c \), where \( c \) ranges over all chambers containing \( b \).

Two simplices \( a, b \) in a spherical Coxeter complex \( A \) are called opposite if they are interchanged by the antipodal map (the opposition involution) of the sphere \(|A|\). In a spherical building, two simplices are called opposite if they are opposite in some (hence every) apartment containing them. If \( a, b \) are opposite, then \( \text{proj}_a : \text{Res}(b) \rightarrow \text{Res}(a) \) is a simplicial isomorphism. The following observations are essentially due to N. Knarr and J. Tits. They were rediscovered by Leeb.

9.2 Perspectives Let \( B \) be a spherical building. We remarked already that if \( a, b \) are opposite simplices in \( B \), then \( \text{proj}_a : \text{Res}(a) \rightarrow \text{Res}(b) \) is a building isomorphism (not necessarily type preserving) between \( \text{Res}(a) \) and \( \text{Res}(b) \). We denote this isomorphism by \([b; a] : \text{Res}(a) \rightarrow \text{Res}(b)\) and call it a projectivity. A concatenation of projectivities is called a projectivity; we write \([c; b] \circ [b; a] = [c; a] : \text{Res}(a) \rightarrow \text{Res}(c)\) etc. The inverse of \([b; a] \) is \([a; b] \). A projectivity is called even if it can be written as a composition of an even number of projectivities.

9.3 Projectivities Recall that a groupoid is a small category where every arrow is an isomorphism. The projectivity groupoid \( \Pi_B \) is the category whose objects are the simplices of \( B \), and whose morphisms are projectivities. It is closely related to the opposition graph \( \text{Opp}(B) \) whose vertices are the simplices of \( B \) and whose edges are unordered pairs of opposite simplices. Every simplicial path in \( \text{Opp}(B) \) induces a projectivity. We denote by \( \Pi_B(a) = \text{Hom}_\Pi(a, a) \) the group of all automorphisms of \( \text{Res}(a) \) induced by maps in \( \Pi_B \). The subgroup \( \Pi_B(a)^+ \) consisting of all even projectivities is a normal subgroup of index 1 or 2. If \( f : B_1 \rightarrow B_2 \) is an isomorphism of spherical buildings, then \( f \) induces an isomorphism between \( \Pi_{B_1} \) and \( \Pi_{B_2} \) in the obvious way.

The following result is due to N. Knarr. We use several facts about galleries and distances which can all be found in [16].

9.4 Theorem Suppose that \( B \) is a thick spherical building and that \( r \) is an \( i \)-panel. If \( i \) is not an isolated node in the Coxeter diagram of \( B \), then \( \Pi_B(r)^+ \) is a 2-transitive permutation group on \( R = \text{Res}(r) \).

Proof. Let \( a, b, b' \) be three distinct chambers in \( R \). We construct a projectivity which fixes \( a \) and maps \( b \) to \( b' \). Let \( j \) denote the type of a neighboring node of \( i \), i.e. \( ij \neq ji \) in \( W \). We choose a nonstammering gallery \( b \rightarrow c \rightarrow d \) in \( B \) (the superscripts indicate the types of panels in the gallery). Since \( ij \neq ji \), this gallery is minimal. Therefore it is contained in some apartment \( A \subseteq B \). Let \( s \) be the panel in \( A \) opposite to the \( j \)-panel \( a \cap c \). Let \( e \) be a chamber in \( \text{Res}(s) \) which is not in \( A \) (here we use that \( B \) is thick). Then \( e \) is opposite to \( a \) and \( c \). There is a unique panel \( t \subseteq e \) which is opposite both to \( r \) and to the panel \( q = c \cap d \). Since \( b \) is not opposite \( e \), \( \text{proj}_b e = e \). Similarly \( \text{proj}_d e = e \), whence \([q; t; r](b) = d\). We claim that \( \text{proj}_j(a) = \text{proj}_j(c) \). Assuming that for the moment that this is true, we have \([t; r](a) = [t; q](c)\), whence \([g; t; r](a) = c\). Applying the same construction to \( b' \rightarrow a \rightarrow c \rightarrow d \), with a second apartment \( A' \) and a panel \( t' \), we obtain the projectivity \([r; t'; q; t; r]\) with the required properties.

It remains to show that \( \text{proj}_j(a) = \text{proj}_j(c) \). Let \( A'' \) denote the apartment spanned by the opposite chambers \( a \) and \( e \). Let \( f \) denote the chamber in \( \text{Res}(t) \cap A'' \) different from \( e \) and let \( k \) denote the gallery distance between \( a \) and \( e \). Then \( a \) and \( f \) have gallery distance \( k - 1 \). Since \( t \) and \( p \) are not opposite, \( g = \text{proj}_t f \) has gallery distance \( k - 2 \) from \( f \). So there is a gallery \((f, \ldots, g, a)\) of length \( k - 1 \), which is therefore minimal. It follows that \((f, \ldots, g, c)\) is also a minimal gallery, and \( f \) has gallery distance \( k - 1 \) from \( c \). Since \( t \) is opposite to \( r \) and \( q \), we have \( \text{proj}_t(a) = f = \text{proj}_t(c) \). \( \square \)
The following construction will become relevant in Section 11. Let \( a, b \) be opposite panels in \( B \) and let \( B(a, b) \) denote the union of all apartments containing \( a \) and \( b \). For a chamber \( c \in B \), let \( c_a = \text{proj}_a c \) and \( c_b = \text{proj}_b c \). If \( A \) is an apartment containing \( a \) and \( b \) and \( c \in A \) is a chamber, then \( c_a, c_b \in A \) and \( \text{proj}_a c_b = c_a \). In particular, each pair of distinct chambers \( c, d \in \text{Res}(a) \) determines a unique apartment in \( B(a, b) \) containing \( a \) and \( b \) and the chambers in \( \text{Res}(a) \) correspond bijectively to the half apartments of \( B(a, b) \) having \( a \) and \( b \) as boundary panels.

9.5 Lemma The subcomplex \( B(a, b) \subseteq B \) is a weak building of the same type as \( B \) and every apartment of this weak building contains \( a \) and \( b \). If \( b' \) is another panel opposite \( a \), then there is a unique simplicial isomorphism \( B(a, b) \longrightarrow B(a, b') \) which fixes \( B(a, b) \cap B(a, b') \).

Proof. Let \( c, d \) be chambers in \( B(a, b) \). Then there exists an apartment \( A \) containing \( a, b, c_a \) and \( d_a \). It follows that \( A \) contains \( c_b \) and \( d_b \), and therefore \( c \) and \( d \). This shows that \( B(a, b) \) is a building and that every apartment of this building contains \( a \) and \( b \).

Suppose now that \( b' \) is also opposite \( a \). For every apartment \( A \) containing \( a, b \), there is a unique apartment \( A' \) containing \( a, b' \), such that \( A \cap \text{Res}(a) = A' \cap \text{Res}(a) \). Since \( A \cap A' \) contains chambers, there is a unique isomorphism \( A \rightarrow A' \) fixing \( A \cap A' \). For \( c \in A \), the image \( c' \in A' \) can be described as follows. It is the unique chamber which has the same \( W \)-valued distances from the two chambers \( c_a \) and \( \text{proj}_a c_b \), as \( c \) has from \( a \) and \( \text{proj}_a c_c \). This description is independent of \( A \) and \( A' \) and shows that we obtain a well-defined map on the chambers. This map is adjacency-preserving on each apartment and hence a building isomorphism. □

9.6 Suppose that \( a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_k \) is a path in the opposition graph. By the previous lemma, we obtain a sequence of canonical isomorphisms \( B(a_0, a_1) \cong B(a_1, a_2) \cong \cdots B(a_{k-1}, a_k) \) fixing the intersections of consecutive buildings. If \( a_0 = a_k \), then the composite is an automorphism of \( B(a_0, a_1) \). If \( k \) is even, then this automorphism fixes \( a \) and its restriction to \( \text{Res}(a) \) coincides with the projectivity \( [a_k; \ldots; a_0] \).

10 The local structure of Euclidean buildings

Now we get back to Euclidean buildings. Each Weyl simplex in a Euclidean building \( X \) comes with a preferred base point, the image \( p = \varphi(0) \) of the the origin under the corresponding chart \( \varphi : \mathbb{R}^n \longrightarrow X \). We call such a Weyl simplex \( p \)-based.

10.1 The residue at a point Let \( p \) be a point in \( X \). Two \( p \)-based Weyl simplices \( a, b \) have equal \( p \)-germs if \( B_r(p) \cap a = B_r(p) \cap b \) holds for some \( r > 0 \). The corresponding equivalence classes form a spherical building \( X_p \) which we call the residue of \( X \) at \( p \); see [22] 1.6]. The Coxeter group of \( X_p \) is \( W \). We call \( p \) thick if the residue \( X_p \) is thick. If \( a \subseteq X \) is a Weyl simplex, then \( \partial a \) has a unique representative which is a \( p \)-based Weyl simplex [22 Cor. 1.9]. We thus obtain a nondegenerate simplicial epimorphism \( \partial A X \longrightarrow X_p \), and a canonical \( 1 \)-Lipschitz map \( EB(\partial A X) \longrightarrow X \). The latter map is surjective and maps the affine apartments of \( EB(\partial A X) \) isometrically onto the affine apartments of \( X \) containing \( p \).

We call an affine wall \( M \subseteq X \) thick if \( M \) is the intersection of three affine apartments. We begin with a local thickness criterion for walls.

10.2 Lemma Let \( A \subseteq X \) be an affine apartment, let \( p \in A \) and let \( M \subseteq A \) be a wall containing \( p \). Let \( r \in X_p \) be a panel in the wall determined by \( M \) in \( X_p \). If \( r \) is contained in three distinct chambers of \( X_p \), then \( M \) is thick.

Proof. Let \( A_p \) denote the apartment induced by \( A \) in \( X_p \), let \( s \in A_p \) be the panel opposite \( r \) and let \( a, b \in A_p \) be the two chambers containing \( s \). We represent \( a, b, r, s \) by \( p \)-based Weyl simplices in \( A \). By assumption, there is a chamber containing \( r \) which is opposite both to \( a \) and to \( b \). By 6.8 we can find a \( p \)-based Weyl chamber \( c \) representing this chamber, such that \( \partial c \) is adjacent to \( \partial A \). Since the panel \( \partial c \cap \partial A \) has a unique \( p \)-based representative, this representative is contained in \( M \). It follows that \( c \cap A \subseteq M \). Let \( A' \) denote the affine apartment spanned by \( c \) and \( a \); see [22 1.12]. Then \( A \cap A' \) is a half space whose boundary is \( M \) and therefore \( M \) is thick. □
10.3 Lemma Let $p$ be a point of an affine apartment $A \subseteq X$. Then $p$ is thick if and only if every wall of $A$ containing $p$ is thick.

Proof. If $p$ is thick, then every wall through $p$ is thick by [10.2]. Conversely, if every wall of $A$ containing $p$ is thick, then all panels in $A_p$ are contained in at least 3 chambers. Thus, $p$ is thick by [6.6]. □

10.4 Lemma Let $A$ be an affine apartment and assume that $L, M \subseteq A$ are non parallel thick walls. Then the reflection of $L$ along $M$ in $A$ is also thick.

Proof. Let $A_\pm \subseteq A$ denote the half spaces bounded by $M$ and let $A_0 \subseteq X$ be a third half space with $A \cap A_0 = M$. There is a unique affine wall $L'$ in the affine apartment $A' = A_+ \cup A_0$ extending $L \cap A_+$. By [10.2], $L'$ is thick. Now let $L''$ be the wall in $A_0 \cup A_-$ which extends $L' \cap A_0$. Again by [10.2], $L''$ is thick. Finally, let $L'''$ be the wall in $A$ which extends $L'' \cap A_-$. A third application of [10.2] yields that $L'''$ is thick. But $L'''$ is precisely the reflection of $L$ along $M$. □

10.5 Lemma The Euclidean building $X$ contains a thick point if and only if $\partial_A X$ is thick.

Proof. By [6.8], thickness of $X_p$ implies thickness of $\partial_A X$. Now suppose that $\partial_A X$ is thick, let $A$ be an affine apartment, let $c$ be a Weyl chamber of $A$ and let $M_1, \ldots, M_n$ be the walls of $A$ bounding $c$. Since $\partial_A X$ is thick, we can choose thick walls $M'_1, \ldots, M'_n$ in $A$ such that $M'_i$ is parallel to $M_i$ for each $i$. The intersection of the thick walls $M'_1, \ldots, M'_n$ contains a point $p$. By [10.3] and [10.4], $p$ is thick. □

10.6 Proposition Suppose that $X$ is an affine building of dimension $n \geq 2$ and that $\partial_A X$ is irreducible and thick. Let $th(X) \subseteq X$ denote the set of thick points. There are the following three possibilities.

(I) There is a unique thick point which is contained in every affine apartment of $X$.

(II) The set of thick points is a closed, discrete and cobounded subset in $X$ and in every apartment of $X$.

(III) The set of thick points is dense in $X$ and in every apartment of $X$.

Proof. We noted in [10.5] that $th(X) \neq \emptyset$. For an affine apartment $A \subseteq X$ we denote by $R(A) \subseteq I(A)$ the group generated by reflections along the thick walls in $A$. If $p$ is a thick point, then $R(A)_p \cong W$. We start with two observations.

(i) Suppose that $p$ is a thick point in an affine apartment $A$, and that $M \subseteq A$ is a thick wall not containing $p$. Such an apartment $A$ exists if $X$ contains at least two thick points. Since the Weyl group $W$ is irreducible and $\dim(A) \geq 2$, the point $p$ is the intersection of thick walls in $A$ which are not parallel to $M$. It follows from [10.2] that the reflection of $p$ along $M$ in $A$ is again a thick point, so the $R(A)$-orbit of $p$ consists of thick points.

(ii) For any two affine apartments $A, A'$, there is a sequence of affine apartments $A = A_0, \ldots, A_k = A'$ such that $A_i \cap A_{i+1}$ is a half space. (Using galleries, this is easily seen to be true for apartments in the spherical building at infinity.) It is clear that the sequence of walls determined by these half spaces is thick.

Suppose first that $th(X) = \{p\}$ consists of a single point and that $A$ is an affine apartment containing $p$. Then (i) shows that the thick walls in $A$ are precisely the ones passing through $p$. If $A'$ is any other affine apartment, then (ii) and a simple induction on $k$ show that all apartments $A_0, \ldots, A_k$ in the sequence connecting $A$ and $A'$ contain $p$. This is case (I).

Suppose now that $th(X)$ contains at least two points $p, q$ and that $A$ is an apartment containing $p$ and $q$. Then the $R(A)$-orbit of $q$ consists of thick points and is cobounded in $A$, because the Weyl group $W$ is irreducible, and because $R(A)$ contains translations. Let $T \subseteq R(A)$ denote the translational part (i.e., the kernel of action of $R(A)$ on $\partial A$). Any closed subgroup of the vector group $\langle \mathbb{R}^n, +\rangle$ is a product of a finite rank free abelian group and a vector group. If $T$ is discrete, then $R(A)$ is an affine reflection group. If $T$ is not discrete, then the closure of $T$ consists of all translations in $A$ (because $W$ acts irreducibly on the set of all translations), so $th(X) \cap A$ is dense.

If $A'$ is any other affine apartment having a half space in common with $A$, then the thick walls in $A$ propagate to thick walls in $A'$. The isometry $A \longrightarrow A'$ also preserves thick points. We obtain a canonical isomorphism $R(A) \cong R(A')$. From (ii) we conclude that the isometry groups $R(A)$ and $R(A')$ are isomorphic for any affine apartment $A'' \subseteq X$, and that this isomorphism maps $th(X) \cap A$ onto $th(X) \cap A''$. In the discrete case, we have (II), whereas the nondiscrete case is (III). □

10.7 Corollary Let $X$ be as in [10.6] (I). Then $X \cong EB(\partial_A X)$ is a Euclidean cone.
Proof. Let \( p \in X \) be the unique thick point. We noted in \([10.4]\) that the map \( EB(\partial_A X) \rightarrow X \) maps the affine apartments of \( EB(\partial_A X) \) isometrically onto the affine apartments of \( X \) containing \( p \). Since every affine apartment of \( X \) contains \( p \), this map is an isometry. \( \square \)

The following observation will not be used, but it illustrates how simplicial affine buildings fit into the picture.

10.8 Corollary Let \( X \) be as in \([10.7]\) (II). Then \( X \) is the metric realization of an affine simplicial building.

Proof. The groups \( R(A) \) are affine Coxeter groups; see \([5\text{ Ch.Vi}]\). In this way, every apartment has a canonical simplicial structure as a Coxeter complex. The axioms of an affine simplicial building follow from \([7.2]\) \( \square \)

In the remaining case (III), \( X \) is a nondiscrete Euclidean building.

11 Panel trees and ecological boundary homomorphisms

In general, a Euclidean building is not determined by its spherical building at infinity. Some additional data are needed, which are encoded in the panel trees. The material in this section is due to Tits \([30]\). A proof of \([11.3]\) with all details filled in can be found in \([13]\).

11.1 Wall trees and panel trees Let \( (X, A) \) be a Euclidean building and let \( a, b \) be a pair of opposite panels in the spherical building at infinity. Let \( X(a, b) \) denote the union over all affine apartments in \( A \) whose boundary contains \( a \) and \( b \). Then \( X(a, b) \) is a Euclidean building, which factors metrically as \( X(a, b) = \mathbb{T} \times \mathbb{R}^{n-1} \), where \( \mathbb{T} \) is a leafless tree; see \([15\text{ 4.8.1}], [27\text{ 3.9}] [30]\). We call this tree the wall tree associated to \((a, b)\), because it depends really only on the unique wall of \( \partial_A X \) containing \( a \) and \( b \). In the notation of Section \([9]\) the spherical building at infinity of \( X(a, b) \) is \((\partial_A X)(a, b); \) see \([9.5]\)

11.2 Lemma If \( b' \) is another panel opposite \( a \), then there is a unique isometry \( X(a, b) \rightarrow X(a, b') \) which fixes \( X(a, b) \cap X(a, b') \) pointwise.

Proof. Let \( A \) be an affine apartment whose boundary contains \( a, b \), and let \( A' \) be the corresponding affine apartment whose boundary contains \( \text{Res}(a) \cap \partial A \) and \( b' \); see \([9.5]\). Since \( A \cap A' \) contains Weyl chambers, there is a unique isometry \( A \rightarrow A' \) fixing \( A \cap A' \). This proves the uniqueness of the isometry. For the existence, we show that these isometries \( A \rightarrow A' \) of the individual affine apartments fit together.

Let \( A_1, A_2 \subseteq X(a, b) \) be two affine apartments containing a point \( x \in A_1 \cap A_2 \). Correspondingly, we have chambers \( c_i, d_i \in \text{Res}(a) \) such that \( \text{Res}(a) \ni x, \text{span} \{c_i \} = \text{span} \{d_i \} \). If two of these four chambers coincide, say \( c_1 = c_2 \), then \( A_1 \) and \( A_2 \) have the same \( x \)-based Weyl chamber \( c \) representing \( c_1 \) in common. Then \( A_1, A_2, A_1' \) and \( A_2' \) have a sub-Weyl chamber of \( c \) in common, and therefore the two isometries \( A_1 \leq A_1' \) and \( A_2 \leq A_2' \) map \( x \) to the same point \( x' \).

If \( c_1, c_2, d_1, d_2 \) are pairwise different, then \( x \) is also in the affine apartment determined by \( c_1, d_2 \) or \( c_1, c_2 \), because \( X(a, b) \) is a product of a tree and Euclidean space. The previous argument, applied twice, shows that the various apartment isometries coincide on \( x \). Thus we have a well-defined bijection which is apartment-wise an isometry. Since any two points are in some affine apartment, the map is an isometry. \( \square \)

11.3 If \( b' \) varies over the panels opposite \( a \), we obtain a family of trees which are pairwise canonically isomorphic. This canonical isomorphism type of a tree is the panel tree \( T_a \) associated to \( a \).

If \( a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_k \) is a path in the opposition graph, then the isomorphisms \( (\partial_A X)(a_0, a_1) \cong (\partial_A X)(a_1, a_2) \cong \cdots \) from \([9.5]\) are accompanied by isometries \( X(a_0, a_1) \cong X(a_1, a_2) \cong \cdots \). In particular, the group of projectivities \( \Pi_{\partial_A X}(a) \) acts on the wall tree and on the panel tree \( T_a \). If we restrict to even projectivities, the action on the Euclidean factor is trivial. If the building is thick and the type of the panel \( a \) is not isolated, then the action of \( \Pi_{\partial_A X}(a) \) on the ends of \( T_a \) is 2-transitive by \([7.4]\).

The branch points in the panel trees correspond to the thick walls in \( X \). Assuming that \( \partial_A X \) is irreducible, we have the following consequence of \([10.3]\): If one panel tree \( T_a \) is of type (I), then every panel tree is of type (I) and \( X \) is an Euclidean cone over \( \partial_A X \) as in \([7.3]\). If one panel tree is of type (II), then every panel tree is of type (II) and \( X \) is simplicial. The remaining possibility is that every panel tree is of type (III).
11.4 Ecological boundary isomorphisms Let $X_1, X_2$ be irreducible Euclidean buildings. Assume that $\partial_{\text{cpl}} X_1$ and $\partial_{\text{cpl}} X_2$ are thick and that $\varphi : \partial_{\text{cpl}} X_1 \longrightarrow \partial_{\text{cpl}} X_2$ is an isomorphism. Assume moreover that for every panel $a \in \partial_{\text{cpl}} X_1$, there is a tree isometry $\varphi_a : T_a \longrightarrow T_{\varphi(a)}$. If for each panel $a$, the map $\partial \varphi_a : \partial T_a \longrightarrow \partial T_{\varphi(a)}$ coincides with the restriction of $\varphi$ to $\text{Res}(a)$, then $\varphi$ is called tree-preserving or ecological. The following is \[30\] Thm. 2]

11.5 Theorem (Tits) Let $(X_1, A_1)$ and $(X_2, A_2)$ be Euclidean buildings. Assume that $\partial_{\text{cpl}} X_1$ is thick and irreducible. If $\varphi : \partial_{\text{cpl}} X_1 \longrightarrow \partial_{\text{cpl}} X_2$ is an ecological isomorphism, then $\varphi$ extends to an isomorphism $\Phi : (X_1, A_1) \longrightarrow (X_2, A_2)$. \hfill \Box

The irreducibility is actually not important for the proof, but the result is stated in this way in \[30\]. For our purposes, the irreducible version suffices.

12 Coarse equivalences are ecological

In this section we assume that $X_1$ and $X_2$ are metrically complete Euclidean buildings whose spherical buildings at infinity $\partial_{\text{cpl}} X_1$ and $\partial_{\text{cpl}} X_2$ are thick. Furthermore, we assume that $f : X_1 \times \mathbb{R}^m \longrightarrow X_2 \times \mathbb{R}^n$ is a coarse equivalence. By 8.3, $f$ induces an isomorphism $\pi_{*} : \partial_{\text{cpl}} X_1 \longrightarrow \partial_{\text{cpl}} X_2$ which is characterized by the fact that for every affine apartment $A \subseteq X_1$, the image $f(A \times \mathbb{R}^n)$ has finite Hausdorff distance from $f_* A \times \mathbb{R}^n$. We denote the dimension of $X_1$ and $X_2$ by $n$. The Euclidean factors also have the same dimension, which we denote by $m = m_1 = m_2$. We put $d(x \times p) = f_1(x \times p) \times f_2(x \times p)$.

Recall from Section 8 that for a closed convex subset $Z$ in a metrically complete CAT(0) space, $\pi_Z$ denotes the nearest point projection onto $Z$. This map is $1$-Lipschitz.

Let $a, b \in \partial_{\text{cpl}} X_1$ be opposite panels. As in Section 11, we denote by $X_1(a, b)$ the union of all affine apartments in $X_1$ whose boundary contains $a$ and $b$. Consider the convex subsets $Y_1 = X_1(a, b) \times \mathbb{R}^m \subseteq X_1 \times \mathbb{R}^m$ and $Y_2 = X_2(f_* a, f_* b) \times \mathbb{R}^m \subseteq X_2 \times \mathbb{R}^m$. The composite $\pi_{Y_2} \circ f$ is a controlled map. If $A \subseteq X_1(a, b)$ is an affine apartment, then $f_* A$ is an affine apartment in $X_2(f_* a, f_* b)$; see 8.3. By 8.2 there is a uniform constant $r_f > 0$ such that $f(y) \pi_{Y_2} f(y) \leq r_f$ for all $y \in Y_1$, so $f Y_1$ and $\pi_{Y_2} \circ f Y_1$ have finite distance. If $g$ is a coarse inverse of $f$, then $\pi_{Y_1} \circ g|_{Y_1}$ is therefore a coarse inverse of $\pi_{Y_2} \circ f|_{Y_2}$, and we obtain a coarse equivalence $Y_1 \longrightarrow Y_2$. By 11.11 $Y_i$ factors for $i = 1, 2$ as a metric product $T_i \times \mathbb{R}^{n-1+m}$ of a tree $T_i$ and Euclidean space.

The ends of the wall tree $T_1$ correspond bijectively to the chambers of $\partial_{\text{cpl}} X_1$ containing the panel $a$. If the type of the panel $a$ is not isolated in the Coxeter diagram of $\partial_{\text{cpl}} X_1$, then $\Pi^+_{\partial_{\text{cpl}} X_1}(a)$ acts 2-transitively on the ends of $T_1$. The boundary map $f_* : \partial_{\text{cpl}} X_1 \longrightarrow \partial_{\text{cpl}} X_2$ is clearly equivariant with respect to the isomorphism $\Pi^+_{\partial_{\text{cpl}} X_1}(a) \longrightarrow \Pi^+_{\partial_{\text{cpl}} X_2}(f_* a)$. In this situation we may apply 5.3 and conclude that $f_*$ extends to an isometry from $T_1$ to $T_2$, possibly after rescaling.

12.1 Proposition If $X_1$ is irreducible, of dimension $n \geq 2$ and if some wall tree of $X_1$ is of type (I), then there is an isometry $\hat{f} : X_1 \longrightarrow X_2$ with $\hat{f}_* = f_*$. \hfill \Box

Proof. If the wall tree $T_1$ is of type (I), then $X_1 = EB(\partial_{\text{cpl}} X_1)$; see 11.3. Since $\partial_{\text{cpl}} X_1$ is thick and irreducible, $\Pi^+_{\partial_{\text{cpl}} X_1}(a)$ acts 2-transitively on the ends of $T_1$ by 9.1. By 5.3 the wall trees of $X_2$ are also of type (I). Therefore $X_2 = EB(\partial_{\text{cpl}} X_2)$. The isomorphism $f_* : \partial_{\text{cpl}} X_1 \longrightarrow \partial_{\text{cpl}} X_2$ extends to an isometry $\hat{f} : X_1 \longrightarrow X_2$ of the respective Coxeter cones, i.e. $f_* = \hat{f}_*$. \hfill \Box

12.2 Proposition If $X_1$ is irreducible, of dimension $n \geq 2$ and if no wall tree is of type (I), then the metric on $X_2$ can be rescaled so that there is an isometry $\hat{f} : X_1 \longrightarrow X_2$ with $\hat{f}_* = f_*$. This isometry $\hat{f}$ is unique and there is a constant $r > 0$ such that $d(f_1(x \times p), \hat{f}(x)) \leq r$ for all $x \times p \in X_1 \times \mathbb{R}^m$.

Proof. Let $(a, b)$ be a pair of opposite panels as in the introduction of this section. Then $\Pi^+_{\partial_{\text{cpl}} X_1}(a)$ acts 2-transitively on the ends of $T_1$ by 9.1. We may apply 5.3 to the coarse equivalence $T_1 \times \mathbb{R}^{n-1+m} \longrightarrow T_2 \times \mathbb{R}^{n-1+m}$, since the equivariance condition is satisfied by our previous discussion. Once and for all, we rescale the metric on $X_2$ in such a way that $f_*$ extends to an equivariant isometry $\tau : T_1 \longrightarrow T_2$.

If $(a', b')$ is any other pair of panels in $\partial_{\text{cpl}} X_1$ of the same type as $(a, b)$, with $X_1(a', b') = T_1' \times \mathbb{R}^{n-1}$, then there is some projectivity which induces an isometry $\varphi_1 : T_1 \longrightarrow T_1'$. Pushing this projectivity forward via $f_*$, we obtain an isometry $\varphi_2 : T_2 \longrightarrow T_2'$, where $X_2(f_* a', f_* b') = T_2' \times \mathbb{R}^{n-1}$. By construction, the maps
\[ \varphi_2 \circ \tau \circ \varphi_1^{-1} \] and \( f_* \) induce the same map \( \partial T'_a \to \partial T'_b \). By Theorem 12.3 the map \( \varphi_2 \circ \tau \circ \varphi_1^{-1} \) is the unique equivariant tree isometry which accompanies the coarse equivalence \( X_1(a', b') \to X_2(f_*a', f_*b') \). This shows that with respect to our metrics on \( X_1 \) and \( X_2 \), the map \( f_* : \partial_{cpl} X_1 \to \partial_{cpl} X_2 \) is ecological for all wall trees which are in the same class as \( T_1 \).

Suppose that \( z \in X_1 \) is a thick point in an affine apartment \( A \subseteq X_1 \). Since \( X_1 \) is irreducible, \( z \) is the intersection of \( n \) walls \( M_1, \ldots, M_n \subseteq A \) which are in the same class as the walls determined by \( (a, b) \). (To see this, we note that the Weyl group orbit of any nonzero vector spans the ambient Euclidean space.) Each of these walls determines a branch point in a panel tree. Let \( M_1, \ldots, M_n, a, b \) be the corresponding walls in \( \mathfrak{f} A \), defined by the isometries between the corresponding wall trees. The intersection \( M_1 \cap \cdots \cap M_n, a, b \) is a point \( z \in \mathfrak{f} A \). By 5.5 there is a uniform constant \( s > 0 \) such that \( d(f_1(z \times p), z_*) \leq s \). If \( z' \in A \) is another thick point and if \( z'_* \) is constructed in the same way, then the \( n \) tree isometries yield \( d(z, z') = d(z_*, z'_*) \), whence \( d(f_1(z \times p), f_1(z'_* \times p) \leq d(z, z') + 2s \). The thick points are bounded in \( X_2 \), so the map \( x \mapsto f_1(x \times p) \) is a rough isometry \( X_1 \to X_2 \). This implies by 5.5 that for no wall tree of \( X_1 \), the accompanying isometry requires any rescaling of \( X_2 \). The accompanying tree isometries therefore fit together with \( f_* \) to an ecological building isomorphism. By Tits’ result 11.5 \( f_* \) is accompanied by an isometry \( \tilde{f} : X_1 \to X_2 \). This isometry maps the thick point \( z \) precisely to the point \( z_* \) described above, \( z_* = \tilde{f}(z) \). It follows that there is a constant \( \epsilon > 0 \) such that \( d(f_1(x \times p), f_1(z)) \leq \epsilon \) holds for all \( x \times p \in X_1 \times \mathbb{R}^m \).

We now decompose the affine building \( X_1 \) into a product \( X_1 \times X_1^{II} \times X_1^{III} \) of affine buildings of types (I), (II) and (III), respectively. Similarly, we decompose \( X_2 \).

12.3 Theorem Let \( f : X_1 \times \mathbb{R}^m \to X_2 \times \mathbb{R}^m \) be a coarse equivalence of affine buildings. Assume that \( \partial_{cpl} X_1 \) and \( \partial_{cpl} X_2 \) are thick and that \( X_1 \) splits off no tree factors. Then the following hold.

(i) The irreducible factors of \( X_2 \) can be rescaled in such a way that there is an isometry \( f_* : X_1 \to X_2 \), with \( f_* = f_* \).

(ii) If \( X_1 = X_1^I \times X_1^{II} \times X_1^{III} \) and \( X_2 = X_2^I \times X_2^{II} \times X_2^{III} \) are decomposed as above, then \( f_* \) factors as a product, \( f(x_1 \times x_1^{II} \times x_1^{III}) = f_1(x_1) \times f_1^{II}(x_1^{II}) \times f_1^{III}(x_1^{III}) \).

(iii) There is a constant \( r > 0 \) such that \( d(f_{N}(x_3), \pi_{X_2}(f_1(x_1 \times x_1^{II} \times x_1^{III} \times p)) \leq r \), for \( N \in \{II, III\} \) and for all \( x_1 \times x_1^{II} \times x_1^{III} \times p \in X_1^I \times X_1^{II} \times X_1^{III} \times \mathbb{R}^m \).

Proof. We proceed by induction on the number irreducible factors of \( X_1 \). The case of one irreducible factor is covered by 12.1 and 12.2. In general, we decompose \( X_1 = Y_1 \times Z_1 \), with \( Z_1 \) irreducible. As \( f_* \) is an isomorphism, we have a corresponding decomposition \( X_2 = Y_2 \times Z_2 \). Fix opposite chambers \( a, b \) in \( \partial_{cpl} Y_1 \), and let \( A = \partial_{cpl} Y \) be the corresponding affine apartment. Then \( A \times Z_1 \) is the union of all affine apartments in \( Y_1 \times Z_1 \) which contain \( a, b \) at infinity. This relation is preserved by \( f_* \). So if \( y \times x \times p \in A \times Z_1 \times \mathbb{R}^m \), then \( f(y \times x \times p) \) has uniform distance from \( f_* A \times Z_1 \times \mathbb{R}^m \). It follows that \( \pi_{f_* A \times Z_1 \times \mathbb{R}^m} \circ f_* \) is a coarse equivalence between \( A \times Z_1 \times \mathbb{R}^m \) and \( f_* A \times Z_2 \times \mathbb{R}^m \). By the induction hypothesis we can rescale the irreducible factors of \( Y_2 \) in such a way that there is an isometry between \( Y_1 \) and \( Y_2 \), and 12.1 and 12.2 give us isometries between \( Z_1 \) and \( Z_2 \), possibly after rescaling \( Z_2 \). These isometries fit together to an isometry \( f : X_1 \to X_2 \), with \( f_* = f_* \). This gives (i). If \( Z_1 \) is of type (II) or (III), then the claims (ii) and (iii) follow, by applying 12.2 to \( A \times Z_1 \times \mathbb{R}^m \) and \( f_* A \times Z_2 \times \mathbb{R}^m \).

13 Appendix

In this section we indicate how certain results by Kleiner-Leeb 15 can be adapted to our setting. The main point is that in 15 it is sometimes assumed that the Euclidean buildings are topologically thick, i.e. that the set of thick points is bounded. The necessary modifications are fortunately small.

13.1 Theorem (Kleiner-Leeb) Let \( f : X_1 \to X_2 \) be a homeomorphism of metrically complete Euclidean buildings. Then \( f \) maps affine apartments to affine apartments (in the respective maximal apartment systems).

Proof. This is 15 6.4.2. By means of the local homology groups, one shows first that \( f \) preserves the local buildings. This implies already that both Euclidean buildings have the same dimension. It follows that for each \( p \in A \subseteq X_1 \) there exists \( r > 0 \) such that \( B_r(f(p)) \cap f(A) \) is contained in some apartment \( A' \) of \( X_2 \). So \( f(A) \) is a complete simply connected flat Riemannian manifold and therefore isometric to \( \mathbb{R}^m \). It follows that \( f(A) \) is an affine apartment.
We remark that the assumption of metric completeness is essential. A homeomorphism between non-complete trees need not preserve apartments.

13.2 Corollary Assume that $X_1$ and $X_2$ are metrically complete Euclidean buildings, and that $\partial_{cpl}X_1$ and $\partial_{cpl}X_2$ are thick. If $f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$ is a homeomorphism, then $m_1 = m_2$.

Proof. Let $p \in X_1$ be a thick point. Then the local building decomposes as $(X_1 \times \mathbb{R}^{m_1})_{p\times 0} = X_{1,p} \ast S^0 \ast \cdots \ast S^0$, and $X_{1,p}$ is thick. Since $f$ preserves the local buildings, it preserves the number $m_1$ of $S^0$-factors in the join.

For asymptotic cones we refer to [13, 2.A] [15, 2.4] [17]. The asymptotic cone of a Euclidean building is again a Euclidean building [19, 5.1].

13.3 Lemma Let $p$ be a thick point in an affine building $X$. Let $\text{Cone}(X)$ denote the asymptotic cone of $X$, with respect to the constant base point sequence $p_n = p$ and arbitrary scaling sequence $c_n$ going to infinity. The $\partial_{cpl}\text{Cone}(X)$ is thick.

Proof. Let $A \subseteq X$ be an affine apartment containing $p$. Let $a, b \subseteq A$ be $p$-based adjacent Weyl chambers. Let $c$ be a third $p$-based Weyl chamber, with $a \cap b = b \cap c = c \cap a$. The metric space $Y = a \cup b \cup c$ is invariant under scaling, so $\text{Cone}(Y) \cong Y$. Therefore every panel in the apartment induced by $\text{Cone}(A)$ in $\text{Cone}(X)_p$ contains distinct chambers. By [10.3] $\text{Cone}(X)_p$ is thick. Since there is an epimorphism $\partial_{cpl}\text{Cone}(X) \longrightarrow \text{Cone}(X)_p$, the spherical building at infinity is also thick.

13.4 Proposition Let $X_1, X_2$ be metrically complete affine buildings whose spherical buildings at infinity are thick. Let $f : X_1 \times \mathbb{R}^{m_1} \longrightarrow X_2 \times \mathbb{R}^{m_2}$ be a coarse equivalence. Then $m_1 = m_2$.

Proof. The coarse equivalence induces a homeomorphism of the asymptotic cones $\text{Cone}(f) : \text{Cone}(X_1 \times \mathbb{R}^{m_1}) \longrightarrow \text{Cone}(X_2 \times \mathbb{R}^{m_2})$. Asymptotic cones preserve product decompositions, so we have a homeomorphism $\text{Cone}(X_1) \times \mathbb{R}^{m_1} \longrightarrow \text{Cone}(X_2) \times \mathbb{R}^{m_2}$. We choose the asymptotic cones as in [13.3]. By [13.3] and [13.2] we have $m_1 = m_2$.

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