ABSTRACT. The behaviour of resonances in the spin–orbit coupling in Celestial Mechanics is investigated. We introduce a Hamiltonian nearly–integrable model describing an approximation of the spin–orbit interaction. A parametric representation of periodic orbits is presented. We provide explicit formulae to compute the Taylor series expansion in the perturbing parameter of the function describing this parametrization. Then we compute approximately the radius of convergence providing an indication of the stability of the periodic orbit. This quantity is used to describe the different probabilities of capture into resonance. In particular, we notice that for low values of the orbital eccentricity the only significative resonance is the synchronous one. Higher order resonances (including 1:2, 3:2, 2:1) appear only as the orbital eccentricity is increased.

KEYWORDS: Resonances, Spin–orbit problem, Periodic orbits, Hamiltonian dynamics.
§1. INTRODUCTION

The dynamics of coupling between revolutional and rotational motions of a satellite is investigated. In particular, we consider an oblate satellite moving on a Keplerian orbit around a central planet and rotating about an internal spin–axis (see Celletti 1990, Celletti 1994, Celletti and Falcolini 1992, Celletti and Chierchia 1998, Goldreich and Peale 1966, 1970, Henrard 1985, Murdock 1978, Peale 1973, Wisdom 1987 and references therein for related papers on this subject). Whenever the ratio between the revolutional and rotational periods is rational, say $\frac{T_{\text{rev}}}{T_{\text{rot}}} = \frac{p}{q}$ for some positive integers $p, q$, a spin–orbit resonance (of order $p : q$) is encountered. Astronomical observations show that most of the evolved satellites or planets of the solar system are trapped in a 1:1 or "synchronous" spin–orbit resonance. The most familiar example is provided by the Moon; due to the equality of the periods of rotation and revolution, the Moon always points the same face to the Earth. The only known exception is provided by Mercury, which moves in a 3:2 resonance, namely after two revolutions around the Sun, Mercury makes three rotations about its spin–axis. An intriguing question is the explanation of the different occurrence between plenty resonances (like the 1:1 or 3:2) and empty resonances (corresponding to any other different ratio of the periods). In this paper we try to address this problem as follows. We introduce a mathematical non–autonomous, one–dimensional model describing a physically relevant approximation of the spin–orbit problem (see §2). The equation of motion takes the form

$$\ddot{x} - \varepsilon f_x(x, t) = 0,$$

for a suitable function $f$ depending also on the orbital eccentricity $e$ of the Keplerian orbit; the parameter $\varepsilon$ is proportional to the equatorial oblateness coefficient of the satellite.

In §3 we introduce a parametric representation of a periodic orbit with frequency $\frac{p}{q}$ as

$$x = \theta + u(\theta) \quad y = v(\theta) \quad \theta \in T \equiv \mathbb{R}/(2\pi\mathbb{Z}),$$

where $u, v$ are suitable functions depending analytically on $\varepsilon$ and $\theta(t) = \theta(0) + \frac{p}{q} t$. We provide explicit formulae for the computation of the coefficients of the Taylor series expansion around $\varepsilon = 0$ of the function $u$, say $u(\theta) = \sum_{j=1}^{\infty} u_j(\theta) \varepsilon^j$. Finally, we compute an approximation of the radius of convergence

$$\varepsilon_\rho(\frac{p}{q}) \equiv \lim_{j \to \infty} \frac{1}{(u_j(\theta_0))^{1/j}}$$

(for a fixed $\theta = \theta_0$). This quantity provides a measure of the regularity of the periodic orbit and is used to provide a qualitative argument toward a higher capture probability of the synchronous resonance. We remark that a recent method based on a quantitative version of the Implicit Function Theorem has been implemented in (Celletti and Chierchia, 1998) to construct Birkhoff periodic orbits.

The results and conclusions are presented in §4. The graphs of $\varepsilon_\rho(\frac{p}{q})$ vs. $\frac{p}{q}$ show that for the Moon and Mercury, the 1:1 and 3:2 resonances respectively seem to be the most suitable
ending states. Moreover we explore the birth of resonances as the orbital eccentricity is increased. In particular, for low values of \( e \) the synchronous resonance is the only one present. As the eccentricity increases, higher order resonances do appear. We relate these results to the evolutionary history of the satellites.

We remark that an exhaustive explanation of this scenario requires the consideration of dissipative forces and their role in stabilizing the resonances. We demand this problem to a future study.

§2. THE SPIN–ORBIT MODEL

We briefly introduce a mathematical model describing the “spin–orbit” interaction in Celestial Mechanics as follows.

Let \( S \) be a triaxial ellipsoidal satellite orbiting around a central planet \( P \). Let \( T_{\text{rev}} \) and \( T_{\text{rot}} \) be the periods of revolution of the satellite around \( P \) and the period of rotation about an internal spin–axis. A \( p : q \) spin–orbit resonance occurs whenever

\[
\frac{T_{\text{rev}}}{T_{\text{rot}}} = \frac{p}{q}, \quad \text{for } p, q \in \mathbb{N}, \ q \neq 0.
\]

In particular, when \( p = q = 1 \) we speak of 1:1 or synchronous spin–orbit resonance, which implies that the satellite always points the same face to the host planet. In the solar system most of the evolved satellites or planets (like, e.g., the Moon) are trapped in a 1:1 resonance (Astronomical Almanac, 1997). The only exception is provided by Mercury which is observed in a nearly 3:2 resonance.

We introduce a mathematical model describing the spin–orbit interaction. In particular we assume that

i) the center of mass of the satellite moves on a Keplerian orbit around \( P \) with semimajor axis \( a \) and eccentricity \( e \) (i.e., we neglect secular perturbations on the orbital parameters);

ii) the spin–axis is perpendicular to the orbit plane (i.e., we neglect the “obliquity”);

iii) the spin–axis coincides with the shortest physical axis (i.e., the axis whose moment of inertia is largest);

iv) dissipative effects as well as perturbations due to other planets or satellites are neglected.

Let \( A < B < C \) be the principal moments of inertia of the satellite. We denote by \( r \) and \( f \), respectively, the instantaneous orbital radius and the true anomaly of the Keplerian orbit. Finally let \( x \) be the angle between the longest axis of the ellipsoid and the periapsis line (see Figure 1). The equation of motion governing the spin–orbit model under the assumptions \( i) – iv) \) may be derived from the standard Euler’s equations for rigid body.
In normalized units (i.e. assuming that the mean motion is one, $2\pi/T_{rev} = 1$) we obtain:

$$\ddot{x} + \varepsilon \left(\frac{1}{r}\right)^3 \sin(2x - 2f) = 0,$$

(2.1)

where $\varepsilon \equiv \frac{3B-A}{C}$ is proportional to the equatorial oblateness coefficient $\frac{B-A}{C}$ (and the dot denotes time differentiation). Notice that (2.1) is trivially integrated when $A = B$ or in the case of zero orbital eccentricity.

A “$p : q$ periodic orbit” (or “Birkhoff periodic orbit with rotation number $p/q$”) is a solution of (2.1) such that

$$x(t + 2\pi q) = x(t) + 2\pi p;$$

the above relation implies that after $q$ orbital revolutions around the central body the satellite makes $p$ rotations about the spin–axis.

Due to assumption $i)$, the quantities $r$ and $f$ are known Keplerian functions of the time; therefore we can expand (2.1) in Fourier series as

$$\ddot{x} + \varepsilon \sum_{m \neq 0, m = -\infty}^{\infty} W\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0,$$

(2.2)

where the coefficients $W\left(\frac{m}{2}, e\right)$ decay as powers of the orbital eccentricity as $W\left(\frac{m}{2}, e\right) \propto e^{|m-2|}$ (see Cayley 1859, for explicit expressions).

A further simplification of the model is performed as follows. According to assumption $iv)$, we neglected dissipative forces and any gravitational attraction beside that of the central planet. The most important contribution comes from the non–rigidity of the satellite, which provokes a tidal torque due to internal friction. Since the magnitude of the dissipative effects is small compared to the gravitational term, we simplify (2.2) retaining only those terms which are of the same order or bigger than the average effect of the tidal torque. Finally we consider an equation of the form

$$\ddot{x} + \varepsilon \sum_{m \neq 0, m = N_1}^{N_2} \tilde{W}\left(\frac{m}{2}, e\right) \sin(2x - mt) = 0,$$

where $N_1$ and $N_2$ are suitable integers (depending on the structural and orbital properties of the satellite), while $\tilde{W}\left(\frac{m}{2}, e\right)$ are truncations of the coefficients $W\left(\frac{m}{2}, e\right)$ (which are power series in $e$).

According to the above discussion, in the case of the Moon–Earth system we consider the following equation of motion:

$$\ddot{x} + \varepsilon \left[\left(-\frac{e}{2} + \frac{e^3}{16}\right) \sin(2x - t) + (1 - \frac{5}{2}e^2 + \frac{13}{16}e^4) \sin(2x - 2t) + \left(\frac{7}{2}e - \frac{123}{16}e^3\right) \sin(2x - 3t) + \left(\frac{17}{2}e^2 - \frac{115}{6}e^4\right) \sin(2x - 4t) + \left(\frac{845}{48}e^3 - \frac{32525}{768}e^5\right) \sin(2x - 5t) + \frac{533}{16}e^4 \sin(2x - 6t) + \frac{228347}{3840}e^5 \sin(2x - 7t)\right] = 0,$$

(2.3)
having taken $N_1 = 1$ and $N_2 = 7$ in (2.2). In what follows we shall always assume that
the leading equation is provided by (2.3), though in principle we should consider different
truncation orders ($N_1$, $N_2$) depending on the intrinsic parameters of the satellite. For
example, in the Mercury–Sun case the above criterion would suggest to take $N_1 = -17$
and $N_2 = 6$. However we checked that our results do not change significantly as new
terms are added to eq. (2.3). Precisely, more terms alter the outputs for any initial
condition, provoking a global rescaling of the results, without conditioning the qualitative
conclusions we shall draw about the behaviour of the resonances. As an illustration we
report in Figure 2 the Poincaré map around the synchronous resonance for $e = 0.004$,
$\varepsilon = 0.2$. The resonance is surrounded by librational surfaces, whose amplitude increases as
the chaotic separatrix is approached. Outside this region rotational surfaces can be found.

§3. PARAMETRIC REPRESENTATION OF PERIODIC ORBITS

In this section we provide a parametric representation of the solution associated to a
periodic orbit with frequency $\omega = \frac{p}{q}$.

We rewrite eq. (2.3) in compact form as

$$
\ddot{x} - \varepsilon f_x(x,t) = 0,
$$

namely

$$
\dot{x} = y
\quad \dot{y} = \varepsilon f_x(x,t). \tag{S}
$$

A periodic orbit with frequency $\omega = \frac{p}{q}$ is defined parametrically by the set of equations

$$
x = \theta + u(\theta) \quad \quad y = \frac{p}{q} + Du(\theta) \quad \quad \theta \in T, \tag{3.2}
$$

where $u = u(\theta)$ is a suitable continuous function, depending analytically on $\varepsilon$, with the
property that the flow in the $\theta$–coordinate is linear, i.e. $\theta \to \theta + \frac{p}{q}t$ after a time $t$; the
operator $D$ acts on a function $u = u(\theta)$ as

$$
Du(\theta) = \omega \frac{du(\theta)}{d\theta} = \frac{p}{q} \frac{du(\theta)}{d\theta}. \tag{3.3}
$$

Inserting (3.2) in the system (3.1) one obtains the second order differential equation

$$
D^2u(\theta) - \varepsilon f_x(\theta + u(\theta), \frac{\theta}{\omega}) = 0. \tag{3.3}
$$
Due to the analyticity in the perturbing parameter $\varepsilon$ of the function $u$, we can expand $u$ in Taylor series around $\varepsilon = 0$ as

$$u(\theta) = \sum_{n=1}^{\infty} u_n(\theta) \varepsilon^n , \quad (3.4)$$

for suitable $p$–periodic functions $u_n(\theta)$. Let us expand the terms $u_n(\theta)$ in Fourier series as

$$u_n(\theta) = \sum_k \hat{u}_k^{(n)} e^{ik\frac{\theta}{p}} ,$$

where the coefficients $\hat{u}_k^{(n)}$ (and therefore $u_n(\theta)$) can be constructed explicitly as follows. Write the function $f_x(x, t)$ as

$$f_x(x, t) = i \sum_{m \in \mathcal{M}} \hat{f}_m \left( e^{i(2x-mt)} - e^{-i(2x-mt)} \right) , \quad (3.5)$$

where in the case of eq. (2.3) the set $\mathcal{M}$ consists of the integers $1, 2, \ldots, 7$ and the $\hat{f}_m$ can be easily identified by comparison with eq. (2.3). From eq. (3.5) it follows that

$$f_x(\theta + u(\theta), \theta \omega) = i \sum_{k = \pm 1, m \in \mathcal{M}} k \hat{f}_m e^{ik(2\theta + 2u(\theta) - m\theta\frac{\omega}{p})} .$$

Define a power series for the exponential as

$$e^{ik(2\theta + 2u(\theta) - m\theta\frac{\omega}{p})} \equiv \sum_{n=0}^{\infty} b_{n, k}^{(m)}(\theta) \varepsilon^n .$$

It can be easily verified that the coefficients $b_{n, k}^{(m)}$ satisfy the following recursive relations:

$$b_{0, k}^{(m)} = e^{ik(2\theta - m\theta\frac{\omega}{p})}$$

$$b_{n, k}^{(m)} = \frac{2ik}{n} \sum_{h=1}^{n} hu_h(\theta) b_{n-h, k}^{(m)} , \quad n \geq 1 .$$

Inserting (3.4) in (3.3) one gets

$$\sum_{n=1}^{\infty} D^2 u_n(\theta) \varepsilon^n = i \sum_{n=1}^{\infty} \varepsilon^n \sum_{k = \pm 1, m \in \mathcal{M}} k \hat{f}_m b_{n-1, k}^{(m)} . \quad (3.6)$$

Moreover, since an explicit computation of the generic term in the summation at the l.h.s. of (3.6) provides

$$D^2 u_n(\theta) = -\frac{1}{q^2} \sum_k k^2 \hat{u}_k^{(n)} e^{ik\frac{\theta}{p}} ,$$

6
a comparison of the above relation with the r.h.s. of (3.6) yields the explicit expression of the coefficients $\hat{u}_k^{(n)}$.

Recasting the above formulae, the function $u(\theta)$ can be written as

$$u(\theta) = \sum_{n=1}^{\infty} u_n(\theta)\varepsilon^n, \quad \text{with} \quad u_n(\theta) = \sum_k \hat{u}_k^{(n)} \sin(n_k \frac{\theta}{p}),$$

for suitable Fourier indexes $n_k$; the fact that $u_n(\theta)$ is a sum of sines is due to the specific form of eq. (2.3) which contains only sine–terms.

As a measure of the regularity of the function $u(\theta)$, we compute for any $\theta = \theta_0$ the radius of convergence of the Taylor series as

$$\varepsilon^\rho \left( \frac{p}{q} \right) \equiv \lim_{j \to \infty} \frac{1}{(u_j(\theta_0))^{\frac{1}{j}}}, \quad (3.7)$$

(which in fact seems not to depend on the choice of $\theta_0$). The radius of convergence provides an indication of the stability of the periodic orbits, showing an approximate value of the perturbing parameter $\varepsilon = \varepsilon^\rho \left( \frac{p}{q} \right)$ at which the transition from elliptic to hyperbolic periodic orbits takes place. A numerical investigation of the phase space suggests that as $\varepsilon$ grows the periodic orbit is surrounded by librational curves with increasing amplitude, until they leave place to a chaotic regime as $\varepsilon$ approaches $\varepsilon^\rho \left( \frac{p}{q} \right)$. We make use of this property in order to study the behaviour of the periodic orbits as the frequency $\frac{p}{q}$ is varied.

§4. RESULTS AND CONCLUSIONS

According to standard evolutionary theories, satellites and planets were rotating fast in the past; a constant decrease of the period of rotation about the spin–axis was provoked by tidal friction due to the internal non–rigidity. Therefore, a common scenario suggests that celestial bodies were slowed down until they reached their actual dynamical configuration, being typically trapped in a 1:1 or 3:2 (for Mercury alone) resonance. This hypothesis implies that higher order resonances (i.e., 2:1, 5:4, 7:3, etc.) were bypassed during the slowing process. However there is actually no convincing explanation concerning the mechanism of escape or capture into a spin–orbit resonance. In this section we argue that there is a greater probability of capture into the synchronous or 3:2 resonance. More precisely, we compute an approximate value of the radius of convergence of the parametric representation of periodic orbits which provides, as remarked in §3, a measure of their stability. We let the eccentricity vary in a reasonable (astronomical) range of values. In particular, we consider the following set of periodic orbits with frequencies $\frac{p}{q}$ where

$$q = 1, \ldots, 14, \quad p = q + 1, \ldots, 15,$$
and

\[ p = 1, \ldots, 14, \quad q = p + 1, \ldots, 15. \]

In order to have a better precision around the main resonances we consider also the periodic orbits with frequencies

\[
\begin{align*}
\frac{1}{2} & \pm \frac{1}{10 \cdot k}, & 1 & \pm \frac{1}{10 \cdot k}, \\
\frac{5}{4} & \pm \frac{1}{10 \cdot k}, & \frac{4}{3} & \pm \frac{1}{10 \cdot k}, \\
\frac{3}{2} & \pm \frac{1}{10 \cdot k}, & \frac{5}{3} & \pm \frac{1}{10 \cdot k}, \\
\frac{7}{4} & \pm \frac{1}{10 \cdot k}, & 2 & \pm \frac{1}{10 \cdot k},
\end{align*}
\]

where \( k = 1, \ldots, 10. \) We computed the quantity \( \frac{1}{(u_j(\theta_0))^2} \) for \( j = 1, \ldots, 30, \) providing a good approximation to the radius of convergence. The choice of the point \( \theta = \theta_0 \) does not influence the result, since the radius of convergence seems to be generally independent on the initial condition.

As an example we report in Figure 3 the radius of convergence \( \varepsilon_{\rho}(\frac{1}{1}) \) as a function of the number of iterations \( j = 1, \ldots, 30 \) (in this case \( \theta_0 \) was set equal to 1.56).

For a fixed value of the eccentricity we compute \( \varepsilon_{\rho}(\frac{p}{q}) \) corresponding to the set of rational numbers \( \frac{p}{q} \) listed above. Figures 4 and 5 show the graphs of \( \varepsilon_{\rho}(\frac{p}{q}) \) versus \( \frac{p}{q} \) for, respectively, the Moon’s eccentricity (i.e., \( e = 0.0549 \)) and Mercury’s eccentricity (\( e = 0.2056 \)). In order to have a more detailed inspection of the behaviour around the resonances of astronomical interest, we report in panel a) the results around the 1:2 resonance, while panel b) corresponds to the synchronous resonance, panel c) to the 3:2 resonance and panel d) to the 2:1 resonance.

A comparison between Figures 4 and 5 suggests that locally the 1:1 and 3:2 resonances have a higher probability of capture. In fact, the periodic orbits around these commensurabilities have very low radii of convergence indicating that the most stable orbits correspond to the periods \( \frac{p}{q} = \frac{1}{1} \) or \( \frac{p}{q} = \frac{3}{2}. \) Moreover, comparing Figure 4b, 4c with Figure 5b, 5c we notice that there is a slightly preference for the Moon to end-up in the synchronous resonance, while Mercury might be trapped in the 1:1 or 3:2 resonance, in agreement with the astronomical observations.

This discussion leads to the question of the existence of attracting tori for a dissipative system (i.e., including tidal forces) corresponding to the most stable resonances. We plan to address this problem in a later study.

Next we analyze the behaviour of the stability of periodic orbits for different values of the eccentricity. Figure 6 shows \( \varepsilon_{\rho}(\frac{p}{q}) \) versus \( \frac{p}{q} \) for \( e = 0.001 \) (Figure 6a), \( e = 0.01 \) (Figure 6b), \( e = 0.06 \) (Figure 6c), \( e = 0.2 \) (Figure 6d). For low values of the eccentricity (Figure 6a) there is only one marked resonance corresponding to the synchronous commensurability. Moreover, the amplitude of the curve around the 1:1 resonance provides a measure of
the size of the region of librational motion surrounding the resonance. For values of $\frac{p}{q}$ bigger than $\frac{5}{2}$, the quantity $\varepsilon_{\rho}(\frac{p}{q})$ increases indefinitely with $\frac{p}{q}$. As $e = 0.01$ (Figure 6b), beside the 1:1 resonance there appears the 3:2, while the 1:2 and 2:1 resonances are already present, but with very small amplitudes of librational motion, which become meaningful as $e = 0.06$ (Figure 6c). For this value of the eccentricity we have the complete set of main resonances, i.e. 1:2, 1:1, 3:2, 2:1. Minor resonances appear as $e$ is increased up to $e = 0.2$ (Figure 6d), where the 5:4 and 7:4 resonances become evident.

These results indicate the existence of a strict relation between the value of the eccentricity and the birth of resonances. Satellites with low eccentricity are suitable candidates for ending-up in the synchronous resonance. However, even when different resonances arise, the 1:1 periodic orbit seems the most likely final state due to the amplitude of librational regime around it. When the eccentricity is highly increased, several new resonances appear. In particular, for Mercury’s eccentricity (compare with Figure 6d) different fates are possible, including the 3:2 resonance.

As a final remark, we suggest that a further development of this study would include the effect of dissipative terms and their role in driving satellites to select their ending states.
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**FIGURE CAPTIONS**

**Figure 1:** The spin–orbit geometry.

**Figure 2:** Poincaré map associated to eq. (2.3) around the synchronous resonance in the \((x, \dot{x})\)-plane for \(e = 0.004, \varepsilon = 0.2\). The initial conditions are \((x, y) = (0, 0.8), (0, 1.5), (0, 1), (0, 1.1), (1.8, 1), (2.1, 1), (1.57, 1)\). Notice that eq. (2.3) is \(\pi\)-periodic in \(x\). **Librational** and **rotational** regions are divided by the **chaotic separatrix**.

**Figure 3:** The radius of convergence \(\varepsilon_{\rho}(\frac{1}{1})\) vs. the degree of approximation of the limit \((3.7), j = 1, \ldots, 30\). Here \(\theta_0 = 1.56\) and \(e = 0.0549\).

**Figure 4:** The radius of convergence \(\varepsilon_{\rho}(\frac{p}{q})\) vs. \(\frac{p}{q}\) for \(e = 0.0549\) (i.e., the eccentricity of the Moon). The dots correspond to the actual computations associated to the frequencies listed in the text. Lines are due to interpolation performed by the graphic program. In abscissa are reported the rotation numbers \(\frac{p}{q}\) (the 1:1 resonance corresponds to 1, the 1:2 to 0.5 and so on).

a) 1:2 resonance, b) 1:1 resonance, c) 3:2 resonance, d) 2:1 resonance.

**Figure 5:** The radius of convergence \(\varepsilon_{\rho}(\frac{p}{q})\) vs. \(\frac{p}{q}\) for \(e = 0.2056\) (i.e., the eccentricity of Mercury). The dots correspond to the actual computations associated to the frequencies listed in the text. Lines are due to interpolation performed by the graphic program. In abscissa are reported the rotation numbers \(\frac{p}{q}\) (the 1:1 resonance corresponds to 1, the 1:2 to 0.5 and so on).

a) 1:2 resonance, b) 1:1 resonance, c) 3:2 resonance, d) 2:1 resonance.

**Figure 6:** Plot of the radius of convergence \(\varepsilon_{\rho}(\frac{p}{q})\) vs. \(\frac{p}{q}\) for the frequencies listed in the text. a) \(e = 0.001\), b) \(e = 0.01\), c) \(e = 0.06\), d) \(e = 0.2\).
