A RESOLVENT APPROACH FOR SOLVING A SET-VALUED VARIATIONAL INCLUSION PROBLEM USING WEAK-RRD SET-VALUED MAPPING

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ABSTRACT. The resolvent operator approach of [2] is applied to solve a set-valued variational inclusion problem in ordered Hilbert spaces. The resolvent operator under consideration is called relaxed resolvent operator and we demonstrate some of its properties. To obtain the solution of a set-valued variational inclusion problem, an iterative algorithm is developed and weak-RRD set-valued mapping is used. The problem as well as main result of this paper are more general than many previous problems and results available in the literature.

1. Introduction

The variational inclusion is an important generalization of the variational inequality and is applicable to solve many problems related to optimization and control, economic and transportation equilibrium, engineering and basic sciences. A lot of work concerned with the ordered variational inequalities and ordered equations is done by H-G Li and his co-authors, see [10,12,13,16,17].

Most of the problems related to variational inclusions are solved by maximal monotone operators and their generalizations such as $H$-monotonicity [6], $H$-accretivity [5] and many more, see e.g., [3,7,8] and reference theirin. Almost all the splitting methods are based on the resolvent
operator of the form \([I + \lambda M]^{-1}\), where \(M\) is a set-valued monotone mapping, \(\lambda\) is a positive constant and \(I\) is the identity mapping.

In this paper, we consider a resolvent operator \(((I - R) + \lambda M)^{-1}\), where \(R\) is a single-valued mapping and is called relaxed resolvent operator, see [2]. We prove that the relaxed resolvent operator is a comparison mapping as well as Lipschitz continuous with respect to operator \(\oplus\). Finally, a set-valued variational inclusion problem is solved by using weak-RRD set-valued mapping.

2. Preliminaries

Let \(X\) be a real ordered Hilbert space equipped with a norm \(\|\cdot\|\) and inner product \(\langle\cdot,\cdot\rangle\), \(d\) be the metric induced by the norm \(\|\cdot\|\), \(2^X\) (respectively, \(CB(X)\)) is the family of nonempty (respectively, closed and bounded) subsets of \(X\), and \(D(\cdot,\cdot)\) is the Hausdorff metric on \(CB(X)\) defined by

\[
D(P,Q) = \max \left\{ \sup_{x \in P} d(x,Q), \sup_{y \in Q} d(P,y) \right\},
\]

where \(P,Q \in CB(X)\), \(d(x,Q) = \inf_{y \in Q} d(x,y)\) and \(d(P,y) = \inf_{x \in P} d(x,y)\).

Let us recall some known concepts and results.

**Definition 2.1.** A nonempty closed convex subset \(C\) of \(X\) is said to be a cone if,

- \(i)\) for any \(x \in C\) and any \(\lambda > 0\), \(\lambda x \in C\);
- \(ii)\) if \(x \in C\) and \(-x \in C\), then \(x = 0\).

**Definition 2.2.** [4] Let \(C\) be the cone of \(X\). \(C\) is said to be normal if and only if there exists a constant \(\lambda_C > 0\) such that \(0 \leq x \leq y\) implies \(\|x\| \leq \lambda_C \|y\|\), where \(\lambda_C\) is called the normal constant of \(C\).

**Definition 2.3.** Let \(C\) be the cone in \(X\). For arbitrary elements \(x, y \in X\), \(x \leq y\) if and only if \(x - y \in C\), then the relation \(\leq\) in \(X\) is a partial ordered relation in \(X\). The Hilbert space \(X\) equipped with the ordered relation \(\leq\) defined by the cone \(C\) is called ordered Hilbert space.

**Definition 2.4.** [19] For arbitrary elements \(x, y \in X\), if \(x \leq y\) (or \(y \leq x\)) holds, then \(x\) and \(y\) are said to be comparable to each other (denoted by \(x \propto y\)).
**Definition 2.5.** [19] For arbitrary elements $x, y \in X$, lub$\{x, y\}$ and glb$\{x, y\}$ mean least upper bound and greatest upper bound of the set $\{x, y\}$. Suppose lub$\{x, y\}$ and glb$\{x, y\}$ exist, some binary operations are defined as follows:

(i) $x \lor y = \text{lub}\{x, y\}$,
(ii) $x \land y = \text{glb}\{x, y\}$,
(iii) $x \oplus y = (x - y) \lor (y - x)$,
(iv) $x \odot y = (x - y) \land (y - x)$.

The operations $\lor, \land, \oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

**Proposition 2.1.** [4] If $x \propto y$, then lub$\{x, y\}$ and glb$\{x, y\}$ exist, $x - y \propto y - x$, and $0 \leq (x - y) \lor (y - x)$.

**Proposition 2.2.** [4] For any positive integer $n$, if $x \propto y_n$ and $y_n \to y^*$ ($n \to \infty$), then $x \propto y^*$.

**Proposition 2.3.** [4, 12] Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold:

(i) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$,
(ii) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$,
(iii) $x \odot \lambda \leq 0$, if $x \propto 0$,
(iv) $0 \leq x \odot y$, if $x \propto y$,
(v) if $x \propto y$, then $x \odot y = 0$ if and only if $x = y$,
(vi) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$,
(vii) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$,
(viii) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, if $x \propto 0$, $\forall x, y, u, v \in X$ and $\alpha, \beta, \lambda \in \mathbb{R}$.

**Proposition 2.4.** [4] Let $C$ be a normal cone in $X$ with normal constant $\lambda_{CN}$, then for each $x, y \in X$, the following relations hold:

(i) $\|0 \odot 0\| = \|0\| = 0$,
(ii) $\|x \lor y\| \leq \|x\| \lor \|y\| \leq \|x\| + \|y\|$, 
(iii) $\|x \odot y\| \leq \|x - y\| \leq \lambda_{CN}\|x \odot y\|$, 
(iv) if $x \propto y$, then $\|x \odot y\| = \|x - y\|$.

**Definition 2.6.** [12] Let $A : X \to X$ be a single-valued mapping.

(i) $A$ is said to be comparison mapping, if for each $x, y \in X$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$.

(ii) $A$ is said to be strongly comparison mapping, if $A$ is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for all $x, y \in X$. 

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Definition 2.7. A mapping $A : X \rightarrow X$ is said to be $\beta$-ordered compression mapping, if $A$ is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$  

Definition 2.8. [11,14] Let $R : X \rightarrow X$ be a strong comparison and $\beta$-ordered compression mapping and $M : X \rightarrow CB(X)$ be a set-valued mapping. Then

(i) $M$ is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, for all $x, y \in X$;

(ii) a comparison mapping $M$ is said to be ordered rectangular, if for each $x, y \in X$, $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \circ v_y, -(x \oplus y) \rangle = 0;$$

(iii) a comparison mapping $M$ is said to be $\gamma_R$-ordered rectangular with respect to $R$, if there exists a constant $\gamma_R > 0$, for any $x, y \in X$, there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that

$$\langle v_x \circ v_y, -(R(x) \oplus R(y)) \rangle \geq \gamma_R \| R(x) \oplus R(y) \|^2,$$

holds, where $v_x$ and $v_y$ are said to be $\gamma_R$-elements, respectively;

(iv) $M$ is said to be a weak comparison mapping with respect to $R$, if for any $x, y \in X$, $x \propto y$, then there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that $x \propto v_x$, $y \propto v_y$ and $v_x \propto v_y$, where $v_x$ and $v_y$ are said to be weak comparison elements, respectively.

(v) $M$ is said to be a $\lambda$-weak ordered different comparison mapping with respect to $R$, if there exists a constant $\lambda > 0$ such that for any $x, y \in X$, there exist $v_x \in M(R(x))$, $v_y \in M(R(y))$, $\lambda(v_x - v_y) \propto (x - y)$ holds, where $v_x$ and $v_y$ are said to be $\lambda$-elements, respectively;

(vi) a weak comparison mapping $M$ is said to be a $(\gamma_R, \lambda)$-weak-RRD mapping with respect to $R$, if $M$ is a $\gamma_R$-ordered rectangular and $\lambda$-weak ordered different comparison mapping with respect to $R$ and $(R + \lambda M)(X) = X$, for $\lambda > 0$ and there exist $v_x \in M(R(x))$ and $v_y \in M(R(y))$ such that $v_x$ and $v_y$ are $(\gamma_R, \lambda)$-elements, respectively.

Remark 2.1. Let $X$ be a real ordered Hilbert space. Let $R : X \rightarrow X$ be a single-valued mapping and $M : X \rightarrow CB(X)$ be a set-valued mapping.
mapping, then the following relation hold, details of which can be found in [14].

(i) Every $\lambda$-ordered monotone mapping is a $\lambda$-weak ordered different comparison mapping.

(ii) If $R = I$ (identity mapping), then a $\gamma_I$-ordered rectangular mapping is an ordered rectangular mapping.

(iii) An ordered RME mapping is $\lambda$-weak-RRD mapping.

Definition 2.9. A set-valued mapping $A : X \rightarrow CB(X)$ is said to be $D$-Lipschitz continuous, if for each $x, y \in X$, $x \preceq y$, there exists a constant $\delta_A$ such that

$$D(A(x), A(y)) \leq \delta_A \|x \oplus y\|, \forall x, y \in X.$$ 

Definition 2.10. Let $M : X \rightarrow CB(X)$ be a set-valued mapping, $R : X \rightarrow X$ be single-valued mapping and $I : X \rightarrow X$ be an identity mapping. Then a weak comparison mapping $M$ is said to be a $(\gamma', \lambda)$-weak-RRD mapping with respect to $(I - R)$, if $M$ is a $\gamma'$-ordered rectangular and $\lambda$-weak ordered different comparison mapping with respect to $(I - R)$ and $[(I - R) + \lambda M](X) = X$, for $\lambda > 0$ and there exist $v_x \in M((I - R)(x))$ and $v_y \in M((I - R)(y))$ such that $v_x$ and $v_y$ are $(\gamma', \lambda)$-elements, respectively.

Example 2.1. Let $X = \mathbb{R}$ with usual inner product. Let $R : X \rightarrow X$ be a mapping defined by

$$R(x) = \frac{x}{2}, \forall x \in X.$$ 

and the set-valued mapping $M : X \rightarrow CB(X)$ is defined by

$$M(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 0, \\ \{1\}, & \text{if } x = 0. \end{cases}$$

Then, it is easy to check that $R$ is 1-ordered compression and $M$ is $(\frac{1}{7}, 1)$-weak-RRD mapping with respect to $R$.

3. Formulation of the problem and some basic properties

Let $X$ be an ordered Hilbert space and $A, B, C : X \rightarrow CB(X)$ be the set-valued mappings and $f, p : X \rightarrow X$ are the single-valued mappings. Suppose that $M : X \rightarrow CB(X)$ be a set-valued mapping. We consider the following problem:
For some $\rho \in X$ and any $\tau > 0$, find $u \in X, w \in A(u), v \in B(u), z \in C(u)$ such that
\[ \rho \in f(w) - p(v) + \tau M(z). \] (3.1)

This problem is called an ordered variational inclusion problem involving weak-RRD set-valued mapping.

Below are some special cases of problem (3.1).

(i) If $\rho = 0$ and $\tau = 1$, then problem (3.1) reduces to the problem of finding $u \in X, w \in A(u), y \in B(u)$ and $z \in C(u)$ such that
\[ 0 \in f(w) - p(y) + M(z). \] (3.2)

Problem (3.2) was introduced and studied by [9].

(ii) If $\rho = 0$, $\tau = 1$, $A = B = I$ (identity mapping) and $C$ is a single-valued mapping, then problem (3.1) reduces to the problem of finding $u \in X$ such that
\[ 0 \in f(u) - p(u) + M(C(u)). \] (3.3)

Problem (3.3) was introduced and studied by [1].

(iii) If $f = p = 0, A = B = 0$ and $C = I$, then problem (3.1) reduces to the problem of finding $u \in X$ such that
\[ \rho \in \tau M(u). \] (3.4)

Problem (3.4) was introduced and studied by [11].

(iv) If $f = p = B = 0$ and $A = C = I$, then problem (3.1) reduces to the problem of finding $u \in X$ such that
\[ \rho \in f(u) + \tau M(u). \] (3.5)

Problem (3.5) was introduced and studied by [15].

**Definition 3.1.** Let $C$ be a normal cone with normal constant $\lambda_{C_X}$ and $M : X \to CB(X)$ be weak-RRD set-valued mapping. Let $I : X \to X$ be the identity mapping and $R : X \to X$ be a single-valued mapping. The relaxed resolvent operator $J_{\lambda,M}^{I-R} : X \to X$ associated with $I, R$ and $M$ is defined by

\[ J_{\lambda,M}^{I-R}(x) = [(I - R) + \lambda M]^{-1}(x), \text{ for all } x \in X \text{ and } \lambda > 0. \] (3.6)

Now, we show that the relaxed resolvent operator defined by (3.6) is single-valued, a comparison mapping as well as Lipschitz continuous.
Proposition 3.1. [2] Let \( R : X \to X \) be a \( \beta \)-ordered compression mapping and \( M : X \to CB(X) \) be the set-valued ordered rectangular mapping. Then the operator \( J_{\lambda,M}^{(I-R)} : X \to X \) is a single-valued, for all \( \lambda > 0 \).

Proposition 3.2. Let \( M : X \to CB(X) \) be a \( (\gamma_R, \lambda) \)-weak-RRD set-valued mapping with respect to \( J_{\lambda,M}^{(I-R)} \). Let \( R : X \to X \) be a strongly comparison mapping with respect to \( J_{\lambda,M}^{(I-R)} \) and \( I : X \to X \) be the identity mapping. Then the resolvent operator \( J_{\lambda,M}^{(I-R)} : X \to X \) is a comparison mapping.

Proof. Let \( M \) be a \( (\gamma_R, \lambda) \)-weak-RRD set-valued mapping with respect to \( J_{\lambda,M}^{(I-R)} \). That is, \( M \) is \( \gamma_R \)-ordered rectangular and \( \lambda \)-weak-ordered different comparison mapping with respect to \( J_{\lambda,M}^{(I-R)} \) so that \( x \prec J_{\lambda,M}^{(I-R)}(x) \). For any \( x, y \in X \), let \( x \prec y \), and let
\[
v_x = \frac{1}{\lambda}(x - (I - R)(J_{\lambda,M}^{(I-R)}(x))) \in M(J_{\lambda,M}^{(I-R)}(x)) \tag{3.7}
\]
and
\[
v_y = \frac{1}{\lambda}(y - (I - R)(J_{\lambda,M}^{(I-R)}(y))) \in M(J_{\lambda,M}^{(I-R)}(y)). \tag{3.8}
\]
Using (3.7) and (3.8), we have
\[
v_x - v_y = \left( \frac{1}{\lambda}(x - (I - R)(J_{\lambda,M}^{(I-R)}(x))) \right) - \left( \frac{1}{\lambda}(y - (I - R)(J_{\lambda,M}^{(I-R)}(y))) \right) = \frac{1}{\lambda}(x - y + (I - R)(J_{\lambda,M}^{(I-R)}(y)) - (I - R)(J_{\lambda,M}^{(I-R)}(x))).
\]
Since \( M \) is \( \lambda \)-weak-ordered different comparison mapping with respect to \( J_{\lambda,M}^{(I-R)} \), we have
\[
\lambda(v_x - v_y) - (x - y) = (x - y) + (I - R)(J_{\lambda,M}^{(I-R)}(y)) - (I - R)(J_{\lambda,M}^{(I-R)}(x)) - (x - y) = (I - R)(J_{\lambda,M}^{(I-R)}(y)) - (I - R)(J_{\lambda,M}^{(I-R)}(x)).
\]
Since \( R \) is strongly comparison mapping with respect to \( J_{\lambda,M}^{(I-R)} \), \( (I - R) \) is also strongly comparison mapping with respect to \( J_{\lambda,M}^{(I-R)} \). Therefore, \( J_{\lambda,M}^{(I-R)}(x) \prec J_{\lambda,M}^{(I-R)}(y) \). The proof is completed. \( \square \)
Proposition 3.3. Let $M : X \to CB(X)$ be a $(\gamma_R, \lambda)$-weak-RRD set-valued mapping with respect to $J^{(I-R)}_{\lambda,M}$. Let $R : X \to X$ be a comparison and $\beta$-ordered compression mapping with respect to $J^{(I-R)}_{\lambda,M}$ with condition $\lambda \gamma_R > \beta + 1$. Then the following condition holds:

$$\|J^{(I-R)}_{\lambda,M}(x) \oplus J^{(I-R)}_{\lambda,M}(y)\| \leq \frac{1}{\lambda \gamma_R - \beta - 1}\|x \oplus y\|.$$

Proof. Let $M$ be a $(\gamma_R, \lambda)$-weak-RRD set-valued mapping with respect to $J^{(I-R)}_{\lambda,M}$. That is, $M$ is $\gamma_R$-ordered rectangular and $\lambda$-weak-ordered different comparison mapping with respect to $J^{(I-R)}_{\lambda,M}$. Then for any $x, y \in X$, set $u_x = J^{(I-R)}_{\lambda,M}(x)$, $u_y = J^{(I-R)}_{\lambda,M}(y)$, and let

$$v_x = \frac{1}{\lambda}(x - (I - R)(J^{(I-R)}_{\lambda,M}(x))) \in M(J^{(I-R)}_{\lambda,M}(x))$$

and

$$v_y = \frac{1}{\lambda}(y - (I - R)(J^{(I-R)}_{\lambda,M}(y))) \in M(J^{(I-R)}_{\lambda,M}(y)).$$

Since $R$ is $\beta$-ordered compression mapping and using Proposition 2.3, we have

$$v_x \oplus v_y = \frac{1}{\lambda}[(x - (I - R)(u_x)) \oplus (y - (I - R)(u_y))]$$

$$\leq \frac{1}{\lambda}[x \oplus y + (I - R)(u_x) \oplus (I - R)(u_y)]$$

$$\leq \frac{1}{\lambda}[x \oplus y + u_x \oplus u_y + R(u_x) \oplus R(u_y)]$$

$$\leq \frac{1}{\lambda}[x \oplus y + u_x \oplus u_y + \beta(u_x \oplus u_y)]$$

$$= \frac{1}{\lambda}[x \oplus y + (1 + \beta)(u_x \oplus u_y)].$$
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Since $M$ is $\gamma_R$-ordered rectangular mapping with respect to $J_{\lambda,M}^{(I-R)}$, we have

$$\gamma_R \|u_x \oplus u_y\|^2 \leq \langle v_x \odot v_y, -(u_x \oplus u_y) \rangle$$

$$= \langle v_x \odot v_y, u_x \oplus u_y \rangle$$

$$\leq \langle \frac{1}{\lambda} (x \oplus y + (1 + \beta)(u_x \oplus u_y)), (u_x \oplus u_y) \rangle$$

$$= \frac{1}{\lambda} \|x \oplus y\|\|u_x \oplus u_y\| + \frac{(1 + \beta)}{\lambda} \|u_x \oplus u_y\|^2.$$

It follows that $\left( \gamma_R - \frac{(1+\beta)}{\lambda} \right) \|u_x \oplus u_y\| \leq \frac{1}{\lambda} \|x \oplus y\|$ and consequently, we have

$$\left\| J_{\lambda,M}^{(I-R)}(x) \oplus J_{\lambda,M}^{(I-R)}(y) \right\| \leq \frac{1}{\lambda \gamma_R - \beta - 1} \|x \oplus y\|.$$ 

The proof is completed. \qed

4. An iterative algorithm and existence result

In this section, we define an iterative algorithm to obtain the solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1).

**Iterative Algorithm 4.1.** Let $A, B, C : X \to CB(X)$ be the set-valued mappings, $R, f, p : X \to X$ be the single-valued mappings and $I : X \to X$ be the identity mapping. Suppose that $M : X \to CB(X)$ is a set-valued mapping.

For any given initial $u_0 \in X$, $w_0 \in A(u_0)$, $v_0 \in B(u_0)$, $z_0 \in C(u_0)$, let

$$u_1 = u_0 - z_0 + J_{\lambda,M}^{(I-R)}[(I - R)(z_0) + \frac{\lambda}{\tau}(\rho - (f(w_0) - p(v_0)))]$$

Since $w_0 \in A(u_0) \in CB(X)$, $v_0 \in B(u_0) \in CB(X)$ and $z_0 \in C(u_0) \in CB(X)$, by Nadler’s theorem [18], there exist $w_1 \in A(u_1)$, $v_1 \in B(u_1)$, $z_1 \in C(u_1)$ and suppose that $u_0 \preceq u_1$, $w_0 \preceq w_1$, $v_0 \preceq v_1$ and $z_0 \preceq z_1$. 


such that
\[ \| w_1 + w_0 \| = \| w_1 - w_0 \| \leq D(A(u_1), A(u_0)), \]
\[ \| v_1 + v_0 \| = \| v_1 - v_0 \| \leq D(B(u_1), B(u_0)), \]
\[ \| z_1 + z_0 \| = \| z_1 - z_0 \| \leq D(C(u_1), C(u_0)). \]

Continuing the above process inductively, we can define the iterative sequences \( \{ u_n \}, \{ w_n \}, \{ v_n \} \) and \( \{ z_n \} \) with the supposition that \( u_n \propto u_{n+1}, w_n \propto w_{n+1}, v_n \propto v_{n+1} \) and \( z_n \propto z_{n+1} \), for all \( n \in \mathbb{N} \). We define the following iterative algorithm schemes:

\[
\begin{align*}
    u_{n+1} &= u_n - z_n + J_{\lambda,M}^{(I-R)}[(I - R)(z_n) + \frac{\lambda}{\tau}(\rho - (f(w_n) - p(v_n)))] , \\
    w_{n+1} &\in A(u_{n+1}), \| w_{n+1} + w_n \| = \| w_{n+1} - w_n \| \leq D(A(u_{n+1}), A(u_n)), \\
    v_{n+1} &\in B(u_{n+1}), \| v_{n+1} + v_n \| = \| v_{n+1} - v_n \| \leq D(B(u_{n+1}), B(u_n)), \\
    z_{n+1} &\in C(u_{n+1}), \| z_{n+1} + z_n \| = \| z_{n+1} - z_n \| \leq D(C(u_{n+1}), C(u_n)),
\end{align*}
\]

where \( \lambda, \rho, \tau > 0 \) are constants.

The fixed point formulation of problem (3.1) is as follows.

**Lemma 4.1.** Let \( u \in X, w \in A(u), v \in B(u) \) and \( z \in C(u) \) be a solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1) if and only if \( (u, w, v, z) \) satisfies the following relation:

\[
    u = u - z + J_{\lambda,M}^{(I-R)}[(I - R)(z) + \frac{\lambda}{\tau}(\rho - (f(w) - p(v)))].
\]

Where

\[
    J_{\lambda,M}^{(I-R)} = [(I - R) + \lambda M]^{-1},
\]

and \( \lambda, \tau, \rho > 0 \) are constants.

**Proof.** The proof directly follows from the definition of the relaxed resolvent operator \( J_{\lambda,M}^{(I-R)} \).

**Theorem 4.1.** Let \( X \) be a real ordered Hilbert space and \( C \) be a normal cone with normal constant \( \lambda_C \). Let \( R, f, P : X \to X \) be the
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single-valued mappings such that $R$ is comparison, $\beta$-ordered compression, $\lambda_f$-ordered compression and $P$ is comparison, $\lambda_p$-ordered compression mappings and $I : X \to X$ is an identity mapping. Let $A,B,C : X \to CB(X)$ be the set-valued mappings such that $A,B$ and $C$ are $D$-Lipschitz continuous mappings with constants $\delta_A, \delta_B$ and $\delta_C$, respectively. Suppose that $M : X \to CB(X)$ is a $(\gamma_R,\lambda)$-weak-RRD set-valued mapping such that the following condition is satisfied

\[
\lambda C_N [\tau \gamma_R (\delta_C + 1) + \lambda \lambda f \delta_A + \lambda \lambda p \delta_B] < \tau \lambda \gamma_R + (\lambda C_N - 1) \tau (\beta + 1). \tag{4.5}
\]

Then, the iterative sequences $\{u_n\}, \{w_n\}, \{v_n\}$ and $\{z_n\}$ generated by Algorithm 4.1 converge strongly to $u, w, v$ and $z$, respectively and $(u, w, v, z)$ is a solution of ordered variational inclusion problem involving weak-RRD set-valued mapping (3.1), where $u \in X$, $w \in A(u)$, $v \in B(u)$ and $z \in C(u)$.

**Proof.** Let us introduce the term $h(u_n) = [(I - R)(z_n) + \frac{1}{\tau} (\rho - f(w_n) - p(v_n))]$. Using Algorithm 4.1 and Proposition 2.3, we obtain

\[
0 \leq u_{n+1} \oplus u_n = \left( u_n - z_n + J_{\lambda,\lambda_M}^{(I-R)}(h(u_n)) \right) \oplus \left( u_{n-1} - z_{n-1} + J_{\lambda,\lambda_M}^{(I-R)}(h(u_{n-1})) \right)
\]

\[
\leq u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + \left( J_{\lambda,\lambda_M}^{(I-R)}(h(u_n)) \oplus J_{\lambda,\lambda_M}^{(I-R)}(h(u_{n-1})) \right). \tag{4.6}
\]

Using Definition 2.2, Proposition 3.3 and from (4.6), we have

\[
\|u_{n+1} \oplus u_n\| \leq \lambda C_N \left\| u_n \oplus u_{n-1} + z_n \oplus z_{n-1} + \left( J_{\lambda,\lambda_M}^{(I-R)}(h(u_n)) \oplus J_{\lambda,\lambda_M}^{(I-R)}(h(u_{n-1})) \right) \right\|
\]

\[
\leq \lambda C_N \left[ \|u_n \oplus u_{n-1}\| + \|z_n \oplus z_{n-1}\| + \left\| J_{\lambda,\lambda_M}^{(I-R)}(h(u_n)) \oplus J_{\lambda,\lambda_M}^{(I-R)}(h(u_{n-1})) \right\| \right]
\]

\[
\leq \lambda C_N \left[ \|u_n \oplus u_{n-1}\| + D(C(u_n), C(u_{n-1})) \right]
\]
Since $R$ is $\beta$-ordered compression mapping, $f$ is $\lambda_f$-ordered compression mapping and $D$-Lipschitz continuity of $A, B$ and $C$ with constants $\delta_A, \delta_B$ and $\delta_C$, respectively, we have

\[
\|h(u_n) \oplus h(u_{n-1})\| \\
= ||[(I - R)(z_n) + \frac{\lambda}{\tau}(\rho - (f(w_n) - p(v_n)))] \\
\oplus [(I - R)(z_{n-1}) + \frac{\lambda}{\tau}(\rho - (f(w_{n-1}) - p(v_{n-1})))]|| \\
\leq ||(I - R)(z_n) \oplus (I - R)(z_{n-1})|| + \frac{\lambda}{\tau}||(\rho - (f(w_n) - p(v_n))) \\
\oplus (\rho - (f(w_{n-1}) - p(v_{n-1})))|| \\
\leq \|z_n \oplus z_{n-1}\| + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\|f(w_n) \oplus f(w_{n-1})\| \\
+ ||p(v_n) \oplus p(v_{n-1})||] \\
\leq D(C(u_n), C(u_{n-1})) + \|R(z_n) \oplus R(z_{n-1})\| + \frac{\lambda}{\tau}[\|f(w_n) \\
\oplus f(w_{n-1})\| + ||p(v_n) \oplus p(v_{n-1})||] \\
\leq \delta_C\|u_n \oplus u_{n-1}\| + \beta D(C(u_n), C(u_{n-1})) \\
+ \frac{\lambda}{\tau}[\lambda_f D(A(u_n), A(u_{n-1})) + \lambda_p D(B(u_n), B(u_{n-1}))] \\
\leq \delta_C\|u_n \oplus u_{n-1}\| + \beta \delta_C\|u_n \oplus u_{n-1}\| + \frac{\lambda}{\tau}[\lambda_f \delta_A\|u_n \oplus u_{n-1}\| \\
+ \lambda_p \delta_B\|u_n \oplus u_{n-1}\|] \\
\leq \left[\delta_C + \beta \delta_C + \frac{\lambda}{\tau}(\lambda_f \delta_A + \lambda_p \delta_B)\right]\|u_n \oplus u_{n-1}\|,
\]

which implies that

\[
\|h(u_n) \oplus h(u_{n-1})\| \leq \left[\delta_C + \beta \delta_C + \frac{\lambda}{\tau}(\lambda_f \delta_A + \lambda_p \delta_B)\right]\|u_n \oplus u_{n-1}\|. \quad (4.8)
\]
Using (4.8), (4.7) becomes
\[
\|u_{n+1} \oplus u_n\| \leq \lambda_{C_N} \left[ 1 + \delta_C + \frac{1}{\lambda \gamma_R} - \delta \left( \delta_C + \beta \delta_C \right) \right.
+ \frac{1}{\tau} (\lambda f \delta_A + \lambda \delta_B) \left]\right] \|u_n \oplus u_{n-1}\| (4.9)
\]
By Proposition 2.4, we have
\[
\|u_{n+1} - u_n\| = \|u_{n+1} \oplus u_n\| \leq \lambda_{C_N} \left[ 1 + \delta_C + \frac{1}{\lambda \gamma_R} - \delta \left( \delta_C + \beta \delta_C \right) \right.
+ \frac{1}{\tau} (\lambda f \delta_A + \lambda \delta_B) \left]\right] \|u_n - u_{n-1}\|
\]
i.e.,
\[
\|u_{n+1} - u_n\| \leq \Theta \|u_n - u_{n-1}\|
\]
where
\[
\Theta = \lambda_{C_N} \left[ 1 + \delta_C + \frac{1}{\lambda \gamma_R} - \delta \left( \delta_C + \beta \delta_C + \frac{1}{\tau} (\lambda f \delta_A + \lambda \delta_B) \right) \right].
\]
By condition (4.5), we have \(0 < \Theta < 1\), thus \(\{u_n\}\) is a Cauchy sequence in \(X\) and since \(X\) is a complete space, there exists \(u \in X\) such that \(u_n \to u\) as \(n \to \infty\). From (4.2), (4.3) and (4.4) of Algorithm 4.1 and \(D\)-Lipschitz continuity of \(A, B\) and \(C\), we have
\[
\|w_{n+1} - w_n\| \leq D(A(u_{n+1}), A(u_n)) \leq \delta \|u_{n+1} - u_n\|, (4.10)
\]
\[
\|v_{n+1} - v_n\| \leq D(B(u_{n+1}), B(u_n)) \leq \delta \|u_{n+1} - u_n\|, (4.11)
\]
\[
\|z_{n+1} - z_n\| \leq D(C(u_{n+1}), C(u_n)) \leq \delta \|u_{n+1} - u_n\|. (4.12)
\]
It is clear from (4.10), (4.11) and (4.12) that \(\{w_n\}, \{v_n\}\) and \(\{z_n\}\) are also Cauchy sequences in \(X\) and so there exist \(w, v\) and \(z\) in \(X\) such that \(w_n \to w, v_n \to v\) and \(z_n \to z\) as \(n \to \infty\). By using the continuity of the operators \(A, B, C, J^{I-R}_{\lambda_M}\) and iterative Algorithm 4.1, we have
\[
u = u - z + J^{I-R}_{\lambda_M}([I - R]z + \frac{\lambda}{\tau} (\rho - (f(w) - p(v))].
\]
By Lemma 4.1, we conclude that \((u, w, v, z)\) is a solution of problem (3.1). It remains to show that \(w \in A(u), v \in B(u)\) and \(z \in C(u)\). In fact
\[
d(w, A(u)) \leq \|w - w_n\| + d(w_n, A(u)) \leq \|w - w_n\| + D(A(u_n), A(u)) \leq \|w - w_n\| + \delta \|u_n - u\| \to 0, \text{ as } n \to \infty.
\]
Hence $w \in A(u)$. Similarly, we can show that $v \in B(u)$ and $z \in C(u)$. This completes the proof. \hfill \Box

References

[1] S. Adly, Perturbed algorithm and sensitivity analysis for a general class of variational inclusions, J. Math. Anal. Appl. 201 (1996), 609–630.
[2] I. Ahmad and R. Ahmad, The relaxed resolvent operator for solving fuzzy variational inclusion problem involving ordered RME set-valued mapping, Caspian J. Math. Sci., (To appear).
[3] S. Chang and N. Huang, Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces, J. Math. Anal. Appl. 158 (1991), 194–202.
[4] Y.H. Du, Fixed points of increasing operators in ordered Banach spaces and applications, Appl. Anal. 38 (1990), 1–20.
[5] Y.P. Fang and N.-J. Huang, $H$-accretive operator and resolvent operator technique for variational inclusions in Banach spaces, Appl. Math. Lett. 17 (6) (2004), 647–653.
[6] Y.P. Fang and N.-J. Huang, $H$-monotone operator and resolvent operator technique for variational inclusions, Appl. Math. Comput. 145 (2-3) (2003), 795–803.
[7] Y.P. Fang and N.-J. Huang, Approximate solutions for non-linear variational inclusions with $(H, \eta)$-monotone operator, Research Report, Sichuan University (2003).
[8] Y.P. Fang, Y.-J. Chu and J.K. Kim, $(H, \eta)$-accretive operator and approximating solutions for systems of variational inclusions in Banach spaces, to appear in Appl. Math. Lett.
[9] N.J. Huang, A new completely general class of variational inclusions with non-compact valued mappings, Comput. Math. Appl. 35 (10) (1998), 9–14.
[10] H.G. Li, L.P. Li, J.M. Zheng and M.M. Jin, Sensitivity analysis for generalized set-valued parametric ordered variational inclusion with $(\alpha, \lambda)$-NODSM mappings in ordered Banach spaces, Fixed Point Theory Appl. 2014 (2014): 122.
[11] H.G. Li, X.B. Pan, Z.Y. Deng and C.Y. Wang, Solving GNOVI frameworks involving $(\gamma_G, \lambda)$-weak-GRD set-valued mappings in positive Hilbert spaces, Fixed Point Theory Appl. 2013 (2013), doi: 10.1186/1687-1812-2013-241.
[12] H.G. Li, A nonlinear inclusion problem involving $(\alpha, \lambda)$-NODM set-valued mappings in ordered Hilbert space, Appl. Math. Lett. 25 (2012) 1384–1388.
[13] H.G. Li, D. Qiu and Y. Zou, Characterizations of weak-ANODD set-valued mappings with applications to approximate solution of GNMOQV inclusions involving $\oplus$ operator in ordered Banach spaces, Fixed Point Theory Appl. 2013 (2013), doi: 10.1186/1687-1812-2013-241.
[14] H.G. Li, Nonlinear inclusion problems for ordered RME set-valued mappings in ordered Hilbert spaces, Nonlinear Funct. Anal. Appl. 16 (1) (2001), 1–8.
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[15] H.G. Li, L.P. Li, M.M. Jin, A class of nonlinear mixed ordered inclusion problems for ordered \((\alpha, \lambda)-ANODM\) set-valued mappings with strong comparison mapping \(A\), Fixed Point Theory Appl. 2014 (2014): 79.

[16] H.G. Li, Approximation solution for general nonlinear ordered variational inequalities and ordered equations in ordered Banach space, Nonlinear Anal. Forum 13 (2) (2008), 205–214.

[17] H.G. Li, Approximation solution for a new class general nonlinear ordered variational inequalities and ordered equations in ordered Banach space, Nonlinear Anal. Forum 14 (2009), 89–97.

[18] S.B. Nadler, Multi-valued contraction mappings, Pacific J. Math. 30 (1969), 475–488.

[19] H.H. Schaefer, Banach lattices and positive operators, Springer-Verlag, Berlin Heidelberg (1974), doi:10.1007/978-3-642-65970-6.

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