SECONDARY HOMOLOGICAL STABILITY FOR MAPPING CLASS GROUPS OF NONORIENTABLE SURFACES

MAX VISTRUP

Abstract. Using the Galatius–Kupers–Randal-Williams framework of cellular $E_2$-algebras, we prove a secondary stability theorem for mapping class groups of nonorientable surfaces. As a corollary, we obtain a new best known stability range for the homology of the mapping class groups of nonorientable surfaces with respect to adding torus holes.

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1. Introduction

For a compact surface $S$, orientable or not, let $\text{Homeo}^\partial(S)$ be the space of homeomorphisms of $S$ fixing $\partial S$ pointwise equipped with the compact–open topology. Composition of homeomorphisms endows $\text{Homeo}^\partial(S)$ with the structure of a topological group. The mapping class group of $S$ is the group of path components

$$\Gamma(S) := \pi_0\text{Homeo}^\partial(S);$$

that is, the group of isotopy classes of boundary-fixing homeomorphisms of $S$. For each $g \geq 1, r \geq 0$, let

$$N_{g,r} := (\mathbb{RP}^2)^g - (\text{Int}(D^2) \times \{1, \ldots, r\})$$

denote the nonorientable surface of genus $g$ with $r$ boundary components. Similarly, let $S_{g,r}$ denote the orientable surface of genus $g$ with $r$ boundary components. As a convention, we let $N_{0,r} := S_{0,r}$ be the $r$-punctured disk. For convenience, we write $\Gamma_{g,r} := \Gamma(S_{g,r})$ and $\text{NT}_{g,r} := \Gamma(N_{g,r})$.

For $g \geq 1$, there are stabilization maps $\Gamma_{g-1,1} \to \Gamma_{g,1}$ given by extending a homeomorphism of $S_{g-1,1}$ along $S_{g-1,1} \hookrightarrow S_{g,1}$ putting the identity $\text{id}_{S_{1,2}}$ outside $S_{g-1,1}$. Harer [Har85] proved that the groups $\Gamma_{g,1}$ exhibit homological stability with respect to the genus $g$. Specifically, he proved that $H_d(\Gamma_{g,1}, \Gamma_{g-1,1})$ vanishes in a range of $d$ increasing with $g$ by slope $\frac{1}{3}$. Subsequent papers have improved this range. Ivanov [Iva89] attained a slope $\frac{2}{3}$ range, Boldsen [Bol12] attained a slope $\frac{2}{3}$ range, or $d \leq \left\lfloor \frac{2g-2}{3} \right\rfloor$ to be precise (see [Wah13] for an exposition of the proof). More recently, Galatius, Kupers, and Randal-Williams [GKRW19a Theorem B(i)] attained the range $d \leq \left\lfloor \frac{2g-4}{5} \right\rfloor \iff \frac{d}{g} < \frac{2}{5}$.

A similar story transpired for mapping class groups $\text{NT}_{g,1}$ of nonorientable surfaces. In this case, there are two relevant stabilization maps that increase genus. First, for $g \geq 3$, there is a torus hole stabilization map

$$\text{NT}_{g-2,1} \to \text{NT}_{g,1},$$

defined as above by taking the boundary sum with the punctured torus $S_{1,1}$. Second, for $g \geq 1$, there is a crosscap stabilization map

$$\text{NT}_{g-1,1} \to \text{NT}_{g,1},$$

defined as before but by taking the boundary sum with the Möbius strip $N_{1,1}$ (a copy of which is known as a crosscap) instead of $S_{1,1}$. These maps are well-defined at least up to an inner automorphism of the target, which means that there is no ambiguity on group homology.

Wahl [Wah07] was the first to prove homological stability for $\text{NT}_{g,1}$. She proved homological stability with respect to the crosscap stabilization map, showing that $H_d(\text{NT}_{g,1}, \text{NT}_{g-1,1}) = 0$ in a range of $d$ increasing with $g$ with slope $\frac{1}{4}$. In terms of Euler characteristic, the genus of an orientable surface is worth twice the genus of a nonorientable surface, and thus this range is analogous to Ivanov’s range for the orientable case discussed above. Randal-Williams [RW16 1.4] improved Wahl’s range to a slope $\frac{1}{4}$ range, which in turn is analogous to the improvement of Boldsen in the orientable case mentioned above.

Moreover, Randal-Williams loc. cit. proved homological stability with respect to the torus hole stabilization map $\text{NT}_{g-2,1} \to \text{NT}_{g,1}$, showing that $H_d(\text{NT}_{g-2,1}, \text{NT}_{g,1})$ is an isomorphism for $g \geq 3d + 6$. We prove that slightly outside this range, there may not be stability, but a secondary stability phenomenon relating torus hole
stabilization to crosscap stabilization occurs. The crosscap stabilization maps \((1.2)\) commute with the torus hole stabilization maps \((1.1)\) up to an inner automorphism of \(N\Gamma g, 1\). For each choice of such inner automorphism, there is an induced map on the relative homology. There is a preferred choice of such induced map, which we describe in Section 1.1.

**Theorem A.** Let \(g \geq 4\). If \(g\) is odd, the secondary stabilization map (cf. Definition 1.1)
\[
H_d(N\Gamma g-1,1, N\Gamma g-3,1) \oplus H_d(\Gamma (g-1)/2,1, \Gamma (g-3)/2,1) \rightarrow H_d(N\Gamma g, 1, N\Gamma g-2,1)
\]
is surjective if \(d < \frac{1}{3}\) and an isomorphism if \(d+1 \leq \frac{1}{3}\). If \(g\) is even, the same is true for the secondary stabilization map
\[
H_d(N\Gamma g-1,1, N\Gamma g-3,1) \rightarrow H_d(N\Gamma g, 1, N\Gamma g-2,1).
\]

We prove this result using the framework of cellular \(E_k\)-algebras developed by Galatius, Kupers, and Randal-Williams [GKRW19b]. Results like Theorem A about the “stability of the failure of stability” are called **secondary stability theorems**. Another result of this kind was obtained by Galatius, Kupers, and Randal-Williams in [GKRW19a] which inspired this paper. There is a qualitative difference between their secondary stability result and ours in the fact that our secondary stabilization map does not increase the homological degree.

Combining the injectivity part of Theorem A with Harer stability [GKRW19a, Theorem B(i)] and homological stability with respect to the torus hole stabilization maps [RW16, 1.4(i) + 1.4(ii)], by repeatedly applying the secondary stabilization map until entering the stable range, we obtain the corollary,

**Corollary B.** Let \(g \geq 3\). If \(d+1 \leq \frac{1}{3}\), then \(H_d(N\Gamma g, 1, N\Gamma g-2,1) = 0\).

This corollary improves the range of Randal-Williams loc. cit. However, it does not extend the range in which the groups \(H_d(N\Gamma g, 1)\) are stable with respect to \((1.1)\), as Randal-Williams already proves that the maps within his range are isomorphisms. Rather, the corollary extends his result by showing that right outside his range the stabilization maps are surjective, as is a typical pattern in algebraic topology.

![Figure 1.1](image.png)

**Figure 1.1.** A table showing the known vanishing and secondary stability ranges of \(H_d(N\Gamma g, 1, N\Gamma g-2,1)\), for small \(g\).

Theorem A admits a sister statement, in which the roles of crosscaps and torus holes have been switched:
**Theorem C.** Let \( g \geq 4 \) be an even integer. Then the secondary stabilization map (cf. Definition 1.2)
\[
H_d(NT_g, NT_{g-1,1}) \to H_d(NT_g, NT_{g-1,1})
\]
is surjective if \( \frac{d}{g} < \frac{1}{3} \) and an isomorphism if \( \frac{d+1}{g} < \frac{1}{3} \).

For \( g \) odd, a more complicated statement may be extracted from Theorem 5.3 (cf. Remark 5.5). As with Theorem A, Theorem C is superseded by actual stability in most cases.

![Figure 1.2. A table showing the known vanishing and secondary stability ranges of \( H_d(NT_g, NT_{g-1,1}) \), for small \( g \).](image)

### 1.1. The secondary stabilization maps.
We now describe the secondary stabilization maps. In the following, we shall work with surfaces inside \([0,n] \times I \times \mathbb{R}^\infty \) that have their boundary in \( \partial([0,n] \times I) \times 0 \subset I^2 \times 0 \) of this boundary. Boundary sum \( \oplus \) of such surfaces is given by horizontal juxtaposition. Fix such models \( N_1 \subseteq I^2 \times \mathbb{R}^\infty \) and \( S_1 \subseteq I^2 \times \mathbb{R}^\infty \) for the Möbius strip and the punctured torus respectively, and define \( N_g := N_1^g \subseteq [0,g] \times I \times \mathbb{R}^\infty \) and \( S_g := S_1^g \subseteq [0,g] \times I \times \mathbb{R}^\infty \). For a surface \( S \subseteq I^2 \times \mathbb{R}^\infty \) again subject to the aforementioned conditions, the square
\[
\begin{array}{ccc}
\Gamma(S) & \xrightarrow{\oplus \text{id}_{N_1}} & \Gamma(S \oplus N_1) \\
\downarrow \text{id}_{S_1} & & \downarrow \text{id}_{S_1} \\
\Gamma(S \oplus S_1) & \xrightarrow{\oplus \text{id}_{N_1}} & \Gamma(S \oplus S_1 \oplus N_1) \\
\end{array}
\]
commutes, where \( \beta : N_1 \oplus S_1 \to S_1 \oplus N_1 \) is the clockwise half Dehn twist (see Figure 2.2), which is well-defined up to isotopy, and \( \Gamma(\text{id} \oplus \beta) \) means conjugation by \( \text{id} \oplus \beta \).

For each homeomorphism \( S_{h,1} \oplus N_{1,1} \cong N_{2h+1,1} \), we get an isomorphism of pairs of groups
\[
(1.4) \quad (\Gamma(S_{h,1} \oplus N_{1,1} \oplus S_1), \Gamma(S_{h,1} \oplus N_{1,1})) \cong (\Gamma(N_{2h+1,1} \oplus S_1), \Gamma(N_{2h+1,1})).
\]
Counting the choice of the homeomorphism, this isomorphism is ambiguous only up to conjugation by an element in \( \Gamma(N_{2h+1,1}) \). Consequently, there is a well-defined isomorphism
\[
(1.5) \quad H_d(\Gamma(S_{h,1} \oplus N_{1,1} \oplus S_1), \Gamma(S_{h,1} \oplus N_{1,1})) \cong H_d(\Gamma(N_{2h+1,1} \oplus S_1), \Gamma(N_{2h+1,1})).
\]
Write
\[ H_d(\Gamma g_{-1}, \Gamma g_{-2,1}) := H_d(\Gamma(N g_{-2,1} \oplus S_{1,1}), \Gamma(N g_{-2,1})), \]
\[ H_d(\Gamma h_{-1}, \Gamma h_{-1,1}) := H_d(\Gamma(S h_{-1,1} \oplus S_{1,1}), \Gamma(S h_{-1,1})). \]

Definition 1.1. The squares (1.3) induce maps on relative group homology
\[ H_d(\Gamma(N g_{1} \oplus S_{1,1}), \Gamma(N g_{1}))) \to H_d(\Gamma(N g_{1} \oplus N_{1,1} \oplus S_{1,1}), \Gamma(N g_{1} \oplus N_{1,1})), \]
and, using the identification (1.5),
\[ H_d(\Gamma(S h_{1} \oplus S_{1,1}), \Gamma(S h_{1}))) \to H_d(\Gamma(N 2h_{1,1} \oplus S_{1,1}), \Gamma(N 2h_{1,1})). \]

Let \( g \geq 4 \). Then these maps induce the secondary stabilization map for Theorem A
\[ H_d(\Gamma(N g_{-1,1}, \Gamma g_{-3,1}) \oplus H_d(\Gamma(g_{-1}/2,1, \Gamma(g_{-3}/2,1)) \to H_d(\Gamma(N g_{1,1}, \Gamma g_{-2,1}), \]
when \( g \) is odd, and
\[ H_d(\Gamma(N g_{-1,1}, \Gamma g_{-3,1}) \to H_d(\Gamma(N g_{1,1}, \Gamma g_{-2,1})) \]
when \( g \) is even.

We now describe the map in Theorem C. For \( g \geq 4 \), the square
\[ \Gamma(N g_{-3,1}) \xrightarrow{\oplus id_{N1,1}} \Gamma(N g_{-3,1} \oplus S_{1,1}) \]
\[ \Gamma(N g_{-3,1} \oplus N_{1,1}) \xrightarrow{\oplus id_{N1,1}} \Gamma(N g_{-3,1} \oplus N_{1,1} \oplus S_{1,1}) \]
commutes by naturality of the braiding. Analogously to (1.5), there are well-defined isomorphisms
\[ H_d(\Gamma(N g_{1} \oplus N_{1,1} \oplus S_{1,1}), \Gamma(N g_{1} \oplus S_{1,1})) \cong H_d(\Gamma(N g_{1,1}, \Gamma g_{-2,1}, \Gamma g_{2,1}), \]

Definition 1.2. Let \( g \geq 4 \) be even. The secondary stabilization map for Theorem C is the map
\[ H_d(\Gamma(N g_{-2,1}, \Gamma g_{-3,1}) \to H_d(\Gamma(N g_{1,1}, \Gamma g_{-1,1}). \]
induced by the squares (1.3) using the identifications (1.7).

1.2. Organization of the paper. The paper is structured as follows.

1. In Section 2, we define an \( E_2 \)-algebra \( R \) of mapping classes and prove that its \( E_2 \)-homology vanishes in a range. In the process, we also prove that the complex of separating arcs on a surface is highly-connected, which may be of independent interest.

2. In Section 3, we construct a homotopy theoretic refinement of the secondary stabilization map of Section 1.1, the relative homology of which measures the failure of secondary stability.

3. In Section 4, we review calculations of the homology of mapping class groups of nonorientable surfaces due to Stukow, Paris, Szepietowski, and others.

4. In Section 5, we construct a “presentation” of \( R \) and prove that it induces isomorphisms on the \( E_2 \)-homology in a range of degrees. Using this, we leverage the Galatius–Kupers–Randal-Williams theory to prove our results.
1.3. Acknowledgements. The content of this paper was mostly extracted from my master’s thesis. I express my gratitude to my advisor, Professor Søren Galatius. As pertains to this project, I am particularly grateful for his suggestion that I should consider the $E_2$-algebra of all surfaces, as opposed to restricting attention to only nonorientable surfaces.

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2. The $E_2$-algebra of surface configurations

In this section, we study an $E_2$-algebra of mapping class groups of surfaces, culminating in a vanishing range for its $E_2$-homology. For $0 < \varepsilon < \frac{1}{2}$, let $\partial_\varepsilon I^2 \subseteq I^2$ denote the open $\varepsilon$-collar neighborhood, that is, the subset of points that have $< \varepsilon$ Euclidean distance to a point in $\partial I^2$.

**Definition 2.1.** The braided strict monoidal groupoid $(\text{MCG}, \oplus, (I^2 \times 0, 0), \beta)$ has

$$\text{Ob}(\text{MCG}) := \{(C \subseteq I^2 \times \mathbb{R}^\infty, r \in \mathbb{R}_{>0}) \mid C \cong S_{g,1} \text{ or } N_{g,1} \text{ for some } g > 1, \quad \partial C = \partial I^2 \times 0, \quad C - \partial C \subseteq (0,1)^2 \times \mathbb{R}^\infty, \quad \exists 0 < \varepsilon < 1/2 : \partial_\varepsilon I^2 \times 0 \subseteq C\}$$

$\cup\{(I^2 \times 0, 0)\}$

as set of objects, and

$$\text{MCG}((C_1, t_1), (C_2, t_2)) := \{f : C_1 \xrightarrow{\text{homeo}} C_2 \mid f|_{\partial I^2 \times 0} = \text{id}\}$$

isotopy rel. $\partial I^2 \times 0$.

as morphisms. In particular, $\text{Aut}_{\text{MCG}}((C, t)) = \Gamma(C)$. The monoidal product $\oplus$ is the strictly associative boundary sum, given by horizontal scaling and horizontal juxtaposition

$$(C_1, t_1) \oplus (C_2, t_2) := \begin{cases} \left(t_1 + \frac{t_1 t_2}{t_1 + t_2} \cdot C_1 \cup \left(t_1 + \frac{t_2}{t_1 + t_2} \cdot C_2\right), t_1 + t_2 > 0, \quad t_1 = t_2 = 0. \right) \\ (I^2 \times 0, 0), \quad \end{cases}$$

The braiding $\beta : (C_1, t_1) \oplus (C_2, t_2) \longrightarrow (C_2, t_2) \oplus (C_1, t_1)$ is the clockwise half Dehn twist. Up to a shrinking of the surfaces (expanding the flat collar inwards), this is
the mapping class given by gluing $\text{id}_{C_1}, \text{id}_{C_2}$ to the clockwise elementary braid $e$ of $S_{0,3}$ that swaps these boundaries; see Figure 2.2.

**Figure 2.2.** Cutting out the surfaces $S_1$ and $S_2$, $e$ is the unique mapping class of a diffeomorphism of $S_{0,3}$ fixing the outer boundary, identically sending the left-hand inner boundary to the right-hand inner boundary, and taking the isotopy class of the arc $a_1$ (respectively $a_2$) to the isotopy class of the arc $a_1'$ (respectively $a_2'$).

**Remark 2.2.** It is tempting to define $\text{Ob}(\text{MCG})$ to be the set of diffeomorphism types of connected surfaces with one boundary component, so that each isomorphism class contains exactly one object. However, doing so, it becomes difficult, if not impossible, to extend the monoid structure of $\text{Ob}(\text{MCG})$ to a monoidal structure on $\text{MCG}$. This has to do with the mixture of orientable and nonorientable surfaces. While there exist diffeomorphisms $N_{1,1}^g \oplus S_{1,1}^{2h} \cong N_{1,1}^{g+2h}$, there are no natural, canonical choices of such, and so it is unclear how one would coherently define the mapping class $f \oplus g \in \Gamma(N_{1,1}^g) \oplus \Gamma(S_{1,1}^{2h})$ given $f \in \Gamma(N_{1,1}^g)$, $g \in \Gamma(S_{1,1}^{2h})$. On the other hand, if we considered only orientable or only nonorientable surfaces, it is easy to define a monoidal structure of boundary sums on the groupoid $\text{MCG}'$ whose objects $\text{Ob}(\text{MCG}') := \{0, 1, 2, \ldots\}$ are diffeomorphism types represented by genera.

By a unital (respectively nonunital) $E_1$-algebra, we mean an algebra over the operad $S$ defined in Appendix A (respectively its nonunital version where $S(0)$ is replaced by $\emptyset$). The corresponding free monad is denoted by $E_1^+$ (respectively $E_1$). By a unital (respectively nonunital) $E_2$-algebra, we mean an algebra over the operad $B$ defined in Appendix A (respectively its nonunital version where $B(0)$ is replaced by $\emptyset$). The corresponding free monad is denoted by $E_2^+$ (respectively $E_2$). Note that these definitions differ from [GKRW19b] which prefers the little $k$-cubes operads (see [GKRW19b, Definition 12.1]), but their formalism is equivalent to ours by Proposition A.3.

**Definition 2.3.** The unital $E_2$-algebra $R'$ in graded simplicial sets $\text{sSet}^{S_0}$ is the classifying space of the braided strict monoidal groupoid $(\text{MCG}, \oplus, (I^2 \times 0, 0), \beta)$. 

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The text continues with more mathematical content, including definitions, theorems, and additional explanations.
as constructed in Appendix A graded (see Remark A.7) via the grading
\[ h : \text{MCG} \rightarrow \mathbb{N}_0, \]
\[ (S,t) \mapsto \begin{cases} 2g, & \text{if } S \cong S_{g,1}, \\ g, & \text{if } S \cong N_{g,1}. \end{cases} \]
The nonunital \( E_2 \)-algebra \( R \in \text{Alg}_{E_2}(\text{sSet}^{N_0}) \) is obtained from \( R' \) by restricting the \( E_2^+ \)-algebra structure to an \( E_2 \)-algebra structure and replacing \( R'(0) \) with \( \emptyset \).

We write \( H_{n,d}(R) := H_d(R(n)) \).

**Lemma 2.4.**
\[ H_{n,d}(R) \cong \begin{cases} 0, & \text{if } n = 0, \\ H_d(\Gamma_{n/2,1}) \oplus H_d(\Gamma_{n,1}), & \text{if } n \geq 1 \text{ and } n \text{ is even}, \\ H_d(\Gamma_{n,1}), & \text{if } n \geq 1 \text{ and } n \text{ is odd}. \end{cases} \]

**Proof.** This is because \( \text{Aut}_{\text{MCG}}((C,t)) = \Gamma(C) \). \hfill \Box

2.1. **Identification of the \( E_1 \)-splitting complex.** The \( E_2 \)-structure on \( R \) forgets down to an \( E_1 \)-structure. Our next objective is to prove a vanishing range for the \( E_1 \)-homology of \( R \) (see [GKRW19b, 10.1.6]).

One of the key features of the Galatius–Kupers–Randal-Williams theory is the \( E_1 \)-splitting complex \( S^{E_1}(g) \), which is a certain semisimplicial set associated to a monoidal groupoid \( \mathcal{G} \) and an object \( g \in \mathcal{G} \). They prove that connectivity estimates for \( S^{E_1}(g) \) for various \( g \in \mathcal{G} \) imply a vanishing estimate for the \( E_1 \)-homology of an \( E_1 \)-algebra associated to \( \mathcal{G} \) (as in Definition A.8). Such an estimate, in turn, can often be “transferred up” to a vanishing estimate for the \( E_2 \)-homology. We refer the reader to [GKRW19b, 17.2 and 14.2] for generalities on these constructions.

We will often notationally suppress length data when denoting objects of \( \text{MCG} \).

**Lemma 2.5.** For \( S_1, S_2 \in \text{MCG} \), the map \( \rightarrow : \Gamma(S_1) \times \Gamma(S_2) \rightarrow \Gamma(S_1 \oplus S_2) \) is injective.

**Proof.** Let \([a]\) be the isotopy class of the arc \( a \) on \( S_1 \oplus S_2 \) placed where the gluing occurred. An application of the isotopy extension theorem shows that the stabilizer of \([a]\) under the action of \( \Gamma(S) \) is \( \Gamma(S-a) \). Therefore, \( \rightarrow \) identifies with the inclusion \( \text{St}([a]) \hookrightarrow \Gamma(S_1 \oplus S_2) \). \hfill \Box

Associated to the monoidal category \( (\text{MCG}, \oplus) \) are the \( E_1 \)-splitting complexes \( S^{E_1}(S) \in \text{ssSet} \) with
\[ S^{E_1}(S)_p := \colim_{(S_0,\ldots,S_{p+1}) \in \text{MCG}^{p+2}} \text{MCG}(S_0 \oplus \cdots \oplus S_{p+1}, S) \]
and the face map \( d_t \) given by changing the subscript
\[ (S_0,\ldots,S_{p+1}) \mapsto (S_0,\ldots,S_t \oplus S_{t+1},\ldots,S_{p+1}). \]

[GRW19b] Assumption 17.1] is satisfied by definition, and [GRW19b] Assumption 17.2] is satisfied by our Lemma 2.5.

**Definition 2.6.** Let \( S \) be an orientable or nonorientable, connected surface with nonempty boundary, and let \( b_0, b_1 \) be distinct points in \( \partial S \) on the same boundary component. The **complex of separating arcs** \( \mathcal{D}(S, b_0, b_1) \) is the simplicial complex whose \( p \)-simplices are collections of \( p + 1 \) distinct isotopy classes of arcs between \( b_0, b_1 \) that admit representatives \( a_0, \ldots, a_p \) such that
(a) for each \(i \neq j\), \(a_i \cap a_j = \{b_0, b_1\}\) and
(b) for each \(i\), \(S - a_i\) consists of two components, none of which are diffeomorphic to \(S_{0,1}\).

The complex of separating arcs is spiritually related to the complex of separating curves treated by Looijenga [Loo13].

![Figure 2.3. A 1-simplex \(⟨x, y⟩ ∈ \mathcal{D}(N_{4,1}, b_0, b_1)\).](image)

We shall temporarily abuse notation and write \(b_0\) for the top left corner of the boundary and \(b_1\) for the bottom left corner of the boundary for any object in \(\text{MCG}\).

Each monoidal product \(S_0 \oplus \cdots \oplus S_{p+1}\) admits a natural system \(σ_{S_0 \oplus \cdots \oplus S_{p+1}} ∈ \mathcal{D}(S_0 \oplus \cdots \oplus S_{p+1}, b_0, b_1)_p\) of separating arcs \(a_0, \ldots, a_p\) such that each \(a_i\) is isotopic (allowing the endpoints to move along the horizontal boundary segments) to the arc where the \(i\)’th gluing occurred. Let \(S ∈ \text{MCG}\). Endow \(\mathcal{D}(S, b_0, b_1)\) with the structure of a semisimplicial set by ordering arcs from left to right at \(b_0\). For each \(p ≥ 0\) and product \(S_0 \oplus \cdots \oplus S_{p+1}\), there is a map

\[
\text{MCG}(S_0 \oplus \cdots \oplus S_{p+1}, S) \to \mathcal{D}(S, b_0, b_1)_p
\]
given by taking the image of \(σ_{S_0 \oplus \cdots \oplus S_{p+1}}\) under diffeomorphisms.

**Lemma 2.7.** The maps described above assemble into an isomorphism of semisimplicial sets

\[
ϕ : S^{E_1}(S) \xrightarrow{\cong} \mathcal{D}(S, b_0, b_1).
\]

**Proof.** Clearly, the maps assemble into a map of semisimplicial sets \(ϕ\). A simplex \(σ ∈ \mathcal{D}(S, b_0, b_1)_p\) cuts \(S\) into surfaces \(S_0, \ldots, S_{p+1}\), ordered from left to right. The diffeomorphism \(S_0 \oplus \cdots \oplus S_{p+1} \to S\), mending \(S\) back together, determines an element \(\overline{σ}\) of \(S^{E_1}(S)_p\), such that \(ϕ(\overline{σ}) = σ\). This shows surjectivity. For injectivity, suppose that \(ϕ(\overline{σ}) = ϕ(\overline{β})\) for \(α ∈ \text{MCG}(S_0 \oplus \cdots \oplus S_{p+1}, S), \ β ∈ \text{MCG}(S'_0 \oplus \cdots \oplus S'_{p+1}, S)\). Since \(α\) and \(β\) must preserve the ordering at \(b_0\) and \(b_1\), and since \(α(σ_{S_0 \oplus \cdots \oplus S_{p+1}}) = β(σ_{S'_0 \oplus \cdots \oplus S'_{p+1}})\), \(α\) and \(β\) differ at most up to precomposition by diffeomorphisms of each summand, hence \(\overline{σ} = \overline{β}\). \(\square\)

### 2.2. Auxiliary complexes

Before we can execute our connectivity arguments, we must define some auxiliary simplicial complexes and review their connectivity estimates. By convention, “\((-1)\)-connected” means nonempty.

**Definition 2.8** ([Wal07, Definition 3.1]). Let \(S\) be an orientable or nonorientable, connected surface with nonempty boundary, and let \(b_0, b_1\) be distinct points in \(∂S\). \(BX(S, b_0, b_1)\) is the simplicial complex whose \(p\)-simplices are collections of \(p + 1\)
distinct isotopy classes of arcs between $b_0, b_1$ that admit representatives $a_0, \ldots, a_p$ such that

(a) $a_i \cap a_j = \{b_0, b_1\}$ for each $i \neq j$ and
(b) $S - (a_0 \cup \cdots \cup a_p)$ is connected.

For convenience, set

$$h(N_{g,r}) := g,$$
$$h(S_{g,r}) := 2g.$$ 

This notation is compatible with the functor $h$ considered earlier.

**Theorem 2.9** ([Wah07, Theorem 3.2]). Let $S$ be an orientable or nonorientable, connected surface with nonempty boundary, let $b_0, b_1$ be distinct points in $\partial S$, and let $i$ be the number of boundary components of $S$ that intersect $\{b_0, b_1\}$. Then $BX(S, b_0, b_1)$ is $(h(S) + i - 3)$-connected.

**Definition 2.10** ([Wah07, Definition 5.3]). Let $S$ be an orientable or nonorientable, connected surface. $C_0(S)$ is the simplicial complex whose $p$-simplices are collections of $p + 1$ distinct isotopy classes of simple closed curves in $S$ that admit representatives $c_0, \ldots, c_p$ such that

(a) $c_i \cap \partial S = \emptyset$ for each $i$,
(b) $c_i \cap c_j = \emptyset$ for each $i \neq j$, and
(c) $S - (c_0 \cup \cdots \cup c_p)$ is connected.

**Theorem 2.11** ([Wah07, Theorem 5.4]). Let $S$ be an orientable or nonorientable, connected surface. Then $C_0(S)$ is $\left\lfloor \frac{h(S) - 3}{2} \right\rfloor$-connected.

### 2.3. Connectivity estimates for the complex of separating arcs

The objective of this subsection is to prove a connectivity estimate for the $E_1$-splitting complexes $S^{E_1}(S)$. We use an induction argument that creates more boundary components in the induction step and will suggest the following auxiliary complex.

**Definition 2.12.** Let $S$ be an orientable or nonorientable, connected surface with $r \geq 1$ boundary components, let $b_0, b_1 \in \partial S$ be distinct points in the same boundary component, and let $\tilde{b}_0$ be an orientation of the boundary near $b_0$. $\mathcal{P}'(S, b_0, \tilde{b}_0, b_1) \subseteq \mathcal{P}(S, b_0, b_1)$ is the subcomplex consisting of those collections $\sigma$ of isotopy classes of arcs that cut $S$ into surfaces $S_0, \ldots, S_{p+1}$ (ordered according to $b_0$ at $b_0$) such that $S_0, \ldots, S_p$ each have only one boundary component.

We remind the reader of the function $h$ defined in Section 2.2 which counts crosscaps by 1 and torus holes by 2.

**Theorem 2.13.** Let $(S, b_0, \tilde{b}_0, b_1)$ be as above. Then

(a) if $r = 1$, $\mathcal{P}'(S, b_0, \tilde{b}_0, b_1)$ is $\left\lfloor \frac{h(S) - 1}{2} \right\rfloor$-connected, or
(b) if $r > 1$, $\mathcal{P}'(S, b_0, \tilde{b}_0, b_1)$ is $\left\lfloor \frac{h(S) - 1}{2} \right\rfloor - 1$-connected.

The proof relies on the following theorem. A poset $\mathcal{X}$ is said to be $n$-connected, if its nerve $N\mathcal{X}$ is $n$-connected. A subposet $\mathcal{Y} \subseteq \mathcal{X}$ is said to be closed if for any $y \in \mathcal{Y}$, $x < y \implies x \in \mathcal{Y}$.
Theorem 2.14 (Nerve Theorem, [GKRW19a, Corollary 4.2]). Let \( X \) be a poset, let \( A \) be another poset, let

\[
F : A^{\text{op}} \longrightarrow \{ \text{closed subposets of } X \}
\]

be a map of posets, let \( t_X : X \longrightarrow \mathbb{Z} \) and \( t_A : A \longrightarrow \mathbb{Z} \) be functions of sets, and let \( n \in \mathbb{Z} \). Suppose that

(1) \( A \) is \((n - 1)\)-connected,  
(2) for every \( a \in A \), \( A_{\leq a} \) is \((t_A(a) - 2)\)-connected and \( F(a) \) is \((n - t_A(a) - 1)\)-connected, and 
(3) for every \( x \in X \), \( X_{\leq x} \) is \((t_X(x) - 2)\)-connected and the subposet

\[
A_x := \{ a \in A \mid x \in F(a) \}
\]

is \(((n - 1) - t_X(x) - 1)\)-connected.

Then \( X \) is \((n - 1)\)-connected.

The following proof is inspired by the proof of [GKRW19a, Theorem 4.9]. We use curves instead of arcs to ensure that cutting always decrease genus, as would not be the case when cutting along one-sided arcs.

Proof of Theorem 2.13. Ordering arcs according to \( \tilde{b}_0 \) at \( b_0 \) endows \( \mathcal{D}'(S, b_0, \tilde{b}_0, b_1) \) with the structure of a semisimplicial set. Furthermore, it gives \( \mathcal{D}'(S, b_0, \tilde{b}_0, b_1)_0 \) the structure of a poset. The canonical map of simplicial sets

\[
s : \mathcal{D}'(S, b_0, \tilde{b}_0, b_1) \longrightarrow N(\mathcal{D}'(S, b_0, \tilde{b}_0, b_1)),
\]

is an isomorphism, so it suffices to show that \( \mathcal{D}'(S, b_0, \tilde{b}_0, b_1) \) is highly-connected.

The cases [a] and [b] are proven simultaneously by induction on \( h(S) \). The base case \( h(S) = 0 \) is vacuous, so suppose \( h(S) \geq 1 \). In each induction step, Case [a] is proven before Case [b], which uses Case [a].

Case [a]. Suppose (a) and (b) are known for surfaces \( S' \) with \( h(S') < h(S) \). For a simplicial complex \( X \), let \( \text{simp}(X) \) denote the poset of simplices. Note that \( \text{simp}(X) \) is \( n \)-connected if and only if \( X \) is \( n \)-connected.

Let \( A := \text{simp}(C_0(S)) \). Consider the map of posets

\[
F : A^{\text{op}} \longrightarrow \{ \text{closed subposets of } \mathcal{D}'(S, b_0, \tilde{b}_0, b_1) \},
\]

\( \langle c_0, \ldots, c_p \rangle \longrightarrow \{ a \mid c_0, \ldots, c_p \text{ are on the right of } a \} \).

By an isotopy class \( c_i \) “being on the right” of an isotopy class \( a \), we mean that there are representatives enjoying this property. Then we show [a] by substituting

\[
n = \left\lfloor \frac{h(S) - 1}{2} \right\rfloor - 1,
\]

\[
t_A(a_0, \ldots, a_p) = p,
\]

\[
t_X(a) = \left\lfloor \frac{h(S_0) - 1}{2} \right\rfloor
\]

for \( S_0 \) the surface left of \( a \) into Theorem 2.14. We now check its assumptions.

[1] \( A \) is \( \left\lfloor \frac{h(S) - 3}{2} \right\rfloor \)-connected by Theorem 2.11 and so [1] follows from the inequality

\[
\left\lfloor \frac{h(S) - 1}{2} \right\rfloor - 2 \leq \left\lfloor \frac{h(S) - 3}{2} \right\rfloor.
\]
For any $p$-simplex $\sigma \in \mathcal{A}$, $\mathcal{A}_{<\sigma}$ is isomorphic to $\partial \Delta^p$, which is $(p - 2)$-connected, showing the first part of (2) on the other hand, the canonical map

$$F(\sigma) \xrightarrow{\cong} \widehat{\mathcal{D}}(S - \sigma, b_0, \bar{b}_0, b_1)$$

admits an inverse induced by the gluing map $S - \sigma \longrightarrow S$. $S - \sigma$ has at least two boundary components and $h(S) \geq h(S) - 2(p + 1)$. Thus, by the induction hypothesis on (b), $F(\sigma)$ is at least $n = t_A(\sigma) - 1$-connected, showing the second part of (2).

Let $x \in \widehat{\mathcal{D}}(S, b_0, \tilde{b}_0, b_1)$. Suppose $x$ separates the surface into $S_0$ on the left and $S_1$ on the right. Then $\widehat{\mathcal{D}}(S, b_0, \tilde{b}_0, b_1) \leq (\widehat{\mathcal{D}}(S_0, b_0, \tilde{b}_0, b_1)$ and $\mathcal{A}_x \cong \text{simp}(C_0(S_1))$. By the induction hypothesis on (a) the first part of assumption (3) is clear. Note that $h(S_0) + h(S_1) = h(S)$, so

$$\left[\frac{h(S_0) - 1}{2}\right] + \left[\frac{h(S_1) - 1}{2}\right] \geq \left[\frac{h(S) - 1}{2}\right] - 1.$$

By Theorem 2.11, $C_0(S_1)$ is therefore

$$\left[\frac{h(S_1) - 1}{2}\right] - 1 \geq \left[\frac{h(S) - 1}{2}\right] - 1 - \left[\frac{h(S_0) - 1}{2}\right] - 1 > (n - 1) - t_X(x) - 1$$

-connected. This shows the second part of (3).

Case (b) Suppose

(i) (a) is known for $S'$ with $h(S') \leq h(S)$,

(ii) (b) is known for $S'$ with $h(S') < h(S)$, and

(iii) (b) is known for $S'$ with $2 \leq r' < r$ boundary components.

Choose $b_2 \in \partial S$ on the interior of the right-hand (according to $\bar{b}_0$) segment between $b_0, b_1$ of the boundary, and choose $b_3 \in \partial S$ on another boundary component. Let $\mathcal{A} := \text{simp}(\text{BX}(S, b_2, b_3))$. Consider the map of posets

$$F : \mathcal{A}^\text{op} \longrightarrow \{\text{closed subposets of } \widehat{\mathcal{D}}(S, b_0, \tilde{b}_0, b_1)\},$$

$$(a_0, \ldots, a_p) \mapsto \{a \mid a_0, \ldots, a_p \text{ is on the right of } a\}.$$ 

Then we show (b) by substituting

$$n = \left\lfloor\frac{h(S) - 1}{2}\right\rfloor,$$

$$t_A(a_0, \ldots, a_p) = p,$$

$$t_X(a) = \left\lfloor\frac{h(S_0) - 1}{2}\right\rfloor$$

for $S_0$ the surface left of $a$ into Theorem 2.14. We now check its assumptions.

1This map is well-defined by standard arguments. For example, one can check this by an induction on intersections and [Eps66] Lemma 3.2.
[1] $A$ is $(h(S) - 2)$-connected by Theorem 2.9 and so (1) follows from the inequality
$$\left\lfloor \frac{h(S) - 1}{2} \right\rfloor - 1 \leq h(S) - 1.$$

(2) For any $p$-simplex $\sigma \in A$, $A_{<\sigma}$ is isomorphic to $\partial \Delta p$, which is $(p - 2)$-connected, showing the first part of [2]. If $S - \sigma$ has only one boundary component, $F(\sigma)$ is contractible as the arc parallel to the right-hand boundary segment between $b_0, b_1$ of $S - \sigma$ constitutes a terminal element, and if $S - \sigma$ has multiple boundary components, $F(\sigma) \cong \bar{D}'(S - \sigma, b_0, \tilde{b}_0, b_1)$ for reasons similar to those in (a) In the former case, there is nothing to show, so suppose that $S - \sigma$ has multiple boundary components. We have $h(S - \sigma) \geq h(S) - 2p$. Therefore, $F(\sigma)$ is at least
$$\left\lfloor \frac{h(S) - 2p - 1}{2} \right\rfloor - 1 = \left\lfloor \frac{h(S) - 1}{2} \right\rfloor - p - 1$$
-connected by induction hypothesis [iii] if $h(S - \sigma) < h(S)$ or [iii] if $S - \sigma$ has fewer boundary components than $S$. This shows the second part of (2).

(3) Let $x \in \bar{D}'(S, b_0, \tilde{b}_0, b_1)$. Suppose $x$ separates the surface into $S_0$ on the left and $S_1$ on the right. Then $\bar{D}'(S, b_0, \tilde{b}_0, b_1) \subseteq \bar{D}'(S_0, b_0, \tilde{b}_0, b_1)$ and $A_{<x} \cong \text{simp}(BX(S_1, b_2, b_3))$. Note that $S_0$ has only one boundary component. By induction hypothesis [1] the first part of assumption (3) is clear. Note that $h(S_0) + h(S_1) = h(S)$. By Theorem 2.9 $BX(S_1, b_2, b_3)$ is therefore
$$h(S_1) - 1 = h(S) - h(S_0) - 1 \geq (n - 1) - t_X(x) - 1$$
-connected. This shows the second part of (3).

Since $D(S, b_0, b_1) = \bar{D}'(S, b_0, \tilde{b}_0, b_1)$ when $S$ has only one boundary component, Theorem 2.13 will suffice for our purpose of giving a vanishing estimate for the $E_2$-homology of $R$. However, the following result might be interesting in its own right.

**Theorem 2.15.** Let $(S, b_0, b_1)$ be as above, where $S$ has $r \geq 1$ boundary components. Then the complex of separating arcs $D(S, b_0, b_1)$ is $((h(S) - 1)/2 - 3 + r)$-connected.

We prove this using Theorem 2.13 and the technique of surgering arcs, which is originally due to Hatcher [Hat91]. We use a particular variant of this technique, which we learned from [Wahl07].

**Proof.** Case: $r = 1$. This is Theorem 2.13(a) for $\tilde{b}_0$ is chosen in any way.

Case: $r > 1$. We apply a surgery argument to reduce to the case $r = 1$. Proceed by induction on $r$. Pick a boundary component $C$ distinct from the one containing $b_0, b_1$. We call a vertex $I$ of $D(S, b_0, b_1)$ special if it separates the surface $S$ into surfaces $S_0, S_1$ one of which is the annulus around $C$. Let $\{I_j\} \subseteq D(S, b_0, b_1)$ be the set of special vertices. The case $(h(S), r) = (0, 2)$ is vacuous. Therefore,
suppose either $h(S) > 0$ or $r > 2$ so that $\{I_j\} \neq \emptyset$. We have
\begin{equation}
\mathcal{D}(S, b_0, b_1) = \bigcup_{j} \mathcal{D}_{sp}(S, b_0, b_1) \cup \text{St}(I_j),
\end{equation}
where $\mathcal{D}_{sp}(S, b_0, b_1) \subseteq \mathcal{D}(S, b_0, b_1)$ is the subcomplex of simplices not containing special vertices, and where each star $\text{St}(I_j)$ is attached to $\mathcal{D}_{sp}(S, b_0, b_1)$ along the link of $I_j$. Indeed, two distinct stars have intersection contained in $\mathcal{D}_{sp}(S, b_0, b_1)$, since any two distinct special arcs intersect. Pick a special vertex $I_1 \in \{I_j\}$.

**Claim.** $X := \mathcal{D}_{sp}(S, b_0, b_1) \cup \text{St}(I_1)$ is contractible.

We say that an arc $a$ on $S$ is **trivial** if $S - a$ has two components, one of which is diffeomorphic to $S_{0,1}$. The definition of "special arcs" was effectively chosen so as to make the following argument go through. In particular, excluding special arcs ensures that the surgery does not create trivial arcs, as is illegal by Definition 2.6(b).

**Proof of claim.** We construct a contraction of $X$ onto the star $\text{St}(I_1)$, which is contractible. Fix a representative $I'_1$ for the isotopy class $I_1$. Let $I$ be a vertex of $X$. If $I \in \text{St}(I_1)$, define $f(I) := I$. If $I \notin \text{St}(I_1)$, any representative of $I$ intersects $I'_1$. Choose a representative $I'$ of $I$ that intersects $I'_1$ transversely with minimal number of intersections. By standard arguments, such a choice is unique up to isotopy through arcs intersecting $I'_1$ transversely and minimally. Inside a fixed model of the $S_{0,2}$ bounded by $I'_1$, label the segments of $I'$ by $\ell_1, \ldots, \ell_n$ ordered starting with the one farthest from $b_0$. We arrange that $\ell_1, \ldots, \ell_{n-1}$ are parallel straight lines through the annulus and, perhaps after relabeling $b_0, b_1$, the remaining $\ell_n$ either meets $b_1$ or is a straight line too. This places us in the situation of one of the following standard pictures.

![Figure 2.4. The standard pictures.](image)

Replace the outermost segment $\ell_1$ of $I'$ with an arc $\ell'_1$ that instead of intersecting $I'_1$ on the interior takes a detour around $I'_1$ to the other side. The resulting arc $\tilde{I}'$ obtained from making this replacement is again separating, nontrivial, and nonspecial, and it will have either $2n - 1 < 2n$ or $2n - 2 < 2n$ intersections with $I'_1$. Repeating this procedure, we eventually earn an arc $J'$, which has no intersections with $I'_1$. Define $f(I) := J$ for $J$ the isotopy class of $J'$.

This defines a simplicial map $f : X \to X$ with image in $\text{St}(I_1)$ and we claim that $|f| \simeq \text{id}_{X_1}$. If $\sigma = \langle a_0, \ldots, a_p \rangle \in X$ is not in $\text{St}(I'_1)$ and $a_i$ contains the rightmost line through the annulus bounded by $I'_1$ among all the lines spanned by $a_0, \ldots, a_p$ in the standard picture as in Figure 2.4, then $\langle a_0, \ldots, a_p, \tilde{a}_i \rangle \in X$ also, where $\tilde{a}_i$
is $a_i$ after a single surgery step. By sliding along the latter simplex in a straight line to replace $a_i$ by $\bar{a}_i$ and iterating this procedure until all intersections with $I'_2$ are resolved, we obtain a well-defined homotopy $H_\sigma : I \times |\sigma| \to |X|$ between the geometric realizations of $f|\sigma$ and $\text{id}_X|\sigma$. After possibly reparameterizing each $H_\sigma(\ldots, x)^2$ all these homotopies patch together to a homotopy

$$f \simeq \text{id}_X : I \times |X| \to |X|.$$ Therefore, $f$ constitutes a homotopy equivalence. \hfill \Box

Note that for each $j$, $\text{Lk}(I_j) \cong \mathcal{D}(S', b_0, b_1)$ for $S' := S \cup_C D^2$. With this, the claim, and the fact that stars are contractible, (2.1) implies that

$$|\mathcal{D}(S, b_0, b_1)| \simeq \bigvee_{j \neq 1} S|\text{Lk}(I_j)| \cong \bigvee_{j \neq 1} S|\mathcal{D}(S', b_0, b_1)|,$$

where $S-$ denotes the unreduced suspension. The latter is $\left(\left\lfloor \frac{h(S) - 1}{2} \right\rfloor - 3 + r \right)$-connected by the induction hypothesis on $r$. This completes the induction step. \hfill \Box

2.4. Consequences for $E_2$-homology. We now harvest vanishing ranges for $E_2$-homology from the connectivity estimates obtained above.

**Corollary 2.16.** $H_{n,d}^{E_1}(R) = 0$ for $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$.

**Proof.** By [GKRW19b, Lemma 17.10], Theorem 2.13 (or Theorem 2.15), and Lemma 2.7, $T^{E_1}(n)$ (see [GKRW19b, Definition 17.3]) is $\left\lfloor \frac{n-1}{2} \right\rfloor$-connected. By [GKRW19b, Corollary 17.4], and [GKRW19b, Theorem 10.13], $S^1 \wedge Q^{E_1}(R)(n) = h_\ast(S^1 \wedge Q^{E_1}(\mathbb{Z}_p U))(n) \Rightarrow \left\lfloor \frac{n-1}{2} \right\rfloor$-connected as well. The difference in the definition of our $R$ from their $R$ is accounted for by Lemma A.9. The statement follows. \hfill \Box

**Corollary 2.17.** $H_{n,d}^{E_2}(R) = 0$ for $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$.

**Proof.** Apply [GKRW19b, Theorem 14.3] with $l = 1, k = 2$, and $\rho(n) = \left\lfloor \frac{n-1}{2} \right\rfloor + 1$. The first assumption of said theorem follows from the inequality

$$\left\lfloor \frac{n_1 - 1}{2} \right\rfloor + 1 + \left\lfloor \frac{n_2 - 1}{2} \right\rfloor + 1 \geq \left\lfloor \frac{n_1 + n_2 - 1}{2} \right\rfloor + 1.$$

The second assumption of said theorem follows from Corollary 2.16. The statement follows. \hfill \Box

3. Identification of the secondary stabilization map

The purpose of this section is to construct a graded simplicial set, the homology of which measures the failure of secondary stability. This object will be the homotopy cofiber of a certain homotopy theoretic refinement of the secondary stabilization map of Section 1.1.

We use a general setting. Let $(\mathcal{G}, \circ, 0)$ be a symmetric monoidal groupoid (for our purposes, $(\mathcal{G}, \circ, 0) = (\text{N}_\mathbb{Z}, +, 0)$ suffices), let $(\mathcal{S}, \otimes, U)$ be a convenient symmetric monoidal, simplicial model category subject to the axioms of [GKRW19b, 2.1 and 7.1] (for example, $(\mathcal{S}, \otimes, U) = (\text{sSet}, \times, *)$), and let $A \in \text{Alg}_{E_2}(\mathcal{S}^\mathcal{G})$ be an $E_2$-algebra. Endow $\mathcal{S}^\mathcal{G}$ with the projective model structure and the Day convolution monoidal product. For any $g \in \mathcal{G}$, define $U_g := g_\ast U \in \mathcal{S}^\mathcal{G}$, the object with

\[\text{2For instance, proceed by induction over the skeleta of } X.\]
\(U_g(x) = \emptyset\) for \(x \not\equiv g\) and \(U_g(x) = U\) for \(x \equiv g\). For any two objects \([x], [y] \in \mathcal{G}\), maps \(x : U_{[x]} \to U^{E_2} A, y : U_{[y]} \to U^{E_2} A\) in \(\mathcal{S}^G\), and a right \(A\)-module \(M \in \text{Alg}_{\otimes A}(\mathcal{S}^G)\) with cofibrant and fibrant underlying object in \(\mathcal{S}^G\), we would like to construct a quotient \(M 
leftrightarrow (x, y)\) that, morally, kills \(x, y\) from the homology of \(M\). One general construction of this sort is given in [GKRW19b, 12.2.3]. Unfortunately, their construction is somewhat inexplicit, which in our application makes it difficult to obtain an explicit description of the secondary stabilization map as in Section 1.1.

To overcome this issue, we give a very concrete, albeit less general, construction. Since \(A\) is an \(E_2\)-algebra, there is a homotopy coherent square in \(\mathcal{S}^G\),

\[
\begin{array}{c}
U^A M \otimes U_{[x]} \otimes U_{[y]} \xrightarrow{\times} U^A M \otimes U_{[y]} \\
\downarrow y & \downarrow y \\
U^A M \otimes U_{[x]} & \xrightarrow{x} U^A M,
\end{array}
\]

(3.1)

where the structure homotopy is the clockwise elementary braid (half rotation) of \(x\) and \(y\), that is, it is induced via the \(A\)-module structure on \(M\) by the composite

\[
\Delta^1 \otimes U_{[x]} \otimes U_{[y]} \xrightarrow{\alpha \otimes x \otimes y} B(2) \otimes U^{E_2} A \otimes U^{E_2} A \xrightarrow{\alpha} U^{E_2} A,
\]

where \(\alpha\) is the map \(\Delta^1 \xrightarrow{} N\text{Braid}_2\) classifying the elementary braid \(e_1 \in B_2\), and the second map is the \(B\)-algebra structure map for \(A\).

**Definition 3.1.** Define \(M \nleftrightarrow y := \text{hocofib}(M \otimes U_{[y]} \to M) \in \text{Ho}(\mathcal{S}^G)\). The homotopy coherent square (3.1) induce a (homotopy class of a) map of homotopy cofibers

\[
\xymatrix{ x : M \nleftrightarrow y \otimes U_{[x]} \ar[r] & M \nleftrightarrow y 
}
\]

Define \(M \nleftrightarrow (x, y) \in \text{Ho}(\mathcal{S}^G)\) to be the homotopy cofiber of this map. Since the homotopy coherent square (3.1) is functorially associated to \(M\), this defines a functor

\[
\xymatrix{ - \nleftrightarrow (x, y) : \text{Alg}_{\otimes A}(\mathcal{S}^G) \ar[r] & \text{Ho}(\mathcal{S}^G). 
}
\]

Recall that we fixed models \(N_{1,1}, S_{1,1} \in \text{MCG}\) in Section 1.1. For a surface \(S \in \text{MCG}\), let \(\text{MCG}_{=S}\) denote the subcategory of those objects isomorphic to \(S\). A homotopy coherent square \(\Box\) rectifies to a strict square \(\Box'\) if there is a zig-zag of weak equivalences of homotopy coherent squares (that is, pointwise weak equivalences which respect the structure homotopies) from \(\Box\) to \(\Box'\).

**Proposition 3.2.** Let \(S \in \text{MCG}\). Then the homotopy coherent square

\[
\begin{array}{c}
\text{NMCG}_{=S} \oplus N_{1,1} \ar[r] & \text{NMCG}_{=S \oplus N_{1,1}} \\
\downarrow \oplus S_{1,1} & \downarrow \oplus S_{1,1} \\
\text{NMCG}_{=S \oplus S_{1,1}} \oplus N_{1,1} \ar[r] & \text{NMCG}_{=S \oplus S_{1,1}, N_{1,1} \oplus S_{1,1}}
\end{array}
\]

(3.2)

in \(\text{sSet}\) with structure homotopy induced by the braiding natural isomorphism rectifies to the nerve functor \(N\) applied to the strict square (1.3).

**Proof.** Consider the zig-zag of homotopy coherent squares,
Here, $\text{MCG}(S \oplus N_1, 1, S \oplus N_1, 1)$ denotes the full subgroupoid of $\text{MCG}$ generated by $S \oplus N_1, 1$ and $S \oplus N_1, 1$. (1) is induced by essentially surjective inclusions of full subgroupoids hence is a weak equivalence. (2) retracts the lower right groupoid $\text{MCG}(S \oplus N_1, 1, S \oplus N_1, 1)$ to its full subgroupoid on the object $S \oplus N_1, 1$, mapping $S \oplus N_1, 1$ and automorphisms thereof identically, and mapping $S \oplus N_1, 1$ to $S \oplus N_1, 1$ and automorphisms of $S \oplus N_1, 1$ to automorphisms of $S \oplus N_1, 1$ by means of the braiding $\beta$.

Set $M := R$ and fix maps
\[
x := \bar{\sigma} : U_1 \rightarrow R,
\]
\[
y := \bar{\tau} : U_2 \rightarrow R
\]
classifying the fixed models $N_1, 1, S_1, 1 \in \text{MCG}$ with length parameter 1. In this setting, the homotopy coherent square (3.1) takes the form
\[
\begin{array}{c}
R \otimes U_1 \otimes U_2 \xrightarrow{\bar{\sigma}} R \otimes U_2 \\
\downarrow \bar{\tau} \quad \downarrow \bar{\tau}
\end{array}
\]
in $\text{sSet}^N$.

**Lemma 3.3.** Let $g \geq 4$. Then the map
\[
- \cdot \bar{\sigma} : \tilde{H}_{g,d}(R \parallel \bar{\tau} \otimes U_1) \rightarrow \tilde{H}_{g,d}(R \parallel \bar{\tau})
\]
defined by (3.3) identifies with the secondary stabilization map (cf. Definition 1.1)
\[
H_d(N \Gamma_{g-1,1}, N \Gamma_{g-3,1}) \oplus H_d(\Gamma_{(g-1)/2,1}, \Gamma_{(g-3)/2,1}) \rightarrow H_d(N \Gamma_{g,1}, N \Gamma_{g-2,1})
\]
when $g$ is odd, and the composite of the secondary stabilization map and the inclusion
\[
H_d(\mathcal{N}_{g-1,1}, \mathcal{N}_{g-3,1}) \longrightarrow H_d(\mathcal{N}_{g,1}, \mathcal{N}_{g-2,1}) \longrightarrow H_d(\mathcal{N}_{g,1}, \mathcal{N}_{g-2,1}) \oplus H_d(\mathcal{N}_{g,1}, \mathcal{N}_{g-2,1})
\]
when $g$ is even.

Proof. Evaluated on an odd $g \geq 4$, (3.3) is
\[
\begin{align*}
\text{NMCG} \cong \mathcal{N}_{g-3,1} \cup \text{NMCG} \cong \mathcal{S}_{g/2-1,1} \oplus \text{NMCG} \cong \mathcal{N}_{g-2,1} \\
\text{NMCG} \cong \mathcal{N}_{g-1,1} \cup \text{NMCG} \cong \mathcal{S}_{g-3/2,1} \oplus \text{NMCG} \cong \mathcal{N}_{g-1,1},
\end{align*}
\]
where the structure homotopy is given by the braiding. Evaluated on an even $g \geq 4$, (3.3) is
\[
\begin{align*}
\text{NMCG} \cong \mathcal{N}_{g-3,1} \cup \text{NMCG} \cong \mathcal{S}_{g-3,1} \oplus \text{NMCG} \cong \mathcal{N}_{g-2,1} \\
\text{NMCG} \cong \mathcal{N}_{g-1,1} \cup \text{NMCG} \cong \mathcal{S}_{g-1,1} \oplus \text{NMCG} \cong \mathcal{N}_{g-1,1},
\end{align*}
\]
where the structure homotopy of the first summand is again given by the clockwise half Dehn twist. The statement now follows from Proposition 3.2. □

Lemma 3.4. Let $g \geq 4$ be an even integer. Then the map
\[
\tilde{\tau} : \tilde{H}_{g,d}(R \parallel \bar{\sigma} \otimes U_2) \longrightarrow \tilde{H}_{g,d}(R \parallel \bar{\sigma})
\]
induced by (3.3) identifies with the map
\[
H_d(\mathcal{N}_{g-2,1}, \mathcal{N}_{g-3,1}) \oplus H_d(\mathcal{N}_{g-2,1}) \longrightarrow H_d(\mathcal{N}_{g,1}, \mathcal{N}_{g-1,1}) \oplus H_d(\mathcal{N}_{g,1}, \mathcal{N}_{g-2,1})
\]
induced by the secondary stabilization map (cf. Definition 1.2) and the torus hole stabilization map (cf. Section 1).

Proof. This follows from the proof of Lemma 3.3 using that the square (1.3) for $S = \mathcal{N}_{g-3,1}$ is isomorphic to the square (1.6) flipped along the diagonal. □

4. Low-dimensional calculations

4.1. Path components. Let $FX$ denote the free commutative semigroup on a set $X$.

Proposition 4.1. The natural map of commutative semigroups
\[
F\{S_{1,1}, N_{1,1}\}/(S_{1,1} \oplus N_{1,1} = N_{1,1}^{\oplus 3}) \longrightarrow \pi_{*,0}(R)
\]
is an isomorphism.

Proof. This follows from the construction of $R$ using that Euler characteristic is additive with respect to boundary sum. □
4.2. **Review of known unstable homology groups of** $N\Gamma_{g,r}$. Aside from the stable homology groups of $N\Gamma_{g,1}$, some of which were computed by Randal-Williams [RW08], not many homology groups beyond $H_1$ are known. This subsection is a review of these unstable calculations.

**Proposition 4.2.** $N\Gamma_{1,1} = 0$.

*Proof.* Consider a parameterized one-sided arc $c : I \rightarrow N_{1,1}$ from some $b_0 \in \partial N_{1,1}$ to itself. Since its complement is $S_0,1$ and $\Gamma_{0,1} = 0$ (the Alexander trick [FM11], Lemma 2.1], the isotopy class of a boundary-fixing diffeomorphism $\varphi : N_{1,1} \rightarrow N_{1,1}$ is determined by the isotopy class of $\varphi_c$. Since the boundary is fixed, $\varphi$ induces the identity $Z \rightarrow Z$ on $\pi_1$ (it must send $2 \rightarrow 2$). Therefore, $c \simeq \varphi_c$, which implies that $c$ and $\varphi_c$ are isotopic by [Eps66, 3.1], and thus $\varphi$ is isotopic to the identity. □

**Definition 4.3.** Fix a model $N_{1,1} \in \text{MCG}$ for the Möbius strip. Write $N_{g,1} := N_{1,1}^{\oplus g}$. The crosscap transposition of the pair of crosscaps $(i, i+1)$ on $N_{g,1}$ is the mapping class

$$\text{id}_{N_{i,1}} \oplus \beta \oplus \text{id}_{N_{g-i-2,1}} : N_{i,1} \oplus N_{g-i-2,1} \xrightarrow{\cong} N_{i,1} \oplus N_{g-i-2,1},$$

where $\beta : N_{1,1}^{\oplus 2} \rightarrow N_{1,1}^{\oplus 2}$ is the unique (by Proposition 4.2) mapping class sending $a_1$ to $a_2$ up to isotopy:

![Figure 4.1. $\beta : [a_1] \mapsto [a_2]$.](image)

**Proposition 4.4.**

$$H_*(N\Gamma_{2,1}) = \begin{cases} 
\mathbb{Z}, & \text{if } * = 0, \\
\mathbb{Z}_2 \oplus \mathbb{Z}_u, & \text{if } * = 1, \\
0, & \text{if } * > 1,
\end{cases}$$

where $b$ is a Dehn twist along a nonseparating, two-sided curve and $u$ is the crosscap transposition.

*Proof.* Stukow [Stu06, A.2] showed that

$$N\Gamma_{2,1} = \langle b, y \mid byb = y = [b, u \mid bab = u] \rangle,$$

where $y := bu$ is the crosscap slide. The statement is now immediate for $* = 0, 1$. For $* > 1$, observe that the Klein bottle $K = N_{2,0}$ is aspherical and has $\pi_1(K) \cong N\Gamma_{2,1}$ hence is a $K(N\Gamma_{2,1})$. However, $H_*(K) = 0$ for $* > 1$. □

---

3Randal-Williams considers $N\Gamma_{\infty,0}$, but this has the same homology as $N\Gamma_{\infty,1}$ by [Wah07, Theorem A(3)].
Proposition 4.5. Let \( a \) denote a Dehn twist along a nonseparating, two-sided curve with nonorientable complement, let \( b \) denote a Dehn twist along a nonseparating, two-sided curve with orientable complement, and let \( u \) denote a crosscap transposition. Then
\[
\begin{align*}
(a) \quad H_1(N\Gamma_{3,1}) &= \mathbb{Z}_2 a \oplus \mathbb{Z}_2 u, \\
(b) \quad H_1(N\Gamma_{4,1}) &= \mathbb{Z}_2 a \oplus \mathbb{Z}_2 b \oplus \mathbb{Z}_2 u, \\
(c) \quad H_1(N\Gamma_{g,1}) &= \mathbb{Z}_2 a \oplus \mathbb{Z}_2 u \text{ for } g \in \{5, 6\}, \text{ and} \\
(d) \quad H_1(N\Gamma_{g,1}) &= \mathbb{Z}_2 u \text{ for } g \geq 7.
\end{align*}
\]

Proof. This is a matter of abelianizing the presentation obtained by Paris and Szepietowski [PS15, Theorem 3.5] combined with the \( H_1 \) calculations of \( \Gamma_{g,1} \) and \( \Gamma_{g,2} \) in [Kor02]. We omit these straightforward calculations. \( \square \)

Remark 4.6. The mapping classes \( a, u \) are uniquely determined up to conjugation hence determine unique classes in \( H_1 \), whereas the conjugacy class of \( b \) is only determined up to inverse, since its complement is orientable and so one cannot construct a diffeomorphism reversing the curve. However, \( b \) is nonetheless uniquely determined in \( H_1 \), as \( 2b = 0 \) in \( H_1 \) per the statements, so the ambiguity up to a sign disappears.

Remark 4.7. The torus hole stabilization map
\[
H_1(N\Gamma_{2,1}) \to H_1(N\Gamma_{4,1})
\]
takes \( b \) to \( b \), whereas the composite of crosscap stabilization maps
\[
H_1(N\Gamma_{2,1}) \to H_1(N\Gamma_{3,1}) \to H_1(N\Gamma_{4,1})
\]
takes \( b \) to \( a \). This shows that the torus hole stabilization map does not always factor as two crosscap stabilization maps.

4.3. Failure of stability on \( H_2 \). We make the following observation for the purpose of compiling the tables in Section 4.

Proposition 4.8. None of
\[
\begin{align*}
(a) \quad H_2(N\Gamma_{3,1}, N\Gamma_{2,1}), \\
(b) \quad H_2(N\Gamma_{5,1}, N\Gamma_{4,1}), \\
(c) \quad H_2(N\Gamma_{7,1}, N\Gamma_{6,1}), \\
(d) \quad H_2(N\Gamma_{4,1}, N\Gamma_{2,1}), \\
(e) \quad H_2(N\Gamma_{6,1}, N\Gamma_{4,1}), \\
(f) \quad H_2(N\Gamma_{7,1}, N\Gamma_{5,1}), \text{ or} \\
(g) \quad H_2(N\Gamma_{8,1}, N\Gamma_{6,1})
\end{align*}
\]
vanish.

Proof. This is immediate from the associated long exact sequences and the results of Section 4.2. \( \square \)

5. A presentation for the \( E_2 \)-algebra \( R \)

In this section, we prove Theorem A using a strategy similar to the one used to prove the generic stability theorem [GKRW19b, Theorem 18.1]. As we are interested in the homology of \( R \), we may as well linearize and consider instead the \( E_2 \)-algebra \( R_\mathbb{Z} \in \text{Alg}_{E_2} (\text{sMod}_\mathbb{Z}) \) in graded simplicial \( \mathbb{Z} \)-modules. We
write $H_{n,d}(R_Z) := \pi_d(R_Z(n))$ and similar for other simplicial modules, reflecting the Dold–Kan correspondence, which then says that

$$H_{n,d}(R_Z) \cong H_{n,d}(R).$$

To simplify the argument, we are going to make use of the operations

$$Q^1_Z : H_{n,0}(A) \to H_{2n,1}(A)$$

for $A \in \textbf{Alg}_{E_2}(\textbf{sMod}_Z)$. These were described in [GKRW19, 3.1] and have two important properties,

1. $Q^1_Z(x) \otimes F_2 = Q^1_Z(x \otimes F_2)$, where $Q^1_Z := \xi : H_{n,0}(A \otimes F_2) \to H_{2n,1}(A \otimes F_2)$ is the Dyer–Lashof top operation, and
2. $2Q^1_Z(x) = -[x, x]$.

Their purpose is to unify arguments that would otherwise need to be repeated first over $F_2$ and then over $F_p$ for $p$ an odd prime.

Recall the homology classes $d, b, u$ from Section 4. Let $\sigma \in H_{1,0}(R_Z)$ denote the generator corresponding to the path-component for $N_{1,1}$, and let $\tau \in H_{2,0}(R_Z)$ denote the generator corresponding to the path-component for $S_{1,1}$. Let $R_{\sigma} \subseteq R$ be the sub-$E_2$-algebra on the orientable surfaces, and let $d \in H_{2,1}(R_{\sigma}) \cong H_1(\Gamma_{1,1})$ denote a class represented by a Dehn twist along a nonseparating curve. (There are two choices for $d$, one for each orientation.) Since Dehn twists along nonseparating curves generate the mapping class group $\Gamma_{g,1}$ (see [FM11, Theorem 5.4]), $d$ generates $H_{2,1}(R_{\sigma})$ and $\tau \cdot d$ generates $H_{4,1}(R_{\sigma})$. Pick $n \in \mathbb{Z}$ such that

$$n \cdot \tau \cdot d = Q^1_Z(\tau) \in H_{4,1}(R_{\sigma}) \subseteq H_{4,1}(R_Z).$$

Fix representative maps $\tilde{d}, \tilde{b}, \tilde{u}$ from spheres to $R_Z$ representing $d, b, u$ respectively. Also, let $\tilde{\sigma}, \tilde{\tau}$ be the maps classifying the models $N_{1,1}, S_{1,1} \in \textbf{MCG}$ that we fixed in Section 4.1. Using these, define a map of nonunital $E_2$-algebras

$$A' := E_2(S^1 \otimes S^2(d + b \oplus Z^1)) \to R_Z.$$  

Fix a map $\tilde{Q}^1_Z(\tau) : S^{4,1} \to A'$ representing $Q^1_Z(\tau) \in H_{4,1}(A')$ and let also $\tilde{Q}^1_Z(\tau) : S^{4,1} \to A' \to R_Z$ denote the composite representing $Q^1_Z(\tau) \in H_{4,1}(R)$. By choosing null homotopies of $\tilde{\sigma} \tilde{\tau} - \tilde{\sigma} \tilde{\tau}$ and $\tilde{Q}^1_Z(\tau) - n \cdot \tilde{\tau} \cdot d$ witnessing their triviality in $H_{4,1}(R_Z)$, we pick an extension

$$A := E_2(S^1 \otimes S^2 \otimes Z^1(d + b \oplus Z^2)) \cup_{\tilde{E}_2} D^3 \cup_{\tilde{E}_2} Q^1_Z(\tau) - n \cdot \tilde{\tau} \cdot d \to R_Z$$

of $A' \to R_Z$. This is the presentation of $R_Z$ we shall use to prove Theorem A.

Here, for a ring $B$, $S^n_B \in \textbf{Mod}_B$ is defined by

$$S^n_B(g) := \begin{cases} 
B[\Delta^n]/B[\partial \Delta^n], & \text{if } g = n, \\
0, & \text{else},
\end{cases}$$

(as opposed to just linearizing the sphere in $\textbf{sSet}$!) and $D^n_B \in \textbf{Mod}_B$ is defined by

$$D^n_B(g) := \begin{cases} 
B[\Delta^n], & \text{if } g = n, \\
0, & \text{else}.
\end{cases}$$

$- \cup_{x} E_2 D^n_B$ denotes $E_2$-cell attachment; see [GKRW19, 6.1.1].
Lemma 5.1. The map $H_*,0(A) \to H_*,0(R\mathbb{Z})$ is an isomorphism of $\mathbb{N}_0$-graded nonunital rings.

Proof. Let $\text{Rng} : \text{Ab}^{\mathbb{N}_0} \to \text{Ab}^{\mathbb{N}_0}$ be the graded, commutative, nonunital ring monad. $H_*,0 : \text{Alg}_{E_2}(s\text{Mod}_{\mathbb{N}_0}^\mathbb{Z}) \to \text{Alg}_{\text{Rng}}(\text{Ab}^{\mathbb{N}_0})$ preserves colimits. Since $H_*,0F_{\text{Rng}} \cong F_{E_2}H_*,0$, we get

$$H_*,0(A) = F_{\text{Rng}}(1\ast \mathbb{Z}\sigma \oplus 2\ast \mathbb{Z}\tau)/(\sigma\tau - \sigma^3).$$

The statement now follows from Proposition 4.1. \qed

Lemma 5.2. $H_{n,d}^{E_2}(R\mathbb{Z}, A) = 0$ for $d/n < 1/3$.

Proof. It is known that $H_{n,d}^{E_2}(A) = 0 = H_{n,d}^{E_2}(R\mathbb{Z})$ for $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$. The first equality is clear from the bidegrees $(n,d)$ of the generators and relators in the presentation. The second equality is Corollary 2.17. From the long exact sequence for $(R\mathbb{Z}, A)$, it follows that $H_{n,d}^{E_2}(R\mathbb{Z}, A) = 0$ when $d \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$.

Since the only $(n,d)$ satisfying $d > \left\lfloor \frac{n-1}{2} \right\rfloor - 1$ and $d/n < 1/3$ are $(n,d) = (1,0), (2,0), (4,1)$, it remains to show vanishing of $H_{n,d}^{E_2}(R\mathbb{Z}, A)$ in these cases.

For any $n \geq 0$, the long exact sequence also gives the exact sequence

$$H_{n,d}^{E_2}(R\mathbb{Z}) \to H_{n,d}^{E_2}(R\mathbb{Z}, A) \to 0,$$

where the first map is an isomorphism by Lemma 5.1 and the fact that $H_*,0 \circ Q^{E_2} = Q^{\text{Rng}} \circ H_*,0$. Thus, $H_{n,0}^{E_2}(R\mathbb{Z}, A) = 0$, resolving the first two cases.

The case $(n,d) = (4,1)$ remains. As was noted above, $\tau \cdot d$ generates $H_1(\Gamma_{2,1})$. By Proposition 4.5, $\sigma^2 \cdot u$, $\tau \cdot b$, and $a = \sigma^2 \cdot d$ generate $H_1(NT_{4,1})$. Therefore, $H_{4,1}(A) \to H_{4,1}(R\mathbb{Z})$ is surjective. We have an exact sequence

$$H_{4,1}(A) \to H_{4,1}(R\mathbb{Z}) \to H_{4,1}(R\mathbb{Z}, A) \to H_{4,0}(A) \cong H_{4,0}(R\mathbb{Z}),$$

where the last map is an isomorphism by Lemma 5.1. Thus, $H_{4,1}(R\mathbb{Z}, A) = 0$. Using [GKRW19b Proposition 11.9] with $c = (0,0,0,\ldots)$, $c_f = (1,1,1,\ldots)$, the Hurewicz
map
\[ H_{4,1}(R_Z, A) \longrightarrow H_{4,1}^F(R_Z, A) \]
is surjective. In particular, \( H_{4,1}^F(R_Z, A) = 0 \) as desired. \( \square \)

5.1. **Proof of Theorem A and Theorem C.** The construction in Section 3 gives us an object \( R \) in \( \text{sMod}_{s_{\text{gr}}}^{N_0} \), which is weakly equivalent to the linearization of the object \( R \) in \( \text{sSet}_{s_{\text{gr}}}^{N_0} \) studied in Section 3.

**Theorem 5.3.** \( \tilde{H}_{n,d}(R_Z \parallel (\bar{\sigma}, \bar{\tau})) = 0 \) for \( d_3 < \frac{1}{3} \).

**Proof.** As a convention, \( F_0 := \mathbb{Q} \). For a prime \( p \) or \( p = 0 \), let \( R_p := R_Z \otimes F_p \in \text{Alg}_{E_2}(\text{sMod}_{s_{\text{gr}}}^{N_0}) \) be the change of coefficients to \( F_p \). In turn, it suffices to show \( \tilde{H}_{n,d}(R_p \parallel (\bar{\sigma}, \bar{\tau})) = 0 \) for each \( p \) and \( d_3 < \frac{1}{3} \).

Fix \( p \). By Lemma 5.2 and [GKRW19b] Theorem 11.21, there is an \( E_2 \)-CW-approximation
\[ A \otimes F_p \longrightarrow Z \xrightarrow{\sim} R_p, \]
with \( E_2 \)-cells \( D_{sp}(n_i, d_i) \times_1 \) with \( d_i/ n_i \geq \frac{1}{3} \) attached to \( A \otimes F_p \). Let \( \text{sk}(Z) \) denote \( Z \) filtered with the nonnegative skeletal filtration. There is a trigraded spectral sequence
\[ E^1_{n,p,q} = H_{n,p+q}(\text{lgr}(\text{sk}(Z) \parallel (\sigma, \tau))) \Rightarrow H_{n,p+q}(Z \parallel (\sigma, \tau)), \]
obtained from the filtered object \( \text{sk}(Z) \parallel (\sigma, \tau) \). Here,
\[ \text{lgr} : \text{Ho}((\text{sMod}_{s_{\text{gr}}}^{N_0})^{\mathbb{Z}_{\geq 0}}) \longrightarrow \text{Ho}((\text{sMod}_{s_{\text{gr}}}^{N_0})^{\mathbb{Z}_{\geq 0}}) \]
is the derived associated graded. Since homotopy cofibers commute with homotopy cofibers, and since \( \text{sk}(Z) \) is filtered by cofibrations, we have
\[ \text{lgr}(\text{sk}(Z) \parallel (\sigma, \tau)) \simeq \text{gr}(\text{sk}(Z)) \parallel (\sigma, \tau). \]

By [GKRW19b] Lemma 12.7(iii) and a variant of [GKRW19b] Theorem 6.14, we have
\[ \text{gr}(\text{sk}(Z)) \cong 0, (A \otimes F_p) \oplus E_2 \left( \bigoplus_i (d_i)_* S_{s_{\text{gr}}}^{n_i, d_i} x_i \right) \]
as \( E_2(\text{s}_1^{1,0} \sigma \oplus \text{s}_2^{2,0} \tau) \)-modules in \( \text{sSet}_{s_{\text{gr}}}^{N_0 \times \mathbb{Z}_{\geq 0}} \). Filtering \( \text{gr}(\text{sk}(Z)) \) by the cell attachment filtration (see [GKRW19b] (6.5)), we get an additional spectral sequence
\[ E^1_{n,p,q} = H_{n,p+q}(\text{lgr}_{\text{cell}}(\text{gr}(\text{sk}(Z) \parallel (\sigma, \tau))(p))) \Rightarrow H_{n,p+q}(\text{gr}(\text{sk}(Z) \parallel (\sigma, \tau))) \]
computing \( E^1_{n,*,*} \), where we decorate each gr with a subscript specifying the filtration. As before,
\[ \text{lgr}_{\text{cell}}(\text{gr}(\text{sk}(Z) \parallel (\sigma, \tau)) \simeq \text{gr}_{\text{cell}}(\text{gr}(\text{sk}(Z))) \parallel (\sigma, \tau) \simeq E_2(X) \parallel (\sigma, \tau) \]
as \( E_2(\text{s}_1^{1,0} \sigma \oplus \text{s}_2^{2,0} \tau) \)-modules, where
\[ X := \text{s}_1^{1,0} \sigma \oplus \text{s}_2^{2,0} \tau \oplus \text{s}_3^{2,1} d \oplus \text{s}_4^{2,1} b \oplus \text{s}_5^{3,1} u \oplus \text{s}_6^{3,1} \lambda \oplus \text{s}_7^{4,2} \rho \oplus \bigoplus_i S_{s_{\text{gr}}}^{n_i, d_i} x_i, \]
forgetting the internal grading. By Cohen’s theorem [GKRW19b] Theorem 16.4,
\[ H_{*,*}(E_2(X)) \cong H_{*,*}(E_2^+ (X)) \cong W_1(H_{*,*}(X)), \]
(5.4)
where \( W_1(-) \) takes free \( W_1 \)-algebra (see [GKRW19b, 16.1]). To be more explicit, this is the bigraded commutative algebra
\[
W_1(\text{H}_*,* (X)) = \Lambda_{\mathbb{F}_{p}} (L)
\]
where \( L \) is the trigraded vector space with homogeneous basis the Dyer–Lashof monomials \( Q^p_{\xi} (y) \) for \( I \) satisfying familiar admissibility and excess conditions and for \( y \) a basic Lie word in \( \{ \sigma, \tau, d, b, u, \lambda, \rho, x_i | i \} \). The homotopy cofibers defining \( E_2(X) \parallel (\sigma, \tau) \) have associated long exact sequences in homology. By (5.4), these degenerate into short exact sequences, which then fit into diagrams

\[
\begin{array}{ccc}
0 & \rightarrow & H_{n,*}(E_2(X)) \\
\downarrow & & \downarrow \sigma \\
0 & \rightarrow & H_{n+1,*}(E_2(X)) \rightarrow H_{n+1,*}(E_2(X) \parallel \sigma) \rightarrow 0 \\
\downarrow \tau & & \downarrow \\
0 & \rightarrow & H_{n+2,*}(E_2(X)) \rightarrow H_{n+3,*}(E_2(X) \parallel \sigma) \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & H_{n+2,3}(E_2(X) \parallel \tau) \rightarrow H_{n+3,*}(E_2(X) \parallel (\sigma, \tau)) \rightarrow 0 \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0 \rightarrow 0 \rightarrow 0.
\end{array}
\]

Therefore,
\[
H_{*,*}(E_2(X) \parallel (\sigma, \tau)) \cong W_1(\text{H}_*,* (X)) / (\sigma, \tau).
\]

We have now computed the \( \overline{E^1} \)-term of (5.3) after forgetting the internal grading.

The slope of a homology class \( x \) in bidegree \( (n,d) \) is the number \( \frac{n}{d} \in \mathbb{Q} \cup \{ \infty \} \). Products \( x \cdot y \) have slope greater than or equal to the least of the slopes of \( x, y \). Browder brackets \([x,y]\) have slope strictly greater than the least of the slopes of \( x, y \). The Dyer–Lashof operations never decrease slope either. \( \sigma \) and \( \tau \) are the only generators in \( X \) that have slope \( < \frac{1}{3} \). Thus, only basis elements of \( L \) involving \( \sigma \) and \( \tau \) can have slope \( < \frac{1}{3} \). In fact, the only basis elements of \( L \) that have slope \( < \frac{1}{3} \) are \( \sigma \in W_1(\text{H}_*,* (X))_{1,0}, \tau \in W_1(\text{H}_*,* (X))_{2,0}, [\tau, \tau] \in W_1(\text{H}_*,* (X))_{4,1}, \) and when \( p = 2, Q^1_{2}(\tau) \in W_1(\text{H}_*,* (X))_{4,1} \).

Claim. \( \overline{E^2}_{n,p,q} = 0 \) for \( \frac{p+q}{n} < \frac{1}{3} \).

Proof of claim. To show this statement, we may forget about the internal grading \( p \) by defining \( \overline{E^2}_{n,k} := \bigoplus_{p+q=k} \overline{E^2}_{n,p,q} \). The claim then becomes that \( \overline{E^2}_{n,k} = 0 \) when \( k < \frac{1}{3} \). \( \overline{E^2}_{*,*} \) is the homology of the free differential graded algebra
\[
(\overline{E^1}_{*,*}, d^1) = (\Lambda_{\mathbb{F}_{p}} (L / (\tau, \sigma)), d^1).
\]
To estimate these homology groups, we make use of an auxiliary filtration. Abuse notation and write \( Q^2_{\xi}(x) := Q^1_{\xi}(x) \otimes \mathbb{F}_{p} \in \text{H}_{0,0}(R_{p}) \). Filter \( \overline{E^1}_{*,*} \) by giving \( Q^1_{\xi}(\tau) \) and \( \rho \) filtration 0, and filtering the remaining basis elements by homological degree, and extending this filtration multiplicatively. This filtration respects the differentials. The desirable effect of this filtration is that taking the associated graded filters
away most of the $d^1$-differential, leaving only the differentials that we need. In particular, the associated graded of this filtration is

$$U := (\Lambda_p(L/(\sigma, \tau, Q^1_2(\tau), \rho)), d^1 = 0) \otimes (\Lambda_p(Q^1_2(\tau), \rho), \delta)$$

where $\delta(\rho) = Q^1_2(\sigma)$ and $\delta(Q^1_2(\sigma)) = 0$ (all generators of degree 0 were killed). The first factor of the tensor product has itself as homology. If $p$ is odd, the homology of the second factor of the tensor product is 0. If $p = 2$, the homology of the second factor of the tensor product is $\Lambda_2[\rho^2]$. Now, there is a trigraded spectral sequence

$$E^1_{n,p,q} = H_{n,p+q,p}(U) \Rightarrow E^2_{n,p+q}.$$

By the considerations preceding the statement of the claim and by properties (1) and (2) of $Q^1_2$ stated in the beginning of the section, all basis elements for $E^1_{n,p,q}$ have slope $p+q \geq \frac{1}{3}$. It follows that $E^1_{n,p,q} = 0$ when $\frac{p+q}{n} < \frac{1}{3}$, and hence that $E^2_{n,k} = 0$ when $\frac{k}{n} < \frac{1}{3}$, as desired.

The statement now follows from the spectral sequences (5.1), (5.3) and the claim.

**Proof of Theorem A.** We have a cofiber sequence

$$R_Z \otimes S^1_{Z} \xrightarrow{\sigma} R_Z \otimes \ldots \rightarrow R_Z \otimes (\sigma, \tau) \xrightarrow{\partial} R_Z \otimes (\sigma, (\sigma, S^1_{Z}$$

in $s\text{Mod}^0$. Since $H_{n,d}(R_{\zeta}) = 0$ for $d < \frac{1}{3}$ by [GKRW9a, Theorem B(i)] and Lemma 3.3, it follows that in the range $\frac{d}{n} < \frac{1}{3}$ and $d \geq 1$, the first map in the cofiber sequence induces the secondary stabilization map of Definition 1.1 on $H_{n,d}$, and so the statement follows from Theorem 5.3.

**Proof of Theorem C.** We have a cofiber sequence

$$R_Z \otimes S^2_{Z} \xrightarrow{\sigma} R_Z \otimes S^2_{Z} \xrightarrow{\sigma} R_Z \otimes (\sigma, (\sigma, S^2_{Z}$$

in $s\text{Mod}^0$. The theorem now follows from Lemma 3.4 and Theorem 5.3.

**Remark 5.4.** The proof of Theorem C in fact recovers the best known version of Harer stability [GKRW9a, Theorem B(i)], due to the orientable terms in Lemma 3.4.

**Remark 5.5.** Theorem 5.3 implies a version of Theorem C for $g$ odd. The resulting statement regards a map of the type

$$H_d(N\Gamma_{g-2,1}, N\Gamma_{g-3,1}, \Gamma_{(g-3)/2,1}) \longrightarrow H_d(N\Gamma_{g,1}, N\Gamma_{g-1,1}, \Gamma_{(g-1)/2,1}).$$

To define the participating relative homology groups, one is forced to pick diffeomorphisms

$$S_{h,1} \otimes N_{1,1} \xrightarrow{\sim} N_{2h+1,1}$$

of which there are no canonical choices (see also Remark 2.2). Due to this ambiguity, we choose not to state a version of Theorem C for $g$ odd, as it seems less useful.
APPENDIX A. Classifying spaces of braided monoidal categories

Let \((\mathcal{C}, \otimes, U, \beta)\) be a braided strict monoidal category. Its classifying space \(B\mathcal{C}\) inherits the structure of a unital \(E_2\)-algebra. One can exhibit this structure in an abstract way by using the formal language of model categories and derived Kan extensions; see [GKRW19b, 17.1]. However, in Section 3 we needed to concretely understand the sense in which the half Dehn twist of 2-cubes corresponds to the braiding \(\beta\). For this purpose, we describe another, more explicit way to construct a unital \(E_2\)-algebra from a strict braided monoidal category.

**Definition A.1.** \(\text{Sym}_n\) is the terminal category. For \(n \geq 1\), \(\text{Sym}_n\) is the discrete category on the object set
\[
\Sigma_n := \{\text{permutations}\ \{1, \ldots, n\} \xrightarrow{\cong} \{1, \ldots, n\}\}.
\]

**Definition A.2.** The (symmetric) operad \(S\) in simplicial sets has
\[
S(n) := N\text{Sym}_n
\]
where \(\Sigma_n\) acts through postcomposition. For \(n \geq 1\), its operadic composition
\[
\text{Sym}_n \times \text{Sym}_{r_1} \times \cdots \times \text{Sym}_{r_n} \rightarrow \text{Sym}_{r_1 + \cdots + r_n},
\]
\[
(\sigma_0, \sigma_1, \ldots, \sigma_n) \mapsto \sigma_{\sigma_0(1)} \oplus \cdots \oplus \sigma_{\sigma_0(n)},
\]
where \(\oplus\) denotes juxtaposition of permutations.

**Definition A.3.** \(\text{Braid}_0\) is the terminal category. For \(n \geq 1\), the groupoid \(\text{Braid}_n\) has \(\text{Ob}(\text{Braid}_n) := \Sigma_n\) as set of objects and
\[
\text{Braid}_n(\sigma_1, \sigma_2) := \{b \in B_n \mid t(b) = \sigma_2 \circ \sigma_1^{-1}\}
\]
as morphisms, where \(B_n\) is the \(n\)th braid group and \(t : B_n \rightarrow \Sigma_n = \text{Ob}(\text{Braid}_n)\) is the canonical homomorphism. Composition is induced by the group operation of \(B_n\).
\[
\circ : \text{Braid}_n(\sigma_2, \sigma_3) \times \text{Braid}_n(\sigma_1, \sigma_2) \rightarrow \text{Braid}_n(\sigma_1, \sigma_3),
\]
\[
(b_1, b_2) \mapsto b_1 b_2 \in B_n.
\]
In particular, for \(n \geq 1\), \(\text{Aut}_{\text{Braid}_n}(\sigma) = B_n\) is the pure braid group.

For \(n \geq 1\), there is a natural action \(\Sigma_n \curvearrowright \text{Braid}_n\) which on objects is given by postcomposing by the acting permutation and on morphisms is given by relabeling braids.

**Definition A.4.** The (symmetric) operad \(B\) in simplicial sets has
\[
B(n) := N\text{Braid}_n
\]
where \(\Sigma_n\) acts through its natural action on \(\text{Braid}_n\). For \(n \geq 1\), its operadic composition
\[
\text{Braid}_n \times \text{Braid}_{r_1} \times \cdots \times \text{Braid}_{r_n} \rightarrow \text{Braid}_{r_1 + \cdots + r_n},
\]
\[
(\sigma_0, \sigma_1, \ldots, \sigma_n) \mapsto \sigma_{\sigma_0(1)} \oplus \cdots \oplus \sigma_{\sigma_0(n)},
\]
\[
(\text{id}_{\sigma_0}, b_1, \ldots, b_n) \mapsto b_{\sigma_0(1)} \oplus \cdots \oplus b_{\sigma_0(n)},
\]
\[
(b, \text{id}, \ldots, \text{id}) \mapsto b
\]
where $\oplus$ denotes juxtaposition of braids or permutations, and $b_\ast \in B_{r_1, \ldots, r_m}$ is block braid induced by $b \in B_n$.

Let $C_k^+$ denote the unital little $k$-cubes operad ([GKW19b Definition 12.1]). There is a natural map of operads

$$f : C_1^+ \rightarrow C_2^+,$$

sending $([a_1, b_1], \ldots, [a_n, b_n])$ to $([a_1, b_1] \times I, \ldots, [a_n, b_n] \times I)$. Furthermore, there is a weak equivalences of operads

$$g : C_1^+ \rightarrow S,$$

which sends $([a_1, b_1], \ldots, [a_n, b_n])$ to the permutation $\sigma \in \Sigma_n$ such that $a_{\sigma(1)} < \ldots < a_{\sigma(n)}$.

**Proposition A.5.** $B$ is $\Sigma$-cofibrant and there is a zig-zag of weak equivalences of pairs of operads

$$(B, S) \xrightarrow{\Sigma} \cdots \xleftarrow{\Sigma} (C_2^+, C_1^+).$$

**Proof.** It is $\Sigma$-cofibrant because each action $\Sigma_n \curvearrowright B(n)_p$ is free. We now exhibit the zig-zag. For each $n \geq 0$, there is a diagram,

$$\begin{array}{cccc}
S(n) & \xleftarrow{\sim} & C_1^+(n) & \xrightarrow{\sim} C_2^+(n) \\
\downarrow & & \downarrow f \circ \eta & \downarrow f \\
B(n) & \xleftarrow{\sim} & \Pi((C_2^+(n), f(C_2^+(n)))) & \xrightarrow{\sim} \Pi(C_2^+(n)).
\end{array}$$

(A.1)

$\Pi(X)$ denotes the fundamental groupoid of $X$ and $\Pi(X, A)$ for $A \subseteq X$ denotes the full subgroupoid of $\Pi(X)$ on the objects $A$. $\eta$ denotes the 1-truncation map $C_1^+(n) \rightarrow \Pi(C_2^+(n))$. (1) is induced by the equivalence of categories $\Pi(C_2^+(n), f(C_2^+(n))) \rightarrow \text{Braid}_n$ that identifies the various vertical configurations corresponding to the same permutation. (2) is induced by the essentially surjective inclusion of the full subcategory $\Pi(C_2^+(n), f(C_2^+(n)))$ into $\Pi(C_2^+(n))$. (3) is the 1-truncation map and is a weak equivalence since its source is an Eilenberg–MacLane space. Ranging $n$, the diagrams assemble into a diagram of operads in $s\text{Set}$, exhibiting the desired zig-zag. \hfill $\square$

**Definition A.6.** Let $(\mathcal{C}, \otimes, U, \beta)$ be a braided strict monoidal category. For each $n \geq 1$, there is a functor

$$\text{Braid}_n \times \mathcal{C}^n \rightarrow \mathcal{C},$$

$$(\sigma, x_1, \ldots, x_n) \mapsto x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

$$(\text{id}_x, f_1, \ldots, f_n) \mapsto f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)},$$

$$(b : \sigma_1 \rightarrow \sigma_2, \text{id}_{x_1}, \ldots, \text{id}_{x_n}) \mapsto b_\ast.$$

Here, $b_\ast : x_{\sigma_1(1)} \otimes \cdots \otimes x_{\sigma_1(n)} \rightarrow x_{\sigma_2(1)} \otimes \cdots \otimes x_{\sigma_2(n)}$ denotes the braiding isomorphism corresponding to $b \in B_n$. There is also a functor

$$\text{Braid}_0 \rightarrow \mathcal{C}$$

sending the unique object in $\text{Braid}_0$ to the unit $U$. Taking the nerve $N$, we get for each $n \geq 0$ a map of simplicial sets

$$N\text{Braid}_n \times N\mathcal{C}^n \rightarrow N\mathcal{C}.$$

These maps endow $N\mathcal{C} \in s\text{Set}$ with the structure of a $B$-algebra.
Remark A.7. The construction also works for graded categories. Let \((\mathbb{N}_0, +, 0)\) denote the discrete, monoidal category whose object set is \(\mathbb{N}_0\) with the additive monoidal structure. A graded braided strict monoidal category \((\mathcal{C}, \otimes, U, \beta, r)\) is a braided strict monoidal category \((\mathcal{C}, \otimes, U, \beta)\) together with a monoidal functor \(r : \mathcal{C} \rightarrow \mathbb{N}_0\). The graded nerve \(N_{\text{gr}} \mathcal{C} \in s\text{Set}^{\mathbb{N}_0}\) has
\[N_{\text{gr}} \mathcal{C}(n) := N(d^{-1}(n)).\]
The construction above gives a map of graded simplicial sets
\[0_* (N\text{Braid}_n) \otimes (N_{\text{gr}} \mathcal{C})^\otimes_n \rightarrow N_{\text{gr}} \mathcal{C},\]
where \(0_* : s\text{Set} \rightarrow s\text{Set}^{\mathbb{N}_0}\) is the left adjoint to the projection to 0, and \(\otimes : s\text{Set}^{\mathbb{N}_0} \times s\text{Set}^{\mathbb{N}_0} \rightarrow s\text{Set}^{\mathbb{N}_0}\) is the Day convolution monoidal product. This gives \(N_{\text{gr}} \mathcal{C}\) the structure of an \(\mathcal{B}\)-algebra in \(s\text{Set}^{\mathbb{N}_0}\).

A.1. Comparison of \(E_1\)-algebras. We now prove a little technical lemma that we need to ascertain that we can make use of the \(E_1\)-splitting complex theory of [GKRW19b] notwithstanding how we define our operads and algebras differently. Let \((\mathcal{G}, \otimes, U, \beta, r)\) be a graded braided strict monoidal category in the sense of Remark A.7. Furthermore, assume that \(\mathcal{G}\) is a groupoid.

Definition A.8. Let \(* \in s\text{Set}^\mathcal{G}\) be the \(\mathcal{G}\)-graded simplicial set with \(* (x) = *\) for all \(x \in \mathcal{G}\). \(*\) admits a unique action from \(\mathcal{C}_1^+\). Define
\[R := Lr_* (\ast) = r_* (c_\ast) \in \text{Alg}_{\mathcal{C}_1^+} (s\text{Set}^{\mathbb{N}_0}),\]
the derived left Kan extension of \(*\) along \(r\). Here,
\[c : \text{Alg}_{\mathcal{C}_1^+} (s\text{Set}^{\mathbb{N}_0}) \rightarrow \text{Alg}_{\mathcal{C}_1^+} (s\text{Set}^{\mathbb{N}_0})\]
denotes a cofibrant replacement functor for the projective model structure.

Recall the map \(g : \mathcal{C}_1^+ \rightarrow S\) defined earlier.

Lemma A.9. There is a zig-zag of weak equivalences between \(R\) and \(g^* N_{\text{gr}} \mathcal{G}\) in \(\text{Alg}_C (s\text{Set}^{\mathbb{N}_0})\) with the projective model structure.

Proof. \(g^* N_{\text{gr}} \mathcal{G}\) may also be described as the left Kan extension along \(r\) of the \(\mathcal{C}_1^+\)-algebra \(T\) arising from the obvious monoid with \(T(x) := N(\mathcal{G}_x)\) the (contractible) nerve of the overcategory. Since \(\mathcal{G}_x\) is a groupoid for each \(x\), the map \(T \rightarrow *\) is a trivial fibration in the projective model structure, hence the map \(c_\ast \rightarrow \ast\) lifts to a weak equivalence \(c_\ast \rightarrow T\), which after left Kan extending along \(r\) descends to a weak equivalence \(R \rightarrow g^* N_{\text{gr}} (\mathcal{G})\) in \(\text{Alg}_{\mathcal{C}_1^+} (s\text{Set}^{\mathbb{N}_0})\) since the underlying objects are cofibrant. \(\square\)
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