On the Factorisation of the Connected Prescription for
Yang-Mills Amplitudes

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Abstract

We examine factorisation in the connected prescription of Yang-Mills amplitudes. The multi-particle pole is interpreted as coming from representing delta functions as meromorphic functions. However, a naive evaluation does not give a correct result. We give a simple prescription for the integration contour which does give the correct result. We verify this prescription for a family of gauge-fixing conditions.

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I. INTRODUCTION

It has been known for some time now that the gauge theory amplitudes in perturbation theory are considerably simpler than the Feynman rules would lead us to believe (see \cite{23} for a review). This has been verified at tree \cite{1,2} and loop \cite{3} levels for ever-increasing numbers of external lines and, in some particular cases such as the MHV (maximally helicity violating) amplitudes, for arbitrary number of external lines. This simplicity is particularly striking for supersymmetric theories but, in some cases, holds true for non-supersymmetric theories.

There is as yet no satisfactory answer to why this is so. After the early work of Nair \cite{5}, Witten \cite{6} tried to explain this simplicity by postulating a duality between the maximally supersymmetric Yang-Mills theory and a string theory in (super-)twistor space. While the details of this duality are not yet fully understood, it has already spurred some considerable advances \cite{4,7,8,21} in computational techniques, especially at tree level. For progress at the loop level, see refs. \cite{9,10,11,12,22,24,25}.

Soon after Witten’s breakthrough, it became clear \cite{18,19,20} that the amplitudes can be computed from a subset of the contributions Witten proposed initially (the connected instantons). We will call this prescription for computing Yang-Mills amplitudes ‘the connected prescription.’ The connected prescription is very elegant, even if it is not the simplest computationally. However, it has the desirable feature of preserving (in a manifest form) a large part of the (sometimes hidden) symmetries of the scattering amplitudes. A number of such arguments for the validity of this prescription were presented in \cite{18}, along with explicit computations in some particular cases. These non-trivial tests leave little doubt that this prescription is correct. For a review covering the early work on the subject, see \cite{13}.

One major drawback of the connected prescription is that the factorisation properties of the amplitudes are not obvious (in the so-called completely disconnected prescription the factorisation properties are obvious but the Lorentz invariance and parity are obscured).

Some arguments that the connected and the completely disconnected prescription are equivalent appeared in ref. \cite{16}. From this point of view, the factorisation can be proven by proving the equivalence with the completely disconnected prescription first. Berkovits and Motl also argued in \cite{17} that factorisation should be a consequence of the possibility of formulating a string field theory (SFT) which, so long as it is consistent, should have the right factorisation properties. This SFT is the off-shell extension of an alternative string
We believe that there should be a simple argument for the factorisation, preferably one which will properly account for particles exchanged in different channels. This is difficult to achieve in this formalism because the formalism is always on-shell, and an off-shell description is not available. Such understanding would be helpful in order to construct loop amplitudes in this picture through use of unitarity.

The organisation of the paper is as follows. First we briefly review the connected prescription and factorisation. We then give an interpretation of the delta functions from the connected prescription in terms of meromorphic functions. Next, we discuss a scaling limit inspired by Berkovits’s string theory interpretation and its consequences. Finally, we argue that this implies factorisation.

II. REVIEW

A. Connected Prescription

In (super-)Yang-Mills theories, tree level $n$-point gluon scattering amplitudes can be colour-decomposed as follows,

$$A_n(\{k_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_n/\mathbb{Z}_n} Tr(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(1)}}) A_n(\sigma(1), h_{\sigma(1)}; \ldots; \sigma(n), h_{\sigma(n)}) ,$$

where $k_i$ is the momentum of the $i$-th particle and $h_i$ is its helicity (we consider all particles to be out-going), $a_i$ label the generators $T^{a_i}$ of the colour algebra, $g$ is the Yang-Mills coupling constant, and $\sigma \in S_n/\mathbb{Z}_n$ instructs us to sum only over cyclically non-equivalent permutations $\sigma$. In the following we will be interested only in the partial amplitudes $A_n$.

In [18, 19, 20] a formula for computing these partial amplitudes was presented. In a slightly different notation, it reads

$$A_n = \int \prod_{k=0}^d d^{d/4} a_k \prod_{i=1}^n d \sigma_i \prod_{i=1}^n \frac{V_i(Z(\sigma_i))}{\sigma_i - \sigma_{i+1}} ,$$

where $I \in \{1, 2, 3, 4|1', 2', 3', 4'\}$, where the external wavefunctions are,

$$V_i(Z) = \int \frac{d \xi_i}{\xi_i} \delta^2(\pi_i - \xi_i \lambda) \exp(i \xi_i \mu, \bar{\pi}_i) \exp(i \xi_i \bar{\psi}^A \eta_i A)$$

In [14] proposed by Berkovits.
and where the twistor-space positions $Z = (\lambda, \mu | \psi)$ are on curves parametrised via,

$$Z^I(\sigma) = \sum_{k=0}^{d} a_k^I \sigma^k,$$

where $\sigma$ and $\xi$ are complex variables, and where $\pi$ is a commuting spinor.

The $\text{vol}(\text{Gl}(2))$ arises from the $\text{Gl}(2)$ symmetry of the integral, which renders it ill-defined. In order to obtain a meaningful result, we must pick a representative from each $\text{Gl}(2)$ orbit, that is fix a gauge.

Eq. (2) yields the scattering amplitude of on-shell gluons with momenta $p_i^\alpha \dot{\pi} = \pi^\alpha \bar{\eta}_i^\alpha$. The helicities are obtained as the coefficients of an expansion in the Grassmann variables $\eta_i$. If there’s no factor of $\eta_i$ present, we take the helicity of $i$-th particle to be $+$, and if there are four factors of $\eta_i$ present ($\eta_1^I \eta_2^J \eta_3^3 \eta_4^4$) we take the helicity of $i$-th particle to be $-$. If $q$ is the number of helicity minus particles, then $d = q - 1$.

One interesting point is that fact that, apart from the momentum conserving delta functions and after gauge-fixing the $\text{Gl}(2)$ symmetry, the number of integrals matches the number of unknowns. Therefore, the integrals above only receive contributions from isolated points. A puzzling fact, already recognised in ref. [18], is that, in order to obtain a correct result, complex solutions to the equations imposed by the above delta functions above had to be included.

The inclusion of these complex solutions is quite unnatural from the point of view of the delta functions and therefore the position in ref. [18] was to not consider the above integrals as ‘real’ integrals, but just as a notation for the procedure of summing a certain ‘Jacobian’ over the roots of the equations imposed by the delta functions.

However, it turns out that this interpretation of eq. (2) is not very useful for displaying the factorisation properties. We will give below an interpretation of these delta functions as meromorphic functions which is suited for proving factorisation.

### B. Factorisation

The partial amplitudes defined above can only have poles when the sum of more than two adjacent momenta goes on-shell. More precisely, if $P \equiv p_1 + \cdots + p_m$ and $P^2 \to 0$, then
we have

\[ A_n(p_1, \ldots, p_n) \sim \sum_{h=\pm} A_{m+1}(p_1, \ldots, p_m, P^h) \frac{i}{P^2} A_{n-m+1}(P^{-h}, p_{m+1}, \ldots, p_n), \]  

(5)

where \( h = \pm \) represents the sum over the two helicities of the internal factorized particle

III. DELTA FUNCTIONS

We now come the the question of how to interpret the delta functions from the connected prescription. The defining property of a delta function is the following property

\[ f(a) = \int dx \delta(x - a)f(x), \]  

(6)

for all functions \( f \).

We propose to take \( \delta(z - z_0) \equiv \frac{1}{2\pi i} \frac{1}{z - z_0} \). Also, the integral should be interpreted as a contour integral along a contour which encircles the point \( z_0 \) in the complex \( z \) plane. Then, the defining property of the delta function results from Cauchy’s theorem (assuming that \( f \) has no poles inside the integration contour).

All the delta functions functions which appear in the connected prescription can be interpreted in this way. This interpretation is compatible with the usual properties of Fourier integrals if we define the Fourier integral to be a complex integral along a contour from zero to infinity, chosen in such a way to insure the convergence. For example, in the case of real \( z \), the Fourier transform of the identity is defined as follows\(^1\)

\[ \int_0^{+\infty} \frac{dk}{2\pi i} e^{ikz} = -\frac{1}{2\pi i} \frac{1}{z} = -\delta(z). \]  

(7)

This kind of contour, from zero to infinity was already used in \([21]\) in a heuristic discussion of twistor-space propagator.

This interpretation is fully compatible with the delta functions manipulations in ref. \([18]\). For example, we have

\[ \int \delta(f(z)) = \frac{1}{2\pi i} \oint \frac{dz}{f(z)} = \sum_{z_i \in \{z | f(z) = 0\}} \frac{1}{f'(z_i)}. \]

(8)

\(^1\) The minus sign might seem a bit strange, but it will be without consequence for our calculation since the phase of the amplitudes in the connected prescription is ambiguous anyway.
FIG. 1: The pinching of the world-sheet in two halves. The crosses mark the position of the world-sheet instantons.

Note also that, multiple roots of \( f(z) = 0 \) do not contribute and the result is obtained by using the Jacobian itself (and not the absolute value of the Jacobian). This is indeed what is required to obtain a correct answer in the connected-prescription computation \[18\].

In what follows, we will leave implicit the replacement of delta functions with their interpretation as meromorphic functions described above.

IV. FACTORISATION

A heuristic picture of when factorisation occurs is clearest in Berkovits’s string. Our discussion below will not depend however on the details of the Berkovits model. An amplitude factorises when the world-sheet (which is topologically a disk for tree amplitudes) can be pinched, separating the vertex operators into two sets. This can be pictured as two disks joined by a very long, thin strip.

In Berkovits’s string the amplitudes are correlation functions of vertex operators in a background gauge field of world-sheet instantons. In the factorisation limit, the vertex operators (instantons) are constrained to be on the border of (in the interior of) the left or right disks. The correlation function of the vertex operators is reduced to an integral over the zero modes of the covariant derivatives in this world-sheet gauge field, which are polynomial in the world-sheet variable \( z \). The integral over the zero modes is then an integral over the coefficients of these polynomials. If the length of the strip is \( L \), which we take to be very
large,\textsuperscript{2} this implies some scaling with $L$ of these coefficients.

We will be interested in the factorisation of an amplitude with $n$ legs and $d + 1$ negative helicity legs into two parts with $n_l$ and $n_r$ legs ($n_l + n_r = n$) and $d_l + 1$ and $d_r + 1$ negative helicity legs ($d_l + d_r = d$).

If we choose the coordinates such that zero is somewhere on the left, (see Fig. 1) the scaling with $L$ of the moduli is such that

$$a_k = \begin{cases} 
\hat{a}_{d_i - k} L^{d_r}, & \text{if } 0 \leq k \leq d_i, \\
\bar{a}_{k - d_i} L^{d - k}, & \text{if } d_l \leq k \leq d,
\end{cases}$$

where $a_k$ is a generic modulus (may be bosonic or fermionic). Note that the two conditions above have an overlap for $k = d_i$. One consequence of this is $\hat{a}_0 = \bar{a}_0$. The hatted and barred variables should be considered of order zero in $L$.

For the positions of the vertex operators, we have

$$\sigma_i = \begin{cases} 
\frac{1}{\sigma_i}, & \text{if } i \text{ is on the left}, \\
\bar{\sigma}_i L, & \text{if } i \text{ is on the right}.
\end{cases}$$

Note that if some $\hat{\sigma} = 0$ or $\bar{\sigma} = 0$, the scaling proposed above does not work. This will become important in the following.

If we also transform the $\xi_i$’s as

$$\xi_i = \begin{cases} 
\hat{\xi}_i \hat{\sigma}_i^{d_i} L^{-d_r}, & \text{if } i \text{ is on the left}, \\
\xi_i L^{-d} \bar{\sigma}_i^{-d_i}, & \text{if } i \text{ is on the right}.
\end{cases}$$

In terms of these variables, the zero modes

$$\xi_i Z^I(\sigma_i) = \xi_i \sum_{k=0}^{d} a_k^I \sigma_i^k,$$  \hspace{1cm} (12)

can be written as

$$\xi_i Z^I(\sigma_i) = \hat{\xi}_i \sum_{k=0}^{d_i} \hat{a}_k^I \hat{\sigma}_i^{d_i} + \xi_i \sum_{k=1}^{d_r} \bar{a}_k^I \bar{\sigma}_i^{-k} L^{-k},$$  \hspace{1cm} (13)

\textsuperscript{2} We will show below that we can make $L \rightarrow \infty$ close to a multi-particle pole. For now, we explore the consequences of this choice.
if \( i \) is on the left side, and
\[
\xi_i Z^I(\sigma_i) = \xi_i \sum_{k=0}^{d_r} \tilde{a}_k^I \tilde{\sigma}_i^k + \xi_i \sum_{k=1}^{d_l} \tilde{a}_k^I \sigma_i^{-k} L^{-k},
\]
(14)

if \( i \) is on the right side.

From these expressions it is obvious that, at the leading order in \( L \), we have a self-similar structure in the left and right sides. If are allowed to take the limit \( L \to \infty \) the variables of integration (\( a_k, \sigma_i \) and \( \xi_i \)) almost factorise. Almost, because we still have \( \tilde{a}_0 = \hat{a}_0 \).

Does the limit \( L \to \infty \) correspond to an internal line going on shell? To see that it does, integrate the moduli corresponding to the \( \mu \) to get equations for the \( \tilde{\lambda} \),
\[
\sum_{i \in \mathbb{L}} \xi_i \hat{\sigma}_i^k \tilde{\lambda}_i^\alpha + \sum_{i \in \mathbb{R}} \xi_i \tilde{\sigma}_i^{d_l-k} L^{-k} \tilde{\lambda}_i^\alpha = 0, \quad \text{for } d_l \geq k > 0,
\]
(15)
\[
\sum_{i \in \mathbb{L}} \xi_i \hat{\lambda}_i^\alpha + \sum_{i \in \mathbb{R}} \xi_i \hat{\lambda}_i^\alpha = 0,
\]
(16)
\[
\sum_{i \in \mathbb{L}} \xi_i L^{-k} \tilde{\sigma}_i^{-k} \tilde{\lambda}_i^\alpha + \sum_{i \in \mathbb{R}} \xi_i \hat{\sigma}_i^{d_l-k} L^{-k} \tilde{\lambda}_i^\alpha = 0, \quad \text{for } d_r \geq k > 0.
\]
(17)

Combining these with the formulae for \( \lambda \) yields
\[
P^{\alpha \dot{\alpha}} = \sum_{i \in \mathbb{L}} \lambda_i^\alpha \tilde{\lambda}_i^{\alpha} = \hat{a}_0^\alpha \sum_{i \in \mathbb{L}} \xi_i \tilde{\lambda}_i^\alpha + \sum_{i \in \mathbb{L}} \sum_{k=1}^{d_r} \hat{\xi}_i \hat{\sigma}_i^a L^{-k} \tilde{\sigma}_i^{d_l-k} L^{-k} \tilde{\lambda}_i^\alpha - \sum_{i \in \mathbb{R}} \sum_{k=1}^{d_l} \xi_i \hat{\sigma}_i^{d_l-k} L^{-k} \tilde{\lambda}_i^\alpha,
\]
(18)

which in the limit \( L \to \infty \), reduces to
\[
P^{\alpha \dot{\alpha}} \to \hat{a}_0^\alpha \sum_{i \in \mathbb{L}} \xi_i \tilde{\lambda}_i^{\alpha},
\]
(19)

which is on shell. We also have
\[
P^2 = \frac{1}{L} \sum_{i,j \in \mathbb{L}} \hat{\xi}_i \hat{\xi}_j [i,j] \langle \hat{a}_0 \hat{a}_1 \rangle \frac{1}{\hat{\sigma}_j} + \frac{1}{L} \sum_{i,j \in \mathbb{R}} \tilde{\xi}_i \tilde{\xi}_j [i,j] \langle \hat{a}_0 \hat{a}_1 \rangle \sigma_j^{d_l-1} + \mathcal{O}(L^{-2}).
\]
(20)

This means that the limit \( L \to \infty \) corresponds to the limit \( P^2 \to 0 \), as expected.

Another important thing to notice is the way the ‘gauge’ symmetry \( \text{Gl}(2) \) acts on the hatted and barred variables. The action on \( \hat{\sigma} \) and \( \tilde{\sigma} \) is easy to find. We have
\[
\hat{\sigma} \to \frac{\delta \hat{\sigma} + \gamma}{\beta \hat{\sigma} + \alpha},
\]
(21)
\[
\tilde{\sigma} \to \frac{\alpha \tilde{\sigma} + \beta}{\gamma \tilde{\sigma} + \delta}.
\]
(22)
The $\text{Gl}(2)$ action on the moduli $a_k$ and on the $\xi_i$ is more complicated but can be obtained by using the action on the $\sigma_i$ along with the invariance of $Z(\sigma_i)$. What is somewhat remarkable is that the hatted and barred variables inherit the same kind of $\text{Gl}(2)$ action as the initial variables. More precisely, at the leading order in $L$, the link between the action on the $\hat{\sigma}$ (or $\bar{\sigma}$) and the action on $\hat{a}$ and $\hat{\xi}$ (or $\bar{a}$ and $\bar{\xi}$) is the same as the link between the action on $\sigma$ and the action on $a$ and $\xi$. (This conclusion is trivially true for the variables on the left-hand side, but not for the variables on the right-hand side. In fact, $\bar{\xi}$ transforms as expected modulo an irrelevant multiplicative factor.)

We can regard this passage from the initial integration variables to the new ones (the barred and the hatted variables) as a change of variables in the integrals forming the amplitude. More precisely, it should be considered as change of variables in a sub-domain of the full integration domain where the scaling of the integration variables with $L$ is as considered above. This excludes for example the possibility that $\hat{\sigma}$ or $\bar{\sigma}$ be zero; in fact, these conditions exclude an open set around zero. Since our purpose here is to display the factorisation on a multi-particle pole, restricting the integration domain need not bother us as long as this integration domain contains all the contributions to this multi-particle pole.

This change of variables for the moduli $a$ has Jacobian one because the bosonic and fermionic contributions cancel, the $\xi$ integrals are homogeneous and do not contribute. The only contribution comes from the $\sigma$s.

Now the product of $\sigma$s can be written

$$
\prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}} = \frac{\prod_{i=1}^{n} d\sigma_i}{(\sigma_1 - \sigma_2) \cdots (\sigma_n - 0)(0 - \sigma_1)} \times \frac{\prod_{i=n+1}^{n} d\sigma_i}{(\sigma_{n+1} - \sigma_{n+2}) \cdots (\sigma_n - 0)(0 - \sigma_{n+1})} \times \frac{\sigma_n(-\sigma_1)\sigma_{n}(-\sigma_{n+1})}{(\sigma_n - \sigma_{n+2})(\sigma_n - \sigma_1)} = \frac{1}{L} \times \prod_{i=1}^{n} d\sigma_i \times \prod_{i=n+1}^{n} d\sigma_i \times \left(1 + O(L^{-1})\right). \tag{23}
$$

Therefore, this part also factorises at the leading order in $L$.

In order to prove factorisation, we need to introduce two more vertex operators corresponding to the internal line going on-shell in the factorisation limit. Use the notation

$$
\Psi_{\pi,\bar{\pi},\eta,\lambda}(\lambda, \mu, \psi) = \int \frac{d\xi}{\xi} \delta^2(\pi - \xi \lambda) \exp(i\xi [\mu, \bar{\pi}]) \exp(i\xi \psi A \eta A). \tag{24}
$$

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3 We have here introduced a $0$, anticipating the fact that the internal line will be attached at $\sigma = 0$. See below.
For these wavefunctions we can prove orthonormality and completeness relations.

Orthonormality\(^4\):

\[
\int \frac{d^2 \lambda d^2 \mu d^4 \psi}{\text{Gl}(1)} \Psi^*_{\pi,\bar{\pi},\eta}(\lambda, \mu, \psi) \Psi_{\pi',\bar{\pi}',\eta'}(\lambda, \mu, \psi) = \\
= \int \frac{d\xi}{\xi} \delta^2(\xi \pi - \pi') \delta^2(\bar{\pi} - \xi \bar{\pi}') \delta^4(\eta - \xi \eta') \equiv \delta_{\pi,\bar{\pi},\eta;\pi',\bar{\pi}',\eta'}. \tag{25}
\]

The Gl(1) group comes from the following symmetry of the integral

\[
\lambda \rightarrow t \lambda, \tag{26}
\]
\[
\mu \rightarrow t \mu, \tag{27}
\]
\[
\psi \rightarrow t \psi, \tag{28}
\]
\[
(\xi, \xi') \rightarrow t^{-1} (\xi, \xi'). \tag{29}
\]

Completeness:

\[
\int \frac{d^2 \pi d^2 \bar{\pi} d^4 \eta}{\text{Gl}(1)} \Psi^*_{\pi,\bar{\pi},\eta}(\lambda, \mu, \psi) \Psi_{\pi',\bar{\pi}',\eta'}(\lambda, \mu, \psi') = \\
= \int \frac{d\xi}{\xi} \delta^2(\lambda - \xi \lambda') \delta^2(\mu - \xi \mu') \delta^4(\psi - \xi \psi') \equiv \delta_{\lambda,\mu,\psi;\lambda',\mu',\psi'}. \tag{30}
\]

The Gl(1) group comes from the following symmetry of the integral

\[
\pi \rightarrow t \pi, \tag{31}
\]
\[
\bar{\pi} \rightarrow t^{-1} \bar{\pi}, \tag{32}
\]
\[
\eta \rightarrow t^{-1} \eta, \tag{33}
\]
\[
(\xi, \xi') \rightarrow t(\xi, \xi'). \tag{34}
\]

The measure transforms like

\[
d^2 \pi d^2 \bar{\pi} d^4 \eta \rightarrow t^4 d^2 \pi d^2 \bar{\pi} d^4 \eta \tag{35}
\]

and a factor in the integrand

\[
\delta^2(\pi - \xi \lambda) \delta^2(\pi - \xi' \lambda') \rightarrow t^{-4} \delta^2(\pi - \xi \lambda) \delta^2(\pi - \xi' \lambda') \tag{36}
\]

(These combine to render the integral invariant.)

\(^4\) The factor \((2\pi)^2\) that seems to be missing has been included in the integration measure.
We use the above formulae to separate the integrals over moduli (remember that we still have the constraint $\hat{a}_0 = \bar{a}_0$). Then, schematically

$$\int d^4(d+1) a \cdots = \int d^{4(d+1)+1} a d^{4(d_r+1)+1} \delta^4(\hat{a}_0 - \bar{a}_0) \cdots =$$

$$= \int d^2 \bar{a} d^4 \eta \bar{a} \delta^4 \Psi^*(\hat{a}_0) \Psi_\pi \bar{\Psi}^{\pi} \Psi_{\pi,\eta}(\hat{a}_0) \cdots, \quad (37)$$

where we have inserted the delta function from the completeness relation and $\text{Gl}(1)$ acts projectively on $\bar{a}$ or $\hat{a}$ moduli. The dots in the above formula stand for a function of the moduli $a$ which is invariant under a scaling of $\bar{a}$ and $\hat{a}$ separately. At the dominant order in $L$ this property is satisfied by the integrands we consider. The delta function really stands for $\delta^{4|4}(\hat{Z}(\sigma) - \bar{Z}(\sigma'))$, understood to be evaluated at $\sigma = \sigma' = 0$.

A. Gauge-fixing

Now we come to the issue of gauge-fixing. We can fix the gauge in several different ways. If we gauge-fix one component of $\hat{a}_0$, say $\hat{a}_{10}$, $\hat{\sigma}_i$, $\hat{\sigma}_j$, and $\bar{\sigma}_p$ we get a Jacobian

$$J = -\frac{1}{L} \hat{a}_{10} \hat{\sigma}_i \hat{\sigma}_j (-1 + L \bar{\sigma}_p \hat{\sigma}_i)(-1 + L \bar{\sigma}_p \hat{\sigma}_j). \quad (38)$$

The $\text{Gl}(1)$ gauge invariance at the right can be gauge-fixed independently and gives a Jacobian $\bar{a}_{10}^L$, for example. Note that if anyone of $\hat{\sigma}_i$, $\hat{\sigma}_j$ or $\bar{\sigma}_p$ is zero, the Jacobian is $J \sim L^{-1}$ and this will not contribute in the factorisation limit (note that in order for a contribution to contribute in the limit $L \to \infty$ it has to cancel the factor in $\frac{1}{L}$ from the product of $\sigma$’s). This is consistent with the interpretation we gave that the internal line has $\hat{\sigma} = \bar{\sigma} = 0$ so, in same sense it already is gauge-fixed at zero.

Alternatively, we could try to gauge-fix three $\sigma$’s and a modulus on the left-hand side which would seem like over-fixing once the internal line is taken into account. However, this gauge-fixing gives a Jacobian of order $L^0$, and thus doesn’t contribute in the factorisation limit.

When $\hat{\sigma}_i$, $\hat{\sigma}_j$ and $\bar{\sigma}_p$ are different from zero we have

$$J = -L \hat{a}_{10}^L \hat{\sigma}_i \hat{\sigma}_j (-1 + L \bar{\sigma}_p \hat{\sigma}_i) + O(1). \quad (39)$$

We want to compare this with the case when the left- and right-hand sides are completely gauge-fixed, however on the right-hand side there are only two $\sigma$’s and a modulus fixed. We
use the fact that $\bar{a}_b \bar{\sigma}_p \bar{\sigma}_q (\bar{\sigma}_p - \bar{\sigma}_q)$ times the right hand integrals where we don’t integrate over $\bar{a}_0$, $\bar{\sigma}_p$, $\bar{\sigma}_q$ is independent of $\bar{\sigma}_q$ and can be taken out of the integral over $\bar{\sigma}_q$.

Dividing the full Jacobian $J$ by the Jacobians needed to recombine the left and right parts into gauge invariant amplitudes gives

$$\frac{J}{L J_l J_r} = \frac{\bar{\sigma}_p}{\bar{\sigma}_q (\bar{\sigma}_p - \bar{\sigma}_q)} = \frac{1}{\bar{\sigma}_q} + \frac{1}{\bar{\sigma}_p - \bar{\sigma}_q},$$

where $J_l = \bar{\sigma}_i \bar{\sigma}_j (\bar{\sigma}_i - \bar{\sigma}_j)$ and $J_r = \bar{\sigma}_p \bar{\sigma}_q (\bar{\sigma}_p - \bar{\sigma}_q)$. We are left with the integral over $\bar{\sigma}_q$

$$\oint d\bar{\sigma}_q \left( \frac{1}{\bar{\sigma}_q} - \frac{1}{\bar{\sigma}_q - \bar{\sigma}_p} \right).$$

This integral is zero if we interpret it in the most naive way possible, by taking a contour around 0 and $\bar{\sigma}_p$ in the $\bar{\sigma}_q$ plane. However, we have to recall that the region in the neighbourhood of $\bar{\sigma}_q = 0$ is special and is not included in our integration domain (at any rate, a contour around zero which is included in our integration domain cannot be shrunk to $\bar{\sigma} = 0$ while staying inside the integration domain). Therefore, we propose to do the next less naive thing possible and take a contour which does not go around $\bar{\sigma} = 0$. The result of the integration is then $-2\pi i$.

B. Testing the Contour Prescription

Since our prescription for the choice of contour is not very well justified, it is important to test it by using different gauge fixing conditions. We test this by gauge-fixing the linear combination $\bar{\sigma}_p + \zeta \bar{\sigma}_q$. This has a Jacobian $J_\zeta$ which is such that

$$\frac{J_\zeta}{L J_l J_r} = \frac{1}{\bar{\sigma}_q} - \frac{1 + \zeta}{\bar{\sigma}_p - \bar{\sigma}_q} + \frac{\zeta}{\bar{\sigma}_p} + \mathcal{O}(L^{-1}).$$

Now suppose $\bar{\sigma}_p + \zeta \bar{\sigma}_q$ is gauge-fixed to a value $\tau$. This is implemented by introducing a delta function $\delta(\bar{\sigma}_p + \zeta \bar{\sigma}_q - \tau)$ and the Jacobian $J_\zeta$ in the integral. After integrating over $\bar{\sigma}_p$ we are left with the following integral over $\bar{\sigma}_q$

$$\oint d\bar{\sigma}_q \left( \frac{1}{\bar{\sigma}_q} - \frac{1}{\bar{\sigma}_q - \tau} + \frac{\zeta}{\bar{\sigma}_q - \tau} \right).$$

In the integrand, the first and the last term correspond to $\bar{\sigma}_q = 0$ and $\bar{\sigma}_p = 0$ respectively. Therefore, as before, we argue that the choice of contour is such that they don’t contribute. The remaining term yields $-2\pi i$. 

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There is however a problem for \( \zeta = -1 \) and, in this particular case, our prescription does not work. It does work however, for the whole family of gauge-fixing conditions where \( \zeta \neq -1 \). (Note that when \( \zeta \to -1 \) the pole which contributes to the integral is sent to infinity and also out of our domain of integration.)

C. The \( \frac{1}{i\pi} \) Pole

We still have the integration \( \int \frac{d^2 \pi d^2 \bar{\pi} d\eta}{G_{\Pi(1)}} \) to perform. Concentrate on the bosonic part. After gauge-fixing \( \pi^1 \) the measure is \( \pi^1 d\pi^2 d^2 \bar{\pi} \). The right and left part each contain a momentum conserving delta function. Denote \( P \) the total momentum at left and \( Q \) the total momentum at right

\[
\int \pi^1 d\pi^2 d^2 \bar{\pi} \delta^4(P^{\alpha\dot{\alpha}} - \pi^\alpha \bar{\pi}^{\dot{\alpha}}) \delta^4(Q^{\alpha\dot{\alpha}} - \pi^\alpha \bar{\pi}^{\dot{\alpha}}) = \delta^4(P - Q) \int \pi^1 d\pi^2 d^2 \bar{\pi} \delta^4(P^{\alpha\dot{\alpha}} - \pi^\alpha \bar{\pi}^{\dot{\alpha}}). \tag{44}
\]

The integral above can be computed straightforwardly and the result is \( \delta(P^2) \).

What we want to do now is to interpret this as a holomorphic delta function \( \delta(z) \equiv \frac{1}{2\pi i} \frac{1}{z} \). Note that this is not an arbitrary assumption. The delta functions from the connected prescription are not real delta functions, but they can be interpreted as complex delta functions (see \[\text{III}\]). Granted this interpretation, we obtain the pole we were looking for

\[
\delta(P^2) \equiv \frac{1}{2\pi i} \frac{1}{P^2}. \tag{45}
\]

D. The Remaining Ingredients

- The fermionic integrals: It is easy to see that the integral over \( \eta \) imposes the constraint that the sum of helicities at the two ends of the internal line be zero. One might be worried that this allows the exchange of fermions and scalars also but this is not so since the amplitudes with one fermion/scalar and all the rest vector particles vanish. Therefore, the only contribution which survives when taking the residue is the exchange of an internal gluon.

- The gauge coupling: it’s \( g^{n-2} \) for the initial amplitude and \( n = n_l + n_r \). Upon factorisation we can write this as \( g^{(n_l+1)-2} \times g^{(n_r+1)-2} \) which is the correct coupling factor.
• The colour factors: They factorise as shown in [23, eq. 6.20]. For $SU(N)$ groups, there is a sub-dominant factor in $\frac{1}{N}$ but it vanishes at the pole.

V. SUMMARY AND DISCUSSION

In this paper we have given some arguments supporting the factorisation of the connected prescription for Yang-Mills amplitudes. This is not a complete proof, however since the choice of contours is not well understood. This is not a new problem. It is already implicit in refs. [18, 19, 20], and also shows up in ambiguities in the contour deformation argument of ref. [16].

In [18], this problem was avoided by summing over all solutions at finite vertex positions and finite moduli, so the contours had to be specified only vaguely by saying that they have to encircle all possible singularities. Here we have followed the same strategy, adding the further constraint of staying inside a specified domain.

It seems to us that there may be several prescriptions for the integration contours which give the same results as far as tree amplitudes are concerned. Berkovits’s model seems to require an ordering of the vertex operators $\sigma_i < \sigma_{i+1}$ on the border of the disk, whereas the moduli are unrestricted. The connected prescription, on the other side, doesn’t seem to impose such ordering restrictions on the $\sigma$s but could impose some restriction on the moduli in keeping with our interpretation of the delta functions (see our discussion above about the contour from zero to infinity and also the discussion in [21]). It might very well be that the contours in one can case can be deformed to the contours in the other case, but a priori they are different. One may wonder whether either of these contour prescriptions can be extended consistently to loop level. Of course, one first needs a better understanding of the contours at tree level. This would also enable a study of corrections to the strict $L = \infty$ limit (corresponding to corrections to $C = 0$ in ref. [16]), yielding a quantitative computation of the pole and its residue.

It would also be interesting to see what the arguments presented in this paper have to say about factorisation in the case of conformal supergravity [15] or Einstein supergravity [26]. Most of the discussion carries over; only the wavefunctions need to be modified.
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[1] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. 56 (1986) 2459.
[2] F. A. Berends and W. T. Giele, Nucl. Phys. B 306, 759 (1988).
[3] Z. Bern, L. J. Dixon and D. A. Kosower, Ann. Rev. Nucl. Part. Sci. 46, 109 (1996) arXiv:hep-ph/9602280.
[4] R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94 (2005) 181602 arXiv:hep-th/0501052.
[5] V. P. Nair, Phys. Lett. B 214 (1988) 215.
[6] E. Witten, Commun. Math. Phys. 252 (2004) 189 arXiv:hep-th/0312171.
[7] T. G. Birthwright, E. W. N. Glover, V. V. Khoze and P. Marquard, JHEP 0505, 013 (2005) arXiv:hep-ph/0503063.
[8] K. Risager, JHEP 0512, 003 (2005) arXiv:hep-th/0508206.
[9] R. Britto, F. Cachazo and B. Feng, super-Yang-Mills, Nucl. Phys. B 725, 275 (2005) arXiv:hep-th/0412103.
[10] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D 72, 065012 (2005) arXiv:hep-ph/0503132.
[11] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D 73, 065013 (2006) arXiv:hep-ph/0507005.
[12] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde and D. A. Kosower, Phys. Rev. D 74, 036009 (2006) arXiv:hep-ph/0604195.
[13] F. Cachazo and P. Svrcek, PoS RTN2005 (2005) 004 arXiv:hep-th/0504194.
[14] N. Berkovits, Phys. Rev. Lett. 93, 011601 (2004) arXiv:hep-th/0402045.
[15] N. Berkovits and E. Witten, JHEP 0408 (2004) 009 arXiv:hep-th/0406051.
[16] S. Gukov, L. Motl and A. Neitzke, arXiv:hep-th/0404085.
[17] N. Berkovits and L. Motl, JHEP 0404, 056 (2004) arXiv:hep-th/0403187.
[18] R. Roiban, M. Spradlin and A. Volovich, Phys. Rev. D 70, 026009 (2004)
[19] R. Roiban and A. Volovich, Phys. Rev. Lett. 93, 131602 (2004) [arXiv:hep-th/0402121].
[20] R. Roiban, M. Spradlin and A. Volovich, JHEP 0404, 012 (2004) [arXiv:hep-th/0402016].
[21] F. Cachazo, P. Svrcak and E. Witten, JHEP 0409, 006 (2004) [arXiv:hep-th/0403047].
[22] I. Bena, Z. Bern, D. A. Kosower and R. Roiban, Phys. Rev. D 71 (2005) 106010 [arXiv:hep-th/0410054].
[23] M. L. Mangano and S. J. Parke, Phys. Rept. 200 (1991) 301 [arXiv:hep-th/0509223].
[24] A. Brandhuber, B. Spence and G. Travaglini, JHEP 0601 (2006) 142 [arXiv:hep-th/0510253].
[25] F. Cachazo, P. Svrcak and E. Witten, JHEP 0410 (2004) 074 [arXiv:hep-th/0406177].
[26] M. Abou-Zeid, C. M. Hull and L. J. Mason, [arXiv:hep-th/0606272]