Q-systems and compact W*-algebra objects

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July 10, 2017

Abstract

We show that given a rigid C*-tensor category, there is an equivalence of categories between normalized irreducible Q-systems, also known as connected unitary Frobenius algebra objects, and compact connected W*-algebra objects. Although this result could be proved as a corollary of our previous article on realizations of algebra objects and discrete subfactors, we prove it here directly via categorical methods without passing through subfactor theory.

1 Introduction

The standard invariant of a finite index subfactor II$_1$ subfactor was first defined as a $\lambda$-lattice [Pop95]. Since, it has been reinterpreted as a planar algebra [Jon99] and a Q-system, or unitary Frobenius algebra object, in a rigid C*-tensor category [Lon89, Müg03, BKLR15]. Using this third language, the standard invariant of a finite index irreducible II$_1$ subfactor is an irreducible Q-system, i.e., the underlying algebra object $A$ is connected, meaning $\dim(\mathcal{C}(1_{\mathcal{C}}, A)) = 1$.

In [JP16], we introduced the notion of a W*-algebra object in a rigid C*-tensor category. In [JP17], we characterized an extremal irreducible discrete inclusion of factors ($N \subseteq M$, $E$) where $N$ is type II$_1$ and $E : M \to N$ is a faithful normal conditional expectation in terms of a connected W*-algebra object $A$ in (an ind-completion of) a rigid C*-tensor category $\mathcal{C}$, and a fully faithful embedding $\mathcal{C} \hookrightarrow \text{Bim}_{\text{bf}}(N)$, the spherical/extremal bifinite bimodules over $N$. This means that we may also view the standard invariant of a finite index irreducible II$_1$ subfactor as a compact connected W*-algebra object, which actually lies not only in the ind-completion of $\mathcal{C}$, but also in the rigid involutive tensor subcategory $\mathcal{C}^\#$ which is obtained from $\mathcal{C}$ simply by forgetting the adjoint. Thus passing through our theorem [JP17, Thm. A and §7.2], we would get an equivalence between irreducible Q-systems and compact connected W*-algebra objects.

The purpose of the present article is to provide a direct proof of the equivalence between irreducible Q-systems and compact connected W*-algebra objects in an arbitrary rigid C*-tensor category without passing through subfactor theory. In doing so, we observe that the correct notion of morphisms between Q-systems is that of involutive algebra morphism from [Vic11, Def. 3.4 and 3.9]. We refer the reader to Section 2.3 below for the definition.

Our main theorem is:

**Theorem A.** Let $\mathcal{C}$ be a rigid C*-tensor category.\footnote{In this article, we assume our rigid C*-tensor category $\mathcal{C}$ has simple unit object and is idempotent complete/semi-simple.} The assignment $A \mapsto L^2 A$ induces an equivalence of categories

\[
\begin{aligned}
\left\{ \text{Compact connected W*-algebra objects } A \in \text{Vec}(\mathcal{C}) \text{ with unital *-algebra natural transformations} \right\} & \cong \\
\left\{ \text{Normalized irreducible Q-systems with involutive algebra morphisms} \right\}.
\end{aligned}
\]
The proof of Theorem A proceeds as follows. In Section 3, starting with an irreducible Q-system \( A \in \mathcal{C} \), we define a compact connected W*-algebra object \( A \in \text{Vec}(\mathcal{C}) \) by \( A(a) = C^\gamma(a, A) \) for all \( a \in \mathcal{C} \). The \( * \)-structure is the conjugation on \( C^\gamma \) as \( A \) is a real (symmetrically self-dual) object by [LR97, §6]. In Section 4, starting with \( A \), we embed the compact Hilbert space object \( L^2A \in \text{Hilb}(\mathcal{C}) \) into the (non-irreducible) Q-system \( L^2A \otimes L^2A \), which gives a canonical projector \( p \in \text{End}_\mathcal{C}(H \otimes \overline{H}) \) where \( H \in \mathcal{C} \) is such that \( L^2A(a) = C(a, H) \) for all \( a \in \mathcal{C} \). This is similar to the strategy of [Vic11, Lem. 3.21], where it is shown that an involutive subalgebra of a unitary Frobenius algebra is again Frobenius. Conceptually, this is analogous to the fact that a unital \(*\)-subalgebra of a finite dimensional von Neumann algebra is again a von Neumann algebra. We then prove the equivalence of categories in Sections 5.1 and 5.2.

While our proof is conceptual from an operator algebraic perspective, and focuses completely on the algebra objects, it is important to note that there is a second conceptual approach that passes through proper module categories. Taking this approach, this result should be viewed as an extension of [NY17, Appendix A]. One could expand the proof of [NY17, Thm. A.1] to prove an equivalence between the category of irreducible Q-systems and the category of proper cyclic \( \mathcal{C} \)-module W*-categories with simple basepoint. One would then apply our equivalence between W*-algebra objects and cyclic \( \mathcal{C} \)-module W*-categories [JP16, Thm. 3.24] to get another proof of Theorem A.

Acknowledgements The authors would like to thank Marcel Bischoff, André Henriques, and Jamie Vicary for helpful discussions. Corey Jones was supported by Discovery Projects ‘Subfactors and symmetries’ DP140100732 and ‘Low dimensional categories’ DP160103479 from the Australian Research Council. David Penneys was supported by NSF grants DMS-1500387/1655912 and 1654159.

The authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Operator Algebras: Subfactors and their Applications where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1.

2 Background

We refer the reader to [HP17, §2.1-2.2] and [JP16, §2.1-2.3] for background on rigid C*-tensor categories and their graphical calculus. Of particular importance is the bi-involution structure consisting of two commuting involutions called the adjoint \( * \) and the conjugate \( \overline{\cdot} \). Applying both to morphisms in \( \mathcal{C} \) gives the contravariant dual functor \( (\cdot)^\vee : \mathcal{C} \rightarrow \mathcal{C} \):

\[
\mathcal{C}(a, b) \ni \psi \mapsto \psi^\vee := \overline{\psi}^* = \overline{\psi^*} = \bigoplus_b \bigcap_a \mathcal{C}(b, a) \in \mathcal{C}(b, a).
\] (1)

The equation (1) implies that we actually have three commuting involutions on morphisms: \( (\cdot)^*, \overline{\cdot}, (\cdot)^\vee \). Here, \( (\cdot)^*, \overline{\cdot} \) are anti-linear and \( (\cdot)^\vee \) is linear, and combining any two in either order gives the third. As usual, we suppress all associators, unitors, and the conjugation structure maps.

2.1 Tensor categories associated to \( \mathcal{C} \)

Fix a rigid C*-tensor category \( \mathcal{C} \), and let \( \text{Irr}(\mathcal{C}) \) be a set of representatives for the isomorphism classes of simple objects of \( \mathcal{C} \).
Definition 2.1 ([LR97, §2]). We say that morphisms $R_c : 1 \to c \otimes c$ and $S_c : 1 \to c \otimes \bar{c}$ are solutions to the conjugate equations if they satisfy the zig-zag axioms. They are called standard solutions if they additionally satisfy the balancing condition

$$R_c^* \circ (\text{id}_c \otimes f) \circ R_c = S_c^* \circ (f \otimes \text{id}_{\bar{c}}) \circ S_c$$

for all $f \in \text{End}_C(c)$.

Definition 2.2. We rapidly recall the tensor categories associated to $C$ which arise in the study of $*$-algebra objects. We refer the reader to [JP16, JP17] for more details on these categories.

- $C^\sharp$ is the rigid involutive tensor category obtained from $C$ by forgetting the adjoint $*$.
- $\text{Vec}(C)$ is the tensor category of linear functors $(C^\sharp)^{\text{op}} \to \text{Vec}$ with linear natural transformations and the Day convolution tensor product. Note that $\text{Vec}(C)$ is equivalent to $\text{ind}(C^\sharp)$.
- $\text{Hilb}(C)$ is the $W^*$-tensor category of linear dagger tensor functors $C^{\text{op}} \to \text{Hilb}$ with bounded linear natural transformations and the Day convolution tensor product. Note that $\text{Hilb}(C)$ is equivalent to the Neshveyev-Yamashita unitary ind category of $C$ [NY16].

Recall that the Yoneda embedding $a \mapsto a := C^\sharp(\cdot, a)$ gives an embedding of involutive tensor categories $C^\sharp \hookrightarrow \text{Vec}(C)$ and similarly, we have an embedding of bi-involutive tensor categories $C \hookrightarrow \text{Hilb}(C)$.

An object $F \in \text{Vec}(C)$ is called compact if $F$ is in the image of the Yoneda embedding, and similarly for $H \in \text{Hilb}(C)$. This means that for $F$ compact, there is a corresponding $F \in C^\sharp$ such that $F(a) = C^\sharp(a, F)$ for all $a \in C$. Likewise, we may identify compact objects of $\text{Hilb}(C)$ with objects in $C$.

Notation 2.3. Typically, we use bold letters for functors, like $F \in \text{Vec}(C)$ and $H \in \text{Hilb}(C)$, and we will use regular letters for objects in $C^\sharp$ and $C$.

2.2 C* Frobenius algebra objects

We now introduce $C^*$ Frobenius algebra objects in rigid $C^*$-tensor categories following [BKLR15, §3.1]. Of particular importance will be Q-systems, which are unitary Frobenius algebra objects discussed in Section 2.3. Some other general references on Frobenius algebras (and Q-systems) in rigid $C^*$-tensor categories include [Lon89, LR97, Müg03, Vic11].

Definition 2.4. An algebra object in a tensor category $\mathcal{T}$ is a triple $(A, m, i)$ where $A \in \mathcal{T}$ and $m \in \mathcal{T}(A \otimes A, A)$ and $i \in \mathcal{T}(1_\mathcal{T}, A)$ are morphisms satisfying the following relations:

- (associativity) $m \circ (m \otimes \text{id}_A) = m \circ (\text{id}_A \otimes m)$.
- (unitality) $m \circ (i \otimes \text{id}_A) = \text{id}_A = m \circ (\text{id}_A \otimes i)$.

A coalgebra object is defined similarly by reversing the arrows.

An algebra object is called connected if $\dim(\mathcal{T}(1_\mathcal{T}, A)) = 1$.

We make heavy use of the graphical calculus for morphisms in tensor categories, in which $m$ is denoted by a trivalent vertex, and $i$ is denoted by a univalent vertex. We now specialize to algebras in a rigid $C^*$-tensor category $\mathcal{C}$.

Definition 2.5. A $C^*$ Frobenius algebra object is an algebra object $(A, m, i) \in \mathcal{C}$ (which automatically implies $(A, m^*, i^*)$ is a coalgebra object) such that the following condition holds:
Lemma 2.7. Suppose \( (A, m, i) \) is an algebra object in \( \mathcal{C} \). The Frobenius condition above is equivalent to the following two conditions:

1. (Self-duality) There exist maps \( \hat{e}_A : A \otimes A \to 1_C \) and \( \coev_A : 1_C \to A \otimes A \) which satisfy the zig-zag axioms, and

2. (Rotational invariance) \( m^* = (\text{id}_A \otimes m) \circ (\coev_A \otimes \text{id}_A) = (m \otimes \text{id}_A) \circ (\text{id}_A \otimes \coev_A) \).

Proof. One passes from the Frobenius condition to the above two conditions by setting \( \hat{e}_A = i^* \circ m \) and \( \coev_A = m^* \circ i \) and using the Frobenius relation. One passes from the two above conditions to the Frobenius condition by trading \( m^* \) for \( m \) using the rotation relation and using associativity. \( \square \)

2.3 Unitary Frobenius algebras/Q-systems and involutive morphisms

Definition 2.8. Given two algebra objects \( (A, m, i), (B, m_B, i_B) \) in a tensor category \( \mathcal{T} \), an morphism \( \theta \in \mathcal{T}(A, B) \) is called an algebra morphism if \( m_B \circ (\theta \otimes \theta) = \theta \circ m_A \) and \( i_B \circ \theta = i_A \).

Given a \( C^* \) Frobenius algebra \( A \in \mathcal{C} \), following [Vic11, Def. 3.4], \( ^2 \) we can define two canonical isomorphisms \( \sigma_A^L, \sigma_A^R \in C(A, \overline{A}) \) given by

\[
\sigma_A^L = \begin{array}{c}
\text{id}_A \\
\downarrow \\
\overline{A}
\end{array} \\
\text{id}_A \\
\sigma_A^R = \begin{array}{c}
\overline{A} \\
\downarrow \\
\text{id}_A
\end{array}
\]

Notice that the cup in the diagram on the left is \( \coev_A \), and the cup in the diagram on the right is \( \ev_A^* \). By [Vic11, Lem. 3.5(13)], \( \sigma_A^L \circ \sigma_A^L = \varphi_A \) and \( \sigma_A^R \circ \sigma_A^R = \varphi_A \), and thus both are involutions. By [Vic11, Lem. 3.11], \( \sigma_A^L = \sigma_A^R \) if and only if either \( \sigma_A^L \) or \( \sigma_A^R \) is unitary.

Lemma 2.9. The following are equivalent for a separable \( C^* \) Frobenius algebra object \( (A, m, i) \) in \( \mathcal{C} \).

1. \( R = S = i_A^* \circ m_A \) are standard solutions to the conjugate equations.
2. \( R^* \circ R = i^* \circ m \circ m^* \circ i = d_A \text{id}_{1_C} \)
3. \( \sigma_A^L \) is unitary, and thus \( \sigma_A^R = \sigma_A^L =: \sigma_A \) by [Vic11, Lem. 3.11].

Proof. (1) implies (2) is immediate, since (2) is satisfied by standard solutions. (2) implies (1) follows from [LR97, §6]. \( ^3 \) To show (1) implies (3), suppose \( R = S = i_A^* \circ m_A \) are standard

\( ^2 \) To translate between our conventions for bi-involution categories and those in [Vic11], our \((\cdot)^*\) is his \((\cdot)^{\dagger}\), our \( \tau \) is his \((\cdot)^\dagger\), and our \((\cdot)^{\gamma}\) is his \((\cdot)^\dagger\).

\( ^3 \) We warn the reader that the proof in [LR97, §6] for a separable \( C^* \) Frobenius algebra involves constructing a subfactor, and thus the Jones index is the minimal index [Lon89].
solutions to the conjugate equations. Then so are \( R' = \text{ev}_A^* \) and \( S' = \text{coev}_A \), and we must have \( \sigma_A^L = (R \otimes \text{id}_A) \circ (\text{id}_A \otimes \text{coev}_A) \) is unitary by \([\text{NT13, Prop. 2.2.13}]\). Finally, to show (3) implies (2), if \( \sigma_A^L \) is unitary, then it is easy to calculate that for \( R = S = i_A^* \circ m_A \),

\[
R^* \circ R = \begin{bmatrix}
A & \sigma_A \\
\sigma_A^* & A
\end{bmatrix} = \begin{bmatrix}
\sigma_A^* \circ A & A \\
A & \sigma_A^* \circ A
\end{bmatrix} = m_A \circ m_A^* = (1) \quad A = d_A \text{id}_C.
\]

\[\Box\]

**Definition 2.10.** A unitary Frobenius algebra or a Q-system is a separable C* Frobenius algebra such that the equivalent conditions in Lemma 2.9 hold \([\text{BKLR15, Def. 3.2 and 3.8}]\). A Q-system is called irreducible if the underlying algebra object \( A \in C \) is connected.

**Remark 2.11.** By condition (1) in Lemma 2.9, the underlying object \( A \in C \) of a Q-system is real or symmetrically self-dual \([\text{Müg03, Rem. 5.6(3)}]\). Note that a Q-system is exactly a unitary \(^\dagger\)-Frobenius monoid in the sense of \([\text{Vic11, Def. 3.9}]\).

**Corollary 2.12.** A connected C* Frobenius algebra is automatically an irreducible Q-system.

**Proof.** Let \((A, m, i)\) be a connected C* Frobenius algebra. By \([\text{BKLR15, Lem. 3.3}]\), \( A \) is separable. Moreover, \( R = S = m^* \circ i \) are automatically standard solutions by \([\text{LR97, §6}]\).\(^4\)

**Example 2.13.** The separable C* Frobenius algebra \((c \otimes \tau, m, i)\) from Example 2.6 is a Q-system since \( R^* \circ R = d_c^2 \). However, one must replace the multiplication and unit by \( m = d_c^{1/2} (\text{id}_c \otimes \text{ev}_c \otimes \text{id}_\tau) \) and \( i = d_c^{-1/2} \text{coev}_c \) to get a normalized Q-system. As before, the Q-system is irreducible if and only if \( c \) is simple.

The next definition is adapted from \([\text{Vic11, Def. 2.21}]\) and gives the correct notion of morphism between irreducible Q-systems for our purposes.

**Definition 2.14.** Suppose \((A, m, i_A), (B, m_B, i_B)\) are algebra objects in \( C \). An algebra morphism \( \theta \in \mathcal{C}(A, B) \) is called involutive if \( \sigma_B \circ \theta = \theta \circ \sigma_A \).

The recent article \([\text{NY17}]\) introduces another notion of isomorphism between (unitary) C*-Frobenius algebras. The say an algebra isomorphism \( \theta \in \mathcal{C}(A, B) \) is a C*-Frobenius algebra isomorphism if

\[
m_A \circ [\text{id}_A \otimes (\theta^* \circ \theta)] = \theta^* \circ \theta \circ m_A.
\]

(2)

We now prove that involutive algebra morphisms satisfy (2).

**Lemma 2.15.** Every involutive algebra morphism \( \theta \in \mathcal{C}(A, B) \) satisfies (2).

**Proof.** First, using that \( \theta \) is involutive and (1), we have

\[
\begin{array}{c|c|c|c|c|c}
A & B & A & B & A & B \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\hline
m_A & m_A & m_B & m_B & m_B & m_B \\
i_A & i_A & i_A & i_A & i_B & i_B
\end{array}
\]

(3)

\[\Box\]

\(^4\)This was first observed by \([\text{Müg03, Rem. 5.6(3)}]\). Note that the proof in \([\text{LR97, §6}]\) relies on subfactor theory. The more general result \([\text{NY17, Thm. 2.9}]\) does not pass through subfactor theory and implies \( R, S \) are standard solutions.
Now using the adjoint of (3), together with Lemma 2.7, and that $\theta^* \in \mathcal{C}(B, A)$ is a coalgebra morphism, we have

\[
\begin{array}{cccc}
A & \overset{m_A}{\longrightarrow} & A & \overset{i_A}{\longrightarrow} A \\
\theta^* & \mapsto & \theta^* & \mapsto \\
\theta & \mapsto & \theta \quad \text{(3)} & \mapsto \\
\end{array}
\]

2.4 Compact connected W*-algebra objects

An algebra object $A \in \text{Vec}(\mathcal{C})$ may be viewed as a lax monoidal functor $(\mathcal{C}_s^*)^{\text{op}} \to \text{Vec}$ by [JP16, Prop. 3.3]. If $A \in \text{Vec}(\mathcal{C})$ is a compact algebra object, the corresponding object $A \in \mathcal{C}_s$ satisfying $A(a) = \mathcal{C}_s(a, A)$ for $a \in \mathcal{C}$ is easily seen to be an algebra object.

**Definition 2.16.** A $*$-structure on an algebra object $A \in \text{Vec}(\mathcal{C})$ is a collection of conjugate linear natural isomorphisms $j_a : A(a) \to A(\bar{a})$ satisfying the following axioms:

- (conjugate naturality) for all $\psi \in \mathcal{C}(b, a)$ and $f \in A(a)$, $j_b(f \circ \psi) = j_a(f) \circ \overline{\psi}$.
- (involutive) for all $f \in A(a)$, $j_\pi(j_a(f)) = f$.
- (unital) $j_1 = i$.
- (monoidal) for all $f \in A(a)$ and $g \in A(b)$, $j_{a \otimes b}(m \circ (f \otimes g)) = m \circ (j_a(g) \otimes j_b(f))$.

**Example 2.17.** Suppose $c \in \mathcal{C}$, and consider the normalized Q-system $(c \otimes \overline{c}, m, i) \in \mathcal{C}$ from Example 2.13. By the Yoneda embedding, $c \otimes \overline{c} : (\mathcal{C}_s^*)^{\text{op}} \to \text{Vec}$ is given by $(c \otimes \overline{c})(a) = \mathcal{C}_s^*(a, c \otimes \overline{c})$. The $*$-algebra structure on $c \otimes \overline{c}$ is given by

\[
\begin{align*}
\mu_{a,b} : (c \otimes \overline{c}) (a) \otimes (c \otimes \overline{c}) (b) & \to (c \otimes \overline{c}) (a \otimes b) \\
\mu_{a,b} = d_c^{1/2} (id_c \otimes ev_c \otimes id_{\overline{c}}) & \circ (f \otimes g) \\
i & \in (c \otimes \overline{c}) (1_c) = \mathcal{C}_s^*(1_c, c \otimes \overline{c}) \\
 i = d_c^{-1/2} \text{coev}_c & \\
j_a : (c \otimes \overline{c}) (a) & \to (c \otimes \overline{c}) (\bar{a}) \\
j_a = f & \mapsto \overline{f}
\end{align*}
\]

This definition of $j$ suppresses the isomorphism between the real object $c \otimes \overline{c}$ and its conjugate. Then $c \otimes \overline{c}$ is a compact $*$-algebra object in $\text{Vec}(\mathcal{C})$ which is connected if and only if $c$ is simple.

The corresponding object in $\mathcal{C}_s^*$ is obviously $c \otimes \overline{c}$.

It was shown in [JP16, Thm. 3.20] that $*$-structures on algebra objects in $\text{Vec}(\mathcal{C})$ are equivalent to dagger structures on the cyclic $\mathcal{C}$-module category $(\mathcal{M}_A, m)$ whose objects are the $c \otimes A$ for $c \in \mathcal{C}$ and whose morphisms are right $A$-module morphisms. Indeed, when $A \in \text{Vec}(\mathcal{C})$ is compact, we have

\[
\mathcal{M}_A(a \otimes A, b \otimes A) \cong \mathcal{C}(a, b \otimes A)
\]

with composition given for $f \in \mathcal{C}(a, b \otimes A)$ and $g \in \mathcal{C}(b, c \otimes A)$ by

\[
ge g \circ_M f := (id_c \otimes m) \circ_C (g \otimes id_A) \circ f.
\]

The dagger structure on $\mathcal{M}_A$ induced by $j$ is given on $f \in \mathcal{C}(a, b \otimes A)$ by

\[
f^* = j_{\overline{c} \otimes a} \left((ev_b \otimes id_A) \circ f \right) \circ (\text{coev}_a \otimes id_{\overline{c}}).
\]
Conversely, given a cyclic $\mathcal{C}$-module dagger category $(\mathcal{M}, m)$, we recover the lax monoidal functor $A : (C^\circ)^{\text{op}} \to \text{Vec}$ by $A(a) = \mathcal{M}(a \otimes m, m)$, and the $\ast$-structure is given by

$$j_a(f) = (\text{ev}_a \otimes \text{id}_m) \circ (\text{id}_a \otimes f^\ast)$$

for $f \in A(a)$. We see that $A$ is compact if and only if $\mathcal{M}$ is proper [NY17] or cofinite [AC15], i.e., $\mathcal{M}(a \otimes m, m)$ is always finite dimensional, and non-zero for only finitely many $a \in \text{Irr}(\mathcal{C})$.

Just as being a C*-algebra is a property of a complex $\ast$-algebra and not extra structure, being a C*-category is a property of a dagger category and not extra structure. We refer the reader to [JP16, §2.1] for more details.

**Definition 2.18.** A $\ast$-algebra object $(A, m, i, j) \in \text{Vec}(\mathcal{C})$ is a C*/W*-algebra object if the cyclic $\mathcal{C}$-module dagger category $\mathcal{M}_A$ is a C*/W*-category. If $A$ is locally finite, meaning $A(a)$ is finite dimensional for all $a \in \mathcal{C}$, then all morphism spaces of $\mathcal{M}_A$ are finite dimensional. In this case, $\mathcal{M}_A$ is a C*-category if and only if $\mathcal{M}_A$ is a W*-category.

**Example 2.19.** The $\ast$-algebra object $c \otimes c = C^\circ(\cdot, c \otimes c) \in \text{Vec}(\mathcal{C})$ from Example 2.17 is a W*-algebra object. To see this, we construct a $\ast$-algebra natural isomorphism $\pi : B(c) \Rightarrow c \otimes c$, where the instance of $c$ inside $B(c)$ is the linear dagger functor $c = C(\cdot, c) : \mathcal{C}^{\text{op}} \to \text{Hilb}$, which is a Hilbert space object under the Yoneda embedding. For $a \in \mathcal{C}$ and $f \in B(c)(a) = C(a \otimes c, c)$, we define

$$\pi_a(f) = d_c^{-1/2} \frac{c}{a} f \frac{c}{c} \in C(a, c \otimes c) = (c \otimes c)(a).$$

The reader can check $\pi$ is a $\ast$-algebra natural isomorphism. (It is similar to, but easier than, the proof of Theorem 5.1 below.) Finally, $B(c)$ is the W*-algebra object corresponding to the cyclic $\mathcal{C}$-module W*-category $(\mathcal{C}, c)$, and thus $c \otimes c$ is W*.

### 2.5 Inner products on hom spaces of $\mathcal{C}$

The morphism spaces of $\mathcal{C}$ come equipped with inner products given by

$$\langle f | g \rangle_{C(a, b)} = \left( \begin{array}{c} f^* \\ g \\ a \\ b \end{array} \right) \pi.$$  \hspace{1cm} (4)

For all $a, b \in \mathcal{C}$, we pick orthonormal bases ONB$(a, b)$ for $C(a, b)$ under the inner product (4).

For each $c \in \text{Irr}(\mathcal{C})$ and $a \in \mathcal{C}$, we let $\text{Isom}(c, a) = \{ \sqrt{a} f | f \in \text{ONB}(c, a) \}$, which is a maximal set of isometries in $C(c, a)$ with mutually orthogonal ranges. Note that $\text{Isom}(c, a)$ is an orthonormal basis for $C(c, a)$ with the modified inner product given by

$$\langle f | g \rangle_{\text{Isom} \text{id}_c} = f^* \circ g.$$  \hspace{1cm} (5)

**Definition 2.20.** The 1-click rotation or Fourier transform $\mathcal{F}$ on $C(a, b \otimes c)$ for $a, b, c \in \mathcal{C}$ is given by

$$\mathcal{C}(c, a \otimes b) \ni f \mapsto \left( \begin{array}{c} a \\ b \end{array} \right) \in \mathcal{C}(\bar{b}, c \otimes a).$$
Note that if \( a, b, c \in \text{Irr}(C) \) and \( f \in \text{Isom}(c, a \otimes b) \), then a weighting is required to produce another isometry:

\[
\text{Isom}(c, a \otimes b) \ni f \mapsto \left( \frac{d_b}{d_c} \right)^{1/2} F(f)
\]

produces an isometry in \( C(\overline{b}, \overline{c} \otimes a) \), and thus

\[
\left\{ \left( \frac{d_b}{d_c} \right)^{1/2} F(f) \middle| f \in \text{Isom}(c, a \otimes b) \right\} \subset C(\overline{b}, \overline{c} \otimes a)
\]

is a new orthonormal basis under inner product (5).

Suppose now that \( A \in \text{Vec}(C) \) is a compact connected W*-algebra object corresponding to \( A \in C^\# \) such that \( A(a) = C^\#(a, A) \) for \( a \in C \). Then \( A \) is tracial by [JP17, Prop. 2.6], and we get an additional inner product on \( A(a) = C^\#(a) \) by

\[
\langle f | g \rangle_a := \pi A \begin{array}{c} j_a(g) \hline a \end{array} A = a \langle g, f \rangle \quad (A \text{ tracial})
\]

(6)

We form the compact Hilbert space object \( L^2_A \in \text{Hilb}(C) \) by \( L^2_A(a) = A(a) \) with inner product (6).

Now \( A \) has a canonical state corresponding to the unique state on \( A(1_C) = C \) via \( i_A : 1_C \mapsto 1_C \). The GNS representation of \( A \) gives a canonical faithful \(*\)-algebra natural transformation \( \lambda : A \Rightarrow B(L^2_A) \) [JP16, Eq. (20)]. Since \( A \) is connected, the inner product (6) satisfies

\[
\langle f | g \rangle_a \pi_{L^2_A} = \pi A \begin{array}{c} \lambda_a(g) \hline a \end{array} L^2_A = \pi A \begin{array}{c} \lambda_a(f)^* \hline a \end{array} L^2_A .
\]

(7)

Finally, given a \(*\)-algebra natural transformation \( \theta : A \Rightarrow B \) of connected W*-algebra objects, notice that \( L^2_A(a) \ni f \mapsto \theta(f) \in L^2_B(a) \) gives a canonical isometry \( L^2 \theta : L^2_A \Rightarrow L^2_B \) since

\[
\langle \theta_a(f) | \theta_a(g) \rangle_a^B = \pi A \begin{array}{c} j^B_a(\theta_a(g)) \hline a \end{array} \begin{array}{c} j^A_a(f) \hline a \end{array} = \pi A \begin{array}{c} \theta_{\pi}(j^A_a(f)) \hline a \end{array} \begin{array}{c} \theta_{\pi}(j^B_a(g)) \hline a \end{array} = \theta_{1_C} \pi A \begin{array}{c} j^A_a(f) \hline a \end{array} A \begin{array}{c} j^B_a(g) \hline a \end{array} = \theta_{1_C} \langle f | g \rangle_a^A \theta_{1_C} = \langle f | g \rangle_a \theta_{1_C} \quad \theta_{1_C} = \theta_{1_C} (\langle f | g \rangle_a \theta_{1_C} = \langle f | g \rangle_a \theta_{1_C})
\]

(8)

Noticing that this argument was independent of the W*-algebra objects being compact, we conclude the following.

**Corollary 2.21.** Any \(*\)-algebra natural transformation \( \theta : A \Rightarrow B \) between connected W*-algebra objects in \( \text{Vec}(C) \) is injective and induces an isometry \( L^2 \theta : L^2_A \Rightarrow L^2_B \) with respect to the right (respectively left) inner products.
3 From Q-systems to W*-algebra objects

In this section, we define a functor $W^*$ from normalized irreducible Q-systems to compact connected W*-algebra objects.

3.1 Objects

Suppose we have a normalized irreducible Q-system $(A, m, i)$ in $\mathcal{C}$. We now define a compact connected W*-algebra object $W^*(A) := (A, m, i, j) \in \mathcal{C}^{\#}$. For $a \in \mathcal{C}^{\#}$, we define $A(a) = \mathcal{C}(a, A)$.

We define the laxitor $\mu_{a,b} : A(a) \otimes A(b) \to A(a \otimes b)$ to be the map $f \otimes g \mapsto m \circ (f \otimes g)$ and the same unit map $i \in A(1_C) = \mathcal{C}(1_C, A)$. It is straightforward to verify $(A, \mu, i)$ is a lax monoidal functor $(\mathcal{C}^{\#})^{op} \to \text{Vec}$. Since $A \in \mathcal{C}$, we have $A$ is compact, and $A(1_C)$ is clearly one dimensional, and thus $A$ is connected.

We now define the $\ast$-structure on $A$.

**Definition 3.1.** We define a $\ast$-structure $j$ on $A$ by

$$C^\sharp(a, A) = A(a) \ni f \mapsto \sigma_A^{-1} \circ f \in \mathcal{C}(a, A),$$

where $\sigma_A \in \mathcal{C}(A, \overline{A})$ is the unitary isomorphism from Lemma 2.9. Using that $\overline{\sigma_A \circ \sigma_A} = \varphi_A$ by [Vic11, Lem. 3.5(13)], it is easy to see $(j_a)_{a \in \mathcal{C}}$ satisfies the axioms of a $\ast$-structure on $A$.

**Proposition 3.2.** The compact connected $\ast$-algebra object $(A, \mu, i, j) \in \text{Vec}(\mathcal{C})$ is $W^*$.

**Proof.** Consider the cyclic $\mathcal{C}$-module dagger category $(\mathcal{M}_A, A)$ of free right $A$-modules in $\mathcal{C}$ with basepoint $A \in \mathcal{C}$ and dagger structure given by the adjoint in $\mathcal{C}$. That is, objects of $\mathcal{M}_A$ are exactly the $c \otimes A$ for $c \in \mathcal{C}$, and the morphisms are right $A$-module morphisms. By Lemma 2.7, for every $g \in \mathcal{M}_A(a \otimes A, b \otimes A)$, $g^* \in \mathcal{M}_A(b \otimes A, a \otimes A)$.

Thus $\mathcal{M}_A$ is a dagger sub-category of $\mathcal{C}$ which is closed under finite direct sums. This means that $\mathcal{M}_A$ is automatically $W^*$ by the finite dimensional bicommutant theorem [Jon10, Thm. 3.2.1] which says a unital $\ast$-subalgebra of a finite dimensional von Neumann algebra is again a von Neumann algebra. Indeed, for all $c \in \mathcal{C}$,

$$\mathcal{M}_A(c \otimes A, c \otimes A) \subseteq \mathcal{C}(c \otimes A, c \otimes A)$$

is a unital $\ast$-subalgebra of a finite dimensional von Neumann algebra. This is sufficient to show $\mathcal{M}_A$ is $W^*$ by Roberts’ $2 \times 2$ trick [GLR85] (see also [JP16, §2.1 and Prop. 3.26]).

We claim that $(A, \mu, i, j)$ is exactly the $\ast$-algebra corresponding to the cyclic $\mathcal{C}$-module $W^*$-category $(\mathcal{M}_A, A)$ under [JP16, Thm. 3.20]. First, the free module functor [KO02, BN11, HPT16] gives for all $a \in \mathcal{C}$ a natural isomorphism

$$A(a) = \mathcal{C}(a, A) \cong \mathcal{M}_A(a \otimes A, A),$$

and this isomorphism is compatible with the multiplication and unit on $A$ and composition and $\text{id}_A$ in $(\mathcal{M}_A, A)$. We now show the $\ast$-structure of $A$ and the dagger structure of $\mathcal{M}_A$ are compatible.

By [KO02, Fig. 4], the isomorphism (9) is given for all $a \in \mathcal{C}$ and $f \in A(a)$ by

$$\mathcal{C}(a, A) \ni f \mapsto \begin{bmatrix} A \downarrow f \\ f \uparrow \end{bmatrix} \in \mathcal{M}_A(a \otimes A, A).$$
By [JP16, Thm. 3.20], the \(\ast\)-structure on the algebra corresponding to \((\mathcal{M}_A, A)\) is given by

\[
\pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{f \ast A a}
\xrightarrow{A} \quad \pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{\sigma^{-1}_A a}
\xrightarrow{A} \quad \pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{A}
\xrightarrow{A}
\end{array}
\]
which is exactly the image of \(j_a(f)\) under the isomorphism (9). We are now finished by the correspondence between cyclic \(\mathcal{C}\)-module \(W^\ast\)-categories and \(W^\ast\)-algebra objects [JP16, Thm. 3.24].

Recall that we define \(L^2A \in \text{Hilb}(\mathcal{C})\) by \(L^2A(a) = A(a)\) with inner product (6). Now \(L^2A \in \text{Hilb}(\mathcal{C})\) is compact, and thus there is an \(H \in \mathcal{C}\) such that \(L^2A(a) = C(a, H)\) for all \(a \in \mathcal{C}\).

**Lemma 3.3.** The object \(H \in \mathcal{C}\) is canonically unitarily isomorphic to \(A\).

**Proof.** By injectivity of the Yoneda embedding \(\mathcal{C} \hookrightarrow \text{Hilb}(\mathcal{C})\), it suffices to show that for all \(a \in \mathcal{C}\), the inner product (4) on \(C(a, A)\) and the inner product (6) on \(C(a, H) = L^2A(a) = C(a, A)\) agree. Indeed, for \(f, g \in C(a, A)\), by Lemma 2.9 and Remark 2.11 together with (1), we have

\[
\langle f | g \rangle_{C(a, A)} = \pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{f^* A a}
\xrightarrow{A} \quad \pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{f \ast A a}
\xrightarrow{A} \quad \pi \begin{array}{c}
\begin{array}{c}
\mathcal{A}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
A
\end{array}
\end{array}
\xleftarrow{A}
\xrightarrow{A}
\end{array}
\]

where \(i^* \circ i = \text{id}_{1_{\mathcal{C}}}\) as in Definition 2.5 since \(A\) is normalized.

**Remark 3.4.** By Lemma 3.3, we see that the cyclic \(\mathcal{C}\)-module \(W^\ast\)-category \(\mathcal{M}_A\) with basepoint \(A\) is equivalent to the cyclic \(\mathcal{C}\)-module \(W^\ast\)-subcategory of \(\mathcal{C}\) generated by \(A\), which in turn is equivalent to the cyclic \(\mathcal{C}\)-module \(W^\ast\)-subcategory of \(\text{Hilb}(\mathcal{C})\) generated by \(L^2A\), which corresponds to the \(W^\ast\)-algebra object \(B(L^2A)\) by [JP16, Ex. 3.35]. Thus (10) provides an explicit description of the canonical GNS faithful \(\ast\)-algebra natural transformation \(\lambda : A \Rightarrow B(L^2A)\).

### 3.2 Morphisms

Suppose \(\theta \in C(A, B)\) is an involutive algebra morphism as in Section 2.3. We now define a \(\ast\)-algebra natural transformation \(W^\ast(\theta) : W^\ast(A) \Rightarrow W^\ast(B)\). We use the notation \(W^\ast(A) = (A, \mu^A, i_A, j^A)\) and recall \(A(a) = C^\ast(a, A)\). We have similar notation for \(W^\ast(B)\).

**Definition 3.5.** Given an involutive algebra morphism \(\theta \in C(A, B)\), we define \(W^\ast(\theta) : A \Rightarrow B\) by defining the \(a\)-component \(W^\ast(\theta)_a\) for \(a \in \mathcal{C}\) by

\[
C^\ast(a, A) = A(a) \ni f \mapsto \theta \circ f \in B(a) = C^\ast(a, B).
\]
Note that \( W^*(\theta) \) is automatically a natural transformation. That \( \theta \) is an algebra morphism implies \( W^*(\theta) \) is an algebra natural transformation, and involutivity of \( \theta \) implies \( W^*(\theta) \) is a \(*\)-algebra natural transformation. Indeed, for \( f \in A(a) \),

\[
j^B_a(W^*(\theta)_a(f)) = j^B_a(\theta \circ f) = \sigma^{-1}_B \circ \theta \circ f = \sigma^{-1}_B \circ \theta \circ \tilde{f} = \theta \circ \sigma^{-1}_A \circ \tilde{f} = \theta \circ j^A_a(f) = W^*(\theta) \pi (j^A_a(f)).
\]

**Proposition 3.6.** \( W^* \) is a functor.

**Proof.** We must prove that \( W^* \) preserves identities and composites. If \( \theta = \text{id}_A \in C(A, A) \), it is obvious that \( W^*(\theta) = \text{id}_A \). Moreover, if we have \( \theta_1 \in C(A, B) \) and \( \theta_2 \in C(B, C) \), then for all \( a \in C \) and \( f \in A(a) \), we have

\[
W^*(\theta_1 \circ \theta_2)_a(f) = \theta_1 \circ \theta_2 \circ f = (W^*(\theta_1) \circ W^*(\theta_2))_a(f).
\]

We are finished. \( \square \)

4 From \( W^* \)-algebra objects to Q-systems

4.1 Objects

Suppose \( A \in \text{Vec}(C) \) is a compact connected \( W^* \)-algebra object, and define \( H := L^2 A \in \text{Hilb}(C) \), the Hilbert space object defined by \( H(a) = L^2 A(a) = A(a) \) with inner product (6). Let \( H \in C \) be the corresponding object satisfying \( H(a) = C(A, H) \) for all \( a \in C \). Recall \( A \) has a canonical state corresponding to the state on \( A(1_C) \cong C \) via \( i_A \mapsto 1_C \). The GNS representation gives a canonical faithful \(*\)-algebra natural transformation \( \lambda : A \Rightarrow B(L^2 A) \).

Now since \( H \in C \), we get a canonical compact \( W^* \)-algebra object \( H \otimes \overline{H} : (C^*)^{\text{op}} \to \text{Vec} \) by \( (H \otimes \overline{H})(a) = C(A, H \otimes \overline{H}) \) as in Examples 2.17 and 2.19, which is isomorphic to the compact \( W^* \)-algebra object \( B(H) \in \text{Vec}(C) \).

**Definition 4.1.** Define \( \pi : A \Rightarrow H \otimes \overline{H} \) by

\[
A(a) \ni f \xrightarrow{\pi_a} d^{-1/2}_H \| \lambda_a(f) \| \left[ \frac{H}{a} \right] \in \text{Hom}_{\text{Hilb}(C)}(a, H \otimes \overline{H}) \cong C(A, H \otimes \overline{H}) = (H \otimes \overline{H})(a).
\]

In particular, we have \( \pi_{1_C}(i_A) = d^{-1/2}_H \coev_H \) since \( \lambda \) is unital.

It is easy to see that \( \pi : A \Rightarrow H \otimes \overline{H} \) is the faithful \(*\)-algebra natural transformation corresponding to \( \lambda : A \Rightarrow B(H) \) under the \(*\)-algebra natural isomorphism \( B(H) \cong H \otimes \overline{H} \).

**Notation 4.2.** For each \( a \in \text{Irr}(C) \), suppose we have \( f, f' \in A(a) \), and define \( \alpha = \pi_a(f) \) and \( \alpha' = \pi_a(f') \). We then have the following identities from (7):

\[
\begin{align*}
H & \xrightarrow{\alpha'} \overline{H} & H \xrightarrow{\alpha} \overline{H} & \xrightarrow{\pi} \overline{H} & \xrightarrow{\alpha^*} H \\
& = d^{-1}_H(f|f')_a \text{id}_H & \langle f|f' \rangle_a & = d^{-1}_a(f|f')_a \text{id}_a \quad (11)
\end{align*}
\]
For each $a \in \text{Irr}(\mathcal{C})$, pick an orthonormal basis $\text{ONB}(\mathbf{A}(a))$ of $\mathbf{A}(a)$ under the inner product (6), and define

$$\mathcal{B}_a^\mathbf{A} = \left\{ d_a^{1/2} \pi_a(f) \bigg| f \in \text{ONB}(\mathbf{A}(a)) \right\}.$$ 

We will suppress the superscript and merely write $\mathcal{B}_a$ when no confusion can arise. By convention, we pick $\text{ONB}(\mathbf{A}(a)) = \{i_A\}$, and thus $\mathcal{B}_1c = \{d_H^{-1/2} \coev_H\}$. This normalization means that for $\alpha, \alpha' \in \mathcal{B}_a$, we have

$$H^\alpha \gamma H^\alpha_{\alpha'} \mathcal{P} = \delta_{\alpha=\alpha'} \id_a.$$  \hfill (12)

Thus $\mathcal{B}_a$ is an orthonormal basis for $\pi_a(\mathbf{A}(a))$ under the new inner product (12).

**Definition 4.3.** We define the following distinguished element of $\mathcal{C}(H \otimes \mathcal{P}, H \otimes \mathcal{P})$:

$$p = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} H^\alpha \gamma H^\alpha_{\alpha'} \mathcal{P}.$$ 

Note that the element $p$ is independent of the choices of orthonormal bases $\mathcal{B}_a$ for $a \in \text{Irr}(\mathcal{C})$ under the inner product (12).

**Lemma 4.4.** The element $p$ is a symmetrically self-dual orthogonal projector. That is, $p^2 = p^* = p$, and $(p \otimes \id_{H \otimes \mathcal{P}}) \circ \coev_{H \otimes \mathcal{P}} = (\id_{H \otimes \mathcal{P}} \otimes p) \circ \coev_{H \otimes \mathcal{P}}$.

**Proof.** It is obvious that $p^* = p$, and it follows readily from (12) that $p^2 = p$. To verify the final condition, for each $a \in \text{Irr}(\mathcal{C})$, there is a unique $\alpha' \in \text{Irr}(\mathcal{C})$ with $\alpha' \cong \pi_a$. We pick a unitary isomorphism $\gamma_a \in \mathcal{C}(a', \pi)$, and we see that

$$(p \otimes \id_{H \otimes \mathcal{P}}) \circ \coev_{H \otimes \mathcal{P}} = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} H^\alpha \gamma H^\alpha_{\alpha'} \mathcal{P} = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} H^\alpha \gamma H^\alpha_{\alpha'} \mathcal{P}.$$ 

Now since $\{\pi_a \circ \gamma_a | \alpha \in \mathcal{B}_a\}$ is an orthonormal basis for $\pi_a(\mathbf{A}(a'))$ under the inner product (12), and since $p$ is independent of the choice of orthonormal bases $\mathcal{B}_a$, we see that the right hand side above is equal to

$$\sum_{a \in \text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} H^\alpha \gamma H^\alpha_{\alpha'} \mathcal{P} = (\id_{H \otimes \mathcal{P}} \otimes p) \circ \coev_{H \otimes \mathcal{P}}.$$  \hfill $\square$

We now define important structure constants which are essentially certain $6j$ symbols.

**Definition 4.5.** For $a, b, c \in \text{Irr}(\mathcal{C})$, $\alpha \in \mathcal{B}_a$, $\beta \in \mathcal{B}_b$, $\gamma \in \mathcal{B}_c$, and $\delta \in \mathcal{C}(c, a \otimes b)$, we define the *tetrahedral structure constant*

$$\Delta \left( \begin{array}{ccc|c} a & b & c \ \alpha & \beta & \gamma \ \delta \end{array} \right) = \mathcal{P}.$$

12
Note that these structure constants have a \( \mathbb{Z}/3\mathbb{Z} \) symmetry:

\[
\Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) = \Delta \left( \begin{array}{ccc} \varphi & a & b \\
\gamma & \beta & \alpha \\
\end{array} | \mathcal{F}(\delta) \right) = \Delta \left( \begin{array}{ccc} b & \varphi & \alpha \\
\gamma & \beta & \alpha \\
\end{array} | \mathcal{F}^2(\delta) \right).
\]

(13)

**Example 4.6.** Using the tetrahedral structure constants, we get the following two important identities. Both are proved by expanding the left hand sides out using the respective inner products.

(1) For all \( a, b, c \in \text{Irr}(\mathcal{C}) \) and \( \alpha \in \mathcal{B}_a \) and \( \beta \in \mathcal{B}_b \), and \( \delta \in \text{Isom}(c, a \otimes b) \), expanding using the inner product (12) on \( \mathcal{B}_c \) yields

\[
\sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) H_{a \beta}^H = d_c^{-1} \sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) H_{\gamma \delta}^H.
\]

(14)

(2) For all \( a, b, c \in \text{Irr}(\mathcal{C}) \), \( \alpha \in \mathcal{B}_a \), \( \beta \in \mathcal{B}_b \), and \( \gamma \in \mathcal{B}_c \), expanding using the inner product on \( \text{Isom}(c, a \otimes b) \) yields

\[
\sum_{\delta \in \text{Isom}(c, a \otimes b)} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) = d_c^{-1} \sum_{\delta \in \text{Isom}(c, a \otimes b)} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) \delta^* a \beta b c.
\]

(15)

**Lemma 4.7.** The following two morphisms in \( \mathcal{C}(H \otimes H \otimes H \otimes H, H \otimes H) \) are equal:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (2,0) {$c$};
\node (d) at (3,0) {$d$};
\node (e) at (4,0) {$e$};
\node (f) at (5,0) {$f$};
\node (g) at (6,0) {$g$};
\node (h) at (7,0) {$h$};
\node (i) at (8,0) {$i$};
\node (j) at (9,0) {$j$};
\node (k) at (10,0) {$k$};
\node (l) at (11,0) {$l$};
\node (m) at (12,0) {$m$};
\node (n) at (13,0) {$n$};
\node (o) at (14,0) {$o$};
\node (p) at (15,0) {$p$};
\node (q) at (16,0) {$q$};
\node (r) at (17,0) {$r$};
\node (s) at (18,0) {$s$};
\node (t) at (19,0) {$t$};
\node (u) at (20,0) {$u$};
\node (v) at (21,0) {$v$};
\node (w) at (22,0) {$w$};
\node (x) at (23,0) {$x$};
\node (y) at (24,0) {$y$};
\node (z) at (25,0) {$z$};
\end{tikzpicture}
\end{array}
= \begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (2,0) {$c$};
\node (d) at (3,0) {$d$};
\node (e) at (4,0) {$e$};
\node (f) at (5,0) {$f$};
\node (g) at (6,0) {$g$};
\node (h) at (7,0) {$h$};
\node (i) at (8,0) {$i$};
\node (j) at (9,0) {$j$};
\node (k) at (10,0) {$k$};
\node (l) at (11,0) {$l$};
\node (m) at (12,0) {$m$};
\node (n) at (13,0) {$n$};
\node (o) at (14,0) {$o$};
\node (p) at (15,0) {$p$};
\node (q) at (16,0) {$q$};
\node (r) at (17,0) {$r$};
\node (s) at (18,0) {$s$};
\node (t) at (19,0) {$t$};
\node (u) at (20,0) {$u$};
\node (v) at (21,0) {$v$};
\node (w) at (22,0) {$w$};
\node (x) at (23,0) {$x$};
\node (y) at (24,0) {$y$};
\node (z) at (25,0) {$z$};
\end{tikzpicture}
\end{array}
\]

Proof. Using (14) and (15) and the fusion relation

\[
\sum_{\alpha \in \text{Irr}(\mathcal{C})} \sum_{\beta \in \mathcal{B}_b} \sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) = d_c^{-1} \sum_{\alpha \in \text{Irr}(\mathcal{C})} \sum_{\beta \in \mathcal{B}_b} \sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) \delta^* a \beta b c,
\]

it is straightforward to show that both morphisms are equal to

\[
\sum_{a,b,c\in\text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} \sum_{\beta \in \mathcal{B}_b} \sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) = d_c^{-1} \sum_{a,b,c\in\text{Irr}(\mathcal{C})} \sum_{\alpha \in \mathcal{B}_a} \sum_{\beta \in \mathcal{B}_b} \sum_{\gamma \in \mathcal{B}_c} \Delta \left( \begin{array}{ccc} a & b & c \\
\alpha & \beta & \gamma \\
\end{array} | \delta \right) \delta^* a \beta b c.
\]

(16)

as \( \text{Isom}(c, a \otimes b) = \sqrt{d_c} \text{ONB}(c, a \otimes b) \).
Remark 4.8. Notice that the morphism (16) is invariant under cyclic rotation, since the sum is independent of the choice of ONB\((c, a \otimes b)\), and the 1-click rotation \(\mathcal{F}\) produces an orthonormal basis of \(C(b, c \otimes a)\) under the standard inner product (4) which caps off all strings. Finally, we use the \(\mathbb{Z}/3\mathbb{Z}\) symmetry of the tetrahedral structure constants (13).

Corollary 4.9. The morphism \(p\) remains unchanged by removing any single \(p\).

Proof. Immediate from Lemmas 4.4 and 4.7 using 2-click rotations.

Lemma 4.10. Capping off \(p\) produces \(\text{id}_H\), i.e., \(\mathcal{P}\) = \(\text{id}_H\).

Proof. Using the definition of \(\mathcal{B}_a\) and applying (11), we have

\[
\mathcal{P} = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{a' \in \mathcal{B}_a} \prod_{H} d_a \text{id}_H = \sum_{a \in \text{Irr}(\mathcal{C})} \dim(\mathcal{A}(a)) \frac{d_a}{d_H} \text{id}_H = \text{id}_H.
\]

A version of the following proposition appears in \[BKLR15\].

Proposition 4.11. The object \(\text{im}(p) \subset H \otimes \overline{H}\) with multiplication and unit morphisms

\[
m = d_H^{1/2} \quad i = d_H^{-1/2}
\]

is a normalized irreducible \(Q\)-system in \(\mathcal{C}\).

Proof. A straightforward calculation using Corollary 4.9 shows that \((\text{im}(p), m, i)\) is a separable \(C^*\) Frobenius algebra. (See [LP15, Lem. 5.8] for a hint.) Using Lemma 4.10, we calculate that \(m \circ m^* = d_H \text{id}_H\) and \(i \circ i^* = \text{id}_{1_C}\), and thus \(\text{im}(p)\) is a normalized \(Q\)-system. It is obvious that \(\text{im}(p)\) is connected as \(\dim(\mathcal{B}_{1_C}) = 1\), and thus the \(Q\)-system is irreducible.

Lemma 4.12. For all \(a \in \mathcal{C}\) and \(g \in \mathcal{A}(a)\), \(p \circ \pi_a(g) = \pi_a(g)\). In particular, the unit for the \(Q\)-system in Proposition 4.11 is given by \(i = d_H^{-1/2}(p \circ \text{coev}_H) = d_H^{-1/2}\text{coev}_H\).

Proof. For \(a \in \mathcal{C}\) and \(g \in \mathcal{A}(a)\), we have

\[
p \circ \pi_a(g) = d_H^{-1/2} \sum_{c \in \text{Irr}(\mathcal{C})} \sum_{a \in \mathcal{B}_c} \prod_{H} d_c \lambda_c(f) \lambda_a(g) = d_H^{-3/2} \sum_{c \in \text{Irr}(\mathcal{C})} \sum_{f \in \text{ONB}(\mathcal{A}(c))} \prod_{H} d_c \lambda_c(f)^* \lambda_a(g)\] (17)
where ONB(\(A(c)\)) is with respect to inner product (6). For \(c \in \text{Irr}(C)\) and \(f \in \text{ONB}(A(c))\), we take inner products against an orthonormal basis Isom\((c,a)\) for inner product (5) on \(C(c,a)\) to see

\[
\lambda_c(f)^* \pi = \sum_{\beta \in \text{Isom}(c,a)} \lambda_c(g \circ \beta)^* \beta^* \pi = d_H \sum_{\beta \in \text{Isom}(c,a)} \langle f | g \circ \beta \rangle_c \beta^*.
\]

by (11). Plugging back in to (17), we obtain

\[
p \circ \pi_a(g) = d_H^{-1/2} \sum_{c \in \text{Irr}(C)} \sum_{f \in \text{ONB}(A(c))} \sum_{\beta \in \text{Isom}(c,a)} \langle f | g \circ \beta \rangle_c \beta^* \pi = d_H^{-1/2} \sum_{h \in \text{ONB}(A(a))} \lambda_a(h)^* \pi = d_H^{-1/2} \lambda_a(g)^* \pi = \pi_a(g).
\]

Now it is straightforward to verify that for all \(f \in A(c)\) and \(\beta \in C(c,a)\), \(\langle f | g \circ \beta \rangle_c = \langle f | \beta^* | g \rangle_a\).

Moreover,

\[
\{f \circ \beta^* | f \in \text{ONB}(A(c)), \beta \in \text{Isom}(c,a), c \in \text{Irr}(C)\}
\]

is an orthonormal basis for \(A(a)\) under inner product (6). Thus (18) simplifies to

\[
p \circ \pi_a(g) = d_H^{-1/2} \sum_{h \in \text{ONB}(A(a))} \langle h | g \rangle_a \lambda_a(h)^* \pi = d_H^{-1/2} \lambda_a(g)^* \pi = \pi_a(g).
\]

4.2 Morphisms

In the last section, to each compact connected W*-algebra object \(A \in \text{Vec}(C)\), we associated a canonical projection \(p_A \in \text{End}_C(H_A \otimes H_A)\) where \(H_A \in C\) is the object representing the compact Hilbert space object \(L^2 A\), i.e., \(L^2 A = C(a,H_A)\). We use the notation \(Q(A) = \text{im}(p_A)\) for the subobject of \(H_A \otimes H_A\).

Suppose now we have a *-algebra natural transformation \(\theta : A \Rightarrow B\) between compact connected W*-algebra objects.

Notation 4.13. Recall that for \(a \in C\), we defined

\[
B^A_a = \left\{ d_a^{1/2} \pi^A_a(f) \middle| f \in \text{ONB}(A(a)) \right\}
\]

where ONB\((A(a))\) is an orthonormal basis for \(A(a)\) under the inner product (6). By a slight abuse of notation, for \(\alpha = d_a^{1/2} \pi^A_a(f) \in B^A_a\), we write \(\theta(\alpha)\) for \(d_a^{1/2} \pi^B_a(\theta(f)) \in B^B_a\). Moreover, by (8), we may choose the orthonormal basis ONB\((B(a))\) so that it contains \(\theta(\text{ONB}(A(a)))\) as a subset. This means that \(\theta(B^A_a) \subset B^B_a\) as orthonormal bases under inner product (12).
**Definition 4.14.** We define the following morphisms in $\mathcal{C}$, which are clearly independent of the choice of orthonormal bases $B^A_a$ for $a \in \text{Irr}(\mathcal{C})$:

$$Q(\theta) = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{a \in B^A_a} H_B(\theta(\alpha)) = \sum_{a \in \text{Irr}(\mathcal{C})} \sum_{a \in B^A_a} H_B(\theta(\alpha)).$$

It is straightforward to verify using (8) together with (11) and (12) that $\theta(\text{p}_A)$ is an orthogonal projection such that $\theta(\text{p}_A) \leq \text{p}_B$, and $Q(\theta)$ is a partial isometry satisfying $Q(\theta)Q(\theta)^* = \theta(\text{p}_A)$ and $Q(\theta)^*Q(\theta) = \text{p}_A$.

We have the following key lemma.

**Lemma 4.15.** For all $a \in \mathcal{C}$ and $f \in A(a)$,

$$d_{H_B}^{-1/2} \begin{array}{c|c}
\lambda^B_a(\theta(a)) \\
\hline
\pi_B
\end{array} = d_{H_B}^{-1/2} \begin{array}{c|c}
\theta(\text{p}_A) \\
\hline
\pi_B
\end{array} = \frac{1}{2} \begin{array}{c|c}
\lambda^B_a(\theta(a)) \\
\hline
\pi_B
\end{array} = \frac{1}{2} \begin{array}{c|c}
\lambda^A_a(\theta(a)) \\
\hline
\pi_B
\end{array}.$$

In particular, $\pi^B_a(\theta(f)) = Q(\theta) \circ \pi^A_a(f)$.

**Proof.** The first equality follows from Lemma 4.12. The second equality follows from the fact that $\pi^B_a(\theta(a))$ is orthogonal to $B^B_a \setminus \theta(B^A_a)$. We omit the proof of the final equality, which is similar to the proof of Lemma 4.12 using (8) together with (11) and (12). □

**Corollary 4.16.** The morphism $Q(\theta) \in \mathcal{C}(Q(A), Q(B))$ is an involutive algebra morphism.

**Proof.** It is easy to see that $Q(\theta)$ is unital, since $\theta_1(i_A) = i_B$. Thus by Lemma 4.15,

$$d_{H_A}^{-1/2} Q(\theta) \circ \coev_{H_A} = Q(\theta) \circ \pi^A_1(i_A) = \pi^B_1(\theta(i_A)) = d_{H_B}^{-1/2} \coev_{H_B}.$$

To show $Q(\theta)$ is multiplicative, we first use multiplicativity of $\theta : A \Rightarrow B$ and $\pi^B : B \Rightarrow L^B \otimes L^B$ and the relationship between the inner products (8), (11), and (12) to prove an identity similar to (15). For all $a, b, c \in \text{Irr}(\mathcal{C})$, $\alpha \in B_a$, $\beta \in B_b$, and $\gamma \in B_c$, expanding using the inner product on $\text{Isom}(c, a \otimes b)$ yields

$$d_{H_B}^{-1/2} \begin{array}{c|c}
\theta(\gamma) \\
\hline
\pi_B
\end{array} = \frac{1}{2} \begin{array}{c|c}
\lambda^B_a(\theta(\alpha)) \\
\hline
\pi_B
\end{array} = \frac{1}{2} \begin{array}{c|c}
\lambda^B_a(\theta(\beta)) \\
\hline
\pi_B
\end{array} = \frac{1}{2} \begin{array}{c|c}
\lambda^B_a(\theta(\gamma)) \\
\hline
\pi_B
\end{array}.$$

(19)
Now, similar to the proof of Lemma 4.7, we have
\[ m_{Q^2}(B) \circ (Q(\theta) \otimes Q(\theta)) \]
\[ = d_{H_B}^{1/2} \sum_{a,b,c \in \text{Irr}(C)} d_c^{-1} \sum_{\alpha \in B_a} \sum_{\beta \in B_b} \sum_{\gamma \in B_c} \Delta \left( \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \delta \end{array} \right) \]
\[ = d_{H_A}^{1/2} \sum_{a,b,c \in \text{Irr}(C)} d_c^{-1} \sum_{\alpha \in B_a} \sum_{\beta \in B_b} \sum_{\gamma \in B_c} \Delta \left( \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \delta \end{array} \right) \]
\[ = d_{H_A}^{1/2} \sum_{a,b,c \in \text{Irr}(C)} d_c^{-1} \sum_{\alpha \in B_a} \sum_{\beta \in B_b} \sum_{\gamma \in B_c} \Delta \left( \begin{array}{ccc} a & b & c \\ \alpha & \beta & \gamma \\ \delta \end{array} \right) \]

Finally, \( \sigma_{Q(A)} = p_A \) and \( \sigma_{Q(B)} = p_B \), so inovitivity of \( Q(\theta) \) is equivalent to \( Q(\theta) = \overline{Q(\theta)} \), which is easily seen to hold by independence of the choice of \( B_a^A \) as in the proof of Lemma 4.4. (This suppresses the isomorphism between real objects of the form \( c \otimes \overline{c} \in C \) and their conjugates as in Example 2.17.)\)

**Proposition 4.17.** \( Q \) is a functor.

**Proof.** It remains to show that \( Q \) preserves identities and composites. First, if \( \theta = \text{id}_A \), then \( Q(\theta) = p_A = \text{id}_{Q(A)} \). Second, if \( \theta_1 : A \Rightarrow B \) and \( \theta_2 : B \Rightarrow C \), then for each \( a \in C \), we choose ONB\((C(a)) \) so as to include \( \theta_2(\text{ONB}(B(a))) \) just as we chose ONB\((B(a)) \) to include \( \theta_1(\text{ONB}(A(a))) \). Now for all \( \alpha \in B_a^A, \theta_1(\alpha) \) is orthogonal to \( B_a^B \setminus \theta_1(B_a^A) \) under inner product (12), and thus

\[ Q(\theta_2) Q(\theta_1) = \sum_{a,b \in \text{Irr}(C)} \sum_{\alpha \in B_a^A} \sum_{\beta \in B_b^B} \theta_2(\theta_1(\alpha)) \]

\[ = Q(\theta_2 \circ \theta_1). \]

5 Equivalence of Q-systems and W*-algebra objects

We now prove the the functors \( W^* \) and \( Q \) constructed in Sections 3 and 4 respectively witness an equivalence of categories.
5.1 From W*-algebras to Q-systems and back

We begin by building a natural isomorphism \( \eta : \text{id} \Rightarrow \mathbf{W}^* \circ Q \).

Suppose we have a compact connected W*-algebra object \((A, \mu^A, i_A, j^A) \in \text{Vec}(C)\) together with its canonical GNS faithful *-algebra natural transformation \( \lambda : A \Rightarrow B(L^2A) \). Recall that \( Q(A) = \text{im}(p_A) \subset H_A \otimes \overline{H_A} \in C \) is an irreducible Q-system by Proposition 4.11. Consider the corresponding compact connected W*-algebra object \( \mathbf{W}^*Q(A) \in \text{Vec}(C) \) as in Proposition 3.2, and notice that for all \( a \in C \), \( \mathbf{W}^*Q(A)(a) = \mathcal{C}(a,Q(A)) \) with multiplication and unit induced from \( Q(A) \), and *-structure given by conjugation, which is induced from the adjoint on the cyclic \( C \)-module W*-category \((C, B)\).

**Theorem 5.1.** The GNS representation \( \lambda : A \Rightarrow B(L^2A) \) induces a canonical *-algebra natural isomorphism \( \eta^A : A \Rightarrow \mathbf{W}^*Q(A) \).

**Proof.** Using the shorthand \( H_A = L^2A \), let \( \pi^A : A \Rightarrow H_A \otimes \overline{H_A} \) be the induced faithful *-algebra natural transformation as in Definition 4.1. Note that \( \pi^A_a(A(a)) \) is exactly equal to

\[
\mathbf{W}^*Q(a) = \mathcal{C}(a,Q(A)) = \mathcal{C}(a,\text{im}(p_A)) \subset \mathcal{C}(a,H_A \otimes \overline{H_A}).
\]

We define \( \eta^A : A \Rightarrow \mathbf{W}^*Q(A) \) by \( \eta^A_a(f) = p_A \circ \pi^A_a(f) \in \mathcal{C}(a,\text{im}(p_A)) \), i.e., \( \eta^A \) is just \( \pi^A \) considered as a morphism \( A \Rightarrow \mathbf{W}^*Q(A) \) rather than \( A \Rightarrow H_A \otimes \overline{H_A} \).

To show that \( \eta^A : A \Rightarrow \mathbf{W}^*Q(A) \) is a *-algebra natural isomorphism, it remains to show that \( \eta^A \) intertwines the unit, multiplication, and *-structures of \( A \) and \( \mathbf{W}^*Q(A) \). As in Lemma 4.12,

\[
\pi^A_{1_c}(i_A) = d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_a(i_A) \\ 1 & 1 \end{vmatrix} \overline{H_A} = d_{H_A}^{-1/2} \text{coev}_{H_A} = i_{\mathbf{W}^*Q(A)}.
\]

If \( f \in A(a) \) and \( g \in A(b) \), then we see that

\[
\begin{align*}
\mathbf{W}^*Q(A)_{a,b} (\pi^A_a(f) \otimes \pi^A_b(g)) &= \frac{d_{H_A}^{1/2}}{d_{H_A}} \begin{vmatrix} H_A & \lambda^A_a(f) \\ a & b \end{vmatrix} \overline{H_A} = d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_a(f) \\ a & b \end{vmatrix} \\
&= d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_{a \otimes b}(\mu^A_{a \otimes b}(f \otimes g)) \\ a & b \end{vmatrix} \overline{H_A} = \pi^A_{a \otimes b}(\mu^A_{a \otimes b}(f \otimes g)).
\end{align*}
\]

Finally, if \( f \in A(a) \), using (1), together with the fact that \( \lambda \) is preserves the *-structure, and the correspondence between the *-structure of \( A \) and the dagger structure of \( \mathcal{M}_A \) from [JP16, Thm. 3.20], we have that

\[
\begin{align*}
\mathbf{W}^*Q(A)_{a} (\pi^A_a(f)) &= d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_a(f) \\ a & 1 \end{vmatrix} \overline{H_A} = d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_a(f)^* \\ a & 1 \end{vmatrix} \\
&= d_{H_A}^{-1/2} \begin{vmatrix} H_A & \lambda^A_a(f)^* \\ a & 1 \end{vmatrix} \overline{H_A} = \pi^A_a(j^A_a(f)).
\end{align*}
\]

This completes the proof. \( \square \)
Proposition 5.2. The *-algebra isomorphism \( \eta^A : A \to W^*Q(A) \) is natural in \( A \).

Proof. Suppose \( \theta : A \to B \) is a natural *-algebra transformation. We must prove that \( W^*Q(\theta) \circ \eta^A = \eta^B \circ \theta \). We calculate that for \( a \in C \) and \( f \in A(a) \), we have

\[
(W^*Q(\theta)_a \circ \eta^A_a)(f) = d_{H_A}^{-1/2} \begin{pmatrix} H_B & H_A \\ \lambda^A_a(f) & H_A \\ \eta^A_a(f) & H_A \\ \end{pmatrix}
\]

and

\[
(\eta^B_a \circ \theta_a)(f) = d_{H_B}^{-1/2} \begin{pmatrix} H_B & H_A \\ \lambda^B_a(\theta(f)) & H_A \\ \eta^B_a(\theta(f)) & H_A \\ \end{pmatrix}
\]

The equality of the above morphisms is exactly the identity \( \pi^B_\alpha(\theta(f)) = Q(\theta) \circ \pi^A_\alpha(f) \) which was established in Lemma 4.15.

\[ \Box \]

5.2 From Q-systems to \( W^* \)-algebras and back

We now build a natural isomorphism \( \zeta : \text{id} \to QW^* \).

Starting with a normalized irreducible Q-system \( (A, m_A, i_A) \in \mathcal{C} \), define the compact connected \( W^*- \)algebra object \( A \in \text{Vec}(\mathcal{C}) \) by \( A(a) = C^2(a, A) \) as in Proposition 3.2. By Lemma 3.3, the compact Hilbert space object \( L^2A \) is given by \( L^2A(a) = C(a, A) \), and by Lemma 4.12, the canonical GNS faithful *-algebra natural transformation \( \lambda : A \to B(L^2A) \) is given by (10). By Proposition 4.11, \( QW^*(A) = \text{im}(p) \subset H \otimes \overline{H} \in \mathcal{C} \) is a normalized irreducible Q-system.

Theorem 5.3. Under the GNS representation \( \lambda : A \to B(L^2A) \), the identity map \( \text{id}_A \in \mathcal{C}(A, A) \) induces a canonical unitary involutive algebra isomorphism \( \zeta_A \in \mathcal{C}(A, QW^*(A)) \).

Proof. Recall that \( \mathcal{C}(A, QW^*(A)) = \mathcal{C}(A, \text{im}(p)) \subset \mathcal{C}(A, A \otimes \overline{A}) \), and that \( \mathcal{C}(A, \text{im}(p)) \) is exactly the image of \( \pi_A : A(A) \to (L^2A \otimes L^2\overline{A})(A) = \mathcal{C}(A, A \otimes \overline{A}) \) by Lemma 4.12. By Remark 3.4,

\[
\lambda_A(\text{id}_A) = \begin{pmatrix} A \\ A \end{pmatrix} \in \mathcal{C}(A \otimes A, A) \quad \Longrightarrow \quad \pi_A(\text{id}_A) = d_A^{-1/2} \begin{pmatrix} A \\ A \end{pmatrix} =: \zeta_A
\]

Denoting \( \zeta_A := \pi_A(\text{id}_A) \), by Lemma 4.12, \( \zeta_A = p \circ \zeta_A \in \mathcal{C}(A, \text{im}(p)) = \mathcal{C}(A, QW^*(A)) \).

We claim \( \zeta_A \) is an involutive unitary isomorphism of Q-systems. First, since \( (A, m_A, i_A) \) is a normalized Q-system, \( m_A \circ m_A^* = d_A \text{id}_A \). Thus \( \zeta_A^* \circ \zeta_A = \text{id}_A \) by construction. Now \( \zeta_A = p \circ \zeta_A \) by Lemma 4.12, and it is easy to see that \( \zeta_A \circ \zeta_A = \zeta_A \circ \zeta_A^* = p = \text{id}_{QW^*(A)} \). Indeed, \( q := \zeta_A \circ \zeta_A^* \) is a subprojection of \( p \) such that \( \text{dim}(\text{im}(q)) = \text{dim}(A) = \text{dim}(\text{im}(p)) \), which means \( q = p \). Thus \( \zeta_A \) is a unitary isomorphism.

It remains to show that \( \zeta_A \) intertwines the unit and multiplications of \( A \) and \( B \). First, by Lemma 4.12,

\[
\zeta_A \circ i_A = d_A^{-1/2} \begin{pmatrix} A \\ i_A \end{pmatrix} = d_A^{-1/2} \text{coev}_A = d_A^{-1/2} (p \circ \text{coev}_A) = i_{QW^*(A)}.
\]

Second, using Corollary 4.9, \( p \circ \zeta_A = \zeta_A \) by Lemma 4.12, and the Frobenius relation,

\[
m_{QW^*(A)} \circ (\zeta_A \otimes \zeta_A) = \frac{d_A^{1/2}}{d_A} \begin{pmatrix} p \\ p \\ p \end{pmatrix} = d_A^{-1/2} \begin{pmatrix} p \\ p \end{pmatrix} = d_A^{-1/2} \begin{pmatrix} p \\ p \end{pmatrix} = d_A^{-1/2} \begin{pmatrix} p \\ p \end{pmatrix} = \zeta_A \circ m_A.
\]
Finally, we show $\zeta_A$ is involutive, i.e., $\sigma_{\im(p)} \circ \zeta_A = \zeta_A \circ \sigma_A$. First, note that by Lemmas 4.4 and 4.7, $\sigma_{\im(p)} = p$ on the nose. Using (1) and Lemma 2.7 we compute

$$d_A^{-1/2} = \zeta_A \circ \sigma_A = (\zeta_A^*)^* \circ \sigma_A = d_A^{-1/2} = \zeta_A.$$

Since $\sigma_{\im(p)} \circ \zeta_A = p \circ \zeta_A = \zeta_A$ by Lemma 4.12, we are finished.

**Proposition 5.4.** The involutive algebra isomorphism $\zeta_A \in \mathcal{C}(A, \mathbf{QW}^*(A))$ is natural in $A$.

**Proof.** Suppose $\theta \in \mathcal{C}(A, B)$ is an involutive algebra morphism. We must prove that $\mathbf{QW}^*(\theta) \circ \zeta_A = \zeta_B \circ \theta$. Expanding the definition of $\theta(\alpha)$ for $\alpha \in \mathcal{B}_a\mathbf{W}^*(A)$, one calculates that

$$\mathbf{QW}^*(\theta) = d_A^{-1/2} d_B^{-1/2} \sum_{c \in \text{Irr}(A)} \sum_{f \in \text{ONB}(A(c))} d_c f_c = d_A^{-1/2} d_B^{-1/2} \sum_{c \in \text{Irr}(A)} \sum_{f \in \text{ONB}(A(c))} d_c f_c = \zeta_B \circ \theta,$$

where in the second equality, we used that $\sqrt{d_c \text{ONB}(A(c))} = \sqrt{d_c \text{ONB}(c, A)} = \text{Isom}(c, A)$. Finally, since $A$ is normalized, we obtain

$$\mathbf{QW}^*(\theta) \circ \zeta_A = d_B^{-1/2} = \zeta_B \circ \theta.$$

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