EXTINCTION OF POPULATIONS AND A TEAM OF LYAPUNOV FUNCTIONS

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\textbf{Abstract.} We investigate the \(d\)-dimensional model,

\[
x_i' = \frac{d}{dt} x_i = x_i \cdot \left( c_i + \sum_{j=1}^{d'} s_{ij} z_j(t) \right), \quad \text{where } i = 1, \ldots, d.
\]

where each \(z_j(t)\) is a time fluctuating resource that can depend on \(x_1, \ldots, x_d\) and even on the weather or stock market. This “nonautonomous” model is a generalization of an autonomous Lotka-Volterra \(d\)-dimensional model. It is nonautonomous and it is not specified how the \(z_j(t)\) are determined. Write \(S^T\) for the transpose of \(S = (s_{ij})\). When the kernel of \(S^T\) has dimension \(k\) and \(k > 0\), we show that for any bounded solution \(X(t) = (x_1, \ldots, x_d)(t)\), at least \(k\) coordinates (or species) will die out and will do so exponentially fast. For the proof, we invent a family of “die-out” Lyapunov functions, a “team” of Lyapunov functions that work together to show that \(k\) species must die. Each die-out Lyapunov function implies one species must die out and provides constraints as to which species must die out. Together they provide a picture of which are likely to die out. We present a “trophic” condition for Lotka-Volterra systems that guarantees that there is a trapping region that is globally attracting.

\section{Introduction}

There is an old principle or rule of thumb for ordinary differential equations models of an ecosystem, that if a number \(b\) of species depends on \(b'\) resources where when \(b > b'\), there is almost always no steady state. We wish to make this more precise.

\textbf{Our (non-autonomous) main model.} Let \(\mathcal{P} := \{(x_1, \ldots, x_d) : x_i > 0\}\) (We sometimes write “:=” instead of “=” in definitions of symbols). Let \(X := (x_1, \ldots, x_d) \in \mathcal{P}, \ C := (c_1, \ldots, c_d) \in \mathbb{R}^d, \) and \(Z = (z_1, \ldots, z_{d'}) \in \mathbb{R}^{d'}, \) and let \(S_{Z}\) or \(S = (s_{ij})\) be a \(d \times d'\) matrix where \(d, d'\) are positive integers. Our “\(Z\)” system is

\begin{equation}
\frac{dx_i'}{dt} = c_i + \sum_{j=1}^{d'} s_{ij} z_j(t), \quad \text{where } i = 1, \ldots, d, \quad \text{and } x_i' = \frac{d}{dt} x_i.
\end{equation}
where \( z_j(t) \) is a continuous function for each \( j \), and each \( z_j(t) \) may depend on \( X(t) \) or even on \( X(t - \tau_j) \) where the \( \tau_j \) are time delays. Each \( x_i \) can be viewed as the population density of the \( i \)th species. Ecologist often refer \( z_j \)'s as the “resources” that the population \( x_i \) depend upon. The \( c_i \)'s are constant per capita growth or death rates. While the right-hand side of the Eq. (1.1) is linear, the left hand side is nonlinear, and the solutions cannot be written in closed form. In some cases, the solution can be chaotic or periodic, or have some coordinates asymptotically going to zero. Smale [1976] showed that Lotka–Volterra systems that have five or more species can exhibit any asymptotic behavior, including a fixed point, a limit cycle, an n-torus, or attractors, and our system is even more general.

Our main result says that if \( X(t) \) is a bounded solution of Eq. (1.1) and the dimension of the kernel of \( S^T \) is \( k \) where \( k > 0 \), then \( k \) “species” or coordinates must die out exponentially fast.

Suppose we had what might appear to be a more general form of Eq. (1.1) where we replace \( z_j(t) \) by \( \dot{z}_j(X(t), X(t - \tau_j), t) \), where \( \tau_j \) is a time delay. Suppose also \( X(t) \) was a solution of this new equation. Then \( X(t) \) would also be a solution of the original Eq. (1.1) after we set \( z_j(t) = \dot{z}_j(X(t), X(t - \tau_j), t) \) using the given \( X(t) \).

For a given solution \( X(t) \), we say \( k \) “species” (or coordinates of \( X(t) \)) die out exponentially fast if there are constants \( a \) and \( b \) (\( b > 0 \)) such that for each \( t \geq 0 \) there are at least \( k \) choices of \( i \) for which \( x_i(t) \leq e^{a-bt} \). Which \( x_i \)'s are small may depend upon \( t \). We can imagine that there is a minimum threshold level of population density below which \( x_i \) cannot go and later recover, below which \( x_i(t) \) will go to 0. We should emphasize that we draw conclusions only about a given bounded solution, if there is one.

Let \( k > 0 \) be the dimension of the kernel of \( S^T \), the transpose of \( S \). Assume there is a solution \( X(t) = (x_1, \ldots, x_d)(t) \) where all \( x_i(t) \) are bounded for \( t \geq 0 \). Theorem 1 addresses the following questions:

1. Must some \( x_i \) die out? (Yes, since the kernel dimension \( k \) is greater than zero.)
2. What is the minimum number of \( x_i \) that must die out? (The answer is \( k \) for almost every choice of \( C \).)
3. Must they die out exponentially fast? (The answer is yes!)
4. Can we tell which species must die out? (The answer depends upon \( C \) and \( S \), and sometime is yes, sometimes no.)

A special case of Eq. (1.1) is the following Lotka-Volterra population model “LV” that occurs in many fields of science and engineering (Gause [1932], Grover et al. [1997]),

\[
\frac{dx_i}{dt} = c_i + \sum_{j=1}^{d} s_{ij} x_j, \quad \text{where } i = 1, \ldots, d,
\]

where \( C = (c_1, \ldots, c_d) \) and \( S_{LV} \) or \( S = (s_{ij}) \) are constants. That it is a special case can be seen by substituting variables \( z_j \) for \( x_j \) on the right-hand side of Eq. (1.1). Then if \( X(t) \) is a solution of Eq. (1.2) and we set \( z_j(t) = x_j(t) \) for all \( j \), \( X(t) \) will be a solution of Eq. (1.1), now with \( d' = d \).

Model Z, Eq. (1.1), allows us to focus on particular coordinates of the dynamics that are primarily responsible for some coordinates dying out, so its matrix \( S_Z \) is a submatrix of \( S_{LV} \) in the more limited model, Eq. (1.2).
One criticism of our model is the over simplicity of the linear interactions between species or coordinates. There is a large literature in which the interactions between species are modeled in more complex ways, Dubey and Upadhyay [2004]. We view our paper as an investigation of the nature of Lyapunov functions rather than ecology and as pure mathematics aimed at the applied scientist. By keeping interactions simple, we hope the model and the methods might be applicable to situations throughout the sciences.

This paper is motivated by connecting the ideas of two papers, from Jahedi et al. [2022a], and Jahedi et al. [2022b], which applies null vectors in their generalized competitive exclusion principle. It is also motivated by Akhavan and Yorke [2020] in which a Lyapunov function of the form $V = \frac{x_1}{x_2}$ was used to show one species must die out in a population model.

McGehee and Armstrong [1977] and Armstrong and McGehee [1980] investigate Lotka-Volterra systems Eq. (1.1) and more general autonomous systems for which they show, the answer to question (1) is yes. They show that the limit set of a trajectory $X(t)$ as $t \to \infty$ cannot contain any points $X_0 \in P$. Hence if $X_0$ is a limit point of $X(t)$, some coordinate of $X_0$ must be zero, i.e., $X_0 \notin P$.

If we add to the results of McGehee and Armstrong [1977] the assumption that there is a bounded trajectory, then their results imply that at least one specie dies out — in the sense that $\min_i x_i(t) \to 0$ as $t \to \infty$. As time passes, at each time $t$, for at least one coordinate $i$, $x_i(t)$ is small, but which specie is small may vary from time to time. None the less, one can argue that in practice, eventually at least, one specie density is so small that it cannot recover and it dies out. They conclude only that eventually, at least one species dies out and provide no information about the speed of the die-off. Our Props. 1 and 19 have a similar spirit, and are included here in part to relate our results to theirs.

Usually a Lyapunov function $V(X)$ is used to establish stability of a steady state, but that is not our approach. We wish to use Lyapunov functions to give the reader the spirit of our methods in attacking the above questions, we present the following result. Notice that this result makes no assumptions about what values $V$ assumes.

**Defining $\dot{V}$ for a Lyapunov function $V$.** Let $U$ be an open set in $\mathbb{R}^d$. Assume $F : U \to \mathbb{R}^d$ and $V : U \to \mathbb{R}$ are $C^1$. For the differential equation

$$\frac{dX}{dt} = F(X),$$

define

$$\dot{V}(X) = \nabla V(X) \cdot F(X) \text{ so that } \frac{d}{dt} V(X(t)) = \dot{V}(X(t)).$$

Hence $\dot{V}(X)$ tells how fast $V(X(t))$ changing when the trajectory is passing through a point $X$. Of course if $F$ depends on $X$ and $t$, $\dot{V}(X,t) = \nabla V(X) \cdot F(X,t)$.

**Proposition 1.** Assume there exists a $C^1$ differential equation (1.3) where $F$ is defined on an open set $U$ and $F$ is $C^1$. Assume

(i) there is a trajectory $X(t) = (x_1, \ldots, x_d)(t) \in U$ for all $t \geq 0$, and

(ii) $V : U \to \mathbb{R}$ is differentiable and $\dot{V}(X) < 0$ for each $X \in U$.

Then

(iii) $X(t)$ has no limit points in $U$ as $t \to \infty$.

(iv) If furthermore $U = P := \{X : \text{for all } i, x_i > 0\}$ and the trajectory $X(t)$ is
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bounded (see below, Def. 3), then

\[
\min_{1 \leq i \leq d} x_i(t) \to 0 \text{ as } t \to \infty.
\]

(1.5)

Such a function \( V \) does not satisfy some of the novel hypotheses so we will call \( V \) “die-out” Lyapunov function when (ii) is satisfied.

In particular, we have made no assumptions about the sign of \( V \) nor whether \( V \) is bounded nor whether it is positive definite. In (ii) we could alternatively assume \( \dot{V}(X) > 0 \) on \( U \) since we need only assume that \( \dot{V}(X) \) is never zero. The result does not require \( U \) to be invariant or positively invariant since it only makes statements about those trajectories that stay in \( U \) for all future time, if there are any. Conclusion (iii) is true since \( \dot{V}(X) \) would be 0 at any such limit point, which contradicts (ii); and (iv) follows from (iii) since if (1.5) is false, there exists a sequence \( t_n \to \infty \) and an \( \varepsilon > 0 \) such that \( \min_{1 \leq i \leq d} x_i(t_n) \geq \varepsilon \). Since \( X(t) \) is bounded, the sequence \( \{X(t_n)\} \) has a limit point \( X^* \) in \( P \) as \( n \to \infty \), which contradicts (iii). See other limit point methods in Barbashin-Krasovskii-LaSalle [Alligood et al., 1996, p. 309].

See Prop. 19 (Sec. 5) for a significant generalization of Prop. 1, though there the domain \( U \) is required to be simply connected.

We construct a die-out Lyapunov function (Definition 2 in Sec. 2) from any null vector \( \nu \) of \( S^T \), i.e., \( \nu S = 0 \).

We define the \( \nu \)-Lyapunov function on \( P := \{x_1, \ldots, x_d : x_i > 0\} \) as an inner product,

\[
V(X) := \nu \cdot (\ln x_1, \ldots, \ln x_d) \text{ on } P.
\]

(1.6)

It has the property that \( V(X) \) is constant for Eqns. (1.1) or (1.2).

Example 1. Suppose

\[
S := \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}.
\]

Its kernel is one-dimensional and it has a null vector \( \nu = [2, -5, 1] \) of \( S^T \). Then

\[
V(X) := 2 \ln x_1 - 5 \ln x_2 + \ln x_3 \text{ on } P,
\]

(1.7)

(which appears in Example 3) and assume (as will be the case in this paper) that

\[
C = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.
\]

\[ \dot{V}(X) = \nu \cdot C = -8 \neq 0 \text{ on } P. \]

Assume \( X(t) \) is a bounded solution of Eq. (1.1). That is, there is \( \beta > 0 \) such that \( x_i(t) \leq \beta \) for all \( t > 0 \) and all \( i \). Since the above constant \( \nu \cdot C \) is negative, \( V(X(t)) \to -\infty \) as \( t \to \infty \). The term \(-5 \ln x_2\) is bounded from below, since \(-5 \ln x_2 \geq -5 \ln \beta\). Hence, \( 2 \ln x_1 + \ln x_3 \to -\infty \), so \( \min\{\ln x_1, \ln x_3\} \to -\infty \), and equivalently,

\[ \min\{x_1(t), x_3(t)\} \to 0 \text{ as } t \to \infty. \]

For a different \( C \), we can have \( \nu \cdot C > 0 \), in which case, then \( V(X(t)) \to +\infty \) as \( t \to \infty \) which implies,

\[ x_2(t) \to 0 \text{ as } t \to \infty. \]
This example is a special case of Example 3 in Sec. 3. In this paper our examples can have a variety of such functions $V$ that depend on different selections of $x_i$. We call the collection of these a “team of Lyapunov functions”. While each $V$ guarantees some coordinate(s) $x_i$ are going to 0 in some sense, the team will allow us to conclude that several are going to 0 in some sense, simultaneously. Our emphasis is on detecting how many are going to 0.

When $\nu \cdot C = 0$. When several species such as rabbits, deer, and cows are feeding on the same resources such as grasses and weeds, they can coexist if their per capita growth rates are “perfectly balanced” with $S$. That condition is that $\nu \cdot C = 0$ for each null vector $\nu$ (see corollary 4).

We use the term “Lyapunov function” loosely for a real-valued function $V(X)$ for which $V(X(t))$ is monotonically decreasing in some specified regions. Usually, Lyapunov functions are used to demonstrate the stability of a steady-state. Instead, we employ a novel type of Lyapunov function. We have two main results.

Our “die-out” result. Assume the matrix $S$ has rank $d - k$ where $k > 0$. The kernel of $S^T$, i.e., $S$ transpose, has dimension $k$. Then for almost every $C$, there is no constant $Z$ for which there is steady-state solution $X$ of Eq. (1.1). Our main result is that if the kernel of $S^T$ is $k$-dimensional, and if there is a bounded solution for $t \geq 0$, then at least $k$ species must die out simultaneously and must do so exponentially fast (Theorem 1). Our proof reveals that there is a set of die-out Lyapunov functions that together guarantee $k$ species will die out. Compare that with the much weaker statement (1.5) where $k = 0$, some species must die out but we do not know how fast they die out.

For the 4-dimensional Lotka-Volterra model in Fig. 2, we will show that $k = 2$ and that there are three die-out Lyapunov functions (see Eqs. (3.2)-(3.4)).

Suppose Fig. 1 represents Lotka-Volterra system for which all species are bounded for $t \geq 0$. Then at least two species must die out exponentially fast. The species dying out are from species numbers 2, 3, and 4.

Fig. 4 (Example 5), is a more complicated example. It is sometimes necessary to look at more than $k$ Lyapunov functions. There, we examine 28 Lyapunov functions to show three species must die out.

Section 3 provides several examples. Next we address the existence of a bounded solution.

Our “trapping region” result. The above results all assume there is a bounded solution in $P$. In Section 4, we use a different kind of Lyapunov function to prove Theorem 2 which gives conditions that guarantee all solutions are bounded.

In contrast with die-out Lyapunov functions, our second application of Lyapunov functions is to establish that for what we call “trophic” Lotka-Volterra systems, there is a globally attracting trapping region. Lorenz [1963] created such a “trapping Lyapunov function” to show his famous differential equations system has a globally attracting region. It says that if a Lotka-Volterra system satisfies our “trophic” condition, Def. 15, then there is a bounded globally attracting region, a region that solutions cannot leave, and all solutions are bounded.

We create a function $V$ of the form $V(X) = \sum_{j=1}^{d} \varepsilon^j x_j$ for some $\varepsilon > 0$, such that (i) there is a positive constant $\lambda$ for which $\dot{V}(X(t)) < 0$ whenever $V(X(t)) \geq \lambda$, and (ii) $V(X) \to \infty$ as $|X| \to \infty$. Then the set of $X$ for which $V(X) \leq \lambda$ is a bounded trapping region.
The Discussion section, Sec. 5, provides additional insights into the themes of the paper.

\[ S = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & 0 & 0 & 0 \\ s_{31} & 0 & 0 & 0 \\ s_{41} & 0 & 0 & 0 \end{bmatrix} \]

**Figure 1.** Three predators and one prey. See Example 2 below in Section 3 for more details. In our figures, each node represents the population density of a species. The existence of an edge from node \( j \) to node \( i \) means species \( j \) directly influences species \( i \), i.e., \( s_{ij} \neq 0 \). Some species \( i \) here may have “self-influence” meaning \( s_{ii} \neq 0 \), indicated by an edge going from its node back to its node. An edge between two nodes with an arrow at each end means each of the two directly influences the other. The graph implies \( S \) has the form shown.

**Figure 2.** Populations of three predators and one prey are plotted. See Example 2. Parameters are given in table (1). In this case, only one predator and the prey survive and tend to the steady state. Two predators die out. In this case, which survives can sometimes be determined from the null vectors of \( S^T \), as we shall show.

2. **Die-out Theorem**

Throughout the discussion we will refer to the following hypotheses.
(H_k): The dimension of the kernel of $S^T$ is $k > 0$.

(H_X): There is a bounded solution $X(t) \in \mathcal{P}$ of (1.1) with bound $\beta$, (see Def. 3).

(H_X D): There is a doubly bounded solution $X$ with bound $\beta$, (see Def. 3).

(H_v): $\nu$ is null vector of $S^T$ such that $\nu \cdot C < 0$.

If $\eta$ is a null vector for which $\eta \cdot C \neq 0$, then either $\eta$ or $-\eta$ has a dot product with $C$ that is negative and satisfies $(H_v)$.

**Definition 2.** We say $k > 0$ species (or coordinates of $X(t)$) die out exponentially fast if there exist $a^*, b^* \in \mathbb{R}$ where $b^* > 0$ such that for each $t$ there are at least $k$ choices of $i$ for which

$$x_i(t) \leq e^{a^* - b^*t}. \tag{2.1}$$

**Theorem 1 (k species die out).** Assume $(H_k)$. The following holds for almost every $C$: if $(H_X)$ is true, then $k$ species (or coordinates of $X(t)$) die out exponentially fast.

**Definition 3.** We say $X(t)$ is a bounded solution of (1.1) (with bound $\beta$ where $x_i(t) \leq \beta$ for all $i \in \{1, \ldots, d\}$ and $t \geq 0$) if $X(t) \in \mathcal{P}$ is defined for all $t \geq 0$ and (1.1) is satisfied for all $t \geq 0$.

We say a bounded solution $X(t)$ is doubly bounded (with bound $\beta > 1$) if for all $t \geq 0, \frac{1}{\beta} \leq x_i(t) \leq \beta$ for all $i \in \{1, \ldots, d\}$.

**Corollary 4.** If for some $C$ there is a doubly bounded solution $X$, then $\nu \cdot C = 0$ for every null vector $\nu$ of $S^T$.

**Proof.** If $X$ is a bounded solution and $\nu$ is a null vector such that $\nu \cdot C$ is non-zero, then some species must die out so $X$ cannot be doubly bounded. But $X$ is doubly bounded so $\nu \cdot C = 0$ for every null vector $\nu$. \hfill \Box

The proof of Theorem 1 is at the end of this section following a lemma, and several “Facts” and definitions that are needed for the proof.

Given a bounded solution $X(t)$, Theorem 1 says that for each $t$ there are at least $k$ choices of $i$ for which Ineq. (2.1) is satisfied. We can get constraints as to which $x_i$’s are small at some times $t$ from study of the null vectors.

To clarify “almost every $C$”, consider the trivial case in which the matrix $S$ is identically zero. Then the differential equation is very simple: $x_i' = c_i x_i$ where $i = 1, \ldots, d$. There is a bounded solutions only if every $c_i$ in $C$ satisfies $c_i \leq 0$. For almost every such $C$, $c_i < 0$ for all $i$. For almost every such $C$, no coordinate of $C$ is zero. Hence for almost every $C$, either there is no bounded solution (when some $c_i$ is positive) or all solutions are bounded and all $x_i(t)$ go to zero as $t \to \infty$ (when $c_i < 0$ for all $i$). Hence we may summarize the trivial system’s behavior by saying “for almost every $C$, if there is a bounded solution, the solution goes to 0”.

**Die-out Lyapunov functions.** Consider the differential equations (1.1) on $\mathcal{P}$. The system of equations (1.1) can be rewritten on $\mathcal{P}$ as

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_d' \end{bmatrix}(t) = C + SZ(t), \tag{2.2}$$

where $Z(t)$ can have the form $Z(X(t), t)$. 
Definition 5. Let \( \Lambda : P \to \mathbb{R} \). In this paper, we will say \( \Lambda \) is a die-out Lyapunov function if \( \dot{\Lambda}(X) < 0 \) on \( \mathcal{P} \).

If there is a die-out Lyapunov function and \( X(t) \) is bounded, then \( \min x_i(t) \to 0 \) as \( t \to \infty \); each limit point of such a trajectory has at least one coordinate = 0, and such a limit point is not in \( \mathcal{P} \).

Definition 6. A vector \( \nu \) is a null vector of the transpose matrix \( S^T \) if \( S^T \nu^T = 0 \), or equivalently \( \nu S = 0 \). We define the \( \nu \)-Lyapunov function on \( \mathcal{P} \),

\[
\Lambda_{\nu}(X) := \nu \cdot \ln X,
\]

(2.3)

i.e., the inner product of \( \nu = (\nu_1, \ldots, \nu_d) \) with \( \ln X := (\ln x_1, \ldots, \ln x_d) \). When \( S \) has a null vector, this function will be applied in Sec. 4 to Eq. (2.2) and to special cases.

What is a “Fact”? The above theorem is a consequence of several key ideas that will guide the reader toward an overview of the proof. The only originality with these facts is in their formulation, not in finding their very short proofs. We call these mini propositions “Facts” so as not to glorify them with full proposition status. And we point out where they are needed.

Fact 7 (\( \dot{\Lambda}_{\nu} \) is constant). Assume \( (H_{\nu}) \). Then for all \( X \in \mathcal{P} \),

\[
\dot{\Lambda}_{\nu}(X) = \nu \cdot C,
\]

(2.4)

This fact is used in the proof of Lemma 9.

Proof. Notice that,

\[
\nabla \Lambda_{\nu}(X) = \nabla \left( \sum_{i=1}^{d} \nu_i \ln x_i \right) = \left( \frac{\nu_1}{x_1}, \ldots, \frac{\nu_d}{x_d} \right),
\]

Since \( \nu \) is a null vector of \( S^T \), \( \nu S = 0 \) and

\[
\nu \cdot (C + SZ) = \nu \cdot C.
\]

(2.5)

Hence, from Eq. (2.2),

\[
\dot{\Lambda}_{\nu}(X) = \sum_{i=1}^{d} \nu_i \frac{x_i'}{x_i} = \nu \cdot C.
\]

\( \square \)

If \( \nu \cdot C \neq 0 \), then either \( \Lambda_{\nu} \) or \( \Lambda_{-\nu} \) is a die-out Lyapunov function. This is a rigorous, stronger version of the competitive exclusion principle (CEP) which asserts that two or more predators cannot coexist if they are only limited by one prey species. The mathematical version says they can not coexist unless they are perfectly balanced in some sense, a situation that would only occur with probability zero. In the theorem below we assume “\( \nu \cdot C \neq 0 \)”, which is precisely the statement that they are not balanced.

Let \( \nu \) be a null vector of \( S^T \). For almost every \( C, \nu \cdot C \neq 0 \). Assume \( \nu \cdot C \neq 0 \). We can assume \( \nu \cdot C < 0 \) since if it was > 0, we could replace \( \nu \) with the null vector \(-\nu\).

2.1. A key Lemma. The following Lemma will be useful in determining which coordinates must die out, and it is crucial for proving Theorem 1.

We need a preliminary fact to determine the exponential rate of decay for a specified \( \nu \). Define

\[
\nu^+ := \sum_{j: \nu_j > 0} \nu_j.
\]

(2.6)
Fact 8 (\(\nu^+\) Positivity). Assume \((H_X)\) and \((H_\nu)\). Then \(\nu^+ \neq 0\) (and it is positive).

This fact is needed to obtain the formula (Eq. (2.8)) for \(a_\nu\) and \(b_\nu\) in the statement of Lemma 9.

Proof. This fact is equivalent to "\(\{j : \nu_j > 0\}\) is non empty" in Eq. (2.6). We need to show \(\nu_i > 0\) for some \(i\). The hypotheses imply that \(\Lambda(X) \to \infty\) as \(t \to \infty\) so there is an \(i\) and a sequence \(t_n \to \infty\) such that \(\nu_i \ln(x_i(t_n)) \to -\infty\) as \(n \to \infty\). Since \(\ln(x_i) \leq \ln(\beta)\), the only way for \(\nu_i \ln(x_i(t_n)) \to -\infty\) to occur is for \(\ln(x_i(t_n)) \to -\infty\) which implies \(\nu_i > 0\). \(\square\)

The null vector \(\frac{\nu}{\nu^+}\) will play an important role. Fact 8 guarantees we are not dividing by 0 in Eqs. (2.8) since \(\nu^+ > 0\).

Lemma 9 ("die-out" Lyapunov Functions). Assume \((H_X)\) and \((H_\nu)\). Then

(i) \(\Lambda_\nu\) is a die-out Lyapunov function for system (1.1) on \(P\).

(ii) There exist \(a_\nu, b_\nu \in \mathbb{R}, b > 0\), such that

\[
\min_{i : \nu_i > 0} x_i(t) \leq e^{a_\nu - b_\nu t} \text{ for all } t \geq 0.
\]

where

\[
\nu^+ := -\ln \beta \sum_{i : \nu_i < 0} \frac{\nu_i}{\nu^+} + \sum_i \nu_i \ln x_i(0), \text{ and } b_\nu := |\frac{\nu}{\nu^+} \cdot C|.
\]

Furthermore \(b_\nu\) and \(a_\nu\) depend only on \(\frac{\nu}{\nu^+}\) and on \(C\) or \(X(0)\) and \(\beta\), respectively.

Proof. From \((H_\nu)\) and Fact 7 and Eq. (2.4),

\[
\tilde{\Lambda}_\nu = \nu \cdot C < 0.
\]

Hence, by definition, \(\Lambda_\nu\) is a die-out Lyapunov function.

Let \(X(t)\) be a solution that, as in the statement of the proposition, is bounded with bound \(\beta\), i.e., there exists \(\beta > 0\) such that \(x_i(t) \leq \beta\) for each \(1 \leq i \leq d\). From Eq. (2.9),

\[
\Lambda_\nu(X(t)) - \Lambda_\nu(X(0)) = (\nu \cdot C)t
\]

For each \(t \geq 0\), let

\[
m_\nu(t) := \min_{i : \nu_i > 0} x_i(t).
\]

Then \(m_\nu \leq x_i \leq \beta\) implies \(\ln m_\nu \leq \ln x_i \leq \ln \beta\). Summing over \(i\) for which \(\nu_i > 0\) and using (2.6),

\[
\nu^+ \ln m_\nu \leq \sum_{i : \nu_i > 0} \nu_i \ln x_i,
\]

For \(\nu_i < 0\), \(\nu_i \ln x_i \geq \nu_i \ln \beta\), so summing over \(i\) for which \(\nu_i < 0\) yields

\[
\ln \beta \sum_{i : \nu_i < 0} \nu_i \leq \sum_{i : \nu_i < 0} \nu_i \ln x_i;
\]

therefore, adding "\(\nu_i > 0\)" and "\(\nu_i < 0\)" terms,

\[
\nu^+ \ln m_\nu + \ln \beta \sum_{i : \nu_i < 0} \nu_i \leq \sum_i \nu_i \ln x_i = \Lambda(X(t)) = \Lambda(X(0)) - |\nu \cdot C| t.
\]
Hence solving for \( \ln m_\nu \) gives
\[
\ln m_\nu(t) \leq a_\nu - b_\nu t,
\]
where
\[
a_\nu := -\ln \beta \sum_{i: \nu_i < 0} \nu_i + \sum_i \nu_i \ln x_i(0) \quad \nu^+,
b_\nu := |\nu \cdot C| \quad \nu^+,
\]
which is equivalent to Eqs. (2.8). Hence Eqs. (2.7) is satisfied.

Therefore we can write \( a_\nu \) and \( b_\nu \) as functions of a normalized \( \nu \) such as \( \frac{\nu}{\nu^+} \). Of course \( a_\nu \) and \( b_\nu \) are also depend upon \( X(0), \beta, \) and \( C \).

### 2.2. A team of Lyapunov functions.

A “team” is a group working together toward a goal. We adapt the word “team” for a group of Lyapunov functions working together.

**The team \( \mathcal{T} \) of null vectors for \( S \).** For simplicity, we will always write null vectors \( \nu \) of \( S^T \) as row vectors despite the fact that they are column vectors, since we often think of \( \nu \) as a left null vector of \( S \), so \( \nu S = 0 \). Then \( \nu \) is a row vector.

**Definition 10.** For a null vector \( \nu \neq 0 \), define the support of \( \nu \),
\[
\text{supp}(\nu) := \{ i : \nu_i \neq 0 \}.
\]
We say a null vector \( \nu \neq 0 \) is a minimal-support vector if there is no null vector with strictly smaller support, i.e., there is no null vector \( \eta \) where \( \text{supp}(\eta) \) is a proper subset of \( \text{supp}(\nu) \).

Let \( \mathcal{T} \) denote the set of minimal-support null vectors of \( S^T \). We call \( \mathcal{T} \) the team (of null vectors) of \( S^T \).

**Fact 11 (Almost every \( C \in \mathbb{R}^d \)).** The set \( \mathcal{T} \cup \{ 0 \} \) consists of a finite number of lines that pass through the origin, one line for each distinct support set \( \text{supp}(\nu) \) for \( \nu \in \mathcal{T} \).

Hence, for almost every \( C \in \mathbb{R}^d \), \( \nu \cdot C \neq 0 \) for all \( \nu \in \mathcal{T} \).

This Fact is used at the beginning of the Proof of Theorem 1.

The simplest matrix is where \( S \) is 0, so every vector is a null vector. We will discuss this case in Example 3. For every vector \( C \) there is a null vector \( \nu \) for which \( \nu \cdot C = 0 \). However, \( \mathcal{T} \) is much smaller than the set of all null vectors when the null space has dimension greater than 1. For the matrix \( S = 0 \), the minimal support null vectors have one coordinate that is non 0. Hence \( \mathcal{T} \cup \{ 0 \} \) consists of \( d \) coordinate axes, and \( \{ C \in \mathbb{R}^d : \nu \cdot C \neq 0 \text{ for all } \nu \in \mathcal{T} \} \) is the set of vectors \( C \) for which no coordinate is 0. Fact 11 is a generalization of this case.

**Proof.** First we show if two minimal-support vectors \( \nu \) and \( \eta \) are linearly independent, then
\[
\text{supp}(\nu) \neq \text{supp}(\eta);
\]
If \( \nu \) is a minimal-support vector, so is \( \alpha \nu \) for each non-zero number \( \alpha \). Hence \( \mathcal{T} \cup \{ 0 \} \) is a finite set of lines through the origin.

If two linearly independent null vectors have the same support \( \sigma \) (see Def. 10), then there are null vectors with strictly smaller supports. Taking linear combinations of the linearly independent null vectors, for each coordinate \( j \in \sigma \) we can find one or
more null vectors with strictly smaller support, one whose support does not include \( j \).

If \( \nu \cdot C = 0 \), then \( C \) is in the subspace that is perpendicular to one of the lines in \( T \). Since there are finitely many lines, the set of such \( C \) is measure 0. \( \square \)

Of course 0 is a null vector but its support is \( \{1, 2, \ldots, d\} \). For each non-zero null vector \( \nu \), the Lyapunov functions \( A_\nu \) tells us some \( x_i \) must die out where \( i \in \text{supp}(\nu) \), (assuming \( \nu \cdot C \neq 0 \)). Therefore the set or “team” of null vectors having the smallest or “minimal” support is most useful, and sometimes the team has many members.

It follows that for each non-zero null vector \( \nu \), there are minimal-support null vectors, the union of whose supports is \( \text{supp}(\nu) \). Furthermore, the original null vector \( \nu \) is a linear combination of the minimal-support null vectors.

**Definition 12.** Let \( k > 0 \) be the dimension of the kernel of \( S^T \), i.e., \((H_k) \) is satisfied. We say a coordinate \( j \) is a “kernel coordinate” if there is some null vector \( \nu \) whose \( j^{th} \) coordinate value \( \nu_j \) is non-zero. Let \( K \) be the set of “kernel coordinates” \( i \), coordinates for which \( \nu_i > 0 \) for some non-zero null vector \( \nu \) of \( S^T \).

Lemma 9 says there are \( a_\nu \) and \( b_\nu \) that tell how fast \( \min_{i: x_i \geq 0} x_i(t) \) dies out. Now we show there are alternative \( a \) and \( b \) that work for all \( \nu \in T \), provided \( \nu \cdot C < 0 \). Because \( a_\nu \) and \( b_\nu \) are only defined when \( \nu \cdot C < 0 \), define:

\[
\begin{align*}
  \mathcal{T}_C & := \{ \nu \in T : \nu \cdot C < 0 \}, \\
  a(\mathcal{T}) & := \sup \{ a_\nu : \nu \in \mathcal{T}_C \}, \text{ and} \\
  b(\mathcal{T}) & := \inf \{ b_\nu : \nu \in \mathcal{T}_C \};
\end{align*}
\]

see Ineq. (2.7) in Lemma 9 for \( a_\nu \) and \( b_\nu \).

**Fact 13** (There exist \( a,b \) that provide a lower bound die out rate for all \( \nu \in T \)).

Assume \((H_k) \) and \((H_X) \). Assume \( \nu \cdot C \neq 0 \) for all \( \nu \in T \). Then \( a(\mathcal{T}) < \infty \) and \( b(\mathcal{T}) > 0 \), and for \( \nu \in \mathcal{T}_C \),

\[
\min_{i: x_i \geq 0} x_i(t) \leq e^{a(\mathcal{T})-b(\mathcal{T})t} \text{ for all } t \geq 0.
\]

Notice that in addition to \( \mathcal{T} \), \( a(\mathcal{T}) \) also depends on \( \beta \) and \( X(0) \), and \( b(\mathcal{T}) \) depends upon \( C \).

Fact 13 is used in the Proof of Theorem 1 to provide an exponential decay rate for \( \nu \in T \). The proof of Fact 13 shows that while there are infinitely many \( \nu \in T \), there are only finitely many values of \( a_\nu \) and \( b_\nu \) for \( \nu \in T \) (see Eqs. (2.12) and (2.13)).

**Proof.** Assume \( \nu \) and \( \eta \) are minimal-support vectors in \( \mathcal{T}_C \). Assume either \( a_\nu \neq a_\eta \) or \( b_\nu \neq b_\eta \). Then \( \frac{\nu}{\eta} \neq \frac{a_\nu}{a_\eta} \) from Eqs. (2.8). Hence \( \nu \) and \( \eta \) are linearly independent. Then \( \text{supp}(\nu) \neq \text{supp}(\eta) \), since if they had the same support \( J \), there would be a non-zero linear combination \( \psi \) of \( \nu \) and \( \eta \) for which \( \psi_i = 0 \) for some \( i \in J \). Hence neither \( \nu \) and \( \eta \) would be minimal-support vectors. Since there are only finitely many subsets of \( 1, \ldots, d \), there are only finitely many distinct values of \( a_\nu \) and \( b_\nu \) in Eqs. (2.12) and (2.13). Hence \( a(\mathcal{T}) < \infty \) and \( b(\mathcal{T}) > 0 \) since each \( b_\nu \) is > 0 for \( \nu \in \mathcal{T}_C \).

Therefore for each \( \nu \in \mathcal{T}_C \),

\[
e^{a_{\nu}-b_\nu t} \leq e^{a(\mathcal{T})-b(\mathcal{T})t} \text{ for all } t \geq 0.
\]
Therefore, Ineq. (2.14) follows from Ineq. (2.7).

We say species \( i \) is **dying out at time** \( t \) if \( x_i(t) \leq e^{a(T)-b(T)t} \). Let \( J \) be a set of \( j \) coordinate numbers such as \( \{1, 3, 5\} \) when \( j = 3 \). Suppose we know that the \( j \) species (or coordinates) in \( J \) are “dying out at time \( t' \)”, and that some species in \( \text{supp}(\nu) \) is “dying out at time \( t' \). When \( J \) and \( \text{supp}(\nu) \) are assumed to be disjoint, we know there are at least \( j+1 \) species that are dying out at time \( t \). This knowledge is what Fact 14 gives us.

**Fact 14.** Assume \((H_k)\). Let \( j < k \). For each set \( J \subset \{1, \ldots, d\} \) of \( j \) coordinates, there is a non-zero null vector \( \nu \in T \) such that \( J \cap \text{supp}(\nu) \) is empty.

This Fact is used in the Proof of Thm. 1 to show \((P_j)\) implies \((P_{j+1})\).

**Proof.** Given \( k \) linearly independent null vectors and any set \( J \) of \( j < k \) coordinates, there is a non-zero linear combination \( \nu \) of those null vectors for which \( \nu_i = 0 \) for each \( i \in J \). \( \square \)

The proof of Theorem 1 uses Lemma 9, Facts 11, 14, and 13.

**Proof of Theorem 1.** Assume \( k > 0 \).

By Fact 11, almost every \( C \) satisfies \( \nu \cdot C \neq 0 \) for all (non-zero) \( \nu \in T \). Choose such a \( C \).

By Fact 13, there exists \( a^* := a(T) \) and \( b^* := b(T) \) (defined in Eqs. (2.12) and (2.13)) such that from Ineq. (2.14), for each \( \nu \in T \),

\[
\min_{i : \nu_i > 0} x_i(t) \leq e^{a^* - b^* t} \leq e^{a - b^* t} \text{ for all } t \geq 0.
\]

For \( j \geq 1 \) we define the following.

**Property \( P_j \):** For each given \( t \), there are \( j \) distinct coordinate numbers \( S_j := \{i_1, \ldots, i_j \} \) such that for each \( i \in S_j \), \( x_i(t) \leq e^{a - b^* t} \).

We will prove \( P_k \) holds by induction. Notice that \( P_1 \) is satisfied by By Lemma 9. We claim that if \( 1 \leq j < k \), then \( P_j \) implies \( P_{j+1} \). To see this, notice that by Fact 14, there is a null vector \( \nu^{(j+1)} \in T \) such that \( \text{supp}(\nu^{(j+1)}) \) contains none of the \( j \) coordinates in \( S_j \). Writing \( \nu = \nu^{(j+1)} \) and applying Lemma 9 to \( \nu \), choose \( i_{j+1} \in \text{supp}(\nu) \) such that \( \nu_{i_{j+1}} > 0 \) and from Ineq. (2.7),

\[
x_{i_{j+1}} = \min_{i : \nu_i > 0} x_i(t) \leq e^{a - b^* t}.
\]

Let \( S_{j+1} := \{i_1, \ldots, i_{j+1} \} \). Hence \( P_{j+1} \) is satisfied, proving the claim.

By an induction that stops at \( P_k \), the result is proved. \( \square \)

### 3. Examples

We use a family of Lyapunov functions of the form Eq. (1.6) to establish that \( k \) species die out exponentially fast (Def. 2).

**Example 2 (A three-predator one-prey model where two species must die out).** Fig. 1 shows a graph of an ecosystem that motivated our investigations of Lyapunov functions. Fig. 2 (see Introduction) displays the behavior of the Lotka-Volterra model. Fig. 2 (A) illustrate that in unequal resource distribution, the less efficient predators can die out. Fig. 2 (B) shows the oscillation of the population.
We can model this system in two ways, either as a 4-dimensional Lotka-Volterra system Eq. (1.2) or as the three-dimensional non-autonomous “Z” system (1.1) where it is not specified how the Z = (z1) is determined. In the Lotka-Volterra system we choose x1 to be the prey density and x2, x3, x4 the three predators. Die-out occurs however Z is defined provided there is a bounded solution. In order to make the coordinates in the two approaches compatible, we write the Z system as follows so that the subscripts of the two approaches are the same, so that the matrix below S_Z is a 3 × 1 submatrix of the 4 × 4 Lotka-Volterra matrix S_{LV}.

\[
(3.1) \quad \begin{pmatrix} \dot{x}^2 & \dot{x}^3 & \dot{x}^4 \\ x^2 & x^3 & x^4 \end{pmatrix}(t) = C + SZ(t); \quad \text{where } S_Z = \begin{bmatrix} s_{21} \\ s_{31} \\ s_{41} \end{bmatrix}.
\]

The number of predators that can survive depends on the nonzero rows of null vectors of the matrix and on C. The kernel coordinates are 2, 3, and 4. There could be additional populations, x1, x5, ..., but as long as there is a bounded solution X(t), the die-out properties of x2, x3, x4 depend only on Eq. (3.1), and are independent of the form of the equations for the additional variables.

Let k be the dimension of the kernel of S. For typical coefficient the matrix S_Z has a kernel of dimension k = 2 with the following null vectors, any two of which are a basis for the kernel.

\[
\nu^{(34)} = \pm \begin{bmatrix} 0 \\ s_{41} \\ -s_{31} \end{bmatrix}, \\
\nu^{(24)} = \pm \begin{bmatrix} -s_{41} \\ 0 \\ s_{21} \end{bmatrix}, \\
\nu^{(23)} = \pm \begin{bmatrix} -s_{31} \\ s_{21} \\ 0 \end{bmatrix}.
\]

In our notation, when a null vector has non-zero coordinates such as j1, j2, and j3, we write it as \(\nu^{(j_1j_2j_3)}\). We use such notation only for null vectors with minimal support.

For almost every C, choose the sign of each \(\nu\) above so that \(\nu \cdot C < 0\) as required by Theorem 1.

Depending on C, the team of die-out Lyapunov functions is:

\[
(3.2) \quad \Lambda_{\nu^{(34)}}(X(t)) = \pm (s_{41} \ln x_3(t) - s_{31} \ln x_4(t)), \\
(3.3) \quad \Lambda_{\nu^{(24)}}(X(t)) = \pm (-s_{41} \ln x_2(t) + s_{21} \ln x_4(t)), \\
(3.4) \quad \Lambda_{\nu^{(23)}}(X(t)) = \pm (-s_{31} \ln x_2(t) + s_{21} \ln x_3(t)).
\]

Let \(X(t)\) be a bounded trajectory.

Let \(\nu^{(34)} = [0, s_{41}, -s_{31}]\); (we have chosen a plus sign here), and assume \(s_{41}, s_{31} > 0\) to simplify calculation. Then \(\Lambda_{\nu^{(34)}} = s_{41} \ln x_3(t) - s_{31} \ln x_4(t)\); and \(\Lambda_{\nu^{(34)}}(X) = s_{41} \ln x_3 - s_{31} \ln x_4\). Suppose \(C \cdot \nu^{(34)} \neq 0\).

If \(C \cdot \nu^{(34)} > 0\), then \(\Lambda_{\nu^{(34)}}(X) \to +\infty\). Since the coordinates of X are bounded, \(s_{41} \ln x_3(t)\) is bounded. Hence, \(-s_{31} \ln x_4(t) \to \infty\), which means \(\ln x_4(t) \to -\infty\), which means \(x_4(t) \to 0\), as \(t \to \infty\).

If however \(C \cdot \nu^{(34)} < 0\), then by similar reasoning, \(x_3(t) \to 0\) as \(t \to \infty\).

More generally, depending on the signs of \(C \cdot \nu\) and \(s_{ij}\),

- \(\Lambda_{\nu^{(23)}}\) tells us \(x_2\) or \(x_3\) must die out, and
- \(\Lambda_{\nu^{(24)}}\) tells us \(x_2\) or \(x_4\) must die out, and
- \(\Lambda_{\nu^{(23)}}\) tells us that for any bounded solution, \(x_3\) or \(x_4\) must die out.
Together the three Λ’s tell us that at least two of the three populations must die out.

In this example knowing the signs of the coefficients \( s_{ij} \) and the signs of the \( C \cdot \nu \)'s, then we can determine which two populations must die out.

### 3.1. Who must die?

Sometimes a team of Lyapunov functions determines who must die (as in the above example) and sometimes it does not. We show a case below where whether it does or does not depend on the constant \( C \). The simplest case of Eq. (1.1) is where (i) the null space is one dimensional, so there is essentially one Lyapunov function, and (ii) where \( Z(t) \) is constant. In the next example, the Lyapunov function is Eq. (1.7).

**Example 3 (The kernel is one dimensional and \( Z \) is constant).** Here we illustrate how the choice of \( C \) affects who dies out in a simple case where \( Z(t) \) is constant consider the following three-dimensional model

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3
\end{bmatrix} =
\begin{bmatrix}
-1 + z_1 + 2z_2 \\
 c_2 + z_1 + z_2 \\
-1 + 3z_1 + z_2
\end{bmatrix} =
\begin{bmatrix}
-1 \\
 c_2 \\
-1
\end{bmatrix} +
\begin{bmatrix}
1 & 2 & \mid z_1 \mid \\
1 & 1 & \mid z_2 \mid \\
3 & 1 & \mid z_3 \mid
\end{bmatrix}
\]

in which \( Z = (z_1, z_2) \) is constant and \( C = (-1, c_2, -1) \) has one free parameter, \( c_2 \), which can be thought of as the per capita death rate of \( x_2 \).

For an ecological interpretation, we might call \( x_1, x_2, \) and \( x_3 \) predators and \( z_1 \) and \( z_2 \) are prey or resources.

We explore how the behavior depends on \( c_2 \). Assume there is a bounded solution \( X(t) \) in \( P \) (with all coordinates strictly positive). Define three half planes,

\[
\begin{align*}
\mathcal{H}_1 & := \{(z_1, z_2) : -1 + z_1 + 2z_2 \leq 0\} \text{ where } \dot{x}_1 \leq 0, \\
\mathcal{H}_2 & := \{(z_1, z_2) : c_2 + z_1 + z_2 \leq 0\} \text{ where } \dot{x}_2 \leq 0, \\
\mathcal{H}_3 & := \{(z_1, z_2) : -1 + 3z_1 + z_2 \leq 0\} \text{ where } \dot{x}_3 \leq 0.
\end{align*}
\]

If \( Z \) is chosen in the interior of half-plane \( H_j \), then \( \frac{\dot{x}_j}{x_j} \) is negative and is constant since \( Z \) is constant, so \( x_j(t) \to 0 \) exponentially fast.

For simplicity we assume \( z_1, z_2 \in P \), and we plot \( G \cap P \) be the shaded set in Fig. 3 where the three planes and \( P \) intersect. For there to be a bounded trajectory, \( Z \) must lie in \( G \) and quite possibly on the boundary of \( G \). The boundary line of each of the above 3 half planes is shown, and the shaded region shows the closed set \( G \) in \( P \) where all three inequalities are satisfied. The dashed line is the boundary of half plane \( (H_3) \).

Since in this example we assume \( Z \) is constant, if \( Z = (z_1, z_2) \) is outside \( G \), all species must die out. When \( Z \in P \) is a vertex of \( G \), two species can persist. Write

\[
S = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}.
\]

The kernel of \( S^T \) is one dimensional, and it contains the null vector \((2, -5, 1)^T \). Its coordinates are the coefficients of Eq. (1.7), and \( C = (-1, c_2, -1) \).

The boundaries of \( H_1 \) and \( H_3 \) intersect at \((\frac{1}{2}, \frac{1}{2})\), which is in \( H_2 \) if \( c_2 \leq -\frac{2}{3} \). Notice \( \nu \cdot C = 0 \) is satisfied when the boundaries of the three half-planes intersect at one point (see Fig. 3).
Figure 3. Where all species die out, Example 3. The shaded (green) area is the intersection of $\mathcal{P}$ and the three half planes $\mathcal{H}_1, \mathcal{H}_2, \text{and } \mathcal{H}_3$ where the inequalities (3.5) are all satisfied. For simplicity we assume $z_1, z_2 \in \mathcal{P}$ and we only examine the part of $G$ that is in $\mathcal{P}$. The arrow for each $i$ points to the boundary of $\mathcal{H}_i$, where $\dot{x}_i = 0$. In the interior of $G$, the inequalities are all negative and all species die out. On the boundary of $G$, one species can survive. At each vertices of $G \in \mathcal{P}$, two species can survive. Panel (A): In the green region in panel (A), $c_2 < -\frac{3}{5}$ and the inequality in the definition of $(\mathcal{H}_2)$ is always negative so $x_2 \to 0$ as $t \to \infty$, and there is one vertex (red dot) where $Z$ is on the boundary of two half-planes. There, $x_1$ and $x_3$ can both persist for that value of $(z_1, z_2)$. Panel (B): There are two vertices on the boundary of $G$. At one (upper blue dot), $x_1$ and $x_2$ can persist while at the other (lower blue dot) $x_2$ and $x_3$ can persist. Panel (C): Only species $x_2$ survives.

Two cases, either $(2, -5, 1) \cdot C = -3 - 5c_2 < 0 \text{ or } > 0$. In Fig. 3, panel (A) is for the case where $(2, -5, 1) \cdot C < 0$ and panel (B) is for $(2, -5, 1) \cdot C > 0$. 


For panel (A), there is one vertex on the boundary of $G$ in $P$, and that corresponds to the vertex where both $x_1$ and $x_3$ survive while $x_2$ dies out.

For panel (B) there are two vertices, the only two possible values of $Z$ for which two species persist. Both are on the boundary of $H_2$. In either case $x_2$ persists while either $x_1$ or $x_3$ dies out.

In panel (C), both $x_1$ and $x_3$ die out.

**Example 4 (Five predators and two prey).** We could consider the 7-dimensional LV model for five predators and two prey with the matrix. Assume $S_{LV} = (s_{ij})$ is the corresponding matrix.

$$S_{LV} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & s_{17} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} & s_{27} \\ s_{31} & s_{32} & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_{71} & s_{72} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  

This matrix has kernel of dimension 3. For almost every choice of $s_{ij}$, every null vector $(\nu_i)$ of $S_{LV}^T$ can only have $\nu_i \neq 0$ if $i \in \{3, 4, 5, 6, 7\}$. To show which species must die out (for a bounded trajectory), we need only consider the $Z$ model whose matrix is

$$S_Z = \begin{bmatrix} s_{31} & s_{32} \\ \vdots & \vdots \\ s_{71} & s_{72} \end{bmatrix}.$$  

This is the smallest submatrix of $S_{LV}^T$ where kernel has dimension 3. Because each null vector of $S_{LV}^T$ can have non-zero coordinates $3, \ldots, 7$ and $S_{LV}^T$ maps those coordinates to coordinates 1 and 2. In other words, $S_Z$ captures the null space behavior of $S_{LV}$ and ignores the rest of the $S_{LV}$.

This transpose matrix $S_{LV}^T$ has a 3-dimensional kernel for almost every choice of the non-zero entries. The coordinates of $Z = (z_1, z_2)$ correspond to either prey species or "resources". These values can even be negative. Theorem 1 asserts that at least three $x_i$ must die out exponentially fast for almost every $C$. Since the kernel coordinates are $3, 4, 5, 6, 7$, three of these must die out. These correspond to the predators in the biological interpretation. Four other species, #s 1 and 2, and two kernel coordinates can coexist for some choices of $S$, $C$, and $Z(t)$.

**Null vectors for Eq. (3.6) and the Die-out Lyapunov functions.** For almost every choice of the non-zero coefficients of $S$, the kernel is three dimensional, and for those cases, the $S_{LV}^T$ null vectors have the form $(0, 0, *, *, *, *, *)$ for model $LV$ where each ‘*’ indicates a coefficient that can be non-zero. For the $Z$ model, the first two coordinates are omitted, so the $S_{LV}^T$ null vectors have the reduced form $(*, *, *, *, *)$ in which the coordinate numbers are still $\{3, 4, 5, 6, 7\}$.

Assuming the kernel is typically $k$-dimensional, we can choose any $k−1$ coordinates, and, taking a linear combination of the kernel vectors, we can create a non-zero null vector whose entries for those $k−1$ coordinates are 0, (Fact 14). It is possible that some other entries would also be 0. Since here the kernel is three-dimensional, for any two kernel coordinates $i$ and $j$, there is a non-zero null vector $\nu$ for which $\nu_i = 0$ and $\nu_j = 0$, leaving at most three non-zero coordinates.

Let the $i^{th}$, $j^{th}$ and $u^{th}$ coordinates be non-zero for $i, j, u \in \{3, 4, 5, 6, 7\}$. There are $\binom{5}{3} = 10$ such choices of null vectors that have 3 non-0 coordinates. We display
Figure 4. A system with 14 species where three must die out. There are 8 blue nodes that have connections only with the 5 nodes (colored red). The numbers on the left indicate the trophic levels. If the Lotka-Volterra system is chosen to be “trophic” (see Definition 15) all solutions are bounded by theorem 2. The position of the nodes on the vertical axis represents their trophic level. We assume higher trophic level species can sometimes survive at the expense of the lower ones. For almost every choice of coefficients in the corresponding Lotka-Volterra system, at least three of the blue-node species must die out simultaneously, exponentially fast.

three of these as samples:

\[ L^{(367)} = \pm[p_3, 0, 0, p_6, p_7], \]
\[ L^{(467)} = \pm[0, q_4, 0, q_6, q_7], \]
\[ L^{(567)} = \pm[0, 0, r_5, r_6, r_7], \]

where

\[ p_3 = s_{61}s_{72} - s_{62}s_{71}, \quad q_4 = s_{61}s_{72} - s_{62}s_{71}, \quad r_5 = s_{61}s_{72} - s_{62}s_{71}, \]
\[ p_6 = s_{32}s_{71} - s_{31}s_{72}, \quad q_6 = s_{42}s_{71} - s_{41}s_{72}, \quad r_6 = s_{52}s_{71} - s_{51}s_{72}, \]
\[ p_7 = s_{31}s_{62} - s_{32}s_{61}, \quad q_7 = s_{62}s_{41} - s_{61}s_{42}, \quad r_7 = s_{62}s_{51} - s_{61}s_{52}. \]

The sign ± of \( \nu \) above is chosen so that \( \nu \cdot C < 0 \), a choice which is possible for almost every \( C \), (see Fact 11). Below we omit the ±, leaving it to the reader.
These three yield the following are the die-out Lyapunov functions:

\[ \Lambda_{\nu}(367)(x_3, x_6, x_7) = p_3 \ln x_3 + p_6 \ln x_6 + p_7 \ln x_7, \]
\[ \Lambda_{\nu}(467)(x_4, x_6, x_7) = q_4 \ln x_4 + q_6 \ln x_6 + q_7 \ln x_7, \]
\[ \Lambda_{\nu}(567)(x_5, x_6, x_7) = r_5 \ln x_5 + r_6 \ln x_6 + r_7 \ln x_7. \]

for which we obtain

\[ \hat{\Lambda}_{\nu}(367) = p_3 c_3 + p_6 c_6 + p_7 c_7, \]
\[ \hat{\Lambda}_{\nu}(467) = q_4 c_4 + q_6 c_6 + q_7 c_7, \]
\[ \hat{\Lambda}_{\nu}(567) = r_5 c_5 + r_6 c_6 + r_7 c_7. \]

Each of the ten \( \Lambda_{\nu} \) with only three non zero coordinates tells us that at least one of its three variables must die out. Knowledge of the signs of the coefficients and of \( \Lambda \cdot C \) will give additional information as to which must die out. Together they guarantee that at least three species must die out for almost every \( C \).

We note that we can also write the die-out Lyapunov functions in the form

\[ V(X) =: \exp(\Lambda_{\nu}(367)(X)) =: x_3^{c_3} x_6^{c_6} x_7^{c_7}. \]

though then \( \hat{V}(x) \) is not constant.

**Example 5 (A 14-species ecosystem and its graph, Fig. 4).** This example is closely related to Example 4 in that the \( S \) in both examples have a 3-dimensional kernel, so at least 3 species must die out.

In Fig. 4, let \( X = (x_1, \ldots, x_8) \) represent the densities of the 8 blue nodes \{4, 5, \ldots, 11\}; and \( Z = (z_{j})_{j=1}^{5} \) (so \( d' = 5 \)) corresponding to the red nodes \{1, 2, 3, 12, 13\}. Each of the edges from 8 blue nodes end at one of the 5 red nodes. The 8 blue species depend only on the 5 red species, and it is this aspect that we capture in the \( Z \)-model. Applying the \( Z \) model to model the graph, the matrix \( S \) is 8 and so has a kernel whose dimension is at least 3. we conclude that of the 8 kernel nodes shown in blue, at least 3 must die out. Note that some red nodes are in a trophic level higher than blue nodes while others are below.

For almost every choice of the matrix \( S \) there are \( \binom{8}{6} = 28 \) ways of choosing a minimal-support null vector \( \nu \) of \( S^T \) so that it has only 6 non-zero coordinates. Hence, there are 28 die-out Lyapunov functions, \( \Lambda_{\nu} \). For almost every \( C \), at least three of the corresponding 8 species must die out. This is generally not enough information to indicate which three of the 8 must die out, but it gives a lot of hints, which species die out depends on the specific values of \( C, S, \) and \( Z(t) \).

### 4. The existence of bounded solutions and a trapping region for Lotka-Volterra models

This paper uses two different kinds of Lyapunov functions. Recall that \( P = \{(x_1, \ldots, x_d) : x_i > 0\} \) is open while \( \bar{P} := \{(x_1, \ldots, x_d) : x_i \geq 0\} \) is closed. Theorem 1 uses die-out Lyapunov functions \( \Lambda_{\nu} \) defined on \( P \) to show species die out. We use it to show there is a globally attracting trapping region, Thm. 2 below, provided the Lotka-Volterra model satisfies our “trophic” condition below. We refer to such a \( V \) as a trapping Lyapunov function.
Here we modify Eq. (1.2) slightly so that solutions can have $x_i = 0$.

\begin{equation}
  x'_i = x_i \left( c_i + \sum_{j=1}^{d} s_{ij} x_j \right), \quad \text{where } i = 1, \ldots, d,
\end{equation}

**Definition 15.** We say a system Eq. (4.1) (or the matrix $S$) is trophic if $S$ and $C$ satisfy the following hypothesis.

(H_tro):

1. (T1) For all $n$ if $c_n \geq 0$, then $s_{nn} < 0$.
2. (T2) For each pair $n, m$, if $s_{nm} > 0$, then $n > m$ and furthermore $s_{mn} < 0$.

(T2) says that if species $n$ benefits from species $m$, then $n > m$ and species $m$ must be hurt by species $n$. It also implies that $s_{mm} \leq 0$ for all $m$, so that species can be self-limiting, i.e., $s_{mm} < 0$.

If (T1) is false, then $c_n > 0$ and $s_{nn} \geq 0$ for some $n$. Hence, if $X(t)$ is a solution with $x_m(t) \equiv 0$ for all $m \neq n$, then $x_n(t) \to \infty$ as $t \to \infty$. Condition (T2) implies $s_{nn}$ can not be positive since $s_{nm} > 0$ implies $m \neq n$.

**Definition 16.** We say $\Gamma$ is a **globally attracting region** (for $\bar{P}$) if every trajectory $X(t)$ in $\bar{P}$ is eventually in $\Gamma$ and if $X(t_0) \in \Gamma$, then $X(t) \in \Gamma$ for all $t \geq t_0$, Meiss [2007].

**Theorem 2** (“Trophic” implies existence of a bounded globally attracting region). Assume (H_tro). Then (1.2) has a bounded globally attracting region.

The proof follows from the following proposition and a lemma.
Proposition 17 (A trapping Lyapunov function $V$ for trophic systems). Assume $(H_{tr})$. Consider system $(4.1)$. Let $\mathcal{E} := (\varepsilon, \varepsilon^2, \ldots, \varepsilon^d)$. Let

\begin{equation}
V(X) := \sum_{n=1}^{d} \varepsilon^n x_n = \mathcal{E} \cdot X.
\end{equation}

Then for some $\varepsilon > 0$ and $\lambda > 0$, $\dot{V}(X) < 0$ when $V(X) \geq \lambda$ (See Fig. 5). Define

$$
\Gamma_{\lambda} := \{X \in \overline{\mathcal{P}} : V(X) \leq \lambda\}.
$$

Then $\Gamma_{\lambda}$ is a bounded globally attracting region for $\overline{\mathcal{P}}$.

The proof will show there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$
\dot{V}(X) \leq \alpha - \beta V(X).
$$

For any initial point $x_0 \in \overline{\mathcal{P}}$ at $t = 0$, we can find an upper bound $v(t)$ for $V(X(t))$ for $t \geq 0$ by setting $v(0) = V(X(0))$ and $v' = \alpha - \beta v$. Hence $v(t) \to \frac{\alpha}{\beta}$ as $t \to \infty$, so $\limsup_{t \to \infty} V(X(t)) \leq \frac{\alpha}{\beta}$. That identifies the trapping region $\mathcal{V} \leq \lambda$ where $\lambda > \frac{\alpha}{\beta}$.

Lemma 18. Assume $c \geq 0$ and $s < 0$. Then there exists an $a$, namely, $a = -\frac{(c+1)^2}{4s} > 0$, such that for all $x$, $cx + sx^2 \leq a - x$.

Proof. Assume $c \geq 0$ and $s < 0$. We need to show

$$
ax + sx^2 + x + \frac{(c+1)^2}{4s} \leq 0,
$$

which is true since the left-hand side is

$$
\frac{1}{s} \left( sx + \frac{c+1}{2} \right)^2 \leq 0.
$$

Proof of Prop. 17. If $s_{nm} > 0$, then by $(T2)$, $n > m$. Hence we can choose $\varepsilon \in (0,1)$ so that for all $n$ for which $s_{nm} > 0$,

$$
\varepsilon^{(n-m)} s_{nm} + s_{mm} \leq 0.
$$

For $x_n'$ in Eq. $(4.1)$ and $V$ in $(4.2)$ $\dot{V} = \sum_{n=1}^{d} \varepsilon^n x_n'$. Let $\mathcal{M} := \{n : c_n \geq 0\}$. Then for $X \in \mathcal{P}$, we can write $\dot{V}(X) = W_1(X) + W_2(X) + W_3(X)$ where

$$
W_1(X) = \sum_{n \in \mathcal{M}} \varepsilon^n (c_n x_n + s_{nn} x_n^2),
$$

$$
W_2(X) = \sum_{n \in \mathcal{M}} \varepsilon^n (c_n x_n + s_{nm} x_m^2) \leq \sum_{n \in \mathcal{M}} \varepsilon^n c_n x_n \leq 0
$$

from $(T1)$ since $c_n < 0$ for $n \notin \mathcal{M}$.

$$
W_3(X) = \sum_{n=1}^{d} \sum_{m=1}^{d} \varepsilon^n s_{nm} x_n x_m \leq \sum_{m=1}^{d} \sum_{n > m} \varepsilon^n s_{nm} + s_{mm} x_n x_m \leq 0
$$

from $(T2)$ since $n > m$ implies $s_{nm} \leq 0$, and if $s_{nm} > 0$, then $s_{mn} < 0$, so for $\varepsilon$ sufficiently small, $\varepsilon^{n-m} s_{nm} + s_{mm} < 0$. 

Hence $V(X) \leq W_1(X)$. In $W_1(X)$, since $n \in \mathcal{M}$, $c_n \geq 0$, so $s_{nn} < 0$. By Lemma 18, there exists $a_n > 0$ such that

$$e_n x_n + s_{nn} x_n^2 \leq a_n - x_n,$$

Then

$$W_1(X) \leq \sum_{n \in \mathcal{M}} e^n (a_n - x_n).$$

Let

$$e_n = \begin{cases} 
  e^n & \text{if } n \in \mathcal{M}, \\
  -e^n c_n & \text{if } n \notin \mathcal{M}.
\end{cases}$$

Notice $e_n > 0$ for all $n$ when $\varepsilon > 0$ since $c_n < 0$ for $n \notin \mathcal{M}$. Let $A := \sum_{n \in \mathcal{M}} e^n a_n$. If $\mathcal{M}$ is the empty set, let $A := 0$.

$$\dot{V}(X) \leq \sum_{n \in \mathcal{M}} e^n (a_n - x_n) + \sum_{n \notin \mathcal{M}} e^n c_n x_n \leq \sum_{n \in \mathcal{M}} e^n (a_n - x_n) - \sum_{n \notin \mathcal{M}} e_n x_n \leq \sum_{n \in \mathcal{M}} e^n a_n - \sum_{n \in \mathcal{M}} e^n x_n - \sum_{n \notin \mathcal{M}} e_n x_n \leq A - \sum_{n=1}^d e_n x_n.$$

For some sufficiently small $B > 0$, $\sum_{n=1}^d e_n x_n \geq BV(X)$, so $\dot{V}(X) \leq A - BV(X)$. Choose $\lambda_0$ so that $A - B\lambda_0 = 0$. Let $\lambda > \lambda_0$. Then $\dot{V}(x) < 0$ when $V(x) \geq \lambda$, so

$$\Gamma_\lambda := \{ X \in \mathcal{P} : \mathcal{E} \cdot X \leq \lambda \}$$

is a bounded global trapping region for $\mathcal{P}$. $\square$

If $\mathcal{M}$ is the empty set, i.e., $c_n < 0$ for all $n$, then $A = 0$, and $\dot{V}(X) < 0$ for all $X \neq 0$ in $\mathcal{P}$. Hence, for every solution $X$, $X(t) \to 0$ as $t \to \infty$.

**Proof of Theorem 2.** By Prop. 17, there exists a $\mathcal{P}$ globally attracting trapping region $\Gamma_\lambda$, for some $\lambda$. No trajectory can leave the region $\Gamma_\lambda$. Hence, each trajectory is bounded. By LaSalle-Barbashin-Krasovskii method, if $\dot{V}(X(t)) < 0$ for all $t$, then $\dot{V} = 0$ at each limit point of $X$ so $X(t) \to \Gamma$ since all limit points of $X(t)$ are in $\Gamma$, and if there exists $t_0 \geq 0$ for which $\dot{V}(X(t_0)) \geq 0$, then $X(t_0) \in \Gamma_\lambda$, and $X(t)$ remains there for $t \geq t_0$. $\square$

**Example 6.** In this example, (T1) is not satisfied and there is no attracting trapping region. The original basic two-dimensional Lotka-Volterra system has prey and predator species with population densities $x_1$ and $x_2$,

$$\begin{align*}
  \dot{x}_1 &= c_1 x_1 + s_{12} x_1 x_2, \\
  \dot{x}_2 &= c_2 x_2 + s_{21} x_1 x_2,
\end{align*}$$

(4.3)

where $c_1 > 0 > c_2$ and $s_{21} > 0 > s_{12}$. This is not a trophic system because $s_{11} = 0$ so (T1) is not satisfied. There is no attracting trapping region because all solutions with $x_1, x_2 > 0$ are periodic except for the steady state $x_1 = -\frac{c_2}{s_{21}}, x_2 = -\frac{c_1}{s_{12}}$. Both numbers are positive.
The standard Lyapunov function for the system (4.3) is
\[ V(x_1, x_2) := s_{21} x_1 + c_2 \ln(x_1) - s_{21} x_2 - c_1 \ln(x_2), \]
where \( x_1, x_2 > 0 \). By direct calculation \( \dot{V}(x_1, x_2) \equiv 0 \). For each constant \( v > 0 \), the set where \( V(x_1, x_2) = v \) is a periodic orbit.

Since \( (T2) \) is satisfied, we conclude \( (T1) \) is essential for Thm. 2.

5. Discussion

In this paper, we introduce a trapping Lyapunov function \( V \) for showing that systems we call trophic must have a bounded globally attracting region (Theorem 2). What is the more novel is the introduction of families of die-out Lyapunov functions, \( \Lambda_v \) (Thm. 1). We believe this technique of finding multiple die-out Lyapunov functions will have applications far beyond ecology.

When trajectories have no limit points in the open set \( \mathcal{P} \). When a bounded trajectory \( X(t) \) has some coordinates dying out as in Thm. 1, each of its limit points \( X^* \) will have some coordinate(s) \( x_i^* = 0 \). Hence, \( X^* \) is in the closed set \( \bar{\mathcal{P}} \) but not in open set \( \mathcal{P} \). As in Prop. 1, assume \( F \) is \( C^1 \) in the differential equation (1.3), where \( F \) is defined on an open set \( U \subset \mathbb{R}^d \). Let \( X(t) = (x_1, \ldots, x_d)(t) \in U \) be a trajectory and let \( V : U \to \mathbb{R} \) be differentiable.

Write \( v(t) = V(X(t)) \). Standard Lyapunov function theorems including Prop. 1 and the Barbashin-Krasovskii-LaSalle Theorem (See [Alligood et al., 1996, p. 309]) conclude that \( v(t) \) is monotonically decreasing. They make assumptions on \( V \) that determine the behavior of \( v(t) \). That is not the only approach.

Taking the \( \dot{V} \) definition Eq. (1.4) a step further, for a differential equation (1.3), define the second Lyapunov derivative
\[ \ddot{V}(X) = \nabla \dot{V}(X) \cdot F(X) \]
so that \( \frac{d^2}{dt^2} V(X(t)) = \ddot{V}(X(t)) \).

Jacobi introduced this concept in 1840 for his “stability criterion” for the N-body problem (See [Wilson and Yorke, 1973, p.118]).

Other higher Lyapunov derivatives are defined analogously. Some papers such as Butz [1969] and Ahmadi and Parrilo [2011] consider the higher order derivatives \( \ddot{V}(X), \dot{V}(X), \) and \( V(X) \) to establish globally asymptotic stability. Also, there are publications that discuss multiple Lyapunov functions but in a different context of the stability of a fixed point (see Branicky [1998], Lakshmikantham et al. [2013]).

The theorem in Yorke [1970] uses the second derivative of \( V(X) \), eliminating assumptions such as “\( V \geq 0 \)” and “\( V \) is unbounded” and “\( \dot{V} \leq 0 \)” while still obtaining the same conclusion about trajectories as in Prop. 1.

The following striking result has conclusions modeled on Prop. 1. We include it to demonstrate how even apparently weak conditions can result in trajectories having no limit points in the interior of the domain of a differential equation.

Proposition 19 (in the spirit of Yorke [1970]). Assume there exists a \( C^1 \) differential equation (1.3) where \( F \) is defined on a simply connected open set \( U \subset \mathbb{R}^d \) and \( F \) is \( C^1 \). Assume

(i) there is a trajectory \( X(t) = (x_1, \ldots, x_d)(t) \in U \) for all \( t \geq 0 \), and
(ii) \( V : U \to \mathbb{R} \) is \( C^2 \), and for each \( X \in U \) either \( \dot{V}(X) \neq 0 \) or \( \ddot{V}(X) \neq 0 \).

Then
(iii) $X(t)$ has no limit points in $U$ as $t \to \infty$.
(iv) If furthermore $U = \mathcal{P}$ and the trajectory is bounded, then $\min_{1 \leq i \leq d} x_i(t) \to 0$ as $t \to \infty$.

This result generalizes Prop. 1 since its requirement that $V < 0$ is a special case of condition (ii) here, and in that we do not assume $U$ is invariant.

This proposition seems to be virtually devoid of assumptions about $V$ that are useful in characterizing $v(t) = V(X(t))$ since $V(X(t))$ is allowed to have even an infinite number of local maxima and minima, as in the following toy example. There is even no assumption that $U$ is invariant.

For a trivial example consider the one-dimensional equation $x' = f(x)$, assume $f : \mathbb{R} \to \mathbb{R}$ is never 0. Assume $X(t)$ is defined for all $t$. Then for $V(x) := \sin(x)$, conditions (i) and (ii) are satisfied since $\dot{V}(x) = \cos(x)f(x)$ and $\dot{V}(x) = -\sin(x)f(x)f'(x)f(x)$. When $\dot{V}(x) = 0$ we have $\dot{V}(x) = -\sin(x)(f(x))^2 \neq 0$. Hence (ii) is satisfied. By (iii), we conclude $X(t)$ has no limit points i.e., $X(t) \to \pm \infty$ as $t \to \infty$.

The proof is based on describing each connected component $Q$ of $\{X \in U : \dot{V}(X) = 0\}$. It is shown that each such component $Q$ separates $\mathcal{P}$ into two pieces, and each trajectory can pass through each $Q$ at most once. Our conditions here are more general in some ways than those in Yorke [1970], but the proof there is easily adapted to our case. For example, here the domain $U$ of the differential equation is only required to be simply connected; it is not assumed to be invariant.

Possible extensions. The $\nu$-Lyapunov functions we use for Eq. (2.2) have the form $\Lambda_{\nu}(x) := \sum_{i} \nu_i \ln x_i$. It employs $\ln x_i$ for each $i$ because the left side of the equation involves $x_i'$ which is the derivative of $\ln x_i$. So the reader may wish to extend the ideas and theorems here to the case where some or all of the $x_i'$ are replaced by the simpler $x_i'$. The corresponding terms in $\nu$-Lyapunov functions would be changed to $\nu_i x_i$ since the derivative of $x_i$ is $x_i'$. The domain of such a variable would be $\mathbb{R}$ instead of $(0, \infty)$. We leave it to the student or researcher to see what kinds of theorems can be created for such a hybrid system.

Many systems of ordinary differential equations will have some coordinates dying out asymptotically. Most writers then write the equations for the remaining coordinates. Here, in some spacial cases we have shown how to conclude that some coordinates die out. We hope that our effort here will encourage others to pursue systems where coordinates die out.

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7. Appendix

Table 1. List of variables in four-dimensional Lotka Volterra model, Fig. (2), \( i, j = 1, \ldots, 4 \).

| Symbol | Variable Name          | Initial Value |
|--------|------------------------|---------------|
| \( x_1 \) | Species (1)           | 60            |
| \( x_2, x_3, x_4 \) | Species (2,3,4)     | 40            |
| \( x_i \) | Species               |               |
| \( c_j \) | Net birth or death rate of species |             |
| \( s_{ij} \) | Trophic coefficient  |               |

| Symbol | Parameter Name          | Typical Value |
|--------|------------------------|---------------|
| \( c_1 \) | Birth rate of prey     | 1.05          |
| \( c_2 \) | Death rate of the first predators | 0.29 |
| \( c_3 \) | Death rate of the first predators | 0.3 |
| \( c_4 \) | Death rate of the first predators | 0.31 |
| \( s_{11} \) | Rate of self-limiting of prey | 0.002 |
| \( s_{12} \) | First predator consumption | 0.008 |
| \( s_{13} \) | Second predator consumption | 0.0075 |
| \( s_{14} \) | Third predator consumption | 0.006 |
| \( s_{21} \) | Rate of change of first predator due to the presence of prey | 0.0023 |
| \( s_{31} \) | Rate of change of second predator due to the presence of prey | 0.0035 |
| \( s_{41} \) | Rate of change of third predator due to the presence of prey | 0.003 |

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