Twisted Dirac Operators over Quantum Spheres

Andrzej Sitarz ∗†
Institute of Physics, Jagiellonian University,
Reymonta 4, 30-059 Kraków, Poland

Abstract
We construct new families of spectral triples over quantum spheres, with a particular attention focused on the standard Podleś quantum sphere and twisted Dirac operators.

1 Introduction
The quantum spaces and, in particular, quantum spheres, are challenging toy models in noncommutative geometry. The standard quantum Podleś sphere [10], which is a quantum homogeneous space, was the first object in the realm of the $q$-deformed manifolds, on which a spectral geometry in the sense of Connes [2] (see also [3] for a review) was constructed [4]. More examples and local index calculations followed [11 5 7].

One of the ad-hoc assumptions of the [4] construction was the existence of the $q \to 1$ limit. This, together with the $\mathcal{U}_q(su(2))$ equivariance enforced the geometric construction of the Hilbert space. However, one may wonder whether in the noncommutative situation we should really be imposing such restrictions, which refer directly to the classical (commutative) case. Moreover, we know from examples that in some cases the axioms of spectral geometry

∗The author acknowledges the Alexander von Humboldt Fellowship at the Mathematical Institute, Heinrich-Heine-Universität Universitätsstrasse 1, 40225 Düsseldorf, Germany
†Partially supported by MNII Grant 115/E-343/SPB/6.PR UE/DIE 50/2005–2008
might be satisfied only with certain accuracy - up to “infinitesimals” within
the algebra of bounded operators.
On the other hand, a closer look at the classical situation of the two-dimensional
sphere [9] shows that apart from the standard Dirac operators, there exists a
family of twisted Dirac operators, with the Hilbert space of spinors twisted
by tensoring it with a line bundle of a nontrivial Chern character.
In this paper we shall explore all these possibilities, focusing our attention
first on the standard Podleś sphere.
Our notation throughout the paper is as follows: 0 < q < 1 is a deformation
parameter, [x] denotes a q-number:

\[ [x] := \frac{q^x - q^{-x}}{q - q^{-1}}. \]

The definitions of the polynomial algebra \( \mathcal{A}(S^2_q) \) and the universal enveloping
algebra \( \mathcal{U}_q(su(2)) \) use the standard presentations, for more details we refer
the reader to [4, 7].

2 The standard Podleś sphere
and its equivariant spectral geometries.

We recall here the definition of the algebra \( \mathcal{A}(S^2_q) \) of the standard Podleś
quantum sphere [10] and its \( \mathcal{U}_q(su(2)) \) symmetry.

**Definition 2.1.** The polynomial algebra of the standard Podleś quantum
sphere, \( \mathcal{A}(S^2_q) \), is a star algebra generated by \( B, B^* \) and \( A = A^* \), with the
relations:

\[
AB = q^2 BA, \quad AB^* = q^{-2} B^* A, \quad BB^* = q^{-2} A(1 - A), \quad B^* B = A(1 - q^2 A). \tag{2.1}
\]

The quantized algebra \( \mathcal{U}_q(su(2)) \) has \( e, f, k, k^{-1} \) as generators of the *-Hopf
algebra, satisfying relations:

\[
e k = q ke, \quad k f = q f k, \quad k^2 - k^{-2} = (q - q^{-1})(fe - ef), \tag{2.2}
\]

with the standard coproduct, counit, antipode and star:

\[
\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f, \quad \\
\epsilon(k) = 1, \quad \epsilon(e) = 0, \quad \epsilon(f) = 0, \quad Sk = k^{-1}, \quad Sf = -q f, \quad Se = -q^{-1} e, \quad k^* = k, \quad e^* = f, \quad f^* = e. \tag{2.3}
\]
From the usual Hopf algebra pairing between $\mathcal{U}_q(su(2))$ and $\mathcal{A}(SU_q(2))$, we obtain an action of $\mathcal{U}_q(su(2))$ on $\mathcal{A}(SU_q(2))$, which when restricted to $\mathcal{A}(S^2_q)$ is given on its generators by:

\begin{align*}
e \triangleright B &= -q^{-\frac{1}{2}}[2]A + q^{-\frac{1}{2}}, \quad e \triangleright B^* = 0, \quad e \triangleright A = q^{-\frac{1}{2}}B^*, \\
k \triangleright B &= qB, \quad k \triangleright B^* = q^{-1}B^*, \quad k \triangleright A = A, \\
f \triangleright B^* &= q^{-\frac{1}{2}}[2]A - q^{-\frac{1}{2}}, \quad f \triangleright B = 0, \quad f \triangleright A = -q^\frac{1}{2}B.
\end{align*}

This action preserves the $*$-structure:

\begin{equation}
h \triangleright (x^*) = ((Sh)^* \triangleright x)^*, \quad \forall h \in \mathcal{U}_q(su(2)), x \in \mathcal{A}(S^2_q). \tag{2.5}
\end{equation}

### 2.1 Equivariant representations of $\mathcal{A}(S^2_q)$.

In the next step we extend the results of [4] and find explicit formulas for a family of equivariant representations of the algebra $\mathcal{A}(S^2_q)$ on a Hilbert space $\mathcal{H}$. The representations derived here are in fact a restriction of the family of representation for all Podleś spheres [1] to the case of the standard Podleś sphere. For completeness, however, we recall here the main steps of the construction.

Let us recall, the definition of an equivariant representation:

**Definition 2.2.** Let $\mathcal{V}$ be an $\mathcal{A}$-module and $\mathcal{H}$ be a Hopf algebra. We say that the representation $\pi$ of $\mathcal{A}$ on $\mathcal{V}$ is $\mathcal{H}$-equivariant if there exists a representation $\rho$ of $\mathcal{H}$ on $\mathcal{V}$ such that:

\begin{equation}
\rho(h)(\pi(a)v) = \pi(h_{(1)} \triangleright a)\rho(h_{(2)})v \quad \forall h \in \mathcal{H}, \ a \in \mathcal{A}, \ v \in \mathcal{V}. \tag{2.6}
\end{equation}

where we have used the Sweedler’s notation for the coproduct of $\mathcal{H}$ and $\triangleright$ for the action of $\mathcal{H}$ on $\mathcal{A}$.

In the construction we use the infinite dimensional linear space, which after completion shall be Hilbert space of the spectral triple construction. However, since the representation of the Hopf algebra of $\mathcal{U}_q(su(2))$ is unbounded, the equivariance relation makes sense only on the dense subspace of $\mathcal{H}$.

To construct suitable modules $\mathcal{V}$ we use the (known) representation theory of $\mathcal{U}_q(su(2))$. The idea and details of derivations are similar as in the case of [4] and [11, 1], therefore we here we present only the result:
Proposition 2.3. For each even $N \in \mathbb{Z}/2$ (non negative integer or half-integer) there exists an irreducible $U_q(su(2))$-equivariant representation of the standard Podleś quantum sphere on the space $V_N$:

$$V_N = \bigoplus_{j=|N|,|N|+1,\ldots} V_j,$$

where $V_j$ is 2$j+1$-dimensional space with the fundamental representation of $U_q(su(2))$ of rank $j$.

The representation $\pi_N$ of $A(S^2_q)$ is given on the basis vectors $|l,m\rangle \in V_l$, $l = |N|,|N|+1,\ldots, m = -l,-l+1,\ldots, l-1,l$, by:

$$\pi_N(B)|l,m\rangle = q^m \sqrt{|l+m+1||l+m+2|} r^+(l)|l+1,m+1\rangle$$
$$+ q^m \sqrt{|l+m+1||l-m|} r^0(l)|l,m+1\rangle$$
$$+ q^m \sqrt{|l-m||l-m-1|} r^-(l)|l-1,m+1\rangle,$$

$$\pi_N(B^*)|l,m\rangle = q^{m-1} \sqrt{|l-m+2||l-m+1|} r^-(l)|l+1,m-1\rangle$$
$$+ q^{m-1} \sqrt{|l+m||l-m+1|} r^0(l)|l,m-1\rangle$$
$$+ q^{m-1} \sqrt{|l+m||l+m-1|} r^+(l)|l-1,m-1\rangle,$$

$$\pi_N(A)|l,m\rangle = -q^{m+l+\frac{1}{2}} \sqrt{|l-m+1||l+m+1|} r^+(l)|l+1,m\rangle$$
$$+ \frac{q^{-\frac{1}{2}}}{1+q^2} (||l-m+1||l+m|$$
$$- q^2|l-m||l+m+1|) r^0(l) + q^{\frac{1}{2}} |l,m\rangle$$
$$+ q^{m-l-\frac{1}{2}} \sqrt{|l-m||l+m|} r^-(l)|l-1,m\rangle,$$

where $r^+(l), r^-(l), r^0(l)$ are:

$$r^0(l) = q^{-\frac{l}{2}} \frac{(q - \frac{1}{q})(l + |N| + 1)(l - |N|) \pm q^{\pm 1}|2N|}{[2l][2l+2]},$$

$$r^+(l) = \frac{q^{l-\frac{3}{2}}}{[2l+2]} \frac{\sqrt{l+N+1}}{\sqrt{2l+1}[2l+3]}$$
$$r^-(l) = -q^l r^+(l-1).$$

Proof. The proof is entirely technical, the derivation of the formulae could be divided into two steps. First, the use of equivariance gives the dependence
on the $m$ parameter (2.8). Then, using the defining relations of the algebra (2.1) leads to two recurrence relations for $r^+(l), r^0(l)$:

$$
r_+^2(l)[2l+1][2l+3]q^{2l+2}(1+q^2)^2 + r_0(l)^2[2l]^2 q^2$$

$$+ r_0(l)[2l]^2 q^2(q^2-1) - q = 0,$$

and

$$
r_+^2(l)\frac{(1+q^2)^2}{1-q^2}[2l+3]q^{4l+4} - r_+(l-1)(1+q^2)^2\frac{1}{1-q^2}q^{2l+2}[2l-1]$$

$$+ r_+(l)[2l+1][2l+3]q^{4l+4}(1+q^2)^2 + r_0(l)^2[2l]^2 q^{2l+4} + r_0(l)[2l]^2 q^2(q^2-1)q^{2l+2} - q^{2l+3} = 0,$$

By solving them and imposing the boundary conditions (that is $l \geq |N|$ we obtain the solutions (2.9)).

\[ \square \]

2.2 Twisted Dirac operators.

Let $\mathcal{H}_{N,1}$ be the completion of the space $\mathcal{V}_N \oplus \mathcal{V}_{N+1}$, for any $N \in \mathbb{Z}/2$. We take the diagonal representation $\pi_N \oplus \pi_{N+1}$ and the natural grading operator taken as 1 on the first and $-1$ on the second component. We have:

**Proposition 2.4.** The following densely defined operator

$$D_N|l, m, \pm\rangle = \sqrt{l - N}|l - N + 1|l, m, \mp\rangle, \quad (2.10)$$

anticommutes with $\gamma$, has bounded commutators with the elements of the algebra $\mathcal{A}(S^2_q)$ and satisfies exactly the order-one condition. The eigenvalues of $D$ are:

$$\lambda_D = \pm \sqrt{l - N}[l + N + 1], \quad N > 0, l = N, N + 1, \ldots$$

and

$$\lambda_D = \pm \sqrt{l - N}[l + N + 1], \quad N < 0, l = |N| - 1, |N|, \ldots$$

with multiplicities $2l+1$. Note that the kernel of $D$ has dimension $2N+1$ if $N \geq 0$ and $2|N| - 1$ for $N < 0$. Thus, the standard Dirac operator $N = -\frac{1}{2}$ has the eigenvalues $[l + \frac{1}{2}]$ and an empty kernel.

The anticommutation with $\gamma$ is an obvious consequence of the definition. We shall postpone the proof of bounded commutators with the elements of the algebra $\mathcal{A}(S^2_q)$ until the next section, here we shall concentrate on the order-one condition.
**Definition 2.5.** We say that the spectral triple \((\mathcal{A}, \mathcal{H}, D)\) extends to a real spectral triple if there exists a spectral triple \((\mathcal{A}, \mathcal{H}', D')\) and an antiunitary isometry \(J : \mathcal{H} \oplus \mathcal{H}' \rightarrow \mathcal{H} \oplus \mathcal{H}'\) such that: \((\mathcal{A}, \mathcal{H} \oplus \mathcal{H}', D \oplus D', J)\) is a real spectral triple.

As a consequence we can postulate:

**Definition 2.6.** The Dirac operator \(D\) satisfies the order one condition if there exists a real extension of the spectral triple with a Dirac operator satisfying the order one condition.

Coming back to our situation of the family of spectral triples over the standard Podleś quantum sphere, we have:

**Lemma 2.7.** Let \((\mathcal{A}(S^2_q), \mathcal{H}_{N,1}, D_N)\) be the spectral triple as defined above. Then, \((\mathcal{A}(S^2_q), \mathcal{H}_{-N-1,1}, D_{-N-1})\) extends it to a real spectral triple and the operator \(D_N\) satisfies the order-one condition.

**Proof.** We define first the reality operator \(J:\)

\[
J|l, m, \pm\rangle_N = i^{2m}|l, -m, \mp\rangle_{-N-1}, \quad |l, m, \mp\rangle_K \in \mathcal{H}_{K,1}. \tag{2.11}
\]

\(J\) is well-defined, since the subspace of \(\mathcal{H}_N\) with eigenvalue of \(\gamma + 1\) is \(\mathcal{V}_N\), whereas the subspace of \(\mathcal{H}_{-N-1}\) with \(\gamma\) eigenvalue \(-1\) is \(\mathcal{H}_{-N}\). It is easy to check that

\[J\gamma = -\gamma J,
\]

and

\[J^2 = \pm 1.
\]

Note that only for a half-integer \(N\) we have the signs of a two-dimensional spectral geometry, whereas for an integer value of \(N\), we have the sign relations corresponding formally to a six-dimensional (modulo 8) real structure. Further, we check that for any two generators \(x, y\) of the algebra \(\mathcal{A}(S^2_q)\):

\[
[J^{-1}\pi(x)J, \pi(y)] = 0.
\]

This could be verified either with an explicit calculations or using the arguments, which identify the space \(\mathcal{H}_N\) with a certain subspace of the Hilbert space of the GNS construction for \(SU_q(2)\) and \(J\) with a conjugation map.

Finally, explicit calculations show that the Dirac operator \(D_N\) satisfies the order-one condition in the sense of extension to a real spectral triple. We skip the lengthy presentation of the calculations.\[1\]

\[1\]The symbolic calculations are available from the author
2.3 More families of spectral triples over $\mathcal{A}(S_q^2)$

In the previous section we have found a good candidate for the equivariant twisted Dirac operator over the standard Podleś quantum sphere. Still, we need to prove that it has bounded commutators with the elements of the algebra.

In this part, in addition to the above construction, we shall briefly sketch the construction of more families of equivariant spectral triples over $\mathcal{A}(S_q^2)$, which satisfy the geometric conditions up to the ideal of compact operators. The boundedness of commutators with the elements of the algebra shall be much easier to show using the approximate representations, so the special case of twisted Dirac operators shall follow as a corollary.

Similarly as in the construction for $SU_q(2)$ we first define the ideal $\mathcal{K}_q$ as an ideal of operators of exponential decay. $\mathcal{K}_q$ could be viewed as an ideal generated by a diagonal operator on $H_N$ with spectrum $q^l$.

Our main tool (as in [7]) is the approximate representation of the algebra $\mathcal{A}(S_q^2)$:

**Proposition 2.8.** The maps $\tilde{\pi}_N$:

$$\tilde{\pi}_N(B)|l,m\rangle_N = \sqrt{1 - q^{2(l+m+2)}} \sqrt{1 - q^{2(l+m+1)}} q^{l-N} |l+1, m+1\rangle_N$$

$$+ q^{l+m} \sqrt{1 - q^{2(l+m+1)}} |l, m+1\rangle_N$$

$$- q^{2(l+m)} q^{l-N} |l-1, m+1\rangle_N + o(q^{2l}),$$

$$\tilde{\pi}_N(B^*)|l,m\rangle_N = - q^{2(l+m)+1} q^{l-N} |l+1, m-1\rangle_N$$

$$+ q^{l+m-1} \sqrt{1 - q^{2(l+m)}} |l, m-1\rangle_N$$

$$+ q^{l-N-1} \sqrt{1 - q^{2(l+m)}} \sqrt{1 - q^{2(l+m-1)}} |l-1, m-1\rangle_N + o(q^{2l}),$$

$$\tilde{\pi}_N(A)|l,m\rangle_N = - q^{l+m} q^{l-N+1} \sqrt{1 - q^{2(l+m+1)}} |l+1, m\rangle_N$$

$$+ q^{2(l+m)} |l,m\rangle_N$$

$$+ q^{l+m} q^{l-N} \sqrt{1 - q^{2(l+m-1)}} |l-1, m\rangle_N + o(q^{2l}),$$

give an approximate representations of $\mathcal{A}(S_q^2)$ on the modules $\mathcal{V}_N$, that is for any $x \in \mathcal{A}(S_q^2)$ the difference $\pi_N(x) - \tilde{\pi}_N(x)$ is in $\mathcal{K}_q$.

It is worth noting that, actually we do have a bit more than an approximate representation, as the above formulas give the approximate representation up to order $q^{2l}$, so that the difference $\pi(x) - \tilde{\pi}(x)$ is of order at least $q^{2l}$. This
shall be important in the calculations concerning the commutators with an unbounded Dirac operator. We have:

**Proposition 2.9.** Let $H_{N,r}$ for $N \in \mathbb{Z}/2$ and $r \in \mathbb{N}$ be the completion of $V_N \oplus V_{N+r}$. Let $\gamma$ be the natural $\mathbb{Z}_2$ grading taken as 1 on the first component and $-1$ on the second. We denote by $\pi$ the diagonal representation of $A(S_q^2)$ on $H_{N,r}$ and by $\tilde{\pi}$ its approximate representation.

The operator:

$$D_K |l, m \rangle = \begin{cases} 0 & \text{if } l < |K + r| \\ d_K q^{-l} |l, m \rangle_{K'}, & \text{if } l \geq |K + r|. \end{cases} \tag{2.12}$$

where $K = N$, $K' = N + r$ or $K = N + r$, $K' = N$, and $d_K$ are complex coefficients has bounded commutators with the algebra $A(S_q^2)$, anticommutes with $\gamma$ and satisfies the order one condition up to the ideal $\mathcal{K}_q$ (in the sense of the real extension of a spectral triple). $D$ is selfadjoint if and only if $(d_N)^* = d_{N+r}$.

**Proof.** First, let us check that the commutators with the elements of the algebra are bounded. We calculate, for example:

$$\begin{align*} &(D_N \tilde{\pi}_N(B) - \tilde{\pi}_{N+r}(B)D_N) |l, m, + \rangle_{N,r} \\
&= (d_N q^{-l-1} q^{l-N} - d_N q^{-l} q^{l-N-r}) \sqrt{1 - q^{2(l+m+2)}} \sqrt{1 - q^{2(l+m+1)}} |l+1, m+1 \rangle_{N+r} \\
&\quad + (d_N q^{-l} - d_N q^{-l-1}) q^{l+m} \sqrt{1 - q^{2(l+m+1)}} |l, m+1 \rangle_{N+r} \\
&\quad - (d_N q^{-l-1} q^{l-N} - d_N q^{-l} q^{l-N-r}) q^{2(l+m)} |l-1, m+1 \rangle_{N+r} + o(q^{2l}), \end{align*}$$

and it is easy to see that the expression remains bounded.

Next, for the real extension of the spectral triple, we notice that the real extension of $H_{N,r}$ is $H_{-N-r,r}$, with $J$ being the antilinear isometry between $V_N$ and $V_{-N}$, and between $V_{N+r}$ and $V_{-N-r}$. Thus, the same arguments as in the case $r = 1$ from the previous section apply.

Finally, for the order one condition, we can use the following argument. Each of the generators $A, B, B^*$ is of the type $T_0 + T_q$, where $T_0$ is bounded, $[T_0, D] = 0$ and $T_q \in \mathcal{K}_q$ is an operator such that both $DT_q$ and $T_q D$ are bounded. Then:

$$\begin{align*} [J(T_0^x + T_q^x)J^{-1}, [D, (T_0^y + T_q^y)]] &= J(T_0^x + T_q^x)J^{-1}, [D, T_q^y] \in \mathcal{K}_q. \end{align*}$$
where in the last estimation we have first used that \([D, T_q^y] \) must be at most bounded and \(JT_q^x J^{-1} \in \mathcal{K}_q \). To estimate the commutator of \(JT_0^xJ \) with \([D, T_q^y] \), we observe first that since \(D \) commutes with \(T_0^x \), it shall be sufficient to prove that \([JT_0^x J^{-1}, T_q^y], D \) is in \(\mathcal{K}_q \).

First, take, for instance the elements, for which the compactness of the commutator is least evident:

\[
T_0^A l, m = q^{2(l+m)}|l, m>, \\
T_q^{B+} l, m = q^{l-N} \sqrt{1 - q^{2l+m+2}} \sqrt{1 - q^{2(l+m+1)}} \sqrt{l+1, m+1}.
\]

We calculate:

\[
|l, m> = q^{2(l-m)}q^{l+m} \sqrt{1 - q^{2l+m+2}} \sqrt{1 - q^{2(l+m+1)}} \sqrt{l+1, m+1} \\
- q^{l+m} \sqrt{1 - q^{2l+m+2}} \sqrt{1 - q^{2(l+m+1)}q^{2(l-m)}} \sqrt{l+1, m+1} \\
= 0 + o(q^{2l}).
\]

Hence, also the order one condition is satisfied but only up to compact operators.

For the other commutators it is worth noting that \(JT_0 J^{-1} \) has always a factor \(q^{l-m} \). When multiplied by \(q^{l+m} \), (which is present in all \(T_q^y \) apart from the above case of \(T_q^y = T_q^{B+} \), we obtain that their product (and hence the commutator) is of order \(q^{2l} \) as most. Therefore multiplying it by \(D \) (or taking a commutator with \(D \)) still gives a result in \(\mathcal{K}_q \).

We have shown in this section that the Dirac operator with the eigenvalues growth \(q^{-l} \) satisfies the modified conditions of (real) spectral geometry. Clearly, a compact perturbation of such \(D \) shall satisfy it as well.

Therefore, for \(r = 1 \), which is the case of previously studied twisted Dirac operators we have:

**Corollary 2.10.** The twisted Dirac operator \((2.10)\) is a compact perturbation of the Dirac operator \((2.12)\) and therefore has bounded commutators with the elements of the algebra \(\mathcal{A}(S^2_q)\).

This follows directly from the estimate, true for big \(l\):

\[
\sqrt{l + N} \sqrt{N - 1} \sim q^{-l} (1 + q^l + \cdots).
\]

Note that in the \(q \to 1 \) limit only the twisted case \(r = 1 \) gives a family of Dirac operators and spectral triples over the two-dimensional sphere, whereas the exotic spectral triples \(r > 1 \) give quasi-Dirac operators studied in [13].
2.4 The Fredholm modules and index pairing

Although the entire construction of the spectral geometry is very similar to the one already presented in [4], we shall see that the obtained spectral geometries fall into a different $K$-homology class. We shall look at the Fredholm module arising from $(\mathcal{A}(S^2_q), \pi, \mathcal{H}_{N,r}, D, \gamma)$ and prove by explicit calculations that the index pairing with an element from $K_0$ group of the standard Podleś sphere depends on $N$ and $r$.

We shall divide the proof into two parts, first, when $\ker D = \emptyset$ and then for the situation when there are harmonic "spinors". To begin with we consider the Fredholm modules for "quasi-Dirac" operators, which corresponds to the case $r = -2N$, where the kernel of $D$ is empty. Using the sign of the Dirac operator $F = D|D|^{-1}$ we have:

Lemma 2.11. The commutators $[F, \pi(a)]$ are trace class for every $a \in \mathcal{A}(S^2_q)$.

The pairing between the cyclic cocycle associated via Chern map to the Fredholm module $(\mathcal{A}(S^2_q), \mathcal{H}_N \oplus \mathcal{H}_{-N}, F, \gamma)$ and the nontrivial projector $e$ of the $K_0(\mathcal{A}(S^2_q))$:

$$e = \frac{1}{2} \begin{pmatrix} 1 - A & qB \\ qB^* & q^2 A \end{pmatrix},$$

is $-2N$.

Proof. By looking at the approximate representation $\tilde{\pi}$ we have already noticed that the difference of representations for different values of $N$ is always in $K_q$, hence it is trace class.

Therefore, the following expression gives a 0-cyclic cocycle:

$$\phi(x) = \text{Tr} \gamma F[F, \pi(x)]. \quad (2.13)$$

We calculate the pairing $<\phi, e>$ explicitly, calculating the trace over $\mathcal{H}_N$:

$$<\phi, e> = - \text{Tr} (1 - q^2)(\pi_N(A) - \pi_{-N}(A))$$

$$= - \frac{1 - q^2}{1 + q^2} \left( 2N \sum_{l=N}^{\infty} \sum_{m=-l}^l \frac{[l-m+1][l+m]-q^2[l-m][l+m+1]}{[2l][2l+2]} \right)$$

$$= - \frac{2}{q} [2N](1 - q^2) \sum_{l=N}^{\infty} \frac{q^{2l+2} - q^{2l+4}}{1 - q^{4l+4}}$$

$$+ 2q[2N](1 - q^2) \sum_{l=N}^{\infty} \frac{q^{2l}}{1 - q^{4l+4}} = \ldots$$
First, let us call \( \frac{q^{2l}}{1-q^4} = a_l \). Then the first component of the sum is:

\[
-2 \frac{2}{q} [2N](1 - q^2) \sum_{l=N}^{\infty} (a_l - a_{l+1}) = -2 \frac{2}{q} [2N](1 - q^2) \left( N a_N + \sum_{l=N+1}^{\infty} a_l \right).
\]

whereas the second component is identified as:

\[
2 \frac{2}{q} [2N](1 - q^2) \sum_{l=N+1}^{\infty} \frac{q^{2l}}{1-q^4} = 2 \frac{2}{q} [2N](1 - q^2) \sum_{l=N+1}^{\infty} a_l.
\]

Therefore we obtain:

\[
< \phi, e > = -2 \frac{2}{q} [2N](1 - q^2) N \frac{q^{2N}}{1-q^4N}
\]

\[
= -2N.
\]

Hence the pairing depends on \( N \), which means that the Fredholm modules obtained for different choices of \( N \) are from different K-homology classes.

In the less trivial case of the twisted Dirac operators, we first need to define a proper Fredholm module. Using the Dirac operator \( D \) and defining \( F \) as 0 on the kernel of \( D \) and the sign of \( D \) the orthogonal complement of the kernel, we obtain only a pre-Fredholm module, with the relation \( F^2 = 1 \) satisfied up to finite rank operator.

Using the procedure of Higson [8], we introduce the Fredholm module with a doubled Hilbert space \( \mathcal{H}_{N,r} \oplus \mathcal{H}_{N,r} \) and the following representation, grading as well as the Fredholm operator \( F' \):

\[
\pi'(x) = \begin{pmatrix} \pi(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad F' = \begin{pmatrix} F & K \\ K & -F \end{pmatrix},
\]

where \( K \) is the orthogonal projection on the kernel of \( D \) and \( F \) is the sign of \( D \) (taken zero on the kernel of \( D \)).

It is easy to verify that this is indeed a Fredholm module, and it again satisfies that \( [F', \pi'(x)] \) is trace class for any \( x \in \mathcal{A}(S^2_q) \). The formula for the cyclic cocycle associated with this Fredholm module reads now:

\[
\phi_{N,r}(x) = \text{Tr} \gamma (F[F, \pi(x)] + K\{K, \pi(x)\}), \quad (2.14)
\]

where the trace is now reduced back to the original Hilbert space \( \mathcal{H}_{N,r} \).
Lemma 2.12. The pairing between the $K$-homology class defined by the generalized twisted Dirac operator and the projector $e$ depends on $n$ and $r$:

$$
\langle \phi_{N,r}, e \rangle = \begin{cases} 
-2(N + r - (r - 1)(2N + r)), & N > 0 \\
-2(N + r - (r - 1)(2N + r)), & N < 0, N + r \geq |N|, \\
2r(r + 2N + 1), & N < 0, 0 < N + r < |N|, \\
2(r + 1)(2N + r), & N < 0, N + r \leq 0, 
\end{cases}
$$

where we always assume $r > 0$.

Proof. The direct and explicit proof, which we have presented in the special case of invertible $D$ is too complicated from the technical point of view. We shall use, however, the result that the pairing should be independent of $q$.

Thus, exploring the pairing in the $q = 0$ limit and assuming that the series, which define the value of the paring converge uniformly in $q$ and the limit exists, we shall be able to make the explicit calculations.

First, we need to recover the $q = 0$ limit of the diagonal matrix element of the appropriate representation of $A$. We have:

$$
\lim_{q \to 0} \langle l, m | \pi_N(A) | l, m \rangle = \begin{cases} 
0 & l > N, l > m, \\
1 & l > N, l = m, \\
1 & l = N.
\end{cases}
$$

To calculate the paring $\langle \phi, e \rangle$ explicitly, we need to consider the relative signs of $N + r$ and $N$. Since we can always assume $r > 0$, we might have $N + r$ and $N$ of the same sign and of different sign. Our choice of $\gamma$ is that it is $+1$ on $H_{N+r}$ and $-1$ on $H_N$. Take, for example $N > 0$, then:

$$
\langle \phi_{N,r}, e \rangle = - \text{Tr} \left( (\pi_{N+r}(A) - \pi_N(A))(1 - K) + 2K\pi_N(1 - A) \right) \\
= - \left( 2(N + r) + (-4N + 4Nr + 2r^2 - 2r) \right) \\
= - \left( 2(N + r - (r - 1)(2N + r)) \right).
$$

Here the trace is taken over $H_N$, then we use the identification $(1 - K)H_N \sim H_{N+r}$. In the most interesting case $r = 1$, which corresponds to the twisted Dirac operators, we have the value of the paring $-2N - 2$.

Similarly, one can consider remaining cases, for instance, if $N < 0$ but $N + r > |N|0$, we again have:

$$
\langle \phi_{N,r}, e \rangle = - \text{Tr} \left( (1 - K)(\pi_{N+r}(A) - \pi_N(A)) - 2K\pi_N(1 - A) \right) \\
= - 2 \left( (N + r - (r - 1)(2N + r)) \right).
$$
3 The twisted Dirac operators over other Podleś quantum spheres

For the other Podleś spheres one cannot expect (as it was shown first in [6] then in [1]) the exactness of commutator relations of the commutant and the order one condition. Therefore, from the beginning we can work with the approximate representation, that is with the representation up to compact operators from the ideal \( \mathcal{K}_q \). The exact families of equivariant representations were already derived and presented in [1], here we present only the approximate ones. The notation is as in the previous part of the paper, however, to distinguish the case from the standard Podleś quantum sphere we denote its generators by \( b, b^*, a \). We have:

\[
\tilde{\pi}_N(a)|l, m\rangle = sq^{l+m}\sqrt{1 - q^{2(l+m+1)}} |l+1, m\rangle \\
+ sq^{l+m-1}\sqrt{1 - q^{2(l+m)}} |l-1, m\rangle \\
+ q^{2(l+m)} |l, m\rangle,
\]

\[
\tilde{\pi}_N(b)|l, m\rangle = s\sqrt{1 - q^{2(l+m+2)}}\sqrt{1 - q^{2(l+m+1)}} |l+1, m+1\rangle \\
- s\delta_{lL}q^{2(l+m)+1} |l-1, m+1\rangle \\
+ q^{l+m+1}\sqrt{1 - q^{2(l+m+1)}} |l, m+1\rangle,
\]

\[
\tilde{\pi}_N(b^*)|l, m\rangle = s\sqrt{1 - q^{2(l+m)}}\sqrt{1 - q^{2(l+m-1)}} |l-1, m-1\rangle \\
+ q^{l+m}\sqrt{1 - q^{2(l+m)}} |l, m-1\rangle \\
- sq^{2(l+m)+1} |l+1, m-1\rangle,
\]

which satisfy the relations:

\[
ab = q^2ba, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Next, we take the definition of the Hilbert space $\mathcal{H}_N$, $N > 0$ as in the previous section and define a representation $\tilde{\pi}_N$ in the following way on the generators $x = a, a^*, b$:

$$\tilde{\pi}_N(x)\langle l, m \rangle = P_N\tilde{\pi}(x)\langle l, m \rangle,$$

where $P_N$ is a projection from the space $V'$ onto $\mathcal{H}_N$. Since this projection has a kernel of finite rank, the maps $\tilde{\pi}(x)$ and $\tilde{\pi}_N(x)$ differ for any $x$ by a finite rank operator, therefore $\tilde{\pi}_N$ is again an approximate representations.

We have then:

**Proposition 3.1.** Let $\mathcal{H}_{N,1}$ be the Hilbert space defined as in the previous section: $\mathcal{H}_{N,1} = \mathcal{H}_N \oplus \mathcal{H}_{N+1}$, with the grading $\gamma$ as defined before. Then the following densely defined operator:

$$D_N\langle l, m \rangle_{N \pm K} = (l - N)\langle l, m \rangle_{N+1 \mp K}, \quad K = 0, 1. \quad (3.1)$$

has bounded commutators with the algebra elements and satisfies the generalized order one condition up to compact operators from the ideal $\mathcal{K}_q$.

The eigenvalues of the Dirac operator are $\pm (l - N)$, $l = N, N + 1, \ldots$ with multiplicities $2l + 1$. Note that, in comparison to the "classical" twisted Dirac operator, $D_N$ is just a compact perturbation, since:

$$\sqrt{l - N} \sqrt{l + N + 1} \sim (l - N) \left(1 + \frac{N + \frac{1}{2}}{l - N} + \cdots\right)$$

for $l - N$ sufficiently big.

The proof of the order-one condition is purely based on the explicit calculations, which we omit. For the boundedness of commutators, first of all, we observe that since the approximate representation does not depend on $N$ (for $l > N$) then the commutators of $D$ with the operator $T_\pm$, where $T_\pm$ changes $l$ by $\pm 1$ are $\pm T_\pm$, whereas the commutators with $T_0$ (operators leaving invariant subspace with fixed $l$) vanish. Since all generators are just finite sums of such bounded operators, hence their commutators with $D$ are bounded as well.

Interestingly enough, the choice $l - N$ is not the only possibility in our case. In fact any linear function of $l$ and $m$ satisfies both the boundedness of commutators as well as the order one condition (up to the compact operators):

$$\tilde{D}_N\langle l, m \rangle_{N \pm K} = \delta l_N (l + \alpha_K m)\langle l, m \rangle_{N+1 \mp K}, \quad K = 0, 1. \quad (3.2)$$
where $\alpha_K = \alpha_{1-K}^*$, $K = 0, 1$ so that $\tilde{D}_N$ is selfadjoint. Note that such operator is clearly not equivariant with respect to the $U_q(su(2))$ symmetry. It could be easily checked that this operator (and its natural extension in the sense of extension of the spectral triple to a real one) satisfies the order one condition up to the compact operators. However, since the eigenvalues are $\pm |l + \alpha_0 m|$ (for $l > N$) only for some values of $\alpha_0$ the operator $\tilde{D}$ has a compact resolvent. A good example of the possible choice of $\alpha_0$ is a pure imaginary number $\alpha_0 = i$. Then the eigenvalues are $\sqrt{l^2 + m^2}$ and we can possibly think of such geometry as corresponding rather to a quantum ellipsoid (roughly speaking) than to a quantum sphere.

4 Conclusions

We have shown in this paper that on the standard Podleś quantum sphere there exist a family of equivariant spectral triples, which are topologically inequivalent and we have extended the construction for other Podleś spheres. Again, similarly as in the ”standard” situation, we see that the spectral properties of the Dirac operator are very different for the standard Podleś compared with the rest of the family. Certainly only some of the presented constructions lead in the $q \to 1$ limit to the classical Dirac operator: this is the case of twisted Dirac operators. The other constructions have only commutative shadows in form of quasi-Dirac geometries [13]. For the standard Podleś sphere the twisted Dirac operators are again singled out by the fact that the order-one condition is satisfied exactly.

From the point of view of abstract spectral geometry the existence of the classical limit shall not be an argument to disqualify certain geometry. Therefore we might be forced to accept that in the $q$-deformed case we might have in some situations many possible topological sectors admitting a geometry in this sense. This is certainly true in the situation of the other members of the family of quantum spheres, where in any case we can satisfy the axioms only up to the ideal compact operators.

A challenging project is to find the description of the twisted Dirac operators using the description, which appears natural in the classical commutative geometry: that is with the help of the ”standard” Dirac operator and connection on the line bundle, which twists the spinor bundle.
References

[1] F. D’Andrea, L. Dąbrowski, G. Landi, E. Wagner, “Dirac operators on all Podles quantum spheres” ArXive:math/0606480.

[2] A. Connes, Noncommutative Geometry and Reality, J. Math. Phys. 36, 619 (1995).

[3] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser, Boston, 2001.

[4] L. Dąbrowski, A. Sitarz, “Dirac operator on the standard Podleś quantum sphere”, in Noncommutative Geometry and Quantum Groups, P. M. Hajac and W. Pusz, eds. (Instytut Matematyczny PAN, Warszawa, 2003), pp. 49–58.

[5] L. Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly, “The Dirac operator on $SU_q(2)$”, Commun. Math. Phys. 259 (2005), 729-759, (2004).

[6] L. Dąbrowski, G. Landi, M. Paschke, A. Sitarz, The spectral geometry of the equatorial Podles sphere., C. R., Math., Acad. Sci. Paris 340, No.11, 819-822 (2005)

[7] Dąbrowski, G. Landi, A. Sitarz, W. van Suijlekom and J. C. Várilly, The local index formula for $SU_q(2)$, math.QA/0501287, to appear in K-Theory

[8] N. Higson, “The Residue Index Theorem of Connes and Moscovici”, Clay Mathematics Proceedings. Volume 6, (2006).

[9] J. A. Mignaco, C. Sigaud, A. da Silva, F.J. Vanhecke, F. J., “Connes-Lott model building on the two-sphere”, Rev.Math.Phys. 13, 1-28, (2001).

[10] P. Podleś, “Quantum spheres”, Lett. Math. Phys. 14, 521–531, (1987).

[11] K. Schmudgen, E. Wagner, “Representations of cross product algebras of Podles quantum spheres”, arXiv:math/0305309
[12] A. Sitarz, “Equivariant spectral triples”, in *Noncommutative Geometry and Quantum Groups*, P. M. Hajac and W. Pusz, eds. Banach Centre Publications 61, IMPAN, Warszawa, 231–263, (2003).

[13] A. Sitarz, “Quasi-Dirac Operators on the Sphere”, ArXive:math-ph/0602030