Proof of a Conjecture on Wiener Index and Eccentricity of a graph due to edge contraction

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Abstract

For a connected graph \(G\), the Wiener index, denoted by \(W(G)\), is the sum of the distance of all pairs of distinct vertices and the eccentricity, denoted by \(\varepsilon(G)\), is the sum of the eccentricity of individual vertices. In [4], the authors posed a conjecture which states that given a graph \(G\) with at least three vertices, the difference between \(W(G)\) and \(\varepsilon(G)\) decreases when an edge is contracted and proved that the conjecture is true when \(e\) is a bridge. In this manuscript, we confirm that the conjecture is true for any connected graph \(G\) with at least three vertices irrespective of the nature of the edge chosen.

Keywords: Wiener index, eccentricity, edge contraction.

MSC: 05C09, 05C12.

1 Introduction

Let \(G = (V(G), E(G))\) be a finite, simple, connected graph with \(V(G)\) as the set of vertices and \(E(G)\) as the set of edges in \(G\). We write \(u \sim v\) to indicate that the vertices \(u\) and \(v\) are adjacent in \(G\). We denote the complete graph on \(n\) vertices by \(K_n\) and the path graph on \(n\) vertices by \(P_n\). A vertex \(u\) is said to be a neighbour of a vertex \(v\) if \(u \sim v\). The collections of all such neighbours of \(v\) in \(G\) is denoted by \(N_G(v)\).

For a given edge \(e\), we write \(G.e\) to denote the graph obtained from \(G\) by contracting the edge \(e\). More precisely, if \(e\) is the edge between two vertices \(x\) and \(y\) in \(G\) then, the vertices \(x\) and \(y\) are merged contracting the edge \(e\) in \(G.e\) and we rename the vertex as \(\alpha\). Note that, due to this graph transformation we have \(N_{G.e}(\alpha) = N_G(x) \cup N_G(y)\). For the vertices \(x\) and \(y\) we will use \(x\) or \(y\) or both.

A uv-path in \(G\) is a path in \(G\) whose end vertices are \(u\) and \(v\). Let \(u \in V(G)\) and \(P\) be a path in \(G\). We say that a path \(P\) uses a vertex \(u\) if \(u \in V(P)\). Similarly, by saying that a path \(P\) uses an edge \(e\) we mean that \(e \in E(P)\).

A connected graph \(G\) is a metric space with respect to the metric \(d\), where \(d_G(u, v)\) equals the length of a minimal uv-path. We set \(d_G(u, u) = 0\) for every vertex \(u\) in \(G\). A uv-path in \(G\) is said to be of minimal length if the length of the path is equal to \(d_G(u, v)\). The Wiener index of a graph \(G\), denoted by \(W(G)\) is defined as

\[
W(G) = \sum_{\{u, v\} \subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u \in V(G)} \sum_{v \in V(G)} d_G(u, v).
\]

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The Wiener index is the oldest topological index studied in mathematical chemistry and is one of the most studied among such indices (for surveys one may refer to [8, 10] and is an active area of research (for details see [1, 4, 6, 9, 11, 12] and the references there in).

If \( u \in V(G) \) then the eccentricity \( \varepsilon_G(u) \) is the distance from \( u \) to a farthest vertex from \( u \). A vertex \( v \) is said to be an eccentric vertex of \( u \) if \( d_G(u, v) = \varepsilon_G(u) \). The eccentricity of a graph \( G \) is

\[
\varepsilon(G) = \sum_{u \in V(G)} \varepsilon_G(u)
\]

which is also known as total eccentricity of a graph. The radius \( \text{rad}(G) \) of \( G \) and the diameter \( \text{diam}(G) \) of \( G \) are the minimum and maximum eccentricity, respectively. For studies related to total eccentricity of a graph and average eccentricity of a graph one may refer to [2]-[5] and [7]. In [1], the Wiener index and the average eccentricity has been studied on strong products of graphs.

In [4], the authors studied the relation between Wiener index and eccentricity for certain classes of graphs and posed the following conjecture:

**Conjecture 1.1.** If \( e \) is an edge of a graph \( G \) with number of vertices at least 3, then

\[
W(G.e) - \varepsilon(G.e) \leq W(G) - \varepsilon(G).
\]

An edge in a connected graph is a bridge if and only if removing it disconnects the graph. In [4], the authors proved the following theorem when the edge is a bridge as partial support for the Conjecture 1.1.

**Theorem 1.2.** If \( e \) is a bridge of a graph \( G \) with at least 3 vertices, then

\[
W(G.e) - \varepsilon(G.e) \leq W(G) - \varepsilon(G).
\]

In this manuscript, we prove that the Conjecture 1.1 is true.

## 2 Proof of Conjecture 1.1

Let \( e \) be an edge in \( G \) between the vertices \( x \) and \( y \) and the graph obtained from \( G \) by contracting the edge \( e \) is denoted by \( G.e \). Let \( \alpha \) be the vertex in \( G.e \), which is formed by merging the vertices \( x \) and \( y \).

**Lemma 2.1.** Let \( u, v \) be two vertices in \( G.e \) which is different from the vertex \( \alpha \) then

\[
d_{G.e}(u, v) = \begin{cases} 
d_G(u, v) & \text{or} \\
d_G(u, v) - 1. & \text{otherwise}
\end{cases}
\]

**Proof.** Let \( P \) be a \( uv \)-path in \( G \) then, we have two possibilities; either \( P \) uses the edge \( e \) or it does not. Suppose all \( uv \)-paths of length \( d_G(u, v) \) does not use the edge \( e \) then, the paths remain preserved in \( G.e \) and we have \( d_{G.e}(u, v) = d_G(u, v) \). If there exists a \( uv \)-path in \( G \) of length \( d_G(u, v) \) that uses the edge \( e \) then, the length of the \( uv \)-path in \( G.e \) is decreased by 1 and we have \( d_{G.e}(u, v) = d_G(u, v) - 1 \).

**Lemma 2.2.** Let \( u \neq \alpha \) be a vertex in \( G.e \) then

\[
d_{G.e}(u, \alpha) = \begin{cases} 
d_G(u, x) & \text{or} \\
d_G(u, x) - 1. & \text{otherwise}
\end{cases}
\]
Proof. Let \( P \) be a path of minimal length in \( G.e \) between \( u \) and \( \alpha \). Let \( u_n \) be the vertex on the path \( P \) which is adjacent to \( \alpha \). Then in \( G \), the vertex \( u_n \) is either adjacent to \( x \) or \( y \) or both. Next, we consider the following cases to complete the proof.

**Case 1:** Suppose for all minimal paths of length \( d_{G,e}(u, \alpha) \), the vertex \( u_n \) adjacent to \( \alpha \) is not adjacent to \( x \) in \( G \). Then \( u_n \sim y \) and the path \( P_1 = u \sim \cdots \sim u_n \sim y \sim x \) from \( u \) to \( x \) is a path of minimal length in \( G \) and hence we have \( d_G(u, x) = d_{G,e}(u, \alpha) + 1 \).

**Case 2:** Suppose there exists a minimal path of length \( d_{G,e}(u, \alpha) \) such that the vertex \( u_n \) is adjacent to \( \alpha \) in \( G.e \) and is adjacent to \( x \) in \( G \). Then it follows that \( d_G(u, x) = d_{G,e}(u, \alpha) \).

\[ \square \]

**Lemma 2.3.** Let \( x \) (or \( y \)) be the eccentric vertex or vertices of \( u \) in \( G \). Then, \( \alpha \) is the eccentric vertex of \( u \) in \( G.e \).

**Proof.** Suppose on the contrary we assume that \( \alpha \) is not the eccentric vertex of \( u \) and \( w \neq \alpha \) is an eccentric vertex of \( u \) in \( G.e \). Then we have

\[
\begin{align*}
\tag{2.1} d_{G,e}(u, \alpha) &\leq d_{G,e}(u, w) - 1 \\
\tag{2.2} d_G(u, w) &\leq d_G(u, x) - 1.
\end{align*}
\]

Thus, combining Eqns. (2.1) and (2.2) we have

\[
d_{G,e}(u, \alpha) + 1 \leq d_{G,e}(u, w) \leq d_G(u, w) \leq d_G(u, x) - 1,
\]

which implies that \( d_{G,e}(u, \alpha) \leq d_G(u, x) - 2 \), but this is a contradiction by Lemma 2.2 and the result follows.

\[ \square \]

**Lemma 2.4.** If there is an eccentric vertex of \( u \) in \( G \) other than \( x \) and \( y \) then there exist an eccentric vertex of \( u \) that is common in both \( G \) and \( G.e \).

**Proof.** We prove the lemma by considering the following two cases:

**Case 1:** Let \( w_1, w_2, \cdots, w_k \) be the eccentric vertices of \( u \) in \( G \), such that none of the \( w_i \)'s are equal to \( x \) or \( y \). Suppose on the contrary, we assume that \( w \neq w_i \) for \( 1 \leq i \leq k \) is an eccentric vertex of \( u \) in \( G.e \). Then, the following holds

\[
\begin{align*}
\tag{2.3} d_{G,e}(u, w_i) &\leq d_{G,e}(u, w) - 1 \text{ for } 1 \leq i \leq k \\
\tag{2.4} d_G(u, w) &\leq d_G(u, w_i) - 1 \text{ for } 1 \leq i \leq k.
\end{align*}
\]

Combining Eqns. (2.3) and (2.4) we have

\[
d_{G,e}(u, w_i) + 1 \leq d_{G,e}(u, w) \leq d_G(u, w) \leq d_G(u, w_i) - 1,
\]

which implies that \( d_{G,e}(u, w_i) \leq d_G(u, w_i) - 2 \), but this is a contradiction by Lemma 2.1 and hence \( w = w_i \) for some \( 1 \leq i \leq k \).

**Case 2:** Let \( w_1, w_2, \cdots, w_k \) be the eccentric vertices of \( u \) in \( G \) other than \( x \) and \( y \). Observe that for \( 1 \leq i \leq k \), none of the minimal \( uw_i \)-paths in \( G \) use the edge \( e \). If a \( uw_i \)-path uses the edge \( e \) then we have either \( u \sim \cdots \sim x \sim y \sim \cdots \sim w_i \) or \( u \sim \cdots \sim y \sim x \sim \cdots \sim w_i \) but in any of the cases \( d_G(u, w_i) > d_G(u, x) \), which is a contradiction. Since the \( uw_i \)-path of minimal length does not use the edge \( e \) in \( G \) the same is preserved in \( G.e \), i.e. \( d_G(u, w_i) = d_{G,e}(u, w_i) \). If there are no other eccentric vertices \( w \) other than \( \alpha \) in \( G.e \) then, \( d_{G,e}(u, w) < d_{G,e}(u, \alpha) \) for all \( w \in V(G.e) \setminus \{\alpha\} \). But all the \( uw \)-paths of minimal length in \( G.e \) are preserved in \( G \) which implies that \( x \) or \( y \) are the only eccentric vertices of \( u \) in \( G \), which is a contradiction. Now suppose \( w \neq w_i \) for \( 1 \leq i \leq k \), be an eccentric vertex of \( u \) in \( G.e \) then by similar arguments as in the previous case we arrive at a contradiction and hence the result follows.

\[ \square \]
Now we are ready to prove the Conjecture 1.1.

Proof of Conjecture 1.1. Let $G$ be a connected graph on $n$ vertices. If $n = 3$ then, $G$ is either the complete graph $K_3$ or the path graph $P_3$ of length 2. In either of the cases the resulting graph after contraction of an edge will lead to a single edge i.e. $K_2$. It is easy to see that $W(K_2) - \varepsilon(K_2) \leq W(G) - \varepsilon(G)$. Thus, the result is true when $n = 3$. Now we consider the case when $n \geq 4$. The difference between the Wiener index and eccentricity of $G$ and $G.e$ can be expressed as

$$W(G) - \varepsilon(G) = \sum_{u \in V(G)} \left( \frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) \right).$$

$$W(G.e) - \varepsilon(G.e) = \sum_{u \in V(G.e)} \left( \frac{1}{2} \sum_{v \in V(G.e)} d_{G.e}(u, v) - \max_{v \in V(G.e)} d_{G.e}(u, v) \right).$$

Let $\tilde{V}$ denote the set of vertices that are common in both $G$ and $G.e$, i.e. $V(G) = \tilde{V} \cup \{x, y\}$ and $V(G.e) = \tilde{V} \cup \{\alpha\}$. We complete the prove by showing that for all $u \in \tilde{V}$

$$\frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) \geq \frac{1}{2} \sum_{v \in V(G.e)} d_{G.e}(u, v) - \max_{v \in V(G.e)} d_{G.e}(u, v) \tag{2.5}$$

and

$$\frac{1}{2} \sum_{v \in V(G)} d_G(x, v) - \max_{v \in V(G)} d_G(x, v) + \frac{1}{2} \sum_{v \in V(G)} d_G(y, v) - \max_{v \in V(G)} d_G(y, v) \geq \frac{1}{2} \sum_{v \in V(G.e)} d_{G.e}(\alpha, v) - \max_{v \in V(G.e)} d_{G.e}(\alpha, v) \tag{2.6}$$

To prove Eqns. (2.5) and (2.6) we consider the following cases.

Case 1: Let $u \in \tilde{V}$.

Subcase 1.1: Let $x$ (or $y$) be the eccentric vertex (or vertices) of $u$ in $G$. Thus, by Lemma 2.3 $\alpha$ is the eccentric vertex of $u$ in $G.e$. To prove Eqn. (2.5) it is enough to show that

$$\frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) - \frac{1}{2} \sum_{v \in V(G.e)} d_{G.e}(u, v) + \max_{v \in V(G.e)} d_{G.e}(u, v) \geq 0.$$

Simplifying the left side of the inequality we have,

$$\frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) - \frac{1}{2} \sum_{v \in V(G.e)} d_{G.e}(u, v) + \max_{v \in V(G.e)} d_{G.e}(u, v)$$

$$= \frac{1}{2} \sum_{v \in \tilde{V}} d_G(u, v) + \frac{1}{2} d_G(u, x) + \frac{1}{2} d_G(u, y) - \max_{v \in \tilde{V}} d_G(u, v)$$

$$- \frac{1}{2} \sum_{v \in \tilde{V}} d_{G.e}(u, v) - \frac{1}{2} d_{G.e}(u, \alpha) + \max_{v \in \tilde{V}} d_{G.e}(u, v)$$

$$= \frac{1}{2} \sum_{v \in \tilde{V}} d_G(u, v) - \frac{1}{2} d_G(u, x) + \frac{1}{2} d_G(u, y) - \frac{1}{2} \sum_{v \in \tilde{V}} d_{G.e}(u, v) + \frac{1}{2} d_{G.e}(u, \alpha)$$

$$= \frac{1}{2} \left( \sum_{v \in \tilde{V}} d_G(u, v) - \sum_{v \in \tilde{V}} d_{G.e}(u, v) \right) + \frac{1}{2} (d_G(u, y) + d_{G.e}(u, \alpha) - d_G(u, x)).$$
Finally, using Lemmas 2.1 and 2.2 and using the fact that \( u \neq y \) the result follows. Note that we have used the fact that \( x \) is an eccentric vertex of \( u \). Similar calculations will follow if \( y \) is an eccentric vertex of \( u \).

**Subcase 1.2:** Let \( w \) be an eccentric vertex of \( u \) in \( G \) other than \( x \) and \( y \). Without loss of generality using Lemma 2.4, we can choose \( w \in \tilde{V} \) such that \( w \) is an eccentric vertex of \( u \) in both \( G \) and \( G.e \). To prove Eqn. (2.5) it is enough to show that

\[
\frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) - \frac{1}{2} \sum_{v \in V(G.e)} d_{G,e}(u, v) + \max_{v \in V(G.e)} d_{G,e}(u, v) \geq 0.
\]

Simplifying the left side of the inequality we have,

\[
\frac{1}{2} \sum_{v \in V(G)} d_G(u, v) - \max_{v \in V(G)} d_G(u, v) - \frac{1}{2} \sum_{v \in V(G.e)} d_{G,e}(u, v) + \max_{v \in V(G.e)} d_{G,e}(u, v)
\]

\[
= \frac{1}{2} \sum_{v \in \tilde{V}} d_G(u, v) + \frac{1}{2} d_G(u, x) + \frac{1}{2} d_G(u, y) - \max_{v \in V(G)} d_G(u, v)
\]

\[- \frac{1}{2} \sum_{v \in \tilde{V}} d_{G,e}(u, v) - \frac{1}{2} d_{G,e}(u, \alpha) + \max_{v \in V(G.e)} d_{G,e}(u, v)
\]

\[
= \frac{1}{2} \left( \sum_{v \in \tilde{V} \setminus \{w\}} d_G(u, v) - \sum_{v \in \tilde{V} \setminus \{w\}} d_{G,e}(u, v) \right)
\]

\[+ \frac{1}{2} (d_G(u, x) + d_G(u, y) - d_G(u, w) - d_{G,e}(u, \alpha) + d_{G,e}(u, w)).
\]

Since \( \sum_{v \in \tilde{V} \setminus \{w\}} d_G(u, v) - \sum_{v \in \tilde{V} \setminus \{w\}} d_{G,e}(u, v) \geq 0 \) follows from Lemma 2.1, it only remains to show that \( d_G(u, x) + d_G(u, y) - d_{G,e}(u, \alpha) \geq d_G(u, w) - d_{G,e}(u, w) \). Note that, using Lemma 2.1 we have \( d_G(u, w) - d_{G,e}(u, w) \) is either 0 or 1. Similarly, by Lemma 2.2 \( d_G(u, x) - d_{G,e}(u, \alpha) \) is either 0 or 1. Combining, we have \( d_G(u, x) + d_G(u, y) - d_{G,e}(u, \alpha) \geq 1 \) since \( u \neq y \) and hence the result follows.

**Case 2:** In this case we prove the inequality (2.6). Let \( w_1 \) and \( w_2 \) be eccentric vertices of \( x \) and \( y \), respectively. Without loss of generality, we assume that \( w_1 \) is an eccentric vertex of \( \alpha \). Then we
have the following
\[
\frac{1}{2} \sum_{v \in V(G)} d_G(x, v) - \max_{v \in V(G)} d_G(x, v) + \frac{1}{2} \sum_{v \in V(G)} d_G(y, v) - \max_{v \in V(G)} d_G(y, v) \\
- \frac{1}{2} \sum_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v) + \max_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v)
\]
\[
= \frac{1}{2} \sum_{v \in V(G) \setminus \{w_1\}} d_G(x, v) - \frac{1}{2} d_G(x, w_1) + \frac{1}{2} \sum_{v \in V(G) \setminus \{w_2\}} d_G(y, v) - \frac{1}{2} d_G(y, w_2)
\]
\[
- \frac{1}{2} \sum_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v) + \frac{1}{2} d_{G,e}(\alpha, w_1)
\]
\[
= \frac{1}{2} \left( \sum_{v \in V(G) \setminus \{w_1, y\}} d_G(x, v) - \sum_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v) \right)
\]
\[
+ \frac{1}{2} \left( \sum_{v \in V(G) \setminus \{w_2\}} d_G(y, v) + d_G(x, y) - d_G(x, w_1) - d_G(y, w_2) + d_{G,e}(\alpha, w_1) \right).
\]

Since the graph $G$ is connected there exists a vertex $w_3$ on the $yw_2$-path of minimal length such that $d_G(y, w_2) = d_G(y, w_3) + 1$. From Lemma 2.2 we have $d_G(x, w_1) - d_{G,e}(\alpha, w_1)$ is at most 1. Thus, to show the fact that
\[
\sum_{v \in V(G) \setminus \{w_2\}} d_G(y, v) + d_G(x, y) - d_G(y, w_2) \geq d_G(x, w_1) - d_{G,e}(\alpha, w_1) \tag{2.7}
\]
it is enough to prove that
\[
\sum_{v \in V(G) \setminus \{w_2, w_3\}} d_G(y, v) \geq 2.
\]
But this is always true since $G$ has at least four vertices. Finally, using Lemma 2.2 and Eqn. 2.7 we have,
\[
\frac{1}{2} \sum_{v \in V(G)} d_G(x, v) - \max_{v \in V(G)} d_G(x, v) + \frac{1}{2} \sum_{v \in V(G)} d_G(y, v) - \max_{v \in V(G)} d_G(y, v)
\]
\[
\geq \frac{1}{2} \sum_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v) - \max_{v \in V(G) \setminus \{w_1\}} d_{G,e}(\alpha, v).
\]
Thus, combining all the above cases and using the inequalities (2.5) and (2.6) we have
\[
W(G,e) - \varepsilon(G,e) \leq W(G) - \varepsilon(G).
\]

\textbf{Remark 2.5.} In [4], the authors posed a second conjecture stating that the difference between the Wiener index of a graph and its eccentricity is largest possible on paths. If $G$ be a graph of order $n$ with $\text{rad}(G) \geq 4$, then
\[
W(G) - \varepsilon(G) \leq \left[ \frac{1}{6} n^3 - \frac{3}{4} n^2 + \frac{1}{3} n + \frac{1}{4} \right]
\]
with equality holding if and only if $G$ is a path. The Conjecture is still open.

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