Degenerate J-flow on compact Kähler manifolds

Tat Dat Tô

Received: 21 February 2022 / Accepted: 13 February 2023 / Published online: 20 March 2023
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract
In this note, we study a degenerate twisted J-flow on compact Kähler manifolds. We show that it exists for all time, it is unique and converges to a weak solution of a degenerate twisted J-equation. In particular, this confirms an expectation formulated by Song–Weinkove for the J-flow. As a consequence, we establish the properness of the Mabuchi K-energy twisted by a certain semi-positive closed (1,1)-form for Kähler classes in a certain subcone.

1 Introduction

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\). We define the space \(\mathcal{H}_\omega\) of Kähler potentials by

\[
\mathcal{H}_\omega = \{ \varphi \in C^\infty(X) : \omega_\varphi := \omega + i \bar{\partial} \partial \varphi > 0 \}.
\]

Let \(\theta\) be another Kähler metric. The J-flow is the parabolic flow defined on \(\mathcal{H}_\omega\) by

\[
\frac{\partial \varphi}{\partial t} = c - \frac{n \omega_\varphi^{n-1} \wedge \theta}{\omega_\varphi^n} = c - \text{tr}_{\omega_\varphi} \theta, \quad \varphi|_{t=0} = \varphi_0,
\]

where \(\varphi_0 \in \mathcal{H}_\omega\) and \(c\) is the constant given by

\[
c = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n}.
\]

A stationary point \(u\) of the J-flow defines a Kähler metric \(\bar{\omega} = \omega + i \partial \bar{\partial} u \in [\omega]\) satisfying the J-equation

\[
\bar{\omega}^{n-1} \wedge \theta = c \bar{\omega}^n.
\]

The J-flow was introduced by Donaldson [7] in the setting of moment maps and by Chen [2] as the gradient flow of the \(J\)-functional appearing in the formula of the Mabuchi K-energy. The critical points of the Mabuchi K-energy are constant scalar curvature Kähler (cscK)
metrics. The existence of cscK metrics in a given Kähler class is a fundamental problem in Kähler geometry and has been extensively studied. Tian [26] conjectured that the properness of the Mabuchi K-energy implies the existence cscK metrics. This conjecture was proved recently by Chen–Cheng [3, 4].

In [24], Song–Weinkove pointed that $J$ being bounded from below is sufficient to imply the properness of the Mabuchi K-energy. A uniform bound from below for the $J$-functional can be obtained as long as the corresponding J-flow exists for all time and converges. For this reason the J-flow has received a lot of attention in order to determine conditions for its convergence. Necessary and sufficient conditions for the regular case (i.e. when $\omega$ and $\theta$ are Kähler) have been obtained by Song–Weinkove [24] extending previous results by Weinkove [30, 31]. An algebraic understanding of these conditions has been developed in [1, 5, 15, 20, 22].

In the degenerate case, conditions for the existence and convergence of weak J-flows in dimension two have obtained in [8, 23]. In particular, Song–Weinkove [23] assumed that $\theta$ is only semi-positive, positive away from a divisor $D$ and big, then under certain condition the weak J-flow exists, is smooth away from $D$ and converges in $C^\infty$ away from $D$ to a weak solution of a degenerate J-equation. They expected that this result can be extended to higher dimension (cf. [23, Remark 3.1]).

In [33], Zheng suggested to study a twisted version of the J-flow defined by

$$\frac{\partial \varphi}{\partial t} = c_\beta - \frac{n\omega_{\varphi}^{n-1} \wedge \theta}{\omega_{\varphi}^{n}} - \beta \frac{\omega^n}{\omega_{\varphi}^{n}} \varphi - \beta \frac{\omega_{\varphi}^{n}}{\omega_{\varphi}^{n}}, \quad \varphi|_{t=0} = \varphi_0,$$  

(1.2)

where $\beta \in [0, +\infty)$, $\varphi_0 \in H_{\omega}$ and $c_\beta$ is the constant given by

$$c_\beta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta.$$

When $\beta = 0$, we obtain the J-flow. In [33], Zheng proved the existence and convergence under a positivity condition (see (1.4) below). A main advantage of this twisted flow is that the necessary and sufficient condition for the existence and convergence for the twisted J-flow is weaker than the one for the J-flow, while it may still imply the properness of the Mabuchi K-energy. We will elaborate on this point in Sect. 2.1.

In this note, motivated by previous works of Song–Weinkove [23, 24] and Zheng [33], we study the existence and convergence for the degenerate twisted J-flow where $\theta$ is no longer a Kähler metric, but a closed $(1, 1)$-form satisfying a certain nonnegativity condition. We extend the results by Song–Weinkove [23] for the J-flow on Kähler surfaces mentioned above to the twisted J-flow in any dimension.

In particular, this confirms the expectation formulated in [23] for the untwisted case: there exists a unique degenerate J-flow for all times which is smooth away from a Zariski set and converges to a weak solution of the degenerate J-equation.

More precisely, we assume that the form $\theta$ is merely semi-positive and satisfies the following condition. We fix an effective divisor $D$ on $X$ and $h$ a Hermitian metric on the line bundle $O_X(D)$. Let $s$ be a holomorphic section of $O_X(D)$ and assume that

$$\theta \geq C_0 |s|^{2\gamma} \omega, \quad \text{and} \quad \theta - \epsilon_0 R_h \geq C_0 \omega,$$

(1.3)

for some positive constants $\gamma, C_0, \epsilon_0$, where $R_h = -i \partial \bar{\partial} \log h$ denotes the curvature form of the Hermitian metric $h$. These conditions are for example satisfied for certain $\theta \in c_1(K_X)$ with $K_X$ is big and nef, i.e $X$ is a smooth minimal model of general type.
We shall moreover assume that there is \( \hat{\omega} \in [\omega] \) such that
\[
(c_\beta \hat{\omega} - (n - 1)\theta) \wedge \hat{\omega}^{n-2} > 0,
\] as an \((n - 1, n - 1)\) form, where
\[
c_\beta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta.
\]
In [33], the author proved that if \( \theta > 0 \), the condition (1.4) is equivalent to the long time existence and the smooth convergence of the twisted J-flow. This result extends the previous one of Song–Weinkove [24] for the J-flow. In particular, the condition (1.4) is also equivalent to the existence of a \( C \)-subsolution introduced by Székelyhidi [25] and Guan [12] for the twisted J-equation as well as a parabolic \( C \)-subsolution introduced by Phong-Tô [19] for the twisted J-flow. As explained in [24, 33], when \( K_X \) is ample and \( \theta \) is a Kähler metric in \(-c_1(K_X)\) this condition then implies the properness of the Mabuchi K-energy on \( H_\omega \). We refer to [6, 14, 16, 21] for recent works studying sufficient conditions so that the Mabuchi K-energy is proper.

Let \( PSH(X, \omega) \) denote the set of \( \omega \)-plurisubharmonic functions: these are functions \( \phi : X \to \mathbb{R} \cup \{ -\infty \} \) which are locally given as the sum of a smooth and a plurisubharmonic function, such that \( \omega + dd^c \phi \geq 0 \) in the weak sense of currents and \( \phi \not\equiv -\infty \). We define
\[
\mathcal{H}_\omega^{weak} = \{ \varphi \in PSH(X, \omega) \cap L^\infty(X) \cap C^\infty(X \setminus D) : \omega + i\partial \bar{\partial} \varphi > 0 \text{ on } X \setminus D \}. 
\]
(1.5)

Our main result is the following.

**Theorem 1.1** Let \((X, \omega)\) be a compact Kähler manifold and \( \beta \) be a non-negative constant. Assume that \( \theta \) is a closed semi-positive \((1,1)\)-form satisfying conditions (1.3) and (1.4).

For any \( \varphi_0 \in \mathcal{H}_\omega \) and for all \( t > 0 \) there exists a unique solution \( \varphi = \varphi(t) \in \mathcal{H}_\omega^{weak} \) to the degenerate twisted J-flow
\[
\frac{\partial \varphi}{\partial t} = c_\beta - \frac{n\omega_{\varphi}^{n-1} \wedge \theta}{\omega_\varphi^n} - \beta \omega_\varphi^n, \quad \varphi|_{t=0} = \varphi_0. 
\]
(1.6)

Moreover, \( \varphi(t) \to \varphi_\infty \) in \( C^\infty_{loc}(X \setminus D) \) as \( t \to \infty \), where \( \varphi_\infty \in \mathcal{H}_\omega^{weak} \) satisfies
\[
c_\beta \omega_\varphi^n = n\omega_{\varphi_\infty}^{n-1} \wedge \theta + \beta \omega^n.
\]
When \( \beta = 0 \), this theorem confirms the expectation of Song–Weinkove [23, Remark 3.1].

Theorem 1.1 also provides a uniform lower bound for the twisted \( J \)-functional \( J_{\omega, \beta}^\theta \).

**Corollary 1.2** Let \((X, \omega)\) be a compact Kähler manifold and \( \beta \) a non-negative constant. Assume that \( \theta \) is a closed semi-positive \((1,1)\)-form satisfying conditions (1.3) and (1.4). Then there exists a constant \( B \) depending only on \( X, \omega, \theta \) such that
\[
J_{\omega, \beta}^\theta(\varphi) \geq B, \quad \forall \varphi \in \mathcal{H}_\omega.
\]

Finally, we have the following criterion for the properness of the Mabuchi K-energy twisted by a semipositive \((1,1)\)-form.

**Corollary 1.3** Let \( X \) be a compact Kähler manifold and \( \eta \) be a smooth closed semipositive \((1,1)\)-form. Assume that \( \theta \in -c_1(X) + [\eta] \) is a semi-positive closed \((1,1)\)-form satisfying (1.3). Suppose that
\[
M_{ent}(\varphi) \geq \alpha_0 J_\omega(\varphi) - C,
\]
(1.7)
for some $\alpha_0 > 0$, where $\mathcal{M}_{ent}$ is the entropy part in the formula of the Mabuchi functional and $J_\omega$ is the Aubin’s $J$-functional (see Sect. 2.1). For any $\beta \in [0, \alpha_0)$, denote by $C$ the cone of all Kähler classes $[\omega_0]$ such that there exists a Kähler form $\omega \in [\omega_0]$ satisfying

$$\left(\frac{n(-c_1(X) + [\eta]) \cdot [\omega]^{n-1}}{[\omega]^n} + \beta\right) \omega - (n - 1)\theta \wedge \omega^{n-2} > 0,$$

as an $(n-1, n-1)$-form.

Then the Mabuchi functional twisted by $\eta$ is proper on every class in $C$. Therefore by [4, Theorem 4.1] there exists a twisted cscK metric in $[\omega]$ which satisfies

$$S(\omega_\varphi) = \mathcal{S} + \text{tr}_{\omega_\varphi} \eta - \eta,$$

where

$$\mathcal{S} = n \frac{c_1(X) \cdot [\omega]^{n-1}}{[\omega]^n} \quad \text{and} \quad \eta = n \frac{[\eta] \cdot [\omega]^{n-1}}{[\omega]^n}.$$

It follows from [27, p.95] that (1.7) holds for $\alpha_0 = \alpha_X([\omega_0])$, namely Tian’s alpha invariant. Corollary 1.3 thus gives a criterion in line with recent results due to [6, 16, 20, 33] (see Proposition 4.1 and Remark 4.2). In particular, when $\eta = 0$, this extends the result in [33] for the properness of the Mabuchi K-energy on compact Kähler manifolds with ample canonical bundle to minimal models of general type.

A key ingredient of the proof of Theorem 1.1 is a uniform $C^0$ estimate for the weak solution of the degenerate twisted J-flow. In [23], for Kähler surfaces, the authors reduced the degenerate $J$-equation to a degenerate Monge–Ampère equation then obtained a uniform $C^0$ bound using Yau’s estimate [32]. It seems delicate to extend this approach to higher dimension.

In this work, we adapt a $C^0$ estimate from [19] which is based on the parabolic Alexandrov–Bakelmann–Pucci (ABP) maximum principle (see [25] for the elliptic version). We can not apply the $C^0$ estimate from [19] directly since $\theta$ is assumed to be positive in [19] while it is merely semi-positive in our case. In [19], with $\theta > 0$, the parabolic equation has been reformulated as

$$\partial_t \varphi = f(\lambda_\theta(\omega_\varphi)),$$

where $\lambda_\theta(\omega_\varphi) \in \mathbb{R}^n$ is the vector of eigenvalues of $\omega_\varphi$ in normal coordinates with respect to $\theta$. In our case, we use normal coordinates for $\omega$ instead of $\theta$, so that $\omega_\varphi$ is diagonal with entries $\lambda_1, \ldots, \lambda_n$. The twisted J-flow can then be written in a new form as

$$\partial_t \varphi = c\beta - \sum_{k=1}^n \frac{\mu_k}{\lambda_1 \cdots \lambda_n} \frac{\beta}{\lambda_1 \cdots \lambda_n},$$

where $\mu_1, \ldots, \mu_n$ denote the diagonal elements of $\theta$ in these coordinates. Then the $C^0$ estimate is deduced using the ABP estimate, a Harnack type inequality as well as a local $C^2$ estimate under the positivity condition (1.4) (cf. Lemma 3.3).

Another ingredient of the proof of Theorem 1.1 is a uniform $C^2$ estimate away from a divisor (Lemma 3.7). We use the maximal principle with a well-known trick due to Tsuji [29], but we do not use the Phong–Sturm’s trick (cf. [18]) as in [23]. In our case, some extra terms also appear in the computation due to the new term in the twisted $J$-flow.

The same idea for $C^0$-estimate can also be used to give a uniform estimate for the twisted $\sigma_{n-1}$ inverse equation where the the uniform bound depends on $\|\theta\|_{C^0(X, \omega)}$ but not on $\|\omega\|_{C^0(X, \varphi)}$ as in [25].

Springer
Proposition 1.4 Let \((X, \omega)\) be a compact Kähler manifold and \(\theta\) be another Kähler metric. Let \(\psi, \rho \in C^\infty(X)\) be smooth functions on \(X\) with \(\psi > 0\). Assume that there is \(\omega' = \omega + i \partial \bar{\partial} u \in [\omega]\) such that
\[
(\psi \omega' - (n - 1)\theta) \wedge \omega^{n-2} > 0,
\]
as an \((n-1,n-1)\) form. Assume \(u\) is a smooth solution of the equation
\[
n \omega_u^{n-1} \wedge \theta + \rho \omega^n = \psi \omega_u^n, \quad \sup_X (u - \bar{u}) = 0.
\]
Then there exists a constant \(C > 0\) which only depends on \(X, \omega, \|u\|_{C^2(X, \omega)}, \|\psi\|_{L^\infty(X)}, \|\rho\|_{L^\infty(X)}\) and \(\|\theta\|_{C^0(X, \omega)}\) such that
\[
\|u\|_{L^\infty(X)} \leq C.
\]

2 Preliminaries

2.1 Mabuchi K-energy

Let \((X, \omega)\) be a \(n\)-dimensional compact Kähler manifold. We define the space \(\mathcal{H}_\omega\) of Kähler potentials by
\[
\mathcal{H}_\omega = \{ \phi \in C^\infty(X) : \omega_\phi = \omega + i \partial \bar{\partial} \phi > 0 \}.
\]
The Mabuchi K-energy functional was introduced by Mabuchi [17] which is characterized by
\[
\frac{dM_\omega(\phi_t)}{dt} = -\int_X \phi_t (S_{\phi_t} - S) \frac{\omega_{\phi_t}^n}{n!},
\]
for any path \((\phi_t)\) in \(\mathcal{H}_\omega\), where \(S_{\phi_t}\) is the scalar curvature of \(\omega_{\phi_t}\) and \(S = \frac{c_1(X, \omega)_{\omega}^{n-1}}{[\omega]^{n-1}}\). The critical points of the Mabuchi K-energy are constant scalar curvature Kähler (cscK) metrics.

The Chen–Tian formula (see [2, 27]), expresses the Mabuchi K-energy as the sum of an entropy part and a pluripotential part
\[
M_\omega(\phi) = M_{\text{ent}}(\phi) + M_{\text{pp}}(\phi),
\]
where
\[
M_{\text{ent}}(\phi) = \int_X \log \left( \frac{\omega_{\phi}^n}{\omega^n} \right) \frac{\omega_{\phi}^n}{n!}
\]
and
\[
M_{\text{pp}}(\phi) = J_\omega^{-Ric(\omega)}.
\]

Here, the twisted \(J\)-functional \(J_\omega^\theta\) associated to a \((1, 1)\)-form \(\theta\) is defined by
\[
J_\omega^\theta(\phi) = \int_0^1 \int_X \phi_t (n \theta \wedge \omega_{\phi_t}^{n-1} - \theta \omega_{\phi_t}^n) \frac{dt}{n!},
\]
\[
= \frac{1}{n!} \int_X \phi \sum_{k=0}^{n-1} \theta \wedge \omega^k \wedge \omega_{\phi}^{n-1-k} - \frac{1}{(n+1)!} \int_X \theta \phi \sum_{k=0}^{n} \omega^k \wedge \omega_{\phi}^{n-k}.
\]
where \((\varphi_t)\) is any path in \(H_\omega\) between 0 and \(\varphi\) and
\[
\theta := \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n}.
\]

Let \(\eta \geq 0\) be a smooth closed \((1, 1)\)-form, the twisted Mabuchi K-energy is defined by
\[
M^\theta_\omega := M_\omega + J_\theta^\omega.
\]
The critical points of \(M^\theta_\omega\) satisfy the following equation
\[
S(\omega_\varphi) = S_\omega + \text{tr}_{\omega_\varphi} \eta - \eta,
\]
which defines twisted cscK metrics in \([\omega]\) (cf. [4, 9, 10]).

A functional \(F\) is called coercive if there are constants \(C_1, C_2\) such that
\[
F(\varphi) \geq C_1 J_\omega(\varphi) - C_2,
\]
for all \(\varphi \in H_\omega\), where \(J_\omega\) is Aubin’s functional defined by
\[
J_\omega(\varphi) = \int_X \varphi \omega^n - \omega_\varphi^n dt.
\]

Coercivity is a strong form of “properness”. The cscK metrics are critical points of the Mabuchi K-energy. Tian [26] conjectured that the properness of Mabuchi functional implies the existence of cscK metrics. This conjecture was proved recently by Chen–Cheng [3, 4]. The similar result for the existence of twisted cscK metrics was also proved in [4, Theorem 4.1].

**Definition 2.1** A function \(\varphi : X \to \mathbb{R} \cup \{-\infty\}\) is \(\omega\)-plurisubharmonic (\(\omega\)-psh for short) if it is locally given as the sum of a smooth and a plurisubharmonic function, and such that \(\omega + dd^c \varphi \geq 0\) in the weak sense of currents. Let \(PSH(X, \omega)\) denote the set of all \(\omega\)-plurisubharmonic functions which are not identically \(-\infty\).

We remark that all functionals above are also well-defined on \(PSH(X, \omega) \cap L^\infty(X)\).

We now recall an observation due to Zheng [33]. Suppose that \(\alpha_0\) is a positive number satisfying
\[
M_{ent}(\varphi) \geq \alpha_0 J_\omega(\varphi) - C, \forall \varphi \in H_\omega
\]
for some \(C > 0\), then for any \(\beta \in [0, \alpha_0)\) we have
\[
M(\varphi) \geq \alpha_0 J_\omega(\varphi) - C + J_{\omega, \beta}^{-Ric(\omega)}
\]
\[
= (\alpha_0 - \beta) J_\omega(\varphi) - C + J_{\omega, \beta}^{-Ric(\omega)}
\]
\[
\geq (\alpha_0 - \beta) J_\omega(\varphi) - C + \inf_{\varphi \in H_\omega} J_{\omega, \beta}^{-Ric(\omega)},
\]
where \(J_{\omega, \beta}^{-Ric(\omega)} = J_{\omega, \beta}^{-Ric(\omega)} + J_\omega\). Therefore, if \(J_{\omega, \beta}^{-Ric(\omega)}\) is uniformly bounded from below on \(H_\omega\) then the Mabuchi K-energy is coercive.

We remark that (2.6) holds for \(\alpha_0 = \alpha_X([\omega])\), namely Tian’s alpha invariant (cf. [27, p.95] or [24, Lemma 4.1]), defined by
\[
\alpha_X([\omega]) = \sup\{\alpha > 0 : \exists C > 0 \text{ such that } \int_X e^{-\alpha(\psi - \sup_X \varphi)} \omega^n \leq C, \forall \varphi \in PSH(X, \omega)\}.
\]

In general, we define the functional
\[
J_{\omega, \beta}^\theta := J_{\omega, \beta}^\theta + J_\omega.
\]
The critical point of \(J_{\omega, \beta}^\theta\) satisfies a new fully nonlinear equation in \(H_\omega\), namely the twisted \(J\)-equation:
\[ n\omega_{\varphi}^{n-1} \wedge \theta + \beta \omega^n = c_\beta \omega_{\varphi}^n, \]

where

\[ c_\beta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta. \]

We also remark that this equation is similar to the equation (4) in [5] in which \( \beta \omega^n \) replaced by \( \beta \theta^n \) assuming \( \theta > 0 \). Moreover, this is also a special case of a general twisted J-equation introduced recently by Chen [1], Theorem 1.11, with \( f = \beta \omega_{\varphi}^n \) and \( \theta > 0 \).

### 2.2 Twisted J-flow

Let \((X, \omega)\) be a Kähler manifold of dimension \( n \) and \( \theta \) be a smooth \((1, 1)\)-forms on \( X \). For \( \beta \in [0, \infty) \), consider the twisted J-flow

\[
\frac{\partial \varphi}{\partial t} = c_\beta - \frac{n\omega_{\varphi}^{n-1} \wedge \theta}{\omega_{\varphi}^n} - \beta \frac{\partial \varphi}{\omega_{\varphi}^n}, \quad \varphi|_{t=0} = \varphi_0. \tag{2.10}
\]

where

\[ c_\beta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta. \]

This is the gradient flow for the twisted \( J \)-functional \( J_{\omega, \beta} = J_\omega + \beta J_\omega \) above.

When \( \theta > 0 \), we have the following result is due to Song–Weinkove [24] for \( \beta = 0 \) and Zheng [33] for \( \beta \geq 0 \):

**Theorem 2.2** [[24, 33]] Let \( (X, \omega) \) be a Kähler manifold of dimension \( n \) and \( \theta \) be a closed positive \((1, 1)\)-forms on \( X \). Assume that there exists a metric \( \hat{\omega} \in [\omega] \) such that

\[
(c_\beta \hat{\omega} - (n - 1) \theta) \wedge \hat{\omega}^{n-2} > 0, \tag{2.11}
\]

where

\[ c_\beta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta. \]

Then for any \( \varphi_0 \in \mathcal{H}_\omega \), the twisted J-flow (2.10) admits a unique solution \( \varphi \in C^\infty(X \times [0, \infty)) \), with \( \varphi(t) \in \mathcal{H}_\omega \), \( \forall t \geq 0 \).

Moreover, \( \varphi(t) \to \varphi_\infty \) in \( C^\infty(X) \) as \( t \to \infty \), where \( \varphi_\infty \in \mathcal{H}_\omega \) satisfies

\[ c_\beta \omega_\infty^n = n\omega_{\varphi_\infty}^{n-1} \wedge \theta + \beta \omega^n. \]

We are now interested in the degenerate case where \( \theta \) is merely semi-positive. We have the following local expression for the degenerate twisted J-flow which will be used to construct weak solutions. We use normal coordinates for \( \omega \) so that \( \omega_{\varphi} \) is diagonal with entries \( \lambda_1, \ldots, \lambda_n \). Let \( \mu_1, \ldots, \mu_n \) denote the diagonal elements of \( \theta \) in these coordinates. The twisted J-flow can then be written as

\[
\partial_t \varphi = c_\beta - \sum_{k=1}^n \frac{\mu_k}{\lambda_k} - \frac{\beta}{\lambda_1 \ldots \lambda_n},
\]
or
\[
\sum_{k=1}^{n} \frac{\mu_k}{\lambda_k} + \frac{\beta}{\lambda_1 \ldots \lambda_n} = c_\beta - \partial_t \varphi.
\]

If there is a uniform constant $\delta_0 > 0$ such that
\[
c_\beta \omega^{n-1} - (n-1)\omega^{n-1} \wedge \theta \geq \delta_0 \omega^{n-1},
\]
then in the coordinates above, this inequality can be rewritten as
\[
c_\beta - \sum_{j \neq k} \mu_j \geq \delta_0.
\]

### 3 Uniform estimates

#### 3.1 Approximating the degenerate twisted J-flow

Let $(X, \omega)$ be a compact Kähler manifold and $\theta$ be a smooth $(1, 1)$-form.

We first assume that $\theta \geq 0$ and $\int_X \theta \wedge \omega^{n-1} > 0.$

Set $\theta_\epsilon = \theta + \epsilon \omega$, for $\epsilon \geq 0$ and, define
\[
c_\epsilon := \frac{n[\theta_\epsilon] \cdot [\omega]^{n-1}}{[\omega]^n} + \beta.
\]

Then $\theta_\epsilon$ is positive for all $\epsilon > 0$. By our assumption in Theorem 1.1, there exists $\hat{\omega} \in [\omega]$ such that
\[
(c_\beta \hat{\omega} - (n-1)\theta) \wedge \hat{\omega}^{n-2} > 0.
\]

As $\epsilon \to 0$, $c_\epsilon \to c_\beta$, so we may choose $\epsilon_0 > 0$ such that for $\epsilon \in [0, \epsilon_0]$ we have
\[
(c_\epsilon \hat{\omega} - (n-1)\theta_\epsilon) \wedge \hat{\omega}^{n-2} \geq \delta_0 \hat{\omega}^{n-1} > 0.
\]

for a uniform constant $\delta_0 > 0$ independent of $\epsilon$. Since $\hat{\omega} \in [\omega]$, $\hat{\omega} = \omega + i \partial \bar{\partial} \phi$ for some $\phi \in C^\infty(X)$, we can replace $\omega$ by $\hat{\omega}$ and still denote it by $\omega$ to get the inequality:
\[
(c_\epsilon \omega - (n-1)\theta_\epsilon) \wedge \omega^{n-2} \geq \delta_0 \omega^{n-1}.
\]

We now consider the family of twisted J-flows
\[
\frac{\partial \varphi_\epsilon}{\partial t} = c_\epsilon - \frac{\omega^{n-1} \wedge \theta_\epsilon}{\omega^{n}_{\varphi_\epsilon}} - \beta \frac{\omega^n}{\omega^{n}_{\varphi_\epsilon}}, \quad \varphi|_{t=0} = \varphi_0,
\]

for $\beta \in [0, \infty)$ and $\epsilon \in (0, \epsilon_0]$.

It follows from Theorem 2.2 that for any $\epsilon \in (0, \epsilon_0]$, with the condition (3.4), the twisted J-flow (3.5) exists for all time and converges to the unique solution of the twisted J-equation
\[
\omega_{u_\epsilon}^{n-1} \wedge \theta_\epsilon + \beta \omega^n = c_\epsilon \omega_{u_\epsilon}^n.
\]
In the next sections we prove uniform \( L^\infty \) bounds for \( \varphi_\epsilon, \dot{\varphi}_\epsilon \) on \( X \) when \( \theta \) is semi-positive with \( \int_X \theta \wedge \omega^{n-1} > 0 \), and \( C^\infty \) estimates for \( \varphi_\epsilon \) away from \( D \) when we assume further that \( \theta \) satisfies the condition (1.4).

### 3.2 Estimate for the time derivative

The \( C^1 \) estimate for \( \varphi_\epsilon \) in the time variable follows easily from the maximal principle.

**Lemma 3.1** There exists a uniform constant \( C \) such that

\[
\| \partial_t \varphi_\epsilon \|_{L^\infty(X)} \leq C
\]

and

\[
\omega_\varphi \geq \frac{1}{C} \theta_\epsilon. \tag{3.7}
\]

**Proof** Differentiating (3.5), we get

\[
\frac{\partial}{\partial t} \left( \frac{\partial \varphi_\epsilon}{\partial t} \right) = \mathcal{L} \left( \frac{\partial \varphi_\epsilon}{\partial t} \right), \tag{3.8}
\]

where

\[
\mathcal{L} = \left( \omega^{k\bar{l}}_\varphi \omega^{i\bar{j}}_\varphi (\theta_\epsilon)_{k\bar{l}} \beta \omega^{n}_\varphi \omega^{i\bar{j}}_\varphi \right) \partial_i \partial_{\bar{j}}.
\]

Here we write \( \omega = \sum_{k,\ell} i\omega_{k\ell} dz^k \wedge d\bar{z}^\ell \) in local coordinates and \( (\omega^{k\bar{l}}) = (\omega_{k\ell})^{-1} \). We infer from the maximum principle that

\[
\min_X \left( c_\epsilon - \frac{\omega^{n-1}_\varphi - \theta_\epsilon}{\omega^{n}_\varphi} - \beta \omega^{n}_\varphi \right) \leq \partial_t \varphi_\epsilon \leq \max_X \left( c_\epsilon - \frac{\omega^{n-1}_\varphi - \theta_\epsilon}{\omega^{n}_\varphi} - \beta \omega^{n}_\varphi \right). \tag{3.9}
\]

This implies that \( |\partial_t \varphi_\epsilon| \leq C \), hence \( \text{tr}_\omega \theta_\epsilon \leq C \) for a uniform constant \( C > 0 \), therefore

\[
\omega_\varphi \geq \frac{1}{C} \theta_\epsilon. \tag{3.10}
\]

### 3.3 \( C^0 \) estimate

We provide here a \( C^0 \) estimate using the ABP maximum principle adapting arguments from [19]. Our situation is different from the general setting in [19] where \( \theta \) is assumed to be positive. In the latter case, the parabolic equation has been reformulated as

\[
\dot{\varphi} = f(\lambda_\theta(\omega_\varphi)),
\]

where \( \lambda_\theta(\omega_\varphi) \in \mathbb{R}^n \) is the vector of eigenvalues of \( \omega_\varphi \) with respect to \( \theta \).

Since we work with a smooth \((1, 1)\)-form \( \theta \) which is only semipositive, this setting is no longer valid. We consider instead \( \lambda = \lambda_\omega(\omega_\varphi) \) and \( \mu = (\mu_1, \ldots, \mu_n) \) the diagonal entries of \( \theta_\epsilon \) with respect to \( \omega \), and reformulate the twisted J-flow as in Sect. 2.2:

\[
\dot{\varphi}_\epsilon = c_\epsilon - \sum_{j=1}^n \frac{\mu_j}{\lambda_j} \beta \frac{1}{\lambda_1 \ldots \lambda_n}.
\]
We first remark that

**Lemma 3.2** Along the twisted J-flow we have

\[
\sum_{j=0}^{n} \int_{X} \varphi_{\epsilon} \omega_{\varphi_{\epsilon}}^{j} \wedge \omega^{k-j} = 0,
\]

hence

\[
sup_{X} \varphi_{\epsilon} \geq 0 \text{ and } \inf_{X} \varphi_{\epsilon} \leq 0.
\]

**Proof** Recall that the functional \( I_{\omega} \) is defined by

\[
I_{\omega}(\phi) = \int_{0}^{1} \int_{X} \frac{\partial \phi}{\partial t} \omega^{n} \phi_{t} \frac{dt}{n!},
\]

for \( \{\phi_{t}\}_{t \in [0,1]} \) a path in between 0 and \( \phi \). Observe that \( I_{\omega}(\varphi_{t}) = 0 \) along the twisted J-flow. On the other hand, we also have (see for example [31, Lemma 3.2])

\[
I_{\omega}(\varphi) = \frac{1}{(n+1)!} \sum_{k=0}^{n} \int_{X} \varphi \omega^{k} \wedge \omega_{\varphi}^{n-k},
\]

so we obtain (3.11).

\( \square \)

The following lemma uses the positivity condition (2.12).

**Lemma 3.3** Let \( \delta_{0} < 1 \), \( C_{0} \) and \( c \) be positive constants. Assume that \( \mu = (\mu_{1}, \ldots, \mu_{n}) \in \mathbb{R}^{n} \) such that \( 0 \leq \mu_{i} \leq C_{0}, \forall i = 1, \ldots, n \) and \( c - \sum_{j \neq k} \mu_{j} \geq \delta_{0}, \forall k = 1, \ldots, n \).

Assume that \( \lambda = (\lambda_{1}, \ldots, \lambda_{n}) \in \Gamma_{n} \) and \( \tau \in \mathbb{R} \) satisfy \( \lambda_{k} \geq 1 - \delta, \forall k > 0, \tau \geq -\delta \) with \( \delta = \delta_{0}/(4c + 4) \) and

\[
\sum_{k=1}^{n} \frac{\mu_{k}}{\lambda_{k}} + f(\lambda_{1}, \ldots, \lambda_{n}) = (c + \tau),
\]

for some continuous function \( f \) with \( \lim_{\lambda_{j} \to +\infty} f(\lambda) = 0, \forall i = 1, \ldots, n \).

Then there exists a uniform constant \( K > 0 \) depending only on \( \delta_{0}, n, c \), such that

\[
|\tau| \leq K \text{ and } \lambda_{k} \leq K, \forall k = 1, \ldots, n.
\]

**Proof** Observe that

\[
-c + \inf_{S} f \leq \tau \leq \frac{n(c - \delta_{0})}{(1 - \delta)} + \sup_{S} f,
\]

where

\[
S = \{\lambda \in \Gamma_{n} : \lambda_{j} \geq 1 - \delta, \forall j = 1, \ldots, n\}.
\]

Assume by contradiction that there is a sequence of \( \lambda^{(\ell)} \in \Gamma_{n} \) and \( k \in \{1, \ldots, n\} \) such that \( \lim_{\ell \to +\infty} \lambda_{k}^{(\ell)} = +\infty \), then there is \( \ell_{0} > 0 \) such that

\[
(c + \tau) - \frac{\delta_{0}}{4} \leq \sum_{j \neq k}^{n} \frac{\mu_{j}}{\lambda_{j}^{(\ell_{0})}} \leq \frac{1}{1 - \delta} \sum_{j \neq k}^{n} \mu_{j} \leq \frac{c - \delta_{0}}{1 - \delta}.
\]
Therefore \((1 - \delta)[(c + \tau) - \delta_0/4] \leq c - \delta_0\), so
\[
\delta_0 \leq c\delta + (1 - \delta)(\frac{\delta_0}{4} - \tau) \leq c\delta + (1 - \delta)(\frac{\delta_0}{4} + \delta) \leq \frac{\delta_0}{4} + \delta(c + 1) = \frac{\delta_0}{2},
\]
so we get a contradiction. \(\square\)

We now deduce a uniform lower bound for \(\varphi_\epsilon\) following an argument in [19].

**Lemma 3.4** There exists a uniform constant \(C\) such that \(\varphi_\epsilon \geq -C\).

**Proof** Since \(\partial_t \varphi_\epsilon\) is uniformly bounded for all time by the constant depending only on \(\varphi_0\), we can consider \(t \geq \delta\). For any \(T > \delta\), for each \(t\) we note

\[
L_\epsilon = \min_{X \times [0, T]} \varphi_\epsilon = \varphi_\epsilon(x_\epsilon, t_\epsilon).
\]

Let \((z_1, \ldots, z_n)\) be a local coordinates centered at \(x_\epsilon\), and \(U = \{z : |z| < 1\}\) such that \(A^{-1} \omega_E \leq \omega \leq A \omega_E\) where \(\omega_E\) is the Euclidean metric and \(A\) is some uniform constant. Denote \(U_\epsilon = U \times \{t : -\delta \leq 2(t - t_\epsilon) < \delta\}\), and define

\[
w_\epsilon = \varphi_\epsilon + \frac{\delta^2}{4} |z|^2 + |t - t_\epsilon|^2,
\]

where \(\delta > 0\).

Then \(w_\epsilon\) attains its minimum on \(U_\epsilon\) at \((z_\epsilon, t_\epsilon)\) and \(w_\epsilon \geq \min_{U_\epsilon} w_\epsilon + \frac{1}{\delta^2}\) on the boundary of \(U_\epsilon\). Now by the ABP inequality due to Tso [28, Proposition 2.1], there exists a constant \(C_n = C(n) > 0\) so that

\[
C_n \delta^{4n+2} \leq \int_S (-\partial_t w_\epsilon) \det((w_\epsilon)_{ij}) dx dt,
\]

where

\[
S := \left\{(x, t) \in U : w_\epsilon(x, t) \leq w_\epsilon(z_0, t_0) + \frac{\delta^2}{4}, |D_x w_\epsilon(x, t)| < \frac{\delta^2}{8}, \text{ and } w_\epsilon(y, s) \geq w_\epsilon(x, t) + D_x w_\epsilon(x, t) (y - x), \forall y \in U, s \leq t\right\}.
\]

Since on \(S\), we have \(D^2 w_\epsilon \geq 0\) and \(\partial_t w_\epsilon \leq 0\), then \(\lambda_\omega(ddc \varphi_\epsilon) \geq -\delta \mathbf{I}\), and \(0 \leq -\partial_t w \leq -\partial_t u + \delta\), hence

\[
\lambda_\omega(\omega + ddc u) \geq (1 - \delta) \mathbf{I} \text{ and } -\partial_t \varphi_\epsilon \geq -\delta. \tag{3.14}
\]

Since \(\varphi_\epsilon\) satisfies the equation (3.5), we have

\[
\sum_{j=1}^n \frac{\mu_k}{\lambda_k} + \frac{\beta}{\lambda_1 \cdots \lambda_n} = (c_\epsilon + \partial_t \varphi_\epsilon), \tag{3.15}
\]

where \(\lambda = \lambda_\omega(\omega + ddc \varphi_\epsilon)\) and \(\mu_1, \ldots, \mu_n\) are diagonal entries of \(\theta_\epsilon\) with respect to normal coordinates of \(\omega\). It follows from Lemma 3.3 that there exists a uniform constant \(C\) such that

\[
|\dot{w}_\epsilon| + \det(D^2 (w_\epsilon)_{jk}) \leq C,
\]

therefore

\[
C_n \delta^{4n+2} \leq C' \int_S dx dt
\]

for a uniform constant \(C' > 0\).
Now on $S$ we have $w_\epsilon \leq L_\epsilon + \frac{\delta^2}{4}$. Since we can assume that $|L_\epsilon| > \delta^2$ so that $L_\epsilon + \frac{\delta^2}{4} < 0$ and $|w_\epsilon| \geq |L_\epsilon|/2$, then we get

$$C_n \delta^{4n+2} \leq C' \int_S dx dt \leq \frac{2C'}{|L_\epsilon|} \int_S |w_\epsilon| dx dt \leq \frac{2C'}{|L_\epsilon|} \int |w_\epsilon| dx dt. \quad (3.16)$$

Next we have $|w_\epsilon| = -w_\epsilon = -\varphi_\epsilon - \frac{\delta^2}{4}|z|^2 - (t - t_0)^2 \leq -\varphi_\epsilon \leq -\varphi_\epsilon + \sup_X \varphi_\epsilon$, since $\sup_X \varphi_\epsilon \geq 0$ by (3.12).

Since $\varphi_\epsilon$ is $\omega$-psh, we have $\|\varphi_\epsilon - \sup_X \varphi_\epsilon\|_{L^1(X, \omega)} \leq C$ for $C$ depending only on $(X, \omega)$ (cf. [13, Proposition 8.5]). Combining this with (3.16) yields

$$C_n \delta^{4n+2} \leq \frac{2C'}{|L_\epsilon|} \int_{|t| < \frac{\delta}{2}} \|\varphi_\epsilon - \sup_X \varphi_\epsilon\|_{L^1(X, \omega)} dt \leq C'' \delta^2 |L_\epsilon|,$$

for a uniform constant $C''$. This implies $\varphi_\epsilon \geq -C$ for a uniform constant $C$.

The uniform bound now follows from a Harnack type inequality for $\varphi_\epsilon$.

**Lemma 3.5** For any $T > 0$ There exists a uniform constant $C$ such that

$$\|\varphi_\epsilon\|_{C^0(X \times [0, T])} \leq C.$$

**Proof** Along the twisted J-flow we have $J_{\omega, \beta}$ is decreasing, hence combining with Lemma 3.2 implies

$$\sum_{j=0}^{n-1} \int_X \varphi_\epsilon \omega^j \wedge \omega^{n-1-j} \wedge \theta_\epsilon \leq C_0,$$

where $C_0$ is a uniform constant. Therefore

$$\int_X \varphi_\epsilon \omega^{n-1} \wedge \theta_\epsilon \leq C_0 - \sum_{j=1}^{n-1} \int_X \varphi_\epsilon \omega^j \wedge \omega^{n-1-j} \wedge \theta_\epsilon$$

$$= C_0 - \sum_{j=1}^{n-1} \int_X (\varphi_\epsilon - \inf_X \varphi_\epsilon) \omega^j \wedge \omega^{n-1-j} \wedge \theta_\epsilon - (n-1) \inf_X \varphi_\epsilon \int_X \omega^{n-1} \wedge \theta_\epsilon$$

$$\leq C_0 - C \inf_X \varphi_\epsilon,$$

where $C = \max_{\epsilon \in [0, \epsilon_0]} (n-1) \int_X \omega^{n-1} \wedge \theta_\epsilon$. So we have

$$\int_X \varphi_\epsilon \omega^{n-1} \wedge \theta \leq \int_X \varphi_\epsilon \omega^{n-1} \wedge \theta \leq C_0 - C \inf_X \varphi_\epsilon.$$

Since $\varphi_\epsilon$ is $\omega$-psh, and $\mu := \omega^{n-1} \wedge \theta / \int_X \omega^{n-1} \wedge \theta$ is a probability measure, then by compactness properties of $\omega$-psh function (cf. [13, Prop. 8.5]) there is a constant $C_\mu > 0$ depending on $\mu$ such that

$$\sup_X \varphi_\epsilon \leq \int_X \varphi_\epsilon \mu + C_\mu,$$

hence

$$\sup_X \varphi_\epsilon \leq C_\mu + \frac{C_0}{\int_X \omega^{n-1} \wedge \theta} - \frac{C}{\int_X \omega^{n-1} \wedge \theta} \inf \varphi_\epsilon.$$

Now combining with Lemma 3.4, we obtain the desired estimate. \qed
3.4 Alternative proof of $C^0$ estimate

We give here a second proof of the $C^0$ estimate to $\varphi_\epsilon$ by proving a uniform $C^0$ estimate for the twisted inverse $\sigma_{n-1}$ equation where the uniform bound does not depend on the norms with respect to $\theta$.

Proposition 3.6 Let $\psi, \rho \in C^\infty(X)$ be smooth functions on $X$ with $\psi > 0$. Assume that there is $\hat{\omega} = \omega + i \partial \bar{\partial} u \in [\omega]$ such that

$$(\psi \hat{\omega} - (n - 1)\theta) \wedge \hat{\omega}^{n-2} > 0. \tag{3.17}$$

Let $u$ be a smooth solution of the equation

$$\rho \omega^n + n \omega_u^{n-1} \wedge \theta = \psi \omega_u^n, \quad \sup_X (u - u) = 0, \tag{3.18}$$

where $\omega$ and $\theta$ are Kähler metrics. There exists a constant $C > 0$ which only depends on $X, \omega, \|\theta\|_{C^0(X,\omega)}, \|\psi\|_{L^\infty}, \|\rho\|_{L^\infty}$ such that

$$\|u\|_{L^\infty(X)} \leq C.$$

Proof The proof uses an ABP estimate, as in [25]. Our $C^0$ bound only depends on the norms with respect to $\omega$ while it depends on the norms with respect to $\theta$ in [25]. We can assume with out loss of generality that $u = 0$ by changing $\omega$. Then max$_X u = 0$, it suffices to get a lower bound for $L = \min_M u = u(x_0)$.

Let $(z_1, \ldots, z_n)$ be a local coordinates centered at $x_0$, $U = \{z : |z| \leq 1\}$, such that $A^{-1} \omega_E \leq \omega \leq A \omega_E$ where $\omega_E$ is the Euclidean metric and $A$ is some uniform constant. Let $w = u + \delta|z|^2$. Then inf$_U w = L = w(0)$ and $w(z) \geq L + \delta$ for $z \in \partial U$. It follows from the ABP maximum principle [11, Lemma 9.2] that there exists a constant $C = C(n)$ so that

$$C(n) \delta^{2n} \leq \int_S \det(D^2 w),$$

where $S$ is defined by

$$S = \left\{ x \in U : |Dw| < \frac{\delta}{2} \text{ and } w(y) \geq w(x) + Dw(x). (y - x), \forall y \in U \right\}. \tag{3.19}$$

Since on $S$ we have $D^2 w \geq 0$, we infer $(u_{jk}) \geq -\delta I_n$, hence $\lambda_\omega(\omega + i \partial \bar{\partial} u) \geq (1 - \delta)I$. Now $u$ satisfies the equation (3.18), hence

$$\frac{\rho}{\lambda_1 \cdots \lambda_n} + \sum_{k=1}^n \frac{\mu_k}{\lambda_k} = \psi. \tag{3.20}$$

As in Lemma 3.4, the condition (3.17) implies that

$$\psi - \sum_{j \neq k} \frac{\mu_k}{\lambda_k} \geq \delta_0.$$

It follows from Lemma 3.3 that there exists a uniform constant $C$ such that $\det(D^2 w_{jk}) \leq C$, hence $C(n)^2 \delta^{2n} \leq C \int_S dx$. 

\textcopyright Springer
Now on $S$ we have $w \leq L + \delta$. Since we can assume that $|L| > 2\delta$ so that $L + \delta < 0$ and $w \leq L/2$. This implies

$$C(n)\delta^{2n} \leq C_1 \int_S dx \leq \frac{2C_1}{|L|} \int_S |w|dx. \tag{3.21}$$

Since $u$ is $\omega$-psh with $\sup_X u = 0$, there exists a uniform constant $C_2 = C(X, \omega)$ such that $\|u\|_{L^1(X)} \leq C_2$, hence $\|w\|_{L^1(U)} \leq C_3$. Combining with (3.21) implies

$$C(n)\delta^{2n} \leq \frac{2C_1}{|L|} \|w\|_{L^1(U)} \leq C_4. \tag{3.22}$$

Therefore $u \geq -C$ for some uniform constant $C$. We now ready to give another proof of Lemma 3.5.

**Proof** We use an argument similar to one given in [23]. Suppose that $u_\epsilon$ is the solution to the equation

$$\beta \omega^n + n\omega_{u_\epsilon}^{n-1} \wedge \theta_\epsilon = c_\epsilon \omega_{u_\epsilon}^n, \tag{3.23}$$

where the existence of unique solution proved in [24]. Now set $\psi_\epsilon = \varphi_\epsilon - u_\epsilon$ and compute

$$\frac{d\psi_\epsilon}{dt} = \frac{d\varphi_\epsilon}{dt} = \frac{n\omega_{u_\epsilon}^{n-1} \wedge \theta_\epsilon}{\omega_{u_\epsilon}^n} + \beta \frac{\omega^n}{\omega_{u_\epsilon}^n} - \frac{n\omega_{\psi_\epsilon}^{n-1} \wedge \theta_\epsilon}{\omega_{\psi_\epsilon}^n} - \beta \frac{\omega^n}{\omega_{\psi_\epsilon}^n} \tag{3.24}$$

where $\omega_s := s\omega_{u_\epsilon} + (1-s)\omega_{\psi_\epsilon}$. We define

$$\eta_s^{k\bar{l}} = \omega_s^{k\bar{j}} \omega_s^{i\bar{j}} \theta_\epsilon^{i\bar{j}} + \beta \frac{\omega^n}{\omega_s^n} \omega_s^{k\bar{l}},$$

which is positive definite. Then we have

$$\frac{d\psi_\epsilon}{dt} = \int_0^1 \frac{d}{ds} \left( \frac{n\omega_s^{n-1} \wedge \theta}{\omega_s^n} + \beta \frac{\omega^n}{\omega_s^n} \right) \omega_s^{k\bar{l}} ds \tag{3.25}$$

$$= \int_0^1 \eta_s^{k\bar{l}} (\omega_{\psi_\epsilon} - \omega_{u_\epsilon}) ds \tag{3.26}$$

$$= \left( \int_0^1 \eta_s^{k\bar{l}} ds \right) \partial_k \partial_{\bar{l}} \psi_\epsilon. \tag{3.27}$$

Since $\left( \int_0^1 \eta_s^{k\bar{l}} ds \right)$ is positive definite tensor, the maximum principle implies that $\psi_\epsilon$ is uniformly bounded by $\sup_X |\psi_\epsilon|$ at $t = 0$. Combining with Lemma 3.6, we have $\psi_\epsilon$ is uniformly bounded. Therefore $\varphi_\epsilon$ is uniformly bounded independent of $\epsilon$. We now assume that $\theta$ also satisfies the condition (1.3): there is an effective divisor $D$ on $X$, such that

$$\theta \geq C_0 |s|^{2y}_h \omega, \text{ and } \theta - \epsilon_0 R_h \geq C_0 \omega. \tag{3.28}$$
for some constants $\gamma$, $\varepsilon_0 > 0$, where $h$ is a hermitian metric on $O_X(D)$ and $s$ is a holomorphic section of $O_X(D)$. Then we have the following $C^2$ estimate for $\varphi_\varepsilon$.

**Lemma 3.7** There exist uniform positive constants $C$, $\alpha$, independent of $\varepsilon$, such that

$$\text{tr}_{\omega}\omega \varphi_\varepsilon \leq \frac{C}{|s|^n_h}.$$  

**Proof** To simplify the notations, we drop all subscripts $\varepsilon$, so we write $\varphi$ (resp. $\theta$) instead of $\varphi_\varepsilon$ (resp. $\theta_\varepsilon$). The constant $C > 0$ below will be independent of $\varepsilon$.

We use Tsuji’s trick [29]: set $\tilde{\varphi} = \varphi - \varepsilon \rho$, where $\rho = \log |s|^2_h$ for some small constant $\varepsilon$ that will be chosen below. Since $\varphi$ is uniformly bounded by Lemma 3.5, $\tilde{\varphi}$ is uniformly bounded from below and tends to $+\infty$ along $D$. We set

$$H = \log \text{tr}_{\omega}\omega - A\tilde{\varphi},$$

where $A > 0$ will be chosen hereafter. It is straightforward that $H$ achieves a maximum at each time $t$ away from $D$. We now prove that $H$ is bounded from above. We will use the maximum principle and follow the computation for the (twisted) J-flow in [23, 24, 33, 34] to simplify $(\partial_t - \mathcal{L})H$, where

$$\mathcal{L} = \left( \omega^j_{\bar{k}}\omega^i_{\bar{l}}\partial_{\bar{k}i} + \beta \frac{\omega^n}{\omega^\varphi}\omega^j_{\bar{\varphi}} \right) \partial_i \partial_{\bar{j}},$$

here we write $\omega = \sum_{k, \bar{k}} i\omega_{k\bar{k}} dz^k \wedge d\bar{z}^\ell$ in local coordinates and $(\omega^{k\bar{l}}) = (\omega_{k\bar{l}})^{-1}$.

Set $\Lambda = \text{tr}_{\omega}\omega_{\varphi}$. Using the flow equation we have

$$\partial_t \Lambda = \omega^{i\bar{j}} \partial_i \omega_{\varphi, i\bar{j}} = \omega^{i\bar{j}} \tilde{\varphi}_{i\bar{j}}$$  

(3.29)

$$ = \omega^{i\bar{j}} \left[ c_{\bar{\varphi}} - \omega^{p\bar{q}}\omega_{\varphi, p\bar{q}} - \beta \frac{\omega^n}{\omega^\varphi} \right]_{i\bar{j}}$$  

(3.30)

$$ = \omega^{i\bar{j}} \left[ \omega^{\bar{q}\bar{\varphi}}\omega_{\varphi, \bar{p}\bar{q}}(\omega_{\varphi, \bar{r}\bar{s}})_{i\bar{j}} - \omega^{\bar{q}\bar{\varphi}}\omega^{p\bar{q}}\omega_{\varphi}^{\bar{a}\bar{b}}(\omega_{\varphi, \bar{a}\bar{b}})_{j}(\omega_{\varphi, \bar{r}\bar{s}})_{i} \right. - \omega^{\bar{p}\bar{a}}\omega_{\varphi}^{\bar{a}\bar{q}}\omega_{\varphi}^{\bar{p}\bar{q}}(\omega_{\varphi, \bar{a}\bar{b}})_{j}(\omega_{\varphi, \bar{r}\bar{s}})_{i} \theta_{p\bar{q}} + 2 Re(\omega^{i\bar{j}}\omega^{\bar{p}\bar{q}}\omega_{\varphi}^{\bar{p}\bar{q}}(\omega_{\varphi, k\bar{l}})(\theta_{p\bar{q}}))_{j} \right)$$  

(3.31)

$$ + \omega^{i\bar{j}}\omega_{\varphi}^{\bar{q}\bar{p}}\theta_{p\bar{q}i\bar{j}} - \beta \omega^{i\bar{j}} \left( \frac{\omega^n}{\omega^\varphi} \right)_{i\bar{j}}.$$  

(3.32)

(3.33)

In the normal coordinates of $\omega$ we have

$$\left( \begin{array}{c} \omega^n \\ \omega^\varphi \end{array} \right)_{i\bar{j}} = -\omega^{k\bar{l}} R_{k\bar{l}ij}(\omega) \frac{\omega^n}{\omega^\varphi} + \omega^{k\bar{l}}\omega_{\varphi}^{p\bar{q}}\omega_{\varphi}^{k\bar{l}}(\omega_{\varphi, p\bar{q}})_{i}(\omega_{\varphi, k\bar{l}})_{j}$$

$$ + \frac{\omega^n}{\omega^\varphi} \omega_{\varphi}^{k\bar{l}}(\omega_{\varphi, p\bar{q}})_{j}(\omega_{\varphi, k\bar{l}})_{i} - \frac{\omega^n}{\omega^\varphi} \omega_{\varphi}^{k\bar{l}}(\omega_{\varphi, k\bar{l}})_{i} \partial_{\bar{j}},$$

and

$$\Lambda_{k\bar{l}} = R_{k\bar{l}ij}(\omega)\omega_{\varphi, p\bar{q}} + \omega_{\varphi}^{p\bar{q}}(\omega_{\varphi, p\bar{q}})_{k\bar{l}}.$$  

Now, we have

$$\left( \frac{\partial}{\partial t} - \mathcal{L} \right) H = \frac{1}{\Lambda} \partial_{t}\Lambda - \omega^{k\bar{l}}\omega^i_{\bar{\varphi}}\partial_{\bar{k}i} \left( \frac{\Lambda_{k\bar{l}}}{\Lambda} - \frac{\Lambda_{k\bar{l}}}{\Lambda^2} \right)$$  

(3.34)
\[
\beta \frac{\omega^n}{\omega^\bar{\psi}} \frac{k^\ell}{\Lambda} \left( \frac{\Lambda_{k \ell}}{\Lambda} - \frac{\Lambda_k \Lambda_\ell}{\Lambda^2} \right) \tag{3.35}
\]
\[
-A(\partial_t - \mathcal{L}) \bar{\psi}. \tag{3.36}
\]

The terms in (3.34) and (3.35) give

\[
\frac{\partial_t \Lambda - \frac{\partial^2}{\partial u^2} \theta_{ij} \Lambda_{k \ell}}{\Lambda} + \frac{\omega^\bar{\psi}_j \omega_i^\varphi \theta_{ij} \Lambda_{k \ell}}{\Lambda^2} \tag{3.37}
\]
\[
-\beta \frac{\omega^n}{\omega^\bar{\psi}} \frac{k^\ell}{\Lambda} \Lambda_{k \ell} \tag{3.38}
\]

Developing (3.37) using previous calculations, we get

\[
\frac{1}{\Lambda} \left\{ \omega^\bar{\psi}_j \omega^\varphi_i \omega^\varphi (\omega \theta_r) \right\} - \frac{\partial^2}{\partial u^2} \theta_{ij} \Lambda_{k \ell} + \omega^\bar{\psi}_j \omega_i^\varphi \theta_{ij} \Lambda_{k \ell} \right\}
\]
\[
-\beta \frac{\omega^n}{\omega^\bar{\psi}} \frac{k^\ell}{\Lambda} \Lambda_{k \ell} \tag{3.39}
\]

Developing (3.38) we obtain

\[
-\beta \frac{\omega^n}{\omega^\bar{\psi}} \frac{k^\ell}{\Lambda} \Lambda_{k \ell} \tag{3.40}
\]

Now we have two following inequalities due to Weinkove [30] and Zheng [33, Lemma 7]:

\[
[\omega^\bar{\psi}_j \omega^\varphi_i \omega^\varphi (\omega \theta_r)] \geq \omega^\bar{\psi}_j \omega_i^\varphi \theta_{ij} \Lambda_{k \ell} \tag{3.41}
\]
\[
[\omega^\bar{\psi}_j \omega^\varphi_i \omega^\varphi (\omega \theta_r)] \geq \omega^\bar{\psi}_j \omega_i^\varphi \theta_{ij} \Lambda_{k \ell} \tag{3.42}
\]

Therefore the last term in (3.43) is dominated by the first term in (3.40) and the last term in (3.44) is dominated by the first term in (3.42). In addition, the last term in (3.42) is canceled by the second term in (3.44) and the second term in (3.43) is canceled by the first term in (3.39).

We now follow [23] to deal with the term involving one derivative of \( \omega^\varphi \) and \( \theta \). We have

\[
G = \omega^\bar{\psi}_j \omega_i^\varphi \omega^\varphi (\omega \theta_r) \geq 0.
\]

where \( \omega^k_i := \omega^\bar{\psi}_k^i \omega^\varphi \theta_r \) and

\[
K_{i \ell} = (\omega \theta_r)_{ij} - \theta_{ij} \omega^k_i \theta_{ij}.
\]

By a direct computation, we get

\[
G = \omega^\bar{\psi}_j \omega_i^\varphi \omega^\varphi (\omega \theta_r) \geq 0.
\]
We infer that

\[ -\omega^{ij} \omega^{kl} \omega_{i}^{pq} \theta_{ij}(\omega_{q,kl})(\omega_{p,q}) + 2 \Re(\omega^{ij} \omega^{kl} \omega_{i}^{pq} \theta_{ij}(\theta_{p,k})) \leq \omega^{ij} \omega^{kl} \theta_{ij} \omega_{p,k}^{ij}, \]

hence the second term in (3.40) minus the second term in (3.39) is dominated by \( \omega^{ij} \omega^{kl} \theta_{ij} \omega_{p,k}^{ij} \).

Finally, we infer that the terms in (3.34) and (3.35) are dominated by

\[ \frac{1}{\Lambda} \left\{ -\omega^{pq} \theta_{pq}^{ij} \omega^{ij} - \beta \omega^{ij} \omega^{kl} \kappa_{ij}(\omega) \omega_{p,q} + \omega^{ij} \omega^{kl} \theta_{ij} \omega_{p,k}^{ij} \right\}. \]

Therefore at a maximum point \((t_0, x_0)\) of \(H\),

\[ 0 \leq \left( \frac{\partial}{\partial t} - L \right) H \leq \frac{1}{\tr_{t_0} \omega_p} \omega^p \omega^q (-\theta_{pq}^{ij} \omega^{ij}) \leq \frac{C \tr_{t_0} \omega}{\tr_{t_0} \omega_p} \leq \frac{1}{|s|^{2\gamma}} \frac{C}{\tr_{t_0} \omega_p}. \]

and

\[ \frac{1}{\tr_{t_0} \omega_p} \omega^{ij} \omega^{kl} \omega^{pq} \theta_{ij} \omega_{p,k}^{ij} \leq \frac{C \tr_{t_0} \omega \tr_{t_0} \theta}{\tr_{t_0} \omega_p} \leq \frac{C \tr_{t_0} \omega \tr_{t_0} \omega}{\tr_{t_0} \omega_p} \leq \frac{1}{|s|^{4\gamma}} \frac{C}{\tr_{t_0} \omega_p}. \]

At \((t_0, x_0)\), we can assume that

\[ \frac{1}{|s|^{4\gamma}} \frac{C}{\tr_{t_0} \omega_p} \leq 1. \]

Otherwise \(\tr_{t_0} \omega_p \leq \frac{C}{|s|^{4\gamma}}\) at \((t_0, x_0)\), then by choosing \(A = 4 \gamma / \delta\) we get the desired estimate.

We also have

\[ \beta \frac{1}{\tr_{t_0} \omega_p} \omega^{ij} \omega^{kl} \kappa_{ij}(\omega) \omega_{p,k}^{ij} \leq \beta \frac{C}{\tr_{t_0} \omega_p} \frac{\omega_{p,k}^{ij}}{\tr_{t_0} \omega_p} = \frac{C}{\tr_{t_0} \omega_p} (c_{\beta} - \hat{\phi} - \tr_{t_0} \omega_p) \leq \frac{C}{\tr_{t_0} \omega_p}, \]

since \(\hat{\phi}\) is uniformly bounded by Lemma 3.1. At \((t_0, x_0)\), we can also assume that

\[ \frac{C}{\tr_{t_0} \omega_p} \leq 1, \]

otherwise we have desired estimate.

The first term in the second line of (3.47) is controlled as

\[ \frac{1}{\tr_{t_0} \omega_p} \omega^{ij} \omega^{kl} \kappa_{ij}(\omega) \omega_{p,k}^{ij} \leq \frac{C \omega^{ij} \omega^{kl} \omega^{pq} \tr_{t_0} \omega_p}{\tr_{t_0} \omega_p} = \frac{C \omega^{ij} \omega^{kl} \omega_{p,k}^{ij} \tr_{t_0} \omega_p}{\tr_{t_0} \omega_p}. \]
For the last term in the second line of (3.47), we have
\[
- \frac{1}{\text{tr}_{\omega_\psi} \omega_\psi} \beta \frac{\omega_\psi}{\omega_\psi^n} R^p_q \omega_\psi (\omega) \omega_\psi, p_q \leq C \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega_\psi \omega_\psi \omega_\psi = C \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega_\psi. 
\] (3.49)

Thus at a maximum point \((t_0, x_0)\) of \(H\),
\[
0 \leq \left( \frac{\partial}{\partial t} - \mathcal{L} \right) H \leq C \left( \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega_\psi + \omega_k^j \omega_i^j \theta_{i j} \omega_k^\ell \right) + 3 - A \left( \frac{\partial}{\partial t} - \mathcal{L} \right) \tilde{\varphi}. 
\] (3.50)

By the condition (1.3) there exists \(\varepsilon'' > 0\) satisfies \(\omega - \varepsilon'' R_h \geq C_0 \omega\) for some \(C_0 > 0\). Hence we can write \(\varepsilon = \varepsilon' \varepsilon''\). Denote by \(\tilde{\omega}\) a positive \((1, 1)\)-form defined by \(\tilde{\omega}^j_\ell = \omega^j_\psi \omega^\ell_\psi \theta_{k \ell}\), then we get
\[
\mathcal{L}(\varepsilon \rho) = \Delta \tilde{\omega}(\varepsilon \rho) + \beta \frac{\omega_\psi}{\omega_\psi^n} \Delta \omega_\psi (\varepsilon \rho), \quad (\Delta_\omega := \text{tr}_{\omega_\psi} \partial \tilde{\omega}) 
\] (3.51)
\[
= \varepsilon' \text{tr}_{\tilde{\omega}} (\omega - \varepsilon'' R_h) + \varepsilon' \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} (\omega - \varepsilon'' R_h) - \varepsilon' (\text{tr}_{\tilde{\omega}} \omega + \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega) \] (3.52)
\[
\geq \varepsilon' C_0 \text{tr}_{\tilde{\omega}} \omega + \varepsilon' C_0 \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega - \varepsilon' (\text{tr}_{\tilde{\omega}} \omega + \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega). \] (3.53)

Combining with
\[
(\partial_t - \mathcal{L}) \varphi = -c \beta - 2 \text{tr}_{\omega_\psi} \theta + \omega_k^j \omega_i^j_\psi \theta_{i j} \omega_k^\ell - \beta (n + 1) \frac{\omega_\psi}{\omega_\psi^n} \frac{n}{\omega_\psi^k} \frac{n}{\omega_\psi^k} \omega_k^\ell \] (3.54)
yields
\[
- A(\partial_t - \mathcal{L}) \tilde{\varphi} = -A[-c \beta - 2 \text{tr}_{\omega_\psi} \theta + \omega_k^j \omega_i^j_\psi \theta_{i j} \omega_k^\ell - \beta (n + 1) \frac{\omega_\psi}{\omega_\psi^n} \frac{n}{\omega_\psi^k} \frac{n}{\omega_\psi^k} \omega_k^\ell] \]
\[
- A[\varepsilon' C_0 \text{tr}_{\tilde{\omega}} \omega + \varepsilon' C_0 \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega - \varepsilon' (\text{tr}_{\tilde{\omega}} \omega + \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega)] \] (3.55)

Choose \(A\) large such that
\[
A \varepsilon' C_0 \geq C, 
\]
so that
\[
C \beta \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega - A \varepsilon' C_0 \frac{\omega_\psi}{\omega_\psi^n} \text{tr}_{\omega_\psi} \omega \leq 0 
\] and
\[
C \omega_k^j \omega_i^j_\psi \theta_{i j} \omega_k^\ell - A \varepsilon' C_0 \text{tr}_{\tilde{\omega}} \omega \leq 0, 
\]
where \(\omega_k^\ell = \omega_k^j \omega_i^j_\psi \theta_{i j}\) and \(C\) is the constant in (3.50). Then we infer from (3.50) and (3.55) that
\[
0 \leq C_1 - A \left\{ c \beta - 2 \text{tr}_{\omega_\psi} \theta + \omega_k^j \omega_i^j_\psi \theta_{i j} \omega_k^\ell - \beta (n + 1) \frac{\omega_\psi}{\omega_\psi^n} - \varepsilon' \text{tr}_{\tilde{\omega}} \omega \right\} \]
\[
= C_1 - A \left\{ c \beta - 2 \text{tr}_{\omega_\psi} \theta + (1 - \varepsilon') \text{tr}_{\tilde{\omega}} \omega - \beta (n + 1) \frac{\omega_\psi}{\omega_\psi^n} \right\}. \] (3.56)

We now consider two cases.
**Case 1:** assume that

\[
c_\beta - 2\text{tr}_{\omega_\psi} \theta + (1 - \epsilon') \text{tr}_{\omega_\psi} \omega \leq \delta_1 = \frac{\delta_0}{4}
\]  

(3.57)

at a maximum point of \(H\), where \(\delta_0\) is the constant in the condition (3.3).

Choosing normal coordinates for \(\omega\) such that \(\omega_\psi\) is diagonal with entries \(\lambda_1, \ldots, \lambda_n\) and \(\theta\) may not be diagonal but we denote its positive diagonal entries by \(\mu_1, \ldots, \mu_n\). Then the above inequality becomes

\[
c_\beta - 2 \sum_{j=1}^{n} \frac{\mu_j}{\lambda_j} + \sum_{j=1}^{n} \left(\frac{1 - \epsilon'}{\lambda_j^2}\right) \mu_j \leq \delta_1.
\]

Now by condition (2.12), we have

\[
(c_\beta \omega - (n - 1)\theta) \wedge \omega^{n-2} \wedge \beta_k > \delta_0 \omega^{n-1} \wedge \beta_k,
\]

where \(\beta_k = i dz^k \wedge \bar{z}^k\) In our coordinates, we get

\[
c_\beta(n - 1)! \beta_1 \wedge \ldots \wedge \beta_n - (n - 1)! \sum_{j \neq k} \mu_j \beta_1 \wedge \ldots \wedge \beta_n > \delta_0(n - 1)! \beta_1 \wedge \ldots \wedge \beta_n,
\]

so

\[
c_\beta - \sum_{j \neq k} \mu_j > \delta_0, \quad \forall k = 1, \ldots, n.
\]

Therefore for any \(k = 0, \ldots, n\) we have

\[
\delta_1 \geq c_\beta + \sum_{j \neq k} \mu_j \left(\sqrt{1 - \epsilon'} - \frac{1}{\sqrt{1 - \epsilon'}}\right)^2 - \frac{1}{1 - \epsilon'} \sum_{j \neq k} \mu_j + \frac{\mu_k}{\lambda_k^2} (1 - \epsilon') - 2 \frac{\mu_k}{\lambda_k}
\]

\[
\geq c_\beta - \frac{1}{1 - \epsilon'} (c_\beta - \delta_0) - 2 \frac{\mu_k}{\lambda_k}.
\]

Hence

\[
2 \frac{\mu_k}{\lambda_k} \geq -\delta_1 + \frac{\delta_0 - c_\beta \epsilon'}{1 - \epsilon'} \geq \frac{\delta_0}{10}, \quad \forall k = 1, \ldots, n,
\]

(3.58)

where we choose \(\epsilon\) small enough such that \(c_\beta \epsilon' \leq \delta_0/4\). This implies that \(\text{tr}_{\theta_\psi} \omega_\psi \leq C\), hence

\[
\text{tr}_{\omega_\psi} \omega_\psi \leq C.
\]

at a maximum point of \(H\) since \(\theta \leq C' \omega\). Therefore \(H\) is bounded from above, and we get the desired estimate for \(\text{tr}_{\omega_\psi} \omega_\psi\).

**Case 2:** assume that at a maximum point of \(H\)

\[
c - 2\text{tr}_{\omega_\psi} \theta + (1 - \epsilon') \text{tr}_{\omega_\psi} \omega > \delta_1,
\]

(3.60)

then (3.56) yields

\[
\beta(n + 1) \frac{\omega^n}{\omega_\psi^n} \geq \delta_1 - \frac{C_1}{A}.
\]

Choosing \(A\) large enough we get

\[
\beta(n + 1) \frac{\omega^n}{\omega_\psi^n} \geq \delta_1/2.
\]
By the equation we have $\text{tr}_{\omega \psi} \theta \leq c_\beta - \psi_0 - \delta_1/2 \leq C$. Therefore

$$\text{tr}_{\omega \phi} \phi \leq \frac{\omega^n}{\omega^{n-1}} (\text{tr}_{\omega \phi} \omega)^{n-1} \leq C \frac{1}{|s|^{2(n-1)\gamma}},$$

(3.61)

(3.62)

here we use the assumption $\theta \geq |s|^{2\gamma} \omega$.

This implies that $H \leq H(t_0, x_0) \leq C - (n-1)\gamma \log |s|^{2} \gamma (x_0) + A\delta \log |s|^{2} \gamma (x_0) \leq C$, since we can choose $A$ is big enough such that $A\delta > (n-1)\gamma$. We thus get the inequality as required.

Higher order estimates follow by standard local parabolic theory. We summarize this section by the following proposition.

**Proposition 3.8** There exists a uniform constant $C$ such that for all $t$ and $\epsilon \in (0, \epsilon_0)$,

$$\|\phi_\epsilon\|_{L^\infty(X)} \leq C$$

and

$$\|\dot{\phi}_\epsilon\|_{L^\infty(X)} \leq C,$$

(3.63)

Moreover, for any compact set $K \subset X \setminus D$ and $k \geq 0$, the exists $C_{K,k} > 0$ such that

$$\|\phi_\epsilon\|_{C^k(K,\omega)} \leq C_{K,k}.$$

(3.64)

**4 Proof of the main result**

The rest of proof of Theorem 1.1 is similar to that of [23, Theorem 1.1].

**Proof of Theorem 1.1** By the estimates in Sect. 3 (Proposition 3.8), we can find a sequence $\epsilon_j \to 0$ such that $\phi_{\epsilon_j}$ converges to $\phi$ in $C^\infty$ on compact sets of $(X \setminus D) \times [0, +\infty)$. Therefore $\phi$ satisfies the degenerate twisted J-flow. Since $\omega + i \partial \bar{\partial} \psi > 0$ on $X \setminus D$ and $\sup_{X \setminus D} |\psi| \leq C$ by Lemma 3.5, we can extend $\phi$ uniquely to bounded $\omega$-psh function on $X$, still denoted by $\phi$.

The $J$-functional

$$J_{\omega, \beta}^\theta (\phi) = \frac{1}{n!} \int_X \phi \sum_{k=0}^{n-1} \frac{\omega^k \wedge \omega_{\psi}^{n-k}}{(n+1)!} \int_X \dot{\phi} \sum_{k=0}^{n-1} \omega^k \wedge \omega_{\psi}^{n-k} - \frac{1}{n+1} \int_X \theta \phi \sum_{k=0}^{n-1} \omega^k \wedge \omega_{\psi}^{n-k},$$

(4.1)

where

$$\theta = \frac{n[\theta] \cdot [\omega]^{n-1}}{[\omega]^n},$$

is well defined on $PSH(X, \omega) \cap L^\infty(X)$ with $J_{\omega, \beta}^\theta (\phi + C) = J_{\omega, \beta}^\theta (\phi)$, $\forall \phi \in PSH(X, \omega) \cap L^\infty(X)$ and for any constant $C$. The uniform $L^\infty$ bound on $\phi$ yields

$$J_{\omega, \beta}^\theta (\phi(t)) \geq -C,$$

(4.2)

for a uniform constant $C$ independent of $t$. We also have
where \( c_\beta = \theta + \beta \). Therefore there exists a constant \( C \) such that

\[
\int_{X \setminus D} (\dot{\phi}(t))^2 \frac{\alpha_n^q}{n!} \leq C.
\]

(4.4)

Then by an argument by contradiction in [8, Proof of Theorem 1.1] shows that \( \dot{\phi}(t) \to 0 \) in \( C^\infty_{\text{loc}}(X \setminus D) \). Since we have uniform \( C^\infty \) bounds for \( \phi \) on compact subsets of \( X \setminus D \), the Arzelà-Ascoli theorem implies there is a sequence \( t_j \) to a \( \psi \in C^\infty_{\text{loc}}(X \setminus D) \) to a \( \varphi_\infty \in \mathcal{H}_w^{\text{weak}} \). Since \( \dot{\phi}(t) \to 0 \), \( \varphi_\infty \) satisfies the equation

\[
\alpha_n = n \alpha_{\varphi_\infty} + \theta + \beta \alpha_n.
\]

(4.5)

We now prove the uniqueness of the solution \( \phi(t) \) to the degenerate J-flow. Define \( \phi_\delta = \phi - \psi - \delta \rho \), for some \( \delta > 0 \) will be chosen later, \( \omega_\delta = s \omega_\psi + (1-s) \omega_\psi \) and \( \eta^{k\bar{l}}_s = \omega_s \alpha_s^{ki} \hat{\theta}_{ij} \beta \omega^{k\bar{l}}_s \). Then on \( X \setminus D \) we have

\[
\frac{\partial \phi_\delta}{\partial t} = \dot{\phi} - \dot{\psi} = \left( \frac{n \alpha_{\psi}^{r-1} + \theta}{\alpha_{\psi}^q} + \beta \frac{\alpha_n^q}{\alpha_{\psi}^q} - \frac{n \alpha_{\psi}^{r-1} + \theta}{\alpha_{\psi}^q} - \beta \frac{\alpha_n^q}{\alpha_{\psi}^q} \right)
\]

(4.6)

\[
= - \int_0^1 \frac{d}{ds} \left( \frac{n \alpha_s^{r-1} + \theta}{\alpha_s^q} + \beta \frac{\alpha_n^q}{\alpha_s^q} \right) ds
\]

(4.7)

\[
= \int_0^1 \eta^{k\bar{l}}_s (\omega_\psi - \omega_\psi) ds
\]

(4.8)

\[
= \left( \int_0^1 \eta^{k\bar{l}}_s ds \right) \frac{\partial_k \partial_{\bar{l}} \phi_\delta - \delta \left( \int_0^1 \eta^{k\bar{l}}_s ds \right) (R_h)_{k\bar{l}} \right).
\]

(4.9)

Since \( \sup_{X \setminus D} |\dot{\psi}| \leq C \) and \( \sup_{X \setminus D} |\dot{\psi}| \leq C \) we have \( \omega_\delta \geq C^{-1} \theta \), and

\[
\beta \frac{\alpha_n^q}{\alpha_s^q} = \varepsilon - u - tr \omega_\theta \leq C, \text{ on } X \setminus D,
\]

where \( u \) is \( \phi \) or \( \psi \). Therefore we also have

\[
\beta \frac{\alpha_n^q}{\alpha_s^q} \leq C, \text{ on } X \setminus D.
\]

It follows that \( \eta^{k\bar{l}}_s \leq C \theta \eta^{k\bar{l}} \) on \( X \setminus D \). Therefore

\[
\delta \left( \int_0^1 \eta^{k\bar{l}}_s ds \right) (R_h)_{k\bar{l}} \leq C \delta g^{k\bar{l}}_s (R_h)_{k\bar{l}} \leq \frac{2C \delta}{\varepsilon_0},
\]

since \( \theta - \varepsilon_0 R_h > 0 \) for a uniform constant \( \varepsilon_0 \). We thus obtain

\[
\frac{\partial}{\partial t} \phi_\delta \geq \left( \int_0^1 \eta^{k\bar{l}}_s ds \right) \frac{\partial_k \partial_{\bar{l}} \phi_\delta - \frac{2C \delta}{\varepsilon_0}}.
\]
The maximum principle, on any time interval \([0, T]\) now yields

\[
\phi_\delta \geq -C_1 \delta t \geq -C_1 \delta T,
\]

for a uniform constant \(C_1\). This implies that \(\phi \geq \psi + \delta |x|_h^2 - C_1 \delta T\), so \(\phi \geq \psi\) on \(X \setminus D\) by letting \(\delta \to 0\). The same argument shows that \(\psi \geq \phi\) on \(X \setminus D\). Therefore \(\phi = \psi\) on \(X \setminus D\), hence on \(X\).

**Proof of Corollary 1.2** For any \(\psi_0 \in \mathcal{H}_\omega\), since \(\mathcal{J}_{\omega, \theta}^0\) is decreasing along the J-flow by (4.3), we have \(\mathcal{J}_{\omega, \theta}^0(\psi_0) \geq \lim_{t \to \infty} \mathcal{J}_{\omega, \theta}^0(\psi_t)\). Arguing as in [8, Lemma 3.2], we obtain

\[
\lim_{t \to \infty} \mathcal{J}_{\omega, \theta}^0(\psi_t) = \mathcal{J}_{\omega, \theta}^0(\psi_\infty).
\]

By Theorem 2.2, for any \(\epsilon \in (0, \epsilon_0]\), \(\psi_\epsilon(t)\) converge uniformly to \(\psi_{\epsilon, \infty}\) which satisfies

\[
\beta \omega^n + \nu_\omega \psi_{\epsilon, \infty}^{n-1} \wedge \theta_\epsilon = c_\epsilon \omega_\psi^n.
\]

It follows from Proposition 3.6 that \(Osc(\psi_{\epsilon, \infty}) := \sup_X \psi_{\epsilon, \infty} \leq C\) since for any constant \(C > 0\) only depends on \(X, \omega, \|\theta\|_{C^0(X, \omega)}, \beta\). This implies that \(Osc(\psi_\infty)\) is uniformly bounded by a constant depending only on \(X, \omega, \|\theta\|_{C^0(X, \omega)}, \beta\).

Since for any constant \(C\), \(\mathcal{J}_{\omega, \theta}^0(\psi + C) = \mathcal{J}_{\omega, \theta}^0(\psi), \forall \psi \in PSH(X, \omega)\), we infer that \(\mathcal{J}_{\omega, \theta}^0(\psi_\infty) \geq B\) for some uniformly constant \(B\) which only depends on \(X, \omega, \|\theta\|_{C^0(X, \omega)}, \beta\). Hence we get the uniform lower bound for \(\mathcal{J}_{\omega, \theta}^0\) on \(\mathcal{H}_\omega\) as required. □

**Proof of Corollary 1.3** Since \(\mathcal{J}_{\omega, \theta}^0 = \mathcal{J}_{\omega, \theta}^{0-\eta} + \mathcal{J}_\omega^\eta\) and \(\mathcal{J}_{\omega, \theta}^{0-\eta} - \mathcal{J}_{\omega, \theta}^{0-\text{Ric}(\omega)} \leq C\) because \(\theta - \eta\) and \(-\text{Ric}(\omega)\) are in the same cohomology class (cf. [5, Proposition 22]), we infer that the Mabuchi K-energy twisted by \(\eta\)

\[
\mathcal{M}_\omega^\eta = \mathcal{M}_\omega + \mathcal{J}_\omega^\eta = (\mathcal{M}_{\text{ent}} - \beta J_\omega) + \mathcal{J}_{\omega, \theta}^{0-\text{Ric}(\omega)} + \mathcal{J}_\omega^\eta \geq (\mathcal{M}_{\text{ent}} - \beta J_\omega) + \mathcal{J}_{\omega, \theta}^{0-\eta} + \mathcal{J}_\omega^\eta - C \geq (\mathcal{M}_{\text{ent}} - \beta J_\omega) + \mathcal{J}_{\omega, \theta}^{0-\eta} - C. \tag{4.11}
\]

We consider the degenerate twisted J-flow with \(\theta = -c_1(X) + \eta\) and \(\omega\) in the assumption. Then by Corollary 1.2, \(\mathcal{J}_{\omega, \theta}^0\) is uniformly bounded from below on \(\mathcal{H}_\omega\). Combining with the fact that \(\mathcal{M}_{\text{ent}} \geq a_0 J_\omega - C\) and \(a_0 > \beta\), we infer that the functional \(\mathcal{M}_\omega^0\) is proper. Then Theorem 4.2 in [4] implies the existence of cscK metric with respect to \(\eta\). □

We remark that another sufficient criterion for the properness of a twisted Mabuchi K-energy can be obtained following the argument for the untwisted version in [16] (see also [14, Lemma 2.1]). We include a sketch of the proof for the reader convenience.

**Proposition 4.1** Let \(X\) be a compact Kähler manifold and \(\eta\) be a smooth closed semipositive \((1, 1)\)-form. Suppose that

\[
\mathcal{M}_{\text{ent}}(\psi) \geq a_1 (I_\omega - J_\omega)(\psi) - C, \forall \psi \in \mathcal{H}_\omega, \tag{4.13}
\]

for some \(a_1 > 0\), where

\[
I_\omega(\psi) = \frac{1}{n!} \int_X \psi(\omega^n - \omega_\psi^n).
\]
If there exists $\epsilon \in [0, \alpha_1)$, a Kähler form $\hat{\omega} \in [\omega]$ and a $(1,1)$-form $\theta \in -c_1(K_X) + [\eta]$ such that $\theta + \epsilon \hat{\omega} > 0$ and

$$\left( \left[ \frac{n(-c_1(X) + [\eta]) \cdot [\omega]^{n-1}}{[\omega]^n} + \epsilon \right] \hat{\omega} - (n-1)\theta \right) \wedge \hat{\omega}^{n-2} > 0. \quad (4.14)$$

Then the Mabuchi functional twisted by $\eta$ is proper on $\mathcal{H}_\omega$. Therefore by [4, Theorem 4.1] there exists a twisted cscK metric in $[\omega]$.

**Proof** We follow the arguments in [16] and [14, Lemma 2.1]. By direct computation we have $\mathcal{J}_\omega = (I_\omega - J_\omega)$. Therefore we obtain

$$\mathcal{M}_{ent}(\varphi) \geq \alpha_1 \mathcal{J}_\omega^\omega - C \geq \mathcal{J}_\omega^{\epsilon \omega} - C \quad (4.15)$$

for any $\epsilon \in [0, \alpha_1)$. On the other hand we have

$$\mathcal{J}_\omega^{\epsilon \omega} + \mathcal{J}_\omega^{-Ric(\omega)} + \mathcal{J}_\omega^\eta = \mathcal{J}_\omega^{\epsilon \omega - Ric(\omega) + \eta}.$$

Since $\epsilon \omega - Ric(\omega) + \eta$ and $\epsilon \hat{\omega} + \theta$ are in the same cohomology class, we infer that

$$|\mathcal{J}_\omega^{\epsilon \omega - Ric(\omega) + \eta} - \mathcal{J}_\omega^{\theta + \epsilon \hat{\omega}}| \leq C$$

for some uniform constant $C$. It follows from the main theorem of Song-Weinkove [24] that the condition (4.14) implies that $\mathcal{J}_\omega^{\theta + \epsilon \hat{\omega}}$ is uniformly bounded from below. Therefore the Mabuchi functional twisted by $\eta$, $\mathcal{M}_\omega^\eta = \mathcal{M}_\omega + \mathcal{J}_\omega^\eta$, is proper on $\mathcal{H}_\omega$. $\square$

**Remark 4.2** It follows from [27, Lemma 6.19, Remark 6.20] that

$$\frac{1}{n} I_\omega \leq J_\omega \leq n J_\omega.$$

Therefore this criterion implies the one in Corollary 1.3 for $\beta \in [0, \alpha_0/n)$.

**Acknowledgements** The author is grateful to Vincent Guedj for support, suggestions and encouragement. We also would like to thank Duong Hong Phong and Zakarias Sjöström Dyrefelt for very useful discussions. The author is partially supported by ANR-21-CE40-0011-01 (research project MARGE) and ANR-11-LABX-0040 (research project HERMETIC). The author would like to thank the referee for useful comments and suggestions.

**References**

1. Chen, G.: The $J$-equation and the supercritical deformed Hermitian–Yang–Mills equation. Invent. math. 225, 529–602 (2021)
2. Chen, X.X.: On the lower bound of the Mabuchi K-energy and its application. Int. Math. Res. Not. 12, 607–623 (2000)
3. Chen, X.X., Cheng, J.: On the constant scalar curvature Kähler metrics (I)-A priori estimates. J. Am. Math. Soc. 34, 909–936 (2021)
4. Chen, X.X., Cheng, J.: On the constant scalar curvature Kähler metrics (II)-Existence results. J. Am. Math. Soc. 34, 937–1009 (2021)
5. Collins, T., Székelyhidi, G.: Convergence of the J-flow on toric manifolds. J. Diff. Geom. 107(1), 47–81 (2017)
6. Dervan, R.: Alpha invariants and coercivity of the Mabuchi functional on Fano manifolds. Ann. Fac. Sci. Toulouse Sér. 6(25), 919–934 (2016)
7. Donaldson, S.K.: Moment maps and diffeomorphisms. Asian J. Math. 3(1), 1–15 (1999)
8. Fang, H., Lai, M., Song, J., Weinkove, B.: The $J$-flow on Kähler surfaces: a boundary case. Anal. PDE 7(1), 215–226 (2014)
9. Fine, J.: Constant scalar curvature Kähler metrics on fibred complex surfaces. J. Differ. Geom. 68(3), 397–432 (2004)
10. Fine, J.: Fibrations with constant scalar curvature Kähler metrics and the CM-line bundle. Math. Res. Lett. 14(2), 239–247 (2007)
11. Gilbarg, D., Trudinger, N. S.: Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, reprint of the 1998 ed., (2001)
12. Guan, B.: Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163(8), 1491–1524 (2014)
13. Guedj, V., Zeriahi, A.: Degenerate complex Monge-Ampère equations, EMS Tracts in Math. (2017), 496p
14. Jian, W., Shi, Y., Song, J.: A remark on constant scalar curvature Kähler metrics on minimal models. Proc. Am. Math. Soc. 147, 3507–3513 (2019)
15. Lejmi, M., Székelyhidi, G.: The J-flow and stability. Adv. Math. 274, 404–431 (2015)
16. Li, H.-Z., Shi, Y.L., Yao, Y.: A criterion for properness of the K-energy in a general Kähler class. Math. Ann. 361(1–2), 135–156 (2015)
17. Mabuchi, T.: Some symplectic geometry on compact Kähler manifolds I. Osaka J. Math. 24, 227–252 (1987)
18. Phong, D.H., Sturm, J.: The Dirichlet problem for degenerate complex Monge-Ampère equations. Comm. Anal. Geom. 18(1), 145–170 (2010)
19. Phong, D.H., Tô, T.D.: Fully non-linear parabolic equations on compact Hermitian manifolds. Ann. Sci. Éc. Norm. Supér. 54(3), 793–832 (2021)
20. Sjöström, Z.: Dyrefelt, Optimal lower bounds for Donaldson’s J-functional. Adv. Math. 374, 107271 (2020)
21. Sjöström, Z.: Dyrefelt, Existence of cscK metrics on smooth minimal models, to appear in Annali della Scuola Normale Superiore di Pisa - Classe di Scienze
22. Song, J.: Nakai-Moishezon criterions for complex Hessian equations, arXiv:2012.07956
23. Song, J., Weinkove, B.: The degenerate J-flow and the Mabuchi energy on minimal surfaces of general type. Univ. Iagel. Acta Math. 2, 89–106 (2013)
24. Song, J., Weinkove, B.: On the convergence and singularities of the J-flow with applications to the Mabuchi energy. Comm. Pure Appl. Math. 61, 210–229 (2008)
25. Székelyhidi, G.: Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differ. Geom. 109(2), 337–378 (2018)
26. Tian, G.: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. 130(1), 1–37 (1997)
27. Tian, G.: Canonical metrics in Kähler geometry, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel (2000)
28. Tso, K.: On an Aleksandrov-Bakel’Man type maximum principle for second-order parabolic equations. Comm. Partial Differ. Equ. 10(5), 543–553 (1985)
29. Tsuji, H.: Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type. Math. Ann. 281(1), 123–133 (1988)
30. Weinkove, B.: Convergence of the J-flow on Kähler surfaces. Comm. Anal. Geom. 12(4), 949–965 (2004)
31. Weinkove, B.: On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy. J. Differ. Geom. 73(2), 351–358 (2006)
32. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. Comm. Pure Appl. Math. 31, 339–411 (1978)
33. Zheng, K.: I-properness of Mabuchi’s K-energy. Calc. Var. Partial Differ. Equ. 54(3), 2807–2830 (2015)
34. Zheng, K.: Existence of constant scalar curvature Kähler cone metrics, properness and geodesic stability, arXiv:1803.09506

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.