Abstract. In this paper, we construct a bijection from a set of bounded free Motzkin paths to a set of bounded Motzkin prefixes that induces a bijection from a set of bounded free Dyck paths to a set of bounded Dyck prefixes. We also give bijections between a set of bounded cornerless Motzkin paths and a set of \( t \)-core partitions, and a set of bounded cornerless symmetric Motzkin paths and a set of self-conjugate \( t \)-core partitions. As an application, we get explicit formulas for the number of ordinary and self-conjugate \( t \)-core partitions with a fixed number of corners.

1. Introduction

The main result of this paper is finding a bijection between two sets of paths in a bounded strip, which have been studied by several researchers (for example, see [1, 5, 6, 7, 10, 13]).

A Motzkin path of length \( n \) is a path from \((0,0)\) to \((n,0)\) which stays weakly above the \( x \)-axis and consists of steps \( u = (1,1) \), \( d = (1,-1) \), and \( f = (1,0) \), called up, down, and flat steps, respectively. A free Motzkin path of length \( n \) is a path which starts at \((0,0)\) or \((0,1)\), ends at \((n,0)\), and consists of \( u \), \( d \), and \( f \). A Motzkin path with no restrictions on the end point is called a Motzkin prefix. For a given path, a peak is a point preceded by an up step and followed by a down step and a valley is a point preceded by a down step and followed by an up step. We say that a path is cornerless if it has no peaks or valleys.

For non-negative integers \( m, r, \) and \( k \), let \( \mathcal{F}(m,r,k) \) be the set of free Motzkin paths of length \( m + r \) with \( r \) flat steps that are contained in the strip \(-\left\lfloor \frac{k}{2} \right\rfloor \leq y \leq \left\lfloor \frac{k+1}{2} \right\rfloor \). We denote \( \mathcal{M}(m,r,k) \) the set of Motzkin prefixes of length \( m + r \) with \( r \) flat steps that are contained in the strip \( 0 \leq y \leq k \). We define \( L_k \) to be one of the boundaries of each path depending on the value of \( k \). More specifically, for \( P \in \mathcal{F}(m,r,k) \), denote \( L_k \) by

\[
y = \begin{cases} 
\left\lfloor \frac{k+1}{2} \right\rfloor & \text{if } k \text{ is odd,} \\
-\left\lfloor \frac{k}{2} \right\rfloor & \text{if } k \text{ is even.}
\end{cases}
\]

Let \( \overline{\mathcal{F}}(m,r,k) \) (resp. \( \overline{\mathcal{M}}(m,r,k) \)) be the set of paths in \( \mathcal{F}(m,r,k) \) (resp. \( \mathcal{M}(m,r,k) \)) which touch the line \( L_k \) (resp. \( y = k \)) so that

\[
\mathcal{F}(m,r,k) = \bigcup_{i=0}^{k} \mathcal{F}(m,r,i) \quad \text{and} \quad \mathcal{M}(m,r,k) = \bigcup_{i=0}^{k} \overline{\mathcal{M}}(m,r,i).
\]

Our main theorem states the following.

**Theorem 1.1.** For given non-negative integers \( m, r, \) and \( k \), there is a bijection between the sets \( \overline{\mathcal{F}}(m,r,k) \) and \( \overline{\mathcal{M}}(m,r,k) \).
To prove Theorem 1.1, we construct a map \( \phi_{m,k} \) and show that it is bijective in Sections 2.1 and 2.2.

Using the adjacency matrices of path graphs, Cigler [5] showed that

\[
|A_{n,k}| = |B_{n,k}| = \sum_{j \in \mathbb{Z}} (-1)^j \left( \frac{n}{n+(k+2)j} \right)
\]

and expected the existence of a simple bijection between \( A_{n,k} \) and \( B_{n,k} \), where \( A_{n,k} \) is the set of paths of length \( n \) which consist of \( u \) and \( d \) only, start at \((0,0)\), end on height 0 or \(-1\), and are contained in the strip \(-\lceil \frac{k+1}{2} \rfloor \leq y \leq \lfloor \frac{k}{2} \rfloor\) of width \( k \), and \( B_{n,k} \) is the set of paths of length \( n \) which consist of \( u \) and \( d \) only, start at \((0,0)\) and are contained in the strip \( 0 \leq y \leq k \). Recently, Gu and Prodinger [10] and Dershowitz [7] found bijections between \( A_{n,k} \) and \( B_{n,k} \) independently. We note that Theorem 1.1 with no flat step (equivalently, \( r = 0 \)) gives a new bijection between \( A_{n,k} \) and \( B_{n,k} \) since \( F(n,0,k) \) can be obtained from \( A_{n,k} \) by mirroring left and right and flipping along the \( x \)-axis, and \( M(n,0,k) = B_{n,k} \) as it is. We should mention that the bijection \( \phi_{m,k} \) is inspired by the bijection due to Gu and Prodinger, but there is a property that \( \phi_{m,k} \) holds whereas Gu and Prodinger’s does not. This property is described in Section 2.3.

Let \( F_c(m,r,k) \) be the set of cornerless free Motzkin paths in \( F(m,r,k) \) that never start with a down (resp. up) step for odd (resp. even) \( m \) and \( M_c(m,r,k) \) be the set of cornerless Motzkin prefixes in \( M(m,r,k) \) that end with a flat step. In Section 3.1, we show that \( \phi_{m,k} \) induces a bijection between \( F_c(m,r,k) \) and \( M_c(m,r,k) \).

In Section 3.2, we combinatorially interpret \( t \)-core partitions by cornerless Motzkin paths. We describe a bijection between a set of cornerless Motzkin paths and a set of \( t \)-core partitions. As an application of this bijection, we count the number of \( t \)-core partitions with \( m \) corners. In Section 3.3, we also count the number of self-conjugate \( t \)-core partitions with \( m \) corners by constructing bijections between any pair of the following sets: a set of cornerless free Motzkin paths, a set of cornerless symmetric Motzkin paths, and a set of self-conjugate \( t \)-core partitions.

2. Bijection

In this section, we recursively define a map

\[
\phi_{m,k} : \bigcup_{r \geq 0} F(m,r,k) \to \bigcup_{r \geq 0} M(m,r,k),
\]

according to the values of \( m \) and \( k \), and then show that it is bijective. For simplicity, we define some notations first. For a path \( P = p_1p_2\ldots p_n \), where each \( p_i \) denotes the \( i \)th step in \( P \), let

\[
\overrightarrow{P} := p_1p_2\ldots p_n \quad \text{and} \quad \overleftarrow{P} := p_np_{n-1}\ldots p_1,
\]

where \( \overrightarrow{u} := d \), \( \overleftarrow{d} := u \), and \( \overrightarrow{f} := f \).

2.1. Map \( \phi_{m,k} \). Now we define the map. Let \( P \) be a path in the set \( F(m,r,k) \) for some \( r \geq 0 \), and \( \gamma \geq 0 \) denote the maximum number such that \( f^\gamma \) is a suffix of \( P \).

**Case 0.** If \( k = 0 \) or \( k = 1 \), then the map is defined as

\[
\phi_{m,k}(P) := \overleftarrow{P}.
\]

We show the bijection \( \phi_{m,1} \) in Figure 1.
Now assume $k > 1$. A special step of $P$ is the first step ending on the line $L_k$.

We write $P$ as

\[
P = A f^\alpha s f^\beta B f^\gamma,
\]

where $s$ is the special step, $\alpha \geq 0$ (resp. $\beta \geq 0$) is the maximum number of consecutive flat steps right before (resp. after) the step $s$, $A$ denotes the prefix of $P$ before the subpath $f^\alpha s$, and $B$ denotes the subpath between the subpaths $sf^\beta$ and $f^\gamma$. Note that $A$ and $B$ never end with a flat step (See Figure 2).

Let the last vertex on the line $L_k$ (resp. $y = k$) be the turning point of a path in $\overline{F}(m, r, k)$ (resp. $\overline{M}(m, r, k)$). We call the first step after the turning point starting from the $x$-axis and heading away from the line $L_k$ the break step, and denote it by $b$. If $P$ has the break step $b$, let $\delta \geq 0$ be the maximum number of consecutive flat steps right before the step $b$ and we write $B$ as $B_1 f^\delta b B_2$.

**Case 1.** Let $m$ and $k$ have the same parity with $k > 1$.

i) If there is no break step, then we write $P$ as (1) and define the map as

\[
\phi_{m,k}(P) := \begin{cases} 
Q & \text{if } k \text{ is odd,} \\
\overline{Q} & \text{if } k \text{ is even,}
\end{cases}
\]

where

\[
Q := f^\gamma \overline{F} f^\alpha s A f^\beta.
\]

Note that $\phi_{m,k}(P)$ ends on the line $y = k$.

ii) If there is the break step $b$, then $P$ can be written as

\[
P = A f^\alpha s f^\beta B_1 f^\delta b B_2 f^\gamma.
\]

Note that $B_1$ is a subpath starting from the line $L_k$ and ending at the $x$-axis with a down (resp. up) step, and $B_2$ is a subpath starting from
the line $y = (-1)^k$ and ending at the $x$-axis with a non-flat step for odd (resp. even) $k$. Define

$$
\phi_{m,k}(P) := \begin{cases} 
Q & \text{if } k \text{ is odd}, \\
QC & \text{if } k \text{ is even}, 
\end{cases}
$$

where

$$Q := f^\alpha B_1 f^\alpha s A f^\beta b$$

and

$$C := \begin{cases} 
\phi_{m',k'}(B_2 f^\delta) & \text{if } k \text{ is odd}, \\
\phi_{m',k'}(B_2 f^\delta) & \text{if } k \text{ is even}. 
\end{cases}$$

Note that $m'$ is odd in this case. The bijection in Case 1 is illustrated in Figure 3.

Case 2. Let $m$ and $k$ have different parity with $k > 1$. In this case we write $A$ as $A_1 a A_2$, where $a$ is the first up (resp. down) step starting from the $x$-axis (resp. $y = 1$) in $P$ for odd (resp. even) $k$. Here, $A_1$ and $A_2$ can be empty. Note that if $A_2$ is non-empty, then it never ends with a flat step. Similar to
the map in Case 1–ii), we define the map as \( M \), where \( Q \) and \( C \) are given as follows.

i) If there is no break step, then \( P \) can be written as

\[
P = A_1 a A_2 f^\alpha s f^\beta B f^\gamma,
\]

and we set

\[
Q := f^n B f^\alpha s A_2 f^\beta \overline{\varepsilon}
\]

and \( C := \begin{cases} \phi_{m',k'}(A_1) & \text{if } k \text{ is odd}, \\ \phi_{m',k'}(A_1) & \text{if } k \text{ is even}. \end{cases} \)

ii) If there is the break step \( b \), then \( P \) can be written as

\[
P = A_1 a A_2 f^\alpha s f^\beta B_1 f^\delta b B_2 f^\gamma,
\]

and we set

\[
Q := f^n B_1 f^\alpha s A_2 f^\beta \overline{\varepsilon}
\]

and \( C := \begin{cases} \phi_{m',k'}(A_1 b B_2 f^\delta) & \text{if } k \text{ is odd}, \\ \phi_{m',k'}(A_1 b B_2 f^\delta) & \text{if } k \text{ is even}. \end{cases} \)

Note that \( m' \) is even in this case. The bijection in Case 2–ii) is illustrated in Figure 4. By regarding \( B_1 \) as \( B \) and \( f^\delta b B_2 \) as \( \varepsilon \) in this figure, we see the bijection in Case 2–i).

**Lemma 2.1.** For given non-negative integers \( m \) and \( k \), the map \( \phi_{m,k} \) is well-defined.

**Proof.** Let \( P \in \overline{\mathcal{F}}(m,r,k) \). In Case 0, it is clear that \( \phi_{m,k}(P) = \overline{P} \in \overline{\mathcal{M}}(m,r,k) \). Now consider Case 1–i). In this case, for a path \( P \) as in (1), we define \( \phi_{m,k}(P) \) as (2). If \( k \) is odd (resp. even), then \( A \) is a subpath of \( P \) that starts from the line \( y = 1 \) (resp. \( y = 0 \)) on the line \( y = (-1)^{k-1}(k-1)/2 \), and is contained in the strip \(-[k-1][k-1)/2] \leq y \leq [k-1]/2 \), while \( B \) is a subpath that starts from the line \( y = (-1)^{k-1}(k+1)/2 \), ends on the \( x \)-axis, and is contained in the strip \(-[k+1]/2 \leq y \leq [(k+1)/2] \). Hence, the prefix \( f^n B f^\alpha \) (resp. \( f^n B f^\alpha \)) of \( \phi_{m,k}(P) \) is a Motzkin prefix that ends at the line \( y = [(k+1)/2] \) and is contained in the strip \( 0 \leq y \leq k \). It follows that the remaining subpath \( A f^\beta \) (resp. \( \overline{A f^\beta} \)) starts from the line \( y = [(k+1)/2] \), ends on the line \( y = k \), and is contained in the strip \( 1 \leq y \leq k \). Hence, the prefix \( f^n B f^\alpha \) (resp. \( f^n B f^\alpha \)) of \( \phi_{m,k}(P) \) is a Motzkin prefix that ends at the line \( y = [(k+1)/2] \) and is contained in the strip \( 0 \leq y \leq k \), and the remaining subpath \( A f^\beta \) (resp. \( \overline{A f^\beta} \)) starts from the line \( y = [(k+1)/2] \), ends on the line \( y = k \), and is contained in the strip \( 1 \leq y \leq k \) for odd (resp. even) \( k \). Therefore, \( \phi_{m,k}(P) \in \overline{\mathcal{M}}(m,r,k) \).

For the remaining cases, we write \( A \) as \( A_1 a A_2 \) and \( B \) as \( B_1 f^\delta b B_2 \) if necessary. Now we use the induction on \( k \). For any \( k' \leq k \), suppose that \( \phi_{m',k'}(P^*) \in \overline{\mathcal{M}}(m',r',k') \) for any paths \( P^* \in \overline{\mathcal{F}}(m',r',k') \). Let \( P \in \overline{\mathcal{F}}(m,r,k) \), \( \phi_{m,k}(P) \) is defined as \( Q C \) or \( \overline{Q C} \), where \( Q \) and \( C \) are of the forms in (1), (3), or (5). In any cases, similar to Case 1–i), \( Q \) or \( \overline{Q} \) is a prefix of \( \phi_{m,k}(P) \) that starts from the \( x \)-axis, touches the line \( y = k' \), ends on the line \( y \leq k - 1 \), and is contained in the strip \( 0 \leq y \leq k \). Since \( C \in \overline{\mathcal{M}}(m',r',k') \) with \( k' < k \), \( \overline{C} \) is a suffix of \( \phi_{m,k}(P) \) that starts from the line \( y = k - 1 \) and is contained in the strip \( k - k' - 1 \leq y \leq k - 1 \) by the induction hypothesis. Thus, we conclude that \( \phi_{m,k}(P) \in \overline{\mathcal{M}}(m,r,k) \). \( \square \)

**Example 2.2.** For given free Motzkin paths, let us apply the map \( \phi_{m,k} \).
Figure 4. The bijection $\phi_{m,k}$ in Case 2-ii)

(a) For the path

$$P_1 = fdudududfdududfdudf \in \mathcal{F}(10, 6, 2),$$

by applying Case 1-ii) and Case 0, we get

$$\phi_{10,2}(P_1) = f f d u d u u f d u f d u f d f \in \mathcal{M}(10, 6, 2)$$

since $A = \emptyset$, $\beta = \delta = 0$, and $C = \phi_{1,1}(B_2 f^3)$ in (3), and $\phi_{1,1}(fd) = \overleftarrow{fd} = uf$.

(b) For the path

$$P_2 = fdudududfdudufudfdudfdudfufudfufudfuf \in \mathcal{F}(20, 11, 3),$$

by applying Case 2-ii), we obtain

$$\phi_{20,3}(P_2) = fufudufududufuufudfufudfufuufudfufudfuf \in \mathcal{M}(20, 11, 3)$$
since \( A_2 = \emptyset, \beta = 0, \) and \( C = \phi_{10,2}(A_1bB_2f^\delta) = \phi_{10,2}(P_1) \) in \([7]\), where \( P_1 \) is the path given in (a).

See Figure 5 for further details.

\[
\begin{align*}
(A) \text{ A bijection } \phi_{10,2} \text{ in Case 1–ii) } & \quad \text{ and } \\
(B) \text{ A bijection } \phi_{20,3} \text{ in Case 2–ii) }
\end{align*}
\]

**Figure 5.** Examples of the map \( \phi_{m,k} \)

### 2.2. Map \( \psi_{m,k} \)

Now we define a map

\[
\psi_{m,k} : \bigcup_{r \geq 0} \mathcal{M}(m, r, k) \rightarrow \bigcup_{r \geq 0} \mathcal{F}(m, r, k)
\]

and show that \( \psi_{m,k} = \phi_{m,k}^{-1} \).

#### Case 0.

For \( k = 0 \) or 1, we define \( \psi_{m,k}(S) = \overleftarrow{S} \).

Recall that the last vertex on the line \( y = k \) is called the turning point of a path in \( \mathcal{M}(m, r, k) \). We define a critical point of \( S \) as the rightmost point on the \( x \)-axis which locates before the turning point.

#### Case 1.

For \( k > 1 \), assume that \( S \) is a path which ends on the line \( y = k \). Note that \( m \) and \( k \) have the same parity and we write

\[
S = f^\gamma B^* f^\alpha u^* A^* f^\beta,
\]

where \( u^* \) is the first up step starting from the line \( y = \lfloor (k + 1)/2 \rfloor \) after the critical point of \( S \), \( \gamma \geq 0 \) (resp. \( \beta \geq 0 \)) is the maximum number of consecutive initial (resp. final) flat steps of \( P \), and \( \alpha \geq 0 \) is the maximum number of consecutive flat steps before the step \( u^* \). Hence, \( B^* \) (resp. \( A^* \)) is the subpath of \( S \) such that it starts from the \( x \)-axis (resp. \( y = \lfloor k/2 \rfloor + 1 \)), ends on the line \( y = \lfloor k/2 \rfloor \) (resp. \( y = k \)), and is contained in the strip \( 0 \leq y \leq k \) (resp. \( 1 \leq y \leq k \)). We define

\[
\psi_{m,k}(S) := \begin{cases} 
A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is odd,} \\
A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is even.}
\end{cases}
\]
Case II. Suppose that $S$ is a path which does not end on the line $y = k$ for $k > 1$.

In this case, we write

$$S = f^\gamma B^* f^\alpha u^* A^* f^\beta d^* C^\gamma,$$

where $u^*, A^*, B^*, \alpha, \beta, \gamma$ is defined as in Case I, $d^*$ is the last down step starting from the line $y = k$, and $C^\gamma$ is a suffix of $S$ after the step $d^*$. Note that $C^\gamma \in \overline{M}(m', r', k')$ for some $k' < k$ since $C^\gamma$ is contained in the strip $0 \leq y \leq k - 1$.

i) Let $m$ and $k$ have the same parity, which follows that $m'$ is odd. We write $\psi_{m', k'}(C^\gamma) = B^* f^\delta$, where $\delta \geq 0$ is the maximum number of consecutive flat steps at the suffix of $\psi_{m', k'}(C^\gamma)$. We set

$$\psi_{m, k}(S) := \begin{cases} A^* f^\alpha u^* f^\beta d^* f^\gamma & \text{if } k \text{ is odd}, \\ A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is even}. \end{cases} \quad (10)$$

ii) Let $m$ and $k$ have different parity. In this case, $m'$ is even. We divide two cases whether $\psi_{m', k'}(C^\gamma)$ goes above the $x$-axis or not.

If $\psi_{m', k'}(C^\gamma)$ does not go above the $x$-axis, then we write $\psi_{m', k'}(C^\gamma) = A^*$ and define

$$\psi_{m, k}(S) := \begin{cases} A^* d^* A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is odd}, \\ A^* d^* A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is even}. \end{cases} \quad (11)$$

If $\psi_{m', k'}(C^\gamma)$ goes above the $x$-axis, then we write $\psi_{m', k'}(C^\gamma) = A^* u^* B^* f^\delta$, where $u^*$ is the first up step starting from the $x$-axis and $\delta \geq 0$ is the maximum number of consecutive flat steps at the suffix of $\psi_{m', k'}(C^\gamma)$. We define

$$\psi_{m, k}(S) := \begin{cases} A^* d^* A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is odd}, \\ A^* d^* A^* f^\alpha u^* f^\beta B^* f^\gamma & \text{if } k \text{ is even}. \end{cases} \quad (12)$$

Lemma 2.3. The map $\psi_{m, k}$ is the inverse map of $\phi_{m, k}$.

Proof. For $k = 0$ or 1, it is clear that $\psi_{m, k}(\phi_{m, k}(P)) = P$ for any path $P \in \mathcal{F}(m, r, k)$ by the construction.

From now on, we set $k > 1$. Let $P \in \mathcal{F}(m, r, k)$ when $m$ and $k$ have the same parity and there is no break step in $P$ so that $P$ is represented as $P = A f^\alpha s f^\beta B f^\gamma$. When $k$ is odd (resp. even), it follows from (2) and (9) that $\psi_{m, k}(\phi_{m, k}(P)) = P$ since $A = A^*$ (resp. $A = A^\top$), $s = u^*$ (resp. $s = \overline{u^*}$), and $B = B^*$ (resp. $B = B^\top$), where $S = \phi_{m, k}(P) \in \overline{M}(m, r, k)$.

Now, we assume that $\psi_{m', k'}(P^*) \in \mathcal{F}(m', r', k')$ for any path $P^*$ with $k' < k$. Let $P \in \mathcal{F}(m, r, k)$ when $m$ and $k$ have the same parity and there is a break step $b$ in $P$ so that $P$ is represented as $P = A f^\alpha s f^\beta B_1 f^\delta b B_2 f^\gamma$. For odd (resp. even) $k$, according to (3) and (10), $\psi_{m, k}(\phi_{m, k}(P)) = P$ since $A = A^*$ (resp. $A = A^\top$), $s = u^*$ (resp. $s = \overline{u^*}$), $B_1 = B^*$ (resp. $B_1 = B^\top$), $b = d^*$ (resp. $b = \overline{d^*}$), and $B_2 = B^*$ (resp. $B_2 = B^\top$).

Similarly, by (6), (8), (11), and (12), we can see that $\psi_{m, k}(\phi_{m, k}(P)) = P$, where $P \in \mathcal{F}(m, r, k)$ when $m$ and $k$ have different parity with $k > 1$. \qed
Example 2.4. For given Motzkin prefixes, 
\[ S_1 = uuufdddufuuf \in \mathcal{M}(9, 4, 3), \]
\[ S_2 = uuufdddfsduufudddfdf \in \mathcal{M}(14, 6, 4), \]
\[ S_3 = fuuufudddfsduufudddfdfuufuuufudddfduuf \in \mathcal{M}(23, 11, 6), \]
we have 
\[ \psi_{9,3}(S_1) = udddfsduufudddf \in \mathcal{F}(9, 4, 3), \]
\[ \psi_{14,4}(S_2) = dfudddfsduufudddfdf \in \mathcal{F}(14, 6, 4), \]
\[ \psi_{23,6}(S_3) = uuuufudddfsduufudddfduufudddfduuf \in \mathcal{F}(23, 11, 6). \]
See Figure 6 for further details.

Figure 6. Examples of the map \( \psi_{m,k} \)
2.3. A property of $\phi_{m,k}$. For a free Motzkin path $P$, a maximal subpath in $P$ with no down (resp. up) step is called an **upward** (resp. **downward**) run if it contains at least one up (resp. down) step. Let $\text{run}(P)$ denote the total number of runs in $P$. If $P$ has no flat step, then the total number of peaks and valleys of $P$ is counted by $\text{run}(P) - 1$. For example, the path $P = uhufuddufuf$ has two upward runs, $uhuf$ and $ufuf$, and one downward run $fudd$ so that $\text{run}(P) = 3$. Note that $\text{run}(P) = 0$ if and only if $P$ is empty or a path consisting of flat steps only, and $\text{run}(P) = \text{run}(P')$. For a path $P \in \mathcal{F}(m, r, k)$, the following proposition shows that $\text{run}(P)$ and $\text{run}(\phi_{m,k}(P))$ are differ by at most 1.

**Proposition 2.5.** For positive integers $m$ and $k$, let $P \in \mathcal{F}(m, r, k)$ be given.

(a) If $P$ starts with an upward run, then
$$\text{run}(\phi_{m,k}(P)) = \text{run}(P) - \{1 - (-1)^m\}/2.$$  

(b) If $P$ starts with a downward run, then
$$\text{run}(\phi_{m,k}(P)) = \text{run}(P) - \{1 + (-1)^m\}/2.$$  

**Proof.** As erasing any number of flat steps do not change the number of runs, it suffices to show that this proposition holds when $r = 0$. We prove it by using induction on $k$.

For the initial step with $k = 1$, we consider Case 0. Recall that $\phi_{m,1}(P) = \mathcal{F}$. If $P$ starts with an up step, $m$ must be even and $\text{run}(\mathcal{F}) = \text{run}(P)$. When $P$ starts with a down step, $m$ is odd and $\text{run}(\mathcal{F}) = \text{run}(P)$.

Now we assume $k > 1$ and suppose that this proposition holds for any $P^* \in \mathcal{F}(m', 0, k')$ with $k' < k$. Here we give a detailed proof for (a) and the proof for (b) comes out similarly.

Suppose that $P \in \mathcal{F}(m, 0, k)$ starts with an up step and let $S = \phi_{m,k}(P)$. We need to show that $\text{run}(S)$ is given by $\text{run}(P) - 1$ (resp. $\text{run}(P)$) if $m$ is odd (resp. even).

In Case 1–i), we write $P = AuB$ (resp. $P = AdB$) and $S = BuA$ (resp. $S = Bu\overline{A}$), where $\overline{B}$ (resp. $B$) ends with an up step if $m$ is odd (resp. even). If $A$ is empty, then $m$ must be odd so that $\text{run}(S) = \text{run}(B) = \text{run}(P) - 1$ as we desire. Now assume that $A$ starts with an up step. In this case, $\text{run}(P) = \text{run}(A) + \text{run}(B)$ and

$$\text{run}(S) = \begin{cases} \text{run}(A) + \text{run}(B) - 1 & \text{if } m \text{ is odd,} \\ \text{run}(A) + \text{run}(B) & \text{if } m \text{ is even,} \end{cases}$$

so we are done.

In Case 1–ii), we write $P = AuB_1dB_2$ (resp. $P = AdB_1uB_2$) and $\overline{B_2} \in \mathcal{F}(m', k')$ (resp. $B_2 \in \mathcal{F}(m', k')$) for some $k' < k$ and odd $m'$, where $m$ is odd (resp. even). Let $r := \text{run}(A) + \text{run}(B_1)$. Note that if $m$ is odd (resp. even), then

$$\text{run}(P) = \begin{cases} r + \text{run}(B_2) & \text{if } B_2 \text{ starts with an up (resp. down) step,} \\ r + \text{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down (resp. up) step.} \end{cases}$$

In this case, if $m$ is odd (resp. even), then $S = B_1uAd\overline{C}$ (resp. $S = B_1uAd\overline{C}$), where $C = \phi_{m', k'}(B_2)$ (resp. $C = \phi_{m', k'}(B_2)$). By the induction hypothesis, if $m$ is...
odd (resp. even), then

\[
\text{run}(C) = \begin{cases} 
\text{run}(B_2) & \text{if } B_2 \text{ starts with an up (resp. down) step}, \\
\text{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down (resp. up) step}.
\end{cases}
\]

Hence, if \( m \) is odd, then \( \text{run}(S) = r + \text{run}(C) - 1 \) so that

\[
\text{run}(S) = \begin{cases} 
\text{run}(B_2) - 1 & \text{if } B_2 \text{ starts with an up step}, \\
\text{run}(B_2) - 2 & \text{if } B_2 \text{ starts with a down step},
\end{cases}
\]

which means that \( \text{run}(S) = \text{run}(P) - 1 \). Similarly, we show that \( \text{run}(S) = \text{run}(P) \) for even \( m \).

The proofs of Case 2–i) and Case 2–ii) are similar, so we only prove Case 2–ii). We divide this case into two cases depending on the parity of \( m \).

When \( m \) is odd, we write \( P = A_1 A_2 d B_1 u B_2 \) and \( S = B_1 u A_2 d C \), where \( A_1 \) starts with an up step and \( C = \phi_{m',k'}(A_1 u B_2) \) for some \( k' < k \) and even \( m' \). Note that \( \text{run}(P) = \text{run}(A_1) + \text{run}(A_2 d B_1) + \text{run}(u B_2) - 2, \text{run}(S) = \text{run}(B_1 u A_2 d) + \text{run}(C) - 1, \text{and run}(C) = \text{run}(A_1 u B_2) - 1 \) by the induction hypothesis. We have

\[
\text{run}(P) = \begin{cases} 
\text{run}(A_1) + \text{run}(A_2 d B_1) + \text{run}(B_2) - 1 & \text{if } B_2 \text{ starts with a down step}, \\
\text{run}(A_1) + \text{run}(A_2 d B_1) + \text{run}(B_2) - 2 & \text{if } B_2 \text{ starts with an up step},
\end{cases}
\]

and

\[
\text{run}(S) = \begin{cases} 
\text{run}(B_1 u A_2 d) + \text{run}(A_1) + \text{run}(B_2) - 2 & \text{if } B_2 \text{ starts with a down step}, \\
\text{run}(B_1 u A_2 d) + \text{run}(A_1) + \text{run}(B_2) - 3 & \text{if } B_2 \text{ starts with an up step}.
\end{cases}
\]

Since \( \text{run}(A_2 d B_1) = \text{run}(B_1 u A_2 d) \) whenever \( A_2 \) starts with \( u \) or \( d \), we get \( \text{run}(S) = \text{run}(P) - 1 \).

For even \( m \), we write \( P = u A_2 A_1 u B_2 \) and \( S = B_1 u A_2 d C \), where \( C = \phi_{m',k'}(u B_2) \) for some \( k' < k \) and even \( m' \). We have \( A_1 = \emptyset \) because \( P \) starts with \( u \). We also get \( \text{run}(S) = \text{run}(P) \) in a similar manner. \( \square \)

**Remark 2.6.** Proposition 2.5 confirms that \( |\text{run}(P) - \text{run}(\phi_{m,k}(P))| \leq 1 \), which shows that the map \( \phi_{m,k} \) is structurally distinguishable from the map from Gu and Prodinger [10]. For example, Gu and Prodinger’s map sends

\[
P = d u u u u u u u u u 
\mapsto S = u u u u u u d u u d u d u d u,
\]

which shows that their map has paths satisfying \( |\text{run}(P) - \text{run}(S)| = 2 \) (one can obtain this example by putting \( A = d d d d d d u u, B = u u, C = d u, \) and \( D = \emptyset \) in Figure 2.5 in [10]).

Let \( \overline{\mathcal{F}}(m, r, k; i) \) denote the set of paths in \( \mathcal{F}(m, r, k) \) with \( \lfloor i/2 \rfloor \) downward (resp. upward) runs for odd (resp. even) \( m \), and let \( \overline{\mathcal{M}}(m, r, k; i) \) denote the set of paths in \( \mathcal{M}(m, r, k) \) with \( i \) runs. By Proposition 2.5 it is straightforward to get the following corollary.

**Corollary 2.7.** For non-negative integers \( i \) and \( m \) of the same (resp. different) parity, the map \( \phi_{m,k} \) induces a bijection between the set \( \overline{\mathcal{M}}(m, r, k; i) \) and the set of paths in \( \overline{\mathcal{F}}(m, r, k; i) \) that end with a downward (resp. upward) run.

**Remark 2.8.** It is clear that the set of paths in \( \overline{\mathcal{M}}(m, 0, k; i) \) is in bijection with the set of symmetric Dyck paths of length \( 2m \) with \( i \) peaks which touch the line \( y = k \) and is contained in the strip \( 0 \leq y \leq k \). By Corollary 2.4 the set of symmetric Dyck
paths of length $2m$ with $i$ peaks corresponds to the set of paths in $\bigcup_{i}F(m,0;k;i)$ that end with a downward (resp. upward) run whenever $i$ and $m$ have the same (resp. different) parity. This one-to-one correspondence gives a combinatorial proof of the well-known fact that the number of symmetric Dyck paths of length $2m$ with $i$ peaks is given by

$$\binom{\left\lfloor \frac{m-i}{2}\right\rfloor}{\left\lfloor \frac{i}{2}\right\rfloor}\binom{\left\lceil \frac{m-i}{2}\right\rceil}{\left\lceil \frac{i}{2}\right\rceil}.$$  

3. Cornerless free Motzkin paths

In this section, we combinatorially interpret cornerless Motzkin paths as a $t$-core partitions. First let us consider the restriction of the map $\phi_{m,k}$.

3.1. Restriction to cornerlessly free Motzkin paths. Recall that $F_c(m,r,k)$ is the set of cornerless free Motzkin paths in $F(m,r,k)$ that never start with a down (resp. up) step for odd (resp. even) $m$, and $M_c(m,r,k)$ is the set of cornerless Motzkin prefixes in $M(m,r,k)$ that end with a flat step. Now we show that the map $\phi_{m,k}$, defined in Section 2.1 gives a one-to-one correspondence between these sets.

Proposition 3.1. For given non-negative integers $m, r$, and $k$, $\phi_{m,k}$ induces a bijection between the sets $F_c(m,r,k)$ and $M_c(m,r,k)$.

Proof. Let $P \in F_c(m,r,k)$ and $\phi_{m,k}(P) \in M_c(m,r,k)$. For $k = 0$ or $1$, $P \in F_c(m,r,k)$ when it is cornerless and starts with a flat step, and $\hat{P} \in M_c(m,r,k)$ when it is cornerless and ends with a flat step. Hence, $P \in F_c(m,r,k)$ if and only if $\phi_{m,k}(P) = \hat{P} \in M_c(m,r,k)$.

Let $m$ and $k$ have the same parity and there is no break step in $P$ with $k > 1$. It follows from (1) and (2) that $P \in F_c(m,r,k)$ and $\phi_{m,k}(P) \in M_c(m,r,k)$ have the same restriction such that $A$ and $B$ are cornerless, $A$ does not start with a down (resp. up) step for odd (resp. even) $m$, and $\beta > 0$. Hence, $P \in F_c(m,r,k)$ if and only if $\phi_{m,k}(P) \in M_c(m,r,k)$.

For the remaining cases, we assume that $\phi_{m',k'}$ induces a bijection between $F_c(m',r',k')$ and $M_c(m',r',k')$ for $k' < k$. We consider the case when $m$ and $k$ have the same parity and there is a break step $b$ in $P$. By (3) and (4), $P \in F_c(m,r,k)$ and $\phi_{m,k}(P) \in M_c(m,r,k)$ have the same condition such that $A$ and $B_1$ are cornerless, $A$ does not start with a down (resp. up) step, $B_2f\delta$ (resp. $B_2f\delta'$) $\in F_c(m',r',k')$ for some $k' < k$ when $m$ is odd (resp. even), and $\beta > 0$. Hence, $P \in F_c(m,r,k)$ if and only if $\phi_{m,k}(P) \in M_c(m,r,k)$.

Similarly, we can show that $P \in F_c(m,r,k)$ if and only if $\phi_{m,k}(P) \in M_c(m,r,k)$ when $m$ and $k$ have different parity by considering (5), (6), (7), and (8). \qed

3.2. Cornerless Motzkin paths and $t$-cores. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a non-increasing positive integer sequence. The Young diagram of $\lambda$ is an array of boxes arranged in left-justified rows with $\lambda_i$ boxes in the $i$th row. An inner corner of a Young diagram is a box that can be removed from the Young diagram and the rest of the Young diagram is still the Young diagram of a partition. We say that $\lambda$ has $m$ corners if its Young diagram has $m$ inner corners. For a given Young diagram, the hook length of a box at the position $(i, j)$, denoted by $h(i, j)$, is the number of boxes on the right, in the below, and itself. For a partition $\lambda$, the beta-set
Theorem 3.3. For non-negative integers $t$, denoted by $\beta(\lambda)$, is the set of hook lengths of boxes in the first column of the Young diagram of $\lambda$. A partition is called a $t$-core if its Young diagram has no box of hook length $t$. We mainly consider $t$-core partitions with $m$ corners and use the abacus diagram introduced by James and Kerber \[12\] to count them. The $t$-abacus diagram is a diagram to be the bottom and left-justified diagram with infinitely many rows labeled by $i \in \mathbb{N} \cup \{0\}$ and $t$ columns labeled by $j = 0, 1, \ldots, t - 1$ whose position $(i, j)$ is labeled by $ti + j$. The $t$-abacus of a partition $\lambda$ is obtained from the $t$-abacus diagram by placing a bead on each position labeled by $h$, where $h \in \beta(\lambda)$. A position without bead is called a spacer. The following lemma is useful to determine whether a given partition is $t$-core or not.

**Lemma 3.2.** \[12, \text{Lemma } 2.7.13\] A partition $\lambda$ is a $t$-core if and only if $h \in \beta(\lambda)$ implies $h - t \notin \beta(\lambda)$ whenever $h > t$. Equivalently, $\lambda$ is a $t$-core if and only if the $t$-abacus of $\lambda$ has no spacer below a bead in any column.

From the above lemma, we easily obtain a simple bijection between the set of $t$-core partitions and the set of non-negative integer sequences $(n_0, n_1, \ldots, n_{t-1})$, where $n_0 = 0$ and $n_j$ is the number of beads in column $j$ for $j = 1, \ldots, t - 1$. Using the bijection between the bar graphs and cornerless Motzkin paths, introduced by Deutsch and Elizalde \[8\], we give a path interpretation of the $t$-core partitions restricted by the number of corners and the first hook length $h(1,1)$.

**Theorem 3.3.** For non-negative integers $t$, $m$, and $k$, there is a bijection between any pair of the following sets.

(a) The set of $t$-core partitions with $m$ corners such that $h(1,1) < kt$.

(b) The set of non-negative integer sequences $(n_0, n_1, \ldots, n_{t-1})$ satisfying that $n_0 = 0$, $n_i \leq k$ for all $i$, and

$$\sum_{i=1}^{t} |n_i - n_{i-1}| = 2m,$$

where we set $n_t := 0$.

(c) The set of cornerless Motzkin paths of length $2m + t - 1$ with $t - 1$ flat steps that are contained in the strip $0 \leq y \leq k$.

**Proof.** Let $A$, $B$, and $C$ be the set described in (a), (b), and (c), respectively. Set the maps $\phi_1 : A \rightarrow B$ and $\phi_2 : B \rightarrow C$. For a partition $\lambda \in A$, let $n_i$ be the number of beads in the $i$th column of the $t$-abacus of $\lambda$. Given $\lambda \in A$, define $\phi_1(\lambda) = (n_0, n_1, \ldots, n_{t-1})$. Then, by the definition of the $t$-abacus and the fact that $h(1,1) < kt$, it is given that $n_0 = 0$ and $n_i \leq k$ for each $i$. Moreover, we get one inner corner for each maximal sequence of consecutive numbers in the beta-set $\beta(\lambda)$. Note that $\sum_{i=1}^{t} \max(n_i - n_{i-1}, 0)$ counts the number of hook lengths which is the smallest among each maximal sequence of consecutive numbers in the beta-set. So we get $\sum_{i=1}^{t} |n_i - n_{i-1}| = 2m$. Let $\psi_1 : B \rightarrow A$ and $\psi_2 : C \rightarrow B$. Define $\psi_1(\bar{n}) = \lambda$, where $\lambda$ is the partition obtained from the $t$-abacus diagram with $n_j$ beads in the $j$th column. We place the beads on the elements of $\beta(\lambda)$ in the $t$-abacus diagram. Then, since column 0 has no bead and each $n_i \leq k$ for all $i$, the largest element in $\beta(\lambda)$ is less than $kt$, meaning that $\lambda$ is a $t$-core partition with $h(1,1) < kt$. Also, the fact that the sum of $|n_i - n_{i-1}|$ is $2m$ implies that there are $m$ piles of beads which are placed on $m$ maximal consecutive numbers, so $\lambda$ has $m$ corners.
For \( \overrightarrow{n} = (n_0, n_1, \ldots, n_{t-1}) \in B \), let \( \phi_2(\overrightarrow{n}) = P_{\overrightarrow{n}} \), where \( P_{\overrightarrow{n}} \) be the cornerless Motzkin path which starts at \((0, 0)\), ends at \((2m + t - 1, 0)\), and has \( t - 1 \) flat steps at height \( n_1, n_2, \ldots, n_{t-1} \) with proper up and/or down steps connecting those flat steps. Due to the fact that \( n_i \leq k \), it is given that \( P_{\overrightarrow{n}} \) is contained in the strip \( 0 \leq y \leq k \).

Let \( \psi_2 : C \to B \) and \( P \in C \). Define \( \psi_2(P) = (n_0, n_1, \ldots, n_{t-1}) \), where \( n_0 = 0 \) and, for \( 1 \leq i \leq t - 1 \), each \( n_i \) represents the height of the \( i \)th flat step in \( P \). We know that \( P \) is contained in the strip \( 0 \leq y \leq k \), which implies \( n_i \leq k \). On the path \( P \), there are \( 2m \) many up and down steps. The number \( |n_i - n_{i-1}| \) represents the difference of the height of the \((i - 1)\)st flat step and the \(i\)th flat step, so it counts the number of up or down steps in between those two flat steps. Since \( \sum_{i=1}^{t} \max(n_i - n_{i-1}, 0) = \sum_{i=1}^{t} |\min(n_i - n_{i-1}, 0)| \), we get \( \sum_{i=1}^{t} |n_i - n_{i-1}| = 2m \).

\[ \square \]

For example, there are sixteen 4-core partitions with 2 corners. By letting \( t = 4 \) and \( m = 2 \) in Theorem 3.3, we get the correspondence between these partitions, abaci, non-negative integer sequences, and cornerless Motzkin paths as described in Figure 7.

We denote that a partition \( \lambda \) is a \((t_1, t_2, \ldots, t_p)\)-core if \( \lambda \) is a \( t_i \)-core for all \( i = 1, \ldots, p \). It is known that the number of \( t \)-core partitions is infinite, and the number of \((t_1, t_2, \ldots, t_p)\)-cores is finite for relatively prime \( t_1, \ldots, t_p \). Huang and Wang [11] enumerated the number of \((t, t+1)\)-cores, \((t, t+1, t+2)\)-cores with the fixed number of corners, where these results are generalized to \((t, t+1, \ldots, t+p)\)-cores in [4]. As far as we know, it seems new to get the formula for the number of \( t \)-core partitions with the fixed number of corners, which we enumerate this by using the path interpretation.

**Proposition 3.4.** The number of \( t \)-core partitions with \( m \) corners is given by

\[ \text{cc}(t, m) := \sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} N(m, i) \binom{t + 2m - 2i}{2m}, \]

where \( N(m, i) = \frac{1}{m} \binom{m}{i} \binom{m}{i-1} \) denotes the Narayana number.

**Proof.** By Theorem 3.3, \( \text{cc}(t, m) \) is equal to the number of cornerless Motzkin paths of length \( 2m + t - 1 \) with \( t - 1 \) flat steps. Let a Dyck path consisting of \( m \) up steps and \( m \) down steps with \( i \) peaks be given. The number of ways of inserting \( t - 1 \) flat steps such that the resultant path becomes a cornerless Motzkin path is \( \binom{2m - 2i}{m} \) since we have to insert at least one flat step at the positions of \( i \) peaks and \( i - 1 \) valleys. As the number of Dyck paths consisting of \( m \) up steps and \( m \) down steps with \( i \) peaks is counted by the Narayana number \( N(m, i) \), the proof is followed. \( \square \)

The numbers of \( t \)-core partitions with \( m \) corners for \( 2 \leq t \leq 6 \) and \( 1 \leq m \leq 8 \) are given in Table 1. Clearly, \( \text{cc}(2, m) = 1 \), \( \text{cc}(3, m) = 2m + 1 \), and \( \text{cc}(4, m) = (5m^2 + 5m + 2)/2 \). See sequences A063490 and A160747 in [13] for more the values of \( \text{cc}(t, m) \) for \( t = 5 \) and \( t = 6 \), respectively.

**3.3. Cornerless symmetric Motzkin paths and self-conjugate \( t \)-cores.** For a partition \( \lambda \), its conjugate is the partition \( \lambda' = (\lambda'_1, \lambda'_2, \ldots) \), where each \( \lambda'_j \) is the number of boxes in the \( j \)th column of the Young diagram of \( \lambda \). A partition \( \lambda \) is called self-conjugate if \( \lambda = \lambda' \). Let \( \text{MD}(\lambda) \) denote the set of the main diagonal hook
### Figure 7. 4-cores with 2 corners and the corresponding objects

### Table 1. The numbers $cc(t, m)$ of $t$-cores with $m$ corners

| $t \backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|
| 2               | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3               | 3 | 5 | 7 | 9 | 11| 13| 15| 17|
| 4               | 6 | 16| 31| 51| 76|106|141|181|
| 5               | 10| 40|105|219|396|650|995|1445|
| 6               | 15| 85|295|771|1681|3235|5685|9325|

Note that if $\lambda$ is a self-conjugate partition, then the elements in $MD(\lambda)$...
are all distinct and odd. Similar to Lemma 3.2, Ford, Mai, and Sze \cite{9} gave a useful result to determine whether a given partition is self-conjugate t-core or not.

**Proposition 3.5.** \cite{9} Proposition 3\] Let \( \lambda \) be a self-conjugate partition. Then \( \lambda \) is a t-core if and only if both of the following hold:

(a) For \( h > t \), if \( h \in \text{MD}(\lambda) \), then \( h - 2t \in \text{MD}(\lambda) \).

(b) If \( h_1,h_2 \in \text{MD}(\lambda) \), then \( h_1 + h_2 \neq 2t \).

We slightly modify the t-abacus to get the t-doubled abacus, which is useful when we deal with a self-conjugate t-core partition. Let the t-doubled abacus diagram is a left-justified diagram with infinitely many rows labeled by \( i \in \mathbb{Z} \) and \( |t/2| \) columns labeled by \( j = 0,1,\ldots,|t/2| - 1 \) whose position \((i,j)\) is labeled by \( |2(ti + j) + 1| \).

The t-doubled abacus of a self-conjugate partition \( \lambda \) is obtained from the t-doubled abacus diagram by placing a bead on each position labeled by \( h \), where \( h \in \text{MD}(\lambda) \).

From Proposition 3.5, we have the following lemma.

**Lemma 3.6.** A self-conjugate partition \( \lambda \) is t-core if and only if the t-doubled abacus diagram of \( \lambda \) satisfies both of the following.

(a) If a bead is placed on position \((i,j)\) with \( i > 0 \) (resp. \( i < 0 \)), then a bead is also placed on position \((0,j)\) (resp. \((-1,j)\)) and there is no spacer between them in any column \( j \).

(b) A bead can be placed on at most one of the two positions \((-1,j)\) and \((0,j)\) in any column \( j \).

From the above lemma, we easily obtain a simple bijection between the set of self-conjugate t-core partitions and the set of integer sequences \( (n_0,\ldots,n_{|t/2| - 1}) \), where the number of beads in column \( j \) is denoted by either \( n_j \) or \(-n_j\) for \( j = 0,1,\ldots,|t/2| - 1 \) if a bead is placed in position \((0,j)\) or not, respectively. Now we give a path interpretation of the self-conjugate t-core partitions restricted by the number of corners and the first hook length \( h(1,1) \). We define

\[
\mathcal{F}_c(m,r,k) := \bigcup_{i=0}^{k} \mathcal{F}_c(m,r,i) \quad \text{and} \quad \mathcal{M}_c(m,r,k) := \bigcup_{i=0}^{k} \mathcal{M}_c(m,r,i).
\]

**Theorem 3.7.** For non-negative integers \( t, m, \) and \( k \), there is a bijection between any pair of the following sets.

(a) The set of self-conjugate t-cores with \( m \) corners such that \( h(1,1) < kt \).

(b) The set of integer sequences \( (n_0,n_1,\ldots,n_{|t/2| - 1}) \) satisfying that for odd (resp. even) \( m, n_0 \) is positive (resp. non-positive); for all \( i, -\lfloor k/2 \rfloor \leq n_i \leq \lfloor (k + 1)/2 \rfloor \); and

\[
\sum_{i=0}^{\lfloor t/2 \rfloor} |n_i - n_{i-1}| = \begin{cases} 
m + 1 & \text{for odd } m, \\
m & \text{for even } m, 
\end{cases}
\]

where we set \( n_{-1} := 0 \) and \( n_{|t/2|} := 0 \).

(c) The set of cornerless free Motzkin paths in \( \mathcal{F}_c(m,\lfloor t/2 \rfloor, k) \).

(d) The set of cornerless Motzkin prefixes in \( \mathcal{M}_c(m,\lfloor t/2 \rfloor, k) \).

(e) The set of cornerless symmetric Motzkin paths of length \( 2m + t - 1 \) with \( t - 1 \) flat steps that are contained in the strip \( 0 \leq y \leq k \).

**Proof.** Let \( A,B,C,D, \) and \( E \) be the set described in (a), (b), (c), (d), and (e), respectively. By similar argument to the proof of Proposition 3.1 we know that
there is a bijection between $C$ and $D$. Now we set $\phi_1 : A \to B$, $\phi_2 : B \to C$, and $\phi_3 : D \to E$ and show that $\phi_1, \phi_2, \phi_3$ are bijections.

Given $\lambda \in A$, let $\phi_1(A) = (n_0, n_1, \ldots, n_{\lfloor t/2 \rfloor} - 1)$, where each $n_i$ is the highest or lowest row that the bead is placed in the $i$th column depending on the sign of $n_i$. We get that $1 \in MD(\lambda)$ when the number of corners $m$ is even and $1 \notin MD(\lambda)$ when the number of corners $m$ is odd. Thus, $n_0$ is positive when $m$ is odd and negative otherwise. This map gives a bijection between $A$ and $B$.

Let $\vec{n} = (n_0, n_1, \ldots, n_{\lfloor t/2 \rfloor} - 1)$. For odd (resp. even) $m$, let $\phi_2(\vec{n})$ be the cornerless free Motzkin path that starts at $(0, 1)$ (resp. $(0, 0)$), ends at $(m + \lfloor t/2 \rfloor, 0)$, has $i$th flat step at height $n_{i-1}$ with proper up and down steps between them. Then, the map $\phi_2$ describes a bijection between $B$ and $C$.

Denote a path by $P = p_1 p_2 \cdots p_{m+\lfloor t/2 \rfloor} \in D$. We set

$$\phi_3(P) = \begin{cases} p_1 p_2 \cdots p_m + \lfloor t/2 \rfloor p_{m+\lfloor t/2 \rfloor} \cdots p_2 p_1 & \text{if } m \text{ is odd}, \\ p_1 p_2 \cdots p_m + \lfloor t/2 \rfloor - 1 p_m + \lfloor t/2 \rfloor p_m + \lfloor t/2 \rfloor - 1 \cdots p_2 p_1 & \text{if } m \text{ is even}. \end{cases}$$

Then, the map $\phi_3$ is a bijection. \qed

Note that Figure 7 shows that there are four self-conjugate 4-core partitions with 2 corners and four cornerless symmetric Motzkin paths of length 7 with 3 flat steps, which are marked by $\ast$ and $\ast\ast$, respectively. The correspondences between the sets described in Theorem 3.7 for $t = 4, m = 2$ and $t = 5, m = 3$ are given in Figure 8.

Although the number of self-conjugate $(t, t + 1, \ldots, t + p)$-cores with the fixed number of corners is unknown in general, it is enumerated in [2, 3] when $p = 1, 2$, and 3. The number of self-conjugate $t$-core partitions with $m$ corners can be counted by using these path interpretations.

**Proposition 3.8.** The number of self-conjugate $t$-core partitions with $m$ corners is given by

$$scc(t, m) := \sum_{i=1}^{\min(m, \lfloor t/2 \rfloor)} \binom{\lfloor t/2 \rfloor}{i} \binom{m/2 - 1}{i} \binom{\lfloor t/2 \rfloor + m - i}{m}$$

for $m > 0$ and $scc(t, 0) = 1$. In addition, $scc(t, m) = scc(t + 1, m)$ for even $t$.

**Proof.** By Theorem 3.7, $scc(t, m)$ also counts the number of cornerless symmetric Motzkin paths of length $2m + t - 1$ with $t - 1$ flat steps. Let a symmetric Dyck path consisting of $m$ up steps and $m$ down steps with $i$ peaks with $2i \leq t$ be given. The number of ways inserting $t - 1$ flat steps such that the resultant path becomes a cornerless symmetric Motzkin path is $\binom{\lfloor t/2 \rfloor + m - i}{m}$. The proof is followed since the number of symmetric Dyck paths consisting of $m$ up steps and $m$ down steps with $i$ peaks is given by [13]. \qed

The numbers of self-conjugate $t$-core partitions with $m$ corners for $2 \leq t \leq 11$ and $1 \leq m \leq 8$ are given in Table 2. Clearly, $scc(2, m) = scc(3, m) = 1$, $scc(4, m) = scc(5, m) = \lceil 3m/2 \rceil + 1$, and $scc(6, m) = scc(7, m) = \left(10m(m + 1) + (-1)^m(2m + 1) + 7\right)/8$. 


Figure 8. Examples of self-conjugate $t$-cores with $m$ corners and the corresponding objects

Table 2. The numbers $\text{scc}(t,m)$ of self-conjugate $t$-cores with $m$ corners

| $t \setminus m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|
| 2,3             | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4,5             | 2 | 4 | 5 | 7 | 8 |10 |11 |13 |
| 6,7             | 3 | 9 |15 |27 |37 |55 |69 |93 |
| 8,9             | 4 |16 |34 |76 |124 |216 |309 |471 |
| 10,11           | 5 |25 |65 |175 |335 |675 |1095 |1875 |

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