Research Article

Approximation by Parametric Extension of Szász-Mirakjan-Kantorovich Operators Involving the Appell Polynomials

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The purpose of this article is to introduce a Kantorovich variant of Szász-Mirakjan operators by including the Dunkl analogue involving the Appell polynomials, namely, the Szász-Mirakjan-Jakimovski-Leviatan-type positive linear operators. We study the global approximation in terms of uniform modulus of smoothness and calculate the local direct theorems of the rate of convergence with the help of Lipschitz-type maximal functions in weighted space. Furthermore, the Voronovskaja-type approximation theorems of this new operator are also presented.

1. Introduction

In the year 1950, a famous mathematician Szász [1] invented the positive linear operators for the continuous function $f$ on $[0, \infty)$ and that were extensively searched rather than Bernstein operators [2]. For $z \in [0, \infty)$ and $f \in C[0, \infty)$, Szász introduced the operators as follows:

$$S_r(f; z) = e^{-rz} \sum_{k=0}^{\infty} \frac{(rz)^k}{k!} f\left(\frac{k}{r}\right),$$

where $C[0, \infty)$ is the space of continuous functions on $[0, \infty)$. In recent years, Szász-Mirakjan operators were introduced by Sucu [3] by proposing an exponential function on Dunkl generalization by including a nonnegative number $\eta \geq 0$, such that

$$S_r^\eta(f; z) = \frac{1}{e^\eta(rz)} \sum_{k=0}^{\infty} \frac{(rz)^k}{k!} \frac{(r+2\eta)^{k+1}}{r} \gamma k(k, r),$$

where $e^\eta(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \gamma k(k, r)$ and a recursion formula for $s = 0, 1, 2, \ldots$

$$y_{\eta'}(\kappa + 1) \frac{1 + 2\eta \theta_{k+1}}{\kappa + 1} = y_{\eta'}(\kappa),$$

$$\theta_k = \begin{cases} 0 & \text{if } \kappa = 2r, r \in \mathbb{N} \cup \{0\}, \\ 1 & \text{if } \kappa = 2r + 1, r \in \mathbb{N} \cup \{0\}. \end{cases}$$

In 1969, Jakimovski and Leviatan introduced the sequence of Szász-Mirakjan-type positive linear operators by the use of Appell polynomials [4], $L(u)e^{uz} = \sum_{k=0}^{\infty} H_k(z)u^k$ such that

$$J_r(h; z) = \frac{e^{-rz}}{L(1)} \sum_{k=0}^{\infty} H_k(rz)f\left(\frac{k}{r}\right),$$

where $L(1) \neq 0$, $L(u) = \sum_{k=0}^{\infty} b_k u^k$, $H_k(z) = \sum_{j=0}^{k} b_j \frac{(z^{k-j})}{(k-j)!} (k \in \mathbb{N})$. Note that, if $L(1) = 1$ in (4), the Szász-Mirakjan
operator (1) is obtained. Most recently, in [5], Nasiruzzaman and Aljohani have introduced the Szász-Mirakjan-Jakimovski-Leviatan-type operators involving the Dunkl generalization for the function \( f \in C[0,\infty) \) by

\[
\mathcal{P}_{\gamma}(f; z) = \frac{1}{L(1)} e_{\gamma}(r) \sum_{n=0}^{\infty} H_n(r) f\left(\frac{k+2n\theta z}{r}\right).
\]

(5)

**Lemma 1.** [5]. For the test function \( y_j = v_j \), if \( j = 0, 1, 2, 3, 4 \), the operators \( \mathcal{P}_{\gamma}(y_j; z) = 1 \), \( \mathcal{P}_{\gamma}(y_1; z) = z + (1/r)((L'(1)/L(1)) + 2\eta) \), and the following identities:

\[
\mathcal{P}_{\gamma}(y_j; z) = z^j + \frac{1}{r} \left( \frac{2L'(1)}{L(1)} + 4\eta + 1 \right) z + \frac{1}{r^2} \left( \frac{L''(1)}{L(1)} + (1 + 4\eta) \frac{L'(1)}{L(1)} + 4\eta^2 \right).
\]

\[
\mathcal{P}_{\gamma}(y_j; z) = z^j + \frac{3}{r} \left( \frac{L'(1)}{L(1)} + 2\eta + 1 \right) z^j + \frac{1}{r^2} \left( \frac{L''(1)}{L(1)} + 6(1 + 2\eta) \frac{L'(1)}{L(1)} + 2 + 6\eta + 12\eta^2 \right) z + \frac{1}{r^3} \left( \frac{L'''(1)}{L(1)} + 3(1 + 2\eta) \frac{L''(1)}{L(1)} + 2 + 6\eta + 12\eta^2 \right) z^j + \frac{1}{r^4} \left( \frac{L''''(1)}{L(1)} + 2(1 + 3\eta + 6\eta^2) \frac{L'(1)}{L(1)} + 8\eta^2 \right).
\]

\[
\mathcal{P}_{\gamma}(y_4; z) = z^4 + \frac{1}{r} \left( \frac{4L'(1)}{L(1)} + 8\eta + 6 \right) z^4 + \frac{1}{r^2} \left( \frac{6L''(1)}{L(1)} + 8(1 + 3\eta) \frac{L'(1)}{L(1)} + 11 + 24\eta + 24\eta^2 \right) z^j + \frac{1}{r^3} \left( \frac{4L'''(1)}{L(1)} + 8(1 + 3\eta) \frac{L''(1)}{L(1)} + 2 + 11 + 24\eta + 24\eta^2 \right) z^j + \frac{1}{r^4} \left( \frac{L'(1)}{L(1)} + 6 + 16\eta + 24\eta^2 + 32\eta^2 \right) z + \frac{1}{r^5} \left( \frac{L'(1)}{L(1)} + 6 + 16\eta + 24\eta^2 + 32\eta^2 \right) z + \frac{1}{r^6} \left( \frac{L'(1)}{L(1)} + 6 + 16\eta + 24\eta^2 + 32\eta^2 \right) z + \frac{1}{r^7} \left( \frac{L'(1)}{L(1)} + 6 + 16\eta + 24\eta^2 + 32\eta^2 \right) z + \frac{1}{r^8} \left( \frac{L'(1)}{L(1)} + 6 + 16\eta + 24\eta^2 + 32\eta^2 \right) z.
\]

(6)

There are several research articles mentioned regarding the Szász-Mirakjan-type operators, for instance, [6–13]. For some further related concepts and approximation, we refer to see [9, 10, 14–20].

### 2. Kantorovich Operators Involving Appell Polynomials and Their Moments

In this section, we construct the generalized operators of recent investigation [5] including the Kantorovich polynomial.

**Proof.** To prove this Lemma, we take into account [5] Lemma 1. Thus, for all \( j = 0, 1, 2, 3, 4 \) and \( y_j = v_j \), we can conclude that...
Thus, from (7) and (9), clearly we can write
\[ R^*_r(\eta /z) = 1, \]
\[ R^*_r(y_1 /z) = \frac{1}{2r} R^*_r(y_0 /z), \]
\[ R^*_r(y_2 /z) = \frac{1}{r} R^*_r(y_1 /z) + \frac{1}{3r} R^*_r(y_0 /z), \]
\[ R^*_r(y_3 /z) = \frac{3}{2r} R^*_r(y_2 /z) + \frac{1}{r} R^*_r(y_1 /z) \]
\[ + \frac{1}{r} R^*_r(y_0 /z), \]
\[ R^*_r(y_4 /z) = \frac{2}{r} R^*_r(y_3 /z) + \frac{2}{r} R^*_r(y_2 /z) \]
\[ + \frac{1}{r} R^*_r(y_1 /z) + \frac{1}{r} R^*_r(y_0 /z). \]

Therefore, by applying Lemma 1, we get the required results.

**Lemma 3.** For the central moments \((y_i - z)^i, i = 1, 2, 4\), we have the following identities:

\[ R^*_r((y_1 - z)^i); z) = \frac{1}{2r} \left( 2 L'(1) L(1) + 4\eta + 1 \right), \]
\[ R^*_r((y_1 - z)^2); z) = \frac{z}{r} \left( 3 L'(1) L(1) + 6(1 + 2\eta) L'(1) L(1) \right. \]
\[ + 12\eta^2 + 6\eta + 1 \),

for \( j = 0, \)

for \( j = 1, \)

for \( j = 2, \)

for \( j = 3, \)

for \( j = 4. \)

\[ R^*_r((y_1 - z)^4); z) = -\frac{7}{r^2} z^2 + \frac{1}{r^3} \left( -8 L''(1) L(1) - 4 L'(1) L(1) \right) \]
\[ + (24\eta + 20) L'(1) \left( 4r^2 + 28\eta + 12 \right) z \]
\[ + \frac{1}{r^2} \left( L'(1) L(1) + 4(3 + 2\eta) L''(1) L(1) \right) \]
\[ + (19 + 36\eta + 24\eta^2) L'(1) L(1) + (13 + 36\eta \]
\[ + 48\eta^2 + 32\eta^3) L'(1) L(1) + 16\eta^4 + 16\eta^3 \]
\[ + 8\eta^2 + 2\eta + 1 \}. \]

\[ (11) \]

### 3. Approximations in Weighted Space

In the present section, we follow the well-known results by Gadziew [21] and recall the results in weighted spaces with some additional conditions precisely, under the analogous of P.P. Korovkin’s theorem holds. In order to define the uniformly approximations, we take \( z \to \phi(z) \) be the kind of functions which is continuous and strictly increasing with the assumptions \( \Phi(z) = 1 + \phi^2(z) \) and \( \lim_{z \to \infty} \phi(z) = \infty \). For this reason, we let \( B_\phi[0, \infty) \) be a set of all such functions which are defined on \((0, \infty)\) and verifying the results

\[ B_\phi[0, \infty) = \{ f : |f(z)| \leq K_f \Phi(z) \}, \]

where \( K_f \) is a constant and depending only on function \( f \) and \( B_\phi[0, \infty) \) equipped the norm with

\[ ||f||_\phi = \sup_{z \in (0, \infty)} \frac{|f(z)|}{\Phi(z)}. \]

\[ (13) \]
Furthermore, we denote the set all continuous functions on \([0, \infty)\) by \(C_0[0, \infty)\) and its subsets by \(C_0[0, \infty), \cap C_0[0, \infty)\).

It is well known for the sequence of linear positive operators \(\{K_r\}_{r \geq 1}\) (see [21]) maps \(C_0[0, \infty)\) into \(B_0[0, \infty)\) if and only if

\[
|K_r(\Phi; z)| \leq M\Phi(z),
\]

where \(M\) is a positive constant. For \(m \in \mathbb{N}\), let us denote

\[
C_0^m[0, \infty) = \left\{ f \in C_0[0, \infty): \lim_{z \to \infty} \frac{f(z)}{\Phi(z)} = c, \text{exists and is finite} \right\}.
\]

**Theorem 4.** Let \(H_j = \{ f: \text{such that } f(z)/\Phi(z) \text{is convergent when } z \to \infty\}. \) Then, for every \(f \in H_j \cap C_0[0, \infty)\), operators (7) are uniformly convergent on each compact subset of \([0, \infty)\) such that

\[
R_{r_n}^*(f; z) \Rightarrow f,
\]

where \(\Rightarrow\) denotes the uniform convergence.

**Proof.** In view of Lemma 2, we use Korovkin’s theorem by [22]; then it is enough to see that for each \(j = 0, 1, 2\)

\[
R_{r_n}^*(y_j; z) \to z^j
\]

uniformly. Thus obviously, we get \(\lim_{r \to \infty} R_{r_n}^*(y_0; z) = 1\), \(\lim_{r \to \infty} R_{r_n}^*(y_1; z) = z\), and \(\lim_{r \to \infty} R_{r_n}^*(y_2; z) = z^2\), which completes the proof of Theorem 4.

**Theorem 5** [21, 23]. Let the positive linear operators \(\{I_r\}_{r \geq 1}\) acting from \(C_0[0, \infty)\) to \(B_0[0, \infty)\) and for every \(f \in H_j \cap C_0[0, \infty)\), if it verifies that \(\lim_{r \to \infty} \|I_r(y_j) - z^j\| \Phi = 0\), then for every \(f \in C_0^m[0, \infty)\) it satisfies

\[
\lim_{r \to \infty} \|I_r(f) - f\| \Phi = 0.
\]

**Theorem 6.** For every \(\varphi \in C_0^m[0, \infty)\), operators \(R_{r_n}^*\) satisfy

\[
\lim_{r \to \infty} \left\| R_{r_n}^*(\varphi) - \varphi \right\| \Phi = 0.
\]

**Proof.** It is enough to prove Theorem 6; we use the well-known Korovkin theorem and show

\[
\lim_{r \to \infty} \| R_{r_n}^*(y_j) - z^j \| \Phi = 0, \quad j = 0, 1, 2.
\]

Taking into account Lemma 2, then it is easy to see that

\[
\| R_{r_n}^*(y_0) - 1 \| \Phi = \sup_{z \in [0, \infty)} \frac{| R_{r_n}^*(1; z) - 1 |}{\Phi(z)} = 0.
\]

For \(j = 1\), we can write here

\[
\| R_{r_n}^*(y_1) - z \| \Phi = \sup_{z \in [0, \infty)} \frac{| R_{r_n}^*(1; z) - 1 |}{\Phi(z)} = 0.
\]

If \(r \to \infty\), then easily we get \(\| R_{r_n}^*(y_1) - z \| \Phi \to 0\). Similarly, for \(j = 2\), we conclude that

\[
\| R_{r_n}^*(y_2) - z^2 \| \Phi = \sup_{z \in [0, \infty)} \frac{| R_{r_n}^*(1; z) - 1 |}{\Phi(z)} = 0.
\]

Thus, we easily get \(\| R_{r_n}^*(y_j) - z^j \| \Phi \to 0\), as \(r \to \infty\).

**Theorem 7.** If \(\varphi \in C_0^m[0, \infty)\). Then, operators \(R_{r_n}^*\) follow that

\[
\lim_{r \to \infty} \sup_{z \in [0, \infty)} \frac{| R_{r_n}^*(\varphi; z) - \varphi(z) |}{(\Phi(z))^{1+\xi}} = 0,
\]

where the number \(\xi \in [0, \infty)\).

**Proof.** By the virtue of \(|\varphi(z)| \leq |\varphi| \Phi(1 + z^2)\) and for any positive real \(z_0\), we easily obtain

\[
\lim_{r \to \infty} \sup_{z \in [0, \infty)} \frac{| R_{r_n}^*(\varphi; z) - \varphi(z) |}{(\Phi(z))^{1+\xi}} \leq \sup_{z \in z_0} \frac{| R_{r_n}^*(\varphi; z) - \varphi(z) |}{(\Phi(z))^{1+\xi}} + \sup_{z \in z_0} \frac{| R_{r_n}^*(\varphi; z) - \varphi(z) |}{(\Phi(z))^{1+\xi}} - \| \varphi \| \Phi \sup_{z \in z_0} \frac{| \varphi(z) |}{(\Phi(z))^{1+\xi}}
\]

\[
= f_1 + f_2 + f_3, \quad (\text{suppose}).
\]

Thus,

\[
f_3 = \sup_{z \in z_0} \frac{| \varphi(z) |}{(\Phi(z))^{1+\xi}} \leq \sup_{z \in z_0} \frac{| \varphi | \Phi (1 + z^2)}{(\Phi(z))^{1+\xi}} \leq \frac{| \varphi | \Phi}{(1 + z_0^2)\xi}.
\]
From Lemma 2, it follows that
\[
\lim_{r \to \infty} \sup_{z \geq z_0} \frac{R_{r, \eta}(1 + t^2; z)}{\Phi(z)} = 1. \tag{27}
\]

Now, for each \( \epsilon > 0 \), there exists \( r_1 \in \mathbb{N} \) for all \( r \geq r_1 \) such that
\[
\sup_{z \geq z_0} \frac{R_{r, \eta}(1 + t^2; z)}{\Phi(z)} \leq \frac{(1 + z_0^2)^{\frac{r}{2}} \epsilon}{3} + 1. \tag{28}
\]
Therefore, for all \( r \geq r_1 \)
\[
J_2 = \frac{\|\phi\| \sup_{z \geq z_0} \frac{R_{r, \eta}(1 + t^2; z)}{(\Phi(z))^{\frac{r}{2}}} \leq \frac{\|\phi\|}{3} + \frac{\epsilon}{3}. \tag{29}
\]
In view of (26) and (29), we get
\[
J_2 + J_3 \leq 2 \left( \frac{\|\phi\|}{3} + \frac{\epsilon}{3} \right). \tag{30}
\]

If we choose any \( z_0 \), so large, such that \( \|\phi\|(1 + z_0^2)^{\frac{r}{2}} \leq \epsilon/6 \), then we get
\[
J_2 + J_3 \leq \frac{2\epsilon}{3}, \quad \text{for all } r \geq r_1. \tag{31}
\]
On the other hand, there exists \( r_2 \geq r \) such that
\[
J_1 = \left\| \frac{R_{r, \eta}(\phi; z) - \phi(z)}{(\Phi(z))^{\frac{r}{2}}} \right\|_{C[0, z_0]} \leq \frac{\epsilon}{3} \tag{32}
\]
Finally, take \( r_3 = \max\{r_1, r_2\} \) and on combining (31) and (32) with the above expression, we get
\[
\sup_{z \in [0, \infty)} \frac{|R_{r, \eta}(\phi; z) - \phi(z)|}{(\Phi(z))^{\frac{r}{2}}} < \epsilon. \tag{33}
\]
This completes the proof of Theorem 7.

**Definition 8.** For every \( \delta > 0 \) and all \( f \in C[0, \infty) \), the modulus of continuity of the uniformly continuous function \( f \) on \([0, \infty)\) defined as
\[
\bar{\omega}(f; \delta) = \sup_{t_1, t_2, \delta} |f(t_1) - f(t_2)|, \quad t_1, t_2 \in [0, \infty),
\]
\[
|f(t_1) - f(t_2)| \leq \left( 1 + \frac{|t_1 - t_2|}{\delta^2} \right) \bar{\omega}(f; \delta). \tag{34}
\]

**Theorem 9** [24]. Let the sequence of positive linear operators \( \{K\}_{r \in \mathbb{N}} : [x, y] \to C[u, v] \) and \([u, v] \subseteq [x, y]\), then

1. for any \( f \in C[x, y] \) and \( z \in [u, v] \), it follows that
\[
|K_r(f; z) - f(z)| \leq |f(z)||K_r(1; z) - 1| + \left\{ K_r(1; z) + \frac{1}{\delta} \sqrt{K_r((t-z)^2; z) K_r(1; z)} \right\} \omega(f; \delta), \tag{35}
\]
2. if any \( \phi' \in C[x, y] \), then for all \( z \in [u, v] \) one has
\[
|K_r(\phi; z) - \phi(z)| \leq |\phi(z)||K_r(1; z) - 1| + |\phi'(z)|
\times |K_r((t-z)^2; z) + K_r((t-z)^2; z) - 2\delta| \omega(\phi'; \delta). \tag{36}
\]

**Theorem 10.** Let \( f \in C_0[0, \infty) \), then for all \( z \in [0, \infty) \) it follows the inequality
\[
|R_{r, \eta}(f; z) - f(z)| \leq 2\omega\left( f; \sqrt{R_{r, \eta}^*(z)} \right). \tag{37}
\]
where \( \delta = \sqrt{R_{r, \eta}^*(z)} = \sqrt{R_{r, \eta}^*((y_1 - z)^2; z)}. \)

**Proof.** If we consider Lemma 2 and Theorem 9, then we can obtain
\[
|R_{r, \eta}(f; z) - f(z)| \leq |f(z)||R_{r, \eta}(1; z) - 1| + \left\{ R_{r, \eta}(1; z) + \frac{1}{\delta} \sqrt{R_{r, \eta}^*((y_1 - z)^2; z) R_{r, \eta}^*(1; z)} \right\} \omega
\times (f; \delta), \tag{38}
\]
where if we take \( \delta = \sqrt{R_{r, \eta}^*(z)} = \sqrt{R_{r, \eta}^*((y_1 - z)^2; z)} \) then we are easily denumerable to get results.

**Theorem 11.** For any \( z \in [0, \infty) \), if \( \phi \in C_0[0, \infty) \), then we have the inequality
\[
|R_{r, \eta}(\phi; z) - \phi(z)| \leq \frac{1}{2\delta} \left( \frac{2L^2}{L(1)} + 4\eta + 1 \right) |\phi'(z)| + 2\delta^2 \omega(\phi'; \sqrt{R_{r, \eta}^*(z)}), \tag{39}
\]
where \( \delta = \sqrt{R_{r, \eta}^*(z)} = \sqrt{R_{r, \eta}^*((y_1 - z)^2; z)}. \)
Proof. If we consider Lemmas 2 and 3 and (2) of Theorem 9, then it is obvious to get that

\[
\left| \mathcal{R}_{r_\eta}(\phi; z) - \phi(z) \right| \\
\leq \left| \mathcal{R}_{r_\eta}(y_0; z) - 1 \right| \left| \phi(z) \right| + \left| \phi'(z) \right| \\
\times \left| \mathcal{R}_{r_\eta}(y_1 - z; z) + \mathcal{R}_{r_\eta}((y_1 - z)^2; z) \right| \\
\times \left\{ \sqrt{\mathcal{R}_{r_\eta}(y_0; z)} + \frac{1}{\delta} \sqrt{\mathcal{R}_{r_\eta}((y_1 - z)^2; z)} \right\} \omega\left( \phi'; \delta \right). 
\]

(40)

Put \( \bar{\delta} = \sqrt{\delta_{r_\eta}(z)} = \sqrt{\mathcal{R}_{r_\eta}((y_1 - z)^2; z)} \), then we easily get our desired results of Theorem 11.

From [25] for an arbitrary \( f \in C^m_{\Omega}(0, \infty) \), \( m \in \mathbb{N} \cup \{0\} \), the weighted modulus of continuity introduced such that

\[
\Omega(f; \delta) = \sup_{x \in [0, \infty), h \leq \delta} \left| \frac{f(x + h) - f(z)}{(1 + h^2)(1 + z^2)} \right|. 
\]

(41)

Two main properties of this modulus of continuity are \( \lim_{\delta \to 0} \Omega(f; \delta) = 0 \) and

\[
|f(t) - f(z)| \leq 2 \left( 1 + \frac{|t - z|}{\delta} \right) \left( 1 + \delta^2 \right) \left( 1 + z^2 \right) \\
\times (1 + (t - z)^2) \Omega(f; \delta), 
\]

(42)

where \( t, z \in [0, \infty) \) and \( \Omega \) weighted modulus of continuity of the function for \( f \in C^m_{\Omega}(0, \infty) \).

**Theorem 12.** Let \( f \in C^m_{\Omega}(0, \infty) \), then for all \( z \in [0, \infty) \) we have the inequality

\[
\sup_{x \in [0,\alpha,(\eta))] \left| \mathcal{R}_{r_\eta}(f; z) - f(z) \right| \\
\leq M(1 + \alpha, (\eta)) \Omega\left( f; \sqrt{\alpha, (\eta)} \right), 
\]

(43)

where \( M = 2(2 + M_1 + \sqrt{M_2}) > 0 \), for \( M_1, M_2 > 0 \) and

\[
\alpha, (\eta) = \max \left\{ \frac{1}{r^3}, \frac{1}{3r^3} \left( \frac{L''(1)}{L(1)} + 6(1 + 2\eta) \frac{L'(1)}{L(1)} \right) \right. \\
+ 12\eta^2 + 6\eta + 1 \right\}. 
\]

(44)

Proof. We use expressions (41) and (42) and applying the Cauchy-Schwarz inequality to operators \( \mathcal{R}_{r_\eta} \), we get

\[
\left| \mathcal{R}_{r_\eta}(f; z) - f(z) \right| \leq 2 \left( 1 + \delta^2 \right) \left( 1 + z^2 \right) \Omega(f; \delta) \\
\times \left\{ 1 + \mathcal{R}_{r_\eta}((y_1 - z)^2; z) + \mathcal{R}_{r_\eta}((y_1 - z)^4; z) \right\}. 
\]

(45)

We know the expression

\[
\mathcal{R}_{r_\eta}\left( (1 + y_1 - z)^2 \frac{|y_1 - z|}{\delta}; z \right) \\
= \frac{1}{\delta} \mathcal{R}_{r_\eta}\left( |y_1 - z|; z \right) + \mathcal{R}_{r_\eta}\left( (y_1 - z)^2 \frac{|y_1 - z|}{\delta}; z \right) \\
\leq \frac{1}{\delta} \left( \mathcal{R}_{r_\eta}\left( (y_1 - z)^2; z \right)^{1/2} + \left( \mathcal{R}_{r_\eta}\left( (y_1 - z)^4; z \right) \right)^{1/2} \right) \\
\times \left\{ \mathcal{R}_{r_\eta}\left( (y_1 - z)^2; z \right)^{1/2} \right\}. 
\]

(46)

In view of Lemma 3, we can obtain

\[
\mathcal{R}_{r_\eta}\left( (y_1 - z)^2; z \right) \leq \alpha, (\eta)(z + 1) \leq M_1(z + 1) \text{ as } r \to \infty, \\
\mathcal{R}_{r_\eta}\left( (y_1 - z)^4; z \right) \leq \beta, (\eta)(z^2 + z + 1) \leq M_2(z^2 + z + 1) \text{ as } r \to \infty, 
\]

(47)

where \( M_1 \) and \( M_2 \) are positive constant and

\[
\alpha, (\eta) = \max \left\{ \frac{1}{r^3}, \frac{1}{3r^3} \left( \frac{L''(1)}{L(1)} + 6(1 + 2\eta) \frac{L'(1)}{L(1)} \right) \right. \\
+ 12\eta^2 + 6\eta + 1 \right\}. 
\]

(48)
Thus, from inequality (45), we get

\[
\begin{align*}
|\mathcal{R}_{r,\eta}(f ; z) - f(z)| \\
& \leq 2 \left[ 1 + \delta^2 \right] (1 + z^2) \bar{\omega}(f ; \delta) \left[ 1 + \mathcal{R}_{r,\eta}^*(\gamma_1 - z)^2 ; z \right] \\
& \quad + \frac{1}{\delta} \left( \mathcal{R}_{r,\eta}^*(\gamma_1 - z)^2 ; z \right)^{1/2} \left\{ 1 + \sqrt{\mathcal{R}_{r,\eta}^*(\gamma_1 - z)^4 ; z} \right\} \\
& \leq 2 \left[ 1 + \delta^2 \right] (1 + z^2) \bar{\omega}(f ; \delta) \left[ 1 + M_1(z + 1) \right] \\
& \quad + \frac{1}{\delta} \sqrt{\alpha_r(\eta)}(z + 1) \left\{ 1 + \sqrt{M_2(z^2 + z + 1)} \right\}.
\end{align*}
\]

(49)

If we choose \( \delta = \sqrt{\alpha_r(\eta)} \) and taking supremum \( z \in [0, \alpha_r(\eta)] \), then we easily get the result.

4. Direct Approximation Results of \( \mathcal{R}_{r,\eta}^* \)

The present section gives some direct approximation results in space of \( K \)-functional and in Lipschitz spaces. We take \( C_0^\infty([0,\infty) \) be the set of all continuous and bounded functions defined on \( [0, \infty) \).

**Definition 13.** For every \( \bar{\delta} > 0 \) and \( f \in C[0,\infty) \) the \( K \)-functional is defined such that

\[
\mathcal{K}_f(f ; \bar{\delta}) = \inf \left\{ \left( \|f - \|C_0^\infty([0,\infty) \) + \bar{\delta}\|f'\|_{C_0^\infty([0,\infty))} \right) : f, f' \in C_0^\infty([0,\infty) \right\},
\]

(50)

\( C_0^\infty([0,\infty) \)

\[
= \left\{ f : f \in C_0^\infty([0,\infty), k \in \mathbb{N} \right\} : \lim_{z \to \infty} \frac{f(z)}{1 + z^k} = m_j < \infty \right\}.
\]

(51)

For an absolute constant \( M > 0 \), one has

\[
\mathcal{K}_f(f ; \bar{\delta}) \leq M \left\{ \bar{\omega}_2(f ; \sqrt{\bar{\delta}}) + \min \left( 1, \bar{\delta} \right) \|f\|_{C_0^\infty([0,\infty))} \right\}.
\]

(52)

Let \( \bar{\omega}_2(f ; \delta) \) denote the modulus of continuity of order two such that

\[
\bar{\omega}_2(f ; \delta) = \sup_{0 < h < \delta} \left\{ \left( f(z + 2h) - 2f(z + h) + f(z) \right) \right\},
\]

(53)

while the classical modulus of continuity is given by

\[
\bar{\omega}(f ; \delta) = \sup_{0 < h < \delta} \left\{ \left| f(z + h) - f(z) \right| \right\}.
\]

(54)

**Theorem 14.** For an arbitrary \( \phi \in C_0^\infty([0,\infty) \), let an auxiliary operator \( \delta_{r,\eta}^* \) be such that

\[
\begin{align*}
\mathcal{K}_{r,\eta}^*(\phi ; z) \\
& = \mathcal{K}_{r,\eta}^*(\phi ; z) + \phi(z) - \phi \left\{ z + \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \right\}.
\end{align*}
\]

(55)

Then, for any \( \phi \in C_0^\infty([0,\infty) \) operators (55), verify the inequality

\[
\left| \mathcal{K}_{r,\eta}^*(\phi ; z) - \phi(v) \right| \leq \left\{ \delta_{r,\eta}^*(\phi) + \frac{1}{4r^2} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right)^2 \right\} \|\phi'\|,
\]

(56)

where \( \delta_{r,\eta}^*(\phi) \) is defined by Theorem 10.

**Proof.** For any \( \phi \in C_0^\infty([0,\infty) \), it is easy to verify that \( \mathcal{K}_{r,\eta}^*(\gamma_0 ; z) = 1 \) and

\[
\mathcal{K}_{r,\eta}^*(\gamma_1 ; z) \\
= \mathcal{K}_{r,\eta}^*(\gamma_1 ; z) + z - \left\{ z + \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \right\} = z.
\]

(57)

We have

\[
\mathcal{K}_{r,\eta}^*(\phi ; z) \leq \|\phi\|,
\]

(58)

For any \( \phi \in C_0^\infty([0,\infty) \), the Taylor series expression gives us

\[
\phi(t) = \phi(z) + (t - z)\phi'(z) + \int_z^t (t - \chi)\phi''(\chi)d\chi.
\]

(59)

Therefore, after applying the operators \( \mathcal{K}_{r,\eta}^* \), on both
sides we get

\[ \mathcal{K}_{r,\eta}^*(\phi; z) - \phi(z) = \phi'(z) \mathcal{K}_{r,\eta}^*(y_1 - z; z) + \mathcal{K}_{r,\eta}^* \left( \int_z^1 (y_1 - \chi) \phi''(\chi) d\chi; z \right) \]

\[ = \mathcal{K}_{r,\eta}^* \left( \int_z^1 (y_1 - \chi) \phi''(\chi) d\chi; z \right) + \int_z^1 (z - \chi) \phi''(\chi) d\chi; z \]

\[ = \mathcal{K}_{r,\eta}^* \left( \int_z^1 (y_1 - \chi) \phi''(\chi) d\chi; z \right) + \int_z^1 (z - \chi) \phi''(\chi) d\chi; z \]

\[ + \int_z^1 (z + \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) - \chi) \phi''(\chi) d\chi; z \]

\[ \leq \mathcal{K}_{r,\eta}^* \left( \int_z^1 (y_1 - \chi) \phi''(\chi) d\chi; z \right) \]

\[ + \int_z^1 \left( z + \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) - \chi \right) \phi''(\chi) d\chi; z \]

\[ (60) \]

We know the inequality

\[ \left| \int_z^1 (t - \chi) \phi''(\chi) d\chi \right| \leq (t - z)^2 \left| \phi'' \right|. \]

\[ \left| \int_z^1 (z + \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) - \chi \right) \phi''(\chi) d\chi \right| \leq \left( \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \right)^2 \left| \phi'' \right|. \]

\[ (61) \]

Thus, we get

\[ \mathcal{K}_{r,\eta}^*(\phi; y) - \phi(y) \]

\[ \leq \mathcal{K}_{r,\eta}^* \left( (y_1 - z)^2; z \right) + \frac{1}{4r^2} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right)^2 \left| \phi'' \right|. \]

\[ (62) \]

This gives the complete proof.

\textbf{Theorem 15.} If \( \phi \in C_{C_0}^0[0,\infty) \), then for any \( f \in C_{C_0}[0,\infty) \) operators \( \mathcal{R}_{r,\eta}^* \) by (7) satisfying

\[ \left| \mathcal{R}_{r,\eta}^*(f; z) - f(z) \right| \leq \mathcal{A} \left\{ \hat{\omega}_2 \left( f; \frac{1}{2r} \delta_{r,\eta}^*(z) + \frac{1}{4r^2} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right)^2 \right) \right\} \]

\[ + \min \left\{ 1; \frac{1}{2r} \left( \delta_{r,\eta}^*(z) + \frac{1}{4r^2} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right)^2 \right) \right\} \]

\[ \times \| f \|_{C_{C_0}[0,\infty)} \right\} + \left( f; \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right)^2 \right), \]

\[ (63) \]

where \( \delta_{r,\eta}^*(z) \) is defined by Theorem 10.

\textbf{Proof.} We prove Theorem 15 in view of Theorem 14. Therefore, for all \( f \in C_{C_0}[0,\infty) \) and \( \phi \in C_{C_0}^0[0,\infty) \), we get

\[ \left| \mathcal{R}_{r,\eta}^*(f; z) - f(z) \right| \leq \mathcal{K}_{r,\eta}^*(f; z) - f(z) + \left| \mathcal{K}_{r,\eta}^*(f; \phi; z) \right| \]

\[ \left| \mathcal{K}_{r,\eta}^*(f; \phi; z) \right| \leq \left( \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \right)^2 \left| \phi'' \right|. \]

\[ (64) \]

If we take infimum for all \( \phi \in C_{C_0}^0[0,\infty) \), then in view of (50) it is easy to conclude that

\[ \left| \mathcal{R}_{r,\eta}^*(f; z) - f(z) \right| \leq \left( \frac{1}{2r} \left( \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \right)^2 \left| \phi'' \right|. \]

\[ (65) \]

The proof is completed here.
Now, we give the local direct estimate for the operators $\mathcal{R}_{r_n}^*$ defined by (7) via the well-known Lipschitz-type maximal function involving the parameters $\mu, \nu > 0$ and number $\lambda \in (0, 1]$. Thus, from [26], we recall that

$$\text{Lip}_x^1 = \left\{ f \in C_b[0, \infty); |f(t) - f(z)| \leq \mathcal{L} \frac{|t - z|}{(\mu z^2 + \nu z + t)^{1/2}} ; z, t \in [0, \infty) \right\} ,$$

(66)

where $\mathcal{L}$ is a positive constant.

**Theorem 16.** For any $f \in \text{Lip}_x^1$ satisfied by (70), operators $\mathcal{R}_{r_n}^*$ hold the inequality

$$|\mathcal{R}_{r_n}^*(f ; z) - f(z)| \leq \mathcal{L} \frac{\delta_{r_n}(z)}{(\mu z^2 + \nu z)^{1/2}} ,$$

(67)

where $\delta_{r_n}(z)$ is obtained by Theorem 10.

**Proof.** Let $f \in \text{Lip}_x^1$ for $0 < \lambda \leq 1$; then, first we verify the results are true when $\lambda = 1$. For any $\mu, \nu \geq 0$, it is easy to use the result $(\mu z^2 + \nu z + t)^{-1/2} \leq (\mu z^2 + \nu z)^{-1/2}$ and then we apply the Cauchy-Schwarz inequality. Thus, we can write

$$|\mathcal{R}_{r_n}^*(f ; z) - f(z)| \leq |\mathcal{R}_{r_n}^*[f(t) - f(z)] ; z| + f(z)(1 ; z) - 1|$$

$$\leq \mathcal{R}_{r_n}^* \left( \frac{|t - z|}{(\mu z^2 + \nu z + t)^{1/2}} ; z \right)$$

$$\leq \mathcal{L} \left( \mu z^2 + \nu z \right)^{-1/2} \mathcal{R}_{r_n}^* |t - z| ; z$$

$$\leq \mathcal{L} \left( \mu z^2 + \nu z \right)^{-1/2} \mathcal{R}_{r_n}^* (y_1 - z)^2 ; z \right)^{1/2} .$$

(68)

From these conclusions, we get that the statement holds for $\lambda = 1$. Now, we check if the statement is valid if $0 < \lambda < 1$. For this reason, we use monotonicity property to $\mathcal{R}_{r_n}^*$ and apply the well-known Hölder inequality

$$|\mathcal{R}_{r_n}^*(f ; z) - f(z)| \leq \mathcal{R}_{r_n}^*[|f(t) - f(z)| ; z]$$

$$\leq \left( \mathcal{R}_{r_n}^* \left( |f(t) - f(z)|^{2/\lambda} ; z \right) \right)^{\lambda/2}$$

$$\leq \left( \mathcal{R}_{r_n}^* (y_0 ; z) \right)^{(2-\lambda)/2}$$

$$\leq \mathcal{L} \left( \frac{\mathcal{R}_{r_n}^* (y_1 - z)^2 ; z}{t + \mu z^2 + \nu z} \right)^{1/2} .$$

which completes the proof.

Here, we obtain the other local approximation results of $\mathcal{R}_{r_n}^*$ in Lipschitz spaces. For all Lipschitz maximal function $f \in C_b[0, \infty), 0 < \lambda \leq 1$ and $t, z \in [0, \infty)$, from [27] we recall that

$$\omega_1(f ; z) = \sup_{t z, t \in [0, \infty)} \frac{|f(t) - f(z)|}{|t - z|^{1/2}} ,$$

(70)

**Theorem 17.** Let $f \in C_b[0, \infty)$, then for all $z \in [0, \infty)$,

$$|\mathcal{R}_{r_n}^*(f ; z) - f(z)| \leq \left( \delta_{r_n}(z) \right)^{\lambda/2} \omega_1(f ; z) ,$$

(71)

where $\omega_1(f ; z)$ is obtained in and $\delta_{r_n}(z)$ is defined by Theorem 10.

**Proof.** From the well-known Hölder inequality, we get

$$|\mathcal{R}_{r_n}^*(f ; z) - f(z)| \leq \mathcal{R}_{r_n}^* |f(t) - f(z)| ; z$$

$$\leq \omega_1(f ; z) \mathcal{R}_{r_n}^* \left( |t - z|^{1/2} ; z \right)$$

$$\leq \omega_1(f ; z) \left( \mathcal{R}_{r_n}^* (y_0 ; z) \right)^{(2-\lambda)/2}$$

$$\cdot \left( \mathcal{R}_{r_n}^* (y_1 - z)^2 ; z \right)^{1/2}$$

$$= \omega_1(f ; z) \left( \mathcal{R}_{r_n}^* (y_1 - z)^2 ; z \right)^{1/2} .$$

(72)

Thus, we get the proof.

### 5. Voronovskaja-Type Approximation Theorems

In this section, we establish a quantitative Voronovskaja-type theorem for the operators $\mathcal{R}_{r_n}^*(f ; z)$. 
Theorem 18. Let $f \in C_b[0,\infty)$, then for each $z \in [0,\infty)$

$$
\lim_{r \to \infty} r \left\{ \mathcal{R}_{r\eta}^*(\psi ; z) - \psi(z) \right\} = \left( 2 \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \frac{\psi'(z)}{2} + \frac{z\psi''(z)}{2},
$$

(73)

where $\psi'(z), \psi''(z) \in C_b[0,\infty)$.

Proof. From the expression of Taylor’s expansion of function $\psi(z)$ in $C_b[0, \infty)$, we write

$$
\psi(t) = \psi(z) + (t - z)\psi'(z) + \frac{1}{2}(t - z)^2\psi''(z) + (t - z)^2Q_z(t),
$$

(74)

where $Q_z(t)$ is the remainder term and $Q_z \in [0, \infty)$ with $Q_z(t) \to 0$ as $t \to z$. On applying the operators $\mathcal{R}_{r\eta}^*(z)$ to (74), then use the Cauchy-Schwarz inequality. Thus, we get

$$
\mathcal{R}_{r\eta}^*(\psi ; z) - \psi(z) = \psi'(z)\mathcal{R}_{r\eta}^*(y_1 - z ; z) + \frac{\psi''(z)}{2} \mathcal{R}_{r\eta}^*

\cdot \left( (y_1 - z)^2 ; z \right) + \mathcal{R}_{r\eta}^*(Q_z(y_1) ; z)

\leq \psi'(z)\mathcal{R}_{r\eta}^*(y_1 - z ; z) + \frac{\psi''(z)}{2} \mathcal{R}_{r\eta}^*

\cdot \left( (y_1 - z)^2 ; z \right)

+ \sqrt{\mathcal{R}_{r\eta}^*(Q_z(y_1) ; z)^2}. \sqrt{\mathcal{R}_{r\eta}^*(Q_z(y_1) ; z)^2}.
$$

(75)

Since we have $\lim_{r \to \infty} \mathcal{R}_{r\eta}^*(Q_z^2(y_1) ; z) = 0$, therefore

$$
\lim_{r \to \infty} r \left\{ \mathcal{R}_{r\eta}^*((y_1 - z)^2Q_z(y_1) ; z) \right\} = 0.
$$

(76)

Thus, we have

$$
\lim_{r \to \infty} r \left\{ \mathcal{R}_{r\eta}^*(\psi ; z) - \psi(z) \right\}

= \lim_{r \to \infty} r \left\{ \mathcal{R}_{r\eta}(y_1 - z ; z)\psi'(z) + \frac{\psi''(z)}{2} \mathcal{R}_{r\eta}^*(y_1 - z)^2 ; z) + \mathcal{R}_{r\eta}^*(Q_z^2(y_1) ; z) \right\},
$$

(77)

which completes the proof.

As a consequence of Theorem 18, we immediately get the corollary.

Corollary 19. For any $\psi \in C[0,\infty)$, we have

$$
\lim_{r \to \infty} r \left\{ \mathcal{R}_{r\eta}^*(\psi ; z) - \psi(z) - \frac{1}{2r} \left( 2 \frac{L'(1)}{L(1)} + 4\eta + 1 \right) \frac{\psi'(z)}{2} \right\}

- \frac{\psi''(z)}{2} = 0.
$$

(78)

6. Conclusion

Motivated by article [5], we have introduced a Kantorovich generalization of the Szász-Mirakian operators by Dunkl analogue involving the Appell polynomials. These types of generalizations enable to give the generalized results rather than the earlier study demonstrations by [3, 5, 7]. Lastly, we have also discussed the Voronovskaja-type approximation theorems of these new operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors are very grateful and declare that they have no competing interest.

Authors’ Contributions

All authors read and agreed to the contents of this research article.

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