On the validity of the LAD and LL classical radiation-reaction equations

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Abstract

The search of the correct equation of motion for a classical charged particle under the action of its electromagnetic (EM) self-field, the so-called radiation-reaction equation of motion, remains elusive to date. In this paper we intend to point out why this is so. The discussion is based on the direct construction of the EM self-potentials produced by a charged spherical particle under the action of an external EM force. In particular we intend to analyze basic features of the LAD (Lorentz-Abraham-Dirac) and the LL (Landau-Lifshitz) equations. Both are shown to lead to incorrect or incomplete results.

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I. INTRODUCTION

An ubiquitous phenomenon which characterizes the dynamics of classical charged particles is the occurrence of self-forces, in particular the electromagnetic (EM) one which is produced by the EM fields generated by the same particles. It is well-known that such a force acts on a charged particle when it is subject also to the action of an arbitrary external force (see Lorentz, 1892 [1]; see also for example Landau and Lifschitz, 1957 [2]). This phenomenon is usually called as radiation reaction (RR) (Pauli [4]) or radiation damping (see [3]), although a proper distinction between the two terms can actually be made [5]. In the sequel we denote as RR problem the treatment of the classical mechanics of a charged particle in the presence of its EM self-field, and RR equation the corresponding equation of motion for such a particle. The RR problem, apart its almost endless physical applications, is of fundamental importance from the theoretical viewpoint for the formulation of consistent relativistic theories, in particular kinetic theory, plasma dynamics and astrophysics. Despite efforts more than a century-long, the problem of its ”exact” theoretical description remains still elusive. This refers, in particular, to the construction of a relativistic equation of motion which results both non-perturbative, in the sense that it does not rely on a perturbative expansion for the electromagnetic field generated by the charged particle and non-asymptotic, i.e., it does not depend on any infinitesimal parameter (such, for example, as the radius of the particle, if it is identified with a spherically-symmetric charge distribution of finite radius, as in the Lorentz approach [1]). The purpose of this paper is to point out that such a problem, since the historical work of Lorentz which first investigated it [1], still remains essentially unsolved to date. To prove the above statement, in this paper we intend to analyze in detail the derivation and basic features of the relativistic equations of motion available so-far, with particular reference to the so-called LAD and to the LL equations, respectively named after Lorentz, Abraham and Dirac and Landau and Lifschitz [2]. A side motivation of this work is also provided by the recent claim (Rohrlich, 2001 [6]; see also Spohn, 2000 [7]) that the LL equation should be regarded as the exact (i.e., both non-perturbative and non-asymptotic) RR equation.
II. A RE-FORMULATION OF THE TRADITIONAL APPROACH TO THE RR PROBLEM

In the weakly-relativistic treatment of classical mechanics, the traditional form of the equations of motion for a charged particle subject to its own EM self-field can be obtained (see for example Ref. [2]) by a suitable asymptotic expansion of the EM self-potentials which generate such a force. It is instructive to discuss here the basic steps of the derivation. For definiteness, we shall consider here the case of a single point-particle with distributed charge density (here denoted as finite-size charge) carrying constant mass and charge, \( m \) and \( q \) (this choice is actually analogous to that adopted in the original Lorentz approach [1]). More precisely, it is assumed that the particle mass is concentrated in the center of the sphere (so that the particle degree of freedom is the same as that of a point particle), while its charge is assumed to be uniformly distributed on a spherical shell of finite radius \( \sigma > 0 \), which carries the homogeneous surface charge density \( \rho = q / 4 \pi \sigma^2 \) (finite-size spherical-shell charge). Thus, letting \( R = |r - r'| \) and denoting respectively \([\rho(r',t), J(r',t)]\) and \([\rho(r',t - \frac{R}{c}), J(r',t - \frac{R}{c})]\) the charge and current densities of the particle at time \( t \) and at the retarded time \( t' = t - \frac{R}{c} \), we shall impose that

\[
\rho(r,t) = \frac{q}{4 \pi \sigma^2} \delta(|r - r(t)| - \sigma), \tag{1}
\]

\[
J(r,t) = \frac{q}{4 \pi \sigma^2} \dot{r}(t) \delta(|r - r(t)| - \sigma). \tag{2}
\]

Then, if \( r(t) \) is at time \( t \) the position of the particle center of mass, the \textit{EM self-force} acting on the same particle is represented by the Lorentz force acting on the particle itself, i.e.,

\[
F^{(self)}(r(t), \dot{r}(t), t) = \int d^3r' \left[ \rho(r',t)E^{(self)}(r',t) + \frac{1}{c}J(r',t) \times B^{(self)}(r',t) \right], \tag{3}
\]

where \( F^{(self)} \) is usually denoted as \textit{RR reaction force} acting on the particle. Here \( \{E^{(self)}, B^{(self)}\} \) denotes the EM self-field generated by the particle itself, related to corresponding EM potentials \( \{\phi^{(self)}, A^{(self)}\} \), here denoted as \textit{EM self-potentials}. By definition \( \{\phi^{(self)}, A^{(self)}\} \) - to be evaluated at time \( t \) and at the position \( r \) [to be later identified with the particle center of mass \( r(t) \)] - can be defined as \( \phi^{(self)}(r,t) = \int d^3r' \frac{1}{R} \left[ \rho(r',t - \frac{R}{c}) - \rho(r',t) \right] \) and \( A^{(self)}(r,t) = \frac{1}{c} \int d^3r' \frac{1}{R} \left[ J(r',t - \frac{R}{c}) - J(r',t) \right] \). We stress that the second terms on the r.h.s. of these equations take into account the charge and current densities evaluated at the same time \( t \) (current time), which manifestly do not
Contribute to the self potentials when they are evaluated at the particle position. In order to evaluate explicitly the EM self-potentials in this case [Eqs. (1), (2)], let us now introduce for them an asymptotic expansion in power series of $\beta = v/c$, assuming that $v/c \ll 1$. Retaining only terms up to third order (in $\beta$) in $\phi^{(self)}(r, t)$ and respectively first order (in $\beta$) in $A^{(self)}(r, t)$, there results

$\phi^{(self)}(r, t) \approx -\frac{1}{c} \frac{\partial}{\partial t} \int d^3 r' \rho(r', t) + \frac{1}{2c^2} \frac{\partial^2}{\partial t^2} \int d^3 r' R \rho(r', t) - \frac{1}{6c^3} \frac{\partial^3}{\partial t^3} \int d^3 r' R^2 \rho(r', t), (4)$

$A^{(self)}(r, t) \approx -\frac{1}{c} \frac{\partial}{\partial t} \int d^3 r' \mathbf{J}(r', t). \tag{5}$

Thus, introducing the EM gauge $\phi^{(self)} = \phi^{(self)} + \frac{1}{c} \frac{\partial}{\partial t} f = 0$, $A^{(self)} = A^{(self)} - \nabla f$, where $f$ is the EM gauge function (to be denoted for later reference as RR gauge)

$f = \int d^3 r' \rho(r', t) - \frac{1}{2c} \frac{\partial}{\partial t} \int d^3 r' R \rho(r', t) + \frac{1}{6c^2} \frac{\partial^2}{\partial t^2} \int d^3 r' R^2 \rho(r', t), \tag{6}$

the transformed vector self-potential $A^{(self)}(r, t)$ becomes

$A^{(self)}(r, t) \approx -\frac{1}{c} \frac{\partial}{\partial t} \int d^3 r' \mathbf{J}(r', t) + \nabla \int d^3 r' \rho(r', t) - \frac{1}{2c} \frac{\partial}{\partial t} \nabla' \int d^3 r' R \rho(r', t) + \frac{1}{6c^2} \frac{\partial^2}{\partial t^2} \nabla' \int d^3 r' R^2 \rho(r', t). \tag{7}$

After straightforward calculations one obtains

$F^{(self)}(r(t), \dot{r}(t), t) = -\frac{2q}{3c^3} \ddot{r}(t) + \alpha q^2 \left[ \frac{\dot{r}(t)}{c^2 \sigma} + \frac{1}{2 \sigma^3} \dot{r}(t) \beta^2 \right], \tag{8}$

where $\beta^2 = \dot{r}(t)/c^2$ and in the case of the finite-size spherical shell here considered there results $\alpha = 1$. Thus, one recovers in this way the weakly-relativistic RR equation or the weakly-relativistic LAD equation (after Lorentz [2], Abraham [8] and Dirac [9]) of the form

$m_R \ddot{r} = F + \mathbf{g}. \tag{9}$

Here the notation is standard. Thus,

$\mathbf{g} \equiv -\frac{2q^2}{3c^3} \mathbf{r} + \frac{\alpha q^2}{2 \sigma^3} \dot{r}(t) \beta^2 \tag{10}$

is the weakly-relativistic EM self-force. Moreover

$$
\begin{cases}
F = q \left[ E^{(ext)} + \frac{1}{c} \mathbf{r} \times B^{(ext)} \right], \\
m_R \equiv m - m_{EM}, \\
m_{EM} \equiv \alpha \frac{q^2}{c^2 \sigma},
\end{cases}
$$

(11)
are respectively the Lorentz force produced by the external EM field, the renormalized mass and the so-called EM mass. Eq. (9) manifestly indicates that the limit $\sigma \to 0$ (point-particle) does not exist. Hence the LAD equation is only defined for a finite-size spherical-shell charge, i.e., for which it results $\sigma > 0$ and possibly also $m_R > 0$ (the vanishing of $m_R$ involves in fact the manifest violation of Newton’s second law). The derivation of the analogous relativistic RR equation is usually achieved (see for example the treatment given in Ref. [2]) under the implicit assumptions that: 1) higher order corrections in $\beta$ are negligible; 2) all previous asymptotic expansions in power series of $\beta$ remain valid for the whole range of values of particle velocities, i.e., even if $\beta \sim o(1)$, namely arbitrary close to the speed of light. In practice, the construction of the equation is achieved formally by writing Eq. (9) in covariant form and by suitably replacing the definitions of the renormalized mass $m_R$ and of the self-force $g$ by an appropriate Lorentz scalar $m_{oR}$ and a 4-vector $g^\mu$. Both are to be defined in such a way that in the weakly-relativistic limit they recover the correct values set by the previous equations. The relativistic RR equation of motion thus obtained, nowadays popularly known as the relativistic LAD equation,

$$m_{oR} c \frac{d^2 r^\mu}{ds^2} = \frac{q}{c} F^{\mu\nu} u^\nu + g^\mu \tag{12}$$

was first presented by Dirac in his famous paper on relativistic radiation reaction in classical electrodynamics [9]. Here, $g^\mu$ is the EM self-force which reads

$$g^\mu = \frac{2q^2}{3c} \left\{ \frac{d^2 u^\mu}{ds^2} - u^\mu u^\nu \frac{d^2 u_\nu}{ds^2} \right\}. \tag{13}$$

Furthermore the relativistic renormalized mass has the form

$$m_{oR} = m_o + m_{EM} + \Delta m_{EM}, \tag{14}$$

with $m_o$, $m_{EM}$ and $\Delta m_{EM}$ respectively, the rest mass, the EM mass and a suitable relativistic correction. Finally, as usual $r^\mu$, $u^\mu$ and $F^{\mu\nu}$ denote the 4-position vector, the 4-velocity vector and the Faraday tensor associated to the external EM field $A^{(ext)\mu}$. Nevertheless, the LAD equation is valid only in the case of the Minkowsky metric. Its generalization for arbitrary curved space-time was later carried out by DeWitt and Brehme [10].
III. DIFFICULTIES WITH RR EQUATIONS

Since Lorentz famous paper \[1\] many textbooks and research articles have appeared on the subject of RR. Among them are \[12, 13, 14, 15, 16\], where one can find the discussion of the related problems: mass renormalization, non-uniqueness, runaway solutions and pre-acceleration. In the literature the usual derivations of relativistic form of the EM self-force are made under the implicit assumption that all the expansions in powers used near the particle trajectory are valid for the whole range of values of particle velocity, in particular, arbitrary close to that of the light. But it is easy to see that this is not true in general.

It is often said that the LAD equation, in its weakly-relativistic form given by Eq.(9), is unsatisfactory because it requires the specification of the initial acceleration \(\ddot{r}(t_0)\), besides the initial state \(\{r(t_0), \dot{r}(t_0)\}\). As a consequence it violates the Newton’s principle of determinacy (NPD), one of the building blocks of classical mechanics. Moreover, another critical issue arises due to the appearance of so-called runaway solutions. These are solutions which by definitions blow up in time for \(t \to +\infty\). One can show that they are present for arbitrary external forces \(F\). In particular, in the case \(F \equiv 0\) the general solution of Eq.(9) takes the form \(r(t) = \ddot{r}(t_0) \exp \left\{ \frac{(t-t_0)}{\tau} \right\}\), where \(\tau > 0\) is the constant parameter \(\tau = \frac{2q^2}{3mc^3}\). This equation is clearly unsatisfactory. In fact, it violates also another fundamental principle of classical mechanics, the Galilei law of inertia, according to which an isolated particle must have a constant velocity in any inertial Galilean frame. In an attempt to circumvent this difficulty it has been suggested to replace Eq.(9) by a suitably-modified integral (or differential) equation. An example is provided by an integro-differential equation which in its weakly-relativistic form reads \[12, 17\]

\[
m_R \ddot{r} = \int_0^\infty F(t+s)e^{-s}ds \equiv \frac{1}{\tau} \int_0^\infty F(x)e^{-x/\tau}dx
\]  

(Haag equation). This equation is manifestly consistent with PND and the law of inertia (since runaway solutions are excluded). However, it exhibits another serious drawback: it violates the principle of causality. Indeed, contrary to the requirement according to which the effect cannot precede the cause, the paradox of pre-acceleration is implied by Eq.\((15)\). Accordingly, a particle obeying Eq.\((15)\) would experience a force before actually turning it on! Eq.\((15)\) clearly violates this principle since if a constant force \(F_o\) is suddenly turned on
at \( t_o \) letting \( F(x) = 0 \) for \( x < t_o \) and \( F(x) = F_o \) for \( x \geq t_o \) (sudden force) it follows

\[
\begin{align*}
\frac{1}{\tau} \int t \to \infty F(x) e^{-\frac{x-t}{\tau}} dx &= F_o \frac{1}{\tau} \int t \to \infty e^{-\frac{x-t}{\tau}} dx = F_o e^{-\frac{t}{\tau}} & t < t_o, \\
\frac{1}{\tau} \int t \to \infty F(x) e^{-\frac{x-t}{\tau}} dx &= F_o \frac{1}{\tau} \int t \to \infty e^{-\frac{x-t}{\tau}} dx = F_o & t \geq t_o.
\end{align*}
\]

(16)

Hence, the RR equation before and after \( t_o \) will be respectively provided by

\[
m_{R\ddot{r}} = \begin{cases} 
F_o e^{-\frac{t}{\tau}} & t < t_o, \\
F_o & t \geq t_o.
\end{cases}
\]

(17)

Hence, the particle should actually experience an acceleration for all \( t < t_o \), a phenomenon usually denoted as pre-acceleration. For this reason neither Eq. (9) nor (15) can be considered satisfactory RR equations.

Another attempt is to adopt an ”iterative approach” whereby the time derivative of the acceleration appearing in the self-force (10) is approximated in terms of the time derivative of the external force (this approach is manifestly admissible only if the self-force can be treated as suitably small). In the weakly-relativistic LAD, letting first \( \frac{2q}{3c} \ddot{r}(t) \ddot{r} \approx \tau \frac{d}{dt} F \), one obtains the reduced equation \( m_{R\ddot{r}} = F + \tau \frac{d}{dt} F \). More generally, by carrying out the full iteration in Eq.(9), one obtains the equation \( m_{R\ddot{r}} = F + \sum_{n=1}^{\infty} \tau^n \frac{d^n}{dt^n} F \). This equation, and the analogous relativistic one which follows in a similar way from Eq.(12), are however useless since they are infinite-order ode’s (and hence have no solutions!).

Nevertheless, this difficult can be circumvented by introducing a ”reduction method” whereby all the derivatives higher than \( \dot{r} \) appearing in each term \( \frac{d^n}{dt^n} F \) are expressed via the same iterative approach which permits to cast \( \frac{d^n}{dt^n} F \) for each \( n \geq 1 \) in the form \( \frac{d^n}{dt^n} F = F_n(r, \dot{r}, t) \). This idea has been exploited by Cook \[18\] who obtained the so-called Cook RR equation

\[
m_{R\ddot{r}} = F + \sum_{n=1}^{\infty} \tau^n F_n(r, \dot{r}, t).
\]

(18)

This procedure was actually first adopted by Landau and Lifshitz \[2\] based on a one-step iteration of the form

\[
m_{R\ddot{r}} \simeq F + \tau F_1(r, \dot{r}, t)
\]

(19)

(which could be denoted as the weakly-relativistic LL equation \[2\]). In the corresponding relativistic formulation (see Ref. \[2\]), applying the one-step reduction process to the relativistic
LAD equation (12) delivers for \( g^\mu \) [as given by Eq.(13)] the approximation

\[
\begin{align*}
g^\mu & \approx \frac{2q^3}{3e^3} \partial F_{\mu k} u_k u^i - \frac{2q^4}{3m_R c^5} F_{\mu k} F_{j k} u^j + \\
& + \frac{2q^4}{2m_R^2 c^5} F_{kl} u^l F^{km} u_m u^n.
\end{align*}
\]

The relativistic LL equation obtained in such a way by Landau and Lifschitz [2] involves, however, also the replacement of the renormalized mass \( m_{oR} \) with the inertial mass only \( m_o \). While in the framework on classical electrodynamics the latter position remains totally unjustified, the resulting equation has been claimed by Rohrlich [6]) to be the exact RR equation. Also in view of the previous discussion, the validity of this statement seems unlikely. In fact the LL equation (both in the weakly-relativistic and fully relativistic versions) has remaining serious problems:

- One reason is that as recalled in Sec.2 the theory is asymptotic in \( \beta \) and does not take into account in a consistent way higher-order finite-\( \beta \) effects;

- A second motivation is that it only applies provided the external force is suitably smooth, i.e., of class \( C^2 \). In fact, if an external force is turned on suddenly (sudden force), as in the example provided above for the weakly-relativistic case, the LL equation becomes manifestly invalid. Since sudden forces cannot be ruled out, purely on first principles, this is actually a major conceptual difficulty (and potential inconsistency) of the part of the LL equation.

- An additional difficulty of the LL equation is that it is non-variational, i.e., the Hamilton variational principle does not apply for such an equation, even if the external force is identified with the Lorentz force [see Eq.(11)]. This feature is actually common to all current RR equations. This is illustrated, for example, by the weakly-relativistic LAD equation Eq.(9) which appears manifestly non-variational. In fact in this case the Lagrangian function \( L \) should actually depend not only on the Lagrangian state \( (r, \dot{r}) \) but also on the acceleration \( \ddot{r} \). Thus, for example, letting introducing the Lagrangian

\[
L(r, \dot{r}, \ddot{r}, t) = \frac{1}{2}m_R \dot{r}^2 - q \left[ \phi^{(ext)}(r, t) - \frac{1}{c^2} \dot{r} \cdot A^{(ext)}(r, t) \right] + f(r, \dot{r}, \ddot{r}, t),
\]

for the validity of
the Hamilton variational principle it should be

\[\delta \int_{t_1}^{t_2} dt L(r(t), \dot{r}(t), \ddot{r}(t), t) = \int_{t_1}^{t_2} dt \delta \dot{r}(t) \cdot \left[ m_R \dddot{r} - F \right] + \]

\[+ \int_{t_1}^{t_2} dt \delta f(r(t), \dot{r}(t), \ddot{r}(t), t) = 0, \quad (21)\]

where \( \delta f(r(t), \dot{r}(t), \ddot{r}(t), t) \) is an exact differential form such that

\[\int_{t_1}^{t_2} dt \delta f(r(t), \dot{r}(t), \ddot{r}(t), t) = \int_{t_1}^{t_2} dt \delta \dot{r}(t) \cdot \frac{2q^2}{3c^3} \dddot{r}(t). \]

This implies, however,

\[\int_{t_1}^{t_2} dt \delta f(r(t), \dot{r}(t), \ddot{r}(t), t) = -\frac{q^2}{3c^3} \int_{t_1}^{t_2} dt \left[ \delta r(t) \cdot \dot{r}(t) - \delta \dot{r}(t) \cdot \dddot{r}(t) \right] \equiv 0, \quad (22)\]

which means that a real function \( f(r(t), \dot{r}(t), \ddot{r}(t), t) \) fulfilling the previous constraint cannot exist.

- Finally, another serious difficulty is related also to the conditions validity of the reduction process indicated above. In fact, it is obvious that the one-step reduction adopted for the derivation of the LL equation provides, at most, an asymptotic approximation for the (still unknown) exact RR equation.

IV. CONCLUSIONS

In this paper we have reviewed aspects of the RR problem. We have shown that:

- the difficulties with the LAD equation are intrinsic, i.e., arise due to the fact that the equation is a third order ode. As a consequence, the Newton’s principle of determinacy and the Galilei law of inertia are potentially violated;

- attempts to circumvent these difficulties, based on various approaches (Haag, LL and Cook equations) fail for different reasons;

- limitations of the current RR equations have been pointed out, with particular reference to the conditions of validity of the LL equation;
all current RR equations are non-variational, implying - contrary to common knowledge in classical mechanics - that the dynamics of a charged particle described by these model equations does not define an Hamiltonian system. This is the reason why relativistic systems of charged particle usually are not (or cannot be) described as Hamiltonian systems. Nevertheless, it not clear whether this feature is only an accident, i.e., is only due to the approximations introduced in the RR equation or is actually intrinsic to the nature of the RR problem. Unfortunately, the precise solution of this dilemma is not yet known, although the prevailing opinion seems directed to the first possibility (see for example Ref. [19]). For a proper discussion of the issue we refer to the accompanying paper [20].

This suggests, in our view, that in many respects the RR problem is still open. Its solution should be based on the search of a relativistic, non-perturbative equation of motion for a particle in the presence of its EM self-field. This problem poses not just an intellectual challenge but a fundamental physical requirement in all applications which involve the description of relativistic dynamics of classical charged particles, such as relativistic kinetic theory, plasma physics and astrophysics. In an accompanying paper an attempt at a possible solution of the RR problem will be discussed [20].

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