SOME GRUSS-TYPE INEQUALITIES USING GENERALIZED KATUGAMPOLA
FRACTIONAL INTEGRAL

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Abstract

The main objective of this paper is to obtain generalization of some Gruss-type inequalities in case of functional bounds by using a generalized Katugampola fractional integral.

Keywords: Gruss inequality; generalized fractional integral.

1 INTRODUCTION:

In 1935, G. Grüss proved the renowned integral inequality [9] (see also [13]):

\[
\left| \frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx - \frac{1}{(b-a)^2} \int_{a}^{b} f(x) \, dx \int_{a}^{b} g(x) \, dx \right| \leq \frac{1}{4} (M-m)(P-p),
\]

where \( f, g \) are two integrable functions on \([a, b]\), satisfying the conditions

\[ m \leq f(x) \leq M \quad \text{and} \quad p \leq g(x) \leq P \quad \text{for all} \quad x \in [a, b], \quad m, M, p, P \in \mathbb{R}. \]

In recent years, the inequalities are getting to play a very vital role in all mathematical fields, especially after the creation of fractional calculus which gave rise to several results and important theories in mathematics, engineering, physics, and other fields of science.

A remarkably large number inequalities of above type involving the special fractional integral (such as the Liouville, Riemann–Liouville, Erdélyi-Kober, Katugampola, Hadamard and Weyl types) have been investigated by many researchers and received considerable attention to it, see ([2], [3], [5], [6], [7], [8], [12], [15], [16], [19], [20]).

Gruss type inequality which has some important applications in a number of mathematical fields, like an integral arithmetic mean, difference equations and h-integral arithmetic mean (see [1], [14]).

Dahmani et al. [4], in (2010), proved the following fractional version inequality
by using Riemann–Liouville fractional integral
\[
\left| \frac{x^\alpha}{\Gamma(\alpha+1)} J^\alpha (fg)(x) - J^\alpha f(x) J^\alpha g(x) \right| \leq \left( \frac{x^\alpha}{\Gamma(\alpha+1)} \right)^2 (M - m) (P - p),
\]
for one parameter, and
\[
\left( \frac{x^\alpha}{\Gamma(\alpha+1)} J^\beta (fg)(x) - J^\alpha f(x) J^\alpha g(x) \right)^2
\]
\[
\leq \left[ \left( M \frac{x^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(x) \right) \left( J^\beta f(x) - m \frac{x^\beta}{\Gamma(\beta+1)} \right) + \left( J^\alpha f(x) - m \frac{x^\alpha}{\Gamma(\alpha+1)} \right) \left( M \frac{x^\beta}{\Gamma(\beta+1)} - J^\beta f(x) \right) \right]
\times \left[ \left( P \frac{x^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(x) \right) \left( J^\beta g(x) - p \frac{x^\beta}{\Gamma(\beta+1)} \right) + \left( J^\alpha g(x) - p \frac{x^\alpha}{\Gamma(\alpha+1)} \right) \left( P \frac{x^\beta}{\Gamma(\beta+1)} - J^\beta g(x) \right) \right],
\]
for two parameters, where \( f, g \) are two integrable functions on \([0, \infty)\), satisfying the conditions
\[
m \leq f(x) \leq M \quad \text{and} \quad p \leq g(x) \leq P \quad \text{for all} \quad x \in [0, \infty), \quad m, M, p, P \in \mathbb{R}. \]

In (2014), Tariboon et al. [18], replaced the constants which appeared as bounds of the functions \( f \) and \( g \) by four integrable functions on \([0, \infty)\), as \( \varphi_1(x) \leq f(x) \leq \varphi_2(x) \) and \( \psi_1(x) \leq g(x) \leq \psi_2(x) \), they obtained inequality
\[
\left| \frac{x^\alpha}{\Gamma(\alpha+1)} J^\alpha (fg)(x) - J^\alpha f(x) J^\alpha g(x) \right| \leq \sqrt{T(f, \varphi_1, \varphi_2) T(g, \psi_1, \psi_2)},
\]
where \( T(u, v, w) \) is defined by
\[
T(u, v, w) = (J^\alpha uv)(x) - (J^\alpha u(x) - J^\alpha v(x))
\]
\[
+ \frac{x^\alpha}{\Gamma(\alpha+1)} J^\alpha (uv)(x) - J^\alpha u(x) J^\alpha v(x)
\]
\[
+ \frac{x^\alpha}{\Gamma(\alpha+1)} J^\alpha (uw)(x) - J^\alpha u(x) J^\alpha w(x)
\]
\[
- \frac{x^\alpha}{\Gamma(\alpha+1)} J^\alpha (v\omega)(x) + J^\alpha v(x) J^\alpha \omega(x).
\]

Motivated from above mentioned results, our purpose in this paper is to establish some new results on Gruss-type inequalities in case of functional bounds using the generalized Katugampola fractional integral.
2 PRELIMINARIES:

In this section, we give some definitions and properties available in literature that will be used in our paper, for more details (see [10], [11], [17]).

Definition 2.1 Consider the space $X^p_c(a,b)$ ($c \in \mathbb{R}, 1 \leq p \leq \infty$), of those complex-valued Lebesgue measurable functions $v$ on $(a,b)$ for which the norm $\|v\|_{X^p_c} < \infty$, such that

$$\|v\|_{X^p_c} = \left( \int_a^b |x^c v|^p \frac{dx}{x} \right)^{\frac{1}{p}}, \quad (1 \leq p < \infty)$$

and

$$\|v\|_{X^\infty} = \sup_{x \in (a,b)} [x^c |v|].$$

In particular, when $c = 1/p$, the space $X^p_c(a,b)$ coincides with the space $L^p(a,b)$.

Definition 2.2 The left- and right-sided fractional integrals of a function $v$ where $v \in X^p_c(a,b)$, $\alpha > 0$, and $\beta, \rho, \eta, k \in \mathbb{R}$, are defined respectively by

$$\rho J_{a+;\eta,k}^{\alpha,\beta} v(x) = \frac{\rho^{1-\beta} x^k}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} v(\tau) \, d\tau, \quad 0 \leq a < x < b \leq \infty,$$

and

$$\rho J_{b-;\eta,k}^{\alpha,\beta} v(x) = \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{k+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} v(\tau) \, d\tau, \quad 0 \leq a < x < b \leq \infty,$$

if the integral exist.

To present and discuss our new results in this paper we use the left-sided fractional integrals, the right sided fractional can be proved similarly, also we consider $a = 0$, in (2.1), to obtain

$$\rho J_{0+;\eta,k}^{\alpha,\beta} v(x) = \frac{\rho^{1-\beta} x^k}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} v(\tau) \, d\tau.$$

The above fractional integral has the following Composition (index) formulae

$$\rho J_{\alpha_1+;\eta_1,k_1}^{\alpha_1,\beta_1} \rho J_{\alpha_2+;\eta_2,-\rho\eta_1}^{\alpha_2,\beta_2} v = \rho J_{\alpha_1+;\eta_2,k_1}^{\alpha_1+\alpha_2,\beta_1+\beta_2} v,$$

and

$$\rho J_{\alpha_1+;\eta_1,k_2}^{\alpha_1,\beta_1} \rho J_{\alpha_2-,;\eta_2,-\rho\eta_1}^{\alpha_2,\beta_2} v = \rho J_{\alpha_1+;\eta_2,k_2}^{\alpha_1+\alpha_2,\beta_1+\beta_2} v.$$

For the convenience of establishing our results we define the following function as in [17]: let $x > 0$, $\alpha > 0$, $\rho, k, \beta, \eta \in \mathbb{R}$, then

$$\Lambda_{x,k}^{\rho,\beta}(\alpha, \eta) = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} x^{\rho-\beta} x^{\rho(\eta+\alpha)}.$$
If $\eta = 0$, $a = 0$, $k = 0$, and taking the limit $\rho \to 1$, the Definition (2.2) reduce to Liouville fractional integral and if $\eta = 0$, $k = 0$, and taking the limit $\rho \to 1$, we can get Riemann-Liouville fractional integral, if $\eta = 0$, $a = -\infty$, $k = 0$, and taking the limit $\rho \to 1$, for Erdélyi-Kober fractional integral, we put $\beta = 0$, $k = -\rho(\alpha + \eta)$, we can also getting Katugampola fractional integral by taking $\beta = \alpha$, $k = 0$, $\eta = 0$. And finally Hadamard fractional integral if $\beta = \alpha$, $k = 0$, $\eta = 0 +$, and taking the limit $\rho \to 1$.

The Definition (2.2) is more generalized and can be reduce to six cases by change its parameters with appropriate choice.

3 Main Results:

Now, we give our main results on Gruss type inequality in case of functional bounds.

**Theorem 3.1** Let $v$ be an integrable function on $[0, \infty)$. Assume that there exist two integrable functions $z_1, z_2$ on $[0, \infty)$ such that

$$z_1(x) \leq v(x) \leq z_2(x) \quad \forall x \in [0, \infty). \quad (3.1)$$

Then, for all $x > 0$, $\alpha > 0$, $\delta > 0$, $\beta, \eta, k, \lambda \in \mathbb{R}$, we have

$$\rho J^{\alpha, \beta}_{\eta, k} z_2(x) \rho J^{\delta, \lambda}_{\eta, k} v(x) + \rho J^{\alpha, \beta}_{\eta, k} v(x) \rho J^{\delta, \lambda}_{\eta, k} z_1(x) \quad (3.2)$$

$$\geq \rho J^{\alpha, \beta}_{\eta, k} v(x) \rho J^{\delta, \lambda}_{\eta, k} v(x) + \rho J^{\alpha, \beta}_{\eta, k} z_2(x) \rho J^{\delta, \lambda}_{\eta, k} z_1(x)$$

**Proof.** From the condition (3.1), for all $\tau \geq 0$, $\sigma \geq 0$, we have

$$(v(\sigma) - z_1(\sigma))(z_2(\tau) - v(\tau)) \geq 0.$$ (3.3)

Therefore

$$v(\sigma) z_2(\tau) + z_1(\sigma) v(\tau) \geq v(\sigma) v(\tau) + z_1(\sigma) z_2(\tau).$$

Multiplying both sides of (3.3) by $\frac{\beta^{-k}}{\Gamma(\alpha)} \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}}$, where $\tau \in (0, x)$ and integrating with respect to $\tau$ over $(0, x)$, we get

$$v(\sigma) \frac{\beta^{-k}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} z_2(\tau) d\tau \quad + z_1(\sigma) \frac{\beta^{-k}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} v(\tau) d\tau \quad \geq v(\sigma) \frac{\beta^{-k}}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} v(\tau) d\tau$$
so we have

$$v(\sigma) \rho^\alpha J_{\eta,k} z_2(x) + z_1(\sigma) \rho^\alpha J_{\eta,k} v(x) \geq v(\sigma) \rho^\alpha J_{\eta,k} v(x) + z_1(\sigma) \rho^\alpha J_{\eta,k} z_2(x).$$

(3.4)

Multiplying both sides of (3.4) by \(\frac{\rho^{1-\beta} x^k}{\Gamma(\delta)} (x^\rho - \sigma^\rho)^{1-\delta} v(\sigma)\), where \(\sigma \in (0, x)\), we obtain

\[
\begin{align*}
\rho^{1-\beta} x^k & \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} (x^\rho - \sigma^\rho)^{1-\delta} v(\sigma) \rho^\alpha J_{\eta,k} z_2(x) \\
+ \rho^{1-\beta} x^k & \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} (x^\rho - \sigma^\rho)^{1-\delta} z_1(\sigma) \rho^\alpha J_{\eta,k} v(x) \\
\geq \rho^{1-\beta} x^k & \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} (x^\rho - \sigma^\rho)^{1-\delta} z_1(\sigma) \rho^\alpha J_{\eta,k} z_2(x).
\end{align*}
\]

(3.5)

Integrating both sides of (3.5) with respect to \(\sigma\) over \((0, x)\), we get

\[
\begin{align*}
\rho^\alpha J_{\eta,k} z_2(x) & \rho^{1-\beta} x^k \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} \int_0^x (x^\rho - \sigma^\rho)^{1-\delta} v(\sigma) d\sigma \\
+ \rho^\alpha J_{\eta,k} v(x) & \rho^{1-\beta} x^k \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} \int_0^x (x^\rho - \sigma^\rho)^{1-\delta} z_1(\sigma) d\sigma \\
\geq \rho^\alpha J_{\eta,k} v(x) & \rho^{1-\beta} x^k \frac{\rho^{\alpha(\delta+1)-1}}{\Gamma(\delta)} \int_0^x (x^\rho - \sigma^\rho)^{1-\delta} z_1(\sigma) d\sigma.
\end{align*}
\]

Hence

\[
\begin{align*}
\rho^\alpha J_{\eta,k} z_2(x) & \rho^\delta \lambda v(x) + \rho^\alpha J_{\eta,k} v(x) \rho^\delta \lambda z_1(x) \\
\geq \rho^\alpha J_{\eta,k} v(x) & \rho^\delta \lambda v(x) + \rho^\alpha J_{\eta,k} z_2(x) \rho^\delta \lambda z_1(x),
\end{align*}
\]

which is inequality (3.2).

**Corollary 3.2** Let \(z\) be an integrable function on \([0, \infty)\) satisfying \(m \leq z(x) \leq M\), for all \(x \in [0, \infty)\) and \(m, M \in \mathbb{R}\). Then, for all \(x > 0\) and \(\alpha > 0, \beta > 0, \delta > 0, \beta, \eta, k, \lambda \in \mathbb{R}\), we have

\[
M \Lambda_{\alpha, \eta}^\beta (x, k) \rho^\alpha J_{\eta,k} v(x) + m \Lambda_{\alpha, \eta}^\beta (x, k) \rho^\alpha J_{\eta,k} v(x) \\
\geq \rho^\alpha J_{\eta,k} v(x) \rho^\delta \lambda v(x) + \rho^\alpha J_{\eta,k} z_2(x) \rho^\delta \lambda z_1(x).
\]
Remark 3.3 If we put $\eta = 0$, $k = 0$, and taking the limit $\rho \to 1$, then theorem (3.1), reduces to theorem 2 and corollary (3.2), reduces to corollary 3 in [18].

Now we give the lemma required for proving our next theorem

Lemma 3.1 Let $v, z_1, z_2$ are integrable functions on $[0, \infty)$ satisfying the condition (3.1) then for all $x > 0$ and $\alpha > 0$, $\rho > 0$, $\beta, \eta, k \in \mathbb{R}$, we have

$$\Lambda^{\rho, \beta}_{x,k} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} v^2(x) - \left( \rho J_{\eta,k}^{\alpha,\beta} v(x) \right)^2 = \left( \rho J_{\eta,k}^{\alpha,\beta} z_2(x) - \rho J_{\eta,k}^{\alpha,\beta} v(x) \right) \left( \rho J_{\eta,k}^{\alpha,\beta} v(x) - \rho J_{\eta,k}^{\alpha,\beta} z_1(x) \right) - \Lambda^{\rho, \beta}_{x,k} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} \left( (z_2(x) - v(x)) (v(x) - z_1(x)) \right) + \Lambda^{\rho, \beta}_{x,k} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} (z_1v)(x) - \rho J_{\eta,k}^{\alpha,\beta} z_1(x) \rho J_{\eta,k}^{\alpha,\beta} v(x)$$

(3.6)

Prove. For any $\tau, \sigma > 0$, we have

$$(z_2(\sigma) - v(\sigma)) (v(\tau) - z_1(\tau)) + (z_2(\tau) - v(\tau)) (v(\sigma) - z_1(\sigma)) - (z_2(\tau) - v(\tau)) (v(\tau) - z_1(\tau)) - (z_2(\sigma) - v(\sigma)) (v(\sigma) - z_1(\sigma)) = v^2(\tau) + v^2(\sigma) - 2v(\tau)v(\sigma) + z_2(\sigma) v(\tau) + z_1(\tau) v(\sigma) - z_1(\tau) z_2(\sigma) + z_2(\tau) v(\sigma) + z_1(\sigma) v(\tau) - z_1(\sigma) z_2(\tau) - z_2(\tau) v(\tau) + z_1(\tau) z_2(\tau) - z_1(\tau) v(\tau) - z_2(\sigma) v(\sigma) + z_1(\sigma) z_2(\sigma) - z_1(\sigma) v(\sigma).$$

(3.7)

Multiplying both sides of (3.7) by $\frac{2^{1-\rho} x^{n-1}}{\Gamma(n)} (x^\rho - x^\sigma)^{\rho-1}$, where $\tau \in (0, x)$ and integrating over $(0, x)$ with respect to the variable $\tau$, we obtain

$$\Lambda^{\rho, \beta}_{x,k} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} v^2(x) + v^2(\sigma) - 2v(\sigma) \rho J_{\eta,k}^{\alpha,\beta} v(x) + z_2(\sigma) \rho J_{\eta,k}^{\alpha,\beta} v(x) + v(\sigma) \rho J_{\eta,k}^{\alpha,\beta} z_1(x) - z_2(\sigma) \rho J_{\eta,k}^{\alpha,\beta} z_1(x) + v(\sigma) \rho J_{\eta,k}^{\alpha,\beta} z_2(x) + z_1(\sigma) \rho J_{\eta,k}^{\alpha,\beta} v(x) - z_1(\sigma) \rho J_{\eta,k}^{\alpha,\beta} z_1(x) - \rho J_{\eta,k}^{\alpha,\beta} (z_2v)(x) + \rho J_{\eta,k}^{\alpha,\beta} (z_1z_2)(x) - \rho J_{\eta,k}^{\alpha,\beta} (z_1v)(x)$$

(3.8)
- \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)z_2(\sigma)v(\sigma) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)z_1(\sigma)z_2(\sigma) - \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)z_1(\sigma)v(\sigma).

Now multiplying both sides of (3.8) by \frac{\rho^{1-\rho}x^{\rho(\rho+1)-1}}{(x^{\rho-2})^{1-\alpha}}$, where $\sigma \in (0, x)$ and integrating over $(0, x)$ with respect to the variable $\sigma$, we obtain

\[
\left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) \right) \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x) \right) \\
+ \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}z_1(x) \right) \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}z_2(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) \right) \\
- \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}[(z_2(x) - v(x))(v(x) - z_1(x))] \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \\
- \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}[(z_2(x) - v(x))(v(x) - z_1(x))] \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta)
\]

\[
= \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x) + \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v^2(x)
\]

Which yields the required identity (3.6).

Our next result is on Gruss type inequality in case of functional bounds with same parameters

**Theorem 3.4** Let $v, u$ be two integrable functions on $[0, \infty)$. Suppose $z_1, z_2, \gamma_1$ and $\gamma_2$ be four integrable functions on $[0, \infty)$ satisfying the condition

\[
z_1(x) \leq v(x) \leq z_2(x) \quad \text{and} \quad \gamma_1(x) \leq u(x) \leq \gamma_2(x) \quad \forall x \in [0, \infty).
\]

Then for all $x > 0$ and $\alpha > 0, \rho > 0, \beta, \eta, k \in \mathbb{R}$, we have

\[
\left[ \Lambda_{x,k}^{\rho,\beta}(\alpha,\eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}(vu)(x) - \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}v(x) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}u(x) \right) \right]^2 \\
\leq T(v, z_1, z_2)T(u, \gamma_1, \gamma_2),
\]

where $T(\varphi, \psi, \omega)$ as in (13), is defined by

\[
T(\varphi, \psi, \omega) = \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x) \right) \left( \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x) \right) \\
+ \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}(\varphi\varphi)(x), \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\psi(x) \\
+ \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}(\varphi\psi)(x) - \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\varphi(x) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x) \\
- \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}(\psi\psi)(x) + \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}(\varphi\omega)(x) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta}\omega(x).\]

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Proof. Define

\[ H(\tau, \sigma) := (v(\tau) - v(\sigma))(u(\tau) - u(\sigma)), \quad \tau, \sigma \in (0, x), \ x > 0. \]  

(3.11)

Multiplying both sides of (3.11) by \( \frac{\beta^{1-\beta} x^k}{\Gamma(\alpha)} \tau^{\alpha(\eta+1)-1} (x^\rho - \tau^\rho)^{-1-\alpha} \), where \( \tau \in (0, x) \) and integrating over \((0, x)\) with respect to the variable \( \tau \), we obtain

\[
\frac{\rho^{\beta} x^k}{\Gamma(\alpha)} \int_0^x \tau^{\alpha(\eta+1)-1} (x^\rho - \tau^\rho)^{-1-\alpha} H(\tau, \sigma) \ d\tau
:= \rho J_{x,k}^{\alpha,\beta} (uv)(x) + A_{x,k}^{\rho,\beta} (\alpha, \eta) v(\sigma) u(\sigma)
- u(\sigma) J_{\eta,k}^{\alpha,\beta} v(x) - v(\sigma) J_{\eta,k}^{\alpha,\beta} u(x). \tag{3.12}
\]

Now multiplying both sides of (3.12) by \( \frac{\rho^{\beta} x^k}{\Gamma(\alpha)} \sigma^{\alpha(\eta+1)-1} (x^\rho - \sigma^\rho)^{-1-\alpha} \), where \( \sigma \in (0, x) \) and integrating the resulting identity over \((0, x)\) with respect to the variable \( \sigma \), we get

\[
\frac{\rho^{2(1-\beta)} x^{2k}}{2\Gamma^2(\alpha)} \int_0^x \int_0^x \tau^{\alpha(\eta+1)-1} (x^\rho - \tau^\rho)^{-1-\alpha} \sigma^{\alpha(\eta+1)-1} (x^\rho - \sigma^\rho)^{-1-\alpha} H(\tau, \sigma) \ d\tau d\sigma
:= A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} (uv)(x) - \rho J_{\eta,k}^{\alpha,\beta} u(x) \rho J_{\eta,k}^{\alpha,\beta} v(x). \tag{3.13}
\]

Applying the Cauchy-Schwarz inequality to (3.13), we can write

\[
\left( A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} (uv)(x) - \rho J_{\eta,k}^{\alpha,\beta} u(x) \rho J_{\eta,k}^{\alpha,\beta} v(x) \right)^2
\leq \left( A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} u^2(x) - \rho J_{\eta,k}^{\alpha,\beta} u(x)^2 \right) \times \left( A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} v^2(x) - \rho J_{\eta,k}^{\alpha,\beta} v(x)^2 \right). \tag{3.14}
\]

Since

\[
(z_2(x) - v(x))(v(x) - z_1(x)) \geq 0,
(\gamma_2(x) - u(x))(u(x) - \gamma_1(x)) \geq 0, \tag{3.15}
\]

for all \( x \in [0, \infty) \), we have

\[
A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} (z_2(x) - v(x))(v(x) - z_1(x)) \geq 0
\]

and

\[
A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} (\gamma_2(x) - u(x))(u(x) - \gamma_1(x)) \geq 0.
\]

Thus, from lemma (3.1), we have

\[
A_{x,k}^{\rho,\beta} (\alpha, \eta) \rho J_{\eta,k}^{\alpha,\beta} v^2(x) - \left( \rho J_{\eta,k}^{\alpha,\beta} v(x) \right)^2
\]
\[
\leq \left( \rho \mathcal{J}^{\alpha,\beta}_{\eta,k} z_2 (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) \right) \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_1 (x) \right)
+ \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (z_1 v) (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_1 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x)
+ \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (z_2 v) (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_2 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x)
- \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (z_1 z_2) (x) + \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_2 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_1 (x)
= T (v, z_1, z_2)
\]

and
\[
\Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} u^2 (x) - \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} u (x) \right)^2
\leq \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_2 (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_1 (x) \right) \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_1 (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_1 (x) \right)
+ \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (\gamma_1 u) (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_1 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} u (x)
+ \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (\gamma_2 u) (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_2 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} u (x)
- \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (\gamma_1 \gamma_2) (x) + \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_2 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} \gamma_1 (x)
= T (u, \gamma_1, \gamma_2) .
\]

Combining the Inequalities (3.16), (3.17) with inequality (3.14), we obtain inequality (3.10). ■

Remark 3.5 If we put \( T (v, z_1, z_2) = T (v, m, M) \) and \( T (u, \gamma_1, \gamma_2) = T (v, p, P) \), in theorem (3.4), where \( m, M, p, P \) are constants, then inequality (3.10) reduces to
\[
\left| \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}^{\alpha,\beta}_{n,k} (vu) (x) - \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} u (x) \right) \right|
\leq \left( \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \right)^2 (M - m) (P - p) .
\]

Which is result given in [17].

Lemma 3.2 Let \( v, z_1, z_2 \) are integrable functions on \([0, \infty)\) satisfying the condition (3.7), then for all \( x > 0 \) and \( \alpha > 0, \delta > 0, \rho > 0, \beta, \lambda, \eta, k \in \mathbb{R}, \) we have
\[
\Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}^{\delta,\lambda}_{n,k} v^2 (x) + \rho \mathcal{J}^{\delta,\lambda}_{n,k} v^2 (x) - 2 \rho \mathcal{J}^{\delta,\lambda}_{n,k} \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x)
= \left( \rho \mathcal{J}^{\delta,\lambda}_{n,k} z_2 (x) - \rho \mathcal{J}^{\delta,\lambda}_{n,k} v (x) \right) \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_1 (x) \right)
+ \left( \rho \mathcal{J}^{\delta,\lambda}_{n,k} v (x) - \rho \mathcal{J}^{\delta,\lambda}_{n,k} z_1 (x) \right) \left( \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_2 (x) - \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) \right)
- \rho \mathcal{J}^{\delta,\lambda}_{n,k} \left[ (z_2 (\tau) - v (\tau)) (v (\tau) - z_1 (\tau)) \right] \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta)
- \rho \mathcal{J}^{\delta,\lambda}_{n,k} \left[ (z_2 (\sigma) - v (\sigma)) (v (\sigma) - z_1 (\sigma)) \right] \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta)
- \rho \mathcal{J}^{\delta,\lambda}_{n,k} z_2 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x) - \rho \mathcal{J}^{\delta,\lambda}_{n,k} v (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_2 (x)
- \rho \mathcal{J}^{\delta,\lambda}_{n,k} v (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} z_1 (x) - \rho \mathcal{J}^{\delta,\lambda}_{n,k} z_1 (x) \rho \mathcal{J}^{\alpha,\beta}_{n,k} v (x)
\]
Proof. In lemma (3.1), multiplying both sides of (3.8) by \( \frac{1 - \lambda x^k}{(x^k)^{\rho(n+1)-1}} \), where \( \sigma \in (0, x) \) and integrating the resulting identity over (0, x) with respect to the variable \( \sigma \), we obtain

\[
\left( \alpha, \beta \right) (x, k) \right) \Lambda^\alpha,\beta_{x,k} (x, \eta, k) + \Lambda^\alpha,\beta_{x,k} (x, \eta, k)
\]

\[
\Lambda^\alpha,\beta_{x,k} (x, \eta, k) + \Lambda^\alpha,\beta_{x,k} (x, \eta, k)
\]

\[
\Lambda^\alpha,\beta_{x,k} (x, \eta, k) + \Lambda^\alpha,\beta_{x,k} (x, \eta, k)
\]

Which gives (3.18) and proves the lemma. ■

In our next theorem we prove the result with different parameters. Here we use our lemma (3.2) to proving the result.

**Theorem 3.6** Let \( v, u \) be two integrable functions on \([0, \infty)\) and suppose \( z_1, z_2, \gamma_1 \) and \( \gamma_2 \) be four integrable functions on \([0, \infty)\) satisfying the condition (3.9), then for all \( x > 0 \) and \( \alpha > 0, \delta > 0, \rho > 0, \beta, \lambda, \eta, k \in \mathbb{R} \), we have

\[
\left| \Lambda^\alpha,\beta_{x,k} (\delta, \eta) \rho J^\alpha,\beta_{y,k} (uv) (x) + \Lambda^\alpha,\beta_{x,k} (\alpha, \eta) \rho J^\alpha,\beta_{y,k} (vu) (x) \right|
\]

\[
\leq \sqrt{K (v, z_1, z_2)} K (u, \gamma_1, \gamma_2),
\]

where \( K (\varphi, \psi, \omega) \) is defined by

\[
K (\varphi, \psi, \omega) = \left( \rho J^\alpha,\beta_{y,k} (\omega) (x) - \rho J^\alpha,\beta_{y,k} (\varphi) (x) \right) \left( \rho J^\alpha,\beta_{y,k} (\psi) (x) - \rho J^\alpha,\beta_{y,k} (\varphi) (x) \right)
\]

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and integrating the resulting identity over \((0, x)\) with respect to the variable \(\sigma\), we obtain

\[
\begin{align*}
\int_{0}^{x} \int_{0}^{x} \frac{\rho^{\alpha}(\eta + 1)^{-1}}{\Gamma(\delta - \alpha \sigma)} \frac{\rho^{\alpha}(\eta + 1)^{-1}}{\Gamma(\delta - \alpha \sigma)} \rho^{\alpha} \varphi(\sigma)(\eta - \sigma) \, d\sigma \, d\tau \\
\end{align*}
\]

\(3.20\)

Applying Cauchy-Schwarz inequality for double integrals, we get

\[
\begin{align*}
&\left[ \Lambda_{x,k}^{(\delta, \eta)} \rho^{\alpha} \varphi(\sigma)(\eta - \sigma) \right] \frac{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)}{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)} \frac{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)}{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)} \frac{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)}{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)} \\
\end{align*}
\]

\(3.21\)

Since

\[
\begin{align*}
(z_{2}(x) - v(x))(v(x) - z_{1}(x)) & \geq 0 \\
(\gamma_{2}(x) - u(x))(u(x) - \gamma_{1}(x)) & \geq 0,
\end{align*}
\]

for all \(x \in [0, \infty)\), therefore

\[
\begin{align*}
&\left( \rho^{\alpha} \varphi(\sigma)(\eta - \sigma) \right) \frac{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)}{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)} \\
\end{align*}
\]

\(3.22\)

and

\[
\begin{align*}
&\left( \rho^{\alpha} \varphi(\sigma)(\eta - \sigma) \right) \frac{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)}{\rho^{\alpha} \varphi(\sigma)(\eta - \sigma)} \\
\end{align*}
\]

\(3.23\)
From the Inequalities (3.22), (3.23) and inequality (3.21), we obtain inequality (3.19).

Thus, from lemma (3.2), we have

\[ \Lambda_{x,k}^{\alpha,\beta}(\delta, \eta) \left[ \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} v^2(x) + \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u^2(x) - 2 \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} v(x) \right] \]
\[ \leq \left( \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_2(x) - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \right) \left( \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u(x) - \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_1(x) \right) \]
\[ + \left( \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} u(x) - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \right) \left( \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_2(x) - \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u(x) \right) \]
\[ - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_2(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u(x) - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} u(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_2(x) \]
\[ - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} u(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_1(x) - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} u(x) \] (3.22)
\[ + \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_2(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_1(x) + \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_2(x) \]
\[ + \Lambda_{x,k}^{\delta, \lambda}(\delta, \eta) \left[ \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_1 (v(x)) + \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} \gamma_2 (u(x)) - \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} (\gamma_1 \gamma_2)(x) \right] \]
\[ + \Lambda_{x,k}^{\delta, \lambda}(\alpha, \eta) \left[ \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1 (u(x)) + \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_2 (u(x)) - \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} (\gamma_1 \gamma_2)(x) \right] \]
\[ = K(u, \gamma_1, \gamma_2) . \]

From the Inequalities (3.22), (3.23) and inequality (3.21), we obtain inequality (3.19).

Now we give the following result

**Theorem 3.7** Let \( v, u \) be two integrable functions on \([0, \infty)\) and suppose \( z_1, z_2, \gamma_1 \) and \( \gamma_2 \) be four integrable functions on \([0, \infty)\) satisfying the condition (3.9), then for all \( x > 0 \) and \( \alpha > 0, \delta > 0, \rho > 0, \beta, \lambda, \eta, k \in \mathbb{R} \), the following inequalities holds:

\[ \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} u(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} z_2(x) + \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} v(x) \]
\[ \geq \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} \gamma_1(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} z_2(x) + \rho \mathcal{J}_{\eta, k}^{\delta, \lambda} u(x) \rho \mathcal{J}_{\eta, k}^{\alpha, \beta} v(x) , \]
\[(b) \quad \rho \mathcal{J}_{n,k}^{\delta,\lambda} z_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) \geq \rho \mathcal{J}_{n,k}^{\delta,\lambda} z_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) ,\]

\[(c) \quad \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) \geq \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_2 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) \]

\[(d) \quad \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) \geq \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} \gamma_1 (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) \]

**Proof.** To prove \((a)\), from the condition \((3.9)\), we have for \(x \in [0, \infty)\) that
\[
\left( z_2 (\tau) - v (\tau) \right) (u (\sigma) - \gamma_1 (\sigma)) \geq 0. \tag{3.24}
\]
Therefore
\[
z_2 (\tau) u (\sigma) + v (\tau) \gamma_1 (\sigma) \geq z_2 (\tau) \gamma_1 (\sigma) + v (\tau) u (\sigma) . \tag{3.25}
\]
Multiplying both sides of \((3.25)\) by \(\frac{\rho^{1-\beta} x^k}{\Gamma(\sigma)}\), where \(\tau \in (0, x)\) and integrating over \(0, x\) with respect to the variable \(\tau\), we obtain
\[
u (\sigma) \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_2 (x) + \rho \mathcal{J}_{n,k}^{\alpha,\beta} v (x) \geq \gamma_1 (\sigma) \rho \mathcal{J}_{n,k}^{\alpha,\beta} z_2 (x) + \rho \mathcal{J}_{n,k}^{\alpha,\beta} v (x) . \tag{3.26}
\]
Now multiplying both sides of \((3.12)\) by \(\frac{\rho^{1-\beta} x^k}{\Gamma(\sigma)}\), where \(\sigma \in (0, x)\) and integrating the resulting identity over \(0, x\) with respect to the variable \(\sigma\), we get the desired inequality \((a)\). To prove \((b)\), \((c)\) and \((d)\), we use the following inequalities:

\[(b) \quad (\gamma_2 (\tau) - u (\tau)) (v (\sigma) - z_1 (\sigma)) \geq 0,\]

\[(b) \quad (z_2 (\tau) - v (\tau)) (u (\sigma) - \gamma_2 (\sigma)) \leq 0,\]

\[(b) \quad (z_1 (\tau) - v (\tau)) (u (\sigma) - \gamma_1 (\sigma)) \leq 0.\]

The next corollary is a special case of Theorem \((3.7)\).

**Corollary 3.8** Let \(v, u\) be two integrable functions on \([0, \infty)\) and suppose that there exist the constants \(n, N, m, M\) satisfying the condition
\[
m \leq v (x) \leq M \quad \text{and} \quad n \leq u (x) \leq N \quad \forall x \in [0, \infty),\]

, then for all \(x > 0\) and \(\alpha > 0, \delta > 0, \rho > 0, \beta, \lambda, \eta, k \in \mathbb{R}\), we have:

\[(A) \quad M \Lambda_{x,k}^{\alpha,\beta} (\alpha, \eta) \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) + n \Lambda_{x,k}^{\alpha,\beta} (\delta, \eta) \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) \geq n M \Lambda_{x,k}^{\alpha,\beta} (\delta, \eta) \Lambda_{x,k}^{\alpha,\beta} (\alpha, \eta) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} u (x) + \rho \mathcal{J}_{n,k}^{\delta,\lambda} v (x) ,\]
\[(B) \quad m \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x) + N \Lambda_{x,k}^{\rho,\lambda} (\alpha, \eta) \rho \mathcal{J}_{\eta,k}^{\delta,\lambda} v (x) \\
\geq m N \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \Lambda_{x,k}^{\rho,\lambda} (\alpha, \eta) + \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x),
\]

\[(C) \quad M N \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} v (x) \rho \mathcal{J}_{\eta,k}^{\delta,\lambda} u (x) \\
\geq M \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x) + N \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} v (x),
\]

\[(D) \quad m n \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x) + n \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} v (x) \\
\geq m \Lambda_{x,k}^{\rho,\beta} (\alpha, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} u (x) + n \Lambda_{x,k}^{\rho,\lambda} (\delta, \eta) \rho \mathcal{J}_{\eta,k}^{\alpha,\beta} v (x).
\]

\textbf{Remark 3.9} If we put $\eta = 0$, $k = 0$, and taking the limit $\rho \to 1$, then theorem (3.7), reduces to theorem 5 and corollary (3.8), reduces to corollary 6 in [18].

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