On classical capacity of Weyl channels

Grigori Amosov

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Abstract
We study an old problem concerning the resolution of the question of whether the supremum of the Holevo upper bound for the output of a quantum channel coincides with the classical capacity of this channel. It is shown that this property takes place for one special case. The additivity of minimal output entropy is proved for the Weyl channel obtained by the deformation of a q-c Weyl channel. The classical capacity of channel is calculated.

Keywords Quantum Weyl channel · Classical capacity of a channel · Holevo upper bound · Additivity conjecture

Mathematics Subject Classification 81P45 · 81P47 · 94A40

1 Introduction
The quantum coding theorem proved independently by A.S. Holevo [1] and B. Schumacher, M.D. Westmoreland [2] posed the task of calculating the Holevo upper bound $C(\Phi^{\otimes N})$ for a tensor product of N copies of quantum channel $\Phi$ because a classical capacity of $\Phi$ is given by the formula

$$C(\Phi) = \lim_{N \to +\infty} \frac{C(\Phi^{\otimes N})}{N}.$$ 

The additivity conjecture asks whether the equality

$$C(\Phi \otimes \Omega) = C(\Phi) + C(\Omega)$$

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Grigori Amosov
gramos@mi-ras.ru

1 Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina St., Moscow, Russia 119991
holds true for the fixed channel $\Phi$ and an arbitrary channel $\Omega$. If the additivity property (1) takes place for $\Phi$, the classical capacity can be calculated as follows:

$$C(\Phi) = \overline{C}(\Phi).$$  \hfill (2)

The same level of interest has the additivity conjecture in the weak form asking whether

$$\overline{C}(\Phi^\otimes N) = N\overline{C}(\Phi)$$

takes place for a fixed channel $\Phi$. The validity of this statement also leads to (2). The additivity conjecture for $\overline{C}$ is closely related to the additivity conjecture for the minimal output entropy of a channel and the multiplicativity conjectures for trace norms of a channel [3]. At the moment, the additivity is proved for many significant cases [4–8] including the solution to the famous problem of Gaussian optimizers [9,10]. On the other hand, there are channels for which the additivity conjecture does not hold true [11]. Recently, the method of majorization was introduced to estimate the Holevo upper bound for Weyl channels [12]. In the present paper, we prove the additivity conjecture for one subclass of Weyl channels that are “deformations” of q-c channels of [1]. Our method is based upon [12].

Throughout this paper, we denote $\mathcal{S}(H)$ the set of positive unit-trace operators (quantum states) in a Hilbert space $H$, $I_H$ is the identity operator in $H$ and $S(\rho) = -Tr(\rho \log \rho)$ is the von Neumann entropy of $\rho \in \mathcal{S}(H)$. Quantum channel $\Phi : \mathcal{S}(H) \to \mathcal{S}(K)$ is a completely positive trace preserving map between the algebras of all bounded operators $B(H)$ and $B(K)$ in Hilbert spaces $H$ and $K$, respectively. Given two $\rho, \sigma \in \mathcal{S}(H)$ for which $supp\rho \subset supp\sigma$ the quantum relative entropy is $S(\rho || \sigma) = Tr(\rho \log \rho) - Tr(\rho \log \sigma)$. The property of non-increasing the relative entropy with respect to the action of a quantum channel $\Phi$ states [13]

$$S(\Phi(\rho) || \Phi(\sigma)) \leq S(\rho || \sigma)$$

for $\rho, \sigma \in \mathcal{S}(H)$.

The Holevo upper bound for a quantum channel $\Phi$ is determined by the formula

$$\overline{C}(\Phi) = \sup_{\pi_j, \rho_j \in \mathcal{S}(H)} \left( S \left( \sum_j \pi_j \Phi(\rho_j) \right) - \sum_j \pi_j S(\Phi(\rho_j)) \right),$$

where the supremum is taken over all probability distributions $(\pi_j)$ on the ensemble of states $\rho_j \in \mathcal{S}(H)$.

## 2 Weyl channels

Here, we use the techniques introduced in [14,15] and developed in [16–19]. Fix an orthonormal basis $(e_j, j \in \mathbb{Z}_n)$ in a Hilbert space $H$ with dimension $dimH = n$, and
consider two unitary operators in $H$ defined by the formula

$$U e_j = e^{\frac{2\pi i}{n}} e_j, \quad V e_j = e_{j+1}, \quad j \in \mathbb{Z}_n.$$  \hfill (3)

Formula (3) determines unitaries $W_{jk} = U^j V^k$ called Weyl operators satisfying the property

$$\sum_{j, k \in \mathbb{Z}_n} W_{jk} \rho W_{jk}^* = n I_H, \quad \rho \in \mathcal{S}(H).$$  \hfill (4)

Quantum channels of the form

$$\Phi(\rho) = \sum_{j, k \in \mathbb{Z}_n} \pi_{jk} W_{jk} \rho W_{jk}^*, \quad \rho \in \mathcal{S}(H),$$  \hfill (5)

where $(\pi_{jk})$ is a probability distribution, are said to be Weyl channels. Given a unitary representation $\lambda$ of $\mathbb{Z}_n$ in $H$ and a probability distribution $(p_k, k \in \mathbb{Z}_n)$ a Weyl channel of the form

$$\Psi_{\lambda}(\rho) = \sum_{k \in \mathbb{Z}_n} p_k \lambda(k) \rho \lambda(k)^*, \quad \rho \in \mathcal{S}(H),$$

is said to be a phase damping channel.

Let us fix a phase damping channel of the form

$$\Psi(\rho) = \sum_{k \in \mathbb{Z}_n} p_k V^k \rho V^{k*}, \quad \rho \in \mathcal{S}(H),$$

where $p = (p_k, k \in \mathbb{Z}_n)$ is a probability distribution. Consider the quantum channel

$$\Phi(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j \Psi(\rho) U^j = \frac{1}{n} \sum_{j, k \in \mathbb{Z}_n} p_k U^j V^k \rho V^{k*} U^{j*}, \quad \rho \in \mathcal{S}(H),$$  \hfill (6)

Formula (6) gives a general form of the Weyl channel invariant with respect to the action of the group $(U^j, \quad j \in \mathbb{Z}_n)$ in the sense

$$U^j \Phi(\rho) U^{j*} = \Phi(\rho), \quad \rho \in \mathcal{S}(H), \quad j \in \mathbb{Z}_n.$$  \hfill (7)

It follows from (7) that

$$\mathbb{E} \circ \Phi = \Phi,$$  \hfill (8)

where the expectation $\mathbb{E}$ to the algebra of fixed elements with respect to the action of $(U^j, \quad j \in \mathbb{Z}_n)$ is given by

$$\mathbb{E}(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j \rho U^{j*}, \quad \rho \in \mathcal{S}(H).$$
Put
\[ \mathcal{E}_k(\rho) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} U^j V^k \rho V^{k*} U^{j*}, \quad \rho \in \mathcal{G}(H), \quad j \in \mathbb{Z}_n, \]
then (6) can be represented as
\[ \Phi(\rho) = \sum_{k \in \mathbb{Z}_n} p_k \mathcal{E}_k(\rho). \]

The property (8) shows that \( \Phi \) is a q-c channel and the additivity of \( \overline{C} \) was shown in [1]. We place the following statement here to calculate the exact value of a classical capacity.

**Proposition 1** Given a quantum channel \( \Omega : \mathcal{G}(K) \to \mathcal{G}(K) \) and a pure state \( |\xi\rangle \langle \xi| \in \mathcal{G}(H \otimes K) \)
\[ \inf_{\rho \in \mathcal{G}(H \otimes K)} S(\Phi \otimes \Omega(|\xi\rangle \langle \xi|)) \geq - \sum_{k \in \mathbb{Z}_n} p_k \log p_k + S(\Omega(Tr_H(|\xi\rangle \langle \xi|))). \]

**Proof** Let us define a c-q channel \( \Upsilon : \mathcal{G}(H) \to \mathcal{G}(H \otimes K) \) by the formula
\[ \Upsilon(\rho) = \sum_{k \in \mathbb{Z}_n} \langle e_k, \rho e_k \rangle (\mathcal{E}_k \otimes \Omega)(|\xi\rangle \langle \xi|), \quad \rho \in \mathcal{G}(H). \]
Put
\[ \rho = \sum_{k \in \mathbb{Z}_n} p_k |e_k\rangle \langle e_k|, \quad \sigma = \frac{1}{n} I_H. \quad \text{(9)} \]
Applying the property of non-increasing the quantum relative entropy with respect to the action of quantum channel, we obtain
\[ S(\Upsilon(\rho) || \Upsilon(\sigma)) \leq S(\rho || \sigma). \quad \text{(10)} \]
It follows from (4) that
\[ \sum_{k \in \mathbb{Z}_n} \mathcal{E}_k(\rho) = I_H, \quad \rho \in \mathcal{G}(H). \]
Hence,
\[ \sum_{k \in \mathbb{Z}_n} (\mathcal{E}_k \otimes \Omega)(|\xi\rangle \langle \xi|) = I_H \otimes \Omega(Tr_H(|\xi\rangle \langle \xi|)) \]
and
\[ \Upsilon(\sigma) = \frac{1}{n} I_H \otimes \Omega(Tr_H(|\xi\rangle \langle \xi|)). \quad \text{(11)} \]
Substituting (9)–(11) to (10), we get

\[- S((\Phi \otimes \Omega)(|\xi\rangle\langle\xi|)) - Tr \left( (\Phi \otimes \Omega)(|\xi\rangle\langle\xi|) \log \left( \frac{1}{n} I_H \otimes \Omega(Tr_H(|\xi\rangle\langle\xi|)) \right) \right) \leq \sum_{k \in \mathbb{Z}_n} p_k \log p_k - Tr(\rho \log \sigma).\]

Taking into account that

\[ Tr \left( (\Phi \otimes \Omega)(|\xi\rangle\langle\xi|) \log \left( \frac{1}{n} I_H \otimes \Omega(Tr_H(|\xi\rangle\langle\xi|)) \right) \right) = - \log n - S(\Omega(Tr_H(|\xi\rangle\langle\xi|))) \]

we obtain the result.

\[ \text{Corollary 1} \]

Given a quantum channel \( \Omega : \mathcal{S}(K) \rightarrow \mathcal{S}(K) \) and the q-c Weyl channel (6), the following equality holds

\[ \inf_{\rho \in \mathcal{S}(H \otimes K)} S((\Phi \otimes \Omega)(\rho)) = \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) + \inf_{\rho \in \mathcal{S}(K)} S(\Omega(\rho)). \]

\[ \text{Proof} \] Notice that

\[ S(\Phi(|e_j\rangle\langle e_j|)) = - \sum_{k \in \mathbb{Z}_n} p_k \log p_k \geq \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) \]

for any \( j \in \mathbb{Z}_n \). It follows from Proposition 1 that

\[ \inf_{\rho \in \mathcal{S}(H \otimes K)} S((\Phi \otimes \Omega)(\rho)) \geq \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) + \inf_{\rho \in \mathcal{S}(K)} S(\Omega(\rho)). \] (12)

On the other hand, the right side in (12) cannot be less than the left hand side. Hence,

\[ \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) = - \sum_{k \in \mathbb{Z}_n} p_k \log p_k \]

and we have the equality in (12). \( \square \)

\[ \text{Corollary 2} \]

The classical capacity of the q-c Weyl channel (6) is given by the formula

\[ C(\Phi) = \log(n) + \sum_{k \in \mathbb{Z}_n} p_k \log p_k. \]
**Proof** The statement can be derived from the fact that

\[
\overline{C}(\Phi^\otimes N) = N \log n - \inf_{\rho \in \mathcal{S}(H^\otimes N)} S(\Phi^\otimes N(\rho))
\]

for covariant channels [20]. It follows from Corollary 1 that

\[
\inf_{\rho \in \mathcal{S}(H^\otimes N)} S(\Phi^\otimes N(\rho)) = N \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)).
\]

In the proof of Corollary 1, we have shown that

\[
\inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) = -\sum_{k \in \mathbb{Z}_n} p_k \log p_k.
\]  \(\text{(13)}\)

\[\square\]

### 3 Majorization

Let \(\mathfrak{J}\) be the index set and \(|\mathfrak{J}| = d < +\infty\). Given a probability distribution \(\lambda = (\lambda_J, \ J \in \mathfrak{J})\), we denote \(\lambda^\downarrow = (\lambda^\downarrow_j, \ 1 \leq j \leq d)\) the probability distribution obtained by sorting \(\lambda\) in the decreasing order,

\[
\lambda^\downarrow_1 \geq \lambda^\downarrow_2 \geq \cdots \geq \lambda^\downarrow_d.
\]

Consider two probability distribution \(\lambda = (\lambda_J, \ J \in \mathfrak{J})\) and \(\mu = (\mu_J, \ J \in \mathfrak{J})\). We shall say that \(\lambda\) majorizes \(\mu\) and write

\[
\mu \prec \lambda
\]

iff

\[
\sum_{j=1}^k \mu_j^\downarrow \leq \sum_{j=1}^k \lambda_j^\downarrow, \ 1 \leq k \leq d.
\]

Let \(H_d\) be a Hilbert space with \(\dim H_d = d\). Denote \(B(H_d)\) the algebra of all bounded operators in \(H_d\). The following statement can be derived from [12] (see Theorem 2).

**Proposition 2** Let \(0 \leq X_J \leq I, \ J \in \mathfrak{J}, \ |\mathfrak{J}| = d^2\), be a set of positive operators in \(B(H_d)\) such that

\[
\sum_{J \in \mathfrak{J}} X_J = d I_{H_d}.
\]
Then, given a probability distribution \( \pi = (\pi_J, J \in \mathcal{J}) \) the eigenvalues \( \lambda = (\lambda_J)_{J=1}^{d} \) of the positive operator

\[
A = \sum_{J \in \mathcal{J}} \pi_J X_J
\]

sorted in the decreasing order \( \lambda \equiv \lambda^\downarrow \) satisfy the relation

\[
\lambda < p,
\]

where

\[
p_j = \sum_{m=1+(j-1)d}^{d+(j-1)d} \pi_m^\downarrow, \quad 1 \leq j \leq d.
\]

**Proof** Let \((e_j)_{j=1}^{d}\) be the unit eigenvectors corresponding to the eigenvalues \((\lambda_J)_{J=1}^{d}\).

Then,

\[
\sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} \langle e_j, Ae_j \rangle = \sum_{j=1}^{k} \sum_{J \in \mathcal{J}} \pi_J \langle e_J, X_J e_j \rangle \leq \sum_{j=1}^{k} p_j, \quad 1 \leq k \leq d.
\]

\(\Box\)

**Corollary 3** The eigenvalues \( \lambda \) of the positive operator \( A \) in Proposition 2 possess the property

\[
- \sum_{j=1}^{d} \lambda_j \log \lambda_j \geq - \sum_{j=1}^{d} p_j \log p_j.
\]

**Proof** Since \( \lambda \) majorizes \( \mu \) due to Proposition 2, we get the result [21]. \(\Box\)

**4 Deformation of q-c Weyl channels**

Let us come back to Weyl channels (5).

**Definition.** Suppose that a probability distribution \((\pi_{jk}, j, k \in \mathbb{Z}_n)\) satisfies the relation

\[
\pi_{00} \geq \pi_{10} \geq \cdots \geq \pi_{n-10} \geq \pi_{01} \geq \cdots \geq \pi_{n-11} \geq \pi_{02} \geq \cdots \geq \pi_{n-1n-1}.
\]

(14)

Put

\[
p_k = \sum_{j \in \mathbb{Z}_n} \pi_{jk}, \quad k \in \mathbb{Z}_n.
\]

(15)
Then (5) is said to be the Weyl channel obtained by the deformation of q-c channel (6).

**Theorem** The Weyl channel $\Phi$ obtained by the deformation of q-c channel satisfies the property

$$\inf_{\rho \in \mathcal{S}(H \otimes N)} S(\Phi \otimes N(\rho)) = -N \sum_{j=1}^{n} p_j \log p_j.$$ 

**Proof** Denote $\mathcal{J}$ the index set $(\mathbb{Z}_n \times \mathbb{Z}_n)^N$ consisting of collections $(j_1, k_1), \ldots, (j_N, k_N), j_s, k_s \in \mathbb{Z}_n$. Let us consider the probability distribution $\Pi = (\Pi_J, J \in \mathcal{J})$ and a set of positive operators $(X_J, J \in \mathcal{J})$ defined by the formula

$$\Pi_J = \prod_{s=1}^{N} \pi_{j_s k_s},$$

$$X_J = \left( \bigotimes_{s=1}^{N} W_{j_s k_s} \right) \rho \left( \bigotimes_{s=1}^{N} W_{j_s k_s}^* \right), \quad J \in \mathcal{J},$$

where $\rho$ is a fixed state in $\mathcal{S}(H \otimes N)$. Then, the conditions of Proposition 2 is satisfied for $(\Pi_J), (X_J)$ and $d = n^N$. Applying Corollary 3, we obtain

$$S(\Phi(\rho)) \geq -N \sum_{j=1}^{n} p_j \log p_j. \quad (16)$$

The equality in (16) is achieved for any

$$\rho = |e\rangle \langle e|,$$

where

$$e = \bigotimes_{s=1}^{N} e_{j_s}, \quad j_s \in \mathbb{Z}_n.$$ 

\[\square\]

**Corollary 4** The classical capacity of the Weyl channel obtained by the deformation of (6) is given by the formula

$$C(\Phi) = \log(n) + \sum_{k \in \mathbb{Z}_n} p_k \log p_k.$$ 

**Proof** The statement can be derived from the fact that

$$\overline{C}(\Phi \otimes N) = N \log n - \inf_{\rho \in \mathcal{S}(H \otimes N)} S(\Phi \otimes N(\rho))$$

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for covariant channels [20]. It follows from Theorem that

\[
\inf_{\rho \in \mathcal{S}(H^\otimes N)} S(\Phi^\otimes N(\rho)) = N \inf_{\rho \in \mathcal{S}(H)} S(\Phi(\rho)) = -N \sum_{j=1}^{N} p_j \log p_j.
\]

\[\Box\]

4.1 Example: qutrits

Because the qubit case \(\dim H = 2\) is completely parsed [4], a simplest example of the introduced techniques can be given for qutrits, \(\dim H = 3\). Let us define two unitary operators \(U\) and \(V\) satisfying (3)

\[
\begin{align*}
U e_0 &= e_0, & U e_1 &= e^{i \frac{2\pi}{3}} e_1, & U e_2 &= e^{i \frac{4\pi}{3}} e_2, \\
V e_0 &= e_1, & V e_1 &= e_2, & V e_2 &= e_0.
\end{align*}
\]

Then, consider the expectation (8)

\[
\mathbb{E}(x) = \frac{1}{3} \sum_{j=0}^{2} U^j x U^{j*}, \quad x \in B(H).
\]

Taking a probability distribution \(\{p_0, p_1, p_2\}\), we can define a qc Weyl channel by the formula

\[
\Phi_{qc}(\rho) = \mathbb{E} \circ \sum_{k=0}^{2} p_k V^k \rho V^{k*}, \quad \rho \in \mathcal{S}(H).
\] (17)

It follows from Corollary 1 and Corollary 2 that

\[
\inf_{\rho \in \mathcal{S}(H^\otimes K)} S((\Phi_{qc} \otimes \Omega)(\rho)) = \inf_{\rho \in \mathcal{S}(H)} S(\Phi_{qc}(\rho)) + \inf_{\rho \in \mathcal{S}(K)} S(\Omega(\rho))
\]

for any quantum channel \(\Omega : \mathcal{S}(K) \to \mathcal{S}(K)\) and the classical capacity is equal to

\[
C(\Phi_{qc}) = n + \sum_{k=0}^{2} p_k \log p_k.
\]

Suppose that \(p_0 \geq p_1 \geq p_2\) and one can pick up positive numbers \(\pi_{jk}, \quad 0 \leq j, k \leq 2\), satisfying the relations

\[
\begin{align*}
\pi_{00} \geq \pi_{10} \geq \pi_{20} \geq \pi_{01} \geq \pi_{11} \geq \pi_{21} \geq \pi_{02} \geq \pi_{12} \geq \pi_{22}, \\
p_k = \pi_{0k} + \pi_{1k} + \pi_{2k}, \quad 0 \leq k \leq 2.
\end{align*}
\]
Then,

\[ \Phi(\rho) = \sum_{j,k=0}^{2} \pi_{jk} U_j V_k \rho V_k^{*} U_j^{*}, \quad \rho \in \mathcal{S}(H), \]

is the Weyl channel obtained by the deformation of (17). Applying Corollary 4, we obtain for a classical capacity

\[ C(\Phi) = \log(3) + p_0 \log p_0 + p_1 \log p_1 + p_2 \log p_2. \]

As a concrete example, one can take

\[ p_0 = \frac{1}{2}, \quad p_1 = \frac{1}{3}, \quad p_2 = \frac{1}{6}. \]

In the case, one of possible deformations is given by

\[ \pi_{00} = \frac{1}{4}, \quad \pi_{10} = \frac{1}{8}, \quad \pi_{20} = \frac{1}{8}, \]
\[ \pi_{01} = \frac{1}{8}, \quad \pi_{11} = \frac{1}{8}, \quad \pi_{21} = \frac{1}{12}, \]
\[ \pi_{02} = \frac{1}{12}, \quad \pi_{12} = \frac{1}{24}, \quad \pi_{22} = \frac{1}{24}. \]

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