Fractional Diffusion Processes: 
Probability Distributions 
and Continuous Time Random Walk

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Abstract. A physical-mathematical approach to anomalous diffusion may be based on generalized diffusion equations (containing derivatives of fractional order in space or/and time) and related random walk models. By the space-time fractional diffusion equation we mean an evolution equation obtained from the standard linear diffusion equation by replacing the second-order space derivative with a Riesz-Feller derivative of order $\alpha \in (0, 2]$ and skewness $\theta$ ($|\theta| \leq \min\{\alpha, 2 - \alpha\}$), and the first-order time derivative with a Caputo derivative of order $\beta \in (0, 1]$. The fundamental solution (for the Cauchy problem) of the fractional diffusion equation can be interpreted as a probability density evolving in time of a peculiar self-similar stochastic process. We view it as a generalized diffusion process that we call \textit{fractional diffusion process}, and present an integral representation of the fundamental solution. A more general approach to anomalous diffusion is however known to be provided by the master equation for a continuous time random walk (CTRW). We show how this equation reduces to our fractional diffusion equation by a properly scaled passage to the limit of compressed waiting times and jump widths. Finally, we describe a method of simulation and display (via graphics) results of a few numerical case studies.

1 Introduction

It is well known that the fundamental solution (or \textit{Green function}) for the Cauchy problem of the linear diffusion equation can be interpreted as a Gaussian (normal) probability density function (\textit{pdf}) in space, evolving in time. All the moments of this \textit{pdf} are finite; in particular, its variance is proportional to the first power of time, a noteworthy property of the standard diffusion that can be understood by means of an unbiased random walk model for the \textit{Brownian motion}.

In recent years a number of master equations have been proposed for random walk models that turn out to be beyond the classical Brownian motion, see e.g. Klafter et al. \textsuperscript{34}. In particular, evolution equations containing fractional derivatives have gained revived interest in that they are expected to provide suitable mathematical models for describing phenomena of anomalous diffusion,
strange kinetics and transport dynamics in complex systems. Recent references include e.g. [1,2,3,24,37,38,42,43,48,51,61].

The paper is divided as follows. In Sect. 2 we introduce our fractional diffusion equations providing the reader with the essential notions for the derivatives of fractional order (in space and in time) entering these equations. More precisely, we replace in the standard linear diffusion equation the second-order space derivative or/and the first-order time derivative by suitable integro-differential operators, which can be interpreted as a space or time derivative of fractional order \(\alpha \in (0,2]\) or \(\beta \in (0,1]\), respectively. The space fractional derivative is required to depend also on a real parameter \(\theta \) (the skewness) subjected to the restriction \(|\theta| \leq \min\{\alpha, 2 - \alpha\}\). Then, in Sect. 3, we pay attention to the fact that the fundamental solutions (or Green functions) of our diffusion equations of fractional order in space or/and in time can be interpreted as spatial probability densities evolving in time, related to certain self-similar stochastic process. We view these processes as generalized (or fractional) diffusion processes to be properly understood through suitable random walk models that have been treated by us in previous papers, see e.g. [15,18,19,20,21,23]. In Sect. 4 we show how such evolution equations of fractional order can be obtained from a more general master equation which governs the so-called continuous time random walk (CTRW) by a properly scaled passage to the limit of compressed waiting times and jump widths. The CTRW structure immediately offers a method of simulation that we roughly describe in Sect. 5 where we also display graphs of a few numerical case studies. Finally, in Sect. 6, we point out the main conclusions and outline the direction for future work.

2 The Space-Time Fractional Diffusion Equation

By replacing in the standard diffusion equation

\[
\frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t), \quad -\infty < x < +\infty, \quad t \geq 0, \tag{1}
\]

where \(u = u(x, t)\) is the (real) field variable, the second-order space derivative and the first-order time derivative by suitable integro-differential operators, which can be interpreted as a space and time derivative of fractional order we obtain a generalized diffusion equation which may be referred to as the space-time-fractional diffusion equation. We write this equation as

\[
iD^\alpha u(x, t) = \mathcal{D}_\beta^\theta u(x, t), \quad -\infty < x < +\infty, \quad t \geq 0, \tag{2}
\]

1 To the topic of strange kinetics a special issue (nowadays in press) of *Chemical Physics* is devoted where the interested reader can find a number of applications of fractional diffusion equations
2 We remind that the term "fractional" is a misnomer since the order can be a real number and thus is not restricted to be rational. The term is kept only for historical reasons, see e.g. [17]. Our fractional derivatives are required to coincide with the standard derivatives of integer order as soon as \(\alpha = 2\) (not as \(\alpha = 1\)) and \(\beta = 1\).