Fitting Correlated Hadron Mass Spectrum Data

C. Michael and A. McKerrell

DAMTP, University of Liverpool, Liverpool, L69 3BX, U.K.

Abstract

We discuss fitting hadronic Green functions versus time $t$ to extract mass values in quenched lattice QCD. These data are themselves strongly correlated in $t$. With only a limited number of data samples, the method of minimising correlated $\chi^2$ is unreliable. We explore several methods of modelling the correlations among the data set by a few parameters which then give a stable and sensible fit even if the data sample is small. In particular these models give a reliable estimate of the goodness of fit.

1 Introduction

We assume that we have $N$ samples of unbiased estimators of quantities $x(t)$ with $t = 1 \ldots D$. Thus the data set is $x^{(n)}(t)$ where $n = 1 \ldots N$. We assume that the samples $x^{(n)}(t)$ are statistically independent versus $n$ for fixed $t$ but may be correlated in $t$. Such a situation arises in lattice gauge theory calculations where there are $N$ independent configurations and $D$ Green functions (vacuum expectation values of combinations of quark propagators and/or gauge links) are measured versus time separation $t$. An introduction to this topic in the context of lattice gauge theory is provided by Toussaint [1].

The aim is to fit a given function $F(t)$ which depends on $P$ parameters $a_p$. This function is to be fitted to the data samples $x(t)$. Thus we require to find the following

- the best values of the parameters $a_p$,
- the errors associated with these best fit parameters,
- the confidence level that the fit represents the data sample.
A general discussion of this problem has been given \[2\] which has shown that with limited data samples \(N\), it may be unrealistic to allow a general form for the \(D \times D\) correlation matrix describing the correlations among the data at the \(D\) different \(t\) values. In particular the estimated \(\chi^2\) was shown to be increased by a factor of \(1 + (D + 1)/N + O(N^{-2})\). This led to the conclusion that \(N > 10(D+1)\) is needed for a reliable use of \(\chi^2\) as a goodness of fit estimator in a correlated \(\chi^2\) fit.

Although this conclusion arose from studying a theoretical distribution which was uncorrelated, the above \(\chi^2\) increase is the same for any true distribution. Indeed the \(\chi^2\) per degree of freedom will have the expected value \((N - 1)/(N - D - 2)\). Furthermore the distribution of \(\chi^2\) can be evaluated theoretically \([3]\) so that confidence levels can be derived. The major consequence of small sample size is that the eigenvalues of the correlation matrix are modified: in particular very small eigenvalues can arise. These correspond to a very narrow distribution in the corresponding eigen-direction and so can bias any fit considerably.

In order to cope with this, it is necessary to use some extra theoretical input. In particular, one needs to assume a form of the correlation matrix or of its eigenvalue spectrum. This has been recognised before, and proposals have been made to truncate the eigenvalues according to the spirit of the SVD inverse of a singular matrix \([4, 5, 6]\). We discuss this proposal and offer our suggestions for an improved treatment. We believe that our approach is more stable.

Another way to attack this problem is to explore the theoretical expectations for the correlation matrix. In the case we are considering, the correlation between meson operators can be expressed in terms of vacuum expectation values of 4-quark operators \([4]\). These can be estimated and this gives a motivation for a direct parametrization of the correlation matrix in terms of exponentials in \(|t - t'|\). If the normalised correlation is given by one such exponential, then its inverse is a tridiagonal matrix. This is a very appealing model since it corresponds to a nearest neighbour linkage in the underlying probability distribution. This one-parameter model is very stable, but does not always give an accurate description of the data. A natural way to extend it is to consider a 5-diagonal inverse as an efficient parametrization. This 2-parameter model corresponds to a correlation matrix given by a specific combination of 2 exponentials and it fits the data well in many cases.

We start by presenting typical data on the hadronic Green functions from quenched lattice studies. This enables us to analyse the behaviour of the correlation matrix which is at the heart of our exploration. We then compare various methods of modelling the correlation matrix and test their stability in correlated \(\chi^2\) fits.
2 Fits with correlated data

The data \( x(t) \) we will consider are Green functions which are vacuum expectations of hadronic operators at times 0 and \( t \). Such data are often referred to as hadron correlators but we will not use this description here since we wish to concentrate on a different correlation: that between the Green functions \( x(t) \) at different \( t \) values.

The data sample themselves give a probability distribution

\[
S(x) = \frac{1}{N} \sum_{n=1}^{N} \delta^{D}(x - x^{(n)})
\]

We shall be interested in estimates of the probability distribution \( Q(X) \) of the averages \( X(t) \) of the data \( x(t) \). A smooth representation of \( Q(X) \) is needed for determining best fit parameters and to estimate the acceptability of such a fit. The natural parametrisation for \( Q \) is suggested by the central limit theorem. Provided that the underlying distributions of \( x(t) \) are sufficiently localised, then for large \( N \), \( X(t) \) will be gaussianly distributed. We are specifically interested in the case where the different components \( x(t) \) are statistically correlated. Thus a general gaussian surface will be needed.

\[
Q(X) = H \exp\left(-\frac{1}{2} \sum_{t,t'} (X(t) - \overline{X}(t))M(t,t')(X(t') - \overline{X}(t'))\right)
\]

with

\[
\overline{X}(t) = \frac{1}{N} \sum_{n=1}^{N} x^{(n)}(t)
\]

\[
M(t,t') = NC^{-1}(t,t') ,\text{where}
\]

\[
C(t,t') = \frac{1}{N-1} \sum_{n=1}^{N} (x^{(n)}(t) - \overline{X}(t))(x^{(n)}(t') - \overline{X}(t'))
\]

To find the best fit parameters then corresponds to maximising

\[
\exp(-\chi^2/2) \text{ where } \chi^2 = \sum_{t,t'} (F(t,a) - \overline{X}(t))M(t,t')(F(t',a) - \overline{X}(t'))
\]

with respect to \( a_p \) for \( p = 1 \ldots P \). This is the usual correlated \( \chi^2 \) method. For sufficiently large \( N \), this is a stable procedure and the expected value of \( \chi^2 \) is the number of degrees of freedom \( D - P \). But for small \( N \), the sampling fluctuation in \( C \) and hence \( M \) can give unreasonable fits. In order to avoid this bias, we aim to make a more stable model for \( C \).

Let us now study the properties of the correlation matrix \( C \). It is a real symmetric positive-definite matrix for any number of samples \( N \) but it has rank \( N - 1 \) so will have \( D - N + 1 \) zero
eigenvalues if \( N \leq D \). In this latter case, its inverse is not defined. A commonly used prescription in such a case is the Singular Value Decomposition (SVD) inverse which corresponds to omitting the eigenmodes corresponding to the zero eigenvectors from the inverse. We will return to discuss the utility of this prescription.

For many purposes, it is simpler to study the normalised correlation matrix which we define as

\[
\tilde{C}(t, t') = \frac{C(t, t')}{\sqrt{C(t, t)C(t', t')}}
\]

### 3 Correlations among hadronic operators

In quenched lattice determinations of hadronic spectra and matrix elements, one studies vacuum expectation values of hadronic operators at times 0 and \( t \). Thus \( x(t) \) is the vacuum expectation value of hadronic operators \( \langle H(0)H(t) \rangle \). These determinations of \( x(t) \) are extracted from the quark propagators derived from inverting the fermion matrix in the given gauge field sample. This propagator inversion is very demanding computationally and is usually only evaluated for one source point \((0,0)\). This implies that quark propagators to all values of \( t \) are equally sensitive to the region around this fixed source site. Thus between different gauge (vacuum) samples, all propagators will tend to be large/small as this fixed region is conducive/resistant to quark propagation. This argument shows that very prominent correlations are expected between hadron Green functions to different \( t \) values. We analyse some data to substantiate this.

We present results for the normalised correlation \( \tilde{C}(t, t') \) in fig 1. Here the data come from a study of the \( \pi \) and \( \rho \) mesons using local hadronic operators from a point source at \((0,0)\) to \((y, t)\) summed over \( y \) to give a zero-momentum observable. There are \( N = 60 \) independent configurations using a hopping parameter \( K = 0.14262 \) with the clover improved action [7]. The lattice has size \( 24^3 \times 48 \) at \( \beta = 6.2 \) and, for orientation, \( m_\pi a \approx 0.17 \) at this hopping parameter value. We use \( t \) values 5 to 24 for this study.

We find that \( \tilde{C} \) decreases with \(|t - t'|\) but is relatively insensitive to \( t + t' \). The decrease with \(|t - t'|\) is illustrated in fig 1. An exponential behaviour versus \(|t - t'|\) is expected from an analysis in terms of hadronic operators.

Let us summarise this argument. Consider the case where \( x(t) \) is the vacuum expectation value of hadronic operators \( \langle H(0)H(t) \rangle \). Then

\[
C(t, t') = \langle H(0)H(0)H(t)H(t') \rangle - \langle H(0)H(t) \rangle \langle H(0)H(t') \rangle
\]

The first term then will have contributions between \( t \) and \( t' \) from intermediate states of lowest energy \( m \) with the quantum numbers created by \( H \), the same as those in \( x(t) \) itself. Between \( 0 \) and \( t \) (for \( t' > t \)), the intermediate state will have the quantum numbers of \( H(0)H(0) \) and
so may have a lower energy which we write as \(2M\). Ignoring for the moment the disconnected term, and assuming that one state only dominates in each case, then
\[
\tilde{C}(t, t') = e^{-(m-M)|t-t'|}
\]

For the case of the \(\rho\) for example: \(m = m_\rho\) and \(M = m_\pi\) since a \(2\pi\) state can couple to \(HH\) (ie to \(\rho\rho\)). Thus a small exponential rate of decrease is to be expected with exponent \(m_\rho - m_\pi\). In this case the disconnected term in \(C\) will be relatively unimportant since it decreases faster than the connected term by \(\exp(-2(m - M)t)\). The contribution of excited intermediate states will modify this simple exponential behaviour except at large \(|t - t'|\) where the lowest state dominates. The curve corresponding to the ground state exponential \(\exp(-(m_\rho - m_\pi)|t-t'|)\) is shown in fig.1, where it is seen to be a reasonable guide to the large \(|t - t'|\) behaviour.

For the case where \(x\) is a pion observable, then both \(m\) and \(M\) are \(m_\pi\) and the correlation \(\tilde{C}(t, t')\) would be constant versus \(|t - t'|\). In this case the disconnected term will be relatively important. The disconnected part will reduce the magnitude of \(\tilde{C}\) especially when relatively more disconnected terms are present - such as with a source summed over spatial position. When the source is at a fixed lattice position \((0,0)\), as above, then the disconnected parts will cancel less completely and we expect \(\tilde{C}\) to remain large for large \(|t - t'|\). Indeed it is larger for \(\pi\) than \(\rho\) correlations as shown in fig. 1.

For baryon spectra, the nucleon is the lowest lying 3 quark state and so \(\tilde{C}\) will be like the pion case above, while the \(\Delta\) will behave analogously to the \(\rho\) above.

### 4 Exponential models for correlations among the data

Since there is some theoretical justification for an exponential decrease of \(\tilde{C}(t, t')\) with \(|t - t'|\), we consider first a simple and robust model with just one exponential. Since \(\tilde{C}(t, t')\) is normalised, this results in a one parameter model with parameter \(a\)
\[
\tilde{C}(t, t') = e^{-a|t-t'|}
\]

In practice the behaviour of \(\tilde{C}(t, t')\) is not exactly exponential, so one must choose a reference value of \(|t - t'| = t_a\) to determine a suitable value of \(a\). We have found that using \(t_a = 4\) is a good choice. Then \(a\) is determined by averaging the \(D - t_a\) values of \(\tilde{C}(t, t + t_a)\) obtained from the sample data. This averaging along the off-diagonal of \(\tilde{C}\) also helps to reduce the sampling fluctuations.

The test of the suitability of a model of \(\tilde{C}(t, t')\) for our purposes is that its inverse \(\tilde{M}\) is stable under fluctuations in the data sample used to model \(\tilde{C}(t, t')\). Thus it is appropriate to study
the inverse of $\tilde{C}(t, t')$. For the case of a single exponential, the inverse is particularly simple: it is tridiagonal. It is given exactly in terms of $\alpha \equiv \exp(-a)$ by

$$\tilde{M}(t, t) = \frac{1+\alpha^2}{1-\alpha^2}, \quad \tilde{M}(t, t \pm 1) = -\frac{-\alpha}{1-\alpha^2}, \quad \tilde{M}(t_m, t_m) = \frac{1}{1-\alpha^2}$$

where $t_m$ is the maximum or minimum $t$ value in the matrix being inverted.

Thus as $a \to 0$, the elements of $\tilde{M}(t, t')$ increase as $a^{-1}$. In the limit of large D, one can estimate the smallest eigenvalue of $\tilde{C}(t, t')$ (largest of $\tilde{M}(t, t')$) which is $a/2$ ($2/a$ respectively) as $a \to 0$. Thus if $a$ is reasonably well determined by the correlated data sample, then the value of $\tilde{M}(t, t')$ will be stable under sample fluctuations. Thus the resultant fits will be stable too. This method provides a stable one parameter model for $\tilde{C}(t, t')$. One drawback of the model is that it does not reproduce very accurately any non-exponential behaviour of $\tilde{C}(t, t')$. This can be taken into account in a straightforward way by considering 2-exponential models for $\tilde{C}(t, t')$.

Rather than consider an arbitrary 2-exponential model, we generalise the tri-diagonal feature of $\tilde{M}(t, t')$ and look for 5-diagonal models instead. The algebra is now somewhat messier, but the conclusion is that a 5-diagonal model for $\tilde{M}(t, t')$ corresponds to a particular 2-exponential model for $\tilde{C}(t, t')$ with:

$$\tilde{C}(t, t') = \frac{(1-\alpha_2^2)\alpha_1|t-t'|+1 - (1-\alpha_1^2)\alpha_2|t-t'|+1}{(\alpha_1-\alpha_2)(1+\alpha_1\alpha_2)}$$

where to have a sensible interpretation we require $0 \leq Re \alpha_1, Re \alpha_2 < 1$.

If $\alpha_1$ and $\alpha_2$ are complex, they must be complex conjugates. In this case the behaviour of $\tilde{C}(t, t')$ versus $|t-t'|$ will be a damped oscillation. Although such a behaviour is not strictly excluded, it seems unreasonable to have anti-correlation at larger $t$ values so we choose not to allow that possibility. Henceforward, we take $\alpha_1$ and $\alpha_2$ to be real.

Thus we obtain $\alpha_1$ and $\alpha_2$ by comparing the above expression for $\tilde{C}(t, t')$ with the sample data. The two parameters can, for instance, be determined by making a least squares fit to $\tilde{C}(t, t')$ at $|t-t'| = 1$ and $t_a$. A fit is needed because in some cases the data may not be reproducible exactly by the expression. We find that this 2-parameter assignment to the sample correlation is stable when inverted for use in fits. This is plausible since a 5-diagonal form of the inverse avoids very small eigenvalues of $\tilde{C}(t, t')$.

We give the explicit formula for the inverse of the 5-diagonal matrix in the Appendix.

5 Eigenvalue smoothing of the correlation matrix

The essence of the problem is that sample values of $\tilde{C}(t, t')$ may have very small eigenvalues and these influence unreasonably the inverse $\tilde{M}(t, t')$ used in modelling the distribution of the
data. An obvious way to proceed is to modify these unreasonable eigenvalues by hand. One suggestion \cite{4} is to remove the largest eigenvalues of $\tilde{C}(t, t')$ since they will have least influence on $\tilde{M}(t, t')$. This seems hard to justify and later suggestions \cite{5, 6} have been to remove the smallest eigenvalues of $\tilde{C}(t, t')$. This latter suggestion is in the spirit of the SVD inverse of a singular matrix: only the contributions from the non-zero eigenvalues are retained in the inverse. This eigenvalue truncation is clearly a rather brutal approximation to $\tilde{M}(t, t')$: its largest components are being removed. Indeed the gaussian surface modelling the probability distribution will be unconstrained in the direction of the deleted eigenmodes. A physical argument, for why this may be acceptable in practice, can be based on the observation \cite{5} that the smallest eigenvalues of $\tilde{C}(t, t')$ usually correspond to eigenvectors which alternate in sign (versus $t$) and so are not very relevant to smooth fit functions.

As we have argued, with a small sample size $N$, the $D$ eigenvalues of the sample correlation matrix will be changed from their true values. The largest relative effect will come when there are several true eigenvalues of similar size - since those eigen directions mix fully. We find that the eigenvalue spectrum is densest at small eigenvalues. This again leads to the conclusion that the smallest eigenvalues of the sample correlation matrix are the most strongly affected.

The SVD approach replaces the smallest (or zero) eigenvalues by very large values (since this corresponds to ignoring those terms in the inverse). We have explored this situation and found a less discontinuous way to modify the smallest eigenvalues $\lambda_i$ of $\tilde{C}(t, t')$. We keep the largest $E$ eigenvalues substantially unchanged and then rearrange the remaining smaller eigenvalues to avoid extremely small ones. Following from the motivation that like eigenvalues mix most, we propose to redistribute the small eigenvalues by replacing them by their average to avoid very small values.

After some experimentation, we propose the following explicit scheme to replace the $D$ eigenvalues $\lambda_i$ of the sample normalised correlation $\tilde{C}(t, t')$ (with convention $\lambda_i \geq \lambda_{i+1}$) with new eigenvalues $\lambda'_i$:

$$
\lambda'_i = K \text{Max}(\lambda_i, \lambda_{min})
$$

where $\lambda_{min} = \frac{1}{D - E} \sum_{i=E+1}^{D} \lambda_i$ and $K^{-1} = \frac{1}{D} \sum_{i=1}^{D} \text{Max}(\lambda_i, \lambda_{min})$

The eigenvectors of $\tilde{C}(t, t')$ and thus of its inverse $\tilde{M}(t, t')$ are retained unchanged. Thus our procedure removes any very small eigenvalues of $\tilde{C}(t, t')$ and replaces them with the average of the $D - E$ smallest eigenvalues while retaining the property that the trace of $\tilde{C}(t, t')$ is $D$. The procedure also ensures a smooth eigenvalue distribution by allowing eigenvalues larger than this average to be retained.

We tested this assignment with the same data as used above. Of course for $E = D - 1$, the method is equivalent to an exact inversion of the sample correlation matrix. For $E > 2,$
this method provides a stable model of the correlation matrix from the sample data. As $E$ is increased, the model reproduces more exactly the sample correlation matrix - but at the expense of including unreasonable fluctuations if the sample size is too small. A compromise is to retain $E \approx \sqrt{N}$ exact eigenvalues when there are $N$ samples.

6 Practical test of models

Here we apply the various models described above to some typical real data. Since we need a large number of samples to give an accurate data set we chose some APE data on $\rho$ meson Green functions $x(t) = \langle H(0)H(t) \rangle$. Here local-source and local-sink operators $H$ are used for creating and destroying a $\rho$ meson. The data set has $N = 420$ samples of $x(t)$ with the clover fermionic action at $K = 0.14190$ on a $18^3 \times 64$ lattice at $\beta = 6.2$.

The data sample is large enough to allow a full correlated fit to the observed $x(t)$. An acceptable fit ($\chi^2$/d.o.f. = 11.11/9) to $x(t)$ with one exponential (actually a cosh is used) is found for the $t$-range 14 to 24. This 2-parameter fit corresponds to requiring a plateau in the effective mass. The eigenvalues of the normalised correlation matrix for this data set are shown in fig 2 by the continuous line.

Here our intention is not to obtain the most precise values of $m_\rho$ but to illustrate the stability of various fitting prescriptions. Thus we take this full data sample as the true result and explore fits using smaller subsets of the 420 data. For example we take 14 blocks of 30 data. For each such set of 30, we perform a fit to the same $t$-range. The intention is to check whether the $\chi^2$/d.o.f. is similar to the true value from the full data set. As well as the average value of $\chi^2$/d.o.f., one may study its distribution so that confidence levels can be extracted. We do not pursue this here.

A valid criterion for goodness of fit is of importance. The usual method is to use $\chi^2$ to decide if a fit is acceptable over a given $t$-range. Thus an erroneous $\chi^2$ value will lead to an increase or decrease in the $t$-range chosen as acceptable. This in turn will bias the fitted parameters such as $m_\rho$. For example, an uncorrelated fit will yield much lower $\chi^2$ which will then suggest lower $t$ values being included. This will tend to increase $m_\rho$ since the effective mass is a decreasing function of $t$ in this case.

Returning now to the comparison: fig 3 shows the results of the average of 14 fits to different samples of size $N = 30$. The full correlated fit ($n = 11$) has a higher $\chi^2$ on average by 40% than the ‘true’ result. This is entirely expected from the factor $1 + (D + 1)/N = 1.39$ increase in $\chi^2$ predicted [3]. The eigenvalues from one such sample fit to $N = 30$ are shown in fig 2. Here the phenomenon of very small eigenvalues appearing for small sample size is clear. Thus a direct use of a correlated fit to $N = 30$ samples is likely to be biassed by those spurious small eigenvalues. We consider now some of the models introduced above to counter this while
retaining a reasonable estimate of the goodness of fit.

As shown in fig 3, the uncorrelated fit has a very much reduced $\chi^2$ as expected. The 1- and 2-exponential models of section 4 do much better in estimating $\chi^2$. The eigenvalue smoothing model of section 5 also does well when 4 to 8 eigenvalues are retained exactly rather than all eigenvalues ($n = 11$). Note that this is indeed consistent with our previous estimate that approximately $\sqrt{N}$ exact eigenvalues can be relied on. The modifications to the eigenvalues from the smoothing model are shown in fig 2 for one sample and 6 exact eigenvalues. Indeed the modification does alter the eigenvalues from the sample in the direction of the true values.

Also shown in fig 3 are the results using an SVD definition of the inverse of $\tilde{C}(t, t')$ in which $n$ eigenvalues are retained exactly and the remainder are discarded. It is possible to estimate the expected value of $\chi^2$ in this approach, yielding $N(n-2)/(N-n-2)$ which is quite close to the values in fig 3. Thus a corrected estimator of the goodness of fit is essential in this approach. The eigenvalue distribution is not smooth since the deletion of eigenmodes is equivalent to replacing the $D - n$ smallest eigenvalues in fig 2 by infinite values.

For the samples of $N = 30$, the situation is that the correlations appear stronger than the true distributions. This can be understood from the earlier discussion of the eigenvalues of $\tilde{C}(t, t')$. The true correlation matrix $\tilde{C}$ has 11 eigenvalues with values in the range 0.00438 to 9.82 whereas from subsets of 30 samples the smallest eigenvalue fluctuates between 0.00052 to 0.002844. These reductions in the magnitude of the smallest eigenvalues correspond to narrower probability distributions and hence stronger correlations. The implications of this for the goodness of fit have been discussed, but we also need to check that the fitting procedures do not upset the fitted parameters themselves. For our chosen $t$-range the correlated fit to the full 420 samples yields $m_\rho a = 0.351(5)$. A range of other fits to the same full sample (ie uncorrelated, 5-diagonal, smoothed, etc) give essentially the same value of $m_\rho$. Thus it is only the goodness of fit that depends on the correlation model used. For the fits to subsets of 30 samples, we find that an uncorrelated fit does give the same result when averaged over the 14 independent blocks. The various correlation models all give a result somewhat higher ($m_\rho a \approx 0.358$). Thus we have some evidence that the uncorrelated fit is the most stable for determining the fit parameters when the sample size is reduced.

7 Conclusion

Data appropriate to hadron propagation in lattice gauge theory calculations are very strongly correlated. A full correlated fit to such data can be biased unless the sample size ($N$) is sufficiently large compared with the number of data points ($D$). The main effect is the appearance of spurious small eigenvalues of the correlation matrix. These increase the correlation in the sample. This increases $\chi^2$ by a factor of $(N-1)/(N-D-2)$. It can also bias the fit parameters.
Here we propose models which can ameliorate this. Such models depend, to some extent, on an understanding of the expected form of the correlation matrix. Thus such methods are not completely general. For the applications considered here, we find two promising avenues. One is to require that the normalised correlation matrix has a tri-diagonal or 5-diagonal inverse. The other is to average the smallest eigenvalues so that spurious small values are removed. Both of these models give reasonable estimates of the goodness of fit even with quite small sample size. The goodness of fit is important because it determines the range of data (eg. the $t$-range) to be fitted. This then will influence the fitted parameters in turn.

For the actual best determination of the fitted parameters for a small sample, we find that an uncorrelated fit is stable and thus an attractive proposition.

In fitting to hadron Green functions, it is preferable to make a simultaneous fit to several operators. For example smeared or fuzzed operators can be used at either sink or source as well as local ones. The observables for different operators will be correlated in general. We have discussed the correlation in $t$, but this correlation among different operators will need somewhat different models. The most attractive route is to use the method of smoothing the eigenvalues of the correlation matrix which is now treated as a $PT \times PT$ matrix if there are $T$ $t$-values and $P$ operators.

8 Acknowledgements

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A Appendix

Consider the matrix given by

$$\tilde{C}(t, t') = \frac{(1 - \alpha_2^2)\alpha_1^{\vert t-t'\vert+1} - (1 - \alpha_1^2)\alpha_2^{\vert t-t'\vert+1}}{(\alpha_1 - \alpha_2)(1 + \alpha_1\alpha_2)}$$

Its inverse $\tilde{M}(t, t')$ is 5-diagonal and is given by the following expressions in terms of $p_1 = \tilde{C}(t, t \pm 1)$ and $p_2 = \tilde{C}(t, t \pm 2)$ with $d = 1 - 2p_1^2 + 2p_1^2p_2 - p_2^2$

$$\tilde{M}(t, t \pm 1) = \frac{p_1(1 - p_2)^2}{d(1 - p_1^2)}, \quad \tilde{M}(t, t \pm 2) = \frac{p_2^2 - p_2}{d}, \quad \tilde{M}(t, t) = 1 - 2p_1\tilde{M}(t, t \pm 1) - 2p_2\tilde{M}(t, t \pm 2),$$
\[ \tilde{M}(t, t) = 1 - \frac{p_1^2}{d}, \quad \tilde{M}(t_m, t_n) = \frac{1}{d} - p_1 \tilde{M}(t, t \pm 1), \quad \tilde{M}(t_m, t_m \pm 1) = -\frac{p_1(1 - p_2)}{d}. \]

where \( t_m \) is the maximum or minimum value of \( t \) and \( t_n \) is the next to maximum or minimum.

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Figure 1: The normalised correlation $\tilde{C}(t,t')$ versus the relative $t$ difference for pion and rho Green functions from the data of section 3. The continuous line is the 2-exponential model, the dotted line is one exponential and the dot-dashed curve is the expected behaviour from theoretical analysis of the 4 point function.
Figure 2: The eigenvalues of the $11 \times 11$ normalised correlation $\tilde{C}(t, t')$ for the data described in section 6. The full line is the result from $N = 420$ configurations. The sample values (diamonds) are from a subset of $N = 30$. The squares represent the smoothed prescription of section 5 with 6 exact eigenvalues.
Figure 3: The $\chi^2$ for the $P = 2$ parameter correlated fit to $D = 11$ data. The full line is the result for the correlated fit to the full data set of $N = 420$. The expected $\chi^2$ is given by the number of degrees of freedom $D - P = 9$. The same fit was made to samples of $N = 30$ data. The average and standard deviation of 14 such independent fits are shown. The uncorrelated, 1-exponential, and 2-exponential models of section 4 are shown by diamonds. The eigenvalue smoothed models of section 5 are shown by squares. For comparison an SVD-truncated model is also shown (bursts).