A NONMEROMORPHIC EXTENSION OF THE
MOONSHINE MODULE VERTEX OPERATOR
ALGEBRA

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Abstract. We describe a natural structure of an abelian intertwining algebra (in the sense of Dong and Lepowsky) on the direct sum of the untwisted vertex operator algebra constructed from the Leech lattice and its (unique) irreducible twisted module. When restricting ourselves to the moonshine module, we obtain a new and conceptual proof that the moonshine module has a natural structure of a vertex operator algebra. This abelian intertwining algebra also contains an irreducible twisted module for the moonshine module with respect to the obvious involution. In addition, it contains a vertex operator superalgebra and a twisted module for this vertex operator superalgebra with respect to the involution which is the identity on the even subspace and is $-1$ on the odd subspace. It also gives the superconformal structures observed by Dixon, Gipsarg and Harvey.

The relation between the modular function $J(q) = j(q) - 744$ and dimensions of certain representations of the Monster was first noticed by McKay and Thompson (see [Th]). Based on these observations, McKay and Thompson conjectured the existence of a natural ($\mathbb{Z}$-graded) infinite-dimensional representation of the Monster group such that its graded dimension as a $\mathbb{Z}$-graded vector space is equal to $J(q)$. In the famous paper [CN] by Conway and Norton, remarkable numerology between McKay-Thompson series (graded traces of elements of the Monster acting on the conjectured infinite-dimensional representation of the Monster) and modular functions was collected, and surprising conjectures about those modular functions occurred were presented. The Monster was constructed by Griess [G] later but the mysterious connection between the Monster and modular functions was still not explained. In [FLM1], Frenkel, Lepowsky and Meurman constructed a natural

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infinite-dimensional representation of the Monster (called the moonshine module and denoted $V^2$) using the method of vertex operators. The moonshine module constructed by them provided a remarkable conceptual framework towards the understanding of monstrous moonshine. In particular, some of the numerology and conjectures in [CN] were expanded and proved in [FLM1]. Motivated partly by [FLM1], Borcherds [B1] developed a general theory of vertex operators based on an even lattice. From this general theory, he axiomized the notion of “vertex algebra” and using the results announced in [FLM1], he stated that the moonshine module $V^2$ has a structure of such an algebra. In [FLM2], Frenkel, Lepowsky and Meurman proved that the moonshine module $V^2$ has a natural structure of vertex operator algebra and the Monster is the automorphism group of this vertex operator algebra. Their proof is very involved and uses triality and some results in group theory. On the other hand, $V^2$ can be viewed as a substructure of a $\mathbb{Z}_2$-orbifold conformal field theory. Using techniques developed in string theory, Dolan, Goddard and Montague [DGM1] gave another proof that $V^2$ has a natural structure of vertex operator algebra. Their proof works for a class of $\mathbb{Z}_2$-orbifold theories and thus allows them to give a further interpretation of Frenkel-Lepowsky-Meurman’s triality [DGM2] [DGM3] (see also [L]). But their proof is still very technical and complicated.

Using the moonshine module $V^2$ constructed by Frenkel, Lepowsky and Meurman, the no-ghost theorem in string theory and the theory of generalized Kac-Moody algebras (or Borcherds algebras), Borcherds [B2] completed the proof of the monstrous moonshine conjecture in [CN]. Part of Borcherds’ proof has been simplified recently by Jurisich [J] and by Jurisich, Lepowsky and Wilson [JLW]. But the last step of Borcherds’ proof uses some case by case identification which is conceptually unsatisfactory. Also it seems that the methods used in the proof cannot be used to prove the generalized moonshine conjecture [N]. In [Tu1], [Tu2] and [Tu3], Tuite showed that the monstrous moonshine conjecture, and the generalized moonshine conjecture in some special cases, can be understood by using Frenkel-Lepowsky-Meurman’s uniqueness conjecture on $V^2$ and some conjectures in the mathematically yet-to-be-established orbifold conformal field theory. In particular, he pointed out that nonmeromorphic operator algebras for orbifold theories of the Leech lattice theory and the moonshine module, which are the foundation of his idea, are still to be constructed, even in the original simplest $\mathbb{Z}_2$-orbifold case. The importance of these nonmeromorphic operator algebras is that they give the whole genus-zero chiral
parts of the orbifold conformal theories. Also the automorphism groups of these nonmeromorphic operator algebras might be of interest.

In this paper, we describe a construction of the nonmeromorphic extension of $V^g$ in the original simplest $Z_2$-orbifold case. The main tools which we use are the decompositions of the untwisted vertex operator algebra associated to the Leech lattice and its irreducible twisted module into direct sums of irreducible modules for a tensor product of the Virasoro vertex operator algebra with central charge $\frac{1}{2}$ obtained by Dong, Mason and Zhu ([DMZ], [D3]), and the theory of tensor products of modules for a vertex operator algebra developed by Lepowsky and the author ([HL1]–[HL5], [H2]). Precisely speaking, we describe a natural structure of an abelian intertwining algebra (in the sense of Dong and Lepowsky [DL]) on the direct sum of the untwisted vertex operator algebra constructed from the Leech lattice and its (unique) irreducible twisted module with respect to an involution induced from a reflection of the Leech lattice (see Theorem 3.8). Our construction and proof are conceptual, that is, every step is natural in the theory of vertex operator algebras. In particular, when restricting ourselves to the moonshine module, we obtain a new and conceptual proof that the moonshine module has a natural structure of vertex operator algebra. This abelian intertwining algebra also contains a (unique) irreducible twisted module (which has also been obtained by Dong and Mason using a different method) for the moonshine module with respect to the obvious involution, a vertex operator superalgebra and a twisted module for this vertex operator superalgebra with respect to the involution which is the identity on the even subspace and is $-1$ on the odd subspace. This abelian intertwining algebra also gives the superconformal structures observed by Dixon, Ginsparg and Harvey [DGH]. We define a superconformal vertex operator algebra to be an abelian intertwining algebra of a certain type equipped with an element which together with the Virasoro element generates a super-Virasoro algebra. Then the abelian intertwining algebra structure above together with any one of the superconformal structures of Dixon, Ginsparg and Harvey is a superconformal vertex operator algebra.

Note that the tensor product theory can be applied to modules for any rational vertex operator algebra satisfying certain conditions described in [H2] and [HL4] (see also Subsection 2.2 below). In the present paper, the results in [DMZ] and [D3] are used to show that the tensor product theory can be used and to calculate the fusion rules. Thus for other orbifold theories and conformal field theories, if the conditions to use the tensor product theory are satisfied and the fusion rules can be calculated, we can also construct the nonmeromorphic operator algebra.
in the same way. In particular, it might be possible to prove the result of [DM] using the tensor product theory.

Since the present paper uses almost all basic concepts and results in the algebraic theory of vertex operator algebras, it is impossible to give a complete exposition to those basic materials which we need. Therefore we assume that the reader is familiar with the basic notions and results in the theory of vertex operator algebras. For details, see [FLM1] and [FHL]. For more material on the axiomatic aspects of the theory of vertex operator algebras, we shall refer the reader to the appropriate references. We shall, however, review briefly the results on the moonshine module and other related results which we need. We shall also give a brief account of the part which we need of the tensor product theory of modules for a vertex operator algebra.

This paper is organized as follows: Section 1 is a review of the constructions of and results on the moonshine module and related structures. Section 2 is a review of a certain part of the tensor product theory for modules for a vertex operator algebra. The construction of the nonmeromorphic extension is sketched in Section 3. Some consequences of this nonmeromorphic extension, including the vertex operator algebra structure on the moonshine module, the twisted module for the moonshine module, a vertex operator superalgebra and its twisted module in this extension, and the superconformal structures of Dixon, Ginsparg and Harvey, are described in Section 4.

The details of the proofs of the results described in this paper will be published elsewhere.

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1. Leech lattice theory and the moonshine module

In this section we review briefly the constructions of the untwisted vertex operator algebra, associated to the Leech lattice, its irreducible twisted module and the moonshine module $V^\natural$. We also review some results on these algebras and modules and related structures. For more details, the reader is referred to [FLM2], [DMZ], [D1], [D2], [D3].

1.1. The Golay code $\mathcal{C}$. Let $\Omega = \{S_1, \ldots, S_n\}$ be a finite set. A (binary linear) code on $\Omega$ is a subspace of the vector space $\mathcal{P}(\Omega) = \bigoplus_{i=1}^n \mathbb{Z}_2 S_i$ over $\mathbb{Z}_2$ spanned by elements of $\Omega$. Any element $S$ of $\mathcal{P}(\Omega)$ is a linear combination of $S_1, \ldots, S_n$. The number of nonzero coefficients
is called the weight of $S$ and is denoted as $|S|$. A code $S$ is said to be of type II if $n \in 4\mathbb{Z}$, $|S| \in 4\mathbb{Z}$ for all $S \in S$ and $S_1 + \cdots + S_n \in S$.

The usual dot product for a vector space with a basis gives a natural nonsingular symmetric bilinear form on $P(\Omega)$. The annihilator of a code $S$ in $P(\Omega)$ with respect to this bilinear form is again a code. It is called the dual code of $S$ and is denoted $S^\circ$. A code is called self-dual if it is equal to its dual code.

**Theorem 1.1.** There is a self-dual code of type II on a 24-element set such that it has no elements of weight 4. It is unique up to isomorphism.

The code in this theorem is called the Golay code and is denoted $C$.

**1.2. The Leech lattice $\Lambda$.** A (rational) lattice of rank $n \in \mathbb{N}$ is a rank $n$ free abelian group $L$ equipped with a rational-valued symmetric $\mathbb{Z}$-bilinear form $\langle \cdot , \cdot \rangle$. A lattice is nondegenerate if its form is nondegenerate.

Let $L$ be a lattice. For $m \in \mathbb{Q}$, we set $L_m = \{ \alpha \in L \mid \langle \alpha, \alpha \rangle = m \}$. The lattice $L$ is said to be even if $L_m = 0$ for any $m \in \mathbb{Q}$ which is not an even integer. The lattice $L$ is said to be integral if the form is integral-valued and to be positive definite if the form is positive definite. Even lattices are integral. Let $L_\mathbb{Q} = L \otimes \mathbb{Q}$. Then $L_\mathbb{Q}$ is an $n$-dimensional vector space over $\mathbb{Q}$ in which $L$ is embedded and the form on $L$ is extended to a symmetric $\mathbb{Q}$-bilinear form on $L_\mathbb{Q}$. The lattice is nondegenerate if and only if this form on $L_\mathbb{Q}$ is nondegenerate. A lattice may be equivalently defined as the $\mathbb{Z}$-span of a basis of a finite-dimensional rational vector space equipped with a symmetric bilinear form.

The dual of $L$ is the set $L^\circ = \{ \alpha \in L_\mathbb{Q} \mid \langle \alpha, L \rangle \subset \mathbb{Z} \}$. This set is a lattice if and if $L$ is nondegenerate, and in this case, $L^\circ$ has as a base the dual base of a given base. The lattice $L$ is said to be self-dual if $L = L^\circ$. This is equivalent to $L$ being integral and unimodular, which means that $| \det((\langle \alpha_i, \alpha_j \rangle)) | = 1$.

Recall that the Golay code $C$ is defined on a 24-element set $\Omega$. Let $\mathfrak{h} = \bigcup_{k \in \Omega} \mathbb{C} \alpha_k$ be a vector space with basis $\{ \alpha_k \mid k \in \Omega \}$ and provided $\mathfrak{h}$ with the symmetric bilinear form $\langle \cdot , \cdot \rangle$ such that $\langle \alpha_k, \alpha_l \rangle = 2\delta_{kl}$ for $k, l \in \Omega$. For $S \subset \Omega$, set $\alpha_S = \sum_{k \in S} \alpha_k$. For any fixed element $k_0$ of $\Omega$, the subset

$$
\Lambda = \sum_{C \in C} \mathbb{Z} \frac{1}{2} \alpha_C + \sum_{k \in \Omega} \mathbb{Z} \left( \frac{1}{4} \alpha_\Omega - \alpha_k \right)
$$

$$
= \sum_{C \in C} \mathbb{Z} \frac{1}{2} \alpha_C + \sum_{k, l} \mathbb{Z} (\alpha_k + \alpha_l) + \mathbb{Z} \left( \frac{1}{4} \alpha_\Omega - \alpha_{k_0} \right),
$$

of $\mathfrak{h}$, equipped with the restriction to $\Lambda$ of the form on $\mathfrak{h}$, is a lattice. This lattice is the Leech lattice.
The Leech lattice $\Lambda$ is an even unimodular lattice such that $\Lambda_2 = \emptyset$. It is unique up to isometry.

The Leech lattice is generated by $\Lambda_4$. It is easy to see that the elements $\pm \alpha_k \pm \alpha_l$, $k, l \in \Omega$, $k \neq l$, are in $\Lambda_4$. Obviously, $\theta : \Lambda \to \Lambda, \alpha \mapsto -\alpha$ is an isometry of the Leech lattice such that $\theta^2 = 1$.

1.3. The untwisted vertex operator algebra $V_\Lambda$. Let $\mathfrak{h}$ be the same vector space as in Subsection 1.2. We view $\mathfrak{h}$ as an abelian Lie algebra and consider the $\mathbb{Z}$-graded untwisted affine Lie algebra $\hat{\mathfrak{h}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c \oplus \mathbb{C}d$, its Heisenberg subalgebra $\mathfrak{h}_Z = \bigoplus_{n \in \mathbb{Z}, n \neq 0} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c$ and the subalgebra $\hat{\mathfrak{h}}_Z = \bigoplus_{n < 0} \mathfrak{h} \otimes t^n$. The symmetric algebra $S(\hat{\mathfrak{h}}_Z)$ over $\hat{\mathfrak{h}}_Z$ is a $\mathbb{Z}$-graded $\mathfrak{h}_Z$-irreducible $\mathfrak{h}$-module. Let $\hat{\Lambda}$ be a central extension of $\Lambda$ by a cyclic group $\langle \kappa \rangle$ of order 2. We denote the projection from $\hat{\Lambda}$ to $\Lambda$ by $\hat{\kappa}$. Define the faithful character $\chi : \langle \kappa \rangle \to \mathbb{C}^\times$ by $\chi(\kappa) = -1$. Denote by $\mathbb{C}_\chi$ the one-dimensional space $\mathbb{C}$ viewed as a $\langle \kappa \rangle$-module on which $\langle \kappa \rangle$ acts according to $\chi$ and denote by $\mathbb{C}\{\Lambda\}$ the induced $\hat{\Lambda}$-module $\mathbb{C}\{\hat{\Lambda}\} \otimes \mathbb{C}\{\langle \kappa \rangle\} \mathbb{C}_\chi$ (where $\mathbb{C}\{\hat{\Lambda}\}$ and $\mathbb{C}\{\langle \kappa \rangle\}$ are the group algebras of $\hat{\Lambda}$ and $\langle \kappa \rangle$, respectively). Set $V_\Lambda = S(\hat{\mathfrak{h}}_Z) \otimes \mathbb{C}\{\Lambda\}$. We regard $S(\hat{\mathfrak{h}}_Z)$ as the trivial $\hat{\Lambda}$-module and $V_\Lambda$ as the corresponding tensor product $\hat{\Lambda}$-module. View $\mathbb{C}\{\Lambda\}$ as a trivial $\hat{\mathfrak{h}}_Z$-module and for $\alpha \in \mathfrak{h}$, define $\alpha(0) : \mathbb{C}\{\Lambda\} \to \mathbb{C}\{\Lambda\}$ by $\alpha(0)(a \otimes 1) = \langle \alpha, a \rangle (a \otimes 1)$ for any $a \in \hat{\Lambda}$. Also define $x^\alpha \in \langle \text{End } \mathbb{C}\{\Lambda\} \rangle[x, x^{-1}]$ for $\alpha \in \Lambda$ by $x^\alpha (a \otimes 1) = x^{\langle \alpha, a \rangle}(a \otimes 1)$ for any $a \in \hat{\Lambda}$. Give $\mathbb{C}\{\Lambda\}$ a $\mathbb{Z}$-gradation (weight) by wt $a \otimes 1 = \frac{1}{2} \langle \alpha, a \rangle$ for all $a \in \hat{\Lambda}$. Then $V_\Lambda$ has a $\mathbb{Z}$-gradation (weight) obtained from the tensor product gradation. Let $d \in \hat{\mathfrak{h}}$ act as the weight operators on $S(\hat{\mathfrak{h}}_Z)$ and on $\mathbb{C}\{\Lambda\}$ and give $V_\Lambda$ the tensor product $\mathfrak{h}$-module structure. We denote the action of $\alpha \otimes t^n$ by $\alpha(n)$ for $\alpha \in \mathfrak{h}$ and denote the element $1 \otimes (a \otimes 1) \in V_\Lambda$ by $\iota(a)$ for $a \in \hat{\Lambda}$.

For $\alpha \in \mathfrak{h}$, let $\alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n)x^{-n-1}$. For any $a \in \hat{\Lambda}$, we define

$$Y_{V_\Lambda}(\iota(a), x) = \frac{e^{\iota(a)} \psi^{\alpha(x)} \circ \psi^{-\alpha(x)}}{x} = \exp \left( -\sum_{n < 0} \frac{\alpha(n)}{n} x^{-n} \right) \exp \left( -\sum_{n > 0} \frac{\alpha(n)}{n} x^{-n} \right) a x^{\alpha}$$

(the (untwisted) vertex operator associated to $\iota(a)$) of $(\text{End } V_\Lambda)[[x, x^{-1}]]$,

where the normal ordering is defined by

$$\circ \alpha_1(m) \alpha_2(n) = \circ \alpha_2(n) \alpha_1(m) = \begin{cases} \alpha_1(m) \alpha_2(n) & m \leq n, \\
\alpha_2(n) \alpha_1(m) & m \geq n, \end{cases}$$

$$\circ \alpha(m) a = \circ a \alpha(m) = a \alpha(m),$$

$$\circ x^\alpha a = \circ ax^\alpha = ax^\alpha.$$
Theorem 1.3. The quadruple $(V_\Lambda, Y_\Lambda, \iota(1), \omega)$ is a vertex operator algebra with central charge (or rank) equal to 24.

When there is no confusion, we shall use $Y$ to denote the vertex operator map $Y_\Lambda$.

The following theorem is a special case of a theorem due to Dong [D1]:

**Theorem 1.4.** Any irreducible $V_\Lambda$-module is isomorphic to $V_\Lambda$ as a $V_\Lambda$-module and any $V_\Lambda$-module is a finite sum of copies of $V_\Lambda$ as a $V_\Lambda$-module.

On $\mathbb{C}\{\Lambda\}$ there is a unique positive definite hermitian form $(\cdot, \cdot)_{C\{\Lambda\}}$ (see [FLM2]) such that

$$(a \otimes 1, b \otimes 1)_{C\{\Lambda\}} = \begin{cases} 0 & \text{if } \bar{a} \neq \bar{b}, \\ 1 & \text{if } a = b. \end{cases}$$

It is easy to see that on $V_\Lambda$ there is a unique bilinear form $(\cdot, \cdot)_{V_\Lambda}$ such that

$$(\iota(a), \iota(b))_{V_\Lambda} = (a \otimes 1, b \otimes 1)_{C\{\Lambda\}},$$

$$(d \cdot u, v)_{V_\Lambda} = (u, d \cdot v)_{V_\Lambda},$$

$$(\alpha(n) \cdot u, v)_{V_\Lambda} = (u, \alpha(n) \cdot v)_{V_\Lambda}. $$

Recall the isometry $\theta$ of the Leech lattice. For $v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a) \in V_\Lambda$, we define $\theta(v) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a^{-1})$. Using linearity, we obtain a linear map $\theta : V_\Lambda \to V_\Lambda$. It is clear that $\theta^2 = 1$ and $\theta$ is an automorphism of the vertex operator algebra $V_\Lambda$. Thus $V_\Lambda = V_\Lambda^+ + V_\Lambda^-$ where $V_\Lambda^\pm$ are the eigenspace of $\theta$ with eigenvalue $\pm 1$. The subspace $V_\Lambda^+$ is a vertex operator algebra and both $V_\Lambda^\pm$ are irreducible $V_\Lambda^+$-modules.
1.4. The twisted module $V^T_\Lambda$ for $V_\Lambda$. Consider the $\mathbb{Z} + \frac{1}{2}$-graded twisted affine Lie algebra $\hat{\mathfrak{h}}[-1] = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathfrak{h} \otimes t^n \oplus \mathbb{C} c \oplus \mathbb{C} d$, its Heisenberg subalgebra $\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}} = \bigoplus_{n \in \mathbb{Z} + \frac{1}{2}} \mathfrak{h} \otimes t^n \oplus \mathbb{C} c$ and the subalgebra $\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}} = \bigoplus_{n < 0} \mathfrak{h} \otimes t^n$. The symmetric algebra $S(\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}})$ over $\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}$ is a $\mathbb{Z} + \frac{1}{2}$-graded $\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}$-irreducible $\hat{\mathfrak{h}}[-1]$-module.

Let $K = \{a^2 \kappa(\bar{a}, a)/2 \mid a \in \hat{\Lambda}\}$. Then $K$ is a central subgroup of $\hat{\Lambda}$. The following result is proved in [FLM2]:

**Theorem 1.5.** The quotient group $\hat{\Lambda}/K$ has a unique (up to equivalence) irreducible module $T$ such that $\kappa K \to -1$ on $T$. Moreover, the corresponding representation of $\hat{\Lambda}/K$ is the unique faithful irreducible representation and $\dim T = 2^{12}$. To construct $T$, let $\Phi$ be any subgroup of $\Lambda$ such that $2\Lambda \subset \Phi \subset \Lambda$, $|\Phi/2\Lambda| = 2^{12}$ and $\frac{1}{2}(\alpha, \alpha) \in 2\mathbb{Z}$ for any $\alpha \in \Phi$. Then the preimage $\hat{\Phi}$ of $\Phi$ under the homomorphism $\gamma : \hat{\Lambda} \to \Lambda$ is a maximal subgroup of $\hat{\Lambda}$ and $\hat{\Phi}/K$ is an elementary abelian 2-group. Let $\Psi : \hat{\Phi}/K \to \mathbb{C}^\times$ be any homomorphism such that $\Psi(\kappa K) = -1$ and denote by $C_\Psi$ the one-dimensional $\hat{\Phi}$-module with the corresponding character. Then viewed as a $\hat{\Lambda}$-module

$$T = C[\hat{\Lambda}] \otimes_{C[\hat{\Phi}]} C_\Psi \cong C[\Lambda/\Phi] \quad \text{(linearly)}.$$ 

For any $a \in \hat{\Lambda}$, the element $a \otimes 1 \in T$ is denoted by $t(a)$.

Set $V^T_\Lambda = S(\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}) \otimes T$. We view $T$ as a $\hat{\Lambda}$-module and regard $S(\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}})$ as the trivial $\hat{\Lambda}$-module and $V_\Lambda$ as the corresponding tensor product $\Lambda$-module. Give $T$ a $\mathbb{Z} + \frac{1}{2}$-gradation (weight) by $\text{wt} t = \frac{24}{16} = \frac{3}{2}$ for all $t \in T$. Then $V^T_\Lambda$ has a $\mathbb{Z} + \frac{1}{2}$-gradation (weight) obtained from the tensor product gradation. Let $d \in \mathfrak{h}$ act as the weight operators on $S(\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}})$ and on $T$. View $T$ as a trivial $\mathfrak{h}$-module and give $V_\Lambda$ the tensor product $\mathfrak{h}[-1]$-module structure. We denote the action of $\alpha \otimes t^n$ for $\alpha \in \mathfrak{h}$ and $n \in \mathbb{Z} + \frac{1}{2}$ by $\alpha(n)$.

For any $\alpha \in \mathfrak{h}$, let $\alpha(x) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \alpha(n)x^{-n-1}$. (Note that though we use the same notation as in Subsection 1.3, $\alpha(x)$ in Subsection 1.3 acts on a different space.) For any $a \in \hat{\Lambda}$, we define the twisted vertex
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operator

\[ Y_{V_{\Lambda}^T}(\iota(a), x) = 2^{-\langle a, a \rangle} \exp \left( \sum_{n \geq 0} \frac{\bar{a}(n + \frac{1}{2})}{n + \frac{1}{2}} x^{-(n + \frac{1}{2})} \right) \cdot \exp \left( \sum_{n > 0} \frac{\bar{a}(n + \frac{1}{2})}{n + \frac{1}{2}} x^{-(n + \frac{1}{2})} \right) a x^{-\langle a, a \rangle}/2 \]

of \((\text{End } V_{\Lambda}^T)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]\), where the normal ordering is defined by

\[ \circ \alpha_1(m) \alpha_2(n) = \circ \alpha_2(n) \alpha_1(m) = \begin{cases} \alpha_1(m) \alpha_2(n) & m \leq n, \\ \alpha_2(n) \alpha_1(m) & m \geq n, \end{cases} \]

for \(\alpha_1, \alpha_2 \in \mathfrak{h}\), \(m, n \in \mathbb{Z}\). For \(v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a) \in V_{\Lambda}\), we define

\[ Y_0(v, x) = \circ \left( \frac{1}{(n_1 - 1)!} \frac{d^{n_1-1} \alpha_1(x)}{dx^{n_1-1}} \right) \cdots \circ \left( \frac{1}{(n_k - 1)!} \frac{d^{n_k-1} \alpha_k(x)}{dx^{n_k-1}} \right) Y_{V_{\Lambda}^T}(\iota(a), x). \]

Let \(c_{mn}\) be the complex numbers determined by the formula

\[ \sum_{n, m \geq 0} c_{mn} x^m y^n = - \log \left( \frac{(1 + x)^{1/2} + (1 + y)^{1/2}}{2} \right). \]

We define the twisted vertex operator associated to \(v\) to be

\[ Y_{V_{\Lambda}^T}(v, x) = Y_0 \left( \exp \left( \sum_{m, n \geq 0} \sum_{i=1}^{24} c_{mn} h_i(m) h_i(n) x^{-m-n} \right) v, x \right) \]

where as in Subsection 1.3 \(\{h_1, \ldots, h_{24}\}\) is an orthogonal basis for \(\mathfrak{h}\).

The following theorem is a special case of a theorem due Frenkel, Lepowsky and Meurman [FLM2]:

**Theorem 1.6.** The pair \((V_{\Lambda}^T, Y_{V_{\Lambda}^T})\) is a \(\theta\)-twisted \(V_{\Lambda}\)-module.

When there is no confusion, we shall use \(Y\) to denote the vertex operator map \(Y_{V_{\Lambda}^T}\).

The following theorem is a special case of a theorem due to Dong [D2]:

**Theorem 1.7.** Any irreducible \(\theta\)-twisted \(V_{\Lambda}\)-module is isomorphic to \(V_{\Lambda}^T\) and any \(\theta\)-twisted \(V_{\Lambda}\)-module is a finite sum of copies of \(V_{\Lambda}^T\).
On $T$ there is a unique positive definite hermitian form $(\cdot, \cdot)_T$ (see [FLM2]) such that

$$(t(a), t(b))_T = \begin{cases} 0 & a \Phi \neq b \Phi, \\ 1 & a = b. \end{cases}$$

Thus on $V_\Lambda$ there is a unique bilinear form $\langle \cdot, \cdot \rangle_{V_\Lambda}^T$ such that

$$\langle 1 \otimes t(a), 1 \otimes t(b) \rangle_{V_\Lambda}^T = (t(a), t(b))_T,$$
$$\langle d \cdot u, v \rangle_{V_\Lambda}^T = \langle u, d \cdot v \rangle_{V_\Lambda}^T,$$
$$\langle \alpha(n) \cdot u, v \rangle_{V_\Lambda}^T = \langle u, \alpha(-n) \cdot v \rangle_{V_\Lambda}^T.$$

For $w = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes t \in V_\Lambda^T$, we define

$$\theta(w) = (-1)^{k+1} \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes t.$$

Using linearity, we obtain a linear map $\theta : V_\Lambda^T \to V_\Lambda^T$. It is clear that $\theta^2 = 1$ and $\theta$ is an automorphism of the twisted $V_\Lambda$-module $V_\Lambda^T$. Thus $V_\Lambda^T = (V_\Lambda^T)^+ + (V_\Lambda^T)^-$ where $(V_\Lambda^T)^\pm$ are the eigenspace of $\theta$ with eigenvalue $\pm 1$. Both $(V_\Lambda^T)^+$ and $(V_\Lambda^T)^-$ are irreducible $V_\Lambda^+$-modules.

1.5. The moonshine module $V^2$. The moonshine module is defined to be $V^2 = V_\Lambda^+ \oplus (V_\Lambda^T)^+$. It is not difficult to show that the generating function of the dimensions of the homogeneous subspaces of $V^2$ is equal to $qJ(q) = q(j(q) - 744)$. The following result is established in [FLM2]:

**Theorem 1.8.** The $\mathbb{Z}$-graded space $V^2$ has a natural vertex operator algebra structure and its automorphism group is the Monster.

The proof of Theorem 1.8 in [FLM2] uses triality and some results in group theory. Though the proof of the theorem above is involved, there is a direct and natural way to define the vertex operator map $Y_{V^2}$ as was carried out in [FHL] for the sum of an arbitrary vertex operator algebra and an arbitrary $\mathbb{Z}$-graded module for the vertex operator algebra. We recall this construction in the case of the moonshine module here: For $u, v \in V_\Lambda^+$, $Y_{V_3}(u, x)v = Y_{V_3}(u, x)v$; for $u \in V_\Lambda^+$, $v \in (V_\Lambda^T)^+$, $Y_{V_3}(u, x)v = Y_{V_\Lambda^T}(u, x)v$; for $u \in (V_\Lambda^T)^+$, $v \in V_\Lambda^+$, $Y_{V_3}(u, x)v = e^{xL(-1)}Y_{V_\Lambda^T}(v, -x)u$; for $u, v \in (V_\Lambda^T)^+$, $Y_{V_3}(u, x)v$ is defined by

$$\langle w, Y_{V_3}(u, x)v \rangle_{V_\Lambda} = \langle Y_{V_\Lambda^T}(w, -x^{-1})e^{xL(1)}(-x^2)^{-L(0)}u, e^{-x^{-1}L(1)}v \rangle_{V_\Lambda^T}$$

for all $w \in V_\Lambda$. The vacuum of $V^2$ is $\iota(1)$ and the Virasoro element is $\omega$.

We now discuss the decompositions of $V_\Lambda^+$ and its modules into direct sums of modules for a tensor product of the Virasoro vertex operator algebra of central charge $\frac{1}{2}$. Let $L(\frac{1}{2}, 0)$ be the rational Virasoro vertex
operator algebra of central charge $\frac{1}{2}$ and $L(\frac{1}{2}, i)$, $i = 0, \frac{1}{2}, \frac{1}{16}$, the irreducible $L(\frac{1}{2}, 0)$-modules (see [FZ] and [DMZ]). The following result is proved by Dong, Mason and Zhu [DMZ]:

**Theorem 1.9.** There exist $\omega_i \in V_+^A$, $i = 1, \ldots, 48$, such that $\omega = \omega_1 + \cdots + \omega_{48}$ and the vertex operator subalgebra $L$ of $V_+^A$ generated by $\omega_i$, $i = 1, \ldots, 48$, is isomorphic to $L(\frac{1}{2}, 0)^{\otimes 48}$. In particular, for $W = V_+^A, (V_+^A)^\perp, (W, Y_W \mid_{L \otimes W})$ are $L(\frac{1}{2}, 0)^{\otimes 48}$-modules.

A vector in an $L(\frac{1}{2}, 0)^{48}$-module $W$ is a lowest weight vector if it is a lowest weight vector when $W$ is regarded as an $L(\frac{1}{2}, 0)$-module for any one of the 48 vertex operator subalgebras $L(\frac{1}{2}, 0)$. The lowest weight of a lowest vector $v$ is defined in the obvious way. It is an array of 48 complex numbers. For any homogeneous subspace $W_n$ of an $L(\frac{1}{2}, 0)^{48}$-module $W$, we denote the subspace spanned by all lowest weight vectors in $W_n$ by $W_n$. Let $a, b, c$ be nonnegative integers such that $a + b + c = 48$. A lowest weight vector $v \in W$ with lowest weight $(h_1, \ldots, h_{48})$ is called a vector of type $(a, b, c)$ if $\#\{h_i \mid h_i = 0\} = a$, $\#\{h_i \mid h_i = \frac{1}{2}\} = b$ and $\#\{h_i \mid h_i = \frac{1}{16}\} = c$. We write $W_n = \Pi_{a,b,c} m_{a,b,c}(a, b, c)$ which means that there are $m_{a,b,c}$ linearly independent vectors of type $(a, b, c)$ in $W_n$. In [DMZ], the following information on the lowest weight vectors in $V_+^A$ and in $V_+^T$ is obtained:

**Theorem 1.10.** We have the following decompositions:

$$
(V_+^A)^l_{(1)} = 24(46, 2, 0),
(V_+^A)^l_{(3/2)} = 2^{12}(24, 0, 24),
(V_+^T)^l_{(2)} = 24 \cdot 2^{12}(23, 1, 24).
$$

The following information on the decompositions of $V_+^A, (V_+^A)^\perp$ into direct sums of irreducible $L(\frac{1}{2}, 0)^{\otimes 48}$-modules can be found in [D3]:

**Theorem 1.11.** Let $W = V_+^A, (V_+^A)^\perp$. As an $L(\frac{1}{2}, 0)^{\otimes 48}$-module,

$$
W = \prod_{h_i = 0, \frac{1}{2}, \frac{1}{16}} c_{h_1 \cdots h_{48}} L(\frac{1}{2}, h_1) \otimes \cdots \otimes L(\frac{1}{2}, h_{48}).
$$

If $W = V_+^A$ and $c_{h_1 \cdots h_{48}} \neq 0$ for $h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}, i = 1, \ldots, 48$, then

$$
(h_{2j-1}, h_{2j}) \in \{(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{16}, \frac{1}{16})\},
$$

\[\text{Proof:}\]
1 \leq j \leq 24. If \( W = (V_A^T)^\pm \) and \( c_{h_1\ldots h_{48}} \neq 0 \) for \( h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}, i = 1, \ldots, 48, \) then

\[
(h_{2j-1}, h_{2j}) \in \{(0, \frac{1}{16}), (\frac{1}{16}, 0), (\frac{1}{2}, \frac{1}{16}), (\frac{1}{16}, \frac{1}{2})\},
\]

1 \leq j \leq 24. Let \( x_1^i, \ldots, x_{i48}^i, i = 1, \ldots, 24, \) be given by \( (x_{2j-1}^i, x_{2j}^i) = (0, 0), j = 1, \ldots, 24, j \neq i, \) and \( (x_{2j-1}^i, x_{2j}^i) = (\frac{1}{2}, \frac{1}{2}), j = 1, \ldots, 24. \) When \( W = V_A^- \), the multiplicities \( c_{x_1^i\ldots x_{48}^i} = 1, i = 1, \ldots, 24. \)

The following results are due to Dong [D3]:

**Theorem 1.12.** The vertex operator algebra \( V_A^+ \) has only the four irreducible modules \( V_A^{\pm}, (V_A^T)^\pm \) (up to isomorphism) and any \( V_A^+ \)-module is a finite sum of irreducible modules.

**Theorem 1.13.** The moonshine module vertex operator algebra \( V^\natural \) has only one irreducible module, \( V^\natural \) itself, (up to isomorphisms) and any \( V^\natural \)-module is a finite sum of irreducible modules.

2. **Tensor products of modules for a vertex operator algebra**

In this section we summarize the basic concepts and constructions in the theory of tensor products of modules for a vertex operator algebra and those results (mainly the associativity) which we need in this paper. Details can be found in [HL2]–[HL3], [HL5], [H2]. This theory is initiated in [HL1]. For the complete picture of the tensor product theory, see [HL4].

2.1. **The definition, some properties and two constructions of \( P(z) \)-tensor products.** Let \( (V, Y, 1, \omega) \) be a vertex operator algebra and \( (W, Y) \) a \( V \)-module. For any \( v \in V \) and \( n \in \mathbb{Z} \), there is a well-defined natural action of \( v_n \) on \( W \). Moreover, for fixed \( v \in V \), any infinite linear combination of the \( v_n \) of the form \( \sum_{n<N} a_n v_n \) (\( a_n \in \mathbb{C} \)) acts on \( W \) in a well-defined way.

Fix \( z \in \mathbb{C}^\times \) and let \( (W_1, Y_1), (W_2, Y_2) \) and \( (W_3, Y_3) \) be \( V \)-modules. A \( P(z) \)-intertwining map of type \( \left( \begin{array}{c} W_3 \\ W_1, W_2 \end{array} \right) \) is a linear map \( F : W_1 \otimes W_2 \to W_3 \) satisfying the condition

\[
x_0^{-1} \delta \left( \frac{x_1 - z}{x_0} \right) Y_3(v, x_1) F(w_1 \otimes w_2) =
= z^{-1} \delta \left( \frac{x_1 - x_0}{z} \right) F(Y_1(v, x_0) w_1 \otimes w_2)
+ x_0^{-1} \delta \left( \frac{z - x_1}{-x_0} \right) F(w_1 \otimes Y_2(v, x_1) w_2)
\]
for \( v \in V, w_{(1)} \in W_1, w_{(2)} \in W_2 \). The vector space of \( P(z) \)-intertwining maps of type \( \frac{w_3}{w_{12}} \) is denoted by \( \mathcal{M}_P(\frac{z}{\tilde{z}})w_{3,\infty,W_3} \). A \( P(z) \)-product of \( W_1 \) and \( W_2 \) is a \( V \)-module \((W_3, Y_3)\) equipped with a \( P(z) \)-intertwining map \( F \) of type \( \frac{w_3}{w_{12}} \) and is denoted by \((W_3, Y_3; F)\) (or simply by \((W_3, F)\)). Let \((W_4, Y_4; G)\) be another \( P(z) \)-product of \( W_1 \) and \( W_2 \). A morphism from \((W_3, Y_3; F)\) to \((W_4, Y_4; G)\) is a module map \( \eta \) from \( W_3 \) to \( W_4 \) such that \( G = \eta \circ F \), where \( \eta \) is the map from \( W_3 \) to \( W_4 \) uniquely extending \( \eta \).

A \( P(z) \)-tensor product of \( W_1 \) and \( W_2 \) is a \( P(z) \)-product

\[
(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})
\]

such that for any \( P(z) \)-product \((W_3, Y_3; F)\), there is a unique morphism from

\[
(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})
\]

to \((W_3, Y_3; F)\). The \( V \)-module \((W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})\) is called a \( P(z) \)-tensor product module of \( W_1 \) and \( W_2 \). A \( P(z) \)-tensor product is unique up to isomorphism.

We have the following properties:

**Proposition 2.1.** Let \( \log z = \log |z| + i \arg z \) such that \( 0 \leq \arg z < 2\pi \) and \( l_p(z) = \log z + 2\pi pi, p \in \mathbb{Z} \). For any value \( p \in \mathbb{Z} \), we have an isomorphism from the vector space \( V_{W_1 W_2}^{W_3} \) of intertwining operators of type \( \frac{w_3}{w_{12}} \) to the vector space \( \mathcal{M}_P(\frac{z}{\tilde{z}})w_{3,\infty,W_3} \). This isomorphism takes an intertwining operator of the type \( \frac{w_3}{w_{12}} \) to the \( P(z) \)-intertwining map of the same type obtained from the intertwining operator by substituting the complex powers of \( e^{l_p(z)} \) for the complex powers of the formal variable.

**Proposition 2.2.** Suppose that \( W_1 \boxtimes_{P(z)} W_2 \) exists. We have a natural isomorphism

\[
\text{Hom}_V(W_1 \boxtimes_{P(z)} W_2, W_3) \xrightarrow{\sim} \mathcal{M}_P(\frac{z}{\tilde{z}})w_{3,\infty,W_3} \\
\eta \mapsto \bar{\eta} \circ \boxtimes_{P(z)}.
\]

(2.1)

**Proposition 2.3.** Let \( U_1, \ldots, U_k \), \( W_1, \ldots, W_l \) be \( V \)-modules and suppose that each \( U_i \boxtimes_{P(z)} W_j \) exists. Then \((\bigotimes_i U_i) \boxtimes_{P(z)} (\bigotimes_j W_j)\) exists and there is a natural isomorphism

\[
\left( \bigotimes_i U_i \right) \boxtimes_{P(z)} \left( \bigotimes_j W_j \right) \xrightarrow{\sim} \bigotimes_{i,j} U_i \boxtimes_{P(z)} W_j.
\]

(2.2)
We consider the following special but important class of vertex operator algebras: A vertex operator algebra $V$ is \textit{rational} if it satisfies the following conditions:

1. There are only finitely many irreducible $V$-modules (up to equivalence).
2. Every $V$-module is completely reducible (and is in particular a finite direct sum of irreducible modules).
3. All the fusion rules (the dimensions of spaces of intertwining operators) for $V$ are finite (for triples of irreducible modules and hence arbitrary modules).

\textbf{Proposition 2.4.} \textit{Let $V$ be rational and let $W_1$, $W_2$ be $V$-modules. Then}

$$(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)}; \boxtimes_{P(z)})$$

\textit{exists and the $P(z)$-tensor product module $W_1 \boxtimes_{P(z)} W_2$ of $W_1$ and $W_2$ is isomorphic to the $V$-module $\bigoplus_{i=1}^{k} (V^{M_i}_{W_{\infty}} \otimes M_i)$ where $\{M_1, \ldots, M_k\}$ is a set of representatives of the equivalence classes of irreducible $V$-modules.}

We now describe the constructions of a $P(z)$-tensor product of two modules. For two $V$-modules $(W_1, Y_1)$ and $(W_2, Y_2)$, we define an action of

$$V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}]$$

on $(W_1 \otimes W_2)^*$ (where as in [LMZ] and [HLZ], $\iota_+$ denotes the operation of expansion of a rational function of $t$ in the direction of positive powers of $t$), that is, a linear map

$$\tau_{P(z)} : V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \to \text{End} (W_1 \otimes W_2)^*, \quad (2.3)$$

by

$$\left( \tau_{P(z)} \left( x_0^{-1} \delta \left( \frac{x_1^{-1} - x}{x_0} \right) Y_t(v, x_1) \right) \right) (w_1 \otimes w_2)$$

$$= z^{-1} \delta \left( \frac{x_1^{-1} - x_0}{z} \right) \lambda(Y_1(e^{x_1 L(1)}(-x_1^{-2} L(0)) v, x_0) w_1 \otimes w_2)$$

$$+ x_0^{-1} \delta \left( \frac{z - x_1^{-1}}{-x_0} \right) \lambda(w_1 \otimes Y^*_2(v, x_1) w_2) \quad (2.4)$$

for $v \in V$, $\lambda \in (W_1 \otimes W_2)^*$, $w_1 \in W_1$, $w_2 \in W_2$, where

$$Y_t(v, x) = v \otimes x^{-1} \delta \left( \frac{t}{x} \right). \quad (2.5)$$
There is an obvious action of
\[ V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \]
on any \( V \)-module. We have:

**Proposition 2.5.** Under the natural isomorphism

\[ \text{Hom}(W'_3, (W_1 \otimes W_2)^*) \cong \text{Hom}(W_1 \otimes W_2, \overline{W}_3), \]

(2.6)

the maps in \( \text{Hom}(W'_3, (W_1 \otimes W_2)^*) \) intertwining the two actions of

\[ V \otimes \iota_+ \mathbb{C}[t, t^{-1}, (z^{-1} - t)^{-1}] \]
on \( W'_3 \) and \( (W_1 \otimes W_2)^* \) correspond exactly to the \( P(z) \)-intertwining maps of type \( \left( \begin{smallmatrix} w_3 \\ w_1 \times w_2 \end{smallmatrix} \right) \).

Write

\[ Y'_{P(z)}(v, x) = \tau_{P(z)}(Y_t(v, x)) \]

(2.7)

and

\[ Y'_{P(z)}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_{P(z)}(n)x^{-n-2}. \]

(2.8)

We call the eigenspaces of the operator \( L'_{P(z)} \) the weight subspaces or homogeneous subspaces of \( (W_1 \otimes W_2)^* \), and we have the corresponding notions of weight vector (or homogeneous vector) and weight.

Let \( W \) be a subspace of \( (W_1 \otimes W_2)^* \). We say that \( W \) is compatible for \( \tau_{P(z)} \) if every element of \( W \) satisfies the following nontrivial and subtle condition (called the compatibility condition) on \( \lambda \in (W_1 \otimes W_2)^* \): The formal Laurent series \( Y'_{P(z)}(v, x_0)\lambda \) involves only finitely many negative powers of \( x_0 \) and

\[ \tau_{P(z)} \left( x_0^{-1} \delta \left( \frac{x_1^{-1} - z}{x_0} \right) Y_t(v, x_1) \right) \lambda = x_0^{-1} \delta \left( \frac{x_1^{-1} - z}{x_0} \right) Y'_{P(z)}(v, x_1) \lambda \quad \text{for all} \quad v \in V. \]

(2.9)

Also, we say that \( W \) is \( (\mathbb{C}) \)-graded if it is \( \mathbb{C} \)-graded by its weight subspaces, and that \( W \) is a \( V \)-module (respectively, generalized module) if \( W \) is graded and is a module (respectively, generalized module, see [HL1] and [HL2]) when equipped with this grading and with the action of \( Y'_{P(z)}(\cdot, x) \). The weight subspace of a subspace \( W \) with weight \( n \in \mathbb{C} \) will be denoted \( W_{(n)} \).
Define
\[ W_1 \mathfrak{S}_{P(z)} W_2 = \sum_{W \in \mathcal{W}_{P(z)}} W = \bigcup_{W \in \mathcal{W}_{P(z)}} W \subset (W_1 \otimes W_2)^*, \tag{2.10} \]
where \( \mathcal{W}_{P(z)} \) is the set of all compatible modules for \( \tau_{P(z)} \) in \( (W_1 \otimes W_2)^* \).

We have:

**Proposition 2.6.** Let \( V \) be a rational vertex operator algebra and \( W_1, W_2 \) \( V \)-modules. Then \( (W_1 \mathfrak{S}_{P(z)} W_2, Y'_{P(z)} \big|_{V \otimes W_1 \mathfrak{S}_{P(z)} W_2}) \) is a module.

Now we assume that \( V \) is rational. In this case, we define a \( V \)-module \( W_1 \boxtimes_{P(z)} W_2 \) by
\[ W_1 \boxtimes_{P(z)} W_2 = (W_1 \mathfrak{S}_{P(z)} W_2)', \tag{2.11} \]
and we write the corresponding action as \( Y_{P(z)} \). Applying Proposition 2.5 to the special module \( W_3 = W_1 \boxtimes_{P(z)} W_2 \) and the identity map \( W_3 \to W_1 \mathfrak{S}_{P(z)} W_2 \), we obtain a canonical \( P(z) \)-intertwining map of type \( \left( W_1 \mathfrak{S}_{P(z)} W_2 \right) \), which we denote
\[ \boxtimes_{P(z)} : W_1 \otimes W_2 \to W_1 \mathfrak{S}_{P(z)} W_2 \]
\[ w_1 \otimes w_2 \mapsto w_1 \boxtimes_{P(z)} w_2. \tag{2.12} \]

We have:

**Proposition 2.7.** The \( P(z) \)-product \( (W_1 \boxtimes_{P(z)} W_2, Y'_{P(z)}; \boxtimes_{P(z)}) \) is a \( P(z) \)-tensor product of \( W_1 \) and \( W_2 \).

Observe that any element of \( W_1 \mathfrak{S}_{P(z)} W_2 \) is an element \( \lambda \) of \( (W_1 \otimes W_2)^* \) satisfying:

**The compatibility condition:**

(a) The lower truncation condition: For all \( v \in V \), the formal Laurent series \( Y'_{P(z)}(v, x)\lambda \) involves only finitely many negative powers of \( x \).

(b) The formula (2.9) holds.

**The local grading-restriction condition:**

(a) The grading condition: \( \lambda \) is a (finite) sum of weight vectors of \( (W_1 \otimes W_2)^* \).

(b) Let \( W_\lambda \) be the smallest subspace of \( (W_1 \otimes W_2)^* \) containing \( \lambda \) and stable under the component operators \( \tau_{P(z)}(v \otimes t^n) \) of the operators \( Y'_{P(z)}(v, x) \) for \( v \in V, n \in \mathbb{Z} \). Then the weight spaces
(W_\lambda)_{(n)}, n \in \mathbb{C}, of the (graded) space W_\lambda have the properties
\begin{align}
\dim (W_\lambda)_{(n)} < \infty & \quad \text{for } n \in \mathbb{C}, \\
(W_\lambda)_{(n)} = 0 & \quad \text{for } n \text{ whose real part is sufficiently small.}
\end{align}
(2.13)\hspace{1cm}(2.14)

We have another construction of \( W_1 \boxtimes_{P(z)} W_2 \) using these conditions:

**Theorem 2.8.** The subspace of \((W_1 \otimes W_2)^*\) consisting of the elements satisfying the compatibility condition and the local grading-restriction condition, equipped with \( Y'_P(z) \), is a generalized module and is equal to \( W_1 \boxtimes_{P(z)} W_2 \).

The following result follows immediately from Proposition 2.6, the theorem above and the definition of \( W_1 \boxtimes_{P(z)} W_2 \):

**Corollary 2.9.** Let \( V \) be a rational vertex operator algebra and \( W_1, W_2 \) two \( V \)-modules. Then the contragredient module of the module \( W_1 \boxtimes_{P(z)} W_2 \), equipped with the \( (z) \)-intertwining map \( \boxtimes_{P(z)} \), is a \( (z) \)-tensor product of \( W_1 \) and \( W_2 \) equal to the structure \((W_1 \boxtimes_{P(z)} W_2, Y_P(z); \boxtimes_{P(z)})\) constructed above.

### 2.2. The associativity.

Given any \( V \)-modules \( W_1, W_2, W_3, W_4 \) and \( W_5 \), let \( \mathcal{Y}_\infty, \mathcal{Y}_\epsilon, \mathcal{Y}_3 \) and \( \mathcal{Y}_4 \) be intertwining operators of type \( (w_4_{W_4}), (w_3_{W_3}), (w_2_{W_2}), \) respectively. Consider the following conditions for the product of \( \mathcal{Y}_\infty \) and \( \mathcal{Y}_\epsilon \) and for the iterate of \( \mathcal{Y}_3 \) and \( \mathcal{Y}_\Delta \):

**Convergence and extension property for products:** There exists an integer \( N \) (depending only on \( \mathcal{Y}_\infty \) and \( \mathcal{Y}_\epsilon \)), and for any \( w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4 \), there exist \( j \in \mathbb{N}, r_i, s_i \in \mathbb{R}, i = 1, \ldots, j, \) and analytic functions \( f_i(z) \) on \(|z| < 1, i = 1, \ldots, j, \) satisfying
\begin{align}
\Re (\text{wt } w_{(1)} + \text{wt } w_{(2)} + s_i) > N, \quad i = 1, \ldots, j,
\end{align}
(2.15)
such that
\begin{align}
\langle w_{(4)}, \mathcal{Y}_\infty(\Xi(\infty), \xi_\epsilon)\mathcal{Y}_\epsilon(\Xi(\epsilon), \xi_\epsilon)\Xi(\delta)\rangle_{W_\Delta}\bigg|_{\delta_\epsilon = 1, \text{log } \delta_\epsilon = 1, \text{log } \delta_\epsilon = 1, \epsilon \in C}
\end{align}
(2.16)
is convergent when \(|z_1| > |z_2| > 0\) and can be analytically extended to the multi-valued analytic function
\begin{align}
\sum_{i=1}^{j} z_2^{r_i} (z_1 - z_2)^{s_i} f_i \left( \frac{z_1 - z_2}{z_2} \right)
\end{align}
(2.17)
when \(|z_2| > |z_1 - z_2| > 0\).
Convergence and extension property for iterates: There exists an integer $\tilde{N}$ (depending only on $\mathcal{Y}_3$ and $\mathcal{Y}_\Delta$), and for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$, $w_{(3)} \in W_3$, $w_{(4)}' \in W_4'$, there exist $k \in \mathbb{N}$, $\tilde{r}_i, \tilde{s}_i \in \mathbb{R}$, $i = 1, \ldots, k$, and analytic functions $\tilde{f}_i(z)$ on $|z| < 1$, $i = 1, \ldots, k$, satisfying

$$\Re(w_{(2)} + w_{(3)} + s_i) > \tilde{N}, \quad i = 1, \ldots, k,$$

(2.18)

such that

$$\langle w_{(4)}', \mathcal{Y}_\Delta(\mathcal{Y}_3(\sqcup_{(\infty)}; \mathbb{R}) \sqcup_{(\varepsilon)}); \mathcal{Y}_\Delta(\mathcal{Y}_3(\sqcup_{(\infty)}; \mathbb{R}) \sqcup_{(\varepsilon)}))w_{\Delta} \rangle_{\tilde{s}_i = \Re \log(t - t_\varepsilon)}^{\tilde{s}_i = \Re \log(t_\varepsilon), \lambda \in \mathbb{C}}$$

(2.19)

is convergent when $|z_2| > |z_1 - z_2| > 0$ and can be analytically extended to the multi-valued analytic function

$$\sum_{i=1}^{k} \tilde{r}_i \tilde{s}_i \tilde{f}_i \left( \frac{z_2}{z_1} \right)$$

(2.20)

when $|z_1| > |z_2| > 0$.

If for any $V$-modules $W_1$, $W_2$, $W_3$, $W_4$ and $W_5$ and any intertwining operators $\mathcal{Y}_\infty$ and $\mathcal{Y}_\varepsilon$ of the types as above, the convergence and extension property for products holds, we say that the products of the intertwining operators for $V$ have the convergence and extension property. Similarly we can define what the iterates of the intertwining operators for $V$ have the convergence and extension property means.

If a generalized $V$-module $W = \prod_{n \in \mathbb{C}} W_n$ satisfying the condition that $W_{(n)} = 0$ for $n$ whose real part is sufficiently small, we say that $W$ is lower-truncated.

Assume that the products and the iterates of the intertwining operators for $V$ are convergent. Let $W_1$, $W_2$ and $W_3$ be three $V$-modules, $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$ and $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. By Proposition 2.1, any $P(z)$-intertwining maps (for $z = z_1, z_2, z_1 - z_2$) can be obtained from certain intertwining operators by substituting complex powers of $e^{\log z}$ for the complex powers of the formal variable $x$. Thus $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ (or $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$) is a product (or an iterate) of two intertwining operators evaluated at $w_{(1)} \otimes w_{(2)} \otimes w_{(3)}$ and with the complex powers of the formal variables replaced by the complex powers of $e^{\log z_1}$ and of $e^{\log z_2}$ (or by the complex powers of $e^{\log (z_1-z_2)}$ and of $e^{\log z_2}$). By assumption, $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$ (or $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$) is a well-defined element of $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$.
(or of $\left(W_1 \boxtimes P(z_1-z_2) W_2 \right) \boxtimes P(z_2) W_3$). The following result is proved in [H2]:

**Theorem 2.10.** Assume that $V$ is a rational vertex operator algebra and all irreducible $V$-modules are $\mathbb{R}$-graded. Also assume that $V$ satisfies the following conditions:

1. Every finitely-generated lower-truncated generalized $V$-module is a $V$-module.
2. The products or the iterates of the intertwining operators for $V$ have the convergence and extension property.

Then for any $V$-modules $W_1$, $W_2$ and $W_3$ and any complex numbers $z_1$ and $z_2$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$, there exists a unique isomorphism $\mathcal{A}^{P(\{\infty\},\{\infty\})}_{P(z_1),P(z_2)}$ from $W_1 \boxtimes P(z_1) (W_2 \boxtimes P(z_2) W_3)$ to $(W_1 \boxtimes P(z_1-z_2) W_2) \boxtimes P(z_2) W_3$ such that for any $w_{(1)} \in W_1$, $w_{(2)} \in W_2$ and $w_{(3)} \in W_3$,

$$
\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} (w_{(1)} \boxtimes P(z_1) (w_{(2)} \boxtimes P(z_2) w_{(3)}))
= (w_{(1)} \boxtimes P(z_1-z_2) w_{(2)}) \boxtimes P(z_2) w_{(3)},
$$

where $\mathcal{A}^{P(z_1-z_2),P(z_2)}_{P(z_1),P(z_2)} : W_1 \boxtimes P(z_1) (W_2 \boxtimes P(z_2) W_3) \rightarrow (W_1 \boxtimes P(z_1-z_2) W_2) \boxtimes P(z_2) W_3$ is the unique extension of $\mathcal{A}^{P(\{\infty\},\{\infty\})}_{P(z_1),P(z_2)}$.

3. The nonmeromorphic extension of $V^3$

We sketch the construction the nonmeromorphic extension of $V^3$ in this section. This nonmeromorphic extension is in fact the algebra of all intertwining operators for the vertex operator algebra $V^+_\Lambda$. We first verify that the conditions for the tensor product theory reviewed in Section 2 are satisfied by $V^+_\Lambda$. Then we calculate the fusion rules for $V^+_\Lambda$. The nonmeromorphic extension is obtained using the fusion rules and the tensor product theory. The details of the proofs of the results stated in Subsections 3.1 and 3.2 are given in [H3].

3.1. Modules for the Virasoro vertex operator algebras and their tensor products. To prove that $V^+_\Lambda$ satisfies the conditions in Theorem 2.10, we shall use the results in [DMZ] and [D3]. So we first have to prove that the tensor product theory can be applied to the vertex operator algebra $L(\frac{1}{2},0)$. The rationality of $L(\frac{1}{2},0)$ is proved in [DMZ]. In general, the rationality of the Virasoro vertex operator algebra $L(c_{p,q},0)$ of central charge $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$ is proved in [W] for an arbitrary pair $p, q$ of relatively prime positive integers larger than 1. We have the following result for $L(c_{p,q},0)$:
Proposition 3.1. Let \( p, q \) be a pair of relatively prime positive integers larger than 1. Then we have:

1. Every finitely-generated lower-truncated generalized \( L(c_{p,q}, 0) \)-module is a module.
2. The products of the intertwining operators for \( L(c_{p,q}, 0) \) have the convergence and extension property.

Sketch of the proof  The first conclusion is an easy exercise on the representations of the Virasoro algebra. The second conclusion is proved using the representation theory of the Virasoro algebra and the differential equations of Belavin, Polyakov and Zamolodchikov (BPZ equations) for correlation functions in the minimal models [BPZ]. We show that

\[
\langle w'(4), \mathcal{Y}_\infty(\Xi) \mathcal{Y}_\varepsilon(\Xi), \mathcal{Y}_\varepsilon(\Xi) \rangle \big|_{\|z_1\| = \|\log \varepsilon\}, \|z_2\| = \|\log \varepsilon\}, \|z_3\| \in \mathbb{C}}
\] (3.1)

satisfies a system of BPZ equations when \( w(1), w(2), w(3) \) and \( w'(4) \) are all lowest weight vectors. The BPZ equation has only regular singular points. Thus using the theory of equations of regular singular points (see, for example, Appendix B of [K]) and the definition of intertwining operator, we can show that (3.1) in this case is convergent when \( |z_1| > |z_2| > 0 \) and can be analytically extended to a function of the form (2.17) when \( |z_2| > |z_1 - z_2| > 0 \). This together with the brackets of \( L(n), n \in \mathbb{C} \), with the intertwining operators and the \( L(-1) \)-derivative property for the intertwining operators shows that the products of the intertwining operators for \( L(c_{p,q}, 0) \) have the convergence and extension property. \( \square \)

Next we discuss tensor products of \( L(c_{p,q}, 0) \).

Lemma 3.2. Let \( n \) be a positive integer, \( (p_i, q_i), i = 1, \ldots, n \), \( n \) pairs of relatively prime positive integers larger than 1, \( V = L(c_{p_1,q_1}, 0) \otimes \cdots \otimes L(c_{p_n,q_n}, 0) \), \( W_i = L(c_{p_i,q_i}, h^{(i)}_1) \otimes \cdots \otimes L(c_{p_i,q_i}, h^{(i)}_n), i = 1, 2, 3 \), irreducible \( V \)-modules and \( \mathcal{Y} \) an intertwining operator of type \( \left( \frac{W_3}{W_1 W_2} \right) \).

Then there exist intertwining operators \( \mathcal{Y}_i \) of type \( \left( \frac{L(c_{p_i,q_i}, h^{(i)}_1)}{L(c_{p_i,q_i}, h^{(i)}_2)} \right) \), \( i = 1, \ldots, n \), such that

\[
\mathcal{Y} = \mathcal{Y}_\infty \otimes \cdots \otimes \mathcal{Y}_n.
\]

This lemma is an easy consequence of the result in [DMZ] expressing the fusion rules of \( V \) in terms of the fusion rules of \( L(c_{p_i,q_i}, 0) \),...
i = 1, . . . , n. It can also be proved directly using the special properties of the Virasoro vertex operator algebras. Using Proposition 3.1, Lemma 3.2 and the methods used to prove results on modules for tensor products of a vertex operator algebra in [FHL], we obtain:

**Proposition 3.3.** For any positive integer n and any n pairs \((p_i, q_i)\) of relatively prime positive integers larger than 1, i = 1, . . . , n, we have:

1. Every finitely-generated lower-truncated generalized \(L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_n, q_n}, 0)\)-module is a module.
2. The products of the intertwining operators for \(L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_n, q_n}, 0)\) have the convergence and extension property.

### 3.2. A class of vertex operator algebras and the associativity of tensor products.

Let n be a positive integer, \((p_i, q_i)\), i = 1, . . . , n, n pairs of relatively prime positive integers larger than 1. A vertex operator algebra \(V\) is said to be in the class \(C_{\sqrt{\infty}, \nu; \cdots \nu; \sqrt{\infty}, \nu}\) if \(V\) has a vertex operator subalgebra isomorphic to \(L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_n, q_n}, 0)\). The work of Dong, Mason and Zhu [DMZ] shows that \(V_A^+\) is in the class \(C_{\infty, \infty; \cdots \infty; \infty, \infty}\) with \(n = 48\).

Using Proposition 3.3, we can prove:

**Proposition 3.4.** Let n be a positive integer, \((p_i, q_i)\), i = 1, . . . , n, n pairs of relatively prime positive integers larger than 1, and let \(V\) be a vertex operator algebra in the class \(C_{\sqrt{\infty}, \nu; \cdots \nu; \sqrt{\infty}, \nu}\). Then we have:

1. Every finitely-generated lower-truncated generalized \(V\)-module is a module.
2. The products of the intertwining operators for \(V\) have the convergence and extension property.

The proof of the second conclusion is easy. The proof of the first conclusion is more subtle than what it seems to be since a finitely-generated generalized \(V\)-module is not obviously finitely-generated as a generalized \(L(c_{p_1, q_1}, 0) \otimes \cdots \otimes L(c_{p_n, q_n}, 0)\)-module.

### 3.3. The fusion rules for \(V_A^+\).

We first quote a result proved in [FHL] and [HL3]:

**Proposition 3.5.** Let \(V\) be a vertex operator algebra and \(W_1, W_2, W_3\) three \(V\)-modules. Then for any permutation \(\sigma\) of three letters, \(N_{W_\infty W_\infty W_\infty}^{W_\infty W_\infty W_\infty} = N_{W_\infty W_\infty W_\infty}^{W_\infty W_\infty W_\infty}\). In particular, if \(W_1, W_2\) and \(W_3\) are all self-dual, that is, are isomorphic to their contragredient modules \(W_1', W_2'\) and \(W_3'\), respectively, then \(N_{W_\infty W_\infty W_\infty}^{W_\infty W_\infty W_\infty} = N_{W_\infty W_\infty W_\infty}^{W_\infty W_\infty W_\infty}\).
We define a vertex operator map \( Y_{W^2} : W^2 \otimes W^2 \to W^2[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]] \) as follows: For \( u, v \in V_\Lambda \), \( W^2(u, x)v = Y_{V_\Lambda}(u, v) \); for \( u \in V_\Lambda \), \( v \in V_\Lambda^+ \), \( Y_{V_\Lambda}(u, x)v = Y_{V_\Lambda}(u, x)v \); for \( u \in V_\Lambda^+ \), \( v \in V_\Lambda \), \( Y_{W^2}(u, x)v = Y_{V_\Lambda}(u, x)v \); for \( u \in V_\Lambda^+ \), \( v \in V_\Lambda \), \( Y_{W^2}(u, x)v = Y_{V_\Lambda}(u, x)v \); for \( u \in V_\Lambda \), \( v \in V_\Lambda^+ \), \( Y_{V_\Lambda}(u, x)v = Y_{V_\Lambda}(u, x)v \); for \( u \in V_\Lambda^+ \), \( v \in V_\Lambda^+ \), \( Y_{W^2}(u, x)v = Y_{V_\Lambda}(u, x)v \); for \( u \in V_\Lambda^+ \), \( v \in V_\Lambda^+ \), \( Y_{V_\Lambda}(u, x)v = Y_{V_\Lambda}(u, x)v \).
$e^{xL(-1)}Y_{V^+}(v,e^{\pi i}x)u$; for $u, v \in V^+_\Lambda$, $Y_{W^z}(u, x)v$ is defined by

$$\langle w, Y_{W^z}(u, x)v \rangle_{V^+} = \langle Y_{V^+}(w, e^{\pi i}x^{-1})e^{xL(1)}(e^{\pi i}x^2)^{-L(0)}u, e^{x-1L(1)}v \rangle_{V^+}$$

for all $w \in V^+$. We have a (nonsymmetric) nondegenerate $\{1, -1\}$-valued $\mathbb{Z}$-bilinear form $\Omega_{SU}$ (the subscript $SU$ means super, see Subsection 4.2) on the finite abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ determined uniquely by

$\Omega_{SU}((1, 0), (1, 0)) = 1, \Omega_{SU}((1, 0), (0, 1)) = -1, \Omega_{SU}((0, 1), (1, 0)) = 1, L_{SU}((0, 1), (0, 1)) = 1$, and the bilinearity. To formulate the main result of this paper, we need the notions of abelian intertwining algebra, whose definition, examples and axiomatic properties can be found in [DL]. In the definition of abelian intertwining algebra, part of the data is an abelian group $G$ and a normalized abelian 3-cocycle $(F, \Omega)$ for the abelian group $G$ with values in $\mathbb{C}^\times$, where $F$ is a normalized 3-cocycle for $G$ as a group. In this paper, $G$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, $F$ is trivial (denoted by 1) and $\Omega$ is equal to $\Omega_{SU}$ defined above.

**Theorem 3.8.** The structure $(W^z, Y_{W^z}, 1, \omega, 2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, 1, \Omega_{SU})$ is an abelian intertwining algebra of central charge 24.

**Sketch of the proof** The proof uses the fusion rules in Theorem 3.7 and the tensor product theory described in Section 2, especially the associativity. We already know that the tensor product theory can be applied to $V^+_\Lambda$. From [H2], we know that the associativity of $P(\cdot)$-tensor products is equivalent to the associativity of the intertwining operators. So in this case, we have the associativity for intertwining operators. By the fusion rules, for every ordered triple of irreducible $V^+_\Lambda$-modules, any two intertwining operators of the type specified by this triple are linearly dependent. On the other hand, $Y_{W^z}$ restricted to any ordered triple of irreducible $V^+_\Lambda$-modules is an intertwining operator and is nonzero if the fusion rule is nonzero. Thus the associativity for intertwining operators gives the associativity for $Y_{W^z}$. It can be verified directly that $Y_{W^z}$ satisfies a version of the skew symmetry for vertex operators. Combining the associativity and the skew symmetry of $Y_{W^z}$, we obtain the commutativity of $Y_{W^z}$. It can be shown that if the fusion algebra for a rational vertex operator algebra satisfying the conditions in Theorem 2.10 is the group algebra of an abelian group, the products and the iterates of intertwining operators among irreducible modules must be appropriate expansions of generalized rational functions (see [DL] for the meaning of generalized rational functions). In our case, the fusion algebra is the group algebra of the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Thus we have the generalized rationalities of both products and iterates for $Y_{W^z}$. □
4. Applications

We give applications of the results obtained in the preceding section. A special case of Theorem 3.8 gives a new and conceptual proof that $V^\sharp$ has a natural vertex operator algebra structure. Other special cases give an irreducible twisted module for the moonshine module with respect to the obvious involution, a vertex operator superalgebra and a twisted module for this vertex operator superalgebra with respect to the involution which is the identity on the even subspace and is $-1$ on the odd subspace. We also use Theorem 3.8 to construct the superconformal structures of Dixon, Ginsparg and Harvey rigorously.

4.1. The moonshine module and its twisted module. When we restrict ourselves to the moonshine module $V^\sharp = V^+_\Lambda \oplus (V^T_\Lambda)^+ \subset W^\sharp$, we immediately obtain the following:

**Theorem 4.1.** The quadruple $(V^\sharp, Y_{V^\sharp}, 1, \omega)$ is a vertex operator algebra.

**Remark 4.2.** Unlike the proof of this theorem given in [FLM2], our proof does not use triality or any result in group theory. It can be proved without using triality or group theory that any automorphism of the Griess algebra can be extended to an automorphism of the vertex operator algebra $V^\sharp$ (this was also observed by Dong). Thus our proof (or any proof without using triality or group theory, for example, the one given in [DGM1]) makes the fact that the automorphism group of the Griess algebra and the automorphism group of the vertex operator algebra $V^\sharp$ are isomorphic, to be logically independent of triality and group theory. This independence allows us to obtain another proof of the theorem saying that the Monster is the (full) automorphism group of the vertex operator algebra $V^\sharp$ based on the theorem saying that the Monster is the (full) automorphism group of the Griess algebra proved by Griess [G] and Tits [Ti1] [Ti2], simplified by Conway [C] and Tits [Ti2] and understood conceptually by Frenkel, Lepowsky and Meurman using the theory of vertex operators and triality [FLMI] [FLM2].

Another immediate consequence is on the irreducible twisted module for $V^\sharp$:

**Theorem 4.3.** Let $\tau : V^\sharp \rightarrow V^\sharp$ be an involution defined by $\tau(v) = v$ if $v \in V^+_\Lambda$ and $\tau(v) = -v$ if $v \in (V^T_\Lambda)^+$. Then the pair

$$(V^-_\Lambda \oplus (V^T_\Lambda)^-, Y_{W^\sharp} |_{V^\sharp\otimes(V^-_\Lambda \oplus (V^T_\Lambda)^-)})$$
is an irreducible \( \tau \)-twisted module for \( V^\natural \). Any irreducible \( \tau \)-twisted module is isomorphic to this one.

Theorem 4.3 is also proved by Dong and Mason using a different method.

**Remark 4.4.** Note that if we are only interested in Theorem 4.1 or Theorem 4.3, we can prove them using only parts of the fusion rules which we calculated for \( V^+_\Lambda \).

### 4.2. The superconformal structures of Dixon, Ginsparg and Harvey

The following result observed first by Dixon, Ginsparg and Harvey is proved using Theorem 3.8 and some concrete calculations of twisted vertex operators:

**Theorem 4.5.** For any \( t \in T \) satisfying \( (1 \otimes t, 1 \otimes t)_{V^+_\Lambda} = 1 \) (for example, \( t(a) \) for any \( a \in \hat{\Lambda} \)), let \( Y_{W^\natural}(2(1 \otimes t), x) = \sum_{n \in \frac{1}{2} \mathbb{Z}} G(n) x^{-n - \frac{3}{2}} \). Then the operators \( L(n), n \in \mathbb{Z} \) and \( G(n), n \in \frac{1}{2} \mathbb{Z} \), satisfies the super-Virasoro relations:

\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{\hat{c}_{W^\natural}}{8}(m^3 - m)\delta_{m+n,0}, \quad (4.1)
\]
\[
[L(m), G(n)] = \left( \frac{m}{2} - n \right) G(m + n), \quad (4.2)
\]
\[
\{G(m), G(n)\} = 2L(m + n) + \frac{\hat{c}_{W^\natural}}{2} \left( m^2 - \frac{1}{4} \right) \delta_{m+n,0} \quad (4.3)
\]

where the super-central charge \( \hat{c}_{W^\natural} = 16 \) and \( \{ \cdot, \cdot \} \) denotes the anti-bracket.

To summarize the superconformal structures on \( W^\natural \), we need the following notions:

**Definition 4.6.** An \((N=1)\) Neveu-Schwarz type superconformal vertex operator algebra of super-central charge (or super-rank) \( \hat{c} \) is a vertex operator superalgebra \((V, Y, 1, \omega)\) (see, for example, [FFR] or [DL]), equipped with an element \( \xi \in V \) such that the components of \( Y(x^{L(0)}\omega, x) \) and \( Y(x^{L(0)}\xi, x) \) satisfies the super-Virasoro relations (4.1)–(4.3) with \( \hat{c}_{W^\natural} \) replaced by \( \hat{c} \).

In [KW], \((N=1)\) Neveu-Schwarz type superconformal vertex operator algebras are studied and are called “\( N=1 \) (NS-type) vertex operator superalgebras.” The \((N=1)\) Neveu-Schwarz type superconformal vertex operator algebra just defined is denoted by \((V, Y, 1, \omega, \xi)\) or simply by \( V \). Homomorphisms, isomorphisms and automorphisms of \((N=1)\) Neveu-Schwarz type superconformal vertex operator algebras are defined in the obvious way.
Let \((V,Y,\mathbf{1},\omega,\xi)\) be a Neveu-Schwarz type superconformal vertex operator algebra of super-central charge \(\hat{c}\). A module for \(V\) is defined to be a module for \(V\) as a vertex operator superalgebra. We define a linear isomorphism \(\sigma\) of \(V\) by linearity and

\[
\sigma(v) = \begin{cases} 
  v, & v \in V^0, \\
  -v, & v \in V^1,
\end{cases}
\]

(4.4)

where \(V^0\) and \(V^1\) are even and odd subspaces of \(V\), respectively. It is clear that \(\sigma\) is an involution and an automorphism of the Neveu-Schwarz type superconformal vertex operator algebra \(V\). Therefore it is natural to consider \(\sigma\)-twisted \(V\)-modules. See [FFR] for a definition of \(\sigma\)-twisted \(V\)-modules. Following physicists’ terminology, we also call an (untwisted) module for \(V\) a Neveu-Schwarz sector for \(V\) and a \(\sigma\)-twisted module for \(V\) a Ramond sector for \(V\).

**Definition 4.7.** An \((N=1)\) superconformal vertex operator algebra of super-central charge (or super-rank) \(\hat{c}\) is a abelian intertwining algebra

\[
(W,Y_W,\mathbf{1},\omega,\mathbb{Z}_2 \oplus \mathbb{Z}_2,\mathbf{1},\Omega_{SU})
\]

(where \(\Omega_{SU}\) is the \(\mathbb{Z}\)-bilinear form on \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) defined in Subsection 3.4), equipped with an element \(\xi \in W\) such that the components of \(Y(\omega,x)\) and \(Y(\xi,x)\) satisfies the super-Virasoro relations (4.1)–(4.3) with \(\hat{c}_W\) replaced by \(\hat{c}\). The element \(\xi\) is called the Neveu-Schwarz-Ramond element. Let \(W = \bigoplus_{(i,j) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2} W^{i,j}\). Then \(W_{NS} = W^{0,0} \oplus W^{1,1}\) is called the Neveu-Schwarz sector and \(W_R = W^{0,1} \oplus W^{1,0}\) is called the Ramond sector.

The abelian intertwining algebra underlying a superconformal vertex operator algebra can be described using its substructures as follows:

**Proposition 4.8.** Let \((W,Y_W,\mathbf{1},\omega,\mathbb{Z}_2 \oplus \mathbb{Z}_2,\mathbf{1},\Omega_{SU})\) be an abelian intertwining algebra of central charge \(\frac{3}{2}\hat{c}\). Then we have:

1. The \(\mathbb{Z}\)-graded vector spaces \(W^{0,0}, W^{0,0} \oplus W^{1,1}, W^{0,0} \oplus W^{1,0}\) with the restrictions of \(Y_W\) as the vertex operator maps, the vacuum \(\mathbf{1}\) and the Virasoro element \(\omega\) are vertex operator algebras of central charge \(\frac{3}{2}\hat{c}\) and \(W_{NS} = W^{0,0} \oplus W^{1,1}\) is a Neveu-Schwarz type superconformal vertex operator algebra of super-central charge \(\hat{c}\).
2. The \(\frac{1}{2}\mathbb{Z}\)-graded vector spaces \(W^{i,j}\), \(i,j \in \mathbb{Z}_2\), are modules for \(W^{0,0}\).
3. The \(\frac{1}{2}\mathbb{Z}\)-graded vector spaces \(W^{1,0} \oplus W^{1,1}\) and \(W^{0,1} \oplus W^{1,1}\) with the restrictions of \(Y_W\) as the vertex operator maps are twisted modules for \(W^{0,0} \oplus W^{0,1}\) and \(W^{0,0} \oplus W^{1,0}\), respectively, with respect to the obvious involutions.
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4. The $\frac{1}{2}\mathbb{Z}$-graded vector spaces $W_R = W^{0,1} \oplus W^{1,0}$ with the restrictions of $Y_W$ as the vertex operator maps is a $\sigma$-twisted module for $W_{NS}$.

Conversely, let $W = \prod_{i,j \in \mathbb{Z}_2} W^{i,j}$ where $W^{i,j}$, $i, j \in \mathbb{Z}_2$, are four $\frac{1}{2}\mathbb{Z}$-graded vector spaces be equipped with a vertex operator map $Y_W : W \otimes W \to W \{x\}$ and two distinguished elements $1$ and $\omega$ such that (i)–(iv) above hold. Then

$$(W, Y_W, 1, \omega, 2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, 1, \Omega_{SU})$$

is an abelian intertwining algebra.

We can summarize the main result and the main applications of the present paper as follows:

**Theorem 4.9.** For any $t \in T$ satisfying $(1 \otimes t, 1 \otimes t)_{V_{W^3}} = 1$ (for example, $t(a)$ for any $a \in \hat{\Lambda}$), $(W^2, Y_{W^2}, 1, \omega, 2, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \Omega_{SU}, 2(1 \otimes t))$ is a superconformal vertex operator algebra of super-central charge $\hat{c}_{W^2} = 16$. In particular, we have:

1. The moonshine module $V^2 = V^+_\Lambda \oplus (V^T_\Lambda)^+$ with the restriction of $Y_{W^2}$ as the vertex operator map, the vacuum $1$ and the Virasoro element $\omega$ is a vertex operator algebra of central charge $\frac{3}{2} \hat{c}_{W^2} = 24$

and $W^2_{NS} = V^+_\Lambda \oplus (V^T_\Lambda)^-$ with the restriction of $Y_{W^2}$ as the vertex operator map is a Neveu-Schwarz type superconformal vertex operator algebra of super-central charge $\hat{c}_{W^2} = 16$.

2. The $\frac{1}{2}\mathbb{Z}$-graded vector spaces $V^-_\Lambda \oplus (V^T_\Lambda)^-$ with the restriction of $Y_{W^2}$ as the vertex operator map is a $\tau$-twisted module for $V^2$.

3. The $\frac{1}{2}\mathbb{Z}$-graded vector spaces $W^2_R = V^-_\Lambda \oplus (V^T_\Lambda)^+$ with the restriction of $Y_{W^2}$ as the vertex operator map is a $\sigma$-twisted module for $W^2_{NS}$.

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