Generating entangled fermions by accelerated measurements on the vacuum

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It is shown that accelerated projective measurements on the vacuum of a free Dirac spinor field results in an entangled state for an inertial observer. The physical mechanism at work is the Davies-Unruh effect. The produced state is always entangled and its entanglement increases as a function of the acceleration, reaching maximal entanglement in the asymptotic limit of infinite acceleration where Bell states are produced.

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Recently, much attention has been given to relativistic effects in the context of quantum information theory. Since relativity is an indispensable component of any complete theoretical model, understanding these effects is important from the viewpoint of fundamental physics. However, it could also be relevant in practical situations in which quantum information processing tasks are implemented by observers in arbitrary relative motion.

Entanglement plays a pivotal role in quantum information — it is a resource for quantum communication and teleportation and for various computational tasks [1]. In a relativistic setting, the role played by entanglement has received much recent attention [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. While it has been shown to be an invariant quantity for observers in uniform relative motion [2, 3, 4, 5, 6], in non-inertial frames the situation is quite different.

The fidelity of teleportation between two parties in relative uniform acceleration decreases as a function of acceleration [7, 8, 13]. From the perspective of a uniformly accelerating observer a communication horizon appears, obstructing access to information about the whole of spacetime. As a consequence, there is a loss of information and a corresponding degradation of entanglement. This has been shown to hold for both scalar fields [9, 10] and spinor fields [11] as viewed by two relatively accelerating observers. The acceleration of the observer effectively introduces an “environmental decoherence” that limits the fidelity of certain quantum information-theoretic processes. Implementation of quantum information processing tasks between accelerating partners thus depends upon a proper and quantitative understanding of such degradation in non-inertial frames.

In this paper we analyze the generation of entanglement between different modes of a Dirac spinor field due to projective measurements on the vacuum by an accelerating observer. For any observer the vacuum is the absence of both particles and antiparticles as measured by that observer’s detector. As a result of the Davies-Unruh effect [14, 15], an accelerating observer will perceive a Fermi-Dirac distribution of particles and antiparticles in what an inertial observer would describe as the vacuum state. We show that if one of these particles is detected, an entangled state is produced in the inertial reference frame. We show that entanglement is always produced and quantify it using the entanglement entropy. We find that larger accelerations produce more entanglement and in the asymptotic limit of infinite acceleration, a maximally entangled Bell state is produced. A similar effect holds for scalar fields [16]; however, further processing is required to extract a Bell state, even in the asymptotic limit of infinite acceleration. We will work in units where \( c = \hbar = k_B = 1 \).

The Davies-Unruh effect for a Dirac spinor field \( \Psi \) of mass \( m \) is a consequence of two inequivalent quantization schemes [17, 18]. For the inertial observer in flat spacetime the appropriate metric is the Minkowski metric \( g_{\mu \nu} = \eta_{\mu \nu} = \text{diag}(1, -1, -1, -1) \). Since this metric is static, the field can be quantized in a straightforward manner by expanding it in terms of a complete set of positive and negative frequency modes (suppressing henceforth the spin degree of freedom for ease of notation)

\[
\Psi = \int dk \left( a_k \psi^+_k + b^\dagger_k \psi^-_k \right),
\]

and imposing the canonical anticommutation relations on the mode operators \( \{a_k, a^\dagger_{k'}\} = \{b_k, b^\dagger_{k'}\} = \delta(k - k') \), with all other anticommutators vanishing.

The key element here is the division of the modes into positive and negative frequency, which is done according to the Minkowski timelike Killing vector \( \partial_t \). The operators \( a_k \) and \( b_k \) are then interpreted as particle creation operators and antiparticle annihilation operators, respectively. With this interpretation, a Fock space can be constructed in the usual manner.

Now consider an observer moving through flat spacetime with uniform acceleration \( a \) in the \( z \) direction. This observer will experience communication horizons that divide the spacetime into four regions denoted \( I, \ II, \ F, \) and

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A separate Rindler coordinate system is needed to cover the whole spacetime, with all other anticommutators vanishing.

The causal structure of Rindler spacetime: the modes $\psi^I_k$ and $\psi^I_{-k}$ have support in region I whereas the modes $\psi^H_k$ and $\psi^H_{-k}$ have support in region II. Each type is divided into positive and negative frequency according to the Rindler timelike Killing vector in the appropriate region. In region I this is given by $\partial_\eta$, however, in region II it is $\partial_{-\eta}$ where the minus sign ensures it is future pointing.

The operators $c^I_k$ and $d^I_{-k}$ annihilate a particle and create an antiparticle in region I while $c^H_k$ and $d^H_{-k}$ annihilate and create particles and antiparticles in region II.

These two quantization schemes are not equivalent [17]. Making the single mode approximation, in which the Rindler observer’s particle detector is sensitive to a narrow bandwidth centered about the perpendicular components of the wavevector $\vec{k}$ (which is the same for a Minkowski observer) [7, 8, 11], the mode operators are related by the following Bogoliubov transformation

$$\begin{pmatrix} a_k \\ \delta_{-k}^I \end{pmatrix} = \begin{pmatrix} \cos r_k & -e^{-i\phi_k} \sin r_k \\ e^{i\phi_k} \sin r_k & \cos r_k \end{pmatrix} \begin{pmatrix} c^I_k \\ d^I_{-k} \end{pmatrix},$$

where the parameter $r_k$ is defined by

$$\cos r_k = [2 \cosh(\pi \omega_k/a)]^{-1/2} \exp(\pi \omega_k/2a),$$

and $\omega_k = \sqrt{k^2 + \vec{k}^2 + m^2}$ is the frequency of the mode. The phase $\phi_k$ can be absorbed into the definitions of the mode operators and will be done so from now on using the sign conventions of [11].

Using these transformations, the Minkowski particle vacuum in mode $k$ can be expressed in terms of Rindler Fock states as

$$|0_k\rangle^+ = \cos r_k \exp(\tan(r_k) c^I_k d^I_{-k}) |0_k\rangle^+_I |0_{-k}\rangle^-_II,$$

where the $+$ ($-$) superscripts denote particle (antiparticle). A formal expression for the total Minkowski particle vacuum is obtained by using Eq. (3) for each mode

$$|0\rangle^+ = N \prod_k \exp(\tan(r_k) c^I_k d^I_{-k}) |0_k\rangle^+_I |0_{-k}\rangle^-_II,$$

where $N = \prod_k \cos r_k$.

Similarly, if the Rindler observer’s antiparticle detector is sensitive to a narrow bandwidth centered about the wavevector $-\vec{k}$ then

$$\begin{pmatrix} b_k \\ \alpha_{-k}^I \end{pmatrix} = \begin{pmatrix} \cos r_k & e^{-i\phi_k} \sin r_k \\ -e^{i\phi_k} \sin r_k & \cos r_k \end{pmatrix} \begin{pmatrix} d^H_{-k} \\ c^H_k \end{pmatrix},$$

and the Minkowski antiparticle vacuum for mode $k$ takes the form

$$|0_k\rangle^- = \cos r_k \exp(-\tan(r_k) d^H_{-k} c^H_k) |0_k\rangle^+_I |0_{-k}\rangle^+_II.$$
We now see that the accelerating observer has a nonzero probability to detect particles and antiparticles in the Minkowski vacuum. The probabilities are given according to the Fermi-Dirac distribution of temperature $T = a/2\pi$, which can be seen by constructing the reduced density matrix for region $I$.

What are the consequences of detecting one of these particles? Suppose two observers, Alice, an inertial observer, and Rob, a uniformly accelerating observer, are moving through the field $\psi$. When the field is in the vacuum state as described by Alice, Rob would describe the state as the thermal state (4) due to the Davies-Unruh effect. Now suppose Rob performs a measurement on this state and detects one particle in mode $k$. Immediately after his measurement, the state will be the projection of (4) onto the single particle state in region $I$. This can be written succinctly as

$$|\psi_+(k)\rangle = P_k N \prod_{k'} \exp(\tan(r_{k'}) c_{k'}^d d_{k'}^-)|0_{k'}\rangle_I |0_{-k'}\rangle_H,$$

where the operator $P_k$ is defined as

$$P_k = \sec(r_k) c_k^d d_k^- \exp(-\tan(r_k) c_k^d d_k^+).$$

Applying the Bogoliubov transformation (1), the state can be simplified to

$$|\psi_+(k)\rangle = [\sin(r_k) + \cos(r_k) a_k^d b_k^-] |0\rangle,$$

from which we see that from Alice’s perspective, the state is a superposition of the vacuum (i.e., no particle emission) and pair production at energy $\omega_k$. This state is entangled in the occupation number of the particle mode $k$ and the antiparticle mode $-k$.

To study the entanglement properties of this state we work in the basis $\{|0\rangle^+, |1\rangle^+, |0\rangle^-, |1\rangle^-\}$ where $|0\rangle^\pm = |0_{\pm k}\rangle$ and $|1\rangle^\pm = |1_{\pm k}\rangle$. The state can then be represented by the density matrix

$$\rho(k) = \begin{pmatrix} \sin^2 r_k & 0 & 0 & \sin r_k \cos r_k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2 r_k \\ \sin r_k \cos r_k & 0 & 0 & \cos^2 r_k \end{pmatrix}.$$ 

We quantify the entanglement of the state in terms of the entanglement entropy, defined as the von Neumann entropy $S(\rho_a) = -\text{Tr}(\rho_a \log_2 \rho_a)$ of $\rho_a$, the reduced density matrix of subsystem $a$. Equivalently, it can be expressed in terms of the eigenvalues $\lambda_i$ of $\rho_a$ as

$$S(\rho_a) = - \sum_i \lambda_i \log_2 \lambda_i.$$ 

For a pure bipartite state, it does not matter which subsystem is traced out as the nonzero eigenvalues of either reduced density matrix are equal. To find the entanglement entropy of (6), we find the reduced density matrix by tracing out the particle states to obtain

$$\rho_-(k) = \text{Tr}_+ \rho(k) = \begin{pmatrix} \sin^2 r_k & 0 \\ 0 & \cos^2 r_k \end{pmatrix}.$$ 

From which we calculate the entropy to be

$$S(\rho_-(k)) = \log_2(\csc^2 r_k) + \cos^2(r_k) \log_2(\tan^2 r_k).$$

Recalling that $r_k$ is defined by (2), we see that the entanglement entropy is nonzero regardless of the frequency detected or the (nonzero) acceleration of the observer. Therefore, the state always contains distillable entanglement with larger accelerations producing more entanglement, as illustrated in Fig. 2.

Note that in order for Alice to use this state in a quantum information processing task, she must know Rob’s acceleration and the momentum of the particle detected. Rob can communicate these parameters to Alice classically since he can always signal to her despite there being a point where he can no longer receive signals from her. It is interesting to note that if Alice does not know Rob’s acceleration, she may be able to deduce it from the resulting quantum state. We leave this for future work.

In the limit $\omega_k/a \to 0$ we have $r_k = \pi/4$ and the state approaches the maximally entangled Bell state

$$\lim_{\omega_k/a \to 0} |\psi_+(k)\rangle = \frac{1}{\sqrt{2}} (|0\rangle^+|0\rangle^- + |1\rangle^+|1\rangle^-),$$

which has an entanglement entropy of 1. This limit corresponds physically to the asymptotic limit of infinite acceleration. However, whenever $\omega_k \ll a$, the state is approximately
\[ |\psi_+ (k)\rangle \approx \sqrt{2} \left( \frac{1}{2} - \frac{\pi \omega_k}{4a} - \frac{\pi^2 \omega_k^2}{16a^2} \right) |0\rangle + |0\rangle - \sqrt{2} \left( \frac{1}{2} - \frac{\pi \omega_k}{4a} + \frac{\pi^2 \omega_k^2}{16a^2} \right) |\bar{1}\rangle + |\bar{1}\rangle, \]

which has an entanglement entropy of

\[ S(a, \omega_k) = 1 - \frac{\pi^2 \omega_k^2}{2 \ln(2) a^2} + O \left( \frac{\omega_k^2}{a^4} \right). \]

In the case of massless fermions, entanglement arbitrarily close to maximal can be generated for finite acceleration by detecting sufficiently low energy modes. However, this is not true in the massive case where accelerations at least much greater than the rest mass energy are required to approximate a Bell state.

While the above analysis is conditioned on Rob detecting a single particle in mode \( k \), it generalizes to other measurement outcomes. If he had instead detected an antiparticle in mode \( k \), the resulting state would be

\[ |\psi_- (k)\rangle = A_k N \prod_{k'} \exp \left( - \tan(r_{k'}) \frac{d^\dagger_{k'} c^\dagger_{-k'}}{d^\dagger_{k'} c^\dagger_{-k'}} |0_{k'}\rangle_I |0_{-k'}\rangle_{II}, \right. \]

where the operator \( A_k \) is defined as

\[ A_k = -\sec (r_k) \frac{d^\dagger_{r_k} c_{-r_k}}{d^\dagger_{r_k} c_{-r_k}} \exp (\tan (r_k) \frac{d^\dagger_{r_k} c_{-r_k}}{d^\dagger_{r_k} c_{-r_k}}). \]

Upon applying the Bogoliubov transformation (5) this simplifies to

\[ |\psi_- (k)\rangle = \left[ \sin (r_k) - \cos (r_k) b^\dagger_{k} a^\dagger_{-k} \right] |0\rangle, \]

which also approaches a Bell state in the asymptotic limit of infinite acceleration.

Noting that the operators \( P_k \) and \( A_{k'} \) only contain an even number of mode operators, they will commute. Therefore, the state after an arbitrary measurement result will be the product of the states \( |\psi_{\pm} (k)\rangle \) for each mode detected. Physically, this would be a superposition of all possible pair productions including no pair production; in the asymptotic limit of infinite acceleration, this approaches a product of Bell states. Regardless of the acceleration, given this state, Alice could use it as a resource in quantum information processing tasks.

In summary, we have shown that accelerated projective measurements on the Minkowskian vacuum of a free Dirac spinor field produce entangled states for inertial observers. These could in principle be used in quantum information processing tasks. The produced states are always entangled and the degree to which they are entangled is a function of the frequency of the detected particles and the acceleration of the observer. The amount of entanglement increases with acceleration and reaches maximal entanglement in the asymptotic limit of infinite acceleration where a product of Bell states is produced.

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