SHARP WEIGHTED NORM INEQUALITIES FOR LITTLEWOOD-PALY OPERATORS AND SINGULAR INTEGRALS

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Abstract. We prove sharp $L^p(w)$ norm inequalities for the intrinsic square function (introduced recently by M. Wilson) in terms of the $A_p$ characteristic of $w$ for all $1 < p < \infty$. This implies the same sharp inequalities for the classical Lusin area integral $S(f)$, the Littlewood-Paley $g$-function, and their continuous analogs $S_\psi$ and $g_\psi$. Also, as a corollary, we obtain sharp weighted inequalities for any convolution Calderón-Zygmund operator for all $1 < p \leq 3/2$ and $3 \leq p < \infty$, and for its maximal truncations for $3 \leq p < \infty$.

1. Introduction

Given a weight (i.e., a non-negative locally integrable function) $w$, its $A_p$, $1 < p < \infty$, characteristic is defined by

$$\|w\|_{A_p} \equiv \sup_Q \left( \frac{1}{|Q|} \int_Q w \, dx \right) \left( \frac{1}{|Q|} \int_Q w^{\frac{1}{p-1}} \, dx \right)^{p-1} ,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

The main conjecture (which is implicit in work of Buckley [2]) concerning the behavior of singular integrals $T$ on $L^p(w)$ says that

$$\|T\|_{L^p(w)} \leq c(T, n, p) \|w\|_{A_p}^{\max\left(\frac{1}{p-1}, 1\right)} \quad (1 < p < \infty). \tag{1.1}$$

For Littlewood-Paley operators $S$ (we specify below the class of such operators we shall deal with) it was conjectured in [12] that

$$\|S\|_{L^p(w)} \leq c(S, n, p) \|w\|_{A_p}^{\max\left(\frac{1}{2}, \frac{1}{p-1}\right)} \quad (1 < p < \infty). \tag{1.2}$$

Observe that the exponents $\max\left(1, \frac{1}{p-1}\right)$ in (1.1) and $\max\left(\frac{1}{2}, \frac{1}{p-1}\right)$ in (1.2) are best possible for all $1 < p < \infty$ (see [2] [12] [13]). Also, by the sharp version of the Rubio de Francia extrapolation theorem [6],

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inequality (1.1) for $p = 2$ implies (1.1) for all $p > 1$; analogously, it is enough to prove (1.2) for $p = 3$.

Currently conjecture (1.1) is proved for: the Hilbert transform and Riesz transforms (Petermichl [16, 17]), the Ahlfors-Beurling operator (Petermichl and Volberg [18]), any one-dimensional Calderón-Zygmund convolution operator with sufficiently smooth kernel (Vagharshakyan [20]). The proofs in [16, 17, 18] are based on the so-called Haar shift operators combined with the Bellman function technique. The main idea in [20] is also based on Haar shifts.

Recently, Lacey, Petermichl and Reguera [11] have established sharp weighted estimates for Haar shift operators without the use of Bellman functions; their proof uses a two-weight 

\[ T_b \]

theorem for Haar shift operators due to Nazarov, Treil and Volberg [15]. This provides a unified approach to works [16, 17, 18, 20]. Very recently, a new, more elementary proof of this result, avoiding the \( T_b \) theorem, was given by Cruz-Uribe, Martell and Pérez [5]; a key ingredient in [5] was a decomposition of an arbitrary measurable function in terms of local mean oscillations obtained in [14].

It was shown in [5] that such a decomposition is very convenient when dealing with certain dyadic type operators. In particular, using these ideas, the authors proved in [5] conjecture (1.2) for the dyadic square function. Note that this is the first result establishing (1.2) for all $p > 1$. Previously (1.2) was obtained for the dyadic square function in the case $p = 2$ by Hukovic, Treil and Volberg [9], and, independently by Wittwer [24]. Also, (1.2) in the case $p = 2$ was proved by Wittwer [24] for the continuous square function. By the extrapolation argument [6], the linear $\|w\|_{A_p}$ bound implies the bound by $\|w\|_{A_p}^{\max(1,1/(p-1))}$ for all $p > 1$. However, for $p > 2$ this is not sharp for square functions. In [13], the linear bound for $p > 2$ was improved to $\|w\|_{A_p}^{p'/2}$ for a large class of Littlewood-Paley operators.

In this paper we show that similar arguments to those developed in [5] work actually for essentially any important Littlewood-Paley operator. To be more precise, we prove conjecture (1.2) for the so-called intrinsic square function $G_\alpha$ introduced by Wilson [22]. As it was shown in [22], the intrinsic square function pointwise dominates both classical square functions and their more recent analogs. As a result, we have the following theorem.

**Theorem 1.1.** Conjecture (1.2) holds for any one of the following operators: the intrinsic square function $G_\alpha(f)$, the Lusin area integral $S(f)$, the Littlewood-Paley function $g(f)$, the continuous square functions $S_\psi(f)$ and $g_\psi(f)$. 
In Section 2 below we give precise definitions of the operators appeared in Theorem 1.1.

We mention briefly the main difference between the proof of Theorem 1.1 and the corresponding proof for the dyadic square function in [5]. Let \( D \) be the set of all dyadic cubes in \( \mathbb{R}^n \). Dealing with the dyadic square function, we arrive to the mean oscillation of the sum \( \sum_{Q \in D} \xi_Q(f) \chi_Q \) on any dyadic cube \( Q_0 \). The corresponding object can be easily handled because of the nice interaction between any two dyadic cubes. Now, working with the intrinsic square function, we have to estimate the mean oscillation of the sum \( \sum_{Q \in D} \xi'_Q(f) \chi_{3Q} \) on any dyadic cube \( Q_0 \). Here we use several tricks. First, as it was shown by Wilson [21], the set \( D \) can be divided into \( 3^n \) disjoint families \( D_k \) such that the cubes \( \{3Q : Q \in D_k\} \) behave essentially as the dyadic cubes. Second, given any dyadic cube \( Q_0 \), one can find in each family \( D_k \) the cube \( Q_k \) such that \( Q_0 \subset 3Q_k \subset 5Q_0 \). This is proved in Lemma 3.2 below. Combining these tricks, we arrive to exactly the same situation as described above for the dyadic square function.

The Littlewood-Paley technique developed by Wilson in [21, 22, 23] along with Theorem 1.1 allows us to get conjecture (1.1) for classical Calderón-Zygmund operators for any \( p \in (1, 3/2] \cup [3, \infty) \). For \( K_\delta(x) = K(x) \chi_{\{|x|>\delta\}} \) let

\[
Tf(x) = \lim_{\delta \to 0} f * K_\delta(x) \quad \text{and} \quad T^* f(x) = \sup_{\delta > 0} |f * K_\delta(x)|,
\]

where the kernel \( K \) satisfies the standard conditions:

\[
|K(x)| \leq \frac{c}{|x|^n}, \quad \int_{r<|x|<R} K(x) \, dx = 0 \quad (0 < r < R < \infty),
\]

and

\[
|K(x) - K(x-y)| \leq \frac{c|y|^{\varepsilon}}{|x|^{n+\varepsilon}} \quad (|y| \leq |x|/2, \varepsilon > 0).
\]

**Theorem 1.2.** Conjecture (1.1) holds for \( T \) for any \( 1 < p \leq 3/2 \) and \( 3 \leq p < \infty \). Also, (1.1) holds for \( T^* \) for any \( p \geq 3 \).

Notice that for the maximal Hilbert, Riesz and Ahlfors-Beurling transforms conjecture (1.1) was recently proved for any \( p > 1 \) by Hytönen et. al. [10]; a different proof for the same operators is given in [5].

The proof of Theorem 1.2 is based essentially on Theorem 1.1, on the pointwise estimate \( S_\psi(Tf)(x) \leq cG_\alpha(f)(x) \) (proved in [22, 23]), and on a version of the Chang-Wilson-Wolff theorem [4] proved in [21].
2. Littlewood-Paley operators

Let \( \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ \) and \( \Gamma_\beta(x) = \{(y, t) \in \mathbb{R}^{n+1}_+ : |y-x| < \beta t \} \). Here and below we drop the subscript \( \beta \) if \( \beta = 1 \).

The classical square functions are defined as follows. If \( u(x, t) = P_t * f(x) \) is the Poisson integral of \( f \), the Lusin area integral \( S_\beta \) and the Littlewood-Paley \( g \)-function are defined respectively by

\[
S_\beta(f)(x) = \left( \int_{\Gamma_\beta(x)} |\nabla u(y, t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2}
\]

and

\[
g(f)(x) = \left( \int_0^\infty t |\nabla u(x, t)|^2 dt \right)^{1/2}.
\]

The modern (real-variable) variants of \( S_\beta \) and \( g \) can be defined in the following way. Let \( \psi \in C^\infty(\mathbb{R}^n) \) be radial, supported in \( \{x : |x| \leq 1\} \), and \( \int \psi = 0 \). The continuous square functions \( S_{\psi, \beta} \) and \( g_\psi \) are defined by

\[
S_{\psi, \beta}(f)(x) = \left( \int_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}
\]

and

\[
g_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}.
\]

In [22] (see also [23, p. 103]), it was introduced a new square function which is universal in a sense. This function is independent of any particular kernel \( \psi \), and it dominates pointwise all the above defined square function. On the other hand, it is not essentially larger than any particular \( S_{\psi, \beta}(f) \). For \( 0 < \alpha \leq 1 \), let \( \mathcal{C}_\alpha \) be the family of functions supported in \( \{x : |x| \leq 1\} \), satisfying \( \int \psi = 0 \), and such that for all \( x \) and \( x' \), \( |\varphi(x) - \varphi(x')| \leq |x - x'|^\alpha \). If \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( (y, t) \in \mathbb{R}^{n+1}_+ \), we define

\[
A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)|.
\]

The intrinsic square function is defined by

\[
G_{\beta, \alpha}(f)(x) = \left( \int_{\Gamma_\beta(x)} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.
\]

If \( \beta = 1 \), set \( G_{1, \alpha}(f) = G_\alpha(f) \).
We mention several properties of $G_\alpha(f)$ (for the proofs we refer to [22] and [23, Ch. 6]). First of all, it is of weak type $(1, 1)$:

\[(2.1) \quad |\{x \in \mathbb{R}^n : G_\alpha(f)(x) > \lambda\}| \leq \frac{c(n, \alpha)}{\lambda} \int_{\mathbb{R}^n} |f| \, dx.\]

Second, if $\beta \geq 1$, then for all $x \in \mathbb{R}^n$,

\[(2.2) \quad G_{\beta,\alpha}(f)(x) \leq c(\alpha, \beta, n) G_\alpha(f)(x).\]

Third, if $S$ is anyone of the Littlewood-Paley operators defined above, then

\[(2.3) \quad S(f)(x) \leq c G_\alpha(f)(x),\]

where the constant $c$ is independent of $f$ and $x$.

3. Dyadic cubes

We say that $I \subset \mathbb{R}$ is a dyadic interval if $I$ is of the form $(\frac{j}{2^k}, \frac{j+1}{2^k})$ for some integers $j$ and $k$. A dyadic cube $Q \subset \mathbb{R}^n$ is a Cartesian product of $n$ dyadic intervals of equal lengths. Let $\mathcal{D}$ be the set of all dyadic cubes in $\mathbb{R}^n$.

Denote by $\ell_Q$ the side length of $Q$. Given $r > 0$, let $rQ$ be the cube with the same center as $Q$ such that $\ell_{rQ} = r \ell_Q$.

The following result can be found in [21, Lemma 2.1] or in [23, p. 91].

**Lemma 3.1.** There exist disjoint families $\mathcal{D}_1, \ldots, \mathcal{D}_{3^n}$ of dyadic cubes such that $\mathcal{D} = \bigcup_{k=1}^{3^n} \mathcal{D}_k$, and, for every $k$, if $Q_1, Q_2$ are in $\mathcal{D}_k$, then $3Q_1$ and $3Q_2$ are either disjoint or one is contained in the other.

Observe that it suffices to prove the lemma in the one-dimensional case. Indeed, if $\mathcal{I}$ is the set of all dyadic intervals in $\mathbb{R}$ and $\mathcal{I} = \bigcup_{j=1}^{3^n} \mathcal{I}_j$ is the representation from Lemma 3.1 in the case $n = 1$, then the required families in $\mathbb{R}^n$ are of the form

$$\mathcal{D}_k = \left\{ \prod_{m=1}^{n} I_m : I_m \in \mathcal{I}_{\alpha_i}, \alpha_i \in \{1, 2, 3\} \right\} \quad (k = 1, \ldots, 3^n).$$

The following property of the families $\mathcal{D}_k$ will play an important role in the proof of Theorem 1.1.

**Lemma 3.2.** For any cube $Q \in \mathcal{D}$ and for each $k = 1, \ldots, 3^n$ there is a cube $Q_k \in \mathcal{D}_k$ such that $Q \subset 3Q_k \subset 5Q$.

**Proof.** Let us consider first the one-dimensional case. Assume that $\mathcal{I} = \bigcup_{j=1}^{3^n} \mathcal{I}_j$ is the representation from Lemma 3.1.

Take an arbitrary dyadic interval $J = (\frac{j}{2^k}, \frac{j+1}{2^k})$. Set $J_1 = J$. Consider the dyadic intervals $J_2 = (\frac{j-1}{2^k}, \frac{j}{2^k})$ and $J_3 = (\frac{j+1}{2^k}, \frac{j+2}{2^k})$. It is easy
to see that each two different intervals of the form $3J_l$ are neither dis-
joint nor one is contained in the other. Therefore, the intervals $J_l$ lie
in the different families $\mathcal{I}_j$. Also, $J \subset 3J_l \subset 5J$ for $l = 1, 2, 3$.

Consider now the multidimensional case. Take an arbitrary cube $Q \in \mathcal{D}$. Then $Q = \prod_{m=1}^n I_m$, where $I_m \in \mathcal{I}$ and $\ell_{I_m} = h$ for each $m$.
Fix $\alpha_i \in \{1, 2, 3\}$. We have already proved that there exists $\tilde{I}_m \in \mathcal{I}_{\alpha_i}$ such that $I_m \subset 3\tilde{I}_m \subset 5I_m$. Observe also that, by the one-dimensional
construction, $\ell_{\tilde{I}_m} = \ell_{I_m} = h$. Therefore, setting $Q_k = \prod_{m=1}^n \tilde{I}_m$, we obtain the required cube from $\mathcal{D}_k$. \hfill \Box

4. LOCAL MEAN OSCILLATIONS

Given a measurable function $f$ on $\mathbb{R}^n$ and a cube $Q$, define the local mean oscillation of $f$ on $Q$ by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} \left((f - c)\chi_Q\right)^*(\lambda|Q|) \quad (0 < \lambda < 1),$$

where $f^*$ denotes the non-increasing rearrangement of $f$. The local sharp maximal function relative to $Q$ is defined by

$$M_{\lambda\mathcal{Q}}^# f(x) = \sup_{x \in Q' \subset Q} \omega_\lambda(f; Q'),$$

where the supremum is taken over all cubes $Q' \subset Q$ containing the point $x$.

By a median value of $f$ over $Q$ we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max \left(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|\right) \leq |Q|/2.$$

It follows from the definition that

$$(4.1) \quad |m_f(Q)| \leq (f\chi_Q)^*(|Q|/2).$$

Given a cube $Q_0$, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with
respect to $Q_0$ (that is, they are formed by repeated subdivision of $Q_0$
and each of its descendants into $2^n$ congruent subcubes). Observe that
if $Q_0 \in \mathcal{D}$, then the cubes from $\mathcal{D}(Q_0)$ are also dyadic in the usual
sense as defined in the previous section.

If $Q \in \mathcal{D}(Q_0)$ and $Q \neq Q_0$, we denote by $\widehat{Q}$ its dyadic parent, that is,
the unique cube from $\mathcal{D}(Q_0)$ containing $Q$ and such that $|\widehat{Q}| = 2^n|Q|$.

The following result has been recently proved in [14].

**Theorem 4.1.** Let $f$ be a measurable function on $\mathbb{R}^n$ and let $Q_0$ be
a fixed cube. Then there exists a (possibly empty) collection of cubes $Q^k_j \in \mathcal{D}(Q_0)$ such that
(i) for a.e. \( x \in Q_0 \),

\[
|f(x) - m_f(Q_0)| \leq 4M_{1/4;Q_0}^# f(x) + 4 \sum_{k=1}^{\infty} \sum_{j} \omega_{\frac{1}{2^{m+2}}} (f; \tilde{Q}_j^k) \chi_{Q_j^k}(x);
\]

(ii) for each fixed \( k \) the cubes \( Q_j^k \) are pairwise disjoint;

(iii) if \( \Omega_k = \bigcup_j Q_j^k \), then \( \Omega_{k+1} \subset \Omega_k \);

(iv) \( |\Omega_{k+1} \cap Q_j^k| \leq \frac{1}{2} |Q_j^k| \).

Remark 4.2. The proof of Theorem 4.1 shows that actually \( M_{1/4;Q_0}^# f \) can be replaced by a smaller dyadic operator, that is, by

\[
M_{1/4;Q_0}^{#,d} f(x) = \sup_{Q \ni x, Q \in \mathcal{D}(Q_0)} \omega_{1/4} (f; Q).
\]

Note that Theorem 4.1 is a development of ideas going back to works of Carleson [3], Garnett-Jones [8] and Fujii [7].

We mention a simple property of local mean oscillations which will be used below.

Lemma 4.3. For any \( k \in \mathbb{N} \) and for each cube \( Q \),

\[
\omega_\lambda \left( \sum_{i=1}^{k} f_i; Q \right) \leq \sum_{i=1}^{k} \omega_{\lambda/k} (f_i; Q) \quad (0 < \lambda < 1).
\]

Proof. It is well known (see, e.g., [1, p. 41]) that

\[
(f + g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2) \quad (t_1, t_2 > 0).
\]

This property is easily extended to any finite sum of functions:

\[
\left( \sum_{i=1}^{k} f_i \right)^*(t) \leq \sum_{i=1}^{k} (f_i)^*(t/k).
\]

Hence, for arbitrary \( \xi_i \in \mathbb{R} \) we have

\[
\omega_\lambda \left( \sum_{i=1}^{k} f_i; Q \right) = \inf_{c \in \mathbb{R}} \left( \left( \sum_{i=1}^{k} f_i - c \right) \chi_Q \right)^*(\lambda|Q|)
\]

\[
= \inf_{c \in \mathbb{R}} \left( (f_1 - c + \sum_{i=2}^{k} (f_i - \xi_i)) \chi_Q \right)^*(\lambda|Q|)
\]

\[
\leq \omega_{\lambda/k} (f_1; Q) + \sum_{i=2}^{k} ((f_i - \xi_i) \chi_Q)^*(\lambda|Q|/k).
\]

Taking the infimum over all \( \xi_i \in \mathbb{R} \) yields (4.2). \( \square \)
5. Proof of Theorems 1.1 and 1.2

5.1. The intrinsic square function $\tilde{G}_\alpha$. For our purposes it will be more convenient to work with the following variant of $G_\alpha$. Given a cube $Q \subset \mathbb{R}^n$, set

$$T(Q) = \{(y, t) \in \mathbb{R}^n : y \in Q, \ell(Q)/2 \leq t < \ell(Q)\}.$$ 

Denote $\gamma_Q(f)^2 = \int_{T(Q)} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}}$ and let

$$\tilde{G}_\alpha(f)(x)^2 = \sum_{Q \in \mathcal{D}} \gamma_Q(f)^2 \chi_Q(x).$$

Lemma 5.1. For any $x \in \mathbb{R}^n$,

$$G_\alpha(f)(x) \leq \tilde{G}_\alpha(f)(x) \leq c(\alpha, n) G_\alpha(f)(x).$$

Proof. For any $x \not\in 3Q$ we have $\Gamma(x) \cap T(Q) = \emptyset$, and hence

$$\int_{\Gamma(x) \cap T(Q)} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}} \leq \gamma_Q(f)^2 \chi_Q(x).$$

Therefore,

$$G_\alpha(f)(x)^2 = \int_{\Gamma(x)} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}} = \sum_{Q \in \mathcal{D}} \int_{\Gamma(x) \cap T(Q)} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}} \leq \tilde{G}_\alpha(f)(x)^2.$$

On the other hand, if $x \in 3Q$ and $(y, t) \in T(Q)$, then $|x - y| \leq 2\sqrt{n\ell(Q)} \leq 4\sqrt{n}t$. Thus,

$$\tilde{G}_\alpha(f)(x)^2 = \sum_{Q \in \mathcal{D}} \gamma_Q(f)^2 \chi_Q(x) \leq \sum_{Q \in \mathcal{D}} \int_{T(Q) \cap \Gamma(4\sqrt{n}t(x))} (A_\alpha(f)(y, t))^2 \frac{dydt}{t^{n+1}} = G_{4\sqrt{n},\alpha}(f)(x)^2.$$ 

Combining this with (2.2), we get the right-hand side of (5.1). \qed

5.2. A local mean oscillation estimate of $\tilde{G}_\alpha$. The key role in our proof will be played by the following lemma.

Lemma 5.2. For any cube $Q \in \mathcal{D}$,

$$\omega_\lambda(\tilde{G}_\alpha(f)^2; Q) \leq c(n, \alpha, \lambda) \left( \frac{1}{|15Q|} \int_{15Q} |f|dx \right)^2.$$
Proof. Applying Lemma 3.1 we can write
\[ \tilde{G}_\alpha(f)(x)^2 = \sum_{k=1}^{3^n} \sum_{Q \in D_k} \gamma_Q(f)^2 \chi_{3Q}(x) \equiv \sum_{k=1}^{3^n} \tilde{G}_{\alpha,k}(f)(x)^2. \]
Hence, by Lemma 4.3,
\[ \omega_\lambda(\tilde{G}_\alpha(f)^2; Q) \leq \sum_{k=1}^{3^n} \omega_{\lambda/3^n}(\tilde{G}_{\alpha,k}(f)^2; Q). \]
By Lemma 3.2 for each \( k = 1, \ldots, 3^n \) there exists a cube \( Q_k \in D_k \) such that \( Q \subset 3Q_k \subset 5Q \). Hence,
\[
\inf_{c \in \mathbb{R}} \left( (\tilde{G}_{\alpha,k}(f)^2 - c) \chi_{3Q_k} \right)^* (\lambda |Q|/3^n) \\
\leq \inf_{c \in \mathbb{R}} \left( (\tilde{G}_{\alpha,k}(f)^2 - c) \chi_{3Q_k} \right)^* (\lambda |Q|/3^n).
\]
Using the main property of cubes from the family \( D_k \) (expressed in Lemma 3.1), for any \( x \in 3Q_k \) we have
\[
(5.2) \quad \tilde{G}_{\alpha,k}(f)(x)^2 = \sum_{Q \in D_k: 3Q \subset 3Q_k} \gamma_Q(f)^2 \chi_{3Q}(x) + \sum_{Q \in D_k: 3Q_k \subset 3Q} \gamma_Q(f)^2.
\]
Arguing as in the proof of Lemma 5.1 we obtain
\[
\sum_{Q \in D_k: 3Q \subset 3Q_k} \gamma_Q(f)^2 \chi_{3Q}(x) \\
\leq \sum_{Q \in D_k: 3Q \subset 3Q_k} \int_{T(Q) \cap T_4 \cap \pi(x)} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \\
\leq \int_{\tilde{T}(3Q_k) \cap T_4 \cap \pi(x)} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}},
\]
where \( \tilde{T}(3Q_k) = \{(y,t): y \in 3Q_k, 0 < t \leq \ell(3Q_k)\} \). For any \( \varphi \) supported in \( \{x: |x| \leq 1\} \) and for \( (y,t) \in \tilde{T}(3Q_k) \) we have
\[ f * \varphi_t(y) = (f \chi_{9Q_k}) * \varphi_t(y). \]
Therefore,
\[
\int_{\tilde{T}(3Q_k) \cap T_4 \cap \pi(x)} (A_\alpha(f)(y,t))^2 \frac{dydt}{t^{n+1}} \leq G_{4\sqrt{n},\alpha}(f \chi_{9Q_k})(x)^2.
\]
Combining the letter estimates with (5.2) and setting
\[ c = \sum_{Q \in D_k: 3Q_k \subset 3Q} \gamma_Q(f)^2, \]
we get

\[ 0 \leq \tilde{G}_{\alpha,k}(f)(x)^2 - c \leq G_{4\sqrt{n},\alpha}(f\chi_{9Q_k})(x)^2 \quad (x \in 3Q_k). \]

From this, by (2.1) and (2.2) (we use also that \(3Q_k \subset 5Q\) implies \(9Q_k \subset 15Q\)),

\[
\inf_{c \in \mathbb{R}} \left( (\tilde{G}_{\alpha,k}(f)^2 - c) \chi_{3Q_k} \right)^* (\lambda|Q|/3^n)
\leq c(n,\alpha)(G_{\alpha}(f\chi_{9Q_k})^* (\lambda|Q|/3^n)^2
\leq c \left( \frac{3^n}{\lambda|Q|} \int_{9Q_k} |f| \right)^2 \leq c \left( \frac{3^n}{\lambda|Q|} \int_{15Q} |f| \right)^2,
\]

which completes the proof. \(\square\)

### 5.3. An auxiliary operator

Let \(\{Q^k_j\}\) be a family of cubes appeared in Theorem 4.1.

Given \(\gamma > 1\), consider the operator \(A_\gamma\) defined by

\[
A_\gamma f(x) = \sum_{k,j} \left( \frac{1}{|\gamma Q^k_j|} \int_{\gamma Q^k_j} |f| \right)^2 \chi_{Q^k_j}(x).
\]

This object will appear naturally after the combination of Lemma 5.2 with Theorem 4.1.

**Lemma 5.3.** For any \(f \in L^3(w)\),

\[
\left( \int_{Q_0} (A_\gamma f)^{3/2} w \, dx \right)^{2/3} \leq c(n,\gamma)\|w\|_{A_3}\|f\|_{L^3(w)}^2.
\]

**Proof.** We follow [5] with some minor modifications. By duality, (5.3) is equivalent to that for any \(h \geq 0\) with \(\|h\|_{L^3(w)} = 1\),

\[
\int_{Q_0} (Af)hw \, dx = \sum_{k,j} \left( \frac{1}{|\gamma Q^k_j|} \int_{\gamma Q^k_j} |f| \right)^2 \int_{Q^k_j} hw \leq c(n,\gamma)\|w\|_{A_3}\|f\|_{L^3(w)}^2.
\]

Let \(E^k_j = Q^k_j \setminus Q_{k+1}\). It follows from the properties (ii)-(iv) of Theorem 4.1 that \(|E^k_j| \geq |Q^k_j|/2\) and the sets \(E^k_j\) are pairwise disjoint. Hence, setting \(A_3(Q) = \frac{w(Q)(w^{-1/2}(Q))^2}{|Q|^3}\) (we use the notion \(\nu(Q) = \int_Q \nu(x) \, dx\),

we have
\[
\left( \frac{1}{|\gamma Q^k_j|} \int_{\gamma Q^k_j} |f| \right)^2 \int_{Q^k_j} hw \leq 2(3\gamma)^n A_3 (3\gamma Q^k_j) \\
\times \left( \frac{1}{w^{-1/2}(3\gamma Q^k_j)} \int_{\gamma Q^k_j} |f| \right)^2 \left( \frac{1}{w(3\gamma Q^k_j)} \int_{\gamma Q^k_j} hw \right) |E^k_j| \\
\leq 2(3\gamma)^n \|w\|_{A_3} \int_{E^k_j} M^c_{w^{-1/2}} (fw^{1/2})^2 M^c_{w} h dx
\]
(here $M^\nu f(x) = \sup_{Q \ni x} \frac{1}{\nu(Q)} \int_Q |f| \nu dx$, where the supremum is taken over all cubes $Q$ centered at $x$).

Applying the latter estimate along with Hölder’s inequality and using the fact (based on the Besicovitch covering theorem) that the $L^p(\nu)$-norm of $M^\nu$ does not depend on $\nu$, we get
\[
\sum_{k,j} \left( \frac{1}{|\gamma Q^k_j|} \int_{\gamma Q^k_j} |f| \right)^2 \int_{Q^k_j} hw \\
\leq 2(3\gamma)^n \|w\|_{A_3} \int_{\mathbb{R}^n} M^c_{w^{-1/2}} (fw^{1/2})^2 M^c_{w} h dx \\
\leq 2(3\gamma)^n \|w\|_{A_3} \|M^c_{w^{-1/2}} (fw^{1/2})\|_{L^3(w^{-1/2})}^2 \|M^c_{w} h\|_{L^3(w)} \\
\leq c(n, \gamma) \|w\|_{A_3} \|f\|_{L^3(w)}^2,
\]
and therefore the proof is complete. \hfill \Box

5.4. Proof of Theorem 1.1. First, by (2.3) and Lemma 5.1, it is enough to prove (1.2) for $\tilde{G}_\alpha$. Second, as we mentioned in the Introduction, the sharp version of the Rubio de Francia extrapolation theorem proved in [6] says that (1.2) for $p = 3$ implies (1.2) for any $p > 1$. Therefore, our aim is to show that
\[
\|\tilde{G}_\alpha(f)\|_{L^3(w)} \leq c(n, \alpha) \|w\|_{A_3}^{1/2} \|f\|_{L^3(w)}.
\]

Further, by a standard approximation argument, it suffices to prove (5.5) for any $f \in L^1(\mathbb{R}^n)$. By the weak type $(1, 1)$ property of $G_\alpha$ (2.1) and by (4.1) and (5.1), for such $f$ we have
\[
\lim_{|Q| \to \infty} |m_Q(\tilde{G}_\alpha(f)^2)| \leq c \lim_{|Q| \to \infty} (G_\alpha f)^*(|Q|/2)^2 = 0.
\]

Now, following [5], denote by $\mathbb{R}^n_i$, $1 \leq i \leq 2^n$ the $n$-dimensional quadrants in $\mathbb{R}^n$, that is, the sets $I^\pm \times I^\pm \times \cdots \times I^\pm$, where $I^+ = [0, \infty)$ and $I^- = (-\infty, 0)$. For each $i, 1 \leq i \leq 2^n$, and for each $N > 0$ let
be the dyadic cube adjacent to the origin of side length $2^N$ that is contained in $\mathbb{R}_i^n$. By (5.6) and by Fatou’s lemma,
\[
\left( \int_{\mathbb{R}_i^n} \tilde{G}_\alpha(f)(x)^3 w(x) dx \right)^{2/3} \leq \liminf_{N \to \infty} \left( \int_{Q_N,i} |\tilde{G}_\alpha(f)(x)|^2 - m_{Q_N,i}(\tilde{G}_\alpha(f)^2)|^{3/2} w(x) dx \right)^{2/3}.
\]

Combining Theorem 4.1 (where $M_{1/4,Q_0}^\# f$ is replaced by $M_{1/4,Q_0}^\#,d f$ from Remark 4.2) with Lemma 5.2 (we use that the cubes $Q_j^k$ are dyadic, and hence the cubes $\hat{Q}_j^k$ are dyadic as well; also, $\hat{Q}_j^k \subset 3Q_j^k$), we get that for all $x \in Q_N,i$,
\[
|\tilde{G}_\alpha(f)(x)|^2 - m_{Q_N,i}(\tilde{G}_\alpha(f)^2) \leq c_n (Mf(x)^2 + A_{45} f(x)),
\]
where
\[
Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy
\]
is the Hardy-Littlewood maximal operator.

It was proved by Buckley [2] that
\[
\|M\|_{L^p(w)} \leq c(p, n)\|w\|_{A_p}^{1/p} \quad (1 < p < \infty).
\]
Therefore,
\[
\left( \int_{Q_N,i} (Mf)^3 w \right)^{2/3} \leq c(n)\|w\|_{A_3}\|f\|_{L^3(w)}^2.
\]

Applying this along with (5.7), (5.8) and Lemma 5.3, we get
\[
\left( \int_{\mathbb{R}_i^n} \tilde{G}_\alpha(f)(x)^3 w(x) dx \right)^{2/3} \leq c(\alpha, n)\|w\|_{A_3}\|f\|_{L^3(w)}^2 \quad (1 \leq i \leq 2^n).
\]

Therefore,
\[
\int_{\mathbb{R}_i^n} \tilde{G}_\alpha(f)(x)^3 w(x) dx = \sum_{i=1}^{2^n} \int_{\mathbb{R}_i^n} \tilde{G}_\alpha(f)(x)^3 w(x) dx \leq 2^n (c(\alpha, n)\|w\|_{A_3})^{3/2}\|f\|_{L^3(w)}^3,
\]
which completes the proof.
5.5. Proof of Theorem 1.2. We use exactly the same approach as in the proof of [13, Corollary 1.4]. The proof is just a combination of several known results.

First, it was proved by Wilson ([22] or [23, p. 155]) that there exists \( \alpha \leq 1 \) (depending on \( T \)) such that for all \( x \in \mathbb{R}^n \),

\[
S_\psi(Tf)(x) \leq c(T, \psi, n)G_\alpha(f)(x).
\]

Exactly the same proof yields

\[
S_{\psi, \beta}(Tf)(x) \leq c(T, \psi, n, \beta)G_\alpha(f)(x) \quad (\beta \geq 1).
\]

Next, define

\[
\|w\|_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w \chi_Q)(x)dx.
\]

It follows easily from (5.9) (see, e.g., [12, Lemma 3.5]) that for any \( p > 1 \),

\[
(5.11) \quad \|w\|_{A_\infty} \leq c(p, n)\|w\|_{A_p}.
\]

Assuming that \( \psi \) additionally satisfies

\[
\int_s^{\infty} |\hat{\psi}(t, 0, \ldots, 0)|^2 \frac{dt}{t} \geq c(1 + s)^{-\xi}
\]

for some \( s > 0 \), it was shown by Wilson [21] that for any \( p > 0 \),

\[
(5.12) \quad \|\mathcal{M}(f)\|_{L^p(w)} \leq c(n, p, \psi)\|w\|_{A_\infty}^{1/2}\|S_{\psi, 3\sqrt{n}}(f)\|_{L^p(w)},
\]

where \( \mathcal{M} \) is the grand maximal function. Note that (5.12) is not contained in [21] in such an explicit form. We refer to [13, Proposition 2.3] for some comments about this. Observe also that the proof of (5.12) is based essentially on the deep theorem of Chang-Wilson-Wolff [4].

Further, it is well-known [19, pp. 67-68] that for all \( x \in \mathbb{R}^n \),

\[
T^*f(x) \leq c(n, T)(\mathcal{M}(Tf) + Mf(x)).
\]

Combining this with (5.10), (5.11) and (5.12), we get

\[
\|T^*f\|_{L^p(w)} \leq c\|w\|_{A_p}^{1/2}\|S_{\psi, 3\sqrt{n}}(Tf)\|_{L^p(w)} + c\|Mf\|_{L^p(w)} \leq c\|w\|_{A_p}^{1/2}\|G_\alpha(f)\|_{L^p(w)} + c\|Mf\|_{L^p(w)}.
\]

This estimate along with Theorem 1.1 and (5.9) for \( p \geq 3 \) yields

\[
\|T^*f\|_{L^p(w)} \leq c\|w\|_{A_p}f_{L^p(w)},
\]

which completes the proof for \( T^* \).

The above estimate for \( T^* \) implies clearly the same estimate for \( T \):

\[
\|T\|_{L^p(w)} \leq c\|w\|_{A_p} \quad (p \geq 3),
\]
which by duality yields

$$\|T\|_{L^p(w)} \leq c\|w^{-\frac{1}{p-1}}\|_{A_{p'}} = c\|w\|_{A_p}^{\frac{1}{p-1}} \quad (1 < p \leq 3/2),$$

and therefore, the theorem is proved.

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