Nonspontaneous Supersymmetry Breaking

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Abstract

A new way of supersymmetry breaking involving a dynamical parameter is introduced. It is independent of particle phenomenology and gauge groups. The only requirement is that Lorentz invariance be valid strictly infinitesimally (i.e. Spin(1, 3) be for some values of the parameter replaced by a compact group $G$ with its Lie algebra $\mathfrak{g} \cong \mathfrak{so}(1, 3)$).

1 Introduction

The particles predicted by supersymmetric field theories failed to appear in experiments, so that within the accessible range of energies supersymmetry must be broken. However, if the Standard Model is to serve as a ($\approx 300$ GeV) low-energy approximation to some as-yet-unknown unified theory, supersymmetry would have to manifest at a higher energy level to allow supermultiplets involving scalar particles to be formed, thus preventing those scalar particles from acquiring large ($\approx 10^{16} - 10^{18}$ GeV) bare masses and bringing corpulence to the presently observable particles \[.\] This disparity of characteristic mass scales, known as the hierarchy problem, and the above-outlined way around it provide the strongest theoretical motivation to keep supersymmetry alive. Another feature of supersymmetric theories that is considered desirable is the presence of sparticles - superpartners of the observable ones. They are natural candidates for exotica such as the missing ‘dark matter’ of the Universe.

Without exception, all mechanisms of supersymmetry breaking hitherto proposed are spontaneous. Generally speaking, spontaneous supersymmetry breaking occurs when the variation of some field under supersymmetry transformations yields nonzero vacuum expectation values:

$$\langle 0 | \delta \text{(field)} | 0 \rangle \neq 0.$$ 

As a result, the vacuum state gains energy, and enters supermultiplets opposite a massless fermion - the goldstino. If gravitation is present, supersymmetry localizes and instead of the goldstino, the gravitino becomes the vacuum’s superpartner.

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Unlike the breaking of electroweak symmetry, a direct coupling of the electroweak force to the resulting Higgs particles is not possible because such a coupling leads to sum rules for the masses of the unobserved superpartners that are excluded. An indirect transmission of supersymmetry breaking to the observable sector is needed. Based on the particulars of coupling, the spontaneous supersymmetry breaking mechanisms are: gauge-mediated (GMSB) [2], anomaly-mediated (AMSB) [3], supergravity (SUGRA) [4], and extra-dimensional [5]. These different mechanisms have characteristic mass spectra and experimental signatures.

The search for sparticles goes on, but the experimental data recently obtained at LEP and Tevatron [6] does not encourage optimism on the subject of plausibility of GMSB and AMSB. SUGRA is unassailable, although its superpotential contains a soft parameter chosen to fit the experimentally confirmed phenomena. That raises the question whether there is a viable alternative to the brane theory treatment of the hierarchy problem in particular and to spontaneous supersymmetry breaking in general.

Our paper is an attempt to explain the why of the hierarchy problem in terms of fundamental space-time symmetries. As far as we know, the first results linking supersymmetry algebras to space-time symmetries were published by Nahm [7]. The anti-de-Sitter space with its $O(3,2)$ symmetries supports all conceivable supersymmetry algebras, whereas the de-Sitter space having $O(4,1)$ as the symmetry group has only $N = 2$ supersymmetry. Thus this Universe evolving from the anti-de-Sitter to the de-Sitter regime may provide a toy model of nonspontaneous $N \neq 2$ supersymmetry breaking. We use the word ‘nonsustentaneous’ advisedly, for no goldstino (or gravitino) is created. It is very instructive to expose the fatal flaw of this model. There is no smooth direct parametric transition from $O(3,2)$ to $O(4,1)$ because $\mathfrak{o}(3,2) \not\simeq \mathfrak{o}(4,1)$, and for some value of the parameter space-time symmetries collapse even infinitesimally.

Therefore to make such a theory work one needs a family of locally isomorphic Lie groups, smoothly depending on a parameter, and differing in their facility to support supersymmetry algebras. Then the parameter may be interpreted as the energy scale, pre- and post-unification values separated by an interval. In the Minkowski $\mathbb{R}^4$ one also requires Lorentz invariance. That could only be satisfied for families of Lie groups locally isomorphic to the Lorentz group, and containing that group as a member. In what follows we find one such family containing, at one extreme Spin$(1, 3)$, and at the other a compact Lie group $G$ which, while allowing to maintain Lorentz invariance, does not support any supersymmetry algebras. Then the parametric evolution from the former to the latter constitutes a mechanism of supersymmetry breaking. The rationale behind our construction is deceptively simple: supersymmetry happens to be broken (or, rather, nonexistent) below the unification mark because the respective $S$-matrix has finite-dimensional blocks - exacting finite-dimensional unitary representations (i.e. compactness) of the (respective) symmetry group. By contrast, above the unification mark more off-diagonal elements of the $S$-matrix become non-zero; consequently, the erstwhile finite-dimensional blocks
coalesce, the representation spaces become infinite-dimensional, and Spin(1,3) takes over.

There are experimentally verifiable effects associated with the symmetry group $G$. The electromagnetic vector potentials transform differently under Spin(1,3), and that difference can be detected.

Lastly, we dispense with the physical constants by setting $h = c = 1$.

2 Mathematical Preliminaries

The Pauli matrices are

$$
\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

(2.1)

The Dirac representation of $SU(2)$, denoted $SU_D(2)$ is generated by

$$
J_1 = \frac{1}{2} \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}, \quad J_2 = \frac{1}{2} \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad J_3 = \frac{1}{2} \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}.
$$

(2.2)

There still exists the twofold covering epimorphism of Lie groups:

$$
A : SU_D(2) \longrightarrow \begin{bmatrix} \text{SO}(3) & 0 \\ 0 & 1 \end{bmatrix}.
$$

(2.3)

Spin(1,3) may be viewed as a complex extension of $SU_D(2)$:

$$
\left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix} \right\} \mapsto \left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \quad K^c_i = \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \right\}.
$$

(2.4)

Fortuitously, there is a class of mutually isomorphic almost complex Lie algebra extensions, of which $so(1,3)$, generated by $\{ J_i, K^c_i \}$ of (2.4) is a member. We are interested mainly in the following almost complex extension:

$$
\left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix} \right\} \mapsto \left\{ J_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \quad K_i = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.
$$

(2.5)

Its relevant properties are summarized in

**Theorem 2.1.** There exists a unique compact semisimple Lie group $G \subset SU(4)$, whose Lie algebra $\mathfrak{g} \cong so(1,3)$ is generated by (2.5).

**Proof.** Every almost complex extension corresponds (up to a nonzero factor) to a matrix

$$
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in U(2), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

$\mathfrak{g} \cong so(1,3)$ implies $ad - bc = 1$. Therefore

$$
\Re a = \Re d = 0, \quad \Im c = \Im b, \quad \Re c = -\Re b.
$$
This allows us to write the most general almost complex extension as

\[ J_i \mapsto \begin{pmatrix} w & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} + u \begin{pmatrix} 0 & i & 0 \\ 0 & -1 & 0 \end{pmatrix} J_i, \quad w^2 + u^2 + v^2 = 1. \]

To ensure compactness, we must have \( \exp i \kappa K \) bounded. Whence \( w = 0, u = 0 \) is the only choice. And this is (2.5).

According to Helgason ([8], Chapter II, §2, Theorem 2.1), there exists a Lie group \( G \), whose Lie algebra is generated by \( \{ J_i, K_i \} \) of (2.5). Its elements are all of the form \( \exp i(\theta^b J_b + \kappa^a K_a) \), which means \( G \) is a Lie subgroup of \( SU(4) \). Now \( G \) has to be closed in the standard matrix topology of \( SU(4) \). That is based on a fundamental result of Mostow [9]: any semisimple Lie subgroup \( H \) of a compact Lie group \( C \) is closed in the relative topology of \( C \). In our case, \( SU(4) \) is compact, \( \mathfrak{g} \) is semisimple.

In the sequel we will work with the homogeneous space \( G/SU_D(2) \).

**Lemma 2.1.**

\[ \pi_1(G/SU_D(2)) = 0. \]

**Proof.** For all Lie groups \( \pi_2(\cdot) = 0 \) [10]; for \( SU_D(2) \), \( \pi_0(SU_D(2)) = 0 \) by connectedness. Also, \( SU_D(2) \) is a closed subgroup of \( SU(4) \) in the ordinary matrix topology. We therefore have the following exact homotopy sequence [10]:

\[ 0 \to \pi_2(SU(4)/SU_D(2)) \to \pi_1(SU_D(2)) \]
\[ \to \pi_1(SU(4)) \to \pi_1(SU(4)/SU_D(2)) \to 0. \]

\( \pi_1(SU(4)) = 0 \) [10] whence

\[ \pi_1(SU(4)/SU_D(2)) \cong \pi_1(SU_D(2)) = \pi_1(\mathbb{S}^3) = 0. \]

Now homotopy is functorial. The embedding \( \xi : G/SU_D(2) \hookrightarrow SU(4)/SU_D(2) \) induces the monomorphism of fundamental groups

\[ \xi_{\ast} : \pi_1(G/SU_D(2)) \to \pi_1(SU(4)/SU_D(2)). \]

**Theorem 2.2.**

\[ G/SU_D(2) \cong \mathbb{S}^3. \]

**Proof.** \( \mathfrak{g} \) decomposes as a vector space into two three-dimensional subspaces,

\[ \mathfrak{g} = \mathfrak{j} \oplus \mathfrak{k}, \]

Based on this decomposition, there is an involutive automorphism

\[ \vartheta : \mathfrak{g} \to \mathfrak{g} \]

defined by

\[ \vartheta(J + K) = J - K, \quad \forall J \in \mathfrak{j}, \quad \forall K \in \mathfrak{k}. \]
\( j \) is the set of fixed points of \( \vartheta \). It is unique ([8], Chapter IV, §3, Proposition 3.5). The pair \((g, \vartheta)\) is an orthogonal symmetric Lie algebra ([8], Chapter IV, §3). There is a Riemannian symmetric pair \((G, SU_D(2))\) associated with \((g, \vartheta)\) so that the quotient \(G/SU_D(2)\) is a complete locally symmetric Riemannian space. Furthermore, its curvature corresponding to any \(G\)-invariant Riemannian structure is given by ([8], Chapter IV, §4, Theorem 4.2):

\[
R(K_{i_1}, K_{i_2})K_{i_3} = -[[[K_{i_1}, K_{i_2}], K_{i_3}] \quad \forall K_{i_1}, K_{i_2}, K_{i_3} \in \mathfrak{t}.
\]

Computing the sectional curvature we see that \(R^{\text{sect}} \equiv 1\). Now a pedestrian version of the Sphere theorem [11] asseverates that a complete simply connected Riemannian manifold with \(R^{\text{sect}} \equiv 1\) is isometric to a sphere of appropriate dimension. In our case the topological condition is satisfied in view of Lemma 2.1.

Consider the natural inclusions of Lie groups

\[
\iota : G \hookrightarrow GL(4, \mathbb{C}), \quad \iota : \text{Spin}(1, 3) \hookrightarrow GL(4, \mathbb{C}). \tag{2.6}
\]

Their images inside \(GL(4, \mathbb{C})\) intersect:

\[
\iota(G) \cap \iota(\text{Spin}(1, 3)) = SU_D(2). \tag{2.7}
\]

Because of (2.7), the set

\[
\text{Ad}_{\iota(G)}(\iota(\text{Spin}(1, 3))) = \bigsqcup_{U \in G} U\text{Spin}(1, 3)U^H, \tag{2.8}
\]

the disjoint union of conjugates of \(\text{Spin}(1, 3)\), has the same cardinality as the set of all boosts in \(G\). Similarly, there is the natural inclusion

\[
\iota : \text{SO}(4) \hookrightarrow GL(4, \mathbb{R}). \tag{2.9}
\]

The set \(\text{Ad}_{\iota(\text{SO}(4))}(\iota(\text{SO}(1, 3)^c))\) is homeomorphic to \(\text{SO}(4)/\text{SO}(3) \cong \mathbb{S}^3\). Combining this with Theorem 2.4 we arrive at:

\[
\begin{array}{ccc}
\text{Ad}_{\iota(G)}(\iota(\text{Spin}(1, 3))) & \longrightarrow & G/SU_D(2) \overset{\cong}{\longrightarrow} \mathbb{S}^3 \\
\downarrow & & \downarrow \\
\text{Ad}_{\iota(\text{SO}(4))}(\iota(\text{SO}(1, 3)^c)) & \longrightarrow & \text{SO}(4)/\text{SO}(3) \overset{\cong}{\longrightarrow} \mathbb{S}^3
\end{array} \tag{2.10}
\]

The double horizontal lines indicate set-theoretic bijective correspondences, the upper \(\cong\) is an isometry, the lower one is a diffeomorphism. Furthermore, the diagram (2.10) commutes and de facto defines the diffeomorphism \(\varphi\). This diffeomorphism is utilized in the sequel to effect an action of \(G\).
3 Equivariant Impulse Operators

$G$ does not act on the Minkowski $\mathbb{R}^4$ by isometries. We have

$$
\begin{aligned}
G \times \mathbb{R}^4 &\to \mathbb{R}^4; \\
(\exp i(\theta^b J_b + \kappa^a K_a), x^\mu) &\mapsto x'^\mu = A(\exp i\theta^b J_b)^\mu_\lambda \varphi(\exp i\kappa^a K_a)^\lambda_\eta x^\eta.
\end{aligned}
$$

(3.1)

In fact, the metric becomes frame-dependent:

$$
\varphi(\exp i\kappa^a(\alpha) K_a) g \varphi(\exp (-i\kappa^a(\alpha) K_a)) \neq g, \quad \alpha \in [0, 2\pi], \quad \alpha \neq \{0, 2\pi\};
$$

with $\alpha$ being the group parameter here. Yet physical quantities must remain frame-independent. Therefore, instead of the standard quantum field theory substitution

$$
P_\mu \mapsto i\partial_\mu,
$$

(3.2)

we employ the rule

$$
P_\mu \mapsto i\nabla_\mu(\alpha) \overset{\text{def}}{=} i(\varepsilon^\mu_\nu(\alpha) \partial_\nu + i\kappa^\mu_\nu(\alpha) K_a),
$$

(3.3)

the exact form of $\varepsilon^\mu_\nu(\alpha)$ and $\kappa^\mu_\nu(\alpha)$ to be determined. $K_a$’s are in keeping with the $(1/2, 1/2)$ representation of $P_\mu$’s. This construction is an equivariant incarnation of the free spin structure due to Plymen and Westbury [12]. Briefly, let $M$ be a 4-dimensional smooth manifold with all the obstructions to the existence of a Lorentzian metric vanishing (for instance, a parallelizable $M$ would do). Let

$$
\Lambda : \text{Spin}(1, 3) \to SO(1, 3)^e
$$

be the twofold covering epimorphism of Lie groups. A free spin structure on $M$ consists of a principal bundle $\zeta : \Sigma \to M$ with structure group Spin$(1, 3)$ and a bundle map $\Lambda : \Sigma \to FM$ into the bundle of linear frames for $TM$, such that

$$
\tilde{\Lambda} \circ \tilde{R}_S = \tilde{R}'_{\zeta \circ \Lambda(S)} \circ \tilde{\Lambda} \quad \forall S \in \text{Spin}(1, 3),
$$

$$
\zeta' \circ \tilde{\Lambda} = \zeta,
$$

$\tilde{R}$ and $\tilde{R}'$ being the canonical right actions on $\Sigma$ and $FM$ respectively, $\iota : SO(1, 3)^e \to GL(4, \mathbb{R})$ the natural inclusion of Lie groups, and $\pi' : FM \to M$ the canonical projection. The map $\Lambda$ is called a spin-frame on Spin$(1, 3)$. This definition of a spin structure induces metrics on $\Sigma$. Indeed, given a spin-frame $\tilde{\Lambda} : \Sigma \to FM$, a dynamic metric $\tilde{g}_\Lambda$ is defined to be the metric that ensures orthonormality of all frames in $\Lambda(\Sigma) \subset FM$. It should be emphasized that within the Plymen and Westbury’s formalism the metrics are built $a \ posteriori$, after a spin-frame has been set by the field equations. In our formalism the metrics are obtained via the $G$-action, and the set of all allowable metrics is $\text{Ad}_{\iota(SO(4))}(\iota(SO(1, 3)^e))$. $\nabla_\mu(\alpha)$ qualifies as a $G$-connection on the principal $G$-bundle over the physical
space-time. Furthermore, we impose an additional condition on (3.3) to ensure validity of the relativistic impulse-energy identity:

\[ P_\mu(\alpha)P_\mu(\alpha) = g^{\nu\lambda}(\alpha)\nabla_\nu(\alpha)\nabla_\lambda(\alpha) \overset{\text{def}}{=} g^{\nu\lambda}(0)\partial_\nu\partial_\lambda = P_\mu(0)P_\mu(0). \]  \hspace{1cm} (3.4)

This translates to some algebraic relations among \( \kappa_\mu^a \)'s and \( \varepsilon_\mu^a \)'s. However, we still need to make the \( G \)-transformation law of \( \kappa_\mu^a \)'s more explicit. First, these operators are natural spinors in the sense that \( SU_\mathfrak{p}(2) \) acts linearly:

\[ U_\gamma^\mu \nabla_\mu U^H = U_\gamma^\mu U^H \varepsilon^\nu_\mu \partial_\nu + i\kappa_\mu^a U_\gamma^\mu U^H U K_a U^H \]
\[ = M_\mu^\gamma \gamma^\rho \varepsilon^\rho_\mu \partial_\rho + M_\mu^\gamma \kappa_\mu^a r^n_a K_n \]
\[ \text{by } [\mathfrak{j}, \mathfrak{t}] = \mathfrak{t}. \]  \hspace{1cm} (3.5)

Here \( M_\mu^\gamma \)'s realize an \( SO(3) \) transformation \( (U \in SU_\mathfrak{p}(2)) \), which is at its most transparent if \( \gamma^0 \) is diagonal. As for \( r^n_a \)'s, they determine how the potentials behave:

\[ \tilde{\kappa}_\mu^a = \kappa_\mu^1 r_1^a + \kappa_\mu^2 r_2^a + \kappa_\mu^3 r_3^a, \quad \text{and} \]
\[ |\varepsilon_1|^2 + |\varepsilon_2|^2 + |\varepsilon_3|^2 = 1, \quad a = \{1, 2, 3\}. \]  \hspace{1cm} (3.6) (3.7)

To see how they are boosted, we treat a prototypical case - that of a boost in the \( x^3 \)-direction. Specifically,

\begin{align*}
\nabla_0 &= \varepsilon_0^0(0)\partial_0 + \varepsilon_0^3(0)\partial_3 + i\kappa_0(0)K_3, \\
\nabla_3 &= \varepsilon_3^0(0)\partial_0 + \varepsilon_3^3(0)\partial_3 + i\kappa_3(0)K_3, \\
\nabla_1 &= \partial_1, \\
\nabla_2 &= \partial_2.
\end{align*}  \hspace{1cm} (3.8) (3.9) (3.10) (3.11)

We look for solutions of

\[ (i\gamma^\mu \nabla_\mu - m)\Psi = 0, \]  \hspace{1cm} (3.12)

modelled on the free spinors

\[ \Psi(\alpha) = s(\alpha)e^{-i(p_0 x^0 + p_3 x^3)}, \]  \hspace{1cm} (3.13)

subject to the relativistic impulse condition \( p_0^2 - p_3^2 = m^2 \). In the standard representation

\[ \gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix}, \]  \hspace{1cm} (3.14)

the equation (3.12) yields the following matrix:

\[ \begin{bmatrix}
\varepsilon_0(\alpha) - m(\alpha) & 0 & -\varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 \\
0 & \varepsilon_0(\alpha) - m(\alpha) & 0 & \varepsilon_3(\alpha) + \kappa_0(\alpha) \\
\varepsilon_3(\alpha) - \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha) & 0 \\
0 & -\varepsilon_3(\alpha) + \kappa_0(\alpha) & 0 & -\varepsilon_0(\alpha) - m(\alpha)
\end{bmatrix}. \]  \hspace{1cm} (3.15)
where the entries are
\[
\begin{aligned}
\varepsilon_0(\alpha) &= \varepsilon_0^0(\alpha)p_0 + \varepsilon_0^3(\alpha)p_3, \\
\varepsilon_3(\alpha) &= \varepsilon_3^0(\alpha)p_0 + \varepsilon_3^3(\alpha)p_3, \\
m(\alpha) &= m + \kappa_3(\alpha).
\end{aligned}
\]
(3.16) (3.17) (3.18)

Its rank has to be 2 for all values of \(\alpha\), thus constraining \(\kappa_0(\alpha)\) and \(\kappa_3(\alpha)\):
\[
\varepsilon_0^2(\alpha) - \varepsilon_3^2(\alpha) = (m + \kappa_3(\alpha))^2 - \kappa_0^2(\alpha).
\]
(3.19)

Evidently \(\kappa_\mu^a(\alpha)\)'s are not identically zero. At the same time, \(\kappa_\mu^a(0) = 0\), \(\forall \mu = \{0, 1, 2, 3\}\). Hence, a boost entails a nonlinear change in the potentials.

Finally, we are in a position to deal with supersymmetry algebras. For the reminder of this section, the impulse operators and all other quantities expressly depend on the parameters introduced in the proof of Theorem 2.1. For convenience, we bundle them into one complex parameter \(z\) via stereographic projection, so that \(K_\mu^a(0) = K_\mu^a\), \(Q_\mu(1), \bar{Q}_\mu(1)\). Adding central charges \(Z_m, Z_m^*\) on the right-hand side would not remedy the situation because these charges commute with the symmetry group generators.

\section{The Relativistic Aharonov-Bohm Effect}

The diffeomorphism between \(G/SU_D(2)\) and \(SO(4)/SO(3)\) established in \(2.10\) induces a vector space isomorphism \(\varphi^*\), taking \(K_i\)'s into the matrices
\[
\begin{bmatrix}
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & 0 & 0 \\
0 & -i & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{bmatrix}.
\]
forming a subspace of \(\mathfrak{so}(4)\). \(\varphi^*\) comes in handy for differentiating vector potentials. Our operators \(\nabla_\mu\) become
\[
\nabla_\mu^\nu \overset{\text{def}}{=} \varepsilon_\mu^\nu \partial_\nu + i\kappa_\nu^a(\varphi^*(K_a)).
\]
(4.1)

Consequently, \(A_\mu\)'s transform via \(A_\mu \mapsto A_\mu'\) such that
\[
\nabla_\mu^\nu A_\nu' = \partial_\mu A_\mu^a A_\mu.
\]
(4.2)
Here $\Lambda^\eta_\eta$ designates a pure Lorentz boost in the direction determined by $\kappa^a_\mu K_a$. Clearly, (4.2) is the only possible way to maintain the Lorentz covariance of the electromagnetic field. But the behavior of $A_\mu$’s is not subject to any other constraints. There are plenty of vector potentials that satisfy (4.2), yet enjoy some freedom: $A'_\mu \neq \Lambda^\eta_\mu A_\eta$.

Now consider the setting of the Aharonov-Bohm experiment [13]. If performed on the ground and on the aircraft moving fast enough to make time slowing detectable, the difference in phase shifts $(\Delta \varphi)_v = 0 - (\Delta \varphi)_{v \neq 0}$ compared with the theoretical values computed using the two transformation laws (G versus Spin(1,3)) would settle the question of which law better describes Nature within the given energy range.

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