(1+1)-dimensional formalism and quasi-local conservation equations

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A set of exact quasi-local conservation equations is obtained in the (1+1)-dimensional description of the Einstein’s equations of (3+1)-dimensional spacetimes. These equations are interpreted as quasi-local energy, linear momentum, and angular momentum conservation equations. In the asymptotic region of asymptotically at spacetimes, it is shown that these quasi-local conservation equations reduce to the conservation equations of Bondi energy, linear momentum, and angular momentum, respectively. When restricted to the quasi-local horizon of a generic spacetime, which is defined without referring to the infinity, the quasi-local conservation equations coincide with the conservation equations on the stretched horizon studied by P. rice and Thorne. All of these quasi-local quantities are expressed as invariant two-surface integrals, and geometrical interpretations in terms of the area of a given two-surface and a pair of null vector fields orthogonal to that surface are given.

I. INTRODUCTION AND KINEMATICS

For the past few decades, there has been enormous progress in general relativity since the pioneering works of the late Professor A. Lichnerowicz on mathematical relativity. His contributions to general relativity are diverse as well as profound, not least because he put the Einstein’s equations on a firm mathematical foundation as partial differential equations and theories of connections. Connections also play important roles in Yang-Mills gauge theories, since gauge theories are nothing but theories of connections coupled to matter fields. Therefore, in this International Conference commemorating Professor A. Lichnerowicz, it seems appropriate to discuss a relatively unknown formalism of general relativity, which is based on the very idea of connections.

This note is about (1+1)-dimensional description of general relativity of (3+1)-dimensional spacetimes, treating the remaining 2-dimensional spatial dimensions as a bire space. In this formalism, all the notions in the theory of bire bundles such as a bire space, connections, and the structure group appear naturally. Instead of going into the details of the formalism itself, however, I will describe the key ideas briefly, mainly to x the notations, and then quickly move on to discuss issues that are more immediate, namely, the problem of defining the quasi-local conservation equations using the (1+1)-dimensional formalism of (3+1)-dimensional spacetimes.

Let us begin by mentioning a few facts about quasi-local conservation equations. In general relativity, there have been many attempts to obtain quasi-local conservation equations. One of the motivations of these attempts is the expectation that quasi-local conservation equations allow us to predict certain aspects of a quasi-local region of a given spacetime without actually solving the Einstein’s equations for that region. Recall that in the Newtonian theory, the conservation of total momentum immediately follows from Newton’s third law,

\[ F_{\text{total}} = \frac{\partial}{\partial t} \left( \sum_i p_i \right) = 0; \]

which is no more than the consistency condition inherent in Newton’s second law. In general relativity, the consistency conditions for evolution are already incorporated into the Einstein’s equations through the constraint equations, from which global conservation equations were found. In this note we will show that, from the Einstein’s equations in the (1+1)-dimensional description, one can nd conservation equations of a stronger form, namely, quasi-local conservation equations. These equations, which are integral differential equations over a compact two-dimensional space, are naturally interpreted as quasi-local energy, linear momentum, and angular momentum conservation equations.

Let us consider the following line element

\[ ds^2 = 2ududv + 2hdu^2 + e_{ab} dy^a + A^a du + A^a dv \]

where + stands for u, v, respectively. To understand the geometry of this metric, it is convenient to introduce the following vector fields,

\[ \theta^a = \delta^a_0 = \delta^a_0 \]

\[ \phi^a = \delta^a_0 \]

which are no more than the consistency condition inherent in Newton’s second law. In general relativity, the consistency conditions for evolution are already incorporated into the Einstein’s equations through the constraint equations, from which global conservation equations were found. In this note we will show that, from the Einstein’s equations in the (1+1)-dimensional description, one can nd conservation equations of a stronger form, namely, quasi-local conservation equations. These equations, which are integral differential equations over a compact two-dimensional space, are naturally interpreted as quasi-local energy, linear momentum, and angular momentum conservation equations.
where we defined the following short-hand notations

\[ \Theta_+ = \frac{\partial}{\partial u}; \quad \Theta = \frac{\partial}{\partial v}; \quad \Theta_a = \frac{\partial}{\partial y^a} (a = 2; 3); \]

(5)

The inner products of the vector fields \( f \Theta \) and \( \Theta_a g \) are given by

\[
\begin{align*}
< \Theta_+ ; \Theta_+ > &= 2h; \quad < \Theta_+ ; \Theta > = 1; \quad < \Theta ; \Theta > = 0; \\
< \Theta ; \Theta_a > &= 0; \quad < \Theta_a ; \Theta_b > = e_{ab};
\end{align*}
\]

(6)

The hypersurface \( u = \text{constant} \) is a constant null hypersurface generated by the outgoing null vector \( e_\Theta \), which is orthogonal to the vector fields \( f \Theta_a g \). Notice that \( v \) is the affine parameter of the outgoing null vector \( e_\Theta \). The hypersurface \( v = \text{constant} \) is generated by the vector \( e_\Theta \), whose norm is \( 2h \), which can be either negative, zero, or positive. The intersection of two hypersurfaces \( u; v = \text{constant} \) defines a spacelike compact two-surface \( N_2 \), which are coordinated by \( y^a \). The metric on \( N_2 \) is decomposed into the area element \( e \) and the conformal two-metric \( ab \), which is normalized to have a unit determinant

\[ \det_{ab} = 1; \]

(7)

In the terminology of the brane bundles, the base manifold is the \((1+1)\)-dimensional space-time coordinated by \((u; v)\), and the brane space is 2-dimensional spacelike space \( N_2 \). The vector fields \( f \Theta_a g \), which are orthogonal to \( f \Theta_a g \), is the horizontal vector \( e_\Theta \), and \( f \Theta_a g \) is tangent to the brane space \( N_2 \). The fields \( A^a \) are the corresponding connections valued in the \( 3 \)-form \( \omega \) of the two-surface \( N_2 \).

For later use, we shall write down the future-directed in-going null vector \( e_\Theta n \) and outgoing null vector \( e_\Theta l \), orthogonal to two-surface \( N_2 \) at each spacetime point. They are given by

\[
\begin{align*}
n &= \Theta_+; \quad h \Theta_+; \\
l &= \Theta; \quad 1 = \Theta;
\end{align*}
\]

(8)

and are normalized such that

\[ < n; l > = 1; \]

(9)

If we further assume that \( A^a = 0 \), then the metric \( \frac{1}{2} \) becomes identical to the metric studied in \([12]\). In this note, however, we shall retain the \( A^a \) fields, since its presence will make the \( N_2 \)-di emorphism invariant Yang-Mills type gauge theory aspect of this formalism transparent. Apart from the \( N_2 \)-di emorphism invariance, there are other residual symmetries that preserve the metric \( \frac{1}{2} \), which are the rescaling transformation of \( u \), and the transformation that shifts the origin of the affine parameter \( v \) at each point of \( N_2 \).

The complete set of the vacuum Einstein’s equations are found to be

\[
\begin{align*}
(a) & \quad e D + e D + e D + 2e (\Theta_+ )D + 2e (\Theta h)D + \frac{1}{2} \theta^{ab} F_{ab} = 0; \\
(b) & \quad e D + e D + e D + 2e (\Theta_+ )D + 2e (\Theta h)D + 2he (\Theta h)D + \frac{1}{2} \theta^{ab} F_{ab} = 0; \\
(c) & \quad 2e (\Theta^2 ) + e (\Theta h)^2 + \frac{1}{2} \theta^{ab} F_{ab} = 0; \\
(d) & \quad D + \theta^{ab} F_{ab} = 0; \\
(e) & \quad D + \theta^{ab} F_{ab} = 0;
\end{align*}
\]

(11)
\[ +2e_{a(b)h} + e_{bcde}(D_{bd})(\theta_{a_c}e) + 2e_{(a}(D_{b)h}) 2\theta_{ab}c_{bcD} = 0; \quad (15) \]
\[ 2e D^2 + 2e D h(\theta_{a}e) + e D a + e D(D_+ + e(D_+)(D_+)) \]
\[ +\frac{1}{2}e_{abcd}(D_+ + e(D_{ab})(D_{bd})) + e_{ab}F_+ c_{F_+} \]
\[ +\frac{1}{2}e_{ab}(D_{ac})(D_{bd}) = 0; \quad (16) \]
\[ \frac{1}{2}e D_+ + D_{ab} + D_{ab} + \frac{1}{2}e_{cd}(D_{ac})(D_{bd} + D_{bc})(D_{ab}) + (D_{bc})(D_{ab}) \]
\[ + e(D_{ab})(D_+ + (D_+)(D_+) = 0; \quad (17) \]

Here \( R_2 \) is the scalar curvature of \( N_2 \), and we defined the \( N_2 \)-covariant derivatives as follow:
\[ F_{+,a} = \theta_{A-a} A_{+a} A_{-a}^{n} (A_{-a})_{-a} \]
\[ D = \theta_{A-a} A_{-a} \]
\[ D_{h} = \theta_{h} A; \]
\[ D_{ab} = \theta_{ab} A; \]
\[ \theta_{ab} = \theta_{ab} A; \]

The bracket \( [A; f]_{ab} \) is the Lie derivative of \( f_{ab} \) along the vector \( \theta_{ab} \).

One can also compute the scalar curvature \( R \) of the metric \( g_{\mu\nu} \) and integrate it over spacetime. It is given by
\[ Z \]
\[ Z = \int du dv d^2 y \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 \int_0 L_0 + \text{surface integrals}; \quad (23) \]

where the \( \text{Lagrangian} \) function \( L_0 \) is given by
\[ L_0 = \frac{1}{2}e_{abcd}F_+ c_{F_+} + e(D_{ab})(D_{bd}) + \int_0 e_{abcd}(D_{ac})(D_{bd}) + \int_0 e_{abcd}(D_{ac})(D_{bd}) \]
\[ 2e(D_{ab})(D_+) = 0. \quad (24) \]

One can easily recognize that this \( \text{Lagrangian} \) function \( L_0 \) is in form of a \((1+1)\)-dimensional \( \theta \)-theory Lagrangian. In geometric terms, the function \( L_0 \) describes how the \((1+1)\)-dimensional spacetime 2-dimensional space is imbedded into an enveloping \((3+1)\)-dimensional spacetime. Each term in \( (24) \) is manifestly \( N_2 \)-invariant, and the \( y^a \)-dependence of each term is completely \( \text{hidden} \) in the Lie derivatives. In this sense we may regard the \( (3+1) \)-dimensional space as in a Yang-Mills theory, with the \( N_2 \)-dimensional group \( \text{diff} \) of \( N_2 \) as the Yang-Mills gauge symmetry. Thus, the above function \( L_0 \) is describable as a \((1+1)\)-dimensional Yang-Mills type gauge theory interacting with \((1+1)\)-dimensional scalar \( \theta \) and \( \theta \)-non-linear sigma models of generic types.

**II. A SET OF QUASI-LOCAL CONSERVATION EQUATIONS**

Notice that the four equations \((11), (12), (13)\) are partial differential equations that are \( r \)-order in \( D \) derivatives. Therefore it is of particular interest to study these four equations, since they are close analogues to the Einstein's constraint equations in the usual \((3+1)\) formalism. Thus, in this formalism, the natural vector \( \theta \)-field...
that denotes the evolution is $D_q$. Then the momenta $i = f_h$: $a; abg$ conjugate to the configuration variables $q_i^2 = fh$: $A^a_i; abg$ are defined as

$$i = \frac{\partial L_0}{\partial \dot{\theta} q_i^2}.$$  

They are found to be

$$h = 2e (\partial \partial_h + 2he (\partial \partial_h) + e (\partial_h),$$

$$a = e^2 abF_{bc}^b;$$

$$ab = he ac bd (\partial \partial_{cd}) \frac{1}{2} e ac bd (\partial_{cd}).$$

Notice that $ab$ is traceless

$$a^a = 0;$$

due to the identities that are direct consequences of the condition (7),

$$abD_{ab} = 0;$$

The Hamiltonian function $H_0$ defined as

$$H_0 = \frac{1}{2} D_q^2 L_0$$

is found to be

$$H_0 = H + \text{total divergences}$$

where $H$ is given by

$$H = \frac{1}{2} e h + \frac{1}{4} he \frac{1}{2} e h^2 \frac{1}{2} e^2 ab a b + \frac{1}{2h} e ac bd ab cd$$

$$+ \frac{1}{2h} (\partial_h) + \frac{1}{2h} ab (\partial_h) + \frac{1}{2h} e cd (\partial_h) + e R_2.$$  

In terms of these canonical variables $f_i; q_i^2$, the first-order equations (25), (26), and (27) can be written as, after a little algebra,

$$abD_{ab} + D_{ab} \quad hD_{ab} h + 2e D_{ab}$$

$$+ \theta_a h \quad A^a_i + 2A^a_i e D_{ab} + 2he ab b + 2 ab \theta_a h = 0;$$

$$\theta_a h + \theta_a A^a_i \quad h + e ab b = 0;$$

$$\theta_a h \theta_a h \quad \theta_a + \theta_a h b c \theta_a b c$$

$$+ \theta_b (b c \theta_a) + \theta_b (b c \theta_a) \theta_a (b c \theta_a) = 0;$$

These are the four first-order equations in the gauge (32), and it is these equations that we are concerned with in this note. Notice that the equations (33) and (34) are divergence-type equations. If we contract the equations (32) by an arbitrary function $\theta_i$ of $h$, $\theta_i^2$ such that

$$\theta_i a = 0;$$

then the resulting equation is also a divergence-type equation,

$$abS_{ab} + S + h S + A^a_i A_{ab} \theta_a (a) + \theta_a a + 2 ab b c b c + A^a_i b b = 0;$$

where $S$ is the Lie derivative along the vector

$\text{a} = \theta_a.$
The integrals of these equations over a compact two-surface \( N \) become, after the normalization by \( \ell = 16 \),

\[
\frac{\partial}{\partial u} \theta_{uv}(a \nu) = \frac{1}{16} \int d^2 y \, ab_{D+} + D_+ \, hD_+ \, h; \\
\frac{\partial}{\partial u} \theta_{uv}(a \nu) = \frac{1}{16} \int d^2 y \, H; \\
\frac{\partial}{\partial u} L(a \nu) = \frac{1}{16} \int d^2 y \, ab_{\pi} + \pi \, h\pi + A_i \pi \quad (\pi^a = 0); 
\]

where in the last integral we used the fact that

\[
\int I d^2 y f = \int d^2 y \theta_{a} (a f) = 0
\]

for a scalar density \( f \) with the weight \( 1 \). Here \( \theta_{uv}(a \nu), \theta_{uv}(a \nu) \), and \( L(a \nu) \) are invariant two-surface integrals defined as

\[
U(a \nu) = \frac{1}{16} \int d^2 y \, h + 2e D_+ + U; \\
P(a \nu) = \frac{1}{16} \int d^2 y \, H + P; \\
L(a \nu) = \frac{1}{16} \int d^2 y \, (\pi) + L \quad (\pi^a = 0);
\]

where \( U, P, \) and \( L \) are undetermined subtraction terms. Notice that these subtraction terms must be \( u \)-independent,

\[
\frac{\partial U}{\partial u} = \frac{\partial P}{\partial u} = \frac{\partial L}{\partial u} = 0; 
\]

in order to satisfy the equations \((40), \(41), \) and \((42)\), respectively. In general the subtraction terms are not unique, and the "right" subtraction term may not even exist at all in a generic situation. One natural criterion for the "right" choice of subtraction term would be that it must be chosen such that the quasi-local physical quantities reproduce "standard" values in the well-known limiting cases.

One can write the r.h.s. of the equation \((42)\) in a more symmetric and suggestive form as follows. To do this, let us contract the equation \((42)\) with \( A_{\pi} \) and integrate over \( N \) to obtain the following equation

\[
\int d^2 y \, A_{\pi} \phi_{\pi} = \int d^2 y \, ab_{\pi} \phi_{\pi} + \phi_{\pi} \, h\phi_{\pi} + A_i \phi_{\pi} \quad (\phi^a = 0);
\]

If we use the definition of \( d N \)-covariant derivatives \( D \) and the equation \((43)\), then the equation \((44)\) can be written as

\[
\frac{\partial}{\partial u} U(a \nu) = \frac{1}{16} \int d^2 y \, ab_{\pi} + \phi_{\pi} \, h\phi_{\pi} + A_i \phi_{\pi} \quad (\phi^a = 0);
\]

where the integrand on the r.h.s. assumes the canonical form of energy-\( \tau \)-flux, which is typically given by

\[
T_0^i \quad \phi_{\pi}^i;
\]

where \( i \) is a generic \( \pi \) and \( i \) is its conjugate momentum. Notice that the r.h.s. of the conservation equations \((42)\) and \((43)\) match exactly, if we interchange the derivatives in the integrands

\[
\int \phi_{\pi}^i \phi_{\pi}^i;
\]

In a region of a spacetime where \( \theta = \phi_{\pi} \) is timelike, these quasi-local equations become quasi-local conservation equations, which relate the instantaneous rates of changes of two-surface integrals at a given \( u \)-time to the associated net \( \tau \)-flux integrals. Let us remark that, unlike the Tamburino-Winicour's quasi-local conservation equations \((42)\) which are "weak" conservation equations since the R-local conditions (i.e., the full vacuum Einstein's equations) were assumed in their derivation, our quasi-local conservation equations are "strong" conservation equations since only the fourth-order equations were used in the derivation.
It is interesting to notice that we can obtain yet another quasi-local conservation equation. This is simply achieved by writing the equation (48) as
\[ \frac{\partial}{\partial u} \int d^2y \left( \frac{a}{a} \right) = \int d^2y \left( a^b A^a - a^b A^a + \frac{1}{2} a^b A^a + \frac{1}{2} a^b A^a \right); \] (52)
which relates the instantaneous u-derivative of the two-surface integral on the l.h.s. to the net flux integral on the right. However, the r.h.s. of this equation is not quite \"canonical\" due to the last term. If we restrict the \( A^a \) such that it satisfies the u-independent condition
\[ a, A^a = 0; \] (53)
which is essentially the same condition (33) that \( a \) satisfies, then the last term in the r.h.s. of the equation (52) drops out, and we obtain the following equation
\[ \frac{\partial}{\partial u} J(u; v) = \frac{1}{16} \int d^2y \left( a^b A^a - a^b A^a + \frac{1}{2} a^b A^a + \frac{1}{2} a^b A^a \right); \] (54)
Here \( J(u; v) \) is defined as
\[ J(u; v) = \frac{1}{16} \int d^2y \left( a^b A^a - a^b A^a + \frac{1}{2} a^b A^a + \frac{1}{2} a^b A^a \right); \] (55)
where \( J \) is an undetermined subtraction term. The r.h.s. of the equation (54) now represents a flux of the canonical form
\[ X_i \left( a^b A^i - a^b A^i + \frac{1}{2} a^b A^i + \frac{1}{2} a^b A^i \right); \] (56)
just as the r.h.s. of the equations (42) and (49) do.

III. GEOMETRICAL INTERPRETATIONS

Remarkably, the two-surface integrals (44), (45), (46), and (53), which were derived using the metric (2), can be expressed geometrically, in terms of the area of the two-surface and a pair of in-going and out-going null vector fields orthogonal to that surface. In order to show this, we need to invoke the definitions of in-going and out-going null vector fields in; let us denote in the section I.

A. Quasi-local energy

Let us first observe that the integral in (44) can be written as the Lie derivative of the scalar density \( e \) along the in-going null vector field \( n \),
\[ \int d^2y \left( h + 2e D_+ \right) = 2 \int d^2y e \; D_+ \; hD \]
\[ = 2 \int d^2y \; n \; e; \] (57)
One finds that
\[ \int d^2y \; n \; e = \int d^2y \; e; \] (58)
where \( A \) is the area of \( N_2 \),
\[ A = \int d^2y \; e; \] (59)
The identity (58) follows trivially from the observation that the null vector \( e_i \) is out of (in fact, orthogonal to) the two-surface, which means that the order of the integration over \( d^2y \) and the Lie derivative \( \delta_n \) in (57) is interchangeable. Thus we have

\[
\frac{1}{16} \int d^2y \ h + 2e \ D_+ \ = \ \frac{1}{8} \delta_n A : \tag{60}
\]

For a reference term \( U \), let us choose

\[
U \ = \ \frac{1}{8} \delta_n A ; \tag{61}
\]

where \( n \) is a future-directed in-going null vector \( e_i \) of a background reference spacetime \( ds^2 \) into which the two-surface \( N_2 \) (with the same metric \( e_{ab} \)) is embedded,

\[
ds^2 = 2du dv \quad 2hdu^2 + e_{ab} \ dy^a + A_a^4 du + A^a dv \quad dy_b + A_b^4 du + A^b dv ; \tag{62}
\]

Notice that \( n \), which is given by

\[
n = \frac{\theta}{\partial u} A_a^4 \ \frac{\theta}{\partial y^a} h \ \frac{\theta}{\partial v} A^a \ \frac{\theta}{\partial y^a} ; \tag{63}
\]

is a function of \( h \) and \( A^a \) only, the embedding degrees of freedom of the two-surface. Thus, the quasi-local energy of a given two-surface \( N_2 \) is defined relative to some fixed background reference spacetime, and is zero when the two-surface under consideration is embedded into the fixed background reference spacetime, i.e., when

\[
n = n : \tag{64}
\]

Therefore, the quantity \( U(u;v) \), which will be interpreted as the quasi-local energy, becomes

\[
U(u;v) = \frac{1}{8} \delta_n A + \frac{1}{8} \delta_n A ; \tag{65}
\]

It is given by the rate of change of the area of a given two-surface along the future-directed in-going null vector \( e_i \), relative to some fixed background null vector \( e_i \). Notice that this definition is entirely geometrical, referring to the area of a given two-surface and the orthogonal null vector \( e_i \) (and \( n \) only).

### B. Quasi-local linear momentum

The two-surface integral (49) can be also written geometrically in a similar way. It becomes

\[
\frac{1}{16} \int d^2y (h) = \frac{1}{8} \int d^2y e D = \frac{1}{8} \int d^2y \delta_1 = \frac{1}{8} \delta : \tag{66}
\]

Therefore, if we choose the reference term \( P \) as

\[
P = \frac{1}{8} \delta_A ; \tag{67}
\]

where \( l \) is a future-directed out-going null vector \( e_i \) of a background reference spacetime, then \( P \) becomes,

\[
P(u;v) = \frac{1}{8} \delta_A + \frac{1}{8} \delta_A ; \tag{68}
\]

Thus, the quasi-local integral \( P(u;v) \), which is to be interpreted as the quasi-local linear momentum, is given by the rate of change of the area of a given two-surface along the future-directed out-going null vector \( e_i \) relative to \( l \).
C. Quasi-local angular momentum

Let us also write down the two-surface integral \((\mathcal{L}_4)\) in a geometrical way. Notice that the Lie bracket of the two null vector fields \(F, h\) is given by

\[ [F, h] = \nabla h \cdot F, \quad \nabla h = \partial h \]

(69)

Thus, \((\mathcal{L}_4)\) becomes

\[
\frac{1}{16} \int d^2y (\begin{pmatrix} a \\ a \end{pmatrix}) = \frac{1}{16} \int d^2y \begin{pmatrix} a \\ a \end{pmatrix} F_+ \begin{pmatrix} a \\ a \end{pmatrix} = \frac{1}{16} \int d^2y \begin{pmatrix} a \\ a \end{pmatrix} h; [h]\\
(70)
\]

where

\[ a = \frac{b^c}{a^c}, \quad b = e^{ab^b}; \]

(71)

This shows that \((\mathcal{L}_4)\) is an invariant two-surface integral of the Lie bracket \([h]\\); \([h]\\) projected to the spacelike vector \(e^a = \nabla_a h\). If we choose the reference term \(L\) as

\[ L = \frac{1}{16} \int d^2y \begin{pmatrix} a \\ a \end{pmatrix} h; [h]\\
(72)
\]

then \((\mathcal{L}_4)\) becomes

\[
L (u; v; ) = \frac{1}{16} \int d^2y \begin{pmatrix} a \\ a \end{pmatrix} h; [h] + \frac{1}{16} \int d^2y \begin{pmatrix} a \\ a \end{pmatrix} h; [h] = 0; \]

(73)

It must be stressed \(a\) is an arbitrary function of \(uv; y^b\). In particular, it need not satisfy any Killing’s equations associated with isometries of a spacetime, or of a two-surface within a given spacetime. The quantity \(L (u; v; )\), a linear functional of \(a\), can be interpreted as the quasi-local angular momentum of a two-surface associated with an arbitrary function \(a\) of \(y^b\), as we will see later.

D. Quasi-local Carter’s constant

Likewise, the fourth integral \((\mathcal{L}_4)\) can be written as

\[
J (u; v) = \frac{1}{16} \int d^2y e^{a} \begin{pmatrix} a \\ a \end{pmatrix} h; [h] + \frac{1}{16} \int d^2y e^{a} \begin{pmatrix} a \\ a \end{pmatrix} h; [h] = 0; \]

(74)

which may be interpreted as a quasi-local, finite, analog of the Carter’s ‘fourth’ constant, as we shall see in the next section.

IV. Asymptotically flat limits

The equations \((\mathcal{L}_4)\), \((\mathcal{L}_4)\), and \((\mathcal{L}_4)\) turn out to be quasi-local energy, linear momentum, and angular momentum conservation equations, respectively \((\mathcal{L}_4)\), and the equation \((\mathcal{L}_4)\) is interpreted as a quasi-local conservation equation of the generalized Carter’s constant \((18, 19, 20)\). In this section we shall evaluate the two-surface integrals \((\mathcal{L}_4)\), \((\mathcal{L}_4)\), \((\mathcal{L}_4)\), and the associated \(v\) integrals in the limiting asymptotically flat region where \(N_2 = S_2\), and show that they all reduce to the well-known Bondi energy, linear momentum, angular momentum, and the corresponding \(v\) integrals defined at the null infinity. The asymptotic form of the ‘fourth’ integral \((\mathcal{L}_4)\) at the null infinity will be also computed, and it will be shown that it is proportional to the total angular momentum squared.

In the limit where the above parameter \(v\) approaches infinity, the asymptotic form of the Kerr metric becomes

\[
ds^2 = 2dudv \left( 2m + \frac{2m}{v^2} + \frac{4m \sin^2 \theta}{v^2} + \frac{4m a^3 \sin^2 \theta \cos^2 \theta}{v^2} + \frac{dud'}{v^2} \right) \\
+ \left( 1 + \frac{a^2 \cos^2 \theta}{v^2} + \frac{d\theta^2 + v^2 \sin^2 \theta}{1 + \frac{a^2}{v^2} + d^\prime} + \frac{\sin^2 \theta}{v^4} + \frac{8m a^3}{v^4} + dv^2 \right)
\]

(75)
where \( \theta = \theta u \) is asymptotic to the timelike Killing vector \( \xi \) at \( u \to \infty \). The asymptotic fall-off rates of the metric coefficients can be read off from the above metrics \[1,2,3,4,5,21,22,23,24,25\]:

\[
\begin{align*}
\eta &= v^2 \sin^2 \theta + O \left( \frac{1}{v^2} \right) ; \\
\# &= 1 + \frac{C_\theta (\#')}{v} + O \left( \frac{1}{v^2} \right) ; \\
\mathcal{C} &= (\sin \#) \left( \frac{C_\theta (\#')}{v} + O \left( \frac{1}{v^2} \right) \right) ; \\
\mathcal{D} &= O \left( \frac{1}{v^2} \right) ; \\
2h &= \frac{2m}{v} + O \left( \frac{1}{v^2} \right) ; \\
A'_+ &= \frac{2m a}{v} + O \left( \frac{1}{v^2} \right) ; \\
A' - &= \frac{2m a^3}{v^2} + O \left( \frac{1}{v^4} \right) ; \\
A' # &= O \left( \frac{1}{v^2} \right) ;
\end{align*}
\]

From these asymptotic behaviors, we can deduce the fall-off rates of the following derivatives,

\[
\begin{align*}
\theta_\nu &= O \left( \frac{1}{v^2} \right) ; \\
\theta &= \frac{2}{v} + O \left( \frac{1}{v^2} \right) ; \\
\theta_{, ab} &= O \left( \frac{1}{v^2} \right) ; \\
\theta_{, ab} &= O \left( \frac{1}{v^2} \right) ; \\
\theta_{, ab} &= O \left( \frac{1}{v^2} \right) ; \\
\theta_{, ab} &= O \left( \frac{1}{v^2} \right) ; \\
\mathcal{D} &= O \left( \frac{1}{v} \right) ; \\
\mathcal{D} &= O \left( \frac{1}{v^2} \right) ; \\
\mathcal{D} &= O \left( \frac{1}{v^4} \right) ; \\
\mathcal{D} &= O \left( \frac{1}{v^2} \right) ;
\end{align*}
\]

A. The Bondi energy-loss relation

Since the integrand of the r.h.s. of \(45\) assumes the typical form of energy-ux, we expect that it represents the energy-ux carried by gravitational radiation crossing \( S_2 \). Then the l.h.s. of \(45\) should be the instantaneous rate of change of the gravitational energy of the region enclosed by \( S_2 \). The energy-ux integral in general does not have a definite sign, since it includes the energy-ux carried by the in-going as well as the out-going gravitational radiation. But in the asymptotically at region, the energy-ux integral turns out to be negative definite, representing the physical situation that there is no in-coming ux coming from the infinity.

Let us now show that the equation \(45\) reduces to the Bondi energy-loss formula \(17\) in the asymptotically at spacetimes. To show this, let us first calculate \( U \left( u; v \right) \) in the limit \( v \to 1 \). Since the nullvector \( \xi \) is n and l asymptotically approach to

\[
\begin{align*}
n ! \theta, \quad \frac{1}{2} \frac{m}{v} \theta ; \\
l ! \theta ;
\end{align*}
\]

the natural background spacetime is the at spacetime so that the embedding degrees of freedom are given by

\[
A^a = 0; \quad 2h = 1.
\]

That is, the background \( \xi \) is n and l become

\[
n = \theta, \quad \frac{1}{2} \theta ; \quad l = \theta ;
\]

Then it follows trivially that the total energy at the null infinity coincides with the Bondi energy \( U_B \left( u \right) \),

\[
\lim_{v \to 1} U \left( u; v \right) = U_B \left( u \right) = \frac{m}{2};
\]
where \( m \) is the Bondi mass of asymptotically at spacetimes. One can further show that the equation (43) is just the Bondi energy-loss formula,

\[
\frac{d}{du} U_B(u) = \lim_{v \rightarrow 1} \frac{1}{32} \int_{s_1} v^2 \, ab \, cd \, \langle \emptyset^+_{ac} \emptyset^+_{bd} \rangle; \tag{89}
\]
or, equivalently,

\[
\frac{d}{du} U_B(u) = \frac{1}{16} \int_{s_2} d \langle \emptyset, C \rangle^2; \tag{90}
\]

where we used the expressions (72) and (73). Notice that the negative-definite energy-ux is a bilinear of the traceless current \( j^a_b \) defined as

\[
j^a_b = a^c \emptyset^+_{bc} (j^a_0 = 0); \tag{91}
\]

representing the shear degrees of freedom of gravitational radiation.

B. The Bondi linear momentum and linear momentum-ux

Let us now evaluate \( P(u; \nu) \) in (53) and the corresponding quasi-local momentum-ux integral in the asymptotic region of asymptotically at spacetimes. We find that the total linear momentum \( P(u; \nu) \) becomes zero in the asymptotic limit,

\[
\lim_{v \rightarrow 1} P(u; \nu) = P_B(u) = 0; \tag{92}
\]

from which we infer that the total momentum-ux is zero,

\[
\frac{d}{du} P_B(u) = 0; \tag{93}
\]

The result (93) can be also obtained by evaluating each term in the Hamiltonian function \( H \) (44). To evaluate the momentum-ux term by term, let us notice that the fourth and the seventh term in (54), which are non-zero individually, add up to zero asymptotically,

\[
\frac{1}{2} e \int_{s_1} \frac{1}{v^2} + \frac{1}{8} e \int_{s_1} \frac{1}{v^2} = \frac{1}{2} e \int_{s_1} \frac{1}{v^2}; \tag{94}
\]

where we used the definition (25) of \( ab \). All other non-vanishing terms are given by

\[
\lim_{v \rightarrow 1} \frac{1}{16} \int_{s_2} d^2 \psi \frac{1}{4} e \int_{s_2} h = \frac{1}{2}; \tag{95}
\]

\[
\lim_{v \rightarrow 1} \frac{1}{16} \int_{s_2} d^2 \psi \frac{1}{4} e \int_{s_2} h = 1; \tag{96}
\]

\[
\lim_{v \rightarrow 1} \frac{1}{16} \int_{s_2} d^2 \psi \frac{1}{4} e \int_{s_2} h = \frac{1}{4}; \tag{97}
\]

where \( = 2 \) for a two-sphere \( S_2 \). Therefore we have

\[
\frac{d}{du} P_B(u) = 0; \tag{98}
\]

C. The Bondi angular momentum and angular momentum-ux

The total angular momentum at the null infinity is naturally defined as the limiting value of the general quasi-local angular momentum \( L(u; \nu) \) in (74),

\[
\lim_{v \rightarrow 1} L(u; \nu) = L_B(u); \tag{99}
\]
Since the background fields $n$ and $l$ in (37) commute,
\[
\{ n; l \} = 0;
\]
(100)
it follows that
\[
L_B (u; l) = 0
\]
(101)
for all $l$. Let be asymptotic to the azimuthal Killing vector field of the Kerr spacetime such that
\[
\phi = ^a \partial_a + ^b \frac{\partial}{\partial \theta};
\]
(102)
Then, we have
\[
L_B (u; ) = \frac{1}{16} Z_{2} Z_{0}^{d'} d^{#} (6m a) \sin^{3} #
\]
\[
= m a;
\]
(103)
which is just the total angular momentum of the Kerr spacetime.
The total angular momentum at the instant $u$ is given by the asymptotically limiting form of the the equation (32), which is
\[
\frac{dL_B}{d u} = \lim_{v \to 1} \frac{1}{16} \int_{S_2} d^2 y \, ab^a_s a + ^b s h_s h A_s = 0;
\]
(104)
Let us evaluate each term in the rhs of this equation. The first term is given by
\[
ab^a_s a = \frac{1}{2} e^{ac bd (\partial_s + \partial_d)} + O (1) s_{ab} = \sin^# (\partial_s C) (\partial C) + O \left( \frac{1}{v} \right);
\]
(105)
so that we have
\[
\int_{S_2} d^2 y \, ab^a_s a = \int_{S_2} d^2 y \, \sin^# (\partial_s C) (\partial C);
\]
(106)
The second term becomes
\[
\frac{n}{v} O s = 2 v \sin^# + O (1) s ;
\]
(107)
Since is asymptotically given by
\[
= 2 \ln v + \ln \sin^# + \ln 1 + O \left( \frac{1}{v^2} \right); \]
(108)
we have
\[
\frac{n}{v} O s = 0 \left( \frac{1}{v^2} \right);
\]
(109)
so that the second term becomes
\[
\int_{S_2} d^2 y \, s = 0 \left( \frac{1}{v} \right) ;
\]
(110)
The third term becomes
\[
h s h = h s h + s (n h) = s \left( 4m \sin^# + h \right) + O \left( \frac{1}{v} \right);
\]
(111)
where we used that
\[ \sin \# = 0; \]  

(112)

Thus we have
\[ \frac{I}{s_2} d^2 y h \sin \# = 0; \]  

(113)

The fourth term is of the order of
\[ A_{s} \sin \# = 0; \]  

(114)

so that
\[ \frac{I}{s_2} d^2 y A_{s} \sin \# = 0; \]  

(115)

If we put together (112), (113), and (114) into (110), then the total angular momentum \( u \) at the null infinity is given by
\[ \frac{dL_B}{du} = \lim_{v \to 1} \frac{I}{32} \frac{1}{s_2} d^2 y \sin \# \left( \frac{1}{v^3} \right) \]  

(116)

or, equivalently,
\[ \frac{dL_B}{du} = \frac{I}{16} \frac{1}{s_2} d \left( \sin \# \right); \]  

(117)

which is precisely the Bondi angular momentum \( u \) at the null infinity.

D. Gravitational Carter's constant

Let us define the asymptotic quantity \( J_B (u) \) as
\[ \lim_{v \to 1} v^3 J (u; v) = J_B (u); \]  

(118)

Because of the equation (110), the subtraction term \( J_B (u) \) is zero,
\[ J_B (u) = 0; \]  

(119)

Then the asymptotic integral \( J_B (u) \) becomes
\[ J_B (u) = 2 \left( m a \right)^2; \]  

(120)

Thus, \( J_B (u) \) is (twice of) the angular momentum squared, deserving the name the gravitational analog of the Carter's "fourth" constant at the null infinity.

V. IN-GOING NULL COORDINATES

One might be also interested in applying this formalism to black holes, and try to obtain quasi-local quantities defined on the black hole horizon and corresponding values incident on that horizon. For this problem, it is appropriate to choose a coordinate system adapted to the in-going null geodesics. Such a coordinate system is described by the metric
\[ ds^2 = + 2udu + 2hdv^2 + e_{ab} dy^a + A_{s} dy^s + A^a dv; \]  

(121)
In this coordinate system, the future-directed out-going and in-going null vector
\( \hat{r}_0 \) and \( \hat{r}_1 \) are given by
\[
\hat{r}_0 = 0 + h\hat{r}_0; \\
\hat{r}_1 = \hat{r}_1; \\
\]
respectively, which are normalized so that
\[
< \hat{r}_0; \hat{r}_1 > = 1.
\]
The inverse relation is given by
\[
\hat{\theta}_+ = n_0 + h\hat{r}_0; \\
\hat{\theta}_- = \hat{r}_1; \\
\]
and the Hamiltonian function \( H \) is given by
\[
H = \frac{1}{2}h + \frac{1}{4}he^2 + \frac{1}{2}h^2 e^{2ab} + \frac{1}{2h}ab + \frac{1}{8h}ab cd ac bd + \frac{1}{2h}D_+ (D_+ + D_. + D_. + D_+) + eR_2.
\]
A new set of quasi-local conservation equations are found to be
\[
\frac{\theta}{\theta u}(u;v) = \frac{1}{16} I d^2 y ab \hat{\theta}_+ + \hat{\theta}_- + h\hat{\theta}_- + A_\alpha \hat{\theta}_+ a a; \\
\frac{\theta}{\theta u}(u;v) = \frac{1}{16} I d^2 y H; \\
\frac{\theta}{\theta L}(u;v) = \frac{1}{16} I d^2 y ab \hat{s}_- + s h\hat{s}_- + A_\alpha \hat{s}_- a a; (\hat{s}_- a = 0); \\
\frac{\theta}{\theta J}(u;v) = \frac{1}{16} I d^2 y ab \hat{s}_- A_\alpha + A_\alpha \hat{s}_- a a; h\hat{s}_- a a (\hat{s}_- a = 0); \\
\]
where \( U(u;v), P(u;v), L(u;v); \), and \( J(u;v) \) are defined as
\[
U(u;v) = \frac{1}{16} I d^2 y (h + 2e D_+) + U; \\
P(u;v) = \frac{1}{16} I d^2 y (h) + P; \\
L(u;v) = \frac{1}{16} I d^2 y (a) + L; \\
J(u;v) = \frac{1}{16} I d^2 y (A_\alpha a) + J; \\
\]
where \( U, P, L, \) and \( J \) are undetermined reference terms as before. Notice that we could also have written the equation
\[
\frac{\theta}{\theta U}(u;v) = \frac{1}{16} I d^2 y ab D_+ + \hat{D}_+ + hD_+ h; \\
\]

\[
(130)
\]

\[
(131)
\]

\[
(132)
\]

\[
(133)
\]

\[
(134)
\]

\[
(135)
\]

\[
(136)
\]

\[
(137)
\]

\[
(138)
\]

\[
(139)
\]
as in \( \mathcal{Q} \). In geometric terms, these quasi-local quantities can be expressed as,

\[
U (u; v) = \frac{1}{8} s_n A + \frac{1}{8} s_n A;
\]

\[
P (u; v) = \frac{1}{8} s_y A + \frac{1}{8} s_y A;
\]

\[
L (u; v) = \frac{1}{16} \int d^2 y e\ \mathbb{A} + \mathbb{L}_e \quad I \quad \frac{1}{16} d^2 y e\ \mathbb{A} + \mathbb{L}_e \quad (\theta_+ v^0 = 0);
\]

\[
J (u; v) = \frac{1}{16} \int d^2 y e^2 \mathbb{A} + \mathbb{L}_e \quad I \quad \frac{1}{16} d^2 y e^2 \mathbb{A} + \mathbb{L}_e \quad (\theta_+ v^0 = 0):
\]

Here \( n^0, \mathfrak{n} \) are future-directed in-going and out-going null vector fields of a background reference spacetime

\[
ds^2 = +2dudv + 2hdu^2 + e_{ab} dy^a + A_{+}^a du + A^a dv + dy^b + A_{+}^b du + A^b dv;
\]

such that

\[
n^0 = \frac{\theta}{\theta u} A_{+}^a \frac{\theta}{\partial y^a} + h \frac{\theta}{\theta v} A^a \frac{\theta}{\partial y^a};
\]

\[
\mathfrak{n}^0 = \frac{\theta}{\theta v} A_{+}^a \frac{\theta}{\partial y^a}:
\]

VI. QUASI-LOCAL HORIZON

Recall that the event horizon is a global concept which is inseparable from the notion of ininity. Therefore, in order to discuss the dynamics of black holes quasi-locally, we have to introduce a new notion of quasi-local horizon, which refers to the quasi-local region only. We define the quasi-local horizon \( \mathcal{H} \) as a three-dimensional hypersurface on which the vector field \( \mathbb{A}_\theta \), which is an arbitrary vector field except that it is asymptotically to the timelike Killing vector at the ininity, has a zero norm,

\[
< \mathbb{A}_\theta, \mathbb{A}_\theta > = 2h = 0:
\]

The location of the quasi-local horizon \( \mathcal{H} \) can be found by solving the equation

\[
h (u; v; y^a) = 0
\]

for \( v \), and the generator of the quasi-local horizon is given by

\[
\mathbb{A}_\theta = \frac{\theta}{\theta u} A_{+}^a \frac{\theta}{\partial y^a};
\]

which is out-going null on \( \mathcal{H} \).

Let us remark a few properties of this quasi-local horizon. First, the quasi-local horizon is defined for generic spacetimes that do not have form eties in general, and its location is not fixed, but varies as much as the choice of the vector field \( \mathbb{A}_\theta \), does. In this sense the quasi-local horizon is not a spacetime invariant but a covariant notion. Even the signature of the vector field \( \mathbb{A}_\theta \), is not determined a priori. In the region where \( \mathbb{A}_\theta \) is non-spacelike, however, the quasi-local horizon may be regarded as a generalization of the Killing horizon, since \( \mathcal{H} \) is defined as the hypersurface where \( \mathbb{A}_\theta \) has a zero norm. For instance, for the Schwarzschild solution, we have

\[
2h = 1 \quad \frac{2m}{v};
\]

so that \( h = 0 \) on the Killing horizon \( v = 2m \). Moreover, since the quasi-local horizon is generated by the out-going null vector fields, all the fluxes crossing the quasi-local horizon are purely in-going, just like the stretched horizon of Price and Thorne.

In this section, we shall delimit our discussions of the quasi-local conservation equations to the quasi-local horizon \( \mathcal{H} \), and that the quasi-local conservation equations on \( \mathcal{H} \) coincides exactly with the quasi-local conservation equations of Price and Thorne [27] defined on the stretched horizon.
A. Surface gravity

In order to discuss the dynamics of quasi-local horizon, it is useful to introduce the notion of the surface gravity to the generic, time-dependent, quasi-local horizon. On the quasi-local horizon $H$ on which $h = 0$, the vector equal to $\theta^\alpha$ becomes a generator of $H$, since we have

$$\mathbf{e} = \frac{\partial}{\partial u}$$

becomes a generator of $H$, since we have

$$\mathbf{e} = 2h_{ij} = 0.$$  \hfill (149)

Hence $r_A (\ \mu)$ is normal to $H$, which means that there exists a function $\theta$ defined on $H$ such that

$$r_A (\ \mu) = 2 \ \theta \ \rho; \ \text{where} \ A \ \text{is a spacetime index such that} \ A = f^+; \ \text{and} \ \rho.$$ \hfill (150)

Notice that this function $\theta$ can be defined on any null hypersurface. When the null hypersurface coincides with the event horizon, this function is the surface gravity of the corresponding black hole. But one may use the same terminology for $H$, since it is a (segment of) null hypersurface beyond which a local observer whose worldline $\theta^\alpha$ has a zero norm on $H$ does not have access to. It may be instructive to notice that, for the K err black hole, $\theta$ is the generator of the event horizon, since it becomes

$$\mathbf{e} = \frac{\partial}{\partial u} \ \text{and} \ \frac{\partial}{\partial \rho}.$$ \hfill (151)

where $\theta^\alpha$ and $\theta^\rho$ are timelike and axial Killing vector fields, and $\rho$ is the angular velocity of the K err horizon relative to the Killing time.

Let us compute on the quasi-local horizon $H$. In the basis $f\theta = \theta^u, \theta^v, \theta^y$, the components of $r_A$ are given by

$$r_A = (1; 0; \ A^a);\ \text{and} \ r_A = g_{ab} \ \rho = (2h_{ij}; 1; 0).$$ \hfill (152)

If we put (152) into (150), then we find that

$$\theta, h_{\mu} = \theta, h_{\nu} = 0;$$ \hfill (153)

and that $\theta$ is given by

$$\theta = D h_{\mu}.$$ \hfill (154)

Notice that, in general, $\theta$ is not constant over $H$ so that

$$\theta, \theta = 0; \ \theta, \phi = 0;$$ \hfill (155)

which reflects the dynamical nature of the quasi-local horizon $H$.

B. Quasi-local energy conservation on $H$

Notice that if we restrict the energy equation (140) to the quasi-local horizon $H$, then it becomes

$$\frac{\partial}{\partial u} U^H = \frac{1}{16} \ D \ h_{i} \ W^1 (\ D^a \ h_{i} \ W^a);$$ \hfill (156)

$$U^H = \frac{1}{8} \ D \ h_{i} \ W^i + U_H.$$ \hfill (157)

This equation is identical to the integral of the following equation

$$\frac{\partial}{\partial u} H + H = \frac{1}{8} \ h^{i} \ W^i + \frac{1}{16} \ h^{ab} \ W^a b;$$ \hfill (158)
over the stretched horizon of Price and Thorne, where the notations are such that

\[ H = \frac{1}{8} H^I, \]
\[ H = D_+; \]
\[ H_{ab} = \frac{1}{2} e D_+ a b; \]
\[ a b_{\alpha \beta} = \alpha \beta D_+ + \alpha \beta_{\alpha \beta}; \]
\[ = D_+ h_{\alpha \beta}; \] 

(159)

This equation was studied in detail in Eq. (6.112,E) in [27].

It is interesting to discuss the limiting case when the quasi-local horizon \( H \) coincides with the event horizon. When this happens, the area \( A_H \) of \( H \) always increases due to the area theorem, so that we have

\[ \frac{dA_H}{du} = I d^2y (e D_+) \; 0; \] 

(160)

Furthermore, if the subtraction term \( U_H \) is chosen zero, then by the equation (157), \( U_H \) is non-positive, and when the black hole no longer expands so that \( D_+ \; \hat{h} = 0 \), then \( U_H \) becomes zero. For instance, for a Schwarzschild or Kerr black hole [27], we have

\[ U_H = 0 \; \text{if} \; U_H \; 0; \] 

(161)

This counter-intuitive aspect is a manifestation of the well-known teleological nature of the event horizon. That is, when the event horizon evolves, its quasi-local energy must be negative so as to cancel out the positive in-\( \times \) energy carried by subsequently in-falling matter or gravitational radiation, leaving \( U_H = 0 \) when the black hole reaches the null stationary state.

C. Quasi-local momentum conservation equation on \( H \)

Let us mention that the momentum equation (133) has a similar structure to the integrated Navier-Stokes equation for a viscous fluid [28],

\[ \frac{\partial P_i}{\partial u} = I d^k p_{ik} + \theta_{V_k} \theta_{0_{ik}}; \] 

(162)

where \( P_i \) and \( \theta_{0_{ik}} \) are the total momentum and the viscous term,

\[ Z \]
\[ P_i = dV (\theta V_i); \] 

(163)

\[ \theta_{0_{ik}} = \frac{\partial v_i}{\partial x^k} + \frac{\partial v_k}{\partial x^i} + \frac{2}{3} \frac{\partial v_i}{\partial x^j} \frac{\partial v_j}{\partial x^i} + \frac{1}{3} \frac{\partial v_i}{\partial x^j} \frac{\partial v_j}{\partial x^j}; \] 

(164)

and \( \theta_{V_k} \) and \( \theta_{0_{ik}} \) are the coefficients of shear and bulk viscosity, respectively. This equation tells us that the rate of the net momentum change of a fluid within a given volume is determined by the net momentum flow across the two-surface enclosing the volume. Notice that the \( \text{Hamiltonian" function} H \) in (133), which is at most quadratic in the conjugate momentum, assumes the form of a momentum-\( \times \) of a viscous fluid. Namely, terms quadratic in \( \theta \), may be viewed as responsible for direct momentum transfer, terms linear in \( \theta \) as viscosity terms, and terms independent of \( \theta \) as pressure terms. From this point of view, one may interpret the \( \text{Hamiltonian" function} H \) as the gravitational momentum \( \times \) and the two-surface integral

\[ P_H = \frac{1}{16} I d^2y (h) + P \] 

(165)

as the quasi-local gravitational momentum within \( N_2 \). On the quasi-local horizon \( H \), the equation (133) becomes,

\[ \frac{\partial}{\partial u} P_H = \frac{1}{16} I d^2y \varepsilon R_2 \; \frac{1}{2} \; \frac{1}{2} e^{\frac{1}{2}} a b \; \frac{1}{2} e^{\frac{1}{2}} h \; \frac{1}{2} h D_+ : \] 

(166)
The first term on the r.h.s. is given by the Euler number,
\[ \frac{1}{16} \int H \ d^2 y \ R_2 = \frac{1}{4}; \] (167)

where \( n = 2 \) for a two-sphere. The second and third terms are quadratic in the momenta, and the last term is linear in the momentum. It is curious that this conservation equation is missing in the work of Price and Thorne. Let us note that, in terms of the conjugation variables, the integrand of the r.h.s. of (166) can be written as
\[ H_H = \alpha R_2 \frac{1}{2} \ e^2 \ abF_{a}^{\ c}F_{c}^{\ b} \ 2e \ D_{\ a} \ R_{a}; \] (168)

D. Quasi-local angular momentum conservation equation on \( H \)

In this section, we shall show that the equation (134) when restricted to the surface \( H \) coincides with the quasi-local angular momentum conservation equation of Price and Thorne on the stretched horizon. Let us first notice that the equation (134) on \( H \) can be written as
\[ \frac{\partial}{\partial u} L_H = \frac{1}{16} \int H d^2 y \ e (2 + D_{\ a} \ R_{a}) + \frac{1}{2} \ e^2 \ ab^{\ cd} (D_{\ a} \ R_{a}) (D_{\ b} \ R_{b}) \ A_{+}^{\ a} \ A_{+}^{\ b}; \] (169)
\[ L_H = \frac{1}{16} \int H d^2 y (\ A_{+}^{\ a}) + L_H; \] (170)

The equation (169) turns out to be identical to the angular momentum conservation equation of Price and Thorne, which is given by
\[ D_{\ a} \ H_{a} + \ H_{a} \ = \ \frac{1}{8} \ \ a \ \ \frac{1}{16} \ H_{a} \ A_{+}^{\ a} + \ \frac{1}{8} \ \ a \ \ b^{\ a} b; \] (171)
\[ H_{a} = \ \frac{1}{16} \ e \ abF_{a}^{\ b}; \] (172)

To show this, let us write the equations (169) and (170) as
\[ \frac{\partial}{\partial u} L_H = \frac{1}{16} \int H d^2 y \ 2e \ S + e \ S (D_{\ a} \ R_{a}) + \frac{1}{2} \ e^2 \ ab^{\ cd} (D_{\ a} \ R_{a}) (D_{\ b} \ R_{b}) + e^2 \ abF_{a}^{\ c}F_{c}^{\ b}; \] (173)
\[ L_H = \frac{1}{16} \int H d^2 y (e^2 \ abF_{a}^{\ c}F_{c}^{\ b}) + L_H; \] (174)

where we used the identity
\[ \frac{\partial}{\partial u} \ d^2 y \ S = \ 0 \] (175)

for any scalar density \( f \) with the weight 1. Using the definitions of \( L_H \) in (174) and \( H_{a} \) in (172), we obtain the following identity,
\[ \frac{\partial}{\partial u} L_H = \frac{1}{16} \int H d^2 y (e^2 \ abF_{a}^{\ c}F_{c}^{\ b}); \] (176)

where we used the di \( N \) -covariant derivative of \( a \),
\[ D_{\ a} \ A_{+}^{\ a} = \ 0; \] (177)
which produce an analysis is the discovery of a set of Bondi-like quasi-local conservation equations for the vacuum general relativity, which reproduce both the well-known conservation equations in the asymptotic null infinity in asymptotically flat spacetimes and the corresponding conservation equations on the inner quasi-local horizon of a generic dynamic spacetime. All of these quasi-local quantities are expressed in geometrically invariant terms such as the area of the two-surface and a pair of null vector fields orthogonal to that surface. It was also found that each quantity has a natural interpretation as the quasi-local energy, linear momentum, and angular momentum of a two-surface and corresponding fluxes crossing that surface.

In addition to the above quasi-local quantities, we also obtained the quasi-local analog of the Carter's \( J^4 \) constant of gravitational field, which is somewhat like the angular momentum squared, measuring the intrinsic angular momentum of a two-surface. The Carter's \( J^4 \) constant is known to exist for spacetimes that possess two commuting Killing vector fields such as the Kerr black hole. But in our analysis, it was found that the quasi-local analog of Carter's constant exists under the condition

\[ \Theta, A_i^a = 0; \]
which is much less restrictive than the existence of two-commuting Killing fields.

Dynamics of the quasi-local horizon was discussed briefly, but it deserves further studies. Applications of this form all to astrophysical problems involving black holes and gravitational radiations are extremely challenging. These problems are left for future works.

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