Kinks versus fermions,
the 2D sine-Gordon versus massive Thirring models,
at $T > 0$ and $\mu \neq 0$

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The two dimensional (1+1) sine-Gordon model finds many applications in condensed matter physics. These in turn provide an experimental means for the study of topological defects, some of which may have had a huge impact on the early universe. As a first step in trying to exploit this analogy and also others which exist with low-energy QCD, we study bosonisation in the massive Thirring and sine-Gordon models at finite temperature $T$ and nonzero fermion chemical potential $\mu$. Both canonical operator and path integral approaches are used to prove the equality of the partition functions of the two models at $T > 0$ and $\mu = 0$, as was recently shown. This enables the relationship between normal ordering and path-integral renormalisation to be specified. Furthermore, we prove that thermal averages of zero-charge operators can be identified as long as one uses the usual canonical operator methods and later on using path integrals.

$\phi$ is a real scalar field, $\gamma_0$, $\alpha_0$ and $\lambda$ are bare parameters to be renormalised later, and $x^\mu = (t, x)$, $\mu = 0, 1$. Notice that apart from the usual kinetic term, the potential term is periodic so that there are an infinite number of degenerate vacua whose value, $\phi_v$, depends on the coupling constant $\lambda$: $\phi_v = 2n\pi/\lambda$ where $n \in \mathbb{Z}$. We comment that the Lagrangian is invariant under $\phi \rightarrow \phi + \phi_v$ and $\phi \rightarrow -\phi$ meaning that the sign of $\alpha_0$ is unimportant, and the name sine-Gordon is related to the fact that the equations of motion contain a sin $\phi$ term.

The massive Thirring (MT) model, on the other hand, is a model with a fermionic field $\psi = (\psi_1, \psi_2)$ and a four Fermi interaction with coupling $g$:

$$L_{Th}[\bar{\psi}, \psi] = i \bar{\psi}(\partial - m_0)\psi + \frac{1}{2}g j^\mu(x)j^\mu(x),$$

where $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$ and $m_0$ is a bare mass. For positive coupling constant, $g > 0$, the interaction term is attractive and there are fermion-antifermion bound states.

Though it maybe somewhat surprising since one is a model with a bosonic field and the other one with a fermionic field, it is very well known that these two theories are linked. In particular, they provide one of the earliest examples of duality in which the weak limit of one theory describes the same physics as the strong limit of the other and conversely since

$$\frac{\lambda^2}{4\pi} = \frac{1}{1 + g/\pi}.$$  \hspace{1cm} (3)

Hence perturbative calculations in one theory tell us about non-perturbative effects in the other. Identity was proved in perturbation theory about $\alpha_0 = 0 = m_0$ (see below) at $T = \mu = 0$ by Coleman using canonical operator methods and later on using path integrals. Note the special value $g = 0$ in which the Thirring model is a free theory of massive fermions, corresponding to $\lambda^2 = 4\pi$ in the SG model. Also the MT bound states exist for $g > 0 \Rightarrow \lambda^2 < 4\pi$.

The duality is directly related to the fact that the models exhibit bosonisation, in which a theory of fermions is equivalent to a theory of bosons. In general the bosonic (fermionic) theory may also have fermionic
(bosonic) excitations; this indeed occurs in the SG and MT models and we will see another example below in low-energy QCD. In the case of the SG and MT models, bosonisation schematically works in the following way (see [3] for a summary as well as a list of the relevant articles). Consider first the SG model. Perturbations in $\lambda$ about one of the minima of the degenerate potential give rise to the usual bosonic simple harmonic oscillator spectrum. The model also has other excitations. Recall that there are topologically non-trivial solitonic solutions to the classical equations of motion, and that these arise from the vacuum degeneracy: there is no reason why at $x \to \infty$ the system should be in the same vacuum as at $x \to -\infty$. As a result one can have finite energy solutions with non-zero charge

$$Q = \int_{-\infty}^{\infty} dx \frac{\lambda}{2\pi} \frac{\partial}{\partial x^1} \phi(x) = \frac{\lambda}{2\pi} [\phi(\infty) - \phi(-\infty)] = \Delta n.$$

(4)

The simplest ‘kink’ solution has $Q = 1$ and an energy proportional to $\alpha_0/\lambda^2$, whilst the anti-kink ($Q = -1$) is obtained from the kink by taking $\phi \to -\phi$. Furthermore, since the system is integrable, exact expressions for multi-kink and anti-kink states are known and one can construct moving kinks by Lorentz transformation. These may then be quantised using semi-classical methods [3,7]. There are also classical solutions with $Q = 0$ corresponding to ‘breather’ solutions in which kinks and anti-kinks oscillate periodically about each other [3]. These can be quantised by WKB methods [3,7]. The relationship between these SG excitations and the fundamental fermionic excitations and bound state excitations of the MT model is summarised schematically in table I.

| MT MODEL | SG MODEL |
|----------|----------|
| fermionic fields | bosonic fields |
| $g$ | $\lambda \sim 1/g$ |
| $f f$ bound states ($g > 0$) | bosons, kink – anti-kink breather solutions ($\lambda^2 < 4\pi$) |
| $f$ | kinks |
| $\mu \gamma^3$ | $m_0 \neq 0 \to$ see later |
| $\pm \mu \gamma^3$ | $m_0 = 0 \to$ topological term [3] |

TABLE I. Table showing schematically some of the links between the SG and MT models at $T = 0$

We see that the kinks themselves are fermion-like excitations corresponding to the fundamental fermions of the Thirring model. For $\lambda^2 < 4\pi$, the bosonic ($Q = 0$) excitations of the SG model correspond to the bound states of the MT model. This is what is meant by bosonisation.

What is the motivation for studying these models at $T > 0$ and $\mu \neq 0$? One first reason is that these models could be used to develop techniques which may then be applicable to more realistic 4D theories, in particular QCD. Recall that at low temperatures, where the quarks and gluons are strongly confined into hadrons, one can describe the system by an effective chiral bosonic lagrangian (CBL) for the lightest mesons (pions, kaons and eta) which are the Nambu Goldstone bosons (NGB) of the chiral symmetry breaking. The relationship between this chiral bosonic theory and the original fermionic QCD has many similarities with the relationship between the SG and MT models (see table II).

| QCD | CBL |
|-----|-----|
| quarks and gluons | bosonic fields |
| $\alpha_s$ (large at low energies) | small expansion parameter $\frac{\alpha_s}{\Lambda}$ |
| $\Lambda$ | $\Lambda \chi \sim 4\pi f_\pi \approx 1 \text{ GeV}$ |
| $\bar{q} q$ bound states | $\pi, K, \eta$ (NGB of chiral symmetry) |
| baryon | skyrmion (topological defect) |
| $\pm \mu$ for baryons | topological term [3] |

TABLE II. Table showing schematically some of the links between QCD and the low energy chiral lagrangian.

For example, the bound states of QCD are the mesons of the chiral theory whilst the baryons of QCD correspond to skyrmions—topological defects [10,11]—in the chiral theory which to lowest order is the non-linear sigma model [12] (c.f. the relationship between the SG and MT models in table I). Note also that the SG Lagrangian in [10] corresponds to a non-linear sigma model in 2D for a single Nambu-Goldstone-like field $\phi$. Although there is no spontaneous symmetry breaking in 2D [3], the potential term in [10] breaks explicitly the symmetry $\phi \to \phi + \phi_c$. These two symmetries are, respectively, the counterparts of the chiral and isospin symmetries for QCD, $\alpha_0$ and $\lambda$ playing the role of the pion mass squared and the inverse of the pion decay constant $f_\pi$ respectively. On the other hand, the chiral symmetry transformations in terms of the Thirring fermion are $\psi \to \exp(i a \gamma_5) \psi$. The massless Thirring model is chiral invariant, the fermion mass term breaking that symmetry in the same way as the $\alpha$ term does in the SG Lagrangian.

A second motivation for studying these models at $T > 0$ and $\mu \neq 0$ comes from cosmology. The SG kinks are the 2D analogues of cosmic strings, line-like defects formed in 4D when a system goes through a symmetry breaking phase transition (say of some group $G$ to a subgroup $H$) for which the first homotopy group $\pi_1(G/H) \neq 0$ [4,5]. The strings trap regions of the unbroken high energy phase and so have energies per unit length which can be very large, depending on the critical temperature. In the context of the early universe and cosmology, cosmic strings may have played an important rôle because as the universe cooled and expanded after the big bang, it went through a number of phase transitions some of which may have led to the formation of...
strings. In particular, strings formed at the GUT phase transition have huge energies per unit length \((\mu \sim 10^{12} \text{ GeV}^2)\) and hence significant gravitational effects, and so it is thought that they may have been responsible for the temperature fluctuations in the cosmic microwave background radiation and for seeding the perturbations which led to the formation of structures such as galaxies [13].

However, in order to make detailed predictions as to their effects, it is important to know the initial distribution of strings and whether or not it contains infinite strings. This is a very difficult task to undertake analytically [1], but recently progress has been made by using the analogy between experimentally observable systems such as \(^3\text{He}\) and \(^4\text{He}\) and the early universe [17]. It would seem, however, that there is an even more simple experimentally accessible system with which one could try to test ideas of defect formation, and that is a Josephson junction [18]. This device consists of two layers of superconductors separated by a thin dielectric barrier, typically of the order of 5nm. Denoting the macroscopic wave function of each of the superconductors by \(\Psi_a = |\Psi_a|^\alpha \exp\{i\theta_a\}\) where \(a = 1, 2\) labels each of the two layers, then Josephson tunnelling of the Cooper pairs across the dielectric layer results in \(\Delta \theta = \theta_1 - \theta_2\) satisfying a SG equation [18]. As the two superconductors are taken through the phase transition, kinks are formed in the junction and these are observed experimentally [19]. Experiments are also done to see what is, for example, the effect of the geometry of the set up on the kinks and their dynamics [20] and these devices are used as sources of some of the highest energy micro-waves finding applications in satellites [18,19]. One idea might therefore be to see whether one can indeed use such experiments with Josephson junctions to test ideas of defect formation. The hope is in particular that the situation can be much simplified in this case because of the duality between the SG and MT models. First, however, one has to check what is the relation between these two models at \(T > 0\) and \(\mu \neq 0\). In particular, could it be possible that the relationship between coupling constants [3] is a function of temperature? As we will comment in the conclusions, these models also provide examples of other phenomena which may too be more easy to study in 2D rather than 4D.

Here we report on our first steps in these directions. In section II we briefly summarise some of the particularities and basic results of 2D which will be useful to bear in mind for the rest of the work. Results are then given in section III. First, in subsection III A, we outline the main steps which must be taken to extend Coleman’s work [9] on the SG model to \(T > 0\) using an operator approach. The results of [21] showing the equivalence of the partition functions are reproduced though the approach is entirely different. Path integral methods are used in the rest of the work. We show that not only the partition functions are equivalent but also that thermal averages of correlators of zero-charge operators evaluated at different space-time points coincide. The relationship between normal ordering and regularisation is specified and we analyse the \(T > 0\) situation in which there is a net number of fermions, \(\mu \neq 0\). The partition function is calculated in that case so extending [3] to the massive case and [21] to \(\mu \neq 0\). The analogy between these models and a classical gas of particles is then noted in the conclusions. Details of the calculations and results presented here may be found in references [3,22].

II. PECULIARITIES OF 2D

The following three basic, though important, results hold for free fields in 2D at \(T > 0\). They will be useful in the following sections.

1) Equality of free massless boson and fermion partition functions

Consider a free bosonic field of mass \(\tilde{\mu}\) and a massless free fermionic field. The partition functions can be obtained by writing the Hamiltonians in terms of a normal ordered part plus an infinite vacuum energy; they are respectively

\[
\ln Z^B_{0}(T) = -L \int^{+\infty}_{-\infty} \frac{dk}{2\pi} \left[ \frac{\beta \omega_k \tilde{\mu}}{2} + \ln \left( 1 - e^{-\beta \omega_k \tilde{\mu}} \right) \right],
\]

\[
\ln Z^{F}_{0}(T) = 2L \int^{+\infty}_{-\infty} \frac{dk}{2\pi} \left[ \frac{\beta |k|}{2} + \ln \left( 1 + e^{-\beta |k|} \right) \right].
\]

Here \(\omega_k \tilde{\mu} = k^2 + \tilde{\mu}^2\), \(\beta\) is the inverse temperature \(\beta = 1/T\), and \(L\) is the spatial dimension which we are taking to infinity. As usual, all the thermodynamic observables are obtained from the logarithm of the partition function and its derivatives by dividing by \(\beta L\).

Observe that when \(\tilde{\mu} = 0\),

\[
\int^{+\infty}_{-\infty} \frac{dk}{2\pi} \ln \left( 1 - e^{-\beta |k|} \right)
= -2 \int^{+\infty}_{-\infty} \frac{dk}{2\pi} \ln \left( 1 + e^{-\beta |k|} \right) = -\frac{\pi T}{6},
\]

and hence it follows that ignoring vacuum terms, the two partition functions \([1,4]\) are equivalent for \(\tilde{\mu} = 0\):

\[
Z^{F}_{0}(T) = Z^{B}_{0}(T) = \exp \left[ \frac{\pi LT}{6} \right].
\]

Thus one cannot tell the difference between the bulk quantities of these two systems: equality \([1]\) is perhaps the simplest example of bosonisation. As will be seen below, the equality of the SG and MT partition functions at \(T > 0\) rests on \([3]\) since in those models we work in perturbation theory, expanding about \(\alpha_0 = 0\) in the SG model and about \(m_0 = 0\) in the MT model (and hence
about massless bosonic and fermionic theories).

2) Free thermal propagators

We will need the free propagators for boson and fermion fields at $T > 0$. Again working with a boson field of mass $\tilde{\mu}$ and a massless fermion field, and calculating in the imaginary time formalism with $\ell = it$, these are found to be respectively [21] 

$$
\Delta_T(x) = -\frac{1}{4\pi} \ln \tilde{\mu}^2 \beta^2 Q^2(x) + K + \mathcal{O}(\tilde{\mu}\beta), \\
S_{\alpha\beta}(x) = -\frac{1}{2\beta} Q_{\alpha\beta}(x), 
$$

where we have expanded the boson propagator about $\tilde{\mu} = 0$, and $K$ is a constant. The $Q$ variable is given by $Q^2 = Q_0^2 + Q_1^2$ where

$$
Q_0(x, \ell) = -\cosh(\frac{\pi x}{\beta}) \sinh(\frac{\pi \ell}{\beta}), \\
Q_1(x, \ell) = -\sinh(\frac{\pi x}{\beta}) \cosh(\frac{\pi \ell}{\beta}), 
$$

so that $Q(x)$ is a Lorentz scalar. The indices $\alpha, \beta$ are Dirac indices and we have worked with the 2D Euclidean formal invariance of the free theories, or as arising from plane, and can be viewed either as a result of the conformal invariance of the free theories, or as arising from solving the Green function equation for the Coulomb potential on a cylinder of radius $\beta$ [23].

3) Ultra-violet divergences and normal ordering

Finally, recall that the UV divergence structure of 2D bosonic theories is much simpler than that of 4D ones. The reason follows from the fact that in $d$ dimensions with an interaction of the form $\phi^r$, a diagram with $n$ vertices and $E$ external lines has a UV degree of divergence $D$ of [23] 

$$
D = d - \left(\frac{d}{2} - 1\right) E + n \left[\frac{r}{2}(d - 2) - d\right].
$$

So with $d = 2$, 

$$
D = 2 - 2n
$$

and the only divergent diagrams $\forall r$ are tadpole diagrams. Since the SG lagrangian contains a term $\cos \lambda \phi = \sum (-1)^r \phi^{2n}/(2n!)$ then in the operator formalism it follows that all UV divergences of the theory should be removed through normal ordering (as this removes tadpole diagrams). In path integral methods there is no operator normal ordering, and the divergences are removed using different methods (see below).

III. RESULTS

A. Bosonisation in the canonical operator approach at $T > 0$ and $\mu = 0$

We have used operator methods to extend the paper of Coleman [4] to $T > 0$ and $\mu = 0$. This enabled us to prove [3] that the partition functions of the SG and MT models are identical

$$
Z_{SG}(T, \mu = 0) = Z_{MT}(T, \mu = 0) \tag{12}
$$

provided the coupling constants of the theories satisfy [3] which is temperature independent. In fact, [13] was already proved using very different path integral methods in [21] (see also section III B).

In the SG model the main steps in the calculation $Z_{SG}(T, \mu = 0)$ are the following:

1. Remove all UV divergences by normal ordering the SG Hamiltonian.

2. Deal with the IR divergence of the propagator [3].

3. Calculate the partition function $Z_{SG}$, a sum of thermal expectation values of free (interaction picture) operators each of which have been normal ordered as a result of step 1.

We now outline the main features of each of these steps, and also comment on the differences between this $T > 0$ case and the $T = 0$ one discussed in [4].

Step 1: Removal of all UV divergences by normal ordering

As stated in point 3 above, the partition function contains thermal expectation values (TEV’s) of normal ordered operators. Therefore to simplify its calculation, we
remove UV divergences by using thermal normal ordering (TNO) introduced in [24] rather than standard zero temperature normal ordering (which places annihilation operators to the left of creation operators). By construction TNO, which is denoted by $N^{ES}_\rho$, guarantees that for any\(^1\) operator (other than the identity) • in the interaction picture,

$$\ll N^{ES}_\rho \bullet \gg_0 = 0,$$

(13)

whereas for usual normal ordered products

$$\ll \bullet : \gg_0 \neq 0.$$

The operation $N^{ES}_\rho \bullet$ was defined in [24] to place the “positive” parts of the operator, • (a combination both of annihilation and creation operators), to the right of the “negative” part, •. See [24] for the exact definitions. In the above equations the angular brackets denote a thermal expectation value, and the subscript zero indicates the above equations the angular brackets denote a thermal expectation value, and the subscript zero indicates the operation TNO, which is denoted by $\Delta_T(x; \rho)$.

Consider therefore

$$\Delta_T(x; \rho) := \Delta_T(x; \rho; \Lambda) := \Delta_T(x; \rho) - \Delta_T(x; \Lambda)$$

(16)

where $\Lambda$ is a large mass. Hence the constant $K$ in the propagator \(13\) cancels, and $\Delta_T(x; \rho; \Lambda)$ is now both non-singular as well as $\beta$ independent for $x \to 0$; $\Delta_T(0; \rho; \Lambda) = -\frac{1}{4\beta} \ln (\rho^2/\Lambda^2)$. Combining \(16\) with \(15\) for $j(x) = \lambda \delta(x - y)$ gives, for example

$$e^{i\lambda \delta(y)}(\rho) = \left( \frac{\rho^2}{\Lambda^2} \right)^{\lambda^2} N^{ES}_\rho \left[ e^{i\lambda \delta(y)} \right].$$

(17)

After similar manipulations, \(14\) can be written as \(13\)

$$\hat{H}_{SG} = -\int_0^L dx \left[ \frac{\pi^2}{2} + \frac{1}{2} \left( \partial_\phi \right)^2 - \frac{\alpha_0}{\lambda^2} \cos \lambda \hat{\phi} - \gamma_0 \right]$$

(14)

which must be divided into a free and interacting part so as to apply TNO. Although the term $\cos \lambda \hat{\phi}$ itself contains a mass term on expansion in powers of $\lambda$, we want to keep $\lambda$ of arbitrary size. Consider therefore

$$\hat{H}_{SG} = \left[ \hat{H}_0 + \int_0^L dx \left( \frac{1}{2} \rho^2 \partial_x^2 \right) \right]$$

$$\ll \bullet : \gg_0 \neq 0.$$

where $\alpha_0$ has been multiplicatively renormalised,

$$\alpha = \alpha_0 \left( \frac{\rho^2}{\Lambda^2} \right)^{\lambda^2}$$

(19)

and $\gamma_0$ has been renormalised according to

$$\gamma = \gamma_0 - E_T(\rho)$$

(20)

where

$$E_T(\rho) = \frac{1}{2} \left\{ [\pi^+, \pi^-] + \left[ (\partial_\phi)^+, (\partial_\phi)^- \right] \right\}$$

$$= E_0(\rho) + L \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{2} N_{k,\rho} \left( \frac{1}{\omega_k,\rho} \right).$$

Here $E_0(\rho)$ is an infinite temperature independent contribution; $E_0(\rho) = L \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{2} N_{k,\rho}$. The temperature dependent part proportional to the Bose-Einstein distribution $N_{k,\rho} = (e^{\beta \omega_{k,\rho}} - 1)^{-1}$ is finite. The coupling $\lambda$ is unchanged.

\(^1\)This could be the field operator $\hat{\phi}$, the momentum operator $\hat{p}$ or any other composite operator.

\(^2\)These equalities may at first sight seem surprising, but they might be clarified by noting that here the advanced and retarded thermal propagators are equal so that $\Delta_{T}(x; y, \rho) = \Delta_{T}(-x, -y, \rho) = [\phi^+(x), \phi^-(y)]$ where the positive and negative parts refer to those of [24].
Thermal normal ordering the SG Hamiltonian has therefore absorbed all UV infinities just as zero temperature normal ordering does \[\text{(4)}\], but it has also introduced some extra $T$-dependent finite terms.

**Steps 2 and 3: IR divergences and calculation of the partition functions**

The IR divergence of the boson propagator \[\text{(8)}\] is removed by introducing a mass $\bar{\mu}$ into the SG Hamiltonian \[\text{(18)}\]. At the end of the calculation we let $\bar{\mu} \to 0$, and hence are free to add the extra mass term within the normal ordering giving the Hamiltonian

$$
\hat{H} = N^ES_{\rho} \left[ \hat{H}_0^\rho - \gamma L - \frac{\alpha}{\lambda^2} \int_0^L dx \left( \cos \lambda \hat{\phi} \right) \right] = N^ES_{\rho} \left[ \hat{H}_0^\rho - \frac{\alpha}{\lambda^2} \int_0^\infty dx \left( \cos \lambda \hat{\phi} \right) \right] =: \hat{A}_0 + \hat{A}_{I}.
$$

(21)

Here $\hat{H}_0^\rho = \hat{H}_0 + \int_0^L dx \mu^2 \hat{\phi}^2$, and $\hat{A}_0$ and $\hat{A}_I$ denote the free ($\alpha = 0$) and interacting Hamiltonians respectively. In perturbation theory the SG partition function is therefore given by

$$
Z_{SG}(T, \mu = 0) = \lim_{\mu \to 0} \text{Tr} \left\{ e^{-\beta \hat{H}} \right\} = \lim_{\mu \to 0} \sum_n \frac{1}{n!} \int_0^\infty dt_1 \cdots \int_0^\infty dt_n \times \Tr \left\{ e^{-\beta \hat{A}_0 \mathcal{T}_c} \left[ \hat{A}_I(t_1) \cdots \hat{A}_I(t_n) \right] \right\},
$$

(22)

so that one only needs calculate free expectation values

$$
\ll \cdots \gg_0 = \text{Tr} \exp(-\beta \hat{A}_0) / \text{Tr} \exp(-\beta \hat{A}_0).
$$

Indeed, the correlator appearing in \[\ll \cdots \gg_0\] can be obtained by showing first that

$$
\ll c_0 = \prod_{j=1}^n N^ES_{\rho} \left[ e^{i \lambda_j \hat{\phi}(x_j)} \right] \gg_0 = \left( \frac{\mu^2}{\rho^2} \right)^{\frac{\lambda^2}{4}} \prod_{j=1}^n [\beta^2 \mu^2 |Q(x_j - x_k)|^2]^{\frac{\lambda \lambda_j}{4 \pi}} \times \prod_{j > k} [\beta^2 \mu^2 |Q(x_j - x_k)|^2]^{\frac{\lambda \lambda_j}{4 \pi}},
$$

(23)

where we have expanded the exponential inside the normal ordered term and noted that all terms which contain $\hat{\phi}^n$ vanish (by \[\ll \cdots \gg_0\]) apart from that with $n = 0$. Then observe that the terms proportional to $\bar{\mu}$ in \[\ll \cdots \gg_0\] have a contribution $\bar{\mu} (\sum \lambda_j)^2 / 4\pi$. Thus in the limit $\bar{\mu} \to 0$, only configurations with $\sum \lambda_j = 0$ contribute (a condition which will become analogous to the fermion chiral selection rule of section IIIB). In \[\ll \cdots \gg_0\] $n$ must therefore be even. If we let $m = n/2$, $y_j = x_j$, for $j = n/2 + 1, \ldots, n$ and define

$$
\hat{A}_\pm = N^ES_{\rho} \left[ e^{\pm i \lambda \hat{\phi}} \right],
$$

(24)

then from \[\ll \cdots \gg_0\]

$$
\ll c_{m} = \prod_{j=1}^m \hat{A}_+(x_j) \hat{A}_-(y_j) \gg_0 = \prod_{j=1}^m \prod_{l_j > k} |Q(x_j - x_k)|^2 |Q(y_j - y_k)|^2 \times |Q(x_1 - y_1)|^2 \frac{\lambda^2}{4 \pi} \prod_{j=1}^n \int_T d^2 x_j d^2 y_j \times
$$

(25)

$$
\prod_{j > k} [Q(x_j - x_k)|^2 |Q(y_j - y_k)|^2 \frac{\lambda^2}{4 \pi} \prod_{j=1}^m \int_T d^2 x_j d^2 y_j \times
$$

(26)

Hence the partition function \[\ll \cdots \gg_0\] is

$$
Z_{SG}(T, \mu = 0) =
$$

(27)

$$
Z_{SG}(T, \mu = 0) = \frac{\prod_{j=1}^m [Q(x_j - x_k)|^2 |Q(y_j - y_k)|^2 \frac{\lambda^2}{4 \pi} \prod_{j=1}^n \int_T d^2 x_j d^2 y_j \times
$$

(26)

$$
\prod_{j > k} [Q(x_j - x_k)|^2 |Q(y_j - y_k)|^2 \frac{\lambda^2}{4 \pi} \prod_{j=1}^m \int_T d^2 x_j d^2 y_j \times
$$

(27)

In fact one has to be extremely careful when considering the precise form of the thermal weight $\exp(-\beta \hat{A}_0)$. The reason is that it contains two different mass scales $\rho$ and $\bar{\mu}$, meaning that $\hat{A}_0$ is not obviously diagonal and so not obviously of the form $\int dk \omega(k) \hat{a}(k) \hat{a}(k)$ as is usually the case and as was assumed in \[\ll \cdots \gg_0\]. See \[\ll \cdots \gg_0\] for details about this point.

\footnote{From the behaviour of the $Q$ variables in \[\ll \cdots \gg_0\], one can see that for $\lambda^2 > 4\pi$ there are extra divergences in \[\ll \cdots \gg_0\]. The treatment of these is commented on in \[\ll \cdots \gg_0\].}
where we have chosen to renormalise the MT model at the scale $\rho$. Here $Z_0^R$ is given in [8],
\[
\kappa^2 = \frac{g}{1 + g^2/\pi} \quad (28)
\]
and $m$ is the renormalised mass (see [8] and also section
III.B)
\[
m = m_0(\Lambda/\rho)\kappa^2/\pi. \quad (29)
\]
Thus term by term (23) and (27) are identical provided that a) the parameters of the two theories are identified as in (8), b) that $\alpha$ and $m$ are related by
\[
\rho m = \frac{\alpha}{\lambda^2} \quad (30)
\]
and c) that $\gamma_0 = 0$ as then $Z_0^R(T) = Z_0^F(T)$ by (1).

We have therefore extended the work of Coleman [21], but also to prove the equivalence of certain sets of correlators of operators evaluated at different space time points. These correlators cannot be obtained from the partition function which only contains information about global thermodynamic observables like the pressure or the condensates, but not about correlators which physically yield, for instance, thermal correlation lengths.

We outline the main points of such PI calculations for the SG model and then the MT model. This will enable the link between normal ordering and PI regularisation to be made. Our results for the correlators will also be stated more precisely.

B. Path Integral bosonisation at $T > 0$ and $\mu = 0$

We have used path integral methods not only to prove the equality of the two partition functions as in (12) [21], but also to prove the equivalence of certain sets of correlators of operators evaluated at different space time points. These correlators cannot be obtained from the partition function which only contains information about global thermodynamic observables like the pressure or the condensates, but not about correlators which physically yield, for instance, thermal correlation lengths.

We outline the main points of such PI calculations for the SG model and then the MT model. This will enable the link between normal ordering and PI regularisation to be made. Our results for the correlators will also be stated more precisely.

1. SG model

As always in path integral methods, one works with the generating functional. Once again it is useful to start with free boson fields and in particular to calculate the correlator (23) which may be obtained from the free boson Euclidean generating functional
\[
Z_0^B[J;T] = N_\beta \int_{\text{periodic}} d\phi \exp \left\{- \int_T d^2 x \frac{1}{2} \times \left[ (\partial_\alpha \phi)^2 + \tilde{\mu}^2 \phi^2 + J(x)\phi(x) \right] \right\} = Z_0^B[0;T] \exp \left\{ \frac{1}{2} \int_T d^2 x \int_T d^2 y \times J(x)\Delta_T(x-y)J(y) \right\}. \quad (31)
\]

Here $N_\beta$ is an infinite $T$-dependent constant arising in the path integral description [26], the propagator is given in [8] for small $\tilde{\mu}$ and the free boson partition function is $Z_0^F(T) = Z_0^B[0;T]$ as in [8]. Note that we have removed the $\tilde{\mu}$ labels on propagator and partition function as there is no longer any possible confusion with other mass scales. Now define
\[
A_\pm = e^{\pm i\lambda \phi}
\]
(remember that we do not normal order the operators in path integral and so this differs from the definition (24)).

Then
\[
\ll T \epsilon \int_1^n A_+(x_j)A_- (y_j) \gg_0 = \frac{Z_0^B[J;T]}{Z_0^F(T)}
\]
with
\[
J(z) = -i\lambda \sum_{j=1}^n \left[ \delta^{(2)}(z-x_j) - \delta^{(2)}(z-y_j) \right].
\]

As opposed to section III.A, here the UV divergence of the propagator (8) is regulated by replacing
\[
Q^2(0,0) \rightarrow Q^2(\varepsilon_0,\varepsilon_1) = T^2\varepsilon^2 + O(\varepsilon^3) \quad (32)
\]
where $\varepsilon_0 \rightarrow 0^+\varepsilon^2$ and $\varepsilon^2 = \pi^2(\varepsilon_0^2 + \varepsilon_1^2)$. From (31), (8) and (23) it follows that
\[
\ll T \epsilon \int_1^n A_+(x_j)A_- (y_j) \gg_0 = (T\epsilon)^{n\lambda^2/2\pi} \prod_{j=1}^n \frac{\int[Q^2(x_j-x_k)Q^2(y_j-y_k)]^{1/4\pi}}{\prod_{k=1}^n [Q^2(x_j-y_k)]^{1/4\pi}}. \quad (33)
\]

The correlator is divergent due to the short-distance (UV) divergent behaviour of the composite operator $\exp[i\alpha \phi(x)]$, which needs to be renormalised. We do this in the usual way through the replacement of $\exp[i\alpha \phi(x)]$, with $a \in \mathbb{R}$ arbitrary, by
\[
[\exp[i\alpha \phi(x)]]_\text{bare} = (\varepsilon \rho)^{1/4\pi} [\exp[i\alpha \phi(x)]]^R. \quad (34)
\]

Here $\rho$ is an arbitrary renormalisation scale and the superscript $R$ will denote renormalised operators. Note that (34) is analogous to (17) with the identification
\[
\varepsilon = \frac{1}{A}. \quad (35)
\]
though in the operator formalism the renormalisation was carried out through TNO. Also observe that the $\rho$’s appear in the same way though they have different origins—in the operator approach $\rho$ corresponded to an arbitrary mass at which normal ordering was performed whereas here it is the arbitrary renormalisation scale.

From equations (33) and (34) observe that (33) reduces to (23) as required.
In the full SG model one again works with the generating function which is expanded formally in powers of $\alpha_0/\lambda^2$ \[\text{(1)}\]. The partition function $Z_{SG}(T, \mu = 0)$ is just obtained by setting the external sources to zero. We find that with the regularisation \[\text{(29)}\] of the propagator, $\alpha_0$ is renormalised just as in \[\text{(19)}\] for all the divergences to be eliminated. Once again the partition function is given by \[\text{(28)}\].

2. MT model

In calculating the MT partition function in perturbation theory about $m_0 = 0$, the correlator analogous to \[\text{(32)}\] in the SG model is just the TEV of insertions of the operators $\sigma_{\pm}(x) = \bar{\psi}(x)P_{\pm}\psi(x)$. Note, however, that the massless fermion theory is invariant under chiral transformations $\psi \rightarrow \exp(i\alpha_T^g)\psi$. Under such transformation $\sigma_{\pm}(x) \rightarrow \exp(\pm 2i\alpha)\sigma_{\pm}(x)$ and therefore the thermal average of a product of $\sigma_{\pm}(x)$ operators will vanish in the massless case unless the number of $\sigma_+$ and $\sigma_-$ is the same. This is the chiral selection rule, which only holds for $m_0 = 0$. Following \[\text{(27)}\], the required correlator is obtained by shifting $\bar{\psi} \rightarrow \bar{\psi}\gamma^0$ so that, naming $\bar{\psi}_a$, with $a = 1, 2$ the two components of the spinor, the free massless theory decouples into two free theories for the spinors $\psi_\alpha$ and we have $\sigma_+ \rightarrow \psi_2\psi_1$ and $\sigma_- \rightarrow \psi_1\psi_2$. One obtains \[\text{(37)}\].

\[
\langle T_c \prod_{j=1}^n \sigma_+(x_j)\sigma_-(y_j) \rangle \approx 0 = (2\beta)^{-n} \prod_{j=1}^n \prod_{k=1}^n \left[ \frac{Q^2(x_j - x_k)Q^2(y_j - y_k)}{\prod_{k=1}^n (Q^2(x_j - y_k))} \right]. \tag{36}
\]

Notice that the above correlator has exactly the same structure as the boson correlator \[\text{(33)}\]—this is another peculiarity of 2D. However, unlike \[\text{(33)}, \text{(35)}\] is finite since it contains no product of fields at the same space-time point and there are no mixing terms between $\psi_1$ and $\psi_2$ in the Lagrangian.

For the MT model one does have to worry about renormalisation (section \[\text{II.A}\]). The reason is that whilst the chiral symmetry is still unbroken so that $\sigma_{\pm}(x)$ correlators still appear in the same number if $g \neq 0$ and $m_0 = 0$, now $(\bar{\psi}\gamma^\mu\psi)^2 \rightarrow 4\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1$ when $\bar{\psi} \rightarrow \bar{\psi}\gamma^0$ and therefore there is mixing between $\psi_1$ and $\psi_2$. Thus products of fields at the same point appear and the $\sigma_{\pm}(x)$ correlator becomes divergent: in the same way as the boson operator exp$(i\alpha\phi(x))$, the $\sigma_{\pm}(x)$ composite operators need renormalisation. Also as in the SG model, those are the only infinities we have to worry about and they are absorbed in the renormalised mass $m$ as in \[\text{(29)}\] whilst

\[
[\sigma_{\pm}(x)]^R = (\epsilon\rho)^{\kappa^2/\pi}[\sigma_{\pm}(x)]\text{bare}. \tag{37}
\]

Given these renormalisations, the MT model partition function is obtained from the generating functional when the sources are set to zero. Its calculation requires some standard manipulations \[\text{(16), (22)}\] (writing the quartic Thirring interaction as a 'gauge-like' interaction as well as calculating the axial anomaly \[\text{(28)}\]) and one again obtains the result \[\text{(27)}\].

3. Zero-charge operators equivalences

So far we have shown the equivalence of the SG and MT partition functions with $\mu = 0$ using both operator and PI techniques. Further equalities between correlators of different operators in each of the two models may also be shown to hold at $T > 0$ and $\mu = 0$—we simply state the results here. Further calculational details may be found in \[\text{[1]}\].

The equivalence of \[\text{(33)}\] and \[\text{(36)}\] in the free massless bosonic and fermionic theories (after renormalisation) may be extended to the SG and MT models where it becomes

\[
\langle T_c \prod_{j=1}^N \sigma_+^R(x_j)\sigma_-^R(y_j) \rangle \approx_M T = \left(\frac{\rho}{2}\right)^{2N} \langle T_c \prod_{j=1}^N A_+^R(x_j)A_-^R(y_j) \rangle \approx_S G.
\]

We have also considered more complicated cases\[5\]. There the calculation goes through in a similar way and leads to the expected result in which

\[
\sigma_+^R(x_j) \rightarrow \left(\frac{\rho}{2}\right) A_+^R(x_j). \tag{38}
\]

When dealing with correlators including insertions of the current operator $j_{\mu}(x)$ in the MT model one is again faced with a product of field operators at the same point, and hence with additional divergences. Using point-splitting regularisation \[\text{(28)}\] and taking particular care to ensure that the Ward identities are satisfied, we have proved that there are no extra divergences to renormalise and

\[
\langle T_c j_{\mu}^R(x)\sigma_+^R(z_1)\sigma_-^R(z_2) \rangle \approx_M T = \left(\frac{\rho}{2}\right)^2 \frac{\lambda}{2\pi} \epsilon_{\mu\nu} \langle T_c \partial_{\nu}\phi(x)A_+^R(z_1)A_-^R(z_2) \rangle \approx_S G
\]

where $j_{\mu}^R(x)$ is the regularised current \[\text{(1)}\]. More generally we find that the identities which hold between different correlators seem to be simply obtainable through the replacements \[\text{[3]}\] and

\[
\text{[5]}\] For example there could be unequal numbers of $\sigma_+$ and $\sigma_-$'s, since the chiral symmetry is broken in the MT model with $m_0 \neq 0$. 

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\[ j_\mu^R(x) = \frac{\lambda}{2\pi} e^{\mu\nu} \partial_\nu \phi(x) \] (39)

which are usually called operator bosonisation identities. We stress, however, that firstly we do not believe these identities to be strong— that is, we only expect them to hold between thermal expectation values—and secondly it is not immediately obvious that they hold inside all expectation values, though it does certainly seem to be the case for the zero fermion charge ones we have considered.

C. Bosonisation of the massive Thirring model at \( T > 0, \mu \neq 0 \)

Finally we have studied the Thirring model at nonzero chemical potential \( \mu \) for the conserved charge \( Q_F = \int dx j^0(t, x) \) which is the net number of fermions minus anti-fermions;

\[ \mathcal{L}_{Th}[^\psi, \bar{\psi}; \mu] = -\bar{\psi}(\partial + m_0)\psi + \frac{1}{2} g^2 j_a(x) j^a(x) + \mu j^0(x), \]

and we have calculated the grand-canonical ensemble partition function \( Z_{Th} \). Recall that now the averaged net fermion number density \( \rho(\mu) = (\beta L)^{-1} \ll Q_F \gg \) is no longer necessarily zero, and so a natural question to ask is what is the bosonised version of this theory. The answer was obtained in \( I \) for the massless case where the free boson partition function acquires an extra \( \mu \)-dependent term. In the massive case with partition function

\[ Z_{Th}(T, \mu) = N_\beta^F \int_{\text{antiper}} d\bar{\psi} d\psi \exp \left[ - \int_T d^2 x \mathcal{L}_{Th}[\bar{\psi}, \psi; \mu] \right] \]

we find in perturbation theory around the massless case, after using some of the results in \( I \) and performing some functional integral manipulations, that

\[ Z_{Th}(T, \mu) = Z_{SG\mu}(T, \mu) \]

where

\[ Z_{SG\mu}(T, \mu) = N_\beta \int_{\text{periodic}} d\phi \exp \left[ - \int_T d^2 x (\mathcal{L}_{SG} - \frac{\mu\lambda}{2\pi} \partial_x^2 \phi(x)) \right]. \]

(See \( I \) for a discussion of the boundary conditions.) That is, the bosonised action is the SG model plus an extra term which is topological in that it only depends on the value of the field at the spatial boundary \( x = \pm \infty \). From \( I \) this term is interpreted as the result of excitations with net kink (fermion) charge being present in the thermal bath and having associated chemical potential \( \mu \) in the grand-canonical ensemble. Recall that an analogous contribution was found in the chiral Lagrangian for low-energy QCD \( I \), where the rôle of kinks (Thirring fermions) is played by the skyrmions (QCD baryons): the chiral Lagrangian at finite baryon density acquires, amongst other things, a new factor \( \mu Q_{SK} \) with \( Q_{SK} \) the skyrmion topological charge.

IV. CONCLUSIONS AND OUTLOOK

After motivating the study of the SG and MT models at \( T > 0 \) and \( \mu \neq 0 \), we have summarised some of the results obtained in \( II \). Firstly, with zero chemical potential we were able to show using both operator methods and path integral methods that the partition functions were equivalent. This also enabled a link to be made between the arbitrary scale \( \rho \) at which we carried out thermal normal ordering in the operator approach and the arbitrary renormalisation scale in the path integral—these are identical. We then studied correlators of operators at different space time points in each of the models. Such results will be of crucial importance for the application of this work to the estimation of the number of topological (kink) defects formed in a phase transition, an extension of this work which we motivated in the introduction \( II \).

Also of relevance to this future work is the relationship between the two models in the presence of a net number of excitations; we stated our results in sections \( III \) 3 and \( III \) C.

Finally, we are studying the link between the sine-Gordon model and a 1D classical gas of positive and negative charges with non-zero fugacity \( \mu_0 \). As a result of this (highly studied system) and of the relationship between the sine-Gordon model, Josephson junctions and the massive Thirring model, we would tentatively suggest that such models should perhaps not be overlooked as ones in which to develop or test calculational methods. For example, one could try to investigate such difficult quantities as the pressure or the fermion number density, which could then give some insight into real physical problems. Similarly we hope to investigate the precise nature of the transition at \( \lambda^2 = 8\pi \).

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