WEIGHTED LEAVITT PATH ALGEBRAS OF FINITE GELFAND-KIRILLOV DIMENSION

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Abstract. We determine the Gelfand-Kirillov dimension of a weighted Leavitt path algebra $L_K(E, w)$ where $K$ is a field and $(E, w)$ a finite weighted graph. Further we show that a finite-dimensional weighted Leavitt path algebra over a field $K$ is isomorphic to a finite product of matrix rings over $K$.

1. Introduction

In a series of papers William Leavitt studied algebras that are now denoted by $L_K(n, n + k)$ and have been coined Leavitt algebras. Let $X = (x_{ij})$ and $Y = (y_{ji})$ be $(n + k) \times n$ and $n \times (n + k)$ matrices consisting of symbols $x_{ij}$ and $y_{ji}$, respectively. Then for a field $K$, $L_K(n, n + k)$ is a $K$-algebra generated by all $x_{ij}$ and $y_{ji}$ subject to the relations $XY = I_{n+k}$ and $YX = I_n$. In [9, p.190] Leavitt studied these algebras for $n = 2$ and $k = 1$, in [10, p.322] for any $n \geq 2$ and $k = 1$ and finally in [11, p.130] for arbitrary $n$ and $k$.

Leavitt path algebras (Lpas) were introduced a decade ago [1, 5], associating a $K$-algebra to a directed graph. For a graph with one vertex and $k + 1$ loops, it recovers the Leavitt algebra $L_K(1, k + 1)$. The definition and the development of the theory were inspired on the one hand by Leavitt’s construction of $L_K(1, k + 1)$ and on the other hand by Cuntz algebras $O_n$ [6] and Cuntz-Krieger algebras in $C^*$-algebra theory [13]. The Cuntz algebras and later Cuntz-Krieger type $C^*$-algebras revolutionised $C^*$-theory, leading ultimately to the astounding Kirchberg-Phillips classification theorem [12]. In the last decade the Lpas have created the same type of stir in the algebraic community. The development of Lpas and its interaction with graph $C^*$-algebras have been well-documented in several publications and we refer the reader to [2] and the references therein.

Since their introductions, there have been several attempts to introduce a generalisation of Lpas which would cover the algebras $L_K(n, n + k)$ for any $n \geq 1$, as well. Ara and Goodearl’s Leavitt path algebras of separated graphs were introduced in [4] which gives $L_K(n, n + k)$ as a corner ring of some separated graphs. The weighted Leavitt path algebras (wLpas) were introduced by R. Hazrat in [7], which gives $L_K(n, n + k)$ for a weighted graph with one vertex and $n + k$ loops of weight $n$. If the weights of all the edges are 1 (i.e., the graph is unweighted), then the wLpas reduce to the usual Lpas.

In [3] linear bases for Lpas were obtained and used to determine the Gelfand-Kirillov dimension of an Leavitt path algebra $L_K(E)$ where $K$ is a field and $E$ a finite directed graph. In [8] linear bases for wLpas were obtained, generalising the basis result for Lpas given in [3]. These bases were used to classify the wLpas which are domains, simple and graded simple rings. In this note we use them to determine the Gelfand-Kirillov dimension of a weighted Leavitt path algebra $L_K(E, w)$ where $K$ is a field and $(E, w)$ a finite weighted graph. Further we show that a finite-dimensional weighted Leavitt path algebra over a field $K$ is isomorphic to a finite product of matrix rings over $K$.

The rest of the paper is organised as follows. In Section 2 we recall some basic facts about wLpas. In Section 3 we prove our first main result, Theorem 22, which gives the Gelfand-Kirillov dimension of a weighted Leavitt path algebra $L_K(E, w)$ where $K$ is a field and $(E, w)$ a finite weighted graph. In Section

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4 we compute the Gelfand-Kirillov dimension of some concrete examples of wLpas. In Section 5 we prove our second main result, the Finite Dimension Theorem 46.

2. Weighted Leavitt path algebras

Throughout this section $R$ denotes an associative, unital ring.

**Definition 1 (Directed graph).** A directed graph is a quadruple $E = (E^0, E^1, s, r)$ where $E^0$ and $E^1$ are sets and $s, r : E^1 \to E^0$ maps. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. If $e$ is an edge, then $s(e)$ is called its source and $r(e)$ its range. $E$ is called row-finite if $s^{-1}(v)$ is a finite set for any vertex $v$ and finite if $E^0$ and $E^1$ are finite sets.

**Definition 2 (Double graph of a directed graph).** Let $E$ be a directed graph. The directed graph $E_d = (E^0_d, E^1_d, s_d, r_d)$ where $E^0_d = E^0$, $E^1_d = E^1 \cup (E^1)^*$ where $(E^1)^* = \{ e^* | e \in E^1 \}$, $s_d(e) = s(e)$, $r_d(e) = r(e)$, $s_d(e^*) = r(e)$ and $r_d(e^*) = s(e)$ for any $e \in E^1$ is called the double graph of $E$. We sometimes refer to the edges in the graph $E$ as real edges and the additional edges in $E_d$ (i.e. the elements of $(E^1)^*$) as ghost edges.

**Definition 3 (Path).** Let $E$ be a directed graph. A path is a nonempty word $p = x_1 \ldots x_n$ over the alphabet $E^0 \cup E^1$ such that either $x_i \in E^1$ ($i = 1, \ldots, n$) and $r(x_i) = s(x_{i+1})$ ($i = 1, \ldots, n-1$) or $n = 1$ and $x_1 \in E^0$. By definition, the length $|p|$ of $p$ is $n$ in the first case and $0$ in the latter case. We set $s(p) := s(x_1)$ and $r(p) := r(x_n)$ (here we use the convention $s(v) = v = r(v)$ for any $v \in E^0$). A closed path is a path $p$ such that $|p| \neq 0$ and $s(p) = r(p)$. A cyclic path is a closed path $p = x_1 \ldots x_n$ such that $s(x_i) \neq s(x_j)$ for any $i \neq j$.

**Definition 4 (Path algebra).** Let $E$ be a directed graph. The quotient $R(E^0 \cup E^1)/I$ of the free $R$-ring $R(E^0 \cup E^1)$ generated by $E^0 \cup E^1$ and the ideal $I$ of $R(E^0 \cup E^1)$ generated by the relations

(i) $vw = \delta_{vw} v$ for any $v, w \in E^0$

(ii) $s(e)e = e = er(e)$ for any $e \in E^1$

is called the path algebra of $E$ and is denoted by $P_R(E)$.

**Remark 5.** The paths in $E$ form a basis for the path algebra $P_R(E)$.

**Definition 6 (Weighted graph).** A weighted graph is a pair $(E, w)$ where $E$ is a directed graph and $w : E^1 \to \mathbb{N} = \{ 1, 2, \ldots \}$ is a map. If $e \in E^1$, then $w(e)$ is called the weight of $e$. A weighted graph $(E, w)$ is called row-finite (resp. finite) if $E$ is row-finite (resp. finite). In this article all weighted graphs are assumed to be row-finite and to have at least one vertex.

**Remark 7.** Let $(E, w)$ be a weighted graph. In [7] and [8], $E^1$ is denoted by $E^{st}$. What is denoted by $E^1$ in [7] and [8] is denoted by $E^{1}$ in this article (see the next definition).

**Definition 8 (Directed graph associated to a weighted graph).** Let $(E, w)$ be a weighted graph. The directed graph $\hat{E} = (\hat{E}^0, \hat{E}^1, \hat{s}, \hat{r})$ where $\hat{E}^0 = E^0$, $\hat{E}^1 := \{ e_1, \ldots, e_{w(e)} | e \in E^1 \}$, $\hat{s}(e_i) = s(e)$ and $\hat{r}(e_i) = r(e)$ is called the directed graph associated to $(E, w)$. We sometimes refer to the edges in the weighted graph $(E, w)$ as structured edges to distinguish them from the edges in the associated directed graph $\hat{E}$.

Until the end of this section $(E, w)$ denotes a weighted graph. A vertex $v \in E^0$ is called a sink if $s^{-1}(v) = \emptyset$ and regular otherwise. The set of all regular vertices is denoted by $E^0_{\text{reg}}$. For a $v \in E^0_{\text{reg}}$ we set $w(v) := \max\{ w(e) | e \in s^{-1}(v) \}$. $\hat{E}_d$ denotes the double graph of the directed graph $\hat{E}$ associated to $(E, w)$.

**Definition 9 (Weighted Leavitt path algebra).** The quotient $P_{\hat{R}}(\hat{E}_d)/I$ of the path algebra $P_{\hat{R}}(\hat{E}_d)$ and the ideal $I$ of $P_{\hat{R}}(\hat{E}_d)$ generated by the relations
(i) \[ \sum_{e \in s^{-1}(v)} e_i e_j^* = \delta_{ij} v \] for all \( v \in E^0_{\text{reg}} \) and \( 1 \leq i, j \leq w(v) \) and

(ii) \[ \sum_{1 \leq i \leq w(v)} e_i^* f_i = \delta_{ef} r(e) \] for all \( v \in E^0_{\text{reg}} \) and \( e, f \in s^{-1}(v) \)

is called \textit{weighted Leavitt path algebra of} \((E, w)\) and is denoted by \( L_R(E, w) \). In relations (i) and (ii), we set \( e_i \) and \( e_i^* \) zero whenever \( i > w(e) \).

**Example 10.** Let \( K \) be a field. It is easy to see that the wLpa of a weighted graph consisting of one vertex and \( n + k \) loops of weight \( n \) is isomorphic to the Leavitt algebra \( L_K(n, n + k) \), for details see [8, Example 4].

**Example 11.** If \( w(e) = 1 \) for all \( e \in E^1 \), then \( L_R(E, w) \) is isomorphic to the usual Leavitt path algebra \( L_R(E) \).

We call a path in the double graph \( \hat{E}_d \) a \textit{d-path}. While the d-paths form a basis for the path algebra \( P_R(\hat{E}_d) \), a basis for the weighted Leavitt path algebra \( L_R(E, w) \) is formed by the nod-paths, which we will define in the next definition.

For any \( v \in E^0_{\text{reg}} \) fix an \( e^v \in s^{-1}(v) \) such that \( w(e^v) = w(v) \). The words

\[ e_i^* (e_j^v) r \quad (v \in E^0_{\text{reg}}, 1 \leq i, j \leq w(v)) \quad \text{and} \quad e_i^* f_i \quad (v \in E^0_{\text{reg}}, e, f \in s^{-1}(v)) \]

over the alphabet \( \hat{E}_d \) are called \textit{forbidden}. If \( A = x_1 \ldots x_n \) is a word over some alphabet, then we call the words \( x_i \ldots x_j \) \( (1 \leq i \leq j \leq n) \) \textit{subwords of} \( A \).

**Definition 12 (Nod-path).** A \textit{normal d-path} or \textit{nod-path} is a d-path such that none of its subwords is forbidden.

**Theorem 13** (Hazrat, Preusser, 2017). \textit{The nod-paths form a basis for} \( L_R(E, w) \).

**Proof.** See [8, Theorem 16] \[ \square \]

### 3. Weighted Leavitt path algebras of polynomial growth

First we want to recall some general facts on the growth of algebras. Let \( K \) be a field and \( A \) an \( K \)-algebra (not necessarily unital), which is generated by a finite-dimensional subspace \( V \). For \( n \geq 1 \) let \( V^n \) denote the span of the set \{ \( v_1 \ldots v_k \mid k \leq n, v_1, \ldots, v_k \in V \) \}. Then \( V = V^1 \subseteq V^2 \subseteq \ldots \), \( A = \bigcup_{n \geq 1} V^n \) and \( d_V(n) := \dim V^n < \infty \). Given functions \( f, g \) from the positive integers \( \mathbb{N} \) to the positive real numbers \( \mathbb{R}^+ \), we write \( f \prec g \) if there is a \( c \in \mathbb{N} \) such that \( f(n) \leq cg(cn) \) for all \( n \). If \( f \prec g \) and \( g \prec f \), then the functions \( f, g \) are called \textit{asymptotically equivalent} and we write \( f \sim g \). If \( W \) is another finite-dimensional subspace that generates \( A \), then \( d_V \sim d_W \). The \textit{Gelfand-Kirillov dimension} or \textit{GK dimension} of \( A \) is defined as

\[ \text{GKdim} \ A := \limsup_{n \to \infty} \log_n d_V(n). \]

The definition of the GK dimension does not depend on the choice of the finite-dimensional generating space \( V \). If \( d_V \leq n^m \) for some \( m \in \mathbb{N} \), then \( A \) is said to have \textit{polynomial growth} and we have \( \text{GKdim} \ A \leq m \). If \( d_V \sim n^m \) for some real number \( a > 1 \), then \( A \) is said to have \textit{exponential growth} and we have \( \text{GKdim} \ A = \infty \).

Until right after the proof of Theorem 22, \( K \) denotes a field and \((E, w)\) a finite weighted graph. Further \( V \) denotes the finite-dimensional subspace of \( L_K(E, w) \) spanned by \( \hat{E}_d^0 \cup \hat{E}_d^1 \) (i.e. spanned by the vertices, real edges and ghost edges).

If \( X \) is a set, we denote by \( \langle X \rangle \) the set of all words over \( X \) including the empty word. Together with juxtaposition \( \langle X \rangle \) is a monoid. If \( A, B \in \langle X \rangle \), then we write \( A|B \) if there is a \( C \in \langle X \rangle \) such that \( AC = B \).

**Definition 14.** Let \( p \) and \( q \) be nod-paths. If there is a nod-path \( o \) such that \( p|o \) and \( poq \) is a nod-path, then we write \( p \equiv_{\text{nod}} q \). If \( pq \) is a nod-path or \( p \equiv_{\text{nod}} q \), then we write \( p \equiv q \).
Definition 15 (Nod²-path, quasi-cycle). A nod²-path is a nod-path $p$ such that $p^2$ is a nod-path. A quasi-cycle is a nod²-path $p$ such that none of the subwords of $p^2$ of length $< |p|$ is a nod²-path. A quasi-cycle $p$ is called selfconnected if $p \Rightarrow p$.

Remark 16. (a) It is easy to see that if $p = x_1 \ldots x_n$ is a quasi-cycle, then $x_i \neq x_j$ for all $i \neq j$ (otherwise there would be a subword of $p^2$ of length $< |p|$ that is a nod²-path). It follows that there is only a finite number of quasi-cycles since $E$ is finite. 
(b) Let $p = x_1 \ldots x_n$ be a quasi-cycle and $\pi \in S^n$ an $n$-cycle. Then $q := x_{\pi(1)} \ldots x_{\pi(n)}$ is a quasi-cycle and we write $p \approx q$. Clearly $\approx$ is an equivalence relation on the set of all quasi-cycles.
(c) Let $p = x_1 \ldots x_n$ be a quasi-cycle. Then $p^* := x_n^* \ldots x_1^*$ is a quasi-cycle.

Example 17. Suppose $(E, w)$ is the weighted graph

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   u →^e_2 v →_f x
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(here $e$ has weight 2 and $f$ and $g$ have weight 1). Then the associated directed graph $\hat{E}$ and its double graph $\hat{E}_d$ are

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   u →_e_1 v →_f_1 x →_g_1 y
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(for ghost edges we draw dashed arrows). One checks easily that $p := e_2 f_1 g_1^* e_2^*$ and $q := v_2 f_1 g_1^* e_1^*$ are quasi-cycles independent of the choice of $e^\nu$. Further $pqp$ and $qpq$ are nod-paths and therefore $p$ and $q$ are selfconnected. This example shows that a quasi-cycle can meet a vertex more than once.

The following lemma shows, that quasi-cycles behave like cycles in a way (one cannot "take a short-cut").

Lemma 18. Let $p = x_1 \ldots x_n$ be a quasi-cycle and $1 \leq i, j \leq n$. Then $x_ix_j$ is a nod-path iff $i < n$ and $j = i + 1$ or $i = n$ and $j = 1$.

Proof. If $i < n$ and $j = i + 1$ or $i = n$ and $j = 1$, then clearly $x_ix_j$ is a nod-path. Suppose now that $x_ix_j$ is a nod-path.

1. Suppose $i = j$. Assume that $n > 1$. Then we get the contradiction that $x_i$ is a nod²-path which is a subword of $p^2$ of length $1 < |p| = n$. Hence $n = 1$ and we have $i = j = 1 = n$.

2. Suppose $i < j$. Then $x_j \ldots x_n x_1 \ldots x_i$ is a nod²-path which is a subword of $p^2$ of length $n - j + 1 + i$. It follows that $j = i + 1$.

3. Suppose $j < i$. Then $x_j \ldots x_i$ is a nod²-path which is a subword of $p^2$ of length $i - j + 1$. It follows that $j = 1$ and $i = n$. \hfill $\square$

Lemma 19. If there is a selfconnected quasi-cycle $p$, then $L_K(E, w)$ has exponential growth.

Proof. Let $o$ be a nod-path such that $p | o$ and $pop$ is a nod-path. Let $n \in \mathbb{N}$. Consider the nod-paths

$$p^{i_1} op^{i_2} \ldots op^{i_k} \quad (1)$$
where \( k, i_1, \ldots, i_k \in \mathbb{N} \)
satisfy
\[
(i_1 + \cdots + i_k)|p| + (k - 1)|r| \leq n.
\] (2)

Let \( A = (k, i_1, \ldots, i_k) \) and \( B = (k', i_1', \ldots, i_k') \) be different solutions of (2). Assume that \( A \) and \( B \) define the same nod-path in (1). After cutting out the common beginning, we can assume that the nod-path defined by \( A \) starts with \( o \) and the nod-path defined by \( B \) with \( p \) or vice versa. Since \( p \neq o \), it follows that \( |p| > |o| \). Write \( p = x_1 \ldots x_m \). Since the next letter after an \( o \) must be a \( p \), we get \( x_{|o|+1} = x_1 \) which contradicts Remark 16(a). Hence different solutions of (2) define different nod-paths in (1). By Theorem 13 the nod-paths in (1) are linearly independent in \( V^n \). The number of solutions of (2) is \( \sim 2^n \) and hence \( L_R(E, w) \) has exponential growth. \( \square \)

**Corollary 20.** If there is a vertex \( v \) and structured edges \( e, f \in s^{-1}(v) \) such that \( w(\alpha), w(\beta) \geq 2 \), then \( L_R(E, w) \) has exponential growth.

**Proof.** Choose \( e^v = e \). First suppose that \( r(f) = v \). Then \( p := f_2 \) is a quasi-cycle. Further \( f_2 f_2^* f_2 \) is a nod-path and therefore \( p \) is selfconnected. Hence, by the previous lemma, \( L_R(E, w) \) has exponential growth. Now suppose that \( r(f) \neq v \). Then \( p := f_2 f_2^* f_2 \) is a quasi-cycle. Further \( f_2 f_2^* f_2 f_2^* f_2 f_2^* \) is a nod-path and therefore \( p \) is selfconnected. Hence, by the previous lemma, \( L_R(E, w) \) has exponential growth. \( \square \)

Let \( E' \) denote the set of all real and ghost edges which do not appear in a quasi-cycle. Let \( P' \) denote the set of all nod-paths which are composed from elements of \( E' \).

**Lemma 21.** \( |P'| < \infty \).

**Proof.** Let \( p' = x_1 \ldots x_n \in P' \). Assume that there are \( 1 \leq i < j \leq n \) such that \( x_i = x_j \). Then \( x_1 \ldots x_{j-1} \) is a nod\(^2\)-path. Since for any nod\(^2\)-path \( q \) which is not a quasi-cycle there is a shorter nod\(^2\)-path \( q' \) such that any letter of \( q' \) already appears in \( q \), we get a contradiction. Hence the \( x_i \)'s are pairwise distinct. It follows that \( |P'| < \infty \) since \( |E'| < \infty \). \( \square \)

A sequence \( p_1, \ldots, p_k \) of quasi-cycles such that \( p_i \neq p_j \) for any \( i \neq j \) is called a **chain of length** \( k \) if \( p_1 \implies p_2 \implies \cdots \implies p_k \).

**Theorem 22.**

(i) \( L_K(E, w) \) has polynomial growth iff there is no selfconnected quasi-cycle.

(ii) If \( L_K(E, w) \) has polynomial growth, then \( \text{GKdim } L_K(E, w) = d \) where \( d \) is the maximal length of a chain of quasi-cycles.

**Proof.** If there is a selfconnected quasi-cycle, then \( L_K(E, w) \) has exponential growth by Lemma 19. Suppose now that there is no selfconnected quasi-cycle. By Theorem 13 the nod-paths of length \( \leq n \) form a basis for \( V^n \). Clearly we can write any nod-path of length \( \leq n \) in the form
\[
o_1 p_1^{l_1} q_1 o_2 p_2^{l_2} q_2 o_3 \ldots o_k p_k^{l_k} q_k o_{k+1}
\] (3)

where \( o_i \in P' \) (\( 1 \leq i \leq k + 1 \)), \( p_1, \ldots, p_k \) is a chain of quasi-cycles, \( l_i \geq 0 \) (\( 1 \leq i \leq k \)) and \( q_i \neq p_i \) is a nod-path such that \( q_i|p_i| \) (\( 1 \leq i \leq k \)) (we allow the \( o_i \)'s and \( q_i \)'s to be the empty word). Clearly \( l_i|p_i| + \cdots + l_k|p_k| \leq n \). This implies that for a fixed chain \( p_1, \ldots, p_k \) of quasi-cycles, the number of the words in (3) is \( \leq n^k \leq n^d \). Since there are only finitely many quasi-cycles, the number of nod-paths of length \( \leq n \) is \( \leq n^d \).

On the other hand, choose a chain \( p_1, \ldots, p_d \) of length \( d \). Then \( p_1 o_1 p_2 \ldots o_{d-1} p_d \) is a nod-path for some \( o_1, \ldots, o_{d-1} \) such that for any \( i \in \{1, \ldots, d-1\} \), \( o_i \) is either the empty word or a nod-path such that \( p_i/o_i \). Consider the nod-paths
\[
p_1^{l_1} o_1 p_2^{l_2} \ldots o_{d-1}^{l_{d-1}} p_d^{l_d}
\] (4)

where \( l_1, \ldots, l_d \in \mathbb{N} \)
satisfy
\[
l_1|p_1| + \cdots + l_d|p_d| + |o_1| + \cdots + |o_{d-1}| \leq n.
\] (5)
Let \( A = (l_1, \ldots, l_d) \) and \( B = (l'_1, \ldots, l'_d) \) be different solutions of (5). Assume that \( A \) and \( B \) define the same nod-path in (4). After cutting out the common beginning, we can assume that the nod-path defined by \( A \) starts with \( o_ip_i+1 \) for some \( i \in \{1, \ldots, d-1\} \) and the nod-path defined by \( B \) with \( p_i^2 \). If \( o_i \) is the empty word, then we get the contradiction \( p_i = p_{i+1} \), since \( p_i \) and \( p_{i+1} \) are quasi-cycles. Suppose now that \( o_i \) is not the empty word. Since \( p_i \neq o_i \), it follows that \( |o_i| < |p_i| \). Further \( |p_i| < |o_i| + |p_{i+1}| \) (otherwise \( p_{i+1} \) would be a subword of \( p_i \) of length \( < |p_i| \)). Write \( p_i = x_1 \ldots x_k \) and \( p_{i+1} = y_1 \ldots y_m \).

**case 1** Assume that \( |p_i| \leq |p_{i+1}| \). Then \( o_i = x_1 \ldots x_j \) and \( p_{i+1} = x_{j+1} \ldots x_kx_1 \ldots x_jy_{k+1} \ldots y_m \) for some \( j \in \{1, \ldots, k-1\} \). By Remark 16(b), \( x_{j+1} \ldots x_kx_1 \ldots x_j \) is a quasi-cycle. It follows that \( k = m \). Hence we get the contradiction \( p_i \approx p_{i+1} \).

**case 2** Assume that \( |p_i| > |p_{i+1}| \). Then \( o_i = x_1 \ldots x_j \) and \( p_{i+1} = x_{j+1} \ldots x_kx_1 \ldots x_j \) for some \( j \in \{1, \ldots, k-1\} \) and \( l \in \{1, \ldots, j-1\} \). But this yields the contradiction that \( p_{i+1} \) is a subword of \( p_i^l \) of length \( < |p_i| \).

Hence different solutions of (5) define different nod-paths in (4). By Theorem 13 the nod-paths in (4) are linearly independent in \( V^n \). The number of solutions of (5) is \( \sim n^d \) and thus \( n^d \ll \) the number of nod-paths of length \( \leq n \).

As a corollary we recover Theorem 5 of [3]. We use the following terminology: Let \( E \) be a directed graph. We denote the set of all cyclic paths by \( CP \). If \( p \in CP \), we denote by \( E(p) \) the subgraph of \( E \) defined by \( p \). A cycle (in the sense of [3]) is a subgraph \( E(p) \) where \( p \in CP \).

**Corollary 23.** Let \( K \) be a field and \( E \) be a finite directed graph. Then:

(i) \( L_K(E) \) has polynomial growth iff two distinct cycles do not have a common vertex.

(ii) If \( L_K(E) \) has polynomial growth, then \( \text{GKdim } L_K(E) = \max(2d_1 - 1, 2d_2) \) where \( d_1 \) is the maximal length of a chain of cycles and \( d_2 \) is the maximal length of a chain of cycles with an exit.

**Proof.** Let \( (E, w) \) be the weighted graph such that \( w \equiv 1 \). Then \( L_K(E) \equiv L_K(E, w) \) (see Example 11). It is easy to see that \( \{p, p^* \mid p \in CP \} \) is the set of all quasi-cycles of \( (E, w) \) (we identify \( E \) with the directed graph \( E^o \) associated to \( (E, w) \)). We will show (i) first and then (ii).

(i) First suppose that \( L_K(E) \) has polynomial growth. Then, by the previous theorem, there is no selfconnected quasi-cycle. Assume that there are two distinct cycles \( C_1 \) and \( C_2 \) with a common vertex \( v \). Then there are \( p, q \in CP \) such that \( E(p) = C_1, E(q) = C_2, s(p) = r(p) = s(q) = r(q) = v \) and \( p \neq q \). By the previous paragraph, \( p \) is a quasi-cycle. Further \( p \) is selfconnected (since \( pqp \) is a nod-path) and hence we arrived at a contradiction. Thus two distinct cycles do not have a common vertex.

Now suppose that two distinct cycles do not have a common vertex. Assume that there is a selfconnected quasi-cycle \( p \). We only consider the case that \( p \in CP \), the case that \( p^* \in CP \) is similar. Let \( o = x_1 \ldots x_n \) be a path such that \( pop \) is a path and \( p \neq o \). Since \( o \) is a closed path, there is a subword \( p' = x_1 \ldots x_j \) of \( o \) (where \( 1 \leq i \leq j \leq n \)) such that \( p' \in CP \). It follows from [3, Lemma 4] that \( E(p') = E(p) \). Assume that \( i = 1 \). Then \( s(p') = r(p') = s(p) \) which implies that \( p' = p \). But that contradicts \( p \neq o \). Hence \( i > 1 \). Set \( o' := x_1 \ldots x_{i-1} \). Then clearly \( p \neq o', po'p \) is a path and \( |o'| = |o| \). We see that we arrive at a contradiction after repeating this step a finite number of times. Hence there is no selfconnected quasi-cycle and thus, by the previous theorem, \( L_K(E) \) has polynomial growth.

(ii) Let \( d \) be the maximal length of a chain of quasi-cycles, \( d_1 \) the maximal length of a chain of cycles and \( d_2 \) is the maximal length of a chain of cycles with an exit. We have to show that \( d = \max(2d_1 - 1, 2d_2) \). Let \( C_1, \ldots, C_{d_2} \) a chain of cycles with an exit. Choose \( p_1, \ldots, p_{d_2} \in CP \) such that \( E(p_i) = C_i \) for any \( 1 \leq i \leq d_2 \). Then \( p_1, \ldots, p_{d_2}, p_{d_2}', \ldots, p_1' \) is a chain of quasi-cycles (see the proof of [3, Theorem 5]) and hence \( d \geq 2d_2 \). Let now \( C_1, \ldots, C_{d_1} \) be a chain of cycles. If it has an exit, then \( d_1 = 2d_2 \) and we have \( d \geq 2d_2 = 2d_1 > 2d_1 - 1 \). Suppose now that \( C_1, \ldots, C_{d_1} \) has no exit. Choose \( p_1, \ldots, p_{d_1} \in CP \) such that
$E(p_i) = C_i$ for any $1 \leq i \leq d_1$. Then $p_1, \ldots, p_{d_1}, p_{d_1-1}, \ldots, p_1^*$ is a chain of quasi-cycles (see the proof of [3, Theorem 5]) and hence $d \geq 2d_1 - 1$. Thus we have shown that $d \geq \max(2d_1 - 1, 2d_2)$. On the other hand it is easy to see that $d \leq \max(2d_1 - 1, 2d_2)$ and hence we have $d = \max(2d_1 - 1, 2d_2)$ as desired. \qed

4. Examples

Throughout this section $K$ denotes a field. We will consider only connected, irreducible weighted graphs (cf. [8, Definitions 21, 23]). While wLpas of reducible weighted graphs are isomorphic to Lpas (cf. [8, Proposition 28]), it is an open question which of the wLpas of irreducible weighted graphs are isomorphic to Lpas.

In general it is not so easy to read off the quasi-cycles from a finite weighted graph. But there is the following algorithm to find all the quasi-cycles: For any vertex $v$ list all the $d$-paths $x_1 \ldots x_n$ starting and ending at $v$ and having the property that $x_i \neq x_j$ for any $i \neq j$ (there are only finitely many of them). Now delete from that list any $p$ such that $p^2$ is not a nod-path. Next delete from the list any $p$ such that $p^2$ has a subword $q$ of length $|q| < |p|$ such that $q^2$ is a nod-path. The remaining $d$-paths on the list are precisely the quasi-cycles starting (and ending) at $v$.

First we consider two trivial examples which show that small changes in the weighted graph can change the GK dimension of its wLpa drastically.

Example 24. Consider the weighted graph

\[(E, w) : u \xrightarrow{e,2} v \xrightarrow{f} x .\]

One checks easily that the length of a nod-path is bounded (the unique longest nod-path is $e_2^*f_1f_1^*e_2$). Hence there is no nod$^2$-path and therefore no quasi-cycle. Thus, by Theorem 22, $\text{GKdim } L_K(E, w) = 0$.

Example 25. Consider the weighted graph

\[(E, w) : u \xrightarrow{e,2} v \xrightarrow{f,2} x .\]

By Corollary 20, $L_K(E, w)$ has exponential growth and hence $\text{GKdim } L_K(E, w) = \infty$.

Through the next example we obtain the GK dimensions of the Leavitt algebras $L_K(n, n+k)$.

Example 26. Let $n \geq 1$ and $k \geq 0$. Consider the weighted graph

\[(E, w) : e^{(n+k),n} \quad v \quad e^{(1),n} \quad e^{(3),n} \quad e^{(2),n} .\]

As mentioned in Example 10, $L_K(E, w)$ is isomorphic to the Leavitt algebra $L_K(n, n+k)$. If $n > 1$, then, by Corollary 20, $L_K(E, w)$ has exponential growth and hence $\text{GKdim } L_K(E, w) = \infty$. If $n = 1$ and $k = 0$, then $\text{GKdim } L_K(E, w) = 1$ by Corollary 23. If $n = 1$ and $k > 0$, then $\text{GKdim } L_K(E, w) = \infty$ by Corollary 23.

Next we consider again the weighted graph from Example 17.

Example 27. Consider the weighted graph

\[u \xrightarrow{e,2} v \quad f \quad x .\]

Then $p = e_2^*f_1g_1^*e_2^*$ is a selfconnected quasi-cycle (see Example 17) and hence, by Lemma 19, $L_K(E, w)$ has exponential growth. Thus $\text{GKdim } L_K(E, w) = \infty$. 
The next example shows, that for any positive integer \( n \) there is a connected, irreducible weighted graph \((E, w)\) such that \( \text{GKdim} \ L_K(E, w) = n \).

**Example 28.** Let \( n \in \mathbb{N} \). Consider the weighted graph

\[
(E, w) : \quad u \overset{e, 2}{\longrightarrow} v \overset{f(1)}{\longrightarrow} x_1 \overset{g(1)}{\longrightarrow} x_2 \overset{f(2)}{\longrightarrow} x_3 \overset{g(2)}{\longrightarrow} \ldots \overset{f(n)}{\longrightarrow} x_n.
\]

One checks easily that the only quasi-cycles are \( p_i := g_1^{(i)} \) and \( p_i^* = (g_1^{(i)})^* \) (1 \( \leq \) \( i \) \( \leq \) \( n \)) and that they are not selfconnected. The longest chains of quasi-cycles are \( p_1, \ldots, p_{n-1}, p_n, p_n^*, \ldots, p_1^* \) and \( p_1, \ldots, p_{n-1}, p_n^*, p_n^*, \ldots, p_1^* \). Hence \( \text{GKdim} \ L_K(E, w) = 2n - 1 \) by Theorem 22. Consider now the weighted graph

\[
(E, w) : \quad u \overset{e, 2}{\longrightarrow} v \overset{f(1)}{\longrightarrow} x_1 \overset{g(1)}{\longrightarrow} x_2 \overset{f(2)}{\longrightarrow} x_3 \overset{g(2)}{\longrightarrow} \ldots \overset{f(n)}{\longrightarrow} x_n \overset{f(n+1)}{\longrightarrow} x_{n+1}.
\]

One checks easily that the only quasi-cycles are \( p_i := g_1^{(i)} \) and \( p_i^* = (g_1^{(i)})^* \) (1 \( \leq \) \( i \) \( \leq \) \( n \)) and that they are not selfconnected. The longest chain of quasi-cycles is \( p_1, \ldots, p_n, p_n, \ldots, p_1 \) and hence \( \text{GKdim} \ L_K(E, w) = 2n \) by Theorem 22.

We finish this section with a last example. We use the following terminology: Let \( x, y \in E_1^1 \). If \( xy \) is a nod-path, then \( y \) is called a *nod-successor* of \( x \). Let \( p = x_1 \ldots x_n \) be a quasi-cycle, \( y \in E_1^1 \) and \( 1 \leq i \leq n \). If \( y \) is a nod-successor of \( x_i \), which is not equal to \( x_{i+1} \) if \( i < n \) resp. to \( x_1 \) if \( i = n \), then the nod-path \( x_i y \) is a called a *nod-exit* of \( p \). Let \([p]\) denote the \( \approx \)-equivalence class of \( p \). If \( q \in [p] \), then clearly \( p \) and \( q \) have the same nod-exits. Hence we can define a *nod-exit of* \([p]\) to be a nod-exit of \( p \).

**Example 29.** Consider the weighted graph

\[
\begin{array}{c}
\text{u} \\
\overset{g}{\nearrow} \\
\text{x} \\
\downarrow \\
\text{f} \\
\end{array} \quad \begin{array}{c}
\text{v} \\
\end{array}
\]

Applying the algorithm described in the second paragraph of this section one gets that the only \( \approx \)-equivalence classes of quasi-cycles are \([e_2 f_1 g_1]\) and \([g_1 f_1^* e_2]\). While \([e_2 f_1 g_1]\) has no nod-exit, \([g_1 f_1^* e_2]\) has the nod-exits

\[
e_2^* e_1, \quad e_2^* e_2, \quad f_1^* e_1^*, \quad g_1 g_1^*.
\]

The only nod-successor of \( e_1 \) is \( f_1 \) and the only nod-successor of \( e_1^* \) and \( g_1^* \) is \( e_2 \). But \( f_1 \) and \( e_2 \) belong to \( e_2 f_1 g_1^* \) which has no nod-exit. We leave it to the reader to conclude that there is no selfconnected quasi-cycle and the maximal length of a chain of quasi-cycles is 2. Thus, by Theorem 22, we have \( \text{GKdim} \ L_K(E, w) = 2 \).

5. Finite-dimensional weighted Leavitt path algebras

Throughout this section \( K \) denotes a field and \((E, w)\) a weighted graph. We call a \( K \)-algebra *finite-dimensional* if it is finite-dimensional as a \( K \)-vector space. The goal of this section is to prove that if \( L_K(E, w) \) is finite-dimensional, then it is isomorphic to a finite product of matrix rings over \( K \).

We call \((E, w)\) *aquasicyclic* if there is no quasi-cycle.

**Lemma 30.** \( L_K(E, w) \) is finite-dimensional iff \((E, w)\) is finite and aquasicyclic.

**Proof.** Follows from Theorems 13 and 22 and the fact that a finitely generated \( K \)-algebra \( A \) is finite-dimensional iff \( \text{GKdim} A = 0 \).


Until right before the Finite Dimension Theorem 46, \((E, w)\) is assumed to be finite and aquasicyclic.

**Definition 31 (Tree).** If \(u, v \in E^0\) and there is a path \(p\) in \(E\) such that \(s(p) = u\) and \(r(p) = v\), then we write \(u \geq v\). Clearly \(\geq\) is a preorder on \(E^0\). If \(u \in E^0\) then \(T(u) := \{ v \in E^0 \mid u \geq v \}\) is called the tree of \(u\). If \(X \subseteq E^0\), we define \(T(X) := \bigcup_{v \in X} T(v)\).

**Definition 32 (Range Weight Forest).** A structured edge \(e \in E^1\) is called weighted if \(w(e) > 1\). The subset of \(E^0\) consisting of all weighted structured edges is denoted by \(E_w\). The set \(\text{RWF}(E, w) := T(r(E^1_w))\) is called the range weight forest of \((E, w)\).

We call \((E, w)\) acyclic if there is no cyclic path in \(E\). We call two structured edges \(e\) and \(f\) in line if \(e = f\) or \(r(e) \geq s(f)\) or \(r(e) \geq s(f)\).

**Lemma 33.** \((E, w)\) is acyclic, any vertex \(v\) emits at most one weighted structured edge, any vertex \(v \in \text{RWF}(E, w)\) emits at most one structured edge and \(T(r(e)) \cap T(r(f)) = \emptyset\) for any \(e, f \in E^1_w\) which are not in line.

**Proof.** That \((E, w)\) is acyclic is clear since any cyclic path in \(\hat{E}\) is a quasi-cycle. Assume there is a vertex \(v\) which emits two distinct weighted structured edges. The proof of Corollary 20 shows that then there is a quasi-cycle and hence we have a contradiction. Suppose now that there is a \(v \in \text{RWF}(E, w)\) such that \(|s^{-1}(v)| \geq 2\). Choose an \(e \in s^{-1}(v) \setminus \{e^*\}\). By the definition of \(\text{RWF}(E, w)\), there is a structured edge \(f \in E^1_w\) and a path \(p\) in \(\hat{E}\) such that \(\hat{s}(p) = r(f)\) and \(\hat{r}(p) = v\). Then \(f_2p_e1e^*_pf_2^*f_2^*\) (resp. \(f_2e_1e^*_pf_2^*\) if \(|p| = 0\)) is a nod^2-path. Since for any nod^2-path \(q\) which is not a quasi-cycle there is a nod^2-path \(q'\) such that \(|q'| < |q|\), the existence of a quasi-cycle follows.

Now let \(e, f \in E^1_w\) be not in line. Assume there is a \(v \in T(r(e)) \cap T(r(f))\). Then there are paths \(p = x_1 \ldots x_m\) and \(q = y_1 \ldots y_n\) in \(\hat{E}\) such that \(\hat{s}(p) = r(e)\), \(\hat{s}(q) = r(f)\) and \(\hat{r}(p) = \hat{r}(q) = v\). W.l.o.g. assume that \(m \geq n\). After cutting off a possible common ending of \(p\) and \(q\) we may assume that we are in one of the following three cases:

1. **Case 1** Assume that \(|p|, |q| \neq 0\) and \(x_m \neq y_n\). Then \(e_2p_qf_2^*f_2qp^*e_2^*\) is a nod^2-path and hence there is a quasi-cycle.

2. **Case 2** Assume that \(|p| \neq 0\) and \(|q| = 0\). Clearly \(x_m \neq f_i\) for any \(1 \leq i \leq w(f)\) since \(e\) and \(f\) are not in line. Hence \(e_2p_f^*f_2p^*e_2^*\) is a nod^2-path and hence there is a quasi-cycle.

3. **Case 3** Assume that \(|p| = |q| = 0\). Since \(e\) and \(f\) are distinct, \(e_2f^*_2f_2e^*_2\) is a nod^2-path and hence there is a quasi-cycle. \(\square\)

We call an \(e \in E^1_w\) weighted structured edge of type A if \(s(e)\) emits only one structured edge (namely \(e\)) and weighted structured edge of type B otherwise.

**Lemma 34.** There is a finite, aquasicyclic weighted graph \((\hat{E}, \hat{w})\) such that \(L_K(\hat{E}, \hat{w}) \cong L_K(E, w)\), \((\hat{E}, \hat{w})\) has at most as many weighted structured edges as \((E, w)\), all weighted structured edges in \((\hat{E}, \hat{w})\) are of type B and their ranges are sinks.

**Proof.** Set \(Z := \text{RWF}(E, w) \cup \{ v \in E^0 \mid \{ e \} \text{ for some } e \in E^1_w \}.\) Define a weighted graph \((\hat{E}, \hat{w})\) by \(\hat{E}^0 = E^0, \hat{E}^1 = \{ e \mid e \in E^1, s(e) \notin Z \} \cup \{ e^{(1)}, \ldots, e^{w(e)} \mid e \in E^1, s(e) \in Z \}, \hat{s}(e) = s(e), \hat{r}(e) = r(e)\) and \(\hat{w}(e) = w(e)\) if \(s(e) \notin Z\) and \(\hat{s}(e^{(i)}) = r(e), \hat{r}(e^{(i)}) = s(e)\) and \(\hat{w}(e^{(i)}) = 1\) if \(s(e) \in Z\) and \(1 \leq i \leq w(e)\). One checks easily that \((\hat{E}, \hat{w})\) has at most as many weighted structured edges as \((E, w)\), all weighted structured edges in \((\hat{E}, \hat{w})\) are of type B and their ranges are sinks. The proof that \(L_K(\hat{E}, \hat{w}) \cong L_K(E, w)\) is very similar to the proof of [8, Proposition 28] and therefore is omitted. That \((\hat{E}, \hat{w})\) is finite and aquasicyclic follows from Lemma 30. \(\square\)
Example 35. Suppose \((E, w)\) is the finite, aquasicyclic weighted graph

\[
\begin{array}{cccccc}
  & k & \rightarrow & u & \rightarrow & x \\
 a & \downarrow & & \downarrow & h & \downarrow \\
 & & v & \rightarrow & y & \rightarrow \\
 & & & \downarrow & \downarrow & \downarrow \\
 s & \rightarrow & f & \rightarrow & g & \rightarrow \\
 & & & \downarrow & \downarrow & \downarrow \\
 & & & & i & j & \rightarrow \ c.
\end{array}
\]

The weighted structured edge \(i\) is of type A and the weighted structured edge \(e\) is of type B. Clearly \(\text{RWF}(E, w) = \{a, u, b, c\}\). Let \(Z\) be defined as in the proof of the previous lemma. Then \(Z = \{a, u, y, b, c\}\). Let \((\tilde{E}, \tilde{w})\) be the weighted graph

\[
\begin{array}{cccccc}
  & k & \rightarrow & u & \rightarrow & x \\
 a & \downarrow & & \downarrow & h & \downarrow \\
 & & v & \rightarrow & y & \rightarrow \\
 & & & \downarrow & \downarrow & \downarrow \\
 s & \rightarrow & f & \rightarrow & g & \rightarrow \\
 & & & \downarrow & \downarrow & \downarrow \\
 & & & & i^{(1)} & j & \rightarrow \ c.
\end{array}
\]

Then \(L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})\). In \((\tilde{E}, \tilde{w})\) there is only one weighted structured edge, namely \(e\), and it is of type \(B\). Further \(u = \tilde{r}(e)\) is a sink.

The next goal is to remove also the weighted structured edges of type \(B\) without changing the \(wLpa\), so that eventually one arrives at an unweighted graph (i.e. a weighted graph \((E', w')\) such that \(w' \equiv 1\)). Recall that a subset \(H \subseteq E^0\) is called hereditary if \(u \geq v\) where \(u \in H\) and \(v \in E^0\) implies \(v \in H\).

**Definition 36 (Weighted Subgraph Defined by Hereditary Vertex Set).** Let \(H \subseteq E^0\) be a hereditary subset. Set \(E_H^0 := H, E_H^1 := \{e \in E^1 \mid s(e) \in H\}\), \(r_H := r|_{E_H^1}\), \(s_H := s|_{E_H^1}\) and \(w_H := w|_{E_H^1}\).

Then \(E_H := (E_H^0, E_H^1, s_H, r_H)\) is a directed graph and \((E_H, w_H)\) a weighted graph. We call \((E_H, w_H)\) the weighted subgraph of \((E, w)\) defined by \(H\).

**Lemma 37.** Suppose that all weighted structured edges in \((E, w)\) are of type \(B\) and their ranges are sinks. Let \(v \in E^0\) be a vertex such that \(v\) is the only element of \(T(v)\) which emits a weighted structured edge. Then there is an unweighted graph \((E', w')\) such that \(L_K(E_T(v), w_{T(v)}) \cong L_K(E', w')\) via an isomorphism which maps vertices to sums of distinct vertices.

**Proof.** Clearly there is an integer \(k \geq 2\), integers \(m, n_1, \ldots, n_m \geq 1\), a vertex \(v \in E^0 \setminus \{v\}\), pairwise distinct vertices \(x_1, \ldots, x_m \in E^0 \setminus \{v\}\) and pairwise distinct structured edges \(e, f^{(ij)} \in E^1 (1 \leq i \leq m, 1 \leq j \leq n_i)\) such that \(s^{-1}(v) = \{e, f^{(ij)} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}\), \(r(e) = u, r(f^{(ij)}) = x_i, w(e) = k\) and \(w(f^{(ij)}) = 1\).

Set \(X := \{x_1, \ldots, x_m\}\). It is easy to see that \(u, v \notin T(X)\) (otherwise there would be a quasi-cycle). Define an unweighted graph \((E', w')\) by

\[
\begin{align*}
(E')^0 := & \{u_i (1 \leq i \leq k), u_{ij} (1 \leq i \leq m, 1 \leq j \leq (k - 1)n_i), \\
u, v_{ij} (1 \leq i \leq m, 2 \leq j \leq n_i), x_i (1 \leq i \leq m), y (y \in T(X) \setminus X)\}, \\
(E')^1 := & \{
\gamma^{(ij)} (1 \leq i \leq m, 1 \leq j \leq (k - 1)n_i), g (g \in E^1, s(g) \in T(X))\}, \\
s'(\alpha^{(1)}) = & v, r'(\alpha^{(1)}) = u_1, s'(\alpha^{(1)}) = u_{i-1}, r'(\alpha^{(1)}) = u_i (i \geq 2), \\
s'(\beta^{(1)}) = & v, r'(\beta^{(1)}) = u_{21}, s'(\beta^{(1)}) = v_{i1}, r'(\beta^{(1)}) = u_{i+1} (1 < j < n_i), s'(\beta^{(1)}) = v_{in_i}, r'(\beta^{(1)}) = x_i, \\
s'(\gamma^{(1)}) = & x_i, r'(\gamma^{(1)}) = u_{i1}, s'(\gamma^{(1)}) = u_{i,j-1}, r'(\gamma^{(1)}) = u_{ij} (j \geq 2), \\
s'(g) = & s(g), r'(g) = r(g) (g \in E^1, s(g) \in T(X) \setminus X), s'(g) = u_{i,(k-1)n_i}, r'(g) = r(g) (g \in E^1, s(g) = x_i).
\end{align*}
\]
Define an algebra homomorphisms $\phi : L_K(E_{T(v)}, w_{T(v)}) \to L_K(E', w')$ by

$$
\phi(u) = \sum_{1 \leq i \leq k} u_i + \sum_{1 \leq i \leq m, 1 \leq j \leq (k-1)n_i} u_{ij},
$$

$$
\phi(v) = v + \sum_{1 \leq i \leq m, 2 \leq j \leq n_i} u_{ij},
$$

$$
\phi(x_i) = x_i \ (1 \leq i \leq m),
$$

$$
\phi(y) = y \ (y \in T(X) \setminus X),
$$

$$
\phi(e_1) = \alpha_1^{(1)}, \quad \phi(e_1^*) = (\alpha_1^{(1)})^*,
$$

$$
\phi(e_l) = \alpha_1^{(1)} \cdots \alpha_1^{(i)} + \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} \beta_1^{(ij)} \cdots \beta_1^{(im_i)} \gamma_1^{(i1)} \cdots \gamma_1^{(i,(l-2)n_i+j)},
$$

$$
\phi(e_1^*) = (\alpha_1^{(i)})^* \cdots (\alpha_1^{(1)})^* + \sum_{1 \leq i \leq m, 1 \leq j \leq n_i} (\gamma_1^{(i,(l-2)n_i+j)})^* \cdots (\gamma_1^{(i1)})(\beta_1^{(im_i)})^* \cdots (\beta_1^{(ij)})^* \ (2 \leq l \leq m),
$$

$$
\phi(f_1^{(ij)}) = \beta_1^{(ij)} \cdots \beta_1^{(im_i)}, \quad \phi((f_1^{(ij)})^*) = (\beta_1^{(im_i)})^* \cdots (\beta_1^{(ij)})^* \ (1 \leq i \leq m, 1 \leq j \leq n_i),
$$

$$
\phi(g_1) = g_1, \quad \phi(g_1^*) = g_1^* \ (g \in E', s(g) \in T(X) \setminus X),
$$

$$
\phi(g_1) = \gamma_1^{(i1)} \cdots \gamma_1^{(i,(k-1)n_i)} g_1, \quad \phi(g_1^*) = g_1^*(\gamma_1^{(i1)})(\gamma_1^{(i,(k-1)n_i)})^* \cdots (\gamma_1^{(i1)})^* \ (g \in E', s(g) = x_i)
$$

and an algebra homomorphism $\psi : L_K(E', w') \to L_K(E_{T(v)}, w_{T(v)})$ by

$$
\psi(u_i) = e_i^* e_1 e_1 e_i \ (1 \leq i \leq k),
$$

$$
\psi(u_{i,(j-2)n_i+1}) = e_i^* f_1^{(ij)}(f_1^{(ij)})^* e_j \ (1 \leq i \leq m, 2 \leq j \leq k, 1 \leq l \leq n_i),
$$

$$
\psi(v) = e_1 e_1^* + \sum_{1 \leq i \leq m} f_1^{(il)}(f_1^{(il)})^*,
$$

$$
\psi(v_{ij}) = f_1^{(ij)}(f_1^{(ij)})^* \ (1 \leq i \leq m, 2 \leq j \leq n_i),
$$

$$
\psi(x_i) = x_i \ (1 \leq i \leq m),
$$

$$
\psi(y) = y \ (y \in T(X) \setminus X),
$$

$$
\psi(\alpha_1^{(i)}) = e_1, \quad \psi((\alpha_1^{(i)})^*) = e_1^*,
$$

$$
\psi(\alpha_1^{(i)}) = e_1 e_1^* e_1, \quad \psi((\alpha_1^{(i)})^*) = e_1^* e_1 e_1^* \ (2 \leq i \leq k),
$$

$$
\psi(\beta_1^{(ij)}) = f_1^{(ij)}(f_1^{(ij+1)})^*, \quad \psi((\beta_1^{(ij)})^*) = f_1^{(ij+1)}(f_1^{(ij)})^* \ (1 \leq i \leq m, 1 \leq j \leq n_i),
$$

$$
\psi(\beta_1^{(im_i)}) = f_1^{(im_i)}(\gamma_1^{(i1)})(\gamma_1^{(i,(j-2)n_i+1)})^* \cdots (\gamma_1^{(i1)})(\gamma_1^{(i,(j-2)n_i+1)})^* (1 \leq i \leq m),
$$

$$
\psi(\gamma_1^{(i1)}) = (f_1^{(i1)})^* e_2, \quad \psi((\gamma_1^{(i1)})^*) = e_2^* f_1^{(i1)} \ (1 \leq i \leq m),
$$

$$
\psi(\gamma_1^{(i,j-2)n_i+1}) = e_j^* f_1^{(im_i)}(f_1^{(i1)})^* e_j, \quad \psi((\gamma_1^{(i,j-2)n_i+1)})^* = e_j^* f_1^{(im_i)}(f_1^{(i1)})^* e_j, \ (1 \leq i \leq m, 3 \leq j \leq k),
$$

$$
\psi(\gamma_1^{(i,j-2)n_i+1}) = e_j^* f_1^{(im_i)}(f_1^{(i1)})^* e_j, \quad \psi((\gamma_1^{(i,j-2)n_i+1)})^* = e_j^* f_1^{(im_i)}(f_1^{(i1)})^* e_j, \ (1 \leq i \leq m, 2 \leq j \leq k, 2 \leq l \leq n_i),
$$

$$
\psi(g_1) = g_1, \quad \psi(g_1^*) = g_1^* \ (g \in (E')^1, s'(g) \in T(X) \setminus X),
$$

$$
\psi(g_1) = e_k^* f_1^{(im_i)} g_1, \quad \psi(g_1^*) = g_1^*(f_1^{(im_i)})^* e_k \ (g \in (E')^1, s'(g) = u_{i,(k-1)n_i}).
$$
It follows from the universal properties of \( L_K(E_{T(v)}, w_{T(v)}) \) and \( L_K(E', w') \) that \( \phi \) and \( \psi \) are well defined. One checks easily that \( \phi \circ \psi = id_{L_K(E', w')} \) and \( \psi \circ \phi = id_{L_K(E_{T(v)}, w_{T(v)})} \). Thus \( L_K(E_{T(v)}, w_{T(v)}) \cong L_K(E', w') \).

**Example 38.** Suppose \( (E, w) \) is the weighted graph

\[
\begin{array}{c}
u \quad (e, 2) \quad v \\
\end{array}
\]

from Example 24. Clearly \( (E, w) = (E_{T(v)}, w_{T(v)}) \). Let \( (E', w') \) be the unweighted graph

\[
\begin{array}{c}
u_2 \quad \alpha^{(2)} \quad u_1 \quad \alpha^{(1)} \quad v \\
\end{array}
\]

Then \( L_K(E, w) \cong L_K(E', w') \) by the previous lemma.

**Example 39.** Suppose \( (E, w) \) is the weighted graph

\[
\begin{array}{c}
\begin{array}{c}
a \quad k \quad u \quad e, 2 \quad v \\
\end{array}
\end{array}
\]

Then \( (E_{T(v)}, w_{T(v)}) \) is the weighted graph

\[
\begin{array}{c}
u \quad (e, 2) \quad v \\
\end{array}
\]

Let \( (E', w') \) be the unweighted graph

\[
\begin{array}{c}
u_2 \quad \alpha^{(2)} \quad u_1 \quad \alpha^{(1)} \quad v \\
\end{array}
\]

Then \( L_K(E_{T(v)}, w_{T(v)}) \cong L_K(E', w') \) by the previous lemma.

We want to show that we can "replace" the subgraph \( (E_{T(v)}, w_{T(v)}) \) in the previous lemma by the unweighted graph \((E', w')\) within \((E, w)\) without changing the wLpa. In order to do that we need some definitions.

**Definition 40** *(Weighted Graph Homomorphism).* Let \((\tilde{E}, \tilde{w})\) be a weighted graph. A morphism \( f : (E, w) \to (\tilde{E}, \tilde{w}) \) consists of maps \( f^0 : E^0 \to \tilde{E}^0 \) and \( f^1 : E^1 \to \tilde{E}^1 \) such that \( \tilde{r}(f^1(e)) = f^0(r(e)) \), \( \tilde{s}(f^1(e)) = f^0(s(e)) \) and \( \tilde{w}(f^1(e)) = w(e) \) for any \( e \in E^1 \).

**Definition 41** *(Complete Weighted Subgraph).* A weighted subgraph of \((E, w)\) is a weighted graph \((\tilde{E}, \tilde{w})\) where \( \tilde{E}^0 \subseteq E^0, \tilde{E}^1 \subseteq E^1, \tilde{s} = s|_{\tilde{E}^1}, \tilde{r} = r|_{\tilde{E}^1} \) and \( \tilde{w} = w|_{\tilde{E}^1} \). A weighted subgraph \((\tilde{E}, \tilde{w})\) of \((E, w)\) is called complete if \( \tilde{s}^{-1}(v) = s^{-1}(v) \) for any \( v \in \tilde{E}^0 \).

**Lemma 42.** Let \((\tilde{E}, \tilde{w})\) denote a complete weighted subgraph of \((E, w)\). Then the canonical graph monomorphism \((\tilde{E}, \tilde{w}) \to (E, w)\) induces an algebra monomorphism \( L_K(\tilde{E}, \tilde{w}) \to L_K(E, w) \).

**Proof.** The existence of an algebra homomorphism \( L_K(\tilde{E}, \tilde{w}) \to L_K(E, w) \) follows from the universal property of \( L_K(E, w) \). That it is injective follows from Theorem 13 since nod-paths in \((\tilde{E}, \tilde{w})\) are mapped to nod-paths in \((E, w)\).
Definition 43 (Replacement graph). Let $H \subseteq E^0$ a hereditary subset, $(E', w')$ a weighted graph and $\phi : L_K(E_H, w_H) \to L_K(E', w')$ an isomorphism which maps vertices to sums of distinct vertices, i.e. for any $v \in H$ there are distinct $u'_{v,1}, \ldots, u'_{v,n_v} \in (E')^0$ such that $\phi(v) = u'_{v,1} + \cdots + u'_{v,n_v}$. The weighted graph $(\tilde{E}, \tilde{w})$ defined by

$$
\tilde{E}^0 = E^0 \setminus H \sqcup (E')^0,
$$

$$
\tilde{E}^1 = \{ e \mid e \in E^1, s(e), r(e) \in E^0 \setminus H \}
\sqcup \{ e^{(1)} \ldots, e^{(n_{r(e)})} \mid e \in E^1, s(e) \in E^0 \setminus H, r(e) \in H \}
\sqcup (E')^1;
$$

$$
\tilde{s}(e) = s(e), \tilde{r}(e) = r(e), \tilde{w}(e) = w(e) \quad (e \in E^1, s(e), r(e) \in E^0 \setminus H),
$$

$$
\tilde{s}(e^{(j)}) = s(e), \tilde{r}(e^{(j)}) = u'_{r(e),j}, \tilde{w}(e^{(j)}) = w(e) \quad (e \in E^1, s(e) \in E^0 \setminus H, r(e) \in H, 1 \leq j \leq n_{r(e)}),
$$

$$
\tilde{s}(e') = s'(e'), \tilde{r}(e') = r'(e'), \tilde{w}(e') = w'(e') \quad (e' \in (E')^1)
$$

is the replacement graph defined by $\phi$.

Replacement Lemma 44. Let $H \subseteq E^0$ be a hereditary subset, $(E', w')$ a weighted graph and $\phi : L_K(E_H, w_H) \to L_K(E', w')$ an isomorphism which maps vertices to sums of distinct vertices. Then $L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})$ where $(\tilde{E}, \tilde{w})$ is the replacement graph defined by $\phi$.

Proof. Clearly $(E', w')$ is a complete weighted subgraph of $(\tilde{E}, \tilde{w})$. By Lemma 42, there is an algebra monomorphism $\psi : L_K(E', w') \to L_K(\tilde{E}, \tilde{w})$. Define an algebra homomorphisms $f : L_K(\tilde{E}, \tilde{w}) \to L_K(E, w)$ by

$$
f(v) = v \quad (v \in E^0 \setminus H),
$$

$$
f(v) = \psi(\phi(v)) \quad (v \in H),
$$

$$
f(e_i) = e_i, f(e_i^*) = e_i^* \quad (e \in E^1, s(e), r(e) \in E^0 \setminus H, 1 \leq i \leq w(e)),
$$

$$
f(e_i) = \sum_{j=1}^{n_{r(e)}} e_i^{(j)}, f(e_i^*) = \sum_{j=1}^{n_{r(e)}} (e_i^{(j)})^* \quad (e \in E^1, s(e) \in E^0 \setminus H, r(e) \in H, 1 \leq i \leq w(e)),
$$

$$
f(e_i) = \psi(\phi(e_i)), f(e_i^*) = \psi(\phi(e_i^*)) \quad (e \in E^1, s(e), r(e) \in H, 1 \leq i \leq w(e))
$$

and an algebra homomorphism $g : L_K(\tilde{E}, \tilde{w}) \to L_K(E, w)$ by

$$
g(v) = v \quad (v \in E^0 \setminus H),
$$

$$
g(v') = \phi^{-1}(v') \quad (v' \in (E')^0),
$$

$$
g(e_i) = e_i, g(e_i^*) = e_i^* \quad (e \in E^1, s(e), r(e) \in E^0 \setminus H, 1 \leq i \leq w(e)),
$$

$$
g(e_i^{(j)}) = e_i \phi^{-1}(u'_{r(e),j}), g((e_i^{(j)})^*) = \phi^{-1}(u'_{r(e),j})e_i^* \quad (e \in E^1, s(e) \in E^0 \setminus H, r(e) \in H, 1 \leq i \leq w(e)),
$$

$$
g(e_i') = \phi^{-1}(e_i'), g((e_i')^*) = \phi^{-1}((e_i')^*) \quad (e' \in (E')^1, 1 \leq i \leq w'(e')).
$$

It follows from the universal properties of $L_K(E, w)$ and $L_K(\tilde{E}, \tilde{w})$ that $f$ and $g$ are well defined. One checks easily that $f \circ g = id_{L_K(\tilde{E}, \tilde{w})}$ and $g \circ f = id_{L_K(E, w)}$. Thus $L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})$. \hfill \Box

Example 45. Suppose $(E, w)$ is the weighted graph

\[ a \xrightarrow{k} u \xrightarrow{e_2} v \xrightarrow{f} x \xrightarrow{h} y \xrightarrow{g} x \xrightarrow{j} b \xrightarrow{j} c. \]
Let \((E', w')\) be the unweighted graph

\[
\begin{array}{c}
\text{u}_2 \xrightarrow{\alpha^{(2)}} \text{u}_1 \xrightarrow{\alpha^{(1)}} \text{v} \xrightarrow{\beta^{(11)}} \text{v}_{12} \xrightarrow{\beta^{(12)}} x \xrightarrow{\gamma^{(11)}} \text{u}_{11} \xrightarrow{\gamma^{(12)}} \text{u}_{12} \xrightarrow{h} y.
\end{array}
\]

Then, as mentioned in Example 39, \(L_K(E_T(v), w_{T(v)}) \cong L_K(E', w')\). Let \(\phi\) be the isomorphism defined in the proof of Lemma 37. Then the replacement graph defined by \(\phi\) is the unweighted graph

\[
(\tilde{E}, \tilde{w}):
\begin{array}{c}
\text{u}_2 \xleftarrow{\alpha^{(2)}} \text{u}_1 \xleftarrow{\alpha^{(1)}} \text{v} \xleftarrow{\beta^{(11)}} \text{v}_{12} \xleftarrow{\beta^{(12)}} x \xleftarrow{\gamma^{(11)}} \text{u}_{11} \xleftarrow{\gamma^{(12)}} \text{u}_{12} \xleftarrow{h} y.
\end{array}
\]

By the previous lemma we have \(L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})\).

Now we are ready to prove the main result of this section.

**Finite Dimension Theorem 46.** Let \(K\) denote a field and \((E, w)\) a weighted graph. Then the following statements are equivalent.

1. \(L_K(E, w)\) is finite-dimensional.
2. \((E, w)\) is finite and aquasicyclic.
3. \(L_K(E, w) \cong \prod_{i=1}^{m} M_{n_i}(K)\) for some \(m, n_1, \ldots, n_m \in \mathbb{N}\).

**Proof.** (i)\iff(ii). Holds by Lemma 30. (ii)\implies(iii). Suppose that \((E, w)\) is finite and aquasicyclic (and hence \(L_K(E, w)\) is finite-dimensional by Lemma 30). By Lemma 34 we may assume that all weighted structured edges in \((E, w)\) are of type \(B\) and their ranges are sinks. Consider the vertices in \(E^0\) which emit weighted structured edges. It is easy to see that at least one of them, say \(v\), has the property that \(v\) is the only element of \(T(v)\) which emits a weighted structured edge (otherwise there would be a cyclic path). By Lemma 37 there is an unweighted graph \((E', w')\) such that \(L_K(E_T(v), w_{T(v)}) \cong L_K(E', w')\) via an isomorphism \(\phi\) which maps vertices to sums of distinct vertices. By the Replacement Lemma 44, \(L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})\) where \((\tilde{E}, \tilde{w})\) is the replacement graph defined by \(\phi\). Clearly \((\tilde{E}, \tilde{w})\) has one weighted structured edge less than \((E, w)\). We see that after a finite number of applications of Lemmas 34, 37 and 44 we arrive at an unweighted graph \((E'', w'')\) such that \(L_K(E'', w'') \cong L_K(E, w)\). Since \(L_K(E, w)\) is finite-dimensional, \(L_K(E'', w'')\) is finite-dimensional. It follows from [2, Theorem 2.6.17] that \(L_K(E, w)\) is isomorphic to a finite product of matrix rings over \(K\). (iii)\implies(i). Clear.

**Example 47.** Suppose \((E, w)\) is the weighted graph

\[
\begin{array}{c}
u \leftarrow e,2 \rightarrow v \xrightarrow{f} x
\end{array}
\]

from Example 24 and Example 38. Let \((E', w')\) be the unweighted graph

\[
\begin{array}{c}
u_2 \xleftarrow{\alpha^{(2)}} \nu_1 \xleftarrow{\alpha^{(1)}} v \xleftarrow{\beta^{(11)}} x \xleftarrow{\gamma^{(11)}} \nu_{11}.
\end{array}
\]

Then \(L_K(E, w) \cong L_K(E', w')\) as mentioned in Example 38. It follows from [2, Theorem 2.6.17] that \(L_K(E, w) \cong M_3(K) \times M_3(K)\).
Example 48. Suppose \((E, w)\) is the weighted graph
\[
\begin{array}{c}
\text{a} \xleftarrow{k} \text{u} \xleftarrow{e, 2} \text{v} \\
\text{i, 2} \xrightarrow{h} \text{y} \xrightarrow{b, j} \text{c} \\
\text{g} \xleftarrow{x} \text{h} \xrightarrow{i, 2} \text{b} \xrightarrow{j} \text{c}
\end{array}
\]
from Example 35. Let \((\tilde{E}, \tilde{w})\) be the unweighted graph
\[
\begin{array}{c}
\text{a} \xleftarrow{k(1)} \text{u_1} \xleftarrow{\alpha^{(1)}} \text{v} \\
\text{i, 2} \xrightarrow{h} \text{y} \xrightarrow{b, j} \text{c} \\
\text{g} \xleftarrow{x} \text{h} \xrightarrow{i, 2} \text{b} \xrightarrow{j} \text{c}
\end{array}
\]
Then \(L_K(E, w) \cong L_K(\tilde{E}, \tilde{w})\) by Examples 35 and 45. It follows from [2, Theorem 2.6.17] that \(L_K(E, w) \cong M_5(K) \times M_{12}(K)\).

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