AN INVERSION METRIC FOR REDUCED WORDS

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Abstract. We study the graph on reduced words with edges given by the Coxeter relations for the symmetric group. We define a metric on reduced words for a given permutation, analogous to Coxeter length for permutations, for which the graph becomes ranked with unique maximal element. We show this metric extends naturally to balanced tableaux, and use it to recover enumerative results of Edelman and Greene and of Reiner and Roichman.

1. Introduction

The symmetric group $\mathfrak{S}_n$ has a Coxeter presentation with generators $s_i$, the simple transpositions interchanging $i$ and $i+1$, and Coxeter relations

$$(1) \quad s_is_j = s_j s_i \text{ for } |i-j| \geq 2,$$

$$(2) \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } 1 \leq i \leq n-2,$$

and $s_i^2$ is the identity. We call (1) a commutation and (2) a Yang–Baxter move.

Given any permutation $w \in \mathfrak{S}_n$, a reduced word for $w$ is a sequence $\rho = (\rho_{\ell(w)}, \ldots, \rho_1)$ such that $w = s_{\rho_{\ell(w)}} \cdots s_{\rho_1}$, where $\ell(w)$ is the length of $w$ given by the number of pairs $(i < j)$ such that $w_i > w_j$.

Tits [10] studied the graph with vertex set given by reduced words and edges connecting two reduced words that differ by a single Coxeter relation. In particular, he showed that the subgraph on reduced words for a given permutation is connected. There has been much research on this graph, in particular for reduced words for the longest permutation $w_0^{(n)}$ of $\mathfrak{S}_n$. In this paper, we add additional structure to this graph, making it into a ranked poset with canonical maximal element. From this we derive an explicit inversion metric on reduced words for the same permutation that precisely gives the minimum number of Coxeter relations needed to transform one into another, along with how many are commutations and how many Yang–Baxter moves. Dehornoy and Autord [4] considered a similar question, phrased as computing the diameter of the graph on reduced words for $w_0^{(n)}$. They used techniques in group theory give a series of bounds and asymptotics, results which can be made explicit with this new metric.

Edelman and Greene [5] introduced balanced tableaux to prove bijectively a result of Stanley [9] equating reduced words for $w_0^{(n)}$ with standard Young tableaux of staircase shape. The poset structure and inversion statistic extend naturally to balanced tableaux, where the constructions simplify greatly. We use this simplified metric on balanced tableaux to give a new, elementary proof of a result of Reiner and Roichman [8] computing the diameter of the graph on reduced words for $w_0^{(n)}$.

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2. Reduced words

Let $R(w)$ denote the set of reduced words for $w$, indexed from right to left to mirror the action of $s_i$ as a function on permutations.

Example A (Reduced words). Take $w$ to be the permutation 42153. Then the word $(\rho_5, \rho_4, \rho_3, \rho_2, \rho_1) = (1, 4, 2, 3, 1)$ is a reduced word for $w$ since

\[
\begin{align*}
s_1s_4s_2s_3s_1 &= s_1s_4s_2s_3s_1 \cdot 12345 \\
&= s_1s_4s_2s_3 \cdot 21345 \\
&= s_1s_4s_2 \cdot 21435 \\
&= s_1 \cdot 24135 \\
&= 42153
\end{align*}
\]

The 11 reduced words in $R(42153)$ are shown in Fig. 1.

\[
(4, 2, 1, 2, 3) \quad (4, 1, 2, 1, 3) \quad (4, 1, 2, 3, 1) \quad (2, 4, 1, 2, 3) \quad (2, 1, 2, 4, 3) \\
(1, 4, 2, 3, 1) \quad (1, 2, 4, 3, 1) \quad (1, 4, 2, 1, 3) \quad (1, 2, 4, 1, 3) \quad (1, 2, 1, 4, 3)
\]

Figure 1. The reduced words for 42153.

Remark 2.1. A pair of indices $(i < j)$ such that $w_i > w_j$ is called an inversion of $w$, and the number of such pairs the inversion number of $w$. We avoid this terminology here, instead referring to the latter as the length of the permutation, in order to avoid confusion with the upcoming definition of inversions for reduced words.

Definition 2.2. The run decomposition of $\rho$, denoted by $(\rho^{(k)})$, partitions $\rho$ into decreasing sequences (read from right to left) of maximal length.

Example B (Run decomposition). The word $\rho = (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$, a reduced word for the permutation $w = 41758236$, has run decomposition

\[
\begin{align*}
\rho^{(5)} &= (5, 6) \\
\rho^{(4)} &= (3, 4, 5, 7) \\
\rho^{(3)} &= 3 \\
\rho^{(2)} &= 1, 4 \\
\rho^{(1)} &= 2, 3, 6
\end{align*}
\]

The following definition for super-Yamanouchi words first appears in [2], where it is shown that the reduced word contributing the unique leading term to a Schubert polynomial is precisely this super-Yamanouchi word. The terminology derives from Yamanouchi words, which capture the unique leading terms for Schur polynomials.

Definition 2.3. A reduced word $\rho$ with run decomposition $(\rho^{(k)})$ is super-Yamanouchi if each $\rho^{(i)}$ is an interval and $\min(\rho^{(k)}) > \cdots > \min(\rho^{(1)})$.

Example C (Super-Yamanouchi). The word $\rho = (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$ from Example B is not super-Yamanouchi since none of $\rho^{(4)}$, $\rho^{(2)}$, $\rho^{(1)}$ is an interval, and since neither $\min(\rho^{(4)}) > \min(\rho^{(3)})$ nor $\min(\rho^{(2)}) > \min(\rho^{(1)})$ holds.

In contrast, the word $\rho = (5, 6, 7, 4, 5, 3, 4, 5, 6, 1, 2, 3)$, another reduced word for the same permutation, is super-Yamanouchi, with run decomposition

\[
\begin{align*}
\rho^{(4)} &= (5, 6, 7) \\
\rho^{(3)} &= 4, 5 \\
\rho^{(2)} &= 3, 4, 5, 6 \\
\rho^{(1)} &= 1, 2, 3
\end{align*}
\]

so each run is an interval and $\min(\rho^{(4)}) > \min(\rho^{(3)}) > \min(\rho^{(2)}) > \min(\rho^{(1)})$. 
Proposition 2.4. For any \( w \), there exists a unique super-Yamanouchi \( \pi \in R(w) \).

Proof. Given \( w \), construct \( \pi \) according to Algorithm 1. To see this is well-defined, the set in line 5 is nonempty whenever \( \ell(v) > 0 \), the set in line 6 is nonempty by construction, and line 8 removes precisely \( (j - 2) - i + 1 = j - i - 1 \geq 1 \) inversions from \( v \), ensuring that algorithm terminates. Line 8 also ensures that the resulting word \( \pi \) will be a word for \( w \) and will be reduced since \( (j - 2) - i + 1 \) inversions are removed when appending \( (j - 2) - i + 1 \) letters to \( \pi \). Each pass through line 7 appends an interval to \( \pi \), so to check the super-Yamanouchi condition, we need only check that a subsequent pass chooses a smaller index at line 5. If \( i \) is chosen in line 5, then after line 8 \( v \) has no inversions weakly beyond index \( i \), ensuring that the maximum in line 5 of the next iteration is strictly less than \( i \). Therefore Algorithm 1 is well-defined and returns a super-Yamanouchi reduced word for \( w \).

Now suppose that \( \rho \neq \pi \) is another super-Yamanouchi reduced word for \( w \). Let \( i \) be the maximum index for which \( \pi_i \neq \rho_i \). Clearly removing the prefix or suffix of a reduced word does not change that it is reduced. Moreover, this also preserves the super-Yamanouchi property since runs must still form intervals and only the leftmost run can have a changed minimum, which necessarily gets weakly larger. Furthermore, removing the same prefix or suffix for two reduced words for the same permutation results again in (shorter) reduced words for the new permutation. Therefore by removing the suffix \( \pi_{\ell}, \pi_{\ell-1} \cdots \pi_{i+1} \) from both \( \pi \) and \( \rho \), we may assume \( i = \ell \).

The interval condition for super-Yamanouchi words ensures that a letter in position \( i \) of \( w \) is moved by success \( s_k \)'s to some position \( j > i \), and the decreasing minimum condition for super-Yamanouchi words ensures that the subsequent letter moved is strictly left of position \( i \). In order to be a reduced word, we must have \( w_{\rho_\ell} > w_{\rho_{\ell+1}} \). Since \( \pi \) is constructed by choosing the maximum \( i \) such that \( w_i > w_{i+1} \), we must have \( \pi_\ell > \rho_\ell \). Since \( \rho \) first selects an index \( \rho_\ell < \pi_\ell \), and since each run of \( \rho \) either fixes the position of the final descent or moves it one position to the left, based on whether or not that run crosses over the descent, there is no way to begin a new run with the final descent without violating the super-Yamanouchi condition. Thus \( \pi \) is the unique super-Yamanouchi word for \( w \). \( \square \)

**Algorithm 1** Super-Yamanouchi reduced word

1: procedure **SUPER** (\( w \))
2: \( v \leftarrow w \)
3: \( \pi \leftarrow () \)
4: while \( \ell(v) > 0 \) do
5: \( i \leftarrow \max \{ k \mid w_k > w_{k+1} \} \)
6: \( j \leftarrow \min \{ k \mid w_i < w_k \} \cup \{ n + 1 \} \)
7: \( \pi \leftarrow (\pi, i, i + 1, \ldots, j - 2) \)
8: \( v \leftarrow s_{j-2} \cdots s_{i+1} s_i v \)
9: end while
10: return \( \pi \)
11: end procedure
Example D (Super-Yamanouchi reduced word). Construct the super-Yamanouchi reduced word for the permutation \( w = 41758236 \) by Algorithm 1 as illustrated in Fig. 2. We initialize with \( v = 41758236 \) and \( \pi = () \), and then

- loop 1: \( i = 5 \) and \( j = 8 + 1 = 9 \), resulting in \( \pi = (5, 6, 7) \) and now \( v = 41752368 \);
- loop 2: \( i = 4 \) and \( j = 7 \), and so \( \pi = (5, 6, 7, 4, 5) \) and \( v = 41723568 \);
- loop 3: \( i = 3 \) and \( j = 8 \), and so \( \pi = (5, 6, 7, 4, 5, 3, 4, 5, 6) \) and \( v = 41235678 \);
- loop 4: \( i = 1 \) and \( j = 5 \), and so \( \pi = (5, 6, 7, 4, 5, 3, 4, 5, 6, 1, 2, 3) \) and \( v = 12345678 \).

Having reached the identity, we terminate. Therefore the unique super-Yamanouchi reduced word for \( w = 41758236 \) is \( \pi = (5, 6, 7, 4, 5, 3, 4, 5, 6, 1, 2, 3) \).

\[
41758236 \xrightarrow{(5,6,7)} 41752368 \xrightarrow{4,5} 41723568 \xrightarrow{3,4,5,6} 41235678 \xrightarrow{1,2,3} 12345678
\]

Figure 2. An illustration of Algorithm 1 for the permutation 41758236.

We define two involutions on reduced words for a given permutation based on the Coxeter relations for the simple transpositions.

Definition 2.5. Given \( w \) and \( 1 \leq i < \ell(w) \), \( c_i \) acts on \( \rho \in R(w) \) by commuting \( \rho_i \) and \( \rho_{i+1} \) whenever \( |\rho_i - \rho_{i+1}| > 1 \) and the identity otherwise.

Definition 2.6. Given \( w \) and \( 1 < i < \ell(w) \), \( b_i \) acts on \( \rho \in R(w) \) by braiding \( \rho_{i-1}\rho_i\rho_{i+1} \) to \( \rho_i\rho_{i\pm1}\rho_i \) whenever \( \rho_{i-1} = \rho_{i+1} = \rho_i \pm 1 \) and the identity otherwise.

We refer to \( c_i \) as a commutation, to \( b_i \) as a Yang–Baxter move, and to either as a Coxeter move. For examples of Coxeter moves on reduced words, see Fig. 3.

It follows from classical work of Tits [10] that the maps \( c_i \) and \( b_i \) are well-defined involutions on \( R(w) \), and that the graph on \( R(w) \) with edges given by \( c_i \) and \( b_i \) is connected. Pushing this further, Fig. 3 suggests a ranked poset structure on reduced words for \( w \) with unique maximal element equal to the super-Yamanouchi reduced word for \( w \). The following definition measures the minimum number of commutations and Yang–Baxter moves needed to get from a given reduced word to the super-Yamanouchi one.
Definition 2.7. Given $\rho \in R(w)$, define the **inversion number of** $\rho$ by

\[
\text{inv}(\rho) = \ell(v(\rho)) - \sum_i (\pi_i - \rho_i),
\]

where $\pi \in R(w)$ is super-Yamanouchi and $v(\rho)$ is defined by Algorithm 2.

**Algorithm 2** Permutation of a reduced word

1: procedure $\text{perm}(\rho)$
2: \hspace*{1em} $\pi \leftarrow$ super-Yamanouchi reduced word for $w$
3: \hspace*{1em} $\text{perm} \leftarrow$ identity permutation of $S_{\ell(w)}$
4: \hspace*{1em} for $i$ from $\ell(w)$ to 1 by $-1$ do
5: \hspace*{2em} $k \leftarrow \pi_i$
6: \hspace*{2em} for $j$ from $\ell(w)$ to 1 by $-1$ do
7: \hspace*{3em} if $\rho_j = k$ and is not already paired then
8: \hspace*{4em} pair $\rho_j$ with $\pi_i$
9: \hspace*{4em} $\text{perm}_i \leftarrow j$
10: \hspace*{3em} break
11: \hspace*{2em} else if $\rho_j = k - 1$ and is not already paired then
12: \hspace*{3em} $k \leftarrow k - 1$
13: \hspace*{2em} next
14: \hspace*{1em} end if
15: \hspace*{1em} end for
16: \hspace*{1em} return $\text{perm}$
17: end procedure

**Example E** (Inversions for reduced words). Let $\rho = (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$. The super-Yamanouchi reduced word is $\pi = (5, 6, 7, 4, 5, 3, 4, 5, 6, 1, 2, 3)$. Following Algorithm 2, the first three iterations of the for loop on line 4 ($i = 12, 11, 10$) will be satisfied by the if condition of line 7, resulting in $\pi_{12} = 5$, $\pi_{11} = 6$, and $\pi_{10} = 7$ paired with $\rho_{12} = 5$, $\rho_{11} = 6$, and $\rho_{10} = 7$, respectively.

On the fourth iteration of the for loop on line 4 ($i = 9$), we set $k = \pi_9 = 4$ on line 5, and on the third iteration of the for loop on line 6 ($j = 10$), the else if condition on line 11 is met, and we decrement $k = 3$. Then, on the seventh iteration of the for loop on line 6 ($j = 6$), the if condition of line 7 is met and we pair $\pi_9 = 4$ with $\rho_6 = 3$. Continuing thus, we pair values of $\pi$ from left to right with values of $\rho$ as illustrated in Fig. 4.

![Figure 4](image)

**Figure 4.** An illustration of the pairings in Algorithm 2 for the reduced word $\rho = (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6)$.

Therefore $\text{perm}(\rho) = 2 \ 3 \ 5 \ 1 \ 8 \ 9 \ 10 \ 4 \ 6 \ 7 \ 11 \ 12$ and so $\text{inv}(\rho) = 13 - 2 = 11$. Note

\[
\rho = c_7 \ c_8 \ c_9 \ c_4 \ c_6 \ b_6 \ b_7 \ c_1 \ c_2 \ c_3 \ pi,
\]

which is a sequence of 11 involutions, exactly 2 of which are Yang–Baxter moves.
Theorem 2.8. For $ρ ∈ R(w)$, $\text{inv}(ρ)$ is a well-defined non-negative integer. Moreover, there exists a sequence $f = f_{\text{inv}(ρ)} \cdots f_1$ of Coxeter moves, i.e. $f_j = c_i$ or $b_i$, such that $f(ρ)$ is super-Yamanouchi, and for any sequence $g = g_m \cdots g_1$ of Coxeter moves such that $g(ρ)$ is super-Yamanouchi, we have $m ≥ \text{inv}(ρ)$.

Proof. We claim the theorem holds for $ρ$ if and only if it holds for $c_i(ρ)$. This is vacuously true if $c_i(ρ)$ acts trivially on $ρ$. Otherwise, $c_i(ρ)$ will have permutation $s_i\text{perm}(ρ)$, and, since the letters of $ρ$ and $c_i(ρ)$ are the same, we have

$$\text{inv}(c_iρ) = \text{inv}(s_i\text{perm}(ρ)) - \sum (π_j - (c_iρ)_j)$$

$$= \text{inv}(\text{perm}(ρ)) ± 1 - \sum (π_j - ρ_j) = \text{inv}(ρ) ± 1.$$ 

Furthermore, $\text{inv}(c_iρ) = \text{inv}(ρ) + 1$ precisely when $i$ is left of $i + 1$ in $\text{perm}(ρ)$.

Next we claim the theorem holds for $ρ$ if and only if it holds for $b_i(ρ)$. If $b_i$ acts trivially on $ρ$, the claim is vacuously true. Otherwise, $b_i(ρ)$ will have permutation $s_i s_{i−1}\text{perm}(ρ)$ or $s_{i−1} s_i\text{perm}(ρ)$, the former when $ρ_{i±1} = ρ_i + 1$ and the latter when $ρ_{i±1} = ρ_i − 1$. Assuming the former, we have

$$\text{inv}(b_iρ) = \text{inv}(s_i s_{i−1}\text{perm}(ρ)) - \sum (π_j - (b_iρ)_j)$$

$$= \text{inv}(\text{perm}(ρ)) + 2 - \sum (π_j - ρ_j) + 1 = \text{inv}(ρ) + 1,$$

and, by the same computation, $\text{inv}(b_iρ) = \text{inv}(ρ) − 1$ in the latter case.

Recall from earlier that any two reduced words for $w$ can be transformed into one another by a sequence of Coxeter moves. Let $m$ be the minimum number of Coxeter moves needed to transform $ρ$ into the super-Yamanouchi reduced word. If $m = 0$, then $ρ$ is super-Yamanouchi, in which case the permutation for $ρ$ is the identity and $\text{inv}(ρ) = 0$, so the theorem holds. Assume for induction, that the theorem holds for any $n < m$, and suppose $ρ = f_m \cdots f_1 π$, where $π$ is super-Yamanouchi and $f_j$ is $c_i$ or $b_i$ for some $i$. By induction, the result holds for $f_{m−1} \cdots f_1 π = f_m π$, and so by the claims, it holds for $ρ$ as well. □

Thus we may define the inversion poset for reduced words as follows.

Corollary 2.9. For $w$ a permutation, the partial order on $R(w)$ given by the transitive closure of covering relations

- $ρ > c_iρ$ if $\text{inv}(c_iρ) = \text{inv}(ρ) + 1$, and
- $ρ > b_iρ$ if $\text{inv}(b_iρ) = \text{inv}(ρ) + 1$

makes $R(w)$ into a ranked partially ordered set with unique maximal element.

Notice the ranking is the co-inversion number, so that the super-Yamanouchi word is the unique maximal element in line with convention from Schubert calculus.

From the proof of Theorem 2.8, can, in fact, count the minimum number of Yang–Baxter moves on any shortest path from a reduced word to the super-Yamanouchi reduced word by considering the offset between the length of the permutation of $ρ$ and the inversion number of $ρ$. More generally, we have the following.

Corollary 2.10. For $ρ, σ ∈ R(w)$, and $f = f_k \cdots f_1$ any minimal length sequence of Coxeter moves, i.e. $f_j = c_i$ or $b_i$, such that $f(ρ) = σ$, the number of Coxeter moves that are Yang–Baxter moves is given by

$$\# \{ j \mid f_j = b_i \text{ for some } i \} = \sum_i |ρ_i - σ(\text{perm}(σ)\text{perm}(ρ)^{-1})_i|.$$
While one can hope to define an explicit metric on reduced words analogous to Kendall’s \( \tau \) metric on permutations \cite{7} by

\[
\text{inv}(\rho, \sigma) = \ell(\text{perm}(\rho, \sigma)) - \sum_i |\rho_i - \sigma_{\text{perm}(\rho, \sigma)}_i|,
\]

where \( \text{perm}(\rho, \sigma) = \text{perm}(\sigma)\text{perm}(\rho)^{-1} \), this does not always give the correct minimum distance between arbitrary reduced words.

\[
\begin{array}{c}
\rho: 1\ 2\ 1\ 3\ 2\ 1 \\
\pi: 3\ 2\ 3\ 1\ 2\ 3 \\
\sigma: 1\ 3\ 2\ 1\ 3\ 2
\end{array}
\Rightarrow
\begin{array}{c}
1\ 2\ 1\ 3\ 2\ 1 \\
1\ 3\ 2\ 1\ 3\ 2
\end{array}
\]

\textbf{Figure 5.} An illustration of the permutation of a pair of reduced words for \( w = 42153 \). Note this does not measure distance.

\textbf{Example F} (Barrier to inversion metric on reduced words). Let \( \rho = (1, 2, 1, 3, 2, 1) \) and \( \sigma = (1, 3, 2, 1, 3, 2) \), both reduced words for the long permutation \( w_0^{(4)} = 4321 \). Then \( \pi = (3, 2, 3, 1, 2, 3) \) is the super-Yamanouchi word, and following Algorithm \( 2 \) we have the two pairings indicated on the left side of Fig. 5. Composing the diagram gives \( \text{perm}(\rho, \sigma) = 51234 \), and so we have

\[
\text{inv}(\rho, \sigma) = \ell(51234) - |1 - 1| - |2 - 2| - |1 - 1| - |3 - 3| - |2 - 2| - |1 - 3| = 4 - 2 = 2.
\]

Observe, from Fig. 6 any shortest path from \( \rho \) to \( \sigma \) has length 4 and uses exactly 2 Yang–Baxter moves. Thus the naive inversion number for arbitrary pairs does not work to give the correct minimum distance.

\textbf{Figure 6.} An illustration of the Coxeter moves on \( R(4321) \).
3. Balanced tableaux

The calculation of the inversion number for a reduced word is admittedly complicated, made more so by the requirement that one first compute the super-Yamanouchi word. By shifting our paradigm to another model for reduced words, this statistic becomes more natural and much simpler to compute.

The Rothe diagram (also called the inversion diagram) of a permutation \( w \), denoted by \( \mathbb{D}(w) \), is the following subset of cells in the first quadrant of the plane,

\[
\mathbb{D}(w) = \{(i, w_j) \mid i < j \text{ and } w_i > w_j\} \subset \mathbb{Z}^+ \times \mathbb{Z}^+.
\]

The Rothe diagram of \( w \) gives a graphical representation of the inversion pairs of \( w \). In particular, the number of cells in \( \mathbb{D}(w) \) is simply \( \ell(w) \).

**Example G** (Rothe diagram). To draw the Rothe diagram for \( w = 41758236 \), we write \( w \) vertically along the \( y \)-axis with \( w_i \) at height \( i \), and we label the cells horizontally along the \( x \)-axis with positive integers, as illustrated in Fig. 7. When computing the cells in row 3, for instance, we consider \( w_3 = 7 \) and place cells in columns 5, 2, 3, 6 since these occur to the right of and are smaller than 7.

![Figure 7. The Rothe diagram \( \mathbb{D}(w) \) for \( w = 41758236 \).](image)

The Rothe diagram of \( w \) provides an alternative method from that described in Proposition 2.4 for computing the super-Yamanouchi reduced word for \( w \).

**Definition 3.1.** For \( w \) a permutation, the row-interval filling for \( \mathbb{D}(w) \) is the positive integer filling with entries \( i, i+1, i+2, \ldots \) in row \( i \), from left to right.

**Example H** (Row-interval filling). The row-interval filling for \( \mathbb{D}(41758236) \) is shown in Fig. 8. Comparing with Ex. C, notice that the row reading word of this filling, i.e. the word obtained by reading the rows from left to right beginning with the highest, is precisely the super-Yamanouchi word for \( w \).

**Proposition 3.2.** The row reading word of the row-interval filling for \( w \) is precisely the super-Yamanouchi reduced word for \( w \).

**Proof.** Following the procedure for computing \( \pi \) in Algorithm 1, the last descent of \( w \) corresponds to the highest occupied row of \( \mathbb{D}(w) \), and the number of positions the letter at that position must move to the right is precisely the number of entries in that row. Thus removing the final descent corresponds to removing the highest occupied row, and the same values are recorded for both processes. \( \square \)
Figure 8. The row-interval filling of \( D(41758236) \).

While this construction applies equally well to any diagram, for a Rothe diagram the columns will be integer intervals as well. In fact, this property uniquely characterizes diagrams as Rothe diagrams.

**Proposition 3.3.** A cell diagram \( D \) in the first quadrant is the Rothe diagram of a permutation if and only if the columns of the row-interval filling form increasing intervals from bottom to top, beginning with \( i \) at the bottom of column \( i \).

**Proof.** From (3.1), one sees that the Rothe diagram for \( w^{-1} \) is the transpose of the Rothe diagram for \( w \). Moreover, transposing the row-interval filling for \( w \) results in the row-interval filling for \( w^{-1} \), so the columns must form intervals as well. \( \square \)

Stanley [9] introduced a new family of symmetric functions indexed by permutations in order to enumerate reduced words. Edelman and Greene [5] introduced balanced labelings of Rothe diagrams in order to prove Stanley’s conjecture that his symmetric functions are Schur positive and to give a precise enumeration of reduced words. We review balanced tableaux here, but give independent, elementary proofs of their bijection with reduced words using the ranked poset structure.

**Definition 3.4** (\([5]\)). A standard balanced tableau is a bijective filling of a Rothe diagram with entries from \( \{1, 2, \ldots, n\} \) such that for every entry of the diagram, the number of entries to its right that are greater is equal to the number of entries above it that are smaller.

Denote the set of standard balanced tableaux on \( D(w) \) by SBT\((w)\).

**Example I** (Balanced tableaux). For \( w = 42153 \), the filling of \( D(w) \) on the left of Fig. 9 is balanced since for each cell (indicated in bold), the cells above and to the right have the same number of entries above that are greater (indicated in circles) as entries to the right that are smaller (also indicated in circles).

![Balanced Tableaux](image)

Figure 9. Checking the balanced condition for a standard tableaux.

The 11 balanced tableaux in SBT(42153) are shown in Fig. 10.

To prove standard balanced tableaux are in bijection with reduced words, first observe there is a canonical super-Yamanouchi standard balanced tableau.
Definition 3.5. A standard balanced tableau $R$ is super-Yamanouchi if its reverse row reading word (right to left from bottom to top) is the identity.

The balanced condition is immediate for the super-Yamanouchi tableau since entries increase in columns from bottom to top and in rows from left to right. For example, the super-Yamanouchi balanced tableau for $41758236$ is shown in Fig. 11.

![Figure 11. The super-Yamanouchi balanced tableau for $D(w)$.](image)

We next define simple analogs of the Coxeter moves for balanced tableaux, where the commutations involve two consecutive values and the Yang–Baxter moves involve three consecutive values. Both act only in certain circumstances.

Definition 3.6. Given $w$ and $1 \leq i < \text{inv}(w)$, $c_i$ acts on $\text{SBT}(w)$ by exchanging $i$ and $i+1$ if they are not in the same row or column and by the identity otherwise.

Definition 3.7. Given $w$ and $1 < i < \text{inv}(w)$, $b_i$ acts on $\text{SBT}(w)$ by exchanging $i-1$ and $i+1$ if one is in the same column and above $i$ and the other is in the same row and right of $i$ and by the identity otherwise.

For examples of Coxeter moves on balanced tableaux, see Fig. 12. Comparing this with Fig. 3 suggests a poset-preserving bijection between reduced words and balanced tableaux, and indeed we will demonstrate this bijection below.

Lemma 3.8. The maps $c_i$ and $b_i$ are well-defined involutions on $\text{SBT}(w)$.

Proof. For $R \in \text{SBT}(w)$, if $i$ and $i+1$ are not in the same row or same column, then interchanging them cannot unbalance the tableau since all other entries compare the same with $i$ and with $i+1$. Thus $c_i(R) \in \text{SBT}(w)$. If $i \pm 1$ is in the same row as $i$ and $i \mp 1$ is in the same column, then swapping them maintains the balance since, again, every $j$ less than $i-1$ or greater than $i+1$ compares with same with both, the two cannot be in the same row or same column as one another, and $i$ has traded the two to maintain its balance.
Remark 3.9. When \( w \) is a permutation with a unique descent, \( D(w) \) has the form of the Young diagram (in English notation) for a partition, and standard balanced tableaux for \( w \) are precisely the standard reverse Young tableau. In this case, the poset structure on \( SBT(w) \) where we consider only the Coxeter–Knuth relations coincides with the dual equivalence graph [1] on standard reverse Young tableaux. For details on this connection and its combinatorial consequences, see [3].

Parallel to the case of reduced words, we introduce a simple statistic on standard balanced tableaux that gives the minimum distance from a standard balanced tableau to the super-Yamanouchi one.

**Definition 3.10.** For \( R \in SBT(w) \), the inversion number of \( R \) is
\[
inv(R) = \# \{(i < j) \mid i \text{ lies in strictly higher row, different column than } j\}.
\]
We call such a pair an inversion of \( R \).

**Example J** (Inversion number of balanced tableaux). The standard balanced tableau in Fig. 13 has 11 inversion pairs as listed to the right. Notice that \( (6, 9) \) and \( (4, 8) \) are not inversions since these pairs occur in the same column.

![Diagram of Coxeter moves on SBT(42153)](image)

**Figure 12.** An illustration of the Coxeter moves on SBT(42153).

![Inversion pairs for a standard balanced tableau](image)

**Figure 13.** The inversion pairs for a standard balanced tableau.
Theorem 3.11. Let $P_w \in SBT(w)$ be the unique super-Yamanouchi tableau. Then for any $R \in SBT(w)$, there exists a sequence $f = f_{\text{inv}(R)} \cdots f_1$ of Coxeter moves such that $f(P_w) = R$, and, for any sequence $g = g_m \cdots g_1$ of Coxeter moves with $g(P_w) = R$, we have $m \geq \text{inv}(R)$.

Proof. We proceed by induction on $\text{inv}(R)$. Clearly $\text{inv}(P_w) = 0$ since it is the unique balanced filling such that all larger entries occur weakly above smaller entries, and the result holds for this case. Moreover, if $R$ has some $i < j$ with $i$ above $j$ and in the same column, then the balanced condition ensures that there is some $k > j$ in the same row as $j$, and so $i < k$ with $i$ and $k$ not in the same column. In particular, $\text{inv}(R) > 0$ for $R \neq P_w$. This establishes the base case.

Let $R \in SBT(w)$ with $\text{inv}(R) > 0$. We claim that there is a pair $(i, i + 1)$ with $i$ above $i + 1$. If not, then for any pair $(i < j)$ with $i$ above $j$ (such a pair exists since $\text{inv}(R) > 0$), there exists $k$ with $i < k < j$ and neither $(i < k)$ nor $(k < j)$ has the smaller strictly above the larger. Thus $k$ is weakly above $i$ and weakly below $j$, an impossibility since $i$ is strictly above $j$. Therefore we may take $i$ such that $i + 1$ lies in a strictly lower row. There are two cases to consider.

If $i$ and $i + 1$ are not in the same column, then $c_i$ acts non-trivially on $R$. Furthermore, $\text{inv}(c_i(R)) = \text{inv}(R) - 1$ since the pair $(i, i + 1)$ is removed from the set of inversions and all other pairs remain but with $i$ and $i + 1$ interchanged. By induction, the result holds for $c_i(R)$, and so, too, for $R$.

If $i$ and $i + 1$ are in the same column for every pair with $i$ above $i + 1$, then take $i$ maximal among all such pairs. We claim that $i + 2$ must lie in the same row and to the right of $i + 1$. If not, then $i + 2$ must lie strictly above $i + 1$, and, by the choice of $i$, $k + 1$ must lie weakly above $k$ for all $k > i + 2$. However, this would mean no larger entry was in the row of $i + 1$, contracting the balanced condition since $i$ is in the same column and above it. Therefore $i + 2$ does lie in the same row as $i + 1$, and so $b_{i+1}$ acts non-trivially on $R$ by interchanging $i$ and $i + 2$. Furthermore, $\text{inv}(b_{i+1}(R)) = \text{inv}(R) - 1$ since the pair $(i, i + 2)$ is removed from the set of inversions and all other pairs remain but with $i$ and $i + 2$ interchanged. By induction, the result holds for $b_{i+1}(R)$, and so it holds for $R$ as well. \hfill $\blacksquare$

Parallel to Corollary 2.10 we can also refine our calculation of Coxeter distance to count only the number of Yang–Baxter moves by considering column inversions.

Corollary 3.12. For $R \in SBT(w)$, and $f = f_k \cdots f_1$ any minimal length sequence of Coxeter moves, i.e. $f_j = c_i$ or $b_i$, such that $f(R)$ is super-Yamanouchi, the number of Coxeter moves that are Yang–Baxter moves is equal to the number of column inversions of $R$, i.e.

$$\# \{j \mid f_j = b_i \text{ some } i \} = \# \{(i < j) \mid i \text{ in higher row, same column as } j\}.$$\hfill (3.3)

Computing the permutation of a balanced tableau is also far simpler.

Definition 3.13. Given $R \in SBT(w)$, define the permutation of $R$, denoted by $\text{perm}(R)$, by sorting the rows of $R$ to be decreasing (read left to right) and taking the reverse row reading word of the result.

Example K (Permutation of balanced tableaux). Letting $R$ be the balanced tableau in Fig. [14] we have $\text{perm}(R) = 2351890467112$.

Note that while $R$ has 11 inversions, its associated permutation has length 13. The difference is precisely the number of steps needed to sort the rows of the
For Theorem 3.11, for any $R \in \text{SBT}(w)$, we have

$$\text{inv}(R) = \ell(\text{perm}(R)) - \sum_r \text{coinv}(\text{row}_r(R)),$$

where $\text{coinv}(\text{row}_r(R))$ is the number of entries $i < j$ with $i$ left of $j$ in row $r$.

**Proof.** Let $I$ be defined by the right hand side of (3.4). Let $R \in \text{SBT}(w)$ and suppose $c_i$ acts non-trivially on $R$. Then $i$ and $i + 1$ lie in different rows and different columns in $R$, so $\text{sort}(R)$ and $\text{sort}(c_i R)$ differ exactly in that $i$ and $i + 1$ have been exchanged, and so $\text{perm}(c_i R) = s_i \text{perm}(R)$. Further, since all letters other than $i$, $i + 1$ compare the same with $i$ and $i + 1$, $R$ and $c_i R$ have the same number of row (co)inversions. In particular, we have

$$I(c_i R) = \ell(s_i \text{perm}(R)) - \sum_r \text{coinv}(\text{row}_r(R)) = I(R) \pm 1,$$

and, moreover, $I(c_i R) = I(R) + 1$ precisely when $i$ is left of $i + 1$ in $v$.

Next suppose that $b_i$ acts non-trivially on $R$, exchanging $i - 1$ and $i + 1$ when $i$ lies directly below the one and directly left of the other. The permutation exchanging $i - 1$ and $i + 1$ is given by $s_{i-1} s_i = s_i s_{i-1}$, but since $i - 1$ and $i + 1$ compare differently with $i$, when the rows are sorted the one in the row of $i$ will flip to the other side of it. Therefore $\text{perm}(b_i R) = s_{i-1} s_i \text{perm}(R)$ if $i + 1$ is above $i - 1$, and $\text{perm}(b_i R) = s_i s_{i-1} \text{perm}(R)$ otherwise, and in the former case we have

$$I(b_i R) = \ell(s_{i-1} s_i \text{perm}(R)) - \sum_r (\text{coinv}(\text{row}_r(R)) + 1) = I(R) + 1,$$

and, by the same computation, $I(b_i R) = I(R) - 1$ in the latter case.

By Theorem 3.11, $\text{inv}(R) = 0$ if and only if $R$ is super-Yamanouchi, in which case $\text{perm}(R)$ is the identity and $R$ has decreasing rows, thus giving $I(R) = 0$ as well. Conversely, if we consider $\hat{v}$ to be the permutation obtained by following Definition 3.13 without first sorting the rows of $R$, then we have $\ell(\hat{v}) = \ell(v) + \sum_r \text{coinv}(\text{row}_r(R))$. In particular, $I(R) = 0$ if and only if $v$ is the identity, in which case $R$ is super-Yamanouchi. Therefore $\text{inv}(R) = I(R)$ whenever either is 0. By Theorem 3.11 for any $R \in \text{SBT}(w)$, we may write $R = f_1 \cdots f_i(P)$, where $P$ is super-Yamanouchi and each $f_i$ is a Coxeter move. The result for $R$ now follows from the analysis of Coxeter moves above.

Comparing Theorem 2.8 with Theorem 3.11 one can anticipate the bijection between reduced words and standard balanced tableaux preserves the permutation and inversion number. Indeed, given the permutation $v$, one can recover the row tableau. Moreover, letting $P$ be the super-Yamanouchi filling, we have

$$R = c_7 c_8 c_4 c_6 b_6 c_7 c_1 c_2 c_3 P,$$

which is a sequence of 11 involutions, exactly 2 of which are Yang–Baxter moves.

**Theorem 3.14.** For $R \in \text{SBT}(w)$, we have

(3.4) \[ \text{inv}(R) = \ell(\text{perm}(R)) - \sum_r \text{coinv}(\text{row}_r(R)), \]

where $\text{coinv}(\text{row}_r(R))$ is the number of entries $i < j$ with $i$ left of $j$ in row $r$.

**Proof.** Let $I$ be defined by the right hand side of (3.4). Let $R \in \text{SBT}(w)$ and suppose $c_i$ acts non-trivially on $R$. Then $i$ and $i + 1$ lie in different rows and different columns in $R$, so $\text{sort}(R)$ and $\text{sort}(c_i R)$ differ exactly in that $i$ and $i + 1$ have been exchanged, and so $\text{perm}(c_i R) = s_i \text{perm}(R)$. Further, since all letters other than $i$, $i + 1$ compare the same with $i$ and $i + 1$, $R$ and $c_i R$ have the same number of row (co)inversions. In particular, we have

$$I(c_i R) = \ell(s_i \text{perm}(R)) - \sum_r \text{coinv}(\text{row}_r(R)) = I(R) \pm 1,$$

and, moreover, $I(c_i R) = I(R) + 1$ precisely when $i$ is left of $i + 1$ in $v$.

Next suppose that $b_i$ acts non-trivially on $R$, exchanging $i - 1$ and $i + 1$ when $i$ lies directly below the one and directly left of the other. The permutation exchanging $i - 1$ and $i + 1$ is given by $s_{i-1} s_i = s_i s_{i-1}$, but since $i - 1$ and $i + 1$ compare differently with $i$, when the rows are sorted the one in the row of $i$ will flip to the other side of it. Therefore $\text{perm}(b_i R) = s_{i-1} s_i \text{perm}(R)$ if $i + 1$ is above $i - 1$, and $\text{perm}(b_i R) = s_i s_{i-1} \text{perm}(R)$ otherwise, and in the former case we have

$$I(b_i R) = \ell(s_{i-1} s_i \text{perm}(R)) - \sum_r (\text{coinv}(\text{row}_r(R)) + 1) = I(R) + 1,$$

and, by the same computation, $I(b_i R) = I(R) - 1$ in the latter case.

By Theorem 3.11, $\text{inv}(R) = 0$ if and only if $R$ is super-Yamanouchi, in which case $\text{perm}(R)$ is the identity and $R$ has decreasing rows, thus giving $I(R) = 0$ as well. Conversely, if we consider $\hat{v}$ to be the permutation obtained by following Definition 3.13 without first sorting the rows of $R$, then we have $\ell(\hat{v}) = \ell(v) + \sum_r \text{coinv}(\text{row}_r(R))$. In particular, $I(R) = 0$ if and only if $v$ is the identity, in which case $R$ is super-Yamanouchi. Therefore $\text{inv}(R) = I(R)$ whenever either is 0. By Theorem 3.11 for any $R \in \text{SBT}(w)$, we may write $R = f_1 \cdots f_i(P)$, where $P$ is super-Yamanouchi and each $f_i$ is a Coxeter move. The result for $R$ now follows from the analysis of Coxeter moves above.

**Figure 14.** Constructing the permutation of a standard balanced tableau.
Definition 4.1. Define the ϕ

Corollary 3.17. The number of reduced words for w is equal to the number of standard balanced tableaux of shape \( \mathcal{D}(w) \).

4. Involution and the long permutation

It is easy to see that if \( \rho \) is a reduced word for \( w \), then the reversal of \( \rho \) is a reduced word for \( w^{-1} \). We give the analogous involution on balanced tableaux.

Definition 4.1. Define the flip map \( \varphi \) on standard balanced tableaux by setting \( \varphi(R) \) to be the transpose of \( R \) composed with replacing entry \( i \) with \( \ell - i + 1 \), where \( \ell \) is the number of cells of \( R \).

Example M (Flip map). The flip map applied to \( R \in SBT(41758236) \) from Example 1 results in \( \varphi(R) \in SBT(26714835) \) shown in Figure 15. As \( R \) corresponds to \( \rho = (5, 6, 3, 4, 5, 7, 3, 1, 4, 2, 3, 6) \) in Example 1 we may also consider the reversal of \( \rho \) given by \( rev(\rho) = (6, 3, 2, 4, 1, 3, 7, 5, 4, 3, 6, 5) \). We can easily compute

\[
\text{perm}(\varphi(R)) = 8 1 4 7 10 2 5 9 11 3 6 12
\]

from Figure 15 and less easily compute by Algorithm 2 that this coincides with \( \text{perm}(rev(\rho)) \), indicating that \( \varphi(R) \) corresponds to \( rev(\rho) \).

Proposition 4.2. The flip map \( \varphi \) is a well-defined involution that maps \( SBT(w) \) to \( SBT(w^{-1}) \) such that \( \varphi(c_i(R)) = c_{\ell-i}(\varphi(R)) \) and \( \varphi(b_i(R)) = b_{\ell-i+1}(\varphi(R)) \).

Proof. By 3.4, the Rothe diagram for \( w^{-1} \) is the transpose of the Rothe diagram for \( w \), and so the flip map \( \varphi \) is a well-defined from \( SBT(w) \) to \( SBT(w^{-1}) \) if its image is balanced. A filling \( R \) is balanced if and only if for each cell \( y \) of \( R \) we have

\[
\# \{ x \in R \mid x < y \text{ and } x \text{ above } y \} = \# \{ z \in R \mid z > y \text{ and } z \text{ right of } y \}
\]
where \( x \) is in the same column and \( z \) is in the same row as \( y \). Transposing \( R \) to \( R^T \) results in a filling such that each cell \( y \) satisfies

\[
\# \{ x \in R^T \mid x < y \text{ and } x \text{ right of } y \} = \# \{ z \in R^T \mid z > y \text{ and } z \text{ above } y \},
\]

where \( x \) is now in the row of \( y \) and \( z \) is in the column of \( y \). Replacing \( i \) with \( \ell - i + 1 \) reverses the relative order of entries, so that each cell \( y \), we have

\[
\# \{ x \in \varphi(R) \mid x > y \text{ and } x \text{ right of } y \} = \# \{ z \in \varphi(R) \mid z < y \text{ and } z \text{ above } y \},
\]

where \( x \) is in the row of \( y \) and \( z \) is in the column of \( y \), i.e. \( \varphi(R) \) is balanced.

Since \( i \) and \( i + 1 \) are not in the row or column in \( R \) if and only if \( \ell - i + 1 \) and \( \ell - i \) are not in the row or column in \( \varphi(R) \), we have \( \varphi(\ell_i(R)) = \ell_{\ell-i}(\varphi(R)) \). Similarly, \( i - 1, i, i + 1 \) form a braid pattern in \( R \) if and only if \( \ell - i + 2, \ell - i + 1, \ell - i \) form a braid pattern in \( \varphi(R) \), showing \( \varphi(\ell_i(R)) = \ell_{\ell-i+1}(\varphi(R)) \).

Using the ranked poset structure on reduced words and balanced tableaux and the observation that \( \mathrm{rev}(\ell_i(\rho)) = \ell_{\ell-i}(\mathrm{rev}(\rho)) \) and \( \mathrm{rev}(\ell_i(\rho)) = \ell_{\ell-i+1}(\mathrm{rev}(\rho)) \), we have the following equivalence of involutions.

**Corollary 4.3.** Given a permutation \( w \), if \( R \in \mathrm{SBT}(w) \) corresponds to \( \rho \in R(w) \), then \( \varphi(R) \in \mathrm{SBT}(w^{-1}) \) corresponds to \( \mathrm{rev}(\rho) \in R(w^{-1}) \).

While these involutions respect the graph structure on reduced words and balanced tableaux, they do not behave particularly well with respect to the ranking. When \( w \) is particularly nice, or rather, when the Rothe diagram of \( w \) is particularly nice, there is a different involution that does respect the poset structure.

**Theorem 4.4.** For the longest permutation \( w_0^{(n)} = n(n-1) \cdots 21 \) of \( S_n \), the map \( \psi \) sending an entry \( i \) to \( \binom{n}{2} - i + 1 \) is an order-reserving involution on \( \mathrm{SBT}(w_0^{(n)}) \).

In particular, \( \mathrm{SBT}(w_0^{(n)}) \) has a unique minimal element \( B \) with

\[
\mathrm{inv}(B) = \frac{(n-2)(n-1)(3n-5)}{24}.
\]

**Proof.** The Rothe diagram \( D(w_0^{(n)}) \) is the staircase diagram \( \delta_{n-1} \) of left-justified rows of lengths 1, 2, \ldots, \( n - 1 \) from top to bottom. Thus every cell \( y \) of \( D(w_0^{(n)}) \) has as many cells above it as to its right. For \( y \) a cell of \( D(w_0^{(n)}) \), let \( \mathrm{leg}(y) \) denote the set of cells above \( y \) in the same column and let \( \mathrm{arm}(y) \) denote the set of cells to the right of \( y \) in the same row. Then

\[
\# \{ x \in \mathrm{leg}(y) \mid x < y \} = \# \mathrm{leg}(y) - \# \{ x \in \mathrm{leg}(y) \mid x > y \},
\]

\[
\# \{ z \in \mathrm{arm}(y) \mid z > y \} = \# \mathrm{arm}(y) - \# \{ z \in \mathrm{arm}(y) \mid z < y \}.
\]
For $R \in \text{SBT}(\ell^{(n)})$, since $\#\text{leg}(y) = \#\text{arm}(y)$ for every $y$, this implies
\[
\#\{x \in \text{leg}(y) \mid x > y\} = \#\{z \in \text{arm}(y) \mid z < y\},
\]
from which it follows that $\psi(R)$ is balanced.

For every pair of cells $x, y$ neither in the same row nor same column, say with $x$ above $y$, the pair $(x, y)$ is an inversion in $R$ if and only if it is not an inversion in $\psi(R)$. In particular, every such pair is an inversion only for $\psi(P)$, where $P$ is the super-Yamanouchi tableau. To compute the number of such pairs, notice that there are \( \binom{k}{2} \) cells above the cell in the $k$th row from the top, and we should not have counted $k - 1$ cells in the first column, $k - 2$ in the second, and so on, giving
\[
\sum_{k=1}^{n-1} \binom{k}{2} - \sum_{k=1}^{n-1} \binom{k}{2} = \frac{1}{4} (3n - 1) \binom{n}{3} - \frac{n}{3} = \frac{(n-2)(n-1)(n)(3n-5)}{24},
\]
where the leftmost summation is the (signless) Stirling numbers of the first kind $s(n, n-2)$ and the rightmost is the tetrahedral numbers.

\[\square\]

\textbf{Example N} (Minimal element of $\text{SBT}(w_0^{(n)})$). The ranked poset on $\text{SBT}(w_0^{(4)})$ is shown in Fig. 16. Notice that the unique minimal element is given by
\[
\psi\left(\begin{array}{cccc}
6 & 5 & 4 & 3 \\
2 & 1 & 4 & 5 \\
3 & 6 & 1 & 2 \\
4 & 3 & 5 & 2
\end{array}\right) = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16
\end{array}
\]
and the number of inversions for this minimum is $11 - 4 = 7$.

\textbf{Figure 16.} An illustration of the Coxeter moves on SBT(4321).

The graph on reduced words for $w_0^{(n)}$ is of particular interest. Dehornoy and Autord [4] proved that the diameter of the graph for $w_0^{(n)}$ grows asymptotically...
like $n^4$. Reiner and Roichman [8] used hyperplane arrangements to prove an exact formula for the diameter that coincides with $\text{inv}(B)$ in Theorem 4.4. We give a new, elementary proof using the inversion metric on balanced tableaux.

**Corollary 4.5.** The maximum distance between two reduced words for $w_0^{(n)}$ is

\[
\max_{\rho, \sigma \in R(w_0^{(n)})} \text{dist}(R, S) = \frac{(n-2)(n-1)(n)(3n-5)}{24}.
\]

**Proof.** Let $P$ denote the super-Yamanouchi balanced tableau for $w_0^{(n)}$, and let $B = \psi(P)$. Given any balanced tableau $R \in \text{SBT}(w_0^{(n)})$, there is an inv-increasing path from $P$ to $R$ and, by considering the reversed poset assured by Theorem 4.4, an inv-decreasing path from $R$ to $B$. Therefore we have

\[
\text{dist}(P, R) + \text{dist}(R, B) = \text{dist}(P, B).
\]

For $R, S \in \text{SBT}(w_0^{(n)})$, the triangle inequality gives

\[
\text{dist}(R, P) + \text{dist}(P, S) \geq \text{dist}(R, S) \leq \text{dist}(R, B) + \text{dist}(B, S).
\]

Combining this with Eq. 4.2 for both $R$ and $S$, we have

\[
2 \text{dist}(R, S) \leq \text{dist}(R, P) + \text{dist}(P, S) + \text{dist}(R, B) + \text{dist}(B, S) = 2 \text{dist}(P, B).
\]

Thus $\text{dist}(R, S) \leq \text{dist}(P, B) = \text{inv}(B)$ for all $R, S \in \text{SBT}(w_0^{(n)})$. In particular, the diameter of the graph is $\text{inv}(B)$, so the result follows from Theorem 4.4. \qed

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