Hydrodynamic-type systems describing 2-dimensional polynomially integrable geodesic flows

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Abstract

Starting from a homogeneous polynomial in momenta of arbitrary order we extract multi-component hydrodynamic-type systems which describe 2-dimensional geodesic flows admitting the initial polynomial as integral. All these hydrodynamic-type systems are semi-Hamiltonian, thus implying that they are integrable according to the generalized hodograph method. Moreover, they are integrable in a constructive sense as polynomial first integrals allow to construct generating equations of conservation laws. According to the multiplicity of the roots of the polynomial integral, we separate integrable particular cases.

Keyword: Integrable geodesic flows, semi-Hamiltonian hydrodynamic systems

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Introduction

This paper is devoted to the following classical problem: how to extract geodesic flows that are integrable according to the Liouville theorem. As usual, the geodesic flow of an n-dimensional (pseudo-)Riemannian manifold $(M, g)$ is locally described by a system of (generally nonlinear) ODEs

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad i = 1, \ldots, n, \]

where $x = (x^i)$ is a system of coordinates of $M$, $\Gamma^i_{jk}(x)$ are the Christoffel symbols of the Levi-Civita connection and $\dot{x}$, $\ddot{x}$ are, respectively, the first and second derivatives of $x$ w.r.t. an external parameter $t$. We denote by $(x^i, p_i)$ the system of coordinates of $T^*M$ induced by coordinates $(x^i)$. It is well known that system (1) can be written in Hamiltonian form

\[ \dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n, \]

where

\[ H(x, p) = \frac{1}{2} g^{km}(x)p_kp_m \]

and $p = (p_i)$. Since the Hamiltonian function $H(x, p)$ does not depend on $t$ explicitly, Hamilton’s equations are Liouville integrable if there exist $n - 1$ first integrals $f_k(x, p)$ in involution, i.e. $\{f_k, H\} = 0$ and $\{f_i, f_k\} = 0$, where

\[ \{f, g\} = \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial x^i} \]

is the usual Poisson bracket.

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In the present paper we shall consider only the case \( n = 2 \). In such a case a first integral \( f(x^1, x^2, p_1, p_2) \) satisfies the equation

\[
\{f, H\} = \frac{\partial f}{\partial x^1} \frac{\partial H}{\partial p_1} - \frac{\partial f}{\partial p_1} \frac{\partial H}{\partial x^1} + \frac{\partial f}{\partial x^2} \frac{\partial H}{\partial p_2} - \frac{\partial f}{\partial p_2} \frac{\partial H}{\partial x^2} = 0.
\]

A central idea is the research of first integrals \( f(x^1, x^2, p_1, p_2) \) that are homogeneous polynomials of degree \( N \) in momenta \( p_i \), i.e. of the form

\[
f = \sum_{m=0}^{N} a_m(x^1, x^2)p_1^{N-m}p_2^m.
\]

By substituting the polynomial ansatz (4) into (3) one can derive a quasi-linear system of first order PDEs (see a similar approach, for instance, in [1]).

The problem of describing 2-dimensional metrics admitting a polynomial integral is classical: its formulation is due at least to Darboux [2] and it has been solved (both locally and globally) in the case when \( N \) is equal either to 1 or 2. For \( N = 2 \) see [3], where metrics admitting quadratic integrals are called of Liouville type: in that paper the author solved the problem posed in [4] of their local characterization (see also [5] for the relationship with superintegrable metrics -in the sense of [6]- and projectively equivalent metrics). For \( N = 3 \) see [7, 8], where the problem of finding such metrics is solved under particular assumptions.

A possible strategy (see, for instance, [9]) is to fix a coordinate system where the metric assumes some special form, for instance: \( ds^2 = a(x^1, x^2)(dx^1)^2 + (dx^2)^2 \) (semi-geodesic coordinates), \( ds^2 = a(x^1, x^2)((dx^1)^2 + (dx^2)^2) \) (isothermal coordinates), \( ds^2 = (dx^1)^2 + a(x^1, x^2)dx^1dx^2 + (dx^2)^2 \) (Chebyshev coordinates), etc. In all these cases condition (3), with \( f \) given by (4), leads to hydrodynamic-type systems of PDEs on the coefficients \( a_m(x^1, x^2) \). Integrability of such hydrodynamic-type systems is a separate question.

In this paper we present an alternative construction. We suppose that polynomial (4) has \( N \) real roots (not necessarily distinct), so that (4) can be written in the following factored form

\[
f = \prod_{m=1}^{N} (\alpha_m p_1 + \beta_m p_2).
\]

By finding a suitable system of coordinates where both (5) and Hamiltonian (2) assume a particular convenient form, by means of condition (3), we arrive to discuss a first-order quasi-linear system of PDEs in the unknown functions \( \alpha_m(x^1, x^2) \), \( \beta_m(x^1, x^2) \) and \( g_{ik}(x^1, x^2) \). Our main achievement is a more general ansatz for polynomial integrals than that presented in [9]: the semi-Hamiltonian hydrodynamic-type system we obtained contains \( N+1 \) equations, while the hydrodynamic-type system considered in [9] contains just \( N \) equations. Furthermore, the system we obtained possesses a simple reduction (i.e. the metric is a linear expression in terms of field variables) to the case considered in [9]. In particular, one can select first order quasi-linear systems according to the multiplicity of roots of polynomial (5).

Notations and conventions

\((M, g)\) will be always a 2-dimensional (pseudo-)Riemannian manifold. The symmetric tensor product is denoted by \( \odot \). We shall denote by \( f_x \) the derivative of \( f \) w.r.t. \( x \). All results presented in the paper are of local character, meaning that we always work in suitable neighborhoods.

1 Polynomial integrals of the geodesic flow

For our purposes we need the following proposition, that is well-known.

**Proposition 1.1.** If the geodesic flow of \((M, g)\) admits a polynomial first integral \( f \) of the following form

\[
f = \mp(\alpha_1 p_1 + \alpha_2 p_2)^N
\]

then it admits also the first linear integral \( \alpha_1 p_1 + \alpha_2 p_2 \). In particular, \((M, g)\) admits a Killing vector field.
Essentially, the case described in Proposition 1.1 is the most simple one that can occur, i.e. when the metric admits a local infinitesimal isometry. Thus, in what follows, we shall focus our attention to the case when the geodesic flow of \((M, g)\) admits a first integral \(f\) with at least two distinct roots. We shall use the results of next Proposition to get a description of a such geodesic flow in terms of a hydrodynamic-type system.

**Proposition 1.2.** Let \(f\) be as in (5). Let us suppose that \(f\) admits at least two distinct roots, i.e.

\[
  f = (\alpha_1^1 p_1 + \alpha_1^2 p_2) (\alpha_2^1 p_1 + \alpha_2^2 p_2) Q
\]

where \(Q\) is a homogeneous polynomial of degree \(N-2\) in momenta, \(\alpha_1^1(p)\alpha_2^2(p) - \alpha_1^2(p)\alpha_2^1(p) \neq 0, (\alpha_1^1(p), \alpha_2^1(p)) \neq 0 \neq (\alpha_2^1(p), \alpha_2^2(p))\) at a point \(p \in M\). Then, in a neighborhood \(U\) of the point \(p\), there exists a system of coordinates \((x_1^{\text{new}}, x_2^{\text{new}})\) such that polynomial \(f\), in the induced system of coordinates \((x_1^{\text{new}}, x_2^{\text{new}}, p_1^{\text{new}}, p_2^{\text{new}})\) of \(T^*M\), assumes the form

\[
  f = p_1 p_2 \prod_{i=1}^{N-2} (\alpha_1^i p_1 + \alpha_2^i p_2), \quad \alpha_j^k = \alpha_j^k (x^1, x^2)
\]

**Proof.** In view of the identification \(p_i \simeq \partial_{x^i}\), we can write polynomial (6) as follows:

\[
  f = X \circ Y \circ \Xi
\]

where \(X = \alpha_1^1 \partial_{x^1} + \alpha_1^2 \partial_{x^2}, Y = \alpha_2^1 \partial_{x^1} + \alpha_2^2 \partial_{x^2}\) and \(\Xi\) is a \((N-2,0)\)-tensor. The two distributions of curves, formed by the integral curves of \(X\) and \(Y\), can always be chosen as coordinate lines: in these coordinates the integral acquires a factor of \(p_1 p_2\), i.e. polynomial (6) assumes the form (7).

**Remark 1.3.** The form of any homogeneous polynomial in momenta (in particular polynomial (7)) does not change under a transformation

\[
  x_1^{\text{new}} = x_1^{\text{new}}(x^1), \quad x_2^{\text{new}} = x_2^{\text{new}}(x^2).
\]

Indeed, both \(\partial_{x^1}\) and \(\partial_{x^2}\) do not change direction under the above transformation.

**Proposition 1.4.** If the geodesic flow of \((M, g)\) admits a polynomial integral \(f\) with the properties described in Proposition 1.2 then there exists a system of coordinates \((x^i, p_i)\) where the Hamiltonian (2) has the form

\[
  H = \frac{1}{2} \epsilon_1 p_1^2 + g^{12} p_1 p_2 + \frac{1}{2} \epsilon_2 p_2^2, \quad g^{12} = g^{12}(x^1, x^2), \quad \epsilon_i \in \{-1, 0, 1\}.
\]

and the polynomial \(f\) the form (7).

**Proof.** Let \(f\) be a polynomial integral of \((M, g)\) with two distinct roots. Then, in view of Proposition 1.2 there exists a system of coordinates where \(f\) has the form (7). Equivalently, it is the same of considering form (4) with \(a_0 = 0 = a_N\). We shall use this latter notation. If we substitute this \(f\) in (4) we obtain

\[
  a_1 g_1^{11} = 0, \quad a_{N-1} g_2^{22} = 0
\]

as they are, respectively, the coefficients of \(p_1^{N+1}\) and \(p_2^{N+1}\) of left hand side term of (4). If both \(a_1\) and \(a_{N-1}\) are not zero, then \(g_1^{11} = 0 = g_2^{22}\), implying that \(g^{11} = g^{11}(x^1)\) and \(g^{22} = g^{22}(x^2)\). If \(a_1 = 0\), then we obtain \(a_2 g_1^{11} = 0 = a_2 g_2^{22}\) is the coefficient of \(p_1^{N+1}\) of the left hand side of (4), that gives either \(g_1^{11} = 0\) or \(a_2 = 0\). If the latter case occurs, then we obtain \(a_3 g_1^{11} = 0\), and so further. A similar reasoning applies, of course, also if we start from \(a_{N-1} g_2^{22}\). We conclude that if the initial polynomial integral is not zero, then \(g_1^{11} = 0 = g_2^{22}\), so that

\[
  g^{-1} = g^{11}(x^1)\partial_{x^1} \circ \partial_{x^1} + 2 g^{12}(x^1, x^2)\partial_{x^1} \circ \partial_{x^1} + g^{22}(x^2)\partial_{x^2} \circ \partial_{x^2}.
\]

Now, in view of Remark 1.3, we can use a changing of coordinates (8) to further simplify (10). In particular, we can find a new system of coordinates such that \(g^{11} = \epsilon_1\) and \(g^{22} = \epsilon_2\), with \(\epsilon_i \in \{-1, 0, 1\}\). Thus, in these new system of coordinates, Hamiltonian \(H\) (see (2)) assumes the form (9) (up to renaming function \(g^{12}\)).
In our case the metric with upper indices looks precisely like the metric with lower indices in the case of the Chebyshev coordinate net, see formula (26) below. Thus, our coordinate system is conformally equivalent to the Chebyshev net. However, our choice of coordinates is very convenient for our further computations. For instance we shall show in Section 1.1 that in this coordinate system all further reductions appear in the most simple way.

To not overload the notation, from now on we shall consider only the Hamiltonian

$$H = \frac{1}{2}p_1^2 + g^{12}p_1p_2 + \frac{1}{2}p_2^2,$$

i.e. (9) with \(\epsilon_1 = \epsilon_2 = 1\), as the other cases can be treated in the same way. As a possible polynomial integral of the above Hamiltonian, without loss of generality, in view of Proposition 1.4, we can consider

$$f = a_1p_1^{N-1}p_2 + a_2p_1^{N-2}p_2^2 + \ldots + a_{N-2}p_1^2p_2^{N-2} + a_{N-1}p_1p_2^{N-1}.$$  

Then substituting (12) into (3) we obtain

$$\begin{align*}
(p_1 + g^{12}p_2)f_x^1 - p_1p_2(g^{12})_x^1f_p_1 + (g^{12}p_1 + p_2)f_x^2 - p_1p_2(g^{12})_x^2f_p_2 &= 0
\end{align*}$$

from which one can derive a quasi-linear system of first order PDEs

$$\begin{align*}
a_{1,x^1} + g^{12}a_{1,x^2} &= a_1(g^{12})_{x^2} \\
0 &= a_{k-1,x^1} + g^{12}a_k - g^{12}a_{k-1,x^2} + ka_k(g^{12})_{x^2} + (N + 1 - k)a_{k-1}(g^{12})_{x^1}, \quad k = 2, \ldots, N - 1 \\
g^{12}a_{N-1,x^1} + a_{N-1,x^2} &= a_{N-1}(g^{12})_{x^1}
\end{align*}$$

with N unknown functions, i.e. the metric coefficient \(g^{12}\) and the \(N - 1\) coefficients \(a_k\) of polynomial ansatz (12).

A very important property of this system is existence of simple reductions based on the multiplicity of the roots of polynomial (12). For instance: \(a_k = 0, k = 1, 2, \ldots, K_1 < N - 1\) and \(a_k = 0, k = N - 1, N - 2, \ldots, K_2 < N - 1\). As an example, if \(N = 5\), we have the following full list of distinguishing reductions: \(a_1 = 0; a_1 = a_2 = 0; a_1 = a_2 = a_3 = 0; a_1 = a_4 = 0; a_1 = a_2 = a_4 = 0\). Thus, instead to investigate each particular case, one can concentrate on the generic system (13) only.

However this system is written in a non-evolutionary form. By this reason, in Section 2 we rewrite it in an evolutionary form by an appropriate reciprocal transformation.

### 1.1 Polynomial integrals of third and fourth degree

Here we consider system (14) in the case of polynomial integrals of third and fourth degree. We also briefly discuss its reductions according to the multiplicity of roots of the polynomial integral.

#### 1.1.1 Geodesic flows admitting homogeneous polynomial integrals of third and fourth degree in momenta

Let \(f\) be a homogeneous polynomial of a third degree in momenta. We already seen that, if \(f\) is a perfect cube, the metric admitting such \(f\) as an integral it admits a Killing vector field (see Proposition 1.1). So, let us then consider the case when polynomial (12) has at least two distinct roots. In this case, in view of Proposition 1.2, a normal form of such polynomial is (12) with \(N = 3\), i.e.

$$a_1p_1^2p_2 + a_2p_1p_2^2, \quad a_1, a_2 \in C^\infty(M).$$

that, in view of system (14), is an integral of Hamiltonian (11) iff

$$\begin{align*}
a_{1,x^1} + a_{1,x^2}g^{12} &= (g^{12})_{x^2}a_1 \\
a_{1,x^1}g^{12} + a_{2,x^1} + a_{1,x^2} + a_{2,x^2}g^{12} &= 2(g^{12})_{x^1}a_1 + 2(g^{12})_{x^2}a_2 \\
a_{2,x^1}g^{12} + a_{2,x^2} &= (g^{12})_{x^1}a_2
\end{align*}$$
If polynomial $f$ admits two coincident roots, then its normal form is \( (15) \) with $a_1 = 0$ (or, equivalently, $a_2 = 0$) and, correspondingly, the quasi-linear system to be considered is \( (16) \) with either $a_1 = 0$ or $a_2 = 0$.

### 1.1.2 Geodesic flows admitting homogeneous polynomial integrals of fourth degree in momenta

Let $f$ be a homogeneous polynomial of fourth degree in momenta. Let us suppose that $f$ admits at least two distinct roots, otherwise \( (M, g) \) admits a Killing vector field (see again Proposition \( 1.1 \)). In this case, in view of Proposition \( 1.2 \) a normal form of such polynomial is \( (12) \) with $N = 4$, i.e.

\[
\begin{align*}
 a_1 p_1^3 p_2 + a_2 p_1^2 p_2^2 + a_3 p_1 p_2^3 \quad &a_1, a_2, a_3 \in C^\infty (M). 
\end{align*}
\]

that, in view of system \( (14) \), is an integral of Hamiltonian \( (11) \) iff

\[
\begin{align*}
 a_{1,x^1} + a_{1,x^2} g^{12} & = a_1 (g^{12})_{x^2} \\
 a_{1,x^2} + a_{2,x^1} + a_{1,x^1} g^{12} + a_{2,x^2} g^{12} & = 2 a_2 (g^{12})_{x^2} + 3 a_1 (g^{12})_{x^1} \\
 a_{3,x^1} + a_{2,x^2} + a_{3,x^2} g^{12} + a_{2,x^1} g^{12} & = 3 a_3 (g^{12})_{x^2} + 2 a_2 (g^{12})_{x^1} \\
 a_{3,x^2} + a_{3,x^1} g^{12} & = a_3 (g^{12})_{x^1}
\end{align*}
\]

According to the multiplicity of roots of polynomial \( (17) \), we have several possible reductions of system \( (18) \). In fact, by arguing as in the end of Section \( 1.1.1 \) if in system \( (18) \) we put $a_2 = a_3 = 0$, then we obtain the system describing metrics admitting a polynomial integral with three coincident roots. If we put $a_1 = a_3 = 0$, we obtain the system describing metrics admitting a polynomial integral with two coincident roots. Finally, if we put $a_3 = 0$, we obtain the system describing metrics admitting a polynomial integral with two coincident roots (and two distinct). If no particular assumption is imposed on $a_1, a_2, a_3$, then, generically, the above system describes metrics admitting polynomial integrals with all distinct roots.

### 2 Integrability

In this section we are going to investigate integrability of hydrodynamic-type system \( (14) \) selected by the Hamiltonian \( (11) \) and the polynomial ansatz \( (12) \). For our further research we need first to reduce this system to an evolutionary form. This is possible by finding an appropriate reciprocal transformation; below we present a constructive algorithm:

- We derive the Liouville equation from commutativity of the Hamiltonian and the first integral \( (8) \);
- we introduce an appropriate reciprocal transformation (to semi–geodesic coordinates);
- under this transformation we recompute the metric, the Hamilton-Jacobi equation, the Liouville equation, momenta and finally the hydrodynamic–type system \( (14) \);
- we discuss the existence of infinite set of conservation laws;
- we prove the diagonalizability of this hydrodynamic–type system;
- as an example, we shall consider the two component case.

Polynomial ansatz \( (12) \) can be written in the form (see more details in \( 10 \| 11 \))

\[
f = \left(p_1^2 + 2 g^{12} p_1 p_2 + p_2^2\right)^{N/2} \lambda(s, x^1, x^2),
\]

where $s = p_2/p_1$ and the function $\lambda(s, x^1, x^2)$ satisfies \( (13) \):

\[
(1 + g^{12} s) \lambda_{x^1} + (g^{12} + s) \lambda_{x^2} + (s g^{12})_{x^1} - s (g^{12})_{x^2} \lambda_s = 0.
\]
Introducing the variable

\[ p = (1 + 2g^{12}s + s^2)^{-1/2} \]

(instead of the variable \( s \)), equation (20) reduces to the canonical form\(^1\)

\[ \lambda_{x^2} = \{ \lambda, H \} = H_p \lambda_{x^1} - \lambda_p H_{x^1}, \]

where now the function \( \lambda \) depends on \( p \) via formula (21) and the Hamiltonian function is

\[ H = \sqrt{(g^{12})^2 - 1} p^2 + 1 - g^{12}p. \]

Equation (22) plays an important role in the theory of integrable hydrodynamic chains and semi-Hamiltonian hydrodynamic-type systems (see, for instance, [13, 14]). The existence of representation (22) for hydrodynamic-type system (14) means that such a system is integrable (or semi-Hamiltonian), i.e. it admits infinitely many conservation laws, commuting flows and particular solutions (see more details in [15, 16]). The integrability procedure means that instead of the function \( \lambda(x^1, x^2, p) \) we consider the function \( \lambda(p, g^{12}, a_1(x^1, x^2), \ldots, a_{N-1}(x^1, x^2)) \) whose existence is equivalent to the integrability of some overdetermined system (now known as the Gibbons-Tsarev system [17]). Substitution of one of its particular solutions (see [12])

\[ \lambda(s, x^1, x^2) = (p_1^2 + 2g^{12}p_1p_2 + p_2^2)^{-N/2} f(x^1, x^2, p_1, p_2) \]

\[ = (p_1^2 + 2g^{12}p_1p_2 + p_2^2)^{-N/2} (a_1p_1^{N-1}p_2 + a_2p_1^{N-2}p_2^2 + \ldots + a_{N-2}p_1^2p_2^{N-2} + a_{N-1}p_1p_2^{N-1}) \]

\[ = (1 + 2g^{12}s + s^2)^{-N/2} (a_1s + a_2s^2 + \ldots + a_{N-2}s^{N-2} + a_{N-1}s^{N-1}) \]

into (22) creates the hydrodynamic-type system (14). The Liouville-type equation (22) takes the form a Hamilton–Jacobi equation

\[ p_{x^2} = H_{x^1} \]

where differentiation of \( H \) (given by (23)) w.r.t. \( x^1 \) means to differentiate not only \( g^{12} \) but also \( p \), that now is a dependent function on \( (x^1, x^2, \lambda) \). So, now the function \( p \) is a generating function of conservation law densities for hydrodynamic-type system (14), while the Hamilton-Jacobi equation (24) plays the role of the generating equation of the corresponding conservation laws (w.r.t. the parameter \( \lambda \)).

### 2.1 Transformation to the Semi-Geodesic Coordinates

In this section we are going to consider the main system of our interest (14) in another coordinate system that is more convenient for our further computations. Instead of our original coordinates (conformally equivalent Chebyshev coordinates, see the discussion before formula (11)), we introduce the so called semi–geodesic coordinates \((x, y)\) (see more details in [11]), which we determine by virtue of the reciprocal transformation

\[ dy = \frac{1}{a_{N-1}} dx^1 - \frac{g^{12}}{a_{N-1}} dx^2, \quad dx = dx^2, \]

where the potential function \( y \) follows from the third equation of system (14) written in the conservative form

\[ \left( \frac{1}{a_{N-1}} \right)_{x^2} + \left( \frac{g^{12}}{a_{N-1}} \right)_{x^1} = 0, \]

i.e.

\[ y_{x^1} = \frac{1}{a_{N-1}}, \quad y_{x^2} = -\frac{g^{12}}{a_{N-1}}. \]

\(^1\)Classifications of such Hamiltonian equations is given in [12].
So, the metric corresponding to the Hamiltonian \((11)\)
\begin{equation}
(26)\quad ds^2 = \frac{1}{1 - \left(g^{12}\right)^2} \left((dx^1)^2 - 2g^{12}dx^1dx^2 + (dx^2)^2\right),
\end{equation}
in these semi–geodesic coordinates \((x, y)\), assumes the form
\begin{equation}
(27)\quad ds^2 = (dx)^2 + \frac{a_{N-1}^2}{1 - \left(g^{12}\right)^2} (dy)^2.
\end{equation}
Correspondingly, the Hamilton–Jacobi equation (see \((24)\))
\begin{equation}
(28)\quad p_x \cdot 2 = \left(\sqrt{((g^{12})^2 - 1)p^2 + 1 - g^{12}p}\right)_{x},
\end{equation}
becomes (see again \([1]\) and other details in \([11]\))
\begin{equation}
(29)\quad \tilde{p}_y = \left(\frac{a_{N-1}}{\sqrt{1 - \left(g^{12}\right)^2}} \sqrt{1 - \tilde{p}^2}\right)_{x},
\end{equation}
where we define \(\tilde{p}\) as follows
\begin{equation}
(30)\quad \tilde{p} = \sqrt{((g^{12})^2 - 1)p^2 + 1}
\end{equation}
(we remind that the variable \(p\) has been defined by \((21)\)). In fact, the conservation law \((28)\) can be written in the potential form
\begin{equation}
d\xi = \tilde{p}_x dx^1 + \left(\sqrt{((g^{12})^2 - 1)p^2 + 1 - g^{12}p}\right) dx^2.
\end{equation}
Under the inverse reciprocal transformation (see \((25)\))
\begin{equation}
dx^1 = g^{12}dx + a_{N-1}dy, \quad dx^2 = dx
\end{equation}
we obtain
\begin{equation}
d\xi = p a_{N-1} dy + \sqrt{((g^{12})^2 - 1)p^2 + 1} dx.
\end{equation}
The compatibility condition \((\xi_x)_y = (\xi_y)_x\) leads to \((29)\) where \(\tilde{p}\) is given by \((30)\). To recompute first integral \((12)\) via new coordinates \((x, y)\) we need first to recompute the corresponding momenta. To do this we use the identity
\begin{equation}
p_1 dx^1 + p_2 dx^2 = \tilde{p}_1 dx + \tilde{p}_2 dy.
\end{equation}
In this case
\begin{equation}
(32)\quad \tilde{p}_1 = p_2 + g^{12}p_1, \quad \tilde{p}_2 = a_{N-1}p_1,
\end{equation}
then the first integral \((12)\) takes the form
\begin{equation}
(33)\quad f = \tilde{p}_2(\tilde{p}_1)^{N-1} + \tilde{a}_1(\tilde{p}_2)^2(\tilde{p}_1)^{N-2} + \tilde{a}_2(\tilde{p}_2)^3(\tilde{p}_1)^{N-3} + ... + \tilde{a}_{N-2}(\tilde{p}_2)^{N-1} \tilde{p}_1 + \tilde{a}_{N-1}(\tilde{p}_2)^N.
\end{equation}
Note that all coefficients \(\tilde{a}_k\) are linear functions with respect to \(a_m\) and polynomial functions with respect to \(g^{12}\) and \((a_{N-1})^{-1}\). Taking into account that first integral \((33)\) can be written in the factorized form
\begin{equation}
f = (\tilde{p}_2)^N \prod_{m=1}^{N-1} \left(\frac{\tilde{p}_1}{\tilde{p}_2} - \tilde{b}_m\right),
\end{equation}
we introduce (cf. \([19]\)) the function
\begin{equation}
\tilde{\lambda}(\tilde{s}, x, y) = \left(\frac{1}{\tilde{s}^2} + \frac{1 - (g^{12})^2}{a_{N-1}^2}\right)^{-N/2} \prod_{m=1}^{N-1} \left(\frac{1}{\tilde{s}} - \tilde{b}_m\right),
\end{equation}
where \( \tilde{s} = \tilde{p}_2/\tilde{p}_1 \). For our further convenience we define an appropriate independent variable \( q \) instead of \( \tilde{s} \):

\[
q = \frac{a_{N-1}}{\sqrt{1 - (g^{12})^2}} \frac{1}{\tilde{s}}.
\]

Then we obtain the equation of the Riemann surface

\[
\tilde{\lambda}(x, y, q) = a^{-1/2}(1 + q^2)^{-N/2} \prod_{m=1}^{N-1} (q - b_m),
\]

where \( \tilde{b}_k = a^{1/2}b_k \) and

\[
a^{-1/2} = \frac{a_{N-1}}{\sqrt{1 - (g^{12})^2}}.
\]

Under the reciprocal transformation (25) Liouville equation (22) takes the form

\[
\tilde{\lambda}_y = a^{-1/2}q\tilde{\lambda}_x + (1 + q^2)\tilde{\lambda}_q(a^{-1/2})_x
\]

where (see (32) and (34))

\[
\tilde{p} = \frac{q}{1 + q^2}, \quad q = \frac{\tilde{p}}{1 - \tilde{p}^2}.
\]

The Liouville-type equation (36) takes again the form of Hamilton–Jacobi equation (29) (see (24)), that was already investigated in [11]. Substitution (35) into (36) yields the hydrodynamic-type system

\[
\begin{align*}
    a_y &= 2a^{1/2} \left( \sum_{m=1}^{N-1} b_m \right) x + a^{-1/2}a_x \left( \sum_{m=1}^{N-1} b_m \right) \\
    (b_k)_y &= a^{-1/2}b_k(b_k)_x - [1 + (b_k)^2](a^{-1/2})_x, \quad k = 1, 2, ..., N - 1.
\end{align*}
\]

As we mentioned above, this system is integrable by the Generalized Hodograph Method (see [15, 16]). This means that any solution of this system determines an integrable geodesic flow (see details in [11]). We emphasize that description of integrable geodesic flows selected by Hamiltonian (11) and by homogeneous polynomial first integral (12) reduces to the integrability of the hydrodynamic–type system (38). Moreover, once solutions of this system are found, one can choose any conservation law density (see (37), here instead of \( p \) we use \( h_k \) and instead of \( q \) we use \( b_k \))

\[
h_k = \frac{b_k}{\sqrt{1 + (b_k)^2}}
\]

to determine elements of the conservation law (for any index \( k \), the density and the flux correspondingly)

\[
g^{12} = h_k, \quad a_{N-1} = \sqrt{1 - (h_k)^2}/a
\]

of the inverse reciprocal transformation (31). Since this transformation can be determined for any index \( k \), we have \( N - 1 \) different choices how to connect metrics (26) and (27). Thus any solution found in semi-geodesic coordinates can be recomputed back to our original coordinates \( (x^1, x^2) \). Here we briefly discuss the integrability of system (38). Its integrability is based on two important properties: existence of Riemann invariants (in these coordinates any hydrodynamic–type system takes a diagonal form) and existence of infinitely many conservation laws. First we explain how to compute these conservation laws. By introducing the so called moments

\[
B^k = \frac{1}{k + 1} \sum_{m=1}^{N} (g^m)^{k+1},
\]

\footnote{Another hydrodynamic-type system was found in [11]. The approach presented in this Section was established in [14] and utilized in [10] for the hydrodynamic-type system derived by V.V. Kozlov in [18], where that hydrodynamic-type system describes classical mechanical systems with one-and-a-half degree of freedom and with polynomial first integrals.}
system (38) assumes the form

\[
\begin{align*}
  a_y &= 2a^{1/2}B_x^0 + a^{-1/2}B^0 a_x \\
  B_y^0 &= a^{-1/2}B_x^1 - (N + 2B_1)(a^{-1/2})_x \\
  B_y^k &= a^{-1/2}B_x^{k+1} + (kB^k - 1 + (k + 2)B^{k+1})(a^{-1/2})_x, \quad k = 1, 2, \ldots
\end{align*}
\]

System (38) has infinitely many polynomial conservation laws whose densities and fluxes depend only on \(a\) and moments \(B^k\). One can derive them either iteratively step by step or from the equation of the Riemann surface (35) (see details below and [10, 11]). For instance, first two conservation laws are

\[
a_y = \left(2a^{1/2}B^0_0\right)_x, \quad \left(B^0 a^{3/2}\right)_y = \left(a \left(\frac{3}{2}(B^0)^2 + B^1 + \frac{N}{2}\right)\right)_x.
\]

Moreover, system (38) is diagonalizable, i.e. it can be put in the form

\[
r^i_t = \mu_i(r)r^i_x \quad i = 1, \ldots, N
\]

where \(r^k\) are Riemann invariants, determined by the condition \(\lambda_q = 0\), i.e., they are branch points of the Riemann surface determined by the equation (35). More precisely, \(r^k(b) = \tilde{\lambda}(b, q)|_{q=q_b(b)}\), where \(b = (b^1, \ldots, b^{N-1})\), i.e., in our case, the \(N\) distinct roots \(q_k(b)\) determined by the condition

\[
N \frac{q}{1 + q^2} = \sum_{m=1}^{N-1} \frac{1}{q - b_m}
\]

and then substituted into the equation of Riemann surface (35) give the Riemann invariants

\[
r^k(x, y) = a^{-1/2}(1 + q_b^2)^{-N/2} \prod_{m=1}^{N-1} (q_k - b_m).
\]

So, if \(q \to q_k(b)\), \(\tilde{\lambda}(b, q) \to r^k(b)\), \(\tilde{\lambda}_q \to 0\), then (36) leads to (39), with characteristic velocities \(\mu_k(r(a, b)) = a^{-1/2}q_k(b)\). Hydrodynamic-type systems which are simultaneously diagonalizable and possess infinitely many conservation laws are integrable by the Generalized Hodograph Method (see [15, 16]). For instance, if \(N = 2\), then characteristic velocities are

\[
\mu_1 = a^{-1/2} \left(b_1 + \sqrt{(b_1)^2 + 1}\right), \quad \mu_2 = a^{-1/2} \left(b_1 - \sqrt{(b_1)^2 + 1}\right);
\]

and the corresponding Riemann invariant are

\[
r^1 = \frac{1}{2b_1 + \sqrt{(b_1)^2 + 1}}, \quad r^2 = \frac{1}{2b_1 - \sqrt{(b_1)^2 + 1}}
\]

Thus, hydrodynamic-type system (39) becomes

\[
r^1_t = -2r^2r_x^1, \quad r^2_t = -2r^1r_x^2.
\]

This system (as well as the corresponding metric written in semi-geodesic coordinates) is discussed in detail in [11].

**Remark 2.1.** The first integral investigated in [11] and [11] is different (cf. (38)):

\[
f = (\tilde{p}_1)^N + \tilde{p}_2(\tilde{p}_1)^{N-1} + \tilde{a}_1(\tilde{p}_2)^2(\tilde{p}_1)^{N-2} + \tilde{a}_2(\tilde{p}_2)^3(\tilde{p}_1)^{N-3} + \ldots + \tilde{a}_{N-2}(\tilde{p}_2)^{N-1}\tilde{p}_1 + \tilde{a}_{N-1}(\tilde{p}_2)^N.
\]

However the integration procedure based on the Generalized Hodograph Method (see more details in [15, 16]) is precisely the same as in [11].
3 Conclusion

In this paper we presented an approach, described in details in Section 1, that allows to reduce the problem of the description of integrable geodesic flows selected by homogeneous polynomial first integrals to the integrability of semi–Hamiltonian hydrodynamic–type systems possessing a variety of inequivalent reductions. In fact, the polynomial ansatz (12) for the first integral depends on two natural numbers: one of them is the degree of the polynomial and the other is the number of its non-zero coefficients. Moreover, the considered hydrodynamic–type systems can be written in evolutionary form by an appropriate reciprocal transformation to semi-geodesic coordinates. All of them can be integrated by the Generalized Hodograph Method. However, even in the two components cases, such solutions can be presented just in implicit form. Nevertheless, this two component case is very interesting since characteristic velocities can be expressed explicitly via Riemann invariants. This particular research will be the topic of a future investigation.

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