FORMAL GROUP RINGS OF TORIC VARIETIES

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Abstract. In this paper we use formal group rings to construct an algebraic model of the $T$-equivariant oriented cohomology of smooth toric varieties. Then we compare our model with known results of equivariant cohomology of toric varieties to justify our construction.

1. Introduction

Let $h$ be an algebraic oriented cohomology theory in the sense of Levine-Morel [14], where examples include the Chow group of algebraic cycles modulo rational equivalence, algebraic $K$-theory, connective $K$-theory, elliptic cohomology, and a universal such theory called algebraic cobordism. It is known that to any $h$ one can associate a one-dimensional commutative formal group law $F$ over the coefficient ring $R = h(\text{pt})$, given by

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2))$$

for any line bundles $L_1, L_2$ on a smooth variety $X$, where $c_1$ is the first Chern class.

Let $T$ be a split algebraic torus, and $T^*$ be the character lattice of $T$. A new combinatorial object, called a formal group ring and denoted by $R[F[T^*]]$, can then be defined by using $R, F$ and $T^*$. It serves as an algebraic model of the completed $T$-equivariant cohomology ring $h_T(\text{pt})^\wedge$ of a point, or of the cohomology ring of the classifying space $h(BT)$ of $T$. Various computations can then be performed on formal group rings to provide algebraic models of the (usual) cohomology ring $h(G/B)$ and the $T$-equivariant cohomology ring $h_T(G/B)$ of a homogeneous space $G/B$, where $G$ is a split semisimple linear algebraic group with a maximal split torus $T$, and $B$ is a Borel subgroup containing $T$. We refer to [6], [7] and [8] for details.

The main goal of this paper is to apply the techniques of formal group rings to a smooth toric variety $X$, so that we obtain algebraic models of the usual cohomology and the $T$-equivariant cohomology of $X$. Since any toric variety can be constructed by gluing affine toric varieties together, our idea is to find a model of the $T$-equivariant cohomology for each affine toric variety and then “glue” them together. This paper is organized as follows. First we establish notation and recall basic facts on toric varieties and formal group rings in Section 2. Then in Section 3 we prove our main result of gluing formal group rings together, and the output

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of gluing will be our models of the usual cohomology and the $T$-equivariant cohomology of $X$. Next we compare our models with known results of equivariant cohomology in Section 4, and we do some explicit computation in Section 5. Finally in Section 6 we define the algebraic counterpart of the pull-back and push-forward homomorphisms of blow-ups.

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2. Notation and Preliminaries

2.1. Toric Varieties. Our references for the theory of toric varieties are [5] and [10], and also [9] for the definition of a toric variety over an arbitrary base.

Let $T$ be a split torus over our base field $k$. The character and cocharacter lattices of $T$ are denoted by $T^*$ and $T_0$ respectively, and there is a perfect pairing $\langle \ , \rangle : T^* \times T_0 \to \mathbb{Z}$. A toric variety $X$ is a normal variety on which the split torus $T$ acts faithfully with an open dense orbit. Recall that $X$ is determined by its fan $\Sigma$ in the lattice $T_0$. In this paper we will always assume $X$ is smooth unless otherwise stated, so that every cone $\sigma$ in $\Sigma$ is generated by a subset of a basis of $T_0$. Let $\Sigma_{max}$ be the set of maximal cones of $\Sigma$. The set of all rays (i.e. one-dimensional cones) in $\Sigma$ is denoted by $\Sigma(1)$, and similarly the set of all rays in $\sigma$ is denoted by $\sigma(1)$ for every cone $\sigma \in \Sigma$.

For every cone $\sigma \in \Sigma$, let $U_{\sigma}$ be the associated open affine subscheme of $X$, and $O_{\sigma}$ be the $T$-orbit corresponding to $\sigma$ under the Orbit-Cone Correspondence. The stabilizer of any geometric point of $O_{\sigma}$ is a subtorus $T_{\sigma} \subseteq T$, so that $O_{\sigma} \cong T/T_{\sigma}$. The character and cocharacter lattices of $T_{\sigma}$ are given by $T_{\sigma}^* = T^*/\sigma^\perp$ and $T_{\sigma*} = \langle \sigma \rangle = \sigma + (\sigma) \subseteq N$ respectively, and $\dim T_{\sigma} = \dim \sigma$. Finally, for every ray $\rho \in \Sigma(1)$, the unique generator of the monoid $\rho$ is denoted by $v_{\rho} \in N$.

2.2. Formal Group Rings. Our main reference for formal group rings is [6].

Let $R$ be a commutative ring, and $F$ be a formal group law over $R$ which we always assume to be one-dimensional and commutative, i.e. $F(x, y) \in R[x, y]$ is a formal power series such that

$$F(x, 0) = 0, \quad F(x, y) = F(y, x), \quad \text{and} \quad F(x, F(y, z)) = F(F(x, y), z),$$

see [14] p.4]. For any nonnegative integer $n$ we use the notation

$$x +_F y = F(x, y), \quad n \cdot_F x = \underbrace{x +_F \cdots +_F x}_{n \text{ copies}}, \quad \text{and} \quad (-n) \cdot_F x = -_F(n \cdot_F x)$$

where $-_F x$ denotes the formal inverse of $x$, i.e. $x +_F (-_F x) = 0$.

Let $M$ be an abelian group, and let $R[x_M]$ denote the polynomial ring over $R$ with variables indexed by $M$. Let $\epsilon : R[x_M] \to R$ be the augmentation map which sends $x_\lambda$ to 0 for every $\lambda \in M$. We denote $R[[x_M]]$ the $\ker(\epsilon)$-adic completion of the polynomial ring $R[x_M]$. Let $J_F \subseteq R[x_M]$ be the closure of the ideal generated by $x_0$ and $x_{\lambda + \mu} - (x_\lambda +_F x_\mu)$ over all $\lambda, \mu \in M$. The formal group ring (also called formal group algebra) is then defined to be the quotient

$$R[M]_F = R[x_M]/J_F.$$
By abuse of notation the class of $x_\lambda$ in $R[[M]]_F$ is also denoted by $x_\lambda$. By definition $R[[M]]_F$ is a complete Hausdorff $R$-algebra with respect to the ker($\epsilon'$)-adic topology, where $\epsilon' : R[[M]]_F \to R$ is the induced augmentation map.

An important subring of $R[[M]]_F$ is the image of $R[x_M]$ under the composition $R[x_M] \to R[[x_M]] \to R[[M]]_F$, denoted by $R[M]_F$. Then $R[[M]]_F$ is the completion of $R[M]_F$ at the ideal ker($\epsilon'$) ∩ $R[M]_F$.

**Example 2.1.** (see [6] Example 2.19) The additive formal group law over $R$ is given by $F(x, y) = x + y$. In this case we have $R$-algebra isomorphisms

$$R[M]_F \cong \text{Sym}_R(M)$$

and $R[[M]]_F \cong \text{Sym}_R(M)^\wedge$,

where Sym$_R(M)$ is the ring of symmetric powers of $M$ over $R$, and the completion is at the kernel of the augmentation map $x_\lambda \mapsto 0$. The isomorphisms are given by sending $x_\lambda$ to $\lambda \in \text{Sym}_R(M)$.

**Example 2.2.** (see [6] Example 2.20) The multiplicative periodic formal group law over $R$ is given by $F(x, y) = x + y - \beta xy$, where $\beta$ is a unit in $R$. Let $R[M]$ be the (usual) group ring $R[M] = \{\sum r_i e^{\lambda_i} \mid r_i \in R, \lambda_i \in M\}$ written in the exponential notation, and $\text{tr} : R[M] \to R$ be the trace map which sends $e^\lambda$ to 0 for every $\lambda \in M$.

Then there are the following $R$-algebra isomorphisms

$$R[M]_F \cong R[M]$$

and $R[[M]]_F \cong R[M]^\wedge$,

where the completion is at ker($\text{tr}$), and the isomorphisms are given by sending $x_\lambda$ to $\beta^{-1}(1 - e^\lambda)$.

Finally we remark that given $\phi : M \to M'$ a homomorphism of abelian groups, it induces ring homomorphisms $R[M]_F \to R[M']_F$ and $R[M]_F \to R[M']_F$ by sending $x_\lambda$ to $x_{\phi(\lambda)}$.

3. **Formal Group Rings of Toric Varieties**

Let $X$ be the smooth toric variety of the fan $\Sigma \subseteq T_*$. For every cone $\sigma \in \Sigma$, since $\sigma$ is smooth we have $U_\sigma = \Lambda_{\dim \sigma} \times T/T_\sigma$. Therefore if $F$ is the formal group law associated to some oriented cohomology theory $h$ over the coefficient ring $R = h(\text{pt})$, by homotopy invariance the formal group ring $R[[T_*]]_F$ can be viewed as an algebraic substitute of the completed equivariant cohomology ring $h_T(U_\sigma)^\wedge$, see [6] Remark 2.22. It is known that a topology can be given to the fan $\Sigma$ by defining the open sets to be subfans of $\Sigma$ (see for example [11] Section 7.2)). Our goal is to “glue” $R[[T_*]]_F$ over all maximal cones $\tau \in \Sigma_{\text{max}}$ together as a sheaf on $\Sigma$, and the ring of global sections of this sheaf will then be an algebraic substitute of $h_T(X)^\wedge$.

For any toric variety $X$, there is a natural isomorphism

$$(1) \quad \text{CDiv}_T(X) \cong \ker \left( \bigoplus_{\tau \in \Sigma_{\text{max}}} T^*_\tau \to \bigoplus_{\tau \neq \tau'} T^*_{{\tau} \cap \tau'} \right)$$

where CDiv$_T(X)$ is the group of $T$-invariant Cartier divisors, and the map on the right hand side is given by the difference of the two natural projections $T^*_\tau : T^*_\tau \to T^*_{{\tau} \cap \tau'}$ on each summand. The idea is that for every $T$-invariant Cartier divisor, its restriction to $U_\tau$ is equal to the divisor of a character, and the obvious compatibility condition holds (see [5] Chapter 4.2)).
For every ray $\rho \in \Sigma(1)$, the closure of the corresponding orbit $\overline{O_\rho}$ is a $T$-invariant prime divisor on $X$, and we will denote it by $D_\rho$. The group of $T$-invariant Weil divisors $\text{Div}_T(X)$ is a lattice generated by $D_\rho$, i.e.

$$\text{Div}_T(X) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho.$$  

Since we assume $X$ is smooth, all Weil divisors are Cartier. Hence

$$\text{CDiv}_T(X) = \text{Div}_T(X) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} \cdot D_\rho.$$  

For every character $\alpha \in T^*$, it determines a $T$-invariant principal Cartier divisor $\text{div}(\alpha) = \sum_{\rho \in \Sigma(1)} \langle \alpha, v_\rho \rangle D_\rho$. This defines a group homomorphism $T^* \to \text{CDiv}_T(X)$. Passing to the formal group rings, we have

$$R[T^*]_F \xrightarrow{\phi} R[\text{CDiv}_T(X)]_F \xrightarrow{| \cdot |_T} R[T^*_\sigma]_F$$

Therefore $R[\text{CDiv}_T(X)]_F$ is a $R[T^*]_F$-algebra. Clearly, $R[T^*_\sigma]_F$ is also a $R[T^*]_F$-algebra under the natural map for every cone $\sigma \in \Sigma$.

Consider the group homomorphism $\text{CDiv}_T(X) \to T^*_\sigma$ defined by

$$\sum_{\rho \in \Sigma(1)} n_\rho D_\rho \mapsto \sum_{\rho \in \sigma(1)} n_\rho \alpha_{\sigma, \rho}$$

where $\{\alpha_{\sigma, \rho} \mid \rho \in \sigma(1)\}$ is the basis of $T^*_\sigma$ dual to $\{v_\rho \mid \rho \in \sigma(1)\}$. We remark that the above homomorphism is different from the one in literature by a minus sign. Again passing to formal group rings we obtain a map $R[\text{CDiv}_T(X)]_F \to R[T^*_\sigma]_F$.

**Lemma 3.1.** The map $R[\text{CDiv}_T(X)]_F \to R[T^*_\sigma]_F$ constructed above is a $R[T^*]_F$-algebra homomorphism.

**Proof.** By functoriality of formal group rings it suffices to show that

$$\text{CDiv}_T(X) \xrightarrow{\phi} T^*_\sigma \xrightarrow{\pi} R[T^*_\tau]_F$$

is commutative. For every $\alpha \in T^*$, we have

$$\sum_{\rho \in \Sigma(1)} \langle \alpha, v_\rho \rangle D_\rho \mapsto \sum_{\rho \in \sigma(1)} \langle \alpha, v_\rho \rangle \alpha_{\sigma, \rho}$$

which is clearly commutative. \qed

Now we are ready to do the “gluing”. Motivated by the isomorphism (1), we consider the following sequence

(2) \quad $R[\text{CDiv}_T(X)]_F \xrightarrow{\psi} \prod_{\tau \in \Sigma_{\max}} R[T^*_\tau]_F \xrightarrow{\pi} \prod_{\tau \neq \tau'} R[T^*_\tau \cap \tau']_F$

where $\psi$ is the product of the maps in Lemma 3.1. $\pi$ is given by the product of the differences of $\text{pr}_{\tau, \tau \cap \tau'} : R[T^*_\tau]_F \to R[T^*_\tau \cap \tau']_F$, $\text{pr}_{\tau', \tau \cap \tau'} : R[T^*_\tau]_F \to R[T^*_\tau \cap \tau']_F$. 
Proposition 3.2. The sequence (2) is an exact sequence of $R[T^*_\Sigma]_F$-modules.

Proof. i) First, it follow from the functoriality of formal group rings that $\text{im}(\psi) \subseteq \ker(\pi)$.

ii) To show $\ker(\pi) \subseteq \text{im}(\psi)$, let $\tau_1, \ldots, \tau_d$ be the maximal cones of $\Sigma$, and let $(f_i)_{i \in \Sigma}$ any element in $\ker(\pi) \subseteq \prod_i R[T^*_\tau i,\Sigma]_F$. Then we define $f_{ij}$ to be the image of $f_i$ in $R[T^*_\tau \cap \tau_j]_F$ (which is the same as the image of $f_j$), $f_{ijk}$ to be the image of $f_i$ in $R[T^*_\tau \cap \tau_j \cap \tau_k]_F$, and so on.

By [6 Corollary 2.13], for every cone $\sigma$ we identify $R[T^*_\sigma]_F$ with the ring of power series $R[x_{\alpha,\rho}]$ with variables $x_{\alpha,\rho}, \rho \in \sigma(1)$, where we recall that $\{\alpha,\rho \mid \rho \in \sigma(1)\}$ is the basis of $T^*_\sigma$ dual to $\{v_\rho \mid \rho \in \sigma(1)\}$. Since $X$ is smooth, under this identification for every face $\mu$ of $\sigma$, the natural maps $R[T^*_\mu]_F \to R[T^*_\mu]_F$ coincides with the canonical projection of the rings of power series $R[x_{\alpha,\rho}] \to R[x_{\alpha,\rho}]$, $x_{\alpha,\rho} \mapsto x_{\alpha,\rho}$ if $\rho \in \mu(1)$, $x_{\alpha,\rho} \mapsto 0$ if $\rho \notin \mu(1)$. Similarly we identify $R[\text{CDiv}_T(X)]_F$ with $R[x_{\alpha,\rho}]$ with variables $x_{\alpha,\rho}, \rho \in \Sigma(1)$. Then $\psi$ coincides with product of the projections $R[x_{\alpha,\rho}] \to R[x_{\alpha,\rho}]$.

For every $i$, let $g_i$ be the unique preimage of $f_i$ under the projection $R[x_{\alpha,\rho}] \to R[x_{\alpha,\rho}]$, such that $g_i$ does not involve any $x_{\alpha,\rho}$ for $\rho \notin \tau_i(1)$. Informally speaking, $g_i$ is obtained from $f_i$ by replacing all $x_{\alpha,\rho}$ with corresponding $x_{\alpha,\rho}$. We define $g_1, \ldots, g_{12\ldots d}$ in the same way.

Finally we define

$$g = \sum_{i} g_i - \sum_{i<j} g_{ij} + \sum_{i<j<k} g_{ijk} - \cdots + (-1)^{d+1} g_{12\ldots d} \in R[x_{D_{\rho}}].$$

Then for example under the projection $R[x_{D_{\rho}}] \to R[x_{\alpha,\rho}]$,

$$g = g_1 + \left( \sum_{1<i} g_i - \sum_{1<j} g_{ij} \right) + \left( - \sum_{1<i<j} g_{ij} + \sum_{1<j<k} g_{ijk} \right) + \cdots + (-1)^{d+1} g_{12\ldots r} \mapsto f_1 + 0 + 0 + \cdots + 0 = f_1.$$

So $(f_i) = \psi(g) \in \text{im}(\psi)$.  

Under the above identifications of the formal group rings with the rings of power series, we can have an explicit description of $\ker(\psi)$.

Proposition 3.3. $\ker(\psi)$ is equal to $I_{\Sigma}$, the ideal generated by the square-free monomial $\prod_{\rho \in S} x_{D_{\rho}}$ over all subsets $S \subseteq \Sigma(1)$ such that $S \notin \sigma(1)$ for any cone $\sigma$.

Proof. i) $\ker(\psi) \supseteq I_{\Sigma}$: Let $x_{D_{\rho_1}} \cdots x_{D_{\rho_t}}$ be a generator of $I_{\Sigma}$. For every maximal cone $\tau_i$, by construction $\rho_j \notin \tau_i(1)$ for some $1 \leq j \leq t$. Therefore $x_{D_{\rho_j}} \mapsto 0 \in R[x_{\alpha,\rho}]$, and $x_{D_{\rho_1}} \cdots x_{D_{\rho_t}} \mapsto 0$ as well.

ii) $\ker(\psi) \subseteq I_{\Sigma}$: For every $f \in \ker(\psi)$, write $f$ as a linear combination of monomials in $x_{D_{\rho}}$. Then for every maximal cone $\tau_i$, $f \mapsto 0 \in R[x_{\alpha,\rho}]$ implies each of these monomials also maps to 0, which means each of them contain some $x_{D_{\rho}}$ for $\rho \notin \tau_i(1)$. Hence these monomials and $f$ are in $I_{\Sigma}$.  

Remark 3.4. $I_{\Sigma}$ is power series version of the standard Stanley-Reisner ideal. The underlying geometric meaning follows from the Orbit-Cone Correspondence: It is known that $\rho$ is a face of $\sigma$ if and only if $O_{\rho} \subseteq D_{\rho}$. Therefore given $S = \{\rho_1, \ldots, \rho_t\} \subseteq \sigma(1)$, we have $S \not\subseteq \sigma(1)$ for any cone $\sigma$ if and only if $D_{\rho_1} \cap \cdots \cap D_{\rho_t} = \emptyset$ in $X$.

Corollary 3.5. $\ker(\pi) = \im(\psi) = R[x_{D_{\rho}}]/I_{\Sigma}$.

Remark 3.6. The above results mean that we have the following exact sequence of $R[T^*]_{F}$-modules

$$0 \longrightarrow R[CDiv_{T}(X)]_{F}/I_{\Sigma} \longrightarrow \prod_{\tau \in \Sigma_{\text{max}}} R[T_{\tau}^*]_{F} \longrightarrow \prod_{\tau \neq \tau'} R[T_{\tau \cap \tau'}^*]_{F}.$$  

Notice that this is precisely the exact sequence of the sheaf axiom, where the subfans induced by $\tau$, $\tau'$ varies over $\Sigma_{\text{max}}$, form an open covering of $\Sigma$. Hence the $R[T^*]_{F}$-algebra $R[CDiv_{T}(X)]_{F}/I_{\Sigma}$ is our algebraic model of $h_{T}(X)^{\wedge}$.

We remark that similar exact sequences for equivariant singular cohomology and equivariant $K$-theory can be found in [3] Chapter 12 and [1] respectively.

Next we would like to study the usual cohomology of $X$. Recall that $R$ is a $R[T^*]_{F}$-algebra via the augmentation map. Then we have an isomorphism

$$(R[CDiv_{T}(X)]_{F}/I_{\Sigma}) \otimes_{R[T^*]_{F}} R \cong R[CDiv_{T}(X)]_{F}/J_{\Sigma},$$

where $J_{\Sigma}$ is the ideal generated $I_{\Sigma}$ and $\sum_{\rho \in \Sigma(1)}[(\alpha, \nu_{\rho})]_{F,x_{D_{\rho}}}$ over all $\alpha \in T^*$. This construction corresponds to the idea that the usual cohomology ring is a quotient of the equivariant cohomology ring, where the corresponding results for Chow group, algebraic $K$-theory and algebraic cobordism are proved in [4 Corollary 2.3], [10 Proposition 28] and [13 Theorem 8.1] respectively.

It is known that there is the following exact sequence

$$T^* \longrightarrow \text{CDiv}_{T}(X) \longrightarrow \text{Pic}(X) \longrightarrow 0$$

where the first homomorphism is defined before Lemma 3.1, and the second homomorphism sends a $T$-invariant Cartier divisor to its class in the Picard group.

Lemma 3.7. $R[CDiv_{T}(X)]_{F} \otimes_{R[T^*]_{F}} R$ is isomorphic to $R[\text{Pic}(X)]_{F}$.

Proof. First, if we identify the lattices $T^*$ and $\text{CDiv}_{T}(X)$ with $\mathbb{Z}^{m}$ and $\mathbb{Z}^{m'}$ respectively, the homomorphism $T^* \to \text{CDiv}_{T}(X)$ is given by a $m' \times m$ matrix with coefficients in $\mathbb{Z}$. Since every matrix with coefficients in $\mathbb{Z}$ has a Smith normal form, it means that we can choose a new $\mathbb{Z}$-basis $\{u_1, \ldots, u_m\}$ of $T^*$ and a new $\mathbb{Z}$-basis $\{u'_1, \ldots, u'_{m'}\}$ of $\text{CDiv}_{T}(X)$ such that the homomorphism is given by

$$u_i \mapsto \begin{cases} a_i u'_i & \text{if } 1 \leq i \leq s \\ 0 & \text{if } s + 1 \leq i \leq m \end{cases}$$

for some integer $s$, and $a_1 | \cdots | a_s$ are positive integers (notice that the value 1 is allowed). Then $\text{Pic}(X)$ is isomorphic to $\mathbb{Z}/a_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_s \mathbb{Z} \oplus \mathbb{Z}^{m'-s}$.

By using Theorem 2.11, Corollary 2.13 and Example 2.15 of [6],

$$R[T^*]_{F} \cong R[x_1, \ldots, x_m]$$

$$R[CDiv_{T}(X)]_{F} \cong R[x'_1, \ldots, x'_{m'}]$$

$$R[\text{Pic}(X)]_{F} \cong R[x'_1, \ldots, x'_{m'}/(a_i \cdot x'_i)_{i = 1, \ldots, s}).$$
The formal group ring homomorphism induced by \( T^* \to \text{CDiv}_T(X) \) is given by

\[
x_i \mapsto \begin{cases} 
a_i \cdot F \ x'_i & \text{if } 1 \leq i \leq s \\
0 & \text{if } s + 1 \leq i \leq m,
\end{cases}
\]

and our lemma follows immediately. \( \square \)

**Corollary 3.8.** The following \( R \)-algebras are isomorphic:

1. \( (R[\text{CDiv}_T(X)]_F/I_{\Sigma}) \otimes_{R[T^*]_F} R \).
2. \( R[\text{CDiv}_T(X)]_F/J_{\Sigma} \).
3. \( R[\text{Pic}(X)]_F/I_{\Sigma} \), where \( I_{\Sigma} \) is the image of \( I_{\Sigma} \) under the surjective homomorphism \( R[\text{CDiv}_T(X)]_F \to R[\text{Pic}(X)]_F \).

It follows from our construction that the \( R \)-algebras in Corollary 3.8 are our algebraic model of \( h(X)^\wedge \).

**Remark 3.9.** If \( F \) is a polynomial formal group law, the subring \( R[T^*_F]_F \) is an algebraic substitute of the equivariant cohomology ring \( h_T(U_n) \) by homotopy invariance and the fact that \( X \) is smooth. Then we want to “glue” \( R[T^*_F]_F \) over all maximal cones \( \tau \in \Sigma_{\text{max}} \) together. By the definition of \( R[T^*_F]_F \) and Remark 3.6, the \( R[T^*]_F \)-algebra obtained from “gluing” is \( R[\text{CDiv}_T(X)]_F/I_{\Sigma} \), which will be our algebraic model of \( h_T(X) \). As a result, \( (R[\text{CDiv}_T(X)]_F/I_{\Sigma}) \otimes_{R[T^*]_F} R \cong R[\text{CDiv}_T(X)]_F/J_{\Sigma} \) is an algebraic model of \( h(X) \).

**Example 3.10.** (see [13 Example 8.3]) As a first example we let \( F \) to be any formal group law, and we consider \( X = \mathbb{P}^n \), where \( \Sigma \subseteq T^* \cong \mathbb{Z}^n \) is the complete fan consisting of the \( n + 1 \) rays \( \rho_1, \ldots, \rho_{n+1} \) generated by \( v_1 = e_1, \ldots, v_n = e_n, v_{n+1} = -e_1 - \cdots - e_n \). Here \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{Z}^n \). Then it is easy to see that

\[
R[\text{CDiv}_T(X)]_F/I_{\Sigma} \cong R[x_1, \ldots, x_{n+1}]/(x_1 \cdots x_{n+1}).
\]

Although the right hand side is independent of the formal group law \( F \), the isomorphism depends on \( F \), see [6, Remark 2.14].

Let \( \{\alpha_1, \ldots, \alpha_n\} \) be the basis of \( T^* \) dual to \( \{e_1, \ldots, e_n\} \). The \( R[T^*]_F \)-algebra structure of \( R[x_1, \ldots, x_{n+1}]/(x_1 \cdots x_{n+1}) \) is given by

\[
x_{\alpha_i} \mapsto \sum_{j=1}^{n+1} [(\alpha_i, v_j)]_F x_j = x_i - F x_{n+1}
\]

\[= x_i - x_{n+1} + x_i x_{n+1} f(x_i, x_{n+1})\]

\[= x_i - u_i x_{n+1}\]

where \( f \) is some power series determined by the formal group law \( F \), and \( u_i = 1 - x_i f(x_i, x_{n+1}) \) is a unit in \( R[x_1, \ldots, x_{n+1}]/(x_1 \cdots x_{n+1}) \). Therefore

\[
R[\text{CDiv}_T(X)]_F/J_{\Sigma} \cong \frac{R[x_1, \ldots, x_{n+1}]}{(x_1 \cdots x_{n+1}, u_1 x_{n+1}, \ldots, u_n x_{n+1})} \cong R[x_{n+1}]/(x_{n+1}^{n+1}) \cong R[x_{n+1}]/(x_{n+1}^{n+1}).
\]

We will see more concrete examples in Section 5.
4. Comparison Results

In the present section we compute the formal group rings of a smooth toric variety $X$ for different formal group laws. Then we compare them with known results of equivariant cohomology of smooth toric varieties, and we also obtain new results.

**Example 4.1.** When $F$ is the additive formal group law $F(x, y) = x + y$ over $R$, we recall that $R[M]_F$ is isomorphic to the ring of symmetric powers $\text{Sym}_R(M)$ over $R$. The corresponding oriented cohomology theory is the Chow ring of algebraic cycles modulo rational equivalence, with coefficient ring $CH^*(pt) = \mathbb{Z}$.

Take $R = \mathbb{Z}$. For every maximal cone $\tau$, $\mathbb{Z}[T^*_\tau]_F$ is isomorphic to $\text{Sym}_\mathbb{Z}(T^*_\tau)$, which can be viewed as the ring of integral polynomial functions on $\tau$. Then the above “gluing” process means that the following $\text{Sym}_\mathbb{Z}(T^*)$-algebras are isomorphic:

1. $\mathbb{Z}[\text{CDiv}_T(X)]_F/\mathcal{I}_\Sigma$.

2. $\mathbb{Z}[x_{D_{\sigma}}]/(\prod_{\rho \in S} x_{D_{\rho}})$, where $x_{D_{\rho}}$ are indeterminates over $\rho \in \Sigma(1)$, and the ideal is generated over all subsets $S \subseteq \Sigma(1)$ such that $S \nsubseteq \sigma(1)$ for any cone $\sigma$.

3. The algebra of integral piecewise polynomial functions on $\Sigma$.

The isomorphism between (b) and (c) is given by mapping $x_{D_{\rho}}$ to the unique piecewise polynomial function $\varphi_{\rho}$ satisfying

1. $\varphi_{\rho}$ is homogeneous of degree 1,

2. $\varphi_{\rho}(v_{\rho}) = 1$, $\varphi_{\rho}(v_{\rho'}) = 0$ for all $\rho' \in \Sigma(1), \rho' \neq \rho$.

This coincides with the description of the equivariant Chow ring $CH^*_T(X)$ by [4] and [17].

**Remark 4.2.** It is known that for any smooth variety $X$, the natural homomorphism $\text{Pic}(X) \rightarrow CH_{n-1}(X)$ is an isomorphism, where $CH_{n-1}(X)$ is the (usual) Chow group of $(n - 1)$-cycles modulo rational equivalence. When $X$ is a smooth toric variety, this isomorphism can be recovered as follows:

First note that since $X$ is smooth, we have $CH^1(X) = CH_{n-1}(X)$. When $F$ is the additive formal group law, we can modify the proof of Lemma 3.7 to show that $R[\text{CDiv}_T(X)]_F \otimes_{R[T^*_\tau]} R \cong R[\text{Pic}(X)]_F$. Therefore $\mathbb{Z}[\text{Pic}(X)]_F/\mathcal{I}_\Sigma$ coincides with $CH^*_T(X) \otimes_{\text{Sym}_\mathbb{Z}(T^*)} \mathbb{Z} = CH^*(X)$, the usual Chow ring of $X$. Then our result follows by comparing the degree 1 elements of the two graded rings, where $R[\text{Pic}(X)]_F/\mathcal{I}_\Sigma$ is given the natural grading as a quotient of a polynomial ring.

**Example 4.3.** When $F$ is the multiplicative periodic formal group law $F(x, y) = x + y - \beta xy$ over $R$, where $\beta \in R^\times$, we have seen that $R[M]_F$ is isomorphic to the group ring $R[M] = \{\sum r_i e^{\lambda_i} \mid r_i \in R, \lambda_i \in M\}$. The corresponding oriented cohomology theory is the $K$-theory that assigns every smooth variety $Y$ to $K^0(Y)[[\beta, \beta^{-1}]]$, where $K^0(Y)$ denotes the Grothendieck group of vector bundles on $Y$. The coefficient ring is $K^0(pt)[[\beta, \beta^{-1}]] = \mathbb{Z}[\beta, \beta^{-1}]$.

Take $R = \mathbb{Z}$ and $\beta = 1$. For every maximal cone $\tau$, $\mathbb{Z}[T^*_\tau]_F$ can be viewed as the ring of integral exponential functions on $\tau$. Therefore the following $\mathbb{Z}[T^*]$-algebras are isomorphic:

1. $\mathbb{Z}[\text{CDiv}_T(X)]_F/\mathcal{I}_\Sigma$.

2. $\mathbb{Z}[e^{D_{\rho}}]/(\prod_{\rho \in S}(1-e^{D_{\rho}}))$, where the ideal is generated over all subsets $S \subseteq \Sigma(1)$ such that $S \nsubseteq \sigma(1)$ for any cone $\sigma$.

3. The algebra of integral piecewise exponential functions on $\Sigma$. 


The isomorphism between (b) and (c) is given by mapping $1 - e^{D_x}$ to the piecewise function $1 - e^{\varphi_v}$, where the notation means that on each cone $\sigma \in \Sigma$,

$$(1 - e^{\varphi_v})_\sigma = 1 - e^{(\varphi_v)_\sigma},$$

and $\varphi_v$ is the piecewise polynomial function defined in the previous example. This description agrees with that of the Grothendieck group of equivariant vector bundles $K_T^0(X)$ by [2] and [13] Theorem 6.4.

**Example 4.4.** Let $K$ be the multiplicative formal group law over $R$, given by $F(x, y) = x + y - vxy$, where $v$ is not required to be a unit. If $v = \beta \in R^\times$, then clearly we obtain the multiplicative periodic formal group law of the previous example. If $v \notin R^\times$, then the multiplicative formal group law is non-periodic. In particular, if $v = 0$ we get the additive formal group law.

The oriented cohomology theory corresponding to $F$ is the connective $K$-theory. It is the universal oriented cohomology theory for Chow ring and $K$-theory, by specializing at $v = 0$ and $v = \beta \in R^\times$ respectively. The coefficient ring for the connective $K$-theory is $\mathbb{Z}[v]$.

The following construction is motivated by the result in [12]. Consider the group ring $R[M] = \{ \sum r_i e^{\lambda_i} | r_i \in R, \lambda_i \in M \}$, and let $\text{tr} : R[M] \rightarrow R$ be the trace map, the $R$-linear map defined by mapping any $e^{\lambda}$ to 1. The ideal $\mathcal{I} = \ker(\text{tr})$ is generated by $1 - e^{\lambda}$ over $\lambda \in M$. Then we consider the Rees ring of $R[M]$ with respect to $\mathcal{I}$

$$\mathfrak{R} = \text{Rees}(R[M], \mathcal{I}) = \sum_{n=-\infty}^{\infty} \mathcal{I}^n t^{-n} = R[M][t, t^{-1}] \subseteq R[M][t, t^{-1}]$$

where $t$ is an indeterminate, and $\mathcal{I}^n = R[M]$ if $n \leq 0$. We have the $R$-algebra isomorphisms

$$R[M]_F \cong \mathfrak{R}/(t-v)\mathfrak{R} \text{ and } R[M]_F \cong \left(\mathfrak{R}/(t-v)\mathfrak{R}\right)^\wedge$$

induced by $x_\lambda \mapsto (1 - e^{\lambda})t^{-1}$, and $e^{\lambda} \mapsto 1 - v x_\lambda, (1 - e^{\lambda})t^{-1} \mapsto x_\lambda$. Here the bar means the image of an element in the quotient ring $\mathfrak{R}/(t-v)\mathfrak{R}$, and $(\mathfrak{R}/(t-v)\mathfrak{R})^\wedge$ is the completion of $\mathfrak{R}/(t-v)\mathfrak{R}$ at the ideal generated by $(1 - e^{\lambda})t^{-1}$. Specializing at $v = 0$ and $v = \beta \in R^\times$, we have

$$\mathfrak{R}/t\mathfrak{R} \cong \text{gr}_1 R[M]$$

$$\mathfrak{R}/(t - \beta)\mathfrak{R} \cong R[M]$$

where $\text{gr}_1 R[M]$ is the associated graded ring of $R[M]$ with respect to $\mathcal{I}$. Notice that $\text{gr}_1 R[M]$ is also isomorphic to $\text{Sym}_R(M)$ via $1 - e^\lambda \mapsto \lambda$. Therefore we recover the previous two examples.

As a simple, concrete example for the case $v \notin R^\times$ and $v \neq 0$, consider $R = \mathbb{Z}$, $v = 2$, and $M = \mathbb{Z}$. By direct computation we see that

$$\mathbb{Z}[\mathbb{Z}]_F \cong \text{Rees}(\mathbb{Z}[\mathbb{Z}], \mathcal{I})/(t-2) \text{Rees}(\mathbb{Z}[\mathbb{Z}], \mathcal{I}) \cong \mathbb{Z}[x, x']/\langle x + x' - 2xx' \rangle$$

where the second isomorphism is induced by $(1 - e^1)t^{-1} \mapsto x, (1 - e^{-1})t^{-1} \mapsto x'$.

Back to our study of toric varieties. Our “gluing” process above shows that $\mathbb{Z}[v][\text{CDiv}_{T}(X)]_F/I_2$ is isomorphic to a ring of tuples of elements in the quotient of Rees rings, where the compatability condition for the tuples of elements hold. This provides a conjecture for the equivariant connective $K$-theory ring of $X$. 

Example 4.5. Let char($R$) ≠ 2, and let $F$ be the Lorentz formal group law over $R$, given by

$$F(x, y) = \frac{x + y}{1 + u^2 xy}$$

for some $u \in R$. If $u = 0$, then we just recover the additive formal group law. If $u \neq 0$, then $F$ is an elliptic formal group law, and the corresponding oriented cohomology theory is an elliptic cohomology with coefficient ring $\mathbb{Z}[u^2]$. We remark that $F$ also appears in the theory of special relativity as the formula of relativistic addition of parallel velocities, where $u$ is taken to be $\frac{1}{c}$, the reciprocal of the speed of light.

Even though $F$ is not a polynomial formal group law, we can still study $R[M]_F$. Once again we consider the group ring $R[M]$. Denote by $S$ the multiplicative subset of $R[M]$ generated by $e^{\lambda} + e^{-\lambda}$ over all $\lambda \in M$, and we let $\mathfrak{H}$ to be the $R$-subalgebra of $S^{-1}R[M]$ generated by 1 and the “hyperbolic tangents” $\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}}$ over all $\lambda \in M$. Finally we let $\mathfrak{I}$ be the ideal of $\mathfrak{H}$ generated by all $e^{\lambda} - e^{-\lambda}$, and similar to the previous example we consider the Rees ring of $\mathfrak{H}$ with respect to $\mathfrak{I}$

$$R = \text{Rees}(\mathfrak{H}, \mathfrak{I}) = \sum_{n=-\infty}^{\infty} I^n t^{-n} = \mathfrak{H}[t, t^{-1}] \subseteq H[t, t^{-1}].$$

We have the $R$-algebra isomorphisms

$$R[M]_F \cong R/(t-u)R \text{ and } R[M]_F \cong (R/(t-u)R)^\wedge$$

induced by

$$x_\lambda \mapsto \frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}t^{-1}}$$

and

$$\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}} \mapsto ux_\lambda, \quad \frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}t^{-1}} \mapsto x_\lambda,$$

where $R/(t-u)R \wedge$ is the completion at the ideal generated by $\frac{e^{\lambda} - e^{-\lambda}}{e^{\lambda} + e^{-\lambda}t^{-1}}$. Specializing at $u = 0$ and $u = \beta \in R^*$, we get

$$R \cong \text{gr}_3 \mathfrak{H}$$

and

$$R \cong \text{gr}_3 \mathfrak{H}.$$

Notice that the associated graded ring $\text{gr}_3 \mathfrak{H}$ is naturally isomorphic to the ring of symmetric powers $\text{Sym}_R(M)$ by construction, hence we recover the example for the additive formal group law again.

Just like the previous example, we see that for a smooth toric variety $X$ and the Lorentz formal group law $F$, $\mathbb{Z}[u^2](\text{CDiv}_T(X))_F/\Sigma$ is isomorphic to a ring of tuples of elements in the quotient of Rees rings, where the compatibility condition for the tuples of elements hold.

Example 4.6. When $F$ is the universal formal group law, the corresponding oriented cohomology theory is the algebraic cobordism (defined over a base field of characteristic 0). The coefficient ring is the Lazard ring $L$.

Similar to the first two examples, our “gluing” process shows that the following $L[T^*]]_F$-algebras are isomorphic:

(a) $L[\text{CDiv}_T(X)]_F/\Sigma$. 

(b) \( L[[x_{D_{\rho}}]]/\prod_{\rho \in S} x_{D_{\rho}} \), where \( x_{D_{\rho}} \) are indeterminates over \( \rho \in \Sigma(1) \), and the ideal is generated over all subsets \( S \subseteq \Sigma(1) \) such that \( S \not\subseteq \sigma(1) \) for any cone \( \sigma \).
(c) the algebra of piecewise power series on \( \Sigma \) with coefficients in \( \mathbb{L} \).

Our result agrees with the description of the equivariant cobordism ring \( \Omega^*_T(X) \) in [11] and [13].

5. Explicit Examples

In this section we look at how the comparison results established in the previous section work in some concrete examples.

Example 5.1. In this example we consider the Hirzebruch surface \( \mathcal{H}_r \), \( r \) a non-negative integer. Its fan \( \Sigma \subseteq T_* \cong \mathbb{Z}^2 \) consists of four 2-dimensional maximal cones \( \sigma_i \), together with all their faces, and the rays \( \rho_i \) are generated by \( v_1 = (0, 1), v_2 = (1, 0), v_3 = (0, -1) \) and \( v_4 = (-1, r) \):

\[
\begin{array}{c}
\sigma_4 \\
\sigma_3 \\
\sigma_2 \\
\sigma_1
\end{array}
\]

Let \( \{x, y\} \) be the basis of \( T^* \) dual to the standard basis \( \{(1, 0), (0, 1)\} \) of \( T_* \cong \mathbb{Z}^2 \). We view \( x, y \) as the coordination functions on \( T_* \). We denote \( (f_1, f_2, f_3, f_4) \) the piecewise function on \( \Sigma \) determined by \( f_i \) on \( \sigma_i \). Then the piecewise polynomial functions defined in Example 4.1 are \( \varphi_{\rho_1} = (y, 0, 0, rx + y), \varphi_{\rho_2} = (x, x, 0, 0), \varphi_{\rho_3} = (0, -y, -rx - y, 0), \varphi_{\rho_4} = (0, 0, -x, -x) \).

(i) When \( F \) is the additive formal group law over \( \mathbb{Z} \), we have \( \mathbb{Z}[T^*]_F \cong \mathbb{Z}[x, y] \), and the following \( \mathbb{Z}[x, y] \)-algebras are isomorphic:
(a) \( CH^*_T(\mathcal{H}_r) \).
(b) \( \mathbb{Z}[CDiv_T(\mathcal{H}_r)]_F / I_\Sigma \).
(c) \( \mathbb{Z}[x_1, x_2, x_3, x_4]/(x_1 x_3, x_2 x_4) \).
(d) the algebra of integral piecewise polynomial functions on \( \Sigma \).

We remark that all Hirzebruch surfaces \( \mathcal{H}_r \) have isomorphic equivariant Chow rings as abstract rings, while the \( \mathbb{Z}[x, y] \)-algebra structure on \( CH^*_T(\mathcal{H}_r) \) depends on \( r \). The action of \( \mathbb{Z}[x, y] \) on the algebra of integral piecewise polynomial functions is simply given by the usual multiplication of polynomials on each cone

\[
g \cdot (f_1, f_2, f_3, f_4) = (gf_1, gf_2, gf_3, gf_4).
\]

Notice that \( (x, x, x, x) = \varphi_{\rho_2} - \varphi_{\rho_4} \), and \( (y, y, y, y) = \varphi_{\rho_1} - \varphi_{\rho_3} + r \varphi_{\rho_4} \). Hence the action of \( \mathbb{Z}[x, y] \) on the algebra of integral piecewise polynomial functions is given by

\[
x \mapsto x_2 - x_4, \quad y \mapsto x_1 - x_3 + rx_4.
\]

Therefore the usual Chow ring of \( \mathcal{H}_r \),

\[
CH^*(\mathcal{H}_r) = CH^*_T(\mathcal{H}_r) \otimes_{\mathbb{Z}[x, y]} \mathbb{Z}
\]

is isomorphic to

\[
\cong \mathbb{Z}[x_1, x_2, x_3, x_4]/(x_1 x_3, x_2 x_4, x_2 - x_4, x_1 - x_3 + rx_4)
\]

\[
\cong \mathbb{Z}[s, t]/(s(s + rt), t^2),
\]
also depends on \( r \).

(ii) When \( F \) is the multiplicative periodic formal group law over \( \mathbb{Z} \) with \( \beta = 1 \), \( \mathbb{Z}[T^+]_F \) is isomorphic to the group ring \( \mathbb{Z}[e^\pm x, e^\pm y] \), and the following \( \mathbb{Z}[e^\pm x, e^\pm y] \)-algebras are isomorphic:

(a) \( K^0_T(\mathcal{H}_r) \).

(b) \( \mathbb{Z}[\text{CDiv}_T(\mathcal{H}_r)]_F / I_\Sigma \).

(c) \( \mathbb{Z}[e^\pm x_1, e^\pm x_2, e^\pm x_3, e^\pm x_4]/\langle 1 - e^{x_1}(1 - e^{x_2}), 1 - e^{x_2-x_4}, 1 - e^{x_1-x_3+r x_4} \rangle \).

(d) the algebra of integral piecewise exponential functions on \( \Sigma \).

Again, as abstract rings all \( K^0_T(\mathcal{H}_r) \) are isomorphic, but the \( \mathbb{Z}[e^\pm x, e^\pm y] \)-algebra structure depends on \( r \). Therefore the usual Grothendieck group of vector bundles \( K^0(\mathcal{H}_r) \),

\[
K^0(\mathcal{H}_r) = K^0_T(\mathcal{H}_r) \otimes_{\mathbb{Z}[e^\pm x, e^\pm y]} \mathbb{Z}
\]

\[
\cong \mathbb{Z}[e^\pm x_1, e^\pm x_2, e^\pm x_3, e^\pm x_4]/\langle 1 - e^{x_1}(1 - e^{x_2}), 1 - e^{x_2-x_4}, 1 - e^{x_1-x_3+r x_4} \rangle
\]

\[
\cong \mathbb{Z}[e^\pm x, e^\pm y]/\langle 1 - e^{x}(1 - e^{s+rf}), 1 - e^{t} \rangle,
\]

depends on \( r \) as well.

(iii) When \( F \) is the non-periodic multiplicative formal group law over \( \mathbb{Z}[v] \) with \( v \) not invertible, \( \mathbb{Z}[v][T^+]_F \) is isomorphic to \( A = \mathbb{Z}[v][x, x', y, y']/\langle x + x' - vxx', y + y' - vyy' \rangle \). The following \( A \)-algebras are isomorphic:

(a) \( \mathbb{Z}[v][\text{CDiv}_T(\mathcal{H}_r)]_F / I_\Sigma \).

(b) \( \mathbb{Z}[v][x_1, x_2, x_3, x_4, x_5]/\langle x_1 + x_5 - vxx', x_1 x_3, x_2 x_4 \rangle \).

(c) the algebra of 4-tuples of elements in \( A \) satisfying the compatability condition given by \( \Sigma \).

This gives us an algebraic substitute for the equivariant connective \( K \)-theory ring of \( \mathcal{H}_r \).

**Example 5.2.** Consider the del Pezzo surface of degree 6 \( dP_6 \), obtained by blowing-up the three \( T \)-fixed points \( p_1, p_2, p_3 \) of \( \mathbb{P}^2 \). We begin by recalling some classical results of \( dP_6 \) (for example, see [13]). The fan \( \Sigma \) of \( dP_6 \), \( \Sigma \subseteq T \cong \mathbb{Z}^2 \), consists of six 2-dimensional maximal cones and all their faces, and the rays are generated by \((0,1), (1,1), (1,0), (0,-1), (-1,-1), (-1,0)\):

\[
\begin{array}{c}
L_1 \\
\downarrow \\
E_1 \\
\downarrow \\
L_3 \\
\downarrow \\
E_2 \\
\downarrow \\
E_3
\end{array}
\]

The \( T \)-invariant divisors corresponding to the six rays are precisely the six exceptional curves \( E_1, E_2, E_3, L_1, L_2, L_3 \) on \( dP_6 \): \( E_i \) is the exceptional curve induced by blowing-up \( p_i \), and \( L_i \) is the strict transform of the unique line in \( \mathbb{P}^2 \) passing through \( p_j \) and \( p_k \), \( i, j, k \) all distinct.
\{E_1, E_2, E_3, L_1, L_2, L_3\} is a basis of the lattice CDiv_T(dP_6), and T* injects into CDiv_T(dP_6) by
\[
x \mapsto E_3 + L_2 - L_3 - E_2
\]
\[
y \mapsto L_1 + E_3 - E_1 - L_3
\]
where \{x, y\} is the basis of T* dual to the standard basis \{(1, 0), (0, 1)\} of T* \cong \mathbb{Z}^2.
It follows that Pic(dP_6) is a rank 4 lattice with basis \{\ell, E_1, E_2, E_3\}, where
\[
\ell = L_1 + E_2 + E_3 = L_2 + E_3 + E_1 = L_3 + E_1 + E_2.
\]
The intersection pairing \langle \, , \, \rangle : Pic(dP_6) \times Pic(dP_6) \rightarrow \mathbb{Z} is determined by
\[
\langle \ell, \ell \rangle = 1, \quad \langle \ell, E_i \rangle = 0, \quad \langle E_i, E_j \rangle = -\delta_{ij}
\]
where \delta_{ij} is the usual Kronecker delta function.

Now we go back to our study of equivariant cohomology on \(dP_6\). For all the expressions below, we always assume the subindices \(i, j \in \{1, 2, 3\}, i \neq j\).

(i) When \(F\) is the additive formal group law over \(\mathbb{Z}\), the following \(\mathbb{Z}[x, y]\)-algebras are isomorphic:
(a) \(CH_T(dP_6)\).
(b) \(\mathbb{Z}[CDiv_T(dP_6)]_F/ I_{\Sigma}\).
(c) \(\mathbb{Z}[L_1, L_2, L_3, E_1, E_2, E_3]/(L_1 E_1, E_2 E_2, E_3 E_3)\).
(d) the algebra of integral piecewise polynomial functions on \(\Sigma\).

As a corollaary, the following rings are isomorphic:
(a) \(CH^*(dP_6)\).
(b) \(\mathbb{Z}[CDiv_T(dP_6)]_F/ I_{\Sigma}\) \(\otimes\mathbb{Z}[x, y] \mathbb{Z}\).
(c) \(\mathbb{Z}[Pic(dP_6)]_F/ I_{2\Sigma}\).
(d) \(\mathbb{Z}[\ell, E_1, E_2, E_3]/(\ell^2 + E_i E_j, E_i E_j, \ell E_i)\).
Notice that the relations \(\ell^2 + E_i E_j = \ell E_i = 0\) in the last ring agree with the values of the intersection pairing on Pic(dP_6).

(ii) When \(F\) is the multiplicative periodic formal group law over \(\mathbb{Z}\) with \(\beta = 1\), the following \(\mathbb{Z}[e^{\pm x}, e^{\pm y}]\)-algebras are isomorphic:
(a) \(K^0(dP_6)\).
(b) \(\mathbb{Z}[CDiv_T(dP_6)]_F/ I_{\Sigma}\) \(\otimes \mathbb{Z}[e^{\pm x}, e^{\pm y}] \mathbb{Z}\).
(c) \(\langle (1 - e^{L_1})(1 - e^{L_2}), (1 - e^{E_1})(1 - e^{E_2}), (1 - e^{L_3})(1 - e^{E_3}) \rangle\).
(d) the algebra of integral piecewise exponential functions on \(\Sigma\).

Similar to part (i), we see that the following rings are isomorphic:
(a) \(K^0(dP_6)\).
(b) \(\mathbb{Z}[CDiv_T(dP_6)]_F/ I_{\Sigma}\) \(\otimes \mathbb{Z}[e^{\pm \ell}, e^{\pm E_1}, e^{\pm E_2}, e^{\pm E_3}] \mathbb{Z}\).
(c) \(\langle (1 - e^{\ell})^2, (1 - e^{L_1})^2, (1 - e^{E_1})(1 - E_2), (1 - e^{L_3})(1 - e^{E_3}) \rangle\).

(iii) In general, for arbitrary formal group law \(F\) over a ring \(R\), we have
\[
R[CDiv_T(dP_6)]_F/ I_{\Sigma} \cong \frac{R[x_{L_1}, x_{L_2}, x_{L_3}, x_{E_1}, x_{E_2}, x_{E_3}]}{(x_{L_1}, x_{L_2}, x_{L_3}, x_{E_1}, x_{E_2}, x_{E_3})}.
\]
There are a couple of useful arithmetic identities in this algebra. Recall that any formal group law $F$ can be expressed as
\[
F(x, y) = x + y - xy \cdot g(x, y)
\]
for some power series $g(x, y)$. Then for example we see that
\[
x_{E_i + E_j} = x_{E_i} + F x_{E_j} = x_{E_i} + x_{E_j} - x_{E_i} x_{E_j} g(x_{E_i}, x_{E_j}) = x_{E_i} + x_{E_j},
\]
and similarly
\[
x_{E_i - E_j} = x_{E_i} - F x_{E_j} = x_{E_i} - x_{E_i} x_{E_j} \chi(x_{E_j})
= x_{E_i} + \chi(x_{E_i}) - x_{E_i} \chi(x_{E_j}) g(x_{E_i}, \chi(x_{E_j}))
= x_{E_i} + x_{E_j},
\]
where $\chi(z)$ is the unique power series such that $z + F \chi(z) = 0$. It follows that we have
\[
(x_{E_i + E_j} + E_k)^n = (x_{E_i} + E_j + E_k)^n = x_{E_i}^n + x_{E_j}^n + x_{E_k}^n
\]
for any positive integer $n$. Clearly we have the corresponding identities for the $X_{L_i}$'s as well.

Tensoring the isomorphism (3) with $R$ over $R[T^*]_F$, we obtain
\[
(4) \quad R[\text{Pic}(dP_6)]_{F/\mathbb{Z}} \cong R[x, x_{E_1}, x_{E_2}, x_{E_3}]/(x_1^2 + \chi(x_{E_1})^2, x_{E_1} x_{E_2}, x_{E_1} x_{E_3}).
\]
By direct computation we see that
\[
x_1^3 = x_{E_1}^3 = x_{E_2}^3 = x_{E_3}^3 = 0
\]
in this ring, which is an expected result as $dP_6$ is a surface. This allows us to simplify the expression in (4). Let $a_{i,j} \in R$ be the coefficients of the formal group law $F$
\[
F(x, y) = \sum_{i,j} a_{i,j} x^i y^j.
\]
Then by (4) Equation (2.3)) the power series $\chi(z)$ is of the form
\[
\chi(z) = -z + a_{1,1} z^2 - (a_{1,1})^2 z^3 + \text{terms of degree } \geq 4.
\]
Hence we have
\[
R[\text{Pic}(dP_6)]_{F/\mathbb{Z}} \cong R[x_1, x_{E_1}, x_{E_2}, x_{E_3}]/(x_1^2 + x_{E_1} x_{E_2}, x_{E_1} x_{E_3}).
\]
We remark again that the isomorphism depends on the formal group law $F$, even though the right hand side is independent of $F$.

6. Pull-back and push-forward formula of blow-up

Let $X$ be the smooth toric variety of the fan $\Sigma$, $\sigma \in \Sigma \setminus \Sigma(1)$ be a cone, and $X'$ be the blow-up of $X$ along the orbit closure $O_\sigma$. Then $X'$ is smooth toric variety whose fan $\Sigma'$ is equal to the star subdivision of $\Sigma$ relative to $\sigma$,
\[
\Sigma' = \{ \theta \in \Sigma \mid \sigma \nsubseteq \theta \} \cup \bigcup_{\sigma \subseteq \theta} \Sigma^*(\theta)
\]
where we let $v_\sigma = \sum_{\rho \in \Sigma(1)} v_\rho$, and
\[
\Sigma^*(\theta) = \{ \text{cone}(S) \mid S \subseteq \{v_\sigma\} \cup \theta(1), \sigma(1) \nsubseteq S \}.
\]
The fan $\Sigma'$ is a refinement of $\Sigma$, and the induced toric morphism $\pi : X' \to X$ is projective. Therefore for every equivariant cohomology theory $h_T$, $\pi$ induces the pull-back homomorphism $\pi^* : h_T(X) \to h_T(X')$ and also the push-forward
homomorphism \( \pi_\ast : h_T(X') \to h_T(X) \). Notice that \( \pi^\ast \) is a \( h_T(\text{pt}) \)-algebra homomorphism, and \( \pi_\ast \) is a \( h_T(X) \)-module homomorphism, where \( h_T(X) \) acts on \( h_T(X') \) via \( \pi^\ast \). Our goal in this section is to define two homomorphisms of formal group rings that will serve as algebraic substitutes of the pull-back and the push-forward homomorphisms. We remark that formulas for the pull-back and the push-forward of equivariant Chow rings are proved in [3, Theorem 2.3].

First, let \( E \) be the \( T \)-invariant prime divisor on \( X' \) corresponding to the ray \( \tilde{\rho} \) generated by \( v_\sigma \). Then we have

\[
\text{CDiv}_T(X') \cong \text{CDiv}_T(X) \oplus \mathbb{Z} \cdot E,
\]

where the \( T \)-invariant prime divisor \( D_{\rho, \Sigma'} \) on \( X' \) corresponding to \( \rho \in \Sigma' \) is the strict transform of the divisor \( D_{\rho, \Sigma} \) on \( X \) corresponding to \( \rho \in \Sigma \). We denote both \( D_{\rho, \Sigma'} \) and \( D_{\rho, \Sigma} \) by \( D_\rho \) if the ambient fan is clear from the context.

Next, we want to define the pull-back homomorphism for formal group rings. Informally speaking from the point of view of piecewise functions on fans, \( \pi^\ast : h_T(X) \to h_T(X') \) is given by treating piecewise functions on \( \Sigma \) as piecewise functions on the refinement \( \Sigma' \). Translating this to the language of formal group rings, we define

\[
\begin{align*}
\pi^\ast : R[\text{CDiv}_T(X)]_{F/I\Sigma} & \to R[\text{CDiv}_T(X')]_{F/I\Sigma'} \\
x_{D_\rho} & \mapsto \begin{cases} 
  x_{D_\rho} & \text{if } \rho \notin \sigma(1) \\
  x_{D_\rho + E} = x_{D_\rho} + F x_E & \text{if } \rho \in \sigma(1).
\end{cases}
\end{align*}
\]

The underlying geometric meaning can again be explained by the Orbit-Cone Correspondence: \( \rho \in \sigma(1) \) if and only if \( \overline{O_\rho} \subseteq D_\rho \).

For the push-forward homomorphism, we impose the condition that \( F \) is the associated formal group law of a birationally invariant theory \( h \), i.e., the push-forward of the fundamental class satisfies \( f_\ast(1_Y) = 1_X \) for any birational projective morphism \( f : Y \to X \) between smooth irreducible varieties. Examples of birationally invariant theories include Chow ring over an arbitrary field, \( K \)-theory over a field of characteristic 0. It is known that the connective \( K \)-theory over a field of characteristic 0 is univeral among all birationally invariant theories, see [13, Theorem 4.3.9] and [6, Example 8.10]. Therefore from now on we assume \( F \) is of the form \( F(x, y) = x + y - vxy \) for some \( v \in R \).

Now we begin the construction of the push-forward homomorphism \( \pi_\ast : R[\text{CDiv}_T(X')]_{F/I\Sigma'} \to R[\text{CDiv}_T(X)]_{F/I\Sigma} \), which is a homomorphism of \( R[\text{CDiv}_T(X)]_{F/I\Sigma} \)-modules. As the blow-up morphism \( \pi \) is birational and projective, by our assumption on \( F \) we define \( \pi_\ast(1) = 1 \). Since \( D_{\rho, \Sigma'} \) is the strict transform of \( D_{\rho, \Sigma} \), we define \( \pi_\ast(x_{D_{\rho, \Sigma'}}) = x_{D_{\rho, \Sigma}} \).

From here we can deduce that \( \pi_\ast(-F x_E) = 0 \), and the proof is as follows: Let \( \rho \in \sigma(1) \) be a ray in \( \sigma \subseteq \Sigma \), and consider \( x_{D_{\rho, \Sigma'}} + F x_E = \pi^\ast(x_{D_{\rho, \Sigma}}) \). By the projection formula,

\[
\pi_\ast(x_{D_{\rho, \Sigma'}} + F x_E) = \pi_\ast(\pi^\ast(x_{D_{\rho, \Sigma}})) = \pi_\ast(1)x_{D_{\rho, \Sigma}} = x_{D_{\rho, \Sigma}}.
\]

On the other hand,

\[
\pi_\ast\left( (x_{D_{\rho, \Sigma'}} + F x_E) - F x_E \right) = \pi_\ast(x_{D_{\rho, \Sigma'}}) = x_{D_{\rho, \Sigma}}.
\]
as well. By subtracting the two equations, we get
\[
\pi_* \left( (-F x_E) - v(x_{D_{\rho, \Sigma}} + F x_E)(-F x_E) \right) = \pi_* \left( (1 - v(x_{D_{\rho, \Sigma}} + F x_E))(-F x_E) \right)
\]
\[
= \pi_* \left( (1 - v x_{D_{\rho, \Sigma}})(-F x_E) \right)
\]
\[
= (1 - v x_{D_{\rho, \Sigma}}) \pi_* (-F x_E)
\]
\[
= 0.
\]
As \(1 - v x_{D_{\rho, \Sigma}}\) is a unit in \(R[\text{CDiv}_T(X)]_F / I_\Sigma\), we have \(\pi_* (-F x_E) = 0\). Notice that in general \(\pi_* (x_E) \neq 0\).

If \(\dim \sigma = 2\), then by the property of a \(R[\text{CDiv}_T(X)]_F / I_\Sigma\)-module homomorphism we can already determine the image of every element in \(R[\text{CDiv}_T(X')]_F / I_\Sigma\). Let \(\sigma(1) = \{\rho_1, \rho_2\} \subseteq \Sigma(1)\). After the star subdivision \(\{\rho_1, \rho_2\} \not\subseteq \theta(1)\) for any cone \(\theta \in \Sigma'\), hence \(x_{D_{\rho_1, \Sigma'}} x_{D_{\rho_2, \Sigma'}} = 0\) in \(R[\text{CDiv}_T(X')]_F / I_\Sigma\). Then for example,
\[
\pi^* (x_{D_{\rho_1, \Sigma}} x_{D_{\rho_2, \Sigma}}) = (x_{D_{\rho_1, \Sigma}} + x_E - v x_{D_{\rho_1, \Sigma'}} x_E) x_{D_{\rho_2, \Sigma'}}
\]
\[
\mapsto \pi_* (\pi^* (x_{D_{\rho_1, \Sigma}} x_{D_{\rho_2, \Sigma'}})) = x_{D_{\rho_1, \Sigma}} x_{D_{\rho_2, \Sigma'}}.
\]
It follows from equations (5) and (6) that
\[
\pi_* (x_{D_{\rho_1, \Sigma}} + F x_E) = x_{D_{\rho_2, \Sigma}} + \pi_* (x_E) - v x_{D_{\rho_1, \Sigma}} x_{D_{\rho_2, \Sigma}}
\]
\[
= x_{D_{\rho_2, \Sigma}}.
\]

**Proposition 6.1.** Let \(\dim \sigma = 2\) and \(\sigma(1) = \{\rho_1, \rho_2\} \subseteq \Sigma(1)\). We use \(x_i\) to denote both \(x_{D_{\rho_1, \Sigma'}}\) and \(x_{D_{\rho_i, \Sigma}}, i = 1, 2\). Then the push-forward homomorphism defined above satisfies
\[
\pi_* : R[\text{CDiv}_T(X')]_F / I_\Sigma' \longrightarrow R[\text{CDiv}_T(X)]_F / I_\Sigma
\]
\[
1 \mapsto 1
\]
\[
x_E \mapsto v x_1 x_2
\]
\[
x_n^* \mapsto v \sum_{i=1}^n x_1^{n+1-i} x_2 - \sum_{i=1}^{n-1} x_1^{n-i} x_2^i
\]
\[
x_a^* x_E^* \mapsto x_a x_b^* (x_a - F x_b)^{s-1},
\]
where \(n \geq 2, s \geq 1, t \geq 0, \) and \(\{a, b\} = \{1, 2\}\).

**Proof.** First, by induction on \(t\) we see that
\[
\pi_* (x_a x_E^t) = \pi_* ((x_b + x_E - v x_b x_E)x_a x_{E}^{t-1})
\]
\[
= \pi_* (\pi^* (x_b) x_a x_{E}^{t-1})
\]
\[
= x_a x_b^t,
\]
where the first equality follows from the fact that \(x_a x_b = 0\) in \(R[\text{CDiv}_T(X')]_F / I_\Sigma'\).

To compute the image of \(x_n^*\), we use induction on \(n\), and compare the image of \(\pi^* (x_a) x_{E}^{n-1}\) computed by two different methods:
\[
\pi_* (\pi^* (x_a) x_{E}^{n-1}) = x_a \pi_* (x_E^{n-1})
\]
and
\[
\pi_* (\pi^* (x_a) x_{E}^{n-1}) = \pi_* ((x_a + x_E - v x_a x_E)x_{E}^{n-1}) = x_a x_b^{n-1} + \pi_*(x_E^n) - v x_a x_b^n.
\]
Next we want to compute the image of $x_a^2$. Let $\chi(z)$ be the unique power series such that $z + F \chi(z) = 0$. Then
\[
\pi_*(x_a^2) = \pi_*((\pi^*(x_a) - F x_E)x_a)
\]
\[
= \pi_*((\pi^*(x_a) + \chi(x_E) - v\pi^*(x_a)\chi(x_E))x_a)
\]
\[
= x_a^2 + \chi(x_b)x_a - vx_a\chi(x_b)x_a
\]
\[
= x_a(x_a - F x_b),
\]
where the third equality follows from equation \ref{eq:triangle}. By using the same trick for $\pi_*(x_a x_b^t)$, we see that
\[
\pi_*(x_a^2 x_b^t) = x_a x_b^t(x_a - F x_b)
\]
for every $t \geq 0$. This allows us to compute $\pi_*(x_a^3)$, and then $\pi_*(x_a x_b^3)$, by the same argument as above. The general case now follows from induction. \hfill \Box

**Remark 6.2.** In our case where the formal group law $F$ is of the form $F(x, y) = x + y - vx y$, we have an explicit description of the power series $\chi(z)$:
\[
\chi(z) = -z - \sum_{i=0}^{\infty} (vz)^i = -\sum_{i=0}^{\infty} v^i z^{i+1}.
\]
Hence $- F x_E = -\sum_{i=0}^{\infty} v^i x_{E}^{i+1}$. Then by using the formula in Proposition 6.1, we recover the result $\pi_*(- F x_E) = 0$ when $\dim \sigma = 2$.

If $\dim \sigma = 3$, let $\sigma(1) = \{\rho_1, \rho_2, \rho_3\} \subseteq \Sigma(1)$ and use $x_i$ to denote both $x_{D_{\rho_i, \Sigma}}$ and $x_{D_{\rho_i, \Sigma}}$, $i = 1, 2, 3$. We further define
\[
\pi_*(x_i x_j) = x_i x_j
\]
for every $i, j \in \{1, 2, 3\}$ such that $i \neq j$. Then the image for the rest of the elements in $R[\text{CDiv}_T(\Sigma')]/F / I_{\Sigma'}$ can be computed by the same technique as above. For example, $\pi_*(x_i x_j x_E) = x_1 x_2 x_3$, $\pi_*(x_i x_j) = vx_1 x_2 x_3$, and $\pi_*(x_E) = v^2 x_1 x_2 x_3$. Finally, we remark that this process of defining the push-forward homomorphism $\pi_*$ extends naturally for arbitrary dimension of $\sigma$.

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