THE DOLD-WHITNEY THEOREM AND THE SATO-LEVINE INVARIANT

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Abstract. We use the Dold-Whitney theorem classifying $\text{SO}(3)$-bundles over a 4-complex to give a mod 4 obstruction to a 2-component link of trivial linking number being slice. It turns out that this coincides with the reduction of the Sato-Levine invariant.

1. Introduction

Let $L$ be a 2-component link in $S^3$ with trivial linking number. Choose a Seifert surface for each component of $L$ that misses the other component and such that the surfaces intersect transversely. The intersection of the two Seifert surfaces gives a framed link in $S^3$. Such a framed link determines a homotopy class of maps $S^3 \to S^2$ by the Pontryagin-Thom construction.

Definition 1.1. The Sato-Levine invariant of $L$ is the corresponding group element of $\pi_3(S^2) = \mathbb{Z}$.

This definition first appears in [5]. The non-vanishing of the Sato-Levine invariant of $L$ provides an obstruction to the link $L$ bounding disjoint locally flat discs in the 4-ball (in other words, an obstruction to $L$ being slice).

In this paper we give a combinatorially-defined obstruction $\phi(L) \in \mathbb{Z}/4\mathbb{Z}$ to $L$ being slice. It turns out to be equal to the modulo 4 reduction of the Sato-Levine invariant.

Nevertheless, the proofs of the well-definedness and properties of $\phi$ are straightforward and direct. The intermediate construction used in the proofs is a flat SO(3) connection on a 4-manifold. The result follows from an application of the Dold-Whitney theorem (which classifies all $\text{SO}(3)$ bundles over a 4-complex by their characteristic classes).

Theorem 1.2 (Dold-Whitney [2]). Let $X$ be a 4-dimensional CW-complex. A principal $\text{SO}(3)$ bundle $E$ over $X$ is determined by the pair consisting of its Pontryagin class $p_1(E) \in H^4(X; \mathbb{Z})$ and second Steifel-Whitney class $w_2(E) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$. Furthermore there is an $\text{SO}(3)$ bundle $E$ realizing $p_1(E) = a$ and $w_2(E) = b$ exactly when

$$\pi = b^2 \in H^4(X; \mathbb{Z}/4\mathbb{Z})$$

where we write $\pi$ for the reduction of $a$ and where the squaring of $b$ is the Pontryagin squaring operation.

In essence, we are giving an essentially 4-dimensional proof of the invariance and properties of a reduction of the Sato-Levine invariant.

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2. Definition and properties

Let $L$ be an oriented link in $S^3$ of trivial linking number comprising two components $K_1$ and $K_2$. Then there certainly exist two disjoint locally flat immersed discs in the 4-ball $B^4$, bounded by $L$, where the discs are boundary-transverse and oriented consistently with $L$. Let $D_1$ and $D_2$ be two such discs.

**Definition 2.1.** To each self-intersection point $p \in B^4$ of $D_1$ or $D_2$ we associate a number $i(p) \in \{-1, 0, 1\}$ as follows.

Let $\{s, t\} = \{1, 2\}$, and suppose that $p$ is a self-intersection point of $D_s$. Choose a loop $l$ which starts and ends at $p \in B^4$, staying on $D_s$ and starting and ending on different branches of the intersection. Then we set

$$w(p) := [l] \in H_1(B^4 \setminus D_t; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}.$$  

Note that this is independent of the choice of $l$.

We define

$$i(p) = w(p)\sigma(p)$$

where $\sigma(p) = \pm 1$ is the sign of the intersection at $p$.

**Definition 2.2.** We define

$$\phi(L, D_1, D_2) = \sum p i(p) \in \mathbb{Z}/4\mathbb{Z}$$

where the sum is taken over all the self-intersections $p$ of $D_1$ and $D_2$.

**Remark 2.3.** The fact that $\phi$ is the reduction of the Sato-Levine invariant may be deduced from this definition and the crossing-change formula due to Jin [3] and Saito [4].

We shall show the following

**Proposition 2.4.** Suppose that $L$ bounds the two pairs of disjoint locally flat immersed discs $(D_1, D_2)$ and $(D'_1, D'_2)$. Then there exists a closed 4-manifold $X$ with a flat $SO(3)$-bundle $E \to X$ with

$$\phi(L, D_1, D_2) - \phi(L, D'_1, D'_2) = w_2^2(E) = p_1(E) = 0 \in \mathbb{Z}/4\mathbb{Z} = H^4(X; \mathbb{Z}/4\mathbb{Z}).$$

From this proposition we immediately obtain a corollary.

**Corollary 2.5.** The quantity $\phi(L, D_1, D_2)$ depends only on the link $L$. So we can write $\phi(L) = \phi(L, D_1, D_2)$. Furthermore, if $\phi(L) \neq 0$ then $L$ does not bound two disjoint embedded locally flat discs in $B^4$. $\square$

We note that the content of the equation in Proposition 2.4 is the first equality sign, the second being the Dold-Whitney theorem (the squaring operation here is the Pontryagin square, a $\mathbb{Z}/4\mathbb{Z}$ lift of the cup product), and the third being a consequence of the flatness of the bundle $E$.

**Remark 2.6.** Work by Saito [4] gives a $\mathbb{Z}/4\mathbb{Z}$-valued extension of the Sato-Levine invariant for links of even linking number. Saito’s invariant is constructed via considering the framed intersection of possibly non-orientable Seifert surfaces, and is distinct from that which we consider.

We devote the following section to the description of the manifold $X$ and the $SO(3)$-bundle $E \to X$. 


3. Construction of a 4-manifold with an $SO(3)$-bundle

Given an immersed locally-flat 2-link $\Lambda \subseteq S^4$ of two components with no intersections between distinct components of the link, we give a construction of a closed diagonal 4-manifold $X_\Lambda$.

Suppose that $\Lambda$ has $n_-$ negative and $n_+$ positive intersection points. Then we blow-up each negative intersection point by taking connect sum with $\mathbb{P}^2$ and each positive intersection point by taking connect sum with $\mathbb{P}^2$. Let $\Lambda \hookrightarrow n_- \mathbb{P}^2 \# n_+ \mathbb{P}^2$ be the proper transform of $\Lambda$.

Because of the way we chose to blow-up the negative and positive intersections respectively, each exceptional sphere intersects $\Lambda$ in two points, once negatively, and once positively. Furthermore, since the self-intersections of $\Lambda$ do not occur between the distinct components of $\Lambda$, each exceptional sphere intersects exactly one component of $\Lambda$.

This means that each component of $\Lambda$ is trivial homologically, and so has a trivial $D^2$-neighborhood. This allows us to do surgery by removing a neighborhood $\Lambda \times D^2$ and gluing in two copies of $D^3 \times S^1$. We call the resulting manifold $X_\Lambda$. Now we collect some information about the algebraic topology of $X_\Lambda$.

**Proposition 3.1.** The 4-manifold $X_\Lambda$ has diagonal intersection form and satisfies

$$H_1(X_\Lambda; \mathbb{Z}) = \mathbb{Z}^2, \quad H_2(X_\Lambda; \mathbb{Z}) = \mathbb{Z}^{n_++n_-},$$

$$b_1^+ = n_+, \quad b_1^- = n_-.$$

**Proof.** We shall display $n_- + n_+$ disjoint embedded tori in $X_\Lambda$, $n_-$ of which have self-intersection $-1$ and $n_+$ of which have self-intersection $+1$. Using a simple argument counting handles and computing Euler characteristics, it is easy then to deduce the statement of the proposition.

Each exceptional sphere $E \subset n_- \mathbb{P}^2 \# n_+ \mathbb{P}^2$ intersects $\Lambda$ transversely in two points. Connect these two points by a path on $\Lambda$. The $D^2$-neighborhood of $\Lambda$ pulls back to a trivial $D^2$-bundle over the path. The fibers over the two endpoints can be identified with neighborhoods on $E$. Removing these neighborhoods from $E$ we get a sphere with two discs removed and we take the union of this with the $S^1$ boundaries of the all the fibers of the $D^2$-bundle over the path.

This either gives a torus or a Klein bottle. Because $E$ intersects $\Lambda$ once positively and once negatively, we see that we in fact get a torus which has self-intersection $\pm 1$. Finally note that we can certainly choose paths on $\Lambda$ for each exceptional sphere which are disjoint. \hfill $\square$

4. A flat connection and the Dold-Whitney theorem

This section considers the characteristic classes of $SO(3)$-bundles, but in fact we shall only be concerned with those bundles whose structure group can be restricted to a small subgroup of $SO(3)$.

**Definition 4.1.** Let $V_4 \subseteq SO(3)$ be the Klein 4-group

$$V_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

In future, we write $x_1, x_2, x_3$ for the non-identity elements.
We begin with a well-known (in certain circles) lemma about a flat $SO(3)$-connection on the torus.

**Lemma 4.2.** Let $T^2$ be a torus and let $\eta : \pi_1(T^2) \to SO(3)$ be defined by $\eta(a) = x_1$ and $\eta(b) = x_2$ where $a, b$ is a basis for $\pi_1(T^2) = H_1(T^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Writing $E_\eta$ for the associated (flat) $SO(3)$-bundle, we have

$$w_2(E_\eta) = 1 \in H^2(T^2; \mathbb{Z}/2) = \mathbb{Z}/2.$$

**Proof.** Note that the matrices of $V_4$ are all diagonal with entries in $\mathbb{Z}/2\mathbb{Z} = O(1)$. Hence, thinking of $E_\eta$ as an $O(3)$-bundle, we can write $E_\eta = L_1 \oplus L_2 \oplus L_3$ where $L_i$ is the (flat) real line bundle determined by the representation

$$\pi_1(T^2) \to V_4 \to \mathbb{Z}/2\mathbb{Z} = O(1),$$

where $p_i$ is given by the $(ii)$ matrix entry.

Each $L_i$ is the pullback of a Möbius line bundle over a circle by a map $T^2 \to S^4$ (depending on $i$) which is a projection map onto an $S^1$ factor of $T^2$. We compute then that

$$w_1(L_1) = \pi, \quad w_1(L_2) = \overline{b}, \quad \text{and} \quad w_1(L_3) = \pi + \overline{b},$$

where we write $\pi, \overline{b} \in H^1(T^2; \mathbb{Z}/2\mathbb{Z})$ for the reductions of the Poincaré duals of $a$ and $b$ respectively.

Then we compute via the cup-product formula for the Stiefel-Whitney class of a sum of bundles:

$$w_2(E_\eta) = \pi \cup \overline{b} + \overline{b} \cup (\pi + \overline{b}) + (\pi + \overline{b}) \cup \pi = \pi \cup \overline{b} = 1 \in H^2(T^2; \mathbb{Z}/2\mathbb{Z}).$$

\[\square\]

**Remark 4.3.** For representations $\eta : \pi_1(T^2) \to V_4$, Lemma 4.2 says that $w_2(E_\eta)$ is non-trivial exactly when $\eta$ is surjective (note that if $\eta$ is not surjective then $E_\eta$ is the pullback of a bundle over a circle).

Suppose now that we are in the situation of the hypotheses of Proposition 2.4. By gluing together the two pairs of disks $(D_1, D_2)$ and $(D'_1, D'_2)$ along their boundary $L \subset S^4$, we get a 2-component locally-flat immersed link $\Lambda \subset S^4$. We write $\Lambda_j$ for the sphere resulting from gluing together $D_j$ and $D'_j$ for $j = 1, 2$. In performing this gluing we of course reverse the orientation of the second 4-ball. This has the effect that positive/negative self-intersections of $(D'_1, D'_2)$ become negative/positive self-intersections of $\Lambda$ respectively. We write $X = X_\Lambda$, and now give a flat $SO(3)$ connection on $X$.

Let $\theta : \pi_1(X) \to SO(3)$ be a representation that factors through an onto map $\overline{\theta} : H_1(X; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \to V_4$. We define $\theta$ by setting $\overline{\theta} : m_j \mapsto x_j$ where $m_j$ is a meridian of $\Lambda_j$ for $j = 1, 2$. We write $E_\theta$ for the associated (flat) $SO(3)$-bundle over $X$. We are interested in the characteristic classes $w_2(E_\theta) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ and $p_1(E_\theta) \in H^4(X; \mathbb{Z})$. In the case we consider in this paper, we know immediately that $p_1(E_\theta) = 0$ since the bundle admits a flat connection.

Proposition 2.4 now follows by computing $w_2^2(E_\theta)$ using our basis of tori representing the second homology of $X$.

**Proof of Proposition 2.4.** As noted before, the content of the proposition is in the first equality sign, namely that we have

$$w_2^2(E_\theta) = \phi(L, D_1, D_2) - \phi(L, D'_1, D'_2) \in H^4(X; \mathbb{Z}/4\mathbb{Z}).$$

We compute $w_2(E_\theta) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$ by pulling back the representation $\theta$ to each torus representing a basis element of $H_2(X; \mathbb{Z})$. Let $T_p \subseteq X$ be a torus as
constructed in Proposition 3.1 coming from a self-intersection point \( p \in \Lambda_j \) for some \( j \in \{1, 2\} \). We wish to give a pair of \( H_1(T_p; \mathbb{Z}) \)-generating circles on \( T_p \).

The first of these circles we take to be a meridian \( m_p \) to \( \Lambda_j \). The other we take to be any circle \( l_p \) on \( T_p \) which is dual to \( m_p \). Then the restriction of \( \theta \) to \( \pi_1(T_p) = H_1(T_p; \mathbb{Z}) \) is determined by \( \overline{\theta}(m_p) \) and \( \overline{\theta}(l_p) \).

We know by the definition of \( \theta \) that we have \( \overline{\theta}(m_p) = x_j \). On the other hand, \( \overline{\theta}(l_p) \) is determined by the class of \( l_p \) in \( H_1(X; \mathbb{Z}/2\mathbb{Z}) \). Consider \( w(p) \) as given in Definition 2.1. If we have \( w(p) = 0 \) then \( \overline{\theta}(l_p) \in \{1, x_j\} \), but if \( w(p) = 1 \) then \( \overline{\theta}(l_p) \notin \{1, x_j\} \). In consequence, \( \theta|_{\pi_1(T_p)} \) maps onto \( V_4 \) if and only if \( w(p) = 1 \).

In light of Remark 4.3 it follows that \( w_2(E_0|_{T_p}) = w(p) \in \mathbb{Z}/2\mathbb{Z} = H^2(T, \mathbb{Z}/2\mathbb{Z}) \).

The equation we wish to prove then follows since, computing in \( H^4(X, \mathbb{Z}/4\mathbb{Z}) \), we have

\[
p_1(E_0) = w_2^2(E_0) = \left( \sum_p (w_2(E_0)|_{T_p})[T_p] \right)^2 = \sum_p (w_2(E_0)|_{T_p})(|T_p| \cup |\overline{T_p}|) = \sum_p w(p)(|T_p| \cup |\overline{T_p}|)
\]

\[
= \phi(L, D_1, D_2) - \phi(L, D_1', D_2'),
\]

where we write \( |T_p| \) for the fundamental class of \( T_p \) and the overline denotes the Poincaré dual. We use here that the Pontryagin square of the \( \mathbb{Z}/2\mathbb{Z} \) reduction of an integral class is the \( \mathbb{Z}/4\mathbb{Z} \) reduction of the usual square of that integral class. \( \square \)

**Remark 4.4.** It is possible to give more a complicated construction along the lines above, which should extend the invariant to 2-component links of even linking number. This recovers the \( \mathbb{Z}/4\mathbb{Z} \) reduction of the Sato-Levine invariant due to Akhmetiev and Repovs [11] for this class of links.

The construction above starts with two pairs of discs \((D_1, D_2)\) and \((D_1', D_2')\). In the case of a link \( L \) of non-zero linking number \( 2n \) we start rather with two immersed concordances from \( L \) to the \((2, 4n)\)-torus link. These may then be glued end-to-end and the resulting immersed surface resolved by blow-up in order to give two embedded tori \( \Lambda \) of self-intersection 0 in a blow-up of \( S^1 \times S^3 \). Surgery may be done on \( \Lambda \) in order to give a closed 4-manifold \( X \).

The main subtleties in this new situation are in performing the surgery so that one obtains \( X \) with the correct algebraic topology, and in dealing with an intersection form that is no longer diagonal.

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