ON EXTREME POINTS OF MEASURES WHICH IMPLEMENT AN ISOMETRIC EMBEDDING OF MODEL SPACES

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Abstract. In 1996 A. Alexandrov solved an isometric embedding problem for model spaces $K_{\Theta}$ with an arbitrary inner function $\Theta$. We find all extreme points of this convex set of measures in the case when $\Theta$ is a finite Blaschke product, and obtain some partial results for generic inner functions.

Introduction

In [2] A. Aleksandrov settled the isometric embedding problem for model spaces $K_{\Theta}$. Precisely, given an arbitrary inner function $\Theta$ on the unit disk $\mathbb{D}$, one seeks for the collection of all finite, positive, Borel measures $\sigma$ on the unit circle $\mathbb{T}$ so that the identity operator (embedding) of the model space $K_{\Theta} := H^2 \ominus \Theta H^2$ to the space $L^2_\sigma(\mathbb{T})$ is isometric. In other words, the equality

$$\langle f, g \rangle_\sigma := \int_{\mathbb{T}} f(t) \overline{g(t)} \, \sigma(dt) = \int_{\mathbb{T}} f(t) \overline{g(t)} \, m(dt) = \langle f, g \rangle_m$$

holds for each $f, g \in K_{\Theta}$. Here $m$ is the normalized Lebesgue measure on $\mathbb{T}$.

Denote this set of measures by $\Sigma(\Theta)$, and the unit ball of $H^\infty$ (the Schur class) by $\mathcal{S}$. The result of Aleksandrov looks as follows.

Theorem A. $\sigma \in \Sigma(\Theta)$ if and only if there is a real number $\beta$ and a Schur function $\omega \in \mathcal{S}$ so that

$$\frac{1 + \Theta(z)\omega(z)}{1 - \Theta(z)\omega(z)} = i\beta + \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma(dt).$$

Relation (0.1) can be viewed as a counterpart of the Nevanlinna parametrization in the Hamburger moment problem, see [1, Theorem 3.2.2].

Remark 0.1. The function $\omega$ in (0.1) is an independent parameter, which runs over the whole class $\mathcal{S}$. Both $\beta$ and $\sigma$ in (0.1) are uniquely determined by $\omega$,

$$\beta = \frac{2 \text{Im} (\omega(0)\Theta(0))}{|1 - \Theta(0)\omega(0)|^2}.$$

Conversely, if two triples $\{\omega_1, \beta_1, \sigma_1\}, \{\omega_2, \beta_2, \sigma_2\}$, satisfy (0.1), then $\omega_1 = \omega_2$ and $\beta_1 = \beta_2$. For instance, $\sigma = m$ enters the only triple $\{0, 0, m\}$. So, equality
generates a bijection $I$

\begin{equation}
I : \Sigma(\Theta) \to \mathcal{S}, \quad I(\sigma) = \omega.
\end{equation}

The set $\Sigma(\Theta)$ is easily seen to be a convex set which is compact in the *-weak topology of the space $\mathcal{M}_+^+(\mathbb{T})$ of finite, positive, Borel measures on $\mathbb{T}$. The study of the set $\Sigma_{\text{ext}}(\Theta)$ of extreme points for $\Sigma(\Theta)$ seems quite natural. This is exactly the problem we address here. A point $\sigma \in \Sigma(\Theta)$ is said to be an extreme point of $\Sigma(\Theta)$ if

\begin{equation}
\sigma = \frac{\sigma_1 + \sigma_2}{2}, \quad \sigma_j \in \Sigma(\Theta) \Rightarrow \sigma_1 = \sigma_2 = \sigma.
\end{equation}

Equivalently, there is no nontrivial representation of $\sigma$ as a convex linear combination of two measures from $\Sigma(\Theta)$.

We say that a measure $\sigma \in \mathcal{M}_+^+(\mathbb{T})$ has a finite support if

\[ \sigma = \sum_{j=1}^{p} s_j \delta(t_j), \quad s_j > 0, \quad \text{supp } \sigma = \{t_j\}_{j=1}^{p}, \quad t_j = t_j(\sigma), \]

and write $|\text{supp } \sigma| = p$ for such measures. Denote by $\Sigma_f(\Theta)$ the set of measures with the finite support in $\Sigma(\Theta)$. It is clear that $\Sigma_f(\Theta)$ is nonempty if and only if both $\Theta = B$ and $\omega$ are finite Blaschke products (FBP).

Here is our main result.

**Theorem 0.2.** Let $B$ be a FBP of order $n \geq 1$. The measure $\sigma \in \Sigma_{\text{ext}}(B)$ if and only if $\sigma \in \Sigma_f(B)$ and

\begin{equation}
n \leq |\text{supp } \sigma| \leq 2n - 1.
\end{equation}

Denote by $\mathcal{S}_{\text{ext}}(\Theta)$ the set of all $\omega \in \mathcal{S}$ so that the corresponding $\sigma$ in (0.1) belongs to $\Sigma_{\text{ext}}(\Theta)$. Equivalently, $\mathcal{S}_{\text{ext}}(\Theta)$ is the image of $\Sigma_{\text{ext}}(\Theta)$ under transformation $I$ (0.2). The above result can be paraphrased as follows.

**Theorem 0.3.** Let $B$ be a FBP of order $n \geq 1$. The set $\mathcal{S}_{\text{ext}}(B)$ agrees with the set of all FBP’s of the order at most $n - 1$.

The case of generic inner functions $\Theta$ is much more delicate. We can supplement the above result with the following

**Theorem 0.4.** Let $\Theta$ be an inner functions with $n$ distinct zeros. Then each FBP of order at most $n - 1$ belongs to $\mathcal{S}_{\text{ext}}(\Theta)$. In particular, each FBP belongs to $\mathcal{S}_{\text{ext}}(\Theta)$ as long as $\Theta$ has infinitely many zeros.

It might be worth comparing this result with [1, Corollary 3.4.3].

Relations (0.1) with unimodular constants $\omega = \alpha \in \mathbb{T}$

\begin{equation}
\frac{1 + \alpha \Theta(z)}{1 - \alpha \Theta(z)} = i\beta_\alpha + \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma_\alpha(dt)
\end{equation}

are well established in the theory of model spaces [3, Chapter 9] and the theory of orthogonal polynomials on the unit circle [6, Chapter 1.3]. The measures $\sigma_\alpha$ in (0.5) are known as the Aleksandrov–Clark measures. Our final result concerns this class of measures.

**Theorem 0.5.** Let $\Theta$ be an arbitrary nonconstant inner function. Then $\sigma_\alpha \in \Sigma_{\text{ext}}(\Theta)$ for all $\alpha \in \mathbb{T}$. 

We examine the class $\sum_f(B)$ of measures with finite support in Section 1 and prove the main result in Section 2. In the last Section 3, given an inner function $\Theta$, we introduce a binary operation $S \times S \to S$ ($\Theta$-product) and show that $\sigma \notin S_{ext}(\Theta)$ if and only if the Schur function $\omega = I(\sigma)$ admits a certain nontrivial factorization with respect to the $\Theta$-product. Thereby, the “$\Theta$-prime functions” $\omega$ correspond to extreme points of $\Sigma(\Theta)$. The results of Theorems 0.4 and 0.5 are obtained along this line of reasoning.

1. Some properties of the class $\Sigma_f(B)$

Given a FBP $B$ of order $n$, we denote the divisor of its zeros by $$\{(z_1, r_1), (z_2, r_2), \ldots, (z_d, r_d)\}, \quad z_i \neq z_j, \quad i \neq j, \quad r_j \in \mathbb{N},$$ so that $$B(z) := \prod_{k=1}^{d} \left( \frac{|z_k|}{z_k} \frac{z_k - z}{1 - \bar{z}_k z} \right)^{r_k}, \quad \deg B = r_1 + \ldots + r_d = n.$$ The model space $$K_B := H^2 \ominus BH^2 = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^{n-1} (1 - \bar{z}_j z)^{r_j}}, \quad \deg P \leq n - 1 \right\},$$ is a finite dimensional space of all rational functions with the poles at the points $1/\bar{z}_j$ of degree at most $r_j$, $\dim K_B = n$. The case $z_d = 0$, i.e., $B(0) = 0$, will be of particular concern. Now $$K_B = \left\{ h(z) = \frac{P(z)}{\prod_{j=1}^{n-1} (1 - \bar{z}_j z)^{r_j}}, \quad \deg P \leq n - 1 \right\},$$ and the monomials $1, z, \ldots, z^{r_d-1} \in K_B$. Put $$\varphi_0(z) = 1, \quad \varphi_k(z) := \frac{1}{1 - \bar{z}_k z}, \quad k = 1, 2, \ldots, d - 1, \quad \varphi_d(z) = z,$$ so the standard basis in $K_B$ is

\begin{equation}
\{ \varphi_1, \varphi_1^2, \ldots, \varphi_1^{r_1}; \ldots; \varphi_{d-1}, \varphi_{d-1}^2, \ldots, \varphi_{d-1}^{r_{d-1}}; \varphi_d, \ldots, \varphi_d^{r_d-1}; \varphi_0 \}. \tag{1.1}
\end{equation}

Sometimes we reorder these functions in a unique sequence $\{e_l\}_{l=1}^{n}$, $e_n = 1$.

The following result is a consequence of Theorem A, but we give a simple, direct proof.

**Proposition 1.1.** The support of each measure $\sigma \in \Sigma$ contains at least $n$ points.

**Proof.** If $\|\text{supp } \sigma\| \leq n - 1$, then $\dim L^2_{\sigma}(\mathbb{T}) \leq n - 1$, and the functions $\{e_l\}_{l=1}^{n}$ in (1.1) are linearly dependent in $L^2_{\sigma}$, so $$\det \|\langle e_j, e_k \rangle_{\sigma}\|_{j,k=1}^{n} = 0.$$ On the other hand, the same system is linearly independent in $L^2_{m}$, so $$\det \|\langle e_j, e_k \rangle_{m}\|_{j,k=1}^{n} \neq 0.$$ The contradiction completes the proof. \qed
It is clear from (0.1) that a measure \( \sigma \in \Sigma_f(B) \) if and only if \( \omega \) is a FBP. Moreover,
\[
|\text{supp } \sigma| = n + \deg \omega,
\]
so \( |\text{supp } \sigma| = n \) if and only if \( \sigma = \sigma_\alpha \) is the Aleksandrov–Clark measure.

It is not hard to write \( \sigma \in \Sigma_f(B) \) explicitly in terms of the corresponding parameters \( \omega \) and \( B \). Indeed, (0.1) now takes the form
\[
(1.2) \quad 1 + \frac{B(z)\omega(z)}{1 - B(z)\omega(z)} = i\beta + \sum_{k=1}^{p} \frac{t_k + z}{t_k - z} s_k,
\]
and
\[
(1.3) \quad \text{supp } \sigma = \{t_j\}_{j=1}^{p} : B(t_j)\omega(t_j) = 1, \quad j = 1, 2, \ldots, p.
\]

The weights \( s_j \) can be determined from the limit relations
\[
2t qs_q = \left(1 + B(t_q)\omega(t_q)\right) \lim_{z \to t_q} \frac{t_q - z}{1 - B(z)\omega(z)} = \frac{2}{[B\omega]'(t_q)},
\]
or, in view of (1.3),
\[
\frac{1}{s_q} = t_q[B\omega]'(t_q) = t_q \frac{B'(t_q)}{B(t_q)} + t_q \frac{\omega'(t_q)}{\omega(t_q)}.
\]

A computation of the logarithmic derivative of a FBP is standard
\[
\frac{B'(z)}{B(z)} = \sum_{k=1}^{d} r_k \frac{1 - |z_k|^2}{(1 - z_k\bar{z})(z - z_k)},
\]
and so
\[
(1.4) \quad \frac{1}{s_q} = \sum_{k=1}^{d} r_k \frac{1 - |z_k|^2}{|t_q - z_k|^2} + \sum_{j=1}^{m} \frac{1 - |w_j|^2}{|t_q - w_j|^2},
\]
where \( w_1, \ldots, w_m \) are all zeros (counting multiplicity) of \( \omega \) in (1.2).

Relation (1.4) provides an answer to the following “extremal mass problem”: given a point \( \tau \in \mathbb{T} \), find a measure \( \sigma_{\max} \in \Sigma_f(B) \) so that
\[
\sigma_{\max}\{\tau\} = \max\{\sigma\{\tau\} : \sigma \in \Sigma_f(B)\}.
\]
Indeed, such measure is exactly the Aleksandrov–Clark measure \( \sigma = \sigma_\alpha \) with \( \alpha = B^{-1}(\tau) \), \( |\text{supp } \sigma_{\max}| = n \), and
\[
\frac{1}{\sigma_{\max}\{\tau\}} = \sum_{k=1}^{d} r_k \frac{1 - |z_k|^2}{|\tau - z_k|^2}.
\]

**Remark 1.2.** As a matter of fact, the above Aleksandrov–Clark measure solves the same extremal problem within the whole class \( \Sigma(B) \). Relation (1.4) holds in the form
\[
\frac{1}{s_q} = \sum_{k=1}^{d} r_k \frac{1 - |z_k|^2}{|\tau - z_k|^2} + |\omega'(\tau)|,
\]
where \( \omega' \) is the angular derivative of \( \omega \) (cf. [3, Section 9.2]).

Here is another simple property of measures from \( \Sigma_f(B) \).
Proposition 1.3. Let \( \{t_j\}_{j=1}^p \) be an arbitrary set of distinct points on \( T \). There is a measure \( \sigma \in \Sigma_f(B) \) so that

1. \( \{t_j\} \in \text{supp} \sigma \);
2. \( |\text{supp} \sigma| \leq n + p - 1 \).

Proof. The proof is based on the interpolation with FBP (see [5, Theorem 1]): there is a FBP \( \omega \) so that \( \deg \omega \leq p - 1 \) and

\[ \omega(t_j) = B^{-1}(t_j), \quad j = 1, \ldots, p. \]

The corresponding measure \( \sigma \) in (1.2) is the one we need. \( \square \)

It turns out that the intersection of supports of two different measures from \( \Sigma_f(B) \) can not be too large. Denote by

\[ \Sigma_{n+k}(B) \colonequals \{ \sigma \in \Sigma_f(B) : |\text{supp} \sigma| = n + k \}, \quad k = 0, 1, \ldots. \]

Lemma 1.4. Let \( \sigma_j \in \Sigma_{n+p_j}(B), j = 1, 2 \), and let \( |\text{supp} \sigma_1 \cap \text{supp} \sigma_2| \geq p_1 + p_2 + 1 \). Then \( \sigma_1 = \sigma_2 \).

Proof. Let \( \omega_j \) be the corresponding FBP in (1.2), \( \deg \omega_j = p_j, j = 1, 2 \). Let

\[ \zeta_1, \ldots, \zeta_{p_1+p_2+1} \in \text{supp} \sigma_1 \cap \text{supp} \sigma_2, \]

so, by (1.3), \( \omega_1(\zeta) = \omega_2(\zeta), l = 1, 2, \ldots, p_1 + p_2 + 1 \). Note that

\[ \omega_j(z) = \gamma_j \frac{Q_j(z)}{Q_j^*(z)}, \quad j = 1, 2, \]

where \( \gamma_j \) are unimodular constants, \( Q_j \) are algebraic polynomials, \( Q_j^* \) are the reversed polynomials, and \( \deg Q_j = p_j, \quad \deg Q_j^* \leq p_j, \quad j = 1, 2 \).

We see that for the polynomial

\[ Q(z) = \gamma_1 Q_1(z) Q_2^*(z) - \gamma_2 Q_2(z) Q_1^*(z), \quad \deg Q \leq p_1 + p_2, \]

the relations

\[ Q(\zeta_l) = 0, \quad l = 1, 2, \ldots, p_1 + p_2 + 1 \]

hold, so \( Q = 0, \omega_1 = \omega_2, \) and \( \sigma_1 = \sigma_2 \) (see Remark 0.1). \( \square \)

Corollary 1.5. If \( \sigma_j \in \Sigma_{n}(B), j = 1, 2, \) and \( \text{supp} \sigma_1 \cap \text{supp} \sigma_2 \neq \emptyset \), then \( \sigma_1 = \sigma_2 \). If \( \sigma_j \in \Sigma_{n+k}(B), k = 0, 1, \ldots, n - 1 \), and \( \text{supp} \sigma_1 = \text{supp} \sigma_2 \), then \( \sigma_1 = \sigma_2 \).

2. Extreme points of \( \Sigma(B) \) for finite Blaschke products

We begin with the result which provides the upper bound in (0.4). It can be viewed as a counterpart of [1, Theorem 2.3.4] for the classical moment problem.

Proposition 2.1. Let \( \sigma \in \Sigma_{\text{ext}}(B) \). Then \( \sigma \in \Sigma_f(B) \) and \( |\text{supp} \sigma| \leq 2n - 1 \).
Proof. Assume first that \( z_d = 0 \). Define an accompanying system of real valued, linearly independent functions on \( \mathbb{T} \):
\[
x_{k,j}(t) := \text{Re} \varphi_{k,j}(t), \quad y_{k,j}(t) := \text{Im} \varphi_{k,j}(t), \quad j = 1, \ldots, r_k, \quad k = 1, \ldots, d-1,
\]
\[
x_{d,j}(t) := \text{Re} \psi_j(t), \quad y_{d,j}(t) := \text{Im} \psi_j(t), \quad j = 1, \ldots, r_d - 1, \quad x_{d,0} = 1.
\]
Reorder them in one sequence \( \{v_i\}_{i=1}^{2n-1} \), and denote by \( E \) their complex, linear span
\[
E := \text{span}_{1 \leq i \leq 2n-1} \{v_i\}, \quad \dim E = 2n - 1.
\]
Clearly,
\[
\varphi_{k,j} = x_{k,j} + iy_{k,j} \in E, \quad \bar{\varphi}_{k,j} = x_{k,j} - iy_{k,j} \in E
\]
(or \( e_m, \bar{e}_m \in E \)) for appropriate values of \( k, j, m \), and \( t^i \in E \) for \( |i| \leq r_d - 1 \). It is a matter of a direct computation to make sure that the product \( e_{m} \bar{e}_{l} \in E \), \( m, l = 1, \ldots, n \). For instance,
\[
\bar{\varphi}_p(t) \varphi_q(t) = \frac{1}{(1 - z_p t)(1 - z_q t)} = \frac{\varphi_p(t) + \bar{\varphi}_q(t) - 1}{1 - \bar{z}_p z_q},
\]
\[
\varphi_p(t) \varphi_q(t) = \frac{1}{(1 - z_p t)(1 - z_q t)^2} = \frac{\varphi_p(t) + \varphi_q(t) - \varphi_{q'}(t)}{1 - \bar{z}_p z_q},
\]
e tc. The rest is a simple induction. We conclude, thereby, that \( \int \mathcal{G} \in E \) for each \( f, g \in K_B \).

Assume next, that \(|\text{supp} \sigma| \geq 2n\). Then the inclusion \( E \subset L^1(\mathbb{T}) \) is proper, so there is a nontrivial, linear functional \( \Phi_0 \) on \( L^1(\mathbb{T}) \), \( \|\Phi_0\| \leq 1 \), which vanishes on \( E \). Equivalently, there is a function \( \varphi_0 \in L^\infty(\mathbb{T}) \) such that \( |\varphi_0| \leq 1 \) \( \sigma \)-almost everywhere, and
\[
\int_{\mathbb{T}} x_{j,k}(t) \varphi_0(t) \sigma(dt) = \int_{\mathbb{T}} y_{j,k}(t) \varphi_0(t) \sigma(dt) = 0
\]
for all appropriate values of \( j, k \). Since the functions \( x_{j,k}, y_{j,k} \) are real valued, the function \( \varphi_0 \) can be taken real valued as well.

Consider now two measures \( \sigma_\pm (dt) := (1 \pm \varphi_0) \sigma(dt) \), \( \sigma_\pm \in \mathcal{M}_+(\mathbb{T}) \). By the construction, \( \sigma_\pm \in \Sigma(B) \), and the representation \( 2\sigma = \sigma_+ + \sigma_- \) is nontrivial. Hence, \( \sigma \) is not an extreme point of \( \Sigma(B) \), as claimed.

It remains to examine the general case when \( B(0) \neq 0 \). The standard trick with the change of variables (see, e.g., [6, pp.140–141]) reduces this case to the one discussed above. Given \( a \in \mathbb{D} \), put
\[
b_a(z) := \frac{z + a}{1 + \bar{a} z}, \quad B_a(z) := B(b_a(z)), \quad \omega_a(z) := \omega(b_a(z)).
\]
If we replace \( z \) with \( b_a(z) \) in (0.1), we have
\[
\frac{1 + B_a(z)\omega_a(z)}{1 - B_a(z)\omega_a(z)} = i\beta + \int_{\mathbb{T}} \frac{t + b_a(z)}{t - b_a(z)} \sigma(dt),
\]
and since
\[
\frac{t + b_a(z)}{t - b_a(z)} = i\beta_a t + \frac{1 - |a|^2}{|t - a|^2} \frac{b_a(t) + z}{b_a(t) - z},
\]
we come to
\[
\frac{1 + B_a(z)\omega_a(z)}{1 - B_a(z)\omega_a(z)} = i\beta_a + \int_{\mathbb{T}} \frac{1 - |a|^2}{|t - a|^2} \frac{b_a(t) + z}{b_a(t) - z} \sigma(dt) = i\beta_a + \int_{\mathbb{T}} \frac{\tau + z}{\tau - z} \sigma_a(d\tau).
\]
It is clear that the map \( \sigma \to \sigma_a \) is a bijection of \( \Sigma(B) \) onto \( \Sigma(B_a) \), which is also the bijection between \( \Sigma_{ext}(B) \) and \( \Sigma_{ext}(B_a) \). Obviously, it is a bijection between \( \Sigma_f(B) \) and \( \Sigma_f(B_a) \), and in this case \( |\text{supp } \sigma| = |\text{supp } \sigma_a| \). But \( B_a(0) = 0 \) with \( a = z_d \), so the above argument applies. The proof is complete. 

Proof of Theorem 0.2.

It remains to show that each measure \( \sigma \in \Sigma_{n+k}(B) \), \( k = 0, 1, \ldots, n - 1 \) is the extreme point of \( \Sigma(B) \). Indeed, let \( 2\sigma = \sigma_1 + \sigma_2 \), then

\[
\sigma_j \in \Sigma_{n+p_j}, \quad j = 1, 2, \quad 0 \leq p_1, p_2 \leq k.
\]

Since \( \text{supp } \sigma = \text{supp } \sigma_1 \cup \text{supp } \sigma_2 \), we have

\[
|\text{supp } \sigma| = |\text{supp } \sigma_1| + |\text{supp } \sigma_2| - |\text{supp } \sigma_1 \cap \text{supp } \sigma_2|,
\]

or

\[
|\text{supp } \sigma_1 \cap \text{supp } \sigma_2| = n + p_1 + n + p_2 - n - k = n + p_1 + p_2 - k \geq p_1 + p_2 + 1.
\]

By Lemma 1.4, \( \sigma_1 = \sigma_2 \), so \( \sigma \) is the extreme point of \( \Sigma(B) \), as claimed.

3. Extreme Points of \( \Sigma(\Theta) \) for Generic Inner Functions

We begin with some basics of the Nevanlinna–Pick interpolation problem in the Schur class. Given \( n \) distinct points \( Z = \{z_1, \ldots, z_n\} \) on the unit disk \( \mathbb{D} \), and \( n \) complex numbers \( W = \{w_1, \ldots, w_n\} \), the problem is to find conditions on the interpolation data \( (Z, W) \) so that the interpolation

\[(3.1) \quad f(z_j) = w_j, \quad j = 1, 2, \ldots, n \]

has at least one solution \( f \in \mathcal{S} \), and to specify such conditions for (3.1) to have a unique solution.

Define a Pick matrix \( P = P_n(Z, W) \) by

\[(3.2) \quad P := \| \frac{1 - \bar{w}_j w_k}{1 - \bar{z}_j z_k} \|^{n}_{j, k = 1}.
\]

The fundamental result of G. Pick (see [4, Theorem I.2.2 and Corollary I.2.3]) looks as follows.

**Theorem P.** The problem (3.1) is solvable in the class \( \mathcal{S} \) if and only if the Pick matrix (3.2) is nonnegative definite, \( P \geq 0 \). It has a unique solution if and only if \( \det P = 0 \).

Our argument leans on a simple consequence of Theorem P.

**Corollary 3.1.** Let \( Z = \{z_1, \ldots, z_n\} \) be \( n \) distinct points on \( \mathbb{D} \), \( s_0 \in \mathcal{S} \), and \( B \) be an FBP of order at most \( n - 1 \). Assume that

\[(3.3) \quad s_0(z_j) = B(z_j), \quad j = 1, \ldots, n.
\]

Then \( s_0 = B \).

**Proof.** It is not hard to see (by the induction on the order) that for an arbitrary FBP \( b \) of order \( m \), and a collection \( \Lambda = \{\lambda_1, \ldots, \lambda_n\} \) of distinct points on \( \mathbb{D} \) with \( n \geq m \), the Pick matrix (3.2) with \( w_j = b(\lambda_j) \) has the rank at most \( m \).

Put \( w_j := B(z_j) \), so \( s_0 \) solves the Nevanlinna–Pick problem (3.1). Since \( \deg B \leq n - 1 = m \), the Pick matrix \( P_n(Z, W) \) has the rank at most \( n - 1 \),
so \( \det \mathcal{P} = 0 \). By Theorem P, interpolation (3.3) has a unique solution in the class \( \mathcal{S} \), so \( s_0 = B \), as claimed. \( \square \)

Let us now define a binary operation on the Schur class. Given an inner function \( \Theta \) and two Schur functions \( s_1, s_2 \), put

\[
(s_1 \circ s_2)_\Theta := \frac{s_0 - \Theta s_1 s_2}{1 - \Theta s_0}, \quad s_0 := \frac{s_1 + s_2}{2}.
\]

\((s_1 \circ s_2)_\Theta\) will be called a \( \Theta \)-product of \( s_1 \) and \( s_2 \).

**Proposition 3.2.** The \( \Theta \)-product is an idempotent, binary operation on the class \( \mathcal{S} \). \((s_1 \circ s_2)_\Theta\) is an inner function if and only if so are both \( s_1 \) and \( s_2 \), provided that \( \Theta \) is nonconstant.

**Proof.** Since \( 1 - \Theta s_0 \) is an outer function, \((s_1 \circ s_2)_\Theta\) belongs to the Smirnov class, so one has to verify that

\[ |(s_1 \circ s_2)_\Theta(t)| \leq 1, \quad t \in \mathbb{T}. \]

This is a matter of direct calculation. Indeed,

\[
|1 - \Theta s_0|^2 - |s_0 - \Theta s_1 s_2|^2 = |s_0|^2 - |s_1 s_2|^2 - |s_1|^2 |s_2|^2 \Re (\Theta s_2) - |s_2|^2 \Re (\Theta s_1),
\]

so

\[
|1 - \Theta s_0|^2 - |s_0 - \Theta s_1 s_2|^2 = 1 - |s_1 s_2|^2 - |s_1|^2 (1 - |s_2|^2) - \Re (\Theta s_1)(1 - |s_2|^2) - \Re (\Theta s_2)(1 - |s_1|^2)
\]

\[
= 1 - |s_1|^2 |s_2|^2 + |s_1|^2 |s_2|^2 - |s_1|^2 |s_2|^2 + |s_1|^2 |s_2|^2 - |s_1|^2 |s_2|^2 + |s_1|^2 |s_2|^2
\]

\[
= (1 - |s_1|^2)(1 - |s_2|^2) + (|s_1| - \Re (\Theta s_1))(1 - |s_1|^2) + (|s_2| - \Re (\Theta s_2))(1 - |s_2|^2)
\]

\[
= (1 - |s_1|^2)(1 - |s_2|^2) + (|s_1| - \Re (\Theta s_1))(1 - |s_1|^2) + (|s_2| - \Re (\Theta s_2))(1 - |s_2|^2)
\]

\[
geq 0,
\]

as needed.

By definition (3.4), \( (s \circ s)_\Theta = s \) for each \( s \in \mathcal{S} \), so the operation is idempotent.

Next, assume that \((s_1 \circ s_2)_\Theta\) is an inner function, but \(|s_1| < 1\) a.e. on a set \( E \subset \mathbb{T} \) of positive measure. It follows from the above calculation that

\[ |s_2| = 1, \quad |s_2| - \Re (\Theta s_2) = 0 \]

a.e. on \( E \). Hence \( \Theta s_2 = 1 \) a.e. on the set of positive measure, so \( \Theta \) is a unimodular constant. The contradiction completes the proof. \( \square \)

**Remark 3.3.** Whereas the original isometric embedding problem make no sense for constant inner functions \( \Theta \), the \( \Theta \)-product is a nontrivial operation already for \( \Theta = 1 \). It is clear from the definition, that for \( s_2 = \Theta = 1 \) one has \((s_1 \circ s_2)_\Theta = 1\) for any \( s_1 \in \mathcal{S} \).

**Definition 3.4.** A function \( s \in \mathcal{S} \) is called \( \Theta \)-prime if

\[
s = (s_1 \circ s_2)_\Theta \implies s = s_1 = s_2.
\]

The \( \Theta \)-product comes in quite naturally in the context of the isometric embedding problem. Specifically, let \( \sigma_1, \sigma_2 \in \Sigma(\Theta) \), and \( \omega_j = I(\sigma_j), j = 1, 2, \)
the map \( I \) is defined in (0.2). Then obviously \( \sigma = \frac{1}{2}(\sigma_1 + \sigma_2) \in \Sigma(\Theta) \). It is a matter of elementary computation based on the relation
\[
\frac{1 + \Theta(z)\omega(z)}{1 - \Theta(z)\omega(z)} = \frac{1}{2} \left( \frac{1 + \Theta(z)\omega_1(z)}{1 - \Theta(z)\omega_1(z)} + \frac{1 + \Theta(z)\omega_2(z)}{1 - \Theta(z)\omega_2(z)} \right), \quad \omega = I(\sigma),
\]
to check that
\[
\omega = (\omega_1 \circ \omega_2)_\Theta.
\]
Thereby, \( \sigma \in \Sigma_{\text{ext}}(\Theta) \) if and only if \( I(\sigma) \) is \( \Theta \)-prime. Equivalently, \( \omega \) belongs to \( S_{\text{ext}}(\Theta) \) if and only if \( \omega \) is \( \Theta \)-prime.

**Proof of Theorem 0.4.**
Let \( z_1, \ldots, z_n \) be \( n \) distinct zeros of \( \Theta \). Given a FBP \( B \) of order at most \( n - 1 \), write
\[
B(z) = (\omega_1 \circ \omega_2)_\Theta(z) = \frac{\omega_0(z) - \Theta(z)\omega_1(z)\omega_2(z)}{1 - \Theta(z)\omega_0(z)}
\]
and so \( B(z_j) = \omega_0(z_j), j = 1, \ldots, n \). By Corollary 3.1, \( \omega_0 = B \), in particular,
\[
\left| \frac{\omega_1(t) + \omega_2(t)}{2} \right| = 1, \quad \forall t \in \mathbb{T}.
\]
But the latter implies \( \omega_1 = \omega_2 = B \), so \( B \) is \( \Theta \)-prime, as claimed.

**Proof of Theorem 0.5.**
Let \( \omega = \gamma \in \mathbb{T} \). Write
\[
\gamma = (\omega_1 \circ \omega_2)_\Theta = \frac{\omega_0 - \Theta \omega_1 \omega_2}{1 - \Theta \omega_0}.
\]
Solve it for \( \omega_0 \)
\[
\omega_0 = \frac{\gamma + \Theta \omega_1 \omega_2}{1 + \gamma \Theta}, \quad \omega_0 - \gamma = \Theta \frac{\omega_1 \omega_2 - \gamma^2}{1 + \gamma \Theta},
\]
and finally,
\[
(\gamma - \omega_0)(1 + \gamma \Theta) = \Theta(\gamma^2 - \omega_1 \omega_2).
\]
Note that both functions \( \gamma - \omega_0 = \gamma(1 - \gamma_0) \) and \( 1 + \gamma \Theta \) are outer, so the left hand side in (3.5) is the outer function, whereas the right hand side in (3.5) has a nontrivial inner factor. Hence,
\[
\gamma^2 = \omega_1 \omega_2, \quad \omega_0 = \gamma \Rightarrow \omega_1 = \omega_2 = \gamma,
\]
so the constant function \( \omega = \gamma \) is \( \Theta \)-prime, as claimed.

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