ABSTRACT

We study a behavior of quantum generalized affine parameter (QGAP), which has been recently proposed by one of the present authors, near the singularity and the event horizon in three and four spacetime dimensions in terms of a minisuperspace model of quantum gravity. It is shown that the QGAP is infinite to the singularity while it remains finite to the event horizon. This fact indicates a possible interpretation that the singularity is wiped out in quantum gravity in this particular model of black hole.
1. Introduction

One of the remarkable achievements in the classical theory of gravity is surely the success of the proof of the well-known singularity theorem by Hawking and Penrose [1]. The theorems show that under rather physically reasonable assumptions spacetime singularities are inevitable, according to the standard theory of general relativity, particularly both in the gravitational collapse of massive stars leading to formation of black holes and in the past era in the big bang.

Even if the singularity theorems are so convincing to us at the present time, since the time of the proof till this day many people have been wondering whether such an approach based on the classical general relativity makes sense to examine singularities at all without using quantum mechanics and/or quantum theory of gravity.

Intuitively a spacetime singularity is a “place” where gravitational fields become extremely large and curvature blows up, by which one would expect that classical theory breaks down anyway while quantum gravitational effects become so important. However, the difficulty lies in making the precise definition of a singularity in quantum theory which is a natural generalization of geodesic incompleteness in classical relativity [2]. Thus it has been hoped that a union of quantum mechanics and general relativity would somehow avoid the formation of all types of singularities, and so would not encounter this difficulty.

Recently one of the present authors has proposed a criterion for a singularity to be quantum mechanically smeared away [3]. As a sort of “order parameter” measuring a physically meaningful length in quantum gravity, he has introduced a quantum mechanical version of Schmidt’s generalization of classical affine parameter [4], which we call quantum generalized affine parameter (QGAP) in what follows. The QGAP is defined in four spacetime dimensions as
\[
|\alpha|_q = \int_0^1 dt \sqrt{\sum_{a,b=0}^3 < [V^\mu \epsilon^b_\mu \left( P \exp \int_0^t \omega \right)_b^a]^2 >},
\]

(1)

where \(< >\) indicates an expectation value with respect to a quantum state, and \(V^\mu\) is the tangent vector along a causal curve, which is not necessarily a geodesic curve. Incidentally, the classical generalized affine parameter (CGAP) is defined like the above but without the expectation value \(< >\). See the previous paper for more details [3]. The key idea behind this definition is that when one approaches a singularity the QGAP would become infinite owing to a large quantum fluctuation of the connection \(\omega^a_{\ b}\) although the CGAP remains finite. By using this definition it has been explicitly shown that this expectation is actually realized in the model of (2+1)-dimensional circular symmetric black hole [3]. At first sight, the definition (1) seems to suffer from the serious drawbacks that it manifestly depends on the initial vierbein as well as the initial point, and furthermore it never be gauge invariant, or BRST invariant, but this is an illusion. The reason is why this definition cannot distinguish two lengths differing by a finite value, but does the finiteness from the infiniteness, and hence gives a useful standard in measuring at least the difference between the finite and infinite Lorentz transformations.

However, in the retrospect we notice that the previous study [3] has some aspects which should be improved. One of them is that an explicit form of a quantum state is just assumed to be a gaussian wave packet spread over a mass a priori. This assumption is certainly too naive since the quantum state should be singled out from the constraint equations to the state a la Dirac [5]. The other is the impossibility of applying the previous method for the more physically interesting (3+1)-dimensional black hole.

In this paper, we shall aim at overcoming these problems. The article is organized as follows. In section 2, we consider a minisuperspace model of (2+1)-dimensional black hole found by Bañados et al. [6]. We perform the canonical quantization of the model where mass and radial function are regarded as the
canonical variables. Here we shall quantize the black hole only in the interior region bounded by the singularity and the event horizon adopting the usual radial coordinate as a time parameter, and then solve the Wheeler-DeWitt equation in order to find the physical quantum state. By using such a selected quantum state, we show that the QGAP is infinite to the singularity while it is finite to the horizon. This fact indicates that the singularity is effectively infinitely far away and therefore the singularity is quantum mechanically wiped out by fluctuation of geometry.

In section 3, the above analysis is extended to the case of the (3+1)-dimensional Schwarzschild black hole where a similar phenomenon can indeed occur. The last section contains our conclusions.

2. (2+1)-dimensional black hole

We will begin by constructing a minisuperspace model of quantum gravity of the circular symmetric (2+1)-dimensional black hole [6]. Let us consider the following spacetime metric

\[
ds^2 = -\frac{1}{M(r) - \phi(r)^2 l^2} dr^2 + (M(r) - \frac{\phi(r)^2}{l^2}) dt^2 + \phi(r)^2 d\theta^2
\]

\[= -(e^0)^2 + (e^1)^2 + (e^2)^2.\]

We have chosen the dreibeins as

\[
e^0 = \frac{1}{\sqrt{M(r) - \frac{\phi(r)^2}{l^2}}} dr = \frac{1}{\sqrt{f(r)}} dr
\]
\[
e^1 = \sqrt{M(r) - \frac{\phi(r)^2}{l^2}} dt = \sqrt{f(r)} dt
\]
\[
e^2 = \phi(r) d\theta,
\]

where we have considered the mass of black hole \(M(r)\) and the circumference radius \(\phi(r)\) to be functions of only the radial coordinate \(r\). Note that the negative
cosmological constant $\Lambda$ is expressed by $\Lambda = -\frac{1}{l^2}$. Moreover, we have defined $f(r)$ to be $M(r) - \frac{\phi(r)^2}{l^2}$ in this section for later convenience. The important point is that the radial coordinate $r$ plays a role of time since we would like to perform a canonical quantization of the present model only inside the event horizon $f(r) > 0$.

To make contact with the Einstein theory of gravity, let us impose the torsion free condition in contrast with the previous work [3] where the model having a torsion off-shell was considered. From the torsion free equation, omitting the wedge product,

$$0 = T^a := de^a + \omega^a_{\ b} e^b,$$  

it is straightforward to derive concrete expressions for the spin connection $\omega^a_{\ b}$ given by

$$\omega^0_{\ 1} = \omega^1_{\ 0} = \frac{\dot{f}}{2\sqrt{f}} e^1,$$

$$\omega^0_{\ 2} = \omega^2_{\ 0} = \frac{\dot{\phi}\sqrt{f}}{\phi} e^2,$$

$$\text{otherwise} = 0,$$  

where the dot denotes the differentiation with respect to $r$. Then the curvature 2-form is easily computed from the second Cartan structure equation. We obtain

$$R^0_{\ 1} := -R^2 = \frac{1}{2} \dot{f} e^0 e^1,$$

$$R^0_{\ 2} := R^1 = \frac{d}{dr}(\dot{\phi}\sqrt{f}) \frac{\sqrt{f}}{\phi} e^0 e^2,$$

$$R^1_{\ 2} := R^2 = \frac{\dot{\phi}}{2\phi} e^1 e^2.$$  

The classical action is now given by

$$S = -\frac{2}{16\pi G} \int (-e_a R^a + \frac{1}{l^2} e^0 e^1 e^2),$$

with $G$ being the Newton constant which is set to be 1 in what follows. By substituting Eqs.(3) and (6) into the right hand side of Eq.(7), this action can be
written to be
\[ S = \int dr\ L, \] (8)

with the Lagrangian \( L \) being
\[ L = T \left( \frac{1}{2} \dot{f} \dot{\phi} - \frac{1}{l^2} \phi \right), \] (9)

Here we have performed the integration with respect to \( t \) with the time interval from \(-2T\) to \(2T\). By solving the Euler-Langrange equations derived from the action (9) we can check that the general solution essentially reproduces the solution \( \phi = r \) and \( M = \text{constant} \) which describes the (2+1)-dimensional black hole solution found by Bañados et al. [6].

Let us now carry out a canonical quantization of the present model. The canonical momenta \( \pi_M \) and \( \pi_\phi \) conjugate to the canonical variables \( M \) and \( \phi \) respectively are given by
\[ \pi_M = \frac{T}{2} \dot{\phi}, \quad \pi_\phi = T \left( \frac{1}{2} \dot{M} - \frac{2}{l^2} \phi \dot{\phi} \right). \] (10)

Then the Hamiltonian becomes
\[ H = \frac{T}{l^2} \phi + \frac{2}{T} \pi_\phi \pi_M + \frac{4}{T} \frac{1}{l^2} \phi \pi_M^2, \] (11)

which leads to the Wheeler-DeWitt (WDW) equation by the standard procedure in canonical gravity,
\[ 0 = H \psi(M, \phi) = \left( \frac{T}{l^2} \phi - \frac{2}{T} \frac{\partial^2}{\partial M \partial \phi} - \frac{4}{T} \frac{1}{l^2} \phi \frac{\partial^2}{\partial M^2} \right) \psi(M, \phi). \] (12)

as the constraint to the state \( \psi(M, \phi) \). Note that as mentioned in the section 1, in the previous work [3] the Hamiltonian identically vanishes so that it is impossible
to set up the physically meaningful WDW equation, however, this time we have the nontrivial WDW equation which is critical in selecting the physical state. Moreover, let us notice that the Hamiltonian (11) includes only the linear term with respect to $\pi_\phi$ which means that the WDW equation (12) has a common feature to the Schrödinger equation in the configuration space [7].

Assuming the general solution of the WDW equation (12) to be the form $\psi(M, \phi) = \psi_1(M)\psi_2(\phi)$, we can solve the equation whose general solution is given by

$$\psi(M, \phi) = B(e^{\alpha M} - Ce^{\beta M})e^{\frac{T A}{4} \phi^2},$$

where $A$, $B$, and $C$ are integration constants, and $\alpha$ and $\beta$ are defined to be

$$\alpha = -\frac{T A l^2}{8} - \frac{T}{2} \sqrt{1 + \left(\frac{A l^2}{4}\right)^2},$$

$$\beta = -\frac{T A l^2}{8} + \frac{T}{2} \sqrt{1 + \left(\frac{A l^2}{4}\right)^2}.$$ (14)

Now that we have a minisuperspace model of the (2+1)-dimensional quantum black hole, we can proceed to an evaluation of the quantum generalized affine parameter (QGAP). Before doing so, let us first consider the classical generalized affine parameter (CGAP) since this analysis brings us a clue leading to an important idea on attacking the QGAP later.

Following the procedure adopted in the previous work [3], let us consider a curve along which

$$e^1 = 0, \quad e^2 = ke^0,$$ (15)

with $k$ being a constant other than zero. The value of $k$ determines the causal property of the path, that is, timelike for $|k| < 1$, null for $|k| = 1$, and spacelike

† As stated in the previous paper [3] the curve should be specified without referring to the metric when we proceed to quantum gravity. So it would be more consistent if we replace Eq.(15) by the expression given in [3] which is written in terms of the coordinate only.
for \(|k| > 1\). Now we would like to compute the CGAP from a point inside the horizon to the singularity. Physically we are interested in a situation where a causal (i.e. timelike or null) curve starting at the point \(r_0\) inside the horizon at \(t = 0\) approaches the singularity \(r = 0\) at \(t = 1\). Thus it is sufficient to consider a curve near the singularity \(r = 0\). After taking the gauge \(\phi = r\), let us define the following quantity for later convenience.

\[
\gamma := \int_0^\infty \omega^2 = \int \frac{\dot{\phi}}{\phi} \sqrt{f} e^{2} = \int \frac{\dot{\phi}}{\phi} k e^0 = k \log r,
\]

(16)

where we have used Eqs.(3), (5) and (15). By using this \(\gamma\), a straightforward calculation gives CGAP as

\[
|\alpha|_c := \int_0^1 dt \left[ \sum_{a,b=0}^2 \left[ V^{\mu} e_b^a (P \exp \int_0^t \omega) \right]^2 \right]^{1/2}
\]

\[
= \int_0^1 dt \sqrt{\left[ V^{\mu} e_0^b (\cosh \gamma - k \sinh \gamma, 0, - \sinh \gamma + k \cosh \gamma) \right]^2}
\]

\[
\sim \int_0^1 dr \frac{1 + |k|}{\sqrt{2M}} r^{-|k|}.
\]

(17)

Thus it turned out that for a timelike curve \(|k| < 1\) the CGAP converges so that the path can reach the singularity \(r = 0\) in a finite length as physically expected, and on the other hand for a null \(|k| = 1\) or a spacelike path \(|k| > 1\) the CGAP diverges [3]. We can also calculate the CGAP from a point inside the event horizon to the event horizon in a perfectly similar way. Because \(\gamma\) is finite at the horizon \(r_H = l\sqrt{M}\) the CGAP becomes convergent for an arbitrary \(k\) except \(k = 0\).

Next let us consider the quantum generalized affine parameter (QGAP). The above classical analysis is very suggestive in the sense that the CGAP to the singularity is convergent for a timelike curve, and divergent for a null or spacelike
curve. Perhaps in taking account of quantum effects the geometry would fluctuate violently particularly near the singularity so that a classically timelike curve may effectively become null or spacelike by which we have a possibility of having the divergent QGAP for an arbitrary curve. Later we will see that this attractive speculation is indeed the case except in the neighborhood of $k = 0$.

In case of quantum theory, if we consider the same path (15) as in classical theory, by using Eq.(10) we can rewrite Eq.(16) to be

$$\gamma = \frac{2k}{T} \int dr \frac{1}{r} \pi_M = -i \frac{2k}{T} \int dr \frac{1}{r} \frac{\partial}{\partial M}. \quad (18)$$

From Eqs.(1) and (17) the QGAP from a point inside the horizon to the singularity is given by

$$|\alpha|_q = \int_0^1 dt \sqrt{\langle V^\mu e^0_\mu (\cosh \gamma - k \sinh \gamma, 0, -\sinh \gamma + k \cosh \gamma) \rangle^2} \quad (19)$$

where $\gamma$ is now given by Eq.(18), and the expectation value $\langle F \rangle$ for some operator $F$ is defined as

$$\langle F \rangle := \frac{1}{\int dM \psi^\dagger(M) \psi(M)} \int dM \psi^\dagger(M) F \psi(M), \quad (20)$$

where $\psi(M)$ denotes the physical quantum state under the gauge $\phi = r$ satisfying the WDW equation (12). For simplicity, we shall confine ourselves to the case of the physical state (13) with $C = 0$, since $C \neq 0$ case can be treated in a similar way without changing the essential results.

As a first step, it is convenient to consider $\langle e^{2\gamma} \rangle$. From Eqs.(13) and (18) one obtains

$$e^{2\gamma} \psi(M) = e^{-i \frac{4k}{T} \alpha \log r} \psi(M), \quad (21)$$

where $\alpha$ is given by Eq.(14). Therefore from the definition of the expectation value
one gets
\[ < e^{2\gamma} > = e^{-i4\alpha \log r}. \] (22)

To have non-zero value in Eq.(22) near \( r \approx 0 \), it is necessary that \( \alpha \) should be a pure imaginary number according to the Riemann-Lebesgue lemma. Hence from Eq.(14) the integration constant \( A \) needs to be expressed as \( A = ia \) with \( a \) being real number satisfying the inequality \( |a| \geq \frac{4}{l^2} \).

Now it is easy to calculate the QGAP to the singularity \( r = 0 \). The result is
\[
|\alpha|_q \approx \int_0^0 dr \frac{1 - |ka|}{\sqrt{2M}} \sqrt{< e^{i|ka|2\gamma} >}
= \int_0^0 dr \frac{1 - |ka|}{\sqrt{2M}} r^{-|ka|/4} t^2 \left[ 1 + \frac{|a|}{a} \sqrt{1 - \left( \frac{4}{a^2} \right)^2} \right].
\] (23)

It is obvious to see that the QGAP to the singularity becomes convergent for the curve with \( |k| < \frac{4}{l^2|a|} \left[ 1 + \frac{|a|}{a} \sqrt{1 - \left( \frac{4}{a^2} \right)^2} \right] \). On the other hand, it does divergent for the curve with \( |k| \geq \frac{4}{l^2|a|} \left[ 1 + \frac{|a|}{a} \sqrt{1 - \left( \frac{4}{a^2} \right)^2} \right] \). In particular, when \( a = -\frac{4}{l^2} \) we obtain exactly the same result as the one in classical theory which suggests no quantum correction in this specific case. This can also be confirmed by checking the fact that the physical observable \( \pi_M \) commuting with the Hamiltonian constraint (11) is equal in both classical and quantum theory just at this value. Now an important point is that for a sufficiently large \( |a| \) the domain of \( |k| \) of divergent QGAP near the singularity is much larger than the one of convergent QGAP. In the limit of \( |a| \to \infty \), the QGAP to the singularity would become divergent for any curve except \( k \neq 0 \): the singularity is infinitely far away because of the strong quantum fluctuation of geometry. However, if we take the limit of \( |a| \to \infty \), the physical state would approach zero. We consider a sufficiently large \( |a| \) but not infinity to have a well-defined physical state. Then it is concluded that the QGAP to the
singularity is divergent for almost any path except in the neighborhood of \( k = 0 \) as mentioned before.

There remains a question of what becomes of paths having small \( |k| \) near zero. Obviously as demonstrated in the above, if \( |k| \) is small the QGAP is finite which means that \( r = 0 \) is still a singularity according to the “classical” definition of the singularity because one has at least one causal curve and thus is bundle-incomplete (b-incomplete) \([1],[2],[3],[4]\). For this problem, the present authors shall take the following attitude. In classical relativity, the spacetime is said to be singular if at least one causal curve is b-incomplete. However, it seems to us that in quantum gravity “at least one causal curve” should be replaced by “a generic curve” since a single curve has a measure zero contribution to the path integral when we quantize the particle trajectory \([3]\).

Here we would like to provide a circumstantial evidence which supports our argument. First, Eqs.\((3)\) and \((15)\) give us the relation \( k \approx \theta \sqrt{M \log r} \) near the singularity \( r = 0 \) for the path \((15)\). Thus by this equality the uncertainty \( \Delta k \) is related to that of mass like \( \frac{\Delta M}{\sqrt{M \log r}} \) and therefore, near \( r \approx 0 \) the mass fluctuation \( \Delta M \) would be huge provided that \( \Delta k \) remains finite. This picture that the mass fluctuation near the singularity \( r = 0 \) becomes large seems to be plausible from physical viewpoint. In other words, we are not able to fix the value of \( k \) to be a certain definite value owing to the quantum fluctuation

Next one expects the Lorentz boost to be so large near the classical singularity that the paths around the null trajectory \((|k| = 1)\) would be most relevant to the contribution for the CGAP. Although the CGAP converges for such paths but \(|k| = 1\), the QGAP tends to diverge if taking a sufficiently large \(|a|\).

Before closing this section, let us make a comment on the QGAP to the event horizon from some point inside the horizon. Its evaluation is straightforward, and is finite for all paths when we take the same physical state adopted in the analysis of the QGAP to the singularity. Thus the arguments done so far imply that in
taking quantum effects into consideration the singularity is far away while the event horizon is not, hence the classical singularity is not the “real” singularity in quantum gravity.

3. (3+1)-dimensional Schwarzschild black hole

We now turn our attention to the more physically interesting Schwarzschild black hole in (3+1)-spacetime dimensions. In this case we can follow almost the same arguments as in the (2+1)-dimensional black hole. Since it is easily proved that the CGAP is finite both to the singularity and to the event horizon, let us concentrate on the analysis of the QGAP. The spacetime metric we now consider is

$$ds^2 = -\frac{1}{-1 + \frac{2M(r)}{\phi(r)}} dr^2 + \left(-1 + \frac{2M(r)}{\phi(r)}\right) dt^2 + \phi(r)^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$= -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2,$$  \hspace{1cm} (24) Where the vierbeins take the form

$$e^0 = \frac{1}{\sqrt{-1 + \frac{2M(r)}{\phi(r)}}} dr = \frac{1}{\sqrt{f(r)}} dr,$$

$$e^1 = \sqrt{-1 + \frac{2M(r)}{\phi(r)}} dt = \sqrt{f(r)} dt,$$

$$e^2 = \phi(r) d\theta,$$

$$e^3 = \phi(r) \sin \theta d\varphi,$$  \hspace{1cm} (25)

where as in the (2+1)-dimensional case we have taken the dynamical variables to be the black hole mass $M(r)$ and the area radius $\phi(r)$. The variable $\phi(r)$ might correspond to the Teichmüller parameter describing the ratio of the area of $S^2$ and the circumference of $S^1$ since the present spacetime has the topology $S^1 \times S^2$. In addition we have used the same symbol $f(r)$ to denote $-1 + \frac{2M(r)}{\phi(r)}$ now instead of $M(r) - \frac{\phi(r)^2}{l^2}$ in three dimensions in the section 2.
Then the torsion free condition (4) yields

\[
\begin{align*}
\omega^0_1 &= \omega^1_0 = \frac{\dot{f}}{2\sqrt{f}}e^1 \\
\omega^0_2 &= \omega^2_0 = \frac{\dot{\phi}\sqrt{f}}{\phi}e^2 \\
\omega^0_3 &= \omega^3_0 = \frac{\dot{\phi}\sqrt{f}}{\phi}e^3 \\
\omega^2_3 &= -\omega^3_2 = -\frac{\cot \theta}{\phi}e^3 \\
\text{otherwise} &= 0.
\end{align*}
\]  

(26)

And the curvature 2-form is of the form

\[
\begin{align*}
R^0_1 &= \frac{1}{2} \dot{f} e^0 e^1 \\
R^0_2 &= \frac{d}{dr}(\dot{\phi}\sqrt{f}) \frac{\sqrt{f}}{\phi} e^0 e^2 \\
R^0_3 &= \frac{d}{dr}(\dot{\phi}\sqrt{f}) \frac{\sqrt{f}}{\phi} e^0 e^3 \\
R^1_2 &= \frac{\dot{\phi}}{2\phi} e^1 e^2 \\
R^1_3 &= \frac{\dot{\phi}}{2\phi} e^1 e^3 \\
R^2_3 &= (\frac{1}{\phi^2} + \frac{\dot{\phi}^2 f}{\phi^2}) e^2 e^3,
\end{align*}
\]

(27)

and the other components of \( R^a_{\ b} \) vanish. By using these expressions the classical action, the Einstein-Hilbert action, given by

\[
S = -\frac{1}{16\pi G} \int \frac{1}{4} \varepsilon_{abcd} e^a e^b R^{cd},
\]

(28)

can be recast to

\[
\begin{align*}
S &= \int dr \ L \\
&= \int dr \frac{T}{4}(-1 + \dot{\phi}\phi + \dot{\phi}^2 f),
\end{align*}
\]

(29)
where $\varepsilon_{abcd}$ denotes the totally antisymmetric symbol with $\varepsilon_{0123} = +1$ and the integration with respect to $t$ has now done from $-\frac{T}{2}$ to $\frac{T}{2}$. The equations of motion derived from the reduced action (29) have in fact the solution $\phi = r$ and $M = \text{constant}$ corresponding to the well-known Schwarzschild black hole.

The canonical quantization is carried out by following the procedure in the last section. The canonical conjugate momenta now have the form

$$\pi_M = \frac{T}{2} \phi, \quad \pi_\phi = T\left(\frac{1}{2} \dot{M} - \frac{1}{2} \dot{\phi}\right),$$

and the Hamiltonian is of the form

$$H = \frac{T}{4} + \frac{2}{T} \pi_\phi \pi_M + \frac{1}{T} \pi_M^2,$$

which leads to the Wheeler-DeWitt (WDW) equation given by

$$0 = H\psi(M, \phi) = \left(\frac{T}{4} - \frac{2}{T} \frac{\partial^2}{\partial M \partial \phi} - \frac{1}{T} \frac{\partial^2}{\partial M^2}\right)\psi(M, \phi).$$

The general solutions for the WDW equation (32) can be searched by the method of separation of variables as done in the previous section. From the method, the following solution can be obtained

$$\psi(M, \phi) = B \left(e^{\alpha M} - Ce^{\beta M}\right)e^{\frac{1}{2}\frac{T}{A}\phi},$$

with $A$, $B$, and $C$ being integration constants, and $\alpha$ and $\beta$ are now defined by

$$\alpha = -\frac{T A}{2} - \frac{T}{2} \sqrt{1 + A^2},$$

$$\beta = -\frac{T A}{2} + \frac{T}{2} \sqrt{1 + A^2}.$$
of the curve (15) let us consider the following curve
\[ e^1 = ke^0, \ e^2 = e^3 = 0, \] (35)
whose causal property is determined by \( k \) as in three dimensions. Under the gauge choice \( \phi = r \), this time \( \gamma \) is defined as
\[ \gamma := \int \omega^1 0 = \int \frac{\dot{f}}{2\sqrt{f}} e^1 = \int \frac{\dot{f}}{2\sqrt{f}} ke^0 \]
\[ = -\frac{k}{T} \int dr \frac{1}{r} \pi_M - \frac{kT}{4} \int dr \frac{1}{2M - r} \pi_M^{-1}, \] (36)
where the Hamiltonian constraint \( H \approx 0 \) was used in deriving the last equation. Again for simplicity, we restrict our consideration to the physical state (33) with \( C = 0 \) without loss of generality. After setting \( A = ia \ (a \in \mathbb{R}) \), from Eqs.(33) and (36), one obtains
\[ e^{2\gamma} \psi(M) = e^{i\frac{2k}{T} \alpha \log r + i\frac{kT}{2a} \log(2M-r)} \psi(M), \] (37)
where \( \alpha \) is given by
\[ \alpha = -i \frac{a + \sqrt{a^2 - 1}}{2} T, \] (38)
with \( |a| \geq 1 \). At this stage, we remark on a subtlety which does not exist in comparison with (2+1)-dimensional black hole. Namely in evaluating (37), we have nontrivial commutation relations between \( 2M \) and \( \pi_M \) or \( \pi_M^{-1} \), which give rise to additional terms in the right hand side of Eq.(37). In this article we have neglected their contributions. More delicate evaluation of them will be certainly needed. Thus from Eqs.(20), (37) and (38) the expectation value of \( e^{2\gamma} \) can be obtained as follows:
\[ < e^{2\gamma} > = e^{k(a + \sqrt{a^2 - 1}) \log r} \frac{1}{\int dM} \int dM e^{-k(a - \sqrt{a^2 - 1}) \log(2M-r)}. \] (39)
Here it is necessary to define the inner product more precisely. Since we are taking account of only the interior region of the event horizon of the black hole, we shall
take the integration region from \( \frac{r}{2} \) to the cutoff of large mass \( M_0 \) in order to keep the norm finite. Under this definition of the inner product, it is easy to calculate the last integration in the right hand side of Eq.(39). The result is

\[
\int_{\frac{r}{2}}^{M_0} dM e^{-k(a-\sqrt{a^2-1})\log(2M-r)} \left. \frac{1}{1 - k(a - \sqrt{a^2 - 1})} x^{1-k(a-\sqrt{a^2-1})} \right|_{x=2M_0}^{x=0},
\]

(40)

where we have performed a change of variable \( x = 2M - r \). The important point is that this integration is finite at \( x = 0 \), i.e., \( r = 2M \) when \( k \) satisfies the inequality

\[
k(a - \sqrt{a^2 - 1}) \leq 1.
\]

(41)

This inequality holds for almost all positive \( k \) if \( |a| \) is sufficiently large. The sign of \( k \) has a simple physical interpretation. Imagine a particle approaching the event horizon or the singularity along the path (35). Substituting (25) into (35), one sees that \( k \) is positive for a path approaching the event horizon, on the other hand, negative for the one to the singularity. For the time being let us assume that the inequality (41) is satisfied.

Now we would like to examine the behavior of the QGAP to the singularity. To do so, it is sufficient to consider only the potentially divergent part of \( \langle e^{2\gamma} \rangle \). We set

\[
\langle e^{2\gamma} \rangle \sim e^{k(a+\sqrt{a^2-1})\log r}.
\]

(42)

Then the QGAP to the singularity can be calculated in a perfectly similar manner.
to the case of 2+1 dimensional black hole. We obtain

$$|\alpha|_q \approx \int_0^1 dt \frac{dr}{dt} \left| \frac{\sqrt{f}}{\sqrt{2}} \right| \sqrt{e^{\frac{|\alpha|}{2}}}$$

$$\propto \int_0^0 dr \sqrt{r} r^\frac{1}{2} |a| |(1 + \frac{|\alpha|}{a} \sqrt{1 - \frac{1}{a^2}} )|$$

$$= \frac{1}{\frac{1}{2} \left[ 3 + k|a|(1 + \frac{|\alpha|}{a} \sqrt{1 - \frac{1}{a^2}} ) \right]}.$$  

Therefore, the QGAP to the singularity diverges if $3 + k|a|(1 + \frac{|\alpha|}{a} \sqrt{1 - \frac{1}{a^2}} ) < 0$
which is satisfied for almost all negative $k$ when $|a|$ is sufficiently large. Moreover, it is easy to show that by using the same physical state the QGAP to the event horizon converges when $|a|$ is so large. At this point, it is remarkable to notice that the QGAP is infinite to the singularity while the QGAP is strictly finite to the event horizon by taking the same quantum state with very large $|a|$. In this way, we have shown that the “classical” singularity $r = 0$ in the Schwarzschild black hole is not a singularity in quantum gravity in the sense of the conventional definition of a singularity [2], [3], [4].

4. Conclusion

In this article, we have shown that the singularities in both three dimensional BTZ balck hole [6] and four dimensional Schwarzschild black hole can be smeared and is infinitely far away if we take an effect of quantum gravity in the specific minisuperspace models. Of course, we do not intend to claim that we have proven a smearing, as a result, a disappearance of the classical singularities in quantum gravity. However, our model seems to reflect essential characteristic features of quantum black holes so that our investigation strongly suggests that the present analysis can be generalized to other minisuperspace models and even to full quantum gravity.
For further development, it would be very interesting to couple various matter fields to the present model and to apply for a proof of the strong cosmic censorship which is now under investigation. And recently, Horowitz and Marolf [8] discussed a possibility of self-adjoint extension of the Laplacian operator in curved spacetimes to a classically singular point at which geodesic is incomplete. If the extension is possible, the singularity is smeared out in the context of quantum field theory in curved spacetime. It might be also interesting to examine the relation between our formalism and theirs.

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