Gravitational Collapse of a Rotating Cylindrical Null Shell in the Cosmic String Spacetime

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Abstract

We study the gravitational collapse of a rotating cylindrical null shell with flat interior and the metric of a spinning cosmic string as the exterior. We see that there is a critical radius, where the energy density of the shell vanishes and beyond which it would be negative, thereby signaling that the matching would be unphysical.

keywords: gravitational collapse; null shell; closed timelike curves.

1 Introduction

Relativistic dynamics of cylindrical symmetric thin shells as sources of gravitational field have been studied during the recent development of general relativity. Such objects are used as idealized models to investigate non-spherical gravitational collapse and the non-linearity of the field equations. A number of papers have been concerned with stationary and non-stationary rotating cylindrical shells in general relativity (see Refs.\([1, 2]\) and references therein). In particular, by considering the collapse of a cylindrical shell made of pressure-free counter-rotating particles in vacuum, the authors showed that even a small amount of rotation can halt the collapse at some minimal non-zero radius.\([3]\) Recently, Mena et al.\([4]\) considered the gravitational collapse of a rotating cylindrical shell in vacuum. Using the matching conditions between a Minkowski interior and the spinning cosmic string exterior through a collapsing, rotating cylindrical shell of null dust, they found that the shell with positive energy density bounces before closed timelike curves (CTCs) can be created. However, it should be noted that they examined the matching conditions across a timelike shell and so applied the Darmois-Israel formalism in their paper.

In this paper we consider the configuration studied in Ref.\([4]\) from a different perspective by looking at a rotating shell whose history is a null hypersurface. For this purpose we use Barrabès-Israel null shell formalism\([5]\) (see also Ref.\([6]\) and references therein) to investigate the matching conditions.

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Natural geometrized units, in which $G = c = 1$ are used throughout the paper. The null hypersurface is denoted by $\Sigma$. The symbol $|\Sigma|$ means “evaluated on the null hypersurface”. We use square brackets $[F]$ to denote the jump of any quantity $F$ across $\Sigma$. Latin indices range over the intrinsic coordinates of $\Sigma$ denoted by $\xi^a$, and Greek indices over the coordinates of the 4-manifolds.

2 Null Shell Formalism

Consider the gravitational collapse of a rotating cylindrical shell in vacuum. We imagine that during the late stages of the collapse, the shell tends to fall at nearly the speed of light such that the history of the shell coincides with a null hypersurface $\Sigma$. Taking spacetime to be flat Minkowski inside the shell ($\mathcal{M}_-$), we write the metric there as

$$ds_-^2 = -dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2,$$

in terms of the Einstein-Rosen canonical cylindrical coordinates $(t, \rho, \varphi, z)$. For the exterior vacuum spacetime of the shell ($\mathcal{M}_+$), we take the metric of a spinning cosmic string in the following form\cite{4, 7}

$$ds_+^2 = -(dT + m d\Phi)^2 + C^2 dr^2 + r^2 d\Phi^2 + dz^2,$$

expressed in the same coordinate $z$ but in terms of distinct coordinates $T, r$ and $\Phi$ in general. The parameters $C$ and $m$ are associated with the mass per unit length and angular momentum per unit length of the string, respectively, such that for positive mass $C > 1$. An inspection of Eq. (2) shows that the region for which $r < m$ contains CTCs.

As seen from $\mathcal{M}_+$, one can obtain the description of the hypersurface by solving the Euler-Lagrange equations. Defining

$$L \equiv \frac{1}{2} \left( -(\dot{T} + m \dot{\Phi})^2 + C^2 r^2 + r^2 \dot{\Phi}^2 \right),$$

where $\dot{q}$ denotes $\frac{dq}{d\lambda}$, with $\lambda$ being an affine parameter on the null generators of the hypersurface, the Euler-Lagrange equations are written as

$$\frac{\partial L}{\partial q} - \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{q}} = 0,$$

where $q$ represents the coordinates $x^\mu_+ = (T, r, \Phi, z)$. Now, equations for the null geodesics are found as\cite{3}

$$\frac{\partial L}{\partial T} = -E \quad \Rightarrow \quad \dot{T} + m \dot{\Phi} = E,$$

$$\frac{\partial L}{\partial \Phi} = K \quad \Rightarrow \quad r^2 \ddot{\Phi} = K + mE,$$

$$L = 0 \quad \Rightarrow \quad C^2 r^2 \dot{r}^2 = E^2 r^2 - (K + mE)^2,$$

where $E$ and $K$ are constants, and it is assumed that the null geodesics are future-directed ($E > 0$) and rotating in the positive direction such that $\dot{\Phi} > 0$. From (7) it is seen that the shell always has a turning point located at

$$b = \frac{K}{E} + m.$$
Solving the Eqs. (5)-(7), we get
\[
\frac{dr}{dT} = -\frac{r\sqrt{r^2 - b^2}}{C(r^2 - mb)}
\] (9)

Integrating Eq. (9) gives the equation of \( \Sigma \) in \( \mathcal{M}_+ \) as
\[
v_+ \equiv T + r^* = 0
\] (10)

where \( r^* = \int \frac{C(r^2 - mb)}{r\sqrt{r^2 - b^2}} \, dr = C\sqrt{r^2 - b^2} - mc \sec^{-1} \left( \frac{r}{b} \right) \). From the Eqs. (5)-(7), we can obtain the tangent-normal vector of \( \Sigma \) as viewed from \( \mathcal{M}_+ \)
\[
n_+^\mu \equiv \frac{dx_+^\mu}{d\lambda} = \left( \frac{E(r^2 - mb)}{r^2}, -\frac{E\sqrt{r^2 - b^2}}{Cr}, \frac{bE}{r^2}, 0 \right) |_{\Sigma}.
\] (11)

We also note that the integration of \( \frac{dr}{d\lambda} = -\frac{E}{Cr} \sqrt{r^2 - b^2} \) yields
\[
\lambda = -\frac{C}{E} \sqrt{r^2 - b^2} |_{\Sigma}.
\] (12)

Furthermore, from Eqs. (6) and (7) one gets
\[
\frac{d\Phi}{dr} = \dot{\Phi} = \frac{-bC}{r \sqrt{r^2 - b^2}} |_{\Sigma}.
\] (13)

Integration of Eq. (13) leads to
\[
\psi = \Phi + C \sec^{-1} \left( \frac{r}{b} \right) |_{\Sigma},
\] (14)

where \( \psi \) is a constant on the null generators. We thus take \( \xi^a = (\lambda, \psi, z) \) as the intrinsic coordinates on \( \Sigma \), and as these are well adapted to the generators we can form the tangent basis vectors \( e_\lambda^\mu = n^\mu, \ e_\psi^\mu = \delta_\psi^\mu, \) and \( e_z^\mu = \delta_z^\mu \). Now, by virtue of \( n_\mu e_\psi^\mu |_{\Sigma} = 0 \), we get \( K = 0 \), so that Eq. (8) reduces to the form \( b = m \). The shell’s intrinsic metric induced from \( \mathcal{M}_+ \) is found as
\[
ds^2 |_{\Sigma} = \frac{\lambda^2 E^2}{C^2} d\psi^2 + dz^2.
\] (15)

The null hypersurface \( \Sigma \) as seen from \( \mathcal{M}_- \) is described by the parametric equations
\[
t + \rho \equiv v_- = \text{const},
\]
\[
\rho = -\lambda,
\]
\[
\varphi = \psi,
\]
\[
z = z.
\] (16)

The tangent-normal vector to \( \Sigma \) as viewed from \( \mathcal{M}_- \) is
\[
n_-^\mu \equiv \frac{dx_-^\mu}{d\lambda} = (1, -1, 0, 0) |_{\Sigma}.
\] (17)
It is then seen that the induced metric on $\Sigma$ from $M_-$ is given by
\[ ds_\Sigma^2 = \lambda^2 d\psi^2 + dz^2. \] (18)

Now from (15) and (18), the requirement of continuity of the induced metric on $\Sigma$ yields the following matching condition
\[ E = C. \] (19)

We may then complete the basis by a transverse null vector $N^\mu$ uniquely defined by the four conditions $n_\mu N^\mu = -1$, $N_\mu e_A^\mu = 0$ ($A = \psi, z$), and $N_\mu N^\mu = 0$. We find for both sides
\[
N_\mu |_- = \frac{1}{2} (-1, +1, 0, 0) |_{\Sigma}, \tag{20}
\]
\[
N_\mu |_+ = \frac{1}{2C} \left( \frac{-r^2}{r^2 - m^2}, \frac{Cr}{\sqrt{r^2 - m^2}} , 0, 0 \right) |_{\Sigma}. \tag{21}
\]

To proceed further, we here need to define a pseudo-inverse of the induced metric $g_{ab}$ on $\Sigma$ as $g^{ac} g_{bc} = \delta^a_b + n^a N_\mu e_A^\mu$, with $n^\alpha = \delta^\alpha_\lambda$[5] leading to $g^{ab} = \text{diag}(0, \frac{1}{\lambda^2}, 1)$. The final junction condition is formulated in terms of the jump in the extrinsic curvature. Using the definition $K_{ab} = e_a^\mu e_b^\nu \nabla_\mu N_\nu$, we can therefore compute the transverse extrinsic curvature tensor[5] on both sides of $\Sigma$. Its non-vanishing component on the minus side is found as
\[ K_{\psi\psi} |_- = \frac{\rho}{2} |_{\Sigma}. \] (22)

The corresponding non-vanishing components on the plus side are computed as
\[ K_{\psi\psi} |_+ = \frac{r^2}{2C^2 \sqrt{r^2 - m^2}} |_{\Sigma}, \tag{23} \]
\[ K_{\lambda\psi} |_+ = \frac{m}{C \sqrt{r^2 - m^2}} \tag{24} \]

The surface energy-momentum tensor of the lightlike shell having the null hypersurface $\Sigma$ as its history is directly related to the jump in the transverse extrinsic curvature. In the tangent basis $e_a$, it can be written in the form[8]
\[ S^{ab} = \mu n^a n^b + p g^{ab} + j^a n^b + j^b n^a, \] (25)

where
\[ \mu = -\frac{1}{8\pi} g^{ab} [K_{ab}] \] (26)
represents the surface energy density,
\[ p = -\frac{1}{8\pi} [K_{ab}] n^a n^b \] (27)
displays the isotropic surface pressure, and
\[ j^a = -\frac{1}{8\pi} g^{ac} [K_{cd}] n^d \] (28)
represents the surface current of the lightlike shell. All these surface quantities are measured by a family of freely-moving observers crossing the null hypersurface. Using the jumps in the extrinsic curvature obtained above, we first notice that the surface pressure term vanishes identically. The energy density and surface current are then calculated as

\[ \mu = \frac{(C^2 - 1)r^2 - C^2m^2}{16\pi C^2(r^2 - m^2)^{\frac{3}{2}}} |\Sigma| , \]

(29)

\[ j^\psi = \frac{m}{8\pi C(r^2 - m^2)^{\frac{3}{2}}} |\Sigma| . \]

(30)

From (29), it is seen that due to the interplay of both gravitational and centrifugal energies, which enter in the expression (29) for the energy density of the shell with opposite signs and different dependence on the shell radius, there is a critical radius where the surface energy density of the shell becomes zero. It is given by

\[ r_c = \frac{mC}{\sqrt{C^2 - 1}} . \]

(31)

For \( r < r_c \), the energy density \( \mu \) is always negative, hence, in order to avoid unphysical negative energy densities, as Dray [9] has suggested, it is more natural to expect that the shell starts expanding at the moment when \( \mu \) becomes zero, but the physical mechanism by which the shell bounces at this critical radius, is unknown. Note that with \( C > 1 \), from (31) it is clear that \( r_c > m \), meaning that the energy density of the shell starts becoming negative before the radius at which CTCs would be created, can be reached.

3 Conclusion

In this paper, we have used Barrabès-Israel null shell formalism to study the relativistic dynamics of a collapsing rotating cylindrical null shell with flat interior and a spinning cosmic string exterior spacetimes. We have seen that the energy density of the shell inevitably becomes negative at radii less than a critical radius given by (31). It turns out that this radius is larger than the radius at which CTCs can be formed in the exterior. Hence, this idealized model, as a test bed for the cosmic censorship conjecture, due to the lack of a physically reasonable distributional matter content on the shell does not rule out this idea. It is remarkable that how our results differ from those of reported in Ref. [4] for a collapsing shell of null dust whose history is a timelike hypersurface and according to the Darmois-Israel matching conditions this shell with the positive definite energy density bounces before CTCs arise.

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