ON THE SCALE-FREENESS OF RANDOM COLORED SUBSTITUTION NETWORKS

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Abstract. Extending previous results in the literature, random colored substitution networks and degree dimension are defined in this paper. The scale-freeness of these networks is proved by introducing a new definition for degree dimension that is associated with Lyapunov exponents. The random colored substitution network hence turns out to be a simple, powerful and promising model to generate random scale-free networks.

Many real-life phenomena are fractals in nature, including growing networks found in biology, brain connections and in social interactions. Previous researchers introduced a mathematical model called substitution networks, to simulate the growth of the networks by iteratively replacing each arc of a network by smaller networks. This model was later expanded by the introduction of arc colors to allow more types of arc replacements. However, these models are deterministic and do not allow for the randomness that real-life growth networks can exhibit. To capture this randomness, we expand the model to what we call random colored substitution networks, by allowing each arc to be replaced by a random choice of network. We describe the properties of the randomly resulting networks, including their number of nodes and arcs and their node degrees. Our main result shows that these random colored substitution networks are almost surely scale-free and that they therefore have a particular type of structure.

1. Introduction

The property of scale-freeness of complex networks was first proposed in 1999 by Albert-Laszlo Barabasi and Reka Albert [2]. Their model presents a graph that grows by the addition of new nodes and their incident edges to existing nodes. The probability that a new node is chosen to be adjacent to an old node depends in this model on the degree of the old node. This model is shown to be scale-free, which refers to the phenomenon of networks having node degrees that obey the power-law distribution. In particular, the fraction $P(k)$ of nodes in the network $G$ that are adjacent to $k$ other nodes is

$$P(k) \sim k^{-\delta}$$

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where \( \sim \) denotes approximation and \( \delta \) is some constant. Scale-free networks have appeared, among other places, in studies on biology \[14,9,15,16\], finance \[7,10,12\] and computer science \[17,21\].

Barabasi and Albert’s model is suitable for modelling many real-world complex networks. However, for modelling growth processes that evolve in fractal-like ways, a more suitable type of complex network model is the substitution network. These networks were first introduced by Xi et al. \[19\] and are models featuring networks whose arcs are at each step replaced by some fixed network. An example of the first four steps of such a substitution network is given in Fig. 1.

![Figure 1. A substitution network](image)

Xi et al. \[19\] proved that substitution networks have the scale-free property, as well as the related fractality property. Li et al. \[13,14\] proved that substitution networks retain the scale-free property in the more general setting of colored arcs and fixed networks to replace each arc of a given color. They also showed that certain of these colored substitution networks have the fractality property. Similarly, substitutions of nodes, rather than arcs, were considered in \[18,22\]. By constructing self-similar networks, Yao et al. \[22\] proved that node substitution networks have the fractality property; see also \[20,23,24\].

Although these substitution networks are suitable for modelling many real-world fractal-like phenomena, they are deterministic and do not reflect real-life randomness.

The purpose of this present paper is to address this limitation. In particular, random colored substitution networks and their degree dimension are defined, and the main result of the paper, Theorem 5.5, extends the scale-freeness of substitution networks to random colored substitution networks.

2. Random colored substitution networks

Consider a directed network \( G^0 \) whose arcs are each colored by one of \( \lambda \) colors. Replace each arc of \( G^0 \) according to its color as follows: if arc \( e \) has color \( i \), then \( e \) will be replaced randomly by a directed network \( R_{ik} \) with probability \( p_{ik} \), among \( q_i \) such directed networks (so \( \sum_{k=1}^{q_i} p_{ik} = 1 \) for each \( i \)). Each directed network \( R_{ik} \) has a node \( A \) and a node \( B \) that respectively replace the beginning node \( A \) and ending node \( B \) of \( e \); this determines exactly how \( R_{ik} \) replaces \( e \). By replacing all arcs in \( G^0 \) randomly by the directed networks \( R_{ik} \), a directed network \( G^1 \) is obtained. This replacement process iteratively defines a directed network \( G^2 \) from \( G^1 \), a directed network \( G^3 \) from \( G^2 \), and so on. After \( t \) such iterations, a directed network \( G^t = (V(G^t), E(G^t)) \) is obtained. These graphs \( R_{ij} \) are called rule graphs.
This iterative process and the resulting directed networks together form a random colored substitution network. Throughout this paper, let $\mathcal{G}$ be the family of all possible sequences $\Gamma = (G^0, G^1, G^2, \ldots)$. When such a sequence $\Gamma$ converges to a network, then we can identify $\Gamma$ as that network, together with the information on how it was generated. That is, $\lim_{t \to \infty} G^t = \Gamma$.

An example of random colored substitution networks with arcs of $\lambda = 2$ colors arcs is given in Fig. 2. The red $(i = 1)$ arcs are each replaced at each step by the directed network $R_{11}$ with probability $\frac{1}{3}$ and the directed network $R_{12}$ with probability $\frac{2}{3}$. Each blue $(i = 2)$ arc is replaced either by $R_{21}$ or by $R_{22}$, with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively. One possible sequence $G^0, G^1, G^2, \ldots$ of the random colored substitution network is shown.

Note that (non-random) colored substitution networks form the particular sub-class of random colored substitution networks for which $q_i = 1$ for all $i$. Note also that it will be assumed in this paper that, for each color $i$, there is at least one integer $k$ and one integer $k'$ such that the distance between nodes $A$ and $B$ in the network $R_{ik}$ is greater than 1 and that, in $R_{ik'}$, the sum of the in-degree and the out-degree of at least one of the nodes $A$ and $B$ is greater than 1.

Remark 2.1. These conditions ensure that the number of nodes in the substitution network and their degrees grow to infinity; this is proved in Section 7.

![Figure 2. A random colored substitution network](image)

3. Defining the scale-free property and the degree dimension

For any undirected graph $G = (E(G), V(G))$, let $\Delta(G)$ be the maximal degree of any node $v$ of $G$ and define the normalised degree of any node $v \in V$ to be $\hat{\deg}_G(v) = \frac{\deg_G(v)}{\Delta(G)}$. This definition is extended to directed networks $G$ by letting $\deg_G(v)$ be given by the underlying undirected graph of $G$; that is, the graph obtained from $G$ by ignoring arc directions.
**Definition 3.1.** Consider a network sequence \((G^0, G^1, G^2, \ldots) \in G\) that converges to a network limit \(\Gamma\). A node \(v \in V(\Gamma)\) is **stationary** if the limit \(\lim_{t \to \infty} \hat{\deg}_{G^t}(v)\) exists, in which case, denote it by \(\hat{\deg}_\Gamma(v)\). The network sequence \(\Gamma\) is (almost) **stationary** if (almost every) node of \(\Gamma\) are stationary. We will only consider almost stationary network sequences \(\Gamma \in G\) in this paper.

Now for each positive real number \(\ell\), define

\[
P_\ell(\Gamma) = \left| \{ v \in V(\Gamma) : \hat{\deg}_\Gamma(v) = \ell \} \right|.
\]

**Definition 3.2.** Define

\[
\dim_D(\Gamma) = \limsup_{\ell \to 0} \frac{\log P_\ell(\Gamma)}{-\log \ell}
\]

where the limit is taken over all values of \(\ell\) such that \(P_\ell(\Gamma) > 0\). The graph \(\Gamma\) is **scale-free** if and only if, taking the limit \(\ell \to 0\) for all \(\ell \) such that \(P_\ell(\Gamma) > 0\),

\[
\dim_D(\Gamma) = \lim_{\ell \to 0} \frac{\log P_\ell(\Gamma)}{-\log \ell}
\]

exists and is positive, in which case the limit is called the **degree dimension** of \(\Gamma\).

**Remark 3.3.** It is interesting, and potentially useful, to note that this definition can be used to define the scale-free property of many other infinite networks besides those arising from random colored substitution networks.

**Lemma 3.4.** If \(\Delta(\Gamma) < \infty\), then \(\dim_D(\Gamma)\) does not exist. If \(|V(\Gamma)| < \infty\), then \(\dim_D(\Gamma) = 0\).

**Proof.** If \(\Delta(\Gamma) < \infty\), then no node has normalised degree \(\ell\) for any \(\ell < \frac{1}{\Delta(\Gamma)}\). Given that \(P_\ell(\Gamma)\) has to be positive, such limit does not exist accordingly. If \(|V(\Gamma)| < \infty\), then \(P_\ell(\Gamma) \leq |V(\Gamma)| < \infty\). It follows that \(\lim_{\ell \to 0} (\log P_\ell(\Gamma))/(-\log \ell) = 0\).

Therefore, \(\Gamma\) is scale-free only when \(\Delta(\Gamma) = \infty\) and \(|V(\Gamma)| = \infty\). By Remark 2.1, the substitution networks in this paper will all feature networks \(\Gamma\) with infinitely many arcs and \(\Delta(\Gamma) = \infty\).

**Notation 3.5.** Let \(\sim\) be defined as asymptotic equivalence. Write \(f(x) \xrightarrow[x \to x_0]{} g(x)\) whenever \(\lim_{x \to x_0} f(x)/g(x) = c\) for some constant \(c > 0\).

**Lemma 3.6.** \(\Gamma\) is scale-free if, taking the limit \(\ell \to 0\) for all \(\ell > 0\) such that \(P_\ell(\Gamma) > 0\),

\[
P_\ell(\Gamma) \xrightarrow[\ell \to 0]{} \ell^{-\dim_D(\Gamma)}.
\]

**Proof.** (3.3) implies \(\dim_D(\Gamma) = \lim_{\ell \to 0} \frac{\log c + \log P_\ell(\Gamma)}{\log \ell} = \lim_{\ell \to 0} \frac{\log P_\ell(\Gamma)}{\log \ell}\).

**Notation 3.7.** For all \(t\), define

\[
P_L(G^t) = \left| \{ v \in V(G^t) : \deg_{G^t}(v) = L \} \right|.
\]
Theorem 3.8. If $|V(G^{t})| \xrightarrow{t \to \infty} \Delta(G^{t})$, then $\Gamma$ is scale-free with degree dimension $\delta = \dim_D(\Gamma)$ if, for all functions $L : \mathbb{N} \to \mathbb{N}$ satisfying $L(t)/\Delta(G^{t}) \xrightarrow{t \to \infty} 0$, 
\begin{equation}
\frac{P_L(t)(G^{t})}{|V(G^{t})|} \xrightarrow{t \to \infty} L(t)\delta.
\end{equation}

Proof. Suppose that Condition (3.4) holds. Let $L(t)$ be any function on $\mathbb{N}$ satisfying $L(t)/\Delta(G^{t}) \xrightarrow{t \to \infty} 0$ and define the function $\ell(t)$ by $\ell(t) = L(t)/\Delta(G^{t})$. Then
\begin{equation}
\frac{P_L(t)(G^{t})}{|V(G^{t})|} \xrightarrow{t \to \infty} \ell(t)^{-\delta} \Delta(G^{t})^{-\delta}.
\end{equation}

Now $\Delta(G^{t}) \xrightarrow{t \to \infty} |V(G^{t})|$, so $P_L(t)(G^{t}) \xrightarrow{t \to \infty} \ell(t)^{-\delta}$. Note that a sequence converges if and only if every subsequence of it converges. Hence, if all possible sequences $L(t)$ satisfy $L(t)/\Delta(G^{t}) \xrightarrow{t \to \infty} 0$ and Condition (3.4), then Condition (3.3) will hold. Therefore,
\[P_L(\Gamma) \xrightarrow{\ell \to 0} 0 \leq \ell^{-\delta},\]
and so $\log P_L(\Gamma) \xrightarrow{\ell \to 0} -\log \ell \delta$. Therefore, $\Gamma$ is scale-free with $\dim_D(\Gamma) = \delta$. \qed

Proposition 3.9. Let $G_1$ and $G_2$ be two subnetworks of $G$. Then
\[\dim_D(G_1 \cup G_2) = \max \{\dim_D(G_1), \dim_D(G_2)\}.\]

Proof. For any $\ell$,
\[2 \max \{P_\ell(G_1), P_\ell(G_2)\} \geq P_\ell(G_1 \cup G_2) \geq \max \{P_\ell(G_1), P_\ell(G_2)\}.\]

As a result,
\[\lim_{\ell \to 0} \frac{\log \max \{P_\ell(G_1), P_\ell(G_2)\}}{-\log \ell} \leq \dim_D(G_1 \cup G_2) \leq \lim_{\ell \to 0} \frac{\log 2 \max \{P_\ell(G_1), P_\ell(G_2)\}}{-\log \ell},\]
while both sides converge to $\max \{\dim_D(G_1), \dim_D(G_2)\}$.

By induction, this property holds for finite unions as well. \qed

4. Stochastic substitution processes

This section introduces a mathematical framework, called a stochastic substitution process. This framework provides the results that are applied in Sections 6 and 7 to study the asymptotic properties of random colored substitution networks regarding Lyapunov exponents.

Notation 4.1. For any vector $x$, let $[x]_i$ be the $i$-th entry of $x$. For each network $G$ with arcs colored in colors $1, \ldots, \lambda$, define $\chi(G)$ to be the vector whose $j$-th entry is the number of $j$-colored arcs in $G$. Let $\|x\|_1$ denote the sum of entries in $x$. For any vectors $x$ and $y$ of equal dimension, write $x \geq y$ if $[x]_i \geq [y]_i$ for all $i$. For any real square matrix $X$, let $\rho(X)$ denote the spectral radius of $X$.

Let $\mathcal{X} = X_1, \ldots, X_N$ be a set of finitely many non-negative square matrices. The set $\mathcal{X}$, together with a probability vector $(p_1, \ldots, p_N)$ where $\Pr(X_i) = p_i$ for $i = 1, \ldots, N$, is called a random matrices set. The notation $\Pr_{\mathcal{X}}(X_i)$ will be used to denote $\Pr(X_i)$, to highlight that these probabilities are associated with $\mathcal{X}$.
Write $\mathcal{L}(\mathcal{X})$ as the maximal Lyapunov exponent of $\mathcal{X}$ defined by

$$\mathcal{L}(\mathcal{X}) := \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \log \| \mathbf{Y}_{i_1} \cdots \mathbf{Y}_{i_n} \| \right),$$

where $\mathbf{Y}_{i_k} \in \mathcal{X}$ is chosen with probability $\Pr_{\mathcal{X}}(\mathbf{Y}_{i_k})$, and where $\mathbb{E}(\cdot)$ is the expectation value. The study of asymptotic behaviours of random matrices product dates back to Bellman [3], Furstenberg and Kesten [5,6], Guivarc [8] and Le Page [11]. A famous theorem by Furstenberg and Kesten [5] asserts that $\mathcal{L}$ exists and that

$$\mathcal{L}(\mathcal{X}) \overset{a.s.}{=} \lim_{n \to \infty} \frac{1}{n} \log \| \mathbf{Y}_{i_1} \cdots \mathbf{Y}_{i_n} \| \overset{a.s.}{=} \lim_{n \to \infty} \frac{1}{n} \log |\mathbf{Y}_{i_1} \cdots \mathbf{Y}_{i_n}|_{jk}$$

if all $\mathbf{X} \in \mathcal{X}$ are primitive.

**Definition 4.2.** Let $\mathcal{X} = \{\mathbf{X}_1, \ldots, \mathbf{X}_N\}$ be a random matrices set. For each $i = 1, \ldots, m$, let $\mathbf{e}_i \in \mathbb{R}^m$ be the $i$-th standard basis unit vector of $\mathbb{R}^m$. Define a random function $T_{\mathcal{X}}^\prime : \{\mathbf{e}_1, \ldots, \mathbf{e}_m\} \rightarrow (\mathbb{Z}^+)^m$ by setting $m$ identical and independent random vectors $T_{\mathcal{X}}^\prime(\mathbf{e}_i)$, each with probability

$$\Pr(T_{\mathcal{X}}^\prime(\mathbf{e}_i) = \mathbf{e}_i \mathbf{X}_j) = p_j.$$

We decompose all $\mathbf{x} = x_1 \mathbf{e}_1 + \cdots + x_m \mathbf{e}_m \in (\mathbb{Z}^+)^m$ ($x_i \in \mathbb{Z}$) through the random function $T_{\mathcal{X}}^\prime : (\mathbb{Z}^+)^m \rightarrow (\mathbb{Z}^+)^m$ by

$$T_{\mathcal{X}}(\mathbf{x}) = \lambda \sum_{j=1}^\lambda \sum_{i=1}^n \mathbf{X}_{i_j}.$$

For simplicity, write $T_{\mathcal{X}}^n = T_{\mathcal{X}} \circ \cdots \circ T_{\mathcal{X}}$. We call such $T_{\mathcal{X}}^n(\mathbf{x})$ a stochastic substitution process. That is because the decomposition of $\mathbf{x}$ represents the independent substitution of each arc, and $T_{\mathcal{X}}^n$ indicates the result of substitution.

**Theorem 4.3.**

$$\lim_{n \to \infty} \frac{1}{n} \log \| T_{\mathcal{X}}^n(\mathbf{x}_0) \| = \mathcal{L}(\mathcal{X}) \quad a.s.$$

**Proof.** Note that $\mathbf{Y}_{i_1}, \ldots, \mathbf{Y}_{i_n}$ forms a stationary stochastic process. By Furstenberg and Kesten’s theorem [5], $\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\| T_{\mathcal{X}}^n(\mathbf{x}_0) \|)$ exists. Hence, for any $\mathbf{x}_0$,

$$\lim_{n \to \infty} \frac{1}{n} \log \| T_{\mathcal{X}}^n(\mathbf{x}_0) \| = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}(\| T_{\mathcal{X}}^n(\mathbf{x}_0) \|)$$

$$= \lim_{n \to \infty} \frac{1}{n} \log \| \mathbf{x}_0 \mathbf{Y}_{i_1} \cdots \mathbf{Y}_{i_n} \|$$

$$= \lim_{n \to \infty} \frac{1}{n} \log \| \mathbf{Y}_{i_1} \cdots \mathbf{Y}_{i_n} \| \overset{a.s.}{=} \mathcal{L}(\mathcal{X}) \quad \square$$

This theorem reveals that the growth rate of stochastic substitution process almost surely follows the Lyapunov exponent of the random matrices product.

**Lemma 4.4.** Let $\mathbf{X}$ be a primitive non-negative $n \times n$ matrix with spectral radius $\rho(\mathbf{X})$. Then, for any positive vector $\mathbf{u} \in (\mathbb{R}^+)^n$,

$$\| \mathbf{uX}^t \|_1 \overset{t \to \infty}{\asymp} \rho(\mathbf{X})^t.$$
Lemma 4.5. Let $\lambda$ be a random matrices set. If $\lambda = \{X\}$, then collect all $M$ that all matrices in $\lambda$ are primitive. Moreover, in this paper, all matrices in $\lambda$ are assumed to be primitive.

Proof. As $X$ is primitive, the Perron-Frobenius Theorem implies that the following limit matrix exists and is positive: $\lim_{t \to \infty} \left( \frac{X}{\rho(X)} \right)^t = G$. Hence,

$$\left\| u \lim_{t \to \infty} \left( \frac{X}{\rho(X)} \right)^t \right\|_1 = \|uG\|_1 = c$$

where $c > 0$ is a constant depending on $u$ and $X$. Thus, $\|uX^t\|_1 \to \infty \rho(X)^t$. □

Lemma 4.5. Let $\mathcal{X}$ be a random matrices set. If $\mathcal{X} = \{X\}$, then $\mathcal{L}(\mathcal{X}) = \log \rho(X)$.

Proof. By definition and Lemma 4.4

$$\mathcal{L}(\mathcal{X}) = \lim_{n \to \infty} \frac{1}{n} \log \|X^n\| = \lim_{n \to \infty} \frac{1}{n} \log c\rho(X)^n = \log \rho(X).$$ □

5. Main Results

The main result of this paper is Theorem 5.5 which states that random colored substitution networks are scale-free under certain natural conditions.

Definition 5.1. For each $i_j \in \{1, \ldots, q_j\}$, define

$$M = \begin{pmatrix} \chi(R_{1i_1}) \\ \vdots \\ \chi(R_{\lambda i_\lambda}) \end{pmatrix}.$$ 

Then collect all $M$ to obtain

$$\mathcal{M} = \{M : i_j \in \{1, \ldots, q_j\}, j \in \{1, \ldots, \lambda\}\}.$$ 

Note that $\mathcal{M}$ has $\prod_{j=1}^\lambda q_j$ elements and that $\text{Pr}_{\mathcal{M}}(M) = \prod_{j=1}^\lambda p_{ij}$. Note also that $\mathcal{M}$ is a random matrices set since $\sum_{M \in \mathcal{M}} \text{Pr}_{\mathcal{M}}(M) = 1$. In this paper, we assume that all matrices in $\mathcal{M}$ are primitive.

Notation 5.2. For a node $v$ of an arc-colored directed network $G$, let $\deg^+_j(G : v)$ and $\deg^-_j(G : v)$ denote the number of $j$-colored out-going arcs $(v, w)$ and the number of $j$-colored in-going arcs $(u, v)$ in $G$, respectively. Let

$$\delta(G : v) := (\deg^+_j(G : v), \deg^-_j(G : v), \ldots, \deg^+_\lambda(G : v), \deg^-_\lambda(G : v))$$

be a $2\lambda$-dimensional non-negative vector.

Definition 5.3. Define the $2\lambda \times 2\lambda$ matrix

$$N = \begin{pmatrix} \delta(R_{1i_1} : A) \\ \delta(R_{1i_1} : B) \\ \vdots \\ \delta(R_{\lambda i_\lambda} : A) \\ \delta(R_{\lambda i_\lambda} : B) \end{pmatrix},$$

where $i_j \in \{1, \ldots, q_j\}$.

Let $\mathcal{N}$ be the set of these random matrices

$$\mathcal{N} = \{N : i_j \in \{1, \ldots, q_j\}, j \in \{1, \ldots, \lambda\}\}$$

and note that $|\mathcal{N}| = \prod_{j=1}^\lambda q_j$. To assign probability for each matrix, let $\text{Pr}_{\mathcal{N}}(N) = \prod_{j=1}^\lambda p_{ij}$. In this way, $\mathcal{N}$ is a random matrices set. Moreover, in this paper, all matrices in $\mathcal{N}$ are assumed to be primitive.
Example 5.4. For the random colored substitution network of Fig. 2,

\[ M = \left\{ \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 5 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 5 & 2 \end{pmatrix} \right\} \]

and

\[ N = \left\{ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right\} . \]

Both \( M \) and \( N \) have associated probability vectors \( \left( \frac{1}{12}, \frac{1}{6}, \frac{1}{3}, \frac{1}{12} \right) \).

The main result of the paper is as follows.

**Theorem 5.5.** For a random colored substitution network, almost every \( \Gamma \in \mathcal{G} \) is stationary and scale-free with associated degree dimension

\[ \dim_D(\Gamma) \overset{a.s.}{=} \frac{\mathcal{L}(M)}{\mathcal{L}(N)}. \]

Theorem 5.5 will be proved in Section 7.

**Corollary 5.6.** For a deterministic colored substitution network, \( \Gamma \) is stationary and scale-free with associated degree dimension

\[ \dim_D(\Gamma) = \frac{\log \rho(M)}{\log \rho(N)}. \]

Here we present a powerful application of random colored substitution networks to the analysis of the degree dimension.

**Proposition 5.7.** There is no fixed inequality between \( \dim_D(G) \) and \( \dim_D(H) \) that holds for \( G \subset H \).

**Proof.** Consider three substitution networks as shown in Fig. 3, where

\[ M_1 = 5, \ M_2 = 4, \ M_3 = 4; \quad N_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ N_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ N_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \]

Their degree dimensions are as follows, by Theorem 5.6

\[ \dim_D(\Gamma_1) = \frac{\log \rho(M_1)}{\log \rho(N_1)} = \frac{\log 5}{\log 2} \approx 2.3219, \]
\[ \dim_D(\Gamma_2) = \frac{\log \rho(M_2)}{\log \rho(N_2)} = \frac{\log 4}{\log 2} = 2, \]
\[ \dim_D(\Gamma_3) = \frac{\log \rho(M_3)}{\log \rho(N_3)} = \frac{\log 4}{\log(\frac{1}{2} \sqrt{5 + \frac{1}{2}})} \approx 2.8808. \]

Even though \( \Gamma_2 \) and \( \Gamma_3 \) are both subnetworks of \( \Gamma_1 \), their degree dimensions are neither both less than, nor both greater than, that of \( \Gamma_1 \). \( \square \)
Example 5.8. The random colored substitution network of Fig. 2 and Example 5.4 has the associated Lyapunov exponents $L(\mathcal{M}) \approx 1.6692$ and $L(\mathcal{N}) \approx 0.9349$. Therefore, the associated degree dimension, as defined by Definition 3.2, is

$$\dim_D(\Gamma) \overset{a.s.}= \frac{L(\mathcal{M})}{L(\mathcal{N})} \approx \frac{1.6692}{0.9349} \approx 1.7854.$$ 

We obtain ten sets of simulated values for when $t = 5$; see Fig. 4. Note that the data is approximately linear on a log-log plot and that the average degree dimension from these ten randomly simulated data sets is $1.7891$, which, despite the low value of $t$ and the small number of simulations, is close to the theoretical asymptotic value $\dim_D(\Gamma) \approx 1.7854$.

![Figure 4. Simulations of scale-freeness for $t = 5$](image)

6. Graph properties

**Notation 6.1.** In this paper, set function $s(t) : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ such that $s(t) \leq t$ for all $t \in \mathbb{Z}^+$, and $s(t) \rightarrow \infty$. Without ambiguity, we will simply denote $s(t)$ by $s$. Let $V^s(G^t)$ be a subset of $V(G^t)$ so that $V^s(G^t) := V(G^t) \setminus V(G^{t-1})$. Also, let $v^t_s$ denote a fixed node in $V^s(G^t)$.

**Theorem 6.2.** Almost every $\Gamma \in \mathcal{G}$ satisfies

$$\deg(v^t_s(t)) \overset{t \rightarrow \infty}{\asymp} \exp(L(N))^{t-s}.$$ 

*Proof.* When an $i$-colored arc connecting $v^t_s$ is substituted randomly according to the rule graphs, the result corresponds to $T^t_N(e_i)$. By considering all arcs connecting $v^t_s$, we obtain

$$\deg(v^{t+1}) = \|T_N^t(\delta(v^t_s))\|_1.$$ 

Let $x_0 = \delta(v^t_s)$, and by induction

$$\deg_{G^t}(v^t_s) = \|T_N^{t-s}(x_0)\|_1.$$ 

Finally by Theorem 4.3 we conclude that $\deg(v^t_{s(t)}) \overset{t \rightarrow \infty}{\asymp} \exp(L(N))^{t-s(t)}$ almost surely. $\square$
Lemma 6.3. \( c^{-1} \leq \frac{\deg(v^t_{(i)})}{\exp(L(N))^{2-2(i)}} \leq c \) almost surely as \( t \to \infty \).

Proof. By Theorem 6.2. \( \square \)

Theorem 6.4. Almost every \( \Gamma \in \mathcal{G} \) satisfies
\[
|E(G^t)| \overset{t \to \infty}{\simeq} \exp(L(M))^t.
\]

Proof. The essence of this proof is similar to that of Theorem 6.2. Whenever an \( i \)-colored arc is substituted by some rule graph, the result corresponds to \( T_{(M)}(e_i) \). Collecting all arcs in \( G_t \), we have, for any fixed \( G_0 \),
\[
|E(G^{t+1})| = ||(G^{t+1})||_1 = ||T_{(M)}(G^t)||_1 = ||T_{(M)}(G^0)||_1.
\]
The proof follows by induction and Theorem 4.3. \( \square \)

Theorem 6.5. Almost every \( \Gamma \in \mathcal{G} \) satisfies
\[
|V(G^t)| \overset{t \to \infty}{\simeq} \exp(L(M))^t.
\]

Proof. An important observation is
\[
|V(G^t)| = \sum_{i=0}^t |V^*(G^i)|.
\]
Define random vectors set \( V \) by
\[
V = \left\{ V = \begin{pmatrix} |V(R_{i1})| - 2 \\ \vdots \\ |V(R_{i\lambda})| - 2 \end{pmatrix} : j_i \in 1, \ldots, q_i, i \in 1, \ldots, \lambda \right\},
\]
with probability \( P(V) = \prod_{i=1}^\lambda p_{ij} \). All new nodes \( V^*(G^i) \) in \( V(G^t) \) are generated by substituting arcs in \( G^{t-1} \), so it follows that, for all \( t \in \mathbb{N} \),
\[
|V^*(G^i)| = T_V(\chi(G^{t-1})).
\]
Consequently, with \( |V(G^0)| < \infty \) and by Furstenberg and Kesten’s theorem [5], \( |V(G^0)| \) equals
\[
\sum_{i=0}^t |V^*(G^i)| = |V(G^0)| + \sum_{i=1}^t T_V(\chi(G^{i-1}))
\]
\[
= |V(G^0)| + \sum_{i=1}^t T^i_V(T_{(M)}^{-1}(\chi(G^0)))
\]
\[
= |V(G^0)| + T_V(\sum_{i=1}^t T_{(M)}^{-1}(\chi(G^0)))
\]
\[
\asymp |V(G^0)| + \sum_{i=1}^t \exp(L(M))
\]
\[
\asymp \exp(L(M))^t.
\]
almost surely as \( T_V \) is bounded. \( \square \)
7. Proof of Theorem 5.5

This section is devoted to proving the scale-freeness for random colored substitution networks.

Proof of Theorem 5.5 First, we prove that $\Delta(\Gamma) = \infty$ almost surely. Recall that we assume all matrices in $\mathcal{M}$ and $\mathcal{N}$ to be primitive. This implies $\min_{\mathcal{N} \in \mathcal{N}} \rho(\mathcal{N}) \geq 2$. As a result, $\mathcal{L}(\mathcal{N}) \geq \log \min_{\mathcal{N} \in \mathcal{N}} \rho(\mathcal{N}) > 0$. This yields that $\deg_G(v)$ grows unboundedly for each node $v \in V(\Gamma)$. Similar arguments also imply that $\mathcal{L}(\mathcal{M})(\mathcal{M} > 0)$.

Now, recall that $V^*(G^t_s) = V(G^s_t) \setminus V(G^{s-1})$ for all $t \in \mathbb{Z}^+$ and let $v^t_s(t)$ denote any node of $V^*(G^t_s)$ in $G^t$ where $s(t) \leq t$. By Theorem 6.2 $\Delta(G^t) \xrightarrow{t \to \infty} \exp(\mathcal{L}(\mathcal{N}))$ almost surely. Therefore,

$$\deg_{G^t}(v^t_s(t)) \xrightarrow{t \to \infty} \frac{\exp(\mathcal{L}(\mathcal{N}))^{t-s(t)}}{\exp(\mathcal{L}(\mathcal{N}))^t} = \exp(\mathcal{L}(\mathcal{N}))^{-s(t)} \in [0,1].$$

This implies that almost every node $v^t_s(t)$ in almost every $\Gamma \in \mathcal{G}$, is stationary.

Finally, we prove that almost every $\Gamma \in \mathcal{G}$ is scale-free. Let $L : \mathbb{N} \to \mathbb{N}$ be a function satisfying $L(t)/\Delta(G^t) \xrightarrow{t \to \infty} 0$. Fix $t$ and a random colored substitution network $G^t$; when $t$ is large enough, we can find a function $k : \mathbb{N} \to \mathbb{N}$ and $k(t) < t$ such that

$$\exp(\mathcal{L}(\mathcal{N}))(L^t) \leq \exp(\mathcal{L}(\mathcal{N}))(L^t+1).$$

By Lemma 6.3 almost surely $c_1^{-1} \exp(\mathcal{L}(\mathcal{N}))(L^t) \leq \deg(v^t_k(L)) \leq c_1 \exp(\mathcal{L}(\mathcal{N}))(L^t)$, where $c_1$ is a constant depending on $\mathcal{N}$ and $v^t_k(L)$. Take $s_0 = \lceil (\log c_1)/(\mathcal{L}(\mathcal{N})) \rceil + 1$ so that, for any integer $s > s_0$,

$$c_1 \exp(\mathcal{L}(\mathcal{N}))(L^t) - s < L(t)$$

and

$$c_1^{-1} \exp(\mathcal{L}(\mathcal{N}))(L^t) + s > L(t).$$

Then $k(t) - s_0 < s < k(t) + s_0$ whenever $s \in \mathbb{Z}$ satisfies $\deg(v^t_k(L)) = L(t)$.

Again, note that Lemma 6.3 implies that $c_2^{-1} \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t)) \leq \deg(v^t_k(L)) \leq c_2 \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t))$. Therefore, if $P_{L(t)}(G^t) \neq 0$, then

$$P_{L(t)}(G^t) \geq c_2^{-1} \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t) - s_0).$$

Also,

$$P_{L(t)}(G^t) \leq |\{v^t_k(L) + s : -s_0 < s < s_0, s \in \mathbb{Z}\}| \leq \sum_{i=-s_0}^{s_0} c_2 \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t)+i).$$

For any sufficiently small $\ell$ with $P_{L(t)}(G^t) > 0$, we can find large $t$ such that $\ell = L(t)/\Delta(G^t)$. With $t$ tending to infinity, $\dim_D(\Gamma)$ equals

$$\lim_{t \to \infty} \frac{\log P_{L(t)}(\Gamma)}{-\log \ell} = \lim_{t \to \infty} \frac{\log P_{L(t)}(G^t)}{-\log \Delta(G^t)}.$$

As discussed above, $P_{L(t)}(G^t) \propto \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t))$. This implies that

$$\lim_{t \to \infty} \frac{\log P_{L(t)}(G^t)}{-\log \Delta(G^t)} = \lim_{t \to \infty} \frac{\log \exp(\mathcal{L}(\mathcal{M}))(L^t-k(t))}{\log \exp(\mathcal{L}(\mathcal{N}))} = \frac{\log \exp(\mathcal{L}(\mathcal{M}))}{\log \exp(\mathcal{L}(\mathcal{N}))}.$$

Hence,

$$\dim_D(\Gamma) \overset{a.s.}{=} \frac{\mathcal{L}(\mathcal{M})}{\mathcal{L}(\mathcal{N})}. \quad \square$$
References

[1] Reka Albert, Scale-free networks in cell biology, J. Cell Sci. 118 (2005), no. 21, 4947–4957.
[2] Albert-László Barabási and Réka Albert, Emergence of scaling in random networks, Science 286 (1999), no. 5439, 509–512. MR2091634
[3] Richard Bellman, Limit theorems for non-commutative operations. I, Duke Math. J. 21 (1954), 491–500. MR02368
[4] Victor M. Eguiluz, Dante R. Chialvo, Guillermo A. Cecchi, Marwan Baliki, and A. Vania Apkarian, Scale-free brain functional networks, Phys. Rev. Letters 94 (2005), no. 1, 018102.
[5] H. Furstenberg and H. Kesten, Products of random matrices, Ann. Math. Statist. 31 (1960), 457–469. MR121828
[6] Harry Furstenberg, Noncommuting random products, Trans. Amer. Math. Soc. 108 (1963), 377–428. MR163345
[7] Diego Garlaschelli, Stefano Battiston, Maurizio Castri, Vito D.P. Servedio, and Guido Caldarelli, The scale-free topology of market investments, Physica A 350 (2005), no. 2-4, 491–499.
[8] Y. Guivarc’h and A. Raugi, Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence, Z. Wahrsch. Verw. Gebiete 69 (1985), no. 2, 187–242. MR779457
[9] Raya Khanin and Ernst Wit, How scale-free are biological networks, J. Comput. Biol. 13 (2006), no. 3, 810–818. MR2255445
[10] H.-J. Kim, I.-M. Kim, Y. Lee, and B. Kahng, Scale-free network in stock markets, J. Korean Phys. Soc. 40 (2002), 1105–1108.
[11] Émile Le Page, Théorèmes limites pour les produits de matrices aléatoires., Probability measures on groups (Oberwolfach, 1981), 1982, pp. 258–303., MR669072
[12] Carlos León and Ron J. Berndsen, Rethinking financial stability: challenges arising from financial networks’ modular scale-free architecture, J. Financ. Stabil. 15 (2014), 241–256.
[13] Ziyu Li, Jialing Yao, and Qin Wang, Fractality of multiple colored substitution networks, Physica A 525 (2019), 402–408. MR3934572
[14] Ziyu Li, Zhouyu Yu, and Lifeng Xi, Scale-free effect of substitution networks, Physica A 492 (2018), 1449–1455. MR375210
[15] Robert M. May and Alun L. Lloyd, Infection dynamics on scale-free networks, Phys. Rev. E 64 (2001), no. 6, 066112.
[16] Wen-Yu Song, Pan Zang, Zhong-Xing Ding, Xin-Yu Fang, Li-Guo Zhu, Ya Zhu, Chang-Jun Bao, Feng Chen, Ming Wu, and Zhi-Hang Peng, Massive migration promotes the early spread of covid-19 in china: a study based on a scale-free network, Infect. Dis. Poverty 9 (2020), no. 1, 1–8.
[17] Dietrich Stauffer, Amnon Aharony, Luciano da Fontoura Costa, and Joan Adler, Efficient hopfield pattern recognition on a scale-free neural network, Eur. Phys. J. B. 32 (2003), no. 3, 395–399.
[18] Lifeng Xi, Bingbin Sun, and Jialing Yao, Fractality of substitution networks, Fractals 27 (2019), no. 3, 1950034, 6. MR3957189
[19] Lifeng Xi, Lihong Wang, Songjing Wang, Zhouyu Yu, and Qin Wang, Fractality and scale-free effect of a class of self-similar networks, Physica A 478 (2017), 31–40. MR3631331
[20] Lifeng Xi and Qianqian Ye, Average distances on substitution trees, Physica A 529 (2019), 121551, 6. MR3956011
[21] Lu-Xing Yang and Xiaofan Yang, The spread of computer viruses over a reduced scale-free network, Physica A 396 (2014), 173–184.
[22] Jialing Yao, Bingbin Sun, and Lifeng Xi, Fractality of evolving self-similar networks, Physica A 515 (2019), 211–216. MR3986869
[23] Qianqian Ye, Jiangwen Gu, and Lifeng Xi, Eigentime identities of fractal flower networks, Fractals 27 (2019), no. 2, 1950008, 8. MR3957163
[24] Qianqian Ye and Lifeng Xi, Average distance of substitution networks, Fractals 27 (2019), no. 6, 1950097, 9. MR4020769
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