On well-posedness of the Cauchy problem for MHD system in Besov spaces

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Abstract

This paper is devoted to the study of the Cauchy problem of incompressible magneto-hydrodynamics system in framework of Besov spaces. In the case of spatial dimension $n \geq 3$ we establish the global well-posedness of the Cauchy problem of incompressible magneto-hydrodynamics system for small data and the local one for large data in Besov space $\dot{B}_{p,r}^{\frac{1}{p}-1}(\mathbb{R}^n)$, $1 \leq p < \infty$ and $1 \leq r \leq \infty$. Meanwhile, we also prove the weak-strong uniqueness of solutions with data in $\dot{B}_{p,r}^{\frac{1}{p}-1}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $\frac{2}{p} + \frac{2}{r} > 1$. In case of $n = 2$, we establish the global well-posedness of solutions for large initial data in homogeneous Besov space $\dot{B}_{p,r}^{\frac{1}{p}-1}(\mathbb{R}^2)$ for $2 < p < \infty$ and $1 \leq r < \infty$.

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1 Introduction

In this paper we consider the $n$-dimensional incompressible magneto-hydrodynamics (MHD) system

\begin{align*}
  u_t - \Delta u + (u \cdot \nabla) u - (b \cdot \nabla) b - \nabla p &= 0 \\
  b_t - \Delta b + (u \cdot \nabla) b - (b \cdot \nabla) u &= 0 \\
  \text{div} u &= 0, \quad \text{div} b = 0
\end{align*}

with initial data

\begin{align*}
  u(0, x) &= u_0(x), \\
  b(0, x) &= b_0(x).
\end{align*}
where \( x \in \mathbb{R}^n, t > 0 \). Here \( u = u(t, x) = (u_1(t, x), \cdots, u_n(t, x)) \) and \( p = p(t, x) \) are non-dimensional quantities corresponding to the flow velocity, the magnetic field, and the pressure at the point \((t, x)\), and \( u_0(x) \) and \( b_0(x) \) are the initial velocity and initial magnetic field satisfying \( \text{div} u_0 = 0, \text{div} b_0 = 0 \), respectively. For simplicity, we have included the quantity \( \frac{1}{2} |b(t, x)|^2 \) into \( p(t, x) \) and we set the Reynolds number, the magnetic Reynolds number, and the corresponding coefficients to be equal to 1.

It is well known that for any initial data \((u_0, b_0) \in L^2(\mathbb{R}^n)\) with \( n \geq 2 \), the MHD equations \((1.1)-(1.5)\) have been shown to possess at least one global \( L^2(\mathbb{R}^n) \) weak solution \((u(t, x), b(t, x)) \in C_b([0, T]; L^2(\mathbb{R}^2)) \cap L^2((0, T]); \tilde{H}^1(\mathbb{R}^2))\) for any \( T > 0 \) such that

\[
\|u(0, b)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \| (\nabla u(s), \nabla b(s))^2 \|^2_{L^2(\mathbb{R}^2)} ds \leq \|u(0, b)\|_{L^2(\mathbb{R}^2)}^2,
\]

but the uniqueness and regularity remain open besides the case of \( n = 2 \). [6, 13]. Usually, we define a Leray weak solution by any \( L^2(\mathbb{R}^n) \) weak solution \((u, b)\) to the MHD \((1.1)-(1.5)\), i.e., which satisfies the MHD equations in distribution sense, and satisfying the energy estimate \((1.6)\).

When \( n = 2 \), for initial data \((u_0(x), b_0(x)) \in L^2(\mathbb{R}^2)\) there exists a unique global solution to MHD system \((1.1)-(1.3)\) with \((u(t, x), b(t, x)) \in C_b([0, \infty); L^2(\mathbb{R}^2)) \cap L^2((0, \infty); \tilde{H}^1(\mathbb{R}^2)) \cap C^\infty((0, \infty) \times \mathbb{R}^2)\), where \( C_b(I) \) denotes the space of bounded and continuous functions on \( I \) [6, 13]. Note that the coupled relation between equations \((1.1)\) and \((1.2)\) as well as the relation \((b \cdot \nabla) b(u) + ((b \cdot \nabla) u, b) = 0, \) for any \( 0 \leq t < \infty \), where \((.,.)\) stands for the inner product in \( L^2 \) with respect to the spatial variables. It follows that the solution \((u, b)\) satisfies the energy equality:

\[
\|u(0, b)\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_0^t \| (\nabla u(s), \nabla b(s))^2 \|^2_{L^2(\mathbb{R}^2)} ds = \|u(0, b)\|_{L^2(\mathbb{R}^2)}^2,
\]

for any \( 0 \leq t < \infty \).

The purpose of this paper can be divided into two aspects. At first, we prove that for initial data \((u_0, b_0) \in \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n), 1 \leq r \leq \infty, 1 \leq p < \infty\), the Cauchy problem \((1.1)-(1.5)\) has the unique local strong solution or global strong small solution in Besov space \( \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n)\). If we further assume that the data \((u_0, b_0)\) is in \( L^2(\mathbb{R}^n)\), the above solution coincides with any Leray weak solution associated with \((u_0, b_0)\). In fact, we shall establish the stability result of the Leray weak solution and strong solution in Section 3 which implies the weak and strong uniqueness.

**Theorem 1.1.** Let \((u_0, b_0) \in \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n), 1 \leq p < \infty, 1 \leq r \leq \infty, 2 < q \leq \infty \) and \( \text{div} u_0 = \text{div} b_0 = 0 \).

(i) For \( 1 \leq r \leq \infty \), there exists \( \varepsilon_0 > 0 \) such that if \( \| (u_0, b_0) \|_{\dot{B}^{n/p-1}_{p,r}} < \varepsilon_0 \), then \((1.1)-(1.5)\) has a unique solution \((u, b)\) satisfying

\[
(u, b) \in C_b(\mathbb{R}^+; \dot{B}^{n/p-1}_{p,r}) \cap \tilde{L}^q(\mathbb{R}^+; \dot{B}^{n/p+2/q}_{p,r}(\mathbb{R}^n)), \quad r < \infty,
\]

or

\[
(u, b) \in C_s(\mathbb{R}^+; \dot{B}^{n/p-1}_{p,\infty}) \cap \tilde{L}^q(\mathbb{R}^+; \dot{B}^{n/p+2/q}_{p,\infty}(\mathbb{R}^n)), \quad r = \infty,
\]
where \( s_p = \frac{n}{p} - 1 > 1 - \frac{4}{q} \) is a real number.

(ii) For \( 1 \leq r < \infty \), there exists a time \( T \) and a unique local solution \((u(t,x),b(t,x))\) to the system \((1.1)-(1.5)\) such that

\[
(u, b) \in C_b([0,T]; \dot{B}^{n/p-1}_{p,r}) \cap \dot{L}^q([0,T]; \dot{B}^{n+\frac{2}{q}-1}_{p,r}(\mathbb{R}^n)), \quad r < \infty,
\]

or

\[
(u, b) \in C_b([0,T]; \dot{B}^{n/p-1}_{p,\infty}) \cap \dot{L}^q([0,T]; \dot{B}^{n+2/q}_{p,\infty}(\mathbb{R}^n)), \quad r = \infty,
\]

where \( p, q \) satisfying \( \frac{p}{q} = \frac{2}{\beta} > 1 \), \( C_* \) denote the continuity in \( t = 0 \) with respect to time \( t \) in weak star sense, \( L^q([0,T]; \dot{B}^{n+\frac{2}{q}-1}_{p,r}(\mathbb{R}^n)) \) denotes the mixed space-time space defined by Littlewood-Paley theory, please refer to Section 2 for details.

**Theorem 1.2.** Let \((u_0,b_0) \in \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\) be a divergence free datum. Assume \( 1 \leq p \leq q \leq p \) and \( 2 \leq r \leq 1 \) such that \( \frac{n}{p} + \frac{2}{\beta} > 1 \). Let \((u,b) \in C([0,T]; \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^+; L^r(\mathbb{R}^n)) \cap L^2(\mathbb{R}^+; H^1(\mathbb{R}^n))\) be the unique solution associated with \((u_0,b_0)\). Then all Leray solutions associated with \((u_0,b_0)\) coincide with \((u,b)\) on the interval \([0,T]\).

Secondly, we shall establish the global well-posedness for the Cauchy problem of the MHD system \((1.1)-(1.5)\) for data in larger space than \( L^2(\mathbb{R}^2) \) space, i.e. the homogeneous Besov space \( \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2) \) for \( 2 < p < \infty \) and \( 1 \leq r < \infty \). Let us give some rough analysis. If \( 1 \leq p < 2 \) and \( 1 \leq r \leq \infty \) or \( p = 2 \) and \( 1 \leq r \leq 2 \), the global well-posedness is trivial because of the embedding relation \( \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2) \); The case \( 2 \leq p < \infty \) and \( 1 \leq r \leq 2 \) can be deduced into the case \( 2 \leq p < \infty \) and \( 2 < r < \infty \) because of Sobolev embedding \( \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2) \hookrightarrow \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2) \) with \( r_1 \leq r_2 \). An interesting question is whether the MHD system \((1.1)-(1.5)\) is global well-posedness for arbitrary data in the Besov space \( \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2) \) for \( 2 \leq p < \infty \), \( r = \infty \).

**Theorem 1.3.** Let \((u_0(x),b_0(x)) \in \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2)\) be divergence free vector field. Assume that \( 2 \leq p < \infty \) and \( 1 \leq r < \infty \). Then there exists a unique solution to the MHD system \((1.1)-(1.5)\) such that \((u,b) \in C([0,T]; \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2))\). Moreover, if \( p, r \) satisfy \( \frac{2}{p} + \frac{2}{r} > 1 \) and \( 1 \leq r < \infty \), the following estimate holds:

\[
\| (u,b) \|_{\dot{B}^{2/p-1}_{p,r}} \leq C \| (u_0,b_0) \|_{\dot{B}^{2/p-1}_{p,r}}^{1+\frac{s}{p}}
\]

for any \( t \geq 0 \), where \( \beta > \frac{2}{p} \).

From the above discussion, it is sufficient to prove the case \( 2 \leq p < \infty \) and \( 2 < r < \infty \) in Theorem 1.3. Since \((u_0,b_0) \in C([0,\infty); \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2))\) has infinite energy, so we have to use Caldrón’s argument \([4, 8]\) and perform an interpolation between the \( L^2\)-strong solution and the solution in \( C([0,\infty); \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2))\) with \( p < \bar{p} < \infty \) and \( r < \bar{r} < \infty \). In detail, let us decompose data

\[
(u_0(x),b_0(x)) = (v_0(x),g_0(x)) + (w_0(x),h_0(x)),
\]

with \((v_0,g_0) \in L^2(\mathbb{R}^n)\) and \((w_0,h_0) \in \dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2)\) for some \( p < \bar{p} < \infty \) and \( r < \bar{r} < \infty \) with small norm. The corresponding solutions are denoted by \((v(t,x),g(t,x))\) and \((w(t,x),h(t,x))\),
where the solutions \((w, h)\) satisfies the MHD system and \((v, g)\) satisfies MHD-like equations. The global existence of solution \((w, h)\) in the Besov space \(L^q((0, \infty); \dot{B}^{n/p+2/q-1}_{p,r}(\mathbb{R}^n))\), \(1 \leq p < \infty, 2 < q \leq \infty\) and \(1 \leq r \leq \infty\) for \(n \geq 2\) can be generally proved. The MHD-like system is locally solved, then by the energy inequality we prove \((v, g)\) is global solvable for \(n = 2\). The idea comes from I. Gallagher and F. Planchon [8] who deal with the Navier-Stokes equations, however we have given a different proof for the strong solutions to the MHD system \((1.1)-(1.5)\) on the mixed time-space Besov spaces \(\tilde{L}^q([0, T]; \dot{B}^{n/p+2/q-1}_{p,r}(\mathbb{R}^n))\).

The remaining parts of the present paper are organized as follows. Section 2 gives some definitions and preliminary tools. In Section 3 we establish some linear estimates and bilinear estimates of the solution in framework of mixed space-time Besov space by Fourier localization and Bony’s para-product decomposition, and by which we complete the proof of Theorem 1.1 and Theorem 1.2. Theorem 1.3 will be proved in Section 4 by Caldrón’s argument in conjunction with the real interpolation method.

We conclude this section by introducing some notations. Denote by \(S(\mathbb{R}^n)\) and \(S'(\mathbb{R}^n)\) the Schwartz space and the Schwartz distribution space, respectively. For any interval \(I \subset \mathbb{R}\) and any Banach space \(X\) we denote by \(C(I; X)\) the space of strongly continuous functions from \(I\) to \(X\), and by \(C_{\sigma}(I; B)\) the time-weighted space-time Banach space as follows

\[
C_{\sigma}(I; X) = \left\{ f \in C(I; B) : \| f \|_{C_{\sigma}(I; X)} = \sup_{t \in I} t^{\frac{\sigma}{n}} \| f \|_X < \infty \right\}.
\]

we denote by \(L^q(I; X)\) and \(L^{q_1, q_2}(I; X)\) the space of strongly measurable functions from \(I\) to \(X\) with \(\| u(\cdot); X \| \in L^q(I)\) and \(\| u(\cdot); X \| \in L^{q_1, q_2}\), respectively. \(L^{q_1, q_2}\) denotes usual Lorentz space, please refer to [1, 9, 14] for details.

**Notation:** Throughout the paper, \(C\) stands for a generic constant. We will use the notation \(A \lesssim B\) to denote the relation \(A \leq CB\) and the notation \(A \approx B\) to denote the relations \(A \lesssim B\) and \(B \lesssim A\). Further, \(\| \cdot \|_p\) denotes the norm of the Lebesgue space \(L^p\) and \(\| (f_1, f_2, \cdots, f_n) \|_X\) denotes \(\| f_1 \|_X + \cdots + \| f_n \|_X\). The time interval \(I\) may be either \([0, T]\) for any \(T > 0\) or \([0, \infty)\).

## 2 Preliminary

In this section we first introduce Littlewood-Paley decomposition and the definition of Besov spaces. Given \(f(x) \in \mathcal{S}(\mathbb{R}^n)\), define the Fourier transform as

\[
\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,
\]

and its inverse Fourier transform:

\[
\hat{f}(x) = \mathcal{F}^{-1} f(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.
\]

Choose two nonnegative radial functions \(\chi, \varphi \in \mathcal{S}(\mathbb{R}^n)\) supported respectively in \(B = \{ \xi \in \mathbb{R}^n, \| \xi \| \leq \frac{\delta}{2}\}\) and \(C = \{ \xi \in \mathbb{R}^n, \frac{3}{4} \leq \| \xi \| \leq \frac{8}{3}\}\) such that

\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.
\]
Set $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and let $h = F^{-1}\varphi$ and $\tilde{h} = F^{-1}\chi$. Define the frequency localization operators:

$$\Delta_j f = \varphi(2^{-j}D)f = 2^{nj}\int_{\mathbb{R}^n} h(2^jy)f(x-y)dy,$$

$$S_j f = \sum_{k\leq j-1} \Delta_k f = \chi(2^{-j}D)f = 2^{nj}\int_{\mathbb{R}^n} \tilde{h}(2^jy)f(x-y)dy.$$ (2.5, 2.6)

Formally, $\Delta_j = S_j - S_{j-1}$ is a frequency projection into the annulus $\{ |\xi| \approx 2^j \}$, and $S_j$ is a frequency projection into the ball $\{ |\xi| \lesssim 2^j \}$. One easily verifies that with the above choice of $\varphi$

$$\Delta_j \Delta_k f \equiv 0 \quad \text{if} \quad |j-k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{if} \quad |j-k| \geq 5.$$ (2.7)

We now introduce the following definition of Besov spaces.

**Definition 2.1.** Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. The homogenous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \{ f \in Z'(\mathbb{R}^n) : \|f\|_{\dot{B}^s_{p,q}} < \infty \},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}}, & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j f\|_p, & \text{for } q = \infty, \end{cases}$$

and $Z'(\mathbb{R}^n)$ can be identified by the quotient space $S'/P$ with the space $P$ of polynomials.

**Definition 2.2.** Let $s \in \mathbb{R}, 1 \leq p, q \leq \infty$. The inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \{ f \in S'(\mathbb{R}^n) : \|f\|_{B^s_{p,q}} < \infty \},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} \left( \sum_{j \geq 0} 2^{jsq} \|\Delta_j f\|_p^q \right)^{\frac{1}{q}} + \|S_0(f)\|_p, & \text{for } q < \infty, \\ \sup_{j \geq 0} 2^{jsq} \|\Delta_j f\|_p + \|S_0(f)\|_p, & \text{for } q = \infty. \end{cases}$$

If $s > 0$, then $B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}$ and $\|f\|_{B^s_{p,q}} \approx \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}$. We refer to [1, 13] for details.

The following Definition 2.3 gives the mixed time-space Besov space dependent on Littlewood-Paley decomposition (cf. [5]).

**Definition 2.3.** Let $u(t,x) \in S'(\mathbb{R}^{n+1}), s \in \mathbb{R}, 1 \leq p, q, \rho \leq \infty$. We say that $u(t,x) \in L^\rho\left(I; \dot{B}^s_{p,q}(\mathbb{R}^n) \right)$ if and only if

$$2^j s \|\Delta_j u\|_{L^\rho(I;L^p)} \in l^q,$$

and we define

$$\|u\|_{L^\rho(I;\dot{B}^s_{p,q})} \triangleq \left( \sum_{j \in \mathbb{Z}} 2^{jsq} \|\Delta_j u\|_{L^\rho(I;L^p)}^q \right)^{1/q}. \quad (2.8)$$
For the convenience we also recall the definition of Bony’s para-product formula which gives the decomposition of the product of two functions \( f(x) \) and \( g(x) \) (cf. [2, 3]).

**Definition 2.4.** The para-product of two functions \( f \) and \( g \) is defined by

\[
T_g f = \sum_{i \leq j \leq 2} \triangle_i g \triangle_j f = \sum_{j \in \mathbb{Z}} S_{j-1} g \triangle_j f. \tag{2.9}
\]

The remainder of the para-product is defined by

\[
R(f, g) = \sum_{|i-j| \leq 1} \triangle_i g \triangle_j f. \tag{2.10}
\]

Then Bony’s para-product formula reads

\[
f \cdot g = T_g f + T_f g + R(f, g). \tag{2.11}
\]

Using Bony’s para-product formula and the definition of homogeneous Besov space, one can prove the following trilinear estimates, for details, see [8].

**Proposition 2.1.** Let \( n \geq 2 \) be the spatial dimension and let \( r \) and \( \sigma \) be two real numbers such that \( 2 \leq r < \infty, \ 2 < \sigma < \infty \) and \( \frac{n}{r} + \frac{2}{\sigma} > 1 \). Define the trilinear form as

\[
T(a, b, c) = \int_0^t \int_{\mathbb{R}^n} (a(s, x) \cdot \nabla b(s, x)) \cdot c(s, x) \, dx \, ds, \tag{2.12}
\]

for \( a, b \in L^\infty([0, \infty); L^2(\mathbb{R}^n)) \cap L^2([0, \infty); \dot{H}^1(\mathbb{R}^n)) \) and \( c \in L^\sigma([0, T]; \dot{B}^{n/r+2/\sigma-1}_r, \sigma)(\mathbb{R}^n) \), \( 0 < t \leq T \). Then \( T(a, b, c) \) is continuous and satisfies estimates as follows:

\[
|T(a, b, c)| \lesssim \|a\|^{1/\sigma}_{L^\infty(\mathbb{R}^+; L^2)} \|\nabla a\|^{1-1/\sigma}_{L^2(\mathbb{R}^+; L^2)} \|b\|^{1/\sigma}_{L^\infty(\mathbb{R}^+; L^2)} \|\nabla b\|^{1-1/\sigma}_{L^2(\mathbb{R}^+; L^2)} \|c\|_{L^\sigma([0, T]; \dot{B}^{n/r+2/\sigma-1}_r, \sigma)}
\]

\[
+ \|\nabla a\|_{L^2(\mathbb{R}^+; L^2)} \|b\|_{L^2(\mathbb{R}^+; L^2)} \|\nabla b\|_{L^2(\mathbb{R}^+; L^2)} \|c\|_{L^\sigma([0, T]; \dot{B}^{n/r+2/\sigma-1}_r, \sigma)}
\]

\[
+ \|a\|^{2/\sigma}_{L^\infty(\mathbb{R}^+; L^2)} \|\nabla a\|^{1-2/\sigma}_{L^2(\mathbb{R}^+; L^2)} \|\nabla b\|_{L^2(\mathbb{R}^+; L^2)} \|c\|_{L^\sigma([0, T]; \dot{B}^{n/r+2/\sigma-1}_r, \sigma)}, \tag{2.13}
\]

and

\[
|T(a, b, c)| \leq C(\varepsilon)(\|\nabla a\|_{L^2(\mathbb{R}^+; L^2)} + \|\nabla b\|_{L^2(\mathbb{R}^+; L^2)})
\]

\[
+ C(\varepsilon)^{-1} \int_0^t (\|a(s)\|^2_{L^2} + \|b(s)\|^2_{L^2}) \|c(s)\|_{\dot{B}^{n/r+2/\sigma-1}_r, \sigma} \, ds. \tag{2.14}
\]

In particular,

\[
|T(a, a, c)| \leq C(\varepsilon)\|\nabla a\|_{L^2(\mathbb{R}^+; L^2)} + C(\varepsilon)^{-1} \int_0^t \|a(s)\|^2_{L^2} \|c(s)\|_{\dot{B}^{n/r+2/\sigma-1}_r, \sigma} \, ds. \tag{2.15}
\]

Here \( C(\varepsilon) \) and \( C(\varepsilon)^{-1} \) are constants that can be arranged by \( \varepsilon \) and \( \frac{1}{\varepsilon} \), respectively, for \( \varepsilon > 0 \).

**Remark 2.1.** In reference [8] authors only proved the estimates (2.13) and (2.15). Actually the proof also implies the estimate (2.14).
Next we give the time-space estimate of the heat semigroup \( u(t, x) = S(t)u_0 \triangleq e^{-t\Delta}u_0(x) \), which has been proved in [5]. But the proof has a misprint that is the inequality (4.3) in [8]. Let Proposition 2.2.

\[
\|u\|_{L^p(I; \dot{B}^s_{2,2})} \lesssim \|u\|_{L^p(I; \dot{B}^s_{2,2})}^{1-\frac{2}{p}} \|\nabla |\nabla|^{\pm\varepsilon} u\|_{L^p(I; \dot{B}^s_{2,2})}^{\frac{2}{p}},
\]

(2.16)

where \( I \subseteq [0, \infty) \) or \( I = [0, \infty) \).

**Proposition 2.2.** Let \( 2 < p < \infty \), \( u_0(x) \in L^2(\mathbb{R}^n) \). Denote \( u(t, x) = S(t)u_0(x) \), then we have

\[
\|u\|_{L^{p,2}(I; L^2)} \leq C\|u_0\|_{L^2},
\]

(2.17)

for \( \frac{2}{p} + \frac{n}{q} = \frac{n}{2} \), \( L^{p,2}(I) \) denotes Lorentz space with respect to \( t \in I \).

The following propositions describe the Hölder’s and Young’s inequalities in Lorentz spaces, which will be used in this paper, for their proofs we refer to [12].

**Proposition 2.3. (Generalized Hölder’s inequality)** Let \( 1 < p_1, p_2, r < \infty \), such that

\[
\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} < 1,
\]

and \( 1 \leq q_1, q_2, s \leq \infty \) with

\[
\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}.
\]

If \( f \in L^{p_1, q_1} \), \( g \in L^{p_2, q_2} \), then \( h = fg \in L^{r, s} \) such that

\[
\|h\|(r,s) \leq r'\|f\|(p_1, q_1)\|g\|(p_2, q_2),
\]

(2.18)

where \( r' \) stands for the dual to \( r \), i.e. \( \frac{1}{r} + \frac{1}{r'} = 1 \).

**Proposition 2.4. (Generalized Young’s inequality)**

Let \( 1 < p_1, p_2, r < \infty \) such that

\[
\frac{1}{p_1} + \frac{1}{p_2} > 1, \quad \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} - 1,
\]

and \( 1 \leq q_1, q_2, s \leq \infty \) with

\[
\frac{1}{q_1} + \frac{1}{q_2} \geq \frac{1}{s}.
\]

If \( f \in L^{p_1, q_1} \), \( g \in L^{p_2, q_2} \), then \( h = fg \in L^{r, s} \) with

\[
\|h\|(r,s) \leq 3r\|f\|(p_1, q_1)\|g\|(p_2, q_2),
\]

(2.19)

In particular, we have the weak Young’s inequality

\[
\|h\|(r,\infty) \leq C(p,q)\|f\|(p,\infty)\|g\|(q,\infty),
\]

(2.20)

where \( 1 < p, q, r < \infty \) and \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \).

**Proposition 2.5.** Let \( 1 \leq q_1 \leq \infty \) and \( 1 \leq q_2 \leq \infty \) satisfy \( \frac{1}{q_1} + \frac{1}{q_2} \geq 1 \), \( p \) and \( p' \) be conjugate indices, i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \). If \( f(x) \in L^{p, q_1} \) and \( g(x) \in L^{p', q_2} \), then \( h(x) = fg \in L^\infty \) such that

\[
\|h\|_\infty \leq \|f\|_{(p, q_1)}\|g\|_{(p', q_2)},
\]

(2.21)
3 Well-posedness in Besov spaces: Case \( n \geq 2 \)

This section is devoted to the proof of Theorem 1.1. One easy sees that (1.1)-(1.5) can be rewritten as

\[
\begin{align*}
    u_t - \triangle u + \mathbb{P} \nabla \cdot (u \otimes u) - \mathbb{P} \nabla \cdot (b \otimes b) &= 0, \\
    b_t - \triangle b + \mathbb{P} \nabla \cdot (u \otimes b) - \mathbb{P} \nabla \cdot (b \otimes b) &= 0, \\
    \nabla u = \nabla b &= 0, \\
    u(0, x) = u_0(x), b(0, x) = b_0(x).
\end{align*}
\]

or their integral form

\[
\begin{align*}
    u &= e^{t\triangle} u_0 - \int_0^t e^{(t-s)\triangle} \left[ \mathbb{P} \nabla \cdot (u \otimes u) - \mathbb{P} \nabla \cdot (b \otimes b) \right] ds, \\
    b &= e^{t\triangle} b_0 - \int_0^t e^{(t-s)\triangle} \left[ \mathbb{P} \nabla \cdot (u \otimes b) - \mathbb{P} \nabla \cdot (b \otimes u) \right] ds.
\end{align*}
\]

Here \( \mathbb{P} \) stands for the Leray projector onto divergence free vector field.

3.1 Linear and nonlinear estimates

To prove the results of global or local well-posedness of the Cauchy problem (3.1)-(3.4) or (3.5)-(3.6) in Besov space \( \dot{B}^{n/p}_{p,r} (\mathbb{R}^n) \), we need to establish linear and nonlinear estimates in framework of mixed space-time space by Fourier localization. First we consider the solution to linear parabolic equation

\[
\begin{align*}
    \begin{cases}
    u_t - \triangle u = f(t, x), \\
    u(0, x) = u_0.
    \end{cases}
\end{align*}
\]

Applying frequency projection operator \( \triangle_j \) to both sides of (3.7), one arrives at

\[
\frac{\partial}{\partial t} (\triangle_j u) + \triangle (\triangle_j u) = \triangle_j f.
\]

Multiplying \( |\triangle_j u|^{p-2} \triangle_j u \) on both sides of (3.8), we obtain

\[
\frac{\partial}{\partial t} |\triangle_j u|^{p-2} \triangle_j u - \Delta \triangle_j u|\triangle u|^{p-2} \triangle_j u = \triangle_j f|\triangle_j u|^{p-2} \triangle_j u.
\]

We integrate both sides of (3.9) and apply the divergence theorem to obtain

\[
\frac{1}{p} \frac{d}{dt} \|\triangle_j u\|_p^p + \int_{\mathbb{R}^n} \nabla \triangle_j u \cdot \nabla (|\triangle_j u|^{p-2} \triangle_j u) dx \leq \|\triangle_j f\|_p \|\triangle_j u\|_p^{p-1}.
\]

Since

\[
\int_{\mathbb{R}^n} \nabla \triangle_j u \cdot \nabla (|\triangle_j u|^{p-2} \triangle_j u) dx = (p-1) \int_{\mathbb{R}^n} |\triangle_j u|^{p-2} |\nabla \triangle_j u|^2 dx
\]

\[
= \frac{4(p-1)}{p^2} \int_{\mathbb{R}^n} |\nabla (|\triangle_j u|^2)|^2 dx = \|\nabla (|\triangle_j u|^2)|_2^2
\]

\[
\geq c_p 2^{2j} \|\triangle_j u\|_p^p
\]
We have
\[
\frac{d}{dt} \| \Delta_j u \|_p + 2^{2j} c_p \| \Delta_j u \|_p \leq \| \Delta_j f \|_p.
\] (3.12)

Integrating both sides of (3.12) with respect to \( t \) we arrive at
\[
\| \Delta_j u \|_p \leq e^{-c_p 2^{2j} t} \| \Delta_j u_0(x) \|_p + e^{-2^{2j} c_p t} (\| \Delta_j f \|_p \chi(t)),
\] (3.13)

where \( \chi(t) \) is a character function
\[
\chi(t) = \begin{cases} 
1, & \text{if } 0 \leq t \\
0, & \text{if others.}
\end{cases}
\] (3.14)

Taking \( L^q \)-norm with respect to \( t \) in interval \( I \) in both sides of (3.14), by Young inequality one has
\[
\| \Delta_j u \|_{L^q(I; L^p)} \leq c_p^{-\frac{1}{q}} 2^{-\frac{2j}{q}} \| \Delta_j u_0(x) \|_p + C(p, q) 2^{-\frac{2j}{p}} \| \Delta_j f \|_{L^{q/2}(I; L^p)}.
\] (3.15)

Here \( \frac{1}{q} + \frac{1}{p} = 1 \). Multiplying \( 2^{j(s+\frac{2j}{q})} \) on both sides of (3.15) and taking \( L^r \)-norm with respect to \( j \) yields
\[
\| u \|_{\dot{B}^s_q(I; B^r_{p,r})} \leq C(p, q) \left( \| u_0 \|_{\dot{B}^s_{p,r}} + \| f \|_{L^{q/2}(I; \dot{B}^{s+4/q-2}_{p,r})} \right).
\] (3.16)

Thus we arrive at

**Lemma 3.1.** Let \( 1 \leq p < \infty, \ 2 \leq q \leq \infty, \ 1 \leq r \leq \infty \) and \( s \in \mathbb{R} \). Assume \( u(t, x) \) is a solution to the Cauchy problem (3.7). Then there exists a constant \( C \) depending on \( p, q, n \) so that
\[
\| u \|_{\dot{B}^s_q(I; B^r_{p,r})} \leq C \left( \| u_0 \|_{\dot{B}^s_{p,r}} + \| f \|_{L^{q/2}(I; \dot{B}^{s+4/q-2}_{p,r})} \right).
\] (3.17)

In particular, if \( \frac{q}{2} \leq r \leq q \) we have
\[
\| u \|_{L^q(I; \dot{B}^s_{p,r})} \leq C \left( \| u_0 \|_{\dot{B}^s_{p,r}} + \| f \|_{L^{q/2}(I; \dot{B}^{s+4/q-2}_{p,r})} \right),
\] (3.18)

by Minkowski inequality.

**Remark 3.1.** For \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \), \( (\dot{B}^s_{p,r}, \| \cdot \|_{\dot{B}^s_{p,r}}) \) is a normed space. It is easy to check that \( (\dot{B}^s_{p,r}, \| \cdot \|_{\dot{B}^s_{p,r}}) \) is a Banach space if and only if \( s < \frac{n}{p} \) or \( s = \frac{n}{p}, \ r = 1 \).

Using Bony’s para-product decomposition we study the bilinear estimates. Consider two tempered distributions \( u(t, x) \) and \( v(t, x) \), then
\[
uv = T_u v + T_v u + R(u, v).
\] (3.19)

First we deal with the para-product term \( T_u v \) or \( T_v u \) as following lemma.
Lemma 3.2. (1) Let $\dot{B}^{s}_{p, r}(\mathbb{R}^n)$ be a Banach space, then
\[
\|T_u v\|_{\tilde{L}^{q/2}(I; \dot{B}^{s}_{p, r})} \leq \|u\|_{L^q(I; L^\infty)} \|v\|_{\tilde{L}^{q/2}(I; \dot{B}^{s}_{p, r})}.
\] (3.20)

(2) Let $s_1 < 0$ and $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{r_2}$, and $\dot{B}^{s_2}_{p, r_2}(\mathbb{R}^n)$ be a Banach space. Then
\[
\|T_u v\|_{\tilde{L}^{q/2}(I; \dot{B}^{s_1+s_2}_{p, r_2})} \leq C \|u\|_{\tilde{L}^{q}(I; \dot{B}^{s_1}_{p, r_1})} \|v\|_{\tilde{L}^{q}(I; \dot{B}^{s_2}_{p, r_2})}.
\] (3.21)

Proof. By the definition of $\tilde{L}^{q}(I; \dot{B}^{s}_{p, r})$ and Hölder inequality, direct computation yields
\[
\|T_u v\|_{\tilde{L}^{q/2}(I; \dot{B}^{s}_{p, r})} = \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|S_j u\|_{L^q(I; L^r)}^r \right)^{1/r} \approx \left( \sum_{j \in \mathbb{Z}} 2^{jsr} \|\triangle_j v\|_{L^r(I; L^q)}^r \right)^{1/r}
\] (3.22)
for $s < 0$ (cf. [3]), we can derive similarly by Hölder inequality
\[
\|T_u v\|_{\tilde{L}^{q/2}(I; \dot{B}^{s_1+s_2}_{p, r_2})} \leq \left( \sum_{j \in \mathbb{Z}} 2^{j(s_1+s_2)r} \|S_j u\|_{L^q(I; L^r)}^r \|\triangle_j v\|_{L^r(I; L^q)}^r \right)^{1/r} \leq C \|u\|_{\tilde{L}^{q}(I; \dot{B}^{s_1}_{p, r_1})} \|v\|_{\tilde{L}^{q}(I; \dot{B}^{s_2}_{p, r_2})}.
\] (3.23)

Next we estimate the remainder of para-product decomposition.

Lemma 3.3. Let $s_1, s_2 \in \mathbb{R}$, $1 \leq p_1, p_2, p, r_1, r_2, r \leq \infty$ and $2 \leq q \leq \infty$ such that
\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}
\] (3.25)
and $\tilde{L}^q(I; \dot{B}^{s_1}_{p_1, r_1})$, $\tilde{L}^q(I; \dot{B}^{s_2}_{p_2, r_2})$ and $\tilde{L}^{q/2}(I; \dot{B}^{s_1+s_2}_{p, r})$ are Banach spaces. Assume $0 < s_1 + s_2 < \frac{n}{p}$, then
\[
\|R(u, v)\|_{\tilde{L}^{q/2}(I; \dot{B}^{s_1+s_2}_{p, r})} \leq C \|u\|_{\tilde{L}^{q}(I; \dot{B}^{s_1}_{p_1, r_1})} \|v\|_{\tilde{L}^{q}(I; \dot{B}^{s_2}_{p_2, r_2})}.
\] (3.26)
Moreover, if $s_1 + s_2 = 0$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$, then one has
\[
\|R(u, v)\|_{\tilde{L}^{q/2}(I; \dot{B}^0_{p, \infty})} \leq C \|u\|_{\tilde{L}^{q}(I; \dot{B}^{s_1}_{p_1, r_1})} \|v\|_{\tilde{L}^{q}(I; \dot{B}^{s_2}_{p_2, r_2})}.
\] (3.27)
If $s_1 + s_2 = \frac{n}{p}$ and $r = 1$, then
\[
\|R(u, v)\|_{\tilde{L}^{q/2}(I; \dot{B}^{s_1+s_2}_{p, 1})} \leq C \|u\|_{\tilde{L}^{q}(I; \dot{B}^{s_1}_{p_1, r_1})} \|v\|_{\tilde{L}^{q}(I; \dot{B}^{s_2}_{p_2, r_2})}.
\] (3.28)
Proof. Write

\[ R(u, v) = \sum_{j' \in \mathbb{Z}} \sum_{k=-1}^{1} \triangle_{j'} u \triangle_{k+j'} v \triangle_{j'} \]  

(3.29)

Since \( \text{supp}[F(\triangle_{j'} u \triangle_{k+j'} v)] \subseteq \{ |\xi| \leq \frac{8}{3} 2^j (1 + 2^k) \} \) and \( \text{supp}[F(\triangle_j f)] \subseteq \{ \frac{8}{3} 2^j \leq |\xi| \leq \frac{8}{3} 2^j \} \), it follows that

\[ \triangle_j R(u, v) = \sum_{j' \geq j-4} \triangle_j R_{j'}. \]  

(3.30)

A straightforward calculation shows that

\[ 2^{j(s_1+s_2)} \| \triangle_j R_{j'} \|_{L^{n/2}(I; L^{p,r})} \leq 2^{j(s_1+s_2)} \sum_{k=-1}^{1} \| \triangle_{j'+k} u \|_{L^q(I; L^{p,1})} \| \triangle_{j'} v \|_{L^q(I; L^{p,2})} \]

\[ \leq \sum_{k=-1}^{1} 2^{-(j'-j)(s_1+s_2)} 2^{j's_1} \| \triangle_{j'+k} u \|_{L^q(I; L^{p,1})} 2^{j's_2} \| \triangle_{j'} v \|_{L^q(I; L^{p,2})}. \]  

(3.31)

Using the estimate (3.31) and the definition (2.8) of mixed time-space Besov space one has

\[ \| R(u, v) \|_{L^{n/2}(I; \dot{B}_{p,r}^{s_1+s_2})} \]

\[ \leq \left( \sum_{j \in \mathbb{Z}} \left( \sum_{l \leq 4} \sum_{k=-1}^{1} 2^{l(s_1+s_2)-s_1 k} 2^{s_1 (j+k-1)} \| \triangle_{j+k-l} u \|_{L^q(I; L^{p,1})} 2^{s_2 (j-l)} \| \triangle_{j-l} v \|_{L^q(I; L^{p,2})} \right)^{1/r} \right) \]

(3.32)

In view of Minkowski and Hölder inequalities, we obtain

\[ \| R(u, v) \|_{\dot{B}_{p,r}^{s_1+s_2}} \leq \sum_{k=-1}^{1} 2^{-s_1 k} \sum_{l \leq 4} 2^{l(s_1+s_2)} \| u \|_{\dot{B}_{l_p,1}^{s_1}(I; \dot{B}_{p,1}^{s_2})} \| v \|_{\dot{B}_{l_p,2}^{s_2}} \]

\[ \leq C \| u \|_{\dot{B}_{l_p,1}^{s_1}(I; \dot{B}_{p,1}^{s_2})} \| v \|_{\dot{B}_{l_p,2}^{s_2}}. \]  

(3.33)

In particular, if \( s_1 + s_2 = 0 \), we first apply Minkowski inequality then Hölder inequality to the right of (3.31), it follows that

\[ \| R(u, v) \|_{L^{n/2}(I; \dot{B}_{p,0}^{s_1})} \leq \sum_{k=-1}^{1} \| u \|_{\dot{B}_{l_p,1}^{s_1}(I; \dot{B}_{p,1}^{s_2})} \| v \|_{\dot{B}_{l_p,2}^{s_2}} \]

\[ \leq C \| u \|_{\dot{B}_{l_p,1}^{s_1}(I; \dot{B}_{p,1}^{s_2})} \| v \|_{\dot{B}_{l_p,2}^{s_2}}. \]  

(3.34)

The proof of Lemma 3.3 is thus complete. \( \square \)

By means of the fact \( L^\infty(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^n) \) and

\[ \dot{L}^q(I; \dot{B}_{p,1}^{s_1/p}(\mathbb{R}^n)) \hookrightarrow L^q(I; \dot{B}_{p,1}^{s_1/p}) \hookrightarrow L^q(I; L^\infty(\mathbb{R}^n)), \quad q \geq 1, \]

we apply Lemma 3.3 with \( p_1 = r_1 = \infty, s_1 = 0 \) and Lemma 3.2 to get the following results.
Corollary 3.1. Let $s$ be a real number such that $s < \frac{n}{p}$, $q \geq 2$ and $1 \leq p$, $r \leq \infty$, one has
\[
\|uv\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,r}^{s})} \leq C\|u\|_{L^{q}(I; L^{\infty})}\|v\|_{\tilde{L}^{q}(I; \dot{B}_{p,r}^{s})} + \|u\|_{\tilde{L}^{q}(I; \dot{B}_{p,r}^{s})}\|v\|_{L^{q}(I; L^{\infty})} \tag{3.35}
\]
and
\[
\|uv\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,1}^{s/p})} \leq C\|u\|_{\tilde{L}^{q}(I; \dot{B}_{p,1}^{s/p})}\|v\|_{\tilde{L}^{q}(I; \dot{B}_{p,1}^{s/p})} \tag{3.36}
\]

Corollary 3.2. Let $s_1$, $s_2 \in \mathbb{R}$, $1 \leq p_k$, $r_k \leq \infty$ and $1 \leq p$, $r \leq \infty$ such that
\[
s_k < \frac{n}{p_k}, \quad \frac{1}{r_1} + 1 = \frac{1}{r}, \quad p \geq \max(p_1, p_2) \tag{3.37}
\]
for $k = 1, 2$. If $s_1 + s_2 > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right)$, then
\[
\|uv\|_{\tilde{L}^{q/2}(I; \dot{B}_{p_1,r_1}^{s_1} + \dot{B}_{p_2,r_2}^{s_2})} \leq C\|u\|_{\tilde{L}^{q}(I; \dot{B}_{p_1,r_1}^{s_1})}\|v\|_{\tilde{L}^{q}(I; \dot{B}_{p_2,r_2}^{s_2})} \tag{3.38}
\]
Proof. Let $\tilde{s}_1 = s_1 - \frac{n}{p_1}$ and $\tilde{s}_2 = s_2 - \frac{n}{p_2} + \frac{n}{p}$, then $\tilde{s}_1 < 0$. Applying Lemma 3.2 to para-product $T_u v$ and $T_v u$, Lemma 3.3 to the remainder $R(u, v)$ we obtain that
\[
\|uv\|_{\tilde{L}^{q/2}(I; \dot{B}_{p_1,r_1}^{\tilde{s}_1} + \dot{B}_{p_2,r_2}^{\tilde{s}_2})} \leq C\|u\|_{\tilde{L}^{q}(I; \dot{B}_{p_1,r_1}^{\tilde{s}_1})}\|v\|_{\tilde{L}^{q}(I; \dot{B}_{p_2,r_2}^{\tilde{s}_2})} \tag{3.39}
\]
Noting the embedding relations
\[
\tilde{L}^{q}(I; \dot{B}_{p_1,r_1}^{\tilde{s}_1}(\mathbb{R}^n)) \hookrightarrow \tilde{L}^{q}(I; \dot{B}_{\infty,r_1}^{\tilde{s}_1}(\mathbb{R}^n)), \quad \tilde{L}^{q}(I; \dot{B}_{p_2,r_2}^{\tilde{s}_2}(\mathbb{R}^n)) \hookrightarrow \tilde{L}^{q}(I; \dot{B}_{\infty,r_2}^{\tilde{s}_2}(\mathbb{R}^n)), \tag{3.40}
\]
we complete the proof of the estimate (3.38). \hfill \Box

3.2 Well-posedness in Besov spaces and uniqueness of weak and strong solutions

In this subsection we first prove Theorem 1.1 i.e. the small global well-posedness of the Cauchy problem (3.1)-(3.4) or (3.5)-(3.6) and local well-posedness in Besov space $\dot{B}_{p,r}^{n/p-1}(\mathbb{R}^n)$ for $n \geq 2$. Then we establish the stability result of the Leray weak solution and strong solution which implies Theorem 1.2.

The proof of Theorem 1.1 Without loss of generality we can prove the case $r \neq \infty$. By Lemma 3.1 we first prove the following bilinear estimate.
\[
\|\nabla \cdot (u \otimes b)\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,r}^{s+4/q-2})} \leq C\|u\|_{\tilde{L}^{q}(I; \dot{B}_{p,r}^{s+2/q})}\|b\|_{\tilde{L}^{q}(I; \dot{B}_{p,r}^{s+2/q})}. \tag{3.41}
\]
Indeed, by the boundedness of Calderón-Zygmund singular integral operator on the space $\tilde{L}^{q/2}(I; \dot{B}_{p,r}^{s+4/q-2})$ and the Sobolev embedding theorem one has
\[
\|\nabla \cdot (u \otimes b)\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,r}^{s+4/q-2})} \leq C\|u \otimes b\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,r}^{s+4/q-1})} \tag{3.42}
\]
Taking $s_1 = s_2 = s_p + \frac{2}{q}$ and $r_1 = r_2 = \frac{2}{p}$ in Corollary 3.2 we have
\[
\|u \otimes b\|_{\tilde{L}^{q/2}(I; \dot{B}_{p,r/2}^{s+4/q-1})} \leq C\|u\|_{\tilde{L}^{q}(I; \dot{B}_{p,r/2}^{s+2/q})}\|b\|_{\tilde{L}^{q}(I; \dot{B}_{p,r/2}^{s+2/q})}. \tag{3.43}
\]
Thus we get estimates of solution to equations (3.5)-(3.6) as follows
\[ \begin{align*}
\|u\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} & \leq C\|u_0\|_{\dot{B}^{s+2/q}_{p,r}} + C\|b\|^2_{L^q(I; \dot{B}^{s+2/q}_{p,r})} + C\|b\|^2_{L^q(I; \dot{B}^{s+2/q}_{p,r})}, \\
\|b\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} & \leq C\|b_0\|_{\dot{B}^{s+2/q}_{p,r}} + 2C\|u\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})}\|b\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})}.
\end{align*} \] (3.44)

For convenience we write
\[ \|u, b\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} = \|u\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} + \|b\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})}, \] (3.45)

thus estimate (3.44) can be written consequently as
\[ \|u, b\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} \leq C\|u_0, b_0\|_{\dot{B}^{s+2/q}_{p,r}} + C\|u, b\|^2_{L^q(I; \dot{B}^{s+2/q}_{p,r})}. \] (3.46)

Let \((u_1, b_1)\) and \((u_2, b_2)\) be two solutions to the Cauchy problem (3.5)-(3.6) with the same data \((u_0, b_0)\). Arguing similarly as in deriving (3.46), one has
\[ \|u_1 - u_2, b_1 - b_2\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} \leq C(\|u_1, b_1\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} + \|u_2, b_2\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})}) \times \|u_1 - u_2, b_1 - b_2\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})}. \] (3.47)

We finally, apply the Banach contraction mapping principle to nonlinear operator \(T\) defined by the right sides of (3.5)-(3.6) in a closed set \(E\)
\[ E = \{(f, g) : \|f, g\|_{L^q(I; \dot{B}^{s+2/q}_{p,r})} \leq 2K_0\}, \] (3.48)

where \(K_0\) is the constant dependent on the local existence time \(T\) in local existence case or on the initial datum norm \(\|(u_0, b_0)\|_{\dot{B}^{n/p-1}_{p,r}} \ll 1\) in global case. In fact, we can choose the existence time \(T\) small enough (cf. estimate (3.13)) or the norm \(\|(u_0, b_0)\|_{\dot{B}^{n/p-1}_{p,r}}\) small enough so that \(T\) is a contraction mapping on \(E\) by (3.46) and (3.47). Thus a standard argument together with Remark 3.2 shows Theorem 1.1 and some further regularity of solution \((u, b)\).

**Remark 3.2.** (1) By Young inequality we have
\[ \|S(t)(u_0, b_0)\|_{L^q(I; \dot{B}^{n/p-1}_{p,r})} \leq \|(u_0, b_0)\|_{\dot{B}^{n/p-1}_{p,r}}. \] (3.49)

Consequently, combining the bilinear estimate (3.42) it follows that
\[ (u, b) \in L^\infty(I; \dot{B}^{n/p-1}_{p,r}) \to L^\infty(I; \dot{B}^{n/p-1}_{p,r}). \] (3.50)

(2) If \(p > n\) and \(1 \leq r \leq \infty\), using the equivalent characterization of negative index homogeneous Besov space [11], we have
\[ \sup_{t > 0} t^{\frac{n}{p} + 1 + \frac{s}{q}} \|\nabla^\alpha S(t)(u_0, b_0)\|_{L^p} \leq C\|\nabla^\alpha (u_0, b_0)\|_{\dot{B}^{n/p-1-\alpha}_{p,r}} \leq C\|(u_0, b_0)\|_{\dot{B}^{n/p-1}_{p,r}}, \] (3.51)

for \(\alpha = 0, 1\). Denote that
\[ B(u, v) = \int_0^t S(t-s)\nabla \cdot (u \otimes v)ds \] (3.52)
A straightforward calculation shows that

$$\sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|B(u, v)\|_{L^p} \leq C \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \int_0^t (t - s)^{-\frac{1}{2} - \frac{n}{2p}} \|u\|_{L^p} \|v\|_{L^p} ds$$

$$\leq C \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|u\|_{L^p} \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|v\|_{L^p},$$

(3.53)

for $\alpha = 0$. Similarly, if $\alpha = 1$, one has

$$\sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|\nabla B(u, v)\|_{L^p} \leq C \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \int_0^t (t - s)^{-1 - \frac{n}{2p}} \|u\|_{L^p} \|v\|_{L^p} ds$$

$$\leq C \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|u\|_{L^p} \sup_{t>0} t^{\frac{1}{2} - \frac{n}{2p}} \|v\|_{L^p}.$$

(3.54)

Arguing similarly as in deriving Theorem 1.1 we can prove that the Cauchy problem (3.1)-(3.4) for small datum enjoys the following estimate

$$\|\nabla^\alpha (u, b)\|_{C_{1/2 - \frac{n}{2p}}(R^+; L^p)} \leq C \|u(0, b_0)\|_{B^{n/p - 1}_{p, r}} \leq \epsilon_0,$$

(3.55)

for $\alpha = 0, 1$, where $\epsilon_0 > 0$ is a small constant.

To prove Theorem 1.2, we establish Proposition 3.1 which describes one stability result for the weak and strong solutions under some suitable conditions. As a direct consequence we get the proof of Theorem 1.2 i.e. weak-strong uniqueness.

**Proposition 3.1.** Let $(u_0, b_0)$ and $(w_0, h_0)$ be the divergence free vector field in $L^2(R^n)$, and $(w_0, h_0)$ be also in $B^{n/p - 1}_{p, r}(R^n)$. Here assume $1 \leq p < \infty$ and $2 < r < \infty$ such that $\frac{n}{2p} + \frac{2}{r} > 1$. Let $(w, g) \in L^\infty(R^+; L^2(R^n)) \cap L^2(R^+; \dot{H}^1(R^n))$ be Leray weak solution associated with $(w_0, h_0)$, let $(u, b)$ be the unique solution associated with $(u_0, b_0)$ with

$$(u, b) \in L^r([0, T]; B^{n/p + \frac{2}{r} - 1}_{p, r}(R^n)) \cap L^\infty([0, T]; L^2(R^n)) \cap L^2([0, T]; \dot{H}^1(R^n)),$$

for some $T > 0$.

Denote $(v, g) = (u, b) - (w, h)$, then we have for $0 < t < T$

$$\|(v, g)\|_{L^2}^2 + \int_0^t \|\nabla(v(s), g(s))\|_{L^2}^2 ds \leq \exp \left( C \int_0^t \|(u(s), b(s))\|_{B^{n/p + \frac{2}{r} - 1}_{p, r}}^r ds \right) \times \|(u_0, b_0) - (w_0, h_0)\|_{L^2}^2.$$

(3.56)

**Proof of Proposition 3.1** To simplify the notation, we write

$$(v_0, g_0) \triangleq (u_0, b_0) - (w_0, h_0),$$

$$(v(s), g(s)) \triangleq \|\nabla(v(s), g(s))\|_{L^2}^2 \triangleq \|\nabla(v(s))\|_{L^2}^2 + \|\nabla g(s)\|_{L^2}^2,$$

$$(u(s), b(s)) \|_{B^{n/p + \frac{2}{r} - 1}_{p, r}} \triangleq \|u(s))\|_{B^{n/p + \frac{2}{r} - 1}_{p, r}}^r + \|b(s))\|_{B^{n/p + \frac{2}{r} - 1}_{p, r}}^r.$$

(3.57)

(3.58)

(3.59)

Subtracting the two equations satisfied by $(u, b)$ and $(w, h)$, respectively, one has

$$\partial_t v - \Delta v + (v \cdot \nabla) u + (w \cdot \nabla) v - [(g \cdot \nabla)b + (h \cdot \nabla)g] - \nabla(p_1 - p_2) = 0,$$

$$\partial_t g - \Delta g + (v \cdot \nabla) b + (w \cdot \nabla) g - [(g \cdot \nabla)u + (h \cdot \nabla)v] = 0.$$
Formally, we may multiply equation (3.60) and (3.61) by \(v\) and \(g\) and integrate with respect to \(x\) on \(\mathbb{R}^n\), respectively, it follows
\[
\frac{1}{2} \partial_t (v, v) + (\nabla v, \nabla v) + (\nabla w, \nabla v) + \left( (g \cdot \nabla b, v) + (h \cdot \nabla v, v) \right) = 0,
\]
\[
\frac{1}{2} \partial_t (g, g) + (\nabla g, \nabla g) + (\nabla w, g) + \left( (g \cdot \nabla u, g) + (h \cdot \nabla v, g) \right) = 0.
\]
Here \((\cdot, \cdot)\) denote the \(L^2\)-inner product. Integrating (3.62) and (3.63), respectively, then summing up them we arrive at
\[
\left\| (v, g) \right\|_{L^2}^2 + 2 \int_0^t \| \nabla (v, g) \|_{L^2}^2 \, ds \leq \left\| (v_0, g_0) \right\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^n} (\nabla v) v u \, dx \, ds
\]
where we have used the divergence free condition and the fact that
\[
\int_{\mathbb{R}^n} (h \cdot \nabla g) v \, dx + \int_{\mathbb{R}^n} (h \cdot \nabla v) g \, dx = 0
\]
and
\[
\int_{\mathbb{R}^n} (w \cdot \nabla v) \, dx = 0, \quad \int_{\mathbb{R}^n} (w \cdot \nabla g) \, dx = 0.
\]
Applying the trilinear estimates (2.14) and (2.15) in Proposition 2.1 to (3.64), it follows that
\[
\left\| (v, g) \right\|_{L^2}^2 + \int_0^t \| \nabla (v, g) \|_{L^2}^2 \, ds \leq \left\| (v_0, g_0) \right\|_{L^2}^2 + C \int_0^t \left\| (v, g) \right\|_{L^2}^2 \| (u(s), b(s)) \|_{\dot{B}_{2,1}^{\frac{2}{p-r}}(\mathbb{R}^n)} \, ds.
\]
The Gronwall inequality yields the estimate (3.56), and the proof Theorem 1.2 is thus complete.

**Remark 3.3.** The above arguments of formal computation can be justified by the standard procedure of multiplying smoothing sequence or frequency localization operator (see [8]).

### 4 Global well-posedness in Besov spaces: Case \(n = 2\)

Now we are in position to prove of Theorem 1.3. As in (1.3) we decompose data \((u_0(x), b_0(x))\) into \((v_0(x), g_0(x)) \in L^2(\mathbb{R}^2)\) and \((u_0(x), h_0(x)) \in \dot{B}_{\frac{3}{2}}^{2/(3-p)}(\mathbb{R}^2)\) for some \(p < \tilde{p} < \infty\) and \(r < \tilde{r} < \infty\) with small norm. By means of Theorem 1.1 we let \((w(t, x), h(t, x)) \in C_b(\mathbb{R}^+; \dot{B}_{2,1}^{2/(3-p)}(\mathbb{R}^2)) \cap \dot{L}^q(\mathbb{R}^+; \dot{B}_{2,1}^{\frac{4}{3}+2/q}(\mathbb{R}^2))\) solve globally the following system:
\[
w_t - \Delta w + \mathbb{P} \nabla \cdot (w \otimes w) - \mathbb{P} \nabla \cdot (h \otimes h) = 0, \quad w(0, x) = w_0(x),
\]
\[
h_t - \Delta h + \mathbb{P} \nabla \cdot (w \otimes h) = 0, \quad \text{div} w = \text{div} h = 0,
\]
\[
w(x, 0) = w_0(x), \quad h(x, 0) = h_0(x).
\]
and satisfies the regularity estimates in Remark 3.2. To prove Theorem 1.3, we are devoted to the study of the global well-posedness to a MHD-like system

\[
v_t - \Delta v + (\mathbb{P} \nabla \cdot (v \otimes v) + \mathbb{P} \nabla \cdot (v \otimes w) + \mathbb{P} \nabla \cdot (w \otimes v)) - (\mathbb{P} \nabla \cdot (g \otimes g) + \mathbb{P} \nabla \cdot (g \otimes h) + \mathbb{P} \nabla \cdot (h \otimes g)) = 0, \tag{4.5}
\]

\[
g_t - \Delta g + (\mathbb{P} \nabla \cdot (v \otimes g) + \mathbb{P} \nabla \cdot (v \otimes h) + \mathbb{P} \nabla \cdot (w \otimes g)) - (\mathbb{P} \nabla \cdot (g \otimes v) + \mathbb{P} \nabla \cdot (h \otimes v) + \mathbb{P} \nabla \cdot (g \otimes w)) = 0, \tag{4.6}
\]

\[
\text{div} v = \text{div} g = 0, \tag{4.7}
\]

\[
v(0, x) = v_0(x), \quad g(0, x) = g_0(x). \tag{4.8}
\]

for \(L^2(\mathbb{R}^2)\) data. To this goal, the first step is to establish the local well-posedness of the MHD-like equations (4.5)-(4.8) for data \((v_0, g_0) \in L^2(\mathbb{R}^2)\) by means of Gallagher-Planchon’s argument [8]. For completeness we give a clear presentation. Let

\[
X(I) = C_{\mathbb{F}}^1(I; L^4(\mathbb{R}^2)) \cap L^4(I; L^4(\mathbb{R}^2)) \cap L^2(I; \dot{H}^1(\mathbb{R}^2)) \cap L^{2r, 2}(I; L^\frac{2r}{r-1}(\mathbb{R}^2))
\]

with norm

\[
\|f\|_X = \sup_{0 < t < T} t^{1/4} \|f\|_{L^4} + \|f\|_{L^4(I; L^4)} + \|\nabla f\|_{L^2(I; L^2)} + \|f\|_{L^{2r, 2}(I; L^{\frac{2r}{r-1}})}.
\]  

(4.10)

Here \(I = [0, T]\) for some time \(T > 0\) and \(r > 1\). Then the local well-posedness result is as follows.

**Theorem 4.1.** Let \((v_0, g_0) \in L^2(\mathbb{R}^2)\). Then there exists a time \(T > 0\) and the unique solution \((v, g) \in X(I)\) to the system (4.5)-(4.8) such that

\[
(v, g) \in C([0, T]; L^2(\mathbb{R}^2)). \tag{4.11}
\]

**Proof.** For convenience we denote by \(\|f\|_{X_1}, \|f\|_{X_2}, \|f\|_{X_3}\) and \(\|f\|_{X_4}\) every part of the norm \(\|f\|_X\) in (4.10). The MHD-like system (4.5)-(4.8) can be represented in the integral form

\[
v(t, x) = S(t)v_0 - \int_0^t S(t - s)(\mathbb{P} \nabla \cdot (v \otimes v) + \mathbb{P} \nabla \cdot (v \otimes w) + \mathbb{P} \nabla \cdot (w \otimes v))ds + \int_0^t S(t - s)(\mathbb{P} \nabla \cdot (g \otimes g) + \mathbb{P} \nabla \cdot (g \otimes h) + \mathbb{P} \nabla \cdot (h \otimes g))ds, \tag{4.12}
\]

\[
g(t, x) = S(t)g_0 - \int_0^t S(t - s)(\mathbb{P} \nabla \cdot (v \otimes g) + \mathbb{P} \nabla \cdot (v \otimes h) + \mathbb{P} \nabla \cdot (w \otimes g))ds + \int_0^t S(t - s)(\mathbb{P} \nabla \cdot (g \otimes v) + \mathbb{P} \nabla \cdot (h \otimes v) + \mathbb{P} \nabla \cdot (g \otimes w))ds. \tag{4.13}
\]

Here \(S(t)\) denotes a semigroup whose kernel \(K_{\sqrt{t}}(x) = t^{-\frac{n}{2}}K(\frac{x}{\sqrt{t}})\), where \(K(\cdot) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\). We first consider the estimates of the free part \(S(t)(v_0, g_0)\). Using Young’s inequality and Marcinkiewicz interpolation theorem [9] or [14], one can easily prove that

\[
\|S(t)(v_0, g_0)\|_{X_i} \leq C\|v_0, g_0\|_{L^2}, \text{ for } i = 1, 2, 3. \tag{4.14}
\]
For details please refer to [7][10]. Choosing \( p = 2r \) and \( q = \frac{3r}{r-1} \) in Proposition 2.2 we obtain
\[
\|S(t)(v_0, g_0)\|_{X_4} \leq C \| (v_0, g_0) \|_{L^2}.
\] (4.15)

It is necessary to point that \(\|S(t)(v_0, g_0)\|_X\) can be small enough provided that \( T \) is small.

Next we estimate the bilinear terms. Observing that the bilinear terms in integral equations (4.12)-(4.13) are composed of two kinds. One is the true nonlinear term \( B(v, v), B(g, g) \), \( B(g, v), B(v, g) \); the other is \( B(v, w), B(w, v), B(g, h), B(h, g), B(v, h), B(h, v) \), \( B(g, w), B(w, g) \), where \( B(v, g) \) is a bilinear form with
\[
B(v, g) = \int_0^t K \sqrt{t-s}(x) \cdot \mathbb{P} \nabla \cdot (v \otimes g)(s) ds,
\] (4.16)
and \( B(v, v), B(g, v), \cdots \) are similarly defined. We consider the two kinds of bilinear forms, respectively.

A straightforward calculation yields
\[
\|B(v, g)\|_{L^4(t)} \leq C \int_0^t (t-s)^{-3/4} s^{-1/2} \|g(s, \cdot)\|_{L^4(s)} \|v\|_{X_1} ds
\]
\[
\leq C \int_0^t (t-s)^{-3/4} s^{-1/2} ds \|v\|_{X_1} \|g\|_{X_1}. \quad (4.17)
\]

Therefore one has
\[
\|B(v, g)\|_{X_1} \leq C \|v\|_{X_1} \|g\|_{X_1}. \quad (4.18)
\]

For the other kind of bilinear forms it can be shown that
\[
\|B(v, w)\|_{L^4(t)} \leq C \int_0^t (t-s)^{-1/2} s^{-1/2} \|v(s, \cdot)\|_{L^4(s)} \|w(s, \cdot)\|_{\infty} ds
\]
\[
\leq C \varepsilon_0 \int_0^t (t-s)^{-1/2} s^{-1/2} \|v(s, \cdot)\|_{L^4(s)} ds \quad \leq \quad C \varepsilon_0 \int_0^t (t-s)^{-1/2} s^{-3/4} ds \|v\|_{X_1}. \quad (4.19)
\]

Here, we have used the estimate (3.55) for \( w \) with \( \alpha = 0 \) and \( p = \infty \). Thus we obtain
\[
\|B(v, w)\|_{X_1} \leq C \varepsilon_0 \|v\|_{X_1}. \quad (4.20)
\]

Applying the generalized Young inequality to the second inequality in (4.17) and the third inequality in (4.19) with respect to time \( t \), respectively, we easily see that
\[
\|B(v, g)\|_{X_2} \leq C \|v\|_{X_1} \|g\|_{X_2}, \quad (4.21)
\]
and
\[
\|B(v, w)\|_{X_2} \leq C \varepsilon_0 \|v\|_{X_2}. \quad (4.22)
\]
Here, we have used the following relations of indices
\[
1 + \frac{1}{4} = \frac{3}{4}, \quad \frac{1}{4} + \frac{1}{4}, \quad \frac{1}{4} + \frac{1}{\infty} = \frac{1}{\infty} + \frac{1}{\infty}, \quad (4.23)
\]
Similarly, to the norm $\| \cdot \|_{X_4}$ one has
\[ \| B(v, g) \|_{X_4} \leq C \| v \|_{X_4} \| g \|_{X_4} \quad \text{or} \quad C \| v \|_{X_4} \| g \|_{X_1}, \] (4.25)
and
\[ \| B(v, w) \|_{X_4} \leq C \| v \|_{X_4}. \] (4.26)

Here, indices can be chosen as
\[ 1 + \frac{1}{2r} = \frac{1}{2} + \frac{1}{2} + (1 - \frac{1}{2r}), \quad \frac{1}{2} < \frac{1}{2} + \frac{1}{2} + \frac{1}{\infty}. \] (4.27)

or
\[ 1 + \frac{1}{2r} = \frac{3}{4} + \frac{1}{4} + \frac{1}{2r}, \quad \frac{1}{2} = \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{2}. \] (4.28)

and
\[ 1 + \frac{1}{2r} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2r}, \quad \frac{1}{2} = \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{2}. \] (4.29)

Finally, we deal with the $X_3$ norm. A straightforward calculation shows that
\[ \| B(\nabla v, g) \|_{L^2(t)} \leq C \int_0^t (t-s)^{-3/4} \| \nabla v(s, \cdot) \|_{L^2} \| g(s, \cdot) \|_{L^4(t)} ds \] (4.30)
\[ \leq C \int_0^t (t-s)^{-3/4} s^{-1/2} \| \nabla v(s, \cdot) \|_{L^2} ds \| g \|_{X_1}. \]

Applying the generalized Young inequality to (4.37), one has
\[ \| B(v, g) \|_{X_3} \leq C(\| v \|_{X_3} \| g \|_{X_1} + \| g \|_{X_3} \| v \|_{X_1}). \] (4.31)

Similar computation follows that
\[ \| B(\nabla v, w) \|_{L^2} \leq C \int_0^t (t-s)^{-\frac{1}{2r} - \frac{1}{2} - \frac{1}{2r} - \frac{1}{2}} \| \nabla v(s, \cdot) \|_{L^2} ds \| w(s) \|_{L^{1, \infty}}(I; L^r), \] (4.32)
and
\[ \| B(v, \nabla w) \|_{L^2} \leq C \int_0^t (t-s)^{-\frac{1}{2r} - \frac{1}{2} - \frac{1}{2r} - \frac{1}{2}} \| v(s, \cdot) \|_{L^{1, \infty}}(I; L^r) ds \| \nabla w(s) \|_{L^{1, \infty}}(I; L^r) ds. \] (4.33)

Using the generalized Young inequality and applying the estimate (3.55) to $w$ with $\alpha = 0$ and $\alpha = 1$, respectively, we arrive at
\[ \| B(v, w) \|_{X_3} \leq C \varepsilon_0(\| v \|_{X_3} + \| v \|_{X_4}). \] (4.34)
Here, the indices chosen satisfy
\[ 1 + \frac{1}{2} = \left( \frac{1}{r} + \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{r} \right) + \frac{1}{2}, \quad \frac{1}{2} = \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{2}, \tag{4.35} \]
and
\[ 1 + \frac{1}{2} = \left( \frac{1}{2r} + \frac{1}{2} \right) + \left( 1 - \frac{1}{r} \right) + \frac{1}{2}, \quad \frac{1}{2} = \frac{1}{\infty} + \frac{1}{\infty} + \frac{1}{2}, \tag{4.36} \]
respectively.

In conclusion, we obtain the two kinds of bilinear estimates as
\[ \|B(v, g)\|_X \leq C\|v\|_X \|g\|_X, \quad \|B(v, w)\|_X \leq C\varepsilon_0 \|v\|_X, \tag{4.37} \]
for some constant C.

Assume \((v_1, g_1)\) and \((v_2, g_2)\) are two solutions \((4.12)\) and \((4.13)\), then we have the difference
\[ v_1 - v_2 = B(g_1 - g_2, g_1) + G(g_2, g_1 - g_2) + B(g_1 - g_2, h) + G(h, g_1 - g_2) \tag{4.38} \]
\[ - (B(v_1 - v_2, v_1) + G(v_2, v_1 - v_2) + B(v_1 - v_2, w) + G(w, v_1 - v_2)), \]
and
\[ g_1 - g_2 = B(g_1 - g_2, v_1) + G(g_2, v_1 - v_2) + B(h, v_1 - v_2) + G(g_1 - g_2, w) \tag{4.39} \]
\[ - (B(v_1 - v_2, g_1) + G(v_2, g_1 - g_2) + B(v_1 - v_2, h) + G(w, g_1 - g_2)). \]

Arguing similarly as in deriving \((4.37)\) one has
\[ \|(v_1 - v_2, g_1 - g_2)\|_X \leq C\|(v_1 - v_2, g_1 - g_2)\|_X (\|(v_1, g_1)\|_X + \|(v_2, g_2)\|_X + \varepsilon_0). \tag{4.40} \]

We can construct a complete metric space as following:
\[ E = \left\{ (f, g) : (f, g) \in X(I) ; \quad \|(f, g)\|_X \leq 2K_0(T) \right\}, \tag{4.41} \]
with metric \(d((f_1, f_2), (g_1, g_2)) = \|(f_1 - f_2, g_1 - g_2)\|_X\), where \(K_0(T) = \|S(t)(v_0, g_0)\|_X\) can be small enough provided we choose the existent time \(T\) small. Thus Applying Banach contraction mapping principle to nonlinear mapping \(T\) defined by the right sides of \((4.12)-(4.13)\) on \(E\), a unique local solution \((v, g)\) to \((4.12)-(4.13)\) is obtained on interval \(0, T\]. Using the fact \((v, g) \in X(I)\) it is not difficult to verify that \((v, g) \in L^\infty([0, T]; L^2(\mathbb{R}^2)) \cap \)
\[ L^2([0, T]; H^1(\mathbb{R}^2)). \]
Noting the imbedding relation \(L^2(\mathbb{R}^2) \hookrightarrow \dot{B}^{2/p-1}_{p,q}(\mathbb{R}^2)\) for \(p > 2\) and \(q \geq 2\), one also has \((v, g) \in C([0, T]; \dot{B}^{2/p-1}_{p,q}(\mathbb{R}^2))\). The proof of Theorem \(4.1\) is thus complete. \(\square\)

To complete the proof of global existence of solution we need the following energy inequality.

**Lemma 4.1.** There exists a time \(t_0\) and constant \(C(\varepsilon_0, t_0)\) such that the solution \((v, g)\) constructed in Theorem \(4.1\) satisfies the inequality
\[ \sup_{t_0 < s < t} \| (v(s), g(s)) \|_{L^2}^2 + \| \nabla (v, g) \|_{L^2([t_0, t]; L^2)}^2 \leq 2C\|(v_0, g_0)\|_{L^2}^2 \left( \frac{t}{t_0} \right)^{\varepsilon_0}. \tag{4.42} \]
In particular, if \(\frac{2}{p} + \frac{2}{q} > 1\), one has
\[ \| (v(s), g(s)) \|_{L^2}^2 + \int_0^t \| \nabla (v, g) \|_{L^2}^2 \, ds \leq C(\varepsilon_0, t_0) \|(v_0, g_0)\|_{L^2}^2. \tag{4.43} \]
Proof. Formally, we multiply (4.5) by \(v(t, x)\), (4.6) by \(g(t, x)\), and integrate both equations with respect to \(x\) on \(\mathbb{R}^2\) and \(t\) on \([0, T]\), then sum the results to arrive at

\[
\| (v(t), g(t)) \|^2_{L^2} + 2 \int_0^t \| \nabla (v(s), g(s)) \|^2_{L^2} ds \leq 2 \| (v_0, g_0) \|^2_{L^2}
\]

\[
-2 \int_0^t \int_{\mathbb{R}^2} (v \cdot \nabla v) w(s, x) dx ds - 2 \int_0^t \int_{\mathbb{R}^2} (g \cdot \nabla g) w(s, x) dx ds
\]

\[
-2 \int_0^t \int_{\mathbb{R}^2} (g \cdot \nabla v) h(s, x) dx ds + 2 \int_0^t \int_{\mathbb{R}^2} (v \cdot \nabla g) h(s, x) dx ds,
\]

where we have made use of the fact

\[
\int_{\mathbb{R}^2} (g \cdot \nabla g) v(t, x) dx + \int_{\mathbb{R}^2} (g \cdot \nabla) g(t, x) dx = 0,
\]

and the divergence free condition. For simplicity of notation, \(\| (v(t), g(t)) \|^2_{L^2}\) denotes \(\| v(t) \|^2_{L^2} + \| g(t) \|^2_{L^2}\) and \(\| (w, h) \|^r_{B^{p+2}_{p+2/r-1}}\) denotes \(\| w \|^r_{B^{p+2}_{p+2/r-1}} + \| h \|^r_{B^{p+2}_{p+2/r-1}}\). Applying estimates (2.14)–(2.15) in Proposition 2.1 to (4.44) it follows that

\[
\| (v(t), g(t)) \|^2_{L^2} + \int_0^t \| \nabla (v(s), g(s)) \|^2_{L^2} ds \leq 2 \| (v_0, g_0) \|^2_{L^2} + C \int_0^t \| (v, g) \|^2_{L^2} \| (w, h) \|^r_{B^{p+2}_{p+2/r-1}} ds.
\]

Noting that \(\int_0^\infty \| (w, h) \|^r_{B^{p+2}_{p+2/r-1}} ds \leq \varepsilon_0\) by Theorem 4.1 and applying the Gronwall inequality yields the estimate (4.43). The above formal arguments can be again justified by standard procedure, see [8].

According to Theorem 4.1 there exists a local solution on \([0, T]\) and there exists a small time \(t_0\) such that the local solution is smoothed out on \([t_0, T]\). So the energy estimate (4.44) is justified if we replace 0 with \(t_0\). Using the fact \(\sup_{0<t<\infty} \sqrt{t} \| (w(t), h(t)) \|_{L^\infty} < \varepsilon_0\) and Hölder and Young inequalities one has

\[
\int_{t_0}^t \int_{\mathbb{R}^2} (v \cdot \nabla v) w(s, x) dx ds \leq \varepsilon_0 \left( \int_{t_0}^t \| \nabla v(s) \|^2_{L^2} ds + \frac{1}{4} \int_{t_0}^t \frac{\| v(s) \|^2_{L^2}}{s} ds \right).
\]

Similarly, we can estimate \(\int_{t_0}^t \int_{\mathbb{R}^2} (g \cdot \nabla v) w(s, x) dx ds, \int_{t_0}^t \int_{\mathbb{R}^2} (g \cdot \nabla g) h(s, x) dx ds\) and \(\int_{t_0}^t \int_{\mathbb{R}^2} (v \cdot \nabla g) h(s, x) dx ds\). Inserting them to the estimate (4.44) we arrive at

\[
\| (v(t), g(t)) \|^2_{L^2} + 2 \int_0^t \| \nabla (v(s), g(s)) \|^2_{L^2} ds \leq 2 \| (v_0, g_0) \|^2_{L^2} + \varepsilon_0 \left( 4 \int_{t_0}^t \| \nabla (v(s), g(s)) \|^2_{L^2} ds + \int_{t_0}^t \frac{\| (v(s), g(s)) \|^2_{L^2}}{s} ds \right).
\]

Applying the Gronwall inequality we thus complete the proof of Lemma 4.1 □
By Lemma 4.1 we obtain that there exists a global solution \((v(t, x), g(t, x))\) to \((4.5)-(4.8)\) for the \(L^2\) data \((v_0(x), g_0(x))\). This together with Theorem 1.1 yields Theorem 1.3 except for the estimate \((1.9)\).

To prove the estimate \((1.9)\) we first recall the concept of the real interpolation method \([1]\). Let \(X_1\), \(X_2\) and \(X\) be three Banach spaces, \(X\) is the real interpolation space of \(X_1\) and \(X_2\) with

\[
X = [X_1, X_2]_{\theta, r}, \text{ with } 0 \leq \theta \leq 1 \text{ and } 1 \leq r \leq \infty.
\]  

(4.50)

Then for any \(f \in X\) one has

\[
\|f\|_X = \left(\sum_{j \in \mathbb{Z}} 2^{j\theta} K(f, j)^r\right)^{1/r},
\]

(4.51)

where \(K(f, j) = \inf_{g \in X_j} (\|f - g\|_{X_j} + 2^{-j}\|g\|_{X_j})\). In our case of Theorem 1.3 we take \(X = B_{\bar{p}, \bar{r}}^{2/\bar{p}-1}(\mathbb{R}^2)\), \(X_1 = B_{p, r}^{2/p-1}(\mathbb{R}^2)\) and \(X_2 = L^2(\mathbb{R}^2)\), where \(p < \bar{p}\) and \(r < \bar{r}\). The following lemma states the property of the interpolation norm, see [8] for details.

**Lemma 4.2.** Let \(X\) be the real interpolation space of two Banach spaces \(X_1\) and \(X_2\) with relation \(X_2 \hookrightarrow X \hookrightarrow X_1\), then there exists a constant \(C\) such that for any integer \(j_0 \geq 1\) and \(f \in X\), one has

\[
\left(\sum_{j \geq j_0} 2^{j\theta} K(f, j)^r\right)^{1/r} \leq \|f\|_X \leq C2^{j_0}\left(\sum_{j \geq j_0} 2^{j\theta} K(f, j)^r\right)^{1/r}.
\]

(4.52)

For any \((u_0, b_0) \in X\), one may decompose \((u_0, b_0)\) as

\[
(u_0, b_0) = (v_0, g_0) + (w_0, h_0),
\]

(4.53)

with \((v_0, g_0) \in X_2\), \((w_0, h_0) \in X_1\) and \(\|(w_0, h_0)\|_{X_1} \leq \varepsilon_0\). The associated solution is \((u, b) = (v, g) + (w, h)\), where \((v, g)\) and \((w, h)\) are the solutions constructed in Theorem 4.1 and 1.1 with initial data \((w_0, h_0)\), respectively. Noting (4.43) in Lemma 4.1 one has the a priori estimates

\[
\|(v, g)\|_{X_2} \leq C(\varepsilon_0)\|(v_0, g_0)\|_{X_2}, \text{ and } \|(w, h)\|_{X_1} \leq 2\|(w_0, h_0)\|_{X_1}.
\]

(4.54)

The first inequality of (4.52) in Lemma 4.2 implies that there exist \((w_0^j, h_0^j) \in X_1\) and \((v_0^j, g_0^j) \in X_2\) such that

\[
\|(u_0, b_0)\|_X \geq \frac{1}{2}\|2^{j\theta}\|(w_0^j, h_0^j)\|_{X_1} + 2^{-j}\|(v_0^j, g_0^j)\|_{X_2}\|v\|_{(j \geq j_0)}.
\]

(4.55)

Thus for any \(j \geq j_0\) one has

\[
\|(w_0^j, h_0^j)\|_{X_1} \leq 2 \cdot 2^{-j\theta}\|(u_0, b_0)\|_X < \varepsilon_0,
\]

(4.56)

where we choose \(j_0\) such that \(2^{-j_0\theta} = \frac{\varepsilon_0}{2\|(u_0, b_0)\|_X}\).
We construct solutions \((v^j, g^j)\) and \((w^j, h^j)\) to the Cauchy problem (4.5)-(4.8) and (3.1)-(3.4) associated with data \((v^j_0, g^j_0)\) and \((w^j_0, h^j_0)\), together with (4.54) and (4.55). It follows that
\[
\| (u_0, b_0) \|_X \geq C^{-1} \| 2^{j\theta} \| (w^j, h^j) \|_{X_1} + 2^{-j} \| (v^j, g^j) \|_{X_2} \|_{r_{(j\geq j_0)}}. \tag{4.57}
\]
In view of the definition of functional \(K(f, j)\) and \((u^j, b^j) = (v^j, g^j) + (w^j, h^j)\) we derive that
\[
\| (u_0, b_0) \|_X \geq C^{-1} \| 2^{j\theta} K((u, b), j) \|_{r_{(j\geq j_0)}}. \tag{4.58}
\]
Noting the right estimate (4.52) in Lemma 4.2 and the choice of \(j_0\) we finally arrive at
\[
\| (u, b) \|_X \leq C(\theta, r) \| (u_0, b_0) \|_{X}^{1+1/\theta}. \tag{4.59}
\]
Here \(\frac{1}{\theta} > \frac{p}{2}, \quad \theta = \frac{2(\beta-p)}{p(p-2)}\) is the interpolation parameter and \(C(\theta, r) \leq C \frac{2^\theta}{(2r-1)^r} \leq C(p, r)\).
Choosing \(\beta = \frac{1}{\theta}\) the estimate (1.9) follows.

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