Conservation laws for even order elliptic systems in the critical dimension - a new approach

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Abstract
We consider elliptic systems of order $2m$ in dimension $2m$ which are generalizations of extrinsic and intrinsic polyharmonic maps. We show the existence of a conservation law for these systems by using a small perturbation of Uhlenbeck’s gauge fixing matrix.

1 Introduction

The regularity of critical points of geometric variational problems for maps between two Riemannian manifolds attracted a lot of attention over the last two decades. The most prominent example are the harmonic maps which are critical points $u \in W^{1,2}(M, N)$ of the Dirichlet energy

$$E(u) = \frac{1}{2} \int_M |\nabla u|^2 \, dv_g,$$

where $(M, g)$ and $(N, h)$ are two smooth and compact manifolds without boundary and $N$ is isometrically embedded into some euclidean space $\mathbb{R}^n$. They solve the elliptic system

$$-\Delta u = A(u)(\nabla u, \nabla u),$$

where $A$ is the second fundamental form of the embedding $N \hookrightarrow \mathbb{R}^n$. The Dirichlet energy is scaling invariant in dimension two, which is called the critical dimension, and it was shown by Hélein [10] that weakly harmonic maps are smooth in this case.
This result was substantially extended by Rivière [15] to more general elliptic systems of the form
\[-\Delta u = \Omega \cdot \nabla u,\]
where \(\Omega \in L^2(B^2, so(n) \otimes \wedge^1(\mathbb{R}^2))\) and \(B^2\) denotes the unit ball in \(\mathbb{R}^2\). Rivière obtained the regularity of weak solutions as a consequence of a conservation law which he derived using the antisymmetry of \(\Omega\). The key ingredient here was the use of the Uhlenbeck gauge fixing result [20], see Theorem 3.2. Note that the Euler-Lagrange equation of all quadratic and conformally invariant variational integrals satisfies an equation of the type above. We sketch a version of this result in Sect. 3.

The regularity result of Hélein was than extended to the so-called weakly biharmonic maps in \(\mathbb{R}^4\), i.e. critical points of the functional
\[E_2(u) = \frac{1}{2} \int_M |\Delta u|^2 d\nu_g\]
by Chang-Wang-Yang [3] for spherical targets and by Wang [21] for general targets. Later, the second author and Rivière [14] were able to show a conservation law for a suitable generalization of the biharmonic map equation in the spirit of the before mentioned paper of Rivière. A modified version of this conservation law was later obtained by Struwe [18].

De Longueville and Gastel [5] recently extended this result to systems of order \(2m\) in the critical dimension. The motivating example behind this system are the \(m\)-polyharmonic maps \(u \in W^{m,2}(B^{2m}, N)\), which are critical points of the functional
\[E_m(u) = \frac{1}{2} \int_{B^{2m}} |\nabla^m u|^2 d\nu_g.\]
The Euler-Lagrange equation for \(E_m\) was calculated by Angelsberg-Pumberger [1] resp. Gastel-Scheven [6]. In the latter paper the authors also showed the regularity for these critical points using Hélein’s moving frame technique.

In the following we consider systems of the from

\[\Delta^m u = \sum_{k=0}^{m-1} \Delta^k (V_k, du) + \sum_{k=0}^{m-2} \Delta^k \delta(w_k du). \tag{1.1}\]

with coefficient functions
\[w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n \times n}) \quad \text{for } k \in \{0, \ldots, m-2\},\]
\[V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m}) \quad \text{for } k \in \{0, \ldots, m-1\}, \text{ where}\]
\[V_0 = d\eta + F, \quad \eta \in W^{2-m,2}(B^{2m}, so(n)), \quad F \in W^{2-m,2/m,1}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m}).\]

It was shown by De Longueville and Gastel [5] that \(m\)-polyharmonic maps are solutions of a system of this type. Note that the definition and the basic properties of the negative Sobolev spaces arising in this equation are collected in Sect. 2.

In our main Theorem 2.13 we establish a new conservation law for systems of the form (1.1). The novelty here is that we use a small perturbation of the gauge fixing matrix \(P\) in a suitable variant of the Uhlenbeck result, see Theorem 4.1.

The paper is organized as follows. In Sect. 2 we recall some basic definitions and properties for negative Sobolev and Lorentz-Sobolev spaces and we show a suitable higher order generalization of the Wente Lemma. Moreover, we state and comment on our main Theorem.
In Sect. 3 we review the second order case of our main result, a proof of which was already sketched by Rivière in [16].

In Sect. 4 we finally show our main Theorem.

2 Lorentz-Sobolev spaces and the main result

In this section we start by recalling the definitions of the relevant function spaces we need in order to obtain the desired conservation law. Moreover, we show a preliminary result on a higher order version of the famous Wente lemma [22] and we state our main result.

2.1 Lorentz- and Lorentz-Sobolev spaces

Important function spaces in our paper are the so called Lorentz spaces. They are interpolation spaces of the classical $L^p$-spaces and in the following we briefly collect a few properties of these spaces. For detailed proofs see for example [5,8,11,12,19,23]. We start with a Lemma on the Hölder inequality for these functions.

**Lemma 2.1 (Hölder inequality) Let $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$ and $p_1, p_2 \in (1, \infty)$, $q_1, q_2 \in [1, \infty]$. Then**

$$||fg||_{L^{p,q}(\mathbb{R}^n)} \leq ||f||_{L^{p_1,q_1}(\mathbb{R}^n)} ||g||_{L^{p_2,q_2}(\mathbb{R}^n)}.$$  

Additionally, we also need the following estimates.

**Lemma 2.2** Let $f : \mathbb{R}^n \to \mathbb{R}$ be measurable.

1. Let $1 < p \leq \infty$ and $1 \leq q < Q \leq \infty$. Then we have

$$||f||_{L^{p,q}(\mathbb{R}^n)} \leq c ||f||_{L^{p,q}(\mathbb{R}^n)}.$$  

2. Let $1 < p < P \leq \infty$, $1 \leq q_1, q_2 \leq \infty$ and let $\Omega \subset \mathbb{R}^n$ be bounded. Then we have

$$||f||_{L^{p,q_1}(\Omega)} \leq c |\Omega|^{\frac{1}{p} - \frac{1}{P}} ||f||_{L^{p,q_2}(\Omega)}.$$  

Next we come to Lorentz-Sobolev spaces. If a function $f \in L^{p,q}(\mathbb{R}^n)$ has derivatives $D^j f \in L^{p,q}(\mathbb{R}^n)$ for all $1 \leq j \leq k \in \mathbb{N}$, then $f$ is an element of the so-called Lorentz-Sobolev space $W^{k,p,q}(\mathbb{R}^n)$.

**Definition 2.3** Let $1 < p \leq \infty$, $1 \leq q \leq \infty$ and $k \in \mathbb{N}$. Let $f \in L^{p,q}(\mathbb{R}^n)$ be $k$ times weakly differentiable and for all multiindices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ let $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} f \in L^{p,q}(\mathbb{R}^n)$. Then $f$ is an element of the Lorentz-Sobolev space $W^{k,p,q}(\mathbb{R}^n)$ with norm

$$||f||_{W^{k,p,q}(\mathbb{R}^n)} := \sum_{0 \leq |\alpha| \leq k} \left\| \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} f \right\|_{L^{p,q}(\mathbb{R}^n)}.$$  

We have a generalized Sobolev embedding theorem for these spaces.

**Lemma 2.4** Let $k, n \in \mathbb{N}$, $1 < p < \frac{n}{k}$ and $1 \leq q \leq \infty$. Then $W^{k,p,q}(B^n) \hookrightarrow L^{p^*,q}(B^n)$ for $\frac{1}{p^*} = \frac{1}{p} + \frac{k}{n}$ with the estimate

$$||f||_{L^{p^*,q}(B^n)} \leq c ||f||_{W^{k,p,q}(B^n)}$$  

for any $f \in W^{k,p,q}(B^n)$.  

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Similar to Lemma 2.1 we have a product estimate for Lorentz-Sobolev functions.

**Lemma 2.5** Let $s, k \in \mathbb{N}$, $p, p', q, q' \in \mathbb{R}$ with $1 < p, p', q, q' < \infty$, $kp < n, sp' < n, s \leq k$, $t := \frac{npp'}{n(p + p') - kpp'} > 1$ and $\frac{1}{u} := \min\{\frac{1}{q} + \frac{1}{q'}, 1\}$. Further let $B^n \subset \mathbb{R}^n$. If $f \in W^{k,p,q}(B^n)$, $g \in W^{s,p',q'}(B^n)$, then

$$f g \in W^{s + t,u}(B^n)$$

and

$$||fg||_{W^{s,t,u}(B^n)} \leq c||f||_{W^{k,p,q}(B^n)}||g||_{W^{s,p',q'}(B^n)}$$

with $c = c(B^n)$.

Furthermore, we need an optimal Sobolev embedding result.

**Lemma 2.6** Let $B^n \subset \mathbb{R}^n$. If $f \in W^{k,\frac{q}{p},1}(B^n)$, then $f$ is continuous on $B^n$.

Later on we also use Lorentz-Sobolev spaces $W^{-k,p',q'}$. These are distribution spaces and for $p, q > 1$ they are the dual spaces of $W^{k,p,q}$.

**Definition 2.7** Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ and $k \in \mathbb{N}$. Then $W^{-k,p,q}(B^n)$ is the space of distributions $\Phi \in (C_c^\infty(B^n))^\prime$ such that

$$||\Phi[f]|| \leq c ||f||_{W^{k,p',q'}(B^n)} \quad \forall f \in C_c^\infty(B^n).$$

Each element of $W^{-k,p,q}$ has a representation in terms of derivatives of Lorentz functions:

**Lemma 2.8** Let $1 < p, q < \infty$, $k \in \mathbb{N}$, $B^n \subset \mathbb{R}^n$ and $f \in W^{-k,p,q}(B^n)$. Then there exist $f_\alpha \in L^{p,q}(B^n)$ so that

$$f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha.$$ 

Note that this representation is not unique. We define the norm on $W^{-k,p,q}(B^n)$ by

$$||f||_{W^{-k,p,q}(B^n)} := \inf \left\{ \sum_{|\alpha| \leq k} ||f_\alpha||_{L^{p,q}(B^n)} : f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha \right\}.$$ 

The definition of negative Lorentz-Sobolev spaces as dual spaces does not hold for $p, q = 1$ since $L^{p,1}$, $L^{p',\infty}$ are not reflexive. In this case we define the space $W^{-k,p,1}$ as follows.

**Definition 2.9** Let $1 < p < \infty$, $k \in \mathbb{N}$. Then

$$W^{-k,p,1}(B^n) := \left\{ f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha : f_\alpha \in L^{p,1}(B^n) \right\}$$

with norm

$$||f||_{W^{-k,p,1}(B^n)} := \inf \left\{ \sum_{|\alpha| \leq k} ||f_\alpha||_{L^{p,1}(B^n)} : f = \sum_{|\alpha| \leq k} \partial^\alpha f_\alpha \right\}.$$ 

Finally we have an embedding theorem and a Hölder inequality.
Lemma 2.10 Let $B^n \subset \mathbb{R}^n$, $1 < p < n$, $1 \leq q \leq p$, $l, s, t \in \mathbb{N}_0$ with $tp < n$ and $f \in W^{-s, p, q}(B^n, \Lambda^{l}\mathbb{R}^n)$. Then $f \in W^{-(s+l), \frac{np}{n-tp}, q}(B^n, \Lambda^{l}\mathbb{R}^n)$ and

$$\|f\|_{W^{-(s+l), \frac{np}{n-tp}, q}(B^n)} \leq c \|f\|_{W^{-s, p, q}(B^n)}.$$

Lemma 2.11 Let $s, t \in \mathbb{N}$, $t \leq s$, $1 < p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} \leq 1$ and $tp < n$, $sp' \leq n$, $1 \leq q, q' \leq \infty$. Let $f \in W^{-t, p, q}(B^n)$ and $g \in W^{s, p', q'}(B^n)$. Then

$$fg \in W^{-t,x,y}(B^n)$$

with $x = \frac{dpp'}{n(p+p')-spp}$ and $\frac{1}{y} = \min\{1, \frac{1}{q}, \frac{1}{q'}\}$. Further

$$\|fg\|_{W^{-t,x,y}(B^n)} \leq c \|f\|_{W^{-t,p,q}(B^n)}\|g\|_{W^{s,p',q'}(B^n)}.$$

More details about these spaces and proofs of the above results can be found in [4].

2.2 A generalized Wente lemma

A key ingredient in the proof of the main Theorem later on will be the following Wente-type lemma in the spirit of Bethuel and Ghidaglia [2]. A fourth order version of this result can already be found in [14].

Lemma 2.12 Let $\sigma > 0$, $f \in L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m}, \mathbb{R}^n)$ for $|\gamma| \leq m - 2$ and $P \in W^{m, 2}(B^{2m}, SO(n))$ with $\|dP\|_{W^{m-1, 2}} \leq \sigma$. There exists $\sigma_0 > 0$ such that if $\sigma < \sigma_0$ there exists a unique solution $u \in W^{2m-1, \frac{2m}{2m-1-|\gamma|}}(B^{2m}, M(n))$ of

$$\begin{align*}
\Delta(\Delta^{m-1}u) &= \delta f & \text{in } B^{2m}, \\
\Delta^j u &= 0 & \text{on } \partial B^{2m} \quad \text{for } j = 0, \ldots, m - 1,
\end{align*}$$

(2.1)

with

$$\|D^{2m-1}u\|_{L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m})} + \|u\|_{L^{\infty}(B^{2m})} \leq c \|f\|_{L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m})}.$$  

Proof The boundary conditions determine a solution $u$ of (2.1) uniquely. To see this we assume there exist solutions $u_1$, $u_2$ and we let $v := u_1 - u_2$. Then $\Delta(\Delta^{m-1}v \cdot P) = 0$. Testing this equation with $\Delta^{m-1}v \cdot P$ and integrating by parts gives

$$0 = \int_{B^{2m}} \Delta(\Delta^{m-1}v \cdot P)(\Delta^{m-1}v \cdot P) = - \int_{B^{2m}} |D(\Delta^{m-1}v \cdot P)|^2.$$

Thus we have $D(\Delta^{m-1}v \cdot P) = 0$ and therefore $\Delta^{m-1}v \cdot P = \text{const}$. Because $P$ is invertible and $\Delta^{m-1}v = 0$ on $\partial B^{2m}$ we get $\Delta^{m-1}v = 0$. Iteratively we get $v = 0$ and thus $u_1 = u_2$.

Now we approximate $f$ by $\tilde{f} \in C_c(\mathbb{R}^{2m})$ so that $\tilde{f} = 0$ on $\mathbb{R}^{2m} \setminus B^{2m}$ and

$$\|\tilde{f}\|_{L^{\frac{2m}{2m-1-|\gamma|}}(\mathbb{R}^{2m})} \leq c \|f\|_{L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m})}.$$

Standard $L^p$-theory and interpolation results (see [11] Theorem 3.3.3) yield

$$\|D(\Delta^{m-1}u P)\|_{L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m})} \leq c \|f\|_{L^{\frac{2m}{2m-1-|\gamma|}}(B^{2m})}.$$
We interchange derivatives and apply the Calderon-Zygmund inequality
\[
\| D \Delta^{m-1} u \|_{L^{2m-1-|\nabla|}^1(B^{2m})} 
\leq c \left( \| f \|_{L^{2m-1-|\nabla|}^1(B^{2m})} + \| D^{2m-2} u \|_{L^{2m-2-|\nabla|}^2(B^{2m})} \| dP \|_{L^{2m,2}(B^{2m})} \right)
\]
\[
\leq c \left( \| f \|_{L^{2m-1-|\nabla|}^1(B^{2m})} + \| u \|_{W^{2m-1,2m-1-|\nabla|}^1(B^{2m})} \| dP \|_{W^{m-1,2}(B^{2m})} \right).
\]

We interchange derivatives and apply the Calderon-Zygmund inequality
\[
\| D^{2m-1} u \|_{L^{2m-1-|\nabla|}^1(B^{2m})} 
\leq c \left( \| f \|_{L^{2m-1-|\nabla|}^1(B^{2m})} + \| u \|_{W^{2m-1,2m-1-|\nabla|}^1(B^{2m})} \| dP \|_{W^{m-1,2}(B^{2m})} \right).
\]

Since \( \| dP \|_{W^{m-1,2}(B^{2m})} < \sigma \) we absorb the second term to the left-hand side. The density of \( C_c^\infty(B^{2m}) \) in \( L^{p,q}(B^{2m}) \) finishes the proof.

### 2.3 The main result

Before we are able to state our main result we introduce some more notation. Let \( \wedge^k \mathbb{R}^{2m} \), \( k \in \mathbb{N}_0 \) be the space of \( k \)-forms on \( \mathbb{R}^{2m} \). Further let
\[
d : W^{1,p}(\mathbb{R}^{2m}, \wedge^k \mathbb{R}^{2m}) \to L^p(\mathbb{R}^{2m}, \wedge^{k+1} \mathbb{R}^{2m})
\]
be the exterior derivative and
\[
\delta : W^{1,p}(\mathbb{R}^{2m}, \wedge^k \mathbb{R}^{2m}) \to L^p(\mathbb{R}^{2m}, \wedge^{k-1} \mathbb{R}^{2m})
\]
the codifferential. We have \( dd = \delta \delta = 0 \) and the Laplacian is given by
\[
\Delta = d \delta + \delta d.
\]

If \( f \) is a function, the exterior derivative of \( f \) is just the gradient \( \nabla f \). Let \( 0 \leq k \leq 2m \) with \( k \in \mathbb{N} \), then we let
\[
\ast : \wedge^k \mathbb{R}^{2m} \to \wedge^{2m-k} \mathbb{R}^{2m}
\]
be the Hodge-Star operator. For a \( k \)-form \( \omega \) we have
\[
\delta \omega = (-1)^{(2m+1)(k+1)} \ast d \ast \omega \quad (2.2)
\]
and
\[
\ast \ast : (-1)^{(k-1)(2m-k)} : \wedge^k \mathbb{R}^{2m} \to \wedge^k \mathbb{R}^{2m}. \quad (2.3)
\]
(see e.g. [13]).

The following is the main result of this paper.

**Theorem 2.13** Assume \( m \geq 2 \), \( n \in \mathbb{N} \). Let coefficient functions be given as
\[
w_k \in W^{2k+2-m,2}(B^{2m}, \mathbb{R}^{n \times n}) \quad \text{for } k \in \{0, \ldots, m-2\},
\]
\[
V_k \in W^{2k+1-m,2}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m}) \quad \text{for } k \in \{0, \ldots, m-1\}, \text{ where}
\]
\[
V_0 = d\eta + F, \quad \eta \in W^{2-m,2}(B^{2m}, so(n)), \quad F \in W^{2-m,2/m+1}(B^{2m}, \mathbb{R}^{n \times n} \otimes \wedge^1 \mathbb{R}^{2m})
\]

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We consider the equation

\[ \Delta^m u = \sum_{k=0}^{m-1} \Delta^k \langle V_k, du \rangle + \sum_{k=0}^{m-2} \Delta^k \delta(w_k du). \tag{2.4} \]

For this equation, the following statements hold.

(i) Let

\[ \sigma := \sum_{k=0}^{m-2} ||w_k||_{W^{2k+2-m,2}(B^{2m})} + \sum_{k=0}^{m-1} ||V_k||_{W^{2k+1-m,2}(B^{2m})} \]

\[ + ||\eta||_{W^{2-m,2}(B^{2m})} + ||F||_{W^{2-m,2m/3+1}(B^{2m})}. \tag{2.5} \]

There is \( \sigma_0 > 0 \) such that whenever \( \sigma < \sigma_0 \), there exist \( \varepsilon \in W^{m,2} \cap L^{\infty}(B^{2m}_{1/2}, M(n)) \) with

\[ ||\varepsilon||_{W^{m,2}(B^{1/2}_{1/2})} + ||\varepsilon||_{L^{\infty}(B^{2m}_{1/2})} \leq c \sigma, \]

a function \( P \in W^{m,2}(B_{1/2}^{1/2}; SO(n)) \) and a distribution \( B \in W^{2-m,2}(B^{2m}_{1/2}, \mathbb{R}^{n \times n} \otimes \wedge^2 \mathbb{R}^{2m}) \) which solves

\[ \delta B = \sum_{k=0}^{m-1} \Delta^k ((id + \varepsilon)P)V_k - \sum_{k=0}^{m-2} d \Delta^k ((id + \varepsilon)P)w_k + d \Delta^{m-1}((id + \varepsilon)P). \]

(ii) A function \( u \in W^{m,2}(B^{2m}_{1/2}, \mathbb{R}^n) \) solves (2.4) weakly if and only if it is a distributional solution of the conservation law

\[ \delta \left[ \sum_{l=0}^{m-1} \Delta^l ((id + \varepsilon)P)\Delta^{m-l-1}du - \sum_{l=0}^{m-2} d \Delta^l ((id + \varepsilon)P)\Delta^{m-l-1}u \right. \]

\[ - \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \Delta^l ((id + \varepsilon)P)\Delta^{k-l-1}d\langle V_k, du \rangle \]

\[ + \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} d \Delta^l ((id + \varepsilon)P)\Delta^{k-l-1} \langle V_k, du \rangle \]

\[ - \sum_{k=0}^{m-2} \sum_{l=0}^{k} \Delta^l ((id + \varepsilon)P)d \Delta^{k-l-1} \delta(w_k du) \]

\[ + \sum_{k=0}^{m-2} \sum_{l=0}^{k-1} d \Delta^l ((id + \varepsilon)P)\Delta^{k-l-1} \delta(w_k du) - \langle B, du \rangle \right] = 0. \tag{2.6} \]

(iii) Every weak solution \( u \) of (2.4) is continuous.

A different variant of this result has been obtained earlier by Lamm and Rivière [14] in the case \( m = 2 \) and by De Longueville and Gastel [5] for general \( m \). The key difference to these papers is that we use a small perturbation \( (id + \varepsilon)P \) of the Uhlenbeck gauge matrix \( P \), see Theorem 4.1, to establish the conservation law. This Ansatz highlights the strong connection between the conservation law and the matrix \( P \) more explicitly than the previous papers. Another new ingredient in our approach is Lemma 2.12, a generalization of an estimate by
Bethuel and Ghidaglia [2], which we use instead of a Wente type result for the poly-Laplace operator. This allows for more general elliptic operators in divergence form and simplifies the argument.

We also remark that in a recent paper by Guo and Xiang [9] it was shown that weak solutions of (2.4) are not only continuous but even Hölder continuous for some positive exponent.

### 3 Second order case

In this section we briefly review the second order case of the main Theorem 2.13. We will not discuss the original proof in [15] but we will focus on Rivière’s subsequent idea to establish a conservation law by using a small perturbation of the Uhlenbeck gauge matrix $P$. This proof was already sketched in [16], chapter VI, but since we will follow the same strategy in the proof of our main Theorem we decided to include this argument here.

**Theorem 3.1** Let $n \in \mathbb{N}$ and $N$ be an oriented submanifold of $\mathbb{R}^n$. Let $u \in W^{1,2}(B^2, N)$ be a solution of

$$-\Delta u = \Omega \cdot \nabla u, \quad (3.1)$$

where $\Omega \in L^2(B^2, so(n) \otimes \wedge^1 \mathbb{R}^2)$ and let $\sigma := ||\Omega||_{L^2}$. There exists $\sigma_0 > 0$ such that whenever $\sigma < \sigma_0$, there exist $\varepsilon \in W^{1,2}(B^2, M(n))$, $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, so(n))$ with

$$||\varepsilon||_{L^\infty(B^2)} + ||\nabla \varepsilon||_{L^2(B^2)} + ||\xi||_{W^{1,2}(B^2)} + ||\nabla P||_{L^2(B^2)} \leq c\sigma,$$

and $B \in W^{1,2}(B^2)$ that solve

$$\nabla \nabla B = \nabla \varepsilon P - (id + \varepsilon)\nabla \nabla \xi P.$$

Further $u$ solves (3.1) if and only if it is a solution of

$$-\text{div}((id + \varepsilon)P \nabla u) = \nabla B \cdot \nabla u$$

and $u$ is continuous.

The proof of Theorem 3.1 relies heavily on Uhlenbeck’s gauge theorem, see for example [15,17,20].

**Theorem 3.2** [Uhlenbeck gauge] There exists $\sigma > 0$ and $c > 0$ such that for every $\Omega \in L^2(B^2, so(n) \otimes \wedge^1 \mathbb{R}^2)$ satisfying $||\Omega||_{L^2(B^2)} < \sigma$ there exist $P \in W^{1,2}(B^2, SO(n))$ and $\xi \in W^{1,2}(B^2, so(n))$ such that

$$\Omega = P^{-1}\nabla \xi P + P^{-1}\nabla P$$

and

$$||\xi||_{W^{1,2}(B^2)} + ||\nabla P||_{L^2(B^2)} \leq c||\Omega||_{L^2(B^2)}.$$

**Proof of Theorem 3.1:** Assume $||\Omega||_{L^2(B^2)} < \sigma$ as in Theorem 3.2. Then we get $P \in W^{1,2}(B^2, SO(n))$, $\xi \in W^{1,2}(B^2, so(n))$ such that

$$\Omega = P^{-1}\nabla \xi P + P^{-1}\nabla P$$

and

$$||\xi||_{W^{1,2}(B^2)} + ||\nabla P||_{L^2(B^2)} \leq c||\Omega||_{L^2(B^2)}.$$
We multiply (3.1) with \((id + \varepsilon)P\), where \(\varepsilon \in W^{1,2} \cap L^\infty(B^2, M(n))\) and \(id\) is the identity matrix in \(\mathbb{R}^n\), and obtain

\[-(id + \varepsilon)P \Delta u = (id + \varepsilon)P \Omega \cdot \nabla u\]

\[-div \ [(id + \varepsilon)P \nabla u] = [-\nabla \varepsilon P + (id + \varepsilon)(-\nabla P + P \Omega)] \cdot \nabla u\]

\[-div \ [(id + \varepsilon)P \nabla u] = \left[-\nabla \varepsilon P + (id + \varepsilon)\nabla \perp \xi P\right] \cdot \nabla u.\]  \(3.2\)

We choose \(\varepsilon \in W^{1,2} \cap L^\infty(B^2, M(n))\) such that

\[div \left[-\nabla \varepsilon P + (id + \varepsilon) \nabla \perp \xi P\right] = 0.\]  \(3.3\)

To do this we apply a fixed point argument. Let

\[\psi : W^{1,2} \cap L^\infty(B^2) \to W^{1,2} \cap L^\infty(B^2)\]

\[\varepsilon \mapsto \text{solution } \lambda \text{ of (3.4)}\]

where

\[
\begin{aligned}
\div [\nabla \lambda P] &= \nabla ((id + \varepsilon)P) \cdot \nabla \perp \xi \quad \text{in } B^2, \\
\lambda &= 0 \quad \text{on } \partial B^2.
\end{aligned}
\]  \(3.4\)

Let \(\varepsilon_1, \varepsilon_2 \in W^{1,2} \cap L^\infty(B^2)\) and \(\psi(\varepsilon_1) = \lambda_1, \psi(\varepsilon_2) = \lambda_2\) be the corresponding solutions of (3.4). Then \(\Lambda := \lambda_1 - \lambda_2\) solves

\[
\begin{aligned}
\div [\nabla \Lambda P] &= \nabla ((\varepsilon_1 - \varepsilon_2)P) \cdot \nabla \perp \xi \quad \text{in } B^2, \\
\Lambda &= 0 \quad \text{on } \partial B^2.
\end{aligned}
\]

Since \(P\) takes values in \(SO(n)\) it satisfies the assumptions of Theorem 1.3 in [2] and we have

\[
||\Lambda||_{L^\infty(B^2)} + ||\nabla \Lambda||_{L^2(B^2)} \leq c \left(||\nabla \varepsilon_1 - \nabla \varepsilon_2||_{L^2(B^2)}||P||_{L^\infty(B^2)} + ||\varepsilon_1 - \varepsilon_2||_{L^\infty(B^2)}||\nabla P||_{L^2(B^2)}\right) \cdot ||\nabla \xi||_{L^2(B^2)} \leq c \sigma \left(||\nabla \varepsilon_1 - \nabla \varepsilon_2||_{L^2(B^2)} + ||\varepsilon_1 - \varepsilon_2||_{L^\infty(B^2)}\right).
\]

For \(\sigma\) small enough we conclude that \(\psi\) is a contraction. To show that \(\psi\) is a self-map from a small ball in \(W^{1,2} \cap L^\infty(B^2)\) into itself, we use again Theorem 1.3 in [2] to get

\[
||\Lambda||_{L^\infty(B^2)} + ||\nabla \Lambda||_{L^2(B^2)} \leq c ||\nabla \xi||_{L^2(B^2)} \left(||\nabla \varepsilon||_{L^2(B^2)} + (1 + ||\varepsilon||_{L^\infty(B^2)})||\nabla P||_{L^2(B^2)}\right).
\]

The Banach fixed point theorem yields a unique \(\varepsilon^* \in W^{1,2} \cap L^\infty(B^2, M(n))\) solving (3.4) and hence also (3.3) and with the estimate above we get

\[
||\varepsilon^*||_{L^\infty(B^2)} + ||\nabla \varepsilon^*||_{L^2(B^2)} \leq c \sigma.
\]

By the Poincaré lemma there exists \(B \in W^{1,2}(B^2)\) such that

\[
\nabla \perp B = -\nabla \varepsilon^* P + (id + \varepsilon^*) \nabla \perp \xi P
\]

and (3.1) is equivalent to

\[-div((id + \varepsilon^*)P \nabla u) = \nabla \perp B \cdot \nabla u.\]
Now that we have our equation in the desired divergence-free form, we can show the continuity of the solution \( u \) using the Hodge decomposition (see Corollary 10.70 in [7])

\[
(id + \varepsilon^*) P \nabla u = \nabla V + \nabla^\perp W
\]

and arguing as in [15]. \( \square \)

### 4 Proof of theorem 2.13

We split the proof of this result into several steps and present each step in a separate subsection.

#### 4.1 Gauge fixing

Following the work of de Longueville and Gastel in the proof of Theorem 4.1 (i) in [5] we repeatedly solve Neumann problems to find \( \Omega_1 \in W^{m-1,2}(B_r, so(n) \otimes \Lambda^1 \mathbb{R}^{2m}) \) such that

\[
\Delta^{m-2} \delta \Omega = -\eta \quad \text{in } B_r \quad \text{and} \quad ||\Omega||_{W^{m-1,2}(B_r)} \leq c ||\eta||_{W^{2-m,2}(B_r)} \leq c\sigma. \tag{4.1}
\]

Next we need the following higher order version of the Uhlenbeck gauge fixing result which is due to De Longueville and Gastel.

**Theorem 4.1** [Theorem 2.4 in [5]] Assume that \( m, n \in \mathbb{N} \) and \( B_r \subset \mathbb{R}^{2m} \) is a ball of radius \( r \). Then there is \( \varepsilon > 0 \) such that for all \( \Omega \in W^{m-1,2}(B_r, so(n) \otimes \Lambda^1 \mathbb{R}^{2m}) \) satisfying

\[
||\Omega||_{W^{m-1,2}(B_r)} < \varepsilon,
\]

there are functions \( P \in W^{m,2}(B_{r/2}; SO(n)) \) and \( \xi \in W^{m,2}(B_{r/2}, so(n) \otimes \Lambda^2 \mathbb{R}^{2m}) \) such that

\[
\Omega = P dP^{-1} + P \delta \xi P^{-1} \quad \text{on } B_{r/2} \tag{4.2}
\]

holds on \( B_{r/2} \). Moreover, we have the estimate

\[
||dP||_{W^{m-1,2}(B_{r/2})} + ||\delta \xi||_{W^{m-1,2}(B_{r/2})} \leq c ||\Omega||_{W^{m-1,2}(B_r)}. \tag{4.3}
\]

We apply this result for \( \sigma > 0 \) sufficiently small, and get \( \xi \in W^{m,2}(B_{1/2}^{2m}, so(n) \otimes \Lambda^2 \mathbb{R}^{2m}) \) and \( P \in W^{m,2}(B_{1/2}^{2m}, SO(n)) \) such that

\[
dP = P\Omega - \delta \xi P \quad \text{and} \quad ||dP||_{W^{m-1,2}(B_{1/2}^{2m})} + ||\delta \xi||_{W^{m-1,2}(B_{1/2}^{2m})} \leq c ||\Omega||_{W^{m-1,2}(B_r)}. \tag{4.4}
\]
4.2 Rewriting the system

We let \( \varepsilon \in W^{m,2} \cap L^{\infty}(B_{1/2}^{2m}, M(n)) \) and we multiply (2.4) with \((id + \varepsilon)P\) and calculate

\[
(id + \varepsilon)P\Delta^mu = (id + \varepsilon)P \left[ \sum_{k=0}^{m-1} \Delta^k(V_k, du) + \sum_{k=0}^{m-2} \Delta^k\delta(w_kdu) \right]
\]

\[
\Leftrightarrow \left[ \sum_{k=0}^{m-1} \Delta^k((id + \varepsilon)P)V_k - \sum_{k=0}^{m-2} d\Delta^k((id + \varepsilon)P)w_k + d\Delta^{m-1}((id + \varepsilon)P) \right] \cdot du
\]

\[
= \delta \left[ \sum_{l=0}^{m-1} \Delta^l((id + \varepsilon)P)\Delta^{m-l-1}du - \sum_{l=0}^{m-2} d\Delta^l((id + \varepsilon)P)\Delta^{m-l-1}u ight.
\]

\[- \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} \Delta^l((id + \varepsilon)P)\Delta^{k-l-1}d(V_k, du)
\]

\[+ \sum_{k=0}^{m-1} \sum_{l=0}^{k-1} d\Delta^l((id + \varepsilon)P)\Delta^{k-l-1}(V_k, du)
\]

\[- \sum_{k=0}^{m-2} \sum_{l=0}^{k} \Delta^l((id + \varepsilon)P)d\Delta^{k-l-1}\delta(w_kdu)
\]

\[+ \sum_{k=0}^{m-2} \sum_{l=0}^{k-1} d\Delta^l((id + \varepsilon)P)\Delta^{k-l-1}\delta(w_kdu) \right].
\]

(4.6)

The right-hand side of this system is already in divergence form, hence in order to obtain a conservation law we need to find \( \varepsilon \in W^{m,2} \cap L^{\infty}(B_{1/2}^{2m}, M(n)) \) such that

\[
\delta \left[ \sum_{k=0}^{m-1} \Delta^k((id + \varepsilon)P)V_k - \sum_{k=0}^{m-2} d\Delta^k((id + \varepsilon)P)w_k + d\Delta^{m-1}((id + \varepsilon)P) \right] = 0
\]

(4.7) on \( B_{1/2}^{2m} \). As in Sect. 3 we want to apply a fixed point argument to solve this problem. However to do this we need to have a certain control on the terms in (4.7) and the terms involving \( V_0 \) are problematic. We know that \( V_0 = d\eta + F \) and we control \( F \in W^{2-m, \frac{2m}{2m+1}}(B_{2m}^{2m}) \) by (2.5) but \( d\eta \in W^{1-m,2}(B_{2m}^{2m}) \) is a priori not bounded. Thus our goal is to remove \( d\eta \).

To do this we take a closer look at \( d\Delta^{m-1}((id + \varepsilon)P) \) and note that we can rewrite the highest order term \((id + \varepsilon)d\Delta^{m-1}P\) so that it cancels \((id + \varepsilon)Pd\eta\) in (4.7). To see this we use (2.2), (2.3) as well as (4.1) and (4.5).

\[
d\Delta^{m-1}P = d\Delta^{m-2}\delta(P\Omega - \delta\xi P)
\]

\[
= d\Delta^{m-2}(dP\Omega) + d\Delta^{m-2}(P\delta\Omega) - d\Delta^{m-2}(\ast d \ast (d \ast \xi P))
\]

\[
= \sum_{i=1}^{2m-2} c_i \nabla i P \nabla^{2m-2-i} \Omega - d(P \eta) + d\Delta^{m-2}(\ast(d \ast \xi \wedge d P))
\]

\[
= \sum_{i=1}^{2m-2} c_i \nabla i P \nabla^{2m-2-i} \Omega - dP \eta - P(V_0 - F) + d\Delta^{m-2}(\ast(d \ast \xi \wedge d P))
\]
with constants \(c_i \in \mathbb{N}_0\), \(1 \leq i \leq 2m - 2\) and

\[
\nabla^k = \begin{cases} 
\Delta \frac{k}{2}, & \text{if } k \text{ even,} \\
\frac{d}{\Delta} \frac{k-1}{2}, & \text{if } k \text{ odd.}
\end{cases}
\]

Plugging this back into (4.7) and rearranging we get

\[
\Delta(\Delta^{m-1} \varepsilon \cdot P) = \delta \left[ - \sum_{j=1}^{2m-2} \tilde{c}_j \nabla^j \varepsilon \nabla^{2m-1-j} P - (id + \varepsilon) \left( \sum_{i=1}^{2m-2} c_i \nabla i P \nabla^{2m-2-i} \Omega \right) - dP \eta + PF + d\Delta^{m-2}(*(d \ast \xi \wedge dP)) \right] - \sum_{k=1}^{m-1} \Delta^k((id + \varepsilon)P)w_k + \sum_{k=0}^{m-2} d\Delta^k((id + \varepsilon)P)w_k \right] \quad \text{in } B_{1/2}^{2m},
\]

(4.8)

where \(\tilde{c}_j\) are constants in \(\mathbb{N}_0\). Now that we have removed the “worst” terms we want to examine this equation further and take a closer look at the function spaces of the summands. We separate the \(\varepsilon\) component from the rest and use the embedding results for Lorentz-Sobolev spaces in Lemma 2.10 and Lemma 2.11 repeatedly. We use the notation \(D^k A \ast D^j B\) for any linear combination of \(D^k A\) and \(D^j B\) and \(D\) denotes the full derivative. For the first term we have

\[
\sum_{j=1}^{2m-2} D^j \varepsilon \ast D^{2m-1-j} P = \sum_{j=1}^{2m-2} W^{m-j,2} \cdot W^{-m+1+j,2},
\]

For the third and fourth term we get

\[
(id + \varepsilon)dP\eta = L^\infty \cdot W^{m-1,2} \cdot W^{2-m,2} \hookrightarrow L^\infty \cdot W^{2-m,\frac{2m}{m+1}},
\]

\[
(id + \varepsilon)PF = L^\infty \cdot L^\infty \cdot W^{2-m,\frac{2m}{m+1}+1}.
\]

The second term is of the from

\[
(id + \varepsilon) \left( \sum_{j=1}^{2m-3} D^j \Omega \ast D^{2m-2-j} P + \Omega \ast D^{2m-2} P \right)
\]

\[
= \sum_{j=1}^{2m-3} L^\infty \cdot W^{m-1-j,2} \cdot W^{-m+2+j,2} + L^\infty \cdot W^{m-1,2} \cdot W^{2-m,2}
\]

\[
\hookrightarrow \sum_{j=1}^{m-2} L^\infty \cdot W^{m-2+j,\frac{2m}{m+1+j}} \cdot L^\infty \cdot W^{m-1-j,\frac{2m}{3m-2-j}} + \sum_{j=m-1}^{m-3} L^\infty \cdot W^{m-1-j,\frac{2m}{3m-2-j}} + L^\infty \cdot W^{2-m,\frac{2m}{m+1}}
\]

where we used Lemma 2.11 in the first step and Lemma 2.10 with \(s = m - 2 - j, p = \frac{2m}{m+1+j}, t = j\) for \(j = 1, \ldots, m - 2\) and \(s = -m + 1 + j, p = \frac{2m}{3m-2-j}, t = 2m - 3 - j\)
for \( j = m - 1, \ldots, 2m - 3 \) in the second step. The fifth term follows in the same way

\[
(id + \varepsilon)d \Delta^{m-2}((dP \wedge d \ast \xi)) = (id + \varepsilon) \sum_{j=1}^{2m-2} D^j \xi \ast D^{2m-1-j} P
\]

\[
= \sum_{j=1}^{2m-2} L^\infty \cdot W^{m-j} \cdot W^{-m+1+j}.
\]

For the last two terms we apply again Lemma 2.11 and 2.10 with \( s = m - 2k - 1, p = \frac{2m}{m+2k-j}, t = 2k - j \) for \( 2k + 1 - m < m - 2k + j \) and \( s = 2k - j - m, p = \frac{2m}{3m-2k-1}, t = 2m - 2k - 1 \) for \( m - 2k + j \leq 2k + 1 - m \).

\[
\sum_{k=1}^{m-1} \Delta^k ((id + \varepsilon)P)V_k = \sum_{k=1}^{m-1} \left( \sum_{j=1}^{2k-1} D^j \varepsilon \ast D^{2k-j}P + (id + \varepsilon)\Delta^k P + \Delta^k \varepsilon P \right)V_k
\]

\[
= \sum_{k=1}^{m-1} \sum_{j=1}^{2k-1} W^{m-j} \cdot W^{m-2k+j} \cdot W^{2k+1-m} + \sum_{k=1}^{m-1} L^\infty \cdot W^{m-2k} \cdot W^{2k+1-m}.
\]

\[
\implies \sum_{j=1}^{2m-3} W^{m-j} \cdot W^{-m+1+j} \cdot L^\infty \cdot W^{2-m}.
\]

and analogously

\[
\sum_{k=0}^{m-2} \nabla \Delta^k ((id + \varepsilon)P)w_k
\]

\[
= \sum_{k=0}^{m-2} \left( \sum_{j=1}^{2k} D^j \varepsilon \ast D^{2k+1-j}P + (id + \varepsilon)\delta \Delta^k P + \delta \Delta^k \varepsilon P \right)w_k
\]

\[
= \sum_{k=0}^{m-2} \sum_{j=1}^{2k} W^{m-j} \cdot W^{m-2k+1-j} \cdot W^{2k+2-m} + \sum_{k=0}^{m-2} L^\infty \cdot W^{m-2k+1} \cdot W^{2k+2-m}.
\]

\[
\implies \sum_{j=1}^{2m-3} W^{m-j} \cdot W^{-m+1+j} + L^\infty \cdot W^{2-m}.
\]

Observe that all terms on the right-hand side of (4.8) consist of products \( W^{m-j} \cdot W^{j+1-m} \), \( j = 1, \ldots, 2m - 2 \) and \( L^\infty \cdot W^{2-m} \). Thus we can simplify (4.8) further...
and write
\[
\Delta (\Delta^{m-1} \varepsilon \cdot P) = \delta \left( \sum_{j=1}^{2m-2} D^j \varepsilon \star K_j + (id + \varepsilon) \star K_0 \right) \tag{4.9}
\]

with \( K_j \in W_j^{1-m,2}(B^{2m}_{1/2}) \), \( K_0 \in W^{2-m,2m/(m+1)}(B^{2m}_{1/2}) \). Moreover with (4.4) and (2.5) we estimate
\[
||K_0||_{W^{2-m,2m/(m+1)}(B^{2m}_{1/2})} + \sum_{j=1}^{2m-2} ||K_j||_{W_j^{1-m,2}(B^{2m}_{1/2})} \leq c \sigma. \tag{4.10}
\]

However the equation still contains distributions. To take care of these we apply the same technique as de Longueville and Gastel and use the representation of negative Lorentz-Sobolev spaces (see Lemma 2.8).

\[
\varepsilon = \sum_{|\alpha| \leq m-2} \partial^\alpha \varepsilon_\alpha, \quad \varepsilon_\alpha \in W^{2m-1,2m/(m+1)}(B^{2m}_{1/2}),
\]

\[
K_0 = \sum_{|\alpha| \leq m-2} \partial^\alpha K_0^\alpha, \quad K_0^\alpha \in L^{2m/(m+1)}(B^{2m}_{1/2}),
\]

\[
K_j = \sum_{|\alpha| \leq m-1-j} \partial^\alpha K_j^\alpha, \quad K_j^\alpha \in L^2(B^{2m}_{1/2}).
\]

Together with (4.10) we get
\[
\sum_{|\alpha| \leq m-1-j} ||K_j^\alpha||_{L^2(B^{2m}_{1/2})} \leq c ||K_j||_{W_j^{1-m,2}(B^{2m}_{1/2})} \leq c \sigma,
\]

\[
\sum_{|\alpha| \leq m-2} ||K_0^\alpha||_{L^{2m/(m+1)}(B^{2m}_{1/2})} \leq c ||K_0||_{W^{2-m,2m/(m+1)}(B^{2m}_{1/2})} \leq c \sigma. \tag{4.12}
\]

Note that we assume \( \varepsilon \in W^{m+1,2m/(m+1)} \) for this representation, which is slightly better than the original assumption \( \varepsilon \in W^{m,2} \cap L^\infty \). We will see that we can solve (4.8) in this better space and since \( W^{m+1,2m/(m+1)}(B^{2m}) \hookrightarrow W^{m,2} \cap L^\infty(B^{2m}) \) we get the desired result.

This new representation allows us to shift derivatives away from the distributional part. Let \( c_{\alpha \gamma}, c_{\beta \gamma} \in \mathbb{Z} \). With the product rule we get for \( j = 1, \ldots, m-2 \)
\[
D^j \varepsilon \star K_j = \sum_{|\alpha| \leq m-2 \atop |\beta| \leq m-1-j} D^j \partial^\alpha \varepsilon_\alpha \star \partial^\beta K_j^\beta = \sum_{|\alpha| \leq m-2 \atop |\beta| \leq m-1-j} \sum_{\gamma \leq m-2} \partial^\gamma (c_{\beta \gamma} \partial^\beta-\gamma \partial^\alpha D^j \varepsilon_\alpha \star K_j^\beta)
\]

The case \( j = 0 \) follows analogously
\[
(id + \varepsilon) \star K_0 = \sum_{|\gamma| \leq m-2} \partial^\gamma K_0^\gamma + \sum_{|\alpha| \leq m-2 \atop |\beta| \leq m-2} \partial^\alpha \varepsilon_\alpha \star \partial^\beta K_0^\beta
\]

\[
= \sum_{|\gamma| \leq m-2} \partial^\gamma K_0^\gamma + \sum_{|\alpha| \leq m-2 \atop |\beta| \leq m-2} \sum_{\gamma \leq m-2} \partial^\gamma (c_{\beta \gamma} \partial^\beta-\gamma \partial^\alpha \varepsilon_\alpha \star K_0^\beta).
\]

For \( j = m-1, \ldots, 2m-2 \) with \( |\alpha| \leq j + 1 - m \) we get
\[
D^j \varepsilon \star K_j = \sum_{|\alpha| \leq m-2} D^j \partial^\alpha \varepsilon_\alpha \star K_j = \sum_{|\alpha| \leq m-2 \atop |\beta| \leq m} \sum_{\gamma \leq m} \partial^\gamma (c_{\alpha \gamma} D^j \varepsilon_\alpha \star \partial^\alpha-\gamma K_j).
\]
If $|\alpha| > j + 1 - m$ we choose $\beta \leq \alpha$ with $|\beta| = j + 1 - m$ and

$$D^j \varepsilon K_j = \sum_{|\alpha| \leq m-2} D^j \partial^\alpha \varepsilon_\alpha K_j$$

$$= \sum_{|\alpha| \leq m-2} \sum_{\gamma \leq \beta \atop |\beta| = j+1-m} \partial^\gamma (c_{\beta\gamma} \partial^\alpha-\beta D^j \varepsilon_\alpha \partial^\beta-\gamma K_j).$$

We rewrite the left-hand side of (4.9) in the same way.

$$\Delta(\Delta^{m-1} \varepsilon \cdot P) = \sum_{|\alpha| \leq m-2} \Delta(\Delta^{m-1} \partial^\alpha \varepsilon_\alpha \cdot P)$$

$$= \sum_{|\alpha| \leq m-2} \sum_{\gamma \leq \alpha} \partial^\gamma \Delta(c_{\alpha\gamma} \Delta^{m-1} \varepsilon_\alpha \partial^\alpha-\gamma P)$$

$$= \sum_{|\gamma| \leq m-2} \partial^\gamma \Delta(\Delta^{m-1} \varepsilon_\gamma \cdot P) + \sum_{|\alpha| \leq m-2} \sum_{\gamma < \alpha} \partial^\gamma \Delta(c_{\alpha\gamma} \Delta^{m-1} \varepsilon_\alpha \partial^\alpha-\gamma P).$$

For the last term note that $P \in W^{m,2}(B_{1/2}^n, \text{SO}(n))$. Thus we identify $P$ with $K_{2m-1}$ and write

$$\sum_{|\alpha| \leq m-2} \sum_{\gamma < \alpha} \partial^\gamma \Delta(\Delta^{m-1} \varepsilon_\gamma \cdot P)$$

$$= \delta \left[ \sum_{|\alpha| \leq m-2} \sum_{\gamma < \alpha} \sum_{i=0}^1 \partial^\gamma \left( c_{\alpha\gamma} D^{2m-2-i} \varepsilon_\alpha \partial^\alpha-\gamma D^{1-i} K_{2m-1} \right) \right].$$

Putting all of this together we get an equation equivalent to (4.7)

$$\sum_{|\gamma| \leq m-2} \partial^\gamma \Delta(\Delta^{m-1} \varepsilon_\gamma \cdot P)$$

$$= \delta \left[ \sum_{|\gamma| \leq m-2} \partial^\gamma K_0^\gamma + \sum_{|\alpha| \leq m-2} \sum_{\gamma < \alpha} \sum_{|\beta| \leq m-2} \partial^\gamma (c_{\beta\gamma} \partial^\alpha-\gamma \partial^\alpha \varepsilon_\alpha K_0^\beta) \right]$$

$$+ \sum_{j=1}^{m-2} \sum_{|\alpha| \leq m-2} \sum_{|\gamma| \leq \beta} \partial^\gamma \left( c_{\beta\gamma} \partial^\alpha-\gamma \partial^\alpha \varepsilon_\alpha \partial^\beta-\gamma D^j \varepsilon_\alpha K_j^\beta \right)$$

$$+ \sum_{j=m-1}^{2m-2} \sum_{|\alpha| \leq j+1-m} \sum_{|\gamma| \leq \alpha} \partial^\gamma \left( c_{\alpha\gamma} D^j \varepsilon_\alpha \partial^\alpha-\gamma K_j \right)$$

$$+ \sum_{j=m-1}^{2m-2} \sum_{|\alpha| > j+1-m} \sum_{|\gamma| \leq \beta} \partial^\gamma \left( c_{\beta\gamma} \partial^\alpha-\beta \partial^\beta-\gamma D^j \varepsilon_\alpha \partial^\beta-\gamma K_j \right)$$

$$+ \sum_{i=0}^{1} \sum_{|\alpha| \leq m-2} \sum_{\gamma < \alpha} \partial^\gamma \left( c_{\alpha\gamma} D^{2m-2-i} \varepsilon_\alpha \partial^\alpha-\gamma D^{1-i} K_{2m-1} \right).$$
We simplify this further by setting
\[
\sum_{|\gamma| \leq m-2} \partial^{\nu} \Delta (\Delta^{m-1} \mathcal{E}_{\nu} \cdot P) =: \delta \left[ \sum_{|\gamma| \leq m-2} \partial^{\nu} \left( \langle \mathcal{E}, K \rangle_{\nu} + K_{0}^{\nu} \right) \right]
\]  \quad (4.13)
with
\[
||K_{0}^{\nu}||_{L^{2m-1-|\nu|} \rightarrow (B_{1/2}^{2m})} + ||\langle \mathcal{E}, K \rangle_{\nu}||_{L^{2m-1-|\nu|} \rightarrow (B_{1/2}^{2m})} \\
\leq c\sigma \left( \sum_{|\alpha| \leq m-2} ||\mathcal{E}_{\alpha}||_{W^{2m-1, \frac{2m}{2m-1-|\alpha|}, 1} (B_{1/2}^{2m})} + 1 \right)
\]  \quad (4.14)
for every \( \gamma \) with \( |\gamma| \leq m-2 \). To see this last inequality we use (4.12) and estimate each term separately
\[
||K_{0}^{\nu}||_{L^{2m-1-|\nu|} \rightarrow (B_{1/2}^{2m})} \leq c||K_{0}^{\nu}||_{L^{2m+1} \rightarrow (B_{1/2}^{2m})} \leq c\sigma;
\]
\( K_{0}^{\nu} \in L^{2m+1} \rightarrow (B_{1/2}^{2m}) \) and \( L^{2m+1} \rightarrow (B_{1/2}^{2m}) \) by Lemma 2.2. Further we have
\[
W^{2m-1-|\beta|+|\nu|-|\alpha|} \rightarrow L^{2m-1-|\nu|} \rightarrow L^{\frac{2m}{2m-1-|\alpha|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\nu|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\gamma|}, 1} \rightarrow (B_{1/2}^{2m})
\]
by Lemma 2.4 and Lemma 2.2 since \( |\beta| \leq m-j-1 \). With Lemma 2.1 and 2.2 we have
\[
L^{\frac{2m}{2m-1-|\gamma|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\nu|}, 1} \quad \text{and since} \quad \gamma \leq \beta
\]
\[
||\partial^{\beta-\nu} \partial^{\alpha} D^{j} \mathcal{E}_{\alpha} K_{\beta}^{\nu}||_{L^{2m-1-|\nu|} \rightarrow (B_{1/2}^{2m})} \\
\leq c||\mathcal{E}_{\alpha}||_{W^{2m-1-j-|\alpha|-|\beta|+|\nu|, \frac{2m}{2m-1-|\alpha|}, 1} (B_{1/2}^{2m})} ||K_{\beta}^{\nu}||_{L^{2} (B_{1/2}^{2m})} \\
\leq c\sigma ||\mathcal{E}_{\alpha}||_{W^{2m-1-\frac{2m}{2m-1-|\alpha|}, 1} (B_{1/2}^{2m})}.
\]
The remaining terms follow in a similar way. With Lemma 2.4
\[
W^{2m-1-|\beta|+|\nu|-|\alpha|, \frac{2m}{2m-1-|\gamma|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\nu|}, 1} \rightarrow (B_{1/2}^{2m})
\]
and by Lemma 2.1 and 2.2 with \( |\beta| \leq m-2 \)
\[
L^{\frac{2m}{2m-1-|\nu|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\beta|+|\nu|+|\gamma|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\gamma|}, 1} \rightarrow (B_{1/2}^{2m}).
\]
With this and \( \gamma \leq \beta \)
\[
||\partial^{\beta-\nu} \partial^{\alpha} \mathcal{E}_{\alpha} K_{0}^{\beta}||_{L^{2m-1-|\nu|} \rightarrow (B_{1/2}^{2m})} \\
\leq c||\mathcal{E}_{\alpha}||_{W^{2m-1-|\alpha|-|\beta|+|\nu|, \frac{2m}{2m-1-|\alpha|}, 1} ||K_{0}^{\beta}||_{L^{2} (2m-1-|\nu|, \frac{2m}{2m-1-|\gamma|}, 1)} \\
\leq c\sigma ||\mathcal{E}_{\alpha}||_{W^{2m-1, \frac{2m}{2m-1-|\alpha|}, 1}}.
\]
For the next term we have with Lemma 2.4 and 2.1
\[
W^{2m-1-j, \frac{2m}{2m-1-|\alpha|}, 1} \rightarrow W^{j+1-m-|\alpha|+|\gamma|, 2} \rightarrow L^{\frac{2m}{2m-1-|\gamma|}, 1} \rightarrow L^{\frac{2m}{2m-1-|\nu|}, 1} \rightarrow (B_{1/2}^{2m})
\]
so that with $\gamma \leq \alpha$

$$||D^j \varepsilon_{\alpha} \cdot \partial^\alpha - \gamma K_j|| L^{2m-1-|\gamma|,1}_{\gamma} \leq c ||\varepsilon_{\alpha}|| W^{2m-1-\frac{j}{2}} \leq c|\varepsilon_{\alpha}|| W^{2m-1, 2m-1-|\gamma|,1}. $$

In the fifth term we use $|\beta| = j + 1 - m$, Lemma 2.4 and 2.1 to get

$$W^{2m-1-|\alpha|+|\beta|-j, \frac{2m}{2m-1-|\gamma|}} \to L^{\frac{2m}{2m-1-|\gamma|}, \frac{2m}{2m-1-|\gamma|}} \to L^{\frac{2m}{2m-1-|\gamma|}, 2^{-m}} \to L^{\frac{2m}{2m-1-|\gamma|}, 1}(B^{1/2}_{1/2})$$

and

$$||\partial^\alpha - \beta D^j \varepsilon_{\alpha} \cdot \partial^\beta \gamma K_j|| L^{2m-1-|\gamma|,1}_{\gamma} \leq c ||\varepsilon_{\alpha}|| W^{2m-1-\frac{j}{2}} \leq c|\varepsilon_{\alpha}|| W^{2m-1, 2m-1-|\gamma|,1}. $$

Finally we estimate for $i = 0, 1$ with (4.4) and $\gamma \leq \alpha$

$$||D^{2m-2-i} \varepsilon_{\alpha} \cdot \partial^\alpha - \gamma D^{1-i} K_{2m-1}|| L^{2m-1-|\gamma|,1}_{\gamma} \leq ||\varepsilon_{\alpha}|| W^{1+i, \frac{2m}{2m-1-|\gamma|},1} ||| P|| W^{m-|\alpha|+|\gamma|-1+i,2} \leq c|\varepsilon_{\alpha}|| W^{2m-1, \frac{2m}{2m-1-|\gamma|},1}. $$

and this proves (4.14).

4.3 The fixed point argument

Instead of solving (4.13) we solve the system

$$\Delta(\Delta^{m-1} \varepsilon_{\gamma} \cdot P) = \delta(\varepsilon, K_{\gamma} + K_{\gamma}^0) \quad \text{for every } \gamma \text{ with } |\gamma| \leq m - 2. \quad (4.15)$$

To do this we apply a fixed point argument: Let $X_\gamma := \{u \in M(n) : ||u|| \leq m, \frac{2m}{2m-1-|\gamma|}, (B^{2m}_{1/2}) < \infty\}$ and $X = \bigoplus_{|\gamma| \leq m-2} X_\gamma$. We define maps $\psi_\gamma : X_\gamma \to X_\gamma$ by

$$\psi_\gamma : \varepsilon_{\gamma} \mapsto \text{solution } \lambda_{\gamma} \text{ of } (4.16)$$

with

$$\begin{cases} \Delta(\Delta^{m-1} \lambda_{\gamma} \cdot P) = \delta(\varepsilon, K_{\gamma} + K_{\gamma}^0) & \text{in } B^{2m}_{1/2}, \\ \Delta^j \lambda_{\gamma} = 0 & \text{on } \partial B^{2m}_{1/2} \text{ for } j = 0, \ldots, m - 1. \end{cases} \quad (4.16)$$

Let $\hat{\lambda} = \sum_{|\gamma| \leq m-2} \lambda_{\gamma}$ and $\hat{\varepsilon} = \sum_{|\gamma| \leq m-2} \varepsilon_{\gamma}$, where $\lambda_{\gamma}$ is a solution of (4.16) for every $\gamma$ with corresponding $\varepsilon_{\gamma}$. Let $\Psi = \bigoplus_{|\gamma| \leq m-2} \psi_{\gamma}$ and

$$\mu := ||\hat{\varepsilon}|| X := \sum_{|\gamma| \leq m-2} ||D^{2m-1} \varepsilon_{\gamma}|| L^{\frac{2m}{2m-1-|\gamma|},1}(B^{2m}_{1/2}).$$
We apply Lemma 2.12 and (4.14) to estimate
\[
||D^{2m-1}\lambda_1\gamma||_{L^{2m-1}B_{1/2}^{2m}} \leq c||\langle\varepsilon, K\rangle_\gamma + K^{\gamma}_0||_{L^{2m-1}B_{1/2}^{2m}}
\]
\[
\leq c\sigma \left( \sum_{|\gamma|\leq m-2} ||\varepsilon_\gamma||_{W^{2m-1, \frac{2m}{2m-1-|\gamma|}}B_{1/2}^{2m}} + 1 \right)
\]
\[
\leq c_1\sigma (\mu + 1).
\]
We choose \( \sigma < \frac{\mu}{2c_1(\mu+1)} \) to get
\[
||\hat{\lambda}\||_X \leq \frac{\mu}{2}.
\]
Next we show that \( \psi_\gamma \) is a contraction. Let \( \lambda_1^\gamma, \lambda_2^\gamma \) be solutions of (4.16) with \( \varepsilon_1^\gamma, \varepsilon_2^\gamma \) respectively. Then \( \Lambda_\gamma : = \lambda_1^\gamma - \lambda_2^\gamma \) is a solution of
\[
\begin{aligned}
\Delta (\Delta^{m-1}\Lambda_\gamma \cdot P) &= \delta (\langle\varepsilon_1^\gamma - \varepsilon_2^\gamma, K\rangle_\gamma) \quad \text{in } B_{1/2}^{2m}, \\
\Delta^{j}\Lambda_\gamma &= 0 \quad \text{on } \partial B_{1/2}^{2m} \text{ for } j = 0, \ldots, m-1.
\end{aligned}
\]
Applying Lemma 2.12 and (4.14) again yields
\[
||D^{2m-1}\lambda_1^\gamma - D^{2m-1}\lambda_2^\gamma||_{L^{2m-1}B_{1/2}^{2m}} \leq c\sigma \sum_{|\gamma|\leq m-2} ||\varepsilon_1^\gamma - \varepsilon_2^\gamma||_{W^{2m-1, \frac{2m}{2m-1-|\gamma|}}B_{1/2}^{2m}}.
\]
With this we have
\[
||\hat{\lambda}_1^\gamma - \hat{\lambda}_2^\gamma||_X \leq c_2\sigma ||\hat{\varepsilon}_1^\gamma - \hat{\varepsilon}_2^\gamma||_X.
\]
Choosing \( \sigma < \min\{\frac{\mu}{2c_1(\mu+1)}, \frac{1}{2c_2}\} \) shows that \( \Psi \) is a contraction. Now we can apply the Banach fixed point theorem which yields a unique \( \hat{\varepsilon}^* \in X \) solving (4.15) and by Lemma 2.12 and (4.14)
\[
\sum_{|\gamma|\leq m-2} ||\varepsilon_\gamma^*||_{W^{2m-1, \frac{2m}{2m-1-|\gamma|}}B_{1/2}^{2m}} \leq c\sigma.
\]
Thus we have
\[
0 = \delta \left( d\Delta^{m-1}\varepsilon_\gamma^* \cdot P - \langle\varepsilon_\gamma^*, K\rangle_\gamma + K^{\gamma}_0 \right)
\]
for every \( \gamma \) with \( |\gamma| \leq m-2 \). What is left to show is that these \( \varepsilon_\gamma^* \) are the Sobolev functions in the representation (4.11) of \( \varepsilon \) and this \( \varepsilon \) solves (4.7).

4.4 Going back to the original system

In order to go back to our original system, we reverse the abbreviations we made at the beginning to get a detailed look at (4.17). To do this we go back to (4.8). As we have seen before, each term of this equation is a product of a distribution and a Sobolev function. More precisely, the terms are of the form \( L^\infty \cdot W^{2m-2m, \frac{2m}{m-1}} \) and \( W^{m-k, 2} \cdot W^{m+1+k, 2} \), \( k = \ldots \)
Then we shift derivatives to get an equation of the form
\[ \sum \] for 1, ..., 2m – 2. We use the following representations for the distributions according to Lemma 2.8

\[ F P - d \Delta^{m - 2} \delta(\Omega P) + d \Delta^{m - 2} \delta \Omega P - d \Delta^{m - 2}(*dP \wedge d * \xi) \]

\[ = \sum_{|\alpha| \leq m - 2} \left( F P - d \Delta^{m - 2} \delta(\Omega P) + d \Delta^{m - 2} \delta \Omega P - d \Delta^{m - 2}(*dP \wedge d * \xi) \right)^\alpha, \]

\[ \left( F P - d \Delta^{m - 2} \delta(\Omega P) + d \Delta^{m - 2} \delta \Omega P - d \Delta^{m - 2}(*dP \wedge d * \xi) \right)^\alpha \in L_{m+1}^{2m+1}(B_{1/2}^{2m}) \]

\[ \Delta^k P \cdot V_k = \sum_{|\alpha| \leq m - 2} \partial^\alpha(\Delta^k PV_k)^\alpha, \quad (\Delta^k PV_k)^\alpha \in L_{m+1}^{2m+1}(B_{1/2}^{2m}), \quad k \neq 0 \]

\[ d \Delta^k P w_k = \sum_{|\alpha| \leq m - 2} \partial^\alpha(\Delta^k P w_k)^\alpha, \quad (d \Delta^k P w_k)^\alpha \in L_{m+1}^{2m+1}(B_{1/2}^{2m}) \]

\[ \nabla^{2k-l} P \cdot V_k = \sum_{|\alpha| \leq m - 1 - l} \partial^\alpha(\nabla^{2k-l} PV_k)^\alpha, \quad (\nabla^{2k-l} PV_k)^\alpha \in L^2(B_{1/2}^{2m}), \quad k \neq 0 \]

\[ \nabla^{2k+1-l} P \cdot w_k = \sum_{|\alpha| \leq m - 1 - l} \partial^\alpha(\nabla^{2k+1-l} P w_k)^\alpha, \quad (\nabla^{2k+1-l} P w_k)^\alpha \in L^2(B_{1/2}^{2m}), \]

\[ \nabla^{2m-1-k} P = \sum_{|\alpha| \leq m - 1 - k} \partial^\alpha(\nabla^{2m-1-k} P)^\alpha, \quad (\nabla^{2m-1-k} P)^\alpha \in L^2(B_{1/2}^{2m}). \]

Then we shift derivatives to get an equation of the form \[ \sum_{|\gamma| \leq m - 2} \partial^\gamma (...) \gamma = 0 \] as in (4.13). Using this we see that (4.17) is equivalent to

\[ 0 = \delta \left[ \sum_{1 \leq k \leq m - 2} \sum_{|\alpha| \leq m - 1 - k} c_{k, \alpha \gamma} \partial^{\alpha - \gamma} \nabla^{k} \partial^{\beta} \varepsilon_{\beta}^{\alpha}(\nabla^{2m-1-k} P)^{\alpha} \right. \]

\[ + \sum_{m - 1 \leq k \leq 2m - 1} \sum_{|\alpha| \leq k + 1 - m} c_{k, \alpha \gamma} \nabla^{k} \varepsilon_{\alpha}^{\alpha} \partial^{\alpha - \gamma} \nabla^{2m-1-k} P \]

\[ + \sum_{m - 1 \leq k \leq 2m - 1} \sum_{|\alpha| \geq m - 1 - k} c_{k, \alpha \beta \gamma} \partial^{\alpha - \gamma} \nabla^{k} \varepsilon_{\alpha}^{\alpha} \partial^{\beta - \gamma} \nabla^{2m-1-k} P \]

\[ + \left( F P - d \Delta^{m - 2} \delta(\Omega P) + d \Delta^{m - 2} \delta \Omega P - d \Delta^{m - 2}(*dP \wedge d * \xi) \right)^\gamma \]

\[ + \sum_{|\alpha|, |\beta| \leq m - 2} c_{\beta \gamma} \partial^{\beta - \gamma} \partial^{\alpha} \varepsilon_{\alpha}^{*} \left( F P - d \Delta^{m - 2} \delta(\Omega P) + d \Delta^{m - 2} \delta \Omega P \right. \]

\[ - d \Delta^{m - 2}(*dP \wedge d * \xi) \left. \right)^\beta \]

\[ + \sum_{k=1}^{m - 1} (\Delta^k PV_k)^\gamma + \sum_{k=0}^{m - 1} \sum_{|\alpha|, |\beta| \leq m - 2} c_{\beta \gamma} \partial^{\beta - \gamma} \partial^{\alpha} \varepsilon_{\alpha}^{*}(\Delta^k PV_k)^{\beta} \]

\[ + \sum_{k=1}^{m - 1} \sum_{1 \leq l \leq m - 2} \sum_{|\alpha| \leq l + 1 - m} c_{l, \alpha \beta} \partial^{\alpha - \gamma} \nabla^{l} \partial^{\beta} \varepsilon_{\beta}^{*}(\nabla^{2k-l} PV_k)^{\alpha} \]
By the Poincaré Lemma (see Lemma 10.68 in [7]) there exist \( B_\gamma \in W^1, \frac{2m}{m+2} - |\gamma| \) \((B^{2m}_{1/2}, \mathbb{R}^{n \times n} \otimes \wedge^2 \mathbb{R}^m)\) for \(|\gamma| \leq m - 2\) such that

\[
\delta B_\gamma = \ldots
\]

Now we transform \( \hat{\varepsilon}^* = \sum_{|\gamma| \leq m-2} \varepsilon^*_\gamma \) and \( \hat{B} = \sum_{|\gamma| \leq m-2} B_\gamma \) back. Then we have \( \varepsilon \in W^{m+1, \frac{2m}{m+2} - 1}_{m-1} (B^{2m}_{1/2}, M(n)) \) with

\[
||\varepsilon||_{W^{m+1, \frac{2m}{m+2} - 1}_{m-1} (B^{2m}_{1/2})} + ||\varepsilon||_{L^\infty (B^{2m}_{1/2})} \leq c_\sigma
\]

and

\[
\varepsilon = \sum_{|\gamma| \leq m-2} \partial^\gamma \varepsilon^*_\gamma \quad \text{solves (4.7).}
\]

Further \( B = \sum_{|\gamma| \leq m-2} \partial^\gamma B_\gamma \in W^{2-m, 2}_{m-2} (B^{2m}_{1/2}, \mathbb{R}^{n \times n} \otimes \wedge^2 \mathbb{R}^m) \) with

\[
\delta B = \sum_{k=0}^{m-1} \Delta^k ((id + \varepsilon) P)V_k - \sum_{k=0}^{m-2} d\Delta^k ((id + \varepsilon) P)w_k + d\Delta^{m-1} ((id + \varepsilon) P)
\]
and

\[
\delta \left[ \sum_{l=0}^{m-1} \Delta^l ((id + \varepsilon)P) \Delta^{m-l-1} du - \sum_{l=0}^{m-2} d\Delta^l ((id + \varepsilon)P) \Delta^{m-l-1} u 
- \sum_{k=0}^{m-1} \sum_{k=0}^{m-1-k-1} \Delta^l ((id + \varepsilon)P) \Delta^{k-l-1} d(V_k, du) 
+ \sum_{k=0}^{m-1} \sum_{l=0}^{m-l-1-k} d\Delta^l ((id + \varepsilon)P) \Delta^{k-l-1} (V_k, du) 
- \sum_{k=0}^{m-2} \sum_{k=0}^{m-2-k-1} \Delta^l ((id + \varepsilon)P) d\Delta^{k-l-1} \delta(w_k du) 
+ \sum_{k=0}^{m-2} \sum_{l=0}^{m-2-k-1} d\Delta^l ((id + \varepsilon)P) \Delta^{k-l-1} \delta(w_k du) - (B, du) \right] = 0.
\]

### 4.5 Regularity

To show (iii) we abbreviate the conservation law (2.6)

\[
\Delta ((id + \varepsilon)P \Delta^{m-1} u) + \delta C = 0 \quad \text{on } B_{1/2}^{2m},
\]

where \( C \in W^{2-m,2m} \cap L^\infty \). Since \( \varepsilon \in W^{m+1,2m} \cap L^\infty \) and \( \Delta^{m-1}u_r \in W^{2-m,2m} \), we have

\[
(id + \varepsilon)P \Delta^{m-1} u \in W^{2-m,2m}. \quad (4.19)
\]

Set \( f = (id + \varepsilon)P \Delta^{m-1} u \). Then

\[
-\Delta f = \delta C \quad \text{on } B_{1/2}^{2m}.
\]

By Theorem 6.2 in [4] we get \( f \in W^{3-m,2m} \cap L^\infty \) on a smaller ball with radius \( 0 < \lambda < 1/2 \). Since \( (id + \varepsilon)P \) is invertible we rewrite (4.19)

\[
\Delta^{m-1} u = [(id + \varepsilon)P]^{-1} f
\]

and \( \Delta^{m-1} u \in W^{3-m,2m} \cap L^\infty \). But this means \( u \in W^{m+1,2m} \cap L^\infty \) and \( B_\lambda^{2m} \) (see Theorem 2.3 in [5]).

Up until now we have assumed that \( \sigma \) is arbitrarily small so that it satisfies the assumptions of Theorem 4.1 and the fixed point argument. A priori this is not true for components \( V_k, w_k \) of a system of the form (2.4). However any solution \( u \) is continuous. To see this we rescale \( u \) (see [4] for a detailed proof). Let \( x_0 \in B_{2m}^{2m} \) and \( r > 0 \) small enough so that \( u_r : B_{2m}^{2m} \to \mathbb{R}^n \), \( u_r(x) := u(x_0 + rx) \) is a solution of (2.4) on \( B_{2m}^{2m} \) with corresponding rescaled components \( V_{k,r} \) and \( w_{k,r} \),

\[
\sigma_r := \sum_{k=0}^{m-2} ||w_{k,r}||_{W^{2k+2-m,2}(B_{2m}^{2m})} + \sum_{k=1}^{m-1} ||V_{k,r}||_{W^{2k+1-m,2}(B_{2m}^{2m})}
+ ||\eta_r||_{W^{-2-m,2}(B_{2m}^{2m})} + ||F_r||_{W^{-2-m,2m} \cap L^1(B_{2m}^{2m})}.
\]
σ_r < σ_0 and B^{2m}_r(x_0) ⊂ B^{2m}. By the above we have \( u_r \in C^0(B^{2m}_\lambda) \) which is the same as \( u \in C^0(B^{2m}_r(x_0)) \). A simple covering argument yields \( u \in C^0(B^{2m}) \). □

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