NON-ARCHIMEDEAN FRÉCHET ALGEBRAS AND THE LOOP SPACE OF A HYPERSURFACE COMPLEMENT

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ABSTRACT. We study the space, $\mathcal{L}^d$, of loops into a hypersurface complement, and show that the corresponding topological algebra of Laurent series with coefficients in $\mathcal{O}(\mathcal{L}^d)$ is a topological localisation of $\mathcal{O}(\mathcal{L}^d)$. This requires introducing a small amount of non-Archimedean functional analysis. In particular we work with topological algebras whose topology is generated by a family of sub-multiplicative, non-Archimedean semi-norms.

1. INTRODUCTION

If $X$ is a variety over a field, $k$, of characteristic 0, then we write $LX$ for its space of algebraic loops. We have by definition $LX(R) = X(R((z)))$. $LX$ can be thought of as roughly the mapping space $\text{Map}(D^*, X)$, where $D^* := \text{spec}(k((z)))$ is the formal punctured disc, and as such is an algebraic analogue of the space $\text{Map}(S^1, M)$ of smooth loops into a manifold. The space $LX$ has been studied by numerous authors from a variety of perspectives, we mention here the works [2], [4] and [3] in particular. A familiar feature is the presence of topologies on the algebras of functions under consideration, notably the so-called Tate topological $k$-vector space $k((z))$, cf. [2] for a discussion of Tate objects in algebraic geometry.

If $X$ is affine then $LX$ is representable by an ind-affine scheme with transition maps closed embeddings. The algebra $\mathcal{O}(LX)$ is thus naturally a topological $k$-algebra with a basis of open neighbourhoods at 0 consisting of ideals. We can form then the algebra $\mathcal{O}(LX)\{z\}$ consisting of two way infinite Laurent series with Laurent tails tending to 0 topologically. There is a natural morphism $\text{ev}_X : \mathcal{O}(X) \to \mathcal{O}(LX)\{z\}$ which for $X = \mathbb{A}^1$ corresponds to the universal Laurent series. We can think of this, informally, as corresponding geometrically to an evaluation map $\text{Map}(D^*, X) \times D^* \to X$.

Our goal in this note is to show that, in a sense that we will make precise, this mapping correspondence is surprisingly local on $X$, at least when $X$ is open inside an affine space $\mathbb{A}^d$ (although we conjecture it is true for smooth varieties...
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More precisely we will prove, writing $A^d$ for the complement of the hypersurface $f = 0$, the following theorem.

**Theorem 1.1.** $\mathcal{O}(LA^d)\{z\}$ is obtained as the Cauchy completion of the localisation of $\mathcal{O}(LA^d)\{z\}$ at the element $ev_{A^d}(f)$.

In order to do this we need to find a convenient category of topological $k$-algebras to which $\mathcal{O}(LA^d)\{z\}$ belongs in which to form the localisation. This is why we will work with non-Archimedean Fréchet algebras. Let us note that this result stands in stark contrast to the case of $\mathcal{O}(LX)$ in place of $\mathcal{O}(LX)\{z\}$. Firstly we lack a candidate element at which to localise, as $LX$ does not naturally live over $X$ because the punctured disc $D^*$ has no $k$-points at which to evaluate a loop. Secondly, if $U$ is open in $X$, then $LU$ is very far from open in $LX$. Indeed $LA^1$ is a vector space and $LG_m$ is highly non-reduced. Note that in the work [4] a variant of the loop space is introduced which has some desirable properties that $LX$ lacks, nonetheless this variant does not have $k$-points $X(k((z)))$, and we will not deal with it in this note.

2. Preliminary Notions

2.1. Loops. We will now introduce the geometric objects of study. The initiated reader can safely skim this section so as to fix notation.

**Definition 2.1.** If $X$ is a $k$-scheme then the functor $LX : \text{Alg}_k \to \text{Sets}$ is defined by $LX(R) := X(R((z)))$ and referred to as the loop space of $X$.

We have the following standard lemma, the proof of which we include so as to fix some notation.

**Lemma 2.1.** If $X$ is a finite type $k$-scheme then $LX$ is representable by an ind-affine scheme which is a countable colimit along closed embeddings.

**Proof.** Let $X$ be given as the zero locus of some polynomials $f_1, \ldots, f_c$ in variables $x^1, \ldots, x^d$. For each $n \geq 0$ consider variables $x^i_j$ for $i = 1, \ldots, d$ and $j \geq -n$. Write $x^i(z) := \sum_{j \geq -n} x^i_j z^j$ with $z$ a formal variable. Let us write $L^nX$ for the closed subset of the affine space with coordinates $x^i_j$ as before, subject to the relations implied by the equations of Laurent series $f_l(x^1(z), \ldots, x^d(z)) = 0$, for $l = 1, \ldots, c$. $L^nX$ is naturally a closed subscheme of $L^{n+1}X$ and we form the ind-scheme $LX := \text{colim} L^nX$. It is easily seen that this represents the desired functor. \qed
Remark. • Note that the schemes $L^n X$ do not have intrinsic meaning as functors of $X$, we had to choose an embedding to make sense of them. Nonetheless the colimit of course has intrinsic meaning, representing as it does the functor $LX$.

• In the particular case of $A^1$ we have that $\mathcal{O}(LA^1) \cong \lim_n k[x_i \mid i \geq -n]$, which we will consider as a topological $k$-algebra in the obvious way.

**Definition 2.2.** Let $A$ be a topological $k$-algebra which is the limit of an inverse system $(A_n)_n$ of topologically discrete algebras. The topological $k$-algebra $A\{z\}$ is defined as $\lim_n A((z))$, with the limit taken inside topological $k$-algebras.

**Remark.** Elements of $A\{z\}$ are explicitly represented as two way infinite sums $\sum_i a_i z^i$, such that $a_{-i} \to 0$ as $i \to +\infty$.

**Definition 2.3.** • For $X$ an affine $k$-scheme we define the topological $k$-algebra $A_X$ to be $\mathcal{O}(LX)\{z\}$.

• The evaluation morphism $ev_{A_1} : \mathcal{O}(A^1) \to A_{A^1}$ is defined by sending $x \in A^1$ to $x(z) := \sum_i x_i z^i$. An identical formula defines $ev_X : \mathcal{O}(X) \to A_X$ for each affine $X$, noting that the relations defining the algebras are obviously preserved.

2.2. **Fréchet Algebras.** We wish to study $A_X$ as a topological algebra. Let us note that is non-Archimedean, which is to say the underlying topological $k$-vector space has a basis of open neighbourhoods at 0 consisting of subspaces. Further more, the underlying topological $k$-vector space is Cauchy complete. Equivalently, an infinite sum $\sum_i a_i$ is convergent iff $a_i \to 0$. We want to be able to perform some familiar algebraic operations on the algebra $A_X$, in particular to localize at an element of the image of $ev_X$. We will see how this can be done in a somewhat ad-hoc manner, that is nonetheless good enough for our purposes, using non-Archimedean Fréchet algebras over $k$. Of course the definition we present is heavily inspired by the corresponding notion in classical functional analysis, as well as some well known constructions in non-Archimedean analysis, cf. [1] for example. Note that we claim no real originality in these definitions, our goal is simply to introduce a context adequate for our purposes.

**Definition 2.4.** A non-Archimedean $\mathbb{R}$ semi-norm topological algebra $A$ is a continuous sub-multiplicative map $\mid - \mid : A \to \mathbb{R}_{\geq 0}$ so that $\mid a + b \mid \leq \max\{\mid a \mid, \mid b \mid\}$. We refer to these as semi-norms henceforth, and leave implicit that they are non-Archimedean. A family $\mid - \mid_i \in I$ of semi-norms on $A$ is said to induce the topology on $A$ if a sequence $a_n \to 0$ iff $\mid a_n \mid_i \to 0$ for all $i \in I$. 

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Remark. Note that by sub-multiplicative we simply mean that $|fg| \leq |f||g|$. We require sub-multiplicativity as we wish to deal with topological algebras such as $A((z))$ where $A$ is possibly not an integral domain, so that the valuation of the product of two elements can potentially be greater than the sum of the valuations of these elements.

**Definition 2.5.** A complete topological $k$-algebra is called a non-Archimedean Fréchet algebra ($nAF_k$ henceforth) if it has a countable family of non-Archimedean semi-norms inducing its topology.

**Lemma 2.2.** For $X$ a finite type affine scheme the topological $k$-algebra $A_X$ is an element of $nAF_k$.

**Proof.** We write $O(LX)$ as a countable limit, $\lim_n A_n$, of topologically discrete algebras. We have for each $n$, a norm on $A_n((z))$ which induces its topology by fixing $0 < \varepsilon < 1$ and setting $|a|_n := \varepsilon^{\text{ord}_z(a)}$, where $\text{ord}_z$ denotes $z$-adic order. This induces a family of $R$ norms on $A_X$ which are easily seen to induce its topology, whence we have exhibited $A_X$ as an element of $nAF_k$. $\square$

**Definition 2.6.** An element $a \in A$, for $A$ in $nAF_k$, is called good if for all $i \in I$ we have $|a|_i \neq 0$ and if further $|a|^n = |a^n|_i$ for all $i$ and $n$. If this is the case, each of the semi-norms extends uniquely to the usual algebraic localisation, which we denote $A[a^{-1}]_{\text{alg}}$. The completion of this with respect to the topology induced by the family of extended norms will be denoted $A[a^{-1}] \in nAF_k$.

**Remark.** We note that (suppressing the index $i$) we require $|a| \neq 0$ so that we can define $|ba^{-n}| := |b||a|^{-n}$ and we require $|a^n| = |a|^n$ so that the extended norms remain sub-multiplicative, which ensures for example the continuity of multiplication.

**Lemma 2.3.** For any $f \in O(A^d)$, the element $\text{ev}_{A^d}(f) \in O(LA^d)$ is good in the sense of definition 2.6.

**Proof.** $O(LA^d)$ is the pro-limit of discrete integral domains, so the multiplicativity of norms is immediate. It remains to check that $\text{ev}_{A^d}(f)$ does not vanish modulo the ideal generated by elements $x_i$ with $i \leq -n$, which is clear. $\square$

We now identify a class of morphisms at which localisations are well behaved.

**Definition 2.7.** A morphism $f : A \to B$ is called bounded above (resp. below) if for all $i \in I$ there exists $C_i > 0$ so that $|f(a)|_i$ is at most (resp. at least) $C_i|a|_i$. If a morphism
is bounded above and below we call it a \textit{quasi-isometric embedding}, and if we can take \( C_i = c_i = 1 \) for all \( i \) then it is called an \textit{isometric embedding}.

\textbf{Remark.} A quasi-isometric embedding is necessarily injective, justifying the name. Further note that the induced topology on the image agrees with the topology on the domain.

**Lemma 2.4.** If \( f : A \to B \) is a quasi-isometric embedding in \( nAF_k \), \( a \in A \) is good and \( f(a) \) is a unit of \( B \), then there is a unique continuous extension of \( f \) to \( A[a^{-1}] \to B \). Further, \( A[a^{-1}] \) agrees with the completion of \( A[a^{-1}]^{\text{alg}} \) inside \( B \), with respect to the induced topology.

\textit{Proof.} First note that \( f(a) \) is automatically good, so behaves multiplicatively with respect to norms. We need to check that \( a_n a^{-n} \) tends to zero, i.e. for all \( i \in I \), \( a_n a^{-n} \) tends to zero with respect to the semi-norm \( | - |_i \), iff \( f(a_n a^{-i}) := f(a_n) f(a)^{-n} \) tends to zero. Again this can be checked on semi-norms, and then the quasi-isometry assumption immediately implies the claim. \( \square \)

**Lemma 2.5.** If \( A \to B \) is a map of pro-discrete algebras, which is a limit of injective maps \( A_j \to B_j \), for discrete \( A_j \) and \( B_j \), then the map in \( nAF_k \), \( A\{z\} \to B\{z\} \) is a quasi-isometric embedding.

\textit{Proof.} Some thought confirms shows that it suffices to prove this for a map \( A(\{z\}) \to B(\{z\}) \), induced from an injection of discrete algebras \( A \to B \). This is now immediate, indeed we can take the constants \( C \) and \( c \) to be 1, and we have an isometric embedding. \( \square \)

3. \textbf{Proof of main theorem}

We are now in a position to formulate and prove the main theorem of this note. We recall here that we write \( A^d_f \) for the complement of the hypersurface \( f = 0 \) inside \( A^d \). We let \( x^1, \ldots, x^d \) denote coordinates on the affine space \( A^d \). We consider \( A^d_f \) as embedded into \( A^{d+1} \), with coordinates \( x^1, \ldots, x^d, y \), as the vanishing locus of \( f(x^1, \ldots, x^d)y = 1 \).

We begin with the following lemma, which is where we crucially have to work with \( A^d \) in place of a general smooth variety.

**Lemma 3.1.** The morphism \( A_A^d \to A^d_f \) is a quasi-isometric embedding in \( nAF_k \).
Proof. By lemma 2.4. above, it suffices to show that the map $O(LA^d) \to O(LA_f^d)$ can be written as the limit of a projective system consisting of injections of topologically discrete algebras. As usual we consider $O(LA^d)$ as the limit of discrete algebras $A_n := k[x_j^i | i = 1, ..., d, j \geq -n]$. We write then $B_n := A_n[y_i | i \geq -n]/(\text{relations})$, where the relations are those implied by the identity of Laurent series $y(z)f(x^1(z), ..., x^d(z)) = 1$, where as usual we have written $y(z) := \sum y_i z^i$. Note that $O(LA_f^d) \cong \lim B_n$.

Now a morphism $B_n \to C$ is equivalent to elements $c^i(z) \in z^{-1}C[[z]]$, for $i = 1, ..., d$, along with an inverse to $f(c^1(z), ..., c^d(z))$ which also lies in $z^{-1}C[[z]]$. Let $f_{\text{min}} \in A_n$ be the coefficient of the highest power of $z^{-1}$ occurring in $f(x^1(z), ..., x^d(z))$, and note that $f_{\text{min}} \neq 0$. Further let $C_n := A_n[f_{\text{min}}^{-1}]$ and note that the map $A_n \to C_n$ is an injection as $A_n$ is obviously an integral domain. Now we observe that there is a map $B_n \to C_n$ which factors the inclusion $A_n \to C_n$, indeed $f(x^1(z), ..., x^d(z))$ is invertible in $C_n((z))$ because its lowest order term is. We deduce thus that $A_n \to B_n$ is an inclusion and thus the lemma is proven. 

The main theorem now follows from a pleasant trick involving differential operators on the punctured disc $D^*$, the essential idea is that differential operators are dense inside all $k$-linear continuous endomorphisms of $O(D^*) = k((z))$. This fact is presumably well known, but we essentially give a proof below anyway.

**Theorem 3.2.** The natural map of non-Archimedean Fréchet algebras, $A_{A^d}[ev_{A^d}(f)^{-1}] \to A_{A_f^d}$, is an equivalence.

**Proof.** We know by lemmas 2.4. and 3.1. that the topology on $A_{A^d}[ev_{A^d}(f)^{-1}]$ agrees with the induced topology for the inclusion into $A_{A_f^d}$. As such we need only show that the image is dense in $A_{A_f^d}$.

In order to prove this let us first note that the algebra, Diff$_{D^*}$, of differential operators on $D^*$ acts continuously on $A_X$, simply by acting on powers of $z$ in the evident manner. We claim now that the sub-algebra spanned by the Diff$_{D^*}$ span of the image of $ev_X$ is dense in $A_X$ for any affine finite type $X$. Indeed it will follow immediately from the case of $A^1$, which we now prove. It suffices to show that each topological $x_i \in A_{A^1}$ is in the desired topological closure.

We write $\nu := z\partial_z$ and define, for each $j$, a sequence of differential operators, $(P_n^j)_{n}$ as follows. We set $P_n^j := \lambda_n^j \prod_{\nu \in [-n, n]\setminus\{j\}}(\nu - i)$, where $\lambda_n^j$ are chosen so that $P_n^j(z) = z^j$. We claim now that $P_n^j(x(z)) \to x_j z^j$, as $n \to \infty$. Indeed let us note that the difference $P_n^j(x(z)) - x_j z^j$ is of the form $\sum c_i x_i z^j$ for some scalars $c_i$, so that $c_i = 0$ for all $i \in [-n, n]$. It follows that this is of the form $I_{x_n} + z^nO(LA^1)[[z]],$
where $I_{\leq n}$ is the ideal generated by the coordinates $x_i$ with $i \leq n$. This difference of course tends to 0, whence we are done.

Now to conclude it suffices to show that the image of $A_{A^d}[ev_{A^d}(f)^{-1}]$ inside $A_{A^d}$ contains the image of $ev_{A^d}$ and is stable under $\text{Diff}_D$. For this it suffices to check that it is stable under $\partial_z$, and it is clear how to extend the derivation to the localisation, so we are done. □

Remark. We expect this result to be true for smooth affine schemes more generally. It is lemma 3.1. which we have been unable to prove in this context.

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