WELL-POSEDNESS OF CAUCHY PROBLEM FOR LANDAU EQUATION IN CRITICAL BESOV SPACE

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Abstract. We study the Cauchy problem for the inhomogeneous non linear Landau equation with Maxwellian molecules. In perturbation framework, we establish the global existence of solution in spatially critical Besov spaces. Precisely, if the initial datum is a small perturbation of the equilibrium distribution in the Chemin-Lerner space $\tilde{L}^2_1(B^{1/2}_{2,1})$, then the Cauchy problem of Landau equation admits a global solution belongs to $L^\infty_t\tilde{L}^2_1(B^{3/2}_{2,1})$. The spectral property of Landau operator enables us to develop new trilinear estimates, which leads to the global energy estimate.

1. Introduction and main result. The Landau equation is a fundamental model in kinetic theory that describes the evolution of the density of particles in a plasma. In this work, we consider the spatially inhomogeneous Landau equation, which is given by

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where \( f = f(t, x, v) \geq 0 \) is the density of particles on position \( x \in \mathbb{R}^3 \) and with velocity \( v \in \mathbb{R}^3 \) at time \( t \geq 0 \). The collision operator \( Q_L \) is a bilinear operator acting only on the velocity variable \( v \) and reads as

\[
Q_L(f, g)(v) = \nabla_v \cdot \left( \int_{\mathbb{R}^3} a(v - v_*) [f_\ast \nabla_v g - \nabla_v f_\ast g] dv_* \right),
\]

where we used the usual shorthand \( f_\ast = f(v_\ast), \nabla_v g = \nabla_v g(v), \nabla_v f_\ast = \nabla_v f(v_\ast) \) and \( g = g(v) \). The matrix-valued function \( a(v) = (a_{i,j}(v))_{1 \leq i,j \leq 3} \) is non-negative, symmetric and depends on the interaction between particles, which is usually assumed by

\[
a(v) = (|v|^2 I - v \otimes v)|v|^\gamma, \quad -3 < \gamma \leq 1,
\]

where \( I = I_{3 \times 3} \) is the unit matrix on \( \mathbb{R}^3 \) and \( v \otimes v = (v_i v_j)_{1 \leq i,j \leq 3} \). One calls hard potentials if \( \gamma \in (0, 1] \), Maxwellian molecules if \( \gamma = 0 \), soft potentials if \( \gamma \in (-3, 0] \) and Coulombian potential if \( \gamma = -3 \). One also divides the soft potentials into two categories: moderately soft potentials if \( \gamma \in (-2, 0) \) and very soft potentials if \( \gamma \in (-3, -2] \). In this paper, we are interested in the Cauchy problem (1) with Maxwellian molecules, since the Landau operator enjoys very nice spectral property in that case.

In what follows, we are concerned with the Landau equation around the absolute Gaussian distribution in \( \mathbb{R}^3 \):

\[
\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.
\]

We can linearize it with the perturbation

\[
f(t, x, v) = \mu(v) + \sqrt{\mu}(v) g(t, x, v).
\]

Then it follows that

\[
\begin{cases}
\partial_t g + v \cdot \nabla_x g + \mathcal{L} g = \mathcal{L}_1 g + \mathcal{L}_2 g, \\
g|_{t=0} = g_0,
\end{cases}
\]

with \( f_0 = \mu(v) + \sqrt{\mu}(v) g_0 \), where

\[
\mathcal{L}(g) = \mathcal{L}_1(g) + \mathcal{L}_2(g)
\]

with

\[
\mathcal{L}_1(g) = -\mu^{-\frac{1}{2}} Q_L(\mu, \sqrt{\mu} g), \quad \mathcal{L}_2(g) = -\mu^{-\frac{1}{2}} Q_L(\sqrt{\mu} g, \mu)
\]

and

\[
\mathcal{L}(g, g) = \mu^{-\frac{1}{2}} Q_L(\sqrt{\mu} g, \sqrt{\mu} g).
\]

We observe (see for example, [23]) that the linear operator \( \mathcal{L} \) is non-negative and its null space has dimension 5, which is given by

\[
\mathcal{N} = \text{span} \{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |v|^2 \sqrt{\mu} \}.
\]

\( \mathbf{P} \) denotes the orthogonal projector onto the null space \( \mathcal{N} \). Furthermore, in case of Maxwellian molecules, one has the explicit form

\[
\mathcal{L} = (2\mathcal{H} - 3 - \Delta_{S^2}) - [2\mathcal{H} - 3 - \Delta_{S^2}] \mathbf{P}_1 - [\Delta_{S^2} + (2\mathcal{H} - 3)] \mathbf{P}_2,
\]

where

\[
\mathcal{H} = -\Delta_v + \frac{|v|^2}{4}.
\]
is the harmonic oscillator and
\[
\Delta_{S^2} = \frac{1}{2} \sum_{1 \leq j, k \leq 3 \atop j \neq k} (v_j \partial_k - v_k \partial_j)^2
\]

is the Laplace-Beltrami operator on the unit sphere $S^2$ and $P_k (k = 1, 2)$ is the orthogonal projection onto the Hermite basis.

Landau equation is a fundamental equation to describe collisions among charged particles interacting with their Coulombic force. There are lots of known results concerning the well-posedness and large-time behavior of solutions to the Landau equation.

In spatially homogeneous case, Villani [33] constructed weak solutions for the Coulombic interaction with $\gamma = -3$ up to some defect measures. Subsequently, in [32], he extended the result to the Landau equation without the presence of spatial dependence. In the Maxwellian molecules case $\gamma = 0$, Villani proved an exponential in time convergence to equilibrium. For the hard potential $\gamma \in (0, 1]$, Desvillettes and Villani [16, 17] investigated the existence, uniqueness and smoothness of classical solutions. They proved a functional inequality for entropy-entropy dissipation that is not linear, from which the polynomial in time convergence of solutions towards equilibrium was also shown. Recently, Carrapatoso [9] proved the optimal exponential decay to equilibrium with the decay rate given by the spectral gap of the associated linearized operator, by using the method developed by Gualdani, Mischler and Mouhot [20]. Morimoto and Xu [28] proved the ultra-analytic effect for the Cauchy problem of linear Landau equation in the case of $\gamma = 0$. Li and Xu [26] studied the nonlinear Landau operator by introducing the spectral analysis.

In spatially inhomogeneous case, Guo [21] proved the global-in-time existence of classical solutions to the Landau equation in a period box. Later, Hsiao-Yu [22] extended Guo’s results [21] to the whole space. Baranger and Mouhot [7] studied the explicit spectral gap estimates to the linearized Landau operator with hard potentials. Mouhot [30] established the coercivity estimates for a general class of interactions including hard potentials and soft potentials. Lerner-Morimoto-Starov-Xu [23] showed that the linearized non-cutoff Boltzmann operator with Maxwellian molecules is exactly equal to a fractional power of the linearized Landau operator which is the sum of the harmonic oscillator and the spherical Laplacian.

To the best of our knowledge, there are various studies concerning the well-posedness of solutions to the Boltzmann and Landau equation, see for example, [1, 2, 3, 8, 16, 17, 25, 27, 31, 32] and references therein. Very recently, Duan, Liu and the last author of this paper [19] first introduced the Chemin-Lerner type spaces involving the microscopic velocity and established the global existence of strong solutions near Maxwellian for the cut-off Boltzmann equation. Subsequently, Morimoto and Sakamoto [29] extended their result to the non-cutoff Boltzmann equation by using the triple norm that was introduced by Alexandre-Morimoto-Ukai-Xu-Yang [2, 5]. However, there are few results concerning the global existence for the Landau equation in spatially critical Besov spaces. So it is very interesting to work a result for (1), since the collision operator between the Boltzmann equation and Landau equation are fundamentally different. As a first step, by using the spectral analysis on the nonlinear Landau operator, we investigate the Cauchy problem (1) with Maxwellian molecules ($\gamma = 0$). The main result is now stated as follows.
Theorem 1.1. There exists a constant $\varepsilon_0 > 0$ such that if $g_0 \in \tilde{L}^2_t(B_{2,1}^{3/2})$ and
$$
\|g_0\|_{\tilde{L}^2_t(B_{2,1}^{3/2})} \leq \varepsilon_0,
$$
then the Cauchy problem (2) admits a global solution satisfying
$$
g \in \tilde{L}^\infty_t \tilde{L}^2_v(B_{2,1}^{3/2}) \quad \text{and} \quad \mathcal{L}^{\frac{3}{2}} g \in \tilde{L}^2_t \tilde{L}^2_v(B_{2,1}^{3/2}).
$$
Moreover, if $f_0(x,v) = \mu(v) + \sqrt{\mu(v)} g_0(x,v) \geq 0$, then $f(t,x,v) = \mu(v) + \sqrt{\mu(v)} g(t,x,v) \geq 0$.

Above norms of the Chemin-Lerner space will be rigorously defined by Appendix B. The Chemin-Lerner space without involving the microscopic velocity was initiated by [12] to establish the global existence of solutions to the incompressible Navier-Stokes equations. Observe that the regularity index $s = 3/2$ that the Besov space is subjected to $B_{2,1}^{3/2}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, but the Sobolev space $H^{3/2}(\mathbb{R}^3)$ is not embedded into $L^\infty$, it thus is critical for the algebra with respect to the spatial variable. Also, we remark that the Chemin-Lerner space $\tilde{L}^\infty_t \tilde{L}^2_v(B_{2,1}^{3/2})$ enjoys stronger topology than the usual mixed space $L_t^\infty L_v^2(B_{2,1}^{3/2})$, which allows to get a good control for nonlinear collision terms. The proof of Theorem 1.1 is separated into several parts. Regarding the local-in-time existence part, we employ the Picard’s iteration scheme to prove the local existence of solution. As a matter of fact, solving the linear Landau equation is quite elaborate, which will be presented by Appendix A for clarity. To get round the difficulty that the dual space of $\tilde{L}^\infty_t \tilde{L}^2_v(B_{2,1}^{3/2})$ is unknown, firstly, we try to find a weak solution in the wider space $L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$ based on the Hahn-Banach extension theorem. Secondly, using various commutators estimates (which are well developed in Section 4) to get the desired solution in $\tilde{L}^\infty_t \tilde{L}^2_v(B_{2,1}^{3/2})$. In the part of global-in-time existence, we establish those trilinear estimates to the nonlinear Landau collision operator, which play a key role achieving the global solution. For that end, our proof heavily depends on spectral properties of Landau operator with Maxwellian molecules (see Section 2 for details) in contrast with [19, 29]. Finally, the standard continuity argument enables us to obtain Theorem 1.1.

Remark 1. The Bony’s para-product decomposition have been widely used in the study of fluid dynamics, see for example [10, 11, 13, 14, 15] and references therein, however, there are few results available that one applies the Besov space theory to the global existence of kinetic equations. The recent works [19, 29] are devoted to the Boltzmann equation. We would like to mention that Theorem 1.1 should be our first effort, and the research of Landau equation in cases of hard potentials and soft potentials, which is left in near future.

The paper is arranged as follows. To make the manuscript self-contained, in Section 2, we recall the spectral analysis properties and some key estimates of Landau collision operator. In Section 3, we establish some crucial estimates for nonlinear estimates, for instance, trilinear estimates for Landau operator. Section 4 is devoted to establish commutator estimates for Landau operator. In Section 5, we prove the global existence of solution to the Landau equation. To do this, we prove the local existence of solutions to (1) by using the iteration Scheme and construct a priori estimates arising from the coercivity of linear Landau operator. However, we cannot deduce the dissipative estimate for the macroscopic part $\mathbf{P} g$ directly. To overcome the difficulty, as in [19, 29], we shall perform the macro-micro decomposition and
deduce a fluid dynamics system of macroscopic projection of $g$. Consequently, by using the standard energy method, we can obtain the estimate on the macroscopic dissipation. Appendix A is dedicated to the solvability of linear Landau equation, which is based on the duality argument and Hahn-Banach extension theorem. Some definitions of Chemin-Lerner type spaces and some inequalities of Besov spaces used in this paper are collected in Appendix B.

2. Preliminary: Analysis of Landau collision operator. For convenience of reader, we present the spectral properties for Landau operator briefly, see [8, 23, 25, 26] for more details. First of all, one has an explicit expression for the linearized Landau operator with Maxwellian molecules.

**Lemma 2.1.** ([23]) The linearized Landau operator with Maxwellian molecules can be written as

$$\mathcal{L}g = -\mu^{-\frac{1}{2}} \left( Q_L(\mu, \sqrt{\mu}g) + Q_L(\sqrt{\mu}g, \mu) \right) := \mathcal{L}_1 g + \mathcal{L}_2 g,$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are equal to

$$\mathcal{L}_1 = (d - 1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2}) - \Delta_{S^{d-1}},$$

$$\mathcal{L}_2 = [\Delta_{S^{d-1}} - (d - 1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2})]P_1 + [-\Delta_{S^{d-1}} - (d - 1)(-\Delta_v + \frac{|v|^2}{4} - \frac{d}{2})]P_2.$$

Here and below, $\Delta_{S^{d-1}}$ stands for the Laplace-Beltrami operator on the unit sphere $S^{d-1}$ and $P_k(k = 1, 2)$ is the orthogonal projection onto the Hermite basis.

There is the algebra property of nonlinear Landau operators on the basis $\{ \varphi_{n,l,m} \}$ (see [25, 26] and references), for $n, l \in \mathbb{N}, m \in \mathbb{Z}, |m| \leq l$, let’s denote

$$\varphi_{n,l,m}(v) = \left( \frac{2}{\sqrt{\pi}} \Gamma(n + l + 3/2) \right)^{1/2} \left( \frac{|v|}{\sqrt{2}} \right)^l e^{-|v|^2} L_n^{(l+1/2)} \left( \frac{|v|^2}{2} \right) \mathcal{Y}_l^m \left( \frac{v}{|v|} \right),$$

where $\Gamma(\cdot)$ is the standard Gamma function, and

- $L_n^{(\alpha)}$ is the Laguerre polynomial of order $\alpha$ and degree $n$,

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^{n-r} \frac{\Gamma(\alpha + n + 1)}{r!(n-r)!\Gamma(\alpha + n - r + 1)} x^{n-r};$$

- $Y_l^m(\sigma)$ is the orthonormal basis of spherical harmonics

$$Y_l^m(\sigma) = \sqrt{\frac{2l + 1}{4\pi} \frac{\Gamma(l - |m|)!}{(l + |m|)!}} P_l^{(|m|)}(\cos \theta)e^{im\phi}, \ |m| \leq l.$$

The notations $\sigma = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$, $N_{l,m}$ are the normalisation factor and $P_l^{(|m|)}$ is the Legendre functions of the first kind of order $l$ and degree $|m|$

$$P_l^{(|m|)}(x) = (1 - x^2)^{\frac{|m|}{2}} \frac{d^{(|m|)}}{dx^{(|m|)}} \left( \frac{1}{2l!} \frac{d^l}{dx^l} (x^2 - 1)^l \right).$$

From [23, 24], we see that those spherical harmonics are equivalent to the real spherical harmonics $\mathcal{Y}_l^m(\sigma)$ for $l \geq 0$ and $-l \leq m \leq l$, which are defined by
Therefore, we deduce that \( \{ \varphi_{n,l,m} \} \subset \mathcal{S}(\mathbb{R}^3) \)

and

\[
\begin{align*}
\varphi_{0,0,0}(v) &= \sqrt{\mu}, \\
\varphi_{0,1,0}(v) &= v_1 \sqrt{\mu}, \\
\varphi_{0,1,1}(v) &= \frac{v_2+i v_3}{\sqrt{2}} \sqrt{\mu}, \\
\varphi_{0,1,-1}(v) &= \frac{v_2-i v_3}{\sqrt{2}} \sqrt{\mu}, \\
\varphi_{1,0,0}(v) &= \sqrt{\frac{3}{2}} \left( \frac{3}{2} - \frac{|v|^2}{2} \right) \sqrt{\mu}.
\end{align*}
\]

In addition, the explicit form of the eigenfunctions \( \{ \varphi_{0,2,m_2}, |m_2| \leq 2 \} \) satisfies that

\[
\begin{align*}
\varphi_{0,2,0}(v) &= \sqrt{\frac{3}{2}} \left( \frac{3}{2} - \frac{1}{2} |v|^2 \right) \sqrt{\mu}, \\
\varphi_{0,2,1}(v) &= \frac{v_2+i v_3}{\sqrt{2}} \sqrt{\mu}, \\
\varphi_{0,2,-1}(v) &= \frac{v_2-i v_3}{\sqrt{2}} \sqrt{\mu}, \\
\varphi_{0,2,-2}(v) &= \left( \frac{v_2-i v_3}{\sqrt{2}} \right) \sqrt{\mu},
\end{align*}
\]

and the eigenfunctions \( \{ \varphi_{1,1,m_1}, |m_1| \leq 1 \} \) are given by

\[
\begin{align*}
\varphi_{1,1,0}(v) &= \frac{1}{\sqrt{10}} \left( 5 - |v|^2 \right) v_1 \sqrt{\mu}, \\
\varphi_{1,1,1}(v) &= \frac{1}{\sqrt{10}} \left( 5 - |v|^2 \right) \frac{v_2+i v_3}{\sqrt{2}} \sqrt{\mu}, \\
\varphi_{1,1,-1}(v) &= \frac{1}{\sqrt{10}} \left( 5 - |v|^2 \right) \frac{v_2-i v_3}{\sqrt{2}} \sqrt{\mu}.
\end{align*}
\]

Next, we present the algebraic property of nonlinear Landau operator.

**Proposition 1.** ([26]) For \( n, l \in \mathbb{N}, |m| \leq l, \) we have

\[
\begin{align*}
(i) \quad & L(\varphi_{n,0,0}, \varphi_{n,l,m}) = - (2 (2 n + l + l + 1)) \varphi_{n,l,m}; \\
(ii) \quad & L(\varphi_{0,1,m_1}, \varphi_{n,l,m}) \\
& = A_{n,l,m,m_1}^{-} \varphi_{n+1,l-1,m_1} + A_{n,l,m,m_1}^{+} \varphi_{n+1,l+1,m_1}, \quad \forall |m_1| \leq 1; \\
(iii) \quad & L(\varphi_{1,0,0}, \varphi_{n,l,m}) = \frac{4 \sqrt{3} \omega + 2 \omega + 3}{3} \varphi_{n+1,l,m}; \\
(iv) \quad & L(\varphi_{0,2,m_2}, \varphi_{n,l,m}) = A_{n,l,m,m_2}^{-} \varphi_{n+2,l-2,m_2} \\
& + A_{n,l,m,m_2}^{+} \varphi_{n+2,l+2,m_2}, \quad \forall |m_2| \leq 2; \\
(v) \quad & L(\varphi_{n,l,m}, \varphi_{n,l,m}) = 0, \quad \forall 2 n + l > 2, \quad |\tilde{m}| \leq \tilde{l},
\end{align*}
\]

where the coefficients are defined as follows

\[
\begin{align*}
A_{n,l,m,m_1}^{-} &= 4 \int_{\mathbb{R}^2} Y_{l,m_1}^{-}(\omega) Y_{l,m_1}^{+}(\omega) Y_{l-1,m_1}^{-1}(\omega) d\omega; \\
A_{n,l,m,m_1}^{+} &= 4 \int_{\mathbb{R}^2} Y_{l-1,m_1}^{+}(\omega) Y_{l,m_1}^{-1}(\omega) Y_{l-1,m_1}^{+}(\omega) d\omega;
\end{align*}
\]
i) For \( n, l \in \mathbb{N}, n \geq 2 \),
\[
\max_{|m^*| \leq l, \sum |m| \leq |m^*| + 2, |m| \leq 2} \sum_{m + m_2 = m^*} \left| A_{n-2,l+2,m,m_2}^1 \right|^2 \leq \frac{16n(n-1)}{3};
\]  
(4)

ii) For \( n, l \in \mathbb{N}, n \geq 1 \),
\[
A_{n-1,0,0,0}^2 = 0;
\]
\[
\max_{|m^*| \leq l, \sum |m| \leq |m^*| + 2} \sum_{m + m_2 = m^*} \left| A_{n-1,l,m,m_2}^2 \right|^2 \leq \frac{4n(2n + 2l + 1)}{3}, \quad \forall \ l \geq 1;
\]  
(5)

iii) For \( n, l \in \mathbb{N}, l \geq 2 \),
\[
\max_{|m^*| \leq l, \sum |m| \leq |m^*| + 2} \sum_{m + m_2 = m^*} \left| A_{n,l-2,m,m_2}^3 \right|^2 \leq \frac{(2n + 2l + 1)(2n + 2l - 1)}{2}.
\]  
(6)

Furthermore, those coefficients \( A^1, A^2, A^3 \) satisfy the following estimates.

**Proposition 2.** ([26]) It holds that

i) For \( n, l \in \mathbb{N}, n \geq 2 \),
\[
A_{n,l,m,m_2}^1 = -4\sqrt{\frac{\pi}{15}} \cdot 3(n+2)(n+1) \int_{S^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l-2}^{-m}(\omega)d\omega;
\]
\[
A_{n,l,m,m_2}^2 = 4\sqrt{\frac{\pi}{15}} \cdot 2(n+1)(2n+2l+3) \times \int_{S^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l-2}^{-m}(\omega)d\omega;
\]
\[
A_{n,l,m,m_2}^3 = -4\sqrt{\frac{\pi}{15}} \cdot (2n+2l+5)(2n+2l+3) \times \int_{S^2} Y_2^{m_2}(\omega)Y_l^m(\omega)Y_{l+2}^{-m}(\omega)d\omega.
\]

In the present paper, the following estimates for the coefficients \( A^- \) and \( A^+ \) will be also used in subsequent proofs.

**Proposition 3.** For the coefficients of the Proposition 1 defined in (3), we have

i) For \( n, l \in \mathbb{N}, n \geq 1 \),
\[
\max_{|m^*| \leq l, \sum |m| \leq |m^*| + 1} \sum_{m + m_1 = m^*} \left| A_{n-1,l+1,m,m_1}^- \right|^2 \leq 4nl(l+1).
\]  
(7)

ii) For \( n, l \in \mathbb{N}, l \geq 1 \),
\[
\max_{|m^*| \leq l, \sum |m| \leq |m^*| + 1} \sum_{m + m_1 = m^*} \left| A_{n,l-1,m,m_1}^+ \right|^2 \leq 2(2n + 2l + 1)(l+1)^2.
\]  
(8)

**Proof.** It follows from (3) that
\[
A_{n-1,l+1,m,m_1}^- = 4l \sqrt{\frac{2n\pi}{3}} \left( \int_{S^2} Y_l^{m_1}(\omega)Y_{l+1}^m(\omega)Y_{l-1}^{-m}(\omega)d\omega \right);
\]
\[
A_{n,l-1,m,m_1}^+ = 4(l+1) \sqrt{\frac{(2n+2l+1)\pi}{3}} \left( \int_{S^2} Y_l^{m_1}(\omega)Y_{l-1}^m(\omega)Y_{l+1}^{-m}(\omega)d\omega \right).
\]
Then
\[
\sum_{|m| \leq l+1, |m_1| \leq 1, m + m_1 = m^*} \left| A^-_{n-1,l+1,m,m_1} \right|^2
\]
\[
= \frac{32nl^2\pi}{3} \sum_{|m| \leq l+1, |m_1| \leq 1} \int_{S^2} \left| Y^{m_1}_l(\omega) Y^m_{l+1}(\omega) Y^{-(m_1-m)}(\omega) d\omega \right|^2
\]
\[
= \frac{32nl^2\pi}{3} \sum_{|m| \leq l+1, |m_1| \leq 1} \sum_{|m| \leq l+1, |m_1| \leq 1, m + m_1 = m^*} \left| Y^{m_1}_l(\omega) Y^m_{l+1}(\omega) Y^{-(m_1-m)}(\omega) d\omega \right|^2
\]
\[
= \frac{32nl^2\pi}{3} \sum_{|m| \leq l+1, |m_1| \leq 1} \sum_{|m| \leq l+1, |m_1| \leq 1, m + m_1 = m^*} \left| Y^{m_1}_l(\omega) Y^m_{l+1}(\omega) Y^{-(m_1-m)}(\omega) Y^{m^*_1}(\omega) Y^{m^*_m}(\omega) d\omega d\sigma. \right.
\]

We recall again that, for \(\sigma, \kappa \in S^2\),
\[
P_k(\sigma \cdot \kappa) = \frac{4\pi}{2k+1} \sum_{|m| \leq k} Y^m_k(\sigma) Y^{-m}_k(\kappa), \quad \forall k \in \mathbb{N}.
\]

Then
\[
\sum_{|m| \leq l+1, |m_1| \leq 1, m + m_1 = m^*} \left| A^-_{n-1,l+1,m,m_1} \right|^2
\]
\[
= \frac{32nl^2\pi}{3} \int_{S^2} \int_{S^2} P_l(\omega \cdot \sigma) P_{l+1}(\omega \cdot \sigma) Y^{-(m_1-m)}_l(\omega) Y^{m^*_1}(\sigma) d\omega d\sigma.
\]

By using the fact that,
\[
P_l(\omega \cdot \sigma) P_{l+1}(\omega \cdot \sigma) = \frac{l+2}{2l+3} P_{l+2}(\omega \cdot \sigma) + \frac{l+1}{2l+3} P_l(\omega \cdot \sigma)
\]
and the orthogonal of \(\{Y^{-(m_1-m)}_l(\omega), l \in \mathbb{N}, |m^*| \leq l\}\) on \(S^2\), one can verify that
\[
\sum_{|m| \leq l+1, |m_1| \leq 1, m + m_1 = m^*} \left| A^-_{n-1,l+1,m,m_1} \right|^2
\]
\[
= \frac{32nl^2\pi}{3} \frac{2l+3}{4\pi} \frac{4\pi}{4\pi} l + 1 \frac{l + 1}{2l+1} 2l + 3 \leq 4nl(l+1).
\]

Hence, we arrive at (7). On the other hand, we have
\[
\sum_{|m| \leq l-1, |m_1| \leq 1, m + m_1 = m^*} \left| A^+_{n,l-1,m,m_1} \right|^2
\]
\[
= \frac{16(2n + 2l + 1)(l+1)^2\pi}{3} \sum_{|m| \leq l-1, |m_1| \leq 1} \int_{S^2} \left| Y^{m_1}_l(\omega) Y^{m_{l-1}}_l(\omega) Y^{-(m_1-m_1) - m}(\omega) d\omega \right|^2
\]
\[
= \frac{16(2n + 2l + 1)(l+1)^2\pi}{3} \sum_{|m| \leq l-1, |m_1| \leq 1} \sum_{|m| \leq l-1, |m_1| \leq 1, m + m_1 = m^*} \left| Y^{m_1}_l(\omega) Y^{m_{l-1}}_l(\omega) Y^{-(m_1-m_1)}(\omega) Y^{m^*_m}(\omega) d\omega \right|^2
\]
Let us begin with bounding the nonlinear term \( (L) \).

**Theorem 3.1.** Let \( g \) for \( 3.1. \) Nonlinear estimates of Landau collision operator. By employing the fact with \( l - 1, \) where \( l \geq 1 \) that,

\[
P_l(\omega \cdot \sigma)P_{l-1}(\omega \cdot \sigma) = \frac{l}{2l-1} P_l(\omega \cdot \sigma) + \cdots
\]

and the orthogonal of the \( \{ Y_l^{-m^*}(\omega), l \in \mathbb{N}, |m^*| \leq l \} \) on \( S^2 \), one can verify that

\[
\sum_{m, m_1 = m^*} \left| A_{n,l-1,m,m_1}^+ \right|^2 = \frac{16(2n + 2l + 1)(l + 1)^2\pi}{3} \frac{2l - 1}{4\pi} \frac{\pi}{2l + 2l - 1} \leq 2(2n + 2l + 1)(l + 1)^2.
\]

This leads to the inequality (8).

3. **Nonlinear estimates of Landau collision operator.** In order to prove Theorem 1.1, we establish nonlinear estimates of Landau collision operator.

3.1. **Trilinear estimates.**

The orthogonal projectors \( \{ S_N, N \in \mathbb{N} \} \) and \( \{ \tilde{S}_N, N \in \mathbb{N} \} \) are defined as follows, for \( g \in \mathcal{S}'(\mathbb{R}^3_v) \),

\[
S_N g = \sum_{0 \leq 2n + l \leq N} \sum_{|m| \leq l} g_{n,l,m} \varphi_{n,l,m} \in \mathcal{S}(\mathbb{R}^3_v)
\]

and

\[
\tilde{S}_N g = \sum_{2 \leq 2n + l \leq N} \sum_{n + l \geq 2} \sum_{|m| \leq l} g_{n,l,m} \varphi_{n,l,m} \in \mathcal{S}(\mathbb{R}^3_v),
\]

where \( g_{n,l,m} = \langle g, \varphi_{n,l,m} \rangle \) and then we have

\[
P g = (S_N - \tilde{S}_N) g = g_{0,0,0} \varphi_{0,0,0} + g_{1,0,0} \varphi_{1,0,0} + \sum_{|m| \leq 1} g_{0,1,m} \varphi_{0,1,m}.
\]

Let us begin with bounding the nonlinear term \( (L(f, g), h)_{L^2} \).

**Theorem 3.1.** Let \( f, g, h \in \mathcal{S}(\mathbb{R}^6_{x,v}) \). It holds that

\[
| (L(f, g), h)_{L^2_{x,v}} | \lesssim \min \left\{ \| S_2 f \|_{L^2_{x,v}}, \| S_2 g \|_{L^2_{x,v}}, \| \tilde{L}^2 h \|_{L^2_{x,v}} \right\} \| \tilde{L}^2 h \|_{L^2_{x,v}},
\]

where the self-adjoint operator \( \tilde{L} \) is defined by

\[
\tilde{L} = 2\mathcal{H} - \Delta_{\mathbb{S}^2}.
\]
Proof. For any temperate functions \( f, g, h \in \mathcal{S}(\mathbb{R}_x) \), one has the decomposition

\[
    z = \sum_{2n+l \geq 0} \sum_{|m| \leq l} z_{n,l,m} \varphi_{n,l,m}, \quad z_{n,l,m} \triangleq \langle z \varphi_{n,l,m} \rangle
\]

with \( z = f, g, h \). It follows from Proposition 1 that

\[
    \left| \langle L(f,g), h \rangle_{L^2_x} \right| \leq J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7
\]

with

\[
    J_1 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} (2(2n+l) + l(l+1)) (f_{0,0,n,l,m} g_{n,l,m} h_{n,l,m})_{L^2_x} \right|
\]

\[
    J_2 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} A_{n-1,l+1,m,m_1}^+ (f_{0,1,m_1,n-1,l+1,m} g_{n,l,m_1+m} h_{n,l,m})_{L^2_x} \right|
\]

\[
    J_3 = \left| \sum_{2n+l \geq 0} \sum_{l \geq 1} A_{n,l-1,m,m_1}^+ (f_{0,1,m_1,n,l-1,m} g_{n,l,m_1+m} h_{n,l,m})_{L^2_x} \right|
\]

\[
    J_4 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} \frac{4\sqrt{3n(2n+2l+1)}}{3} (f_{1,0,n-1,l,m} g_{n,l,m})_{L^2_x} \right|
\]

\[
    J_5 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} A_{n-2,l+2,m,m_2}^+ (f_{0,2,m_2,n-2,l+2,m} g_{n,l,m+m_2} h_{n,l,m+m_2})_{L^2_x} \right|
\]

\[
    J_6 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} A_{n-1,l,m,m_2}^+ (f_{0,2,m_2,n-1,l,m} g_{n,l,m+m_2} h_{n,l,m+m_2})_{L^2_x} \right|
\]

\[
    J_7 = \left| \sum_{2n+l \geq 0} \sum_{|m| \leq l} A_{n-2,l,m,m_2}^+ (f_{0,2,m_2,n-2,l,m} g_{n,l,m+m_2} h_{n,l,m+m_2})_{L^2_x} \right|
\]

For \( J_1 \), by using the Cauchy-Schwarz inequality, we get

\[
    J_1 \lesssim \sum_{2n+l \geq 0} \sum_{|m| \leq l} (2(2n+l) + l(l+1)) \| f_{0,0,n,l,m} \|_{L^2_x} \| g_{n,l,m} \|_{L^2_x} \| h_{n,l,m} \|_{L^2_x}
\]

\[
    \lesssim \sum_{2n+l \geq 0} \sum_{|m| \leq l} (2(2n+l) + l(l+1)) \| f_{0,0,0} \|_{L^2_x} \| g_{n,l,m} \|_{L^2_x} \| h_{n,l,m} \|_{L^2_x}
\]

\[
    \lesssim \| f_{0,0,0} \|_{L^2_x} \| g_{n,l,m} \|_{L^2_x} \| h_{n,l,m} \|_{L^2_x}
\]

\[
    \lesssim \| f_{0,0,0} \|_{L^2_x} \| g_{n,l,m} \|_{L^2_x} \| h_{n,l,m} \|_{L^2_x}.
\]
By exchanging the order in the last summation, we deduce from (7) in Proposition 3 that

\[ J_4 \lesssim \|f_{0,0,0}\|_{L^2} \|\tilde{L}^2 g\|_{L^2_{\infty}(L^\infty_x)} \|\tilde{L}^2 h\|_{L^2_{\infty,v}}. \]

Next, we turn to estimate \( J_2 \). Precisely,

\[
J_2 \lesssim \sum_{2n+l \geq 0} \sum_{n \geq 1} \left( \sum_{|m| \leq l+1, |m_1| \leq l} \|f_{0,1,m_1}\|_{L^2_x}^2 \|g_{n-1,l+1,m}\|_{L^\infty_x} \|A_{n-1,l+1,m_1}^\star h_{n,l,m_1+m}\|_{L^2_x} \right)^{1/2} \times \left( \sum_{|m| \leq l+1, |m_1| \leq l} \|g_{n-1,l+1,m}\|_{L^\infty_x}^2 \right)^{1/2} \times \left( \sum_{|m| \leq l+1, |m_1| \leq l} \|A_{n-1,l+1,m_1}^\star h_{n,l,m_1+m}\|_{L^2_x} \right)^{1/2}
\]

By exchanging the order in the last summation, we deduce from (7) in Proposition 3 that

\[
\sum_{|m| \leq l+1, |m_1| \leq l} \|A_{n-1,l+1,m_1}^\star h_{n,l,m_1+m}\|_{L^2_x}^2
\]

\[
= \sum_{|m^\star| \leq l} \left( \sum_{|m| \leq 1, |m| \leq l+1, m_1 + m = m^\star} |A_{n-1,l+1,m_1}^\star|^2 \right) \|h_{n,l,m^\star}\|_{L^2_x}^2 \leq \left( \max_{|m^\star| \leq l} \sum_{|m| \leq l+1, m_1 + m = m^\star} |A_{n-1,l+1,m_1}^\star|^2 \right) \sum_{|m^\star| \leq l} \|h_{n,l,m^\star}\|_{L^2_x}^2 \leq 4(n+1)(l+1) \sum_{|m^\star| \leq l} \|h_{n,l,m^\star}\|_{L^2_x}^2.
\]

Then we obtain

\[
J_2 \lesssim \sum_{2n+l \geq 0} \sum_{n \geq 1} \left( \sum_{|m| \leq 1} \|f_{0,1,m_1}\|_{L^2_x}^2 \right)^{1/2} \sqrt{4(n+1)(l+1)}
\]
Theorem 3.2. Let frequency packets of comparable sizes. Indeed, one has

\[ \sum_{n,l,m} \| g_{n-1,l+1,m} \|_{L_x^\infty}^2 \bigg( \sum_{m^* \leq l} \| h_{n,l,m^*} \|_{L_x^2}^2 \bigg)^{1/2} \]

\[ \lesssim \left( \sum_{|m| \leq t+1} \| f_{0,1,m} \|_{L_x^2}^2 \right)^{1/2} \| \tilde{L}^2 g \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 h \|_{L_{x,v}^2} . \]

Those terms \( J_3, J_5, J_6 \) and \( J_7 \) may be treated along the same line as \( J_2 \) with aid of (4), (5), (6) in Proposition 2 and (8) in Proposition 3. Therefore, we can deduce that

\[ J_3 \lesssim \left( \sum_{|m| \leq 1} \| f_{0,1,m} \|_{L_x^2}^2 \right)^{1/2} \| \tilde{L}^2 g \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 h \|_{L_{x,v}^2} \]

and

\[ J_5, J_6, J_7 \lesssim \left( \sum_{|m| \leq 2} \| f_{0,2,m} \|_{L_x^2}^2 \right)^{1/2} \| \tilde{L}^2 g \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 h \|_{L_{x,v}^2} . \]

Together with those estimates on \( J_1-J_7 \), we get

\[ \left| (L(f,g), h)_{L_x^2, v} \right| \lesssim \| S_2 f \|_{L_{x,v}^2} \| \tilde{L}^2 g \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 h \|_{L_{x,v}^2} . \]

Similarly, when estimating the terms \( J_1-J_7 \), taking \( L^\infty \) norm on the position variable \( x \) for \( f \) and taking \( L^2 \) norm on the position variable \( x \) for \( g \), we obtain

\[ \left| (L(f,g), h)_{L_x^2} \right| \lesssim \| S_2 f \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 g \|_{L_{x,v}^2} \| \tilde{L}^2 h \|_{L_{x,v}^2} . \]

Combining (10)-(11) gives the desired inequality (9). Hence, the proof of Theorem 3.1 is finished. \( \square \)

Furthermore, optimal information will be obtained if splitting the functions into frequency packets of comparable sizes. Indeed, one has

**Theorem 3.2.** Let \( f, g, h \in S(\mathbb{R}_{x,v}^6) \). It holds that

\[ \left| (\Delta_j L(f, g), \Delta_j h)_{L_x^2, v} \right| \]

\[ \lesssim \sum_{|p-j| \leq 4} \| \Delta_p S_2 f \|_{L_{x,v}^2} \| \tilde{L}^2 g \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 \Delta_j h \|_{L_{x,v}^2} \]

\[ + \sum_{p \geq j} \| S_2 f \|_{L_x^2(L^\infty_x)} \| \tilde{L}^2 g \|_{L_{x,v}^2} \| \tilde{L}^2 \Delta_j h \|_{L_{x,v}^2} \]

(12)

for any \( j \geq -1 \).

**Proof.** For \( f, g, h \in S(\mathbb{R}_{x,v}^6) \), it follows from Proposition 1 and the orthogonal property of \( \{ \varphi_{n,l,m}; n, l \in \mathbb{N}, |m| \leq l \} \), that

\[ \left| (\Delta_j L(f, g), \Delta_j h)_{L_x^2, v} \right| \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 \]

with

\[ I_1 = \sum_{2n+l \geq 0} \sum_{|m| \leq l} \left| (2(2n+l) + l(l+1)) (\Delta_j (f_{0,0,0} g_{n,l,m}) , \Delta_j h_{n,l,m})_{L_x^2} \right| , \]
To estimate $I_1$, Bony’s decomposition comes into play in our context. The product of $u$ and $v$ can be decomposed into

$$uv = T_u v + T_v u + R(u,v)$$

with

$$T_u v = \sum_p S_{p-1} u \Delta_p v, \quad R(u,v) = \sum_{|p-p'| \leq 1} \Delta_p' u \Delta_p v,$$

for $u, v \in S'({\mathbb{R}^3})$.

The above operators $T$ and $R$ are called “paraproduct” and “remainder”, respectively. Moreover, it follows from Lemma 7.1 that

$$\Delta_j T_u v = \sum_p \Delta_j (S_{p-1} u \Delta_p v) = \sum_{|p-j| \leq 4} \Delta_j (S_{p-1} u \Delta_p v),$$

$$\Delta_j T_v u = \sum_p \Delta_j (S_{p-1} v \Delta_p u) = \sum_{|p-j| \leq 4} \Delta_j (S_{p-1} v \Delta_p u),$$

$$\Delta_j R(u,v) = \sum_{|p-p'| \leq 1} \Delta_j (\Delta_{p'} u \Delta_p v) = \sum_{\max(p,p') \geq j-2} \sum_{|p-p'| \leq 1} \Delta_j (\Delta_{p'} u \Delta_p v).$$
Consequently, we are led to the inequality
\[ I_1 \leq \sum_{2n+l \geq 0} (2(2n + l) + l(l + 1)) \left| (\Delta_j (f_{0,0,0}g_{n,t,m}), \Delta_j h_{n,t,m})_{L^2(R^n)} \right| \]
\[ \leq \sum_{|p-j| \leq 4} \sum_{2n+l \geq 0} (2(2n + l) + l(l + 1)) \times \left\| \Delta_j (S_{p-1}f_{0,0,0} \Delta_p g_{n,t,m} + S_{p-1}g_{n,t,m} \Delta_p f_{0,0,0}) \right\|_{L^2} \left\| \Delta_j h_{n,t,m} \right\|_{L^2} \]
\[ + \sum_{p \geq j-3} \sum_{2n+l \geq 0} (2(2n + l) + l(l + 1)) \times \left\| \Delta_j (\Delta_p f_{0,0,0} \Delta_p g_{n,t,m}) \right\|_{L^2} \left\| \Delta_j h_{n,t,m} \right\|_{L^2}. \]

Furthermore, with aid of Lemma 7.2, we arrive at
\[ I_1 \lesssim \sum_{|p-j| \leq 4} \sum_{2n+l \geq 0} \sum_{|m| \leq l} (2(2n + l) + l(l + 1)) \times \left( \left\| f_{0,0,0} \right\|_{L^p} \left\| \Delta_p g_{n,t,m} \right\|_{L^2}, \left\| g_{n,t,m} \right\|_{L^p} \left\| \Delta_p f_{0,0,0} \right\|_{L^2} \right) \left\| \Delta_j h_{n,t,m} \right\|_{L^2} \]
\[ + \sum_{p \geq j-3} \sum_{2n+l \geq 0} \sum_{|m| \leq l} (2(2n + l) + l(l + 1)) \left\| f_{0,0,0} \right\|_{L^p} \left\| \Delta_p g_{n,t,m} \right\|_{L^2} \left\| \Delta_j h_{n,t,m} \right\|_{L^2}. \]

The Cauchy-Schwarz inequality enables us to get
\[ I_1 \leq \sum_{|p-j| \leq 4} \left\| f_{0,0,0} \right\|_{L^p} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} \]
\[ + \sum_{|p-j| \leq 4} \left\| \Delta_p f_{0,0,0} \right\|_{L^2} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} \]
\[ + \sum_{p \geq j-3} \left\| f_{0,0,0} \right\|_{L^p} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} \]
\[ \lesssim \sum_{|p-j| \leq 4} \left\| \Delta_p f_{0,0,0} \right\|_{L^2} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} \]
\[ + \sum_{p \geq j-4} \left\| f_{0,0,0} \right\|_{L^p} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2}. \]

Similar to the proof of the term \( I_1 \), one can verify that
\[ I_4 \lesssim \sum_{|p-j| \leq 4} \left\| \Delta_p f_{1,0,0} \right\|_{L^2} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2} \]
\[ + \sum_{p \geq j-4} \left\| f_{1,0,0} \right\|_{L^p} \left\| \Delta_p g \right\|_{L^2} \left\| \Delta_j h \right\|_{L^2}. \]

We now estimate the remaining terms \( I_2, I_3, I_5, I_6, I_7 \). The process of the proofs with respect to these five terms are almost the same, so we can take \( I_5 \) as the example.
\[ I_5 \leq \sum_{|p-j| \leq 4} \sum_{2n+l \geq 0} \sum_{|m| \leq l+2,|m| \leq 2} A_{n-2,l+2,m,m_2} \]
\[ \times \left( \Delta_j (S_{p-1}f_{0,2,m_2} \Delta_p g_{n-2,l+2,m}), \Delta_j h_{n,t,m+m_2} \right)_{L^2}. \]
where the last summation can be estimates by (4) in Proposition 2,

\[ I \]

Regarding the term \( I_{51} \), we use Cauchy-Schwarz inequality again and get

\[
\sum_{|m| \leq t+2, |m'| \leq 2} \sum_{m+2, m \leq l} A_{n-2, t+2, m, m'}^1 \left| \Delta_j \left( \frac{1}{n} \sum_{t+2, m} \Delta_p g_{n-2, t+2, m}, \Delta_j h_{n, l, m, m'} \right) \right| L^2
\]

\[
\leq \left( \sum_{|m| \leq t+2, |m'| \leq 2} \left\| f_{0, 2, m} \right\|_{L^\infty}^2 \left\| \Delta_p g_{n-2, t+2, m} \right\|_{L^2} \left\| A_{n-2, t+2, m, m'} \Delta_j h_{n, l, m, m'} \right\|_{L^2} \right)^{1/2}
\]

where the last summation can be estimates by (4) in Proposition 2,

\[ I_{51} \leq \sum_{|m| \leq t+2, |m'| \leq 2} \left\| f_{0, 2, m} \right\|_{L^\infty}^2 \left\| A_{n-2, t+2, m, m'} \right\|_{L^2}^2 \]

\[ = \sum_{|m| \leq t+2} \left( \sum_{|m'| \leq 2} \left\| A_{n-2, t+2, m, m'} \right\|_{L^2}^2 \right) \left\| \Delta_j h_{n, l, m} \right\|_{L^2}^2 \]

\[ \leq \left( \max_{|m| \leq t+2} \sum_{|m'| \leq 2} \left\| A_{n-2, t+2, m, m'} \right\|_{L^2}^2 \right) \sum_{|m| \leq t} \left\| \Delta_j h_{n, l, m} \right\|_{L^2}^2 \]

Then it follows that

\[ I_{51} \leq \sum_{|p-j| \leq 4} \left( \sum_{|m| \leq 2} \left\| f_{0, 2, m} \right\|_{L^\infty}^2 \right)^{1/2} \sum_{n+2, l \geq 0} \sum_{n \geq 2} \frac{16(n-1)}{3} \]
By using Theorem 3.2, we establish crucial estimates for Landau collision operator in the framework of Besov space, which are used to achieve the global-in-time existence.
Theorem 3.3. Assume $s > 0$, $0 < T \leq +\infty$. Let $f = f(t, x, v)$, $g = g(t, x, v)$ and $h = h(t, x, v)$ be three suitable functions, then it holds that,

$$
\sum_{j \geq -1} 2^j s \left[ \int_{0}^{T} |(\Delta_j L(f, g), \Delta_j h)| dt \right]^{1/2} \\
\lesssim \|S_2 f\|^{1/2}_{L_T^\infty L_x^s(B_{s,1})} \|\tilde{\Delta}^+ g\|^{1/2}_{L_T^2 L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ h\|^{1/2}_{L_T^2 L_x^s(B_{s,1})} \\
+ \|S_2 f\|^{1/2}_{L_T^\infty L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ g\|^{1/2}_{L_T^2 L_x^s(B_{s,1})} \|\tilde{\Delta}^+ h\|^{1/2}_{L_T^2 L_x^s(B_{s,1})}.
$$

(13)

Proof. It follows from Theorem 3.2 and Cauchy-Schwarz inequality that

$$
\sum_{j \geq -1} 2^j s \left[ \int_{0}^{T} |(\Delta_j L(f, g), \Delta_j h)|_{L_{s,v}} dt \right]^{1/2} \\
\lesssim \sum_{j \geq -1} 2^j s \left[ \sum_{p \geq -j \leq 4} \int_{0}^{T} \|\Delta_p S_2 f\|_{L_{s,v}^2} \|\tilde{\Delta}^+ g\|_{L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ \Delta_j h\|_{L_{s,v}^2} dt \right]^{1/2} \\
+ \sum_{j \geq -1} 2^j s \left[ \sum_{p \geq -j \leq 4} \int_{0}^{T} \|S_2 f\|_{L_{s,v}^2(L^\infty_v)} \|\tilde{\Delta}^+ \Delta_p g\|_{L_{s,v}^2} \|\tilde{\Delta}^+ \Delta_j h\|_{L_{s,v}^2} dt \right]^{1/2} \\
\lesssim \sum_{j \geq -1} 2^j s \left( \sum_{p \geq -j \leq 4} \|\Delta_p S_2 f\|_{L_{s,v}^\infty L_x^2(L^\infty_v)} \right)^{1/2} \|\tilde{\Delta}^+ g\|^{1/2}_{L_T^2 L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ \Delta_j h\|^{1/2}_{L_T^2 L_x^s(L^\infty_v)} \\
+ \sum_{j \geq -1} 2^j s \|S_2 f\|^{1/2}_{L_{s,v}^\infty L_x^2(L^\infty_v)} \left( \sum_{p \geq -j \leq 4} \|\tilde{\Delta}^+ \Delta_p g\|_{L_T^2 L_x^s(L^\infty_v)} \right)^{1/2} \|\tilde{\Delta}^+ \Delta_j h\|^{1/2}_{L_T^2 L_x^s(L^\infty_v)}.
$$

By changing the order of the summation, we have

$$
\sum_{j \geq -1} 2^j s \left[ \int_{0}^{T} |(\Delta_j L(f, g), \Delta_j h)|_{L_{s,v}} dt \right]^{1/2} \\
\lesssim \|S_2 f\|^{1/2}_{L_T^\infty L_x^s(B_{s,1})} \|\tilde{\Delta}^+ g\|^{1/2}_{L_T^2 L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ h\|^{1/2}_{L_T^2 L_x^s(B_{s,1})} \left( \sum_{j \geq -1} \sum_{p \geq j - 4} 2^{(j-p)s} c(p) \right)^{1/2} \\
+ \|S_2 f\|^{1/2}_{L_T^\infty L_x^s(L^\infty_v)} \|\tilde{\Delta}^+ h\|^{1/2}_{L_T^2 L_x^s(B_{s,1})} \left( \sum_{j \geq -1} \sum_{p \geq j - 4} 2^{js} \|\tilde{\Delta}^+ \Delta_p g\|_{L_T^2 L_x^s(L^\infty_v)} \right)^{1/2},
$$

where

$$
c(p) = \frac{2^{ps} \|\Delta_p S_2 f\|_{L_T^\infty L_x^s(B_{s,1})}}{\|S_2 f\|_{L_T^\infty L_x^s(B_{s,1})}}.
$$

fulfills $\|c(p)\|_{\ell^1} \leq 1$. Hence, by Fubini’s theorem and Young’s inequality, we get

$$
\sum_{j \geq -1} \sum_{p \geq j - 4} 2^{(j-p)s} c(p) = \sum_{j \geq -1} \sum_{|p| \leq 4} 2^{s} [1_{|p| \leq 4}] c(p) (J) \\
\leq \|1_{|p| \leq 4}\|_{\ell^1} c(p) \|c(p)\|_{\ell^1} < +\infty.
$$
We exchange the order in the summation that
\[
\sum_{j \geq -1} \sum_{p \geq j-4} 2^{js} \| \tilde{L}^j \Delta_p g \|_{L^2_x L^2_v} = \sum_{p \geq -1} 2^{ps} \left( \sum_{-1 \leq j \leq p+4} 2^{(j-p)s} \right) \| \tilde{L}^j \Delta_p g \|_{L^2_x L^2_v}.
\]
It is obviously that, for \( s > 0 \),
\[
\sum_{-1 \leq j \leq p+4} 2^{(j-p)s} = 2^{-(p+1)s} + 2^{-ps} + \ldots + 2^{4s} = \frac{2^{4s}(1 - 2^{-ps})}{1 - 2^{-s}} < +\infty,
\]
then it follows that
\[
\sum_{j \geq -1} \sum_{p \geq j-4} 2^{js} \sum_{p \geq -1} 2^{ps} \| \tilde{L}^j \Delta_p g \|_{L^2_x L^2_v} \lesssim \sum_{p \geq -1} 2^{ps} \| \tilde{L}^j \Delta_p g \|_{L^2_x L^2_v} = \| \tilde{L}^j g \|_{L^2_x L^2_v(B_{2,1}^s)}.
\]
Consequently, we end the proof of (13). \( \square \)

3.2. Coercivity of linear Landau operator.

In order to obtain energy estimates, the coercivity of linear operator which indicates the microscopic dissipation plays a key role. The lower estimate of \((\mathcal{L}_1 g, g)\) and the upper estimate of \((\mathcal{L}_2 f, g)\) are shown in the following theorem.

**Theorem 3.4.** For the linear operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \), it holds that
\[
(\mathcal{L}_1 g, g)_{L^2_v} \geq \frac{1}{2} (\mathcal{L}_g, g)_{L^2_v} = \frac{1}{2} \| \tilde{L}^j g \|_{L^2_v}^2 \geq \frac{1}{2} \| \tilde{L}^j g \|_{L^2_v}^2 - C\| g \|_{L^2_v},
\]
\[
(\mathcal{L}_2 f, g)_{L^2_v} \lesssim \| S_2 f \|_{L^2_v} \| S_2 g \|_{L^2_v}
\]
for some positive constant \( C \).

**Proof.** By Proposition 1, we have
\[
\mathcal{L}_1 g = -\mathcal{L} (\varphi_{0,0,0}, g)
\]
\[
= - \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+\infty} g_{n,l,m} \mathcal{L} (\varphi_{0,0,0}, \varphi_{n,l,m})
\]
\[
= \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=-l}^{+\infty} \left( 2(2n + l) + l(l + 1) \right) \varphi_{n,l,m} g_{n,l,m},
\]
\[
\mathcal{L}_2 g = -\mathcal{L} (g, \varphi_{0,0,0})
\]
\[
= - \sum_{2n+l \leq 2} g_{n,l,m} \mathcal{L} (\varphi_{n,l,m}, \varphi_{0,0,0})
\]
\[
= - \left\{ \sum_{|m_1| \leq 1} A_{n,0,0,m_1}^1 g_{0,1,m_1} \varphi_{n,1,m_1} + 4 g_{1,0,0} \varphi_{1,0,0} + \sum_{|m_2| \leq 2} (A_{n,0,0,m_2}^2 g_{0,2,m_2} \varphi_{n,2,m_2} + A_{0,0,0,m_2}^1 g_{0,2,m_2} \varphi_{0,2,m_2}) \right\}
\]
\[
= -4 \sum_{|m_1| \leq 1} g_{0,1,m_1} \varphi_{n,1,m_1} - 4 g_{1,0,0} \varphi_{n,1,m_1} + 2 \sum_{|m_2| \leq 2} g_{0,2,m_2} \varphi_{n,2,m_2}.
\]
Proposition 4. Let

\[(L_1g,g)_{L^2_w} = \sum_{n=0}^{+\infty} \sum_{l=0}^{+\infty} (2(2n + l) + (l + 1)) |g_{n,l,m}|^2\]

\[= 4 \sum_{|m| \leq 1} |g_{0,1,m}|^2 + 4 |g_{1,0,0}|^2 + 10 \sum_{|m| \leq 2} |g_{0,2,m}|^2\]

\[+ \sum_{2n+l>2} \sum_{|m| \leq l} (2(2n + l) + (l + 1))|g_{n,l,m}|^2,\]

\[(L_2f,g)_{L^2_w} = -4 \sum_{|m| \leq 1} f_{0,1,m}g_{0,1,m} - 4f_{1,0,0}g_{1,0,0} + 2 \sum_{|m| \leq 2} f_{0,2,m}g_{0,2,m},\]

\[(L_2g,g)_{L^2_w} = -4 \sum_{|m| \leq 1} |g_{0,1,m}|^2 - 4 |g_{1,0,0}|^2 + 2 \sum_{|m| \leq 2} |g_{0,2,m}|^2.\]

Thus, keeping in mind that \((L(g),g)_{L^2_w} = \|L^2g\|_{L^2_w}^2\), we can obtain (14) with aid of Young’s inequality. The proof of Theorem 3.4 is completed. \(\Box\)

Moreover, we have the direct consequence of Theorem 3.4.

Corollary 1. For the linear operators \(L_1\) and \(L_2\), it holds that

\[(\Delta_j L_1g,\Delta_j g)_{L^2_{w,v}} = \frac{1}{2} (\Delta_j Lg,\Delta_j g)_{L^2_{w,v}} \geq \frac{1}{2} \|L_j^2\Delta_j g\|_{L^2_{w,v}}^2,\]

\[(\Delta_j L_2f,\Delta_j g)_{L^2_{w,v}} \lesssim \|S_j f\|_{L^2_{w,v}} \|S_j g\|_{L^2_{w,v}}\]

for each \(j \geq -1\). Furthermore, we have

\[\sum_{j \geq -1} 2^{js} \left( \int_0^T (\Delta_j Lg,\Delta_j g)_{L^2_{w,v}} dt \right)^{1/2} = \frac{1}{2} \|L_j^2 g\|_{L^2_t L^2_{w,v}(B^2_{w,v})}\]

for \(s > 0\) and \(0 < T \leq \infty\).

3.3. Macro projections of nonlinear operator.

For the nonlinear Landau operator \(L(g,g)\), we have the following macro projections.

Proposition 4. Let \(\varphi_{n,l,m}\) be the set of eigenfunctions. For \(g,h \in \mathcal{S}(\mathbb{R}^3)\), we have

\[(L(g,g), Ph)_{L^2(\mathbb{R}^3)} = 0;\]

\[(L(g,g), \varphi_{0,2,m})_{L^2(\mathbb{R}^3)} = -12g_{0,0,0}g_{0,2,m}\]

\[+ \sum_{|m| \leq 1,|m'| \leq 1} 4\sqrt{15}\pi \left( \int_{\mathbb{S}^2} Y^m_1 Y^{m'}_1 Y^m_2 g_{0,1,m} g_{0,1,m'} \right) g_{0,1,m} g_{0,1,m'}, \quad \text{for } |m| \leq 2;\]

\[(L(g,g), \varphi_{1,1,m})_{L^2(\mathbb{R}^3)} = -8g_{0,0,0}g_{1,1,m} + \frac{8\sqrt{15}}{3} g_{1,0,0} g_{0,1,m}\]

\[+ \sum_{|m'| \leq 2,|m| \leq 1} \frac{8\sqrt{15}}{3} \left( \int_{\mathbb{S}^2} Y^{m'}_1 Y^{m'}_2 Y^{-m}_1 g_{0,1,m} g_{0,2,m'} \right) g_{0,1,m} g_{0,2,m'}, \quad \text{for } |m| \leq 1.\]
Proof. For temperate functions \( f, g \in S(\mathbb{R}^3) \), it follows from Proposition 1 that
\[
(L(f, g), \mathcal{P}h)_{L^2(\mathbb{R}^3)} = -4f_{0,0,0}g_{1,0,0}h_{1,0,0} - 4 \sum_{|m| \leq 1} f_{0,0,0}g_{0,1,m}h_{0,1,m}
+ \sum_{|m| \leq 1, |m| \leq 1} A_{0,1,m,m}f_{0,1,m}g_{0,1,m}h_{1,0,0}
+ \sum_{|m| \leq 1} A_{0,0,0,m}f_{0,1,m}g_{0,0,0}h_{0,1,m} + 4f_{1,0,0}g_{0,0,0}h_{1,0,0}.
\]
We need to compute \( A_{0,0,0,m}, A_{0,1,m,m} \) for \(|m| \leq 1 \) and \(|m| \leq 1 \). The direct computation shows that
\[
A_{0,0,0,m} = 4, \quad A_{0,1,m,m} = 0.
\]
Therefore, if we take the case of \( f = g \), then we obtain
\[
(L(g, g), \mathcal{P}h)_{L^2(\mathbb{R}^3)} = -4g_{0,0,0}g_{1,0,0}h_{1,0,0} - 4 \sum_{|m| \leq 1} g_{0,0,0}g_{0,1,m}h_{0,1,m}
+ \sum_{|m| \leq 1} 4g_{0,1,m}g_{0,0,0}h_{0,1,m} + 4g_{1,0,0}g_{0,0,0}h_{1,0,0} = 0,
\]
which is just (15).

Now we prove the equality (16). For \(|m| \leq 2\),
\[
(L(g, g), \varphi_{0,2,m})_{L^2(\mathbb{R}^3)}
= -10g_{0,0,0}g_{0,2,m} + \sum_{|m| \leq 1, |m'| \leq 1 \atop m + m' = m} A_{0,1,m,m'}g_{0,1,m}g_{0,1,m'} + A_{0,0,0,m}g_{0,2,m}g_{0,0,0}.
\]
It is not difficult to find that
\[
A_{0,1,m,m} = 4\sqrt{15\pi}\left( \int_{\mathbb{R}^2} Y_1^m Y_2^{-m'} \right), \quad A_{0,0,0,m} = -2.
\]
Substituting into the above formula, we get
\[
(L(g, g), \varphi_{0,2,m})_{L^2(\mathbb{R}^3)}
= -12g_{0,0,0}g_{0,2,m} + \sum_{|m| \leq 1, |m'| \leq 1 \atop m + m' = m} 4\sqrt{15\pi}\left( \int_{\mathbb{R}^2} Y_1^m Y_2^{-m'} \right)g_{0,1,m}g_{0,1,m'}.
\]
On the other hand, by a direct computation, we obtain
\[
(L(g, g), \varphi_{1,1,m})_{L^2(\mathbb{R}^3)} = -8g_{0,0,0}g_{1,1,m} + \sum_{|m'| \leq 2, |m| \leq 1 \atop m' + m = m} A_{0,2,m,m'}g_{0,1,m}g_{0,2,m'}
+ A_{1,0,0,m}g_{0,1,m}g_{1,0,0} + \frac{4\sqrt{15}}{3} g_{1,0,0}g_{0,1,m}
+ \sum_{|m| \leq 2, |m'| \leq 1 \atop m + m' = m} A_{0,1,m,m'}g_{0,2,m}g_{0,1,m'}.
\]
Consequently, we deduce that

\[ A^+_1,0,0,m_1 = \frac{4\sqrt{15}}{3}; \]
\[ A^-_{0,2,m',m_1} = \frac{4\sqrt{6\pi}}{3} \left( \int_{\mathbb{S}^2} Y^{m_1} Y^{m'-m_1}; \right); \]
\[ A^0_{0,1,m',m_2} = \frac{4\sqrt{6\pi}}{3} \left( \int_{\mathbb{S}^2} Y^{m_2} Y^{m'-m_2}; \right). \]

Consequently, we deduce that

\[ (L(g,g), \varphi_{1,1,m})_{L^2(\mathbb{R}^3)} = -8g_{0,0,0}g_{1,1,m} + \frac{8\sqrt{15}}{3} g_{1,0,0}g_{0,1,m} \]
\[ + \sum_{|m'| \leq 2, |m_1| \leq 1 \atop m' + m_1 = m} \frac{8\sqrt{6\pi}}{3} \left( \int_{\mathbb{S}^2} Y^{m_1} Y^{m'-m}; \right) g_{0,1,m_1}g_{0,2,m'}. \]

Hence, the proof of Proposition 4 is completed.

Based on Proposition 4, we have the following estimate in spatially Besov spaces.

**Proposition 5.** Let \( s > 0 \) and \( \phi(v) \) be the finite combination of the eigenfunctions

\[ \{ \varphi_{0,2,m_2}(v), \varphi_{1,1,m_1}(v) \}_{|m_1| \leq 1, |m_2| \leq 2}. \]

Then it holds that

\[ \sum_{j \geq -1} 2^{js} \left( \int_0^T \| (\Delta_j L(g,g), \phi(v))_{L^2} \|^2 dt \right)^{1/2} \leq \| \mathcal{S}_3 g \|_{L^2_T L^2_x(\mathbb{R}^3)} \| \mathcal{P} g \|_{L^2_T L^2_x(B_{2,1})} + \| \mathcal{P} g \|_{L^2_T L^2_x(L^\infty)} \| \mathcal{S}_3 g \|_{L^2_T L^2_x(B_{2,1})} \]

for any \( T > 0 \).

**Proof.** We need to prove (18) for all \( \{ \varphi_{0,2,m_2}(v), \varphi_{1,1,m_1}(v) \} \) with \( |m_1| \leq 1, |m_2| \leq 2 \) one by one. For \( |m| \leq 2 \), it follows from (16) that

\[ \left\| (\Delta_j L(g,g), \varphi_{0,2,m})_{L^2_x(\mathbb{R}^3)} \right\|^2_{L^2_T} \leq \sum_{j \geq -1} \left\| (\Delta_j g_{0,0,0}, \varphi_{0,2,m})_{L^2_T} \right\|^2_{L^2} \]
\[ + \sum_{|m_1| \leq 1, |m'| \leq 1 \atop m_1 + m' = m} \left( \int_{\mathbb{S}^2} Y_1^{m_1} Y_1^{m'} Y_2^{m-2}; \right) \Delta_j (g_{0,1,m_1}, g_{0,1,m'}) \left\|^2_{L^2_T} \right. \]
\[ \leq \left. \sum_{|m_1| \leq 1} \sum_{|m'| \leq 1} \left\| \Delta_j (g_{0,1,m_1}, g_{0,1,m'}) \right\|^2_{L^2_T} \right. \]
\[ \leq \left. \sum_{|m_1| \leq 1} \sum_{|m'| \leq 1} \left\| \Delta_j (\Delta_j (S_{p-1} g_{0,0,0} \Delta_p g_{0,2,m} + \Delta_p g_{0,0,0} S_{p-1} g_{0,2,m})) \right\|^2_{L^2_T} \right. \]
\[ + \sum_{\max(p,p') \geq j-2 \atop |p-p'| \leq 1} \left\| \Delta_j (\Delta_j (S_{p-1} g_{0,0,0} \Delta_p g_{0,2,m})) \right\|^2_{L^2_T} \]
\[ \leq \sum_{|p-j| \leq 4} \left\| \Delta_j (S_{p-1} g_{0,0,0} \Delta_p g_{0,2,m} + \Delta_p g_{0,0,0} S_{p-1} g_{0,2,m}) \right\|^2_{L^2_T} + \sum_{\max(p,p') \geq j-2 \atop |p-p'| \leq 1} \left\| \Delta_j (\Delta_j (S_{p-1} g_{0,0,0} \Delta_p g_{0,2,m})) \right\|^2_{L^2_T} \]
\[ \left. \right. \]
4. **Commutators estimates.** To improve the regularity of weak solution, we need delicate commutator estimates involving the nonlinear term $L(f, g)$ and cut-off functions with respect to variables $x$ and $v$. Notice that

$$L(f, g) = \sum_{i, j=1}^{3} \partial_i \int_{\mathbb{R}^3} a^{ij}(v - v_\star)\mu^{1/2}(v_\star) f(v_\star) \partial_j g(v) dv_\star$$

$$- \frac{1}{2} \sum_{i, j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v - v_\star) v_{\star i} \mu^{1/2}(v_\star) f(v_\star) \partial_j g(v) dv_\star$$

and

$$\|\Delta_j (g_{0,1,m}, g_{0,1,m'})\|^2_{L^2_x} \leq \sum_{|p-j| \leq 4} \|g_{0,1,m}\|^2_{L^2_x} \|\Delta_p g_{0,1,m'}\|^2_{L^2_x} + \sum_{p \geq j-4} \|\Delta_p g_{0,1,m}\|^2_{L^2_x} \|g_{0,1,m'}\|^2_{L^\infty_x}.$$
\[- \sum_{i,j=1}^{3} \partial_i \int_{\mathbb{R}^3} a^{ij} (v - v_*) \mu^{1/2}(v_*) (\partial_j f)(v_*) g(v) dv_*\]
\[+ \frac{1}{2} \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij} (v - v_*) v_{*i} \mu^{1/2}(v_*) (\partial_j f)(v_*) g(v) dv_*\]

(20)

with \( a^{ij}(v) = \delta_{ij} |v|^2 - v_i v_j \) (\( \delta_{ij} \) is Kronecker’s delta), where we used the fact that
\[
\sum_i a^{ij}(v - v_*)(v_i - v_{*i}) = \sum_j a^{ij}(v - v_*)(v_j - v_{*j}) = 0.
\]

In addition, we have
\[
(\mathcal{L} g, g)_{L^2} = \| \mathcal{L}^2 g \|_{L^2}^2 = 2 \| \nabla_v (I - P) g \|_{L^2}^2 + \frac{1}{2} \| v (I - P) g \|_{L^2}^2 - 3 \| (I - P) g \|_{H^1}^2
\]
\[+ \frac{1}{2} \sum_{1 \leq j, k \leq 3, j \neq k} \| (v_j \partial_k - v_k \partial_j)(I - P) g \|_{L^2}^2,
\]

which leads to
\[
\| \nabla_v (I - P) g \|_{L^2}^2 + \| (I - P) g \|_{L^2}^2 \lesssim \| \mathcal{L}^2 g \|_{L^2}^2 \lesssim \| (I - P) g \|_{H^1}^2
\]

and
\[
\| \mathcal{L}^2 g \|_{L^2} \leq \| \mathcal{L}^2 P g \|_{L^2} + \| \mathcal{L}^2 P g \|_{L^2} \leq \| \mathcal{L}^2 g \|_{L^2} + C \| g \|_{L^2}.
\]

4.1. Commutators with moments.

Let
\[
W_{\delta'}(v) = (\delta' v)^{-2} = \frac{1}{1 + |\delta' v|^2}
\]
for \( 0 < \delta' < 1 \). Here and below, we agree with the norm
\[
\| f \|_{L^2} = \left( \int_{\mathbb{R}^3} |f(v)|^2 (1 + |v|^2) dv \right)^{1/2}.
\]

**Proposition 6.** For \( 0 < \delta' < 1 \) and \( f, g, h \in \mathcal{S}(\mathbb{R}^3) \), one has
\[
\left| (W_{\delta'} \mathbf{L}(f, g) - \mathbf{L}(f, W_{\delta'} g), h)_{L^2} \right| \lesssim \| f \|_{L^2} \left( \| W_{\delta'} g \|_{L^2} \| h \|_{L^2} + \| \nabla_v W_{\delta'} g \|_{L^2} \| h \|_{L^2} + \| W_{\delta'} g \|_{L^2} \| \nabla_v h \|_{L^2} \right).\]

\[
(22)
\]

**Proof.** It follows from (20) that
\[
W_{\delta'} \mathbf{L}(f, g) - \mathbf{L}(f, W_{\delta'} g)
\]
\[= \sum_{i,j=1}^{3} W_{\delta'} \partial_i \int_{\mathbb{R}^3} a^{ij} (v - v_*) \mu^{1/2}(v_*) f(v_*) (2\delta^2 v_j W_{\delta'} g + W_{\delta'}^{-1} \partial_j (W_{\delta'} g(v))) dv_*
\]
\[+ \frac{1}{2} \sum_{i,j=1}^{3} W_{\delta'} \int_{\mathbb{R}^3} a^{ij} (v - v_*) v_{*i} \mu^{1/2}(v_*) f(v_*) (2\delta^2 v_j W_{\delta'} g + W_{\delta'}^{-1} \partial_j (W_{\delta'} g(v))) dv_*
\]
which implies that

\[ W_{\partial_s} \mathbf{L}(f, g) - \mathbf{L}(f, W_{\partial_s} g) \]

\[ = \sum_{i,j=1}^{3} \partial_{t_i} \left( \int_{\mathbb{R}^3} a^{ij}(v - v_*) \mu^{1/2}(v_*) (\partial_j f)(v_*) g(v) dv_* \right) + \frac{1}{2} \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} a^{ij}(v - v_*) v_i \mu^{1/2}(v_*) f(v_*) \partial_j (W_{\partial_s} g(v)) dv_* , \]

Therefore, we arrive at

\[ (W_{\partial_s} \mathbf{L}(f, g) - \mathbf{L}(f, W_{\partial_s} g), h)_{L^2} \]

\[ = - \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} a^{ij}(v - v_*) \mu^{1/2}(v_*) f(v_*) 2\delta^2 v_j W_{\partial_s}^2 g(v) dv_* \right) \partial_t h(v) dv + \sum_{i,j=1}^{3} \int_{\mathbb{R}^6} a^{ij}(v - v_*) \mu^{1/2}(v_*) f(v_*) 2\delta^2 v_i v_j W_{\partial_s}^2 g(v) h(v) dv dv_* \]

\[ + \sum_{i,j=1}^{3} \int_{\mathbb{R}^6} a^{ij}(v - v_*) \mu^{1/2}(v_*) f(v_*) 4\delta^4 v_i v_j W_{\partial_s}^3 g(v) h(v) dv dv_* \]

\[ + \sum_{i,j=1}^{3} \int_{\mathbb{R}^6} a^{ij}(v - v_*) \mu^{1/2}(v_*) (\partial_j f)(v_*) 2\delta^2 v_j W_{\partial_s}^2 g(v) h(v) dv dv_* \]

\[ - \sum_{i,j=1}^{3} \int_{\mathbb{R}^6} a^{ij}(v - v_*) v_i \mu^{1/2}(v_*) f(v_*) 2\delta^2 v_j W_{\partial_s}^2 gh(v) dv dv_* \]

\[ - \sum_{i,j=1}^{3} \int_{\mathbb{R}^6} a^{ij}(v - v_*) \mu^{1/2}(v_*) (\partial_j f)(v_*) 2\delta^2 v_i W_{\partial_s}^2 g(v) h(v) dv dv_* \]

\[ \triangleq A_1 + A_2 + A_3 + A_4 + A_5. \]

In the following, \( A_1, \ldots, A_5 \) can be estimated one by one. For \( A_1 \), by using integration by parts and Cauchy-Schwarz inequality, we have
As in [1, 29], let us introduce Proposition 7. Let

$$|A_1| \lesssim \sum_{i,j=1}^{3} \left| \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} a^{ij}(v - v_s)^2 \mu(v_s) dv_s \right)^{1/2} \left( \int_{\mathbb{R}^3} |f(v_s)|^2 dv_s \right)^{1/2} \times \delta^2|v| |W_{33} g(v) \partial_v h(v)| dv \right|$$

$$\lesssim \sum_{i=1}^{3} \left| \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1 + |v|)^2 \delta^{22|v|} |W_{33} g(v) \partial_v h(v)| dv \right|$$

$$\lesssim \|f\|_{L^2_v} \left( \int_{\mathbb{R}^3} (1 + |v|)^2 |W_{33} g|^2 dv \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla_v h|^2 dv \right)^{1/2}$$

where we used the fact

$$(1 + |v|)^2 |W_{33} | W_{33} = \frac{\delta^{22|v|}}{1 + |\delta v|^2} + \frac{\delta^{22|v|}}{1 + |\delta v|^2} \leq \delta^2 + 1 \leq 2.$$ 

Next, we proceed the similar procedures for $A_2$-$A_5$ and obtain

$$|A_2| \lesssim \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1 + |v|)^2 \delta^{22|v|} |W_{33} g(v) h(v)| dv$$

$$\lesssim \|f\|_{L^2_v} \|W_{33} g\|_{L^2_v} \|h\|_{L^2_v} ;$$

$$|A_3| \lesssim \sum_{j=1}^{3} \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1 + |v|)^2 \delta^{22|v|} |W_{33} | \partial_j (W_{33} g(v)) h(v)| dv$$

$$\lesssim \|f\|_{L^2_v} \|\nabla_v W_{33} g\|_{L^2_v} \|h\|_{L^2_v} ;$$

$$|A_4| \lesssim \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1 + |v|)^2 \delta^{22|v|} |W_{33} g(v) h(v)| dv$$

$$\lesssim \|f\|_{L^2_v} \|W_{33} g\|_{L^2_v} \|h\|_{L^2_v} ;$$

$$|A_5| \lesssim \int_{\mathbb{R}^3} \|f\|_{L^2_v} (1 + |v|)^2 \delta^{22|v|} |W_{33} g(v) h(v)| dv$$

$$\lesssim \|f\|_{L^2_v} \|W_{33} g\|_{L^2_v} \|h\|_{L^2_v} .$$

Hence, (22) follows from (23)-(24) directly.

4.2. Commutators with a mollifier in the $x$ variable.

Inspired by [1, 29], we can obtain the commutator of the collision operator with a mollifier in the $x$ variable.

**Proposition 7.** Let $S \in C_c^\infty(\mathbb{R})$ satisfying $0 \leq S \leq 1$ and

$$S(\tau) = 1, \quad |\tau| \leq 1; \quad S(\tau) = 0, \quad |\tau| \geq 2.$$ 

Let $S_\delta(D_x) = S(\delta D_x)$ for $\delta > 0$. Then for $f, g, h \in S(\mathbb{R}^6_{x,v})$, we have

$$\int_0^T \left( S_\delta(D_x) L(f, g) - L(f, S_\delta(D_x) g), h \right)_{L^2_v} dt$$

$$\lesssim \delta \|\nabla_x S_\delta f\|_{L^\infty([0,T] \times \mathbb{R}^6_{x,v})} \|\tilde{L}^4 g\|_{L^2([0,T] \times \mathbb{R}^6_{x,v})} \|\tilde{L}^4 h\|_{L^2([0,T] \times \mathbb{R}^6_{x,v})} .$$

**Proof.** As in [1, 29], let us introduce

$$K_\delta(z) = -\frac{z}{\delta^2} F^{-1}(S)\left(\frac{z}{\delta}\right).$$
Then, we have
\[
\left| \int_0^T (S_\delta(D_x)\mathbf{L}(f,g) - \mathbf{L}(f,S_\delta(D_x)g), h)_{L^2(\mathbb{R}^6_x)} \, dt \right|
\]
\[
= \left| \int_0^1 \left( \int_{\mathbb{R}^3_x \times \mathbb{R}^3_y} K_\delta(x - y) \times \int_0^T (\mathbf{L}(\nabla_x f(t, x + \tau(y - x), v), \delta g(t, y, v)), h(t, x, v))_{L^2(\mathbb{R}^6_y)} \, dt \, dx \, dy \right) \right| \, dt
\]
\[
\lesssim \delta \| \nabla_x \mathbb{S}_2 f \|_{L^\infty([0,T] \times \mathbb{R}^3_x ; L^2_y)} \int_0^T \int_{\mathbb{R}^3_y} \left( |K_\delta| * \| \mathcal{L}^\frac{1}{2} g \|_{L^2_y} \right) (x) \| \mathcal{L}^\frac{1}{2} h \|_{L^2_y} \, dx \, dt
\]
\[
\lesssim \delta \| \nabla_x \mathbb{S}_2 f \|_{L^\infty([0,T] \times \mathbb{R}^3_x ; L^2_y)} \| \mathcal{L}^\frac{1}{2} g \|_{L^2([0,T] \times \mathbb{R}^6_y)} \| \mathcal{L}^\frac{1}{2} h \|_{L^2([0,T] \times \mathbb{R}^6_y)},
\]
which is just (25) and where we used Theorem 3.1 and \( \|K_\delta\|_{L}_2 = \|K_1\|_{L}_1^2 \).

4.3. Commutators with a mollifier in the \( v \) variable.

Let \( M^\delta(D_v) = \frac{1}{1-\delta v} \) for \( 0 < \delta \leq 1 \), which is a pseudo-differential operator of symbol \( M^\delta(\xi) = \frac{1}{1+|\xi|^2} \).

**Proposition 8.** Let \( 0 < \delta \leq 1 \), and \( f, g, h \in \mathcal{S}(\mathbb{R}^3) \). It holds that
\[
\left| (M^\delta \mathbf{L}(f,g) - \mathbf{L}(f,M^\delta g), h)_{L^2} \right| \lesssim \| f \|_{L^2} \left( \| M^\delta g \|_{L^2} \| \nabla_v h \|_{L^2} + \| M^\delta g \|_{L^2} \| h \|_{L^2} \right).
\]

**Proof.** By using the nonlinear term \( \mathbf{L}(f,g) \) in (20) and integration by parts, we have
\[
(M^\delta \mathbf{L}(f,g) - \mathbf{L}(f,M^\delta g), h)_{L^2}
= - \sum_{i,j=1}^3 \int_{\mathbb{R}^6} \left[ M^\delta, a^{ij}(v - v_*) \right] \partial_j g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) \, dv, \, dv
- \frac{1}{2} \sum_{i,j=1}^3 \int_{\mathbb{R}^6} \left[ M^\delta, a^{ij}(v - v_*) \right] \partial_j g(v) \mu^{1/2}(v_*) f(v_*) h(v) \, dv, \, dv
+ \sum_{i,j=1}^3 \int_{\mathbb{R}^6} \left[ M^\delta, a^{ij}(v - v_*) \right] g(v) \mu^{1/2}(v_*) (\partial_j f)(v_*) \partial_i h(v) \, dv, \, dv
+ \frac{1}{2} \sum_{i,j=1}^3 \int_{\mathbb{R}^6} \left[ M^\delta, a^{ij}(v - v_*) \right] g(v) \mu^{1/2}(v_*) (\partial_j f)(v_*) h(v) \, dv, \, dv
\triangleq B_1 + B_2 + B_3 + B_4.

For \( B_1 \), note that \( (M^\delta)^{-1} = 1 - \delta \Delta_v \), we get
\[
B_1 = - \sum_{i,j=1}^3 \int_{\mathbb{R}^6} M^\delta \left[ a^{ij}(v - v_*) \right] (M^\delta)^{-1} M^\delta \partial_j g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) \, dv, \, dv
= \sum_{i,j=1}^3 \int_{\mathbb{R}^6} M^\delta \left[ 1 - \delta \Delta_v, a^{ij}(v - v_*) \right] \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_i h(v) \, dv, \, dv.
\]
Owing to the fact

\[ [1 - \delta \Delta_v, a^{ij}(v - v_*)] = -\delta \left( 2 \sum_{l=1}^{3} \partial_l a^{ij}(v - v_*) \partial_l + \sum_{l=1}^{3} \partial_l^2 a^{ij}(v - v_*) \right), \]

furthermore, one can deduce that

\[ B_1 = -2\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} M^3 \partial_l a^{ij}(v - v_*) \partial_l \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_l h(v) dv_* dv \]

\[ - \delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} M^\delta \partial_l^2 a^{ij}(v - v_*) \partial_j M^\delta g(v) \mu^{1/2}(v_*) f(v_*) \partial_l h(v) dv_* dv, \]

where we agree with \( \partial_l = \partial_{v_l} \). The direct calculation enables us to get

\[ M^\delta \partial_l a^{ij}(v - v_*) = \partial_l a^{ij}(v - v_*) M^\delta + [M^\delta, \partial_l a^{ij}(v - v_*)] \]

\[ = \partial_l a^{ij}(v - v_*) M^\delta - M^\delta \left[ 1 - \delta \Delta_v, \partial_l a^{ij}(v - v_*) \right] M^\delta \]

\[ = \partial_l a^{ij}(v - v_*) M^\delta + 2\delta M^\delta \sum_{k=1}^{3} \partial_k \partial_l a^{ij}(v - v_*) \partial_k M^\delta. \]

It is not difficult to see that

\[ \partial_l a^{ij}(v - v_*) = 2\delta_{ij}(v_l - v_{l*}) - (\delta_{l_l}(v_{j} - v_{j*}) + \delta_{l_l}(v_i - v_{i*})) ; \]

\[ \partial_l^2 a^{ij}(v - v_*) = 2\delta_{ij} - 2\delta_{l_l}\delta_{l_{ij}}, \quad l = 1, 2, 3; \]

\[ \partial_l \partial_k a^{ij}(v - v_*) = -\delta_{l_k}\delta_{l_{ij}}, \quad k \neq l, \]

where the last two derivations are constants. Consequently, we are led to

\[ B_1 = -2\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l a^{ij}(v - v_*) \partial_l \partial_j (M^\delta)^2 g(v) \mu^{1/2}(v_*) f(v_*) \partial_l h(v) dv_* dv \]

\[ - 4\delta^2 \sum_{i,j=1}^{3} \sum_{l=1}^{3} \sum_{k=1}^{3} \int_{\mathbb{R}^6} \partial_k \partial_l a^{ij}(v - v_*) \partial_l \partial_j \partial_k (M^\delta)^3 g(v) \mu^{1/2}(v_*) f(v_*) \partial_l h(v) dv_* dv \]

\[ - \delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l^2 a^{ij}(v - v_*) \partial_j (M^\delta)^2 g(v) \mu^{1/2}(v_*) f(v_*) \partial_l h(v) dv_* dv. \]

So, by Cauchy-Schwarz inequality, we conclude that

\[ |B_1| \leq \delta \|f\|_{L^2} \sum_{i,j,l=1}^{3} \int_{\mathbb{R}^3} (1 + |v|) |\partial_l \partial_j (M^\delta)^2 g(v)||\partial_l h(v)| dv \]

\[ + \delta^2 \|f\|_{L^2} \sum_{i,j,l,k=1}^{3} \int_{\mathbb{R}^3} |\partial_l \partial_j \partial_k (M^\delta)^3 g(v)||\partial_l h(v)| dv \]

\[ + \delta \|f\|_{L^2} \sum_{i,j,l=1}^{3} \int_{\mathbb{R}^3} |\partial_j (M^\delta)^2 g(v)||\partial_l h(v)| dv \]

\[ \leq \|f\|_{L^2} \|M^\delta g\|_{L^4} \|\nabla h\|_{L^2}, \]
where the following estimates are used:
\[
\delta \| (1 + |\nu|) \partial l \partial l_j (\delta^2 u) \|_{L^2(\mathbb{R}^3)} \leq \| u \|_{L^2}^2;
\]
\[
\delta^2 \| \partial l \partial l_j \partial k (\delta^2 u) \| \leq \| u \|_{L^2} \leq \| u \|_{L^2}^2;
\]
\[
\delta \| \partial_j (\delta^2 u) \| \leq \| u \|_{L^2} \leq \| u \|_{L^2}^2.
\]
for \( u \in L^2(\mathbb{R}^3) \).

By employing the similar calculations as \( B_1 \), we have
\[
B_2 = -\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) \partial_l (\delta^2 g(v) \mu^{1/2}(\nu) f(\nu) h(v) dv, dv
\]
\[
- 2\delta^2 \sum_{i,j=1}^{3} \sum_{l=1}^{3} \sum_{k=1}^{3} \int_{\mathbb{R}^6} \partial_k \partial_l (v - \nu) \partial_l (\delta^3 g(v) \mu^{1/2}(\nu) f(\nu) h(v) dv, dv
\]
\[
- \frac{1}{2}\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) \partial_j (\delta^2 g(v) \mu^{1/2}(\nu) f(\nu) h(v) dv, dv,
\]
which gives
\[
|B_2| \lesssim \| f \|_{L^2} \| \delta g \|_{L^2} \| h \|_{L^2}.
\]

Similarly, for \( B_3, B_4 \), we obtain
\[
B_3 = 2\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) \partial_l (\delta^2 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv
\]
\[
+ 4\delta^2 \sum_{i,j=1}^{3} \sum_{l=1}^{3} \sum_{k=1}^{3} \int_{\mathbb{R}^6} \partial_k \partial_l (v - \nu) \partial_l (\delta^3 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv
\]
\[
+ \delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) (\delta^2 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv
\]
and
\[
B_4 = \delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) \partial_l (\delta^2 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv
\]
\[
+ 2\delta^2 \sum_{i,j=1}^{3} \sum_{l=1}^{3} \sum_{k=1}^{3} \int_{\mathbb{R}^6} \partial_k \partial_l (v - \nu) \partial_l (\delta^3 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv
\]
\[
+ \frac{1}{2}\delta \sum_{i,j=1}^{3} \sum_{l=1}^{3} \int_{\mathbb{R}^6} \partial_l (v - \nu) (\delta^2 g(v) \mu^{1/2}(\nu) \partial_l f(\nu) \partial_h (v) dv, dv.
\]

Integration by parts with respect to the variable \( \nu \), we arrive at
\[
|B_3| \lesssim \| f \|_{L^2} \| \delta g \|_{L^2} \| \nabla \nu \|_{L^2},
\]
\[
|B_4| \lesssim \| f \|_{L^2} \| \delta g \|_{L^2} \| \nu \|_{L^2}.
\]

Putting the above estimates for \( B_1-B_4 \) together, we achieve (26) eventually.

Based on Propositions 6-8, we obtain commutator estimates involving the non-linear term \( L \) and various cut-off functions.
Proposition 9. For the nonlinear term, for any \( f, g \in \mathcal{S}(\mathbb{R}^6_{x,v}) \), it holds that

\[
\int_0^T \left| (M^8 S_\delta W_{\delta'} L(f, g) - L(f, M^8 S_\delta W_{\delta'} g), M^8 S_\delta W_{\delta'} g)_{x,v} \right| dt \\
\lesssim \|f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \|W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} + \|\nabla_v W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
+ \|f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} + \epsilon \|M^8 S_\delta W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
+ \delta C_\epsilon \|\nabla_v S_2 f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \left\| \tilde{L}^2_2 M^8 S_\delta W_{\delta'} g \right\|^2_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})},
\]

for \( 0 < \delta, \delta' < 1 \) and \( \epsilon > 0 \), where \( W_{\delta'} \), \( S_\delta \), \( M^8 \) are defined by Propositions 6-8 respectively.

Proof. Note that

\[
(M^8 S_\delta W_{\delta'} L(f, g) - L(f, M^8 S_\delta W_{\delta'} g), M^8 S_\delta W_{\delta'} g)_{x,v} \\
= (W_{\delta'} L(f, g) - L(f, W_{\delta'} g), S_\delta (M^8)^2 S_\delta W_{\delta'} g)_{x,v} \\
+ (S_\delta L(f, W_{\delta'} g) - L(f, S_\delta W_{\delta'} g), (M^8)^2 S_\delta W_{\delta'} g)_{x,v} \\
+ (M^8 L(f, S_\delta W_{\delta'} g) - L(f, M^8 S_\delta W_{\delta'} g), M^8 S_\delta W_{\delta'} g)_{x,v} \\
= J_1 + J_2 + J_3.
\]

For \( J_1 \), it follows from Proposition 6 that

\[
\int_0^T \left| J_1 \right| dt \lesssim \|f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \\
\times \left( \|W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \|M^8 S_\delta W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
+ \|\nabla_v W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \|M^8 S_\delta W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
+ \|W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \|\nabla_v M^8 S_\delta W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \right) \\
\lesssim \|f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \left\| M^8 S_\delta W_{\delta'} g \right\|^2_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
\times \left( \|W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \|\nabla_v W_{\delta'} g\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \right).
\]

For \( J_2 \), thanks to Proposition 7 and Young’s inequality, we get

\[
\int_0^T \left| J_2 \right| dt \lesssim \delta \|\nabla_v S_2 f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \left\| \tilde{L}^2_2 W_{\delta'} g \right\|^2_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
\times \left\| \tilde{L}^2_2 M^8 S_\delta W_{\delta'} g \right\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
\lesssim \delta C_\epsilon \|\nabla_v S_2 f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \left\| \tilde{L}^2_2 W_{\delta'} g \right\|^2_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})} \\
+ \epsilon \left\| M^8 S_\delta W_{\delta'} g \right\|^2_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})},
\]

for \( \epsilon > 0 \), where \( C_\epsilon \) is some constant depending only on \( \epsilon \). For \( J_3 \), by Proposition 8, we obtain

\[
\int_0^T \left| J_3 \right| dt \lesssim \|f\|_{L^\infty([0,T]\times \mathbb{R}^3_v; L^2_2)} \left\| M^8 S_\delta W_{\delta'} g \right\|_{L^2([0,T]\times \mathbb{R}^3_v; L^2_{x,v})}.
\]
In addition, we can get the second inequality according to Theorem 3.1.

The first inequality in Proposition 9 follows from above estimates for \( J_1, J_2 \) and \( J_3 \).

In addition, we can get the second inequality according to Theorem 3.1.

4.4. Commutators for linear operator.

Moreover, we also need some commutators with the linear operator \( \mathcal{L}_1 \).

**Proposition 10.** For \( 0 < \delta' < 1 \), it holds that

\[
\left| (W_{\delta'} \mathcal{L}_1 \gamma - \mathcal{L}_1 W_{\delta'} \gamma, h)_{L^2_\omega} \right| \lesssim \|W_{\delta'} \gamma\|_{L^2_\omega} (\|h\|_{L^2_\omega} + \|v_{\omega} h\|_{L^2_\omega}).
\]  

(27)

**Proof.** By using Lemma 2.1, we have

\[
(W_{\delta'} \mathcal{L}_1 \gamma - \mathcal{L}_1 W_{\delta'} \gamma, h)_{L^2_\omega} = (2 \[\Delta_v, W_{\delta'} \gamma, h\] + (\[\Delta_{\delta'}, W_{\delta'} \gamma, h\])_{L^2_\omega} = D_1 + D_2.
\]

For the term \( D_1 \), integration by parts allows us to get

\[
D_1 = 2 \sum_{l=1}^{3} \int_{\mathbb{R}^3} (2(\partial_l W_{\delta'}) \partial_l g + (\partial_l^2 W_{\delta'}) g) h dv
\]

\[
= -4 \sum_{l=1}^{3} \int_{\mathbb{R}^3} (\partial_l W_{\delta'}) (\partial_l h) g dv - 2 \sum_{l=1}^{3} \int_{\mathbb{R}^3} (\partial_l^2 W_{\delta'}) gh dv.
\]

Furthermore, we obtain

\[
|D_1| \leq 4 \sum_{l=1}^{3} \int_{\mathbb{R}^3} (\partial_l W_{\delta'}) \partial_l h dv + 2 \sum_{l=1}^{3} \int_{\mathbb{R}^3} (\partial_l^2 W_{\delta'}) gh dv
\]

\[
\lesssim \|W_{\delta'} \gamma\|_{L^2_\omega} \|v_{\omega} h\|_{L^2_\omega} + \|W_{\delta'} \gamma\|_{L^2_\omega} \|h\|_{L^2_\omega}.
\]

On the other hand, it follows from the direct calculation that

\[
D_2 = \frac{1}{2} \sum_{1 \leq j, k \leq 3} \int_{\mathbb{R}^3} \left( (v_j \partial_k - v_k \partial_j)^2 (W_{\delta'} \gamma) - W_{\delta'} (v_j \partial_k - v_k \partial_j)^2 g \right) h dv
\]

\[
= \frac{1}{2} \sum_{1 \leq j, k \leq 3} \int_{\mathbb{R}^3} \left( (v_j \partial_k - v_k \partial_j)^2 W_{\delta'} \right) g + 2 (v_j \partial_k - v_k \partial_j) W_{\delta'} (v_j \partial_k - v_k \partial_j) g h dv
\]

\[
= 0,
\]

where we used the relation

\[
(v_j \partial_k - v_k \partial_j) W_{\delta'} = -2\delta'^2 v_j v_k W_{\delta'}^2 + 2\delta'^2 v_j v_k W_{\delta'}^2 = 0.
\]

We obtain (27) directly. Therefore, the proof of Proposition 10 is completed.

**Proposition 11.** For \( 0 < \delta \leq 1 \), it holds that

\[
\left| (M^\delta \mathcal{L}_1 \gamma - \mathcal{L}_1 M^\delta \gamma, h)_{L^2_\omega} \right| \lesssim \|M^\delta \gamma\|_{L^2_\omega} \|h\|_{L^2_\omega}.
\]  

(28)
Proof. From Lemma 2.1, we obtain
\[
(M^\delta \mathcal{L}_1 g - \mathcal{L}_1 M^\delta g, h)_{L^2_v} = \frac{1}{2} \left( \left[ M^\delta, v^2 \right] g, h \right)_{L^2_v} - \left( \left[ M^\delta, \Delta g \right] g, h \right)_{L^2_v} = E_1 + E_2.
\]

For \( E_1 \), by the direct calculation, we arrive at
\[
E_1 = \frac{1}{2} \int_{\mathbb{R}^3} M^\delta \left[ (1 - \delta\Delta), v^2 \right] M^\delta gh dv
= \delta \int_{\mathbb{R}^3} \sum_{l=1}^3 v_l \partial_l (M^\delta)^2 gh dv + 2\delta^2 \int_{\mathbb{R}^3} \sum_{l,k=1}^3 \delta_{kj,l} \partial_l \partial_k (M^\delta)^3 gh dv + \frac{1}{2} \delta \int_{\mathbb{R}^3} \sum_{l=1}^3 (M^\delta)^2 gh dv,
\]
which indicates that
\[
|E_1| \lesssim \|M^\delta g\|_{L^2_v}^2 \|h\|_{L^2_v}.
\]
Note that
\[
(v_j \partial_k - v_k \partial_j)^2 = v_j^2 \partial_k^2 - 2v_k v_j \partial_k \partial_j + v_k^2 \partial_j^2 - v_k \partial_k - v_j \partial_j,
\]
then for \( 1 \leq k \neq j \leq 3 \), we get
\[
\Delta (v_j \partial_k - v_k \partial_j)^2 u - (v_j \partial_k - v_k \partial_j)^2 \Delta u
= \sum_{l=1}^3 \partial_l^2 \left[ (v_j^2 \partial_k^2 - 2v_k v_j \partial_k \partial_j + v_k^2 \partial_j^2 - v_k \partial_k - v_j \partial_j) u \right]
- 3 \sum_{l=1}^3 (v_j^2 \partial_k^2 - 2v_k v_j \partial_k \partial_j + v_k^2 \partial_j^2 - v_k \partial_k - v_j \partial_j) \partial_l^2 u
= 2 \sum_{l=1}^3 \left( \delta_{lj} \partial_k^2 + \delta_{lk} \partial_j^2 - \delta_{lk} \partial_l \partial_l - \delta_{lj} \partial_j \partial_l \right) u
+ 4 \left( \delta_{lj} v_j \partial_k^2 - \delta_{lj} v_k \partial_j \partial_j \right) \partial_l u + \left( \delta_{lk} v_k \partial_j^2 - \delta_{lk} v_j \partial_j \partial_j \right) \partial_l u
= 2 \left( \partial_k^2 + \partial_j^2 - \partial_l^2 \right) u.
\]
Which shows that
\[
E_2 = -\sum_{1 \leq j,k \leq 3} \left( M^\delta \left[ (1 - \delta\Delta), (v_j \partial_k - v_k \partial_j)^2 \right] M^\delta g, h \right)_{L^2_v} = 0.
\]
The inequality (28) is followed directly. Hence, the proof of Proposition 11 is completed. \(\square\)

As a consequence of Propositions 10-11, we get the commutator estimate.

**Proposition 12.** Let \( g \in \mathcal{S}(\mathbb{R}^6_{x,v}) \). There exist some positive constants \( C_\epsilon > 0 \) independent of \( T > 0 \) such that
\[
\int_0^T \left| \left( \mathcal{L}_1 g, W_{\delta^\epsilon} S_\delta (M^\delta)^2 S_\delta W_{\delta^\epsilon} g \right)_{x,v} - \left( \mathcal{L}_1 M^\delta S_\delta W_{\delta^\epsilon} g, M^\delta S_\delta W_{\delta^\epsilon} g \right)_{x,v} \right| dt
\leq \epsilon \left\| M^\delta S_\delta W_{\delta^\epsilon} g \right\|_{L^2_x (\mathbb{R}^6_{x,v})}^2 + \epsilon T \left\| S_\delta W_{\delta^\epsilon} g \right\|_{L^\infty (\mathbb{R}^6_{x,v})}^2 + C_\epsilon T \left\| M^\delta S_\delta W_{\delta^\epsilon} g \right\|_{L^2_x (\mathbb{R}^6_{x,v})}^2,
\]
where \( x \in [0,T] \times \mathbb{R}^6 \).
where $\epsilon > 0$ is sufficiently small.

Proof. Obviously, we see that

$$ (L_1g, W^\theta S_\delta (M^\delta)^2 S_\delta W^\theta g)_{x,v} - (L_1 M^\delta S_\delta W^\theta g, M^\delta S_\delta W^\theta g)_{x,v} $$

$$ = (S_\delta W^\theta L_1 g - L_1 S_\delta W^\theta g, (M^\delta)^2 S_\delta W^\theta g)_{x,v} $$

$$ + (M^\delta L_1 S_\delta W^\theta g - L_1 M^\delta S_\delta W^\theta g, M^\delta S_\delta W^\theta g)_{x,v} $$

$$ = F_1 + F_2. $$

For $F_1$, it follows from the Proposition 10 that

$$ \int_0^T F_1 dt \lesssim \sqrt{T} \| S_\delta W^\theta g \|_{L^\infty([0,T];L^2(\mathbb{R}^3_\nu))} \| M^\delta S_\delta W^\theta g \|_{L^2([0,T] \times \mathbb{R}^3;L^2(\mathbb{R}^3))} $$

$$ + \sqrt{T} \| S_\delta W^\theta g \|_{L^\infty([0,T];L^2(\mathbb{R}^3_\nu))} \| \nabla_x M^\delta S_\delta W^\theta g \|_{L^2([0,T] \times \mathbb{R}^3;L^2(\mathbb{R}^3))} $$

$$ \leq \frac{\epsilon}{2} \| ||| M^\delta S_\delta W^\theta g \| ||^2_{L^2([0,T] \times \mathbb{R}^3)} + C_T \| S_\delta W^\theta g \|_{L^\infty([0,T];L^2(\mathbb{R}^3_\nu))}. $$

For $F_2$, thanks to Proposition 11, we obtain

$$ \int_0^T F_2 dt \lesssim \sqrt{T} \| M^\delta S_\delta W^\theta g \|_{L^\infty([0,T];L^2(\mathbb{R}^3_\nu))} \| M^\delta S_\delta W^\theta g \|_{L^2([0,T] \times \mathbb{R}^3;L^2(\mathbb{R}^3))} $$

$$ \leq \frac{\epsilon}{2} \| ||| M^\delta S_\delta W^\theta g \| ||^2_{L^2([0,T] \times \mathbb{R}^3)} + C_T \| M^\delta S_\delta W^\theta g \|_{L^\infty([0,T];L^2(\mathbb{R}^3_\nu))}. $$

Therefore, we finish the proof of Proposition 12. \qed

5. The existence of solution. In this section, we are devoted to obtain the global existence of the solution to the inhomogeneous Landau equation.

5.1. The local-in-time existence.

We first establish the local-in-time existence of solution to the Cauchy problem (2).

Proposition 13. For a sufficiently small constant $\epsilon_0 > 0$, there exists $T > 0$ such that

$$ \| g_0 \|_{\tilde{L}^2_T(\mathbb{R}^3)} \leq \epsilon_0, $$

then the Cauchy problem (2) admits a local solution $g(t, v, x) \in (0, T) \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying

$$ g \in \tilde{L}^{\infty}_T \tilde{L}^2_T(\mathbb{R}^3) \quad \text{and} \quad \tilde{\nabla} g \in \tilde{L}^2_T \tilde{L}^2_T(\mathbb{R}^3). $$

Proof. Firstly, we construct the following sequence of iterating approximate solutions:

$$ \begin{cases} 
\partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + L_1 g^{n+1} = L(g^n, g^{n+1}) - L_2 g^n, \ t > 0, \ v \in \mathbb{R}^3, \\
g^{n+1}(t, x, v)|_{t=0} = g_0(x, v),
\end{cases} $$

starting from $g^0(t, x, v) \equiv g_0(x, v)$. Taking $g = g^{n+1}, f = g^n$ and $T = \min\{T_0, 1/(4C_0^2)\}$ in Theorem 6.1 gives

$$ \| g^n \|_{\tilde{L}^{\infty}_T \tilde{L}^2_T(\mathbb{R}^3)} + \| \tilde{\nabla} g^n \|_{\tilde{L}^2_T \tilde{L}^2_T(\mathbb{R}^3)} \leq \epsilon, $$

where $\epsilon_0 > 0$ is chosen such that $2C_0 \epsilon_0 \leq \epsilon$. Secondly, it suffices to prove the convergence of the sequence $\{g^n\}$ in the space $Y$, which is defined by

$$ Y = \{ g | g \in \tilde{L}^{\infty}_T \tilde{L}^2_T(\mathbb{R}^3), \tilde{\nabla} g \in \tilde{L}^2_T \tilde{L}^2_T(\mathbb{R}^3) \}. $$


Set \( w_n = g^{n+1} - g^n \). It follows from (29) that
\[
\frac{\partial w_n}{\partial t} + v \cdot \nabla_x w_n + \mathcal{L}_t w_n = \mathbf{L}(g^n, w^n) + \mathbf{L}(w^{n-1}, g^n) - \mathcal{L}_2 w^{n-1}
\]
with \( w_0 = 0 \). By employing the similar energy estimate leading to (55), we get
\[
\| w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} + \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
\leq C \sqrt{T} \| w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} + C \| S_2 g^n \|^{1/2}_{\tilde{L}_T^2(B_2^{3/2}(v))} \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
+ C \| S_2 w^{n-1} \|^{1/2}_{\tilde{L}_T^2(B_2^{3/2}(v))} \| \tilde{L}^2 g^n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
+ C \sqrt{T} \| S_2 w^{n-1} \|^{1/2}_{\tilde{L}_T^2(B_2^{3/2}(v))} \| w_n \|^{1/2}_{\tilde{L}_T^2(B_2^{3/2}(v))} \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
\leq C \sqrt{T} \| w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} + C \sqrt{\varepsilon} \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
+ C (\sqrt{T} + \sqrt{\varepsilon}) \| S_2 w^{n-1} \|_{\tilde{L}_T^2(B_2^{3/2}(v))}
\]
Furthermore, if \( \varepsilon \) and \( T \) are sufficiently small (for example, taking \( C \sqrt{\varepsilon}, C \sqrt{T} < \frac{1}{8} \)), then
\[
\| w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} + \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \leq \lambda \| S_2 w^{n-1} \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \| \tilde{L}^2 w_n \|_{\tilde{L}_T^2(B_2^{3/2}(v))} \\
\leq \lambda^{n-1} \| S_2 w^1 \|_{\tilde{L}_T^2(B_2^{3/2}(v))}
\]
for some \( 0 < \lambda < 1 \). Clearly, we see that \( \{ g^n \} \) is a Cauchy sequence in \( Y \), so there is some limit function \( g \in Y \) such that \( g^n \to g \) as \( n \to \infty \). The standard procedure enables us to know that \( g \) is the desired solution to the Cauchy problem (2) satisfying
\[
g(t, v, x) \in \tilde{L}_T^\infty(B_2^{3/2}(v)) \quad \text{and} \quad \tilde{L}^2 g(t, v, x) \in \tilde{L}_T^2(B_2^{3/2}(v)).
\]
The proof of Proposition 13 is finished.

**The non-negativity of the solution.**

For the solution of the Cauchy problem (2) that obtained in Proposition 13, which is the limit of the sequence of (29), coming back to the original Landau equation, it is also the limit of a sequence constructed successively by the following linear Cauchy problem
\[
\begin{cases}
\frac{\partial f^{n+1}}{\partial t} + v \cdot \nabla_x f^{n+1} = Q_L(f^n, f^{n+1}), \\
f^{n+1} |_{t=0} = f_0 = \mu + \sqrt{\varepsilon} g_0 \geq 0.
\end{cases}
\]
Then, the non-negativity of the solution to the Cauchy problem (1) can be proved by the same methods as in [4, 21].

**5.2. The global-in-time solution.**

We prove now that we can extend the above local-in-time solution to a global-in-time solution, which heavily depends on the key a priori estimate. For this end, we define the energy functional
\[
\mathcal{E}_T(g) = \| g \|_{\tilde{L}_T^2(B_2^{3/2}(v))}
\]
and the dissipation functional
\[
\mathcal{D}_T(g) = \| \nabla_x (a, b, c) \|_{\tilde{L}_T^2(B_2^{3/2}(v))} + \| \tilde{L}^2 g \|_{\tilde{L}_T^2(B_2^{3/2}(v))}
\]
respectively. Furthermore, it is shown that
Proposition 14. Let $g \in Y$ be the solution to the Cauchy problem (2). It holds that
\[ \mathcal{E}_T(g) + \mathcal{D}_T(g) \leq C \|g_0\|_{L^2_{t,v}(B^{3/2}_{2,1})} + C \left( \mathcal{E}_T(g) + \sqrt{\mathcal{E}_T(g)} \right) \mathcal{D}_T(g) \] (30)
for any $T > 0$, where $C > 0$ is some constant independent of $T$.

Based on the above global a priori estimate (30) and the local existence result (Proposition 13), Theorem 1.1 is followed by the standard continuity argument. We feel free to skip the procedure. The interested reader is referred to [19] for similar details.

We split the proof of Proposition 14 into several parts.

5.3. Estimate on the macroscopic dissipation.

In this part, we bound the macroscopic dissipation arising from Landau collision operator. We by $\mathbf{P}$ denote the projection operator on $\ker \mathcal{N}$, which is given by
\[ \mathbf{P} g = (a(t,x) + v \cdot b(t,x) + (|v|^2 - 3)c(t,x)) \sqrt{\mu}. \] (31)
In terms of the macro-micro decomposition, the distribution function $g(t,v,x)$ can be decomposed as
\[ g = \mathbf{P} g + (\mathbf{I} - \mathbf{P}) g. \]
Precisely, the macroscopic dissipation of $g$ is included in the following proposition.

Proposition 15. It holds that
\[ \| \nabla_x (a, b, c) \|_{L^2_t L^2_v(B^{3/2}_{2,1})} \leq \|g_0\|_{L^2_t L^2_v(B^{3/2}_{2,1})} + \mathcal{E}_T(g) \]
\[ + \| \mathcal{L}_T^2 g \|_{L^2_t L^2_v(B^{3/2}_{2,1})} + \mathcal{E}_T(g) \mathcal{D}_T(g) \] (32)
for any $T > 0$.

Proof. We take the velocity moments
\[ \sqrt{\mu}, \; v_i \sqrt{\mu}, \; \frac{1}{6} (|v|^2 - 3) \sqrt{\mu}, \; (v_i v_j - \delta_{ij}) \sqrt{\mu}, \; \frac{1}{\sqrt{10}} (|v|^2 - 5) v_i \sqrt{\mu} \]
with $i, j = 1, 2, 3$ for the Landau equation (2). Define
\[ \phi_{i,j} = (v_i v_j - \delta_{ij}) \sqrt{\mu}, \; \phi_{1,1,i} = \frac{1}{\sqrt{10}} (|v|^2 - 5) v_i \sqrt{\mu} \]
and
\[ A_{i,j}(g) = (g, \phi_{i,j}), \; B_i(g) = (g, \phi_{1,1,i}). \]
Noticing that
\[ \phi_{i,j} \in \text{span} \{ \varphi_{0,0,0}, \varphi_{1,0,0}, \varphi_{0,2,0}, \varphi_{0,2,0}, \varphi_{0,2,1}, \varphi_{0,2,1}, \varphi_{0,2,2}, \varphi_{0,2,2} \} \]
and
\[ \phi_{1,1,1} = -\varphi_{1,1,0}; \; \phi_{1,1,2} = -\frac{\varphi_{1,1,1} + \varphi_{1,1,-1}}{\sqrt{2}}; \; \phi_{1,1,3} = -\frac{\varphi_{1,1,1} - \varphi_{1,1,-1}}{\sqrt{2}}, \]
we deduce that
\[ \mathcal{L} \phi_{i,j} = 12 \sqrt{2} \phi_{i,j} \text{ with } |C_{i,j}| \leq 2; \; \mathcal{L} \phi_{1,1,i} = 8 \phi_{1,1,i}, \; i, j = 1, 2, 3. \]

With aid of the orthogonal of $\{ \varphi_{n,m} \}$, we infer that $(a, b, c)$ which is the coefficient of the macroscopic component $\mathbf{P} g$ (31) satisfies the fluid-type system
\[
\begin{aligned}
\partial_t a + \nabla_x \cdot b &= 0, \\
\partial_t b_i + \partial_{x_i}(a + 2c) + \sum_{j=1}^{3} \partial_{x_j} A_{i,j}((I - P)g) &= 0, \\
\partial_t c + \sum_{i=1}^{3} \partial_{x_i} B_i((I - P)g) &= 0, \\
\partial_t (A_{i,j}((I - P)g) + 2c\delta_{i,j}) + \partial_{x_i} b_j + \partial_{x_j} b_i + 12\left(\tilde{S}_2 g, \phi_{i,j}\right) &= A_{i,j}(L(g, g) - v \cdot \nabla_x (I - P)g), \\
\partial_t B_i((I - P)g) + \sqrt{10} \partial_{x_i} c + 8B_i(g) &= B_i(L(g, g) - v \cdot \nabla_x (I - P)g),
\end{aligned}
\]

where \(i, j = 1, 2, 3\). Applying the cut-off operator \(\Delta_p\) with \(p \geq -1\) to the system \(33\) implies that

\[
\begin{aligned}
\partial_t \Delta_p a + \nabla_x \cdot \Delta_p b &= 0, \\
\partial_t \Delta_p b_i + \partial_{x_i}(\Delta_p a + 2\Delta_p c) + \sum_{j=1}^{3} \partial_{x_j} A_{i,j}(\Delta_p(I - P)g) &= 0, \\
\partial_t \Delta_p c + \sum_{i=1}^{3} \partial_{x_i} B_i(\Delta_p(I - P)g) &= 0, \\
\partial_t (A_{i,j}(\Delta_p(I - P)g) + 2\Delta_p c\delta_{i,j}) + \partial_{x_i} \Delta_p b_j + \partial_{x_j} \Delta_p b_i + 12\left(\tilde{S}_2 \Delta_p g, \phi_{i,j}\right) &= A_{i,j}(\Delta_p L(g, g) - v \cdot \nabla_x \Delta_p (I - P)g), \\
\partial_t B_i(\Delta_p(I - P)g) + \sqrt{10} \partial_{x_i} \Delta_p c + 8B_i(\Delta_p g) &= B_i(\Delta_p L(g, g) - v \cdot \nabla_x \Delta_p (I - P)g).
\end{aligned}
\]

As in [18, 19], we denote the temporal interactive functionals as follows:

\[
\begin{aligned}
\mathcal{E}_0(t) &= \sum_{i=1}^{3} \left(\partial_{x_i} \Delta_p c, B_i(\Delta_p(I - P)g)\right)_{L^2_x}, \\
\mathcal{E}_1(t) &= \sum_{i,j=1}^{3} \left(\partial_{x_i} \Delta_p b_j + \partial_{x_j} \Delta_p b_i, A_{i,j}(\Delta_p(I - P)g) + 2\Delta_p c\delta_{i,j}\right)_{L^2_x}, \\
\mathcal{E}_2(t) &= \sum_{i=1}^{3} \left(\partial_{x_i} \Delta_p a, \Delta_p b_i\right)_{L^2_x}.
\end{aligned}
\]

Set \(\mathcal{E}_p^{int}(g(t)) = \mathcal{E}_0(t) + \kappa_1 \mathcal{E}_1(t) + \kappa_2 \mathcal{E}_2(t)\), where \(0 < \kappa_2 < \kappa_1 \ll 1\) are some constants (to be confirmed below).

By multiplying the fifth equality of \(34\) by \(\partial_{x_i} \Delta_p c\) and summing up on \(i\) with \(1 \leq i \leq 3\), one can get

\[
\begin{aligned}
\frac{d\mathcal{E}_0(t)}{dt} - \sum_{i=1}^{3} \left(\partial_{x_i} \Delta_p \partial_t c, B_i(\Delta_p(I - P)g)\right)_{L^2_x} &= + \sqrt{10} \sum_{i=1}^{3} \left\|\partial_{x_i} \Delta_p c\right\|^2_{L^2_x} \\
+ 8 \sum_{i=1}^{3} \left(\partial_{x_i} \Delta_p c, B_i(\Delta_p g)\right)_{L^2_x} &= = \sum_{i=1}^{3} \left(\partial_{x_i} \Delta_p c, B_i(\Delta_p L(g, g) - v \cdot \nabla_x \Delta_p (I - P)g)\right)_{L^2_x}.
\end{aligned}
\]
Using the third equality of (34) and Young’s inequality, we arrive at
\[
\sum_{i=1}^{3} \left| \left( \partial_{x_i} \Delta_p \partial_t c, B_i(\Delta_p(I - P)g) \right) \right|_{L^2}
\]
\[
= \sum_{i=1}^{3} \left| \left( \frac{1}{3} \nabla_x \cdot \Delta_p b + \frac{\sqrt{10}}{6} \sum_{j=1}^{3} \partial_{x_j} B_i(\Delta_p(I - P)g), \partial_{x_i} B_i(\Delta_p(I - P)g) \right) \right|_{L^2}
\]
\[
\leq \epsilon_{01} \left\| \nabla_x \Delta_p b \right\|_{L^2}^2 + \frac{C}{\epsilon_{01}} \left\| \nabla_x (I - P) \Delta_p g \right\|_{L^2}^2
\]
for \( \epsilon_{01} > 0 \), where \( C_{\epsilon_{01}} \) is a constant depending on \( \epsilon_{01} \). Furthermore, by using Young’s equality again, we are led to
\[
\frac{dE_0(t)}{dt} + \lambda_1 \left\| \nabla_x \Delta_p c \right\|_{L^2}^2
\]
\[
\leq \epsilon_{01} \left\| \nabla_x \Delta_p b \right\|_{L^2}^2 + \frac{C}{\epsilon_{01}} \left\| \nabla_x (I - P) \Delta_p g \right\|_{L^2}^2
\]
\[
+ C_{\epsilon_{01},\epsilon_{02}} \left( \left\| \nabla_x (I - P) \Delta_p g \right\|_{L^2}^2 + \left\| (I - P) \Delta_p g \right\|_{L^2}^2 \right)
\]
for \( \lambda_1 > 0 \) and \( \epsilon_{01}, \epsilon_{02} > 0 \), where \( C_{\epsilon_{01}} \) and \( C_{\epsilon_{01},\epsilon_{02}} \) are some constants depending on \( \epsilon_{01}, \epsilon_{02} \).

Multiplying the fourth equality of (34) by \( \partial_{x_i} \Delta_p b_j + \partial_{x_j} \Delta_p b_i \) and summing up \( 1 \leq i, j \leq 3 \), we obtain
\[
\frac{dE_1(t)}{dt} - \sum_{i,j=1}^{3} \left( \partial_{x_i} \Delta_p \partial_t b_j + \partial_{x_j} \Delta_p \partial_t b_i, A_{i,j} (\Delta_p(I - P)g) + 2 \Delta_p c \delta_{i,j} \right)_{L^2}
\]
\[
+ 2 \left( \left\| \nabla_x \Delta_p b \right\|_{L^2}^2 + \left\| \nabla_x \cdot \Delta_p b \right\|_{L^2}^2 \right) + \sum_{i,j=1}^{3} 12 \left( \partial_{x_i} \Delta_p b_j + \partial_{x_j} \Delta_p b_i, \left( \mathcal{S}_p \Delta_p g, \phi_{i,j} \right) \right)_{L^2}
\]
\[
= \sum_{i,j=1}^{3} \left( \partial_{x_i} \Delta_p b_j + \partial_{x_j} \Delta_p b_i, A_{i,j} (\Delta_p(I - P)g) - v \cdot \nabla_x \Delta_p(I - P)g \right)_{L^2}
\]
Then, substitute the second equality of (34) to eliminate \( \Delta_p \partial_t b \) and get
\[
\sum_{i,j=1}^{3} \left| \left( \partial_{x_i} \Delta_p \partial_t b_j, A_{i,j} (\Delta_p(I - P)g) + 2 \Delta_p c \delta_{i,j} \right) \right|_{L^2}
\]
\[
= \sum_{i,j=1}^{3} \left| \left( \partial_{x_i} (\Delta_p a + 2 \Delta_p c) + \sum_{j=1}^{3} \partial_{x_j} A_{i,j} (\Delta_p(I - P)g), \partial_{x_i} A_{i,j} (\Delta_p(I - P)g) + 2 \partial_{x_i} \Delta_p c \delta_{i,j} \right) \right|_{L^2}
\]
\[
\leq \epsilon_{11} \left\| \nabla_x \Delta_p a \right\|_{L^2}^2 + C_{\epsilon_{11}} \left\| \nabla_x \Delta_p c \right\|_{L^2}^2 + C_{\epsilon_{11}} \sum_{i,j=1}^{3} \left\| \nabla_x A_{i,j} (\Delta_p(I - P)g) \right\|_{L^2}^2
\]
for \( \epsilon_{11} > 0 \), where \( C_{\epsilon_{11}} \) is a constant depending on \( \epsilon_{11} \). Consequently, there exist some constant \( \lambda_2 > 0 \) such that the following inequality holds
\[
\frac{dE_1(t)}{dt} + \lambda_2 \| \nabla_x \Delta_p b \|_{L^2_x}^2 \\
\leq \epsilon_{11} \| \nabla_x \Delta_p a \|_{L^2_x}^2 + C_{\epsilon_{11}} \| \nabla_x \Delta_p c \|_{L^2_x}^2 + \epsilon_{12} \left( \| \nabla_x \Delta_p (I - P) g \|_{L^2_t L^2_x}^2 + \| \Delta_p (I - P) g \|_{L^2_t L^2_x}^2 \right) \\
+ \sum_{i,j=1}^3 \| A_{ij} (\Delta_p L(g,g)) \|_{L^2_x}^2 \),
\]

where \(C_{\epsilon_{11}}\) and \(C_{\epsilon_{11}, \epsilon_{12}}\) are positive constants depending on \(\epsilon_{11}, \epsilon_{12} > 0\). Multiplying the second equality by \(\partial_t \Delta_p a\) and summing up \(1 \leq i \leq 3\), by similar calculations as above, we get

\[
\frac{dE_2(t)}{dt} + \lambda_3 \| \nabla_x \Delta_p a \|_{L^2_x}^2 \leq C_{\epsilon_{21}} \| \nabla_x \Delta_p (b,c) \|_{L^2_x}^2 + C_{\epsilon_{21}} \| \nabla_x (I - P) \Delta_p g \|_{L^2_t L^2_x}^2
\]

for \(\lambda_3 > 0\), where \(C_{\epsilon_{21}}\) is a positive constant depending on \(\epsilon_{21}\).

Put above energy estimates together and choose \(\epsilon_{01}, \epsilon_{11}, \kappa_1, \kappa_2\) small enough. Consequently, there exists a positive constant \(\lambda > 0\) such that

\[
\frac{d}{dt} E^\text{int}_p (g(t)) + \lambda \| \nabla_x \Delta_p (a,b,c) \|_{L^2_x}^2 \\
\leq \| \nabla_x (I - P) \Delta_p g \|_{L^2_t L^2_x}^2 + \| \Delta_p (I - P) g \|_{L^2_t L^2_x}^2 + \| \Delta_p g \|_{L^2_t L^2_x}^2 \\
+ \sum_{i,j=1}^3 \| A_{ij} (\Delta_p L(g,g)) \|_{L^2_x}^2 + \sum_{i=1}^3 \| B_i (\Delta_p L(g,g)) \|_{L^2_x}^2 .
\]

Integrating the above inequality with respect to \(t\) over \([0, T]\) and taking square roots on both sides give

\[
\left( \int_0^T \| \nabla_x \Delta_p (a,b,c) \|_{L^2_x}^2 dt \right)^{1/2} \\
\leq \sqrt{\| E^\text{int}_p (g(T)) \|} + \sqrt{\| E^\text{int}_p (g(0)) \|} + \| \nabla_x (I - P) \Delta_p g \|_{L^2_t L^2_x} + \| (I - P) \Delta_p g \|_{L^2_t L^2_x} \\
+ \sum_{i,j=1}^3 \| A_{ij} (\Delta_p L(g,g)) \|_{L^2_x} + \sum_{i=1}^3 \| B_i (\Delta_p L(g,g)) \|_{L^2_x} .
\]

Multiplying the above inequality by \(2^\frac{p}{2}\) and then taking the summation over \(p \geq -1\) to get

\[
\| \nabla_x (a,b,c) \|_{\bar{L}^2_{p} (B^{1/2}_{2,1})} \\
\leq \sum_{p \geq -1} 2^\frac{p}{2} \sqrt{\| E^\text{int}_p (g(T)) \|} + \sum_{p \geq -1} 2^\frac{p}{2} \sqrt{\| E^\text{int}_p (g(0)) \|} + \| (I - P) g \|_{\bar{L}^2_{p} (B^{p/2}_{2,1})} \\
+ \sum_{p \geq -1} \sum_{i,j=1}^3 2^\frac{p}{2} \| A_{ij} (\Delta_p L(g,g)) \|_{L^2_x} + \sum_{p \geq -1} \sum_{i=1}^3 2^\frac{p}{2} \| B_i (\Delta_p L(g,g)) \|_{L^2_x} .
\]

Clearly, it is not difficult to check that

\[
\sum_{p \geq -1} 2^\frac{p}{2} \sqrt{\| E^\text{int}_p (g(t)) \|} \lesssim \| \nabla_x (a,b,c) \|_{B^{1/2}_{2,1}} + \| g \|_{\bar{L}^2_{p} (B^{p/2}_{2,1})} \\
\lesssim \| (a,b,c) \|_{\bar{L}^2_{p} (B^{p/2}_{2,1})} + \| g \|_{\bar{L}^2_{p} (B^{p/2}_{2,1})} ,
\]
which implies that
\[
\sum_{p \geq -1} 2^\frac{p}{2} \sqrt{|\mathcal{E}_p^{\text{int}}(g(T))|} \lesssim \mathcal{E}_T(g), \quad \sum_{p \geq -1} 2^\frac{p}{2} \sqrt{|\mathcal{E}_p^{\text{int}}(g(0))|} \lesssim \|g_0\|_{L_x^{3/2}(\mathbb{R}_x^3)}.
\] (36)

By using Proposition 5 (taking \(s = \frac{1}{2}\)) and the Sobolev embedding \(B^{3/2}_{2,1}(\mathbb{R}_x^3) \hookrightarrow \tilde{B}^{3/2}_{2,1}(\mathbb{R}_x^3) \hookrightarrow L^\infty(\mathbb{R}_x^3)\), we obtain
\[
\sum_{p \geq -1, i,j=1} 3 2^\frac{p}{2} \|A_{ij}(\Delta_p L(g,g))\|_{L^2_x} \lesssim \|\mathcal{S}_3 g\|_{L_x^2 L_t^\infty} \lesssim \|\mathcal{P} g\|_{L_x^2 L_t^\infty} + \|\mathcal{S}_3 g\|_{L_x^2 L_t^\infty} \lesssim \mathcal{E}_T(g) D_T(g),
\]
where we have used the following estimates:
\[
\|\mathcal{P} g\|_{L_x^2 L_t^\infty} \lesssim \|\mathcal{P} g\|_{L_x^2 L_t^\infty} \lesssim \|\nabla_x (a,b,c)\|_{L^2_x (B^{3/2}_{2,1}(\mathbb{R}_x^3))} \lesssim D_T(g),
\]
\[
\|\mathcal{S}_3 g\|_{L_x^2 L_t^\infty} \lesssim \|\mathcal{P} g\|_{L_x^2 L_t^\infty} + \|\mathcal{S}_3 g\|_{L_x^2 L_t^\infty} \lesssim \|\nabla_x (a,b,c)\|_{L^2_x (B^{3/2}_{2,1}(\mathbb{R}_x^3))} + \|\mathcal{S}_3 g\|_{L_x^2 L_t^\infty} \lesssim D_T(g).
\]

By inserting (36)-(37) into (35), we conclude that
\[
\|\nabla_x (a,b,c)\|_{L^2_x (B^{3/2}_{2,1}(\mathbb{R}_x^3))} \lesssim \|g_0\|_{L_x^2 (B^{3/2}_{2,1}(\mathbb{R}_x^3))} + \|\mathcal{L}_2 g\|_{L^2_x L_t^\infty} \lesssim D_T(g).
\]
which is just (32).

\[\square\]

5.4. Estimate on the nonlinear term \(L(g,g)\).

In this section, we estimate the nonlinear term \(L(g,g)\). It follows from (15) in Proposition 4 that
\[(L(g,g), P g)_{L^2(\mathbb{R}^3)} = 0, \quad \forall g \in \mathcal{S} (\mathbb{R}^3),
\]
so it suffices to bound
\[(L(g,g), (I - P) g)_{L^2(\mathbb{R}^3)}.
\]

Proposition 16. Let \(g = g(t,x,v)\) be suitably smooth function. It holds that
\[
\sum_{p \geq -1} 2^\frac{p}{2} \left( \int_0^T |(\Delta_p L(g,g), \Delta_p (I - P) g)| dt \right)^{1/2} \lesssim \mathcal{E}_T(g) D_T(g)
\]
for \(0 < T \leq +\infty\).

Proof. With aid of the macro-micro decomposition, we split \(L(g,g)\) into four terms:
\[L(g,g) = L(P g, g) + L(g, (I - P) g) + L((I - P) g, P g) + L((I - P) g, (I - P) g).
\]
Then it follows from Theorem 3.3 (taking \(s = \frac{3}{2}\)) that
\[
\sum_{p \geq -1} 2^\frac{p}{2} \left( \int_0^T |(\Delta_p L(P g, g), \Delta_p (I - P) g)| dt \right)^{1/2} \lesssim \|P g\|_{L^2_x L_t^\infty}^{1/2} \|\mathcal{L}_x P g\|_{L^2_x L_t^\infty}^{1/2} \|\mathcal{L}_x (I - P) g\|_{L^2_x L_t^\infty}^{1/2} \|\mathcal{L}_x (I - P) g\|_{L^2_x L_t^\infty}^{1/2}
\]
...
Hence, we conclude that

\[ \sum_{p \geq -1} 2^{3p} \left( \int_0^T \left( |(\Delta_p L, (I-P)g)| + |(\Delta_p L, (I-P)g)| \right) dt \right)^{1/2} \lesssim \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})}, \]

where

\[ \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \lesssim \left\| (a, b, c) \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \lesssim \left\| \nabla_x (a, b, c) \right\|_{L^2(B^{3/2}_{2,1})} \lesssim \left\| \nabla_x (a, b, c) \right\|_{L^2(B^{3/2}_{2,1})} \leq \sqrt{\mathcal{D}_T(g)}. \]

Hence, we conclude that

\[ \sum_{p \geq -1} 2^{3p} \left( \int_0^T \left( |(\Delta_p L, (I-P)g)| + |(\Delta_p L, (I-P)g)| \right) dt \right)^{1/2} \lesssim \sqrt{\mathcal{E}_T(g)} \sqrt{\mathcal{D}_T(g)}. \]

In fact, other collision terms can be estimated at a similar way. Precisely,

\[ \sum_{p \geq -1} 2^{3p} \left( \int_0^T \left( |(\Delta_p L, (I-P)g)| + |(\Delta_p L, (I-P)g)| \right) dt \right)^{1/2} \lesssim \left\| (I-P)g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \times \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})}, \]

and

\[ \sum_{p \geq -1} 2^{3p} \left( \int_0^T \left( |(\Delta_p L, (I-P)g)| + |(\Delta_p L, (I-P)g)| \right) dt \right)^{1/2} \lesssim \left\| (I-P)g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})} \left\| \mathcal{P} g \right\|_{L^1_T L^2(B^{3/2}_{2,1})}. \]

Therefore, combine those estimates to finish the proof of Proposition 16. \( \square \)
Having Propositions 15-16, we can establish a priori estimate (30).

Proof of Proposition 14.

Proof. Applying \( \Delta_p (p \geq -1) \) to (2) and taking the inner product with \( \Delta_p g \) over \( \mathbb{R}^3_x \times \mathbb{R}^3_v \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \Delta_p g \|^2_{L^2_x L^2_v} + \| \mathcal{L}^\frac{1}{2} \Delta_p g \|^2_{L^2_x L^2_v} \leq (\Delta_p L(g, g), \Delta_p (I - P) g)_{L^2_x L^2_v},
\]

where we used \( (v \cdot \Delta_p \nabla_x g, \Delta_p g)_{L^2} = 0 \). Integrate the above inequality with respect to the time variable over \([0, t]\) with \( 0 \leq t \leq T \) and then take the square root of both sides of the resulting inequality. Consequently, we obtain

\[
2^{\frac{1}{2}p} \| \Delta_p g \|_{L^2_x L^2_v} + 2^{\frac{1}{2}p} \| \mathcal{L}^\frac{1}{2} \Delta_p g \|_{L^2_x L^2_v} \\
\leq 2^{\frac{1}{2}p} \| \Delta_p g_0 \|_{L^2_x L^2_v} + 2^{\frac{1}{2}p} \left( \int_0^t \left| (\Delta_p L(g, g), \Delta_p g)_{L^2_x L^2_v} \right| \, dt \right)^{1/2},
\]

which implies that (by using Proposition 16)

\[
\| g \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} + \| \mathcal{L}^\frac{1}{2} g \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} \leq C \| g_0 \|_{\tilde{L}^{3/2}_x(B^{3/2}_2)} + \sqrt{\mathcal{E}_T(g)} D_T(g). \tag{38}
\]

It follows from Proposition 15 that (performing the calculation \( \alpha \times (32) + (38) \) in fact) that

\[
(1 - \alpha) \mathcal{E}_T(g) + (1 - \alpha) \| \mathcal{L}^\frac{1}{2} g \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} + \alpha \| \nabla_x (a, b, c) \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} \\
\lesssim \| g_0 \|_{\tilde{L}^{3/2}_x(B^{3/2}_2)} + (\sqrt{\mathcal{E}_T(g)} + \mathcal{E}_T(g)) D_T(g),
\]

which leads to (30) directly if taking \( \alpha > 0 \) sufficiently small. \( \square \)

6. Appendix A. This section can be regarded as an independent one in regard to the present paper, which is devoted to the local existence in spatially critical Besov spaces for linearized Landau equation.

Theorem 6.1. There exist \( C_0 > 1, \epsilon_0 > 0 \) and \( T_0 > 0 \) such that for all \( 0 < T \leq T_0, f \in \tilde{L}^\infty_T \tilde{L}^2_x(\mathbb{R}^3_v) \), \( g_0 \in \tilde{L}^2_x(\mathbb{R}^3_v) \) satisfying

\[
\| f \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} \leq \epsilon_0,
\]

then the Cauchy problem

\[
\begin{cases}
\partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = L(f, g) - \mathcal{L}_2 f, \\
g(t, x, v)|_{t=0} = g_0(x, v),
\end{cases}
\tag{39}
\]

admits a weak solution \( g \in L^\infty([0, T]; L^2(\mathbb{R}^6_{x,v})) \) satisfying

\[
\| g \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} + \| \mathcal{L}^\frac{1}{2} g \|_{L^p_T \tilde{L}^{2^*/2}_x(B^{3/2}_2)} \\
\leq C_0 \left( \| g_0 \|_{\tilde{L}^{3/2}_x(B^{3/2}_2)} + \sqrt{T} \| S_2 f \|_{L^p_T \tilde{L}^{2}_x(B^{3/2}_2)} \right). \tag{40}
\]

Due to the fact that the dual space of \( \tilde{L}^\infty_T \tilde{L}^2_x(B^{3/2}_2) \) is unknown, the proof of Theorem 6.1 is a little bit complicated. For clarity, we divided it into several parts.
6.1. The local existence of weak solution.

Firstly, we establish the local existence of weak solution to the Cauchy problem (39) by using the duality argument and Hahn-Banach extension theorem.

**Proposition 17.** There exist $\epsilon_1 > 0$ and $T_0 > 0$ such that for all $0 < T \leq T_0$, $f \in L^\infty([0, T] \times \mathbb{R}_x^3; L^2(\mathbb{R}_v^6))$, $g_0 \in L^2(\mathbb{R}_x^3)$ satisfying $\|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^2)} \leq \epsilon_1$, then the Cauchy problem (39) admits a weak solution $g \in L^\infty([0, T]; L^2(\mathbb{R}_x^3))$.

**Proof.** The strategy of that proof was originated from [5], and well developed by [29]. We consider the joint operator $G = -\partial_t + (v \cdot \nabla_x + L_1 - L(f, \cdot))^*$, where $(\cdot)^*$ is taken with respect to the scalar product in $L^2(\mathbb{R}_x^3)$. For all $h \in C^\infty([0, T], S(\mathbb{R}_x^3))$ with $h(T) = 0$, we have

$$\text{Re} (Gh(t), h(t))_{x,v}$$

$$= -\frac{1}{2} \frac{d}{dt} (\|h(t)\|_{L^2_{x,v}}^2) + \text{Re} (v \cdot \nabla_x h, h)_{x,v} + \text{Re} (L_1 h, h)_{x,v} - \text{Re} (L(f, h), h)_{x,v}$$

$$\geq -\frac{1}{2} \frac{d}{dt} (\|h(t)\|_{L^2_{x,v}}^2) + \frac{1}{C} \|\tilde{\nabla}^x h\|_{L^2_{x,v}}^2 - C \|h\|_{L^2_{x,v}}^2 - C \|S_2 f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^2)} \|\tilde{\nabla}^x h\|_{L^2_{x,v}}^2$$

for $0 \leq t \leq T$, where we used the fact that $\text{Re}(v \cdot \nabla_x h, h) = 0$.

Since $\|f\|_{L^\infty([0, T] \times \mathbb{R}_x^3; L^2)}$ is sufficiently small, we obtain

$$-\frac{d}{dt} (e^{2Ct} \|h(t)\|_{L^2_{x,v}}^2) + \frac{1}{C} e^{2Ct} \|\tilde{\nabla}^x h\|_{L^2_{x,v}}^2 \leq 2e^{2Ct} \|Gh\|_{L^2_{x,v}}^2.$$

Furthermore, we arrive at

$$\|h(t)\|_{L^2_{x,v}}^2 + \frac{1}{C} \|\tilde{\nabla}^x h\|_{L^2([0, T]; L^2_{x,v})}^2$$

$$\leq 2 \int_0^T e^{2C(\tau-t)} \|h(\tau)\|_{L^2_{x,v}} \|Gh(\tau)\|_{L^2_{x,v}} d\tau$$

$$\leq 2e^{2CT} \|h\|_{L^\infty([0, T], L^2_{x,v})} \|Gh\|_{L^1([0, T], L^2_{x,v})},$$

which leads to

$$\|h\|_{L^\infty([0, T], L^2_{x,v})} \leq 2e^{2CT} \|Gh\|_{L^1([0, T], L^2_{x,v})}.$$  \hspace{1cm} (41)

In the following, we consider the vector subspace

$$U = \{ u = Gh : h \in C^\infty([0, T], S(\mathbb{R}_x^3)), h(T) = 0 \} \subset L^1([0, T], L^2(\mathbb{R}_x^3));$$

Indeed, the above inclusion is true due to similar calculations in Theorem 3.1. For $g \in L^2_{x,v}$, we get

$$|\langle L(f, \cdot)^* h, g \rangle_{L^2_{x,v}}| = |\langle h, L(f, g) \rangle_{L^2_{x,v}}| \leq \|S_2 f\|_{L^\infty(L^2)} \|g\|_{L^2_{x,v}} \|\tilde{\nabla}^x h\|_{L^2_{x,v}},$$

which implies that

$$\|L(f, \cdot)^* h\|_{L^2_{x,v}} \leq \|S_2 f\|_{L^\infty(L^2)} \|\tilde{\nabla}^x h\|_{L^2_{x,v}}.$$
for any $t \in [0, T]$. For $g_0 \in L^2(\mathbb{R}^6_{x,v})$, we define the linear functional as follows

$$Q : \mathbb{U} \rightarrow \mathbb{C}$$

$$u = \mathcal{G}h \mapsto (g_0, h(0))_{L^2_{x,v}} - (L_2f, h)_{L^2([0,T], L^2_{x,v})},$$

where $h \in C^\infty([0, T], \mathcal{S}(\mathbb{R}^6_{x,v}))$ with $h(T) = 0$. It follows from (41) that the operator $\mathcal{G}$ is injective. The linear functional $Q$ is hence well-defined. We obtain

$$|Q(u)| \leq \|g_0\|_{L^2_{x,v}} \|h(0)\|_{L^2_{x,v}} + C_T \|S_2f\|_{L^\infty([0,T], L^2_{x,v})} \|\mathcal{E}h\|_{L^2_{x,v}}$$

$$\leq C_T \left(\|g_0\|_{L^2_{x,v}} + \|S_2f\|_{L^\infty([0,T], L^2_{x,v})}\right) \|\mathcal{G}h\|_{L^2_{x,v}}$$

$$= C_T \left(\|g_0\|_{L^2_{x,v}} + \|S_2f\|_{L^\infty([0,T], L^2_{x,v})}\right) \|u\|_{L^2_{x,v}}.$$ 

Hence, $Q$ is a continuous linear form on $(\mathbb{U}, \|\cdot\|_{L^1([0,T], L^2_{x,v})})$. By using the Hahn-Banach theorem, $Q$ can be extended as a continuous linear form on $L^1([0,T]; L^2(\mathbb{R}^6_{x,v}))$. It follows that there exists $g \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$ satisfying

$$\|g\|_{L^\infty([0,T], L^2_{x,v})} \leq C_T \left(\|g_0\|_{L^2_{x,v}} + \|S_2f\|_{L^\infty([0,T], L^2_{x,v})}\right),$$

such that

$$\forall u \in L^1([0,T]; L^2(\mathbb{R}^6_{x,v})), \quad Q(u) = \int_0^T (g(t), u(t))_{L^2_{x,v}} dt.$$ 

It implies that for all $h \in C^\infty_0((-\infty, T), \mathcal{S}(\mathbb{R}^6_{x,v}))$,

$$Q(\mathcal{G}h) = \int_0^T (g(t), \mathcal{G}h(t))_{L^2_{x,v}} dt$$

$$= (g_0, h(0))_{L^2_{x,v}} - \int_0^T (L_2f(t), h(t))_{L^2_{x,v}} dt.$$ 

Therefore, $g \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$ is a weak solution of the Cauchy problem (39). The proof of Proposition 17 is completed.

In next steps, we need to improve the regularity of weak solution $g \in L^\infty([0,T]; L^2(\mathbb{R}^6_{x,v}))$ in both velocity and position variables.

6.2. Regularity of weak solution in velocity variable.

To do this, we smooth out the function $f$. Set $f_N = S_N f$ for $N \in \mathbb{N}$. Then, we have $f_N \in \tilde{L}^\infty_T \tilde{L}^2_2(B_{2,1}^{3/2})$ and the following property.

**Lemma 6.2.** For any $f \in \tilde{L}^\infty_T \tilde{L}^2_2(B_{2,1}^{3/2})$, it holds that

(i) $\{f_N\}$ is a Cauchy sequence in $\tilde{L}^\infty_T \tilde{L}^2_2(B_{2,1}^{3/2})$;

(ii) For $0 < s \leq 3/2$, $f_N$ satisfies $\|f_N\|_{\tilde{L}^s_T \tilde{L}^2_2} \leq C_s \|f\|_{\tilde{L}^\infty_T \tilde{L}^2_2}$ and

$$\|f_N\|_{L^s_T \tilde{L}^2_2} \leq C_s f_N \tilde{L}^2_2(B_{2,1}^{3/2}) \leq C_2 \|f\|_{\tilde{L}^\infty_T \tilde{L}^2_2(B_{2,1}^{3/2})},$$

where $C_1, C_2, C_3 > 0$ are some constants independent of $N$. 

Proof. Firstly, for any $M', M'' \in \mathbb{N}$, we have

$$\|f_{M'} - f_{M''}\|_{L^\infty_t L^2_x(B_{2,1}^{3/2})} = \sum_{q \geq -1} 2^{3q/2} \|\Delta_q(f_{M'} - f_{M''})\|_{L^\infty_t L^2_x L^2_v}$$

$$= \sum_{q \geq -1} 2^{3q/2} \sum_{p=0}^{M'-1} \Delta_q \Delta_p f \|_{L^\infty_t L^2_x L^2_v} \sum_{p=0}^{M''-1} \Delta_q \Delta_p f \|_{L^\infty_t L^2_x L^2_v}$$

$$\leq \sum_{q=p+1}^{M'-1} \sum_{p=M''-1}^{p-1} 2^{3q/2} \|\Delta_q \Delta_p f\|_{L^\infty_t L^2_x L^2_v} \leq 3 \times 2^{3/2} \sum_{p=M''-1}^{p+1} 2^{3p/2} \|\Delta_p f\|_{L^\infty_t L^2_x L^2_v},$$

where we used $\Delta_q \Delta_p f = 0$ if $|p-q| \geq 2$. Since $f \in \tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})$, by the definition of the norm $\tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})$, that is, $\{2^{3/2} \|\Delta_p f\|_{L^\infty_t L^2_x L^2_v}\} \in \ell^1$, this deduces that $\{f_N, N \in \mathbb{N}\}$ is a Cauchy sequence in $\tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})$.

It follows from Lemma 7.2 that $\|f_N\|_{L^\infty_t L^2_x L^2_v} \leq C \|f\|_{L^\infty_t L^2_x L^2_v}$. For the left hand side of (42), it can be obtained from Lemma 7.5. For the right hand side, we have

$$\|f_{N}\|_{L^\infty_t \tilde{L}_x^2(B_{2,1}^{3/2})} = \sum_{q \geq -1} 2^{qs} \|\Delta_q f_N\|_{L^\infty_t L^2_x L^2_v}$$

$$\leq \sum_{q \geq -1} 2^{qs} \sum_{p \geq -1} 2^{(q-p)s} \|\Delta_p f\|_{L^\infty_t L^2_x L^2_v}$$

$$\leq \sum_{q \geq -1} 2^{qs} \sum_{p \geq -1} 2^{(q-p)s} c(p) \|f\|_{\tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})} \leq C_3 \|f\|_{\tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})},$$

where $c(p) = 2^{ps} \|\Delta_p f\|_{L^\infty_t L^2_x L^2_v} \|f\|_{\tilde{L}_T^\infty \tilde{L}_x^2(B_{2,1}^{3/2})}$ and we used the estimate

$$\sum_{q \geq -1} \sum_{|p-q| \leq 1} 2^{(q-p)s} c(p) \leq \sum_{q \geq -1} (1_{|p| \leq 2ps} * c(p)) \leq \|1_{|p| \leq 2ps}\|_{\ell^1} \|c(p)\|_{\ell^1} < +\infty.$$

Therefore, the proof of Lemma 6.2 is completed.

Then, according to the commutator estimate in Section 4, we have the following estimate for the weak solution.

**Proposition 18.** Set $f_N = S_N f$ for each $N \in \mathbb{N}$. There exist $\epsilon_1 > 0$ and $T_0 > 0$ such that for all $0 < T \leq T_0$, $g_0 \in L^2(\mathbb{R}^n_x, v), f \in L^\infty([0, T] \times \mathbb{R}^3_x; L^2(\mathbb{R}^3_v))$ satisfying

$$\|f\|_{L^\infty([0, T] \times \mathbb{R}^3_x; L^2_v)} \leq \epsilon_1,$$

then the Cauchy problem

$$\begin{cases}
\partial_t g_N + v \cdot \nabla_x g_N + L_1 g_N = L(f_N, g_N) - L_2 f_N, \\
g_N(t, x, v)|_{t=0} = g_0(x, v),
\end{cases} \quad (43)$$

admits a weak solution $g_N(t, x, v) \in L^\infty([0, T]; L^2(\mathbb{R}^n_x, v))$ satisfying

$$\|g_N\|_{L^\infty_t L^2_x L^2_v} + \|\tilde{L}_x^2 g_N\|_{L^\infty_t L^2_x L^2_v} \leq C \|g_0\|_{L^2_x L^2_v} + C \sqrt{T} \|S_2 f\|_{L^\infty_t L^2_x L^2_v}. \quad (44)$$
Proof. It follows from Proposition 17 that the Cauchy problem (43) admits a weak solution \(g_N(t, x, v) \in L^\infty([0, T]; L^2(\mathbb{R}^6))\). In what follows, we show (44) under the assumption that \(\|f_N\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)}\) is sufficiently small (independent of \(N\)) and \(\|\nabla_x f_N\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)} < +\infty\).

Let \(0 < \delta, \delta' < 1\). We use a weighted function \(W_\delta(v) = (\delta'v)^{-2}\) and mollifiers \(M^\delta(D_v), S_\delta(D_x)\) defined as in Section 4. Taking the inner products of (43) with \(W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N \in H^+_x H^4_v\), and integrating the resulting equation with respect to the time \(t \in [0, T]\) and \((x, v) \in \mathbb{R}^6\). We obtain

\[
\|M^\delta S_\delta W_\delta g_N\|_{L^2([0, T]; L^2(\mathbb{R}^6_v))}^2 + \int_0^T (v \cdot \nabla_x g_N + L_1 g_N, W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)\ dt
= \int_0^T (L(f_N, g_N) - L_2 f_N, W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)\ dt + \|M^\delta S_\delta W_\delta g_0\|_{L^2(\mathbb{R}^6_v)}^2.
\]

To the term \(v \cdot \nabla_x g_N\), we bound it as

\[
(v \cdot \nabla_x g_N, W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)_{x,v} = \left(\left[ M^\delta, v \right] \cdot \nabla_x S_\delta W_\delta g_N, M^\delta S_\delta W_\delta g_N\right)_{x,v} \leq 2 \|M^\delta S_\delta W_\delta g_N\|_{L^2_x v}^2,
\]

where we have used \(|(\nabla_x M^\delta(\xi) \cdot \eta) S_\delta(\eta)| \leq M^\delta(\xi)|\delta\eta|S(\delta\eta) \leq 2M^\delta(\xi)S_\delta(\eta)|\delta\eta|\). It follows from Proposition 9 that

\[
\int_0^T (L(f_N, g_N), W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)_{x,v}\ dt \leq \left(\left\|f_N\right\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)} + \epsilon\right) \|M^\delta S_\delta W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2 + \|f_N\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)} \|W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2 + \|\nabla_x S_\delta f_N\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)} \|W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2.
\]

On the other hand, regarding linear terms \(L_1, L_2\), we obtain, by Proposition 12 and Lemma 3.4,

\[
\int_0^T (L_1 g_N, W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)_{x,v}\ dt \geq \left(\frac{1}{2} - \epsilon\right) \|M^\delta S_\delta W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2 - C\|S_\delta W_\delta g_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))}^2 - C\|S_\delta W_\delta g_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))}^2.
\]

and

\[
\int_0^T (L_2 f_N, W_\delta S_\delta(M^\delta)^2 S_\delta W_\delta g_N)\ dt \leq CT\|S_\delta f_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))} \|M^\delta S_\delta W_\delta g_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))},
\]

where \(\epsilon > 0\) is sufficiently small.

Combining (45)–(46), it is shown that

\[
\|M^\delta S_\delta W_\delta g_N\|_{L^2([0, T]; L^2(\mathbb{R}^6_v))}^2 + \|\nabla_x S_\delta W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2 \leq C\|M^\delta S_\delta W_\delta g_0\|_{L^2(\mathbb{R}^6_v)}^2 + CT\|S_\delta W_\delta g_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))}^2 + CT\|S_\delta f_N\|_{L^\infty([0, T]; L^2(\mathbb{R}^6_v))}^2 + \|f_N\|_{L^\infty([0, T] \times \mathbb{R}^6; L^2)} \|W_\delta g_N\|_{L^2([0, T] \times \mathbb{R}^6)}^2.
\]
since \( \|f_N\|_{L^\infty([0,T] \times \mathbb{R}^6_x)} \) is small enough. Taking \( T \) sufficiently small (for example, \( CT < \frac{1}{2} \)) and passing to the limit \( \delta \to 0 \), we get

\[
\|M^\delta S_1 W_\delta' g_N\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))} + \|\tilde{L}_\delta^2 W_\delta' g_N\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))} \\
\leq C \|g_0\|_{L^2(\mathbb{R}_x^6)}^2 + CT \|S_2 f_N\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))} \\
+ C \|f_N\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))} \|W_\delta' g_N\|_{L^2([0,T];L^2(\mathbb{R}_x^6))}^2 \\
\leq C \|g_0\|_{L^2(\mathbb{R}_x^6)}^2 + CT \|S_2 f\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))}^2 \\
+ C \|f\|_{L^\infty([0,T];L^2(\mathbb{R}_x^6))} \|\tilde{L}_\delta^2 W_\delta' g_N\|_{L^2([0,T];L^2(\mathbb{R}_x^6))}^2,
\]

where we used the fact \( \|\tilde{L}_\delta^2 g\|_{L^2_x} \sim \|g\| \) and Proposition 6.2. Since \( \|f\|_{L^\infty([0,T] \times \mathbb{R}_x^2;L^2_x)} \) and \( T \) are both small, we obtain

\[
\|W_\delta' g\|_{L^2_x([0,T] \times \mathbb{R}_x^6)} \leq \|g\|_{L^2_x([0,T] \times \mathbb{R}_x^6)}, \quad \text{for } 0 < \delta' < 1.
\]

Now letting \( \delta' \to 0 \) and taking square root to the resulting inequality give the desired (44) for a weak solution \( g_N \in L^\infty([0,T];L^2(\mathbb{R}_x^6)) \). Hence, the proof of Proposition 18 is finished.

6.3. Regularity of weak solution in position variable.

In the following, we need to obtain the regularity of \( g_N \) with respect to the position variable \( x \).

**Lemma 6.3.** Let \( 0 < s \leq 3/2 \) and \( 0 < T \leq \infty \). Set \( f_N = S_N f \) for each \( N \in \mathbb{N} \). If \( g_N \) satisfies

\[
g_N \in L^\infty([0,T];L^2(\mathbb{R}^6_x)), \quad \tilde{L}_\delta^2 g_N \in L^2([0,T] \times \mathbb{R}_x^6),
\]

then there exists a constant \( C > 0 \) (independent of \( N \)) such that

\[
\sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T \left| \langle \Delta_p L(f_N, g_N), \Delta_p g_N \rangle_{x,v} \right| dt \right)^{1/2} \\
\leq C \|S_2 f_N\|_{L^2_T \tilde{L}^2_x(\mathbb{R}^6_x)}^{1/2} \|\tilde{L}_\delta^2 g_N\|_{L^2_T \tilde{L}^2_x(\mathbb{R}^6_x)}^{1/2} + C_N \|S_2 f_N\|_{L^2_T \tilde{L}^2_x(\mathbb{R}^6_x)}^{1/2} \|\tilde{L}_\delta^2 g_N\|_{L^2_T \tilde{L}^2_x L^2_x} (47)
\]

for any \( \kappa > 0 \), where

\[
\|g_N\|_{L^2_T \tilde{L}^2_x(\mathbb{R}^6_x)} \triangleq \sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \|\Delta_p g_N\|_{L^2_T L^2_x L^2_x}
\]

and \( C_N > 0 \) is a constant depending only on \( N \).

**Proof.** Due to \( \tilde{L}_\delta^2 g_N \in L^2([0,T] \times \mathbb{R}_x^6) \), we obtain

\[
\|\tilde{L}_\delta^2 g_N\|_{L^2_T \tilde{L}^2_x(\mathbb{R}^6_x)} = \sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \|\Delta_p \tilde{L}_\delta^2 g_N\|_{L^2_T \tilde{L}^2_x L^2_x} \\
\leq C_\kappa \sum_{p \geq 1} 2^{-ps} \|\tilde{L}_\delta^2 g_N\|_{L^2_T L^2_x L^2_x} \leq C_\kappa \|\tilde{L}_\delta^2 g_N\|_{L^2_T L^2_x L^2_x} < +\infty,
\]

where \( C_\kappa > 0 \) is a constant depending only on \( \kappa > 0 \).
By using the Lemma 3.2 and Bony’s decomposition, we divide the inner product into three terms:

\[
(\Delta_p L(f_N, g_N), \Delta_p g_N) = \left( \Delta_p \left( \sum_j L(S_{j-1}f_N, \Delta_j g_N) \right), \Delta_p g_N \right)
\]

\[
+ \left( \Delta_p \left( \sum_j L(\Delta_j f_N, S_{j-1}g_N) \right), \Delta_p g_N \right)
\]

\[
+ \left( \Delta_p \left( \sum_j \sum_{|j-j'|\leq 1} L(\Delta_{j'} f_N, \Delta_j g_N) \right), \Delta_p g_N \right)
\]

\[
\triangleq H_1 + H_2 + H_3.
\]

For the term \( H_1 \), noticing that

\[
\Delta_p \sum_j (S_{j-1}f_N, \Delta_j g_N) = \Delta_p \sum_{|j-p|\leq 4} (S_{j-1}f_N, \Delta_j g_N),
\]

we have

\[
|H_1| \leq \sum_{|j-p|\leq 4} \|S_2 S_{j-1}f_N\|_{L^2_x L^\infty_t} \|\tilde{L}^2 \Delta_j g_N\|_{L^2_x L^2_v} \|\tilde{L}^2 \Delta_p g_N\|_{L^2_x L^2_v}.
\]

Hence, it follows that

\[
\sum_{p \geq -1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T |H_1| \, dt \right)^{1/2}
\]

\[
\lesssim \|S_2 f_N\|_{L^\infty_T L^2_x(B^{s,0}_{2,1})} \left( \sum_{p \geq -1} \sum_{|j-p|\leq 4} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T \|\tilde{L}^2 \Delta_j g_N\|_{L^2_x L^2_v}^2 \, dt \right)^{1/2} \right)
\]

\[
\times \left( \sum_{p \geq -1} \sum_{|j-p|\leq 4} \frac{1 + \kappa 2^{2js}}{1 + \kappa 2^{2ps}} \cdot \left( \int_0^T \|\tilde{L}^2 \Delta_p g_N\|_{L^2_x L^2_v}^2 \, dt \right)^{1/2} \right)^{1/2}
\]

\[
\lesssim \|S_2 f_N\|_{L^\infty_T L^2_x(B^{s,0}_{2,1})} \|\tilde{L}^2 g_N\|_{L^2_x L^2_v(B^{s,0}_{2,1})}
\]

\[
\times \left( \sum_{p \geq -1} \sum_{|j-p|\leq 4} \frac{1 + \kappa 2^{2js}}{1 + \kappa 2^{2ps}} \cdot \left( \int_0^T \|\tilde{L}^2 g_N\|_{L^2_x L^2_v}^2 \, dt \right)^{1/2} \right)^{1/2}
\]

\[
\lesssim \|S_2 f_N\|_{L^\infty_T L^2_x(B^{s,0}_{2,1})} \|\tilde{L}^2 g_N\|_{L^2_x L^2_v(B^{s,0}_{2,1})},
\]

where we used Lemmas 7.2, 7.5 and 7.7 and the following estimate

\[
\sum_{p \geq -1} \sum_{|j-p|\leq 4} \frac{1 + \kappa 2^{2js}}{1 + \kappa 2^{2ps}} \cdot 2^{(p-j)s} c(j)
\]

\[
\leq C \sum_{p \geq -1} \sum_{|j-p|\leq 4} 2^{(p-j)s} c(j) \leq C \sum_{p \geq -1} \left( \mathbf{1}_{|j|\leq 4} \ast c(j) \right)(p)
\]

\[
\leq C \|\mathbf{1}_{|j|\leq 4} \ast c(j)\|_{L^1} < +\infty
\]
with
\[
c(j) := \frac{2^{j^2}}{1 + \kappa 2^{2ps}} \|L^j \Delta_j g_N \|_{L^2_t L^2_x} \|L^j g_N \|_{L^2_t L^2(B^2_{1,\infty})}
\]
satisfying \( \|c(j)\|_{\ell^1} \leq 1 \).

For \( H_2 \), we get
\[
\sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T |H_2| \, dt \right)^{1/2}
\leq \sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T \sum_{|j| \leq 4} \|S_2 \Delta_j f_N \|_{L^2_t L^\infty_x} \left\|L^j S_{j-1} g_N \right\|_{L^2_x} \left\|L^j \Delta_p g_N \right\|_{L^2_t L^2_x} \, dt \right)^{1/2}
\leq N \sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \|S_2 f_N \|_{L^p_t L^2_x(B^2_{1,1})} \left\|L^j g_N \right\|_{L^2_t L^2_x} \left\|L^j \Delta_p g_N \right\|_{L^2_t L^2_x} \right)^{1/2}
\leq C_N \|S_2 f_N \|_{L^p_t L^2_x(B^2_{1,1})} \left\|L^j g_N \right\|_{L^2_t L^2_x} \left\|L^j \Delta_p g_N \right\|_{L^2_t L^2_x},
\]
where \( C_N = CN 2^{N_s} \). Owing to
\[
\Delta_p \left( \sum_{j \sim j'} (\Delta_j f_N \Delta_j g_N) \right) = \Delta_p \left( \sum_{\max(j,j') \geq p-2} \sum_{|j-j'| \leq 1} (\Delta_{j'} f_N \Delta_j g_N) \right)
= 0, \quad \text{if } p \geq N + 3.
\]
Finally, \( H_3 \) can be estimated as follows
\[
\sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T |H_3| \, dt \right)^{1/2}
= \sum_{p \geq 1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T \left| (\Delta_p L^3(f_N, g_N), \Delta_p g_N)_{x,v} \right| \, dt \right)^{1/2}
\leq \sum_{p \geq 1} \sum_{l \leq p+1} \frac{2^{ps}}{1 + \kappa 2^{2ps}} \left( \int_0^T \|S_2 \Delta_j f_N \|_{L^2_t L^\infty_x} \left\|L^j \Delta_j g_N \right\|_{L^2_x} \left\|L^j \Delta_p g_N \right\|_{L^2_t L^2_x} \, dt \right)^{1/2}
\leq C_N \|S_2 f_N \|_{L^p_t L^2_x(B^2_{1,1})} \left\|L^j g_N \right\|_{L^2_t L^2_x} \left\|L^j \Delta_p g_N \right\|_{L^2_t L^2_x}.
\]
Together with above three inequalities, we achieve (47). This ends the proof of Lemma 6.3.

Based on Proposition 18 and Lemma 6.3, we can obtain the regularity of the weak solution \( g_N \) to the Cauchy problem (48) and get the corresponding energy estimate dependent of \( N \).
Proposition 19. There exist $\epsilon_2 > 0$ and $T_0 > 0$ such that for all $0 < T \leq T_0$, $f \in L^\infty_T L^p_x(B_{2,1}^{3/2})$, $g_0 \in L^p_x(B_{2,1}^{3/2})$ satisfying
\[
\|f\|_{L^\infty_T L^p_x(B_{2,1}^{3/2})} \leq \epsilon_2,
\]
then the Cauchy problem
\[
\begin{aligned}
\partial_t g_N + v \cdot \nabla_x g_N + \mathcal{L} g_N &= L(f_N; g_N) - \mathcal{L} f_N, \\
g_N(t, x, v)|_{t=0} &= g_0(x, v),
\end{aligned}
\]
admits a weak solution $g_N \in L^\infty([0, T]; L^2(\mathbb{R}^6_{x,v}))$ satisfying
\[
\|g_N\|_{L^\infty_T L^p_x(B_{2,1}^{3/2})} + \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_T L^2_x(B_{2,1}^{3/2})} \leq C \|g_0\|_{L^p_x(B_{2,1}^{3/2})} + C \sqrt{T} \|S_2 f_N\|_{L^\infty_T L^2_x(B_{2,1}^{3/2})} + C_N \|S_2 f_N\|^{1/2}_{L^p_T \tilde{L}^2_x(B_{2,1}^{3/2})} \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_T L^2_B(B_{2,1}^{3/2})},
\]
where $C_N > 0$ is a constant depending on $N$ and $f_N = S_N f$.

Proof. We consider a weak solution $g_N \in L^\infty([0, T]; L^2(\mathbb{R}^6_{x,v}))$ to the above Cauchy problem (48). To do this, applying $\Delta_p (p \geq -1)$ to (48) and taking the inner product with $\Delta_p g_N$ over $\mathbb{R}^4_x \times \mathbb{R}^3_v$ give
\[
\frac{d}{dt} \|\Delta_p g_N\|_{L^2_v}^2 + \|\tilde{\mathcal{L}}^\frac{p}{2} \Delta_p g_N\|_{L^2_v}^2 \\
\leq 2 (\Delta_p L(f_N; g_N), \Delta_p g_N)_{L^2_v} + C (\|\Delta_p S_2 f_N\|_{L^2_v}^2 + \|\Delta_p g_N\|_{L^2_v}^2),
\]
where we used Lemma 3.4 and Corollary 1. Integrating the above inequality with respect to the time variable over $[0, t]$ with $0 \leq t \leq T$ and taking the square root of both sides of the resulting inequality. Multiplying the resulting inequality by $\frac{2^{3p/2}}{1 + \kappa 2^{3p}}$ gives:

\[
\frac{2^{3p/2}}{1 + \kappa 2^{3p}} \|\Delta_p g_0\|_{L^2_v}^2 + \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \left( \int_0^t \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_v}^2 dt \right)^{1/2} \\
\leq \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \|\Delta_p g_0\|_{L^2_v}^2 + \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \left( \int_0^t \|\Delta_p L(f_N, g_N), \Delta_p g_N\|_{L^2_v}^2 dt \right)^{1/2} \\
+ C \sqrt{T} \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \|S_2 \Delta_p f_N\|_{L^\infty_T L^2_v L^2_x} + \|\Delta_p g_N\|_{L^\infty_T L^2_v L^2_x} \\
\leq C 2^{3p/2} \|\Delta_p g_0\|_{L^2_v}^2 + \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \left( \int_0^t \|\Delta_p L(f_N, g_N), \Delta_p g_N\|_{L^2_v}^2 dt \right)^{1/2} \\
+ C \sqrt{T} \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \|S_2 \Delta_p f_N\|_{L^\infty_T L^2_v L^2_x} + C \sqrt{T} \frac{2^{3p/2}}{1 + \kappa 2^{3p}} \|\Delta_p g_N\|_{L^\infty_T L^2_v L^2_x}.
\]

Taking supremum over $0 \leq t \leq T$ on the left side and summing up over $p \geq -1$, we obtain
\[
\|g_N\|_{L^\infty_T \tilde{L}^2_x(B_{2,1}^{3/2})} + \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_T \tilde{L}^2_x(B_{2,1}^{3/2})} \\
\leq C \|g_0\|_{L^p_x(B_{2,1}^{3/2})} + C \|S_2 f_N\|^{1/2}_{L^p_T \tilde{L}^2_x(B_{2,1}^{3/2})} \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_T \tilde{L}^2_x(B_{2,1}^{3/2})} \\
+ C \|S_2 f_N\|^{1/2}_{L^p_T \tilde{L}^2_x(B_{2,1}^{3/2})} \|\tilde{\mathcal{L}}^\frac{p}{2} g_N\|_{L^2_T \tilde{L}^2_x(B_{2,1}^{3/2})} + C \sqrt{T} \|S_2 f_N\|_{L^p_T \tilde{L}^2_x(B_{2,1}^{3/2})} \\
+ C \sqrt{T} \|g_N\|_{L^p_T \tilde{L}^2_x(B_{2,1}^{3/2})}.
satisfies the Proposition 18. Then, for the small constant $T > g$ we have

$$
\text{Proof. The proof is similar to that of Lemma 6.3. With aid of Bony's decomposition, then there exists a $g$ such that the regularity of the weak solution $g_N$ satisfies Proposition 19. Then, for the small constant $T > 0$ and the small norm $\|f\| \lesssim g_N L^\infty T$, (for example, taking $C \|f\| \lesssim g_N L^\infty T$, $C \sqrt{T} < \frac{1}{2}$), we obtain}
$$
\begin{align*}
\|g_N\|_{L^\infty T L^2(B^{3/2}_{2,1})} + \|\tilde{L}^2 g_N\|_{L^2 T L^2(B^{3/2}_{2,1})} \\
\leq C \|g_0\|_{L^\infty T L^2(B^{3/2}_{2,1})} + C \sqrt{T} \|S_2 f\|_{L^\infty T L^2(B^{3/2}_{2,1})} + C_N \|S_2 f\|^{1/2}_{L^\infty T L^2(B^{3/2}_{2,1})} \|\tilde{L}^2 g_N\|_{L^2 T L^2 L^2}.
\end{align*}

Letting $\kappa \to 0$, we deduce that

$$
\|g_N\|_{L^\infty T L^2(B^{3/2}_{2,1})} + \|\tilde{L}^2 g_N\|_{L^2 T L^2(B^{3/2}_{2,1})} \\
\leq C \|g_0\|_{L^\infty T L^2(B^{3/2}_{2,1})} + C \sqrt{T} \|S_2 f\|_{L^\infty T L^2(B^{3/2}_{2,1})} + C_N \|S_2 f\|^{1/2}_{L^\infty T L^2(B^{3/2}_{2,1})} \|\tilde{L}^2 g_N\|_{L^2 T L^2 L^2}.
$$

which ends the proof of Proposition 19.

\begin{proof}
\end{proof}

### 6.4. Energy estimates in Besov space.

From (49) in Proposition 19, we see that $g_N$ satisfies

$$
\|g_N\|_{L^\infty T L^2(B^{3/2}_{2,1})} + \|\tilde{L}^2 g_N\|_{L^2 T L^2(B^{3/2}_{2,1})} < +\infty.
$$

That is, the regularity of the weak solution $g_N$ has been improved. However, the upper bound for $g_N$ depends on $N$. In the following, we perform a technical procedure to eliminate the dependence of $N$.

**Lemma 6.4.** Let $0 < T \leq \infty$. Set $f_N = S_N f$ for each $N \in \mathbb{N}$. If $g_N$ satisfies

$$
g_N \in L^\infty T L^2(B^{3/2}_{2,1}), \quad \tilde{L}^2 g_N \in L^2 T L^2(B^{3/2}_{2,1}),
$$

then there exists a $C > 0$ independent of $N$ such that

$$
\sum_{p \geq -1} 2^{\frac{3}{2} p} \left( \int_0^T |(\Delta_p L(f_N, g_N), \Delta_p g_N)_{x,v}| \, dt \right)^{1/2} \\
\leq C \|S_2 f_N\|^{1/2}_{L^\infty T L^2(B^{3/2}_{2,1})} \|\tilde{L}^2 g_N\|_{L^2 T L^2 L^2}.
$$

**Proof.** The proof is similar to that of Lemma 6.3. With aid of Bony's decomposition, we have

$$
\sum_{p \geq -1} 2^{\frac{3}{2} p} \left( \int_0^T |H_1| \, dt \right)^{1/2} \\
\lesssim \|S_2 f_N\|^{1/2}_{L^\infty T L^2 L^\infty} \left( \sum_{p \geq -1} \sum_{|j| \leq 4} 2^{\frac{3}{2} p} \|\tilde{L}^2 \Delta_j g_N\|_{L^2 T L^2} \right)^{1/2} \\
\times \left( \sum_{p \geq -1} 2^{\frac{3}{2} p} \|\tilde{L}^2 \Delta_p g_N\|_{L^2 T L^2} \right)^{1/2}.
$$
Thus Proposition 20.

For $H_2$, we get

$$\sum_{p \geq 1} 2^{2p} \left( \int_0^T |H_2| \, dt \right)^{1/2} \lesssim \left\| \tilde{L}_2^2 g_N \right\|_{L^2_t L^2_x}^{1/2} \left( \sum_{p \geq 1} \sum_{|j-j'| \leq 1} 2^{2p} \left\| \tilde{L}_2^2 \Delta_p g_N \right\|_{L^2_t L^2_x} \right)^{1/2},$$

where we used $B_{3/2}^1(\mathbb{R}^3) \leftrightarrow L^\infty(\mathbb{R}^3)$ and Lemma 7.2. Owing to

$$\Delta_p \left( \sum_{j} \sum_{|j-j'| \leq 1} (\Delta_{j'} f_N \Delta_j g_N) \right) = \Delta_p \left( \sum_{\max|j,j'| \geq 2} \sum_{|j-j'| \leq 1} (\Delta_{j'} f_N \Delta_j g_N) \right).$$

Thus $H_3$ can be estimated as follows:

$$\sum_{p \geq 1} 2^{2p} \left( \int_0^T |H_3| \, dt \right)^{1/2} \lesssim \left\| \tilde{L}_2^2 g_N \right\|_{L^2_t L^2_x}^{1/2} \left( \sum_{p \geq 1} \sum_{j \geq -3} 2^{2p} \left\| \tilde{L}_2^2 \Delta_j g_N \right\|_{L^2_t L^2_x} \right)^{1/2} \times \left( \sum_{p \geq 1} 2^{2p} \left\| \tilde{L}_2^2 \Delta_p g_N \right\|_{L^2_t L^2_x} \right)^{1/2} \lesssim \left\| \tilde{L}_2^2 f_N \right\|_{L^2_t L^2_x(B_{3/2}^3)} \left\| \tilde{L}_2^2 g_N \right\|_{L^2_t L^2_x(B_{3/2}^3)}.$$

Therefore, we can obtain (50). \qed

Based on Lemma 6.4, we obtain the energy estimate for the weak solution $g_N$, which is independent of $N$.

**Proposition 20.** There exist $\epsilon_2 > 0$ and $T_0 > 0$ such that for all $0 < T \leq T_0$, $f \in \tilde{L}^\infty_t \tilde{L}_0^2(B_{2,1}^{3/2})$, $g_0 \in \tilde{L}_0^2(B_{2,1}^{3/2})$ satisfying

$$\| f \|_{\tilde{L}^\infty_t \tilde{L}_0^2(B_{2,1}^{3/2})} \leq \epsilon_2,$$

then the Cauchy problem

\[
\begin{aligned}
\partial_t g_N + v \cdot \nabla_x g_N + \mathcal{L}_1 g_N &= \mathbf{L}(f_N, g_N) - \mathcal{L}_2 f_N, \\
g_N(t, x, v)|_{t=0} &= g_0(x, v),
\end{aligned}
\]  
(51)

admits a weak solution $g_N \in L^\infty([0, T]; L^2(\mathbb{R}^6))$ satisfying

$$\| g_N \|_{\tilde{L}^\infty_t \tilde{L}_0^2(B_{2,1}^{3/2})} + \| \tilde{L}_2^2 g_N \|_{L^2_t L^2_x(B_{3/2}^3)}$$
where $C > 0$ is a constant independent of $N$.

Proof. We consider the weak solution $g_N \in L^\infty([0, T]; L^2(\mathbb{R}^d))$ to (51). Applying $\Delta_p(p \geq -1)$ to (51) and taking the inner product with $\Delta_p g_N$ over $\mathbb{R}^d_x \times \mathbb{R}^d_\nu$ give

$$
\frac{d}{dt} \| \Delta_p g_N \|^2_{L^2_x} + \| L^2 \Delta_p g_N \|^2_{L^2_x} \\
\leq 2 (\Delta_p \mathbf{L}(f_N, g_N), \Delta_p g_N)_{L^2_x} + C (\| \Delta_p S_2 f_N \|^2_{L^2_x} + \| \Delta_p g_N \|^2_{L^2_x}),
$$

where we used Lemma 3.4 and Corollary 1. Integrating the above inequality with respect to the time variable over $[0, t]$ with $0 \leq t \leq T$ and taking the square root of both sides of the resulting inequality. Multiplying the resulting inequality by $2^{2p}$ gives

$$
2^{2p} \| \Delta_p g_N \|_{L^2_x} + 2^{2p} \| L^2 \Delta_p g_N \|_{L^2_x} \\
\leq 2^{2p} \| \Delta_p g_0 \|_{L^2_x} + 2^{2p} \left( \int_0^t \left| (\Delta_p \mathbf{L}(f_N, g_N), \Delta_p g_N)_{L^2_x} \right| dt \right)^{1/2} \\
+ C \sqrt{T} 2^{2p} (\| S_2 f_N \|_{L^2_x} + \| \Delta_p g_N \|_{L^2_x}),
$$

Take supremum over $0 \leq t \leq T$ on the left side and sum up over $p \geq -1$, we obtain

$$
\| g_N \|_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} + \| \tilde{L}^2 g_N \|_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} \\
\leq C \| g_0 \|_{L^2_x} + C \| S_2 f_N \|^2_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} + C \sqrt{T} \| S_2 f_N \|_{L^2_x} + \| g_N \|_{L^2_x} \\
+ C \| g_0 \|_{L^2_x} + C \| f \|^2_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} + C \sqrt{T} \| S_2 f \|_{L^2_x} + \| g_N \|_{L^2_x},
$$

where we used the Proposition 6.2 and Lemma 6.4 because the weak solution $g_N$ satisfies the Proposition 19. Then, for a small $T > 0$ and small $\| f \|_{L^2_x \tilde{L}^2 (B^{3/2}_2)}$ (taking $C \sqrt{T}, C \| f \|^2_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} < 1$), we get

$$
\| g_N \|_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} + \| \tilde{L}^2 g_N \|_{L^\infty_T \tilde{L}^2 (B^{3/2}_2)} \\
\leq C \| g_0 \|_{L^2_x} + C \sqrt{T} \| S_2 f \|_{L^2_x},
$$

which is the desired (52). Hence, the proof of Proposition 20 is finished. \qed

The proof of the Theorem 6.1

Proof. As a first step, let us show that

$$
g(t, x, v) \in \tilde{L}^\infty_T \tilde{L}^2_v (B^{3/2}_2) \quad \text{and} \quad \tilde{L}^2 g(t, x, v) \in \tilde{L}^2_T \tilde{L}^2_v (B^{3/2}_2).
$$

Based on Proposition 20, it suffices to prove the convergence of the sequences

$$
\{ g_N, N \in \mathbb{N} \} \subset \tilde{L}^\infty_T \tilde{L}^2_v (B^{3/2}_2) \quad \text{and} \quad \{ \tilde{L}^2 g_N, N \in \mathbb{N} \} \subset \tilde{L}^2_T \tilde{L}^2_v (B^{3/2}_2).
$$
Set $w_{M,M'} = g_M - g_{M'}$, $M, M' \in \mathbb{N}$. Then it follows that (51) that
\[
\partial_t w_{M,M'} + v \cdot \nabla_x w_{M,M'} + \mathcal{L}_1 w_{M,M'} = \mathbf{L}(f_{M'}, w_{M,M'}) + \mathbf{L}(f_M - f_{M'}, g_N) - \mathcal{L}_2(f_M - f_{M'}). \tag{53}
\]
Applying $\Delta_p(p \geq -1)$ to (53) and taking the inner product with $2^{3p} \Delta_p w_{M,M'}$ over $\mathbb{R}^3 \times \mathbb{R}^3$, we get
\[
\frac{d}{dt} 2^{3p} \| \Delta_p w_{M,M'} \|^2_{L^2_{x,v}} + 2^{3p} \| \tilde{\mathcal{L}}^2 \Delta_p w_{M,M'} \|^2_{L^2_{x,v}} \\
\leq C 2^{3p} \| \Delta_p w_{M,M'} \|^2_{L^2_{x,v}} + 2^{3p+1} (\Delta_p \mathbf{L}(f_{M'}, w_{M,M'}), \Delta_p w_{M,M'})_{L^2_{x,v}} \\
+ 2^{3p+1} (\Delta_p \mathbf{L}(f_M - f_{M'}, g_M), \Delta_p w_{M,M'})_{L^2_{x,v}} \\
+ C 2^{3p} \| \Delta_p \mathbf{S}_2(f_M - f_{M'}) \|_{L^2_{x,v}} \| \Delta_p w_{M,M'} \|_{L^2_{x,v}}. \tag{54}
\]
Integrating (54) with respect to the time variable over $[0, t]$ with $0 \leq t \leq T$, taking the square root of both sides of the resulting inequality and summing up over $p \geq -1$. Following from Lemma 3.3, we obtain
\[
\| w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
\leq C \sqrt{T} \| w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + C \| \mathbf{S}_2 f_M \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
+ C \| \mathbf{S}_2 (f_M - f_{M'}) \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + C \sqrt{T} \| \mathbf{S}_2 (f_M - f_{M'}) \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \| \Delta_p w_{M,M'} \|_{L^2_{x,v}} \tag{55}
\]
It follows from the Proposition 20 and Young’s inequality that
\[
\| w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
\leq C \sqrt{T} \| w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + C \| f \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
+ C \left( \| g_0 \|_{L^2_2(B_{1,1}') + \sqrt{T} \| f \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \right) \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
+ \sqrt{T} \| \mathbf{S}_2 (f_M - f_{M'}) \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \tag{56}
\]
Thanks to the smallness of $\| g_0 \|_{L^2_2(B_{1,1}')}$, $\| f \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))}$ and $T$, we can choose
\[
C \left( \| g_0 \|_{L^2_2(B_{1,1}') + \sqrt{T} \| f \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \right) \leq \frac{1}{8}.
\]
Then it is shown that
\[
\| w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} + \| \tilde{\mathcal{L}}^2 w_{M,M'} \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))} \\
\leq \tilde{\lambda} \| \mathbf{S}_2 (f_M - f_{M'}) \|_{L^p_T \tilde{L}^2_2(B_{1,1}'(B_{2,2}'))}
\]
for $0 < \tilde{\lambda} < 1$. As $\{ f_N \}$ is a Cauchy sequence in $L^\infty_T \tilde{L}^2_2(B_{2,1}')$, we deduce that $\{ g_N \}$ is also true and satisfies
\[
\{ g_N, N \in \mathbb{N} \} \subset L^\infty_T \tilde{L}^2_2(B_{2,1}') \quad \text{and} \quad \{ \tilde{\mathcal{L}}^2 g_N, N \in \mathbb{N} \} \subset L^2_T \tilde{L}^2_2(B_{2,1}').
\]
Set $g = \lim_{N \to \infty} g_N$. Therefore we obtain
\[
g(t, v, x) \in L^\infty_T \tilde{L}^2_2(B_{2,1}') \quad \text{and} \quad \tilde{\mathcal{L}}^2 g(t, v, x) \in L^2_T \tilde{L}^2_2(B_{2,1}').
\]
Next, we prove (40). Applying $\Delta_p(p \geq -1)$ to (39), taking the inner product with $2^{3p}\Delta_p g$ over $\mathbb{R}^3_+ \times \mathbb{R}^3_+$ and using Lemma 3.4 and Corollary 1, we get

$$\frac{d}{dt} 2^{3p} \|\Delta_p g\|_{L^2_+}^2 + 2^{3p} \|\tilde{L}^2 \Delta_p g\|_{L^2_+}^2 \leq 2^{3p+1} (\Delta_p L(f, g), \Delta_p g)_{L^2_+} + 2^{3p} C(\|\Delta_p S_2 f\|_{L^2_+}^2 + \|\Delta_p g\|_{L^2_+}^2).$$

Integrating (56) with respect to the time variable over $[0, t]$ with $0 \leq t < T$, taking the square root of both sides of the resulting inequality and summing up over $p \geq -1$. It follows from Lemma 3.3 that

$$\|g\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)} + \|\tilde{L}^2 g\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)} \leq \tilde{C} \|S_2 f\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)}^{1/2} \|\tilde{L}^2 g\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)}^{1/2} + C \left( \|g_0\|_{L^2(B_{2^3/4}^3)} + \sqrt{T} \|S_2 f\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)} + \sqrt{T} \|\tilde{L}^2 g\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)} \right).$$

Since $\|f\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)}$ and $T$ are small (taking $\tilde{C} \|S_2 f\|_{L^2_T \tilde{L}^2(B_{2^3/4}^3)} C \sqrt{T} < \frac{1}{4}$), we get the desired inequality (40). □

7. Appendix B. For convenience of reader, we recall the Littlewood-Paley decomposition and definition of Besov spaces. The reader is also referred to [6] for more details.

Let $(\varphi, \chi)$ be a couple of smooth functions valued in the closed interval $[0, 1]$ such that $\varphi$ is supported in the shell $C(0, \frac{4}{3}, \frac{8}{3}) = \{ \xi \in \mathbb{R}^3 : \frac{4}{3} \leq |\xi| \leq \frac{8}{3} \}$ and $\chi$ is supported in the ball $B(0, \frac{4}{3}) = \{ \xi \in \mathbb{R}^3 : |\xi| \leq \frac{4}{3} \}$. In terms of the two functions, one has the unit decomposition

$$\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1, \quad \forall \xi \in \mathbb{R}^3.$$

The inhomogeneous dyadic blocks are defined by

$$\Delta_n u \doteq \chi(D) u, \quad \Delta_q u \doteq \varphi(2^{-q} D) u, \quad q \geq 0$$

for $u = u(x) \in \mathcal{S}'(\mathbb{R}^3)$. The Littlewood-Paley decomposition of a general tempered distribution $u$ reads

$$u = \sum_{q \geq 1} \Delta_q u.$$

It is also convenient to introduce the low-frequency cut-off:

$$S_q u = \sum_{p \leq q-1} \Delta_p u.$$

The Littlewood-Paley decomposition is “almost” orthogonal.

Lemma 7.1. For any $u \in \mathcal{S}'(\mathbb{R}^d)$ and $v \in \mathcal{S}'(\mathbb{R}^d)$, the following properties hold:

$$\Delta_q \Delta_q' u = 0, \quad \text{if} \quad |p - q| \geq 2,$$

$$\Delta_q (S_{p-1} u \Delta_p v) = 0, \quad \text{if} \quad |p - q| \geq 5.$$

Additionally, it is crucial that we have

Lemma 7.2. Let $1 \leq p \leq \infty$ and $u \in L^p$, then there exists a constant $C > 0$ independent of $p, q$ and $u$ such that

$$\|\Delta_q u\|_{L^p} \leq C \|u\|_{L^p}, \quad \|S_q u\|_{L^p} \leq C \|u\|_{L^p}.$$

Now, we turn to the definition of the main functional spaces and norms in the present paper.
Definition 7.3. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov space $B^s_{p,r}$ is defined by
\[ B^s_{p,r} := \{ u \in \mathcal{S}'(\mathbb{R}^3) : \| u \|_{B^s_{p,r}} < \infty \}, \]
where
\[ \| u \|_{B^s_{p,r}} := \| \langle 2^k \| \Delta_k u \|_{L^p} \rangle \|_{L^r}. \]

For the distribution $f = f(t, v, x)$, we define the Banach space
\[ L^p_T \dot{L}^p_v L^q_x \triangleq L^p([0, T] ; L^p(\mathbb{R}^3_x ; L^q(\mathbb{R}^3_v))) \]
for $0 < T \leq \infty$, $1 \leq p_1, p_2, p_3 \leq \infty$, where the norm is given by
\[ \| u \|_{L^p_T \dot{L}^p_v L^q_x} = \left( \int_0^T \left( \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |u(t, x, v)|^{p_3} \, dx \right)^{p_2/p_3} \, dv \right)^{p_1/p_2} \, dt \right)^{1/p_1} \]
with the usual convention if $p_1, p_2, p_3 = \infty$.

Next, we present the definition of the Chemin-Lerner type space, which were initiated in [12].

Definition 7.4. Let $s \in \mathbb{R}$ and $1 \leq q_1, q_2, p, r \leq \infty$. For $0 < T \leq \infty$, the space $\tilde{L}^{q_1}_{v} \tilde{L}^{q_2}_{v}(B^s_{p,r})$ is defined by
\[ \tilde{L}^{q_1}_{v} \tilde{L}^{q_2}_{v}(B^s_{p,r}) = \left\{ u(t, v, x) \in \mathcal{S}' : \| u \|_{\tilde{L}^{q_1}_{v} \tilde{L}^{q_2}_{v}(B^s_{p,r})} < \infty \right\}, \]
where
\[ \| u \|_{\tilde{L}^{q_1}_{v} \tilde{L}^{q_2}_{v}(B^s_{p,r})} = \left( \sum_{q \geq -1} \left\langle 2^q \| \Delta_q u \|_{L^{q_1}_{v} L^{q_2}_{v}} \right\rangle^r \right)^{1/r} \]
with the usual convention for $q_1, q_2, p, r = \infty$. Similarly, one also denote
\[ \| u \|_{\tilde{L}^{q_1}_{v} \tilde{L}^{q_2}_{v}(B^s_{p,r})} = \left( \sum_{q \in \mathbb{Z}} \left\langle 2^q \| \Delta_q u \|_{L^{q_1}_{v} L^{q_2}_{v}} \right\rangle^r \right)^{1/r} \]
with the usual convention for $q_1, q_2, p, r = \infty$.

In addition, we also use the weighted Sobolev spaces $H^s_{x,v}(\mathbb{R}^6_{x,v})$. For $s, \ell \in \mathbb{R}$, one define
\[ H^s_{x,v}(\mathbb{R}^6_{x,v}) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6_{x,v}) : \langle v \rangle^s g \in H^s(\mathbb{R}^6_{x,v}) \right\}, \]
where the weight is with respect to the velocity variable $v \in \mathbb{R}^3$.

The following embedding properties in Besov spaces have been used several times.

Lemma 7.5. Let $s \in \mathbb{R}$.
(1) If $s > 0$, then $B^s_{2,1} = L^2 \cap \tilde{B}^s_{2,1}$; (2) If $s < 0$, then $B^s_{2,1} \hookrightarrow \tilde{B}^s_{2,1}$.

According to [34], we have the following topology between homogeneous Chemin-Lerner spaces and nonhomogeneous Chemin-Lerner spaces.

Lemma 7.6. Let $1 \leq q, q, r \leq \infty$ and $s > 0$. It holds that
\[ \| \nabla_x \cdot \tilde{L}^p_v(B^s_{p,r}) \sim \| \cdot \|_{\tilde{L}^p_v(B^s_{p,r})}, \quad \| \cdot \|_{\tilde{L}^p_v(B^s_{p,r})} \lesssim \| \cdot \|_{\tilde{L}^p_v(B^s_{p,r})}. \]

Furthermore, it can be shown that
Proposition 21. Let $s \in \mathbb{R}$ and $1 \leq \varrho_1, \varrho_2, p, r \leq \infty$.

1. It holds that
   
   $$L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} \cap \overline{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})} \subset \overline{L^{\varrho_1}_{\tilde{v} T} L^{p}_{x} (B^s_{p,r})}.$$

2. Furthermore, if $s > 0$ and $r \leq \min \{\varrho_1, \varrho_2\}$, then
   
   $$L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} \cap \overline{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})} = \overline{L^{\varrho_1}_{\tilde{v} T} L^{p}_{x} (B^s_{p,r})}$$

for any $T > 0$.

Finally, it follows from [19] that

Lemma 7.7. Let $s \in \mathbb{R}$ and $1 \leq \varrho_1, \varrho_2, p, r \leq \infty$.

1. If $r \leq \min \{\varrho_1, \varrho_2\}$, then it holds that
   
   $$\|u\|_{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})} \leq \|u\|_{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})}.$$

2. If $r \geq \max \{\varrho_1, \varrho_2\}$, then it holds that
   
   $$\|u\|_{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})} \geq \|u\|_{L^{p_1}_{\tilde{v} T} L^{\varrho_2}_{x} (B^s_{p,r})}.$$

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