Plancherel Theorem on the Symplectic Group
\( SP(4, \mathbb{R}) \)

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Abstract
Let \( SL(4, \mathbb{R}) \) be the 15− dimensional connected semisimple Lie group and let \( SL(4, \mathbb{R}) = KAN \) be the Iwasawa decomposition. Let \( \mathbb{R}^4 \rtimes SL(4, \mathbb{R}) \) be the group of the semidirect product of \( SL(4, \mathbb{R}) \) with the real vector group \( \mathbb{R}^4 \). The goal of this paper is to define the Fourier transform on \( SL(4, \mathbb{R}) \) in order to obtain the Plancherel theorem on \( SL(4, \mathbb{R}) \) and so on \( \mathbb{R}^4 \rtimes SL(4, \mathbb{R}) \). Since the symplectic group \( SP(4, \mathbb{R}) \) is a subgroup of \( SL(4, \mathbb{R}) \), then it will be easy to get the Plancherel theorem on \( SP(4, \mathbb{R}) \) and so on its inhomogeneous group. To this end, we obtain some interesting results on its nilpotent symplectic group.

Key words: Semisimple Lie group \( SL(4, \mathbb{R}) \), Symplectic Lie group \( SP(4, \mathbb{R}) \), Nilpotent Symplectic Group, Fourier transform and Plancherel Theorem on \( SL(4, \mathbb{R}) \), Plancherel Theorem on \( SP(4, \mathbb{R}) \) and on its Inhomogeneous group.

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1 Introduction

1. As well known the connected semisimple Lie group \( SL(n, \mathbb{R}) \), consists of the following matrices

\[
SL(n, \mathbb{R}) = \{ A = GL(n, \mathbb{R}); \det A = 1 \}
\] (1)
The group $Sp(2n, \mathbb{R})$ is a subgroup of $SL(2n, \mathbb{R})$, which is

$$SP(2n, \mathbb{R}) = \{g \in SL(2n, \mathbb{R}); g^tg = j\}$$

where $j$ is the symplectic matrix defined by

$$j = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

and 0 and $I$ are the $n \times n$ zero and identity matrices. It is clear $det \ j = 1$, $j^2 = I$, and $j^t = j^{-1} = -j$.

The 10-dimensional symplectic group $SP(4, \mathbb{R})$. If $g \in SP(4, \mathbb{R})$, then

$$g = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{14} & x_{24} \\ x_{31} & x_{32} & -x_{11} & -x_{21} \\ x_{32} & x_{42} & -x_{12} & -x_{22} \end{pmatrix}, x_{ij} \in \mathbb{R}, 1 \leq i, j \leq 4$$

The goal of this paper is to define the Fourier transform in order to obtain the Plancherel theorem on the group $SP(4, \mathbb{R})$ and its inhomogeneous group. Therefore, I will define the Fourier transform on $SL(4, \mathbb{R})$, and I will prove its Plancherel theorem. Besides, I will demonstrate the existence theorem and hypoellipticity of the partial differential equations on its nilpotent symplectic

2 Notation and Results

2.1. In the following and far away from the representations theory of Lie groups we use the Iwasawa decomposition of $SL(4, \mathbb{R})$, to define the Fourier transform and to get the Plancherel formula on the connected real semisimple Lie group $SL(4, \mathbb{R})$. Therefore let $SL(4, \mathbb{R})$ be the complex Lie group, which is

$$SL(4, \mathbb{R}) = \{A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} : a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 4 \ and \ det \ A = 1\}$$

(5)
Let $G = SL(4, \mathbb{R}) = KNA$ be the Iwasawa decomposition of $G$, where

$$K = SO(4)$$

$$N = \{ \begin{pmatrix} 1 & x_1 & x_3 & x_6 \\ 0 & 1 & x_2 & x_5 \\ 0 & 0 & 1 & x_4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_i \in \mathbb{R}, 1 \leq i \leq 6 \}$$

$$A = \{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \mid a_i \in \mathbb{R}_+^*, 1 \leq i \leq 4, a_1a_2a_3a_4 = 1 \} \quad (6)$$

Hence every $g \in SL(4, \mathbb{R})$ can be written as $g = kan \in SL(4, \mathbb{R})$, where $k \in K$, $a \in A$, $n \in N$. We denote by $L^1(SL(4, \mathbb{R}))$ the Banach algebra that consists of all complex valued functions on the group $SL(4, \mathbb{R})$, which are integrable with respect to the Haar measure $dg$ of $SL(4, \mathbb{R})$ and multiplication is defined by convolution product on $SL(4, \mathbb{R})$, and we denote by $L^2(SL(4, \mathbb{R}))$ the Hilbert space of $SL(4, \mathbb{R})$. So we have for any $f \in L^1(SL(4, \mathbb{R}))$ and $\phi \in L^1(SL(4, \mathbb{R}))$

$$\phi \ast f(h) = \int_G f(g^{-1}h)\phi(g)dg \quad (7)$$

The Haar measure $dg$ on a connected real semi-simple Lie group $G = SL(4, \mathbb{R})$, can be calculated from the Haar measures $dn$, $da$ and $dk$ on $N; A$ and $K$; respectively, by the formula

$$\int_{SL(4, \mathbb{R})} f(g)dg = \int_A \int_N \int_K f(ank)dadndk \quad (8)$$

2.2. Keeping in mind that $a^{-2\rho}$ is the modulus of the automorphism $n \rightarrow ana^{-1}$ of $N$ we get also the following representation of $dg$

$$\int_{SL(4, \mathbb{R})} f(g)dg = \int_A \int_N \int_K f(ank)dadndk = \int_N \int_A \int_K f(nak)a^{-2\rho}dndadk \quad (9)$$

where

$$\rho = 2^{-1} \sum_{\alpha \neq 0} m(\alpha)\alpha$$
and \( m(\alpha) \) denotes the multiplicity of the root \( \alpha \) and \( \rho = \) the dimension of the nilpotent group \( N \). Furthermore, using the relation \( \int_G f(g)dg = \int_G f(g^{-1})dg \), we receive

\[
\int_{\text{SL}(4,\mathbb{R})} f(g)dg = \int_K \int_A \int_N f(kan)a^{2\rho}dnddk
\] (10)

### 3 Fourier Transform and Plancherel Theorem

#### On \( N \)

3.1. Let \( N \) be the real group consisting of all matrices of the form

\[
\begin{pmatrix}
1 & x_1 & x_3 & x_6 \\
0 & 1 & x_2 & x_5 \\
0 & 0 & 1 & x_4 \\
0 & 0 & 0 & 1
\end{pmatrix}
\] (11)

where \((x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6\). The group can be identified with the group \((\mathbb{R}^3 \times \mathbb{R}^2) \rtimes \mathbb{R}\) be the semidirect product of the real vector groups \(\mathbb{R}, \mathbb{R}^2\) and \(\mathbb{R}^3\), where \(\rho_2\) is the group homomorphism \(\rho_2 : \mathbb{R}^2 \rightarrow Aut(\mathbb{R}^3)\), which is defined by

\[
\rho_2(x_3, x_2)(y_6, y_5, y_4) = (y_6 + x_3y_4, y_5 + x_2y_4, y_4)
\] (12)

and \(\rho_1\) is the group homomorphism \(\rho_1 : \mathbb{R} \rightarrow Aut(\mathbb{R}^3 \rtimes \mathbb{R}^2)\), which is given by

\[
\rho_1(x_1)(y_6, y_5, y_4, y_3, y_2) = (y_6 + x_1y_5, y_4, y_3 + x_1y_2, y_2)
\] (13)

where \(Aut(\mathbb{R}^3) (resp. Aut(\mathbb{R}^3 \rtimes \mathbb{R}^2))\) is the group of all automorphisms of \(\mathbb{R}^3\) (resp.\(\mathbb{R}^3 \rtimes \mathbb{R}^2\)), see [6].
Let \( L = \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \) be the group with law:

\[
X,Y = (x_6, x_5, x_4, x_3, x_2, t_3, t_2, x_1, t_1)(y_6, y_5, y_4, y_3, y_2, s_3, s_2, y_1, s_1)
\]

\[
= ((x_6, x_5, x_4, x_3, x_2, t_3, t_2)(\rho_1(t_1)(y_6, y_5, y_4, y_3, y_2, s_3, s_2), x_1, s_1 + t_1)
\]

\[
= ((x_6, x_5, x_4, x_3, x_2, t_3, t_2)(y_6 + t_1 y_5, y_4, y_3, s_3 + t_1 s_2, x_2 + y_2, y_1 + x_1, s_1 + t_1)
\]

\[
= ((x_6, x_5, x_4) + (y_6 + t_1 y_5 + t_3 y_4, y_4, x_3 + y_3, s_3 + t_1 s_2,
\]

\[
s_2 + t_2, x_2 + y_2, y_1 + x_1, s_1 + t_1)
\]

\[
= (x_6 + y_6 + t_1 y_5 + t_3 y_4, x_5 + y_5 + t_2 y_4, x_4 + y_4, x_3 + y_3, t_3 + s_3 + t_1 s_2,
\]

\[
y_2 + x_2, s_2 + t_2, y_1 + x_1, s_1 + t_1)
\]

(14)

for all \((X, Y) \in L^2\). In this case the group \( N \) can be identified with the closed subgroup \( \mathbb{R}^3 \times \{0\} \times \mathbb{R}^2 \times \{0\} \times \mathbb{R} \) of \( L \) and \( B \) with the closed subgroup \( \mathbb{R}^3 \times \mathbb{R}^2 \times \{0\} \times \mathbb{R} \times \{0\} \) of \( L \), where \( B = \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R} \) the group, which is the direct product of the real vector groups \( \mathbb{R} \), \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

Let \( C^\infty(N), D(N), D'(N), E'(N) \) be the space of \( C^\infty \)-functions, \( C^\infty \) with compact support, distributions and distributions with compact support on \( N \) respectively. We denote by \( L^1(N) \) the Banach algebra that consists of all complex valued functions on the group \( N \), which are integrable with respect to the Haar measure of \( N \) and multiplication is defined by convolution on \( N \), and we denote by \( L^2(N) \) the Hilbert space of \( N \).

**Definition 3.1.** For every \( f \in C^\infty(N) \), one can define function \( \tilde{f} \in C^\infty(L) \) as follows:

\[
\tilde{f}(x, x_3, x_2, t_3, t_2, x_1, t_1) = f(\rho_1(x_1)(\rho_2(x_3, x_2)(x), t_3 + x_3, t_2 + x_2), t_1)
\]

(15)

for all \((x, x_3, x_2, t_3, t_2, x_1, t_1) \in L, \) here \( x = (x_6, x_5, x_4) \in \mathbb{R}^3 \).

**Remark 3.1.** The function \( \tilde{f} \) is invariant in the following sense:

\[
\tilde{f}(((\rho_1(h)((\rho_2(r, k)(x), x_3 - r, x_2 - k, t_3 + r, t_2 + k), x_1 - h, t_1 + h)
\]

(16)

for any \((x, x_3, x_2, t_3, t_2, x_1, t_1) \in L, h \in \mathbb{R} \) and \((r, k) \in \mathbb{R} \), where \( x = (x_6, x_5, x_4) \in \mathbb{R}^3 \). So every function \( \psi(x, x_3, x_2, x_1) \) on \( N \) extends uniquely as an invariant function \( \tilde{\psi}(x, x_3, x_2, t_3, t_2, x_1, t_1) \) on \( L \).

**Theorem 3.1.** For every function \( F \in C^\infty(L) \) invariant in sense (16) and for every \( \varphi \in D(N) \), we have

\[
u * F(x, x_3, x_2, t_3, t_2, x_1, t_1) = u * F(x, x_3, x_2, t_3, t_2, x_1, t_1)
\]

(17)
for every \((X, x_3, x_2, t_3, t_2, x_1, t_1) \in L\), where \(*\) signifies the convolution product on \(N\) with respect the variables \((x, t_3, t_2, t_1)\) and \(\ast_c\) signifies the commutative convolution product on \(B\) with respect the variables \((x, x_3, x_2, x_1)\).

**Proof:** In fact we have

\[
\varphi \ast F(x, x_3, x_2, t_3, t_2, x_1, t_1) = \int_N F \left[ (y, y_3, y_2, s)^{-1}(X, x_3, x_2, t_3, t_2, x_1, t_1) \right] u(y, y_3, y_2, s)dy_3dy_2ds
\]

\[
= \int_N F \left[ (\rho_1(s^{-1})(y, y_3, y_2)^{-1}, -s)(x, x_3, x_2, t_3, t_2, x_1, t_1) \right] u(y, y_3, y_2, s)dy_3dy_2ds
\]

\[
= \int_N F[(\rho_1(s^{-1})(\rho_2(y_3, y_2)^{-1}((-y) + (x))), x_3, x_2, t_3 - y_3, t_2 - y_2, x_1, t_1 - s)]
\]

\[
\varphi \ast_c F(x, x_3, x_2, t_3, t_2, x_1, t_1)
\]  

(18)

Since \(F\) is invariant in sense (16), then for every \((x, x_3, x_2, t_3, t_2, x_1, t_1) \in L\) we get

\[
\varphi \ast F(x, x_3, x_2, t_3, t_2, x_1, t_1) = \int_N F \left[ (\rho_1(s^{-1})(\rho_2(y_3, y_2)^{-1}(-y + x), x_3, x_2, t_3 - y_3, t_2 - y_2), x_1, t_1 - s) \right] u(y, y_3, y_2, s)dy_3dy_2ds
\]

\[
= \int_N F[x - y, x_3 - y_3, x_2 - y_2, t_3, t_2, x_1 - s, t] u(y, y_3, y_2, s)dy_3dy_2ds
\]

\[
= \varphi \ast_c F(x, x_3, x_2, t_3, t_2, x_1, t_1)
\]  

(19)

As in [6], we will define the Fourier transform on \(G\). Therefore let \(S(N)\) be the Schwartz space of \(N\) which can be considered as the Schwartz space of \(S(B)\), and let \(S'(N)\) be the space of all tempered distributions on \(N\).

**Definition 3.2.** If \(f \in S(N)\), one can define its Fourier transform \(\mathcal{F}f\) by the Fourier transform on its vector group:

\[
\mathcal{F}f \ (\xi) = \int_N f(X) e^{-i \langle \xi, X \rangle} dX
\]  

(20)
for any $\xi = (\xi_6, \xi_5, \xi_4, \xi_3, \xi_2, \xi_1) \in \mathbb{R}^6$, and $X = (x_6, x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^6$, where $\langle \xi, X \rangle = \xi_6 x_6 + \xi_5 x_5 + \xi_4 x_4 + \xi_3 x_3 + \xi_2 x_2 + \xi_1 x_1$ and $dX = dx_6 dx_5 dx_4 dx_3 dx_2 dx_1$ is the Haar measure on $N$. The mapping $f \rightarrow \mathcal{F}f$ is isomorphism of the topological vector space $S(N)$ onto $S(\mathbb{R}^6)$.

**Theorem 3.2.** The Fourier transform $\mathcal{F}$ satisfies:

$$\varphi \ast f(0) = \int_{\mathbb{R}^4} \mathcal{F}f(\xi)\overline{\mathcal{F}u(\xi)}d\xi \tag{21}$$

for every $f \in S(N)$ and $\varphi \in S(N)$, where $\varphi(X) = \overline{u(X^{-1})}$, $\xi = (\xi_6, \xi_5, \xi_4, \xi_3, \xi_2, \xi_1)$, $d\xi = d\xi_1 d\xi_2 d\xi_3 d\xi_4 d\xi_5 d\xi_6$, is the Lebesgue measure on $\mathbb{R}^6$, and $\ast$ denotes the convolution product on $N$.

**Proof:** By the classical Fourier transform, we have:

$$\varphi \ast f(0) = \int_{\mathbb{R}^4} \mathcal{F}(\varphi \ast f)(\xi)d\xi$$

$$= \int_{\mathbb{R}^6} \int_{N} \varphi(X) e^{-i\langle \xi,X \rangle} \, dXd\xi$$

$$= \int_{\mathbb{R}^6} \int_{N} f(YX) \overline{u(Y)}e^{-i\langle \xi,Y \rangle} \, dY \, dX \, d\xi. \tag{22}$$

By change of variable $YX = X'$ with $Y = (x_6, x_5, x_4, x_3, x_2, x_1)$ and $X' = (y_6, y_5, y_4, y_3, y_2, y_1)$, we get

$$X = Y^{-1}X' = (x_6, x_5, x_4, x_3, x_2, x_1)^{-1}(y_6, y_5, y_4, y_3, y_2, y_1)$$

$$= (y_6 - x_6 + x_1 x_5 - x_1 y_5 - x_1 x_2 x_4 + x_3 x_4 - x_3 y_4 + x_1 x_2 y_4, y_5 - x_5 + x_2 y_4 - x_2 x_4, x_4 + y_4, y_3 - x_3 - x_1 y_2 + x_1 x_2, y_2 - x_2, y_1 - x_1)$$

and

$$-i\langle \xi, X \rangle$$

$$= -i\langle \xi, Y^{-1}X' \rangle$$

$$= -i[(y_6 - x_6 + x_1 x_5 - x_1 y_5 - x_1 x_2 x_4 + x_3 x_4 - x_3 y_4 + x_1 x_2 y_4)\xi_6 + (y_5 - x_5 + x_2 y_4)\xi_5$$

$$-x_2 x_4 \xi_5 + (y_4 - x_4)\xi_4 + (y_3 - x_3 - x_1 y_2 + x_1 x_2)\xi_3 + (y_2 - x_2)\xi_2 + (y_1 - x_1)\xi_1]$$

$$= -i[(x_6 \xi_6 - y_6 \xi_6) + (-x_2 x_4 \xi_6 + x_2 y_4 \xi_6 - y_2 \xi_3 + x_2 \xi_3 + x_3 \xi_6 - y_5 \xi_6 - \xi_1) x_1 - y_1 \xi_1$$

$$+(y_5 \xi_5 - x_5 \xi_5) + (y_4 \xi_5 - x_4 \xi_5 - \xi_2) x_2 + y_2 \xi_2 + y_3 \xi_3 + (x_4 \xi_6 - y_4 \xi_6 - \xi_3) x_3 + (y_1 - x_4)\xi_4$$
So we obtain

\[ e^{-i(y_6y_5x_1x_3x_4x_5x_1x_2x_3y_4x_1y_4x_1x_2y_4)x_6} e^{-i((y_3+y_1x_4y_4x_1x_2y_4) + (y_4-x_4)x_4)} \]
\[ e^{-i((y_3x_3x_1y_2x_1x_2x_3y_3x_1x_3y_5x_6(y_1-x_1)x_1)} \]
\[ = e^{-i(y_6x_0y_0x_0) + (-x_2x_4x_6 + x_2y_4x_6 + y_2x_3 + x_2x_3 + x_3x_0 + y_5x_6(y_1-x_1)x_1)} \]
\[ e^{-i((y_5x_5(x_5x_5)) + (y_4x_5(x_5x_5))x_2 + y_2x_2) - i(y_5x_3x_3 + x_4x_6 + y_4x_6x_3)x_3 + (y_4-x_4)x_4)} \]

By the invariance of the Lebesgue's measures \( d\xi_1, d\xi_2 \) and \( d\xi_3 \), we get

\[
\varphi \ast f(0) = \int_N \int_N \int_{\mathbb{R}^6} f(X') e^{-i(y_6x_0y_0x_0) + (-x_2x_4x_6 + x_2y_4x_6 + y_2x_3 + x_2x_3 + x_3x_0 + y_5x_6(x_5x_5))x_1 + y_1x_1)} \]
\[ e^{-i((y_5x_5(x_5x_5)) + (y_4x_5(x_5x_5))x_2 + y_2x_2) - i(y_5x_3x_3 + x_4x_6 + y_4x_6x_3)x_3 + (y_4-x_4)x_4)} \]
\[ = \int_N \int_N \int_{\mathbb{R}^6} f(X') e^{-i(y_6x_0y_0x_0) + y_5x_5x_5 + y_4x_4x_4 + y_3x_3x_3 - x_2x_2 + y_2x_2 - x_1x_1 + y_1x_1)} \]
\[ \varphi(Y) dY dX' d\xi \]
\[ = \int_{\mathbb{R}^6} \mathcal{F} f(\xi) \overline{\varphi(\xi)} d\xi \]

for any \( Y = (x_6, x_5, x_4, x_3, x_2, x_1) \in \mathbb{R}^6 \) and \( X' = (y_6, y_5, y_4, y_3, y_2, y_1) \in \mathbb{R}^6 \), where \( 0 = (0, 0, 0, 0, 0, 0) \) is the identity of \( N \). The theorem is proved.

**Corollary 3.1.** In theorem 3.2, if we replace \( \varphi \) by \( f \), we obtain the Plancherel's formula on \( N \)

\[ f \ast f(0) = \int_N |f(X)|^2 dX = \int_{\mathbb{R}^6} |\mathcal{F} f(\xi)|^2 d\xi \quad (23) \]

4 Fourier Transform and Plancherel Theorem On \( SL(4, \mathbb{R}) \)

4.1. Let \( \mathfrak{k} \) be the Lie algebra of \( K = SO(4) \). Let \( (X_1, X_2, X_3, X_4) \) a basis of \( \mathfrak{k} \), such that the both operators

\[ \Delta = \sum_{i=1}^{3} X_i^2, \quad D_q = \sum_{0 \leq l \leq q} \left( -\sum_{i=1}^{3} X_i^2 \right)^l \quad (24) \]
are left and right invariant (bi-invariant) on $K$, this basis exist see [4, p.564]. For $l \in \mathbb{N}$. Let $D^l = (1 - \Delta)^l$, then the family of semi-norms $\{\sigma_l, l \in \mathbb{N}\}$ such that
\[
\sigma_l(f) = \int_K |D^l f(y)|^2 \, dy \frac{1}{2}, \quad f \in C^\infty(K)
\] (25)
define on $C^\infty(K)$ the same topology of the Frechet topology defined by the semi-norms $\|X^\alpha f\|_2$ defined as
\[
\|X^\alpha f\|_2 = \int_K (|X^\alpha f(y)|^2 \, dy)^\frac{1}{2}, \quad f \in C^\infty(K)
\] (26)
where $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, see [4, p.565]

Let $\hat{K}$ be the set of all irreducible unitary representations of $K$. If $\gamma \in \hat{K}$, we denote by $E_\gamma$ the space of the representation $\gamma$ and $d_\gamma$ its dimension then we get

**Definition 4.1.** The Fourier transform of a function $f \in C^\infty(K)$ is defined as
\[
Tf(\gamma) = \int_K f(x) \gamma(x^{-1}) \, dx
\] (27)
where $T$ is the Fourier transform on $K$

**Theorem (A. Cerezo [4]) 4.1.** Let $f \in C^\infty(K)$, then we have the inversion of the Fourier transform
\[
f(x) = \sum_{\gamma \in \hat{K}} d_\gamma tr[Tf(\gamma) \gamma(x)]
\] (28)
\[
f(I_K) = \sum_{\gamma \in \hat{K}} d_\gamma tr[Tf(\gamma)]
\] (29)
and the Plancherel formula
\[
\|f(x)\|_2^2 = \int_K |f(x)|^2 \, dx = \sum_{\gamma \in \hat{K}} d_\gamma \|Tf(\gamma)\|_{H.S}^2
\] (30)
for any $f \in L^1(K)$, where $I_K$ is the identity element of $K$ and $\|Tf(\gamma)\|_{H.S}^2$ is the Hilbert-Schmidt norm of the operator $Tf(\gamma)$
Definition 4.2. For any function \( f \in \mathcal{D}(G) \), we can define a function \( \Upsilon(f) \) on \( G \times K = G \times SO(4) \) by

\[
\Upsilon(f)(g, k_1) = \Upsilon(f)(kna, k_1) = f(gk) = f(knak_1)
\]

(31)

for \( g = kna \in G \), and \( k_1 \in K \). The restriction of \( \Upsilon(f) \ast \psi(g, k_1) \) on \( K(G) \) is \( \Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = f(nak_1) = f(g) \in \mathcal{D}(G) \), and \( \Upsilon(f)(g, k_1) \downarrow_{SO(4)} = f(kna) \in \mathcal{D}(G) \).

Remark 4.1. \( \Upsilon(f) \) is invariant in the following sense

\[
\Upsilon(f)(gh, h^{-1}k_1) = \Upsilon(f)(g, k_1)
\]

(32)

Definition 4.3. If \( f \) and \( \psi \) are two functions belong to \( \mathcal{D}(G) \), then we can define the convolution of \( \Upsilon(f) \) and \( \psi \) on \( G \times SO(4) \) as

\[
\Upsilon(f) \ast \psi(g, k_1) = \int_G \Upsilon(f)(gg^{-1}, k_1)\psi(g_2)dg_2
\]

\[
= \int_{SO(4)} \int_N \int_A \Upsilon(f)(kna^{-1}n^{-1}k^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2
\]

and so we get

\[
\Upsilon(f) \ast \psi(g, k_1) \downarrow_{K(G)} = \Upsilon(f) \ast \psi(IKna, k_1)
\]

\[
= \int_{SO(4)} \int_N \int_A f(naa^{-1}n^{-1}k^{-1}k_1)\psi(k_2n_2a_2)dk_2dn_2da_2
\]

\[
= \Upsilon(f) \ast \psi(na, k_1)
\]

where \( g_2 = k_2n_2a_2 \).

Definition 4.3. For \( f \in \mathcal{D}(G) \), let \( \Upsilon(f) \) be its associated function, we define the Fourier transform of \( \Upsilon(f)(g, k_1) \) by

\[
\mathcal{F}\Upsilon(f)(I_{SO(4)}, \xi, \lambda, \gamma)
\]

\[
= \int_{SO(3)} \int_N \int_A (TT(f)(I_{SO(4)}na, k_1)\gamma(k_1^{-1})dk_1]
\]

\[
a^{-i\lambda}e^{-i\langle\xi, n\rangle}dadn
\]

\[
= \int_{SO(3)} \int_N \int_A [\Upsilon(f)(I_{SO(3)}na, k_1)a^{-i\lambda}e^{-i\langle\xi, n\rangle} \gamma(k_1^{-1})dadnk_1
\]

(33)
where $\mathcal{F}$ is the Fourier transform on $AN$ and $T$ is the Fourier transform on $SO(4), I_{SO(3)}$ is the identity element of $SO(4)$, and $n = (x_1, x_2, x_3, x_4, x_5, x_6), n_2 = (y_1, y_2, y_3, y_4, y_5, y_6), \xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6), a = b_1b_2b_3$ and $a = a_1a_2a_3$

Plancherel’s Theorem 4.2. For any function $f \in L^1(G) \cap L^2(G)$, we get

$$
\int_G |f(g)|^2 dg = \int_A \int_N \int_{SO(4)} |f(kna)|^2 d\alpha d\gamma dk = \sum_{\gamma \in \widehat{SO(4)}} d\gamma \int \int \|T\mathcal{F} f(\alpha, \xi)\|_2^2 d\alpha d\xi
$$

(34)

$$
f(I_AI_NI_{S^1}) = \int_A \int_N \sum_{\gamma \in \widehat{K}} d\gamma \mathcal{F} f(\alpha, \xi) d\alpha d\xi = \sum_{\gamma \in \widehat{K}} d\gamma \int \int \mathcal{F} f(\alpha, \xi) d\alpha d\xi
$$

(35)

where $I_A, I_N$, and $I_K$ are the identity elements of $A, N$ and $K$ respectively, $\mathcal{F}$ is the Fourier transform on $AN$ and $T$ is the Fourier transform on $K$.

Proof: First let $\mathcal{F}$ be the function defined by

$$
f(kna) = f((kna)^{-1}) = f(a^{-1}n^{-1}k^{-1})
$$

(36)

Then we have

$$
\int_{SL(4,\mathbb{R})} |f(g)|^2 dg
= \mathcal{Y}(f) * \mathcal{F}(I_{SO(4)}I_NI_A, I_{SO(4)})
= \int_G \mathcal{Y}(f)(I_{SO(4)}I_NI_Ag_2^{-1}, I_{SO(4)}) \mathcal{F}(g_2)dg_2
$$

$$
= \int_A \int_N \int_{SO(4)} \mathcal{Y}(f)(a_2^{-1}n_2^{-1}k_2^{-1}, I_{SO(4)}) f(k_2n_2a_2) da_2dn_2dk_2
$$

$$
= \int_A \int_N \int_{SO(4)} f(a_2^{-1}n_2^{-1}k_2^{-1}) f((k_2n_2a_2)^{-1}) da_2dn_2dk_2
$$

$$
= \int_A \int_N \int_{SO(4)} |f(a_2n_2k_2)|^2 da_2dn_2dk_2
$$

(37)
Secondly

$$\Upsilon(f) \ast \overset{\vee}{f}(I_{SO(4)}I_N I_A, I_{SO(4)})$$

$$= \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \text{tr}(\Upsilon(f) \ast \overset{\vee}{f}(I_{SO(4)}na, k_1)\gamma(k_1^{-1}))$$

$$a^{-i\alpha}e^{-i(\xi, n)}d\text{ad}n d\lambda d\xi$$

$$= \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \text{tr}[\Upsilon(f) \ast \overset{\vee}{f}(I_{SO(3)na}, k_1)dk a^{-i\alpha}e^{-i(\xi, n)}\gamma(k_1^{-1})]d\text{ad}n d\lambda d\xi$$

$$= \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \text{tr}[\Upsilon(f)(I_{SO(4)nb}^{-1}n_2^{-1}k_2^{-1}, k_1)\overset{\vee}{f}(k_2n_2b)\gamma(k_1^{-1})dk_1]$$

$$a^{-i\alpha}e^{-i(\xi, n)}d\text{nd}a d\text{db} d\lambda d\xi$$

where

$$e^{-i(y_6-x_6+x_1x_5-x_1y_5-x_1x_2x_4+x_3x_4-x_3y_4+x_1x_2y_4)\xi_6}e^{-i((y_5-x_5+x_2y_4-x_2x_4)\xi_5+(y_4-x_4)\xi_4)}$$

$$e^{-i((y_3-x_3+y_2x_2)\xi_3+(y_2-x_2)\xi_2+(y_1-x_1)\xi_1)}$$

$$e^{-i(y_6\xi_6-x_6\xi_6+y_5\xi_5-x_5\xi_5+y_4\xi_4-x_4\xi_4+y_3\xi_3-x_3\xi_3-x_2\xi_2+y_2\xi_2-\xi_1x_1+y_1\xi_1)}$$

$$n = (x_1, x_2, x_3, x_4, x_5, x_6), n_2 = (y_1, y_2, y_3, y_4, y_5, y_6), \xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6), a = b_1b_2b_3 \text{ and } a = a_1a_2a_3$$

Using the fact that

$$\int \int \int_{SO(4)} f(kn)a d\text{ad}nk = \int \int \int_{SO(4)} f(ka)a^2 d\text{nd}a dk \quad (38)$$
\[
\int \int \int \int_{\mathbb{R}^6} f(kn) e^{-i(\xi, n)} d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int \int \int_{\mathbb{R}^6} f(kn) e^{-i(\xi, an_1^{-1})} a^2 d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int \int \int_{\mathbb{R}^6} f(kn) e^{-i(a_\xi a^{-1}, n)} a^2 d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int \int \int_{\mathbb{R}^6} f(kn) e^{-i(\xi, n)} d\xi d\eta d\lambda d\Omega
\] (39)

Then we have

\[
\Upsilon(f) * f(I_{SO(4)} I_N I_A, I_{SO(4)})
\]
\[
= \int \int_{SO(4)} \int \int_{SO(4)} \sum_{\gamma \in \hat{SO}(3)} d_\gamma \int_{SO(4)} f(nab^{-1}n_2^{-1}k_2^{-1}, k_1) f(k_2n_2b) \gamma(k_1^{-1}) d\xi d\eta d\lambda d\Omega
\]
\[
a^{-i\lambda} e^{-i(\xi, n)} d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int_{SO(4)} \int \int_{SO(4)} \sum_{\gamma \in \hat{SO}(4)} d_\gamma \int_{SO(4)} f(ank_2^{-1}k_1^{-1}) f(k_2n_2b) \gamma(k_1^{-1}) d\xi d\eta d\lambda d\Omega
\]
\[
a^{-i\lambda} e^{-i(\xi, n)} d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int_{SO(4)} \int \int_{SO(4)} \sum_{\gamma \in \hat{SO}(4)} d_\gamma \int_{SO(4)} f(ank_2^{-1}k_1^{-1}) f(k_2n_2b) \gamma(k_1^{-1}) d\xi d\eta d\lambda d\Omega
\]
\[
a^{-i\lambda} e^{-i(\xi, n_2)} d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int_{SO(4)} \int \int_{SO(4)} \sum_{\gamma \in \hat{SO}(4)} d_\gamma \int_{SO(4)} f(ank_2^{-1}k_1^{-1}) f(k_2n_2b) \gamma(k_1^{-1}) d\xi d\eta d\lambda d\Omega
\]
\[
a^{-i\lambda} e^{-i(\xi, n_2)} d\xi d\eta d\lambda d\Omega
\]
\[
= \int \int_{SO(4)} \int \int_{SO(4)} \sum_{\gamma \in \hat{SO}(4)} d_\gamma \int_{SO(4)} f(ank_2^{-1}k_1^{-1}) f(k_2n_2b) \gamma(k_1^{-1}) d\xi d\eta d\lambda d\Omega
\]
\[
a^{-i\lambda} e^{-i(\xi, n_2)} d\xi d\eta d\lambda d\Omega
\]
So, we get
\[
\mathcal{Y}(f) \ast \hat{\mathcal{f}}(I_{SO(4)}I_NI_A, I_{SO(4)}) = \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \int_{SO(4)} f(ank_1^{-1}) f(k_2n_2b) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1dk_2 \\
\quad a^{-i\lambda} b^{-i\lambda} e^{-i(\xi, n_n) + i(\xi, n_2)} d\eta d\lambda d\xi
\]
\[
= \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \int_{SO(4)} f(ank_1^{-1}) f(b^{-1}n_2^{-1}k_2^{-1}) \gamma(k_1^{-1}) \gamma(k_2^{-1}) dk_1dk_2 \\
\quad a^{-i\lambda} e^{-i(\xi, n_n) b^{-i\lambda} e^{-i(\xi, n_2)}} d\eta d\lambda d\xi
\]
\[
= \int \sum_{\gamma \in SO(4)} d\gamma \int_{SO(4)} \int_{SO(4)} f(ank_1^{-1}) f(bn_2k_2) \gamma(k_2^{-1}) \gamma(k_1^{-1}) dk_1dk_2 \\
\quad a^{-i\lambda} e^{-i(\xi, n_n) b^{-i\lambda} e^{-i(\xi, n_2)}} d\eta d\lambda d\xi
\]
\[
= \int \sum_{\gamma \in SO(4)} d\gamma T\mathcal{F} f(\lambda, \xi, \gamma) \overline{T\mathcal{F} f(\lambda, \xi, \gamma)} d\lambda d\xi = \int \sum_{\gamma \in SO(4)} d\gamma |T\mathcal{F}(\lambda, \xi, \gamma)|^2 d\lambda d\xi
\]
Hence theorem of Plancherel on $SL(4, \mathbb{R})$ is proved.
Let $SP(4, \mathbb{R}) = KNA$ be the Iwasawa decomposition of the symplectic $SP(4, \mathbb{R})$. My state result is

**Corollary 4.1.** For any function $f \in L^1(SP(4, \mathbb{R})) \cap L^2(SP(4, \mathbb{R}))$, we get
\[
\int_{SP(4, \mathbb{R})} |f(v, g)|^2 dv dg = \int \int \sum_{\gamma \in K} d\gamma \|\mathcal{F}_{R^2} T\mathcal{F}(\xi, \lambda, \gamma)\|^2 d\eta d\lambda d\xi
\]
(40)
Which is the Plancherel theorem on the symplectic $SP(4, \mathbb{R})$

5 Plancherel Theorem on Group $\mathbb{R}^4 \rtimes SL(4, \mathbb{R})$

Let $P = \mathbb{R}^4 \rtimes_{\rho} SL(4, \mathbb{R})$ be the 14—dimensional affine group. Let $(v, g)$ and $(v', g')$ be two elements belong $P$, then the multiplication of $(v, g)$ and $(v', g')$
is given by
\[(v, g)(v', g') = (v + \rho(g)(v'), gg') = (v + gv', gg')\] (41)
for any \((v, v') \in \mathbb{R}^4 \times \mathbb{R}^4\) and \((g, g') \in SL(4, \mathbb{R}) \times SL(4, \mathbb{R})\), where \(gv' = \rho(g)(v')\). To define the Fourier transform on \(P\), we introduce the following new group

**Definition 5.1.** Let \(Q = \mathbb{R}^3 \times SL(4, \mathbb{R}) \times SL(4, \mathbb{R})\) be the group with law:

\[
X \cdot Y = (v, h, g)(v', h', g')
= (v + gv', hh', gg')
\] (42)
for all \(X = (v, h, g) \in Q\) and \(Y = (v', h', g') \in Q\). Denote by \(A = \mathbb{R}^4 \times SL(4, \mathbb{R})\) the group of the direct product of \(\mathbb{R}^4\) with the group \(SL(4, \mathbb{R})\). Then the group \(A\) can be regarded as the subgroup \(\mathbb{R}^3 \times SL(4, \mathbb{R}) \times \{I_{SL(4,\mathbb{R})}\}\) of \(Q\) and \(P\) can be regarded as the subgroup \(\mathbb{R}^4 \times \{I_{SL(4,\mathbb{R})}\} \times SL(4, \mathbb{R})\) of \(Q\).

**Definition 5.2.** For any function \(f \in \mathcal{D}(P)\), we can define a function \(\tilde{f}\) on \(Q\) by

\[
\tilde{f}(v, g, h) = f(gv, gh)
\] (43)

**Remark 5.1.** The function \(\tilde{f}\) is invariant in the following sense

\[
\tilde{f}(q^{-1}v, g, q^{-1}h) = \tilde{f}(v, gq^{-1}, h)
\] (44)

**Theorem 5.1.** For any function \(\psi \in \mathcal{D}(P)\) and \(\tilde{f} \in C^\infty(Q)\) invariant in sense (32), we get

\[
\psi \ast \tilde{f}(v, h, g) = \tilde{f} \ast_c \psi(v, h, g)
\] (45)

where \(\ast\) signifies the convolution product on \(P\) with respect the variable \((v, g)\), and \(\ast_c\) signifies the convolution product on \(A\) with respect the variable \((v, h)\)
Proof: In fact for each $\psi \in \mathcal{D}(P)$ and $\tilde{f} \in C^\infty(Q)$, we have

$$
\psi * \tilde{f}(v, h, g)
= \int_{\mathbb{R}^4} \int_{SL(4, \mathbb{R})} \tilde{f}((v', g')^{-1}(v, h, g))\psi(v', g')dv'dg'
= \int_{\mathbb{R}^4} \int_{SL(4, \mathbb{R})} \tilde{f}([g'^{-1}(-v'), g'^{-1})(v, h, g)]\psi(v', g')dv'dg'
= \int_{\mathbb{R}^4} \int_{SL(4, \mathbb{R})} \tilde{f}([g'^{-1}(-v'), g'^{-1})(v, h, g)]\psi(v', g')dv'dg'
= \int_{\mathbb{R}^4} \int_{SL(4, \mathbb{R})} \tilde{f}([g'^{-1}(-v'), g'^{-1})(v, h, g)]\psi(v', g')dv'dg'
= \psi *_c \psi(v, h, g)
$$

(46)

(47)

Corollary 5.1. From theorem 5.1, the equation turns as

$$
\psi * \tilde{f}(v, h, I_G)
= \psi *_c \tilde{f}(v, h, I_{SL(3, \mathbb{R})}) = \int_{\mathbb{R}^2} \int_{SL(3, \mathbb{R})} \tilde{f}[v - v', hg'^{-1}, g]\psi(v', g')dv'dg'
= \int_{\mathbb{R}^4} \int_{SL(4, \mathbb{R})} f[vg'^{-1}(v - v'), hg'^{-1}]\psi(v', g')dv'dg'
= h(f) *_c \psi(v, h)
$$

(48)

where

$$
h(f)(v, g) := f(gv, g)
$$

(49)

Definition 5.3. Let $\Upsilon F$ be the function on $P \times SL(4, \mathbb{R})$ defined by

$$
\Upsilon F(v, (g, k_1)) = F(v, gk_1)
$$

(50)

Definition 5.4. Let $\psi \in \mathcal{D}(P)$ and $F \in \mathcal{D}(P)$, then we can define a
convolution product on the Affine group \( P \) as

\[
\psi * c \Upsilon F(v, (g, k_1))
= \int_{R^4} \int_{SL(4, \mathbb{R})} \Upsilon F(v - v', (gg^{-1}, k_1))\psi(v', g')dv'dg'
= \int_{R^4} \int_{K} \int_{N} \int_{A} F(v - v', kna(k'n'a')^{-1}k_1)\psi(v', k'n'a')dv'dk'dn'da'
\]

where \( g = kna \) and \( g' = k'n'a' \)

**Corollary 5.2.** For any function \( F \) belongs to \( \mathcal{D}(P) \), we obtain

\[
\psi * c \Upsilon h(F)(v, (g, k_1))
= \int_{R^4} \int_{SL(4, \mathbb{R})} \Upsilon h(F)(v - v', (gg^{-1}, k_1))\psi(v', g')dv'dg'
= \int_{R^4} \int_{SL(4, \mathbb{R})} \Upsilon h(F)(v - v', (gg^{-1}, k_1))\psi(v', g')dv'dg'
= \int_{R^4} \int_{SL(4, \mathbb{R})} h(F)(v - v', gg^{-1}k_1)\psi(v', g')dv'dg'
= \int_{R^4} \int_{SL(4, \mathbb{R})} F(gg^{-1}k_1(v - v'), gg^{-1}k_1)\psi(v', g')dv'dg'
\]

**Corollary 5.3.** For any function \( F \) belongs to \( \mathcal{D}(P) \), we obtain

\[
F \ast \Upsilon h(F)^\vee(0, (I_{SL(4, \mathbb{R})}, I_K)) = \int_{R^4} \int_{SL(4, \mathbb{R})} |f(v, g)|^2dgdv = \|f\|^2
\] (51)

**Proof:** If \( F \in \mathcal{D}(P) \), then we get
\[ F \ast \Upsilon h(F)(0, (I_{SL(4,\mathbb{R})}, I_K)) \]
\[ = \int \int_{SL(4,\mathbb{R})} \Upsilon h(F)((0 - v), (I_G^{-1})_{I_S})]F(v, g)dgdv \]
\[ = \int \int_{SO(3) \times \mathbb{R}^3} h(F)((0 - v), I_{G^{-1}I_S})F(v, g)dgdv \]
\[ = \int \int_{SL(4,\mathbb{R})} F[g^{-1}(-v), g^{-1}]F(v, g)dgdv \]
\[ = \int \int_{SL(4,\mathbb{R})} \overline{F}[g^{-1}(-v), g^{-1}]F(v, g)dgdv \]
\[ = \int \int_{SL(4,\mathbb{R})} |F[v, g]|F(v, g)dgdv = \int \int_{SL(4,\mathbb{R})} |f(v, g)|^2 dgdv \]

**Definition 5.5.** Let \( f \in \mathcal{D}(P) \), we define its Fourier transform by

\[ \mathcal{F}_{\mathbb{R}^4}T\mathcal{F} f(\eta, \gamma, \xi, \lambda) = \int \int_{\mathbb{R}^4} \int_{A} \int_{N} \int_{K} f(v, kna)e^{-i(\eta, v)\gamma(k^{-1})a^{-i\lambda}e^{-i(\xi, n)}}dkdvdn \]

where \( \mathcal{F}_{\mathbb{R}^4} \) is the Fourier transform on \( \mathbb{R}^3 \), \( kna = g, \eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{R}^4 \), \( v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4 \), and \( dv = dv_1dv_2dv_3dv_4 \) is the Lebesgue measure on \( \mathbb{R}^4 \) and

\[ \langle (\eta_1, \eta_2, \eta_3, \eta_4), (v_1, v_2, v_3, v_4) \rangle = \sum_{i=1}^{4} \eta_i v_i \]  \hspace{1cm} (52)

**Plancherel’s Theorem 5.2.** For any function \( f \in L^1(P) \cap L^2(P) \), we get

\[ \int_{P} |f(v, g)|^2 dv dg = \int_{\mathbb{R}^4} \sum_{\gamma \in SO(4)} d_\gamma \| \mathcal{F}_{\mathbb{R}^2}T\mathcal{F} f(\eta, \gamma, \xi, \lambda) \|^2 d\eta d\lambda d\xi \]  \hspace{1cm} (53)
Proof: Let $\Upsilon h(\vec{v})$ be the function defined as
\[
\Upsilon h(\vec{v}) (v; (g, k_1)) = h(\vec{F})(v; g k_1)
\]
\[
= \vec{F}(g k_1 v; g k_1) = \vec{F}(g k_1 v; g k_1)^{-1}
\]
then, we have
\[
\int \int \int_{\mathbb{R}^2 \times \mathbb{R}^3} \mathcal{F}_2 \mathcal{F}(\Upsilon h(\vec{F}))(\eta, (I_{SO(4)} h, \lambda, I_{SO(4)})) d\lambda d\xi d\eta
\]
\[
= \int \int \sum_{\gamma \in SO(4)} d_{\gamma} \int_{SO(4)} \Upsilon h(\vec{F})(v, (I_{SO(4)} h, \lambda, I_{SO(4)})) \gamma(\vec{k}_1) d\lambda d\xi d\eta
\]
\[
eq \int \int \sum_{\gamma \in SO(4)} d_{\gamma} \int_{SO(4)} \Upsilon h(\vec{F})(v, (I_{SO(4)} h, \lambda, I_{SO(4)})) \gamma(\vec{k}_1) d\lambda d\xi d\eta
\]
\[
= \int \int \sum_{\gamma \in SO(4)} d_{\gamma} \int_{SO(4)} \Upsilon h(\vec{F})(v, (I_{SO(4)} h, \lambda, I_{SO(4)})) \gamma(\vec{k}_1) d\lambda d\xi d\eta
\]
\[
eq \int \int \sum_{\gamma \in SO(4)} d_{\gamma} \int_{SO(4)} \Upsilon h(\vec{F})(v, (I_{SO(4)} h, \lambda, I_{SO(4)})) \gamma(\vec{k}_1) d\lambda d\xi d\eta
\]
So, we get
\[\begin{align*}
\int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \sum_{\gamma \in \mathcal{S}(4) \cap \mathcal{O}(4)} d_{\gamma} \int \left( \hat{F}(\text{ank}_1 v, \text{ank}_1) \gamma(k_1) dk_1 F(w, k_2 n_2 a_2) \gamma(k_2) dk_2 
\right. \\
\left. e^{-i(\eta, v)} e^{-i(\xi, a_1)} e^{-i(k_1)} e^{-i(k_2)} dk_2 n_2 da_2 d\lambda d\xi d\eta \right)
= \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \sum_{\gamma \in \mathcal{S}(4) \cap \mathcal{O}(4)} d_{\gamma} \int \left( \hat{F}(\gamma(k_1) F(w, k_2 n_2 a_2) \gamma(k_2) \right) \\
\left. dk_1 dk_2 
\right) \\
= \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \sum_{\gamma \in \mathcal{S}(4) \cap \mathcal{O}(4)} d_{\gamma} \int \left( \hat{F}(\gamma(k_1) F(w, k_2 n_2 a_2) \gamma(k_2) \right) \\
\left. \right) \\
= \int_{\mathbb{R}^4} \sum_{\gamma \in \mathcal{S}(4)} d_{\gamma} \left\| \mathcal{F}_{\mathbb{R}^4} T \mathcal{F}(\gamma, \eta, \xi, \lambda) \right\|^2_{\mathcal{H}, \mathcal{S}} d\eta d\lambda d\xi
\end{align*}\]

Hence the theorem is proved on the $\mathbb{R}^4 \rtimes SL(4, \mathbb{R})$.

**Corollary 5.3.** For any function $f \in L^1(\mathbb{R}^4 \rtimes \rho, SP(4, \mathbb{R})) \cap L^2(\mathbb{R}^4 \rtimes \rho, SP(4, \mathbb{R}))$, we get
\[\int_{\mathbb{R}^4 \rtimes \rho, SP(4, \mathbb{R})} |f(v, g)|^2 dv dg = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \sum_{\gamma \in \mathcal{S}(4)} d_{\gamma} \left\| \mathcal{F}_{\mathbb{R}^4} T \mathcal{F}(\eta, \gamma, \xi, \lambda) \right\|^2_{\mathcal{H}, \mathcal{S}} d\eta d\lambda d\xi
\]  

(55)

Which is the Plancherel theorem on the inhomogeneous group $\mathbb{R}^4 \rtimes \rho$, $SP(4, \mathbb{R})$ of the symplectic $SP(4, \mathbb{R})$, where $KNA$ is the Iwasawa decomposition of the symplectic group $SP(4, \mathbb{R})$.

### 6 Hypoellipticity of Differential Operators on the Symplectic

**6.1.** Denote by $SP_N$ the nilpotent symplectic subgroup of the group $SP(4, \mathbb{R})$ consists of all matrices of the form
\[SP_N = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & z - xt & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \right\}, (x, y, z, t) \in \mathbb{R}^4\]  

(56)
We denote by $N$ the nilpotent symplectic subgroup of $SP_N$, formed by the following matrix

$N = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & z & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix}, (x, y, z) \in \mathbb{R}^3 \right\}$ \hspace{1cm} (57)

The group $N$ is isomorphic onto the group $G = \mathbb{R}^3 \rtimes \rho \mathbb{R}$ semidirect of two groups $\mathbb{R}^2$ and $\mathbb{R}$, where $\rho : \mathbb{R} \to Aut(\mathbb{R}^2)$ is the group homomorphism defined by $\rho(x)(z, y) = (z + xy, y)$. The multiplication of two elements $X = (z, y, x)$ and $Y = (c, b, a)$ is given by

$$(z, y, x)(c, b, a) = (z + c + xb - ay, y + b, x + a)$$

$$(z + c + \begin{vmatrix} x & y \\ a & b \end{vmatrix}, y + b, x + a)$$ \hspace{1cm} (58)

Our aim is to prove the solvability and hypoellipticity of the following Lewy operators

$L = (-\partial_x - i\partial_y - 2y\partial_z + 2ix\partial_z) \hspace{1cm} (59)$

$L_* = (-\partial_x + i\partial_y - 2y\partial_z - 2ix\partial_z) \hspace{1cm} (60)$

**Definition 6.1.** One can define a transformation $h : \mathcal{D}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3)$

$h\Psi(z, y, x) = \Psi(z - 2xy, y, -x)$ \hspace{1cm} (61)

It results from this definition that $h^2 = h$.

**Theorem 6.1.** Let $Q = \partial_x - i\partial_y$ be the Cauchy-Riemann operator, then we have for any $f \in C^\infty(\mathbb{R}^3)$

$$(Lf)(z, y, -x) = hQhf(z, y, -x)$$ \hspace{1cm} (62)

For the proof of this theorem see [6].

**Corollary 6.1.** The Lewy operator $L$ is solvable
Proof: In fact the Cauchy-Riemann operator \( Q = \partial_x - i\partial_y \) is solvable, because \( QC^\infty(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \), and \( \hbar C^\infty(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \). So, I have \( LC^\infty(H) = C^\infty(H) \).

**Definition 6.1.** Let \( G \) be a Lie group an operator \( \Gamma : \mathcal{D}'(G) \to \mathcal{D}'(G) \) is called hypoelliptic if

\[
\Gamma \varphi \in C^\infty(G) \implies \varphi \in C^\infty(G)
\]  

for every distribution \( \varphi \in \mathcal{D}'(G) \).

**Theorem 6.2.** The Lewy operator is hypoelliptic

Proof: First the operator \( \hbar \) is hypoelliptic, and the Cauchy-Riemann operator \( \partial_x - i\partial_y \) is hypoelliptic. So if \( \varphi \in \mathcal{D}'(\mathbb{R}^3) \) and if \( L\varphi(z, y, -x) = \hbar Q\varphi(z, y, -x) \in C^\infty(\mathbb{R}^3) \), then I get

\[
L\varphi \in C^\infty(\mathbb{R}^3) \implies \hbar Q\varphi \in C^\infty(\mathbb{R}^3) \\
\implies Q\hbar \varphi \in C^\infty(\mathbb{R}^3) \implies \hbar \varphi \in C^\infty(\mathbb{R}^3) \\
\implies \varphi \in C^\infty(\mathbb{R}^3)
\]

**Theorem 6.3.** Let \( Q_* \) be the operator

\[
L_* = (-\partial_x + i\partial_y - 2y\partial_z - 2ix\partial_z)
\]

\[
Q_* = \partial_x + i\partial_y
\]

then for every \( \varphi \in C^\infty(\mathbb{R}^3) \), I have

\[
\hbar (\partial_x - i\partial_y)(\partial_x + i\partial_y)h\varphi(z, y, -x) = \hbar \Delta h\varphi(z, y, -x)
\]

\[
= \left[(-\partial_x - 2y\partial_z) + (-i\partial_y + 2ix\partial_z)((-\partial_x - 2y\partial_z) + (i\partial_y - 2ix\partial_z))\varphi(z, y, -x)
\]

\[
= \Delta L_* \varphi(z, y, -x)
\]

where \( \Delta \) and \( L_* \) are the operators

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

\[
L_* = (i\partial_y - 2ix\partial_z) + (-\partial_x - 2y\partial_z)
\]

\( L_* \) is called the conjugate of the Lewy operator, which can be considered another form of the Lewy operator. As in theorem 6.2, we can easily see
that \( L \ast C^\infty(\mathbb{R}^3) = C^\infty(\mathbb{R}^3) \). The operator \( LL \ast \) can be regarded as the square of the Lewy operator on the 3–dimensional Heisenberg group.

**Corollary 6.1.** The operators \( LL \ast \) and \( L \ast \) are hypoelliptic

**Proof:** From the above we deduce the following

\[
L \ast \phi \in C^\infty(\mathbb{R}^3) \implies hQ \ast h\phi \in C^\infty(\mathbb{R}^3)
\implies Q \ast h\phi \in C^\infty(\mathbb{R}^3) \implies h\phi \in C^\infty(\mathbb{R}^3)
\implies \phi \in C^\infty(\mathbb{R}^3)
\]

(70)

In other hand we have

\[
LL \ast \phi \in C^\infty(\mathbb{R}^3) \implies hQQ \ast h\phi \in C^\infty(\mathbb{R}^3)
\implies QQ \ast h\phi \in C^\infty(\mathbb{R}^3) \implies Q \ast h\phi \in C^\infty(\mathbb{R}^3)
\implies h\phi \in C^\infty(\mathbb{R}^3) \implies \phi \in C^\infty(\mathbb{R}^3)
\]

(71)

**Theorem 6.4.** The following left invariant differential operators on \( G \)

\[
y\partial_z + \partial_x + i\partial_y + ix\partial_z
\]

(72)

\[
\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - 2x\frac{\partial}{\partial z} \frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z} \frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial x^2}
\]

(73)

are solvable and hypoelliptic

**Proof:** The solvability results from theorem 6.1. For the hypoellipticity, we consider the mapping \( \Gamma : \mathcal{D}'(G) \to \mathcal{D}'(G) \) defined by

\[
\Gamma \phi(z, y, x) = \phi(z - xy, y, x)
\]

(74)

The operator \( \Gamma \) is hypoelliptic and its inverse is

\[
\Gamma^{-1} \phi(z, y, x) = \phi(z + xy, y, x)
\]

(75)

thus we get

\[
\Gamma(\partial_z + i\partial_y)\Gamma^{-1} \phi(z, y, x) = (y\partial_z + \partial_x + i\partial_y + ix\partial_z)\phi(z, y, x)
\]

(76)

and

23
\[
\Gamma \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Gamma^{-1} \phi(z, y, x) \quad (77)
\]

\[
= \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - 2x \frac{\partial}{\partial z} \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \frac{\partial}{\partial x} + \left(y^2 - x^2\right) \frac{\partial^2}{\partial z^2} \right) \phi(z, y, x) \quad (78)
\]

Since the operators \( \Gamma, \partial_x + i \partial_y, \frac{\partial^2}{\partial x^2} \) and \( \Gamma^{-1} \) are hypoelliptic, then the hypoellipticity of the operators \( (y \partial_z + \partial_x + i \partial_y + ix \partial_z) \) and \( \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2} - 2x \frac{\partial}{\partial z} \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z} \frac{\partial}{\partial x} + \left(y^2 - x^2\right) \frac{\partial^2}{\partial z^2} \) is fulfilled.

**Hormander condition for the hypoellipticity**

By the sufficient condition of the hypoellipticity given by the Hormander theorem [3, page 11], we oblige already quoted the sublaplacian

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4x \frac{\partial}{\partial z} \frac{\partial}{\partial y} - 4y \frac{\partial}{\partial z} \frac{\partial}{\partial x} + 4(y^2 + x^2) \frac{\partial^2}{\partial z^2} \quad (79)
\]

which is hypoelliptic by the Hormander theorem, while the operator

\[
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4x \frac{\partial}{\partial z} \frac{\partial}{\partial y} - 4y \frac{\partial}{\partial z} \frac{\partial}{\partial x} + 4(y^2 + x^2) \frac{\partial^2}{\partial z^2} - 4i \frac{\partial}{\partial z} \quad (80)
\]

is not hypoelliptic because the Hormander condition is not fulfilled. By contrast all our results, which are obtained by above theorems, contradict the Hormander conditions for the solvability and the hypoellipticity.

The basis of the Lie algebra of the group \( N \) is given by the following vector fields \( Z = \frac{\partial}{\partial z}, Y = (x \frac{\partial}{\partial z} + \frac{\partial}{\partial y}), X = (-y \frac{\partial}{\partial z} + \frac{\partial}{\partial x}) \). Since \([X, Y] = 2Z\), and \( X, Y, [X, Y] \) span the Lie algebra of \( N \). Then the Hormander theorem in [5], gives the hypoellipticity of the operator

\[
X^2 + Y^2 = \left(x \frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right)^2 + \left(-y \frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right)^2 \quad (81)
\]

While my results prove the solvability and hypoellipticity operators

\[
X^2 + Y^2 + Z^2 = \left(x \frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right)^2 + \left(-y \frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right)^2 + \frac{\partial^2}{\partial z^2} \quad (82)
\]
As well known the Laplace operator

$$\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

(83)

on the real vector group $\mathbb{R}^3$ is solvable and hypoelliptic. This operator as a left invariant differential on the group $N$ is nothing but the following operator

$$\Delta_{h_1} = \sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} + (x \frac{\partial}{\partial z} + \frac{\partial}{\partial y})^2 + (-y \frac{\partial}{\partial z} + \frac{\partial}{\partial x})^2$$

(84)

and as a right invariant on $N$ is the operator

$$\Delta_{h_2} = \sum_{i=1}^{3} \frac{\partial^2}{\partial z_i^2} + (-x \frac{\partial}{\partial z} + \frac{\partial}{\partial y})^2 + (y \frac{\partial}{\partial z} + \frac{\partial}{\partial x})^2$$

(85)

where $\Delta_{h_1}$ (resp. $\Delta_{h_2}$) is the left (resp. right) invariant differential operator associated to $\Delta$. The operators $\Delta_{h_1}$ and $\Delta_{h_2}$ can be regarded as the Laplacian operators on the 3–dimensional Symplectic Nilpotent group $N$.

My aim result is

**Theorem 6.4.** The Laplace operators $\Delta_{h_1}$ and $\Delta_{h_2}$ on the Heisenberg group are hypoelliptic.

**Proof:** We consider the following mappings from $\mathcal{D}'(N) \to \mathcal{D}'(N)$ defined by

$$\Lambda \Psi(z, y, x) = \Psi(z + xy, y, x)$$

(86)

$$\tau \Psi(z, y, x) = \Psi(z + xy, -y, x)$$

(87)

$$\pi \Psi(z, y, x) = \Psi(z + xy, y, -x)$$

(88)

These operators have the property of hypoellipticity, because if $\Lambda \Psi(z, y, x) = \Psi(z + xy, -y, x) \in C^\infty(N)$, then $\Psi(z, y, x) \in C^\infty(N)$, so on $\tau$ and $\pi$. In other side we have

$$\tau \Delta \Lambda \Psi(z, y, x) = \Delta_{h_1} \Psi(z, -y, x)$$

(89)

$$\pi \Delta \Lambda \Psi(z, y, x) = \Delta_{h_2} \Psi(z, y, -x)$$

(90)

Since $\Delta$, $\Lambda$, $\tau$ and $\pi$ are hypoelliptic, then the hypoellipticity of $\Delta_{h_1}$ and $\Delta_{h_2}$ are accomplished.
7 On the Existence Theorem on $N$

Out of the proofs of my book [6], I solve here by different method the equation

$$PC^\infty(G) = C^\infty(G)$$

For this, I introduce two groups: The first is the group $G \times \mathbb{R}$, which is the direct product of the group $G$ with the real vector group $\mathbb{R}$. The second is the group $E = \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ with law:

$$g \cdot g' = (X, x, y)(X', x', y') = (X + X' + yX', x + x', y + y')$$  \hspace{1cm} (91)

for all $g = (X, x, y) \in \mathbb{R}^4, g' = (X', x', y') \in \mathbb{R}^4, X \in \mathbb{R}^2$ and $X' \in \mathbb{R}^2$. In this case the group $G$ can be identified with the closed sub–group $\mathbb{R}^2 \times \{0\} \times \mathbb{R}$ of $E$ and the group $A = \mathbb{R}^2 \times \mathbb{R}$, direct product of the group $\mathbb{R}^2$ by the group $\mathbb{R}$ with the closed sub–group $\mathbb{R}^2 \times \mathbb{R} \times \{0\}$ of $E$.

**Definition 7.1.** For every $\phi \in C^\infty(G)$, one can define a functions $\tau \phi$belong to $C^\infty(G \times \mathbb{R})$, and $\iota \phi$ belong to $E$, as follows:

$$\tau \phi(X, x, y) = \phi(x^{-1}X, x + y)$$  \hspace{1cm} (92)

$$\iota \phi(X, x, y) = \phi(xX, x + y)$$  \hspace{1cm} (93)

for any $(X, x, y) \in G \times \mathbb{R}^m$. The functions $\tau \phi$ and $\iota \phi$ are invariant in the following sense

$$\tau \phi(kX, x + k, y - k) = \phi(z, x, y)$$  \hspace{1cm} (94)

$$\iota \phi(kX, x - k, y + k) = \phi(z, x, y)$$  \hspace{1cm} (95)

Now, I state my theorem

**Theorem 7.1.** Let $P$ be a right invariant differential on $G$, and let $u$ be the distribution associated to $P$. Then the equation

$$P\phi(X, x) = u \ast \phi(X, x) = \int_G \phi((w, v)^{-1}(X, x)u(X, x)dwdv = \varphi(X, x)$$  \hspace{1cm} (96)

has a solution $\phi \in C^\infty(G)$, for any function $\varphi \in C^\infty(G)$, where $\ast$ signifies the convolution product on $G$.  

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Proof: Consider the operator $P$ as a differential operator $Q$ on the abelian group $A = \mathbb{R}^2 \times \{0\} \times \mathbb{R}$. By the theory of partial differential equations with constant coefficients on $\mathbb{R}^2 \times \mathbb{R}$, then for any function $g \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$, there exist a function $\psi$ on $\mathbb{R}^2 \times \mathbb{R}$, such that

$$Q\psi(X, x) = u_\ast \psi(X, x) = \int_{\mathbb{R}^3} \psi(X - a, x - b)u(X, x)dadu = g(X, x) \quad (97)$$

Using the extension of the function $\psi$ on the group $G \times \mathbb{R}$, then for each $f \in C^\infty(\mathbb{R}^2 \times \mathbb{R})$, I get

$$= (u_\ast \tau\psi)(X, 0, y) \downarrow_A = f(X, y) \quad (98)$$

Let $\tau f$ be the extension of the function $f$ on the group $G \times \mathbb{R}$, that means

$$Q\tau\psi(X, 0, y) \downarrow_A = (u_\ast \tau\psi)(X, 0, y) \downarrow_A = \tau f(X, 0, y) \downarrow_A = f(X, y) \quad (99)$$

where

$$= (u_\ast \psi)(X, y) = \int_{\mathbb{R}^3} \psi(X - a, y - b)u(X, y)dadu \quad (100)$$

$$= \tau f(X, 0, y) \downarrow_A = f(X, y) \quad (101)$$

Let $\top_x$ be the right translation of the group $G$, which is defined as

$$= \top_x \Psi(X, t) = \Psi((X, t)((0, x)) = \Psi(X, t + x) \quad (102)$$

Then I have $\tau\psi$ is the solution of the equation

$$(u_\ast \tau\psi)(X, x, 0) \downarrow_G = f(x^{-1}X, x) \quad (103)$$

In fact, we have

$$= \top_x (u_\ast \tau\psi)(X, 0, 0)$$

$$= (u_\ast \tau\psi)(X, x, 0) \downarrow_G = (u_\ast \tau\psi)(X, x, 0)$$

$$= \top_x \tau f(X, 0, 0) \downarrow_G = \tau f(X, x, 0) = f(x^{-1}X, x) \quad (104)$$

So I get, if $\psi$ is the solution of the equation on the abelian group $A = \mathbb{R}^2 \times \mathbb{R}$

$$(Q\psi)(X, y) = f(X, y) \quad (105)$$
on the abelian group \( A = \mathbb{R}^2 \times \mathbb{R} \), then the function \( \tau \psi \) is the solution of the equation
\[
(P \tau \psi)(X, x) = f(x^{-1}X, x)
\] (106)
on the group \( G \). Let \( \tilde{\psi}(X, x) \) be the function, which is defined as
\[
\tilde{\psi}(X, x) = \psi(xX, x)
\] (107)
In the same way, I have proved by in [6], if \( \tilde{\psi}(X, x) \) is the solution of the equation
\[
Q \tilde{\psi}(X, x) = \tilde{\varphi}(X, x)
\] (108)
on the group \( A \), then the function \( \psi \) is the solution of the equation
\[
P\psi(X, x) = \varphi(X, x)
\] (109)
on the group \( G \).

**Corollary 7.1.** The Lewy equation is solvable in the sense, for any \( g \in C^\infty(\mathbb{R}^3) \) there is a function \( f \in C^\infty(\mathbb{R}^3) \), such that
\[
L = (-\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} + 2i(x + iy)\frac{\partial}{\partial z})f = g
\] (110)
The Lewy equation is invariant on the 3-dimensional nilpotent symplectic group \( N = \mathbb{R}^2 \rtimes \rho \mathbb{R} \). So it is solvable.

**The Example of Hormander for the non solvability**

Hormander had considered in his book [16, p.156], another form of the Lewy operator, which is
\[
P(x, D) = (i\partial_x + \partial_y - 2x\partial_z - 2iy\partial_z)
\] (111)
He constructed his example the operator of real variable coefficients, which is
\[
Q(x, D) = P(x, D)P(x, D)P(x, D)P(x, D)P(x, D)
\] (112)
and proved \( Q(x, D) \) is unsolvable see [16, p164], where \( P(x, D) \) is the operator defined by
\[
P(x, D) = (i\partial_x + \partial_y - 2x\partial_z + 2iy\partial_z)
\] (113)
My result is:

**Theorem 7.2.** The operator \( Q(x, D) \) is solvable
**Proof:** Let \( R \) be the following Cauchy-Riemann operator

\[
R = -i \partial_x + \partial_y
\]  
(114)

and let \( \phi \) be any function infinitely differentiable on \( \mathbb{R}^3 \), then we get

\[
\begin{align*}
\hbar(-i \partial_x)h\phi (z, y, -x) &= -i \partial_x h\phi(z + 2yx, y, x) \\
&= (\frac{d}{dt})_0 h\phi (z + 2yx, y, x + t) \\
&= (\frac{d}{dt})_0 \phi (z - 2yt), y, -x - t) \\
&= (-i \partial_x - 2yi \partial_z) \phi (z, y, -x)
\end{align*}
\]
(115)

and

\[
\hbar(\partial_y)h\phi (z, y, -x) = \partial_y h\phi (z + 2xy, y, x)
\]
\[
\begin{align*}
&= (\frac{d}{ds})_0 h\phi (z + 2yx, y + s, x) \\
&= (\frac{d}{ds})_0 \phi (z - 2sx, y + s, -x) \\
&= (\partial_y - 2x \partial_z) \phi (z, y, -x)
\end{align*}
\]

So, we get

\[
(P(x, D)\phi)(z, y, -x) = (-i \partial_x + \partial_y - 2x \partial_z - 2iy \partial_z)\phi = \hbar R \phi(z, y, -x)
\]
(116)

In the same manner, I prove

\[
(P(x, D)\phi)(z, y, -x) = (i \partial_x + \partial_y - 2x \partial_z + 2iy \partial_z)\phi = R_* \hbar \phi(z, y, -x)
\]
(117)

where \( R_* \)

\[
R_* = i \partial_x + \partial_y
\]
(118)

Finally, I find

\[
((P(x, D)(P(x, D)\phi)(z, y, -x) = \hbar R_* R \phi(z, y, -x)
\]
(119)

\[
((P(x, D)(P(x, D))\phi)(z, y, -x) = \hbar RR_* \phi(z, y, -x)
\]
(120)
\[
(((P(x, D)P(x, D)P(x, D))))(P(x, D)\phi)(z, y, -x) = hRR_*R_*Rh\phi(z, y, -x) = Q(x, D)\phi(z, y, -x)
\] (121)

Hence the solvability of the operator \(Q(x, D)\). Also the operator

\[
X + iY - 4iZ = ix\frac{\partial}{\partial z} + iy\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} - 4i\frac{\partial}{\partial z}
\] (122)

is solvable. So the invalidity of the Hormander condition for the non solvability

\section{Conclusion}

\subsection{Any invariant differential operator has the form}

\[
P = \sum_{\alpha,\beta} a_{\alpha,\beta} X^\alpha Y^\beta
\] (123)

on the Lie group \(G = \mathbb{R}^2 \times \rho \mathbb{R}\), where \(X^\alpha = (X_1^{\alpha_1}, X_2^{\alpha_2}), Y^\beta, \alpha_i \in \mathbb{N} \in \mathbb{N}\) (1 \(\leq i \leq 2\) and \(X = (X_1, X_2)\), are the invariant vectors field on \(G\), which are the basis of the Lie algebra \(g\) of \(G\) and \(a_{\alpha,\beta} \in \mathbb{C}\). Any invariant partial differential equation on the \(3\)-dimensional group \(G = \mathbb{R}^2 \times \rho \mathbb{R}\) is solvable. So the invalidity of the Hormander condition for the non existence.

Over fifty years ago where there are a lot of books and lot of published papers by many mathematicians as \([2, 5, 9, 17, 18, 20, 21, 25]\), are all based on a non careful mathematical ideas. Especially those research published after 2006 the date of opening my new way in Fourier analysis on non abelian Lie groups. Unfortunately, some of those research books and articles were published in the famous scientific centers such as Springer \([3, 16]\), Elsevier \([23]\), AMS \([22, 27]\), Wiley \([26]\), Francis & Taylor \([1]\), etc.

\subsection{Open questions.} The operator \([3, p.2]\)

\[
(y^2 - z^2)\frac{\partial^2 u}{\partial x^2} + (1 + x^2)\left(\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2}\right) - xy\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 (xyu)}{\partial x \partial y} + xz\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 (xyu)}{\partial x \partial z}
\] (124)
can be solved

8.3. Open questions. Is the operator

$$X^2 + Y^2 - 4iZ = \left(x \frac{\partial}{\partial z} + \frac{\partial}{\partial y}\right)^2 + \left(-y \frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right)^2 - 4i \frac{\partial}{\partial z}$$  \hspace{1cm} (125)$$

solvable and hypoelliptic, and the operator

$$L - \alpha i \partial_z = \partial_x + 2y \partial_z + i \partial_y - 2ix \partial_z - \alpha i \partial_z, \alpha \in \mathbb{R}$$  \hspace{1cm} (126)$$

is hypoelliptic

Open Question. Consider the Kannai operators [3, p.5]

$$D_1 = \frac{\partial}{\partial x} + x \frac{\partial^2}{\partial y^2}, \hspace{0.5cm} D_2 = \frac{\partial}{\partial x} - x \frac{\partial^2}{\partial y^2}$$  \hspace{1cm} (127)$$

I believe, the first can be solved on the 3—dimensional Heisenberg group $H$ and the second can be hypoelliptic.

References

[1] E. Barletta , S. dragomir, On Lewy’s Unsolvability Phenomenon, in Complex variables and Elliptic Equations- January 2011, Publisher Francis &Taylor

[2] U. N. Bassey and M. E. Egwe, “Non Solvability of Heisenberg Laplacian by Factorization,” Journal of Mathematical Sciences, Vol. 21, No. 1, 2010, pp. 11-15.

[3] M. Bramanti, An Invitation to Hypoelliptic Operators and Hormander’s Vector Fields, Series: Springer Briefs in Mathematics, 2014.

[4] A. Cerezo and F. Rouviere, (1969)”Solution elemetaire d’un operator differentielle lineare invariant a gauch sur un group de Lie reel compact” Annales Scientiques de E.N.S. 4 serie, tome 2, n°4,p 561-581.

[5] L. Corwin, L.P. Rothschild, Necessary Conditions for Local Solvability of Homogeneous Left Invariant Operators on Nilpotent Lie Groups, Acta Math., 147 (1981), pp. 265–288.
[6] K. El- Hussein., (2015), Abstract Harmonic Analysis on Poincare Space-Time, Book, LAP Lambert Academic

[7] L. Ehrenpreis, Solution of Some Problem Division, (I,II, III) Am. J. Math , vol 76,78,82.

[8] M. E. Egwe, On Some Properties of the Heisenberg Laplacian, Advances in Pure Mathematics, 2012, 2, 354-357.

[9] A. El Hamidi, M. Kirane, Nonexistence Results of Solutions to Systems of Semilinear Differential Inequalities on the Heisenberg Group. Abstract and Applied Analysis 2004, No 2, 2004, 155-164.

[10] G. B. Folland, Subelliptic Estimates and Function Spaces on Nilpotent Lie Groups, Ark. Mat. 13, 1975, 161-207

[11] J. Gallier; (2014), Notes on Differential Geometry and Lie Groups, Department of Computer and Information Science, University of Pennsylvania Philadelphia, PA 19104, USA.

[12] Harish-Chandra; (1952), Plancherel formula for $2 \times 2$ real unimodular group, Proc. nat. Acad. Sci. U.S.A., vol. 38, pp. 337-342.

[13] Harish-Chandra, The Plancherel formula for complex semisimple Lie groups, Trans. Amer. Math. Soc., Vol. 76, No. 3, 1954, 458-528.

[14] S. Helgason., (2005), The Abel, Fourier and Radon Transforms on Symmetric Spaces. Indagationes Mathematicae. 16, 531-551.

[15] S. Helgason, (1984), Groups and Geometric Analysis, Academic Press.

[16] L. Hormander, Linear Partial Differential Operators I, Springer-Verlag, Berlin, 1963.

[17] K. Haouam 1, M. Sfaxi, Non Existence Results of Solutions of Semilinear Differential Inequalities with Temporal Fractional Derivative on the Heisenberg Group, Fractional Calculus & Applied Analysis, V12, No1, 2009.

[18] M, Jleli, Mokhtar Kirane, Bessem Samet, Nonexistence Results for a Class of Evolution Equations in the Heisenberg Group, Fractional Calculus and Applied Analysis Volume 18, Issue 3, Jun 2015.
[19] Y. Kannai, An Unsolvable Hypoelliptic Differential Operator, Israel Journal of Mathematics. September 1971, Volume 9, Issue 3, pp 306-315.

[20] H. Lewy, An Example of a Smooth Linear Partial Differential Operator without Solution, Annals of Mathematics, Vol. 66, No. 2, 1957, pp. 155-158.

[21] N. Lerner, A Tribute to Lars Hormander, Matapli100, 25/4/2013.

[22] D. Müller and M. Peloso, Non-Solvability for a Class of Left-Invariant Second-Order Differential Operators on the Heisenberg Group, Transactions of the American Mathematical Society Volume 355, Number 5, Pages 2047-2064 S 0002-9947(02)03232-4 Article electronically published on December 18, 2002.

[23] D. Müller, Marco M. Peloso, F. Ricci, On the Solvability of Homogeneous Left Invariant Differential Operators on the Heisenberg Group, Journal of Functional Analysis Volume 148, Issue 2, 15 August 1997, Pages 368–383.

[24] B. Malgrange, Existence and Approximation des Solutions des équations aux Derivées Partielles et des équations de Convolution, Ann. Inst. Fourier Grenoble, 6, 271, 1955.

[25] A. Mater, On Solvability of PDEs Studiorum Universita di Bologna, Anno Accademico 2011/2012.

[26] L. Nireberg, F. Treves, Solvability of a first order partial Differential Equations, Communication on Pure and Applied Mathematics, VOL. XVI, 331-336, 1963.

[27] L. P. Rothschild, Local Solvability of Left Invariant Differential Operators on the Heisenberg Group, Proceedings of the American Mathematical Society, Vol. 74, No. 2, 1979, pp. 383-388.

[28] Sundaram Thangavelu, Harmonic Analysis on the Heisenberg Group, 2012, Mathematics books. google.

[29] F. Treves, Linear Partial Differential Equations with Constant Coefficients, Gordan and Breach, 1966.
[30] L. Venieri, Hypoelliptic Differential Operators in Heisenberg Group, Università di Bologna, Anno Accademico 2012/2013.