Inflationary $\alpha$—attractors and $F(R)$ gravity

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We consider a generic class of the so-called inflationary $\alpha$ attractor models and compute the cosmological observables in the Einstein and Jordan frames, of the corresponding $F(R)$—gravity theory. We find that the two sets coincide (to within errors from the use of the slow-roll approximation) for moderate and large values of the number of e-foldings $N$, which is the novel result of this paper, generalizing previous results on the subject (see e.g. Ref. 24). We briefly comment on the possible generalizations of these results.

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1. Introduction

Inflationary cosmology is the main theoretical description of the Early Universe in the context of which it was possible to address and solve the main theoretical problems of the Standard Big Bang description of our Universe (see Refs. 1–3). Other theoretical attempts to solve the Early Universe puzzles include the so-called bouncing cosmological models (see Refs. 4–10). In what follows in this paper we will be comparing our results for the cosmological observables, namely the spectral index of primordial curvature perturbations $n_s$ and the tensor-to-scalar ratio $r$, with those of the Planck observational data.11

Recently an interesting class of models was discovered in Ref. 12, called the $\alpha$—attractors models with the property that the cosmological observables are identical for all the members of the $\alpha$—class, in the large $N$—limit, where $N$ is the number of e-foldings. Subsequently these models were studied more extensively (see Refs. 13–23). Also a recent study is that of Ref. 24 which the present study follows and is a generalization of it.

The above $\alpha$—attractor inflationary potentials have a large flat plateau for large values of the inflaton scalar field and in the small $\alpha$—limit are asymptotically quite similar to the hybrid inflation scenarios.27 Well known inflationary models are special limiting cases of the $\alpha$—attractors models, such as the Starobinsky model.28
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In this paper we compute the cosmological observation parameters of the spectral index of primordial curvature perturbations \( n_s \) and the tensor-to-scalar ratio \( r \) for a more general class of inflationary potentials with the \( \alpha \)–attractors property, than those examined in Refs. [24], [12]. This potential was suggested in [14]. More specifically we compute these cosmological observables in the so-called Einstein frame [23] where the form of the potential in explicitly given in the action functional. Then we find the same cosmological observables [24] in the so-called Jordan frame, where the action functional is that of the corresponding \( F(R) \)–gravity theory (see Refs. [32]–[36]) and we explicitly compute the corresponding \( F(R) \) gravity theory.

Then we want to compare these two sets of values for the cosmological observables in order to test whether the two frames’ descriptions are equivalent observationally. The equivalence of the descriptions in the two frames was explicitly shown in Ref. (25). See however Ref. (26) for an important exception. We thus find that these two sets of observables coincide (to within computational errors from the use of the so-called slow-roll approximation), for moderate and large values of the number of e-foldings \( N \), a result that generalizes those of Ref. [24], in a novel way. This occurs for reasonable values of the set of parameters of the inflationary potential and for small to moderate \( \alpha \)–values. For more references related to inflation in \( F(R) \) gravity theories see Refs. [37]–[42].

This paper in organized as follows: In section 2 we refer to the basic facts about inflationary \( \alpha \)–attractor models and compute the corresponding \( F(R) \)–gravity theory. In section 3 we compute the cosmological observables in the Einstein frame and in section 4 in the Jordan frame. Finally in section 5 we present the results and discuss them.

In this paper we assume that the metric of the cosmological spacetime is that of the flat FRW model

\[
ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]
\]

where \( a(t) \) is the scale factor and \( \dot{H} = \dot{a}/a \) is the Hubble parameter. Also we assume that the connection is a symmetric, metric compatible and torsion-less affine connection, namely the so-called Leci-Civita connection and the corresponding Ricci-scalar curvature is given by

\[
R = 6(\dot{H} + 2H^2)
\]

Finally we use units where \( \kappa^2 = 8\pi G = 1 \) and \( h = c = 1 \).

2. Basics of Inflationary \( \alpha \)–attractors and \( F(R) \) gravity

In this section we introduce the basic discussion concerning the inflationary \( \alpha \)–attractor models, which are classes of inflationary potentials with large flat potential plateaus, and their relation to the cosmological observables. Thus we consider the \( F(R) \) gravity action in the so-called Jordan frame

\[
\hat{S} = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} F(\hat{R})
\]
where \( \hat{g}_{\mu\nu} \) is the metric in the Jordan frame and \( \hat{R} \) the corresponding Ricci scalar curvature. Introducing the auxiliary scalar field \( A \), one can write the above action as\(^{24}\)

\[
\hat{S}_1 = \frac{1}{2} \int d^4x \sqrt{-\hat{g}} \left[ F'(A) (\hat{R} - A) + F(A) \right]
\]

(4)

By varying the action \( \hat{S}_1 \) with respect to \( A \) we obtain \( A = \hat{R} \) and verify thus the equivalence of the actions \(^{3},^{1}\). Making the conformal transformation

\[
\hat{g}_{\mu\nu} = e^{\Phi} g_{\mu\nu}
\]

(5)

and introducing the canonical transformation

\[
\Phi = \sqrt{3} \phi \quad e^{\Phi} F'(A) = 1
\]

(6)

corresponding to Eq. (20) of Ref. \(^{24}\) the action is transformed to \( \hat{S}_1 \rightarrow S_1 \), namely to the action in the Einstein frame

\[
S_1 = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) \right]
\]

(7)

where

\[
V(\Phi) := 2 \left[ \frac{A}{F'(A)} - \frac{F(A)}{(F'(A))^2} \right] = e^{\Phi/\sqrt{3}} \hat{R}(e^{-\Phi/\sqrt{3}}) - \frac{1}{2} e^{2\Phi/\sqrt{3}} F(\hat{R}(e^{-\Phi/\sqrt{3}}))
\]

(8)

Here the dependence of \( \hat{R} \) on \( \Phi \) is found by solving the second of Eqs. (6) with respect to \( A = \hat{R} \). Also we obtain from this

\[
\frac{d\Phi}{d\hat{R}} = -\sqrt{3} \frac{F''(\hat{R})}{F'(\hat{R})}
\]

(9)

and using Eq. (8) we finally obtain (see Eq. (24) of Ref. \(^{24}\))

\[
\hat{R} F' = -2\sqrt{3} \frac{d}{d\Phi} \left[ \frac{V(\Phi)}{e^{2\Phi/\sqrt{3}}} \right]
\]

(10)

In our conventions and notation we have \(-\infty < \Phi \leq 0\) for the scalar field. For example, for the Starobinsky model\(^{28}\) which in our notational conventions is given by

\[
V(\Phi) = \mu^2 (1 - e^{\Phi/\sqrt{3}})^2
\]

(11)

we obtain the corresponding \( F(\hat{R}) \) gravity description as\(^{24}\)

\[
F(\hat{R}) = \hat{R} + \frac{\hat{R}^2}{4\mu^2}
\]

(12)
The above action of Eq. (7) can also occur from an action, in the so-called $\phi$–Jordan frame with a non-canonically coupled scalar field of the form

$$S_n = \int d^4x \sqrt{-g} \left[ R - \frac{\partial \mu \partial \nu \phi}{2 \left( 1 - \frac{\phi}{\alpha} \right)^2} - V(\phi) \right]$$

Making the transformation

$$\frac{d\phi}{\left( 1 - \frac{\phi}{\alpha} \right)} = d\Phi$$

we finally obtain the action of Eq. (7), namely

$$S_1 = \int d^4x \sqrt{-g} \left[ R - \frac{1}{2} (\partial \Phi)^2 - V\left( \sqrt{6} \tanh \left( \frac{\Phi}{\sqrt{6} \alpha} \right) \right) \right]$$

In the present paper we consider the potential $(-\infty < \Phi \leq 0)$

$$V(\Phi) = V_0 \frac{[\tanh(\Phi/\sqrt{6} \alpha)]^{2n}}{[1 - \tanh(\Phi/\sqrt{6} \alpha)]^{2m}}$$

This is a generalization of the potential proposed in Fig. 2 of Ref. (12), where the parameter $\alpha$ is introduced, which is inversely proportional to the curvature of the inflaton Kähler manifold. The parameters $m, n$ are not necessarily integers. As in the previous case of the Starobinsky model the slow-roll regime corresponds to the case of $\Phi \rightarrow -\infty$ where $F_h = 2e^{-\Phi/\sqrt{3}} \gg 1$. The choice of the above potential for our study is partially based on the fact that it is quite generic, possesses a large horizontal flat plateau for large negative $\Phi$–values for the slow-roll inflation and possesses many limiting cases as special cases, for example the Starobinsky model or the Higgs inflationary model and so on. The potential of Eq. (16) is shown in Figs. (1)-(2).

Using the expansions (for $z := e^{\sqrt{2/3} \alpha \Phi} \ll 1$)

$$ (1 - z)^N \simeq 1 - Nz + \frac{N(N - 1)}{2} z^2 $$

$$ \left( \frac{1}{1 + z} \right)^N \simeq 1 - Nz + \frac{N(N + 1)}{2} z^2 $$

we obtain in the slow-roll regime

$$ V(\Phi) \simeq \frac{V_0}{2^{2m}} \left[ 1 - A(n, m)z + B(n, m)z^2 \right] $$

where $A(n, m) := 2(2n - m)$ and $B(n, m) := 8n^2 - 8nm + 2m^2 - m$. From Eq. (16)
we then obtain

\[
\hat{R}F_R - \frac{V_0}{2m} \left[ 4 \left( \frac{F_R}{2} \right)^2 + 2A \left( \sqrt{\frac{2}{\alpha}} - 2 \right) \left( \frac{F_R}{2} \right)^{2-\sqrt{2}} \right] \\
- 4B \left( \sqrt{\frac{2}{\alpha}} - 1 \right) \left( \frac{F_R}{2} \right)^{2-2\sqrt{2}} = 0
\]

(19)

This equation will be needed in the analysis that follows in section 5.

In the main part of the analysis that will follow we will be concerned with the large curvature limit \((\alpha \to 0)\). Also, although the analysis that follows can be generalized to arbitrary values of \(m, n\), from now on we will focus on the specific subclass of models where \(m = 2n\). This is an exceptional and interesting case that is trackable for analytical treatment. Then \(A = 0\), \(B = -2n\) and we have from Eq. (19)

\[
\hat{R} - c_0 F_R - c_1 F_R^{-\delta} = 0
\]

(20)

where \(c_0 := \frac{V_0}{2m}\), \(c_1 := c_0 2^{(\delta+1)} n^{(\delta-1)}\) and \(\delta := 2\sqrt{\frac{2}{\alpha}} - 1 \geq 0\).

3. Cosmological Parameters in the Einstein frame

For the potential of Eq. (16) and for \(m = 2n\) one can calculate the cosmological observables of the spectral index of primordial curvature perturbations \(n_s\) and the scalar-to-tensor ratio \(r\), as they occur in the Einstein frame of Eq. (7), from

\[
n_s^{(E)} = 1 - 6\epsilon + 2\eta
\]

\[
r^{(E)} = 16\epsilon
\]

(21)

where the slow-roll parameters are given by

\[
\epsilon := \frac{1}{2} \left( \frac{V'}{V} \right)^2
\]

\[
\eta := \frac{V''}{V}
\]

(22)

The calculation is exact as it was done in Eqs. (5.2)-(5.4) of Ref. [14], without invoking the slow-roll hypothesis, but only assuming that inflation ends when \(\epsilon \simeq 1\). The results are as follows: We define the number of e-foldings \(N\) as

\[
N := \int_{\Phi_i}^{\Phi_e} \frac{V(\Phi)}{V'(\Phi)} d\Phi
\]

(23)

where \(\Phi_i, \Phi_e\) are the initial and end values of the inflaton scalar field. Along with the definitions \(\xi := \tanh(\Phi/\sqrt{6\alpha})\) and \(g := \sqrt{\frac{4\alpha}{m}}\) we obtain

\[
N = 3\alpha \left[ \frac{\Phi_i - \Phi_e}{\sqrt{6\alpha}} + \frac{1}{1 + \xi_e} - \frac{1}{1 + \xi_i} - \frac{1}{(1 + \xi_e)^2} + \frac{1}{(1 + \xi_i)^2} \right]
\]

(24)
On the other hand from the requirement that inflation ends when $\epsilon \simeq 1$, we obtain from the first of Eqs. (22)

$$
\xi_e = \left[ -(1 + g) + \sqrt{g(g + 2)} \right] < 0 \quad (25)
$$

Now remembering that $y = \tanh^{-1}(x) = \frac{1}{2} \ln \frac{1 + x}{1 - x}$ and observing that in the limit of $\xi_i \to -1 + 0$ the sixth term in Eq. (24) dominates the with respect to the fourth and the second terms, we obtain

$$
N = \frac{3\alpha}{4n} \left[ -\frac{1}{2} \ln \left( \frac{1 + \xi_o}{1 - \xi_o} \right) + \frac{1}{(1 + \xi_i)^2} + \frac{1}{1 + \xi_e} - \frac{1}{(1 + \xi_e)^2} \right] \quad (26)
$$

Now using Eq. (25) into Eq. (26) and defining

$$
f(N, n, \alpha) := \frac{4nN}{3\alpha} - \frac{1}{4} \left[ \ln \left( 1 + \frac{2}{g} \right) - \frac{2}{g} \right]
$$

$$
\xi_o := \frac{1}{\sqrt{f}} - 1 \quad (27)
$$

we obtain after a lengthy and careful calculation that

$$
\epsilon = \frac{3n^2 (1 + \xi_o)^4}{\alpha \xi_o^2}
$$

$$
\eta = \left[ \frac{2n(2n - 1)}{6\alpha} \frac{(1 - \xi_o)^2}{\xi_o^2} + \frac{16n^2 (1 + \xi_o)^2}{6\alpha \xi_o} \right] \quad (28)
$$

and the cosmological observation parameters are given by Eq. (21), using Eqs. (28).

4. Cosmological Parameters in the Jordan frame

It can be easily shown that an approximate solution to Eq. (20), valid in the slow-roll regime ($F_R \gg 1$) is given by

$$
F_R = \frac{1}{c_0} \dot{R} - c_1 c_0^{(\delta - 1)} \dot{R}^{\delta}
$$

$$
F(R) = \frac{\dot{R}^2}{2c_0} + \frac{c_2 c_0^{(\delta - 1)}}{(\delta - 1)} \ddot{R}^{1 - \delta} + \Lambda \quad (29)
$$

where $\Lambda$ is a positive cosmological constant. Now varying the action of Eq. (3) we obtain $\dot{R}$ (is given by Eq. (4) where $H = \dot{a}/a$ is the Hubble parameter and we drop hats for simplicity, since it is clear that we work in the Jordan frame of Eq. (3))

$$
6H^2F_R = RF_R - F - 6\dot{H}F_R + (2\dot{H} + 3H^2)F_R = F_{RR}(\dot{R})^2 + 2HF_{RR}\dot{R} + F_{RR}\ddot{R} + \frac{F - RF_R}{2} \quad (30)
$$
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These reproduce Eq. (31) of Ref. (24) in the proper limit. In the slow-roll regime, where $(\dot{H} \ll H^2)$ we may approximate

$$R^{-\delta} = (12H^2)^{-\delta} \left(1 + \frac{\dot{H}}{2H^2}\right)^{-\delta} \simeq \frac{1}{(12H^2)\delta} \left(1 - \delta \frac{\dot{H}}{2H^2}\right)$$

(31)

Using Eqs. (29) we obtain then from the first of Eqs. (30) we obtain after a slightly lengthy calculation

$$36H\ddot{H} + \Lambda c_0 - 18\dot{H}^2 + 108H^2\dot{H} +$$

$$+ \frac{6c_1c_0^\delta}{(12H^2)\delta} \left[\left(\frac{\delta + 1}{\delta - 1}\right)H^2 + \frac{3}{2}\delta\dot{H} - \delta(\delta - 1) \frac{\dot{H}^2}{H^2}\right] = 0$$

(32)

This generalizes Eq. (32) of Ref. (24). It is quite interesting and unlike the case of Eq. (32) of Ref. (24), where in order to obtain a non-trivial solution they had to take the derivative of Eq. (31) in order to obtain Eq. (36) and finally Eq. (37) (see Ref. (24)), that, in our case, an exact solution to Eq. (32) is given by (in the slow-roll regime where $H_1 \ll H_0$)

$$H(t) = H_0 - H_1(t - t_k)$$

(33)

where $H_0$, $t_k$ are arbitrary integration constants of Eq. (32). We assume that the cosmological constant is given by $\Lambda c_0 = 108H_0^2H_1 > 0$. Then the third and the last three terms of Eq. (32) are equated to zero and this gives

$$H_1 = \frac{1}{2k_0} \left[-k_0 + \sqrt{k_0^2 - 4k_0k_2}\right] > 0$$

$$k_0 = 3 \left[1 + \frac{4\delta(\delta - 1)c_1c_0^\delta}{(12H_0^2)^{\delta+1}}\right]$$

$$k_1 = \frac{3}{2} \frac{\delta c_1c_0^\delta}{(12H_0^2)^{\delta+1}}$$

$$k_2 = -\left(\frac{\delta + 1}{\delta - 1}\right)H_0^2 \frac{c_1c_0^\delta}{(12H_0^2)^{\delta+1}}$$

$$c_0 = \frac{V_0}{24n}$$

$$c_1 = c_0n^{(\delta-1)2(\delta+1)}$$

(34)

Specifically the time $t_k$ is assumed to be the time where the horizon crossing for the comoving wavenumber $k = a(t)H(t)$ occurred. For the action of Eq. (3) the cosmological observables corresponding to Eq. (21) are given by (see Refs. (24) and (31))

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4 \simeq 1 - \frac{2\epsilon_1}{H(t)\epsilon_1}$$

$$r = 48\epsilon_1^2$$

(35)
where

$$\epsilon_1 = -\frac{H}{H^2}, \quad \epsilon_2 = 0, \quad \epsilon_3 \simeq \epsilon_1, \quad \epsilon_4 \simeq -3\epsilon_1 + \frac{\dot{H}}{H}\epsilon_1$$

(36)

If we assume that the slow-roll regime (and essentially the inflation also) ends when $\epsilon_1 \simeq O(1)$, at $t = t_f$, so that $H(t_f) \equiv H_f$ we have $H_f = \sqrt{H_1}$ and also $(t_f - t_k) \simeq (H_0/H_1)$. Defining the number of e-foldings of inflation as

$$N = \int_{t_k}^{t_f} H(t)dt$$

(37)

we end up with

$$(t_f - t_k) \simeq \frac{2N}{H_0}$$

(38)

Thus finally the cosmological observables of Eq. (35), as they occur in the Jordan frame, are given by

$$n_s^{(J)} \simeq 1 - \frac{4H_1}{\left(H_0 - 2H_1N/H_0\right)^2}$$

$$r^{(J)} \simeq \frac{48H_1^2}{\left(H_0 - 2H_1N/H_0\right)^4}$$

(39)

Although these equations are similar in form with Eq. (50) of Ref. (24), they are essentially different in the fact that they depend additionally on the parameter $\alpha$ and on the freely specified parameter $H_0$. Hereafter we choose $H_0 = 1$. In all the numerical examples below we found that in Eq. (34), $H_1 \leq 0.02$ in practically all the cases, so that the slow-roll approximation ($H_1 \ll H_0$) is indeed satisfied in all of the relevant cases.

5. Discussion

In this section we are ready to compare the observational indices as they occur in the Einstein and Jordan frames of Eqs. (21) and Eqs. (39) respectively. In order to make compatible our results with Eqs. (15), (22) of Ref. (11) namely

$$n_s = 0.9645 \pm 0.0049$$

$$r < 0.10$$

(40)

we choose to have $n_s^{(E)}(N = 60) = 0.9645$ in Fig. 3. This was easily achieved by choosing $\alpha = 0.0625$, as it is referred to in the caption of Fig (3). This is shown by the black horizontal line. Also we observe that the two curves are practically asymptote to this value for all values of $N \geq 50$ (to within errors due to the use of the slow-roll approximation), namely the two sets of observables coincide. This is the main result of the present paper that generalizes the results of Ref. (24). The curves of Fig. 3 as has been referred to, correspond to the case of $\alpha = 0.0625$, $n = 1$. This is in our
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viewpoint a novel result, unlike the case of Ref. (24), where the observational indices in the two frames coincide only in the large-$N$ limit and in the small-$\alpha$ limit. In our case this happens also for $\alpha \leq 0.1$ also for moderate $N-$values. In the same manner we obtain also Fig. (4) where we find that $r^{(E)}(N = 60) \simeq 0.00072038 < 0.10,$ for the parameter value of $V_0 = 45.$ When the value of the $\alpha-$parameter increases however, (namely for $\alpha = 0.125$) we obtain the curves of Figs. (5)-(6). Here the observational indices in the two frames do not asymptote to a common value unless the number of e-foldings is larger than $N \geq 300.$ Therefore the two sets of observational indices, corresponding to the two frames coincide for practically all relevant values of the number of e-foldings $N,$ when $\alpha \leq 0.1$ and $n \simeq 1.$ This is also evident from Figs. (7)-(8). It is quite possible, although difficult to ascertain analytically, that without invoking the condition $m = 2n$ in the potential of Eq. (16), equivalence of the two frame descriptions would occur for an even vaster range of parameter values.

Regarding the crucial issue of whether these attractors and the observational indices connected with them can be used to distinguish between the two frames (namely the Einstein and Jordan frames) the author considers this to be a very deep question and a definite answer cannot so easily be given. However according to the authors viewpoint and relevant work on this subject (see Refs. 26, 29, 43), in the case considered in this paper, although the two frames are mathematically equivalent, as they are connected by a conformal transformation, there exists a physical non-equivalence of the two frames, and by using the results obtained here, regarding the observational indices, one may be able to distinguish between the two frames.

Since it is quite difficult to obtain a definite answer for the most generic case, regarding the equivalence of the descriptions of the observational indices in the two frames (Einstein and Jordan frames), it would be interesting to try to check this postulate for more realistic and general inflationary potentials, as those for example suggested by appropriate limits of certain supergravity and/or string theory actions (see Ref. (2) and references therein). Work along these lines is in progress.

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Fig. 1. The potential of Eq. (16) for $V_0 = 1$, $n = m = 1$, for various values of the parameter $\alpha$.

Fig. 2. The potential of Eq. (16) normalized to its value at $\Phi \to -\infty$, namely $V_\infty = V_0/2^{4n}$, for $n = 1$ and for various values of the parameter $m$. 
Fig. 3. The spectral index of primordial curvature perturbations $n_s$, in the Einstein and Jordan frames, as a function of the number of e-foldings $N$, namely that of Eqs. (21) and (39), for $n = 1, m = 2n, \alpha = 0.0625$. In black it is shown the constant value of Eq. (40).

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Fig. 4. The tensor-to-scalar ratio $r$, in the Einstein and Jordan frames, as a function of the number of e-foldings $N$, namely that of Eqs. (21) and (39), for $n = 1$, $m = 2n$, $\alpha = 0.09$.
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Fig. 5. The spectral index of primordial curvature perturbations $n_s$, in the Einstein and Jordan frames, as a function of the number of e-foldings $N$, namely that of Eqs. (21) and (39), for $n = 1, m = 2n, \alpha = 0.125$. In black it is shown the constant value of $n_{sa} = 0.995$.

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Fig. 6. The tensor-to-scalar ratio $r$, in the Einstein and Jordan frames, as a function of the number of e-foldings $N$, namely that of Eqs. (21) and (39), for $n = 1$, $m = 2n$, $\alpha = 0.125$.

Fig. 7. The spectral index of primordial curvature perturbations $n_s$, in the Einstein and Jordan frames, as a function of the parameter $\alpha$, namely that of Eqs. (21) and (39), for $N = 60$ and $n = 1$, $m = 2n$. 
Fig. 8. The spectral index of primordial curvature perturbations $n_s$, in the Einstein and Jordan frames, as a function of the parameter $n$, namely that of Eqs. (21) and (39), for $m = 2n$, $N = 80$ and $\alpha = 0.09$. 