Causal horizons in a bouncing universe

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Abstract

As our understanding of the past in a bouncing universe is limited, it becomes difficult to propose a cosmological model which can give some understanding of the causal structure of the bouncing universe. In this article we address the issue related to the particle horizon problem in the bouncing universe models. It is shown that in many models the particle horizon does not exist, and consequently the horizon problem is trivially solved. In some cases a bouncing universe can have a particle horizon and we specify the conditions for its existence. In the absence of a particle horizon the Hubble surface specifies the causal structure of a bouncing universe. We specify the complex relationship between the Hubble surface and the particle horizon when the particle horizon exists. The article also addresses the issue related to the event horizon in a bouncing universe. A toy example of a bouncing universe is first presented where we specify the conditions which dictate the presence of a particle horizon. Next we specify the causal structures of three widely used bouncing models. The first case is related to quintom matter bounce model, the second one is loop quantum cosmology based bounce model and lastly $f(R)$ gravity induced bounce model. We present a brief discussion on the horizon problem in bouncing cosmologies. We point out that the causal structure of the various bounce models fit our general theoretical predictions.

1 Introduction

The issue of causality is at the heart of relativistic physics. In cosmology, where it is assumed that we live in an expanding universe modelled on the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the causal nature of the universe is understood by the properties of the particle horizon and the event horizon. In Big-Bang cosmology it is assumed that there is a finite age of the universe and intuitively one can visualize that during this time light has travelled only a finite region and consequently Big-Bang cosmology predicts a calculable particle horizon for any observer in the present universe. On the other hand if there was no Big-Bang to start with, as in non-singular bouncing cosmological models [1,2], then it becomes very difficult to say whether a particle horizon exists for any observer. In this article we will show that for non-singular bouncing cosmologies, where a contracting phase precedes the expansion phase there may exist certain conditions which dictate when a particle horizon will exist. Like the particle horizon the event horizon plays an important role in gravitational theories. In general thermodynamic behavior of a system can be linked with the event horizon as is done in black hole physics. Some authors have even tried to associate thermodynamic properties of the universe with the cosmic event horizon [3]. In the present article we will show that some of our examples of bouncing cosmologies do have event horizons.

Except the particle and event horizons the Hubble radius also plays an important role [4,5] in determining the causal structure of the universe. The Hubble radius defines the Hubble sphere and the surface of the Hubble sphere is called the Hubble surface. Many authors use Hubble horizon to describe the boundary of the Hubble sphere although in this article we will not associate the word horizon with the Hubble surface. Later we will see that in bouncing
cosmologies the Hubble surface affects the causal structure of spacetime in a very subtle way. A general discussion illuminating the relationship of the particle horizon and the Hubble surface, in expanding spacetime, can be found in Ref. [6]. The referred article addresses various misconceptions related to the actual role of the Hubble surface.

Although the concepts of the particle horizon, event horizon and the Hubble surface are commonly used and well discussed topics in Big-Bang cosmology [7–9] they are rarely discussed in the bouncing universe paradigm where a contracting phase of the universe changes course and starts to expand again giving rise to the expanding universe we observe. The non-singular bouncing models are interesting because they do not contain any singularities [1,2,10,13]. Once one includes a contracting universe, a universe which does not have any initial time to start, the concept of the particle horizon becomes much more involved. Some discussion on the causality issue in bouncing cosmological scenario is included in Ref. [14]. In a bouncing universe the scale-factor of the FLRW metric is not a power law function of time during the bounce and consequently the difference between Hubble radius and the particle horizon distance (if particle horizon exists) diverge maximally near the bounce point. Before we end the our discussion on the causal structure of purely bouncing models we want to briefly specify that there are certain models of the early universe which employs a non-singular bounce as well as cosmological inflation. In Ref. [15] the authors specify the necessity of inflation after bounce and in Ref. [16] the author explicitly gives a model of matter bounce followed by inflation. We will show in our article that matter bounce models, where the earliest phase of the universe during the contraction phase was matter dominated, lack particle horizons and in these models the whole of the universe can be in causal contact. In matter bounce models one does not require inflation to solve the horizon problem (the problem is solved) but one may require inflation for other cosmological purpose. In this article we will in general solely concentrate on the causal structure of bouncing models which do not include inflation in the beginning of the expansion phase. The material in the article is presented in the following way. In the next section we describe the preliminary theoretical tools which we will employ throughout the article. In section 3 we quantitatively define the concepts of the horizons and the Hubble radius. In section 4 we specify a toy bounce model where the bouncing universe accommodates two phase transitions during which the scale-factors of the metric change. In the subsequent section 5 we present three kind of bouncing scenarios widely used by authors to design a cosmological bounce. The first scenario deals with quintom matter bounce, the second one with loop quantum cosmology induced bounce and the third one is related to $f(R)$ gravity induced cosmological bounce. In section 6 we address the horizon problem in bouncing cosmologies. We discuss about the results obtained in this article in section 7 and finally conclude the paper in the subsequent section.

2 Requirements of a cosmological bounce

In this article we will use the homogeneous and isotropic FLRW spacetime. We will particularly work with the spatially flat FLRW metric and its form in the spherical polar coordinates is given by

$$ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta \, d\phi^2) \right] , \quad (1)$$

where $r$ is comoving radial coordinate. We are using units where the velocity of light $c = 1$. Here $a(t)$ is the scale-factor for the FLRW spacetime. The Einstein equation in the cosmological setting can be expressed in terms of the Hubble parameter $H \equiv \dot{a}/a$, where a dot over a quantity specifies a time derivative of that quantity. The relevant equations are

$$H^2 = \frac{\kappa}{3} \rho , \quad \dot{H} = -\frac{\kappa}{2}(\rho + P) , \quad (2,3)$$

where $\rho$ is the energy density and $P$ is the pressure of matter which pervades the universe. In the above equations $\kappa = 8\pi G$ where $G = 1/M_P^2$, $M_P = 1.2 \times 10^{19}$ GeV, being the Planck mass. We are using units where $\hbar = 1$ where $\hbar$ is the reduced Planck constant. We have assumed that the cosmological constant term to be zero. The energy density and pressure are related as

$$P = \omega \rho , \quad (4)$$

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where the value of $\omega$ specifies a particular barotropic equation of state. During a contraction phase, the above equations always predicts a spacetime singularity as $t \to 0$ (from the negative side) if the matter content of the universe satisfies the strong energy condition (SEC). One can evade the singularity, by introducing a bouncing universe where the scale-factor remains finite as $t \to 0$, by violating the null energy condition (NEC) in general relativity. Till now we have specified the cosmological dynamics via the equations of general relativity (GR), later in the article we will discuss $f(R)$ gravity based cosmologies. The dynamical equations like Eq. (2) and Eq. (3) will only get modified in $f(R)$ gravity based cosmology. The other important point regarding $f(R)$ theories is related to the energy conditions. The energy conditions specify a bounce strictly in GR based models and we do not give much importance to the energy conditions in modified gravity theories as in these theories it becomes difficult to formally define the energy conditions.

If one wants to avoid the Big-Bang singularity near $t = 0$ then one can model a universe where the scale-factor $a(t)$ never becomes zero at $t = 0$. In these models the universe contracts during the time $-\infty < t \leq 0$ and expands during $t \geq 0$ and the cosmological bounce happens at $t = 0$ when $a(t = 0) \neq 0$ and $\dot{a}(t = 0) = 0$. The global minima of the scale-factor is at the bounce point. The universe expands after the bounce, implying $\dot{a}$ becomes positive and increases after the bounce. This second condition implies that $\ddot{a}(t = 0) > 0$. The above conditions can also be specified in terms of the Hubble parameter as:

$$H|_{t=0} = 0, \quad \dot{H}|_{t=0} > 0.$$  

The bounce conditions stated above and Eq. (3) immediately shows that in the flat FLRW spacetime, during bounce

$$[\rho + P]|_{t=0} < 0.$$  

The above condition specifies that the NEC has to be violated at the bounce point. The violation of the NEC, in the cosmological perspective, require exotic matter as was inferred in Ref. [17], later [18] also shows how to conceive of NEC violating matter in the early universe. One may even change the gravitational theory to accommodate a cosmological bounce [19, 20]. In this article we will not go into the details of the complexities of theories which generates a cosmological bounce, assuming satisfactory solutions of these difficult issues exist in principle. We will concentrate on the main issue of the article which is related to the causality question in bouncing cosmologies.

3 Particle Horizon, Event Horizon and Hubble Radius in bouncing cosmological models

In the context of, spatially flat, FLRW spacetime let us think of a light ray which travels from a point $(t_i, R, \theta, \phi)$ to $(t_0, 0, \theta, \phi)$. Light travels along a null geodesic and in our particular case we have assumed a null, radial geodesic which serves our purpose. From the line-element we see that for such a null geodesic

$$ds^2 = -dt^2 + a^2(t)dr^2 = 0.$$  

This above equation gives,

$$\int_{t_i}^{t_0} \frac{dt}{a(t)} = R,$$  

which gives the comoving distance between the emitter and the observer. In the above the subscript '$i$' refers to the time of emission of the light signal from the source, and the subscript '0' refers to the time of reception of the light signal by the observer.

If the time $t_0$ is the present cosmological time when the observer is observing the universe, the physical distance to the the emitter in the observers frame will be $R_P(t_0) = a(t_0)R$. The regions from where light could reach the observer at $t = t_0$ forms the region which may have any causal effect on the present condition of the universe and regions outside this region can have no causal effect on the present day universe. All the regions from which light
has reached the present observable universe is enclosed by the causal horizon for the central observer. The causally connected region is enclosed by a 2-dimensional spacelike spherical surface whose radial physical coordinate at \( t = t_0 \) is called \( R_P \), and this spacelike surface is called the particle horizon. In the bouncing universe, \( t_i \to -\infty \), and we can write,

\[
R_P(t_0) \equiv a(t_0) \int_{-\infty}^{t_0} \frac{dt}{a(t)} .
\] (8)

We will use the above definition of the particle horizon throughout this article.

If there exists a finite distance which light can travel in infinite time in the future, emanating from a spatial point at some time, then an event horizon can exist. The event horizon is defined by a spatial two-dimensional spherical surface, whose radius is given by the finite distance travelled by light in infinite time in the future. The formal mathematical definition (of the radius) of the event horizon is:

\[
R_E(t_0) \equiv a(t_0) \int_{t_0}^{\infty} \frac{dt}{a(t)} ,
\] (9)

where again \( t_0 \) is the present time of the observer.

In bouncing cosmology it may happen that \( R_P(t_0) \to \infty \) for some finite \( t_0 \). In that case an observer at any cosmic time can get information about the past universe without any bound and one says that the particle horizon does not exist. Similarly when one says the event horizon does not exist one means that \( R_E(t_0) \to \infty \) for some finite \( t_0 \).

The Hubble radius is the radial coordinate of the boundary of the Hubble sphere which is a closed two dimensional spatial surface at any cosmological time. The Hubble radius is defined as:

\[
R_H(t_0) \equiv \frac{1}{|H(t_0)|}.
\] (10)

The center of the Hubble sphere is located at the observers position. The Hubble sphere does not depend upon the past history or the future of the universe. In the expression of \( R_H \) we have deliberately used the modulus of \( H \) to accommodate a contracting phase of the universe when \( H < 0 \).

From the definitions of the horizons one can deduce

\[
\dot{R}_P = 1 + HR_P ,
\]

\[
\dot{R}_E = -1 + HR_E .
\] (11) (12)

In the Big-Bang paradigm it is seen that \( R_P \) always increases superluminally as \( HR_P \) is positive definite. In the Big-Bang paradigm the rate of increment of the particle horizon is more than the expansion rate of the universe. The galaxies on the particle horizon are receding with a speed \( HR_P \) where as the particle horizon is receding relatively faster and the size of the observable universe increases with time. In the bouncing scenario more interesting things can happen.

In the contracting phase of the universe \( H < 0 \) and Eq. (11) shows that during this time

\[
0 \leq \dot{R}_P < 1 , \quad \text{or} \quad \dot{R}_P < 0 .
\]

The particle horizon distance can increase with time when \( HR_P > -1 \) and the opposite can happen when \( HR_P < -1 \). If \( HR_P = -1 \) at an instant of time, then at that instant \( \dot{R}_P = 0 \) and consequently bouncing cosmologies may have a minimum of the particle horizon distance. In general

\[
\ddot{R}_P = H + R_P(\dot{H} + H^2) .
\]

When \( HR_P = -1 \) the double time derivative of the particle horizon is given by

\[
\ddot{R}_P = - \frac{\dot{H}}{H} ,
\] (13)
which show that $HR_P = -1$ indeed predicts a minima of the particle horizon distance, in the contracting phase when $H < 0$, if $\dot{H} > 0$. The condition $\dot{H} > 0$ holds true near bouncing time and in this article we will see that the minima of the particle horizon distance for most bouncing cosmologies (which admits a finite $R_P$) appear in the contracting phase of the universe near the bounce time. When the particle horizon distance increases with time, during the contraction phase, one must have $|H|R_P < 1$ or

$$0 \leq \dot{R}_P < 1, \quad \text{if } R_P < R_H. \quad (14)$$

Similarly, when the particle horizon distance decreases with time, during the contraction phase, one must have

$$\dot{R}_P < 0, \quad \text{if } R_P > R_H. \quad (15)$$

The above equations show that during the contraction phase of the universe the particle horizon can increase subluminally or may decrease at any rate. On the other hand if the condition $R_P = R_H$ holds for a certain time period then during that period $\dot{R}_P = 0$.

From the above discussion one can predict some properties related to the particle horizon and the Hubble radius. The points are as follows:

1. If the particle horizon distance tends to a constant, non-singular, value as $t_0 \to -\infty$ then the Hubble radius must be equal to the particle horizon distance as $t_0 \to -\infty$.

2. If $R_P(t_0) \to \infty$ as $t_0 \to -\infty$ then in the initial phase, or the full phase, of contraction one must have $R_P > R_H$.

One can easily prove the above statements. If $R_P$ tends to a constant as $t_0$ tends to large negative time then $\dot{R}_P \sim 0$ during a prolonged time period in the far past and consequently $R_P \sim R_H$ as $t_0 \to -\infty$. The second statement can be proved by noting the fact that if $R_P$ is maximum as $t_0 \to -\infty$ then for later times the particle horizon distance can only decrease. When the particle horizon distance decreases with time one must have $R_P > R_H$.

As we see that in bouncing cosmological models the particle horizon distance may not always be increasing in the contracting phase a natural question arises about the fate of wavelengths of cosmological perturbations (or simply length scales). Can a physical wavelength $\lambda_p$ once less than $R_P$ during contraction exceed the particle horizon distance in the future? The answer to this question was given in Ref. [14]. Due to the importance of this question we reproduce the result briefly in this section. Suppose the physical wavelength of the mode at some time $t_c$ is given by $\lambda_p(t_c) = 2\pi a(t_c)/k$ where $k$ is the comoving wave number. Let the particle horizon distance $R_P(t_c)$ be greater than $\lambda_p(t_c)$ at time $t_c$. Once we know this we can now write for any time $t_0 > t_c$

$$\frac{R_P(t_0)}{\lambda_p(t_0)} = \frac{k}{2\pi} \int_{-\infty}^{t_0} \frac{dt}{a(t)} = \frac{k}{2\pi} \int_{-\infty}^{t_c} \frac{dt}{a(t)} + \frac{k}{2\pi} \int_{t_c}^{t_0} \frac{dt}{a(t)},$$

$$= \frac{R_P(t_c)}{\lambda_p(t_c)} + \frac{k}{2\pi} \int_{t_c}^{t_0} \frac{dt}{a(t)}. \quad (16)$$

The first term on the right hand side of the second line is always greater than one by assumption and the term added to it is positive definite. Consequently once a physical length scale is inside the particle horizon it should always be inside the particle horizon. On the other hand a physical length scale once lesser than the Hubble radius may become super-Hubble during the contraction phase. The ratio

$$\frac{R_H(t_0)}{\lambda_p(t_0)} = \frac{k}{2\pi|H(t_0)|a(t_0)}, \quad (17)$$

can change from values greater than one to lesser than one, for some values of $k$, when $R_H$ decreases during the contraction process. These modes again re-enter the Hubble sphere near $t \to 0$ when $R_H \to \infty$. Consequently modes which are sub-Hubble can become super-Hubble during the contraction phase of the universe but such changes cannot happen if the initial modes are smaller then $R_P$. 

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The way the event horizon grows with time is given in Eq. (12). In the contracting phase, where $HR_E$ is negative definite, it can be easily seen that the event horizon will steeply decrease with time. All the above points will become apparent in various bouncing models in the later part of this article. It must be noted that the properties of the horizons and Hubble radius discussed in this section do not have any intrinsic connection with GR, the results discussed in this section remains true even in scalar-tensor theories of gravity.

4 A toy model calculation of the horizons

In this section we present a toy calculation which shows how the causal structures evolve in bouncing cosmologies based on GR. We use the word “toy model calculation” because in this section we do not specify the kind of “matter” which produces the cosmological bounce. We primarily concentrate on the nature of the scale-factor near bounce and simply assume that there must be some exotic matter which induces the bouncing behavior. In the next sections we will present various phenomenological models of cosmological bounce. The discussion in this section will set the paradigm which will be used to interpret later results.

In this section we assume at least three transformations (or phase transitions in the matter sector) in the bouncing universe. The first transformation occurs during the contracting phase of the universe. We assume that the contracting universe was dominated by some form of matter which gave rise to a scale-factor whose functional form was given by a power law. A power law scale-factor cannot lead to a cosmological bounce and consequently we assume that at some time $t < 0$ the nature of the matter content/geometry of the universe changed. During $t < t < t_0$ where $t_0 > 0$, the scale-factor of the universe changed from the power law form and the new scale-factor accommodates a bounce at $t = 0$. Ultimately the universe comes out of the bouncing phase at $t_0$ and the scale-factor again transforms to a power law function. This is a simple description of a cosmological bounce as we know that the scale-factor of the bouncing universe must have changed to a power law form after the bounce. From our understanding of the expanding phase of the universe and symmetry arguments it is natural to think that some power law contraction phase may precede the bouncing phase of the universe. The geometry of the universe changes at $t'$ and $t''$ and in our simplistic model the change happens instantaneously. During such a phase transition we assume that the junction conditions in GR are satisfied such that the scale-factor and its time derivative remains continuous across the junction. The change of the scale-factor in a typical bounce in our toy model is shown in Fig. 1. The parameters specifying the bounce are introduced later in this section.

We will assume that our model universe has the three following scale-factors in the three different phases as:

$$a(t) = \begin{cases} 
    c_0 (-t)^m, & t_i \leq t < t', \\
    a_0 + b_0 t^{2p}, & t' < t < t'', \\
    d_0 t^n, & t'' < t \leq \infty.
\end{cases}$$

In the above equation $t_i (t < 0)$ is assumed to be some initial time which will ultimately tend to $-\infty$. The quantities $m$ and $n$ are positive real constants and $m$ is in general not equal to $n$. The constant $p$ takes positive integer values. Out of the three constants we will assume that $0 < n < 1$ as because in an expanding flat FLRW model this constraint is generally followed. If the scale-factor is given by a power law where $n$ is constrained in the above way then the integral in Eq. (9) diverges and the event horizon does not exist. The coefficients $c_0$, $a_0$, $b_0$ and $d_0$ are also positive real constants out of which $a_0$ normalizes the scale-factor at the bounce time. The Hubble radius during the three phases are:

$$R_H(t_0) = \frac{1}{H(t_0)} = \begin{cases} 
    \frac{|t_0|}{m}, & t_i \leq t_0 < t', \\
    \frac{|a_0 + b_0 t_0^{2p}|}{2b_0 p t_0^{2p-1}}, & t' < t_0 < t'', \\
    \frac{t_0}{\pi}, & t'' < t_0 \leq \infty.
\end{cases}$$

1In this article we do not consider the case where the bouncing universe culminates in an expanding inflationary universe.

2A nice discussion on the junction conditions in general relativity can be found in Ref. [21].
Because of the junction conditions, on the metric, at \( t' \) and \( t'' \) one can easily verify that \( R_H(t_0) \) changes continuously at the junctions. The particle horizon at any time during expansion can be written as:

\[
R_P(t_0) = d_0 t_0^n \left[ \int_{t_i}^{t'} \frac{dt}{c_0(1-t^m)} + \int_{t'}^{t''} \frac{dt}{a_0 + b_0 t^{2p}} + \int_{t''}^{t_0} \frac{dt}{d_0 t^n} \right]. \quad (t'' < t_0 \leq \infty)
\]

In the present case we have to specify the particle horizons in all the three phases of development of the universe as the scale-factors change during these phases. The particle horizon radius during the power law contraction phase is given by

\[
R_P(t_0) = \frac{1}{1-m} \left[ (-t_0)^m (-t_i)^{1-m} + t_0 \right]. \quad (t_i \leq t_0 < t')
\]

Similarly, the particle horizon radius during the bouncing phase is given by

\[
R_P(t_0) = (a_0 + b_0 t^{2p}) \left[ \frac{1}{c_0(1-m)} \left( (-t_i)^{1-m} - (-t')^{1-m} \right) + \int_{t'}^{t_0} \frac{dt}{a_0 + b_0 t^{2p}} \right]. \quad (t' < t_0 < t'')
\]

The expression of the particle horizon distance during the expansion phase is

\[
R_P(t_0) = d_0 t_0^n \left[ \int_{t_i}^{t'} \frac{dt}{c_0(1-t^m)} \left( (-t_i)^{1-m} - (-t')^{1-m} \right) + \int_{t'}^{t''} \frac{dt}{a_0 + b_0 t^{2p}} + \int_{t''}^{t_0} \frac{dt}{d_0 t^{1-n}} \right]. \quad (t'' < t_0 \leq \infty)
\]

We will specify the integral containing the term \( 1/(a_0 + b_0 t^{2p}) \) later, at present we concentrate on the junction conditions. Applying the junction conditions at \( t' \) we get

\[
a_0 = \left( \frac{2p}{m} - 1 \right) b_0 t^{2p}, \quad c_0(-t')^m = \frac{a_0}{1 - (m/2p)}.
\]

Similarly applying the junction conditions at \( t'' \) one gets,

\[
a_0 = \left( \frac{2p}{n} - 1 \right) b_0 t'^{2p}, \quad d_0 t'^{n} = \frac{a_0}{1 - (n/2p)}.
\]
Comparing the above conditions one easily gets
\[
\left( \frac{t''}{t'} \right)^{2p} = \frac{2p - 1}{2p - 1},
\] (26)
which sets a relationship between the matching times and the parameters appearing in the scale-factors. Henceforth whenever we specify the expression of the particle horizon and the Hubble radius we will assume that the constants appearing in those expressions satisfy the above junction conditions.

The expressions of the particle horizon distances, as given in Eqs. (21), (22) and (23), shows that \( R_p(t_0) \) for all the phases is finite (as \( t_i \to -\infty \)) only when \( m > 1 \). Consequently, the existence of the particle horizon in the realistic case depends upon the value of \( m \).

### 4.1 Nature of particle horizon

In this subsection we will consider \( m > 1 \) and assume \( t_i \to -\infty \). In this article we will present the results for two values of \( p \). In the first case \( p = 1 \) and in the second case \( p = 2 \) both of which gives rise to a symmetric bouncing phase.

#### 4.1.1 The case where \( p = 1 \)

When \( p = 1 \) one can write the expressions for particle horizon distance as:

\[
R_p(t_0) = \begin{cases}
\frac{t_0}{1-m}, & (-\infty \leq t_0 < t') \\
\frac{1}{c_0(m-1)}(-t')^{1-m} - \frac{1}{\sqrt{a_0b_0}} \arctan\left(\frac{b_0}{a_0}t'\right) + \frac{1}{\sqrt{a_0b_0}} \arctan\left(\frac{b_0}{a_0}t_0\right), & (t' < t_0 < t'') \\
\frac{1}{a_0(m-1)}(-t')^{1-m} - \frac{1}{\sqrt{a_0b_0}} \arctan\left(\frac{b_0}{a_0}t'\right) + \frac{1}{\sqrt{a_0b_0}} \arctan\left(\frac{b_0}{a_0}t''\right), & (t'' < t_0 \leq \infty)
\end{cases}
\] (27)

Finally we have to choose the constants appearing in the above expressions judiciously such that the junction conditions are satisfied. We take \( a_0 = 1 \) and \( n = 1/2 \) assuming a radiation dominated universe just after the bouncing phase. The time instants where the scale-factors change are assumed to be \( t' = -2 \) and \( t'' = 1 \) in some units. One can indeed express all time variables as multiples of a fiducial time \( t_f \), which can be a relevant microphysical one such as some multiple of the Planck or GUT scale. One may choose \( t_f = 10^8t_p \) where the \( t_p \sim 10^{-43}s \) is the Planck time and \( k > 0 \). Typically \( k \) may be between 3 to 5. Once \( t_f \) is specified all the time labels as \( t' \) or \( t'' \) will actually mean \( t't_f \) or \( t''t_f \). In this paper we will always assume that all time intervals or time labels are actually expressed in terms of the fiducial time unit \( t_f \). From the junction conditions one can now easily obtain

\[
m = \frac{8}{7}, \quad b_0 = \frac{1}{3}, \quad c_0 = \frac{7}{3}2^{-\frac{4}{7}}, \ \text{and} \ \frac{d_0}{a_0} = \frac{4}{3}.
\]

Using these values we can write the particle horizon radius at any phase of evolution of the universe. Particle horizon distance at any time during power law contraction is simply given by

\[
R_p(t_0) = -7t_0.
\] (28)

In this phase \( t_0 < 0 \). Particle horizon distance in the intermediate bouncing phase comes out to be:

\[
R_p(t_0) = \left(1 + \frac{t_0^2}{3}\right) \left[6 + \sqrt{3}\left(\arctan\left(\frac{t_0}{\sqrt{3}}\right) + \arctan\left(\frac{2}{\sqrt{3}}\right)\right)\right].
\] (29)

Finally the particle horizon radius during the expanding phase of the universe is:

\[
R_p(t_0) = \frac{4}{3}t_0^{\frac{1}{2}} \left[6 + \sqrt{3}\left(\arctan\left(\frac{1}{\sqrt{3}}\right) + \arctan\left(\frac{2}{\sqrt{3}}\right)\right) + \frac{3}{2}(t_0^{1/2} - 1)\right].
\] (30)

The particle horizon and the Hubble radius are plotted for the case \( p = 1 \) in Fig. 2. In the plot the Hubble radius is
drawn in blue and the particle horizon is shown by orange dashed curve. The plot shows that most of the time the Hubble sphere is causally connected except very near to the bounce point where the Hubble radius diverge. We have plotted the behavior of the particle horizon distance and the Hubble radius near the bounce point and consequently the plots do not convey the complete information about these distance scales away from the bounce point. As one goes back in time the Hubble radius increases and so does the particle horizon distance. The important feature which comes out of the plot is that the particle horizon decreases initially and then it attains a minimum value near the bounce and increases in the expanding phase.

4.1.2 The case where $p = 2$

In the present case the integral involving the term $1/(a_0 + b_0 t^{2p})$ yields \([22]\)

$$
\int \frac{dt}{a_0 + b_0 t^4} = \frac{\alpha}{4\sqrt{2}a_0} \left[ \ln \left( \frac{t^2 + \sqrt{2}\alpha t + \alpha^2}{t^2 - \sqrt{2}\alpha t + \alpha^2} \right) + 2 \arctan \frac{\sqrt{2}\alpha t}{\alpha^2 - t^2} \right],
$$

where $\alpha = (\frac{a_0}{b_0})^{\frac{1}{4}}$. Using the above result we can write the particle horizon distance in the three cases as:

$$
R_p(t_0) = \begin{cases} 
\frac{t_0}{1-m}, & (-\infty < t < t') \\
\left( a_0 + b_0 t_0^4 \right)^{\frac{1}{2}} \left[ \frac{1}{\cos(m-1)} \left( -t' \right)^{1-m} + \frac{\alpha}{4\sqrt{2}a_0} \left( \ln \left( \frac{t'^2 - \sqrt{2}\alpha t' + \alpha^2}{t'^2 + \sqrt{2}\alpha t' + \alpha^2} \right) \times \frac{t_0^2 + \sqrt{2}\alpha t_0 + \alpha^2}{t_0^2 - \sqrt{2}\alpha t_0 + \alpha^2} \right) + 2 \arctan \frac{\sqrt{2}\alpha t'}{\alpha^2 - t'^2} \right], & (t' < t_0 < t'') \\
d_0 t_0^{n} \left[ \frac{1}{\cos(m-1)} \left( -t'' \right)^{1-m} + \frac{\alpha}{4\sqrt{2}a_0} \left( \frac{t''^2 - \sqrt{2}\alpha t'' + \alpha^2}{t''^2 + \sqrt{2}\alpha t'' + \alpha^2} \right) \times \frac{t_0^2 + \sqrt{2}\alpha t_0 + \alpha^2}{t_0^2 - \sqrt{2}\alpha t_0 + \alpha^2} \right] + 2 \arctan \frac{\sqrt{2}\alpha t''}{\alpha^2 - t''^2} \right], & (t'' < t_0 \leq \infty)
\end{cases}
$$

(31)

In the present we set $a_0 = 1$ and $n = 1/2$ as done in the previous case. The time instants where the scale-factors change are assumed to be the same as in the previous case. The junction conditions now predict

$$
m = \frac{64}{23}, \quad b_0 = \frac{1}{7}, \quad a_0 = \frac{23}{7} 2^{-\frac{2m}{n}}, \quad d_0 = \frac{8}{7}.
$$

The resulting particle horizon distance is plotted in orange dashed curve in Fig.\([2]\). The Hubble radius at each instant is plotted in blue curve. The present plot is qualitatively same as the one in Fig.\([2]\). The only difference between them is that for the case $p = 2$ the radiation dominated universe can have a certain region where the Hubble radius exceeds the particle horizon distance. Both the curves show that the particle horizon follows a smooth curve which has a minima near the bounce point.
In both the above cases we observe that during the contraction phase the Hubble surface lies within the particle horizon. From our discussion in section 3 one can infer that in such cases the particle horizon size must decrease with time.

4.2 Effect of the various parameter choices on the toy bounce model

The generic features of the particle horizon were discussed in section 3. Except the generic features many properties of the cosmological system may depend on parameter choices. We will like to end our discussion on the toy model by presenting a more general discussion on the dependence of the system on various parameters appearing in the model.

From the expression of the scale-factors in the various phases, as given in Eq. (18), we see that our model has in total nine parameters as: $n$, $p$, $m$, $d_0$, $a_0$, $b_0$, $c_0$, $t'$ and $t''$. Most of these parameters have some conditions as $m$, $n$ and $p$ are positive real constants and $t' < 0$ and $t'' > 0$. The definition of the various cosmological phases justifies the above constrains. More over as because the scale-factor cannot be negative we must have $d_0$ and $c_0$ to be positive and real constants. For a proper bouncing behavior near $t = 0$ one must also require $a_0$ and $b_0$ to be positive real constants.

Out of the nine parameters discussed above only five can be independently chosen (all of which satisfies the conditions discussed in the last paragraph) because of the relations appearing in Eq. (24) and Eq. (25). In the results presented we have independently chosen $p$, $a_0$, $n$, $t'$ and $t''$. The non existence of event horizons in our case was a result of choosing $0 < n < 1$. If we had chosen $n > 1$ our models will also have event horizons. We have chosen $a_0 = 1$ for both the cases as this choice normalizes the scale-factor at the bounce point. We will like to use this normalization as no new physics emerges by changing this normalization. Except this choice we have also chosen $p$ to be positive integers as 1 and 2. While in principle $p$ can take non-integer values, we will show below that taking $p$ to be a non-integer value does not alter the physics of the model.

The important thing to be discussed in this section is that the particle horizon exists only when $m > 1$. In this subsection we will see that the choice of values of $t'$ and $t''$ does have very interesting consequences as far as the existence of the horizon is concerned. If we call the ratio $t''/t' = t_r$ then Eq. (26) gives us

$$t_r^{2p} = \left(\frac{n}{m}\right) \frac{2p - m}{2p - n},$$

then it immediately gives some more constrains on the parameters, as $2p > m$ and $2p > n$ simultaneously or $2p < m$ and $2p < n$ simultaneously. More over the condition for the existence of particle horizon translates to

$$t_r^{2p} < n \left(\frac{2p - 1}{2p - n}\right).$$

The above relation shows that $t_r$, $p$ and $n$ must satisfy an inequality if a particle horizon exists in the toy bouncing model. The parameter choices in the previous subsections all satisfy the above conditions.

During the expansion phase in standard (Big-bang) model of cosmology the Hubble radius remains proportional to the particle horizon distance when the scale-factor is a power law function of time. In general the particle horizon distance always remains greater than or equal to the Hubble radius when the power law index (of the scale-factor) is a fraction, in Big-bang cosmologies. For the radiation dominated early universe one gets $R_P = R_H$. For power law expansion phases one never encounters a situation when $R_H$ becomes greater than $R_P$ in standard cosmology. If in the bouncing models of cosmology one demands that the radiation dominated phase just after bounce develops similarly to the standard cosmological models then the $p = 2$ case as shown in our earlier subsection has to be rejected. If one does not reject the cases as studied above then one may obtain interesting results which may have observational signatures. We discuss this issue in the section devoted to the horizon problem in bouncing cosmological models.

One can choose parameters in the model such that $R_H$ never becomes bigger than $R_P$ in the expansion phase of the toy models. The result of one of such parameter choices is shown in Fig. 4. In this case all the parameters except the value of $t'$ remains the same as in the case of sub-subsection 4.1.2. Till no new evidence of bouncing cosmologies are observed from the cosmic microwave background radiation (CMBR) data one may be tempted to
avoid bouncing models where at any time $t_0$ during the expansion phase of the universe one gets $R_P(t_0) = R_H(t_0)$ and $\dot{R}_P(t_0) = \dot{R}_H(t_0)$. In such cases the bouncing models will be further constrained.

5 The horizons in various different bouncing models

After the general discussion on the toy bounce model we will discuss some specific models of cosmological bounce in this section. The first subsection deals with bounce influenced by quintom matter. The second subsection deals with loop quantum cosmology influenced cosmological bounce. In the third subsection we do present a particular bouncing solution in $f(R)$ cosmology. We do not claim that our examples are exhaustive in character but the results presented in this section does indeed reveal the causal structure of some important cosmological bounce models. Unlike the toy bounce model we do not specify various phase transitions of the universe, undergoing cosmological bounce, in the cases discussed. For simplicity we have not separated the exact phases and scale-factors in the examples given in this section although the connection with the results of the toy-model will become apparent and will be pointed out at various positions.

5.1 The causal structure in quintom bounce model

In the quintom bounce model the “matter” in the universe transits from a $\omega > -1$ phase during contraction to the bouncing phase where $\omega < -1$ and NEC is violated. In the initial expanding phase after bounce the equation of state again changes from $\omega < -1$ to $\omega > -1$ and the universe enters a hot Big Bang era after the bounce. There are many ways one can build quintom models, perhaps the simplest model consists of two scalar fields which produces the quintom nature. Out of the two scalar fields one is similar in character to the quintessence field and the other one is similar to the phantom field [23–25].

In the quintom bounce model one can use an equation of state described by

$$\omega = -r - \frac{s}{t^2}, \quad (33)$$

where $t$ is cosmological time. Here $r$ is a dimensionless constant where as $s$ is a dimensional constant [23]. In general $0 < r < 1$ and $s > 0$ such that away from bounce one has $\omega > -1$ and near the bouncing point, $t = 0$, $\omega \ll -1$. 

Figure 4: The case where $p = 2$ but $t' = -3$ and $t'' = 1$. In this case the Hubble radius remains smaller that $R_P$ during the expansion phase away from the bouncing point.
Solving the Friedmann equations in GR using the above equation of state one gets:

\[ a(t) = \left[ t^2 + \frac{s}{1-r} \right]^{\frac{1}{\pi(1-r)}} , \]  

which shows the signature of a non-singular bounce at \( t = 0 \) when the equation of state diverges. If we compare the form of the scale-factor in the quintom bounce case with the bouncing scale-factor in the toy model, given by the middle expression in Eq. (18), we see that near the bounce point the above scale-factor is similar in form to the one given in the toy model with \( p = 1 \). From Eq. (34) we see that as \( t \to -\infty \) the scale-factor behaves as \( a(t) \sim (-t)^m \) where

\[ m = \frac{2}{3(1-r)} , \]

which shows that \( m > 1 \) for \( r > 1/3 \). We know from the toy example that in such a case the universe can admit a particle horizon:

\[
R_P(t_0) = \left( t_0^2 + \frac{s}{1-r} \right)^{\frac{1}{\pi(1-r)}} \int_{-\infty}^{t_0} \frac{dt}{a(t)}
\]

\[ = \left( t_0^2 + \frac{s}{1-r} \right)^{\frac{1}{\pi(1-r)}} \int_{-\infty}^{t_0} \left( t^2 + \frac{s}{1-r} \right)^{-\frac{2}{\pi(1-r)}} dt \]  

Consequently the quintom models may or may not accommodate a particle horizon. The criterion for existence of the particle horizon depends solely upon the numerical value of the dimensionless constant \( r \). In this case the expression of the Hubble parameter is

\[ H(t) = \frac{2}{3} \left[ \frac{t}{(1-r)t^2 + s} \right] , \]

which helps us to specify the nature of the Hubble surface through out the contraction phase of the universe. In Fig. 5 we show the nature of variation of the particle horizon distance and the Hubble radius during the bounce. In this case we have used \( s = 1 \) and \( r = 2/3 \). The causal structure of the quintom matter influenced bounce show that the particle horizon distance, in general, is always greater than the Hubble radius except at a narrow region near the bouncing point where the condition \( R_P = R_H \) is satisfied. The particle horizon radius in Fig. 6 does not say the general nature of all quintom matter influenced cosmological bounces, as there is the other case where \( r < 1/3 \) when the particle horizons does not exist. In those cases the Hubble radius specifies the causal structure of the quintom bounce models. From Fig. 6 we see clearly that the minimum of the particle horizon distance happens when \( R_P = R_H \) during the contraction phase, as discussed in section 3.
5.2 The evolution of the Hubble radius in loop quantum cosmology

Loop quantum gravity (LQG) is a background independent non-perturbative quantized theory of gravity. Loop quantum cosmology (LQC) tries to understand the cosmological evolution, near the initial singularity, of our universe in the light of some simple ideas coming from LQG. In loop quantum cosmology (LQC), spacetime is quantized by using the holonomies of SU(2) group. From the Hamiltonian constraint one can derive the effective equations of motion, guiding the dynamics of the universe, that include the leading order quantum corrections to the classical equations of general relativity. In LQC the evolution of the universe takes place in the following way. The early time singularity is avoided by quantum effects and the universe is guided by quantum laws during this phase. The early quantum epoch takes into account the discreteness of spacetime near singularity, and it was shown that the initial singularity can be avoided to achieve both inflationary and bouncing cosmology ([26], [27], [28]). In LQC the singularity is avoided by a cosmological bounce guided by quantum gravity laws. Next there appears a semiclassical epoch when some of the quantum correlations transform into classical signals. Ultimately the universe transforms into a classical phase.

For semi-classical states the quantum gravity effects are well approximated by a set of effective equations. For the flat FLRW universe the effective Friedmann equations which captures the quantum effects are [29,30],

\[
H^2 = \frac{\kappa \rho}{3} \left(1 - \frac{\rho}{\rho_c}\right),
\]

(37)

\[
\dot{H} = -\frac{\kappa}{2} (\rho + P) \left(1 - \frac{2\rho}{\rho_c}\right),
\]

(38)

where the critical energy density \(\rho_c \sim M_P^4\). In the above effective equations the other terms have their conventional meaning and \(H\) is still defined as the ratio of the time derivative of the scale-factor to the value of the scale-factor. When the energy density of the universe reaches \(\rho_c\) the universe undergoes a cosmological bounce. The matter variables \(\rho, P\) can specify conventional matter as dust or radiation. In both [29,31] the authors talk about matter bounce, where in the long past of the contraction era the universe was filled with pressure-less matter and then there was a phase transition at some negative time (much before bounce) when the universe entered the ekpyrotic phase. The ekpyrotic phase is obtained by using some specific form of scalar field potential during the contraction phase. In [30] we notice that as one moves away from the bounce in the past the scale-factor becomes proportional to \((-t)^{2/3}\) when \(\omega = 0\). Both in the ekpyrotic models and the model in [30] we see far away in the past we have the scale-factor behaving as \((-t)^{2/3}\) and consequently in these LQC motivated matter bounce models we will not have any particle horizon, a point established in our analysis of the toy bounce model study. From Ref. [30] we see that the scale-factor of the
universe in the immediate vicinity of $t = 0$, when $\omega = 0$, is given by
\[ a(t) = \left(1 + \frac{3}{4} \kappa \rho_c t^2\right)^{1/3}, \] (39)
from which we can specify the Hubble radius near about the bounce point. Near $t = 0$ we see that the above scale-factor becomes similar in form to the scale-factor, given by the middle expression on the right hand side in Eq. (18), in the toy model bounce with $p = 1$. In Fig. 6 we plot the effective nature of variation of the Hubble radius near the bounce point. In this plot we have used $\kappa \rho_c = 1$, which becomes natural if one sets $8\pi M_P^2 = 1$. In absence of any particle horizon the Hubble radius sets the causal structure of these cosmologies.

5.3 The causal structure of $f(R)$ theory induced bounces

In $f(R)$ gravity, most of the basics we have discussed in sections 2, 3 remain the same except the dynamical equations of cosmology as given in Eq. 2 and Eq. 3. The new equations replacing them are:
\[ H^2 = \frac{\kappa}{3} (\rho + \rho_{\text{curv}}), \] (40)
\[ \dot{H} = -\frac{\kappa}{f'(R)} (\rho + P + \rho_{\text{curv}} + P_{\text{curv}}), \] (41)

where a prime over a quantity designates a derivative with respect to the Ricci scalar $R$. The curvature dependent energy-density and pressure are
\[ \rho_{\text{curv}} = \frac{R f' - f}{2\kappa} - \frac{3H f'' R}{\kappa}, \] (42)
\[ P_{\text{curv}} = \frac{R f'''}{\kappa} + 2H R f''' + \frac{\dot{R}f'}{\kappa} - \frac{R f' - f}{2\kappa}. \] (43)

The bouncing conditions remain the as in GR. Simple forms of $f(R)$ may accommodate cosmological bouncing solutions. The simplest of them may be when $f(R) = \lambda + R + \alpha R^2$ where $\alpha < 0$ for a bounce. The cosmological bounce in this model where $\lambda = 0$ was studied earlier [32]. Later a thorough analysis of such a bounce was also presented in [19]. Bouncing in quadratic gravity with $\lambda \neq 0$ has been studied in [33]. In the last case the authors note that a solution of the bouncing problem in quadratic gravity with non-zero $\lambda$ can be written as
\[ a(t) = a_0 e^{At^2}, \] (44)

where $a_0$ and $A$ are constants. The quadratic gravity model which accommodates a cosmological bounce is in general gravitationally unstable. The instability arises because of the fact that $f''(R) < 0$ for $\alpha < 0$. There can be another instability, when $f'(R) < 0$ in some domain of the Ricci scalar. One can suitably choose the parameters such that $f'(R) > 0$ during the bouncing regime. The above form of scale-factor can also give a cosmological bounce when $f(R) = \frac{1}{\beta} e^{\beta R}$ where $\beta > 0$ is some constant parameter of the theory. This theory does not have the above mentioned instabilities. There can be many other forms of $f(R)$ which admits Eq. (44) as a solution. In this section we will specify the causal structure of an universe, where the underlying theory of gravity is $f(R)$ gravity and the scale-factor of the universe is as given in Eq. (44). In this case also we notice that the scale-factor used becomes similar in form to the middle expression on the right hand side in Eq. (18) in the toy model bounce with $p = 1$ [4] The functional form of the scale-factor show that the bounce is symmetrical in time. The particle horizon is given as
\[ R_P(t_0) = e^{\frac{A t_0^2}{2}} \int_{-\infty}^{t_0} e^{-At^2} dt, \] (45)

\footnote{It must be noted that in the present case one cannot match the scale-factors at various time instants as was done in the toy bounce model. The matching of scale-factors should be more involved in the present case as the junction conditions in general change in $f(R)$ gravity when compared with the same conditions in GR.}
assumed the scale-factor remained the same till $t \to -\infty$. The integral on the right hand side can be evaluated by using the properties of the Gaussian integrals. After doing the integral one obtains

$$
\int_{-\infty}^{t_0} e^{-At^2} dt = \sqrt{\frac{\pi}{A}} \left[ 1 - \frac{1}{2} \text{erfc}(\sqrt{A} t_0) \right],
$$

(46)

where $\text{erfc}(\sqrt{A} t_0)$ is the complimentary error function. To see how the above integral behaves in the extreme cases where $t \to \pm \infty$ one requires the following properties of the complementary error function,

$$
\lim_{x \to -\infty} \text{erfc}(x) = 2, \quad \lim_{x \to \infty} \text{erfc}(x) = 0.
$$

It must be noted that the above limits saturates near $x = 0$, this information will help us to figure out the behavior of the particle horizon at the extreme limits. With all these information we can now write the expression for particle horizon as:

$$
R_P(t_0) = e^{At_0^2} \sqrt{\frac{\pi}{A}} \left[ 1 - \frac{1}{2} \text{erfc}(\sqrt{A} t_0) \right].
$$

(47)

The limits and their properties of the complimentary error function listed above shows that the value of the particle horizon remains finite at both the extremities of the time variable.

The above expression shows that the particle horizon quickly vanishes as one goes back in negative time and the particle horizon increases indefinitely as time evolves, $t \to \infty$. The plot of the particle horizon, represented in orange dashed curve, is shown in Fig. 7. The above result can be interpreted in a simple way. As the observer moves back in time, $t \to -\infty$, the observer does not receive any light from the other parts of the universe from the past as light emitting particles are infinitely distant from the observer. After a long time the observer first receives light from its past, and a causal connection is made. At this moment, $t_0$, the particle horizon $R_P(t_0)$ comes into account. The amount of light which the observer receives is coming from the closest sources in the past. As time proceeds more and more regions from the past are coming in causal contact with the observer and the process continues forever.

In this case we have $|H| = 2A|t|$ and consequently one can easily see that $R_P(t_0)$ is never proportional to $1/|H(t_0)|$. In the present case we see that near $t \to -\infty$ both $R_P(t_0)$ and $1/|H(t_0)|$ vanishes. At $t = 0$, $R_P(0)$ remains finite whereas $1/|H(t_0)| \to \infty$. The nature of the Hubble surface, plotted in the blue, is shown in Fig. 7. The interesting thing to note is the relative behavior of the Hubble surface and particle horizon before the bounce. In the present model the Hubble radius is always greater than the particle horizon radius before the bounce, unlike the toy bounce model case, and consequently the particle horizon size increases with time. Here $R_H > R_P$ and the two distance scales differ maximally near the bounce point $t = 0$. 

![Figure 7: Plot of Hubble radius represented by continuous blue curve, particle horizon radius in orange dashed curve and event horizon radius in green dotted curve for $a = e^{t^2}$ bounce model. Here $a_0 = 1$ sets the scale-factor at the bounce point and $A$ is assumed to have unit magnitude. For more explanation see the text below.](image)
We know that the scale-factor changes from the exponential form to other forms in the expanding phase of the universe. From a purely mathematical point of view if we assume that the exponential scale-factor keeps the same functional form as 
\[ t \to \infty \] 
then this bouncing universe do admit an event horizon. In this case we have

\[
R_E(t_0) = e^{At_0^2} \int_{t_0}^{\infty} e^{-At^2} \, dt 
= \frac{1}{2} \sqrt{\frac{\pi}{A}} e^{At_0^2} \text{erfc}(\sqrt{A} t_0). 
\] (48)

The behavior of the event horizon is plotted in green in Fig. 7. The plot shows that the event horizon diverges at 
\[ t_0 \to -\infty, \] but in general it has a finite value for all other times and decreases smoothly as time evolves. As predicted in the last section, the event horizon steeply decreases during the contracting phase. In this toy model the event horizon and particle horizon has the same radius at the bounce point \( t = 0 \). The nature of the horizons as plotted in Fig. 7 show an interesting feature. As the bounce is symmetrical in time the particle horizon and the event horizon are symmetrical in time. The particle horizon transforms into the event horizon if one changes the direction of time and vice versa.

6 On the horizon problem in bouncing cosmologies

Inflationary cosmology solves the horizon problem by invoking the initial inflationary phase during which period the particle horizon grew exponentially. As a consequence all the regions on the CMBR sky observed now were in actual causal contact during inflation. Bouncing models tackle the horizon problem in different ways. One of the ways to solve the horizon problem in bouncing cosmological models is to propose that such models may not have any particle horizon, as a result of which \( R_P(t_0) \to \infty \) for any finite time \( t_0 \). If the particle horizon does not exist then the observed CMBR sky was in causal contact, solving the horizon problem of standard Big-Bang cosmology. The authors of Refs. 2, 14, 34 strongly favors this idea for solving the horizon problem in cosmological bouncing scenarios. In this scenario the whole of the universe is in principle causally related, the Hubble radius only limits our observational ability (in the expanding phase) at any time \( t_0 \). If the idea regarding the possibility of the absence of particle horizon in cosmological bouncing models is elevated to a principle about bouncing models then we see that our present work shows some counterexamples where the principle is violated. Our present work shows that some of the bouncing cosmological models can indeed accommodate particle horizons for some range of model parameters. If one still wants to keep \( R_P(t_0) \to \infty \) for bouncing models then our work shows the kind of model parameters which one has to exclude to propose a bouncing model. A bouncing model may not always satisfy the condition of nonexistence of the particle horizon for some range of parameters in the model and consequently one has to avoid the specific parameter ranges which gives finite \( R_P \) to come to a model where \( R_P(t_0) \to \infty \) for all finite values of \( t_0 \). It must be noted that any cosmological bounce which starts with a matter dominated phase, as in Ref. 16, one must have \( R_P(t_0) \to \infty \) for all finite \( t_0 \)

In the previous paragraph we have specified the simplest way bouncing cosmologies can tackle the horizon problem. Unlike the inflationary paradigm where the horizon problem is solved in all inflationary models because of the exponential expansion of the particle horizon (irrespective of model particularities), the bouncing models a priori do not always give infinite \( R_P \). Some models do not have particle horizon (as LQC model discussed in the article) and some bouncing models may have particle horizon for some parameter choice (as the quintom bounce model). If one does not universally accept that all bouncing models must have \( R_P(t_0) \to \infty \) then the horizon problem acquires new features. If \( R_P \) remains always greater than \( R_H \) (except at the bounce point) and \( R_P(t_0) \to \infty \) as \( t_0 \to -\infty \) in the bouncing universe then the horizon problem can be solved by assuming that the whole CMBR sky was in causal contact in the contracting phase of the bouncing universe.\footnote{In such cases \( R_P \) must always be greater than \( R_H \) during the contraction phase (except very near to \( t = 0 \)) was shown in section 5.} The whole observable universe at the present moment was in causal contact long before the bounce solving the horizon problem. During the contracting phase the cosmological
perturbation modes left the Hubble horizon and these modes re-entered the Hubble sphere during the expansion phase producing fluctuations on the CMBR sky. In this case the Hubble sphere was always causally connected. One can use the above points to solve the horizon problems in the toy model cases for where the behavior of the particle horizon are as shown in Figs. 3, 4 and 5.

If the particle horizon distance is bounded in the far past as \( t \to -\infty \), as it happens according to Eq. (47), then the part of the universe initially causally connected remains much smaller and has a finite length scale. In such cases the cosmological perturbations which will ultimately re-enter the Hubble sphere later in the expansion phase may produce fluctuations which are not nearly scale-invariant and more over acausal effects may show up in the bouncing models [14]. The acausality appears because one cannot apply proper initial conditions to all the modes with various wavelengths at the early phase of contraction. Particularly in the bouncing case as depicted in Fig. 7 although the particle horizon distance grows and is always more than the Hubble radius during the expansion phase, the Hubble sphere was not inside the particle horizon during the contracting phase. Just after the bounce there are regions inside the Hubble sphere which were never causally connected before and consequently in this case the horizon problem is not ideally solved. The cosmological case studied in subsection 5.3 has other serious difficulties as in this case the event horizon shrinks in the expanding phase and the Hubble radius never increases after bounce. The main difficulty in this particular case is that we have used the same scale-factor throughout all the phases of the bouncing universe. The scale-factor decreases or increases very rapidly away from \( t = 0 \) making the cosmological model pathological. Ideally bouncing cosmological models should avoid scenarios where the causal structure is as given in subsection 5.3 because of the nearly scale-invariant nature of the scalar perturbations on the CMBR sky and our inherent belief on causality. In all the toy model examples and the quintom bounce model discussed previously one can see that \( R_P \) diverges as \( t_0 \to -\infty \) such that one can apply initial conditions on all the wavelengths and the issue of initial scale dependence does not arise.

On the other hand if the particle horizon distance becomes smaller than the Hubble radius during the expansion phase, as shown in Fig. 3 for the \( p = 2 \) case, then deep inside the radiation dominated phase during expansion (all length scales inside) the Hubble sphere does not remain causally connected. As long as \( R_P(t_0) \to \infty \) as \( t_0 \to -\infty \), as it happens for the above case, the horizon problem is solved but another important problem arises. In this case all the physics related to the formation of CMBR will be guided by the causal patch predicted by the actual particle horizon size \( R_P \) during the radiation domination. As magnitude of \( R_P \) during radiation domination is not defined locally but depends on the global history of the bouncing model the imprints on CMBR can vary between various bouncing models. If such a case arises one has to check properly that the given bouncing model reproduces all the known features of CMBR spectrum.

The above discussion in this section specifies the various probable solutions of the horizon problem in the bouncing universe models. There can be multiple ways one can solve the horizon problem in such cases. We will now present a general discussion of the previous results in the next section.

7 Discussion

In the toy cosmological bounce presented in the paper we see that if the contracting phase prior to the bouncing phase has a power law scale-factor then the particle horizon exists only when \( m > 1 \). As in general relativity the exponent in the power law is related with the barotropic equation of state \( \omega \) via

\[
\omega = \frac{2 - 3m}{3m},
\]

it is seen that \( \omega \geq 0 \) only when \( m \leq 2/3 \), where the equation of state becomes zero at \( m = 2/3 \). If \( m > 2/3 \) the barotropic ratio becomes negative. As a consequence it follows that if there is a power law contraction phase before the bouncing phase then the condition \( \omega < 0 \) is a necessary condition for a finite particle horizon. The condition that \( m > 1 \) translates to

\[
-1 < \omega < -\frac{1}{3},
\]

(49)
which specifies that in such a case $\rho + 3P < 0$ during the contraction phase. This result shows that the particle horizon in such cases can only exist if the SEC is violated during the power law contraction phase. This conclusion can also be applied in the quintom bounce case where the particle horizon can only exist if $r > 1/3$ and consequently $\rho + 3P < 0$ far away from the bouncing point. Near the bouncing point $\rho + 3P$ becomes a function of time in the quintom bounce models. At the bounce point one must have to break the NEC to have a proper bounce in GR based models.

In the toy bounce model we have used various forms of the scale-factors in the various evolutionary phases of the universe. The different metric smoothly transforms from one form to the other because of the junction conditions. The junction conditions and our choice of the bouncing scale-factor combines to produce an interesting effect. It is apparent from Eq. (26) that if one chooses $t'' = 1$, $t' = -1$ and $n = 1/2$ then $m$ also turns out to be 1/2 when $p$ is an integer. The junction condition and symmetric matching times combine to predict a symmetrical evolution of the universe through bounce. If we want to have an asymmetrical evolution of the universe the matching times should be different which will lead to dissimilar values of $m$ and $n$ or one may choose to have $m \neq n$ which will give asymmetric matching times.

In $f(R)$ gravity induced bounce we saw the minima of the particle horizon appears near $t \rightarrow -\infty$. The particle horizons then grows monotonically. The particle horizon radius displays such a behavior because the scale-factor at $t \rightarrow -\infty$ diverge too fast as one moves back in time and consequently considerable amount of light cannot reach any region of the universe. In the toy bounce model and the quintom bounce model, where particle horizon exists, we observe that the minima of the particle horizon distance is always near the bounce point. In these cases the scale-factor during the contraction phase is given by a power law function of time which diverges as $t \rightarrow -\infty$ but this divergence is much milder than the divergence of the scale-factor in $f(R)$ gravity induced bounce. As one moves back in time a much wider part of the universe seems to be causally connected in the toy bounce model and the quintom bounce model. But as in these cases the particle horizon distance exceeds the Hubble radius the surface defining the particle horizon, at a particular time, is radially moving inward in a superluminal way. Photons which are moving radially inward from outside this surface will not be able to reach this surface. As the surface contracts with time the particle horizon distance diminishes with time. It is to be noted that a contracting particle horizon does not imply that causal regions of the universe are moving out of the horizon, it implies that as the universe contracts no new causal regions are entering the particle horizon.

The minima of the particle horizon appears near the bounce point in some models, as in the toy bounce model and quintom bounce model. This happens when $R_P = R_H$. In the example given in this paper the condition $R_P = R_H$ happens at an instant and consequently the particle horizon attains a minima at that instant. If the two surfaces, describing the particle horizon and the Hubble surface, remain identical for some period of time then during that period the particle horizon distance will remain constant. This will happen because light cannot enter radially inward into the particle horizon as the surface defining the particle horizon, at a particular time, contracts with the speed of light. More over as the spatial surface defining the particle horizon, at a particular time, is moving inwards with just the velocity of light all the emitters inside this surface move inwards subluminally and the light they emit can reach the observer in due time. In this case neither the particle horizon distance increases nor it decreases with time\footnote{It must be noted that the surface, with definite physical radius, which define the particle horizon or the Hubble surface at a particular time, $t$, may not define the particle horizon or the Hubble surface at a later time $t' > t$. Another surface, with a different physical radius, will define the particle horizon at $t'$. As an example, the surface which coincides with the particle horizon at time $t$, in the case where $R_P < R_H$, contracts with time as its physical radial coordinate shrinks with time. On the other hand the particle horizon distance increases with time.}

Although in the Big-Bang paradigm the particle horizon distance and the Hubble radius are proportional to each other when the scale-factor of expansion is given by a power law function in bouncing models this fact does not hold anymore. In the bouncing models the power law expansion phase may accommodate a particle horizon and the relationship between the particle horizon distance and the Hubble radius is much more complex. Although the present authors have not seen any article addressing the issue solely related to the particle horizons in bouncing models, a recent publication [35] discuss the effect of bouncing models on luminosity distance in cosmology.

If a bouncing universe does not have a finite particle horizon then the only compact surface which determines
the causal structure of the universe is the Hubble surface, as in the case of loop quantum cosmology based bounce studied in this article. Although the Hubble radius diverges during the bounce time, the Hubble surface is the only compact 2-dimensional surface which can have any say on the causal structure of the universe during the contracting and expanding phases.

8 Conclusion

The present article addresses the topic related to causality in general bouncing cosmological models based on the flat FLRW solution. Keeping the standard definitions of the particle horizon, event horizon and Hubble radius the present article generalizes their meaning in a bouncing universe which accommodates an infinitely stretched (in time) contraction phase. It is shown that in many bouncing models the particle horizon may not exist for some parameter values. When the particle horizon does not exist it means \( R_P(t_0) \to \infty \) for finite \( t_0 \). In such a case any observer at any time instant can in principle get light rays from infinite distance away (in the past) and consequently the horizon problem is solved in a straightforward manner. If in a bouncing model the particle horizon does not exist then the Hubble surface defines the causal structure of this universe. If matter content of the universe, during the contraction phase, violates some energy conditions then the particle horizon can exist in some simple toy models based on GR. It is shown that even if the particle horizon exists one can tackle the horizon problem in bouncing cosmological models in various cases.

The quintom bounce model shows that under certain circumstances the quintom universe can have a particle horizon. The casual properties of the quintom model is similar in nature to toy bounce model. Loop quantum cosmology induced bounces may not admit any particle horizons. The bounce model based on \( f(R) \) gravity has a single scale-factor during contraction, bounce and expansion phases. In this simple model the \( f(R) \) gravity based bounce may accommodate the particle horizon and the event horizon. In reality the \( f(R) \) based model studied in this article will have a particle horizon and the Hubble radius defined for all time up to the bounce. Depending upon the future scale-factor (after the bouncing phase) one has to decide whether the event horizon can exist. In this article we have simply used the same scale-factor throughout time to show the symmetry between the particle and event horizons. The example shown in our paper do not solve the horizon problem and one must reject such cases while studying bouncing cosmological models.

The toy bounce, which accommodates three phases of evolution of the universe, exposes the difficulty in calculating the particle horizon distance as the properties of particle horizon becomes dependent on the earliest history of the bouncing models. In the toy model calculation, presented in the article, the contraction and the expansion phases are guided by a power law scale-factor where as the bouncing scale-factor is assumed to be an even function of time. The simple calculations show that the criterion for existence of particle horizons depends upon the energy conditions followed by matter during contraction. We have emphasized the special role of the Hubble surface which affects the time evolution of the particle horizon.

We have presented a brief discussion on the solution of the horizon problem in bouncing universes. It is shown than there are number of ways in which the horizon problem can be solved in bouncing cosmological models. Some of the probable cosmological models must have to be modified or abandoned depending upon how successfully they solve the horizon problem. The present work shows that the causality problem in bouncing universe is intrinsically related to an understanding of the various phases of the universe during the contraction phase. As our understanding of the contraction phase is purely speculative at present the models we use to figure out the nature of particle horizon remains over simplistic. The present authors believe that although the causality problem in bouncing universe models are far from being solved the present article shows the qualitative and quantitative difficulties one must have to circumvent in the future to produce more meaningful results.
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