Semiclassical asymptotics of the solution of the Helmholtz equation in a three-dimensional layer of variable thickness with a localized right-hand side

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Abstract. We construct a semiclassical asymptotics of the solution of the Helmholtz equation in a three-dimensional layer of variable thickness with a localized right-hand side, assuming the absence of “trap” states and the fulfillment of radiation conditions at infinity (such as the Sommerfeld condition). Wave part of the asymptotic solution can be represented as a decomposition into a finite number of modes, each mode is connected with a pair of Lagrangian manifolds.

1. Introduction

The traditional modeling of the sound propagation in shallow water is done by solving the Helmholtz equation in a three-dimensional layer of variable thickness with δ-function on the right-hand side. In the underwater acoustics several methods for solving such a problem are developed, among these methods one can note the method of the parabolic equation[1, 2], the method of Gaussian beams[3] and methods based on separating the vertical variable and obtaining equations describing the change in modal amplitudes in the horizontal plane[4, 5].

Recently, a new approach for constructing asymptotic solutions of inhomogeneous stationary problems with localized right-hand sides was proposed in [6]. This approach is based on using the Maslov canonical operator on a pair of Lagrangian manifolds and generalizes results [7, 8, 9]. In this paper we develop the approach [6] for constructing asymptotic formulae for the solution of the Helmholtz equation in a three-dimensional layer $D(\xi_1, \xi_2) \leq z_0 \leq 0$ with slowly varying boundaries with a localized right-hand side

$$\Delta u + \omega^2 \frac{c^2}{\mu} u = F\left(\frac{x_1 - \xi_1}{\mu}, \frac{x_2 - \xi_2}{\mu}\right) g\left(\frac{z - z_0}{\mu}\right), \quad u|_{z=0} = 0, \quad u|_{z=0} = 0. \quad (1)$$

Here $z$-axis is directed perpendicular upwards from the horizontal plane $(x_1, x_2)$, $\mu$ is a small parameter defining characteristic source size, values $(\xi_1, \xi_2, z_0)$ define coordinates of the source, the source is located inside the layer $(D(x_1, x_2) \leq z_0 \leq 0)$, $F$ and $g$ are smooth rapidly decaying at infinity functions.
First we apply the adiabatic reduction [10] of the 3-D problem to a family of 2-D (plane) problems in \( \mathbb{R}_x^2 \), describing the waves corresponding to various horizontal modes. Then we apply the approach [6] to obtain 2-D problems. Note that it is possible to present the asymptotic solution in a standard form as the convolution of the right-hand side with the Green function asymptotics [7, 8, 9], but the calculation of the corresponding integral is a sufficiently complicated problem, and the realization of the approach [6] seems much more pragmatic and convenient.

As we mentioned before the asymptotics is related to a pair of Lagrangian manifolds, one of them is connected with a localized ("singular") part of the solution in the neighborhood of the point \( (x_1 = \xi_1, x_2 = \xi_2) \) and the second manifold defines the oscillating ("wave") part of the solution in the entire layer (taking into account the possible appearance of caustics and focal points). For the realization of obtained formulae, it is necessary to construct rays (or trajectories in the phase space), their ends determine wave fronts, and restore the wave field on them using sufficiently effective formulae. Note that our consideration assumes the absence of so-called "trapped" (horizontal) modes.

2. Adiabatic reduction to 2-D problems

Let us pass to dimensionless variables by the following way: \( x' = x/l_0, z' = z/d_0 \), where \( d_0 \) is the characteristic thickness of the layer, and \( l_0 \) is some horizontal size. We assume that \( l_0 \gg d_0 \) and introduce the small parameter \( h = d_0/l_0 \). The equation in dimensionless variables has the form

\[
\left( h^2 \Delta x' + \frac{\partial^2}{\partial z'^2} + n^2 \right) u = f \left( \frac{x' - \xi'}{\mu'} \right) g \left( \frac{z' - z'_0}{\mu''} \right) \left. u \right|_{z' = D'(x')} = 0, \quad u \big|_{z' = 0} = 0, \tag{2}
\]

where \( n^2 = \frac{d_0^2}{l_0^2} \), \( f \left( \frac{x' - \xi'}{\mu'} \right) g \left( \frac{z' - z'_0}{\mu''} \right) = d_0^2 F \left( \frac{x - \xi}{\mu} \right) g \left( \frac{z - z_0}{\mu} \right) \), \( D'(x') = \frac{D(x/l_0)}{d_0}, \mu'' = \frac{\mu}{d_0}, \mu' = \frac{\mu'}{l_0} \).

Further all hatches are omitted, and, for simplicity, we consider Eq. (2) with one parameter \( (h = \mu' = \mu'') \).

We are looking for a solution in the form:

\[
u = \sum_{\nu} \chi_{\nu} \left( \frac{2}{x}, -i h \nabla_x, z, h \right) \psi_{\nu}(x, h), \tag{3}
u\]

where \( \hat{\chi}_\nu \) are pseudodifferential operators with the small parameter \( h \), these are operator analogs of mode functions, and numbers above operators denote the order of operator action[11]. Functions \( \psi_{\nu}(x, h) \) satisfy two-dimensional equations

\[
L_{\nu} \left( \frac{2}{x}, -i h \nabla_x, h \right) \psi_{\nu}(x, h) = \left( g \left( \frac{z - z_0}{h} \right), \chi_{\nu}^0(x, p, z, h) \right) f \left( \frac{x - \xi}{h} \right), \tag{4}
\]

where \( \hat{p} = -i h \nabla_x \), brackets \( \left( v(z), w(z) \right) \) define the scalar product \( \left( v(z), w(z) \right) = -\frac{2}{D(x)} \int_D(v(z)w(z))dz \). \( L_{\nu}(x, p, h) \) and \( \chi_{\nu}(x, p, z, h) \) are symbols of operators \( L_{\nu}(x, -i h \nabla_x, h) \) and \( \chi_{\nu}(x, -i h \nabla_x, z, h) \). We suppose that functions \( \chi_{\nu}, L_{\nu} \) are smooth, and these can be represented in terms of the asymptotic expansion with respect to the parameter \( h \).

\[
\begin{align*}
\chi_{\nu} &= \chi_{\nu}^0(x, p, z) + h \chi_{\nu}^1(x, p, z) + h^2 \chi_{\nu}^2(x, p, z) + \ldots, \\
L_{\nu} &= L_{\nu}^0(x, p) + h L_{\nu}^1(x, p) + h^2 L_{\nu}^2(x, p) + \ldots
\end{align*}
\tag{5}
\]

With the help of perturbation theory we obtain symbols of operators \( \hat{\chi}_{\nu}, \hat{L}_{\nu} \) \[10\]

\[
L_{\nu} = -\hat{p}^2 + n^2 - \frac{(\pi \nu)^2}{D^2(x)} + i h \left( \frac{p \cdot \nabla_x D(x)}{D(x)} \right) + O(h^2), \\
\chi_{\nu} = \sin \left( \frac{\pi \nu z}{D(x)} \right) + i h \left( \frac{p \cdot \nabla_x D(x)}{D(x)} \right) z^2 + \frac{\pi \nu z}{2D(x)} \sin \left( \frac{\pi \nu z}{D(x)} \right) + O(h^2). \tag{6}
\]

2
The equation in the three-dimensional layer was reduced to the family of problems in $\mathbb{R}^2$

$$
\left(\hbar^2 \Delta - \frac{(\pi \nu)^2}{D(x)^2} + n^2 + i\hbar \left( p \nabla_x D(x) \right) + O(h^2) \right) \psi_\nu(x, h) = \left( f \left( \frac{x - \xi}{\hbar} \right) g \left( \frac{z - \eta}{\hbar} \right) \sin \left( \frac{\pi \nu z}{D(x)} \right) \right)
$$

(7)

3. Representation of rapidly decaying functions in terms of the Maslov canonical operator and elliptic modes

It is reasonable to represent right-hand sides in Eq. (7) in the form of the Maslov canonical operator, after that we can use the developed theory [6] to construct an asymptotic solution. Such functions correspond the following Lagrangian manifold(see [12])

$$
\Lambda = \left\{ p = \alpha, x = \xi, \alpha \in \mathbb{R}^2 \right\}, \quad d\mu = dp_1 \wedge dp_2.
$$

(8)

Thus function $F\nu \left( \frac{x - \xi}{\hbar} \right)$ can be represented in the form:

$$
F\nu \left( \frac{x - \xi}{\hbar} \right) = \frac{\hbar}{i} K^h_{(\Lambda, d\mu)} [A\nu(\alpha)], \quad A\nu(\alpha) = \frac{1}{2\pi} \int_{\mathbb{R}^2} F\nu(y) e^{-i\alpha y} dy,
$$

where $K^h_{(\Lambda, d\mu)}$ is the Maslov canonical operator [13].

If the principal symbol $H\nu(x, p) = L\nu(x, p, 0)$ of the operator $\hat{L}\nu$ does not go to zero on the vertical Lagrangian manifold $\Lambda$, i.e. $p^2 + \left( \frac{\pi \nu}{D(x)} \right)^2 - n^2 \neq 0$, the operator $\hat{L}\nu$ is invertible on functions $F\nu$[14]

$$
\psi_\nu = \frac{\hbar}{i} \hat{L}\nu^{-1} K^h_{(\Lambda, d\mu)} [A\nu(\alpha)] = \frac{\hbar}{i} \left( K^h_{(\Lambda, d\mu)} \left[ \frac{A\nu(\alpha)}{H\nu(\alpha, \xi)} \right] + O(h) \right).
$$

(10)

Functions $\psi_\nu$ are localized in the neighborhood of the point $\left( x_1 = \xi_1, x_2 = \xi_2 \right)$. If $H\nu|_{\Lambda} \neq 0$, the solution of the three-dimensional problem (2) can be represented in the form:

$$
\begin{align*}
  u &= \sum_{\nu=1}^{\infty} \left[ \frac{\hbar}{i} \left( K^h_{(\Lambda, d\mu)} \left[ \frac{A(\alpha)}{H\nu(\alpha, \xi)} \right] \sin \left( \frac{\pi \nu z}{D(x)} \right) + O(h) \right) \right] = \\
  &+ \sum_{\nu=1}^{\infty} \left[ \frac{1}{2\pi} \sin \left( \frac{\pi \nu z}{D(x)} \right) \int_{\mathbb{R}^2} \frac{A\nu(p)e^{\frac{i\pi \nu (x-\xi)}{D(x)}} dp}{H\nu(p, \xi)} + O(h^2) \right].
\end{align*}
$$

(11)

It is the formal asymptotic solution, it is necessary to prove that the corresponding series does converge, and the cumulative correction is the value of the order $O(h^2)$.

4. Hyperbolic modes and their asymptotics

The formula (10) is not correct in some neighborhood of the submanifold $L_0 = \left\{ (x, p) \in \Lambda : H\nu(x, p) = 0 \right\}$. The operator $\hat{L}\nu$ becomes hyperbolic on $L_0$, and wave part of the solution appears. The vanishing of the principal symbol $H\nu(x, p) = L\nu(x, p, 0)$ of the operator $\hat{L}\nu$ on some submanifold $L_0$ of $\Lambda$ is necessary for the appearance of wave part of the solutions. Emphasize that the identity $p^2 + \left( \frac{\pi \nu}{D(x)} \right)^2 - n^2 = 0$ performed for a finite number of modes or not fulfilled at all, thus wave part of the solution contains a finite number of modes. Wave part of the solution is connected with a new Lagrangian manifold, thus solution is connected with a pair of Lagrangian manifolds.

Under the assumption that the mode with number $\nu$ is hyperbolic, we use the construction of the asymptotic solution from [6]. Let us introduce needed objects to present the corresponding proposition:
• $t_0$ is a sufficiently small number, and $\alpha^* \in L_0$ is the initial point on $\Lambda$.
• $\hat{\Lambda}$ is a small neighborhood of the submanifold $L_0$ in $\Lambda$. $[\Lambda_t = g_H^t(\hat{\Lambda}), \ t \leq t_0]$ is the Lagrangian manifold with the measure $d\mu_t = (g_H^{t*})^{-1}d\mu$.
• $\Lambda_+ = \cup_{t \geq 0} g^0_H(L_0)$ is the two-dimensional Lagrangian manifold in $\mathbb{R}^4$ with the boundary $L_0$, $d\mu_+ = dt \wedge (g_H^0)^{-1}d\sigma$ is the measure on $\Lambda_+$, where $d\sigma$ is the measure on $L_0$.
• Let $\rho(\alpha) \in C^\infty(\Lambda)$ and $\theta(t) \in C_0^\infty(\mathbb{R}^1)$ be cut-off functions such that $\rho(\alpha) = 1$ in the neighborhood of the manifold $L_0$, and $\text{supp}(\rho(\alpha)) \subseteq \Lambda$, $\theta(t) = 1$ if $t \leq t_0$, $\theta(t) = 0$ if $t \geq t_0$.

We define the function $\hat{A}$ as a solution of the Cauchy problem for the transport equation on trajectories of the Hamiltonian system outgoing from points $(x_0, p_0) \in \Lambda$

$$\frac{d\hat{A}_\nu}{dt} + H_{\text{sub}}(X(x_0, p_0, t), (P(x_0, p_0, t)))\hat{A}_\nu = 0, \quad \hat{A}_\nu|_{t=0} = A_\nu(x_0, p_0), \quad (12)$$

where $A_\nu$ is the amplitude in the formula (9), and $H_{\text{sub}}(x, p) = i \frac{\partial L_0}{\partial \dot{q}}(x, p, 0) - \frac{i}{2} \sum_{n=1}^{n} \frac{\partial^2 H_0}{\partial x_n \partial p_n}(x, p)$ is the subprincipal symbol of the operator $\hat{L}_\nu$. $A_{\nu+}$ is restrictions of functions $A_\nu$ on the manifold $\Lambda_+$. In our problem $H_{\text{sub}}(x, p) = - \frac{(p, \nabla D(x))}{D(x)}$. The solution of the transport equation can be obtained in the explicit form:

$$A_{\nu+}(x_0, p_0, t) = A_\nu(x_0, p_0) \exp\left(\int_0^t \frac{(P, \nabla X D(X))}{D(X)} dt\right) = A_\nu(x_0, p_0)$$

$$\exp\left(-\int_0^t \frac{(X, \nabla X D(X))}{2D(X)} dt\right) = A_\nu(x_0, p_0) \exp\left(-\int \frac{D(X(x_0, p_0, t))}{D(X)} dD(X)\right)$$

$$= A_\nu(x_0, p_0) \frac{\sqrt{D(X(x_0, p_0, t))}}{\sqrt{D(X)}} = A_\nu(x_0, p_0) \frac{\sqrt{D(\xi)}}{\sqrt{D(X)}} \quad (13)$$

Now we can apply the Theorem 1 from [6].

**Proposition** Eq. (7) with right-hand side(9) has an asymptotic solution $\psi_\nu = \psi_\nu(x, h)$ which satisfy the asymptotic limit absorption principle and can be represented up to lower order terms

$$\psi_\nu = \frac{h}{i} K^h_{(\Lambda, d\mu)} \left(\frac{1 - \rho(\alpha) A_\nu}{H_\nu}\right) + \frac{2\pi}{h} e^{i\pi/2} K^h_{(\Lambda_+, d\mu_+)} \left(\frac{1 - \Theta(t)}{H_\nu} A_{\nu+}\right) + \frac{i}{h} \int_0^{t_0} \exp\left(\int_{a_x}^{g_{ht}(a_x)} p dx\right) K^h_{(\Lambda_+, d\mu_+)} \left(\Theta(t) \rho(\alpha) A_\nu(t)\right) dt + O(h^{1/2}) \quad (14)$$

The first summand corresponds elliptic modes, the second summand is the wave ”hyperbolic” part of the solution, and the third summand is the solution in the small neighborhood of the Lagrangian manifold $L_0$.

**Remark:** Let the right-hand side(9) be finite and localized in the neighborhood of the point $x = \xi$ function. Then asymptotic solution of the Eq. (7) in the region $|x - \xi| > Ch^\beta$ for any constants $C > 0$, $\beta \in \left(\frac{1}{2}, 1\right)$ for a sufficiently big $L > 0$ can be represented in the form

$$\psi_\nu(x, h) = 2\pi \left[K^h_{(\Lambda_+, d\mu_+)} \left(1 + \Theta(Lt/h^\beta)\right) A_{\nu+}\right](x, h) + O(h^{2\beta-1/2}) \quad (15)$$

The solution of two-dimensional problem can be simplified, if we apply the representation of the canonical operator developed in the work[15]. The Maupertuis-Jacobi principle allows us to pass
from one Hamiltonian system to another, if equations $H(x, p) = E$ and $F(H(x, p), x, p) = E$ define the same surface in the phase space. We pass to the Hamiltonian

$$
\tilde{H} = \frac{|p|}{\sqrt{\frac{\pi^2 \nu^2}{D^2(x)} + n^2}} - 1.
$$

(16)

Main features of such passage are following

- The Jacobian depends on differentiation with respect to only the one variable \( \tilde{J} = \frac{1}{\sqrt{-\pi^2 \nu^2/D^2(X)}}|X_\psi| \), where \( \psi \) is the coordinate on the circle \( L_0 \).
- The phase is the “proper” time of the new Hamiltonian system \( s(\tau, \psi) = \tau \) and wave fronts are ends of trajectories of the Hamiltonian system.

Consider now Eq. (7) with the right-hand side with a small parameter \( \mu \). If \( \mu \geq h \), the right-hand side can be represented in the form of the canonical operator with the small parameter \( h \)

$$
F_\nu\left(\frac{x - \xi}{\mu}\right) = \frac{h}{i} R^{h}_{(\lambda, d\mu)}[A_\nu(\alpha)], \quad A_\nu(\alpha) = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_\nu\left(\frac{y}{\mu}\right) e^{-i\alpha y} dy.
$$

(17)

Previous formulae are applicable in this case. The case \( \mu < h \) is not considered in this paper.

5. Example

Consider the problem (7) with the right-hand side as the Gaussian function

$$
F = \exp\left(\frac{-(x_1 + 3)^2}{2h^2} - x_2^2\right)
$$

and with \( D(x) = -1 - \exp(-x_1^2 - x_2^2) \). This problem describes the change in modal amplitudes of sound signal generated by the (time-) monochromatic source located near the point \((\xi_1, \xi_2) = (-3, 0)\) in shallow water with bathymetry function \( D(x) \). The Hamiltonian systems have been solved for different (wave) modes, and corresponding trajectories and wave fronts are presented at Fig. (1,2). Rays are projections of characteristics from the 4-D phase space to the configuration space (2-D plane). The ambiguity of the preimages of points \( x \) on the Lagrangian manifold leads to intersections of wave fronts. The solution is the sum of corresponding functions connected with all wave fronts. We use the representation of the Maslov canonical operator in terms of Airy and Pearcy functions[16, 17] near caustics and focal points.

![Figure 1](image1.png)

**Figure 1.** Rays and wave fronts for \( \nu = 3, n = 13, h = 0.01, \psi \in \left[-\frac{\pi}{3}; \frac{\pi}{3}\right] \).

![Figure 2](image2.png)

**Figure 2.** Rays and wave fronts for \( \nu = 4, n = 13, h = 0.01, \psi \in \left[-\frac{\pi}{3}; \frac{\pi}{3}\right] \).
Figure 3. Rays and wave fronts for $\nu = 4$, $n = 13$, $h = 0.01$ in the neighborhood of the caustics.

Figure 4. The solution of Eq. (7) for $\nu = 4$ in the neighborhood of the caustic.

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