Modeling of thermal stresses in thin-walled composite structures with phase transformation

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Abstract. An asymptotic theory modified version for calculating thin elastic multilayer plates made of high-temperature composites with multistage phase transformations, considering the composites shear characteristics finite values has been developed. Proposed in this work method version does not contain any assumptions about the kind of the unknown functions and allows ones to find all 6 stress tensor components while ensuring the high mathematical accuracy that symbolic for asymptotic method. Using the developed model, it became possible to calculate temperature non-stationary field and thermal stresses in thin multilayer plates of fabric composite materials on an inorganic binder. For material properties calculation were used multiscale averaging method and finite element modeling for periodical cells. Some change stresses numerical simulation results for composite plate during heating and loading are given.

1. Introduction

In modern technology a special attention is paid to structures and shells made of composite materials with high indicative mechanical properties (elasticity, hardness, strength). [1-3]. Recently, composite materials based on inorganic binders are used in heat-loaded constructions. Such materials are used as heat-insulating and fire-resistant structures in construction equipment, metallurgy, as well as high-temperature electrical insulators in electrical engineering. At high temperatures up to 1500 °C, complex phase transformations occur in the matrix and fibers of such composite materials, leading to non-monotonous irreversible changes in all the thermal and mechanical material properties. To describe the deformation processes in composites with phase transformations under high temperature, it is necessary to develop refined calculation methods that provide an acceptable accuracy in calculating interlayer shear stresses and transverse stresses. The purpose of this paper is the development of constitutive relations for a composite material with high-temperature multistage phase transitions, based on microscopic analysis of the stress-strain state, and development of a modified asymptotic theory for calculating thin plates from composites of this type.

An internal structure multiscale model of the inorganic matrix composite (IMC) was proposed in [3,4]. According to the general modeling composites methodology [6] IMC are considered as a multi-level structure contains 4 structural levels, each level consists of a large number of periodicity cells. The first structural level is formed by the periodicity cells, which consist of two elements: the fiber and the matrix. At the second level the composite material matrix consists of a binder and a filler from dispersed particles. At high temperatures, the matrix is a multiphase medium; therefore, a third structural level has been identified, at which the periodicity cell consists of two phases with the symbols (a) and (b), since it is assumed that phase transformations in the material proceed along two separate chains because of the different dispersed particles at 2-d level. At 4-th level each of the phases (a) and (b) is represented as a five-phase system. When heated to high temperatures at 4-th level, a redistribution of the relationships between the phases occurs in the periodicity cell for each chain (a) and (b).

The textile threads at the second level are composed of a large number of monofilaments surrounded by a matrix. During heating in fibers undergo phase transformations, the initial fiber phase volume concentration decreases, and a new phase appears. A periodicity cells model with a cubic shape phases was used for cells geometry description at the 3-thd and 4-th levels. Effective properties at these levels are calculated using the stacking principle. [3,4,10] The effective properties at the 1-st and 2-nd structural levels for periodicity cells with a known phase geometry are determined by solving a series of local problems by the asymptotic averaging method and calculations based on finite-element method [6, 7, 8].
2. The mathematical formulation of the problem of calculating the thermal stresses in a composite plate under unsteady heating

A thin plate of constant thickness made of a fabric composite material on inorganic binder with multistage high-temperature phase transformations is considered. Global dimensionless coordinates and a small parameter $\chi$ (the ratio of the total plate thickness to the characteristic size of the entire plate) are introduced as in [5]. The three-dimensional problem of the linear theory of thermoelasticity for a thin multilayer plate with temperature-dependent properties is written in dimensionless form, and the kinematic relations system for the phase concentrations is added.

$$\frac{C(\Psi)}{\chi} \frac{\partial \theta(x, y, z)}{\partial t} = -\frac{\partial q_i}{\partial x_i}, \quad q_i = -\lambda_y(\theta) \frac{\partial \theta(x, y, z)}{\partial x_j},$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \sigma_{ij} = A_{ijkl}(\Psi)(\epsilon_{ij} - \epsilon_{ij}^0)$$

$$\Sigma_{3b} : \sigma_{33} = -\chi \rho_{33} \delta_{33},$$

$$q_j = \pm q_{ej},$$

$$\Sigma_T : q_j n_j = 0, \quad u_i = u_{ei}$$

$$\Sigma_S : [\theta] = 0, \quad [q_i] = 0, \quad [u_3] = 0, \quad \sigma_{33} = 0$$

$$\rho_{sw} \frac{\partial \varphi_{sw}}{\partial t} = J_{sw} \varphi_{sw}, \quad w = \{a, b\}, \quad s, v = 1, N$$

$$\rho_h \frac{\partial \varphi_h}{\partial t} = J_h \varphi_h f_h(\theta), \quad \sum_{\gamma=1}^N \varphi_{\gamma w} = \varphi_{(w)}, \quad \varphi_h + \varphi_i = \varphi_{r2}$$

$$\varphi_{\gamma w}(0) = \varphi_{\gamma w}^0, \quad \varphi_h(0) = \varphi_{r2}, \quad \theta(0) = \theta^0$$

The plate ends are considered to be thermally insulated. At the interface between plate layers ideal contact conditions are satisfied. The following quantities are indicated in system (1): $u_{ei}$ – displacement vector components that given at the plate ends $\Sigma_T$, $p_h$ – pressure and $q_{0e}$ – plate top $\Sigma_3$, and bottom $\Sigma_3$ – surface heat flux. Here dimensional values are indicated with a cap on top and basic values by an index 0: $\theta$ – temperature, $t = t / t_0$ – time, $u_j = \dot{u}_j / L$ – displacement vector components, $\rho = \rho / \rho_0$ – density, $\sigma_{ij} = \dot{\sigma}_y / \sigma_0$ – stress tensor components, $c = c / c_0$ – heat capacity, $q_i = \dot{q}_i L / \lambda \theta_0$ – heat flux vector components, $x_i = \tilde{x}_i / L$ – Cartesian coordinates, $L$, $h$ - plate length and thickness. Capital Latin index take the values 1 and 2, and the lowercase index takes values from 1 to 3. Equations (1) also denote: $\epsilon_{ij}$ – small strain tensor components, $\epsilon_{ij}^T$ – thermal strain tensor components, which are temperature difference functions $\Delta \theta = \theta - \theta_0$, where $\theta_0$ is the initial temperature, $\alpha_{ki}$ are the thermal expansion tensor components. The elastic modulus tensor components $A_{ijkl}$, thermal conductivity $\lambda_y$, thermal deformation $\varepsilon_{ij}^T$, as well as the mass heat capacity $C = \rho c / F_0 h$ all of them depend on variable plate material phase composition. $\varphi = (\varphi_{sw}, \varphi_{sw}, \varphi_h, \varphi_i)$, $w = \{a, b\}$ – the phases concentrations vector, includes the phases concentrations that contained in the binder at the transformation chains (a) and (b), and fiber’s phases concentrations. $f(\theta)$ – the function describes the reaction rate depending from temperature, where is the phase activation energy, for example, for the fiber $h$-phase it will have the form $f_h(\theta) = \exp(-E_h / \theta)$, where $E_h$ is a $h$-phase activation energy in relation to gas constant. $\varphi_{sw}^0$ – the values of the phases relative volumetric concentrations for the
chain (w) at the initial time point. $J_{svw}$ – matrix of multistage phase transformations rates, example for $N=4$ is given in [11].

The following assumptions are made in system (1): 1) the parameter $\chi$ is small $\chi << 1$, 2) the pressure on the outer and inner surfaces of the plate has the third order of smallness $O(\chi^3)$ (i.e. $\sigma_{ij} = -\chi^3 p_{ij}$), 2) the heating time is not long, in the sense that the Fourier criterion $F_0 = \lambda_d / \rho_c C_L$ of the heating process has one order of smallness with $\chi^2$ (i.e., $-F_0 = \chi^2 F_0$, where the $F_0$ number is of the order of 1: $F_0 = O(1)$; 3) pressure and heat flux vary little over distances of the order of plate thickness.

3. Asymptotic expansions

To describe the changing plate properties over its thickness, we will use a local dimensionless coordinate such that $\xi = x / \chi$. The problem solution (1) will be sought in the form of asymptotic expansions in terms of a small parameter $\chi$

$$
\begin{align*}
\phi &= \phi^{(0)}(x, \xi) + \chi \phi^{(1)}(x, \xi) + \chi^2 \phi^{(2)}(x, \xi) + \chi^3 \phi^{(3)} + \ldots \\
\theta &= \theta^{(0)}(x, \xi) + \chi \theta^{(1)}(x, \xi) + \chi^2 \theta^{(2)}(x, \xi) + \chi^3 \theta^{(3)} + \ldots \\
u_k &= u_k^{(0)}(x) + \chi u_k^{(1)}(x, \xi) + \chi^2 u_k^{(2)}(x, \xi) + \chi^3 u_k^{(3)}(x, \xi) + \ldots \\
\sigma_{ij} &= \sigma_{ij}^{(0)} + \chi \sigma_{ij}^{(1)} + \chi^2 \sigma_{ij}^{(2)} + \ldots
\end{align*}
$$

Substituting expansions (2) into problem (1) and grouping the terms by degrees $\chi$, we obtain a recurrent sequence of local problems of thermoelasticity with allowance for multistage phase transformations. The problem in the zeroth approximation has the form:

$$
\rho_w \frac{\partial \phi^{(0)}}{\partial t} = J_{svw}(0) \phi^{(0)}(0), \quad w = \{a, b\}, \quad s, v = 1, 4, \quad \phi_{rs}^{(0)}(0) = \phi_{rs}^0, \quad \phi_{r}^{(0)}(0) = \phi_{r}^0
$$

$$
\rho_h \frac{\partial \phi^{(0)}_h}{\partial t} = J_{hv}^{(0)}(0) \theta^{(0)}(0), \quad \sum_{r=1}^{N} \phi_{ht}^{(0)}(0) = \phi_{ht}^{(0)}, \quad \phi_{ht}^{(0)}(0) + \phi_{ht}^{(1)}(0) = \phi_{ht}^{f}
$$

$$
\begin{align*}
\frac{C(\phi^{(0)}_h) \partial \theta^{(0)}_h}{\partial t} + \frac{\partial q^{(1)}_{i}(\xi)}{\partial \xi} = 0, \lambda_{i} q^{(1)}_{i}(\xi) = -q^{(1)}_{i}(\xi), q^{(1)}_{i}(\xi) = \frac{\partial \theta^{(0)}_h}{\partial \xi} \\
\frac{\partial \sigma_{ij}^{(0)}}{\partial \xi} = 0, \quad \sigma_{ij}^{(0)} = A_{kr} \left(\phi^{(0)}_h \right) e_{ij}^{(0)} + A_{kr} \left(\phi^{(0)}_h \right) e_{ij}^{(1)}, \\
e^{(0)} = \frac{1}{2} \left( \frac{\partial u^{(1)}_j}{\partial \xi} + \frac{\partial u^{(1)}_j}{\partial x} \right), \quad e^{(1)} = \frac{1}{2} \left( \frac{\partial u^{(0)}_j}{\partial \xi} + \frac{\partial u^{(1)}_j}{\partial x} \right), \quad e^{(0)} = \frac{\partial u^{(0)}_j}{\partial \xi}, \\
\Sigma_{i} q^{(1)}_{i} = \pm q_{i3}, \quad \sigma_{ij}^{(0)} = 0; \quad \int_{-\Delta}^{\Delta} u^{(0)}_i d\xi = 0, \quad \Sigma_{i} q^{(1)}_{i} = 0, \quad \theta^{(0)} = 0, \quad \sigma_{ij}^{(0)} = 0, \quad u^{(1)}_i = 0;
\end{align*}
$$

The system (3) contains kinematic equations for the phases in the zero approximation; the kinematic equations for the phases will not be included in the problem for the first and second approximations, since the expansion of the functions present in them is limited by the first term of the series. Due to the fact that local problems are one-dimensional at the local coordinate their decision can be found analytically. The problem of thermoelasticity is unrelated, then the solution to the problem of unsteady heat conduction and the calculation of the concentration can be found separately from the problem of mechanical equilibrium.

4. Averaged equations of plates with multi-stage phase transformations

We denote the forces $N_{ij}$, transvers forces $Q_l$ and moments $M_{ij}$ in the plate

$$
\begin{align*}
N_{ij} &= \int_{-0.5}^{0.5} \sigma_{ij}^{(0)} d\xi + \chi \int_{-0.5}^{0.5} \sigma_{ij}^{(1)} d\xi + \ldots, \\
Q_l &= \chi \int_{-0.5}^{0.5} \phi^{(0)}_l d\xi + \chi^2 \int_{-0.5}^{0.5} \phi^{(1)}_l d\xi + \ldots
\end{align*}
$$
\[ M_{ij} = x^0_{0.5} \xi \sigma_{ij}^{(0)} d \xi + x^2_{0.5} \xi \sigma_{ij}^{(1)} d \xi + \ldots, \]

and give the equilibrium equations:
\[
\frac{\partial N_{ij}}{\partial x_j} = 0, \quad \frac{\partial Q_j}{\partial x_j} = \Delta \bar{p}, \quad \frac{\partial M_{ij}}{\partial x_j} - Q_i = 0,
\]

where \( \Delta \bar{p} = \chi^2 \Delta p \).

Asymptotic expansions for general three-dimensional equations of the elasticity theory in the framework of taking into account the first order only of smallness lead to the conclusion that the longitudinal displacements of fabric composite material with phase transformations linearly depend on the thickness coordinate.

5. Timoshenko-type plates asymptotic theory for multiphase composite materials based on the 2nd approximation displacements

The algorithm developed above for this stage of calculations corresponds to the asymptotic theory of plates of the Kirchhoff-Love type (asymptotic theory of the 1st order of execution with respect to displacement) [5]. To find displacements \( u_j^{(0)} \) and \( u_3^{(0)} \) in this theory, averaged equilibrium equations (5) are used.

Using the kinematic relation from the local problem for the first approximation (3), we find the displacement \( u_j^{(2)} \)
\[
u_j^{(2)} = < e_{3i}^{(0)} >_\xi = < \xi Z_{3kk} >_\xi \frac{\partial^2 u_i^{(0)}}{\partial x_k \partial x_L} + < Z_{3kk} e_{iL}^{(1)} >_\xi,
\]

where
\[ Z_{ikl}(\phi^{(0)}) = A^{-1}_{ij,kl}(\phi^{(0)}) A_{33kl}(\phi^{(0)}) \],
\[ \langle f(\xi) \rangle_\xi = \frac{1}{2} \int_{-1/2}^{1/2} f(\xi) d\xi - \frac{1}{2} \int_{-1/2}^{1/2} f(\xi) d\xi. \]

To improve the accuracy of the theory, we will additionally consider the displacements of not only the zero and first, but also the 2nd approximation \( u_l^{(2)} \) and the 3rd approximation \( u_3^{(3)} \).

Suppose that the displacements \( u_i^{(2)} \) have a similar linear dependence on the coordinate \( \xi \) as \( u_i^{(1)} \), and the displacement \( u_3^{(3)} \) is similar to \( u_3^{(2)} \).
\[
u_j^{(2)} = \xi \beta_j^{(2)}, \quad u_3^{(3)} = -< \xi Z_{3kk} >_\xi \frac{\partial \beta_k^{(2)}}{\partial x_L} + < Z_{3kk} e_{iL}^{(2)} >_\xi.
\]

where \( \beta_j^{(2)}(x_j) \) – new unknown functions of global coordinates.

Then, using (7), we find the deformations \( \epsilon_{ij}^{(2)} \), \( \epsilon_{35}^{(2)} \) and \( \epsilon_{i3}^{(1)} \)
\[
\epsilon_{ij}^{(2)} = \frac{1}{2} \left( \frac{\partial u_i^{(2)}}{\partial x_j} + \frac{\partial u_j^{(2)}}{\partial x_i} \right) = \frac{1}{2} \left( \frac{\partial \beta_j^{(2)}}{\partial x_i} + \frac{\partial \beta_i^{(2)}}{\partial x_j} \right),
\]
\[
\epsilon_{35}^{(2)} = \frac{\partial u_3^{(2)}}{\partial x_5} = -< \xi Z_{3kk} >_\xi \frac{\partial \beta_k^{(2)}}{\partial x_L} + Z_{3kk} e_{iL}^{(2)},
\]
\[
\epsilon_{i3}^{(1)} = \frac{1}{2} \left( \frac{\partial u_i^{(1)}}{\partial x_3} + \frac{\partial u_3^{(1)}}{\partial x_i} \right) = \frac{1}{2} \left( U_{i3k} \frac{\partial \beta_k^{(1)}}{\partial x_L} + \beta_i^{(1)} - U_i^{(1)} \right). \]
6. Relationships for higher approximations stresses

Substituting (8) into the constitutive relations for stresses, we find the 2nd approximation stresses

\[ \sigma_{ij}^{(2)} = A_{ijkl}^{(0)} \epsilon_{kl}^{(0)} + A_{ijkl}^{(1)} \epsilon_{kl}^{(1)} - A_{ijkl}^{(0)} \epsilon_{ij}^{(2)} = \xi A_{ijkl}^{(0)} \frac{\partial \epsilon_{ij}^{(2)}}{\partial x_l} - A_{ijkl}^{(0)} \epsilon_{ij}^{(2)} \]

And also 1st order shear stresses

\[ \tilde{\sigma}_{ij}^{(1)} = 2 A_{ijkl}^{(0)} \epsilon_{Ml}^{(1)} - A_{ijkl}^{(1)} \epsilon_{ij}^{(1)} = \]

\[ = A_{ijkl}^{(1)} \left( U_{3kl} \frac{\partial \epsilon_{ij}^{(1)}}{\partial x_M} + \beta_{ij}^{(2)} \right) - A_{ijkl}^{(1)} \left( \frac{\partial U_T^{(1)}}{\partial x_M} - A_{ijkl}^{(1)} \epsilon_{ij}^{(1)} \right) \]

7. Averaged constitutive relations for plates with phase transformations

Let us present expressions (4) taking into account formulas (9) and (10):

\[ N_{ij} = < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(0)} + \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \left( -\frac{\partial \epsilon_{ij}^{(0)}}{\partial x_i} + \chi \frac{\partial \epsilon_{ij}^{(2)}}{\partial x_i} \right) - \]

\[ - < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} > \]

\[ M_{ij} = \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(0)} + \chi^2 < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \left( -\frac{\partial \epsilon_{ij}^{(0)}}{\partial x_i} + \chi \frac{\partial \epsilon_{ij}^{(2)}}{\partial x_i} \right) - \]

\[ - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} - \chi^2 < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} > \]

\[ Q_i = \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \phi_{ij}^{(0)} U_{3kl} > \frac{\partial \epsilon_{ij}^{(0)}}{\partial x_M} - \]

\[ - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \left( \frac{\partial U_T^{(1)}}{\partial x_M} - \right) > - \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > \]

\[ U_{3kl} \text{ – function depending on the elastic modulus tensor components, Angle brackets denote the thickness averaging operation.} \]

Introduce designations for membrane, mixed, bending, and shear stiffesses for plate:

\[ \tilde{A}_{ijkl}^{(0)} \phi_{ij}^{(0)} = < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \tilde{B}_{ijkl}^{(0)} \phi_{ij}^{(0)} > \tilde{G}_{ijkl}^{(0)} \phi_{ij}^{(0)} > \tilde{R}_{ijkl}^{(0)} \phi_{ij}^{(0)} = \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > U_{3kl} > \]

and for thermal forces \( N_{ij}^T \), moments \( M_{ij}^T \), and shear forces \( Q_i^T \) in plate

\[ N_{ij}^T = < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(T)} + \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > + \chi^2 < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} > \]

\[ M_{ij}^T = \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \left( \frac{\partial U_T^{(1)}}{\partial x_M} > \right) > + \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > \]

\[ Q_i^T = \chi < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(T)} > + \chi^2 < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(1)} > + \chi^3 < A_{ijkl}^{(0)} \phi_{ij}^{(0)} > \epsilon_{ij}^{(2)} > \]

8. Average kinematic relations

The rotation angles of the normal to the plate middle surface are introduced in the form:
\[ \beta_i = -\frac{\partial u_i^{(0)}}{\partial x_i} + \chi \cdot \beta_i^{(2)} \]  

(16)

Then
\[ \chi \cdot \beta_i^{(2)} = \beta_i + \frac{\partial u_i^{(0)}}{\partial x_i} \]  

(17)

Introduce the curvature of the plate middle surface
\[ \eta_{ij} = \frac{1}{2} \left( \frac{\partial \beta_{ij}}{\partial x_i} + \frac{\partial \beta_{ij}}{\partial x_j} \right), \]

\[ \eta_{ij} = \frac{1}{2} \left( \beta_i + \frac{\partial u_i^{(0)}}{\partial x_i} \right). \]  

(18)

Then the expressions for the deformation gradients \( \varepsilon_{ij}^{(0)} \), middle surface and deformations \( \varepsilon_{kl}^{(0)} \), that depends on global coordinate functions \( u_i^{(0)}, u_3^{(0)} \) will look like
\[ \varepsilon_{ij}^{(0)} = \frac{1}{2} \left( \frac{\partial u_i^{(0)}}{\partial x_j} + \frac{\partial u_i^{(0)}}{\partial x_j} \right) \]

\[ \varepsilon_{kl,u}^{(0)} = \frac{1}{2} \left( \frac{\partial^2 u_l^{(0)}}{\partial x_k \partial x_j} + \frac{\partial^2 u_l^{(0)}}{\partial x_k \partial x_j} \right) \]  

(19)

9. Equilibrium equations averaged system for a Timoshenko-type plate with phase transformations

In view of (14) - (19), the averaged constitutive relations (11) - (13) take the form
\[ N_{ij} = \overline{A}_{ijkl} \left( \phi^{(0)} \right) \varepsilon_{kl}^{(0)} + B_{ijkl} \left( \phi^{(0)} \right) \eta_{kl} - N_{ij}^T \]  

(20)

\[ M_{ij} = B_{ijkl} \left( \phi^{(0)} \right) \varepsilon_{kl}^{(0)} + \overline{B}_{ijkl} \left( \phi^{(0)} \right) \eta_{kl} - M_{ij}^T \]  

(21)

\[ Q_i = G_{ijm} \left( \phi^{(0)} \right) \eta_{ijm} + R_{ijkl} \left( \phi^{(0)} \right) \frac{\partial \varepsilon_{kl}^{(0)}}{\partial x_m} - Q_i^T \]  

(22)

These expressions differ from the traditional relations of Timoshenko plates only by the term
\[ R_{ijkl} \left( \phi^{(0)} \right) \frac{\partial \varepsilon_{kl}^{(0)}}{\partial x_m} \] in the expression for the shearing force and by the thermal terms.

Substituting further expressions (20) - (22) and (19) into system (5), taking into account (16) - (18), we obtain a system of 5 equations for 5 unknown functions \( u_i^{(0)}, u_3^{(0)}, \beta_i \):

\[ \overline{A}_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} + B_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 \beta_k}{\partial x_i \partial x_j} - \frac{\partial N_{ij}^T}{\partial x_j} = 0 \]

\[ B_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} + \overline{B}_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 \beta_k}{\partial x_i \partial x_j} - \frac{1}{2} G_{ijm} \left( \phi^{(0)} \right) \left( \beta_i + \frac{\partial u_i^{(0)}}{\partial x_i} \right) + \]

\[ + R_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} - \frac{\partial M_{ij}^T}{\partial x_j} - Q_i^T = 0 \]  

(23)

\[ \frac{1}{2} G_{ijm} \left( \phi^{(0)} \right) \left( \frac{\partial \beta_i}{\partial x_j} + \frac{\partial \beta_i}{\partial x_j} \right) + R_{ijkl} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} - \frac{\partial Q_i^T}{\partial x_j} - \frac{\partial Q_i^T}{\partial x_j} = \chi^2 \Delta p \]

10. Interlaminar shear and transverse stresses in a plate with phase transformations

If functions \( u_i^{(0)}, u_3^{(0)}, \beta_i \) are obtained from the averaged system (23), then the deformations are calculated from the averaged kinematic relations (19), and the stresses \( \sigma_{ij}^{(0)} \) can be obtained.
The zero-order shear stress $\sigma_{33}^{(0)}$ and the zero-order transverse stress $\sigma_{33}^{(0)}$ were found to be identically zero in the plate. Nonzero shear stress appear at the next term in asymptotic expansion. For transverse stress, the first nonzero value in asymptotic is the value $\sigma_{33}^{(2)}$.

$$\sigma_{33} = -\chi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial \sigma_{33}^{(1)}}{\partial x_j} - \frac{\partial \sigma_{33}^{(1)}}{\partial x_j} \right) d\xi + \chi \left( -p - \Delta p \left( \frac{\xi}{2} + \frac{1}{2} \right) + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial \sigma_{33}^{(2)}}{\partial x_j} - \frac{\partial \sigma_{33}^{(2)}}{\partial x_j} \right) d\xi \right)$$  \hspace{1cm} (24)

$$\sigma_{13} = \chi \sigma_{13}^{(1)} + \chi^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial \sigma_{13}^{(2)}}{\partial x_j} - \frac{\partial \sigma_{13}^{(2)}}{\partial x_j} \right) d\xi$$  \hspace{1cm} (25)

11. Thermoelasticity equations system for composite plates with multistage phase transformations

The final equations system includes: the equations system describing changes in the phase compound composite material in the zero approximation, the heat conduction equation in the zero approximation, the averaged equations for the plates:

$$\rho_m \frac{\partial \phi_m^{(0)}}{\partial t} = J_{sw}^{(0)} \phi_m^{(0)}, \quad w = \{a, b\}, \quad s,v = 1,4$$

$$\rho_s \frac{\partial \phi_s^{(0)}}{\partial t} = J_s^{(0)} f_s \left( \theta^{(0)} \right)$$

$$C \left( \phi^{(0)} \right) \frac{\partial \theta^{(0)}}{\partial t} - \frac{\lambda_s}{\partial \xi} \left( \frac{\partial \theta^{(0)}}{\partial \xi} \right) = 0,$$

$$\tilde{A}_{iKL} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} + B_{iKL} \left( \phi^{(0)} \right) \frac{\partial^2 \beta_k}{\partial x_i \partial x_j} - \frac{\partial N_i^T}{\partial x_j} = 0$$

$$B_{iKL} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} + \tilde{B}_{iKL} \left( \phi^{(0)} \right) \frac{\partial^2 \beta_k}{\partial x_i \partial x_j} - \frac{1}{2} G_{IM} \left( \phi^{(0)} \right) \left( \frac{\partial \theta^{(0)}}{\partial x_i} + \frac{\partial ^2 u_i^{(0)}}{\partial x_j \partial x_j} \right) +$$

$$+ R_{IM} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} - \frac{\partial M_i^T}{\partial x_j} - Q_i^F = 0$$  \hspace{1cm} (26)

$$\frac{1}{2} G_{IM} \left( \phi^{(0)} \right) \left( \frac{\partial \beta_k}{\partial x_i} + \frac{\partial ^2 u_i^{(0)}}{\partial x_j \partial x_j} \right) + R_{IM} \left( \phi^{(0)} \right) \frac{\partial^2 u_k^{(0)}}{\partial x_i \partial x_j} - \frac{\partial^2 \beta_k}{\partial x_i \partial x_j} - \frac{\partial Q_i^F}{\partial x_j} = \chi^2 \Delta p$$

System (26) is supplemented by initial conditions for concentrations, initial and boundary conditions for the heat equation, and also boundary conditions at the ends of the plate.

12. Numerical calculation results for a thin plate under heating and bending

Figures 1 – 9 show the results of numerical simulation for the problem of plate deflection under the action of a uniformly distributed pressure on the upper surface ($\xi = 0.5$) and uniformly distributed temperature field. The left end of the plate is pinched, the right one is freely supported, the lower surface ($\xi = -0.5$) is thermally insulated. The plate parameter is $\chi = 0.025$.

Figure 1 shows temperature distribution over heating time at the different distance from the heating surface. Figure 2 shows the change in temperature over thickness at relative times $t_1 < t_2 < t_3 < t_4 (t_2 = 2 \cdot t_1, t_3 = 3 \cdot t_1, t_4 = 4 \cdot t_1)$. Figure 3 shows the change in the binder and fiber phases concentrations over the thickness at the time $t_3$ for the section $x_L = 0.75$. Matrix and fiber elasticity modulus over the thickness of the plate at the moments of time $t_1 < t_2 < t_3$ are presented at the figure 4. (the value of the modulus is referred to the characteristic value). Elastic modulus tensor components of fabric composite are given at figure 5 at the time $t_2$. 
Figure 1. Temperature dependence of time in section $\xi = 0.5, \xi = 0.25, \xi = 0, \xi = -0.5$

Figure 2. Temperature dependence over thickness at time points $t_1 < t_2 < t_3 < t_4$

Figure 3. Phases concentration over thickness at the time point $t_3$

Figure 4. Matrix and fiber dimensionless elasticity modulus at three different points in time

Figure 5. Elasticity modulus tensor components $A_{ijkl}$ for fabric composite material at the time point $t_2$ (dimensionless)

At the time $t_1$ at the point $\xi = 0.3$, the binder elastic modulus is significantly higher than on the upper surface of the plate, this shows that there is a relationship between the phases of the binder in a certain temperature range, at which the elastic properties will be greater than at the maximum and minimum values of temperature. This is a feature of the proposed model of binder phase transformations, which is characterized
by an increase the binder elastic properties in the temperature range 800-1100 K.

Figures 6 - 8 show the distribution of stresses $\sigma_{11}$, $\sigma_{33}$ and $\sigma_{13}$ over the thickness of the plate at times $t_1 < t_2 < t_3 < t_4$. Figure 9 shows the change in the deflection along the length of the plate at $t_1 < t_2 < t_3 < t_4$ time points.

Figure 6. Stress distribution over thickness in the section $x_1 = 0.75$ at 4 different points in time

Figure 7. Stress distribution over thickness in the section $x_1 = 0.75$ at 4 different points in time

Figure 8. Stress distribution $\sigma_{13}$ over thickness in the section $x_1 = 0.75$ at 4 different points in time

Figure 9. Deflection along the length of the plate at 4 different points in time

13. Conclusions

A modified version of the asymptotic theory for calculating thin multilayer plates made of composites with multistage phase transformations, which considers the finite values of the shear characteristics of the composite, has been developed. The developed mathematical model makes it possible to predict the stresses change in time in a thin inorganic composites plate during unsteady uneven heating to high temperatures and loading by uniform distributed pressure. The performed calculations of the stress-strain state of a thin composite plate made with multistage phase transformations made it possible to establish the effect of the influence of phase transformations on the kinetics of the change in the stress-strain state of the plate at different points in time, in particular, the appearance of internal zones of hardening of the material after preliminary heating before the onset of intermediate stages of phase transformations, and, as a consequence, an increase in the level of bending stresses in these zones.

References

[1] Abzgil’din F Yu, Tesvyatskiy S G 1980 Asbo-phosphate materials. Kiev. Naukova dumka 99 p
[2] Kopeykin V A et al 1974 Technology and properties of phosphate materials Moscow, Stroyizdat 224 p
[3] Dimitrienko Yu I, Sborschikov S V, Egoleva E S, Matveeva A A 2013 Modeling of thermo-elastic properties of composites with aluminocromic phosphate matrices Science and Education. Electronic scientific and technical Journal 11 pp 497-518

[4] Dimitrienko Yu I, Sborschikov S V, Gubareva E A, Egoleva E S, Matveeva A A 2015 Multiscale Modeling of High Temperature elastic-strength properties of inorganic matrix Composites Mathematical Modeling 27(11) pp 3-20

[5] Dimitrienko Yu I, Yakovlev D O 2015 The Asymptotic Theory of Thermoeelasticity of Multilayer Composite Plates Composites: Mechanics, Applications. An International Journal. 6(1) pp 13-51

[6] Dimitrienko Yu I 2015 Thermomechanics of Composites Structures under High Temperatures Springer. 367 p

[7] Dimitrienko Yu I, Gubareva E A, Sborschikov S V 2015 Numerical simulation of composite material thermal expansion by homogenization method Engineering Journal: Science and Innovation 12(48)

[8] Dimitrienko Yu I, Sborschikov S V, Belenovskaya Yu V, Aniskovich V A, Perevislov S N 2013 Modeling of Microdestruction and Strength of ceramic Composites on the base of reaction bonded SiC Science and Education. Electronic scientific and technical Journal. 11 pp 475-496

[9] Dimitrienko Yu I, Yurin Yu V 2017 Finite Element Modeling of Damage and Durability of Composite Structures with Local Delaminations Mathematical Modeling and Computational Methods 3 pp 49–70

[10] Dimitrienko Yu I, Sborschikov S V, Egoleva E S, Yakovlev D O 2019 Modeling of thermal stresses in inorganic matrix composite plates based on the asymptotic theory IOP Conference Series: Material Science and Engineering 683 № 012010

[11] Dimitrienko Yu I, Egoleva E S, Yakovlev D O and Sborschikov S V 2020 Modeling of stresses in inorganic composite plates under non uniform high temperature heating IOP Conference Series: Material Science and Engineering 934(2020) 012015