Fractal Generation in Modified Jungck–S Orbit

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ABSTRACT The aim of this paper is to modify the Jungck–S iterative scheme by adding the idea of $s$-convexity. We define and analyze the modified Jungck–S orbit (MJSO) with $s$-convex combination and derive the escape criterion for MJSO. Moreover, we establish the algorithms to visualize some Julia sets, Mandelbrot sets, and biomorphs in this orbit. In the biomorph generation algorithm, we did not fix the threshold radius of proposed orbit (i.e., MJSO) as fixed in literature earlier. We also discuss the graphical behavior of some complex polynomials in the generation of Julia sets, Mandelbrot sets, and biomorphs in MJSO.

INDEX TERMS Jungck–S orbit, biomorph, escape radius, $s$-convexity.

I. INTRODUCTION

In many fields of social sciences like complex graphics, biology, mathematics, computer and physics, fractal geometry plays an important role. The most archetypal studied fractals kinds were the Julia sets, Mandelbrot sets and then biomorphs respectively. Many different generalizations in fractals generation have been made in last few years. The use of one, two or multi step iterative scheme is one of such generalization. In these generalizations, researcher defined the orbit of iteration and proved the escape criterion for each feedback iteration. In the history of fractals, first time in 1970’s Benoît Mandelbrot studied the complex graph of connected Julia set for a complex function $z^2 + c$, where $z$ is a complex variable and $c \in \mathbb{C}$ as an input parameter. His work established a new field of mathematics to the world. Our universe full of beautiful natural scenes and fractal geometry exhibited these scene on computer with the use of mathematical algorithms. Fractal geometry provides the patterns of natural things, such as sky, living objects, mountains and galaxies, that establishes the new scheme of mathematical modeling. The complex graphical behavior of fractals discussed for complex polynomial $z^p + c$, where $p \geq 2$, in orbits of one-step, two-steps and higher-steps iterations which is the application of fixed point theory in fractal geometry.

The idea is to develop escape criterion for different steps iterative schemes given in [1], [2], and [3]. Some generalized Julia and Mandelbrot sets generated in [4] and [5]. Noor iterative scheme used to visualized the graphs of Julia sets, and also the images of Mandelbrot sets in [6]. The concept of $s$-convex function [7] with the combination of different iterative schemes demonstrated in many articles. The used of implicit iteration schemes in the visualization of fractals studied in [8], furthermore Nazeer et al. [9] proved new escape criterion for generalized Jungck iterations. The authors generated complex fractals via Noor iteration with $s$-convexity and $S$-iteration with $s$-convexity in [10] and [11]. Recently, Kwun et al. [12] also generalized the Jungck-CR iteration and visualized some beautiful fractals.

Biomorphism is a new architectural style in fractal geometry. The generated images have resembled with biological objects like bacteria, unicellular objects and many other living organism. Pickover [13] generated biomorphs in 1986, while he was generating Julia sets. A mistake made by Pickover in Julia sets algorithm [14] caused the appearance of biomorphs in fractal geometry. He found that biomorphs generated by him, were the modified Julia sets. Since then they were applied in design and art to create artful images [15]. Furthermore, many biological images found as applications of biomorphs. Mojica et al. [16] studied Darwinian theory of evolution via Pickover biomorphs and discussed the different steps of population evolution. Levin [17] applied biomorphs to develop morphologies form some basic law.
In 2016, Gdawiec et al. [18] studied some generalizations of the biomorphs generator algorithm established the Pickover by using some modified iterative schemes by choosing \( R \in \mathbb{R}_+ \) as threshold. In the generation of biomorphs authors used \( R = 10 \) as fixed input parameter.

In this article we introduce the Jungck-S iteration with \( s \)-convexity in the generation of fractals. We establish escape time criterion for the complex function \( z^p - d_1z + d_2 \). We also discuss the graphical behavior of Julia sets Mandelbrot sets and biomorphs in MJSO through some experiments. In biomorph generation we did not fixed the threshold radius in MJSO as 10. We generate them in modified Jungck-S orbit of general threshold radius. Because the orbital radius in fractal generation depends upon the complex polynomial \( f(z) \).

We systemized the rest of article as follows: if there exists \( f \in \mathbb{C} \) we study some basic concepts used in the article. The Sec. III and biomorphs for \( f \) mine whether the orbit escapes or not we take escape radius whether the orbital sequence tends to infinity or not. To determine the number of iterates which is necessary to examine \( K \) time algorithm. This algorithm depend on \( s \)-debrt sets and biomorphs in MJSO. The last section (Sec. V), we draw some conclusions.

II. PRELIMINARIES

Definition 1 (Julia Set [19]): Let \( F_f \) be the set of points in \( \mathbb{C} \) such that \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a complex polynomial of degree \( \geq 2 \). The set \( F_f \) is called filled Julia set, when the orbits of \( F_f \rightarrow \infty \) as \( p \rightarrow \infty \), i.e.,

\[
F_f = \{ z \in \mathbb{C} : (|f^p(z)|)_{p=0}^{\infty} \text{ is bounded} \}.
\]

The boundary of filled Julia set is called simply Julia set.

Definition 2 (Mandelbrot Set [20]): The set which consists of all connected Julia sets is called Mandelbrot set \( M \), i.e.,

\[
M = \{ c \in \mathbb{C} : f_c \text{ is connected} \},
\]

or we can define Mandelbrot set equivalently as follows [21]:

\[
M = \{ c \in \mathbb{C} : |f^p(0)| \text{ does not tend to } \infty \text{ as } p \rightarrow \infty \}.
\]

\( f \) has only critical point 0 (i.e., \( f'(0) = 0 \)). So authors choose 0 as a initial point.

Definition 3 (Biomorph): An object resemble with the living organism is called biomorph.

For the generation of fractals like Julia sets, Mandelbrot sets and biomorphs, authors use many different algorithms [22], [23]. In this article we use only the escape time algorithm. This algorithm depend on \( K \) i.e., the maximum number of iterates which is necessary to examine whether the orbital sequence tends to infinity or not. To determine whether the orbit escapes or not we take escape radius derived in criterion. Usually, for Julia sets, Mandelbrot sets and biomorphs for \( f(z) = z^2 + d_2 \), the escape criteria is as follows: if there exists \( i \geq 0 \) such that

\[
|f^i(z)| > \max(|d_2|, 2),
\]

then \( f^i(z) \rightarrow \infty \) as \( i \rightarrow \infty \).

We call max\(|d_2|, 2\) the escape threshold radius. The escape radius for \( f \) is different for each step iteration. The escape radius plays a primary role in the visualization of fractals.

We establish escape time algorithms to generate some Julia sets, Mandelbrot sets and biomorphs. The algorithms are presented in Algorithm 1, 2 and 3, respectively.

Algorithm 1: Visualization of Julia Set

Input: \( f : \mathbb{C} \rightarrow \mathbb{C} \) – polynomial function, \( d_2 \in \mathbb{C} \) – parameter, \( A \subset \mathbb{C} \) – area, \( K \) – maximum number of iterations, \( \text{colourmap}[0..C-1] \) – colourmap with \( C \) colours.

Output: \( A \) is area for Julia set.

1. \( R = \) calculate the escape threshold radius
2. for \( z_0 \in A \) do
3. \( i = 0 \)
4. while \( i \leq K \) do
5. \( z_{i+1} = f_c(z_i) \)
6. if \( |z_{i+1}| > R \) then
7. \( \text{break} \)
8. \( i = i + 1 \)
9. \( n = \lfloor (C-1)/2 \rfloor \)
10. color \( z_0 \) with \( \text{colourmap}[n] \)

Algorithm 2: Visualization of Mandelbrot

Input: \( f : \mathbb{C} \rightarrow \mathbb{C} \) – polynomial function, \( A \subset \mathbb{C} \) – area, \( K \) – maximum number of iterations, \( \text{colourmap}[0..C-1] \) – colourmap with \( C \) colours.

Output: \( A \) is area for Mandelbrot set.

1. for \( c \in A \) do
2. \( R = \) calculate the escape threshold radius
3. \( i = 0 \)
4. \( z_0 = \) critical point of \( f \)
5. while \( i \leq K \) do
6. \( z_{i+1} = f_c(z_i) \)
7. if \( |z_{i+1}| > R \) then
8. \( \text{break} \)
9. \( i = i + 1 \)
10. \( n = \lfloor (C-1)/2 \rfloor \)
11. color \( c \) with \( \text{colourmap}[n] \)

Algorithm 3: Jungck Iteration [24]

Let \( P, Q : X \rightarrow X \) be the two maps such that \( P \) is one to one and \( Q \) is differentiable of degree greater and equal to 2. For any \( x_0 \in X \) the Jungck iteration is defined in the following way

\[
P(x_{i+1}) = Q(x_i),
\]

where \( i = 0, 1, \ldots \).
Algorithm 3: Visualization of Biomorphs

Input: \( f_c : \mathbb{C} \rightarrow \mathbb{C} \) - polynomial function, \( d_2 \in \mathbb{C} \) - parameter, \( A \subset \mathbb{C} \) - area, \( K \) - maximum number of iterations, \( \text{colourmap}[0..C-1] \) - colourmap with \( C \) colours.

Output: \( A \) is area for biomorphs.

1. \( R = \) calculate the escape threshold radius
2. for \( z_0 \in A \) do
   3. \( i = 0 \)
   4. while \( i \leq K \) do
      5. \( z_{i+1} = f(z_i) \)
      6. if \( |z_{i+1}| > R \) then
         7. break
      8. \( i = i + 1 \)
   9. if \( |\Re(z_i)| < R \lor |\Im(z_i)| < R \) then
      10. \( n = \lfloor (C - 1) \frac{|z_i|}{R} \rfloor \)
      11. colour \( z_0 \) with \( \text{colourmap}[n] \)
   12. else
      13. colour \( z_0 \) with \( \text{colourmap}[0] \)

Definition 5 (Jungck-Mann Iteration With s-Convexity [9]): Let \( P, Q : \mathbb{C} \rightarrow \mathbb{C} \) be the two complex maps such that \( Q \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( P \) is injective. For any \( x_0 \in \mathbb{C} \) the Jungck-Mann iteration with s-convexity is defined as:
\[
P(x_{i+1}) = (1 - a)^iP(x_i) + a^iQ(x_i),
\]
where \( a, s \in (0, 1], i = 0, 1, 2, \ldots \).

Definition 6 (Jungck-Isikawa Iteration With s-Convexity [9]): Let \( P, Q : \mathbb{C} \rightarrow \mathbb{C} \) be the two complex maps such that \( Q \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( P \) is one to one. For any \( x_0 \in \mathbb{C} \) the Jungck-Isikawa iteration with s-convexity is defined in the following way
\[
\begin{align*}
P(x_{i+1}) &= (1 - a)^iP(x_i) + a^iQ(x_i), \\
P(y_i) &= (1 - b)^iP(x_i) + b^iQ(u_i),
\end{align*}
\]
where \( a, b, s \in (0, 1] \) and \( i = 0, 1, 2, \ldots \).

Definition 7 (Jungck-Noor Iteration With s-Convexity [9]): Let \( P, Q : \mathbb{C} \rightarrow \mathbb{C} \) be the two complex maps such that \( Q \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( P \) is one to one. For any \( x_0 \in \mathbb{C} \) the Jungck-Noor iteration with s-convexity is defined as:
\[
\begin{align*}
P(x_{i+1}) &= (1 - a)^iP(x_i) + a^iQ(x_i), \\
P(y_i) &= (1 - b)^iP(x_i) + b^iQ(u_i), \\
P(u_i) &= (1 - c)^iP(x_i) + c^iQ(x_i),
\end{align*}
\]
where \( a, b, c, s \in [0, 1] \) and \( i = 0, 1, 2, \ldots \).

Definition 8 (Jungck-CR Iteration With s-Convexity [?]?): Let \( P, Q : \mathbb{C} \rightarrow \mathbb{C} \) be the two complex maps such that \( Q \) is a complex polynomial of degree greater and equal to 2, also differentiable and \( P \) is one to one. For any \( x_0 \in \mathbb{C} \) the Jungck-CR iteration with s-convexity is defined in the following way
\[
\begin{align*}
P(x_{i+1}) &= (1 - a)^iP(x_i) + a^iQ(x_i), \\
P(y_i) &= (1 - b)^iP(x_i) + b^iQ(x_i),
\end{align*}
\]
where \( a, b, c, s \in [0, 1] \) and \( i = 0, 1, 2, \ldots \).

III. MAIN RESULT

Generally, in the literature authors used Picard iteration to visualized the fractals. Authors replaced the Picard iteration with some implicit iterative schemes, e.g., with some iterative schemes discussed in Sec. II [7], [9]. In this section we define Jungck-S iterative scheme with s-convexity and prove escape criterion to determine the escape radius of modified Jungck-S orbit (MISO).

Here we take a start with the definition of the Jungck-S iterative scheme.

Definition 9 (Jungck-S Iteration [24]): Let \( P, Q : \mathbb{X} \rightarrow \mathbb{X} \) be to maps, where \( P \) one to one and \( Q \) is differentiable. Consider \( x_0 \in \mathbb{X} \) be the initial point. The Jungck-S iterative scheme is defined as:
\[
\begin{align*}
P(x_{i+1}) &= (1 - a)Q(x_i) + aQ(y_i), \\
P(y_i) &= (1 - b)P(x_i) + bQ(u_i),
\end{align*}
\]
where \( a, b \in (0, 1] \) and \( i = 0, 1, 2, \ldots \).

In each of the two steps of the Jungck-S iterative scheme we use a convex combination of two elements. In the literature we can find some generalizations of the convex combination. One of such generalizations is the s-convex combination.

Definition 10 (s-Convex Combination [7]): Let \( z_1, z_2, \ldots, z_t \in \mathbb{C} \) and \( s \in (0, 1] \). The s-convex combination is defined in the following way:
\[
\theta_1 z_1 + \theta_2 z_2 + \ldots + \theta_t z_t,
\]
where \( \theta_i \geq 0 \) for \( i \in \{1, 2, \ldots, p\} \) and \( \sum_{i=1}^{p} \theta_i = 1 \).

We analyze that the s-convexity for \( s = 1 \) reduces to the standard convex combination. This type of combination was successfully used in the generation of fractals [2], [8], [9]. Furthermore, it was also used in the generation of other type of fractals generated in the complex plane, namely in the methods that use root finding of complex polynomials [25].

Now, we will change the convex combination in the Jungck-S iterative scheme with the s-convex one.

Definition 11 (Jungck-S Iterative Scheme With s-Convexivity): Let \( X \) be a metric space and \( P, Q : X \rightarrow X \) be mappings, where \( P \) one to one, and let \( x_0 \in X \) be an initial point. The Jungck-S iterative scheme with s-convexity is defined as follows:
\[
\begin{align*}
P(x_{i+1}) &= (1 - a)^iQ(x_i) + a^iQ(y_i), \\
P(y_i) &= (1 - b)^iP(x_i) + b^iQ(x_i),
\end{align*}
\]
where \( a, b, s \in (0, 1] \) and \( i = 0, 1, 2, \ldots \).
We have made the following observations about Jungck-S iterative scheme with \( s \)-convex combination:
- Reduce to Picard iterative scheme when \( P \) is identity map.
- Reduce to Mann iterative scheme when \( P \) is identity map, \( b = 0 \) and \( s = 1 \).
- Reduce to \( S \) iterative scheme, when we choose \( S \) as a identity map and \( s = 1 \).
- Reduce to Jungck Mann iterative scheme when \( b = 0 \) and \( s = 1 \) and
- Reduce to Jungck-S iterative scheme, when we choose \( s = 1 \).

Therefore, we will establish new orbits for this proposed iteration and in the constitution of new fractals.

We break down \( f \) into two maps \( P, Q : \mathbb{C} \rightarrow \mathbb{C} \) and \( \{z_i\}_{i \in \mathbb{N}} \) be the sequence of iterates for any initial point \( z \in \mathbb{C} \) defined as follows:

\[
\begin{align*}
P(z_{i+1}) &= (1 - a)z_i + aQ(z_i), \\
Q(z_i) &= bP(z_i) + \beta Q(z_i) \quad i = 0, 1, 2, \ldots ,
\end{align*}
\] (13)

where \( a, b, \beta \in (0, 1] \). Then the sequence \( z_{i+1} \) of equation (13) is known as modified Jungck-S orbit.

In next three subsections we derive escape radius of MJSO for some complex polynomials.

### A. ESCAPE CRITERION FOR THE QUADRATIC COMPLEX POLYNOMIAL

Let \( f(z) = z^2 - d_1z + d_2 \) be a quadratic complex polynomial. We break down the function \( f \) in such way that: \( Q(z) = z^2 + d_2 \) and \( P(z) = d_1z \), where \( d_1, d_2 \in \mathbb{C} \).

**Theorem 1:** Suppose that \( |z| \geq |d_2| > \frac{2(1 + |d_1|)}{sb} \), \( |z| \geq |d_2| > \frac{2(1 + |d_1|)}{ab} \), \( |z| \geq |d_2| > \frac{2(1 + |d_1|)}{sab} \), where \( a, b, \beta \in (0, 1] \), then the sequence \( \{z_i\}_{i \in \mathbb{N}} \) of iterates as \( i \to \infty \) we have

\[
|P(z_0)| = \left|(1 - b)^sP(z_0) + b^sQ(z_0)\right| = \left|(1 - b)^s|d_1z| + (1 - b)^s(z^2 + d_2)\right|
\]

Expanding up to first term of \( b \) and \( 1 - b \), and using \( s < 1 \) we get

\[
|d_1|y_0| \geq (1 - s(1 - b))|z|^2 + d_2| - (1 - sb)|d_1z| \geq (s - s(1 - b))(z^2 + d_2) - |(1 - sb)d_1z|.
\]

Using facts \( |z| \geq |d_2| \) and \( sb < 1 \) we get

\[
|d_1y_0| \geq sb|z|^2 - sb|d_2| - (1 - sb)|d_1z|\]

\[
= sb|z|^2 - |d_1z| + sb|d_1z| \\
\geq sb|z|^2 - |d_1z| + sb|d_1z| \geq |z|(sb|z| - (1 + |d_1|)).
\]

As follows

\[
|y_0| \geq |z| \left(\frac{sb|z|}{1 + |d_1|} - 1\right).
\]

Since \( |z| > \frac{2(1 + |d_1|)}{sb} \), this implies \( |z| > |z|^2 \left(\frac{sb|z|}{1 + |d_1|} - 1\right)^2 > |z|^2 > sb|z|^2 \), and \( 1 + |d_1| \geq 1 \). We get

\[
|d_1y_0| \geq sb|z|^2 - (1 + |d_1|)|z|.
\]

Thus

\[
|y_0| \geq |z| \left(\frac{sb|z|}{1 + |d_1|} - 1\right).
\]

Now in next step of iteration we have

\[
|P(z_1)| = \left|(1 - a)^sQ(z_0) + a^sQ(y_0)\right| \\
|d_1z_1| = \left|(1 - a)^s(z^2 + d_2) + a^s(y^2 + d_2)\right| \\
\geq \left|(1 - a)^s(z^2 + d_2) + (1 - (1 - a)^s(y^2 + d_2)\right| \\
\geq \left|(1 - (1 - a)^s(z^2 + d_2) + (1 - (1 - a)^s(y^2 + d_2)\right|.
\]

Since \( (1 - sa) \geq 0, (1 - (1 - a)) \geq sa \) and \( |y|^2 \geq sb|z|^2 \) we get

\[
|d_1z_1| \geq s^2|abz|^2 + d_2 \\
\geq s^2ab|z|^2 - |d_2| \geq s^2ab|z|^2 - |z| \geq |z|(s^2ab|z|^2 - (1 + |d_1|)).
\]

Hence

\[
|z_1| \geq |z| \left(\frac{s^2ab|z|^2}{1 + |d_1|} - 1\right).
\]

Since \( |z| > \frac{2(1 + |d_1|)}{sb} \), \( |z| > \frac{2(1 + |d_1|)}{ab} \), so \( |z| > \frac{2(1 + |d_1|)}{sab} \) and this implies \( s^2ab|z|^2 - 1 \geq 1 \). Therefore there exist \( \theta > 0 \) such that \( \theta \geq \frac{|z|^2}{ab|z|^2 - 1} \). Consequently \( |z_1| > (1 + \theta)|z| \). In particular \( |z_1| > |z| \). So we may iterate to find \( |z_1| > (1 + \theta)^s|z| \). Hence, the orbit of \( z \) tends to infinity and this completes the proof.

**Corollary 1:** Assume that

\[
|d_2| > 2(1 + |d_1|) \quad \text{and} \quad |d_1| > 2(1 + |d_1|),
\]

then MJSO escapes to infinity.

Hence this corollary clarify our results.

**Corollary 2:** Let \( a, b, s \in (0, 1] \) and

\[
|z| > \max \left\{ \left|d_2\right|, \left|\frac{2(1 + |d_1|)}{sa}, \frac{2(1 + |d_1|)}{sb}\right\} .
\]

\[
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\]
therefore there exist \( \theta > 0 \) such that \(|z_i| > (1 + \theta)^i|z|\) and \(|z_i| \to \infty\) as \(i \to \infty\).

Corollary 3: Assume that

\[ |z_m| > \max\left\{|d_2|, \frac{2(1 + |d_1|)}{sa}, \frac{2(1 + |d_1|)}{sb}\right\} \]

for some \(m \geq 0\). Thus there exist \( \theta > 0 \) such that \(|z_{m+1}| > (1 + \theta)^i|z_m|\) and \(|z_i| \to \infty\) as \(i \to \infty\).

B. ESCAPE CRITERION FOR THE CUBIC COMPLEX POLYNOMIAL

Let \(f(z) = z^3 - d_1 z + d_2\), where \(d_1, d_2 \in \mathbb{C}\). We break down the function \(f\) in such a way that: \(Q(z) = z^3 + d_2\) and \(S(z) = d_1 z\).

Theorem 2: Suppose that \(|z| \geq |d_2| > \frac{2(1+|d_1|)}{sb}\), \(|z| \geq |d_2| > \frac{2(1+|d_1|)}{sa}\), \(|z| \geq |d_2| > \frac{2(1+|d_1|)}{sa}\), \(|z| \geq |d_2| > \frac{2(1+|d_1|)}{sa}\).

\[
\begin{align*}
P(z+1) &= (1 - a)^i Q(z) + a^i Q(y) \\
P(y) &= (1 - b)^i P(z) + b^i Q(z) \quad i = 0, 1, 2, \ldots, (15)
\end{align*}
\]

where \(z_0 = z\) and \(y_0 = y\). Then \(|z_i| \to \infty\) as \(i \to \infty\).

Proof: Since \(Q(z) = z^3 + d_2\), \(P(z) = d_1 z\), \(z_0 = z\) and \(y_0 = z\) we have

\[ |P(y_0)| = |(1 - a)^i P(z) + b^i Q(z)| = |(1 - a)^i d_1 z + (1 - b)^i (z_3 + d_2)| \]

Expanding up to first term of \(b\) and \(1 - b\), and using \(s < 1\) we get

\[ \begin{align*}
|d_1 y_0| &\geq |(s - s(1-b))(z_3 + d_2)| - |(1-sb)|d_1 z| \\
&\geq |(s - s(1-b))(z_3 + d_2)| - |(1-sb)|d_1 z|.
\end{align*} \]

Since \(|z| \geq |d_2|\) and \(sb < 1\) we have

\[ \begin{align*}
|d_1 y_0| &\geq sb|z|^3 - sb|d_2| - |(1-sb)|d_1 z| \\
&= sb^2|z|^3 - sb|d_2| - |d_1 z| + sb|d_1 z| \\
&\geq sb|z|^3 - |z| - |d_1 z|. \\
&= |z|(sb|z|^2 - (1 + |d_1|)).
\end{align*} \]

This yields

\[ |y_0| \geq |z| \left(\frac{sb|z|^2}{1 + |d_1|} - 1\right). \]

Since \(|z| > |d_2|\), this follows \(|z|^3 \left(\frac{sb|z|^2}{1 + |d_1|} - 1\right) > |z|^3\). Therefore \(|y|^3 > |z|^3 \left(\frac{sb|z|^2}{1 + |d_1|} - 1\right) > |z|^3 > sb|z|^3\), and \(1 + |d_1| \geq 1\). We obtain

\[ |d_1 y_0| \geq sb|z|^3 - (1 + |d_1|)|z|. \]

So

\[ |y_0| \geq |z| \left(\frac{sb|z|^2}{1 + |d_1|} - 1\right). \]

Know in next step of iteration we get

\[ \begin{align*}
|P(z_1)| &= |(1 - a)^i Q(z_0) + a^i Q(y)| \\
|d_1 z_1| &= |(1 - a)^i (z_3 + d_2) + a^i (y^3 + d_2)| \\
&\geq |(1 - a)^i (z_3 + d_2) + (1 - (1-a)^i (y_3 + d_2))| \\
&\geq |(1 - sa)(z_3 + d_2) + (1 - s(1-a))(y_3 + d_2)|.
\end{align*} \]

Since \((1 - sa) > 0\), \((1 - s(1-a)) > sa\) and \(|y|^3 > sb|z|^3\), we have

\[ \begin{align*}
|d_1 z_1| &\geq |sb|z|^3 + d_2| \\
&\geq |sb^2|z|^3| - |d_2| \\
&\geq |sb^2|z|^3| - |z| \\
&= |z|\left(|sb^2|z|^2 - (1 - |d_1|)\right).
\end{align*} \]

Hence

\[ |z| \geq |z|\left(|sb^2|z|^2 - (1 - |d_1|)\right). \]

Since \(|z| > |d_2|\), this implies \(sb^2|z|^2 > 1 > 1 + \theta\). Consequently \(|z| > (1 + \theta)|z|\). In particular \(|z| > |z|\). So we may iterate to find \(|z| > (1 + \theta)^i|z|\). Hence, the orbit of \(z\) tends to infinity and this completes the proof.

Corollary 4: Assume that

\[ |d_2| > \left(\frac{2(1 + |d_1|)}{sa}\right)\frac{1}{2} \quad \text{and} \quad |d_2| > \left(\frac{2(1 + |d_1|)}{sb}\right)\frac{1}{2}, \]

then MJSO escapes to infinity.

Hence the corollary clarify our results.

Corollary 5: Let \(a, b, s \in (0, 1]\) and

\[ |z| > \max\left\{|d_2|, \left(\frac{2(1 + |d_1|)}{sa}\right)\frac{1}{2}, \left(\frac{2(1 + |d_1|)}{sb}\right)\frac{1}{2}\right\}, \]

therefore there exist \( \theta > 0 \) such that \(|z_i| > (1 + \theta)^i|z|\) and \(|z_i| \to \infty\) as \(i \to \infty\).

Corollary 6: Assume that

\[ |z_m| > \max\left\{|d_2|, \left(\frac{2(1 + |d_1|)}{sa}\right)\frac{1}{2}, \left(\frac{2(1 + |d_1|)}{sb}\right)\frac{1}{2}\right\}, \]

for some \(m \geq 0\). Thus there exist \( \theta > 0 \) such that \(|z_{m+1}| > (1 + \theta)^i|z_{m+1}|\) and \(|z_i| \to \infty\) as \(i \to \infty\).

C. ESCAPE CRITERION FOR HIGHER DEGREE COMPLEX POLYNOMIALS

Let \(f(z) = z^3 - d_1 z + d_2\), where \(d_1, d_2 \in \mathbb{C}\). We break down the function \(f\) in such a way that: \(Q(z) = z^3 + d_2\) and \(P(z) = d_1 z\).
Theorem 3: Suppose that $|z| \geq |d_2| > \frac{2(1+|d_1|)}{sa}$, $|z| \geq |d_2| > \frac{2(1+|d_1|)}{sb}$ where $a, b, s \in (0, 1]$, then the sequence $\{z_i\}_{i \in \mathbb{N}}$ define as follows

$\begin{align*}
P(z_{i+1}) &= (1-a)^i Q(z_0) + a^i Q(y_0) \\
P(y_0) &= (1-b)^i P(z_0) + b^i Q(z_0) \quad i = 0, 1, 2, \ldots
\end{align*}$

(16)

where $z_0 = z$ and $y_0 = y$. Then $|z_i| \to \infty$ as $i \to \infty$.

Proof: Since $Q(z) = z^p + d_2$, $P(z) = d_1 z$, $z_0 = z$ and $y_0 = y$ we get

$$|P(y_0)| = |(1-b)^i P(z) + b^i Q(z)| = |(1-b)^i d_1 z + (1-(1-b))^i| > 0$$

Expanding up to first term of $b$ and $1-b$, and using $s < 1$ we obtain

$$|d_1 y_0| \geq (s-1-b)|z| - (1-sb)|d_1|z| \geq 1 + |d_1||z|.$$

Since $|z| \geq |d_2|$ and $sb < 1$ we get

$$|d_1 y_0| \geq sb |z^p| - sb|d_2| - (1-sb)|d_1|z|,$$

and $1 + |d_1| \geq 1$. We have

$$|d_1 y_0| \geq sb |z^p| - (1+|d_1|)|z|.$$

So

$$|y_0| \geq |z| \left( \frac{sb |z^p|-1}{1+|d_1|} \right).$$

Therefore

$$|z_1| \geq |z| \left( \frac{s^2 ab |z^p|-1}{1+|d_1|} \right).$$

Since $|z| > \frac{2(1+|d_1|)}{sa}$, this implies $|z| > \frac{2(1+|d_1|)}{sb}$ and this implies $s^2 ab |z^p|-1 > 1$.

Therefore there exist $\theta > 0$ such that $s^2 ab |z^p|-1 > 1 + \theta$. Consequently $|z_1| > (1+\theta)|z|$. In particular $|z_1| > |z|$. So we may iterate to find $|z_i| > (1+\theta^i)|z|$. Hence, the orbit of $z$ tends to infinity and this completes the proof.

Corollary 7: Assume that

$$|d_2| > \left( \frac{2(1+|d_1|)}{sa} \right)^{\frac{1}{\theta}} \text{ and } |d_2| > \left( \frac{2(1+|d_1|)}{sb} \right)^{\frac{1}{\theta}},$$

then MJSO escapes to infinity.

Corollary 8: Let $a, b, s \in (0, 1]$ and

$$|z| > \frac{2(1+|d_1|)}{sa}, \quad (1+1+\theta)|z|$$

therefore there exist $\theta > 0$ such that $|z| > (1+\theta)|z|$ and $|z| \to \infty$ as $i \to \infty$.

Corollary 9: Assume that

$$|z_m| > \frac{2(1+|d_1|)}{sa}, \quad \text{for some } m \geq 0. \text{ Thus there exist } \theta > 0 \text{ such that } |z_{m+i}| > (1+\theta)|z_m| \text{ and } |z| \to \infty \text{ as } i \to \infty.$$

Theorem 4: Suppose that $\{z_i\}_{i \in \mathbb{N}}$ be the sequence of points in MJSO for complex polynomial $z^p - d_1 z + d_2$ with $p \geq 2$ such that $|z_i| \to \infty$ as $i \to \infty$, then $|z| \geq |d_2| > \left( \frac{2(1+|d_1|)}{sa} \right)^{\frac{1}{\theta}},$

$$|z| \geq |d_2| > \left( \frac{2(1+|d_1|)}{sa} \right)^{\frac{1}{\theta}}, \quad \text{where } a, b, s \in (0, 1].$$

Proof: Since $\{z_i\}_{i \in \mathbb{N}}$ is the sequence of points in MJSO for complex polynomial $z^p - d_1 z + d_2$ with $p \geq 2$ such that $|z_i| \to \infty$ as $i \to \infty$, therefore there exist $\theta > 0$ such that

$$|z| > (1+\theta)|z|.$$
Since for fractal generation it must be true $|z| \geq |d_2|$. Also $sb < 1$ we obtain

$$\begin{align*}
|d_1|_{0} & \geq sb \left| z^p - sb|d_2| - (1 - sb)|d_1|z \right| \\
& = sb \left| z^p - sb|d_2| - |d_1|z + sb|d_1|z \right| \\
& \geq sb \left| z^p - |d_1|z \right| \\
& = |z|(sb \left| z^{p-1} \right| - (1 + |d_1|)).
\end{align*}$$

This yields

$$\begin{align*}
|y_0| & \geq |z| \left( \frac{sb \left| z^{p-1} \right| - 1}{1 + |d_1|} \right) \\
|y_0| & \geq sb |z|.
\end{align*}$$

Since Julia and Mandelbrot sets are bounded, therefore

$$\left( \frac{sb \left| z^{p-1} \right|}{1 + |d_1|} - 1 \right) \geq 1.$$ 

For next step we have

$$\begin{align*}
|P(z_1)| & = |(1 - a)^y (z_0) + a^p Q(y_0)| \\
|a_z| & = |(1 - a)^y (z_0 + d_2) + a^p (y_0 + d_2)| \\
& \geq |(1 - a)^y (z_0 + d_2) + (1 - (1 - a)^y (y_0 + d_2))| \\
& \geq |(1 - a)^y (z_0 + d_2) + (1 - s(1 - a))(y_0 + d_2)| \\
& \geq |s| y^p + d_2| \\
& \geq |s^2 ab z^p + d_2| \\
& \geq |s^2 ab z^p| - |d_2| \\
& \geq |z|(|s^2 ab z^{p-1}| - 1).
\end{align*}$$

This follows that

$$|z| \geq \left( \frac{s^2 ab \left| z^{p-1} \right| - 1}{1 + |d_1|} \right). \quad (18)$$

Comparing (17) and (18), we have

$$\begin{align*}
\frac{s^2 ab \left| z^{p-1} \right| - 1}{1 + |d_1|} & = 1 + \theta \\
\frac{s^3 ab \left| z^{p-1} \right| - 1}{1 + |d_1|} & > 1,
\end{align*}$$

because $\theta > 0$. This implies

$$|z| > \left( \frac{2(1 + |d_1|)}{s^2 ab} \right)^{\theta^{\text{pars}}}.$$ 

As a result, we obtain $|z| > \left( \frac{2(1 + |d_1|)}{s^2 ab} \right)^{\theta^{\text{pars}}}$ and $|z| > \left( \frac{2(1 + |d_1|)}{s^2 ab} \right)^{\theta^{\text{pars}}}$ where $p \geq 2$ and $a, b, s \in (0, 1]$. To visualize complex fractal $|z| \geq |d_2|$ must exist, because for any given point $|z| < |d_2|$, we have to compute the MJSO of $z$. If for some $i$, $|z_i|$ lies outside the circle of radius $\max \left\{ |d_2|, \frac{2(1 + |d_1|)}{s^2 ab}, \frac{2(1 + |d_1|)}{s^2 ab} \right\}$, we observed that MJSO escapes. Hence, $z$ is not in the Julia sets and also, is not in Mandelbrot sets. But if the sequence $|z_i|$ is bounded to obey $|z| \geq |d_2|$, then by definition of complex fractals, the sequence $|z_i|$ lies in MJSO. Hence the result. □

### IV. GENERATION OF FRACTALS

Here we demonstrate some complex graphs of Julia sets, Mandelbrot sets and biomorphs in improved Jungck-S orbit with the help of algorithms established in Sec. III. The generated graphs are discuss in three different subsections. We used 32-bit operating system with specifications: Intel(R) Core(TM)i5-3320M CPU @ 2.60 GHz and 4GB RAM to run the algorithms in Mathematica to generate images. The color variation appeared in images due to change of input parameters. The color variation depends upon the number of iterations also.

#### A. GENERATION OF JULIA SETS

The first three figures of complex graphs are the results of experiments for Julia sets generated in MJSO with algorithm 1 for the complex polynomial $f(z) = z^p - d_1 z + d_2$. The maximum number of iterations was fixed for all three experiments of Julia sets, and we fixed the iterations at 15. The quadratic, cubic and quartic Julia sets are generated in Figs. 1–3 and following were input limitations for Figs. 1–3:

- Fig. 1: $p = 2$, $d_2 = 0.73 - 2.6i$, $d_1 = -1 - i$, $a, b, s = 0.9$, $A = [-2.1, 2.1] \times [-2, 2]$. 
- Fig. 2: $p = 3$, $d_2 = 0.5 - 1.6i$, $d_1 = 1 + i$, $a, b, s = 0.9$, $A = [-1.75, 1.3] \times [-1.6, 1.5]$. 
- Fig. 3: $p = 4$, $d_2 = 1 + 0.8i$, $d_1 = \sqrt{5} - i$, $a, b, s = 0.1$, $A = [-1.5, 1.5]^2$.

#### B. GENERATION OF MANDELBROT SETS

In next three experiments, we presented Mandelbrot sets in MJSO with algorithm 2 for the complex $f(z) = z^p - d_1 z + d_2$.

**FIGURE 1.** Quadratic Julia set generated in MJSO.

**FIGURE 2.** Cubic Julia set generated in MJSO.
We fixed the iterations at 15. The quadratic, cubic and quartic Mandelbrot sets are generated in Figs. 4–6 and following were input limitations for Figs. 4–6:

- Fig. 4: \(p = 2, d_1 = -1 - i, a, b, s = 0.9, A = [-5.1, 2] \times [-3.5, 3.5]\),
- Fig. 5: \(p = 3, d_1 = 1 + i, a, b, s = 0.9, A = [-2.3, 2.3] \times [-3, 3]\),
- Fig. 6: \(p = 4, d_1 = 2 - i, a, b, s = 0.1, A = [-2.3, 2.3]^2\).

C. GENERATION OF BIOMORPHS

In this experiment images of biomorphs generated MJSO for the complex polynomial \(f_z = z^2 - d_1z + d_2\) with algorithm 3. Similar to the experiments for Julia and Mandelbrot sets, the maximum number of iterations were fixed and we stopped the iteration at 15 for all images of biomorphs. We observed from figures presented in Figs. 7–15 that the image of quadratic biomorph for the complex polynomial \(f_z = z^2 - d_1z + d_2\) changes with change of input parameters. The complex graphs of quadratic biomorphs are presented in Fig. 7–15 and following were input limitations for Fig. 7–15:

- Fig. 7: \(d_2 = -1 - i, d_1 = 1 + i, a = 0.3, b = 0.4, s = 0.5, A = [-13, 13] \times [-20.5, 8]\),
- Fig. 8: \(d_2 = 0, d_1 = 1 + i, a = 0.3, b = 0.4, s = 0.5, A = [-13, 13] \times [-25.5, 10]\),
- Fig. 9: \(d_2 = 0, d_1 = 2, a = 0.3, b = 0.4, s = 0.5, A = [-43, 13] \times [-25.5, 25]\),
- Fig. 10: \(d_2 = 0.5 + 0.5i, d_1 = 1 + i, a = 0.3, b = 0.4, s = 0.5, A = [-13, 13] \times [-30.5, 8]\),
- Fig. 11: \(d_2 = -1 + i, d_1 = 1 + i, a = 0.3, b = 0.4, s = 0.5, A = [-13, 13] \times [-15.5, 23]\),
- Fig. 12: \(d_2 = -1 + i, d_1 = 1 + i, a = 0.9, b = 0.9, s = 0.9, A = [-2.5, 2.5] \times [-7.8, 1.5]\),
- Fig. 13: \(d_2 = 0, d_1 = -1 - i, a = 0.9, b = 0.9, s = 0.9, A = [-3.3, 3.3] \times [-5.3, 2.2]\),
- Fig. 14: \(d_2 = -1 + 2i, d_1 = 1 + i, a = 0.9, b = 0.9, s = 0.9, A = [-3.3, 10] \times [-5.2, 10.3]\),
- Fig. 15: \(d_2 = 5i, d_1 = -1 - i, a = 0.9, b = 0.9, s = 0.9, A = [-23.3, 63.3] \times [-35.3, 32.2]\).

In this experiment we presented the images of biomorphs generated in MJSO for the complex polynomial \(f_z = z^3 - d_1z + d_2\) by using algorithm 3. Similar to the previous
experiments, the maximum number of iterations was fixed and we stopped the iteration at 15 for all images of biomorphs. We observed from figures presented in Figs. 16–24 that the image of cubic biomorph for the complex polynomial \( f_z = z^3 + d_1 z + d_2 \) also changes with change of input parameters. The generated images of cubic biomorphs are
presented in Fig. 16–24 and following were input limitations for Fig. 16–24:

- Fig. 16: $d_2 = 1 + i, d_1 = 1 + i, a = 0.9, b = 0.9, s = 0.9, A = [-8, 8] \times [-8.5, 10]$.
- Fig. 17: $d_2 = -1 - i, d_1 = 1 + i, a = 0.9, b = 0.9, s = 0.9, A = [-12, 10.5] \times [-10.5, 16]$.
- Fig. 18: $d_2 = -1 - i, d_1 = 1 + i, a = 0.5, b = 0.5, s = 0.5, A = [-12, 10.5] \times [-10.5, 16]$.
- Fig. 19: $d_2 = -2 + i, d_1 = 1 + i, a = 0.5, b = 0.5, s = 0.5, A = [-8, 18.5] \times [-22.5, 5]$.
- Fig. 20: $d_2 = -2 + i, d_1 = 3/2, a = 0.5, b = 0.5, s = 0.5, A = [-8, 18.5] \times [-24.5, 3]$.
- Fig. 21: $d_2 = -2 + i, d_1 = 1, a = 0.5, b = 0.5, s = 0.5, A = [-5, 14.5] \times [-2.5, 0]$.
- Fig. 22: $d_2 = 0, d_1 = 1 + i, a = 0.9, b = 0.9, s = 0.9, A = [-8, 8] \times [-8.5, 10]$.
- Fig. 23: $d_2 = -2i, d_1 = 1 + 3i, a = 0.9, b = 0.9, s = 0.9, A = [-12, 12] \times [-28.5, 10]$.
- Fig. 24: $d_2 = -2 + i, d_1 = 1 + i, a = 0.5, b = 0.5, s = 0.5, A = [-22, 38.5] \times [-32.5, 20]$.

In last three experiments we present some biomorphs generated in the Jungck-S orbit with $s$-convexity by using the algorithm 3 for the complex polynomial $f(z) = z^4 + d_1 z + d_2$, $f(z) = z^5 + z^4 + z^3 + z^2 + z + d_2$ and $f(z) = z^5 + z^4 + d_1 z + d_2$ respectively. In Fig. 27 we choose $p = \max|z|$, 6 in algorithm to generate the resulting image. In all three experiments the maximum number of iterations was fixed and we stopped the iteration at 15 for all images of biomorphs.

The generation of biomorphs are presented in Figs. 25–27 and the parameters used to generate them were the following:
V. CONCLUSIONS

We introduced the Jungck-S iterative scheme with \( s \)-convexity in the study of fractals. We defined the Modified Jungck-S orbit (MJSO) and proved the new escape criterion for complex quadratic, cubic and \( p \)th degree polynomials. We also established three algorithms for fractal generation in (MJSO) and implemented them in Mathematica to visualize some Julia sets, Mandelbrot sets and biomorphs. We presented some examples of Julia sets, Mandelbrot sets and biomorphs. The obtained images of biomorphs resembled with some biological objects. We also showed the image of a biomorph changes with the change of input parameters. With the use of different input parameters, we observed that color difference appeared in images. We hope, the results of this research can be helpful for those whose research activities or hobbies are linked to aesthetic pattern and image encryption.

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