Multiple Mertens evaluations

Tianfang Qi
Department of Mathematics
Nanjing University
Nanjing 210093, China
E-mail: 15914449424@163.com

Su Hu
Department of Mathematics
South China University of Technology
Guangzhou 510640, China
E-mail: mahusu@scut.edu.cn

Abstract

The Mertens’ first theorem gives us the following asymptotic formula

\[ \sum_{\substack{p \leq x \text{ prime}}} \frac{\log p}{p} = \log x + O(1), \]

and the Mertens’ second theorem indicates that there exists a constant \( B \approx 0.261 \), named the Mertens constant, such that

\[ \sum_{\substack{p \leq x \text{ prime}}} \frac{1}{p} = \log_2 x + B + O\left(\frac{1}{\log x}\right). \]

In this paper, by using the Abel summation formula and Dirichlet’s hyperbola method, we extend them to multiple cases.

1 Introduction

Throughout this paper we need the following notation. Denote \( \log_2 x = \log(\log x) \) as the iterated natural logarithm, and \( p \) as a prime number. For a fixed number \( a \in \mathbb{R} \cup \{-\infty\} \)

2010 Mathematics Subject Classification: 11N05, 11N37.

Key words and phrases: Mertens’ first theorem, Mertens’ second theorem, Arithmetic function, Riemann zeta function, Polylogarithm.
and real-valued functions \( g : (a, \infty) \to [0, \infty) \), \( f : (a, \infty) \to \mathbb{R}, f(x) = O(g(x)) \) or \( f(x) \ll g(x) \) means there exists \( M > 0, b \geq a \), such that \( |f(x)| \leq Mg(x) \) for any \( x \geq b \).

In 1874, Mertens \([M74]\) proved the following two interesting and beautiful theorem ([A98, p. 89–90, Theorem 4.10 and 4.12]).
Mertens’ first theorem:

\[
\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1);
\]

Mertens’ second theorem:

\[
\sum_{p \leq x} \frac{1}{p} = \log_2 x + B + O \left( \frac{1}{\log x} \right),
\]

where \( B \approx 0.261 \) is the Mertens constant.

The above Mertens’ theorems have many applications in modern number theory and have appeared in many works (for example, \([A98, \text{Theorem } 4.12]\) and \([EW05, \text{p. 13}]\)). One direction on investigating Mertens’ second theorem (1.2) is to increase the number of prime variables. In 2002, Saidak \([S02]\) presented the following double Mertens type evaluation

\[
\sum_{pq \leq x} \frac{1}{pq} = (\log_2 x + B)^2 - \frac{\pi^2}{6} + O \left( \frac{\log_2 x}{\log x} \right)
\]

and in 2014, Popa \([P14]\) also obtained the following evaluation

\[
\sum_{pq \leq x} \frac{1}{pq} = (\log_2 x + B)^2 - \log^2 2 + 2 \int_{0+0}^{\frac{1}{2}} \frac{\log(1-x)}{x}dx + O \left( \frac{\log_2 x}{\log x} \right).
\]

From the equality \(- \log^2 2 + 2 \int_{0+0}^{\frac{1}{2}} \frac{\log(1-x)}{x}dx = -\frac{\pi^2}{6}\) \([L81, \text{p. 5, (1.11)}]\), we see that (1.4) implies (1.3).

In 2016, Popa \([P16]\) further proved the following triple Mertens evaluation

\[
\sum_{pqr \leq x} \frac{1}{pqr} = (\log_2 x + B)^3 - \frac{\pi^2}{2}(\log_2 x + B) + 2\zeta(3) + O \left( \frac{(\log_2 x)^2}{\log x} \right),
\]

where

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re}(s) > 1
\]
is the Riemann zeta function.

In this paper, by using the Abel summation formula and Dirichlet’s hyperbola method, we extend the above Mertens’ formulas (1.1) and (1.2) to multiple cases. Our main results are as follows.
Theorem 1.1. Set $P_0(y) = 1$ and $P_1(y) = y + B$. For any $k \geq 2$, let $P_k(x)$ be given in (3.2). Then for any positive integers $k$ and $s$, the following evaluation holds

\begin{align*}
\sum_{p_1 \cdots p_k \leq x} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k} &= \sum_{l=0}^{k-1} (-1)^l \frac{A_{k-1-l}}{s^{l+1}} P_{k-1-l}(\log_2 x) \cdot \log^s x + f(2) \log^{s-1} 2 \\
&+ O(\log^{s-1} x \cdot (\log_2 x)^k),
\end{align*}

where

$$f(x) = \sum_{l=0}^{k-1} (-1)^l A_{k-1-l} P_{k-1-l}(\log_2 x) \cdot \log x.$$ 

and the combinatorial number $A_k = \binom{k}{l} \cdot l!$.

In the case $s = 1$, $k = 1$, Theorem 1.1 reduces to Mertens’ first theorem (1.1), and $s = 1$, $k = 2$ is Theorem 3.3 in Bănescu-Popa [B18]. Moreover, the following results follow immediately.

Corollary 1.2. We have the following evaluation

$$\frac{1}{(\log x)^s} \sum_{p_1 \cdots p_k \leq x} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k} = \sum_{l=0}^{k-1} (-1)^l \frac{A_{k-1-l}}{s^{l+1}} P_{k-1-l}(\log_2 x) + O \left( \frac{(\log_2 x)^k}{\log x} \right).$$

Corollary 1.3. We have the following evaluation

$$\frac{1}{(\log x)^s} \sum_{p_1 \cdots p_k \leq \sqrt{x}} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k} = \frac{1}{2^s} \sum_{l=0}^{k-1} (-1)^l \frac{A_{k-1-l}}{s^{l+1}} P_{k-1-l}(\log_2 \sqrt{x}) + O \left( \frac{(\log_2 x)^k}{\log x} \right).$$

Theorem 1.4. For any positive integer $k$, we have

\begin{align*}
\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} &= (\log_2 x + B)^k + \sum_{m=2}^{k} C_k^m a_m (\log_2 x + B)^{k-m} \\
&+ O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\end{align*}

where $\{a_n\}$ is a sequence related to the Riemann zeta function $\zeta$, that is,

\begin{align*}
a_2 &= -\zeta(2), a_3 = 2\zeta(3), a_4 = 3\zeta(2)^2 - 6\zeta(4), \\
a_k &= \sum_{i=1}^{k-3} (-1)^i C_{k-1-i}^i \zeta(i+1) a_{k-1-i} + (-1)^{k-1}(k-1)! \zeta(k) \quad (k > 4).
\end{align*}

and $C_k^l = \binom{k}{l}$. 

Remark 1.5. The cases $k = 1, 2$ and 3 are Mertens’ second theorem \cite{P12, P14} Theorem 1 and \cite{P16} Theorem 11, respectively.

Remark 1.6. Let $\Gamma$ be the Euler gamma function and $\gamma$ be the Euler constant. Based on the argument of the Selberg-Delange method and the complex function theory, in 2016, Tenenbaum obtained the following multiple Mertens evaluation for $x \geq 3$ (see \cite{T17} Theorem 1)

\[
\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} = P_k(\log_2 x) + O\left(\frac{(\log_2 x)^k}{\log x}\right),
\]

where $P_k(X) := \sum_{0 \leq j \leq k} \lambda_{j,k}X^j$, and

\[
\lambda_{j,k} := \sum_{0 \leq m \leq k-j} \binom{k}{m,j,k-m-j} (B-\gamma)^{k-m-j} \left(\frac{1}{\Gamma}\right)^{(m)} \quad (0 \leq j \leq k).
\]

In a recent version of the above article \cite{T19}, Tenenbaum showed that his method may provide the same error term as Theorem 1.4.

The main term in (1.9) is expressed by the higher order derivatives of the gamma function, while the main term in our formula (1.7) is related to the special values of Riemann zeta function. There seems to be no direct connections between them. But a recent work \cite{B-K-L21} shows that (1.7) and (1.9) are equivalent (see \cite{B-K-L21} Sec. 6). Our approach is based on the Abel summation formula (see \cite{B18}) and Dirichlet’s hyperbola method, it is elementary and makes no use of complex function theory. Furthermore, during our approach we also get a multiple generalization of Dirichlet’s hyperbola method (see Proposition 2.1).

Remark 1.7. Korolev \cite{K16} p. 17–33] also calculated some other types multiple sums with primes.

Remark 1.8. Recently, building on (1.7) and (1.9), Bayless, Kinlaw and Lichtman \cite{B-K-L21} gave elementary proofs of precise asymptotics for the reciprocal sum of $k$-almost primes,

\[
R_k(x) = \sum_{\Omega(n)=k} \frac{1}{n} = \sum_{n \leq x} \frac{1}{p_1 \cdots p_k},
\]

where for a positive integer $n$, $\Omega(n)$ denotes the number of prime factors of $n$, counted with multiplicity.
2 Preliminaries

2.1 The hyperbola method of Dirichlet for a multiple sum

The main purpose of this subsection is to extent the classical hyperbola method of Dirichlet (see [A98 Theorem 3.17]), especially the triple sum given in [P16] to multiple cases.

Proposition 2.1. (1) Let \( \psi : \mathbb{N}^k \to \mathbb{R} \) be a real-valued function. Then for each \( 0 < y < x \), we have the following identity

\[
\sum_{i_1 \cdots i_k \leq x} \psi(i_1, \cdots, i_k) = \sum_{i_k \leq y} \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k) + \sum_{i_1 \cdots i_{k-1} \leq \frac{y}{i_k}} \sum_{i_k \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k).
\]

(2) Let \( f_1, \cdots, f_k : \mathbb{N} \to \mathbb{R} \) be real-valued functions and define \( S_{f_k}, S_{f_1, \cdots, f_{k-1}} : (0, \infty) \to \mathbb{R} \) by

\[
S_{f_k}(x) = \sum_{i_k \leq x} f_k(i_k), \quad S_{f_1, \cdots, f_{k-1}}(x) = \sum_{i_1 \cdots i_{k-1} \leq x} f_1(i_1) \cdots f_{k-1}(i_{k-1}).
\]

Then for each \( 0 < y < x \), we have the following identity

\[
\sum_{i_1 \cdots i_k \leq x} f_1(i_1) \cdots f_k(i_k) = \sum_{i_k \leq y} f_k(i_k) S_{f_1, \cdots, f_{k-1}} \left( \frac{x}{i_k} \right) + \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} f_1(i_1) \cdots f_{k-1}(i_{k-1}) S_{f_k} \left( \frac{x}{i_1 \cdots i_{k-1}} \right) - S_{f_k}(y) S_{f_1, \cdots, f_{k-1}} \left( \frac{x}{y} \right).
\]

Proof. Our method employed here is analogue to [P16 Proposition 3].

(1) We have

\[
\sum_{i_1 \cdots i_k \leq x} \psi(i_1, \cdots, i_k) = \sum_{i_k \leq y} \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k) + \sum_{i_1 \cdots i_{k-1} \leq \frac{y}{i_k}} \sum_{i_k \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k),
\]

since

\[
\{(i_1, \cdots, i_k) \in \mathbb{N}^k \mid i_1 \cdots i_k \leq x\} = \{(i_1, \cdots, i_k) \in \mathbb{N}^k \mid i_1 \cdots i_k \leq x, \ i_k \leq y\} \\
\cup \{(i_1, \cdots, i_k) \in \mathbb{N}^k \mid i_1 \cdots i_k \leq x, \ y < i_k\}
\]

and the sets on the right side of the equation are disjoint.

Clearly

\[
\sum_{i_1 \cdots i_k \leq x, i_k \leq y} \psi(i_1, \cdots, i_k) = \sum_{i_k \leq y} \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k),
\]
which follows that
\[
\sum_{i_1 \cdots i_k \leq x} \psi(i_1, \cdots, i_k) = \sum_{i_k \leq y} \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} \psi(i_1, \cdots, i_k) + \sum_{i_1 \cdots i_k \leq x} \sum_{y < i_k} \psi(i_1, \cdots, i_k).
\]
It is easy to check that
\[
\{(i_1, \cdots, i_k) \in \mathbb{N}^k \mid i_1 \cdots i_k \leq x, \ y < i_k\}
= \{(i_1, \cdots, i_k) \in \mathbb{N}^k \mid y < i_k \leq \frac{x}{i_1 \cdots i_{k-1}}, \ i_1 \cdots i_{k-1} \leq \frac{x}{y}\}.
\]
Thus we obtain
\[
\sum_{i_1 \cdots i_k \leq x} \sum_{y < i_k} \psi(i_1, \cdots, i_k) = \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{y}} \sum_{y < i_k \leq \frac{x}{i_1 \cdots i_{k-1}}} \psi(i_1, \cdots, i_k).
\]
Now we have completed the proof for the first part by substituting (2.4) into (2.3).

(2) By (2.1), we have
\[
\sum_{i_1 \cdots i_k \leq x} f_1(i_1) \cdots f_k(i_k) = \sum_{i_k \leq x} f_k(i_k) \left( \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{i_k}} f_1(i_1) \cdots f_{k-1}(i_{k-1}) \right)
+ \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{y}} f_1(i_1) \cdots f_{k-1}(i_{k-1}) \left( \sum_{y < i_k \leq \frac{x}{i_1 \cdots i_{k-1}}} f_k(i_k) \right)
= \sum_{i_k \leq x} f_k(i_k) S_{f_1, \cdots, f_{k-1}}(\frac{x}{i_k})
+ \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{y}} f_1(i_1) \cdots f_{k-1}(i_{k-1}) \left( S_{f_k} \left( \frac{x}{i_1 \cdots i_{k-1}} \right) - S_{f_k}(y) \right)
= \sum_{i_k \leq x} f_k(i_k) S_{f_1, \cdots, f_{k-1}}(\frac{x}{i_k})
+ \sum_{i_1 \cdots i_{k-1} \leq \frac{x}{y}} f_1(i_1) \cdots f_{k-1}(i_{k-1}) S_{f_k} \left( \frac{x}{i_1 \cdots i_{k-1}} \right)
- S_{f_k}(y) S_{f_1, \cdots, f_{k-1}}(\frac{x}{y}),
\]
which completes the proof.

Letting $f_1 = \cdots = f_k = f$ and $y = \sqrt{x}$ in Proposition 2.1, we immediately have the following corollaries.

**Corollary 2.2.** Let $f : \mathbb{N} \to \mathbb{R}$ be a real-valued function, and define $S_1, S_{k-1} : (0, \infty) \to \mathbb{R}$ by
\[
S_1(x) = \sum_{i \leq x} f(i) \quad \text{and} \quad S_{k-1}(x) = \sum_{i_1 \cdots i_{k-1} \leq x} f(i_1) \cdots f(i_{k-1}),
\]
we have
\[
\sum_{i_1 \cdots i_k \leq x} f(i_1) \cdots f(i_k) = \sum_{i_k \leq \sqrt{x}} f(i_k) S_{k-1}(\frac{x}{i_k}) + \sum_{i_1 \cdots i_{k-1} \leq \sqrt{x}} f(i_1) \cdots f(i_{k-1}) S_1(\frac{x}{i_1 \cdots i_{k-1}})
- S_1(\sqrt{x}) S_{k-1}(\sqrt{x}).
\]

**Corollary 2.3.** Let \( \mathbb{P} \) be the set of all prime numbers and \( u : \mathbb{P} \to \mathbb{R} \) be a real-valued function on \( \mathbb{P} \). Define \( V_1, V_{k-1} : (0, \infty) \to \mathbb{R} \) by
\[
V_1(x) = \sum_{p \leq x} u(p) \quad \text{and} \quad V_{k-1}(x) = \sum_{p_1 \cdots p_{k-1} \leq x} u(p_1) \cdots u(p_{k-1}),
\]
we have
\[
\sum_{p_1 \cdots p_k \leq x} u(p_1) \cdots u(p_k) = \sum_{p_k \leq \sqrt{x}} u(p_k) V_{k-1}(\frac{x}{p_k}) + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} u(p_1) \cdots u(p_{k-1}) V_1(\frac{x}{p_1 \cdots p_{k-1}}) - V_1(\sqrt{x}) V_{k-1}(\sqrt{x}).
\]

**Proof.** Set \( f = u \chi_\mathbb{P} \), \( f(n) = \begin{cases} u(n) & n \in \mathbb{P} \\ 0 & n \notin \mathbb{P} \end{cases} \) in Corollary 2.2 \( \square \)

### 2.2 Polylogarithmic functions

Polylogarithmic functions \( \text{Li}_n : [0, 1] \to \mathbb{R} \) are defined by (see [L81])
\[
\text{Li}_2(x) = -\int_0^x \log(1-t) \frac{dt}{t} = \sum_{k=1}^{\infty} \frac{x^k}{k^2},
\]
\[
\text{Li}_n(x) = \int_0^x \frac{\text{Li}_{n-1}(t)}{t} dt = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \quad (n \geq 3).
\]

They will appear in a natural way, as in [P16], in the proof of multiple Mertens evaluations.

We will need the following integral later.

**Proposition 2.4.** Denote by \( a = \log 2 \). For any natural number \( m \), we have
\[
\int_{0+0}^{\frac{1}{2}} \log^m(1-x) \frac{dx}{x} = (-a)^{m+1} + (-1)^m m! \zeta(m+1) + (-1)^{m-1} \sum_{s=1}^m A_m a^{m-s} \text{Li}_{s+1}(\frac{1}{2}).
\]

(2.5)
Proof. From the formula of integration by parts and log(1 - x) = - ∑_{k=1}^{∞} \frac{x^k}{k}, we obtain
\[
\int_{1/2}^{1} \frac{\log^m(1-x)}{x} \, dx = \log x \log^m(1-x) \bigg|_{0+0}^{1/2} + m \int_{1/2}^{1} \frac{\log^{m-1}(1-x)}{1-x} \, dx
\]
\[
= (-a)^{m+1} + m \int_{1/2}^{1} \frac{\log^{m-1} x \log(1-x)}{x} \, dx
\]
\[
= (-a)^{m+1} + m \int_{1/2}^{1} \frac{\log^{m-1} x}{x} \left( - \sum_{k=1}^{\infty} \frac{x^k}{k} \right) \, dx
\]
\[
= (-a)^{m+1} - m \sum_{k=1}^{\infty} \frac{1}{k} \int_{1/2}^{1} x^{k-1} \log^{m-1} x \, dx.
\]
Using the following indefinite integral, which can also be derived from the formula of integration by parts,
\[
\int x^{k-1} \log^{m-1} x \, dx = \sum_{s=1}^{m} (-1)^{s-1} A_{s-1}^{m-1} \frac{x^k \log^{m-s} x}{k^s} + \text{constant},
\]
we have
\[
\int_{1/2}^{1} x^{k-1} \log^{m-1} x \, dx = (-1)^{m-1}(m-1)! \frac{1}{k^m} - \sum_{s=1}^{m} (-1)^{s-1} A_{s-1}^{m-1} \frac{(-a)^{m-s}}{k^s 2^k}.
\]
By the definition of polylogarithmic functions \( \text{Li}_n \) (\( n \geq 2 \)), we get
\[
\int_{0+0}^{1/2} \frac{\log^m(1-x)}{x} \, dx
\]
\[
= (-a)^{m+1} - m \sum_{k=1}^{\infty} \frac{1}{k} \left( (-1)^{m-1}(m-1)! \frac{1}{k^m} - \sum_{s=1}^{m} (-1)^{s-1} A_{s-1}^{m-1} \frac{(-a)^{m-s}}{k^s 2^k} \right)
\]
\[
= (-a)^{m+1} + (-1)^m m! \sum_{k=1}^{\infty} \frac{1}{k^{m+1}} + (-1)^{m-1} \sum_{s=1}^{m} A_{s}^{m-s} \sum_{k=1}^{\infty} \frac{1}{k^{s+1} 2^k}
\]
\[
= (-a)^{m+1} + (-1)^m m! \zeta(m+1) + (-1)^{m-1} \sum_{s=1}^{m} A_{s}^{m-s} \text{Li}_{s+1} \left( \frac{1}{2} \right),
\]
which completes the proof. 

3 Proofs of the main results

In this section, we will prove our main results (Theorems 1.1 and 1.4 above).

For \( k = 1, 2, \) and \( 3 \), it is easy to verify the result (compare with \([14, 16]\)). We proceed by induction on \( k \). Assume that the result holds for any positive integer \( < k \), we desire to show that it holds for \( k \).
For simplification of notations, we set

\[ \sum_{p_1 \cdots p_s \leq x} \frac{1}{p_1 \cdots p_s} = P_s(\log_2 x) + R_s(x), \]

where, with \( B \) is the Mertens constant as before,

\[ P_s(y) = (y + B)^s + \sum_{m=2}^{s} C_s^m a_m (y + B)^{s-m}, \quad R_s(x) = O\left(\frac{(\log_2 x)^{s-1}}{\log x}\right). \]

For example, \( P_1(y) = y + B \), \( P_2(y) = (y + B)^2 - \zeta(2) \), \( P_3(y) = (y + B)^3 - 3\zeta(2)(y + B) + 2\zeta(3) \), they have been appeared in Popa’s papers [P14] and [P16].

It is clear that

\[ P_s(y) = P_1(y)^s + \sum_{m=2}^{s} C_s^m a_m P_1(y)^{s-m}, \]

and with a natural notation \( P_0(y) = 1 \),

\[ P_s(y-a) = \sum_{t=0}^{s} C_s^t (-1)^t a^t P_{s-t}(y). \]

It should be noted that our proof is always under the inductive hypothesis, and we will not repeat it.

First, we prove the following evaluation.

**Proposition 3.1.** Suppose that \( p_1, \cdots, p_k \) are primes. Then we have

\[
\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} P_{k-1}\left(\frac{\log_2 \left(\frac{x}{p_1}\right)}{p_1}\right)
\]

\[
+ \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} P_1\left(\frac{\log_2 \left(\frac{x}{p_1 \cdots p_{k-1}}\right)}{p_1 \cdots p_{k-1}}\right)
\]

\[
- P_1(\log_2 \sqrt{x}) P_{k-1}(\log_2 \sqrt{x}) + O\left(\frac{(\log_2 x)^{k-1}}{\log x}\right)
\]

\[= A + B - C + O\left(\frac{(\log_2 x)^{k-1}}{\log x}\right), \]

with the notations

\[ A = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} P_{k-1}\left(\frac{\log_2 \left(\frac{x}{p_1}\right)}{p_1}\right), \]

\[ B = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} P_1\left(\frac{\log_2 \left(\frac{x}{p_1 \cdots p_{k-1}}\right)}{p_1 \cdots p_{k-1}}\right), \]

\[ C = P_1(\log_2 \sqrt{x}) P_{k-1}(\log_2 \sqrt{x}). \]
Proof. Define $V_1, V_{k-1} : (0, \infty) \to \mathbb{R}$ by

$$V_1(x) = \sum_{p \leq x} \frac{1}{p} \quad \text{and} \quad V_{k-1}(x) = \sum_{p_1 \cdots p_{k-1} \leq x} \frac{1}{p_1 \cdots p_{k-1}}.$$  

Using Corollary 2.3 and the simplified notations (3.2), we have

$$\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} V_{k-1} \left( \frac{x}{p_1} \right) + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} V_1 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) - V_1(\sqrt{x}) V_{k-1}(\sqrt{x})$$

$$= \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \left[ P_{k-1} \left( \log_2 \left( \frac{x}{p_1} \right) \right) + R_{k-1} \left( \frac{x}{p_1} \right) \right]$$

$$+ \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} \left[ P_1 \left( \log_2 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) \right) + R_1 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) \right]$$

$$- \left( P_1(\log_2 \sqrt{x}) + R_1(\sqrt{x}) \right) \left( P_{k-1}(\log_2 \sqrt{x}) + R_{k-1}(\sqrt{x}) \right)$$

$$= \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} P_{k-1} \left( \log_2 \left( \frac{x}{p_1} \right) \right) + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} P_1 \left( \log_2 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) \right)$$

$$- P_1(\log_2 \sqrt{x}) P_{k-1}(\log_2 \sqrt{x}) + R_k(x),$$

where

$$R_k(x) = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} R_{k-1} \left( \frac{x}{p_1} \right) + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} R_1 \left( \frac{x}{p_1 \cdots p_{k-1}} \right)$$

$$- P_{k-1}(\log_2 \sqrt{x}) R_1(\sqrt{x}) - P_1(\log_2 \sqrt{x}) R_{k-1}(\sqrt{x}) - R_1(\sqrt{x}) R_{k-1}(\sqrt{x}).$$

Since $p_i \geq 2$ ($1 \leq i \leq k - 1$) and

$$\sum_{p_i \leq \sqrt{x}} \frac{1}{p_i \log(x/p_i)} \leq \sum_{p_i \leq \sqrt{x}} \frac{1}{p_i \log(x/\sqrt{x})} = O \left( \frac{\log_2 x}{\log x} \right),$$

we obtain

$$\sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} R_{k-1} \left( \frac{x}{p_1} \right) = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} O \left( \frac{(\log_2(x/p_1))^{k-2}}{\log(x/p_1)} \right) = O \left( \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \frac{(\log_2(x/p_1))^{k-2}}{\log(x/p_1)} \right)$$

$$= O \left( \frac{(\log_2 x)^{k-2}}{\log x} \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1 \log(x/p_1)} \right) = O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right).$$
and

\[
\sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} R_1 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} O \left( \frac{1}{\log \left( \frac{x}{p_1 \cdots p_{k-1}} \right)} \right) = O \left( \frac{1}{\log x} \right). 
\]

Notice that

\[
P_{k-1}(\log_2 \sqrt{x}) R_1(\sqrt{x}) = P_{k-1}(\log_2 \sqrt{x}) O \left( \frac{1}{\log x} \right) = O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

\[
P_1(\log_2 \sqrt{x}) R_{k-1}(\sqrt{x}) = P_1(\log_2 \sqrt{x}) O \left( \frac{(\log_2 x)^{k-2}}{\log x} \right) = O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

\[
R_1(\sqrt{x}) R_{k-1}(\sqrt{x}) = O \left( \frac{1}{\log x} \right) O \left( \frac{(\log_2 x)^{k-2}}{\log x} \right) = O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

we have

\[
R_k(x) = O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

which completes the proof.

To evaluate the terms on the right hand side of (3.3), we need the following propositions.

**Proposition 3.2.** The following evaluation holds

\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log \left( 1 - \frac{\log p}{\log x} \right) \right]^m = \int_{0+0}^{\frac{1}{2}} \frac{\log^m(1-x)}{x} dx + O \left( \frac{1}{\log x} \right).
\]

**Proof.** Using the Generalized Newton binomial formula, we have

\[
\left( x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right)^m = \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1 \cdots k_m} \frac{x^{k_1}}{k_1} \cdots \frac{x^{k_m}}{k_m} = \sum_{k=1}^\infty b_k x^k
\]

where \( b_1 = \cdots = b_{m-1} = 0, \)

\[
b_k = \sum_{k_1 + \cdots + k_m = k} \binom{k}{k_1 \cdots k_m} \frac{1}{k_1 \cdots k_m}.
\]

Thus for any \(|x| < 1\), we have \( \log^m(1-x) = (-1)^m \sum_{k=1}^\infty b_k x^k \) and from the evaluation (see [P14] Proposition 1))

\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \frac{\log p}{\log x} \right)^k = \frac{1}{k \cdot 2^k} + \frac{1}{2^{k-1}} O \left( \frac{1}{\log x} \right),
\]
we obtain
\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log \left( 1 - \frac{\log p}{\log x} \right) \right]^m = (-1)^m \sum_{k=1}^{\infty} b_k \sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \frac{\log p}{\log x} \right)^k
\]
\[\text{(3.4)}\]
\[
= (-1)^m \sum_{k=1}^{\infty} b_k \left( \frac{1}{k \cdot 2^k} + \frac{1}{2^{k-1}} O \left( \frac{1}{\log x} \right) \right)
\]
\[
= (-1)^m \left( \sum_{k=1}^{\infty} \frac{b_k}{k \cdot 2^k} + \left( \sum_{k=1}^{\infty} \frac{b_k}{2^{k-1}} \right) O \left( \frac{1}{\log x} \right) \right).
\]

Since
\[
\int_0^{\frac{1}{x}} \left( \log^m (1 - x) \right) dx = (-1)^m \int_0^{\frac{1}{x}} \sum_{k=2}^{\infty} b_k (k-1) x^{k-2} dx
\]
\[
= (-1)^m \sum_{k=2}^{\infty} b_k (k-1) \int_0^{\frac{1}{x}} x^{k-2} dx
\]
\[
= (-1)^m \sum_{k=2}^{\infty} \frac{b_k}{2^{k-1}},
\]
we see that the series \(\sum_{k=1}^{\infty} \frac{b_k}{2^{k-1}}\) is convergent. Finally substituting the following integral into (3.4)
\[
\int_0^{\frac{1}{x}} \log^m (1 - x) dx = (-1)^m \int_0^{\frac{1}{x}} \log (1 - x) dx = (-1)^m \sum_{k=1}^{\infty} b_k \int_0^{\frac{1}{x}} x^{k-1} dx
\]
\[
= (-1)^m \sum_{k=1}^{\infty} \frac{b_k}{k \cdot 2^k},
\]
we obtain
\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log \left( 1 - \frac{\log p}{\log x} \right) \right]^m = \int_0^{\frac{1}{x}} \log^m (1 - x) dx + O \left( \frac{1}{\log x} \right),
\]
which is the desired result. \(\square\)

**Proposition 3.3.** The following evaluation holds
\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log_2 \left( \frac{x}{p} \right) \right]^m = (\log_2 x)^m P_1(\log_2 \sqrt{x}) + \sum_{t=1}^{m} C_t^m (\log_2 x)^{m-t} \int_0^{\frac{1}{x}} \frac{\log^t (1 - x)}{x} dx + O \left( \frac{(\log_2 x)^m}{\log x} \right).
\]

**Proof.** Using Newton binomial formula, we obtain
\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log_2 \left( \frac{x}{p} \right) \right]^m = \sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \log (\log x - \log p) - \log_2 x + \log_2 x \right)^m.
\]
Multiple Mertens evaluations

\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \log_2 x + \log \left( 1 - \frac{\log p}{\log x} \right) \right)^m = \sum_{p \leq \sqrt{x}} \frac{1}{p} \sum_{t=0}^{m} C_t^m (\log_2 x)^{m-t} \left( \log \left( 1 - \frac{\log p}{\log x} \right) \right)^t
\]

\[
= (\log_2 x)^m \sum_{p \leq \sqrt{x}} \frac{1}{p} + \sum_{t=1}^{m} C_t^m (\log_2 x)^{m-t} \sum_{p \leq \sqrt{x}} \frac{1}{p} \left( \log \left( 1 - \frac{\log p}{\log x} \right) \right)^t.
\]

By Proposition 3.2, we have

\[
\sum_{p \leq \sqrt{x}} \frac{1}{p} \left[ \log_2 \left( \frac{x}{p} \right) \right]^m = (\log_2 x)^m \left( P_1(\log_2 \sqrt{x}) + O\left( \frac{1}{\log x} \right) \right) + \sum_{t=1}^{m} C_t^m (\log_2 x)^{m-t} \int_{0+0}^{1/2} \frac{\log^t (1-x)}{x} dx + O\left( \frac{1}{\log x} \right)
\]

\[
= (\log_2 x)^m P_1(\log_2 \sqrt{x}) + \sum_{t=1}^{m} C_t^m (\log_2 x)^{m-t} \int_{0+0}^{1/2} \frac{\log^t (1-x)}{x} dx + O\left( \frac{(\log_2 x)^m}{\log x} \right),
\]

which completes our proof.

Using Proposition 3.2 and Proposition 3.3, we can evaluate the sum A on the right hand side of (3.3).

**Proposition 3.4.** The following evaluation holds

\[
A = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} P_{k-1} \left( \log_2 \left( \frac{x}{p_1} \right) \right)
\]

\[
= P_{k-1}(\log_2 x) P_1(\log_2 \sqrt{x}) + \sum_{c=1}^{k-1} C_{k-1}^c P_{k-1-c}(\log_2 x) \int_{0+0}^{1/2} \frac{\log^c (1-x)}{x} dx
\]

\[
+ O\left( \frac{(\log_2 x)^{k-1}}{\log x} \right).
\]

**Proof.** Using the simplified notations (3.2) and Newton binomial formula, we obtain

\[
A = \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \left[ \log_2 \left( \frac{x}{p_1} \right) + B \right]^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^m a_m \left( \log_2 \left( \frac{x}{p_1} \right) + B \right)^{k-1-m}
\]

\[
= \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \sum_{t=0}^{k-1} C_{k-1}^t B^t \left( \log_2 \left( \frac{x}{p_1} \right) \right)^{k-1-t}
\]

\[
+ \sum_{m=2}^{k-1} C_{k-1}^m a_m \sum_{s=0}^{k-1-m} C_{k-1-m}^s B^s \left( \log_2 \left( \frac{x}{p_1} \right) \right)^{k-1-m-s}\]

\[
\]
Applying Proposition 3.3 we have

\[
A = \sum_{t=0}^{k-1} C_{k-1}^t B^t \left( \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \left( \log_2 \left( \frac{x}{p_1} \right) \right)^{k-1-t} \right) + \sum_{m=2}^{k-1-m} \sum_{s=0}^{k-1-m} C_{k-1-m}^s B^s \left( \sum_{p_1 \leq \sqrt{x}} \frac{1}{p_1} \left( \log_2 \left( \frac{x}{p_1} \right) \right)^{k-1-m-s} \right)
\]

Applying Proposition 3.3, we have

\[
A = \sum_{t=0}^{k-1} C_{k-1}^t B^t \left( \log_2 x \right)^{k-1-t} P_1 \left( \log_2 \sqrt{x} \right) + \sum_{m=2}^{k-1-m} \sum_{s=0}^{k-1-m} C_{k-1-m}^s B^s \left( \log_2 x \right)^{k-1-m-s} P_1 \left( \log_2 \sqrt{x} \right)
\]

\[
= \sum_{t=0}^{k-1} C_{k-1}^t B^t \left( \log_2 x \right)^{k-1-t} P_1 \left( \log_2 \sqrt{x} \right) + \sum_{m=2}^{k-1-m} \sum_{s=0}^{k-1-m} C_{k-1-m}^s B^s \left( \log_2 x \right)^{k-1-m-s} P_1 \left( \log_2 \sqrt{x} \right)
\]

\[
= \left( \log_2 x + B \right)^{k-1} + \sum_{m=2}^{k-1} \sum_{s=0}^{k-1-m} C_{k-1-m}^s B^s \left( \log_2 x \right)^{k-1-m-s} P_1 \left( \log_2 \sqrt{x} \right)
\]

\[
A = \sum_{t=0}^{k-1} C_{k-1}^t B^t \left( \log_2 x \right)^{k-1-t} P_1 \left( \log_2 \sqrt{x} \right)
\]
Multiple Mertens evaluations

\[
\begin{align*}
&= \left( (\log_2 x + B)^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^m a_m (\log_2 x + B)^{k-1-m} \right) P_1(\log_2 \sqrt{x}) \\
&\quad + \sum_{c=1}^{k-1} C_{k-1}^c \left( (\log_2 x + B)^{k-1-c} + \sum_{m=2}^{k-1-c} C_{k-1-c}^m a_m (\log_2 x + B)^{k-1-c-m} \right) \int_{\frac{1}{2}}^1 \frac{\log^c(1-x)}{x} dx \\
&\quad + O\left( \frac{(\log_2 x)^{k-1}}{\log x} \right).
\end{align*}
\]

The result is now seen from the simplified notations \((3.2)\).

Now we are at the position to prove Theorem 1.1.

**Proof of Theorem 1.1**

The method used here is analogue to that in \([B18, \text{Theorem 3.3}]\). If \(s = 1\), then first we claim that

\[
\sum_{p_1, \ldots, p_k \leq x} \frac{\log(p_1 \cdots p_k)}{p_1 \cdots p_k} = f(x) + A(x),
\]

where

\[
f(x) = \sum_{l=0}^{k-1} (-1)^l A_{k-1}^{l+1} P_{k-1-l}(\log_2 x) \cdot \log x
\]

and

\[
(3.5) \quad A(x) = O \left( (\log_2 x)^k \right).
\]

In fact, by the Abel summation formula (see \([B18, \text{p. 5}]\), we have

\[
\begin{align*}
&\sum_{p_1, \ldots, p_k \leq x} \frac{\log(p_1 \cdots p_k)}{p_1 \cdots p_k} \\
&= \left( \sum_{p_1, \ldots, p_k \leq x} \frac{1}{p_1 \cdots p_k} \right) \log x - \int_2^x \left( \sum_{p_1, \ldots, p_k \leq t} \frac{1}{p_1 \cdots p_k} \right) (\log t)' \, dt \\
&= (P_k(\log_2 x) + R_k(x)) \log x - \int_2^x (P_k(\log_2 t) + R_k(t)) (\log t)' \, dt \\
&= P_k(\log_2 x) \log x + O \left( (\log_2 x)^{k-1} \right) - \left( \log t \cdot P_k(\log_2 t) \right) \bigg|_2^x - \int_2^x \frac{P_k'(\log_2 t)}{t} \, dt \\
&\quad - \int_2^x R_k(t)(\log t)' \, dt \\
&= \int_2^x \frac{P_k'(\log_2 t)}{t} \, dt + \log 2 \cdot P_k(\log_2 2) - \int_2^x R_k(t)(\log t)' \, dt + O \left( (\log_2 x)^{k-1} \right) \\
&= \int_{\log_2^x}^{\log x} P_k'(\log t) \, dt + O \left( (\log_2 x)^k \right),
\end{align*}
\]
since the inner term
\[ \int_2^x R_k(t)(\log t)\,dt = \int_2^x O\left(\frac{(\log_2 t)^{k-1}}{\log t}\right)(\log t)\,dt = O\left(\int_2^x \frac{(\log_2 t)^{k-1}}{t\log t}\,dt\right) \]
\[ = O\left(\int_{\log_2 x}^{\log x} \frac{\log^{k-1} t}{t}\,dt\right) = O\left(\int_{\log_2 x}^{\log x} \log^{k-1} td(\log t)\right) \]
\[ = O\left(\frac{\log^k}{k}\right) = O\left((\log_2 x)^k\right). \]

Note that
\[ P'_k(y) = \left((y + B)^k + \sum_{m=2}^{k} C^m_k a_m(y + B)^{k-m}\right) \]
\[ = k(y + B)^{k-1} + \sum_{m=2}^{k-1} C^m_k a_m(k - m)(y + B)^{k-1-m} \]
\[ = k \left((y + B)^{k-1} + \sum_{m=2}^{k-1} C^m_{k-1} a_m(y + B)^{k-1-m}\right). \]

Hence
\[ \int_{\log_2 x}^{\log x} P'_k(\log t)\,dt = k \int_{\log_2 x}^{\log x} \left((\log t + B)^{k-1} + \sum_{m=2}^{k} C^m_{k-1} a_m(\log t + B)^{k-1-m}\right)\,dt \]
\[ = k \int_{\log_2 x}^{\log x} \left(\sum_{s_1=0}^{k-1} C^s_{k-1} B^{s_1}(\log t)^{k-1-s_1} \right. \]
\[ + \sum_{m=2}^{k} C^m_{k-1} a_m \sum_{s_2=0}^{k-1-m} C^s_{k-1-m} B^{s_2}(\log t)^{k-1-m-s_2} \left.\right)\,dt \]
\[ = k \left(\sum_{s_1=0}^{k-1} C^s_{k-1} B^{s_1} \int_{\log_2 x}^{\log x} (\log t)^{k-1-s_1}\,dt \right. \]
\[ + \sum_{m=2}^{k} C^m_{k-1} a_m \sum_{s_2=0}^{k-1-m} C^s_{k-1-m} B^{s_2} \int_{\log_2 x}^{\log x} (\log t)^{k-1-m-s_2}\,dt \right). \]

By the indefinite integral
\[ \int \log^a t\,dt = \sum_{i=0}^{a} (-1)^i A_i^t (\log t)^{a-i} + \text{constant}, \]
the first term on the right hand side of (3.6) becomes to
\[ \sum_{s_1=0}^{k-1} C^s_{k-1} B^{s_1} \left(\sum_{l_1=0}^{k-1-s_1} (-1)^{l_1} A_{k-1-s_1}^{l_1} t(\log t)^{k-1-s_1-l_1}\right) \int_{\log_2 x}^{\log x} \log \,dt \]
\[
\sum_{l_1=0}^{k-1} (-1)^{l_1} A_{k-1}^{l_1} \left( \sum_{s_1=0}^{k-1-l_1} C_{k-1-l_1}^{s_1} B^{s_1} (\log t)^{k-1-s_1-l_1} \right) \cdot t \mid_{\log 2}^{\log x} \\
= \sum_{l_1=0}^{k-1} (-1)^{l_1} A_{k-1}^{l_1} (\log t + B)^{k-1-l_1} \cdot t \mid_{\log 2}^{\log x}
\]
and the second term becomes to
\[
\sum_{m=2}^{k} C_{k-1}^{m} a_m \sum_{s_2=0}^{k-1-m} C_{k-1-m}^{s_2} B^{s_2} \left( \sum_{l_2=0}^{k-1-l_2} (-1)^{l_2} A_{k-1-l_2}^{l_2} t (\log t)^{k-1-m-s_2-l_2} \right) \mid_{\log 2}^{\log x} \\
= \sum_{l_2=0}^{k-1} (-1)^{l_2} A_{k-2}^{l_2} \sum_{m=2}^{k-1-l_2} C_{k-1-l_2}^{m} a_m \left( \sum_{s_2=0}^{k-1-m-l_2} C_{k-1-l_2-m}^{s_2} B^{s_2} (\log t)^{k-1-m-s_2-l_2} \right) \cdot t \mid_{\log 2}^{\log x} \\
= \sum_{l_2=0}^{k-1} (-1)^{l_2} A_{k-1}^{l_2} \sum_{m=2}^{k-1-l_2} C_{k-1-l_2}^{m} (\log t + B)^{k-1-m-l_2} \cdot t \mid_{\log 2}^{\log x}.
\]
Hence
\[
\int_{\log 2}^{\log x} P_k'(\log t) dt \\
= k \sum_{l=0}^{k-1} (-1)^{l} A_{k-1}^{l} \left( (\log t + B)^{k-1-l} + \sum_{m=2}^{k-1-l} C_{k-1-l}^{m} a_m (\log t + B)^{k-1-l-m} \right) \cdot t \mid_{\log 2}^{\log x} \\
= k \sum_{l=0}^{k-1} (-1)^{l} A_{k-1}^{l} P_{k-1-l}(\log t) \cdot t \mid_{\log 2}^{\log x} \\
= k \sum_{l=0}^{k-1} (-1)^{l} A_{k-1}^{l} P_{k-1-l}(\log_2 x) \cdot \log x - k \sum_{l=0}^{k-1} (-1)^{l} A_{k-1}^{l} P_{k-1-l}(\log_2 2) \cdot \log 2,
\]
and the claim follows.

If \( s \geq 2 \), then by the Abel summation formula (see [B18]), we have
\[(3.7) \sum_{p_1 \cdots p_k \leq x} \frac{\log^s(p_1 \cdots p_k)}{p_1 \cdots p_k} \]
\[
= \left( \sum_{p_1 \cdots p_k \leq x} \frac{\log(p_1 \cdots p_k)}{p_1 \cdots p_k} \right) \log^{s-1} x - \int_{2}^{x} \left( \sum_{p_1 \cdots p_k \leq t} \frac{\log(p_1 \cdots p_k)}{p_1 \cdots p_k} \right) (\log^{s-1} t)' dt \\
= (f(x) + A(x)) \log^{s-1} x - \int_{2}^{x} (f(t) + A(t))(\log^{s-1} t)' dt \\
= f(x) \log^{s-1} x + A(x) \log^{s-1} x - \left( f(t) \log^{s-1} t \mid_{2}^{x} - \int_{2}^{x} f'(t) \log^{s-1} t dt \right) - \int_{2}^{x} A(t)(\log^{s-1} t)' dt \\
= \int_{2}^{x} f'(t) \log^{s-1} t dt + f(2) \log^{s-1} 2 + N(x),
\]
where
\[ N(x) = A(x) \log^{s-1} x - (s - 1) \int_2^x A(t) \frac{\log^{s-2} t}{t} dt. \]

Now we evaluate the right hand side of (3.7) term-wise.

For \( N(x) \), by (3.5) we have

\[ N(x) \ll |A(x)| \log^{s-1} x + (s - 1) \int_2^x |A(t)| \frac{\log^{s-2} t}{t} dt \ll \log^{s-1} x \cdot ((\log_2 x)^k) + (s - 1) \int_2^x (\log_2 t)^k \log^{s-2} t dt. \]

and from the integral

\[ \int (\log t)^k t^s dt = \sum_{l=0}^k (-1)^l A_k^l (s+1)^{l+1} + \text{constant}, \]

we have

\[ \int_2^x (\log_2 t)^k \log^{s-2} t dt \ll \log^{s-1} x \cdot (\log_2 x)^k \]

and

(3.8) \quad N(x) = O (\log^{s-1} x \cdot (\log_2 x)^k). \]

For \( \int_2^x f'(t) \log^{s-1} t dt \), first we notice that

\[ f'(x) = \sum_{l=0}^{k-2} (-1)^l A_k^{l+1} \left( (k-1-l)(\log_2 x + B)^{k-l-1} \frac{k-2-l}{x \log x} + \sum_{m=2}^{k-2-l} C_{k-1-l}^m (k-1-l-m)(\log_2 x + B)^{k-2-l-m} \right) \log x \]

\[ + \frac{1}{x} \sum_{l=1}^{k-1} (-1)^l A_k^{l+1} \left( \left( \log_2 x + B \right)^{k-1-l} + \sum_{m=2}^{k-1-l} C_{k-1-l}^m \left( \log_2 x + B \right)^{k-1-l-m} \right) \]

\[ = \frac{1}{x} \sum_{l=1}^{k-1} (-1)^l A_k^{l+1} \left( \left( \log_2 x + B \right)^{k-1-l} + \sum_{m=2}^{k-1-l} C_{k-1-l}^m \left( \log_2 x + B \right)^{k-1-l-m} \right) \]

\[ + \frac{1}{x} \sum_{l=0}^{k-1} (-1)^l A_k^{l+1} \left( \left( \log_2 x + B \right)^{k-1-l} + \sum_{m=2}^{k-1-l} C_{k-1-l}^m \left( \log_2 x + B \right)^{k-1-l-m} \right) \]

\[ = k \cdot \frac{1}{x} \left( \left( \log_2 x + B \right)^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^m \left( \log_2 x + B \right)^{k-1-m} \right). \]

This implies

\[ \int_2^x f'(t) \log^{s-1} t dt = k \int_2^x \frac{(\log_2 t + B)^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^m (\log_2 t + B)^{k-1-m}}{t} \log^{s-1} t dt \]
\[ = k \int_{\log 2}^{\log x} \left( (\log t + B)^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^m a_m (\log t + B)^{k-1-m} \right) t^{s-1} dt \]

\[ = k \int_{\log 2}^{\log x} \left( \sum_{t=0}^{k-1} C_{k-1}^t B^t (\log t)^{k-1-t} \right. \]

\[ + \sum_{m=2}^{k-1} C_{k-1}^m a_m \sum_{t=0}^{k-1-m} C_{k-1-m}^t B^t (\log t)^{k-1-m-t} t^{s-1} dt \]

\[ = k \left( \sum_{t=0}^{k-1} C_{k-1}^t B^t \int_{\log 2}^{\log x} (\log t)^{k-1-t} t^{s-1} dt \right. \]

\[ + \sum_{m=2}^{k-1} C_{k-1}^m a_m \sum_{t=0}^{k-1-m} C_{k-1-m}^t B^t \int_{\log 2}^{\log x} (\log t)^{k-1-m-t} t^{s-1} dt \right) \]

If we denote

\[ A_s = \sum_{l=0}^{k-1} (-1)^{l+1} A_{k-1-l}^l \frac{A_{k-1-l}^l}{s^{l+1}} P_{k-1-l} (\log_2 2) \log^s 2, \]

then

\[ \int_2^x f'(t) \log^{s-1} t dt \]

\[ = k \left[ \sum_{t=0}^{k-1} C_{k-1}^t B^t \sum_{l=0}^{k-1-t} (-1)^{l} A_{k-1-t}^l \frac{t^s (\log t)^{k-1-t-l}}{s^{l+1}} \right. \]

\[ + \sum_{m=2}^{k-1} C_{k-1}^m a_m \sum_{t=0}^{k-1-m} C_{k-1-m}^t B^t \sum_{l=0}^{k-1-m-t} (-1)^{l} A_{k-1-m-t}^l \frac{t^s (\log t)^{k-1-m-t-l}}{s^{l+1}} \right) \]

\[ = k \left[ \sum_{t=0}^{k-1} (-1)^{t} \frac{A_{k-1}^t}{s^{t+1}} \left( \sum_{l=0}^{k-1-t} C_{k-1-l}^t B^t (\log t)^{k-1-l-t} \right) t^s \right. \]

\[ + \sum_{m=2}^{k-1} (-1)^{t} \frac{A_{k-1}^t}{s^{t+1}} \sum_{m=2}^{k-1-m} C_{k-1-m}^m (\sum_{l=0}^{k-1-m-t} C_{k-1-m-t}^l B^t (\log t)^{k-1-m-t-l} \right) t^s \]

\[ = \sum_{l=0}^{k-1} (-1)^{t} \frac{A_{k-1}^t}{s^{t+1}} P_{k-1-t} (\log_2 2) \cdot \log^s x + A_s. \]

Therefore

\[ \sum_{p_1 \cdots p_k \leq x} \log^s (p_1 \cdots p_k) \frac{1}{p_1 \cdots p_k} = \sum_{l=0}^{k-1} (-1)^{l} \frac{A_{k-1}^l}{s^{l+1}} P_{k-1-l} (\log_2 2) \cdot \log^s x + f(2) \log^{s-1} 2 \]

\[ + O \left( \log^{s-1} x \cdot (\log_2 x)^k \right), \]

the proof of Theorem 1.1 is completed.
Proposition 3.5. The following evaluation holds

\[ B = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} P_1 \left( \log_2 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) \right) \]

\[ = P_1 (\log_2 x) P_{k-1} (\log_2 \sqrt{x}) + \sum_{l=0}^{k-2} (-1)^{l+1} A_{k-1}^{l+1} P_{k-2-l} (\log_2 \sqrt{x}) Li_{l+2} \left( \frac{1}{2} \right) + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right). \]

**Proof.** The simplified notation \( [3.2] \) gives

\[ B = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} P_1 \left( \log_2 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) \right) \]

\[ = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} \left( \log_2 \left( \frac{x}{p_1 \cdots p_{k-1}} \right) + B \right) \]

\[ = \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} \left( \log \left( \log x - \log(p_1 \cdots p_k) \right) - \log_2 x + \log_2 x + B \right) \]

\[ = (\log_2 x + B) \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} \log \left( 1 - \frac{\log(p_1 \cdots p_{k-1})}{\log x} \right). \]

Then from the series expansion \( \log(1 - x) \), we have

\[ B = P_1 (\log_2 x) P_{k-1} (\log_2 \sqrt{x}) + \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{1}{p_1 \cdots p_{k-1}} \left( -\sum_{s=1}^{\infty} \frac{\log^s(p_1 \cdots p_{k-1})}{s(\log x)^s} \right) \]

\[ + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right) \]

\[ = P_1 (\log_2 x) P_{k-1} (\log_2 \sqrt{x}) - \sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{1}{(\log x)^s} \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{\log^s(p_1 \cdots p_{k-1})}{p_1 \cdots p_{k-1}} \right) \]

\[ + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right) \]

\[ = P_1 (\log_2 x) P_{k-1} (\log_2 \sqrt{x}) + M + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right), \]

where

\[ M = -\sum_{s=1}^{\infty} \frac{1}{s} \left( \frac{1}{(\log x)^s} \sum_{p_1 \cdots p_{k-1} \leq \sqrt{x}} \frac{\log^s(p_1 \cdots p_{k-1})}{p_1 \cdots p_{k-1}} \right) \]

\[ = -\sum_{s=1}^{\infty} \frac{1}{s} \sum_{l=0}^{k-2} (-1)^l \frac{A_{k-1}^{l+1}}{s^{l+1}} P_{k-2-l} (\log_2 \sqrt{x}) + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right) \].
the last equation holds by using Corollary [1.3]. Since the series \( \sum_{s=1}^{\infty} \frac{1}{2^{s-1}} \) is convergent, we obtain

\[
M = \sum_{l=0}^{k-2} (-1)^{l+1} \frac{A_{k-1}^{l+1}}{s^{l+1}} P_{k-2-l}(\log_2 \sqrt{x}) \sum_{s=1}^{\infty} \frac{1}{s^{l+2} 2^s} + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right)
\]

\[
= \sum_{l=0}^{k-2} (-1)^{l+1} \frac{A_{k-1}^{l+1}}{s^{l+1}} P_{k-2-l}(\log_2 \sqrt{x}) \Li_{l+2} \left( \frac{1}{2} \right) + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right)
\]

and the result is established. \( \square \)

Finally, we prove Theorem [1.4]

**Proof of Theorem [1.4]** By Proposition [3.1] we have

\[
\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} = A + B - C + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

Denote by \( a = \log 2 \). Then substituting the results of Propositions [3.4] and [3.5] into (3.9), we have

\[
\sum_{p_1 \cdots p_k \leq x} \frac{1}{p_1 \cdots p_k} = P_k(\log_2 x) + O \left( \frac{(\log_2 x)^{k-1}}{\log x} \right),
\]

where

\[
P_k(y) = P_1(y-a)P_{k-1}(y) + \sum_{c=1}^{k-1} C_{k-1}^c P_{k-1-c}(y) \int_{0+0}^{1/2} \frac{\log^c(1-x)}{x} dx
\]

\[
+ P_1(y)P_{k-1}(y-a) + \sum_{l=0}^{k-2} (-1)^{l+1} A_{k-1}^{l+1} P_{k-2-l}(y-a) \Li_{l+2} \left( \frac{1}{2} \right)
\]

\[
- P_1(y-a)P_{k-1}(y-a)
\]

\[
= P_1(y-a)P_{k-1}(y) + aP_{k-1}(y-a) + \sum_{c=1}^{k-1} C_{k-1}^c P_{k-1-c}(y) \int_{0+0}^{1/2} \frac{\log^c(1-x)}{x} dx
\]

\[
+ \sum_{l=1}^{k-1} (-1)^{l} A_{k-1}^{l} P_{k-1-l}(y-a) \Li_{l+1} \left( \frac{1}{2} \right).
\]

Now we evaluate \( P_k(y) \) carefully. Notice that for any \( s \leq k-1 \),

\[
P_s(y-a) = \sum_{t=0}^{s} C_s^t (-1)^t a^t P_{s-t}(y)
\]

and from the integral (2.5), we obtain

\[
P_k(y) = P_1(y-a)P_{k-1}(y) + a \sum_{l=0}^{k-1} C_{k-1}^l (-1)^l a^l P_{k-1-l}(y)
\]
\[
\begin{align*}
&\sum_{t=1}^{k-1} C_{k-1}^{t} P_{k-1-t}(y) \left( (-a)^{t+1} + (-1)^t t! \zeta(t+1) + (-1)^{t-1} \sum_{s=1}^{t} A_s^t a^{t-s} L_{s+1} \left( \frac{1}{2} \right) \right) \\
&+ \sum_{t=1}^{k-1} (-1)^t A_{k-1}^t P_{k-1-t}(y-a)L_{t+1} \left( \frac{1}{2} \right).
\end{align*}
\]

Consider the identity given by operations on combinatorial numbers,
\[
\begin{align*}
&\sum_{t=1}^{k-1} (-1)^{t-1} C_{k-1}^{t} P_{k-1-t}(y) \sum_{s=1}^{t} A_s^t a^{t-s} L_{s+1} \left( \frac{1}{2} \right) \\
&+ \sum_{t=1}^{k-1} (-1)^{t-1} A_{k-1}^t a^{t-s} P_{k-1-t}(y) L_{s+1} \left( \frac{1}{2} \right) \\
&= \sum_{s=1}^{k-1} \sum_{t=s}^{k-1} (-1)^{t-s} C_{k-1}^{t+s} A_{s+1}^t \sum_{s=1}^{t} (-1)^s C_{k-1-s}^{t-s} a^{t-s} P_{k-1-t-s}(y)L_{s+1} \left( \frac{1}{2} \right) \\
&= \sum_{s=1}^{k-1} (-1)^{s-1} A_{k-1}^s \left( \sum_{t=0}^{k-1} (-1)^t C_{k-1-s}^{t-s} a^{t} P_{k-1-t-s}(y) \right) L_{s+1} \left( \frac{1}{2} \right) \\
&= \sum_{s=1}^{k-1} (-1)^{s-1} A_{k-1}^s P_{k-1-s}(y-a)L_{s+1} \left( \frac{1}{2} \right).
\end{align*}
\]

Then substitution it to the above identity for \( P_k(y) \), we obtain
\[
\begin{align*}
P_k(y) &= P_1(y) P_{k-1}(y) - a P_{k-1}(y) + a P_{k-1}(y) + \sum_{t=1}^{k-1} C_{k-1}^{t} (-1)^t a^{t+1} P_{k-1-t}(y) \\
&+ \sum_{t=1}^{k-1} C_{k-1}^{t} (-1)^{t+1} a^{t+1} P_{k-1-t}(y) + \sum_{t=1}^{k-1} C_{k-1}^{t} (-1)^t t! \zeta(t+1) P_{k-1-t}(y) \\
&+ \sum_{s=1}^{k-1} (-1)^{s-1} A_{k-1}^s P_{k-1-s}(y-a)L_{s+1} \left( \frac{1}{2} \right) \\
&+ \sum_{t=1}^{k-1} (-1)^t A_{k-1}^t P_{k-1-t}(y-a)L_{t+1} \left( \frac{1}{2} \right) \\
&= P_1(y) P_{k-1}(y) + \sum_{t=1}^{k-1} C_{k-1}^{t} (-1)^t t! \zeta(t+1) P_{k-1-t}(y).
\end{align*}
\]

Reusing the simplified notations (3.2), we have
\[
\begin{align*}
P_k(y) &= P_1(y) \left( P_1(y)^{k-1} + \sum_{m=2}^{k-1} C_{k-1}^{m} a_m P_1(y)^{k-1-m} \right) \\
&+ \sum_{t=1}^{k-1} C_{k-1}^{t} (-1)^t t! \zeta(t+1) \left( P_1(y)^{k-1-t} + \sum_{m=2}^{k-1-t} C_{k-1-t}^{m} a_m P_1(y)^{k-1-t-m} \right)
\end{align*}
\]
Thus from the definition of the sequence \( \{a_n\} \) (1.8), we get

\[
P_k(y) = P_1(y)^k + \sum_{m=2}^{3} \left( C_{k-1}^m a_m + C_{k-1}^{m-1} (-1)^{m-1} (m-1)! \zeta(m) \right) P_1(y)^{k-m} \\
+ \sum_{m=4}^{k-1} \left( C_{k-1}^m a_m + C_{k-1}^{m-1} (-1)^{m-1} (m-1)! \zeta(m) \right) P_1(y)^{k-m} \\
+ \sum_{t=2}^{k-2} \left( (-1)^{t-1} C_{k-1}^{t-1} (t-1)! \zeta(t) C_{k-t}^{m-t} a_{m-t} \right) P_1(y)^{k-m} \\
+ \sum_{t=1}^{k-3} C_{k-1}^{t} (-1)^t t! \zeta(t+1) a_{k-1-t} + (-1)^{k-1}(k-1)! \zeta(k).
\]

Then by using the equation

\[
\sum_{t=2}^{m-2} (-1)^{t-1} C_{k-1}^{t-1} (t-1)! \zeta(t) C_{k-t}^{m-t} a_{m-t} = \sum_{t=1}^{m-3} (-1)^t C_{k-1-t}^{t} \zeta(t+1) C_{k-t-1}^{m-t-1} a_{m-1-t-1},
\]

we obtain

\[
P_k(y) = P_1(y)^k + \sum_{m=2}^{3} \left( C_{k-1}^m a_m + C_{k-1}^{m-1} a_m \right) P_1(y)^{k-m} \\
+ \sum_{m=4}^{k-1} \left( C_{k-1}^m a_m + C_{k-1}^{m-1} (-1)^{m-1} (m-1)! \zeta(m) \right) P_1(y)^{k-m} \\
+ C_{k-1}^{m-1} \sum_{t=1}^{m-3} (-1)^t C_{m-t}^{t} t! \zeta(t+1) a_{m-1-t} P_1(y)^{k-m} + a_k \\
= P_1(y)^k + \sum_{m=2}^{k-1} C_{k}^m a_m P_1(y)^{k-m} + a_k,
\]

since \( C_{k-1}^m + C_{k-1}^{m-1} = C_k^m \). This completes the proof of Theorem 1.4.
Acknowledgement

We would like to thank Professor M. A. Korolev for his interested in this paper and for sending his work to us.

References

[A98] T. M. Apostol, Introduction to Analytic Number theory, Undergraduate Texts in Mathematics, Springer–Verlag, 1998.

[B18] M. Bănescu, D. Popa, A multiple Abel summation formula and asymptotic evaluations for multiple sums, Int. J. Number Theory, 14 (4) (2018), 1197–1210.

[B-K-L21] J. Bayless, P. Kinlaw, J. D. Lichtman, Higher Mertens constants for almost primes, preprint (2021) arXiv:2103.09866v1

[EW05] G. Everest, T. Ward, An introduction to number theory, Graduate Texts in Mathematics, 232. Springer–Verlag London, Ltd., London, 2005.

[I00] A. E. Ingham, The Distribution of Prime Numbers, Cambridge University Press, 1900.

[K16] M. A. Korolev, Gram’s law in the theory of the Riemann zeta-function. Part 1, Proc. Steklov Inst. Math. 292 (2) (2016), 1–146.

[L53] E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, 2 Bände. (German) 2d ed. With an appendix by Paul T. Bateman. Chelsea Publishing Co., New York, 1953.

[L97] P. Lindqvist, J. Peetre, On the reminder in a series of Mertens, Expo. Math. 15 (5) (1997), 467–478.

[L81] L. Lewin, Polylogarithms and Associated Functions, North Holland, New York, 1981.

[M74] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie, Journal füe Math., 78 (1874), 46–62.

[N00] M. B. Nathanson, Elementary Methods in Number Theory, Graduate Texts in Mathematics, vol. 195, Springer–Verlag, 2000.

[P84] D. P. Parent, Exercises in Number Theory, Translated from the French. Problem Books in Mathematics. Springer–Verlag, New York, 1984.
Multiple Mertens evaluations

[P14] D. Popa, *A double Mertens type evaluation*, J. Math. Anal. Appl. 409 (4) (2014) 1159–1163.

[P16] D. Popa, *A triple Mertens evaluation*, J. Math. Anal. Appl. 444 (1) (2016) 464–474.

[S02] F. Saidak, *An elementary proof of a theorem of Delange*, C.R. Math. Acad. Sci. Soc. R. Can. 24 (4) (2002) 144–151.

[T17] G. Tenenbaum, *Generalized Mertens sums*, in Analytic number theory, modular forms and $q$-hypergeometric series, 733–736, Springer Proc. Math. Stat., 221, Springer, Cham, 2017.

[T19] G. Tenenbaum, *Generalized Mertens sums*,

https://arxiv.org/abs/1910.02781