A SHORT NOTE ON BIHARMONIC SUBMANIFOLDS IN NON-SASAKIAN CONTACT METRIC 3-MANIFOLDS

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Abstract. We characterize biharmonic anti-invariant surfaces in 3-dimensional generalized \((\kappa, \mu)\)-manifolds with non-zero constant mean curvature by means of the scalar curvature of the ambient space and the mean curvature. In addition, we give a method for constructing infinity many examples of biharmonic submanifolds in a certain 3-dimensional generalized \((\kappa, \mu)\)-manifold. Moreover, we determine 3-dimensional generalized \((\kappa, \mu)\)-manifolds which admit a certain kind of proper biharmonic foliation.

1. Introduction

The notion of biharmonic submanifolds is a natural extension of the notion of minimal submanifolds from a variational point of view. Non-minimal biharmonic submanifolds are said to be proper. Considerable advancement has been made in the study of proper biharmonic submanifolds in manifolds with special metric properties (e.g., real space forms, complex space forms, Sasakian space forms, conformally flat spaces, etc.) since the beginning of this century.

When the dimension of the ambient space is three, it seems worthwhile and interesting to construct and classify proper biharmonic submanifolds in contact metric 3-manifolds. Some classification results for proper biharmonic submanifolds in Sasakian 3-manifolds have been obtained (see, for example, [3], [5], [6]). As an extension of the notion of a Sasakian manifold, Koufogiorgos and Tsichlias [7] introduced the notion of a generalized \((\kappa, \mu)\)-manifold for two real functions \(\kappa\) \((\leq 1)\) and \(\mu\). If \(\kappa \equiv 1\), then it is a Sasakian manifold. Markellos and Papantoniou [10] studied proper biharmonic Legendre curves and anti-invariant surfaces of 3-dimensional non-Sasakian generalized \((\kappa, \mu)\)-manifolds. However, they provided no examples of such submanifolds. As far as the author know, there have been no examples of proper biharmonic submanifolds in non-Sasakian contact metric 3-manifolds.

In this short note, we first give a method for constructing infinity many examples of proper biharmonic Legendre curves in a certain non-Sasakian generalized \((\kappa, \mu)\)-manifold. This result corrects an error in [10]. We then give a characterization of biharmonic anti-invariant surfaces in 3-dimensional non-Sasakian generalized \((\kappa, \mu)\)-manifolds with non-zero constant mean curvature, by means of the scalar curvature of the ambient space and the mean curvature. Applying the result, we obtain a method for constructing infinity many examples of proper biharmonic anti-invariant surfaces in a certain 3-dimensional non-Sasakian generalized \((\kappa, \mu)\)-manifolds. Moreover, we determine 3-dimensional non-Sasakian generalized \((\kappa, \mu)\)-manifolds which admit a certain kind of proper biharmonic anti-invariant foliation.

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2. Preliminaries

Let $M^n$ be an $n$-dimensional submanifold of a Riemannian manifold $\tilde{M}$. Let us denote by $\nabla$ and $\tilde{\nabla}$ the Levi-Civita connections on $M^n$ and $\tilde{M}$, respectively. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_XY = \nabla_X Y + B(X,Y),$$
$$\tilde{\nabla}_X N = -A_N X + D_X N$$

for tangent vector fields $X$, $Y$ and normal vector field $N$, where $B$, $A$ and $D$ are the second fundamental form, the shape operator and the normal connection.

The mean curvature vector field $H$ is defined by

$$(2.1)$$
$$H = (1/n)\text{trace } B.$$

The function $|H|$ is called the mean curvature. If it vanishes identically, then $M$ is called a minimal submanifold.

3. Biharmonic submanifolds

Let $f : M \to N$ be a smooth map of an $n$-dimensional Riemannian manifold into another Riemannian manifold. The tension field $\tau(f)$ of $f$ is a section of the induced vector bundle $f^*TN$ defined by

$$\tau(f) = \sum_{i=1}^n \{\nabla^f_{e_i} df(e_i) - df(\nabla e_i e_i)\}$$

for a local orthonormal frame $\{e_i\}$ on $M$, where $\nabla^f$ and $\nabla$ denote the induced connection and the Levi-Civita connection of $M$, respectively. If $f$ is an isometric immersion, then we have

$$(3.1)$$
$$\tau(f) = nH.$$

A smooth map $f$ is called a harmonic map if it is a critical point of the energy functional

$$E(f) = \int_\Omega |df|^2 dv_g$$

over every compact domain $\Omega$ of $M$, where $dv_g$ is the volume form of $M$. A smooth map $f$ is harmonic if and only if $\tau(f) = 0$ at each point on $M$.

Eells and Lemaire introduced the notion of biharmonic maps as a natural generalization of the notion of harmonic maps from a variational point of view.

Definition 3.1 (\cite{4}). A smooth map $f : M \to \tilde{M}$ is called a biharmonic map if it is a critical point of the bienergy functional defined by

$$E_2(f) = \int_\Omega |\tau(f)|^2 dv_g$$

with respect to all variations with compact support.

The Euler-Lagrange equation for $E_2$ is given by

$$(3.2)$$
$$\tau_2(f) := -\Delta f(\tau(f)) + \text{trace} \tilde{R}(\tau(f), df) df = 0,$$

where $\Delta f = -\text{trace}(\nabla^f \nabla^f - \nabla f)$ and $\tilde{R}$ is the curvature tensor of $\tilde{M}$ which is defined by

$$\tilde{R}(X,Y)Z = [\tilde{\nabla}_X, \tilde{\nabla}_Y]Z - \tilde{\nabla}_{[X,Y]}Z$$

for the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$. 
If $f$ is a biharmonic isometric immersion, then $M$ or $f(M)$ is called a biharmonic submanifold in $M$. It follows from (3.1) and (3.2) that any minimal submanifold is biharmonic. However, the converse is not true in general. Non-minimal biharmonic submanifolds are called proper biharmonic submanifolds.

4. Generalized $(\kappa, \mu)$-manifolds

A differentiable manifold $\tilde{M}$ is called an almost contact manifold if it admits a unit vector field $\xi$, a one-form $\eta$ and a $(1,1)$-tensor field $\phi$ satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi.$$  

Every almost contact manifold admits a Riemannian metric $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

If, in addition, the condition

$$d\eta(X, Y) := (1/2)(\eta(YX) - Y(\eta(X)) - \eta([X, Y])) = g(X, \phi Y)$$

holds, then $(\tilde{M}, \xi, \eta, \phi, g)$ is called a contact metric manifold. A contact metric manifold is called a Sasakian manifold if it satisfies

$$[\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$ which is defined as $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi(X), \phi(Y)] - \phi[\phi(X), Y] - \phi[X, \phi(Y)]$.

Koufogiorgos and Tsichlias [7] introduced the notion of generalized $(\kappa, \mu)$-manifold, which is defined as contact metric manifold whose curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)\xi = (\kappa I + \mu h)(\eta(Y)X - \eta(X)Y)$$

for any vector field $X$ and $Y$, where $2h$ is the Lie differentiation of $\phi$ with respect to $\xi$, and $\kappa, \mu$ are smooth functions. If $\kappa$ and $\mu$ are constant, then the manifold is simply called a $(\kappa, \mu)$-manifold (\tilde{2}). Sasakian manifolds are $(\kappa, \mu)$-manifolds with $\kappa = 1$ and $h = 0$.

In [7], it was proved that if the dimension of generalized $(\kappa, \mu)$-manifold is greater than 3, then $\kappa$ and $\mu$ must be constant, and if the dimension is 3, then there exist examples of generalized $(\kappa, \mu)$-manifold with $\kappa, \mu$ non-constant smooth functions.

Perrone [12] proved that a contact metric 3-manifold $\tilde{M}^3$ is a generalized $(\kappa, \mu)$-manifold on an everywhere dense open subset if and only if the vector field $\xi$ defines a harmonic map from $\tilde{M}^3$ into its unit tangent bundle equipped with the Sasaki metric.

Any 3-dimensional generalized $(\kappa, \mu)$-manifold $\tilde{M}^3$ with $\kappa < 1$ admits three mutually orthogonal distributions $D(0) = \text{Span}\{\xi\}$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of $h$, where $\lambda = \sqrt{1 - \kappa}$. Furthermore, on such a manifold the following relations hold (see Lemma 3.3 of [9] and its proof):

\begin{align}
\tilde{R}(\phi X, \phi X) & = \left(-\frac{\Delta \lambda}{2\lambda^2} - \frac{||\text{grad} \lambda||^2}{2\lambda^2} - \kappa - \mu\right) \phi X, \\
S & = -\frac{\Delta \lambda}{\lambda} - \frac{||\text{grad} \lambda||^2}{\lambda^2} + 2(\kappa - \mu),
\end{align}

where $X$ is a unit vector lying in $D(\pm \lambda)$, $\Delta \lambda = -\sum_{i=1}^{3} \{e_i(e_i \lambda) - (\nabla_{e_i} e_i) \lambda\}$, and $S$ denotes the scalar curvature.

Below, we shall exhibit examples of 3-dimensional generalized $(\kappa, \mu)$-manifolds which are not $(\kappa, \mu)$-manifolds.
Example 4.1 ([9]). Let \( \lambda : I \subset \mathbb{R} \to \mathbb{R} \) be a non-constant positive function defined on an open interval \( I \). We put \( \lambda' = \frac{d\lambda}{dx} \). Consider a manifold \( \tilde{M}^3 = \{(x, y, z) \in \mathbb{R}^2 \times I \subset \mathbb{R}^3\} \). The vector fields

\[
(4.3) \quad e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = (\pm 2y + f(z)) \frac{\partial}{\partial x} + \left(2\lambda x - \frac{\lambda'}{2\lambda} y + h(z)\right) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( \tilde{M}^3 \), where \( f(z) \) and \( h(z) \) are arbitrary functions of \( z \). Let \( g \) be the Riemannian metric defined by \( g(e_i, e_j) = \delta_{ij} \), \( i, j = 1, 2, 3 \), and \( \eta \) the dual 1 form to \( e_1 \). If we define the \((1,1)\)-tensor field \( \phi \) by \( \phi e_1 = 0 \), \( \phi e_2 = \pm e_3 \) and \( \phi e_3 = \mp e_2 \), then the manifold \( (\tilde{M}^3, \phi, e_1, \eta, g) \) is a generalized \((\kappa, \mu)\)-manifold with

\[
(4.4) \quad \kappa = 1 - \lambda^2, \quad \mu = 2(1 \pm \lambda),
\]

where double signs correspond to \( \pm 2y \) in \([1,3]\). Let us denote the manifold by \( \tilde{M}(1 - \lambda^2, 2(1 \pm \lambda)) \).

The following classification result was obtained by Koufogiorgos and Tsichlias.

Theorem 4.1 ([9]). Let \( \tilde{M}^3 \) be a 3-dimensional generalized \((\kappa, \mu)\)-manifold which is not a \((\kappa, \mu)\)-manifold. If \( \tilde{M}^3 \) satisfies \( \xi \mu = 0 \), then it is locally given by \( \tilde{M}(1 - \lambda^2, 2(1 \pm \lambda)) \).

Remark 4.1. Every generalized \((\kappa, \mu)\)-manifold satisfies \( \xi \kappa = 0 \) (see [8]).

Remark 4.2. Hereafter, except \([6,1]\), all double signs correspond to that of \( \tilde{M}(1 - \lambda^2, 2(1 \pm \lambda)) \).

On \( \tilde{M}(1 - \lambda^2, 2(1 \pm \lambda)) \), the following relations are true (see the proof of Theorem 4.2 in [9]):

\[
(4.5) \quad \tilde{\n} e_1 e_1 = \tilde{\n} e_2 e_3 = 0,
\]

\[
(4.6) \quad \tilde{\n} e_2 e_2 = \frac{\lambda'}{2\lambda} e_3,
\]

\[
(4.7) \quad \tilde{\n} e_2 e_3 = -\frac{\lambda'}{2\lambda} e_2 + (\lambda \pm 1)e_1,
\]

\[
(4.8) \quad he_2 = \pm \lambda e_2.
\]

5. Biharmonic Legendre curves in 3-dimensional generalized \((\kappa, \mu)\)-manifolds

Let \( \tilde{M}^3 \) be a contact metric 3-manifold. A curve \( \gamma : I \subset \mathbb{R} \to \tilde{M}^3 \) parametrized by arclength is called a Legendre curve if \( \eta(\gamma'(s)) = 0 \) holds identically.

For Legendre curves in 3-dimensional \((\kappa, \mu)\)-manifolds, we have

Theorem 5.1 ([10]). Let \( \gamma \) be a non-geodesic Legendre curve in a 3-dimensional \((\kappa, \mu)\)-manifold with \( \kappa < 1 \). Then \( \gamma \) is biharmonic if and only if either \( \tilde{\n} \gamma(\gamma') || \phi \gamma' \) and in this case \( \gamma \) is a helix satisfying \( k_g^2 + \tau_g^2 = -(\kappa + \mu) \) or \( \tilde{\n} \gamma(\gamma') || \xi \) and \( \gamma \) is a helix satisfying \( k_g^2 + \tau_g^2 = \kappa + \delta \mu \), where \( k_g \) (resp. \( \tau_g \)) denotes the geodesic curvature (resp. the geodesic torsion), and \( \delta := g(\gamma'(\gamma'), \gamma') \) is constant.

On the other hand, in [10, Theorem 3.2] the authors claimed the following:

Let \( \gamma \) be a Legendre curve satisfying \( \tilde{\n} \gamma(\gamma') || \phi \gamma' \) in a 3-dimensional generalized \((\kappa, \mu)\)-manifold which is not a \((\kappa, \mu)\)-manifold. Then, \( \gamma \) is biharmonic if and only if it is a geodesic.
However, this claim is incorrect. In fact, there exist infinity many non-geodesic biharmonic Legendre curves in $\tilde{M}(1-\lambda^2, 2(1+\lambda))$, as shown below.

For two constants $b$ and $c$, we consider the Legendre curve in $\tilde{M}(1-\lambda^2, 2(1+\lambda))$ given by
\begin{equation}
(5.1) \quad \gamma_{(b,c)}(s) = (b, s, c).
\end{equation}
It follows from (4.6) that the curve (5.1) satisfies $\nabla_{\gamma'}\gamma'' \parallel \phi\gamma'$.

**Proposition 5.1.** A Legendre curve $\gamma(s)$ in $\tilde{M}(1-\lambda^2, 2(1+\lambda))$ given by $\gamma_{(b,c)}(s) = (b, s, c)$ is proper biharmonic if and only if $c$ satisfies $\lambda'(c) \neq 0$ and
\begin{equation}
(5.2) \quad \lambda'' - 2(\lambda')^2 - 8\lambda^2(1+\lambda)|_{z=c} = 0.
\end{equation}

**Proof.** The unit tangent vector field of $\gamma$ is given by $e_2$. Equation (4.6) yields that $\gamma$ is a geodesic if and only if $c$ satisfies $\lambda'(c) = 0$. Equation (5.2) can be written as
\begin{equation}
(5.3) \quad \tilde{\nabla}_{e_2}\tilde{\nabla}_{e_2} + \tilde{R}(\tilde{\nabla}_{e_2}e_2, e_2)e_2 = 0.
\end{equation}

Since (4.8) is satisfied, we put $X = e_2$ in (4.1). Then, using (4.5)-(4.7) and (4.1), by a straightforward computation we can show that (5.3) is equivalent to (5.2). This finishes the proof.

**Example 5.1.** If we put $\lambda = z^{-n}$ for $z > 0$ and $n \in (0, 1)$, then $\lambda > 0$, $\lambda' \neq 0$ and (5.2) becomes
\[8c^2(1+e^{-n}) = n(1-n).\]
For any $n \in (0, 1)$, this equation for $c$ has a positive solution, because $\lim_{c \to 0} c^2(1+e^{-n}) = 0$ and $\lim_{c \to \infty} c^2(1+e^{-n}) = \infty$.

6. **Biharmonic anti-invariant surfaces in 3-dimensional generalized $(\kappa, \mu)$-manifolds**

A submanifold $M$ in a contact metric manifold is said to be anti-invariant if it is tangent to $\xi$ and satisfies $g(X, \phi Y) = 0$ for all tangent vectors $X$ and $Y$ of $M$. For proper biharmonic anti-invariant surfaces in 3-dimensional $(\kappa, \mu)$-manifolds, we have the following result.

**Theorem 6.1 ([1]).** Let $f : M^2 \to \tilde{M}^3$ be an anti-invariant isometric immersion of a Riemannian 2-manifold into a 3-dimensional $(\kappa, \mu)$-manifold. Then $f$ is proper biharmonic if and only if $\kappa = 1$; that is, $\tilde{M}^3$ is a Sasakian manifold, and moreover, $|H|^2 = (1/8)(S - 6) = \text{constant } (\neq 0)$ on $M^2$.

This motivates us to study proper biharmonic anti-invariant surfaces in 3-dimensional generalized $(\kappa, \mu)$-manifolds which are not $(\kappa, \mu)$-manifolds. The following theorem characterizes such surfaces with constant mean curvature by means of $S$ and $H$.

**Theorem 6.2.** Let $f : M^2 \to \tilde{M}^3$ be an anti-invariant immersion of a Riemannian 2-manifold into a 3-dimensional generalized $(\kappa, \mu)$-manifold which is not a $(\kappa, \mu)$-manifold. Assume that $M^2$ has non-zero constant mean curvature. Then, $f$ is biharmonic if and only if $S$ is constant on $M^2$ and
\begin{equation}
(6.1) \quad h(\phi H) = \left(\pm \sqrt{\frac{S}{6} - \frac{4}{3}|H|^2 - 1}\right)\phi H
\end{equation}
holds at any point of $M^2$. 
Proof. Let $M^2$ be an anti-invariant surface in a 3-dimensional generalized $(\kappa, \mu)$-manifold which is not a $(\kappa, \mu)$-manifold. Assume that $M^2$ has non-zero constant mean curvature $\alpha$. Let $\{\xi, e_1, \phi e_1\}$ be a local orthonormal frame of $M^2$ such that $H = \alpha \phi e_1$. We put

$$\beta = 1 + g(he_1, e_1), \quad \gamma = g(he_1, \phi e_1).$$

By the same way as in the proof of Theorem 4.3 in [10] and using (4.9) of Proposition 4.1, we see that $M^2$ is biharmonic if and only if $\beta$ is constant on $M^2$, and moreover, the following relations hold:

$$\gamma = 0, \quad -4\alpha^2 - 2\beta^2 + (S/2) - \kappa - \mu(\beta - 1) = 0. \quad (6.3)$$

For any anti-invariant surface in a 3-dimensional generalized $(\kappa, \mu)$-manifold, the following relations are true (see Lemma 4.1 of [10]):

$$\gamma = 0, \quad -4\alpha^2 - 2\beta^2 + (S/2) - \kappa - \mu(\beta - 1) = 0. \quad (6.4)$$

$$\kappa = 2\beta - \beta^2, \quad \mu = 2\beta. \quad (6.5)$$

Since $\kappa < 1$, it follows from (6.4) and (6.5) that the first equation in (6.3) implies

$$\kappa = 2\beta - \beta^2, \quad \mu = 2\beta. \quad (6.6)$$

By (6.2), we see that the first equation in (6.3) is equivalent to

$$h(\phi H) = (\beta - 1)\phi H. \quad (6.7)$$

Substituting (6.6) into the second equation of (6.3), we obtain

$$-4\alpha^2 - 3\beta^2 + (S/2) = 0. \quad (6.8)$$

Combining (6.7) and (6.8) leads to (6.1). Accordingly, we conclude that (6.3) is equivalent to (6.1). The proof is finished.

Applying Theorem 6.2, we obtain a method for constructing of infinity many examples of proper biharmonic anti-invariant surfaces in 3-dimensional generalized $(\kappa, \mu)$-manifolds which are not $(\kappa, \mu)$-manifolds, as shown below.

For a constant $c$, we consider the immersion

$$f_c(x, y) = (x, y, c)$$

of $\mathbb{R}^2$ into $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$. Suppose that $\mathbb{R}^2$ is equipped with the induced metric $f_c^*g$. Then, $\{f_c^*e_1, f_c^*e_2\}$ forms an orthonormal frame of $f_c$. It is clear that $f_c$ is an anti-invariant immersion whose unit normal vector field $N$ is $\phi(f_c^*e_2) = f_c^*e_3$, and $\{f_c(\mathbb{R}^2)|c \in I\}$ defines a codimension one foliation on $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$.

**Proposition 6.1.** The anti-invariant isometric immersion $f_c : \mathbb{R}^2 \to \tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$ given by $f_c(x, y) = (x, y, c)$ is proper biharmonic if and only if $c$ satisfies $\lambda'(c) \neq 0$ and

$$\lambda'' - 2(\lambda')^2 - 8\lambda^2(1 \pm \lambda)^2|_{z=c} = 0. \quad (6.9)$$

**Proof.** It follows from (4.4)-(4.6) that (4.2) can be rewritten as

$$S = \frac{\lambda''}{\lambda} - \frac{3(\lambda')^2}{4\lambda^2} - 2(1 \pm \lambda)^2. \quad (6.10)$$

By (2.1), (2.2) and (4.6) we have

$$|H|^2 = (\lambda')^2/(16\lambda^2)|_{z=c}. \quad (6.11)$$
Thus, the surface $f_c(\mathbb{R}^2)$ is minimal if and only if $X'(c) = 0$. By using (6.10), (6.11) and (18), we can show that (6.1) is equivalent to (6.9). The proof is finished. }

**Example 6.1.** If we put $\lambda = z^{-n}$ for $z > 0$ and $n \in (0, 1)$, then $\lambda > 0$, $\lambda' \neq 0$ and (6.9) becomes

$$8c^2(1 \pm e^{-n})^2 = n(1 - n).$$

Similarly to Example 5.1, for any $n \in (0, 1)$ this equation for $c$ has a positive solution.

**Remark 6.1.** A generalized $(\kappa, \mu)$-manifold with $||\text{grad} \kappa|| = a$ (constant)$\neq 0$ is locally given by $M(1 - \lambda^2, 2(1 + \lambda))$ or $M(1 - \lambda^2, 2(1 - \lambda))$, where $\lambda = \sqrt{1 - a z - b}$ for some constant $b$ (see [7], Proposition 4.4) and [8] Theorem 5). In this case, since $\lambda$ does not satisfy (6.9) for any $c$, there exists no anti-invariant surface $f_c(\mathbb{R}^2)$ which is proper biharmonic. This implies that anti-invariant surfaces described in [10] Example 4.1 is not proper biharmonic. Hence, Proposition 6.1 provides the first examples of proper biharmonic surfaces in non-Sasakian contact metric 3-manifolds.

The following theorem provides us with many examples of 3-dimensional generalized $(\kappa, \mu)$-manifolds foliated by proper biharmonic surfaces.

**Theorem 6.3.** Let $\tilde{M}^3$ be a 3-dimensional generalized $(\kappa, \mu)$-manifold which is not a $(\kappa, \mu)$-manifold. If $\tilde{M}^3$ admits an integrable distribution $D(0) \oplus D(\pm \lambda)$ whose leaves are proper biharmonic surfaces, then $\tilde{M}^3$ is locally given by $\tilde{M}(1 - \lambda^2, 2(1 \pm \lambda))$, where $\lambda$ is a positive solution of

$$(\lambda')^2 = \beta \lambda^4 + 16 \lambda^2 \ln \lambda \mp 32 \lambda^2 - 8 \lambda^2 \neq 0$$

for some constant $\beta$.

**Proof.** Let $\{\xi, X, \phi X\}$ be a local orthonormal frame such that

$$hX = \lambda X, \quad \hphi X = -\lambda \phi X.$$ 

Then, by Lemma 3.1 and 3.2 of [9] we have

$$[\xi, X] = (1 + \lambda - \mu/2)\phi X, \quad [\xi, \phi X] = (\lambda - 1 + \mu/2)X.$$ 

Hence, $D(0) \oplus D(1 + \lambda)$ or $D(0) \oplus D(1 - \lambda)$ is integrable if and only if $\mu = 2(1 + \lambda)$ or $\mu = 2(1 - \lambda)$, respectively. By applying Theorem 4.1 and Remark 4.1, we obtain that if $\tilde{M}^3$ admits an integrable distribution $D(0) \oplus D(\lambda)$ or $D(0) \oplus D(-\lambda)$, then $\tilde{M}^3$ is locally given by $\tilde{M}(1 - \lambda^2, 2(1 + \lambda))$ or $\tilde{M}(1 - \lambda^2, 2(1 - \lambda))$, respectively.

In this case, each leaf is given by $f_c$. All leaves of a foliation $\{f_c(\mathbb{R}^2) | c \in I\}$ are biharmonic surfaces if and only if (6.9) is satisfied for all $c \in I$. By solving the ODE, we have (6.12).

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