On the Finiteness of the Capacity of Continuous Channels

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Abstract

Evaluating the channel capacity is one of many key problems in information theory. In this work we derive rather-mild sufficient conditions under which the capacity of continuous channels is finite and achievable.

These conditions are derived for generic, memoryless and possibly non-linear additive noise channels. The results are based on a novel sufficient condition that guarantees the convergence of differential entropies under point-wise convergence of probability density functions.

Perhaps surprisingly, the finiteness of channel capacity holds for the majority of setups, including those where inputs and outputs have possibly infinite second-moments.

I. INTRODUCTION

Over continuous-alphabets channels, a common belief is that with “sufficient” power, one is capable of transmitting at arbitrarily large rates. Stated differently, if an input of infinite power is allowed, the channel capacity is infinite. This belief is perhaps inspired from the well-known Additive White Gaussian Noise (AWGN) and linear Gaussian channels for example.

However, recent studies have suggested that for some channels this is not true: even if an infinite power input is allowed, the achievable rates are not arbitrarily large:

• In [1], the authors studied a linear additive-noise channel where the noise is heavy-tailed –modeled using alpha-stable statistics. They showed that even if the input constraint does

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allow for an infinite-power input, the channel capacity is finite. Actually, the authors found the optimal input to be surprisingly of finite power.

- In [2], the authors studied an additive Cauchy-distributed noise, and the constraints did allow as well for infinite-power input signals. The capacity was proven to be finite despite the fact that the optimal input was found in this case to have infinite power.

The natural question that arises is: “under which conditions does one have a finite channel capacity?”, the answer to which does clearly depend on the input constraints, but also on the noise statistics. In this work, we study the interaction between the input constraints, the input-output relationship and the noise distribution and derive conditions on the triplet under which the channel capacity is finite.

This guarantee of finiteness is of high significance as it is typically the first step one would undertake in order to quantify the capacity of a channel at hand. Consider for example an additive Gaussian noise channel where the output $Y$ is related to its input as follows:

$$Y = X + N,$$

and where the input is subject to a constraint $E \left[ \ln^2 (1 + |X|) \right] \leq 1$. This channel is equivalent to the following channel:

$$Y = \text{sgn}(U) \left( e^{\|U\|} - 1 \right) + N$$

where $U$ is average power constrained $E [U^2] \leq 1$. At a first look, it is not clear whether the capacity of such a channel is finite or not. Indeed, in some sense the channel is “exponentially amplifying” the input by more than what the cost is constraining it. An appropriately-chosen Cauchy distributed input $X$ will satisfy the constraint but will have an infinite second moment. The average of $Y^2$ will be infinite as well. Is the capacity of this channel finite? Our result provides an unexpected positive answer to this question.

More formally, the notion of capacity of a discrete memoryless channel was defined in the early works of Shannon [3], [4] to be “the largest” rate at which one can communicate over a channel with an arbitrarily low probability of error. Through a coding theorem, Shannon proved that the capacity is given by the solution to an optimization problem, whereby the mutual information between the input and output of the channel is maximized. When it comes to continuous channels the inputs of which are potentially constrained, the results were extended (see for example [4]–[6]) and the channel capacity was also tied to a constrained optimization problem.
Naturally, in both setups it is implicitly assumed that the optimization problem is “well-defined”, for otherwise relating the channel capacity to a solution of a maximization of mutual information is problematic. In this work we tackle this assumption and provide a sufficient condition for such an optimization problem to be both well-defined and yielding a finite and achievable solution for a wide range of channels.

We consider a generic average-constrained channel model where the noise is additive and absolutely continuous. We prove in Section III that under very mild conditions on the noise and the constraint the channel capacity is indeed finite and achievable.

We start by deriving sufficient conditions that ensure that mutual information is finite—and hence well-defined— and we make use of the extreme value principle [7] to ensure that the maximization problem yields a finite and achievable solution. This could be achieved by enforcing two characteristics:

1- The input space of feasible distribution functions is compact.
2- The mutual information between the input and the output of the channel is continuous in the input distribution function.

We emphasize that these two properties are intimately related to the channel model and the input constraints if any.

The generic model adopted in this work encompasses multiple channel models found in the literature: We consider input-output relationships that are possibly non-linear; A generic average cost function $C(\cdot)$ is imposed on the input; The absolutely continuous additive noise has a finite “super-logarithmic” moment as is the case for Gaussian, uniform, generalized Gaussian, generalized t, Pareto, Gamma, alpha-stable distributions, and their mixtures. We show that whenever the input cost function has a ”super-logarithmic” growth, the channel capacity is finite and achievable.

Establishing the continuity of mutual information under any “super-logarithmic” input constraint is achieved using a novel result on the convergence of differential entropies. While numerous studies have tackled this subject (see for example [8], [9]), the conditions presented in Section III are among the weakest that insure this convergence whenever Probability Density Functions (PDFs) converge point-wise.

The rest of the paper is organized as follows: In Section III a preliminary theorem concerning the convergence of differential entropies is listed and proved. The primary problem and the main
result are presented in Section III, where we describe the channel model and state the conditions under which our result holds. Some extensions are listed in Section IV and Section V concludes the paper.

II. CONVERGENCE OF DIFFERENTIAL ENTROPIES

In this section we establish a sufficient condition for the convergence of differential entropies whenever there is point-wise convergence of the corresponding PDFs. More precisely, we prove a theorem that guarantees this convergence under some rather-mild sufficient conditions. In layman terms, this theorem states that whenever the PDFs satisfy a super-logarithmic type of moment, point-wise convergence will imply convergence of differential entropies. We emphasize that the new conditions are weaker than some of those derived by Godavarti and al. [9, Thm 1]. Alternative conditions found in [9, Thm 4] are not directly related to those presented hereafter.

Theorem 1. Let the sequence of PDFs on $\mathbb{R}$, $\{p_m(y)\}_{m \geq 1}$ and $p(y)$ satisfy the following conditions:

C1- The PDFs $\{p_m(y)\}_m$ and $p(y)$ are uniformly upperbounded:

$$\sup_{y \in \mathbb{R}, m \geq 1} \left\{ p_m(y), p(y) \right\} \leq M.$$  \hspace{1cm} (1)

C2- There exists a non-negative and non-decreasing function $l : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $l(y) = \omega(\ln(y))$ and

$$\sup_m \left\{ E_{p_m} [l(|Y|)], E_p [l(|Y|)] \right\} \leq L.$$ \hspace{1cm} (2)

for some positive (finite) value $L$.

Under these conditions, $h(p_m) \rightarrow h(p)$ whenever the PDFs $p_m(y) \rightarrow p(y)$ point-wise.

Proof: We start by noting that the differential entropies $h(p)$ and $\{h(p_m)\}_{m \geq 1}$ exist and are finite by virtue of the fact that the PDFs are upperbounded and have a finite logarithmic moment [10, Proposition 1].

Assume now that the conditions of the theorem hold and that $p_m$ converges to $p$ point-wise. If the upperbound $M$ is larger than one, consider the change of variables, $Z = MY$ (for

*We say that $f(x) = \omega(g(x))$ if and only if $\forall \kappa > 0, \exists c > 0$ such that $f(x) \geq \kappa g(x), \forall x \geq c.$
which \( h(Z) = h(Y) + \ln M, \) or equivalently the PDFs,

\[
d(y) = \frac{1}{M} p \left( \frac{y}{M} \right), \quad d_m(y) = \frac{1}{M} p_m \left( \frac{y}{M} \right), \quad m \geq 1.
\]

These densities are upperbounded by one and the sequence \( \{d_m(y)\} \) converges point-wise to \( d(y) \). Furthermore, the function \( l'(y) = l(y/M) \) is non-negative, non-decreasing and \( l'(y) = \omega \left( \ln(y) \right) \). Additionally,

\[
E_{d_m} [l'(|Y|)] = E_{p_m} [l'(|MY|)] = E_{p_m} [l(|Y|)] \leq L.
\]

The conditions of the theorem therefore hold for the laws \( \{d_m, d\} \) and in what follows we assume Without Loss Of Generality (WLOG) that \( M \leq 1 \), and the differential entropies are all non-negative.

Let \( \tilde{y} \) be any positive scalar such that \( l(\tilde{y}) > 0 \), and denote by \( q(y) = \frac{1}{\pi} \frac{1}{1 + y^2} \) the Cauchy density. Then, using the convention “\( 0 \ln 0 = 0 \)” and the fact that \( y \ln y \geq -\frac{1}{e} \) for \( y > 0 \), we can write

\[
- \int_{|y| \geq \tilde{y}} p(y) \ln p(y) \, dy = - \int_{|y| \geq \tilde{y}} p(y) \ln q(y) \, dy + \int_{|y| \geq \tilde{y}} q(y) \frac{p(y)}{q(y)} \ln \frac{q(y)}{p(y)} \, dy
\]

\[
\leq \ln \pi \int_{|y| \geq \tilde{y}} l(|y|) p(y) \, dy + \int_{|y| \geq \tilde{y}} \ln \left[ 1 + y^2 \right] p(y) \, dy + \frac{1}{e} \int_{|y| \geq \tilde{y}} q(y) \, dy
\]

\[
\leq \ln \pi \frac{1}{l(\tilde{y})} \int_{|y| \geq \tilde{y}} l(|y|) p(y) \, dy + \int_{|y| \geq \tilde{y}} \ln \left[ 1 + y^2 \right] p(y) \, dy + \frac{1}{e} \ln \left[ 1 + \tilde{y}^2 \right] \int_{|y| \geq \tilde{y}} q(y) \, dy,
\]

where equation (5) is due to the fact that \( l(\cdot) \) is non-decreasing. Hence,

\[
- \int_{|y| \geq \tilde{y}} p(y) \ln p(y) \, dy \leq \ln \pi \frac{E_p [l(|Y|)]}{l(\tilde{y})} + 2 \int_{|y| \geq \tilde{y}} \ln \left[ 1 + |y| \right] p(y) \, dy + \frac{1}{e} \frac{E_q [\ln (1 + Y^2)]}{\ln (1 + \tilde{y}^2)}
\]

\[
\leq \frac{L \ln \pi}{l(\tilde{y})} + 2 \sup_{|y| \geq \tilde{y}} \left\{ \frac{\ln \left[ 1 + |y| \right]}{l(|y|)} \right\} \int_{|y| \geq \tilde{y}} l(y) p(y) \, dy + \frac{\ln 4}{e \ln (1 + \tilde{y}^2)}
\]

\[
\leq \frac{L \ln \pi}{l(\tilde{y})} + 2L \sup_{|y| \geq \tilde{y}} \left\{ \ln \left[ 1 + |y| \right] \right\} + \frac{\ln 4}{e \ln (1 + \tilde{y}^2)},
\]

where equation (4) is justified since \( l(\tilde{y}) \) is positive and \( l(y) \) is non-negative. In order to write equation (5) we use the identity \( E_q [\ln (1 + Y^2)] = \ln 4 \) [11, Sec.3.1.3, p.51]. The supremum in equations (5) and (6) is finite –and goes to 0– for \( \tilde{y} \) large-enough because \( l(y) = \omega \left( \ln y \right) \).
Since the upperbound (6) also holds for any \( p_m(y) \), then for every \( \delta > 0 \), there exists a \( \tilde{y} > 0 \) such that for all \( m \geq 1 \):

\[
\left| \int_{|y| \geq \tilde{y}} p_m(y) \ln p_m(y) \, dy \right| < \delta \quad \& \quad \left| \int_{|y| \geq \tilde{y}} p(y) \ln p(y) \, dy \right| < \delta.
\]

It remains to show that

\[
\lim_{m \to +\infty} -\int_{|y| < \tilde{y}} p_m(y) \ln p_m(y) \, dy = -\int_{|y| < \tilde{y}} p(y) \ln p(y) \, dy,
\]

which is guaranteed by the Dominated Convergence Theorem (DCT) since \( |p_m(y) \ln p_m(y)| \leq \frac{1}{e} \) by virtue of the fact that \( p_m(y) \leq 1 \) for all \( m \), which completes the proof. 

III. A SUFFICIENT CONDITION FOR FINITENESS OF CHANNEL CAPACITY

In what follows we derive a sufficient condition for a memoryless additive-noise channel to have a finite and achievable capacity. More specifically, we consider a generic discrete-time real and memoryless noisy communication channel where the noise is additive and where the input and output are possibly non-linearly related as follows:

\[ Y = f(X) + N, \tag{7} \]

where \( Y \in \mathbb{R} \) is the channel output and where the input is denoted \( X \) and is assumed to have an alphabet \( \mathcal{X} \subseteq \mathbb{R} \). The channel’s input is distorted according to the deterministic and possibly non-linear function \( f(x) \). Additionally, the communication channel is subjected to an additive noise –that is independent of the input– that is absolutely continuous with PDF \( p_N(\cdot) \).

Finally, we assume that the input is subject to an average cost constraint of the form: \( \mathbb{E} \left[ C(|X|) \right] \leq A \), for some \( A \in \mathbb{R}^+ \) and where \( C(\cdot) \) is some cost function: \( C : \mathbb{R}^+ \rightarrow \mathbb{R} \).

Accordingly, we define for \( A > 0 \)

\[ \mathcal{P}_A = \left\{ \text{Prob. distributions } F : \int C(|x|) \, dF(x) \leq A \right\}, \tag{8} \]

the set of all distribution functions satisfying the average cost constraint.

The primary question that we would like to answer is whether can one reliably get an arbitrarily large number of bits per use over this channel. Said differently, are the achievable rates over this channel bounded? The answer to this question follows from those of the following two questions:
• Is the mutual information between a feasible input and the corresponding output always finite?
• In case it is, can this mutual information be arbitrarily large?

A positive answer to the first question allows by the coding theorem [12] to state that the channel capacity is the supremum of the mutual information $I(\cdot)$ between the input $X$ and output $Y$ over all input probability distributions $F$ that meet the constraint $\mathcal{P}_A$:

$$C = \sup_{F \in \mathcal{P}_A} I(F).$$

For the channel at hand (7), we note that the channel transition probability law is absolutely continuous with density function given by

$$p_{Y|X}(y|x) = p_N(y - f(x)), \quad y \in \mathbb{R}, \ x \in \mathcal{X}. \tag{9}$$

and the mutual information may be expressed as [6]

$$I(F) = \int \int p_N(y - f(x)) \ln \left[ \frac{p_N(y - f(x))}{p(y; F)} \right] dy \, dF(x), \tag{10}$$

where $p(y; F) = \int p_N(y - f(x)) \, dF(x)$ is the output PDF.

A. Sufficient conditions

We make the following rather-mild assumptions:

- The cost function $C(\cdot)$:
  - A1- The cost function is lower semi-continuous.
  - A2- The cost function is non-decreasing.
  - A3- $C(|x|) = \omega (\ln |f(x)|)$.

Without loss of generality, one may assume that $C(\cdot)$ is non-negative. For if it were not, define $C'(|x|) = C(|x|) - C(0)$ and adjust the input constraint accordingly.

- The function $f(\cdot)$:
  - A4- The function is continuous.
  - A5- The absolute value of the function $|f(\cdot)|$ is an non-decreasing function of $|x|$ and $|f(x)| \to +\infty$ as $|x| \to +\infty$.

- The noise PDF $p_N(\cdot)$:
A6- The PDF is continuous on \( \mathbb{R} \).
A7- The PDF is upperbounded.
A8- There exits a non-decreasing function

\[
C_N : \mathbb{R}^+ \longrightarrow \mathbb{R}
\]

such that \( C_N(|n|) = \omega (\ln |n|) \), and

\[
\mathbb{E}_N [C_N(|N|)] = L_N < \infty.
\]

As an example, this condition holds true for any noise PDF whose tail is "faster" than \( \frac{1}{x(\ln x)^3} \).

Conditions A7 and A8 guarantee that the noise differential entropy \( h_N \), exists and is finite \([10, \text{Proposition 1}]\).

Since from an information theoretic perspective, the general channel model (7) is invariant with respect to output scaling, we consider WLOG that the noise PDF is upperbounded by one.

Also WLOG, one may assume that \( C_N(\cdot) \) is non-negative. Otherwise, one may adopt \( C'_N(|x|) = C_N(|x|) - C_N(0) \).

The above assumptions are sufficient conditions on the triplet \( f(\cdot), C(\cdot), \) and \( p_N(\cdot) \) that guarantee the finiteness and the achievability of the capacity of channel (7). A result which we formally state in the following Theorem.

**Theorem 2.** Under conditions A1 through A8, the capacity of the average-cost constrained channel (7) is finite and achievable. Furthermore, the maximum is achieved by a unique \( F^* \) in \( \mathcal{P}_A \) if and only if the output PDF is injective in \( F \).

Theorem 2 is established using the extreme value principle which we state next for completeness and can be found in [7]:

**Theorem.** If \( I(\cdot) \) is a real-valued, weak continuous functional on a compact set \( \Omega \subseteq \mathcal{F} \), then \( I(\cdot) \) achieves its maximum on \( \Omega \).

If furthermore \( \Omega \) is convex, and \( I(\cdot) \) is strictly concave, then the maximum

\[
C = \max_{F \in \Omega} I(F),
\]
is achieved by a unique \( F^* \) in \( \Omega \).

In order to apply this principle in our work, we use techniques that have been first developed in [6] and later adopted in various works on mutual information maximization as in [13]: Denote by \( \mathcal{F} \) the space of all probability distribution functions on \( \mathbb{R} \). We adopt weak convergence [14, III-1, Def.2, p.311] on \( \mathcal{F} \), and use the Levy metric to metrize this weak convergence [15, Th.3.3, p.25]. The optimization is carried out in this metric topology.

We proceed by showing in Section III-B that the set \( \mathcal{P}_A \) is convex and compact (Theorem 3). Then, we show in Section III-C that the mutual information \( I(F) \) is finite, concave and continuous (Theorems 4 and 5).

When it comes to uniqueness, it is due to the fact that strict concavity of \( I(\cdot) \) follows from the strict concavity of the output differential entropy in \( p_Y(\cdot) \), which is the case if and only if \( p_Y(\cdot) \) is injective in \( F \).

In the remainder of this section we state and prove Theorems 3, 4 and 5.

B. Optimization set properties

**Theorem 3.** Whenever conditions A1, A2, A3 and A5 are satisfied, the set \( \mathcal{P}_A \) defined in (8) is convex and compact.

**Proof:**

We note first that the theorem was shown to hold for cost functions of the form \( C(|x|) = |x|^r \), for \( r > 1 \) in [1], [13]. We adopt the same methodologies to generalize the results presented hereafter.

**Convexity:** Let \( F_1 \) and \( F_2 \) be two probability distribution functions in \( \mathcal{P}_A \), and \( \lambda \) some scalar between 0 and 1. Define \( F = \lambda F_1 + (1 - \lambda) F_2 \). It is clear that \( F \) is a probability distribution function because it is non decreasing, right continuous, \( F(-\infty) = 0 \) and \( F(+\infty) = 1 \). Additionally,

\[
\int_{\mathbb{R}} C(|x|) \, dF = \int_{\mathbb{R}} C(|x|) \, d(\lambda F_1 + (1 - \lambda) F_2) \\
= \lambda \int_{\mathbb{R}} C(|x|) \, dF_1 + (1 - \lambda) \int_{\mathbb{R}} C(|x|) \, dF_2 \\
\leq \lambda A + (1 - \lambda) A = A.
\]
Therefore, \( F \in \mathcal{P}_A \) and \( \mathcal{P}_A \) is convex.

**Compactness:** Consider a RV \( X \) with probability distribution function \( F \in \mathcal{P}_A \). Applying Markov’s inequality to random variable \( C(|X|) \) yields,

\[
\Pr\{C(|X|) \geq \alpha\} \leq \frac{\mathbb{E}[C(|X|)]}{\alpha}, \quad \forall \alpha > 0.
\]

Now let

\[
K = \inf \{x \in \mathbb{R}^+ \text{ s.t. } C(x) \geq \alpha\} + 1,
\]

which is always greater or equal to 1. For any finite value of \( \alpha \), such a \( K \) exists since \( C(x) \) increases to \( +\infty \) as \( x \to +\infty \) by virtue of properties A3 and A5. Additionally, since \( C(\cdot) \) is non-decreasing by property A2,

\[
\Pr\{C(|X|) \geq \alpha\} \geq \Pr\{|X| > K - 1\} \geq \Pr\{|X| \geq K\} \\
\geq F(-K) + [1 - F(K)].
\]

Hence, for all \( F \in \mathcal{P}_A \), we obtain

\[
F(-K) + [1 - F(K)] \leq \frac{\mathbb{E}[C(|X|)]}{\alpha} \leq \frac{A}{\alpha}.
\]

Therefore, for every \( \epsilon > 0 \), there exists a \( K_\epsilon > 0 \), namely

\[
K_\epsilon = \min \{x \in \mathbb{R}^+ \text{ s.t. } C(x) \geq (A/\epsilon)\} + 1,
\]

such that

\[
\sup_{F \in \mathcal{P}_A} [F(-K_\epsilon) + [1 - F(K_\epsilon)]] \leq \epsilon.
\]

This implies that \( \mathcal{P}_A \) is *tight* [14, III-2, Def.2, p.318]. By Prokhorov’s Theorem [14, III-2, Th.1, p.318], \( \mathcal{P}_A \) is therefore relatively sequentially compact and every sequence \( \{F_n\} \) of distribution functions in \( \mathcal{P}_A \) has a convergent sub-sequence \( \{F_{n_j}\} \) where the limit \( F^\ast \) does not necessarily belong to \( \mathcal{P}_A \). If we prove that \( F^\ast \in \mathcal{P}_A \), the latter will be sequentially compact and hence compact since the space is metrizable [16, Th.28.2, p.179]. In order to show that the limiting distribution function \( F^\ast \) is in \( \mathcal{P}_A \), it must satisfy the cost constraint which is the case. In fact,

\[
\int C(|u|) dF^\ast(u) \leq \lim_{n_j \to \infty} \inf \int C(|u|) dF_{n_j} \leq A,
\]
where the first inequality holds because \( C(|u|) \) is lower semi-continuous (property A1), and is bounded from below by \( C(0) \) for all \( u \in \mathbb{R} \) (property A2) [17, Th. 4.4.4]. In addition, the second inequality is valid since the sub-sequence \( \{F_{n_j}\} \) is in \( P_A \) and therefore satisfies the cost constraint \( \forall n_j \). Finally, \( F^* \in P_A \) and \( P_A \) is compact.

\section{C. Properties of the mutual information, \( I(\cdot) \)}

In this Section we prove the finiteness, concavity and continuity of \( I(\cdot) \) on \( P_A \) through Theorems 4 and 5.

\textbf{Theorem 4.} Whenever conditions A3, A7 and A8 hold, the mutual information \( I(F) \) between the input and output of channel (7) is finite for all input distribution functions \( F \) such that \( E[C(|X|)] \) is finite.

\textbf{Proof:} Since \( Y = f(X) + N \),
\[
\ln[1 + |Y|] \leq \ln[1 + |f(X)|] + \ln[1 + |N|],
\]
and \( E[\ln[1 + |Y|]] \) if finite because both \( E[\ln[1 + |f(X)|]] \) and \( E[\ln[1 + |N|]] \) are finite (by properties A3 and A8).

Moreover, and since \( p_Y(y) \) is upperbounded (by A7) by one \( h_Y(F) = - \int p(y; F) \ln p(y; F) \, dy \), the differential entropy of \( Y \), is well defined [10, Proposition 1] and \( 0 \leq h_Y(y) < +\infty \).

The differential entropy \( h_N \) of the noise being finite (due to properties A7 and A8), the mutual information (10) can therefore be written as the difference of two terms:
\[
I(F) = h_Y(F) - h_{Y|X}(F) = h_Y(F) - h_N, \tag{11}
\]
both of which are finite and this completes the proof.

\textbf{Theorem 5.} Assume that conditions A2 through A8 hold. Under a cost constraint
\[
\int C(|X|) \, dF(x) \leq A, \quad A > 0,
\]
the mutual information \( I(F) \) between the input and the output of channel (7) is concave and continuous whenever \( C(|x|) = \omega (\ln |f(x)|) \).
Before we proceed with the proof, we note that under the conditions of the Theorem, the mutual information $I(F)$ between the input and the output of channel (7) is finite by virtue of Theorem 4.

Proof:

Concavity: The output differential entropy $h_Y(F)$ is a concave function of $F$ on $\mathcal{F}$. In fact,

$$h_Y(F) = -\int p_Y(y; F) \ln p_Y(y; F) \, dy$$

exists (by Theorem 4) and is a concave function of $p_Y(\cdot)$ because $-x \ln x$ is concave in $x$. Since $p_Y(F)$ is linear in $F$, $I(F) = h_Y(F) - h_N$ is concave on $\mathcal{P}_A$.

Continuity: To prove the continuity of $I(F)$, it suffices to show that $h_Y(F)$ is continuous by virtue of equation (11). To this end, we let $F \in \mathcal{P}_A$ and let $\{F_m\}_{m \geq 1}$ be a sequence of probability measures in $\mathcal{P}_A$ that converges weakly to $F$.

In order to apply Theorem 1 and show the convergence of $h_Y(F_m)$ to $h_Y(F)$ and hence the weak continuity of $h_Y(F)$ on $\mathcal{P}_A$, we establish that the appropriate conditions are satisfied:

- By definition of weak convergence, since $p_N(y - x)$ is bounded and continuous (properties A6 and A7), then $p(y; F_m) = \int p_N(y - f(x)) \, dF_m(x)$ converges point-wise to $p(y; F) = \int p_N(y - f(x)) \, dF(x)$.
- The induced output PDF $p(y; F_m)$ is also bounded by one.
- It remains to find a non-negative and non-decreasing function, $l : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $l(y) = \omega(\ln(y))$, and a scalar $L > 0$ such that equation (2) holds for $p(y; F_m)$, $m \geq 1$ and $p(y; F)$, a task which we fulfill in what follows.

For any $y \geq |f(0)|$, let $\mathcal{S} = f^{-1}([|f(0)|, y])$ be the inverse image by $f(\cdot)$ of the closed interval $[|f(0)|, y]$. Since $f(\cdot)$ is continuous (A4), the set $\mathcal{S}$ is closed. It is also bounded because $|f(x)|$ is non-decreasing in $|x|$ and tends to infinity (A5). Therefore any element in $\mathcal{S}$ is smaller than a positive $t_u$ such that $|f(t_u)| = 2y$ and greater than a negative $t_b$ such that $|f(t_b)| = 2y$. Such $t_u$ and $t_b$ exist because $f(\cdot)$ is continuous.

The set is compact and has a maximal value that we denote $z(y) = \max\{z : z \in \mathcal{S}\}$. Note that $|f(z(y))| = y$.

Define the function $C_{\min}(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ as follows:

$$C_{\min}(y) = \begin{cases} 
\min \{C(z(y)) ; C_N(y)\} & y \geq |f(0)| \\
0 & \text{otherwise,}
\end{cases}$$
where \( C_N(\cdot) \) is defined in A8. The function \( C_{\min}(y) \) is non-negative and non-decreasing on \( \mathbb{R}^+ \) since both \( C(y) \) and \( C_N(\cdot) \) are non-negative and non-decreasing by properties A2 and A8 and \( z(y) \) is non-decreasing for \( y \geq |f(0)| \). Additionally, \( C_{\min}(y) = \omega \ln y \) because \( C(x) = \omega \ln|f(x)| \) (A3) and \( C_N(x) = \omega \ln x \) (A8).

Now, for any \( X \) with distribution \( F \in \mathcal{P}_A \),

\[
E_Y \left[ C_{\min} \left( \frac{|Y|}{2} \right) \right] = E_{X,N} \left[ C_{\min} \left( \frac{|f(X)| + N}{2} \right) \right] \leq E_{X,N} \left[ C_{\min} \left( \frac{|f(X)| + |N|}{2} \right) \right] \tag{12} \\
= E_{X,N} \left[ C_{\min} \left( \frac{|f(X)| + |N|}{2} \right) \right] \left[ |f(X)| \leq |N| \right] P \left( |f(X)| \leq |N| \right) \\
+ E_{X,N} \left[ C_{\min} \left( \frac{|f(X)| + |N|}{2} \right) \right] \left[ |f(X)| > |N| \right] P \left( |f(X)| > |N| \right) \\
\leq E_{X,N} \left[ C_{\min}(|N|) \right] \left[ |f(X)| \leq |N| \right] P \left( |f(X)| \leq |N| \right) \\
+ E_{X,N} \left[ C_{\min}(|f(X)|) \right] \left[ |f(X)| > |N| \right] P \left( |f(X)| > |N| \right) \tag{13} \\
\leq E_N \left[ C_{\min}(|N|) \right] + E_X \left[ C_{\min}(|f(X)|) \right] \tag{14} \\
\leq E_N \left[ C_N(|N|) \right] + E_X \left[ C(|X|) \right] \\
\leq L_N + A = L < \infty. \tag{15}
\]

where \( 0 \leq L_N = E_N \left[ C_N(|N|) \right] < \infty \) by property A8. Equations (12) and (13) are justified since \( C_{\min}(|x|) \) is non-decreasing in \( |x| \) and (14) is due to the fact that \( C_{\min}(|x|) \) is non-negative. Since the value \( 0 \leq L < \infty \) is independent of the input distribution function \( F \in \mathcal{P}_A \), then (15) holds for any output variable \( Y \), i.e for all \( p(y; F) \) where \( F \in \mathcal{P}_A \). Letting \( l(y) = C_{\min} \left( \frac{y}{2} \right), \ y \in \mathbb{R}^+ \), then equation (2) is satisfied for \( p(y; F_m), m \geq 1 \) and \( p(y; F) \). Therefore, Theorem 1 holds and \( h_Y(F_m) \) converges to \( h_Y(F) \) and hence \( h_Y(F) \) is continuous which concludes the proof.

\[ \blacksquare \]

### IV. Extensions

The results may be extended to the case where the noise PDF is not necessarily continuous on \( \mathbb{R} \). In fact, we weaken condition A6 and we show that Theorem 2 also holds for noise PDFs which are piece-wise continuous on a countable number of pieces. Note that under this category...
fall absolutely continuous noise variables with a compact support \( \text{such as the uniform, and also ones that are one-sided such as the Gamma or the Pareto RVs.} \) We start by noting the following:

- It can be seen from the proof of Theorem 1 that almost everywhere (a.e.) point-wise convergence with respect to the Lebesgue measure (in addition to C1 and C2) is sufficient in order to have convergence of differential entropies.
- According to the definition of weak convergence [18, p. 700], one can replace continuous bounded test functions by \( F \)-a.e. continuous functions where \( F \) is the limit distribution.

We show now that if \( p_N(\cdot) \) has a countable number of discontinuities then weak convergence of the input distributions implies Lebesgue-a.e. point-wise convergence of the output PDFs: Denote by \( \{a_i\}_{i \geq 1} \) the countable discontinuities of \( p_N(\cdot) \) and by \( \{x_i\}_{i \geq 1} \) the discontinuity points of \( F \), which are necessarily countable (see Jordan decomposition lemma in [19, p. 40]). Point-wise convergence of the PDFs holds except at values of \( y \) of the form \( y_{ij} = a_i - f(x_j), i, j \geq 1 \). The fact that the \( \{y_{ij}\} \)'s are countable proves our assertion.

V. Conclusion

Tangible models for communication channels implicitly assume a finite value for the channel capacity. Knowing that maximizing the transmission rates is directly related to a constrained maximization problem, we have derived sufficient conditions for finiteness and achievability of the capacity of generic memoryless additive noise channels. The involved conditions on the input-output relationship, the input cost function and the type of the noise define a wide collection of models for which finding codes that strive toward achieving maximum transmission rates is sensible. The result is applicable to possibly non-linear channels, to nearly all the widely known additive noise models and for cost functions that are "super-logarithmic". Interestingly, communications at finite rates is not directly related to an input average-power constraint. Even when signaling strategies are allowed to have an infinite second moment on average, transmission rates could not be arbitrarily large. We mention that while searching for sufficiency,

\[ \text{we define the support of a RV as being the set of its points of increase, i.e., the set } \{x \in \mathbb{R} : \Pr(x - \eta < X < x + \eta) > 0 \text{ for all } \eta > 0\}. \]

\[ \text{We say that a property holds almost everywhere with respect to a measure } \mu \text{ and we denote it } \mu\text{-a.e. iff the measure by } \mu \text{ of the set where the property fails is equal to zero.} \]
intermediately we derived conditions under which point-wise convergence of PDFs implies convergence of differential entropies.

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