Modeling of stress state of a perforated cement sheath in a production well

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Abstract. Modeling of stress state of a perforated cement sheath in a production well is performed. Long-term well operation is considered. The slightly compressible fluid flow model is used to calculate the pore pressure of a fluid. The linear-elastic body model and finite volume method with multipoint stress approximation are used to calculate the stress state of the cement sheath and production casing. The numerical model was verified by comparing the calculation results with a calculation in the Fenics open-source computing platform. It is shown that the maximum von Mises stress value falls on the perforation zone at the junction of the cement sheath and production casing. Stresses are shown to slightly decrease during long-term well operation. It is also shown that an increase in the Young's modulus of the production casing leads to a decrease in the von Mises stress in the cement sheath.

1. Introduction

One of the main problems in the oil and gas industry is the destruction of the cement sheath between the rock and the production casing. This leads to premature water breakthrough into the production well and environmental pollution. In this paper, the strength of the cement sheath is modeled by calculating the von Mises stresses at each point of the cement sheath and production casing. Areas with the highest von Mises stress are least stable.

Modeling the strength of the cement sheath is performed using weakly coupled linear elasticity and Darcy filtration equations. The following assumptions are used:

- the formation rock is undeformable;
- the cement sheath and the production casing are described by isotropic linear elasticity model;
- the conditions of perfect contact are met at the junction of the cement sheath and the production casing;
- bottomhole pressure inside the well acts on the inner wall of the production casing;
- the outer wall of the cement sheath is affected only by the pore pressure of the fluid in the reservoir;
- deformations of the cement sheath and production casing are small and do not affect the fluid movement in the reservoir and in the well.

The algorithm for solving the problem is to successively perform two actions for all time points:
1. Calculate the reservoir pressure distribution at the current time using the transient filtration equation.

2. Calculate the stress state of the cement sheath and production casing at the current time using the solution of the steady-state equation of linear elasticity, in which the reservoir pressure calculated in step 1 is set as a boundary condition.

Spatial discretization of the equations is carried out using the construction of a 2.5-dimensional Voronoi computational grid [1] for the reservoir, the cement sheath and the production casing. To solve the filtration problem, a finite volume method with a two-point flux approximation is used, and a finite volume method with multipoint stress approximation [2] is used to solve the elasticity problem.

2. The filtration problem

The filtration problem is described by the following equations:

\[
\frac{\phi c}{\mu} \frac{\partial p}{\partial t} + \frac{k}{\Delta} \Delta p = 0,
\]

\[
p(0) = P_{\text{init}},
\]

\[
p(t)_{\text{perforations}} = P_{\text{well}}, \forall t,
\]

\[
\frac{\partial p}{\partial n}(t)_{\text{boundary}} = 0, \forall t.
\]

Where \( p \) = reservoir pressure; \( \phi \) = rock porosity; \( c \) = fluid compressibility; \( k \) = rock permeability; \( \mu \) = fluid viscosity. The boundary of the reservoir includes the inner part and the outer part. The outer part of the reservoir boundary describes the boundary of the drainage area. The inner part of the reservoir boundary is the outer wall of the cement sheath of the well.

3. The elasticity problem

The cement sheath and the production casing are a single hollow cylinder, as shown in figure 1.

![Figure 1](image)

**Figure 1.** Cement sheath and production casing. The outer wall of the cement sheath is blue, the inner wall of the production casing is green, the perforation walls are red, and the ends of the cement sheath and the production casing are gray.

The stress state of the cement sheath and production casing is described by the following equations:

\[
\nabla \cdot \sigma + \dot{f} = 0,
\]

\[
\sigma = Ce,
\]

\[
\varepsilon = \frac{1}{2} \left( \nabla \ddot{u} + (\nabla \ddot{u})^T \right).
\]
\[
\begin{align*}
\vec{f}|_{S_1} &= -p_{\text{reservoir}} \vec{n}, \\
\vec{f}|_{S_2} &= -p_{\text{well}} \vec{n}, \\
\vec{f}|_{S_3} &= -p_{\text{well}} \vec{n}, \\
\vec{u}|_{S_4} &= \vec{0}.
\end{align*}
\]

Where \( \sigma \) = symmetric stress tensor (2nd order); \( \vec{f} \) = external forces; \( C \) = stiffness tensor (4th order); \( \varepsilon \) = symmetric strain tensor (2nd order); \( \vec{u} = (u_x, u_y, u_z)^T \) = displacement vector; \( u_x, u_y, u_z \) = displacements along the axes \( x, y, z \), respectively; \( S_1 \) = outer wall of the cement sheath (figure 1); \( S_2 \) = inner wall of the production casing; \( S_3 \) = perforation walls in the cement sheath and production casing; \( S_4 \) = ends of the cement sheath and production casing; \( \vec{n} \) = field of external normals to the cement sheath and production casing.

Let us demonstrate the solution of the problem of elasticity. Integrating the equation of elasticity in the cell \( i \), we obtain:

\[
\int_{V_i} \nabla \cdot \sigma \, dV + \int_{\partial V_i} \vec{f} \, d\vec{n} = 0,
\]

\[
\sum_{j \in \Psi(i)} \int_{S_{ij}} \sigma \cdot \vec{n} \, dS + \vec{f}_i = 0.
\]

Where \( V_i \) = volume of the cell \( i \); \( \vec{f}_i \) = average value of the external force acting on the cell \( i \); \( \Psi(i) \) = set of cells adjacent to the \( i \)-th cell; \( S_{ij} \) = face between the \( i \)-th and \( j \)-th cells. Let \( (\sigma \cdot \vec{n})_{ij} \) be the average normal stress on the face between the \( i \)-th and \( j \)-th cells. Then the equation will be

\[
\sum_{j \in \Psi(i)} \left| S_{ij} \right| (\sigma \cdot \vec{n})_{ij} + \vec{f}_i = 0.
\]

We also introduce the mean value \( \vec{u}_i \) of displacement in a cell \( i \). The multipoint stress approximation method (MPSA) [2] consists in looking for a value \( \left| S_{ij} \right|(\sigma \cdot \vec{n})_{ij} \) on the face between the \( i \)-th and \( j \)-th cells as a linear combination of \( \vec{u}_k \):

\[
\left| S_{ij} \right|(\sigma \cdot \vec{n})_{ij} = \sum_{v \in V(i,j) \cap C(v)} \sum_{k} t_{ijkv} \vec{u}_k.
\]

Here \( t_{ijkv} \) = 2nd order tensors; \( V(i,j) \) = set of all vertices of the face \( ij \); \( C(v) \) = set of all cells adjacent to the vertex \( v \). Each vertex \( v \) of the computational grid corresponds to one cell of the dual grid. For each dual cell, a special local problem is solved to determine the coefficients \( t_{ijkv} \) within this dual cell. When all coefficients \( t_{ijkv} \) are calculated, the elasticity equations in all cells can be represented as a system of linear equations for unknowns \( \vec{u}_i \).

### 3.1. Solution of the MPSA local problem

For each vertex of the grid, a separate local problem is solved. Consider a vertex \( v \) and a cell \( i \) incident to this vertex. Consider all the edges and all the faces of the grid, which are incident to both the vertex \( v \) and the cell \( i \). The polyhedron formed by the vertex \( v \), the centroid of the cell \( i \) and the centroids of
all of these edges and faces is called a subcell and is denoted \((i, v)\) (figure 2). The face of a subcell \((i, v)\) that is incident to some other cell \(j\) is called a subface and is denoted \((i, j, v)\). Obviously, all the cells of the grid can be represented as a union of a certain set of subcells, and all faces as a union of a certain set of subfaces.

![Figure 2. Cells incident to the vertex v. The subcell \((i,v)\) is shaded gray.](image)

Let at each point \(\vec{r}\) of the subcell \((i, v)\) each component of the displacement vector 
\[
\vec{u}(\vec{r}) = \left( u_x(\vec{r}), u_y(\vec{r}), u_z(\vec{r}) \right)^T
\]
be a linear function of the coordinates:
\[
\begin{align*}
  u_x(\vec{r}) &= u_{i,x} + \nabla u_{i,x} \cdot (\vec{r} - \vec{r}_i), \\
  u_y(\vec{r}) &= u_{i,y} + \nabla u_{i,y} \cdot (\vec{r} - \vec{r}_i), \\
  u_z(\vec{r}) &= u_{i,z} + \nabla u_{i,z} \cdot (\vec{r} - \vec{r}_i).
\end{align*}
\]

Where \(\vec{u}_i = \left( u_{i,x}, u_{i,y}, u_{i,z} \right)^T\) = mean value of the displacement in the cell \(i\); \(\nabla u_{i,x}, \nabla u_{i,y}, \nabla u_{i,z}\) = gradients of the components of the displacement vector (these are vectors); \(\vec{r}_i\) = center of the cell \(i\). We assume that the gradients \(\nabla u_{i,x}, \nabla u_{i,y}, \nabla u_{i,z}\) in this subcell are constants, i.e. they do not depend on the coordinate \(\vec{r}\). Therefore, the stress tensor \(\sigma_{iv}\) in the subcell \((i, v)\) is also independent of \(\vec{r}\).

Let a vertex \(v\) be incident to \(N\) cells and \(M\) faces. On each subface \(m\) (varies from 1 to \(M\)) we enforce the stress continuity (on the entire subface) and the displacement continuity (at a single point \(\vec{r}_m\) on the subface):
\[
\begin{align*}
  \sigma \cdot \vec{n}_{ijv} &= -\sigma \cdot \vec{n}_{jiv}, \quad (i, j) \in \text{Neighbours}(m), \\
  \vec{u}_i + \nabla \vec{u}_{iv} \cdot (\vec{r}_m - \vec{r}_i) &= \vec{u}_j + \nabla \vec{u}_{jv} \cdot (\vec{r}_m - \vec{r}_j), \quad (i, j) \in \text{Neighbours}(m).
\end{align*}
\]

These are \(2M\) vector equations, that is, \(6M\) scalar equations. Unknown variables are gradients \(\nabla \vec{u}_{iv}\) in each subcell. We have \(N\) subcells, which means \(N\) unknown gradients, i.e. \(9N\) unknown scalars. This system of equations has a unique solution if \(2M = 3N\). For simple three-dimensional grids consisting of parallelepipeds, prisms or tetrahedra, this condition is always satisfied. But for grids consisting of polyhedra of arbitrary shape (in particular, for the Voronoi grid in general), this condition is not satisfied.

Let \(\sigma = \{\sigma_{iv,pq}\}, \quad C = \{c_{iv,pqrs}\}, \quad \epsilon = \{\epsilon_{iv,rs}\}\), where \(p, q, r, s\) are the indices of the components of tensors, taking values from 1 to 3. Then \(\sigma_{iv,pq} = \sum_{r=1}^3 \sum_{s=1}^3 c_{iv,pqrs} \epsilon_{iv,rs}\). One can construct a system of equations in the form:
\[
\begin{pmatrix}
C_{iv} \cdot \bar{\bar{\nu}}_{ij} & 0 \\
(r_m - \bar{r}) \pm 1 & 0
\end{pmatrix} \begin{pmatrix}
\nabla \bar{u}_{iv} \\
\bar{u}_i
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

or

\[\mathbf{A} \bar{\bar{\nu}} = \bar{\bar{b}}.\]

By calculating the pseudo-inverse matrix \(\mathbf{A}^{-1}\), we can represent the unknown gradients \(\nabla \bar{u}_{iv}\) as linear combinations of variables \(\bar{u}_i\):

\[\nabla \bar{u}_{iv,rs} = \sum_{k=1}^{N} \alpha_{iv,rs,k} \cdot \bar{u}_k,\]

Where \(\alpha_{iv,rs,k}\) is the vector of length 3 (taken from the matrix \(\mathbf{A}^{-1}\)). Thus, for arbitrary three-dimensional grids, gradients \(\nabla \bar{u}_{iv}\) are expressed via \(\bar{u}_i\) in terms of least squares. For simple three-dimensional grids consisting of parallelepipeds, prisms or tetrahedra, they are expressed in terms of \(\nabla \bar{u}_{iv}\) exactly.

Substitute the obtained \(\nabla \bar{u}_{iv}\) into the expression for the stress on the surface to express this stress in terms of unknown variables \(\bar{u}_k\):

\[
\sigma_{iv} \cdot \bar{\bar{\nu}}_{ij} = \begin{pmatrix}
\sum_{r=1}^{3} \sum_{s=1}^{3} \sum_{k=1}^{N} C_{iv,1r}C_{iv,rs}C_{iv,rs,1s} & \sum_{r=1}^{3} \sum_{s=1}^{3} \sum_{k=1}^{N} C_{iv,1r}C_{iv,rs}C_{iv,rs,1s} & \sum_{r=1}^{3} \sum_{s=1}^{3} \sum_{k=1}^{N} C_{iv,1r}C_{iv,rs}C_{iv,rs,1s}
\end{pmatrix} \cdot \bar{\bar{\nu}}_{ij}
\]

Each vector \(\alpha_{iv,rs,k}\) can be written as a row, and then \(\sigma_{iv} \cdot \bar{\bar{\nu}}_{ij}\) becomes a 3x3 matrix. Thus, we presented the stress on the surface in the form of a linear combination of \(\bar{u}_k\):

\[\sigma_{ij} \cdot \bar{\bar{\nu}}_{ij} = \sum_{k=1}^{N} t_{ijk} \bar{u}_k\]

The stress across the face \(ij\) will be

\[\sigma_{ij} \cdot \bar{\bar{\nu}}_{ij} = \sum_{v \in \mathcal{V}(i,j)} \sigma_{ij} \cdot \bar{\bar{\nu}}_{ij} = \sum_{v \in \mathcal{V}(i,j)} \sum_{k \in \mathcal{C}(v)} t_{ijk} \bar{u}_k\]

In this paper, to solve local problems at the grid boundary, the zero normal stress condition at the boundary is used. For this, a layer of auxiliary cells with a zero stiffness tensor is created around the grid boundary, and all local problems are solved in the manner described above for the inner vertices of the grid.

3.2. Solution of the linear elasticity problem

For each cell, we obtain a linear equation for the unknown displacements \(\bar{u}_i\) in each cell:
\[
\sum_{j \in \Psi(i)} \sum_{v \in \Lambda(i,j)} \sum_{k \in C(v)} t_{ijkv} \tilde{u}_k + \tilde{f}_i = \tilde{0}
\]

Since the gradient \( \nabla u_{iv,rs} \) in each subcell is known, it is possible to calculate the stress tensor \( \sigma_{iv} \) and the von Mises stress \( \sigma_{m,iv} \) in each subcell:

\[
S_{iv} = \sigma_{iv} - \frac{1}{3} \text{trace}(\sigma_{iv}) I
\]

\[
\sigma_{m,iv} = \sqrt{\frac{3}{2} S_{iv} : S_{iv} = \sqrt{\frac{3}{2} \sum_{p=1}^{3} \sum_{q=1}^{3} s_{iv,pq} s_{iv,pq}}}
\]

The von Mises stress in the cell \( i \) is taken equal to the average value of the von Mises stress across all subcells \( (i,v) \).

4. Numerical experiment

The production and injection vertical wells located at a distance of 500 m and operating with constant bottomhole pressures of 50 and 150 bar, respectively. The stress state of the cement sheath and production casing for the production well is calculated. The thickness of the reservoir is 3 m, the inner radius of the production casing is 6 cm, the thickness of the production casing is 1.8 cm, the thickness of the cement sheath is 3 cm, the initial reservoir pressure is 100 bar, the porosity of the reservoir is 0.2, the permeability of the reservoir is 100 millidarcy, fluid compressibility is 1e-4, fluid viscosity is 1 centipoise, Poisson's ratio of the cement sheath and production casing is 0.25, the Young's modulus of the cement sheath is 5 GPa, the Young's modulus of the production casing are 5 and 200 GPa (in two different experiments).

The computational grid consists of reservoir cells, cells of the cement sheath and cells of the production casing for both wells (figure 3). Perforations are voids in the cement sheath and production casing, having the form of parallelepiped with sides of 5 cm. The number of reservoir cells is 97000, the number of cells of the cement sheath and production casing (for production well only) is 10000. Figure 4 shows the distribution of reservoir pressure near the production well at different times.

![Figure 3. Voronoi grid with radial refinement near two wells. 1 - injection well, 2 - production well.](image-url)
4.1. Homogeneous case
First, we consider the homogeneous case: suppose that the Young's modulus of the cement sheath and the production casing are the same and equal to 5 GPa.

**Figure 5.** Displacements (m) in the homogeneous case along the horizontal axis 1 hour after the start of the wells.

In figure 5 it can be seen that the displacements of the cement sheath and production casing in the horizontal plane are approximately the same across the entire height of the reservoir.

**Figure 6.** Von Mises stress (bar) in the homogeneous case 1 hour after the start of the wells.
In figure 6 it can be seen that the highest von Mises stress is in the perforated zone (blue areas). At a later time (24 hours after the well start-up), the gradient of reservoir pressure near the well decreases (figure 4), which leads to a decrease in the maximum values of the von Mises stress at all points of the cement sheath and production casing by 8%. In this case, the spatial distribution of the von Mises stress distribution does not change.

4.2. Nonhomogeneous case

In the nonhomogeneous case, it is assumed that the Young's modulus of the cement sheath and the production casing are different and equal to 5 GPa and 200 GPa, respectively.

![Figure 7](image1.png)

**Figure 7.** Displacements (m) in the nonhomogeneous case along the horizontal axis 1 hour after the start of the wells.

In figure 7 it can be seen that the cement sheath is displaced much more than the production casing. Compared to the homogeneous case, the absolute magnitude of the displacement decreased by about one order of magnitude.

![Figure 8](image2.png)

**Figure 8.** Von Mises stress (bar) in the nonhomogeneous case 1 hour after the start of the wells.

Figure 8 shows that the von Mises stress in the cement sheath is significantly higher than the von Mises stress in the production casing. Compared to the homogeneous case, the absolute value of this stress decreased by approximately 3 times.

The maximum von Mises stress falls on the end of the cement sheath, because the ends are fixed. However, at the junction of the cement sheath and the production casing in the perforated zone, the von Mises stress is also close to the maximum. If the von Mises stress exceeds the yield strength of the cement, the cement sheath will begin to deform plastically in this zone. Cracks can form from the...
perforations on the inner wall of the cement sheath, which will eventually become the cause of behind-the-casing flows.

As in the homogeneous case, after some time (24 hours after starting the well), the maximum values of the von Mises stress at all points of the cement sheath and production casing decrease by 5%.

5. Verification
To verify the numerical method, the problem of beam deformation under its own weight is considered. The beam consists of an isotropic material and is fixed at one end. Each point of the beam is affected by gravity.

The calculation obtained using the developed numerical method is compared with the calculation in the open-source Fenics modeling package [3], which uses the finite element method. A structured grid (hex mesh) consisting of 3840 cells of a cubic shape was used in our calculation, and a tetrahedral grid consisting of 3840 * 6 elements was used in the Fenics package.

Figure 9 shows the deformed state of the beam and the von Mises stress at each point of the beam, obtained in two calculations. It can be seen that the deformations and von Mises stresses are close to each other in two calculations at each point of the beam.

![Figure 9. Von Mises stress at each point of the deformed beam. On the left is the calculation in the Fenics package, on the right is our calculation.](image)

6. Conclusion
The stress state of a perforated cement sheath adjacent to the production well is modeled. It is shown that the maximum von Mises stress value falls on the perforation zone at the junction of the cement sheath and the production casing. Stresses decrease slightly during long-term well operation. It is also shown that an increase in the Young's modulus of the production casing leads to a decrease in the von Mises stress in the cement sheath.

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References
[1] Palagi C L and Aziz K SPE Advanced Technology Series 1994 2(2) 69–77
[2] Nordbotten J M International Journal for Numerical Methods in Engineering 2014 100(6) 399–418
[3] Alnaes M S, Blechta J, Hake J, Johansson A, Kehlet B, Logg A, Richardson C, Ring J, Rognes M E and Wells G N Archive of Numerical Software 2015 3(100)