Strategies of Voting in Stochastic Environment: Egoism and Collectivism

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Abstract—Consideration was given to a model of social dynamics controlled by successive collective decisions based on the threshold majority procedures. The current system state is characterized by the vector of participants' capitals (utilities). At each step, the voters can either retain their status quo or accept the proposal which is a vector of the algebraic increments in the capitals of the participants. In this version of the model, the vector is generated stochastically. Comparative utility of two social attitudes—egoism and collectivism—was analyzed. It was established that, except for some special cases, the collectivists have advantages, which makes realizable the following scenario: on the conditions of protecting the corporate interests, a group is created which is joined then by the egoists attracted by its achievements. At that, group egoism approaches altruism. Additionally, one of the considered variants of collectivism handicaps manipulation of voting by the organizers.

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1. INTRODUCTION

As was shown by A.V. Malishevskii [1, pp. 92–95] at a turn of the 1970’s, for any distribution of riches in society one can formulate a number of proposals such that each of them enriches 99% members of the society, but as a result of successive acceptance of these proposals each member of the society becomes very poor. Bluntly speaking, these proposals lie in successive dispossession of all participants one after another. At that, part of the participant’s property is divided between the rest of them including the already dispossessed ones, another part is passed to the person formulating the proposal. This algorithm can be used in a number of votings, provided that the participants vote according to their valuable interests and completely ignore the interests of the others. One can readily see that the heart of the problem remains unchanged if instead of defending their own interests the voters defend the interests of their electors, clans, industries, countries (in the UN), and so on. This is due to the fact that if the voters are guided by any particular interests dividing them into smaller groups, then voting becomes easily manipulatable by those who have priority in formulating the proposals submitted to voting (see also [2, 3]).

How to eliminate this manipulation? It is difficult to restrict the powers of the “Presidium” and the influence of the political backstage spin doctors. The passage from shorthanded protection by the voters of their private interests to the universal principle of “all for one and one for all” would be a reliable safeguard, but as a rule it is unlikely. First, the “voters will not understand.” Second, “self likes itself best.” From the practical point of view, the participant is interested only in “all for one,” and only if this “one” is himself. Yet the participant may hope never to be in the role

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of the “one” needing protection, especially if he intends to remain truthful to the “Presidium.” Moreover, he understands that if sometimes he will be in this position, the others may betray him, no matter how arduously he defended their interests in the past. Therefore, the immediate utility of the everyday mission of “being for all” for which he will have to pay a pretty penny is very doubtful for him.

However, it is not always the case that the collective decisions are made in the conditions of selfish or malevolent manipulation by the proposal makers. In essence, the agenda is often defined not by them, but rather by external—natural, man-caused, economical, political, demographic, and so on—phenomena. In what concerns these phenomena, it is possible to discuss only the general trends, rather than the times of occurrence and the scope of the future events. In the first approximation this reality may be modeled by a sequence of stochastically generated proposals. Each proposal is advantageous to some participants and disadvantageous to some other participants. In the simplistic model, this proposal may be identified with a vector of algebraic increments in some abstractly understood capitals (utilities) of the participants.

A model of this sort is discussed below. The main subject matter is represented by the social attitudes of the voting participants. The main attitudes are egoistic and collectivistic. Like in the Malishevskii model, the egoistic participant supports any proposal providing an increment to its capital. There are also other participants making up (still one) group and voting jointly for the proposals that are beneficial, at least minimally, to the group as a whole. They sacrifice to some extent their private interests to the interests of their group. This kind of collectivism can be justly called the group egoism. The present paper is concerned mostly with the attitude—egoistic or collectivistic—that is beneficial to an individual participant. More specifically, consideration is given to the dynamics of the mean capital of the egoists and the group members.

The following hypothetical mechanism is of special interest. Let under certain conditions the collectivistic attitude be more advantageous. Then the egoists have an incentive to join the group. If the group puts no obstacles in this way and grows by including more and more members, then its group egoism resembles more and more the actions in the interests of all, that is, the basically altruistic interests. It deserves noting that in this decision model the entire society participates in decision making, rather than the “Parliament” that represents it.

Under the assumption of utility of the collectivistic attitude, this mechanism enables the following scenario based on the pragmatic interest and not on the altruistic ideals of the participants. Let some participants make up a group taking upon itself the responsibility to vote not for their own interests, but for those of their group. Formulation of the latter is, of course, a problem per se. The group is open to new members who join it little by little as they see that the capital of its participants on the average grows faster than that of the egoists. The conditions under which this scenario is feasible are the same as those under which the collectivist attitude is preferable to the egoist one. These conditions which are defined by certain values of the model inputs are the subject matter of our study.

Why we discuss only a single group whereas in practice we mostly observe confrontation of more than one group? This is due to the fact that in the environment where collectivism brings the best results it is more beneficial to the groups to unite, rather than to compete. In the politics, the main obstacle to this is represented by the ideological differences, ambitions of the leaders, and the desire to find its own political “niche” (see, for example, [4]). Somehow or other, but the mechanisms of interaction of several groups within the framework of a model undoubtedly deserve investigation; and this investigation is projected. We also plan to consider socially oriented groups which support their poorest members in order to prevent their ruin, the model variants where the increments of capital depend on their current values, the variants using mechanisms for collection of taxes and “party dues” within the groups, and so on. At the same time, we do not plan to model purely
economic mechanisms of reproduction and drain of capital because we aim at analyzing the social, rather than economic phenomena. We again repeat that in the model the “capital” is meant in the most general (like “utility”) and not economic sense.

2. MODEL OF SOCIAL DYNAMICS

2.1. Basic Model

There are \( n \) participants among which \( n_e \) are egoists and \( n_g = n - n_e \) belong to the group. It will be convenient in what follows to define the relation of the participants by the parameter \( \beta = n_e/2n \), half of the fraction of egoists among all participants. The “society” coincides with the set of participants. Its state at each instant is described by the \( n \)-dimensional vector of capitals whose \( i \)th component is a real number characterizing the capital of the \( i \)th participant. Defined is the vector of initial capitals; by the “proposal of the environment” is meant the vector \((d_1, \ldots, d_n)\) of capital increments, where \( d_i \) is the algebraic increment in the capital of the \( i \)th participant according to the proposal of the environment.

It is assumed below as in [5–7] that the proposals of the environment are generated stochastically and their distribution remains unchanged from step to step. Namely, we assume that \( d_i \) is a normally distributed random variable whose mean value and variance are denoted, respectively, by \( \mu \) and \( \sigma^2 \). The values \( d_1, \ldots, d_n \) are assumed to be independent in the aggregate.

We denote by \( \xi_e \) and \( \xi_g \) the random variables representing, respectively, the fractions of those egoists among all egoists and those group members among all group members to whom a random proposal of the environment provides a positive increment of the capital. These random variables take values over the interval \([0, 1]\). We do not rule out the possibility of zero increment and at calculating \( \xi_e \) and \( \xi_g \) use the coefficient 0.5 to take into account the numbers of participants getting the zero increment.

At each step the participants learn the next proposal and vote in line with their principles. If the \( i \)th participant is egoist, he votes for a proposal if and only if \( d_i > 0 \); if \( d_i < 0 \), he votes against; and if \( d_i = 0 \), he abstains from voting (gives half-vote “for” and half-vote “against”). All group members vote identically in compliance with the group principle of voting. In the present paper, we consider two principles, “A” and “B”.

The group votes “for” a proposal \((d_1, \ldots, d_n)\) if and only if…

- Principle A: . . . as the result of accepting it the number of group members getting a positive increment in the capital exceeds the number of group members getting the negative increment: \( \xi_g > 0.5 \);
- Principle B: . . . the sum of the increments in the capitals of group members is positive: \( \sum d_i > 0 \) (the sum is taken over the participants of the group).

We denote by \( \xi \) the fraction of participants voting for the given random proposal. Since the group votes jointly,

\[
\xi = \begin{cases} 
(n_e\xi_e + n_g)/n & \text{if the group supports the proposal,} \\
\ n_e\xi_e/n, & \text{otherwise,} 
\end{cases}
\]

\[
= \begin{cases} 
2\beta\xi_e + (1 - 2\beta) & \text{if the group supports the proposal,} \\
2\beta\xi_e, & \text{otherwise.} 
\end{cases}
\]

Decision making follows the “\( \alpha \)-majority” procedure according to which a proposal is accepted if and only if \( \xi > \alpha \), where \( \alpha \in [0, 1] \) is the decision threshold. Consideration will be given not only to the voting thresholds \( \alpha \geq 0.5 \), but also to \( \alpha < 0.5 \) which are used in the practice of voting for various initiatives such as organization of a new parliamentary group, sending a letter of inquiry...
to the Constitutional Court, initiating a referendum, putting a question on the agenda, and so on. Such initiative decisions often play in the social life a role not smaller than the majority decisions.

If a proposal is accepted, then the corresponding vector of increments \((d_1, \ldots, d_n)\) is added to the current vector of capitals; otherwise, the latter remains unchanged. Passage is made to the next step where a new proposal is considered. Thus, the voting trajectory is constructed. We are going to consider the dynamics of the mean capital of the egoists and the group on such trajectories and again emphasize that in the model at hand the environmental proposals are random, that is, consideration is given to the utility of the egoistic and collectivistic attitudes under stochastic uncertainty, rather than to deliberate manipulation of voting by its organizers. Nevertheless, the problem of manipulatability will not drop out of sight. So, one can note straightway that in fact the voting principle A does not contribute to solving this problem. Indeed, if the group demonstrates its efficiency and all egoists join it, then the situation will return to the initial one where no group existed at all: the group will continue to vote jointly exactly for those decisions for which the simple majority would vote if they were egoists. It is namely this voting that is most readily manipulatable. Nevertheless, it is of interest to compare the dynamics of the mean capitals of the egoists and the group guided by principle A under stochastic uncertainty. Principle B resembles in many respects principle A, but excludes—in the case where the group coincides with the entire society—manipulation by the organizers as described by A.V. Malishevskii. Indeed, according to principle B the group supports only those proposals that replenish its common stock. Therefore, after a series of such decisions the capitals of all participants cannot decrease.

2.2. Additional Options of the Model

It is of interest to consider dynamics of participant’s ruin. To analyze this phenomenon, provided was a variant of the model where the participant with the capital falling down to a negative value “leaves the field,” is disregarded in the new proposal of the environment, and does not vote. Additionally, the model provides as option the possibility for egoists to join or leave the group. Two types of conditions for joining or leaving the group were considered. (i) The egoist is ready to join a group if its capital remains smaller than the mean capital of the group members during \(s_1\) successive steps, where \(s_1\) is a parameter. Correspondingly, a member of the group is ready to leave it if its capital is smaller than the mean capital of the egoists over \(s_2\) moves; \(s_1\) and \(s_2\) can differ. (ii) The same comparison is carried out for the capital increments rather than for the capitals themselves. For the transitions to be smoother and less “mechanistic,” it is assumed that if the condition for the first or second type of transition is satisfied, then the transition does not occur of necessity but with a certain probability which is the model parameter.

3. SOME EXAMPLES

The present paper considers the basic model which disregards ruin, joining the group, and exit from it. Some regularities of these phenomena were described in brief in [6]. In the following examples the number of participants is 200, the rms deviation of the capital increment is \(\sigma = 10\), and the group adheres to principle B. We consider first the case of the neutral environment \((\mu = 0)\), the group of 50% of all participants \((2\beta = 0.5)\), and the decision threshold 50% \((\alpha = 0.5)\). Let the initial capital of all participants be \(a = 700\). The typical dependence of the capitals of the group members and the egoists vs. the step number\(^2\) on a voting trajectory is depicted in Fig. 1a.

The mean capital of the egoists is practically time-independent, and that of the group grows uniformly. Variation of the voting threshold within rather wide limits does not affect the picture. Is

\(^2\) One should not assume that hundreds of steps are always required for appreciable changes in the capitals. The parameters are deliberately taken here in such a way that the changes are slow and the height of the graph steps is insignificant on purpose not to dim the general trend by random fluctuations.
it possible that simultaneously the capitals of one category of the participants grow and those of the other, diminish? This example is shown in Fig. 1b where the environment is unfavorable ($\mu = -1$), the group is small ($2\beta = 0.92$), and the voting threshold is much higher than one half ($\alpha = 0.48$).

Does the fact that decisions may be made against the opinion of majority play here the key role? No, an increase in the threshold up to $\alpha = 0.5$ entails no essential changes (Fig. 1c). One more example of similar dynamics arises under an unfavorable environment ($\mu = -1$), overwhelming majority of the group ($2\beta = 0.08$), and extremely low decision threshold $\alpha = 0.07$ (Fig. 2a).

At first sight it is surprising that even a greater reduction in the decision threshold ($\alpha = 0.04$) makes the curves practically change places, the capital of the egoists grow and that of the group decrease (Fig. 2b). Finally, we discuss an example of high decision threshold. For $\alpha = 0.97$, very large group ($2\beta = 0.08$), and the favorable environment ($\mu = 0.5$), the egoists also are ahead (Fig. 3): the mean capital of the group member grows much slower than that of the egoists. Consideration of these few examples convinces us that the regularities of the social dynamics corresponding to the given model are not obvious; therefore, it is of interest to analyze them by means of mathematical tools.
4. ON RANDOM VARIABLES IN THE MODEL

In the given model, the proposal of the environment is the vector \((d_1, \ldots, d_n)\) of the capital increments of all participants, where \(d_1, \ldots, d_n\) are independent random variables with the distribution \(N(\mu, \sigma^2)\). The corresponding one-dimensional density and the distribution function will be denoted by \(f_{\mu,\sigma}(\cdot)\) and \(F_{\mu,\sigma}(\cdot)\); \(f(\cdot)\) and \(F(\cdot)\) denote the density and the distribution function of the normal distribution with the center 0 and variance 1; \(M(\eta)\) and \(\sigma(\eta)\) denote the expectation and the rms deviation of any random variable \(\eta\) under consideration.

Each egoistic participant \(i\) votes for a proposal if and only if \(d_i > 0\). The probability of this event is as follows:

\[
p = P\{d_i > 0\} = 1 - F_{\mu,\sigma}(0) = F\left(\frac{\mu}{\sigma}\right);
\]

the probability of voting “against” is as follows:

\[
q = 1 - p = P\{d_i < 0\} = F_{\mu,\sigma}(0) = 1 - F\left(\frac{\mu}{\sigma}\right) = F\left(-\frac{\mu}{\sigma}\right).
\]

According to the model, the probability that the participant abstains from voting is zero because the normal distribution is continuous. Therefore, the voting of each egoistic participant is the Bernoulli test with the parameter \(p\). Then, since the values \(d_i\) are independent, the number of egoists voting “for” is distributed binomially with the parameters \(n_e\) and \(p\). The mean value and the variance of this distribution are, respectively, \(pn_e\) and \(pqn_e\). Normalization by dividing by \(n_e\) provides the aforementioned random variable \(\xi_e\), the fraction of egoists voting “for”. It has the mean \(p\) and the variance \(n_e^{-2}n_epq = n_e^{-1}pq\). The fraction \(\xi_g\) of the group members getting a positive increment in capital according to the random proposal of the environment has the same distribution with the mean \(p\) and the variance \(n_g^{-1}pq\), \(\xi_e\) and \(\xi_g\) being independent.

The confidence interval—usually symmetrical or centered at \(M(\eta)\)—which the random variable \(\eta\) hits with the probability 0.995 will be called for brevity the concentration zone of \(\eta\). In the case of normal random variable with the parameters \(\mu\) and \(\sigma^2\), this interval lies inside the segment \([\mu - 3\sigma, \mu + 3\sigma]\) where about 99.73% of the normal distribution are concentrated. Exit of the random variable from the concentration zone will be regarded as a highly improbable event. In a series of, say, 1000 steps, such events happen several times, but make no appreciable contribution to the mean indices, which justifies their screening-out. The following sections will be devoted to analysis of the social dynamics under various model parameters.
5. CASE OF NEUTRAL ENVIRONMENT

We first assume that \( \mu = 0 \), that is, the environment is neutral and the distribution of its proposals is symmetrical about 0. At that, \( p = q = 0.5 \); the distributions of the random variables \( \xi_e \) and \( \xi_g \) are symmetrical about 0.5. We assume for the time being, unless the contrary is allowed, that the group is guided by the voting principle A. The effects of ruin and passage of the participant from one category to another are disregarded.

5.1. Decision Threshold \( \alpha = 1 - \beta \)

Let us assume that the decision threshold is set as \( \alpha = 1 - \beta \) and that \( 2\beta < 2/3 \). Then the voices even of all egoists will be insufficient to make decision. It is necessary and sufficient that the proposal be supported by the group and at least one half of the egoists, that is, that the events \( \xi_e > 0.5 \) and \( \xi_g > 0.5 \), be realized (by default the group uses the voting principle A). As was noted above, the probabilities of each of these events are 0.5, and the events are independent. Therefore, their joint probability is 0.25, and, therefore, the asymptotic (for a great number of steps) value of the fraction of accepted proposals will be 0.25.

Let us consider now the dynamics of the mean capital of egoists and the group members. As it was just established, the proposal is accepted if and only if the total increment in the capitals of its members is positive. For a neutral environment, this condition, as that of principle A, is satisfied with the probability 0.5. Since the mean magnitude of the deviation from the mean, since the symmetrical binomial distribution is well approximated by the normal distribution even for a relatively small number of tests (usually this approximation is used for a number of Bernoulli tests exceeding \( 9(pq)^{-1} \), in order to pass from the rms deviation to the mean deviation magnitude we make use of the same coefficient \( \sqrt{2/\pi} \) as for the normal distribution, that is, estimate the mean deviation magnitude by \( \sqrt{0.5n_e/\pi} \). Then, under the condition that more participants voted "for," the mean deviation of the number of "fors" over the number of "againsts" is estimated by \( \sqrt{2n_e/\pi} \). By multiplying it by the estimated positive increment (this value and the number of positive increments are independent), we get that the total increment for the egoists is \( 2\sigma_e \pi \sqrt{n_e} \). Then, \( 2\sigma_e / \pi \sqrt{n_e} \) is the estimate of the capital increment of one egoist. Since on the whole a quarter of proposals is accepted, the mean increment for each egoist participant in one step is estimated as \( \sigma_e / \pi \sqrt{n_e} \). If the total number of steps is \( s \), then after this series \( \sigma_e s / 2\pi \sqrt{n_e} \) is the estimate of the expected capital increment of the egoist. For example, if \( \sigma_e = 10 \) and \( n_e = 50 \), then the estimated increment for one participant in 100 steps is about 22.5. Similarly, the estimated mean capital increment in a series of \( s \) moves for a member of the group adhering to principle A is \( \sigma_g s / 2\pi \sqrt{n_g} \). Therefore, the growth rate of the mean capital of each category is in inverse proportion to the root of its quantity (according to the well-known pseudo-scientific aphorism, the speed of a human group is also in inverse proportion to its size). In particular, for \( n_g = n_e \) the capital dynamics of the group and egoists is the same, and for \( n_g \neq n_e \) the smaller category is in a better position. However, one must bear in mind that the considered decision threshold \( \alpha = 1 - \beta \) itself depends on the relation between the quantities.

Let now the group adhere to principle B and give all its votes for the proposal of the environment if and only if the total increment in the capitals of its members is positive. For a neutral environment, this condition, as that of principle A, is satisfied with the probability 0.5. Since the
increments in the capitals of the group and the egoists are independent, a quarter of the proposals is accepted asymptotically as before. How the group dynamics changes at that? For an arbitrary proposal of the environment, the mean increment in the capital of the group member is normally distributed with the mean 0 and variance $\sigma^2/n_g$. The mean increment in the capital of the group member, provided that it is positive as required by principle B, is equal to the mean magnitude of deviation, that is, differs from the rms deviation by the coefficient $\sqrt{2}/\pi$ and is equal to $\sigma\sqrt{2/\pi n_g}$.

By estimating again the fraction of the accepted proposals as one quarter, we obtain for a series of $s$ moves the mean increment of the group member equal to $\sigma s\sqrt{2/\pi n_g}$ over a series of $s$ steps. The same results are obtained for the group: the mean capital increment is $\sigma s\sqrt{2/\pi n_g}$ depending on which principle, A or B, is used. The only difference of the general dynamics lies in the fact that the presence of “counterbalancing” decisions, that is, providing the mean total zero of the “quarters” (fractions of 25%), increases the spread as compared to the case of the threshold $\alpha = 1 - \beta$ where they are absent. Now, both in the group and among the egoists a stronger stratification in incomes is observed. Therefore, the voting thresholds $\alpha = 1 - \beta$ and $\alpha = \beta$ that are symmetrical about 0.5 provide different (by the factor of three) numbers of the

Note 1. The assumption of $2\beta < 2/3$ made at the beginning of this section can be relaxed using the notion of concentration zone of the random variable $\xi_e$. Indeed, to disable the egoists to make a decision by their own forces without approval by the group, it is sufficient to the accuracy of a highly improbable event that the threshold $\alpha = 1 - \beta$ be above the right boundary of the concentration zone of $2\beta \xi_e$. This boundary is estimated by $2\beta (M(\xi_e) + 3\sigma(\xi_e))$, where $M(\xi_e) = 0.5$, $\sigma(\xi_e) = 0.5/\sqrt{n_e}$. By solving the inequality $2\beta (0.5 + 3 \cdot 0.5/\sqrt{n_e}) < \alpha = 1 - \beta$, we get

$$2\beta < \frac{1}{1 + 3\sigma(\xi_e)} = \frac{\sqrt{n_e}}{\sqrt{n_e} + 1.5}. \quad (4)$$

For example, for $n_e = 225$ this condition provides $2\beta < 10/11$, thus relaxing the initial constraint $2\beta < 2/3$.

5.2. Decision Threshold $\alpha = \beta$

Now we consider the “mirror” case of voting with the threshold $\alpha = \beta$ and assume for a start that $2\beta < 2/3$. At that, sufficient is not only approval of the proposal by the majority of egoists, but also by the group. If in the above case a conjunction of both conditions (support by the group and the majority of egoists) was required, here it suffices to satisfy their disjunction. Since the environment is neutral, the probability of none of the events is 1/4; consequently, the disjunction of the conditions is satisfied in 3/4 of cases. These 75% are divided into 50% where the majority of the egoists are “for” and 25% where the majority are “against.” These 25% are fully symmetrical to half of the first of 50% and “balance” them in the sense of the mean capital increment (the sum of two means is zero). We apply to the remaining half of 50% the same reasoning as in the case of the threshold $\alpha = 1 - \beta$ and obtain the same result: the mean capital increment of the egoist is $\sigma s\sqrt{2/\pi n_e}$ over a series of $s$ steps. The same results are obtained for the group: the mean capital increment is $\sigma s\sqrt{8\pi n_g}$ or $\sigma s\sqrt{8\pi n_g}$ depending on which principle, A or B, is used. The only difference of the general dynamics lies in the fact that the presence of “counterbalancing” decisions, that is, providing the mean total zero of the “quarters” (fractions of 25%), increases the spread as compared to the case of the threshold $\alpha = 1 - \beta$ where they are absent. Now, both in the group and among the egoists a stronger stratification in incomes is observed. Therefore, the voting thresholds $\alpha = 1 - \beta$ and $\alpha = \beta$ that are symmetrical about 0.5 provide different (by the factor of three) numbers of the
accepted proposals and different spreads of the incomes in the group and among the egoists, but the same dynamics of the mean values. As will be shown below, in the case of neutral environment the situation with the mean values is always the same for \( \alpha = \alpha' \) and \( \alpha = \alpha'' \) if \( \alpha' + \alpha'' = 1 \).

Note 2. As in the above section, we relax the initial constraint \( 2\beta < 2/3 \). Now we need that the approval by the group should suffice (to within a highly improbable event) for accepting the proposal of the environment. By assuming that support of the proposal by the egoists does not diminish the left boundary of the concentration zone of \( \xi \), which is equal to \( M(\xi_e) - 3\sigma(\xi_e) \), we obtain that the minimal total support of the proposal as approved by the group is \( 2\beta(M(\xi_e) - 3\sigma(\xi_e)) + (1 - 2\beta) \) because \( 1 - 2\beta \) is the fraction of the group among the participants. Therefore, the condition for sufficiency of support by the group for approval of the proposal looks like \( 2\beta(M(\xi_e) - 3\sigma(\xi_e)) + (1 - 2\beta) > \alpha = \beta \). By substituting \( M(\xi_e) = \beta \) and \( \sigma(\xi_e) = 0.5/\sqrt{n_e} \), we get the same condition (4) as in the preceding case.

5.3. Other Values of the Decision Threshold

Since the symmetrical binomial distribution is well approximated by the normal distribution even for a relatively small number of tests, we still estimate the concentration zone of the symmetrical random variable \( \xi \) by the interval with the boundaries \( (M(\xi_e) \pm 3\sigma(\xi_e)) \), where in this case \( M(\xi_e) = p = 0.5 \) and \( \sigma(\xi_e) = \sqrt{pq/n_e} = 0.5/\sqrt{n_e} \), that is, the interval with the boundaries \( 0.5(1 \pm 3/\sqrt{n_e}) \). Then, according to (1), it is unlikely that the random variable \( \xi \) misses the union of the interval with the boundaries

\[
\beta \pm \frac{3\beta}{\sqrt{n_e}} \quad \text{(if the group does not support the proposal)}
\]

and the interval with the boundaries

\[
1 - \beta \pm \frac{3\beta}{\sqrt{n_e}} \quad \text{(if the group supports the proposal)},
\]

For example, if \( n_e = n_g = 225 \), then \( \beta = 1/4 \) and the values of \( \xi \) are concentrated in the domain \([0.2;0.3] \cup [0.7;0.8] \).

Let us consider the five cases of localization of the decision thresholds \( \alpha \).

Zone 1. \( \alpha < \beta - 3\beta/\sqrt{n_e} \). Here, almost always \( \xi > \alpha \), and practically all proposals of the environment are accepted. Since the environment is neutral, the expectation of the mean increment in capital in the series of \( s \) steps is extremely close to zero both for the egoists and the group members, and the rms deviation of this value is \( \sqrt{s/n_e}\sigma \) for the egoists and \( \sqrt{s/n_g}\sigma \) for the group.

Zone 2. \( \alpha > 1 - \beta + 3\beta/\sqrt{n_e} \). Then, almost always \( \xi < \alpha \), and actually the proposals of the environment are never accepted. At that, the capital of the participants does not vary, but the extremely rare accepted proposals increase the capitals of both the group members and the egoists (the increase of the latter is greater because they have a higher decision threshold than the group).

Zone 3. \( \beta - 3\beta/\sqrt{n_e} < \alpha < 1 - \beta - 3\beta/\sqrt{n_e} \). For these decision thresholds, \( \xi > \alpha \) is satisfied to an accuracy of low-probable events if and only if the group supports the proposal. If the group uses the voting principles A or B, then this happens on the average in one half of cases, which is twice as frequent as in the case discussed in Sec. 5.1 (page 317), the rest being the same because the expected value of the increment in the capital of the group member in the \( s \)-step series is \( \frac{\sigma s}{\sqrt{2\pi n_g}} \) for principle A and \( \frac{\sigma s}{\sqrt{2\pi n_g}} \) for principle B. The egoists do not affect the voting; therefore, for them.
Fig. 4. The capitals averaged by the categories and steps vs. the threshold $\alpha$. One realization, 450 participants, $\mu = 0$, $\sigma = 10$, principle B, $2\beta = 0.5$.

the proposals of the environment are neutral on the average and the expected increment in their mean capital in an $s$-step series is slightly greater than zero (nevertheless, slightly greater owing to the rare improbable events) and has the rms deviation $\sqrt{s/2n_e \sigma}$. This zone of variations of the threshold $\alpha$ exists if the condition $\beta + 3\beta/\sqrt{n_e} < 1 - \beta - 3\beta/\sqrt{n_e}$, which is equivalent to the condition $2\beta < \sqrt{n_e}/(n_e + 3\beta)$, is satisfied. For example, for $n_e = 225$ the zone exists for $\beta < 5/6$.

**Zone 4.** $1 - \beta - 3\beta/\sqrt{n_e} \leq \alpha \leq 1 - \beta + 3\beta/\sqrt{n_e}$. With an increase of the decision threshold $\alpha$ from $1 - \beta - 3\beta/\sqrt{n_e}$ to $1 - \beta$ (zone “4a”), the egoists exerts more and more influence on the decisions and consistently reject more and more proposals that are unadvantageous for them on the average. At that, the mean increment in the capital (in what follows, simply “mean increment”) of an egoist in an $s$-step series grows from zero (see case 3) to $\sigma_s/\pi\sqrt{n_g}$ (Sec. 5.1, page 317). At the same time, the increment of the group member in an $s$-step series is halved from $\sigma_s/\pi\sqrt{n_g}$ (zone 3) to $\sigma_s/2\pi\sqrt{n_g}$ (for principle A) or from $\sigma_s/\sqrt{2\pi n_g}$ to $\sigma_s/2\sqrt{2\pi n_g}$ (for principle B).

With further growth of the threshold from $1 - \beta$ to $1 - \beta + 3\beta/\sqrt{n_e}$ (zone “4b”), the mean increment of the group member continues to vanish because the number of accepted proposals vanishes. For the egoists, the mean increment also diminishes because now not all but only the most advantageous proposals are accepted. An interesting effect is observed here. If in zone 4b $n_g = n_e$ ($2\beta = 0.5$) and the group adheres to principle A, then the mean increment is higher for the egoists than for the group. This effect was mentioned when discussing zone 2, and one can readily see that it is retained also when the group adheres to principle B.

**Zone 5.** $\beta - 3\beta/\sqrt{n_e} \leq \alpha \leq \beta + 3\beta/\sqrt{n_e}$. Here, the dynamics of the mean increments is a mirror reflection of that observed in zone 4. When $\alpha$ varies from $\beta - 3\beta/\sqrt{n_e}$ to $\beta$ (zone “5a”), less and less proposals for which $\xi \in (\beta - 3\beta/\sqrt{n_e}, \beta)$ are accepted. They are not supported either by the group or by the majority of the egoists. Therefore, in what concerns the expected mean increments in the capital, they are unfavorable to all, and a reduction in the number of such accepted proposals leads to higher mean increments both for the group (principles A and B) and the egoists. Similar to the case of the “mirror” zone 4b, here also for $n_g = n_e$ ($2\beta = 0.5$) the mean capital increments
are higher for the egoists than for the group, which is accounted for by the fact that the rejected proposals here are *just* unfavorable for the group ($\xi_g \leq 0.5$) and *especially unfavorable* for the egoists for which the stricter condition $\xi_e \leq \alpha/\beta < 0.5$ is satisfied. Therefore, the egoists gain more from their rejection.

With further increase of $\alpha$ from $\beta$ to $\beta + 3\frac{\beta}{\sqrt{n_e}}$ (zone “5b”), the set of the accepted proposals contracts even more (their expected fraction decreases from 75% to 50%) at the expense of the proposals rejected by the group but supported by the majority of the egoists ($\xi > \beta$, consequently, $\xi_e > 0.5$). Therefore, the mean increment for the group continues to grow from the values reached at $\alpha = \beta$ (Sec. 5.2, page 318) to the value of zone 3, and the mean increment for the egoists decreases from $\frac{\sigma_a}{2\pi\sqrt{n_e}}$ (Sec. 5.2, page 318) to zero.

The results of the computer-aided modeling are illustrated in Figs. 4 and 5 where the mean capitals of the egoists and the group members in a series of $s = 1000$ steps corresponding to various values of the decision threshold are laid off on the vertical axis. The initial capital of each participant is 3000, the environment is neutral ($\mu = 0$), $\sigma = 10$, $n = 450$; the ruin and passage of the egoists to the group and exit from the group are disregarded. Averaging is carried out both over the participants and steps. By virtue of stationarity of the distribution of the capital increments, the expectation of the step mean differs from the initial value half as many as the expectation of the capital after the $s$th step.
These and the following figures depict the mean values for one realization of the process which also reflect the spread in values about the expectation, rather than the capital expectations (the data for it are presented above, the general analytical expressions will be published later). For example, if the voting threshold lies in zone 3, the expectation of the capital of egoists almost does not differ from their initial capital, but their mean capital is appreciably (by 10 units) smaller because of the spread in the graph (Fig. 4).

Figure 4 shows the case of as many egoists as there are members in the group, \( n_e = n_g \) (2\( \beta = 0.5 \)). The first graph of Fig. 5 refers to the case of small group, \( 2\beta = 0.92 \), in which case the condition of “Case 3” cannot be met and no characteristic horizontal segment exists on the curves of the mean capital of the egoists and the group.

In the example at hand, the value of \( \beta \) is slightly higher than the threshold of Notes 1 and 2 in which case for \( \alpha = 1 - \beta \) a very small fraction of decisions is made only by the egoists without approval of the group and for \( \alpha = \beta \) the proposal is not accepted despite support of the group in a small fraction of cases. The second graph of Fig. 5 shows the opposite case of a very large group, \( 2\beta = 0.08 \). Here, the greater part of the entire range of \( \alpha \) lies in zone 3. Since \( n_e < n_g \), for \( \alpha = \beta \) and \( \alpha = 1 - \beta \), the mean capitals of the egoists are higher than those of the group members. Additionally, owing to a low value of \( n_e \), the capitals of the egoists have a substantial spread for the values of \( \alpha \) related to zones 1 and 3. This spread accounts for the noticeable difference between the observed mean capital of the egoists in zones 1 and 3 and the expected capital coinciding with the initial capital (3000 units).

5.4. Some More About the Zones where the Egoists Have Advantages

In the case of neutral environment \( n_e = n_g \), the above analysis shows that in what concerns the expected increment in the capital the egoists have advantage over the group members only in zones 4b and 5a. In zone 4b, for all proposals advantageous for the group the egoists take only those that are most advantageous to them. In the “mirror” case 5a, the egoists block approval of the proposals that are most disadvantageous for them, whereas the group cannot do so.

We note that for \( n_e = n_g \) in these two zones the group can easily bring to nothing the egoists’ advantage by using the following procedure. We define for the group a special “internal voting threshold” \( \alpha' \) depending on \( \alpha \) and \( \beta \).

\[
\alpha' = \begin{cases} 
\frac{1}{2} - \frac{\delta}{2\beta}, & \text{where } \delta = \beta - \alpha \quad \text{if } \alpha < \beta; \\
\frac{1}{2} + \frac{\delta}{2\beta}, & \text{where } \delta = \alpha - (1 - \beta) \quad \text{if } \alpha > 1 - \beta; \\
\frac{1}{2}, & \text{if } \beta \leq \alpha \leq 1 - \beta 
\end{cases}
\]

\[
= \begin{cases} 
\frac{\alpha}{2\beta}, & \text{if } \alpha < \beta; \\
1 - \frac{1 - \alpha}{2\beta}, & \text{if } \alpha > 1 - \beta; \\
\frac{1}{2}, & \text{if } \beta \leq \alpha \leq 1 - \beta.
\end{cases}
\]

Let us consider principle A’.

**Principle A’.** The group votes for the proposal of the environment if and only if in the case of its approval the fraction \( \xi_g \) of its members getting a positive increment in capital exceeds the threshold \( \alpha' \) defined by (7).

Voting by principle A’ offers to the group the same possibilities of influencing the decisions made in zones 4b and 5a as enjoyed by the egoists. Indeed, in zone 4b \( \alpha = 1 - \beta + \delta \). In the case of
accepting the proposal of the environment, \((1 - 2\beta)n\) votes are given by the group; consequently, it is necessary that among all participants the fraction of egoists supporting the proposal exceed 
\[ \alpha - (1 - 2\beta) = \beta + \delta. \]
Consequently, the condition \(\xi_e > \frac{\alpha - \delta}{2\beta} = \frac{1}{2} + \frac{\delta}{2\beta}\) must be satisfied for approval of the decision. If the group establishes for itself the same threshold of voting, that is, decision, it will be in the same position as the egoists: they will have identical “utility thresholds” for the supported proposals and, therefore, identical expected dynamics of the capital.

Similarly, in zone 5a \(\alpha = \beta - \delta\). To approve a proposal by the efforts of egoists, that is, without participation of the group, it is necessary that they provide more than \(\alpha n = (\beta - \delta)n\) votes. Therefore, among all egoists the fraction of those voting “for” must exceed \(\frac{\beta - \delta}{2\beta} = \frac{1}{2} - \frac{\delta}{2\beta}\). Establishment of the same “internal threshold” smaller than 0.5 puts the group in the same conditions as the egoists: if the votes collected in the group exceed this threshold, then the proposal will be accepted for sure, and this will happen as frequently as the votes of the egoists exceed the same threshold. As the result, the expected dynamics of capital in the group and among the egoists again will be the same.

If the number of egoists is smaller than one half, then under the above changes in the intragroup threshold they will be again ahead in zones 5a and 4b, but their advantage will decrease substantially.

Further reduction of the intragroup threshold in zone 5a and increase in zone 4b\(^3\) will offer advantage to the group over the egoists. However, this advantage will be relative, that is, the group really decreases its increment in capital almost for all values of the threshold \(\alpha\), but the reduction of the egoists will be even greater. If for the group to “live better than the egoists” is preferable just to just “living better,” then it can reach this aim.

6. CASES OF FAVORABLE AND UNFAVORABLE ENVIRONMENT

The case of \(\mu > 0\) corresponds to the favorable environment, the case of \(\mu < 0\), to the unfavorable environment. Reasoning providing conclusions about the nature of the dynamics of mean capitals are in this case similar to those above. Figure 6 illustrates the case of \(n_e = n_g (2\beta = 0.5)\), the rest of the parameters being the same as before.

For small deviations of \(\mu\) from 0 (Fig. 6a,b), the graphs have five zones like those considered above. In zone 2 (enumeration as above), that is, for the decision thresholds close to 1, the capital is equal to the initial capital because the proposals are rejected.

On the contrary, in zone 1 almost all proposals are accepted. Therefore, the mean increment in capital after an \(s\)-step series is \(\mu s\) (we recall that the capitals of the participants are averaged on the graphs also over the steps, therefore the values shown there are half as many) for the rms deviation \(\sqrt{s}\sigma\).

In zone 3, only those decisions are made that are supported by the group. Let us estimate their frequency assuming that the group adheres to principle B. The mean increment in capital of a group member in one step \(\bar{d}_G\) has the distribution \(N(\mu, (\sigma')^2)\), where \(\sigma' = \frac{\sigma}{\sqrt{n_g}}\). The expectation of the frequency of decisions made in zone 3 is equal to the probability of positiveness of this value, that is,

\[ P\left\{\bar{d}_G > 0\right\} = F\left(\frac{\mu}{\sigma'}\right) = F(\mu'), \]  

\(^3\) This may be done in a most natural way by establishing an intragroup threshold equal to the decision threshold \(\alpha\).
where as before $F(\cdot)$ is the standard normal distribution function and $\mu'$ stands for $\mu/\sigma'$. Let us determine the expectation of $\tilde{d}_G$, provided that it is positive. After integration we get

$$M\left(\tilde{d}_G \mid \tilde{d}_G > 0\right) = \left(P\{\tilde{d}_G > 0\}\right)^{-1} \int_0^\infty x f_{\mu,\sigma'}(x) \, dx = \mu' + \sigma' f(\mu') F(\mu').$$

(9)

Since the unconditional expectation of $\tilde{d}_G$ is as follows:

$$M(\tilde{d}_G) = P\{\tilde{d}_G > 0\} M\left(\tilde{d}_G \mid \tilde{d}_G > 0\right) = \sigma' f(\mu') + \mu F(\mu'),$$

(10)

the expectation of an increment in the capital of a group member after $s$ steps is as follows:

$$sM(\tilde{d}_G) = s(\sigma' f(\mu') + \mu F(\mu')).$$

(11)

The increment of the capital of an egoist in zone 3 after $s$ steps is estimated by

$$sM(\tilde{d}_E) = sP\{\tilde{d}_G > 0\} M\left(\tilde{d}_E \mid \tilde{d}_G > 0\right) = s\mu F(\mu').$$

(12)
Therefore, in zone 3 the estimated expectation of the difference in the capitals of the group member and the egoist after \( s \) steps is as follows:

\[
s \left( M \left( \tilde{d}_G \right) - M \left( \tilde{d}_E \right) \right) = s \sigma' f(\mu'). \tag{13}
\]

As can be seen in Fig. 6, this value and \( f(\mu') \) rapidly decrease with increase in \(|\mu|\): as compared to the case of \(|\mu| = 0.2\), for \(|\mu| = 1\) the graphs of the mean capital of the group members and the egoists approach closely each other and actually fuse for \(|\mu| = 2\). The case of \( n_e \neq n_g \) is illustrated in Fig. 7.

The cases of \( \beta = 0.92 \) and \( \beta = 0.08 \) are shown here for convenience of comparing with Fig. 5. It goes without saying that the regularities are the same: with an increase in \( \mu > 0 \), to the right of zone 3 the graphs make a higher and higher “step” and a lower and lower step to its left. On the contrary, with a decrease of \( \mu < 0 \), the graphs have a higher “step” to the left of zone 3 and a lower one to the right of it. With increase in \(|\mu|\), the graphs of the mean capitals of the egoists and the group members draw together until almost complete fusion.
Although the present paper models the “capital dynamics,” the term “capital” should not be misleading. It is not an economic model that is considered above, but that concerning only the social and political sciences. Indeed, it does not cover reproduction of capital by investing into production or using the financial tools, as well as changes in the capital that are related with other factors independent of collective decisions. All this is unnecessary because it is required here to study the dynamics defined by the mechanism of democratic voting with the built-in basic social attitudes of egoism and collectivism, rather than the economic or uncontrollable random mechanisms. Therefore the term “capital” refers here to any countable resource (utility) controlled by the collective decisions and having nothing to do with special regularities of reproduction or waste. In what follows, we summarize some conclusions obtained by analyzing the model.

1. The voting-dependent social dynamics is defined basically by the decision threshold. For higher thresholds (zone 2), the proposals are not accepted and the status quo is retained. For lower values (zone 1), actually all proposals are accepted, the dynamics being identical both for the group and the egoists defined by the mean and the variance of distribution of the proposals of the environment. The zone boundaries depend on the number of egoists and the group members, as well as on the parameters of the environment. With increase in the total number of the participant and the fraction of egoists, zones 1 and 2 widen; for lower values of these parameter they contract. Inside the middle (zone 3), which exists if the group is not too small, the dynamics is independent of the decision threshold and the group has superiority over the egoists. For the neutral or a moderately unfavorable environment, it manages to realize decisions that on the average are advantageous for its members. At that, the egoists do not influence the decisions, and for them the accepted proposals do not differ from a random sample of the proposals of the environment. There are two zones on either side of zone 3 (zone 5 to the left and zone 4 to the right) where the egoists influence the decisions. There are two peaks (maxima) of the increment of the egoists’ capital which, however, vanish in the case of apparently favorable and apparently unfavorable environments. The group characteristics within these zones vary monotonically from the values in the extreme zones 1 and 2 to the value in the middle zone. For the neutral environment, the expected increment in the capital reached by the egoists at the two maxima—for \( \alpha = 1 - \beta \) and \( \alpha = \beta \)—are the same as for a group of the same size for the same thresholds; for different sizes, the advantage belongs to the smaller category. If the egoists make 20% of the total number of participants, the increment in their capital reaches at the maxima the value which the group has in the middle zone in the case of voting by principle A. If the egoists make about 14% of the participants, then at the maxima they reach the value which the group has in the middle zone in the case of the voting principle B. If the number of egoists is still smaller, then for the neutral environment, the increment in their capital at the maxima exceeds all increments reachable by the group. For a small group, zones 4 and 5 merge, and zone 3 degenerates. At the interfaces of zones 4 and 5 both the group and the egoists have maximal increments in the capital, but the group maximum is higher.

2. As for the scenario described in the Introduction—egoists join the group and thus approach the group egoism to altruism,—it is absolutely realistic. The group is especially attractive for the mean values of the decision threshold to which the simple-majority threshold \( \alpha = 0.5 \) belongs. For such thresholds, the group retains its attractiveness even if it includes the majority of the participants; the egoists can have advantage only for high and low thresholds.

3. As was noted in Item 1 of this Summary, a smaller category of the participants has advantage in zones 4 and 5. Explanation of this curious phenomenon is related with the law of large numbers: since the variance of the sample mean decreases with sample volume, the proposals that are “very good” in the sense of the mean increment of capital are less frequent for the larger category. The
“very bad” proposals are less frequent as well, but this fact does not affect the dynamics because these proposals are rejected by voting.

4. Principle B which better protects the group against the manipulations of the organizers is preferable to principle A also in the sense of the mean increment in capital, which can be readily explained by the fact that it is namely the positiveness of the mean (total) increment in capital, rather than satisfaction of the majority of the group (as it is the case with principle A), that is declared by principle B as the group utility. For example, in the case of the neutral environment, the ratio of the increments in capitals of the groups adhering to principles B and A is \( \sqrt{\pi/2} \approx 1.25 \).

5. For changes in \( \mu \) and \( \sigma \) that retain their ratio \( \mu/\sigma \), the graphs of the capital increments extend/contract along the \( Y \)-axis in proportion to \( \sigma \). Therefore, the impact of the parameter \( \sigma \) on the increments in capital is defined by \( \mu \). Namely, the passage from \( \sigma \) to \( \sigma' = \rho \sigma \) (\( \rho > 0 \)) provides graphs extended by the factor of \( \rho \) that are obtained by passing from \( \mu \) to \( \mu/\rho \).

6. The maxima of the increment in the egoists’ capitals lie in the middles of zones 4 and 5. In the more distant, “external” parts of these zones, the egoists have advantage over the group for smaller, equal, and even somewhat higher their number. The group can reduce or, for a sufficiently high relative number of the egoists, even bring to nothing this effect by passing to the voting principle \( A' \). By changing its “internal voting threshold,” the group can even achieve advantage over the egoists. At the same time, in the majority of cases it reduces its mean increment in capital, but the mean increment in the egoists’ capital reduces even more. Is it advantageous to the group? There is no unambiguous answer. In the social practice, the question “What is more attractive, to live better than before or to live better then the rest?” always remains open. It is only clear that as a disciplined unit the group can choose an answer, whereas the egoists do not have such a possibility.

8. CONCLUSIONS

The paper analyzed the model of social dynamics defined by voting in the stationary stochastic environment. Time uniformity of the environment parameter defines the specificity of the results and distinguishes the case at hand from the situation of voting which the organizers try to manipulate. As was shown by analysis, for wider domains in the parameter space, the group has a better dynamics of capital than the egoists, which makes realistic the scenario where the egoists join the group and the group egoism approaches altruism. The narrow domains where the egoists have advantage over the group were identified. The main results of analysis were interpreted in terms of the social sciences.

REFERENCES

1. Mirkin, B.G., Problema gruppovogo vybora (Problem of Group Choice), Moscow: Nauka (Fizmatlit), 1974.
2. Aizerman, M.A., Dynamic Aspects of the Voting Theory (Review of the Problem), Autom. Telemekh., 1981, no. 12, pp. 103–118.
3. Chebotarev, P.Yu., Some Properties of Trajectories in the Dynamic Problem of Voting, Autom. Telemekh., 1986, no. 1, pp. 133–138.
4. Aleskerov, F.T. and Orteshuk, P., Vyborg. Golosovanie. Partii (Elections. Voting. Parties), Moscow: Akademiya, 1995.
5. Borzenko, B.I., Lezina, Z.M., Lezina, I.B., et al., Model of Social Dynamics Defined by Collective Decisions and Random Environmental Changes, in: II Mezhdunar. konf. po probl. upr. Tez. dokl. (II Int, Conf Control, Abstracts), Moscow: IPU RAN, 2003, p. 120.
6. Chebotarev, P.Yu., Borzenko, V.I., Lezina, Z.M., et al., Model of Social Dynamics Controlled by Collective Decisions, in: Tr. In-ta probl. upr. RAN. (Proc. Int. Control Probl.), vol. XXIII, Moskva, 2004, pp. 102–109.
7. Chebotarev, P.Ju., Borzenko, V.I., Lezina, Z.M., et al., A model of Social Dynamics Governed by Collective Decisions, in: Proc. Int. Conf. “Math. Modelling of Social and Economic Dynamics” (MMSED-2004), June 23–25, 2004, Moscow: RSSU, 2004, pp. 80–83.

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