Field condensation and non-critical string for $c>1$

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Abstract

Quantum theory of 2d gravity for $c > 1$ is examined as a non-critical string theory by taking account of the loop-correction of open strings whose end points are on the 2d world surface of the closed string. This loop-correction leads to a conformal anomaly, and we obtain a modified target-space action which implies a new phase of the non-critical closed-string. In this phase, the dual field of the gauge field, which lives on the boundary, condenses and the theory can be extended to $c > 1$ without any instability.
1. Introduction

The quantized 2d gravity can be formulated as a conformal invariant nonlinear sigma-model on the 2d manifold \([1]\). The non-critical string theory is also formulated in the same way since it can be regarded as a 2d gravity coupled to several scalar-fields with conformal coupling, where the number \((c)\) of the scalar-fields is related to the dimension of the target-space as \(d = c + 1\).

The vacuum of the theories is determined by solving the conditions of zero \(\beta\)-functions of the nonlinear sigma-model by the \(\alpha'\)-expansion \([2]\). In the sense of \(\alpha'\)-expansion, some exact solutions are given according to a simple ansatz \([3]\) other than the well-known linear dilaton vacuum. Quantum fluctuations of matter fields on these vacua have been examined, and we found that the properties like the renormalization group equations of matter-sector are insensitive to the details of the vacuum configurations \([4]\). This result could be understood such that the properties like the renormalization group equations are determined by the short distance behaviors on the 2d manifold and they do not depend on the global aspect of the vacuum.

While there is a serious problem called as \(c = 1\) wall which means that the 2d manifold becomes unstable for \(c > 1\), where \(c\) is the central charge of the theory. This instability is observed as the complex string susceptibility of the surface, and the numerical simulations \([5]\) indicate the branched polymer phase of 2d manifold for \(c > 1\). This surface instability would heavily depend on the structure of the vacuum. In terms of the non-critical string theory, the ground state of the string-state becomes tachyonic, then the theory is unstable for \(c > 1\) or \(d > 2\). And this difficulty could not be removed in any vacuum state found until now. In other words, any consistent theory of 2d gravity is not found still for \(c > 1\) except for the special cases, \(c = 7, 13, 19\) \([6]\). But the cases of \(c = 7, 13\) and 19 do not include the physical degrees of freedom and they are topological theory in this sense. So, to try resolving this problem is very challenging.

It is possible to obtain a real string susceptibility for \(c > 1\) by adding the curvature-squared term \([7]\) to the world-sheet action or by considering a scalar-field which couples non-minimally to the scalar-curvature of the world sheet \([8]\). The reason why the complex susceptibility is avoided is that the high-curvature configuration is suppressed due to these terms, but these theories sacrifice the unitarity. On the one hand, the touching interactions have been examined \([9]\) in the recent matrix model. This interaction can be regarded as the wormhole interactions of the continuum theory. Then the model including these interactions was expected as a possible theory for \(c > 1\), but these interactions could not extend the theory to the region \(c > 1\) and a new phase of a special susceptibility has been found for \(c \leq 1\). Many other attempts to resolve this problem have been tried both in the discretized- and in the continuum-models, but no one has found an unitary-
theory extended to the case of $c > 1$ without the instability of the tachyon. Here we approach to this problem from the non-critical string theory by taking account of the loop-corrections. From the viewpoint of the 2d gravity, this approach is equivalent to consider the interacting many universes. In this sense, this approach is similar to the matrix model mentioned above. But a new degrees of freedom related to the open-string state appears in our continuum approach differently from the matrix model, and this new freedom plays an important role in extending the theory to $c > 1$ region.

Our purpose is to show the existence of a phase where the theory is stable even in the region $c > 1$. This phase is found by taking account of the loop-correction of open-strings whose end points couple to the boundary on the world sheet of the closed-string. The importance of this kind corrections in considering the vacuum of the string theory was pointed out previously in [10], and the calculational technique has been developed in [11, 12] in the superstring theory. Here this technique is applied to the non-critical string case. In this case, we must take care about the Liouville-field part in the world-sheet action.

The analysis is performed around the linear dilaton vacuum, but the tachyon might be shifted by a constant due to the loop-correction. And we assume that the string coupling constant $g_s$ is small. Namely, we are considering in the small coupling region. The guiding principle of our analysis is the conformal invariance or the BRST invariance of the theory.

2. Tree Vacuum State

We set up a vacuum state as a basis to calculate the string-loop corrections. Such a vacuum is obtained by imposing the conformal invariance on the world-sheet action, which is written in the following nonlinear $\sigma$-model,

$$
S_{2d} = \frac{1}{4\pi} \int d^2 z \sqrt{\hat{g}} \left[ \frac{1}{2} G_{\mu\nu}(X) \hat{g}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + \hat{R} \Phi(X) + T(X) \right] + \hat{S}_{gh},
$$

where the ghost action $\hat{S}_{gh}$ is expressed as,

$$
\hat{S}_{gh} = \frac{1}{2\pi} \int d^2 z \sqrt{\hat{g}} \hat{g}^{\alpha\beta} \hat{c}^\gamma \nabla_\alpha b_{\beta\gamma}.
$$

Here the theory is quantized by the conformal gauge, $g_{\alpha\beta} = e^{2\phi} \hat{g}_{\alpha\beta}$, where $\hat{g}_{\alpha\beta}$ is some fiducial metric. The conformal mode ($\phi$) and $c$-scalar fields ($x^i$, $i = 1 \sim c$) in the world sheet action are denoted by the target space coordinates, $X^\mu = \{\phi, x^i\}$, where $\mu = 0, i$. The naive form of the target-space action is obtained from the $\alpha'$-expansion as [4, 4],

$$
S_T = \frac{1}{4\pi} \int d^d X \sqrt{G} e^{-2\phi} \left\{ R - 4(\nabla \Phi)^2 + (\nabla T)^2 - 2T^2 - \frac{25 - c}{3} \right\},
$$
where \( d = 1 + c \). And we do not include the nonlinear terms coming from \( T\)-expansion \([13, 14]\) since it gives a minor change in our analysis and does not affect on the conclusion as discussed below. While, it might be possible to consider that (2.2) represents the action obtained after the field redefinitions to leave \( T^2 \) as the only possible non-derivative term of \( T \) \([13, 14]\). However there is an argument \([17]\) that it is impossible to rewrite the power-series of \( T \) in such a way. But this point is not important here.

Solving the equations derived from (2.2), we obtain the following vacuum solution,

\[
G_{\mu\nu} = G_{\mu\nu}^{(0)} \equiv \delta_{\mu\nu}, \quad \Phi = \Phi^{(0)} \equiv \frac{1}{2}Q\phi, \quad T = 0,
\]

(2.3)

where \( Q = \sqrt{(25 - c)/3} \). Then, \( S_{2d} \) can be written as

\[
S_{2d} = \frac{1}{8\pi} \int d^2z \sqrt{g} \left[ g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu + Q\hat{R}\phi \right] + \hat{S}_{gh},
\]

(2.4)

Although (2.2) is not an exact form of the target-space action since higher derivative terms are neglected, but the solution (2.3) is exact in the sense that (2.4) is exactly conformal invariant. In terms of (2.4), the mode expansion of the fields \( X^\mu \) is performed by noticing that the Coulomb gas picture \([18]\) is applicable to the Liouville mode \( \phi = X^0 \) which couples to the background charge. It is obtained as follows,

\[
X^\mu(z, \bar{z}) = \varphi^\mu(z) + \bar{\varphi}^\mu(\bar{z}),
\]

(2.5)

\[
\partial \varphi^\mu(z) = -i \sum_m \alpha^\mu_m z^{-m-1},
\]

(2.6)

\[
\bar{\partial} \bar{\varphi}^\mu(\bar{z}) = -i \sum_m \bar{\alpha}_m^\mu \bar{z}^{-m-1},
\]

(2.7)

where \( z = \exp(\tau + i\sigma) \) and

\[
[\alpha^\mu_m, \alpha^{\nu}_n] = \delta^\mu_\nu m \delta_{m+n,0}.
\]

(2.8)

The vacuum for these bosonic fields is defined as,

\[
\alpha^\mu_m | 0 > = \bar{\alpha}^\mu_m | 0 > = \begin{cases} 0 & \text{for } m > 0 \text{ and } m = 0, \mu = i > 0 \\ -\frac{1}{2}Q & \text{for } m = \mu = 0 \end{cases}
\]

(2.9)

For the ghost fields, we obtain

\[
c(z) = \sum_m c_m z^{-m+1},
\]

\[
b(z) = \sum_m b_m z^{-m-2}, \quad \{c_n, b_m\} = \delta_{n+m,0}.
\]

(2.10)

Similar formula are obtained for \( \bar{b}(\bar{z}) \) and \( \bar{c}(\bar{z}) \).
The vacuum of the ghost would be given in the next section. And the stress tensors for each field are obtained as follows,

\[ T^\phi(z) = -\frac{1}{2} : \partial \phi \partial \phi : + \frac{Q}{2} \partial^2 \phi , \]  
(2.11)

\[ T^X(z) = \sum_{i=1}^c -\frac{1}{2} : \partial X^i \partial X^i : . \]  
(2.12)

\[ T^{bc}(z) = : c \partial b + 2 \partial cb : . \]  
(2.13)

These formula are used to construct the boundary state in the next section.

3. Boundary State

Here we construct the loop-amplitude of the open-string. This is achieved by connecting the tube, which is the propagator of the closed string, to the boundary on the surface of the world sheet since this tube can be identified as the loop of the open-string whose end point is sewing the boundary of the world surface. Another end point disappears in the vacuum or couples to the external string configurations. The boundary state with a tube made in this way is not conformal invariant and it leads to a conformal anomaly when the closed-string state propagating through the tube is massless. We consider this formulation for the non-critical string case by applying the procedure given for the superstring case [12].

First, consider the boundary (at time \( \tau \) and boundary coordinate \( \sigma \)) on the world sheet, where the following boundary action is assumed,

\[ S_B = -\frac{i}{4\pi} \int_{\partial M} d\sigma A_\mu(X) \partial_\sigma X^\mu . \]  
(3.1)

Here \( \partial M \) represents the boundary (or boundaries) on the 2d manifold \( M \), and \( A_\mu(X) \) is the gauge field living on the open-string and on the boundary. We can consider this gauge field in two ways; (i) It is the usual gauge field defined on the open string. In this case, \( S_B \) could contain other fields of open-string states, but we concentrate here our attention on this gauge field only. (ii) It is an auxiliary field which is needed for the gauge invariance on the boundary of the antisymmetric field \( B_{\mu\nu} \) defined on the world sheet as, \( \int d^2 z \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu . \)

In any case, we assume here that its field strength, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), is varying very slowly with \( X^\mu \). In this case, \( S_B \) can be approximated as

\[ S_B = \frac{i}{8\pi} F_{\mu\nu} \int_{\partial M} d\sigma X^\mu(X) \partial_\sigma X^\nu . \]  
(3.2)

And the following boundary conditions are needed

\[ \partial_\tau X^\mu + i F_{\mu\nu} \partial_\sigma X^\nu |_\tau = -\frac{i}{2} Q \delta_0^\mu . \]  
(3.3)
The condition for $\mu = 0$ in (3.3) represents the one for the Liouville field. It is consistent with (2.9), and it can be corresponded to the boundary condition for the non-critical open-string [19, 20].

The condition (3.3) leads to the following operator relations,

$$\alpha^\mu_m = - \left( \frac{1 - F}{1 + F} \right)^\mu^\nu \bar{\alpha}^\nu_m e^{-2m\tau}$$

(3.4)

for $m \neq 0$ and

$$\alpha^i_0 = \bar{\alpha}^i_0 = 0, \quad \alpha^0_0 = \bar{\alpha}^0_0 = -\frac{i}{2}Q.$$  

(3.5)

By determining the normalization of the boundary state according to the path integral method [12] with our boundary action (3.1), we obtain the following boundary state for the bosonic part,

$$|B >_{\text{boson}} = \sqrt{\det(G + F)} \exp \left( \sum_{m=1}^{\infty} e^{2m\tau} \alpha^\mu_m \left( \frac{1 - F}{1 + F} \right)^\mu^\nu \bar{\alpha}^\nu_m \right)|0 >,$$

(3.6)

where the bosonic vacuum is defined in (2.9).

The operator relations for the ghost part are derived from the requirement of the BRST invariance of the boundary state, $(d + \bar{d})|B >= 0$. The BRST operator $d$ is defined as

$$d = \frac{1}{2\pi i} \oint J(z)dz,$$

$$J(z) = : T^\phi(z) + T^X(z) + \frac{1}{2} T^bc(z) :,$$

(3.7)

and $\bar{d}$ can be obtained similarly. As a result, the following relations are obtained,

$$c_n = -\bar{c}_{-n}, \quad b_n = \bar{b}_{-n}.$$  

(3.8)

Then the boundary state for the ghost part is obtained as

$$|B >_{\text{gh}} = \exp \left\{ \sum_{n=1}^{\infty} e^{2n\tau} [\bar{c}_{-n}b_{-n} + c_{-n}\bar{b}_{-n}] \right\} (c_0 + \bar{c}_0)|\downarrow\downarrow >,$$

(3.9)

where $|\downarrow\downarrow >$ is the Siegel vacuum [12] and $<\uparrow\uparrow |\downarrow\downarrow > = 1$. While the left eigenvector, which is consistent with (3.8) for $n < 0$, is found as,

$$<\uparrow\uparrow |(b_0 - \bar{b}_0) \exp \left\{ \sum_{n=1}^{\infty} e^{2n\tau} [\bar{c}_n b_n + c_n \bar{b}_n] \right\},$$

(3.10)

In this way, we obtain the boundary state $|B >$ as

$$|B >= |B >_{\text{boson}} |B >_{\text{ghost}}$$
The next step is to add the cylinder, the propagator of the closed-string, $[L_0 + \bar{L}_0 - 2]^{-1}$, to the boundary. In order to have a non-zero cylinder-amplitude for the vacuum boundary state, we demand that the propagator should be accompanied with the zero modes of $b$ in the form, $-(b_0 + \bar{b}_0)$. Then, we arrive at the following state with a tube of the closed string,

$$|\Psi >_B = -(b_0 + \bar{b}_0)[L_0 + \bar{L}_0 - 2]^{-1} \{ |B > + |C > \}, \quad (3.11)$$

where $|C >$ denotes the boundary state of Mobius strip which gives a different normalization coefficient from that of the annuls $|B >$.

It is easy to derive the relation,

$$(d + \bar{d})|\Psi >_B = |B >_0 + |C >_0, \quad (3.12)$$

where $|B >_0$ and $|C >_0$ denote the zero-mass closed-string state of $|B >$ and $|C >$ respectively. Then if the tachyon is massless, we obtain

$$|B >_0 + |C >_0 = \kappa \sqrt{\det(G + F)(c_0 + \bar{c}_0)} \downarrow \downarrow >, \quad (3.13)$$

where $\kappa$ denotes the product of the string coupling constant and the numerical factor depending on the details of the open-string model considered here.

### 4. Loop Corrected Phase

Before examining the loop-corrected action, we consider the tree level action. The mass-square of the tachyon-field is derived from (2.2) with the vacuum (2.3), and it is given as follows,

$$m_T^2 = 1 - \frac{c}{12}. \quad (4.1)$$

It becomes negative for $c > 1$ as is well-known. However, this situation can be changed if the improved action is considered instead of (2.2), and it is possible to obtain massless tachyon even if $c$ is larger than one as shown below. The reason why this is possible is that the loop-correction given above modifies the tachyon part of the target-space action if we assume the masslessness of the tachyon in the considering region of $c(>1)$. And the consistency of this assumption is assured by examining the tachyon mass which is derived from the corrected target-space action.

The improved action is obtained so that the loop-corrected field equations are derived from this action. The corrected equations of the string-fields are obtained by demanding the cancellation of the conformal anomalies between the one coming...
from the loop-correction and the one obtained by the usual $\alpha'$-expansion. Then we obtain the following equation of the BRST invariance,
\[ (d + \bar{d})|\Psi > = (d + \bar{d})(|\Psi >_{T} + |\Psi >_{B}) = 0, \]
(4.2)
where $|\Psi >_{T}$ represents the fluctuation around the vacuum (2.3),
\[ |\Psi >_{T} = \{ T(x) + h_{\mu\nu}(x)\tilde{\alpha}_{\mu - 1}\alpha_{\nu - 1} + \tilde{\Phi}(x)[\tilde{c}_{-1}\tilde{b}_{-1} + c_{-1}\tilde{b}_{-1}] \} | \downarrow \downarrow >. \]
(4.3)
It is noticed here that the equations obtained from $\alpha'$-expansion for the string fields, $T$, $h_{\mu\nu}$ and $\tilde{\Phi}$, are derived by the equation, $(d + \bar{d})|\Psi >_{T} = 0$. And it implies the target-space action $S_{T}$ of (2.2). While eqs.(4.2) and (3.13) lead to the following equations,
\[ (-\bar{\nabla}^{2} + \frac{1}{4}Q^{2} - 2)T = -\kappa\text{det}^{1/2}(1 + F), \]
(4.4)
\[ (-\bar{\nabla}^{2} + \frac{1}{4}Q^{2})h_{\mu\nu} = (-\bar{\nabla}^{2} + \frac{1}{4}Q^{2})\tilde{\Phi} = 0. \]
(4.5)
\[ \partial^{\mu}h_{\mu\nu} = Qh_{0\nu} + \partial_{\nu}\tilde{\Phi}. \]
(4.6)
where $\bar{\nabla}^{2} = \sum_{i=1}^{c}\partial_{i}^{2} + (\partial_{0} - Q/2)^{2}$ and the third equation denotes the gauge fixing condition in this scheme. Here we notice that the zero-th component of the momentum above has the following correspondence, $p^{0} = \alpha^{0} + iQ/2$. On the other hand, the differential operator has the correspondence, $\partial^{0} = i\alpha^{0}$.

These equations can be obtained from the following target space action,
\[ S_{T}^{\text{eff}} = S_{T} + 2\kappa \frac{1}{4\pi} \int d^{d}X \sqrt{\text{det}(G + F)}\tilde{T}, \]
(4.7)
where $S_{T}$ is given in (2.2) and we set as
\[ \tilde{T} = \exp(-2\Phi)T. \]
(4.8)
This form of $\tilde{T}$ is the simplest one, but it should be noticed that there are many other possibilities of the form of $\tilde{T}$ since the necessary condition of $\tilde{T}$ being satisfied is $\tilde{T} = \exp(-2\Phi(0))T$ when we set the dilaton-field by the vacuum, $\Phi = \Phi(0)$. For example, we might take as $\tilde{T} = \exp(-\Phi - \Phi(0))T [21]$. Then other conditions would be necessary to remove the ambiguity of the form of $\tilde{T}$. In the case of superstring theory, we should take as $\kappa\tilde{T} = e^{-\Phi}T$ since $e^{\Phi}$ represents the string coupling constant which is denoted here by $\kappa$. But we are considering here the vacuum, $\Phi = \Phi(0)$, so the coupling constant part is separated here.

This modified action includes the gauge field $A_{\mu}$, so we must obtain the solution of its variational equation, $\delta S_{T}^{\text{eff}}/\delta A_{\mu} = 0$, simultaneously with $G_{\mu\nu}$, $\Phi$ and $T$. If we consider $A_{\mu}$ as the auxiliary field as mentioned in the previous section, we must integrate it. So the procedure given here is considered as the integration by a
saddle point approximation. The equation for $A_\mu$ is solved under the following ansatz,

$$G_{\mu\nu} = G_{\mu\nu}^{(0)} \equiv \delta_{\mu\nu}, \quad \Phi = \Phi^{(0)} \equiv \frac{1}{2} Q \phi, \quad T = \text{const.}, \quad (4.9)$$

Under this ansatz, the variational equation is written as

$$-Q F_{0\nu} + \sqrt{\det(1 + F)} \partial_\mu \left( \frac{F_{\mu\nu}}{\sqrt{\det(1 + F)}} \right) = 0.$$ 

Here we search for the solution restricting it to the constant field-strength, which is denoted by $\tilde{F}_{\mu\nu}$. Then we obtain the following solution,

$$\tilde{F}_{0i} = 0, \quad \tilde{F}_{ij} = \text{const.} \quad (4.10)$$

and $i, j \neq 0$. The solution (4.10) is corresponding to the magnetic-field condensation in the four dimensional case. Then we substitute (4.10) into $S_T^{\text{eff}}$, and the effective potential for $T$ is obtained as,

$$v^{\text{eff}}(T) = -2T^2 + \tilde{\kappa} T, \quad (4.11)$$

where $\tilde{\kappa} = 2\kappa \sqrt{\det(1 + \tilde{F})}$.

It should be noticed that this result is the same form with the one obtained without the gauge field if we consider $\tilde{\kappa}$ as the modified string coupling constant. Since this potential contains the linear term, $T$ should be shifted by a constant, $T_1$, to remove the tadpole of the tachyon. $T_1$ is obtained by solving the equation, $v''_{\text{eff}}(T_1) = dv_{\text{eff}}/dT |_{T=T_1} = 0$, and we find $T_1 = \tilde{\kappa}/4$. And the mass of the tachyon is obtained as

$$m_T^2 = \frac{25 - c}{12} + \frac{1}{2} v''_{\text{eff}}(T_1) = \frac{1 - c}{12}. \quad (4.12)$$

This is the same with (4.1), so it is impossible to extend the magnetic-field condensed phase to the region $c > 1$. If we consider other form of potential, which is improved at the tree level and includes $T^3$ in (2.2), then we find a mass shift of $T$ of the order $\kappa$ of (21). Then it might be possible to exceed $c = 1$ in this phase. In fact, if we add $T^3/6$ term to (2.2) for example, we obtain $m_T^2 = \frac{1 - c}{12} + \tilde{\kappa}/2$. Then $c = 1 + 6\kappa$ is obtained if we set $m_T = 0$. However, it seems to be impossible to go some finite value of $c(>1)$ within the small-$\kappa$ approximation. As for the solutions of non-constant field strength, we do not examined about them here and the problem related to those solutions is remained open. Here we consider another possibility which is obtained by the condensation of the dual field, whose definition is given below, of $A_\mu$.

In order to search for a new phase, we rewrite (4.7) in terms of the dual field. It is enough to consider only the second term of (4.7) to do this, and it can be rewritten by introducing the auxiliary fields, $\Lambda_{\mu\nu}$ and $f_{\mu\nu}$ as follows,

$$\exp \left( -2\kappa \frac{1}{4\pi} \int \text{d}^4X \sqrt{\det(G + F)} e^{-2\Phi T} \right) =$$
\[ \int Df_{\mu\nu} D\Lambda_{\mu\nu} \exp \left\{ -2\kappa \frac{1}{4\pi} \int d^dX \sqrt{\det(G + f)} e^{-2\Phi T} + \frac{1}{2} i \Lambda^{\mu\nu} (f_{\mu\nu} - 2\partial_{\mu} A_{\nu}) \right\}. \] (4.13)

And the square root in (1.13) can be removed in terms of the formula,
\[ \int_{-\infty}^{\infty} e^{-\frac{a}{2x^2} - bx^2} dx = \sqrt{\frac{\pi}{b}} e^{-\frac{2\sqrt{ab}}{\sqrt{a}}} \] (4.14)
as follows,
\[ \exp \left( -2\kappa \frac{1}{4\pi} \int d^dX \sqrt{\det(G + f)} e^{-2\Phi T} \right) = \int Dv \exp \left\{ - \int d^dX \left[ \frac{1}{2v^2} \det(G + f) + \frac{1}{2} \nu^2 (e^{-2\Phi \frac{\kappa}{2\pi} T})^2 \right] \right\}. \] (4.15)

In this rewriting, the measure of the path integral gains the factor \( \Pi_i (e^{-2\Phi_i T_i}) \), where \( i \) denotes the discretized space-time label. But this factor is regularized out in the dimensional scheme since it gives a volume divergent term in the action, and we can neglect this factor. The determinant part, \( \det(G_{\mu\nu} + f_{\mu\nu}) \), is expressed as
\[ \det(G + f) = \ \begin{cases} \det G + \frac{1}{2} f_{\mu\nu}^2 & \text{for } d=2 \\ \det G + f_{\mu\nu} \Omega_{\mu\nu,\alpha\beta} f_{\alpha\beta} & \text{for } d=3 \end{cases} \] (4.16)
where \( \Omega \) is the \( 3 \times 3 \) matrix depending on the metric \( G_{\mu\nu} \) and it becomes the unit matrix for \( G_{\mu\nu} = \delta_{\mu\nu} \). For \( d = 4 \), the quartic terms of \( f_{\mu\nu} \) appear, so it is difficult to integrate over \( f_{\mu\nu} \) exactly. We discuss this case and the cases of \( d > 4 \) afterward in the weak field approximation, and the explicit form of their determinant are not given here.

As a first example, we show the explicit calculation for \( d=2 \). From eqs. (4.13) \( \sim \) (4.16), we can integrate \( f_{\mu\nu} \) by rewriting the terms depending on it as,
\[ \frac{1}{4v^2} f_{\mu\nu}^2 + \frac{i}{2} \Lambda^{\mu\nu} f_{\mu\nu} = \frac{1}{4v^2} (f_{\mu\nu} + iv^2 \Lambda_{\mu\nu})^2 + \frac{v^2}{4} \Lambda_{\mu\nu}^2. \] (4.17)
After that, we perform the \( v \)-integration, and we obtain
\[ - \int d^dX \left[ \sqrt{\det G} \sqrt{\left( e^{-2\Phi \frac{\kappa}{2\pi} T} \right)^2 + \frac{1}{2} \Lambda_{\mu\nu}^2} - i \Lambda^{\mu\nu} \partial_{\mu} A_{\nu} \right]. \] (4.18)
Then \( \Lambda_{\mu\nu} \) is solved by the constraint, \( \partial_{\mu} \Lambda^{\mu\nu} = 0 \), which is obtained from the integration of \( A_{\mu} \). The solution is found as,
\[ \Lambda^{\mu\nu} = e^{\mu\nu} \tilde{a} \]
where \( \tilde{a} \) is an arbitrary constant. Then we obtain the following correction term,
\[ - \frac{2\kappa}{4\pi} \int d^dX e^{-2\Phi} \sqrt{\det G} \sqrt{T^2 + \left( \frac{2\pi}{\kappa} \tilde{a} \right)^2} e^{i\Phi}. \] (4.19)
And the effective action is obtained by replacing the second term of (4.7) by (4.19). For \( d = 2 \), the dual field is a constant \( \bar{a} \), and we obtain \( \bar{a} = 0 \) by solving its variational equation. Then the effective potential of \( T \) is obtained as

\[
v_{\text{eff}}(T) = -2T^2 + \kappa T
\]

which is the same form with (4.11) obtained for \( F_{\mu \nu} \) condensation phase. Then the tachyon is massless and the consistency with the calculation given here is assured. But we can not find any difference from the magnetic field condensation given above.

Since our interest is in the region of \( c > 1 \), then we consider the three dimensional case \( (c = 2) \) nextly. After the procedure similar to the 2d case, we obtain the following correction term for \( d = 3 \),

\[
-2\kappa \frac{1}{4\pi} \int d^d X e^{-2\Phi} \sqrt{\det G} \left( T^2 + \left( \frac{2\pi}{\kappa} \right)^2 \Lambda_0^{0\mu\nu} \Omega^{\mu\nu,\alpha\beta} \Lambda_0^0 \epsilon^{4\Phi} \right), \quad (4.20)
\]

where

\[
\Lambda_0^{\mu\nu} = \epsilon^{\mu\nu\lambda} \partial_\lambda a(X), \quad (4.21)
\]

and \( a(X) \) is the dual field, which is not a constant but a scalar function in this case. This scalar field \( a(X) \) is solved by the following variational equation,

\[
\partial_k \left( \frac{e^{2\Phi(0)}}{\sqrt{T^2 + \left( \frac{2\pi}{\kappa} \right)^2 \partial_\mu a(2)e^{4\Phi(0)}}} \right) = 0. \quad (4.22)
\]

This equation is obtained by substituting \( G_{\mu\nu} = \delta_{\mu\nu}, \Phi = \Phi(0) \) into the original variational-equation since we are restricting the type of the solution to this kind. Further, we assume the constancy of \( T \) and the condensation of the dual field in the form,

\[
\tilde{a}_k^2 = \tilde{a}_k \tilde{a}^k = (\frac{\Lambda_0^0}{2\pi})^2, \quad (4.23)
\]

where \( \tilde{a}_k = e^{2\Phi(0)} \partial_k a \). Then (4.22) is written as,

\[
Q \partial_\theta a + \partial^2 a = 0. \quad (4.24)
\]

A simple solution of this equation is obtained as follows,

\[
a = e^{\beta \phi} f(X_i), \quad \partial_i^2 f(X_i) = -\beta(\beta + Q)f, \quad (4.25)
\]

where \( \beta \) is a constant and \( f(X_i) \) is a function of \( X_i \) with \( i \neq 0 \). Although it is easy to solve \( f(X_i) \) and to write its general solution, but we abbreviate it since it is not necessary hereafter. The important is the fact that there is an explicit solution of
(4.22) with the condition of (4.23). Taking into account of (4.23), we arrive at the following effective potential,

\[ v_{\text{eff}}(T) = -2T^2 + 2\kappa \sqrt{T^2 + \left(\frac{\Lambda_0}{\kappa}\right)^2} . \]  

(4.26)

According to the discussion given above we must find the shift \((T_1)\) of the tachyon from the above potential. It is given by \(v'_{\text{eff}}(T_1) = dv_{\text{eff}}/dT \big|_{T=T_1} = 0\). We find two branches of the solution,

\[ T_1 = \begin{cases} 0 & \text{(A)} \\ \frac{\kappa}{2} \sqrt{1 - \lambda^2} & \text{(B)} \end{cases} \]  

(4.27)

where \(\lambda \equiv 2\Lambda_0/\kappa^2\). Here we assume that \(\Lambda_0\) is the order of \(\kappa^2\) so that \(\lambda\) becomes finite. As for \(T_1\), it is the order of \(\kappa\) or zero. This setting is consistent with our approximation of the calculation. For these solutions, the tachyon-mass is obtained as

\[ m_T^2 = \frac{25 - c}{12} + \frac{1}{2} v''_{\text{eff}}(T_1) \]  

(4.28)

where

\[ v''_{\text{eff}}(T_1) = \begin{cases} \frac{1-c}{12} + \frac{2}{\lambda} & \text{(A)} \\ \frac{1-c}{12} + 2\lambda^2 & \text{(B)} \end{cases} \]  

(4.29)

Since the above calculation has been done for \(d = 3\) \((c = 2)\) and \(m_T = 0\), then we obtain

\[ \lambda = \begin{cases} \frac{24}{c-1} = 24 & \text{(A)} \\ \frac{1}{2\sqrt{6}} & \text{(B)} \end{cases} \]  

(4.30)

for each branch. Both cases are acceptable as far as we consider the \(d = 3\) case. While, if we try to extend the above results to the \(c = 1\) limit, the branch (B) is smoothly connected to the previous \(c = 1\) result, \(T_1 = \kappa/2\) and \(\Lambda_0 = 0\) at the limit \(\lambda = 0\). On the other hand, the branch (A) produces a large gap at \(c=1\) with the solution obtained at \(d = 2\). In this sense, the (B)-branch is preferable in order to extend this phase to the wider region of \(c\), and it seems to be expandable to the region of \(25 \geq c \geq 1\). The upper bound of \(c\) comes from the reality condition of \(T_1\). We can interpret this phenomenon as a phase transition where the transition point is \(c = 1\) and the order parameter is not \(T_1\) but \(\Lambda_0\).

Of course, there are other solutions of (4.22) which do not lead to the potential (4.26). One simple example is obtained as follows. Rewrite (4.22) as

\[ \partial_k E^k = 0 , \]  

(4.31)

where \(E_k = \tilde{a}_k/\sqrt{T^2 + (\frac{2\pi \kappa}{\kappa})^2 \tilde{a}_k^2}\) and \(\tilde{a}_k = e^{2\Phi(\eta)} \partial_k a\) which is given above. Then the solution is obtained as

\[ E^k = e^{kij} \partial_i \eta_j \]
where $\eta_i$ is an arbitrary function, and we obtain the following equation,

$$\tilde{a}_k^2 = T^2 \frac{E_k^2}{1 - \left(\frac{2\pi}{\kappa}\right)^2 E_k^2}.$$

Here,

$$E_k^2 = E_k E_k = \frac{1}{2} f_{ij}^2(\eta)$$

$$f_{ij}(\eta) = \partial_i \eta_j - \partial_j \eta_i. \tag{4.33}$$

Next, we assume the condensation of $f_{ij}(\eta)$ such that

$$< E_k E_k > = \alpha^2$$

where $\alpha$ is a constant. Then we obtain,

$$\tilde{a}_k^2 = T^2 \frac{\alpha^2}{1 - \left(\frac{2\pi}{\kappa}\right)^2 \alpha^2}. \tag{4.34}$$

Substituting this into the target-space action, we get the same form of potential with (4.11) but the coefficient of the linear term of $T$ is different. So this solution leads to the negative mass-squared of the tachyon.

Now we turn to the case of $d = 4$ (or $c = 3$), we obtain in this case

$$\det(G + f) = \det G + f_{\mu\nu} \omega^{\mu\nu\alpha\beta} f_{\alpha\beta} + f_{\text{qrt}}, \tag{4.35}$$

where $\omega$ is 6×6 matrix written by the metric $G_{\mu\nu}$, and $f_{\text{qrt}}$ represents the quartic terms of $f_{\mu\nu}$. Eq.(4.35) can be explicitly written for the flat metric, $G_{\mu\nu} = \delta_{\mu\nu}$, as follows

$$\det(G + f) = \det G + \frac{1}{2} f_{\mu\nu}^2 + \left(\frac{1}{2} f_{ijkl} f_{ijkl}\right)^2, \tag{4.36}$$

where $i, j, k = 1 \sim 3$. Due to the quartic term of $f_{\mu\nu}$, it is difficult to perform the complete integration over $f_{\mu\nu}$. Then we concentrate our attention on the weak field region where $f_{\mu\nu}$ is small and the quartic term can be neglected compared to the quadratic one. In this case, we can proceed the similar calculation to the one performed in $d = 2, 3$ cases, and the same form of the result with (4.20) is obtained by replacing the matrix $\Omega$ and $\Lambda_{\mu\nu}^0$ in (4.20) by $\omega$ and

$$\Lambda_{\mu\nu}^0 = \epsilon^{\mu\nu\lambda\sigma} \partial_{\lambda} \tilde{A}_\sigma, \tag{4.37}$$

respectively. Here $\tilde{A}_\mu$ represents the dual gauge field. Similarly to the case of $d = 3$, we obtain the following variational equation with respect to $\tilde{A}_\mu$,

$$\partial_\mu \left( \frac{e^{2\phi(0)} \tilde{f}_{\mu\nu}}{\sqrt{T^2 - \frac{2\pi^2}{\kappa} \left(\frac{e^{4\phi(0)}}{8}(f_{\mu\nu})^2 e^{4\phi(0)}\right)}} \right) = 0, \tag{4.38}$$
where \( \tilde{f}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \). As in the previous case, we solve this equation under the condition of constant \( T \) and

\[
e^{2\Phi} \tilde{f}_{\mu\nu} = \epsilon_{\mu\nu\lambda\sigma} \tilde{f}^{\lambda\sigma}, \quad \pi^2 \tilde{f}_{\mu\nu} = \Lambda_0^2, \tag{4.39}\]

where \( \Lambda_0 \) is a constant. This ansatz is taken so that we can obtain the same form of potential with (4.26). Then (4.38) is written as

\[
\epsilon_{\mu\nu\lambda\sigma} \partial_\mu \tilde{f}^{\lambda\sigma} = 0
\]

Then a simple solution is obtained as \( \tilde{f}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu \), where \( \tilde{A}_\mu \) is an arbitrary function. Since we do not need the explicit form of \( \tilde{A}_\mu \), it is not necessary to specify it here as in the \( d=3 \) case. The important is the fact that we can arrive at the same effective-potential with (4.26), and the values of \( T_1 \) and \( m_T \) are given by the same formula with (4.27) and (4.29). The difference is the value of \( c \), which is taken at \( c = 3 \) here. As in the case of \( c = 2 \), both branches are possible, but we prefer (B)-branch since it can be connected to \( c = 1 \) as stated above. As a result, the massless tachyon phase is realized also in \( d = 4 \) due to the condensation of the dual field.

Within the weak field approximation, it is straightforward to extend the analysis of \( d = 4 \) to the cases of \( d > 4 \) by taking only the quadratic term of \( f_{\mu\nu} \) into account. In this way, we can extend the formula (4.26) \( \sim \) (4.30) to the region of \( d \geq 4 \) or \( c \geq 3 \) within the weak field approximation.

Finally we remind the vacuum configuration of the other fields. It is derived from \( S_{\text{eff}}^T \) as follows,

\[
G_{\mu\nu} = \delta_{\mu\nu}, \quad \Phi = \frac{1}{2} Q \phi, \quad T = T_1 . \tag{4.40}\]

Here \( Q \) and \( T_1 \) vary with \( c \), but this form of (4.40) is not changed for \( c \geq 1 \). We should notice that non-zero value of \( T_1 \) is not essencial to the vacuum structure. The essential point is the value of \( \Lambda_0 \). The origin of \( \Lambda_0 \) is however hidden in the target-space action, which controls the dynamics of the closed string and determines the form of its world-sheet action. From the viewpoint of the 2d gravity, the above analysis implies the importance of the interaction of open universe and the closed universe by taking a picture of interacting many universes. This picture could open the way to arrive a 2d gravitational theory of \( c > 1 \).

5. Concluding remarks

Quantum fluctuation, which is corresponding to the loop-correction of the open string, of 2d surface is considered to find a vacuum of the non-critical string theory for \( c > 1 \). The calculational technique developed in the critical super-string theory
is applied. The loop-correction is given in terms of the boundary state accompanying a tube of the closed string propagator. The important point is that this correction provides a conformal anomaly for the massless-state channel of the closed string states propagating the tube. As a result, the field equation of the tachyon (the ground state of the closed-string) is modified if the tachyon is assumed to be massless even for $c > 1$. This assumption is justified by obtaining the zero-mass tachyon state from the effective target-space action which is improved by the loop-correction.

The essential factor to get this justification is the the condensation of the dual field of the gauge field, which is in the corrected action of the Born-Infeld type. The gauge field lives on the boundary of the world-sheet and is also needed from the gauge invariance on the world-sheet with boundaries for the antisymmetric tensor of the closed-string states. However, we could not obtain a phase, where there is no $c=1$ wall, by the condensation of this gauge field directly. Such a phase has been found in the vacuum, where the dual field of this gauge field condenses. It is found that the ground state of the closed string is the massless tachyon in this phase. This is seen in terms of the dual transformed target-space action. For $d = 2, 3$, the dual transformation is exact. But, it is performed for $d \geq 4$ by the weak field approximation. Within this approximation, the phase found here seems continuing to exist up to the critical dimension $c = 25$, where the Liouville field disappears. So the condensation of the dual field is essential to this phase where there is no $c = 1$ wall.

Acknowledgment: The author thanks for the encouragement of T.Yukawa and H.Kawai at the conference held in October, 1996 at Shikanosima. He also thanks K.Inoue for useful discussions.
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