A practically feasible entanglement assisted quantum key distribution protocol

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We give an entanglement assisted scheme for quantum key distribution. The scheme requires the maximally entangled 2-qubit state but does not require any quantum storage. The protocol is unconditionally secure under whatever type of Eve’s attack. The threshold of the channel error rate for the protocol to produce larger than zero final bit depends on the noisy pattern. In particular, given the symmetric channel with independent errors, our scheme can tolerate a bit error rate up to 26% in the 4-state case and 30% in the 6-state case, respectively. These values are higher than those of all currently known two-level-state schemes without using a quantum storage.

I. INTRODUCTION

Due to the Heisenberg uncertainty principle, quantum key distribution (QKD) is different from classical cryptography in that an unknown quantum state is in principle not known unless it is disturbed, rather than the conjectured difficulty of computing certain functions. The first published protocol, proposed in 1984 [1], is called BB84 after its inventors (C. H. Bennett and G. Brassard.) For a history of the subject, one may see e.g. [2]. In this protocol, the participants (Alice and Bob) wish to agree on a secret key about which no eavesdropper (Eve) can obtain significant information. Alice sends each bit of the secret key in one of a set of conjugate bases which Eve does not know, and this key is protected by the impossibility of measuring the state of a quantum system simultaneously in two conjugate bases. Since then, studies on QKD are extensive.

Most of the entanglement assisted QKD protocols such as the Lo-Chau protocol [3] are not so practical compared with the prepare-and-measure protocols such as BB84 [1], 6-state protocol [4], the Gottesman-Lo protocol [6,16] and so on because normally the entanglement assisted protocols require a quantum storage which is generally believed to be technically difficult. However, producing the maximally entangled pairs is not a problem by our current technology. Maximally entangled pairs in polarization space can be robustly produced by the type two spontaneous parametric down conversion (SPDC) [5]. The proposed QKD protocol in this work does not require any quantum storage, though it uses the entanglement pairs and the collective measurement. The collective measurement can be done by using a polarizing beam splitter as we shall show in the section of “experimental realization”. Therefore it is practically implementable by the currently existing technology.

Normally, the channel for quantum bit transmission is noisy. The error to a qubit caused by the channel noise can be divided into $\sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ errors, which represent for a bit flip error only, a phase flip error only and both error, respectively. The detected bit (or phase) flip error rate is the summation of $\sigma_z$ (or $\sigma_y$) error rate and $\sigma_y$ error rate. The most natural noisy channel is the symmetric channel, i.e., the $\sigma_x, \sigma_z, \sigma_y$ errors are equally distributed.

We assume that the channel noise is symmetric, one must use 6-state protocol. In the BB84 protocol, since the $\sigma_y$ error can never be detected or deduced, we have to assume the worst case of zero $\sigma_y$ error [6]. This is why the 4-state protocol can only tolerate a lower error rate than 6-state protocol. This is the reason that the largest tolerable bit flip error rate or phase flip error rate for BB44 protocol is 25%, lower than that of 6-state protocol, which is 33.3% [6]. However, in our entanglement assisted protocol, the channel flipping rate upper bound of 25% for the 4-state protocol is broken. The tolerable channel bit-flip and phase-flip rate is raised to 26% for the symmetric channel. This is not a strange result if we consider an entanglement purification protocol with bit flip error rejection and phase flip error rejection alternately: even though $\sigma_y$ error is never detected or deduced, the tolerable bit flip rate or phase flip rate can be as high as 33.3% for the entanglement distillation. In such a case one does not have to make sure of the $\sigma_y$ rate, he can check the bit flip rate and phase flip rate after the purification. If initially the error rate is indeed symmetrically distributed and the total flipping rate is less than 50%, he must be able to distill highly pure entangled state finally. This means, with the real entanglement purification, a 4-state protocol can tolerate a flipping rate of 33.3% if the channel noise is symmetric. (One does not have to test whether the channel noise is symmetric in the protocol, Alice and Bob just go ahead to make the purification even though the rate of $\sigma_x$ and the rate of $\sigma_z$ are

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larger than 25%. If the channel is symmetric, he will be able to verify that both bit flip rate and phase flip rate are 0 after the purification). In our entanglement assisted protocol, the tolerable bit flip rate or phase flip rate for the 6-state protocol is also raised to 30%. Our entanglement assisted protocol can tolerate the highest error rate among all proven secure prepare-and-measure protocols raised so far for both 4-state case and 6-state case.

Practically speaking, a quantum channel is normally noisy therefore the raw key string with Bob and the string with Alice are normally not identical. Moreover, it is possible that, in the case Alice and Bob try to make the quantum key distribution, the channel noise could actually come from Eve. In such a case the raw key can be significantly correlated with Eve’s quantum state. Even in the case that no bit flip or phase flip is detected on the subset of the check bits, the raw key is still not completely reliable or secure because there could be still a few bit flips or phase flips for the code bits which are unchecked. To obtain the highly reliable and secure key, one has to take the error correction (EC) and privacy amplification (PA) to the raw key and then use the final key which should be unconditionally secure and perfectly reliable. Although the classical EC can be used to remove all bit-value errors (with a high probability) therefore to help Alice and Bob obtain a reliable key, it’s not transparent that whether the classical PA can really work here: Eve may first store her qubits which are correlated with the raw key. After Alice and Bob complete the EC and PA, she then takes an optimized measurement to her qubits directly attacking the final key. A strict mathematical proof for the unconditional security is non-trivial [7–9]. The security proof of QKD is greatly simplified if one connects this with the quantum entanglement purification protocol (EPP) [10–12]. The main idea is conceptually simple and clear: Alice and Bob first share a number of raw entanglement pairs and then purify them to almost maximally entangled pairs and measure each of them to obtain the final key [12]. The strict mathematical security proof with EPP was given by Lo and Chau [3]. Interestingly, it was then shown by Shor and Preskill [13] that Lo-Chau entanglement purification based protocol can be reduced to the quantum error correction (QEC) protocol with one-way communication and the QEC protocol is equivalent to BB84 protocol followed by the classical EC and PA done by decoding a classical CSS code. Shor-Preskill protocol [13] works as long as the measured bit flip error and phase flip error rates are less than 11%, the point at which the Shannon rate hits 0. Note that this threshold is lower than that of certain EPP protocols with two-way communications (2-EPP): the 2-EPP with error rejection works as long as the summation of measured bit error and flip error rates are less than a 50%. In such a case, Alice and Bob randomly choose two pairs and measure the parity of 2 qubits in each side and discard both pairs if the parities disagree and keep one pair and discard the other pair if the parities agree with each other. In such a case, the bit flip error is reduced if they measure the parity in Z basis; the phase flip error is reduced if they measure the parity in X basis. X, Y and Z represents the basis in the eigenstates of operator $\sigma_x, \sigma_y$ and $\sigma_z$ respectively. The vector representation for the two level state is $| 0 \rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $| 1 \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This 2-EPP with error rejection done in both Z basis and X basis cannot be reduced to a classical protocol. In a classical protocol, Alice just sends Bob the qubits prepared in X or Z (or Y) basis randomly and then they carry out EPP task as if they were sharing a number of raw entangled pairs. Or equivalently, in a classicalized protocol, Alice had measured her halves of the entangled pairs before the protocol was started. They can still do the parity comparison on Z basis, but they will not be able to do so in X basis: they are never able to know what the parity values should be if they had really used entangled pairs here therefore they don’t know whether they should discard both bits or keep one bit. Very recently, motivated for higher bit error rate tolerance and higher efficiency, Gottesman and Lo [6] studied the classicalization of 2-EPP. It has been shown there that a 2-EPP can be classicalized iff their action after the phase parity comparison is deterministic. In such a case they can carry out the task as if they were doing the 2-EPP on entangled pairs. Based on this observation, a new QKD protocol was given there [6] with partially two way communications: in removing the bit flip errors, the error-rejection method is used with two way communication; in removing the phase flip error, the error correction method is used with one way communication, i.e., Alice asked Bob to measure the syndrome of certain randomly chosen 3 qubits and Bob will use the majority rule to decide whether to take a phase flip operation to one of the 3 qubits. This method has increased the tolerable bit error rate of noisy channel to 18.9% and 26.4% for 4-state QKD and 6-state QKD, respectively. Very recently, these values have been upgraded to 20% and 27.4% by Chau [16].

In this paper, we propose a revised scheme which is also unconditionally secure and which can further increase these thresholds on bit error rates given the independent channel errors. We propose to let Alice send Bob the quantum states randomly chosen from \{\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), |00\rangle, |11\rangle\}. As we shall see, these states are just the quantum phase-flip error-rejection (QPFER) code for the BB84 state \{\{0\rangle, |1\rangle, \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}. When Bob receives them, he first decodes each two-qubit code to the one-qubit state (a BB84 state) and then carry out the rest tasks of EC and PA. In decoding, Bob discards those codes which contains one bit-flip error therefore after decoding the phase-flip error for the accepted bits are greatly decreased. As we are going to show, we may take this advantage to increase the threshold of the tolerable bit error rate caused by the s-channel. The advantage of a higher tolerable error rate is limited to the cases where the 2-qubit QPFER code works, e.g., in the case that the channel noise is uncorrelated. But security part of our protocol is unconditioned, i.e., it is secure under whatever types of
attack. In other words, Alice and Bob don’t care about the actual noise pattern (correlated or uncorrelated) in the QKD, they just test the error rate after decoding and then go ahead to distill the final key by our protocol. The final key is always secure no matter what type of noise has been actually happened to the raw qubits. Subtle points for the unconditional security with conditional advantage of the protocol is given in the following section. In particular, the protocol is totally different from the almost useless protocol which is only secure with uncorrelated channel noise.

II. SUBTILITIES OF UNCONDITIONAL SECURITY WITH CONDITIONAL ADVANTAGE OF OUR PROTOCOL

To a QKD protocol, one may evaluate it by several criterion, e.g., the security, the efficiency and the key rate. The unconditional security is the first important thing. Without this property, the protocol is almost useless even if it has other advantages such as a high key rate. After all, the reason we use quantum key distribution is due to the temptation of its unconditional security. In this logic, a conditional secure QKD protocol with unverifiable condition for security is useless, because it is actually insecure and has lost the assumed advantage to a classical key distribution.

For example if a new protocol is proposed with advantages in efficiency or economy, but the security is only proved based on the assumption of individual attack, this new protocol is useless unless the proof of unconditional security is completed, i.e., the proof of the security under whatever type of attack.

An unconditionally secure QKD protocol means that Eve’s information to the final key is exponentially close to 0. To Alice and Bob, if they carry out that protocol, the worst result there is that Alice and Bob may get 0 bit for the final key, i.e., when the channel noise (including the disturbance caused by Eve’s attack) is higher than the threshold of the protocol itself. The “efficiency” is defined by the threshold of the channel noise for the protocol. If the channel noise is larger than that, Alice and Bob will obtain zero bit for their final key. In almost all proven unconditionally secure protocols so far, the efficiency is dependent on the channel error rate, but it is independent of the channel error pattern itself, i.e., no matter it is a coherent error or uncorrelated error, the threshold is the same. These protocols are unconditional secure with threshold value of noise which is universal to all noise patterns, correlated or uncorrelated.

Here in our protocol, the threshold is dependent on the channel error pattern while the security is independent of the error pattern (which is required by the term “unconditional security”). Note that our protocol is totally different from the ones with a conditional security. Our protocol is unconditionally secure with a conditional advantage in noise threshold.

Security proof of a QKD protocol is normally related to the channel noise. Here we shall use the term “s-channel” for the physical channel which is supposed to transmit the qubits from Alice to Bob. This s-channel can be an optical fiber, or can be just the nature. We shall use the term “r-channel” for the actual channel in the QKD, the noise of “r-channel” is supposed to be under control of Eve. in the security proof. In particular, if there is no Eve., r-channel is just s-channel. Suppose Alice and Bob decide to do QKD through a certain s-channel (e.g., optical fiber F) by using Gottesman-Lo protocol [6,16]. Before they do so they have been very sure of the properties of their s-channel noise. They find the channel noise is all independent to each qubit and the bit error rate of that s-channel is 21%.

(It is to many people’s intuition that the errors should be independent if the different qubits are spatially separated sufficiently. Also we can choose to use the type of fiber which only causes independent errors to qubits.) In such a case, if we are limited to the previously proposed 4-state prepare-and-measure protocols, we can do nothing for the quantum key distribution because the error rate is larger than the thresholds of all those protocols. However, in our protocol, Alice will first encode each BB84 qubit by a 2-qubit quantum error-rejection code and then send each quantum code to Bob. Bob will first take the parity check and decoding after he receives the codes. He will test the error rate and distill the final key with Alice after decoding. Due to the fault tolerance property of the error rejection code, the error rate of the raw bits after decoding is expected to be reduced and the net effect here is that our 4-state protocol can tolerate an error rate up to 26% for the s-channel. In the supposed case of 21% error rate for the symmetric s-channel where all errors are independent, our protocol will still work.

The advantage of a higher error threshold is not an unconditional advantage. It is only for the cases where the 2-qubit error rejection code indeed works. One example is that the channel only causes uncorrelated errors. However, the security of our protocol is unconditional. That is to say, no matter what type of attack Eve. may use, the final key obtained by our protocol is always secure. In a real QKD process the qubits could be transmitted through Eve. or the property of s-channel could be changed due to Eve’s attack therefore the 2-qubit error-rejection code does not work properly as what is expected. In such a case, after decoding, Alice and Bob will find that the error rate is unexpectedly high and they will choose to either abort all the raw bits or continue the rest steps of the protocol and obtain a shorter final key to which Eve.’s information is exponentially close to 0. But in cases there is no Eve., or Eve.’s attack is insignificant, the 2-bit error-rejection code works perfectly or almost perfectly, the advantage of our protocol appears: one can obtain the final key in the case that the s-channel noise is higher than the thresholds of all previously known secure prepare-and-measure protocols.
If Eve always disturbs the qubits very significantly, our protocol will not work at all even though the error rate of s-channel is lower than the threshold. However, in such a case no protocol will work. A quantitative comparison of different protocols on Eve’s cost to destroy a protocol or to shorten the final key is not a topic of this work.

III. GOTTESMAN-LO PROTOCOL: ERROR-REJECTION AND ERROR-CORRECTION

There are many ways to do entanglement purification. However, not all of them can be used for the security proof of a QKD protocol without using quantum storage. To carry out such a task one must first study an entanglement purification based QKD protocol and then show that the protocol is classicalizable, i.e., to show that it is equivalent to the case where Alice measures all her halves of entangled pairs initially and continue with all other steps in the protocol. Different types of EPP may tolerate different flipping rates of the channel. We now first analyse the reason why the currently existing protocols [6,16] does not reach the theoretically allowed threshold of the channel flipping rate. To prove a secure QKD protocol, one may first consider an entanglement purification protocol and then find out the corresponding QKD protocol based on that. In general, one has two simple ways to purify entangled pairs shared by spatially separated parties, Alice and Bob. One method is the error-rejection [10]: The raw pairs are randomly divided into many 2-pair groups. To each group, Alice and Bob measure the parity on each side. If they obtain the same parity value, they discard the target pair and keep the control pair (see Fig. 1). If the parity values are different, they discard both pairs. In doing the parity check, they can choose the measurement basis of $Z_1Z_2$ to reduce the bit flip errors or the basis of $X_1X_2$ to reduce the phase flip errors. Suppose the initially shared raw pairs between Alice and Bob bear the $\sigma_x, \sigma_y, \sigma_z$ errors are $p_x, p_y, p_z$, respectively. Let $p_1 = 1 - p_x - p_y - p_z$. After one round of bit-flip error-rejection, the error rate for the survived pairs is changing by the following iteration formula as given by Chau [16]:

$$
\begin{align}
  p_{1EP} &= \frac{p_1^2 + p_z^2}{(p_1 + p_z)^2 + (p_x + p_y)^2}, \\
  p_{xEP} &= \frac{p_x^2 + p_y^2}{(p_1 + p_z)^2 + (p_x + p_y)^2}, \\
  p_{yEP} &= \frac{2p_xp_y}{(p_1 + p_z)^2 + (p_x + p_y)^2}, \\
  p_{zEP} &= \frac{2p_1p_z}{(p_1 + p_z)^2 + (p_x + p_y)^2}.
\end{align}
$$

Exchange $p_x$ and $p_z$ above we can obtain the iteration formula by phase error-rejection operation. By taking the error-rejection operation alternately in $Z$ basis and $X$ basis for many rounds, one can always distill out maximally entangled pairs asymptotically from the raw state $\rho$ if $\langle \Phi^+ | \rho | \Phi^+ \rangle > 1/2$, where $| \Phi^+ \rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. This is to say, with the error-rejection method, the theoretical upper bound of tolerable channel flipping rate for QKD distribution can be reached. For the symmetric channel, the bit-flip rate or the phase-flip rate can be as high as 33.3% no matter whether we use the 4-state protocol or 6-state protocol. However, the error-rejection method above cannot be classicalized to the prepare-and-measure QKD protocol since if Alice had measured her halves in $Z$ basis initially, Alice and Bob would have no way to take phase-flip error-rejection. Note that the error-rejection method means they have to decide whether they should keep one bit or discard both bits according to the measurement result.

To overcome this barrier, Gottesman and Lo proposed to use the error-correction method to reduce the phase-flip error: they correct a possible error by $[3,1,3/2]$ code instead of discarding the the corrupted pairs, once a possible error is detected. To do so, Alice and Bob randomly choose 3 pairs in one group. They each measure the parity of qubits 2,3 and qubit 1,2 and compare both values (Fig. 2). After the comparison, they decide whether to take a flip operation to pair 1. They keep pair 1 and discard pair 2 and pair 3. With a high probability that pair one is now free of phase flip error if the parity checks are done in $X$ basis. However the tolerable initial error rate is decreased with this method towards the distillation of maximally entangled pairs. This implies that the error-rejection can be more effective than error-correction in the entanglement distillation.

In Gottesman-Lo QKD protocol, Alice and Bob distill the classical bits as if they were distilling the entangled pairs: no one knows whether Alice had measured her halves of the entangled pairs initially. Therefore the security of the protocol is equivalent to that in a real entanglement distillation protocol, which has been proved to be unconditionally secure [3].
Gottesman-Lo protocol reduces the bit flip error by error-rejection and reduces the phase flip error by error-correction. Although in the case of real entanglement distillation, Alice may often need to phase flip certain qubit according to the parity comparison result for phase-flip error-correction in the entanglement purification protocol. However, if the final purpose is to set up the faithful and secure key only, Alice need not really take any phase flip. Instead, she may simply use the parity of randomly chosen 3 bits as the new bit value after “error-correction” (see Fig.2).

The phase flip does not affect the final bit result therefore omitting the phase flip will not affect the reliability of the final key. That is, the final key is as faithful as that in the case Alice takes phase flip to her qubits as required by the standard entanglement purification. Moreover, one can also find that omitting the phase flip operation does not affect the security either. Consider the case that Alice never takes phase flip to her qubit but she keeps the information in the mind that which qubit should be phase flipped and then continue the distillation, they will finally obtain \(|\chi_1\rangle \otimes |\chi_2\rangle \cdots |\chi_n\rangle\). Each \(|\chi_i\rangle\) is either \(|\Phi^+\rangle\) or \(|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\) and Alice explicitly knows the state of each pair (with a probability exponentially close to 1). That is to say, if Alice never takes any phase flip, Alice and Bob will share a product of different maximally entangled state. Note that the shared pairs are in a pure state. Therefore those shared pairs, with a probability exponentially close to 1, are unentangled with any third party. The security in such a case is totally equivalent to the case where all shared pairs are in state \(|\Phi^+\rangle\). Further, even Alice never keeps any information in her mind on which qubit should be phase flipped, the security is unchanged. Because even in such a case, the finally distilled pairs are also unentangled with any third party. The only thing that is important here is that Alice would be able to know the state of finally distilled pairs if she had remembered the information on which qubit should be phase flipped.

One may further reduce the above protocol. Taking the fact that the phase flip operation can actually be ignored, Alice can then choose to measure each of her halves of EPR pair in Z basis initially and send the other halves to Bob and then carry out the same entanglement purification scheme as if they were sharing the entangled pairs. This reduction should not affect the reliability of Bob’s final key because it does not affect any bit value in Z basis. Note that the operation commutes with all operations taken by Alice latter on. This reduction should be as secure as before because Eve will never know whether Alice had measured her halves of EPR pairs in the beginning, even with the full collaboration with Bob. Suppose the reduction at this step is insecure. We have already known that the final key is secure if Alice had not measured her halves of pairs . Then we shall get the confliction that Eve. and Bob will be able to know whether Alice had measured her qubits in the beginning by checking whether Bob’s final key is significantly correlated with Eve’s assumed key.

From the above analysis we spot one important fact in Gottesman-Lo protocol [6] and its modified version [16]: Alice and Bob are only allowed to take error-correction but not allowed to take error-rejection operation to treat the phase flip error, otherwise it cannot be classicalized. But error-rejection operation can be more effective in removing errors. This is the reason why the error rate threshold cannot reach the theoretically allowed value, i.e., the threshold for the real entanglement distillation with all error-rejection method.

IV. PHASE-FLIP ERROR REJECTION WITH 2-QUBIT ERROR REJECTION CODE.

To improve the tolerable channel error rate threshold in quantum key distribution, one may naturally consider the possibility of using error-rejection method to reduce the phase error also. Naively, one may consider the real 2-EPP. But that requires the quantum storage which is obviously impractical with our current technology. However, with the quantum error-rejection code as we are going into, one can take the error-rejection quantumly therefore any quantum storage is unnecessary. However, if we want to use quantum phase error-rejection code to remove the phase flip error in all rounds, we must then use a large concatenated quantum code which is also impractical. Keeping this point in mind, we choose to use quantum error-rejection only at the first round therefore we only need a 2-bit quantum code to encode each initial qubits with Alice:

\[
\begin{align*}
|0\rangle|0\rangle &\rightarrow (|00\rangle + |11\rangle)/\sqrt{2} \\
|1\rangle|0\rangle &\rightarrow (|00\rangle - |11\rangle)/\sqrt{2}.
\end{align*}
\] (2)

Here the second \(|0\rangle\) state qubit in the left hand side of the arrow is the ancilla for the encoding. This code is not assumed to reduce the errors to qubits in all cases. But in the case that the channel noise is uncorrelated or nearly uncorrelated, it works effectively. We shall consider the ideal case that the channel errors are uncorrelated here. Therefore the initial perfect EPR pair \(|\Phi^+\rangle\) with Alice is encoded by the following formula before sending half of it Bob over noisy channel:

\[
E[|\phi^+\rangle] = \frac{1}{2} \left[ |0\rangle_A(|0\rangle_B 1|0\rangle_{B2} + |1\rangle_B |1\rangle_{B1} |1\rangle_{B2} + |1\rangle_A |0\rangle_B |0\rangle_{B2} - |1\rangle_B |1\rangle_{B1} |1\rangle_{B2} \right]
\] (3)
Alice then sends qubits B1 and B2 to Bob. In receiving them, Bob first takes a parity check, i.e. measures $Z_1 Z_2$. Note that this measurement does not destroy the code state itself. Moreover, any linear superposition of $|00\rangle$ and $|11\rangle$ is an eigenstate of $Z_1 Z_2$ with eigenvalue 0; any linear superposition of $|01\rangle$ and $|10\rangle$ is an eigenstate of $Z_1 Z_2$ with eigenvalue 1. If he obtains 1, Alice and Bob discards the code, if he obtains 0, Bob decodes the code: he takes a measurement in $X$ basis to one qubit (say, $B_1$), if he obtains $|+\rangle$, he takes a Hadamard transformation $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ to the other qubit ($B_2$); if he obtains $|-\rangle$ for qubit $B_1$, he takes the Hadamard transformation to qubit $B_2$ and then flips qubit $B_2$ in $Z$ basis. In such a way, Alice and Bob may share the raw pairs of qubit A and B2. The phase flip error to the survived raw pairs will be decreased. Suppose the channel error rates of $\sigma_x, \sigma_y, \sigma_z$ types are $p_{x0}, p_{y0}, p_{z0}$, respectively. Let $p_{10} = 1 - p_{x0} - p_{y0} - p_{z0}$. With a probability of $p_{10}$ there is no error to both qubit $B_1$ and $B_2$. In such a case, the state after decoding is exactly $|\Phi^+\rangle$. Explicitly, with no flip happens the total state for $A, B_1$ and $B_2$ before decoding is

$$E[\Phi^+] = |0\rangle_A |(+\rangle B_1 |(+\rangle B_2 + |−\rangle B_1 |−\rangle B_2) + |1\rangle_A |(+\rangle B_1 |−\rangle B_2 + |−\rangle B_1 |+\rangle B_2).$$

(4)

Note that the total state is symmetric to qubit $B_1$ and $B_2$. Suppose Bob takes measurement in $X$ basis to qubit $B_1$ for the decoding. From the formula above we can see that the state for qubits A and B2 will be collapsed to $|0\rangle |+\rangle + |1\rangle |−\rangle$ if he obtains $|+\rangle$ for qubit $B_1$, after a Hadamard transformation, the shared pair is changed to $|\Phi^+\rangle$ for sure. Qubits A and B2 will be collapsed to $|0\rangle |+\rangle - |1\rangle |−\rangle$ if he obtains $|−\rangle$ for qubit $B_1$. In such a case, he will first take a Hadamard transformation to qubit B2 and then flip it in Z basis. After these operations, the shared pair is also changed back to state $|\Phi^+\rangle$ for sure. If one of the transmitted qubits bears a bit flip error (including both $\sigma_x$ and $\sigma_y$ error) while the other transmitted qubit does not bear the bit flip error (e.g., it can be of no error or of $\sigma_z$ error), the code will be discarded for sure after the parity check. The probability for this type of event is $2(p_{x0} + p_{y0})(p_{10} + p_{z0})$. If both transmitted qubits bear bit flip errors with arbitrary phase flip errors, or none of them suffer a bit flip error with arbitrary phase flips, the 2-qubit code will pass the parity check and be accepted. In such a case Alice and Bob will share a wrong state after decoding. However, the probability for such cases are small. We list the probability distribution for all possible states after decoding in the following table:

| joint channel error | probability | state after decoding | raw pair error |
|---------------------|-------------|---------------------|---------------|
| $I \otimes I$       | $p_{10}$    | $|\Phi^+\rangle$    | $I$           |
| $\{I \otimes \sigma_x\}$ | $2p_{10}p_{z0}$ | $|\psi^-\rangle$ | $\sigma_x$ |
| $\sigma_y \otimes \sigma_z$ | $p_{y0}$ | $|\phi^-\rangle$ | $\sigma_z$ |
| $\{\sigma_x \otimes \sigma_y\}$ | $2p_{z0}$ | $|\phi^-\rangle$ | $\sigma_z$ |
| $\sigma_x \otimes \sigma_z$ | $2p_{z0}$ | $|\psi^-\rangle$ | $\sigma_y$ |

We use $\{\alpha \otimes \beta\}$ to denote both $\alpha \otimes \beta$ and $\beta \otimes \alpha$ in the most left column above. There are other types of joint errors to the two transmitted qubits besides those listed in the above table. However, all codes with those types of unlisted joint channel errors will be discarded after Bob’s parity check operation. After renormalize the probability distribution for each term in the above table we can obtain the error rate distribution for the survived raw pairs shared by Alice and Bob.

$$
\begin{align*}
\begin{cases}
    p_1 = \frac{p_{10}^2 + p_{z0}^2}{(p_{10} + p_{z0})^2 + (p_{x0} + p_{y0})^2}, \\
p_z = \frac{p_{z0}^2 + p_{y0}^2}{(p_{10} + p_{z0})^2 + (p_{x0} + p_{y0})^2}, \\
p_y = \frac{2p_{x0}p_{y0}}{(p_{10} + p_{z0})^2 + (p_{x0} + p_{y0})^2}, \\
p_x = \frac{2p_{10}p_{z0}}{(p_{10} + p_{z0})^2 + (p_{x0} + p_{y0})^2}.
\end{cases}
\end{align*}
$$

(5)

With this formula, the phase flip error to the shared raw pairs is obviously reduced. The formula is similar to the error-rejection formula by directly measuring the parity of two pairs on each side. Therefore such a code can be more effective than $[3, 1, 3]_2$ error correction code in reducing phase-flip errors and can cause advantages to the QKD protocols. Note that it is a bit subtle that the phase error to the decoded state is caused by bit-flip channel errors.
V. ENTANGLEMENT ASSISTED PROTOCOL.

We first consider the following entanglement purification based protocol with quantum storages:

0 Before they carry out the protocol, Alice and Bob test their channels by the error rejection code in above section. They find Eq.(5) almost always (approximately) holds.

1 Alice prepares \(N\) copies of EPR state \(|\Phi^{+}\rangle, |\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B|\rangle).\) She also prepares \(N\) ancilla qubits which are all in state \(|0\rangle\). We shall use the subscript \(B2\) for each of theses \(N\) ancilla qubits. To each pair, she will keep qubit \(A\) with herself.

2 Alice takes unitary transformation given in Eq.(2) to qubits \(B1\) and \(B2\). \(B1\) and \(B2\) are now the phase-flip quantum error-rejection code for the original state of qubit \(B1\). Now the total state of qubit \(A\), \(B1\), \(B2\) is in the form of Eq.(3).

3 Alice sends each 2-qubit code \((B1,B2)\) to Bob and keeps qubits \(A\).

4 Bob takes parity check to each 2-bit code in basis of \(|0\rangle\langle 0|, |1\rangle\langle 1|\). That is, Bob measures \(Z_{B1}Z_{B2}\). He discards both of them whenever he obtains 1 and stores both qubits whenever he obtains 0.

5 Bob decodes each of the survived 2-bit code: To each code that has passed the parity check, he measures one qubit (e.g., qubit \(B1\)) in \(X\) basis. If he obtains \(|+\rangle\), he takes a Hadamard transformation to the other qubit (qubit \(B2\)) and saves it as the decoded qubit; if he obtains \(|-\rangle\) as the measurement result for \(B1\), he will first take a Hadamard transformation to \(B2\) and then flip \(B2\) in \(Z\) basis and save it as the decoded qubit.

6 Bob announces which codes have passed the parity check and have been decoded. Alice discards those qubits with her which were entangled with Bob’s discarded codes. Now Alice and Bob share a number of raw entangled pairs with each of them containing qubit \(A\) and qubit \(B2\). Alice then randomly chooses a number of her qubits and measure each of them in either \(Z\) basis or \(X\) basis. She announces which qubits have been chosen and the measurement basis and results of each of them. According to her announcement, Bob measures his halves of those raw pairs in the same basis as chosen by Alice. He compares his measurement results with those announced by Alice. If too many of them disagree, they abort the game and then restart from the beginning. If the error rate is acceptable, they continue the game with the following steps [18].

7 Alice and Bob then take the bit-flip error-rejection [6]: they divide the rest of the survived raw pairs into many 2-pair groups. The two pairs for each individual groups are randomly chosen. They measure the parity of each group in \(Z\) basis at each side and compare the results. They discard those groups whose parities on two sides disagree; they discard the target pair and keep the control pair (see Fig.1) if the results at two sides agree with each other. They can repeatedly take this bit-flip error correction for \(g\) round.

8 They can then take the \([r,1,r]_2\) phase-flip error-correction code as proposed in [6] to reduce the phase-flip error. They should choose the appropriate \(g\) and \(r\) value so that both bit-flip error and phase-flip error are small enough therefore the shared pairs can be distilled to maximally entangled by a certain classicalizable distillation method in the next step.

9 They further purify the survived pairs by a classicalizable distillation method so that the error rate of their shared pairs are exponentially small. The classicalizable distillation method means the method which can be reduced to an equivalent prepare-and-measure scheme [6] where no quantum storage is required.

Note that this scheme requires the quantum storage. We shall classicalize it to an equivalent form latter therefore a quantum storage is unnecessary. For the moment we first calculate the largest tolerable channel error rate of the protocol.

The two-qubit error-rejection code decreases the phase flip error and increases the bit flip error of the shared raw pairs. After the quantum decoding Alice and Bob will reduce the bit flip error rate by bit flip error-rejection. The error rate for the survived pairs after one round bit flip error-rejection is given by Eq.(1). This is the formula which is used iteratively in the multi-rounds bit-flip error-rejection. In doing the error rejection for bit-flip error, the phase flip rate and also perhaps the total error rate is increased. After some rounds of bit flip error rejection, one must then reduce to the phase flip error rate by the scheme of phase flip error-correction. In the original Gottesman-Lo protocol, \([3,1,3]_2\) code is used. Actually, any \([r,1,r]_2\) error correction code can be used for that task and any \([r,1,r]_2\) can be classicalized [16]. In our protocol, we have followed Chau’s protocol: carry out the bit flip error-rejection repeatedly.
and use an appropriate $[r, 1, r]_2$ code to reduce the phase error. After the phase error correction, the new error rates satisfy the following inequalities provided that $p_I > 1/2$.

\[
\begin{align*}
& p_x^{PEC} + p_y^{PEC} \leq r(p_x + p_y), \\
& p_y^{PEC} + p_z^{PEC} \leq [4(p_I + p_z)(p_x + p_y)]^{1/2} \leq e^{-2r(0.5 - p_z - p_y)^2}.
\end{align*}
\]  

This shows that, given $p_{x0}, p_{y0}, p_{z0}$, if there exits a finite number $k$, after $k$ rounds of bit-flip error-rejection, we can find a $r$ which satisfy

$$r(p_x + p_y) \leq 5\%$$

$$e^{-2r(0.5 - p_z - p_y)^2} \leq 5\%.$$  

One can then obtain the unconditional secure and faithful final key with a further purification through any classicalizable methods.

To the above 4-state protocol, even though $p_{y0}$ value is not detected, we don’t have to assume $p_{y0} = 0$. We actually need not to know it. What we need to know is each types of error rates to the raw pairs, i.e. $p_x, p_y, p_z$. Based on the information of $p_x, p_y, p_z$ we then decide whether the shared raw pairs are distillable by the remained steps in the scheme. In other words, this is equivalent to the case where Alice directly sends the EPR halves to Bob through an un-symmetric noisy channel with flip rates of $p_x, p_y, p_z$ instead of $p_{x0}, p_{y0}, p_{z0}$. That is to say, the role of the quantum phase flip error-rejection code is to replace the natural channel error rate $p_{x0}, p_{y0}, p_{z0}$ by the effective channel error rate of $p_x, p_y, p_z$ which are linked by eq.(5). Note that a quantum bit-flip error-rejection code will not help to improve the tolerable error rate of Gottesman-Lo protocol [6] because the effect of that is equivalent to that of the classical bit-flip error-rejection as used there. In the 4-state protocol, we don’t detect the $\sigma_y$ error therefore we have to assume $p_y = 0$ after the quantum parity check and decoding. This shows that, if the channel noise is symmetric, after the quantum decoding, both $\sigma_x$ error and $\sigma_y$ error are reduced, i.e., the detectable phase error rate (including both $\sigma_x$ and $\sigma_y$ error rate) has been reduced in a rate as it should be in the case of symmetric noisy channel. We then start from the un-symmetric error rate with assumption $p_y = 0$ and $p_x, p_z$ being the detected bit-flip rate and phase-flip rate, respectively. After the calculation, we find that the tolerable error rate of bit flip or phase flip is 26% for the 4-state protocol. Moreover, in the case that the channel error itself is un-symmetric, e.g., $p_{y0} = 0; p_{x0} = p_{z0}$, the tolerable channel error rate for our protocol is $p_{x0} = p_{z0} \leq 21.7\%$.

VI. THE PROTOCOL WITHOUT QUANTUM STORAGE

The above QKD scheme requires a quantum storage for both Alice and Bob. This is impractical by our current technology. However, the scheme can be reduced to an equivalent scheme which does not require a quantum storage. We now show it in details.

In the protocol above, the phase-flip quantum error-correction code of $[r, 1, r]$ is used [16]. This is equivalent to the classical method of replacing the $r$ bits by one new bit whose value is just the parity of those $r$ bits [16]. For example, one may consider the case of $r = 3$. As it was shown in Ref. [6], the parity measurement in X basis can be replaced by the equivalent one in Z basis, see Fig.2. The actual operation is just to replace the bit value of qubit 1 by the parity of 3 qubits. More generally, the $[3, 1, 3]_2$ error correction code can be replaced by $[r, 1, r]_2$ code with the majority rule, i.e., replacing the original one bit value by the parity of $r$ bits [16]. The initial EPR pairs prepared by Alice will be treated in three different ways: some of them will be used as the check pairs; some of them will be discarded after the parity check before decoding; some of them will be used for the entanglement distillation. To those check pairs, Alice’s only operation to qubit A is just a measurement in either Z basis or X basis, there is no other operation therefore Alice can choose to measure qubit A before encoding qubit B1. After Bob’s parity check before decoding, some quantum codes will be discarded. The discarded codes do not affect any results of final key, therefore Alice may choose to measure qubit A initially in any basis to each of those qubits. To those qubits which will be used for the distillation, Alice’s all operations are done in Z basis, therefore Alice can choose to measure all those qubits A in Z basis before encoding B1. Moreover, instead of preparing each single qubits and then encoding them with ancilla by Hadamard transformation and C-NOT gate, Alice may directly prepare the error rejection code and put down the bit value corresponding to the code. That is, code $|00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ correspond to single qubit state $|+,\rangle, |-,\rangle, |0\rangle, |1\rangle$ respectively.

With the above arguments, the protocol can be revised to the following protocol without a quantum storage:

1. Alice prepares $N$ 2-qubit quantum codes with each of them randomly chosen from the set $\{ |00\rangle, |11\rangle, \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)\}$. Among all of them, $N/4$ of them are prepared in $|00\rangle$ or $|11\rangle$ with equal probability and $3N/4$ of them
are prepared in $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ with equal probability. She records the “preparation basis” as X basis for code $|00\rangle$ or $|11\rangle$, and as Z basis for code $\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ [19]. And she records the bit value of 0 for the code $|00\rangle$ or $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$; bit value 1 for the code $|11\rangle$ or $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$. She sends each randomly chosen 2-qubit codes to Bob.

2 Bob takes parity check to each 2-qubit code in Z basis. That is, Bob measures $Z_{B1}Z_{B2}$. He discards the codes whenever he obtains 1 and he takes the following measurement if he obtains 0 in the parity check: he measures one qubit in X basis and the other qubit in either X basis or Z basis with equal probability. Bob makes a record of his “measurement basis” to the decoded qubit as Z basis (\{\{0\},\{1\}\}) if he measures “the other qubit” in basis X basis ($\{|\pm\}\}$); and he records his “measurement basis” as X basis if he measures “the other qubit” in Z basis [20]. If he obtains $|+\rangle|+\rangle$, $|+\rangle|0\rangle$, $|-\rangle|-\rangle$ or $|\rangle|0\rangle$, he records bit value 0 for that code; if he obtains $|-\rangle|+\rangle$, $\langle-\rangle|1\rangle$, $|+\rangle|-\rangle$ or $|+\rangle|1\rangle$, he records bit value 1 for that code.

3 Alice announces her records of “preparation basis” of each qubit; Bob compares his measurement basis with Alice’s announced “preparation basis”. He discards those bit values whose measurement basis disagree with Alice’s announcement. Bob announces which qubits are survived. Bob announces the bit value for those qubits whose “preparation basis” and “measurement basis” are X. He also randomly chooses the same number of bits whose “preparation basis” and “measurement basis” are Z. He announces their bit values. If too many of them disagree with Alice’s record, they abort the protocol.

4 They reduce the bit flip rate by the following way: they randomly group all their unchecked bits with each group containing 2 bits. They compare the parity of each group. If the results are different, they discard both bits. If the results are same, they discard one bit and keep the other. They repeatedly do so for a number of rounds until they believe that both bit flip rate and phase flip rate will be reduced to less than 5% with the next step being taken.

5 They then randomly group the remained bits with each group containing r bits. They use the parity of each group as the new bits.

6 They use any other proven secure classical methods to further reduce the error rate until both bit flip rate and phase flip rate are negligible, e.g. less than $2^{-50}$. Here “other classical methods” includes the Gottesman-Lo method [6], the classical CSS code [13], the concatenated 7-bit code [16] and so on.

The above classicalized protocol is totally equivalent to the one based on entanglement purification in the previous section, therefore it tolerates the same channel error rate as that of the entanglement purification based protocol. That is to say, this classicalized 4-state protocol can tolerate a channel flipping rate of 26%. Although the quantum storage is removed now, it still requires the parity check and decoding operation for Bob. We now show how to make it with ordinary linear optics devices.

VII. EXPERIMENTAL REALIZATION.

The 2-qubit codes in our protocol above can be robustly produced form SPDC process [5]. In such a polarization space, $|0\rangle, |1\rangle$ are represented by the horizontal polarization state and vertical polarization state. In the “classicalized” protocol above, Bob need to take the operation of quantum parity check and quantum decoding to the codes received from Alice. In practice, Bob can take the two operation together with a polarizing beam splitter and obtain the result by post-selection. Bob’s operation to the incoming 2-qubit code is shown in Fig.(3). If he obtains nothing in either D1 or D2 measurement, he aborts the code; if there is one photon on each side of the PBS, he records the bit value and “measurement basis” according to the correspondence rule in step 2 in the protocol. Note that a PBS transmits the horizontal qubits and reflects the vertical qubits. In Alice’s initial preparation of the quantum code, the two qubits have the same polarization. If one of them is flipped in Z basis by the channel, the code will contain two different polarizations and both photons will be on the same side of the PBS, i.e., either D1 or D2 will be vacuum, therefore the code will be discarded. To verify the fact of one photon on each side of the PBS, Bob only needs to see that both photon detectors on each side of the PBS click. Note that we only need yes-no photon detectors here. The low efficiency of the detectors does not affect the security of the protocol.
Our protocol can obviously be extended to the 6-state protocol. In doing so, Alice just change the initially random codes by adding $N/4$ codes from $\{\frac{1}{\sqrt{2}}((|00\rangle+|11\rangle)\pm i(|00\rangle-|11\rangle)\}$. This is equivalent to $\frac{1}{\sqrt{2}}\{(|00\rangle+i|11\rangle)\}$. She regards all this type codes as Y bits. In decoding the codes, Bob’s “measurement basis” is randomly chosen from 3 basis, with the basis of $\frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ being added. All decoded X bits, Y bits and the same number of decoded Z bits which are randomly chosen will be used as the check bits. Since the Hadamard transformation will switch the two eigenstates of $\sigma_y$, therefore in classicalized protocol, after the decoding if Bob choose to take the measurement to B2 qubit in Y basis, he shall records his “measurement basis” as Y but the “measurement outcome” should be flipped in the Y basis so that to obtain everything the same as that in the 2-EPP with quantum storages. In such a way, if the channel is symmetric, Bob will find $p_y \neq 0$. And he will know $p_x, p_y, p_z$ exactly instead of assuming $p_y = 0$. This will increase the tolerable error rate accordingly. We find that the tolerable bit flip rate and phase flip rate are 30% in such a case, i.e., our 6-state protocol can tolerate a total flip rate of 45% with the symmetric noisy channel.

IX. ANOTHER VIEWPOINT OF OUR PROTOCOLS

To further clarify the concept of unconditional security with conditional advantage in efficiency of our protocols, here we view our scheme from a another viewpoint. We denote channel between Alice and Bob by operator $\hat{C}$, i.e. a transmitted qubit will be interacted with channel by that operator. We can now compare Chau protocol [16] with our protocol given the same channel $\hat{C}$. In our protocol, if we can also regard the Alice’s quantum encoding part and Bob’s quantum decoding part as part of the channel interaction. In such a case, our protocol with channel $\hat{C}$ is actually just the Chau protocol [16] with a lossy channel $\hat{C}' = \hat{D}\hat{C}\hat{E}$ (8)

where the operators $\hat{E}, \hat{D}$ represent for encoding and decoding respectively. Note that so far all the known secure protocols are also secure with a lossy channel, since the purification scheme can be done with a lossy channel, with the device imperfections such as dark counting being ignored here. The error rates for channel $\hat{C}'$ are functions of the error rates of channel $\hat{C}$. In particular, they are related by Eq.(5) if errors caused by $\hat{C}$ are uncorrelated. To whatever channel, the error rate should satisfy the same convergence condition to produce non zero bits of final key. Specifically, we can use the following formula for the convergence condition:

$$f(p_{x0}, p_{y0}, p_{z0}) \leq \eta$$

(9)

for channel $\hat{C}$ and

$$f(P_x, P_y, P_z) \leq \eta$$

(10)

for channel $\hat{C}'$ where $(p_{x0}, p_{y0}, p_{z0})$ and $(P_x, P_y, P_z)$ are detected error rates for channel $\hat{C}$ and channel $\hat{C}'$, respectively. $\eta$ is a small number, say 5%. Though the above 2 equations have the same form, Eq.(10) requires a different convergence condition to $(p_{x0}, p_{y0}, p_{z0})$. If the error rates of Channel $\hat{C}'$ and that of channel $\hat{C}$ are related by eq.(5), then we can solve the condition of formula (10) numerically and obtain the thresholds to $(p_{x0}, p_{y0}, p_{z0})$, which is just the claimed threshold in our protocols.

Roughly speaking, our protocol is equivalent to the one which first investigates the channel properties and then find a scheme to reduce the s-channel noise based on the investigated properties and then carry out Chau protocol [16]. The channel noise is not unconditionally reduced, it is based on the investigated properties of the s-channel itself. At the time we are really carrying out QKD, Eve. may attack it and the assumed properties is not necessarily true. But we still believe that the protocol will work in the way as it is assumed in most of the cases for the reasons we have addressed before. It’s a reasonable assumption that it should be an advantage if we have a way to decrease the noise of the s-channel, even though in principle Eve. may do everything to the channel in QKD.

X. CONCLUDING REMARK.

We have proposed an entanglement assisted protocol for quantum key distribution. The protocol is unconditionally secure with a conditional advantage of an improved threshold of the channel error rate. Given the uncorrelated noise of
s-channel, in 4-state case, the tolerable bit flip or phase flip rate is 26%; in the 6 state case, the values are increased to 30%, if the noise is symmetric. The initial 2-qubit entangled state can be produced through SPDC process [5]. Bob’s quantum parity check and decoding can be done with a polarizing beam splitter. Obviously, all devices appeared in our proposed scheme are just ordinary devices in linear optics.

There are also some loosen ends for our protocol in practical realization. In the SPDC process, there are also some multi-pair emission with small probability. This type of emission will affect the security that should be taken into consideration for the security proof for a practical entanglement assisted QKD protocol with SPDC process. We believe this can be solved in the similar idea for the case of imperfection of the single photon source for the BB84 scheme. We have neither considered the devices imperfections, e.g., the dark counting of the photon detectors. Neither have we considered the Trojan Horse attack to the protocol. Similar to other QKD proposals [6,16], here the tolerable upper bound for channel error rate is calculated asymptotically. In practice, the number of qubits sent to Bob is always finite. With the statistical fluctuation being taken into consideration, the tolerable flip rate of the channel is decreased in practice with a finite number of initial qubits.

**Comparison of measurement results by two way communication**

![Diagram](image-url)

**Alice’s operation**

![Diagram](image-url)

**Bob’s operation**

![Diagram](image-url)

FIG. 1. Controlled-not (C-NOT) operation used for the bit-flip error-rejection. The horizontal lines marked by 1 and 2 are pair 1 and 2 respectively. Alice and Bob compare the measurement outcomes of the target qubit 2. If they choose Z basis for C-NOT operation and the measurement, it is equivalent to measure $Z_1 Z_2$; if they choose to use X basis, it is equivalent to measure $X_1 X_2$ on each side.

![Diagram](image-url)

FIG. 2. Two equivalent ways to measure the operators $X_1 X_2$ and $X_1 X_3$. This figure is taken from the paper by Gottesman and Lo.
FIG. 3. Bob’s operation for quantum parity check and decoding. The two horizontal lines are the 2-bit code sent by Alice. PBS is a polarizing beam splitter which transmits horizontally polarized photons and reflects vertically polarized photons. D represents for measurement. D1(+,−) means a measurement in basis |±⟩, D2[(+,−);(0,1)] means that Bob takes that measurement in either {±} basis or {0,1} basis.

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[18] We do not have to require that the detected error rate satisfy or approximately satisfy Eq.(5) as the condition of an acceptable error rate. Even the detected error rates are significantly larger than the expected values given by Eq.(5), they can still continue the protocol. In such a case they will obtain a much shorter final key to which Eve’s information is exponentially close to zero. In any case they can carry out the whole protocol and accept the final key provided that they believe the final key in that length deserves the cost of carrying out the protocol.
The “preparation basis” here is defined by the basis of the original qubit which corresponds the 2-qubit code by our encoding method.

Here Bob has omitted the Hadamard transformation as appeared in the EPP, therefore the measurement basis to B2 and Bob’s recorded “measurement basis” to the decoded qubit are different.