STABILITY OF TRAVELING WAVES OF MODELS FOR IMAGE PROCESSING WITH NON-CONVEX NONLINEARITY

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ABSTRACT. We establish the existence and stability of smooth large-amplitude traveling waves to nonlinear conservation laws modeling image processing with general flux functions. We innovatively construct a weight function in the weighted energy estimates to overcome the difficulties caused by the absence of the convexity of fluxes in our model. Moreover, we prove that if the integral of the initial perturbation decays algebraically or exponentially in space, the solution converges to the traveling waves with rates in time, respectively. Furthermore, we are able to construct another new weight function to deal with the degeneracy of fluxes in establishing the stability.

1. Introduction. Nonlinear partial differential equations (PDEs) have been applied to denoise images, detect edges and inpaint images in image processing and computer vision (e.g. see [23, 24, 27]).

This paper is concerned with existence and asymptotic stability of smooth traveling wave solutions to nonlinear conservation laws with general flux functions $f$ arising from image processing

$$u_t + f(u)_x = (g(u_x)u_x)_x$$

for $x \in \mathbb{R}$ and $t > 0$, where $u(x, t)$ is the gray scale of images and

$$g(s) = \frac{1}{1 + s^2}.$$ 

The initial data, the original image, is given by

$$u(x, 0) = u_0(x) \rightarrow \begin{cases} u_- & \text{as } x \to -\infty \\ u_+ & \text{as } x \to \infty \end{cases}$$

where $u_-$ and $u_+$ are nonnegative constants.

Before the development of denoising techniques based on nonlinear PDEs [9, 23], the linear filtering PDEs method was introduced by Marr and Hildereth [16], and was developed by [25, 8, 3]. However, after denosing image by the linear PDEs methods, it is hard to recognize the original important contents such as edges in...
the denoised image. To resolve such blurriness caused by the linear methods, Perona and Malik [23] introduced the nonlinear PDEs

$$u_t = \nabla \cdot (g(|\nabla u|)\nabla u)$$ (4)

where \(u(\cdot, t) : \Omega \rightarrow \mathbb{R}, t > 0\) and \(\Omega \subset \mathbb{R}^2\) denotes an image domain. A typical \(g\) in (4) is given in (2) and it plays a role in preserving edges in the image while reducing overall noise of the image.

In [9], Kurganov et al. introduced a convection term in (4)

$$u_t + uu_x = (g(u_x)u_x)_x$$ (5)

for \(x \in \mathbb{R}\) and \(t > 0\), where \(g(s)\) is given in (2) with the initial data (3). In fact, such a convection motion has been applied to image inpainting in a region where the image data is lost and is filled within the image information surrounded the region (e.g. see [1, 2, 5]). Kurganov et al. [9] and Goodman et al. [4] investigated shock or jump type behavior of solutions to (5) which are typical edges in images.

The existence of smooth traveling wave solutions to the Burgers-type problem (5) was studied by authors in [9, 26, 5]. Rigorous proofs of its stability were achieved by Wu [26] and Li and Park [15]. Wu [26] obtained the stability of the smooth traveling waves in some exponential weighted space by spectral analysis. In [15], the authors proved nonlinear stability of traveling waves for general small initial perturbations. The authors in [15] also used a weighted energy method to show that if the initial perturbation decays algebraically or exponentially as \(|x| \rightarrow \infty\), then the Cauchy problem solution approaches to the traveling wave at corresponding rates as \(t \rightarrow \infty\). It is noticed that equation (5) is equation (1) with \(f(u) = \frac{u^2}{2}\) which is convex.

In the current paper, we prove existence and stability of smooth large-amplitude traveling wave solutions to (1)-(3) with non-convex fluxes \(f \in C^2\). To show the existence of smooth traveling solutions to (1)-(3), we impose the entropy condition [17]

$$h(u) := -c(u - u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0 & \text{if } u_+ < u < u_- \\ > 0 & \text{if } u_- < u < u_+ \end{cases}$$ (6)

where \(c\) satisfies the Rankine-Hugoniot condition

$$-c(u_+ - u_-) + f(u_+) - f(u_-) = 0.$$ (7)

We show the stability with or without decay rates for both non-degenerate case

$$f'(u_+) < c < f'(u_-)$$ (8)

and degenerate case

$$f'(u_+) = c \quad \text{or} \quad f'(u_-) = c.$$ (9)

In establishing the stability theorems, the absence of convexity of \(f\) causes difficulties in energy estimates. To overcome such obstacles, Matsumura and Nishihara [17] used weighted energy estimate methods for scalar viscous conservation laws. Their weight function has been widely adopted to obtain stability with non-convex \(f\), for example, [11, 13, 14, 19]. However the above weight function does not apply to our problem (1)-(3) with large-amplitude traveling waves due to the different dissipative term. In order to prove the stability for general fluxes \(f\) and for arbitrary wave strengths, we construct a new weight function which is piecewisely defined on bounded and unbounded domains, see (47). On the bounded domains where \(f\) changes its convexity, we set up \(w\) in (47) so that the \(L^2\)-integral of \(\phi\)
does not appear in the $L^2$-estimate (52). On the unbounded domains, using the entropy condition (6), we derive the desired $L^2$-estimates (52). It is worth mentioning that in proving our stability results, we only need the entropy condition (6) instead of imposing additional conditions on the fluxes as in the previous related results [7, 10, 11].

Combining the same weight function with algebraic and exponential weights, we prove our stability with decay rates. For related previous results, see [6, 12, 15, 17, 18, 19, 21, 22]. For the degenerate case (9), we construct a new unbounded weight function (81) whose essential feature at far fields comes from the degeneracy. Moreover, using the weight function (47), we are able to improve the previous results in [15] where $f$ is convex. See Remarks 3 and 4.

This paper is organized as follows. In Section 2, we show existence of smooth traveling wave solutions for (1)-(3), and describe properties of the traveling wave solutions. We also state our main results. In Section 3, we reformulate our problem in terms of perturbation quantities. Our main stability theorems are proved in Section 4 for the non-degenerate case (8) and for the degenerate case (9) in Section 5. In Section 6, we investigate algebraic and exponential decay rates for the non-degenerate case (8).

**Notation.** We denote $C > 0$ as a generic constant so each $C$ can be a different number at different context. We write $f(x) \sim g(x)$ as $x \to a$ if $C^{-1}g < f < Cg$ in a neighborhood of $a$ for some $C > 0$. Let $\|f\|$ be the $L^2$-norm of a function $f \in L^2$ which is

$$\|f\| = \left( \int |f(x)|^2 dx \right)^{\frac{1}{2}}.$$  

We denote $H^k$ the usual Sobolev space $W^{k,2}$ with the norm $\|f\|_k$ for any function $f \in H^k$, $k \geq 1$,

$$\|f\|_k = \left( \int \sum_{i=0}^k \left| \frac{d^i f(x)}{dx^i} \right|^2 dx \right)^{\frac{1}{2}}.$$  

We also denote $L^2_w$ the space of measurable functions $f$ on $\mathbb{R}$ which satisfy $\sqrt{w(x)} f \in L^2$, where $w(x) \geq 0$ is a weight function. Then the space is endowed with the norm

$$\|f\|_w = \left( \int w(x) |f(x)|^2 dx \right)^{\frac{1}{2}}.$$  

Similarly, $H^k_w$ denotes the weighted Sobolev space of $L^2_w$-functions $f$ whose derivatives $\partial_i^k f$, $i = 1, \ldots, k$, are $L^2_w$-functions, with the norm

$$\|f\|_{k,w} = \left( \sum_{i=0}^k \|\partial_i^k f\|_{w}^2 \right)^{\frac{1}{2}}.$$  

We define

$$\langle x \rangle = \sqrt{1 + x^2}. \tag{10}$$

Weight functions in each sections and subsections differ. Particularly, a weight function in Subsection 6.1 is of the form $w(x) \sim \langle x \rangle^\alpha$ for $\alpha \geq 0$ and we denote

$$L^2_w = L^2_\alpha. \tag{11}$$
2. Preliminaries and main results.

2.1. Existence of smooth traveling wave solutions and their properties.

Let us consider smooth traveling wave solutions of (1)-(3)

\[ u(x,t) = U(x - ct) = U(z) \quad \text{and} \quad U(z) \to u_\pm, U_z(z) \to 0 \quad \text{as} \quad z \to \pm \infty \]

where \( c \) is the traveling wave speed and \( z = x - ct \) is the traveling wave variable. Constants \( u_\pm \) and \( c \) satisfy the Rankine-Hugoniot condition (7) and the entropy condition (6). The condition (6) implies

\[ f'(u_+) \leq c \leq f'(u_-). \]

We consider a non-convex function \( f(u) \) and the case \( u_+ < u_- \). It follows from (6) that

\[ h(u) < 0 \quad \text{for} \quad u_+ < u < u_- \]

Substituting (12) into (1) and using the definition of \( h(u) \) in (6), we have

\[ (g(U_z)U_z)_z = -cU_z + f(U)_z = h'(U)U_z. \]

Integrating (15) over \((\pm \infty, z)\) and using the Rankine-Hugoniot condition (7), one obtains

\[ \frac{U_z}{1 + U_z^2} = -c(U - u_+) + f(U) - f(u_+) = h(U). \]

Solving for \( U_z \), we have

\[ U_z = \frac{1 \pm \sqrt{1 - 4h^2(U)}}{2h(U)} \]

where \( 0 < |h(U)| \leq \frac{1}{2} \). Since the critical threshold of discontinuity of solutions of (16) occurs at \( |h(U)| = \frac{1}{2} \), in order to have smooth traveling waves, we need a condition

\[ 0 < |h(U)| < \frac{1}{2}. \]

This is analogous to the analysis of existence of smooth traveling wave solutions in [9].

Let \( b > 0 \) be the upper bound of the strength of smooth traveling wave solutions, namely,

\[ 0 < |u_+ - u_-| < b. \]

For example, \( b=2 \) in [9, 15]. Then for each \( u_\pm \) satisfying (19), there is a constant \( 0 < b_0 < \frac{1}{2} \) such that

\[ -\frac{1}{2} < -b_0 \leq h(U) < 0, \quad u_+ < U < u_- \]

Under condition (19), we have a unique smooth solution of (16) and the following theorem.

**Theorem 2.1.** Let \( f(u) \), \( u_\pm \), \( c \) satisfy the Rankine-Hugoniot condition (7) and the entropy condition (14) and (19). If there are nonnegative integers \( n_\pm \) such that

\[ h'(u_\pm) = \cdots = h^{(n_\pm)}(u_\pm) = 0 \quad \text{and} \quad h^{(n_\pm+1)}(u_\pm) \neq 0, \]

there exists a unique smooth traveling wave solution \( U(z) \) of (1)-(2).
Here the non-degenerate case (8) implies \( n_\pm = 0 \), and the degenerate one (9) implies \( n_\pm \geq 1 \).

Now we present some properties of the smooth traveling wave solution \( U(z) \) obtained from Theorem 2.1. The properties in the following proposition provide the main mechanism of our stability results.

Proposition 1. Under the same assumptions in Theorem 2.1, the smooth traveling wave solution \( U(z) \) satisfies

1. \( U(z) \) is monotonically decreasing over \((-\infty, \infty)\), namely,
\[
U_z < 0, \quad z \in \mathbb{R}.
\]

2. There are positive constants \( m \) and \( m_0 \) such that
\[
0 < |U_z| \leq \frac{2b_0}{1 + \sqrt{1 - 4b_0^2}} =: m < 1
\]
where \( 0 < b_0 < \frac{1}{2} \) is such that (20) holds, and
\[
0 < m_0 := \min_{z \in \mathbb{R}} \left\{ \frac{1 - U_z^2}{(1 + U_z^2)^2} \right\} < 1.
\]

3. The second and third order derivatives of \( U(z) \) can be expressed as
\[
U_{zz} = h'(U)U_z \frac{(1 + U_z^2)^2}{1 - U_z^2}
\]
and
\[
U_{zzz} = (h'(U))^2U_z \frac{1 + U_z^4}{(1 - U_z^2)^2} \left( 1 + \frac{2U_z^2(3 - U_z^2)}{1 - U_z^4} \right) + h''(U)U_z^2 \frac{(1 + U_z^2)^2}{1 - U_z^2}.
\]

4. For the non-degenerate case (8), there are \( c_\pm > 0 \) such that
\[
|U(z) - u_\pm| \sim e^{-c_\pm|z|} \quad \text{as} \quad z \to \pm \infty.
\]

For the degenerate case (9) and (21), it holds
\[
|U(z) - u_\pm| \sim |z|^{-1/n_\pm} \quad \text{as} \quad z \to \pm \infty
\]
where \( n_\pm > 0 \) are integers satisfying (21).

Proof. (1) By (19) and (20), \( U_z < 0 \) for \( z \in \mathbb{R} \).

(2) From (16) and (22), \( U_z \) is bounded by (23) under the condition (20). (24) follows from (23). In fact, (23) and (24) are key properties to establish the stability theorems in this paper.

(3) It is straightforward to show (25) by differentiating (16) with respect to \( z \). Similarly, it is easy to show (26).

(4) From (16) and (21), \( |h(U)| \sim |U - u_\pm|^{n_\pm + 1} \) as \( U \to u_\pm \), which shows (27) and (28). \( \square \)

2.2. The main results. To investigate the stability of smooth traveling wave solution \( U \), we assume that \( u_0 - U \) is integrable over \( \mathbb{R} \), where \( u_0 \) is the initial data in (3). There exists a unique \( x_0 \) satisfying
\[
\int_{-\infty}^{\infty} (u_0(x) - U(x))dx = x_0(u_+ - u_-)
\]
and the conservation law (1) implies that
\[
\frac{d}{dt} \int_{-\infty}^{\infty} (u(x,t) - U(x - ct + x_0)) \, dx = 0.
\]
Hence, we arrive at
\[
\int_{-\infty}^{\infty} (u(x,t) - U(x - ct + x_0)) \, dx = \int_{-\infty}^{\infty} (u_0(x) - U(x)) \, dx + \int_{-\infty}^{\infty} (U(x) - U(x + x_0)) \, dx + \int_{-\infty}^{\infty} (u_0(x) - U(x)) \, dx - x_0(u_+ - u_-) = 0.
\]

Now we decompose the solution \( u \) to the Cauchy problem (1)-(3) into the traveling wave \( U \) and its perturbation as
\[
u(x,t) = U(x - ct + x_0) + \phi(x,t), \quad x \in \mathbb{R}, \quad t \geq 0
\] where
\[
\phi(x,t) = \int_{-\infty}^{x} (u(y,t) - U(y - ct + x_0)) \, dy.
\]

From (29) we have
\[
\phi(\pm \infty, t) = \int_{-\infty}^{\infty} (u_0(x,t) - U(x - ct + x_0)) \, dx = 0, \quad t \geq 0.
\]

Without loss of generality, let \( x_0 = 0 \). The initial data of \( \phi \) is given by
\[
\phi(x,0) = \phi_0(x) = \int_{-\infty}^{x} (u_0(y) - U(y)) \, dy.
\]

Our main theorems are stated as follows.

**Theorem 2.2** (Non-degenerate case (8)). Let \( U(z) \) be a smooth traveling wave solution obtained in Theorem 2.1. Let \( u_0 - U \) be integrable and \( \phi_0 \in H^2 \). There is a constant \( \varepsilon_1 > 0 \) such that if \( \|\phi_0\|_2 < \varepsilon_1 \), then the Cauchy problem (1)-(3) has a unique global solution \( u(x,t) \) satisfying
\[
u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2).
\]

In addition, it holds
\[
\sup_{x \in \mathbb{R}} |u(x,t) - U(x - ct)| \to 0 \quad \text{as} \quad t \to \infty.
\]

**Theorem 2.3** (Degenerate case (9)). Let \( U(z) \) be a smooth traveling wave solution obtained in Theorem 2.1. Let \( u_0 - U \) be integrable and \( \phi_0 \in H^2 \cap L^2([z]^{1+1/n_+}) \). Then there exists a constant \( \varepsilon_2 > 0 \) such that if \( \|\phi_0\|_{[z]^{1+1/n_+}} < \varepsilon_2 \), where \( n_+ \) is given in (28), then the Cauchy problem (1)-(3) has a unique global solution \( u(x,t) \) satisfying
\[
u - U \in C^0([0, \infty); H^1) \cap L^2([0, \infty); H^2 \cap L^2([z]^{1+1/n_+})).
\]

and
\[
\sup_{x \in \mathbb{R}} |u(x,t) - U(x - ct)| \to 0 \quad \text{as} \quad t \to \infty.
\]
where
\[ \langle z \rangle_+ = \begin{cases} \sqrt{1 + z^2}, & z \geq 0 \\ 1, & z < 0 \end{cases} \tag{34} \]

**Remark 1.** For \( f'(u_+) < c = f'(u_-) \) or \( f'(u+) = c = f'(u_-) \), we have the same stability theorem as Theorem 2.3 by replacing \( \langle z \rangle_+ \) by
\[ \langle z \rangle_- = \begin{cases} \sqrt{1 + z^2}, & z \leq 0 \\ 1, & z > 0 \end{cases} \]
or
\[ \langle z \rangle = \sqrt{1 + z^2}, \quad z \in \mathbb{R} \]
respectively.

In addition, we investigate algebraic and exponential decay rates for the non-degenerate case (8).

**Theorem 2.4** (Algebraic decay rates for the non-degenerate case (8)). Under the same assumptions of Theorem 2.2, let \( \phi_0 \in H^2 \cap L^2_\alpha \) for any \( \alpha > 0 \), where \( L^2_\alpha \) is defined in (11). Then the solution \( u(x,t) \) satisfies
\[ \sup_{x \in \mathbb{R}} |u(x,t) - U(x - ct)| \leq C(1 + t)^{-\frac{\theta}{2}} (\|\phi_0\|_\alpha + \|\phi_{z,0}\|_1) \]
for some \( C > 0 \) and for any \( t \geq 0 \).

**Theorem 2.5** (Exponential decay rates for the non-degenerate case (8)). Under the same assumptions in Theorem 2.2, let \( \phi_0 \in H^2 \cap L^2_\alpha \) the weight function \( w_\alpha(z) \) satisfying \( w_\alpha(z) \sim e^{d|z|} \) as \( z \to \pm \infty \) for some \( d > 0 \). Then there are constants \( \theta = \theta(d, U_z) > 0 \) and \( C > 0 \) such that
\[ \sup_{x \in \mathbb{R}} |u(x,t) - U(x - ct)| \leq Ce^{-\frac{\theta}{2}t}(\|\phi_0\|_w + \|\phi_{z,0}\|_1) \]
for any \( t \geq 0 \).

3. **Reformulation of the problem.** As discussed in Subsection 2.1, we look for a solution of (1)-(3) of the form
\[ u(x,t) = U(x - ct) + \phi_z(x,t) = U(z) + \bar{\phi}_{z}(z,t), \quad z = x - ct \tag{35} \]
in certain solution spaces which are different in each sections. For simplicity of notation, we omit the bar in \( \bar{\phi}_z \) throughout this paper. Without loss of generality, we assume \( u_+ = 0 \) and \( f(u_+) = 0 \).

Substituting (35) into the problem (1)-(3), we have
\[ \phi_{zt} - c\phi_{zz} + (f(U + \phi_z) - f(U))_z = \left( \frac{U_z + \phi_{zz}}{1 + (U_z + \phi_{zz})^2} - \frac{U_z}{1 + U_z^2} \right)_z. \tag{36} \]
Integrating (36) with respect to \( z \) over \((z, \infty)\), we have the reformulated problem
\[ \phi_t - c\phi_z + f(U + \phi_z) - f(U) = \left( \frac{U_z + \phi_{zz}}{1 + (U_z + \phi_{zz})^2} - \frac{U_z}{1 + U_z^2} \right)_z \tag{37} \]
with the initial data
\[ \phi(z,0) = \phi_0(z) = \int_{-\infty}^{z} (u_0 - U)(y) dy. \tag{38} \]
By (16), equation (37) is rewritten as
\[ \phi_t + h'(U)\phi - \frac{1 - U_z^2}{(1 + U_z^2)^2} \phi_{zz} = F(U, \phi) + G(U, \phi_{zz}) \] (39)
where
\[ F(U, \phi) := -(f(U + \phi) - f(U) - f'(U)\phi) \] (40)
and
\[ G(U, \phi_{zz}) := -\frac{3U_z - U_z^3 + (1 - U_z^2)\phi_{zz}}{(1 + U_z^2)^2(1 + (U_z + \phi_{zz})^2)} \phi_{zz}. \] (41)
Here, \( F = O(\phi_z^2) \) and \( G = O(\phi_{zz}^2) \) provided that \( \phi_z^2 \) and \( \phi_{zz}^2 \) are small.

We then state the following theorems.

**Theorem 3.1** (Non-degenerate case (8)). Under the assumptions of Theorem 2.2, there exists a constant \( \delta_1 > 0 \) such that if \( \|\phi_0\|_2 < \delta_1 \), the problem (39)-(38) has a unique global solution satisfying
\[ \phi \in C([0, \infty); H^2) \quad \text{and} \quad \phi_z \in L^2([0, \infty); H^2) \]
and
\[ \|\phi(\cdot, t)\|_2^2 + \int_0^t \int_{\mathbb{R}\setminus (R_1, R_2)} \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) |\phi|^2 d z d \tau + \int_0^t \|\phi_z(\cdot, \tau)\|_{L^2(\mathbb{R})}^2 d \tau \leq C\|\phi_0\|_2^2 \] (42)
for some \( C > 0 \), and constants \( a > 0 \), \( R_1 < 0 \) and \( R_2 > 0 \) are determined in Section 4.

Consequently,
\[ \sup_{z \in \mathbb{R}} |\phi_z(z, t)| \to 0 \quad \text{as} \quad t \to \infty. \] (43)

**Theorem 3.2** (Degenerate case (9)). Under the assumptions of Theorem 2.3, there exists a constant \( \delta_2 > 0 \) such that if \( \|\phi_0\|_2 < \delta_2 \), the problem (39)-(38) has a unique global solution satisfying
\[ \phi \in C([0, \infty); H^2 \cap L^2_{(z)_{+}^{1+1/n_+}}) \quad \text{and} \quad \phi_z \in L^2((0, \infty); H^2 \cap L^2_{(z)_{+}^{1+1/n_+}}) \]
and
\[ \|\phi(\cdot, t)\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2 + \|\phi_z(\cdot, t)\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2 \]
\[ + \int_0^t \int_{-\infty}^R \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) |\phi|^2 d z d \tau + \int_0^t \|\phi_z(\cdot, \tau)\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2 d \tau + \int_0^t \|\phi_{zz}(\cdot, \tau)\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2 d \tau \]
\[ \leq C\left(\|\phi_0\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2 + \|\phi_z(0, \cdot)\|_{L^2_{(z)_{+}^{1+1/n_+}}}^2\right) \] (44)
for some \( C > 0 \), where \( a > 0 \) and \( R < 0 \) are determined in Section 5, and \( n_+ \) is given in (82).

Consequently,
\[ \sup_{z \in \mathbb{R}} |\phi_z(z, t)| \to 0 \quad \text{as} \quad t \to \infty. \]

Theorems 2.2 and 2.3 are consequences of Theorems 3.1 and 3.2 respectively. Theorems 2.4 and 2.5 are proved in Section 6.
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4. The non-degenerate case - proof of Theorem 2.2. In this section, we prove Theorem 2.2 by proving Theorem 3.1. To achieve our stability result, we introduce a new weight function for the non-degenerate case (8).

For a small constant $0 < a < m$, where $m$ is defined in (23), there are constants $R_1 < 0$ and $R_2 > 0$ satisfying

$$|U_z(R_1)| = |U_z(R_2)| = a$$  \hspace{1cm} (45)

and

$$0 < |U_z(z)| < a, \quad z \in \mathbb{R}\setminus(R_1, R_2).$$  \hspace{1cm} (46)

With such constants $a, R_1,$ and $R_2$, we now define our new weight function $w(z)$

$$w = w(z) = \begin{cases} 
w_1(z) := \frac{(1 + U_z^2)^2}{1 - U_z^2} \left( \frac{1}{a^3} U_z^2 + \frac{3}{a^2} U_z + \frac{3}{a} \right) & \text{if } z \in \mathbb{R}\setminus(R_1, R_2), \\
w_2(z) := \frac{(1 + U_z^2)^2}{1 - U_z^2} \left( -\frac{1}{U_z} \right) & \text{if } z \in (R_1, R_2). 
\end{cases}$$  \hspace{1cm} (47)

From Proposition 1, one can easily check that $w(z)$ is positive and uniformly bounded for all $\mathbb{R}$, namely,

$$0 < w(z) \leq M, \quad z \in \mathbb{R}$$  \hspace{1cm} (48)

for some $M > 0$. One can also check that $w(z)$ belongs to $C^2(\mathbb{R})$.

In previous works [7, 10, 11], authors imposed conditions for the fluxes to be almost convex. In this paper, piecewisely defined $w$ in (47) enables us to deal with non-convexity of fluxes. Due to our construction of $w$, the $L^2$-integral of $\phi$ on the bounded domain does not appear in the $L^2$-estimate of $\phi$ in (52). Now by manipulating the size of $a > 0$ in (47) and using the entropy condition (6), we complete the $L^2$-estimates for $\phi$. It is worthwhile to mention that we only need the entropy condition (6) instead of imposing additional conditions on the fluxes.

Now we define the solution space as

$$X(0, T) = \{ \phi \in C([0, T); H^2), \quad \phi_z \in L^2([0, T); H^2) \}$$  \hspace{1cm} (49)

with $0 < T \leq \infty$. If we let

$$N(t) = \sup_{0 \leq \tau \leq t} \{ \| \phi(\cdot, \tau) \|_2 \},$$  \hspace{1cm} (50)

the Sobolev embedding theorem gives us

$$\sup_{z \in \mathbb{R}} \{ |\phi|, |\phi_z| \} \leq CN(t)$$  \hspace{1cm} (51)

for some $C > 0$.

The global existence of $\phi$ in Theorem 3.1 is a consequence of the following two propositions, the local existence and the \textit{a priori} estimates, by the continuation arguments.

**Proposition 2** (Local existence). \textit{For any $\delta > 0$, there is a constant $T > 0$ depending on $\delta$ such that if $\phi_0 \in H^2$ and $2N(0) < \delta$, the problem (39)-(38) has a unique solution $\phi \in X(0, T)$ satisfying}

$$N(t) < 2N(0)$$

\textit{for any $0 \leq t \leq T$.}
Proposition 3 (A priori estimates). Assume that $\phi \in X(0,T)$ is a solution obtained from Proposition 2 for a constant $T > 0$. Then there exists a constant $\delta_2 > 0$, which is independent of $T$, such that if
\[ N(t) < \delta_2 \]
for any $0 \leq t \leq T$, then the solution $\phi$ of (39)-(38) satisfies
\[ \|\phi(\cdot,t)\|^2_2 + \int_0^t \int_{\mathbb{R}\setminus(R_1,R_2)} \left| \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right| \phi^2 \, dz \, dt + \int_0^t \|\phi(\cdot,\tau)\|^2_2 \, d\tau \leq CN^2(0) \]
for some constants $C > 0$, $R_1 < 0$, $R_2 > 0$, $0 < a < m$, where $m$ is given in $(23)$.

The local existence can be shown in a standard way as in [20], so we omit the proof. In this section, we intend to establish the a priori estimates.

Lemma 4.1 ($L^2$-estimates). Under the conditions of Theorem 3.1, if $\phi \in X(0,T)$ is a solution of (39)-(38), there is a constant $C > 0$ such that for all $0 \leq t \leq T$,
\[ \|\phi(\cdot,t)\|^2_2 + \int_0^t \int_{\mathbb{R}\setminus(R_1,R_2)} \left| \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right| \phi^2 \, dz \, dt + \int_0^t \|\phi(\cdot,\tau)\|^2_2 \, d\tau \leq C\|\phi_0\|^2_2 + C \int_0^t \int_{\mathbb{R}} (|F\phi| + |G\phi|) \, dz \, d\tau \]
for some small $0 < a < m$, where $m$ is defined in (23) and $R_1 < 0$ and $R_2 > 0$ are defined in (45).

Proof. Multiplying (39) by $2w\phi$, we have
\[
\begin{align*}
(w\phi_\cdot)_t + (h'(U)w\phi^2 - 2\frac{1-U_z^2}{(1+U_z^2)^2}w\phi_\tau + \left( \frac{1-U_z^2}{(1+U_z^2)^2}w \right) z \phi^2) = 2wF\phi + 2wG\phi
\end{align*}
\]
where $w$ is defined in (47).

After integrating (53) with respect to $z$ over $\mathbb{R}$, we have
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}} w\phi^2 \, dz - \int_{\mathbb{R}} \left( (h'(U)w)_z \phi^2 + \left( \frac{1-U_z^2}{(1+U_z^2)^2}w \right) z \phi^2 \right) \, dz + 2 \int_{\mathbb{R}} \frac{1-U_z^2}{(1+U_z^2)^2} w\phi_\tau^2 \, dz = 2 \int_{\mathbb{R}} (wF\phi + wG\phi) \, dz.
\end{align*}
\]

Now we estimate the second term on the left hand side of (54) by considering two cases: $z \in \mathbb{R}\setminus(R_1, R_2)$ and $z \in (R_1, R_2)$, where $R_1$ and $R_2$ are defined in (45).

Case 1. $z \in \mathbb{R}\setminus(R_1, R_2)$, so $w(z) = w_1(z)$ from (47).

From the definition of $w(z)$ in (47) and equation (25), direct calculation gives us
\[
h'(U)w = U_{zz} \left( \frac{1}{a^4} U_z + \frac{3}{a^2} + \frac{3}{a} \frac{1}{U_z} \right)
\]
and
\[
\left( \frac{1-U_z^2}{(1+U_z^2)^2}w \right)_z = \left( \frac{1}{a^4} U_z^2 + \frac{3}{a^2} U_z + \frac{3}{a} \right)_z = U_{zz} \left( \frac{2}{a^3} U_z + \frac{3}{a^2} \right).
\]
Combining (55) and (56), we obtain
\[ h'(U)w + \left( \frac{1 - U_z^2}{(1 + U_z^2)^2} \right)_z = U_{zzz} \left( \frac{3}{a^3} U_z + \frac{6}{a^2} + \frac{3}{a} U_z \right). \]  
(57)

Differentiating (57) with respect to \( z \), and substituting (25) and (26), it follows
\[ \left( h'(U)w + \left( \frac{1 - U_z^2}{(1 + U_z^2)^2} \right)_z \right)_z = U_{zzz} \left( \frac{3}{a^3} U_z + \frac{6}{a^2} + \frac{3}{a} U_z \right) \]
[\[ = h'(U)^2 \left[ \frac{1}{(1 + U_z^2)^2} \right] \left( 1 + \frac{2U_z^2(3 - U_z^2)}{1 - U_z^2} \right) \left( \frac{3}{a^3} U_z^2 + \frac{6}{a^2} U_z + \frac{3}{a} U_z \right) + h''(U) \left( \frac{1 + U_z^2}{1 - U_z^2} \right)^2 \left( \frac{3}{a^3} U_z^2 - \frac{3}{a} \right). \]
(58)

By (22) \( U_z < 0 \) and by (46) \( \frac{U_z}{a} + 1 > 0 \) for \( z \in \mathbb{R} \setminus (R_1, R_2) \), we obtain from (58) that
\[ \left( h'(U)w + \left( \frac{1 - U_z^2}{(1 + U_z^2)^2} \right)_z \right)_z \leq (h'(U))^2 \frac{U_z}{a} \left( \frac{U_z}{a} + 1 \right) \left( \frac{1 + U_z^2}{1 - U_z^2} \right)^2 \left[ - \frac{1}{a} + \frac{2|U_z|(3 - U_z^2)}{1 - U_z^2} + \frac{|h''(U)|}{(h'(U))^2} \frac{1 - U_z^2}{1 + U_z^2} \right]. \]  
(59)

In order to establish \( L^2 \)-estimate (52), we show the following inequality
\[ \frac{2|U_z|(3 - U_z^2)}{1 - U_z^2} + \frac{|h''(U)|}{(h'(U))^2} \frac{1 - U_z^2}{1 + U_z^2} < \frac{1}{2a} \]  
(60)

for any \( z \in \mathbb{R} \setminus (R_1, R_2) \) with some choice of \( a \). To prove the inequality (60) with some choice of \( a \), we find an upper bound of the left hand side of (60). Indeed, by (23) and (24), the first term on the left hand side of (60) is bounded by \( \frac{6m}{m^2} \).

For the second term on the left hand side of (60), noting that \( h'(U) = f'(U) - c \) from (6), \( h''(U) = f''(U) \), and there is an \( M > 0 \) such that \( |f''(U)| \leq M \) for any \( z \in \mathbb{R} \). Moreover, the non-degeneracy condition (8) yields that \( h'(U) = f'(U) - c \) is nonzero for any \( z \in \mathbb{R} \setminus (R_1, R_2) \). Hence, the second term is bounded by \( \frac{M}{(f'(U) - c)^2} \).

Therefore, the left hand side of (60) is bounded by
\[ \frac{6m}{1 - m^4} + \frac{M}{(f'(U) - c)^2}. \]  
(61)

Now if we let \( a' = \frac{m}{2} \), there is an \( \alpha' > 0 \) such that \( |f'(U) - c| \geq \alpha' > 0 \) for any \( z \in \mathbb{R} \setminus (R_1', R_2') \), where \( R_1' \) and \( R_2' \) are defined in (45) with \( a' \). Plugging \( |f'(U) - c| \geq \alpha' \) into (61), we are able to find an \( a_0 > 0 \) such that
\[ \frac{6m}{1 - m^4} + \frac{M}{\alpha'^2} < \frac{1}{2a_0}. \]  
(62)

For any \( a \) satisfying
\[ 0 < a \leq \min\{a_0, a'\}, \]  
(63)
there exist an \( \alpha \geq \alpha' \) such that \( |f'(U) - c| \geq \alpha \geq \alpha' > 0 \) for all \( z \in \mathbb{R} \setminus (R_1, R_2) \). Therefore,

\[
\frac{2|U_z|(3 - U_z^2)}{1 - U_z^4} + \frac{|h''(U)|}{(h'(U))^2} \left( 1 - \frac{U_z^2}{1 + U_z^2} \right)^2 \leq \frac{6m}{1 - m^4} + \frac{M}{\alpha^2} < \frac{1}{2a}
\]

for \( z \in \mathbb{R} \setminus (R_1, R_2) \). Hence (60) is proved with \( a \) satisfying (63).

Now plugging (60) into (59), we deduce that

\[
\int_{\mathbb{R} \setminus (R_1, R_2)} \left( h'(U) w + \left( \frac{1}{1 + U_z^2} - w z \right) \right) dz \leq -3 \int_{\mathbb{R} \setminus (R_1, R_2)} \left( h'(U) \right)^2 \frac{|U_z|}{a^2} \left( \frac{U_z}{a} + 1 \right) \left( 1 + U_z^2 \right)^4 \frac{\phi^2}{(1 - U_z^2)^2} dz
\]

(64)

for some \( 0 < C_0 < \frac{3(h'(U))^2}{m_0} \), where \( m_0 \) is given in (24).

**Case 2.** \( z \in (R_1, R_2) \), so \( w = w_2 \) in (47).

We first calculate \( (h'(U) w + \left( \frac{1}{1 + U_z^2} - w z \right) \right) \). By the definition of \( w(z) \) in (47) and equation (25), it holds

\[
h'(U) w = h'(U) \left( \frac{1 + U_z^2}{1 - U_z^2} \right) \left( - \frac{1}{U_z} \right) = -U_{zz} U_z^2.
\]

(65)

On the other hand, by (47), we obtain

\[
\left( \frac{1}{1 + U_z^2} - w z \right) z = \left( \frac{1}{U_z} \right) z = U_{zz}. \]

(66)

Thus, we have

\[
(h'(U) w + \left( \frac{1}{1 + U_z^2} - w z \right) \right)_z = 0.
\]

(67)

Hence, we arrive at

\[
\int_{R_1}^{R_2} \left( (h'(U) w)_z + \left( \frac{1}{1 + U_z^2} - w z \right) \right) \phi^2 dz = 0.
\]

(68)

Combining the estimates (64) and (68), and using (23), (24), (48), one deduces that

\[
\frac{d}{dt} \| \phi(\cdot, t) \|^2 + \int_{\mathbb{R} \setminus (R_1, R_2)} \left| \frac{U_z}{a} \left( \frac{U_z}{a} + 1 \right) \right| \phi^2 dz + \| \phi(\cdot, t) \|^2 \leq C \int_R (|F\phi| + |G\phi|) dz
\]

(69)

for some \( C > 0 \). Consequently, we complete the proof of (52) by integrating (69) with respect to \( t \).

**Remark 2.** By choosing the constant \( a > 0 \) in (45) satisfying (63), the difficulty in dealing with the non-convexity of \( f \) is overcome, see (60). It is worth mentioning that the existence of such an \( a \) is guaranteed only when \( f'(u(t)) \neq c \). Since (60) no longer holds for the degenerate case (9) and (21), we need to reconstruct the weight function \( w(z) \) in Section 5.
Next, we derive estimates of the first order derivative of \( \phi \).

**Lemma 4.2 (H^1-estimates).** Under the same assumptions in Lemma 4.1, there is a constant \( C > 0 \) such that

\[
\| \phi_z(\cdot, t) \|^2 + \int_0^t \| \phi_{zzz}(\cdot, \tau) \|^2 d\tau \\
\leq C \| \phi_0 \|^2 + C \int_0^t \int_{\mathbb{R}} (|F\phi| + |G\phi| + F_z \phi_z + G_z \phi_z + F_{zzz} \phi_{zzz} + G_{zzz} \phi_{zzz}) dz d\tau
\]

for all \( 0 \leq t \leq T \).

**Proof.** Differentiating (39) with respect to \( z \), multiplying the result by \( 2\phi \), and integrating the resulting equation with respect to \( z \) over \( \mathbb{R} \), we have

\[
\frac{d}{dt} \int_R \phi_z^2 + 2 \int_R \frac{1 - U_z^2}{(1 + U_z^2)^2} \phi_{zzz}^2 dz \\
\leq \int_R (|h'(U)|)_{zzz} |\phi_z|^2 dz + 2 \int_R (F_z \phi_z + G_z \phi_z) dz.
\]

Integrating (71) with respect to \( t \), using (23) and (24), and combining the \( L^2 \)-estimates (52), we obtain the desired \( H^1 \)-estimates (70).

We shall estimate the second order derivative of \( \phi \).

**Lemma 4.3 (H^2-estimates).** Under the same assumptions in Lemma 4.1, there is a constant \( C > 0 \) such that

\[
\| \phi_{zz}(\cdot, t) \|^2 + \int_0^t \| \phi_{zzzz}(\cdot, \tau) \|^2 d\tau \\
\leq C \| \phi_0 \|^2 + C \int_0^t \int_{\mathbb{R}} (|F\phi| + |G\phi| + F_z \phi_z + G_z \phi_z + F_{zzz} \phi_{zzz} + G_{zzz} \phi_{zzz}) dz d\tau
\]

for all \( 0 \leq t \leq T \).

**Proof.** Differentiating (39) with respect to \( z \) twice, multiplying the result by \( 2\phi_{zz} \), and integrating the resulting equation with respect to \( z \) over \( \mathbb{R} \), we obtain

\[
\frac{d}{dt} \int_R \phi_{zz}^2 + 2 \int_R \frac{1 - U_z^2}{(1 + U_z^2)^2} \phi_{zzzz}^2 dz \\
\leq \int_R (|h'(U)|)_{zzzz} |\phi_z|^2 dz + 2 \int_R (F_z \phi_z + G_z \phi_z) dz.
\]

Integrating (73) with respect to \( t \), using (23) and (24), and combining the \( L^2 \)-estimates (52) and \( H^1 \)-estimates (70), the proof of (72) is completed.

In conclusion, from the \( L^2 \)-estimates (52), \( H^1 \)-estimates (70) and \( H^2 \)-estimates (72), there is a constant \( C > 0 \) such that

\[
\| \phi_0 \|^2 + \int_0^t \int_{\mathbb{R} \setminus (R_1, R_2)} \left| \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right| |\phi|^2 dz d\tau + \int_0^t \| \phi_z(\cdot, \tau) \|^2 d\tau \\
\leq C \| \phi_0 \|^2 + C \int_0^t \int_{\mathbb{R}} (|F\phi| + |G\phi| + F_z \phi_z + G_z \phi_z + F_{zzz} \phi_{zzz} + G_{zzz} \phi_{zzz}) dz d\tau
\]

for all \( 0 \leq t \leq T \), where \( R_1, R_2 \) and \( a \) are given in (45) and (60).
Indeed, all the cubic terms in (74) are bounded by $CN$ by using integration by parts, the Cauchy-Schwarz inequality and (75) as follows

\[
|F| = O(1)(\phi_z^2), \quad |F_z| = O(1)(\phi_z^2 + |\phi_z\phi_{zz}|),
\]

\[
|F_{zz}| = O(1)(\phi_z^2 + \phi_{zz}^2 + |\phi_z\phi_{zzz}| + |\phi_z\phi_{zzzz}|),
\]

\[
|G| = O(1)(\phi_{zz}^2), \quad |G_z| = O(1)(\phi_{zz}^2 + |\phi_{zz}\phi_{zzz}|),
\]

\[
|G_{zz}| = O(1)(\phi_{zz}^2 + \phi_{zzz}^2 + |\phi_{zz}\phi_{zzzz}| + |\phi_{zz}\phi_{zzzzz}|).
\]

(75)

Indeed, all the cubic terms in (74) are bounded by $CN(t) \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau$ for some $C > 0$, where $N(t)$ is defined in (50). For example, $\int_R^t \int_R G_{zzz} \phi_{zzz} d\tau$ is estimated by using integration by parts, the Cauchy-Schwarz inequality and (75) as follows

\[
\left| \int_R^t \int_R G_{zzz} \phi_{zzz} d\tau \right| \leq \int_R^t \int_R |G_{zzz} \phi_{zzz}| d\tau \leq C \int_R^t \int_R (|\phi_{zzz}^2\phi_{zzz}| + |\phi_{zzz}\phi_{zzzz}|) d\tau \leq CN(t) \int_0^t \|\phi_{zzz}(\cdot, \tau)\|_2^2 d\tau
\]

for some $C > 0$. Therefore, substituting the estimated results of the cubic terms into (74), we derive that

\[
\|\phi(\cdot, t)\|_2^2 + \int_0^t \int_R (U_z^2 \left( \frac{U_z}{a} + 1 \right) |\phi|^2 dz d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau \leq C\|\phi_0\|_2^2 + CN(t) \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau
\]

(77)

for some $C > 0$.

Taking $\delta_2 > 0$ in Proposition 3 such that $4C\delta_2 < 1$, from (77) we conclude that for any $0 \leq t \leq T$, where $T > 0$ is from Proposition 3,

\[
\|\phi(\cdot, t)\|_2^2 + \int_0^t \int_R (U_z^2 \left( \frac{U_z}{a} + 1 \right) |\phi|^2 dz d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_2^2 d\tau \leq CN^2(0)
\]

(78)

for some $C > 0$ as desired in Proposition 3. Therefore, the proof of (42) in Theorem 3.1 is completed by the continuation arguments based on the local existence in Proposition 2 and the a priori estimates in Proposition 3.

It remains to prove (43) in Theorem 3.1. From (42), we derive

\[
\|\phi_z(\cdot, t)\|_1^2 \to 0 \quad \text{as} \quad t \to \infty,
\]

and it follows for all $z \in \mathbb{R}$ that

\[
\phi_z^2(z, t) = 2 \int_{-\infty}^z \phi_y \phi_{yy}(y, t) dy \leq 2 \left( \int_{-\infty}^\infty \phi_z^2 dz \right)^{1/2} \left( \int_{-\infty}^\infty \phi_{zz}^2 dz \right)^{1/2} \to 0 \quad \text{as} \quad t \to \infty.
\]

(79)

We finally complete the proof of Theorem 3.1. Consequently, Theorem 2.2 is proved.
Remark 3. In the case \( f(u) = \frac{u^2}{2} \), authors in [9, 26, 5] proved that there exist smooth traveling wave solutions for \( 0 < |u_- - u_+| < 2 \) and solutions become discontinuous at \( |u_- - u_+| = 2 \). Stability of smooth traveling waves were shown when \( 0 < |u_- - u_+| \leq 1.8 \) in [15]. Applying the method developed in this paper to [15], we can improve the stability results in [15] from \( 0 < |u_- - u_+| \leq 1.8 \) to \( 0 < |u_- - u_+| < 2 \).

5. The degenerate case - proof of Theorem 2.3. We prove Theorem 2.3 by proving Theorem 3.2 in this section. We assume that \( f'((u_+) = c < f'(u_-) \) and \( n_+ \) satisfies (21). For simplicity of notation, we replace \( n_+ \) by \( n \) throughout this section.

We show global existence of \( \phi \) to the problem (39)-(38) by using the weighted energy estimates. However, due to the degeneracy condition \( f'((u_+) = c \), the weight function \( w(z) \) in (47), which is introduced for the non-degenerate case (8), does not work for \( z \) in a neighborhood of \( \infty \). Thus, to overcome such a difficulty, we need to construct a new weight function \( \tilde{w}(z) \).

On the other hand, since \( c < f'((u_-) \) which is nondegenerate, we can still use similar arguments in defining the weight function \( w(z) \) in (47) when \( z \) is bounded above.

For a small \( 0 < a < m \), where \( m \) is defined in (23) and \( a \) is to be determined in Lemma 5.1, there is an \( R < 0 \) such that

\[
|U_z(R)| = a \quad \text{and} \quad 0 < |U_z(z)| < a, \quad z < R. \tag{80}
\]

With \( a \) and \( R \) satisfying (80), we introduce a new weight function \( \tilde{w}(z) \) as follows.

\[
\tilde{w} = \tilde{w}(z) = \begin{cases} 
\tilde{w}_1(z) := \frac{(1 + U_z^2)^2}{1 - U_z^2} \left( \frac{1}{a^3}U_z^2 + \frac{3}{a^2}U_z + \frac{3}{a} \right) & \text{if } z \leq R, \\
\tilde{w}_2(z) := \frac{(1 + U_z^2)^2}{1 - U_z^2} \left( -\frac{1}{U_z} \right) & \text{if } z > R.
\end{cases} \tag{81}
\]

One can easily check that \( \tilde{w}(z) \in C^2(\mathbb{R}) \). Moreover, there is a significant difference between \( \tilde{w}(z) \) in (81) and \( w(z) \) defined for the non-degeneracy in (47). While \( w(z) \) is bounded (48), \( \tilde{w}(z) \) is unbounded and the rate is determined by the degeneracy of \( f \) at far fields

\[
\tilde{w}(z) \sim |U_z|^{-1} \sim |z|^{1+1/n} \sim (z)_+^{1+1/n} \quad \text{as} \quad z \to \infty, \tag{82}
\]

see (28).

Now we define the solution space as

\[
X_{\tilde{w}}(0,T) = \{ \phi \in C([0,T); H^2 \cap L^2_{\tilde{w}}), \quad \phi_z \in L^2([0,T); H^2 \cap L^2_{\tilde{w}}) \} \tag{83}
\]

with \( 0 < T \leq \infty \). By the Sobolev embedding theorem, if we set

\[
N_{\tilde{w}}(t) = \sup_{0 \leq \tau \leq t} \{ \| \phi(\cdot, \tau) \|_{\tilde{w}} + \| \phi_z(\cdot, \tau) \|_{1} \}, \tag{84}
\]

then

\[
\sup_{z \in \mathbb{R}} \{ \sqrt{\tilde{w}}|\phi_z|, |\phi_z| \} \leq C N_{\tilde{w}}(t) \tag{85}
\]

for some \( C > 0 \).

We then prove Theorem 3.2 by establishing the following \textit{a priori} estimates.
Lemma 5.1 ($L^2$-estimates). Under the conditions of Theorem 3.2, if $\phi \in X_0(0,T)$ is a solution of (39)-(38), then there is a constant $C > 0$ such that

$$\|\phi(\cdot, t)\|_{2_w}^2 + \int_0^t \int_{-\infty}^t \left| \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right| \phi^2 dz d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_{2_w}^2 d\tau$$

$$\leq C\|\phi_0\|_{2_w}^2 + C \int_0^t \int_{\mathbb{R}} \left( \hat{w}F\phi + \hat{w}G\phi \right) dz d\tau$$

(86)

for all $0 \leq t \leq T$ and for some small $0 < a < m$, where $m$ is defined in (23) and $R < 0$ is defined in (80).

Proof. Multiplying (39) by $2\hat{w}\phi$, and integrating the resulting equation with respect to $z$ over $\mathbb{R}$, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}} \hat{w}\phi^2 dz - \int_{\mathbb{R}} \left( (h'U)\hat{w} \right)_z + \left( \frac{1-U_z^2}{1+U_z^2} \hat{w} \right) \phi^2 dz = 2 \int_{\mathbb{R}} \left( \hat{w}F\phi + \hat{w}G\phi \right) dz$$

(87)

where $\hat{w}$ is defined in (81). In order to estimate the second term of (87), due to the structure of $\hat{w}$ in (81), we consider two cases: $z \leq R$ and $z > R$.

For the non-degenerate part $z \leq R$, we choose a small $a > 0$ in a similar way in obtaining (60) in Lemma 4.1. Similar to the derivation of (64) gives us

$$\int_{-\infty}^R \left( (h'U)\hat{w} \right)_z + \left( \frac{1-U_z^2}{1+U_z^2} \hat{w} \right) \phi^2 dz \leq C \int_{-\infty}^R \frac{|U_z|}{a} \left( \frac{U_z}{a} + 1 \right) \phi^2 dz$$

(88)

for some $C > 0$.

For the degenerate part $z > R$, from (65)-(67), it holds

$$\int_R^\infty \left( (h'U)\hat{w} \right)_z + \left( \frac{1-U_z^2}{1+U_z^2} \hat{w} \right) \phi^2 dz = 0.$$  

(89)

From (88) and (89), the second term of (87) is estimated as

$$\int_{-\infty}^R \left( (h'U)\hat{w} \right)_z + \left( \frac{1-U_z^2}{1+U_z^2} \hat{w} \right) \phi^2 dz$$

$$= \int_{-\infty}^R \left( (h'U)\hat{w} \right)_z + \left( \frac{1-U_z^2}{1+U_z^2} \hat{w} \right) \phi^2 dz$$

$$\leq C \int_{-\infty}^R \frac{|U_z|}{a^2} \left( \frac{U_z}{a} + 1 \right) \phi^2 dz$$

(90)

for some $C > 0$.

Substituting (90) into (87), one has

$$\frac{d}{dt} \|\phi(\cdot, t)\|_{2_w}^2 + \int_{-\infty}^R \left| \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right| \phi^2 dz + \|\phi_z(\cdot, t)\|_{2_w}^2 \leq C \int_{\mathbb{R}} \left( \hat{w}F\phi + \hat{w}G\phi \right) dz$$

(91)

for some $C > 0$.

Integrating (91) with respect to $t$, the proof of (86) is completed. \hfill \Box

Moreover the following estimates are proved. We omit details.
Lemma 5.2 \((H^1\)-estimates\). Under the assumptions in Lemma 5.1, it holds
\[
\| \phi_z(\cdot,t) \|^2 + \int_0^t \| \phi_{zz}(\cdot,\tau) \|^2 \, d\tau \\
\leq C \| \phi_0 \|^2_{\bar{w}} + C \| \phi_{z,0} \|^2 \quad (92)
\]
for some \( C > 0 \) and any \( 0 \leq t \leq T \).

Lemma 5.3 \((H^2\)-estimates\). Let the assumptions in Lemma 5.1 hold. There exists a constant \( C > 0 \) such that
\[
\| \phi_{zz}(\cdot,t) \|^2 + \int_0^t \| \phi_{zzz}(\cdot,\tau) \|^2 \, d\tau \\
\leq C \| \phi_0 \|^2_{\bar{w}} + C \| \phi_{z,0} \|^2 \quad (93)
\]
for any \( 0 \leq t \leq T \).

Now, in order to derive the desired \textit{a priori} estimates for (44), we shall combine all the results from the weighted \(L^2\)-estimates (86) and \(H^1\)- and \(H^2\)-estimates (92), (93). We then conclude that there is a \( C > 0 \) such that
\[
\| \phi(\cdot,t) \|^2_{\bar{w}} + \| \phi_z(\cdot,t) \|^2_{\bar{w}} \\
+ \int_0^t \int_{-\infty}^t |\frac{U_z}{a} \left( \frac{U_z}{a} + 1 \right) - \phi^2 \, dz \, d\tau + \int_0^t \| \phi_z(\cdot,\tau) \|^2_{\bar{w}} \, d\tau + \int_0^t \| \phi_{zz}(\cdot,\tau) \|^2 \, d\tau \\
\leq C \| \phi_0 \|^2_{\bar{w}} + C \| \phi_{z,0} \|^2 \quad (94)
\]
for all \( 0 \leq t \leq T \), where \( a > 0 \) is determined in Lemma 5.1 and \( R < 0 \) is defined in (80). The third term on the right hand side of (94) can be estimated by the following inequalities
\[
\int_0^t \int_{-\infty}^t |\tilde{\omega} F \phi| \, dz \, d\tau \leq C N_{\bar{w}}(t) \int_0^t \| \phi_z(\cdot,\tau) \|^2_{\bar{w}} \, d\tau \quad (95)
\]
and
\[
\int_0^t \int_{-\infty}^R |\tilde{G} \phi| \, dz \, d\tau \leq C N_{\bar{w}}(t) \int_0^t \sqrt{\phi_z^2} \, dz \, d\tau \\
\leq C N_{\bar{w}}(t) \int_0^t \phi_z^2 \, dz \, d\tau + C N_{\bar{w}}(t) \int_0^t \sqrt{\phi_z^2} \, dz \, d\tau \\
\quad (96)
\]
for some $C > 0$, where $N_w(t)$ is defined in (84). In fact, by (82), $z^{\frac{1}{2} + \frac{\alpha}{a}} \phi_z^2 = (z^{\frac{1}{2} + \frac{\alpha}{a}} \phi_z \phi_{zz})_z - (z^{\frac{1}{2} + \frac{\alpha}{a}} \phi_z \phi_{zzz})_z - z^{\frac{1}{2} + \frac{\alpha}{a}} \phi_z \phi_{zzz}$, and the Cauchy-Schwarz inequality, the second term of (96) is further estimated as

$$CN_w(t) \int_0^t \int_\mathbb{R} \sqrt{z^{1 + \frac{\alpha}{a}} \phi_z^2} dz \, d\tau$$

$$\leq CN_w(t) \int_0^t \int_\mathbb{R} (\sqrt{w|\phi_z \phi_{zz}} + \sqrt{w|\phi_z \phi_{zzz}}) dz \, d\tau$$

$$\leq CN_w(t) \int_0^t \left( \|\phi_z(\cdot, \tau)\|_{L^\infty}^2 + \|\phi_{zz}(\cdot, \tau)\|_{L^1}^2 \right) d\tau$$

for some $C > 0$. It is noticed that the last term on the right hand side of (94) is estimated by $CN_w(t) \int_0^t \|\phi_z(\cdot, \tau)\|_{L^2}^2 d\tau$ for some $C > 0$.

If $N_w(0) < \delta_w$ for some $\delta_w > 0$ small enough, the same arguments in obtaining the a priori estimates (78) from the $L^2$-estimates (52), $H^1$-estimates (70) and $H^2$-estimates (72) yield

$$\|\phi(\cdot, t)\|_{H^2}^2 + \|\phi_z(\cdot, t)\|_{L^2}^2$$

$$+ \int_0^t \int_{-\infty}^\infty \left[ \frac{U_z}{a} \left( \frac{U_z}{a} + 1 \right) \right] \phi^2 \, dz \, d\tau + \int_0^t \|\phi_z(\cdot, \tau)\|_{L^2}^2 \, d\tau + \int_0^t \|\phi_{zz}(\cdot, \tau)\|_{L^1}^2 \, d\tau$$

$$\leq CN_w^2(0)$$

for some $C > 0$ and all $0 \leq t \leq T$. By (82) we finally obtain the desired estimates (44).

6. Decay rates for the non-degenerate case. In this section, we investigate algebraic and exponential rates of convergence for the non-degenerate case (8).

6.1. Algebraic decay rates. In this subsection, we prove that the perturbation converges algebraically in time if the initial perturbation decays algebraically in space. Kawashima and Matsumura [6] and Matsumura and Nishihara [17] introduced the rates of asymptotic speed based on the weighted energy estimates. We also refer to [15, 19, 21, 22] for the improved techniques.

For $\alpha > 0$, we define the solution space as

$$X_\alpha(0, T) = \{ \phi \in C([0, T); H^2 \cap L_\alpha^2), \phi_z \in L^2([0, T); H^2 \cap L_\alpha^2) \}$$

(98)

with $0 < T \leq \infty$, where $L_\alpha^2$ is defined in (11). If we set

$$N_\alpha(t) = \sup_{0 \leq \tau \leq t} \{ \|\phi(\cdot, \tau)\|_\alpha + \|\phi_z(\cdot, \tau)\|_1 \},$$

(99)

then the Sobolev embedding theorem yields

$$\sup_{z \in \mathbb{R}} \{ \sqrt{z^{\alpha}}|\phi|, |\phi_z| \} \leq CN_\alpha(t)$$

(100)

for some $C > 0$, where $\langle z \rangle^\alpha$ is defined in (10).

Theorem 2.4 is a consequence of the following theorem.

Theorem 6.1. Suppose that the assumptions of Theorem 2.4 hold. Let $\alpha > 0$. Then there is a constant $\delta_\alpha > 0$ such that if

$$N_\alpha(0) < \delta_\alpha,$$
the problem (39)-(38) has a unique global solution \( \phi \in X_\alpha(0, \infty) \) satisfying

\[
(1 + t)^\alpha \|\phi(., t)\|_2^2 + \int_0^t (1 + \tau)^\alpha \|\phi_z(., \tau)\|_2^2 d\tau \leq CN_\alpha^2(0)
\]  

(101)

and for any \( \varepsilon > 0 \)

\[
(1 + t)^\alpha \|\phi(., t)\|_2^2 + (1 + t)^{-\gamma} \int_0^t (1 + \tau)^{\alpha+\gamma} \|\phi_z(., \tau)\|_2^2 d\tau \leq CN_\alpha^2(0)
\]  

(102)

for some \( C > 0 \) and for all \( 0 \leq t \leq T \).

Theorem 6.1 is shown by the continuation arguments based on the local existence and the \( a \ priori \) estimates in a similar framework in proving Theorem 3.1. Hence, we devote to establish the \( a \ priori \) estimates in this subsection.

First, we derive \( L^2 \)-estimates for \( \phi \) on \( |z| > r \) for some \( r > 0 \) in the following lemma.

**Lemma 6.2.** Let \( \alpha > 0 \), \( \gamma \geq 0 \) and \( \beta \) be any number in \([0, \alpha]\). Under the assumptions of Theorem 6.1, for some \( r > \max\{\|R_1\|, R_2\} \) large enough, where \( R_1 \) and \( R_2 \) are given in (45), we have for any \( 0 \leq t \leq T \)

\[
(1 + t)^\gamma \|\phi(\cdot, t)\|_\beta^2 + \beta \int_0^t \int_{|z|>r} (1 + \tau)^\gamma \langle z \rangle^{\beta-1} \hat{\phi}^2 d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}\setminus(\mathbb{R}_1, \mathbb{R}_2)} (1 + \tau)^\gamma \langle z \rangle^\beta \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \|\phi^2 d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|_3^2 d\tau
\]

\[
\leq C \|\phi_0\|_\beta^2 + C \gamma \int_0^t (1 + \tau)^\gamma \|\phi(\cdot, \tau)\|_3^2 d\tau
\]

\[
+ C \int_0^t \int_{\mathbb{R}} (1 + \tau)^\gamma \langle z \rangle^\beta (|F\phi| + |G\phi|) dz d\tau
\]

(103)

for some \( C > 0 \) which is independent of \( \beta \) and \( \gamma \).

**Proof.** With \( w \) defined in (47), multiplying (39) by \( 2(1 + t)^\gamma \langle z \rangle^\beta w \phi \), and integrating the resulting equation with respect to \( z \) over \( \mathbb{R} \), we have

\[
\frac{d}{dt} \int_{\mathbb{R}} (1 + t)^\gamma \langle z \rangle^\beta w \phi^2 dz
\]

\[
+ \int_{\mathbb{R}} (1 + t)^\gamma \left( I_1(z, t) + I_2(z, t) \right) dz + 2 \int_{\mathbb{R}} (1 + t)^\gamma \langle z \rangle^\beta \frac{1 - U_z^2}{(1 + U_z^2)^2} w \phi_z^2 dz
\]

\[
= \gamma \int_{\mathbb{R}} (1 + t)^{\gamma-1} \langle z \rangle^\beta w \phi^2 dz - 2\beta \int_{\mathbb{R}} (1 + t)^\gamma \langle z \rangle^{\beta-2} z \frac{1 - U_z^2}{(1 + U_z^2)^2} w \phi_z dz
\]

\[
+ 2 \int_{\mathbb{R}} (1 + t)^\gamma \langle z \rangle^\beta (w F\phi + w G\phi) dz
\]

where

\[
I_1(z, t) := -\beta \langle z \rangle^{\beta-2} z \left( h'(U)w + \left( \frac{1 - U_z^2}{(1 + U_z^2)^2} w \right)_z \right) \phi^2,
\]

\[
I_2(z, t) := -\langle z \rangle^\beta \left( (h'(U)w)_z + \left( \frac{1 - U_z^2}{(1 + U_z^2)^2} w \right)_{zz} \right) \phi^2.
\]

(105)

To estimate the second term on the left hand side of (104), we shall estimate \( \int_{\mathbb{R}} I_1(z, t)dz \) and \( \int_{\mathbb{R}} I_2(z, t)dz \).
Let us estimate $\int_R I_1(z,t)dz$ by dividing into two cases: $z \in \mathbb{R}\backslash(R_1, R_2)$ and $z \in (R_1, R_2)$. Applying the results (55)-(57) and (25) to the case $z \in \mathbb{R}\backslash(R_1, R_2)$ and the results (65) and (66) to $z \in (R_1, R_2)$, one derives

$$
\int_R I_1(z,t)dz = -\beta \int_{\mathbb{R}\backslash(R_1, R_2)} \langle z \rangle^{\beta-2} z h'(U) \left(\frac{1+U_z^2}{1-U_z^2} 3 \left(\frac{U_z}{a} + 1\right)^2\right) \phi^2 dz. \quad (106)
$$

It is noticed that the non-degeneracy condition (8) implies

$$-zh'(U) \geq c_0 > 0, \quad z \in \mathbb{R}\backslash(R_1, R_2) \quad (107)$$

for some $c_0 > 0$. Furthermore, choose $r > \max\{|R_1|, R_2\} > 0$ large enough so that we have, for any $|z| > r$,

$$\left(\frac{U_z}{a} + 1\right)^2 \geq c_1 > 0 \quad (108)$$

for some $c_1 > 0$. Then substituting (107) and (108) into (106), and using (107), we conclude that

$$\int_R I_1(z,t)dz \geq c_2 \beta \int_{|z| > r} \langle z \rangle^{\beta-1} \phi^2 dz \quad (109)$$

for some $c_2 > 0$.

Similarly, the results (64) for $z \in \mathbb{R}\backslash(R_1, R_2)$ and (68) for $z \in (R_1, R_2)$ yield

$$\int_R I_2(z,t)dz \geq C_0 \int_{\mathbb{R}\backslash(R_1, R_2)} \langle z \rangle^{\beta} \left| U_z \right|^2 \left(\frac{U_z}{a} + 1\right) \phi^2 dz \quad (110)$$

where $C_0$ is given in (64).

Now substituting (109) and (110) into (104), and using (48) and (24), we deduce

$$\frac{d}{dt} \int_R (1+t)^\gamma \langle z \rangle^{\beta} w \phi^2 dz + \beta \int_{|z| > r} (1+t)^\gamma \langle z \rangle^{\beta-1} \phi^2 dz \leq C\gamma \int_R (1+t)^{\gamma-1} \langle z \rangle^{\beta} \phi^2 dz + C\beta \int_R (1+t)^\gamma \langle z \rangle^{\beta-1} \phi_z dz + C \int_R (1+t)^\gamma \langle z \rangle^{\beta} (|F\phi| + |G\phi|) dz \quad (111)$$

for some $C > 0$.

Furthermore, with $r > 0$ large enough so that (108) holds, the second term on the right hand side of (111) can be estimated by the following estimates

$$C\beta \int_R \langle z \rangle^{\beta-1} |\phi\phi_z| dz = C\beta \int_{|z| \leq r} \langle z \rangle^{\beta-1} |\phi\phi_z| dz + C\beta \int_{|z| > r} \langle z \rangle^{\beta-1} |\phi\phi_z| dz \leq C\beta \int_{|z| \leq r} \langle z \rangle^{\beta-1} \phi^2 dz + C\beta \int_{|z| \leq r} \langle z \rangle^{\beta-1} \phi_z^2 dz \quad (112)$$

$$+ \frac{\beta}{2} \int_{|z| > r} \langle z \rangle^{\beta-1} \phi^2 dz + C\beta \int_{|z| > r} \langle z \rangle^{\beta} \phi_z^2 dz$$
for some $C > 0$. By the boundedness of $(z)^{\beta-1}$ for $|z| \leq r$ and some choice of $\hat{r} > r$, the second and fourth terms on the right hand side of (112) can be further estimated by

$$C\beta \|\phi_z(\cdot, t)\|^2 + \frac{1}{2} \int_{|z| > \hat{r}} (z)^{\beta-1} \phi_z^2 dz \leq C\beta \|\phi_z(\cdot, t)\|^2 + \frac{1}{2} \|\phi_z(\cdot, t)\|_{\beta}^2$$

(113)

for some $C > 0$.

Combining (112) with (113), inserting the resulting estimate of $C\beta \int_R (z)^{\beta-1} |\phi_z| dz$ into (111), and using (48), we finally arrive at

$$\frac{d}{dt} (1 + t)^\gamma \|\phi(\cdot, t)\|_{\beta}^2 + \beta \int_0^t \int_{|z| > r} (1 + t)^\gamma (z)^{\beta-1} \phi^2 dz dr$$

$$+ \int_{\mathbb{R}\setminus (R_1, R_2)} (z)^{\beta-1} \left( \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right) \phi^2 dz$$

$$\leq C\gamma (1 + t)^{-1} \|\phi(\cdot, t)\|_{\beta}^2 + C\beta \int_0^t (1 + t)^\gamma (z)^{\beta-1} \phi^2 dz$$

$$+ C\beta \|\phi_z(\cdot, t)\|^2 + C \int_\mathbb{R} (1 + t)^\gamma (\phi(\cdot, t))^2 dz$$

(114)

for some $C > 0$.

Integrating the resulting inequality (114) with respect to $t$, we finally obtain the desired (103).

Different from the $L^2$-estimates without weights (52), the weighted $L^2$-estimates (103) are not enough to conclude our algebraic decay rate results. Indeed, we need to obtain the weighted $L^2$-estimate for $\phi$ on $(-r, r)$ to be added to (103) so that we can yield the desired estimate (130) as in [6, 17].

To do this, we first present the following estimate

$$(1 + t)^\gamma \|\phi(\cdot, t)\|^2$$

$$+ \int_0^t \int_{\mathbb{R}\setminus (R_1, R_2)} (1 + \tau)^\gamma \left( \frac{U_z}{a^2} \left( \frac{U_z}{a} + 1 \right) \right) \phi^2 dz d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|^2 d\tau$$

$$\leq C \|\phi_0\|^2 + C\gamma \int_0^t (1 + \tau)^{-1} \|\phi(\cdot, t)\|^2 d\tau + C \int_0^t (1 + \tau)^\gamma (\|F\phi| + |G\phi|) dz d\tau$$

(115)

for some $C > 0$, where $a > 0$ is given in (60) and $R_1 < 0$ and $R_2 > 0$ are defined in (45). Estimate (115) can be proved similarly as in proving (52), so we omit the proof for brevity. Next, using the above estimate, we derive the desired weighted $L^2$-estimate for $\phi$ on $(-r, r)$ in the following lemma.

**Lemma 6.3.** Let $\beta, \gamma \geq 0$ be given in Lemma 6.2 and $r > \max \left\{ |R_1|, |R_2|, \frac{1}{16c_\pm} \right\} > 0$ large enough, where $R_1, R_2$ and $c_\pm$ are given in (45) and (27). Under the assumptions of Theorem 6.1, there is a constant $C > 0$ such that

$$\int_0^t \int_{|z| \leq r} (1 + \tau)^\gamma (z)^{\beta-1} \phi^2 dz d\tau$$

$$\leq C \|\phi_0\|^2 + C\gamma \int_0^t (1 + \tau)^{-1} \|\phi(\cdot, \tau)\|^2 d\tau + C \int_0^t (1 + \tau)^\gamma (\|F\phi| + |G\phi|) dz d\tau$$

(116)
for any $0 \leq t \leq T$.

**Proof.** Since we have

$$\int_0^t \int_{|z| \leq r} (1 + \tau)^{\gamma - 1} \phi^2 dz d\tau \leq (1 + r^2)^{\frac{\gamma - 1}{2}} \int_0^t \int_{|z| \leq r} (1 + \tau)\phi^2 dz d\tau,$$

it is sufficient to show that $\int_0^t \int_{|z| \leq r} (1 + \tau)\phi^2 dz d\tau$ is bounded by the right hand side of (116).

For a given $r > 0$ large enough satisfying

$$r > \max \left\{ |R_1|, R_2, \frac{1}{16c_\pm} \right\} > 0$$

(117)

where $R_1, R_2$ and $c_\pm$ are given in (45) and (27), we define

$$p = p(r) := \frac{1}{8r^2} > 0.$$  

(118)

Now multiplying (39) by $(1 + t)^\gamma e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi$, and integrating the resulting equation with respect to $z$ over $\mathbb{R}$, we obtain

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi^2 dz \\
+ \int_\mathbb{R} (1 + t)^\gamma (p - 2p^2z^2)e^{-pz^2}\phi^2 dz + \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2}\phi^2 dz \\
= \frac{\gamma}{2} \int_\mathbb{R} (1 + t)^{-1}e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi^2 dz - \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{h'(U)}{1 - U_z^2} \phi \phi_z dz \\
+ \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi' \phi dz.
\end{align*}$$

(119)

Here, by the Cauchy-Schwarz inequality, the second term on the right hand side of (119) can be further estimated as

$$\begin{align*}
\left| \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} h'(U) \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi \phi_z dz \right| \\
\leq \frac{p}{2} \int_{|z| \leq r} (1 + t)^\gamma e^{-pz^2} \phi^2 dz + \frac{p}{2} \int_{|z| > r} (1 + t)^\gamma e^{-pz^2} \phi^2 dz \\
+ \frac{1}{2p} \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{h'(U)^2}{1 - U_z^2} \phi \phi_z^2 dz.
\end{align*}$$

(120)

Plugging (120) into (119), we arrive at

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi^2 dz + \frac{p}{4} \int_{|z| \leq r} (1 + t)^\gamma e^{-pz^2} \phi^2 dz \\
\leq \frac{\gamma}{2} \int_\mathbb{R} (1 + t)^{-1}e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi^2 dz + \int_\mathbb{R} (1 + t)^\gamma (-\frac{p}{2} + 2p^2z^2)e^{-pz^2} \phi^2 dz \\
+ \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{1}{2p} \frac{(h'(U)^2)}{1 - U_z^2} \phi \phi_z^2 dz \\
+ \int_\mathbb{R} (1 + t)^\gamma e^{-pz^2} \frac{(1 + U_z^2)^2}{1 - U_z^2} \phi' \phi dz.
\end{align*}$$

(121)
where we have split the second integral on the left hand side of (119) into two parts: \(|z| \leq r\) and \(|z| > r\). We have also used (118) to have \(\frac{p}{2} - 2p^2z^2 \geq \frac{p}{2} - 2p^2r^2 = \frac{p}{2}\) for \(|z| \leq r\).

Now we shall estimate the second term on the right hand side of (121). For \(r > 0\) large enough, there is a constant \(C > 0\) such that

\[
\frac{-p}{2} + 2p^2z^2 \leq Ce^{c|z|} \quad (122)
\]

for \(|z| > r\). By denoting \(c := \max\{c_{\pm}\} > 0\) and by (117), for \(r < |z| < 16cr^2\), since \(e^{-p(z^2-r^2)} \leq 1 \leq e^{c(32cr^2-2|z|)}\), we have

\[
e^{-p|z|^2} \leq (e^{-p^2} + 32c^2r^2)e^{-2c|z|}. \quad (123)
\]

On the other hand, for \(|z| \geq 16cr^2\), (118) leads to

\[
e^{-p|z|^2} \leq e^{-2c|z|}. \quad (124)
\]

Thus, noticing that \(\left|\frac{U_z}{a^2} \left(\frac{U_z}{a} + 1\right)\right| \sim e^{-c|z|}\), from (122)-(124) we conclude that there is a constant \(C > 0\) such that

\[
\int_{|z| > r} e^{-p|z|^2} \phi^2 dz \leq C \int_{R \setminus (R_1, R_2)} \left|\frac{U_z}{a^2} \left(\frac{U_z}{a} + 1\right)\right| \phi^2 dz. \quad (125)
\]

Now substituting (125) into (121), integrating (121) with respect to \(t\), and combining the resulting inequality with (115), the proof of (116) is finally completed. 

For a given \(\alpha > 0\) and for any \(0 \leq \beta \leq \alpha\) and \(\gamma \geq 0\), adding (103) and \(\beta \times (116)\), we derive that for any \(0 \leq t \leq T\),

\[
(1 + t)^\gamma \|\phi(\cdot, t)\|^2_\beta + \beta \int_0^t (1 + \tau)^\gamma \|\phi(\cdot, \tau)\|^2_{\beta-1} d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|^2_{\beta} d\tau
\]

\[
\leq C\|\phi_0\|^2_\beta + C\gamma \int_0^t (1 + \tau)^{\gamma-1} \|\phi(\cdot, \tau)\|^2_\beta d\tau + C \int_0^t \int_R (1 + \tau)^\gamma \langle z \rangle^\beta (|\Phi \phi| + |G \phi|) dz d\tau
\]

for some \(C > 0\) which is independent of \(\beta\) and \(\gamma\).

Moreover, the last term on the right hand side of (126) is estimated by

\[
\int_0^t \int_R (1 + \tau)^\gamma \langle z \rangle^\beta |\Phi \phi| d\tau \leq CN_\alpha(t) \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|^2_\beta d\tau \quad (127)
\]

and

\[
\int_0^t \int_R (1 + \tau)^\gamma \langle z \rangle^\beta |G \phi| dz d\tau \leq CN_\alpha(t) \int_0^t \int_R (1 + \tau)^\gamma \langle z \rangle^2 \phi_z^2 dz d\tau \quad (128)
\]

for some \(C > 0\). Furthermore, using

\[
\langle z \rangle^2 \phi_z^2 = (\langle z \rangle^2 \phi_z \phi_{zz})_z - \frac{\beta}{2} (\langle z \rangle^2 z^2 \phi_z \phi_{zz} - \langle z \rangle^2 \phi_z \phi_{zzz})
\]

and the Cauchy-Schwarz inequality, the right hand side of (128) is majored by

\[
CN_\alpha(t) \int_0^t (1 + \tau)^\gamma (\|\phi_z(\cdot, \tau)\|^2_\beta + \|\phi_{zz}(\cdot, \tau)\|^2_\beta) d\tau \quad (129)
\]

for some \(C > 0\).

Therefore, we have the following.
Lemma 6.4 (Weighted $L^2$-estimates). Under the assumptions of Theorem 6.1, for any $\alpha > 0$ and $\gamma \geq 0$, $0 \leq \beta \leq \alpha$, it holds for any $0 \leq t \leq T$

$$(1 + t)^\gamma \|\phi(\cdot, t)\|_\beta^2 + \beta \int_0^t (1 + \tau)^\gamma \|\phi(\cdot, \tau)\|_{\beta - 1}^2 d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|_{\beta}^2 d\tau$$

$$\leq C \|\phi_0\|_\beta^2 + C \gamma \int_0^t (1 + \tau)^{\gamma - 1} \|\phi(\cdot, \tau)\|^2 d\tau + C N_\alpha(t) \int_0^t (1 + \tau)^\gamma (\|\phi_z(\cdot, \tau)\|^2 + \|\phi_{zz}(\cdot, \tau)\|_{\beta}^2) d\tau$$

(130)

for some $C > 0$ which is independent of $\gamma$ and $\beta$.

Next we establish the $H^1$- and $H^2$-estimates for $\phi$ when $\beta = 0$. In obtaining the $H^1$- and $H^2$- estimates, we multiply $(1 + t)^\gamma$ to perform an iteration argument in Lemma 6.6 with the weighted $L^2$-estimates. We omit the proofs.

Lemma 6.5 ($H^1, H^2$-estimates with $\beta = 0$). Under the assumption of Theorem 6.1, for any $\gamma \geq 0$, it holds

$$(1 + t)^\gamma \|\phi_z(\cdot, t)\|_\beta^2 + \int_0^t (1 + \tau)^\gamma \|\phi_{zz}(\cdot, \tau)\|_{\beta}^2 d\tau$$

$$\leq C \|\phi_0\|_\beta^2 + C \gamma \int_0^t (1 + \tau)^{\gamma - 1} \|\phi(\cdot, \tau)\|^2 d\tau$$

$$+ C N_\alpha(t) \int_0^t (1 + \tau)^\gamma (\|\phi_z(\cdot, \tau)\|^2 + \|\phi_{zz}(\cdot, \tau)\|_{\beta}^2) d\tau$$

(131)

for some $C > 0$ and any $0 \leq t \leq T$.

If $N_\alpha(0) < \delta_\alpha$ for some $\delta_\alpha > 0$ small, it follows from the same arguments in obtaining the a priori estimates (78) from the $L^2$-estimates (52), $H^1$-estimates (70) and $H^2$-estimates (72) that for any $0 \leq t \leq T$

$$(1 + t)^\gamma \|\phi(\cdot, t)\|_\beta^2 + (1 + t)^\gamma \|\phi_z(\cdot, t)\|_\beta^2 + \beta \int_0^t (1 + \tau)^\gamma \|\phi(\cdot, \tau)\|_{\beta - 1}^2 d\tau$$

$$+ \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|_{\beta}^2 d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_{zz}(\cdot, \tau)\|_{\beta}^2 d\tau$$

(132)

$$\leq C N_\alpha^2(0) + C \gamma \int_0^t (1 + \tau)^{\gamma - 1} \|\phi(\cdot, \tau)\|_{\beta}^2 d\tau$$

for some $C > 0$ which is independent of $\beta$ and $\gamma$.

From now on, we prove (101) and (102) in Theorem 6.1. First, by applying the principle of induction in [15, 22] to (132), we establish the desired estimates (101) when $0 \leq \gamma \leq [\alpha]$.

Lemma 6.6. Let $\alpha > 0$ and $\gamma = 0, 1, \cdots, [\alpha]$. Then it holds

$$(1 + t)^\gamma \|\phi(\cdot, t)\|_{\alpha - \gamma}^2 + (1 + t)^\gamma \|\phi_z(\cdot, t)\|_1^2 + (\alpha - \gamma) \int_0^t (1 + \tau)^\gamma \|\phi(\cdot, \tau)\|_{\alpha - \gamma - 1}^2 d\tau$$

$$+ \int_0^t (1 + \tau)^\gamma \|\phi_z(\cdot, \tau)\|_{\alpha - \gamma}^2 d\tau + \int_0^t (1 + \tau)^\gamma \|\phi_{zz}(\cdot, \tau)\|_1^2 d\tau$$

$$\leq C N_\alpha^2(0)$$
for some $C > 0$ and all $0 \leq t \leq T$. Consequently, for any $0 \leq \gamma \leq [\alpha]$,
\[
(1 + t)^\gamma \|\phi(\cdot, t)\|_2^2 + \int_0^t (1 + \tau)^\gamma \|\phi_\varepsilon(\cdot, \tau)\|_2^2 d\tau \leq CN_\alpha(0)^2
\]
for some $C > 0$ and all $0 \leq t \leq T$.

Next, a technique improved by [19, 21] enables us to conclude the desired estimates (102) as follows.

**Lemma 6.7.** Let $\alpha > 0$ and $\varepsilon > 0$. Then it holds
\[
(1 + t)^{\alpha \varepsilon} \|\phi(\cdot, t)\|_2^2 + \int_0^t (1 + \tau)^{\alpha \varepsilon} \|\phi_\varepsilon(\cdot, \tau)\|_2^2 d\tau \leq C(1 + t)^\varepsilon N_\alpha^2(0)
\]
for some $C > 0$ and any $0 \leq t \leq T$. Consequently,
\[
(1 + t)^{\alpha} \|\phi(\cdot, t)\|_2^2 + (1 + t)^{-\varepsilon} \int_0^t (1 + \tau)^{\alpha + \varepsilon} \|\phi_\varepsilon(\cdot, \tau)\|_2^2 d\tau \leq CN_\alpha^2(0)
\]
(133)
for some $C > 0$ and any $0 \leq t \leq T$.

**Remark 4.** For a non-integer $\alpha > 0$, the authors in [15] obtained the decay rate result, $\sup_{z \in \mathbb{R}} |\phi_\varepsilon(z, t)| \leq C_\varepsilon(1 + t)^{-\frac{d}{2} + \varepsilon} N_\alpha^2(0)$, where $C_\varepsilon \to \infty$ if $\varepsilon \to 0$. By applying the technique in [19, 21], the result can be improved as shown in (133).

6.2. **Exponential decay rates.** In this subsection, we prove that if the initial perturbation decays exponentially in space, then the solution to the problem (39)-(38) converges exponentially to the smooth traveling wave solution asymptotically in time. We refer to [12, 15, 18, 19] for details.

We define a new weight function
\[
w_\varepsilon = w_\varepsilon(z) = e^{d(z)}
\]
(134)
for some $d > 0$, where $\langle z \rangle$ is defined in (10).

To investigate the exponential time decay rate, we define the solution space as
\[
X_\varepsilon(0, t) = \{\phi \in \mathcal{C}([0, T); H^2 \cap L^2_{w_\varepsilon}), \ \phi_\varepsilon \in L^2([0, T); H^2 \cap L^2_{w_\varepsilon})\}
\]
(135)
with $0 < T \leq \infty$. If we let
\[
N_\varepsilon(t) = \sup_{0 \leq \tau \leq t} \{\|\phi(\cdot, \tau)\|_{w_\varepsilon} + \|\phi_\varepsilon(\cdot, \tau)\|_1\},
\]
(136)
the Sobolev embedding theorem leads to
\[
\sup_{z \in \mathbb{R}} \{\sqrt{w_\varepsilon} \phi, |\phi_\varepsilon|\} \leq CN_\varepsilon(t)
\]
(137)
for some $C > 0$.

We have the following theorem.

**Theorem 6.8.** Suppose the assumptions of Theorem 2.5 hold. Then there is a constant $\delta_\varepsilon > 0$ such that if
\[
N_\varepsilon(0) < \delta_\varepsilon,
\]
the problem (39)-(38) has a unique global solution $\phi \in X_\varepsilon(0, \infty)$ satisfying
\[
\|\phi(\cdot, t)\|_{w_\varepsilon}^2 + \|\phi_\varepsilon(\cdot, t)\|_1^2 + \theta \int_0^t \|\phi(\cdot, \tau)\|_{1, w_\varepsilon}^2 d\tau + \theta \int_0^t \|\phi_\varepsilon(\cdot, \tau)\|_1^2 d\tau \leq CN_\varepsilon^2(0)
\]
(138)
for some constants $C > 0$ and $\theta > 0$. Consequently, it holds
\[
\sup_{z \in \mathbb{R}} \vert \phi_z(z, t) \vert \leq C N_0(0) e^{-\frac{\theta}{2} t}
\] (139)
for some $C > 0$.

The above theorem is proved by the standard procedure in obtaining Theorem 3.1. Since the derivation of the \textit{a priori} estimates is similar to one in Subsection 6.1, we omit the proofs.

7. Conclusion. We have studied the existence and stability of smooth traveling wave solutions arising from image processing. Different from the previous works on image processing [9, 26, 5, 15] with convex flux function $f(u) = \frac{u^2}{2}$, we dealt with general fluxes $f$. In particular, we established the asymptotic stability theorems with general fluxes by constructing weight functions to overcome the non-convexity of the fluxes and to achieve for arbitrary wave strengths. Using our weight functions, the proofs require only the entropy condition, which allows us to show the stability results for any arbitrary wave strengths without imposing additional conditions on the general fluxes [7, 10, 11].

Nonlinear stability was proved for two cases: the non-degenerate case (8) and the degenerate case (9), which occur when $f$ is non-convex. For the non-degenerate case (8), we constructed a new weight function in (47) which is piecewisely defined on bounded and unbounded domains. Due to our construction of weight function, the $L^2$-integral of $\phi$ on the bounded domain disappears in the $L^2$ estimate. On the unbounded domain, by manipulating $a > 0$ in (47) and using the entropy condition, we achieved the $L^2$-estimates (52). Combining the same weight function with algebraic or exponential weights, we showed that the solution converges to the traveling wave with respective rates in time. For the degenerate case (9), we defined another new weight function in (81) whose essential feature at far fields comes from the degeneracy.

Moreover, our current results of stability and decay rates for the non-degenerate case improved the previous results in [15] where $f(u) = \frac{u^2}{2}$. The techniques for construction of the weight functions from this paper could be adapted to prove stability for other problems. We plan to solve such problems in the future.

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