EDGE-TRANSITIVE CORE-FREE NEST GRAPHS

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Abstract. A finite simple graph $\Gamma$ is called a Nest graph if it is regular of valency 6 and admits an automorphism $\rho$ with two orbits of the same length such that at least one of the subgraphs induced by these orbits is a cycle. We say that $\Gamma$ is core-free if no non-trivial subgroup of the group generated by $\rho$ is normal in $\text{Aut}(\Gamma)$. In this paper we show that, if $\Gamma$ is edge-transitive and core-free, then it is isomorphic to one of the following graphs: the complement of the Petersen graph, the Hamming graph $H(2, 4)$, the Shrikhande graph and a certain normal 2-cover of $K_{3,3}$ by $\mathbb{Z}_4^2$.

1. Introduction

All groups in this paper will be finite and all graphs will be finite and simple. A graph admitting an automorphism with two orbits of the same length is called a bicirculant. Symmetry properties of bicirculants have attracted considerable attention (see, e.g., [1, 5, 7, 16, 22, 23, 25, 29]). Following [17], for an integer $d \geq 3$, we denote by $\mathcal{F}(d)$ the family of regular graphs having valency $d$ and admitting an automorphism with two orbits of the same length such that at least one of the subgraphs induced by these orbits is a cycle. Jajcay et al. [11] initiated the investigation of the edge-transitive graphs in the classes $\mathcal{F}(d)$, $d \geq 6$. The families $\mathcal{F}(d)$ with $3 \leq d \leq 5$ were studied under different names. The graphs in $\mathcal{F}(3)$ were introduced by Watkins [27] under the name generalised Petersen graphs, the graphs in $\mathcal{F}(4)$ by Wilson [28] under the name Rose Window graphs, and the graphs in $\mathcal{F}(5)$ by Arroyo et al. [2] under the name Tabac\'ijn graphs. The automorphism groups of these graphs form the subject of the papers [10, 15, 9, 18], and the question which of them are edge-transitive has been answered in [10, 15, 2].

Jajcay et al. [11] asked whether there exist edge-transitive graphs in $\mathcal{F}(d)$ for $d \geq 6$. Following [26], they call the graphs in $\mathcal{F}(6)$ Nest graphs. Several infinite families of edge-transitive Nest graphs were exhibited, which turn out to have interesting properties (e.g., half-arc-transitivity). However, no edge-transitive graph of valency larger than 6 was found. Recently, it was proved by the author and Ruff [17] that the complement of the Petersen graph is the only edge-transitive graph in $\mathcal{F}(d)$ with $d \geq 6$, which has twice an odd number of vertices. The main result of [11] is the classification of the edge-transitive Nest graphs of girth 3 (see [11, Theorem 8]), and the task to classify all edge-transitive Nest graphs was posed as [11, Problem 2]. In what follows, the Nest graphs will be described via their representation due to [11, Construction 3], which goes as follows. Let $n \geq 4$ and let $a, b, c, k \in \mathbb{Z}_n$ such that each of them is distinct from 0 (the zero element of $\mathbb{Z}_n$), the elements

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a, b and c are pairwise distinct, and in the case when n is even, \( k \neq n/2 \). Then the Nest graph \( \mathcal{N}(n; a, b, c; k) \) is defined to have vertex set \( \{ u_i : i \in \mathbb{Z}_n \} \cup \{ v_i : i \in \mathbb{Z}_n \} \), and three types of edges such as

- \( \{ u_i, u_{i+1} \} \) for \( i \in \mathbb{Z}_n \) (rim edges),
- \( \{ v_i, v_{i+k} \} \) for \( i \in \mathbb{Z}_n \) (hub edges),
- \( \{ u_i, v_i \}, \{ u_i, v_{i+a} \}, \{ u_i, v_{i+b} \} \) and \( \{ u_i, v_{i+c} \} \) for \( i \in \mathbb{Z}_n \) (spoke edges),

where the sums in the subscripts are computed in \( \mathbb{Z}_n \). It is easy to see that the permutation \( \rho \) of \( V(\Gamma) \), defined as \( \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}) \), is an automorphism of \( \Gamma \) with orbits \( \{ u_i : i \in \mathbb{Z}_n \} \) and \( \{ v_i : i \in \mathbb{Z}_n \} \), and the subgraph induced by the former orbit is a cycle. It is not hard to show that all the graphs \( \mathcal{N}(n; a, b, c; k) \) comprise the whole family \( \mathcal{F}(6) \).

In the case of both the Rose Window and the Tabačjn graphs, the classification of the edge-transitive graphs was obtained in two main steps. The so called core-free graphs were found first and the rest was retrieved from the core-free graphs using covering techniques (see \[13\][2]). Here is the formal definition of a core-free Nest graph.

**Definition 1.1.** Let \( \Gamma = \mathcal{N}(n; a, b, c; k) \) be a Nest graph and \( \rho \) be the permutation of \( V(\Gamma) \) defined as \( \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}) \). Then \( \Gamma \) is core-free if no non-trivial subgroup of \( \langle \rho \rangle \) (the group generated by \( \rho \)) is normal in \( \text{Aut}(\Gamma) \).

**Remark.** The term “core-free” comes from group theory. For a subgroup \( A \leq B \), the core of \( A \) in \( B \) is the largest normal subgroup of \( B \) contained in \( A \). In the case when \( A \) has trivial core, it is also called core-free. In this context, Definition 1.1 can be rephrased by saying that \( \Gamma \) is core-free if and only if \( \langle \rho \rangle \) is core-free in \( \text{Aut}(\Gamma) \).

Our goal in this paper is to determine the edge-transitive core-free Nest graphs. For an explanation why this task is a more subtle than in the case of Rose Window and Tabačjn graphs, we refer to \[11\] p. 9. The edge-transitive non-core-free Nest graphs are handled in the paper \[14\].

The main result of this paper is the following theorem.

**Theorem 1.2.** If \( \mathcal{N}(n; a, b, c; k) \) is an edge-transitive core-free graph, then it is isomorphic to one of the following graphs:

\[
\mathcal{N}(5; 1, 2, 3; 2), \mathcal{N}(8; 1, 3, 4; 3), \mathcal{N}(8; 1, 2, 5; 3) \text{ and } \mathcal{N}(12; 2, 4, 8; 5).
\]

**Remark.** The fact that each of the Nest graphs in Theorem 1.2 is core-free was mentioned by Jajcay et al., see \[11\] p. 9. The first three of them are well-known strongly regular graphs. The Nest graph \( \mathcal{N}(5; 1, 2, 3; 2) \) is the complement of the Petersen graph, \( \mathcal{N}(8; 1, 3, 4; 3) \) is the Hamming graph \( H(2, 4) \), and \( \mathcal{N}(8; 1, 2, 5; 3) \) is the Shrikhande graph. The fourth Nest graph \( \mathcal{N}(12; 2, 4, 8; 5) \) is not strongly-regular, it can be described as a normal 2-cover of the complete bipartite graph \( K_{3,3} \) by \( \mathbb{Z}_4^2 \) (for the definition of a normal 2-cover, see the 2nd paragraph of Subsection 2.1).

The paper is organised as follows. Section 2 contains the needed results from graph and group theory. In Section 3 we review some results about Nest graphs obtained in \[11,17\]. Section 4 is devoted to the Nest graphs in the form \( \mathcal{N}(2m; 2, m, 2 + m; 1) \), \( m \) is odd. The
main result (Proposition 4.1) is a characterisation, which was mentioned in [11] without a proof, and which is needed for us in the proof Theorem 1.2. The latter proof is presented in Section 5.

2. Preliminaries

2.1. Graph theory. Given a graph $\Gamma$, let $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\text{Aut}(\Gamma)$ denote its vertex set, edge set, arc set and automorphism group, respectively. The number $|V(\Gamma)|$ is called the order of $\Gamma$. The set of vertices adjacent with a given vertex $v$ is denoted by $\Gamma(v)$. If $G \leq \text{Aut}(\Gamma)$ and $v \in V(\Gamma)$, then the stabiliser of $v$ in $G$ is denoted by $G_v$, the orbit of $v$ under $G$ by $v^G$, and the set of all $G$-orbits by $\text{Orb}(G, V(\Gamma))$. If $B \subseteq V(\Gamma)$, then the setwise stabiliser of $B$ in $G$ is denoted by $G_{\{B\}}$. If $G$ is transitive on $V(\Gamma)$, then $\Gamma$ is said to be $G$-vertex-transitive, and $\Gamma$ is simply called vertex-transitive when it is $\text{Aut}(\Gamma)$-vertex-transitive; $(G)$-edge- and $(G)$-arc-transitive graphs are defined correspondingly.

Let $\pi$ be an arbitrary partition of $V(\Gamma)$ and for a vertex $v \in V(\Gamma)$, let $\pi(v)$ denote the class containing $v$. The quotient graph of $\Gamma$ with respect to $\pi$, denoted by $\Gamma/\pi$, is defined to have vertex set $\pi$, and edges $\{\pi(u), \pi(v)\}$, where $\{u, v\} \in E(\Gamma)$ such that $\pi(u) \neq \pi(v)$. Now, if there exists a constant $r$ such that

$$\forall\{u, v\} \in E(\Gamma) : \pi(u) \neq \pi(v) \text{ and } |\Gamma(u) \cap \pi(v)| = r,$$

then $\Gamma$ is called an $r$-cover of $\Gamma/\pi$. The term cover will also be used instead of 1-cover. In the special case when $\pi = \text{Orb}(N, V(\Gamma))$ for an intransitive normal subgroup $N \triangleleft \text{Aut}(\Gamma)$, $\Gamma/N$ will also be written for $\Gamma/\pi$ and when $\Gamma$ is also an $r$-cover (cover, respectively) of $\Gamma/N$, then the term normal $r$-cover (normal cover, respectively) will also be used. It is well-known that this is always the case when $\Gamma$ is edge-transitive. More precisely, if $\Gamma$ is a $G$-edge-transitive graph, $\Gamma$ is regular with valency $\kappa$, and $N \triangleleft G$ is intransitive, then $\Gamma$ is a normal $r$-cover of $\Gamma/N$ for some $r$ and $r$ divides $\kappa$.

A graph admitting a regular cyclic group of automorphisms is called circulant. A recursive classification of finite arc-transitive circulants was obtained independently by Kovács [13] and Li [19]. The paper [13] also provides an explicit characterisation (see [13, Theorem 4]), which was rediscovered recently by Li et al. [20]. The characterisation presented below follows from the proof of [13, Theorem 4] or from [20, Theorem 1.1].

In what follows, given a cyclic group $C$ and a divisor $d$ of $|C|$, we denote by $C_d$ the unique subgroup of $C$ of order $d$.

**Theorem 2.1.** ([13]) Let $\Gamma$ be a connected arc-transitive graph of order $n$ and of valency $\kappa$ and suppose that $C \leq \text{Aut}(\Gamma)$ is a regular cyclic subgroup. Then one of the following holds.

(a) $\Gamma$ is the complete graph.
(b) $C$ is normal in $\text{Aut}(\Gamma)$.
(c) $B = \text{Orb}(C_d, V(\Gamma))$ is a block system for $\text{Aut}(\Gamma)$ for some divisor $d$ of $\gcd(n, \kappa)$, $d > 1$. $\Gamma$ is a normal $d$-cover of $\Gamma/B$, and $\Gamma/B$ is a connected arc-transitive circulant of valence $\kappa/d$.
(d) $B_1 = \text{Orb}(C_d, V(\Gamma))$ and $B_2 = \text{Orb}(C_{n/d}, V(\Gamma))$ are block systems for $\text{Aut}(\Gamma)$ for some divisor $d$ of $n$ such that $d > 1$, $\gcd(d, n/d) = 1$ and $d-1$ divides $\kappa$. $\Gamma/B_1$ is a connected
arc-transitive circulant of valency \( \kappa/(d-1) \), \( \Gamma/B_2 \cong K_d \), and
\[
\text{Aut}(\Gamma) = G_1 \times G_2,
\]
where \( C_d \leq G_1, G_1 \cong S_d, C_{n/d} < G_2, \) and \( G_2 \cong \text{Aut}(\Gamma/B_1) \).

**Remark.** Although not used later, it is worth mentioning that the graph \( \Gamma \) in part (c) is isomorphic to the lexicographical product \( \Gamma/B[K_d] \), where \( K_d \) is the edgeless graph on \( d \) vertices, and the graph \( \Gamma \) in part (d) is isomorphic to the tensor (direct) product \( K_d \times \Gamma/B_1 \) (see [13, 20]).

In the rest of the section we restrict ourselves to arc-transitive circulants of small valency.

**Lemma 2.2.** ([3, part (ii) of Corollary 1.3]) Let \( \Gamma \) be a connected arc-transitive graph of order \( n \) and of valency \( \kappa \), where \( \kappa = 3 \) or \( 4 \), and suppose that \( C \leq \text{Aut}(\Gamma) \) is a regular cyclic subgroup. Then one of the following holds.

1. \( \Gamma \) is isomorphic to one of the graphs: \( K_4, K_5, K_{3,3} \) and \( K_{5,5} - 5K_2 \).
2. \( \kappa = 4 \) and \( C \) is normal in \( \text{Aut}(\Gamma) \).
3. \( \kappa = 4, n \) is even, \( B = \text{Orb}(C_3, V(\Gamma)) \) is a block system for \( \text{Aut}(\Gamma) \), and \( \Gamma \) is a normal 2-cover of \( \Gamma/B \), which is a cycle.

**Lemma 2.3.** Let \( \Gamma \) be a connected arc-transitive graph of order \( n > 14 \) and of valency \( 6 \), and suppose that \( C \leq \text{Aut}(\Gamma) \) is a regular cyclic subgroup. Then \( \text{Aut}(\Gamma) \) contains a normal subgroup \( N \) such that one of the following holds.

1. \( N = C \), or
2. \( n \equiv 4 \pmod{8} \) and \( N = C_{n/4} \), or
3. \( N \cong \mathbb{Z}_\ell^3 \) for \( \ell \geq 2 \) and \( C_3 < N \).

**Proof.** \( \Gamma \) belongs to one of the families (a)–(d) in Theorem 2.1.

Family (a): This case cannot occur as \( n > 14 \).

Family (b): Part (1) follows.

Family (c): In this case \( \text{Orb}(C_d, V(\Gamma)) \) is a block system for \( \text{Aut}(\Gamma) \), where \( d = 2 \) or \( d = 3 \). Let \( B = \text{Orb}(C_d, V(\Gamma)) \).

If \( d = 2 \), then \( \Gamma/B \) has valency 3. It follows from Lemma 2.2 that \( n \leq 12 \), but this is excluded.

If \( d = 3 \), then choose \( N \) to be the Sylow 3-subgroup of the kernel of the action of \( \text{Aut}(\Gamma) \) on \( B \). It follows that \( N \cong \mathbb{Z}_\ell^3 \) for some \( \ell \geq 2 \). Also, \( N \) is characteristic in the latter kernel, which implies that \( N \lhd \text{Aut}(\Gamma) \). Finally, \( \text{Orb}(N, V(\Gamma)) = B = \text{Orb}(C_3, V(\Gamma)) \), and so \( C_3 < N \), i.e., part (3) holds.

Family (d): In this case it follows from the assumption that \( n > 14 \) that \( \text{Orb}(C_4, V(\Gamma)) \) and \( \text{Orb}(C_{n/4}, V(\Gamma)) \) are block systems for \( \text{Aut}(\Gamma) \) and \( n \equiv 4 \pmod{8} \). Furthermore,
\[
\text{Aut}(\Gamma) = G_1 \times G_2,
\]
where \( C_4 < G_1, G_1 \cong S_4, C_{n/4} < G_2, G_2 \cong \text{Aut}(\Gamma/B_1) \), where \( B_1 = \text{Orb}(C_4, V(\Gamma)) \). The graph \( \Gamma/B_1 \) is connected of valency 2, hence it is a cycle of length \( n/4 \). It follows that \( C_{n/4} \) is characteristic in \( G_2 \), and as \( G_2 \lhd \text{Aut}(\Gamma) \), part (2) follows. \( \square \)
2.2. **Group theory.** Our terminology and notation are standard and we follow the books [8, 12]. The *socle* of a group $G$, denoted by $\text{soc}(G)$, is the subgroup generated by the set of all minimal normal subgroups (see [8, p. 111]). The group $G$ is called *almost simple* if $\text{soc}(G) = T$, where $T$ is a non-abelian simple group. In this case $G$ is embedded in $\text{Aut}(T)$ so that its socle is embedded via the inner automorphisms of $T$, and we also write $T \leq G \leq \text{Aut}(T)$ (see [8, p. 126]).

Our proof of Theorem 1.2 relies on the classification of primitive groups containing a cyclic subgroup with two orbits due to Müller [24]. Here we need only the special case when the cyclic subgroup is semiregular.

**Theorem 2.4.** ([24, Theorem 3.3]) Let $G$ be a primitive permutation group of degree $2n$ containing an element with two orbits of the same length. Then one of the following holds, where $G_0$ denotes the stabiliser of a point in $G$.

1. **(Affine action)** $\mathbb{Z}_2^m \vartriangleleft G \leq \text{AGL}(m, 2)$, where $n = 2^{m-1}$. Furthermore, one of the following holds.
   (a) $n = 2$, and $G_0 = \text{GL}(2, 2)$.
   (b) $n = 2$, and $G_0 = \text{GL}(1, 4)$.
   (c) $n = 4$, and $G_0 = \text{GL}(3, 2)$.
   (d) $n = 8$, and $G_0$ is one of the following groups: $\mathbb{Z}_5 \times \mathbb{Z}_4$, $\Gamma \text{L}(1, 16)$, $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_4$, $\Sigma \text{L}(2, 4)$, $\Gamma \text{L}(2, 4)$, $A_6$, $\text{GL}(4, 2)$, $(S_3 \times S_3) \times \mathbb{Z}_2$, $S_5$, $S_6$ and $A_7$.

2. **(Almost simple action)** $G$ is an almost simple group and one of the following holds.
   (a) $n \geq 3$, $\text{soc}(G) = A_{2n}$, and $A_{2n} \leq G \leq S_{2n}$ in its natural action.
   (b) $n = 5$, $\text{soc}(G) = A_5$, and $A_5 \leq G \leq S_5$ in its action on the set of $2$-subsets of $\{1, 2, 3, 4, 5\}$.
   (c) $n = (q^4 - 1)/(q - 1)$, $\text{soc}(G) = \text{PSL}(d, q)$, and $\text{PSL}(d, q) \leq G \leq \text{PGL}(d, q)$ for some odd prime power $q$ and even number $d \geq 2$ such that $(d, q) \neq (2, 3)$.
   (d) $n = 6$ and $\text{soc}(G) = G = M_{12}$.
   (e) $n = 11$, $\text{soc}(G) = M_{22}$, and $M_{22} \leq G \leq \text{Aut}(M_{22})$.
   (f) $n = 12$ and $\text{soc}(G) = G = M_{24}$.

If $G$ is a group in one of the families (a)-(f) in part (2) above, then it follows from [8, Theorem 4.3B] that $\text{soc}(G)$ is the unique minimal normal subgroup of $G$. Therefore, we have the following corollary.

**Corollary 2.5.** Let $G$ be a primitive permutation group in one of the families (a)-(f) in part (2) of Theorem 2.4 and let $N \triangleleft G$, $N \neq 1$. Then $N$ is also primitive.

For a transitive permutation group $G \leq \text{Sym}(\Omega)$, the *subdegrees* of $G$ are the lengths of the orbits of a point stabiliser $G_\omega$, $\omega \in \Omega$. Since $G$ is transitive, it follows that the subdegrees do not depend on the choice of $\omega$ (see [8, p. 72]). The number of orbits of $G_\omega$ is called the *rank* of $G$. The actions of a group $G$ on sets $\Omega$ and $\Omega'$ are said to be *equivalent* if there is a bijection $\varphi : \Omega \to \Omega'$ such that

$$\forall \omega \in \Omega, ~ \forall g \in G : \varphi(\omega^g) = (\varphi(\omega))^g.$$  

Now, suppose that $G$ is a group in one of the families (a)-(f) in part (2) of Theorem 2.4. If $G$ is in family (a), then the action is unique up to equivalence and $G$ is clearly 2-transitive. If $G$ is in family (b), then the action is unique up to equivalence and the subdegrees are
Let \( G \) be in family (c). The semiregular cyclic subgroup of \( G \) with two orbits is contained in a regular cyclic group, called the Singer subgroup of \( \text{PGL}(d, q) \) (see [12, Chapter 2, Theorem 7.3]). In this case the action is unique up to equivalence if and only if \( d = 2 \). If \( d \geq 4 \), then the action of \( G \) is equivalent to either its natural action on the set of points of the projective geometry \( \text{PG}(d-1, q) \), or to its natural action on the set of hyperplanes of \( \text{PG}(d-1, q) \). In both actions \( G \) is 2-transitive. Finally, if \( G \) is in the families (d)-(f), then the action is unique up to equivalence and \( G \) is 2-transitive (this can also be read off from [6]). All this information is summarised in the lemma below.

**Lemma 2.6.** Let \( G \) be a primitive permutation group in one of the families (a)-(f) in part (2) of Theorem 2.4.

1. \( G \) is 2-transitive, unless \( G \) belongs to family (b). In the latter case the subdegrees are 1, 3 and 6.
2. The action of \( G \) is unique up to equivalence, unless \( G \) is in family (c) and \( d \geq 4 \). In the latter case \( G \) admits two inequivalent faithful actions, namely, the natural actions on the set of points and the set of hyperplanes, respectively, of the projective geometry \( \text{PG}(d-1, q) \).

The following result about \( G \)-arc-transitive bicirculants can be found in Devillers et al. [7]. The proof works also for the edge-transitive bicirculants, in fact, it is an easy consequence of Theorem 2.4.

**Proposition 2.7.** ([7, part (1) of Proposition 4.2]) Let \( \Gamma \) be a \( G \)-edge-transitive bicirculant such that \( G \) is a primitive group. Then \( \Gamma \) is one of the following graphs:

1. The complete graph, and \( G \) is one of the 2-transitive groups described in part (2) of Theorem 2.4.
2. The Petersen graph or its complement, and \( A_5 \leq G \leq S_5 \).
3. The Hamming graph \( H(2, 4) \) or its complement, and \( G \) is a rank 3 subgroup of \( \text{AGL}(4, 2) \).
4. The Clebsch graph or its complement, and \( G \) is a rank 3 subgroup of \( \text{AGL}(4, 2) \).

Using the computational result that there exists no edge-transitive graph in \( F(d) \) with \( 7 \leq d \leq 10 \) and of order at most 100, one can easily deduce which of the graphs in the families (1)–(4) above belongs also to the family \( F(d) \) for some \( d \geq 3 \).

**Corollary 2.8.** Let \( \Gamma \in F(d) \) be a \( G \)-edge-transitive graph for some \( d \geq 3 \). If \( G \) is primitive on \( V(\Gamma) \), then \( \Gamma \) is isomorphic to one of the graphs: \( K_6 \), the Petersen graph and its complement, and the Hamming graph \( H(2, 4) \).

Finally, we also need a result of Lucchini [21] about core-free cyclic subgroups (this serves as a key tool in [15, 2] as well). For the definition of a core-free subgroup, see the remark following Definition 1.1.

**Theorem 2.9.** ([21]) If \( C \) is a core-free cyclic proper subgroup of a group \( G \), then \( |C|^2 < |G| \).

3. Nest graphs

In this section we review some previous results about Nest graphs, which were obtained in [11, 17].
Lemma 3.1. ([11] Lemma 4) Let $\Gamma = \mathcal{N}(n; a, b, c; k)$ and suppose that $c = a + b$ (in $\mathbb{Z}_n$). Then $\Gamma$ is edge-transitive if and only if it is also arc-transitive.

The next result establishes some obvious isomorphisms.

Lemma 3.2. ([11] Lemma 5) The graph $\mathcal{N}(n; a, b, c; k)$ is isomorphic to $\mathcal{N}(n; a', b', c'; k)$, where $\{a, b, c\} = \{a', b', c'\}$, as well as to any of the graphs:

\[
\mathcal{N}(n; a, b, c; -k), \mathcal{N}(n; -a, -b, -c; k) \text{ and } \mathcal{N}(n; -a, b - a, c - a; k).
\]

The graphs in the next lemma will be further studied in the next section.

Lemma 3.3. ([11] Lemma 6) If $m \geq 3$ is an odd integer, then the graph $\mathcal{N}(2m; 2, m, 2 + m; 1)$ is arc-transitive having vertex stabilisers of order 12. Furthermore, the stabiliser of $u_0$ in $\text{Aut}(\Gamma)$ is the dihedral group $D_6$ of order 12 generated by the involutions $\varphi$ and $\eta$ defined by

\[
u_i^\varphi = \begin{cases} u_{i-1} & \text{if } i \text{ is even,} \\ v_{i+1} & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad v_i^\varphi = \begin{cases} u_{i+1} & \text{if } i \text{ is even,} \\ v_{i+2} & \text{if } i \text{ is odd,} \end{cases}
\]

and $u_i^0 = u_i$ and $v_i^0 = v_{i+m}$ for every $i \in \mathbb{Z}_n$.

Suppose that $\Gamma$ is a $G$-edge-transitive Nest graph. In the next two lemmas we consider block systems for $G$. A block system $\mathcal{B}$ is said to be minimal if it is non-trivial, and no non-trivial block for $\Gamma$ is contained properly in a block of $\mathcal{B}$ (by non-trivial we mean that the block is neither a singleton subset nor the whole vertex set). We say that $\mathcal{B}$ is normal if $\mathcal{B} = \text{Orb}(N, V(\Gamma))$ for some $N \trianglelefteq G$. Furthermore, we say that $\mathcal{B}$ is cyclic if any block in $\mathcal{B}$ is contained in either $\{u_i : i \in \mathbb{Z}_n\}$ or $\{v_i : i \in \mathbb{Z}_n\}$.

Lemma 3.4. ([17] Lemma 4.1) Let $\Gamma$ be a $G$-edge-transitive Nest graph of order $2n$ such that

\[C < G, \ C = \langle \rho \rangle, \ \text{and } \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}),\]

and let $\mathcal{B}$ be a cyclic block system for $\Gamma$ with blocks of size $d$, $d < n/2$. Then the following hold.

1. The kernel of the action of $G$ on $\mathcal{B}$ is equal to $C_d$ (the subgroup of $C$ of order $d$).
2. $\Gamma$ is a normal cover of $\Gamma/\mathcal{B}$.
3. $\Gamma/\mathcal{B}$ is a $G$-edge-transitive Nest graph of order $2n/d$, where $\bar{G}$ is the image of $G$ induced by its action on $\mathcal{B}$.

Remark. Suppose that the graph $\Gamma$ in the lemma above is given as $\Gamma = \mathcal{N}(n; a, b, c; k)$ for $a, b, c, k \in \mathbb{Z}_n$. Note that, then $\Gamma/\mathcal{B} \cong \mathcal{N}(n/d; f(a), f(b), f(c); f(k))$, where $f$ is the homomorphism from $\mathbb{Z}_n$ to $\mathbb{Z}_{n/d}$ such that $f(1) = 1$.

Lemma 3.5. ([17] Lemma 4.2) Let $\Gamma$ be a $G$-edge-transitive Nest graph of order $2n$ such that

\[C < G, \ C = \langle \rho \rangle, \ \text{and } \rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1}),\]

and let $\mathcal{B}$ be a non-cyclic block system for $\Gamma$ with blocks of size $d$. Then the following hold.

1. The number $d$ is even and any block in $\mathcal{B}$ is a union of two $C_{d/2}$-orbits.
2. The group $C$ acts transitively on $\mathcal{B}$ and the kernel of the action of $C$ on $\mathcal{B}$ is equal to $C_{d/2}$.
3. If $d > 2$ and $\mathcal{B}$ is minimal, then $\mathcal{B}$ is normal.
4. A property of the graphs $N(2m; 2, m, 2 + m; 1)$, $m$ is odd

In this section we give the following characterisation of the Nest graphs in the title. As we said in the introduction, this was mentioned already in [11] without a proof.

**Proposition 4.1.** Let $\Gamma = N(n; a, b, c; k)$ be an edge-transitive graph such that $n > 8$ and suppose that there exists a non-identity automorphism of $\Gamma$, which fixes all vertices $u_i, i \in \mathbb{Z}_n$. Then $\Gamma \cong N(2m; 2, m, 2 + m; 1)$ for some odd number $m$.

We prove first an auxiliary lemma.

**Lemma 4.2.** Let $\Gamma = N(2m; a, m, a + m; k)$, where $m > 2, a = 2$ or $m - 2$, and $k = 1$ or $m - 1$. Then $\Gamma$ is edge-transitive if and only if $m$ is odd and $\Gamma \cong N(2m; 2, m, 2 + m; 1)$.

**Proof.** The “if” part follows from Lemma 3.3.

For the “only if” part, assume that $\Gamma$ is edge-transitive. Applying Lemma 3.2 to $\Gamma$, we find that

$$\Gamma \cong \Gamma' := N(2m; 2, m, 2 + m; k),$$

where $k = 1$ or $m - 1$. We have to show that $m$ is odd and $k = 1$.

Assume on the contrary that $m$ is even or $k = m - 1$. Moreover, let us choose $m$ so that it is the smallest number for which this happens. A quick check with the computer algebra package MAGMA [4] shows that $m \geq 12$.

Define the binary relation $\sim$ on $V(\Gamma')$ by letting $u \sim v$ whenever $|\Gamma'(u) \cap \Gamma'(v)| = 4$ for any $u, v \in V(\Gamma')$. Using that $m \geq 12$, it is not hard to show that $\sim$ is an equivalence relation with classes in the form

$$B_i := \{u_i, u_{i+m}, v_{i+1}, v_{i+1+m}\}, \quad i \in \mathbb{Z}_n.$$

Clearly, $\sim$ is invariant under $\text{Aut}(\Gamma')$, so the classes above form a block system for $\text{Aut}(\Gamma')$. Part of the graph with $k = m - 1$ is shown in Figure 1.

![Figure 1. The Nest graph $N(2m; 2, m, 2 + m, m - 1)$.](image-url)
Let $K$ be the kernel of the action of $\text{Aut}(\Gamma')$ on the latter block system. We prove next that $K$ is faithful on every block. Suppose that $g$ fixes pointwise the block $B_i$ for some $i \in \mathbb{Z}_n$. Any pair of vertices in $B_{i+1}$ are contained in a unique 4-cycle intersecting $B_i$ at two vertices (see Figure 1). This means that $g$ maps any pair to itself, implying that $g$ fixes pointwise $B_{i+1}$. Repeating the argument, we conclude that $g$ fixes pointwise each block $B_i$, i.e., $g$ is the identity automorphism.

Let $n = 2m$, $C = \langle \rho \rangle$, where $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$. Using the facts that $K$ is faithful on every block $B_i$ and the quotient graph $\Gamma / \mathcal{B}$ is an $n/2$-cycle whose automorphism group is isomorphic to the dihedral group $D_{n/2}$ of order $n$, we obtain the bound

$$|\text{Aut}(\Gamma')| \leq |K| \cdot n = 24n.$$  

Thus $|C|^2 = n^2 \geq |\text{Aut}(\Gamma')|$ because $n = 2m \geq 24$. By Theorem 2.9, $C$ has a non-trivial core in $\text{Aut}(\Gamma')$, let this core be denoted by $N$.

Assume first that $|N|$ is even. Then as $C_2$ is characteristic in $N$ and $N \lhd \text{Aut}(\Gamma')$, we obtain that $C_2 \lhd \text{Aut}(\Gamma')$. Thus $\text{Orb}(C_2, V(\Gamma'))$ is a block system for $\text{Aut}(\Gamma')$. But this is impossible because $u_0$ has one neighbour from the orbit $\{u_1, u_{1+m}\}$ and two from the orbit $\{v_0, v_m\}$.

Let $|N|$ be odd and choose an odd prime divisor $p$ of $|N|$. It follows as above that $C_p \lhd \text{Aut}(\Gamma')$. Clearly, $p$ divides $m$.

Assume that $m = p$ or $2p$. If $m = p$, then by our initial assumptions, $k = p - 1$. But this means that the edge $\{v_0, v_k\}$ is contained in a $C_p$-orbit, contradicting that $C_p \lhd \text{Aut}(\Gamma')$ and $\Gamma'$ is edge-transitive. If $m = 2p$, then $u_0$ has one neighbour from the $C_p$-orbit $\{u_{4i+1} : 0 \leq i \leq p - 1\}$ and two from the $C_p$-orbit $\{v_{4i} : 0 \leq i \leq p - 1\}$ (namely, $v_0$ and $v_{2+m} = v_{2+2p}$), which is a contradiction again.

Let $m > 2p$. By Lemma 3.4(3) and the remark after the lemma,

$$\Gamma' / C_p \cong \mathcal{N}(2m/p; f(2), f(m), f(2 + m), f(k)),$$

where $f$ is the homomorphism from $\mathbb{Z}_{2m}$ to $\mathbb{Z}_{2m/p}$ such that $f(1) = 1$. Since $m > 2p$ and $p$ is odd, it follows that

$$f(2) = 2, \ f(m) = m/p \text{ and } f(2 + m) = 2 + m/p.$$  

Furthermore, $f(k) = 1$ if $k = 1$ and $f(k) = m/p - 1$ if $k = m - 1$. By the minimality of $m$, we see that $m/p$ is odd and $f(k) = 1$. This, however, contradicts that $m$ is even or $k = m - 1$. \hfill \Box

Proof of Proposition 4.1. Let $H$ and $N$ be the setwise and the pointwise stabiliser, respectively, of the set $\{u_i : i \in \mathbb{Z}_n\}$ in $\text{Aut}(\Gamma)$. Then $N \neq 1$ and $N \lhd H$. It follows that the $N$-orbits contained in $V := \{v_i : i \in \mathbb{Z}_n\}$ form a block system for the action of $H$ on $V$, implying that $\text{Orb}(N, V) = \text{Orb}(C_d, V)$ for some $d > 1$. This yields that $n = 2m$ and we may write w.l.o.g. that

$$a < m, \ b = m, \ c = a + m, \text{ and } k < m.$$

Let $\eta$ be the permutation of the vertex set acting as

$$u_i^\eta = u_i \text{ and } v_i^\eta = v_{i+m} \ (i \in \mathbb{Z}_n).$$

It is easy to check that $\eta \in \text{Aut}(\Gamma)$.  

Note that, by Lemma 3.1, $\Gamma$ is arc-transitive, so $\text{Aut}(\Gamma)_w$ is transitive on $\Gamma(u_0)$. Let $s = |\Gamma(v_0) \cap \Gamma(v_m)|$. It is easy to see that $s \geq 4$. Define the graph $\Delta$ as follows:

$$V(\Delta) = \Gamma(u_0) \text{ and } E(\Delta) = \{\{w, w'\} : |\Gamma(w) \cap \Gamma(w')| = s\}.$$ 

Note that $\Delta$ is vertex-transitive, in particular, it is regular.

Assume for the moment that $u_1$ and $u_{-1}$ are adjacent in $\Delta$. This means $|\Gamma(u_1) \cap \Gamma(u_{-1})| = s \geq 4$. Since $\Gamma(u_1) \cap \Gamma(u_{-1}) \cap \{u_i : i \in \mathbb{Z}_n\} = \{u_0\}$, we conclude that

$$|\{1, 1 + a, 1 + m, 1 + a + m\} \cap \{-1, -1 + a, -1 + m, -1 + a + m\}| \geq 3. \quad (2)$$

At least one of $1$ and $1 + m$ is in the intersection. If it is $1$, then $1 = -1 + a$ or $-1 + a + m$ because $n > 4$. As $a < m$, we find that $a = 2$. Similarly, if $1 + m$ is in the intersection, then $1 + m = -1 + a$ or $-1 + a + m$, and we find again that $a = 2$. Now, substituting $a = 2$ in (3), a contradiction arises because $n > 8$.

Therefore, $u_1$ and $u_{-1}$ are not adjacent in $\Delta$. Using also that $\eta \in \text{Aut}(\Gamma)$, see above, we obtain that $u_1$ must be adjacent with $v_0$ or $v_4$.

Assume first that $u_1$ and $v_0$ are adjacent. Then $\Gamma(u_1) \cap \Gamma(v_0) = \{u_0, u_2, v_k, v_{-k}\}$, hence $2 \in \{0, n - a, m, n - a + m\}$ and $k \in \{1, 1 + a, 1 + m, 1 + a + m\}$. Since $a, k < m$ and $n > 4$, we find in turn that $a = m - 2$, and $k = 1$ or $m - 1$.

Now, if $u_1$ and $v_a$ are adjacent, then $\Gamma(u_1) \cap \Gamma(v_a) = \{u_0, u_2, v_{a+k}, v_{a-k}\}$, whence $2 \in \{a, 0, a + m, m\}$ and $a + k \in \{1, 1 + a, 1 + m, 1 + a + m\}$. Since $a, k < m$ and $n > 4$, we find in turn that $a = 2$, and $k = 1$ or $m - 1$.

To sum up, $a = 2$ or $m - 2$ and $k = 1$ or $m - 1$, and so the proposition follows from Lemma 4.2.

**Corollary 4.3.** Let $\Gamma$ be a $G$-edge-transitive Nest graph of order $2n$ such that $n > 8$ and $C < G$, $C = \langle \rho \rangle$, and $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$,

and suppose that $\text{Orb}(C_{n/2}, V(\Gamma))$ is a block system for $G$. Then $C_{n/2} \triangleleft G$.

**Proof.** Let $K$ be the kernel of the action of $G$ on the block system $\text{Orb}(C_{n/2}, V(\Gamma))$, and let $K^*$ and $(C_{n/2})^*$ denote the image of $K$ and $C_{n/2}$, respectively, induced by their action on $U := \{u_i : i \in \mathbb{Z}_n\}$. Since the subgraph of $\Gamma$ induced by $U$ is a cycle of length $n > 8$, it follows that $(C_{n/2})^*$ is characteristic in $K^*$. Therefore, if $K$ is faithful on $U$, then $C_{n/2}$ is characteristic in $K$ and as $K \triangleleft G$, we obtain that $C_{n/2} \triangleleft G$, as required.

If $K$ is not faithful on $U$, then by Proposition 1.1 $n = 2m$, $m$ is odd, and $G \cong \Gamma' := N(2m; 2, m, 2+m; 1)$. Consider the group $\langle C, \varphi, \eta \rangle$, where $\varphi$ and $\eta$ are defined in Lemma 3.3. This is transitive on $V(\Gamma')$ and also contains the stabiliser of $u_0$ in $\text{Aut}(\Gamma')$, therefore, $\text{Aut}(\Gamma') = \langle C, \varphi, \eta \rangle$. A straightforward computation shows that $\varphi \rho^2 \varphi = \rho^{-2}$ and $\eta \rho^2 = \rho^2 \eta$, and hence $C_{n/2} \triangleleft \text{Aut}(\Gamma')$. All these show that $C_{n/2} \triangleleft G$ holds in this case as well. \qed

5. **Proof Theorem 1.2**

Throughout this section we keep the following notation.

**Hypothesis 5.1.**

$\Gamma = N(n; a, b, c; k)$ is a Nest graph of order $2n$, $n \geq 4$,

$C = \langle \rho \rangle$, where $\rho = (u_0, u_1, \ldots, u_{n-1})(v_0, v_1, \ldots, v_{n-1})$,
\[ G \leq \text{Aut}(\Gamma) \text{ such that } \rho \in G, \text{ } G \text{ acts transitively on } E(\Gamma), \text{ and } \text{core}_G(C) = 1. \]

Instead of Theorem 1.2 we show the following slightly more stronger theorem. The proof will be given in the end of the section.

**Theorem 5.2.** Assuming Hypothesis 5.1, \( \Gamma \) is isomorphic to one of the graphs: \( \mathcal{N}(5; 1, 2, 3; 2), \mathcal{N}(8; 1, 3, 4; 3), \mathcal{N}(8; 1, 2, 5; 3) \) and \( \mathcal{N}(12; 2, 4, 8; 5) \).

We start with a computational result, which we retrieved from [11, Table 1] with the help of MAGMA [4]. Here we use the obvious facts that \( C \) is also core-free in \( \text{Aut}(\Gamma) \) and that \( \text{Aut}(\Gamma) \) is primitive whenever so is \( G \).

**Lemma 5.3.** Assuming Hypothesis 5.1, if \( n \leq 50 \), then the following hold.

1. \( \Gamma \) is isomorphic to one of the graphs:
   \( \mathcal{N}(5; 1, 2, 3; 2), \mathcal{N}(8; 1, 3, 4; 3), \mathcal{N}(8; 1, 2, 5; 3) \) and \( \mathcal{N}(12; 2, 4, 8; 5) \).
2. \( G \) is either primitive and \( \Gamma \cong \mathcal{N}(5; 1, 2, 3; 2) \) or \( \mathcal{N}(8; 1, 3, 4; 3) \); or for \( N := \text{soc}(\text{Aut}(\Gamma)) \), \( N \cong \mathbb{Z}_2^3 \) if \( n = 8 \) and \( N \cong \mathbb{Z}_4^2 \) if \( n = 12 \), and the \( N \)-orbits have length 4.

The existence of a non-trivial non-cyclic block system is established next.

**Lemma 5.4.** Assuming Hypothesis 5.1, suppose that \( n > 8 \). Then \( G \) admits a non-trivial non-cyclic block system.

**Proof.** Observe that, if \( G \) is primitive, then Corollary 2.8 shows that \( \Gamma \cong \mathcal{N}(5; 1, 2, 3; 2) \) or \( \mathcal{N}(8; 1, 3, 4; 3) \). As \( n > 8 \), \( G \) is imprimitive.

Let \( \mathcal{B} \) be a non-trivial block system with blocks of size \( d \). If \( \mathcal{B} \) is cyclic, then by Lemma 3.4(1) and Corollary 4.3, \( C_d \triangleleft G \), where \( C_d \) is the subgroup of \( C \) of order \( d \). This contradicts our assumption that \( \text{core}_G(C) = 1 \), so \( \mathcal{B} \) is non-cyclic. \( \square \)

In the next two lemmas we study non-cyclic block systems with blocks of size 2.

**Lemma 5.5.** Assuming Hypothesis 5.1, suppose that \( n > 50 \) and \( \mathcal{B} \) is a non-cyclic block system for \( G \) with blocks of size 2. Then \( \Gamma/\mathcal{B} \) has valency 12.

**Proof.** Let \( K \) be the kernel of the action of \( G \) on \( \mathcal{B} \), and for a subgroup \( X \leq G \), denote by \( \bar{X} \) the image of \( X \) induced by its action on \( \mathcal{B} \). For a block \( B \in \mathcal{B} \), we write \( B = \{u_B, v_B\} \), where \( u_B \in \{u_i : i \in \mathbb{Z}_n\} \) and by \( v_B \in \{v_i : i \in \mathbb{Z}_n\} \), and define the permutation \( \tau \) of \( V(\Gamma) \) as
\[
\tau := \prod_{B \in \mathcal{B}} (u_B v_B). \tag{3}
\]
Observe that \( \tau \) commutes with any element of \( G \).

Now define the graph \( \Gamma' \) as
\[
V(\Gamma') := V(\Gamma) \text{ and } E(\Gamma') := \{\{u_0, u_1\}^x : x \in \langle G, \tau \rangle\} \tag{4}
\]
Then \( E(\Gamma) = \{\{u_0, u_1\}^x : x \in G\} \subseteq E(\Gamma') \). Also, \( \langle \tau, G \rangle \leq \text{Aut}(\Gamma') \), hence \( \Gamma' \) is both vertex- and edge-transitive. Since \( \tau \) commutes with every element of \( G \), it follows that \( E(\Gamma') = E(\Gamma) \cup E(\Gamma)^\tau \) and \( E(\Gamma) = E(\Gamma')^\tau \). Notice that
\[
\Gamma/\mathcal{B} = \Gamma'/\mathcal{B},
\]
hence we are done if show that \( \Gamma'/\mathcal{B} \) has valency 12.
Denote by \( d \) and \( d' \) the valency of \( \Gamma' \) and \( \Gamma'/B \), respectively. Now, \( d = |E(\Gamma')|/n = 6|E(\Gamma')|/|E(\Gamma)| \). This shows that \( d = 6 \) if \( E(\Gamma) = E(\Gamma') \), and \( d = 12 \) otherwise.

Assume for the moment that \( d = 6 \), i.e., \( E(\Gamma) = E(\Gamma') \). In this case \( \tau \in K \), hence \( B \) is normal and \( \Gamma' \) is a normal \( r \)-cover of \( \Gamma'/B \) and \( r = 1 \) or \( r = 2 \).

If \( r = 2 \), then \( d' = 3 \). As \( n > 50 \), this is impossible due to Lemma 2.2. Here we use the facts that \( \Gamma'/B \) is edge-transitive and \( \bar{C} \) is regular on \( V(\Gamma'/B) \). It is well-known that an edge-transitive circulant graph is also arc-transitive. Thus \( r = 1 \) and \( d' = 6 \). As \( n > 50 \), Lemma 2.3 can be applied to \( \Gamma'/B \) and \( \bar{C} \). This says that \( \text{Aut}(\Gamma'/B) \) has a normal subgroup \( N \) such that

1. \( N \equiv \bar{C} \), or
2. \( n \equiv 4 \mod 8 \) and \( N = \bar{C}_{n/4} \), or
3. \( N \cong \mathbb{Z}_q^4 \) for \( q \geq 2 \) and \( \bar{C}_3 \leq N \).

In case (1), \( N < \bar{G} \), hence \( KC \not< G \). The condition \( r = 1 \) yields that \( K = \langle \tau \rangle \). Thus \( KC \) is abelian and \( \langle x^2 : x \in KC \rangle = C_n \) if \( n \) is odd and \( C_{n/2} \) if \( n \) is even. Using that the latter group is characteristic in \( KC \) and \( KC \not< G \), we obtain that \( \text{core}_G(C) \neq 1 \), a contradiction.

In case (2), \( N < \bar{G} \), hence \( KC_{n/4} \not< G \). Since \( KC \) is abelian, it follows that \( C_{n/4} \) is characteristic in \( KC \), implying that \( C_{n/4} \not< G \), a contradiction.

In case (3), \( \bar{C}_3 \leq G \cap N \). Since \( G \cap N < \bar{G} \), it follows that \( G \) contains a normal subgroup \( M \) such that \( M = \langle \tau \rangle \times S \), where \( S \cong \mathbb{Z}_q^{2\ell} \) for some \( \ell \geq 1 \). Thus \( S \) is normal in \( G \), and we obtain that \( \text{Orb}(S,V(\Gamma)) \) is a non-trivial cyclic block system for \( G \). This contradicts Lemma 5.4 and we conclude that \( d = 12 \).

The graph \( \Gamma' \) is a normal \( r \)-cover of \( \Gamma'/B \), where \( r = 1 \) or \( r = 2 \), and we have \( d' = d/r = 12/r \). If \( r = 2 \), then \( u_0 \) is adjacent with 6 vertices that are contained in the set \( \{ u_i : i \in \mathbb{Z}_n \} \). It follows from the definition of \( \Gamma' \) that this impossible. Thus \( r = 1 \), and so \( d' = 12 \).

**Lemma 5.6.** Assuming Hypothesis 5.4, suppose that \( n > 50 \) and \( B \) is a non-cyclic block system for \( G \) with blocks of size 2. Then there is a normal non-cyclic block system for \( G \) with blocks of size 4.

**Proof.** Let \( K \) be the kernel of the action of \( G \) on \( B \), and for a subgroup \( X \leq G \), denote by \( \bar{X} \) the image of \( X \) induced by its action on \( B \). For the sake of simplicity we write \( \bar{\Gamma} \) for \( \Gamma/B \).

By Lemma 5.5, \( \bar{\Gamma} \) has valency 12. This implies that \( K = 1 \). As \( \bar{C} \leq \text{Aut}(\bar{\Gamma}) \) and it is regular on \( V(\bar{\Gamma}) \), Theorem 2.1 can be applied to \( \bar{\Gamma} \) and \( \bar{C} \). As \( n > 50 \), \( \bar{\Gamma} \) cannot be the complete graph. Also, if \( \bar{C} \not< \text{Aut}(\bar{\Gamma}) \), then \( C \not< G \) because \( K = 1 \). This is also impossible, hence \( \bar{\Gamma} \) is in one of the families (c) and (d) of Theorem 2.1.

**Case 1.** \( \bar{\Gamma} \) is in family (c).

In this case \( \text{Orb}(\bar{C}_d,V(\bar{\Gamma})) \) is a block system for \( \text{Aut}(\bar{\Gamma}) \), hence for \( \bar{G} \) as well, where \( d \in \{ 2, 3, 4, 6 \} \). Let \( N \) be the unique subgroup of \( G \) for which \( \bar{N} \) is the kernel of the action of \( \bar{G} \) on \( \text{Orb}(\bar{C}_d,V(\bar{\Gamma})) \). Note that \( \bar{N} \not< \bar{G} \) and \( N \cong N \) because \( K = 1 \). Let \( B' = \text{Orb}(N,V(\Gamma)) \). It follows that \( B' \) is non-cyclic and it has blocks of size \( 2d \).

Let \( d = 2 \). Then \( B' \) is normal with blocks of size 4, so the conclusion of the lemma holds.

Let \( d = 3 \). Then the Sylow 3-subgroup of \( N \) is normal in \( \bar{G} \). It follows in turn that, the Sylow 3-subgroup of \( N \) is normal in \( G \), the orbits of the latter subgroup form a non-trivial cyclic block system for \( G \). This contradicts Lemma 5.4.
Let $d = 4$. Then $\Gamma$ has valency 3. It follows from Lemma 2.2 that $n \leq 6$, but this is excluded.

Finally, let $d = 6$. Let $\tau$ be the permutation of $V(\Gamma)$ defined in (3) and $\Gamma'$ be the graph defined in (4). Let $\Delta$ be the subgraph of $\Gamma'$ induced by the set $u_0^N \cup u_1^N$. It is not hard to show that $\Delta$ is a bipartite graph, it has valency 6, and it is also edge-transitive. Moreover, if $B \in \mathcal{B}$ such that $u_0 \notin B$ and $B \subset u_0^N \cup u_1^N$, then

$$|\Delta(u_0) \cap B| = 1. \quad (5)$$

Since $\langle \tau \rangle \times C_6 \leq \text{Aut}(\Delta)$, it follows that $\Delta$ is uniquely determined by $\Delta(u_0)$. It follows from the definition of $\Gamma'$ that $|\Gamma'(u_0) \cap \{u_i : i \in \mathbb{Z}_n\}| = 4$. Therefore, replacing $u_0^N \cup u_1^N$ with $u_0^N \cup u_{n-1}^N$ if necessary, we may assume w.l.o.g. that $|\Delta(u_0) \cap \{u_i : i \in \mathbb{Z}_n\}| \leq 2$. This together with (5) show that there are 6 possibilities for $\Delta$. A computation with MAGMA shows that none of these 6 graphs is edge-transitive.

Case 2. $\Gamma$ is in family (d).

We finish the proof by showing this case does not occur. Theorem 2.1 shows that $\mathcal{B}_1 := \text{Orb}(\overline{C}_d, V(\Gamma))$ and $\mathcal{B}_2 := \text{Orb}(\overline{C}_{n/d}, V(\Gamma))$ are blocks for $\overline{G}$ for some divisor $d$ of $n$ such that $d \in \{4, 5, 7\}$ and $\gcd(d, n/d) = 1$. Furthermore,

$$\overline{C}_d \times \overline{C}_{n/d} < \overline{G} \leq \text{Aut}(\Gamma') = G_1 \times G_2,$$

where $\overline{C}_d \leq G_1$, $G_1 \cong S_d$, $\overline{C}_{n/d} < G_2$ and $G_2 \cong \text{Aut}(\Gamma'/\mathcal{B}_1)$.

Let $d = 4$. Then $\overline{\Gamma}/\mathcal{B}_1$ has valency 4. Using also that $n/d$ is odd and that $n > 20$, it follows from Lemma 2.2 that $\overline{C}_{n/d} < G$, hence $\overline{C}_{n/d} < G$, a contradiction.

Let $d = 5$. Then $\overline{\Gamma}/\mathcal{B}_1$ has valency 3, hence $n \leq 30$ by Lemma 2.2, which is excluded.

Finally, let $d = 7$. Then $\overline{\Gamma}/\mathcal{B}_1$ is a cycle of length $n/7$, implying that $\overline{C}_{n/d} < G$, so $\overline{C}_{n/d} < G$, a contradiction. \hfill \square

Before the proof of Theorem 5.2 we need two more lemmas dealing with non-cyclic block systems with blocks of size at least 4.

**Lemma 5.7.** Assuming Hypothesis 5.1, suppose that $n > 50$ and $\mathcal{B}$ is a minimal non-cyclic block system for $G$ with blocks of size at least 4, and let $B \in \mathcal{B}$ be any block. Then the permutation group of $B$ induced by $G_{\{B\}}$ is an affine group.

**Proof.** For a subgroup $X \leq G_{\{B\}}$, denote by $X^*$ the image of $X$ induced by its action on $B$. As $B$ is minimal, $(G_{\{B\}})^*$ is a primitive permutation group. Also, $(C_{\{B\}})^*$ is a semiregular cyclic subgroup of $(G_{\{B\}})^*$ with 2 orbits, hence Theorem 2.4 can be applied to $(G_{\{B\}})^*$. This shows that $(G_{\{B\}})^*$ is either an affine group or it is one of the groups in the families (a)-(f) in part (2) of Theorem 2.4. Assume that the latter case occurs. We drive in three steps that this leads to a contradiction. Let $K$ be the kernel of the action of $G$ on $\mathcal{B}$.

**Step 1.** $K$ acts faithfully on every block in $\mathcal{B}$.

Since $K < G_{\{B\}}$, it follows that $K^* < (G_{\{B\}})^*$. By Corollary 2.5, $K^*$ is primitive and belongs to the same family as $(G_{\{B\}})^*$.

Assume on the contrary that $K$ is not faithful on every block. Using the connectedness of $\Gamma$, it is easy to show that there are blocks $B, B'$ in $\mathcal{B}$ with the following properties: The kernel of the action of $K$ on $B$ is non-trivial on $B'$, and $\Gamma$ has an edge $\{w, w'\}$ such that
By Lemma 3.5, assuming Hypothesis 5.1, suppose that \( n > 50 \). Then \( \Gamma \| B \), and the second orbit consists of the remaining hyperplanes. Clearly, the minimum of these numbers is bounded above by the valency of \( \Gamma \), implying that \( q^{d-1} - 1 \leq 6(q - 1) \), and hence \( q^{d-2} < 6 \). This is impossible because \( d \geq 4 \).

Step 2. The action of \( K \) on \( B \) is equivalent with its action on \( B' \).

Assume on the contrary that the actions are inequivalent. Due to Lemma 2.6(2), \( K \) belongs to family (c) in Theorem 2.4(2) with \( d \geq 4 \), and the elements in \( B \) and \( B' \) correspond to the points and the hyperplanes of the projective geometry \( PG(d-1, q) \), respectively. The set \( B' \) splits into two \( K_{u_0} \)-orbits of lengths

\[
(q^{d-1} - 1)/(q - 1) \text{ and } q(q^{d-1} - 1)/(q - 1).
\]

The first orbit consists of the hyperplanes of \( PG(d-1, q) \) through the point represented by \( u_0 \), and the second orbit consists of the remaining hyperplanes. Clearly, the minimum of these numbers is bounded above by the valency of \( \Gamma \), implying that \( q^{d-1} - 1 \leq 6(q - 1) \), and hence \( q^{d-2} < 6 \). This is impossible because \( d \geq 4 \).

Step 3. \( \text{core}_C(C) \neq 1 \).

Since \( K \) acts equivalently on \( B \) and \( B' \), it follows that \( K_{u_0} = K_v \) for some vertex \( v \in B' \) (see [8] Lemma 1.6B)]. Define the binary relation \( \sim \) on \( V(\Gamma) \) by letting \( u \sim v \) if and only if \( K_u = K_v \). It is not hard to show, using that \( K \leq G \), that \( \sim \) is a \( G \)-congruence (see [8] Exercise 1.5.4]), and so there is a block for \( G \) containing \( u_0 \) and \( v \). Also, as \( K \) is not regular, this block is non-trivial, and this shows that \( v \neq u_1 \).

By Lemma 2.6(1), \( K \) is 2-transitive on \( B' \), unless \( |B'| = 10 \), \( K = A_5 \) or \( S_5 \), and it has subdegrees 1, 3 and 6.

Assume first that \( K \) is 2-transitive on \( B' \). Then the orbit \( u_0^{K_{u_0}} = u_1^{K_v} = |B'|-1 \) and each vertex in \( u_0^{K_v} \) is adjacent with \( u_0 \). Hence \( u_0 \) has \( |B'|-1 \) neighbours in \( B' \). On the other hand, as \( B \) is normal, this number divides 6, so \( |B'| = 4 \), contradicting that \( G'_{(B)} \) is an almost simple group.

We are left with the case that \( |B'| = 10 \), \( K = A_5 \) or \( S_5 \), and it has subdegrees 1, 3 and 6. Consequently, \( u_0 \) has 3 or 6 neighbours in \( B' \).

If \( u_0 \) has 6 neighbours, then it is clear that \( n = 10 \), which is excluded.

Now assume that \( u_0 \) has 3 neighbours in \( B' \). In this case \( \Gamma \) is a normal 3-cover of a cycle of length \( n/5 \). Since \( \Gamma/B \) is a cycle of length \( n/5 \), it follows that \( |G| \leq |K| \cdot 2n/5 = 48n \). Using that \( n > 50 \), Theorem 2.9 shows that \( \text{core}_C(C) \neq 1 \).

Lemma 5.8. Assuming Hypothesis 5.1, suppose that \( n > 50 \) and \( B \) is a minimal non-cyclic block system for \( G \) with blocks of size at least 4. Then the blocks have size 4.

Proof. Let \( K \) be the kernel of the action of \( G \) on \( B \), and let \( B' \in B \) be the block containing \( u_0 \). Denote by \( (G_{(B)})^* \) the permutation group of \( B \) induced by \( G_{(B)} \). By Lemma 5.7 \( (G_{(B)})^* \) is an affine group, and thus it is one of the groups in the families (a)–(d) in part (1) of Theorem 2.4. In particular, \( |B'| \in \{4, 8, 16\} \). Assume on the contrary that \( |B'| > 4 \). By Lemma 3.5 \( B \) is normal, hence \( \Gamma \) is a normal \( r \)-cover of \( \Gamma/B \) for \( r \in \{1, 2, 3\} \).
Case 1. \( r = 1 \).

In this case \( K \) is regular on every block, in particular, \( K \cong \mathbb{Z}_4 \) or \( \mathbb{Z}_5 \). On the other hand, by Lemma 3.5, (2), \( C_{|B|/2} < K \), a contradiction.

Case 2. \( r = 2 \).

In this case \( \Gamma/B \) has valency 3. It follows from Lemma 2.2 that \( n \leq 48 \), but this is excluded.

Case 3. \( r = 3 \).

Then \( \Gamma/B \) is a cycle of length \( 2n/|B| \). This implies that the action of \( G_{u_0} \) on \( \Gamma(u_0) \) admits a block system consisting of two blocks of size 3. Consequently, the restriction of \( G_{u_0} \) to \( \Gamma(u_0) \) is a \( \{2,3\} \)-group. This together with the fact that \( \Gamma \) is connected yield that \( G_{u_0} \) is also a \( \{2,3\} \)-group. Now checking the stabilisers in part (1) of Theorem 2.4, we find that \( |B| = 16 \) and

\[
(G_{(B)}^*)^* \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4 \text{ or } (S_3 \times S_3) \rtimes \mathbb{Z}_2 \tag{6}
\]

Assume for the moment \( K \) is not faithful on \( B \). Then there exist adjacent blocks \( B' \) and \( B'' \) such that the apkernel of the action of \( K \) on \( B' \) is non-trivial on \( B'' \). Denote this kernel by \( L \). The \( L \)-orbits contained in \( B'' \) have the same size, which is equal to \( 2^s \) for some \( 1 \leq s \leq 4 \). On the other hand, for \( w \in B' \), the set \( \Gamma(w) \cap B'' \) is \( L \)-invariant, implying that \( |\Gamma(w) \cap B''| \) is equal to some power of 2, a contradiction. Thus \( K \) is faithful on \( B \).

The group \( K \) contains a normal subgroup \( E \) such that \( E \cong \mathbb{Z}_4 \). Note that \( B = \text{Orb}(E,V(\Gamma)) \). Let \( P \) be the Sylow 3-subgroup of \( G_{(B)} \). Since \( \Gamma/B \) is a cycle, it follows that \( P \leq K \). This also shows that \( P \cong \mathbb{Z}_3^2 \). Also, \( C_8 \leq K \), and in view of (6), we obtain that \( |(G_{(B)}^*)^*: K| \leq 2 \) and if the index is equal to 2, then \( (G_{(B)}^*)^* \cong (S_3 \times S_3) \rtimes \mathbb{Z}_2 \). A direct check by MAGMA [4] shows that in the latter case \( (G_{(B)}^*)^* \) has a unique subgroup of index 2 containing an element of order 8, which is also primitive. All these show that \( K \) is primitive on \( B \).

Denote by \( \Delta \) be the subgraph of \( \Gamma \) induced by \( u_0^F \cup u_1^F \). Using that \( E \cong \mathbb{Z}_4 \) acting regularly on both \( u_0^F \) and \( u_1^F \), it is not hard to show that \( \Delta \) is the union of four 3-dimensional cube \( Q_3 \). If \( \Delta_1 \) is a component of \( \Delta \), then \( |V(\Delta_1) \cap B| = 4 \) (note that \( B = u_0^F \)) and \( V(\Delta_1) \cap B \) is a block for \( K \). This, however, contradicts the fact that \( K \) is primitive on \( B \).

We are ready to settle Theorem 5.2 and therefore, Theorem 1.2 as well.

Proof of Theorem 5.2. In view of Lemma 5.3, we may assume that \( n > 50 \). It follows from Lemmas 5.4, 5.8 that \( G \) admits a normal non-cyclic block system with blocks of size 4. Denote this block system by \( B \). Let \( K \) be the kernel of the action of \( G \) on \( B \), and for a subgroup \( X \leq G \), denote by \( \bar{X} \) the image of \( X \) induced by its action on \( B \). As \( B \) is normal, \( \Gamma \) is a normal \( r \)-cover of \( \Gamma/B \) for some \( r \in \{1,2,3\} \). We exclude below all possibilities case-by-case.

Case 1. \( r = 1 \).

In this case \( |K| = 4 \) and \( K \cap C = C_2 \). If \( K \cong \mathbb{Z}_4 \), then \( C_2 \) is characteristic in \( K \), and therefore, it is normal in \( G \). This is impossible because \( \text{core}_G(C) = 1 \), hence \( K \cong \mathbb{Z}_2^2 \).
The graph \( \Gamma / \mathcal{B} \) is edge-transitive, it has valency 6, and \( \mathcal{C} \) is regular on \( V(\Gamma / \mathcal{B}) \). As \( n > 50 \), Lemma 2.3 can be applied to \( \Gamma / \mathcal{B} \) and \( \mathcal{C} \). It follows that \( \text{Aut}(\Gamma / \mathcal{B}) \) has a normal subgroup \( N \) such that

1. \( N = \mathcal{C} \), or
2. \( n \equiv 4 \pmod{8} \) and \( N = \mathcal{C}_{n/4} \), or
3. \( N \cong \mathbb{Z}_3^\ell \) for \( \ell \geq 2 \) and \( \mathcal{C}_3 \leq N \).

In case (1), we obtain that \( KC \vartriangleleft G \), whereas in case (2), \( KC_{n/4} \vartriangleleft G \). In either case, \( |KC : C| = 2 \), and therefore, for the derived subgroup \((KC)'/C\), \((KC)'/C \leq C\). Thus \((KC)'/C = \text{core}_G(C)\), and so \((KC)' = 1\), i.e., \( KC \) is an abelian group. Then we obtain that \( C_{n/2} \vartriangleleft G \) in case (1), and \( C_{n/4} \vartriangleleft G \) in case (2). None of these is possible because \( \text{core}_G(C) = 1 \).

In case (3), \( G \) contains a normal subgroup \( M \) such that \( KC_3 \leq M \) and \( M/K \cong \mathbb{Z}_{3'}^\ell \) for some \( \ell' \geq 1 \). Then \( M \) can be written as \( M = KS \) where \( C_3 \leq S \) and \( S \cong \mathbb{Z}_{3'}^\ell \). As \( C_2 \leq K \), we obtain that \( C_3 \) commutes with \( K \), and so \( C_3 \leq O_3(M) \), where \( O_3(M) \) denotes the largest normal 3-subgroup of \( M \). As \( O_3(M) \) is characteristic in \( M \), \( O_3(M) \vartriangleleft G \). This yields that \( \text{Orb}(O_3(M), V(\Gamma)) \) is a non-trivial cyclic block system, a contradiction to Lemma 5.4.

Case 2. \( r = 2 \).

In this case \( \Gamma / \mathcal{B} \) has valency 3, hence \( n \leq 12 \) by Lemma 2.2 which is excluded.

Case 3. \( r = 3 \).

The group \( K \) is faithful on every block of \( \mathcal{B} \). This can be shown by copying the argument that has been used in Case 3 in the proof of Lemma 5.8. Since \( \Gamma / \mathcal{B} \) is a cycle of length \( n/2 \), it follows that \( |G| \leq |K| \cdot n = 24n \). Using that \( n > 50 \), Theorem 2.9 shows that \( \text{core}_G(C) \neq 1 \), a contradiction. \( \square \)

References

[1] I. Antončič, A. Hujdurović, and K. Kutnar, A classification of pentavalent arc-transitive bicirculants, J. Algebraic Combin. 41 (2015) 643–668.
[2] A. Arroyo, I. Hubard, K. Kutnar, E. O’Reilly, and P. Šparl, Classification of symmetric Tabacićn graphs, Graphs Combin. 31 (2015) 1137–1153.
[3] Y.-G. Baik, Y.-Q. Feng, H. S. Sim, and M. Xu, On the normality of Cayley graphs of abelian groups, Algebra Colloq. 5 (1998) 297–304.
[4] W. Bosma, C. Cannon, and C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput. 24 (1997) 235–265.
[5] M. Conder, J.-X. Zhou, Y.-Q. Feng, and M.-M. Zhang, Edge-transitive bi-Cayley graphs, J. Combin. Theory Ser. B 145 (2020) 264–306.
[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, Atlas of finite groups, Clarendon Press, Oxford 1985.
[7] A. Devillers, M. Giudici, and W. Jein, Arc-transitive bicirculants, J. London Math. Soc. 105 (2022) 1–23.
[8] J. D. Dixon and B. Mortimer, Permutation groups, Graduate Text in Mathematics 163, Springer-Verlag, New York 1996.
[9] E. Dobson, I. Kovács, and Š. Miklavič, The automorphism groups of non-edge-transitive rose window graphs, Ars. Math. Contemp. 9 (2015) 63–75.
[10] R. Frucht, J. E. Graver, and M. E. Watkins, The group of the generalized Petersen graphs, Proc. Camb. Philos. Soc. 70 (1971) 211–218.
[11] R. Jajcay, Š. Miklavič, P. Šparl, and G. Vasiljević, On certain edge-transitive bicirculants, Electron. J. Combin. 26 (2) (2019), #P2.6.

[12] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin Heidelberg New York 1967.

[13] I. Kovács, Classifying arc-transitive circulants, J. Algebraic Combin. 20 (2005) 353–358.

[14] I. Kovács, Classification of edge-transitive Nest graphs, manuscript.

[15] I. Kovács, K. Kutnar, and D. Marušič, Classification of edge-transitive rose window graphs, J. Graph Theory 65 (2010) 216–231.

[16] I. Kovács, B. Kuzman, A. Malnič, and S. Wilson, Characterization of edge-transitive 4-valent bicirculants, J. Graph Theory 69 (2012) 441–463.

[17] I. Kovács and J. Ruff, On certain edge-transitive bicirculants of twice odd order, to appear in Electron. J. Combin., preprint arXiv:2111.07982v1 [math.CO], 2021.

[18] K. Kutnar, D. Marušič, Š. Miklavič, and R. Strašek, Automorphisms of Tabačn graphs, Filomat 27 (2013) 1157–1164.

[19] C. H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, J. Algebraic Combin. 21 (2005) 131–136.

[20] C. H. Li, B. Xia, and S. Zhou, An explicit characterisation of arc-transitive circulants, J. Combin. Theory Ser. B 150 (2021) 1–16.

[21] A. Lucchini, On the order of transitive permutation groups with cyclic point-stabiliser, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 9 (1998) 241–243.

[22] A. Malnič, D. Marušič, P. Šparl, and B. Frelih, Symmetry structure of bicirculants, Discrete Math. 307 (2007) 409–414.

[23] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, Croat. Chem. Acta 73 (2000) 969–981.

[24] P. Müller, Permutation groups with a cyclic two-orbits subgroup and monodromy groups of Laurent polynomials, Ann. Sc. Norm. Super. Pisa CI Sci. 12 (2013) 369–438.

[25] T. Pisanski, A classification of cubic bicirculants, Discrete Math. 307 (2007) 567–578.

[26] G. Vasiljević, O simetričnih Nest grafih, MSc thesis, University of Ljubljana, 2017.

[27] M. E. Watkins, A Theorem on Tait colorings with an application to the generalized Petersen graphs, J. Combin. Theory 6 (1969), 152–164.

[28] S. Wilson, Rose window graphs, Ars. Math. Contemp. 1 (2008) 7–18.

[29] J.-X. Zhou and M.-M. Zhang, The classification of half-arc-regular bi-circulants of valency 6, European J. Combin. 64 (2017) 45–56.

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