Cut-elimination for SBL

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Abstract

In this paper we give a terminating cut-elimination procedure for a logic calculus SBL. SBL corresponds to the second order arithmetic $\Pi^1_2$-Separation and Bar Induction.

1 Introduction

Let $\Pi^1_2$-Sep+BI(= $\Delta^1_2$-CA+BI) denote the subsystem of second order arithmetic with $\Pi^1_2$-Separation and Bar Induction. $\Pi^1_2$-Sep+BI is proof-theoretically equivalent to the set theory KP$i$ for recursively inaccessible universes. K. Schütte [11] gives an upper bound $\psi_0 I$ for the proof theoretic ordinal of $\Pi^1_2$-Sep+BI. The ordinal $\psi_0 I$ is the order type of an initial segment of the recursive notation system $T(I)$ of ordinals introduced by W. Buchholz and Schütte [5]. G. Jäger [8] shows the wellfoundedness up to each ordinal $< \psi_0 I$ in the S. Feferman’s [6] constructive theory $T_0$, which is interpretable in $\Pi^1_2$-Sep+BI. Thus the proof-theoretic ordinal of $\Pi^1_2$-Sep+BI and of $T_0$ is shown to be equal to $\psi_0 I$. Jäger’s proof is based on Ausgezeichnete Klass introduced by Buchholz [3].

The analysis of the derivations in $\Pi^1_2$-Sep+BI due to Schütte is based on the Buchholz’s $\Omega_{\alpha+1}$-rule, and the system $(T(I), <)$ is utilized indirectly: in fact the totally defined collapsing functions $d$ and $d_\sigma$ appear in the analysis, which are also introduced in [5].

On the other side G. Takeuti [13] uses his systems of ordinal diagrams directly for a proof theory of $\Pi^1_1$-Comprehension. The definition of ordinal diagrams is closely related to the cut-elimination procedure due to him. But unfortunately Takeuti’s systems of ordinal diagrams are equipped with many order relations and are too small to handle such a strong theory $\Pi^1_2$-Sep+BI.

Turning to the problem of the cut-elimination in second order, and higher order logic calculi (known as Takeuti’s Fundamental Conjecture), W. Tait [12] proves the cut-eliminability (Hauptsatz) for the classical second order (full impredicative) logic calculus based on the Schütte’s [10] reformulation of it by means of a semantical notion, semivaluation.

Given these advances in 1980’s, we had introduced a system $(O(I), <)$ of ordinal diagrams and proved a cut-elimination theorem for a logic calculus SBL.
in the style of Gentzen-Takeuti [7, 13] by transfinite induction on the system. This was done in the original version of this paper written in 1988. The system $O(I, <)$ of ordinal diagrams was obtained as a kind of mixture of totally defined collapsing functions $d_\sigma$ in [5] and Takeuti’s ordinal diagrams. Specifically, $d_\sigma$ is a primitive constructor of ordinal terms in $O(I)$, whereas it is a derived term in [5]. In the original version of this paper it was shown that each initial segment determined by $\alpha < \Omega_1 \in O(I)$ is well-founded. The proof is formalizable in $T_0$ as in the Jäger’s proof [8]. This was a starting point for us to construct larger notation systems of ordinals, e.g., in [1].

$SBL$ corresponds to the system $\Pi^1_2$-Sep + BI in the sense that the Hauptsatz (normal form theorem) for $SBL$ is equivalent to the 1-consistency of $\Pi^1_2$-Sep + BI over a weak theory, e.g., over $T^0 \Sigma_1$. The proof of the cut-elimination in the original version was inspired from Schütte’s proof in [11].

In the present version let us update the original proof via the partially defined collapsing functions $\psi_\sigma$ and the operator controlled derivations both due to Buchholz [4].

In section 2 let us recall the collapsing functions $\psi_\sigma$ up to $\sigma \leq I$, where $I$ is the least weakly inaccessible cardinal. A wellfounded proof in $T_0$ is omitted in the present version since it should not be hard. In subsection 2.1 we define an essentially less than relation $\alpha \ll \beta \{\eta\}$ for ordinals $\alpha, \beta, \eta$ in terms of Skolem hulls $H_\gamma(\psi_\sigma \gamma)$. In section 3 a second order logic calculus $SBL$ is introduced. In section 4 we introduce a stratified logic calculus $SBL'$ following Schütte [11]. $SBL$ is then embedded in $SBL'$, and a cut-free proof in $SBL'$ denotes essentially a cut-free proof in $SBL$. For each proof $P$ in $SBL'$ we assign an ordinal $o(P)$ such that $\psi_{\Omega_1 \in I + 1} \leq o(P)$ and we can construct another proof $P'$ of the same end sequent in $SBL'$ such that $o(P') < o(P)$ (Main Lemma 4.17). It turns out that each proof appearing in the cut-elimination procedure enjoys some conditions on assigned ordinals, which are spelled out in Definition 4.15.3. Restrictions similar to these conditions are found in [4]. So our proof seems to be a finitary analogue to the proof through operator controlled derivations.

The final section 5 is devoted to a proof of Main Lemma 4.17.

### 2 Collapsing functions $\psi_\sigma$

Let $I$ denote the least weakly inaccessible cardinal, and $\Omega_\alpha := \omega_\alpha$ for $0 < \alpha < I$. Put $\Omega_0 := 0$ and $R := \{\Omega_{\alpha + 1} : \alpha < I\}$.

In this section let us recall the collapsing functions $\psi_\sigma (\sigma \in R)$ due to W. Buchholz [4].

**Definition 2.1** $H_\alpha(X)$ denote the Skolem hull of the set $X \cup \{0, I\}$ of ordinals under the functions $+, \beta \mapsto \omega, \beta \mapsto \Omega_\beta$ and $(\sigma, \beta) \mapsto \psi_\sigma \beta (\beta < \alpha)$.

$$\psi_\sigma \alpha = \min(\{\beta < \sigma : \sigma \in H_\alpha(\beta) \& H_\alpha(\beta) \cap \sigma \subset \beta\} \cup \{\sigma\}).$$

The following facts are shown in Lemma 4.5 of [4]. We see that $\psi_\sigma \alpha < \sigma$ from the regularity of $\sigma$ and $\alpha + 1 < \Omega_\alpha + 1$ for $\alpha < I$. When $\alpha = \Omega_\alpha + 1$,
we have \( \mu \leq \Omega_\mu < \psi_\alpha \alpha < \Omega_{\mu+1} \). Hence \( \sigma \in H_\alpha(\psi_\alpha \alpha) \). If \( \alpha_0 < \alpha_1 \), then \( H_{\alpha_0}(\beta) \subset H_{\alpha_1}(\beta) \), and \( \psi_\sigma \alpha_0 \leq \psi_\sigma \alpha_1 \). Also \( H_{\alpha}(\beta) \) is closed under the natural sum \( \gamma \# \delta \) and the functions \( \gamma \mapsto \omega^\gamma \) and \( \gamma \mapsto \Omega_\gamma \) in the reverse direction, i.e., \( \gamma \# \delta \in H_{\alpha}(\beta) \Rightarrow \{ \gamma, \delta \} \subset H_{\alpha}(\beta) \), \( \omega^\gamma \in H_{\alpha}(\beta) \Rightarrow \gamma \in H_{\alpha}(\beta) \) and \( \Omega_\gamma \in H_{\alpha}(\beta) \Rightarrow \gamma \in H_{\alpha}(\beta) \).

\( \varepsilon_{I+1} \) denotes the next epsilon number above \( I \).

\( H_{\alpha+1}(0) \) is the notation system of ordinals generated from \( \{0, I\} \) by \( +, \beta \mapsto \omega^\beta, \beta \mapsto \Omega_\beta \) and \( (\sigma, \beta) \mapsto \psi_\sigma \beta (\beta < \varepsilon_{I+1}) \).

The computability of \( H_{\alpha+1}(0) \) together with the relation \( < \) on it is seen from the following facts. \( \psi_\sigma \alpha \in H_{\varepsilon_{I+1}(0)}(0) \) iff \( \{\sigma, \alpha\} \subset H_{\varepsilon_{I+1}(0)}(0) \cap H_{\alpha}(\psi_\sigma \alpha) \).

\( \gamma \in H_{\alpha}(\beta) \Leftrightarrow G_\beta(\gamma) < \alpha \), where \( G_\beta(0) = G_\beta(I) = \emptyset \), \( G_\beta(\alpha_0 + \ldots + \alpha_n) = \bigcup \{ G_\beta(\alpha_i) : i \leq n \} \), \( G_\beta(\omega^\alpha) = G_\beta(\Omega_\alpha) = G_\beta(\alpha) \).

\[
G_\beta(\psi_\sigma \alpha) = \begin{cases} 
\emptyset & \text{if } \psi_\sigma \alpha < \beta \\
G_\beta(\sigma) \cup G_\beta(\alpha) \cup \{ \alpha \} & \text{otherwise}
\end{cases}
\]

1. \( Fx := \{ \alpha \in H_{\varepsilon_{I+1}(0)} : \Omega_\alpha = \alpha > 0 \} = \{ \psi_I \alpha : \alpha \in H_{\varepsilon_{I+1}(0)}, \alpha \in H_{\alpha}(\psi_1 \alpha) \} \cup \{ I \} \).
2. \( \Omega_\alpha < \psi_{\Omega_{\alpha+1}} \beta < \Omega_{\alpha+1} \).
3. \( \psi_{\Omega_{\alpha+1}} \beta < \psi_I \gamma \Leftrightarrow \alpha < \psi_I \gamma \).
4. \( \psi_I \alpha < \psi_I \beta \Leftrightarrow \alpha < \beta \).

In what follows \( \alpha, \beta, \gamma, \delta, \ldots \) range over ordinal terms in \( H_{\varepsilon_{I+1}(0)} \), and \( \sigma, \tau, \ldots \) over elements in the set \( R = \{ I \} \cup \{ \Omega_\mu + 1 : \mu \in H_{\varepsilon_{I+1}(0)} \} \).

2.1 Relations \( \alpha \ll \beta \{ \eta \} \)

In this subsection an essentially less than relation \( \alpha \ll \beta \{ \eta \} \) is defined through Skolem hulls \( H_{\gamma}(\psi_\sigma \gamma) \).

**Definition 2.2** For ordinal terms \( \delta_0, \delta_1, \eta \in H_{\varepsilon_{I+1}(0)} \),

\[
\delta_0 \ll \delta_1 \{ \eta \} \quad : \quad \delta_0 < \delta_1 \land \forall \sigma \forall \alpha([\delta_1, \eta] \subset H_{\alpha}(\psi_\sigma \alpha) \Rightarrow \delta_0 \in H_{\alpha}(\psi_\sigma \alpha)] \]

\[
\delta_0 \ll \delta_1 \{ \eta \} \quad : \quad \delta_0 = \delta_1 \lor (\delta_0 \ll \delta_1 \{ \eta \}) \]

\[
\delta_0 \ll \delta_1 \quad : \quad \delta_0 \ll \delta_1 \{ 0 \} \]

\[
\delta_0 \ll \delta_1 \quad : \quad \delta_0 = \delta_1 \lor \delta_0 \ll \delta_1
\]

**Proposition 2.3**

1. \( \delta_0 \ll \delta_1 \{ \eta \} \Rightarrow \delta_0 \# \alpha \ll \delta_1 \# \alpha \{ \eta \} \).
2. \( \delta_0 \ll \delta_1 \{ \eta \} \Rightarrow \omega^{\delta_0} \ll \omega^{\delta_1} \{ \eta \} \).
3. Assume \( \{ \gamma, \delta_1, \eta \} \subset H_{\gamma}(\psi_\sigma (\gamma \# \delta_1)) \). Then \( \delta_0 \ll \delta_1 \{ \eta \} \Rightarrow \psi_\sigma (\gamma \# \delta_0) \ll \psi_\sigma (\gamma \# \delta_1) \{ \eta \} \).
4. Assume \( \gamma \in H_{\gamma}(\psi_{\Omega_{\mu+1}} \gamma) \). Then \( \alpha \ll \gamma \& \alpha \leq \Omega_\mu \Rightarrow \alpha \ll \psi_{\Omega_{\mu+1}} \gamma \).
5. Assume $\alpha \ll \psi \gamma \{\eta\}, \gamma \in \mathcal{H}_s(\psi \gamma)$ and $\forall \beta \gamma \exists \gamma \sigma \{\gamma, \tau\} \subset \mathcal{H}_s(\psi \sigma \beta) \Rightarrow \eta \in \mathcal{H}_s(\psi \sigma \beta)$. Then $\alpha \ll \psi \gamma$.

6. Assume $\delta_0 \ll \delta_1 \{\eta\}, \psi (\gamma \# \delta_0) < \psi (\gamma \# \delta_1)$, and $\{\psi (\gamma \# \delta_1), \eta\} \subset \mathcal{H}_s(\psi \sigma \alpha)$. Then $\psi (\gamma \# \delta_0) \in \mathcal{H}_s(\psi \sigma \alpha)$.

Proof. Assume $\{\gamma, \delta_1, \eta\} \subset \mathcal{H}_s(\psi \sigma (\gamma \# \delta_1))$ and $\delta_0 \ll \delta_1 \{\eta\}$. Then $\delta_0 \in \mathcal{H}_s(\psi \sigma (\gamma \# \delta_1))$, and $\{\gamma, \delta_0, \sigma\} \subset \mathcal{H}_s(\psi \sigma (\gamma \# \delta_1))$. Hence $\psi \sigma (\gamma \# \delta_0) \in \mathcal{H}_s(\psi \sigma (\gamma \# \delta_1))$. We show $\psi \sigma (\gamma \# \delta_0) \in \mathcal{H}_s(\psi \sigma \alpha)$. We can assume $\psi \sigma (\gamma \# \delta_1) > \psi \sigma \alpha$. Then $\{\sigma, \gamma, \delta_1, \eta\} \subset \mathcal{H}_s(\psi \sigma \alpha)$ and $\gamma \# \delta_1 < \alpha$. These yield $\psi \sigma (\gamma \# \delta_0) \in \mathcal{H}_s(\psi \sigma \alpha)$. □

Let $Var = \{U, V, \ldots\}$ be a countable set of (unary) second-order free variables, and $Var' := \{U^n : U \in Var, n \in Fx\}$. Also $\Sigma = \{0, 1, +, \omega, \Omega, \psi\} \cup Var'$. $Var(t)$ denotes the set of variables occurring in $t \in \Sigma^*$ (the set of finite sequences over symbols $\Sigma$).

**Definition 2.4** Let max be a symbol not in $\Sigma$.

1. $S \subset (\Sigma \cup \{max\})^*$ and ordinals $od(s) \leq I$ for $s \in S$ are defined recursively.

   (a) Each $U^n \in Var'$ is in $S$. $od(U^n) = \eta$.

   (b) $Fx \cup \{0\} \subset S$. $od(\eta) = \eta$ for $\eta \in (Fx \cap I) \cup \{0\}$.

   (c) $s \in S \Rightarrow s + 1 \in S$. $od(s + 1) = \min(od(s) + 1, I)$.

   (d) $s_1, s_2 \in S \Rightarrow \max(s_1, s_2) \in S$.

   $od(\max(s_1, s_2)) = \min(I, \max(od(s_1), od(s_2)))$.

2. For $s \in S$, a finite non-empty set $I(s) \subset Fx \cup \{0\}$ is defined recursively.

   (a) $I(U^n) = \{\eta\}$.

   (b) $I(s) = \{s\}$ for $s \in Fx \cup \{0\}$.

   (c) $I(s + 1) = I(s)$.

   (d) $I(\max(s_1, s_2)) = I(s_1) \cup I(s_2)$.

Note that $od(s) = I$ iff a free variable $U^I$ with index $I$ occurs in $s$. In particular if no free variable occurs in $s$, then $od(s) < I$.

**3 The logic calculus SBL**

In this section a second-order logic calculus SBL is introduced. $\mathcal{L}$ denotes a second-order language consisting of logical symbols $\forall, \land, \exists, \forall$, individual constants $c, \ldots$, function symbols $f, \ldots$, first-order free variables $a, b, \ldots$, first-order bound variables $x, y, \ldots$, relation symbols $R, \ldots$, second-order free variables...
Let us assume that each relation symbol and each second-order (free or bound) variable is unary for simplicity, and \( L \) contains an individual constant \( c \) and a (unary) relation symbol \( R \), cf. pure variable condition in Definition 4.8.

Let \( T \) stand for either a (unary) second-order free variable or a predicate constant. \( Tt \) and \( \neg Tt \) are prime formulas (literals) for terms \( t \). Formulas are generated from literals by means of \( \lor \), \( \land \) and first-order and second-order quantifications \( \exists, \forall \) as usual. Negations \( \neg A \) of formulas \( A \) are defined recursively through de Morgan’s law and the elimination of double negations \( \neg(\neg Tt) \equiv (Tt) \):

For formal expressions \( E, s, t \) such as terms and proofs \( E[s/t] \) denotes the expression obtained from \( E \) by replacing some occurrences of the expression \( t \) in \( E \) by the expression \( s \). Let \( F \) be a formula\(^1\) with a second-order bound variable \( X \), and \( A \) a formula with a variable \( x \). Then \( F[A/X] \) denotes the formula obtained from \( F \) by replacing each occurrence of \( Xt \) by \( A[t/x] \) and each occurrence of \( \neg Xt \) by \( \neg A[t/x] \).

**Definition 3.1**

1. For formulas \( A \), \( VT(A) \) denotes the set of second-order free variables occurring in a scope of a second-order quantifier in \( A \). \( U \in VT(A) \) iff \( U \) is tied by a second-order quantifier in \( A \) in the sense of [13].

2. Let \( A \) be a formula.

\[
\begin{align*}
\forall X A[X/U] & \in \Pi_1^1 \iff A \in \Pi_2^1 \\
\exists X A[X/U] & \in \Sigma_2^1 \iff A \in \Sigma_1^2 \\
\forall X A[X/U] & \in \Pi_2^1 \iff A \in \Pi_1^1 \cap \Sigma_2^1 \land U \notin VT(A) \\
\exists X A[X/U] & \in \Pi_2^1 \iff A \in \Pi_1^1 \cap \Sigma_2^1 \land U \notin VT(A)
\end{align*}
\]

\( A \in \Pi_1^1 \cap \Sigma_2^1 \) iff \( A \) is isolated in the sense of [13].

3. An occurrence of a second-order quantifier \( QX \) in a formula \( QXA[X/U] \) is said to be distinguished if either \( Q = \forall \), \( \forall X A[X/U] \in \Pi_2^1 \) and \( U \notin VT(A) \), or \( Q = \exists \), \( \exists X A[X/U] \in \Sigma_2^1 \) and \( U \notin VT(A) \).

**Definition 3.2** The logic calculus SBL.

Axioms or initial sequents are

\[
(Ax) \quad (\Gamma, \neg L, L \quad \text{for prime } L)
\]

Inference rules are the followings.

\[
\frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma} \quad (\lor) \quad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma} \quad (\land)
\]

where in the rule \( (\lor) \), \( A_0 \lor A_1 \) is the main formula of \( (\lor) \) and is in the lower sequent \( \Gamma \). The formula \( A_i \) \((i = 0, 1)\) in the upper sequent is the minor formula

\(^1\)Strictly speaking, we should say that \( F \) is a semi-formula as in [13], but for simplicity let us call semi-formulas and semi-terms with bound variables as formulas and terms, resp.
of the \( (\lor) \). In the rule \( (\land) \), \( A_0 \land A_1 \) is the \textit{main formula}, and is in the lower sequent \( \Gamma \). Formulas \( A_i \), \( (i = 0, 1) \) in the upper sequents are the minor formulas of the \( (\land) \).

\[
\frac{\Gamma, F[t/x]}{\Gamma} \quad (\exists_1) \quad \frac{\Gamma, F[a/x]}{\Gamma} \quad (\forall_1)
\]

where in the rule \( (\exists)_1 \), \( \exists x F \) is the main formula of \( (\exists)_1 \) and is in the lower sequent \( \Gamma \). The formula \( F[t/x] \) in the upper sequent is the minor formula of the \( (\exists)_1 \). In the rule \( (\forall)_1 \), \( \forall x F \) is the main formula, and is in the lower sequent \( \Gamma \). The \( F[a/x] \) in the upper sequent is the minor formula of the \( (\forall)_1 \), and the free variable \( a \) is the \textit{eigenvariable} of the \( (\forall)_1 \), which does not occur in the lower sequent \( \Gamma \).

\[
\frac{\Gamma, \neg C \quad C, \Delta}{\Gamma, \Delta} \quad (cut)
\]

\( C \) is the \textit{cut formula} of the \( (cut) \).

\[
\frac{\Gamma, F[T/X]}{\Gamma} \quad (\exists_2) \quad \frac{\Gamma, F[U/X]}{\Gamma} \quad (\forall_2)
\]

where in the rule \( (\exists)_2 \), \( \exists X F \) is the main formula of \( (\exists)_2 \) and is in the lower sequent \( \Gamma \). The formula \( F[T/X] \) in the upper sequent is the minor formula of the \( (\exists)_2 \). In the rule \( (\forall)_2 \), \( \forall X F \) is the main formula, and is in the lower sequent \( \Gamma \). The \( F[U/X] \) in the upper sequent is the minor formula of the \( (\forall)_2 \), and the free variable \( U \) is the \textit{eigenvariable} of the \( (\forall)_2 \), which does not occur in the lower sequent \( \Gamma \).

\[
\frac{\Gamma, F[A/X]}{\Gamma} \quad (BI)
\]

where \( \exists X F \) is the main formula of the \( (BI) \) and is in the lower sequent \( \Gamma \), \( F[A/X] \) is the minor formula, and either

\( (BI)_1 \quad \exists X F \in \Pi^1_2 \cap \Sigma^1_2 \), or

\( (BI)_2 \quad A \in \Pi^1_2 \cap \Sigma^1_2 \).

\[
\frac{\Gamma, A \subset B}{\Gamma} \quad (\Pi^1_2\text{-Sep})
\]

where \( \exists X (A \subset X \subset B) \) is the main formula the \( (\Pi^1_2\text{-Sep}) \) and is in the lower sequent \( \Gamma \), \( A \subset B \) is the minor formula, and \( A \in \Pi^1_2 \) and \( B \in \Sigma^1_2 \).

**Theorem 3.3** Cut-elimination theorem for SBL.

There is a rewriting procedure \( r \) on derivations in SBL such that for any SBL-derivation \( P \) of a sequent, if \( P \) contains a \( (cut) \), then \( r(P) \) is an SBL-derivation of the same sequent, and there is an \( n \) such that its \( n \)-th iterate \( r^{(n)}(P) \) is cut-free.
Definition 3.4  
1. A formula is said to be first-order if no second-order quantifier occurs in it.
2. A sequent is first-order if every formula in it is first-order.
3. We say that the cut-elimination theorem holds for derivations in SBL ending with first-order sequents if there is a rewriting procedure $r$ on derivations of first-order sequents for which Theorem 3.3 holds.

Proposition 3.5 If the cut-elimination theorem holds for derivations in SBL ending with first-order sequents, then the cut-elimination theorem holds for SBL.

Proof. This is seen by cut-elimination by absorption combined with the joker translation due to P. Päppinghaus [9]. Note that
\[
\begin{align*}
\Gamma, \neg C & \quad \Gamma, C \\
\Gamma, -B_0 & \quad \Gamma, \exists Y F(Y)
\end{align*}
\]
are admissible rules in the presence of the inference rules $(BI)_1$ and $(BI)_2$ for $B_0 \equiv (\forall X \forall y[X(y) \rightarrow X(y)]) \in \Pi^1_2 \cap \Sigma^2_1$, cf. Theorem 1.3 and Lemma 1.5(ii) in [9], resp.

Remark. Let $SBL_1$ denote temporarily the calculus $SBL$ without the rule $(BI)_2$. Namely in $SBL_1$, the rule $(BI)$ is restricted to the case when the main formula $\exists X F \in \Pi^1_2 \cap \Sigma^1_2$. $SBL_1$ is equivalent to $SBL$ with respect to derivable sequents since $\exists X \forall y[X(y) \leftrightarrow A(y)] \in \Pi^1_2 \cap \Sigma^1_2$ for $A \in \Pi^1_2 \cap \Sigma^1_2$. The reason why we introduce the superfluous $(BI)_2$ in $SBL$ is as follows: when we operate our cut-elimination procedure to an $SBL_1$-derivation, then we obtain a cut-free $SBL$-derivation with rules $(BI)_2$ since we need ‘infer $\exists Y F(Y)$ from $F(\{x : B_0\})$ in replacing the joker $J_0$ by $B_0$ in Lemma 1.5(ii) of [9]. In other words, we don’t have an ‘inner’ cut-elimination theorem for $SBL_1$. It is open for us whether or not the ‘inner’ cut-elimination theorem for $SBL_1$ holds besides its intrinsic interests.

4 The stratified logic calculus $SBL'$

A stratified (in German: geschichtet) calculus $SBL'$ is introduced.

Definition 4.1 A stratified language $L'$ is obtained from a second-order language $L$ by modifying relation symbols and second-order variables as follows.
1. (unary) relation symbols with index 0: $R^0$.
2. (unary second-order) unstratified bound variables: $X, Y, \ldots$.
3. stratified variables:
(a) free variables with index \( s \) : \( U^s \) for \( s \in S \) and free variables \( U \) in \( L \).

(b) bound variables with index \( \eta \) : \( X^\eta \) for \( \eta \in Fx \) and bound variables \( X \) in \( L \).

When \( T \) denotes a predicate constant \( R \), \( T^s := R^0 \), i.e., \( s = 0 \).

**Definition 4.2** \( L' \)-formula \( A \) is obtained from an \( L \)-formula \( A \) by attaching indices as follows.

1. Attach the index 0 to each predicate constant \( R \) occurring in \( A \).

2. Attach an index \( s \in S \) to every occurrence of each free variable \( U \) occurring in \( A \). The indices may depend on free variables.

3. Attach an index \( \eta \in Fx \) to all undistinguished quantifiers. In a formula each undistinguished quantifier receives the same index. Also leave distinguished quantifiers without indices.

\( A^C \) denotes the (unstratified) \( L \)-formula obtained from an \( L' \)-formula \( A \) by erasing all indices. Conversely \( A' \) denotes ambiguously an \( L' \)-formula obtained from an \( L \)-formula \( A \) by attaching some indices.

\( A' \in \Pi^1_1[\Sigma^1_2] \) iff \( A \in \Pi^1_1[\Sigma^1_2] \), resp. \( \forall^\exists, \exists^\forall \) denote stratified quantifiers \( \forall X^\nu, \exists X^\nu \) for a bound variable \( X \).

**Definition 4.3** \( Gr(A), gr(A) < \omega \) for \( L' \)-formulas \( A \).

1. (a) \( Gr(A) = 0 \) if neither \( \forall^I \) nor \( \exists^I \) occurs in \( A \).

   In what follows assume that either \( \forall^I \) or \( \exists^I \) occurs in \( A \).

   (b) \( Gr(A) = \max\{Gr(A_0), Gr(A_1)\} + 1 \) if \( A \in \{A_0 \lor A_1, A_0 \land A_1\} \).

   (c) \( Gr(A) = Gr(B) + 1 \) if \( A \in \{\forall x B[x/u], \exists x B[x/u]\} \).

   (d) \( Gr(A) = 1 \) if \( A \in \{\forall X^I F, \exists X^I F\} \).

   (e) \( Gr(A) = \max\{2, Gr(F[R^0/X]) + 1\} \) if \( A \in \{\forall X^I F, \exists X^I F\} \).

2. (a) \( gr(A) = 0 \) if \( A \) is either a prime formula or of the form \( QXF \).

   (b) \( gr(A_0 \lor A_1) = gr(A_0 \land A_1) = \max\{gr(A_0), gr(A_1)\} + 1 \).

   (c) \( gr(\exists x B[x/u]) = gr(\forall x B[x/u]) = gr(B) + 1 \).

   (d) \( gr(QX^\nu F) = gr(F[R^0/X]) + 1 \).

**Definition 4.4** Let \( A \) be an \( L' \)-formula.

1. \( A \in \Sigma^I \) (in Schütte’s terminology ‘\( A \) ist klein’) if \( \forall^I \) does not occur in \( A \).

2. An occurrence of a free variable \( U^\nu \in Var' \) in \( A \) is said to be

   (a) in an index if the occurrence is in an index of a stratified (free) variable, and

   (b) an occurrence as a part of a formula otherwise.
3. $A$ is said to be stratified if for each index $s \in S$ of a free variable $U^s$ occurring as a part of the formula $A$, $\text{Var}(s) = \emptyset$ and $\text{od}(s) < I$.

**Definition 4.5** For $\mathcal{L}'$-formulas $A$, $st_\Pi(A) \in S$ if $A \in \Pi_2^1$, and $st_\Sigma(A) \in S$ if $A \in \Sigma_2^1$ are defined. Let $\Lambda \in \{\Pi, \Sigma\}$.

1. $st_\Lambda(T^s t) = st_\Lambda(\neg T^s t) = s$.
2. $st_\Lambda(A_0 \circ A_1) = \max(st_\Lambda(A_0), st_\Lambda(A_1))$ for $\circ \in \{\wedge, \lor\}$.
3. $st_\Lambda(Qx B[x/u]) = st_\Lambda(B)$ for $Q \in \{\forall, \exists\}$.

In what follows let $F_0 \equiv F[R^0/X]$.

4. $st_\Pi(\forall X F) = st_\Pi(F_0)$; $st_\Sigma(\forall X F) = st_\Pi(F_0) + 1$
   $st_\Sigma(\exists X F) = st_\Sigma(F_0)$; $st_\Pi(\exists X F) = st_\Sigma(F_0) + 1$

5. $st_\Pi(\forall X^\eta F) = \max(\eta, st_\Pi(F_0))$
   $st_\Sigma(\exists X^\eta F) = \max(\eta, st_\Sigma(F_0))$

Let $A$ be an $\mathcal{L}'$-formula. For a variable $U^\eta \in \text{Var}'$ and $s \in S$ let $A^{[s/U^\eta]}$ denote the $\mathcal{L}'$-formula obtained from $A$ by replacing every occurrence of $U^\eta$ in an index by $s$. $\Delta^{[s/U^\eta]} = \{A^{[s/U^\eta]} : A \in \Delta\}$ for sequents $\Delta$, and $P^{[s/U^\eta]}$ the tree of sequents obtained from a preproof $P$ by replacing each sequent $\Delta$ in $P$ by $\Delta^{[s/U^\eta]}$.

**Proposition 4.6** Let $A$ be an $\mathcal{L}'$-formula such that $A \in \Pi_2^1$.

1. $st_\Sigma(\neg A) = st_\Pi(A)$.
2. $\text{Var}(st_\Pi(A)) = \text{Var}(A)$.

In what follows assume that $A$ is stratified.

3. Let $\eta$ denotes the index of an undistinguished quantifier in $A$ if such a quantifier occurs. Otherwise let $\eta = 0$. Also let $\nu = \max I(A)$. Then there is a $k < \omega$ such that $st_\Pi(A) = \max\{\eta, \nu + k\}$.

4. Let $A \equiv (\forall X F)$, and $U$ a variable not occurring in $A$. Then $st_\Pi(F[U^s/X]) = st_\Pi(\forall X F)$ for $s = st_\Pi(\forall X F)$.

**Definition 4.7** Axioms and inference rules in $\text{SBL}'$.

$$(Ax) \quad \Gamma, \neg A, A \text{ with } Gr(A) = 0$$

$(\wedge), (\lor), (\forall), (\exists), (cut)$ are the same as in $\text{SBL}$.

$$\frac{\Gamma, \Delta}{\Gamma, \Delta} (th)$$

is the thinning.
1. critical rule.

\[ \frac{\Gamma, \exists X \eta F, F(T^s)}{\Gamma, \exists X \eta F} \] (c)

where \( \eta \neq I \Rightarrow Var(s) = \emptyset \) & \( od(s) < \eta \). \( s \) the index, and \( \eta \) type of the inference.

2. distinguished rules.

   (a)

\[ \frac{\Gamma, \exists X (A \subset X \subset B), A \subset B \text{, } \Gamma, \exists X (A \subset X \subset B)}{\Gamma, \exists X (A \subset X \subset B)} \] (d1)

where \( Gr(\exists X (A \subset X \subset B)) \neq 0 \).

(b)

\[ \frac{\Gamma, \exists X F, F(T^s)}{\Gamma, \exists X F} \] (d2)

where \( Gr(\exists X F) \neq 0 \).

3.

\[ \frac{\Gamma, \exists X F, F(A)}{\Gamma, \exists X F} \] (BI)

where

(a) if \( \exists X F \not\in \Pi_2 \), then \( \exists X F \) is stratified, and

(b) \( Gr(\exists X F) = 0 \).

4. strong rules.

   (a)

\[ \frac{\Gamma, \forall X F, F(U^s)}{\Gamma, \forall X F} \] (s1)

where \( Gr(\forall X F) \neq 0 \), \( U \) does not occur in the lower sequent, and \( s \) is obtained from \( st_{\Pi}(\forall X F) \) by replacing occurrences of \( I \) corresponding to an undistinguished quantifier in \( \forall X F \) by \( U^I \). \( st_{\Pi}(\forall X F) = s[I/U^I] \). (s1) is of type \( I \).

(b)

\[ \frac{\Gamma, \forall X \eta F, F(U^{\eta s})}{\Gamma, \forall X \eta F} \] (s2)

where \( U \) does not occur in the lower sequent. (s2) is of type \( \eta \).
5. weak rule.

\[ \Gamma, \forall X F, F(U^s) \quad (w) \]

where

(a) if \( \forall X F \notin \Sigma_2^1 \), then \( \forall X F \) is stratified.
(b) \( \text{Gr}(\forall X F) = 0 \).
(c) \( s = st\Pi(\forall X F) \).
(d) \( U \) does not occur in the lower sequent.

\( s \) is the index of the \((w)\).

6. substitution of level \( s \).

\[ \Gamma \quad (\text{sub})^s \]

where

(a) \( \text{Var}(s) = \emptyset \) and \( \text{od}(s) < I \).
(b) \( A \) is a stratified formula such that \( B[A/U^s] \) is an \( L' \)-formula for \( B \in \Gamma \).
(c) \( U \) does not occur in the lower sequent.
(d) any \( B \in \Gamma \) enjoys the followings.
   i. \( B \) is a stratified \( \Pi_2^1 \)-formula such that \( st\Pi(B) \leq \text{od}(s) \).
   ii. \( U \notin VT(B^c) \).

7. \( \forall I \)-reduction of type \( \eta < I \).

\[ \Delta_0, \Delta_1 \quad (\forall I \text{-red})^\eta \]

where each \( A \in \Delta_0 \cup \Delta_1 \) is either \( A \in \Sigma^I \) or \( \neg A \in \Sigma^I \), and \( A[\forall \eta/\forall^I] \) denotes the \( L' \)-formula obtained from \( A \) by replacing \( \forall X^I \) by \( \forall X^\eta \).

8. \( \exists I \)-reduction of type \( \eta < I \).

\[ \Delta_0, \Delta_1 \quad (\exists I \text{-red})^\eta \]

where \( \Delta_0 \cup \Delta_1 \subset \Sigma^I \) and \( A[\exists \eta/\exists^I] \) denotes the \( L' \)-formula obtained from \( A \) by replacing \( \exists X^I \) by \( \exists X^\eta \).

Inference rules without main formulas are \((\text{cut}), (\text{th}), (\text{sub}), (\forall I \text{-red})\) and \((\exists I \text{-red})\).
Definition 4.8 A preproof is a finite tree with \((Ax)\) and inference rules in \(SBL'\).

A preproof \(P\) enjoys the pure variable condition if all eigenvariables are distinct each other, each eigenvariable does not occur in the end-sequent of \(P\) and if a free variable occurs in an upper sequent of a rule, but not in the lower sequent, then the variable is the eigenvariable of the rule.

Let \(P\) be a preproof with the pure variable condition, and \(U^s\) a second-order free variable occurring in \(P\). Then either the stratified variable \(U^s\) occurs in the end-sequent, or an eigenvariable of one of rules \((s), (w), (sub)\) \(J\). Consider the latter case, and let \(V^\eta\) be a variable with index \(\eta\) occurring in the index \(s\) of \(U^s\). Then the rule \(J\) is either an \((s)\) or a \((w)\). When \(J\) is either an \((s1)\) or a \((w)\) with its main formula \(\forall XF\), then either \(s[I/U^I] = st_\Pi(\forall XF)\) or \(s = st_\Pi(\forall XF)\), cf. Definitions 4.7.4a and 4.7.5. Hence either \(V^\eta \equiv U^I\) corresponds to an undistinguished quantifier in \(\forall XF\), or \(V^\eta\) occurs in the index of a variable occurring in the main formula \(\forall XF\). Arguing inductively, this means that either the variable \(V^\eta\) occurs in the end-sequent, or corresponds to an undistinguished quantifier in a main formula of an \((s1)\), or an eigenvariable of an \((s2)\) with the index \(\eta\).

\[
\Gamma, \forall Y^\eta G, G(V^\eta) \quad (s2)
\]

Definition 4.9

1. The degree \(dg(A) < \omega + \omega\).

\[dg(A) = \begin{cases} gr(A) & \text{if } Gr(A) = 0 \\ \omega + (Gr(A) - 1) & \text{otherwise} \end{cases}\]

2. The height \(h(\Gamma) = h(\Gamma; P)\) of a sequent \(\Gamma\) in a preproof \(P\).

(a) \(h(\Gamma) = 0\) if \(\Gamma\) is the end-sequent of \(P\).

(b) \(h(\Gamma) = 0\) if \(\Gamma\) is an upper sequent of a \((sub)\).

(c) \(h(\Gamma) = \omega\) if \(\Gamma\) is an upper sequent of a \((Q^I\)-red\).

In what follows assume that \(\Gamma\) is an upper sequent of a rule \(J\) other than \((sub), (Q^I\)-red\) with the lower sequent \(\Delta\).

(d) \(h(\Gamma) = \max\{h(\Delta), dg(A)\}\) if \(J\) is either a \((cut)\) with the cut formula \(A\), or a \((BI)\) with the auxiliary formula \(A\).

(e) \(h(\Gamma) = h(\Delta)\) in other cases.

Relations between occurrences \(A, B\) of formulas in a preproof such as ‘\(A\) is a descendant of \(B\)’ or equivalently ‘\(B\) is an ancestor of \(A\)’, and ‘an occurrence of inference rule is implicit or explicit’ are defined as in [2][3].

Definition 4.10 Let \(P\) be a preproof.

1. Let \(\Delta\) be a sequent in \(P\).
(a) $\Delta$ is in the *explicit part* of $P$ if every rule below $\Delta$ is either a explicit rule or a $(th)$, and $\Delta$ is either an $(Ax)$ or a lower sequent of an explicit rule or a $(th)$.

(b) $\Delta$ is a *bar sequent* of $P$ if $\Delta$ is not in the explicit part of $P$, and either $\Delta$ is the end-sequent or an upper sequent of an explicit rule or a $(th)$ whose lower sequent is in the explicit part of $P$.

2. Let $\Delta_0$ be a bar sequent of $P$. The *end-piece* of $\Delta_0$ consists of the following sequents in $P$: $\Delta_0$ is in the end-piece. If a lower sequent of a rule other than implicit rule is in the end-piece, then its upper sequents are in the end-piece.

3. An implicit rule is *boundary rule* if its lower sequent is in an end-piece of $P$.

4. A triple $(J_1, J_2, J)$ of rules in $P$ is a *suitable triangle* if $J_i$ is a boundary rule with its main formula $A_i$ for $i = 1, 2$, and $J$ is a $(cut)$:

$$
\Gamma, \neg A, A, \Delta \quad (cut) J
$$

where $\neg A$ is a descendant of $A_1$, $A$ is a descendant of $A_2$ and $\{\neg A, A\} \cap (\Gamma \cup \Delta) = \emptyset$.

$A$ is said to be a *suitable cut formula*.

**Proposition 4.11** For a preproof $P$, $P$ contains no bar sequent iff $P$ consists solely of explicit rules and $(th)$’s.

In what follows a closed $s \in S$ is identified the ordinal $od(s)$.

**Definition 4.12** Let $P$ be a preproof enjoying the pure variable condition. A *stack function* $sck$ for $P$ assigns an ordinal $sck(J)$ (the *stack* of $J$) to each occurrence $J$ of rules $(\exists^f$-red) and $(sub)$ in $P$.

Given a stack function $sck$, we assign ordinals $o(\Delta) = o(\Delta; P, sck)$, $o(J) = o(J; P, sck)$ to each sequent $\Delta$ and each line of a rule $J$ recursively as follows.

1. $o(\Delta) = 1$ if $\Delta$ is an $(Ax)$.

In what follows let $\Delta$ be a lower sequent of a rule $J$ with upper sequents $\Gamma$ (and $\Gamma'$).

2. $o(J) = o(\Gamma)$ if $J$ is oner of rules $(Q$-red), $(sub)$, $(th)$.

3. $o(J) = o(\Delta) + 1$ if $J$ is one of rules $(\lor), (\forall_1), (\exists_1), (c), (d), (s)$ and $(w)$.

4. $o(J) = o(\Gamma) \# o(\Gamma')$ if $J$ is either a $(\land)$ or a $(cut)$.

5. Let $J$ be a $(BI)$ with its main formula $\exists XF$.

$$
o(J) = \begin{cases} 
\Omega_{s+1} \# o(\Gamma) & \text{if } \exists XF \text{ is stratified and } s = st_{\Sigma}(\exists XF) \\
I \# o(\Gamma) & \text{otherwise}
\end{cases}
$$
6. If \( J \) is a \(( sub)^* \) of level \( s \), then \( o(\Delta) = \psi_{\Omega^{s+1}}(\gamma \# \omega^\alpha) \) with \( \gamma = sck(J) \) and 
\( \alpha = o(J) = o(\Gamma) \).

7. If \( J \) is an \(( \exists I^I \)-red \), then \( o(\Delta) = \psi_I(\gamma \# \omega^\alpha) \) with \( \gamma = sck(J) \) and 
\( \alpha = o(J) = o(\Gamma) \).

8. Let \( J \) be a rule other than \(( sub)\), \(( \exists I^I \)-red \).

(a) \( o(\Delta) = \omega_m(o(J)) \) where \( h(\Gamma) = h(\Delta) + m \) for an \( m < \omega \).

(b) \( o(\Delta) = 0 \) if \( h(\Delta) < \omega \leq h(\Gamma) \), cf. the condition in Definition 4.15.2.

Finally \( o(P) = o(\Gamma_{end}; P, sck) \) for the end sequent \( \Gamma_{end} \) of \( P \).

Note that we have \( Gr(\exists XF) = 0 \) for the main formula \( \exists XF \) of a \(( BI) \), and 
\( \exists XF \) is stratified if \( \exists XF \not\in \Pi_2 \), cf. Definition 4.7.3. Then \( od(st_{\Sigma}(\exists XF)) = I \) iff 
\( \exists XF \in \Pi_2 \) and a free variable \( U^s \) occurs as a part of the formula \( \exists XF \) such
that \( od(s) = I \), while for an \( s \in S \), \( od(s) = I \) iff \( s \) contains a free variable \( V^I \) with the index \( I \). Suppose that the rule \(( BI) \) is in a preproof with the pure variable condition and the condition \( \[ \] \) in Proposition 4.13 is fulfilled for the
preproof. Then \( od(st_{\Sigma}(\exists XF)) = I \) iff \( \exists XF \in \Pi_2 \) and a free variable \( U^s \) occurs
as a part of the formula \( \exists XF \) such that \( s \) contains the eigenvariable \( V^I \) of an
\(( s2) \) with type \( I \).

\[
\Delta, \exists XF(U^s(V^I)) \quad (BI) \\
\vdots \\
\Gamma, G(V^I) \quad (s2)
\]

**Proposition 4.13** Let \( P \) be a preproof enjoying the pure variable condition. Let \( \Delta \) be a sequent in \( P \) and 
\( Var(\Delta) = \bigcup \{ Var(A) : A \in \Delta \} \) for the set \( Var(A) \) of
variables occurring in an index in the formula \( A \). Then the followings hold.

1. Each variable \( U \in Var(\Delta) \) is either an eigenvariable of a strong rule below
\( \Delta \), or \( U \in Var(\Gamma_{end}) \) with the end-sequent \( \Gamma_{end} \) of \( P \).

2. Let \( U^s \) be a variable other than eigenvariables of strong rules in \( P \), and
\( s \in S \) such that \( Var(s) = \emptyset \) and \( s < I \). For \( P^{(s/U)} \), cf. Definition 4.5,
\( P^{(s/U)} \) is a preproof enjoying the pure variable condition.

3. Let \( U \) and \( V \) be variables other than eigenvariables of strong rules in \( P \). Assume that \( V \) does not occur as a part of a formula in \( P \). For each formula 
\( A \) in \( P \) let \( A[V/U] \) denote the formula obtained from \( A \) by replacing every occurrence of the variable \( U \) as a part of a formula by the variable \( V \). 
\( \Delta[V/U] = \{ A[V/U] : A \in \Delta \} \), and \( P[V/U] \) be the tree of sequents obtained from \( P \) by replacing each sequent \( \Delta \) in \( P \) by \( \Delta[V/U] \).

Then \( P[V/U] \) is a preproof enjoying the pure variable condition.
Proof. Proposition 4.13.1 is seen inductively from below to above.
Propositions 4.13.2 and 4.13.3 are shown by induction on the depth of $P$ using Proposition 4.6.

Proposition 4.14 Let $P$ be a preproof enjoying the pure variable condition. Assume that $P$ satisfies the following condition:

The end-sequent of $P$ is a first-order sequent $\Gamma_{end}$ such that
any $A \in \Gamma_{end}$ is stratified and $st_{II}(A) = 0$ (1)

Let $\Delta$ be a sequent in $P$.

1. If $h(\Delta; P) < \omega$, then $dg(A) < \omega$, i.e., $Gr(A) = 0$ for any $A \in \Delta$.
2. Let $U^s$ be a stratified variable occurring in $\Delta$. Then $I(s) \cap Fx < I$. In particular
   (a) if the main formula $A$ of a (BI) is stratified, then $st_{2}(A) < I$, and
   (b) if the main formula $A$ of a (w) is stratified, then $st_{II}(A) < I$.
3. For any upper sequent $\Delta$ of a (sub), $Var(o(\Delta)) = \emptyset$ and $o(\Delta) < I$.

Proof. Proposition 4.14.1 is seen inductively from below to above. If $\Delta$ is an upper sequent of a (sub)$^s$ of level $s < I$ and $A \in \Delta$, then $st_{II}(A) \leq s$, and hence $Gr(A) = 0$.

Proposition 4.14.2 is shown inductively from below to above. If $\Delta$ is an upper sequent of a (sub)$^s$ of level $s < I$ with the eigenvariable $U$, then $Var(s) = \emptyset$. Hence $I(s) \cap F = I(s) < I$.

If $\Delta$ is an upper sequent of (w) with the eigenvariable $U$ and the main formula $\forall XF$, then $I(s) \cap F = I(\forall XF)$. The assertion follows from III.

If $\Delta$ is an upper sequent of (s1) with the eigenvariable $U$ and the main formula $\forall XF$, then by III, we have $I(s) \cap F = I(\forall XF) < I$.

Proposition 4.14.3 is seen from Proposition 4.13.

Definition 4.15 Let $P$ be a preproof enjoying the pure variable condition and $sck$ a stack function for $P$. $P$ together with $sck$ is said to be a proof (in SBL') if the following conditions are satisfied:

1. The end-sequent of $P$ is a first-order sequent $\Gamma_{end}$ such that
any $A \in \Gamma_{end}$ is stratified and $st_{II}(A) = 0$ (2)
2. Let \( J \) be a rule with its lower sequent \( \Delta \) and an upper sequent \( \Gamma \) such that
\[ h(\Delta) < \omega \leq h(\Gamma). \]
Then the rule \( J \) is a vacuous (\( \exists^I \)-red).

Any rule \( (Q^I\text{-red}) \) \( J \) occurring in \( P \) is in a series \( (J_0, \ldots, J_n) \) of rules
\( (Q^I\text{-red}) \), where \( J = J_{i_0} \) for an \( i_0 \leq n \), each \( J_{i+1} \) is immediately below \( J_i \),
there is a \( k \) with \( 0 \leq k \leq n \) such that each \( J_i \) (\( i < k \)) is an (\( \forall^I \)-red),
while each \( J_i \) (\( i \geq k \)) is an (\( \exists^I \)-red), and there is no rule (\( Q^I\text{-red} \)) above \( J_0 \) nor
below \( J_n \).

3. Let \( J \) be either an (\( \exists^I \)-red) or a (\( \text{sub}^\alpha \)), \( \Delta \) the upper sequent of \( J \), \( \alpha = o(\Delta; P, sck) \) and and \( \gamma = sck(J) \) the stack of \( J \) with respect to the stack
function \( sck \). Let \( \sigma = I \) when the rule is an (\( \exists^I \)-red), and \( \sigma = \Omega_{\alpha+1} \) when
it is a rule (\( \text{sub}^\alpha \)). Then for any index \( s \) occurring above \( J \)
\[ s \in H_\gamma(\psi_\sigma \gamma) \quad (2) \]
and
\[ \{\gamma, \alpha\} \subset H_\gamma(\psi_\sigma \gamma) \quad (3) \]
where by an index \( s \) occurring above \( J \) we mean
(a) \( T^s_0 \) occurs above \( J \) with \( s \in I(s_0) \), or
(b) \( X^s \) occurs above \( J \), or
(c) there is a rule (\( \exists^I \)-red)* occurring above \( J \), or
(d) there is a rule (\( \exists^I \)-red) \( J_0 \) occurring above \( J \) such that \( s = \psi_I(\alpha \# \omega^\beta) \)
with \( \alpha = sck(J_0) \) and \( \beta = o(J_0; P, sck) \).

4. Let \( J \) be an (\( \exists^I \)-red) of type \( \eta \) with the stack \( \gamma = sck(J) \), \( \Delta \) the upper
sequent of \( J \) with \( \alpha = o(\Delta; P, sck) \). Then
\[ \eta \geq \psi_I(\gamma \# \omega^\alpha) \quad (4) \]

5. every (\( \text{sub} \)) is in an end-piece of a bar sequent.

6. the eigenvariable of a (\( \text{sub} \)) does not occur in any explicit formula in the
upper sequent of the (\( \text{sub} \)).

7. each bar sequent \( \Gamma \) is the lower sequent of a vacuous (\( \text{sub}^0 \)) of level 0. The
vacuous (\( \text{sub} \)) is of the form
\[ \Gamma \vdash \Gamma \quad (\sub^0) \]
with an eigenvariable \( U^0 \) not occurring in \( \Gamma \).

Clearly for any proof \( P \), \( o(P) < \Omega_1 \).

For a first-order sequent \( \Gamma \) in the language \( L \), let \( \Gamma^0 \) denote the sequent in
\( L' \) obtained from \( \Gamma \) by attaching the index 0 to every second-order free variable
and predicate constant occurring in \( \Gamma \).
Proposition 4.16 Let $\Gamma$ be a first-order sequent $\Gamma$ in $\mathcal{L}$.

1. If $\Gamma$ is derivable in $\text{SBL}$, then so is $\Gamma^0$ in $\text{SBL}'$.

2. If there is a proof in $\text{SBL}'$ ending with $\Gamma^0$ and containing no bar sequent, then $\Gamma$ is (cut-free) derivable in the first-order sequent calculus $\text{LK}$.

Proof. Proposition 4.16.1 Let $P$ be a $\text{SBL}$-derivation of the first-order sequent $\Gamma$. We can assume that $P$ enjoys the pure variable condition, $P$ contains no rule $(BI)_2$, cf. Remark after Proposition 3.5, and any main formula $\exists X (A \subset X \subset B)$ of a $(\Pi^2_2\text{-Sep})$ is not $\Pi^2_2$, for otherwise $\exists X (A \subset X \subset B) \in \Pi^2_2 \cap \Sigma^2_2$, and it is derivable by the rule $(BI)_1$ and $A \subset B$, i.e., from, e.g., $A \subset A \subset B$.

Then construct a proof $P^0$ of $\Gamma^0$ from $P$ as follows: attach the index $I$ to every undistinguished quantifiers, attach the index 0 to every predicate constant, attach suitable indices to every second-order free variable from below to above. Clearly the condition (1) in Definition 4.15.1 is enjoined, and $st_P(A) = 0$ for any $A$ in the end-sequent $\Gamma^0$, which is first-order.

In the resulting preproof $P_0$, insert vacuous $(\exists^l\text{-red})$ immediately below a (cut) such that $h(\Delta_0, \Delta_1; P_0) < \omega \leq h(\Delta_0, \neg C; P_0)$ for the lower sequent $\Delta_0$, $\Delta_1$ and an upper sequent $\Delta_0, \neg C$ of the (cut). Namely change

$$
\frac{\Delta_0, \neg C, C, \Delta_1}{\Delta_0, \Delta_1} \quad \text{ (cut)}
$$

to

$$
\frac{\Delta_0, \neg C, C, \Delta_1}{\Delta_0, \Delta_1} \quad \text{ (cut)}
$$

Then the condition in Definition 4.15.2 is fulfilled. Note that $\text{dg}(A) < \omega$, i.e., $Gr(A) = 0$ for any $A \in \Delta_0 \cup \Delta_1$ since the end-sequent $\Gamma^0$ is first-order. In particular no undistinguished quantifier $Q^l$ occurs in $\Delta_0 \cup \Delta_1$.

Moreover insert vacuous $(\text{sub})^0$ at bar sequents. Note that any formula $B$ in any bar sequent is first-order, and hence $st_P(B) = 0$.

The resulting preproof is denoted $P^0$. Any main formula of rules $(BI)$ and $(w)$ in $P^0$ is in $\Pi^2_2 \cap \Sigma^2_2$, and $\eta = I$ for any main formula $\exists X^c \forall$ of rules (c) in $P^0$.

A stack function $sck^0$ together with types of vacuous $(\exists^l\text{-red})$ is defined as follows. First put $sck^0(J_0) = 0$ for any $(\exists^l\text{-red}) J_0$. Then the condition (2) is fulfilled since any index $s$ occurring in $P^0$ is in $\{ n, I + n : n < \omega \}$, and there is no rules $(\exists^l\text{-red})$ nor $(\text{sub})$ in $P^0$. Next the type $\eta$ of $(\exists^l\text{-red})$ $J_0$ is defined to be $\delta_0 = \psi_I(0 \# \alpha_0)$ for $\alpha_0 = o(J_0; P^0, sck^0)$. Obviously $\alpha_0 \in \mathcal{H}_0(0) \subset \mathcal{H}(\psi_I0)$, and the normality condition (3) is fulfilled for $J_0$. Then assign ordinals up to upper sequents of $(\text{sub})^0 J_1$. Let $\alpha_1 = o(J_1; P^0, sck^0)$, and pick an $n < \omega$ so that $\alpha_0 < \omega_n(I + 1)$ for any $(\exists^l\text{-red}) J_0$ occurring above $J_1$ with $\alpha_0 = o(J_0; P^0, sck^0)$. Then let $sck^0(J_1) = \omega_n(I + 1)$, and $\delta_1 = o(\Delta; P^0, sck^0) = \psi_{\omega_n(I + 1)}(\omega_n(I + 1) \# \alpha_1)$ for the lower (bar) sequent $\Delta$ of the $(\text{sub})^0 J_1$. Then $\psi_I(0 \# \alpha_0), \omega_n(I + 1) \# \alpha_1 \in \mathcal{H}_{\omega_n(I + 1)}(\psi_{\omega_n(I + 1)})$. Hence the conditions (2) and (3) are fulfilled for $J_1$.  

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Thus $P^0$ is a proof in $\text{SBL}'$.

Proposition 4.16.2 is seen from Proposition 4.11. Namely erase all the indices 0 from the proof of $\Gamma^0$ without bar sequent. Then the result is a cut-free $\text{LK}$-derivation of $\Gamma$. □

By Propositions 3.5 and 4.16, and the well-foundedness of $(\mathcal{H}_{e_{i+1}}(0) \cap \Omega_1, <)$ it suffices to show the following.

**Main Lemma 4.17** For any proof $P$ and a stack function $sck$ in $\text{SBL}'$ with a bar sequent, we can construct another proof $P'$ and stack function $sck'$ with the same end-sequent such that $o(P') < o(P)$.

Main Lemma is proved in the next section 5.

### 5 Proof of Main Lemma 4.17

Throughout this section $P$ together with a stack function $sck$ denotes a proof with a bar sequent. For simplicity let us suppress stack functions in ordinals attached to sequents and rules. Namely $o(\Gamma; P, sck) \mid o(J; P, sck)$ is denoted by $o(\Gamma; P) \mid o(J; P)$, resp.

Each reduction, i.e., rewriting step is performed within the end-piece of a bar sequent $\Delta_0$. By Definition 4.15.7 the bar sequent $\Delta_0$ is the lower sequent of a vacuous $(\text{sub})^0$ with its stack $\gamma$.

\[
\frac{\Delta_0; \alpha_0}{\Delta_0; \psi_{\Omega_1}(\gamma \# \omega^\alpha)} \quad (\text{sub})^0; \alpha_0
\]

where and everywhere in this section, $\Gamma; \alpha$ designates that $o(\Gamma; P) = \alpha$ for sequents $\Gamma$ in $P$, and $J; \alpha$ that $o(J; P) = \alpha$ for rules $J$ in $P$. Also we see from (1) in Definition 4.15.1 and the pure variable condition that each formula in an end-piece is stratified.

When $P$ is rewritten to another $P'$ below, a stack function $sck'$ for $P'$ is defined in an obvious way except otherwise stated explicitly. Namely a rule $J'$ in $P'$ receives the same stack as one for the corresponding rule in $P$ in most cases. In each step we need to verify that $P'$ is a proof and $o(P') < o(P)$. In most cases this amounts to show that $P'$ together with a stack function $sck'$ fulfills the conditions in (2), (3), and (4).

**Case 1.** An explicit rule is in an end-piece of a bar sequent $\Delta_6$ in $P$:

Let $J_0$ be one of the lowest explicit rule in the end-piece of $\Delta_6$. By (4) in Definition 4.15.1, the end-sequent of $P$ is a first-order sequent, and hence $J_0$ is one of rules $(\land), (\lor), (\forall_1), (\exists_1)$. Consider the case when $J_0$ is a rule $(\forall_1)$, and
let $P$ be the following:

$$
\begin{align*}
\frac{\Gamma, B(a); \beta}{\Gamma; \beta + 1} & \quad (\forall_1) J_0 \\
\vdots & \\
\frac{\Delta_b; \alpha}{\Delta_b; \psi_{\Omega_1}(\gamma\#\alpha^\omega)} & \quad J
\end{align*}
$$

where $\forall x B(x) \in \Gamma \cap \Delta_b$ and $\Delta_b; \alpha$ for the upper sequent $\Delta_b$ of the vacuous $(sub)^0 J$ with its lower sequent $\Delta_b$ and its stack $\gamma = sck(J)$. Note that by Definition 4.15.6 no $(sub)$ change explicit formulas, and the end-piece ends with a vacuous $(sub)$ by Definition 4.15.7.

Let $P'$ be the following.

$$
\begin{align*}
\frac{\Gamma, B(a); \beta}{\Delta_b, B(a); \alpha'} & \quad J' \\
\frac{\Delta_b, B(a); \alpha'}{\Delta_b; \psi_{\Omega_1}(\gamma\#\alpha'^\omega)} & \quad (\forall_1)
\end{align*}
$$

We see from $\beta \ll \beta + 1$ and Proposition 2.3 that $\alpha' \ll \alpha$. From this we see $o(P') < o(P)$. Let us verify that $P'$ is a proof. The condition $(4)$ on rules $(\exists I)-red$ in $P'$ is fulfilled by $\beta \ll \beta + 1$. We have $\alpha \in H_{\gamma}(\psi_{\Omega_1})$ by $(3)$ for $J$. Hence $\alpha' \in H_{\gamma}(\psi_{\Omega_1})$ for the stack $\gamma = sck(J')$ of the vacuous $(sub)^0 J'$ in $P'$. Similarly we see that the conditions $(2)$ and $(3)$ on rules $(\exists I)-red$, $(sub)$ in $P'$ are fulfilled. Therefore $P'$ is a proof.

**Case 2.** $\{\neg A, A\} \subset \Delta_b$ for a formula $A$ and a bar sequent $\Delta_b$:

By $(1)$ in Definition 4.15.1 $A$ is a first-order formula, and $Gr(A) = 0$.

$$
P = \frac{\Delta_b, \neg A, A; \psi_{\Omega_1}(\gamma\#\alpha^\omega)}{\Delta_b, \neg A, A; 1} \quad (Ax)
$$

$$
P' = \frac{\Delta_b, \neg A, A; \alpha'}{\Delta_b, \neg A, A, 1} \quad (Ax)
$$

**Case 3.** The end-piece of a bar sequent $\Delta_b$ contains a $(cut)$ of the following form:

$$
P = \frac{\Gamma, \neg A; \alpha A, \Delta; \beta}{\Gamma, \Delta; \gamma} \quad (cut); \alpha\#\beta
$$

where $\neg A \in \Gamma \cup \Delta$. By Proposition 4.14.4 we have $h(\Gamma, \Delta) < \omega \Rightarrow h(\Gamma, \neg A) < \omega$. In other words, $h(\Gamma, \neg A) = h(\Gamma, \Delta) + m$ for an $m < \omega$. Thus $\gamma = \omega_m(\alpha\#\beta)$. Let $P'$ be the following.

$$
P' = \frac{\Gamma, \neg A; \alpha'}{\Gamma, \Delta; \alpha'} \quad (th)
$$
Note that the height of an upper sequent of a \((\text{sub})\) is defined to be 0 in Definition 4.9.2b and the height of an upper sequent of an \((\exists^l\text{-red})\) is equal to \(\omega\) by Definition 4.9.2d. Hence there is no (\text{sub}) nor \((\exists^l\text{-red})\) in the height lowering part in \(P'\). Thus we see that \(\alpha' \leq \omega_m(\alpha) \leq \gamma\), and \(P'\) is a proof such that \(o(P') < o(P)\). By virtue of Case 1-3 we can assume that any end-piece of \(P\) contains no explicit rule nor axiom. Then we see as in Sublemma 12.9 of [13] that \(P\) contains a suitable triangle.

Before reducing suitable triangles, let us consider the following cases. Cases 4-6 when a descendant of the main formula of a boundary rule \(J_0 = (d),(c)\) is changed by a \(J_1 = (\exists^l\text{-red})\), and Cases 7-8 when a descendant of the main formula of a boundary rule \(J_0 = (s)\) is changed by a \(J_1 = (\forall^l\text{-red})\). In each of these cases, \(J_0\) and \(J_1\) are exchanged. When \(J_0 = (d)\), the distinguished rule \((d)\) is changed to a \((BI)\). When \(J_0 = (s1)\), the strong rule \((s1)\) is changed to a weak rule \((w)\).

**Case 4.** A descendant of the main formula of a boundary rule \((d1)\) is changed by an \((\exists^l\text{-red})\): Let \(P\) be the following.

\[
P = \frac{\Delta_0, \exists XF, A \subset B; \gamma}{\Delta_0, \exists XF; \gamma + 1} (d1)
\]

where the lower rule \((\exists^l\text{-red})\) is a vacuous one such that \(h(\Gamma_1, \exists XF') < \omega = h(\Gamma_0, \exists XF'), F \equiv (A \subset X \subset B)\) with \((\exists XF) \in \Sigma^l\) and \(Gr(\exists XF) = 1\). Also \(F' \equiv F[\exists^l/\exists^l] \equiv (A' \subset X \subset B') \equiv (A[\forall^l/\forall^l] \subset X \subset B[\forall^l/\forall^l])\). Hence \(Gr(A') = Gr(B') = Gr(\exists XF') = 0\), and \(\exists XF'\) is stratified. Note that there is no (\text{sub}) between the boundary \((d1)\) and the \((\exists^l\text{-red})\) since the formula \(\exists XF\) with \(Gr(\exists XF) \neq 0\), and hence with \(s_{t1}(\exists XF) \geq I > s\) is not in the upper sequent of a (\text{sub}). All of these are seen from Definition 4.7.

Let \(P'\) be the following.

\[
P' = \frac{\Delta_0, \exists XF, A \subset B; \gamma}{\Delta_0, \exists XF; \delta} (\exists^l\text{-red})^n
\]

\[
\frac{\Delta_0, \exists XF, A \subset B; \delta'}{\Delta_1, \exists XF', A' \subset B'} (\exists^l\text{-red})^n
\]

\[
\frac{\Gamma_0, \exists XF, A \subset B; \delta'}{\Gamma_1, \exists XF', A' \subset B'} (\exists\text{-red})
\]

\[
\frac{\Gamma_0, \exists XF, A \subset B; \delta'}{\Gamma_1, \exists XF', A' \subset B'} (\forall\text{-red})
\]

\[
\frac{\Gamma_1, \exists XF', A' \subset B'}{\Gamma_1, \exists XF'} (BI)
\]

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where \( A' \subset A' \) is derived from the axiom \( \neg A'(u), A'(u) \) with \( Gr(A'(u)) = 0 \) by two \( \forall \)'s followed by a \( \forall \).

It is easy to see that \( o(\Gamma, \exists XF, A \subset B; P') = \gamma \) since \( Gr(F'(A')) = 0 \) and \( dg(F'(A')) = gr(F'(A')) < \omega = h(\Gamma_0, \exists XF', A' \subset B'; P') = h(\Gamma_0, \exists XF'; P) \). \( \delta' \) is an ordinal such that \( \delta' \ll \delta \) by Proposition 2.3. In particular \( \psi_1(\alpha\#\omega^\delta) < \psi_1(\alpha\#\omega^\delta) \leq \eta \) for the stack \( \alpha \) of the rules \((\exists\ell\text{-red})^\eta \) by the condition 4.

Let \( s \) be an index occurring in the formula \( \exists XF \). Then by the condition (2) we have \( s \in H_\alpha(\psi_1(\alpha)) \), and hence \( s \in H_{\alpha\#\omega^\delta}(\psi_1(\alpha\#\omega^\delta)) \) for \( \eta \). Hence \( st_{\Sigma}(\exists XF) = \eta \).

Let us show \( o(\Gamma_1, \exists XF'; P') \ll o(\Gamma_1, \exists XF'; P) \{\eta\} \). Let \( \beta \) be the stack of the lower vacuous rule \((\exists\ell\text{-red})^\eta \). Then \( o(\Gamma_1, \exists XF'; P) = \psi_1(\beta\#\omega^\delta) \), while \( o(\Gamma_1, \exists XF'; P') = \omega_m(4\#\psi_1(\beta\#\omega^\delta)\#\Omega_{\eta+1}) \) for an \( m < \omega \).

We see \( o(\Gamma_1, \exists XF'; P') \ll o(\Gamma_1, \exists XF'; P) \) from \( \psi_1(\beta\#\omega^\delta) \ll \psi_1(\beta\#\omega^\delta) \) and \( \eta \ll \psi_1(\beta\#\omega^\delta) \). The latter follows from (2), i.e., from \( \eta \in H_\beta(\psi_1(\beta) \cap I \subset H_{\beta\#\omega^\delta}(\psi_1(\beta\#\omega^\delta)) \cap I = \psi_1(\beta\#\omega^\delta) \). This yields \( \Omega_{\eta+1} \ll \psi_1(\beta\#\omega^\delta) \{\eta\} \), and \( o(\Gamma_1, \exists XF'; P') \ll o(\Gamma_1, \exists XF'; P) \{\eta\} \). We see that the conditions 3 and 4 is fulfilled for rules \((\exists\ell\text{-red}) \) in \( P' \).

Consider the condition (2) for rules \((\exists\ell\text{-red}) \), e.g., for the lower vacuous rule \((\exists\ell\text{-red})^\eta \). There occur new indices, e.g., \( \psi_1(\alpha\#\omega^\delta) \) for the rule \((\exists\ell\text{-red})^\eta \) in \( P' \), we need to show \( \psi_1(\alpha\#\omega^\delta) \in H_\beta(\psi_1(\beta)) \). We have \( \psi_1(\alpha\#\omega^\delta) \in H_\beta(\psi_1(\beta)) \) for the stack \( \beta \) of the vacuous rule. From \( \psi_1(\alpha\#\omega^\delta) \ll \psi_1(\alpha\#\omega^\delta) \), \( \delta' \ll \delta \) and Proposition 2.3.6 we see that \( \psi_1(\alpha\#\omega^\delta) \ll \psi_1(\alpha\#\omega^\delta) \). In particular \( \psi_1(\alpha\#\omega^\delta) \in H_\beta(\psi_1(\beta)) \).

Let \( \Pi_0 \) be an upper sequent of a \(( sub )^\mu \) occurring below \( \Gamma_1, \exists XF' \in P \) with its lower sequent \( \Pi_1 \) and its stack \( \beta = stc(k)(J) \). Assume \( \alpha_0 \ll \alpha \{\eta\} \) for \( \alpha_0 = o(\Pi_0; P') \) and \( \alpha_0 = o(\Pi_0; P) \). Let the stack \( \beta = stc^k(J') \) of the corresponding rule \(( sub )^\mu \) \( J' \) in \( P' \). We see that the condition (2) on \(( sub )^\mu \) \( J' \) is fulfilled as above from Proposition 2.3.6. For \( \sigma = \Omega_{\eta+1} \), let \( \alpha_1 = o(\Pi_1; P) = \psi_1(\beta\#\omega^\delta) \) and \( \alpha_1' = o(\Pi_1; P') = \psi_1(\beta\#\omega^\delta) \). Then \( \alpha_1' \ll \alpha \{\eta\} \) follows from Proposition 2.3.6 and (2), \( \eta \in H_\beta(\psi_1(\beta)) \). Hence \( o(P') \ll o(\Pi; \{\eta\}) \) and \( o(P') < o(P) \).

**Case 5.** A descendant of the main formula of a boundary rule \(( d2 \) (d2) is changed by an \((\exists\ell\text{-red}) \): Let \( P \) be the following.

\[
P = \frac{\Delta_0, \exists XF, F(T^s); \gamma}{\Delta_0, \exists XF} \quad (d2)
\]

\[
P = \frac{\Delta, \exists XF; \delta}{\Delta', \exists XF'} \quad (\exists\ell\text{-red})^\eta
\]

\[
\frac{\Gamma_0, \exists XF'}{\Gamma_1, \exists XF'} \quad (\exists\ell\text{-red})^\eta
\]

where the lower rule \((\exists\ell\text{-red})^\eta \) is a vacuous one such that \( h(\Gamma_1, \exists XF') < \omega = h(\Gamma_0, \exists XF'), F' \equiv F[\exists^\eta/\exists^\ell], Gr(\exists XF) \neq 0 \), and \( T^s \) is either a predicate constant \( R^0 \) or a stratified free variable \( U^s \) with \( Var(s) = \emptyset \). Also \( o(\Delta_0, \exists XF; P) = 21 \).
\( \gamma + 1 \) and \( Gr(\exists XF') = 0 \) with stratified \( \exists XF' \). Similarly as in Case 4 we see that \( st_\Sigma(\exists XF') = \eta \), and the following \( P' \) is a proof such that \( o(P') < o(P) \).

\[
P' = \Delta_0, \exists XF, F(T^*)
\]

\[
P' = \Delta, \exists XF, F(T^*) \quad (\exists'^I\text{-red})^\eta
\]

\[
\Delta', \exists XF', F'(T^*) \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_0, \exists XF', F'(T^*) \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_1, \exists XF'
\]

**Case 6.** A descendant of the main formula of a boundary rule \((c)\) is changed by an \((\exists'^I\text{-red})\): Let \( P \) be the following.

\[
P = \Delta_0, \exists X^I F, F(T^*); \gamma + 1 \quad (c)
\]

\[
P = \Delta, \exists X^I F \quad (\exists'^I\text{-red})^\eta
\]

\[
\Delta', \exists X^I F' \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_0, \exists X^I F'; \delta \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_1, \exists X^n F'
\]

where the lower rule \((\exists'^I\text{-red})\) is a vacuous one such that \( h(\Gamma_1, \exists XF') < \omega = h(\Gamma_0, \exists XF') \), \( F' \equiv F[\exists^n / \exists^I] \). There is no \((\text{sub})\) between the boundary \((c)\) and the \((\exists'^I\text{-red})\) since \( Gr(\exists X^I F) \neq 0 \) as in Case 4. By (2) and (4) we have \( s \in H, (\psi I \gamma) \) and \( \psi I \gamma \leq \eta \) with the stack of rules \((\exists'^I\text{-red})^\eta \). Hence \( s < \eta \), and the rule \((c)\) in the following \( P' \) is a legitimate one.

\[
P' = \Delta_0, \exists X^I F, F(T^*); \gamma
\]

\[
P' = \Delta, \exists X^I F \quad (\exists'^I\text{-red})^\eta
\]

\[
\Delta', \exists X^I F', F'(T^*) \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_0, \exists X^n F', F'(T^*); \delta' \quad (\exists'^I\text{-red})^\eta
\]

\[
\Gamma_1, \exists X^n F'
\]

We have \( o(\Gamma_1, \exists X^n F'; P) = \psi_I(\alpha \# \omega^\delta) \) and \( o(\Gamma_1, \exists X^n F'; P') = \psi_I(\alpha \# \omega^\delta') + 1 \) for the stack \( \alpha \) of lower vacuous rules \((\exists'^I\text{-red})\). From \( \delta' \ll \delta \) we see \( \psi_I(\alpha \# \omega^\delta) + 1 \ll \psi_I(\alpha \# \omega^\delta) \). We see easily that \( P' \) is a proof such that \( o(P') < o(P) \).
Case 7. A descendant of the main formula of a boundary rule (s1) is changed by an ($\forall^l$-red): Let $P$ be the following.

$$P = \frac{\Delta_0, \forall X \bar{F}, F(U^\ast); \gamma}{\Delta_0, \forall X \bar{F}; \gamma + 1} \quad (s1)$$

where the lower rule ($\exists^l$-red) is a vacuous one such that $h(\Gamma_1, \forall X F') < \omega = h(\Gamma_0, \forall X F'), F' \equiv F[\eta/\forall^l], Gr(\forall X F) \neq 0$ and $s[I/U^l] = st_{\Pi}(\forall X F)$. Since $\forall X \bar{F} \in \Pi^1_2$ and $Gr(\forall X F') \neq 0$, $\forall^l$ occurs in $\forall X F$, i.e., $\forall X F \notin \Sigma^l$. Therefore there occurs no (sub) between the boundary (s1) and the ($\forall^l$-red). On the other, $Gr(\forall X F') = 0$ and $\forall X F'$ is stratified. Hence the rule $(w)$ in the following $P'$ is a legitimate one.

$$P' = \frac{\Delta_0, \forall X \bar{F}, F(U^\ast); \gamma'}{\Delta', \forall X F'; \delta'} \quad (\forall^l$-red)$^\eta$

$$\frac{\Delta', \forall X F'; \delta'}{\Delta', \forall X F'; \delta'} \quad (\exists^l$-red)

$$\frac{\Delta_0, \forall X \bar{F}, F(U^\ast); \gamma'}{\Delta_0, \forall X \bar{F}; \gamma + 1} \quad (w)$$

where $s' = st_{\Pi}(\forall X F') = s[\eta/U^l]$. $P'_0$ is obtained from $P_0$ by $P'_0 = P_0[\eta/U^l]$.

We have $\gamma' \ll \gamma \{\eta\}$, and $\gamma' \ll \gamma \{\eta\}$ if in $P'_0$, there is a rule (BI) with a main formula $\exists Y B$ such that the variable $U^\ast$ occurs as a part of $\exists Y B$, or $s$ occurs in an index of a free variable in $\exists Y B$. At such a rule (BI), $I$ is added in $P$, while $\Omega_{\mu+1}$ is added in $P'$ for $\mu = st_{\Sigma}(\exists Y B)[U^\ast/U^l])$.

Let $S_{\Omega_{\mu}}$ be the set of all indices $s_0$ such that either a free variable $V^\ast$ or a bound variable $Y_{s_0}$ occurs in a main formula of a (BI) in $P_0$. Let $\eta' = \eta \# \{I(s_0) : s_0 \in S_{\Omega_{\mu}}\}$. Then $\Omega_{\mu+1} \ll I \{\eta'\}$ for each such (BI). Hence $\delta' \ll \delta \{\eta'\}$. On the other hand we have $\{\alpha, \delta, \eta'\} \subset H_{\alpha}(\psi_{\eta}\alpha)$ for the stack $\alpha$ of the lower vacuous rules ($\exists^l$-red) by Proposition 2.3.4 and Definitions 3.1.6.3a, 3.1.6.3b. Hence by Proposition 2.3.4 we obtain $\psi_{\eta}(\alpha \# \omega^\delta) \ll \psi_{\eta}(\alpha \# \omega^\delta) \{\eta'\}$ for $o(\Gamma_1, \forall X F'; P) = \psi_{\eta}(\alpha \# \omega^\delta)$ and $o(\Gamma_1, \forall X F'; P') = \psi_{\eta}(\alpha \# \omega^\delta) + 1$. Thus $o(\Gamma_1, \forall X F'; P') \ll o(\Gamma_1, \forall X F'; P) \{\eta'\}$, and we obtain $o(P') \ll o(P) \{\eta'\}$, and $o(P') < o(P)$. 23
Let us verify that $P'$ is a proof. Although $\eta$ is a new index in the upper part of $\Delta, \forall XF, F(U^*)$, there is no rule ($\exists I$-red) nor ($\text{sub}$) in the part since there is no ($\text{sub}$) above the boundary ($s_1$) by Definition 4.15.5. Hence the conditions (2) and (3) are enjoyed for the upper part. The condition in (4) is fulfilled as we saw above. We see that the conditions (2) and (3) are fulfilled below ($\forall I$-red) in $P'$ by Proposition 2.3.6 and (2) for $P$.

**Case 8.** A descendant of the main formula of a boundary rule ($s_2$) is changed by an ($\forall I$-red): Let $P$ be the following.

\[
\frac{\Delta_0, \forall X^I F, F(U^I); \gamma}{\Delta_0, \forall X^I F; \gamma + 1} \quad (s_2)
\]

\[
P = \frac{\Delta, \forall X^I F}{\Delta', \forall X^\eta F'} \quad (\forall I\text{-red})^\eta
\]

\[
\frac{\Gamma_0, \forall X^\eta F'}{\Gamma_1, \forall X^\eta F'} \quad (\exists I\text{-red})
\]

where the lower rule ($\exists I$-red) is a vacuous one such that $h(\Gamma_1, \forall X^\eta F') < \omega = h(\Gamma_0, \forall X^\eta F'), F' \equiv F[\forall \eta/\forall I]$.

Let $P'$ be the following.

\[
P' = \frac{\Delta_0, \forall X^I F, F(U^\eta)}{\Delta_0, \forall X^I F; \gamma'} \quad (s_2)
\]

\[
\frac{\Delta, \forall X^I F, F(U^\eta)}{\Delta', \forall X^\eta F', F'(U^\eta)} \quad (\forall I\text{-red})^\eta
\]

\[
\frac{\Gamma_0, \forall X^\eta F', F'(U^\eta)}{\Gamma_1, \forall X^\eta F', F'(U^\eta)} \quad (\exists I\text{-red})
\]

In $P'$, the index $U^I$ is replaced by $U^\eta$. As in Case 7 we see that $P'$ is a proof such that $o(P') < o(P)$.

In the following cases let us reduce suitable triangles $(J_1, J_2, J)$, where descendants of main formulas of $J_1$ and $J_2$ are not changed by any rules ($Q^I$-red) by virtue of Cases 4-8.
Case 9. $J_1$ is an (s1) and $J_2$ is a (d): Let $P$ be the following.

$$
\begin{array}{c}
\Delta_0, \forall X \neg F(U^\alpha) \\
\Delta_0, \forall X \neg F \\
\vdots \\
\Delta, \forall X \neg F: \alpha \\
\Delta, \Gamma; \alpha \# \beta \\
\Pi; \delta \\
\Phi; \psi_I(\alpha_4 \# \omega^\beta) \\
\hline
\end{array}
\begin{array}{c}
(s1) J_1 \\
(d) J_2 \\
\ldots \\
J \\
J_3 \\
J_4 \\
\Phi; \psi_I(\alpha_4 \# \omega^\beta) \\
\hline
\end{array}
$$

where $Gr(\exists XF) = 1$, i.e., $dg(\exists XF) = \omega = h(\Pi)$, $J_2$ is either a (d1) with $G \equiv (A \subset B)$ and $F \equiv (A \subset X \subset B)$, or a (d2) with $G \equiv F(T^\alpha)$. $\Pi$ denotes the upper sequent of the uppermost $(\exists^l$-red) $J_3$ below $J$. $\Phi$ denotes the lower sequent of the lowest vacuous rule $(\exists^l$-red) $J_4$. In other words $\Phi$ is the uppermost sequent below the (cut) $J$ such that $h(\Phi) < \omega$. Let $\alpha_n = sek(J_n)$ be the stack of the rule $J_n$ for $n = 3, 4$.

Note that no (sub) occurs between $J$ and $\Phi$ since the height of the upper sequent of a (sub) is defined to be 0, cf. Definition 3.19. Furthermore there is no (sub) between the (s1) $J_1$ and (cut) $J$, and no (sub) between the (d) $J_2$ and $J$ since $Gr(\exists XF) \neq 0$.

Let $P'$ be the following.

$$
\begin{array}{c}
\Delta, \forall X \neg F: \alpha \\
\Delta, \Gamma, \forall X \neg F \\
\vdots \\
\Pi, \forall X \neg F; \delta_1 \\
\Pi, \forall X \neg F' \\
\vdots \\
\Phi, \forall X \neg F', \psi_I(\alpha_4 \# \omega^\beta) \\
\hline
\Delta, \forall X \neg F: \alpha \\
\Delta, \Gamma, \forall X \neg F \\
\vdots \\
\Pi, \forall X \neg F; \delta_1 \\
\Pi, \forall X \neg F' \\
\vdots \\
\Phi, \forall X \neg F', \psi_I(\alpha_4 \# \omega^\beta) \\
\hline
\end{array}
\begin{array}{c}
\exists XF; \Gamma; \beta \\
\exists XF; \Delta, \Gamma \\
\vdots \\
\exists XF; \Pi; \delta_2 \\
\exists XF'; \Pi \\
\vdots \\
\Phi, \forall X \neg F', \psi_I(\alpha_4 \# \omega^\beta) \\
\hline
\end{array}
$$

where $F' \equiv F[\exists^\eta / \exists^l]$. Hence $dg(\exists XF') = gr(\exists XF') < \omega = h(\Pi; P)$. Then $o(\Delta, \forall X \neg F; P) = \alpha = o(\Delta, \forall X \neg F; P')$ and $o(\exists XF, \Gamma; P) = \beta = o(\exists XF, \Gamma; P')$.

From $\alpha, \beta \ll \alpha \# \beta$ we see that

$$
\delta_1, \delta_2 \ll \delta
$$

(5)

The stack of the new rule $(\exists^l$-red)$^\eta$ is defined to be $\alpha_3 = sek(J_3)$, and the type $\eta$ of the new rules $(\forall^l$-red)$^\eta$ and of $(\exists^l$-red)$^\eta$ is defined to be $\eta = \psi_I(\alpha_3 \# \omega^\beta)$ with $\delta_2 = o(\exists XF, \Pi; P')$. Let us verify the conditions 2, 3 and
by (3) for $J_3$. The condition (3), $\alpha_3, \delta_2 \in \mathcal{H}_{\alpha_3}(\psi_1\alpha_3)$ follows from this and (1).

Next let us increase stacks of the rules $(\exists^I\text{-red}) J_3i$ by $\omega^\omega + 1$. The stack of the rules $(\exists^I\text{-red}) J_3i_1$ and of $J_3i_2$ is defined to be $\alpha'_3 = \text{sck}'(J_3i_1) = \text{sck}'(J_3i_2) = \alpha_3\#\omega^\omega \#1$. We see that the conditions (2), (3) and (4) are fulfilled for $J_3i_1$ with $i = 1, 2$ as follows. The new index $\eta \in \mathcal{H}_{\alpha_3}(\psi_1\alpha'_3)$ for (2). This is seen from (3), (4) and $\alpha_3\#\omega^\omega < \alpha'_3$. For $i = 1, 2$, we see $\alpha'_i, \delta_i \in \mathcal{H}_{\alpha_3}(\psi_1\alpha'_3)$ and $\psi_1(\alpha'_i\#\omega^\omega) \ll \psi_1(\alpha_3\#\omega^\omega)$ from (5) and (6). Thus the conditions (3) and (4) are enjoyed for rules $J_3i$.

Let $K$ be a rule $(\exists^I\text{-red})$ occurring below $J_3$ in $P$, and $\gamma = \text{sck}(K)$ its stack. Then the stack $\gamma'$ of the corresponding rules $K'$ in $P'$ is defined to be $\text{sck}'(K') = \text{sck}(K) = \gamma$, and let $\delta' = o(K'; P') \in \{\delta_1, \delta_2\}$. In particular the stack $\text{sck}'(J_4) = \text{sck}'(J_4_1) = \text{sck}'(J_4_2) = \alpha_4$ of the rules $(\exists^I\text{-red}) J_4_1$ and of $J_4_2$. We obtain $\gamma, \omega^\omega \in \mathcal{H}_\gamma(\psi_1\gamma)$ and $\psi_1(\gamma\#\omega^\omega) \ll \psi_1(\gamma\#\omega^\omega)$ from (5) and (6). In particular the conditions (2), (3) and (4) are enjoyed for rules $K'$.

Consider (2) for $K'$. Let $\mu$ be the type of $J_3$. Then we have $\mu \in \mathcal{H}_\gamma(\psi_1\gamma)$ by Definition 1.13.34 for $K$. On the other hand we have $\eta < \psi_1\alpha'_3 < \psi_1(\alpha_4\#\omega^\omega) \leq \mu$ by (3) for $J_3$. Hence $\eta < \mu \in \mathcal{H}_\gamma(\psi_1\gamma) \cap I = \psi_1\gamma$ and $\eta \in \mathcal{H}_\gamma(\psi_1\gamma)$. Moreover we have $\psi_1(\alpha'_3\#\omega^\omega) \in \mathcal{H}_\gamma(\psi_1\gamma)$ by $\psi_1(\alpha'_3\#\omega^\omega) < \psi_1(\alpha_3\#\omega^\omega)$, (4), Proposition 2.3.6 and $\psi_1(\alpha_3\#\omega^\omega) \in \mathcal{H}_\gamma(\psi_1\gamma)$. Thus (2) is fulfilled for $K'$.

For $i = 1, 2$, we obtain $\psi_1(\alpha'_i\#\omega^\omega) \ll \psi_1(\alpha_4\#\omega^\omega)$. Hence $o(\Phi; P') = \omega_m(\psi_1(\alpha_4\#\omega^\omega)\#\psi_1(\alpha_4\#\omega^\omega)) \ll o(\Phi; P)$ for an $m < \omega$.

Finally let $S$ be a $(\text{sub})_n$ occurring below $\Phi$ in $P$, and $S'$ be the corresponding rule in $P'$. Let $\gamma = \text{sck}'(K') = \text{sck}(K)$, and $\sigma = \Omega_{\sigma+1}$.

We have $\psi_1(\alpha_3\#\omega^\omega) \in \mathcal{H}_\gamma(\psi_1\gamma)$ by Definition 1.13.34 for $K$. Proposition 2.3.6 yields $\psi_1(\alpha'_i\#\omega^\omega) \in \mathcal{H}_\gamma(\psi_1\gamma)$ for $i = 1, 2$, and $\eta \in \mathcal{H}_\gamma(\psi_1\gamma)$ by (5) and $\eta < \psi_1(\alpha'_i\#\omega^\omega) < \psi_1(\alpha_3\#\omega^\omega)$.

Thus the conditions (2) and (3) for $K'$ are seen from $o(K'; P') \ll o(K; P)$ using Proposition 2.3.6 as above.

**Case 10.** $J_1$ is a $(w)$ and $J_2$ is a $(BI)$: Let $P$ be the following.

$$
P = \frac{\Delta_0, \forall X \neg F, \neg F(U^*) \beta}{\Delta_0, \forall X \neg F; \beta + 1} (w) J_1 \frac{F(A), \exists X F, \Gamma_0; \alpha}{\exists X F, \Gamma_0; \alpha_1} (BI) J_2 \frac{\Delta, \forall X \neg F}{\exists X F, \Gamma} \frac{\Delta}{\Pi; \delta \text{ (sub)} J_3}
$$

where $Gr(\exists X F) = 0$, $s = st_H(\forall X \neg F) = st_E(\exists X F)$, and let $\sigma = \Omega_{\sigma+1}$. Then $o(J_2; P) = \sigma \# \alpha$ and $\alpha_1 = \omega_m(\sigma \# \alpha)$ for $m$ such that $h(F(A), \exists X F, \Gamma_0) = \Delta_0, \forall X \neg F, \neg F(U^*) \beta$.
max\{h(∃XF,∆₀), dg(F(A))\} = h(∃XF,∆₀) + m. We see s < I from Proposition 4.1.4. Also \(\Pi\) denotes the upper sequent of the uppermost \((\text{sub}) J₃\) of level \(\leq s\) below \(J\).

Note that no \((\text{sub})\) changes the descendants of \(∀X ∼ F\) nor of ∃XF by the condition in Definition 4.7.6(d)ᵢ of the rule \((\text{sub})\).

From Proposition 4.1.4 we see that \(s = st_{Π}(¬F(U^s))\), and \(st_{Π}(∃XF) = s + 1\).

Hence from the Definition 4.7.6(d)ᵢ of the rule \((\text{sub})\), we see that the level \(ν\) of any \((\text{sub})''\) occurring between ∃XF; \(Γ₀\) and \(Π\) is larger than \(s\), \(ν > s\). In particular no eigenvariable of a \((\text{sub})\) occurring between \(Δ, Γ\) and \(Π\) occurs in \(¬F\).

Let \(P'\) be the following.

\[
\Delta₀, ∀X ∼ F, ¬F(U^s); β
\]

\[
\Delta, ¬F(U^s), ∀X ∼ F \quad ∃XF, Γ
\]

\[
Δ, Γ, ¬F(U^s)
\]

\[
Π, ¬F(U^s); δ'
\]

\[
Π, ¬F(A); γo(γ#ω^δ') (\text{sub}) F(A), ∃XF, Γ₀; α (\text{cut}) J'_2
\]

\[
ΞXF, Γ₀; α'₁
\]

\[
J'_3
\]

where \(h(Π, ¬F(U^s); P') = h(Π; P) = 0\) for the upper sequent \(Π, ¬F(U^s)\) of the new \((\text{sub})^s\), \(h(F(A), ∃XF, Γ₀; P') = h(F(A), ∃XF, Γ₀; P)\). The rules occurring above \(Π, ¬F(U^s)\) in \(P'\) receives the same stack of the corresponding rule in \(P\). From \(β = o(Δ₀, ∀X ∼ F, ¬F(U^s); P') = o(Δ₀, ∀X ∼ F, ¬F(U^s); P) \leq o(Δ₀, ∀X ∼ F; P) = β + 1\), we see for \(δ' = o(Π, ¬F(U^s); P')\) that

\[
δ' \ll δ
\]

The stack \(γ\) of the new \((\text{sub})^s\) is defined to be the stack \(γ = sck(J₃)\) of the \((\text{sub}) J₃\) in \(P\). Then \(\{γ, δ'\} \in \mathcal{H}_γ(ψ_γ)\), \(o(J'_2; P') = ψ_γ(γ#ω^δ')#α\), and for \(α'₁ = ω_m(ψ_γ(γ#ω^δ')#α)\)

\[
α'₁ \ll α₁ \{ψ_γ(γ#ω^δ')\}
\]

Rules occurring between \(J'_2\) and \(J'_3\) in \(P'\) receive the same stacks of the corresponding rules in \(P\). Then the condition (8) is enjoyed for these \((\text{sub})'\)’s
by [5]. Note that the level \( \nu \) of any \((\text{sub})^\nu\) between \( J_2' \) and \( J_3' \) is higher than \( s \), \( \nu > s \). Then \( \psi_\sigma(\gamma\#\omega^{\delta'}) < \sigma \leq \Omega_\nu \) and for the stack \( \lambda \) of such a rule \((\text{sub})^\nu\),

\[
\psi_\sigma(\gamma\#\omega^{\delta'}) \in \mathcal{H}_\lambda(\psi_{\Omega_{\nu+1}} \lambda)
\]

(9)

Since no eigenvariable of a \((\text{sub})\) between \( \Delta, \Gamma \) and \( \Pi \) occurs in \( \neg F \), such \((\text{sub})\) does not change the descendants of \( \neg F(\Phi^*) \).

The stack of the \((\text{sub})J_2' \) is increased by \( \omega^{\delta'} \#1 \), i.e., \( \gamma' = \text{sc}k'(J_2') = \gamma\#\omega^{\delta'} \#1 \). Let \( \tau = \Omega_{\nu+1} \) with the level \( \mu \) of \( J_3 \). Then we have \( \{ \gamma, \delta \} \subset \mathcal{H}_\tau(\psi_\tau \gamma) \), and \( \delta' \in \mathcal{H}_\tau(\psi_\tau \gamma) \) by (7). Hence \( \gamma' \in \mathcal{H}_\tau(\psi_\tau \gamma) \). Next from [5], [9] and Proposition 2.3.3 we obtain

\[
\delta'' \ll \delta \{ \psi_\tau(\gamma\#\omega^{\delta'}) \}
\]

(10)

On the other hand we have \( \sigma \ll \alpha \), and hence \( \sigma \ll \delta \) by Proposition 2.3.4. Hence

\[
\psi_\sigma(\gamma\#\omega^{\delta'}) \in \mathcal{H}_\gamma(\psi_\tau \gamma)
\]

(11)

and \( \delta'' \in \mathcal{H}_\gamma(\psi_\tau \gamma) \) by (10). Therefore we obtain \( \{ \gamma', \delta'' \} \subset \mathcal{H}_\gamma(\psi_\tau \gamma) \). Thus the condition (5) is enjoyed by \( J_3' \).

The condition (2) for \( J_3' \) is enjoyed by (11) since no essentially new index occurs above \( J_3' \).

Let us show \( o(\Phi; P') = \psi_\tau(\gamma'\#\omega^{\delta''}) \ll \psi_\tau(\gamma\#\omega^{\delta'}) = o(\Phi; P) \). We have \( \psi_\tau(\gamma'\#\omega^{\delta''}) \ll \psi_\tau(\gamma\#\omega^{\delta'}) \{ \psi_\tau(\gamma\#\omega^{\delta'}) \} \) by (10), Proposition 2.3.3 and (11). Moreover we see from (7) that for any \( \alpha > \gamma\#\omega^{\delta'} \) and any \( \rho \), if \( \{ \gamma, \delta, \sigma \} \subset \mathcal{H}_\alpha(\psi_\rho \alpha) \), then \( \psi_\tau(\gamma\#\omega^{\delta'}) \in \mathcal{H}_\alpha(\psi_\rho \alpha) \). \( \psi_\tau(\gamma'\#\omega^{\delta''}) \ll \psi_\tau(\gamma\#\omega^{\delta'}) \) is seen from Proposition 2.3.4.

The stacks of \((\text{sub})'s\) below \( J_3' \) remain the same. (2) and (3) are fulfilled for these \((\text{sub})'s\) by \( \psi_\tau(\gamma'\#\omega^{\delta''}) \ll \psi_\tau(\gamma\#\omega^{\delta'}) \).

**Case 11.** \( J_1 \) is an \((s2) \) and \( J_2 \) is a \((c) \) with a main formula \( \exists X^I F \): Let \( P \) be the following.

\[
P = \begin{array}{c}
\Delta_0, \forall X^I \neg F; \neg F(U^{U^I}); \delta \\
\Delta_0, \forall X^I \neg F; \delta + 1 \\
\vdots \\
\Delta, \forall X^I \neg F \\
\Delta, \Gamma \\
V, \beta \\
\Phi; \beta \\
\Phi; \gamma \\
\Phi; \alpha \\
P_1 \\
\vdots \\
F(V^*), \exists X^I F, \Gamma_0; \xi + 1
\end{array}
\]

\[
\begin{array}{c}
(s2) J_1 \\
\exists X^I F, \Gamma_0; \xi + 1 \\
\hat{J}_0 \\
\exists X^I F, \Gamma
\end{array}
\]

where \( \beta = o(\Phi) \), \( \alpha = o(J_0) \), \( \gamma = o(J) \), and \( \delta = o(\Delta_0, \forall X^I \neg F; \neg F(U^{U^I})) \), and the lower sequent \( \Phi \) of the rule \( J_0 \) denotes the uppermost sequent below \( \Delta, \forall X^I \neg F \) such that \( h(\Phi) < h(\Delta, \forall X^I \neg F) \).

By Definition 4.3 \( Gr(\forall X^I \neg F) > 1 \), and hence \( h(\Delta, \forall X^I \neg F; P) > \omega \). Since the height of upper sequents of \((Q^I\text{-red})\) is defined to be \( \omega \), we see that there is no \((Q^I\text{-red})\) between \( J \) and \( J_0 \) in \( P \).
Note that no \((\text{sub})\) changes the descendants of \(\forall X^I \neg F\) nor of \(\exists X^I F\) by the condition in Definition \[6(d)\].

From \(\exists X^I F \not\in \Pi^1_2\), we see that there is no \((\text{sub})\) between \(J_2\) and \(J_0\) in \(P\) since the height of the upper sequents of any \((\text{sub})\) is defined to be 0.

Let \(P'\) be the following.

\[
\begin{array}{c}
\Delta_0, \forall X^I \neg F, \neg F(V^s); \delta' \\
\vdots \\
\hdots \\
\Delta, \neg F(V^s), \forall X^I \neg F; \exists X^I F, \Gamma \\
\vdots \\
\Phi, \neg F(V^s); J_{01}; \alpha_1 \\
\vdots \\
\Phi; \beta' \\
\end{array}
\]

where \(P'_1 = (P'_1[\text{sub}])[V/U]\), i.e., in \(P_1\), replace first the occurrences of the variable \(U^I\) in an index by \(s\), and then replace the occurrences of the variable \(U\) as a part of formula by the variable \(V\), cf. Definition \[4.5\].

Let \(\beta' = o(\Phi; P'_{1}), \beta_1 = o(\neg F(V^s); P'_{1}), \beta_2 = o(\Phi; P'_{1}), \alpha_1 = o(J_{01}; P'_{1}), \alpha_2 = o(J_{02}; P'_{1}).\) Also \(\delta' = o(\Delta_0, \forall X^I \neg F; \neg F(V^s); P'_{1})\) and \(\gamma' = o(J_{1}; P'_{1}).\)

For \(P'_{1}\) to be a proof, we need to verify the condition on rules \((\text{sub})\) in Definition \[4.7(6(d))]\) the condition \[\theta\] on rules \((\exists^f\text{-red})\), and the conditions \[\eta\] and \[\zeta\] on rules \((\text{sub})\), \((\exists^f\text{-red})\).

First consider the condition on rules \((\text{sub})\) in Definition \[4.7(6(d))]\). Since there is no \((\text{sub})\) between \(\exists X^I F, \Gamma_0\) and \(\Phi\) in \(P\), it suffices to examine \((\text{sub})\) occurring between \(\Delta_0, \forall X^I \neg F\) and \(\Delta, \forall X^I \neg F\) with the added formula \(\neg F(V^s)\) in \(P'\). From the same condition for the \((\text{sub})\) in \(P\) we see that \(\forall X^I \neg F \in \Pi^1_2\), and hence \(s_{\Pi}(\neg F(V^s)) \leq s_{\Pi}(\forall X^I \neg F)\) by \(s < I\).

Next consider the conditions on rules \((\text{sub})\), \((\exists^f\text{-red})\) in \(P'_{1}\). Let \(K\) be a rule in \(P\), which is either a \((\text{sub})\) or an \((\exists^f\text{-red})\). Assume that \(K\) occurs either in \(P_1\) or between \(J_1\) and \(J_0\). We saw that \(K\) is not between \(J\) and \(J_0\). From \(Gr(\forall X^I \neg F) > 1\) we see that \(\forall X^I \neg F\) is not in an upper sequent of a \((\text{sub})\), which is in an end-piece. Hence \(K\) is not a \((\text{sub})\). Also from \(h(\Delta, \forall X^I \neg F) > \omega\) and \(h(\Delta_0, \forall X^I \neg F, \neg F(U^I)) > \omega\), we see that \(K\) is not an \((\exists^f\text{-red})\). Therefore there is no such rule \(K\).

Let \(\mathcal{S}_{P_1}\) be the set of all indices \(s_1\) such that either a free variable \(W^{s_1}\) or a bound variable \(Y^{s_1}\) occurs in a main formula of a \((BI)\) in \(P_{1}\). Let \(s' = s \# \bigcup\{I(s_1) : s_1 \in \mathcal{S}_{P_1}\}\).

We have \(\delta' \ll \delta \{s'\}\) and \(\gamma' \ll \gamma \{s'\}\). Hence \(\alpha_1 \ll \alpha \{s'\}\) and \(\alpha_2 \ll \alpha\). Let \(\Pi\) denote an upper sequent of \(J_0\), and let \(h = h(\Pi; P)\). Then \(h = h(\Phi) + m\) for an \(m < \omega\). From \(d_{\Pi}(F(V^s)) < d_{\Pi}(\exists X^I F) \leq h\) and \(h(F(V^s), \Phi; P') < h\), we see \(\beta' \ll \beta \{s'\}\).

Let \(K\) be the uppermost \((\exists^f\text{-red})\) below \(J_0\) in \(P\), and \(K'\) the corresponding rule in \(K'\) with their stacks \(\gamma = sck(K) = sck'(K')\). Consider the conditions
\(3\) and \(4\) on \(K\). We have \(\alpha_{K'} = o(K'; P') \ll o(K; P) = \alpha_K \{s'\}\), and \(s' \in \mathcal{H}_{\gamma}(\psi_1\gamma)\) since the indices \(s, s_1\) occur above \(K\). From \(\{\alpha_K, \gamma\} \in \mathcal{H}_s(\psi_1\gamma)\) we obtain \(\alpha_{K'} \in \mathcal{H}_s(\psi_1\gamma),\) and \(\psi_1(\gamma\#\omega^{\alpha_{K'}}) < \psi_1(\gamma\#\omega^{\alpha_{K}})\). Similarly we see that rules \((3^2\text{-red})\) below \(J_{01}\) enjoy the conditions \(2\), \(3\) and \(4\).

Next assume that \(K\) is a \((\text{sub})^\mu\) occurring below \(J_0\), and consider the conditions \(2\) and \(3\) on \(K\). Let \(\alpha_K = o(K; P)\) and \(\alpha_{K'} = o(K'; P')\). We need to show that \(\alpha_{K'} \in \mathcal{H}_s(\psi_\sigma\gamma)\), where \(\gamma = \text{sek}(K) = \text{sek}(K')\) and \(\sigma = \Omega_{\mu+1}\). We have \(\alpha_{K'} \ll \alpha_K \{s'\}\), and \(s' \in \mathcal{H}_s(\psi_\sigma\gamma)\) by \(2\) for \(K\). Then \(\alpha_{K'} \in \mathcal{H}_s(\psi_\sigma\gamma)\).

**Case 12.** \(J_1\) is an \((s2)\) and \(J_2\) is a \((c)\) with a main formula \(\exists^\eta F\) for an \(\eta < I\): Let \(P\) be the following.

\[
P = \frac{\frac{\Delta_0, \forall^\eta \neg F, \neg F(U^\eta); \delta}{\Delta_0, \forall^\eta \neg F; \delta + 1}}{\Delta, \forall^\eta \neg F; \Phi; \beta} \mid J_{01}; \alpha^{(s2)} J_1\]

\[
P = \frac{\frac{F(V^s); \exists^\eta F, \Gamma_0; \xi}{\exists^\eta F, \Gamma_0; \xi + 1}}{\Delta, \forall^\eta \neg F; \Phi; \beta} \mid J_{02}; \alpha^{(c)} J_2\]

where \(\beta = o(\Phi), \alpha = o(J_0), \gamma = o(J),\) and \(\delta = o(\Delta_0, \forall^\eta \neg F, \neg F(U^\eta))\), and the lower sequent \(\Phi\) of the rule \(J_0\) denotes the uppermost sequent below \(\Delta, \forall^\eta \neg F\) such that \(h(\Phi) < h(\Delta, \forall^\eta \neg F)\).

Note that no \((\text{sub})\) changes the descendants of \(\forall^\eta \neg F\) nor of \(\exists^\eta F\) by the condition in Definition \(1.10\) (b(d)ii).

From \(\exists^\eta F \notin \Pi_1^1\), we see that there is no \((\text{sub})\) between \(J_2\) and \(J_0\) in \(P\) since the height of the upper sequents of any \((\text{sub})\) is defined to be 0.

By Definition \(4.17\) and Proposition \(4.14.2\) we obtain \(s < \eta\). Then Proposition \(4.6.3\) with a limit \(\eta\) yields \(\text{st}_\Pi(\neg F(V^s)) \leq \text{st}_\Pi(\forall^\eta \neg F)\) when \(\forall^\eta \neg F \in \Pi_1^1\).

Let \(P'\) be the following.

\[
P' = \frac{\frac{\Delta_0, \forall^\eta \neg F, \neg F(V^s); \delta'}{\Delta, \neg F(V^s); \Phi; \beta_1} \mid J_{01}; \alpha_{1}^{(s^2)} P_1}{F(V^s); \exists^\eta F, \Gamma_0; \xi} \mid J_{02}; \alpha_{2}^{(c)} J_2}\]

\[
\frac{\Delta, \neg F(V^s), \forall^\eta \neg F; \Phi, \beta_2}{\Delta, \forall^\eta \neg F; \Phi; \beta} \mid J_{02}; \alpha_{2}^{(c)} J_2
\]

where \(P'_1 = (P'_1[s/U^\eta])[V/U]\), i.e., in \(P_1\), replace first the occurrences of the variable \(U^\eta\) in an index by \(s\), and then replace the occurrences of the variable \(U\) as a part of formula by the variable \(V\), cf. Definition \(4.5\).
Let $\beta' = o(\Phi; P')$, $\beta_1 = o(\Phi, \neg F(V^\ast); P')$, $\beta_2 = o(F(V^\ast), \Phi; P')$, $\alpha_1 = o(J_0; P')$, $\alpha_2 = o(J_0; P')$. Also $\delta' = o(\Delta_0, \forall X\eta \neg F; P')$ and $\gamma' = o(J'; P')$.

For $P'$ to be a proof, we need to verify the condition on rules $(\text{sub})$ in Definition 4.7.6(d)3, the condition (4) on rules $(\exists'$-red), and the conditions (2) and (3) on rules $(\text{sub}),(\exists'$-red).

We see that the condition on rules $(\text{sub})$ in Definition 4.7.6(d)3 is fulfilled in $P'$ as in Case 11.

Next consider the conditions on rules $(\text{sub}),(\exists'$-red) in $P'$. Let $K$ be a rule in $P$, which is either a $(\text{sub})$ or an $(\exists'$-red). Assume that $K$ occurs either in $P_1$ or between $J_1$ and $J_0$. Let $K'$ be the corresponding rule occurring in the left part of $J'$ in $P'$. If the eigenvariable $U_{\eta}$ does not occur above $K$, then the new index $s$ does not occur above $K'$ except it occurs already above $K$, and the ordinal remains the same. In this case there is nothing to prove. Assume that $U_{\eta}$ occurs above $K$.

Let $\sigma = I$ when $K$ is an $(\exists'$-red), and $\sigma = \Omega_{\mu+1}$ when $K$ is a $(\text{sub})\mu$. Then $\eta < \sigma$ is seen from $\eta < I$ when $K$ is an $(\exists'$-red), which is in $P_1$. Also $\eta < \sigma$ is seen from $\eta \leq st_{\Omega}(\forall X\eta \neg F) \leq \mu < \sigma$ when $K$ is a $(\text{sub})\mu$, which is between $J_1$ and $J_0$, and the formula $\forall X\eta \neg F$ is in the upper sequent of $K$, cf. Definition 4.7.6(d)3.

Let $\gamma = sck(K) = sck'(K')$ be the stack of the rule $K$ in $P$, and of the rule $K'$ in $P'$. Let $S_{P_1}$ be the set of all indices $s_1$ such that either a free variable $W^{s_1}$ or a bound variable $Y^{s_1}$ occurs in a main formula of a $(BI)$ in $P_1$. Let $s' = s \# \bigcup (I(s_1) : s_1 \in S_{P_1})$. Since the variable $U_{\eta}$, i.e., the index $\eta$ as well as indices $s_1$ in $P_1$ occurs above $K$, we have $\{\eta\} \cup \bigcup (I(s_1) : s_1 \in S_{P_1}) \subseteq H_{\gamma}(\psi_{\sigma \gamma})$ by Definition 4.15.3a. Then $s < \eta \in H_{\gamma}(\psi_{\sigma \gamma}) \cap \sigma = \psi_{\sigma \gamma}$, and

$$s' \in H_{\gamma}(\psi_{\sigma \gamma})$$  \hspace{1cm} (12)

Thus (2) is fulfilled for $K$.

Next let $\alpha_K = o(K; P)$ and $\alpha_{K'} = o(K'; P')$. We have $\alpha_{K'} \ll \alpha_K \{s'\}$. By (3) we have $\alpha_K \in H_{\gamma}(\psi_{\sigma \gamma})$. Hence $\alpha_{K'} \in H_{\gamma}(\psi_{\sigma \gamma})$ by (12). Thus (3) is fulfilled for $K'$.

Finally let us show $o(P') < o(P)$. We have $\beta' \ll \beta \{s'\}$. Consider a $(\text{sub})\mu K$ occurring between $J_1$ and $J$. Then $s < \eta \leq st_{\Omega}(\forall X\eta \neg F) \leq \mu < \Omega_{\mu+1} = \sigma$, and $\eta \in H_{\gamma}(\psi_{\sigma \gamma}) \cap \sigma$ with the stack $\gamma = sck(K)$. Hence $s' \in H_{\gamma}(\psi_{\sigma \gamma})$, and this yields $\gamma' \ll \gamma \{s'\}$. We see that $\beta' \ll \beta \{s'\}$ as in Case 11.

Case 13. The case when the suitable cut formula is a disjunction $A \lor B$.

Case 14. The case when the suitable cut formula is an existential formula $\exists x A[x/u]$.

These cases are seen as in Case 11.

This completes a proof of Main Lemma 4.17.
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