BISPECTRAL OPERATORS OF RANK 1 AND DUAL ISOMONODROMIC DEFORMATIONS†

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Abstract. A comparison is made between bispectral operator pairs and dual pairs of isomonodromic deformation equations. Through examples, it is shown how operators belonging to rank one bispectral algebras may be viewed equivalently as defining 1–parameter families of rational first order differential operators with matricial coefficients on the Riemann sphere, whose monodromy is trivial. By interchanging the rôles of the two variables entering in the bispectral pair, a second 1-parameter family of operators with trivial monodromy is obtained, which may be viewed as the dual isomonodromic deformation system.

1. Bispectral Operators.

1.1 Bispectral Pairs.

Consider pairs of differential operators $L(x, \partial_x), \Lambda(z, \partial_z)$ in two variables $x$ and $z$, for which there exists a function $\psi(x, z)$ that is simultaneously a parametric family of eigenfunctions of $L$ and $\Lambda$.

\[
L\psi(x, z) = f(z)\psi(x, z) \tag{1.1a}
\]

\[
\Lambda\psi(x, z) = \phi(x)\psi(x, z), \tag{1.1b}
\]

where the eigenvalues $f(z)$ and $\phi(x)$ are nonconstant functions of the variables $z$ and $x$. Such pairs of operators were named bispectral and studied by Duistermaat and Grünbaum in [DG]. Choosing a normalization in which both $L$ and $\Lambda$ have unit leading coefficients and constant next to leading coefficients (which is always possible, up

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to reparametrization), they were able to draw a number of conclusions about these operators. In particular, the coefficients of $L$ and $\Lambda$ are rational in the variables $x$ and $z$, respectively, and the functions $f(x)$ and $\phi(z)$ are polynomials. Furthermore, for the case of second order operators:

$$L = \frac{d^2}{dx^2} + u(x)$$

[DG] were able to determine all possible $u(x)$; namely (up to translations in $x$ or addition of a constant),

$$u(x) = x \ (\text{Airy}) \quad \text{or} \quad u(x) = \frac{c}{x^2} \ (\text{Bessel})$$

or anything that can be obtained from the two cases

$$u(x) = 0, \quad u(x) = -\frac{1}{4x^2}$$

through the application of rational Darboux transformations.

1.2 Bispectral Algebras of Rank 1.

Wilson [W1, W2] considered the case when $L$ is embedded in a commutative algebra $A$ of differential operators, sharing the same family of eigenfunctions $\psi(x, z)$, such that the orders of the elements of $A$ are relatively prime (a rank 1 algebra). Such algebras can be characterized by the associated spectral data, consisting of an algebraic curve, the spectral curve of $A$, denoted $\text{spec}(A)$ and, in general, a line bundle whose fibres are the joint eigenvectors. This data may be viewed as determining a point $W \in \text{Gr}$ in the Hilbert space Grassmannian of Sato [SS] and Segal–Wilson [SW], such that the bispectral wave function $\psi(x, z)$ is the corresponding Baker–Akhiezer function $\psi_W(t, z)$ at the point $t := (t_1, t_2, \ldots)$ with $(t_1 = x, t_i = 0, i \geq 2)$. We recall that the Baker-Akhiezer function is the unique function of the form

$$\psi_W(t, z) = \gamma(t) \left( 1 + \sum_{i=1}^{\infty} \frac{b_i(t)}{z^i} \right),$$

where $\gamma(t) := e^{\sum_{j=1}^{\infty} t_j z^j}$, taking values in the subspace $W$ of the Hilbert space $H := L^2(S^1, \mathbb{C})$ of square integrable functions on the unit circle in the complex $z$–plane. Here $t = (t_1, t_2, \ldots)$ is an infinite component vector and $\gamma(t)$ may be viewed as an element of the infinite abelian group $\Gamma_+$ consisting of elements of $L^2(S^1, \mathbb{C})$ admitting a holomorphic continuation to the interior of $S^1$ in the complex $z$–plane and taking
value 1 at the origin. The Baker–Akhiezer function may, in turn, be expressed in terms of the Sato tau function associated to \( W \), by the formula [DJKM]

\[
\psi_W(t, z) = \gamma(t) \frac{\tau_W(t - [z])}{\tau_W(t)}, \tag{1.11}
\]

where the components of \([z] = (z_1, z_2, \ldots)\) are

\[
z_k := \frac{1}{k z_k}. \tag{1.12}
\]

Geometrically, \( \tau_W \) is understood as the determinant, defined up to normalization, of the projection \( P_+ : \gamma W \to \mathcal{H}_+ \), where \( \gamma W \) is the image under \( \gamma \in \Gamma_+ \) of the subspace \( W \subset \mathcal{H} \) and \( \mathcal{H}_+ \subset \mathcal{H} \) is the subspace consisting of elements admitting a holomorphic extension to the exterior of \( S^1 \).

Wilson [W1] gave the following equivalent characterizations of such \( W \)'s corresponding to bispectral algebras of rank 1.

**Theorem (Wilson [W1]).** The algebra \( \mathcal{A} \) is bispectral and of rank 1 if and only if the following equivalent conditions hold:

(i) \( \text{spec}(\mathcal{A}) \) is a rational algebraic curve with only cusp singularities.

(ii) Within the proper normalization, the Baker–Akhiezer function

\[
\psi(x, z) = \psi_W(x = t_1, t_j = 0, j \geq 2, z) = e^{xz} \left( 1 + \sum_{i=1}^{\infty} b_i(x) \frac{z^i}{i!} \right) \tag{1.13}
\]

belongs to a plane \( W \) belonging to a subvariety \( \text{Gr}^{\text{ad}} \) of the rational Segal–Wilson Grassmannian \( \text{Gr}_{\text{rat}} \), called the adelic Grassmannian (defined below).

The adelic Grassmannian \( \text{Gr}^{\text{ad}} \) is shown in [W1] to consist of the \( W \)-planes in the Hilbert space \( \mathcal{H} \) of the form

\[
W = \frac{1}{q(z)} W^\varepsilon_+ \tag{1.14}
\]

where \( q(z) \) is a polynomial of suitable degree, rendering the projection \( P_+ : W \to \mathcal{H}_+ \) to the subspace \( \mathcal{H}_+ \subset \mathcal{H} \) a Fredholm operator of index 0, and the subspace \( W^\varepsilon_+ \subset \mathcal{H}_+ \) is the intersection of the kernels of a finite number of linear forms \( \{ c_i^\mu \in \mathcal{H}_+^\varepsilon, \} \), each of which has support at one point \( \lambda_i \) in the complex \( z \)-plane. That is, each \( c_i^\mu \) determines a condition of the form

\[
c_i^\mu(g) = \sum_{a=1}^{m_i^\mu} c^\mu_{ia} g^{(a)}(\lambda_i) = 0, \tag{1.15}
\]
where \( g^{(a)} \) denotes the \( a \)-th derivative of \( g \in \mathcal{H}_+ \) and \( \{ c^a_{ia} \} \) is a finite set of coefficients. The \( \lambda_i \)'s are just the roots of the polynomial \( q(z) \), with multiplicities coinciding with the number of conditions localized at that root.

Wilson also showed that the property of bispectrality could be viewed as a consequence of the existence of a “bispectral involution”

\[
\begin{align*}
   b : \text{Gr}^{\text{ad}} & \rightarrow \text{Gr}^{\text{ad}} \\
   b : W & \mapsto W'
\end{align*}
\]

such that

\[
\psi_{W'}(x, z) = \psi_W(z, x)
\]

The larger, rational Grassmannian \( \text{Gr}^{\text{rat}} \) contains all rational and soliton solutions to the KP hierarchy. It was shown by Krichever [Kr], following earlier work of Airault, Mckean and Moser [AMM] on rational solutions to the KdV equation, that rational solutions of the KP equation which tend to zero as \( x \to \infty \) can be expressed as

\[
u(x, y, t) = -\sum_{i=1}^{n} \frac{2}{(x - x_i(y, t))^2},
\]

where the location of the poles \( x_i(y, t) \) is determined by the fact that they satisfy the equations of the first two commuting flows of the Calogero-Moser system [M]. It follows that these are associated to bispectral tau functions \( \tau_W \). In [W2], Wilson showed that, by suitably completing the complexified Calogero-Moser phase space so as to allow for particle collisions, this extends to a complete parametrization of the adelic Grassmannian \( \text{Gr}^{\text{ad}} \), and hence the set of rank 1 bispectral algebras. More precisely, let \( C_n \) denote the completed \( n \)-particle rational Calogero–Moser phase space, defined as the space of pairs \( (X, Z) \) of complex \( n \times n \) matrices whose commutator \( [Z, X] \) is a rank 1 perturbation of the identity matrix \( I \), quotiented by the natural action of the general linear group \( \mathfrak{gl}(n, \mathbb{C}) \) on such pairs;

\[
g : (X, Z) \mapsto (gXg^{-1}, gZg^{-1}), \quad g \in \mathfrak{gl}(n, \mathbb{C}).
\]

Define a map from the union of the spaces \( C_n \) to the adelic Grassmannian

\[
\begin{align*}
   W : \cup_n C_n & \rightarrow \text{Gr}^{\text{ad}} \\
   W : [(X, Z)] & \mapsto W(X, Z),
\end{align*}
\]
where \( W(X, Z) \in \text{Gr}^{\text{ad}} \) is determined by the following formula for the corresponding Baker function at \( t = (x, 0, \ldots) \).

\[
\psi_{W(X,Z)}(x, z) = e^{xz} \det \left( I - (X + xI)^{-1}(Z + zI)^{-1} \right).
\] (1.21)

(Note that \( \psi_W(x, z) \) and its derivatives at \( x = 0 \) span \( W \).) Equivalently, the corresponding tau function may be expressed

\[
\tau_{W(X,Z)} = \det(X + xI + \sum_{j=2}^{\infty} jt_j(-Z)^{j-1}).
\] (1.22)

Wilson showed in [W2] that this identification gives an isomorphism of affine, nonsingular, irreducible algebraic varieties. It is clear from (1.16), (1.21) that under this identification, the bispectral involution is equivalent to the symplectic involution \( \tilde{b} : \mathbb{C}^n \rightarrow \mathbb{C}^n \)

\[
\tilde{b} : (X, Z) \mapsto (Z^t, X^t)
\] (1.23)
on the completed Caloger–Moser phase space. (This was noted for the case of the original real Calogero–Moser phase by Kasman [K], with \( \tilde{b} \) viewed as Moser’s linearizing involution [M].)

This therefore provides a complete determination of all wave functions \( \psi_W \) corresponding to rank 1 bispectral algebras. The following are two of the simplest examples of these. (In each case, the algebra \( \mathcal{A} \) is identified with the space of polynomials which, under multiplication, leave \( W(X, Z) \) invariant. Only the simplest representative elements \( L \) and \( \Lambda \), are given.)

**Example (i)**

\[
\begin{align*}
n &= 1, \quad X = (\alpha), \quad Z = (0) \\
\psi_{W(X,Z)}(x, z) &= e^{xz} \left( 1 - \frac{1}{(x+\alpha)z} \right) \\
L &= \frac{d^2}{dx^2} - \frac{2}{(x+\alpha)^2}, \quad f(z) = z^2 \\
\Lambda &= \frac{d^2}{dz^2} + 2\alpha \frac{d}{dz} - \frac{2}{z^2}, \quad \phi(x) = x^2 + 2\alpha x
\end{align*}
\] (1.24a-d)
Example (ii)

\[ n = 2, \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \]  

(1.25a)

\[ \psi_{W(X,Z)}(x,z) = e^{xz} \left( 1 - \frac{2}{xz} + \frac{2}{x^2 z^2} \right) \]  

(1.25b)

\[ L = \frac{d^3}{dx^3} - 6 \frac{d}{x^2} + 12 \frac{1}{x^2}, \quad f(z) = z^3 \]  

(1.25c)

\[ \Lambda = \frac{d^3}{dz^3} - 6 \frac{d}{z^2} + 12 \frac{1}{z^2}, \quad \phi(x) = x^3 \]  

(1.25d)

2. Isomonodromic Deformations.

2.1 Rational Covariant Derivatives Operators.

Consider a parametric family of matrix first order differential operators

\[ D_\lambda(\tau) := \frac{\partial}{\partial \lambda} - B(\lambda, \tau) - \sum_{i=1}^{n} \sum_{a=1}^{m_i} \frac{N_{ia}(\tau)}{(\lambda - \alpha_i(\tau))^a}, \]  

(2.1)

where \( B(\lambda, \tau) \) is an \( r \times r \) matrix valued polynomial in \( \lambda \) that may depend also on a deformation parameter \( \tau \), \( \{N_{ia}\}_{1 \leq i \leq n, 1 \leq a \leq m_i} \) is a set of \( r \times r \) matrices, also dependent on \( \tau \), and \( \{\alpha_i(\tau) \in \mathbb{C}\}_{1 \leq i \leq n} \) are the location of the finite poles, also possibly \( \tau \) dependent. This may be viewed as a \( \tau \)-parametric family of rational covariant derivative operators defined on a rank \( r \) vector bundle over the Riemann sphere, punctured at the poles (including possibly \( \infty \)). The generalized monodromy data associated to the operator \( D_\lambda \) consists of the local monodromy matrices about each of the poles, together with the Stokes matrices and connection matrices [JMU]. Assuming local differentiability in the deformation parameter \( \tau \), it is a basic fact that this monodromy data is invariant under variations in \( \tau \) provided there exists a second differential operator

\[ D_\tau := \frac{\partial}{\partial \tau} - E(\lambda, \tau), \]  

(2.2)

where \( E(\lambda, \tau) \) is also a \( \tau \)-parametric family of \( r \times r \) matrices, rational in \( \lambda \), such the commutativity condition

\[ [D_\tau, D_\lambda] = 0 \]  

(2.3)

is satisfied. This means that the data \( \{B(\lambda, \tau), \alpha_i(\tau), N_{ia}(\tau)\} \) determining \( D_\lambda(\tau) \) satisfy a set of first order ODE’s in the deformation parameter \( \tau \). It also implies that, locally
at least, an invertible matrix valued function $\Psi(\lambda,t)$ exists, uniquely determined by any given set of initial values, simultaneously satisfying

$$D_\lambda \Psi = 0 \quad (2.4a)$$
$$D_\tau \Psi = 0. \quad (2.4b)$$

The simplest case, studied since the beginning of this century, is when the operator $D_\lambda$ is Fuchsian; i.e., it has only regular singular points. This means that all the finite poles are simple ($m_i = 1, \forall i$) and $B(\lambda, \tau) \equiv 0$, so $D_\lambda$ is of the form

$$D_\lambda = \frac{\partial}{\partial \lambda} - \sum_{i=1}^{n} \frac{N_i}{\lambda - \alpha_i}. \quad (2.5)$$

For this case, it has long been known that the only way the residue matrices $\{N_i\}_{i=1,...,n}$ can vary is through their dependence on the pole locations $\{\alpha_i\}_{i=1,...,n}$, and they are constrained to satisfy an overdetermined system of PDE’s. Choosing the $\alpha_i$’s as the deformation parameters, the associated infinitesimal isomonodromic deformation operators are

$$D_{\alpha_i} := \frac{\partial}{\partial \alpha_i} + \frac{N_i}{\lambda - \alpha_i}. \quad (2.6)$$

The corresponding commutativity conditions

$$[D_\lambda, D_{\alpha_i}] = 0, \quad [D_{\alpha_i}, D_{\alpha_j}] = 0, \quad 1 \leq i, j \leq n \quad (2.7)$$

are mutually compatible (i.e. integrable in the Frobenius sense), and equivalent to the following system of equations, known as the Schlesinger equations

$$\frac{\partial N_i}{\partial \alpha_j} = \frac{[N_i, N_j]}{\alpha_i - \alpha_j}, \quad i \neq j \quad (2.8a)$$
$$\frac{\partial N_i}{\partial \alpha_i} = -\sum_{j=1}^{n} \frac{[N_i, N_j]}{\alpha_i - \alpha_j}. \quad (2.8b)$$

(In particular, the simplest Fuchsian case involving nontrivial isomonodromic deformations is when $n = 3$ and $r = 2$, for which these reduce to the equations of the sixth Painlevé transcendent $P_{VI}$.)

The corresponding determination of all equations of the type (2.3) generating isomonodromic deformations of operators of the general form (2.1) was made in [JMU].
For the simplest nonfuchsian case, where the finite poles remain first order, but an irregular singularity of Poincaré index 1 is added at infinity by allowing $B$ to be a nonzero diagonal matrix $B(\tau) = \text{diag}(\beta_1(\tau), \ldots \beta_r(\tau))$ with distinct eigenvalues that are independent of $\lambda$, we have

$$D_\lambda(\tau) = \frac{\partial}{\partial \lambda} - N(\lambda, \tau)$$

(2.9)

where

$$N(\lambda, \tau) := B(\tau) + \sum_{i=1}^{n} \frac{N_i(\tau)}{\lambda - \alpha_i}.$$  

(2.10)

In this case, as shown in [JMMS], besides the original $\alpha_i$’s, the possible deformations parameters include the eigenvalues $\{\beta_a\}_{1 \leq a \leq r}$ of the matrix $B$. The corresponding infinitesimal isomonodromic deformation operators are

$$D_{\alpha_i} := \frac{\partial}{\partial \alpha_i} + \frac{N_i}{\lambda - \alpha_i}, \quad i = 1, \ldots, n$$

(2.11a)

$$D_{\beta_a} := \frac{\partial}{\partial \beta_a} - \lambda E_a - \sum_{b=1, b \neq a}^{r} \frac{E_a N_\infty E_b + E_b N_\infty E_a}{\beta_a - \beta_b}, \quad a = 1, \ldots, r,$$

(2.11b)

where $E_a$ is the elementary matrix with 1 in the $aa$ position and 0’s elsewhere, and

$$N_\infty := \sum_{i=1}^{n} N_i.$$  

(2.12)

Again the mutual commutativity of the operators $\{D_{\alpha_i}, D_{\beta_a}, D_\lambda\}_{i=1,\ldots,n,a=1,\ldots,r}$,

$$[D_{\alpha_i}, D_{\alpha_j}] = [D_{\alpha_i}, D_{\beta_a}] = [D_{\beta_a}, D_{\beta_b}] = [D_\lambda, D_{\alpha_i}] = [D_\lambda, D_{\alpha_i}] = 0,$$

(2.13)

gives a Frobenius integrable system for the residue matrices $N_i$, locally determining their dependence on the parameters $\{\alpha_i, \beta_a\}$.

2.2 $R$-Matrix theory and Duality.

The above isomonodromic deformation equations may be viewed as nonautonomous Hamiltonian equations ([JM], [H]) on the space $(\mathfrak{gl}(r,\mathbb{C}))^n$ of $n$-tuples of $r \times r$ matrices $\{N_1, \ldots N_n\}$, with respect to the Lie Poisson bracket structure

$$\{(N_i)_{ab}, (N_j)_{cd}\} = \delta_{ij} [\delta_{ab} N_{cb} - \delta_{bc} N_{da}].$$

(2.14)
Equivalently, the rational matrix valued functions \( N(\lambda) \), for different values of the complex parameters \( \lambda \) and \( \mu \), satisfy

\[
\{N(\lambda) \otimes N(\mu)\} = [r(\lambda - \mu), N(\lambda) \otimes I + I \otimes N(\mu)],
\]

where \( r(\lambda - \mu) \) is the rational \( R \)-matrix, viewed as an element of \( \text{End}(C^n \otimes C^n) \), defined by

\[
r(\lambda) := \frac{P_{12}}{\lambda},
\]

where

\[
P_{12}(u \otimes v) = v \otimes u
\]

is the transposition endomorphism. This defines a Poisson bracket structure on the space \( \mathfrak{gl}_{\text{rat}}(r, C) \) of rational \( r \times r \) matrix valued functions of \( \lambda \). The \( \alpha_i \)'s and \( \beta_a \)'s may be viewed as multi–time parameters associated to the \( n + r \) nonautonomous Hamiltonians \( \{H_i, K_a\}_{i=1,..,n, a=1,..,r} \) defined by

\[
H_i := \frac{1}{4\pi i} \oint_{\lambda=\alpha_i} \text{tr} \left( N^2(\lambda) \right) d\lambda = \text{tr}(BN_i) + \sum_{j=1}^{n} \frac{\text{tr}(N_i N_j)}{\alpha_i - \alpha_j},
\]

\[
K_a := \frac{1}{4\pi i} \oint_{\lambda=\infty} \oint_{z=\beta_a} d\lambda d\beta \left[ \text{tr} \left( (B - zI)^{-1} N(\lambda) \right)^2 - z \text{tr} \left( (B - zI)^{-1} N(\lambda) \right) \right] = \sum_{j=1}^{n} \alpha_j (N_j)_{aa} + \sum_{b \neq a}^{r} \frac{(N_{\infty})_{ab} (N_{\infty})_{ba}}{\beta_a - \beta_b}.
\]

Defining a 1–form \( \theta \) on the parameter space by

\[
\theta := \sum_{i=1}^{n} H_i d\alpha_i + \sum_{a=1}^{r} K_a d\beta_a,
\]

the following results, which may be verified directly [JMU], are a consequence of the general \( R \)-matrix theory, adapted to the nonautonomous setting [H].

**Theorem.** The isomonodromic deformation equations (2.13) are Hamilton’s equations for the set of Hamiltonians \( \{H_i, K_a\} \) contained in the 1–form \( \theta \)

\[
dN_i := \sum_{j=1}^{n} \frac{\partial N_i}{\partial \alpha_j} d\alpha_j + \sum_{a=1}^{r} d\beta_a \frac{\partial N_i}{\partial \beta_a} = \{N_i, \theta\}.
\]
Furthermore, the Hamiltonians \( \{ H_i, K_a \} \) all Poisson commute amongst themselves, and hence \( \theta \) is a closed form on the parameter space

\[
d\theta = 0, \quad (2.21)
\]

and there exists, at least locally, a function \( \tau \) (the tau function) such that

\[
\theta = d(\ln \tau). \quad (2.22)
\]

The Hamiltonian equations determining these isomonodromic deformations may in fact be lifted to an associated symplectic vector space, and then projected to another space of rational differential operators, giving rise to a second, dual representation of these equations as isomonodromic deformations [H]. It is easy to see that all \( \mathcal{N}(\lambda) \)'s of the form (2.10) may be expressed as

\[
\mathcal{N}(\lambda) = B + G^T (A - \lambda \mathbf{I})^{-1} F, \quad (2.23)
\]

where \( A \) is a diagonal \( N \times N \) matrix,

\[
A := \text{diag} (\alpha_1, \ldots, \alpha_n), \quad (2.24)
\]

with \( N = \sum_{i=1}^{n} k_i \), \( k_i = \text{rk}(N_i) \), whose eigenvalues \( \{ \alpha_i \} \) are at the poles of \( \mathcal{N}(\lambda) \), and have multiplicity \( k_i \), and \((F,G)\) are a pair of rectangular \( N \times r \) matrices. If the space \( \mathcal{M}^{N \times r} := \{(F,G)\} \) of such pairs is given the canonical symplectic structure

\[
\omega = \text{tr}(dF^T \wedge dG), \quad (2.25)
\]

the map \( J^A_B : \mathcal{M}^{N \times r} \to \mathfrak{gl}_r(\mathbb{C}) \) to the space \( \mathfrak{gl}_r(\mathbb{C}) \) defined, for fixed \( A \) and \( B \), by

\[
J^A_B : (F,G) \mapsto \mathcal{N}(\lambda), \quad (2.26)
\]

is a Poisson map. In fact, it may be viewed as defining a Poisson quotient of \( \mathcal{M}^{N \times r} \) by the canonical Hamiltonian action of the stability subgroup \( \mathfrak{g} \mathfrak{g}_A \subset \mathfrak{gl}(N, \mathbb{C}) \) of \( A \), given by

\[
\mathfrak{g} \mathfrak{g}_A \times \mathcal{M}^{N \times r} \longrightarrow \mathcal{M}^{N \times r}
\]

\[
(g, (F,G)) \mapsto (gF, (g^T)^{-1}G). \quad (2.27)
\]

(The \( k_i \times r \) blocks \((F_i,G_i)\) within the matrices \( F \) and \( G \) corresponding to the eigenvalues \( \alpha_i \) determine the local monodromy of the operator \( \mathcal{D}_\lambda \) about these points.)
Denoting the pull-back of the 1–form $\theta$ under $J_B^A$ by $\tilde{\theta}$, the isomonodromic deformation equations (2.13) may be viewed as the projection to $\mathfrak{gl}_{rat}(r, \mathbb{C})$ of the following nonautonomous Hamiltonian equations on the auxiliary space $\mathcal{M}^{N \times r}$:

$$dF = \{F, \tilde{\theta}\}, \quad dG = \{G, \tilde{\theta}\},$$

where

$$d = \sum_{j=1}^{n} d\alpha_j \frac{\partial}{\partial \alpha_j} + \sum_{a=1}^{r} d\beta_a \frac{\partial}{\partial \beta_a}.$$  \hspace{1cm} (2.29)

To obtain the dual set of isomonodromic deformation equations, we define a new Hamiltonian quotient of $\mathcal{M}^{N \times r}$, this time by the stabilizer $\mathfrak{g}l_B \subset \mathfrak{g}l(r, \mathbb{C})$ of the element $B \in \mathfrak{g}l(r)$, by the dual Poisson map $J_B^A : (F, G) \rightarrow \mathcal{M}(z)$, where

$$\mathcal{M}(z) := A + F(B - zI_r)^{-1}G^T = A + \sum_{a=1}^{r} \frac{M_a}{z - \beta_a}.$$ \hspace{1cm} (2.30)

Defining the associated set of differential operators

$$\tilde{D}_z := \frac{\partial}{\partial z} - \mathcal{M}(z)$$

$$\tilde{D}_{\alpha_i} := \frac{\partial}{\partial \alpha_i} - zE_i - \sum_{j=1}^{n} \frac{E_iM_\infty E_j + E_jM_\infty E_i}{\alpha_i - \alpha_j}$$

$$\tilde{D}_{\beta_a} := \frac{\partial}{\partial \beta_a} + \frac{M_a}{z - \beta_a},$$

Hamilton’s equations (2.28) are projectible under the map $J_B^A$ and imply the commutativity conditions

$$[\tilde{D}_{\alpha_i}, \tilde{D}_{\alpha_j}] = [\tilde{D}_{\alpha_i}, \tilde{D}_{\beta_a}] = [\tilde{D}_{\beta_a}, \tilde{D}_{\beta_b}] = [\tilde{D}_z, \tilde{D}_{\alpha_i}] = [\tilde{D}_z, \tilde{D}_{\beta_a}] = 0,$$ \hspace{1cm} (2.32)

which, in turn, imply the invariance of the monodromy data of the operator $\tilde{D}_z$ (resp. $\tilde{D}_z$) under changes in the parameter $x$ (resp. $z$). These equations may also be interpreted as Hamiltonian equations on the space of rational $N \times N$ matrix valued functions of $\mathcal{M}(z)$, generated by the set of Poisson commuting, nonautonomous Hamiltonians obtained by projection:

$$\tilde{K}_a := \frac{1}{4\pi i} \oint_{z=\beta_a} \text{tr} \left( \mathcal{M}^2(z) \right) dz = \text{tr}(AB_a) + \sum_{b=1}^{r} \frac{\text{tr}(M_aM_b)}{\beta_a - \beta_b}$$ \hspace{1cm} (2.33a)

$$\tilde{H}_i := \frac{1}{4\pi i} \oint_{z=\infty} dz \oint_{\lambda = \alpha_i} d\lambda \left[ \text{tr} \left( (A - \lambda I_N)^{-1}\mathcal{M}(z) \right) \right]^2 - \lambda \text{tr} \left( (A - \lambda I_N)^{-1}\mathcal{M}(z) \right)$$

$$= \sum_{b=1}^{r} \beta_b(M_b)_{ii} + \sum_{j=1}^{n} \frac{(M_\infty)_{ij}(M_\infty)_{ji}}{\alpha_i - \alpha_j}.$$ \hspace{1cm} (2.33b)
The duality map
\[(F, G, A, B, \lambda, z) \rightarrow (G^T, F^T, B, A, z, \lambda)\] (2.34)
may be viewed, after projection, as transforming the system of isomonodromic deformations equations (2.13) into the dual system (2.32).

This duality map is reminiscent of the bispectral involution (1.23) expressed in terms of the matrices \((X, Z)\) determining the Calogero–Moser system. A full explanation of the relationship between these two maps is not yet developed, but one certainly exists. What will be provided in the next section will be a reformulation of the two examples of bispectral pairs given in section 1 in terms of an equivalent, dual pair of isomonodromic deformation equations - in the very particular case where the monodromy happens to be trivial. These examples in fact illustrate a correspondence that can be made between the bispectral pairs of the type studied by Wilson; i.e., those belonging to rank 1 bispectral algebras, and dual pairs of parametric families of matrix differential operators of first order, depending rationally on both parameters, and having trivial monodromy.

3. Relation between Bispectrality and Dual Isomonodromy.

The general structure of bispectral equations and infinitesimal isomonodromic deformation equations, both involving a wave function depending on two variables that simultaneously satisfies a linear differential equation in each of them, suggests that the two may be related. The similarity between the duality map (2.34) and the bisectral involution (1.23) further suggests a relation between bispectrality and duality. The deeper relation between bispectral pairs and dual isomonodromic deformations involves the theory of dressing transformations and the tau function, and will be not developed here. However, to demonstrate that such a relation does exist, we consider the two examples of bispectral rank 1 operators introduced in section 1.2, and show how an equivalent infinitesimal isomonodromic deformation system can be derived for each.

3.1 Examples.

(i) For the case of the Baker type bispectral wave function
\[\psi(x, z) = e^{xz} \left(1 - \frac{1}{(x + \alpha)z}\right)\] (3.1)
there exists a second bispectral wave function for the same pair of operators \(L\) and \(\Lambda\) given in (1.24c,d), namely
\[\psi_1 := \psi(-x - 2\alpha, z) = e^{-2\alpha z} \psi(x, -z)\] (3.2)
Forming the Wronskian matrix

\[ \Psi := \begin{pmatrix} \psi & \psi_1 \\ \psi_x & \psi_{1,x} \end{pmatrix}, \tag{3.3} \]

we may associate the following pair of rational 2 \times 2 covariant derivative operators

\[ \mathcal{D}_x := \frac{\partial}{\partial x} - \begin{pmatrix} z^2 + \frac{2}{(\alpha+x)^2} & 1 \\ \frac{\alpha+x}{\alpha} & 0 \end{pmatrix}, \tag{3.4a} \]
\[ \mathcal{D}_z := \frac{\partial}{\partial z} - \begin{pmatrix} -\alpha & \frac{\alpha+x}{z} \\ (\alpha + x)z + \frac{2}{(\alpha+x)z} & \frac{1}{z} - \alpha \end{pmatrix}, \tag{3.4b} \]

which simultaneously annihilate \( \Psi \),

\[ \mathcal{D}_x \Psi = 0, \quad \mathcal{D}_z \Psi = 0, \tag{3.5} \]

This is equivalent to the bispectral conditions (1.1a,b) for the operators \( L, \Lambda \) defined in (1.24c,d) and implies the commutativity condition

\[ [\mathcal{D}_x, \mathcal{D}_z] = 0. \tag{3.6} \]

It follows that the monodromy of either of these operators, say \( \mathcal{D}_z \), is invariant under the deformations determined by varying the other parameter, \( x \). Note however the fact that there is not just one single-valued bispectral wave function, but a basis of single valued bispectral wave functions, which implies in this case that the operators \( \mathcal{D}_x \) and \( \mathcal{D}_z \) actually have trivial monodromy for all parameter values.

To obtain the dual system of (trivial) isomonodromic deformation equations, we just define new operators similarly, using the Wronskian matrix formed by differentiating with respect to the variable \( z \):

\[ \tilde{\Psi} := \begin{pmatrix} \psi & \psi_1 \\ \psi_z & \psi_{1,z} \end{pmatrix}. \tag{3.7} \]

This is simultaneously annihilated by the operators

\[ \tilde{\mathcal{D}}_x := \frac{\partial}{\partial x} - \begin{pmatrix} \frac{\alpha z}{\alpha+x} & \frac{z}{\alpha+x} \\ \frac{(\alpha^2+2\alpha x)z^2}{\alpha+x} + \frac{2+\alpha z}{(\alpha+x)z} & \frac{1-\alpha z}{\alpha+x} \end{pmatrix}, \tag{3.8a} \]
\[ \tilde{\mathcal{D}}_z := \frac{\partial}{\partial z} - \begin{pmatrix} 0 & 1 \\ \frac{2}{z^2} + x^2 + 2\alpha x & -2\alpha \end{pmatrix}, \tag{3.8b} \]
implying the commutativity condition

\[ [\tilde{D}_x, \tilde{D}_z] = 0, \quad (3.9) \]

from which again follows that the monodromy of either of these operators is invariant under the deformations determined by varying the other parameter.

Note however that the singularity structure of the operators \( D_x, D_z, \tilde{D}_x \) and \( \tilde{D}_z \) is such that the Hamiltonian \( \mathcal{R} \)-matrix formulation discussed in section 2.2 does not suffice. In order to include these, the \( \mathcal{R} \)-matrix approach must be extended (cf. [HTW], [HR], [HW1], [HW2]) to allow for singularities of higher order at \( \infty \).

(ii) For the case of the bispectral wave function

\[ \psi_{W(x,z)}(x,z) = e^{xz} \left( 1 - \frac{2}{xz} + \frac{2}{x^2z^2} \right), \quad (3.10) \]

because of the invariance of the eigenvector equations for the operators \( L \) and \( \Lambda \) in (1.25c,d) under the cyclic group generated by

\[ z \mapsto \omega z, \quad \omega := e^{\frac{2\pi i}{3}}, \quad (3.11) \]

we may define a basis \( \{\psi_0, \psi_1, \psi_2\} \) of bispectral wave functions by

\[ \psi_j(x, z) := \psi(x, \omega^j z), \quad j = 0, 1, 2. \quad (3.12) \]

The Wronskian matrix

\[ \Psi := \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_{0,x} & \psi_{1,x} & \psi_{2,x} \\ \psi_{0,xx} & \psi_{1,xx} & \psi_{2,xx} \end{pmatrix} \quad (3.13) \]

is then simultaneously annihilated by the operators

\[ D_x := \frac{\partial}{\partial x} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^3 - \frac{12}{x^3} & \frac{6}{x^2} & 0 \end{pmatrix}, \quad (3.14a) \]
\[ D_z := \frac{\partial}{\partial z} - \begin{pmatrix} 0 & \frac{1}{x} & 0 \\ 0 & \frac{1}{x^2} & \frac{1}{xz} \\ xz^2 - \frac{12}{x^3} & \frac{6}{x^2} & \frac{5}{z} \end{pmatrix}. \quad (3.14b) \]

This is equivalent to the eigenvector equations for the operators \( L \) and \( \Lambda \) of eqs. (1.25c,d), and implies the commutativity of the operators \( D_x, D_z \). Hence the monodromy of each of these operators (which again is trivial because of the single valuedness
of the matrix $\Psi$), remains invariant under changes in the parameter values. Because
of the symmetry of the bispectral wave functions under the interchange $x \leftrightarrow z$, the
preliminary “dual” isomonodromic system is obtained by simply interchanging the
two variables in the definitions of the operators $D_x$, $D_z$. As in the previous example,
the proper treatment of these systems within the $R$–matrix framework requires the
inclusion of higher order singularities at $\infty$.

Proceeding similarly, we may associate to every rank 1 bispectral pair two mutually
dual 1–parameter families of differential operators having trivial monodromy for
all parameter values, since in each case we may form a basis of single–valued bispectral
wave functions. This procedure may also be applied to the case of higher rank bispec-
tral operators, except that the monodromy of the resulting dual families of operators
is no longer necessarily trivial, and we obtain instead 1-parameter families of operators
with constant monodromy under variation of the deformation parameters. These more
general cases, as well as their relations to the evaluation of certain limits of Fredholm
determinants, will be dealt with elsewhere.

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