Automorphisms of the type $II_1$ Arveson system of Warren’s noise

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Abstract

Motions of the plane (shifts and rotations) correspond to automorphisms of the type $I$ Arveson system of white noise. I prove that automorphisms corresponding to rotations cannot be extended to the type $II$ Arveson system of Warren’s noise.

Introduction

...we lack information about the gauge groups of non-type I examples.
W. Arveson [1, Sect.2.10]

This is a noise richer than white noise: in addition to the increments of a Brownian motion $B$ it carries a countable collection of independent Bernoulli random variables which are attached to the local minima of $B$.
J. Warren [5, the end]

The simplest example of a type $II_1$ Arveson system emerges from the simplest example of a nonclassical noise, Warren’s noise of splitting (see [5] and [2], sections 2c, 2e, 4d, 6g). The classical part of the noise, the white noise, corresponds to the classical part of the Arveson system, a type $I_1$ system. The group of automorphisms of the type $I_1$ system, described by Arveson, is basically the group of motions of the plane (shifts and rotations). It is easy to extend to the nonclassical system the action of the shifts. It is also easy to see that the rotation by $\pi$ cannot be extended. However, what happens to other rotations (say, by $2\pi/3$)? It is shown here that only the trivial (by 0) rotation can be extended.

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1 Automorphisms of the type $I_1$ Arveson system of the white noise

This section summarizes Arveson’s theory of automorphisms of type $I$ systems, specialized to the white noise, that is, the standard Brownian motion in $\mathbb{R}$. The latter is the Gaussian process $(B_t)_{t \in [0, \infty)}$ with $E B_t = 0$ and $E (B_s B_t) = s$ for $0 \leq s \leq t < \infty$. Let $\Omega_t$ be the probability space of the Brownian paths on the time interval $[0, t]$. Then

$$\Omega_{s+t} = \Omega_s \times \Omega_t$$

up to the natural isomorphism of probability spaces. Thus,

$$L_2(\Omega_{s+t}) = L_2(\Omega_s) \otimes L_2(\Omega_t)$$

up to the natural isomorphism of Hilbert spaces (just a unitary operator). It means that these $L_2(\Omega_t)$ form an Arveson system (as defined in [1, Sect. 3.1, Def. 3.1.1]).

The space $L_2(\Omega_t)$ may be thought of as the exponential of $L_2(0,t)$ (see [1, Sect. 2.1], especially (2.7)). To this end we define a map $\text{Exp}: L_2(0,t) \to L_2(\Omega_t)$ by

$$\text{Exp}(f) = \exp \left( \int_0^t f(s) \, dB_s - \frac{1}{2} \int_0^t f^2(s) \, ds \right)$$

(here $\exp x$ means the usual $e^x$, not to be confused with $\text{Exp}$), using the well-known unitary operator $f \mapsto \int f \, dB$ from $L_2(0,t)$ onto a subspace of $L_2(\Omega_t)$ (in fact, the subspace of all measurable linear functionals of $B$). Linear combinations of these $\text{Exp}(f)$ are dense in $L_2(\Omega_t)$, and

$$\langle \text{Exp}(f), \text{Exp}(g) \rangle = \exp(\langle f, g \rangle),$$

since $E \left( \exp(\int f \, dB - \frac{1}{2} \int f^2 \, ds) \exp(\int g \, dB - \frac{1}{2} \int g^2 \, ds) \right) = \exp(-\frac{1}{2} \int (f^2 + \overline{g}^2) \, ds) \exp(\int (f + \overline{g}) \, dB)$, and $E \exp(\int (f + \overline{g}) \, dB) = \exp(\frac{1}{2} \int (f + \overline{g})^2 \, ds) \exp(\int f \, dB)$. It means that the Arveson system is (isomorphic to) the exponential product system of rank 1 [1, Sect. 3.1, Def. 3.1.6], therefore it is of type $I_1$ (in other words, of type $I$ and index 1).

The units of the Arveson system (see [1, Sect. 3.6], especially (3.21)) are of the form $u^{(a, \zeta)}$ for $a, \zeta \in \mathbb{C}$; here

$$u^{(a, \zeta)}(t) = e^{at} \exp(\zeta \cdot \chi_{(0,t)}) = \exp(\zeta B_t - \frac{1}{2} \zeta^2 t + at).$$

Isomorphisms between Arveson systems are defined in [1, Sect. 3.1, Def. 3.1.3]; isomorphisms to itself are called automorphisms. An automorphism $\theta$ consists of unitary operators $\theta_t : L_2(\Omega_t) \to L_2(\Omega_t)$ such that
\( \theta_{s+t} = \theta_s \otimes \theta_t \). The automorphisms of the type \( I_1 \) system (see \([1, \text{Sect. 3.8}]\), especially (3.37) and the proof of Th. 3.8.4) are of the form \( \theta^{(\lambda, \xi,U)} \) for \( \lambda \in \mathbb{R} \), \( \xi \in \mathbb{C} \), \( U \in \mathbb{C} \), \( |U| = 1 \); they act on the units as follows:

\[
\theta^{(\lambda, \xi,U)} u^{(a,\zeta)}(t) = u^{(a',\zeta')} \left( a' = a + i\lambda - \frac{1}{2}|\xi|^2 - U\bar{\xi} \right.
\]

See \([1, \text{Sect. 3.8}]\) for the composition formula (for two automorphisms) in terms of \( \lambda, \xi, U \).

Parameters \( a \) and \( \lambda \) are trivial in the sense that

\[
u^{(a,\zeta)}(t) = e^{at} u^{(0,\zeta)}(t), \quad \theta^{(\lambda, \xi,U)} = e^{i\lambda t} \theta^{(\xi,U)}
\]

where \( u^{(\zeta)} = u^{(0,\zeta)} \) and \( \theta^{(\xi,U)} = \theta^{(0,\xi,U)} \). Accordingly,

\[
\theta^{(\xi,U)} u^{(\zeta)}(t) = \exp\left( -\frac{1}{2}|\xi|^2 t - U\bar{\xi}t \right) u^{(U\bar{\xi}+\zeta)}(t).
\]

Denoting for convenience

\[
\theta^{\text{shift}}(\xi) = \theta^{(\xi,1)}, \quad \theta^{\text{rotat}}(U) = \theta^{(0,\xi,U)}
\]

we have

\[
\theta^{\text{rotat}}(U) \theta^{\text{rotat}}(V) = \theta^{\text{rotat}}(UV), \quad \theta^{\text{rotat}}(U) u^{(\zeta)} = u^{(U\zeta)}(t),
\]

\[
\theta^{\text{shift}}(\xi) u^{(\zeta)}(t) = \exp\left( -\frac{1}{2}|\xi|^2 t - U\bar{\xi}t \right) u^{(U\bar{\xi}+\zeta)}(t).
\]

For \( \lambda, \mu \in \mathbb{R} \),

\[
\theta^{\text{shift}}(i\lambda) \theta^{\text{shift}}(i\mu) = \theta^{\text{shift}}(i(\lambda+\mu)), \quad \theta^{\text{rotat}}(\lambda) \theta^{\text{shift}}(\mu) = \theta^{\text{shift}}(\lambda+\mu),
\]

\[
\theta^{\text{shift}}(\lambda) u^{(\zeta)}(t) = \exp\left( -\frac{1}{2}\lambda^2 t + i\lambda \zeta t \right) u^{(\zeta+i\lambda)}(t),\]

\[
\theta^{\text{shift}}(\lambda) u^{(\zeta)}(t) = \exp\left( -\frac{1}{2}\lambda^2 t - \lambda \zeta t \right) u^{(\zeta+\lambda)}(t),
\]

which leads to canonical commutation relations (CCR) \([1\text{ Remark 3.8.2}]\)

\[
\theta^{\text{shift}}(i\lambda) \theta^{\text{shift}}(i\mu) = e^{2i\lambda\mu t} \theta^{\text{shift}}(i\lambda + i\mu).
\]

Combining \([1,1] \) and \([1,7] \) we get \( \theta^{\text{shift}}(i\lambda) \exp(\zeta B_t) = \exp(i\lambda B_t) \exp(\zeta B_t) \),

thus identifying the automorphism \( \theta^{\text{shift}}(i\lambda) \) with the automorphism formed by multiplication operators,

\[
\theta^{\text{shift}}(i\lambda) f = \exp(i\lambda B_t) f \quad \text{for } f \in L_2(\Omega_t).
\]
2 An inequality related to CCR

Let \( P, Q \) be selfadjoint operators on a separable Hilbert space. The canonical commutation relations \([P, Q] = -i\) will be treated as an abbreviation of the Weyl relations

\[
\forall \lambda, \mu \in \mathbb{R} \quad e^{i\lambda P} e^{i\mu Q} = e^{i\lambda \mu} e^{i\mu Q} e^{i\lambda P}.
\]

If \([P, Q] = -i\) then \( P + Q \) is well-defined and \([Q, -(P + Q)] = -i\). Thus, we may speak about three operators \( P, Q, R \) such that \( P + Q + R = 0 \) and \([P, Q] = -i, [Q, R] = -i, [R, P] = -i\) (these three relations being in fact mutually equivalent).

2.1 Theorem. Let selfadjoint operators \( P, Q, R \) be such that \( P + Q + R = 0 \) and \([P, Q] = [Q, R] = [R, P] = -i\). Then

\[
\| \text{sgn} P + \text{sgn} Q + \text{sgn} R \| < 3 .
\]

(Here ‘sgn \( P \)’ is the discontinuous sign function applied to the operator \( P \).) The proof is given in [4, Th. 2.1] for the irreducible representation of CCR (unique up to unitary equivalence). The general case follows easily, since every representation decomposes into irreducible representations (von Neumann’s theorem).

Note that \( \| \text{sgn} P + \text{sgn} Q + \text{sgn} R \| \) is an absolute constant (since all irreducible triples \( P, Q, R \) are mutually unitarily equivalent). According to a numerical computation [4, Sect. 1], the constant is approximately 2.1.

Returning to the context of Sect. 1 we introduce the generator \( Q_t \) of the unitary group \( \left( \theta_{t}^\text{shift(i\lambda)} \right)_{\lambda \in \mathbb{R}} \),

\[
e^{i\lambda Q_t} = \theta_t^\text{shift(i\lambda)} ;
\]

by (1.10), it is the multiplication by \( B_t : \Omega_t \to \mathbb{R} \),

\[
Q_t f = B_t f \quad \text{for } f \in L_2(\Omega_t) \text{ such that } B_t f \in L_2(\Omega_t).
\]

By (1.4), the operator \( \theta_t^\text{rotat(U\lambda)} Q_t (\theta_t^\text{rotat(U\lambda)})^{-1} \) is the generator of the unitary group \( \left( \theta_t^\text{shift(i\mu\lambda)} \right)_{\lambda \in \mathbb{R}} \). Especially, the operator

\[
P_t = \theta_t^\text{rotat(i)} Q_t \theta_t^\text{rotat(-i)}
\]

satisfies

\[
e^{-i\mu P_t} = \theta_t^\text{shift(\mu)} ,
\]

and we may rewrite (1.9) as

\[
[P_t, Q_t] = -2ti .
\]
More generally, for $\alpha \in \mathbb{R}$
\begin{equation}
\theta_t^{\text{rotat}(e^{i\alpha})} Q_t \theta_t^{\text{rotat}(e^{-i\alpha})} = Q_t \cos \alpha + P_t \sin \alpha.
\end{equation}

2.4 Lemma. There exists $\varepsilon > 0$ such that for every $t \in (0, \infty)$, $\alpha \in (\pi/2, \pi]$ and $f \in L_2(\Omega_t)$,
\[\langle \text{sgn} Q_t \rangle_f + \langle \text{sgn} Q_t \rangle_g + \langle \text{sgn} Q_t \rangle_h \leq (3 - \varepsilon) \|f\|^2;\]
here $g = \theta_t^{\text{rotat}(e^{i\alpha})} f$, $h = \theta_t^{\text{rotat}(e^{-i\alpha})} f$, and $\langle A \rangle_f$ stands for $\langle Af, f \rangle$.

Proof. Using the general relations
\[\langle A \rangle_g = \langle \theta_t^{\text{rotat}(e^{i\alpha})} A \theta_t^{\text{rotat}(e^{i\alpha})} \rangle_f\]
and
\[\theta_t^{\text{rotat}(e^{-i\alpha})} (\text{sgn} A) \theta_t^{\text{rotat}(e^{i\alpha})} = \text{sgn} (\theta_t^{\text{rotat}(e^{i\alpha})} A \theta_t^{\text{rotat}(e^{i\alpha})})\]
we get by (2.3)
\[\langle \text{sgn} Q_t \rangle_g = \langle \text{sgn}(Q_t \cos \alpha - P_t \sin \alpha) \rangle_f,\]
\[\langle \text{sgn} Q_t \rangle_h = \langle \text{sgn}(Q_t \cos \alpha + P_t \sin \alpha) \rangle_f.\]

If $\alpha = \pi$ then $\langle \text{sgn} Q_t \rangle_f + \langle \text{sgn} Q_t \rangle_g = 0$, thus, the inequality holds (for $\varepsilon = 2$). Otherwise, Theorem 2.1 may be applied to the operators $P = aQ_t$, $Q = b(Q_t \cos \alpha - P_t \sin \alpha)$, $R = c(Q_t \cos \alpha + P_t \sin \alpha)$ provided that $a, b, c \in (0, \infty)$ are chosen appropriately (namely, $a = \sqrt{-\cos \alpha}/(t \sin \alpha)$ and $b = c = 1/\sqrt{-4t \sin \alpha \cos \alpha}$). We get
\[\langle \text{sgn} Q_t \rangle_f + \langle \text{sgn}(Q_t \cos \alpha + P_t \sin \alpha) \rangle_f + \langle \text{sgn}(Q_t \cos \alpha - P_t \sin \alpha) \rangle_f \leq \| \text{sgn} Q_t + \text{sgn}(Q_t \cos \alpha + P_t \sin \alpha) + \text{sgn}(Q_t \cos \alpha - P_t \sin \alpha) \| \cdot \|f\|^2 \leq (3 - \varepsilon) \|f\|^2,
\]
where $3 - \varepsilon$ is the absolute constant given by Theorem 2.1.

3 The first superchaos of the type $II_1$ system

Probability spaces denoted by $\Omega_t$ in sections 1, 2 will be denoted by $\Omega_t^{\text{white}}$ in sections 3, 4. Similarly, other objects relating to the white noise will be marked ‘white’, because we turn to Warren’s noise of splitting, richer than the white noise.

A path $\omega_1$ of the noise of splitting on the time interval $[0, 1]$ consists of a Brownian path $\omega_1^{\text{white}} \in \Omega_1^{\text{white}}$ and a map $\eta_1 : \text{LocMin}(\omega_1^{\text{white}}) \rightarrow \{-1, +1\}$;
here $\text{LocMin}(\omega^\text{white}_1)$ is the set of all local minimizers of the path $\omega^\text{white}_1 : [0, 1] \to \mathbb{R}$, and $\Omega^\text{white}_1 \subset C_0[0, 1]$ is a Borel set of full Wiener measure such that for every $\omega^\text{white}_1 \in \Omega^\text{white}_1$ the set $\text{LocMin}(\omega^\text{white}_1)$ is a dense countable subset of $(0, 1)$, and all the local minima are strict. We may choose a measurable enumeration of local minimizers, that is, a sequence of measurable maps $\tau_1, \tau_2, \cdots : \Omega^\text{white}_1 \to (0, 1)$ such that

$$\text{LocMin}(\omega^\text{white}_1) = \{\tau_1(\omega^\text{white}_1), \tau_2(\omega^\text{white}_1), \ldots\}$$

for every $\omega^\text{white}_1 \in \Omega^\text{white}_1$, and these $\tau_k(\omega^\text{white}_1)$ are pairwise different.

Every measurable enumeration $(\tau_k)_k$ of the local minimizers on $(0, 1)$ gives us a one-to-one correspondence

$$\Omega_1 \leftrightarrow \Omega^\text{white}_1 \times \{-1, +1\}^\infty,$$

$$\omega_1 = (\omega^\text{white}_1, \eta_1) \leftrightarrow (\omega^\text{white}_1, (\eta_1(\tau_1(\omega^\text{white}_1)), \eta_1(\tau_2(\omega^\text{white}_1)), \ldots)) ;$$

here $\Omega_1$ is the set of all paths of the noise of splitting on the time interval $[0, 1]$, and $\{-1, +1\}^\infty$ is the set of all infinite sequences of $\pm 1$. We equip $\{-1, +1\}^\infty$ with the product measure $m^\infty$, where $m$ gives to $-1$ and $+1$ equal probabilities $1/2, 1/2$. Further, we equip $\Omega^\text{white}_1 \times \{-1, +1\}^\infty$ with the Wiener measure multiplied by $m^\infty$. Finally, using the one-to-one correspondence, we transfer the probability measure (and the underlying $\sigma$-field) to $\Omega_1$, getting $P_1$. The choice of an enumeration $(\tau_k)_k$ does not matter, since $m^\infty$ is invariant under permutations.

Probability spaces $\Omega_t = (\Omega_t, P_t)$ for $t \in (0, \infty)$ are constructed similarly; they satisfy $\Omega_{s+t} = \Omega_s \times \Omega_t$.

The general form of a function $f \in L_2(\Omega^\text{white}_1 \times \{-1, +1\}^\infty)$ is

$$f(\omega^\text{white}_1, (\sigma_1, \sigma_2, \ldots)) = \sum_{n=0}^{\infty} \sum_{k_1 < \cdots < k_n} \sigma_{k_1} \cdots \sigma_{k_n} f_{k_1, \ldots, k_n}(\omega^\text{white}_1),$$

$$f_{k_1, \ldots, k_n} \in L_2(\Omega^\text{white}_1), \quad \sum_{n=0}^{\infty} \sum_{k_1 < \cdots < k_n} \|f_{k_1, \ldots, k_n}\|^2 = \|f\|^2 < \infty;$$

of course, $\sigma_k = \pm 1$. Therefore the general form of $f \in L_2(\Omega_1)$ is

$$f(\omega_1) = \sum_{n=0}^{\infty} \sum_{k_1 < \cdots < k_n} \eta_1(\tau_{k_1}(\omega^\text{white}_1)) \cdots \eta_1(\tau_{k_n}(\omega^\text{white}_1)) f_{k_1, \ldots, k_n}(\omega^\text{white}_1),$$

$$f_{k_1, \ldots, k_n} \in L_2(\Omega^\text{white}_1), \quad \sum_{n=0}^{\infty} \sum_{k_1 < \cdots < k_n} \|f_{k_1, \ldots, k_n}\|^2 = \|f\|^2 < \infty.$$
For \( n = 0 \) we get the natural embedding \( L_2(\Omega_1^{\text{white}}) \subset L_2(\Omega_1) \).

The Hilbert spaces \( H_t = L_2(\Omega_t) \) for \( t \in (0, \infty) \) are an Arveson system. Its automorphisms \( \theta \) consist of unitary operators \( \theta_t : H_t \to H_t \). The subspace \( H_t^{\text{white}} = L_2(\Omega_t^{\text{white}}) \) of \( H_t \) is invariant under \( \theta_t \) (for every automorphism \( \theta \)) since, first, the classical (in other words: type I; completely spatial; decomposable) part of an Arveson system is invariant under automorphisms, and second, the classical part of the system \( (H_t)_t \) is the system \( (H_t^{\text{white}})_t \) (see \[2\] Sections 4d, 6g).

The set \( \text{Finite}(0, 1) \) of all finite subsets of \((0, 1)\) is a Borel space. Every bounded Borel function \( \varphi : \text{Finite}(0, 1) \to \mathbb{R} \) leads to an operator \( \mathcal{E}_\varphi : H_1 \to H_1 \), given in terms of (3.1) by

\[
(\mathcal{E}_\varphi f)(\omega_1) = \sum_{n=0}^{\infty} \sum_{k_1 < \cdots < k_n} \eta_1(\tau_{k_1}(\omega_1^{\text{white}})) \cdots \eta_1(\tau_{k_n}(\omega_1^{\text{white}})) \\
\cdot f_{k_1, \ldots, k_n}(\omega_1^{\text{white}}) \varphi\left(\{\tau_{k_1}(\omega_1^{\text{white}}), \ldots, \tau_{k_n}(\omega_1^{\text{white}})\}\right).
\]

Thus, the commutative algebra of all bounded Borel functions on \( \text{Finite}(0, 1) \) acts on \( H_1 \). Its action commutes with automorphisms; this fact is a special case of a more general statement \[3\] Sect. 3, but I give a streamlined proof here.

3.2 Lemma. Operators \( \mathcal{E}_\varphi \) and \( \theta_1 \) commute, whenever \( \theta = (\theta_t)_{t \in (0, \infty)} \) is an automorphism of the Arveson system \( (H_t)_{t \in (0, \infty)} \) and \( \varphi : \text{Finite}(0, 1) \to \mathbb{R} \) is a bounded Borel function.

Proof. The orthogonal projection onto the subspace \( H_s \otimes H_{t-s}^{\text{white}} \otimes H_{1-t} \subset H_1 \) (for \( 0 < s < t < 1 \)) commutes with \( \theta_t = \theta_s \otimes \theta_{t-s} \otimes \theta_{1-t} \) and is of the form \( \mathcal{E}_{\varphi'} \); namely, \( \varphi'(C) = 1 \) if \( C \cap (s, t) = \emptyset \), otherwise \( \varphi'(C) = 0 \). Thus, the lemma holds for these special \( \varphi' \).

The Borel sets of the form \( \{C \in \text{Finite}(0, 1) : C \cap (s, t) = \emptyset\} \) generate the Borel \( \sigma \)-field of \( \text{Finite}(0, 1) \). Proof: restricting ourselves to rational \( s, t \) we get a countable collection of Borel sets separating points of \( \text{Finite}(0, 1) \), therefore, generating the Borel \( \sigma \)-field.

It means that the lemma holds for all \( g \) taking on the values 0, 1 only. The general case follows.

Thus, a subspace of \( H_1 \) corresponds to every Borel subset of \( \text{Finite}(0, 1) \). Especially, the classical part, \( H_1^{\text{white}} \), corresponds to \( \{C : |C| = 0\} = \{\emptyset\} \) (just \( n = 0 \) in (3.1)). The subspace corresponding to \( \{C : |C| = 1\} = \{\{t\} : 0 < t < 1\} \) (just \( n = 1 \) in (3.1)) is the so-called first superchaos space.
$H_1^{(1)} \subset H_1$;

\[ f \in H_1^{(1)} \iff f(\omega_1) = \sum_k \eta_k(\tau_k(\omega_1^{\text{white}})) f_k(\omega_1^{\text{white}}), \]

\[ f_k \in L_2(\Omega_1^{\text{white}}), \quad \sum_k \|f_k\|^2 = \|f\|^2 < \infty. \]

(Subspaces $H_i^{(1)} \subset H_i$ appear similarly.) Automorphisms leave $H_1^{(1)}$ invariant. The commutative algebra of bounded Borel functions $\chi : (0, 1) \to \mathbb{R}$ acts on $H_1^{(1)}$,

\[ (\mathcal{A}_\chi f)(\omega_1) = \sum_k \eta_k(\tau_k(\omega_1^{\text{white}})) f_k(\omega_1^{\text{white}}) \chi(\tau_k(\omega_1^{\text{white}})), \]

and commutes with automorphisms restricted to $H_1^{(1)}$. For example, taking $\chi(\cdot) = 1$ on $(0, t)$ and $\chi(\cdot) = 0$ on $(t, 1)$ we get the orthogonal projection onto the subspace $H_1^{(1)} \otimes H_1^{\text{white}} \subset H_1$.

Operators $\mathcal{A}_\chi$ commute also with the natural action of the algebra $L_\infty(\Omega_1^{\text{white}})$ on $H_1^{(1)}$,

\[ (\mathcal{B}_\varphi f)(\omega_1) = \sum_k \eta_k(\tau_k(\omega_1^{\text{white}})) f_k(\omega_1^{\text{white}}) \varphi(\omega_1^{\text{white}}). \]

However, the action $\varphi \mapsto \mathcal{B}_\varphi$ does not commute with automorphisms. We may join the actions $\mathcal{A}, \mathcal{B}$ into an action $\mathcal{C}$ of the commutative algebra of bounded Borel functions $\psi : (0, 1) \times \Omega_1^{\text{white}} \rightarrow \mathbb{R}$ on $H_1^{(1)}$,

\[ (\mathcal{C}_\psi f)(\omega_1) = \sum_k \eta_k(\tau_k(\omega_1^{\text{white}})) f_k(\omega_1^{\text{white}}) \psi(\tau_k(\omega_1^{\text{white}}), \omega_1^{\text{white}}). \]

In particular, consider the function

\[ \psi(t, \omega_1^{\text{white}}) = \begin{cases} 
\text{sgn}(B_1(\omega_1^{\text{white}}) - B_{0.5}(\omega_1^{\text{white}})) & \text{if } t < 0.5, \\
0 & \text{otherwise}.
\end{cases} \]

(Of course, $B_t(\omega_1^{\text{white}})$ is just another notation for $\omega_1^{\text{white}}(t)$.) It acts on $H_1^{(1)} = H_1^{(1)} \otimes H_0^{\text{white}} \oplus H_0^{\text{white}} \otimes H_0^{(1)}$ as follows (recall (2.2)):

\[ \mathcal{C}_\psi = 1_{0.5} \otimes \text{sgn } Q_{0.5} \quad \text{on } H_1^{(1)} \otimes H_0^{\text{white}}, \]

\[ \mathcal{C}_\psi = 0 \quad \text{on } H_0^{\text{white}} \otimes H_0^{(1)}. \]

Each function $f \in H_1^{(1)}$ leads to a finite positive Borel measure $\mu_f$ on $(0, 1) \times \Omega_1^{\text{white}}$ such that for every bounded Borel $\psi$,

\[ \int \psi \, d\mu_f = \langle \mathcal{C}_\psi f \rangle_f = \sum_k \int_{\Omega_1^{\text{white}}} |f_k(\omega_1^{\text{white}})|^2 \psi(\tau_k(\omega_1^{\text{white}}), \omega_1^{\text{white}}) P_1^{\text{white}}(d\omega_1^{\text{white}}). \]
Clearly, $\mu_f((0,1) \times \Omega_1^{\text{white}}) = \|f\|^2$, and $t \in \text{LocMin}(\omega_1^{\text{white}})$ for $\mu_f$-almost all pairs $(t, \omega_1^{\text{white}})$.

4 Main result

4.1 Theorem. For every $\alpha \in (0,2\pi)$ the automorphism $\theta_{\text{rotat}(e^{i\alpha})}$ of the classical part $(H_t^{\text{white}})$ of the Arveson system $(H_t)$ cannot be extended to an automorphism of the whole system.

Assume the contrary: the extension $\theta$ exists for some $\alpha \in (0,2\pi)$. We also assume that $\alpha \in \left(\frac{\pi}{2}, \pi]\right.$ (otherwise we may use $n\alpha$ for an appropriate $n \in \mathbb{Z}$). As before, $\langle A \rangle_f$ stands for $\langle Af, f \rangle$. The operator $Q_t$ acts on $H_t^{\text{white}}$, recall (2.2).

4.2 Lemma. Let $s, t > 0$, $f \in H_s^{(1)} \otimes H_t^{\text{white}} \subset H_{s+t}$. Then

$$\langle 1_s \otimes \text{sgn} Q_t \rangle_f + \langle 1_s \otimes \text{sgn} Q_t \rangle_g + \langle 1_s \otimes \text{sgn} Q_t \rangle_h \leq (3 - \varepsilon)\|f\|^2,$$

where $1_s$ is the identity operator on $H_s$, $g = \theta_{s+t} f$, $h = \theta_{s+t}^{-1} f$ and $\varepsilon$ is the same as in Lemma 2.4 (a positive absolute constant).

Proof. We repeat the proof of Lemma 2.4, taking into account that $\theta_{s+t}^{-1}(1_s \otimes Q_t)\theta_{s+t} = (\theta_{s+t}^{-1} \otimes \theta_t^{\text{rotat}(e^{i\alpha})})(1_s \otimes Q_t)(\theta_s \otimes \theta_t^{\text{rotat}(e^{i\alpha})}) = 1_s \otimes (Q_t \cos \alpha - P_t \sin \alpha)$ on $H_s^{(1)} \otimes H_t^{\text{white}}$.

The following construction is the key to the proof of Theorem 4.1. For any $n \in \{1, 2, \ldots\}$ and $\delta \in (0,0.5)$ we define Borel functions $\psi_{n,\delta} : (0,1) \times \Omega_1^{\text{white}} \to [-1,1]$ by

$$\psi_{n,\delta}(t, \omega_1^{\text{white}}) = \sum_{k=1}^{n} \chi_{n,k}(t) \varphi_{n,k,\delta}(\omega_1^{\text{white}}),$$

where

$$\chi_{n,k}(t) = \begin{cases} 1 & \text{if } t \in \left[\frac{k-1}{2n}, \frac{k}{2n}\right), \\ 0 & \text{otherwise;} \end{cases}$$

$$\varphi_{n,k,\delta}(\omega_1^{\text{white}}) = \text{sgn}\left( B^{\frac{k}{2n}}_{\omega_1^{\text{white}}} - B^{\frac{k+\delta}{2n}}_{\omega_1^{\text{white}}}(\omega_1^{\text{white}}) \right).$$

Note that the functions $\chi_{n,k}(t) \varphi_{n,k,\delta}(\omega_1^{\text{white}})$ are similar to (3.3) and act similarly to (3.4).
4.3 Lemma. For every $f \in H^{(1)}_{0,5} \otimes H^{white}_{0,5}$,
\[ \liminf_{n \to \infty} \langle C_{\psi_n,\delta} \rangle_f \to \|f\|^2 \quad \text{as } \delta \to 0^+. \]

Proof. We define Borel sets $U_\delta \subset (0,0.5) \times \Omega^{white}_1$ by
\[ (t, \omega^{white}_1) \in U_\delta \iff B_{t+\delta}(\omega^{white}_1) > B_t(\omega^{white}_1). \]
For $\mu_f$-almost all pairs $(t, \omega^{white}_1)$ we have $t \in (0,0.5) \cap \text{LocMin}(\omega^{white}_1)$, therefore, $(t, \omega^{white}_1) \in U_\delta$ for all $\delta$ small enough. It follows that
\[ \mu_f(U_\delta) \to \|f\|^2 \quad \text{as } \delta \to 0^+. \]
If $(t, \omega^{white}_1) \in U_\delta$ then (by continuity of Brownian paths) $\psi_{n,\delta}(t, \omega^{white}_1) = +1$ for all $n$ large enough. Therefore
\[ \int_{U_\delta} \psi_{n,\delta} \, d\mu_f \to \mu_f(U_\delta) \quad \text{as } n \to \infty. \]
However, \[ \langle C_{\psi_n,\delta} \rangle_f = \int \psi_{n,\delta} \, d\mu_f \geq \int_{U_\delta} \psi_{n,\delta} \, d\mu_f - \mu_f(\overline{U_\delta}) \]
(since $\psi_{n,\delta}() \geq -1$), thus,
\[ \liminf_{n \to \infty} \langle C_{\psi_n,\delta} \rangle_f \geq \mu_f(U_\delta) - \mu_f(\overline{U_\delta}) \to \|f\|^2 \quad \text{as } \delta \to 0^+. \]
Also, \[ \langle C_{\psi_n,\delta} \rangle_f \leq \|f\|^2 \quad \text{(since } \psi_{n,\delta}() \leq 1). \]

4.4 Lemma. For all $f \in H^{(1)}_{0,5} \otimes H^{white}_{0,5}$ and all $n, \delta$
\[ \langle C_{\psi_n,\delta} \rangle_f + \langle C_{\psi_n,\delta} \rangle_g + \langle C_{\psi_n,\delta} \rangle_h \leq (3 - \varepsilon)\|f\|^2, \]
where $g = \theta_1 f$, $h = \theta_1^{-1} f$, and $\varepsilon$ is the same as in Lemma 2.4 (a positive absolute constant).

Proof. Applying Lemma 4.2 (or rather, its evident generalization) to $A_{\chi_{n,k}} f$ (in place of $f$) and taking into account that $A_{\chi_{n,k}} g = \theta_1 A_{\chi_{n,k}} f$, $A_{\chi_{n,k}} h = \theta_1^{-1} A_{\chi_{n,k}} f$ (since $A_{\chi_{n,k}}$ commutes with $\theta_1$) we get
\[ \langle A_{\chi_{n,k}} B_{\varphi_{n,k,\delta}} \rangle_f + \langle A_{\chi_{n,k}} B_{\varphi_{n,k,\delta}} \rangle_g + \langle A_{\chi_{n,k}} B_{\varphi_{n,k,\delta}} \rangle_h \leq (3 - \varepsilon)\|A_{\chi_{n,k}} f\|^2. \]
We sum up in $k$ and note that $\sum_k A_{\chi_{n,k}} B_{\varphi_{n,k,\delta}} = C_{\psi_n,\delta}$ and $\sum_k\|A_{\chi_{n,k}} f\|^2 = \|f\|^2$.

Applying Lemma 4.3 to $f$, $g = \theta_1 f$ and $h = \theta_1^{-1} f$ we get
\[ \liminf_{n \to \infty} \langle C_{\psi_n,\delta} \rangle_f + \langle C_{\psi_n,\delta} \rangle_g + \langle C_{\psi_n,\delta} \rangle_h \to 3\|f\|^2 \quad \text{as } \delta \to 0^+ \]
in contradiction to Lemma 4.4 which completes the proof of Theorem 4.1.
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