Predictions from the logotropic model: the universal surface density of dark matter halos and the present proportion of dark matter and dark energy

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The logotropic model [P.H. Chavanis, Eur. Phys. J. Plus 130, 130 (2015)] may be an interesting alternative to the ΛCDM model. It is able to account for the present accelerating expansion of the universe while solving at the same time the core-cusp problem of the CDM model. In the logotropic model, there is a single dark fluid. Its rest-mass plays the role of dark matter and its internal energy plays the role of dark energy. We highlight two remarkable predictions of the logotropic model. It yields cored dark matter halos with a universal surface density equal to Σ_{dm,0} = 2.669 ± 0.08. Using the measured present proportion of baryonic matter Ω_{bs,0} = 0.0486 ± 0.0010, we find that the values of the present proportion of dark matter and dark energy are Ω_{dm,0} = 0.141 ± 0.0486 and Ω_{de,0} = 0.6911 ± 0.0062 within the error bars. These theoretical predictions are obtained by advocating a mysterious strong cosmic coincidence (dubbed “dark magic”) implying that our epoch plays a particular role in the history of the universe. We review the three types of logotropic models introduced in our previous papers depending on whether the equation of state is expressed in terms of the energy density, the rest-mass density, or the pseudo-rest mass density of a complex scalar field. We discuss the similarities and the differences between these models. Finally, we point out some intrinsic difficulties with the logotropic model similar to those encountered by the Chaplygin gas model and discuss possible solutions.

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I. INTRODUCTION

Baryonic (visible) matter constitutes only 5% of the content of the universe today. The rest of the universe is made of approximately 25% dark matter (DM) and 70% dark energy (DE) [1, 2]. DM can explain the flat rotation curves of the spiral galaxies. It is also necessary to form the large-scale structures of the universe. DE does not cluster but is responsible for the late time acceleration of the universe revealed by the observations of type Ia supernovae, the cosmic microwave background (CMB) anisotropies, and galaxy clustering. Although there have been many theoretical attempts to explain DM and DE, we still do not have a robust model for these dark components that can pass all the theoretical and observational tests.

The most natural and simplest model is the ΛCDM model which treats DM as a nonrelativistic cold pressureless gas and DE as a cosmological constant Λ (originally introduced by Einstein [3]) possibly representing vacuum energy [4, 5]. The effect of the cosmological constant is equivalent to that of a fluid with a constant energy density ϵ_{Λ} = Λc^2/8πG and a negative pressure P_{de} = -ϵ_{de}. Therefore, the ΛCDM model is a two-fluid model comprising DM with an equation of state P_{dm} = 0 and DE with an equation of state P_{de} = -ω_{de}P_{de}. When combined with the energy conservation equation [see Eq. (18)], the equation of state P_{dm} = 0 implies that the DM density decreases with the scale factor as ϵ_{dm} = ϵ_{dm,0}a^{-3} and the equation of state P_{de} = -ω_{de} implies that the DE density is constant: ϵ_{de} = ϵ_{Λ}. Baryonic matter can also be modeled as a pressureless fluid (P_{b} = 0) whose density decreases as ϵ_{b} = ϵ_{b,0}a^{-3}. Therefore, the total energy density of the universe (baryons + DM + DE) evolves as

$$\epsilon = \frac{\epsilon_{m,0}}{a^{3}} + \epsilon_{\Lambda},$$

(1)

where ϵ_{m,0} = ϵ_{dm,0} + ϵ_{b,0} is the present density of (baryonic + dark) matter. Matter dominates at early times when the density is high (ϵ ∝ ϵ_{m,0}/a^3) and DE dominates at late times when the density is low (ϵ → ϵ_{Λ}). The scale factor increases algebraically as a ∝ t^{2/3} during the matter era (Einstein-de Sitter regime) and exponentially as a ∝ exp(√Λ/3t) during the DE era (de Sitter regime). As a result, the universe undergoes a decelerated expansion followed by an accelerating expansion. At the present epoch, both baryonic matter, DM and DE are important in the energy budget of the universe. Introducing the Hubble constant $H = \dot{a}/a = (8\pi G c^3/3c^2)^{1/2}$ [see Eq. (15)], we can rewrite Eq. (1) as

$$\frac{H^2}{H_0^2} = \frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{a^3} + \Omega_{de,0},$$

(2)

where $\epsilon_0 = 3H_0^2c^2/8\pi G$ is the present particle energy density of the universe, $\Omega_{m,0} = \epsilon_{m,0}/\epsilon_0$ is the present proportion of matter and $\Omega_{de,0} = \epsilon_{\Lambda}/\epsilon_0$ is the present proportion of DE. From the observations, we get $H_0 = 2.195 \times 10^{-18}$ s^{-1}, $\epsilon_0 = 7.75 \times 10^{-7}$ g m^{-1} s^{-2}, $\epsilon_0/c^2 = \Omega_{de,0} = \epsilon_{\Lambda}/\epsilon_0 = 0.6911 ± 0.0062$. The probability distribution of the quantity P_{dm} is given by the expression

$$P_{dm} = \frac{1}{a^{3}} = \frac{1}{(1 + \Omega_{m,0}/\epsilon_0)^{3/2}} = \frac{1}{(1 + \epsilon_{m,0}/\epsilon_0)^{3/2}},$$

(3)

where $\Omega_{m,0} = \epsilon_{m,0}/\epsilon_0$ is the present proportion of matter and $\epsilon_{m,0}$ is the present particle energy density of the universe. The probability distribution of $\Omega_{m,0}$ is given by the expression

$$P_{\Omega_{m,0}} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\Omega_{m,0}} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\Omega_{m,0}} \frac{d\Omega_{m,0}}{d\epsilon_{m,0}} \frac{d\epsilon_{m,0}}{d\epsilon},$$

(4)

where $\Omega_{m,0} = \epsilon_{m,0}/\epsilon_0$. The probability distribution of $\epsilon_{m,0}$ is given by the expression

$$P_{\epsilon_{m,0}} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon_{m,0}} \frac{d\epsilon_{m,0}}{d\epsilon_{dm,0}} \frac{d\epsilon_{dm,0}}{d\epsilon},$$

(5)

where $\epsilon_{m,0} = \epsilon_{dm,0}/\epsilon_0$. The probability distribution of $\epsilon_{dm,0}$ is given by the expression

$$P_{\epsilon_{dm,0}} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon_{dm,0}} \frac{d\epsilon_{dm,0}}{d\epsilon_{dm,0}} \frac{d\epsilon_{dm,0}}{d\epsilon},$$

(6)

where $\epsilon_{dm,0} = \epsilon_{dm,0}/\epsilon_0$. The probability distribution of $\epsilon_{dm,0}$ is given by the expression

$$P_{\epsilon_{dm,0}} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon} = \frac{1}{\epsilon_0} \frac{d\epsilon}{d\epsilon_{dm,0}} \frac{d\epsilon_{dm,0}}{d\epsilon_{dm,0}} \frac{d\epsilon_{dm,0}}{d\epsilon},$$

(7)
8.62 \times 10^{-24} \text{g m}^{-3}, \Omega_{k0} = 0.0486, \Omega_{\text{dm0}} = 0.2589 \text{ and } \Omega_{\text{de0}} = 0.6911. \text{ This gives } \epsilon_{\Lambda} = \Omega_{\text{de0}} \epsilon_{0} = 5.35 \times 10^{-7} \text{ g m}^{-1} \text{s}^{-2}. \text{ Therefore, the value of the cosmological density } \rho_{\Lambda} = \epsilon_{\Lambda}/c^{2} \text{ is }

\rho_{\Lambda} = \frac{\Lambda}{8\pi G} = 5.96 \times 10^{-24} \text{g m}^{-3} \quad (3)

\text{ and the value of the cosmological constant is } \Lambda = 1.00 \times 10^{-35} \text{ s}^{-2}.

The CDM model faces important problems at the scale of DM halos such as the core-cusp problem [6], the missing satellite problem [7, 8], and the “too big to fail” problem [10]. This leads to the so-called small-scale crisis of CDM [11]. Basically, this is due to the assumption that DM is pressureless so there is nothing to balance the gravitational attraction at high densities. As a result, classical N-body simulations lead to DM halos exhibiting central cusps where the density diverges as \( r^{-1} \) [12] while observations reveal that they have constant density cores [13]. A possibility to solve these problems is to take into account quantum mechanics. Fermionic and bosonic models of DM halos display quantum cores, even at \( T = 0 \), that replace the cusp (see, e.g., [12, 13] and references therein). For self-gravitating fermions, the quantum pressure (leading to a fermion ball) is due to the Heisenberg uncertainty principle. However, these quantum pressure terms therein). For self-gravitating fermions, the quantum pressure (leading to a soliton) is due to the quantum pressure (leading to a fermion ball) is due to the Heisenberg uncertainty principle. However, these quantum models cannot explain the observation that DM halos have a constant surface density [16, 18].

\[ \Sigma_{0} = \rho_{0} r_{h} = 141^{+83}_{-52} M_{\odot}/\text{pc}^{2}. \quad (4) \]

Indeed, in fermionic and bosonic DM models, the mass decreases as the radius increases (see Appendix L of [14]) instead of increasing according to \( M_{h} \propto \Sigma_{0} r_{h}^{3} \) as implied by the constancy of the surface density. \[ \text{On the other hand, although the ΛCDM model is perfectly consistent with current cosmological observations, it faces two main problems. The first problem is to explain the tiny value of the cosmological constant } \Lambda \text{ as } 1.00 \times 10^{-35} \text{ s}^{-2}. \text{ Indeed, if DE can be attributed to vacuum fluctuations, quantum field theory predicts that } \Lambda \text{ should correspond to the Planck scale which is associated with the Planck density }

\[ \rho_{P} = \frac{c^{5}}{h^{2} c^{2}} = 5.16 \times 10^{99} \text{ g/m}^{3}. \quad (5) \]

Now, the ratio between the Planck density [5] and the cosmological density [3] is

\[ \frac{\rho_{P}}{\rho_{\Lambda}} = \frac{8\pi c^{5}}{h G \Lambda} \sim 10^{123}. \quad (6) \]

Therefore, the observed cosmological constant lies 123 orders of magnitude below the theoretical value. This is called the cosmological constant problem [19, 20]. The second problem is to explain why DM and DE are of similar magnitudes today (within a factor 3) although they scale differently with the universe’s expansion. This is the cosmic coincidence problem [21, 22], which is a fine-tuning problem, frequently triggering anthropic explanations.

For these reasons, other types of matter with negative pressure that can behave like a cosmological constant at late time have been considered as candidates of DE: fluids of topological defects (domain walls, cosmic strings...) [23, 27]. X-fluids with a linear equation of state \( P = w_{X} \epsilon \) with a coefficient \( w_{X} < -1/3 \) triggering an accelerating expansion of the universe [28, 30], a time-varying cosmological constant \( \Lambda(t) \) [31, 32], quintessence fields in the form of an evolving self-interacting scalar field (SF) minimally coupled to gravity [35, 41], k-essence fields corresponding to a SF with a noncanonical kinetic term [45, 47] and even phantom or ghost fields [48, 49] represented by a SF with a negative kinetic term implying that the energy density of the universe increases with the scale factor. However, these models still face the cosmic coincidence problem because they treat DM and DE as distinct entities.

Indeed, in the standard ΛCDM model and in the above-mentioned models, DM and DE are two independent components introduced to explain the clustering of matter and the cosmic acceleration, respectively. However, DM and DE could be two different manifestations of a single underlying substance (a dark fluid) called “quartessence” [50]. The most famous example is the Chaplygin gas [51] or generalized Chaplygin gas (GCG) [52] in which the pressure depends on a power of the density. The generalized Chaplygin equation of state can be viewed as a polytropic equation of state with a negative pressure [53, 54]

\[ P = K \rho^{\gamma} \quad (K < 0). \quad (7) \]

The original Chaplygin gas corresponds to \( \gamma = -1 \) [51]. This dark fluid behaves as DM at early times and as DE at late times. It provides therefore a simple unification of DM and DE. This is an example of unified dark matter and dark energy (UDM) model [50]. This dual behavior avoids fine-tuning problems since the dark fluid can be interpreted as an entangled mixture of DM and DE. The ΛCDM model can be seen as a UDM model where the pressure is a negative constant [54, 58, 57]

\[ P = -\rho_{\Lambda} c^{2}. \quad (8) \]

Indeed, by combining this equation of state with the energy conservation equation we recover Eq. (1) with just one fluid. This is a particular case of the generalized Chaplygin gas corresponding to \( \gamma = 0 \) and \( K = -\rho_{\Lambda} c^{2} \).

Recently, we have introduced the notion of logotropic dark fluid (LDF) [58, 63] (see also [64, 71]) where the

\[ \text{In Refs. [13, 14], the law } M_{h} \propto \Sigma_{0} r_{h}^{2} \text{ is heuristically accounted for by the presence of an isothermal envelope (surrounding the quantum core) whose temperature changes with } r_{h} \text{ according to } k_{B} T/m \propto G \Sigma_{0} r_{h}. \]
pressure depends on the logarithm of the density as\(^2\)

\[
P = A \ln \left( \frac{\rho_p}{\rho_A} \right),
\]

where \(\rho_p\) is the Planck density and \(A\) is a new fundamental constant of physics superseding Einstein’s cosmological constant \(\Lambda\). We will show that its value is given by

\[
A/c^2 = \frac{\rho_{\Lambda}}{\ln \left( \frac{\rho_p}{\rho_{\Lambda}} \right)} = 2.10 \times 10^{-26} \text{ g m}^{-3},
\]

where \(\rho_{\Lambda}\) is the cosmological density from Eq. 9. Therefore, the logotropic equation of state reads

\[
P = -\frac{\rho_A c^2}{\ln \left( \frac{\rho_p}{\rho_{\Lambda}} \right)} \ln \left( \frac{\rho_p}{\rho} \right).
\]

We note that \(P = -\rho_A c^2\) when \(\rho = \rho_{\Lambda}\). It is convenient to write the fundamental constant \(A\) under the form

\[
A = B \rho_{\Lambda} c^2,
\]

where \(B\) is the dimensionless number

\[
B = \frac{1}{\ln \left( \frac{\rho_p}{\rho_{\Lambda}} \right)} = 3.53 \times 10^{-3}.
\]

Rewriting Eq. 13 as

\[
\frac{\rho_p}{\rho_{\Lambda}} = e^{1/B}
\]

and comparing this expression with Eq. 6, we see that \(B \approx 1/[123 \ln(10)]\) is essentially the inverse of the famous number 123 (up to a conversion factor from neperian to decimal logarithm). We note that \(B\) has a small but nonzero value. This is because \(B\) depends on the Planck constant \(h\) through the Planck density \(\rho_p\) in Eq. 13 and because \(h\) has a small but nonzero value. In the classical (nonquantum) limit \(h \to 0\), we find that \(\rho_p \to +\infty\) and \(B \to 0\). In that case, we recover the \(\Lambda\)CDM model.

Indeed, when \(\rho_p \to +\infty\), the logotropic equation of state reduces to the constant equation of state \(\Lambda\). The fact that \(B\) is nonzero means that quantum effects \((h \neq 0)\) play a fundamental role in the logotropic model. Since the effects of \(B\) manifest themselves in the late universe, this implies (surprisingly!) that quantum mechanics affects the late acceleration of the universe.

For a UDM model the equation of state can be specified in different manners depending on whether the pressure \(P\) is expressed in terms of the energy density \(\epsilon\) (model of type I), the rest-mass density \(\rho_{\text{dm}} = nm\) (model of type II), or the pseudo-rest mass density \(\rho = (m^2/h^2)|\phi|^2\) associated with a complex SF (model of type III). In the nonrelativistic regime, these three formulations coincide and \(\rho = \rho_{\text{dm}} = \epsilon/c^2\) represents the mass density. However, in the relativistic regime, they lead to different models. The relation between these different models has been discussed in detail in [62]. For the logotropic equation of state, each of these models has been studied exhaustively in a specific paper (the logotropic model of type II has been discussed in [58–61] and the logotropic models of type I and III have been discussed in [63]). In the present paper, we provide a brief comparison between these models and stress their main properties. We also emphasize the main predictions of the logotropic model [58–63].

(i) At small (galactic) scales, the logotropic model is able to solve the small-scale crisis of the CDM model. Indeed, contrary to the pressureless CDM model, the logotropic equation of state provides a pressure gradient that can balance the gravitational attraction and prevent gravitational collapse. As a result, logotropic DM halos present a central core rather than a cusp, in agreement with the observations.

(ii) Remarkably, the logotropic model implies that DM halos have a constant surface density and it predicts its universal value \(\Sigma^\text{th}_0 = 0.01955 c\sqrt{\Lambda}/G = 133 M_\odot /\text{pc}^2\) without adjustable parameter (here \(\Lambda = 1.00 \times 10^{-35} \text{ s}^{-2}\) is interpreted as an effective cosmological constant). This theoretical value is in good agreement with the value \(\Sigma_{0}^\text{obs} = 141^{+83}_{-52} M_\odot /\text{pc}^2\) obtained from the observations [18].

(iii) As a corollary, the logotropic model implies that the mass of dwarf galaxies enclosed within a sphere of fixed radius \(r_a = 300 \text{ pc}\) has a universal value \(M^{\text{th}}_{300} = 1.82 \times 10^7 M_\odot\), i.e. \(\log(M^{\text{th}}_{300}/M_\odot) = 7.26\), in agreement with the observations giving \(\log(M^{\text{obs}}_{300}/M_\odot) = 7.0^{+0.3}_{-0.4}\) [73]. The logotropic model also reproduces the Tully-Fisher relation \(M_\odot \propto v_t^2\), where \(M_\odot\) is the baryonic mass and \(v_t\) the circular velocity at the halo radius, and predicts a value of the ratio \((M_\odot/v_t^2)^{\text{th}} = 46.4 M_\odot \text{ km}^{-4} \text{s}^4\) which is close to the observed value \((M_\odot/v_h^2)^{\text{obs}} = 47 \pm 6 M_\odot \text{ km}^{-4} \text{s}^4\) [74].

(iv) At large (cosmological) scales, the logotropic model is able to account for the transition between DM and DE and for the present acceleration of the universe. Remarkably, it predicts the present ratio of DE and DM to be the pure number \(\Omega^{\text{th}}_{\text{de},0}/\Omega^{\text{th}}_{\text{dm},0} = \epsilon = 2.71828\)... in very good agreement with the observations giving \(\Omega^{\text{obs}}_{\text{de},0}/\Omega^{\text{obs}}_{\text{dm},0} = 2.660 \pm 0.08\). This then yields the values of the present proportion of DM and DE as \(\Omega^{\text{obs}}_{\text{dm},0} = \frac{1}{1 + \epsilon}(1 - \Omega_{\text{de},0}) = 0.2559\) and \(\Omega^{\text{obs}}_{\text{de},0} = \frac{1}{1 + \epsilon}(1 - \Omega_{\text{dm},0}) = 0.6959\) (where we have used \(\Omega^{\text{obs}}_{\text{dm},0} = 0.0486 \pm 0.0010\) and \(\Omega^{\text{obs}}_{\text{de},0} = 0.2589 \pm 0.0057\) and \(\Omega^{\text{obs}}_{\text{dm},0} = 0.6911 \pm 0.0062\) within the error bars.

\(\text{2 The logotropic equation of state can be obtained from the polytropic equation of state (7) in the limit } \gamma \to 0\text{ and } K \to \infty\text{ with } A = K' \gamma \text{ finite}[65, 62, 63, 72]. \text{ It is interesting that the Planck density appears in this equation of state in order to make the argument of the logarithm dimensionless.}\)
II. LOGOTROPIC EQUATION OF STATE OF TYPE I

In this section, we consider a relativistic barotropic fluid described by an equation of state of type I where the pressure \( P = P(\epsilon) \) is specified as a function of the energy density.

A. Friedmann equations

We consider an expanding homogeneous universe and adopt the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In that case, the Einstein field equations reduce to the Friedmann equations \[ H^2 = \frac{8\pi G}{3c^2} \epsilon, \] (15)

\[ 2\dot{H} + 3H^2 = -\frac{8\pi G}{c^2} P, \] (16)

where \( H = \dot{a}/a \) is the Hubble constant and \( a(t) \) is the scale factor. To obtain Eq. (15), we have assumed that the universe is flat \((k = 0)\) in agreement with the inflation paradigm \[76\] and the observations of the cosmic microwave background (CMB) \[1, 2\]. On the other hand, we have set the true cosmological constant to zero \((\Lambda_{\text{true}} = 0)\) since, in our model, DE will be taken into account in the equation of state \( P \). Equation (16) can also be written as

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (3P + \epsilon), \] (17)

showing that the expansion of the universe is decelerating when \( P > -\epsilon/3 \) and accelerating when \( P < -\epsilon/3 \).

B. Energy conservation equation

Combining Eqs. (15) and (16), we obtain the energy conservation equation

\[ \frac{d\epsilon}{dt} + 3H (\epsilon + P) = 0. \] (18)

The energy density increases with the scale factor when \( P > -\epsilon \) and decreases with the scale factor when \( P < -\epsilon \). The latter case corresponds to a phantom behavior \[18, 19\].

For a given equation of state \( P(\epsilon) \) we can solve Eq. (18) to get

\[ \ln a = -\frac{1}{3} \int \frac{d\epsilon}{\epsilon + P(\epsilon)}. \] (19)

This equation determines the relation between the energy density \( \epsilon \) and the scale factor \( a \). We can then solve the Friedmann equation \[15\] with \( \epsilon(a) \) to obtain the temporal evolution of the scale factor \( a(t) \). We note that the function \( a(\epsilon) \) is univalued. As a result, an equation of state of type I describes either a normal behavior or a phantom behavior but it cannot describe the transition from a normal to a phantom behavior.

C. Logotropic equation of state

For the logotropic equation of state of type I:

\[ P = A\ln \left( \frac{\epsilon}{\rho_0c^2} \right), \] (20)

the energy conservation equation \[19\] reads

\[ \ln a = -\frac{1}{3} \int \epsilon \frac{de'}{e'} + A \ln \left( \frac{\epsilon}{\rho_0c^2} \right), \] (21)

where \( \epsilon_0 \) denotes the present energy density of the universe (when \( a = 1 \)). This equation determines the evolution of the energy density \( \epsilon(a) \) as a function of the scale factor. When \( a \to 0 \), Eq. (21) reduces to

\[ \ln a \sim -\frac{1}{3} \int \epsilon \frac{de'}{e'}, \] (22)

implying that the energy density decreases like \( \epsilon \propto a^{-3} \) as the universe expands. This corresponds to the DM regime. The scale factor increases in time like \( a \propto t^{2/3} \) (Einstein-de Sitter). When \( a \to +\infty \), the energy density tends to a constant \( \epsilon_{\text{min}} \), which is the solution of the equation

\[ \epsilon_{\text{min}} + A \ln \left( \frac{\epsilon_{\text{min}}}{\rho_0c^2} \right) = 0. \] (23)

This corresponds to the DE regime. The scale factor increases exponentially rapidly in time like \( a \propto \exp(\sqrt{8\pi G\epsilon_{\text{min}}/3c^2} t) \) (de Sitter).

D. The value of \( A \)

The behavior of the energy density in the logotropic model of type I is similar to the one in the ΛCDM model. If we identify \( \epsilon_{\text{min}} \) with the DE density \( \rho_\Lambda c^2 \) in the ΛCDM model, which is equal to the asymptotic value of \( \epsilon \) for \( a \to +\infty \), we get

\[ A = \frac{\rho_\Lambda c^2}{\ln \left( \frac{\rho_\Lambda}{\rho_0c^2} \right)}. \] (24)

This determines the value of the constant \( A \) in the logotropic model as given by Eq. \[10\]. We then find that the pressure decreases monotonically from \( +\infty \) to \( P_{\text{min}} = -\epsilon_\Lambda \) as the universe expands. The pressure \( P \)
is positive when \( \epsilon > \rho_P c^2 \) and negative when \( \epsilon < \rho_P c^2 \). It vanishes at \( \epsilon = \rho_P c^2 \). Since the logotropic model is a unification of DM and DE, it is not expected to be valid in the early universe. Therefore, the pressure is always negative in the regime of interest (\( \epsilon \ll \rho_P c^2 \)) where the logotropic model is valid.

### E. Evolution of the universe

Setting \( x = \epsilon / \epsilon_0 \) and \( A = B \Omega_{de,0} \epsilon_0 \) we can rewrite Eq. (21) as

\[
\ln \frac{\epsilon}{\epsilon_0} = -\frac{1}{3} \int \frac{dx}{x + B \Omega_{de,0} (\ln x - \ln \Omega_{de,0} - \frac{1}{3})}.
\]

The function \( \epsilon / \epsilon_0 (a) \) is plotted in Fig. 1. We have taken \( \Omega_{de,0} = 0.6911 \) and \( B = 3.53 \times 10^{-3} \). The logotropic model of type I behaves similarly to the \( \Lambda \)CDM model. The two models coincide in the limit \( B \rightarrow 0 \) where Eq. (25) returns Eq. (2). For the simplicity of the presentation, we have ignored here the contribution of baryonic matter but it is straightforward to take it into account.

![Fig. 1](image)

**FIG. 1:** Normalized energy density \( \epsilon / \epsilon_0 \) as a function of the scale factor \( a \) for the logotropic model of type I. It is compared with the \( \Lambda \)CDM model. The two curves are indistinguishable on the figure.

### III. LOGOTROPIC EQUATION OF STATE OF TYPE II

In this section, we consider a relativistic barotropic fluid described by an equation of state of type II where the pressure \( P = P(\rho_{dm}) \) is specified as a function of the rest-mass density. The notation \( \rho_{dm} \) for the rest-mass density will soon become clear.

#### A. First principle of thermodynamics

The first principle of thermodynamics for a relativistic gas can be written as

\[
d \left( \frac{\epsilon}{\rho_{dm}} \right) = -P d \left( \frac{1}{\rho_{dm}} \right) + T d \left( \frac{s}{\rho_{dm}} \right),
\]

where

\[
\epsilon = \rho_{dm} c^2 + u
\]

is the energy density including the rest-mass energy density \( \rho_{dm} c^2 \) (where \( \rho_{dm} = nm \) is the rest-mass density) and the internal energy density \( u \), \( s \) is the entropy density, \( P \) is the pressure, and \( T \) is the temperature. We assume that \( T d(s/\rho_{dm}) = 0 \). This corresponds to a cold \((T = 0)\) or isentropic \((s/\rho_{dm} = \text{cst})\) gas. In that case, Eq. (26) reduces to

\[
d \left( \frac{\epsilon}{\rho_{dm}} \right) = -P d \left( \frac{1}{\rho_{dm}} \right) = \frac{P}{\rho_{dm}^2} d\rho_{dm}.
\]

Assuming that \( P = P(\rho_{dm}) \) and integrating Eq. (28) we obtain Eq. (27) with

\[
u(\rho_{dm}) = \rho_{dm} \int^{\rho_{dm}} P(P') \rho'^{-2} d\rho'.
\]

This relation determines the internal energy as a function of the equation of state \( P(\rho_{dm}) \). Inversely, the equation of state is determined by the internal energy \( u(\rho_{dm}) \) through the relation

\[
P(\rho_{dm}) = \rho_{dm} u'(\rho_{dm}) - u(\rho_{dm}).
\]

Let us apply these equations in a cosmological context, namely for a spatially homogeneous fluid in an expanding background. Combining the energy conservation equation (18) with the first principle of thermodynamics (28) which can be rewritten as

\[
d\epsilon = \frac{P + \epsilon}{\rho_{dm}} d\rho_{dm},
\]

we find that the rest-mass density satisfies the equation

\[
\frac{d\rho_{dm}}{dt} + 3H \rho_{dm} = 0.
\]

This equation can be integrated into

\[
\rho_{dm} = \rho_{dm,0} \frac{a^3}{a^3},
\]

where \( \rho_{dm,0} \) is the present value of the rest-mass density. Equations (32) and (33) express the conservation of the rest-mass of the dark fluid. As argued in our previous papers \[58, 59\], the rest-mass energy density \( \rho_{dm} c^2 \) plays
the role of DM and the internal energy $u$ plays the role of DE. Therefore, we have

$$\epsilon_{dm} = \rho_{dm} c^2 = \frac{\Omega_{dm,0} \epsilon_0}{a^3}$$  \hspace{1cm} (34)

and

$$\epsilon_{de} = u(\rho_{dm}) = u \left( \frac{\Omega_{dm,0} \epsilon_0}{c^2 a^3} \right).$$  \hspace{1cm} (35)

The total energy density of the dark fluid then reads

$$\epsilon = \frac{\Omega_{dm,0} \epsilon_0}{a^3} + u \left( \frac{\Omega_{dm,0} \epsilon_0}{c^2 a^3} \right).$$  \hspace{1cm} (36)

The decomposition $\epsilon = \rho_{dm} c^2 + u = \epsilon_{dm} + \epsilon_{de}$ provides a simple and nice interpretation of DM and DE in terms of the rest-mass energy and internal energy of a single DF \[58, 59\]. For given $P(\rho_{dm})$ or $u(\rho_{dm})$ the relation between the energy density and the scale factor is determined by Eq. (36). We can then solve the Friedmann equation \[58, 59\] with $\epsilon(a)$ to determine the temporal evolution of the scale factor $a(t)$.

**Remark:** The equation of state parameter $w = P/\epsilon$ is given by

$$w = \frac{\rho_{dm} u'(\rho_{dm}) - u(\rho_{dm})}{\rho_{dm} c^2 + u(\rho_{dm})}. \hspace{1cm} (37)$$

For a barotropic equation of state of type II the universe exhibits a normal behavior ($w > -1$) when $1 + (1/c^2) u'(\rho_{dm}) > 0$ and a phantom behavior ($w < -1$) when $1 + (1/c^2) u'(\rho_{dm}) < 0$. An equation of state of type II can describe the transition from a normal to a phantom behavior.

**B. Logotropic equation of state and logarithmic internal energy**

For the logotropic equation of state of type II \[58, 59\]

$$P = A \ln \left( \frac{\rho_{dm}}{\rho_P} \right), \hspace{1cm} (38)$$

the internal energy obtained from Eq. (29) reads

$$u = -A \left[ 1 + \ln \left( \frac{\rho_{dm}}{\rho_P} \right) \right]. \hspace{1cm} (39)$$

Therefore, the energy density of the LDF is

$$\epsilon = \rho_{dm} c^2 - A \left[ 1 + \ln \left( \frac{\rho_{dm}}{\rho_P} \right) \right] = \epsilon_{dm} + \epsilon_{de}, \hspace{1cm} (40)$$

where the first term (rest-mass) is interpreted as DM and the second term (internal energy) as DE. Our model provides a simple unification of these two entities. Eliminating $\rho_{dm}$ between Eqs. (38) and (40), the equation of state $P(\epsilon)$ is given in the reversed form $\epsilon(P)$ by

$$\epsilon = e^{P/A} \rho_P c^2 - P - A. \hspace{1cm} (41)$$

This is the equation of state of type I corresponding to the logotropic model of type II \[62\]. Following \[58, 59\], we define the dimensionless parameter $B$ through the relation

$$\frac{\rho_p}{\rho_{dm,0}} = e^{1+1/B}. \hspace{1cm} (42)$$

Using Eqs. (39) and (42), the pressure and the total energy density can be expressed as a function of the scale factor as

$$P = -A \left( 1 + \frac{1}{B} + 3 \ln a \right), \hspace{1cm} (43)$$

$$\epsilon = \frac{\Omega_{dm,0} \epsilon_0}{a^3} + A \left( \frac{1}{B} + 3 \ln a \right). \hspace{1cm} (44)$$

**C. The value of A**

According to Eqs. (40) and (44), the DE density is given by

$$\epsilon_{de} = A \left( \frac{1}{B} + 3 \ln a \right). \hspace{1cm} (45)$$

Applying this relation at the present time ($a = 1$), we obtain the relation

$$A = BO_{de,0}de_0. \hspace{1cm} (46)$$

Explicating the expression of $B$ from Eq. (42) we get

$$A = \frac{\Omega_{de,0} \epsilon_0}{\ln \left( \frac{\rho_{dm} c^2}{\Omega_{dm,0} \epsilon_0} \right) - 1}. \hspace{1cm} (47)$$

If we view $A$ as a fundamental constant of physics, this equation gives a relation between the present fraction $\Omega_{dm,0}$ of DM, the present fraction $\Omega_{de,0}$ of DE and the present energy density $\epsilon_0$. Inversely, we can use the measured values of $\Omega_{dm,0}$, $\Omega_{de,0}$ and $\epsilon_0$ to determine the expression of the constant $A$ appearing in the logotropic equation of state. Therefore, there is no free (undetermined) parameter in our model.

As in our previous papers \[58, 63\], it is convenient to give a special name to the present density of DE and write it as

$$\epsilon_A = \rho_A c^2 = \Omega_{de,0} \epsilon_0. \hspace{1cm} (48)$$

In the $\Lambda$CDM model, the DE density $\rho_A = \Lambda/8\pi G$ is constant. It represents the cosmological density which is determined by Einstein’s cosmological constant $\Lambda$. The present DE density of the universe coincides with the constant DE density of the universe in the $\Lambda$CDM model. This justifies the notation from Eq. (48). With this notation we can rewrite Eq. (47) as

$$A = \frac{\rho_A c^2}{\ln \left( \frac{\rho_P}{\rho_A} \right) + \ln \left( \frac{\Omega_{de,0}}{\Omega_{dm,0}} \right) - 1}. \hspace{1cm} (49)$$
Recalling the value of the ratio $\rho_p/\rho_\Lambda$ from Eq. (4), we find that $\ln (\rho_p/\rho_\Lambda) = 283$. On the other hand, using the measured values of $\Omega_{\text{obs}}^{\Lambda}=0.6911$ and $\Omega_{\text{obs}}^{\text{dm}}=0.2589$, we get $\ln (\Omega_{\text{obs}}^{\text{de}}/\Omega_{\text{obs}}^{\text{dm}})-1 = -0.0182$. Therefore, the first term in the denominator of Eq. (49) is much larger than the second term so that, in very good approximation, we have

$$A = \frac{\rho_\Lambda c^2}{\ln \left( \frac{\rho_c}{\rho_\Lambda} \right)}, \quad (50)$$

as given by Eq. (10). Similarly, $B$ is given in very good approximation by Eq. (13).

### D. Evolution of the universe

Using Eq. (46), the pressure $P$ and the energy density $\epsilon$ can be rewritten as

$$P = -\Omega_{\text{de},0}\epsilon_0 (B + 1 + 3B \ln a), \quad (51)$$

$$\frac{H^2}{H_0^2} = \frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\text{dm},0}}{a^4} + \Omega_{\text{de},0} (1 + 3B \ln a). \quad (52)$$

In Eq. (52) we have combined the relation $\epsilon(a)$ with the Friedmann equation (15) to obtain a differential equation determining the temporal evolution of the scale factor. If we take into account the presence of baryons, we must add a term $\epsilon_b = \Omega_{b,0}\epsilon_0/a^3$ in the energy density. The \Lambda CDM model is recovered for $B = 0$ corresponding to the limit $h \to 0$ or $\rho_p \to +\infty$. In that case, the internal energy is constant ($u = \epsilon_A$) and Eqs. (51) and (52) return Eqs. (1), (2) and (6). The evolution of the universe in the logotropic model of type II has been discussed in detail in [55-60]. Below, we just recall the main results of these studies.

Starting from $+\infty$, the energy density $\epsilon$ first decreases, reaches a minimum $\epsilon_M = A \ln(\rho_p c^2/A) > 0$ at $\rho_{\text{dm}} = A/c^2$, then increases to $+\infty$. The minimum energy density $\epsilon_M = 0.707\epsilon_0 = 1.02\epsilon_A$ is achieved at a scale factor $a_M = 5.01$ corresponding to a time $t_M = 2.81H_0^{-1} = 40.6$ Gyrs (the age of the universe is $t_0 = 13.8$ Gyrs). At early times, the pressure is negligible with respect to the energy density and the LDF is equivalent to a pressureless fluid like in the CDM model. The energy density decreases as $\epsilon \sim \Omega_{\text{dm},0}\epsilon_0/a^3$ and the scale factor increases algebraically as $a \propto t^{2/3}$ (Einstein-de Sitter). This leads to a decelerated expansion of the universe. At later times, the negative pressure of the LDF becomes efficient and explains the acceleration of the universe that we observe today. Ultimately, DE dominates over DM (more precisely the internal energy of the LDF dominates over its rest-mass energy) and the energy density increases logarithmically as $\epsilon \sim \Omega_{\text{de},0}\epsilon_0 (1 + 3B \ln a)$. The scale factor has a super de Sitter behavior $a \propto \exp[3B\Omega_{\text{de},0}H_0^2 t^2/4]$ [55-60]. This corresponds to a phantom regime since the energy density increases as the universe expands. In the late universe, we have $\epsilon \sim \Omega_{\text{de},0}\epsilon_0 (1 + 3B \ln a)$ and $P \sim -\Omega_{\text{de},0}\epsilon_0 (B + 1 + 3B \ln a)$, which implies that the equation of state $P(\epsilon)$ behaves asymptotically as $P \sim -\epsilon - A$.

As the universe expands, the pressure decreases from $+\infty$ to $-\infty$. The pressure $P$ is positive when $\rho_{\text{dm}} > \rho_p$ and negative when $\rho_{\text{dm}} < \rho_p$. It vanishes at $\rho_{\text{dm}} = \rho_p$. Since the logotropic model is a unification of DM and DE, it is not expected to be valid in the early universe. Therefore, the pressure is always negative in the regime of interest ($\rho_{\text{dm}} \ll \rho_p$) where the logotropic model is valid.

The DE density $\epsilon_{\text{de}}$ increases from $-\infty$ to $+\infty$. The DE is negative when $\rho_{\text{dm}} > \rho_p/e$ and positive when $\rho_{\text{dm}} < \rho_p/e$ (its value at $\rho_{\text{dm}} = \rho_p$ is $\epsilon_{\text{de}} = -A < 0$). In the logotropic model, since the DE density corresponds to the internal energy density $u$ of the LDF, it can very well be negative as long as the total energy density $\epsilon$ is positive. In the regime of interest ($\rho_{\text{dm}} \ll \rho_p$) where the logotropic model is valid, the DE density $\epsilon_{\text{de}}$ is positive.

The function $\epsilon/\epsilon_0(a)$ is plotted in Fig. 2. We have taken $\Omega_{m,0} = 0.3075$, $\Omega_{\text{de},0} = 0.6911$ and $B = 3.53 \times 10^{-3}$. The logotropic model is able to account for the transition between a DM era where the expansion of the universe is decelerated and a DE era where the expansion of the universe is accelerating. It is indistinguishable from the \Lambda CDM model up to the present time for what concerns the evolution of the cosmological background.

The two models will differ in about 27 Gyrs when the logotropic model is valid. The DE density $\epsilon_{\text{de}}$ increases from $-\infty$ to $+\infty$. The DE is negative when $\rho_{\text{dm}} > \rho_p/e$ and positive when $\rho_{\text{dm}} < \rho_p/e$ (its value at $\rho_{\text{dm}} = \rho_p$ is $\epsilon_{\text{de}} = -A < 0$). In the logotropic model, since the DE density corresponds to the internal energy density $u$ of the LDF, it can very well be negative as long as the total energy density $\epsilon$ is positive. In the regime of interest ($\rho_{\text{dm}} \ll \rho_p$) where the logotropic model is valid, the DE density $\epsilon_{\text{de}}$ is positive.

The energy density $\epsilon$ with a in
the logotropic model is slow (logarithmic). As a result, there is no future finite time singularity, i.e., there is no “big rip” where the energy density and the scale factor become infinite in a finite time [77]. In the logotropic model of type II the energy density and the scale factor become infinite in infinite time. This is called “little rip” [78].

E. Two-fluid model

In the model of type II, we have a single dark fluid with an equation of state \( P = P(\rho_{dm}) \). Still, the energy density \( \epsilon \) given by Eq. (27) is the sum of two terms, a rest-mass density term \( \rho_{dm} \) which mimics DM and an internal energy term \( u(\rho_{dm}) \) which mimics DE. It is interesting to consider a two-fluid model which leads to the same results as the single dark fluid model, at least for what concerns the evolution of the homogeneous background. In this two-fluid model, one fluid corresponds to pressureless DM with an equation of state \( P_{dm} = 0 \) and a density \( \rho_{dm} c^2 = \Omega_{dm,0} \epsilon_0 / a^3 \) determined by the energy conservation equation for DM, and the other fluid corresponds to DE with an equation of state \( P_{de}(\epsilon_{de}) \) and an energy density \( \epsilon_{de}(a) \) determined by the energy conservation equation for DE. We assume that the two fluids are independent from each other. We can obtain the equation of state of DE yielding the same results as the one-fluid model by taking

\[
P_{de} = P(\rho_{dm}), \quad \epsilon_{de} = u(\rho_{dm}),
\]

and eliminating \( \rho_{dm} \) from these two relations. In other words, the equation of state \( P_{de}(\epsilon_{de}) \) of DE in the two-fluid model corresponds to the relation \( P(u) \) in the single fluid model. We note that although the one and two-fluid models are equivalent for the evolution of the homogeneous background, they may differ for what concerns the formation of the large-scale structures of the universe.

In the two-fluid model associated with a logotrope of type II, the DE has an affine equation of state [55] [60]

\[
P_{de} = -\epsilon_{de} - A,
\]

which is obtained by eliminating \( \rho_{dm} \) between Eqs. (38) and (39), and by identifying \( P(u) \) with \( P_{de}(\epsilon_{de}) \). Solving the energy conservation equation (19) with the equation of state of DE from Eq. (54), we recover Eq. (49). However, this two-fluid model (with \( \rho_{dm} = 0 \) and \( P_{de} = -\epsilon_{de} - A \)) does not determine the value of \( A \), contrary to the one-fluid model. This is a huge advantage of the one-fluid model.

F. Present proportion of DM and DE: Dark magic

We now come to a remarkable and very intriguing result. We have seen that the logotropic model of type II determines a relation between the present fraction \( \Omega_{dm,0} \) of DM, the present fraction \( \Omega_{de,0} \) of DE and the present energy density \( \epsilon_0 \). This relation is given by Eq. (47) or, equivalently, by Eq. (49) where we have introduced the notation from Eq. (48). Then, using observational results, we have shown that Eq. (49) can be written in very good approximation as Eq. (50). We note that this approximation is valid as long as

\[
\frac{\Omega_{de,0}}{\Omega_{dm,0}} \ll 10^{124}.
\]

This shows that the expression of \( A \) is essentially independent from the present ratio of DM and DE. Now, we ask ourselves the following question: What do we get if we assume that Eq. (50) is exactly satisfied? In that case, we find from Eq. (49) that the ratio between the present proportion of DM and DE is given by the pure number

\[
\frac{\Omega_{th,0}^{dm}}{\Omega_{th,0}^{de}} = e = 2.71828...
\]

If we neglect baryonic matter \( \Omega_{b,0} = 0 \) we obtain the pure numbers \( \Omega_{th,0}^{dm} = 0.731059... \) and \( \Omega_{th,0}^{de} = 0.268941... \) which give the correct proportions 70% and 25% of DE and DM [61]. If we take baryonic matter into account and use the measured value of \( \Omega_{b,0} = 0.0486 \pm 0.0010 \), we get \( \Omega_{th,0}^{dm} = 0.6955 \pm 0.0007 \) and \( \Omega_{th,0}^{de} = 0.2559 \pm 0.0003 \) which are very close to the observational values \( \Omega_{obs}^{dm,0} = 0.6911 \pm 0.0002 \) and \( \Omega_{obs}^{de,0} = 0.2589 \pm 0.0057 \) within the error bars. The argument leading to Eq. (56) means that the present DE density \( \rho_A \) is such that \( \rho_A c^2 / \ln(\rho_P/\rho_A) \) is equal to the fundamental constant \( A \) (with infinite precision). This can be viewed as a strong cosmic coincidence [61] giving to our epoch a central place in the history of the universe.

3 Of course, this argument gives nothing in the framework of the ΛCDM model since \( \rho_P \) is always equal to Einstein’s cosmological constant \( \Lambda / (8\pi G) \). The argument leading to Eq. (56) cannot be advocated in the ΛCDM model where \( \rho_P \rightarrow +\infty \) and \( A \rightarrow 0 \) because Eq. (49) degenerates into Eq. (50).
the scale factor as
\[ \frac{\Omega_{4e}}{\Omega_{dm}} = e a^3 (1 + 3B \ln a). \tag{58} \]
It changes algebraically rapidly with the scale factor. This ratio is plotted as a function of time in Fig. 3. It is only at the present epoch \( a = 1 \) that \( \Omega_{4e}/\Omega_{dm} = e. \)

FIG. 3: Ratio between the proportion of DM and DE as a function of time. Some arguments based on the logotropic model suggests that this ratio might be equal to \( e = 2.71828... \) at the present epoch. This is in agreement with the observational value \( 2.669 \pm 0.08. \)

IV. LOGOTROPIC EQUATION OF STATE OF TYPE III

In this section, we consider a relativistic barotropic fluid described by an equation of state of type III where the pressure \( P = P(\rho) \) is specified as a function of the pseudo rest-mass density of a complex SF in the Thomas-Fermi (TF) approximation.

A. Klein-Gordon-Einstein equations

Let us consider a relativistic complex SF \( \varphi(x^\mu) = \varphi(x, y, z, t) \) which is a continuous function of space and time. It can represent the wavefunction of a relativistic fluid described by an equation of state of type III where \( \varphi \) is the relativistic pseudo rest-mass density of a complex SF in the Thomas-Fermi (TF) approximation.

The action of the system, which is the integral of the Lagrangian density of the SF, can be written as
\[ S = \int \frac{c^4}{16\pi G} R + \mathcal{L} \sqrt{-g} \, d^4x, \tag{59} \]
where \( R \) is the Ricci scalar curvature, \( \mathcal{L} = \mathcal{L}(\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*) \) is the Lagrangian density of the SF, and \( g = \det(g_{\mu\nu}) \) is the determinant of the metric tensor. We consider a canonical Lagrangian density of the form
\[ \mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi - \frac{m^2 c^2}{2\hbar^2} |\varphi|^2 - V(|\varphi|^2), \tag{60} \]
where the first term is the kinetic energy, the second term is minus the rest-mass energy term and the third term is minus the self-interaction energy term.

The least action principle \( \delta S = 0 \) yields the Klein-Gordon-Einstein (KGE) equations
\[ \square \varphi + \frac{m^2 c^2}{\hbar^2} \varphi + \frac{2}{d|\varphi|^2} \varphi = 0, \quad \tag{61} \]
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad \tag{62} \]
where \( \square = D_\mu D^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \) is the d’Alembertian operator in a curved spacetime, \( R_{\mu\nu} \) is the Ricci tensor and \( T_{\mu\nu} \) is the energy-momentum (stress) tensor of the SF given by
\[ T_{\mu\nu} = \frac{1}{2} (\partial_\mu \varphi^* \partial_\nu \varphi + \partial_\nu \varphi^* \partial_\mu \varphi) - g_{\mu\nu} \mathcal{L}. \tag{63} \]

The conservation of the energy-momentum tensor, which results from the invariance of the Lagrangian density under continuous translations in space and time (Noether theorem), reads
\[ D_\mu T^{\mu\nu} = 0. \tag{64} \]

This conservation law can be directly obtained from the Einstein field equations by using the Bianchi identities.

The current of charge of the complex SF is given by
\[ J_\mu = -\frac{m}{2\hbar} (\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*). \tag{65} \]

Using the KG equation (61), one can show that
\[ D_\mu J^\mu = 0. \tag{66} \]

This equation expresses the local conservation of the charge of the SF. The total charge of the SF is \( Q = \frac{1}{m\epsilon_0} \int J^0 \sqrt{-g} \, d^3x \), where the elementary charge has been set to unity. In that case, the charge \( Q \) is equal to the number \( N \) of bosons provided that antibosons are counted negatively [80]. Therefore, Eq. (66) also expresses the local conservation of the boson number \( N \) or its rest-mass \( N m \). This conservation law results via the Noether theorem from the global \( U(1) \) symmetry of the Lagrangian, i.e., from the invariance of the Lagrangian density under a global phase transformation \( \varphi \rightarrow e^{i\theta} \varphi \) (rotation) of the complex SF.

B. Hydrodynamic representation

We can write the KG equation (61) under the form of hydrodynamic equations by using the de Broglie transformation [81][83]. To that purpose, we write the SF as
\[ \varphi = \frac{\hbar}{m} \sqrt{\rho} e^{iS_{\text{tot}}/\hbar}, \tag{67} \]
where $\rho$ is the pseudo rest-mass density defined by
\begin{equation}
\rho = \frac{m^2}{\hbar^2} |\phi|^2, \quad (68)
\end{equation}
and $S_{\text{tot}}$ is the action. Substituting Eq. (67) into the Lagrangian density \[63\], we obtain
\begin{equation}
L = \frac{1}{2} g^{\mu\nu} \rho \frac{\partial \mu S_{\text{tot}}}{m} \frac{\partial \nu S_{\text{tot}}}{m} + \frac{\hbar^2}{8m^2\rho} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - \frac{1}{2} \rho c^2 - V(\rho). \quad (69)
\end{equation}
The equations of motion resulting from the least action principle $\delta S = 0$ read \[62\] \[79\]
\begin{equation}
D_\mu \left( \rho \frac{\partial^\mu S_{\text{tot}}}{m} \right) = 0, \quad (70)
\end{equation}
\begin{equation}
\partial_\mu S_{\text{tot}} \partial^\mu S_{\text{tot}} = \frac{\hbar^2}{2m} \left( \nabla_\mu \sqrt{\rho} \right)^2 + m^2 c^2 + 2m^2 V'(\rho). \quad (71)
\end{equation}
These equations can also be obtained by substituting the de Broglie transformation from Eq. (67) into the KG equation \[61\], and by separating the real and the imaginary parts. Equation (70) can be interpreted as a continuity equation and Eq. (71) can be interpreted as a classical relativistic Hamilton-Jacobi (or Bernoulli) equation. Note that the continuity equation is not affected by the TF approximation.

In the TF approximation, the energy-momentum tensor \[73\] reduces to
\begin{equation}
T_{\mu\nu} = \frac{\rho}{m} \frac{\partial \mu S_{\text{tot}}}{m} \frac{\partial \nu S_{\text{tot}}}{m} - g_{\mu\nu} L. \quad (78)
\end{equation}
We introduce the fluid quadrivelocity
\begin{equation}
u_\mu = - \frac{\partial \mu S_{\text{tot}}}{\sqrt{m^2 c^2 + 2m^2 V'(\rho) c}} c, \quad (79)
\end{equation}
which satisfies the identity $\nu_\mu \nu^\mu = c^2$. Combining Eqs. \[78\] and \[79\], we get
\begin{equation}
T_{\mu\nu} = \rho \left[ 1 + \frac{2}{c^2} V'(\rho) \right] \nu_\mu \nu_\nu - g_{\mu\nu} L. \quad (80)
\end{equation}
The energy-momentum tensor \[80\] can be written under the perfect fluid form
\begin{equation}
T_{\mu\nu} = (\epsilon + P) \frac{\nu_\mu \nu_\nu}{c^2} - P g_{\mu\nu}, \quad (81)
\end{equation}
where $\epsilon$ is the energy density and $P$ is the pressure, provided that we make the identifications $P = L$, $\epsilon + P = \rho c^2 + 2\rho V'(\rho)$. \[82\]
Therefore, the Lagrangian density plays the role of the pressure of the fluid. Combining Eq. \[75\] with the Hamilton-Jacobi (or Bernoulli) equation \[77\], we get
\begin{equation}
L = \rho V'(\rho) - V(\rho). \quad (83)
\end{equation}

Therefore, according to Eqs. \[82\] and \[83\], the energy density and the pressure of the SF in the TF approximation are given by \[63\] \[83\]
\begin{equation}
\epsilon = \rho c^2 + \rho V'(\rho) + V(\rho), \quad (84)
\end{equation}
\[ P = \rho V'(\rho) - V(\rho). \] (85)

Eliminating \( \rho \) between Eqs. \[84\] and \[85\], we obtain the equation of state \( P(\epsilon) \). On the other hand, Eq. \[85\] can be integrated into \[62\]

\[ V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} \, d\rho. \] (86)

Equation \[85\] determines \( P(\rho) \) as a function of \( V(\rho) \). Inversely, Eq. \[86\] determines \( V(\rho) \) as a function of \( P(\rho) \).

In the TF approximation, using Eqs. \[74\] and \[79\], we can write the current as

\[ J_\mu = \rho \sqrt{1 + \frac{2}{c^2} V'(\rho)} \, u_\mu. \] (87)

The rest-mass density \( \rho_{\text{dm}} = nm \) (which is equal to the charge density) is such that

\[ J_\mu = \rho_{\text{dm}} u_\mu. \] (88)

The continuity equation \[66\] can then be written as

\[ D_\mu (\rho_{\text{dm}} u^\mu) = 0. \] (89)

Comparing Eq. \[87\] with Eq. \[88\], we find that the rest-mass density of the SF is given by

\[ \rho_{\text{dm}} = \rho \sqrt{1 + \frac{2}{c^2} V'(\rho)}. \] (90)

In general, \( \rho_{\text{dm}} \neq \rho \) except when \( V \) is constant, corresponding to the \text{ACDM} model (see below), and in the nonrelativistic limit \( c \to +\infty \).

Remark: in the TF approximation, the equation of state parameter \( w = P/\epsilon \) is given by

\[ w = \frac{\rho V'(\rho) - V(\rho)}{\rho c^2 + \rho V'(\rho) + V(\rho)}. \] (91)

For a barotropic equation of state of type III, the universe exhibits a normal behavior \( (w > -1) \) when \( 1 + (2/c^2) V'(\rho) > 0 \) and a phantom behavior \( (w < -1) \) when \( 1 + (2/c^2) V'(\rho) < 0 \). In the latter case, the Lagrangian of the SF involves a negative kinetic term. The SF has either a normal behavior (positive kinetic term) or a phantom behavior (negative kinetic term) but cannot pass from a normal to a phantom regime. Therefore, a barotropic equation of state of type III cannot describe the transition from a normal to a phantom behavior. Here, we only consider the normal behavior where \( 1 + (2/c^2) V'(\rho) > 0 \).

\section*{D. Spatially homogeneous SF}

For a spatially homogeneous SF in an expanding universe with a Lagrangian

\[ L = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 - \frac{m^2 c^2}{2h^2} |\varphi|^2 - V(|\varphi|^2), \] (92)

the KG equation \[61\] becomes

\[ 1 \frac{d^2 \varphi}{dt^2} + \frac{3H d\varphi}{c^2} + \frac{m^2 c^2}{2h^2} |\varphi|^2 + 2 \frac{dV}{d|\varphi|^2} \varphi = 0, \] (93)

while the Einstein field equations \[62\] reduce to the Friedmann equations of Sec. \[II A\]. The energy density and the pressure of the SF are given by

\[ \epsilon = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 + \frac{m^2 c^2}{2h^2} |\varphi|^2 + V(|\varphi|^2), \] (94)

\[ P = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 - \frac{m^2 c^2}{2h^2} |\varphi|^2 - V(|\varphi|^2). \] (95)

We can easily check that the KG equation \[63\] with Eqs. \[94\] and \[95\] imply the energy conservation equation \[18\] (see Appendix G of \[62\]). In the following, we use the hydrodynamic representation of the SF (see Secs. \[IV B\] and \[IV C\]). The total energy of the SF (including its rest mass energy \( mc^2 \)) is

\[ E_{\text{tot}}(t) = - \frac{dS_{\text{tot}}}{dt}. \] (96)

For a spatially homogeneous SF, the continuity equation \[70\] expressing the conservation of the charge of the SF can be written as \[84\]

\[ \frac{d}{dt} \left( \frac{E_{\text{tot}}}{mc^2} a^3 \right) = 0. \] (97)

It can be integrated into

\[ \rho \frac{E_{\text{tot}}}{mc^2} = \frac{Q m}{a^3}, \] (98)

where \( Q \) is the charge of the SF. The rest-mass density \( \rho_{\text{dm}} \) of a spatially homogeneous SF is given by \[62\]

\[ \rho_{\text{dm}} = \rho \frac{E_{\text{tot}}}{mc^2}. \] (99)

It is equal to \( \rho_{\text{dm}} = J_0/c \), where \( J_0 = -\rho \partial_\alpha S_{\text{tot}}/m \) is the time component of the current of charge. This formula is only valid for a spatially homogeneous SF. Comparing Eqs. \[98\] and \[99\], we get

\[ \rho_{\text{dm}} = \frac{Q m}{a^3}. \] (100)

The rest-mass density (or the charge density) of the SF decreases as \( a^{-3} \). This expresses the conservation of the charge of the SF or, equivalently, the conservation of the boson number.\(^5\) As in Sec. \[III\] the rest-mass density of

\(^5\) Inversely, Eq. \[99\] can be directly obtained from Eq. \[98\] by using Eq. \[100\].
the SF may be interpreted as DM. Identifying Eq. (33) with Eq. (100) we obtain
\[ Qmc^2 = \rho_{dm,0}c^2 = \Omega_{dm,0}\epsilon_0. \]  
(101)

Therefore, the constant \( Qm \) (charge) is equal to the present density \( \rho_{dm,0} \) of DM (rest-mass density).

On the other hand, in the TF approximation, the Hamilton-Jacobi (or Bernoulli) equation from Eq. (77) becomes
\[ E_{tot}^2 = m^2c^4 + 2m^2c^2V'(\rho). \]  
(102)

It can be rewritten as
\[ E_{tot} = mc\sqrt{1 + \frac{2}{c^2}V'(\rho)}. \]  
(103)

Combining Eqs. (98) and (103), we obtain
\[ \rho c^2 \sqrt{1 + \frac{2}{c^2}V'(\rho)} = \frac{\Omega_{dm,0}\epsilon_0}{a^3}. \]  
(104)

This relation can also be obtained from Eqs. (90) and (104). We can then solve the Friedmann equation (15) with \( \epsilon(a) \) to obtain the temporal evolution of the scale factor \( a(t) \).

E. Logarithmic potential and logotropic equation of state

We now consider a relativistic complex SF with a logarithmic potential of the form
\[ V(|\varphi|^2) = -A\ln\left(\frac{m^2|\varphi|^2}{\hbar^2\rho_p}\right) - A. \]  
(105)

The corresponding KG equation reads
\[ \Box \varphi + \frac{m^2c^2}{\hbar^2}\varphi - \frac{2A}{|\varphi|^2}\varphi = 0. \]  
(106)

This is called the logotropic KG equation. As detailed in [63], we argue that the potential (105) is not a specific attribute of the SF (such as its mass \( m \) or self-interaction constant \( \lambda \)) but that it is an intrinsic property of the wave equation itself. In other words, we argue that the logarithmic term involving the fundamental constant \( A \) is always present in the wave equation (106) even if, in many situations, it can be neglected leading to the ordinary KG equation (corresponding to \( A = 0 \)). In this sense, the nonlinear wave equation (106) is more fundamental than the linear KG equation.

Using the hydrodynamic variables introduced in Sec. IV.B, the SF potential can be written as
\[ V(\rho) = -A\ln\left(\frac{\rho}{\rho_p}\right) - A. \]  
(107)

In the TF approximation, using Eqs. (85) and (107), we find that the pressure is given by the logotropic equation of state of type III [63]
\[ P = A\ln\left(\frac{\rho}{\rho_p}\right). \]  
(108)

On the other hand, using Eqs. (84) and (107), we obtain the energy density
\[ \epsilon = \rho c^2 - A\ln\left(\frac{\rho}{\rho_p}\right) - 2A. \]  
(109)

Eliminating \( \rho \) between Eqs. (108) and (109), the equation of state \( P(\epsilon) \) is given in the reversed form \( \epsilon(P) \) by
\[ \epsilon = e^{P/A}\rho_p c^2 - P - 2A. \]  
(110)

This is the equation of state of type I corresponding to the logotropic model of type III [62].

For a spatially homogeneous SF in an expanding universe, the pseudo rest-mass density \( \rho \) evolves according to [see Eq. (104)]
\[ \rho c^2 \sqrt{1 - \frac{2A}{\rho c^2}} = \frac{\Omega_{dm,0}\epsilon_0}{a^3}. \]  
(111)

Equation (111) is a second degree equation for \( \rho \) which can be solved explicitly to give
\[ \rho c^2 = A + \sqrt{A^2 + \frac{(\Omega_{dm,0}\epsilon_0)^2}{a^6}}. \]  
(112)

Substituting Eq. (112) into Eq. (109), the energy density can be expressed in terms of the scale factor as
\[ \epsilon = -A + \frac{\sqrt{A^2 + \frac{(\Omega_{dm,0}\epsilon_0)^2}{a^6}}}{\rho_p c^2} - \frac{\Omega_{dm,0}\epsilon_0}{a^3}. \]  
(113)

F. The value of \( A \)

Applying Eq. (113) at the present time \( (a = 1) \) and subtracting the present contribution \( \Omega_{dm,0}\epsilon_0 \) of DM we obtain
\[ \Omega_{de,0}\epsilon_0 = -A + \sqrt{A^2 + \frac{(\Omega_{dm,0}\epsilon_0)^2}{a^6}} \left[ \frac{\sqrt{A^2 + \frac{(\Omega_{dm,0}\epsilon_0)^2}{a^6}}}{\rho_p c^2} - \frac{\Omega_{dm,0}\epsilon_0}{a^3} \right]. \]  
(114)

6 Inversely, we could start from the equation of state (108) and integrate Eq. (86) to obtain the potential \( V(\rho) \).
Assuming that $A$ is a universal constant, this equation gives a relation between $\Omega_{dm,0}$, $\Omega_{de,0}$ and $\epsilon_0$. Inversely, we can use Eq. (114) and the measured values of $\epsilon_0$, $\Omega_{dm,0}$ and $\Omega_{de,0}$ to determine the constants of our model. Therefore, there is no free (undetermined) parameter in our model.

Introducing the notation from Eq. (48) and writing

$$A = B\rho_{\Lambda}\epsilon^2,$$  \hspace{1cm} (115)

where $B$ is a dimensionless constant, Eq. (114) can be rewritten as

$$1 + \frac{\Omega_{dm,0}}{\Omega_{de,0}} = -B + \sqrt{B^2 + \left(\frac{\Omega_{dm,0}}{\Omega_{de,0}}\right)^2}$$

$$-B\ln\left[B + \sqrt{B^2 + \left(\frac{\Omega_{dm,0}}{\Omega_{de,0}}\right)^2}\right] + B\ln\left(\frac{\rho_P}{\rho_\Lambda}\right).$$  \hspace{1cm} (116)

Using the fact that $B \ll 1$ which can be checked a posteriori, the foregoing equation reduces to

$$B = \frac{1}{\ln\left(\frac{\rho_P}{\rho_\Lambda}\right) - 1 - \ln\left(\frac{\Omega_{dm,0}}{\Omega_{de,0}}\right)}.$$  \hspace{1cm} (117)

We can then redo the discussion from Sec. III. If we use the measured values of $\Omega_{dm,0}$ and $\Omega_{de,0}$, we find that $A$ and $B$ are given in very good approximation by Eqs. (10) and (13). These results are largely independent from the ratio $\Omega_{dm,0}/\Omega_{de,0}$. Inversely, if we impose that $A$ and $B$ are exactly given by Eqs. (10) and (13), we find that the ratio $\Omega_{dm,0}/\Omega_{de,0}$ is given by Eq. (56) in very good agreement with the observed value.

Remark: Combining Eqs. (48) and (56), we find that the charge $Qmc^2 = \Omega_{dm,0}\epsilon_0$ of the SF [see Eq. (101)] is given by

$$Qmc^2 = \frac{\rho_{\Lambda}}{\epsilon}.$$  \hspace{1cm} (118)

G. Evolution of the universe

Using Eqs. (112) and (115), the pressure $P$ and the energy density $\epsilon$ can be expressed in terms of the scale factor as

$$P = B\Omega_{de,0}\epsilon_0\ln\left[B + \sqrt{B^2 + \left(\frac{\Omega_{dm,0}}{\Omega_{de,0}}\epsilon_0\right)^2}\right] - \Omega_{de,0}\epsilon_0,$$  \hspace{1cm} (119)

$$\frac{H^2}{\Omega_{de,0}} = \frac{\epsilon}{\epsilon_0} = -B\Omega_{de,0} + \sqrt{\left(B\Omega_{de,0}\right)^2 + \left(\frac{\Omega_{dm,0}}{a^3}\right)^2}$$

$$-B\Omega_{de,0}\ln\left[B + \sqrt{B^2 + \left(\frac{\Omega_{dm,0}}{\Omega_{de,0}}\epsilon_0\right)^2}\right] + \Omega_{de,0}.$$  \hspace{1cm} (120)

In Eq. (120), we have combined the relation $\epsilon(a)$ with the Friedmann equation (15) to obtain a differential equation determining the temporal evolution of the scale factor. If we take into account the presence of baryons, we must add a term $\epsilon_0 = \Omega_{b,0}\epsilon_0/a^3$ in the energy density. The $\Lambda$CDM model is recovered for $B = 0$ corresponding to the limit $h \to 0$ or $\rho_P \to +\infty$. In that case, the self-interaction potential is constant ($V = \epsilon_\Lambda$)\(^7\) and Eqs. (119) and (120) return Eqs. (1), (2) and (8). The evolution of the universe in the logotropic model of type III has been discussed in detail in [63]. Below, we just recall the main results of this study.

Starting from $+\infty$, the energy density $\epsilon$ decrease monotonically and tend to a constant value when $a \to +\infty$. The pseudo rest-mass density has a similar behavior. At early times, the pressure is negligible with respect to the energy density and the LDF is equivalent to a pressureless fluid like in the CDM model. The energy density decreases as $\epsilon \sim \rho c^2 \sim \Omega_{dm,0}\epsilon_0/a^3$ and the scale factor increases algebraically as $a \propto t^{2/3}$ (Einstein-de Sitter). This leads to a decelerated expansion of the universe. At later times, the negative pressure of the LDF becomes efficient and explains the acceleration of the universe that we observe today. Ultimately, DE dominates over DM and the energy density tends to a constant $\epsilon_{\min} = \epsilon_\Lambda[1 - B\ln(2B)] = 1.02\,\epsilon_\Lambda$. Similarly, the pseudo

\(^7\)In the complex SF representation of the $\Lambda$CDM model viewed as a UDM model (see Appendix E of [63]), the total potential is

$$V_{tot} = \frac{m^2\epsilon^2}{2\hbar^2} |\psi|^2 + \epsilon_\Lambda.$$  \hspace{1cm} (121)

The constant energy density $\epsilon_\Lambda$ does not appear explicitly in the KG equation which only involves the gradient of the potential. However, it appears in the energy density and in the pressure. In the TF approximation, using Eqs. (84), (85) and (90) with $V = \epsilon_\Lambda$ we get $\epsilon = \rho c^2 + \epsilon_\Lambda$ with $\rho = \rho_{dm}$ and $P = -\epsilon_\Lambda$. 

![FIG. 4: Normalized energy density $\epsilon/\epsilon_0$ as a function of the scale factor $a$ for the logotropic model of type III. It is compared with the $\Lambda$CDM model. The two curves are indistinguishable at the present time but their asymptotes differ by a factor 1.02.](image-url)
rest-mass density tends to \( \rho_{\text{min}} = 2B\rho \). The scale factor has a de Sitter behavior \( a \propto \exp(\sqrt{8\pi G\epsilon_{\text{min}}/3c^2} t) \) with a quantum-modified cosmological constant (i.e. it depends on \( B \) hence on \( \rho_p \) or \( h \)). The function \( \epsilon/\epsilon_0(a) \) is plotted in Fig. 4.

As the universe expands, the pressure decreases monotonically from \( +\infty \) to a minimum value \( P_{\text{min}} = -\epsilon_{\text{min}} \). The pressure \( P \) is positive when \( \rho > \rho_p \) and negative when \( \rho < \rho_p \). It vanishes at \( \rho = \rho_p \). Since the logotropic model is a unification of DM and DE, it is not expected to be valid in the early universe. Therefore, the pressure is always negative in the regime of interest \( (\rho \ll \rho_p) \) where the logotropic model is valid.

H. Rest-mass density (DM) and internal energy (DE)

The energy density can be written as

\[
\epsilon = \rho_{\text{dm}}c^2 + u = \epsilon_{\text{dm}} + \epsilon_{\text{de}}, \tag{122}
\]

where the first term is the rest-mass energy and the second term is the internal energy. As explained in Sec. \text{III} the rest-mass density \( \rho_{\text{dm}} \) represents DM and the internal energy \( u \) represents DE. From Eqs. \text{[34], [84] and [90]}, we have

\[
\epsilon_{\text{dm}} = \rho_{\text{dm}}c^2 = \Omega_{\text{dm},0}\epsilon_0/\alpha^4 = \rho c^2 \left( 1 + \frac{2}{c^2}V'(\rho) \right), \tag{123}
\]

\[
\epsilon_{\text{de}} = u = \epsilon - \epsilon_{\text{dm}} = \rho c^2 + V(\rho) + \rho V'(\rho) - \rho \left( 1 + \frac{2}{c^2}V'(\rho) \right). \tag{124}
\]

From these equations we can obtain \( u = u(\rho_{\text{dm}}) \) and \( P = P(\rho_{\text{dm}}) \).

For the LDF, the rest-mass density \( \rho_{\text{dm}} \) (DM) is related to the pseudo rest-mass density \( \rho \) by

\[
\rho_{\text{dm}} = \rho \left( 1 - \frac{2A}{\rho c^2} \right). \tag{125}
\]

This equation can be inverted to give

\[
\rho = \frac{A}{c^2} + \sqrt{\frac{A^2}{c^4} + \rho_{\text{dm}}^2}. \tag{126}
\]

On the other hand, the internal energy (DE) is given by

\[
\epsilon_{\text{de}} = u = \rho c^2 - A \ln \left( \frac{\rho}{\rho_p} \right) - 2A - \rho c^2 \left( 1 - \frac{2A}{\rho c^2} \right). \tag{127}
\]

Substituting Eq. \text{[126]} into Eq. \text{[127]} we get

\[
\epsilon_{\text{de}} = u = A \sqrt{\frac{A^2}{c^4} + \rho_{\text{dm}}^2 c^4} - \rho_{\text{dm}} c^2 - A \ln \left[ \frac{A}{\rho_p c^2} + \sqrt{\frac{A^2}{\rho_p c^4} + \frac{\rho_{\text{dm}}^2}{\rho_p^2}} \right]. \tag{128}
\]

which determines \( u(\rho_{\text{dm}}) \). On the other hand, according to Eqs. \text{[108]} and \text{[126]}, we obtain the equation of state \( P(\rho_{\text{dm}}) \) of the SF in terms of the rest-mass density as

\[
P = A \ln \left[ \frac{A}{\rho_p c^2} + \sqrt{\frac{A^2}{\rho_p c^4} + \frac{\rho_{\text{dm}}^2}{\rho_p^2}} \right]. \tag{129}
\]

This is the equation of state of type II corresponding to the logotropic model of type III \text{[62]}.

I. Two-fluid model

In the two-fluid model (see Sec. \text{II}E), the DE has an equation of state \( P_{\text{de}}(\epsilon_{\text{de}}) \) which is obtained by eliminating \( \rho \) between Eqs. \text{[55] and [124]}, and by identifying \( P(u) \) with \( P_{\text{de}}(\epsilon_{\text{de}}) \). For the logotropic model, it can be written in inverse form as

\[
\epsilon_{\text{de}} = \rho_p c^2 e_{\text{de}}/A - P_{\text{de}} - 2A - \rho_p c^2 \sqrt{2e_{\text{de}}/A - \frac{2A}{\rho_p c^2} e_{\text{de}}/A}. \tag{130}
\]

V. LOGOTROPIC DM HALOS

In this section, we describe the structure of logotropic DM halos and determine their universal surface density. We use a nonrelativistic approach that is appropriate to DM halos. For the sake of generality, we consider the logotropic model of type III which is based on a complex SF theory. However, after giving general results, we shall make the TF approximation. In that case, the models of type I, II and III are equivalent in the nonrelativistic limit so our results are valid for all these models. In the nonrelativistic limit, the mass density is equal to \( \rho = \rho_{\text{dm}} = \epsilon/c^2 \) and the SF potential \( V \) coincides with the internal energy \( u \) \text{[62]}.

A. Logotropic GP equation

Basically, a complex SF is governed by the KGE equations \text{[61] and [62]}. In the nonrelativistic limit \( c \rightarrow +\infty \), making the Klein transformation \text{[79, 86, 87]}

\[
\varphi(r, t) = \frac{\hbar}{m} e^{-imc^2t/\hbar} \psi(r, t), \tag{131}
\]

\text{8} This relation can be obtained from Eqs. \text{[108] and [127]}. It can also be obtained by solving Eq. \text{[129]} to get \( \rho_{\text{dm}}(P) \) and by using Eqs. \text{[109] and [122]}.
the KGE equations (61) and (62) reduce to the GPP equations

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi + m \frac{dV}{d|\psi|^2} \psi, \]

(132)

\[ \Delta \Phi = 4\pi G|\psi|^2, \]

(133)

where \( \psi \) is the wavefunction such that \( \rho = |\psi|^2 = (m/\hbar^2)|\varphi|^2 \). We refer to [79, 88] for a detailed derivation of these equations. For the logarithmic potential

\[ V(|\psi|^2) = -A \ln \left( \frac{|\psi|^2}{\rho_p} \right) - A, \]

(134)

corresponding to Eq. (107), we obtain the logotropic GP equation [58]

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m\Phi \psi - Am \frac{\rho}{|\psi|^2} \psi. \]

(135)

For \( A = 0 \), corresponding to a constant potential (see Sec. [IV]), we recover the Schrödinger-Poisson equations of the fuzzy dark matter (FDM) model [91, 92].

### B. Madelung transformation

Writing the wave function as

\[ \psi(r, t) = \sqrt{\rho(r, t)} e^{iS(r, t)/\hbar}, \]

(136)

where \( S(r, t) \) is the action, and making the Madelung [95] transformation

\[ u = \nabla S/m, \]

(137)

where \( u(r, t) \) is the velocity field, the GPP equations (132) and (133) can be written under the form of hydrodynamic equations as

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0, \]

(138)

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{m} \nabla Q_B - \frac{1}{\rho} \nabla \Phi - \nabla P, \]

(139)

\[ \Delta \Phi = 4\pi G \rho, \]

(140)

where

\[ Q_B = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right], \]

(141)

is the Bohm quantum potential taking into account the Heisenberg uncertainty principle and \( P(\rho) \) is the pressure determined by the potential according to Eq. (83). For the logarithmic potential (107), we obtain the logotropic equation of state (108).

### C. Condition of hydrostatic equilibrium

In this section, we make the TF approximation which amounts to neglecting the quantum potential. In that case, the equilibrium state of a DM halo results from the balance between the gravitational attraction and the repulsion due to the pressure force. It is described by the classical equation of hydrostatic equilibrium

\[ \nabla P + \rho \nabla \Phi = 0 \]

(142)

coupled to the Poisson equation

\[ \Delta \Phi = 4\pi G \rho. \]

(143)

These equations can be combined into a single differential equation

\[ -\nabla \cdot \left( \frac{\nabla P}{\rho} \right) = 4\pi G \rho, \]

(144)

which determines the density profile of a DM halo. For the logotropic equation of state (108), it becomes

\[ A \Delta \left( \frac{1}{\rho} \right) = 4\pi G \rho. \]

(145)

If we define

\[ \theta = \frac{\rho_0}{\rho}, \quad \xi = \left( \frac{4\pi G \rho_0^2}{A} \right)^{1/2} r, \]

(146)

where \( \rho_0 \) is the central density and \( r_0 = (A/4\pi G \rho_0^2)^{1/2} \) is the logotropic core radius, we find that Eq. (145) reduces to the Lane-Emden equation of index \( n = -1 \) [96]:

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = \frac{1}{\theta}, \]

(147)

with the boundary conditions \( \theta = 1 \) and \( \theta' = 0 \) at \( \xi = 0 \). This equation has been studied in detail in [58, 72]. There exists an exact analytical solution \( \theta_s = \xi/\sqrt{2} \), corresponding to \( \rho_s = (A/8\pi G)^{1/2} r^{-1} \), called the singular logotropic sphere. The regular logotropic density profiles must be computed numerically. The normalized density profile \( \rho/\rho_0(r/r_0) \) is universal as a consequence of the homology invariance of the solutions of the Lane-Emden equation. It is plotted in Fig. 18 of [58]. The density profile of a logotropic DM halo has a core \( \rho \rightarrow c \) st when \( r \rightarrow 0 \) and decreases at large distances as \( \rho \sim r^{-1} \). More precisely, for \( r \rightarrow +\infty \), we have

\[ \rho \sim \left( \frac{A}{8\pi G} \right)^{1/2} \frac{1}{r}, \]

(148)

---

9 We consider here a static background \( (a = 1) \) since we will discuss these equations in the context of DM halos where the expansion of the universe can be neglected (see [88, 89] for generalizations).

10 The effect of the quantum potential is discussed in [63]. In that case, the DM halos exhibit an additional quantum core corresponding to a noninteracting self-gravitating BEC (soliton) like in the FDM model [93, 94].
like for the singular logotropic sphere. This profile has an infinite mass because the density does not decrease sufficiently rapidly with the distance. This implies that, in the case of real DM halos, the logotropic equation of state \([108]\) or the logotropic profile determined by Eq. \([145]\) cannot be valid at infinitely large distances (corresponding to very low densities). Actually, the logotropic core is expected to be surrounded by an extended envelope resulting from a process of violent collisionless relaxation \([63, 97, 98]\). In the envelope, the density decreases more rapidly than \(r^{-1}\), typically like \(r^{-2}\) corresponding to the isothermal sphere \([63, 97, 98]\) or like \(r^{-3}\) similar to the NFW \([12]\) and Burkert \([13]\) profiles. In the following, we shall consider the logotropic profile up to a few halo radii \(r_h\) so we do not have to consider the effect of the envelope.

**D. Halo mass**

The halo radius \(r_h\) is defined as the distance at which the central density \(\rho_0\) is divided by 4. For logotropic DM halos, using Eq. \([146]\), it is given by

\[
 r_h = \left( \frac{A}{4\pi G \rho_0^2} \right)^{1/2} \xi_h, \tag{149}
\]

where \(\xi_h\) is determined by the equation

\[
 \theta(\xi_h) = 4. \tag{150}
\]

The normalized density profile \(\rho/\rho_0(r/r_h)\) of logotropic DM halos is plotted in Fig. 5. The halo mass \(M_h\), which is the mass \(M_h = \int_0^{r_h} \rho(r') 4\pi r'^2 dr'\) contained within the sphere of radius \(r_h\), is given by \([58]\)

\[
 M_h = 4\pi \frac{\theta'(\xi_h)}{\xi_h} \rho_0 r_h^3. \tag{151}
\]

Solving the Lane-Emden equation of index \(n = -1\) [see Eq. \((147)\)], we numerically find \([58]\)

\[
 \xi_h = 5.85, \quad \theta_h = 0.693. \tag{152}
\]

This yields

\[
 r_h = 5.85 \left( \frac{A}{4\pi G} \right)^{1/2} \frac{1}{\rho_0} \tag{153}
\]

and

\[
 M_h = 1.49 \rho_0 r_h^3. \tag{154}
\]

**E. Constant surface density**

Eliminating the central density between Eqs. \([153]\) and \([154]\), we obtain the logotropic halo mass-radius relation

\[
 M_h = 8.71 \left( \frac{A}{4\pi G} \right)^{1/2} r_h^2. \tag{155}
\]

Since \(M_h \propto r_h^2\) we see that the surface density \(\Sigma_0\) is constant.\(^{12}\) This is a very important property of logotropic DM halos \([58]\). From Eq. \((153)\), we get

\[
 \Sigma_0 = \rho_0 r_h = 5.85 \left( \frac{A}{4\pi G} \right)^{1/2}. \tag{156}
\]

Therefore, all the logotropic DM halos have the same surface density, whatever their size, provided that \(A\) is interpreted as a universal constant. With the value of \(A/c^2 = 2.10 \times 10^{-26} \text{g m}^{-3}\) obtained (without free parameter) from the cosmological considerations of Secs. \([11, 14]\) we get \(\Sigma_0^{133} = 133 \text{M}_\odot/\text{pc}^2\) in very good agreement with the value \(\Sigma_0^{\text{obs}} = \rho_0 r_h = 141^{+83}_{-52} \text{M}_\odot/\text{pc}^2\) obtained from the observations \([18]\). On the other hand, Eq. \((154)\) may be rewritten as

\[
 M_h = 1.49 \Sigma_0 r_h^2 = 1.49 \frac{\Sigma_0^3}{\rho_0^2}. \tag{157}
\]

We note that the ratio \(M_h/(\Sigma_0 r_h^2) = 1.49\) in Eq. \[(157)\] is in good agreement with the ratio \(M_h/(\Sigma_0 r_h^2) = 1.60\) obtained from the observational Burkert profile (see Appendix D.4 of \([14]\)). This is an additional argument in favor of the logotropic model.

\(^{11}\) In the quantum logotropic model based on the logotropic GPP equations \([133]\) and \([135]\), the envelope is due to quantum interferences of excited states, like in the FDM model \([93, 94]\).

\(^{12}\) This is a consequence of the fact that the density of a logotropic DM halo decreases as \(r^{-1}\) at large distances.
F. Alternative expressions of the universal surface density

We can write the universal surface density of DM halos

\[ \Sigma^\text{th}_0 = \left( \frac{A}{4\pi G} \right)^{1/2} \xi_h = 133 M_\odot/\text{pc}^2 \]  

(158)

in terms of the Einstein cosmological constant \( \Lambda \) interpreted as an effective constant in our model (defined in terms of the present DE density). Using \( A = B\rho_\Lambda c^2 \) and \( \rho_\Lambda = \Lambda/(8\pi G) \), we get

\[ \Sigma^\text{th}_0 = \left( \frac{B}{32} \right)^{1/2} \xi_h \frac{c^3 \sqrt{\Lambda}}{G} = 0.0195 \frac{c^3 \sqrt{\Lambda}}{G}, \]  

(159)

where we have used the numerical values of \( B = 3.53 \times 10^{-3} \) and \( \xi_h = 5.85 \). Recalling that \( B \) is given by Eq. \( 13 \) with \( \rho_\rho / \rho_\Lambda = 8\pi c^3/h\Lambda \), we also have

\[ \Sigma^\text{th}_0 = 0.329 \frac{c^3 \sqrt{\Lambda}}{G} \frac{1}{\sqrt{\ln \left( \frac{8\pi c^3}{h\Lambda} \right)}}. \]  

(160)

Since \( \rho_\Lambda \) represents the present density of DE, it may be more relevant to express \( \Sigma^\text{th}_0 \) in terms of the present value of the Hubble constant \( H_0 \). Using \( \Lambda = 3\Omega_{de,0} H_0^2 \) obtained from Eqs. 3, 15 and 45, we get

\[ \Sigma^\text{th}_0 = 0.0281 \frac{H_0 c}{G}. \]  

(161)

These identities express the universal surface density of DM halos in terms of the fundamental constants of physics \( G, c, \) and \( \Lambda \) (or \( A \)). We stress that the prefactors are determined by our model so there is no free parameter. We note that the identities from Eqs. 158-161, which can be checked by a direct numerical application, are interesting in themselves even in the case where the logotropic model would turn out to be wrong. Furthermore, as observed in 61, the surface density of DM halos is of the same order of magnitude as the surface density of the universe and, more surprisingly, as the surface density of the electron. As a result, the identities from Eqs. 159-161 allow us to express the mass of the electron\(^{13} \) in terms of the cosmological constant and the other fundamental constants of physics as 61

\[ m_e \sim \left( \frac{M^4}{G^2 c^2} \right)^{1/6}, \]  

(162)

or as

\[ m_e \sim \left( \frac{H_0 c^2}{G} \right)^{1/3}. \]  

(163)

This returns the empirical Eddington-Weinberg relation 73, 69 obtained from different considerations. This relation provides a curious connection between microphysics and macrophysics (i.e. between atomic physics and cosmology) which is further discussed in 61, 100. Curiously, the Weinberg relation 163 involves the present value of the Hubble constant (see the conclusion).

G. The gravitational acceleration

We can define an average DM halo surface density by the relation

\[ \langle \Sigma \rangle = \frac{M_h}{\pi r_h^2}. \]  

(164)

For logotropic DM halos, using Eq. 157, we find

\[ \langle \Sigma \rangle^\text{th} = \frac{M_h}{\pi r_h^2} = 1.49 \frac{\Sigma^\text{th}_0}{\pi} = 63.1 M_\odot/\text{pc}^2. \]  

(165)

This theoretical value is in good agreement with the value \( \langle \Sigma \rangle_{\text{obs}} = 72_{-2}^{+42}, M_\odot/\text{pc}^2 \) obtained from the observations 101.

The gravitational acceleration at the halo radius is

\[ g = g(r_h) = \frac{G M_h}{r_h^2} = \pi G \langle \Sigma \rangle. \]  

(166)

For logotropic DM halos, we find

\[ g_{th} = \pi G \langle \Sigma \rangle^\text{th}_0 = 1.49 G \Sigma^\text{th}_0 = 2.76 \times 10^{-11} \text{ m/s}^2. \]  

(167)

Again, this theoretical value is in good agreement with the measured value \( g_{\text{obs}} = \pi G \langle \Sigma \rangle_{\text{obs}} = 3.2_{-1.2}^{+4.5} \times 10^{-11} \text{ m/s}^2 \) of the gravitational acceleration 101.

The circular velocity at the halo radius is

\[ v_h^2 = \frac{G M_h}{r_h}. \]  

(168)

Using Eqs. 165-167, we obtain the relation

\[ v_h^4 = G g M_h = \pi \langle \Sigma \rangle G^2 M_h = 1.49 \Sigma_0 G^2 M_h, \]  

(169)

where \( g \) and \( \Sigma_0 \) are universal constants. This relation is connected to the Tully-Fisher relation 102 which involves the baryon mass \( M_b \) instead of the DM halo mass \( M_h \) via the cosmic baryon fraction \( f_b = M_b/M_h \sim 0.17 \). This yields \( (M_b/v_h^4)^{1/5} = 46.4 M_\odot \text{ km}^{-4}\text{s}^4 \) which is close to the observed value \( (M_b/v_h^4)_{\text{obs}} = 47 \pm 6 M_\odot \text{ km}^{-4}\text{s}^4 \) 74. The Tully-Fisher relation is also a prediction of the MOND (modification of Newtonian dynamics) theory 103. Using Eqs. 167 and 168, we obtain

\[ g_{th} = 0.0291 \sqrt{3\Omega_{de,0} H_0 c} = 0.0419 H_0 c. \]  

(170)

This relation explains why the fundamental constant \( a_0 = g/f_b \) that appears in the MOND theory is of order \( H_0 c/4 = 1.65 \times 10^{-10} \text{ m/s}^2 \) (see the Remark in Sec. 3.3. of 61 for a more detailed discussion). Note, however, that our model is completely different from the MOND theory.

\(^{13} \) We remain here at a very qualitative level and ignore dimensionless prefactors. The mass scale appearing in Eqs. 162 and 163 may correspond to the mass of the electron, nucleon (proton or neutron), pion...
VI. CONCLUSION

In this paper, we have first discussed the similarities and the differences between three types of logotropic models. The logotropic model of type I where the pressure is proportional to the logarithm of the energy density is indistinguishable from the ΛCDM model, at least for what concerns the evolution of the homogeneous background. The logotropic model of type II where the pressure is proportional to the logarithm of the rest mass density will differ from the ΛCDM model in about 27 Gyr years. At that moment it will present a phantom behavior in which the energy density increases with the scale factor (leading to a little rip) while the energy density of the ΛCDM model tends to a constant. Finally, the logotropic model of type III where the pressure is proportional to the logarithm of the pseudo rest mass density of a complex SF is similar to the ΛCDM model except that its asymptotic energy density differs from the asymptotic energy density of the ΛCDM model by a factor 1.02.

We have then emphasized two main predictions of the logotropic model. These predictions were made in our previous papers but we have improved and generalized our argumentation.

The first prediction is the universality of the surface density of DM halos. This is a direct consequence of the logotropic equation of state which implies an asymptotic density profile of the form \( \rho \propto r^{-1} \) or a mass-radius relation of the form \( M_h \propto r_h^2 \). The surface density is determined by the logotropic constant \( A \). If this constant is interpreted as a fundamental constant of physics, it immediately explains the universality of the surface density of DM halos. Interestingly, we have determined the value of this constant from cosmological (large scale) considerations without free parameter. The constant \( A \) is related to the cosmological density \( \rho_\Lambda \), defined in our model as the present DE density, by Eq. (10). This value ensures that the logotropic model explains the evolution of the homogeneous background up to the present time as well as the ΛCDM model. Then, using the value of \( A \) to determine the universal surface density of DM halos (small scales) we obtained Eq. (158) which is in very good agreement with the observational value. Therefore, the fundamental constant \( A \) appearing in the logotropic model is able to account both for the large scale (cosmological) and the small scale (astrophysical) properties of our universe. Indeed, it explains both the acceleration of the universe and the universality of the surface density of DM halos. Intriguingly, there also seems to exist a connection between cosmological, astrophysical and atomic scales which manifests itself in the commensurability of the surface density of DM halos (or the surface density of the universe) and the surface density of the electron [61].

This "coincidence" may shed a new light on the mysterious Eddington-Weinberg relation which relates the mass of the electron to the cosmological constant or to the present value of the Hubble constant [61, 100].

The second prediction of the logotropic model is even more intriguing. We have argued that the present ratio \( \Omega_{\text{de},0}/\Omega_{\text{dm},0} \) of DE and DM is equal to the pure number \( e = 2.71828... \). This prediction lies in the error bars of the measured value \( \Omega_{\text{obs}} = 0.30 \pm 0.08 \). This result is striking because the proportion of DE and DM changes with time so it is only at the present epoch that their ratio is equal to \( e \). Indeed, to obtain this result, we have assumed that the present proportion of DE \( \rho_\Lambda \) is related to the fundamental constant \( A \) (independent of time) by the relation from Eq. (10). This gives to our present epoch a particular place in the history of the universe. This coincidence is mysterious and disturbing. It is almost mystical (dark magic). It can be viewed as a refined form of the well-known cosmic coincidence problem [21, 22], namely why \( \Omega_{\text{de}}/\Omega_{\text{dm}} \) is of order one today. If our result is confirmed, we have to explain not only why \( \Omega_{\text{de},0}/\Omega_{\text{dm},0} \) is of order one but, more precisely, why it is equal to \( e \). We call it the strong cosmic coincidence problem. This new coincidence may be related in some sense to the coincidence of large numbers noted by Eddington, Dirac and Weinberg. For example, the Weinberg relation (163) relates the mass of the electron to the present value of the Hubble constant (or equivalently to the present age of the universe). Therefore, it also gives to our present epoch a particular place in the history of the universe. To avoid this coincidence, Dirac [103] proposed that the fundamental constants of physics (e.g. the gravitational constant) change with time so that the relation between the large numbers that we observe today is always valid. For example, the Weinberg relation would always be valid if the gravitational constant changes with time as \( G \sim t^{-1} \). Unfortunately, the observations do not support Dirac’s theory about a time-varying gravitational constant [105, 106]. As a result, these coincidences and the special place of our epoch in the history of the universe remain a mystery. We must either accept that our epoch plays a particular role in the cosmological evolution (which would give to mankind a central place in the universe like in the old geocentric theory) or reformulate the laws of physics and cosmology so that what appears to be a coincidence at the present epoch finds a natural justification. The relation \( \Omega_{\text{de},0}/\Omega_{\text{dm},0} = e \) may correspond to a fixed point in a more sophisticated theory.

Despite these interesting and intriguing results, there are two main issues with the logotropic model:

(i) The density profile of logotropic DM halos decreases at large distances as \( r^{-1} \) so that their mass is infinite. This may not be a too serious problem because isothermal DM halos also face the same infinite mass problem while they have often been used to model DM halos. Actually, the logotropic or the isothermal equation of state is only valid in the core of DM halos. In practice, this

14 By contrast, the ΛCDM model works well at large scales but faces a small scale crisis [11] (see the introduction).
core region is surrounded by an extended envelope resulting from a process of (incomplete) violent relaxation where the density decreases more rapidly than $r^{-1}$ and ensures a finite mass. Logotropic DM halos have a constant surface density (with the correct value) determined by the fundamental constant $\Lambda$ while isothermal DM halos have not a constant surface density unless the temperature $T$ changes from halo to halo in a rather ad hoc manner (in the absence of a more solid justification) according to the law $k_{B}T/m \propto G\Sigma_{0}r_{h}$ [11, 12].

(ii) The speed of sound $c_{s}$ in logotropic DM halos increases as the density decreases [58–63]. This can inhibit the formation of large scale structures by Jeans instability and produce damped oscillations in the matter power spectrum (similar to baryonic or acoustic oscillations) that are not observed. The Chaplygin gas faces similar difficulties. This is actually a problem of all UDM models (see the discussion in Sec. XVI of [63]). These difficulties have been evidenced at the level of linear perturbation theory [57]. It is not obvious if they persist in the fully nonlinear problem [63, 107]. This will be an important point to clarify in the future but it requires sophisticated numerical simulations.

We would like to conclude this paper by stressing again one of our most striking results. We know from observations that the present universe contains approximately 70% DE and 25% DM. Their ratio is $\sim 2.8$. Our line of investigation based on the logotropic model suggests that this ratio may be equal to $e = 2.7182818...$. This “prediction” is interesting in itself, independently from the logotropic model. It would be of considerable interest to test this prediction with more accurate measurements and try to understand its meaning and the strong cosmic coincidence that it implies.

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