OPERS ON THE PROJECTIVE LINE, FLAG MANIFOLDS AND BETHE ANSATZ

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To Boris Feigin on his 50th birthday

1. Introduction

1.1. Our starting point is the Gaudin model associated to a simple finite-dimensional Lie algebra $\mathfrak{g}$. Let us introduce some notation. For any integral dominant weight $\lambda$, denote by $V_\lambda$ the irreducible finite-dimensional representation of $\mathfrak{g}$ of highest weight $\lambda$. Let $z_1, \ldots, z_N$ be a set of distinct complex numbers and $\lambda_1, \ldots, \lambda_N$ a set of dominant integral weights of $\mathfrak{g}$. Set

$$V_{(\lambda_i)} = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}.$$ 

Let $\{J_a\}, a = 1, \ldots, d$, be a basis of $\mathfrak{g}$ and $\{J^a\}$ the dual basis with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$.

The Gaudin hamiltonians are linear operators on $V_{(\lambda_i)}$:

$$\Xi_i = \sum_{j \neq i} \sum_{a=1}^d \frac{J_a(i) J^a(j)}{z_i - z_j}, \quad i = 1, \ldots, N,$$

They commute with the diagonal action of $\mathfrak{g}$ on $V_{(\lambda_i)}$ and hence their action is well-defined on the subspace of highest weight vectors in $V_{(\lambda_i)}$ of an arbitrary dominant integral weight $\mu$ with respect to the diagonal $\mathfrak{g}$-action. We may decompose $V_{(\lambda_i)}$ with respect to the diagonal action of $\mathfrak{g}$ as

$$V_{(\lambda_i)} = \bigoplus_{\mu} V_\mu \otimes \text{Hom}_\mathfrak{g}(V_\mu, V_{(\lambda_i)}).$$

Then the space of highest weight vectors of weight $\mu$ is identified with $\text{Hom}_\mathfrak{g}(V_\mu, V_{(\lambda_i)})$, or, equivalently, with

$$V_{(\lambda_i), \lambda_\infty} = (V_{(\lambda_i)} \otimes V_{\lambda_\infty})^G,$$

if we write $\mu = -w_0(\lambda_\infty)$, where $w_0$ is the longest element of the Weyl group of $\mathfrak{g}$.

Consider the problem of simultaneous diagonalization of the Gaudin hamiltonians in $V_{(\lambda_i)}$ (or equivalently, in all spaces $V_{(\lambda_i), \lambda_\infty}$). Set

$$|0\rangle = v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_N} \in V_{(\lambda_i)}.$$
It is an eigenvector of the \( \Xi \)'s. Other eigenvectors are constructed by a procedure known as the Bethe Ansatz. We explain it for \( \mathfrak{g} = \mathfrak{sl}_2 \). Let \( \{e, h, f\} \) be the standard basis of \( \mathfrak{sl}_2 \) and set

\[
f(w) = \sum_{i=1}^{N} \frac{f(i)}{w - z_i}.
\]

Define the Bethe vector

\[
|w_1, \ldots, w_m\rangle = f(w_1) f(w_2) \ldots f(w_m) |0\rangle.
\]

It is easy to show that it is an eigenvector of the Gaudin hamiltonians if and only if the following equations are satisfied:

\[
\sum_{i=1}^{N} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0, \quad j = 1, \ldots, m.
\]

These are the Bethe Ansatz equations for \( \mathfrak{g} = \mathfrak{sl}_2 \).

One can write analogous systems of equations for a general simple Lie algebra (or, more generally, a Kac-Moody algebra) \( \mathfrak{g} \). They are equations on the set of points \( w_1, \ldots, w_m \) colored by simple roots of \( \mathfrak{g} \), which we denote by \( \alpha_i, \ldots, \alpha_m \):

\[
\sum_{i=1}^{N} \frac{\langle \lambda_i, \check{\alpha}_i \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \alpha_s, \check{\alpha}_j \rangle}{w_j - w_s} = 0, \quad j = 1, \ldots, m.
\]

If they are satisfied, then one can construct the corresponding Bethe vector (see formula (1.2)) which is an eigenvector of the Gaudin hamiltonians (see [BaFl, FFR, RV]). This is a highest weight vector of weight

\[
\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{ij},
\]

so it can only be non-zero if

\[
\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{ij} = \mu,
\]

where \( \mu \) is a dominant integral weight which we write again as \( \mu = -w_0(\lambda_\infty) \). Then it belongs to \( V^G_{(\lambda_i), \lambda_\infty} \), considered as the subspace of highest weight vectors of weight \( -w_0(\lambda_\infty) \) in \( V_{(\lambda_i)} \).

For \( \mathfrak{g} = \mathfrak{sl}_2 \) it has been proved by I. Scherbak and A. Varchenko [SV] that for generic \( z_i \)'s the Bethe vectors form an eigenbasis in the space of highest weight vectors in \( V_{(\lambda_i)} \) (some important results in this direction have been obtained earlier by E. Sklyanin [Sk] using the so-called functional Bethe Ansatz).
1.2. In [FFR], B. Feigin, N. Reshetikhin and myself have given an interpretation of the Bethe Ansatz procedure using the spaces of conformal blocks for representations of the affine Kac-Moody algebra $\hat{g}$ associated to $g$. We showed that the Gaudin Hamiltonians naturally arise from central elements of the completed universal enveloping algebra of $\hat{g}$ at the critical level. This center has been identified by Feigin and myself (see [FF3, F2]) with the algebra of functions on the space of $L^G$-opers on the punctured disc. Here $L^G$ is the group (of adjoint type) which is Langlands dual to the (simply-connected) Lie group of $G$. Recall that passing from $G$ to $L^G$ means switching the sets of weights and coweights, roots and coroots of $G$ (with respect to a maximal torus), and at the level of Lie algebras it corresponds to taking the transpose of the Cartan matrix.

An $L^G$-oper on a smooth curve (or the formal disc) $X$ are triples $(\mathcal{F}, \nabla, \mathcal{F}_{L^B})$, where $\mathcal{F}$ is a $L^G$-bundle on $X$ equipped with a connection $\nabla$ and a reduction $\mathcal{F}_{L^B}$ to a Borel subgroup $L^B$ of $L^G$. In more concrete terms, opers may be described as gauge equivalence classes of first order differential operators of a certain form. They were defined in this way first by V. Drinfeld and V. Sokolov in their study [DS] of generalized KdV hierarchies, and later this definition was made more geometric and coordinate-independent by A. Beilinson and V. Drinfeld [BD2].

For example, in the case when $g = sl_n$ an oper on a smooth affine curve (or on the disc) is an equivalence class of operators of the form

$$\partial_t + \begin{pmatrix} * & * & * & \cdots & * \\ -1 & * & * & \cdots & * \\ 0 & -1 & * & \cdots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & * \end{pmatrix},$$

with respect to the gauge action of the group $N$ of the upper triangular matrices with 1’s on the diagonal. It is easy to see that each gauge class contains a unique operator of the form

$$\partial_t + \begin{pmatrix} 0 & v_1 & v_2 & \cdots & v_{n-1} \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

But giving such an operator is the same as giving a scalar $n$th order differential operator

$$L = \partial_t^n + v_1(t)\partial_t^{n-2} + \ldots + v_{n-1}(t)$$

(taking into account its transformation properties under changes of variables, we obtain that it must act from $\Omega^{-(n-1)/2}$ to $\Omega^{(n+1)/2}$). So the space of $PGL_n$-opers is the space of operators of the form (1.4), which is incidentally the phase space of the $n$th KdV hierarchy introduced by Adler and Gelfand–Dickey when $X$ is the disc.

The interpretation of the Gaudin model in terms of the affine Kac-Moody algebra of critical level allows us to construct a large commutative algebra of Hamiltonians acting on $V^{\mathbb{Z}}_{(\lambda_1), \lambda_\infty}$, which includes the Gaudin Hamiltonians (1.1), and to view their
eigenvalues as $L^G$–opers. The first main result of this paper (see Theorem 4.7) is a precise statement as to what kind of opers may appear as the eigenvalues of the generalized Gaudin hamiltonians (in the case when $g = \mathfrak{sl}_2$ this was proved in my paper [F1]).

**Theorem 1.** There is an injective map from the spectrum of the generalized Gaudin hamiltonians acting on $V^G_{(\lambda_1),\lambda_{\infty}}$ to the set of $L^G$–opers on $\mathbb{P}^1$ with regular singularities at $z_1, \ldots, z_N, \infty$ that have residues $\lambda_1, \ldots, \lambda_N, \lambda_{\infty}$ and trivial monodromy representation.

Thus, eigenvalues of the generalized Gaudin hamiltonians are encoded by $L^G$–opers on $\mathbb{P}^1$ with prescribed singularities and trivial monodromy. We remark that if we remove the “no monodromy” condition, then we obtain a description of the $L^G$–opers corresponding to the eigenvalues of the generalized Gaudin hamiltonians acting on the tensor product of the Verma modules $M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_N}$ (this description in fact holds for arbitrary weights $\lambda_1, \ldots, \lambda_N$).

Though this subject is beyond the scope of this paper, it is worth noting here that the correspondence between the eigenvectors of the generalized Gaudin hamiltonians and $L^G$–opers on $\mathbb{P}^1$ is an example of the geometric Langlands correspondence. This is a correspondence between $L^G$–local systems on a smooth projective curve $X$ over $\mathbb{C}$ (possibly, with ramifications at marked points) and certain sheaves (D–modules) on the moduli spaces of $G$–bundles on $X$ (possibly, with additional structures at the marked points). In the case at hand, $X = \mathbb{P}^1$ and the local system is represented by a $L^G$–oper on $\mathbb{P}^1$. The corresponding D–module on the moduli space of $G$–bundles on $\mathbb{P}^1$ with parabolic structures at $z_1, \ldots, z_N$ and $\infty$ is represented by the Gaudin system, according to a general construction of Beilinson and Drinfeld [BD1] (see [F1] for more details on this connection).

1.3. Recall that only special solutions of the Bethe Ansatz equations (1.2) may give rise to non-zero eigenvectors of the generalized Gaudin hamiltonians, namely, the ones which satisfy the condition (1.3). The corresponding eigenvalues are then encoded by a $L^G$–oper on $\mathbb{P}^1$.

Now we want to describe all solutions of the Bethe Ansatz equations in geometric terms. It turns out that general solutions are parameterized by Miura opers.

While a $L^G$–oper is a triple $(\mathcal{F}, \nabla, \mathcal{F}_L)$, a Miura $L^G$–oper is by definition a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_L, \mathcal{F}_B)$ where $\mathcal{F}_B$ is another $L^B$–reduction of $\mathcal{F}$, which is preserved by $\nabla$. The space of Miura opers on a curve $X$ (or on the disc) whose underlying oper has a regular singularities and trivial monodromy representation (so that $\mathcal{F}$ is isomorphic to the trivial bundle) is isomorphic to the flag manifold $L^G/LB$ of $L^G$. Indeed, in order to define the $L^B$–reduction $\mathcal{F}_B$ of such $\mathcal{F}$ everywhere, it is sufficient to define it at one point $x$ in $X$ and then use the connection to “spread” it around. But choosing a $L^B$–reduction at one point means choosing an element of the twist of $L^G/LB$ by $\mathcal{F}_x$, and so we see that the space of all reductions is isomorphic to the flag manifold of $L^G$.

The flag manifold is the union of Schubert cells which are the $L^B$–orbits. They are parameterized by the Weyl group $W$ of $g$. The cell attached to $1 \in W$ is open and dense. It corresponds to those reductions $\mathcal{F}_B$ which are in generic position with $\mathcal{F}_L$. Then there are cells of codimension one labeled by the simple reflections $s_i$, etc.
Suppose now that we are given a Miura oper corresponding to a $L^G$–oper on $\mathbb{P}^1$ with regular singularities at $z_1, \ldots, z_N, \infty$. Let $L^H = L^B/[L^B, L^B]$ and $L^\mathfrak{h}$ be its Lie algebra. We construct an $L^H$–bundle on $\mathbb{P}^1$ equipped with a connection with regular singularities, i.e., an operator of the form $\partial_t + u(t)$, where $u(t)$ is an $L^\mathfrak{h}$–valued function which has poles of order at most one. Namely, we intersect $\mathcal{F}_{L^B}$ with $\mathcal{F}'_{L^B} w_0$ – this will be an $L^H$–bundle, and it inherits a connection from $\mathcal{F}'_{L^B}$. This map gives us a bijection between Miura $L^G$–opers and $L^H$–connections. The corresponding map from $L^H$–connections to $L^G$–opers is called the Miura transformation.

For example, for $\mathfrak{g} = \mathfrak{sl}_n$ we have $u(t) = (u_1(t), \ldots, u_n(t))$, and the Miura transformation is given by the formula

$$L = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t)),$$

where $L$ is the operator (1.4). Hence for $\mathfrak{g} = \mathfrak{sl}_2$ we have

$$\partial_t^2 - v(t) = (\partial_t - u(t))(\partial_t + u(t)),$$

i.e.,

$$v(t) = u(t)^2 - u'(t)$$

(note that this is the Poisson map intertwining the KdV and mKdV hierarchies of soliton equations discovered by R. Miura).

The reductions $\mathcal{F}_{L^B}$ and $\mathcal{F}'_{L^B}$ are going to be in generic position everywhere on $\mathbb{P}^1$ except at finitely many points (see Lemma 2.6 below). Denote these points by $w_1, \ldots, w_m \in \mathbb{P}^1$. At these points the $L^H$–connection will develop a regular singularity. For a generic Miura oper the relative positions at $w_j$’s will correspond to simple reflections from $W$. An explicit computation then shows that the corresponding $L^H$–connection will have regular singularity with residue $\alpha_{ij}$. In addition, our $L^H$–connection will have regular singularity at $z_i$ with residue $-\lambda_i$. So the connection will look like this:

$$\partial_t - \sum_{i=1}^N \frac{\lambda_i}{t - z_i} + \sum_{j=1}^m \frac{\alpha_{ij}}{t - w_j}.$$  

But the $L^G$–oper underlying our $L^H$–connection has singularities only at the points $z_1, \ldots, z_N$ and no singularity at $w_1, \ldots, w_m$. Therefore these singularities must be somehow erased by the Miura transformation.

We have shown in [FFR] that the oper obtained by applying the Miura transformation to (1.5) has no singularity if and only if the Bethe Ansatz equations are satisfied. So we obtain an interpretation of the Bethe Ansatz equations as the conditions that the singularities of our $L^H$–connection at $w_1, \ldots, w_m$ be erased by the Miura transformation.

Our connection also has a regular singularity at $\infty$, which is determined by the relative position of $\mathcal{F}_{L^B}$ and $\mathcal{F}'_{L^B}$ at $\infty$. If the relative position is $y \in W$, then the residue is $-y(-w_0(\lambda_\infty) + \rho) + \rho$. The transformation properties of the connection
determine the residue at $\infty$, so we obtain the “charge conservation law”

$$\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{ij} = y(-w_0(\lambda_\infty) + \rho) - \rho.$$  \hspace{1cm} (1.6)

This leads to the following statement (implicit already in [F1]), which is the second main result of this paper (see Corollary 3.3):

**Theorem 2.** The set of those solutions of the Bethe Ansatz equations which correspond to a fixed $^L G$–oper is in bijection with an open and dense subset of the flag manifold $^L G/^L B$.

Further, every solution must satisfy the equation (1.6) for some $y \in W$, and the solutions which satisfy this equation with fixed $y \in W$ are in bijection with an open subset of the Schubert cell $^L B w_0 y w_0 ^L B \subset ^L G/^L B$.

In particular, a solution for which we have $y = 1$ corresponds to the one-point Schubert cell in the flag manifold. If this point is contained in the open dense subset of the flag manifold from Theorem 2, then this solution gives rise to a Bethe eigenvector. It was shown in [FFR] that the eigenvalues of the Gaudin hamiltonians on this vector are encoded precisely by the $^L G$–oper obtained by applying the Miura transformation to the $^L H$–connection (1.3). This follows immediately from the construction of the Bethe eigenvectors using conformal blocks of Wakimoto modules presented in [FFR].

1.4. Let us summarize the results: the eigenvalues of the hamiltonians of the Gaudin model associated to a simple Lie algebra $g$ are encoded by $^L G$–opers on $\mathbb{P}^1$, where $^L G$ is the Langlands dual group of $G$, which have regular singularities at the marked points and trivial monodromy. We attach to each solution of the Bethe Ansatz equations (1.2) an $^L H$–connection on $\mathbb{P}^1$ with regular singularities. There is a special map from $^L H$–connections to $^L G$–opers which is called the Miura transformation. The Bethe Ansatz equations naturally arise as the conditions that the Miura transformation erases the singularities of the corresponding $^L H$–connection. The set of all solutions of the Bethe Ansatz equations (1.2) is the union of certain open dense subsets of the flag manifold of the Langlands dual group, one for each oper of the above type. If the open subset corresponding to an oper $\tau$ contains the one-point Schubert cell, then the corresponding solution gives rise to a Bethe eigenvector of the Gaudin hamiltonians whose eigenvalues are encoded by $\tau$.

One can easily write down the Bethe Ansatz equations for an arbitrary Kac-Moody algebra $g$, and it is natural to ask whether the set of solutions is again the union of open dense subsets of the flag manifold associated to the Langlands dual group. We show that this is indeed the case. The subtle point is which flag manifold appears here, because for infinite-dimensional Kac-Moody algebras there are non-isomorphic flag manifolds: the “thick” flag variety, which is a proalgebraic variety, and the “thin” one, which is an ind-scheme. It turns out that the relevant flag variety is the thin one, $^L G/^L B$. Here $^L G$ is an ind-group corresponding to the Lie algebra $^L g$ whose Cartan matrix is the transpose of the Cartan matrix of $g$, and $^L B$ is the Borel subgroup of $^L G$ that is a proalgebraic group (see, e.g., [KM], Ch. VII, for the precise definition).
To establish the connection between solutions of the Bethe Ansatz equations and points of the ind-flag variety, we proceed in the same way as in the finite-dimensional case. First, we introduce suitable notions of opers and Miura opers for Kac-Moody algebras. (Note that in the case when \( g \) is an untwisted affine Kac-Moody algebra, the notions of opers and Miura opers have been introduced earlier by D. Ben-Zvi and myself \[BeFr\]; however, those notions are different from the ones we introduce here, see Remark 5.1.) Using them, we show that the set of solutions of the Bethe Ansatz equations is an open and dense subset in the set of Miura opers on the projective line with prescribed residues at marked points. We then show that, as in the finite-dimensional case, this set is in bijection with a disjoint union of the sets of points of certain open dense subsets of the ind-flag variety \( L^G/LB_− \). Thus, we generalize Theorem 2 to the case of an arbitrary Kac-Moody algebra (see Theorem 5.7).

In the case of a symmetrizable Kac-Moody algebra \( g \) it is also easy to write down analogues of the Gaudin hamiltonians acting on the tensor product of integrable representations of \( g \). Then to any solution of the Bethe Ansatz equations one associates a Bethe eigenvector of these hamiltonians in the same way as in the finite-dimensional case. But the connection between the eigenvalues of the Gaudin hamiltonians on these vectors and opers on the projective line is not obvious in the infinite-dimensional case. Recall that in the finite-dimensional case it was based on the concept of conformal blocks for modules over the affinization of \( g \) at the critical level. It is not immediately clear what should be the analogue of this Lie algebra when \( g \) is an infinite-dimensional Kac-Moody algebra. This as a very interesting open problem, which in fact served as one of our motivations. The fact that our results on the solutions of the Bethe Ansatz equations apply to general Kac-Moody algebras indicates that this question may have a good answer. We hope to return to it in a future publication.

1.5. As a corollary of the above description of the solutions of the Bethe Ansatz equations we obtain a (rational) action of the group \( L^G \) on the set of solutions of the Bethe Ansatz equations. It is easy to write down explicitly the action of the one-parameter subgroups corresponding to the generators \( e_i \) of the nilpotent Lie algebra \( L^n \). Taking the closure of an orbit of such a subgroup, we obtain for each simple root a procedure for producing a projective line (minus finitely many points) worth of new solutions of the Bethe Ansatz equations from a given one (these projective lines are precisely the ones appearing in the Bott-Samelson resolutions of the closures of the Schubert cells).

These procedures were introduced independently and in a different way by E. Mukhin and A. Varchenko \[MV\]. They defined what they called a “population” of solutions of the Bethe Ansatz equations as the closure of the set of all solutions obtained from a given one by iterating these procedures. In the case when \( g \) is of types \( A_n \), \( B_n \) or \( C_n \) they proved (by a method different from ours) that a population of solutions is isomorphic to the flag manifold of \( L^G \) (for \( g = sl_2 \) this had been proved earlier by Scherbak and Varchenko \[SV\], and for \( g = G_2 \) this was subsequently proved by Borisov and Mukhin \[BM\]). In Section 5.4 it is shown that the reproduction procedures of \[MV\] are equivalent to the action of one-parameter subgroups of \( LN \) on Miura opers.
1.6. **Plan of the paper.** The paper is organized as follows. We start in Section 2 by defining opers and Miura opers following [BD2] and [F2]. We describe opers with regular singularities, explain the connection between Miura opers and Cartan connections and establish the correspondence between relative positions of Borel reductions in a Miura oper and residues of the corresponding Cartan connection (this last result is borrowed from a forthcoming joint work with D. Gaitsgory [FG]). Next, we explain in Section 3 the connection between solutions of Bethe Ansatz equations and non-degenerate Miura $G$–opers on $\mathbb{P}^1$ with prescribed singularities at marked points. Here, in order to simplify our notation, we consider the Bethe Ansatz equations which correspond to the Gaudin model of $Lg$. Then the solutions of these equations correspond to Miura $G$–opers, rather than $Lg$–opers (as discussed in this Introduction) and hence to points of the flag manifold of $G$ rather than $Lg$. Our main result is that the set of solutions of the Bethe Ansatz equations is isomorphic to a union of open dense subsets of the flag manifold $G/B$ (one for each $G$–oper on $\mathbb{P}^1$ with prescribed singularities).

Section 4 is devoted to the Gaudin model. We recall how the Bethe Ansatz equations arise naturally in the problem of diagonalization of the Gaudin hamiltonians. We explain the construction of the generalized Gaudin hamiltonians as central elements of the vertex algebra corresponding to $\hat{g}$ acting on a suitable space of coinvariants on $\mathbb{P}^1$. We then show that the eigenvalues of the generalized Gaudin hamiltonians are encoded by $Lg$–opers on $\mathbb{P}^1$ with fixed singularities and trivial monodromy representation. We discuss an application of this result to the problem of completeness of Bethe Ansatz. Finally, in Section 5, we generalize our results on solutions of the Bethe Ansatz equations to the case of an arbitrary Kac-Moody algebra. We introduce suitable notions of opers and Miura opers and exhibit the connection between the solutions of the Bethe Ansatz equations and Miura opers on $\mathbb{P}^1$, much like in the finite-dimensional case.

1.7. **Acknowledgments.** It is a great pleasure to dedicate this paper, with gratitude and admiration, to Boris Feigin, my teacher, friend and collaborator of many years. Especially so, since the results of this paper are based on or motivated by the results of our previous joint works.

I thank E. Mukhin and A. Varchenko for stimulating discussions, which encouraged me to revisit my earlier work concerning the Bethe Ansatz equations and led me to consider these equations when the underlying Lie algebra is infinite-dimensional.

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2. Opers and Miura opers

In this section we first recall the notions of opers and Miura opers (see [DS, BD1, F2]). We will then discuss the spaces of opers and Miura opers with singularities.

2.1. **Opers.** Let $G$ be a simple algebraic group of adjoint type, $B$ a Borel subgroup and $N = [B, B]$ its unipotent radical, with the corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$. There is an open $B$–orbit $O \subset \mathfrak{n}, \mathfrak{n}^\perp / \mathfrak{b} \subset \mathfrak{g}/\mathfrak{b}$, consisting of vectors which are stabilized by the radical $N \subset B$, and such that all of their negative simple root components, with respect to the adjoint action of $H = B/N$, are non-zero. This orbit may also be
described as the $B$–orbit of the sum of the projections of simple root generators $f_i$ of any nilpotent subalgebra $\mathfrak{n}_-$, which is in generic position with $\mathfrak{b}$, onto $\mathfrak{g}/\mathfrak{b}$. The torus $H = B/N$ acts simply transitively on $O$, so $O$ is an $H$–torsor.

We will often choose a splitting $H → B$ of the homomorphism $B → H$ and the corresponding splitting $\mathfrak{h} → \mathfrak{b}$ at the level of Lie algebras. Then we will have a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- ⊕ \mathfrak{h} ⊕ \mathfrak{n}$. We will choose generators $\{e_i\}, i = 1, \ldots, \ell$, of $\mathfrak{n}$ and generators $\{f_i\}, i = 1, \ldots, \ell$ of $\mathfrak{n}_-$ corresponding to simple roots, and denote by $\hat{\rho} \in \mathfrak{h}$ the sum of the fundamental coweights of $\mathfrak{g}$. Then we will have the following relations: $[\hat{\rho}, e_i] = 1, [\hat{\rho}, f_i] = -1$.

Suppose we are given a principal $G$–bundle $\mathcal{F}$ on $X$, which is a smooth curve, or a disc $D ≃ \text{Spec } \mathbb{C}[[t]]$, or a punctured disc $D^\times ≃ \text{Spec } \mathbb{C}((t))$, together with a connection $\nabla$ (automatically flat) and a reduction $\mathcal{F}_B$ to the Borel subgroup $B$ of $G$. Then we define the relative position of $\nabla$ and $\mathcal{F}_B$ (i.e., the failure of $\nabla$ to preserve $\mathcal{F}_B$) as follows. Locally, choose any flat connection $\nabla'$ on $\mathcal{F}$ preserving $\mathcal{F}_B$, and take the difference $\nabla − \nabla'$. It is easy to show that the resulting local sections of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} ⊗ \Omega$, where $\Omega$ is the canonical line bundle of $X$, are independent of $\nabla'$, and define a global $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$–valued one-form on $X$, denoted by $\nabla/\mathcal{F}_B$.

Let $X$ be as above. A $G$–oper on $X$ is by definition a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal $G$–bundle $\mathcal{F}$ on $X$, $\nabla$ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a $B$–reduction of $\mathcal{F}$, such that the one–form $\nabla/\mathcal{F}_B$ takes values in $O_{\mathcal{F}_B} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$.

This definition is due to A. Beilinson and V. Drinfeld [BD1] (in the case when $X$ is the punctured disc opers were first introduced in [DS]). Note that $O$ is $\mathbb{C}^\times$–invariant, so that $O ⊗ \Omega$ is a well-defined subset of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} ⊗ \Omega$.

Equivalently, the above condition may be reformulated as follows. Let $U$ be an open subset of a smooth curve $X$ (in the analytic or Zariski topology) which admits a coordinate $t : U → \mathbb{A}^1$ (analytic or étale, respectively) and a trivialization of $\mathcal{F}_B$, then with respect to this coordinate and this trivialization the connection will have the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t) f_i + v(t),$$

where each $\psi_i(t)$ is a nowhere vanishing function, and $v(t)$ is a $\mathfrak{b}$–valued function. If we change the trivialization of $\mathcal{F}_B$, then this operator will get transformed by the corresponding gauge transformation from the group $B(R)$, where $R$ is the ring of functions (analytic or algebraic, respectively) on $U$. This observation allows us to describe opers on $U$ in more concrete terms.

Namely, we obtain from the above description that the space $\text{Op}_G(U)$ of $G$–opers on $U$ is the quotient of the space of all operators of the form (2.1), where $\psi_i(t) \in R$ is nowhere vanishing, and $v(t) \in \mathfrak{b}(R)$, by the action of the group $B(R)$ by gauge transformations:

$$g \cdot (\partial_t + A(t)) = \partial_t + g A(t) g^{-1} - \partial_t g \cdot g^{-1}.$$ 

The same description applies if $U = D$ or $U = D^\times$, with $t$ being the topological generator of $R$, which is equal to $\mathbb{C}[[t]]$ or $\mathbb{C}((t))$, respectively.
Since the $B$–orbit $O$ is an $H$–torsor, we can use the $H$–action to make all functions $\psi_i(t)$ equal to 1 (or any other non-zero constant). Thus, we obtain that $Op_G(U)$ is equal to the quotient of the space $\tilde{Op}_G(U)$ of operators of the form

$$\nabla = \partial_t + \sum_{i=1}^{\ell} f_i + v(t), \quad v(t) \in b(R),$$

by the action of the group $N(R)$.

The operator $\text{ad} \rho$ defines the principal gradation on $b$, with respect to which we have a direct sum decomposition $b = \bigoplus_{i \geq 0} b_i$. Set

$$p_{-1} = \sum_{i=1}^{\ell} f_i.$$

Let $p_1$ be the unique element of degree 1 in $n$, such that $\{p_{-1}, 2\rho, p_1\}$ is an $\mathfrak{sl}_2$–triple. Let $V_{\text{can}} = \bigoplus_{i \in E} V_{\text{can},i}$ be the space of $\text{ad} p_1$–invariants in $n$. Then $p_1$ spans $V_{\text{can},1}$. Choose a linear generator $p_j$ of $V_{\text{can},d_j}$ (if the multiplicity of $d_j$ is greater than one, which happens only in the case $g = D^{(1)}_{2n}$, $d_j = 2n$, then we choose linearly independent vectors in $V_{\text{can},d_j}$).

**Lemma 2.1** ([DS]). The gauge action of $N(R)$ on $\tilde{Op}_G(\text{Spec} R)$ is free, and each gauge equivalence class contains a unique operator of the form $\nabla = \partial_t + p_{-1} + v(t)$, where $v(t) \in V_{\text{can}}(R)$, so that we can write

$$v(t) = \sum_{j=1}^{\ell} v_j(t) \cdot p_j.$$

**Proof.** The operator $\text{ad} p_{-1}$ acts from $b_{i+1}$ to $b_i$ injectively for all $i \geq 0$ and we have $b_i = [p_{-1}, b_{i+1}] \oplus V_{\text{can},i}$. In particular, $V_0 = 0$. We claim that each element of $\partial_t + p_{-1} + v(t) \in Op_G(\text{Spec} R)$ can be uniquely represented in the form

$$\nabla = \partial_t + p_{-1} + v(t) = \exp (\text{ad} M) \cdot (\partial_t + p_{-1} + c(t)),$$

where $M \in n \otimes R$ and $c(t) \in V_{\text{can}} \otimes R$. To see that, we decompose with respect to the principal gradation: $M = \sum_{j \geq 0} M_j$, $v(t) = \sum_{j \geq 0} v_j(t)$, $c(t) = \sum_{j \in E} c_j(t)$. Equating the homogeneous components of degree $j$ on both sides of (2.3), we obtain that $c_i + [M_{i+1}, p_{-1}]$ is expressed in terms of $v_i, c_j, j < i$, and $M_j, j \leq i$. The injectivity of $\text{ad} p_{-1}$ then allows us to determine uniquely $c_i$ and $M_{i+1}$. Hence $M$ and $c$ satisfying equation (2.3) may be found uniquely by induction, and the lemma follows. \hfill \Box

If we choose another coordinate $s$ such that $t = \varphi(s)$, then the operator (2.2) will become

$$\nabla = \partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot v(\varphi(s)).$$

In order to bring it back to the form (2.2) we need to apply the gauge transformation by $\rho(\varphi'(s))$, where we choose a splitting $H \to B$ of the homomorphism $B \to H$ and
view $\tilde{\rho}$ as a homomorphism $\mathbb{C}^\times \to H$. We have

$$\tilde{\rho}(\varphi'(s)) \cdot \left( \partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \nu(\varphi(s)) \right)$$

(2.4) \[ = \partial_s + \sum_{i=1}^{\ell} f_i + \varphi'(s) \tilde{\rho}(\varphi'(s)) \cdot \nu(\varphi(s)) \cdot \tilde{\rho}(\varphi'(s))^{-1} - \tilde{\rho} \cdot \frac{\varphi''(s)}{\varphi'(s)}. \]

This formula allows us to glue together opers defined on various open subsets of a general curve $X$ and thus describe the space $\text{Op}_G(X)$ in terms of first order differential operators. It also allows us to describe the space of opers on the disc $D_x = \text{Spec} O_x$, where $O_x$ is the completion of the local ring of $U$ at $x$, or on the punctured disc $D_x^\times = \text{Spec} \mathcal{K}_x$, where $\mathcal{K}_x$ is the field of fractions of $O_x$.

In particular, we obtain the following result. Consider the $H$–bundle $\Omega^{\tilde{\rho}}$ on $D$. It is uniquely determined by the following property: for any character $\lambda : H \to \mathbb{C}^\times$, the line bundle $\Omega^{\tilde{\rho}} \times H \lambda$ associated to the corresponding one-dimensional representation of $H$ is $\Omega_{H,\lambda}^{\tilde{\rho}}$.

**Lemma 2.2** ([F2], Lemma 10.1). The $H$–bundle $\mathcal{F}_H = \mathcal{F}_B \times H = \mathcal{F}_B/N$ is isomorphic to $\Omega^{\tilde{\rho}}$.

Moreover, it is easy to find transformation formulas for the canonical representatives of opers. Indeed, by Lemma 2.1, there exists a unique operator $\partial_s + p_{-1} + \overline{\nu}(s)$ with $\overline{\nu}(s) \in V_{\text{can}}(R)$ and $g \in B(R)$, such that

$$\partial_s + p_{-1} + \overline{\nu}(s) = g \cdot \left( \partial_s + \varphi'(s) \sum_{i=1}^{\ell} f_i + \varphi'(s) \cdot \nu(\varphi(s)) \right).$$

(2.5) \[ \text{It is straightforward to find that (see [F2])} \]

$$g = \exp \left( \frac{1}{2} \varphi'' \cdot p_1 \right) \tilde{\rho}(\varphi'),$$

(2.6) \[ \overline{\nu}_1(s) = v_1(\varphi(s)) (\varphi')^2 - \frac{1}{2} \{ \varphi, s \}, \]

$$\overline{\nu}_j(s) = v_j(\varphi(s)) (\varphi')^{d_j+1}, \quad j > 1,$$

where

$$\{ \varphi, s \} = \frac{\varphi''}{\varphi'} - \frac{3}{2} \left( \frac{\varphi''}{\varphi'} \right)^2$$

is the Schwarzian derivative.

The above formulas describe the transition functions of the bundle $\mathcal{F}_B$ and hence of $\mathcal{F}$. Namely, they are equal to

$$\exp \left( \frac{1}{2} \varphi'' \cdot p_1 \right) \tilde{\rho}(\varphi'),$$

where $\varphi(s)$ is the change of coordinate function. Thus, we find that the bundles $\mathcal{F}$ and $\mathcal{F}_B$ are the same for all opers.
These formulas also imply that under changes of variables, $v_1$ transforms as a projective connection, and $v_j, j > 1$, transforms as a $(d_j + 1)$–differential on $U$. Thus, we obtain an isomorphism

$$(2.7) \quad \mathcal{O}_{PG}(X) \simeq \mathcal{P}roj(X) \times \bigoplus_{j=2}^{\ell} \Gamma(X, \Omega^{\otimes(d_j+1)}),$$

where $\mathcal{P}roj(X)$ is the $\Gamma(X, \Omega^{\otimes2})$–torsor of projective connections on $X$ (see, e.g., [FB], Sect. 8.2).

2.2. **Opers for classical Lie groups.** For Lie groups of classical types opers may be described in terms of scalar differential operators. Consider first the case of $\mathfrak{g} = \mathfrak{sl}_n$.

Then the space of opers on $U = \text{Spec } R$ is the quotient of the space of operators of the form

$$(2.8) \quad \partial_t + \left( \begin{array}{cccc}
* & * & * & \cdots & *\\
-1 & * & * & \cdots & * \\
0 & -1 & * & \cdots & * \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & *
\end{array} \right),$$

where the stars stand for elements of $R$, by the gauge action of the group $N(R)$ of upper triangular matrices over $R$ with 1’s on the diagonal. It is easy to see that each gauge orbit contains a unique operator of the form

$$(2.9) \quad \partial_t + \left( \begin{array}{cccc}
0 & v_1 & v_2 & \cdots & v_{n-1} \\
-1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 0
\end{array} \right).$$

But giving such an operator is the same as giving a scalar $n$th order differential operator

$$(2.10) \quad L = \partial_t^n + v_1(t)\partial_t^{n-2} + \ldots + v_{n-1}(t).$$

Thus, we obtain representatives of the gauge equivalence classes that is different from those described by Lemma 2.1. If we look at how these operators transform under changes of variables, we find that they transform as operators acting from $-(n - 1)/2$–densities, i.e., section of the $-(n - 1)/2$th power of $\Omega$, to the $(n + 1)/2$–densities. This completely describes the transformation formulas of the $v_i(t)$’s. Note that if $n$ is even, we need to choose a square root of $\Omega$, but the resulting space of differential operators will not depend on this choice. For example, if $n = 2$, we obtain the space of projective connections, i.e., operators of the form $\partial_t^2 + v(t)$ acting from $\Omega^{-1/2}$ to $\Omega^{3/2}$. Under changes of coordinates $v(t)$ transforms according to formula (2.6).

For the classical Lie algebras $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n+1}$ opers may also be realized as scalar differential operators, as explained by Drinfeld and Sokolov [DS], Sect. 8 (see also [BD2], Sect. 2). Observe that using the residue pairing we can identify the dual space of the space of sections of the line bundle $\Omega^m$ on the punctured disc with that of
Then the adjoint of a differential operator from $\Omega^m$ to $\Omega^k$ acts from $\Omega^{1-k}$ to $\Omega^{1-m}$. Now the space $\text{Op}_{sp_{2n}}(D^x)$ (resp., $\text{Op}_{so_{2n+1}}(D^x)$) is realized as the space of self-adjoint differential operators $L: \Omega^{-n+1/2} \to \Omega^{n+1/2}$ of order $2n$ (resp., anti-self adjoint operators $L: \Omega^{-n} \to \Omega^{n+1}$ of order $2n+1$) with the principal symbol 1.

In the case of $\mathfrak{g} = \mathfrak{so}_{2n}$ opers may be realized as scalar pseudo-differential operators (see [DS, BD2]).

2.3. Opers with regular singularities. Let $x$ be a point of a smooth curve $X$ and $D_x = \text{Spec} \mathcal{O}_x, D_x^x = \text{Spec} \mathcal{K}_x$, where $\mathcal{O}_x$ is the completion of the local ring of $x$ and $\mathcal{K}_x$ is the field of fractions of $\mathcal{O}_x$. Choose a formal coordinate $t$ at $x$, so that $\mathcal{O}_x \simeq \mathbb{C}[t]$ and $\mathcal{K}_x = \mathbb{C}((t))$. Recall that the space $\text{Op}_G(D_x)$ (resp., $\text{Op}_G(D_x^x)$) of $G$–opers on $D_x$ (resp., $D_x^x$) is the quotient of the space of operators of the form (2.1) where $\psi_i(t)$ and $\nu(t)$ take values in $\mathcal{O}_x$ (resp., in $\mathcal{K}_x$) by the action of $B(\mathcal{O}_x)$ (resp., $B(\mathcal{K}_x)$).

A $G$–oper on $D_x$ with regular singularity at $x$ is by definition (see [BD1], Sect. 3.8.8) a $B(\mathcal{O}_x)$–conjugacy class of operators of the form

$$\nabla = \partial_t + t^{-1} \left( \sum_{i=1}^\ell \psi_i(t) f_i + \nu(t) \right),$$

where $\psi_i(t) \in \mathcal{O}_x, \psi_i(0) \neq 0$, and $\nu(t) \in \mathfrak{b}(\mathcal{O}_x)$. Equivalently, it is an $\mathcal{N}(\mathcal{O}_x)$–equivalence class of operators

$$\nabla = \partial_t + \frac{1}{t} (p_{-1} + \nu(t)), \quad \nu(t) \in \mathfrak{b}(\mathcal{O}_x).$$

Denote by $\text{Op}_G^{RS}(D_x)$ the space of opers on $D_x$ with regular singularity. It is easy to see (BD1 or Proposition 2.3) that the natural map $\text{Op}_G^{RS}(D_x) \to \text{Op}_G(D_x^x)$ is injective. Therefore an oper with regular singularity may be viewed as an oper on the punctured disc. But to an oper with regular singularity one can unambiguously attach a point in

$$\mathfrak{g}/G := \text{Spec} \mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W =: \mathfrak{h}/W,$$

its residue, which in our case is equal to $p_{-1} + \nu(0)$.

In particular, the residue of a regular oper $\partial_t + p_{-1} + \nu(t)$, where $\nu(t) \in \mathfrak{b}(\mathcal{O}_x)$, is equal to $-\bar{\rho}$ (see [BD1]). Indeed, a regular oper may be brought to the form (2.12) by using the gauge transformation with $\bar{\rho}(t) \in B(\mathcal{K}_x)$, after which it takes the form

$$\partial_t + \frac{1}{t} (p_{-1} - \bar{\rho} + t \cdot \bar{\rho}(t) \nu(t) \bar{\rho}(t)^{-1}).$$

If $\nu(t)$ is regular, then so is $\bar{\rho}(t) \nu(t) \bar{\rho}(t)^{-1}$. Therefore the residue of this oper in $\mathfrak{h}/W$ is equal to $-\bar{\rho}$.

Given $\lambda \in \mathfrak{h}$, we denote by $\text{Op}_G^{RS}(D_x)_\lambda$ the subvariety of $\text{Op}_G^{RS}(D_x)$ which consists of those opers that have residue $-\lambda - \bar{\rho} \in \mathfrak{h}/W$ (in particular, $\text{Op}_G(D_x) = \text{Op}_G^{RS}(D_x)_0$).

Denote by $\mathfrak{g}_{\text{can}}$ the affine subspace of all elements of the form

$$p_{-1} + \sum_{j \in E} y_j p_j.$$
Recall from [Ko] that the adjoint orbit of any regular element in the Lie algebra \( \mathfrak{g} \) contains a unique element that belongs to \( \mathfrak{g}_{\text{can}} \), and the corresponding morphism \( \mathfrak{g}_{\text{can}} \to \mathfrak{h}/W \) is an isomorphism.

**Proposition 2.3** ([BD1], Prop. 3.8.9). The canonical representatives of opers with regular singularities have the form

\[
(2.13) \quad \partial_t + p_{-1} + \sum_{j \in E} t^{-j-1}c_j(t)p_j, \quad c_j(t) \in \mathbb{C}[[t]].
\]

Moreover, the residue of this oper is realized in \( \mathfrak{g}_{\text{can}} \) as

\[
(2.14) \quad p_{-1} + \left( c_1(0) + \frac{1}{4} \right) p_1 + \sum_{j \in E, j > 1} c_j(0)p_j.
\]

Let \( (\mathcal{F}, \nabla, \mathcal{F}_R) \in \text{Op}_G^{RS}(D_x) \). For each finite-dimensional representation \( V \) of \( G \), consider the system of differential equations with regular singularities \( \nabla \cdot \phi_V(t) = 0 \), where \( \phi_V(t) \) takes values in \( V \). For varying \( V \) the solutions of these equations give rise to a well-defined solution with values in \( G \), whose monodromy around \( x \) is a well-defined conjugacy class in \( G \).

Now let \( \lambda \) be a dominant integral coweight of \( \mathfrak{g} \). Following Drinfeld, introduce the variety \( \text{Op}_G(D_x\lambda) \) as the quotient of the space of operators of the form

\[
(2.15) \quad \nabla = \partial_t + \sum_{i=1}^\ell \psi_i(t)f_i + \upsilon(t),
\]

where

\[
\psi_i(t) = t^{(a_i, \lambda)}(\kappa_i + t(\ldots)) \in \mathcal{O}_x, \quad \kappa_i \neq 0
\]

and \( \upsilon(t) \in \mathfrak{b}(\mathcal{O}_x) \), by the gauge action of \( B(\mathcal{O}_x) \). Equivalently, \( \text{Op}_G(D_x\lambda) \) is the quotient of the space of operators of the form

\[
(2.16) \quad \nabla = \partial_t + \sum_{i=1}^\ell t^{(a_i, \lambda)}f_i + \upsilon(t),
\]

where \( \upsilon(t) \in \mathfrak{b}(\mathcal{O}_x) \), by the gauge action of \( N(\mathcal{O}_x) \). Considering the \( N(\mathcal{K}_x) \)-class of such an operator, we obtain an oper on \( D_x^\lambda \). Thus, we have a map \( \text{Op}_G(D_x\lambda) \to \text{Op}_G(D_x^\lambda) \).

**Lemma 2.4.** The map \( \text{Op}_G(D_x\lambda) \to \text{Op}_G(D_x^\lambda) \) is injective and its image is contained in the subvariety \( \text{Op}_G^{RS}(D_x)^\lambda \). Moreover, the points of \( \text{Op}_G(D_x\lambda) \) are precisely those \( G \)-opers with regular singularity and residue \( \lambda \) which have no monodromy around \( x \).

**Proof.** By using the gauge transformation with \( (\lambda + \tilde{\rho})(t) \), we bring the operator \( \mathcal{F} \) to the form \( (2.11) \), with

\[
(2.17) \quad \upsilon(t) = - (\lambda + \tilde{\rho}) + \upsilon_0(t) + \sum_{\alpha \in \Delta^+} \upsilon_\alpha(t),
\]

\[
\upsilon_0 \in \mathfrak{h} \otimes t\mathbb{C}[[t]], \quad \upsilon_\alpha(t) \in \mathfrak{n}_\alpha \otimes t^{(\alpha, \lambda + \tilde{\rho})}\mathbb{C}[[t]],
\]

and the \( N[[t]] \)-equivalence class of \( (2.15) \) is mapped to the \( (\lambda + \tilde{\rho})(t)N[[t]](\lambda + \tilde{\rho})(t)^{-1} \) class of the conjugate operator. It is then easy to see that the subgroup of \( N[[t]] \) which
preserves the operators with $v(t)$ of the form (2.17) is precisely $(\lambda + \hat{\rho})(t)N[[t]](\lambda + \hat{\rho})(t)^{-1}$. This proves the first statement.

To prove the second statement, observe that the monodromy of $\nabla$ is trivial if and only if $\nabla$ is gauge equivalent, under the gauge action of the entire loop group $G((t))$, to a regular connection (not necessarily an oper). Therefore the second statement is equivalent to the statement that an oper $\tau \in \mop_{G}(D_{x})_{\lambda}$ is gauge equivalent to a regular connection if and only if it belongs to $\mop_{G}(D_{x})_{\lambda}$. But $G((t)) = G[[t]]B((t))$, and the gauge action of $G[[t]]$ preserves the space of regular connections. Therefore if an oper is gauge equivalent to a regular connection, then its $B((t))$ gauge class already must contain a regular connection. The oper condition then implies that this gauge class contains a connection operator of the form (2.15), where $\psi_{i}(t) = \gamma^{(v_{i},\mu)}(\kappa_{i} + t(\ldots)) \in \mathcal{O}_{x}, \kappa_{i} \neq 0$ and $v(t) \in \mathcal{B}(\mathcal{O}_{x})$ for some integral dominant coweight $\mu$ of $\mathfrak{g}$. But according to the above calculation, the residue of such an oper is equal to $-\mu - \hat{\rho}$. This gives us the second statement of the lemma. \hfill \Box

2.4. Miura opers. By definition (see [F2], Sect. 10.3), a Miura $G$–oper on $X$ (which is a smooth curve or a disc) is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{B}, \mathcal{F}_{B,x})$, where $(\mathcal{F}, \nabla, \mathcal{F}_{B})$ is a $G$–oper on $X$ and $\mathcal{F}_{B}$ is another $B$–reduction of $\mathcal{F}$ which is preserved by $\nabla$.

We denote the space of Miura $G$–opers on $X$ by $\mop_{G}(X)$.

A $B$–reduction of $\mathcal{F}$ which is preserved by the connection $\nabla$ is uniquely determined by a $B$–reduction of the fiber $\mathcal{F}_{x}$ of $\mathcal{F}$ at any point $x \in X$ (in the case when $U = D$, $x$ has to be the origin $0 \in D$). The set of such reductions is the $\mathcal{F}_{x}$–twist

$$(2.18) \quad (G/B)_{\mathcal{F}} = \mathcal{F}_{x} \times G/B = \mathcal{F}_{B,x} \times G/B = (G/B)_{\mathcal{F}_{B,x}}$$

of the flag manifold $G/B$. If $X$ is a curve or a disc and the oper connection has a regular singularity and trivial monodromy representation, then this connection gives us a global (algebraic) trivialization of the bundle $\mathcal{F}$. Then any $B$–reduction of the fiber $\mathcal{F}_{x}$ gives rise to a global (algebraic) $B$–reduction of $\mathcal{F}$. Thus, we obtain:

**Lemma 2.5.** Suppose that we are given an oper $\tau$ on a curve $X$ (or on the disc) such that the oper connection has a regular singularity and trivial monodromy. Then for each $x \in X$ there is a canonical isomorphism between the space of Miura opers with the underlying oper $\tau$ and the twist $(G/B)_{\mathcal{F}_{B,x}}$.

Recall that the $B$–orbits in $G/B$, known as the Schubert cells, are parameterized by the Weyl group $W$ of $G$. Let $w_{0}$ be the longest element of the Weyl group of $G$. Denote the orbit $Bw_{0}B \subset G/B$ by $S_{w}$ (so that $S_{1}$ is the open orbit). We obtain from the second description of $(G/B)_{\mathcal{F}}$ given in formula (2.17) that $(G/B)_{\mathcal{F}}$ decomposes into a union of locally closed subvarieties $S_{w,\mathcal{F}_{B,x}}$, which are the $\mathcal{F}_{B,x}$–twists of the Schubert cells $S_{w}$. The $B$–reduction $\mathcal{F}_{B,x}$ defines a point in $(G/B)_{\mathcal{F}_{B,x}}$. We will say that the $B$–reductions $\mathcal{F}_{B,x}$ and $\mathcal{F}_{B,x}'$ are in relative position $w$ if $\mathcal{F}_{B,x}$ belongs to $S_{w,\mathcal{F}_{B,x}}$. In particular, if it belongs to the open orbit $S_{1,\mathcal{F}_{B,x}}$, we will say that $\mathcal{F}_{B,x}$ and $\mathcal{F}_{B,x}'$ are in generic position.

A Miura $G$–oper is called *generic* at the point $x \in X$ if the $B$–reductions $\mathcal{F}_{B,x}$ and $\mathcal{F}_{B,x}'$ of $\mathcal{F}$ are in generic position. In other words, $\mathcal{F}_{B,x}$ belongs to the stratum
\( \text{Op}_G(X) \times S_1 \cdot \mathcal{F}_{B,x} \subset \text{MOp}_G(X) \). Being generic is an open condition. Therefore if a Miura oper is generic at \( x \in X \), then there exists an open neighborhood \( U \) of \( x \) such that it is also generic at all other points of \( U \). We denote the space of generic Miura opers on \( U \) by \( \text{MOp}_G(U)_{\text{gen}} \).

**Lemma 2.6.** Suppose we are given a Miura oper on the disc \( D_x \) around a point \( x \in X \). Then its restriction to the punctured disc \( D_x^\times \) is generic.

**Proof.** Since being generic is an open condition, we obtain that if a Miura oper is generic at \( x \), it is also generic on the entire \( D_x \). Hence we only need to consider the situation where the Miura oper is not generic at \( x \), i.e., the two reductions \( \mathcal{F}_{B,x} \) and \( \mathcal{F}'_{B,x} \) are in relative position \( w \neq 1 \). Let us trivialize the \( B \)–bundle \( \mathcal{F}_B \), and hence the \( G \)–bundle \( \mathcal{F}_G \) over \( D_x \). Then \( \nabla \) gives us a connection on the trivial \( G \)–bundle which we can bring to the canonical form

\[
\nabla = \partial_t + p_{-1} + \sum_{j=1}^{\ell} v_j(t) \cdot p_j
\]

(see Lemma 2.1). It induces a connection on the trivial \( G/B \)–bundle. We are given a point \( gB \) in the fiber of the latter bundle which lies in the orbit \( S_w = Bw_0wB \), where \( w \neq 1 \). Consider the horizontal section whose value at \( x \) is \( gB \), viewed as a map \( D_x \to G/B \). We need to show that the image of this map lies in the open \( B \)–orbit \( S_1 = Bw_0B \) over \( D_x^\times \), i.e., it does not lie in the orbit \( S_y \) for any \( y \neq 1 \).

Suppose that this is not so, and the image of the horizontal section actually lies in the orbit \( S_y \) for some \( y \neq 1 \). Since all \( B \)–orbits are \( H \)–invariant, we obtain that the same would be true for the horizontal section with respect to the connection \( \nabla' = h \nabla h^{-1} \) for any constant element of \( H \). Choosing \( h = \rho(a) \) for \( a \in \mathbb{C}^\times \), we can bring the connection to the form

\[
\partial_t + a^{-1}p_{-1} + \sum_{j=1}^{\ell} a^{d_j}v_j(t) \cdot p_j.
\]

Changing the variable \( t \) to \( s = a^{-1}t \), we obtain the connection

\[
\partial_s + p_{-1} + \sum_{j=1}^{\ell} a^{d_j+1}v_j(t),
\]

so choosing small \( a \) we can make the functions \( v_j(t) \) arbitrarily small. Therefore without loss of generality we can consider the case when our connection operator is \( \nabla = \partial_t + p_{-1} \).

In this case our assumption that the horizontal section lies in \( S_y, y \neq 1 \), means that the vector field \( \xi_{p_{-1}} \) corresponding to the infinitesimal action of \( p_{-1} \) on \( G/B \) is tangent to an orbit \( S_y, y \neq 1 \), in the neighborhood of some point \( gB \) of \( S_w \subset G/B, w \neq 1 \). But then, again because of the \( H \)–invariance of the \( B \)–orbits, the vector field \( \xi_{h_{p_{-1}}h^{-1}} \) is also tangent to this orbit for any \( h \in H \). For any \( i = 1, \ldots, \ell \), there exists a one-parameter subgroup \( h^{(i)}_\epsilon \), \( \epsilon \in \mathbb{C}^\times \) in \( H \), such that \( \lim_{\epsilon \to 1} \epsilon^{-1}p_{-1} - 1 = f_i \). Hence we obtain that each of the vector fields \( \xi_{f_i}, i = 1, \ldots, \ell \), is tangent to the orbit \( S_y, y \neq 1 \), in the neighborhood of \( gB \in S_w, w \neq 1 \). But then all commutators of these vectors fields are also tangent
to this orbit. Hence we obtain that all vector fields of the form \( \xi_p, p \in \mathfrak{n}_- \), are tangent to \( S_y \) in the neighborhood of \( gB \in S_ω \).

Consider any point of \( G/B \) that does not belong to the open dense orbit \( S_1 \). Then the quotient of the tangent space to this point by the tangent space to the \( B \)–orbit passing through this point is non-zero and the vector fields from the Lie algebra \( \mathfrak{n}_- \) map surjectively onto this quotient. Therefore they cannot be tangent to the orbit \( S_y, y \neq 1 \), in a neighborhood of \( gB \). Therefore our Miura oper is generic on \( D_x^\times \). □

This lemma shows that any Miura oper on any smooth curve \( X \) is generic over an open dense subset.

Consider the \( H \)–bundles \( \mathcal{F}_H = \mathcal{F}_B/N \) and \( \mathcal{F}_H' = \mathcal{F}_B'/N \) corresponding to a generic Miura oper \( (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B') \) on \( X \). If \( \mathcal{F} \) is an \( H \)–bundle, then applying to it the automorphism \( w_0 \) of \( H \), we obtain a new \( H \)–bundle which we denote by \( w_0^*(\mathcal{F}_H) \).

**Lemma 2.7** ([F2], Lemma 10.3). For a generic Miura oper \( (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B') \) the \( H \)–bundle \( \mathcal{F}_H' \) is isomorphic to \( w_0^*(\mathcal{F}_H) \).

**Proof.** Consider the vector bundles \( \mathfrak{g}_\mathcal{F} = \mathcal{F} \otimes \mathfrak{g}, \mathfrak{b}_\mathcal{F}_B = \mathcal{F}_B \otimes \mathfrak{b} \) and \( \mathfrak{b}_\mathcal{F}_B' = \mathcal{F}_B' \otimes \mathfrak{b} \). We have the inclusions \( \mathfrak{b}_\mathcal{F}_B, \mathfrak{b}_\mathcal{F}_B' \subset \mathfrak{g}_\mathcal{F} \) which are in generic position. Therefore the intersection \( \mathfrak{b}_\mathcal{F}_B \cap \mathfrak{b}_\mathcal{F}_B' \) is isomorphic to \( \mathfrak{b}_\mathcal{F}_B'/[\mathfrak{b}_\mathcal{F}_B, \mathfrak{b}_\mathcal{F}_B] \), which is the trivial vector bundle with the fiber \( \mathfrak{h} \). It naturally acts on the bundle \( \mathfrak{g}_\mathcal{F} \) and under this action \( \mathfrak{g}_\mathcal{F} \) decomposes into a direct sum of \( \mathfrak{h} \) and the line subbundles \( \mathfrak{g}_{F,\alpha}, \alpha \in \Delta \). Furthermore, \( \mathfrak{b}_\mathcal{F}_B = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{F,\alpha} \), \( \mathfrak{b}_\mathcal{F}_B' = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{F,w_0(\alpha)} \). Since the action of \( B \) on \( \mathfrak{h} / [\mathfrak{n}, \mathfrak{n}] \) factors through \( H = B/N \), we find that

\[
\mathcal{F}_H \times_{\mathcal{H}} \bigoplus_{i=1}^\ell \mathbb{C}_{\alpha_i} \cong \bigoplus_{i=1}^\ell \mathfrak{g}_{\mathcal{F},\alpha_i}, \quad \mathcal{F}_H' \times_{\mathcal{H}} \bigoplus_{i=1}^\ell \mathbb{C}_{\alpha_i} \cong \bigoplus_{i=1}^\ell \mathfrak{g}_{\mathcal{F},w_0(\alpha_i)}.
\]

Therefore we obtain that

\[
\mathcal{F}_H \times_{\mathcal{H}} \mathbb{C}_{\alpha_i} \cong \mathcal{F}_H' \times_{\mathcal{H}} \mathbb{C}_{w_0(\alpha_i)}, \quad i = 1, \ldots, \ell.
\]

Since \( G \) is of adjoint type by our assumption, the above associated line bundles completely determine \( \mathcal{F}_H \) and \( \mathcal{F}_H' \), and the above isomorphisms imply that \( \mathcal{F}_H' \cong w_0^*(\mathcal{F}_H) \). □

Since the \( B \)–bundle \( \mathcal{F}_B' \) is preserved by the oper connection \( \nabla \), we obtain a connection \( \nabla \) on \( \mathcal{F}_H' \) and hence on \( \mathcal{F}_H \cong \Omega^\delta \). Therefore we obtain a morphism \( \mathfrak{a} \) from the variety \( \text{MOp}_G(U)_{\text{gen}} \) of generic Miura operas on \( U \) to the variety of connections \( \text{Conn}_U \) on the \( H \)–bundle \( \Omega^\delta \) on \( U \).

Explicitly, connections on \( \Omega^\delta \) may be described as follows. If we choose a local coordinate \( t \) on \( U \), then we trivialize \( \Omega^\delta \) and represent the connection as an operator \( \partial_t + \mathfrak{u}(t) \), where \( \mathfrak{u}(t) \) is an \( \mathfrak{b} \)–valued function on \( U \). If \( s \) is another coordinate such that \( t = \varphi(s) \), then this connection will be represented by the operator

\[
(2.19) \quad \partial_s + \varphi'(s)\mathfrak{u}(\varphi(s)) = \tilde{\rho} \frac{\varphi''(s)}{\varphi'(s)}.
\]
We define the relative positions of $F$ with regular singularity which belongs to $\text{Op}_{\mathcal{H}}^\lambda$ of coweight $\lambda$ by $M\text{Op}_{G}(U)$ in particular, if $\lambda$ which is preserved by $\mathcal{F}$ transformation by a constant element of $M\text{Op}(U)$. Likewise, rescaling of the generators $b_i$ of $\mathcal{F}$ transformation by this element. Therefore it will not change the underlying Miura oper structure. Thus, the morphism $b$ by setting $b(\nabla) = (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B')$.

This map is independent of the choice of a splitting $H \to B$ and of the generators $f_i, i = 1, \ldots, \ell$. Indeed, changing the splitting $H \to B$ amounts to conjugating of the old splitting by an element of $N$. This is equivalent to applying to $\nabla$ the gauge transformation by this element. Therefore it will not change the underlying Miura oper structure. Likewise, rescaling of the generators $f_i$ may be achieved by a gauge transformation by a constant element of $H$, and this again does not change the Miura oper structure. Thus, the morphism $b$ is well-defined. It is clear from the construction that $a$ and $b$ are mutually inverse isomorphisms.

More generally, we define, for any dominant integral coweight $\tilde{\lambda} \in \mathfrak{h}$, Miura $G$--opers of coweight $\tilde{\lambda}$ on $D_x$ as quadruples $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B')$, where $(\mathcal{F}, \nabla, \mathcal{F}_B)$ is a $G$--oper on $D_x^\times$ with regular singularity which belongs to $\text{Op}_G(D_x)_{\tilde{\lambda}}$ and $\mathcal{F}_B'$ is another $B$--reduction of $\mathcal{F}$ which is preserved by $\nabla$.

We denote the space of Miura $G$--opers of coweight $\tilde{\lambda}$ on $D_x$ by $M\text{Op}_G(D_x)_{\tilde{\lambda}}$. In particular, if $\tilde{\lambda} = 0$ we obtain the old definition of Miura opers on $D_x$. It is clear that we have an isomorphism $M\text{Op}_G(D_x)_{\lambda} \simeq \text{Op}_G(D_x)_{\lambda} \times (G/B)_{\mathcal{F}_B'}$.

We define the relative positions of $\mathcal{F}_B$ and $\mathcal{F}_B'$ in the same way as for $\tilde{\lambda} = 0$ and denote by $M\text{Op}_G(D_x)_{\tilde{\lambda}, \text{gen}}$ the variety of generic Miura opers of coweight $\tilde{\lambda}$.

Let $\text{Conn}_{D_x, \tilde{\lambda}}$ be the variety of connections on the $H$--bundle $\Omega^\phi$ over $D_x$ with regular singularity at $x$ and residue $-\tilde{\lambda}$. With respect to a coordinate $t$ at $x$, the corresponding
connection operator has the form
\[ \nabla = \partial_t + \lambda t^{-1} + u(t), \quad u(t) \in \mathfrak{h}[[t]]. \]
Denote by \( \Omega^\rho(-\lambda \cdot x) \) the \( H \)-bundle on \( D_x \) determined by the associated line bundles
\[ \Omega^\rho(-\lambda \cdot x) \times \mathfrak{c}_{\lambda_i} = \Omega(-\langle \alpha_i, \lambda \rangle x). \]
We have a morphism
\[ b_\lambda : \text{Conn}^{RS}_{D_{x, \lambda}} \to \text{MOp}_G(D_x)_{\lambda, \text{gen}} \]
sending such a connection \( \nabla \) to the triple \( (\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B) \), where
\[ \mathcal{F} = \Omega^\rho \times G, \quad \mathcal{F}_B = \Omega^\rho x B, \quad \mathcal{F}'_B = \Omega^\rho \times w_0 B, \]
and \( \nabla = \nabla + p_{-1} \), or equivalently,
\[ \mathcal{F} = \Omega^\rho(-\lambda \cdot x) \times G, \quad \mathcal{F}_B = \Omega^\rho(-\lambda \cdot x) \times B, \quad \mathcal{F}'_B = \Omega^\rho(-\lambda \cdot x) \times w_0 B, \]
\[ \nabla = \lambda(t)^{-1}(\nabla + p_{-1})\lambda(t) = \partial_t + \sum_{i=1}^n f^{(\alpha_i, \lambda)} f_i + u(t) \]
(the corresponding oper does not depend on the choice of \( t \)). We prove in the same way as in Proposition 2.3 that \( b_\lambda \) is an isomorphism.

2.5. **Miura transformation.** Under the isomorphism of Proposition 2.3, the natural forgetful morphism \( \text{MOp}_G(U)_{\text{gen}} \to \text{Op}_G(U) \) becomes a map \( \text{Conn}_U \to \text{Op}_G(U) \). We call this map the **Miura transformation.** The origin of this terminology is as follows. In Section 2.2 we described a realization of opers for Lie algebras of classical types in terms of scalar differential operators. These realizations may be used to describe explicitly the Miura transformation as well.

In the case of \( \mathfrak{s}l_n \) the space \( \text{Op}_{\mathfrak{s}l_n}(D^\times) \) consists of differential operators of the form \( p \). The space \( \text{Conn}_{D^\times} \) consists of the operators \( \partial_t + u(t) \), where \( u(t) \in \mathfrak{h}(t) \) may be viewed as an \( n \)-tuple \( (u_1(t), \ldots, u_n(t)) \) such that \( \sum_{i=1}^n u_i(t) = 0 \). The Miura transformation sends \( \partial_t + u(t) \) to the operator
\[ L = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t)). \]
In particular, for \( \mathfrak{g} = \mathfrak{s}l_2 \) we obtain a map sending a connection \( \partial_t + u(t) \) to the projective connection \( \partial_t^2 - v(t) \) where
\[ \partial_t^2 - v(t) = (\partial_t - u(t))(\partial_t + u(t)), \]
i.e.,
\[ u(t) \mapsto v(t) = u(t)^2 - u'(t). \]
This map was first introduced by R. Miura as the Poisson map from the phase space of the mKdV hierarchy (the space \( \text{Conn}_{D^\times} \) in our notation) to the phase space of the KdV hierarchy (the space \( \text{Op}_G(D^\times) \) in our notation). This is the reason why we call this map (for an arbitrary \( \mathfrak{g} \)) the Miura transformation.

One can also write down explicit formulas for the Miura transformation for other simple Lie algebras of classical types. As we have seen in Section 2.2 (following [DS]),
in the case of the Lie algebras \( sp_{2n} \) and \( so_{2n+1} \) the spaces of opers consist of self-adjoint differential operators \( L: \Omega^{-n+1/2} \rightarrow \Omega^{n+1/2} \) of order \( 2n \) (resp., anti-self adjoint operators \( L: \Omega^{-n} \rightarrow \Omega^{n+1} \) of order \( 2n+1 \)) with principal symbol 1. Identifying the Cartan subalgebras of these Lie algebras with \( \mathbb{C}^n \), we obtain an identification of the corresponding space of connections with the space of \( n \)–tuples \( (u_1(t), \ldots, u_n(t)) \), where \( u_i(t) \in \mathbb{C}((t)) \). Then the Miura transformation takes the form

\[
L = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t))(\partial_t - u_n(t)) \ldots (\partial_t - u_1(t))
\]

for \( g = sp_{2n} \) and

\[
L = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t))\partial_t(\partial_t - u_n(t)) \ldots (\partial_t - u_1(t))
\]

for \( g = so_{2n+1} \).

Finally, in the case of \( g = so_{2n} \) the Miura transformation is realized by the formula

\[
L = (\partial_t + u_1(t)) \ldots (\partial_t + u_n(t))\partial_t^{-1}(\partial_t - u_n(t)) \ldots (\partial_t - u_1(t))
\]

(see [DS], Sect. 8).

2.6. Singularities of Miura opers. Now suppose that we are given a Miura oper of coweight \( \lambda \) on the disc \( D_x \) such that the reduction \( \mathcal{F}_{B,x} \) has relative position \( w \) with \( \mathcal{F}_{B,x} \) at \( x \). The restriction of this Miura oper to the punctured disc \( D_x^\times \) is generic by Lemma 2.6 (which is easily generalized to the case of an arbitrary \( \lambda \)), and hence it corresponds, by Proposition 2.8, to a connection \( \nabla \) on the \( H \)–bundle \( \Omega^p \) over \( D_x^\times \). We would like to describe the singularity of this connection at \( x \).

First of all, we claim that \( \nabla \) has a regular singularity at \( x \). Indeed, the corresponding \( G \)–oper may be represented by the connection operator \( \nabla = \nabla + p_{-1} \). Let us choose a coordinate \( t \) at \( x \) and write \( \nabla = \partial_t + u(t) \), where \( u(t) \in \mathfrak{h}(\!(t)\!) \). Then

\[
(2.21) \quad \nabla = \partial_t + p_{-1} + u(t), \quad u(t) \in \mathfrak{h}(\!(t)\!).
\]

The corresponding \( G \)–oper should be regular, i.e., there should exist an element \( g \in N(\!(t)\!) \) such that \( g\nabla g^{-1} \) has no singularity at \( t = 0 \), so that the equation \( g\nabla g^{-1} \cdot \phi(t) = 0 \) has solutions in \( G[\![t]\!] \) for arbitrary initial conditions in \( G \). But then the equation \( \nabla \phi(t) = 0 \) would have solutions in \( G(\!(t)\!) \) (for arbitrary initial conditions in \( G \)). This implies that \( \nabla \) has at most regular singularity. Suppose that this is not so. Then there would exist a dominant integral weight \( \chi \) such that \( \langle \chi, u(t) \rangle \) has a pole of order higher than 1. But then consider the equation \( \nabla \phi(t) = 0 \), where \( \phi(t) \) takes values in \( V_{-\mathfrak{w}_0(\chi)} \). Clearly, the component of the solution lying in the subspace of lowest weight \( -\chi \) would not belong to \( \mathbb{C}(\!(t)\!) \), which is a contradiction.

Thus,

\[
u(t) = \hat{\mu}t^{-1} + \text{reg}
\]

for some integral coweight \( \hat{\mu} \). Using the gauge transformation with \( \hat{\rho}(t) \in B(\!(t)\!) \), we obtain that the operator \( \nabla \) given by (2.21) is gauge equivalent to the operator

\[
\partial_t + \frac{1}{t}(p_{-1} - \hat{\rho} + \hat{\mu} + t(\ldots)).
\]

Therefore the \( G \)–oper corresponding to \( \nabla \) is an oper with regular singularity (see Section 2.8), whose residue in \( \mathfrak{h}/W \) is equal to the image of \( -\hat{\rho} + \hat{\mu} \in \mathfrak{h} \). But by our
assumption this oper belongs to $\text{Op}_G(D_x)_{\lambda}$, hence its residue is the image of $-\lambda - \rho$ in $\mathfrak{h}/W$. Therefore we obtain that there exists $y \in W$ such that $-\rho + \mu = -y(\lambda + \rho)$, i.e., $\mu = \rho - y(\lambda + \rho)$.

We wish to show that $y = w$, where $w$ is the relative position of $\mathcal{F}'_{B,x}$ with $\mathcal{F}_{B,x}$. Let us make a more precise statement.

Denote by $\text{Conn}_{RS}^{D_x,\lambda,w}$ the variety of all connections on the $H$–bundle $\Omega^\rho$ with regular singularity at $x$ and residue $-w(\lambda + \rho) + \rho$. We have a morphism

$$\mathbf{b}_{\lambda,w} : \text{Conn}_{RS}^{D_x,\lambda,w} \to \text{Op}_G^{RS}(D_x)$$

defined as in Section 2.4, namely, we send a connection $\nabla \in \text{Conn}_{RS}^{D_x,\lambda,w}$ to the oper $(\mathcal{F}, \nabla, \mathcal{F}_B)$ where we set $\mathcal{F} = \Omega^\rho \times G$, $\mathcal{F}_B = \Omega^\rho \times B$ and $\nabla = \nabla + p$.

Explicitly, after choosing a coordinate $t$ on $D$, we can write $\nabla$ as $\partial_t + t^{-1}u(t)$, where $u(t) \in \mathfrak{h}[[t]]$. Then the corresponding oper with regular singularity is the $N([t])$–equivalence class of the operator

$$\nabla = \partial_t + p - t^{-1}u(t),$$

which is the same as the $N[[t]]$–equivalence class of the operator

$$\rho(t)\nabla \rho(t)^{-1} = \partial_t + t^{-1}(p - \rho + u(t))$$

(so it is indeed an oper with regular singularity).

Denote by $\text{Conn}_{reg}^{D_x,\lambda,w}$ the reduced part of the preimage of $\text{Op}_G(D_x)_{\lambda} \subset \text{Op}_G^{RS}(D_x)$ under this morphism. Then we have a morphism

$$\mathbf{b}_{\lambda,w} : \text{Conn}_{reg}^{D_x,\lambda,w} \to \text{MOp}_G(D_x)_{\lambda},$$

which sends $\nabla \in \text{Conn}_{reg}^{D_x,\lambda,w}$ to the Miura oper $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}'_B)$, where $(\mathcal{F}, \nabla, \mathcal{F}_B)$ are as above and $\mathcal{F}'_B = \Omega^\rho \times w_0 B$, where $w_0$ is the longest element of the Weyl group. Recall that in Section 2.4 we have established an isomorphism between the variety $\text{MOp}_G(D_x)_{\lambda}$ of Miura opers of coweight $\lambda$ on $D_x$ and the product $\text{Op}_G(D_x)_{\lambda} \times (G/B)\mathcal{F}'_{B,x}$. Denote by $\text{MOp}_G(D_x)_{\lambda,w} \subset \text{MOp}_G(D_x)_{\lambda}$ the subvariety of those Miura opers of coweight $\lambda$ which have relative position $w$ at $x$. Then $\text{MOp}_G(D_x)_{\lambda,w} \simeq \text{Op}_G(D_x)_{\lambda} \times S_w\mathcal{F}'_{B,x}$.

We wish to show that each map $\mathbf{b}_{\lambda,w}$ is an isomorphism between $\text{Conn}_{reg}^{D_x,\lambda,w}$ and $\text{MOp}_G(D_x)_{\lambda,w}$. The following result is proved by joint work [FG] with D. Gaitsgory in a more general setting.

**Proposition 2.9.** For each $w \in W$ the morphism $\mathbf{b}_{\lambda,w}$ is an isomorphism between the varieties $\text{Conn}_{reg}^{D_x,\lambda,w}$ and $\text{MOp}_G(D_x)_{\lambda,w}$.

**Proof.** First we observe that at the level of points the map defined by $\mathbf{b}_{\lambda,w}, w \in W$, from the union of $\text{Conn}_{reg}^{D_x,\lambda,w}, w \in W$, to $\text{MOp}_G(D_x)_{\lambda}$, is a bijection. Indeed, by Proposition 2.8 we have a map taking a Miura oper from $\text{MOp}_G(D_x)_{\lambda}$, considered as a Miura oper on the punctured disc $D_x^\times$, to a connection $\nabla$ on the $H$–bundle $\Omega^\rho$ over
We have shown above that $\nabla$ has regular singularity at $x$ and that its residue is of the form $-w(\lambda + \rho) + \tilde{\rho}$, $w \in W$. Thus, we obtain a map from the set of points of $\text{MOp}_G(D_x)^*$ to the union of $\text{Conn}^{\text{reg}}_{D_x,\lambda,w}$, $w \in W$, and by Proposition 2.8 it is a bijection.

It remains to show that if the Miura oper belongs to $\text{MOp}_G(D_x)^*_{\lambda,w}$, then the corresponding connection has residue precisely $-w(\lambda + \rho) + \tilde{\rho}$.

Thus, we are given a $G$–oper $(\mathcal{F}, \nabla, \mathcal{F}_B, \mathcal{F}_B')$ of coweight $\lambda$. Let us choose a trivialization of the $B$–bundle $\mathcal{F}_B$. Then the connection operator reads

$$\nabla = \partial_t + \sum_{i=1}^{\ell} t^{(\alpha_i,\lambda)} f_i + v(t), \quad v(t) \in \mathfrak{b}[[t]].$$

Suppose that the horizontal $B$–reduction $\mathcal{F}_B'$ of our Miura oper has relative position $w$ with $\mathcal{F}_B$ at $x$ (see Section 2.4 for the definition of relative position). We need to show that the corresponding connection on $\mathcal{F}_B' \simeq \Omega^b$ has residue $-w(\lambda + \rho) + \tilde{\rho}$.

This is equivalent to the following statement. Let $\Phi(t)$ be the $G$–valued solution of the equation

$$(2.23) \quad \left( \partial_t + \sum_{i=1}^{\ell} t^{(\alpha_i,\lambda)} f_i + v(t) \right) \Phi(t) = 0,$$

such that $\Phi(0) = 1$. Since the connection operator is regular at $t = 0$, this solution exists and is unique. Then $\Phi(t)w^{-1}w_0$ is the unique solution of the equation (2.23) whose value at $t = 0$ is equal to $w^{-1}w_0$.

By Lemma 2.6 we have

$$\Phi(t)w^{-1}w_0 = X_w(t)Y_w(t)Z_w(t)w_0,$$

where $X_w(t) \in N((t))$, $Y_w(t) \in H((t))$, $Z_w(t) \in N_-((t))$.

We can write $Y_w(t) = \tilde{\mu}_w(t)\tilde{Y}_w(t)$, where $\tilde{\mu}_w$ is a coweight and $\tilde{Y}_w(t) \in H[[t]]$.

Since the connection $\nabla$ preserves

$$\Phi(t)w_0 \mathfrak{b}_+ w_0 \Phi(t)^{-1} = \Phi(t)\mathfrak{b}_- \Phi(t)^{-1},$$

the connection $X(t)^{-1}_w \nabla X_w(t)$ preserves

$$Y_w(t)Z_w(t)\mathfrak{b}_- Z_w(t)^{-1}Y_w(t)^{-1} = \mathfrak{b}_-,$$

and therefore has the form

$$\partial_t + \sum_{i=1}^{\ell} t^{(\alpha_i,\lambda)} f_i - \frac{\tilde{\mu}_w}{t} + u(t), \quad u(t) \in \mathfrak{b}[[t]].$$

By conjugating it with $\tilde{\lambda}(t)$ we obtain a connection

$$\partial_t + p_- - \frac{\tilde{\lambda} + \tilde{\mu}_w}{t} + u(t), \quad u(t) \in \mathfrak{b}[[t]].$$

Therefore we need to show that

$$(2.24) \quad \tilde{\mu}_w = w(\tilde{\lambda} + \tilde{\rho}) - (\tilde{\lambda} + \tilde{\rho}).$$
To see that, let us apply the identity \( \Phi(t)w^{-1} = X_w(t)Y_w(t)Z_w(t) \) to a non-zero vector \( v_{w_0(\nu)} \) of weight \( w_0(\nu) \) in a finite-dimensional irreducible \( g \)-module \( V_\nu \) of highest weight \( \nu \) (so that \( v_{w_0(\nu)} \) is a lowest weight vector and hence is unique up to scalar). The right hand side will then be equal to a \( P(t)v_{w_0(\nu)} \) plus the sum of terms of weights greater than \( w_0(\nu) \), where \( P(t) = ct^{(w_0(\nu),\mu_\omega)}, c \neq 0 \), plus the sum of terms of higher degree in \( t \). Applying the left hand side to \( v_{w_0(\nu)} \), we obtain \( \Phi(t)v_{w^{-1}w_0(\nu)} \), where \( v_{w^{-1}w_0(\nu)} \in V_\nu \) is a non-zero vector of weight \( w^{-1}w_0(\nu) \) which is also unique up to a scalar.

Thus, we need to show that the coefficient with which \( v_{w_0(\nu)} \) enters \( \Phi(t)v_{w^{-1}w_0(\nu)} \) is a polynomial in \( t \) whose lowest degree is equal to

\[
\langle w_0(\nu), w(\tilde{\lambda} + \tilde{\rho}) - (\tilde{\lambda} + \tilde{\rho}) \rangle,
\]

because if this is so for all dominant integral weights \( \nu \), then we obtain the desired equality \( \text{(2.24)} \). But this formula is easy to establish. Indeed, from the form \( \text{(2.22)} \) of the oper connection \( \nabla \) it follows that we can obtain a vector proportional to \( v_{w_0} \) by applying the operators \( t^{(\alpha_i, \tilde{\lambda})} f_i, i = 1, \ldots, \ell \), to \( v_{w^{-1}w_0(\nu)} \) in some order. The linear combination of these monomials appearing in the solution is the term of the lowest degree in \( t \) with which \( v_{w_0(\nu)} \) enters \( \Phi(t)v_{w^{-1}w_0(\nu)} \). It follows from Lemma \( \text{(2.10)} \) that it is non-zero. The corresponding power of \( t \) is nothing but the difference between the \( (\tilde{\lambda} + \tilde{\rho}) \)-degrees of the vectors \( v_{w^{-1}w_0} \) and \( v_{w_0} \), i.e.,

\[
\langle w^{-1}w_0(\nu), \tilde{\lambda} + \tilde{\rho} \rangle - \langle w_0(\nu), \tilde{\lambda} + \tilde{\rho} \rangle = \langle w_0(\nu), w(\tilde{\lambda} + \tilde{\rho}) - (\tilde{\lambda} + \tilde{\rho}) \rangle,
\]

as desired. This completes the proof.

Suppose we are given a Miura oper on \( D_x \) with \( \tilde{\lambda} = 0 \) that has relative position \( s_i \) at \( x \). Then the corresponding connection on \( \Omega^B \) has residue \(-s_i(\tilde{\rho}) + \tilde{\rho} = \bar{\alpha}_i \). Choosing a coordinate \( t \) at \( x \), we write this connection as

\[
(2.25) \quad \nabla = \partial_t + \frac{\bar{\alpha}_i}{t} + u(t), \quad u(t) \in \mathfrak{g}[[t]].
\]

**Lemma 2.10.** A connection of the form \( \text{(2.25)} \) belongs to \( \text{Conn}^{\text{reg}}_{D_x,s_i} \) (i.e., the corresponding \( G \)-oper is regular at \( x \)) if and only if \( \langle \alpha_i, u(0) \rangle = 0 \).

**Proof.** Let \( V_{\omega_i} \) be the \( i \)th fundamental representation of \( g \). It contains a one-dimensional subspace \( L_{\omega_i} \) of \( B \)-invariants. There is a canonical two-dimensional subspace \( W_{\omega_i} \) of \( V_{\omega_i} \) stable under \( B \), containing \( L_{\omega_i} \), and on which the \( SL_2 \) subgroup corresponding to the \( i \)th simple root acts irreducibly. Moreover, the generators \( f_j, j \neq i \), act on \( W_{\omega_i} \) by \( 0 \). Consider the vector bundle

\[ V_{\omega_i, F} = F \times V_{\omega_i} = F_B \times V_{\omega_i} \]

and the corresponding rank two subbundle \( W_{\omega_i, F_B} \). The connection \( \nabla \) preserves \( W_{\omega_i, F_B} \), and its restriction to \( W_{\omega_i, F_B} \) is equal to

\[
\partial_t + \left( \begin{array}{cc} \frac{1}{2} + \frac{1}{2}u_i(t) & 0 \\ \frac{1}{2} & -1 - \frac{3}{2}u_i(t) \end{array} \right),
\]

where \( u_i(t) = \langle \alpha_i, u(t) \rangle \).
The corresponding equation \( \nabla \Phi = 0 \) has two linearly independent solutions:

\[
\Phi_1 = \begin{pmatrix} 0 \\ t e^{\int u_i(t) dt} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} -t^{-1} e^{-\int u_i(t) dt} \\ t e^{\int u_i(t) dt} \int t^{-2} e^{-2 \int u_i(t) dt} dt \end{pmatrix}.
\]

Hence the monodromy of these solutions is equal to

\[
\begin{pmatrix} 1 & -4\pi i u_i(0) \\ 0 & 1 \end{pmatrix}.
\]

This implies that this oper is regular only if \( u_i(0) = 0 \).

Conversely, if \( u_i(0) = 0 \), then applying to the connection \( \nabla \) the gauge transformation with \( \exp(-e_i/t) \) we obtain a regular connection. Hence the corresponding oper is regular. This completes the proof. \( \square \)

The above calculation also implies that the scheme-theoretic preimage of \( \text{Op}_G(D_x) \) under the morphism \( \text{Conn}^{\text{RS}}_{D_x,s_i} \to \text{Op}_G^{\text{RS}}(D_x) \) is in fact reduced, and therefore it is equal to \( \text{Conn}^{\text{reg}}_{D_x,s_i} \).

3. Bethe Ansatz equations and Miura opers on \( \mathbb{P}^1 \)

In this section we consider Miura opers and the corresponding \( H \)-connections on \( \mathbb{P}^1 \). We show that the Miura opers having the simplest possible degenerations are described by the solutions of the so-called Bethe Ansatz equations. This will allow us eventually to describe the set of solutions of the Bethe Ansatz equations as an open dense subset of the flag variety.

3.1. Miura opers on \( \mathbb{P}^1 \). Let us fix a set of distinct points \( z_1, \ldots, z_N \) on \( \mathbb{P}^1 \) such that \( z_i \neq \infty \) for all \( i = 1, \ldots, N \), and a set of dominant coweights \( \lambda_1, \ldots, \lambda_N \) of \( g \). Let \( \text{Op}_G^{\text{RS}}(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty} \) be the set of \( G \)-opers on \( \mathbb{P}^1 \) which are regular at all points other than \( z_1, \ldots, z_N, \infty \) and have regular singularities at \( z_1, \ldots, z_N, \infty \) with the residues \( \lambda_1, \ldots, \lambda_N, \lambda_\infty \). More precisely, this is the subset of the set \( \text{Op}_G(\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}) \) consisting of those opers whose restriction to the punctured disc \( D^\times_{z_i} \) at the point \( z_i \) (resp., \( D^\times_{\infty} \)) belongs to \( \text{Op}_G^{\text{RS}}(D_{z_i})_{\lambda_i} \) for all \( i = 1, \ldots, N \) (resp., to \( \text{Op}_G^{\text{RS}}(D_{\infty})_{\lambda_\infty} \)).

Let \( \lambda_\infty \) be another dominant coweight of \( g \). Introduce a subset

\[
\text{Op}_G(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty} \subset \text{Op}_G^{\text{RS}}(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty}
\]
of those \( G \)-opers whose restriction to \( D^\times_{z_i} \) belongs to \( \text{Op}_G(D_{z_i})_{\lambda_i} \subset \text{Op}_G^{\text{RS}}(D_{z_i})_{\lambda_i} \) for all \( i = 1, \ldots, N \) and whose restriction to \( D^\times_{\infty} \) belongs to \( \text{Op}_G(D_{\infty})_{\lambda_\infty} \subset \text{Op}_G^{\text{RS}}(D_{\infty})_{\lambda_\infty} \). Denote by \( \text{MO}_G(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty} \) the space of Miura opers on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \) whose underlying opers belong to \( \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty} \).

Let \( \tau = (\mathcal{F}, \nabla, \mathcal{F}_B) \) be an oper from \( \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty; (\lambda_i), \lambda_\infty} \). The above conditions mean that the oper bundle \( \mathcal{F} \), which is a priori defined on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \), has a canonical extension to the entire \( \mathbb{P}^1 \). By Lemma 2.3, the monodromy around each of the points \( z_1, \ldots, z_N, \infty \) is trivial. Therefore the flat connection \( \nabla \) has the trivial monodromy representation and therefore defines a global trivialization of the oper bundle \( \mathcal{F} \). Hence, by Lemma 2.3, the space \( \text{MO}_G(\mathbb{P}^1)_\tau \) of Miura \( G \)-opers on \( \mathbb{P}^1 \) whose underlying oper is \( \tau \) is isomorphic to the flag variety \( G/B \) of \( G \). Indeed, any \( B \)-reduction of
the fiber of this bundle at an arbitrary point \( x \) of \( \mathbb{P}^1 \) uniquely extends to a horizontal \( B \)-reduction of \( \mathcal{T} \) on the entire \( \mathbb{P}^1 \). But a \( B \)-reduction of the fiber of the trivial bundle at \( x \) is the same as a point of \( (G/B)_x \), which is isomorphic to \( G/B \).

On the other hand, let \( \text{Conn}(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \) be the space of connections on the \( H \)-bundle \( \Omega^\rho \) on \( \mathbb{P}^1 \) with regular singularities at the points \( z_1, \ldots, z_N, \infty \) and a finite number of other points \( w_1, \ldots, w_m \) such that the residue at \( z_i \) (resp., \( \infty, w_j \)) is equal to \(-y_i(\lambda_i + \bar{\rho}) + \bar{\rho} \) (resp., \(-y_\infty(\lambda_\infty + \bar{\rho}) + \bar{\rho}, -y'_j(\bar{\rho}) + \bar{\rho} \)) for some elements \( y_i, y_\infty, y'_j \in W \).

Such a connection then has the form

\[
\partial_t - \sum_{i=1}^N \frac{y_i(\lambda_i + \bar{\rho})}{t - z_i} - \sum_{j=1}^m \frac{y'_j(\bar{\rho}) - \bar{\rho}}{t - w_j}
\]

on \( \mathbb{A}^1 = \mathbb{P}^1 \setminus \infty \). According to formula \((2.19)\), connection \( \partial_t + f(t) \) on \( \Omega^\rho \) over \( \mathbb{A}^1 \) has the following expansion on the disc around \( \infty \in \mathbb{P}^1 \) with respect to the coordinate \( u = t^{-1} \):

\[
\partial_u - u^{-2} f(u^{-1}) + 2\bar{\rho} u^{-1}.
\]

Therefore the residue of the connection \((3.1)\) at \( \infty \) is equal to

\[
2\bar{\rho} + \sum_{i=1}^N (y_i(\lambda_i + \bar{\rho}) - \bar{\rho}) + \sum_{j=1}^m (y'_j(\bar{\rho}) - \bar{\rho}).
\]

On the other hand, by our assumption, it should be equal to \(-y_\infty(\lambda_\infty + \bar{\rho}) + \bar{\rho} \) for some \( y_\infty \in W \). Denoting \( y'_\infty \) by \( y'_\infty w_0 \), we obtain the following equation relating the residues of our connection:

\[
\sum_{i=1}^N (y_i(\lambda_i + \bar{\rho}) - \bar{\rho}) + \sum_{j=1}^m (y'_j(\bar{\rho}) - \bar{\rho}) = y'_\infty (-w_0(\lambda_\infty) + \bar{\rho}) - \bar{\rho}.
\]

To a connection of this form we associate a \( G \)-oper on \( \mathbb{P}^1 \) with regular singularities at \((z_i), (\infty, w_j))\) in the same way as above. Namely, we set

\[
\mathcal{T} = \Omega^\rho \times G, \quad \mathcal{T}_B = \Omega^\rho \times B, \quad \nabla = \nabla + p_{-1}.
\]

This oper belongs to the set

\[
\text{Op}_G(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \quad (z_i), (\infty, (\lambda)), (\lambda) \in \mathbb{P}^1
\]

defined in the same way as at the beginning of this section. Note that by construction the residue of this oper at \( z_i \) is in the \( W \)-orbit of \( \lambda_i \), whereas at \( w_j \) it is in the \( W \)-orbit of \( 0 \).

Thus, we have a map

\[
\text{Conn}(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \rightarrow \text{Op}_G(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}.
\]

Let \( \text{Conn}(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \) be the subset of \( \text{Conn}(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \) consisting of those connections for which the resulting oper \( \tau \) on \( \mathbb{P}^1 \) belongs to \( \text{Op}_G(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}} \). The resulting map

\[
\mathcal{F}_{(z_i), (\lambda)} : \text{Conn}(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)} \rightarrow \text{Op}_G(\mathbb{P}^1)_{RS}^{(\infty),(\lambda)}
\]
may be lifted to a map
\[ b_{(z_i),\infty;\lambda_i,\lambda_\infty} : \text{Conn}(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \to \text{MOp}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty}. \]

Here \( \text{MOp}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \) is the space of Miura opers on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \) such that the underlying oper belongs to \( \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \). Namely, we give an oper \( \tau \) that is in the image of \( \overline{\mathcal{F}}_{(z_i),\infty;\lambda_i,\lambda_\infty} \) the structure of a Miura oper by defining a horizontal \( B \)-reduction \( \mathcal{F}'_B \) of \( \mathcal{F} \) by the formula
\[ \mathcal{F}'_B = \Omega^H \times w_\lambda B. \]

Next, we construct the map
\[ a_{(z_i),\infty;\lambda_i,\lambda_\infty} : \text{MOp}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \to \text{Conn}(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty}. \]

Any Miura oper on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \) from \( \text{MOp}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \) becomes generic after removing finitely many points \( w_j \). Hence it gives rise to a connection \( \nabla \) on the bundle \( \mathcal{F}_H \simeq \Omega^H \) over \( \mathbb{P}^1 \setminus \{(z_i), (w_j), \infty\} \). The restrictions of this connection to the discs around the points \( z_i, w_j \) and \( \infty \) must have regular singularities with the residues being in the \( W \)-orbits of \( \lambda_i, 0 \), and \( \lambda_\infty \), respectively. Therefore this connection must be of the form \( (3.1) \). This defines a map \( a_{(z_i),\infty;\lambda_i,\lambda_\infty} \). In the same way as in Proposition 2.8 and Proposition 2.9 we show that the maps \( a_{(z_i),\infty;\lambda_i,\lambda_\infty} \) and \( \overline{\mathcal{F}}_{(z_i),\infty;\lambda_i,\lambda_\infty} \) are mutually inverse bijections.

Let us fix an oper
\[ \tau \in \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \]
and trivialize the underlying \( G \)-bundle \( \mathcal{F} \) by identifying the fiber at the point \( \infty \in \mathbb{P}^1 \) with \( G \). Then the connection trivializes the bundle \( \mathcal{F} \) by identifying all fibers with the fiber at \( \infty \) and hence with \( G \). Therefore we also obtain a trivialization of the corresponding \( G/B \)-bundle, and so the reduction \( \mathcal{F}_B \) gives us a map \( \phi_\tau : \mathbb{P}^1 \to G/B \).

Note that giving \( \tau \) the structure of a Miura oper amounts to picking a point in the flag variety \( G/B \). If this point belongs to the \( B \)-orbit \( S_{y_\infty} = B y_\infty^{-1} w_\lambda B \subset G/B \) (i.e., if the corresponding \( B \)-reduction and the oper reduction of the fiber at \( \infty \) are in relative position \( y_\infty \); see the definition in Section 2.3), then the connection has the residue \( -y_\infty (\lambda_\infty + \tilde{\rho}) + \tilde{\rho} \) at \( \infty \). Furthermore, identifying the fiber of \( \mathcal{F} \) at \( \infty \) with the fiber of \( \mathcal{F} \) at \( z_i \), we obtain a reduction of \( \mathcal{F}_{z_i} \) to \( B \). According to Proposition 2.9 this reduction then has relative position \( y_i \) with the oper reduction \( \mathcal{F}_{B,z_i} \), precisely when the residue of our connection at \( z_i \) is equal to \( -y_i (\lambda_i + \tilde{\rho}) + \tilde{\rho} \).

Likewise, the points \( w_j \)'s are the points where our \( B \)-reduction is not in generic position with the oper reduction, and it is then in relative position \( y_j' \) precisely when the residue of our connection at \( w_j \) is equal to \( -y_j' (\tilde{\rho}) + \tilde{\rho} \), by Proposition 2.9. All of these residues must satisfy the relation \( (3.2) \). Thus, we obtain the following result.

**Theorem 3.1.** The map \( b_{(z_i),\infty;\lambda_i,\lambda_\infty} \) is a bijection at the level of points. Thus, the set of all connections from \( \text{Conn}(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \) which correspond to a fixed \( G \)-oper \( \tau \in \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty;\lambda_i,\lambda_\infty} \) is isomorphic to the set of points of the flag variety \( G/B \).
Moreover, the residues of these connections at the points $z_i$ (resp., the points $w_j$) are equal to $R_{z_i} = -y_i(\lambda_i + \hat{\rho}) + \hat{\rho}$ (resp., $R_{w_j} = -y'_j(\hat{\rho}) + \hat{\rho}$) for some elements $y_i, y'_j \in W$ and they must satisfy the relation
\[
\sum_{i=1}^n R_{z_i} + \sum_{j=1}^m R_{w_j} = -y'_\infty(-w_0(\lambda_\infty) + \hat{\rho}) + \hat{\rho}
\]
for some $y'_\infty \in W$. The set of those connections which satisfy this relation is in bijection with the Schubert cell $Bw_0y'_\infty w_0B$ in $G/B$.

3.2. Bethe Ansatz equations. Let $\tau$ be again an oper from $\text{Op}_G(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ and suppose we have a Miura oper projecting onto $\tau$ under the Miura transformation. Let $\phi_\tau : \mathbb{P}^1 \to G/B$ be the map corresponding to the reduction $\mathcal{F}_B$. Recall that its value at $x \in \mathbb{P}^1$ is $\mathcal{F}_{B,x}$ considered as a point of $(G/B)_{\mathcal{F}_{B,x}} \simeq (G/B)_{\mathcal{F}_{B,\infty}} \simeq G/B$, where the first isomorphism is obtained from the identification of the fibers of $\mathcal{F}_B$ induced by the oper connection and the second isomorphism corresponds to a choice of trivialization of $(G/B)_{\mathcal{F}_{B,\infty}}$. Consider the subvariety $(G/B)_\tau$ of $G/B$ whose points $p$ satisfy the following conditions:

1. $\phi_\tau(z_i)$ is in generic position with $p$ for all $i = 1, \ldots, N$;
2. the relative position of $\phi_\tau(x)$ and $p$ is either generic or corresponds to a simple reflection $s_i \in W$ for all $x \in \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$.

It is clear that $(G/B)_\tau$ is an open and dense subvariety of $G/B$. Indeed, $(G/B)_\tau$ is contained in the intersection $U_\tau$ of finitely many open and dense subsets, namely, the sets of points of $G/B$ which are in generic relative position with $\phi_\tau(z_i)$ (each is isomorphic to the big Schubert cell). The complement of $(G/B)_\tau$ in $U_\tau$ is a subvariety of codimension one. This subvariety consists of all points in $G/B$ which are in relative position $w$ with $\phi_\tau(x), x \in \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$, where $w$ runs over the subset of $W$ of all elements of length $l(w) \geq 2$. The subvariety of these points for fixed $x$ has codimension two, and therefore their union, as $x$ moves along the curve $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$, has codimension (at least) one.

Let
\[
\text{MOP}_G(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}^\text{gen} \subset \text{MOP}_G(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}
\]
be the open dense subvariety which is the union of $(G/B)_\tau, \tau \in \text{Op}_G(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$.

Note that equation (3.2) now reads
\[
(3.3) \quad \sum_{i=1}^N \lambda_i - \sum_{j=1}^m \alpha_{ij} = y(-w_0(\lambda_\infty) + \hat{\rho}) - \hat{\rho},
\]
where we write $y = y'_\infty$ to simplify notation.

Consider the image of $\text{MOP}_G(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}^\text{gen}$ in $\text{Conn}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ under the bijection $b_{(z_i),\infty;(\lambda_i),\lambda_\infty}$. We denote it by $\text{Conn}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}^\text{gen}$. Then according to
Lemma 2.10. \( \text{Conn}(\mathbb{P}^1)^{\text{gen}}_\lambda \) is precisely the set of all connections of the form

\[(3.4) \quad \nabla = \partial_t - \sum_{i=1}^{N} \frac{\dot{\lambda}_i}{t - z_i} + \sum_{j=1}^{m} \frac{\dot{\alpha}_{ij}}{t - w_j},\]

where \( w_1, \ldots, w_m \) are points of \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \) and \( i_j \in I \) for all \( j = 1, \ldots, m \), such that if

\[\partial_t + \frac{\dot{\alpha}_{ij}}{t - w_j} + u_j(t - w_j), \quad u_j(u) \in \mathfrak{h}[u],\]

is the expansion of the connection \((3.3)\) at the point \( w_j \), then \( \langle \alpha_{ij}, u_j(0) \rangle = 0 \) for all \( j = 1, \ldots, m \). Explicitly, these equations read

\[(3.5) \quad \sum_{i=1}^{N} \frac{\langle \alpha_{ij}, \dot{\lambda}_i \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \alpha_{ij}, \dot{\alpha}_{is} \rangle}{w_j - w_s} = 0, \quad j = 1, \ldots, m.\]

They are called the Bethe Ansatz equations. We have an obvious action of a product of symmetric groups permuting the points \( w_j \), that is, we can permute the \( w_j \). Furthermore, it follows from Theorem 3.1 that those elements of \( \text{MOp}_G(\mathbb{P}^1)^{\text{gen}}_\lambda \) which satisfy formula \((3.3)\) correspond to points that lie in the Schubert cell \( Bw_0w_0B \)

The set of solutions of the Bethe Ansatz equations \((3.5)\) is in bijection with the set of points of \( \text{MOp}_G(\mathbb{P}^1)^{\text{gen}}_\lambda \). Furthermore, it follows from Theorem 3.1 that those elements of \( \text{MOp}_G(\mathbb{P}^1)^{\text{gen}}_\lambda \) which satisfy formula \((3.3)\) correspond to points that lie in the Schubert cell \( Bw_0w_0B \). Therefore we obtain

**Theorem 3.2.** The set of solutions of the Bethe Ansatz equations \((3.5)\) is in bijection with the set of points of \( \text{MOp}_G(\mathbb{P}^1)^{\text{gen}}_\lambda \). Note that except for the big cell \( Bw_0B \), the intersection between the Schubert cell \( Bw_0w_0B \) and the open dense subset \( (G/B)_{\tau} \) could be either an open dense subset of \( Bw_0w_0B \) or empty.\(^1\) Therefore we obtain

**Corollary 3.3.** The set of those solutions of the Bethe Ansatz equations which correspond to a fixed \( G \)-oper \( \tau \) is in bijection with the set of points of an open and dense subset \( (G/B)_{\tau} \) of the flag variety \( G/B \). Further, every solution must satisfy the equation \((3.3)\) for some \( y \in W \), and the solutions which satisfy this equation with fixed \( y \in W \) are in bijection with an open subset of the Schubert cell \( Bw_0w_0B \subseteq G/B \).

\(^1\)For example, it follows from the results of Mukhin and Varchenko in [MV2] that sometimes this open set may not contain the one point Schubert cell \( B \subset G/B \) even if we allow \( z_1, \ldots, z_N \) to be generic.
3.3. The action of $N$ on solutions of the Bethe Ansatz equations. The group $N$ naturally acts on $G/B$, and thus we obtain an action of $N$ on the set of solutions of the Bethe Ansatz equations which correspond to a fixed $G$–oper. This action is however rational, because solutions of the Bethe Ansatz equations correspond to points of an open dense subset of $G/B$, not the entire $G/B$.

Let us identify the set of solutions of the Bethe Ansatz equations with an open dense subset of the flag variety by using the fiber at $0 \in \mathbb{P}^1$, instead of $\infty \in \mathbb{P}^1$. Then the action of $g \in N$ is given by gauge transformations on a connection of the form

$$\partial_t + p_{-1} + u(t), \quad u(t) \in \mathfrak{h},$$

by a rational $N$–valued function $g(t)$ such that $g(0) = g$ and

$$g(t)(\partial_t + p_{-1} + u(t))g(t)^{-1} = \partial_t + p_{-1} + \tilde{u}(t),$$

where $\tilde{u}(t)$ is again in $\mathfrak{h}$. Clearly, $g(t)$ is uniquely determined by these conditions.

By our assumptions, the connection $\partial_t + p_{-1} + u(t)$ has trivial monodromy representation. Therefore there exists a (unique) polynomial $G$–valued solution $\Phi(t)$ of the equation $(\partial_t + p_{-1} + u(t))\Phi(t) = 0$ with the initial condition $\Phi(0) = 1$. Because of the form of the connection, we find that $\Phi(t)$ actually takes values in $B_-$. Further, for any constant element $M$ of $G$, the solution of the equation $(\partial_t + p_{-1} + u(t))\Psi(t) = 0$ with the initial condition $\Psi(0) = M$ is $\Psi(t) = \Phi(t)M$.

Now if $\tilde{\Phi}(t)$ is the solution of the equation $(\partial_t + p_{-1} + \tilde{u}(t))\tilde{\Phi}(t) = 0$ with the initial condition $\tilde{\Phi}(0) = 1$ (like $\Phi(t)$, it takes values in $B_-$), we obtain the following equation:

$$\tilde{\Phi}(t)g^{-1} = g(t)^{-1}\tilde{\Phi}(t).$$

Thus, to find $g(t)$, we need to find $\Phi(t)$ and to project the function $\Phi(t)g^{-1}$ onto $N$ considered as an open dense subset of $G/B_-$ (in general, this may only be done for generic values of $t$).

Let us consider more explicitly the case when $g = \exp(\alpha_i)$, $i \in I, a \in \mathbb{C}$ (these one-parameter subgroups generate the action of the group $N$). We claim that $g(t)$ is then necessarily of the form $g(t) = \exp(f(t)e_i)$, where $f(t)$ is a rational function in $t$ such that $f(0) = a$.

Indeed, since $\tilde{\Phi}(t) \in B_-$, we obtain that the left hand side of (3.7) belongs to the $i$th minimal parabolic subgroup of $G$ generated by $B_-$ and the $SL_2$ subgroup corresponding to the $i$th simple root. Hence the left hand side must also belong to this parabolic subgroup, and therefore $g(t)$ necessarily has the form $\exp(f(t)e_i)$ for some rational function $f(t)$ satisfying $f(0) = a$.

Let us compute $f(t)$. We have

$$\exp(f(t)e_i)(\partial_t + p_{-1} + u(t))\exp(-f(t)e_i) =$$

$$\partial_t + (u(t) + f(t)\alpha_i) - (f'(t) + f(t)^2 + f(t)u_i(t))e_i,$$

where $u_i(t) = \langle \alpha_i, u(t) \rangle$. Therefore $f(t)$ has to be a rational solution of the differential equation

$$f'(t) + f(t)^2 + f(t)u_i(t) = 0$$

This equation can be solved by standard methods, and we obtain

$$f(t) = a_0 + b_0 t + \cdots + a_n t^n + b_n t^n$$

where $a_i, b_i \in \mathbb{C}$. Thus

$$\Phi(t) = \exp(f(t)e_i) = \exp(a_0 + b_0 t + \cdots + a_n t^n + b_n t^n).$$
with the initial condition \( f(0) = a \). Note that from the previous discussion we already know that such a solution exists and is unique. Then in the new connection we will have \( \tilde{u}(t) = u(t) + f(t)\tilde{\alpha}_i \). For generic values of \( a \) the function \( \tilde{u}(t) \) will have the same form as \( u(t) \). In particular, only the positions of the poles \( w_j \) such that \( i_j = i \) will be changed in \( u(t) \), and hence its poles will again give us a solution of the Bethe Ansatz equations \( (3.5) \).

Thus we obtain a rational action of the elements of the form \( \exp(a\lambda_i) \) on the set of solutions of the Bethe Ansatz equations. These actions generate a rational action of the group \( N \) on the set of solutions of the Bethe Ansatz equations corresponding to a fixed \( G \)-oper \( \tau \). By our construction, this action becomes the natural action of \( N \) on the flag variety \( G/B \) under the embedding of the set of solutions of \( (3.5) \) into the flag variety as an open dense subset.

3.4. Comparison with results of Mukhin and Varchenko. In [MV], Mukhin and Varchenko associated to each solution \( \{w_1, \ldots, w_m\} \) of the Bethe Ansatz equations \( (3.5) \) an \( I \)-tuple of polynomials \( (y_i(x)), i \in I = \{1, \ldots, \ell\} \), each defined up to a scalar, such that the roots of \( y_i(x) \) are precisely those \( w_j \)'s for which \( i_j = i \). Thus, they obtained an embedding of the set of solutions of \( (3.5) \) into the product of \( \ell \) copies of \( \mathbb{P}(\mathbb{C}[x]) \). Then they defined the “\( i \)th reproduction procedure” of solutions as follows. Set

\[
T_i(x) = \prod_{j=1}^{N} (x - z_j)^{\langle \lambda_j, \alpha_i \rangle}
\]

and let \( \tilde{y}_i \) be a new polynomial that has the form

\[
\tilde{y}_i(x) = y_i(x) \int_{x}^{x} \prod_{j \in I} y_j(t)^{-\langle \lambda_j, \alpha_i \rangle} dt.
\]

The closure of the set of polynomials of this form in \( \mathbb{P}(\mathbb{C}[x])^\ell \) is isomorphic to a projective line. It is proved in [MV] that for all but finitely many points of this line the \( I \)-tuple \( (y_1(x), \ldots, \tilde{y}_i(x), \ldots, y_\ell(x)) \) will again correspond to a solution of the Bethe Ansatz equations \( (3.5) \), and thus they obtain a rational map from \( \mathbb{P}^1 \) to the set of solutions of \( (3.5) \).

Let us show that the image of this map coincides with the closure of the orbit of the group \( \{\exp(a\lambda_i)\} \) acting on the set of solutions of \( (3.5) \) as explained above (this observation is due to Mukhin, Varchenko and myself). Indeed, given an \( I \)-tuple \( (y_i(x))_{i \in I} \) encoding a solution of the equations \( (3.5) \), the corresponding connection is given by the formula \( \partial_t + p_{-1} + u(t) \), where

\[
u(t) = -\sum_{j=1}^{N} \frac{\tilde{\lambda}_j}{t - z_j} + \sum_{i \in I} \tilde{\alpha}_i \frac{d}{dt} \log y_i(t),
\]

so that

\[
u_i(t) = \langle \alpha_i, u(t) \rangle = -\frac{d}{dt} \log \left( T_i(t) \prod_{j \in I} y_j(t)^{-\langle \lambda_j, \alpha_i \rangle} \right).
\]
But (3.9) implies that \( \tilde{y}_i(t) \) satisfies the equation
\[
\frac{d}{dt} \log \tilde{y}_i(t) = \frac{d}{dt} \log y_i(t) + \frac{d}{dt} \log \int_t^T y_j(x)^{-\langle \tilde{\alpha}_j, \alpha_i \rangle} \, dx.
\]

Therefore the connection corresponding to the new \( I \)-tuple \( (y_1(x), \ldots, \tilde{y}_i(x), \ldots, y_\ell(x)) \) has the form \( \partial_t + p_{-1} + \tilde{u}(t) \), where \( \tilde{u}(t) = u(t) + f(t) \tilde{\alpha}_i \), and \( f(t) \) satisfies the differential equation (3.8).

Therefore the \( i \)th reproduction procedure of [MV] on the set of solutions of (3.5) coincides with the action of the group \( \{ \exp(\alpha e_i) \} \) on the image of this set in \( G/B \). Mukhin and Varchenko use the reproduction procedures to construct “populations” of solutions of (3.5). A population is by definition the closure in \( \mathbb{P}(\mathbb{C}[x])^\ell \) of the set of all polynomials obtained by applying consecutively all possible reproduction procedures to an \( I \)-tuple of polynomials corresponding to a particular solution of (3.5). Conjecture 3.10 in [MV] then asserts that each population is isomorphic to the flag variety \( G/B \) and that the subset of the population corresponding to the \( I \)-tuples of polynomials of fixed degrees is isomorphic to a Schubert cell in \( G/B \). This has been proved in [MV] for \( g \) of types \( A_n, B_n \) and \( C_n \) and in [BM] for \( G_2 \) (note that in the convention of [MV] the Bethe Ansatz equations (3.5) correspond to the Langlands dual group \( L^G \), and so the relevant flag manifold is \( L^G/L^B \) rather than \( G/B \)).

Now this assertion immediately follows for an arbitrary simple Lie algebra \( g \) from Corollary 3.3 and the above discussion. Indeed, we have found in Corollary 3.3 that the set of solutions of the Bethe Ansatz equations is identified with an open dense subset of \( G/B \) and that those solutions which correspond to \( I \)-tuples of polynomials of fixed degrees correspond to points of Schubert cells in \( G/B \). Moreover, we have identified the reproduction procedures with the action of the one-parameter subgroups \( \{ \exp(\alpha e_i) \} \) on this set inside \( G/B \). But the closure of the union of the consecutive orbits of these subgroups is equal to the entire flag manifold \( G/B \). Hence we obtain that any population of solutions (in the terminology of [MV]) is indeed isomorphic to \( G/B \).

4. The Gaudin model and the Bethe Ansatz

In this section we consider a simple Lie algebra \( g \) and its Langlands dual Lie algebra \( L^G \) (whose Cartan matrix is the transpose of that of \( g \)). We will identify the set of roots of \( g \) with the set of coroots of \( L^G \) and the set of weights of \( g \) with the set of coweights of \( L^G \). The results on opers and Miura opers from the previous sections will be applied here to the Lie algebra \( L^G \).

4.1. The definition of the Gaudin model. Here we recall the definition of the Gaudin model and the realization of the Gaudin Hamiltonians in terms of the spaces of conformal blocks for affine Kac-Moody algebras of critical level. We follow closely the paper [FFR].

For a dominant integral weight \( \lambda \) denote by \( V_\lambda \) the irreducible representation of \( g \) of highest weight \( \lambda \). Choose a non-degenerate invariant inner product \( \kappa_0 \) on \( g \). Let \( \{ J_a \}, a = 1, \ldots, d, \) be a basis of \( g \) and \( \{ J^a \} \) the dual basis with respect to \( \kappa_0 \). Denote
by $\Delta$ the quadratic Casimir operator from the center of $U(\mathfrak{g})$:

$$\Delta = \frac{1}{2} \sum_{a=1}^{d} J_a J^a.$$  

Let $(\lambda_i)$ be a set of of dominant highest weights of $\mathfrak{g}$. Denote by $V_{(\lambda_i)}$ the tensor product $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$. Let $z_1, \ldots, z_N$ be a set of distinct complex numbers. The Gaudin hamiltonians are the linear operators

$$\Xi_i = \sum_{j=1}^{d} \sum_{a=1}^{d} \frac{J_a^{(i)}}{z_i - z_j}, \quad i = 1, \ldots, N,$$

acting on $V_{(\lambda_i)}$. Note that

$$\sum_{i=1}^{N} \Xi_i = 0.$$  

These operators commute with the diagonal action of $\mathfrak{g}$ on $V_{(\lambda_i)}$ and hence their action is well-defined on the subspace of highest weight vectors in $V_{(\lambda_i)}$ of an arbitrary dominant integral weight $\mu$ with respect to the diagonal $\mathfrak{g}$–action. Writing $\mu = -w_0(\lambda_\infty)$ where $\lambda_\infty$ is another dominant integral weight, we identify this subspace with $(V_{(\lambda_i)} \otimes V_{\lambda_\infty})^G$.

Consider the problem of simultaneous diagonalization of the Gaudin hamiltonians. Set

$$|0\rangle := v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_N} \in V_{(\lambda_i)}.$$  

Clearly, it is an eigenvector of the $\Xi$’s. Other eigenvectors are constructed by a procedure known as the Bethe Ansatz.

Let

$$F_j(w) = \sum_{i=1}^{N} \frac{F_j^{(i)}}{w - z_i}, \quad j = 1, \ldots, \ell.$$  

For a set of distinct complex numbers $w_1, \ldots, w_m$ and a collection of labels $i_1, \ldots, i_m \in I$ we introduce the Bethe vector

$$|w_1^{i_1}, \ldots, w_m^{i_m}\rangle = \sum_{p=(1, \ldots, N)} \prod_{j=1}^{\ell} \frac{F_j^{(i_j)}}{(w_{j_1} - z_{i_1})(w_{j_2} - z_{i_2}) \ldots (w_{j_a} - z_{i_a})} |0\rangle.$$  

Here the summation is taken over all ordered partitions $I^1 \cup I^2 \cup \ldots \cup I^N$ of the set $\{1, \ldots, m\}$, where $I^j = \{i_1^j, i_2^j, \ldots, i_a^j\}$. Note that one can consider vector (4.2) as an element of the tensor product of Verma modules $M_{\lambda_1} \otimes \ldots \otimes M_{\lambda_N}$ with arbitrary highest weights $\lambda_1, \ldots, \lambda_N$.

The following result is proved in [BaFl, FFR, RV].

**Proposition 4.1.** The vector $|w_1^{i_1}, \ldots, w_m^{i_m}\rangle$ is an eigenvector of the Gaudin hamiltonians $\Xi_i, i = 1, \ldots, \ell$, if and only if the Bethe Ansatz equations

$$\sum_{i=1}^{N} \langle \lambda_i, \tilde{a}_{i_j} \rangle w_j^{i_j} - z_i - \sum_{s \neq j} \langle \alpha_{j_s}, \tilde{a}_{i_j} \rangle w_j^{i_j} - w_s^{i_s} = 0, \quad j = 1, \ldots, m.$$
are satisfied.

Note that the equations (4.3) are nothing but the equations (3.5) for the Langlands dual Lie algebra $Lg$.

One checks also that if this vector is an eigenvector, then it is automatically a highest weight vector of weight $\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{ij}$. Hence for this vector to be non-zero, we must have

$$\sum_{i=1}^{N} \lambda_i - \sum_{j=1}^{m} \alpha_{ij} = -w_0(\lambda_\infty)$$

for some dominant integral weight $\lambda_\infty$. Note that this relation is nothing but a special case of equation (3.3) for $Lg$ (when $y = 1$).

4.2. Gaudin model and coinvariants. In [FFR] Proposition 4.1 is proved using the following interpretation of the Gaudin hamiltonians.

Let $\tilde{g}$ be the affine Kac-Moody algebra corresponding to $g$. It is the extension of the Lie algebra $g \otimes \mathbb{C}[[t]]$ by the one-dimensional center $\mathbb{C}K$. The commutation relations in $\tilde{g}$ read

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes fg - \kappa_c(A, B) \text{Res}_{t=0} fdg \cdot K,$$

where $\kappa_c(\cdot, \cdot)$ is the critical invariant inner product on $g$ defined by the formula

$$\kappa_c(A, B) = -\frac{1}{2} \text{Tr}_g \text{ad} A \text{ad} B.$$

Denote by $\tilde{g}_+$ the Lie subalgebra $g \otimes \mathbb{C}[[t]] \oplus \mathbb{C}K$ of $\tilde{g}$. We extend the action of $g$ on the finite-dimensional representation $V_\lambda$ to $\tilde{g}_+$ in such a way that $g \otimes t \mathbb{C}[[t]]$ acts trivially and $K$ acts as the identity. Denote by $\mathbb{V}_\lambda$ the Weyl module which is the induced representation of $\tilde{g}$

$$\mathbb{V}_\lambda = U(\tilde{g}) \otimes_{U(\tilde{g}_+)} V_\lambda.$$

These are the representations of critical level. In the normalization of [K], the central element $K$ acts as minus the dual Coxeter number.

Consider the projective line $\mathbb{P}^1$ with a global coordinate $t$ and $N$ distinct finite points $z_1, \ldots, z_N \in \mathbb{P}^1$. In the neighborhood of each point $z_i$ we have the local coordinate $t - z_i$ and in the neighborhood of the point $\infty$ we have the local coordinate $t^{-1}$. Set $\tilde{g}(z_i) = g \otimes \mathbb{C}((t - z_i))$ and $\tilde{g}(\infty) = g \otimes \mathbb{C}((t^{-1}))$. Let $\tilde{g}_N$ be the extension of the Lie algebra $\bigoplus_{i=1}^{N} \tilde{g}(z_i) \oplus \tilde{g}(\infty)$ by a one-dimensional center $\mathbb{C}K$ whose restriction to each summand $\tilde{g}(z_i)$ or $\tilde{g}(\infty)$ coincides with the above central extension. The Lie algebra $\tilde{g}_N$ naturally acts on the tensor product

$$\mathbb{V}(\lambda_1) \otimes \cdots \otimes \mathbb{V}(\lambda_N) \otimes \mathbb{V}(\lambda_\infty);$$

in particular, $K$ acts as the identity.

Let $g(z_i) = g_{z_1, \ldots, z_N}$ be the Lie algebra of $g$-valued regular functions on $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$ (i.e. rational functions on $\mathbb{P}^1$, which may have poles only at the points $z_1, \ldots, z_N$ and $\infty$). Clearly, such a function can be expanded into a Laurent power series in
the corresponding local coordinates at each point $z_i$ and at $\infty$. Thus, we obtain an embedding
\[
\mathfrak{g}(z_i) \hookrightarrow \bigoplus_{i=1}^{N} \mathfrak{h}(z_i) \oplus \mathfrak{h}(\infty).
\]
It follows from the residue theorem and formula (4.6) that the restriction of the central extension to the image of this embedding is trivial. Hence this embedding lifts to an embedding $\mathfrak{g}(z_i) \rightarrow \hat{\mathfrak{g}}_N$.

Denote by $H(\lambda_i,\lambda_\infty)$ the space of coinvariants of $V(\lambda_i,\lambda_\infty)$ with respect to the action of the Lie algebra $\mathfrak{g}(z_i)$. By construction, we have a canonical embedding of the finite-dimensional representation $V_\lambda$ into the module $V_\lambda$:
\[
x \in V_\lambda \rightarrow 1 \otimes x \in V_\lambda,
\]
which commutes with the action of $\mathfrak{g}$ on both spaces (where $\mathfrak{g}$ is embedded into $\hat{\mathfrak{g}}$ as the constant subalgebra). Thus we have an embedding $V(\lambda_i,\lambda_\infty) = V(\lambda_i) \otimes V_{\lambda_\infty}$ into $V(\lambda_i,\lambda_\infty)$. We will use the same notation $V(\lambda_i,\lambda_\infty)$ for the image of this embedding. Denote by $V^G(\lambda_i,\lambda_\infty)$ the subspace of $G$–invariants (equivalently, $\mathfrak{g}$–invariants) in $V(\lambda_i,\lambda_\infty)$ with respect to the diagonal action.

**Lemma 4.2** ([FFR], Lemma 1). The composition of the embedding $V^G(\lambda_i,\lambda_\infty) \hookrightarrow V(\lambda_i,\lambda_\infty)$ and the projection $V(\lambda_i,\lambda_\infty) \twoheadrightarrow H(\lambda_i,\lambda_\infty)$ is an isomorphism.

Let $V_0$ be the representation of $\hat{\mathfrak{g}}$, which corresponds to the one-dimensional trivial $\mathfrak{g}$–module $V_0$; it is called the **vacuum module**. Denote by $v_0$ the generating vector of $V_0$. We assign the vacuum module to a point $u \in \mathbb{P}^1$ which is different from $z_1, \ldots, z_N, \infty$. Denote by $H((\lambda_i,\lambda_\infty),0)$ the space of $\hat{\mathfrak{g}}(z_i)$–invariant functionals on $V(\lambda_i,\lambda_\infty) \otimes V_0$ with respect to the Lie algebra $\mathfrak{g}(z_i,u)$. Lemma 4.2 tells us that the composition of the embedding $V^G(\lambda_i,\lambda_\infty) \otimes v_0 \hookrightarrow V(\lambda_i,\lambda_\infty) \otimes V_0$ and the projection $V(\lambda_i,\lambda_\infty) \otimes V_0 \twoheadrightarrow H((\lambda_i,\lambda_\infty),0)$ is an isomorphism.

Let $v$ be an arbitrary vector in $V_0$. For any $x \in V(\lambda_i,\lambda_\infty)$ consider the vector $x \otimes v \in V(\lambda_i,\lambda_\infty) \otimes V_0$. By Lemma 4.2 the projection of this vector onto $H((\lambda_i,\lambda_\infty),0)$ is equal to the projection of a vector of the form $(\Psi_v(u) \cdot x) \otimes v_0$, where $\Psi_v(u) \cdot x \in V(\lambda_i,\lambda_\infty)$. Thus we obtain a well-defined linear operator $\Psi_v(u)$ on $V(\lambda_i,\lambda_\infty)$ corresponding to any $v \in V_0$ and any point $u \in \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$.

For $A \in \mathfrak{g}$ and $m \in \mathbb{Z}$, denote by $A^m$ the element $A \otimes t^m \in \hat{\mathfrak{g}}$. Now introduce the following Segal-Sugawara vector in $V_0$:
\[
(4.6) \quad S = \frac{1}{2} \sum_{a=1}^{d} J_{a,-1} J_{a,1} v_0.
\]
This vector defines a linear operator $\Psi_S(u)$ on $V(\lambda_i,\lambda_\infty)$.

Denote by $\Delta(\lambda)$ the scalar by which the Casimir operator $\Delta$ acts on $V_\lambda$.

**Proposition 4.3** ([FFR], Prop. 1). We have
\[
\Psi_S(u) = \sum_{i=1}^{N} \frac{\Xi_i}{u - z_i} + \sum_{i=1}^{N} \frac{\Delta(\lambda_i)}{(u - z_i)^2},
\]
where the \( \Xi _i \)'s are the Gaudin operators \( ^{11} \).

Now consider the subspace \( \mathfrak{g}(\mathfrak{g}) \) of all \( \mathfrak{g}_+ \)-invariant vectors in \( V_0 \). One checks that \( S \in \mathfrak{g}(\mathfrak{g}) \).

**Proposition 4.4** ([FFR], Prop. 2). For any \( Z_1, Z_2 \in \mathfrak{g}(\mathfrak{g}) \) and any points \( u_1, u_2 \in \mathbb{P}^1 \setminus \{ z_1, \ldots, z_N, \infty \} \) the linear operators \( \Psi_{Z_1}(u_1) \) and \( \Psi_{Z_2}(u_2) \) commute.

Taking the coefficients in the expansions of the operators of the form \( \Psi_Z(u) \) at \( z_1, \ldots, z_N \) we obtain a family of commuting linear operators on \( V_{(\lambda_1), \infty}^{G} \) which includes the Gaudin hamiltonians. It is natural to call them the generalized Gaudin hamiltonians.

### 4.3. The center of \( V_0 \) and \( ^L G \)-opers.

In order to describe the algebra of generalized Gaudin hamiltonians and its spectrum we need to recall the description of \( \mathfrak{g}(\mathfrak{g}) \) from [FF3, F2].

First, observe that each element \( v \) of \( \mathfrak{g}(\mathfrak{g}) \) gives rise to an endomorphism of \( V_0 \) commuting with the action of \( \mathfrak{g} \) which sends the generating vector \( v_0 \) to \( v \). Conversely, any \( \mathfrak{g} \)-endomorphism of \( V_0 \) is uniquely determined by the image of \( v_0 \) which necessarily belongs to \( \mathfrak{g}(\mathfrak{g}) \). Thus, we obtain an isomorphism \( \mathfrak{g}(\mathfrak{g}) \simeq \text{End}_{\mathfrak{g}}(V_0) \) which gives \( \mathfrak{g}(\mathfrak{g}) \) an algebra structure. The opposite algebra structure on \( \mathfrak{g}(\mathfrak{g}) \) coincides with the algebra structure induced by the identification of \( V_0 \) with the algebra \( U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \).

The realization of \( \mathfrak{g}(\mathfrak{g}) \) as \( \text{End}_{\mathfrak{g}}(V_0) \) allows us to interpret the action of \( \mathfrak{g}(\mathfrak{g}) \) on \( V_{(\lambda_1), \lambda_\infty}^{G} \) as follows. We identify \( H_{(\lambda_i), \lambda_\infty, 0} \) with \( V_{(\lambda_i), \lambda_\infty}^{G} \). By functoriality, any endomorphism of \( V_0 \) gives rise to an endomorphism of \( H_{(\lambda_i), \lambda_\infty, 0} \), and hence of \( V_{(\lambda_i), \lambda_\infty}^{G} \). In particular, we see immediately that the map \( \Psi : \mathfrak{g}(\mathfrak{g}) \to \text{End} V_{(\lambda_i), \lambda_\infty}^{G} \) is an algebra homomorphism with respect to the algebra structure on \( \mathfrak{g}(\mathfrak{g}) \) that we introduced above.

Let \( \text{Der} \mathfrak{O} = C[[t]] \partial_t \) be the Lie algebra of continuous derivations of the topological algebra \( \mathfrak{O} = C[[t]] \). The action of its Lie subalgebra \( \text{Der}_0 \mathfrak{O} = tC[[t]] \partial_t \) on \( \mathfrak{O} \) exponentiates to an action of the group \( \text{Aut} \mathfrak{O} \) of formal changes of variables. Both \( \text{Der} \mathfrak{O} \) and \( \text{Aut} \mathfrak{O} \) naturally act on \( V_0 \) in a compatible way, and these actions preserve \( \mathfrak{g}(\mathfrak{g}) \). They also act on the space \( \text{Op}_{\ell G}(D) \) of \( \ell G \)-opers on the disc \( D = \text{Spec} C[[t]] \).

Denote by \( \text{Fun} \text{Op} \ell G(D) \) the algebra of regular functions on \( \text{Op}_{\ell G}(D) \). In view of Lemma 2.4, it is isomorphic to the algebra of functions on the space of \( \ell \)-tuples \( (v_1(t), \ldots, v_{\ell}(t)) \) of formal Taylor series, i.e., the space \( C[[t]]^\ell \). If we write \( v_i(t) = \sum_{n \geq 0} v_{i,n} t^n \), then we obtain

\[
\text{Fun} \text{Op}_{\ell G}(D) \simeq C[v_{i,n}]_{i \in I, n \geq 0}.
\]

Note that the vector field \( -t \partial_t \) acts naturally on \( \text{Op}_{\ell G}(D) \) and defines a \( \mathbb{Z} \)-grading on \( \text{Fun} \text{Op}_{\ell G}(D) \) such that \( \text{deg} v_{i,n} = d_i + n + 1 \). The vector field \( -\partial_t \) acts as a derivation such that \( -\partial_t \cdot v_{i,n} = -(d_i + n + 1)v_{i,n+1} \).

**Theorem 4.5** ([FF3, F2]). There is a canonical isomorphism

\[
\mathfrak{g}(\mathfrak{g}) \simeq \text{Fun} \text{Op}_{\ell G}(D)
\]

of algebras which is compatible with the action of \( \text{Der} \mathfrak{O} \) and \( \text{Aut} \mathfrak{O} \).
The module $\mathcal{V}_0$ has a natural $\mathbb{Z}$-grading defined by the formulas $\deg v_0 = 0$, $\deg J^n_\alpha = -n$, and it carries a translation operator $T$ defined by the formulas $Tv_0 = 0$, $[T, J^n_\alpha ] = -nJ^n_{\alpha - 1}$. Theorem 4.5 and the isomorphism (1.7) imply that there exist non-zero vectors $S_i \in \mathcal{V}_0$ of degrees $d_i + 1, i \in I$, such that

$$\mathfrak{z}(\hat{g}) = \mathbb{C}[[T^n_{\alpha}]_{i \in I, n \geq 0}v_0].$$

Then under the isomorphism of Theorem 4.5 we have $S_i \mapsto v_{i,0}$, the $\mathbb{Z}$-gradings on both algebras get identified and the action of $T$ on $\mathfrak{z}(\hat{g})$ becomes the action of $-\partial_0$ on $\text{Fun } \text{Op}_- G \mathcal{V}(D)$. Note that the vector $S_1$ is nothing but the vector (4.6), up to a non-zero scalar.

Recall from [FB] that $\mathcal{V}_0$ is a vertex algebra, and $\mathfrak{z}(\hat{g})$ is its commutative vertex subalgebra; in fact, it is the center of $\mathcal{V}_0$. Consider the corresponding enveloping algebra $U(\mathfrak{z}(\hat{g}))$ as defined in [F2]. It is shown in [F2] that $U(\mathfrak{z}(\hat{g}))$ is isomorphic to the algebra of functions on the space $\text{Op}_- G \mathcal{V}(D^\times)$ of $L^G$-opers on the punctured disc. Moreover, $U(\mathfrak{z}(\hat{g}))$ is the center $\mathfrak{Z}(\hat{g})$ of the completed universal enveloping algebra of $\hat{g}$ at the critical level (see [BD1]). For each integral dominant weight $\lambda$ we have a homomorphism $\text{Fun } \text{Op}_- G \mathcal{V}(D^\times) \simeq Z(\hat{g}) \rightarrow \text{End}_{\mathfrak{z}}(\mathcal{V}_0)$. The following result is proved in [F2].

**Theorem 4.6.** The homomorphism $\text{Fun } \text{Op}_- G \mathcal{V}(D^\times) \rightarrow \text{End}_{\mathfrak{z}}(\mathcal{V}_0)$ is surjective. Moreover, it identifies $\text{End}_{\mathfrak{z}}(\mathcal{V}_0)$ with the algebra $\text{Fun } \text{Op}_- G \mathcal{V}(D)_\lambda$ so that this homomorphism becomes the natural surjection $\text{Fun } \text{Op}_- G \mathcal{V}(D^\times) \rightarrow \text{Fun } \text{Op}_- G \mathcal{V}(D)_\lambda$ induced by the embedding $\text{Op}_- G \mathcal{V}(D)_\lambda \hookrightarrow \text{Op}_- G \mathcal{V}(D^\times)$.

In particular, if $\lambda = 0$ we obtain the statement of Theorem 4.5 because $\text{End}_{\mathfrak{z}}(\mathcal{V}_0) = \mathfrak{z}(\hat{g})$.

### 4.4. Eigenvalues of the generalized Gaudin hamiltonians and $L^G$-opers

Now suppose we have an eigenvector $A \in V^G_{(\lambda), \lambda_\infty}$ of the generalized Gaudin hamiltonians. The action of $\mathfrak{z}(\hat{g})$ on $A$ defines, for any $u \in \mathbb{P}^1$, a homomorphism $\mathfrak{z}(\hat{g}) \simeq \text{Fun } \text{Op}_- G \mathcal{V}(D_u) \rightarrow \mathbb{C}$, i.e., a $L^G$-oper on $D_u$. Let us denote this oper by $\eta_{A,u}$.

The following theorem asserts that these opers on the discs $D_u$ for different values of $u$ are restrictions of one and the same regular $L^G$-oper on $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$. Moreover, this oper has regular singularities at $z_1, \ldots, z_N, \infty$ with residues $-\lambda_1 - \rho, \ldots, \lambda_N - \rho, -\lambda_\infty - \rho$ at those points, and it has trivial monodromy.

**Theorem 4.7.** The $L^G$-opers $\eta_{A,u}$ on $D_u$ corresponding to the eigenvalues of the generalized Gaudin hamiltonians on $A \in V^G_{(\lambda), \lambda_\infty}$ are restrictions to the respective discs of a unique (regular) $L^G$-oper $\eta_A$ on $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$. Moreover, the oper $\eta_A$ belongs to the space $\text{Op}_- G \mathcal{V}(\mathbb{P}^1)_{(z_1, \infty): (\lambda), \lambda_\infty}$. In particular, it has trivial monodromy representation.

**Proof.** We will give first an abridged version of the proof and then explain the details.

In [FB] we defined, for any quasi-conformal vertex algebra $V$, a smooth projective curve $X$, a set of points $x_1, \ldots, x_N \in X$ and a collection of $V$–modules $M_1, \ldots, M_N$, the space of coinvariants $H_V(X, (x_i), (M_i))$ and its dual space, the space of conformal blocks $C_V(X, (x_i), (M_i))$. This construction (which is recalled below) is functorial: if $W \rightarrow V$
is a homomorphism of vertex algebras, then we have natural maps $H_W(X, (x_i), (M_i)) \to H_V(X, (x_i), (M_i))$ and $C_V(X, (x_i), (M_i)) \to C_W(X, (x_i), (M_i))$.

In the case of the vertex algebra $V_0$ associated to the affine Kac-Moody algebra $\hat{g}$, the curve $\mathbb{P}^1$, the points $z_1, \ldots, z_N, \infty$ and the modules $V_{\lambda_1}, \ldots, V_{\lambda_N}, V_{\lambda_\infty}$, the space of coinvariants is nothing but the space $H_{(\lambda_i),\lambda_\infty}$, which we have identified with $V_{(\lambda_i),\lambda_\infty}$ in Lemma 4.2.

The subspace $\mathfrak{z}(\hat{g})$ of $\mathfrak{g}[t]$–invariant vectors in $V_0$ is a commutative vertex subalgebra of $V_0$; in fact, it is the center of $V_0$ (see [FB]). The embedding $\mathfrak{z}(\hat{g}) \to V_0$ then gives rise to a map

$$H_{\mathfrak{z}(\hat{g})}(\mathbb{P}^1; (z_i), \infty; (V_{\lambda_i}), V_{\lambda_\infty}) \to H_{(\lambda_i),\lambda_\infty}.$$  

Applying the results of [FB], Sect. 8.4, (as explained below) we obtain that each eigenvector $A$ of the generalized Gaudin hamiltonians in $V_{(\lambda_i),\lambda_\infty} \simeq H_{(\lambda_i),\lambda_\infty}$ gives rise to a character (i.e., an algebra homomorphism)

$$\text{Fun Op}_{L,G}(\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}) \to \mathbb{C},$$

i.e., to a $L^G$–oper on $\mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\}$. This is precisely the $L^G$–oper $\eta_A$ that we are looking for. Next, we apply Theorem 4.6 to show that $\eta_A$ actually belongs to the space $\text{Op}_{L,G}(\mathbb{P}^1; (z_i), \infty; (\lambda_i), \lambda_\infty)$.

Let us now explain all of this in detail. First, we recall the definition of the space of conformal blocks from [FB], Ch. 8. For that we will need a coordinate-independent description of the structure of a module over a vertex algebra given in [FB], Ch. 5 and Sect. 6.3.9, where we refer the reader for more details.

Let $X$ be a smooth algebraic curve and $\text{Aut}_X$ be the principal $\text{Aut} \mathcal{O}$–bundle over $X$ whose fiber $\text{Aut}_x$ at $x \in X$ is the space of formal coordinates at $x$. Let $V$ be a quasi-conformal vertex algebra (see [FB], Sect. 5.2.4). It then carries an action of $\text{Aut} \mathcal{O}$. We define a vector bundle $\mathcal{V} = V_X$ on $\mathbb{P}^1$ as the twist $\text{Aut}_X^\mathcal{O} \times V$. This bundle carries a (flat) connection. If we choose a coordinate $t$ and trivialize $\text{Aut}_X$ and $\mathcal{V}$ using this coordinate, then the connection operator reads $\nabla = \partial_t + T$.

Let $M$ be a $V$–module which carries an action of $\text{Der}_0 \mathcal{O} = t\mathbb{C}[t] \partial_t$ compatible with that of $V$ such that the action of $-t \partial_t$ is semi-simple and the eigenvalues belong to the union of the sets $\kappa_i + \mathbb{Z} \epsilon_i$, where $\{\kappa_i\}$ is a finite set of complex numbers. The action of the Lie algebra $\text{Der}_+ \mathcal{O} = t^2 \mathbb{C}[t] \partial_t$ on $M$ may be exponentiated to an action of the group $\text{Aut}_+ \mathcal{O}$ consisting of the formal coordinate changes of the form $z \mapsto z + z^2(\ldots)$. Let us fix a non-zero tangent vector $\tau$ at $x$ and consider the $\text{Aut}_+ \mathcal{O}$–torsor $\mathcal{A}_{x,\tau}$ consisting of all formal coordinates at $x$ whose one-jet is equal to $\tau$. We define the twist $\mathcal{M}_x = \text{Aut}_{x,\tau} \times M$ of $M$ at $x \in X$.

Let us pick a formal coordinate $t_x$ at $x$ whose one-jet is equal to $\tau$. We use this coordinate to trivialize $\mathcal{V}|_{D_x}$ and $\mathcal{M}_x$ and to define an $\text{End} \mathcal{M}_x$–valued section $\mathfrak{y}_x^M$ of $\mathcal{V}^*|_{D_x}$ as follows. The value $\langle \varphi, \mathfrak{y}_x^M \cdot v \rangle$ of this section on $v \in \mathcal{M}_x \simeq M$, $\varphi \in \mathcal{M}_x^* \simeq M^*$ and the constant section $s_A$ of $\mathcal{V}|_{D_x}$ corresponding to the vector $A \in V$ with respect to our trivialization, is equal to $\langle \varphi, Y^M(A, t_x) v \rangle$. It is proved in [FB] that the section $\mathfrak{y}_x^M$ is well-defined, i.e., independent of the choice of the coordinate $t_x$. Moreover, this
section is horizontal with respect to the connection on \( V^* \) which is the transpose of the connection \( \nabla \) on \( V \) (see [FB], Theorem 5.5.3).

Let \( x_1, \ldots, x_N \) be a collection of distinct points on \( X \). We will fix once and for all a non-zero tangent vector \( \tau_i \) at \( x_i \) for each \( i = 1, \ldots, N \). The space of conformal blocks \( C_V(X, (x_i), (M_i)) \) is by definition the space of linear functionals \( \varphi \) on \( M_{i,x_1} \otimes \ldots \otimes M_{N,x_N} \) satisfying the following condition: for any \( A_i \in M_{i,x_i}, i = 1, \ldots, N \), there exists a regular section of \( V^* \) on \( X \setminus \{x_1, \ldots, x_N\} \) such that for all \( i = 1, \ldots, N \) its restriction to \( D_{x_i}^X \) is equal to

\[
\langle \varphi, A_1 \otimes \ldots \otimes y_{x_i}^M_i \cdot A_i \otimes \ldots \otimes A_N \rangle.
\]

This regular section is then automatically horizontal. This section may be constructed explicitly as follows. In [FB], Theorem 9.3.1, we established, for all \( y \in X \), \( u \neq x_i \), an isomorphism

\[
C_V(X, (x_i), (M_i)) \simeq C_V(X; (x_i), u; (M_i), V).
\]

This means that the space of conformal blocks does not change if we insert the vacuum module \( V \) at a point \( u \in X \) different from all the \( x_i \)'s; note that we have considered in Section 4.2 a special case of this isomorphism. Let \( \bar{\varphi} \) be the functional in \( C_V(X; (x_i), y; (M_i), V) \) corresponding to \( \varphi \in C_V(X, (x_i), (M_i)) \) under this isomorphism. Then the value of our section at \( y \in X \) on an element \( A \in V_y \) is precisely equal to \( \langle \bar{\varphi}, A_1 \otimes \ldots \otimes A_N \otimes A \rangle \).

Now we set \( V \) to be the affine Kac-Moody vertex algebra \( V_0, X = \mathbb{P}^1 \), with the marked points \( z_1, \ldots, z_N, \infty \), and take as the modules attached to these points the Weyl modules \( V_{\lambda_1}, \ldots, V_{\lambda_N}, V_{\lambda_{\infty}} \). Our global coordinate \( t \) on \( \mathbb{P}^1 \) gives rise to the coordinate \( t - z_i \) at each point \( z_i \) and the coordinate \( t^{-1} \) at \( \infty \). Hence we obtain an identification of \( V_{\lambda_i,z_i} \) with \( V_{\lambda_i} \). It is proved in [FB] (see Theorem 8.3.3 and Remark 8.3.10) that the corresponding space of conformal blocks is the space of \( \mathfrak{g}(z_i) \)-invariant functionals on \( V_{(\lambda_i),\lambda_{\infty}} \), i.e., the dual space to \( H_{(\lambda_i),\lambda_{\infty}} \).

Let \( \varphi \) be a linear functional on \( V_{(\lambda_i),\lambda_{\infty}} \). For each vector \( A_1 \otimes \ldots \otimes A_N \otimes A_{\infty} \in V_{(\lambda_i),\lambda_{\infty}} \) we then obtain a section

\[
\langle \varphi, A_1 \otimes \ldots \otimes y_{z_i}^{\lambda_i} \cdot A_i \otimes \ldots \otimes A_N \otimes A_{\infty} \rangle
\]

of \( V_{0,\mathbb{P}^1}|_{D_{z_i}^X} \) for all \( i = 1, \ldots, N \), and likewise at the point \( \infty \). According to the above discussion, the functional \( \varphi \) is \( \mathfrak{g}(z_i) \)-invariant if and only if the above sections are restrictions to the respective punctured discs of a single rational section of \( V_{0,\mathbb{P}^1}^* \) with poles only at the points \( z_1, \ldots, z_N \) and \( \infty \), which is horizontal with respect to the connection \( \nabla \). Moreover, the value of this section at \( u \in \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \) may be obtained as explained above, by inserting \( V_0 \) at \( u \).

For any eigenvector \( A \) of the Gaudin Hamiltonians in \( V_{(\lambda_i),\lambda_{\infty}}^G \simeq H_{(\lambda_i),\lambda_{\infty}} \), there is a linear functional on \( H_{(\lambda_i),\lambda_{\infty}} \), taking a non-zero value on \( A \), which is an eigenvector of the transposed Gaudin operators and has the same eigenvalues. We view this functional as a conformal block. Then it satisfies the above condition, namely, that the sections \( \mathcal{E}_z \) are restrictions to the respective punctured discs of a single horizontal section of \( V_{0,\mathbb{P}^1}^* \) that is regular on \( \mathbb{P}^1 \setminus \{z_1, \ldots, z_N, \infty\} \). We will denote this section by \( s_\varphi \). Evaluating \( s_\varphi \) on an arbitrary section of \( V_{0,\mathbb{P}^1} \), we obtain a rational function on \( \mathbb{P}^1 \) with poles at \( z_1, \ldots, z_N, \infty \).
Consider the subbundle $\mathcal{Z}_{P^1} \subset V_{P^1}$ obtained by twisting $\mathfrak{g}(\mathfrak{h}) \subset V_0$. The fiber $\mathcal{Z}_u$ of $\mathcal{Z}_{P^1}$ at $u \in P^1$ is just the algebra of functions on the space $Op_{L}(D_u)$ of $L\bar{G}$–opers on $D_u$. Therefore $\mathcal{Z}_{P^1}$ is nothing but the algebra of functions on the scheme $Op_{L}(X)$ of jets of $L\bar{G}$–opers on $P^1$, whose fiber at $u \in P^1$ is the space $Op_{L}(D_u)$. This scheme carries a natural connection and the corresponding conncetion on $\mathcal{Z}_{P^1}$ coincides with the connection $\nabla$ described above. Note that horizontal sections of $Op_{L}(P^1)$ over $U \subset P^1$ are the same as the regular $L\bar{G}$–opers on $U$.

We now evaluate $s_\varphi$ on sections of $\mathcal{Z}_{P^1}$. According to our construction of Section 4.2, the value of $s_\varphi$ on $v \in \mathcal{Z}_u$ at the point $u \in P^1\{z_1, \ldots, z_N, \infty\}$ is precisely the eigenvalue of the generalized Gaudin hamiltonian $\Psi(u)$ on our eigenvector. Moreover, these eigenvalues are multiplicative with respect to the commutative algebra structure on the bundle $\mathcal{Z}_{P^1}$, which is inherited from that on $\mathfrak{g}(\mathfrak{h})$. Therefore these eigenvalues define an algebra homomorphism $\mathcal{Z}_u \to \mathbb{C}$ for all $u \in P^1\{z_1, \ldots, z_N, \infty\}$. This is the same as an algebra homomorphism from the sheaf of algebras

$$\mathcal{Z}_{P^1\{z_1, \ldots, z_N, \infty\}} \cong \text{Fun} Op_{L}(P^1\{z_1, \ldots, z_N, \infty\})$$

to $\mathbb{C}$ (considered as the constant sheaf over $P^1\{z_1, \ldots, z_N, \infty\}$). Moreover, according to the general results on conformal blocks (see above), this homomorphism must be horizontal. But such a homomorphism is the same as a horizontal section of the bundle $Op_{L}(P^1)$ of jets of $L\bar{G}$–opers on $P^1\{z_1, \ldots, z_N, \infty\}$, which is the same as a regular $L\bar{G}$–oper on $P^1\{z_1, \ldots, z_N, \infty\}$. This is the desired oper $\eta_1$. By construction, its restriction to $D_u$ for each $u \in P^1\{z_1, \ldots, z_N, \infty\}$ equals to the $L\bar{G}$–oper on $D_u$ which records the eigenvalues of the generalized Gaudin hamiltonians corresponding to the point $u$.

Let us now look at the restrictions of the oper $\eta_1$ to the punctured discs around the points $z_1, \ldots, z_N$ and $\infty$. By definition of the section $s_\varphi$, these restrictions are equal to the sections 4.3, where the vertex operations $\frac{\partial}{\partial z_i}$ are restricted to $\mathfrak{g}(\mathfrak{h})_{z_i}$ and $\mathfrak{g}(\mathfrak{h})_{\infty}$ (which are the twists of $\mathfrak{g}(\mathfrak{h})$ by $Aut_{z_i}$ and $Aut_{\infty}$, respectively). Each section gives rise to a homomorphism $Z(\mathfrak{g})_{z_i} \to \mathbb{C}$, and hence to a point in Spec $Z(\mathfrak{g})_{z_i} = Op_{L}(D^\vee_{z_i})$. But we know from Theorem 4.6 that the action of the center $Z(\mathfrak{g})$ on $\mathcal{V}_X$ factors through the algebra of functions on $Op_{L}(D_\lambda_1)$. Therefore this point belongs to $Op_{L}(D_{z_1})_{\lambda_1} \subset Op_{L}(D^\times_{z_1})$. Hence we find that the restriction of our $L\bar{G}$–oper on $P^1\{z_1, \ldots, z_N, \infty\}$ to the disc $D^\vee_{z_i}$ (resp., $D^\times_{z_i}$) belongs to $Op_{L}(D_{z_i})_{\lambda_i}$ (resp., $Op_{L}(D_{\infty})_{\lambda_\infty}$). Therefore this oper belongs to $Op_{L}(P^1)_{z_1, \ldots, z_N, \infty; (\lambda_i), (\lambda_\infty)}$, which is what we wanted to prove.

In more concrete terms, the oper $\eta_1$ may be described as follows. From the description of $\mathfrak{g}(\mathfrak{h})$ we know that all eigenvalues are encoded in the rational functions $v_i^\lambda(u)$ which are the eigenvalues of the operators $\Psi_{S_i}(u)$, $i = 1, \ldots, \ell$, on $A$. The corresponding $L\bar{G}$–oper connection then reads (with respect to our trivialization of $\mathcal{F}$ and the global coordinate $t$ on $P^1$)

$$\nabla = \partial_t + p_{-1} + \sum_{i \in I} v_i^A(t)p_i.$$  

(4.9)
4.5. Completeness of the Bethe Ansatz. According to Theorem 4.7, each point in the spectrum of the generalized Gaudin hamiltonians occurring in \( V^G_{(\lambda_i),\lambda_\infty} \) (i.e., a collection of joint eigenvalues of these operators on \( V^G_{(\lambda_i),\lambda_\infty} \)) is encoded by a \( L^G \)-oper on \( \mathbb{P}^1 \) with regular singularities at \( z_1, \ldots, z_N, \infty \) which has trivial monodromy. Moreover, two different points of the spectrum give rise to different opers. Thus, we obtain the following

**Corollary 4.8.** There is an injective map from the spectrum of the generalized Gaudin hamiltonians on \( V^G_{(\lambda_i),\lambda_\infty} \) (not counting multiplicities) to the set \( \text{Op}_{L^G}(\mathbb{P}^1)(z_i),\infty; (\lambda_i),\lambda_\infty \) of \( L^G \)-opers on \( \mathbb{P}^1 \) with regular singularities at \( z_1, \ldots, z_N, \infty \) which have trivial monodromy.

On the other hand, suppose that we are given a \( L^G \)-oper \( \tau \) in \( \text{Op}_{L^G}(\mathbb{P}^1)(z_i),\infty; (\lambda_i),\lambda_\infty \). Consider the (unique) Miura oper structure on it for which the horizontal Borel reduction coincides with the oper reduction at the point \( \infty \). Suppose that this Miura oper satisfies the conditions (1) and (2) from Section 3.2, i.e., it belongs to the space \( \text{MOp}_{L^G}(\mathbb{P}^1)_{(\lambda_i),\infty; (\lambda_i),\lambda_\infty} \). Then we will call \( \tau \) a non-degenerate oper.

According to Theorem 3.2, there is a bijection between the points of the space \( \text{MOp}_{L^G}(\mathbb{P}^1)_{(\lambda_i),\infty; (\lambda_i),\lambda_\infty} \) and the set of solutions of the Bethe Ansatz equations. Note that we have switched to the Langlands dual group \( L^G \), and so these equations are given by formula (4.3). Our Miura \( L^G \)-oper, for which the horizontal Borel reduction coincides with the oper reduction at the point \( \infty \), gives rise to a unique solution of equations (4.3) which satisfies the condition (4.4). We will refer to it as the special solution corresponding to the non-degenerate \( L^G \)-oper \( \tau \) from \( \text{Op}_{L^G}(\mathbb{P}^1)(z_i),\infty; (\lambda_i),\lambda_\infty \).

According to Proposition 4.1, we associate to this special solution of the Bethe Ansatz equations an eigenvector of the Gaudin hamiltonians by formula (4.2). Let us denote this eigenvector by \( v_\tau \).

A natural question is what are the eigenvalues of the generalized Gaudin hamiltonians on \( v_\tau \). By Theorem 4.7 these eigenvalues are encoded by a \( L^G \)-oper in \( \text{Op}_{L^G}(\mathbb{P}^1)(z_i),\infty; (\lambda_i),\lambda_\infty \). Not surprisingly, the answer is that this oper is \( \tau \) itself (see [FPR], Theorem 3):

**Proposition 4.9.** The eigenvalues of the generalized Gaudin hamiltonians acting on the Bethe eigenvector \( v_\tau \) constructed from the special solution of the Bethe Ansatz equations corresponding to \( \tau \in \text{Op}_{L^G}(\mathbb{P}^1)(z_i),\infty; (\lambda_i),\lambda_\infty \) are encoded precisely by the \( L^G \)-oper \( \tau \).

Thus, the \( L^G \)-oper on \( \mathbb{P}^1 \) corresponding to the eigenvalues of the Gaudin hamiltonians on a given Bethe vector (4.2) may be found by applying the Miura transformation (see Section 3.5) to the \( LH \)-connection

\[
\partial_t - \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} + \sum_{j=1}^{m} \frac{\alpha_j}{t - w_j},
\]

where \( w_1, \ldots, w_m \) satisfy the Bethe Ansatz equations (4.3) and the condition (4.4). For example, in the case of \( \mathfrak{sl}_n \), the \( PGL_n \)-oper is nothing but an \( n \)th order differential
operator $\partial_t - \sum_{i=1}^{N} \frac{\lambda_i}{t - z_i} + \sum_{j=1}^{m} \frac{\alpha_{ij}}{t - w_j} = \partial_t + \sum_{k=1}^{n} u_k(t) \epsilon_k,$

where we identify the dual Cartan subalgebra of $\mathfrak{sl}_n$ with the hyperplane $\sum_{k=1}^{n} \epsilon_k = 0$ of the vector space span$\{\epsilon_k\}_{k=1,\ldots,n}$. Then the corresponding $PGL_n$–oper is given by formula (2.20). One obtains similarly the opers for other simple Lie algebras of classical types.

Let us assume from now on that all $L^G$–opers in $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ are non-degenerate. Then each $L^G$–oper $\tau$ in $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ corresponds to a special solution of the Bethe Ansatz equations and hence a Bethe vector $v_\tau$. If all Bethe vectors $v_\tau$ are non-zero, then we obtain an inverse map to the map of Corollary 4.8, which assigns to $\tau \in \text{Op}_{L^G}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ the point in the spectrum corresponding to the eigenvector $v_\tau$ (note that a priori it could happen that there are other eigenvectors with the same eigenvalues $\tau$). This leads us to the following result.

**Proposition 4.10.** Suppose that all $L^G$–opers in $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ are non-degenerate and that all Bethe vectors obtained from solutions of the Bethe Ansatz equations (4.13) satisfying the condition (4.14) are non-zero. Then there is a bijection between the spectrum of the generalized Gaudin hamiltonians on $V^G_{(\lambda_1),\lambda_\infty}$ (not counting multiplicities) and the set $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ of $L^G$–opers on $\mathbb{P}^1$ with regular singularities at $z_1,\ldots,z_N,\infty$ which have trivial monodromy.

Moreover, if in addition the Gaudin hamiltonians are diagonalizable and have simple spectrum on $V^G_{(\lambda_1),\lambda_\infty}$, then the Bethe vectors constitute an eigenbasis of $V^G_{(\lambda_1),\lambda_\infty}$.

The last statement of this proposition that the Bethe vectors constitute an eigenbasis of $V^G_{(\lambda_1),\lambda_\infty}$ is referred to as the completeness of the Bethe Ansatz (sometimes completeness is taken to mean that the Bethe vectors span $V^G_{(\lambda_1),\lambda_\infty}$, but we use this term to mean that they form an eigenbasis).

For $\mathfrak{g} = \mathfrak{sl}_2$ and generic values of the $z_i$'s it was proved by Scherbak and Varchenko in [SV] (see also [RV]) that the Bethe vectors are all non-zero. It also follows from [SV] that in the case of $\mathfrak{sl}_2$ all opers are non-degenerate when $z_1,\ldots,z_N$ are in generic position. Hence we obtain a bijection between the spectrum of the Gaudin hamiltonians on $V^{SL_2}_{(\lambda_1),\lambda_\infty}$ and the set $\text{Op}_{PGL_2}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ for generic $z_i$'s. Moreover, Scherbak [S1] has shown that the eigenvalues of the Gaudin hamiltonians have no multiplicities on the Bethe vectors for generic $z_i$'s, so we obtain the completeness of the Bethe Ansatz as well.

We note that the completeness of the Bethe Ansatz has been previously proved for $\mathfrak{g} = \mathfrak{sl}_2$ and generic values of $z_1,\ldots,z_N$ by Varchenko and Scherbak [SV] by other methods. In addition, it follows from the results of Mukhin and Varchenko [MV] and Scherbak [S2] that for $\mathfrak{g} = \mathfrak{sl}_n$ the number of points of $\text{Op}_{PGL_n}(\mathbb{P}^1)_{(z_1),\infty;(\lambda_1),\lambda_\infty}$ is less than or equal to the dimension of $V^{SL_n}_{(\lambda_1),\lambda_\infty}$.

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$^2$It follows from the results of Mukhin and Varchenko in [MV2] that for some $\lambda_1,\ldots,\lambda_N,\lambda_\infty$ this may not be the case even for generic values of $z_1,\ldots,z_N$. 
In the general case we have the following conjecture.

**Conjecture 1.** For generic values of $z_1,\ldots,z_N$ the generalized Gaudin hamiltonians are diagonalizable on $V^G_{(\lambda_i),\lambda_\infty}$ and have simple spectrum, and the Bethe vectors corresponding to the solutions of the Bethe Ansatz equations (4.3) are all non-zero.

If the statement of Conjecture 1 is true and all $L^G$-opers in $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ are non-degenerate, then we obtain from Proposition 4.10 the completeness of the Bethe Ansatz and a bijection between the spectrum of the Gaudin hamiltonians, counted with multiplicity, and the set $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$.

Suppose now that there are degenerate opers in $\text{Op}_{L^G}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$. Consider again the unique Miura oper structure on one of the degenerate opers for which the horizontal Borel reduction coincides with the oper reduction at the point $\infty$. According to Theorem 3.1, this Miura oper corresponds to a connection in $\text{Conn}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ which has the form (3.1):

$$
(4.10) \quad \partial_t - \sum_{i=1}^N y_i(\lambda_i + \rho) - \rho = \sum_{j=1}^m y'_j(\rho) - \rho
$$

for some $y_i,y'_j \in W$ satisfying the relation (3.2) with $y_\infty = w_0$:

$$
\sum_{i=1}^N (y_i(\lambda_i + \rho) - \rho) + \sum_{j=1}^m (y'_j(\rho) - \rho) = -w_0(\lambda_\infty).
$$

The fact that $\tau$ is not generic means that either some of the elements $y_i$ are not equal to 1 or some of the elements $y'_j$ have lengths greater than 1 (i.e., are not simple reflections).

We expect that if $z_1,\ldots,z_N$ are generic, then for every $\tau \in \text{Op}_{L^G}(\mathbb{P}^1)_{(z_i),\infty;(\lambda_i),\lambda_\infty}$ we have $y_i = 1$ for all $i = 1,\ldots,N$ in formula (4.10). In other words, we expect that for generic $z_1,\ldots,z_N$ this Miura oper still satisfies condition (1) from Section 3.2 but may not satisfy condition (2), that is at least one of the $y'_j$’s is not a simple reflection.

Then we can still attach to the connection (4.10) an eigenvector of the generalized Gaudin hamiltonians in $V^G_{(\lambda_i),\lambda_\infty}$ by generalizing the procedure of [FFR]. We expect that for generic $z_1,\ldots,z_N$ all of these vectors are non-zero and that they provide an eigenbasis for the generalized Gaudin hamiltonians in $V^G_{(\lambda_i),\lambda_\infty}$. For more on this, see Sect. 5.5 of [F3].

5. **Opers and Bethe Ansatz equations for arbitrary Kac-Moody algebras**

In this section we generalize some of the results of the previous sections to the situation where $\mathfrak{g}$ is an arbitrary Kac-Moody algebra. One can easily write down the Bethe Ansatz equations in this general setting and try to describe the set of solutions of these equations. We show that, just as in the case of a simple finite-dimensional Lie algebra, this set is an open subset of the (ind-)flag variety of $\mathfrak{g}$. For that we introduce the notions of opers and Miura opers for an arbitrary Kac-Moody algebra and show that the set of solutions of the Bethe Ansatz equations is an open and dense subset in the set of Miura opers on the projective line with prescribed residues at marked points (as in the finite-dimensional case).
5.1. Oper and Miura o pérs for general Kac-Moody algebras. Let \( g \) be the Kac-Moody algebra associated to a Cartan matrix \( A \) of size \( \ell \times \ell \) (not necessarily symmetrizable) and \( h \) be its (extended) Cartan subalgebra of dimension \( \ell + d \), where \( \ell - d \) is the rank of \( A \). We use the same notation as before for coroots and roots of \( g \), which are vectors in \( h \) and \( h^* \), respectively. We have the Cartan decomposition \( g = n_+ \oplus h \oplus n_- \), and the generators \( \{ e_i \}_{i=1,\ldots,\ell} \) and \( \{ f_i \}_{i=1,\ldots,\ell} \) of \( n_+ \) and \( n_- \), respectively. The Lie subalgebra \( n_- \) has a natural descending filtration by Lie ideals of finite codimension. We consider its completion with respect to this filtration and the corresponding completion of \( g \). From now on we will use the symbols \( n_- \) and \( g \) to denote these completions.

For example, in the case of untwisted affine algebras, the completed Lie algebra \( g \) has the form \( \overline{g}(t^{-1}) \oplus Ck \oplus Cd \), where \( \overline{g} \) is a finite-dimensional simple Lie algebra, \( K \) is the central element and \( d \) is the vector field \( t\partial_t \).

Let \( \tilde{G} \) be the algebraic group associated to \( g \) in [Ka]. If \( g \) is infinite-dimensional, then \( \tilde{G} \) is not a group scheme, but a group ind-scheme. We denote by \( G \) the quotient of \( \tilde{G} \) by its center (which belongs to the Cartan subgroup of \( G \) corresponding to the Lie subalgebra \( h \)). It comes with the lower unipotent and Borel subgroups \( N_- \) and \( B_- \) (which are proalgebraic groups) corresponding to \( n_- \) and \( b_- = h/\ell \oplus n_- \), respectively, and the upper unipotent and Borel subgroups \( N_+ \) and \( B_+ \) (which are group ind-schemes) corresponding to \( n_+ \) and \( b_+ = h/\ell \oplus n_+ \), respectively (here \( \ell \) is spanned by those elements \( x \) of \( h \) that satisfy \( (\alpha, x) = 0, i = 1, \ldots, \ell \)). We denote by \( H \) the intersection \( B_+ \cap B_- \). It is isomorphic to \( B_+/N_+ \) and to \( B_-/N_- \).

We wish to define the spaces of \( G \)-opers and Miura \( G \)-opers on \( X \) (which is again a smooth curve or a disc or a punctured disc).

First we need to introduce the notion of a \( G \)-bundle on \( X \) and a connection on such a bundle. A \( G \)-bundle on \( X \) is an ind-scheme \( F \) over \( X \) equipped with fiberwise simply transitive action of \( G \), which is locally trivial in the Zariski topology. This means that \( X \) may be covered by Zariski open subsets \( U_i \) such that the restriction of \( F \) to each \( U_i \) is isomorphic to the trivial bundle \( U_i \times G \). Two such trivializations differ by a morphism \( U_i \to G \) called the change of trivializations. If \( U = \text{Spec} \, R \), then the changes of trivializations on \( U \) form the group \( G(R) \).

To define a connection on a \( G \)-bundle it suffices to define the notion of a connection on the trivial \( G \)-bundle on an affine curve \( X = \text{Spec} \, R \) and explain how to act on these connections by the changes of trivializations. Without loss of generality we may assume that we are given an étale coordinate \( t : X \to \mathbb{A}^1 \) on \( X \). Let \( \partial_t \) be the vector field on \( X \) induced by a fixed translation vector field on \( \mathbb{A}^1 \). Then a connection on the trivial bundle is by definition an operator \( \nabla = \partial_t + A(t) \), where \( A(t) \in g(R) \). If \( g \in G(R) \) is a change of trivialization, then it acts on \( \nabla \) by the usual formula

\[
\nabla \mapsto \partial_t + gA(t)g^{-1} - (\partial_t g)g^{-1}.
\]

Under a change of coordinates \( t = \varphi(s) \) the operator \( \nabla \) transforms in the usual way:

\[
\nabla \mapsto \partial_s + \varphi'(s)A(\varphi(s)).
\]

It is easy to render this definition into the setting of analytic topology.

We will say that a connection \( \nabla \) on \( F \) gives rise to a trivialization of \( F \) over \( X \) if there is a trivialization of \( F \) over \( X \) with respect to which \( \nabla = \partial_t \).
Note that for ind-groups, such as $G$, a connection does not necessarily give rise to a trivialization of $\mathcal{F}$, even locally analytically. The usual correspondence between connections on $G$–bundles and local trivializations of the $G$–bundles does not exist in this case, because we do not have the exponential map from the Lie algebra $\mathfrak{g}$ to the group $G$ (though this correspondence exists if $X$ is a formal disc). However, in what follows we will consider Miura opers which carry a reduction to the proalgebraic group $B_-$ preserved by the connection. In this case a connection does give rise to local analytic trivializations of the underlying $B_-$–bundle, and hence the $G$–bundle as well.

The definition of $G$–opers is similar to the definition given in Section 2.4 in the finite-dimensional case.

A $G$–oper on $X$ is a triple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$, where $\mathcal{F}$ is a principal $G$–bundle $\mathcal{F}$ on $X$, $\nabla$ is a connection on $\mathcal{F}$ and $\mathcal{F}_{B_+}$ is a $B_+$–reduction of $\mathcal{F}$, such that locally, with a choice of a coordinate $t$ and a trivialization of $\mathcal{F}_{B_+}$, the connection operator has the form

\begin{equation}
\nabla = \partial_t + \sum_{i=1}^{\ell} \psi_i(t)f_i + v(t),
\end{equation}

where each $\psi_i(t)$ is a nowhere vanishing function, and $v(t)$ is a $\mathfrak{b}_+$–valued function. We denote the set of $G$–opers on $X$ by $\text{Op}_G(X)$.

The changes of trivialization amount in this case to the gauge action by $B_+$, so when $X = \text{Spec } R$ is affine, a $G$–oper is a gauge equivalence class of operators \[5.1\], where $v(t) \in \mathfrak{b}_+(R)$, with respect to the group of gauge transformations by $B_+(R)$. This is the same as an $N_+(R)$ gauge equivalence class of operators of the form

\begin{equation}
\nabla = \partial_t + p_- + v(t), \quad v(t) \in \mathfrak{b}_+(R),
\end{equation}

where, as before, $p_- = \sum_{i=1}^{\ell} f_i$.

Next, we give the definition of Miura $G$–opers. A Miura $G$–oper on $X$ is a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, where $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$ is a $G$–oper on $X$ and $\mathcal{F}_{B_-}$ is a $B_-$–reduction of $\mathcal{F}$ which is preserved by $\nabla$. We denote the set of Miura $G$–opers on $X$ by $\text{MOp}_G(X)$.

In the case when $\mathfrak{g}$ is finite-dimensional, this definition is equivalent to our old definition from Section 2.4. Indeed, $B_-$ is conjugate to $B_+$. Therefore a $B_-$–reduction $\mathcal{F}_{B_-}$ gives rise to a $B_+$–reduction $\mathcal{F}_{B_+}$, which is preserved by the connection. We may then take this $B_+$–reduction as the reduction $\mathcal{F}_{B_+}'$ of our old definition. But in the infinite-dimensional case the groups $B_+$ and $B_-$ are not conjugate to each other (in fact, one of them is not even a group scheme but a group ind-scheme) and there is an essential difference between asking for a horizontal $B_+$–reduction or a horizontal $B_-$–reduction.

In fact, for the purposes of the present paper it is essential that the horizontal reduction be to a proalgebraic subgroup $B_-$ and the oper reduction be to an ind-subgroup $B_+$. Indeed, we wish to relate our Miura opers to Cartan connections (see Proposition 5.4 below). We will do this by intersecting the two reductions inside $\mathcal{F}$, so they need to be “opposite” to each other. Next, since the connection operators of the Miura opers preserve a $B_-$–bundle $\mathcal{F}_{B_-}$, and $B_-$ is a proalgebraic group (not an ind-group), it makes sense to talk about parallel transport and horizontal sections on $\mathcal{F}_{B_-}$ (and hence on the induced $G$–bundle $\mathcal{F}$) over an arbitrary curve. Hence we can trivialize
locally a $G$–bundle equipped with a connection and a horizontal $B_-$–reduction. Then a reduction to $B_+$ gives rise to locally defined maps to $G/B_+$ which is a scheme of infinite type.

If we were to switch $B_+$ and $B_-$, our horizontal reduction would be to an ind-group $B_+$, and the notion of parallel transport would only make sense over a formal disc. But in what follows we need to use this notion for arbitrary curves (particularly, for $\mathbb{P}^1$), and this forces us to define opers and Miura opers in this fashion.

Remark 5.1. In [BeFr] Ben-Zvi and the author have already defined “affine opers” and “affine Miura opers”. However, these objects are different from the $G$–opers and Miura $G$–opers for an untwisted affine Kac-Moody algebra $g$ that we consider here, because in [BeFr] we considered the completion of $n_+$ rather than $n_-$, so that $g = \mathfrak{g}(t) \oplus Ck \oplus Cd$ (in the case of opers, we had also chosen in addition to the above data a reduction to the subgroup $G[t^{-1}]$ of $G$). In other words, in [BeFr] the roles of $B_+$ and $B_-$ were switched in the sense that in [BeFr] the group $B_+$ was a proalgebraic group and $B_-$ was an ind-group. □

Next, we define $G$–opers on the disc $D_x$ with regular singularity at $x$ following Section 2.3: these are the $N_+((t))$–equivalence classes of operators of the form

\[(5.3) \quad \nabla = \partial_t + \frac{1}{t} (p_{-1} + v(t)), \quad v(t) \in b_+[[t]].\]

Denote by $\text{Op}^\text{RS}_G(D_x)$ the space of opers on $D_x$ with regular singularity. By definition, it is a subspace of $\text{Op}_G(D_x^\times)$.

Finally, we define, for any dominant integral coweight $\tilde{\lambda} \in h/\mathfrak{c}$, the notion of a $G$–oper of coweight $\tilde{\lambda}$ on $D_x$ as an $N_+(\mathcal{X}_x)$–gauge equivalence class of operators of the form

\[(5.4) \quad \nabla = \partial_t + \sum_{i=1}^{\ell} t^{(\alpha_i, \tilde{\lambda})} f_i + v(t),\]

where $v(t) \in b_+[[t]]$. Denote the set of $G$–opers of coweight $\tilde{\lambda}$ on $D_x$ by $\text{Op}_G(D_x)_{\tilde{\lambda}} \subset \text{Op}^\text{RS}_G(D_x)$.

5.2. Miura opers and Cartan connections. We generalize the results of Section 2.4 to the case of an arbitrary Kac-Moody algebra.

Consider the flag variety $G/B_-$. This is an ind-scheme with the ind-scheme structure defined as follows. As a set, $G/B_-$ decomposes into a disjoint union of $B_-$–orbits parameterized by the Weyl group $W$ of $G$. We denote the orbit corresponding to $w$ by $S^w$. These orbits are finite-dimensional, and the closure of $S^w$ is the union of the orbits $S^y$ corresponding to the elements $y \in W$ which are less than or equal to $w$ with respect to the Bruhat order on $W$ (see [Kui], Ch. VII, for more details). Each of these closures, $\overline{S^w}$, has the structure of a finite-dimensional (in general, singular) algebraic variety. We have a collection of closed embeddings $\overline{S^y} \hookrightarrow \overline{S^w}$ of these varieties into each other corresponding to the Bruhat order. This collection defines the structure of a (strict) ind-scheme on $G/B_-$. 
We will say that $U \subset G/B_-$ is an open (resp., dense) subset if for sufficiently large $w \in W$, with respect to the Bruhat order, the intersection $U \cap S^w$ is open (resp., dense) in $U \cap S^w$.

The $B_+$–orbits in $G/B_-$ are also parameterized by the Weyl group. We denote the $B_+$–orbit $B_+w^{-1}B_+ \subset G/B_-$ by $S_w$, so that $S_1$ is the open dense orbit.

Let $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$ be a Miura $G$–oper on a curve $X$. Then we have the following analogue of Lemma 2.5.

**Lemma 5.2.** Let $\mathcal{F}$ be a local coordinate $t$, that of the bundle $\mathcal{F}$ given in formula (5.6) that $\mathcal{F}$ is isomorphic to both $\mathcal{F}_{B_+}$ and $\mathcal{F}_{B_-}$-twists of the flag variety $G/B_-$, which coincides with its $\mathcal{F}_{B_+,x}$-twists,

$$ (G/B_-)_{\mathcal{F}_x} = \mathcal{F}_x \times G/B_- = \mathcal{F}_{B_+,x} \times G/B_- = (G/B_-)_{\mathcal{F}_{B_+,x}}. $$

We obtain from the second description of $(G/B_-)_{\mathcal{F}_x}$ given in formula (5.5) that $(G/B_-)_{\mathcal{F}_x}$ decomposes into a union of the $\mathcal{F}_{B_+,x}$-twists of the $B_+$–orbits $S_w$ which we denote by $S_w,\mathcal{F}_{B_+,x}$. We will say that $\mathcal{F}_{B_+,x}$ and $\mathcal{F}_{B_-}$ are in relative position $w$ if $\mathcal{F}_{B_+,x}$, considered as point of $(G/B_-)_{\mathcal{F}_x}$, belongs to $S_w,\mathcal{F}_{B_+,x}$ (this agrees with the definition given in Section 2.4 in the finite-dimensional case). In particular, if it belongs to the open orbit $S_1,\mathcal{F}_{B_+,x}$, we will say that $\mathcal{F}_{B_+,x}$ and $\mathcal{F}_{B_-}$ are in generic position.

A Miura $G$–oper is called *generic* on $U \subset X$ if the reductions $\mathcal{F}_{B_+,x}$ and $\mathcal{F}_{B_-}$ of $\mathcal{F}_x$ are in generic position for all $x \in U$. We denote the set of generic Miura opers on $U$ by $\text{MOp}_G(U)_{\text{gen}}$.

Consider the $H$–bundles $\mathcal{F}_H = \mathcal{F}_{B_+}/N_+$ and $\mathcal{F}'_H = \mathcal{F}_{B_-}/N_-$ corresponding to a generic Miura oper $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$ on $X$. Then we have the following result (compare with Lemma 2.7 in the finite-dimensional case):

**Lemma 5.3.** For a generic Miura oper $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$ the $H$–bundles $\mathcal{F}_H$ and $\mathcal{F}'_H$ are isomorphic.

*Proof.* Since $\mathcal{F}_{B_+}$ and $\mathcal{F}_{B_-}$ are in generic position, their intersection $\mathcal{F}_{B_+} \cap \mathcal{F}_{B_-}$ inside $\mathcal{F}$ is isomorphic to both $\mathcal{F}_H$ and $\mathcal{F}'_H$. Hence we obtain that $\mathcal{F}_H \simeq \mathcal{F}'_H$. 

Since the $B_-$–bundle $\mathcal{F}_{B_-}$ is preserved by the oper connection $\nabla$, we obtain a connection $\nabla$ on $\mathcal{F}'_H$ and hence on $\mathcal{F}_H$. We prove, in exactly the same way as in the proof of Lemma 2.2, that $\mathcal{F}_H \simeq \Omega_{\tilde{\rho}}$, where $\tilde{\rho}$ is the unique cocharacter $C^\times \to H$ such that $\langle \alpha_i, \tilde{\rho} \rangle = 1, i = 1, \ldots, \ell$. Therefore we obtain a map $a$ from the set of $\text{MOp}_G(U)_{\text{gen}}$ of generic Miura opers on $U$ to the set of connections $\text{Conn}_U$ on the $H$–bundle $\Omega_{\tilde{\rho}}$ on $U$.

Connections on $\Omega_{\tilde{\rho}}$ are described in the same way as in the finite-dimensional case. If we choose a local coordinate $t$ on $U$, then we trivialize $\Omega_{\tilde{\rho}}$ and represent the connection as an operator $\partial_t + u(t)$, where $u(t)$ is an $h/e$-valued function on $U$. If $s$ is another coordinate such that $t = \varphi(s)$, then this connection will be represented by the operator

$$ \partial_s + \varphi'(s)u(\varphi(s)) - \tilde{\rho}' \frac{\varphi''(s)}{\varphi'(s)}. $$
Proposition 5.4. The map $a : \text{MOp}_G(U)_{\text{gen}} \to \text{Conn}_U$ is an isomorphism.

Proof. We define a map $b$ in the opposite direction, similarly to the finite-dimensional case. Suppose we are given a connection $\nabla$ on the $H$-bundle $\Omega^b$ on $D$. We associate to it a generic Miura oper as follows. We set $\mathcal{F} = \Omega^b \times G$, $\mathcal{F}_{B_\pm} = \Omega^b \times B_\pm$, where we consider the adjoint action of $H$ on $G$ and on $B_\pm$.

The space of connections on $\mathcal{F}$ is isomorphic to the direct product

$$\text{Conn}_U \times \bigoplus_{\alpha \in \Delta} \Gamma(U, \Omega^{\alpha(b)+1}).$$

Its subspace corresponding to negative simple roots is isomorphic to $\left( \bigoplus_{i=1}^\ell g_{-\alpha_i} \right) \otimes R$.

Having chosen a basis element $f_i$ of $g_{-\alpha_i}$ for each $i = 1, \ldots, \ell$, we now construct an element $p_{-1} = \sum_{i=1}^\ell f_i \otimes 1$ of this space. Now we set $\nabla = \nabla + p_{-1}$. By construction, $\nabla$ has the correct relative position with the $B_\pm$-reduction $\mathcal{F}_{B_\pm}$ and preserves the $B_\pm$-reduction $\mathcal{F}_{B_-}$. Therefore the quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$ is a generic Miura oper on $U$. We define the map $b$ by setting $b(\nabla) = (\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$. It is clear that this map is independent of the choice of the generators $f_i, i = 1, \ldots, \ell$, and that $a$ and $b$ are mutually inverse maps.

A Miura $G$-oper of coweight $\lambda$ on $D_x$ is defined as a quadruple $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$, where $(\mathcal{F}, \nabla, \mathcal{F}_{B_+})$ is a $G$-oper on $D_x^\times$ which belongs to $\text{Op}_G(D_x)_\lambda$ and $\mathcal{F}_{B_-}$ is a $B_-$-reduction of $\mathcal{F}$ which is preserved by $\nabla$. We denote the set of Miura $G$-opers of coweight $\lambda$ on $D_x$ by $\text{MOp}_G(D_x)_\lambda$. In particular, if $\lambda = 0$, then we obtain the old definition of Miura opers. All of the above definitions and results can be easily carried over to the case of an arbitrary integral $\lambda$.

5.3. Bethe Ansatz equations and Miura opers. Now we establish a connection between the Bethe Ansatz equations and Miura opers on $\mathbb{P}^1$, following Section 4.2.

Let us fix, as in the finite-dimensional case, a set of distinct complex numbers $z_1, \ldots, z_N$ (which we will view as points of $\mathbb{P}^1 \setminus \infty$) and a set of dominant integral coweights $\lambda_1, \ldots, \lambda_N \in \mathfrak{h}/\mathfrak{c}$ (a dominant integral coweight is by definition an element $\lambda$ of $\mathfrak{h}/\mathfrak{c}$ such that $\langle \alpha_i, \lambda \rangle \in \mathbb{Z}_+$ for all $i = 1, \ldots, \ell$).

In what follows we will consider the Miura opers on $\mathbb{P}^1$ rather than opers. The reason is that, as explained above, an oper connection does not in general allow us to identify the nearby fibers of the oper bundle; this is because the group $G$ is an ind-group. However, if in addition to an oper structure we are given a horizontal $B_-$-reduction $\mathcal{F}_{B_-}$, i.e., if we are given the structure of a Miura oper, then, because $B_-$ is a proalgebraic group, we can identify nearby fibers of $\mathcal{F}_{B_-}$ (and hence of $\mathcal{F}$) using the oper connection. This makes Miura opers much easier to handle.

Let $\text{MOp}_G(\mathbb{P}^1)_{(z_i)_{\lambda_i}}$ be the set of all Miura $G$-opers $(\mathcal{F}, \nabla, \mathcal{F}_{B_+}, \mathcal{F}_{B_-})$ on $\mathbb{P}^1$ without the points $z_1, \ldots, z_N, \infty$, whose restrictions to the punctured discs $D_x^\times$ around the points $z_i$ belong to $\text{MOp}_G(D_x)_{\lambda_i}$, and such that the restriction of the underlying oper to the disc around $\infty$ has regular singularity. Then the connection induced by $\nabla$ on the $B_-$-bundle $\mathcal{F}_{B_-}$ is regular everywhere on $\mathbb{A}^1$. Since $B_-$ is a proalgebraic group, we obtain that the connection identifies the fibers of the bundle $\mathcal{F}_{B_-}$ (and hence of
\( \mathcal{F} \) at all points of the affine line. Let us trivialize the fiber of \( \mathcal{F}_{B_+} \) at some point \( z_0 \) of \( \mathbb{A}^1 \). Then we obtain a trivialization of the bundle \( \mathcal{F} \) over \( \mathbb{A}^1 \). The oper reduction \( \mathcal{F}_{B_+} \) gives rise to a map \( \phi : \mathbb{A}^1 \to G/B_+ \). Two opers underlying Miura opers from \( \text{MOp}_{G/B_+}^{(\mathbb{P}^1)}(z_1,\ldots,z_N) \) are isomorphic if and only if the corresponding maps \( \phi \) differ by an element \( g \in G \) acting on \( G/B_+ \).

On the other hand, the choice of horizontal reduction \( \mathcal{F}_{B_-} \) corresponds to the choice of a reduction to \( B_- \) in the fiber of \( \mathcal{F} \) at \( z_0 \) (see Lemma 5.2). Since we have trivialized this fiber, the latter is nothing but a point of \( G/B_- \). Thus, we obtain that the space \( \text{MOp}^{G/B_-}_G(\mathbb{P}^1)(z_1,\ldots,z_N) \) of all Miura opers with the underlying oper map \( \phi \) is isomorphic to \( G/B_- \).

Recall that any pair of points \( yB_+ \in G/B_+ \) and \( pB_- \in G/B_- \) have a well-defined relative position. Namely, we will say that they have relative position \( w \in W \) if \( pB_- \in (yB_+y^{-1})w^{-1}B_- \). We will say that they are in generic position if \( w = 1 \).

Consider the subset \( (G/B_-)_\phi \) of \( G/B_- \) whose points \( pB_- \) satisfy the following conditions (as in Section 5.2):

1. \( \phi(z_i) \) is in generic position with \( pB_- \) for all \( i = 1,\ldots,N \);
2. the relative position of \( \phi(x) \) and \( pB_- \) is either generic or corresponds to a simple reflection \( s_i \in W \) for all \( x \in \mathbb{A}^1 \{z_1,\ldots,z_N\} \).

As in the finite-dimensional case, it is clear that \( (G/B_-)_\phi \) is an open and dense subset of \( G/B_- \). We claim that there is a bijection between this set and the set of solutions of the Bethe Ansatz equations (5.7)

\[
\sum_{i=1}^{N} \frac{\langle \alpha_{ij}, \lambda_i \rangle}{w_j - z_i} - \sum_{s \neq j} \frac{\langle \alpha_{ij}, \tilde{\alpha}_{is} \rangle}{w_j - w_s} = 0, \quad j = 1,\ldots,m.
\]

As in the finite-dimensional case, we have an obvious action of a product of symmetric groups permuting the points \( w_j \) corresponding to simple roots of the same kind. As before, by a solution of the BetheAnsatz equations we will understand a solution defined up to these permutations. We will also adjoin to the set of all solutions associated to all possible collections \( \{\alpha_{ij}\} \) of simple roots of \( \mathfrak{g} \), the unique “empty” solution, corresponding to the empty set of simple roots.

We start with the following

**Lemma 5.5.** Suppose that we are given a regular Miura oper on the disc \( D_x \) such that \( \mathcal{F}_{B_+} \) and \( \mathcal{F}_{B_-} \) are in relative position \( s_i \). Then the oper connection on \( D_x^\infty \) may be brought to the form

\[
\partial_t + p_{-1} + \frac{\tilde{\alpha}_i}{t} + u(t), \quad u(t) \in \mathfrak{b}[t]\]

(with respect to a coordinate \( t \) at \( x \), where \( \langle \alpha_i, u(0) \rangle = 0 \).

**Proof.** First, we observe that the two reductions \( \mathcal{F}_{B_+} \) and \( \mathcal{F}_{B_-} \) are in generic relative position on the punctured disc \( D_x^\infty \). Indeed, let \( V_{-\omega_i} \) be the lowest weight integrable module over \( \mathfrak{g} \) with lowest weight \( -\omega_i \). Consider its two-dimensional submodule over the \( \mathfrak{sl}_2 \) subalgebra corresponding to the \( i \)th simple root, generated by a lowest weight vector \( v_{-\omega_i} \). Let \( v_{-\omega_i + \alpha_i} = e_i v_{-\omega_i} \). By our assumption, the oper connection has the
form
\[ \nabla = \partial_t + \sum_{i=1}^{\ell} f_i + v(t), \quad v(t) \in \mathfrak{b}_+ [[t]]. \]

Let \( \Phi(t) \) be the unique solution of the equation \( \nabla \Phi(t) = 0 \) such that \( \Phi(0) = 1 \). Such a solution exists by our assumption that our oper carries a horizontal \( B_- \)-reduction. The above statement is equivalent to the assertion that \( \Phi(t) \cdot v_{-\omega_i + \alpha_i} \in V_{-\omega_i}[[t]] \) is a linear combination of weight vectors which contains the lowest weight vector \( v_{-\omega_i} \) with a non-zero coefficient. But this follows immediately from the observation that \( f_i v_{-\omega_i + \alpha_i} = -v_{-\omega_i} \).

This further implies, in the same way as in the proof of Proposition 2.9 that by gauge transformation with an element of \( N_+((t)) \) we can bring the oper connection \( \nabla \) to the form
\[ \partial_t + p_{-1} + \frac{\tilde{\lambda}_i}{t - z_i} + u(t), \quad u(t) \in \mathfrak{h}[[t]]. \]

We associate to this Miura \( G \)-oper a Miura \( SL_2 \)-oper in the same way as in Lemma 2.10. Following the argument used in the proof of Lemma 2.10, we find that the monodromy of this oper is non-trivial unless \( \langle \alpha_i, u(0) \rangle = 0 \). This completes the proof. \( \square \)

Now consider the Miura oper in \( \text{MOp}_{G}^\phi(\mathbb{P}^1)_{(z_i)_(\tilde{\lambda}_i)} \) corresponding to a point of the subset \( (G/B_-)_\phi \). Then by Lemma 5.5, the corresponding connection operator may be brought to the form \( \partial_t + p_{-1} + u(t) \), where \( u(t) \) is a rational function on \( \Lambda^1 \) which is regular at all points other than \( z_1, \ldots, z_N \), and whose expansion at \( z_i \) has the form
\[ \partial_t + p_{-1} - \frac{\tilde{\lambda}_i}{t - z_i} + \text{reg.}, \]

and the expansion at \( w_j \) has the form
\[ \partial_t + p_{-1} + \frac{\tilde{\alpha}_{ij}}{t - w_j} + u_j(t - w_j), \quad u_j(t - w_j) \in \mathfrak{h}[[t - w_j]], \]

and \( \langle \alpha_{ij}, u_j(0) \rangle = 0 \). By our assumption, our oper has regular singularity at \( \infty \), which implies that
\begin{equation}
\nabla = \partial_t + p_{-1} - \sum_{i=1}^{N} \frac{\tilde{\lambda}_i}{t - z_i} + \sum_{j=1}^{m} \frac{\tilde{\alpha}_{ij}}{t - w_j}.
\end{equation}

The condition \( \langle \alpha_{ij}, u_j(0) \rangle = 0 \) from Lemma 5.5 is precisely the \( j \)-th Bethe Ansatz equation (5.7). Thus, we obtain a map from \( (G/B_-)_\phi \) to the set of solutions of equations (5.7).

Let us construct the inverse map. Given a solution of the Bethe Ansatz equations, we define a Miura \( G \)-oper in \( \text{MOp}_{G}^\phi(\mathbb{P}^1)_{(z_i)_(\tilde{\lambda}_i)} \). We set \( \mathcal{F} = \Omega^\phi_H \times G, \mathcal{F}_{B_\pm} = \Omega^\phi_H \times B_\pm \) and define the connection operator by formula (5.8). Clearly, the two reductions \( \mathcal{F}_{B_\pm} \) satisfy the conditions of a Miura oper. It remains to show that its restriction to the punctured disc at \( z_i \) (resp., \( w_j \)) belongs to \( \text{MOp}_{G}(D_{z_i})_{\tilde{\lambda}_i} \) (resp., \( \text{MOp}_{G}(D_{w_j}) \)), and that it has regular singularity at \( \infty \).
The expansion at $z_i$ of the connection \( \partial_t + p_{-1} - \frac{\lambda_i}{t-z_i} + \text{reg.,} \)

which after conjugation by $\tilde{\lambda}_i(t-z_i)^{-1}$ becomes

\[
\partial_t + \sum_{k=1}^\ell (t-z_i)^{\langle \alpha_k, \tilde{\lambda}_i \rangle} f_k + \text{reg.}
\]

Therefore the restriction to the punctured disc at $z_i$ belongs to $\text{MOp}_G(D_{z_i})\tilde{\lambda}_i$ as desired.

Next, consider the expansion at the point $w_j$. We find that it has the form

\[
\partial_t + p_{-1} + \frac{\tilde{\alpha}_{ij}}{t-w_j} + u_j(t-w_j), \quad u(t-w_j) \in \mathfrak{h}[[t-w_j]].
\]

Moreover, we find that $\langle \alpha_{ij}, u_j(0) \rangle$ is given by the expression appearing in the $j$th Bethe Ansatz equation. The Bethe Ansatz equation means that $\langle \alpha_{ij}, u(0) \rangle = 0$, which ensures that this connection becomes regular after conjugation with $\exp(-e_{ij}/(t-w_j))$ (compare with Lemma \[2.10\] in the finite-dimensional case). Therefore the restriction to the punctured disc at $w_j$ belongs to $\text{MOp}_G(D_{w_j})$ as desired.

Finally, using the transformation formula for the oper connection that is identical to the one obtained in the finite-dimensional case (see formula \[2.4\]), we find the restriction of the oper \( \partial_t + p_{-1} \) to the punctured disc at $\infty$:

\[\text{(5.9)}\]

\[
\partial_u + p_{-1} + u^{-1} \left( \sum_{i=1}^N \tilde{\lambda}_i - \sum_{j=1}^m \tilde{\alpha}_{ij} + 2\hat{\rho} \right) + \text{reg.,}
\]

where $u = t^{-1}$. Thus, it has regular singularity at $\infty$.

Thus, we obtain a bijection between the set of solutions of the Bethe Ansatz equations and the union of the sets of points of open dense subsets $(G/B_-)_\phi$ of the flag variety $G/B_-$, just as in the finite-dimensional case. Now we show that the residues of the connection at $\infty$ correspond to the $B_-$-orbits in $G/B_-$.

Consider the action of the group $N_+$ on $G/B_-$. It translates into a rational action of $N_+$ on the set of solutions of the Bethe Ansatz equations. Let $SL_2^{(i)}$ be the $SL_2$ subgroup of $G$ corresponding to the $i$th simple root and set $N_{\alpha_i} = N_+ \cap SL_2^{(i)}$, $B_{\alpha_i} = B_- \cap SL_2^{(i)}$. Note that $N_{\alpha_i}$ is the one-parameter additive subgroup $\{\exp(ae_i)\}_{a \in \mathbb{C}} \subset N_+$.

Observe that the $SL_2^{(i)}$-orbits in $G/B_-$ give us a partition of $G/B_-$ into a disjoint union of $\mathbb{P}^1 \simeq G/B_{\alpha_i}$. Furthermore, if $pB_-$ is a point in the Schubert cell $B_- y B_- \subset G/B_-$, then there are two possibilities. The first case is that the intersection of the $SL_2^{(i)}$-orbit passing through this point and $B_- y B_-$ is an affine line. Then $l(s_i y) < l(y)$ and the remaining point of this $SL_2^{(i)}$-orbit belongs to the smaller Schubert cell $B_- s_i y B_-$ which is in the closure of $B_- y B_-$. This point is then stable under $B_{\alpha_i}$. The second case is that this intersection is the point $pB_-$, which is therefore stable under $B_{\alpha_i}$. Then $l(s_i y) > l(y)$ and the remaining part of the $SL_2^{(i)}$-orbit belongs to the larger Schubert cell $B_- s_i y B_-$ which contains $B_- y B_-$ in its closure.
On the other hand, the action of $N_{\alpha_i}$ on solutions of the Bethe Ansatz equations may be computed explicitly as in Section 3.3. Namely, we find that the element $\exp(\alpha_i)$ acts on the connection $\partial_t + p_{-1} + u(t)$ by sending $u(t)$ to $\tilde{u}(t) = u(t) + f(t)\tilde{\alpha}_i$, where $f(t)$ is the solution of the equation
\begin{equation}
(5.10) \quad f'(t) + f(t)^2 + f(t)u_i(t) = 0, \quad u_i(t) = \langle \alpha_i, u(t) \rangle,
\end{equation}
with the initial condition $f(0) = a$. Now observe that
\begin{equation}
(5.11) \quad u_i(t) = -\sum_{k=1}^{N} \frac{\langle \alpha_i, \lambda_k \rangle}{t - z_i} + \sum_{j \notin S_i} \frac{\langle \alpha_i, \tilde{\alpha}_j \rangle}{t - w_j} + \sum_{j \in S_i} \frac{2}{t - w_j},
\end{equation}
where $S_i \subset \{1, \ldots, m\}$ is the set of those $j$’s for which $i_j = i$. Hence (5.10) looks exactly like the corresponding equation for the action of $\exp(\alpha_i)$ on the solution of the Bethe Ansatz equation in the case of $\mathfrak{sl}_2$ corresponding to the connection $\partial_t + p_{-1} + u_i(t)$, where $u_i(t)$ is given by (5.11).

This solution corresponds to the situation where we have dominant coweights $\langle \alpha_i, \tilde{\lambda}_k \rangle$ of $\mathfrak{sl}_2^{(i)}$ attached to the point $z_k$ for $k = 1, \ldots, N$, dominant coweights $\langle \alpha_i, \tilde{\alpha}_j \rangle$ attached to the points $w_j$ with $j \in \{1, \ldots, m\} \setminus S_i$, and the variables of the Bethe Ansatz equations are $w_j, j \in S_i$. Clearly, the Bethe Ansatz equations (5.7) with $j \in S_i$ imply that these $w_j$’s, $j \in S_i$ indeed solve the Bethe Ansatz equations for $\mathfrak{sl}_2^{(i)}$ in the above situation. Hence we find that the action of $\exp(\alpha_i)$ on our solution can be read off of the action of $\exp(\alpha_i)$ on the corresponding solution of the Bethe Ansatz equation for $\mathfrak{sl}_2$. But we know from Corollary 3.3 and the discussion of Section 3.3 that the latter corresponds to the action of the unipotent subgroup of $SL_2$ on the flag manifold $SL_2/B_- = \mathbb{P}^1$. The closure of the orbit of any solution of the $SL_2$ Bethe Ansatz equation under the action of $N$ coincides with this $\mathbb{P}^1$. Further, it has two $B_-$-orbits: a point and an affine line. Suppose that the one point orbit belongs to the open subset of $SL_2/B_-$ of points corresponding to the solutions of the Bethe Ansatz equations (this will be so for a generic collection of points $z_1, \ldots, z_N$). Using Corollary 3.3 we find that then this point corresponds to the unique solution for which
\begin{equation}
(5.12) \quad \left\langle \alpha_i, \sum_{k=1}^{N} \tilde{\lambda}_k - \sum_{j=1}^{m} \tilde{\alpha}_{ij} \right\rangle
\end{equation}
is a non-negative integer; we denote this integer by $n_i$. The points of the other, one-dimensional, cell correspond to a one-parameter family of solutions for which the number (5.12) is a negative integer equal to $-n_i - 2$. Moreover, the number (5.12) is always an integer and is never equal to $-1$.

For any solution of the Bethe Ansatz equation call the expression
\begin{equation}
(5.13) \quad \sum_{k=1}^{N} \tilde{\lambda}_k - \sum_{j=1}^{m} \tilde{\alpha}_{ij}
\end{equation}
the residue of the solution at $\infty$. Note that it can be obtained from the expansion (5.9) of the connection (5.8) around $\infty$.

The above analysis leads us to the following conclusion.
Lemma 5.6. Let \( pB_- \) be a point of \((G/B_-)_\phi\) which belongs to the Schubert cell \(B_-yB_-\). Denote the residue of the corresponding solution of the Bethe ansatz equations by \( \tilde{\mu}_\infty \).

Then \( \langle \alpha_i, \tilde{\mu}_\infty \rangle \) is an integer not equal to \(-1\). It is non-negative if and only if \( l(s_iz) > l(y) \) and negative if and only if \( l(s_iz) > l(y) \).

This lemma implies that we may set up our bijection between solutions of the Bethe Ansatz equations corresponding to a fixed oper and the set of points of an open dense subset of \( G/B_- \) in such a way that the residue of the solution at \( \infty \) is always equal to

\[
\sum_{k=1}^{N} \lambda_k - \sum_{j=1}^{m} \alpha_{ij} = y(\lambda_\infty + \tilde{\rho}) - \tilde{\rho}
\]

for some dominant integral coweight \( \lambda_\infty \) and \( y \in W \), and in this case the corresponding point of \( G/B_- \) belongs to the \( B_- \)-orbit \( B_-yB_- \) of \( G/B_- \).

Consider first the simplest case when \( \lambda_\infty = \sum_{k=1}^{N} \lambda_k \). Then to \( y = 1 \) corresponds the “empty” solution of the Bethe Ansatz equations, when the set of the \( w_j \)'s is empty.

This solution has residue \( \sum_{k=1}^{N} \lambda_k \) and hence indeed corresponds to the one-point \( B_- \)-orbit \( B_- \in G/B_- \). Now we apply induction on the length \( l(y) \) of \( y \). Suppose we have proved the result for all \( y \) whose length is less than or equal to \( N \). Let us prove it for those elements whose length is \( N + 1 \). Those may be written in the form \( s_iy \), where \( s_i \) runs over the list of all simple reflections which satisfy \( l(s_iy) > l(y) \). This is equivalent to the following property: for any dominant integral coweight \( \lambda_\infty \) we have \( \langle \alpha_i, y(\lambda_\infty + \tilde{\rho}) - \tilde{\rho} \rangle = n_i \in \mathbb{Z}_+ \). Then \( \langle \alpha_i, s_iy(\lambda_\infty + \tilde{\rho}) - \tilde{\rho} \rangle = -n_i - 2 \). In this case the union of the \( B_- \)-orbits \( B_-yB_- \) and \( B_-s_iyB_- \) is the union of the closures of the orbits \( N_{\alpha_i} \cdot gB_-yB_- \). Our inductive assumption and the above computation then shows that the solutions corresponding to the points of \( B_-s_iyB_- \) have residue \( s_iy(\lambda_\infty + \tilde{\rho}) - \tilde{\rho} \).

Consider now the open subset \((G/B_-)_\phi\) corresponding to a general oper \( \phi \). Suppose that the one-point \( B_- \)-orbit \( B_- \in G/B_- \) belongs to this subset. Then we claim that the residue \( \tilde{\mu}_\infty \) of the corresponding solution of the Bethe Ansatz equations is a dominant coweight, i.e., \( y = 1 \) in formula (5.14). Indeed, for each \( i = 1, \ldots, \ell \) the point \( B_- \in G/B_- \) is the one point \( B_{\alpha_i} \)-orbit in the \( SL_2^{(i)} \)-orbit passing through \( B_- \) for all \( i = 1, \ldots, \ell \). Hence it follows from Lemma 5.6 that \( \langle \alpha_i, \tilde{\mu}_\infty \rangle \geq 0 \) for all \( i = 1, \ldots, \ell \).

Next, we consider the solutions whose residue belongs to the orbit of \( \tilde{\mu}_\infty \) under the action of the Weyl group. Using induction on the length in the same way as above, we obtain that the points that belong to \( B_-yB_- \cap (G/B_-)_\phi \) correspond to solutions satisfying (5.14).

Finally, suppose that the one-point \( B_- \)-orbit \( B_- \in G/B_- \) does not belong to \((G/B_-)_\phi\). Then we pick a point of \((G/B_-)_\phi\) that belongs to the Schubert cell of the smallest possible dimension. Consider the closures of the orbits of this point under the action of the subgroups \( SL_2^{(i)} \). Then we consider the \( SL_2^{(i)} \)-orbits of the points obtained this way, and so on. As the result, we can reach any point of \( G/B_- \) in finitely many steps. According to (5.14), each time we cross from a smaller Schubert cell \( B_-yB_- \) to a larger one \( B_-s_iyB_- \) via the \( SL_2^{(i)} \)-orbit, the residue of the corresponding solutions of the Bethe Ansatz equations changes from being dominant with respect to the \( i \)th
simple root, say, $n_i \in \mathbb{Z}_+$, to being anti-dominant $-2 - n_i$, while the pairing with the roots $\alpha_j, j \neq i$, remains unchanged. Hence consistency with Lemma 5.6 requires that the solutions corresponding to the points of $B_-yB$ have residue $y(\lambda_\infty + \rho) - \rho$.

Therefore we obtain the following result.

**Theorem 5.7.** There is a bijection between the set of solutions of the Bethe Ansatz equations corresponding to the same underlying $G$–oper $\phi$ and an open dense subset of the ind-flag variety $G/B_-$ such that the set of solutions which satisfy (5.14) is in bijection with an open subset of the $B_-$–orbit $B_-yB \subset G/B_-$.

We also have an analogue of Theorem 3.1 establishing a bijection between the set of all points of $G/B_-$ and a certain set of connections on the $H$–bundle $\Omega^\rho$ over $\mathbb{A}^1$ of the form

$$\partial_t - \sum_{i=1}^N \frac{y_i(\tilde{\lambda}_i + \tilde{\rho}) - \tilde{\rho}}{t - z_i} - \sum_{j=1}^m \frac{y'_j(\tilde{\rho}) - \tilde{\rho}}{t - w_j}.$$

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