Universal classes in the category of simple graphs

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Abstract. In this article, we show that the concept of graph compression is important for the study of universal equivalence of simple graphs (not necessarily finite) and graph algebraic structures.

1. Introduction
Several categories of algebraic structures are associated with the category of simple graphs. One of them is the category of so-called graph algebraic structures. We will present a concept of a graph algebraic system as it was done in [1]. Let $\mathbb{M}$ be an arbitrary variety of algebraic structures of a language $L$, such that $L$ contains an associative binary operation. Let $\Gamma$ be a simple graph with a set of vertices $X = \{x_1, \ldots, x_n\}$ and a set of edges $E(\Gamma)$. We define a graph $\mathbb{M}$-algebra $A_\Gamma$ using generators and defining relations, by the following way:

$$A_\Gamma = \langle x_1, \ldots, x_n | x_i \cdot x_j = x_j \cdot x_i \forall (x_i, x_j) \in E(\Gamma) \rangle_{\mathbb{M}},$$

where the dot denotes a binary operation.

This correspondence allows many results from graph theory to be used or transferred to the theory of corresponding algebraic systems. The methods of graph theory serves as a powerful tool for the study of such systems. For example, in papers [1, 2, 3, 4, 5, 6] universal classes, automorphisms, centralizer dimension of graph structures in different varieties of groups and algebras were investigated using methods from graph theory. On the other hand, research of the so-called limit combinatorics and limit graphs has intensified recently. In this direction, graph theory, in our opinion, plays a major role. In the note [10] you can find links to papers in this direction. We will follow [7] for notions and terms of the graph theory.

In this paper, we introduce several new definitions and prove universal equivalence results for the category of simple graphs (not necessarily finite), meaning using them later in group theory. Due to the limited volume of the paper, using the results for the study of universal classes of graph groups and limit graph groups, as well as for solving systems of equations in these groups, is beyond the scope of this article.

2. Universal relations
Let $L$ be the graph theory language without constants which contains the equality predicate $x = y$ and adjoin predicate $E(x, y)$. The category of simple graphs is defined by the following axioms:
\[
\forall x \sim E(x, x) \text{ a graph does not contains loops; } \forall x, y (x \neq y \land E(x, y) = E(y, x))
\]

Assume that \(\Gamma\) be a simple graph with vertices set \(V(\Gamma) = \{x_1, \ldots\}\) and edges \(E(\Gamma) = \{(x_i, x_j)\}\). The set \(\text{Th}_V(\Gamma)\) of all universal sentences of the language \(L\) which holds on \(\Gamma\) is called a universal theory of graph \(\Gamma\). By definition, two graphs \(\Gamma_1\) and \(\Gamma_2\) are universally equivalent if \(\text{Th}_V(\Gamma_1) = \text{Th}_V(\Gamma_2)\). In this case we will write \(\Gamma_1 \equiv \Gamma_2\). The principal universal class \(\text{ucl}(\Gamma)\) generated by graph \(\Gamma\) consists graphs \(\Gamma'\) such that \(\text{Th}_V(\Gamma') \subseteq \text{Th}_V(\Gamma')\).

The next proposition is an adaptation for the category of simple graphs of the criterion about universal equivalence of algebraic structures from [8, 9].

**Proposition 1.** Let \(\Gamma_1\) and \(\Gamma_2\) be simple graphs. Then \(\Gamma_1 \equiv \Gamma_2\) if and only if any finite full subgraph of \(\Gamma_1\) is isomorphic to some finite full subgraph of \(\Gamma_2\) and vice versa.

Let us define basic universal predicates which will be used in the article.

- Let \(\Gamma\) be a finite graph with vertex set \(V(\Gamma) = \{x_1, \ldots, x_n\}\) and edge set \(E(\Gamma)\). Let \(\phi(\Gamma)\) be a quantifier-free formula, which defines the graph \(\Gamma\):

\[
\phi(\Gamma) = \bigwedge_{i,j,i\neq j} x_i \neq x_j \bigwedge_{i,j,i\neq j, (x_i, x_j) \in E(\Gamma)} E(x_i, x_j) \bigwedge_{i,j,i\neq j, (x_i, x_j) \notin E(\Gamma)} \neg E(x_i, x_j).
\]

Then universal relation \(\forall x_1, \ldots, x_n \sim \phi(\Gamma)\) defines a universal class of graphs \(S\), which does not contains the graph \(\Gamma\) as a full subgraph of \(S\).

- \(x \perp y = \forall z (E(x, z) \to E(y, z) \land E(y, z) \to E(x, z) \land E(x, y))\).
- \(x \circ y = \forall z (E(x, z) \to E(y, z) \land E(y, z) \to E(x, z) \land \neg E(x, y))\).
- \(x \sim y = x \perp y \lor x \circ y\).

### 3. Compressed graphs and lattice of closed subsets

Recall some results of [2] about the compression graph \(\Gamma^c\) of the graph \(\Gamma\). Let \(\Gamma\) be a simple graph with vertices set \(X = \{x_1, \ldots\}\). For any connected vertices \(x, y\) of graph \(\Gamma\) we define \(d(x, y)\) as minimum of length of all paths connected \(x\) and \(y\). For subset \(Y \subseteq X\) we will define orthogonal complement:

\[
Y^\perp = \{x \in X | d(x, y) \leq 1 \forall y \in Y\}.
\]

We put \(\emptyset^\perp = X\). Define as \(cl(Y) = (Y^\perp)^\perp\). Not hard to check that \(cl(Y) \supseteq Y\) and hence, \(cl\) is a closure operator. Thus we have the following definition:

**Definition 1.** A set \(Y \subseteq X\) is called closed set if \(cl(Y) = Y\).

Let \(L(\Gamma)\) denotes the set of all closed subsets of \(X\). The set of all closed sets \(L(\Gamma)\) has a lattice structure by inclusion of elements and we can define a height \(h(L(\Gamma))\) of the \(L(\Gamma)\) as a length of maximal path in the lattice \(L(\Gamma)\).

Using \(\perp\) operator introduced above let us define several equivalence relations on a vertices set \(X\) of a graph \(\Gamma\). We put \(x \sim_{\perp} y\) if and only if \(x^\perp = y^\perp\). Denote by \([x]_\perp\) a class of \(\perp\)-equivalence of \(x\). Then we put \(x \sim_o y\) if and only if \(x^\perp \setminus x = y^\perp \setminus y\). Denote by \([x]_o\) a class of \(o\)-equivalence of vertex \(x\).

The following lemma reflects basic properties of introduced equivalences:

**Lemma 1.** [2].

(i) \([x]_\perp\) is clique for any \(x \in X\).
(ii) \([x]_\perp \cap [x]_o = \{x\}\) for any \(x \in X\).
(iii) If $|x| \geq 2$ then $|x_o| = 1$.
(iv) If $|x_o| \geq 2$, then $|x| = 1$.

Define $\sim$ relation on $X$ as $x \sim y$ if and only if $x \sim \bot y$ or $x \sim_o y$. From lemma 1 follows that $\sim$ is an equivalence relation. Denote by $[x]$ a class of equivalence of vertex $x \in X$ and let $[x_1], \ldots, [x_m]$ a set of all equivalence classes of vertices from $X$.

**Definition 2.** Compression of a graph $\Gamma$ is the graph $\Gamma^c$ with vertices $X^c = \{[x] | x \in X\}$ and vertices $[x]$ and $[y]$ are connected, if and only if vertices $x$ and $y$ are connected in graph $\Gamma$.

Direct reasoning proves the following facts:

**Proposition 2.** The mapping $c : X \rightarrow X^c$ by the rule $c(x) = [x]$, $x \in X$ induces a surjective graph homomorphism $c : \Gamma \rightarrow \Gamma^c$. The mapping $c : \Gamma \rightarrow \Gamma^c$ induces a lattice isomorphism $c : L(\Gamma) \rightarrow L(\Gamma^c)$.

We will consider a graph $\Gamma^c$ in the category of labeled graphs. To do this we will split $X$ in a disjunctive union of the following sets:

\[
X_1 = \{x \in X | [x] = [x_o] = [x]_{\bot}\},
\]
\[
X_{\bot} = \{x \in X | [x]_{\bot} = r_x \geq 2\},
\]
\[
X_0 = \{x \in X | [x]_o = l_x \geq 2\}.
\]

If $x \in X_1$, then mark of $x$ is $\mu(x) = \{1\}$. If $x \in X_{\bot}$, then mark of $x$ is $\mu(x) = \{\bot, r_x\}$ and if $x \in X_0$ then $\mu(x) = \{o, l_x\}$.

**Theorem 1.** Two graphs $\Gamma_1$ and $\Gamma_2$ such that at least one of them has finite compression graph are universally equivalent in the language of the graph theory without constants if and only if the compressed graphs $\Gamma_1^c$ and $\Gamma_2^c$ are isomorphic in the category of labeled graphs.

**Proof.** Let the graphs $\Gamma_1$ and $\Gamma_2$ be universally equivalent and the graph $\Gamma_1^c$ be a finite graph. We show that graphs $\Gamma_1^c$ and $\Gamma_2^c$ are the same. First, we prove that they are isomorphic in the category of simple unlabeled graphs. Let $|V(\Gamma_1^c)| = k \in \mathbb{N}$, then the formula $\psi(z_1, \ldots, z_k, z) = \forall z z_1 \sim z \lor \ldots \lor z_k \sim z$ is holds on the graph $\Gamma_1^c$ if and only if $z_1, \ldots, z_k$ are vertices of the compressed graph $\Gamma_1^c$. Let $\phi(\Gamma_1^c)$ be a quantifier-free formula defining the graph $\Gamma_1^c$. Then the universal formula $\Phi_1 = \forall x_1, \ldots, x_k, x \phi(\Gamma_1^c) \rightarrow \psi(x_1, \ldots, x_k, x)$ is holds on the graph $\Gamma_1$. The graph $\Gamma_1^c$ is a full subgraph of $\Gamma_2$ by the proposition 1. Therefore, the formula $\Phi_2 = \exists z_1, \ldots, z_k \forall z \phi(\Gamma_1^c) \land \psi(z_1, \ldots, z_k, z)$ is holds on the graph $\Gamma_2$. This implies, by definition of a compressed graph, the graph $\Gamma_1^c$ is a compressed graph for $\Gamma_2$, and therefore, the graphs $\Gamma_1^c$ and $\Gamma_2^c$ coincide in the category of simple unlabeled graphs.

We now prove that the labels of the graphs $\Gamma_1^c$ and $\Gamma_2^c$ are the same. To do this, it suffices to show that the labels of the vertices of a squeezed graph are determined by universal formulas. Let us examine three types of equivalent vertices:

(i) If $x \in X_1$, then the formula defining the label of the vertex $x$ is as follows: $\phi(\mu(x)) = \forall \exists (x \bot z) \land (x \circ y) \rightarrow x = z$

(ii) If $x \in X_{\bot}$ and $\mu(x) = \{\bot, r \in \mathbb{N}\}$, then $\phi(\mu(x)) = \forall z_1, \ldots, z_r (\bigwedge_{i=1}^r x \bot z_i) \rightarrow x = z_1 \lor \ldots \lor x = z_r$.

(iii) If $x \in X_0$ and $\mu(x) = \{o, l \in \mathbb{N}\}$, then $\phi(\mu(x)) = \forall z_1, \ldots, z_l (\bigwedge_{i=1}^l x \circ z_i) \rightarrow x = z_1 \lor \ldots \lor x = z_l$. 


Since the obtained formulas are universal, therefore the formula $\Phi^c = \forall x_1, \ldots, x_k, x(\phi(\Gamma^c_1)) \rightarrow \\
\psi(x_1, \ldots, x_k, x) \land \bigwedge_{x \in V(\Gamma^c_1)} \phi(\mu(x))$ is also universal. Hence $\Phi^c$ is satisfied on the graph $\Gamma_2$.

Therefore, the compressed graphs $\Gamma^c_1$ and $\Gamma^c_2$ coincide in the category of labeled graphs.

Let the graphs $\Gamma'_1$ and $\Gamma'_2$ coincide as labeled graphs. It is easy to see the graphs $\Gamma'_1$ and $\Gamma'_2$ restored by $\Gamma^c_1$ and $\Gamma^c_2$ graphs will be isomorphic. Since the set of all finite full subgraphs of the graph $\Gamma'_1$ coincides with the set of all finite full subgraphs of the graph $\Gamma_1$ then by proposition 1 two graphs $\Gamma'_1$ and $\Gamma_1$ are universally equivalent. Therefore the graphs $\Gamma_1$ and $\Gamma_2$ are universally equivalent. ■

4. Conclusion
Since graphs appear in the notions and proofs of the paper is finite, all our results are well formalized using programming languages. Therefore, these results will be useful for procedures for solving systems of equations over graph structures.

Acknowledgments
The work was supported by Russian Science Foundation, grant 18-71-10028

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