Polynomial invariants and Vassiliev invariants

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Abstract We give a criterion to detect whether the derivatives of the HOMFLY polynomial at a point is a Vassiliev invariant or not. In particular, for a complex number $b$ we show that the derivative $P^{(m,n)}_K(b,0) = \frac{\partial^m}{\partial a^m} \frac{\partial^n}{\partial x^n} P_K(a,x)|_{(a,x)=(b,0)}$ of the HOMFLY polynomial of a knot $K$ at $(b,0)$ is a Vassiliev invariant if and only if $b = \pm 1$. Also we analyze the space $V_n$ of Vassiliev invariants of degree $\leq n$ for $n = 1, 2, 3, 4, 5$ by using the $\bar{r}$--operation and the $\ast$--operation in [5]. These two operations are unified to the $\hat{r}$--operation. For each Vassiliev invariant $v$ of degree $\leq n$, $\hat{v}$ is a Vassiliev invariant of degree $\leq n$ and the value $\hat{v}(K)$ of a knot $K$ is a polynomial with multi--variables of degree $\leq n$ and we give some questions on polynomial invariants and the Vassiliev invariants.

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1 Introduction

In 1990, V. A. Vassiliev introduced the concept of a finite type invariant of knots, called Vassiliev invariants [13]. There are some analogies between Vassiliev invariants and polynomials. For example, in 1996 D. Bar–Nat an showed that when a Vassiliev invariant of degree $m$ is evaluated on a knot diagram having $n$ crossings, the result is approximately bounded by a constant times of $n^m$ [2] and S. Willerton [15] showed that for any Vassiliev invariant $v$ of degree $n$, the function $p_n(i,j): = v(T_{i,j})$ is a polynomial of degree $\leq n$ for each variable $i$ and $j$. Recently, we [4] defined a sequence of knots or links induced from a double dating tangle and showed that any Vassiliev invariant has a polynomial growth on this sequence.

J. S. Birman and X.–S. Lin [3] showed that each coefficient in the Maclaurin series of the Jones, Kauffman, and HOMFLY polynomial, after a suitable
change of variables, is a Vassiliev invariant, and T. Kanenobu [7, 8] showed that some derivatives of the HOMFLY and the Kauffman polynomial are Vassiliev invariants. For the question whether the \( n \)–th derivatives of knot polynomials are Vassiliev invariants or not, we [5] gave complete solutions for the Jones, Alexander, Conway polynomial and a partial solution for the \( Q \)–polynomial. Also we introduced the \( \overline{\ } \)–operation and the \( * \)–operation to obtain polynomial invariants from a Vassiliev invariant of degree \( n \). From each of these new polynomial invariants, we may get at most \( (n + 1) \) linearly independent numerical Vassiliev invariants.

In this paper, we find a line and two points in the complex plane where the derivatives of the HOMFLY polynomial can possibly be Vassiliev invariants and analyze the space \( V_n \) of Vassiliev invariants for \( n \leq 5 \) by using the \( \overline{\ } \)–operation and the \( * \)–operation.

Throughout this paper all knots or links are assumed to be oriented unless otherwise stated. For a knot \( K \) and \( i \in \mathbb{N} \), \( K^i \) denotes the \( i \)–times self–connected sum of \( K \) and \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C} \) denote the sets of nonnegative integers, integers, rational numbers, real numbers and complex numbers, respectively.

A knot or link invariant \( v \) taking values in an abelian group can be extended to a singular knot or link invariant by taking the difference between the positive and negative resolutions of the singularity. A knot or link invariant \( v \) is called a Vassiliev invariant of degree \( n \) if \( n \) is the smallest nonnegative integer such that \( v \) vanishes on singular knots or links with more than \( n \) double points. A knot or link invariant \( v \) is called a Vassiliev invariant if \( v \) is a Vassiliev invariant of degree \( n \) for some nonnegative integer \( n \).

**Definition 1.1** [4] Let \( J \) be a closed interval \([a, b]\) and \( k \) a positive integer. Fix \( k \) points in the upper plane \( J^2 \times \{b\} \) of the cube \( J^3 \) and their corresponding \( k \) points in the lower plane \( J^2 \times \{a\} \) of the cube \( J^3 \). A \((k, k)\)–tangle is obtained by attaching, within \( J^3 \), to these \( 2k \) points \( k \) curves, none of which should intersect each other. A \((k, k)\)–tangle is said to be oriented if each of its \( k \) curves is oriented. Given two \((k, k)\)–tangles \( S \) and \( T \), roughly the tangle product \( ST \) is defined to be the tangle obtained by gluing the lower plane of the cube containing \( S \) to the upper plane of the cube containing \( T \). The closure \( \overline{T} \) of a tangle \( T \) is the unoriented knot or link obtained by attaching \( k \) parallel strands connecting the \( k \) points and their corresponding \( k \) points in the exterior of the cube containing \( T \). When the tangles \( S \) and \( T \) are oriented, the oriented tangle \( ST \) is defined only when it respects the orientations of \( S \) and \( T \) and the closure \( \overline{S} \) has the orientation inherited from that of \( S \) and \( \overline{ST} \) is the oriented knot or link obtained by closing the \((k, k)\)–tangle \( ST \).
Definition 1.2 [4] An oriented \((k, k)\)-tangle \(T\) is called a double dating tangle (DD-tangle for short) if there exist some ordered pairs of crossings of the form \((\ast)\) in Figure 1, so that \(T\) becomes the trivial \((k, k)\)-tangle when we change all the crossings in the ordered pairs, where \(i\) and \(j\) in Figure 1, denote components of the tangle. Note that a DD-tangle is always an oriented tangle.

Since every \((1, 1)\)-tangle is a double dating tangle, every knot is a closure of a double dating \((1, 1)\)-tangle. But there is a link which is not the closure of any DD-tangle since the linking number of two components of the closure of a DD-tangle must be 0.

Definition 1.3 [4] Given an oriented \((k, k)\)-tangle \(S\) and a double dating \((k, k)\)-tangle \(T\) such that the product \(ST\) is well-defined, we have a sequence of links \(\{L_i(S, T)\}_{i=0}^{\infty}\) obtained by setting \(L_i(S, T) = \overline{ST^i}\) where \(T^i = TT \cdots T\) is the \(i\)-times self-product of \(T\) and \(T^0\) is the trivial \((k, k)\)-tangle. We call \(\{L_i(S, T)\}_{i=0}^{\infty}\) \(\{\{L_i\}_{i=0}^{\infty}\}\) for short) the sequence induced from the \((k, k)\)-tangle \(S\) and the double dating \((k, k)\)-tangle \(T\) or simply a sequence induced from the double dating tangle \(T\).

In particular, if \(\overline{S}\) is a knot for a \((k, k)\)-tangle \(S\), then \(L_i(S, T) = \overline{ST^i}\) is a knot for each \(i \in \mathbb{N}\) since \(T^i\) can be trivialized by changing some crossings.

Theorem 1.4 [5] Let \(\{L_i\}_{i=0}^{\infty}\) be a sequence of knots induced from a DD-tangle. Then any Vassiliev knot invariant \(v\) of degree \(n\) has a polynomial growth on \(\{L_i\}_{i=0}^{\infty}\) of degree \(\leq n\).

Corollary 1.5 [5] Let \(L\) and \(K\) be two knots. For each \(i \in \mathbb{N}\), let \(K_i = K^iL^i \cdots L^i\) be the connected sum of \(K\) to the \(i\)-times self-connected sum of \(L\). If \(v\) is a Vassiliev invariant of degree \(n\), then \(v|_{\{K_i\}_{i=0}^{\infty}}\) is a polynomial function in \(i\) of degree \(\leq n\).

The converse of Corollary 1.5 is not true. In fact, the maximal degree \(u(K)\) of the Conway polynomial \(\nabla_K(z)\) for a knot \(K\) is a counterexample.
2 The derivatives of the HOMFLY polynomial and Vassiliev invariants.

From now on, the notations $3_1$, $4_1$, $5_1$ and $6_1$ will mean the knots in the Rolfsen’s knot table [11]. For the definitions of the HOMFLY polynomial $P_L(a, z)$ and the Kauffman polynomial $F_L(a, x)$ of a knot or link $L$, see [10].

Note that the Jones polynomial $J_L(t)$, the Conway polynomial $\nabla_L(z)$, and the Alexander polynomial $\Delta_L(t)$ of a knot or link $L$ can be defined from the HOMFLY polynomial $P_L(a, z) \in \mathbb{Z}[a, a^{-1}, z, z^{-1}]$ via the equations $J_L(t) = P_L(t, t^{1/2} - t^{-1/2})$, $\nabla_L(z) = P_L(1, z)$ and $\Delta_L(t) = P_L(1, t^{1/2} - t^{-1/2})$ respectively and that the $Q$–polynomial $Q_L(x)$ can be defined from the Kauffman polynomial $F_L(a, x)$ via the equation $Q_L(x) = F_L(1, x)$.

By using the skein relations, we can see that $P_L(a, z)$ and $F_L(a, x)$ are multiplicative under the connected sum. i.e. $P_{L_1 \sharp L_2}(a, z) = P_{L_1}(a, z) P_{L_2}(a, z)$ and $F_{L_1 \sharp L_2}(a, x) = F_{L_1}(a, x) F_{L_2}(a, x)$ for all knots or links $L_1$ and $L_2$. So the Jones, Conway, Alexander and $Q$–polynomials are also multiplicative under the connected sum.

It is well known that $P_K(a, z) \in \mathbb{Z}[a^2, a^{-2}, z^2]$ and $F_K(a, x) \in \mathbb{Z}[a, a^{-1}, x]$ for a knot $K$. For each $i \in \mathbb{N}$ and each knot $K$, we denote by $F_i(K; a)$ and $P_{2i}(K; a)$ the coefficient of $x^i$ in $F_K(a, x)$ and the coefficient of $z^{2i}$ in $P_K(a, z)$, respectively, which are polynomials in $a$.

Throughout this section, knot polynomials are always assumed to be multiplicative under the connected sum.

We consider 1–variable knot polynomials first and then 2–variable knot polynomials.

Lemma 2.1 [5] Let $f_K(x)$ be a knot polynomial of a knot $K$ such that $f_K(x)$ is infinitely differentiable in a neighborhood of a point $a$ and assume that $f_K^{(1)}(a) \neq 0$. Then there exists a unique polynomial $p(x)$ of degree $m$ such that $f_K^{(m)}(a) = (f_K(a))^i p(i)$ for $i > m$. 

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Theorem 2.2 [5] For each $n \in \mathbb{N}$, we have

1. $J_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 1$.

2. $\nabla_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 0$.

3. $\Delta_K^{(n)}(a)$ is a Vassiliev invariant if and only if $a = 1$.

4. $Q_K^{(n)}(a)$ is not a Vassiliev invariant if $a \neq -2, 1$.

Theorem 2.3 Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function at $x = a$ with $g^{(1)}(a) \neq 0$. Assume that $f_K(x)$ is a knot polynomial which is infinitely differentiable in a neighborhood of $g(a)$ for all knots $K$ and that there exists a knot $L$ such that $f_L(g(a)) \neq 0, 1$ and $f_L^{(1)}(g(a)) \neq 0$. Then each coefficient of $(x - a)^n$ in the Taylor expansion of $f_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant.

Proof Consider a sequence $\{L^i\}_{i=0}^\infty$ of knots. By Lemma 2.1, we see that $(f_L(g(x)))^{(n)}|_{x=a} = (f_L(g(a)))^i p(i)$, where $p(i)$ is a polynomial in $i$ of degree $n$, and hence the coefficient of $(x - a)^n$ does not have a polynomial growth on $\{L^i\}_{i=0}^\infty$.

It follows from Corollary 1.5 that the coefficient of $(x - a)^n$ in the Taylor expansion of $f_K \circ g(x)$ is not a Vassiliev invariant.

J. S. Birman and X.–S. Lin [3] showed that each coefficient in the Maclaurin series of $J_K(e^x)$ is a Vassiliev invariant. As a generalization of Birman and Lin’s type of changing variables, we have

Theorem 2.4 Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function at $x = a$. Assume that $g^{(1)}(a) \neq 0$. Then

1. each coefficient of $(x - a)^n$ in the Taylor expansion of $J_K \circ g(x)$ at $x = a$, is a Vassiliev invariant if and only if $g(a) = 1$,

2. each coefficient of $(x - a)^n$ in the Taylor expansion of $\nabla_K \circ g(x)$ at $x = a$, is a Vassiliev invariant if and only if $g(a) = 0$,

3. each coefficient of $(x - a)^n$ in the Taylor expansion of $\Delta_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant if and only if $g(a) = 1$ and

4. if $g(a) \neq -2, 1$ then each coefficient of $(x - a)^n$ in the Taylor expansion of $Q_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant.
Proof (1) Let \( A_K = \{ t | J_K(t) = 0, 1 \} \cup \{ t | J_K^{(1)}(t) = 0 \} \) for a knot \( K \). Then \( A_3 \cap A_4 = \{ 1 \} \). Thus if \( g(a) \neq 1 \), then \( g(a) \notin \mathbb{R} \setminus (A_3 \cap A_4) \). Take \( L = 3 \) in Theorem 2.3 if \( g(a) \in \mathbb{R} \setminus A_3 \), and \( L = 4 \) in Theorem 2.3 if \( g(a) \in \mathbb{R} \setminus A_4 \). Then \( J_L(g(a)) \neq 0, 1 \) and \( J_L^{(1)}(g(a)) \neq 0 \). So by Theorem 2.3, each coefficient of \((x - a)^n\) in the Taylor expansion of \( J_K \circ g(x) \) is not a Vassiliev invariant. Conversely, assume that \( g(a) = 1 \) and that \( n \in \mathbb{N} \). Since the coefficient of \((x - a)^n\) in the Taylor expansion of \( J_K(g(x)) \) is a linear combination of \( 1, J_K^{(1)}(1), \ldots, J_K^{(n)}(1) \), by Theorem 2.2, it is a Vassiliev invariant. The proofs of (2), (3) and (4) are similar.

Example 2.5 Take \( f(x) = \sin(x) \) for \( x \in \mathbb{R} \). Then \( f(0) \neq 1 \) and \( f^{(1)}(0) \neq 0 \). Thus each coefficient in the Maclaurin series of \( J_K(\sin(x)) = J_K(f(x)) \) is not a Vassiliev invariant. But each coefficient in the Maclaurin series of \( \nabla_K(\sin(x)) = \nabla_K(f(x)) \) is a Vassiliev invariant, since it is a finite linear combination of the coefficients of the Conway polynomial \( \nabla_K(z) \) of a knot \( K \).

Now we will deal with 2–variable knot polynomials such as the HOMFLY polynomial \( P_K(a, z) \in \mathbb{Z}[a, a^{-1}, z] \) and the Kauffman polynomial \( F_K(a, x) \in \mathbb{Z}[a, a^{-1}, x] \). For a 2–variable Laurent polynomial \( g(x, y) \) which is infinitely differentiable on a neighborhood of \((a, b)\), we denote \( \frac{\partial^n}{\partial x^m \partial y^n} g(x, y) \big|_{(x, y) = (a, b)} \) by \( g^{(m, n)}(a, b) \) for each pair \((m, n) \in \mathbb{N}^2\).

Theorem 2.6 [5] Let \( g_K(x, y) \) be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of \((a, b)\) for all knots \( K \). If there exists a knot \( L \) such that \( g_L(a, b) \neq 0, 1 \), \( g_L^{(1, 0)}(a, b) \neq 0 \) and \( g_L^{(0, 1)}(a, b) \neq 0 \) then \( g_K^{(m, n)}(a, b) \) is not a Vassiliev invariant for all \( m, n \in \mathbb{N} \).

Lemma 2.7 Let \( g_K(x, y) \) be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of \((a, b) \in \mathbb{C}^2\) for all knots \( K \) and \( k, n \in \mathbb{N} \). If there exists a knot \( L \) such that \( g_L(a, b) \neq 0, 1 \), \( g_L^{(1, 0)}(a, b) \neq 0 \), \( g_L^{(0, 1)}(a, b) = 0 \) and \( g_L^{(0, 2)}(a, b) \neq 0 \) then there exists a polynomial \( p(i) \) of degree \( m + n \) such that \( g_L^{(m, 2n)}(a, b) = (g_L(a, b))^i p(i) \) for \( i > m + 2n \).

Proof It is similar to that of Theorem 2.12 in [5].

Lemma 2.8 Let \( g_K(x, y) \) be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of \((a, b) \in \mathbb{C}^2\) for all knots \( K \). If there exists a knot \( L \) such that \( g_L(a, b) \neq 0, 1 \), \( g_L^{(1, 0)}(a, b) \neq 0 \), \( g_L^{(0, 1)}(a, b) = 0 \) and \( g_L^{(0, 2)}(a, b) \neq 0 \) then \( g_K^{(m, 2n)}(a, b) \) is not a Vassiliev invariant for all \( m, n \in \mathbb{N} \).
Proof It follows from Lemma 2.7 and Corollary 1.5.

Theorem 2.9 Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$. $P_{2i}^{(n)}(K; a)$ is a Vassiliev invariant if and only if $a = \pm 1$.

Proof Note that $P_{2i}^{(n)}(K; a) = (2i)!P_{K}^{(n,2i)}(a,0)$. Since $P_{K}(a,z) \in \mathbb{Z}[a^2, a^{-2}, z^2]$ for all knots $K$, $P_{K}^{(n,1)}(a,0) = 0$ for all $a \in \mathbb{C}$ and all knots $K$. For each knot $K$, let $A_{1}^{K} = \{a \in \mathbb{C} \mid P_{K}(a,0) = 0 \}$, $A_{2}^{K} = \{a \in \mathbb{C} \mid P_{K}^{(1,0)}(a,0) = 0 \}$, $A_{3}^{K} = \{a \in \mathbb{C} \mid P_{K}^{(2,0)}(a,0) = 0 \}$ and $A_{K} = A_{1}^{K} \cup A_{2}^{K} \cup A_{3}^{K}$. Since $P_{3i}(a,z) = (-a^{-4} + 2a^{-2}) + a^{-2}z^{-2}$ and $P_{4i}(a,z) = (a^{-2} - 1 + a^2) - z^2$, we have $A_{3i} = \{\pm \sqrt{2}/2, \pm 1\}$, $A_{4i} = \{\pm (\sqrt{3} + 1)/2, \pm (\sqrt{3} - 1)/2, \pm 1, \pm \sqrt{-1}\}$ and hence $A_{3i} \cap A_{4i} = \{\pm 1\}$. Thus if $a \neq \pm 1$, then, by Lemma 2.8, $P_{2i}^{(n)}(K; a)$ is not a Vassiliev invariant. Conversely, T. Kanenobu [8] showed that $P_{2i}^{(n)}(K; 1)$ is a Vassiliev invariant. Since $P_{2i}^{(n)}(K; -1) = (-1)^n P_{2i}^{(n)}(K; 1)$, $P_{2i}^{(n)}(K; -1)$ is also a Vassiliev invariant.

By Theorem 2.9, for $b \in \mathbb{C}$, $P_{K}^{(m,n)}(b,0)$ is a Vassiliev invariant if and only if $n$ is odd or $b = \pm 1$. For $(b,y) \in \mathbb{C}^2$ with $y \neq 0$, we have the following

Theorem 2.10 Let $m,n$ be nonnegative integers. If $(b,y) \in \mathbb{C}^2$ with $y \neq 0$ such that $P_{K}^{(m,n)}(b,y)$ is a Vassiliev invariant, then $(b,y) = (b, \pm (b - b^{-1}))$, $(\pm \sqrt{-1}, \sqrt{-3})$ or $(\pm \sqrt{-1}, -\sqrt{-3})$.

Proof By direct calculations, $P_{3i}(a,z) = (-a^{-4} + 2a^{-2}) + a^{-2}z^{-2}$, $P_{4i}(a,z) = (a^{-2} - 1 + a^2) - z^2$ and $P_{6i}(a,z) = (a^{-4} - a^{-2} + a^2) + z^2(-a^{-2} - 1)$. Let $A_{1}^{K} = \{(b,y) \mid P_{K}(b,y) = 0 \}$, $A_{2}^{K} = \{(b,y) \mid P_{K}^{(1,0)}(b,y) = 0 \}$, $A_{3}^{K} = \{(b,y) \mid P_{K}^{(0,1)}(b,y) = 0 \}$ and $A_{K} = A_{1}^{K} \cup A_{2}^{K} \cup A_{3}^{K}$ for each knot $K$. Then $A_{3i} \cap A_{4i} = (A_{3i} \cap A_{4i}) \cup (A_{3i} \cap A_{6i}) \cup \cdots \cup ((A_{3i} \cap A_{4i}) \cap A_{6i}) = \{(\pm \sqrt{-1}, 2\sqrt{-1}), (\pm \sqrt{-1}, -2\sqrt{-1})\}$ $\cup \{(\pm \sqrt{-1}, \sqrt{-3}), (\pm \sqrt{-1}, -\sqrt{-3}), (\pm 1, \sqrt{-1}), (\pm 1, -\sqrt{-1})\}$ $\cup \{(-1 \pm \sqrt{5}/2, \sqrt{1 \pm \sqrt{5}}), (-1 \pm \sqrt{5}/2, -\sqrt{1 \pm \sqrt{5}})\}$ $\cup \{(b,y) \mid y = \pm (b - b^{-1})\}$. So we get $A_{3i} \cap A_{4i} \cap A_{6i} = ((A_{3i} \cap A_{4i}) \cap A_{6i}) \cup ((A_{3i} \cap A_{4i}) \cap A_{6i}) \cup ((A_{3i} \cap A_{4i}) \cap A_{6i}) = \{(b,y) \mid y = \pm (b - b^{-1})\} \cup \{(\pm \sqrt{-1}, \sqrt{-3}), (\pm \sqrt{-1}, -\sqrt{-3})\}$. 

Geometry & Topology Monographs, Volume 4 (2002)
If \((b,y) \in \mathbb{C}^2 \setminus (A_3 \cap A_4 \cap A_{61})\), then, by Theorem 2.6, \(P_K^{(m,n)}(b,y)\) is not a Vassiliev invariant.

Whether a finite product of the derivatives of knot polynomials at some points is a Vassiliev invariant or not can be detected by using Lemma 2.1, Theorem 2.6, Lemma 2.7 and Corollary 1.5. For example if there is a knot \(L\) such that \(J_L^{(1)}(a) \neq 0, Q_L^{(1)}(b) \neq 0, P_L^{(1,0)}(c, y) \neq 0, P_L^{(0,1)}(c, y) \neq 0\) and \(J_L(a)Q_L(b)P_L(c, y) \neq 0, 1\), then the product \(J_K^{(k)}(a)Q_K^{(l)}(b)P_K^{(m,n)}(c, y)\) is not a Vassiliev invariant for any \(k, l, m, n \in \mathbb{N}\).

Since \(Q_K^{(1)}(-2) = J_K^{(2)}(1)\) (T. Kanenobu [6]), \(Q_K^{(1)}(-2)\) is a Vassiliev invariant of degree \(\leq 2\). Note that \(Q_K^{(0)}(1) = 1\) for any knot \(K\) and hence \(Q_K^{(0)}(1)\) is a Vassiliev invariant of degree 0, but \(Q_K^{(1)}(1)\) and \(Q_K^{(2)}(1)\) are not Vassiliev invariants [5].

**Open Problem** (A. Stoimenow [12]) Is \(Q_K^{(n)}(-2)\) a Vassiliev invariant for \(n \geq 2\)?

**Question 2.11** Is \(Q_K^{(n)}(1)\) a Vassiliev invariant for \(n \geq 3\)?

The above two problems are the only remaining unsolved problems in one variable knot polynomials [5].

**Question 2.12** Find all the points at which the derivatives of the Kauffman polynomial are Vassiliev invariants.

**Question 2.13** Find all linear combinations of any finite products of derivatives of knot polynomials, which are Vassiliev invariants.

### 3 New polynomial invariants from Vassiliev invariants

In this section, a Vassiliev invariant \(v\) always means a Vassiliev invariant taking values in a numerical number field \(\mathbb{F} = \mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C}\). We begin with introducing the constructions of new polynomial invariants from a given Vassiliev invariant (see [4]) and then we will define a new polynomial invariant unifying the polynomial invariants obtained from the constructions in [4]. The new polynomial
invariant is also a Vassiliev invariant and so we get various numerical Vassiliev invariants from the coefficients of the new polynomial invariant.

Let $K$ and $L$ be two knots and let $\{L_i\}_{i=0}^\infty$ be a sequence of knots induced from a DD–tangle. Since any $(1,1)$–tangle is a DD–tangle, we get two sequences $\{L^*_iK^i\}_{i=0}^\infty$ and $\{K^*_iL_i\}_{i=0}^\infty$ of knots induced from DD–tangles.

Let $v$ be a Vassiliev invariant of degree $n$ and fix a knot $L$. Then by Corollary 1.5, for each knot $K$ there exist unique polynomials $p_K(x)$ and $q_K(x)$ in $F[x]$ with degrees $\leq n$ such that $v(L^*_iK^i) = p_K(i)$ and $v(K^*_iL_i) = q_K(i)$. We define two polynomial invariants $\bar{v}$ and $v^*$ as follows: $\bar{v}: \{\text{knots}\} \to F[x]$ by $\bar{v}(K) = p_K(x)$ and $v^*: \{\text{knots}\} \to F[x]$ by $v^*(K) = q_K(x)$. Then $\bar{v}(K)|_{x=0} = p_K(j) = v(L^*_iK^j)$ and $v^*(K)|_{x=0} = q_K(j) = v(K^*_iL_j)$ for all $j \in \mathbb{N}$.

Then we have the following

**Theorem 3.1** [5]. Let $v$ be a Vassiliev invariant of degree $n$ taking values in a numerical field $F$.

1. For a fixed knot $L$, $\bar{v}$ is a Vassiliev invariant of degree $\leq n$ and the degree of $x$ in $\bar{v}(K)$ is $\leq n$. In particular if $L$ is the unknot, $\bar{v}$ is a Vassiliev invariant of degree $n$ and $\bar{v}(K)|_{x=1} = v(K)$.

2. For a fixed sequence $\{L_i\}_{i=0}^\infty$ of knots induced from a DD–tangle, $v^*$ is a Vassiliev invariant of degree $\leq n$ and the degree of $x$ in $v^*(K)$ is $\leq n$. In particular if $L_j$ is the unknot for some $j \in \mathbb{N}$, then $v^*$ is a Vassiliev invariant of degree $n$ and $v^*(K)|_{x=j} = v(K)$.

Given a Vassiliev invariant $v$ of degree $n$, we may get at most $(n+1)$ linearly independent numerical Vassiliev invariants which are the coefficients of the polynomial invariants $\bar{v}$ and $v^*$ respectively and then apply $-$–operation and $^*$–operation repeatedly on these new Vassiliev invariants to get another new Vassiliev invariants. Inductively we may obtain various Vassiliev invariants.

We note that for a Vassiliev invariant $v$ of degree $n$, since $\bar{v}(K)$ and $v^*(K)$ are polynomials of degrees $\leq n$ for any knot $K$, the polynomial invariants $\bar{v}$ and $v^*$ are completely determined by $\{\bar{v}(K)|_{x=i} \mid 0 \leq i \leq n\}$ and $\{v^*(K)|_{x=i} \mid 0 \leq i \leq n\}$ respectively.

Let $V_n$ be the space of Vassiliev invariants of degrees $\leq n$ and let $A_n \subset V_n$. For each nonnegative integer $j$, define $A^j_n$ as follows. Set $A^0_n = A_n$ and define inductively $A^j_n$ to be the set of all Vassiliev invariants obtained from the coefficients of the new polynomial invariants $\bar{v}$ and $v^*$ ranging over all $v \in A^{j-1}_n$. 





all knots $L$ and all sequences $\{L_i\}_{i=0}^{\infty}$ induced from all DD–tangles in Theorem 3.1.

Define $A_n^* = \bigcup_{j=0}^{\infty} A_n^j$. We ask ourselves the following:

**Question** [5] Find a minimal finite subset $A_n$ of $V_n$ such that $\text{span}(A_n^*) = V_n$.

Let $V_n$ be the space of Vassiliev invariants of degree $\leq n$. Then the dimension of $V_n/V_{n-1}$ is $0, 1, 1, 3, 9, 14$ for $n = 1, 2, 3, 4, 5, 6, 7$ [1].

**Proposition 3.2** [7, 8] For each nonnegative integer $k$ and $l$,

1. $P_{2k}^{(l)}(K; 1)$ is a Vassiliev invariant of degree $\leq 2k + l$.
2. $(\sqrt{-1})^{k+l} F_k^{(l)}(K; \sqrt{-1})$ is a Vassiliev invariant of degree $\leq k + l$.

If $v_n$ and $v_m$ are Vassiliev invariants of degrees $n$ and $m$ respectively, then the product $v_nv_m$ is a Vassiliev invariant of degree $\leq n + m$ [1, 14].

We get a base for each $V_n$ ($n \leq 5$) from the results of J. S. Birman and X.–S. Lin (citeBL, D. Bar–Natan [1] and T. Kanenobu [9]).

**Theorem 3.3** [9, 3, 1] Let $V_n$ be the space of Vassiliev invariants of degree $\leq n$. Then

1. $\{1\}$ is a basis for $V_0 = V_1$, where 1 is the constant map with image $\{1\}$.
2. $\{a_2(K)\}$ is a basis for $V_2/V_1$.
3. $\{J_{K}^{(3)}(1)\}$ is a basis for $V_3/V_2$.
4. $\{(a_2(K))^2, a_4(K), J_{K}^{(4)}(1)\}$ is a basis for $V_4/V_3$.
5. $\{a_2(K)P_0^{(3)}(K; 1), P_0^{(5)}(K; 1), P_4^{(1)}(K; 1), \sqrt{-1} F_4^{(1)}(K; \sqrt{-1})\}$ is a basis for $V_5/V_4$.

We can easily see that the Vassiliev invariants $a_2(K)$, $\sqrt{-1} F_4^{(1)}(K; \sqrt{-1})$ and $J_{K}^{(3)}(1)$ are additive. If $v$ is an additive Vassiliev invariant, then, from the coefficients of the polynomial invariants $\pi$ and $v^*$, we cannot get Vassiliev invariants other than linear combinations of $v$ and the constant Vassiliev invariants.

Let $v$ be a Vassiliev invariant of degree $n$ and $L$ a knot. Define $v_L^i$ to be the Vassiliev invariant defined by $v_L^i(K) = v(L^i K)$ and define $v_L$ to be the Vassiliev invariant defined by $v_L(K) = v(L^* K)$ [5]. Then we can see that the
Vassiliev invariants obtained from the coefficients of $\overline{\tau}$ and $\nu^*$ are contained in the spans of the sets $\{v^i_L \mid L$ is a knot, $i = 0, 1, 2, \cdots, n\}$ and $\{v_L \mid L$ is a knot$\}$ respectively.

Take the trivial knot, $3_1$, $4_1$ and $5_1$ for $L$ and $(3_1)^i$, $(4_1)^i$ and $(5_1)^i$ for $L_i$ in Theorem 3.1. Then all linearly independent Vassiliev invariants obtained by applying the $\overline{\tau}$–operations and the $\overline{*}$–operations for the non–additive Vassiliev invariants of degree $\leq 5$ in Theorem 3.3 can be found as follows.

\[
(a_2(K))^2 \rightarrow \{a_2(K)\}
\]
\[
a_4(K) \rightarrow \{a_2(K), (a_2(K))^2\}
\]
\[
J^{(4)}_K(1) \rightarrow \{a_2(K), (a_2(K))^2\}
\]
\[
a_2(K)P_0^{(3)}(K; 1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}, \quad a_2(K)P_0^{(3)}(K; 1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}
\]
\[
a_2(K)J^{(3)}_K(1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}
\]
\[
P_0^{(5)}(K; 1) \rightarrow \{a_2(K)P_0^{(3)}(K; 1), J^{(3)}_K(1)\}; \quad P_0^{(5)}(K; 1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}
\]
\[
P_4^{(1)}(K; 1) \rightarrow \{a_2(K)P_2^{(1)}(K; 1), J^{(3)}_K(1)\}; \quad P_4^{(1)}(K; 1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}
\]
\[
a_2(K)P_2^{(1)}(K; 1) \rightarrow \{a_2(K), J^{(3)}_K(1)\}
\]

For simplicity, for each Vassiliev invariant $\nu$, we unlist the Vassiliev invariants obtained from $\nu^*$ if they can be obtained from $\overline{\tau}$ and we also exclude the constant map $1$ whose image is $\{1\}$ and $v$ itself in the list of Vassiliev invariants obtained from $\overline{\tau}$ and $\nu^*$.

Thus we get the following

Theorem 3.4 Let $A_n$ be a subset of the space $V_n$ of the Vassiliev invariants of degree $\leq n$ such that $\text{span}(A_n^* \backslash \nu^*) = V_n$. Then $A_n$ can be chosen as follows.

1. $A_0 = A_1 = \{1\}$, where $1$ denotes the constant map with image $\{1\}$.
2. $A_2 = \{a_2(K)\}$.
3. $A_3 = \{a_2(K), J^{(3)}_K(1)\}$.
4. $A_4 = \{J^{(3)}_K(1), a_4(K), J^{(4)}_K(1)\}$.
5. $A_5 = \{P_0^{(5)}(K; 1), I_4^{(1)}(K; 1), \sqrt{-1}I_4^{(1)}(K; \sqrt{-1}), a_4(K), J^{(4)}_K(1)\}$.

Let $v$ be a Vassiliev invariant of degree $n$. In [5], the authors generalized the one–variable knot polynomial invariants $\overline{\nu}$ and $\nu^*$ to two–variable knot polynomial invariants $\overline{\nu}$ and $\nu^*$, respectively with the same notation.
Now we want to generalize the two–variable knot polynomial invariants $\bar{v}$ and $v^*$ in Theorem 3.1 simultaneously to a multi–variable knot polynomial invariant $\hat{v}$ by unifying both $\bar{v}$ and $v^*$ to a multi-variable polynomial invariant $\hat{v}$ whose proof is analogous to that of Theorem 3.1. See [5].

Given sequences $\{L_i^{(1)}\}_{i=0}^{\infty}, \ldots, \{L_i^{(k)}\}_{i=0}^{\infty}$ of knots induced from DD–tangles, for each knot $K$, there exists a unique polynomial

$$p_K(x_0, x_1, \ldots, x_k) \in \mathbb{F}[x_0, x_1, \ldots, x_k]$$

such that for all $(i_0, i_1, \ldots, i_k) \in \mathbb{N}^{k+1}$, $v(K^{i_0} L_i^{(1)} \# \cdots \# L_i^{(k)}) = p_k(i_0, i_1, \ldots, i_k)$.

Now we define a new polynomial invariant $\hat{v}: \{\text{knots}\} \to \mathbb{F}[x_0, \ldots, x_k]$ by

$$\hat{v}(K) = p_K(x_0, \ldots, x_k).$$

Then by applying the similar argument to the case of $\bar{v}$ and $v^*$ [5], we can see that $\hat{v}$ is a Vassiliev invariant of degree $\leq n$ and the degree of each variable $x_i$ in $\hat{v}(K)$ is $\leq n$. Thus we get the following

**Theorem 3.5** Let $v$ be a Vassiliev invariant of degree $n$ taking values in a numerical field $\mathbb{F}$ and let $\{L_i^{(1)}\}_{i=0}^{\infty}, \ldots, \{L_i^{(k)}\}_{i=0}^{\infty}$ be sequences of knots induced from DD–tangles. Then $\hat{v}: \{\text{knots}\} \to \mathbb{F}[x_0, \ldots, x_k]$ is a Vassiliev invariant of degree $\leq n$ and the degree of each variable $x_i$ in $\hat{v}(K)$ is $\leq n$.

For a Vassiliev invariant $v$, let $C_v: = \{\text{the coefficients of the polynomial } \hat{v}(K)\}$. Then, in Theorem 3.5, $\hat{v}$ is completely determined by $C_v$. Since the degree of each variable in $\hat{v}$ is $\leq n$, we see that

$$\text{span}(C_v) = \text{span} \{\{\hat{v}(K)\}_{(x_0, \ldots, x_k)=(i_0, \ldots, i_k)} \mid 0 \leq i_0, \ldots, i_k \leq n\}.$$ 

**Question 3.6** Let $v$ be a Vassiliev invariant of degree $n$. Find sequences $\{L_i^{(1)}\}_{i=0}^{\infty}, \ldots, \{L_i^{(k)}\}_{i=0}^{\infty}$ of knots induced from DD–tangles such that $\text{span}(C_v) = \text{span}(\{v\}^*)$ where $C_v$ is the set of coefficients of the polynomial invariant $\hat{v}$ induced from $v$ and $\{L_i^{(1)}\}_{i=0}^{\infty}, \ldots, \{L_i^{(k)}\}_{i=0}^{\infty}$.

**References**

[1] D Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995) 423–472

[2] D Bar-Natan, *Polynomial invariants are polynomials*, arXiv:q-alg/960625

[3] J S Birman, X-S Lin, *Knot polynomials and Vassiliev’s invariants*, Invent. Math. 111 (1993) 225–270

*Geometry & Topology Monographs, Volume 4 (2002)*
Polynomial invariants and Vassiliev invariants

[4] M-J Jeong, C-Y Park, Vassiliev invariants and double dating tangle, J. of Knot Theory and Its Ramifications 11 (2002) 527–544

[5] M-J Jeong, C-Y Park, Vassiliev invariants and knot polynomials, to appear in Topology and Its Applications

[6] T Kanenobu, An evaluation of the first derivative of the $Q$–polynomial of a link, Kobe J. Math. 5 (1988) 179–184

[7] T Kanenobu, Kauffman polynomials as Vassiliev link invariants, from: “Proceedings of Knots 96”, (S Suzuki, editor), World Sci. Publ. Co. Singapore (1997) 411–431

[8] T Kanenobu, Y Miyazawa, HOMFLY polynomials as Vassiliev link invariants, from: “Knot Theory”, Banach Center Publications 42, (VFR Jones, J Kania–Bartoszyńska, JH Przytycki, P Traczyk and VG Turaev, editors), Institute of Mathematics, Polish Academy of Science, Warszawa (1998) 165–185

[9] T Kanenobu, Vassiliev knot invariants of order 6, J. of Knot Theory and its Ramifications 10 (2001) 645–665

[10] A Kawauchi, A Survey of Knot Theory, Birkhäuser Verlag (1996)

[11] D Rolfsen, Knots and Links, Publish or Perish Inc. (1990)

[12] A Stoimenow, Problem Session Notes, available from his webpage: http://guests.mpim-bonn.mpg.de/alex

[13] V A Vassiliev, Cohomology of knot spaces, from: “Theory of Singularities and Its Applications”, (V I Arnold, editor) Advances in Soviet Mathematics, Vol. 1, AMS (1990)

[14] S Willerton, Vassiliev invariants and the Hopf algebra of chord diagrams, Math. Proc. Camb. Phil. Soc. 119 (1996) 55–65

[15] S. Willerton, Cabling the Vassiliev invariants, preprint

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