Conservation Laws in Higher-Order Nonlinear Optical Effects

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ABSTRACT

Conservation laws of the nonlinear Schrödinger equation are studied in the presence of higher-order nonlinear optical effects including the third-order dispersion and the self-steepening. In a context of group theory, we derive a general expression for infinitely many conserved currents and charges of the coupled higher-order nonlinear Schrödinger equation. The first few currents and charges are also presented explicitly. Due to the higher-order effects, conservation laws of the nonlinear Schrödinger equation are violated in general. The differences between the types of the conserved currents for the Hirota and the Sasa-Satsuma equations imply that the higher-order terms determine the inherent types of conserved quantities for each integrable cases of the higher-order nonlinear Schrödinger equation.

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In the ultrafast optical signal system, the higher-order nonlinear effects such as the third-order dispersion, the self-steepening, and the self-frequency shift become important if the pulses are shorter than $T_0 \leq 100 \text{fs}$ [1]. The use of optical pulses with distinct polarizations and/or frequencies also require the consideration of nonlinear cross-couplings between different modes of pulses. Inclusion of both the higher-order and the cross-coupling effects lead to the study on the coupled higher-order nonlinear Schrödinger equations (CHONSE) which are not in general integrable except for special cases of coupling constants. Those integrable cases of coupling constants have been classified in association with Hermitian symmetric spaces [2]. It is also well known that soliton equations which can be integrated by the inverse scattering method possess infinite number of conserved quantities. For example, the nonlinear Schrödinger equation (NSE) has infinite number of conserved charges in addition to the ones corresponding to the energy and the intensity-weighted mean frequency. However, the effect of the higher-order and the cross coupling terms on the conservation laws has not been considered up to now.

In this paper, we make a systematic study on the conservation laws in the presence of the higher-order and the cross-coupling terms. We first indicate that except for the energy conservation, other conservation laws of the NSE such as the conservation of the intensity-weighted mean frequency do not hold due to the higher-order effects any more, unless the higher-order terms are of a unique type. In the case of integrable CHONSE, we derive general expressions of infinite number of conserved currents and charges from the Lax pair formulation utilizing the properties of the Hermitian symmetric space. From the general expressions, explicit forms of the first few conserved currents and the associated charges of the Hirota and the Sasa-Satsuma equations are calculated. We then explain the correlations of conservation laws between the two integrable cases of the higher-order extension of the NSE.

In order to illustrate the issue, we first consider the NSE including the higher-order terms. In a mono-mode optical fiber, the propagation of a ultrashort pulse is governed by the higher-order nonlinear Schrödinger equation [3]

$$\bar{\partial} \psi = i(\gamma_1 \partial^2 \psi + \gamma_2 |\psi|^2 \psi) + \gamma_3 \partial^3 \psi + \gamma_4 \partial(|\psi|^2 \psi) + \gamma_5 \partial(|\psi|^2) \psi, \quad (1)$$

where $\bar{\partial} \equiv \partial/\partial \bar{z}$ and $\partial \equiv \partial/\partial z$ are derivatives in retarded time coordinates ($\bar{z} = x, z = t - x/v$), and $\psi$ is the slowly varying envelope function. The real coefficients $\gamma_i \ (i = 1, 2, 3, 4)$ in the first four terms on the right hand side of Eq. (1) specify in sequence the effects
of the group velocity dispersion, the self-phase modulation, the third order dispersion, and the self-steepening. With appropriate scalings of space, time, and field variables, one can readily normalize Eq. (1) so that \( \gamma_1 = 1, \gamma_2 = 2, \gamma_3 = 1 \) which we assume from now on. The remaining coefficient \( \gamma_5 \) in the last term is complex in general. The real and the imaginary parts of \( \gamma_5 \) are due to the effect of the frequency-dependent radius of fiber mode and the effect of the self-frequency shift by stimulated Raman scattering, respectively. It is well known that the above equation becomes integrable if \( \gamma_4 = -\gamma_5 = 6 \) (Hirota case) \([4]\) or \( \gamma_4 = -2\gamma_5 = 6 \) (Sasa-Satsuma case) \([5]\). In the absense of higher-order terms \((\gamma_3 = \gamma_4 = \gamma_5 = 0)\), Eq. (1) possesses infinite number of conserved charges among which the first three charges \([6]\) are

\[
Q_1 = \int_{-\infty}^{\infty} |\psi|^2 dt ,
Q_2 = i \int_{-\infty}^{\infty} (\psi^* \partial \psi - \partial \psi^* \psi) dt ,
Q_3 = \int_{-\infty}^{\infty} (\partial \psi^* \partial \psi - |\psi|^4) dt ,
\]

where \( Q_1 \) represents conserved energy, and \( Q_2 \) the mean frequency weighted by the intensity of optical pulses. In the conventional NSE where the time and the space coordinates are interchanged, \( Q_1, Q_2 \) and \( Q_3 \) respectively correspond to conserved mass, momentum and energy. If we include higher-order terms, \( Q_i \) are not necessarily conserved but subject to the relations;

\[
\bar{\partial} Q_1 = 0 ,
\bar{\partial} Q_2 = 2i(\gamma_4 + \gamma_5) \int_{-\infty}^{\infty} \partial |\psi|^2 , (\psi^* \partial \psi - \partial \psi^* \psi) dt 
\bar{\partial} Q_3 = (3\gamma_4 + 2\gamma_5 - 6) \int_{-\infty}^{\infty} \partial |\psi|^2 \partial \psi^* \partial \psi dt .
\]

The calculations indicate that the charge \( Q_1 \) which corresponds to energy is conserved for all values of \( \gamma_4, \gamma_5 \) while \( Q_2 \) and \( Q_3 \) are conserved provided \( \gamma_4 + \gamma_5 = 0 \) and \( 3\gamma_4 + 2\gamma_5 = 6 \), respectively. Note that \( Q_2 \) and \( Q_3 \) are conserved simultaneously only for the specific value \( \gamma_4 = -\gamma_5 = 6 \) that is precisely the Hirota case. It is interesting to observe that integrability does not always imply the same types of conserved currents in the presence of higher-order terms. Another integrable case of the Sasa-Satsuma equation, where \( \gamma_4 = -2\gamma_5 = 6 \), in fact does not have \( Q_2 \) and \( Q_3 \) in Eq. \([3]\) as the conserved charges. This consequence is rather remarkable in view of the fact that integrable equations possess infinite number of
conserved quantities. We will show, however, the Sasa-Satsuma equation also possesses infinitely many conserved charges of different types other than the ones of the Hirota equation.

In case we include both the higher-order and the cross-coupling nonlinear effects, the propagating system is governed by a CHONSE. Without understanding physical settings, it would be meaningless to write down any general expression of the CHONSE. However, as explicitly derived in [2], there exists a group theoretic specification which admits a systematic classification of integrable cases of the CHONSE. In the following, we consider a group theoretic generalization of the NSE and define the CHONSE in association with a Hermitian symmetric space. By solving the linear Lax equations iteratively, we derive infinite number of conserved currents and charges for the CHONSE. For the later use, now we briefly review the definition of Hermitian symmetric spaces [7, 8] and the generalization of the NSE [2, 9] according to the Hermitian symmetric spaces.

A symmetric space is a coset space $G/K$ for Lie groups $G \supseteq K$ whose associated Lie algebras $g$ and $k$, with the decomposition: $g = k \oplus m$, satisfy the commutation relations:

$$[k, k] \subset k, \quad [m, m] \subset k, \quad [k, m] \subset m$$

(4)

A Hermitian symmetric space is the symmetric space $G/K$ equipped with a complex structure. One can always find an element $T$ in the Cartan subalgebra of $g$ whose adjoint action defines a complex structure and also the subalgebra $k$ as a kernel, i.e., $k = \{V \in g : [T, V] = 0\}$. That is, the adjoint action $J \equiv \text{ad}T = [T, \ast]$ is a linear map $J : m \rightarrow m$ that satisfies the complex structure condition, $J^2 = -I$, or $[T, [T, M]] = -M$ for $M \in m$. Then, we define a CHONSE as

$$\partial E = \partial^2 \tilde{E} - 2E^2 \tilde{E} + \alpha(\partial^3 E + \beta_1 E^2 \partial E + \beta_2 \partial EE^2)$$

(5)

where $E$ and $\tilde{E} \equiv [T, E]$ are extended field variables belonging to $m$. The arbitrary constant $\alpha$ may be normalized to 1 by an appropriate scaling but we keep it in order to exemplify the higher-order effects. Also the cross-coupling effects between different modes of polarizations or frequencies are accommodated in the matrix form of $E$ which is determined by each Hermitian symmetric space. For example, in the case where $G/K = A\!II\!I, C\!I$ and $D\!II\!I$ only so that the expression of CHONSE becomes simplified [2].
$SU(N+1)/U(N)$, matrices $E$ and $T$ are represented as

$$E = \begin{pmatrix} 0 & \psi_1 & \cdots & \psi_N \\ -\psi_1^* & 0 & 0 \\ \vdots \\ -\psi_N^* & 0 & \cdots & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \frac{i}{2} & 0 & \cdots & 0 \\ 0 & -\frac{i}{2} & 0 \\ \vdots \\ 0 & \cdots & 0 & -\frac{i}{2} \end{pmatrix},$$

and the CHONSE becomes the higher-order vector nonlinear Schrödinger equation,

$$\bar{\partial}\psi_k = i\left[\partial^2\psi_k + 2\left(\sum_{j=1}^{N}|\psi_j|^2\right)\psi_k\right] - \alpha \left[\beta_1\left(\sum_{j=1}^{N}|\psi_j|^2\right)\partial_\psi_k + \beta_2\left(\sum_{j=1}^{N}\psi_j^*\partial_\psi_k\right)\psi_k - \partial_3\psi_k\right] \quad k = 1, 2, \ldots, N. \quad (7)$$

This equation is an obvious generalization of Eq. (1) to the multi-component case. It is easy to see that Eq. (7) with $N = 1$ and $\beta_1 = \beta_2 = -3$ is precisely the Hirota equation. As verified in [2], Eq. (7) is integrable if $\beta_1 = \beta_2 = -3$ because in such a case the CHONSE admits a Lax pair. That is, Eq. (7) with $\beta_1 = \beta_2 = -3$ arises from the compatibility condition ($[L_z, \bar{L}_\bar{z}] = 0$) of the associated linear equations,

$$L_z \Psi \equiv [\partial + E + \lambda T]\Psi = 0,$$
$$L_{\bar{z}} \Psi \equiv [\bar{\partial} + U_K^0 + U_M^0 + \lambda(U_K^1 + U_M^1) + \lambda^2(U_M^2 + T) - \alpha\lambda^2 T] \Psi = 0, \quad (8)$$

which holds for all values of the spectral parameter $\lambda$. The entities $U_K^1$ and $U_M^1$ in $L_z$ are given by

$$U_K^0 = -E\tilde{E} - \alpha[E, \partial E], \quad U_M^0 = \partial\tilde{E} + \alpha(\partial^2 E - 2E^3),$$
$$U_K^1 = \alpha E\tilde{E}, \quad U_M^1 = E - \alpha\partial\tilde{E}, \quad U_M^2 = -\alpha E. \quad (9)$$

Here, the subscripts $K$ and $M$ signify that they belong to the subalgebra $k$ and the remaining complement $m$, respectively. The algebraic decomposition can be also extended to a more general case including the matrix solution $\Psi = \Psi_K + \Psi_M$ with the properties that $[T, \Psi_K] = 0, \quad [T, \Psi_M] \in m$, and the following multiplication properties;

$$[T, \Psi_K^1 \Psi_K^2] = [T, \Psi_M^1 \Psi_M^2] = 0, \quad [T, \Psi_K^1 \Psi_M^2] \in m \quad (10)$$

The adjoint action of the element $T$ in the Cartan subalgebra together with the complex structure condition, if applied to the decomposition, lead to a couple of general identities for any $M_1, M_2 \in m$;

$$[T, M_1 M_2] = \tilde{M}_1 M_2 + M_1 \tilde{M}_2 = 0, \quad \tilde{M}_1 \tilde{M}_2 = M_1 M_2 \quad (11)$$
These identities are useful for many calculations, for example, in deriving conserved currents or in verifying that the CHONSE in Eq. (3) is equivalent to the compatibility condition of the Lax pair in Eq. (8).

Having presented necessary ingredients, we are now ready to derive infinitely many conserved currents and charges of the integrable CHONSE by solving the associated linear equations in Eq. (8). In order to make use of the algebraic properties of Hermitian symmetric spaces, we make a change of the variable \( \Phi \) in Eq. (8) by

\[
\Phi = \Psi \exp[(\lambda z - \alpha \lambda^3)\bar{z}T],
\]

which results in the change of the multiplicative term \( T\Psi \) to the commutative term \([T, \Phi]\) in the linear equations. The adjoint action, \([T, \Phi]\), allows the splitting of the linear equations for \( \Phi \) into the \( K \)- and the \( M \)-components as explained below. Let us first assume that the linear equations can be solved iteratively in terms of

\[
\Phi(z, \bar{z}, \lambda) \equiv \sum_{n=0}^{\infty} \frac{1}{\lambda^n} \left( \Phi^n_K(z, \bar{z}) + \Phi^n_M(z, \bar{z}) \right),
\]

where \( \Phi^n_K \) and \( \Phi^n_M \) denote the decomposition of a coefficient \( \Phi^n \) satisfying the properties in Eq. (10). Then, the \( n \)-th order equation \( (n \geq 0) \) separates into the \( K \)- and the \( M \)-components such as

\[
\partial \Phi^n_K + E \Phi^n_M = 0,
\]

\[
\partial \Phi^n_M + E \Phi^n_K + [T, \Phi^n_{M+1}] = 0,
\]

while the \( \bar{\partial} \)-part of the linear equation becomes

\[
\bar{\partial} \Phi^n_K + U^0_K \Phi^n_K + U^0_M \Phi^n_M + U^1_K \Phi^{n+1}_K + U^1_M \Phi^{n+1}_M + U^2_M \Phi^{n+2}_M = 0,
\]

\[
\bar{\partial} \Phi^n_M + U^0_K \Phi^n_M + U^0_M \Phi^n_M + U^1_K \Phi^{n+1}_K + U^1_M \Phi^{n+1}_K + U^2_K \Phi^{n+2}_K + [T, \Phi^n_{M+2}] - \alpha [T, \Phi^n_{M+3}] = 0.
\]

In addition, there are equations arising from the positive powers of \( \lambda \), which can be given by Eqs. (14)-(17) provided that \( n = -1, -2, -3 \) and \( \Phi^{n<0}_K = \Phi^{n<0}_M = 0 \) are defined. These equations can be solved recursively for \( \Phi^n_K, \Phi^n_M \ (n \geq 0) \) starting from a consistent set of initial conditions;

\[
\Phi^0_M = 0, \quad \Phi^0_K = -iI, \quad \Phi^1_M = -i\bar{E}.
\]
Note that Eq. (13) can be solved for $\Phi_{M+1}^n$ by using the complex structure condition. That is, $\Phi_{M+1}^n = -[T, [T, \Phi_{M+1}^n]] = [T, \partial \Phi_M^n] + \bar{E} \Phi_K^n$. Thus, $\Phi_{M+1}^n$ is obtained directly provided that $\Phi_K^n$ and $\Phi_M^n$ are determined. Contrary to $\Phi_{M+1}^n$ which is obtained algebraically, $\Phi_{K+1}^n$ can be obtained by a direct integration of Eq. (14). In fact, $\Phi_{K+1}^n$ is over-determined due to Eq. (10) as well. Thus, in order for $\Phi_{K+1}^n$ to be integrable, the compatibility condition that $[\partial, \bar{\partial}] \Phi_K^n = 0$ should be required. The condition is satisfied provided the integrable CHONSE holds. In this case the compatibility condition gives rise to infinitely many conserved currents labeled by integer $n$ such that $\partial J_K^n + \bar{\partial} \bar{J}_K^n = 0$;

$$J_K^n = -\partial \Phi_K^n = E \Phi_M^n$$

$$\bar{J}_K^n = \bar{\partial} \Phi_K^n = \bar{\partial} (E \Phi_M^n)$$

In order to derive the local currents explicitly, we solve the recurrence relations in Eqs. (14)-(17) with the initial conditions as in Eq. (18). The first few conserved currents are listed below;

$$J_1^1 = -iE \tilde{E},$$

$$\bar{J}_1^1 = -i[E, \partial E] + i\alpha([\partial^2 E, \tilde{E}] + \partial \tilde{E} \partial E - 3E^3 \tilde{E}),$$

for $n = 1$, and

$$J_2^1 = -i \partial \Phi_{K+1}^1 \Phi_{K}^1 + iE \partial E,$$

$$\bar{J}_2^1 = +i \bar{\partial} \Phi_{K+1}^1 \Phi_{K}^1 - i(E \partial^2 \tilde{E} + \bar{\partial} \tilde{E} \partial E - E^3 \tilde{E})$$

$$-i\alpha(E \partial^3 E + [\partial^2 E, \partial E] + \partial E E^3 - 2E \partial E E^2 - E^2 \partial E E - 4E^3 \partial E),$$

$$J_3^2 = -i(\partial \Phi_{K+1}^2 \Phi_{K}^2 + \partial \Phi_{K+1}^2 \Phi_{K}^2 - i \partial \Phi_{K+1}^1 \Phi_{K}^1 \Phi_{K}^1) + i(E \partial^2 \tilde{E} - E^3 \tilde{E}),$$

$$\bar{J}_3^2 = i(\bar{\partial} \Phi_{K+1}^2 \Phi_{K}^2 + \bar{\partial} \Phi_{K+1}^2 \Phi_{K}^2 - i \bar{\partial} \Phi_{K+1}^1 \Phi_{K}^1 \Phi_{K}^1)$$

$$+ i(E \partial^3 E - \partial E \partial^2 E + \partial E E^3 - 2E \partial E E^2 - E^2 \partial E E - 2E^3 \partial E)$$

$$- i\alpha(E \partial^4 \tilde{E} + \partial E \partial^2 \tilde{E} + \partial E \tilde{E} \partial^3 E - 5E^3 \partial^2 \tilde{E} - E^2 \partial^2 E \tilde{E} - 3E \partial^2 \tilde{E} E^2$$

$$+ \partial^2 \tilde{E} E^3 - 2E \partial \tilde{E} \partial E E^2 - \partial \tilde{E} E \partial E E - 2E \tilde{E} E^2 \partial E - 3E \partial E \partial \tilde{E} \partial E$$

$$- 5E \partial E \partial \tilde{E} E - 3E^2 \partial E \partial \tilde{E} + 4E^5 \tilde{E}),$$

for $n = 2$ and $n = 3$, respectively. Note that currents $J_K^n$ and $\bar{J}_K^n$ for $n \geq 2$ contain non-local terms $\Phi_K^m$ with $m < n$. Fortunately, these non-local terms can be separated
from the conservation law if we consider a scalar expression of the conserved current by taking an appropriate trace as follows;

\[ S^K_n = \text{Tr}(P J^K_n), \quad \bar{S}^n_K = \text{Tr}(P \bar{J}^n_K) \]  

(24)

The parameter \( P \) is any matrix entity which commutes with matrices \( \Phi^K_m \), or we may choose \( P = c_1 I + c_2 T \) for arbitrary constants \( c_1 \) and \( c_2 \). For instance, we have for \( n = 2, 3 \)

\[ S^K_2 = -\partial(\text{Tr} \{i \frac{1}{2} (\Phi^K_1)^2\}) + \text{Tr} P(i E \partial E), \]

\[ \bar{S}^2_K = \bar{\partial}(\text{Tr} P \{i \frac{1}{2} (\Phi^K_1)^2\}) + \text{Tr} P\{-i(E \partial^2 \bar{E} + \partial \bar{E} \partial E - E^3 \bar{E}) - i\alpha(E \partial^3 E + [\partial^2 E, \partial E] - 6 E^3 \partial E)\}, \]

(25)

\[ S^K_3 = -\partial(\text{Tr} P \{i(\Phi^K_1 \Phi^K_2 - \frac{1}{3} (\Phi^K_1)^3)\}) + \text{Tr} P\{i(E \partial^2 \bar{E} - E^3 \bar{E})\}, \]

\[ \bar{S}^3_K = \bar{\partial}(\text{Tr} P \{i(\Phi^K_1 \Phi^K_2 - \frac{1}{3} (\Phi^K_1)^3)\}) \]

\[ + \text{Tr} P\{i(E \partial^3 E - \partial E \partial^2 E - 4 E^3 \partial E) - i\alpha(E \partial^4 \bar{E} + \partial^2 E \partial^2 \bar{E} + \partial \bar{E} \partial^3 E - 8 E^3 \partial^2 \bar{E}

+ 2 \partial^2 \bar{E} E^3 + E^2 \partial \bar{E} \partial E - \partial \bar{E} E \partial EE + \partial \bar{E} E \partial E - 5 E \partial \bar{E} E \partial E + 4 E^5 \bar{E})\}. \]

(26)

The derivations show that the non-local terms appear as total derivative terms thus they are conserved separately. Dropping the non-local terms and integrating over the time coordinate, we obtain infinite number of global charges which are conserved in space, i.e.

\[ \bar{\partial} Q^K_n = 0 \]

where

\[ Q^K_n \equiv \int_{-\infty}^{+\infty} dt S^K_n. \]  

(27)

For the case of \( G/K = SU(N+1)/U(N) \) as mentioned in Eq. (3), we work out explicitly and obtain the conserved charges

\[ Q^K_1 = \int_{-\infty}^{+\infty} dt \sum_{k=1}^{N} \psi_k^* \psi_k, \]  

(28)

for \( n = 1 \) and

\[ Q^K_2 = i \int_{-\infty}^{+\infty} dt \sum_{k=1}^{N} (\psi_k^* \partial \psi_k - \partial \psi_k^* \psi_k), \]

(29)

\[ Q^K_3 = \int_{-\infty}^{+\infty} dt \left[ \sum_{k=1}^{N} \partial \psi_k^* \partial \psi_k - (\sum_{k=1}^{N} \psi_k^* \psi_k)^2 \right], \]  

(30)

for \( n = 2 \) and \( n = 3 \), respectively. Conserved charges for other cases of integrable CHONSE can be similarly obtained from the specification of \( E \) and \( T \) as classified in [4].
As noted in Eq. (3), the types of charges $Q_2$ and $Q_3$ are not conserved in the Sasa-Satsuma case where $\gamma_4 = -2\gamma_5 = 6$. Nevertheless, the Sasa-Satsuma equation equivalently possesses infinitely many conserved charges of different types as well. This seemingly contradicting characteristics can be explained by the fact that the Sasa-Satsuma equation arises from the discrete $Z_2$-reduction of the $SU(3)/U(2)$ CHONSE combined with a point transformation [4]. In this case, matrices $E$ and $T$ can be denoted as

$$E = \begin{pmatrix} 0 & \psi & \psi^* \\ -\psi^* & 0 & 0 \\ -\psi & 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} \frac{i}{2} & 0 & 0 \\ 0 & -\frac{i}{2} & 0 \\ 0 & 0 & -\frac{i}{2} \end{pmatrix}. \quad (31)$$

Since the charge $Q_k^a$ in Eq. (27) is invariant under the point transformation, we can also calculate the first few conserved charges of the Sasa-Satsuma equation using the expressions $E$ and $\tilde{E} \equiv [T, E]$ given in Eq. (31). The resulting charges of the Sasa-Satsuma equation are

$$Q_1^k = \int_{-\infty}^{+\infty} dt \psi^* \psi,$$
$$Q_2^k = 0,$$
$$Q_3^k = \int_{-\infty}^{+\infty} dt [3 \partial \psi^* \partial \psi - 6(\psi^* \psi)^2 - i(\psi^* \partial \psi - \partial \psi^* \psi)]. \quad (32)$$

If the charges in Eq. (32) are compared with those of the Hirota type in Eq. (2) (or equivalently Eq. (28)-(30) for $N = 1$), we note that the current for $n = 1$, which corresponds to energy, is the same but other currents are of different types. Remarkably, in Eq. (32) the current for $n = 2$ turns out to be trivial while the current for $n = 3$ is a new type that is seemingly combination of currents for $n = 2, 3$ in Eq. (2). From Eq. (1) with normalized coefficients $\gamma_1 = \gamma_2/2 = \gamma_3 = 1$, one can readily confirm that the current $S_3^k = 3 \partial \psi^* \partial \psi - 6(\psi^* \psi)^2 - i(\psi^* \partial \psi - \partial \psi^* \psi)$ is conserved only if $\gamma_4 + \gamma_5 = 3$ and $3\gamma_4 + 2\gamma_5 = 12$. Solving the equations results in $\gamma_4 = -2\gamma_5 = 6$ that definitely leads to the Sasa-Satsuma case, to be compared with Eq. (3) for the Hirota case.

To summarize, using the properties of Hermitian symmetric space we have constructed the Lax pair formalism of the coupled higher-order nonlinear Schrödinger equation and derived a general expression of infinite number of conservation laws. Remarkably, the conserved currents and charges for both the Hirota and the Sasa-Satsuma equations are calculated from the general expressions, accompanying the reduction procedure. We have shown that, except for the Hirota case, the current conservations of the nonlinear
Schrödinger equation are in general broken by the higher-order effects. The types of conserved currents and charges for the Sasa-Satsuma case are different from the types for the Hirota case except for the energy conserved irrespective of all the higher-order effects. These differences may leave scope for more physical explanations and applications in the further study of higher-order effects including numerical analysis.

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