A P-LOCAL DELOOPING MACHINE

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ABSTRACT. We show that for spaces $A$ that satisfy a certain smallness condition, there is a Lawvere theory $T_A$ so that a space $X$ has the structure of a $T_A$-algebra if and only if $X$ is weakly equivalent to a mapping space out of $A$. In particular, spheres localized at a set of primes satisfy this condition.

1. INTRODUCTION

It is a classical result that the existence of certain algebraic structures on a space can determine whether or not it is an $n$-fold loop space (where here $n$ is possibly $\infty$). For instance, Beck [5] showed that a space is an $n$-fold loop space if and only if it is an algebra over the monad $\Omega^n\Sigma^n$. In [19] May found a much simpler description using monads which come from operads. Such monads have the useful property that they are finitary: completely determined by their restriction to finite sets. Finitary monads and their algebras correspond to (Lawvere) theories and their algebras [7, II.4].

Definition 1.1. A theory $T$ is a based simplicial category with objects $t_0, t_1, t_2, ...$ such that $t_i$ is the $i$-fold product of $t_1$. In particular, $t_0$ is the basepoint. An algebra over $T$ is a based, product preserving functor $X$ from $T$ into based simplicial sets. The underlying space of $X$ is $X(t_1)$.

So an algebra over a theory consists of a space $X(t_1)$ along with operations $X(t_1)^n \to X(t_1)$ that satisfy certain relations which are parameterized by the theory $T$. The above discussion is saying that loop spaces are detected by theories, in the following sense.

Definition 1.2. A space $A$ is detectable if there is a theory $T$ with the property that a space $X$ is weakly equivalent to the underlying space of a $T$-algebra if and only if $X$ is weakly equivalent to $\text{Map}_*(A, Y)$ for some $Y$.

One of the upshots of a space being detectable is that algebras over a theory are closed under various operations. For example, let $F$ be a functor from spaces to spaces which preserves weak equivalences and preserves products up to weak equivalences. Then [2, Cor 1.4] says that if $X$ is weakly equivalent to an algebra over a theory $T$, so is $F(X)$. So if $A$ is detectable, applying such a functor $F$ to a mapping space of the form $\text{Map}_*(A, Y)$ gives a space which is weakly equivalent to $\text{Map}_*(A, Z)$ for some space $Z$.

In particular, if $A$ is detectable, then the localization $L_f \text{Map}_*(A, Y)$ with respect to any map $f$ is weakly equivalent to $\text{Map}_*(A, Z)$ for some space $Z$. In [4], it is shown that if $A$ is a finite, pointed CW-complex with the property that its mapping spaces are closed under localization in this way, then $A$ has the rational homotopy type of a wedge of spheres which are all the same dimension.
Based on this result, it is not clear if any spaces are detectable other than wedges of $n$-spheres. On the contrary, we show that any space which satisfies a certain smallness condition is detectable. In particular, this includes spheres localized at a set of primes. Our main result is the following:

**Theorem 1.3.** Let $S^n_P$ be the $P$-local sphere for $P$ a set of primes, and $n \geq 2$. There is a theory $T_{S^n_P}$ with the property that a space $X$ is weakly equivalent to $\text{Map}_*(S^n_P, Y)$ for some $Y$ if and only if $X$ is weakly equivalent to the underlying space of an $T_{S^n_P}$-algebra.

As in [3], we will actually prove the stronger statement that there is a Quillen equivalence between the category of algebras over the theory $T_{S^n_P}$ and the right Bousfield localization of spaces with respect to $S^n_P$.

**Notation**

- We work in the category of pointed simplicial sets, which we denote by $\text{sSet}_*$. 
- For a set of primes $P$ and $n \geq 2$, the $P$-local sphere $S^n_P$ is the singularization of the mapping telescope:

$$\text{Tel}(S^n \overset{l_1}{\rightarrow} S^n \overset{l_2}{\rightarrow} S^n \rightarrow ...)$$

where $\{l_i\}$ are the maps whose degree are the positive integers relatively prime to the primes in $P$.
- We freely use the language of model categories. For an introduction see [11].

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2. **The Canonical Theory of a Space and the Homotopy Theory of Algebras**

The category of pointed simplicial sets is equipped with a pointed mapping space given by:

$$\text{Map}_*(A, X) = \text{Hom}_{\text{sSet}_*}(\Delta^*_n \wedge A, X)$$

where the simplicial structure is encoded by the cosimplicial structure of $\Delta^*_n$, which is the pointed cosimplicial space sending each $[n] \in \Delta$ to the standard $n$-simplex with a disjoint basepoint. For any pointed simplicial set $A$, we define a theory $T_A$ by:

$$\text{Hom}_{T_A}(t_n, t_m) = \text{Map}_*(\bigvee_m A, \text{Sing}\big|\bigvee_n A|)$$

That this actually is a theory is probably easiest seen by the fact that the adjunction between pointed simplicial sets and topological spaces induces isomorphisms of simplicial mapping spaces:

$$\text{Map}_*(\bigvee_m A, \text{Sing}\big|\bigvee_n A|) \cong \text{Map}_*(\bigvee_m |A|, \bigvee_n |A|)$$
Thus we may view our theory as the opposite of the full subcategory of the (pointed simplicial) category of pointed topological spaces consisting of wedges of copies of $|A|$. Since $\bigvee^m |A|$ is the $n$-fold coproduct of $|A|$, it becomes the $n$-fold product in the opposite category.

**Definition 2.1.** Let $A$ be a space. The theory $T_A$ just described is called the **Canonical Theory Associated to $A$**.

In [3], it was shown that for $A = S^n$, the canonical theory detects spheres in the sense of definition 1.1. Indeed, for any space $A$, the mapping space $\text{Map}_*(A, Y)$ is the underlying space of the $T_A$ algebra:

$$\Omega^A(Y) : T_A \to \text{sSet}_* \quad t_i \mapsto \text{Map}_*(\bigvee_i A, Y)$$

The content of [3] was that, in the case of a sphere, every algebra over the canonical theory $T_{S^n}$ has underlying space weakly equivalent to an $n$-fold loop space.

We recall the homotopy theory of algebras over a theory $T$. Let $\text{Alg}^T$ denote the category of algebras over a theory $T$, where the morphisms are natural transformations. Then taking the underlying space gives a functor $U : \text{Alg}^T \to \text{sSet}_*$ from the category of algebras over $T$ to pointed simplicial sets. It has a left adjoint $F$, the free algebra functor, and this adjunction is used to lift the Quillen model category of pointed simplicial sets to the category of algebras.

**Theorem 2.2.** [21] There is a model category structure on $\text{Alg}^T_A$, where a map $\phi : X \to Y$ between algebras is a:

1. Weak equivalence if $U(\phi)$ is a weak equivalence in $\text{sSet}_*$.
2. Fibration if $U(\phi)$ is a fibration in $\text{sSet}_*$.
3. Cofibration if $\phi$ has the left lifting property with respect to the acyclic fibrations.

Moreover, with this model structure, the free/forgetful adjunction $F \dashv U$ is a Quillen adjunction.

The fibrant algebras are the algebras whose underlying spaces are fibrant simplicial sets. In the remainder of this section, we define a useful cofibrant replacement of an algebra.

Let $X$ be an algebra over a theory $T$. Define the simplicial $T$-algebra $FU_* X$ to be $(FU)^{n+1}(X)$ in simplicial degree $n$. The face and degeneracy maps are defined using the unit $\eta : 1 \to UF$ and counit $\epsilon : FU \to 1$ of the adjunction. Specifically, we have

$$d_i := (FU)^k X \xrightarrow{(FU)^i \epsilon (FU)^{k-i}} (FU)^{k-1} X$$
$$s_i := (FU)^k X \xrightarrow{(FU)^i \eta U(FU)^{k-i}} (FU)^{k+1} X$$

The counit $\epsilon : FU X \to X$ induces a map $\epsilon_* : |FU_* X| \to X$ of $T$ algebras.

**Definition 2.3.** The map $\epsilon_* : |FU_* X| \to X$ is called **Bar Resolution of $X$**.

**Theorem 2.4.** The bar resolution $\epsilon_* : |FU_* X| \to X$ is a cofibrant replacement in the category of algebras.

**Proof.** The fact that $\epsilon_*$ is a weak equivalence follows from a standard extra degeneracy argument, as in [19, Proposition 9.8]. To complete the proof, we need to show that $FU_* X$ is cofibrant. We will do this by showing that $FU_* X$ is a Reedy cofibrant algebra in the category of simplicial algebras. Then geometric realization,
being a left Quillen adjoint, sends Reedy cofibrant simplicial algebras to cofibrant algebras.

Let $\Delta^\text{op}_0$ be the full subcategory of $\Delta^\text{op}$ consisting of the degeneracy maps and only positive face maps. Like $\Delta^\text{op}$, this is a Reedy category. Consider the functor $UF_*X$ from $\Delta^\text{op}_0$ to $s\text{Set}$, which is $(UF)^nUX$ in simplicial degree $n$ and whose face maps and degeneracy maps are similar to $FU_*X$. All of the latching maps of $UF_*X$ are cofibrations, and applying $F$ we see that the restriction of $FU_*X$ to $\Delta^\text{op}$ is Reedy cofibrant. By [16, Proposition 3.17], the simplicial algebra $FU_*X$ is Reedy cofibrant as well. □

3. CELLULAR SPACES AND HOMOTOPY PROJECTIVITY

The mapping space algebra $\Omega^A$ in the previous section defines a functor from $s\text{Set}_*$ to the category of algebras $\text{Alg}^{T_A}$ whose morphisms are natural transformations. This functor has a left adjoint, which we call $B^A$, and this adjunction passes to the level of homotopy categories.

**Theorem 3.1.** Let $A$ be a space, and let $T_A$ be the canonical theory from definition 2.1. Then the adjunction

$$s\text{Set} : \xrightarrow{B^A} \text{Alg}^{T_A}$$

is Quillen.

**Proof.** It suffices to show that the right adjoint preserves fibrations between fibrant objects and acyclic fibrations. If $f : X \to Y$ is a fibration between fibrant objects, or an acyclic fibration, then so is the induced map

$$\text{Map}_*(A, X) \to \text{Map}_*(A, Y)$$

In other words $U(f)$ is a fibration or acyclic fibration, and by definition of the model structure on algebras, so is $\Omega(f)$. □

We will see that for spaces $A$ which satisfy a smallness condition (see definition 3.5) we can define a model category structure on $s\text{Set}_*$, where a map $f : X \to Y$ is a:

1. Weak equivalence if the induced map $\text{Map}_*(A, |\text{Sing}(X)|) \to \text{Map}_*(A, |\text{Sing}(Y)|)$ is a weak equivalence.
2. Fibration if it is a Kan fibration.
3. Cofibration if it has the left lifting property with respect to all acyclic fibrations.

**Theorem 3.2.** [14, Theorem 5.1.1] Let $A$ be a pointed simplicial set. There is a model category structure on $s\text{Set}_*$, where a map $f : X \to Y$ is a:

Proof. This follows from a general theorem on existence of right Bousfield localizations [14, Theorem 5.1.1], since the model category of pointed simplicial sets is proper and cellular [14, Proposition 5.1.8] □
By [14, 5.1.5], the cofibrant spaces in $R^A sSet_*$ are the $A$-cellular spaces, which we define now.

**Definition 3.3.** A nonempty class of simplicial sets is said to be **closed** if it is closed under homotopy colimits and weak equivalences. The smallest closed class which contains a simplicial set $A$ is called the class of $A$-**cellular spaces**, denoted $\text{Cell } A$.

The $A$-cellular spaces are roughly the spaces built from $A$. For example, the $S^n$-cellular spaces are the $(n-1)$-connected simplicial sets. For us, the most important example is the following.

**Lemma 3.4.** Let $n \geq 2$, and $P$ be a set of primes. Then $\text{Cell}(S^n_P)$ is the class of $(n-1)$-connected $P$-local cell-complexes.

**Proof.** By definition, $\text{Cell}(S^n_P)$ is the smallest class of simplicial sets containing $S^n_P$ which is closed under weak equivalences and homotopy colimits. The constructions in [24] show that it is possible to build any $P$-local space by homotopy colimits. It is shown in [14, 5.3.7] that it is enough to consider filtered colimits and homotopy pushouts. Since the property of being $(n-1)$-connected and $P$-local is determined by reduced homology, any homotopy pushout or filtered colimit of $(n-1)$-connected $P$-local spaces is still $(n-1)$-connected and $P$-local. □

The final ingredient in our recognition principle is that $A$ needs to satisfy a smallness condition.

**Definition 3.5.** Let $C$ be a closed class. A pointed simplicial set $A$ is called **homotopy projective relative to** $C$ if $\text{Map}_*(A, -)$ commutes with homotopy colimits of filtered or simplicial diagrams which take their value in $C$. A space $A$ is called **homotopy self-projective** if it is homotopy projective relative to $\text{Cell } A$.

**Proposition 3.6.** The spheres $S^n$ are homotopy self-projective.

**Proof.** $\text{Map}_*(S^n, -)$ commutes with homotopy filtered colimits because $S^n$ is a finite simplicial set. The class $\text{Cell } S^n$ is exactly the class of $(n-1)$-connected simplicial spaces, so it follows from the Bousfield-Friedlander theorem [8, Theorem B.4] that $\text{Map}_*(S^n, -)$ commutes with homotopy colimits of $(n-1)$-connected simplicial spaces. □

**Proposition 3.7.** For $n \geq 2$ and $P$ a set of primes, the $P$-local sphere $S^n_P$ is homotopy self-projective.

**Proof.** The map:

$$\text{Map}_*(S^n_P, K) \to \text{Map}_*(S^n, K)$$

induced from the localization map $L_P : S^n \to S^n_P$ is a weak equivalence for $P$-local spaces $K$. Let $\bar{K} : D \to sSet_*$ be either a filtered diagram or else a simplicial diagram which takes its values in $\text{Cell } S^n_P$. We have the following commutative diagram:

$$\begin{array}{ccc}
\text{hocolim}_D \text{Map}_*(S^n_P, \bar{K}) & \xrightarrow{\nu P} & \text{Map}_*(S^n_P, \text{hocolim } \bar{K}) \\
L_P \downarrow & & \downarrow L_P \\
\text{hocolim}_D \text{Map}_*(S^n, \bar{K}) & \xrightarrow{\nu} & \text{Map}_*(S^n, \text{hocolim } \bar{K})
\end{array}$$
The left and right maps are weak equivalences because \( \tilde{K} \) takes its value in \( P \)-local spaces. The bottom map is a weak equivalence because \( S^n \) is homotopy self-projective and \( \tilde{K} \) takes its values in \((n - 1)\)-connected spaces. Hence the top map is also a weak equivalence.

These are actually the only homotopy self-projective spaces we know of, and we think it is likely that wedges of \( n \)-spheres are the only finite spaces which are homotopy self-projective. On the other hand, we can say that the property of being homotopy self-projective is stable.

**Proposition 3.8.** If \( A \) is connected and homotopy self-projective, so is \( \Sigma A \).

**Proof.** A space \( X \) is in \( \text{Cell}(\Sigma A) \) if and only if \( \Omega X \) is in \( \text{Cell}(A) \) [12]. So if \( \tilde{K} : D \to S_\ast \) is some filtered or simplicial diagram which takes its values in \( \text{Cell}(\Sigma A) \), then the diagram \( \Omega \tilde{K} \) obtained by postcomposition takes its values in \( \text{Cell}(A) \).

Since \( A \) is homotopy self-projective

\[
\text{hocolim}_D \text{Map}_\ast(A, \tilde{K}) \cong \text{hocolim}_D \text{Map}_\ast(A, \Omega \tilde{K})
\]

is a weak equivalence. Since \( \tilde{K} \) takes its values in connected spaces, and since \( S^1 \) is homotopy self-projective there is a weak equivalence

\[
\text{Map}_\ast(A, \text{hocolim}_D \Omega \tilde{K}) \to \text{Map}_\ast(A, \Omega(\text{hocolim}_D \tilde{K})) \cong \text{Map}_\ast(\Sigma A, \text{hocolim}_D \tilde{K})
\]

\( \square \)

4. Proof of the Main Theorem

In this section we prove theorem 1.3. It will follow easily from the following

**Theorem 4.1.** Let \( A \) be a homotopy self-projective space. Then the Quillen adjunction

\[
R^A s\text{Set}_\ast \xRightarrow{\eta^A} \text{Alg}^T_A
\]

is a Quillen equivalence.

**Proof.** We first need to show that this is still a Quillen adjunction after passing from \( s\text{Set}_\ast \) to \( R^A s\text{Set}_\ast \). Note that the identity functor:

\[
\text{Id} : s\text{Set}_\ast \to R^A s\text{Set}_\ast
\]

is a right Quillen adjoint, so that this is not automatic. However \( R^A s\text{Set}_\ast \) is a simplicial model category with the same simplicial structure as \( s\text{Set}_\ast \) [14, Theorem 5.1.2]. So \( \Omega^A \) preserves fibrations between fibrant objects and acyclic fibrations.

Next, since \( \Omega^A \) reflects weak equivalences between fibrant objects, it is enough to show that the unit map \( \eta : X \to \Omega^A(\text{Sing}[B^A X]) \) is a weak equivalence for all cofibrant \( X \). We split the proof into cases:

1. Consider first the case when \( X = F(n_+) \), where \( n = \{0, 1, \ldots, n\} \) is the set based at 0. For any space \( K \), the composition \( U^A \Omega^A(K) \) is the mapping space \( \text{Map}_\ast(A, X) \), so that in the other direction \( B^A F(K) \) is \( A \land K \). Thus, applying the forgetful functor to the map \( X \to \Omega^A(\text{Sing}[B^A X]) \) is isomorphism:

\[
F(n_+) \to \text{Map}_\ast(A, \text{Sing}[B^A F(n_+)] \to \text{Map}_\ast(A, \text{Sing}[V_n A])
\]
2. Next, let $S$ be a set, and let $X = F(S)$. Then we can write $X$ as the filtered homotopy colimit of over the finite subsets of $S$:

$$X = \text{hocolim}_{L \subseteq S} F(L)$$

We will write the unit as a composition. First there is the map

$$\text{hocolim}_{L \subseteq S} F(L) \xrightarrow{\eta} \text{hocolim}_{L \subseteq S} \Omega^A(\text{Sing}|B^A F(L)|)$$

obtained by applying the unit degreewise. This map is a weak equivalence since by part 1 it is an objectwise weak equivalence. We compose this with the map

$$\text{hocolim}_{L \subseteq S} \Omega^A(\text{Sing}|B^A F(L)|) \xrightarrow{\nu} \Omega^A(\text{hocolim}_{L \subseteq S} \text{Sing}|B^A F(L)|)$$

which is a weak equivalence because $A$ is homotopy projective. The final map in the composition is

$$\Omega^A(\text{hocolim}_{L \subseteq S} \text{Sing}|B^A F(L)|) \xrightarrow{\eta} \Omega^A(\text{hocolim}_{L \subseteq S} \text{Sing}|B^A F(L)|)$$

This is a weak equivalence because $\text{hocolim}_{L \subseteq S} \text{Sing}|B^A F(L)| \rightarrow \text{Sing}|B^A \text{hocolim}_{L \subseteq S} F(L)|$ is a weak equivalence, by [14, Proposition 18.9.12].

3. Next suppose that $X = \text{hocolim}_{\Delta^{op}} K_n$, where each $K_n$ is a free algebra $F(S)$.

Then a similar argument as in step 2 shows that the unit is a weak equivalence in this case.

4. Finally, let $X$ be a cofibrant algebra. Then the bar resolution $\epsilon_* : \text{hocolim}_{\Delta^{op}} F(U_*(X)) \rightarrow X$ fits into the commutative diagram:

$$\begin{array}{ccc}
\text{hocolim}_{\Delta^{op}} F(U_*(X)) & \xrightarrow{\eta_*} & \Omega^A \text{Sing}|B^A \text{hocolim}_{\Delta^{op}} F(U_*(X))| \\
\epsilon_* & & \Omega^A \text{Sing}|B^A \epsilon_*| \\
X & \xrightarrow{\eta} & \Omega^A \text{Sing}|B^A X|
\end{array}$$

The bar resolution has the form from step 3, so that $\eta_*$ is a weak equivalence. By theorem, the map $\epsilon_*$ is a weak equivalence. Finally, $\Omega^A$ preserves weak equivalences between fibrant objects, so that $\eta$ is a weak equivalence as well.

**Proof of Theorem 1.3.** We showed in section 2 that if a space $X$ is weakly equivalent to $\text{Map}_*(A,Y)$ for some $Y$, then $X$ is weakly equivalent to the underlying space of the algebra $\Omega^A(Y)$. Conversely, suppose a space $X$ is weakly equivalent to the underlying space of an algebra $M$, then $X$ is also weakly equivalent to the underlying space of a cofibrant replacement $M^c$ of $M$. By 4.1, there is a weak equivalence of algebras $M^c \rightarrow \Omega^A(\text{Sing}|B^A M^c|)$. By the definition of weak equivalences between algebras, it follows that $X$ is weakly equivalent to $\text{Map}_*(A, \text{Sing}|B^A M^c|)$. □

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