Off-perturbative states in disordered systems

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Abstract
The systematic approach for the off-perturbative calculations in disordered systems is developed. The proposed scheme is applied for the random temperature and the random field ferromagnetic Ising models. It is shown that away from the critical point, in the paramagnetic phase of the random temperature model, and in the ferromagnetic phase of the random field one, the free energy contains non-analytic contributions which have the form of essential singularities. It is demonstrated that these contributions appear due to localized in space instanton-like excitations.

Key words: Quenched disorder, instantons, replicas, mean-field, non-analytic functions.

1 Introduction

In very simplified terms, studies of classical statistical systems involves two main domains: (1) investigation of the ground state, and (2) summation over fluctuations around this ground state. Although formally according to the definition of the partition function, one has to perform summation over the whole configurational space of a system, in reality it is never done. And it is not that we are doing something wrong. The point is that in most of the cases only very limited part of the configurational space which is relevant for observable thermodynamics. Very often, the question, what this "relevant part" is (which involves the choice of the so called "relevant variables"), which is the most difficult. Studies of the systems containing quenched disorder, in addition to the two items mentioned above, involves the third one (although, technically, very often it turns into the starting one), which is the averaging of self-averaging quantities over random parameters. Nevertheless, at a qualitative level, the situation here remains the same: only very limited part of the configurational space is relevant for observable thermodynamics.

However, in some statistical systems, besides the ground state another local minimum (or minima) could exists. Let us consider extremely simplified situation, schematically shown in Fig.1, when in addition to the ground state the system has another local minimum located in the configurational space "far away" from the ground states, and separated from it by a big (compared to the temperature) energy barrier, which, however, remains finite in the thermodynamic limit. If we are dealing with the system which contains no quenched disorder, then the thermodynamic contribution due to this another state with an exponential accuracy will be simply of order of exp(−βΔE) (provided ΔE ∝ T), where ΔE = E₁ − E₀.

The crucial point, however, is that to get the above exponential contribution, one has to know about existence of the other local minimum, otherwise, its contribution would be just missing. In other words, summing up the perturbation theory around the ground state, and even taking into account all non-linear terms of the Hamiltonian, responsible for the existence of the other local minimum, would not recover the contribution exp(−βΔE) of the other state, which is located "beyond barrier". It is these type of contributions which are usually called "off-perturbative".

In the studies of the effects produced by the quenched disorder, conditionally, one can distinguish two main domains of research: strongly disordered systems, like spin-glasses (where the disorder in the dominant factor), and the systems containing some kind of weak disorder which is supposed not to destroy the ground states properties of the corresponding pure system. Traditionally, magnetic statistical systems containing weak disorder, such as random bond ferromagnetic Ising models, are studied focusing mainly on modifications introduced into their critical properties at the phase transition point[1, 2]. In fact, as was pointed out by Griffith many years ago[3], modification of the critical behavior, is not the only qualitative physical phenomenon which can be produced here.

Let us come back to the example shown in Fig.1. The presence of weak quenched disorder here, provided it does not ruin the global structure of the phase space, would just require supplementary averaging of the above exponential factor, exp(−βΔE) (since both the energy of the ground state E₀ and the energy
of the excited state $E_1$ are now the functions of the disorder parameters), but qualitatively, it would not modify the situation too much. Completely different and new physical phenomena comes into play when the structure of the phase space similar to that shown in Fig.1 is created by the presence of randomness. In other words, this is the situation when weak quenched disorder, although it does not modify the ground state of the system, creates something completely new (absent in the corresponding pure system), namely, the local minima states, somewhere at the periphery of the phase space, "far away" from the ground state of the system.

According to the original observation by Griffiths[3] and later studies[4, 5], the presence of such off-perturbative states in the disordered ferromagnetic Ising model makes its free energy to be non-analytic function of the external magnetic field $h$ in a whole temperature interval above the ferromagnetic phase transition point. Moreover, at least in some cases, this non-analyticity has the form of essential singularity in the limit $h \to 0$, [4, 6]. It has to be stressed that all such contributions are just missing in the traditional (perturbative) RG treatment of the problem[2]. In more spectacular way the off-perturbative effects manifest themselves in the dynamical properties, producing the slowing down of the relaxation processes[7], as well as in the quantum systems (see e.g. [8] and references therein).

Although at a qualitative level the origin of the off-perturbative contributions is more or less clear, their technical implementation, namely, the derivation of e.g. the non-analytic part of the free energy, turns out to be rather tricky problem. Usually, analytic calculations in disordered systems are performed in terms of the replica method. Many years ago Parisi has suggested[9], that the presence of additional local minima configurations in weakly disordered systems is related, in the replica approach, to the existence of localized in space and breaking replica symmetry instanton-like excitations (translation invariance and replica symmetry is recovered by taking into account all possible excitations of this kind). Later on there were several attempts of concrete realization of this idea for the random temperature[5] and the random field[10,11] Ising models where it has been demonstrated that the corresponding saddle-point equations may indeed have instanton-like solutions. Next step has been done when the systematic method of summation over all such type of solutions, breaking symmetry in the replica vector order parameter, has been developed[12, 13]. In terms of this method the explicit form of the off-perturbative contributions in the random temperature Ising model has been derived[14].

In the present paper (following the recent study the original Griffith problem[6]) the systematic approach for the off-perturbative calculations and its relation with the method of the vector replica symmetry breaking[12,13] is formulated (section II). Then, proposed scheme is applied for the random temperature (section III) and the random field (section IV) ferromagnetic Ising models. It is shown that in both
systems at temperatures away from the critical point, namely, in the paramagnetic phase of the random
temperature model, and in the ferromagnetic phase of the random field one, the free energy contains
non-analytic contributions which (as the functions of the disorder parameters) have the form of essential
singularities. It is demonstrated that these contributions appear due to localized in space instanton-like
configurations. Physical discussion of the obtained results is given in Section V.

2 General scheme of calculations

Let us consider a general (continuous) D-dimensional random system described by a Hamiltonian \( H [\phi(x); \xi(x)] \),
where \( \phi(x) \) is a field which defines the microscopic state of the system, and \( \xi(x) \) are quenched random
parameters. Let us suppose that in addition to the ground state, this system has another thermodynamically
relevant (Griffith) region of the configurational space located ”far away” from the ground state and
separated from it by a finite barrier of the free energy (see Fig.1). In other words, it is

called relevant (Griffith) region of the configurational space located “far away” from the ground state and

\[ F = -\frac{1}{\beta} \ln Z = \mathcal{F}_0 - \frac{1}{\beta} \ln \left[ 1 + Z_1 Z_0^{-1} \right] \]

The second term in the above equation, which will be denoted by \( F_G \), can be represented in the form of the series:

\[ F_G = -\frac{1}{\beta} \sum_{m=1}^{\infty} \left( -1 \right)^{m-1} \frac{Z_1^m Z_0^{-m}}{m} = -\frac{1}{\beta} \lim_{n \to 0} \sum_{m=1}^{\infty} \left( -1 \right)^{m-1} \frac{Z_n(m)}{m} \]

where

\[ Z_n(m) = \prod_{b=1}^{m} \int \mathcal{D} \phi_b^{(1)} \prod_{c=1}^{n-m} \int \mathcal{D} \phi_c^{(0)} e^{-\beta H_n[\phi_1^{(1)},...,\phi_n^{(1)};\phi_1^{(0)},...,\phi_n^{(0)}]} \]

is the replica partition function (\( H_n[\phi] \) is the corresponding replica Hamiltonian), in which the replica
symmetry in the \( n \)-component vector field \( \phi_a (a = 1, ..., n) \) is assumed to be broken. Namely, it is supposed
that the saddle-point equations

\[ \frac{\delta H_n[\phi]}{\delta \phi_a(x)} = 0 \], \( a = 1, ..., n \)

have non-trivial solutions with the RSB structure

\[ \phi_a^*(x) = \begin{cases} \\ \phi_1(x) & \text{for } a = 1, ..., m \\ \phi_0(x) & \text{for } a = m + 1, ..., n \end{cases} \]

with \( \phi_1(x) \neq \phi_0(x) \), so that the integration in the above partition function, eq.(4), goes over fluctuations
in the vicinity of these components:

\[ \phi_b^{(1)}(x) = \phi_1(x) + \varphi_b(x), \quad (b = 1, ..., m) \]
\[ \phi_c^{(0)}(x) = \phi_0(x) + \chi_c(x), \quad (c = 1, ..., n - m) \]

It should be stressed that to be thermodynamically relevant, the RSB saddle-point solutions, eq.(6),
should satisfy the following tree crucial conditions:

1. the solutions should be localized in space, so that they are characterized by finite space sizes \( R(m) \);
in this case the partition function, eq.(4), will be proportional to the entropy factor \( V/R^D(m) \) (where \( V \)
is the volume of the system), and the corresponding free energy contribution \( F_G \), eq.(3), will be extensive
quantity;
2. they should have finite energies \( E(m) = H_n[\phi^*] \);
3. the corresponding Hessian matrix of these solutions should have all eigenvalues positive.
Thus, in the systematic calculations one should find all saddle-point RSB solutions \( \phi^*_a(\mathbf{x}) \), eq.(6), (satisfying the above three requirements), after that one has to compute their energies \( E(m) \) (for \( n \to 0 \)), next one has to integrate over the fluctuations in the vicinity of these solutions, and finally one has to sum up the series

\[
F_G = -\frac{V}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} R^{-D}(m) \left( \beta \det \hat{T} \right)^{-1/2} \sum_{n=0}^{\infty} e^{-\beta n E(m)}
\]  \tag{8}

where \( \hat{T} \) is the \((n \times n)\) matrix

\[
T_{aa'} = \frac{\delta^2 H [\phi]}{\delta \phi_a \delta \phi_{a'}} \bigg|_{\phi = \phi^*}
\]  \tag{9}

Note that in the present approach the procedure of analytic continuation \( n \to 0 \) is quite similar to that in the usual replica theory [15]: whenever the parameter \( n \) becomes an algebraic factor (and not the summation parameter, or the matrix size, etc.), it can safely be set to zero right away.

The above scheme of calculations can be easily generalized for an arbitrary number of the Griffiths regions. For example, let us consider the situation which is qualitatively represented in Fig.2, when in addition to the ground state, the system has two thermodynamically relevant Griffiths states. In this case instead of eq.(1) we will have

\[
Z = \int \mathcal{D}\phi(x)e^{-\beta H} = e^{-\beta F_0} + e^{-\beta F_1} + e^{-\beta F_2} \equiv Z_0 + Z_1 + Z_2
\]  \tag{10}

and correspondingly, instead of eq.(3) we find

\[
F_G = -\frac{1}{\beta} \ln \left[ 1 + Z_1 Z_0^{-1} + Z_2 Z_0^{-1} \right] = -\frac{1}{\beta} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left( Z_1^k Z_2^{m-k} Z_0^{-m} \right)
\]

\[
= -\frac{1}{\beta} \lim_{n \to 0} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} Z_n(k,m)
\]  \tag{11}

Here, in the replica partition function
\[ Z_n(k, m) = \prod_{b=1}^{k} \int \mathcal{D}\phi_{b}^{(1)} \prod_{c=1}^{m-k} \int \mathcal{D}\phi_{c}^{(2)} \prod_{d=1}^{n-m} \int \mathcal{D}\phi_{d}^{(0)} \ e^{-\beta H_n[\phi^{(1)}, \phi^{(2)}, \phi^{(0)}]} \] \tag{12}

the integration is supposed to be performed in the vicinity of the saddle-point replica vector
\[
\phi_*^a(x) = \begin{cases} 
\phi_1(x), & \text{for } a = 1, \ldots, k \\
\phi_2(x), & \text{for } a = k + 1, \ldots, m \\
\phi_0(x), & \text{for } a = m + 1, \ldots, n
\end{cases}
\tag{13}
\]

(where \(\phi_1(x) \neq \phi_2(x) \neq \phi_0(x)\)) which is the solution of the saddle-point equations (5). Finally, for the Griffiths free energy contribution, instead of eq.(8) one obtain
\[
F_G = -V \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{\beta m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \ R^{-D} \left( \beta \det \hat{T} \right)^{-1/2} e^{-\beta E(k, m)} \] \tag{14}

where \(E(k, m) = H_{n \to 0}[\phi_*]\) is the energy of a given solution, eq.(13), and \(\hat{T}\) is the Hessian matrix, eq.(9).

It is interesting to note that one can arrive to the same representations for the off-perturbative free energy contributions, eq.(8) or eq.(14), following the so called vector replica symmetry breaking scheme [12, 13]. The starting point here is the standard replica definition for the averaged over disorder free energy,
\[
\mathcal{F} = -\frac{1}{\beta} \lim_{n \to 0} \frac{Z_n^{1/n} - 1}{n} \] \tag{15}

where the replica partition function \(Z_n^{1/n} \equiv Z_n\) is formally defined by the integration over all configurational space:
\[
Z_n = \prod_{a=1}^{n} \int \mathcal{D}\phi_{a} \ e^{-\beta H_n[\phi_1, \ldots, \phi_n]} \] \tag{16}

Now, let us suppose that in addition to the usual replica symmetric (RS) ground state configuration, the saddle-point equations (5) have another types of solutions, which are well separated in the configurational space from the RS state. In this case (again, denoting their contributions by the label “G”) the replica partition function, eq.(16), can be decomposed into two parts:
\[
Z_n = Z_{RS} + Z_{G} \] \tag{17}

Here \(Z_{RS}\) contains all “routine” perturbative contributions in the vicinity of the ground state, and, as usual, (in the limit \(n \to 0\)) this partition function can be represented in the form:
\[
Z_{RS} = e^{-\beta n F_0} \] \tag{18}

Thus, according to eq.(15) for the total free energy we get:
\[
\mathcal{F} = F_0 + F_G \] \tag{19}

where
\[
F_G = -\lim_{n \to 0} \frac{1}{\beta n} Z_G \] \tag{20}

contains all non-replica-symmetric contributions (if any). As an example, let us suppose that the saddle-point eqs.(5) have non-trivial solutions with the three groups structure like in eq.(13). Moreover, let us suppose that these solutions possess three crucial properties: (1) they are localized in space and characterized by finite spatial sizes \(R_n(m, k)\); (2) they have finite energies \(E_n(k, m)\); and (3) their Hessian matrices \(T_{ab}\), eq.(9), have all the eigenvalues positive. Than taking into account all possible permutations of the tree replica vector components the above free energy \(F_G\) can be represented in the form
\[
F_G = -\lim_{n \to 0} \frac{1}{\beta n} \sum_{m=1}^{n} \frac{n!}{m!(n-m)!} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \ V R_{ab}^{Dk}(k, m) \left( \beta \det \hat{T} \right)^{-1/2} e^{-\beta E_n(k, m)} \] \tag{21}
To perform the analytic continuation \( n \to 0 \) in the above expression the parameter \( n \) must enter as an algebraic factor, and not as the parameter of summation. This can be achieved if we represent the above series in the following way:

\[
F_G = - \lim_{n \to 0} \frac{1}{\beta n} \sum_{m=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \frac{V}{\beta \det \hat{T}}^{1/2} e^{-\beta E_n(k,m)} (22)
\]

Here the summation over \( m \) is extended beyond \( m = n \) limit since the gamma function \( \Gamma(z) \) is equal to infinity both at \( z = 0 \) and at all negative integers. Now using the relation:

\[
\Gamma(-z) = -\frac{\pi}{z\Gamma(z) \sin(\pi z)} \quad (23)
\]

and referring to "good" analytical properties of the Gamma functions, we can perform the analytic continuation \( n \to 0 \):

\[
\frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(n-m+1)} = \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma[-(m-1-n)]} = \frac{\Gamma(n+1)(m-1-n)\sin[\pi(m-1)-\pi n]}{\pi\Gamma(m+1)} = \frac{\sin(\pi n)}{\pi} \frac{\Gamma(n+1)\Gamma(m-n)}{\Gamma(m+1)} \bigg|_{n \to 0} = n \frac{(-1)^{m-1}}{m} \quad (24)
\]

Substituting this into eq.(22) we obtain eq.(14) (where \( R(k,m) \equiv R_{n=0}(k,m) \) and \( E(k,m) \equiv E_{n=0}(k,m) \)).

Now, in the next two sections I am going to demonstrate how the above general scheme works in the concrete cases of the random temperature and the random field ferromagnetic Ising models.

### 3 Random temperature Ising model

Let us consider weakly disordered \( D \)-dimensional Ising model described by the continuous Ginzburg-Landau Hamiltonian:

\[
H = \int d^Dx \left[ \frac{1}{2}(\nabla \phi(x))^2 + \frac{1}{2}(\tau - \delta \tau(x))\phi^2(x) + \frac{1}{4}g\phi^4(x) \right] \quad (25)
\]

The disorder is modeled here by a random function \( \tau(x) \) which is described by the Gaussian distribution,

\[
P[\delta \tau] = p_0 \exp \left( -\frac{1}{4u} \int d^Dx (\delta \tau(x))^2 \right), \quad (26)
\]

where \( u \) is the small parameter which describes the strength of the disorder, and \( p_0 \) is the normalization constant. We are going to consider this system in the paramagnetic phase away from the critical point, so that the reduced temperature parameter \( \tau \) will be taken to be positive and not too small (it will be demonstrated below that in dimensions \( D < 4 \) the limitation on the value of \( \tau \) is given by the usual Ginzburg-Landau condition, \( \tau \gg g^2/(4-D) \)).

Weakly disordered systems described in terms of the continuous Ginzburg-Landau Hamiltonian have been usually studied in the framework of the renormalization-group (RG), (perturbative) treatment. In this approach one is able to perform the systematic integration over all fluctuations at the background of the homogeneous state up to the scales of the correlation length \( R_c(\tau) \). In some cases, this makes possible to derive the leading singularities of the thermodynamical functions in the critical point, at \( \tau \to 0 \), where \( R_c(\tau) \) diverges[2].

However, considering paramagnetic phase of this system, intuitively, it is clear that the contributions of non-homogeneous local minima configurations at scales bigger that the correlation length, which exist
due to rare localized in space "ferromagnetic islands" with negative effective value of the local temperature parameter \((\tau - \delta \tau)\) are missing in the traditional RG treatment of the problem. To what extend these states are relevant for the critical behavior \((\tau \rightarrow 0)\) is still unclear[16]. In the present study, however, I am going to address much more simple question: what is the explicit form of the free energy contributions due to such off-perturbative states away from \(T_c\) \((\tau \gg \tau_g)\).

In fact, at purely heuristic level, it is not so difficult to estimate form of these contributions. Let us consider the spatial island of the linear size \(L\) characterized by the typical value of the "local temperature" \((\tau - \delta \tau) = -\xi < 0\) Its probability is exponentially small,

\[
\mathcal{P}[L, \xi] \sim \exp\left(-\frac{(\tau + \xi)^2}{4u} L^D\right),
\]

and therefore such islands are well separated from each other and can be considered non-interacting.

It has to be noted that the island with small (negative) value of the local temperature parameter \(-\xi\) can be characterized by the mean-field "up" and "down" states only if its size is bigger than its local correlation length \(\xi_0(\xi) \sim \xi^{-1/2}\). Thus, the contribution to the free energy coming from the local ferromagnetic states of such islands with the exponential accuracy can be estimated by their probability:

\[
F_G \sim \int_0^\infty d\xi \int_{R_0(\xi)}^\infty dL \exp\left[-\frac{1}{4u}(\tau + \xi)^2 L^D\right] \sim \int_0^\infty d\xi \exp\left[-(\text{const})\frac{(\tau + \xi)^2}{u} \xi^{-D/2}\right]
\]

Here in the integration over \(\xi\) the leading contribution comes from the vicinity of the saddle-point value

\[
\xi_* = \frac{D}{4-D} \tau
\]

(which is positive in dimensions \(D < 4\), and \(\xi_* \gg \tau_g\) provided \(\tau \gg \tau_g\)). In this way we obtain the following estimate for the off-perturbative contributions coming from rare locally ferromagnetic islands:

\[
F_G \sim \exp\left[-(\text{const})\frac{\tau(4-D)/2}{u}\right]
\]

Now let us consider how this result (including the value of the \((\text{const})\) factor) can be derived analytically in terms of the systematic approach developed in the previous section. This derivation has been already reported elsewhere[6, 14]. Here I am going to give some more details about the corresponding replica instanton solutions, but in general this section can be considered just as a "warming up" exercise before passing to more difficult calculations for the random field model considered in the next section.

Performing the standard Gaussian integration over random parameters \(\delta \tau(x)\), for the replica partition function one gets

\[
Z_n = \prod_{a=1}^n \int \mathcal{D}\phi_a(x) e^{-\beta H_n[\phi]}
\]

where

\[
H_n[\phi] = \int d^D x \left\{ \frac{1}{4} \sum_{a=1}^n (\nabla \phi_a)^2 + \frac{\tau}{2} \sum_{a=1}^n \phi_a^2 + \frac{g}{4} \sum_{a=1}^n \phi_a^4 - \frac{u}{4} \sum_{a=b=1}^n \phi_a^2 \phi_b^2 \right\}
\]

is the corresponding replica Hamiltonian. The saddle-point configurations of the fields \(\phi_a(x)\) are defined by the equations

\[
-\Delta \phi_a(x) + \tau \phi_a(x) + g \phi_a^3(x) - u \phi_a(x) \left( \sum_{b=1}^n \phi_b^3(x) \right) = 0
\]

Below we are going to demonstrate that besides the trivial solution \(\phi_a(x) = 0\) these equations have non-trivial localized in space instanton-like solutions with the RSB (two groups) structure:

\[
\phi_a^*(x) = \begin{cases} 
\phi_1(x) & \text{for } a = 1, \ldots, m \\
0 & \text{for } a = m+1, \ldots, n 
\end{cases}
\]
Substituting this ansatz into the saddle-point eqs(33) and into the Hamiltonian, eq.(32), we find that (in the limit $n \to 0$) the instanton configuration $\phi_1(x)$ is defined by the equation

$$-\Delta \phi_1(x) + \tau \phi_1(x) - \lambda(m)\phi_1(x)^3 = 0$$

(35)

which is controlled by the parameter

$$\lambda(m) = um - g$$

(36)

and the energy of this configuration is

$$E(m) = m \int d^Dx \left[ \frac{1}{2}(\nabla \phi_1)^2 + \frac{1}{2}\tau \phi_1^2 - \frac{1}{4} \lambda(m)\phi_1^4 \right]$$

(37)

In what follows the parameter $\lambda(m)$ will be assumed to be positive. In other words, the solution, which we are going to derive below, exists only for $m$ such that $m > \lfloor g/u \rfloor$ (where $\lfloor \ldots \rfloor$ denotes the integer part). It has to be noted that one should not be confused by the "wrong" sign of the coupling $\phi^4$ term in the above equations. In fact, it can be shown that the integration over the replica fluctuations around considered solution in the limit $n \to 0$ yields the Hessian matrix which has all the eigenvalues positive (this is the usual situation for the replica theory, where the minima of the physical quantities in the limit $n \to 0$ turns into maxima of the corresponding replica quantities[13, 15]).

Rescaling the fields,

$$\phi_1(x) = \sqrt{\frac{\tau}{\lambda(m)}} \psi(x\sqrt{\tau})$$

(38)

and introducing $z \equiv x\sqrt{\tau}$, instead of eq.(35) one get the differential equation which contains no parameters:

$$-\Delta \psi(z) + \psi(z) - \psi^3(z) = 0$$

(39)

Correspondingly, for the energy of this configuration, eq.(37), one obtains:

$$E(m) = \frac{m}{um - g} \tau^{(4-D)/2} E_0(D)$$

(40)

where the quantity $E_0(D)$ depends only on the dimensionality of the system:

$$E_0(D) = \int d^Dz \left[ \frac{1}{2}(\nabla \psi(z))^2 + \frac{1}{2}\psi^2(z) - \frac{1}{4} \psi^4(z) \right]$$

(41)

It can be shown (see e.g. [17]) that in dimensions $D < 4$ eq.(39) has the smooth (with $\psi'(0) = 0$) spherically symmetric instanton-like solution $\psi(r)$ (where $r = |z|$) such that:

$$\psi(r \leq 1) \sim \psi(r = 0) \equiv \psi_0 \sim 1,$n

$$\psi(r \gg 1) \sim e^{-r} \to 0.$$n

(42)

The energy $E_0(D)$ of this solution is a finite and positive number. In particular, in dimensions $D = 3$, $\psi_0 \simeq 4.34$ and $E_0 \simeq 18.9$ (see Fig.3).

As the dimension parameter $D$ approaches the upper critical dimensionality $D_c = 4$, from below the value of the field at the origin $\psi_0(D)$ tend to infinity (see Fig.4), while the energies of the corresponding instanton configurations $E_0(D)$ approach the finite universal value $E_0(D \to 4) = E_* \simeq 26.3$. Above dimensions $D = 4$ eq.(39) has no smooth instanton-like solutions. In other words, described in terms of the dimensions parameter $D$, when passing the critical value $D_c = 4$ from below, the instanton solution disappears in the discontinuous way.
Note that according to the rescaling, eq.(38), the size of these instanton solutions in terms of the original fields $\phi_1(x)$ is $R_c(\tau) = \tau^{-1/2}$ (which is the usual correlation length of the Ginsburg-Landau theory) and it does not depends on $m$. Note also that due to obvious symmetry property $\phi_a \rightarrow -\phi_a$ of the original saddle-point eqs.(33) (which is valid for all non-zero replica field components independently), the above instanton solution $\phi_a^*(x)$, eq.(34), has additional degeneracy factor $2^m$.

The final step is the integration over fluctuations $\varphi_a(x)$ at the background of the above instanton solution. Substituting $\phi_a(x) = \phi_a^*(x) + \varphi_a(x)$, and expanding the Hamiltonian up to the second order in $\varphi_a(x)$, one has to perform the standard Gaussian integration. These calculations, although slightly cumbersome, are quite straightforward (for the details see Refs.[6, 14]). In the result for the Hessian factors one gets

$$\left(\det \hat{T}\right)^{-1/2}_{n \rightarrow 0} \simeq \exp \left[\frac{3m}{2(um - g)}g\psi_0^2\right]$$  \hfill (43)

Comparing this with factor $\exp[-E(m)]$, where $E(m)$ is the instanton energy, eq.(40), we see, that under condition

$$\tau \gg \tau_g = g^{2/(4-D)}$$  \hfill (44)

the contribution of fluctuations can be neglected. This is not surprising because eq.(44) is nothing else, but the Ginzburg-Landau criterion which defines the temperature region away from $T_c$, where the critical fluctuations are irrelevant. On the other hand, it has to be stressed that in the close vicinity of $T_c$ (at $\tau \leq \tau_g$), where the critical fluctuations are relevant, the Gaussian approximation used for obtaining the result, eq.(43), can not be valid anymore, and to derive the corresponding fluctuations contribution one would have to start some kind of RG procedure which would properly take into account non-Gaussian interactions. Thus, the above result for the fluctuations contribution, eq.(43), either can be considered as the small correction (at $\tau \gg \tau_g$), or otherwise (if it is not small), it is not valid (at $\tau \leq \tau_g$).

Thus, substituting the value of the instanton energy, eq.(40), its size $R = \tau^{-1/2}$ as well as its degeneracy factor $2^m$ into the series, eq.(8), we get

$$F_G \simeq -V_D^{D/2} \sum_{m=[g/u]+1}^{\infty} \frac{(-1)^{m-1}}{m} 2^m \exp \left[-E_0(D)\frac{m}{um - g}r^{(4-D)/2}\right]$$  \hfill (45)

The exact summation of this series seems to be rather tricky problem, but with the exponential accuracy it can be estimated in a very simple way. One can easily see that in the limit of weak disorder, at $u \ll g$, the leading contribution in this summation comes from the region $m \gg g/u \gg 1$ (where the exponential
factor in eq.(45) becomes $m$-independent) and this contribution is

$$F_G \sim \exp\left(-E_0(D)\frac{\tau^{(4-D)/2}}{u}\right)$$

(46)

We see that obtained off-perturbative (Griffith-like) part of the free energy, as the function of the disorder parameter in the limit $u \to 0$, has the form of the essential singularity. Note again that this contribution exists only in dimensions $D < 4$. As discussed above, at $D \to 4$ (the upper critical dimensions), the dimensionless instanton energy factor $E_0(D)$ approaches the finite universal limiting value $E_\ast \simeq 26.3$. In our world, in three dimensions, $E_0(D = 3) \simeq 18.9$.

### 4 Random field Ising model

To study the off-perturbative effects in the random field Ising model we are going to use again the Ginzburg-Landau continuous representation:

$$H = \int d^D x \left[ \frac{1}{2} (\nabla \phi(x))^2 + \frac{1}{2} \tau \phi^2(x) + \frac{1}{4} g \phi^4(x) - h(x)\phi(x) \right]$$

(47)

Here the random function $h(x)$ is described by the Gaussian distribution,

$$P[h(x)] = p_0 \exp\left(-\frac{1}{2h_0} \int d^D x h^2(x)\right),$$

(48)

where $h_0$ is the small parameter which describes the effective strength of the random field, and $p_0$ is the normalization constant. Unlike the random temperature model, considered in the previous section, here we are going to consider the system in the low-temperature ferromagnetic phase (supposing that the dimensionality $D$ is such that this phase exists), so that the reduced temperature parameter $\tau$ will be taken to be negative, $\tau = -|\tau|$. Again, we will place the system away from the critical point, assuming that the absolute value $|\tau|$ is not too small. As usual, to avoid the effects of the critical fluctuations (in dimensions $D < 4$) we impose the condition $|\tau| \gg g^{2/(4-D)}$. 
Let us suppose that in the absence of the random fields the ferromagnetic ground state of the system, eq.(47), is "up". This state (at the mean-field level) is characterized by the order parameter
\[ \phi_0 = +\sqrt{\frac{|\tau|}{g}} \]  
and the energy density
\[ \epsilon_0 = -\frac{\tau^2}{4g} \]  
In the usual perturbative approach the effects produced by the random field term of the Hamiltonian, eq.(47), together with the thermal fluctuations could be calculated in the systematic way in terms of the RG procedure (see e.g. [18] and references therein). This approach is designed to take into account all degrees of freedom at scales less that the correlation length, \( R_c(\tau) \sim |\tau|^{-1/2} \).

![Figure 5: Two alternative local minima field configurations in a spatial "island" of the linear size L, where the average value of the random field \( h \) is negative and its absolute value is not too small.](image)

On the other hand, at scales bigger that the correlation length we can observe completely different type of thermal excitations. Let us consider a spatial island of the linear size \( L \), where the average value of the field \( h \) is negative and its absolute value is not too small. Then, in addition to the state "up" (with slightly modified value of the order parameter), another local minimum with orientation "down" can exist in this island (Fig.5). To be stable, the gain in the energy due to the interaction with the field,
\[ E_h \sim -L^D|h|\phi_0 \]  
should overrun the loss of energy due to the creation of the domain wall,
\[ E_{d.w.} \sim L^{D-1}\frac{\phi_0^2}{R_c} \]  
Thus, such double-state situation in the considered island is created provided
\[ |h| > \frac{\phi_0}{LR_c} \sim \frac{|\tau|}{L\sqrt{g}} \equiv h_c \]  
According to eq.(48) the probability to find an island of the size \( L \) with the average value of the field \( h \) is
\[ P(L,h) \sim \exp \left[ -\frac{h^2}{2\phi_0^2} L^D \right] \]  
Then the contribution to the free energy of such rare "flipped" states can be estimated by their probability:
\[ F_G \sim \int_{R_c}^{\infty} dL \int_{h_c}^{\infty} dh \exp \left[ -\frac{h^2}{2\phi_0^2} L^D \right] \]  
\[ \sim \int_{R_c}^{\infty} dL \exp \left[ -(const) \frac{\tau^2}{h_c^2 g} L^{D-2} \right] \]  
\[ \sim \exp \left[ -(const) \frac{\tau^{\alpha-D}}{h_c^2 g} \right] \]  
(55)
Note that to obtain this result in the above integration over \( L \) the dimensionality of the system \( D \) must be bigger than two (otherwise the integral will become divergent). This is nothing else but slightly modified version of the good old Imri-Ma arguments[19] which tells that in dimension \( D \leq 2 \) flipping of magnetizations in big spatial islands can become energetically favorable, which indicate the instability of the global ferromagnetic state. Here we assume that the ferromagnetic state is stable, and we see that rare off-perturbative flipping excitations produce non-analytic contribution to the free energy, which in the limit \( h_0 \to 0 \) has the form of essential singularity.

Now we are going to re-derive the above prediction, eq.(55), in terms of much more rigorous systematic procedure described in section II. Coming back to the original Hamiltonian, eq.(47), after the Gaussian averaging of the replicated partition function over the random function \( h(\mathbf{x}) \), one obtains the replica Hamiltonian

\[
H_n [\phi] = \int d^Dx \left[ \frac{1}{2} \sum_{a=1}^{n} (\nabla \phi_a)^2 - \frac{1}{2} |\tau| \sum_{a=1}^{n} \phi_a^2 + \frac{1}{4} g \sum_{a=1}^{n} \phi_a^4 - \frac{1}{2} h_0^2 \sum_{a,b=1}^{n} \phi_a \phi_b \right] (56)
\]

The saddle-point configurations of the fields \( \phi_a(x) \) are defined by the equations

\[
-\Delta \phi_a(x) - |\tau| \phi_a(x) + g \phi_a^3(x) - h_0^2 \left( \sum_{b=1}^{n} \phi_b(x) \right) = 0 \tag{57}
\]

Below we are going to demonstrate that besides the obvious (replica symmetric) ferromagnetic solution \( \phi_a(x) = \sqrt{|\tau|/g} \) these equations have non-trivial localized in space instanton-like solutions with the RSB two-groups structure:

\[
\phi_a^*(x) = \begin{cases} 
\sqrt{\frac{|\tau|}{g}} \psi_1(x \sqrt{|\tau|}) & \text{for } a = 1, \ldots, m \\
\sqrt{\frac{|\tau|}{g}} \psi_0(x \sqrt{|\tau|}) & \text{for } a = m+1, \ldots, n
\end{cases} \tag{58}
\]

Substituting these rescaled fields into the saddle-point eqs(57) and into the Hamiltonian, eq.(56), we find that (in the limit \( n \to 0 \)) the instanton configuration \{\( \psi_1(z), \psi_0(z) \)\} (where \( z = x \sqrt{|\tau|} \)) is defined by the two equations

\[
-\Delta \psi_1 - \psi_1 + \psi_1^3 + \lambda(m) (\psi_1 - \psi_0) = 0 \\
-\Delta \psi_0 - \psi_0 + \psi_0^3 - \lambda(m) (\psi_1 - \psi_0) = 0
\]

and its energy is

\[
E(m) = m \frac{|\tau|^{2-D/2}}{g} \int d^Dz \left[ \frac{1}{2} \left( (\nabla \psi_1)^2 - (\nabla \psi_0)^2 \right) - \frac{1}{2} \left[ \psi_1^2 - \psi_0^2 \right] + \frac{1}{4} \left[ \psi_1^4 - \psi_0^4 \right] - \frac{1}{2} \lambda(m) \left( \psi_1 - \psi_0 \right)^2 \right] \tag{60}
\]

where

\[
\lambda(m) = \frac{h_0^2 m}{|\tau|} \tag{61}
\]

We are looking for the localized in space (spherically symmetric) solutions of the eqs.(59), such that the two functions \( \psi_1(r) \) and \( \psi_0(r) \) (where \( r = |z| \)) are different from each other in a finite region of space, and at large distances they both sufficiently quickly approach the same value \( \psi = 1 \), so that the integral in eq.(60) will be converging. Simple analysis of the structure of the "potential energy"

\[
U(\psi_1, \psi_0) = -\frac{1}{2} \left[ \psi_1^2 - \psi_0^2 \right] + \frac{1}{4} \left[ \psi_1^4 - \psi_0^4 \right] - \frac{1}{2} \lambda \left[ \psi_1 - \psi_0 \right]^2 \tag{62}
\]

shows that until the parameter \( \lambda \) is small (so that the last coupling term in the above expression is just a small correction), the potential \( U(\psi_1, \psi_0) \) has 9 saddle-points (in the vicinity of the points \( (0;0), (0; \pm 1), (\pm 1; 0), (\pm 1, \pm 1) \) and \( (\pm 1; \mp 1) \)). In this situation the two fields \( \psi_1 \) and \( \psi_0 \) are effectively
in the leading order in $\lambda$ starting from

$$\lambda > \lambda_c \simeq 0.23$$

(63)

only 5 saddle points of the potential $U(\psi_1,\psi_0)$ remains in the plain $(\psi_1;\psi_0)$. They have coordinates: $(0;0), (\pm 1; \pm 1)$ and $(\pm \psi_1^*(\lambda); \pm \psi_0^*(\lambda))$, where $0 < \psi_0^* < 1$ (in particular, $\psi_1^*(\lambda_c) \simeq 0.17$ and $\psi_0^*(\lambda_c) \simeq 0.90$). It is crucial that at the points $(\pm \psi_1^*; \pm \psi_0^*)$ the potential $U(\psi_1,\psi_0)$ has the maxima. It is due to the existence of these maxima that at $\lambda > \lambda_c$ the instanton solutions become possible.

Let us consider the limit $\lambda(m) \gg 1$, or

$$m \gg m_c = \left[ \lambda_c \frac{|\tau|}{h_0^2} \right] + 1$$

(64)

In this limit, according to eqs.(59), the two fields $\psi_1$ and $\psi_0$ must be close to each other. Redefining,

$$\psi_1(r) = \psi(r) + \frac{1}{\lambda} \chi(r)$$

$$\psi_0(r) = \psi(r) - \frac{1}{\lambda} \chi(r)$$

(65)

in the leading order in $\lambda^{-1}$ instead of eqs.(59) we get much more simple equations:

$$-\Delta \psi - \psi + \psi^3 - 2\chi = 0$$

$$-\Delta \chi + (3\psi^2 - 1)\chi = 0$$

(66)

which contain no parameters. For the energy of the configurations described by the two fields $\psi(r)$ and $\chi(r)$ instead of eq.(60) (again, in the leading order in $\lambda^{-1}$) we find the value, which does not depend on the summation parameter $m$,

$$E = \frac{|\tau|^{\frac{6-D}{2}}}{h_0^{2-D}} E_0(D)$$

(67)

where

$$E_0(D) = \int d^Dz \left[ (\nabla \psi)(\nabla \chi) + (\psi^3 - \psi)\chi - \chi^2 \right]$$

(68)

is the universal quantity which depends only on the dimensionality of the system.

It turns out that in dimensions $D < 3$, the system of eqs.(66), indeed has smooth instanton-like spherically symmetric solution which has finite and positive energy $E_0(D)$. Within the limited spatial region $r \leq r_c \sim 1$, the values of the fields $\psi(r)$ and $\chi(r)$ are finite and of the order of their values at the origin, $\psi_0 \sim 1$ and $\chi_0 \sim 1$. On the other hand, at $r \gg r_c$ the function $\psi(r)$ exponentially quickly approaches 1, while the function $\chi(r)$ exponentially tents to zero. The illustration of the instanton solution in the dimension $D = 2.9$ is given in Fig.6, where $\psi_0 \simeq -0.818$, $\chi_0 \simeq -0.284$

Fig.7 demonstrates the corresponding "trajectories" of the instanton solutions in the plane $(\psi,\chi)$ at various values of the dimension. As the dimension $D$ approaches the value $D_c = 3$ from below, the starting values $\psi_0 \to -1$ and $\chi_0 \to 0$. Above three dimension the instanton solution disappears. Thus we have to conclude that $D = 3$ is the upper critical dimension for the considered Griffiths phenomena in the RFIM, in agreement with the earlier suggestion [10, 11] as well as with the recent studies of similar instanton-like configurations in the presence of external magnetic field [20].

Let us come back to the general expression for the off-perturbative part of the free energy, the series, eq.(8), where the summation over $m$ starts now from $m = m_c$, eq.(64). Noting that the instanton energy $E(m)$ is the decreasing function of $m$, we can conclude that with the exponential accuracy this converging series can be estimated by its asymptotic part at $m \gg m_c$. Thus, substituting here the value of the
The instanton solution $\psi(r), \chi(r)$ of eqs.(66) in dimensions $D = 2.9$.

The instanton energy, eq.(67), (and neglecting the critical fluctuations), with the exponential accuracy we get the result

$$F_G \sim \exp \left[ -E_0(D) \frac{|r|^\frac{D-4}{2}}{h_0^2g} \right]$$

which perfectly agrees with the "hand-waving" estimate, eq.(55). This non-analytic in $h_0$ contribution, which has the form of essential singularity at $h_0 \to 0$, is valid only in dimensions $D < 3$, and at temperatures not too close to the critical point, at $|\tau| \gg g^2/(4-D)^2$. The dimension $D = 3$ is marginal for this kind of phenomena. Therefore the investigation of the Griffith-like contributions in the three-dimensional RFIM requires much more discreet analysis.

## 5 Discussion

In this paper the systematic method for computing off-perturbative thermodynamic contributions in disordered systems has been proposed. It has been tested on two the most popular classical statistical systems containing quenched disorder: the random temperature and the random field Ising models. In both cases the off-perturbative contributions as the functions of the parameters which describe the strength of the disorder have the form of the essential singularities, eqs.(46), (69).

Of course, thinking about possible experimental or numerical tests, the validity of the obtained results is rather limited. On one hand, since the consideration has been done in terms of the continuous Ginsburg-Landau Hamiltonian, one has to place the system sufficiently close to the phase transition point, so that the correlation length would be large compared to the lattice spacing. On the other hand, since the present study completely neglects the critical fluctuations, the system has to be sufficiently far from the critical point. Formally, in terms of the Ginsburg-Landau Hamiltonian, these two requirements can be easily satisfied: it is sufficient to demand that (1) the coupling parameter $g$ is small, and (2) the reduced temperature parameter $\tau$ is bounded by the condition $g^{2/(4-D)} \ll |\tau| \ll 1$ (where $D$ is the system dimensionality, $D < 4$). Besides, the strength of the disorder must be small: $u \ll g$ in the random temperature model, and $h_0 \ll \sqrt{|\tau|}$ in the random field one.

On the other hand, keeping in mind more general perspectives, the proposed approach, in my view, may open the way to study the nature of the phase transitions in the considered systems. It is generally believed that it is the off-perturbative states which makes the study of the phase transition in the random field Ising model so difficult[18]. It is remarkable, that according to the present study, quite similar off-perturbative contributions are also present in the random temperature Ising systems where, at least at the qualitative level, the nature of the phase transition was traditionally believed to be well understood.
Figure 7: The instanton solution "trajectories" in the plane $(\psi, \chi)$ in various dimensions:
(a) $D = 2$, $\psi_0 \simeq 0.001$, $\chi_0 \simeq -0.5173$; (b) $D = 2.5$, $\psi_0 \simeq -0.325$, $\chi_0 \simeq -0.5776$; (c) $D = 2.7$, $\psi_0 \simeq -0.520$, $\chi_0 \simeq -0.5300$; (d) $D = 2.9$, $\psi_0 \simeq -0.818$, $\chi_0 \simeq -0.2844$; (e) $D = 2.92$, $\psi_0 \simeq -0.988$, $\chi_0 \simeq -0.0212$.

(see e.g. [2,13]). This supports recent suspects[16] that the off-perturbative effects could be quite relevant for the critical properties of the disordered ferromagnetic systems.

Another interesting observation is that in terms of the considered off-perturbative contributions, the situation, when approaching $T_c$ from above and from below, looks totally asymmetric both in the random field and in the random temperature systems. The considered effects are present in the ferromagnetic phase, at $T < T_c$, of the random field model, while they are absent in its paramagnetic phase at $T > T_c$. On the other hand, the situation in the random temperature model is similar, although "reversed": the off-perturbative contributions are present in the paramagnetic phase, at $T > T_c$, while they are absent in the ferromagnetic phase, at $T < T_c$. May be it is this asymmetry, which makes the nature of the phase transitions in these systems to be so non-trivial?

One more qualitative observation is that according to the present study, the off-perturbative contributions are absent in the random temperature model at dimensions $D > 4$, and in the random field model at dimensions $D > 3$. As for the random temperature systems, this is not surprising: all the previous studies were definite that at $D > 4$ the disorder is irrelevant for the phase transition. What does this mean for the random field model is much less clear, because here it is well established that its upper critical dimensionality is equal to 6 (the dimensionality above which the critical behavior is described by the Gaussian theory, and the presence of the random fields is irrelevant for the phase transition). Well, of course, the absence of the off-perturbative contributions in dimensions $3 < D \leq 6$ does not mean, that the random fields are irrelevant. Probably it indicates that here all the random field effects can be taken into account in the framework of the perturbative RG procedures.

To answer all the above questions (as well as many other important questions which were not formulated here), the only thing which remains to be done is to find the way to overcome the Ginzburg-Landau limitation $|\tau| \gg g^{2/(4-D)}$, and to take the limit $\tau \to 0$. To do that one has just to formulate a theory which would properly take into account the critical fluctuations on top of the instanton-like configurations described in this paper.

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