INVARIANT IDEALS AND ITS APPLICATIONS TO THE TURNPIKE THEORY

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Abstract. In this paper the turnpike property is established for a non-convex optimal control problem in discrete time. The functional is defined by the notion of the ideal convergence and can be considered as an analogue of the terminal functional defined over infinite time horizon. The turnpike property states that every optimal solution converges to some unique optimal stationary point in the sense of ideal convergence if the ideal is invariant under translations. This kind of convergence generalizes, for example, statistical convergence and convergence with respect to logarithmic density zero sets.

1. Introduction

The turnpike theory investigates an important property of dynamical systems. It can be considered as a theory that justifies the importance of some equilibrium/stationary states. For example, in macroeconomic models the turnpike property states that, regardless of initial conditions, all optimal trajectories spend most of the time within a small neighborhood of some optimal stationary point when the planning period is long enough. Obviously, in the absence of such a property, using some of optimal stationary points as a criteria for “good” policy formulation might be misleading. Correspondingly, the turnpike property is in the core of many important theories in economics.

Many real-life processes are happening in an optimal way and have the tendency to stabilize; that is, the turnpike property is expected to hold for a broad class of problems. It provides valuable insights into the nature of these processes by investigating underlying principles of evolution that lead to stability. It can also be used to assess the “quality” of mathematical modeling and to develop more adequate equations describing system dynamics as well as optimality criteria.

The first result in this area is obtained by John von Neumann [30] for discrete time systems. The phenomenon is called the turnpike property after Chapter 12, 19 by Dorfman, Samuelson and Solow. For a classification of different definitions for this property, see [22, 28, 36], as well as [6] for the so-called exponential turnpike property. Possible applications in Markov Games can be found in a recent study [16].

The approaches suggested for the study of the turnpike property involve continuous and discrete time systems. Some convexity assumptions are sufficient for discrete time systems [22, 28]; however, rather restrictive assumptions are usually
required for continuous time systems. The majority of them deal with the (discounted and undiscounted) integral functionals. We mention here the approaches developed by Rockafellar \[33, 34\], Scheinkman, Brock and collaborators (see, for example, \[21, 35\]), Cass and Shell \[4\], Leizarowitz \[18\], Mamedov \[24\], Montrucchio \[29\], Zaslavski \[37, 38, 39\] (we refer to \[2, 36\] for more references).

In this paper we consider an optimal control problem in discrete time. It extends the results obtained in \[23\] where a special class of terminal functionals is introduced as a lower limit at infinity of utility functions. This approach allowed to establish the turnpike property for a much broader class of optimal control problems than those involving integral functionals (discounted and undiscounted).

Later, this class of terminal functionals was used to establish a connection between the turnpike theory and the notion of statistical convergence \[25, 32\]; as a result, the convergence of optimal trajectories is proved in terms of the statistical ("almost") convergence. These terminal functionals also allowed the extension of the turnpike theory to time delay systems; the first results in this area have been established in several recent papers \[13, 26\]. Moreover, some generalizations based on the notion of the \(A\)-statistical cluster points have been obtained in \[7\].

The main purpose of this paper is to formulate the optimality criteria by using the notion of ideal convergence. As detailed in the next section, the ideal convergence is a more general concept than the statistical convergence as well as the \(A\)-statistical convergence. In this way the turnpike property is established for a broad class of non-convex optimal control problems where the asymptotical stability of optimal trajectories is formulated in terms of the ideal convergence.

Recently (and independently) Leonetti and Caprio in \[19\] considered turnpike property for ideals invariant under translation in the context of normed vector spaces. We discuss our approaches in Section 4.

The rest of this article is organized as follows. In the next section the definition of the ideal, its properties and some particular cases, including the statistical convergence, are provided. In Section 3 we formulate the optimal control problem and main assumptions. The main results of the paper — the turnpike theorems are provided in Section 4. The proof of the main theorem is in Section 5.

2. CONVERGENCE WITH RESPECT TO IDEAL VS STATISTICAL CONVERGENCE

Let \(x = (x_n)_{n \in \mathbb{N}}\) be a sequence of elements of \(\mathbb{R}^m\). For the sake of simplicity, we will consider the Euclidean norm \(\|\cdot\|\). The classical definition of convergence of \(x\) to \(a\) says that for every \(\varepsilon > 0\) the set of all \(n \in \mathbb{N}\) with \(\|x_n - a\| \geq \varepsilon\) is finite, i.e. it is "small" in some sense. If we understand the word "small" as "of asymptotic density zero" then we obtain the definition of statistical convergence (Def. \[3\]). The same method can be used to formulate the definition of statistical cluster point. The classical one says that \(a\) is a cluster point of \(x\) if for every \(\varepsilon > 0\) the set of all \(n \in \mathbb{N}\) with \(\|x_n - a\| < \varepsilon\) is infinite, i.e. it has "many" elements. If "many" means "not of asymptotic density zero" then we obtain the definition of statistical cluster point (see e.g. \[12\]).

One of the possible generalizations of this kind of being "small" (having "many" elements) is "belonging to the ideal" ("be an element of co-ideal").

The cardinality of a set \(X\) is denoted by \(#X\). \(\mathcal{P}(\mathbb{N})\) denotes the power set of \(\mathbb{N}\).
**Definition 1.** An ideal on $\mathcal{P}(\mathbb{N})$ is a family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ which is non-empty, hereditary and closed under taking finite unions, i.e. it fulfills the following three conditions:

1. $\emptyset \in \mathcal{I}$;
2. $A \in \mathcal{I}$ if $A \subset B$ and $B \in \mathcal{I}$;
3. $A \cup B \in \mathcal{I}$ if $A, B \in \mathcal{I}$.

**Example 2.** By $\text{Fin}$ we denote the ideal of all finite subsets of $\mathbb{N} = \{1, 2, \ldots\}$.

There are many examples of ideals considered in the literature, e.g.

1. the ideal of sets of asymptotic density zero
   $$\mathcal{I}_d = \{ A \subset \mathbb{N} : \overline{d}(A) = 0 \} ,$$
   where $\overline{d} : \mathcal{P}(\mathbb{N}) \to [0, 1]$ is given by the formula
   $$\overline{d}(A) = \limsup_{n \to \infty} \frac{\#(A \cap \{1, 2, \ldots, n\})}{n}$$
   is the well-known definition of upper asymptotic density of the set $A$;
2. the ideal of sets of logarithmic density zero
   $$\mathcal{I}_{\log} = \{ A \subset \mathbb{N} : \limsup_{n \to \infty} \sum_{k \in A \cap \{1, 2, \ldots, n\}} \frac{1}{k} \leq 0 \} ;$$
3. the ideal
   $$\mathcal{I}_{1/n} = \{ A \subset \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty \} ;$$
4. the ideal of arithmetic progressions free sets
   $$\mathcal{W} = \{ W \subset \mathbb{N} : W \text{ does not contain arithmetic progressions of all lengths} \} .$$

Ideals $\mathcal{I}_d$ and $\mathcal{I}_{\log}$ belongs to the wider class of Erdős-Ulam ideal’s (defined by submeasures of special kind, see [14]). Ideal $\mathcal{I}_{1/n}$ is an representant of the class of summable ideals (see [27]). The fact that $\mathcal{W}$ is an ideal follows from the non-trivial theorem of van der Waerden (this ideal was considered by Kojman in [15]). One can also consider trivial ideals $\mathcal{I} = \mathcal{P}(\mathbb{N})$, $\mathcal{I} = \{ \emptyset \}$, or principal ideals $\mathcal{I}_n = \{ A \subset \mathbb{N} : n \notin A \}$, however they are not interesting from our point of view. If not explicitly said we assume that all considered ideals are proper (i.e. $\mathcal{I} \neq \mathcal{P}(\mathbb{N})$) and contain all finite sets (i.e. $\text{Fin} \subset \mathcal{I}$). The inclusions between abovementioned families are shown on Figure 1. The only non-trivial inclusions are: $\mathcal{I}_{1/n} \subset \mathcal{I}_d$ (a folklore application of Cauchy condensation test), $\mathcal{W} \subset \mathcal{I}_d$ (the famous theorem of Szemerédi), and $\mathcal{I}_d \subset \mathcal{I}_{\log}$ (by well-known inequalities between upper logarithmic density and upper asymptotic density). It is easy to observe that $\mathcal{I}_{1/n} \not\subset \mathcal{W}$, but the status of the inclusion $\mathcal{W} \subset \mathcal{I}_{1/n}$ is unknown (“Erdős conjecture on arithmetic progressions” says that the van der Waerden ideal $\mathcal{W}$ is contained in the ideal $\mathcal{I}_{1/n}$.)

### 2.1. $\mathcal{I}$-convergence and $\mathcal{I}$-cluster points.

The notion of the ideal convergence is dual (equivalent) to the notion of the filter convergence introduced by Cartan in 1937 ([3]). The notion of the filter convergence has been an important tool in general topology and functional analysis since 1940 (when Bourbaki’s book [1] appeared). Nowadays many authors prefer to use an equivalent dual notion of the ideal convergence (see e.g. frequently quoted work [17]).
Figure 1. Inclusions of ideals, implications between \( I \)-convergence, and inclusions of sets of \( I \)-cluster points for ideals from Example 2. Arrow “\( I \rightarrow J \)” means that “\( I \subset J \)”, and for every sequence \( x \), “\( x \rightarrow_I a \Rightarrow x \rightarrow_J a \)”, “\( \Gamma(x) \supset \Gamma_J(x) \)”.

**Definition 3.** A sequence \( (x_n)_{n \in \mathbb{N}} \) of elements of \( \mathbb{R}^m \) is said to be \( I \)-convergent to \( a \in \mathbb{R}^m \) (a = \( I - \lim x_n \), or \( x_n \rightarrow_I a \), in short) if and only if for each \( \varepsilon > 0 \)
\[
\{ n \in \mathbb{N} : \|x_n - a\| \geq \varepsilon \} \in I.
\]
The sequence \( (x_n) \) is convergent to \( a \) if and only if it is Fin-convergent to \( a \).

It is also easy to see that for any sequence \( x = (x_n) \) and two ideals \( I, J \), if \( I \subset J \) then \( x \rightarrow_I a \) implies that \( x \rightarrow_J a \) (see Figure 1).

**Definition 4.** The \( a \in \mathbb{R}^m \) is an \( I \)-cluster point of a sequence \( x = (x_n)_{n \in \mathbb{N}} \) of elements of \( \mathbb{R}^m \) if for each \( \varepsilon > 0 \)
\[
\{ n \in \mathbb{N} : \|x_n - a\| < \varepsilon \} \notin I.
\]
By \( I \)-cluster set of \( x \) we understand the set
\[
\Gamma_I(x) = \{ a \in \mathbb{R}^m : a \text{ is an } I \text{-cluster point of } x \}.
\]
Recall that \( \Gamma(x) = \Gamma_{\text{Fin}}(x) \) is a set of classical cluster (limit) points of \( x \).

**Proposition 5.** For any bounded sequence \( x = (x_n) \),

1. \( \Gamma_I(x) \neq \emptyset \) \( (\text{[M]} \] \), and
2. \( \Gamma_I(x) \) is closed \( (\text{[17]} \) \), and
3. \( \Gamma_I(x) = \{ a \} \) if and only if \( x \rightarrow_I a \).

Moreover, if \( I \subset J \) then \( \Gamma_J(x) \subset \Gamma_I(x) \) \( (\text{[31]} \) \), see Figure 1).

Part (3) follows from the folklore argument: \( a \) is the unique \( I \)-cluster point of \( x \), iff \( \{ n : \|x_n - a\| \geq \varepsilon \} \in I \) for every \( \varepsilon > 0 \), iff \( x_n \rightarrow_I a \).

**2.2. \( I \)-convergence vs statistical convergence.** The notion of the ideal convergence is a common generalization of the classical notion of convergence and statistical convergence. The concept of statistical convergence was introduced by Fast \( \text{[10]} \) and then it was studied by many authors.

**Definition 6 \( (\text{[10]} \) \).** A sequence \( x = (x_n)_{n \in \mathbb{N}} \) of elements of \( \mathbb{R}^m \) is said to be statistically convergent to an \( a \in \mathbb{R}^m \) if for each \( \varepsilon > 0 \) the set of all indices \( n \) such that \( \{ n \in \mathbb{N} : \|x_n - a\| \geq \varepsilon \} \) has upper asymptotic density zero, i.e.
\[
\overline{d}(\{ n \in \mathbb{N} : \|x_n - a\| \geq \varepsilon \}) = 0, \text{ for all } \varepsilon > 0.
\]

Obviously, \( x \) is statistically convergent to \( a \) if and only if \( x \rightarrow_{d,a} \). Following the concept of a statistically convergent sequence Fridy in \( \text{[12]} \) introduced the notion of a statistical cluster point, which—using our notation—is equal to the notion of \( \mathcal{I}_{d}\)-cluster point. Proposition \( 3 \) in case of statistical convergence was proved in \( \text{[12]} \).
Since statistical convergence is a particular case of \( I \)-convergence, each theorem which has an ideal variant is also true in its statistical version. However, in the sequel we will use some lemmas which were formulated in the literature for the case of statistical convergence and statistical cluster points.

An open \( \varepsilon \)-neighbourhood of a given set \( A \subset \mathbb{R}^m \) will be denoted by
\[
B(A, \varepsilon) = \{ y \in \mathbb{R}^m : \exists a \in A \ | \ a - y | < \varepsilon \}.
\]
For each \( a \in \mathbb{R}^m \) we do not distinguish between \( B(\{a\}, \varepsilon) \) and \( B(a, \varepsilon) \).

**Lemma 7** ([32]). Let \( x = (x_k)_{k \in \mathbb{N}} \) be a bounded sequence. Then for any \( \varepsilon > 0 \)
\[
\overline{d}(\{ k \in \mathbb{N} : x_k \notin B(\Gamma_I(x), \varepsilon) \}) = 0.
\]

The ideal version of the above lemma can be proved using the same method as in [32], but we give a short proof using [5, Le. 3.1].

**Lemma 8** ([5]). Suppose that \( I \) is an ideal, \( (x_n) \) is a sequence and \( K \subset \mathbb{R}^m \) is compact. If \( \{ n \in \mathbb{N} : x_n \notin K \} \notin I \) then \( K \cap \Gamma_I(x) \neq \emptyset \).

**Lemma 9** (Ideal version of Lemma 7). Let \( x = (x_k)_{k \in \mathbb{N}} \) be a bounded sequence. Then for any ideal \( I \) and \( \varepsilon > 0 \)
\[
\{ k \in \mathbb{N} : x_k \notin B(\Gamma_I(x), \varepsilon) \} \in I.
\]

**Proof.** Since \( x \) is bounded, there exists a compact set \( C \) such that \( x_n \in C \) for all \( n \). If we assume that \( \{ k \in \mathbb{N} : x_k \notin B(\Gamma_I(x), \varepsilon) \} \notin I \), then the set \( K = C \setminus B(\Gamma_I(x), \varepsilon) \) is compact and \( \{ n \in \mathbb{N} : x_n \in K \} \notin I \). By Lemma 8, \( K \cap \Gamma_I(x) \neq \emptyset \), a contradiction. \( \square \)

2.3. Ideals invariant under translations. By \( \mathbb{Z} \) we denote the set of all integers.

**Definition 10.** We say that an ideal \( I \) is invariant under translations if for each \( A \in I \) and \( i \in \mathbb{Z} \),
\[
A + i \in I, \text{ where } A + i = \{ a + i : a \in A \} \cap \mathbb{N}.
\]

All ideals considered in Example 2 are invariant under translations. For the proof of this fact and other examples see [11].

Our main results from Section 4 are valid for ideals invariant under translations. The key argument for this fact is the following property of \( I \)-cluster sets for such ideals.

**Lemma 11.** Suppose that \( I \) is invariant under translations, \( x = (x_k)_{k \in \mathbb{N}} \) is a sequence in \( \mathbb{R}^m \). Then for any non-empty \( G \subset \Gamma_I(x) \), \( i \in \mathbb{Z} \) and \( \delta_1, \delta_2 > 0 \):
\[
\{ k \in \mathbb{N} : x_k \in B(G, \delta_1) \text{ and } x_{k+i} \in B(\Gamma_I(x), \delta_2) \} \notin I.
\]
In particular, this set is non-empty.

**Proof.** Let \( K_{\delta_1}^1 = \{ k \in \mathbb{N} : x_k \in B(G, \delta_1) \} \) and \( K_{\delta_2}^2 = \{ k \in \mathbb{N} : x_k \in B(\Gamma_I(x), \delta_2) \} \). Since \( G \subset \Gamma_I(x) \) and \( G \neq \emptyset \), \( K_{\delta_1}^1 \notin I \). By Lemma 8, \( \mathbb{N} \setminus K_{\delta_2}^2 \in I \).

Consider the set \( K_{\delta_1}^1 + i = \{ k + i : k \in K_{\delta_1}^1 \} \). \( I \) is invariant under translations, so \( K_{\delta_1}^1 + i \notin I \). Let \( K = (K_{\delta_1}^1 + i) \cap K_{\delta_2}^2 \). Since \( K \) is an intersection of two sets, one from the coideal (i.e. not from the ideal) and second from the dual filter (i.e. its complement belongs to the ideal), \( K \notin I \). Consider the set \( K - i = \{ k - i : k \in K \} \).
Again, since \( \mathcal{I} \) is invariant under translations, \( K - i \notin \mathcal{I} \). For each \( k \in K - i \), \( x_k \in B(G, \delta_1) \) and \( x_{k+1} \in B(\Gamma(x), \delta_2) \). \( \square \)

If \( \mathcal{I} \) is invariant under translations then either \( \mathcal{I} \) is a trivial ideal \( \{\emptyset\} \), or \( \mathcal{I} \) contains all finite sets (i.e. \( \text{Fin} \subset \mathcal{I} \)). Indeed, if there is a non-empty set \( F \in \mathcal{I} \), then \( \{n\} \in \mathcal{I} \) for each \( n \in F \). From the invariance of \( \mathcal{I} \) it follows that \( \{k\} \in \mathcal{I} \) for every \( k \in \mathbb{N} \). Since \( \mathcal{I} \) is closed on finite unions, each finite set belongs to \( \mathcal{I} \).

3. OPTIMAL CONTROL PROBLEM AND MAIN ASSUMPTIONS

Consider the problem

\[
x_{n+1} = f(x_n, u_n), x_1 = \zeta^0, u_n \in U, \tag{*}
\]

\[
J_\mathcal{I}(x) = \mathcal{I}- \liminf \phi(x_n) \to \max. \quad (\mathcal{I}/\ast \ast)
\]

Here \( \zeta^0 \) is a fixed initial point, function \( f: \mathbb{R}^m \times \mathbb{R}^t \to \mathbb{R}^m \) is continuous, \( U \subset \mathbb{R}^t \) is a compact set, \( \phi: \mathbb{R}^m \to \mathbb{R} \) is a continuous function, and for any sequence of reals \( y = (y_n) \)

\[
\mathcal{I}- \liminf y = \sup \{y_0 \in \mathbb{R} : \{n \in \mathbb{N} : y_n < y_0\} \in \mathcal{I}\}.
\]

The pair \( \langle u, x \rangle \) is called a process if the sequences \( x = (x_n) \) and \( u = (u_n) \) satisfy \((\ast)\) for all \( n \in \mathbb{N} \) (\( x \) is called a trajectory and \( u \) is called a control).

In the sequel we will use the following characterization of the functional \( J_\mathcal{I} \) (Le. 4.1]) that is a generalization of Lemma 3.1 in [32] established for the statistical convergence, as well as the corresponding result from [20] established for classical convergence (see also [19] Cor. 3.3]).

**Lemma 12.** For any bounded trajectory \( x = (x_n)_{n \in \mathbb{N}} \) the following representation is true

\[
J_\mathcal{I}(x) = \min \Gamma_\mathcal{I}(\phi(x)) = \min_{\zeta \in \mathcal{F}_\mathcal{I}(x)} \phi(\zeta).
\]

We assume that there is a compact (bounded and closed) set \( C \subset \mathbb{R}^m \) such that \( x_n \in C \) for all trajectories; that is, we assume that trajectories are uniformly bounded.

\( \zeta \in \mathbb{R}^m \) is called a stationary point if there exists \( u_0 \in U \) such that \( f(\zeta, u_0) = \zeta \).

We denote the set of stationary points by \( M \). It is clear that \( M \) is a closed set. \( \zeta^* \in M \) is called an optimal stationary point if

\[
\phi(\zeta^*) = \phi^* = \max_{\zeta \in M} \phi(\zeta).
\]

We will assume that the set of all optimal stationary points is non-empty. This is not a restrictive assumption since function \( \phi \) is continuous and the set \( M \) is closed; for example, it is satisfied if \( M \) is in addition bounded.

Define the set

\[
M^* = \{\zeta^* \in M : \zeta^* \text{ is an optimal stationary point}\},
\]

and

\[
D^* = \{\zeta \in C : \phi(\zeta) \geq \phi^*\}.
\]

We assume that the set \( C \) is large enough to accommodate \( M^* \); that is, \( M^* \subset C \).

Then clearly, \( M^* = M \cap D^* \).

Consider the following three conditions.

\((C1)\): optimal stationary point \( \zeta^* \) is unique, i.e. \( M^* = \{\zeta^*\} \);

\((\mathcal{I}/C2)\): there exists a process \( \langle u^*, x^* \rangle \) such that \( \Gamma_\mathcal{I}(x^*) \subset D^* \);
(C3): there exists a continuous function $P: \mathbb{R}^m \to \mathbb{R}$ such that

$$P(f(x_0, u_0)) < P(x_0)$$

for all $x_0 \in D^* \setminus M^*$, $u_0 \in U$,

and

$$P(f(x_0, u_0)) \leq P(x_0)$$

for all $x_0 \in D^*$, $u_0 \in U$.

One can also consider condition $(C2) = (\text{Fin}/C2)$:

(C2): there exists a process $\langle u^*, x^* \rangle$ such that any limit point of the sequence $x^*$ is in $D^*$.

Note that if the (unique) optimal stationary point $\zeta^*$ belongs to the interior of $D^*$, then proof of turnpike property is not difficult and can be regarded as a “trivial” case where condition $(C3)$ ensures the existence of some Lyapunov function, with derivative $P$, defined on a small neighborhood of $\zeta^*$.

The most interesting case is when an optimal stationary point $\zeta^*$ belongs to a boundary of $D^*$; that is, the both sets $D^*$ and $D^{*-} = \{ \zeta \in C: \phi(\zeta) < \phi^* \}$ have nonempty intersection with any small neighborhood of $\zeta^*$. In this case the inequality $P(f(x_0, u_0)) > P(x_0)$ may hold for some $x_0 \in D^{*-}$; that is, condition $(C3)$ does not guarantee the existence of Lyapunov functions.

Note also that condition $(I/C2)/(C2)$ can be formulated equivalently “there exists a process $\langle u^*, x^* \rangle$ such that $J_I(x^*) \geq \phi^{**}$” (see [19, (A6)]), or stronger “there exists a process $\langle u^*, x^* \rangle$ such that $x^* \rightarrow_I \zeta^{**}$” (see [23, 25, 32]).

Recall that if $I \subset J$ then $I$-convergence is stronger than $J$-convergence, thus by Proposition 5, $(C2)$ is stronger than $(I/C2)$ for each non-trivial $I$ which is invariant under translations. Example 13 shows that these two conditions are really different, i.e. there exists a system for which $(C1)$, $(I_d/C2)$, $(C3)$ hold, but $(C2)$ does not hold (see also [19, Ex. 2.5]).

Example 13. Consider the middle-third Cantor set $T$. It is homeomorphic to the space $\{0, 1\}^\mathbb{N}$ with the product (Tychonoff) topology; for example, the formula

\[
\sum_{i=1}^{\infty} \frac{2 \cdot a_i}{3^i} \quad \text{for any } a = \langle a_1, a_2, \ldots \rangle \in \{0, 1\}^\mathbb{N}
\]

gives us a homeomorphism between $\{0, 1\}^\mathbb{N}$ with Tychonoff topology and middle-third Cantor set. In this example we will not distinguish between $T$ and $\{0, 1\}^\mathbb{N}$ with appropriate topologies.

For any $a = \langle a_1, a_2, \ldots \rangle \in \{0, 1\}^\mathbb{N} = T$ consider the shift map $\sigma$ given by the formula (8):

$$\sigma(a) = \langle a_2, a_3, \ldots \rangle.$$ 

Since $T$ is a closed subspace of $[0, 1]$, by Tietze’s extension theorem it can be extended to some continuous function $f_0: [0, 1] \to [0, 1]$.

Let

$$S = \left\{ \frac{1}{2}, 1, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{9}, \ldots \right\};$$

that is, it is the set of centers of most left intervals removed from $[0, 1]$ during the classical construction of the middle-third Cantor set. Since $\sigma(0) = 0$ and $\sigma$ is continuous, we can assume also that $f_0(s) = 0$ for each $s \in S$ (we can multiply original $f_0$ by the continuous function which is equal to identity on $T$ and equals 0 on $S$).
Let \( M = t = 1, C = [0, 1], U = \{0\} \). Define \( f: C \times U \to C \) by the formula
\[
f(x_0, u_0) = f_0(x_0).
\]
Additionally, let \( \zeta^0 \in \{0, 1\}^\mathbb{N} = T \subset C \) be given by the formula
\[
\zeta^0 = (1, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, \ldots)
\]
(the sequence of \( n \) zeros and one, followed by \( n + 1 \) zeros and one, and so on). In terms of mapping \([1]\),
\[
\zeta^0 = 2 \cdot \sum_{i=2}^{\infty} \left( \frac{1}{3} \right)^{i-1} \in [0, 1].
\]
Let \( P(x_0) = x_0 \) for each \( x_0 \in [0, 1] \), and \( \phi: [0, 1] \to [0, 1] \) be a continuous function such that \( \phi(x_0) = 1 \) for \( x_0 \in S \cup \{0\} \), and \( \phi(x_0) < 1 \) otherwise.

Note that for the problem defined in Section 3:
\begin{itemize}
\item 0 \in M and \( S \cap M = \emptyset \);
\item \( \zeta^* = 0, M^* = \{ \zeta^* \}; \)
\item \( D^* = S \cup \{0\} \).
\end{itemize}

Thus
\begin{enumerate}
\item the condition \((C1)\) holds: the optimal stationary point \( \zeta^* \) is unique, i.e. \( M^* = \{ \zeta^* \} \);
\item the condition \((C3)\) holds: for every \( \zeta \in S \) and \( u \in U \),
\[
P(f(\zeta, u)) = f(\zeta, u) = 0 < \zeta = P(\zeta) \quad \text{and} \quad P(f(\zeta^*, u)) = f(\zeta^*, u) = 0 = \zeta^* = P(\zeta^*).
\]
\end{enumerate}

Observe, that for any path \( x \) for the system \((*)\):
\begin{itemize}
\item \( \Gamma(x) = \{0, \langle 1, 0, 0, 0, \ldots \rangle, \langle 0, 1, 0, 0, 0, \ldots \rangle, \langle 0, 0, 1, 0, 0, \ldots \rangle, \ldots \} \); in terms of mapping \([1]\), \( \Gamma(x) = \{0, 2/3, 2/9, 2/27, \ldots \} \).
\item \( \Gamma_{x_0}(x) = \{0\} = I_{x_0} \lim x. \)
\end{itemize}

Therefore, the condition \((I_3/C2)\) holds (take \( x^* = (\zeta^*, \sigma(\zeta^*), \sigma(\sigma(\zeta^*)), \ldots) \)), but \((C2) = (\text{Fin}/C2) \) does not hold. \( \square \)

4. Main results

The main result of this paper is presented next. The proof of this theorem is provided in Section 5.

Theorem 14. Suppose that \( I \) is invariant under translations, \((C1), (I_3/C2), (C3)\) hold and \((\mathfrak{u}_{\text{opt}}, x_{\text{opt}})\) is an optimal process in the problem \((*)\), \((I_3/\**\)**). Then \( x_{\text{opt}} \to_I \zeta^* \), where \( \zeta^* \) is the unique optimal stationary point from \((C1)\).

Note that from part \((3)\) of Proposition 5 the assertion \( "x_{\text{opt}} \to_I \zeta^*" \) is equivalent to \( "\Gamma_I(x) = \{\zeta^*\}" \).

It is also easy to see that the assertion of Theorem 14 is true if \( D^* \) is a singleton (i.e. \( D^* = \{\zeta^*\} \)). However, the following example shows that if \( \zeta^* \) is an isolated point of \( D^* \) (if we assume only the first part of condition \((C3)\)) then Theorem 14 may not be true.

Example 15. Let \( I = \text{Fin}, m = 1, C = U = [0, 1], \) and for each \( x \in C \):
\[
f_0(x) = \begin{cases} 
x & \text{for } 0 \leq x \leq \frac{1}{2} + \delta; \\
\frac{x - (\frac{1}{2} + \delta)}{\frac{1}{2} - (\frac{1}{2} + \delta)} + \left(\frac{1}{2} + \delta\right) & \text{for } \frac{1}{2} + \delta < x \leq 1,
\end{cases}
\]

\[\]
Figure 2. Graph of the quantity $\phi$ and $f_0, f_1$ for Example 15.

$$f_1(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq \frac{1}{3}; \\
\frac{\left((\frac{2}{3} - \delta) - 1\right)(x - \frac{1}{4}) + 1}{\frac{1}{2} - \frac{5}{9}} & \text{for } \frac{1}{3} < x \leq \frac{2}{3} - \delta; \\
\frac{\left((\frac{2}{3} - \delta) - \frac{2}{3}\right)(x - (\frac{2}{3} - \delta)) + (\frac{2}{3} - \delta)}{1 - (\frac{2}{3} - \delta)} & \text{for } \frac{2}{3} - \delta < x \leq 1,
\end{cases}$$

where $\delta < \frac{1}{12}$ (for example, on Figure 2, $\delta = 0.05$). Define $f : C \times U \to C$ by the affine formula

$$f(x, u) = f_0(x) \cdot (1 - u) + f_1(x) \cdot u.$$ 

Additionally, let $\zeta^0 = \frac{1}{3}$ and $P(x) = x$ for each $x \in C$. For the definition of $\phi$ and visualization of $f_0, f_1$ see Figure 2.

Note that for the problem defined in Section 3:

1. optimal stationary point $\zeta^* = \frac{1}{3}$ is unique;
2. the condition (C2) also holds, for example for $x^* = (\zeta^0, \zeta^0, \zeta^0, \ldots)$, $u^* = (0, 0, 0, \ldots)$;
3. the first part of condition (C3) holds: for every $\zeta \in \left[\frac{2}{3}, 1\right]$ and $u \in U$,

$$P(f(\zeta, u)) = f(\zeta, u) \leq f_1(\zeta) < \zeta = P(\zeta).$$ 

In this example, the process $(u^{opt}, x^{opt})$, where $x^{opt} = (\frac{1}{3}, 1, \frac{1}{3}, 1, \frac{1}{3}, 1, \ldots)$ and $u^{opt} = (0, 1, 0, 1, 0, 1, \ldots)$, is an optimal process, however $x^{opt}$ does not converge to $\zeta^*$ in the sense of $I$-convergence (which is equivalent to Fin-convergence).

Example 15 works for classical convergence, statistical convergence, and for general ideal convergence. It shows that additional assumption about “density” of $D^*$ in $\zeta^*$ (i.e. the second part of condition (C3)) is necessary in [7], as well as in [25].

Recently Leonetti and Caprio in [19] proposed another way to bypass the problem indicated in the Example 15.
(C3-LC): there exists a linear (and therefore continuous) function $P: \mathbb{R}^m \to \mathbb{R}$ such that
\[ P(f(x_0, u_0)) < P(x_0) \text{ for all } x_0 \in D^*, u_0 \in U, \langle x_0, f(x_0, u_0) \rangle \neq \langle \zeta^*, \zeta^* \rangle, \]
where $\zeta^*$ is an optimal stationary point. It follows from the above condition that $\zeta^*$ is the unique optimal stationary point, and it is easy to see that (C3-LC) imply (C3). However, we do not have any example of the system with (C1) + (C3) and without (C3-LC).

4.1. Special cases. In this section, we consider two special cases of the ideal convergence; that is, classical convergence and statistical convergence.

Classical convergence. Consider the classical convergence in the problem $(\ast)$, $(\mathcal{I}/* \ast)$. In this case
\[ \Gamma(x) = \Gamma_{\text{Fin}}(x) = \{ a \in \mathbb{R}^m : (x_{n_k})_{k \in \mathbb{N}} \to a \text{ for some subsequence } (x_{n_k}) \text{ of } x \} \]
is the set of $\omega$-limit points. Condition $(\mathcal{I}/C2)$ is in the form $(C2)$ and functional $(\mathcal{I}/* \ast)$ is represented in the form
\[ (**): J(x) = J_{\text{Fin}}(x) = \liminf_{k \to \infty} \phi(x_k) \to \max. \]

Corollary 16. Let $(C1)$, $(C2)$, $(C3)$ hold and $\langle u^{opt}, x^{opt} \rangle$ is an optimal process in the problem $(\ast), (**)$ = $(\text{Fin}/* \ast)$. Then $x^{opt}$ converges to $\zeta^*$.

Statistical convergence. Now consider the statistical convergence instead of ideal convergence in the problem $(\ast), (\mathcal{I}/* \ast)$. Functional $(\mathcal{I}/* \ast) = (\mathcal{I}_d/* \ast)$ in this case can be defined as follows
\[ (\mathcal{I}_d/* \ast): J_{\mathcal{I}_d}(x) = C - \liminf_{k \to \infty} \phi(x_k) \to \max, \]
where $C - \liminf_{k \to \infty} \phi(x_k) = \mathcal{I}_d - \liminf x$ stands for the minimal element in the set of statistical cluster points. Recall also that according to Example 13, condition $(C2)$ is stronger than $(\mathcal{I}_d/C2)$.

Corollary 17. Let $(C1)$, $(\mathcal{I}_d/C2)$, $(C3)$ hold and $\langle u^{opt}, x^{opt} \rangle$ is an optimal process in the problem $(\ast), (\mathcal{I}_d/* \ast)$. Then $x^{opt}$ statistically converges to $\zeta^*$.

5. Proof of Theorem 14

For every $r \in \mathbb{R}$ define the set
\[ D_r = \{ \zeta \in C : \phi(\zeta) \geq r \}. \]
Clearly, $D^* = D_{\phi^*}$. For any continuous function $P: \mathbb{R}^m \to \mathbb{R}$ let
\[ E_P = \{ \zeta \in \mathbb{R}^m : P(f(\zeta, u_0)) < P(\zeta) \text{ for all } u_0 \in U \}, \]
and
\[ \overline{E}_P = \{ \zeta \in \mathbb{R}^m : P(f(\zeta, u_0)) \leq P(\zeta) \text{ for all } u_0 \in U \}. \]
It is clear that $M \cap E_P = \emptyset$. If $A \subset \mathbb{R}^m$ is compact then
\[ \arg \min_{\zeta \in A} P(\zeta) = \left\{ \zeta_1 \in A : P(\zeta_1) = \min_{\zeta \in A} P(\zeta) \right\}. \]
Analogously we define operator arg max.
Lemma 18. Assume that \( \mathcal{I} \) is invariant under translations, \( r \in \mathbb{R} \) and \( \langle u, x \rangle \) is a process in the problem (\( * \)), \((\mathcal{I}/* \)) with \( J_{\mathcal{I}}(x) \geq r \). If \( P: \mathbb{R}^m \to \mathbb{R} \) is a continuous function then

\[
\arg \min_{\zeta \in \Gamma_{\mathcal{I}}(x)} P(\zeta) \subset D_r \setminus E_P.
\]

Proof. As \( J_{\mathcal{I}}(x) \geq r \), by Lemma 12, \( J_{\mathcal{I}}(x) = \min_{\zeta \in \Gamma_{\mathcal{I}}(x)} \phi(\zeta) \geq r \). Thus \( \Gamma_{\mathcal{I}}(x) \subset D_r \), and so \( \arg \min_{\zeta \in \Gamma_{\mathcal{I}}(x)} P(\zeta) \subset D_r \).

Let

\[
F(\zeta) = \max_{u_0 \in U} P(f(\zeta, u_0)) - P(\zeta).
\]

It is clear that \( F: \mathbb{R}^m \to \mathbb{R} \) is continuous, and

\[
F(\zeta) < 0 \quad \text{for all} \quad \zeta \in E_P.
\]

Suppose that there exists \( \zeta_1 \in \Gamma_{\mathcal{I}}(x) \) such that \( \zeta_1 \in E_P \) and

\[
\min_{\zeta \in \Gamma_{\mathcal{I}}(x)} P(\zeta) = P(\zeta_1).
\]

Denote \( \delta = -F(\zeta_1)/8 \). Clearly \( \delta > 0 \) thanks to (2).

Since functions \( F \) and \( P \) are continuous and \( \Gamma_{\mathcal{I}}(x) \) is a compact set, there exists \( \gamma > 0 \) such that

\[
(3) \quad \forall \zeta \in B(\zeta_1, \gamma) F(\zeta) \leq -4\delta, \quad \text{and}
\]

\[
(4) \quad \forall \zeta \in B(\zeta_1, \gamma) P(\zeta) \leq P(\zeta_1) + \delta, \quad \text{and}
\]

\[
(5) \quad \forall \zeta \in B(\Gamma_{\mathcal{I}}(x), \gamma) P(\zeta) \geq \min_{y \in \Gamma_{\mathcal{I}}(x)} P(y) - \delta.
\]

If \( x_k \in B(\zeta_1, \gamma) \) then \( F(x_k) \leq -4\delta \), i.e. \( P(f(x_k, u_0)) \leq P(x_k) - 4\delta \) for each \( u_0 \in U \) and in particular for \( u_k \in U \) that leads to \( P(x_{k+1}) \leq P(x_k) - 4\delta \). Moreover, from (4) we have \( P(x_k) \leq P(\zeta_1) + \delta \) and therefore

\[
P(x_{k+1}) \leq P(\zeta_1) - 3\delta.
\]

On the other hand, (5) implies

\[
\forall \zeta \in B(\Gamma_{\mathcal{I}}(x), \gamma) P(\zeta) \geq \min_{y \in \Gamma_{\mathcal{I}}(x)} P(y) - \delta = P(\zeta_1) - \delta > P(\zeta_1) - 3\delta.
\]

Thus \( x_{k+1} \notin B(\Gamma_{\mathcal{I}}(x), \gamma) \). By the above considerations we get

\[
(6) \quad x_k \in B(\zeta_1, \gamma) \implies x_{k+1} \notin B(\Gamma_{\mathcal{I}}(x), \gamma).
\]

This contradicts with Lemma 11. □

Lemma 19. Assume that \( \mathcal{I} \) is invariant under translations, \( r \in \mathbb{R} \) and \( \langle u, x \rangle \) is a process in the problem (\( * \)), \((\mathcal{I}/* \)) with \( J_{\mathcal{I}}(x) \geq r \). If \( P: \mathbb{R}^m \to \mathbb{R} \) is a continuous function and \( D_r \setminus E_P \subset E_{P} \) then

\[
\arg \max_{\zeta \in \Gamma_{\mathcal{I}}(x)} P(\zeta) \cap (D_r \setminus E_P) \neq \emptyset.
\]

Proof. As in the proof of Lemma 18 observe that \( \Gamma_{\mathcal{I}}(x) \subset D_r \setminus E_P \), \( \arg \max_{\zeta \in \Gamma_{\mathcal{I}}(x)} P(\zeta) \subset D_r \), and define

\[
F(\zeta) = \max_{u_0 \in U} P(f(\zeta, u_0)) - P(\zeta).
\]

Again, \( F: \mathbb{R}^m \to \mathbb{R} \) is continuous, and

\[
(7) \quad F(\zeta) < 0 \quad \text{for all} \quad \zeta \in E_P.
\]
and assume (contrary to the lemma assertion) that
\[ \min_{\zeta \in \Gamma(x)} P(\zeta) < \min_{\zeta \in Z_2} P(\zeta) \] (in fact, \( P \upharpoonright Z_2 \) is constant and equal to the maximum value of \( P \) on \( \Gamma(x) \); if it is equal to \( \max_{\zeta \in Z_1} P(\zeta) \) then it follows from the definition of \( Z_2 \) that \( Z_1 \cap Z_2 \neq \emptyset \). Since (by the assumption of the lemma) \( D_r \setminus E_P \subseteq \Sigma \), it gives us \( F \upharpoonright Z_1 = 0 \).

Denote also
\[ p_1 = \max_{\zeta \in Z_1} P(\zeta), \quad p_2 = \min_{\zeta \in Z_2} P(\zeta). \]

Let
\[ a = \frac{p_2 - p_1}{8} > 0. \]

1. Since functions \( F, P \) are continuous and \( F \upharpoonright Z_1 = 0 \), there exists \( \gamma > 0 \) such that
\[ \forall_{\zeta \in B(Z_1, \gamma)} F(\zeta) \leq 4a, \quad \text{and} \]
\[ \forall_{\zeta \in B(Z_1, \gamma)} P(\zeta) \leq p_1 + a, \quad \text{and} \]
\[ \forall_{\zeta \in B(Z_2, \gamma)} P(\zeta) \geq p_2 - a. \]

Let \( x_{k-1} \in B(Z_1, \gamma) \). Then from (11), (12) the following two relations hold
\[ P(x_k) - P(x_{k-1}) = P(f(x_{k-1}, u_{k-1})) - P(x_{k-1}) \leq F(x_{k-1}) \leq 4a; \]
\[ P(x_{k-1}) \leq p_1 + a. \]

From these inequalities we have
\[ P(x_k) \leq p_1 + 5a. \]

From (10) it follows \( p_1 = p_2 - 8a \) and then
\[ P(x_k) \leq p_2 - 8a + 5a < p_2 - a. \]

According to (13) this means that \( x_k \notin B(Z_2, \gamma) \). Therefore we conclude that
\[ x_k \in B(Z_2, \gamma) \implies x_{k-1} \notin B(Z_1, \gamma). \]

2. We fix the number \( \gamma \) and consider the set
\[ \Gamma = \Gamma(x) \setminus B(Z_1, \gamma). \]

From (10), (12) and (13) and the fact that \( Z_2 \subseteq \Gamma(x) \) it follows that \( Z_2 \subseteq \Gamma \). Moreover, \( \Gamma \subseteq D_r \cap E_P \), and (7) implies that \( F(\zeta) < 0 \) for all \( \zeta \in \Gamma \). Denote
\[ \delta = \max_{\zeta \in \Gamma} F(\zeta) > 0. \]

Take any number \( \varepsilon > 0 \) satisfying
\[ 4\varepsilon < \delta. \]
Since functions $F$, $P$ are continuous there exists a sufficiently small number $\eta \in (0, \gamma)$ such that

\begin{align}
\forall \zeta \in B(\Gamma, \eta) & F(\zeta) \leq -\delta + \varepsilon, \text{ and} \\
\forall \zeta \in B(Z_2, \eta) & P(\zeta) \geq p_2 - \varepsilon, \text{ and} \\
\forall \zeta \in B(\Gamma_I(x), \eta) & P(\zeta) \leq p_2 + \varepsilon. 
\end{align}

We show by contradiction that there is no $k$ such that

\begin{align}
x_k & \in B(Z_2, \eta) \text{ and } x_{k-1} \in B(\Gamma, \eta).
\end{align}

Suppose that $k$ fulfills (20). From (17) we have

\begin{align*}
P(x_k) - P(x_{k-1}) \leq F(x_{k-1}) \leq -\delta + \varepsilon,
\end{align*}

or

\begin{align*}
P(x_{k-1}) \geq P(x_k) + \delta - \varepsilon.
\end{align*}

Then, from (18) it follows

\begin{align*}
P(x_{k-1}) \geq (p_2 - \varepsilon) + \delta - \varepsilon = p_2 + \delta - 2\varepsilon,
\end{align*}

and by (16)

\begin{align*}
P(x_{k-1}) > p_2 + 2\varepsilon.
\end{align*}

On the other hand (19) yields

\begin{align*}
P(x_{k-1}) \leq p_2 + \varepsilon.
\end{align*}

The last two inequalities lead to a contradiction. This proves that the relations $x_k \in B(Z_2, \eta)$ and $x_{k-1} \in B(\Gamma, \eta)$ cannot be satisfied at the same time. Therefore the following is true:

\begin{align}
x_k \in B(Z_2, \eta) \Rightarrow x_{k-1} /\in B(\Gamma, \eta).
\end{align}

3. Now since $\eta < \gamma$, it is not difficult to observe that the relation $B(\Gamma_I(x), \eta) \subset B(\Gamma, \eta) \cup B(Z_1, \gamma)$ holds. Then (14) and (21) implies that

\begin{align}
x_k \in B(Z_2, \eta) \Rightarrow x_{k-1} /\in B(\Gamma_I(x), \eta).
\end{align}

The above implication contradicts with Lemma 11. \hfill \Box

**Proof of Theorem 14.** Let $r = \phi^*$. Then $D_r = D_{\phi^*} = D^*$. By (I/C2) for the process $\langle u^*, x^* \rangle$, $J_I(x^*) \geq r$.

Fix the function $P$ like in (C3). Then

\[ D_r \setminus E_P = D^* \setminus E_P \subset M^* \subset \overline{E_P} \setminus E_P. \]

Since $J_I(x^*) = r$, the maximal value of the functional $(I/\ast \ast)$ is not less than $r$. As $\langle u^{opt}, x^{opt} \rangle$ is an optimal process, $J_I(x^{opt}) \geq r$. Thus, by Lemma 18

\[ \arg \min_{\zeta \in \Gamma_I(x^{opt})} P(\zeta) \subset D^* \setminus E_P \subset M^* = \{ \zeta^* \}, \]

where $\zeta^*$ is the unique optimal stationary point from (C1). Then, by Lemma 19

\[ \zeta^* \in \arg \max_{\zeta \in \Gamma_I(x^{opt})} P(\zeta) \cap M^*. \]
Thus $P(\zeta) = P(\zeta^*)$ for all $\zeta \in \Gamma(x^{opt})$. It follows that

$$\Gamma(x^{opt}) = \text{arg min}_{\zeta \in \Gamma(x^{opt})} P(\zeta) \subset M^* = \{\zeta^*\}.$$ 

From part (3) of Proposition we obtain $x^{opt} \rightarrow I \zeta^*$. □

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References

[1] Nicolas Bourbaki. *Éléments de mathématique. Part I. Les structures fondamentales de l'analyse. Livre III. Topologie générale. Chapitres I et II*. Actual. Sci. Ind., no. 858. Hermann & Cie., Paris, 1940.

[2] D.A. Carlson, A.B. Haurie, and A. Leizarowitz. *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*. Springer-Verlag, Berlin, 1991, 2nd edition.

[3] Henri Cartan. *Filtres et ultrafiltres*. C. R. Acad. Sci. Paris, 205:777–779, 1937.

[4] D. Cass and K. Shell. The structure and stability of competitive dynamical systems. *J. Econ. Theory*, 12:31–70, 1976.

[5] Juraj Činčura, Tibor Šalát, Martin Slziak, and Vladimír Toma. Sets of statistical cluster points and $I$-cluster points. *Real Anal. Exchange*, 30(2):565–580, 2004/05.

[6] T. Damm, L. Gröne, M. Stieler, and K. Worthmann. An exponential turnpike theorem for averaged optimal control. to appear in: *SIAM Journal on Control and Optimization*.

[7] Pratulananda Das, Sudipta Dutta, S. A. Mohiuddine, and Abdullah Alotaibi. $A$-Statistical Cluster Points in Finite Dimensional Spaces and Application to Turnpike Theorem. *Abstr. Appl. Anal.*, pages Art. ID 354846, 7, 2014.

[8] R.L. Devaney. An introduction to chaotic dynamical systems. Benjamin-Cummings, Menlo Park, 1986.

[9] R. Dorfman, P.A. Samuelson, and R.M. Solow. *Linear programming and economic analysis*. New York: McGraw-Hill, 1958.

[10] H. Fast. Sur la convergence statistique. *Colloquium Math.*, 2:241–244 (1952), 1951.

[11] Rafał Filipów and Piotr Szuca. Density versions of Schur’s theorem for ideals generated by submeasures. *J. Combin. Theory Ser. A*, 117(7):943–956, 2010.

[12] J. A. Fridy. Statistical limit points. *Proc. Amer. Math. Soc.*, 118(4):1187–1192, 1993.

[13] A.F. Ivanov, M.A. Mammadov, and S.I. Trofimchuk. Global stabilization in nonlinear discrete systems with time-delay. *Journal of Global Optimization*, 56(2):251–263, 2013.

[14] Winfried Just and Adam Krawczyk. On certain Boolean algebras $P(\omega)/I$. *Trans. Amer. Math. Soc.*, 285(1):411–429, 1984.

[15] Menachem Kojman. van der Waerden spaces. *Proc. Amer. Math. Soc.*, 130(3):631–635 (electronic), 2002.

[16] V. Kokkotis and W. Yang. Turnpike theorems for Markov games. *Dynamic Games and Applications*, 2(3):294–312, 2012.

[17] Pavel Kostyrko, Tibor Šalát, and Władysław Wilczyński. $T$-convergence. *Real Anal. Exchange*, 26(2):669–685, 2000/01.

[18] A. Leizarowitz. Optimal trajectories on infinite horizon deterministic control systems. *Appl. Math. and Opt.*, 19:11–32, 1989.

[19] Paolo Leonetti and Michele Caprio. Turnpike in infinite dimension. *Canad. Math. Bull.*, 65(2):416–430, 2022.

[20] A. N. Lyapunov. Asymptotical optimal paths for convex mappings. *Optimal Models in System Analysis, Moskow, VNII*, (9):74–80, 1983.

[21] M.J.P. Magill and J.A. Scheinkman. Stability of regular equilibria and the correspondence principle for symmetric variational problems. *International Economic Review*, pages 297–315, 1979.

[22] V.L. Makarov and A.M. Rubinov. *Mathematical theory of economic dynamics and equilibria*. Springer-Verlag, New York, 1977.

[23] M.A. Mamedov. Asymptotical optimal paths in models with environment pollution being taken into account. *Optimization (Novosibirsk)*, 36(53):101–112, (in Russian), 1985.
[24] M.A. Mamedov. Turnpike theorems in continuous systems with integral functionals. English transl. In: Russian Acad. Sci. Dokl. Math., 45(2):432–435, 1992.
[25] M.A. Mamedov and S. Pehlivan. Statistical cluster points and turnpike theorem in nonconvex problems. Journal of mathematical analysis and applications, 256(2):686–693, 2001.
[26] M.A. Mammadov. Turnpike theorem for an infinite horizon optimal control problem with time delay. SIAM Journal on Control and Optimization, 52(1):420–438, 2014.
[27] Krzysztof Mazur. $F_2$-ideals and $\omega_{1}\omega_{1}^*$-gaps in the Boolean algebras $P(\omega)/I$. Fund. Math., 138(2):103–111, 1991.
[28] L.W. McKenzie. Turnpike theory. Econometrica: Journal of the Econometric Society, pages 841–865, 1976.
[29] L. Montrucchio. A turnpike theorem for continuous-time optimal-control models. Journal of Economic Dynamics and Control, 19(3):599–619, 1995.
[30] J.V. Neumann. A model of general economic equilibrium. Review of Economic Studies, 13:1–9, 1945–46.
[31] Fatih Nuray and William H. Ruckle. Generalized statistical convergence and convergence free spaces. J. Math. Anal. Appl., 245(2):513–527, 2000.
[32] S. Pehlivan and M.A. Mamedov. Statistical cluster points and turnpike. Optimization, 48(1):91–106, 2000.
[33] R.T Rockafellar. Saddle points of hamiltonian systems in convex problems of lagrange. Journal of Optimization Theory and Applications, 12(4):367–390, 1973.
[34] R.T Rockafellar. Saddle points of hamiltonian systems in convex lagrange problems having a nonzero discount rate. Journal of Economic Theory, 12(1):71–113, 1976.
[35] J.A. Scheinkman. On optimal steady states of n-sector growth models when utility is discounted. J. Econ. Theory, 12:11–30, 1976.
[36] A.J. Zaslavski. Turnpike properties in the calculus of variations and optimal control. Springer, 2006.
[37] A.J. Zaslavski. A turnpike property of approximate solutions of an optimal control problem arising in economic dynamics. Dynamic Systems and Applications, 20(2-3):395–422, 2011.
[38] A.J. Zaslavski. Necessary and sufficient conditions for turnpike properties of solutions of optimal control systems arising in economic dynamics. Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms, 20(4):391–420, 2013.
[39] A.J. Zaslavski. Turnpike properties of approximate solutions in the calculus of variations without convexity assumptions. Communications on Applied Nonlinear Analysis, 20(1):97–108, 2013.

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