Finite Difference Schemes for Linear Stochastic Integro-Differential Equations

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Abstract

We study the rate of convergence of an explicit and an implicit-explicit finite difference scheme for linear stochastic integro-differential equations of parabolic type arising in non-linear filtering of jump-diffusion processes. We show that the rate is of order one in space and order one-half in time.

Keywords: Finite differences, Lévy processes, SPDEs, Integro-differential

1. Introduction

Let $\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, be a complete filtered probability space such that the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. Let $\{w^\vartheta\}_{\vartheta = 1}^{\infty}$ be a sequence of independent real-valued $\mathbb{F}$-adapted Wiener processes and let $\tilde{\mu}(dz, dt) = \mu(dz, dt) - \nu(dz)dt$ be a compensated $\mathbb{F}$-adapted Poisson random measure on $\mathbb{R}^d \times \mathbb{R}_+$, where $\nu(dz)$ is a Borel $\sigma$-finite measure on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} |z|^2 \wedge |z| \nu(dz) < \infty. \quad (1)$$

Let $T > 0$ be an arbitrary fixed constant. On $[0, T] \times \mathbb{R}^d$, we consider finite difference approximations for the following stochastic integro-differential

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equation (SIDE)

\[
du_t = ((L_t + I)u_t + f_t) \, dt + \sum_{\varrho=1}^{\infty} (M_{t,\varrho}^\varrho u_t + g_{t,\varrho}^\varrho) \, dw_t^\varrho + \int_{\mathbb{R}^d} J(z)u_{t-\tilde{\mu}}(dz, dt)
\]

(2)

with initial condition

\[u_0(x) = \psi(x), \quad x \in \mathbb{R}^d,\]

where the operators are given by

\[
L_t \phi(x) := \sum_{i,j=0}^{d} a_{t}^{ij}(x) \partial_i \partial_j \phi(x),
\]

\[
I \phi(x) := \int_{\mathbb{R}^d} \left( \phi(x+z) - \phi(x) - \sum_{j=1}^{d} z_j \partial_j \phi(x) \right) \nu(dz),
\]

(3)

\[
M_t^\varrho \phi(x) := \sum_{i=0}^{d} b_{t,\varrho}^i \partial_i \phi(x), \quad J(z) \phi(x) = \phi(x+z) - \phi(x).
\]

In the above, we denote the identity operator by \( \partial_0 \).

Equation (2) arises naturally in non-linear filtering of jump-diffusion processes. We refer the reader to [6] and [7] for more information about non-linear filtering of jump-diffusions and the derivation of the Zakai equation. Various methods have been developed to solve stochastic partial differential equations (SPDEs) (e.g. (2) with \( \nu \equiv 0 \)) numerically. For SPDEs driven by continuous martingale noise see, for example, [5], [8], [9], [14], [21], [20] and [23] and for SPDEs driven by discontinuous martingale noise, see [18], [19], [1], and [2]. Among the various methods considered in the literature is the method of finite differences. For second order linear SPDEs driven by continuous martingale noise it is well-known that the \( L^p(\Omega) \)-pointwise error of approximation in space is proportional to the parameter \( h \) of the finite difference (see, e.g., [24]). In [14], I. Gyöngy and A. Millet consider abstract discretization schemes for stochastic evolution equations driven by continuous martingale noise in the variational framework and, as a particular example, show that the \( L^2(\Omega) \)-pointwise rate of convergence of an Euler-Maruyuma (explicit and implicit) finite difference scheme is of order one in space and one-half in time. More recently, it was shown by I. Gyöngy and N.V. Krylov
that under certain regularity conditions, the rate of convergence in space of a semi-discretized finite difference approximation of a linear second order SPDE driven by continuous martingale noise can be accelerated to any order by Richardson’s extrapolation method. For the non-degenerate case, we refer to [12] and [13], and for the degenerate case, we refer to [10]. In [16] and [17], E. Hall proved that the same method of acceleration can be applied to implicit time-discretized SPDEs driven by continuous martingale noise.

While finite difference schemes for SPDEs driven discontinuous martingale noise have not been explicitly considered in the literature, finite element, spectral, and, more generally, Galerkin schemes have. One of the earliest works in this direction is a paper [18] by E. Hausenblas and I. Marchis concerning $L^p(\Omega)$-convergence of Galerkin approximation schemes for abstract stochastic evolution equations in Banach spaces driven by Poisson noise of impulsive-type. As an application of their result, they study a spectral approximation of a linear SPDE (i.e. $\nu \equiv 0$) in $L^2([0,1])$ with Neumann boundary conditions driven by Poisson noise of impulsive-type and derive $L^p(\Omega)$-error estimates in the $L^2([0,1])$-norm. In [19], E. Hausenblas considers finite element approximations of linear SPDEs in polyhedral domains $D$ driven by Poisson noise of impulsive-type and derives $L^p(\Omega)$ error estimates in the $L^p(D)$-norm. In a more recent work [1], A. Lang studied semi-discrete Galerkin approximation schemes for SPDEs of advection diffusion type (i.e. $\nu \equiv 0$) in bounded domains $D$ driven by càdlàg square integrable martingales in a Hilbert Space. A. Lang showed that the rate of convergence in the $L^p(\Omega)$ and almost-sure sense in the $L^2(D)$-norm is of order two for a finite-element Galerkin scheme. In [2], A. Lang and A. Barth derive $L^2(\Omega)$ and almost-sure estimates in the $L^2(D)$-norm for the error of a Milstein-Galerkin approximation scheme for the same equation considered in [1] and obtain convergence of order two in space and order one in time. It is worth mentioning that in the articles [1], [2], [18], and [19], the authors make use of the semigroup theory of SPDEs (mild solution) and only consider SPDEs with the principal part of the operator in the drift non-random. The principal part of the operator in the drift of the Zakai equation is, in general, random-adapted, and hence numerical schemes that approximate SPDEs or SIDEs with random-adapted principal part are of importance. In this paper, since we use the variational framework ($L^2$-theory) of SPDEs, we are easily able to treat the case of random-coefficients, and hence the diffusion coefficients $a_{ij}^t(x)$ appearing in (2) are random-adapted.

In dimension one, a finite difference scheme for deterministic degenerate
integro-differential equations has been studied by R. Cont and E. Voltchkova in [3]. The authors in [3] first approximate the integral operator near the origin with a second derivative operator. The resulting PDE is then non-degenerate and has an integral operator of order zero. The error of this approximation is obtained by means of the probabilistic representation of the solution of both the original equation and the non-degenerate equation. In the second step of their approximation, R. Cont and E. Voltchkova consider an implicit-explicit finite difference scheme and obtain pointwise error estimates of order one in space. As a consequence of the two-step approximation scheme, there are two separate errors for the approximation.

In this paper, we consider the non-degenerate stochastic integro-differential equation (2) with random coefficients and apply the method of finite differences in the time and space variables. To the best of our knowledge, this article is the first to use the finite difference method to approximate stochastic integro-differential equations with random coefficients. The approximations of the non-local integral operators in the drift and in the noise of (2) we choose are both natural and easy to implement. In particular, we are able to treat the singularity of the integral operators near the origin directly, unlike [3]. We consider a fully-explicit time-discretization scheme and an implicit-explicit time-discretization scheme, where we treat part of the approximation of the integral operator in the drift explicitly. To obtain error estimates for our approximations, we use the approach in [24], where the discretized equations are first solved as time-discretized SDE’s in Sobolev spaces over \( \mathbb{R}^d \) and an error estimate is obtained in Sobolev norms. After obtaining \( L^2(\Omega) \) error estimates in Sobolev norms, the Sobolev embedding theorem is used to obtain \( L^2(\Omega) \)-pointwise error estimates. Using this approach, we are easily able to deduce that the more regularity on the coefficients and data we have, the stronger the error estimates we can obtain.

The paper is organized as follows. In the next section, we introduce the notation that will be used throughout the paper and state the main results. In the third section, we prove auxiliary results that will be used in the proof of the main theorems. In the fourth section, we prove the main theorems of the paper.

2. Notation and the main results

For \( x \in \mathbb{R}^d \), denote by \( |x| \) the Euclidean norm of \( x \). For \( i \in \{1, \ldots, d\} \), let \( \partial_{-i} = -\partial_i \), and let \( \partial_0 \) be the identity. For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \), let
\( \mathbb{N} \cup \{0\} \) of length \( |\alpha| = \alpha_1 + \ldots + \alpha_d \), set \( \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \). Let \( \ell_2 \) be the space of all square-summable real sequences \( b = (b^e)^\infty_{e=1} \). For an \( \ell_2 \)-valued function \( f \) on \( \mathbb{R}^d \), the derivative of \( f \) with respect to \( x^i \) is denoted by \( \partial_i f \).

Let \( C_c^\infty(\mathbb{R}^d) \) be the space of all smooth real-valued functions on \( \mathbb{R}^d \) with compact support. We write \((\cdot, \cdot)_0 \) for the inner product and \( \| \cdot \|_0 \) for the norm in \( L_2(\mathbb{R}^d) =: H^0 \). For \( m \in \mathbb{N} \), denote by \( H^m \) the Sobolev space of all functions \( u \in L_2(\mathbb{R}^d) \) having distributional derivatives up to order \( m \) in \( L_2(\mathbb{R}^d) \). We denote by

\[
(\cdot, \cdot)_m := \sum_{|\alpha| \leq m} (\partial^\alpha, \partial^\alpha)_0
\]

the inner product in \( H^m \) and by \( \| \cdot \|_m \) the corresponding norm. Define \( H^{-1} \) to be the completion of \( C_c^\infty(\mathbb{R}^d) \) with respect to the norm \( \| \cdot \|_{-1} = \|(1 - \Delta)^{-1/2} \cdot \|_0 \), where \( \Delta \) is the Laplace operator. It is easy to see that for all \( u \in H^1 \) and \( v \in H^0 \), \( (u,v)_0 \leq \|u\|_1 \|v\|_{-1} \). Since \( H^1 \) is dense in \( H^{-1} \), we may define the pairing \( [\cdot, \cdot]_0 : H^1 \times H^{-1} \to \mathbb{R} \) by \( [v, v']_0 = \lim_{n \to \infty} (v, v_n)_0 \) for all \( v \in H^1 \) and \( v' \in H^{-1} \), where \( (v_n)_{n=1}^\infty \subset H^1 \) is such that \( \|v_n - v'\|_{-1} \to 0 \) as \( n \to \infty \). The mapping from \( H^{-1} \) to \( (H^1)^* \) given by \( v' \mapsto [\cdot, v']_0 \) is an isometric isomorphism. For more details, see [22]. For an integer \( m \geq 0 \), we write \( H^m(\ell_2) \) for the space of all \( \ell_2 \)-valued functions \( g(x) = (g^e(x))_{e=1}^\infty \) on \( \mathbb{R}^d \) such that for each \( g, g^e \in H^m \) and

\[
\|g\|_{m, \ell_2}^2 := \sum_{e=1}^\infty \|g^e\|_m^2 < \infty.
\]

On \([0, T] \times \mathbb{R}^d \), we consider the stochastic integro-differential equation

\[
du_t = ((L_t + I)u_t + f_t) \, dt + \sum_{e=1}^\infty (M_t^e u_t + g_t^e) \, dw_t^e + \int_{\mathbb{R}^d} J(z) u_t - \tilde{\mu}(dz, dt)
\]

with initial condition

\[
u_0(x) = \psi(x), \quad x \in \mathbb{R}^d.
\]

Denote the predictable sigma-algebra on \( \Omega \times [0, T] \) relative to \( \mathbb{F} \) by \( \mathcal{P}_T \). Let \( m \geq 0 \) be an integer.

**Assumption 1.** For \( i, j \in \{0, \ldots, d\} \), \( a_t^{ij} = a_t^{ij}(x) \) are real-valued functions defined on \( \Omega \times [0, T] \times \mathbb{R}^d \) that are \( \mathcal{P}_T \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable and \( b_t^i = \ldots \)
\((b_i^e(x))_{i=1}^\infty\) are \(\ell_2\)-valued functions that are \(\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R}^d)\)-measurable. Moreover,

(i) for each \((\omega, t) \in \Omega \times [0, T]\), the functions \(a_{ij}^e\) are \(\max(m, 1)\)-times continuously differentiable in \(x\) for all \(i, j \in \{1, \ldots, d\}\), \(a_i^0\) and \(a_i^\alpha\) are \(m\)-time continuously differentiable in \(x\) for all \(i \in \{0, 1, \ldots, d\}\), and \(b_i^e\) are \(m\)-times continuously differentiable in \(x\) as \(\ell_2\)-valued functions for all \(i \in \{0, \ldots, d\}\). Furthermore, there exists a constant \(K > 0\) such that for all \((\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d\),

\[
|\partial^\alpha a_{ij}^e| \leq K, \quad \forall i, j \in \{1, \ldots, d\}, \quad \forall |\alpha| \leq \max(m, 1),
\]

\[
|\partial^\alpha a_i^0| + |\partial^\alpha a_i^\alpha| + |\partial^\alpha b_i^e|_2 \leq K, \quad \forall i \in \{0, \ldots, d\}, \quad \forall |\alpha| \leq m;
\]

(ii) there exists a positive constant \(\kappa > 0\) such that for all \((\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^d\) and \(\eta \in \mathbb{R}^d\)

\[
\sum_{i,j=1}^d \left( 2a_{ij}^e - \sum_{\theta=1}^\infty b_{ij}^\theta b_t^\theta \right) \eta_i \eta_j \geq \kappa |\eta|^2.
\]

We define the following spaces:

\[
\mathbb{H}_m := L_2(\Omega \times [0, T], \mathcal{P}_T; H^m), \quad \mathbb{H}_m^e(\ell_2) := L_2(\Omega \times [0, T], \mathcal{P}_T; H^m(\ell_2))
\]

\[
\mathbb{H}_m^e(\nu) := L_2(\Omega \times [0, T] \times \mathbb{R}^d, \mathcal{P}_T \times \mathcal{B}(\mathbb{R}^d), dP \times dt \times d\nu; H^m).
\]

**Assumption 2.** The initial condition \(\psi\) is \(\mathcal{F}_0\)-measurable with values in \(H^m\) such that \(E|\psi|_m^2 < \infty\). Moreover, \(f \in \mathbb{H}_m^{m-1}\) and \(g \in \mathbb{H}_m^e(\ell_2)\). Set

\[
\kappa_m^2 = E|\psi|_m^2 + E \int_{[0,T]} \left( \|f_s\|^2_{m-1} + \|g_s\|^2_{m,\ell_2} \right) ds.
\]

For a real-valued twice continuously differentiable function \(\phi\) on \(\mathbb{R}^d\), it is easy to see that for all \(x, z \in \mathbb{R}^d\),

\[
\phi(x + z) - \phi(x) - \sum_{j=1}^d z^j \partial_j \phi(x) = \int_0^1 \sum_{i,j=1}^d z^i z^j \partial_i \partial_j \phi(x + \theta z)(1 - \theta) d\theta. \quad (5)
\]

Fix \(\delta > 0\) such that

\[
\zeta(\delta) := \int_{|z| \leq \delta} |z|^2 \nu(dz) < \kappa^2. \quad (6)
\]
and notice that
\[ \int_{|z| > \delta} |z| \nu(dz) + \nu(\{|z| > \delta\}) < \infty. \] (7)

We write \( I = I_\delta + I_{\delta^c} \), where
\[ I_\delta \phi(x) = \int_{|z| \leq \delta} \int_0^1 \sum_{i,j=1}^d z^i z^j \partial_i \partial_j \phi(x + \theta z)(1 - \theta) d\theta \nu(dz) \]
and where \( I_{\delta^c} \) is defined as in (3) with integration over \( \{|z| > \delta\} \) instead of \( \mathbb{R}^d \).

**Definition 2.1.** An \( H^0 \)-valued càdlàg adapted process \( u \) is called a solution of (4) if

(i) \( u_t \in H^1 \) for \( dP \otimes dt \)-almost-every \((\omega, t) \in \Omega \times [0, T]\);

(ii) \( E \int_{[0,T]} \|u_s\|^2_1 ds < \infty \);

(iii) there exists a set \( \tilde{\Omega} \subset \Omega \) of probability one such that for all \((\omega, t) \in [0, T] \times \tilde{\Omega} \) and \( \phi \in C_c(\mathbb{R}^d) \),
\[
(u_t, \phi)_0 = (\psi, \phi)_0 + \int_{[0,t]} \left( \sum_{i,j=1}^d (\partial_j u_s, \partial_i(a^i_j \phi))_0 + [\phi, f_s]_0 \right) ds \\
+ \int_{[0,t]} \int_{|z| \leq \delta} \int_0^1 \sum_{i,j=1}^d \left( \partial^j \partial_i u_s (\cdot + \theta z), z^i \partial^j \phi \right)_0 (1 - \theta) d\theta \nu(dz) ds \\
+ \int_{[0,t]} \int_{|z| > \delta} \left( u_s(\cdot + z) - u_s - \sum_{j=1}^d \partial^j u_s, \phi \right)_0 \nu(dz) ds \\
+ \sum_{e=1}^\infty \int_{[0,t]} \sum_{i=0}^d (b^e_i \partial_i u_s + g^e_s, \phi)_0 dw^e_s \\
+ \int_{[0,t]} \int_{\mathbb{R}^d} (u_{s-}(\cdot + z) - u_{s-}, \phi)_0 \tilde{\mu}(dz, ds). \] (8)
In the above definition, instead of $\delta$ we may choose any other positive constant. The following existence theorem is a consequence of Theorems 2.9, 2.10, and 4.1 in [11] and will be verified in Section 4. The notation $N = N(\cdot, \cdot, \cdot, \cdot, \cdot)$ is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter.

**Theorem 2.1.** If Assumptions 1 and 2 hold with $m \geq 0$, then there exist a unique solution $u$ of (4). Furthermore, $u$ is a cádlág $H^m$-valued process with probability one and there exists a constant $N = N(d, m, \kappa, K, T, \nu)$ such that

$$E \sup_{t \leq T} \|u_t\|_{m}^2 + E \int_{[0, T]} \|u_s\|_{m+1}^2 ds \leq N\kappa_m^2.$$  \hfill (9)

The following proposition is needed to establish the rate of convergence in time of our approximation scheme and is proved in Section 4.

**Proposition 2.2.** Let Assumptions 1 and 2 hold with $m \geq 1$ and $u$ be the solution of (4). Moreover, assume that $\sup_{t \leq T} E\|g_t\|_{m-1, \ell_2}^2 \leq K$. Then there exists a constant $\lambda = \lambda(d, m, K, T, \kappa^2_m, \nu)$ such that for all $s, t \in [0, T]$,

$$E\|u_t - u_s\|_{m-1}^2 \leq \lambda|t - s|.$$  \hfill (10)

**Assumption 3.** For $m \geq 3$, in addition to Assumption 2, there exists a random variable $\xi$ with $E\xi < K$ such that for all $\omega \in \Omega$, $s, t \in [0, T], i, j \in \{0, 1, \ldots, d\}$,

$$\|g_t\|_{m-1, \ell_2}^2 \leq \xi \|g_t - g_s\|_{m-2, \ell_2}^2 \leq \xi |t - s|.$$ 

**Assumption 4.** For $m \geq 3$, in addition to Assumption 2 (i), there exists a constant $C$ such that for all $(\omega, x) \in \Omega \times \mathbb{R}^d$, $s, t \in [0, T], i, j \in \{0, 1, \ldots, d\}$,

$$|\partial^\alpha (a_s^{ij} - a_t^{ij})|^2 + |\partial^\alpha (b_s^i - b_t^i)|^2_{\ell_2} \leq C|t - s|, \quad \forall|\alpha| \leq m - 2.$$ 

We turn our attention to the discretisation of equation (4). For each $h \in \mathbb{R} - \{0\}$ and standard basis vector $e_i$, $i \in \{1, \ldots, d\}$, of $\mathbb{R}^d$ we define the first-order difference operator $\delta_{h,i}$ by

$$\delta_{h,i} \phi(x) := \frac{\phi(x + he_i) - \phi(x)}{h},$$
for all real-valued functions $\phi$ on $\mathbb{R}^d$. We define $\delta_{h,0}$ to be the identity operator. Notice that for all $\psi, \phi \in H^0$, we have

$$
\langle \phi, \delta_{-h,i}\psi \rangle_0 = -\langle \delta_{h,i}\phi, \psi \rangle_0.
$$

(11)

Set

$$
\delta^h_i := \frac{1}{2}(\delta_{h,i} + \delta_{-h,i})
$$

and observe that for all $\phi \in H^0$,

$$
\langle \phi, \delta^h_i\phi \rangle_0 = 0.
$$

(12)

We introduce the grid $G_h := \{hz_k : z_k \in \mathbb{Z}^d, k \in \mathbb{N} \cup \{0\}, z_0 = 0\}$ with step size $|h|$. Let $\ell_2(G_h)$ be the Hilbert space of real-valued functions $\phi$ on $G_h$ such that

$$
\|\phi\|_{\ell_2(G_h)}^2 := |h|^d \sum_{x \in G_h} |\phi(x)|^2 < \infty.
$$

We approximate the operators $L$ and $M^\theta$ by

$$
L^h_t \phi(x) := \sum_{i,j=0}^d a_{ij}^t(x)\delta_{h,i}\delta_{-h,j}\phi(x) \quad \text{and} \quad M^h_{t,\theta} \phi(x) := \sum_{i=0}^d b_i^t(x)\delta_{h,i}\phi(x),
$$

respectively. In order to approximate $I$, we approximate $I_{\delta}$ and $I_{\delta^c}$ separately. For each $k \in \mathbb{N} \cup \{0\}$ and $h \neq 0$, define the rectangles in $\mathbb{R}^d$

$$
A^h_k := \left(z_k^i|h| - \frac{|h|}{2}, z_k^i|h| + \frac{|h|}{2}\right] \times \cdots \times \left(z_k^d|h| - \frac{|h|}{2}, z_k^d|h| + \frac{|h|}{2}\right],
$$

where $z_k^i, i \in \{1, \ldots, d\}$, are the coordinates of $z_k \in \mathbb{Z}^d$, and set

$$
B^h_k := A^h_k \cap \{|z| \leq \delta\}, \quad \bar{B}^h_k := A^h_k \cap \{|z| > \delta\}.
$$

We approximate $I_{\delta^c}$ by

$$
I^h_{\delta^c} \phi(x) := \sum_{k=0}^{\infty} \left(\phi(x + hz_k) - \phi(x)\right)\bar{\zeta}_{h,k} - \sum_{i=1}^d \xi_{h,k,i}^i \delta^h_i \phi(x),
$$

(13)

where

$$
\bar{\zeta}_{h,k} := \nu(\bar{B}^h_k) \quad \text{and} \quad \xi_{h,k,i}^i := \int_{\bar{B}^h_k} z^i \nu(dz).
$$
We continue with the approximation of the operator $I_\delta$. By (5), for all $x \in \mathbb{G}_h$,

$$I_\delta \phi(x) = \sum_{k=0}^\infty \int_{B_k^h} \int_0^1 \sum_{i,j=1}^d z^i z^j \partial_i \partial_j \phi(x + \theta z)(1 - \theta) d\theta \nu(dz),$$

where there are only a finite number of non-zero terms in the infinite sum over $k$. The closest point in $\mathbb{G}_h$ to any point $z \in B_k^h$ is clearly $h z_k$. This simple observation leads us to the following (intermediate) approximation of $I_\delta \phi(x)$:

$$\sum_{k=0}^\infty \int_0^1 \sum_{i,j=1}^d \int_{B_k^h} z^i z^j \nu(dz) \partial_i \partial_j \phi(x + \theta h z_k)(1 - \theta) d\theta.$$

However, in order to ensure that our approximation is well-defined for functions $\phi \in \ell_2(\mathbb{G}_h)$, we need to approximate the integral over $\theta \in [0,1]$ carefully. For fixed $h \neq 0$ and $k \in \mathbb{N} \cup \{0\}$, there exist $\sigma(h,k) \in \mathbb{N}$, $r_{i,l}^h, k \in \mathbb{N} \cup \{0\}$, for $l \in \{0, \ldots, \sigma(h,k) - 1\}$, and real numbers $(\theta_{i,l,k}^{h,k})_{l=0}^{\sigma(h,k)}$ satisfying $0 = \theta_{0,k}^h \leq \ldots \leq \theta_{\sigma(h,k),k}^h = 1$, such that the line segment $\{\theta h z_k\}_{\theta \in [0,1]}$ is contained in the set $\cup_{l=0}^{\sigma(h,k)-1} A_{r_l,k}^h$, and for $\theta \in (\theta_{i,l,k}^{h,k}, \theta_{i+l+1,k}^{h,k})$, we have $\theta h z_k \in A_{r_l,k}^h$. In particular, for $k = 0$, we have $\sigma(h,0) = 1$, $r_0^h = 0$, $\theta_0^h = 0$, and $\theta_1^h = 1$. Since the diagonal of a $d$-dimensional hypercube with side length $|h|$ has length $\sqrt{d} h$, for each $k \in \mathbb{N} \cup \{0\}$, $z \in B_k^h$, and $l \in \{0, \ldots, \sigma(h,k) - 1\}$,

$$|\theta z - h z_{r_l,k}^h| \leq |\theta z - \theta h z_k| + |\theta h z_k - h z_{r_l,k}^h| \leq \sqrt{d|h|}, \quad (14)$$

for all $\theta \in (\theta_{i,l,k}^{h,k}, \theta_{i+l+1,k}^{h,k})$. Set

$$\zeta_{i,j}^{h,k} = \int_{B_k^h} z^i z^j \nu(dz), \quad \bar{\theta}_{i,l,k}^h = \int_{\theta_{i,l,k}^{h,k}}^{\theta_{i+l+1,k}^{h,k}} (1 - \theta) d\theta$$

and define the operator

$$I_{\delta}^h \phi(x) = \sum_{k=0}^\infty \sum_{l=0}^{\sigma(h,k)-1} \bar{\theta}_{i,l,k}^h \sum_{i,j=1}^d \zeta_{i,j}^{h,k} \delta_{h,i} \delta_{h,j} \phi(x + h z_{r_l,k}^h). \quad (15)$$

where there are only a finite number of non-zero terms in the infinite sum over $k$. Set $I^h = I_{\delta}^h + I_{\hat{\delta}}^h$ and introduce the martingales

$$p_{t}^{h,k,i} = \int_{[0,t]} \int_{B_k^h} z^i \hat{\mu}(dz, dt), \quad \hat{\theta}_{t}^{h,k} = \hat{\mu}(B_k^h, [0,t]).$$

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Moreover, set
\[ \tilde{\theta}^{h,k}_l := \theta^{h,k}_{l+1} - \theta^{h,k}_l. \]

Let \( T \geq 1 \) be an integer and set \( \tau = T/T \) and \( t_n = n\tau \) for \( i \in \{0,1,\ldots,T\} \). For any \( \mathcal{F} \)-martingale \( p \), we use the notation \( \Delta p_{n+1} := p_{n+1} - p_n \). Define recursively the \( \ell_2(\mathbb{G}_h) \)-valued random variables \( \{\hat{u}^{h,\tau}_n\}_{n=0}^T \) by
\[
\hat{u}^{h,\tau}_n(x) = \hat{u}^{h,\tau}_{n-1}(x) + \left((L_{t_{n-1}}^h + I^h)\hat{u}^{h,\tau}_{n-1}(x) + f_{t_{n-1}}(x)\right)\tau
\]
\[ + \sum_{k=0}^{\infty} \left(M_{t_{n-1}}^{h,0}\hat{u}^{h,\tau}_{n-1}(x) + g_{t_{n-1}}^{0}(x)\right)\Delta w_n^0
\]
\[ + \sum_{k=0}^{\infty} \sum_{i=1}^{d} \sum_{l=0}^{\sigma(h,k)-1} \left( \tilde{\theta}^{h,k}_l \delta_{h,i} \hat{u}^{h,\tau}_{n-1}(x + h z_{r_l,k}) \right) \Delta p_{n}^{h,k,i}
\]
\[ + \sum_{k=0}^{\infty} \left( \hat{u}^{h,\tau}_{n-1}(x + h z_k) - \hat{u}^{h,\tau}_{n-1}(x) \right) \Delta p_{n}^{h,k}, \quad n \in \{1,\ldots,T\},
\]
(16)

with initial condition
\[ \hat{u}^{h,\tau}_0(x) = \psi(x), \quad x \in \mathbb{R}^d. \]

It is clear that \( \hat{u}^{h,\tau}_n \) is \( \mathcal{F}_{t_n} \)-measurable for every \( n \in \{0,1,\ldots,T\} \). Define the operators
\[ \tilde{L}^h \phi = \sum_{i,j=0}^{d} a_{ij} \delta_{h,i} \delta_{-h,j} \phi - \nu(\{|z| > \delta\}) \phi - \sum_{i=1}^{d} \int_{|z|>\delta} z^i \nu(dz) \delta_{i}^h \phi \]
and
\[ \tilde{I}^h_{\delta} \phi = \sum_{k=0}^{\infty} \phi(x + h z_k) \zeta_{h,k} \]
and note that \( \tilde{L}^h + \tilde{I}^h_{\delta} + I_{\delta} = L^h + I^h \). On \( \mathbb{G}_h \), we also consider the following
implicit-explicit discretization scheme of (4):

\[ \hat{v}_{n}^{h,\tau}(x) = \tilde{v}_{n-1}^{h,\tau}(x) + \left( (\bar{I}_{t_{n}}^{h} + I_{\delta}^{h}) \hat{v}_{n-1}^{h,\tau}(x) + \bar{I}_{\delta} \hat{v}_{n-1}^{h,\tau}(x) + f_{t_{n}}(x) \right) \tau \]

\[ + \mathbb{I}_{n>1} \sum_{e=1}^{\infty} (M_{t_{n-1}}^{h,\theta} \hat{v}_{n-1}^{h,\tau}(x) + g_{t_{n-1}}^{\theta}(x)) \Delta w_{n}^{e} \]

\[ + \mathbb{I}_{n>1} \sum_{k=0}^{\infty} \sum_{i=1}^{d} \left( \sum_{l=0}^{\sigma(h,k)-1} \delta_{h,i}^{k,l} \hat{v}_{n-1}^{h,\tau}(x + hz_{i,k}) \right) \Delta F_{n}^{h,k,i} \]

\[ + \mathbb{I}_{n>1} \left( \hat{v}_{n-1}^{h,\tau}(x + hz_{k}) - \hat{v}_{n-1}^{h,\tau}(x) \right) \Delta F_{n}^{h,k}, \quad n \in \{1, \ldots, T\}, \quad (17) \]

with initial condition

\[ \hat{v}_{0}^{h,\tau}(x) = \psi(x), \quad x \in \mathbb{R}^{d}, \]

where \( \mathbb{I}_{n>1} = 0 \) if \( n = 1 \) and \( \mathbb{I}_{n>1} = 1 \) if \( n \geq 2 \). A solution \( (\hat{v}_{n}^{h,\tau})_{n=0}^{M} \) of \((17)\) is understood as a sequence of \( \ell_{2}(G_{h}) \)-valued random variables such that \( \hat{v}_{n}^{h,\tau} \) is \( \mathcal{F}_{t_{n}} \)-measurable for every \( n \in \{0,1, \ldots, M\} \) and satisfies \((17)\).

**Remark 2.1.** Under Assumptions 2 and 3 for \( m > 2 + d/2 \), by virtue of the embedding \( H^{m-2} \hookrightarrow \ell_{2}(G_{h}) \), the free-terms \( f \) and \( g \) are continuous \( \ell_{2}(G_{h}) \) valued processes, and consequently the above schemes make sense. Moreover, for \( 0 < |h| < 1 \), there exists a constant \( N \) independent of \( h \) such that

\[ ||\phi||_{\ell_{2}(G_{h})} \leq N ||\phi||_{m-2}. \quad (18) \]

**Assumption 5.** The parameters \( h \neq 0 \) and \( T \) are such that

\[ \frac{d}{h^{2}} \frac{\tau}{2} < \frac{\kappa - 2\zeta(\delta)}{\left( 2 \left( \sup_{t,x,\omega} \sum_{i,j=1}^{d} |a_{ij}^{t,\omega}(x)|^{2} \right)^{1/2} + \zeta(\delta) \right)^{2}}. \]

The following are our main theorems.

**Theorem 2.3.** Let Assumptions 1 through 4 hold with \( m > 2 + d/2 \) and let Assumption 5 hold. Let \( u \) be the solution of \((14)\) and let \( (\hat{u}_{n}^{h,\tau})_{n=0}^{T} \) be defined by \((16)\). Then there exists a constant \( N = N(d, m, \kappa, K, C, \lambda, \kappa^{2}, \delta, \nu) \) such that for any real number \( h \) with \( 0 < |h| < 1 \),

\[ E \max_{0 \leq n \leq T} \sup_{x \in G_{h}} |u_{t_{n}}(x) - \hat{u}_{n}^{h,\tau}(x)|^{2} \]

\[ + E \max_{0 \leq n \leq T} \|u_{t_{n}} - \hat{u}_{n}^{h,\tau}\|_{\ell_{2}(G_{h})}^{2} \leq N \left( |h|^{2} + \tau \right). \]
Theorem 2.4. Let Assumptions 1 through 4 hold with \( m > 2 + \frac{d}{2} \) and let \( u \) be a solution of (4). There exists a constant \( R = R(d, m, \kappa, K, \delta, \nu) \) such that if \( T > R \), then there exists a unique solution \( (\hat{v}^n\tau_n)_{n=0}^T \) of (17) and a constant \( N = N(d, m, \kappa, K, T, C, \lambda, \kappa_m^2, \delta, \nu) \) such that for any real number \( h \) with \( 0 < |h| < 1 \),

\[
E \max_{0 \leq n \leq T} \sup_{x \in \mathbb{G}_h} |u_{tn}(x) - \hat{v}^n\tau_n(x)|^2 \\
+ E \max_{0 \leq n \leq T} \|u_{tn} - \hat{v}^n\tau_n\|_{\ell^2(\mathbb{G}_h)}^2 \leq N \left(|h|^2 + \tau\right).
\]

Remark 2.2. Note that for any fixed \( \delta > 0 \) satisfying (6), both the explicit scheme \((\hat{u}^n\tau_n)_{n=0}^T\) and the implicit-explicit \((\hat{v}^n\tau_n)_{n=0}^T\) scheme converge with rate one as \( h \) tends to zero and rate one-half as \( \tau \) tends to infinity.

3. Auxiliary results

In this section, we present some results that will be needed for the proof of Theorems 2.3 and 2.4. Introduce the operators

\[
J^h_\delta(z)\phi(x) := \sum_{k=0}^\infty \int_{B_k^h}(z) \sum_{i=0}^{\sigma(h,k)-1} \sum_{l=1}^d \tilde{\theta}^h_{l,l} \zeta_l \delta_{h,i} \phi(x + h\zeta_l h,k), \tag{19}
\]

\[
J^h_\delta(c)\phi(x) := \sum_{k=0}^\infty \int_{B_k^h}(z) \left[ \phi(x + h\zeta_k) - \phi(x) \right], \tag{20}
\]

\[
J^h(z)\phi(x) := J^h_\delta(z)\phi(x) + J^h_\delta(c)\phi(x). \tag{21}
\]

Consider the following explicit and implicit-explicit schemes in \( H^0 \):

\[
u_{tn}^h = u_{tn}^{h,\tau} + \left( (L_{tn}^h + I_{tn}) u_{tn-1}^{h,\tau} + f_{tn-1} \right) \tau \\
+ \sum_{k=1}^\infty (M_{tn}^{h,\theta} u_{tn}^{h,\tau} + g_{tn}^e) \Delta u_n^e \\
+ \int_{\mathbb{R}^d} J^h(z) u_{tn-1}^{h,\tau} q(dz, [t_{n-1}, t_n]), \quad n \in \{1, \ldots, T\}, \tag{22}
\]
and
\[ v_n^{h,\tau} = v_{n-1}^{h,\tau} + \left( (\tilde{L}_n^h + I_{\delta}^h) v_n^{h,\tau} + \tilde{f}_n^{h,\tau} \right) \tau \]
\[ + \sum_{i=1}^{\infty} (M_i^{n-1} v_{n-1}^{h,\tau} + g_i^{n-1} \Delta w_n^i) \]
\[ + \sum_{i=1}^{\infty} \int_{\mathbb{R}^d} j^h(z) v_{n-1}^{h,\tau} q(dz, t_{n-1}, t_n), \quad n \in \{1, \ldots, T\}, \]
\[ (23) \]

with initial condition
\[ v_0^{h,\tau}(x) = v_0^{h,\tau}(x) = \psi(x), \quad x \in \mathbb{R}^d. \]

We now prove some lemmas that will help us to establish the consistency of our approximations. The following lemma is well-known and we omit the proof (see, e.g., [12]).

**Lemma 3.1.** For each integer \( m \geq 0 \), there exists a constant \( N = N(d,m) \) such that for all \( u \in H^{m+2} \) and \( v \in H^{m+3} \),
\[ \| \delta_h u - \partial_i u \|_m \leq \frac{1}{2} |h| \|u\|_{m+2}, \]
\[ \| \delta_h \delta - \partial_i \partial_j v \|_m \leq N |h| \|v\|_{m+3}. \]

**Lemma 3.2.** For each integer \( m \geq 0 \), there exists a constant \( N = N(d,m,\delta,\nu) \) such that for all \( u \in H^{m+3} \), we have
\[ \| I_u - I^h u \|_m \leq N |h| \|u\|_{m+3}. \]
\[ (24) \]

**Proof.** It suffices to show \[(24)\] for \( u \in C_c^\infty(\mathbb{R}^d) \). We begin with \( m = 0 \). A simple calculation shows that
\[ I_\delta u(x) - I_{\delta^h} u(x) = \sum_{k=0}^{\infty} \int_{B_k^h} \left( u(x+z) - u(x + h z_k) - \sum_{i=1}^{d} z^i (\partial_i u(x) - \delta_i^h u(x)) \right) \nu(dz) \]
\[ = \sum_{k=0}^{\infty} \int_{B_k^h} \left( \sum_{i=1}^{d} (z^i - h z_k^i) \partial_i u(x + h z_k + \theta(z - h z_k)) \right) d\nu(dz) \]
\[-\sum_{k=0}^{\infty} \int_{B_h^k} \sum_{i=1}^{d} z^i (\partial_i u(x) - \delta_i^h u(x)) \nu(dz)\]

By Minkowski’s inequality, we get

\[
\|I_{\delta} u - I_{\delta}^h u\|_0 \leq \sum_{k=0}^{\infty} \int_{B_h^k} \sum_{i=1}^{d} |z^i| \|\partial_i u\|_0 \nu(dz) \\
+ \sum_{k=0}^{\infty} \int_{B_h^k} \sum_{i=1}^{d} |z^i| \|\partial_i u(x) - \delta_i^h u(x)\|_0 \nu(dz) \\
\leq N|h|\|u\|_3 + N\sum_{i=1}^{d} \|\partial_i u(x) - \delta_i^h u(x)\|_0,
\]

since $|z - hz_k| \leq |h| \sqrt{d}/2$ and (7) holds. Thus, by Lemma 3.1, we have

\[
\|I_{\delta} u - I_{\delta}^h u\|_0 \leq N|h|\|u\|_3.
\] (25)

We also have

\[
I_{\delta} u(x) - I_{\delta}^h u(x) = \\
\sum_{k=0}^{\infty} \int_{B_h^k} \sum_{i=0}^{\sigma^{(h,k)}-1} \int_{q^{h,i,k}_l}^{q^{h,i,k}_{l+1}} \sum_{i,j=1}^{d} z^i z^j [\partial_i \partial_j u(x + \theta z) - \delta_i h \delta_j h u(x + h z_{r_{i,k}})] (1 - \theta) d\theta \nu(dz).
\] (26)

Note that

\[
\partial_i \partial_j u(x + \theta z) - \delta_i h \delta_j h u(x + h z_{r_{i,k}}) = \partial_i \partial_j u(x + \theta z) - \partial_i \partial_j u(x + h z_{r_{i,k}}) + \partial_i \partial_j u(x + h z_{r_{i,k}}) - \delta_i h \delta_j h u(x + h z_{r_{i,k}})
\]

\[
= \int_0^{1} \sum_{q=1}^{d} (\theta z^q - h z^q_{r_{i,k}}) \partial_q \partial_i \partial_j u \left( x + h z_{r_{i,k}} + \rho(\theta z - h z_{r_{i,k}}) \right) d\rho
\]

\[
+ \partial_i \partial_j u(x + h z_{r_{i,k}}) - \delta_i h \delta_j h u(x + h z_{r_{i,k}}).
\]

By (4), we have $|\theta z^q - h z^q_{r_{i,k}}| \leq N|h|$. Hence, substituting the above relation in (26), using Minkowski’s inequality, (6), and Lemma 3.1, we obtain

\[
\|I_{\delta} u - I_{\delta}^h u\|_0 \leq |h| N\|u\|_3.
\] (27)
Combining (25) and (27), we have (24) for $m = 0$. The case $m > 0$ follows from the case $m = 0$, since for a multi-index $\alpha$, we have
\[
\partial^\alpha (Iu - I^h u) = I\partial^\alpha u - I^h \partial^\alpha u.
\]

**Lemma 3.3.** For each integer $m \geq 0$, there exists a constant $N = N(d, m, \delta, \nu)$, there exists a constant such that for all $u \in H^{m+2}$, we have
\[
\int_{\mathbb{R}^d} \|J^h(z)u - J(z)u\|_m^2 \nu(dz) \leq N|h|^2\|u\|_m^{2m+2}.
\] (28)

**Proof.** It suffices to prove the lemma for $u \in C^\infty_c(\mathbb{R}^d)$ and $m = 0$. We have
\[
J_\delta(z)u(x) - J^h_\delta(z)u(x) =
\sum_{k=0}^{\infty} \|J^h_\delta(z)u - J_\delta(z)u\|_0^2 \leq \int_{\mathbb{R}^d} |z|\nu(dz) \leq N|h|^2\|u\|_2^2,
\]
and hence by (6), we obtain
\[
\int_{\mathbb{R}^d} \|J^h_\delta(z)u - J_\delta(z)u\|_0^2 \nu(dz) \leq N|h|^2\|u\|_2^2.
\] (29)

We also have
\[
|J_\delta(z)u(x) - J^h_\delta(z)u(x)| = \sum_{k=0}^{\infty} \|J^h_\delta(z)|u(x+z) - u(x+hz)\|
\]
\[
\leq \sum_{k=0}^{\infty} \mathbb{I}_{\mathcal{B}_k}(z) \int_0^1 \sum_{i=1}^d |\partial_i u(x + h z_k + \rho(z - h z_k))| z^i - h z^i_k|d\rho.
\]
Consequently,
\[
\|J_h^b(z)u - J_{\delta^c}(z)u\|_0^2 \leq \mathbb{I}_{|z| > \delta} N|h|^2 \|u\|_1^2;
\]
which implies by (7) that
\[
\int_{\mathbb{R}^d} \|J_h^b(z)u - J_{\delta^c}(z)u\|_0^2 \nu(dz) \leq N|h|^2 \|u\|_1^2. \tag{30}
\]
Combining (30) and (29), we have (28) for \(m = 0\). The case \(m > 0\) follows from the case \(m = 0\), since for a multi-index \(\alpha\), we have
\[
\partial^\alpha(Ju - J^hu) = J\partial^\alpha u - J^h\partial^\alpha u.
\]
\[\square\]

**Lemma 3.4.** If Assumption 1 holds for some \(m \geq 0\), then for any \(\epsilon \in (0, 1)\) there exists constants \(N_1 = N_1(d, m, \kappa, K, \delta, \nu, \epsilon)\) and \(N_2 = N_2(d, m, \kappa, K, \delta, \nu, \epsilon)\) such that for any \(u \in H^m\),
\[
Q_t^{(m)}(u) := 2(u, L_t^hu)_m + \|M_t^hu\|_{m,\ell}^2 + 2(u, I_t^hu)_m
\]
\[
+ \int_{\mathbb{R}^d} \|J^h(z)u\|_m^2 \nu(dz) \leq - (\kappa - 2\varsigma(\delta) - \epsilon) \sum_{i=1}^d \|\delta_{h,i} u\|_m^2 + N_1 \|u\|_m^2, \tag{31}
\]
and
\[
(u, \tilde{L}_t^hu)_m + (u, I_t^hu)_m \leq - (\kappa - \varsigma(\delta) - \epsilon) \sum_{i=1}^d \|\delta_{h,i} u\|_m^2 + N_2 \|u\|^m_n. \tag{32}
\]

**Proof.** By virtue of Lemma 3.1 and Theorem 3.2 in [13], under Assumption 1 there exists a constant \(N = N(d, m, \kappa)\) such that for any \(u \in H^m\) and \(\epsilon > 0\),
\[
2(u, L_t^hu)_m + \|M_t^hu\|_{m,\ell}^2 \leq - (\kappa - \epsilon) \sum_{i=1}^d \|\delta_{h,i} u\|_m^2 + N \|u\|^2_m.
\]
Therefore, it suffices to show that there exists a constant $N = N(\delta, \nu)$ such that for all $u \in C_c^\infty(\mathbb{R}^d)$,

$$2(u, I^h u)_m + \int_{\mathbb{R}^d} \|J^h(z)u\|^2 m \nu(dz) \leq 2 \varsigma(\delta) \sum_{i=1}^d \|\delta_{h,i} u\|^2_m + N\|u\|^2_m. \quad (33)$$

We start with $m = 0$. Since

$$(u, I^h u)_0 = \sum_{k=0}^{\infty} \int_{B_k^h} \sum_{l=1}^d \sum_{i,j=1} \theta_{i,j}^h z^i z^j \int_{\mathbb{R}^d} \delta_{h,i}\delta_{h,j} u(x + h z_{i,k})u(x)dx \nu(dz)$$

and

$$\int_{\mathbb{R}^d} \delta_{h,i}\delta_{h,j} u(x + h z_{i,k})u(x)dx = -\int_{\mathbb{R}^d} \delta_{h,i}u(x + h z_{i,k})\delta_{h,j}u(x)dx,$$

by Hölder’s inequality, we get

$$2(u, I^h u)_0 \leq \int_{|z| \leq \delta} |z|^2 \nu(dz) \sum_{i=1}^d \|\delta_{h,i} u\|^2_0 = \varsigma(\delta) \sum_{i=1}^d \|\delta_{h,i} u\|^2_0. \quad (34)$$

In addition, owing to Hölder’s inequality and (12), we have

$$2(u, I^h u)_0 \leq \sum_{k=0}^{\infty} \int_{B_k^h} \int_{\mathbb{R}^d} [u(x + h z_k) - u(x) - \sum_{i=1}^d z^i \delta_{i,h} u(x)]u(x)dx \nu(dz) \leq 0. \quad (35)$$

By Minkowski’s inequality, we have

$$\|J^h(z)u\|^2 \leq \sum_{k=0}^{\infty} \mathbb{I}_{B_k^h}(z)|z|^2 \sum_{i=1}^d \|\delta_{h,i} u\|^2_0 \quad \text{and} \quad \|J^h(z)u\|^2_0 \leq 4 \sum_{k=0}^{\infty} \mathbb{I}_{B_k^h}(z)\|u\|^2_0$$

and hence

$$\int_{\mathbb{R}^d} \|J^h(z)u\|^2_0 \nu(dz) \leq \varsigma(\delta) \sum_{i=1}^d \|\delta_{h,i} u\|^2_0 + 4\nu(|z| > \delta)\|u\|^2_0, \quad (36)$$

which proves (33) for $m = 0$. The case $m > 0$ follows by replacing $u$ with $\partial^\alpha u$ for $|\alpha| \leq m$. This proves (31), which implies (32). \qed
Remark 3.1. It follows $m \geq 0$, there exists a constant $N_5 = N_5(d, m, K, \delta, \nu)$ such that for any $u \in H^m$,

$$
\|M_n h u\|_m^2 + \int_{\mathbb{R}^d} \|J^h(z)u\|_m^2 \nu(dz) \leq N_5 \sum_{i=0}^d \|\delta_h,i u\|_m^2
$$

(37)

$$
\leq N_5 \left(1 + \frac{4d}{h^2}\right) \|u\|_m^2.
$$

(38)

Lemma 3.5. For any $m \geq 0$ and $u \in H^m$,

$$
\|\tilde{I}_h^e u\|_m^2 \leq \nu(\{|z| > \delta\})^2 \|u\|_m^2.
$$

(39)

Moreover, if Assumption \[ \Box \] holds for some $m \geq 0$, then for any $\epsilon > 0$ and $u \in H^m$,

$$
\|(L_h^t + I_h^s)u\|_m^2 \leq (1 + \varepsilon) \frac{N_3 d}{h^2} \sum_{i=0}^d \|\delta_h,i u\|_m^2 + N_4 \left(1 + \frac{1}{h^2}\right) \|u\|_m^2
$$

(40)

where

$$
N_3 := \left(2 \left( \sup_{t,x,\omega} \sum_{i,j=1}^d |a^{ij}(x)|^2 \right)^{1/2} + \varsigma(\delta) \right)^2
$$

and $N_4$ is a constant depending only on $d, m, K, \delta, \nu$, and $\epsilon$.

Proof. It suffices to prove the lemma for $u \in C^\infty_c(\mathbb{R}^d)$. It follows that

$$(L_h^t + I_h^s)u(x) = \sum_{k=0}^\infty \sum_{l=0}^{\sigma(h,k)-1} \tilde{g}_{h,k}^l \sum_{i,j=1}^d \tilde{\zeta}_{t,h,k}^{ij}(x) \delta_{h,i} \delta_{-h,j} u(x + hz_{l,i}^k)$$

$$
+ \sum_{i,j=0}^d a^{ij} \delta_{h,i} \delta_{-h,j} u(x)
$$

where $\tilde{\zeta}_{t,h,k}^{ij}(x) := \zeta_{t,h,k}^{ij}$ for $k \neq 0$ and $\tilde{\zeta}_{t,h,0}^{ij}(x) := \zeta_{t,h,0}^{ij} + 2a^{ij}(x)$. Moreover, for each multi-index $\alpha$ with $1 \leq |\alpha| \leq m$,

$$
\partial^\alpha (L_h^t + I_h^s)u(x) = \sum_{k=0}^\infty \sum_{l=0}^{\sigma(h,k)-1} \tilde{g}_{h,k}^l \sum_{i,j=1}^d \tilde{\zeta}_{t,h,k}^{ij}(x) \partial^\alpha \delta_{h,i} \delta_{-h,j} \partial^\alpha u(x + hz_{l,i}^k)
$$

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\[ + \sum_{\{\beta : \beta < \alpha\}} N(\beta, \alpha) \sum_{i,j=1}^{d} (\partial^{\alpha-\beta} a_{ij}^t(x)) \delta_{h,i} \delta_{-h,j} \partial^\beta u(x) \]

\[ + \sum_{\{\beta : \beta \leq \alpha\}} N(\beta, \alpha) \sum_{i,j=0}^{d} \left( (\partial^{\alpha-\beta} a_{ij}^t(x)) \delta_{h,i} \delta_{-h,j} \partial^\beta u(x) \right) \]

\[ =: [A_1(\alpha) + A_2(\alpha) + A_3(\alpha)] u(x), \]

where \( N(\beta, \alpha) \) are constants depending only on \( \beta \) and \( \alpha \). By Young’s inequality and Jensen’s inequality, for any \( \epsilon \in (0, 1) \), we have

\[ \| (L^h_t + I^h_t) u \|_m^2 \leq (1 + \epsilon) \sum_{|\alpha| \leq m} \| A_1(\alpha) u \|_0^2 \]

\[ + 3 \left( 1 + \frac{1}{\epsilon} \right) \left[ \sum_{|\alpha| \leq m} (\| A_2(\alpha) u \|_0^2 + \| A_3(\alpha) u \|_0^2) + \| I^h_t \|_m^2 \right]. \]

Applying Minkowski’s inequality and the Cauchy-Bunyakovsky-Schwarz inequality, we obtain

\[ \| A_1(\alpha) u \|_0^2 \leq \]

\[ \sum_{k=0}^{\infty} \sum_{l=0}^{\sigma(h,k)-1} \tilde{\theta}^{h,k} \left( \sup_{t,x,\omega} \sum_{i,j=1}^{d} |\tilde{\zeta}^{ij}_{t,h,k}(x)|^2 \right)^{1/2} \left( \sum_{i,j=1}^{d} \| \delta_{h,i} \delta_{-h,j} u(\cdot + h z_{i,h,k}) \|_m^2 \right)^{1/2} \]

\[ \leq \frac{\sqrt{d}}{h} \sum_{k=0}^{\infty} \left( \sup_{t,x,\omega} \sum_{i,j=1}^{d} |\tilde{\zeta}^{ij}_{t,h,k}(x)|^2 \right)^{1/2} \left( \sum_{i=1}^{d} \| \delta_{h,i} \partial^\alpha u \|_0^2 \right)^{1/2} \]

and

\[ \sum_{k=0}^{\infty} \left( \sup_{t,x,\omega} \sum_{i,j=1}^{d} |\tilde{\zeta}^{ij}_{t,h,k}(x)|^2 \right)^{1/2} = \sum_{k=1}^{\infty} \left( \sum_{i,j=1}^{d} \left| \int_{B_h^k} z^{i} z^{j} \nu(dz) \right| \right)^{1/2} \]

\[ + \left( \sup_{t,x,\omega} \sum_{i,j=1}^{d} \left| \int_{B_h^0} z^{i} z^{j} \nu(dz) + 2 a_{ij}^t(x) \right| \right)^{1/2} \]
\[ \leq \varsigma(\delta) + 2 \left( \sup_{t, x, \omega} \sum_{i,j=1}^{d} |a_{ij}(x)|^2 \right)^{1/2}. \]

Thus,
\[ \sum_{|\alpha| \leq m} \| A_1(\alpha) u \|_0^2 \leq \frac{N_3 d}{h^2} \sum_{i=1}^{d} \| \partial_{h,i} u \|_m^2. \]

Another application of the Cauchy-Bunyakovsky-Schwarz inequality and Minkowski’s inequality, combined with the inequalities
\[ \| \delta_{h,i} \partial^\beta u \|_0 \leq \| \partial_i \partial^\beta u \|_0; \forall i \in \{0, 1, \ldots, d\}, \forall |\beta| \leq m - 1, \]
\[ \| \delta_{h,i} \delta_{h,j} \partial^\beta u \|_0 \leq \| \partial_i \partial_j \partial^\beta u \|_0; \forall i, j \in \{0, 1, \ldots, d\}, \forall |\beta| \leq m - 2, \]
and
\[ \| \delta_{h,i} \delta_{h,j} \partial^\beta u \|_0 \leq \frac{2}{h} \| \delta_{h,i} u \|_m, \forall i, j \in \{0, 1, \ldots, d\}, \forall |\beta| = m - 1, \]
yields
\[ \sum_{|\alpha| \leq m} \left( \| A_2(\alpha) u \|_0^2 + \| A_3(\alpha) \|_0^2 \right) \leq N \left( 1 + \frac{1}{h^2} \right) \| u \|_m^2. \]

By Minkowski’s integral inequality, we have
\[ \| f^h \|_m \leq \int_{\mathbb{R}^d} \sum_{k=0}^{\infty} \Pi_{\mathbb{R}_k} \| u(\cdot + hz_k) - \partial^\alpha u - z^i \delta^i_{h} \partial^\alpha u \|_m \nu(dz) \]
\[ \leq 3 \left( \nu(\{|z| > \delta\}) + \frac{2d\int_{|z|>\delta} |z| \nu(dz)}{h^2} \right) \| \partial^\alpha u \|_0. \]

It is also easy to see that (39) holds. Combining above inequalities, we obtain (40).

The following theorem establishes the stability of the explicit approximate scheme (22).
Theorem 3.6. Let Assumption 1 hold with \( m \geq 0 \) and Assumption 2 hold. Let \( f^i \in \mathbb{H}^m \) for \( i \in \{0, \ldots, d\} \), \( g \in \mathbb{H}^m(\ell_2) \) and \( r \in \mathbb{H}^m(\nu) \). Consider the following scheme in \( \mathbb{H}^m \):

\[
    u_n^{h,\tau} = u_{n-1}^{h,\tau} + \int_{[t_{n-1}, t_n]} \left( (L_{t_{n-1}} + I^h) u_{n-1}^{h,\tau} + \sum_{i=0}^{d} \delta_{h,i} f_t^i \right) dt + \sum_{q=1}^{\infty} \int_{[t_{n-1}, t_n]} \left( M_{t_{n-1}}^{h,q} u_{n-1}^{h,\tau} + g_t^q \right) dw_t^q
\]

\[
    + \int_{[t_{n-1}, t_n]} \int_{\mathbb{R}^d} \left( J^h(z) u_{n-1}^{h,\tau} + r_t(z) \right) \tilde{\mu}(dz, dt), \quad n \in \{1, \ldots, T\}, \quad (41)
\]

for any \( \mathbb{H}^m \)-valued \( \mathcal{F}_0 \)-measurable initial condition \( \psi \). Moreover, if \( E\|\psi\|_m^2 < \infty \), then there exists a constant \( N = N(d, m, \varkappa, K, T, \delta, \nu) \) such that

\[
    E \max_{0 \leq n \leq T} \| u_n^{h,\tau} \|_m^2 + E \sum_{n=0}^{T} \tau \sum_{i=0}^{d} \| \delta_{h,i} u_n^{h,\tau} \|_m^2 \leq NE\|\psi\|_m^2
\]

\[
    + NE \int_0^T \left( \sum_{i=0}^{d} \| f_t^i \|_m^2 + \| g_t \|_m^2 + \int_{\mathbb{R}^d} \| r_t(z) \|_m^2 \nu(dz) \right) dt. \quad (42)
\]

Proof. If \( E\|\psi\|_m^2 < \infty \), then proceeding by induction on \( n \) and using Young’s and Jensen’s inequality, Itô’s isometry, (40), and (38), we get that for all \( n \in \{0, 1, \ldots, T\} \), \( E\|u_n^{h,\tau}\|_m^2 < \infty \). Applying the identity \( \|y\|_m^2 - \|x\|_m^2 = 2(x, y - x)_m + \|y - x\|_m^2 \), for each \( n \in \{1, \ldots, T\} \), we obtain

\[
    \| u_n^{h,\tau} \|_m^2 = \| u_{n-1}^{h,\tau} \|_m^2 + \sum_{i=1}^{6} I_i(t_n), \quad (43)
\]

where

\[
    I_1(t_n) := 2\tau (u_{n-1}^{h,\tau}, (L_{t_{n-1}} + I^h) u_{n-1}^{h,\tau})_m + \| \eta(t_n) \|_m^2,
\]

\[
    I_2(t_n) := 2 \int_{[t_{n-1}, t_n]} \sum_{i=0}^{d} (u_{n-1}^{h,\tau}, \delta_{h,i} f_t^i)_m dt,
\]

\[
    I_3(t_n) := \left\| \tau (L_{t_{n-1}} + I^h) u_{n-1}^{h,\tau} + \int_{[t_{n-1}, t_n]} \sum_{i=0}^{d} \delta_{h,i} f_t^i dt \right\|_m^2,
\]

\[
    \sum_{i=1}^{6} I_i(t_n)
\]

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\[ I_4(t_n) := 2 \sum_{\varrho=1}^{\infty} \int_{[t_{n-1}, t_n]} \left( u_{n-1}^{h, \tau}, M_{t_{n-1}}^{h, \varrho} u_{n-1}^{h, \tau} + g_\varrho^t \right)_m d\omega_\varrho, \]
\[ I_5(t_n) := 2 \int_{[t_{n-1}, t_n]} \int_{\mathbb{R}^d} \left( u_{n-1}^{h, \tau}, J^h(z) u_{n-1}^{h, \tau} + r_t(z) \right)_m \bar{\mu}(dz, dt), \]
\[ I_6(t_n) := 2 \left( \tau(L_{t_{n-1}}^h + J^h) u_{n-1}^{h, \tau}, \eta(t_n) \right)_m + 2 \left( \int_{[t_{n-1}, t_n]} \sum_{i=0}^{d} \delta_{h,i} f^i_t dt, \eta(t_n) \right)_m \]
and where
\[ \eta(t_n) := \sum_{\varrho=1}^{\infty} \int_{[t_{n-1}, t_n]} \left( M_{t_{n-1}}^{h, \varrho} u_{n-1}^{h, \tau} + g_\varrho^t \right)_m d\omega_\varrho \]
\[ + \int_{[t_{n-1}, t_n]} \int_{\mathbb{R}^d} \left( J^h(z) u_{n-1}^{h, \tau} + r_t(z) \right) \bar{\mu}(dz, dt) . \]

By virtue of Assumption 5, we fix \( \tilde{q} > 0 \) small enough such that
\[ \tilde{q} := \tau - 2\sigma(\delta) - (1 + \epsilon)(1 + \tilde{q})N_3 d \frac{\tau}{h^2} - (\epsilon + \tilde{q}) > 0, \quad (44) \]
where \( N_3 \) is the constant in (3.5). Since the two stochastic integrals that define \( \eta \) are orthogonal square-integrable martingales, by Young's inequality and (37), for all \( q > 0 \),
\[ E \| \eta(t_n) \|_m^2 \leq E \tau \| M_{t_{n-1}}^{h, \tau} u_{n-1}^{h, \tau} \|_{m, \ell_2}^2 + E \tau \int_{\mathbb{R}^d} \| J^h(z) u_{n-1}^{h, \tau} \|_m^2 \nu(dz) + qE \tau \sum_{i=0}^{d} \| \delta_{h,i} u_{n-1}^{h, \tau} \|_m^2 \]
\[ + \left( 1 + \frac{N_5}{q} \right) E \int_{[t_{n-1}, t_n]} \left( \| g_t \|_{m, \ell_2}^2 + \int_{\mathbb{R}^d} \| r_t(z) \|_m^2 \nu(dz) \right) dt. \quad (45) \]
Thus, taking \( q = \frac{\tilde{q}}{3} \) in (43), we have
\[ EI_1(t_n) \leq E \tau \mathcal{Q}^{(m)}_{t_{n-1}}(u_{t_{n-1}}^{h, \tau}) + \frac{\tilde{q}}{3} E \tau \sum_{i=0}^{d} \| \delta_{h,i} u_{n-1}^{h, \tau} \|_m^2 \]
\[ + \left( 1 + \frac{3N_5}{\tilde{q}} \right) E \int_{[t_{n-1}, t_n]} \left( \| g_t \|_{m, \ell_2}^2 + \int_{\mathbb{R}^d} \| r_t(z) \|_m^2 \nu(dz) \right) dt. \]
Using (38) and Young’s inequality, we obtain

\[ EI_2(t_n) \leq \frac{\tilde{q}}{3} E\tau \sum_{i=0}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2 \leq \frac{3}{q} E\int_{|t_{n-1},t_n|} \sum_{i=0}^{d} \|f_i^l\|_m^2 dt. \]

An application of Young’s inequality and (40) yields

\[ EI_3(t_n) \leq (1+\epsilon)(1+\tilde{q})N_2 \frac{\tau}{h^2} E\tau \sum_{i=1}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2 \]

\[ + (1+\tilde{q})N_4 \left( \tau + \frac{\tau}{h^2} \right) E\tau \|u_{n-1}^{h,\tau}\|_m^2 \]

\[ + (d+1) \left( 1 + \frac{1}{q} \right) E \int_{[t_{n-1},t_n]} \left( \tau \|f_i^0\|_m^2 + \frac{4d\tau}{h^2} \sum_{i=1}^{d} \|f_i^l\|_m^2 \right) dt. \]

Making use of the estimate (38) and noting that \( E\|u_{n-1}^{h,\tau}\|_m^2 < \infty \), \( g \in \mathbb{H}^m(\ell_2) \), and \( r \in \mathbb{H}^m(\nu) \), we obtain \( EI_4(t_n) = EI_5(t_n) = 0 \). Moreover, as \((L^h_{t_{n-1}} + I^h) u_{n-1}^{h,\tau}\) is \( \mathcal{F}_{t_{n-1}} \)-measurable and \( E(\eta(t_n)|\mathcal{F}_{t_{n-1}}) = 0 \), the expectation of first term in \( I_6(t_n) \) is zero, and hence by Young’s inequality, for any \( q_1 > 0 \),

\[ EI_6(t_n) \leq q_1 E\|\eta(t_n)\|_m^2 + \frac{1}{q_1} E \left| \int_{[t_{n-1},t_n]} \sum_{i=0}^{d} \delta_{h,i} f_i^l dt \right|_m^2. \]

Moreover, by Jensen’s inequality, (45), and (37), for any \( q_1, q > 0 \),

\[ EI_6(t_n) \leq (q_1 q + q_1 N_5) \frac{\tau}{h} E \sum_{i=0}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2 \]

\[ + q_1 \left( 1 + \frac{N_5}{q} \right) E \int_{[t_{n-1},t_n]} \left( \|g_i\|_m \|f_i^l\|_m^2 + \frac{4d(d+1)\tau}{q_1 h^2} \sum_{i=1}^{d} \|f_i^l\|_m^2 \right) dt \]

\[ + q_1 \left( 1 + \frac{N_5}{q} \right) E \int_{[t_{n-1},t_n]} \left( \|r_i(z)\|_m^2 \|f_i^l\|_m^2 \right) dt. \]

We choose \( q \) and \( q_1 \) such that \( q_1 q + q_1 N_5 \leq \tilde{q}/3 \). Thus, owing to (31), we have

\[ E(\mathbb{Q}_{t_{n-1}}^{(m)} (u_{n-1}^{h,\tau}) + (\tilde{q} + (1+\epsilon)(1+\tilde{q})N_3 d \frac{\tau}{h^2}) \frac{\tau}{h} E \sum_{i=1}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2 \]

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\[ -\tau E\| u_{n}^{\text{h,}\tau} \|_{m}^{2} \leq E\| \psi \|_{m}^{2} - \tau \sum_{l=1}^{n} \| \delta_{h,i} u_{l-1}^{\text{h,}\tau} \|_{m}^{2} \]

Taking the expectation of both sides of (43), summing-up, and combining
the above inequalities and identities, we find that there exists a constant
\( N = N(d, m, \kappa, K, \delta, \nu) \) such that for all \( n \in \{0, 1, \ldots, T\} \),

\[
E\| u_{n}^{\text{h,}\tau} \|_{m}^{2} \leq E\| \psi \|_{m}^{2} - \tau \sum_{l=1}^{n} \| \delta_{h,i} u_{l-1}^{\text{h,}\tau} \|_{m}^{2} + N_{1} E\| u_{n-1}^{\text{h,}\tau} \|_{m}^{2}.
\]

Therefore, by discrete Gronwall’s inequality, there exists a constant
\( N = N(d, m, \kappa, K, T, \delta, \nu) \) such that

\[
E\text{max}_{0 \leq n \leq T} \sum_{l=1}^{n} \| \delta_{h,i} u_{l}^{\text{h,}\tau} \|_{m}^{2} \leq N E\| \psi \|_{m}^{2}
\]

Now that we have proved (46), we will show (42). Estimating as we did
above, we get that there exists a constant \( N = N(d, m, \kappa, K, T, \delta, \nu) \) such that

\[
E\text{max}_{0 \leq n \leq T} \sum_{l=1}^{n} \| \delta_{h,i} u_{l}^{\text{h,}\tau} \|_{m}^{2} \leq N E\| \psi \|_{m}^{2} + N E\int_{[0,T]} \left( \sum_{i=0}^{d} \| f_{i}^{t} \|_{m}^{2} + \| g_{t} \|_{m,t_{2}}^{2} + \int_{R^{d}} \| r_{t}(z) \|_{m}^{2} \nu(dz) \right) dt.
\]

Applying the Burkholder-Davis-Gundy inequality and Young’s inequality, we
obtain

\[
E\text{max}_{0 \leq n \leq T} \sum_{l=1}^{n} I_{5}(t_{l})
\]
\[
\begin{align*}
\leq 6 E \left| \sum_{i=1}^{n} \int_{t_{n-1}, t_n} \int_{\mathbb{R}^d} \left( v_{n-1}^{h, \tau}, J_{n-1}^{h}(z)v_{n-1}^{h, \tau} + r_t(z) \right)^2 \frac{\nu(dz)}{m} dt \right|^{1/2}
\end{align*}
\]

\[
\leq \frac{1}{4} E \max_{0 \leq n \leq T} \|u_n^{h, \tau}\|_m^2 + N \left( E \sum_{l=0}^{T-1} \tau E \| \delta_{h, l} u_{l}^{h, \tau}\|_m^2 + E \sum_{l=0}^{T-1} \tau E \| u_{l}^{h, \tau}\|_m^2 \right)
\]

\[
+ NE \int_{[0, T]} \int_{\mathbb{R}^d} \| r_t(z) \|_m^2 \nu(dz) dt.
\]

We can estimate \( E \max_{0 \leq n \leq T} \sum_{l=1}^{n} I_1(t_l) \) in similar way. Combining the above \( E \max_{0 \leq n \leq T} \)-estimates and (46), we obtain (42).

The following theorem establishes the existence and uniqueness of a solution to (23) and the stability of the implicit-explicit approximation scheme.

**Theorem 3.7.** Let Assumption 1 hold with \( m \geq 0 \). Let \( f^i \in H^m \) for \( i \in \{0, ..., d\} \), \( g \in H^m(\mathbb{L}_2) \) and \( r \in H^m(\nu) \). Then there exists a constant \( R = R(d, m, \kappa, K, \delta, \nu) \) such that if \( T > R \), then for any \( h \neq 0 \), there exists a unique \( H^m \)-valued solution \( (v_n^{h, \tau})_{n=0}^{T} \) of

\[
\begin{align*}
v_n^{h, \tau} &= v_{n-1}^{h, \tau} + \int_{[t_{n-1}, t_n]} \left( (\tilde{L}_{n-1}^h + I_{\kappa}^h) v_n^{h, \tau} + \tilde{I}_{h, \delta} v_n^{h, \tau} + \sum_{i=0}^{d} \delta_{h, i} f^i \right) dt \\
&\quad + \sum_{\ell=1}^{\infty} \int_{[t_{n-1}, t_n]} \left( \mathbb{I}_{n>1} M_{n-1}^{h, \delta} v_{n-1}^{h, \tau} + g^\ell \right) d\nu^\ell \\
&\quad + \int_{[t_{n-1}, t_n]} \int_{\mathbb{R}^d} \left( \mathbb{I}_{n>1} J^h(z) v_{n-1}^{h, \tau} + r_t(z) \right) \bar{\mu}(dz, dt),
\end{align*}
\]

for \( n \in \{1, \ldots, T\} \), for any \( H^m \)-valued \( \mathcal{F}_0 \)-measurable initial condition \( \psi \).

Moreover, if \( E \| \psi \|_m^2 < \infty \), then there exists a constant \( N = N(d, m, \kappa, K, T, \delta, \nu) \) such that

\[
\begin{align*}
E \max_{0 \leq n \leq T} \|v_n^{h, \tau}\|_m^2 + E \sum_{n=0}^{T} \tau \sum_{i=0}^{d} \| \delta_{h, i} v_n^{h, \tau} \|_m^2 &\leq NE \| \psi \|_m^2 \\
+ NE \int_{0}^{T} \left( \sum_{i=0}^{d} \| f^i \|_m^2 + \| g^i \|_m^2 + \int_{\mathbb{R}^d} \| r_t(z) \|_m^2 \nu(dz) \right) dt.
\end{align*}
\]
Proof. For each \( n \in \{1, \ldots, T\} \), we write (47) as
\[
D_n(v_h^{h, n}) = y_{n-1},
\]
where \( D_n \) is the operator defined by
\[
D_n(\phi) := \phi - \tau \left( \bar{L}_h^{h, n} + I_\delta^h \right) \phi
\]
and
\[
y_{n-1} := v_h^{h, n} + \int_{[t_{n-1}, t_n]} \left( \bar{I}_h^{h, n} + \sum_{i=0}^d \delta h_i f_i \right) dt
\]
\[
+ \sum_{\varrho=1}^{\infty} \int_{[t_{n-1}, t_n]} \left( \bar{I}_{n>1} M_{t_{n-1}} f_h^{h, \varrho} + g_{t}^\varrho \right) dw_t^\varrho
\]
\[
+ \int_{[t_{n-1}, t_n]} \int_{\mathbb{R}^d} \left( \bar{I}_{n>1} J_h(z) v_h^{h, n} + r_t(z) \right) \tilde{\mu}(dz, dt).
\]
Fix an \( \epsilon_1, \epsilon_2 \in (0, 1) \) such that
\[
\bar{q}_1 := \kappa - \varsigma(\delta) - \epsilon_1 > 0.
\]
and
\[
\bar{q}_2 := \kappa - 2\varsigma(\delta) - \epsilon_2 > 0.
\]
Owing to Lemma 3.5, there exists a constant \( N = N(d, m, K, \delta, \nu) \) such that for all \( \phi \in H^m \),
\[
\|D_n \phi\|_m^2 \leq N \left( 1 + \tau^2 \left( \frac{1}{h^2} + \frac{1}{h^4} \right) \right) \|\phi\|_m^2. \tag{49}
\]
Assume \( T > T_N \). By (32), for all \( \phi \in H^m \), we have
\[
(\phi, D_n \phi)_m \geq (1 - \tau N_2) \|\phi\|_m^2 + \bar{q}_1 \tau \sum_{i=1}^d \|\delta h_i \phi\|_m^2 \geq (1 - \tau N_2) \|\phi\|_m^2. \tag{50}
\]
Using Jensen’s inequality and (39), we get
\[
\|y_0\|_m^2 \leq 5 \left( 1 + \nu(\{|z| > \delta\})^2 \tau^2 \right) \|\psi\|_m^2
\]
\[
+ \frac{20\tau}{h^2} \sum_{i=0}^d \|f_i\|_m^2 dt + 5 \left\| \sum_{\varrho=1}^{\infty} \int_{[0, t_1]} g_{t}^\varrho dw_t^\varrho \right\|_m^2
\]
\[
+ 5 \left\| \int_{[0, t_1]} \int_{\mathbb{R}^d} r_t(z) \tilde{\mu}(dz, dt) \right\|_m^2. \tag{51}
\]
Since \( \psi \in H^m, f^i \in \mathbb{H}^m, i \in \{0, 1, \ldots, d\}, g \in \mathbb{H}^m(\ell_2), \) and \( r \in \mathbb{H}^m(\nu) \), it follows that \( y_0 \in H^m \). By (49) and (50), owing to Proposition 3.4 in [15] \((p = 2)\), there exists a unique \( v_1^{h, \tau} \) in \( H^m \) such that \( D_1 v_1^{h, \tau} = y_0 \), and moreover
\[
\|v_1^{h, \tau}\|_m^2 \leq 1 + \frac{\|y_0\|_m^2}{(1 - \tau N_2)^2} < \infty. \tag{52}
\]
Proceeding by induction on \( n \in \{1, \ldots, T\} \), one can show that there exists a unique \( v_n^{h, \tau} \) in \( H^m \) such that \( D_n v_n^{h, \tau} = y_{n-1} \), and moreover
\[
\|v_n^{h, \tau}\|_m^2 \leq 1 + \frac{\|y_{n-1}\|_m^2}{(1 - \tau N_2)^2} < \infty. \tag{53}
\]
Assume that \( E\|\psi\|_m^2 \leq \infty \). By (51) and (52) and the fact that \( f^i \in \mathbb{H}^m, i \in \{0, 1, \ldots, d\}, g \in \mathbb{H}^m(\ell_2), \) and \( r \in \mathbb{H}^m(\nu) \), it follows that \( E\|v_1^{h, \tau}\|_m^2 \leq \infty \). By Jensen’s inequality, (39), and (38), we have
\[
E\|y_{n-1}\|_m^2 \leq 7N \left( 1 + \nu(\{z > \delta\})^2 \right) \tau^2 + \|y_{n-1}\|_m^2 \left( 1 + \frac{1}{h^2} \right) + \frac{28\tau}{h^2} E \int_{|0,t_1|} \sum_{i=0}^d \|f^i_t\|_m^2 dt + 7E \int_{|0,t_1|} \|g_t\|_{m, \ell_2} dt + 7E \int_{|0,t_1|} \int_{\mathbb{R}^d} \|r_t(z)\|_m^2 \nu(dz) dt. \tag{54}
\]
Proceeding by induction on \( n \) and combining (53) and (54), we obtain
\[
E\|v_n^{h, \tau}\|_m^2 < \infty, \quad \forall n \in \{0, 1, \ldots, T\}. \tag{55}
\]
Applying the identity \( \|y\|_m^2 - \|x\|_m^2 = 2\langle x, y - x \rangle_m + \|y - x\|_m^2, x, y \in H^m \), for any \( n \in \{1, \ldots, T\} \), we have
\[
\|v_n^{h, \tau}\|_m^2 = \|v_{n-1}^{h, \tau}\|_m^2 + \sum_{i=1}^6 I_i(t_n), \tag{56}
\]
where
\[
I_1(t_n) := 2\tau(v_n^{h, \tau}, (\bar{I}_{t_n} + I_{\delta}^h) v_n^{h, \tau})_m + 2\tau(v_{n-1}^{h, \tau}, (\bar{I}_{t_n} + I_{\delta}^h) v_{n-1}^{h, \tau})_m + \|\eta(t_n)\|_m^2,
\]
\[
I_2(t_n) := 2 \int_{|t_{n-1}, t_n|} \sum_{i=0}^d (v_n^{h, \tau}, \delta h, i f^i) d_0 m dt,
\]
\[
I_3(t_n) := \int_{|t_{n-1}, t_n|} \sum_{i=0}^d (v_n^{h, \tau}, \delta h, i f^i) d_1 m dt,
\]
\[
I_4(t_n) := \int_{|t_{n-1}, t_n|} \sum_{i=0}^d (v_n^{h, \tau}, \delta h, i f^i) d_2 m dt,
\]
\[
I_5(t_n) := \int_{|t_{n-1}, t_n|} \sum_{i=0}^d (v_n^{h, \tau}, \delta h, i f^i) d_3 m dt,
\]
\[
I_6(t_n) := \int_{|t_{n-1}, t_n|} \sum_{i=0}^d (v_n^{h, \tau}, \delta h, i f^i) d_4 m dt.
\]
Assume $T$ we get that there exist a constant $\kappa$ and where

As in the proof Theorem 3.6, by Young’s inequality, (31), and (39), we have

Set $E \parallel (v_{t_n}^{h,\tau}, \eta_{t_n}) \parallel^2 R m \leq 2 (1 + 2 \nu(\{ |z| > \delta \})) E \parallel \psi \parallel^2 m - \bar{q}_2 E \sum_{l=1}^n \tau \sum_{i=1}^d \parallel \delta_{h,i} v_{t_l}^{h,\tau} \parallel^2 m

I_4(t_n) := 2 \sum_{\tau=1}^\infty \int_{[t_{\tau-1}, t_\tau]} \left( v_{t_\tau}^{h,\tau}, \eta_{t_\tau} \right) \parallel \sum_{i=1}^d \left( \delta_{h,i} f_i^t \right) dt + \parallel \delta_{h,i} v_{t_l}^{h,\tau} \parallel^2 m \tau^2$,

$I_5(t_n) := 2 \int_{[t_{\tau-1}, t_\tau]} \int_{\mathbb{R}^d} \left( v_{t_\tau}^{h,\tau}, \eta_{t_\tau} \right) \parallel \int_{\mathbb{R}^d} \left( \sum_{i=1}^d \left( \delta_{h,i} f_i^t \right) dt \right) \mu(dz, dt),

I_6(t_n) := \left( \tau \delta_{h,i} v_{t_l}^{h,\tau}, \eta(t_n) \right) m,$

and where

$\eta(t_n) := \sum_{\tau=1}^\infty \int_{[t_{\tau-1}, t_\tau]} \left( \sum_{i=1}^d \left( \delta_{h,i} f_i^t \right) dt \right) \mu(dz, dt),

As in the proof Theorem 3.6 by Young’s inequality, (31), and (39), we have

$E \parallel v_{t_n}^{h,\tau} \parallel^2 m \leq (1 + 2 \nu(\{ |z| > \delta \})) E \parallel \psi \parallel^2 m - \bar{q}_2 E \sum_{l=1}^n \tau \sum_{i=1}^d \parallel \delta_{h,i} v_{t_l}^{h,\tau} \parallel^2 m

+ E \sum_{l=1}^n \tau \left( N_2 + 2 \nu(\{ |z| > \delta \}) + \tau \nu(\{ |z| > \delta \})^2 \right) \parallel v_{t_l}^{h,\tau} \parallel^2 m

+ N E \int_{[0, t_n]} \left( \sum_{i=1}^d \parallel f_i^t \parallel^2 m dt + \parallel g_i \parallel^2 m \right) + \int_{\mathbb{R}^d} \parallel r_i(z) \parallel^2 m \nu(dz) \right) dt.

Set

$Z := N_2 + 2 \nu(\{ |z| > \delta \})$,

$R := \max \left( \frac{2 \nu(\{ |z| > \delta \})^2}{\sqrt{Z^2 + 4 \nu(\{ |z| > \delta \})^2} - Z}, N_2 \right) T.$

Assume $T > R$. Making use of (55) and applying discrete Gronwall’s lemma, we get that there exist a constant $N(d, m, K, \kappa, T, \delta, \nu)$ such that

$E \parallel v_{t_n}^{h,\tau} \parallel^2 m + E \sum_{l=1}^n \tau \sum_{i=0}^d \parallel \delta_{h,i} v_{t_l}^{h,\tau} \parallel^2 m \leq N E \parallel \psi \parallel^2 m$
+ \frac{NE}{2} \int_{[0,T]} \left( \sum_{i=0}^{d} \|f^i_t\|_{m}^2 + \sum_{i,j=0}^{d} \|a^{ij}_t\|_{m,\ell_2}^2 + \int_{\mathbb{R}^d} \|r_t(z)\|_{m}^2 \nu(dz) \right) dt. \tag{57}

Using (39) instead of (40), we obtain (48) from (57) in the same manner as Theorem 3.6. Note that no bound on $\tau/h^2$ is needed in this case. 

\section{Proof of the main results}

\textbf{Proof of Theorem 2.1.} By virtue of Theorems 2.9, 2.10, and 4.1 in [11], in order to obtain the existence, uniqueness, regularity, and the estimate (9), we only need to show that (4) may be realized as an abstract stochastic evolution equation in a Gelfand triple and that the growth condition and coercivity condition are satisfied. Indeed, since (4) is a linear equation, the evolution equation in a Gelfand triple and that the growth condition and monotonicity follows directly from the coercivity condition. By Holder’s inequality and Assumption 1 (i), for $u, v \in H^1$, we have

\[
\sum_{i,j=0}^{d} \left( \partial_j u, (v \partial_{-i} a^{ij}_t + a^{ij} \partial_{-i} v) \right)_0 + \int_{|z|>\delta} \left( u(\cdot + z) - u - \sum_{j=1}^{d} z^j \partial_j u, v \right)_0 \nu(dz)
+ \int_{|z|\leq\delta} \sum_{i,j=1}^{d} \left( z^j \partial_j u(\cdot + \theta z), z^j \partial_{-i} v \right)_0 (1 - \theta)d\theta \nu(dz) \leq N \|u\|_1 \|v\|_1. \tag{58}
\]

Therefore, since the pairing $[\cdot, \cdot]_0$ brings $(H^1)^*$ and $H^{-1}$ into isomorphism, for each $(\omega, t) \in [0, T] \times \Omega$, there exists a linear operator $\hat{A}_t : H^1 \to H^{-1}$ such that $[u, \hat{A}_t u]_0$ agrees with the left-hand-side of the above inequality and for $u, v \in H^1$, $\|A_t u\|_{-1} \leq N \|u\|_1$. By Assumption 2 the operator $A$ defined by $A(u) = \hat{A} u + f$, maps $H^1$ to $H^{-1}$ and for $u \in H^1$, $\|A_t(u)\|_{-1} \leq N(\|u\|_1 + \|f\|_{-1})$. 

For an integer $m \geq 2$, with abuse of notation, we write

\[
(\cdot, \cdot)_m = ((1 - \Delta)^{m/2}\cdot, (1 - \Delta)^{m/2}\cdot)_0.
\]

and $\|\cdot\|_m$ for the corresponding norm in $H^m$. It is well known that the above inner product and norm are equivalent to the ones introduced in Section 1. For each $m \geq 1$ and for all $u \in H^{m+1}$ and $v \in H^m$, we have $(u, v)_m \leq \|u\|_{m+1} \|v\|_{m-1}$. Since $H^{m+1}$ is dense in $H^{m-1}$, we may define the pairing $[\cdot, \cdot]_m : H^{m+1} \times H^{m-1} \to \mathbb{R}$ by $[v, v']_m = \lim_{n \to \infty} (v, v_n)_m$ for all $v \in H^{m+1}$ and $v' \in H^{m-1}$, where $(v_n)_{n=1}^{\infty} \subset H^{m+1}$ is such that $\|v_n - v'\|_{m-1} \to 0$ as

\[30\]
$n \to \infty$. It can be shown that the mapping from $H^{m-1}$ to $(H^{m+1})^*$ given by $v' \mapsto \langle \cdot, v' \rangle_m$ is an isometric isomorphism. For more details, see [22]. Therefore, for all $m \geq 0$, $(H^{m+1}, H^m, H^{m-1})$ forms a Gelfand triple with the pairing $\langle \cdot, \cdot \rangle_m$.

For $m \geq 1$, using integration by parts, for $u \in H^{m+1}$ and $v \in H^m$, we have $[v, A_t(u)]_0 = ((L_t + I_t)u + f, v)_0 = [v, (L_t + I_t)u + f]_0$. Since this is true for all $v \in H^m$, which is dense in $H^1$ when $m \geq 1$, the restriction of $A$ to $H^{m+1}$ coincides with $L + I + f$. Moreover, when $m \geq 1$, it can easily be shown under Assumptions I(i) and II that for $u, v \in H^{m+1}$, $\|A_t(u)\|_{m-1} \leq N\|u\|_{m+1} + \|f\|_{m-1}$, where $N$ is a constant depending only on $m, d, K, \nu$, which shows that $A$ satisfies the growth condition. For $u \in H^m$, define $B_t^\varphi(u) = b_t^\varphi \partial_z u + g_t^\varphi$ and $B_t = (B_t^\varphi)_{\varphi=1}^\infty$ and $C(z, u) = u(\cdot + z) - u$. Owing to Assumption I(i), $B_t$ is an operator from $H^{m+1}$ to $H^m(\ell_2)$ and it is easy to see that for each $z \in \mathbb{R}^d$, $C(z, \cdot)$ is a bounded linear operator on $H^m$.

Under the current assumptions, $A, B_t$, and $C$ are appropriately measurable. Thus, (I) may be realized as the following stochastic evolution equation in the Gelfand triple $(H^{m+1}, H^m, H^{m-1})$:

$$u_t = u_0 + \int_{[0,t]} A_s(u_s)ds + \sum_{\varphi=1}^\infty \int_{[0,t]} B_s^\varphi(u_s)dw_s^\varphi + \int_{[0,t]} C(z, u_{s-})\tilde{\mu}(dz, ds), \quad (59)$$

for $t \in [0, T]$. Let $u \in C^\infty_c(\mathbb{R}^d)$. By (7), we have

$$\int_{\mathbb{R}^d} \|u(\cdot + z) - u\|^2_m \nu(dz) \leq N(\delta')\|u\|^2_{m+1} + N\|u\|^2_m, \quad (60)$$

where $N(\delta') \to 0$ as $\delta' \to 0$ and where $N$ is a constant depending only on $\nu$. Applying Holder’s inequality and the identity $(u, \partial_j u) = 0$, we obtain

$$\int_{|z| > \delta'} \left( u(\cdot + z) - u - \sum_{j=1}^d z_j \partial_j u, u \right)_m \nu(dz) \leq 0,$$

and by Holder’s inequality, we have

$$\int_{|z| \leq \delta'} \int_0^1 \sum_{i,j=1}^d \left( z_j \partial_j u(\cdot + \theta z), z_i \partial_{-i} u \right)_m \theta (1 - \theta) d\theta \nu(dz) \leq N(\delta')\|u\|^2_{m+1},$$

where $N(\delta')$ is as in (60). As in Theorem 4.1.2 in [22] and Lemma 3.1, using Holder’s and Young’s inequalities, the above estimates, and Assumption I
there exists a constant $N = N(d, m, \kappa, K, T, \nu)$ such that for all $(\omega, t) \in \Omega \times [0, T],
\begin{align*}
2[u, A_t(u)]_m + \|B_t(u)\|_{m,t}^2 + \int_{\mathbb{R}^d} \|C(z, u)\|_m^2 \nu(dz) + \frac{K}{8} \|u\|_{m+1}^2
\leq L\|u\|_m^2 + \tilde{f}_t,
\end{align*}
where $\tilde{f} = N(\|f\|_{m-1} + \|g\|_{m, \ell_2})$. When $m \geq 1$, using the self-adjointness of $(1 - \Delta)^{1/2}$, the properties of the CBF $[\cdot, \cdot]_m$, and Assumption 2 for $v \in C_c^\infty(\mathbb{R}^d)$ and $u \in H^{m+1}$, we have
\begin{align*}
[v, A(u)]_m = ((L + I)u, (1 - \Delta)^m v)_0 + (f, (1 - \Delta)^m v)_0.
\end{align*}
Owing to (61) and the denseness of $(1 - \Delta)^{-m} C_c^\infty(\mathbb{R}^d)$ in $H^1$, from Theorems 2.9, 2.10, and 4.1 in [11], we obtain the existence and uniqueness of a solution $u$ of (1), such that $u$ is a càdlàg $H^m$-valued process satisfying (61).

Proof of Proposition 2.2. Let $A$, $B$, and $C$ be as in (59). Owing to Assumption 1, the boundedness of the $m - 1$-norm of $g$ in expectation, and estimate (9), using Jensen’s inequality and Itô’s isometry, for $s, t \in [0, T]$, we get
\begin{align*}
E \left[ \left\| \int_{[s,t]} A_r(u_r) ds \right\|_{m-1}^2 \right]
\leq & \ |t - s| \left( NE \int_{[0,T]} \|u_t\|_{m+1}^2 dt + E \int_{[0,T]} \|f_t\|_{m-1}^2 dr \right) \leq N|t - s|,
\end{align*}
\begin{align*}
E \left[ \left\| \int_{[s,t]} \sum_{q=1}^{\infty} B_r^q(u_r) dw^q_r \right\|_{m-1}^2 \right] = E \int_{[s,t]} \|B_r(u_r)\|_{m-1, \ell_2}^2 dr
\leq & \ N|t - s| \left( \sup_{t \leq T} E\|u_t\|_m^2 + \sup_{t \leq T} E\|g_t\|_{m, \ell_2} \right) \leq N|t - s|,
\end{align*}
and
\begin{align*}
E \left[ \left\| \int_{[s,t]} \int_{\mathbb{R}^d} C(z, u_r) q(dr, dz) \right\|_{m-1}^2 \right] = E \int_{[s,t]} \int_{\mathbb{R}^d} \|C(z, u_r)\|_{m-1}^2 \nu(dz) ds
\leq & \ N|t - s| \sup_{t \leq T} E\|u_t\|_m^2 \leq N|t - s|,
\end{align*}
which completes the proof of the proposition. \hfill \Box
Theorem 4.1. Let Assumptions 1 through 5 hold for some $m \geq 2$. Let $u$ be the solution of (4) and $(u_n^{h,\tau})_{n=0}^T$ be defined by (22). Then there exists a constant $N = N(d, m, \kappa, K, T, \lambda, \kappa_m^2, \delta, v)$ such that

$$E \max_{0 \leq n \leq T} \| u_n - u_n^{h,\tau} \|_{m-2}^2$$

$$+ E \sum_{t=0}^T \sum_{i=0}^d \| \delta_{h,i} u_{ti} - \delta_{h,i} u_{ti}^{h,\tau} \|_{m-2}^2 ds \leq N(\tau + |h|).$$

(62)

**Proof.** For $t \in [0, T]$, let $\kappa_1(t) := t_{n-1}$ for $t \in [t_{n-1}, t_n]$, and set $e_n^{h,\tau} := u_n^{h,\tau} - u_{tn}$. One can easily verify that $e_n^{h,\tau}$ satisfies in $H^{m-2}$,

$$e_n^{h,\tau} = e_{n-1}^{h,\tau} + \int_{t_{n-1}, t_n} \left( (L^h_{t_{n-1}} + I^h_{t_{n-1}}) e_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_{t_i}^i \right) dt$$

$$+ \sum_{\theta=1}^{\infty} \int_{t_{n-1}, t_n} \left( M^h_{t_{n-1}} e_{n-1}^{h,\tau} + G_{t_i}^\theta \right) du_{t_i}$$

$$+ \int_{t_{n-1}, t_n} \int_{\mathbb{R}^d} \left( J^h(z) e_{n-1}^{h,\tau} + R_t(z) \right) \tilde{\mu}(dz, dt),$$

(63)

where

$$F_{t_i}^0 := (L^h_{\kappa_1(t)} - L_{\kappa_1(t)}) u_t + (L_{\kappa_1(t)} - L_t) u_t$$

$$+ (I^h - I) u_t + (f_{\kappa_1(t)} - f_t) + I^h_{\kappa_1(t)} (u_{\kappa_1(t)} - u_t)$$

$$+ \sum_{j=1}^d a_{\kappa_1(t)}^0 \delta_{h,j} (u_{\kappa_1(t)} - u_t) + \sum_{i=0}^d a_{\kappa_1(t)}^i \delta_{h,i} (u_{\kappa_1(t)} - u_t)$$

$$- \sum_{i,j=1}^d \delta_{h,j} (u_{\kappa_1(t)} - u_t)(\cdot + h) \delta_{h,i} a_{\kappa_1(t)}^{ij},$$

$$F_{t_i}^i := \sum_{j=1}^d a_{\kappa_1(t)}^i \delta_{h,j} (u_{\kappa_1(t)} - u_t)$$

$$+ \sum_{k=0}^{\sigma(h,k)-1} \sigma(h,k-1) \delta_{h,k} (u_{\kappa_1(t)} - u_t)(\cdot + h) z_i^k, k, k,$$
\[ G^g_t := (M^g_{\kappa_1(t)} - M^e_t)u_t + (M^{g,e}_{\kappa_1(t)} - M^e_t)u_t + M^e_{\kappa_1(t)}(u_{\kappa_1(t)} - u_t) + (g^e_{\kappa_1(t)} - g^g_t) \]

\[ R^h_t(z) := \left(J^h(z) - J(z)\right) u_t - J^h(z)(u_{\kappa_1(t)} - u_t). \]

By Theorem 3.6, we have

\[ E \max_{0 \leq n \leq T} \|e_{n}^{h,\tau}\|^{2}_{m-2} + E \sum_{n=0}^{M} \sum_{i=0}^{d} \|\delta_{h,i}e_{n}^{h,\tau}\|^{2}_{m-2} \]

\[ \leq NE \int_{[0,T]} \left( \sum_{i=0}^{d} \|F_{t,i}\|^{2}_{m-2} + \|G_{t}\|^{2}_{m-2,\ell_2} + \int_{\mathbb{R}^{d}} \|R_{t}(z)\|^{2}_{m-2,\ell_2} \nu(dz) \right) dt. \quad (64) \]

Using Lemmas 3.1, 3.2, and 3.3 and Assumptions 1 (i) and 4, the right-hand-side of the above relation can be estimated by

\[ NE \int_{[0,T]} \left( \|h\|^{2}_{m-2} + \|\kappa_{1}(t) - t\|u_{t}\|^{2}_{m} + \|u_{\kappa_1(t)} - u_{t}\|^{2}_{m-1} \right) dt \]

\[ + NE \int_{[0,T]} \left( \|f_{\kappa_1(t)} - f_{t}\|^{2}_{m-2} + \|g_{\kappa_1(t)} - g_{t}\|^{2}_{m-2,\ell_2} \right) dt \]

where \( N \) depends only on \( d, m, \kappa, K, \lambda, T, \delta \) and \( \nu \). By virtue of (3), Proposition 2.2, and Assumption 3, we obtain (62), which completes the proof. □

**Theorem 4.2.** Let Assumptions 1 through 4 hold with \( m \geq 2 \) and let \( u \) be the solution of (4). There exists a constant \( R = R(d, m, \kappa, K, \delta, \nu) \) such that if \( T > R \), then there exists a unique solution \((v^{h,\tau})_{n=0}^{T}\) of (23) in \( H^{m-2} \). Moreover, there exists a constant \( N = N(d, m, \kappa, K, T, C, \lambda, K_{m}, \delta, \nu) \) such that

\[ E \max_{0 \leq n \leq T} \|u_{t_{n}} - v_{n}^{h,\tau}\|^{2}_{m-2} \]

\[ + E \sum_{i=0}^{T} \sum_{i=0}^{d} \|\delta_{h,i}u_{t_{i}} - \delta_{h,i}v_{i}^{h,\tau}\|^{2}_{m-2}ds \leq N(\tau + \|h\|^{2}). \quad (65) \]

**Proof.** The existence and uniqueness follows directly from Theorem 3.7. Let \( \kappa_1(t) \) be as in the previous proof and set \( \kappa_2(t) = t_n \) for \( t \in [t_{n-1}, t_n] \). Let \( G \) and \( J \) be defined as in Theorem 4.1 and define \( F^{i} \) to be \( F^{i} \) with \( \kappa_1(t) \)
replaced with $\kappa_2(t)$. Set $e_n^{h,\tau} = v_n^{h,\tau} - u_{t_n}$. As in the proof of Theorem 4.1 we have
\[
e_n^{h,\tau} = e_{n-1}^{h,\tau} + \int_{t_{n-1},t_n} \left( (\tilde{L}_n^h + I_n^h)e_n^{h,\tau} + \tilde{I}_{\delta_n}^{h}e_{n-1}^{h,\tau} + \sum_{i=0}^{d} \delta_{h,i}\tilde{F}_t \right) dt \\
+ \sum_{\varrho=1}^{\infty} \int_{[t_{n-1},t_n] \cap \mathbb{T}_{\varrho}} \left( \mathbb{I}_{n>1} M_{t_n-1}^{h,\varrho} e_{n-1}^{h,\tau} + \tilde{G}_t^{\varrho} \right) d\tilde{w}_t^{\varrho} \\
+ \int_{[t_{n-1},t_n]} \int_{\mathbb{R}^{d}} \left( \mathbb{I}_{n>1} J_n^h(z) e_{n-1}^{h,\tau} + \tilde{R}_t(z) \right) \tilde{p}(dz,dt),
\]
where
\[
\tilde{F}_t = F_t, \text{ for } i \neq 0, \quad \tilde{F}_0 = F_0 + \tilde{I}_{\delta_0}^h(u_{\kappa_1(t)} - u_{\kappa_2(t)}), \\
\tilde{G}_t^{\varrho} = \mathbb{I}_{t\leq t_1} (M_t^{\varrho} u_t + g_t^{\varrho}) + \mathbb{I}_{t> t_1} G_t^{\varrho} \\
\tilde{R}_t = \mathbb{I}_{t\leq t_1} J_t u_- + \mathbb{I}_{t> t_1} R_t.
\]
By Theorem 3.7 we have
\[
E \max_{0 \leq n \leq T} \|e_n^{h,\tau}\|_{m-2}^2 + E \sum_{n=0}^{M} \tau \sum_{i=0}^{d} \|\delta_{h,i} e_n^{h,\tau}\|_{m-2}^2 \leq \\
+ NE \int_{[0,T]} \sum_{i=0}^{d} \|\tilde{F}_t^i\|^2_{m-2} dt \\
+ N \int_{[t_1,T]} \left( \|G_t\|^2_{m-2,\ell_2} + \int_{\mathbb{R}^{d}} \|R_t(z)\|^2_{m-2,\nu} \nu(dz) \right) dt \\
+ NE \int_{[0,T]} \|\tilde{I}_{\delta_0}^h(u_{\kappa_1(t)} - u_{\kappa_2(t)})\|^2_{m-2} dt \\
+ NE \int_{[0,t_1]} \left( \|M_t u_t + g_t\|^2_{m-2,\ell_2} + \int_{\mathbb{R}^{d}} \|J_n^h(z) u_t\|^2_{m-2,\nu} \nu(dz) \right) dt \\
=: A_1 + A_2 + A_3.
\]
As in the proof of Theorem 4.1 we have $A_1 \leq N(\tau + |h|^2)$. By Proposition 2.2 we get
\[
A_2 \leq NE \int_{0}^{T} \|u_{\kappa_1(t)} - u_{\kappa_2(t)}\|^2_{m-1} dt \leq N\tau.
\]
Owing to (3), we have

\[ A_3 \leq NE \int_0^{t_1} \left( \|u_t\|_{m-1}^2 + \|g_t\|_{m-2,\ell^2}^2 \right) dt \]

\[ \leq N\tau E \int_0^{t_1} \left( \sup_{t \leq T} \|u_t\|_{m-1}^2 + \xi \right) dt \leq N\tau. \]

Combining these estimates gives (65).

By virtue of Sobolev’s embedding theorem and (18), as in [12], we obtain the following corollaries of Theorem 4.1 and Theorem 4.2.

**Corollary 4.3.** Suppose the assumptions of Theorem 4.1 hold with \( m > n + 2 + d/2 \), where \( n \) is an integer with \( n \geq 0 \). Then for all \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \{1, \ldots, d\}^n \) and \( \delta_{h,\lambda} = \delta_{h,\lambda_1} \cdots \delta_{h,\lambda_n} \), there exists a constant \( N = N(d, m, k, K, T, C, \lambda, \kappa_m, \delta, \nu) \) such that

\[ E \max_{0 \leq n \leq T} \sup_{x \in \mathbb{R}^d} |\delta_{h,\lambda} u_{tn}(x) - \delta_{h,\lambda} u_{tn}^h(x)|^2 \]

\[ + E \max_{0 \leq n \leq T} \|\delta_{h,\lambda} u_{tn} - \delta_{h,\lambda} u_{tn}^h\|_{\ell^2(\mathcal{G}_h)}^2 \leq N(\tau + |h|^2). \]

**Corollary 4.4.** Suppose the assumptions of Theorem 4.2 hold with \( m > n + 2 + d/2 \), where \( n \) is an integer with \( n \geq 0 \). Then for all \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \{1, \ldots, d\}^n \) and \( \delta_{h,\lambda} = \delta_{h,\lambda_1} \cdots \delta_{h,\lambda_n} \), there exists a constant \( N = N(d, m, k, K, T, C, \lambda, \kappa_m, \delta, \nu) \) such that

\[ E \max_{0 \leq n \leq T} \sup_{x \in \mathbb{R}^d} |\delta_{h,\lambda} u_{tn}(x) - \delta_{h,\lambda} v_{tn}^h(x)|^2 \]

\[ + E \max_{0 \leq n \leq T} \|\delta_{h,\lambda} u_{tn} - \delta_{h,\lambda} v_{tn}^h\|_{\ell^2(\mathcal{G}_h)}^2 \leq N(\tau + |h|^2). \]

**Proof of Theorems 2.3 and 2.4.** Let \((\tilde{\phi}_n^h)_{n=0}^M\) be defined by (10). Denote by \((\cdot, \cdot)_{\ell^2(\mathcal{G}_h)}\) the inner product of \( \ell^2(\mathcal{G}_h) \). There exists a constant \( \epsilon = \epsilon(\alpha, \delta, \nu) \) such that

\[ \overline{\theta} := \alpha - \sigma(\delta) - \epsilon > 0. \]

As in (32), there exists a constant \( N_6 = N_6(d, \kappa, K, \delta, \nu) \) such that for all \( \phi \in \ell^2(\mathcal{G}_h) \),

\[ (\phi, \tilde{L}_t^h \phi)_{\ell^2(\mathcal{G}_h)} + (\phi, \tilde{L}_0^h \phi)_{\ell^2(\mathcal{G}_h)} \leq -\overline{\theta} \sum_{i=1}^d \|\delta_{h,i} \phi\|_{\ell^2(\mathcal{G}_h)}^2 + N_6 \|\phi\|_{\ell^2(\mathcal{G}_h)}^2. \]
Following the arguments in the beginning of the proof of Theorem 3.7 we conclude that if $T > N_6 T$, then there exists a unique solution $(\hat{v}^{h,\tau}_{n=0})$ in $\ell_2(\mathbb{G}_h)$ of (17). It is easy to see that $N_6 < N_2$ (for the same choice of $\epsilon$) for all $m > 0$, where $N_2$ is the constant appearing on the right-hand-side of (32), and hence $N_6 < R$, where $R$ is as in Theorem 3.7. Let $(u^{h,\tau}_{n=M})$ be defined by (22). By Theorem 4.2 there exists a unique solution $(v^{h,\tau}_{n=1})$ of (23). It suffices to show that almost surely, 

$$u^{h,\tau}_{n}(x) = \hat{u}^{h,\tau}_{n}(x)$$

and

$$v^{h,\tau}_{n}(x) = \hat{v}^{h,\tau}_{n}(x),$$

for all $n \in \{0, ..., M\}$ and $x \in \mathbb{G}_h$. Let $\mathcal{S} : H^{m-2} \rightarrow \ell_2(\mathbb{G}_h)$ denote the embedding from Remark 2.1. Applying $\mathcal{S}$ to both sides of (22), one can see that $\mathcal{S}u^{h,\tau}$ and $\hat{u}^{h,\tau}$ satisfy the same recursive relation in $\ell_2(\mathbb{G}_h)$ with common initial condition $\psi$, and hence (67) follows. Similarly, $\mathcal{S}v^{h,\tau}$ and $\hat{v}^{h,\tau}$ satisfy the same equation in $\ell_2(\mathbb{G}_h)$ and (68) follows from the uniqueness of the $\ell_2(\mathbb{G}_h)$ solution of (17).

Remark 4.1. It follows from Corollaries 4.3, 4.4, and relations (67) and (68) that if more regularity is assumed of the coefficients and the data of the equation (4), then better estimates can be obtained than the ones presented in Theorems 2.3 and 2.4.

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