Generalized $q$-Bernoulli polynomials generated by Jackson $q$-Bessel functions

S.Z.H. Eweis$^a$ and Z.S.I. Mansour$^b$

$^a$Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt.
$^b$Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.

ARTICLE HISTORY
Compiled January 26, 2022

ABSTRACT
In this paper, we introduce the polynomials $B^{(k)}_{n,\alpha}(x;q)$ generated by a function including Jackson $q$-Bessel functions $J^{(k)}_{\alpha}(x;q)$ ($k = 1, 2, 3$), $\alpha > -1$. The cases $\alpha = \pm \frac{1}{2}$ are the $q$-analogs of Bernoulli and Euler’s polynomials introduced by Ismail and Mansour for $(k = 1, 2)$, Mansour and Al-Towalib for $(k = 3)$. We study the main properties of these polynomials, their large $n$ degree asymptotics and give their connection coefficients with the $q$-Laguerre polynomials and little $q$-Legendre polynomials.

KEYWORDS
$q$-Bessel functions, $q$-Bernoulli polynomials and numbers, asymptotic expansions, cauchy residue theorem.

AMS CLASSIFICATION
05A30, 11B68, 30E15, 32A27

1. Introduction and Preliminaries
The Bernoulli polynomials $(B_n(x))_n$ are defined by the generating function

$$
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi.
$$

In a series of papers, Frappier [9, 11] studied the generalized Bernoulli polynomials $B_{n,\alpha}(x)$, defined by the generating function

$$
\frac{e^{(x-\frac{1}{2})t}}{g_\alpha(\frac{t}{2})} = \sum_{n=0}^{\infty} B_{n,\alpha}(x) \frac{t^n}{n!}, \quad |t| < 2j_{1,\alpha}, \quad (1)
$$

where

$$
g_\alpha(t) = 2^\alpha \Gamma(\alpha + 1) \frac{J_{\alpha}(t)}{t^\alpha},
$$

$J_{\alpha}(t)$ is the Bessel function of the first kind of order $\alpha$, and $j_{1,\alpha}$ is the smallest positive zero of $J_{\alpha}(t)$. Ismail and Mansour, see [19], introduced $q$-pair of analogs of the Bernoulli polynomials.
by the generating functions

\[
\frac{t e_q(x t)}{e_q(\frac{x}{t})e_q(\frac{x}{t}) - 1} = \sum_{n=0}^{\infty} b_n(x; q |n|_q)^n,
\]

(2)

\[
\frac{t E_q(x t)}{e_q(\frac{x}{t})e_q(\frac{x}{t}) - 1} = \sum_{n=0}^{\infty} B_n(x; q |n|_q)^n.
\]

They also defined a pair of \( q \)-analogs of the Euler polynomials by the generating functions

\[
\frac{2 e_q(x t)}{E_q(\frac{x}{t})e_q(\frac{x}{t}) + 1} = \sum_{n=0}^{\infty} C_n(x; q |n|_q)^n,
\]

(3)

\[
\frac{2 E_q(x t)}{E_q(\frac{x}{t})e_q(\frac{x}{t}) + 1} = \sum_{n=0}^{\infty} E_n(x; q |n|_q)^n,
\]

where

\[|n|_q = \frac{(q; q)_n}{(1 - q)^n}, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbb{N},\]

and \( a \in \mathbb{C} \), see [12]. The functions \( E_q(x) \) and \( e_q(x) \) are the \( q \)-analogs of the exponential functions defined by

\[
E_q(x) := (-x(1 - q); q)_\infty = \sum_{n=0}^{\infty} q^{n(n-1)/2} (1 - q)^n x^n, \quad x \in \mathbb{C},
\]

(4)

\[
e_q(x) := \frac{1}{(x(1 - q); q)_\infty} = \sum_{n=0}^{\infty} (1 - q)^n x^n (q; q)_n, \quad |x| < \frac{1}{1 - q}
\]

see e.g. [12]. In [22], Mansour and Al-Towalib introduced \( q \)-analogs of Bernoulli and Euler polynomials by the generating functions

\[
\frac{t \exp_q(x t) \exp_q(\frac{x}{t})}{\exp_q(\frac{x}{t}) - \exp_q(\frac{x}{t})} = \sum_{n=0}^{\infty} \tilde{B}_n(x; q |n|_q)^n,
\]

(5)

\[
\frac{2 \exp_q(x t) \exp_q(\frac{x}{t})}{\exp_q(\frac{x}{t}) + \exp_q(\frac{x}{t})} = \sum_{n=0}^{\infty} \tilde{E}_n(x; q |n|_q)^n,
\]

where

\[
\exp_q(x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} x^n |n|_q!, \quad x \in \mathbb{C},
\]

is a \( q \)-analogue of the exponential function. This \( q \)-exponential function has the property

\[
\lim_{q \to 1} \exp_q(x) = e^x \quad \text{for} \quad x \in \mathbb{C}.
\]

It is an entire function of \( x \) of order zero, see [12, Eq. (1.3.27), p. 12].

In this paper, we use \( \mathbb{N} \) to denote the set of positive integers and \( \mathbb{N}_0 \) to denote the set of non-negative integers. Throughout this paper, unless otherwise is stated, \( q \) is a positive number that is less than one. We follow Gasper and Rahman [12] to define the \( q \)-shifted factorial, the \( q \)-binomial coefficients, and the \( q \)-gamma function. The \( q \)-integer number \( |n|_q \) is defined by

\[|n|_q = \frac{1 - q^n}{1 - q}, \quad n \in \mathbb{N}_0.\]
Jackson in [20] defined the $q$-difference operator by

$$D_qf(z) = \frac{f(qz) - f(z)}{z(q - 1)}, \quad z \neq 0.$$  

The symmetric $q$-difference operator is defined by, see [8,12],

$$\delta_{q,z}f(z) = \frac{f(q^2z) - f(q^{-2}z)}{(q^2 - q^{-2})z}, \quad z \neq 0.$$  

The $q$-trigonometric functions $\sin_qz$, $\cos_qz$, $Sin_qz$ and $Cos_qz$ are defined by

$$\sin_qz = \frac{e_q(iz) - e_q(-iz)}{2i}, \quad \cos_qz = \frac{e_q(iz) + e_q(-iz)}{2}, \quad |z| < 1,$$

$$Sin_qz = \frac{E_q(iz) - E_q(-iz)}{2i}, \quad Cos_qz = \frac{E_q(iz) + E_q(-iz)}{2}, \quad z \in \mathbb{C},$$

see [5,12]. The $q$-sine and cosine functions $S_q(z)$, $C_q(z)$ are defined by the $q$-Euler's formula

$$\exp_q(iz) := C_q(z) + iS_q(z),$$

where

$$C_q(z) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{|2n|_q!} z^{2n}, \quad S_q(z) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{|2n+1|_q!} z^{2n+1},$$

cf. [8, P. 2]. The hyperbolic functions $Sh_q(z)$ and $Ch_q(z)$ are defined for $z \in \mathbb{C}$ by

$$Sh_q(z) := -iS_q(iz) = \frac{\exp_q(z) - \exp_q(-z)}{2},$$

$$Ch_q(z) := C_q(iz) = \frac{\exp_q(z) + \exp_q(-z)}{2}.$$  

(6)

There are three known $q$-analogs of the Bessel function that are due to Jackson [20]. These are denoted by $J^{(k)}_\alpha(t; q) (k = 1, 2, 3)$ and defined by

$$J^{(1)}_\alpha(t; q) = \frac{(q^{\alpha+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{t}{q})^{2n+\alpha}}{q^n q^{\alpha+1} q^n n}, \quad (|t| < 2),$$

$$J^{(2)}_\alpha(t; q) = \frac{(q^{\alpha+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (\frac{t}{q})^{2n+\alpha}}{q^n q^{\alpha+1} q^n n}, \quad (t \in \mathbb{C}),$$

$$J^{(3)}_\alpha(t; q) = \frac{(q^{\alpha+1}; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (\frac{t}{q})^{2n+\alpha}}{q^n q^{\alpha+1} q^n n}, \quad (t \in \mathbb{C}).$$

For convenience, we set

$$J^{(k)}_\alpha(t; q) := \frac{(q; q)_{\infty}}{(q^{\alpha+1}; q)_{\infty}} \frac{1}{2} (\frac{t}{q})^{-\alpha} J^{(k)}_\alpha(t; q) \quad (k = 1, 2),$$

$$J^{(3)}_\alpha(t; q) := \frac{(q; q)_{\infty}}{(q^{\alpha+1}; q)_{\infty}} t^{-\alpha} J^{(3)}_\alpha(t; q).$$  

(7)

The functions $J^{(k)}_\alpha(t; q) (k = 1, 2, 3)$ are called the modified Jackson $q$-Bessel functions. From now on, we use $(J^{(k)}_{\alpha(m)}(\cdot; q))_{m=1}^\infty$ to denote the positive zeros of $J^{(k)}_\alpha(\cdot; q^2)$ arranged in increasing order of magnitude. Consequently, $j^{(k)}_{1,\alpha}$ is the smallest positive zero of $J^{(k)}_\alpha(\cdot; q^2) (k = 1, 2, 3).$
This paper is organized as follows. In Section 2, we introduce three $q$-analogs of the generalized Bernoulli polynomials defined in \(1\). The generating functions of these $q$-analogs include the three $q$-analogs of Jackson $q$-Bessel functions mentioned above. We also include the main properties of these $q$-analogs. Section 3 introduces a $q$-Fourier expansion for the generalized Bernoulli numbers related to the first and second Jackson $q$-Bessel functions. Also, their large $n$ degree asymptotics is derived. Finally, in Section 4 as an application, we introduce the connection coefficients between $q$-analogs and certain $q$-orthogonal polynomials.

2. Generalized $q$-Bernoulli polynomials generated by Jackson $q$-Bessel functions

This section introduces three $q$-analogs of the generalized Bernoulli polynomials introduced by Frappier in \(9\) \(11\).

**Definition 2.1.** The generalized $q$-Bernoulli polynomials $B_{n,\alpha}^{(k)}(x; q) \ (k = 1, 2, 3)$ are defined by the generating functions

\[
\frac{e_{q}(xt)e_{q}(\frac{t}{2})}{g_{\alpha}^{(1)}(it; q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^{n}}{[n]_{q}!}, \quad |t| < \frac{j_{1,\alpha}^{(1)}}{1 - q},
\]

\[
\frac{E_{q}(xt)E_{q}(\frac{t}{2})}{g_{\alpha}^{(2)}(it; q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^{n}}{[n]_{q}!}, \quad |t| < \frac{j_{1,\alpha}^{(2)}}{1 - q},
\]

\[
\frac{\exp_{q}(xt)\exp_{q}(\frac{t}{2})}{g_{\alpha}^{(3)}(it; q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(3)}(x; q) \frac{t^{n}}{[n]_{q}!}, \quad |t| < \frac{2q^{\frac{1}{2}}j_{1,\alpha}^{(3)}}{1 - q},
\]

where $g_{\alpha}^{(k)}(t; q) \ (k = 1, 2, 3)$ are the functions defined for $(k = 1, 2)$ by

\[
g_{\alpha}^{(k)}(t; q) := (1 + q)^{\alpha} \Gamma_{q^{2}}(\alpha + 1) (t/2)^{-\alpha} J_{\alpha}^{(k)}(t(1 - q); q^{2}) = J_{\alpha}^{(k)}(t(1 - q); q^{2}),
\]

and

\[
g_{\alpha}^{(3)}(t; q) := (1 + q)^{\alpha} \Gamma_{q^{2}}(\alpha + 1) (q^{-\frac{1}{2}}t/2)^{-\alpha} J_{\alpha}^{(3)}(\frac{t}{2}(1 - q)q^{-\frac{1}{2}}; q^{2}) = J_{\alpha}^{(3)}(\frac{t}{2}(1 - q)q^{-\frac{1}{2}}; q^{2}).
\]

Since the generating functions in \(2\), \(3\), and \(5\) can be written as

\[
\frac{te_{q}(xt)e_{q}(\frac{t}{2})}{2\sinh_{q} \frac{t}{2}} = \sum_{n=0}^{\infty} b_{n}(x; q) \frac{t^{n}}{[n]_{q}!},
\]

\[
\frac{tE_{q}(xt)E_{q}(\frac{t}{2})}{2\sinh_{q} \frac{t}{2}} = \sum_{n=0}^{\infty} B_{n}(x; q) \frac{t^{n}}{[n]_{q}!},
\]

\[
\frac{e_{q}(xt)e_{q}(\frac{t}{2})}{\cosh_{q} \frac{t}{2}} = \sum_{n=0}^{\infty} e_{n}(x; q) \frac{t^{n}}{[n]_{q}!},
\]

\[
\frac{E_{q}(xt)E_{q}(\frac{t}{2})}{\cosh_{q} \frac{t}{2}} = \sum_{n=0}^{\infty} E_{n}(x; q) \frac{t^{n}}{[n]_{q}!},
\]
and

\[ \frac{t \exp_q(xt) \exp_q(\frac{t}{2})}{2 \exp_q(xt) \exp_q(\frac{t}{2})} = \sum_{n=0}^{\infty} \tilde{B}_n(x; q) \frac{t^n}{[n]_q!}, \]

\[ \frac{\exp_q(xt) \exp_q(\frac{t}{2})}{C_{\exp_q}(\frac{t}{2})} = \sum_{n=0}^{\infty} \tilde{E}_n(x; q) \frac{t^n}{[n]_q!}, \]

(13)

then, if we substitute with \( \alpha = \pm \frac{1}{2} \) in [8], [9], and [10], we obtain the \( q \)-Bernoulli and Euler polynomials defined in [11], [12] and [13], respectively.

Lemma 2.2. For \( n \in \mathbb{N}_0 \) and \( \text{Re} \alpha > -1 \),

\[ \frac{e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \frac{E_q(\frac{t}{2})}{g_\alpha^{(2)}(it; q)}, \quad |t| < \frac{1}{1-q} \min\{j_{1,\alpha}^{(1)}, j_{1,\alpha}^{(2)}\}. \]

Proof. Hahn in [14] proved the identity

\[ J_\alpha^{(2)}(t; q) = \left( \frac{-t^2}{4}; q \right)_\infty J_\alpha^{(1)}(t; q), \quad |t| < 2. \]

(14)

Since

\[ g_\alpha^{(k)}(it; q) = (1 + q^\alpha \Gamma_q(x + 1) \left( \frac{it}{2} \right)^{-\alpha} J_\alpha^{(k)}(it(1 - q); q^2) (k = 1, 2), \]

(15)

then, substituting from [13] into [12], we conclude that

\[ g_\alpha^{(2)}(it; q) = \left( \frac{-t^2}{4}; q^2 \right)_\infty g_\alpha^{(1)}(it; q) = E_q\left( \frac{t}{2} \right)E_q\left( \frac{-t}{2} \right) g_\alpha^{(1)}(it; q). \]

(16)

Hence

\[ \frac{E_q(\frac{t}{2})}{g_\alpha^{(2)}(it; q)} = \frac{E_q(\frac{-t}{2})}{g_\alpha^{(1)}(it; q)} = \frac{e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)}, \]

which completes the proof. \( \square \)

Definition 2.3. The generalized \( q \)-Bernoulli numbers \( \beta_{n,\alpha}(q) \), \( \beta_{n,\alpha}^{(3)}(q) \) are defined respectively in terms of the generating functions

\[ \frac{e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \frac{E_q(\frac{-t}{2})}{g_\alpha^{(2)}(it; q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}, \]

(17)

\[ \frac{\exp_q(\frac{t}{2})}{g_\alpha^{(3)}(it; q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}^{(3)}(q) \frac{t^n}{[n]_q!}. \]

(18)

Proposition 2.4. For \( n \in \mathbb{N} \), we have

\[ B_{2n+1,\alpha}(\frac{1}{2}; q) = 0 \quad (k = 1, 2, 3). \]

Proof. If we substitute with \( x = \frac{1}{2} \) in Equations [3]-[10], we find that their left hand side are even functions. Therefore, the coefficients of the odd powers of \( t^n \) on the right hand sides of Equations [3]-[10] vanish. This proves the proposition. \( \square \)

Proposition 2.5. For \( k \in \{1, 2, 3\} \) and \( n \in \mathbb{N} \), the polynomials \( B_{n,\alpha}^{(k)}(x; q) \) have the representation

\[ B_{n,\alpha}^{(k)}(x; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \beta_{n-k,\alpha}(q)x^k, \]

(19)
\[ B_{n,\alpha}^{(2)}(x; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\frac{k(k-1)}{2}} \beta_{n-k,\alpha}(q)x^k, \]  
\[ B_{n,\alpha}^{(3)}(x; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q^{\frac{k(k-1)}{3}} \beta_{n-k,\alpha}(q)x^k. \]  

**Proof.** We prove the case \((k = 1)\). The proofs for \((k = 2, 3)\) are similar and are omitted. Substituting with the series representation of \(e_q(x)\) from (14) into (8) gives

\[ \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} = \frac{e_q(x)}{g_\alpha^{(1)}(it; q)} e_q(x), \]

\[ = \left( \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} \frac{(xt)^n}{[n]_q!} \right). \]

Hence

\[ \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q \beta_{n-k,\alpha} x^k, \]

where we applied the Cauchy product formula. Equating the \(n\)th power of \(t\) in (22), we obtain (19).

**Proposition 2.6.** For \(n \in \mathbb{N}\) and \(k \in \{1, 2, 3\}\), the polynomials \(B_{n,\alpha}^{(k)}(x; q)\) satisfy the \(q\)-difference equations

\[ D_{q,x} B_{n,\alpha}^{(1)}(x; q) = [n]_q B_{n-1,\alpha}^{(1)}(x; q), \]  
\[ D_{q^{-1},x} B_{n,\alpha}^{(2)}(x; q) = [n]_q B_{n-1,\alpha}^{(2)}(x; q), \]  
\[ \delta_{q,x} B_{n,\alpha}^{(3)}(x; q) = [n]_q B_{n-1,\alpha}^{(3)}(x; q). \]

**Proof.** We only prove the case \((k = 1)\) and the proofs of \((k = 2, 3)\) are similar. Calculating the \(q\)-derivative of both sides of (8) with respect to the variable \(x\) and taking into consideration that

\[ D_{q,x} e_q(xt) = te_q(xt), \]

we obtain

\[ \frac{te_q(xt)e_q(x)}{g_\alpha^{(1)}(it; q)} = \sum_{n=1}^{\infty} D_{q,x} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!}. \]

Therefore,

\[ \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^{n+1}}{[n]_q!} = \sum_{n=1}^{\infty} D_{q,x} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!}, \]

Equating the corresponding \(n\)th power of \(t\) in (26), we obtain (23).

**Corollary 2.7.** Let \(n \in \mathbb{N}\) and \(k\) be a positive integer such that \(k \leq n\). Then for \(x \in \mathbb{C}\),

\[ D_{q,x}^{k} B_{n,\alpha}^{(1)}(x; q) = \frac{B_{n-k,\alpha}^{(1)}(x; q)}{[n-k]!}, \]
\[ D_{q^{-1},x}^{k} B_{n,\alpha}^{(2)}(x; q) = \frac{B_{n-k,\alpha}^{(2)}(x; q)}{[n-k]!}, \]
\[ \delta_{q,x}^{k} B_{n,\alpha}^{(3)}(x; q) = \frac{B_{n-k,\alpha}^{(3)}(x; q)}{[n-k]!}. \]

**Proof.** The proofs follow from Proposition 2.6 and the mathematical induction.
Proposition 2.8. For \(|t| < \frac{1}{1-q} \min\{j_{1,\alpha}, j_{1,\alpha}^{(2)}, 2\}\),

\[
\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(\frac{1}{2}; q) \frac{t^n}{[n]_q!} = \frac{1}{g^{(2)}(it; q)}.
\] (27)

\[
\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(\frac{1}{2}; q) \frac{t^n}{[n]_q!} = \frac{1}{g^{(1)}(it; q)}.
\] (28)

**Proof.** Set \(x = \frac{1}{2}\) in (8), we obtain

\[
\frac{e_q(\frac{1}{2}) e_q(\frac{1}{2})}{g^{(1)}(it; q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(\frac{1}{2}; q) \frac{t^n}{[n]_q!}.
\] (29)

Substituting from (16) into (29), we obtain (27). Similarly, we can prove (28). \(\square\)

The following Lemma from [22] gives the reciprocal of \(\exp_q(z)\) in a certain domain.

**Lemma 2.9.** Let \(z \in \Omega = \{z \in \mathbb{C} : |1 - \exp_q(-z)| < 1\}\). Then

\[
\frac{1}{\exp_q(z)} := \sum_{n=0}^{\infty} c_n z^n,
\]

where

\[
c_n = \sum_{k=1}^{n} (-1)^k \sum_{s_1 + s_2 + \cdots + s_k = n} q^{\sum_{i=1}^{k} s_i(1-1)/4} \frac{[s_1]_q [s_2]_q \cdots [s_k]_q!}{[n-k]_q!}.
\] (30)

**Proposition 2.10.** For \(\Re \alpha > -1\) and \(t \in \Omega = \{t \in \mathbb{C} : |1 - \exp_q(-t)| < 1\}\),

\[
\frac{1}{g^{(3)}(it; q)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k c_k \frac{\beta^{(3)}_{n,k,\alpha}(q)}{[n-k]_q!} t^n n_{k,\alpha}(q),
\] (31)

where \(c_n\) is defined in (30).

**Proof.** Substitute with \(x = 0\) in Equation (10). This gives

\[
\frac{\exp_q(\frac{1}{2})}{g^{(3)}(it; q)} = \sum_{n=0}^{\infty} \beta^{(3)}_{n,\alpha}(q) \frac{t^n}{[n]_q!}.
\]

From Lemma 2.9

\[
\frac{1}{g^{(3)}(it; q)} = \frac{1}{\exp_q(\frac{1}{2})} \sum_{n=0}^{\infty} \beta^{(3)}_{n,\alpha}(q) \frac{t^n}{[n]_q!} = \left(\sum_{n=0}^{\infty} c_n \frac{(-1)^n t^n}{2^n}\right) \left(\sum_{n=0}^{\infty} \beta^{(3)}_{n,\alpha}(q) \frac{t^n}{[n]_q!}\right).
\]

Applying the Cauchy product formula, we obtain (31) and completes the proof. \(\square\)

**Theorem 2.11.** For \(n \in \mathbb{N}_0\) and \(x \in \mathbb{C}\),

\[
\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] q B_{k,\alpha}^{(1)}(-x; q) B_{n-k,\alpha}^{(2)}(x; q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right] q \beta_{k,\alpha}(q) \beta_{n-k,\alpha}(q).
\]
Proof. If we replace $x$ by $-x$ in \([8]\), then
\[
e_{q}(-xt) = \sum_{n=0}^{\infty} \frac{B_{n,\alpha}^{(1)}(-x; q) t^{n}}{[n]_{q}!}.
\]  
(32)

Since $e_{q}(-xt) E_{q}(xt) = 1$, then multiplying \([31]\) by \([32]\) gives
\[
\frac{E_{q}(-\frac{t}{q}) e_{q}(-\frac{t}{q})}{g_{\alpha}^{(2)}(it; q) g_{\alpha}^{(1)}(it; q)} = \left( \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(-x; q) \frac{t^{n}}{[n]_{q}!} \right) \left( \sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^{n}}{[n]_{q}!} \right).
\]

From \([17]\), we obtain
\[
\left( \sum_{n=0}^{\infty} \frac{\beta_{n,\alpha}(q) t^{n}}{[n]_{q}!} \right)^{2} = \sum_{n=0}^{\infty} \frac{t^{n}}{[n]_{q}!} \sum_{k=0}^{n} \frac{n}{k} B_{k,\alpha}^{(1)}(-x; q) B_{n-k,\alpha}^{(2)}(x; q).
\]

Hence
\[
\sum_{n=0}^{\infty} \frac{t^{n}}{[n]_{q}!} \sum_{k=0}^{n} \frac{n}{k} \beta_{k,\alpha}(q) \beta_{n-k,\alpha}(q) = \sum_{n=0}^{\infty} \frac{t^{n}}{[n]_{q}!} \sum_{k=0}^{n} \frac{n}{k} B_{k,\alpha}^{(1)}(-x; q) B_{n-k,\alpha}^{(2)}(x; q).
\]

(33)

So, equating the nth power of $t$ in \([33]\), we obtain the required result.

Proposition 2.12. For $n \in \mathbb{N}_{0}$, $x \in \mathbb{C}$ and $q \neq 0$,
\[
B_{n,\alpha}^{(2)}(x; q) = q^{\frac{n(n-1)}{2}} B_{n,\alpha}^{(1)}(x; \frac{1}{q}).
\]

(34)

In particular,
\[
\beta_{n,\alpha}(q) = q^{\frac{n(n-1)}{2}} \beta_{n,\alpha}(\frac{1}{q}).
\]

(35)

Proof. Replacing $q$ by $\frac{1}{q}$ on the generating function in \([8]\) and use $E_{q}(x) = e_{\frac{1}{q}}(x)$, we obtain
\[
\frac{E_{q}(xt) E_{q}(-\frac{t}{q})}{g_{\alpha}^{(1)}(it; \frac{1}{q})} = \sum_{n=0}^{\infty} \frac{B_{n,\alpha}^{(1)}(x; \frac{1}{q}) t^{n}}{[n]_{\frac{1}{q}}!}.
\]

(36)

Since
\[
g_{\alpha}^{(1)}(it; \frac{1}{q}) = \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^{2(n+\frac{1}{2})} 2^{n}}{(q^{-2}; q^{-2})_{n} (q^{-2\alpha+2}; q^{-2})_{n}} \frac{t^{n}}{[n]_{\frac{1}{q}}!}
\]
\[
= \sum_{n=0}^{\infty} \frac{(1 - q^{2n})^{2(n+\alpha)(\frac{1}{2})}}{(q^{2}; q^{2\alpha+2}; q^{2})_{n}} = g_{\alpha}^{(2)}(it; q),
\]

where we used the identity $(a; q^{-1})_{n} = (a^{-1}; q)_{n} (-a)^{n} q^{-\frac{n(n-1)}{2}}$. Since $[n]_{1/q}! = q^{\frac{n(n-1)}{2}} [n]_{1/q}!$, then \([36]\) takes the form
\[
\frac{E_{q}(xt) E_{q}(-\frac{t}{q})}{g_{\alpha}^{(2)}(it; q)} = \sum_{n=0}^{\infty} \frac{B_{n,\alpha}^{(1)}(x; \frac{1}{q}) t^{n}}{[n]_{\frac{1}{q}}!}.
\]

Therefore,
\[
\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^{n}}{[n]_{q}!} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; \frac{1}{q}) \frac{t^{n}}{[n]_{\frac{1}{q}}!}.
\]

(37)

Equating the coefficients of $t^{n}$ in \([37]\) gives \([34]\) and substituting with $x = 0$ into \([34]\) yields directly \([35]\).
Al-Salam, in [3], introduced the polynomials

\[ H_n(x) := \sum_{k=0}^{n} \binom{n}{k}_q x^k, \quad G_n(x) := \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2-nk} x^k. \]

(38)

He also proved that

\[ E_q(x)E_q(-x) = \sum_{n=0}^{\infty} q^{n(n-1)/2} G_n(-1) \frac{x^n}{[n]_q!}, \quad x \in \mathbb{C}, \]

(39)

\[ e_q(x)e_q(-x) = \sum_{n=0}^{\infty} H_n(-1) \frac{x^n}{[n]_q!}, \quad |x| < \frac{1}{1-q}. \]

(40)

The following theorem introduces connection relations between the polynomials \(B_{n,\alpha}^{(1)}(x; q)\) and \(B_{n,\alpha}^{(2)}(x; q)\).

**Theorem 2.13.** For \(n \in \mathbb{N}_0\),

\[ B_{n,\alpha}^{(1)}(x; q) = \sum_{k=0}^{n} \binom{n}{k}_q x^k H_k(-1) B_{n-k,\alpha}^{(2)}(x; q), \]

(41)

\[ B_{n,\alpha}^{(2)}(x; q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{k(k-1)/2} x^k G_k(-1) B_{n-k,\alpha}^{(1)}(x; q). \]

(42)

**Proof.** Since \(E_q(x)e_q(-xt) = 1\), \(|xt| < \frac{1}{1-q}\), then from (10), the generating function of \(B_{n,\alpha}^{(1)}(x; q)\) can be represented as

\[ \frac{e_q(xt)e_q(-xt)}{g_{\alpha}^{(1)}(it; q)} = \frac{E_q(xt)E_q(-xt)}{g_{\alpha}^{(2)}(it; q)} e_q(xt)e_q(-xt). \]

From (8), (9) and (40), we obtain

\[ \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} = \left( \sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} H_n(-1) \frac{x^n}{[n]_q!} \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}_q x^k H_k(-1) B_{n-k,\alpha}^{(2)}(x; q) \right) \frac{t^n}{[n]_q!}. \]

(43)

Therefore, equating the coefficients of the \(n\)th power of \(t\) in the series of the outside parts of (43) gives (44). The proof for \(B_{n,\alpha}^{(2)}(x; q)\) follows similarly from the generating function of \(B_{n,\alpha}^{(2)}(x; q)\) and the identity (39), and is omitted. \(\square\)

**Theorem 2.14.** Let \(n\) be a positive integer and \(x, \alpha\) be complex numbers such that \(\text{Re} \alpha > -1\). Then

\[ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (1-q)^{2k} B_{n-2k,\alpha}^{(1)}(-\frac{x}{2}; q) \frac{[k]!}{[n-k]!} (q^2, q^{2\alpha+1}, q^2)_k = \frac{(-1/2)^n}{[n]_q!} H_n(x), \]

(44)

\[ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (1-q)^{2k} q^{2k(k+\alpha)} B_{n-2k,\alpha}^{(2)}(-\frac{x}{2}; q) \frac{[k]!}{[n-k]!} (q^2, q^{2\alpha+2}, q^2)_k = \frac{(-1/2)^n}{[n]_q!} \frac{q^{n(n-1)/2}}{([n]_q!)^2} G_n(x), \]

\[ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (1-q)^{2k} q^{k^2+k/2} B_{n-2k,\alpha}^{(3)}(-\frac{x}{2}; q) \frac{[k]!}{[n-k]!} (q^2, q^{2\alpha+2}, q^2)_k = \frac{(-1/2)^n}{[n]_q!} \frac{q^{n(n-1)/2}}{([n]_q!)^2} (-xq^{1/2} q^n)\).

(44)
Proof. We can write the generating function of the polynomials \( B_{n,\alpha}^{(1)}(x; q) \) as

\[
e_q(x) e_q\left(\frac{-t}{2}\right) = g^{(1)}_\alpha(it; q) \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x;q) \frac{t^n}{[n]_q}.
\]

(45)

Hence

\[
e_q(x) e_q\left(\frac{-t}{2}\right) = \left(\sum_{n=0}^{\infty} \frac{(1-q)^n t^{2n}}{2^{2n} [n]_q [q, q^{2\alpha+2}; q^2]_n}\right) \left(\sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q}ight).
\]

On one hand, applying the Cauchy product formula in (45), we obtain

\[
e_q(x) e_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{t^n}{[n]_q} \frac{(1-q)^{2k} B_{n-2k,\alpha}^{(1)}(x; q)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k},
\]

On the other hand, using the series representation of \( e_q(x) \) in (44) followed by the Cauchy product formula, and using (38) yields

\[
e_q(x) e_q\left(\frac{-t}{2}\right) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q} (-\frac{1}{2})^n H_n(-2x).
\]

(46)

Hence

\[
\sum_{n=0}^{\infty} \frac{t^n}{[n]_q} (-\frac{1}{2})^n H_n(-2x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-q)^{2k} B_{n-2k,\alpha}^{(1)}(x; q)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k},
\]

equating the coefficients of \( t^n \) in (47), we get

\[
\frac{(-1/2)^n}{[n]_q} H_n(-2x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-q)^{2k} B_{n-2k,\alpha}^{(1)}(x; q)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k}.
\]

(48)

Replacing \( x \) by \( \frac{-x}{2} \) in (48) gives

\[
\frac{(-1/2)^n}{[n]_q} H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-q)^{2k} B_{n-2k,\alpha}^{(1)}\left(\frac{-x}{2}; q\right)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k},
\]

which readily completes the proof for \( B_{n,\alpha}^{(1)}(x; q) \). The proofs for \( B_{n,\alpha}^{(2)}(x; q) \) and \( B_{n,\alpha}^{(3)}(x; q) \) are similar and are omitted.

If we set \( x = 0 \) in (44), we obtain the following recurrence relations for \( \beta_{n,\alpha}(q) \) and \( \beta_{n,\alpha}^{(3)}(q) \),

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-q)^{2k} \beta_{n-2k,\alpha}(q)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k} = \frac{(-1)^n}{[n]_q}, \quad n \in \mathbb{N},
\]

(49)

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(1-q)^{2k} q^{k^2+k/2} \beta_{n-2k,\alpha}^{(3)}(q)}{2^{2k} [n-2k]_q [q, q^{2\alpha+2}; q^2]_k} = \frac{q^{-n-1} (-1)^n}{[n]_q}, \quad n \in \mathbb{N}.
\]

As a consequence of the recursive relations in (49), and the fact that

\[
\beta_{0,\alpha}(q) = \beta_{0,\alpha}^{(3)}(q) = 1,
\]

we can prove that
\[ \beta_{1,\alpha}(q) = -\frac{1}{2}, \quad \beta_{2,\alpha}(q) = \frac{q(1-q^{2\alpha+1})}{4(1-q^{2\alpha+2})}, \quad \beta_{3,\alpha}(q) = \frac{-q^3(1-q^{2\alpha-1})}{8(1-q^{2\alpha+2})}, \]

\[ \beta_{4,\alpha}(q) = \frac{1}{16} \frac{(q + q^3)(1 - q^3)(1 - q^{2\alpha+1})}{16(1-q^{2\alpha+2})^2} - \frac{(1-q)(1-q^3)}{16(q^{2\alpha+2}; q^2)_2}, \]

\[ \beta_{5,\alpha}(q) = \frac{(1 + q^2)(1 - q^5)(q^3 - q^{2\alpha+2})}{32(1-q^{2\alpha+2})^2} + \frac{(1 - q^3)(1 - q^5)}{32(q^{2\alpha+2}; q^2)_2} - \frac{1}{32}, \]

and

\[ \beta_{4,\alpha}^{(3)}(q) = \frac{q^3(2^{2\alpha+2}; q^2)_2(1 - q^{2\alpha+2})}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} - \frac{[3]_q q^5(1 - q^3)(1 - q^{2\alpha+2})}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} \]

\[ + \frac{[4]_q [3]_q q^3(1 - q^{2\alpha+4}) (q^{1/2}(1-q^{2\alpha+2}) - q^{3/2}(1-q^3))}{16(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})}, \]

\[ \beta_{5,\alpha}^{(3)}(q) = \frac{[5]_q q^3(1-q)(1+q^2)(q^3-q^{2\alpha+2})(1-q^{2\alpha+4})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} \]

\[ + \frac{[5]_q q^5(1-q)(1-q^3)(1-q^{2\alpha+4})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})} \frac{q^5(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})}{32(1-q^{2\alpha+2})^2(1-q^{2\alpha+4})}. \]

**Theorem 2.15.** For \( n \in \mathbb{N}_0 \) and complex numbers \( a \) and \( x \),

\[ B_{n,\alpha}^{(1)}(x; q) = \sum_{k=0}^{n} \binom{n}{k} (a;q)_k x^k B_{n-k,\alpha}^{(1)}(ax; q), \]  \( (50) \)

\[ B_{n,\alpha}^{(2)}(x; q) = \sum_{k=0}^{n} \binom{n}{k} (-a)^k (1/a;q)_k x^k B_{n-k,\alpha}^{(2)}(ax; q). \]  \( (51) \)

**Proof.** The proof of \( (50) \) follows from the generating function \( (8) \) since

\[ \frac{e_q(tx)}{g_\alpha^{(1)}(it; q)} = \frac{e_q(tax)}{g_\alpha^{(1)}(it; q)} \frac{e_q(tx)}{e_q(atx)}, \quad |tx| < \frac{1}{1-q}. \]

From the \( q \)-binomial theorem (see [12, Eq.(1.3.2), P. 8]), we can prove that

\[ \frac{e_q(tx)}{e_q(atx)} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} ((1-q)t)^n, \quad |tx| < \frac{1}{1-q}. \]

Therefore,

\[ \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} = \left( \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(ax; q) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} ((1-q)t)^n \right) \]

\[ = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k} (a; q)_k x^k B_{n-k,\alpha}^{(1)}(ax; q), \]  \( (52) \)
where we used the Cauchy product formula. Equating the coefficients of \( t^n \) in (52), we obtain \( (50) \). The proof for \( B_{n, \alpha}^{(2)}(x; q) \) is similar and is omitted.  

\[ (55) \]

**Lemma 2.16.** For \( n \in \mathbb{N}_0 \), \( \text{Re} \ \alpha > -1 \), and \( |(1-q)t| < 1 \),

\[
g_{\alpha}^{(1)}(it; q)E_q(t/2) = g_{\alpha}^{(2)}(it; q)e_q(t/2) = 2\phi_1 \left( q^{\alpha+\frac{1}{2}}, -q^{\alpha+\frac{1}{2}}, q^{2\alpha+1}; q, \frac{(1-q)t}{2} \right). \tag{53} \]

**Proof.** From Lemma 2.12, we conclude that

\[
g_{\alpha}^{(2)}(it; q)e_q(t/2) = g_{\alpha}^{(1)}(it; q)E_q(t/2). \]

From the series representations of \( E_q(x) \) and \( g_{\alpha}^{(1)}(it; q) \) in (44) and (45), respectively, we obtain

\[
g_{\alpha}^{(1)}(it; q)E_q(t/2) = \left\{ \sum_{n=0}^{\infty} \frac{(1-q)^n t^{2n}}{2^{2n}(q^2, q^{2\alpha+2}; q^2)_n} \left( \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(1-q)^n t^{2n}}{2^{2n}(q; q)_n} \right) \right\} \]

\[
= \sum_{n=0}^{\infty} \frac{(1-q)^n q^{n(n-1)/2} t^{2n}}{2^{2n}(q; q)_n} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{2k^2 - 2nk + k} \frac{q^{2k}q^{1-n}}{(q^2, q^{2\alpha+2}; q^2)_k}.
\]

where we used the identity, see [12] Eq. (1.2.32), P. 6,

\[ (a; q)_n = \frac{(a; q)_n}{(a^{-1}q^{1-n}; q)_k} \] \( (k = 0, 1, \ldots, n) \). \tag{54} \]

Therefore, using the identity \( (a; q)_{2n} = (a; q^n)(aq; q^n)_n \) yields

\[
g_{\alpha}^{(1)}(it; q)E_q(t/2) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(1-q)^n (q; q)_n}{(q^2, q^{2\alpha+2}; q^2)_n} \sum_{k=0}^{\lfloor n/2 \rfloor} q^{2k}q^{1-n} \frac{q^{2k}q^{1-n}}{(q^2, q^{2\alpha+2}; q^2)_k} \]

\[
= \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(1-q)^n (q; q)_n}{(q^2, q^{2\alpha+2}; q^2)_n} 2\phi_1 \left( q^{1-n}, q^{-n+1}, q^{2\alpha+2}; q^2, q^2 \right).
\]

Since

\[ 2\phi_1 \left( q^{1-n}, q^{-n+1}, q^{2\alpha+2}; q^2, q^2 \right) = \frac{(q^2; q^n)_n q^{-n(n-1)}}{(b^2; q)_n} \] \( (n \in \mathbb{N}) \),

see [12] P. 26, then

\[
g_{\alpha}^{(1)}(it; q)E_q(t/2) = \sum_{n=0}^{\infty} \frac{(q^{\alpha+\frac{1}{2}}; q)_n (-q^{\alpha+\frac{1}{2}}; q)_n (1-q)^n t^{2n}}{(q^2, q^{2\alpha+1}; q)_n} \]

\[
= 2\phi_1 \left( q^{\alpha+\frac{1}{2}}, -q^{\alpha+\frac{1}{2}}, q^{2\alpha+1}; q, \frac{(1-q)t}{2} \right). \tag{55} \]

Hence from Lemma 2.12 and (55), we obtain (50) and completes the proof.  

\[ \square \]
Theorem 2.17. Let \( \alpha \) be a complex number such that \( \Re \alpha > -1 \). Then

\[
\sum_{m=0}^{n} \left[ \frac{n}{m} \right] \frac{(q^{2\alpha+1}; q^2)_m B_{n-m,\alpha}^{(1)}(x; q)}{2^m(q^{2\alpha+1}; q)_m} = x^n,
\]

\[
\sum_{m=0}^{n} \left[ \frac{n}{m} \right] \frac{(q^{2\alpha+1}; q^2)_m B_{n-m,\alpha}^{(2)}(x; q)}{2^m(q^{2\alpha+1}; q)_m} = q^{\frac{n(n-1)}{2}} x^n,
\]

\[
\sum_{m=0}^{n} \left( \frac{(-1)^m}{2^m} \sum_{k=0}^{\infty} \left( \frac{q}{k!} \right)^2 k^{k/2} (1 - q)^{2k} c_{m-2k} \right) \frac{B_{n-m,\alpha}^{(3)}(x; q)}{(n-m)q!} = q^{\frac{n(n-1)}{2}} x^n,
\]

where \((c_k)_k\) are the coefficients defined in \( (56) \).

Proof. We can write Equation (8) in the form

\[
e_q(xt) = E_q\left(\frac{t}{2}\right) g_{\alpha}^{(1)}(it; q) \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!},
\]

\[
= \left( \sum_{n=0}^{\infty} d_n t^n \right) \left( \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} \right).
\]

From Lemma 2.10, we obtain

\[
g_{\alpha}^{(1)}(it; q) E_q\left(\frac{t}{2}\right) = \sum_{n=0}^{\infty} d_n t^n,
\]

where

\[
d_n = \frac{(1 - q)^n (q^{2\alpha+1}; q^2)_n}{2^n(q; q)_n (q^{2\alpha+1}; q)_n}.
\]

Now, applying the Cauchy product formula in \( (56) \) gives

\[
e_q(xt) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} d_m \frac{B_{n-m,\alpha}^{(1)}(x; q)}{[n-m]_q!} = \sum_{n=0}^{\infty} (xt)^n.
\]

Equating the coefficients of the \( nt \)th power of \( t \) in \( (59) \) gives

\[
\sum_{m=0}^{n} d_m \frac{B_{n-m,\alpha}^{(1)}(x; q)}{[n-m]_q!} = x^n.
\]

Substituting from \( (58) \) into \( (60) \), we get the result for \( B_{n,\alpha}^{(1)}(x; q) \). Similarly, we can prove the result for \( B_{n,\alpha}^{(k)}(x; q) \) \( (k = 2, 3) \). \( \square \)

Theorem 2.18. Let \( n \) be a positive integer and \( x \) be a complex number. If \( \Re \alpha > -1 \), then

\[
B_{n,\alpha}^{(1)}(x; q) - (-1)^n B_{n,\alpha}^{(1)}(-x; q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] \left( \left( \frac{1}{2} \right)^k H_k(-2x) - \left( \frac{1}{2} \right)^k H_k(2x) \right) B_{n-k,\alpha}^{(2)}(\frac{1}{2}; q),
\]

\[
B_{n,\alpha}^{(2)}(x; q) - (-1)^n B_{n,\alpha}^{(2)}(-x; q) = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] \left( \left( \frac{1}{2} \right)^k G_k(-2x) - \left( \frac{1}{2} \right)^k G_k(2x) \right) B_{n-k,\alpha}^{(1)}(\frac{1}{2}; q).
\]
Proof. We give only the proof of (61) since the proof of (62) is similar. From (8),

\[ \frac{e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} - \frac{e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(-it; q)} = \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} - \sum_{n=0}^{\infty} B_{n,\alpha}^{(1)}(-x; q) \frac{(-t)^n}{[n]_q!}. \] (63)

Since \( g_\alpha^{(1)}(-it; q) = g_\alpha^{(1)}(it; q), \) then Equation (63) can be written as

\[ \frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \sum_{n=0}^{\infty} \left[ B_{n,\alpha}^{(1)}(x; q) - (-1)^n B_{n,\alpha}^{(1)}(-x; q) \right] \frac{t^n}{[n]_q!}. \] (64)

Replacing \( x, t \) by \( -x, -t, \) respectively in (64) gives

\[ e_q(xt)e_q(\frac{t}{2}) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} (\frac{1}{2})^n H_n(2x). \] (65)

From (46) and (65), the left hand side of (64) can be written as

\[ \frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \frac{1}{g_\alpha^{(1)}(it; q)} \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \left( (-\frac{1}{2})^n H_n(-2x) - (\frac{1}{2})^n H_n(2x) \right). \]

Therefore, by (28) and the Cauchy product formula, we get

\[ \frac{e_q(xt)e_q(\frac{-t}{2}) - e_q(xt)e_q(\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} \left\{ \frac{n}{k} \right\} \left( (-\frac{1}{2})^k H_k(-2x) - (\frac{1}{2})^k H_k(2x) \right) B_{n-k,\alpha}^{(2)} \frac{1}{[\frac{1}{2}]_q!}. \] (66)

Since the left hand side of (64) and (66) are equal, then equating the coefficients of \( t^n \) on the right hand sides of (64) and (66) yields (61) and completes the proof.

**Proposition 2.19.** If \( \alpha_0 > -1 \) satisfies the condition

\[ q^{2(\alpha_0+1)}(1 - q^2)^2 < (1 - q^2)(1 - q^{2\alpha_0+2}), \] (67)

then \( (t/2)^{-\alpha} J_{\alpha}^{(2)}(t(1 - q); q^2) \) has no zeros in \( |t| \leq 1 \) for all \( \alpha \geq \alpha_0. \)

**Proof.** Set

\[ F(t) := \frac{(q; q)_\infty}{(q^{\alpha_0+1}; q)_\infty} (t/2)^{-\alpha} J_{\alpha}^{(2)}(t(1 - q); q^2) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k(\alpha+\alpha)}(1 - q^{2\alpha})^{2k}}{2^{2k}(q^2, q^{2\alpha+2}, q^2)_k} t^{2k}, \]

and

\[ a_\alpha := \frac{q^{2k(\alpha+\alpha)}(1 - q^{2\alpha})^{2k}}{2^{2k}(q^2, q^{2\alpha+2}, q^2)_k}. \]

Then, under hypothesis (67) and since \( 0 < q < 1, \)

\[ q^{2(\alpha+1)}(1 - q^2)^2 \leq q^{2(\alpha_0+1)}(1 - q^2)^2 < (1 - q^2)(1 - q^{2\alpha_0+2}) \leq (1 - q^2)(1 - q^{2\alpha_2+2}), \]

holds whenever \( \alpha \geq \alpha_0. \) Hence

\[ \frac{a_{\alpha+1}}{a_\alpha} = \frac{q^{4k+2(\alpha+1)}(1 - q)^2}{4(1 - q^{2k+2})(1 - q^{2\alpha+2})} \leq \frac{q^{2(\alpha+1)}(1 - q)^2}{4(1 - q^2)(1 - q^{2\alpha_0+2})} < 1, \]
for \(t \in \mathbb{R}, |t| \leq 1\)

\[
F(t) = \sum_{k=0}^{\infty} t^{2k}(a_{2k} - a_{2k+1}t^2) \geq (a_0 - a_1t^2) \geq (a_0 - a_1) > 0.
\]

This proves that \(F(t)\) has no zeros on \([-1, 1]\), since \(F(t)\) has only real zeros, then \(F(t)\) has no zeros in the unit disk. i.e \(|F(t)| > 0\) for \(|t| \leq 1\).

**Corollary 2.20.** There exists \(\alpha_0 > -1\) such that \(J^{(2)}_\alpha(t(1 - q); q^2)\) has no zeros in the unit disk for all \(\alpha \geq \alpha_0\).

**Proof.** Since for a fixed \(q \in (0, 1)\),

\[
\lim_{\alpha \to \infty} q^{2\alpha+2} = 0, \quad \lim_{\alpha \to \infty} (1 - q^2)(1 - q^{2\alpha+2}) = (1 - q^2),
\]

then there exists \(\alpha_0 > -1\) such that the condition \(67\) holds for all \(\alpha \geq \alpha_0\). Consequently from Proposition 2.19, \(J^{(2)}_\alpha(t(1 - q); q^2)\) has no zeros in the unit disk for all \(\alpha \geq \alpha_0\).

**Theorem 2.21.** For \(n \in \mathbb{N}\),

\[
\lim_{\alpha \to \infty} B^{(2)}_{n, \alpha}(x; q) = (-\frac{1}{2})^n q^{\frac{n(n-1)}{2}} G_n(-2x), \quad (68)
\]

\[
\lim_{\alpha \to \infty} B^{(1)}_{n, \alpha}(x; q) = x^n(\frac{1}{2x}; q)_n. \quad (69)
\]

**Proof.** Taking the limit on both sides of Equation \(69\) as \(\alpha \to \infty\) we get

\[
\lim_{\alpha \to \infty} \frac{E_q(xt)E_q(\frac{-t}{x})}{g^{(2)}_\alpha(it; q)} = \lim_{\alpha \to \infty} \sum_{n=0}^{\infty} B^{(2)}_{n, \alpha}(x; q) \frac{t^n}{[n]_q!}. \quad (70)
\]

From Corollary 2.20, there exists \(\alpha_0 > -1\) such that \(g^{(2)}_\alpha(it; q)\) has no zeros in \(|t| \leq 1\) for all \(\alpha \geq \alpha_0\). This means that \(\frac{E_q(xt)E_q(\frac{-t}{x})}{g^{(2)}_\alpha(it; q)}\) is analytic in \(|t| \leq 1\) for all \(\alpha \geq \alpha_0\). Therefore, we can interchange the limit with the summation in \(70\) when \(|t| \leq 1\) to obtain

\[
\lim_{\alpha \to \infty} \frac{E_q(xt)E_q(\frac{-t}{x})}{g^{(2)}_\alpha(it; q)} = \sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B^{(2)}_{n, \alpha}(x; q) \frac{t^n}{[n]_q!}.
\]

Since

\[
\lim_{\alpha \to \infty} g^{(2)}_\alpha(it; q) = 1, \quad \lim_{\alpha \to \infty} \frac{E_q(xt)E_q(\frac{-t}{x})}{g^{(2)}_\alpha(it; q)} = \frac{1}{\frac{1}{2} q^{\frac{n(n-1)}{2}} t^{\frac{n}{2}}} G_n(-2x).
\]

then from \(39\)

\[
\sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B^{(2)}_{n, \alpha}(x; q) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} t^{\frac{n}{2}}}{[n]_q!} (-\frac{1}{2})^n G_n(-2x). \quad (71)
\]

Equating the coefficients of \(t^n\) in \(71\) gives \(68\). The proof of \(68\) follows directly from the relation \(16\) since

\[
1 = \lim_{\alpha \to \infty} g^{(2)}_\alpha(it; q) = E_q(\frac{t}{2})E_q(\frac{-t}{2}) \lim_{\alpha \to \infty} g^{(1)}_\alpha(it; q).
\]

Hence

\[
\lim_{\alpha \to \infty} g^{(1)}_\alpha(it; q) = e_q(\frac{t}{2})e_q(\frac{-t}{2}), \quad |t(1 - q)| < 2.
\]
Therefore, computing the limit in both sides of (3) gives

\[
e_q(xt) e_q(\frac{t}{2}) = \sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!}
\]

From the \(q\)-binomial theorem (see \[12\] Eq.(1.3.2), P. 8), we have

\[
e_q(xt) e_q(\frac{t}{2}) = (\frac{q}{(1 - q); q}_\infty) (xt(1 - q))^n, \quad |xt(1 - q)| < 1.
\]

Hence

\[
\sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B_{n,\alpha}^{(1)}(x; q) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} (\frac{tx}{(q; q)_n}) \frac{t^n}{[n]_q!} \frac{1}{2x^n} q^n, \quad (72)
\]

equating the coefficients of \(t^n\) in (72) yields the required result.

\[\Box\]

Corollary 2.22. For \(n \in \mathbb{N}\),

\[
\lim_{\alpha \to \infty} \beta_{n,\alpha}(q) = (-1)^n 2^n - n q^{\frac{n(n - 1)}{2}}.
\]

\[\textbf{Proof.}\] Since

\[
\lim_{x \to 0} x^n \frac{1}{2x^n} q^n = \lim_{x \to 0} x^n \prod_{k=0}^{n-1} (1 - \frac{q^k}{2x^n}) = \lim_{x \to 0} \prod_{k=0}^{n-1} (x - \frac{q^k}{2}) = (-1)^n 2^n - n q^{\frac{n(n - 1)}{2}},
\]

then substituting with \(x = 0\) into (73) yields (74).

\[\Box\]

Lemma 2.23. Let \(\alpha_0 > -1\). If \(q^{3/2}(1 - q)^2 < (1 - q^2)(1 - q^{2\alpha_0 + 2})\), then \((q^{3/2}t/2)^{-\alpha} J_{n,m}^{(3)}(\frac{1}{2}(1 - q)q^{-\frac{3}{2}}; q^2)\) has no zeros in \(|t| \leq 1\) for all \(\alpha \geq \alpha_0\).

\[\textbf{Proof.}\] The proof is similar to the proof of Proposition 2.19 and is omitted.

\[\Box\]

Theorem 2.24. For \(n \in \mathbb{N}\),

\[
\lim_{\alpha \to \infty} B_{n,\alpha}^{(3)}(x; q) = q^{\frac{n(n+1)}{2}} \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(n-k+3)}(q^{-n}; q)^{2k}}{(q^2; q^2)_k}(2xq^{1-n} \frac{1}{2^n})^{q^{n+1}},
\]

\[
\lim_{\alpha \to \infty} \beta_{n,\alpha}(q) = q^{\frac{n(n+1)}{2}} \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(n-k+3)}(q^{-n}; q)^{2k}}{(q^2; q^2)_k}.
\]

\[\textbf{Proof.}\] Taking the limit as \(\alpha \to \infty\) on both sides of (10), we obtain

\[
\lim_{\alpha \to \infty} \frac{\exp_q(xt) \exp_q(\frac{1}{q})}{g_{\alpha}^{(3)}(it; q)} = \lim_{\alpha \to \infty} \sum_{n=0}^{\infty} B_{n,\alpha}^{(3)}(x; q) \frac{t^n}{[n]_q!}.
\]

(76)

We can choose \(\alpha_0 > -1\) such that

\[
q^{3/2} \leq \frac{1 - q^2}{1 - q} \frac{(1 - q^{2\alpha_0 + 2})}{1 - q} \leq \frac{1 - q^2}{1 - q} \frac{(1 - q^{2\alpha_0 + 2})}{1 - q},
\]

for all \(\alpha \geq \alpha_0\). Hence from Lemma 2.23, the function \(g_{\alpha}^{(3)}(it; q)\) does not vanish on the unit disk, and the left hand side of (76) is analytic for \(|t| \leq 1\). Therefore, we can interchange the
Since
\[ \lim_{\alpha \to \infty} g_\alpha^{(3)}(it; q) = \sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B_{n,\alpha}^{(3)}(x; q) \frac{t^n}{[n]_q!}. \]

Hence
\[ \frac{\exp_q(xt) \exp_q(-\frac{t}{2})}{(-q^{\frac{1}{4}}(1-q)^{2/4}; q^2)_{\infty}} = \sum_{n=0}^{\infty} \lim_{\alpha \to \infty} B_{n,\alpha}^{(3)}(x; q) \frac{t^n}{[n]_q!}. \]

But
\[ \exp_q(xt) \exp_q(-\frac{t}{2}) = \sum_{n=0}^{\infty} \left( \frac{-t}{2} \right)^n q^{\frac{n(n-1)}{2}} \frac{t^n}{[n]_q!} (2x q^{\frac{1-n}{2}}; q)_n. \]

Therefore,
\[ \frac{\exp_q(xt) \exp_q(-\frac{t}{2})}{(-q^{\frac{1}{4}}(1-q)^{2/4}; q^2)_{\infty}} = \left( \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}} (1-q)^{2n}}{(q^2; q^2)_n} \right) \left( \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} (1-q)^{2n}}{[n]_q!} (2x q^{\frac{1-n}{2}}; q)_n \right) \]
\[ = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \left( \frac{-t}{2} \right)^n \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k} q^{2k-nk+k/2}}{(q^2; q^2)_k (q^2; q^2)_{n-2k}} (2x q^{\frac{1-n}{2}}; q)_{n-2k}. \]

Substituting from (78) into (77) and equating the coefficients of \( t^n \) yields (74). The proof of (75) follows directly by setting \( x = 0 \) in (72).

**Theorem 2.25.** Let \( \alpha \) be a complex number such that \( \Re \alpha > -1 \). Then for \( n \in \mathbb{N}, \ n \geq 2 \),

\[ \beta_{n,\alpha}(q) = -\frac{[n]_q! (1-q)^2}{4} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{((1-q)/2)^{2k} \beta_{n-2k-2,\alpha}(q)}{[n-2k-2]_q! (q^2; q^{2n+2}; q^2)_{k+1}} + \frac{(-1)^n}{2^n}, \]

\[ \beta_{n,\alpha}^{(3)}(q) = -\frac{[n]_q! q^{3/2} (1-q)^2}{4} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{q^{k+5k/2} ((1-q)/2)^{2k} \beta_{n-2k-2,\alpha}^{(3)}(q)}{[n-2k-2]_q! (q^2; q^{2n+2}; q^2)_{k+1}} + \frac{(-1)^n}{2^n}. \]

**Proof.** We give in detail the proof of (79). The proof for \( \beta_{n,\alpha}^{(3)}(q) \) is similar. Since
\[ \frac{\exp_q(xt) \exp_q(-\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}, \]
then
\[ \frac{\exp_q(xt) \exp_q(-\frac{t}{2})}{g_\alpha^{(1)}(it; q)} - \frac{\exp_q(-\frac{t}{2})}{g_\alpha^{(1)}(it; q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!} - \frac{\exp_q(-\frac{t}{2})}{g_\alpha^{(1)}(it; q)}. \]
Consequently, from the series representation of \( c_q(t) \) in (43), we get
\[
\frac{c_q(\frac{t}{2})}{g^{(1)}_\alpha(it; q)} \left(1 - g^{(1)}_\alpha(it; q)\right) = \sum_{n=0}^{\infty} \left(\beta_{n,\alpha}(q) - \frac{(-1)^n}{2^n}\right) \frac{t^n}{[n]_q!}.
\] (82)

Since
\[
\left(g^{(1)}_\alpha(it; q) - 1\right) = t^2 \sum_{m=0}^{\infty} \frac{(-1)^2m_2 t^{2m}}{2^{2m+2}(q^2, q^{2\alpha+2}; q^2)_{m+1}},
\] (83)
then substituting from (83) into (82) and using (81), we obtain
\[
\left(\sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{t^n}{[n]_q!}\right) \left(-t^2 \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n+2}(q^2, q^{2\alpha+2}; q^2)_{n+1}}\right) = \sum_{n=0}^{\infty} \left(\beta_{n,\alpha}(q) - \frac{(-1)^n}{2^n}\right) \frac{t^n}{[n]_q!}.
\] (84)

Equating the coefficient of \( t^n \) in (84), we get (79) and the theorem follows.

The following theorem gives a recursive relations between the polynomials \( B^{(k)}_{n,\alpha}(x; q) \) and \( B^{(k)}_{n,\alpha+1}(x; q) \) \( (k = 2, 3) \).

**Theorem 2.26.** If \( Re \alpha > -1 \), \( x \in \mathbb{C} \), and \( k \in \mathbb{N} \), then
\[
\frac{B^{(r)}_{n,\alpha}(x; q)}{[n]_q!} = 2(1 - q^{2\alpha+2}) \sum_{k=0}^{|\frac{x}{2}|} (-1)^k \frac{(1 - q)^{2k}}{[n - 2k]_q!} B^{(r)}_{n - 2k,\alpha+1}(x; q) \quad (r = 2, 3),
\]
where
\[
h^{(r)}_k(q^2) = \sum_{m=1}^{\infty} \frac{(-2)^r J^{(r)}_{\alpha+1}(j^{(r)}_m; q^2)}{\prod_{j=0}^{r-1} J^{(j)}_{\alpha+1}(j^{(j)}_m; q^2)_{j^{(j)}_m}} \left(\frac{1}{j^{(r)}_m}\right)^{2k}.
\]
and \( (j^{(r)}_m)_{m=1}^{\infty} \) \( (r = 2, 3) \) are the positive zero of \( J^{(r)}_{\alpha+1}(\cdot; q^2) \).

**Proof.** We start with the proof of the case \( (r = 2) \). From (613), we have the identity
\[
\frac{J^{(2)}_{\alpha+1}(t; q)}{J^{(2)}_\alpha(t; q)} = \sum_{n=1}^{\infty} h^{(2)}_n(q^2) t^{2n-1},
\] (85)
where
\[
h^{(2)}_n(q^2) = \sum_{m=1}^{\infty} \frac{-2 J^{(2)}_{\alpha+1}(j^{(2)}_m; q^2)}{\prod_{j=0}^{1} J^{(j)}_{\alpha+1}(j^{(j)}_m; q^2)_{j^{(j)}_m}} \left(\frac{1}{j^{(2)}_m}\right)^{2n}.
\]
Replacing \( t \) by \( it(1 - q) \) and \( q \) by \( q^2 \) in (85), we obtain
\[
\frac{1}{J^{(2)}_\alpha(it(1 - q); q^2)} = \frac{1}{J^{(2)}_{\alpha+1}(it(1 - q); q^2)} \sum_{n=1}^{\infty} h^{(2)}_n(it(1 - q))^{2n-1}.
\] (86)

Multiplying (86) by \( E_q(x)E_q(\frac{1}{t}) \) to obtain
\[
\frac{E_q(x)E_q(\frac{1}{t})}{J^{(2)}_\alpha(it(1 - q); q^2)} = \frac{E_q(x)E_q(\frac{1}{t})}{J^{(2)}_{\alpha+1}(it(1 - q); q^2)} \sum_{n=1}^{\infty} h^{(2)}_n(q^2)(it(1 - q))^{2n-1}.
\] (87)
Substituting from (15) into (87), we get
\[
\frac{E_q(xt) E_q(\frac{t}{q})}{g_{\alpha}^{(2)}(it; q)} = \frac{(1 + q)[\alpha + 1]q^2}{(\frac{q}{2})} \frac{E_q(xt) E_q(\frac{t}{q})}{g_{\alpha + 1}(it; q)} \sum_{n=1}^{\infty} h_n^{(2)}(q^2)(it(1 - q))^{2n-1}. \tag{88}
\]

Consequently,
\[
\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^n}{[n]_q!} = \frac{2(1 + q)[\alpha + 1]q^2}{it} \left( \sum_{n=0}^{\infty} B_{n,\alpha+1}^{(2)}(x; q) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} h_n^{(2)}(q^2)(it(1 - q))^{2n-1} \right)
\]
\[
= 2(1 - q^{2\alpha+2}) \left( \sum_{n=0}^{\infty} B_{n,\alpha+1}^{(2)}(x; q) \frac{t^n}{[n]_q!} \right) \left( \sum_{n=0}^{\infty} h_n^{(2)}(q^2)(it(1 - q))^{2n} \right)
\]
\[
= 2(1 - q^{2\alpha+2}) \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(1 - q)^{2k}}{[n - 2k]_q!} h_{k+1}^{(2)}(q^2) B_{n-2k,\alpha+1}^{(2)}(x; q).
\tag{89}
\]

Hence
\[
\sum_{n=0}^{\infty} B_{n,\alpha}^{(2)}(x; q) \frac{t^n}{[n]_q!} = 2(1 - q^{2\alpha+2}) \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{(1 - q)^{2k} h_{k+1}^{(2)}(q^2)}{[n - 2k]_q!} B_{n-2k,\alpha+1}^{(2)}(x; q).
\]

Equating the coefficients of \( t^n \) in (89), we get the result for \((r = 3)\). The proof of the case \((r = 2)\) follows from the identity (see [1, Eq. (4.3), P. 6]),
\[
\frac{J_{\alpha+1}^{(3)}(t; q)}{J_{\alpha}^{(3)}(t; q)} = \sum_{n=1}^{\infty} h_n^{(3)}(q) t^{2n-1},
\]
where
\[
h_n^{(3)}(q) = \sum_{m=1}^{\infty} \frac{-2J_{\alpha+1}^{(3)}(j_{m,\alpha}; q^2)}{J_{\alpha}^{(3)}(j_{m,\alpha}; q^2)} \left( \frac{1}{J_{\alpha}^{(3)}(j_{m,\alpha})} \right)^2 2n,
\]
and by using the same technique.

\[\square\]

3. Asymptotic relations for the generalized \(q\)-Bernoulli numbers

In this section, we derive asymptotic relations for the generalized \(q\)-Bernoulli numbers defined in (17).

**Theorem 3.1.** Let \(n\) be a non-negative integer and \(\alpha\) be a complex number such that \(\text{Re} \alpha > -1\). Then for \(n \in \mathbb{N}\),
\[
\beta_{2n,\alpha}(q) = 2(-1)^{n+1}(q; q)_{2n} \sum_{k=1}^{\infty} \frac{\cos_q(j_{k,\alpha}^{(2)})}{(2k+1)} \frac{d}{dz} \frac{1}{\gamma_{\alpha}^{(2)}(z; q^2)} \bigg|_{z=j_{k,\alpha}^{(2)}}.
\tag{90}
\]
\[
\beta_{2n+1,\alpha}(q) = 2(-1)^n(q; q)_{2n+1} \sum_{k=1}^{\infty} \frac{\sin_q(j_{k,\alpha}^{(2)})}{(2k+1)} \frac{d}{dz} \frac{1}{\gamma_{\alpha}^{(2)}(z; q^2)} \bigg|_{z=j_{k,\alpha}^{(2)}}.
\]

where \(\gamma_{\alpha}^{(2)}(z; q)\) is defined in (7).

**Proof.** Since
\[
G(z) := \frac{E_q(-z)}{g_{\alpha}^{(2)}(iz; q)} = \sum_{n=0}^{\infty} \beta_{n,\alpha}(q) \frac{z^n}{[n]_q!}, \quad |z| < \frac{j_{1,\alpha}^{(2)}}{1 - q},
\]
since
\[ \beta_{n,a}(q) = \frac{G(n)(q)}{n!}, \quad n \in \mathbb{N}_0. \]

Now, we integrate \( f(z) := \frac{G(z)}{z^{n+1}} \), \( G(z) = \frac{E_q\left(\frac{z}{2}\right)}{g(z)} \) on the contour \( \Gamma_m \), where \( \Gamma_m \) is a circle of radius \( R_m \), \( |z_m| < R_m < |z_{m+1}|. \) From the Cauchy Residue Theorem, see \([2]\),
\[ \int_{\Gamma_m} f(z) \, dz = 2\pi i \sum \text{Res}(f, z_k), \]
where \( \{z_k\} \) are the poles of \( f \) that lie inside \( \Gamma_m \). The function \( f(z) \) has a pole at \( z = 0 \) of order \( n + 1 \) and simple poles at \( \pm z_k \) where \( z_k = i \frac{j_{k,\alpha}}{1-q}, \, k \in \mathbb{N}. \) Consequently,
\[ I_m = \frac{1}{2\pi i} \int_{\Gamma_m} f(z) \, dz = \text{Res}(f(z), 0) + \sum_{k=1}^m \text{Res}(f(z), \pm z_k). \quad (91) \]

Since
\[ \text{Res}(f, 0) = \frac{f^n(0)}{n!} = \frac{\beta_{n,a}(q)}{[n]_q^!}, \]
\[ \text{Res}(f, z_k) = \frac{E_q\left(\frac{z_k}{2}\right)}{d\frac{g(z)}{dz}(i\alpha; q)} \left. \right|_{z=z_k} (z_k)^n + 1 = \frac{E_q\left(\frac{i j_{k,\alpha}^{(2)}}{2(1-q)}\right)}{d\frac{g(z)}{dz}(i\alpha; q)} (j_{k,\alpha}^{(2)})^n, \]
and
\[ \text{Res}(f, -z_k) = \frac{E_q\left(\frac{-z_k}{2}\right)}{d\frac{g(z)}{dz}(i\alpha; q)} \left. \right|_{z=-z_k} (-z_k)^n + 1 = \frac{E_q\left(\frac{i j_{k,\alpha}^{(2)}}{2(1-q)}\right)}{d\frac{g(z)}{dz}(i\alpha; q)} (j_{k,\alpha}^{(2)})^n. \]

Then Equation (91) can be written as
\[ I_m = \frac{\beta_{n,a}(q)}{[n]_q^!} + \sum_{k=1}^m 2\text{Re} \left( -i \right)^n E_q\left(\frac{i j_{k,\alpha}^{(2)}}{2(1-q)}\right) (1-q)^n \frac{(j_{k,\alpha}^{(2)})^n + 1}{d\frac{g(z)}{dz}(i\alpha; q)} (j_{k,\alpha}^{(2)})^n. \quad (92) \]

Substituting into (92) with \( -i = e^{-\frac{\pi i}{2}} \) gives
\[ I_m = \frac{\beta_{n,a}(q)}{[n]_q^!} + 2(1-q)^n \cos \frac{n\pi}{2} \sum_{k=1}^m \frac{C_{q}(\frac{j_{k,\alpha}^{(2)}}{2(1-q)})}{d\frac{g(z)}{dz}(i\alpha; q)} \left. \right|_{z=j_{k,\alpha}^{(2)}} (j_{k,\alpha}^{(2)})^n + 1.
- 2(1-q)^n \sin \frac{n\pi}{2} \sum_{k=1}^m \frac{S_{q}(\frac{j_{k,\alpha}^{(2)}}{2(1-q)})}{d\frac{g(z)}{dz}(i\alpha; q)} \left. \right|_{z=j_{k,\alpha}^{(2)}} (j_{k,\alpha}^{(2)})^n + 1. \]

Now, we show that the integral \( I_m \to 0 \) as \( m \to \infty \). Bergweiller and Hayman \([7]\) introduced the asymptotic relation for \( E_q(z) \),
\[ |M(r; E_q)| := \sup\{|E_q(z)| : |z| = r\} \sim e^{-\frac{(\log r)^2}{2}}, \quad \text{when} \quad r = |z| \to \infty. \]

In \([4]\), Annaby and Mansour proved that for \( r = |z| \to \infty \)
\[ z^{-\nu} J_{v}^{(2)}(z; q) \sim \exp \left(-\frac{(\log r)^2}{2\log q} - \frac{\log 2}{\log q} \log r \right). \]
Hayman in [15] introduced the higher order asymptotics of $J^{(2)}_\nu(z; q)$. Then, Annaby and Mansour, see [4], pointed out that the first order asymptotics of the zeros of $\delta_{J_m}^{(2)}(z; q)$ is given by

$$J_{m, \nu}^{(2)} = 2q^{-2m}q^{-\nu+1}(1 + O(q^{2m})), \quad (m \to \infty).$$

Hence if $z_m$ are the positive zero of $g^{(2)}_\alpha(iz; q)$, then

$$\lim_{m \to \infty} \frac{z_m}{z_{m+1}} = \lim_{m \to \infty} \frac{J_{m, \nu}^{(2)}}{J_{m+1, \nu}^{(2)}} = q^2, \quad \lim_{m \to \infty} z_m = \infty. \quad (93)$$

Let $0 < \epsilon < (q^{-1} - 1)$. There exists $M_0 \in \mathbb{N}$ such that if $m \in \mathbb{N}$, $m \geq M_0$, then

$$q^2(1 - \epsilon) < \frac{z_m}{z_{m+1}} < q^2(1 + \epsilon).$$

Hence $z_m < qz_{m+1}$ for all $m \geq M_0$. We can choose $R_m, \delta := q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}}$ such that $(z_m < \delta R_m < qz_{m+1} < R_m)$. Indeed,

$$\delta = q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}} \geq q^{-1} \lim_{m \to \infty} \frac{z_m}{z_{m+1}} = q^{-1}q^2 = q.$$ 

Also $\delta = q^{-1} \sup_{m \geq M_0} \frac{z_m}{z_{m+1}} < q(1 + \epsilon) < 1$. Hence $1 > \delta > q$ and so by

$$z_m < R_m < \frac{q}{\delta}z_{m+1} < z_{m+1}, \quad (94)$$

the annulus $\delta R_m < |z| < R_m$ has no zeros of the function $g^{(2)}_\alpha(iz; q)$. Hence, from the minimum modulus principle we have

$$\left| g^{(2)}_\alpha(iz; q) \right| \geq c_1 e^{\frac{(\log R_m)^2}{2 \log q} - \frac{\log 2}{\log q} \log \delta R_m}, \quad c_1 > 0.$$

$$\left| E_q(\frac{z}{2}) \right| \leq c_2 e^{-\frac{(\log R_m)^2}{2 \log q}}, \quad c_2 > 0. \quad (95)$$

Therefore, from (95), we conclude that

$$\left| \frac{E_q(\frac{z}{2})}{g^{(2)}_\alpha(iz; q)} \right| \leq \frac{c_2}{c_1} e^{\frac{(\log R_m)^2}{2 \log q} - \frac{\log 2}{\log q} \log \delta R_m} \leq \frac{c_2}{c_1} e^{\frac{1}{\log q} ((\log R_m)^2 - (\log R_m)^2)} + \frac{\log 2}{\log q} \log \delta R_m \leq \frac{c_2}{c_1} e^{\frac{2 \log 2 \log R_m}{\log q} + \frac{\log 4 \log R_m}{\log q}},$$

where

$$K = \frac{1}{2 \log q} ((\log 2)^2 - (\log 2)^2 + 2 \log 2 \log \delta).$$

21
Now, using the ML-inequality (see \[2\]) to obtain
\[
|I_m| = \left| \int_{\Gamma_m} f(z)dz \right| \leq (2\pi R_m)|M(r; f(z))|
\leq \frac{2\pi R_m c_2}{c_1} e^K e^{2 \log R_m + \log \delta, \log R_m \frac{1}{R_m^{\delta-n}}} (96)
\]
From (93) and (94), we have \( R_m \to 0 \). Also, since \( 0 < q < 1 \) and \( 1 > \delta > q \) then
\[
\lim_{m \to \infty} \frac{2 \log R_m}{R_m^{\delta-n}} \to 0 \quad \text{and} \quad \lim_{m \to \infty} \frac{\log \delta}{R_m^{\delta-n}} \to 0 \quad \text{as} \quad m \to \infty.
\]
Hence \( \lim_{m \to \infty} I_m = 0 \). Consequently,
\[
\frac{\beta_{n,\alpha}(q)}{\lfloor n \rfloor!} = -2(1-q)^n \cos \frac{n\pi}{2} \sum_{k=1}^{\infty} \frac{\cos_q(\frac{2^j}{2(1-q)})}{\frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{k,\alpha}} \frac{1}{(j_{k,\alpha})^{n+1}}
+ 2(1-q)^n \sin \frac{n\pi}{2} \sum_{k=1}^{\infty} \frac{\sin_q(\frac{2^j}{2(1-q)})}{\frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{k,\alpha}} \frac{1}{(j_{k,\alpha})^{n+1}}.
\]
Therefore,
\[
\beta_{2n,\alpha}(q) = 2(-1)^{n+1}(q; q)_{2n} \sum_{k=1}^{\infty} \frac{\cos_q(\frac{2^j}{2(1-q)})}{\frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{k,\alpha}},
\]
\[
\beta_{2n+1,\alpha}(q) = 2(-1)^{n}(q; q)_{2n+1} \sum_{k=1}^{\infty} \frac{\sin_q(\frac{2^j}{2(1-q)})}{\frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{k,\alpha}},
\]
which completes the proof of the theorem.

\[ \square \]

Remark 1. If we substitute with \( \alpha = \frac{1}{2} \) in the second equation in \( (90) \), then \((z_k)_k\) will be the positive zeros of \( \sin_q(z) \) and consequently, the series in the left hand side vanishes which coincide with the known result that the odd Bernoulli numbers vanish \((\beta_{2n+1}(q) = 0, n \geq 1) \) (see \( 19 \)). Similarly, if we set \( \alpha = -\frac{1}{2} \) in the first equation in \( (90) \), the series in the left hand side vanishes and this coincide with the fact that the even Euler’s numbers are zero \((E_{2n}(q) = 0, n \geq 1) \) (see \( 19 \)).

Corollary 3.2. The asymptotic relations of the generalized q-Bernoulli numbers \((\beta_{n,\alpha}(q))_n\),
\[
\beta_{2n,\alpha}(q) = 2(-1)^{n+1}(q; q)_{2n} \frac{\cos_q(\frac{2^j}{2(1-q)})}{(j_{1,\alpha})^{2n+1} \frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{1,\alpha}} (1 + o(1)),
\]
\[
\beta_{2n+1,\alpha}(q) = 2(-1)^n(q; q)_{2n+1} \frac{\sin_q(\frac{2^j}{2(1-q)})}{(j_{1,\alpha})^{2n+2} \frac{\partial^2}{\partial z} \mathcal{J}_\alpha(2)\left(z; q^2\right)}|_{z=j_{1,\alpha}} (1 + o(1)),
\]
where \( \mathcal{J}_\alpha(2)(z; q) \) is defined in \( 7 \).

\[ \square \]

Proof. The proof follows directly from Theorem 3.1.
4. Applications of the generalized $q$-Bernoulli polynomials

In this section, we introduce connection relations between the generalized $q$-Bernoulli polynomials $B_n^{(k)}(x; q) \ (k = 1, 2, 3)$ and the $q$-Laguerre and the little $q$-Legendre polynomials.

The $q$-Laguerre polynomials $L_n^\alpha(x; q)$ of degree $n$ are defined by

$$L_n^\alpha(x; q) : = \frac{1}{(q; q)_n} 2^\varphi_1 \left( q^{n-1}; q^n \right)_0 \frac{(q^{n+\alpha+1}; q)_k}{(q^\alpha+1; q)_k} \sum_{k=0}^{\infty} \frac{(-1)^k (q^{n+\alpha+1})^k x^k}{(q^n; q)_k}.$$  \hspace{1cm} (97)

The Rodrigues formula is given by

$$L_n^\alpha(x; q) = \frac{(1-q)^n}{(q; q)_n} (-x; q)_{\infty} x^{-\alpha} D_q^{n} \left( \frac{x^{\alpha+n}}{(-x; q)_{\infty}} \right),$$  \hspace{1cm} (98)

and the orthogonality relation is

$$\int_0^\infty \frac{x^\alpha}{(-x; q)_\infty} L_m^\alpha(x; q) L_n^\alpha(x; q) dx = \frac{(q^{-\alpha}; q)_{\infty} (q^{n+1}; q)_n}{(q; q)_\infty (q; q)_n q^n} \Gamma_q(-\alpha) \Gamma_q(\alpha+1) \delta_{mn}, \quad \alpha > -1,$$  \hspace{1cm} (99)

where $\delta_{mn}$ is the Kronecker delta function, see [17,21]. The $q$-Laguerre polynomials $L_n^\alpha(x; q)$ satisfy three term recurrence relation

$$-x a_n L_n^\alpha(x; q) = L_{n+1}^\alpha(x; q) - b_n L_n^\alpha(x; q) + d_n L_{n-1}^\alpha(x; q),$$

where

$$a_n = \frac{q^{2n+\alpha+1}}{1-q^{\alpha+1}}, \quad b_n = 1 + \frac{q (1-q^{n+\alpha})}{1-q^{\alpha+1}}, \quad d_n = \frac{q (1-q^{\alpha+n})}{1-q^{\alpha+1}}.$$  

In the following, let $\alpha > -1$ and $\mathbb{P}_n = \{ p(x) : \deg p(x) \leq n \}$ with the inner product

$$\langle p(x), g(x) \rangle = \int_0^\infty \frac{x^\alpha}{(-x; q)_{\infty}} p(x) g(x) dx,$$

where $p(x), g(x) \in \mathbb{P}_n$. From [99], we note that $\{ L_0^\alpha(x; q), L_1^\alpha(x; q), \ldots, L_n^\alpha(x; q) \}$ is an orthogonal basis for $\mathbb{P}_n$.

**Theorem 4.1.** Let $p(x) \in \mathbb{P}_n$. Then $p(x)$ can be expanded as

$$p(x) = \sum_{m=0}^{n} C_m L_m^\alpha(x; q),$$

where

$$C_m = q^m (1-q)^m (q^{\alpha+m+1}; q)_\infty \int_0^\infty D_q^{m} \left( \frac{x^{\alpha+m}}{(-x; q)_{\infty}} \right) p(x) dx.$$  

**Proof.** Since

$$p(x) = \sum_{m=0}^{n} C_m L_m^\alpha(x; q),$$

in order to calculate the constant $C_m$, we use [99] to obtain

$$\langle p(x), L_k^\alpha(x; q) \rangle = \sum_{m=0}^{n} C_m \langle L_m^\alpha(x; q), L_k^\alpha(x; q) \rangle = \sum_{m=0}^{n} C_m \langle L_m^\alpha(x; q), L_k^\alpha(x; q) \rangle.$$  

23
Then

\[ \langle p(x), L_m^\alpha(x; q) \rangle = C_m \left( L_m^\alpha(x; q), L_m^\alpha(x; q) \right) = C_m \left( \frac{q^{\alpha+1}; q)_m}{q^{\alpha}; q)_m (1 - q)^{1+\alpha} \Gamma_q(\alpha + 1) \right). \]

Therefore,

\[ C_m = \frac{q^m(q; q)_m}{(q^{\alpha+1}; q)_m (1 - q)^{1+\alpha} \Gamma_q(\alpha + 1)} \int_0^\infty \frac{x^\alpha}{(-x; q)_{\infty}} L_m^\alpha(x; q)p(x)dx. \quad (100) \]

Using (98) with \( n \) replaced by \( m \), we obtain

\[ C_m = \frac{q^m(1 - q)^{m-1}(q^{\alpha+m+1}; q)_\infty}{(q; q)_\infty} \int_0^\infty D_q^m \left( \frac{x^\alpha+1}{(-x; q)_{\infty}} \right) p(x)dx, \]

and the theorem follows. \( \square \)

The following Lemma, see [18], is essential in the proof of Theorem 4.3.

**Lemma 4.2.** Let the functions \( f \) and \( g \) be defined and continuous on \([0, \infty]\). Assume that the improper Riemann integrals of the functions \( f(x)g(x) \) and \( f(x/q)g(x) \) exist on \([0, \infty]\). Then

\[ \int_0^\infty f(x)D_qg(x)dx = \frac{f(0)g(0)}{1 - q} \ln q - \frac{1}{q} \int_0^\infty g(x)D_qf(x)dx = \frac{f(0)g(0)}{1 - q} \ln q - \int_0^\infty g(x)D_qf(x)dx. \]

**Theorem 4.3.** If \( n \in \mathbb{N} \) and \( x \in \mathbb{C} \), then

\[
B_{n,\alpha}^{(1)}(x; q) = \sum_{m=0}^n A_m \left( \sum_{k=m}^n q^{k(n-k+1)}(q^{-n}; q)_k(q^{-k}; q)_m(q^{-\alpha-k}; q)_k \beta_{n-k,\alpha}(q) \right) L_m^\alpha(x; q),
\]

\[
B_{n,\alpha}^{(2)}(x; q) = \sum_{m=0}^n A_m \left( \sum_{k=m}^n q^{k(n-k+1)}(q^{-n}; q)_k(q^{-k}; q)_m(q^{-\alpha-k}; q)_k \beta_{n-k,\alpha}(q) \right) L_m^\alpha(x; q),
\]

\[
B_{n,\alpha}^{(3)}(x; q) = \sum_{m=0}^n A_m \left( \sum_{k=m}^n q^{k(n-k+1)}(q^{-n}; q)_k(q^{-k}; q)_m(q^{-\alpha-k}; q)_k \beta_{n-k,\alpha}(q) \right) L_m^\alpha(x; q),
\]

where

\[ A_m = \frac{-q^m(q^{\alpha+m+1}, q^{-\alpha}; q)_\infty}{(1 - q)^{2}(q; q)_\infty} \frac{\pi}{\sin(\alpha \pi)}. \]

**Proof.** We prove the identity for \( B_{n,\alpha}^{(1)}(x; q) \) and the proofs for \( B_{n,\alpha}^{(k)}(x; q) \) \( (k = 2, 3) \) are similar. Substitute with \( p(x) = B_{n,\alpha}^{(1)}(x; q) \) in (100). This gives

\[ C_m = \frac{q^m(q; q)_m}{(q^{\alpha+1}; q)_m (1 - q)^{1+\alpha} \Gamma_q(\alpha + 1)} \int_0^\infty \frac{x^\alpha}{(-x; q)_{\infty}} L_m^\alpha(x; q)B_{n,\alpha}^{(1)}(x; q)dx. \quad (101) \]

Since \( \{L_m^\alpha(x; q)\}_{n \in \mathbb{N}} \) is an orthogonal polynomials sequence then \( C_m = 0 \) for \( m > n \), and

\[ B_{n,\alpha}^{(1)}(x; q) = \sum_{m=0}^n C_m L_m^\alpha(x; q). \]

Now, we calculate \( C_m \). Using (139) in (101) gives

\[ C_m = \frac{q^m(q; q)_m}{(q^{\alpha+1}; q)_m (1 - q)^{1+\alpha} \Gamma_q(\alpha + 1)} \sum_{k=0}^n \beta_{n-k,\alpha}(q) \int_0^\infty \frac{x^{\alpha+k}}{(-x; q)_{\infty}} L_m^\alpha(x; q)dx. \]
Since
\[ \int_0^\infty \frac{x^\alpha}{(−x; q)_\infty} L_m^\alpha(x; q)x^k dx = 0, \quad \text{for } k < m, \]
then
\[ C_m = \frac{q^m(q; q)_\infty}{(q; q)_\infty} \sum_{k=m}^n \left[ \binom{n}{k} q^{-i-k} \right] \frac{[k]_q^{-1}}{[k-m]_q^i} \beta_{n-k, \alpha}(q) \int_0^\infty \frac{x^{α+k}}{(−x; q)_\infty} L_m^\alpha(x; q) dx. \]

From (98), we get
\[ C_m = \frac{q^m(1−q)^{m-1}(q^{α+1}; q)_\infty}{(q; q)_\infty} \sum_{k=m}^n \left[ \binom{n}{k} q^{-i-k} \right] \frac{[k]_q^{-1}}{[k-m]_q^i} \beta_{n-k, \alpha}(q) \int_0^\infty \frac{x^{α+k}}{(−x; q)_\infty} dx, \]
then applying the \( q \)-integration by part introduced in Lemma 4.2 \( m \) times, we obtain
\[ C_m = \frac{(−1)^m q^m(1−q)^{m-1}(q^{α+m+1}; q)_\infty}{(q; q)_\infty} \sum_{k=m}^n \left[ \binom{n}{k} q^{-i-k} \right] \frac{[k]_q^{-1}}{[k-m]_q^i} \beta_{n-k, \alpha}(q) \int_0^\infty \frac{x^{α+k}}{(−x; q)_\infty} dx. \]

From [16, Eq. (5.4), P. 465],
\[ \frac{1}{Γ_q(z)} = \frac{\sin π z}{π} \int_0^\infty \frac{t^z}{(−t(1−q); q)_\infty} dt, \quad \text{Re } z > 0. \]
Then
\[ \int_0^\infty \frac{x^{α+k}}{(−x; q)_\infty} dx = \frac{π}{\sin(−α−k)π} \frac{1}{Γ_q(−α−k)}(1−q)^{α+k}. \]

Therefore,
\[ C_m = \frac{(−1)^m q^m(q^{α+m+1}; q)_\infty}{(1−q)^2(q; q)_\infty} \sum_{k=m}^n \left[ \prod_{i=0}^{m-1} q^{-i-k} \right] \frac{q^{−α−k}}{(q; q)_{n−k}(q; q)_{k−m}} \frac{π}{\sin(−α−k)π} β_{n-k, \alpha}(q). \]

Since
\[ \frac{π}{\sin(−α−k)π} = (-1)^{k−1} \frac{π}{\sin(απ)}, \quad \prod_{i=0}^{m−1} q^{-i-k} = q^{\frac{m(m−1)}{2}} q^{−km}, \]
then substituting from (103) into (102), we get
\[ C_m = \frac{(−1)^m q^m q^{m(m−1)/2}(q^{α+m+1}; q)_\infty(q^{−α}; q)_\infty}{(1−q)^2(q; q)_\infty} \frac{π}{\sin(απ)} \sum_{k=m}^n \left[ \prod_{i=0}^{m−1} q^{-i-k} \right] \frac{(q; q)_{n−k}(q; q)_{k−m}}{(q; q)_{n−k}(q; q)_{k−m}} β_{n-k, \alpha}(q). \]

Using the relation [54], we obtain
\[ C_m = \frac{−q^m(q^{α+m+1}; q^{−α}; q)_\infty}{(1−q)^2(q; q)_\infty} \frac{π}{\sin(απ)} \sum_{k=m}^n \frac{q^{k(m−1)}(q^{−n; q)k(q−k; q)_m(q^{−α−k}; q)_k}}{(q; q)_k} β_{n-k, \alpha}(q), \]
and this completes the proof of the theorem. \( \square \)

25
The little $q$-Legendre polynomials $(P_n(x|q))_n$ are defined by

$$P_n(x|q) = 2\varphi_1 \left( \frac{q^{-n}}{q}, \frac{q^{n+1}}{q}; qx \right) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (q^{n+1}; q)_k}{(q; q)_k} \frac{q^k x^k}{(q; q)_k}.$$ 

They satisfy the Rodrigues formula

$$P_n(x|q) = \frac{q^n(n-1)/2(1-q)^n}{(q; q)_n}D_n^{q^{-1}}(x^n(qx; q)_n), \quad \text{for } n \geq 0,$$  

and the orthogonality relation

$$\int_0^1 P_m(x|q) P_n(x|q) dq x = \frac{(1-q)}{(1-q^{2n+1})} \delta_{mn}, \quad \text{for } m, n \geq 0,$$  

see [21]. Let $\mathbb{P}_n = \{g(x) : \deg g(x) \leq n\}$ with the inner product

$$\langle g(x), p(x) \rangle = \int_0^1 g(x)p(x) dq x,$$

where $p(x), g(x) \in \mathbb{P}_n$.

**Theorem 4.4.** Let $g(x) \in \mathbb{P}_n$. Then $g(x)$ can be represented by

$$g(x) = \sum_{k=0}^{n} C_k P_k(x|q),$$

where

$$C_k = \frac{q^{k(k-1)/2}(1-q)^{k-1}(1-q^{2k+1})}{(q; q)_k} \int_0^1 D_k^{q^{-1}}(x^k(qx; q)_k)g(x) dq x.$$  

**Proof.** Since

$$g(x) = \sum_{k=0}^{n} C_k P_k(x|q),$$

then by the orthogonality relation (105), we obtain

$$C_k = \frac{(1-q^{2k+1})}{(1-q)} \langle g(x), P_k(x|q) \rangle = \frac{(1-q^{2k+1})}{(1-q)} \int_0^1 P_k(x|q)g(x) dq x.$$  

(106)

By using (104), we get

$$C_k = \frac{q^{k(k-1)/2}(1-q)^{k-1}(1-q^{2k+1})}{(q; q)_k} \int_0^1 D_k^{q^{-1}}(x^k(qx; q)_k)g(x) dq x,$$

which readily gives the result. □
Theorem 4.5. For \( n \in \mathbb{N} \) and \( x \in \mathbb{C} \),

\[
B_{n,\alpha}^{(1)}(x; q) = \sum_{k=0}^{n} \lambda_k \left( \sum_{m=k}^{n} (-1)^m q^{m(2m-n+1)/2} \frac{(q^{-n}; q)_m (q^{-m}; q)_k}{(q; q)_{m+k+1}} \beta_{n-m,\alpha}(q) \right) P_k(x|q),
\]

\[
B_{n,\alpha}^{(2)}(x; q) = \sum_{k=0}^{n} \lambda_k \left( \sum_{m=k}^{n} (-1)^m q^{mn} \frac{(q^{-n}; q)_m (q^{-m}; q)_k}{(q; q)_{m+k+1}} \beta_{n-m,\alpha}(q) \right) P_k(x|q),
\]

\[
B_{n,\alpha}^{(3)}(x; q) = \sum_{k=0}^{n} \lambda_k \left( \sum_{m=k}^{n} (-1)^m q^{m(2m-n+1)/2} \frac{(q^{-n}; q)_m (q^{-m}; q)_k \beta_{n-m,\alpha}(q)}{(q; q)_{m+k+1}} \right) P_k(x|q),
\]

where

\[
\lambda_k = q^{-k(k-3)/2}(1 - q^{2k+1}).
\]

Proof. Substitute with \( g(x) = B_{n,\alpha}^{(1)}(x; q) \) in (109), we obtain

\[
C_k = \frac{(1 - q^{2k+1})}{(1 - q)} \int_0^1 P_k(x|q)B_{n,\alpha}^{(1)}(x; q) d_q x.
\]  

(107)

Since the polynomials \( \{P_k(x|q)\} \) are orthogonal, then \( C_k = 0 \) for \( k > n \), and

\[
B_{n,\alpha}^{(1)}(x; q) = \sum_{k=0}^{n} C_k P_k(x|q).
\]  

(108)

Set

\[
B_{n,\alpha}^{(1)}(x; q) = \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] q \beta_{n-m,\alpha}(q)x^m.
\]

From (107),

\[
C_k = \frac{(1 - q^{2k+1})}{(1 - q)} \sum_{m=0}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] q \beta_{n-m,\alpha}(q) \int_0^1 P_k(x|q)x^m d_q x,
\]

\[
= \frac{(1 - q^{2k+1})}{(1 - q)} \sum_{m=k}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] q \beta_{n-m,\alpha}(q) \int_0^1 P_k(x|q)x^m d_q x,
\]

since

\[
\int_0^1 P_k(x|q)x^m d_q x = 0 \quad for \ m < k.
\]

Hence, by the Rodrigues formula in (104), we obtain

\[
C_k = \frac{(1 - q^{2k+1})q^{k(k-1)/2}(1 - q)^{k-1}}{(q; q)_k} \sum_{m=k}^{n} \left[ \begin{array}{c} n \\ m \end{array} \right] q \beta_{n-m,\alpha}(q) \int_0^1 D_{q^{-1}}^k(x^k(qx; q)_k)x^m d_q x.
\]  

(109)

Using the \( q^{-1} \)-integration by parts

\[
\int_0^{a/q} f(t)D_{q^{-1}} g(t)d_q t = q \left( (fg)(\frac{a}{q}) - (fg)(0) \right) - \int_0^{a} g(t)D_{q^{-1}} f(t)d_q t,
\]

(110)

where \( f \) and \( g \) are continuous functions at zero, see [3]. This gives

\[
\int_0^1 D_{q^{-1}}^k(x^k(qx; q)_k)x^m d_q x = q \left[ x^m D_{q^{-1}}^{k-1}(x^k(qx; q)_k) \right]_0^a - [m]_q q^{1-m} \int_0^1 x^{m-1} D_{q^{-1}}^{k-1}(x^k(qx; q)_k)d_q x.
\]  

(111)
The first term on the right hand side of (111) vanishes because
\[ D_{q^{-1}}(x^k(qx; q)_k) = [k]_{q^{-1}}x^{k-1}(x; q)_k + x^k D_{q^{-1}}(qx; q)_k, \]
and
\[ D_{q^{-1}}(qx; q)_k \mid _{x=\frac{1}{q}} = a^k \frac{[k]_q!}{[k-j]_q!} (1; q)_{k-j} = 0, \quad \text{for } j = 0, 1, \ldots, k - 1. \]
Therefore,
\[ \int_{0}^{1} D_{q^{-1}}^k(q^x(qx; q)_k)x^mdx = -[m]_q q^{-m-1} \int_{0}^{1} x^{m-1} D_{q^{-1}}^k(q^x(qx; q)_k)dx. \quad (112) \]
Now, applying \[110\] \( k - 1 \) times on the right hand side of \[112\], and using that
\[ D_{q^{-1}}^m(q^x(qx; q)_k) = 0 \text{ at } x = 0, x = \frac{1}{q} (m = 0, 1, \ldots, k - 1) \] yields
\[ \int_{0}^{1} D_{q^{-1}}^k(q^x(qx; q)_k)x^mdx = (-1)^k \left( \prod_{j=0}^{k-1} q^{1-m-j} \right) \frac{[m]_q!}{[m-k]_q!} \int_{0}^{1} x^m (qx; q)_k dx. \]
Since
\[ B_q(x, y) = \int_{0}^{1} t^{x-1}(qt; q)_{y-1} dt = \int_{0}^{1} t^{x-1} \left( \frac{tq; q}{tq^y; q} \right) dt, \quad \text{Re} (x) > 0, \text{Re} (y) > 0, \]
see \[3, \text{Eq. (1.58), P. 22}, \] then
\[ \int_{0}^{1} D_{q^{-1}}^k(q^x(qx; q)_k)x^mdx = (-1)^k q^{-\frac{k(k-1)}{2}} q^{-mk} \frac{[m]_q!}{[m-k]_q!} B_q(m + 1, k + 1) \]
\[ = (-1)^k q^{-\frac{k(k-3)}{2}} q^{-mk} \frac{[m]_q! \Gamma_q(m + 1) \Gamma_q(k + 1)}{[m-k]_q! \Gamma_q(m + k + 2)} \quad (113) \]
\[ = (-1)^k q^{-\frac{k(k-3)}{2}} q^{-mk} \frac{(m)_{q+1} [m-k]_q! (m+k+1)_q!}{(m)_{q+1} [m-k]_q! (m+k+1)_q!}. \]
Substituting from \[113\] into \[109\] yields
\[ C_k = (-1)^k q^k (1 - q^{2k+1}) \sum_{m=k}^{\infty} \frac{[q^m]_q!}{[q^{m-k}]_q!} \frac{(q; q)_m (q; q)_m}{(q; q)_{m-k} (q; q)_{m+k+1}} \beta_{n-m, \alpha}(q) \]
\[ = q^{-\frac{k(k-3)}{2}} (1 - q^{2k+1}) \sum_{m=k}^{\infty} (-1)^{m} \frac{m}{(m)_{q+1}} \frac{(q^{-m}; q)_m (q^{-m}; q)_m}{(q; q)_{m+k+1}} \beta_{n-m, \alpha}(q), \quad (114) \]
where we used the identity in \[54\]. Therefore, from \[114\] and \[108\], we get the required result for \( B_{n, \alpha}^{(k)}(x; q) \). Similarly, we can prove the result for \( B_{n, \alpha}^{(k)}(x; q) \) \( k = 2, 3 \). \[\square\]

References

[1] L. D. Abreu. A \( q \) -sampling theorem related to the \( q \) -Hankel transform. \textit{Proc. Amer. Math. Soc.}, 133:1197–1203, 2004.
[2] L. Ahlfors. \textit{Complex analysis. An introduction to the theory of analytic functions of one complex variable}. New York-Toronto-London:McGraw-Hill, 1953.
[3] W. A. Al-Salam. \( q \)-Bernoulli numbers and polynomials. \textit{Math. Nachr.}, 17:239–260, 1959.
[4] M. H. Annaby and Z. S. Mansour. On the zeros of the second and third Jackson \( q \)-Bessel functions and their associated \( q \)-Hankel transforms. \textit{Math. Proc. Cambridge Philos. Soc.}, 147:47–67, 2009.
[5] M. H. Annaby and Z. S. Mansour. *q-Fractional Calculus and Equations*. Lecture Notes in Mathematics 2056. Springer-Verlag, Berlin, 2012.

[6] M. H. Annaby, Z. S. Mansour, and O. A. Ashour. Sampling theorems associated with biorthogonal $q$-Bessel functions. *J. Phys. A*, 43(29):15 pp, 2010.

[7] W. Bergweiler and W. K. Hayman. Zeros of Solutions of a Functional Equation. *Comput. Methods Funct. Theory.*, 3:55–78, 2004.

[8] J. L. Cardoso. Basic Fourier series convergence on and outside the $q$-Linear grid. *J. Fourier Anal. Appl.*, 17(1):96–114, 2011.

[9] C. Frappier. Representation formulas for entire functions of exponential type and generalized Bernoulli polynomials. *J. Austr. Math. Soc. Ser.*, 64:307–316, 1998.

[10] C. Frappier. Generalized Bernoulli polynomials and series. *Bull. Austral. Math. Soc.*, 61:289–304, 2000.

[11] C. Frappier. A unified calculus using the generalized Bernoulli polynomial. *J. Approx. Theory.*, 109:279–313, 2001.

[12] G. Gasper and M. Rahman. *Basic Hypergeometric Series*. Cambridge University Press, Cambridge, second edition, 2004.

[13] A. M. L. El Guindy and Z. S. Mansour. On $q$-zeta functions associated with a pair of $q$-analogue of Bernoulli numbers and polynomials. *J. Quaest. Math.*, pages 1–28, 2021.

[14] W. Hahn. *Beiträge zur Theorie der Heineschen Rei-hen*. *Math. Nachr.*, 2:340–379, 1949.

[15] W. K. Hayman. On the Zeros of $q$-Bessel Function. *Contemp. Math.*, 382:205–216, 2005.

[16] M. E. H. Ismail. The Basic Bessel Functions and Polynomials. *J. Math. Anal.*, 12(3):454–468, 1982.

[17] M. E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2005.

[18] M. E. H. Ismail, S. J. Johnston, and Z. S. Mansour. Structure relations for $q$-polynomials and some applications. *Applicable Analysis*, 90:747–767, 2011.

[19] M. E. H. Ismail and Z. S. Mansour. $q$-Analogue of Lidstone expansion theorems, two-point Taylor expansions theorems and Bernoulli polynomials. *Analysis and Applications*, 17:853–895, 2019.

[20] F. H. Jackson. The basic gamma function and elliptic functions. *Proc. Roy. Soc. A*, 76:127–144, 1905.

[21] R. Koekoek and R. Swarttouw. *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*. Reports of the Faculty of Technical Mathematics and Information, 1998.

[22] Z. S. Mansour and M. Al-Towalib. New types of $q$-Lidstone expansion theorems, submitted.