Decay of the stochastic linear Schrödinger equation in $d \geq 3$ with small multiplicative noise

December 18, 2018

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Abstract

We give decay estimates of the solution to the linear Schrödinger equation in dimension $d \geq 3$ with a small noise which is white in time and colored in space. As a consequence, we also obtain certain asymptotic behaviour of the solution. The proof relies on the bootstrapping argument used by Journé-Soffer-Sogge for decay of deterministic Schrödinger operators.

1 Introduction

1.1 Statement of the result

Let $d \geq 3$ and $V \in S(\mathbb{R}^d)$ be a Schwartz function. Consider the Schrödinger equation

\[ i \partial_t \Psi + \Delta \Psi = \delta V \Psi \dot{B} - \frac{i}{2} \delta^2 V^2 \Psi, \quad \Psi(0, \cdot) = f \in S(\mathbb{R}^d), \tag{1.1} \]

where $B$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t \geq 0}$, the product between $\Psi$ and $\dot{B}$ is in the Itô sense, and $\delta > 0$ is a small number to be specified later. The linear correction term $- \frac{i}{2} \delta^2 V^2 \Psi$ makes the $L^2$ norm of $\Psi$ conserved pathwise.

For every $\rho \geq 1$ and $q \geq 1$, we write $L^\rho_{\omega} L^q_x := L^\rho(\Omega, L^q(\mathbb{R}^d))$. The main estimate is the following.

Theorem 1.1. Let $\Psi$ be the solution to (1.1). For every $\rho \geq 1$, $q \in [2, +\infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$, there exists $\delta_0 > 0$ such that for all $\delta < \delta_0$, we have

\[ \|\Psi(t)\|_{L^\rho_{\omega} L^q_x} \lesssim_{\rho, q} t^{-\frac{d}{2} - \frac{1}{4}} \|f\|_{L^p_x} \tag{1.2} \]

for all $t > 0$ and $f \in S(\mathbb{R}^d)$. The proportionality constant is independent of $t$ and $\delta$.

The estimates in Theorem 1.1 allows us to start with initial data in $L^p$ for any $p \in (1, 2]$. Another consequence of the decay estimate is the asymptotic behaviour of the solution.
Proposition 1.2. For every \( f \in L^2(\mathbb{R}^d) \), there exists \( g \in L_\infty^p L_x^q \) such that

\[
\lim_{t \to +\infty} \| \Psi(t) - e^{it\Delta} g \|_{L_p^p L_x^q} = 0
\]

for every \( \rho \geq 1 \).

Remark 1.3. For simplicity of presentation, we choose the noise to be of the form \( W(t, x) = V(x) \tilde{B}(t) \). This factorisation or finite dimensionality is not essential, and the argument still works through as long as \( W \) is white in time and sufficiently nice in space.

Remark 1.4. We shall see later that the assumption on \( V \) could be relaxed. In fact, one only needs \( \delta(\|V\|_{L_1} + \|\tilde{V}\|_{L_1}) \) to be small. We however write it in terms of \( \delta \) times a Schwartz function to avoid appearance of various norms in the bounds later.

1.2 Background and motivation

It is well known ([Tao06], [Caz03]) that the free Schrödinger operator \( e^{it\Delta} \) satisfies the dispersive estimate

\[
\|e^{it\Delta}\|_{L_p^p L_x^q} \lesssim t^{-d\left(\frac{1}{2} - \frac{1}{q}\right)}, \quad q \in [2, +\infty], \quad \frac{1}{p} + \frac{1}{q} = 1 \quad (1.3)
\]

in any dimension \( d \). The estimate for the pair \((p, q) = (2, 2)\) is an immediate consequence of the unitarity of \( e^{it\Delta} \) in \( L^2 \). The other extreme case \((1, +\infty)\) follows from the explicit representation of the integration kernel of \( e^{it\Delta} \). All intermediate situations then follow from Riesz-Thorin interpolation.

For the deterministic linear operator \( e^{it(\Delta + V)} \), which corresponds to the linear equation

\[
i\partial_t u + \Delta u + V(x)u = 0,
\]

it is known that when \( V \) is small and \( d \geq 3 \), one also has the dispersive estimate

\[
\|e^{it(\Delta + V)}\|_{L_p^p L_x^q} \lesssim t^{-d\left(\frac{1}{2} - \frac{1}{q}\right)}, \quad q \in [2, +\infty], \quad \frac{1}{p} + \frac{1}{q} = 1. \quad (1.4)
\]

This estimate corresponds to a simple special case considered in [JSS91], which could be derived via perturbation around the free bound (1.3) for the pair \((1, +\infty)\). The reason (1.4) can hold only for \( d \geq 3 \) is that the bound \( \|e^{it\Delta}\|_{L_1^1 \to L_\infty^\infty} \) for the free Schrödinger operator in (1.3) is integrable for large \( t \) if and only if \( d \geq 3 \).

These dispersive estimates can be used to derive Strichartz estimates via the \( T^*T \) method (see for example the survey [Sch07] for more details), and they are essential for the study of long time behaviour of the solution to the nonlinear Schrödinger equation.

As for the stochastic case, there have been a series of well-posedness results on the stochastic nonlinear Schrödinger equation with multiplicative noise on whole space ([dBDo99, dBDo03, BRZ14, BRZ16, Hor18, FX18a, FX18b, Zha18]). However, none of the above results gives information on long time behaviour of the solution unless one puts on the noise an an extra fast decay in time (see [HRZ18] for a scattering statement for noise of finite quadratic variation in time). A first step towards the understanding the long time behaviour of the solution to the nonlinear equation would be to establish certain decay properties of the solutions to the corresponding linear equation. This is the main
motivation of the current article, and we have chosen the simplest possible model in the
set-up of Theorem 1.1.
It will also be of interest to establish a stochastic version of the Strichartz estimate for
the solution \( \Psi \) to (1.1), which would be of the form
\[
\| \Psi \|_{L_t^\infty L_x^q} \lesssim \| \Psi(0, \cdot) \|_{L_x^2}
\]
for admissible pairs \((q, r)\) satisfying the Strichartz relation. But different from the
deterministic case, since Brownian motion is not time reversible, it is not clear at this stage
how such an estimate could be obtained from the dispersive estimate in Theorem 1.1.

1.3 Outline of the proof
Proposition 1.2 is a simple consequence of Theorem 1.1. As for Theorem 1.1, we first note
that since \( \| \Psi(t) \|_{L_x^2} = \| f \|_{L_x^2} \) for all \( t \) almost surely, if we can prove the statement for
all sufficiently large \( q \) (or equivalently, all \( p \) sufficiently close to 1), then the theorem will
follow from interpolation ([BL12, Theorems 5.1.1 and 5.1.2]). Note that the interpolation
statements used here are for mixed norm spaces, and are more general than Riesz-Thorin.

We refer to [Cal63], [Cal64] and [Cal66] for more details.

It then remains to prove Theorem 1.1 for \( q \) sufficiently large such that
\[
d \left( \frac{1}{2} - \frac{1}{q} \right) > 1.
\]
This is always possible since \( d \geq 3 \). We follow the strategy in [JSS91] and use a bootstrap
argument. The key is to establish a bootstrap relation
\[
t^{d \left( \frac{1}{q} - \frac{1}{2} \right)} \| \Psi(t) \|_{L_t^\infty L_x^q} \leq C_1(\rho, q) \| f \|_{L_x^p} + C_2(\rho, q, \delta) \sup_{r \in [0, t]} \left( t^{d \left( \frac{1}{q} - \frac{1}{2} \right)} \| \Psi(r) \|_{L_t^\infty L_x^q} \right)
\]
for some \( C_1, C_2 \) independent of \( t \), and \( C_2(\rho, q, \delta) \to 0 \) as \( \delta \to 0 \). This would allow us to
absorb the second term on the right hand side into the left, and obtain the claim.

Throughout, we will frequently use the following Burkholder inequality ([Bur73, BP99,
Brz97]) to control the stochastic integral.

**Proposition 1.5** (Burkholder). Let \( \Phi \) be progressively measurable with respect to \((\mathcal{F}_t)\).
Then for every \( q \in [1, +\infty) \) and every \( \rho \geq 2 \), we have
\[
\left\| \int_0^t e^{i(t-s)\Delta} \Phi(s) dB_s \right\|_{L_t^{\rho} L_x^q} \lesssim_{\rho, q} \left( \int_0^t \left\| e^{i(t-s)\Delta} \Phi(s) \right\|_{L_x^2}^2 ds \right)^{\frac{1}{2}}.
\]
As a consequence, by triangle inequality, we have
\[
\left\| \int_0^t e^{i(t-s)\Delta} \Phi(s) dB_s \right\|_{L_t^{\rho} L_x^q} \lesssim_{\rho, q} \left( \int_0^t \left\| e^{i(t-s)\Delta} \Phi(s) \right\|_{L_x^2}^2 ds \right)^{\frac{1}{2}}.
\]

**Remark 1.6.** In order to apply Proposition 1.5 directly, we will restrict to \( \rho \geq 2 \) below. As
for Theorem 1.1 and Proposition 1.2, the case \( \rho \in (1, 2) \) follows from that of \( \rho \geq 2 \).

**Remark 1.7.** The reason the end-point case \((p, q) = (1, +\infty)\) is excluded from Theorem 1.1
is that the Burkholder inequality does not hold for the space \( L_x^\infty \).
**Organisation of the article**

The rest of the article is organised as follows. In Section 2, we show how Proposition 1.2 follows from Theorem 1.1. In Section 3, we give some preliminary lemmas needed for the proof of Theorem 1.1. Section 4 is devoted to the proof of the main theorem in $d = 3$ and $q$ sufficiently large. We briefly explain in Section 5 how the arguments could be modified to cover higher dimensions. By interpolation, this completes the proof of Theorem 1.1.

**Notation**

We fix $d \geq 3$. We use $L^q_\rho$ to denote $L^q(\mathbb{R}^d)$, and write $L^\rho_\omega L^q_\rho = L^\rho(\Omega, L^q(\mathbb{R}^d))$. For any $r \in \mathbb{R}$, we write $\langle r \rangle = 1 + |r|$. Also, since the statements are for every fixed pair $(p, q)$, we use $\alpha$ to denote

$$\alpha = d \left( \frac{1}{2} - \frac{1}{q} \right).$$

When non-commutative products are involved, we write

$$\prod_{j=1}^m A_j = A_m \cdots A_1. \quad (1.6)$$

Typically these $A_j$'s will be operators. Finally, according to Remark 1.6, we assume without loss of generality that $\rho \geq 2$.

**Acknowledgment**

We thank Zihua Guo and Carlos Kenig for discussion, in particular on interpolation. WX gratefully acknowledges the support from the Engineering and Physical Sciences Research Council through the fellowship EP/N021568/1.

## 2 Proof of Proposition 1.2

Since $e^{it\Delta}$ is unitary and $\Psi$ has pathwise mass conservation, it suffices to prove the proposition for $f \in \mathcal{S}(\mathbb{R}^d)$. We need to show that $e^{-it\Delta}\Psi(t)$ has a limit in $L^\rho_\omega L^2_x$ as $t \to +\infty$, and equivalently, $\{e^{-it\Delta}\Psi(t)\}_t$ is Cauchy in $L^\rho_\omega L^2_x$. To see this, we write down the Duhamel formula

$$e^{-it\Delta}\Psi(t) - e^{-is\Delta}\Psi(s) = -i\delta \int_s^t e^{-ir\Delta}(V\Psi(r))dB_r - \frac{\delta^2}{2} \int_s^t e^{-ir\Delta}(V^2\Psi(r))dr. \quad (2.1)$$

We need to control the $L^\rho_\omega L^2_x$-norm of the two terms on the right hand side. For the first one, by Burkholder and triangle inequalities (Proposition 1.5), we have

$$\left\| \int_s^t e^{-ir\Delta}(V\Psi(r))dB_r \right\|_{L^\rho_\omega L^2_x}^2 \lesssim \rho \int_s^t \left\| e^{-ir\Delta}(V\Psi(r)) \right\|_{L^\rho_\omega L^2_x}^2 dr.$$

The integrand on the right hand side then can be controlled as

$$\|e^{-ir\Delta}(V\Psi(r))\|_{L^\rho_\omega L^2_x}^2 \lesssim V \|\Psi(r)\|_{L^\rho_\omega L^q_x}^2 \lesssim V \|\Psi(r)\|_{L^\rho_\omega L^q_x}^2 \lesssim V \|r^{-2d(\frac{1}{2} - \frac{1}{q})}\|f\|_{L^q_x}^2,$$

where we have used the unitary property of $e^{-ir\Delta}$ and the decay estimates in Theorem 1.1 and the bound in the middle follows from $q \geq 2$ and Hölder. Note that the bound above
holds for every \( q \geq 2 \). If we choose \( q \) sufficiently large such that (1.5) holds, one can immediately deduce that
\[
\left\| \int_s^t e^{-ir\Delta} (V\Psi(r)) dB_r \right\|_{L^2_t L^2_r}^2 \to 0
\]
as \( s, t \to +\infty \). The second term on the right hand side of (2.1) can be controlled in a similar way with \( q \) in the range of (1.5). One can see the limit belongs to \( L^\infty \omega L^2 x \) since \( e^{-it\Delta} \) is unitary and the \( L^2 \)-norm of \( \Psi(t) \) is conserved.

The situation for general \( f \) follows from the unitarity of \( e^{it\Delta} \) and pathwise mass conservation. This completes the proof of Proposition 1.2.

Remark 2.1. One can see from the proof that the Cauchy property for the stochastic term only needs \( q \) to satisfy
\[
2d\left( \frac{1}{2} - \frac{1}{q} \right) > 1,
\]
while control of the third term on the right hand side of (2.1) requires \( q \) to satisfy (1.5).

3 Preliminary bounds

We give some preliminary bounds that will be used in the rest of the article. Throughout, \( \Psi \) denotes the solution to (1.1) with initial data \( f \). We also fix arbitrary \( q \in [2, +\infty) \) and \( p \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Recall
\[
\alpha = d\left( \frac{1}{2} - \frac{1}{q} \right)
\]
and the dispersive estimate for the free Schrödinger operator
\[
\| e^{it\Delta} \|_{L^p_x \to L^q_x} \lesssim |t|^{-\alpha}, \quad t \in \mathbb{R}.
\] (3.1)
We do not impose conditions on \( \alpha \) in this section. Also recall (1.6) for the non-commutative product. We have the following lemma.

Lemma 3.1. For every \( m \in \mathbb{N} \) and every \( u_1, \ldots, u_m \in \mathbb{R} \) with \( \sum_j u_j \neq 0 \), there exists \( \theta, \eta, \zeta \in \mathbb{R} \) such that
\[
e^{i\sum_{j=1}^m u_j \Delta} \prod_{j=1}^{m-1} e^{i\langle \xi_j, \cdot \rangle} e^{i\langle \eta_j \cdot \rangle} e^{i\langle \sum_{j=1}^m u_j \rangle \Delta} e^{i\langle \zeta, \cdot \rangle}.
\] (3.2)
As a consequence, we have
\[
\left\| e^{i\sum_{j=1}^m u_j \Delta} \prod_{j=1}^{m-1} (V_j e^{i\eta_j \Delta}) \right\|_{L^p_\omega L^q_\omega} \lesssim \left| \sum_{j=1}^m u_j \right|^{-\alpha} \prod_{j=1}^m \| V_j \|_{L^1}.
\] (3.3)
Proof. The assertion (3.2) is precisely [JSS01] Lemma 2.4. The second assertion (3.3) is a direct consequence of the first one and the dispersive estimate (3.1) for the free Schrödinger operator. \( \square \)
If we know a lower bound of $|\sum_j u_j|$, we may improve the above lemma to the following.

**Lemma 3.2.** For every $\varepsilon > 0$, we have

$$
\left\| e^{iu_0} \prod_{m=1}^m (V_j e^{iu_{j-1}}) \right\|_{L_t^\infty \rightarrow L_t^p} \leq \varepsilon \prod_{m=1}^m \left( \prod_{j=1}^{m-1} (\|V_j\| \|V_j\|_{L_t^p}) \right),
$$

uniformly over the points $u_1, \ldots, u_m \in \mathbb{R}$ such that $|\sum_j u_j| > \varepsilon$.

**Proof.** This is the content of the first bound in [JSS91, Lemma 2.6].

We now look at bounds that involve $\Psi$, the solution to (1.1). We now require all the times $u_j$ involved are positive.

**Lemma 3.3.** For $0 \leq s \leq t$, let

$$
F_t(s) = \sup_{\xi \in \mathbb{R}^d} \| e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} \Psi(s) \|_{L_t^\infty L_x^q}.
$$

Then, there exists $C = C(\rho, q)$ such that

$$
F_t(s) \leq C \cdot t^{-\alpha} \| f \|_{L_t^\infty} \exp \left( C(\delta^2 \|\hat{V}\|_{L_t^1}^2 + \delta^4 \|\hat{V}\|_{L_t^1}^2) \right),
$$

for all $t > 0$ and all $s \in [0, t]$.

**Proof.** We expand $\Psi(s)$ as

$$
\Psi(s) = e^{is\Delta} f - i\delta \int_0^s e^{i(s-r)\Delta} (V \Psi(r)) \, dB_r - \frac{\delta^2}{2} \int_0^s e^{i(s-r)\Delta} (V^2 \Psi(r)) \, dr.
$$

Hence, we have the bound

$$
F_t(s) \leq (I) + (II) + (III),
$$

where

$$(I) = \sup_{\xi \in \mathbb{R}^d} \| e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} f \|_{L_t^\infty L_x^q},
$$

$$(II) = \delta \sup_{\xi \in \mathbb{R}^d} \left\| \int_0^s e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} e^{i(s-r)\Delta} V \Psi(r) \, dB_r \right\|_{L_t^\infty L_x^q},
$$

$$(III) = \frac{\delta^2}{2} \sup_{\xi \in \mathbb{R}^d} \left\| \int_0^s e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} e^{i(s-r)\Delta} V^2 \Psi(r) \, dr \right\|_{L_t^\infty L_x^q}.
$$

We control them one by one. For the first one, a direction application of Lemma 3.1 shows that it is controlled by

$$(I) \leq C t^{-\alpha} \| f \|_{L_t^\infty}.
$$

For the third one, another application of (3.2) shows that the integrand satisfies

$$
\left\| e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} e^{i(s-r)\Delta} V^2 \Psi(r) \right\|_{L_t^\infty L_x^q} \leq \left\| \int_{\mathbb{R}^d} \hat{V}(\eta) e^{i(t-s)\Delta} e^{i\xi \cdot \cdot \cdot} e^{i(s-r)\Delta} e^{i\eta \cdot \cdot \cdot} \Psi(r) \, d\eta \right\|_{L_t^\infty L_x^q}
$$

$$
\leq \|\hat{V}\|_{L_t^2} \sup_{\eta \in \mathbb{R}^d} \| e^{i(t-r)\Delta} e^{i\eta \cdot \cdot \cdot} \Psi(r) \|_{L_t^\infty L_x^q}
$$

$$
= \|\hat{V}\|_{L_t^2}^2 F_t(r),
$$

The proof is completed.

\[\square\]
which is true for all $\xi \in \mathbb{R}^d$. Hence, we have

$$(\text{III}) \leq \frac{\delta^2}{2} \|\hat{V}\|_{L^2}^2 \int_0^s F_t(r)dr.$$ 

As for (II), by Burkholder and then triangle inequalities, we have

$$(\text{II}) \leq C\delta \sup_{\xi \in \mathbb{R}^d} \left( \int_0^s \|e^{i(t-s)\Delta}e^{i(s-r)\Delta}V\Psi(r)\|_{L^2(L^2)}^2 dr \right)^{\frac{1}{2}} \leq C\delta \|\hat{V}\|_{L^1} \left( \int_0^s (F_t(r))^2 dr \right)^{\frac{1}{2}},$$

where we used the previous bound for (III) with $\hat{V}^2$ replaced by $\hat{V}$. Combining the above three bounds, we have

$$F_t(s) \leq C \left[ t^{-\alpha} \|f\|_{L^\infty L^p} + \delta \|\hat{V}\|_{L^1} \left( \int_0^s (F_t(r))^2 dr \right)^{\frac{1}{2}} + \delta^2 \|\hat{V}\|_{L^1}^2 \int_0^s F_t(r)dr \right].$$

Let $K_t(s) = (F_t(s))^2$, the above bound then implies

$$K_t(s) \leq C \left( t^{-2\alpha} \|f\|_{L^\infty L^p}^2 + \delta^2 \|\hat{V}\|_{L^1}^2 \int_0^s K_t(r)dr + s\delta^4 \|\hat{V}\|_{L^1}^4 \int_0^s K_t(r)dr \right),$$

where the additional factor $s$ in front of the last integral comes from Hölder. The desired bound then follows from Grönwall and then taking square root of $K_t$. \hfill $\square$

Combining (3.2) and Lemma 3.3, we have the following consequence.

**Corollary 3.4.** For $m \geq 1$ and $u_1, \ldots, u_m \in \mathbb{R}^+$, we have

$$\left\| e^{i\sum_{j=1}^m V_{m-1}} \cdots V_2 e^{i\sum_{j=1}^2 V_1 \Psi(u_1)} \right\|_{L^\infty L^2_2} \leq C \left( \sum_{j=1}^m u_j \right)^{-\alpha} \|f\|_{L^p} \cdot \left( \prod_{j=1}^{m-1} \|\hat{V}_j\|_{L^1} \right) \exp \left( C(\delta^2 \|\hat{V}\|_{L^1}^2 u_1 + \delta^4 \|\hat{V}\|_{L^1}^4 u_1^2) \right),$$

where $C = C(\rho, p, d)$ is independent of $m$ and the $u_j$’s.

The following lemma is analogous to Lemma 3.2 but with $\Psi$ involved and requiring all $u_j$’s being positive.

**Lemma 3.5.** Let $m \geq 2$. We have

$$\left\| e^{i\sum_{j=1}^m V_{m-1}} \cdots V_2 e^{i\sum_{j=1}^2 V_1 \Psi(u_1)} \right\|_{L^\infty L^2_2} \lesssim_m \left( \prod_{j=1}^m (u_j)^{-\alpha} \right) \left( \prod_{j=1}^{m-1} (\|\hat{V}_j\|_{L^1} + \|V_j\|_{L^2_{\infty, \rho}}) \right) \|f\|_{L^p},$$

uniformly over the points $u_1, \ldots, u_m \in \mathbb{R}^+$ satisfying $\sum_j u_j > 2$ and $u_1 < 1$. 
Proof. Note that the claim does not follow directly from Corollary \ref{cor:second_bound}, since we do not want to have a singularity in \(u_1\) when it is small. This lemma is similar to the second bound in \cite[Lemma 2.6]{JSS91} but not identical, so we give detailed arguments here.

We distinguish two cases: \(u_2 > 2^{2m}\) and \(u_2 \leq 2^{2m}\), starting with the first one. In this case, let \(k_1 = 2\), and define recursively
\[
k_{\ell+1} := m \land \inf \{ j > k_\ell : u_j > 2^{2m-j} \}
\]
until it reaches \(m\). This gives an increasing (finite) sequence of integers \(2 = k_1 < \cdots < k_{N+1} = m\), and partitions \(\{u_j\}_{j=2}^m\) into \(N\) blocks of the form \((u_{k_1}, \ldots, u_{k_{\ell+1}-1})\) for \(\ell = 1, \ldots, N\). In each block, we have
\[
\left( \sum_{j=k_\ell}^{k_{\ell+1}-1} u_j \right)^{-a} \lesssim_m \prod_{j=k_\ell}^{k_{\ell+1}-1} \langle u_j \rangle^{-a} \quad \text{and} \quad \left( \sum_{j=1}^{k_2-1} u_j \right)^{-a} \lesssim_n \prod_{j=1}^{k_2-1} \langle u_j \rangle^{-a}. \tag{3.5}
\]
Hence, for \(\ell \geq 2\), we have the bound
\[
\left\| e^{iu_{k_{\ell+1}-1} \Delta} \left( \prod_{j=k_\ell}^{k_{\ell+1}-2} V_j e^{iu_j \Delta} \right) V_{k_{\ell+1}-1} \right\|_{L^2_t L^q_x} \lesssim_m \left( \prod_{j=k_\ell-2}^{k_{\ell+1}-1} \langle u_j \rangle^{-a} \right) \prod_{j=k_\ell-1}^{k_{\ell+1}-2} \left( \| \hat{V}_j \|_{L^q_x} + \| V_j \|_{L^{q-1}_x} \right). \tag{3.6}
\]
As for \(\ell = 1\), Fourier expanding \(V_1, \ldots, V_{k_2-2}\) and applying \ref{cor:fourier_bound}, Corollary \ref{cor:second_bound} and the second bound in \ref{eq:second_bound}, we have
\[
\left\| e^{iu_{k_2-1} \Delta} V_{k_2-1} \cdots V_2 e^{iu_2 \Delta} V_1 \Psi(u_1) \right\|_{L^p_t L^q_x} \lesssim_m \left( \prod_{j=1}^{k_2-1} \langle u_j \rangle^{-a} \right) \prod_{j=1}^{k_2-1} \| \hat{V}_j \|_{L^q_x} \| f \|_{L^p_t}. \tag{3.7}
\]
Multiplying \ref{eq:fourier_bound} and \ref{eq:second_bound} gives the desired bound in the case \(u_2 > 2^{2m}\).

We now turn to the case \(u_2 \leq 2^{2m}\). If \(u_j \leq 2^{2m-j}\) for all \(j\), then we have
\[
\left( \sum_{j=1}^m u_j \right)^{-a} \lesssim_m \prod_{j=1}^m \langle u_j \rangle^{-a}, \tag{3.8}
\]
and the bound follows. If not, then let
\[
k = \inf \{ j \geq 3 : u_j > 2^{2m+j} \}.
\]
If \(k = m\), or \(k < m\) and \(|u_j| \leq 1\) for all \(k < j \leq m\), then we still have \ref{eq:fourier_bound}, and the bound follows in the same way. Otherwise, let
\[
k' = \inf \{ j > k : u_j > 1 \}.
\]
Proof of Theorem 1.1

Then \( k' < m \). Since \( |u_{k'}| > 1 \), by Lemma 3.2, we have

\[
\| e^{iu_{k'} \Delta} \left( \prod_{j=k'}^{m-1} V_j e^{i \mu_j \Delta} \right) V_{k'-1} \|_{L^q_x \to L^q_x} \lesssim_m \left( \prod_{j=k'}^{m} \left| \langle u_j \rangle - \alpha \right| \right) \prod_{j=k'-1}^{m-1} \left( \| \hat{V}_j \|_{L^1} + \| V_j \|_{L_{\mu_j}^\infty} \right).
\]

For the term \( \| e^{iu_{k'-1} \Delta} V_{u_{k'-2}} \cdots V_2 e^{iu_2 \Delta} \Psi(u_1) \|_{L^p_x L^q_x} \), one can control it in the same way as (3.7) since we have

\[
\left( \sum_{j=1}^{k'-1} u_j \right)^{-\alpha} \lesssim_m \prod_{j=1}^{k'-1} \langle u_j \rangle^{-\alpha}.
\]

This completes the proof. \( \square \)

4 Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. We consider \( d = 3 \) in this section, and will briefly explain in the next section how the situation for general \( d \) can be established with minor modification of the argument. Recall the notation

\[
\alpha = d \left( \frac{1}{2} - \frac{1}{q} \right), \quad d = 3.
\]

As mentioned in Section 1.3, it suffices to consider (arbitrary) \( q \) in the range (1.5). Also since \( q < +\infty \), we have

\[
1 < \alpha < \frac{3}{2}. \tag{4.1}
\]

Theorem 1.1 is equivalent to the bound

\[
\sup_{t \in \mathbb{R}^+} \left( t^\alpha \| \Psi(t) \|_{L^p_x L^q_x} \right) \lesssim_{\rho, q} \| f \|_{L^p_x}. \tag{4.2}
\]

To achieve this, we need to establish an inequality of the form

\[
t^\alpha \| \Psi(t) \|_{L^p_x L^q_x} \leq C_1(\rho, q) \| f \|_{L^p_x} + C_2(\rho, q, \delta) \sup_{r \in [0, t]} \left( r^\alpha \| \Psi(r) \|_{L^p_x L^q_x} \right) \tag{4.3}
\]

for all \( t \), where both \( C_1 \) and \( C_2 \) are independent of \( t \), and \( C_2(\rho, q, \delta) \to 0 \) as \( \delta \to 0 \) for every fixed \( \rho \) and \( q \). This will allow us to absorb the second term on the right hand side into the left and establish (4.2).

By Corollary 3.4, the bound (4.3) is true with \( C_2 = 0 \) if \( t \leq 2 \). Hence, we only need to prove (4.3) for \( t > 2 \). Following the strategy in [JSS91], we expand \( \Psi \) with Duhamel formula twice to obtain

\[
\Psi(t) = e^{it \Delta} f - i \delta \int_0^t e^{i(t-s) \Delta} V e^{is \Delta} f dB_s - \frac{\delta^2}{2} \int_0^t e^{i(t-s) \Delta} V^2 e^{is \Delta} f ds + (I) + (II) + (III) + (IV), \tag{4.4}
\]

\( \square \)
Proof of Theorem 1.1

where

\[ (I) = \frac{\delta^4}{4} \int_0^t \int_0^s e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V^2 \Psi(r) dr ds, \]

\[ (II) = \frac{\delta^3}{2} \int_0^t \int_0^s e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V \Psi(r) dB_r ds, \]

\[ (III) = \frac{\delta^3}{2} \int_0^t \int_0^s e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V^2 \Psi(r) dr dB_s, \]

\[ (IV) = -\delta^2 \int_0^t \int_0^s e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V \Psi(r) dB_r dB_s. \]

We need to show that each term in the right hand side of (4.4) is bounded by the right hand side of (4.3). Since \( \rho \) and \( q \) are fixed, for simplicity of notation, in what follows we write “\( \lesssim \)” instead of “\( \lesssim_{\rho,q} \)”. Also, we restrict to \( t > 2 \) from now on.

4.1 “Constant” terms

We first treat the terms without \( \Psi(r) \), that is, those on the first line of the right hand side of (4.4). The bound for \( e^{i t \Delta} f \) is the standard dispersive estimate for \( e^{i t \Delta} \).

As for the second term, since \( \rho \geq 2 \), by Burkholder and triangle inequalities (Proposition 1.5), we have

\[
\left\| \int_0^t e^{i(t-s)\Delta} V e^{is\Delta} f dB_s \right\|_{L_x^p L_t^\rho} \lesssim \left( \int_0^t \left\| e^{i(t-s)\Delta} V e^{is\Delta} f \right\|_{L_x^p L_t^{\rho,\rho}}^2 ds \right)^{\frac{1}{2}}.
\]

Since \( t > 2 \), by Lemma 3.2 the integrand satisfies the bound

\[
\left\| e^{i(t-s)\Delta} V e^{is\Delta} f \right\|_{L_x^p} \lesssim \langle t-s \rangle^{-\alpha} \langle s \rangle^{-\alpha} \left\| f \right\|_{L_x^p}.
\]

Since \( 2\alpha > 1 \), we get

\[
\left\| \int_0^t e^{i(t-s)\Delta} V e^{is\Delta} f dB_s \right\|_{L_x^p L_t^\rho} \lesssim t^{-\alpha} \left\| f \right\|_{L_x^p}.
\]

The third term can be controlled in the same way (but requiring \( \alpha > 1 \)). This completes the three terms in the first line of the right hand side of (4.4).

The rest of the section is devoted to the four terms in (4.5). We start with the first one.

4.2 Term (I)

Let

\[ G_t(r, s) = \left\| e^{i(t-s)\Delta} V^2 e^{i(s-r)\Delta} V^2 \Psi(r) \right\|_{L_x^p L_t^\rho}, \]

so we have

\[
t^\alpha \left\| (I) \right\|_{L_x^p L_t^\rho} \leq \delta^4 t^\alpha \int_0^t \int_0^s G_t(r, s) dr ds. \tag{4.6}
\]

Recall that \( t > 2 \). In the domains \( r \in [0, 1] \), \( r \in [1, t-1] \) and \( r \in [t-1, t] \), by Lemmas 3.5, 3.2 and 3.1 respectively, we can bound the integrand \( G_t \) pointwise by

\[
G_t(r, s) \lesssim \begin{cases} 
\langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \left\| f \right\|_{L_x^p}, & r \in [0, 1] \\
\langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \left\| \Psi(r) \right\|_{L_x^p L_t^{\rho,\rho}}, & r \in [1, t-1] \\
(t-r)^{-\alpha} \left\| \Psi(r) \right\|_{L_x^p L_t^{\rho,\rho}}, & r \in [t-1, t].
\end{cases} \tag{4.7}
\]
All the proportionality constants above are independent of $r$, $s$ and $t$. Note that the factor in the last bound is $(t - r)^{-\alpha}$ rather than $\langle t - r \rangle^{-\alpha}$, so it has a singularity at $r \approx t$.

According to (4.7), we decompose the integral on the right hand side of (4.6) into three disjoint regions as

$$
\int_0^t \int_0^s G_t dr ds = \int_0^t \int_0^{s^{\Lambda_1}} G_t dr ds + \int_t^1 \int_1^{s^{\Lambda_1(t-1)}} G_t dr ds + \int_1^t \int_t^s G_t dr ds.
$$

For the first one, using the first bound in (4.7) and $\alpha > 1$, we have

$$
\int_0^t \int_0^{s^{\Lambda_1}} G_t dr ds \lesssim \|f\|_{L_p} \int_0^t \int_0^{s^{\Lambda_1}} \langle t - s \rangle^{-\alpha} \langle s - r \rangle^{-\alpha} dr ds \lesssim t^{-\alpha} \|f\|_{L_p}, \quad (4.8)
$$

For the second one, using the second bound in (4.7), we get

$$
\int_t^1 \int_1^{s^{\Lambda_1(t-1)}} G_t dr ds \lesssim \int_t^1 \int_1^{s^{\Lambda_1(t-1)}} \langle t - s \rangle^{-\alpha} \langle s - r \rangle^{-\alpha} \|\Psi(r)\|_{L_r^p} dr ds
\lesssim \sup_{r \in [0, t]} \left( r^\alpha \|\Psi(r)\|_{L_r^p} \right) \int_t^1 \int_1^s \langle t - s \rangle^{-\alpha} \langle s - r \rangle^{-\alpha} dr ds \lesssim t^{-\alpha} \sup_{r \in [0, t]} \left( r^\alpha \|\Psi(r)\|_{L_r^p} \right), \quad (4.9)
$$

where in the last bound we have also used $\alpha > 1$. For the third term, we have

$$
t^\alpha \int_{t-1}^t \int_{t-1}^s G_t dr ds \lesssim t^\alpha \int_{t-1}^t \int_{t-1}^s (t - r)^{-\alpha} \|\Psi(r)\|_{L_r^p} dr ds
\lesssim \sup_{r \in [t-1, t]} \left( r^\alpha \|\Psi(r)\|_{L_r^p} \right) \int_{t-1}^t \int_{t-1}^s (t - r)^{-\alpha} dr ds \quad (4.10)
$$

Here, for the second line above, we used $r \in [t-1, t]$ and $t > 2$ so that we replaced $t^\alpha$ with $r^\alpha$ with a universal proportionality constant. Also, since $\alpha < 2$, the integral on the second line above is finite if and only if $\alpha < 2$, which is indeed the case. Hence one obtains the last bound.

Combining (4.6), (4.8), (4.9) and (4.10), we conclude that

$$
t^\alpha \|\Theta\|_{L_r^p L_s^2} \lesssim \delta^4 \|f\|_{L_r^p} + \delta^4 \sup_{r \in [0, t]} \left( r^\alpha \|\Psi(r)\|_{L_r^p} \right),
$$

which is of the form (4.3). This completes Term (I).

4.3 Term (II)

We now turn to the second term in (4.5). By Burkholder and triangle inequalities, we have the bound

$$
|| (\Pi) ||_{L_r^p L_s^2} \lesssim \delta^3 \int_0^t \left( \int_0^s |G_t(r, s)|^2 dr \right)^{\frac{1}{2}} ds,
$$

Proof of Theorem 4.1
where this time \( G_t \) has the expression
\[
 G_t(r, s) = \|e^{i(t-s)\Delta}V^2e^{i(s-r)\Delta}V\Psi(r)\|_{L^q_t L^3_x},
\]
but still satisfies exactly the same bound as in (4.7). Similar as before but with an inequality, we split the integral by
\[
\int_0^t \left( \int_0^s G_t^2 dr \right)^{\frac{1}{2}} ds \leq \int_0^t \left( \int_0^{s/2} G_t^2 dr \right)^{\frac{1}{2}} ds + \int_1^t \left( \int_1^{s/2} G_t^2 dr \right)^{\frac{1}{2}} ds + \int_1^t \left( \int_1^s G_t^2 dr \right)^{\frac{1}{2}} ds.
\]
The first two terms above can be controlled in exactly the same way as in Term (I) since they contain no singularity. For the third one, also similar as before, we have
\[
t^\alpha \int_{t-1}^t \left( \int_{t-1}^s G_t^2 dr \right)^{\frac{1}{2}} ds \lesssim \sup_{r \in [t-1, t]} \left( r^\alpha \|\Psi(r)\|_{L^q_t L^3_x} \right) \int_{t-1}^t \left( \int_{t-1}^s (t-r)^{-2\alpha} dr \right)^{\frac{1}{2}} ds.
\]
This time, the integral on the right hand side above has a worse singularity, but it is still integrable
\[
\frac{1}{2}(2\alpha - 1) < 1 \iff \alpha < \frac{3}{2}.
\]
Hence, \( t^\alpha \|\Pi \|_{L^q_t L^3_x} \) is also bounded by the right hand side of (4.3) with \( C_1 \) and \( C_2 \) proportional to \( \delta^3 \). This completes the bound for Term (II).

**Remark 4.1.** As we can see, if there were no square root for the inner-integral, then bound for \( \int_{t-1}^t \int_{t-1}^s G_t^2 dr ds \) would have a non-integrable singularity. This is precisely the case for Term (IV), for which we need to expand one more time to reduce this singularity to make it integrable.

### 4.4 Term (III)

For Term (III), we have
\[
\|\Pi \|_{L^q_t L^3_x} \lesssim \delta^3 \left[ \int_0^t \left( \int_0^s G_t(r, s) dr \right)^2 ds \right]^{\frac{1}{2}},
\]
where
\[
G_t(r, s) = \|e^{i(t-s)\Delta}V^2e^{i(s-r)\Delta}V^2\Psi(r)\|_{L^q_t L^3_x}
\]
satisfies the same bound as in (4.7). Again, we split (the square of) the integral above as
\[
\int_0^t \left( \int_0^s G_t dr \right)^2 ds \lesssim \int_0^t \left( \int_0^{s/2} G_t dr \right)^2 ds + \int_1^t \left( \int_1^{s/2} G_t dr \right)^2 ds + \int_1^t \left( \int_1^s G_t dr \right)^2 ds.
\]
The first two terms on the right hand side can be controlled in the same way as in (I) and (II). The third one is also essentially the same except a slightly different but still integrable singularity. It can be controlled by
\[
t^{2\alpha} \int_{t-1}^t \left( \int_{t-1}^s G_t dr \right)^2 ds \lesssim \sup_{r \in [t-1, t]} \left( r^{2\alpha} \|\Psi(r)\|_{L^q_t L^3_x}^2 \right) \int_{t-1}^t \left( \int_{t-1}^s (t-r)^{-\alpha} dr \right)^2 ds.
\]
The integral on the right hand side above is finite since \( 2(\alpha - 1) < 1 \). Term (III) is then complete by taking square root of the above bounds.
4.5 Term (IV)

As explained in Remark 4.1 if we control Term (IV) in the same way as before, then we will end up with a non-integrable singularity at \( s \approx t \), so we need to expand the term one more time to make it integrable.

Expanding \( \Psi(r) \) as
\[
\Psi(r) = e^{ir\Delta} f - i\delta \int_0^r e^{i(r-u)\Delta} V \Psi(u) dB_u - \frac{\delta^2}{2} \int_0^r e^{i(r-u)\Delta} V^2 \Psi(u) du,
\]
and substituting it into (IV) in (4.5), we get
\[
(IV) = (IV-i) + (IV-ii) + (IV-iii),
\]
where
\[
(IV-i) = -\delta^2 \int_0^t \int_0^r e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f dB_r dB_s,
(IV-ii) = i\delta^3 \int_0^t \int_0^r e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V \Psi(u) dB_u dB_r dB_s, \tag{4.11}
(IV-iii) = \frac{\delta^4}{2} \int_0^t \int_0^r e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V^2 \Psi(u) du dB_r dB_s.
\]

Term (IV-i) is straightforward. Indeed, by Burkholder and triangle inequalities, we have
\[
\| (IV-i) \|_{L^2_t L^2_x} \lesssim \delta^2 \left( \int_0^t \int_0^r \| e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f \|_{L^2_x}^2 dr ds \right)^{\frac{1}{2}},
\]
Since \( t > 2 \), by Lemma 3.2 the integrand satisfies a pointwise bound
\[
\| e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{ir\Delta} f \|_{L^2_x} \lesssim \langle t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r \rangle^{-\alpha} \| f \|_{L^2_x},
\]
and hence the desired bound for (IV-i) follows immediately. As for (IV-ii), we have
\[
\| (IV-ii) \|_{L^2_t L^2_x} \lesssim \delta^3 \left( \int_0^t \int_0^r \int_0^r |G_t(u, r, s)|^2 du dr ds \right)^{\frac{1}{2}},
\]
where again by Lemmas 3.5 3.2 and 3.1 the integrand
\[
G_t(u, r, s) = \| e^{i(t-s)\Delta} V e^{i(s-r)\Delta} V e^{i(r-u)\Delta} V \Psi(u) \|_{L^2_t L^2_x}
\]
satisfies the pointwise bound
\[
G_t(u, r, s) \lesssim \begin{cases} 
\| t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r-u \rangle^{-\alpha} \| f \|_{L^2_x}, & u \in [0, 1] \\
\| t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r-u \rangle^{-\alpha} \| \Psi(u) \|_{L^2_t L^2_x}, & u \in [1, t-1] \\
\| t-s \rangle^{-\alpha} \langle s-r \rangle^{-\alpha} \langle r-u \rangle^{-\alpha} \| \Psi(u) \|_{L^2_t L^2_x}, & u \in [t-1, t].
\end{cases} \tag{4.12}
\]

We now split the integration into three disjoint sub-domains: \( \{u \leq 1\} \), \( \{u \in [1, t-1]\} \) and \( \{u \geq t-1\} \). The bound for the first two domains are similar as before. The only one containing a singularity is the third one, which we have the bound
\[
t^2 \alpha \int_{t-\epsilon < u < r < c < t} G_t^2 du dr ds \lesssim \sup_{u \in [t-\epsilon, t]} \left( u^{2\alpha} \| \Psi(u) \|_{L^2_t L^2_x} \right) \int_{t-\epsilon}^t \int_{t-\epsilon}^r \int_{t-\epsilon}^r (t-u)^{-2\alpha} du dr ds.
\]
This time, the integral on the right hand side above is finite since the exponent of the singularity satisfies \( 2\alpha - 2 < 1 \). One can then conclude the desired bound for (IV-ii).

Finally, the bound for Term (IV-iii) is essentially the same. This completes the case for (IV) as well as the proof of Theorem 1.1 in \( d = 3 \).
5 Higher dimensions

The argument for higher dimensions is essentially the same, except that when the inner-most integration variable is close to \(t\), the singularity is worse since \(\alpha\) becomes larger. Hence, we need to expand more times with the Duhamel formula in order to make the singularity integrable.

In dimension \(d\), we expand \(\Psi(t)\) to the \(d\)-th order so that it can be expressed as a linear combination of terms of the form

\[
\left\{ \int_{0<s_1<\cdots<s_d<t} e^{i(t-s_j)\Delta} V_j \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 e^{i s_1 \Delta} f dX_{s_1} \cdots dX_{s_d} \right\}_{j=1,\ldots,d} \tag{5.1}
\]

and

\[
\left\{ \int_{0<s_1<\cdots<s_d<t} e^{i(t-s_d)\Delta} V_d \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 \Psi(s_1) dX_{s_1} \cdots dX_{s_d} \right\}, \tag{5.2}
\]

where each \(X_{s_j}\) is either \(s_j\) or \(B_{s_j}\), and each \(V_j\) is either \(\delta V\) or \(\delta^2 V^2\). To control these terms, we first note that if \(q\) is sufficiently close to \(1\), then

\[
\alpha = d \left( \frac{1}{2} - \frac{1}{q} \right) = \frac{d}{2} - \kappa
\]

for some sufficiently small \(\kappa > 0\).

The bound for the terms in (5.1) are straightforward. As for (5.2), when \(s_1 < t - 1\), the proof are exactly the same as before since the pointwise decay estimates are even better. Singularity occurs when \(s_1 > t - 1\), and the worst case is that all \(X_{s_j}\) in (5.2) are \(B_{s_j}\), and we need to control

\[
\delta^d \int_{t-1<s_1<\cdots<s_d<t} \left\| e^{i(t-s_d)\Delta} V \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 \Psi(s_1) \right\|_{L^q_t L^\infty_x}^2 ds_1 \cdots ds_d \tag{5.3}
\]

By Lemma 3.4 we have the pointwise bound

\[
\left\| e^{i(t-s_d)\Delta} V \cdots V_2 e^{i(s_2-s_1)\Delta} V_1 \Psi(s_1) \right\|_{L^q_t L^\infty_x}^2 \lesssim (t - s_1)^{-2\alpha} \left\| \Psi(s_1) \right\|_{L^q_t L^\infty_x}^2
\]

for the integrand, so that (5.3) is controlled by

\[
\delta^d \sup_{r \in [t-1,t]} \left( r^{\alpha} \left\| \Psi(r) \right\|_{L^q_t L^\infty_x} \right) \left( \int_{t-1<s_1<\cdots<s_d<t} (t - s_1)^{-2\alpha} ds_1 \cdots ds_d \right)^{\frac{1}{2}}.
\]

The integral on the right hand side above is finite since \(-2\alpha + (d - 1) > -1\). This bound is of the form (4.3) with both \(C_2\) proportional to \(\delta^d\). The proof is then complete.

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