On the existence of canonical gyrokinetic variables for chaotic magnetic fields§

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(Dated: June 30, 2008)

Abstract

The gyrokinetic description of particle dynamics faces a basic difficulty when a special type of canonical variables is sought, i.e., the so-called gyrokinetic canonical variables. These are defined in such a way that two of them are respectively identified with the gyrophase-angle, describing the fast particle gyration motion around magnetic field lines, and its canonically conjugate momentum. In this paper we intend to discuss the conditions of existence for these variables.

PACS numbers: 52.30.Gz,52.30.-q
I. INTRODUCTION

The gyrokinetic description of particle dynamics concerns the representation - in terms of generally non-canonical variables - of the state of a single classical charged point-particle immersed in an electromagnetic (EM) field. In classical electrodynamics - for arbitrary EM fields and provided the EM self-force is neglected (according to the customary interpretation; see, however, the related discussion in Ref. [1]) - it is well known that this defines an Hamiltonian dynamical system. As such, locally in phase-space, it can always be represented in canonical form. Nevertheless, although well-known in the literature, the gyrokinetic problem presents a basic difficulty related specifically to the construction of a special set of canonical variables. As described below, this arises in connection with the construction of the so-called canonical gyrokinetic variables. In this paper we intend to show that, unless special symmetry conditions hold (in particular for the magnetic field), such variables do not exist.

For a proper formulation of the problem it is important to stress that the gyrokinetic description of particle dynamics concerns in principle two possible viewpoints:

A) an exact representation (of the particle state) realized by suitably prescribing an appropriate phase-space diffeomorphism [see below Eq. (1)];

B) an approximate representation, obtained by means of a suitable asymptotic expansion.

In the exact gyrokinetic treatment (Approach A) it is assumed that there exists a phase-space mapping (i.e., a diffeomorphism, which is assumed to exist at least in a suitable subset of phase-space) of the form

\[(r, v) \rightarrow (y', \phi')\] (1)

between the Newtonian particle state \((r, v)\) (with \(r\) and \(v\) denoting respectively the particle position and velocity) and the (real) gyrokinetic state \((y', \phi')\). The latter is defined in such a way that equations of motion for the transformed variables take the form

\[
\begin{cases}
\frac{dy'}{dy} = Y(y', t), \\
\frac{d\phi'}{dt} = F(y', t).
\end{cases}
\] (2)

By construction \(Y(y', t)\) and \(F(y', t)\), to be identified with suitably smooth real functions of the gyrokinetic state, are both assumed independent of the variable \(\phi'\). Hence, its canonically conjugate momentum \(p_{\phi'}\) is necessarily a first integral of motion. Here \(\phi'\) and \(y'\) are the
so-called gyrokinetic (or guiding-center) variables, representing respectively an angle (the so-called gyrophase) which describes the particle gyration motion around the magnetic flux lines and an arbitrary 5-dimensional real vector, representing a reduced non-canonical gyrokinetic state. As an example, in particular, the vector \( y' \) may be identified with the non-canonical variables

\[
y' \equiv (r', \xi', p_{\phi'}),
\]

where \( r' \) and \( \xi' \) denote respectively the guiding-center position vector and an additional (independent) velocity-space gyrokinetic variable.

However, unless the electromagnetic field is specially prescribed (i.e., for example, it is a constant), the exact gyrokinetic transformation \( y' \) cannot generally be achieved. Nevertheless, it is still possible - under suitable assumptions - to determine it in an approximate sense by means of an appropriate asymptotic approximation for the (canonical or non-canonical) particle state (Approach B). This is obtained by introducing an asymptotic expansion (i.e., a truncated perturbative expansion in terms of an appropriate infinitesimal dimensionless parameter \( \varepsilon \)) of the form

\[
(r, v) \rightarrow (y', \phi') \approx \sum_{k=0}^{N} \varepsilon^k (y_k', \phi_k') + o(\varepsilon^{N+1})
\]

where the integer \( N > 1 \) (to be suitably prescribed) denotes the "order" of the asymptotic approximation. In particular, by assuming that the magnetic field in which the particle is immersed is suitably “intense”, the infinitesimal dimensionless parameter can be defined as

\[
\varepsilon = r_L/L << 1.
\]

Here the notation is standard. Thus, \( L \) and \( r_L \) are respectively a characteristic scale length of the EM fields (to be suitably defined, see below) and the velocity-dependent particle Larmor radius \( r_L = \frac{\omega'}{\Omega'} \). In particular, all primed quantities are evaluated at the guiding-center position \( r' \), which requires that the diffeomorphism

\[
r \rightarrow r' \approx \sum_{k=0}^{N} \varepsilon^k r_k' + o(\varepsilon^{N+1})
\]

is assumed to exist. For example \( \Omega' \equiv \Omega'(r', t) \), where \( \Omega' = \frac{\omega'}{m q} \) is the Larmor frequency and \( q, m, B \) are respectively the charge and mass of a point particle and the magnitude of
the magnetic field. Moreover, \( \mathbf{w}' \) is the orthogonal component of the particle velocity to be evaluated - in a suitable reference frame - at the same position \( \mathbf{r}' \).

The first author who systematically investigated the gyrokinetic problem, based on the explicit construction of an asymptotic expansion of the form (4) and (6), was Alfven [2] who pointed out the existence of an adiabatic invariant, the magnetic moment \( \mu' \equiv \frac{qc}{m} p'_\phi \), in the sense:

\[
\frac{d}{dt} \ln \mu' \sim O(\varepsilon). \tag{7}
\]

After subsequent work which dealt with direct construction methods of gyrokinetic variables [3, 4, 5], a significant step forward was made by Kruskal [6] who, first, established the consistency of the Alfven approach by proving, under suitable assumptions on the EM fields, that the magnetic moment can be constructed correct at any order \( N \) in \( \varepsilon \) in such a way that, denoting \( M' \) such a dynamical variable, it is an adiabatic invariant of order \( N \), namely in the sense

\[
\frac{d}{dt} \ln M' \sim O(\varepsilon^N). \tag{8}
\]

\[
M' = \mu' + \varepsilon \mu'_1 + \ldots + \varepsilon^N \mu'_N. \tag{9}
\]

A modern picture of the Hamiltonian formulation, which makes easier the formulation of higher order perturbative theories, was given only later in terms of Lie-transform methods [7, 8, 9]. However, it was only with the adoption of non-canonical Lie-transform methods [9] that the approach was given a general formulation. As a motivation to his non-canonical approach, Littlejohn [9, 10] pointed out what in his views was a critical point of purely canonical formulations such as previously developed Lie transform approaches [7, 8], namely the ambiguity in the separation of the unperturbed and perturbed contributions in the Hamiltonian due the presence of the vector potential \( \mathbf{A} \) in the canonical momenta. He showed that this difficulty can be circumvented by making use of suitable non-canonical variables which include the canonical pair \((\phi', p'_\phi)\).

The possibility of constructing canonical gyrokinetic variables has relied, since, on two possible methods

- the Darboux reduction algorithm, based first on the construction of a set of non-canonical gyrokinetic variables [11];
• the direct construction of canonical gyrokinetic variables, either in terms of mixed-
variables generating functions [3], canonical Lie-transform methods [7, 8, 12] or based
on the adoption of the hybrid Hamilton variational principle [11, 13, 14].

The first approach, and probably the most popular in the literature [15, 16, 17, 18], is the
based on the use of Darboux theorem which allows, in principle, the construction of canoni-
cal variables for an arbitrary differential 1-form. The canonical 1-form expressed in terms
of the canonical variables is then obtained by applying recursively the so-called “Darboux re-
duction algorithm” as pointed out by Littlejohn, which is obtained by a suitable combination
dynamical gauge and coordinate transformations. Nevertheless this approach leads to
potential complications and ambiguities due to the fact that the so-called canonical gyroki-
etic coordinates (see below) are field-related. Therefore it would be highly desirable to be
able to construct gyrokinetic canonical variables which result independent of the magnetic
field geometry. In a previous work Tessarotto and Nicolini, 2006 [19]) a possible solution to
this problem has been pointed out by adopting superabundant canonical variables. Purpose
of this work is - instead - to address the problem of the construction of essential gyrokinetic
canonical variables.

II. LAGRANGIAN APPROACH AND CANONICAL GYROKINETIC REDUC-
TION

Starting point for the application of the Darboux reduction method is the standard
Lagrangian formulation for gyrokinetic particle dynamics, expressed in non-canonical gy-
rokinetic variables. For definiteness, let us briefly recall its formulation. Let us assume for
this purpose that the EM potentials \( \Phi \), \( \mathbf{A} \) are analytic functions of \( \varepsilon \) and hence can be
represented in power series of \( \varepsilon \)

\[
\Phi = \sum_{i=-1}^{N} \varepsilon^i \Phi_i(\mathbf{r}, t),
\]

(10)

\[
\mathbf{A} = \sum_{i=-1}^{N} \varepsilon^i \mathbf{A}_i(\mathbf{r}, t).
\]

(11)

Here \( \varepsilon \) is the infinitesimal dimensionless parameter defined above [II]. In particular, the
characteristic scale length \( L \) entering its definition is identified with the minimum of
the gradient-scale lengths for the perturbations of the EM potentials \((\Phi_i, A_i)\), namely as
\[
L \leq \min \left( \frac{1}{|A_i|} \left| \frac{\partial A_i}{\partial r} \right| \right)^{-1}, \quad \min \left( \frac{1}{|\Phi_i|} \left| \frac{\partial \Phi_i}{\partial r} \right| \right)^{-1}
\]
for \(i = -1, N\). In addition, denoting where \(b(r, t) = B(r, t)/B(r, t)\), the magnitudes of the particle velocity \(|v|\) and of the electric drift velocity \(v_E = cE \times b/B\) are assumed of the same order, in the sense
\[
|v| / |v_E| \sim o(1)
\]  
and consequently the parallel electric field is similarly ordered as
\[
b \cdot E \sim o(1)
\]  
\textbf{(condition of small parallel electric field)}. In validity of these hypotheses the construction of the standard gyrokinetic variables is well known and has been achieved by several authors (see for example \[9\]). For definiteness, let us identify the reduced gyrokinetic state \(y'\) with \(y' = (r', u', p'_\phi)\). Here \(u'\) denotes the parallel velocity
\[
u' = b' \cdot (v' - v'_E),
\]
where \(v'\) is the guiding-center velocity and \(v'_E = -\frac{e}{B'} b' \times \nabla' \phi'\) is the \(E \times B\)-drift velocity evaluated at the guiding-center position. In this case - and in the presence of slowly varying EM fields - the fundamental Lagrangian differential 1-form expressed in terms of gyrokinetic variables reads
\[
d\Gamma' \equiv dt L'(y', v', \dot{r}', \dot{\phi}', t) = dG' - d\phi' p'_\phi - dt H'
\]
where \(dG'\) and \(H'\) are respectively the exchange term
\[
dG' \equiv a(y', t) \cdot dr',
\]
\[
a(y', t) \equiv \frac{q}{\varepsilon c} A^*(y', t),
\]
and the gyrokinetic Hamiltonian
\[
H' \equiv \frac{m}{2} v'^2 + \mu' B' + \frac{q}{\varepsilon} \Phi^*(y', t).
\]
Moreover, for definiteness, let us identify the reduced state \(y'\) with a suitable "effective" vector potential \(A^*(y', t)\), i.e., it reads . Here, both the exchange term and gyrokinetic Hamiltonian, in particular the effective EM potentials \((\Phi^*, A^*)\) are expressed as functions only of the non-canonical reduced gyrokinetic state \(y'\) defined above \[3\]. In the following the
gyrokinetic differential 1-form \( d\Gamma' \) will be considered either exactly prescribed (Approach A) or determined in terms of an asymptotic approximation of order \( o(\epsilon^{N+1}) \), namely neglecting corrections of order \( o(\epsilon^{N+1}) \) to \( d\Gamma' \) (Approach B).

Let us now seek a diffeomorphism, to be assumed at least locally defined in the relevant phase-space, of the form

\[
(r', u', p_{\phi'}, \phi') \rightarrow \left( q'^1, q'^2, p'_1, p'_2, p_{\phi'}, \phi' \right),
\]

where \( y'^1, y'^2, p'_1 \) and \( p'_2 \) are assumed smooth real functions only of \( y' \) (and hence by definition as gyrokinetic variables). Provided the differential 1-form \( dG' \) when expressed in terms of \( (q'^1, q'^2, p'_1, p'_2, p_{\phi'}) \) takes the canonical form

\[
dG' = p'_1 dq'^1 + p'_2 dq'^2,
\]

(canonical gyrokinetic reduction) the variables \( z' = (q'^1, q'^2, p'_1, p'_2, p_{\phi'}, \phi') \) are manifestly canonical gyrokinetic variables. In fact, it is immediate to prove that the Euler-Lagrange equations corresponding to the variational differential form (15) expressed in the variables \( z' \) are canonical.

Particle dynamics expressed in terms of the canonical variables \( z' \) denotes the so-called canonical gyrokinetic treatment (CGKT). The explicit construction of these variables has been first pointed out by Littlejohn [11], adopting the so-called Darboux reduction method, by considering the vector potential \( A \), and hence the associated magnetic field \( B \) (equilibrium magnetic field), as stationary. However, the proof - achieved in this way - of the local existence of the diffeomorphism (19) and hence of the gyrokinetic canonical variables defined above \( z' \), is not generally applicable to general situations. In fact, to reach it in Ref. [11] it was assumed that the magnetic field admits, at least locally (in configuration space), a family of nested toroidal magnetic surfaces.

This raises, therefore, the issue of the general validity of such a conclusion. In fact, the question is whether it applies only in the case of equilibrium magnetic fields which are symmetric, i.e., which possess at least one ignorable coordinate, or - at most - exhibit suitably small deviations from a symmetric equilibrium. In fact, it is well known that the proof of existence of smooth MHD equilibria with good magnetic surfaces (namely which admit a family of locally nested toroidal magnetic surfaces in a finite subset of configuration space) can only be achieved for symmetric equilibria (1) or at most for magnetic fields which
are asymptotically close, in some sense, to equilibria of this type (2). As an example, in Ref. [18] to obtain the canonical variables with the Darboux reduction method, consistent with the requirement (2), it was assumed a magnetic fields almost axi-symmetric, i.e., allowing actually only infinitesimally small deviations from axi-symmetric toroidal geometry.

III. ON THE EXISTENCE OF CANONICAL GYROKINETIC VARIABLES

For definiteness let us pose, in this Section, the problem of the existence of the canonical gyrokinetic variables in the framework of the exact gyrokinetic formulation (Approach A). In order to solve the related problem let us analyze the conditions of validity of the canonical gyrokinetic reduction (20) in the particular case in which there results $a = a(y')$ in $dG'$ [see Eq.(16)]. For this purpose let us seek a diffeomorphism

$$r' \rightarrow q'(y'), q' \equiv (q'^1, q'^2, q'^3)$$

denoting in principle arbitrary real and gyrokinetic variables. These can be defined, in particular, in such a way that

$$(r', u', p_{\phi'}, \phi') \rightarrow (q'^1, q'^2, q'^3, \xi', p_{\phi'}, \phi'),$$

with $\xi'$ to be suitably defined, is a phase-space diffeomorphism. Hence it follows that the differential 1-form $dG'$ has necessarily the general form

$$dG' = f'_i dq^i \equiv dG''$$

where $f'_i = f'_i (q', \xi', p_{\phi'})$. The analysis of the conditions of validity of the dynamical reduction - under which the differential 1-form $dG''$, as given by Eq.(23), can be brought to its canonical form (20) - is straightforward. Let us first establish the following lemma

**Lemma - Reduced form for $dG''$**

Let us assume that the real functions $f'_i = f'_i (q', \xi', p_{\phi'})$ (for $i = 1, 2, 3$):

1) are suitably smooth (i.e., at least $C^{(2)}$);

2) the set of gyrokinetic variables $(q'^1, q'^2, f'_1, f'_2, p_{\phi'}, \phi')$ are defined so that they all independent;

3) are defined so that for at least an index $i$ (for $i = 1, 2, 3$) there results

$$\frac{\partial f'_i (q', \xi', p_{\phi'})}{\partial \xi'} = 0$$

(24)
only in isolated points of the gyrokinetic phase-space spanned by the vector \((q', \xi', p_{\phi'})\).

Then a necessary and sufficient condition that the differential 1-form \(dG'' \equiv f'_i dq'^i\) can be represented in the reduced form

\[
dG'' = f'_1 dq'^1 + f'_2 dq'^2
\]  

(25)

is that \(f'_3 = f'_3(q', \xi', p_{\phi'})\) is a first integral of motion, i.e., there results

\[
df'_3(q', \xi', p_{\phi'}) = 0.
\]  

(26)

PROOF

Both the necessary and sufficient conditions are trivial. In fact, if \(f'_3\) is a first integral, since the Lagrangian 1-form is defined up to an arbitrary gauge it follows

\[
f'_2 dq'^2 = d(f'_2 q'^2) - q'_3 df'_3(q', \xi', p_{\phi'}) = 0.
\]  

(27)

On the other hand, if up to an arbitrary gauge transformation, the equation \(f'_2 dq'^2 = 0\) holds identically in a finite subset [neighborhood] of gyrokinetic phase-space, it follows necessarily \(Eq.(26)\).

Provided the hypotheses of the lemma hold the following theorem has the flavor of:

**Theorem 1 - Existence of canonical gyrokinetic variables**

In validity of the hypotheses of the Lemma, provided the gyrokinetic transformation

\[
(r', u', p_{\phi'}, \phi') \rightarrow (q'^1, q'^2, f'_1, f'_2, p_{\phi'}, \phi')
\]  

(28)

is a \(C^{(2)}\)-diffeomorphism, it follows that:

A) it is always possible to identify

\[
p'_1 = f'_1,
\]  

(29)

\[
p'_2 = f'_2.
\]  

(30)

in \(Eq.(25)\);

B) the transformed variables \((q'^1, q'^2, p'_1 = f'_1, p'_2 = f'_2, p_{\phi'}, \phi')\) are canonical gyrokinetic variables.

PROOF

To prove the theorem one has to realize, first, that the assumptions of the Lemma are indeed satisfied by the gyrokinetic Lagrangian defined by \(Eq.15\). Then the proof is an immediate consequence of the Lemma.
A basic consequence of the theorem here pointed out is that the adoption of canonical gyrokinetic variables in gyrokinetic theory is only permitted if the gyrokinetic Lagrangian, besides $\phi'$, has an additional ignorable coordinate, $q'^3$ and hence it admits necessarily two first integrals of motion $p'_\phi$ and $p'_3$. In turn, one can show that this condition implies that both the electric and magnetic fields (as well the corresponding EM potentials $\Phi$, $A$) must be symmetric [20]. This implies that if the equilibrium magnetic field $B$ is non-symmetric, or more generally is locally chaotic (i.e., it does not admit locally a family of nested magnetic surfaces), the gyrokinetic transformation (19) - in the sense of approach A - does not exist.

**IV. DISCUSSION AND CONCLUSIONS**

In this paper the conditions of existence of the canonical gyrokinetic variables for a classical charged point-particle have been investigated. We have shown that - in the framework of an exact gyrokinetic treatment (Approach A) - these variables can only be achieved provided the particle gyrokinetic Lagrangian is symmetric. This means, actually, that it must have generally two ignorable coordinates ($\phi'$ and $q'^3$).

The extension of these results to the asymptotic gyrokinetic treatment (Approach B) is non-trivial. In fact, even small perturbations of the EM field can in principle produce significant local (and even non-local) stochastic effects. Nevertheless, near an axi-symmetric MHD equilibrium, i.e., for magnetic fields which are weakly non-symmetric (and weakly-turbulent) - in the sense that they are characterized by suitably small deviations from a symmetric equilibrium - one should expect CGKT to hold locally, at least, in an asymptotic sense, a result earlier pointed out by White [18].

However, these conclusions cannot be extended to general situations. As an example, in Stellarators magnetic surfaces may only exist locally namely in the neighborhood of nested magnetic surfaces only. Therefore it would be highly desirable to be able to construct gyrokinetic canonical variables which result independent of the magnetic field geometry and apply also to the case of chaotic magnetic fields. An example is given by so-called quasi-symmetric [21] MHD equilibria which arise in Stellarators. These equilibria - which actually may be strongly non-symmetric - are expected to be characterized, at most, by a family of isolated nested toroidal magnetic surface. Typically, in the intermediate regions between these surfaces the magnetic field is chaotic. Another typical situation is that arising in
the presence of local MHD/kinetic turbulence, in which EM perturbations may give rise to local chaotic behavior of the magnetic field. These results are potentially relevant for their implications for theoretical investigations and numerical simulations of magnetized plasmas. In fact, the regularity conditions on the EM fields, to be imposed for the validity of CGKT, may be locally violated in typical MHD equilibria. For example, a consistent kinetic description or a numerical gyrokinetic particle simulation of a magneto-plasma in these variables cannot be achieved unless the EM field is weakly non-symmetric in whole the domain occupied by the plasma.

It should be stressed that there is a simple alternative to the description based on canonical gyrokinetic variables. This is represented by the super-abundant canonical gyrokinetic treatment (super-abundant CGKT) pointed out in Ref. [12], which preserves both the Hamiltonian character of the equations and - unlike CGKT - is applicable also in the presence of a chaotic magnetic field. Basic features of this approach are in fact that: 1) no symmetry (or quasi-symmetry) assumption is required for the magnetic field, so that it holds also in the case of chaotic magnetic fields; 2) the Hamilton equations for the canonical pair \((r', p_{r'})\) are in vector form. Its formulation is summarized by the following constrained variational principle,

**Theorem 2 - Superabundant CGKT**

Let \(x = (r, p_r)\) be the canonical state of a charged point particle described by the Hamiltonian

\[
H(r, p_r, t) = \frac{1}{2m} \left[ p_r - \frac{q}{\varepsilon c} A \right]^2 + \frac{q}{\varepsilon} \Phi
\]

and introduce the diffeomorphism

\[
x = (r, p_r) \rightarrow x' = (r', p_{r'}, \phi', p_{\phi'}),
\]

\((x' \equiv \text{superabundant canonical gyrokinetic state})\), where

\[
p_{r'} \equiv \frac{q}{\varepsilon c} A^* = \frac{\partial \mathcal{L}'}{\partial \left( \frac{d}{dt} r' \right)} \equiv m v' + \frac{q}{\varepsilon c} A^*,
\]

and the gyrokinetic Hamiltonian \((18)\) is represented in the form

\[
K(x', t) = -p_{\phi'} \dot{\Omega}' + \frac{1}{2m} \left[ p_{r'} - \frac{q}{\varepsilon c} A^* (r', u', p_{\phi'}, t) \right]^2 + \frac{q}{\varepsilon} \Phi^* r', u', p_{\phi'}, t).
\]

It follows that: 1) \(x'(t)\) is the extremal curve of the functional \(S(x') = \int_{t_1}^{t_2} dt \left\{ r' \cdot p_{r'} - \phi' p_{\phi'} - K \right\} \) which satisfies the synchronous variational principle \(\delta S(x') = 0\),
with the variations $\delta x' \equiv (\delta r', \delta p_r', \delta \phi', \delta p_\phi')$ to be taken as linearly independent and the function $u'(t)$ to be considered extremal with respect to $\delta p_\phi'$ and $\delta r'$, i.e., such that there results identically

$$\delta p_{r'} \cdot \frac{\partial u'}{\partial p_{r'}} \equiv 0; \quad (35)$$

$$\delta r' \cdot \frac{\partial u'}{\partial r'} = 0; \quad (36)$$

2) $x'(t)$ is canonical with respect to the gyrokinetic Hamiltonian

**PROOF**

The proof is straightforward. In particular, by taking the variations with respect to $p_{r'}$ and $r'$, the Euler-Lagrange equations for $r'$ and $p_{r'}$ are simply

$$\frac{d}{dt} r' = \frac{\partial}{\partial p_{r'}} K(x', t) = \frac{1}{m} \left[ p_{r'} - \frac{q}{\varepsilon c} A^* \right]. \quad (37)$$

$$\frac{d}{dt} p_{r'} = -\frac{\partial}{\partial r'} K(x', t). \quad (38)$$

Finally the equations for $p_{\phi'}$ and $\phi'$ are manifestly

$$\frac{d}{dt} p_{\phi'} = -\frac{\partial}{\partial \phi'} K(x', t) = 0, \quad (39)$$

$$\frac{d}{dt} \phi' = \frac{\partial}{\partial p_{\phi'}} K(x', t). \quad (40)$$

It is immediate to prove that these equations coincide with the equations of motion obtained from the Lagrangian (18).

**Acknowledgments**

Work developed in cooperation with the CMFD Team, Consortium for Magneto-fluid-dynamics (Trieste University, Trieste, Italy). Research developed in the framework of the MIUR (Italian Ministry of University and Research) PRIN Programme: *Modelli della teoria cinetica matematica nello studio dei sistemi complessi nelle scienze applicate*. The support COST Action P17 (EPM, *Electromagnetic Processing of Materials*) and GNFM (National Group of Mathematical Physics) of INDAM (Italian National Institute for Advanced Mathematics) is acknowledged.
Notice

§ contributed paper at RGD26 (Kyoto, Japan, July 2008).
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