Studying Radiative Baryon Decays with the SU(3) Flavor Symmetry

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The weak and electromagnetic radiative baryon decays of octet $T_8$, decuplet $T_{10}$, single charmed anti-triplet $T_{c3}$ and sextet $T_{c6}$, single heavy bottomed anti-triplet $T_{b3}$ and sextet $T_{b6}$ are investigated by using SU(3) flavor symmetry irreducible representation approach. We analyze the contributions from a single quark transition $q_1 \rightarrow q_2 \gamma$ and $W$ exchange transitions, and find that the amplitudes could be easily related by SU(3) flavor symmetry in the $T_{b3, b6}$ weak radiative decays, $T_{c3, c6}$ weak radiative decays, $T_{10} \rightarrow T_8 \gamma$ weak decays and $T_{10} \rightarrow T_0^\prime \gamma$ electromagnetic decays. Nevertheless, the amplitude relations are a little complex in the $T_8 \rightarrow T_8^\prime \gamma$ weak decays due to quark antisymmetry in $T_8$ and $W$ exchange contributions. Predictions for branching ratios of $\Lambda_0^b \rightarrow n \gamma$, $\Xi^-_b \rightarrow \Xi^- \gamma$, $\Xi_b^0 \rightarrow \Sigma^- \gamma$, $\Xi_b^0 \rightarrow \Sigma^0 \gamma$, $\Xi_b^0 \rightarrow \Lambda^0 \gamma$, $\Xi_b^0 \rightarrow \Xi^0 \gamma$, $\Xi^- \rightarrow \Xi \gamma$, $\Sigma^{*0} \rightarrow \Sigma^0 \gamma$, $\Delta^0 \rightarrow n \gamma$ and $\Delta^+ \rightarrow p \gamma$ are given. The results in this work can be used to test SU(3) flavor symmetry approach in the radiative baryon decays by the future experiments at BESIII, LHCb and Belle-II.

I. INTRODUCTION

Radiative weak decays have attracted a lot of attention for a long time in both theory and experiment, since they could give us a chance to study the interplay of the electromagnetic, weak and strong interactions, to test the standard model and to probe new physics. A large number of bottomed baryons, charmed baryon and hyperons are produced at the LHC [1–3], significant experimental progresses about $\Lambda_0^b$ baryon rare decays have been achieved recently at LHCb, and one of them is that radiative decay $\Lambda_0^b \rightarrow \Lambda^0 \gamma$ has been observed with a branching ratio of $(7.1 \pm 1.5 \pm 0.6 \pm 0.7) \times 10^{-6}$ for the first time [4]. Furthermore, many radiative weak decays of strange baryons have been measured [5], and there are longstanding theoretical difficulties to explain the experimental data [6, 7]. Now the sensitivity for measurements of hyperon decays is in the range of $10^{-5} – 10^{-8}$ at the BESIII [8–11]. Therefore, more baryon radiative decays will be detected by the experiments in the near future, so it’s feasible to explore these decays now.

Theoretically, due to our poor understanding of QCD at low energy regions, theoretical calculations of decay amplitudes are not well understood. SU(3) flavor symmetry has attracted a lot of attentions. The SU(3) flavor symmetry approach, which is independent of the detailed dynamics, offers an opportunity to relate different decay modes. Nevertheless, it cannot determine the size of the amplitudes by itself. However, if experimental data are enough, one may use the data to extract the amplitudes, which can be viewed as predictions based on symmetry. There are two popular ways of the SU(3) flavor symmetry. One is to construct the SU(3) irreducible representation amplitude by decomposing effective Hamiltonian. Another way is topological diagram approach, where decay amplitudes are
represented by connecting quark line flows in different ways and then relate them by the SU(3) symmetry. The SU(3) irreducible representation approach (IRA) shows a convenient connection with the SU(3) symmetry, the topological diagram approach gives a better understanding of dynamics in the different amplitudes. The SU(3) flavor symmetry works well in bottomed hadron decays [12–25], charmed hadron decays [26–40] and hyperon decays [41–45].

Many weak radiative decays have been studied by chiral perturbation theory [46], perturbative QCD [47], quark model approach [48], Bethe-Salpeter equation approach [49], relativistic quark model [50], light-cone sum-rule [51], single universal extra dimension scenario [52] and effective Lagrangian approach [53], etc. And some electromagnetic radiative baryon decays have been also studied in Refs. [54, 55]. In this work, we will study the weak radiative baryon decays with a single quark transitions (\(b \rightarrow d \gamma\), \(b \rightarrow s \gamma\), \(c \rightarrow u \gamma\), \(s \rightarrow d \gamma\)) and corresponding \(W\) exchange transitions as well as the electromagnetic radiative decays of \(T_{10} \rightarrow T_8 \gamma\) by using the SU(3) IRA. We will firstly construct the SU(3) irreducible representation amplitudes for different kinds of radiative baryon decays, secondly obtain the decay amplitude relations between different decay modes, then use the available data to extract the SU(3) irreducible amplitudes, and finally predict the not-yet-measured modes for further tests in experiments.

This paper is organized as follows. In Sec. II, we will collect the representations for the baryon multiplets and the branching ratio expressions of the radiative baryon decays. In Sec. III, we will analyze the weak radiative decays of \(T_{b,6}, T_{c,6}\) and \(T_{8,10}\) as well as the electromagnetic radiative decays \(T_{10} \rightarrow T_8 \gamma\). Our conclusions are given in Sec. IV.

II. Theoretical Frame for \(B_1 \rightarrow B_2 \gamma\)

A. Baryon multiplets

The light baryons octet \(T_8\) and decuplet \(T_{10}\) under the SU(3) flavor symmetry of \(u,d,s\) quarks can be written as

\[
T_8 = \begin{pmatrix}
\frac{\Lambda^0}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ & p \\
\Sigma^- & \frac{\Lambda^0}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} & n \\
\Xi^- & \Xi^0 & -2\frac{\Lambda^0}{\sqrt{6}}
\end{pmatrix},
\]

\[
T_{10} = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{3} \Delta^{++} & \Delta^+ & \Sigma^+ \\
\Delta^+ & \Delta^0 & \frac{\Sigma^0}{\sqrt{2}} \\
\Sigma^+ & \frac{\Sigma^0}{\sqrt{2}} & \Xi^0
\end{pmatrix},
\]

\[
T_{10} = \begin{pmatrix}
\frac{\Sigma^+}{\sqrt{2}} & \frac{\Sigma^0}{\sqrt{2}} & \Xi^0 \\
\Sigma^0 & \Sigma^0 & \Xi^0 \\
\Xi^0 & \Xi^0 & \Xi^0
\end{pmatrix}.
\]

The single charmed anti-triplet \(T_{c3}\) and sextet \(T_{c6}\) can be written as

\[
T_{c3} = (\Xi^0_c, -\Xi^+_c, \Lambda^+_c), \quad T_{c6} = \begin{pmatrix}
\frac{\Sigma^+}{\sqrt{2}} & \frac{\Sigma^+}{\sqrt{2}} & \Xi^0_c \\
\frac{\Sigma^0}{\sqrt{2}} & \Sigma^0 & \Xi^0_c \\
\frac{\Xi^0}{\sqrt{2}} & \frac{\Xi^0}{\sqrt{2}} & \Omega_c
\end{pmatrix}.
\]
The anti-triplet $T_{b3}$ and sextet $T_{b6}$ with a heavy b quark have a similar form to $T_{c3}$ and $T_{c6}$, respectively,

$$T_{b3} = (\Xi_b^-, -\Xi_b^0, \Lambda_b^0), \quad T_{b6} = \left( \begin{array}{ccc} \Sigma_b^+ & \frac{1}{\sqrt{2}} \Sigma_b^0 & \frac{1}{\sqrt{2}} \Xi_b^0 \\ \frac{1}{\sqrt{2}} \Sigma_b^0 & \Sigma_b^- & \frac{1}{\sqrt{2}} \Xi_b^- \\ \frac{1}{\sqrt{2}} \Xi_b^0 & \frac{1}{\sqrt{2}} \Xi_b^- & \Omega_b \end{array} \right).$$

(4)

B. Decay branching ratio of $B_1 \rightarrow B_2 \gamma$

In the standard model, the weak radiative baryon decays $B_1 \rightarrow B_2 \gamma$ with $q_1 \rightarrow q_2 \gamma$ transition can proceed via loop Feynman diagrams as shown in Fig. 1. The effective Hamiltonian for $q_1 \rightarrow q_2 \gamma$ transition shown in Fig. 1 can be written as [56]

$$H_{\text{eff}}(q_1 \rightarrow q_2 \gamma) = -\frac{G_F}{\sqrt{2}} e \lambda_{q_1 q_2} C_7^{\text{eff}} m_{q_1} \left( \bar{q}_2 \sigma^{\mu \nu} P_R q_1 \right) \epsilon_{\mu \nu},$$

(5)

where $P_R = (1 + \gamma_5)/2$, $\sigma^{\mu \nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) q_{\nu}$ with $q = p_1 - p_2$, $\epsilon_{\mu \nu}$ is the polarization vectors of photon, and the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $\lambda_{q_1 q_2} = V_{tb}^* V_{ts}$, $V_{tb}^* V_{td}$, $-V_{us} V_{ud}$, $V_{cb}^* V_{ub}$ for $b \rightarrow s \gamma$, $b \rightarrow d \gamma$, $s \rightarrow d \gamma$, $c \rightarrow u \gamma$, respectively.

Then the decay amplitudes can be written as

$$\mathcal{M}(B_1 \rightarrow B_2 \gamma) = \langle B_2 | H_{\text{eff}}(q_1 \rightarrow q_2 \gamma) | B_1 \rangle = -\frac{i G_F}{\sqrt{2}} e \lambda_{q_1 q_2} C_7^{\text{eff}} m_{q_1} \epsilon_{\mu \nu} \langle B_2 | \bar{q}_2 \sigma^{\mu \nu} P_R q_1 | B_1 \rangle.$$  

(6)

The baryon matrix elements $\langle B_2 | \bar{q}_2 \sigma^{\mu \nu} P_R q_1 | B_1 \rangle$ can be parameterized by the form factors, but not all relevant form factors have been calculated and there is no very reliable method to calculate some form factors at present. Nevertheless, the baryon matrix elements also can be obtained by the SU(3) IRA. In terms of the SU(3) flavor symmetry, baryon states and quark operators can be parameterized into SU(3) tensor forms, while the polarization vectors $\epsilon_{\mu \nu}$ are invariant under SU(3) flavor symmetry. The decay amplitudes in terms of the SU(3) IRA are given in later Tab. I, Tab. III, Tab. IV and Tab. V for $T_{b3,66} \rightarrow T_{8,10} \gamma$, $T_{b3,66} \rightarrow T_{c3,c6} \gamma$, $T_{c3,c6} \rightarrow T_{8,10} \gamma$ and $T_{8,10} \rightarrow T_8' \gamma$ weak decays, respectively.

![FIG. 1: Feynman diagram for $B_1 \rightarrow B_2 \gamma$ weak decays via a single quark emission in the standard model.](image-url)
The branching ratios of the $B_1 \rightarrow B_2 \gamma$ weak decays can be obtained by the decay amplitudes

$$\mathcal{B}(B_1 \rightarrow B_2 \gamma) = \frac{\alpha e}{16\pi m_{B_1}^2} m_{B_1}^2 |\mathcal{M}(B_1 \rightarrow B_2 \gamma)|^2.$$  

(7)

After extracting the masses, the Wilson Coefficients, etc, from $|\mathcal{M}(B_1 \rightarrow B_2 \gamma)|^2$, the branching ratios of the $T_{b3,c3,8} \rightarrow T'_{c3,c6,8,10} \gamma$ and $T_{b6,c6,10} \rightarrow T_{c3,8} \gamma$ weak decays are [57, 58]

$$\mathcal{B}(B_1 \rightarrow B_2 \gamma) = \frac{\tau_{B_1}}{64\pi^4} G^2_F m_{B_1}^2 |\lambda_{q_1q_2}|^2 |C_{\gamma}^{eff}(m_{q_1})|^2 \left(1 - \frac{m_{B_2}^2}{m_{B_1}^2}\right)^3 |A(B_1 \rightarrow B_2 \gamma)|^2,$$

(8)

where $A(B_1 \rightarrow B_2 \gamma)$ may be given by the form factors as $|A(B_1 \rightarrow B_2 \gamma)|^2 = K(|f_{TV}^2(0)|^2 + |f_{TA}^2(0)|^2) = K(|\hat{h}_\perp(0)|^2 + |\tilde{h}_\perp(0)|^2)$ with $K = 1$ for the $T_{b3,c3,8} \rightarrow T'_{c3,c6,8,10} \gamma$ weak decays and $K = 1/2$ for the $T_{b6,c6,10} \rightarrow T_{c3,8} \gamma$ weak decays.

As for the $T_{b6,c6} \rightarrow T'_{c6,10} \gamma$ weak decays, the branching ratios are [59]

$$\mathcal{B}(B_1 \rightarrow B_2 \gamma) = \frac{\tau_{B_1}}{384\pi^4} G^2_F m_{B_1}^2 m_{B_2}^5 \left(1 - \frac{m_{B_2}^2}{m_{B_1}^2}\right)^3 |\lambda_{q_1q_2}|^2 |C_{\gamma}^{eff}(m_{q_1})|^2 |A(B_1 \rightarrow B_2 \gamma)|^2.$$

(9)

For the electromagnetic $T_{10} \rightarrow T_{8} \gamma$ decays, the expressions of their branching ratios are different from Eq. (8), and the following relations will be used to obtain the results [55]

$$\mathcal{B}^E(T_{10} \rightarrow T_{8} \gamma) \propto \tau_{T_{10}} \left(\frac{m_{T_{10}}^2 - m_{T_8}^2}{m_{T_{10}}^2}\right)^2 \left|m_{T_{10}} - m_{T_8}\right|^2 |A^E(T_{10} \rightarrow T_{8} \gamma)|^2.$$

(10)

In addition, according to Refs. [6, 60, 61], other three kinds of Feynman diagrams might contribute to the weak baryon decays. The example for $s + u \rightarrow u + d + \gamma$ is displayed in Fig. 2. Fig. 2 (a-b) are two-quark and three-quark transitions with the $W$ exchange, which have been discussed, for examples, in Refs. [60, 62]. Since Fig. 2 (c) is suppressed by the two $W$ propagators, and its contribution can be safely neglected. We will consider the W-exchange contributions in Fig. 2 (a-b) in later analysis of SU(3) flavor symmetry.

![Diagram](image)

FIG. 2: Other quark diagrams for weak radiative $B_1 \rightarrow B_2 \gamma$ weak decays. (a) two-quark bremsstrahlung, (b) three-quark transition, (c) internal radiation.

III. Results and Analysis

The theoretical input parameters and the experimental data within the $2\sigma$ errors from Particle Data Group [5] will be used in our numerical results.
A. $T_{b3,b6}$ weak radiative decays

The SU(3) flavor structure of the relevant $b \rightarrow s,d$ Hamiltonian can be found, for instance, in Refs. [23, 63, 64]. The SU(3) IRA decay amplitudes for $T_{b3,b6} \rightarrow T_{8,10}\gamma$ decays via $b \rightarrow s/d\gamma$ can be parameterized as

$$A(T_{b3} \rightarrow T_8\gamma) = a_1(T_{b3})^{ij}T(\bar{3})^k(T_8)^{ij}k + a_2(T_{b3})^{ij}T(\bar{3})^k(T_8)^{ij}k,$$

$$A(T_{b6} \rightarrow T_8\gamma) = a'_1(T_{b6})^{ij}T(\bar{3})^k(T_8)^{ij}k,$$

$$A(T_{b6} \rightarrow T_{10}\gamma) = a''_1(T_{b6})^{ij}T(\bar{3})^k(T_{10})^{ij}k,$$

with $T(\bar{3}) = (0,1,1)$, which denotes the transition operators $(\bar{q}_2 b)$ with $q_2 = d,s$. The coefficients $a_1^{(i)}, a'_1^{(i)}, a''_1^{(i)}$, which contain information about QCD dynamics, include the single quark emission contributions in Fig. 1 and $T_{b3} \rightarrow T_8\psi_i$ and $T_{b6} \rightarrow T_{8,10}\psi_i$ ($\psi_i$ are the set of all $J = 1, l = 0$ ($c\bar{c}$)) long distance contributions [65, 66] (the similar in following $b_i^{(i)}, c_i^{(i)}, d_i$ in Eqs. (18-29)), nevertheless, the long distance contributions in the b-sector are small and under control [65, 66]. Noted that $T_{b3} \rightarrow T_{10}\gamma$ (and later $T_{c3} \rightarrow T_{10}\gamma$) weak decays are not allowed by the quark symmetry.

The SU(3) IRA amplitudes of $T_{b3} \rightarrow T_8\gamma$ and $T_{b6} \rightarrow T_{8,10}\gamma$ weak decays are given in Tab. I. And the information of relevant CKM matrix elements $V_{cb}V_{ts}^\ast$ ($V_{cb}V_{td}^\ast$) for $b \rightarrow s\gamma$ ($b \rightarrow d\gamma$) transition is not shown in Tab. I.

Now, we discuss the $T_{b3} \rightarrow T_8\gamma$ weak decays. Decays $\Lambda_0^0 \rightarrow \Lambda^0\gamma$, $\Lambda_3^0 \rightarrow \Sigma^0\gamma$, $\Xi_6^0 \rightarrow \Xi^0\gamma$ and $\Xi_8^0 \rightarrow \Xi^{-}\gamma$ ($\Lambda_0^0 \rightarrow n\gamma$, $\Xi_6^0 \rightarrow \Lambda^0\gamma$, $\Xi_8^0 \rightarrow \Sigma^0\gamma$ and $\Xi_8^0 \rightarrow \Sigma^{-}\gamma$) proceed via the $b \rightarrow s\gamma$ ($b \rightarrow d\gamma$) flavor changing neutral current transition. From Tab. I, one can see that the 7 decay amplitudes of $T_{b3} \rightarrow T_8\gamma$ can be related by one parameter $A_1$. Noted that our $|A(T_{b3} \rightarrow T_8\gamma)|$ via the $b \rightarrow s/d\gamma$ transitions are consistent with ones of $T_{b3} \rightarrow T_8J/\psi$ in Ref. [67] and ones of the CKM-leading part results $T_{b3} \rightarrow T_8J/\psi$ in Ref. [12]. Among these 7 decay modes, only $B(\Lambda_0^0 \rightarrow \Lambda\gamma)$ has been measured at present, which is listed in the second column of Tab. II. Using data of $B(\Lambda_0^0 \rightarrow \Lambda\gamma)$ and the expression of $B(B_1 \rightarrow B_2\gamma)$ in Eq. (8) to get $|A_1|$, and then other 6 branching ratios are obtained, which are given in the third column of Tab. II. Previous predictions for $B(\Lambda_0^0 \rightarrow \Lambda\gamma)$ in the light-cone sum rules and for $B(\Lambda_0^0 \rightarrow n\gamma)$ in the relativistic quark model and in the Bethe-Salpeter equation approach are listed in the last column of Tab. II.

Our SU(3) IRA prediction of $B(\Lambda_0^0 \rightarrow n\gamma)$ agrees with ones in the relativistic quark model or in the Bethe-Salpeter equation approach [49, 68]. More experimental data about the $T_{b3} \rightarrow T_8\gamma$ decays in the further could test the SU(3) flavor symmetry approach.

As given in Tab. I, the decay amplitudes of $T_{b6} \rightarrow T_8\gamma$ and $T_{b6} \rightarrow T_{10}\gamma$ weak decays can be parameterized by only one parameter $a'_1$ and $a''_1$, respectively. And our $|A(T_{b6} \rightarrow T_8\gamma)|$ via the $b \rightarrow s\gamma$ transition are consistent with ones of $T_{b6} \rightarrow T_8J/\psi$ in Ref. [67]. Unfortunately, none of the $T_{b6} \rightarrow T_8\gamma$ and $T_{b6} \rightarrow T_{10}\gamma$ weak decays has been measured at present. Any measurement of the $T_{b6} \rightarrow T_8\gamma$ ($T_{b6} \rightarrow T_{10}\gamma$) will give us chance to predict other 10 (11) decay modes.

In addition, $T_{b3,b6}$ baryons also can decay to $T_{c3,c6}$ by only $W$ exchange $b + u \rightarrow c + s/d + \gamma$ [53]. The SU(3) IRA decay amplitudes for the $T_{b3,b6} \rightarrow T_{c3,c6}\gamma$ decays are

$$A(T_{b3} \rightarrow T_{c3}\gamma) = \tilde{a}_1V_{cb}V_{q_jq_i}(T_{b3})^{ij}(T_{c3})^{id},$$

$$A(T_{b3} \rightarrow T_{c6}\gamma) = \tilde{a}'_1V_{cb}V_{q_jq_i}(T_{b3})^{ij}(T_{c6})^{id},$$

$$A(T_{b6} \rightarrow T_{c3}\gamma) = \tilde{a}''_1V_{cb}V_{q_jq_i}(T_{b6})^{ij}(T_{c3})^{id},$$

$$A(T_{b6} \rightarrow T_{c6}\gamma) = \tilde{a}'''_1V_{cb}V_{q_jq_i}(T_{b6})^{ij}(T_{c6})^{id},$$
TABLE I: The SU(3) IRA amplitudes of the $T_{b3,b6} \rightarrow T_{8,10}\gamma$ weak decays by the $b \rightarrow s/d\gamma$ transitions, and $A_1 \equiv a_1 + a_2$.

| Decay modes | $A(T_{b3,b6} \rightarrow T_{8,10}\gamma)$ |
|-------------|------------------------------------------|
| $T_{b3} \rightarrow T_{8}\gamma$ via the $b \rightarrow s\gamma$ transition: | |
| $\Lambda_b^0 \rightarrow \Lambda^0\gamma$ | $-2A_1/\sqrt{6}$ |
| $\Lambda_b^0 \rightarrow \Sigma^0\gamma$ | 0 |
| $\Xi_b^0 \rightarrow \Xi^0\gamma$ | $-A_1$ |
| $\Xi_b^- \rightarrow \Xi^-\gamma$ | $A_1$ |
| $T_{b3} \rightarrow T_{8}\gamma$ via the $b \rightarrow d\gamma$ transition: | |
| $\Lambda_b^0 \rightarrow n\gamma$ | $A_1$ |
| $\Xi_b^0 \rightarrow \Lambda^0\gamma$ | $-A_1/\sqrt{6}$ |
| $\Xi_b^0 \rightarrow \Sigma^0\gamma$ | $-A_1/\sqrt{2}$ |
| $\Xi_b^- \rightarrow \Sigma^-\gamma$ | $A_1$ |
| $T_{b6} \rightarrow T_{8}\gamma$ via the $b \rightarrow s\gamma$ transition: | |
| $\Sigma_b^+ \rightarrow \Sigma^+\gamma$ | $-a_1'$ |
| $\Sigma_b^0 \rightarrow \Lambda^0\gamma$ | 0 |
| $\Sigma_b^0 \rightarrow \Sigma^0\gamma$ | $-a_1'$ |
| $\Sigma_b^- \rightarrow \Sigma^-\gamma$ | $a_1'$ |
| $\Xi_b^0 \rightarrow \Xi^0\gamma$ | $-a_1'/\sqrt{2}$ |
| $\Xi_b^- \rightarrow \Xi^-\gamma$ | $a_1'/\sqrt{2}$ |
| $T_{b6} \rightarrow T_{8}\gamma$ via the $b \rightarrow d\gamma$ transition: | |
| $\Sigma_b^+ \rightarrow p\gamma$ | $a_1'$ |
| $\Sigma_b^0 \rightarrow n\gamma$ | $a_1'$ |
| $\Xi_b^0 \rightarrow \Lambda^0\gamma$ | $-\sqrt{2} a_1'$ |
| $\Xi_b^0 \rightarrow \Sigma^0\gamma$ | $a_1'/2$ |
| $\Xi_b^- \rightarrow \Sigma^-\gamma$ | $-a_1'/\sqrt{2}$ |
| $\Omega_b \rightarrow \Xi^-\gamma$ | $-a_1'$ |
| $T_{b6} \rightarrow T_{10}\gamma$ via the $b \rightarrow s\gamma$ transition: | |
| $\Sigma_b^+ \rightarrow \Sigma^{++}\gamma$ | $a_1''$ |
| $\Sigma_b^0 \rightarrow \Sigma^{*0}\gamma$ | $a_1'/\sqrt{2}$ |
| $\Sigma_b^- \rightarrow \Sigma^{*-}\gamma$ | $\sqrt{3} a_1''$ |
| $\Xi_b^0 \rightarrow \Xi^{0}\gamma$ | $a_1''/\sqrt{2}$ |
| $\Xi_b^- \rightarrow \Xi^{*-}\gamma$ | $a_1''/\sqrt{2}$ |
| $\Omega_b \rightarrow \Omega\gamma$ | $a_1''$ |
| $T_{b6} \rightarrow T_{10}\gamma$ via the $b \rightarrow d\gamma$ transition: | |
| $\Sigma_b^+ \rightarrow \Delta^{++}\gamma$ | $a_1''$ |
| $\Sigma_b^0 \rightarrow \Delta^{0}\gamma$ | $a_1'/\sqrt{2}$ |
| $\Sigma_b^- \rightarrow \Delta^{*-}\gamma$ | $\sqrt{3} a_1''$ |
| $\Xi_b^0 \rightarrow \Sigma^{*0}\gamma$ | $a_1''/2$ |
| $\Xi_b^- \rightarrow \Sigma^{*-}\gamma$ | $a_1''/\sqrt{2}$ |
| $\Omega_b \rightarrow \Xi^{*-}\gamma$ | $a_1''$ |
TABLE II: Branching ratios of the $T_{b3} \rightarrow T_{b} \gamma$ decays via the $b \rightarrow s/d \gamma$ transitions.

| Observables | Experimental data [5] | Our SU(3) IRA predictions | Other predictions |
|-------------|-----------------------|----------------------------|-------------------|
| $b \rightarrow s$ | | | |
| $B(\Lambda_b^0 \rightarrow \Lambda \gamma)(10^{-6})$ | $7.1 \pm 1.7$ | $7.1 \pm 3.4$ | $7.3 \pm 1.5$ [69] |
| $B(\Lambda_b^0 \rightarrow \Sigma^0 \gamma)$ | $\cdots$ | $0$ | | |
| $B(\Xi^-_b \rightarrow \Xi^- \gamma)(10^{-5})$ | $\cdots$ | $1.23 \pm 0.64$ | | |
| $B(\Xi^0_b \rightarrow \Sigma^0 \gamma)(10^{-5})$ | $\cdots$ | $1.16 \pm 0.60$ | | |
| $b \rightarrow d$ | | | |
| $B(\Lambda_b^0 \rightarrow n \gamma)(10^{-7})$ | $\cdots$ | $5.03 \pm 2.67$ | $3.69^{+3.76}_{-1.95}$ [49, 68] |
| $B(\Xi_b^0 \rightarrow \Sigma^0 \gamma)(10^{-8})$ | $\cdots$ | $9.17 \pm 5.10$ | | |
| $B(\Xi_b^- \rightarrow \Sigma^- \gamma)(10^{-7})$ | $\cdots$ | $2.71 \pm 1.50$ | | |
| $B(\Xi_b^+ \rightarrow \Lambda^+ \gamma)(10^{-6})$ | $\cdots$ | $5.74 \pm 3.21$ | | |

TABLE III: The SU(3) IRA amplitudes of the $T_{b3,b6} \rightarrow T_{c3,c6} \gamma$ weak decays by the $W$ exchange $b + u \rightarrow c + s/d + \gamma$.

| Decay modes | $A(T_{b3,b6} \rightarrow T_{c3,c6} \gamma)$ |
|-------------|------------------------------------------|
| $T_{b3} \rightarrow T_{c3} \gamma$ | | |
| $\Xi_b^0 \rightarrow \Xi_c^0 \gamma$ | $\tilde{a}_1$ | |
| $\Lambda_b^0 \rightarrow \Xi_c^0 \gamma$ | $-\tilde{a}_1 s_c$ | |
| $T_{b6} \rightarrow T_{c3} \gamma$ | | |
| $\Sigma_b^+ \rightarrow \Lambda_c^+ \gamma$ | $\tilde{a}_1^{''}$ | |
| $\Xi_b^0 \rightarrow \Xi_c^0 \gamma$ | $-\tilde{a}_1^{''}/\sqrt{2}$ | |
| $\Sigma_b^+ \rightarrow \Xi_c^+ \gamma$ | $\tilde{a}_1'' s_c$ | |
| $\Sigma_b^0 \rightarrow \Xi_c^0 \gamma$ | $\tilde{a}_1'' s_c/\sqrt{2}$ | |
| $T_{b6} \rightarrow T_{c6} \gamma$ | | |
| $\Sigma_b^+ \rightarrow \Sigma_c^+ \gamma$ | $\tilde{a}_1''/\sqrt{2}$ | |
| $\Sigma_b^0 \rightarrow \Sigma_c^0 \gamma$ | $\tilde{a}_1''/\sqrt{2}$ | |
| $\Xi_b^0 \rightarrow \Xi_c^0 \gamma$ | $\tilde{a}_1''/2$ | |
| $\Sigma_b^+ \rightarrow \Xi_c^+ \gamma$ | $\tilde{a}_1'' s_c/\sqrt{2}$ | |
| $\Sigma_b^0 \rightarrow \Xi_c^0 \gamma$ | $\tilde{a}_1''/s_c/2$ | |
| $\Xi_b^0 \rightarrow \Omega_c \gamma$ | $\tilde{a}_1'' s_c/\sqrt{2}$ | |
A(T_{b6} \rightarrow T_{c6}\gamma) = \bar{a}''_1 V_{cb} V_{q_i, q_j} (T_{b6})^{ij} (T_{c6})_{ik}. \tag{17}

Since the CKM matrix element $V_{cb}$ occur for all processes, we will absorb it in the coefficients $\bar{a}_1$, $\bar{a}_1'$, $\bar{a}_1''$ as well as $\bar{a}_1'''$, and only keep $V_{ud} \approx 1$ and $V_{us} \approx \lambda = s_c \equiv \sin \theta_C \approx 0.22453$ representing the Cabbibo angle $\theta_C$ \cite{5}. The SU(3) IRA amplitudes of $T_{b3, b6} \rightarrow T_{c3, c6}\gamma$ weak decays are summarized in Tab. III, in which we keep $V_{ud}$ and $V_{us}$ information for comparing conveniently, and we may easily see the amplitude relations in this table. Just none of these decays has been measured yet.

B. $T_{c3, c6}$ weak radiative decays

$T_{c3, c6} \rightarrow T_{8,10}\gamma$ via $c \rightarrow u\gamma$ transition are similar to the $T_{b3, b6} \rightarrow T_{8,10}\gamma$ via $b \rightarrow s/d\gamma$ transition. Nevertheless, the short distance contributions from $c \rightarrow u\gamma$ could be negligible, and the dominant contributions in charmed baryon decays mainly from the W-exchange contributions similar as Fig. 2 (a-b) \cite{25}. The SU(3) flavor structure of the $T_{c3, c6}$, $c$ for quark decays into light quarks in terms denote the contributions from $c \rightarrow u\gamma$ shown in Eq.1 and the long distance contribution from the real $q\bar{q}$ intermediate state $\rho$, $\omega$ and $\phi$. The $\bar{b}_1^{(i)}$, $\bar{b}_1^{(i)'}, \bar{b}_1^{(i)''}$ terms denote the W exchange contributions similar as Fig. 2. Noted that $H(6)^{lk}_j$ ($H(15)^{lk}_j$) related to $(\bar{q}\tilde{q}^{i})(\bar{q}_b c)$ operator is antisymmetric (symmetric) in upper indices. The non-vanish $H(6)^{lk}_j$ and $H(15)^{lk}_j$ for $c \rightarrow sud, dus, udd, uss$ transitions can be found in Ref. \cite{37}. Using $l, k$ antisymmetric in $H(6)^{lk}_j$ and $l, k$ symmetric in $H(15)^{lk}_j$, we have

$$\bar{b}_2^{(i)} = -\bar{b}_1^{(i)}, \quad \bar{b}_3^{(i)} = \bar{b}_4^{(i)}, \quad \bar{b}_6^{(i)} = 0.$$ \tag{18}

Since none of the down-type quarks are heavy, the Glashow-Iliopoulos-Maiani (GIM) mechanism suppression is obvious in the charm sector. The $\bar{b}_1^{(i)'}, \bar{b}_1^{(i)''}$ terms related to the short and long distance contributions of $c \rightarrow u\gamma$ transition are strongly suppressed by the GIM mechanism. As for the W exchange transition, there are three kinds of charm quark decays into light quarks

$$d + c \rightarrow u + s + \gamma, \quad d + c \rightarrow u + d + \gamma \quad (s + c \rightarrow u + s + \gamma), \quad s + c \rightarrow u + d + \gamma, \tag{19}$$
which are related to $H(6, 15)^{13}_2$, $H(6, 15)^{12}_2$ ($H(6, 15)^{13}_1$), and $H(6, 15)^{12}_1$ are proportional to $V_{cs}^*V_{ud}$, $V_{cd}^*V_{ud}$ ($V_{cs}^*V_{us}$), and $V_{cd}^*V_{us}$, respectively. The relevant CKM matrix elements can be written by the Wolfenstein parameterization [5]

\[
V_{cs}^*V_{ud} = (1 - \lambda^2/2)^{\frac{1}{2}} \approx 1, \\
V_{cd}^*V_{ud} = -\lambda(1 - \lambda^2/2) \approx -s_c, \\
V_{cs}^*V_{us} = \lambda(1 - \lambda^2/2) \approx s_c, \\
V_{cd}^*V_{us} = -\lambda^2(1 - \lambda^2/2) \approx -s_c^2. 
\]  

(23)

So three kinds of charm quark decays in Eq. (22) are called Cabibbo allowed, singly Cabibbo suppressed, and doubly Cabibbo suppressed decays, respectively.

The SU(3) IRA amplitudes of $T_{c3} \to T_8 \gamma$ and $T_{c6} \to T_{8,10} \gamma$ weak decays are given in the second column of Tab. IV. For well understanding, we also show the relevant CKM matrix information in Tab. IV, too. In addition, the contribution of $H(\bar{6})$ to the decay branching ratio can be about 5.5 times larger than one of $H(15)$ due to Wilson Coefficient suppressed, for examples, see [33, 34]. If ignoring the GIM strongly suppressed $c \to u\gamma$ transition contributions and the Wilson Coefficient suppressed $H(15)$ term contributions, the decay amplitudes of $T_{c3} \to T_8 \gamma$, $T_{c6} \to T_8 \gamma$ and $T_{c6} \to T_{10} \gamma$ are related by only one parameter $\tilde{B} \equiv \tilde{b}_1 - \tilde{b}_3$, $\tilde{B}' \equiv \tilde{b}_1' - \tilde{b}_3'$ and $\tilde{b}_1''$, respectively. The simplifications resulting are listed in the last column of Tab. IV. Just all baryon weak radiative decays of $T_{c3,c6}$ baryons have not been measured yet.

C. $T_{8,10}$ weak radiative decays

The SU(3) flavor structure of the relevant $s \to d$ Hamiltonian can be found in Ref. [42]. The decay amplitudes of the $T_8$ and $T_{10}$ weak radiative decays can be parameterized as

\[
A(T_8 \to T_8' \gamma) = c_1(T_8)_{[ij]n}T''(\bar{3})^k(kT_8'_{[ij]k} + c_2(T_8)_{[ij]n}T'(3)^k(T_8)_{[ik]j} \\
+ c_3(T_8)_{[in]m}T''(\bar{3})^k(T_8'_{[ik]j} + c_4(T_8)_{[in]m}T'(3)^k(T_8)_{[ik]j} + c_5(T_8)_{[in]m}T''(\bar{3})^k(T_8)_{[ik]j}], \\
+ \bar{c}_1(T_8)_{[ij]n}(T_8'_{[ik]j}H(4)^{jk} + \bar{c}_2(T_8)_{[in]m}(T_8'_{[ik]j}H(4)^{jk} + \bar{c}_3(T_8)_{[in]m}(T_8'_{[ik]j}H(4)^{jk} \\
\]  

(24)

\[
A(T_8 \to T_{10}' \gamma) = c_1'(T_8)_{[in]j}T''(\bar{3})^k(T_{10})_{[ik]j} + \bar{c}_1'(T_8)_{[in]j}(T_{10})_{[ik]j}H(4)^{ik} + \bar{c}_2'(T_8)_{[in]j}(T_{10})_{[ik]j}H(4)^{ik} + \bar{c}_3'(T_8)_{[in]j}(T_{10})_{[ik]j}H(4)^{ik}, \\
\]  

(25)

\[
A(T_{10} \to T_8 \gamma) = c_1''(T_{10})_{[ij]n}T''(\bar{3})^k(T_8)_{[ik]j} + c_1''(T_{10})_{[ij]n}(T_8)_{[ik]j}H(4)^{ik}, \\
\]  

(26)

\[
A(T_{10} \to T'_{10}) = c_1'''(T_{10})_{[ij]n}T''(\bar{3})^k(T_{10})_{[ik]j} + c_1'''(T_{10})_{[ij]n}(T_{10})_{[ik]j}H(4)^{ik}, \\
\]  

(27)

where $n \equiv 3$ for s quark, $T''(\bar{3}) = (0, 1, 0)$ related to the transition operator $(\bar{d}s)$, and $H(4)^{ik}$ related to $(\bar{q}_iq'j)(\bar{q}_ks)$ operator is symmetric in upper indices [42]. In Eqs. (24-27), the $c_i''''(\cdots)$ terms denote the contributions from $s \to d\gamma$ shown in Fig.1 and the long distance contribution from $\psi_i$, $\rho$ and $\omega$ [25] (for $T_{8,10}$ weak radiative decays, the long distance contributions may be significantly larger than the short distance ones), the $c_i''''(\cdots)$ terms denote the W exchange contributions shown in Fig. 2 (a-b), and the internal radiation contributions in Fig. 2 (c) are neglected in this work.
TABLE IV: The SU(3) IRA amplitudes of the $T_{c3,c6} \rightarrow T_{8,10\gamma}$ weak decays, $B_1 \equiv b_1 + b_2$, $\bar{B}_1 \equiv \bar{b}_1 - \bar{b}_3 + \bar{b}_4$, $B_2 \equiv b_1 - \bar{b}_3 - \bar{b}_4$, $\bar{B}_1 \equiv \bar{b}_1 - \bar{b}_3 + \bar{b}_4$, $\bar{B}_2 \equiv \bar{b}_1 - \bar{b}_3 - \bar{b}_4$ and $\bar{B}^{'} \equiv \bar{b}_1^{(')} - \bar{b}_3^{(')}$.

| Decay modes | $\lambda_{q_1 q_2} T_{c3,c6} \rightarrow T_{8,10\gamma}$ | approximative $\lambda_{q_1 q_2} T_{c3,c6} \rightarrow T_{8,10\gamma}$ |
|-------------|-------------------------------------------------|-------------------------------------------------|
| Cabibbo allowed $T_{c3} \rightarrow T_{8\gamma}$: | | |
| $\Lambda_+ \rightarrow \Sigma^+ \gamma$ | $-\bar{B}_1$ | $-\bar{B}$ |
| $\Xi^0 \rightarrow \Xi^0 \gamma$ | $-\bar{B}_2$ | $-\bar{B}$ |
| singly Cabibbo suppressed $T_{c3} \rightarrow T_{8\gamma}$: | | |
| $\Lambda_+ \rightarrow \Sigma^+ \gamma$ | $[B_1 - (\frac{3}{4} \bar{B}_1 - \frac{1}{4} \bar{B}_2)] s_c$ | $-\frac{1}{2} \bar{B}_s_c$ |
| $\Xi^0 \rightarrow \Lambda^0 \gamma$ | $[-B_1 - (\frac{3}{4} \bar{B}_1 - \frac{1}{4} \bar{B}_2)] s_c$ | $-\frac{1}{2} \bar{B}_s_c$ |
| $\Xi^0 \rightarrow \Sigma^0 \gamma$ | $[B_1 + 3\bar{B}_1 - \frac{3}{2} \bar{B}_2] s_c/\sqrt{2}$ | $-\frac{1}{2} \bar{B}_s_c/\sqrt{2}$ |
| doubly Cabibbo suppressed $T_{c3} \rightarrow T_{8\gamma}$: | | |
| $\Xi^0 \rightarrow \Sigma^0 \gamma$ | $B_1 s_c^2$ | $\bar{B}_s_c^2$ |
| $\Xi^0 \rightarrow n \gamma$ | $\bar{B}_s_c^2$ | $\bar{B}_s_c^2$ |
| Cabibbo allowed $T_{c6} \rightarrow T_{8\gamma}$: | | |
| $\Sigma^+ \rightarrow \Sigma^+ \gamma$ | $-\bar{B}_1^{(')}/\sqrt{2}$ | $-\bar{B}^{(')}/\sqrt{2}$ |
| $\Sigma^0 \rightarrow \Lambda^0 \gamma$ | $-\bar{B}_2^{(')}/\sqrt{6}$ | $-\bar{B}^{(')}/\sqrt{6}$ |
| $\Sigma^0 \rightarrow \Sigma^0 \gamma$ | $-\bar{B}_1^{(')}/\sqrt{2}$ | $-\bar{B}^{(')}/\sqrt{2}$ |
| $\Xi^0 \rightarrow \Xi^0 \gamma$ | $-\bar{B}_2^{(')}/\sqrt{2}$ | $-\bar{B}^{(')}/\sqrt{2}$ |
| singly Cabibbo suppressed $T_{c6} \rightarrow T_{8\gamma}$: | | |
| $\Sigma^+ \rightarrow \Sigma^+ \gamma$ | $[B_1 - (\frac{3}{4} \bar{B}_1^{(')} - \frac{1}{4} \bar{B}_2^{(')})] s_c^{(')}/\sqrt{2}$ | $-\frac{1}{2} \bar{B}_s_c^{(')}/\sqrt{2}$ |
| $\Sigma^0 \rightarrow \Sigma^0 \gamma$ | $[-B_1^{(')} - (\frac{3}{4} \bar{B}_1^{(')} - \frac{1}{4} \bar{B}_2^{(')})] s_c^{(')}/\sqrt{2}$ | $-\frac{1}{2} \bar{B}_s_c^{(')}/\sqrt{2}$ |
| $\Xi^0 \rightarrow \Lambda^0 \gamma$ | $[B_1^{(')} - (\frac{1}{4} \bar{B}_1^{(')} - \frac{3}{4} \bar{B}_2^{(')})] s_c^{(')}/\sqrt{2}$ | $-\frac{1}{2} \bar{B}_s_c^{(')}$ |
| $\Xi^0 \rightarrow \Sigma^0 \gamma$ | $[B_1^{(')} + (\frac{1}{4} \bar{B}_1^{(')} - \frac{3}{4} \bar{B}_2^{(')})] s_c^{(')}$ | $-\frac{1}{2} \bar{B}_s_c^{(')}$ |
| $\Omega_0 \rightarrow \Xi^0 \gamma$ | $[2B_1^{(')} + (\frac{1}{4} \bar{B}_1^{(')} - \frac{3}{4} \bar{B}_2^{(')})] s_c^{(')}$ | $-\frac{1}{2} \bar{B}_s_c^{(')}$ |
| doubly Cabibbo suppressed $T_{c6} \rightarrow T_{8\gamma}$: | | |
| $\Xi^0 \rightarrow \Sigma^0 \gamma$ | $B_1^{(')} s_c^{(')}/\sqrt{2}$ | $\bar{B}_s_c^{(')}/\sqrt{2}$ |
| $\Xi^0 \rightarrow n \gamma$ | $B_1^{(')} s_c^{(')}/\sqrt{2}$ | $B_1^{(')} s_c^{(')}/\sqrt{2}$ |
| $\Omega_0 \rightarrow \Lambda^0 \gamma$ | $-(\bar{B}_1^{(')} + \bar{B}_2^{(')}) s_c^{(')}/\sqrt{6}$ | $-2\bar{B}_s_c^{(')}/\sqrt{6}$ |
| $\Omega_0 \rightarrow \Xi^0 \gamma$ | $(\bar{B}_1^{(')} - \bar{B}_2^{(')}) s_c^{(')}/\sqrt{2}$ | $0$ |
| Cabibbo allowed $T_{c6} \rightarrow T_{10\gamma}$: | | |
| $\Sigma^+ \rightarrow \Sigma^+ \gamma$ | $\bar{B}_1^{(')}/\sqrt{2}$ | $\bar{B}_1^{(')}/\sqrt{2}$ |
| $\Sigma^0 \rightarrow \Sigma^0 \gamma$ | $\bar{B}_1^{(')}/\sqrt{2}$ | $\bar{B}_1^{(')}/\sqrt{2}$ |
| $\Xi^0 \rightarrow \Xi^0 \gamma$ | $\bar{B}_1^{(')}/\sqrt{2}$ | $\bar{B}_1^{(')}/\sqrt{2}$ |
| singly Cabibbo suppressed $T_{c6} \rightarrow T_{10\gamma}$: | | |
| $\Sigma^+ \rightarrow \Delta^+ \gamma$ | $\sqrt{3}B_1^{(')} s_c$ | $0$ |
| $\Sigma^+ \rightarrow \Delta^+ \gamma$ | $(B_1^{(')} - \frac{3}{4} \bar{B}_1^{(')} s_c) s_c/\sqrt{2}$ | $-\frac{1}{2} \bar{B}_s_c/\sqrt{2}$ |
| $\Sigma^0 \rightarrow \Delta^0 \gamma$ | $(B_1^{(')} - \frac{3}{4} \bar{B}_1^{(')} s_c) s_c$ | $-\frac{1}{2} \bar{B}_s_c$ |
| $\Xi^0 \rightarrow \Sigma^0 \gamma$ | $(B_1^{(')} + \frac{3}{4} \bar{B}_1^{(')} s_c) s_c/\sqrt{2}$ | $\frac{1}{2} \bar{B}_s_c/\sqrt{2}$ |
| $\Omega_0 \rightarrow \Xi^0 \gamma$ | $(B_1^{(')} + \frac{3}{4} \bar{B}_1^{(')} s_c)$ | $\frac{3}{4} \bar{B}_s_c$ |
| doubly Cabibbo suppressed $T_{c6} \rightarrow T_{10\gamma}$: | | |
| $\Sigma^+ \rightarrow \Sigma^+ \gamma$ | $\bar{B}_1^{(')} s_c^{(')}/\sqrt{2}$ | $\bar{B}_s_c^{(')}/\sqrt{2}$ |
| $\Sigma^0 \rightarrow \Sigma^0 \gamma$ | $\bar{B}_1^{(')} s_c^{(')}/\sqrt{2}$ | $\bar{B}_s_c^{(')}/\sqrt{2}$ |
| $\Xi^0 \rightarrow \Xi^0 \gamma$ | $\bar{B}_1^{(')} s_c^{(')}/\sqrt{2}$ | $\bar{B}_s_c^{(')}/\sqrt{2}$ |
The SU(3) IRA amplitudes of the $T_{8,10} \rightarrow T'_{8,10}$ weak decays are summarized in Tab. V, in which the information of the same CKM matrix elements $V_{us}V_{ud}^*$ is not shown. From Tab. V, one can see that the amplitudes of $\Xi^- \rightarrow \Sigma^-\gamma$, $\Xi^- \rightarrow \Sigma^*\gamma$, $\Sigma^- \rightarrow \Delta^-\gamma$, $\Omega^- \rightarrow \Xi^-\gamma$, $\Xi^- \rightarrow \Sigma^-\gamma$, $\Omega^- \rightarrow \Xi^*\gamma$, $\Xi^* \rightarrow \Sigma^*\gamma$, $\Sigma^* \rightarrow \Delta^\gamma$ only contain coefficients $c_1^{(i)(ii)(iii)}$, which means that the $W$ exchange transitions don’t contribute to these decays since the initial baryons don’t contain $u$ quark. Otherwise, the $W$ exchange contributions are canceled in $\Xi^0 \rightarrow \Lambda^0\gamma$ and $\Xi^{*0} \rightarrow \Lambda^0\gamma$ decays. So

| Decay modes | $A(T_{8,10} \rightarrow T'_{8,10}\gamma)$ |
|-------------|----------------------------------------|
| $T_8 \rightarrow T'_8\gamma$ weak decays: | |
| $\Xi^- \rightarrow \Sigma^-\gamma$ | $C_1$ |
| $\Xi^0 \rightarrow \Lambda^0\gamma$ | $(C_1 + 2C_2)/\sqrt{6}$ |
| $\Xi^0 \rightarrow \Sigma^0\gamma$ | $(C_1 + 2\tilde{C}_A)/\sqrt{2}$ |
| $\Lambda^0 \rightarrow n\gamma$ | $-(2C_1 + C_2 + 2\tilde{C}_A + \tilde{C}_B)/\sqrt{6}$ |
| $\Sigma^- \rightarrow \Delta^-\gamma$ | $2\sqrt{3}c'_1$ |
| $\Sigma^+ \rightarrow \Delta^+\gamma$ | $-(2c'_1 + \tilde{C}_A)$ |
| $\Lambda^0 \rightarrow \Delta^0\gamma$ | $(\tilde{C}_A + \tilde{C}_B)/\sqrt{6}$ |
| $\Sigma^0 \rightarrow \Delta^0\gamma$ | $-(2c'_1 + \tilde{C}_A)/\sqrt{2}$ |

| Decay modes | $A(T_{10} \rightarrow T'_{10}\gamma)$ |
|-------------|----------------------------------------|
| $T_{10} \rightarrow T'_{10}\gamma$ weak decays: | |
| $\Omega^- \rightarrow \Xi^-\gamma$ | $-c''_1$ |
| $\Xi^0 \rightarrow \Xi^0\gamma$ | $-c''_1/\sqrt{3}$ |
| $\Xi^{*0} \rightarrow \Lambda^0\gamma$ | $-c''_1/\sqrt{2}$ |
| $\Xi^{*0} \rightarrow \Sigma^0\gamma$ | $(c''_1 + 2\tilde{c''}_1)/\sqrt{6}$ |
| $\Sigma^{*0} \rightarrow n\gamma$ | $(c''_1 - \tilde{c''}_1)/\sqrt{6}$ |
| $\Sigma^{*+} \rightarrow p\gamma$ | $(c''_1 + \tilde{c''}_1)/\sqrt{3}$ |

| Decay modes | $A(T_{10} \rightarrow T'_{10}\gamma)$ |
|-------------|----------------------------------------|
| $T_{10} \rightarrow T'_{10}\gamma$ weak decays: | |
| $\Omega^- \rightarrow \Xi^-\gamma$ | $\sqrt{3}c''''_1$ |
| $\Xi^0 \rightarrow \Xi^0\gamma$ | $c''''_1$ |
| $\Xi^{*0} \rightarrow \Sigma^{*0}\gamma$ | $(c''''_1 + \tilde{c''''}_1)/\sqrt{2}$ |
| $\Sigma^{*+} \rightarrow \Delta^+\gamma$ | $c''''_1 + \tilde{c''''}_1$ |
| $\Sigma^{*0} \rightarrow \Delta^0\gamma$ | $(c''''_1 + \tilde{c''''}_1)/\sqrt{2}$ |
| $\Sigma^{*-} \rightarrow \Delta^-\gamma$ | $\sqrt{3}c''''_1$ |
above decays could be used to explore the short distance and long distance contributions. Other decay amplitudes
contained both \(c_1^+\) and \(c_2^+\) could proceed from the short distance contributions, long distance contributions
and W-exchange contributions.

For the weak \(T_8 \to T'_8 \gamma\) decays, all decay modes expect for \(\Sigma^0 \to n \gamma\) have been measured and paid a lot of attentions,
experimental data are listed in the second column of Tab. VI, and there are longstanding theoretical difficulties to
explain the experimental data of the weak \(T_8 \to T'_8 \gamma\) decays. In the \(T_8 \to T'_8 \gamma\) weak decays, they may decay via the
\(s \to d \gamma\) single quark emission, corresponding long distance effects and the W-exchange transition. Since the quarks
are antisymmetric in both the initial states \(T_8\) and the final states \(T'_8\), there are more independent parameters than
ones in \(T_{k,3,c3} \to T_8 \gamma\) weak decays. The relevant SU(3) flavor parameters could be complex, and we set \(C_1 = \text{real}
\) and add relative phases \(\delta_1, \delta_1, \delta_1, \delta_1\) for \(C_2, \tilde{C}_A\) and \(\tilde{C}_B\), respectively. And then \(7\) independent parameters given by
\[
C_1, C_2 e^{i \delta_{12}}, \tilde{C}_A e^{i \delta_{1A}}, \tilde{C}_B e^{i \delta_{1B}}.
\] (28)

Two cases for the \(T_8 \to T'_8 \gamma\) weak decays will be considered in our analysis. In case \(S_A\), we will only consider the
\(s \to d \gamma\) single quark emission and long distance effects, i.e., set \(\tilde{C}_A = \tilde{C}_B = 0\). In case \(S_B\), we will consider all effects.

In case \(S_A\), there are \(3\) independent parameters \(C_1\) and \(C_2 e^{i \delta_{12}}\). Firstly, we use three data of \(B(\Xi^- \to \Sigma^- \gamma)\), \(B(\Lambda^0 \to n \gamma)\) and \(B(\Sigma^+ \to p \gamma)\) to constrain the parameters as well as obtain that \(C_1 = 32.27 \pm 6.23, C_2 = 48.96 \pm 9.74\) and \(\delta_{12} = (0.08 \pm 21.58)^\circ\), and noted that \(C_2\) is slightly larger than \(C_1\). Then, we use the obtained \(C_1, C_2\) and \(\delta_{12}\) to predict other three branching ratios, and the results are given in the third column of Tab. VI. One can see that the predictions of \(B(\Xi^0 \to \Lambda \gamma)\) and \(B(\Xi^0 \to \Sigma^0 \gamma)\) in \(S_A\) are not inconsistent with their data. In case \(S_A\), since \(\frac{B(\Xi^0 \to \Sigma^0 \gamma)}{B(\Xi^0 \to \Sigma^- \gamma)} \approx 1.8\), one have that \(\frac{B(\Xi^0 \to \Sigma^0 \gamma)}{B(\Xi^0 \to \Sigma^- \gamma)} \approx 0.9\), which are far away from the experimental one \(\frac{B_{exp}(\Xi^0 \to \Sigma^0 \gamma)}{B_{exp}(\Xi^0 \to \Sigma^- \gamma)} \approx 26.2\). In addition, the SU(3) IRA prediction of \(B(\Xi^0 \to \Lambda \gamma)\) is about \(2.6\) times larger than its data. So it’s necessary to considering the \(W\) exchange contributions.

In case \(S_B\), we use all five data of the branching ratios to constrain seven parameters. We obtain that \(C_1 = 28.04 \pm 8.77, C_2 = 41.32 \pm 21.95, \tilde{C}_A = 77.82 \pm 24.29, \tilde{C}_B = 45.82 \pm 45.48, \delta_{12} = (-3.05 \pm 176.86)^\circ, \delta_{1} = (4.30 \pm 171.46)^\circ\) and \(\delta_{1B} = (-0.03 \pm 179.36)^\circ\). One can see that, after satisfying all present data within \(2\sigma\), three phases \(\delta_{12}, \delta_{1A}, \delta_{1B}\) are almost unlimited, and the \(C_1, C_2\) and \(\tilde{C}_A\) terms give the same magnitude contributions. \(\tilde{C}_B\) lies in \([0.33, 91.30]\), and its contribution might be similar to (or smaller than) ones of \(C_1, C_2\) and \(\tilde{C}_A\).

| Observables | Experimental data [5] | Our predictions in \(S_A\) | Our predictions in \(S_B\) |
|-------------|-----------------------|--------------------------|--------------------------|
| \(T_8 \to T'_8 \gamma\); | | | |
| \(B(\Xi^- \to \Sigma^- \gamma)(\times 10^{-4})\) | 1.27 \pm 0.23 | 1.67 \pm 0.06 | 1.27 \pm 0.46 |
| \(B(\Xi^0 \to \Lambda \gamma)(\times 10^{-3})\) | 1.17 \pm 0.07 | 3.05 \pm 0.29 | 1.17 \pm 0.14 |
| \(B(\Xi^0 \to \Sigma^0 \gamma)(\times 10^{-3})\) | 3.33 \pm 0.10 | 0.14 \pm 0.01 | 3.33 \pm 0.20 |
| \(B(\Lambda^0 \to n \gamma)(\times 10^{-3})\) | 1.75 \pm 0.15 | 1.48 \pm 0.03 | 1.75 \pm 0.30 |
| \(B(\Sigma^+ \to p \gamma)(\times 10^{-3})\) | 1.23 \pm 0.05 | 1.27 \pm 0.06 | 1.23 \pm 0.10 |
| \(B(\Sigma^0 \to n \gamma)(\times 10^{-13})\) | \(\ldots\) | 6.12 \pm 1.35 | 3.146 \pm 3.143 |
In both $S_A$ and $S_B$ cases, the branching ratio of $\Sigma^0 \to n\gamma$ is too small to see in the experiments, and this weak decay completely overwhelmed by the simpler electromagnetic decay $\Sigma^0 \to \Lambda^0\gamma$.

For the baryon decuplet radiative weak decays, only $\Omega$ baryon has a sufficiently long lifetime, is accessible to experimental study. Using the experimental upper limit $B(\Omega^- \to \Xi\gamma) < 4.60 \times 10^{-4}$, we obtain that $B(\Xi^{-} \to \Sigma^{-}\gamma) < 1.82 \times 10^{-16}$ and $B(\Xi^0 \to \Lambda^0\gamma) < 3.53 \times 10^{-16}$. The upper limits of the two branching ratios are very tiny since $\Xi^{0,-}$ have very short lifetime, so the $\Xi^{-} \to \Sigma^{-}\gamma$ and $\Xi^0 \to \Lambda^0\gamma$ decays are difficult to be measured in the experiments.

D. $T_{10} \to T_8\gamma$ electromagnetic radiative decays

In addition, the baryon decuplet $T_{10}$ also can decay only through electromagnetic interactions by the $q_n \to q_{n'}\gamma (n = n')$ transition at the quark level. The SU(3) IRA amplitudes of the $T_{10} \to T_8\gamma$ electromagnetic decays can be parameterized as

$$A^E(T_{10} \to T_8\gamma) = d_1(T_{10})^i{}^j{}^n(T_8)[i'n'j].$$

Three cases are considered in calculating the electromagnetic decay amplitudes. $S_1$: assuming all three quarks in $T_{10}$ baryon can emit photon, $S_2$: assuming $d$ and $s$ quarks in $T_{10}$ baryon can emit photon, and $S_3$: assuming the heaviest quark in $T_{10}$ baryon can emit photon. The SU(3) IRA amplitudes of $T_{10} \to T_8\gamma$ electromagnetic decays in three cases are listed in Tab. VII.

For $T_{10} \to T_8\gamma$ electromagnetic decays, $B(\Sigma^{++} \to \Sigma^{+}\gamma)$ and $B(\Sigma^0 \to \Lambda^0\gamma)$ have been measured, $B(\Xi^* \to \Xi\gamma)$ and $B(\Sigma^{*-} \to \Sigma^{-}\gamma)$ have been upper limited, and the relevant experimental data are listed in the second column of Tab. VIII. Comparing the amplitudes in three cases with the data, the $S_1$ and $S_3$ cases are eliminated, and we will use the IRA amplitudes in $S_2$ case in the following analysis.

Using Eq. (10), the branching ratios will be obtained by $\mathcal{M}(T_{10} \to T_8\gamma) = A^E(T_{10} \to T_8\gamma)$, and the results are listed in the last column of Tab. VIII. The SU(3) IRA predictions for $B(\Sigma^{++} \to \Sigma^{+}\gamma)$ and $B(\Sigma^0 \to \Lambda^0\gamma)$ are quite consistent with present data. Noted that the estimated result $B(\Delta \to N\gamma) = (5.5 - 6.5) \times 10^{-3}$ from PDG [5], which

| Decay modes | $A^E(T_{10} \to T_8\gamma)$ in $S_1$ | $A^E(T_{10} \to T_8\gamma)$ in $S_2$ | $A^E(T_{10} \to T_8\gamma)$ in $S_3$ |
|-------------|-------------------------------------|-------------------------------------|-------------------------------------|
| $\Xi^{*-} \to \Xi^{-}\gamma$ | 0 | 0 | $-2d_1/\sqrt{3}$ |
| $\Xi^0 \to \Xi^0\gamma$ | 0 | $-2d_1/\sqrt{3}$ | $-2d_1/\sqrt{3}$ |
| $\Delta^+ \to p\gamma$ | 0 | $2d_1/\sqrt{3}$ | $2d_1/\sqrt{3}$ |
| $\Delta^0 \to n\gamma$ | 0 | $2d_1/\sqrt{3}$ | $2d_1/\sqrt{3}$ |
| $\Sigma^{*-} \to \Sigma^{-}\gamma$ | 0 | 0 | $2d_1/\sqrt{3}$ |
| $\Sigma^0 \to \Lambda^0\gamma$ | 0 | $-d_1$ | 0 |
| $\Sigma^0 \to \Sigma^0\gamma$ | 0 | $-d_1/\sqrt{3}$ | $-2d_1/\sqrt{3}$ |
| $\Sigma^{++} \to \Sigma^{+}\gamma$ | 0 | $-2d_1/\sqrt{3}$ | $-2d_1/\sqrt{3}$ |
is covered by our prediction. The branching ratio predictions are at the order of $10^{-3}$, and they might be measured at the BESIII or LHC experiments in near future. So these $T_{10} \rightarrow T_8 \gamma$ electromagnetic decays could be used to test the SU(3) flavor symmetry.



| Observables                  | Experimental data [5] | Our predictions |
|------------------------------|------------------------|------------------|
| $\mathcal{B}(\Xi^+ \rightarrow \Sigma\gamma)\times10^{-3}$ | $<37$                  | $2.64 \pm 1.06$ |
| $\mathcal{B}(\Sigma^{\ast+} \rightarrow \Sigma^+\gamma)\times10^{-3}$ | $7.0 \pm 1.7$          | $6.48 \pm 2.80$ |
| $\mathcal{B}(\Sigma^{\ast 0} \rightarrow \Lambda^0\gamma)\times10^{-2}$ | $1.25^{+0.13}_{-0.12}$ | $1.26 \pm 0.25$ |
| $\mathcal{B}(\Sigma^{\ast 0} \rightarrow \Sigma^0\gamma)\times10^{-3}$ | $\ldots$               | $1.57 \pm 0.31$ |
| $\mathcal{B}(\Sigma^{\ast -} \rightarrow \Sigma^-\gamma)\times10^{-4}$ | $<2.4$                 | $0$              |
| $\mathcal{B}(\Delta^0 \rightarrow n\gamma)\times10^{-3}$ | $\ldots$               | $8.25 \pm 3.64$ |
| $\mathcal{B}(\Delta^+ \rightarrow p\gamma)\times10^{-3}$ | $\ldots$               | $8.36 \pm 3.68$ |



IV. Conclusions

Baryon radiative decays give us a chance to study the interplay of the electromagnetic, weak and strong interactions. Some baryon radiative decay modes have been measured and some others could be studied at BESIII, LHCb and Belle-II experiments. In this work, we have analyzed baryon radiative decays of the octet $T_8$, decuplet $T_{10}$, single charmed anti-triplet $T_{c3}$, single charmed sextet $T_{c6}$, single bottomed anti-triplet $T_{b3}$ and single bottomed sextet $T_{b6}$ by using the irreducible representation approach to test the SU(3) flavor symmetry. Our main results are given in order:

- $T_{b3,b6}$ weak radiative decays:
  Each kind of the decay amplitudes can be related by only one parameter in the $T_{b3} \rightarrow T_8 \gamma$, $T_{b6} \rightarrow T_8 \gamma$ and $T_{b6} \rightarrow T_{10} \gamma$ weak decays via $b \rightarrow s/d \gamma$ as well as the $T_{b3} \rightarrow T_{c3} \gamma$, $T_{b3} \rightarrow T_{c6} \gamma$, $T_{b6} \rightarrow T_{c3} \gamma$ and $T_{b6} \rightarrow T_{c6} \gamma$ via the $W$ exchange transitions. Using the only measured $\mathcal{B}(\Lambda_b^0 \rightarrow \Lambda \gamma)$, we have predicted other six decay branching ratios of $T_{b3} \rightarrow T_8 \gamma$ weak decays, and they might be measured by the experiments in near future. Unfortunately, none of $T_{b6} \rightarrow T_{8,10} \gamma$ and $T_{b3,b6} \rightarrow T_{c3,c6} \gamma$ weak decays has been measured at present. Any measurement of $T_{b6} \rightarrow T_{8,10} \gamma$ will give us chance to predict many other decay branching ratios.

- $T_{c3,c6}$ weak radiative decays:
  $T_{c3,c6}$ weak radiative decays are quite different from $T_{b3,b6}$ weak radiative decays, they may receive the contributions of both the $c \rightarrow u \gamma$ and the $W$ exchange transitions. After ignoring the GIM strongly suppressed $c \rightarrow u \gamma$ transition contributions and the Wilson Coefficient suppressed $H(15)$ term contributions, the decay amplitudes of $T_{c3} \rightarrow T_8 \gamma$, $T_{c6} \rightarrow T_8 \gamma$ and $T_{c6} \rightarrow T_{10} \gamma$ are also related by only one parameter $\tilde{B}$, $\tilde{B}'$ and $\tilde{B}''$, respectively. Just none of $T_{c3,c6}$ weak radiative decays has been measured at present.
• $T_{8,10}$ weak radiative decays:
As for $T_{8,10}$ weak radiative decays, some decays only receive the short distance and long distance $s \rightarrow d\gamma$ transition contributions, other decays could receive both the $s \rightarrow d\gamma$ transition and the $W$-exchange transition contributions. For the $T_8 \rightarrow T_8\gamma$ weak decays, all decay modes expect for $\Sigma^0 \rightarrow n\gamma$ have been measured, and we have found that only considering the short and long distance $s \rightarrow d\gamma$ transition contributions can not explain all current data by SU(3) IRA. Present all data could be explained by considering both the $s \rightarrow d\gamma$ transition contributions and the $W$-exchange contributions. $B(\Sigma^0 \rightarrow n\gamma)$ has been predicted, just this branching ratio is very tiny. For the $T_{10} \rightarrow T_8\gamma$ weak decays, we have used the upper limit of $B(\Omega^- \rightarrow \Xi\gamma)$ to obtain the upper limit predictions of $B(\Xi^+ \rightarrow \Sigma^-\gamma)$ and $B(\Xi^0 \rightarrow \Lambda^0\gamma)$, just both upper limit predictions are tiny.

• $T_{10} \rightarrow T_8\gamma$ electromagnetic radiative decays:
All decay amplitudes of $T_{10} \rightarrow T_8\gamma$ electromagnetic radiative decays could be related by only one parameter $d_1$, the SU(3) IRA predictions for $B(\Sigma^{*+} \rightarrow \Sigma^{+}\gamma)$ and $B(\Sigma^{*0} \rightarrow \Lambda^0\gamma)$ are quite consistent with present data. Other branching ratio predictions are at the order of $10^{-3}$, and they might be measured by the experiments in near future.

Flavor SU(3) symmetry could provide us very useful information about the decays. According to our predictions, some branching ratios are accessible to the experiments at BESIII, LHCb and Belle-II. Our results in this work can be used to test SU(3) flavor symmetry approach in the radiative baryon decays by the future experiments.

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