Frequency distributions have played an important part in many investigations in very different contexts. For example, distribution of income, frequency of words in a long text, forest fires, scientific citations, world-wide web surfing, ecology, solar flares, population distribution in large cities, economic index, epidemics in isolated populations, among others. In a comparative study, the q-exponential and Weibull distributions are employed to investigate frequency distributions of basketball baskets, cyclone victims, brand-name drugs by retail sales, and highway length. In order to analyze the intermediate cases, a distribution, the q-Weibull one, which interpolates the q-exponential and Weibull ones, is introduced. It is verified that the basketball baskets distribution is well described by a q-exponential, whereas the cyclone victims and brand-name drugs by retail sales ones are better adjusted by a Weibull distribution. On the other hand, for highway length the q-exponential and Weibull distributions do not give satisfactory adjustment, being necessary to employ the q-Weibull distribution. Furthermore, the introduction of this interpolating distribution gives an illumination from the point of view of the stretched exponential against inverse power law (q-exponential with $q > 1$) controversy.

PACS numbers: 89.20.-a, 89.65.-s, 89.90.+n

I. INTRODUCTION

One of the simplest generalizations of a power law which leads $\int_0^\infty p(x)dx$ to a finite value is the Zipf-Mandelbrot law, $p_{zm}(x) = p_0/(1 + sx)^q$ ($s > 0$), which was employed by Mandelbrot to investigate frequency of words in long texts. More recently, this distribution was generalized in order to incorporate negative $\alpha$ values, i.e.,

$$p_q(x) = \frac{p_0}{x_0} \exp_q \left( -\frac{x}{x_0} \right), \quad (1)$$

where $\exp_q(x) \equiv [1 - (1 - q)x]^{1/(1-q)}$ if $1 - (1 - q)x \geq 0$ and $\exp_q(x) \equiv 0$ if $1 - (1 - q)x < 0$. Thus, for $q > 1$ we have a long-tailed function since $p_q(x) \propto x^{-1/(q-1)}$ for $x \gg x_0/(q-1)$; for $q < 1$ we obtain a short-tailed function since $p_q(x) = 0$ for $x \geq x_0/(1-q)$; and for $q = 1$ we have an exponential since $\lim_{q \to 1} \exp_q(x) = \exp(-x)$. Note also that a convenient identification of the parameters in Eq. (1) immediately leads to the Zipf-Mandelbrot law when $q > 1$. This distribution for $q > 1$ has been employed, for instance, in connection with scientific citations, goal distribution, and population distribution in a country.

Eq. (1) can be viewed as a generalization of the exponential function since it basically replaces the exponential one in the canonical ensemble of the Tsallis statistics. More precisely, $p_q(x)$ can be obtained from a maximum entropic principle. These facts motivate us to refer to $p_q(x)$ as q-exponential distribution. We remark that the Tsallis statistics has been employed in connection with several anomalous systems (for a recent review see Ref. 1).

The Weibull distribution

$$p_w(x) = \frac{p_0}{x_0^r} \exp \left[ -\left( \frac{x}{x_0} \right)^r \right] \quad (2)$$

has been employed to accomplish a systematic curvature in a log-log graphics. The parameter $r$ is taken positive and when $r = 1$ Eq. (2) reduces to an exponential.

The distribution (2) was firstly employed in a statistical theory of the strength of material, and has been used in many contexts such as dielectric failure, cardiac contraction, aspects of road accident death, wind speed, profile of foliage area, porcelain strength, among others. In particular, this distribution has been related with multiplicative process by Laherrere and Sornette, who refers to $p_w(x)$ as stretched exponential.

It must be stressed that the parameters $x_0$ and $r$ control the tail behavior of $p_w(x)$ as well as $x_0$ and $q$ dictate the $p_q(x)$ shape. Furthermore, if we restrict our analysis to a finite range for $x$ it is very common that $p_w(x)$ and $p_q(x)$ indistinctly adjust a restrict set of data very well. Thus, it is natural to investigate whether distribution, $p_w(x)$ or $p_q(x)$, gives a better adjustment for a large set of data. This work is addressed to investigate this question by focusing the frequency distributions of basketball baskets in a championship, tropical cyclone victims,
brand-name drugs by retail sales, and highway length. In particular, for the highway length distribution neither the $q$-exponential nor the Weibull ones lead to a satisfactory adjustment. In order to accommodate this case and the others in a unified framework, we introduce a distribution which smoothly interpolates the $q$-exponential and the Weibull ones. Moreover, this approach gives an illumination from the point of view of the stretched exponential against inverse power law ($q$-exponential with $q > 1$) controversy.

II. $q$-WEIBULL DISTRIBUTION

We refer to this interpolating function as $q$-Weibull distribution, and it is given by

$$p_{qw}(x) = p_0 \frac{r}{x_0^r} x^{r-1} \exp_q \left[ - \left( \frac{x}{x_0} \right)^r \right].$$

Note that for $r = 1$ and $q \neq 1$ we recover the $q$-exponential. Analogously, in the limit $q \to 1$ with $r \neq 1$ or $r = 1$, we obtain the Weibull or the exponential distributions, respectively. Furthermore, when $q > 1$ and $x \gg x_0/(q - 1)^{1/r}$, we verify that $p_{qw}(x) \propto x^{-u}$ with $u = r[(2 - q)/(q - 1)] + 1$. In contrast, when $q < 1$ and $x \geq x_0/(1 - q)^{1/r}$, we have $p_{qw}(x) = 0$.

In order to obtain a sufficiently smooth curve, it is a common practice to employ the cumulative distribution

$$R(x) = \int_x^\infty p(y) dy$$

(4)

to investigate the frequency distribution for a set of data. In our case, we obtain

$$R_{qw}(x) = p'_0 \exp_q \left[ - \left( \frac{x}{x_0} \right)^{q'} \right],$$

(5)

with $q' = 1/(2 - q)$, $x'_0 = x_0/(2 - q)^{1/r}$, and $p'_0 = p_0/(2 - q)$. It must be emphasized that $R_{qw}(x)$ exists only if $q < 2$. Moreover, from our construction, the cumulative distributions, $R_q(x)$ or $R_w(x)$, follow from Eq. (4) by taking the appropriate limits ($r \to 1$ or $q \to 1$). Therefore, the cumulative distribution of a power law is a power again, of a $q$-exponential is another $q$-exponential, of a Weibull distribution is a stretched exponential, and of a $q$-Weibull is a $q$-stretched.

Before starting our empirical investigation of data by using these distributions, we would like to remark that the functions $R_q(x)$, $R_w(x)$, and $R_{qw}(x)$ can be related with diffusion equations. In fact, $R_q(x)$ satisfies the generalized decay equation $dR_q(x)/dx = -a[R_q(x)]^q$ with $a = (1 - q)p_0^{1-q}/x'_0$; $R_w(x)$ is intimately related with the point source solution of the anomalous diffusion proposed by O’Shaughnessy and Procaccia; and $R_{qw}(x)$ occurs in the solution of the nonlinear diffusion equation proposed in Ref. 27. Moreover, $R_{qw}(x)$ obeys the generalized decay equation $dR_{qw}(x)/dx = -b x^{r-1}[R_{qw}(x)]^q$ with $b = r(1 - q')p_0^{1-q'}/x'_0^r$.

III. EMPIRICAL ANALYSIS

In this section, we are going to apply the $q$-exponential, Weibull, and $q$-Weibull distributions to analyze the data about basketball baskets in a championship, tropical cyclone victims, brand-name drugs by retail sales, and highway length. To obtain the best set of values for the parameters, we use the least square minimum method. Complementary, as an additional tool in the analysis of how good the adjustment is, we use the residual error analysis. In this kind of analysis, a good adjustment is related to a gaussian residual frequency distribution and a random distribution of errors.

Naturally, since the $q$-Weibull distribution the $q$-exponential or the Weibull ones has as limit case; in general, it gives a better adjustment than the two others. However,
there are cases where the adjustment with the \( q \)-Weibull distribution does not give a significative improvement when compared with that obtained from one of the two other distributions. In such cases, either the parameter \( r \) is close to one leading to a \( q \)-exponential distribution or the parameter \( q \) is close to one leading to a Weibull distribution. As we are going to verify below, there are cases where such limits are not a good approximation, therefore indicating that the \( q \)-Weibull distribution gives a significative improvement in the adjustment. Now we describe the systems to be analyzed.

It is natural to investigate aspects related to sports since it has attracted interest of a large number of people in our society. In fact, many studies concerning sports have been developed. For instance, those related to frequency distributions in soccer [13], baseball [28, 29, 30], golf [29], hockey [29, 30], football [29, 30], and basketball [29, 31]. Here, we concentrate our attention on basketball. More precisely, we consider the distribution of two-point basket among approximately four hundred players of 1999 NBA championship [32].

In our analysis about basketball baskets, we verified that the \( q \)-exponential function gives a better adjustment than the Weibull distribution since the parameter \( r \), obtained from the \( q \)-Weibull one, is close to one (\( r = 0.91 \)). The basketball basket cumulative distribution, as well as the corresponding fits, is presented in Fig. 1. Table I contains the parameters employed in the fits presented in Fig. 1. We remark that in the previous examples about \( q \)-exponential distribution (Refs. [12, 13, 14]) the parameter \( q \) is bigger than one, in contrast with the present example where \( q = 0.68 \). As far as we are concerned this is the first system whose adjustment leads to a \( q \)-exponential distribution with \( q < 1 \). Thus, our adjustment suggests the existence of a cutoff for the maximum number of baskets \( x_\rho \) for 1999 NBA championship. This \( x_\rho = x^\prime_\rho/(1 - q^\prime) = 9.9 \times 10^2 \) is compatible with the true maximum number of baskets, 839. In contrast to this case, now we focus on systems, which are well described by a Weibull distribution.

Sudden inundations, such as those due to cyclones, are known as the most significative natural catastrophe [33] leading along the years to massive death and loss of property. Studies related with such phenomena enclose the spatial and temporal distribution of cyclone for a given region [34]. Furthermore, attempts to identify the climatologic process that leads to cyclone formation have employed meteorologic satellites [35]. However, since such technology is only recently disposable, it is difficult to investigate cyclones that have occurred for a long time. Thus, in order to analyze at least in part the effects of cyclones, including those occurred many years ago, we consider the Atlantic tropical fatal victims number from 1492 to 1999 [36]. In this period, two hundred and sixty-three Atlantic tropical cyclones with deaths were catalogued.

| System       | \( q \)-exponential | Weibull | \( q \)-Weibull |
|--------------|---------------------|---------|----------------|
| Basketball   | 0.68                | -       | 0.51           |
|             | \( r \)              | -       | 1.15           |
|             | \( x_0 \)            | 313     | 212            |
|             | \( p_0 \)            | 520     | 387            |
| Cyclone     | 1.38                | -       | 0.95           |
|             | \( r \)              | -       | 0.23           |
|             | \( x_0 \)            | 24.9    | 4.41           |
|             | \( p_0 \)            | 145     | 1.14 \times 10^3 |
| Drug        | 1.39                | -       | 0.88           |
|             | \( r \)              | -       | 0.35           |
|             | \( x_0 \)            | 1.30 \times 10^4 | 7.03 \times 10^2 |
|             | \( p_0 \)            | 266     | 1.41 \times 10^3 | 2.49 \times 10^3 |
| Highway     | 1.12                | -       | 0.077         |
|             | \( r \)              | -       | 0.87           |
|             | \( x_0 \)            | 608     | 712            |
|             | \( p_0 \)            | 181     | 215            | 465           |

FIG. 3: Cumulative distribution of brand-name drugs by retail sales. The parameters employed in \( R(x) \) [Eq. 5] are obtained from those presented in Tab. I.

FIG. 4: Length highway cumulative distributions in USA. The parameters employed in \( R(x) \) [Eq. 5] are obtained from those presented in Tab. I.
In the analysis of cumulative distribution of deaths as a consequence of tropical cyclones, the adjustment with a q-Weibull distribution leads to q close to one \((q = 0.95)\). This indicates that Weibull distribution can be in a good agreement with the data. In Fig. 3, we plot the cumulative distribution of cyclone fatal victims versus the number of deaths related to it by using the three distributions employed in this work. In Tab. 1, we show the correspondent parameters.

An important part of the expense in our society is invested in health. In this context, the use of pharmaceutical services plays a special role. This decisively contributes to an enormous competition among pharmacute industries, motivating the appearance of new drugs every year. For these reasons, studies have been performed, for instance, in the relationship between pharmaceutical expenditures and income. In this work, we investigate another aspect, the distribution of brand-name drugs. Here we are considering the top two hundred brand-name drugs by retail sales in 1999 in USA. As the above system, the adjustment with q-Weibull gives q close to one \((q = 0.88)\), so the Weibull distribution leads to a better adjustment than the q-exponential one. This fact can be visualized in Fig. 5 and the optimal parameters are presented in Tab. 1.

In the above systems, the q-Weibull distribution does not give a significative improvement since the systems can be well described by q-exponential or Weibull ones. But this may not occur in general. This is just the case of the following example.

In general, roads strategically connect different parts of a country. In particular, understanding the myriad of aspects related with road transport is of great value since the economy of a region is strongly connected with the transport infrastructure. Possible aspects related to road transport are the study of mortality and the morbidity due to road accidents and the influence of the length of the road segment in the statistical description of accident counts and density. Here, we focus attention on the distribution of the highway length, by analyzing one hundred and ninety-eight USA highways. In Fig. 4, we can see that neither q-exponential nor Weibull distributions are able to give a satisfactory adjustment. In this system, the interpolating q-Weibull distribution is necessary to have a good fit. In fact, the parameters \(r\) and \(q\) are not close to one \((r = 0.57\) and \(q = 0.077\)).

In every system, we had applied the mean square minimum method to obtain the optimal parameters. We also used the residual errors analysis to confirm the goodness of the adjustments. To exemplify such procedure, we present the residual analysis to highway length distribution in Fig. 6. The graphic of frequency distribution of residues and the graphic of predicted versus residual values to q-Weibull distribution [Fig. 5(c)] are better than the corresponding ones referent to q-exponential [Fig. 5(a)] and Weibull distributions [Fig. 5(b)]. In fact, the histogram for the residual errors seems closer to a gaussian for q-Weibull distribution than those ones to q-exponential and Weibull distributions. The analogous improvement occurs in the randomness observed in the graphics of predicted versus residual values.

An alternative distribution that generalizes the q-exponential one is the distribution that arises from the differential equation

\[
\frac{dp}{dx} = -\mu p^r - (\lambda - \mu)p^q,
\]

where \(q, r, \mu, \lambda\) and the initial condition \(p_0\) are real parameters to be adjusted. The q-exponential distribution is recovered if \(\mu = 0\) or if \(r = q\). However, \(p(x)\) obtained from Eq. 6 does not contain the Weibull distribution as particular case, desirable feature since, as emphasized in this work, many systems are well adjusted by the Weibull distribution. The fact that Eq. 6 does not contain the Weibull one as a particular case is clear since it satisfies the differential equation

\[
\frac{dp}{dx} = \left[\frac{(c - 1)}{x} - \frac{ce^{-x/c}}{x_0^c}\right] p.
\]

In addition, we mention some practical difficulties in the full application of a distribution dictated by Eq. 6. First, Eq. 6 does not have analytical solutions for all \(r\) and \(q\), so we need to employ a numerical approach. Second, it contains five parameters to be adjusted (one more than q-Weibull). Due to these facts, the analysis and adjustment become cumbersome. On the other hand, when the distribution exhibits two slopes, the solution of Eq. 6 seems appropriate since neither q-exponential nor q-Weibull is able to reproduce this behavior. This characteristic has been explored in the study of folded proteins. Another possibility of application of Eq. 6 occurs when the distribution presents a well definite slope in the intermediate region. This situation can be verified in linguistics, when investigating deviation from the Zipf-Mandelbrot law. Beside these difficulties (five parameters and numerical solution of Eq. 6), we have investigated the adjustment of our data by using \(p(x)\) from Eq. 6. We mention first that standard programs of adjustment can not be used since we do not have an analytical form to the distribution from Eq. 6. Furthermore, due to the large number of parameters, a method that not only does not strongly depend on initial set of parameters but also presents fast convergence is necessary. In this direction, we applied the simulated annealing method, as employed in Ref. 14, to obtain a good set of parameters. However, even this method was not good enough to lead to a satisfactory result.
IV. CONCLUSION

In several areas of nature Weibull and Zipf-Mandelbrot (\(q\)-exponential with \(q > 1\)) have an important role to describe many frequency distributions. For a short range, both distributions can lead to an apparent agreement. Thus, to decide which distribution gives a better description, it is necessary to consider a sufficiently large range. In this work, in order to investigate the referred controversy, we focalized systems just taking a sufficiently large range of data into account. Moreover, as to obtain a further illumination on this controversy, we introduce a new distribution, \(q\)-Weibull. Such distribution slowly interpolates the \(q\)-exponential and the Weibull ones. When either the \(q\)-exponential or the Weibull distribution is able to give a good description of a system, its respective parameter, \(r\) or \(q\), is close to one. We applied \(q\)-Weibull function to analyze distributions of basketball baskets, cyclone victims, brand-name drugs by retail sales, and highway length. We verified that the basketball baskets distribution is well described by a \(q\)-exponential with \(q < 1\). For cyclone victims and brand-name drugs by retail sales, the Weibull distribution gives a good adjustment. On the other hand, for length of highway, \(q\)-Weibull distribution gives a satisfactory adjustment. We hope that this new distribution can be useful in other situations where neither \(q\)-exponential nor Weibull ones leads to a satisfactory result.

We thank CAPES and CNPq (Brazilian agencies) for partial financial support.
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