BOWEN ENTROPY FOR ACTIONS OF AMENABLE GROUPS

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Abstract. Bowen introduced a definition of topological entropy of subset inspired by Hausdorff dimension in 1973 [1]. In this paper we consider the Bowen’s entropy for amenable group action dynamical systems and show that under the tempered condition, the Bowen entropy of the whole space equals to the topological entropy for a given Følner sequence. For the proof of this result, we establish a variational principle related to the Bowen entropy and the Brin-Katok’s local entropy formula for amenable group action dynamical systems.

1. Introduction

Let \((X, G)\) be a \(G\)-action topological dynamical system, where \(X\) is a compact metric space with metric \(d\) and \(G\) a topological group. In this paper, we assume \(G\) is a discrete countable amenable group. Recall that the group \(G\) is amenable if it admits a left invariant mean (a state on \(\ell^\infty(G)\) which is invariant under left translation by \(G\)). This is equivalent to the existence of a sequence of finite subsets \(\{F_n\}\) of \(G\) which are asymptotically invariant, i.e.,

\[
\lim_{n \to +\infty} \frac{|F_n \triangle gF_n|}{|F_n|} = 0, \text{ for all } g \in G.
\]

Such sequences are called Følner sequences. For the detail of amenable group actions, one may refer to Ornstein and Weisss pioneering paper [9].

The topological entropy of \((X, G)\) is defined in the following way.

Let \(U\) be an open cover of \(X\), the topological entropy of \(U\) is

\[
h_{\text{top}}(G, U) = \lim_{n \to +\infty} \frac{1}{|F_n|} \log N(U_{F_n}),
\]

where \(U_{F_n} = \bigvee_{g \in F_n} g^{-1}U\). It is shown that \(h_{\text{top}}(G, U)\) is not dependent on the choice of the Følner sequences \(\{F_n\}\). And The topological entropy of \((X, G)\) is

\[
h_{\text{top}}(X, G) = \sup_{U} h_{\text{top}}(G, U),
\]

where the supremum is taken over all the open covers of \(X\).

Bowen [1] introduced a definition of topological entropy on subsets inspired by Hausdorff dimension. For an amenable group action dynamical system \((X, G)\), we define the Bowen type topological entropy in the following way.

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Let \( \{F_n\} \) be a Følner sequence in \( G \) and \( \mathcal{U} \) be a finite open cover of \( X \). Denote \( \text{diam}(\mathcal{U}) := \max\{\text{diam}(U) : U \in \mathcal{U}\} \). For \( n \geq 1 \) we denote by \( \mathcal{W}_F(U) \) the collection of families \( \mathcal{U} = \{U_g\}_{g \in F_n} \) with \( U_g \in \mathcal{U} \). For \( \mathcal{U} \in \mathcal{W}_F(U) \) we call the integer \( m(\mathcal{U}) = |F_n| \) the length of \( \mathcal{U} \) and define

\[
X(\mathcal{U}) = \bigcap_{g \in F_n} g^{-1}U_g
= \{x \in X : gx \in U_g \text{ for } g \in F_n\}.
\]

For \( Z \subseteq X \), we say that \( \Lambda \subseteq \bigcup_{n \geq 1} \mathcal{W}_F(U) \) covers \( Z \) if \( \bigcup_{U \in \Lambda} X(\mathcal{U}) \supseteq Z \). For \( s \in \mathbb{R} \), define

\[
\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) = \inf_{\Lambda} \left\{ \sum_{U \in \Lambda} \exp(-s m(\mathcal{U})) \right\}
\]

and the infimum is taken over all \( \Lambda \subseteq \bigcup_{j \geq N} \mathcal{W}_F(U) \) that covers \( Z \). We note that \( \mathcal{M}(\cdot, \mathcal{U}, N, s, \{F_n\}) \) is a finite outer measure on \( X \), and

\[
\mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) = \inf_{C} \{ \mathcal{M}(C, \mathcal{U}, N, s, \{F_n\}) : C \text{ is an open set that contains } Z \}.
\]

\( \mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\}) \) increases as \( N \) increases. Define

\[
\mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, \mathcal{U}, N, s, \{F_n\})
\]

and

\[
h_{top}^B(\mathcal{U}, Z, \{F_n\}) = \inf\{s : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = 0\}
= \sup\{s : \mathcal{M}(Z, \mathcal{U}, s, \{F_n\}) = +\infty\}.
\]

Set

\[
h_{top}^B(Z, \{F_n\}) = \sup_{\mathcal{U}} h_{top}^B(\mathcal{U}, Z, \{F_n\}),
\]

where \( \mathcal{U} \) runs over finite open covers of \( Z \). We call \( h_{top}^B(Z, \{F_n\}) \) the Bowen topological entropy of \( (X, G) \) restricted to \( Z \) or the Bowen topological entropy of \( Z \) (w.r.t. the Følner sequence \( \{F_n\} \)).

Similar to the Bowen topological entropy of subsets (see, for example, Pesin [10]), it is easy to show that

\[
h_{top}^B(Z, \{F_n\}) = \lim_{\text{diam}(\mathcal{U}) \to 0} h_{top}^B(\mathcal{U}, Z, \{F_n\}),
\]

where \( \text{diam}(\mathcal{U}) = \sup\{\text{diam}(U) : U \in \mathcal{U}\} \), the diameter of the cover \( \mathcal{U} \). So the Bowen topological entropy can be defined in an alternative way.

For a finite subset \( F \) in \( G \), we denote by

\[
B_F(x, \epsilon) = \{y \in X : d_F(x, y) < \epsilon\}
= \{y \in X : d(gx, gy) < \epsilon, \text{ for any } g \in F\}.
\]

(1.1)

For \( Z \subseteq X, s \geq 0, N \in \mathbb{N}, \{F_n\} \) a Følner sequence in \( G \) and \( \epsilon > 0 \), define

\[
\mathcal{M}(Z, N, \epsilon, s, \{F_n\}) = \inf \sum_{i} \exp(-s|F_n|),
\]
where the infimum is taken over all finite or countable families \( \{B_{F_n}(x_i, \epsilon)\} \) such that \( x_i \in X, n_i \geq N \) and \( \bigcup_i B_{F_n}(x_i, \epsilon) \supseteq Z \). The quantity \( \mathcal{M}(Z, N, \epsilon, s, \{F_n\}) \) does not decrease as \( N \) increases and \( \epsilon \) decreases, hence the following limits exist:

\[
\mathcal{M}(Z, \epsilon, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{M}(Z, N, \epsilon, s, \{F_n\}), \mathcal{M}(Z, s, \{F_n\}) = \lim_{\epsilon \to 0} \mathcal{M}(Z, \epsilon, s, \{F_n\}).
\]

Bowen topological entropy \( h_{\text{top}}^B(Z, \{F_n\}) \) can be equivalently defined as the critical value of the parameter \( s \), where \( \mathcal{M}(Z, s, \{F_n\}) \) jumps from \( \infty \) to 0, i.e.

\[
\mathcal{M}(Z, s, \{F_n\}) = \begin{cases} 
0, & s > h_{\text{top}}^B(Z, \{F_n\}), \\
\infty, & s < h_{\text{top}}^B(Z, \{F_n\}). 
\end{cases}
\]

In [1] Bowen showed that \( h_{\text{top}}(X, T) = h_{\text{top}}^B(X, T) \) for any compact metric dynamical system \( (X, T) \). It nature to ask: Does this result also hold for the amenable group action system \( (X, G) \)?

In this paper, we will prove this under certain condition on the Følner sequences. Although this is a topological problem, our proof uses a measure-theoretic way.

A Følner sequence \( \{F_n\} \) in \( G \) is said to be tempered (see Shulman [11]) if there exists a constant \( C \) which is independent of \( n \) such that

\[
|\bigcup_{k<n} F^{-1}_k F_n| \leq C |F_n|, \text{ for any } n.
\]

(1.2)

In Lindenstrauss [5], (1.2) is also called Shulman Condition.

Now we state our main theorem as follows.

**Theorem 1.1 (Main result).** Let \( (X, G) \) be a compact metric \( G \)-action topological dynamical system and \( G \) a discrete countable amenable group, then for any tempered Følner sequence \( \{F_n\} \) in \( G \) with the increasing condition

\[
\lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty,
\]

we have

\[
h_{\text{top}}^B(X, \{F_n\}) = h_{\text{top}}(X, G).
\]

2. Local entropy and Brin-Katok’s entropy formula

In this section, we will prove Brin-Katok’s entropy formula [2] for amenable group action dynamical systems. The statement of this formula is the following.

**Theorem 2.1 (Brin-Katok’s entropy formula: ergodic case).** Let \( (X, G) \) be a compact metric \( G \)-action topological dynamical system and \( G \) a discrete countable amenable group. Let \( \mu \) be a \( G \)-ergodic Borel probability measure on \( X \) and \( \{F_n\} \) a tempered Følner sequence in \( G \) with the increasing condition \( \lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty \), then for \( \mu \) almost everywhere
$x \in X$,
\[
\lim_{\delta \to 0} \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) = \lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) = h_{\mu}(X, G).
\]

Since this formula gives an alternative definition for metric entropy–local entropy, we give the following definition of local entropy in amenable group action case.

**Definition 2.2.** Let $(X, G)$ be a compact metric $G$–action topological dynamical system and $G$ a discrete countable amenable group. Denote by $M(X)$ the collection of Borel probability measures $X$. For any $\mu \in M(X)$, $x \in X$, $n \in \mathbb{N}$, $\epsilon > 0$ and $\{F_n\}$ any Følner sequence in $G$, denote by
\[
h_{\mu}^{\text{loc}}(x, \epsilon, \{F_n\}) = \lim_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \epsilon)).
\]
Then the (lower) local entropy of $\mu$ at $x$ (along $\{F_n\}$) is defined by
\[
h_{\mu}^{\text{loc}}(x, \{F_n\}) = \lim_{\epsilon \to 0} h_{\mu}^{\text{loc}}(x, \epsilon, \{F_n\})
\]
and the (lower) local entropy of $\mu$ is defined by
\[
h_{\mu}^{\text{loc}}(\{F_n\}) = \int_X h_{\mu}^{\text{loc}}(x, \{F_n\}) d\mu.
\]
Similarly, we can define the upper local entropy.

For the proof of Theorem 2.1 we need the following classic results in ergodic theory for amenable group actions.

Let $(X, G, \mu)$ be a measure-theoretic $G$–action dynamical system where $G$ is a discrete countable amenable group and $\mu$ is a $G$–ergodic probability measure on $X$. The ergodic theorem states that,

**Theorem 2.3** (E. Lindenstrauss [5], see also B. Weiss [13]).

Let $(X, G, \mu)$ be an ergodic $G$–system, $\{F_n\}$ be a tempered Følner sequence in $G$ and $f \in L^1(X, \mathcal{B}, \mu)$. Then
\[
\lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} f(gx) = \int_X f(x) d\mu,
\]
almost everywhere and in $L^1$.

Let $\mathcal{P}$ be a finite measurable partition of $X$. For a finite subset $F$ in $G$, we denote by $\mathcal{P}_F = \bigvee_{g \in F} g^{-1} \mathcal{P}$. Then the classical measure-theoretical entropy of $\mathcal{P}$ is defined by
\[
h_{\mu}(G, \mathcal{P}) = \liminf_{n \to +\infty} \frac{1}{|F_n|} H(\mathcal{P}_{F_n}),
\]
where $\{F_n\}$ is any Følner sequence in $G$ and the definition is independent of the specific Følner sequence $\{F_n\}$. The measure-theoretical entropy of the system $(X, G, \mu)$, $h_{\mu}(X, G)$, is the supremum of $h_{\mu}(G, \mathcal{P})$ over $\mathcal{P}$. 
For \( x \in X \), let \( \mathcal{P}(x) \) denote the atom in \( \mathcal{P} \) that contains \( x \). Now we recall the classical Shannon-McMillan-Breiman theorem for ergodic \( G \)-systems.

**Theorem 2.4** (Shannon-McMillan-Breiman(SMB) theorem, see \([5, 13]\)). Let \( (X, G, \mu) \) be an ergodic \( G \)-system. For any tempered Folner sequence \( \{F_n\} \) in \( G \) with \( \lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty \) and \( \mathcal{P} \) a finite measurable partition of \( X \),

\[
\lim_{n \to +\infty} \frac{1}{|F_n|} \log \mu(\mathcal{P}_{F_n}(x)) = h_\mu(G, \mathcal{P}),
\]

for \( \mu \) almost everywhere \( x \in X \).

Now we give the proof of Brin-Katok’s entropy formula for amenable group action systems.

**Proof of Theorem 2.1**. Let \( h = h_\mu(X, G) \). We first prove

\[
\lim_{\delta \to 0} \limsup_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \leq h_\mu(X, G).
\]

Let \( \delta > 0 \) be given and let \( \xi \) be a finite measurable partition of \( X \) such that the diameter of every set in \( \xi \) is less than \( \delta \). Then by SMB theorem, for \( \mu \)-a.e. \( x \in X \),

\[
\lim_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(\xi_{F_n}(x)) = h_\mu(G, \xi) \leq h_\mu(X, G).
\]

Since \( \xi_{F_n}(x) \subset B_{F_n}(x, \delta) \), we have that

\[
\limsup_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \leq h_\mu(X, G)
\]

for ever \( \delta \).

Now we are to show that

\[
\lim_{\delta \to 0} \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \geq h_\mu(X, G).
\]

For any \( \epsilon > 0 \), let \( \xi \) be a finite measurable partition of \( X \) such that \( h_\mu(G, \xi) = \tilde{h} > h - \epsilon \) and \( \mu(\partial \xi) = 0 \). Then for sufficiently small \( \delta > 0 \), the \( \delta \)-neighboorhood of \( \partial \xi \) (denoted by \( U_\delta \)) can have measure less than \( \epsilon \). By The ergodic theorem 2.3, \( \frac{1}{|F_n|} \sum_{g \in F_n} \chi_{U_\delta}(gx) \) converges to \( \mu(U_\delta) \) a.e.. Since

\[
\{x \in X : \lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \chi_{U_\delta}(gx) = \mu(U_\delta)\}
\]

\[
\subset \{x \in X : \lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \chi_{U_\delta}(gx) < \epsilon\}
\]

\[
= \bigcup_{N \in \mathbb{N}} \{x \in X : \forall n > N, \frac{1}{|F_n|} \sum_{g \in F_n} \chi_{U_\delta}(gx) < \epsilon\}
\]

\[
\triangleq \bigcup_{N \in \mathbb{N}} E_N
\]
and $E_N$ increases, for sufficiently large $N$, whence $n > N$,
\begin{equation}
\mu(\{x \in X : \forall n' \geq n, \sum_{g \in F_{n'}} \chi_{U_{g'}}(gx) < \epsilon |F_{n'}|\}) > 1 - \epsilon.
\end{equation}

By the SMB Theorem 2.4, $-\frac{1}{|F_n|} \log \mu(\xi_{F_n}(x))$ converges to $\tilde{h}$ a.e. Hence by the same argument as above, for sufficiently large $N$, whence $n > N$,
\begin{equation}
\mu(\{x \in X : \forall n' \geq n, -\frac{1}{|F_{n'}|} \log \mu(\xi_{F_{n'}}(x)) > \tilde{h} - \epsilon\}) > 1 - \epsilon.
\end{equation}

Let
\begin{equation}
E = \left\{x \in X : \forall n' \geq n, \sum_{g \in F_{n'}} \chi_{U_{g'}}(gx) < \epsilon |F_{n'}|\right\}
\end{equation}

whence
\begin{equation}
\bigcap \{x \in X : \forall n' \geq n, -\frac{1}{|F_{n'}|} \log \mu(\xi_{F_{n'}}(x)) > \tilde{h} - \epsilon\}.
\end{equation}

Then for any $n > N$, $\mu(E) > 1 - 2\epsilon$.

Let $w_{\xi,F_n}(x) = (\xi(gx))_{g \in F_n}$ be the $(\xi, F_n)$-name of $x$. For any $y \in B(x, \delta)$, we have that either $\xi(x) = \xi(y)$ or $x \in U_\delta(\xi)$. Hence if $x \in E$ and $y \in B_{F_n}(x, \delta)$, then the Hamming distance between $w_{\xi,F_n}(x)$ and $w_{\xi,F_n}(y)$ is less than $\epsilon$. This implies that whence $x \in E$,
\[
B_{F_n}(x, \delta) \subset \bigcup \{\xi_{F_n}(y) : w_{\xi,F_n}(y) \text{ is } \epsilon \text{-close to } w_{\xi,F_n}(x) \text{ under Hamming metric}\}.
\]

By Stirling’s formula, the total number of such $(\xi, F_n)$-names, denoted by $L_n$, can be estimated by:
\[
L_n \leq \sum_{j=0}^{\lfloor |\epsilon|F_n\rfloor} C_{|F_n|}^j (\#\xi - 1)^j \leq \exp(K\epsilon |F_n|),
\]
where $K$ can be chosen as
\[
K = \epsilon + \epsilon \log(\#\xi - 1) - \epsilon \log \epsilon - (1 - \epsilon) \log(1 - \epsilon) + 2.
\]

For the calculation of $K$, one may refer to [6] or [2].

We now note that $K > 2$ is a constant only dependent on $\#\xi$ and $\epsilon$ but independent of $x$ and $n$ and moreover, $K\epsilon$ tends to 0 while $\epsilon$ tends to 0.

Let
\[
D_n = \{x \in E : \mu(B_{F_n}(x, \delta)) > \exp((-\tilde{h} + 3K\epsilon)|F_n|)\}.
\]

If we can prove that $\sum_{n=N}^{\infty} \mu(D_n) < \infty$, then apply the Borel-Cantelli Lemma: for a.e. $x \in E$,
\[
\lim \inf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \geq \tilde{h} - 3K\epsilon \geq h - (3K + 1)\epsilon.
\]

Hence we can obtain
\[
\lim \lim \inf_{\delta \to 0 \ n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \delta)) \geq h_u(X, G).
\]
Now we estimate the measure of $D_n$.

For any $x \in D_n$, in those $L_n$—many $(\xi, F_n)$—names which are $\epsilon$—close to $W_{\epsilon, F_n}(x)$ in Hamming distance, there exists at least one corresponding atom of $\xi_{F_n}$ whose measure is greater than $\exp((-\tilde{h} + 2K\epsilon)|F_n|)$. The total number of such atoms will not exceed $\exp(\tilde{h} - 2K\epsilon|F_n|)$. Hence $Q_n$, the total number of elements in $\xi_{F_n}$ that intersect $D_n$, satisfies:

$$Q_n \leq L_n \exp((\tilde{h} - 2K\epsilon)|F_n|) \leq \exp((\tilde{h} - K\epsilon)|F_n|).$$

Let $S_n$ denote the total measure of such $Q_n$ elements of $\xi_{F_n}$ whose intersections with $E$ have positive measure. Then from $(2.3)$,

$$S_n \leq Q_n \exp((-\tilde{h} + \epsilon)|F_n|) \leq \exp((-K\epsilon + \epsilon)|F_n|),$$

which follows that

$$\mu(D_n) \leq S_n \leq \exp((-K\epsilon + \epsilon)|F_n|).$$

From the increasing condition $(1.3)$, for sufficiently large $N$, whenever $n \geq N$, $\frac{|F_n|}{\log n} \geq \frac{2}{(K - 1)\epsilon}$ holds. Then $\exp((-K\epsilon + \epsilon)|F_n|) \leq n^{-2}$ and hence $\sum_{n=N}^{\infty} \mu(D_n) \leq \sum_{n=N}^{\infty} S_n < \infty$. Thus the proof is completed.  

\[\square\]

3. A variation principle for Bowen entropy

In this section we will show that there exists a variational principle between Bowen topological entropy and (lower) local entropy.

Theorem 3.1. Let $(X, G)$ be a compact metric $G$—action topological dynamical system and $G$ a discrete countable amenable group. If $K \subseteq X$ is non-empty and compact and $\{F_n\}$ a sequence of finite subsets in $G$ with the increasing condition $\lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty$, then

$$h_{\text{top}}^B(K, \{F_n\}) = \sup\{\mu^\text{loc}(\{F_n\}) : \mu(K) = 1\},$$

where the supremum is taken over $\mu \in M(X)$, the Borel probability measures on $X$.

For the proof of this theorem, we use the idea from [4], which is a dynamical corresponding of the classical result in fractal geometry.

We first introduce the so-called weighted entropy for system $(X, G)$ in the following way. Let $\{F_n\}$ a sequence of finite subsets in $G$ with $|F_n|$ tends to infinity. For any function $f : X \to [0, +\infty)$, $N \in \mathbb{N}$ and $\epsilon > 0$, define

$$\mathcal{W}(f, N, \epsilon, s, \{F_n\}) = \inf \sum_i c_i \exp(-s|F_{n_i}|),$$

where the infimum is taken over all finite or countable families $\{(B_{F_{n_i}}(x_i, \epsilon), c_i)\}$ such that $0 < c_i < +\infty, x_i \in X, n_i \geq N$ and $\sum_i c_i 1_{B_{F_{n_i}}(x_i, \epsilon)} \geq f$.

For $Z \subset X$ and $f = \chi_Z$ we set $\mathcal{W}(Z, N, \epsilon, s, \{F_n\}) = \mathcal{W}(\chi_Z, N, \epsilon, s, \{F_n\})$. The quantity $\mathcal{W}(Z, N, \epsilon, s, \{F_n\})$ does not decrease as $N$ increases and $\epsilon$ decreases, hence the following limits exist:

$$\mathcal{W}(Z, \epsilon, s, \{F_n\}) = \lim_{N \to +\infty} \mathcal{W}(Z, N, \epsilon, s, \{F_n\}), \mathcal{W}(Z, \epsilon, \{F_n\}) = \lim_{\epsilon \to 0} \mathcal{W}(Z, \epsilon, s, \{F_n\}).$$
Clearly, there exists a critical value of the parameter $s$, which we will denote by $h_{\text{top}}^{WB}(Z, \{F_n\})$, where $\mathcal{W}(Z, s, \{F_n\})$ jumps from $+\infty$ to 0, i.e.

$$\mathcal{W}(Z, s, \{F_n\}) = \begin{cases} 0, s > h_{\text{top}}^{WB}(Z, \{F_n\}), \\ +\infty, s < h_{\text{top}}^{WB}(Z, \{F_n\}). \end{cases}$$

We call $h_{\text{top}}^{WB}(Z, \{F_n\})$ the weighted Bowen’s topological entropy along $\{F_n\}$ restricted to $Z$ or, simply, the weighted Bowen’s topological entropy of $Z$ along $\{F_n\}$.

Now we will consider the relation between the Bowen topological entropy and the weighted Bowen’s topological entropy. It is clear that if we take $f = \chi_Z$ and $c_i = 1$, then the following holds.

**Proposition 3.2.** $\mathcal{W}(Z, N, \epsilon, s, \{F_n\}) \leq \mathcal{M}(Z, N, \epsilon, s, \{F_n\})$, for any $s, \epsilon > 0$ and $N \in \mathbb{N}$.

By alternating some parameters, we can get

**Proposition 3.3.** $\mathcal{M}(Z, N, 6\epsilon, s + \delta, \{F_n\}) \leq \mathcal{W}(Z, N, \epsilon, s, \{F_n\})$, for any $s, \epsilon, \delta > 0$ and sufficient large $N \in \mathbb{N}$.

The proof is similar to Proposition 3.2 of [1]. But we should remark here that condition [1.3] ensures the existence of the parameter $N$.

**Proof of Proposition 3.3.** From the increasing condition [1.3], there exists $N > 2$, such that for $n \geq N$, $|F_n|_{\text{lin}} \geq 2\delta^{-1}$. Then $e^{-\delta|F_n|} \leq n^{-2}$ and hence $\sum_{n=N}^{+\infty} e^{-\delta|F_n|} < 1$.

Let $\{(B_{F_n}(x_i, \epsilon), c_i)\}_{i \in \mathcal{I}}$ be a countable family such that $\mathcal{I} \subset \mathbb{N}, x_i \in X, 0 < c_i < +\infty, n_i \geq N$ and

$$\sum_{i} c_i \chi_{B_i} \geq \chi_Z,$$

where $B_i := B_{F_n_i}(x_i, \epsilon)$. Then we will show that

$$\mathcal{M}(Z, N, 6\epsilon, s + \delta, \{F_n\}) \leq \sum_{i \in \mathcal{I}} c_i \exp(-s|F_n_i|),$$

and hence

$$\mathcal{M}(Z, N, 6\epsilon, s + \delta, \{F_n\}) \leq \mathcal{W}(Z, N, \epsilon, s, \{F_n\}).$$

Decompose $\mathcal{I}$ into subsets $\mathcal{I}_n := \{i \in \mathcal{I} : n_i = n\}$ and the finite subsets $\mathcal{I}_{n,k} = \{i \in \mathcal{I}_n : i \leq k\}$ for $n \geq N$ and $k \in \mathbb{N}$. Write for brevity $B_i := B_{F_n_i}(x_i, \epsilon)$ and $5B_i := B_{F_n_i}(x_i, 5\epsilon)$ for $i \in \mathcal{I}$. We may assume $B_i$’s are mutually different. For $t > 0$, set

$$Z_{n,t} = \{x \in Z : \sum_{i \in \mathcal{I}_n} c_i \chi_{B_i}(x) > t\}$$

and

$$Z_{n,k,t} = \{x \in Z : \sum_{i \in \mathcal{I}_{n,k}} c_i \chi_{B_i}(x) > t\}.$$

For $Z_{n,k,t}$, we may assume that each $c_i$ is a positive integer. Since $\mathcal{I}_{n,k}$ is finite and by approximating the $c_i$’s from above, we may first assume $c_i$’s are positive rational numbers.
numbers. Also notice that \( Z_{n,k,t} \) for \( dc_i \)'s is equal to \( Z_{n,k,t} \) for \( c_i \)'s, so multiplying with a common denominator \( d \), we may assume that each \( c_i \) is a positive integer. Let \( m \) be the least integer with \( m \geq t \). Denote \( B = \{ B_i : i \in I_{n,k} \} \) and define \( u : B \to Z \) by \( u(B_i) = c_i \). We define by induction integer-valued functions \( v_0, v_1, \cdots, v_m \) on \( B \) and sub-families \( B_1, \cdots, B_m \) of \( B \) starting with \( v_0 = u \). Using the classical \( 5r \)-coving Lemma in fractal geometry (see, for example, Theorem 2.1 of [7]), taking the metric \( d_{F_n} \) instead of \( d \), there exists a pairwise disjoint subfamily \( B_1 \) of \( B \) such that \( \bigcup_{B \in B_1} B \subset \bigcup_{B \in B_1} 5B \), and hence \( Z_{n,k,t} \subset \bigcup_{B \in B_1} 5B \). Repeating this process, we can define inductively for \( j = 1, \cdots, m \), disjoint subfamilies \( B_j \) of \( B \) such that

\[
B_j \subset \{ B \in B : v_{j-1}(B) \geq 1 \}, Z_{n,k,t} \subset \bigcup_{B \in B_j} 5B
\]

and the functions \( v_j \) such that

\[
v_j(B) = \begin{cases} 
v_{j-1}(B) - 1 & \text{for } B \in B_j, \\
v_{j-1}(B) & \text{for } B \in B \setminus B_j. 
\end{cases}
\]

Since \( j \leq m \), \( Z_{n,k,t} \subset \{ x : \sum_{B \in B, B \supset x} v_j(B) \geq m - j \} \), whenever every \( x \in Z_{n,k,t} \) belongs to some ball \( B \in B \) with \( v_j(B) \geq 1 \), the above inductive process works. Thus

\[
\sum_{j=1}^m \#(B_j) e^{-s|F_n|} = \sum_{j=1}^m \sum_{B \in B_j} (v_{j-1}(B) - v_j(B)) e^{-s|F_n|} \\
\leq \sum_{B \in B} \sum_{j=1}^m (v_{j-1}(B) - v_j(B)) e^{-s|F_n|} \\
\leq \sum_{B \in B} u(B) e^{-s|F_n|} = \sum_{i \in I_{n,k}} c_i e^{-s|F_n|}. 
\]

Choose \( j_0 \in \{1, \cdots, m\} \) such that \( \#(B_{j_0}) \) is the smallest. Then

\[
\#(B_{j_0}) e^{-s|F_n|} \leq \frac{1}{m} \sum_{i \in I_{n,k}} c_i e^{-s|F_n|} \leq \frac{1}{t} \sum_{i \in I_{n,k}} c_i e^{-s|F_n|}. 
\]

This shows that: for each \( n \geq N, k \in \mathbb{N} \) and \( t > 0 \), there exists a finite set \( J_{n,k,t} \subset I_{n,k} \) such that the balls \( B_i \) \((i \in J_{n,k,t})\) are pairwise disjoint, \( Z_{n,k,t} \subset \bigcup_{i \in J_{n,k,t}} 5B_i \) and

\[
\#(J_{n,k,t}) e^{-s|F_n|} \leq \frac{1}{t} \sum_{i \in I_{n,k}} c_i e^{-s|F_n|}. 
\]

Now assume \( Z_{n,t} \neq \emptyset \). Since \( Z_{n,k,t} \uparrow Z_{n,t} \), \( Z_{n,k,t} \neq \emptyset \) when \( k \) is large enough. Let \( J_{n,k,t} \) be the sets constructed above. Then \( J_{n,k,t} \neq \emptyset \) when \( k \) is large enough. Define \( E_{n,k,t} = \{ x_i : i \in J_{n,k,t} \} \). Note that the family of all non-empty compact subsets of \( X \) is compact with respect to the Hausdorff distance. It follows that there is a subsequence \( \{ k_j \} \) of natural numbers and a non-empty compact set \( E_{n,t} \subset X \) such that \( E_{n,k_j,t} \) converges to \( E_{n,t} \) in the Hausdorff distance as \( j \to +\infty \). Since any two points in \( E_{n,k,t} \)
have a distance (with respect to $d_{F_n}$) not less than $\epsilon$, so do the points in $E_{n,t}$. Thus $E_{n,t}$ is a finite set, moreover, $|(E_{n,kj,t})| = |(E_{n,t})|$ when $j$ is large enough. Hence
\[
\bigcup_{x \in E_{n,t}} B_{F_n}(x, 5.5\epsilon) \supset \bigcup_{x \in E_{n,kj,t}} B_{F_n}(x, 5\epsilon) = \bigcup_{i \in J_{n,kj,t}} 5B_i \supset Z_{n,kj,t}
\]
when $j$ is sufficiently large, and thus $\bigcup_{x \in E_{n,t}} B_{F_n}(x, 6\epsilon) \supset Z_{n,t}$. Since $|(E_{n,kj,t})| = |(E_{n,t})|$ when $j$ is large enough, we have $|(E_{n,t})|e^{-s|F_n|} \leq \frac{1}{t} \sum_{i \in I_n} c_i e^{-s|F_n|}$. This forces
\[
\mathcal{M}(Z_{n,t}, N, 6\epsilon, s + \delta, \{F_n\}) \leq |(E_{n,t})|e^{-(s+\delta)|F_n|}
\]
\[
\leq \frac{1}{e^{\delta|F_n|t}} \sum_{i \in I_n} c_i e^{-s|F_n|}.
\]
Since $\sum_{n=1}^\infty e^{-\delta|F_n|} < 1$, we can deduce that $Z \subset \bigcup_{n=1}^\infty Z_{n,e^{-\delta|F_n|t}}$ for any $t \in (0, 1)$.
And also note that $\mathcal{M}(Z, N, \epsilon, s, \{F_n\})$ is an outer measure of $X$, we have
\[
\mathcal{M}(Z, N, 6\epsilon, s + \delta, \{F_n\}) \leq \sum_{n=N}^\infty \mathcal{M}(Z_{n,e^{-\delta|F_n|t}}, N, 6\epsilon, s + \delta, \{F_n\})
\]
\[
\leq \sum_{n=N}^\infty \frac{1}{t} \sum_{i \in I_n} c_i e^{-s|F_n|}
\]
\[
= \frac{1}{t} \sum_{i \in I} c_i e^{-s|F_n|}.
\]
Hence
\[
\mathcal{M}(Z, N, 6\epsilon, s + \delta, \{F_n\}) \leq \sum_{i \in I} c_i e^{-s|F_n|}.
\]
This finishes the proof of the proposition. \qed

The following is a dynamical Frostman’s lemma related to the weighted Bowen entropy in the amenable group action case.

**Lemma 3.4.** Let $K$ be a non-empty compact subset of $X$. Let $s \geq 0, N \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $c := \mathcal{W}(K, N, \epsilon, s, \{F_n\}) > 0$. Then there is a Borel probability measure $\mu$ on $X$ such that $\mu(K) = 1$ and
\[
\mu(B_{F_n}(x, \epsilon)) \leq \frac{1}{c} e^{-s|F_n|}, \forall x \in X, n \geq N.
\]

**Proof.** Since $c < \infty$, we can define a function $p$ on $C(X)$ (the space of continuous real-valued functions on $X$) by
\[
p(f) = (1/c)\mathcal{W}(\chi_K : f, N, \epsilon, s, \{F_n\}).
\]
Let $1 \in C(X)$ denote the constant function $1(x) \equiv 1$. It is easy to verify that

1. $p(f + g) \leq p(f) + p(g)$ for any $f, g \in C(X)$;
2. $p(tf) = tp(f)$ for any $t \geq 0$ and $f \in C(X)$;
3. $p(1) = 1, 0 \leq p(f) \leq \|f\|_{\infty}$ for any $f \in C(X)$, and $p(g) = 0$ for $g \in C(X)$ with $g \leq 0$. 


By the Hahn-Banach theorem, we can extend the linear functional \( t \mapsto tp(1), t \in \mathbb{R} \), from the subspace of the constant functions to a linear functional \( L : C(X) \to \mathcal{R} \) satisfying
\[
L(1) = p(1) = 1 \quad \text{and} \quad p(-f) \leq L(f) \leq p(f) \quad \text{for any} \; f \in C(X).
\]
If \( f \in C(X) \) with \( f \geq 0 \), then \( p(-f) = 0 \) and so \( L(f) \geq 0 \). Hence combining the fact \( L(1) = 1 \), we can use the Riesz representation theorem to find a Borel probability measure \( \mu \) on \( X \) such that \( L(f) = \int f \, d\mu \) for \( f \in C(X) \).

For any compact set \( E \subset X \setminus K \), by the Uryson lemma, there is \( f \in C(X) \) such that \( 0 \leq f \leq 1, f(x) = 1 \) for \( x \in E \) and \( f(x) = 0 \) for \( x \in K \). Then \( f \cdot \chi_K \equiv 0 \) and thus \( p(f) = 0 \). Hence \( \mu(E) \leq L(f) \leq p(f) = 0 \). This shows \( \mu(X \setminus K) = 0 \), i.e., \( \mu(K) = 1 \).

For any compact set \( E \subset B_{F_n}(x, \epsilon) \), by the Uryson lemma again, there exists \( f \in C(X) \) such that \( 0 \leq f \leq 1 \), \( f(y) = 1 \) for \( y \in E \) and \( f(y) = 0 \) for \( y \in X \setminus B_{F_n}(x, \epsilon) \). Then \( \mu(E) \leq L(f) \leq p(f) \). Since \( f \cdot \chi_K \leq \chi_{B_{F_n}(x, \epsilon)} \) and \( n \geq N \), we have \( W(\chi_K \cdot f, N, \epsilon, s, \{F_n\}) \leq e^{-s|F_n|} \) and thus \( p(f) \leq \frac{1}{c} e^{-s|F_n|} \). Therefore \( \mu(E) \leq \frac{1}{c} e^{-s|F_n|} \). It follows that
\[
\mu(B_{F_n}(x, \epsilon)) = \sup \{ \mu(E) : E \text{ is a compact subset of } B_{F_n}(x, \epsilon) \} \leq \frac{1}{c} e^{-s|F_n|}.
\]

**Proof of Theorem 3.1.** For any \( \mu \in M(X) \) with \( \mu(K) = 1, x \in X, n \in \mathbb{N} \) and \( \epsilon > 0 \), recall that
\[
h_{\mu}^{loc}(x, \epsilon, \{F_n\}) = \liminf_{n \to +\infty} -\frac{1}{|F_n|} \log \mu(B_{F_n}(x, \epsilon)).
\]
Since \( h_{\mu}^{loc}(x, \epsilon, \{F_n\}) \) is nonnegative and increases as \( \epsilon \) decreases, by the monotone convergence theorem,
\[
\lim_{\epsilon \to 0} \int_X h_{\mu}^{loc}(x, \epsilon, \{F_n\}) d\mu = \int_X h_{\mu}^{loc}(x, \{F_n\}) d\mu = h_{\mu}^{loc}(\{F_n\}).
\]

Fix \( \epsilon > 0 \) and \( \ell \in \mathbb{N} \). Denote \( u_{\ell} = \min \{ \ell, \int_X h_{\mu}^{loc}(x, \epsilon, \{F_n\}) d\mu - \frac{1}{2} \} \). Then there exist a Borel set \( A_\ell \subset X \) with \( \mu(A_\ell) > 0 \) and \( N \in \mathbb{N} \) such that
\[
\mu(B_{F_n}(x, \epsilon)) \leq e^{-u_{\ell}|F_n|}, \forall x \in A_\ell, n \geq N.
\]
Now let \( \{B_{F_{n_i}}(x_i, \epsilon/2)\} \) be a countable or finite family such that \( x_i \in X, n_i \geq N \) and \( \bigcup_i B_{F_{n_i}}(x_i, \epsilon/2) \supseteq K \cap A_\ell \). We may assume that for each \( i \), \( B_{F_{n_i}}(x_i, \epsilon/2) \cap K \cap A_\ell \neq \emptyset \), and choose \( y_i \in B_{F_{n_i}}(x_i, \epsilon/2) \cap K \cap A_\ell \). Then we have
\[
\sum_i e^{-u_{\ell}|F_{n_i}|} \geq \sum_i \mu(B_{F_{n_i}}(y_i, \epsilon)) \geq \sum_i \mu(B_{F_{n_i}}(x_i, \epsilon/2)) \geq \mu(K \cap A_\ell) = \mu(A_\ell) > 0.
\]
It follows that \( \mathcal{M}(K, u_{\ell}, \{F_n\}) \geq \mathcal{M}(K, u_{\ell}, N, \epsilon/2, \{F_n\}) \geq \mathcal{M}(K \cap A_\ell, u_{\ell}, N, \epsilon/2, \{F_n\}) \geq \mu(A_\ell) \). Therefore \( h_{\mu}^{B}(K, \{F_n\}) \geq u_{\ell} \). Letting \( \ell \to +\infty \), we have the inequality \( h_{\mu}^{B}(K, \{F_n\}) \geq \int_X h_{\mu}^{loc}(x, \epsilon, \{F_n\}) d\mu \). Hence \( h_{\mu}^{B}(K, \{F_n\}) \geq h_{\mu}^{loc}(\{F_n\}) \).

Now we will show that
\[
h_{\mu}^{B}(K, \{F_n\}) \leq \sup \{ h_{\mu}^{loc}(\{F_n\}) : \mu \in M(X), \mu(K) = 1 \}.
\]
We can assume that \( h_{\text{top}}^B(K, \{ F_n \}) > 0 \). By Proposition 3.2 and 3.3, \( h_{\text{top}}^B(K, \{ F_n \}) = h_{\text{top}}^B(K, \{ F_n \}) \). Let \( 0 < s < h_{\text{top}}^B(K, \{ F_n \}) \). Then there exist \( \epsilon > 0 \) and \( N \in \mathbb{N} \) such that \( c := W(K, N, \epsilon, s, \{ F_n \}) > 1 \). By Lemma 3.4, there exists \( \mu \in M(X) \) with \( \mu(K) = 1 \) such that \( \mu(B_{F_n}(x, \epsilon)) \leq \frac{1}{e^{-s|F_n|}} \) for any \( x \in X \) and \( n \geq N \). Clearly \( h_{\mu}^{\text{loc}}(x, \{ F_n \}) \geq h_{\mu}^{\text{loc}}(x, \epsilon, \{ F_n \}) \geq s \) for each \( x \in X \) and hence \( \mu_{\mu}^{\text{loc}}(\{ F_n \}) = \int_X h_{\mu}^{\text{loc}}(x, \{ F_n \}) d\mu(x) \geq s \). This finishes the proof of the theorem.

\[ \square \]

4. Proof of the main results

Now we give the proof of the Theorem 1.1.

We first prove that for any Følner sequence \( \{ F_n \} \) and any open cover \( \mathcal{U} \),

\[ h_{\text{top}}^B(\mathcal{U}, X, \{ F_n \}) \leq h_{\text{top}}^B(G, \mathcal{U}). \]

Let \( \mathcal{V} \) be a subcover of \( \mathcal{U}_{F_n} \) with minimal cardinality. We can write \( \mathcal{V} = \{ X(U) : U \in \Lambda \} \), where \( \Lambda \subset W_{F_n}(\mathcal{U}) \). Then the cardinality of \( \Lambda \) equals to \( N(\mathcal{U}_{F_n}) \). Hence

\[ \sum_{U \in \Lambda} \exp(-sm(U)) = N(\mathcal{U}_{F_n})e^{-s|F_n|}, \]

which implies that

\[ \mathcal{M}(X, \mathcal{U}, s, \{ F_n \}) \leq \exp \left( (-s + \frac{1}{|F_n|}) \log N(\mathcal{U}_{F_n}) \right). \]

If \( s \) is larger than \( h_{\text{top}}(G, \mathcal{U}) \), \( \mathcal{M}(X, \mathcal{U}, s, \{ F_n \}) = 0 \). So \( h_{\text{top}}^B(\mathcal{U}, X, \{ F_n \}) \leq h_{\text{top}}^B(G, \mathcal{U}). \)

In the following we show that \( h_{\text{top}}^B(X, \{ F_n \}) \geq h_{\text{top}}(X, G) \) for tempered Følner sequence \( \{ F_n \} \) with \( \lim_{n \to +\infty} \frac{|F_n|}{\log n} = \infty \). For the proof we need the following classical variational principle for amenable group action dynamical systems, see [8, 12].

**Theorem 4.1** (Variational principle for topological entropy).

\[ h_{\text{top}}(X, G) = \sup_{\mu \in M(X, G)} h_{\mu}(X, G) = \sup_{\mu \in E(X, G)} h_{\mu}(X, G), \]

where \( M(X, G) \) and \( E(X, G) \) are the collection of \( G \)-invariant and \( G \)-ergodic Borel probability measures of \( X \) respectively.

Then by Theorem 2.1, 3.1 and 4.1

\[ h_{\text{top}}(X, G) = \sup_{\mu \in E(X, G)} h_{\mu}(X, G) = \sup_{\mu \in E(X, G)} h_{\mu}(X, G), \]

\[ = \sup \{ h_{\mu}(X, G, \{ F_n \}) : \mu \in E(X, G) \} \]

\[ = \sup \{ h_{\mu}^{\text{loc}}(\{ F_n \}) : \mu \in E(X, G) \} \]

\[ \leq \sup \{ h_{\mu}^{\text{loc}}(\{ F_n \}) : \mu \in M(X) \} \]

\[ = h_{\text{top}}^B(X, \{ F_n \}). \]

This completes the proof of Theorem 1.1.

To end this paper, we ask the following question:
Question 4.2. Can Theorem 4.1 be proved through a pure topological way?

Acknowledgements

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