We investigate generalized thermalization in an isolated free Fermionic chain evolving from an out of equilibrium initial state through a sudden quench. We consider the quench where a Fermionic chain is broken into two disjoint chains. We focus on the evolution of the local observables namely, occupation number, information sharing and out-of-time-order correlations after the quench and study the relaxation of the observable, leading to generalized Gibbs ensemble for the system in the thermodynamic limit. We obtain the light cone formed by the evolution of the observables along the Fermionic lattice chain due to the sudden quench which abides by the Lieb-Robinson bound in quantum systems. We also analytically study a simpler model which captures the essential features of the system. Our analysis strongly suggest that the internal interactions within the system do not remain of much importance once the quench is sufficiently strong.

I. INTRODUCTION

The study to understand a definite connection between quantum mechanics and statistical mechanics — two mighty frameworks in physics — is an area which has actively received attention for many years. Large enough systems left on their own, over large times seem to settle into a thermodynamic configuration. However, the ideas of thermalization and unitary evolution of the underlying quantum theory do not go hand in hand [1]. A quantum evolution keeps a pure state as pure throughout, whereas thermalization demands a mixed state description. The initial contributions to resolve this apparent conflict in this area came in the 1920’s from von Neumann about thermalization in isolated many-body quantum systems, proposing that demand on thermalization on large systems may be relaxed to the demand that only the expectation values of macroscopic observables need to thermalize [2, 3]. Thus, for large enough systems the late time expectations should closely resemble those of a thermalized system, and that is about it! The system then thermalizes without really thermalizing [4].

However the problem of explicit verification of this idea remained dormant for nearly eight decades because of the analytical complexity; as the thermalization is supposed to work for large systems and the Hilbert space dimension increases exponentially as the number of degrees of freedom increases, making the analytic handling almost intractable.

When a system thermalizes, we expect the macroscopic properties of the system to equilibrate to its corresponding statistical ensemble predictions. Classically a system is called integrable if it has \( N \) independent constants of motion in a 2\( N \) dimensional phase space; by doing a canonical transformation to its corresponding action-angle coordinates, the action is conserved, and the angle evolves linearly in time. Hence for an integrable system, its dynamics can be predicted at all times, and it will never be ergodic. However, dynamics for non-integrable systems is governed by non-linearity and chaos, making the evolution ergodic thereby resulting in thermalization [5–7].

The isolated quantum many-body systems have a different mechanism for thermalization owing to its unitary time evolution. A pure initial state would never evolve into a mixed thermal state density matrix through unitary evolution in an isolated system. However as pointed out by von Neumann, one needs to compare the expectation value of macroscopic observables in the thermodynamic limit, and not the density matrix themselves [2]. Isolated quantum systems with short-range interactions are said to thermalize if, after a long time, the expectation values of few-body observables equilibrate to a steady state predicted by statistical mechanics [7]. Srednicki and Deutsch proposed a mechanism suggesting that the thermalization occurs at the level of eigenstates [8, 9]. This mechanism is referred to as the Eigenstate Thermalization Hypothesis. Eigenstate thermalization holds for non-integrable systems where the expectation value of observables settles down to the value given by the ensemble description.

In order to look at how unitary time evolution generated by an arbitrary Hamiltonian \( \hat{H} \) in an isolated quantum system could lead to thermalization, let us consider the dynamics of an initial state \( \rho_I = |\psi(0)\rangle\langle\psi(0)| \) (where \( [\rho_I, \hat{H}] \neq 0 \)) and compare the expectation value of the macroscopic observables with the value given by statistical ensemble. For now, let the Hamiltonian be the only physically relevant conserved quantity for the system.

Let \( E_n \) and \( |n\rangle \) be the eigenvalues and eigenvectors of \( \hat{H} \). The energy of the system is a conserved quantity and is set by the initial state given by \( E = \text{Tr} (\rho_I \hat{H}) \). The fluctuations in energy is given by \( \delta E = \sqrt{\text{Tr}(\rho_I \hat{H}^2) - E^2} \). We choose \( \rho_I \) such that \( \delta E \) is subextensive. The time
evolution of the state $|\psi(t)\rangle$ is given by

$$|\psi(t)\rangle = \exp(-i\hat{H}t)|\psi(0)\rangle = \sum_{n=1}^{D} C_n e^{-iE_n t}|n\rangle$$

where $C_n = \langle n | \psi(0) \rangle$ and $D$ is the dimension of the Hilbert space. (Throughout this work, we set Planck constant $\hbar = 1$ and Boltzmann constant $k_B = 1$.)

The expectation value of an observable is given by

$$\langle \hat{O}(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle = \sum_{n} |C_n|^2 \langle n | \hat{O} | n \rangle + \sum_{n,m,n \neq m} C_m^* C_n e^{(-i(E_m - E_n)t)} \langle m | \hat{O} | n \rangle$$

For non-integrable systems, after a reasonably long time, it is phenomenologically observed that $\langle \hat{O}(t) \rangle$ equilibrates to a steady state given by the Gibb’s (microcanonical) ensemble i.e.,

$$\lim_{t \to \infty, L \to \infty} \langle \hat{O}(t) \rangle \approx O(\bar{E})$$

where $O(\bar{E}) = \frac{1}{\Omega} \sum_{n} O_{nn}$ where $\Omega$ is the number of energy eigenstates with energies within the window $[\bar{E} - \Delta E, \bar{E} + \Delta E]$ with $\Delta E << \bar{E}$, [7, 10, 11]. It is important to note that this is not a mathematically proven result, and is referred to as Eigenstate Thermalization Hypothesis (ETH) [8, 9]. However, it has to be studied and verified in a variety of non-integrable systems [7, 10, 12].

In the case of integrable systems — which is the focus of this work — the expectation of observables do not thermalize to Gibb’s ensemble. This is because such systems have other conserved quantities (totaling $N$) and can relax only to a steady state predicted by the generalized Gibb’s ensemble (GGE) [13–17]. The notion of generalized thermalization was obtained in integrable systems by generalizing the statistical ensemble description for integrable systems by including all the integrals of motion.

We can arrive at GGE by maximizing entropy subject to constraints imposed by the integrals of motion i.e.,

$$\hat{\rho}_{GGE} = \frac{e^{(-\sum_{m} \lambda_m \hat{I}_m)}}{\text{Tr}[e^{(-\sum_{m} \lambda_m \hat{I}_m)}]},$$

where $\hat{I}_m$ is the set of all integrals of motion and $\lambda_m$ are the Lagrange multipliers which are fixed using initial condition $\text{Tr}[\hat{\rho}_{GGE} \hat{I}_m] = \langle \hat{I}_m \rangle (t = 0)$. Using which we arrive at the state to which the diagonal ensemble settle down. Just like ETH, there is no general proof for generalized thermalization [18, 19].

Though a complete analytical understanding of the generalized thermalization is still lacking, recent advances in ultracold atom experiments and computational techniques made it possible to simulate dynamics of nearly isolated quantum systems and study out of equilibrium dynamics [17, 20–30]. Also, there have been various studies in the literature for Fermionic and Bosonic integrable systems to understand thermalization for integrable systems [18, 31–34]. Most of the Fermionic system’s studies have focused on the 1-dimensional Fermionic chain that makes a transition from a non-integrable to an integrable configuration.

The primary goal of this work is to study the equilibration of observables in isolated integrable 1-D free non-number conserving Fermionic lattice chain and compare it with the GGE prediction. We study the quench [20, 25–27] where a Fermionic lattice chain is broken into two smaller disjoint chains which result in moving the initial system out of equilibrium. Thus a Fermionic system jumps from one integrable set up to another integrable setting. We verify whether the systems lands into a GGE owing to this quench. We calculate expectation values for observables which play the role of conserved charges. To visualize the dynamical evolution of the system into a GGE description, we calculate the information content in bits per Fermion before and after the quench [35] and out of time ordered correlators (OTOC) [36]. We show that information content per Fermion provides crucial information about thermalization of the isolated system under quench.

It is essential to compare and contrast the current work with the earlier works: To visualize the thermalization in integrable systems, one of the most analyzed models is the 1-dimensional Fermionic chain that makes a transition from non-integrable to integrable configuration [37] (see also, [38–58]). In our case, the Fermionic system jumps from one integrable set up to another integrable setting.

The current framework nearly mirrors a setting of gravitational origin, where the issue of thermalization turns out to be more related to the formation of black hole. Once a black hole forms, the Hilbert space of the initial data gets bifurcated into Hilbert spaces of interior and exterior, where the exterior appears to be put (at late times) in the thermal environment. Thus, the current model helps us get insights about the settings where one part of the systems is dynamically decoupled from another part. The Fermionic nature of the system controls the dimension of the Hilbert space involved.

In Ref. [59] two of the present authors studied a Bosonic system jumping from integrable to the integrable setting. However, quench action in that system was joining of two disjoint chains; inverse of the present system of study. It was shown that the system tends towards the GGE as the at-large times and as the system size is increased. In the Bosonic case, the physical quantities can be computed only up to leading order. However, in the present case, we can compute the quantities precisely for all orders.

The rest of the paper is organized as follows: In section II, we introduce the model Hamiltonian, the quench protocol and the observables of interest. In section III, we compare the long time evolution of the observable quantities against the corresponding GGE value by varying the system size and the time of evolution. Through var-
ious estimators, we demonstrate that the system quickly settles into a GGE configuration with increasing size. In section IV, we provide the analytical calculations for the observable quantities of interest in a similar yet simpler model to attain a better understanding of the general characters observed. Finally, section V sums up the findings and the discuss the implications for the field theoretic setup.

II. MODEL AND SETUP

A. Model Hamiltonian

The aim is to study the dynamics of local quench in analytically soluble one dimensional spinless Fermionic system. The model we consider is the non-number conserving free Fermionic model whose Hamiltonian is [60]:

\[ H = -\frac{J}{2} \sum_{j=1}^{2N} (\hat{a}^\dagger_{j+1} \hat{a}_j + \hat{a}^\dagger_j \hat{a}_{j+1}) - J \sum_{j=1}^{2N} \hat{a}^\dagger_j \hat{a}_j - h \sum_{j=1}^{2N} \hat{a}^\dagger_j \hat{a}_j \] (5)

where \( \hat{a}^\dagger_j(\hat{a}_j) \) creates (annihilates) Fermion at lattice site \( j \). As mentioned earlier, this is an integrable model. For the ease of computations, we assume a periodic boundary condition for the Fermionic chain. In Sec. IV, we provide an analytic study of a simpler yet similar number conserving system, tight binding model with Fermions. This model will provide a way to understand which features are generic for these class of spin chains.

Hamiltonian (5) can be diagonalized to normal modes by a Fourier transformation followed by the Bogoliubov transformation. Under the Fourier transformations,

\[ \hat{b}^\dagger_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} \hat{f}^\dagger_j e^{\frac{2\pi i j k}{2N}}; \quad \hat{b}_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} \hat{f}_j e^{-\frac{2\pi i j k}{2N}} \]

the Hamiltonian (5) gets transformed to

\[ H = \sum_{k=1}^{N} \omega_k (\hat{b}^\dagger_k \hat{b}_k + \hat{b}^\dagger_k \hat{b}_k) + \sum_{k=1}^{N} i \Delta_k (\hat{b}^\dagger_k \hat{b}^\dagger_{-k} - \hat{b}_{-k} \hat{b}_k) \] (6)

where

\[ \omega_k = -h - J \cos \left( \frac{2\pi k}{2N} \right) \quad \text{and} \quad \Delta_k = J \sin \left( \frac{2\pi k}{2N} \right) \]

Performing the Bogoliubov transformation:

\[ \hat{\gamma}_{k1} = \alpha_k \hat{b}_k + i \beta_k \hat{b}^\dagger_{-k}; \quad \hat{\gamma}_{k2} = \alpha_k \hat{b}_{-k} - i \beta_k \hat{b}^\dagger_k \]

the Hamiltonian (5) becomes

\[ H = \sum_{k=1}^{N} E_k (\hat{\gamma}^\dagger_{k1} \hat{\gamma}_{k1} + \hat{\gamma}^\dagger_{k2} \hat{\gamma}_{k2}) - (E_k - \omega_k) \] (7)

where

\[ \alpha_k^2 = \frac{1}{2} \left( 1 + \frac{\omega_k}{E_k} \right), \quad \beta_k^2 = \frac{1}{2} \left( 1 - \frac{\omega_k}{E_k} \right), \]

and \( E_k = \sqrt{J^2 + h^2 + 2 J h \cos \frac{2\pi k}{2N}} \), \( k = 1, 2, ..., N \) are the normal mode frequencies.

B. The quench and covariance matrix

The initial Hamiltonian \( H_I \) is \( H_I = H_{2N+2M} \) where \( H_{2N+2M} \) describes the Fermionic lattice of size \( (2N + 2M) \) with periodic boundary condition. The quench action corresponds to (i) simultaneously switching off the interaction between \( 1^{\text{st}} \) and \( (2N + 2M)^{\text{th}} \) sites and \( 2N^{\text{th}} \) and \( (2N + 1)^{\text{th}} \) sites of \( H_{2N+2M} \) and (ii) introducing the interaction (with coupling constant \( J \)) between \( 1^{\text{st}} \) and \( 2N^{th} \) site resulting in \( H_{2N} \) and \( (2N + 1)^{th} \) and \( (2N + 2M)^{th} \) site resulting in \( H_{2M} \). In other words, we break the chain of lattice size \( (2N + 2M) \) into two independent chains of sizes \( (2N) \) and \( (2M) \), resulting in the quenched Hamiltonian \( H_f = H_{2N} \oplus H_{2M} \).

FIG. 1: An illustration of the quench that breaks an initial chain of size \( 2N + 2M \) into two disjoint chains with periodic boundary condition for all the chains.

Since the initial and the final Hamiltonian can be diagonalized, the system is described by non-interacting quasi-particles. The non-interacting nature implies that all the information about the system can be obtained from the expectation value of the two-point correlators between various lattice points which can be compactly arranged in the covariance matrix \( \tilde{G} \) defined as the outer product of \( \tilde{A} \) and \( \tilde{A}^T \), i.e., \( G = \tilde{A}\tilde{A}^T \) where the column vector \( \tilde{A} = [a_1 a_1^\dagger a_2 a_2^\dagger ... a_N a_N^\dagger] \), leading to

\[ G = \begin{bmatrix} \langle a_1 a_1 \rangle & \langle a_1 a_1^\dagger \rangle & ... & \langle a_1 a_N \rangle & \langle a_1 a_N^\dagger \rangle \\ \langle a_1^\dagger a_1 \rangle & \langle a_1^\dagger a_1^\dagger \rangle & ... & \langle a_1^\dagger a_N \rangle & \langle a_1^\dagger a_N^\dagger \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \langle a_N a_1 \rangle & \langle a_N a_1^\dagger \rangle & ... & \langle a_N a_N \rangle & \langle a_N a_N^\dagger \rangle \\ \langle a_N^\dagger a_1 \rangle & \langle a_N^\dagger a_1^\dagger \rangle & ... & \langle a_N^\dagger a_N \rangle & \langle a_N^\dagger a_N^\dagger \rangle \end{bmatrix} \] (8)

Hence a symplectic transformation (linear transformations that preserve Fermionic anti-commutation relation)
of the creation and annihilation operators like, \( \vec{\gamma} = U \vec{a} \)
would cause the covariance matrix to transform as
\[
G' = U G U^T \tag{9}
\]
and one can use the transformed covariance matrix to obtain correlators. The creation and annihilation operators for Fermions in the energy eigenstates of the quenched Hamiltonian evolves in time as \( \gamma_k(t) = e^{-iE_k t} \gamma_k \) and \( \gamma_k^\dagger(t) = e^{iE_k t} \gamma_k^\dagger \) where \( E_k \) are the energy eigenvalues of \( H_f \).

C. Observables and GGE

In this work, we have used three estimators to quantify thermalization. As we will show in the next section, these three estimators provide complementary information about how the system drives to a generalized Gibbs ensemble.

1. Occupancy of a site The macroscopic observable of our interest is the number density per lattice site in real space. The expectation value of the time evolved number operator can be obtained from the time evolved covariance matrix \( G \), given by \( \langle n_i(t) \rangle = \langle a_i(t) a_i(t) \rangle \). The main aim would be to verify if the longtime expectation value of the number operator per lattice site in the real space would converge to GGE.

In order to find the GGE ensemble, we need to find the conserved quantities of the system. Since the system is non-interacting in the normal modes, the occupation number of each normal mode after the quench is conserved. Hence the independent conserved quantities are \( n_k = \gamma_k^\dagger \gamma_k \) where \( k = 1, 2 \cdots (N + M) \). The GGE density matrix is given by
\[
\hat{\rho} = \frac{\exp \left( -\sum_{k=1}^{N+M} \lambda_k \hat{n}_k \right)}{\text{Tr}[\exp \left( -\sum_{k=1}^{N+M} \lambda_k \hat{n}_k \right)]} \tag{10}
\]
where \( \lambda_k \) are the Lagrange multipliers which are fixed using the initial condition \( \text{Tr}[\hat{\rho} \hat{n}_k] = \langle \hat{n}_k(0) \rangle \). Using the initial condition, we obtain the Lagrange multipliers to be
\[
\lambda_k = \ln \left( \frac{1 - \langle \hat{n}_k(0) \rangle}{\langle \hat{n}_k(0) \rangle} \right) \tag{11}
\]
where \( \langle \hat{n}_k(0) \rangle \) can be obtained from the covariance matrix in the normal modes.

2. Bits per Fermion Another important quantity we calculate is the information content in bits per Fermion in each normal mode and compare its profile before and after the quench. The von Neumann entropy for the density matrix \( \rho \) is given by
\[
S(\rho) = -\text{Tr}(\rho \ln \rho) \]

3. Out-of-time-order correlator Out of time order correlator (OTOC) corresponding to \( [\hat{x}(t), \hat{p}(0)] \) measures the quantum analog of the classical quantity \( \delta x(t)/\delta x(0) \) for Bosonic systems, identifying the measure of chaos in the system. If the system turns chaotic, this quantity should gradually rise with time, while for a system landing in a pre-ascribed configuration, the strength of OTOC should remain within bounds for large times [61], whereas for Fermionic systems it shows a tendency of flattening out [62]. An OTOC can be constructed for any two non commuting observables. We will be considering an out-of-time-order correlator for the chain, given by
\[
F_{ij}(t) = \frac{1}{2} \langle [\hat{x}_i(t), \hat{p}_j(0)]^2 \rangle \tag{14}
\]
where we define Hermitean observables
\[
\hat{x}_i(t) = \frac{\hat{a}_i(t) + \hat{a}_i(t)}{\sqrt{2}}; \quad \hat{p}_j(0) = \frac{\hat{a}_j(0) - \hat{a}_j(0)}{\sqrt{2}} \tag{15}
\]
in analogy to the Bosonic case (but keeping in mind that in Fermionic systems they satisfy anticommutation relation). We will now calculate these estimators in order to robustness of generalized thermalization and confirm whether the jump of a system from one integral configuration to another integral configuration with causal disruption does not make it chaotic [63].

III. RESULTS

In this section, we present semi-analytical results for the model Hamiltonian (5) with the quench action. As mentioned earlier, we use three estimators — occupancy at a site, Bits per Fermion and OTOC — to identify the late-time evolution of the initial state.

A. Occupation Number

To study equilibration of a local observable, we look at the evolution of expectation value of the number operator at a particular lattice site in real space. For the verification, we have plotted the mean value of the evolved
number operator and the value given by the GGE. For all the plots, we have fixed the parameters $h/J = -2$ and the nearest neighbor interaction $J$ to be 0.5. The initial state is chosen to be a thermal state with the lattice chain at temperature $T/J = 0.5$, i.e., inverse temperature $\beta T = 1/4$. We calculate the energy in the unit of the onsite coupling constant $h = -1$. This value sets the unit of time to be $1/h$.

The fluctuations become smaller, and the system tends to equilibrates to the GGE value and has tend to decay in time, and the expectation value of the system goes out of equilibrium. Second, the fluctuations of quench reaches the site, the value fluctuates, and the remains in the initial thermal state. As soon as the effect of quench reaches the particular site of observation, it lattice. We infer the following: First, until the effect situation where the quench happens at the center of the site of quench as a function of time. We have taken a time average of $\langle n_i(t) \rangle$ as a function of time for the case when $2N = 2M = 300$, (b) $2N = 2M = 400$, (c) $2N = 2M = 500$, (d) $2N = 2M = 600$. GGE value and time average of $\langle n_i(t) \rangle$ is also plotted in each case.

In Fig. 2, we plot the expectation value of the number operator at a site slightly away from the quenching site $\langle n_i(t) \rangle$ as a function of time for the case when $2N = 2M$, i.e., (a) $2N = 2M = 300$, (b) $2N = 2M = 400$, (c) $2N = 2M = 500$, (d) $2N = 2M = 600$. GGE value and time average of $\langle n_i(t) \rangle$ is also plotted in each case.

As mentioned above, from Fig. 2, we observe that sufficiently long time average value matches the GGE value. In order to substantiate the same, we evaluate the time average of the observable:

$$\pi(\tau) = \frac{1}{\tau} \int_0^\tau \langle \dot{n}_i(t) \rangle dt .$$

and calculate the relative deviation of $\pi(\tau)$ ($\Delta n$)

$$\Delta n = \frac{\langle |\pi(\tau) - \langle n_i \rangle_{GGE} | \rangle}{\langle \langle n_i \rangle_{GGE} \rangle}$$

from the GGE value as a function of $\tau$. Fig. 3 contains the plot of $\Delta n(\tau)$ as a function of $\tau$. The figure explicitly shows a power-law decay of the relative deviation. Thus, in the infinite time limit, the relative deviation vanishes.

To further quantify, we evaluate the relative deviation of the lattice occupation number $\langle n_i(t) \rangle$ from the GGE value, i.e.,

$$\delta n_i(t) = \frac{|\langle n_i(t) \rangle - \langle n_i \rangle_{GGE}|}{\langle n_i \rangle_{GGE}}$$

as a function of time. From the Fig. 4, we infer the following: Initially $\delta(n_i(t))$ relaxes to zero after quench and at later times, starts showing fluctuations. As the number of lattice sites increases, the time of initiation of the late time fluctuations is delayed in a linear fashion and the magnitude of fluctuation also reduces.

To investigate further, in Fig. 5, we plot the expectation value of the number operator at the site where quench happens as a function of time. Fig. 5 shows the same trend as in Fig. 2. In other words, as the lattice size increases, the deviation from the GGE is small and better relaxation is observed. The plot indicates that the observable will relax to GGE in the thermodynamic limit. Also, the late time fluctuations get delayed linearly.
also shows the formation of a light cone gate along the lattice with constant speed. The figure following features: First, number density peaks propagate along the quenched lattice in time. We observe the color intensity map shows how the disturbance travels along lattice position (in the x-axis) and time (in the y-axis). To understand the time-delay for the quench to reach a lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice size increases confirming the finite size effect.

To understand the time-delay for the quench to reach a lattice site, in Fig. 6, we plot the expectation of number operator (as a color intensity) as a function of the lattice position (in the x-axis) and time (in the y-axis). The color intensity map shows how the disturbance travels along the quenched lattice in time. We observe the following features: First, number density peaks propagate along the lattice with constant speed. The figure also shows the formation of a light cone (in analogy to causal propagation) which marks the existence of maximum speed for the information propagation. This result is consistent with the Lieb-Robinson bound [64] for short-range interactions which give a theoretical limit for the speed of propagation of information in non-relativistic quantum systems. Second, the fluctuations are caused by the interference between different light cones due to finite size effect and periodic boundary condition. Once the effect of quench reaches the lattice, it goes out of equilibrium and then equilibrates to an almost steady state which matches with the GGE value but later starts to fluctuate due to the finite size effect. As the lattice size increases, the fluctuations from the average value reduces. In the thermodynamic limit, we then expect the system to equilibrate to the GGE value thus verifying generalized relaxation.

In Fig. 7, connected correlation \( \langle n_i n_j \rangle - \langle n_i \rangle \langle n_j \rangle \) between two sites each in the disconnected chains after quench is plotted. We consider the two cases where \( i \) and \( j \) are the two nearby sites before breaking and the case where they are far apart. In the first case, the connected correlation is large when the quench happens and then decays to zero since the two sites are in independent chains after the quench. When the two sites are far away
FIG. 6: This plot shows the expectation of number operator (color intensity) as a function of the position of lattice (x-axis) and time (y-axis) for the four cases: (a) $2N = 2M = 300$, (b) $2N = 2M = 400$, (c) $2N = 2M = 500$, (d) $2N = 2M = 600$.

from each other and the quench, the connected correlation is minimal initially and then goes to zero after the quench. Any fluctuation which might occur at a later time is due to the finite size.

To see how the expectation value of the number operator at a site equilibrates to GGE, in Fig. 8, we plot the logarithm of $\delta n_i(t)$ defined in Eq. (18) against logarithm of time. For immediate comparison, the figure contains a linear plot with coefficient $-1$. Following points are worth noting regarding the above figure: First, as the lattice size increases, $\delta n_i(t)$ decays as a power-law with exponent close to $-1$. Second, the exponent $-1$ is indicative of ballistic behavior rather than diffusive behavior where the exponent needs to be $-0.5$ [65]. Classically, ballistic behavior arises due to the collisionless transport of particles whereas the system under consideration is composed of non-interacting quasiparticles, mimicking the classical behavior.

B. Information content in bits per Fermion

In the previous subsection, to evaluate the occupation number in a lattice site, we fixed $J = 0.5$ and $\beta_t = 4$. To obtain information about the response of the system for
different parameters, in Fig. 9, we plot the information content in bits per Fermion $I(k)$ for each normal-mode $k/L$ (where $L$ is the length of the lattice chain and $k = 1, 2, \cdots$) before and after the quench for different $T/J$ parameters. Note that we have fixed the values of $J$ and $h/J$ to be 0.5 and $-2$, respectively.

We see two distinct features: When $T/J$ is high, corresponding to a high temperature initial thermal state, $I(k)$ before and after the quench almost overlaps. However, when the temperature is low, say at $T/J = 0.01$, the profile shows a different trend. We thus infer the following: First, for the low-temperature initial state, the information content per Fermion after the quench is distributed evenly to all the normal modes in contrast to before the quench distribution. Second, the information content per Fermion in each mode after quench is smaller compared to before the quench. However, the total entropy of the system increases after the quench, consistent with the second law of thermodynamics. Third, it implies that the initial state of the system before the quench for small $T/J$ and large $T/J$ is not identical.

To further investigate this, we calculate fidelity which is a measure of closeness or overlap between two quantum states [66, 67]. Fidelity between two density matrices $\rho_1$ and $\rho_2$ can be written as:

$$F(\rho_1, \rho_2) = \sum_i \sqrt{p_i q_i}$$  \hspace{1cm} (19)

where $p_i$ and $q_i$ are the eigenvalues of two density matrices, i.e.,

$$\rho_1 = \sum_i p_i |i><i| \quad \text{and} \quad \rho_2 = \sum_i q_i |i><i|$$  \hspace{1cm} (20)

for some orthonormal basis $|i>$. Note that we have taken the situation where the two density matrices can be simultaneously diagonalized by unitary matrices.

Fig. 10 contains the plot of the fidelity between $\rho_T$ and $\rho_{GS}$ — between the initial thermal (with non-zero $T/J$) and ground state — as a function of $T/J$. We see that as $T/J$ decreases, the fidelity goes closer to unity. Thus, the information content in bits per Fermion can be used as an indicator to identify quantum phases [68, 69]. To our knowledge, this is the first time that the information content in bits per Fermion can possibly be used as a tool to identify quantum phase transition.
The initial value of OTOC i.e., $F_{ij}(0) = -0.5$ for any $i, j$ since the operators are Fermionic in nature. From the OTOC analysis (see Fig. 11) we can see that OTOC parameter, after being affected by the quench, decays with time. Theoretically, it indicates that due to some initial perturbation at $j$–th site, the effect on the configuration at $i$–th site (recalling $[\hat{x}_i(t), \hat{p}_j(0)]$ measures $\delta x_i(t)/\delta x_j(0)$) settles over time, for any pair $(i,j)$, to the value $-0.5$ indicating that at long time, the system homogenizes as fit for a thermalized set-up. In the Fig. 11, the long time average value obtained is $-0.4983$ which is close to $-0.5$. Therefore, in the thermodynamic limit, the OTOC is expected to settle down to steady value to $-0.5$ indicating homogeneity.

IV. ANALYTICAL UNDERSTANDING

The results in the previous section are exact for the model Hamiltonian (5). Since it is impractical to obtain an analytical expression for these observables and to have a better understanding of the features of the above model, we now analytically study a qualitatively similar yet simpler model namely the tight-binding model [70]. In the rest of this section, we explicitly write down the analytic expressions for the three observables in this model and discuss the essential features.

Hamiltonian of the tight-binding model is

$$H_{TB} = -\hbar \sum_{j=1}^{2N} a_j^\dagger a_j - \frac{J}{2} \sum_{j=1}^{2N} a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}$$

(21)

where $a_j^\dagger(a_j)$ creates(annihilates) Fermion at lattice site $j$. Comparing the above Hamiltonian with (5), it is clear that this model is number conserving.

This Hamiltonian with periodic boundary condition can be diagonalized by the Fourier transformation:

$$\tilde{b}_k = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} e^{2\pi ijk} a_j^\dagger$$

$$\tilde{b}_k^\dagger = \frac{1}{\sqrt{2N}} \sum_{j=1}^{2N} e^{-2\pi ijk} a_j^\dagger$$

(22)

Substituting the Fourier transforms in the Hamiltonian (21), we get

$$H_{TB} = \sum_{k=1}^{2N} \omega_k \tilde{b}_k^\dagger \tilde{b}_k$$

(23)

where

$$\omega_k = -\hbar - J \cos \frac{2\pi k}{2N}, \quad k = 1, 2, ..., 2N.$$  

(24)

As compared to the Hamiltonian (5), the tight-binding model does not require one to perform Bogoliubov transformation to diagonalize the Hamiltonian.

A. $\langle n_i(t) \rangle$ for the tight-binding model

Like in the earlier case, we assume the state to be in a thermal state for the initial Hamiltonian, $H_i = H_{TB}^{\frac{2N+2M}{2}}$. The density matrix for the initial state is

$$\hat{\rho}_i = \frac{e^{-\beta_i H_i}}{Tr[e^{-\beta_i H_i}]}$$

(25)
\[ [\hat{\rho}_f, \hat{H}_f] = 0 \]  

At \( t = 0 \), we quench the system by splitting into two spin chains of sizes \( 2N \) and \( 2M \). (See the illustration in Fig. 1.) The Hamiltonian changes to \( \hat{H}_F^{TB} = H_T^{TB} \oplus H_T^{TB} \).

Our aim is to analytically evaluate the time evolution of average occupation number in real space for the quenched system, i.e., after the chain is broken:

\[
\langle n_j(t) \rangle = \langle a_j^\dagger(t) a_j(t) \rangle \quad j = 1, \cdots, 2N
\]  

Substituting the inverse Fourier transform from (22) in the above expression and using the time-evolution of the operators \( \langle b_k(t) = e^{-i\omega_k t} b_k(0) \rangle \), we get,

\[
\langle n_j(t) \rangle = \sum_{k=1}^{2N} \sum_{k'=1}^{2N} e^{i\omega_k t} e^{i2\pi(j-k') \frac{2N}{N}} \langle b_k^\dagger(0) b_{k'}(0) \rangle
\]  

where \( k, k' = 1, \cdots, 2N \). From the above expression, we see that the disturbance propagates as a plane wave with the speed

\[
v = 2N \left( \omega_k - \omega_{k'} \right) / 2\pi(k - k') = 2N \left( \cos \frac{2k'}{2N} - \cos \frac{2k}{2N} \right) \]

In the limit of \( k \to k' \), the maximum speed of propagation is \( J \) when \( k = N/2 \). Comparing this result with Fig. 6, we see that for the Hamiltonian (5), the maximum speed of propagation is achieved for \( k = N/2 \). In other words, the number density peaks propagate along the lattice with constant speed and forms a light cone; implying the existence of maximum speed for the information propagation.

To obtain the time evolution of the occupation number, we substitute the Fourier transform (22) in equation (28):

\[
\langle b_k^\dagger(0) b_{k'}(0) \rangle = \frac{1}{2N} \sum_{m=1}^{2N} \sum_{m'=1}^{2N} e^{-i2\pi(km-m'm')/2N} \langle a_m^\dagger(0) a_{m'}(0) \rangle.
\]  

To evaluate \( \langle a_m^\dagger(0) a_{m'}(0) \rangle \), we perform the Fourier transform (22) for the chain of size \( 2N + 2M \), we get,

\[
\langle a_m^\dagger(0) a_{m'}(0) \rangle = \frac{1}{2N + 2M} \sum_{k, k'=1}^{2N+2M} e^{i2\pi(mk - m'k')/(2N + 2M)} \langle b_{k'}^\dagger(0) b_k(0) \rangle
\]

where,

\[
\langle b_{k'}^\dagger(0) b_k(0) \rangle = 0 \quad \text{for} \quad k \neq k'
\]

\[
\langle b_{k'}^\dagger(0) b_k(0) \rangle = \frac{1}{1 + e^{2\pi k_k}} \quad \text{for} \quad k = k'
\]

\( E_{k_k} \) corresponds to the energy Eigenvalues for the chain \( 2N + 2M \) and \( k_k = 1, \cdots, 2N + 2M \).

FIG. 12: The initial state with the particle number \( n_I \) chosen to be 0, 24, 50, 76 and 100 for the initial lattice chain of size \( 2N + 2M = 100 \). The expectation value of the number operator at a site where quench happens is plotted against time. The GGE value is also plotted.

The advantage of working with the number conserving Hamiltonian is that we can control the total number of particles in the real space since it would be the same as the number of particles in the normal mode. We can study what happens when we change the number of particles in the initial state.

In Fig. 12, we have constructed the initial state with the particle number \( n_I \) chosen to be 0, 24, 50, 76 and 100 for the initial lattice chain of size \( 2N + 2M = 100 \). The expectation value of the number operator at a site where quench happens is plotted against time. We infer the following: First, the minimum fluctuations are obtained for half the number of sites \( (n_I = 50) \). Second, when we have particles in half the number of sites, the Hilbert space dimension is \( (L/2)^5 \) which is the largest for the system to span, hence resulting in better equilibration to GGE. Third, in Fermionic systems like the ground state, the highest excited state is also unique; therefore, post-quench there is no freedom left to (re)distribute the population, resulting in a perfect matching with GGE expectation.

For both the models, we see that the expectation of number operator equilibrates to the GGE value in the thermodynamic limit. In the case of the tight-binding model also there is the evolution of a light cone due to the local quench indicating maximum speed which we have analytically calculated to be \( J \). The relaxation to GGE in the case of the tight-binding model also follows the power law with exponent \(-1\). In other words, GGE
behavior of the Hamiltonian (5) can be inferred from the tight-binding model.

Fig. 13 is the plot of $I(k)$ for the tight-binding model. Comparing these plots with the plots of $I(k)$ in Fig. 9, we conclude the following: For both the models, with the lower-temperature initial state, the information content per Fermion after the quench is distributed evenly across all the normal modes in contrast to before the quench distribution. Also, the information content per Fermion in each mode after the quench is smaller compared to before the quench. Like the average expectation value of the number operator, $I(k)$ behavior of the Hamiltonian (5) can be inferred from the tight-binding model.

C. OTOC for the tight-binding model

Out-of-time-order correlator is given by (14). Substituting (15) in (14) and simplifying the expression, we get,

$$F_{ij}(t) = 2 \left[ \text{Re} \left( \frac{i}{2} \left( \langle a_j^\dagger(t) a_i^\dagger(0) \rangle - \langle a_j(0) a_i(0) \rangle \right) - \langle a_j(t) a_i(0) \rangle - \langle a_j(0) a_i(t) \rangle \right) \right]^2 - \frac{1}{2} \tag{35}$$

In the case of tight binding model, $\langle a_j^\dagger(t) a_i^\dagger(0) \rangle = 0 = \langle a_j(t) a_i(0) \rangle$. So we are left to calculate the expressions for $\langle a_j(t) a_i(0) \rangle$ and $\langle a_j^\dagger(t) a_i(0) \rangle$.

Assuming $j$ and $l$ to be on the broken chain of size $2N$ after the quench, we get,

$$\langle a_j(t) a_i^\dagger(0) \rangle = \frac{1}{2N} \sum_{k,k'} e^{iE_k t} e^{-\frac{\pi i (k-k')^2}{2N}} \langle b_k(0) b_{k'}^\dagger(0) \rangle \tag{36}$$

where,

$$\langle b_k(0) b_{k'}^\dagger(0) \rangle = \frac{1}{2N} \sum_{m,m'} e^{\frac{i2\pi (m - m') k}{2N}} \langle a_m(0) a_{m'}^\dagger(0) \rangle \tag{37}$$

Since the quench happens at $t=0$,

$$\langle a_m(0) a_{m'}^\dagger(0) \rangle = \frac{1}{2N + 2M} \sum_{k_1,k_1'} e^{-\frac{\pi i (m - m') k_1 k_1'}{2N + 2M}} \langle b_{k_1}(0) b_{k_1'}^\dagger(0) \rangle \tag{38}$$

where,

$$\langle b_{k_1}(0) b_{k_1'}^\dagger(0) \rangle = 0 \quad \text{for} \quad k_1 \neq k_1' \tag{39}$$

$$\langle b_{k_1}(0) b_{k_1'}^\dagger(0) \rangle = 1 - \langle b_{k_1}(0) b_{k_1'}(0) \rangle \tag{40}$$

$$= 1 - \left( \frac{1}{1 + e^{\beta E_{k_1}}} \right) \quad \text{for} \quad k_1 = k_1' \tag{41}$$

$E_{K_{ij}}$ corresponds to the energy Eigenvalues for the chain $2N + 2M$ and $k_1 = 1...2N + 2M$.

**B. $I(k)$ for the tight-binding model**

The information content in bits per Fermion $I(k)$, for a given normal mode $k$, is given by (13). Since the Hamiltonian (21) is number conserving, average expectation value of the number operator ($\langle \hat{n}_k \rangle$) is a conserved quantity. Once we obtain $\langle \hat{n}_k \rangle = \langle b_k^\dagger b_k \rangle$, we can calculate $I(k)$. The Fourier transform leads to:

$$\langle b_k^\dagger b_k \rangle = \frac{1}{2N} \sum_{m=1}^{2N} \sum_{m'=1}^{2N} e^{-\frac{i2\pi (m - m') k}{2N}} \langle a_m^\dagger a_{m'} \rangle \tag{34}$$

where $\langle a_m^\dagger(0) a_{m'}(0) \rangle$ is given by (31).
chain of size 2N models have similar features. More specifically, the evolution in the thermodynamic limit. Gibb’s generalized ensemble with the fluctuations vanishes. The observable is seen to equilibrate to the value given by the quench. Like in the earlier case, at long-times the parameter \( F \) approaches \(-0.5\). Similarly, we can obtain:

\[
\langle a_j^\dagger(t)a_i(0) \rangle = \frac{1}{2N} \sum_{k,k'=1}^{2N} e^{i\omega_k t} e^{i\pi(k-k')} \langle b_k^\dagger(0)b_{k'}(0) \rangle \tag{42}
\]

where \( \langle b_k^\dagger(0)b_{k'}(0) \rangle \) is given by (30).

Fig. 14 contains the plot of OTOC for the tight-binding model. Comparing this with Fig. 11, we see that both models have similar features. More specifically, the evolution of the light cone in just one half of the entire initial chain of size \( 2N + 2M \) showing the correlations between the two chains vanishes with the quench. Like in the earlier case, at long-times the parameter \( F_{ij}(t) \) approaches \(-0.5\).

V. CONCLUSIONS

In this work, we considered a Fermionic lattice chain breaks into two smaller disconnected chains due to sudden quench. The time evolution of local lattice occupation number following quench is calculated using the quenched Hamiltonian. Thus, the system jumps between two integrable configurations. The expectation value of the observable is seen to equilibrate to the value given by Gibb’s generalized ensemble with the fluctuations vanishing in the thermodynamic limit.

We have also obtained a light-cone like evolution with which one can accurately track the evolution of initial data. The light-cone confirms the existence of the maximum limit for the speed of propagation of the information of the quench. This result is consistent with the Lieb-Robinson bound in quantum systems. We have also seen that the relaxation to GGE goes as a power law with exponent approaching \(-1\) indicating ballistic dynamics. We also calculated the connected correlation between two sites each in the disconnected chains after the quench which gradually vanishes.

Further, we calculated the information content in bits per Fermion \( I(k) \) per normal mode \( k/L \) before and after the quench. We observed exciting trends in the distribution of \( I(k) \) before and after the quench. We see from the plot of \( I(k) \) against \( k/L \) that for initial low-temperature thermal states, the information content per Fermion after quench is smaller and spreads evenly for all normal modes compared to before the quench, in the spirit of thermalization. However, the total entropy of the system increases after the quench, consistent with the second law of thermodynamics.

Another measure of predictability of evolution is OTOC. If OTOC grows in time, the evolution becomes chaotic and final state cannot be ascribed with accuracy. The dynamics of the systems demonstrates that post-quench the OTOC parameter decays once the quench hits the system and the gradually settles to a value close to \(-0.5\).

The model we study suffers from the limitation of analytical studies. Therefore, we also study a closely resembling analytically tractable model, which captures the essential features of the system. In this model we also analytically demonstrated the march of the system towards a GGE configuration, strongly suggesting that internal interactions within the system do not remain of much importance once the quench is sufficiently strong.

The system under study also has a close resemblance to specific gravitational systems: quantum systems in space-time with horizons. Development of the horizon disrupts the causal communication between parts of the quantum system, one specific example being the black hole. Thus the present study potentially reflects the evolution of an initial data which spontaneously develops a horizon, which through entanglement is considered to be thermalizing the parts the system. For this purpose, we will be carrying out the field theoretic generalization of the present work with geometric features elsewhere.

VI. ACKNOWLEDGEMENT

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