Relation between the number of leaves of a tree and its diameter*

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Abstract

Let \(L(n,d)\) denote the minimum possible number of leaves in a tree of order \(n\) and diameter \(d\). In 1975 Lesniak gave the lower bound \(B(n,d) = \lceil (2(n-1)/d) \rceil\) for \(L(n,d)\). When \(d\) is even, \(B(n,d) = L(n,d)\). But when \(d\) is odd, \(B(n,d)\) is smaller than \(L(n,d)\) in general. For example, \(B(21,3) = 14\) while \(L(21,3) = 19\). We prove that for \(d \geq 2\),

\[
L(n,d) = \begin{cases} 
\lceil \frac{2(n-1)}{d} \rceil & \text{if } d \text{ is even;} \\
\lceil \frac{2(n-2)}{d-1} \rceil & \text{if } d \text{ is odd.}
\end{cases}
\]

The converse problem is also considered. Let \(D(n,f)\) be the minimum possible diameter of a tree of order \(n\) with exactly \(f\) leaves. We prove that

\[
D(n,f) = \begin{cases} 
2 & \text{if } n = f + 1; \\
2k + 1 & \text{if } n = kf + 2; \\
2k + 2 & \text{if } kf + 3 \leq n \leq (k+1)f + 1.
\end{cases}
\]

Key words. Leaf; diameter; tree

A leaf in a graph is a vertex of degree 1. For a real number \(r\), \(\lfloor r \rfloor\) denotes the largest integer less than or equal to \(r\), and \(\lceil r \rceil\) denotes the least integer larger than or equal to \(r\). Let \(L(n,d)\) denote the minimum possible number of leaves in a tree of order \(n\) and diameter

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In 1975 Lesniak [1, Theorem 2 on p.285] gave the lower bound $B(n, d) = \lceil 2(n - 1)/d \rceil$ for $L(n, d)$. When $d$ is even, $B(n, d) = L(n, d)$. But when $d$ is odd, $B(n, d)$ is smaller than $L(n, d)$ in general. For example, $B(21, 3) = 14$ while $L(21, 3) = 19$.

In this note we first determine $L(n, d)$. We use an idea different from that in [1]. The proof also makes it clear why $L(n, d)$ has such an expression. We then determine the minimum possible diameter of a tree with given order and number of leaves.

We make the necessary preparation. For terminology and notation we follow the books [3] and [2]. We denote by $V(G)$ the vertex set of a graph $G$ and by $d(u, v)$ the distance between two vertices $u$ and $v$. For vertices $x$ and $y$, an $(x, y)$-path is a path with end vertices $x$ and $y$. We denote by $\text{deg}(v)$ the degree of a vertex $v$.

Let $P$ be a path in a tree $T$ and we call $P$ the stem of $T$. For every vertex $x \in V(T)$, there is a unique $(x, y)$-path $Q$ such that $V(Q) \cap V(P) = \{y\}$. We say that $x$ originates from $y$. Note that by definition, a vertex on the stem originates from itself. A diametral path of a tree $T$ is a path of length equal to the diameter of $T$.

A spider is a tree with at most one vertex of degree larger than 2 and this vertex is called the branch vertex. If no vertex has degree larger than 2, then any vertex may be specified as the branch vertex. Thus, a spider is a subdivision of a star. A leg of a spider is a path from the branch vertex to a leaf.

We will need the following lemma.

**Lemma 1.** [2, p.63] A path $P = v_0v_1v_2...v_k$ in a tree is a diametral path if and only if for every vertex $x$, $$d(x, v_i) \leq \min\{i, k - i\}$$ where $x$ originates from $v_i$ with $P$ as the stem.

The case $d = 1$ for $L(n, d)$ is trivial, since the only tree of diameter 1 is $K_2$ which has two leaves. Thus it suffices to consider the case $d \geq 2$.

**Theorem 2.** Let $L(n, d)$ denote the minimum possible number of leaves in a tree of order $n$ and diameter $d$ with $d \geq 2$. Then

$$L(n, d) = \begin{cases} \left\lfloor \frac{2(n-1)}{d} \right\rfloor & \text{if } d \text{ is even;} \\ \left\lfloor \frac{2(n-2)}{d-1} \right\rfloor & \text{if } d \text{ is odd.} \end{cases}$$

**Proof.** The idea is to show that for any tree $T$, there is a corresponding spider with
the same order, diameter and number of leaves as $T$. Hence, to determine $L(n, d)$ it suffices to consider spiders.

If $d = n - 1$, then the tree must be a path which has two leaves. In this case the formula for $L(n, d)$ is true. Note also that a path is a spider. Next we assume $d \leq n - 2$.

Let $T$ be a tree of order $n$ and diameter $d$. Choose a diametral path $P = v_0v_1v_2...v_d$ as the stem. Suppose that $x$ is a leaf of $T$ outside $P$ originating from $y$. There is a unique $(x, y)$-path $Q$. Since $P$ is a diametral path, $y \neq v_0, v_d$. Hence $\deg(y) \geq 3$. We define the first big vertex of $x$, denoted by $b(x)$, to be the first vertex of degree at least 3 from $x$ to $y$ on $Q$.

Denote $c = \lfloor d/2 \rfloor$. Then $c = d/2$ if $d$ is even and $c = (d-1)/2$ if $d$ is odd. Let $z = v_c$. If $T$ has a leaf $u$ outside $P$ with $b(u) \neq z$, let $w$ be the neighbor of $b(u)$ on the $(b(u), u)$-path. Since $T$ is a tree, $w$ and $z$ are not adjacent. We delete the edge $wb(u)$ and add the edge $wz$ to obtain a new tree $T_1$. Since $\min\{i, d-i\} \leq \min\{c, d-c\}$ for any $0 \leq i \leq d$, by Lemma 1 we deduce that $P$ remains a diametral path of $T_1$. Clearly $T_1$ and $T$ have the same set of leaves. Hence $T_1$ and $T$ have the same order, diameter and number of leaves. We still designate $P$ as the stem of $T_1$. If $T_1$ has a leaf outside $P$ whose first big vertex is not $z$, perform the above operation on $T_1$ to obtain a tree $T_2$. Repeating this operation in the resulting trees successively finitely many times, we obtain a tree in which every leaf outside $P$ originates from $z$ and with $z$ as its first big vertex. Such a tree is a spider. An example of the above transformations is depicted in Figure 1.

![Figure 1 Transforming a general tree to a spider](image-url)
The above analysis shows that $L(n, d)$ can be attained at a spider $S$ with a diametral path $P = v_0v_1v_2 \ldots v_d$ where $z = v_c$ is the branch vertex. Clearly the number of leaves in $S$ is equal to the number of legs of $S$. To make the number of legs as small as possible, we need to make each leg as long as possible. Since the diameter of $S$ is $d$, except the leg $v_cv_{c+1} \ldots v_d$ when $d$ is odd, every other leg has length at most $c$. Thus the minimum possible number of legs of such a spider is $\lceil (n - 1)/c \rceil$ when $d$ is even and is $\lceil (n - 2)/c \rceil$ when $d$ is odd. This completes the proof. □

Next we consider the converse problem: Determine the minimum possible diameter of a tree of order $n$ with exactly $f$ leaves. It suffices to treat the case when $n \geq f + 1$, since $K_2$ is the only tree with $n \leq f$.

**Theorem 3.** Let $D(n, f)$ be the minimum possible diameter of a tree of order $n$ with exactly $f$ leaves. Then

$$D(n, f) = \begin{cases} 2 & \text{if } n = f + 1; \\ 2k + 1 & \text{if } n = kf + 2; \\ 2k + 2 & \text{if } kf + 3 \leq n \leq (k + 1)f + 1. \end{cases}$$

**Proof.** In the proof of Theorem 2, we showed that for any tree $T$, there is a corresponding spider with the same order, diameter and number of leaves as $T$. Thus, it suffices to consider spiders. Note that the number of leaves of a spider is equal to its number of legs, which is also true for the case when the spider is a path (corresponding to $f = 2$) if we take a central vertex of the path as its branch vertex. Let $S$ be a spider of order $n$ with exactly $f$ legs whose lengths are $x_1 \geq x_2 \geq \cdots \geq x_f$ arranged in nonincreasing order. Then the diameter of $S$ is $x_1 + x_2$. Hence our problem is equivalent to minimizing $x_1 + x_2$ under the constraint

$$x_1 + x_2 + x_3 + \cdots + x_f = n - 1 \quad (1)$$

where $x_1 \geq x_2 \geq \cdots \geq x_f$ are positive integers.

If $n = f + 1$, then (1) becomes $x_1 + x_2 + x_3 + \cdots + x_f = f$, which has the only solution $x_1 = x_2 = x_3 = \cdots = x_f = 1$. Hence $x_1 + x_2 = 2$.

Let $n = kf + 2$. If $x_1 + x_2 \leq 2k$, then $x_2 \leq k$ and consequently $x_i \leq k$ for each $i = 3, \ldots, f$. It follows that

$$x_1 + x_2 + x_3 + \cdots + x_f \leq (x_1 + x_2) + (f - 2)k \leq 2k + (f - 2)k = fk = n - 2,$$
contradicting (1). This shows that \( D(n, f) \geq 2k + 1 \). On the other hand, the values \( x_1 = k + 1, x_2 = \cdots = x_f = k \) satisfy (1) and \( x_1 + x_2 = 2k + 1 \). Hence \( D(n, f) = 2k + 1 \).

Now consider the third case \( kf + 3 \leq n \leq (k + 1)f + 1 \). We have \( kf + 2 \leq n - 1 \leq kf + f \). Thus there exists an integer \( r \) with \( 2 \leq r \leq f \) such that \( n - 1 = kf + r \). We first show \( D(n, f) \geq 2k + 2 \). If \( x_1 + x_2 \leq 2k + 1 \), then \( x_2 \leq k \) and consequently each \( x_i \leq k \) for \( i = 3, \ldots, f \). It follows that

\[
\begin{align*}
    x_1 + x_2 + x_3 + \cdots + x_f & \leq (x_1 + x_2) + (f - 2)k \\
    & \leq 2k + 1 + (f - 2)k \\
    & = fk + 1 \\
    & < fk + r = n - 1,
\end{align*}
\]

contradicting (1). Hence \( D(n, f) \geq 2k + 2 \). On the other hand, the values \( x_1 = x_2 = \cdots = x_r = k + 1 \) and \( x_{r+1} = \cdots = x_f = k \) satisfy (1) and \( x_1 + x_2 = 2k + 2 \), which shows \( D(n, f) = 2k + 2 \). This completes the proof. \( \square \)

Finally we remark that the maximum problem corresponding to Theorem 2 or Theorem 3 is trivial. The maximum possible number of leaves in a tree of order \( n \) and diameter \( d \) is \( n - d + 1 \) and the maximum possible diameter of a tree of order \( n \) with exactly \( f \) leaves is \( n - f + 1 \).

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