The Algebra of Physical Observables in Nonlinearly Realized Gauge Theories

Andrea Quadri

Physikalisches Institut, Albert-Ludwigs-Universität Freiburg
Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany

and

Phys. Dept. University of Milan, via Celoria 16, I-20133 Milan, Italy

Abstract

We classify the physical observables in spontaneously broken nonlinearly realized gauge theories in the recently proposed loopwise expansion governed by the Weak Power-Counting (WPC) and the Local Functional Equation. The latter controls the non-trivial quantum deformation of the classical nonlinearly realized gauge symmetry, to all orders in the loop expansion. The Batalin-Vilkovisky (BV) formalism is used. We show that the dependence of the vertex functional on the Goldstone fields is obtained via a canonical transformation w.r.t. the BV bracket associated with the BRST symmetry of the model. We also compare the WPC with strict power-counting renormalizability in linearly realized gauge theories. In the case of the electroweak group we find that the tree-level Weinberg relation still holds if power-counting renormalizability is weakened to the WPC condition.

1E-mail address: andrea.quadri@mi.infn.it
1 Introduction

The theoretical understanding of the mass generation mechanism in non-Abelian gauge theories, which will be experimentally probed in the coming years at the LHC, is a challenging open issue in today’s high-energy physics.

The spontaneous symmetry breaking realization based on the Higgs mechanism \[1\] is a sound option which leads to the phenomenologically successful Standard Model (SM) of particle physics \[2\], a theory which is both physically unitary and power-counting renormalizable. In models based on the Higgs mechanism, at least one additional physical scalar particle is present in the perturbative spectrum.

Due to the lack of experimental evidence for the Higgs resonance, other possibilities have nevertheless been investigated. Higher dimensional models have been intensively studied \[3\]. Higgsless models based on modified energy-dependent running coupling constants have been considered \[4\].

Chiral models have been proposed since a long time \[5\]. They are formulated in the presence of a (classical) non-linearly realized non-Abelian gauge symmetry. The perturbative treatment of these theories is usually performed in the momentum expansion, leading to the low-energy Higgsless effective field theory of the chiral electroweak lagrangian \[6\].

More recently an approach based on the perturbative loop expansion of models endowed with a nonlinearly realized gauge symmetry has been investigated. The discovery of the Local Functional Equation (LFE) \[7\] has provided a key tool in the program of taming the divergences of the nonlinearly realized theories recursively in the loop number. The LFE encodes the invariance of the path-integral Haar measure under local nonlinearly realized gauge transformations. It provides a consistent way to handle the non-trivial deformation of the classical chiral symmetry induced by radiative corrections \[8\]. The LFE enforces a hierarchy among 1-PI Green functions: those containing at least one Goldstone field (descendant amplitudes) are fixed in terms of amplitudes which do not involve any Goldstone leg (ancestor amplitudes) \[7\]. Applications to the nonlinear sigma model in \(D = 4\) have been given in \[7\]-\[12\]. The massive nonlinearly realized SU(2) Yang-Mills theory has been studied in \[13\]-\[15\] and the nonlinearly realized electroweak (EW) model has been formulated in \[16\]-\[18\].

The LFE technique has also found applications to nonlinearly realized field transformations, like e.g. polar coordinates in the complex free field theory in \(D = 4\) \[19\].

The present paper is devoted to the cohomological characterization of the algebra of physical observables in the nonlinearly realized (non-anomalous) gauge theories, in the framework of the quantization procedure based on the LFE and the loop expansion.

In view of the availability of an all-orders mathematically consistent formulation of (quantum deformed) gauge symmetry via the LFE, it is very important to be able to classify all the physical observables in nonlinearly realized gauge models on symmetry grounds and to connect them with their classical counterpart.

The appropriate framework for such a task is provided by the Batalin-Vilkovisky (BV)
formalism \cite{20,21}. For the sake of simplicity we work with the non-linearly realized SU(2) Yang-Mills theory, but the results can be easily extended to more general gauge groups. We make use of cohomological techniques \cite{22} in order to analyze the physical observables of the theory. The dependence of the 1-PI vertex functional on the Goldstone fields, dictated by the LFE, turns out to be generated via a canonical transformation w.r.t. the BV bracket induced by the BRST symmetry.

The zero ghost number cohomology $H_0(S^\Gamma)$ of the full linearized BV bracket $S^\Gamma$ (which takes into account the effects of the deformation of the non-linearly realized gauge symmetry) is shown to be isomorphic to the zero ghost number cohomology $H_0(S_0)$ of the classical linearized BV bracket $S_0$. The latter is given for the massive SU(2) Yang-Mills theory based on the non-linearly realized gauge group by the set of all possible global $SU(2)_R$-invariant polynomials in the bleached variable $a_\mu$ (the classical gauge-invariant combination of the gauge field $A_\mu$ and of the Goldstone fields $\phi_a$ which reduces at $\phi_a = 0$ to $A_\mu$) and its ordinary derivatives which do not vanish on the tree-level equation of motion for $a_\mu$.

However, not all these (integrated) operators are allowed in the tree-level vertex functional. In fact a Weak Power-Counting (WPC) condition holds \cite{12,14,17} for the model at hand. The WPC states that only a finite number of divergent ancestor amplitudes exists order by order in the loop expansion. The validity of the WPC provides in turn a very restrictive selection criterion for the operators which can be introduced in the tree-level vertex functional. Only the standard Yang-Mills field strength squared plus the St"uckelberg mass term are compatible with the WPC and all the symmetries of the theory \cite{14}.

The situation is more involved for the non-linearly realized SU(2) $\otimes$ U(1) EW model. There the WPC predicts \cite{16,17} the same couplings in the gauge and fermionic matter sectors (at zero Goldstone fields) as in the Standard Model in the unitary gauge. It however allows for two independent mass invariants in the vector meson sector, i.e. the tree-level Weinberg relation $M_Z = M_W/c_W$ (where $M_Z$ and $M_W$ are the Z the W masses and $c_W$ is the cosine of the Weinberg angle) does not hold.

It should be emphasized that in the subtraction scheme controlled by the WPC and the LFE the existence of a second mass invariant is intimately related to the nature (linear or non-linear) of the gauge group realization. We will indeed prove that the tree-level Weinberg relation still holds in the presence of the linearly realized EW gauge symmetry if power-counting renormalizability is weakened to the WPC condition.

The paper is organized as follows. In Sect. 2 we discuss the non-linearly realized SU(2) massive Yang-Mills theory and its symmetries. The BV bracket is defined in Sect. 3. The master equation is derived in the same Section. In Sect. 4 we study the cohomologies in ghost number zero of the quantum and classical linearized BV brackets and prove that they are isomorphic. In Sect. 5 we compute the cohomology in ghost number zero for the classical linearized BV bracket. In Sect. 6 we show that the bleached variables (invariant under the linearized LFE) are generated via a canonical transformation. In Sect. 7 we
compare the allowed interaction terms, compatible with the WPC, in the framework of the linearly vs. the nonlinearly realized SU(2) gauge theory. Sect. 8 extends this analysis to the SU(2) ⊗ U(1) gauge group. Conclusions are finally given in Sect. 9.

2 The Model and its Symmetries

We consider pure massive Yang-Mills theory based on the nonlinearly realized SU(2) gauge group [14]. By imposing the relevant symmetries of the theory (Slavnov-Taylor (ST) identity, LFE, ghost equation, global SU(2)\(_R\) invariance) and the requirement of the WPC a unique tree-level vertex functional arises [14]

\[
\Gamma^{(0)} = \Lambda^{D-4} \int d^Dx \left( -\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu}_a + \frac{M^2}{2} (A_{a\mu} - F_{a\mu})^2 
+ B_a D^\mu [V](A - V)_{a\mu} - \bar{c}_a (D^\mu [V] D_\mu [A])_a + c_a (D^\mu [A] \Theta_\mu)_a 
+ A_{a\mu} s A_{a\mu} + \phi_a^0 s \phi_0 + \phi_a^0 s \phi_a - c_a^* s c_a + K_0 \phi_0 \right). \tag{1}
\]

The gauge bosons acquire a mass via the St"uckelberg mechanism [23, 24].

In the above equation \(G_{\mu\nu}\) is the non-Abelian field strength

\[
G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \epsilon_{abc} A_{b\mu} A_{c\nu}.
\tag{2}
\]

The covariant derivative is defined as

\[
D_\mu [A]_{ab} = \delta_{ab} \partial_\mu + \epsilon_{acb} A_{c\mu}.
\tag{3}
\]

\(g\) denotes the gauge coupling constant.

The BRST differential \(s\) acts as follows on the fields of the theory

\[
\begin{align*}
  s A_{a\mu} &= \partial_\mu c_a + \epsilon_{abc} A_{b\mu} c_c, \\
  s c_a &= -\frac{1}{2} \epsilon_{abc} c_b c_c, \\
  s \phi_a &= \frac{1}{2} \phi_0 c_a + \frac{1}{2} \epsilon_{abc} \phi_b c_c, \\
  s \phi_0 &= -\frac{1}{2} \phi_a c_a, \\
  s \bar{c}_a &= B_a, \\
  s B_a &= 0.
\end{align*}
\tag{4}
\]

\(c_a\) are the ghost fields, \(\bar{c}_a\) the antighost fields and \(B_a\) the Nakanishi-Lautrup [25, 26] multiplier fields. \(\phi_0\) is the solution of the nonlinear constraint

\[
\phi_0^2 + \phi_a^2 = v^2, \quad \phi_0 = \sqrt{v^2 - \phi_a^2}.
\tag{5}
\]

For the sake of simplicity and conciseness we have adopted the Landau gauge. The extension to an arbitrary ’t Hooft gauge is discussed in [15].

\(V_{a\mu}\) is the background connection necessary for the implementation of the LFE. It is paired in the usual fashion [27] with the background ghost \(\Theta_{a\mu}\) into a BRST doublet [28]

\[
\begin{align*}
  s V_{a\mu} &= \Theta_{a\mu}, \\
  s \Theta_{a\mu} &= 0.
\end{align*}
\tag{6}
\]

\(K_0\) is the scalar source coupled to the nonlinear constraint \(\phi_0\) in eq. [10]. While the background ghost \(\Theta_{a\mu}\) is not needed in order to formulate the LFE and the ST identity, it is
an expedient technical tool in order to show that physical observables do not depend on
the background connection $V_{a\mu}$, as explained in Refs. [27].

The antifields $A^*_a$, $\phi^*_a$, $c^*_a$ and $\phi^*_0$ are external sources coupled to the nonlinear BRST variations [21, 29] of the corresponding fields $A_{a\mu}$, $\phi_a$, $c_a$ and $\phi_0$. The antifield $c^*_a$ has an extra minus sign w.r.t. the conventions adopted in [14]. This choice turns out to be
convenient in the definition of the Batalin-Vilkovisky bracket in Sect. 3.

Notice that the presence of the scalar source $K_0$ (required for the formulation of the LFE) forces the introduction of the antifield $\phi^*_0$ in order to derive the ST identity, despite the fact that $\phi^*_0$ is not an elementary field. The source $K_0$ is paired with $\phi^*_0$ into a BRST doublet as follows

$$s\phi_0^* = -K_0, \quad sK_0 = 0. \quad (7)$$

$\Lambda$ is a mass scale for continuation in $D$ dimensions. In the present paper we choose to
factor out $\Lambda$ in front of the full tree-level vertex functional $\Gamma^{(0)}$. This will simplify the
notations in the discussion of the BV formalism in Sect. 3.

The following functional identities hold for the 1-PI vertex functional $\Gamma$ [14]:

- the ST identity

$$S(\Gamma) = \int d^Dx \left[ \frac{1}{\Lambda^{(D-4)}} \left( \frac{\delta \Gamma}{\delta A_{a\mu}} \frac{\delta \Gamma}{\delta A^*_a} + \frac{\delta \Gamma}{\delta \phi_a} \frac{\delta \Gamma}{\delta \phi^*_a} - \frac{\delta \Gamma}{\delta c^*_a} \frac{\delta \Gamma}{\delta c_a} \right) \right.
+ B_a \frac{\delta \Gamma}{\delta c_a} + \Theta_{a\mu} \frac{\delta \Gamma}{\delta V_{a\mu}} - K_0 \frac{\delta \Gamma}{\delta \phi_0} \right] = 0 \quad (8)$$

- the LFE

$$W_a(\Gamma) = -\partial_\mu \frac{\delta \Gamma}{\delta V_{a\mu}} + \epsilon_{abc} V_{b\mu} \frac{\delta \Gamma}{\delta V^*_{c\mu}} - \partial_\mu \frac{\delta \Gamma}{\delta A_{a\mu}} + \epsilon_{abc} A_{c\mu} \frac{\delta \Gamma}{\delta A^*_{a\mu}}
+ \frac{1}{2 \Lambda^{D-4}} \delta K_0 \delta \phi_a
+ \frac{1}{2} \epsilon_{abc} \frac{\delta \Gamma}{\delta \phi^*_c} \frac{\delta \Gamma}{\delta \phi^*_b} + \epsilon_{abc} \frac{\delta \Gamma}{\delta \phi^*_c} \frac{\delta \Gamma}{\delta \phi^*_b}
+ \epsilon_{abc} \frac{\delta \Gamma}{\delta \Theta_{c\mu}} + \epsilon_{abc} A^*_{c\mu} \frac{\delta \Gamma}{\delta A_{b\mu}} + \epsilon_{abc} \frac{\delta \Gamma}{\delta c^*_b} \frac{\delta \Gamma}{\delta c^*_b}
+ \frac{1}{2} \frac{\delta \Gamma}{\delta \phi_0} \frac{\delta \Gamma}{\delta \phi_a} + \frac{1}{2} \epsilon_{abc} \phi^*_a \frac{\delta \Gamma}{\delta \phi^*_b} - \frac{1}{2} \frac{\delta \Gamma}{\delta \phi_0} \frac{\delta \Gamma}{\delta \phi_a} = -\Lambda^{D-4} \frac{1}{2} K_0 \phi_a \quad (9)$$

- the Landau gauge equation

$$\frac{\delta \Gamma}{\delta B_a} = \Lambda^{D-4} (D^\mu [V] (A - V)_{\mu})_a \quad (10)$$

and the ghost equation (which holds as a consequence of the ST identity and the Landau gauge equation)

$$\frac{\delta \Gamma}{\delta \bar{c}_a} = -D^\mu_{ab} [V] \frac{\delta \Gamma}{\delta A^*_{b\mu}} + \Lambda^{D-4} (D^\mu [A]\Theta_{\mu})_a. \quad (11)$$
Physical unitarity follows from the ST identity \([8]\). Since
\[
\frac{\delta \Gamma^{(0)}}{\delta K^0} = \Lambda^{D-4} \phi_0 = \Lambda^{D-4} \left( v - \frac{1}{2} \phi_a^2 + \ldots \right)
\]
is invertible, the LFE \([11]\) fixes the dependence of \(\Gamma\) on the \(\phi\)'s once the ancestor amplitudes are known, order by order in the loop expansion. Explicit integration techniques for the LFE, to all orders in the loop expansion, have been studied in \([8]\).

### 3 Batalin-Vilkovisky bracket

In order to elucidate the meaning of the hierarchy in terms of canonical transformations we need to make use of the Batalin-Vilkovisky (BV) formalism \([20, 21]\) for the model at hand. This requires to introduce the antifields \(V^*_a, \Theta^*_a\) paired with \(V^*_a, \Theta^*_a\) and the antifields \(\tilde{c}^*_a, B^*_a\) paired with \(\tilde{c}_a, B_a\). Moreover one also needs the antifield \(K^*_0\) paired with \(K_0\) and the field \(\phi_0^{**}\) paired with the antifield \(\phi_0^*\). \(\phi_0^{**}\) is needed because in the nonlinear theory \(\phi_0\) is not an elementary field.

The BV bracket is defined according to the conventions of \([21]\)
\[
(X, Y) = \int d^D x \sum_I \left[ (-1)^{\epsilon_x + 1} \frac{\delta X}{\delta \Phi^*_I} \frac{\delta Y}{\delta \Phi^*_I} - (-1)^x \frac{\delta X}{\delta \Phi^*_I} \frac{\delta Y}{\delta \Phi^*_I} \right].
\]

\(\Phi_I, \Phi^*_I\) is a collective notation for the fields \(\{A_{a\mu}, \phi_a, c_a, V_{a\mu}, \Theta_{a\mu}, \tilde{c}_a, B_a, K_0, \phi_0^*\}\) and antifields \(\{A^*_a, \phi^*_a, \Theta^*_a, V^*_a, \Theta^*_a, \tilde{c}^*_a, B^*_a, K^*_0, \phi_0^{**}\}\) respectively. \(\epsilon_x\) denotes the statistics of \(x\) (0 for bosons, 1 for fermions). We always use left derivatives.

The couplings of \(\tilde{c}^*_a, V^*_a\) and \(\phi_0^{**}\) are fixed by the BRST transform of their partner in the second line of eq. \([8]\). This leads us to consider the following tree-level vertex functional
\[
\hat{\Gamma}^{(0)} = \Gamma^{(0)}|_{\varepsilon = B = V = \Theta = K = \phi_0^{**} = 0} + \Lambda^{D-4} \int d^D x \left( -\tilde{c}^*_a B_a + V^*_a \Theta^*_a - K^*_0 \phi_0^{**} \right). \tag{14}
\]
The ghost number is assigned as follows. \(A_{a\mu}, \phi_a, V_{a\mu}, B_a, K_0, \phi_0^{**}\) have ghost number zero, \(c_a, \Theta_{a\mu}\) have ghost number one, \(\tilde{c}_a\) and all the other antifields with the exception of \(\tilde{c}^*_a, \Theta^*_a\) have ghost number \(-1\), while \(c^*_a\) and \(\Theta^*_a\) have ghost number \(-2\). \(\hat{\Gamma}^{(0)}\) has ghost number zero.

A canonical transformation (i.e. a transformation preserving the BV bracket in eq. \([13]\)) connects \(\hat{\Gamma}^{(0)}\) to the original tree-level vertex functional in eq. \([11]\).

In order to prove this result it is convenient to use finite canonical transformations of the second type \([31]\). They are obtained from a fermionic generating functional \(F(\Phi, \Phi^*)\) depending on the old fields \(\Phi\) and the transformed antifields \(\Phi^*_I\) according to
\[
\Phi'_I = \frac{\delta F(\Phi, \Phi^*)}{\delta \Phi^*_I}, \quad \Phi^*_I = \frac{\delta F(\Phi, \Phi^*)}{\delta \Phi_I}. \tag{15}
\]

Then the generating functional of the canonical transformation by which \(\Gamma^{(0)}\) is recovered from \(\hat{\Gamma}^{(0)}\) (upon setting in the end \(V^*_a = \phi_0^{**} = \tilde{c}^*_a = 0\) is given by
\[
\mathcal{F} = \int d^D x \left( \phi_0^* (\phi_0^{**} + \phi_0) + \tilde{c}_a (\tilde{c}_{a'}^* - D[V_{\mu}] (A^\mu - V^\mu)_a) \right). \tag{16}
\]
where the prime denotes the new variables. We do not explicitly write in eq. (16) the obvious terms yielding the identity transformation on the relevant fields and antifields.

The second term in eq. (16) is the usual gauge-fixing generating functional (in the background Landau gauge) \[21\]. The first term takes into account the necessity of introducing a source for the nonlinear constraint in order to formulate the LFE.

The ST identity for \( \hat{\Gamma}(0) \) can be finally written as
\[
S(\hat{\Gamma}(0)) = \frac{1}{2\Lambda^{D-1}}(\hat{\Gamma}(0), \hat{\Gamma}(0)) = 0.
\]
This is the master equation \[20, 21\] of the nonlinear theory.

### 4 Quantum and Classical Linearized BV Brackets

We denote by \( \hat{\Gamma} \) the effective action containing the Feynman rules of the theory (tree-level plus counterterms)
\[
\hat{\Gamma} = \sum_{j=0}^{\infty} \hat{\Gamma}^{(j)}.
\]
(18)

Since the theory is non-anomalous and we assume to work in a symmetric regularization scheme, the effective action \( \hat{\Gamma} \) obeys the master equation
\[
(\hat{\Gamma}, \hat{\Gamma}) = 0.
\]
(19)

The operator \( S_{\hat{\Gamma}} = (\hat{\Gamma}, \cdot) \) is nilpotent
\[
S_{\hat{\Gamma}}^2 = 0.
\]
(20)

This follows from the master equation \[19\] and the (graded) Jacobi identity for the BV bracket \[21\]
\[
((X, Y), Z) + (-1)^{(e_x+1)(e_y+e_z)}((Y, Z), X) + (-1)^{(e_x+1)(e_x+e_y)}((Z, X), Y) = 0.
\]
(21)

\( S_{\hat{\Gamma}} \) can be filtered w.r.t. the number of loops
\[
S_{\hat{\Gamma}} = \sum_{j=0}^{\infty} S_j, \quad S_j = (\hat{\Gamma}^{(j)}, \cdot).
\]
(22)

The quantum BV master equation \[19\] can be cast as follows
\[
S_{\hat{\Gamma}} \hat{\Gamma} = 0.
\]
(23)

Notice that the lowest order operator \( S_0 \) in eq. (22) is also nilpotent. This can be seen either by using eq. (17) and the Jacobi identity \[21\] or by taking the lowest order in the expansion of eq. (20) according to the loop number.

This allows to define a mapping \( \mathcal{R} \) between the cohomology classes \([X] \in H_0(S_{\hat{\Gamma}})\) and \([X^{(0)}] \in H_0(S_0)\) at zero ghost number, where \( X = \sum_{j=0}^{\infty} X^{(j)} \) is a local function with
ghost number zero graded according to the loop number. \( X^{(j)} \) denotes the coefficient of order \( j \) in such an expansion.

The cohomology classes \([X]\) of a nilpotent differential operator \( \delta \) are defined by the equivalence relation

\[
X \sim Y \iff \delta X = 0, \ \delta Y = 0, \ \ X = Y + \delta Z
\]

for some functional \( Z \). \( \delta \) is assumed to increase the ghost number by one, as is the case for \( \hat{S}_\Gamma \) and \( S_0 \). If \( X \) and \( Y \) have ghost number zero (and thus \( Z \) has ghost number \( -1 \)), we speak of the cohomology \( H_0(\delta) \) in zero ghost number. The equivalence class of \( X \) is denoted by \([X]\) whenever it is clear to which operator the cohomology class must be referred.

We set

\[
\mathcal{R}[X] = [X^{(0)}].
\]

The mapping \( \mathcal{R} \) is well-defined in cohomology, i.e. \( \mathcal{R}[0_{H_0(\hat{S}_\Gamma)}] = [0_{H_0(S_0)}] \), where \([0_{H_0(\hat{S}_\Gamma)}] \) is the null cohomology class of \( H_0(\hat{S}_\Gamma) \) resp. \( H_0(S_0) \). In fact by expanding \( X = S_\Gamma Y \) according to the loop number one finds

\[
X = X^{(0)} + X^{(1)} + \ldots = (S_0 + S_1 + \ldots)(Y^{(0)} + Y^{(1)} + \ldots).
\]

Therefore at lowest order one gets \( X^{(0)} = S_0 Y^{(0)} \) and thus

\[
\mathcal{R}[X] = [S_0 Y^{(0)}] = [0_{H_0(S_0)}].
\]

The mapping \( \mathcal{R} \) is an isomorphism. This can be proven by using standard methods in homological perturbation theory \[22]\ [28]. A short proof of this result is sketched in Appendix A.

5 Classifying Physical Observables

Since \( H_0(\hat{S}_\Gamma) \) is isomorphic to \( H_0(S_0) \), the computation of \( H_0(S_0) \) is sufficient in order to classify the local physical operators of the theory. In order to carry out this task, we first notice that the perturbation theory based on the tree-level vertex functional in eq.(14) coincides with the one generated by \( \Gamma^{(0)} \) in eq.(1), once the canonical transformation induced by the functional \( F \) in eq.(16) is performed.

In fact the dependence on \( \bar{c}^*_a, V^*_a, \phi^*_0 \) is confined at tree-level due to the validity of the following identities for the vertex functional \( \Gamma \)

\[
\frac{\delta \Gamma}{\delta \bar{c}^*_a} = -\Lambda^{D-4} B_a, \quad \frac{\delta \Gamma}{\delta V^{*}_a} = \Lambda^{D-4} \Theta^*_a, \quad \frac{\delta \Gamma}{\delta \phi^{**}_0} = -\Lambda^{D-4} K^*_0.
\]

Since \( \Gamma^{(n)} \), \( n \geq 1 \) does not depend on \( \bar{c}^*_a, V^*_a, \phi^*_0 \) (as a consequence of eq.(28)) and on \( \Theta^*_a, B^*_a, K^*_0 \) (since they do not enter into \( \hat{\Gamma}^{(0)} \)) we can limit ourselves to the local functional space spanned by \( \{A_{a\mu}, \phi_a, c_a, V_{a\mu}, \Theta_{a\mu}, \bar{c}_a, B_a, K_0\} \) and \( \{A^*_a, \phi^*_a, c^*_a, \phi^*_0\} \).
It is convenient to introduce a matrix notation and set

\[ A_\mu = A_{a\mu} \frac{\tau_a}{2}, \]  

(29)

where \( \tau_a \) are Pauli matrices. The Goldstone fields \( \phi_a \) and the nonlinear constraint \( \phi_0 \) are gathered into the SU(2) matrix

\[ \Omega = \frac{1}{v}(\phi_0 + i\tau_a\phi_a), \quad \Omega^\dagger \Omega = 1, \quad \det \Omega = 1, \phi_0^2 + \phi_a^2 = v^2. \]  

(30)

The SU(2) flat connection is defined in terms of \( \Omega \) by

\[ F_\mu = F_{a\mu} \tau_a = i\Omega \partial_\mu \Omega^\dagger. \]  

(31)

\( F_{a\mu} \) reads in components

\[ F_{a\mu} = \frac{2}{v^2}(\phi_0 \partial_\mu \phi_a - \partial_\mu \phi_0 \phi_a + \epsilon_{abc} \partial_\mu \phi_b \phi_c). \]  

(32)

A finite SU(2)\(_L\) gauge transformation acts as follows:

\[ \Omega' = U_L \Omega, \quad A'_\mu = U_L A_\mu U_L^\dagger + iU_L \partial_\mu U_L^\dagger, \quad F'_\mu = U_L F_\mu U_L^\dagger + iU_L \partial_\mu U_L^\dagger. \]  

(33)

The computation of the cohomology \( H_0(S_0) \) is simplified if one moves to gauge-invariant (bleached) variables, which automatically satisfy the classical linearized LFE [14].

From eq.(33) one sees that the following combination is invariant under a local SU(2)\(_L\) transformation

\[ a_\mu = \Omega^\dagger (A_\mu - F_\mu) \Omega = \Omega^\dagger A_\mu \Omega - i\Omega^\dagger \partial_\mu \Omega. \]  

(34)

We call \( a_\mu \) the bleached counterpart of the original gauge connection \( A_\mu \).

The bleached counterpart of the background connection \( V_\mu \) is

\[ v_\mu = \Omega^\dagger (V_\mu - F_\mu) \Omega = \Omega^\dagger V_\mu \Omega - i\Omega^\dagger \partial_\mu \Omega. \]  

(35)

The bleached counterparts of the ghost field \( c = c_a \frac{\tau_a}{2} \), the ghost background source \( \Theta_\mu = \Theta_{a\mu} \frac{\tau_a}{2} \), the ghost antifield \( c^* = c^*_a \frac{\tau_a}{2} \) are defined by

\[ \tilde{c} = \Omega^\dagger c \Omega, \quad \tilde{\Theta}_\mu = \Omega^\dagger \Theta_\mu \Omega, \quad \tilde{c}^* = \Omega^\dagger c^* \Omega. \]  

(36)

By exploiting the ghost equation (11) or alternatively by performing the canonical transformation in eq.(16) one sees that the vertex functional \( \Gamma \) only depends on the combination

\[ \tilde{A}^*_a = A^*_{a\mu} + (D_\mu[V] \tilde{c})_a. \]  

(37)
The bleached counterpart of \( \tilde{A}_\mu^s = \hat{A}_\mu^s \frac{Z_a}{2} \) is
\[
\tilde{A}_\mu^s = \Omega^\dagger \hat{A}_\mu^s \Omega .
\]  
(38)

Unlike in \([14]\) we do not work with the bleached variables of \( \phi^s_a \). In fact the canonical transformation in eq. (16) generates the combination
\[
\hat{\phi}^s_a = \phi^s_a - \frac{\phi_a^s}{\phi_0} \phi_0^s
\]  
(39)

Then by explicit computation one finds that its \( S_0 \)-variation is
\[
S_0 \hat{\phi}_a^s = \left. \frac{\delta \Gamma^{(0)}}{\delta \phi_a} \right|_{K_0=0}
\]  
(40)

where the R.H.S. is expressed as a function of \( \hat{\phi}_a^s \). I.e. the canonical transformation in eq. (16) allows to recover precisely the tangent space of the group SU(2). This geometrical property is of course expected and becomes transparent in the approach based on the BV formalism. Moreover we notice that \( K_0 \) and \( \phi^s_0 \) form a \( S_0 \)-doublet [28] and consequently they do not enter into the non-trivial cohomology classes of \( H(S_0) \).

By direct computation one then obtains the \( S_0 \)-transforms of the other variables [14]:
\[
S_0 a_\mu = 0 , \quad S_0 \tilde{c} = -\frac{i}{2} \{ \tilde{c}, \bar{\tilde{c}} \} ,
\]
\[
S_0 v_\mu = \tilde{\Theta}_\mu - D_\mu[v] \tilde{c} , \quad S_0 \tilde{\Theta}_\mu = -i \{ \tilde{c}, \tilde{\Theta}_\mu \} ,
\]
\[
S_0 \hat{A}_\mu^s = \Lambda^{b-4} \left[ \frac{1}{g^2} D^\rho G_\rho [a] + M^2 a_\mu \right] ,
\]
\[
S_0 \tilde{c}^s = (D^\rho [a] \hat{A}_\rho^s ) - \frac{i}{4} \Omega^{s\dagger} \Omega + \frac{i}{8} \text{Tr}[\Omega^{s\dagger} \Omega] 1 .
\]  
(41)

\( \Omega^s \) is a \( 2 \times 2 \) matrix defined by
\[
\Omega^s = \phi_0^s + i \phi_a^s \tau_a .
\]  
(42)

The combinations
\[
\tilde{\Theta}_\mu' = \tilde{\Theta}_\mu - D_\mu[v] \tilde{c} , \quad -\frac{i}{4} \Omega^{s\dagger} \equiv -\frac{1}{4} \phi_a^{s\dagger} \tau_a = (D^\mu [a] \hat{A}_\mu^s ) - \frac{i}{4} \Omega^{s\dagger} \Omega + \frac{i}{8} \text{Tr}[\Omega^{s\dagger} \Omega] 1
\]  
(43)

form \( S_0 \)-doublets with \( v_\mu \) and \( \tilde{c}^s \) respectively. Moreover the change of variables \( \tilde{\Theta}_\mu^s \rightarrow \tilde{\Theta}_\mu^s \), \( \phi_a^s \rightarrow \phi_a^{s\dagger} \) is invertible. Thus the pairs \((v_\mu, \tilde{\Theta}_\mu^s ), (\tilde{c}^s, -\frac{i}{4} \Omega^{s\dagger} )\) cannot contribute to the non-trivial cohomology classes of \( H_0(S_0) \). Hence the latter only depend on \( a_\mu, \tilde{c}, \hat{A}_\mu^s \).

The cohomology \( H_0(S_0) \) in this space has been computed in [14, 32] and is given by all possible local polynomials built out from \( a_\mu \) and its ordinary derivatives modulo the ideal generated by the transformation of \( \hat{A}_\mu^s \) (which yields the classical equation of motion for the gauge field \( a_\mu \)) plus cohomologically trivial \( S_0 \)-exact terms. In particular the whole dependence on all variables but \( a_\mu \) is confined to the \( S_0 \)-exact sector.

Since the theory is also invariant under global \( SU(2)_R \) symmetry, only global \( SU(2)_R \)-invariant operators need to be considered.
The construction of bleached variables is not limited to the fields and antifields of pure Yang-Mills theory. As an example, for a fermion matter doublet \( L \) transforming in the fundamental representation of SU(2)

\[
L' = U_L L
\]

its bleached counterpart is

\[
\tilde{L} = \Omega \dagger L.
\]

The construction can be generalized to fields in arbitrary representations of the gauge group along the lines of [32].

6 Canonical Transformations For The Bleached Variables

It remains to be shown that the bleached variables discussed in the previous Section can indeed be obtained via a canonical transformation. For that purpose the easiest way is to provide the relevant generating functional, which looks as follows

\[
F_1 = \int d^D x \left( 2 \, T_r[\tilde{A}_\mu^\dagger (\tilde{\phi}, A_\nu)] + 2 \, T_r[\tilde{c}^\dagger \tilde{c} (\tilde{\phi}, c)] + 2 \, T_r[\tilde{V}_\mu^\dagger v_\mu (\tilde{\phi}, V_\nu)] + 2 \, T_r[\tilde{\Theta}_\mu^\dagger \tilde{\Theta}_\mu (\tilde{\phi}, \Theta)] \right).
\]

The associated canonical transformation automatically induces the bleaching transformation on the antifields. Moreover the redefinition of \( \phi^*_a \) is the one required in order to generate the transformation properties of the bleached variables, as expected.

As an example, let us work out in detail the term proportional to \( \tilde{A}^*_\mu \) of such a redefinition. One finds

\[
\phi^*_a(z) = \frac{\delta F_1}{\delta \phi_a(z)}
= \frac{\delta}{\delta \phi_a(z)} \int d^D x \, T_r[\tilde{A}^*_\mu(i \Omega \dagger \partial^\mu \Omega + \Omega \dagger A^\mu \Omega)]
= 2 \int d^D x \, T_r[\tilde{A}^*_\mu(i \Omega \dagger \partial^\mu \Omega + \Omega \dagger \frac{\delta}{\delta \phi_a(z)} \partial^\mu \Omega]
+ \frac{\delta \Omega \dagger}{\delta \phi_a(z)} A^\mu \Omega + \Omega \dagger A^\mu \frac{\delta \Omega}{\delta \phi_a(z)}],
\]

\[
= 2 \int d^D x \, T_r[\tilde{A}^*_\mu(i \Omega \dagger \partial^\mu \Omega + \Omega \dagger \frac{\delta}{\delta \phi_a(z)} \partial^\mu \Omega]
+ \frac{\delta \Omega \dagger}{\delta \phi_a(z)} (\Omega \partial_{\mu \dagger} + i \Omega \partial_{\mu \dagger} \Omega)]
+ \Omega \dagger (\Omega \partial_{\mu \dagger} + i \Omega \partial_{\mu \dagger} \Omega) \frac{\delta \Omega}{\delta \phi_a(z)}].
\]

By using the nonlinear constraint one gets

\[
0 = \frac{\delta}{\delta \phi_a(z)} (\Omega \dagger \Omega) = \frac{\delta \Omega \dagger}{\delta \phi_a(z)} \Omega + \Omega \dagger \frac{\delta \Omega}{\delta \phi_a(z)}.
\]
By using eq.(48) into eq.(47) we end up with

$$\phi^*_a(z) = 2 \int d^D x T_r[A^*_\mu i\Omega^\dagger \delta/\delta\phi_a(z) \delta^{\mu\Omega} + i\delta\mu\Omega^\dagger \delta\phi_a(z) + [a_\mu, \Omega^\dagger \delta\phi_a(z)]]]$$

This has to be inserted back into the piece

$$\int d^D x \phi^*_a s_\phi_a$$

of the tree-level vertex functional. By plugging eq.(49) into eq.(50) one finally obtains

$$2i \int d^D x T_r[A^*_\mu D^\mu[a] (\Omega^\dagger \delta/\delta\phi_a(z)) = 2i \int d^D x T_r[A^*_\mu D^\mu[a] (\Omega^\dagger \delta\phi_a(z))]$$

which exactly cancels the $\tilde{A}^*$-dependent term in $\Gamma^{(0)}$. This cancellation corresponds to the fact that the bleached variable $a_\mu$ is $S_0$-invariant and therefore there should be no dependence on its antifield in $\Gamma^{(0)}$ (after the canonical transformation generated by $F_1$ is implemented).

7 Comparison With The Linear Theory

As it has been shown in Sect. 5, the physical observables of the theory are classified by $H_0(S_0)$, which in turn is given by all possible global $SU(2)_R$-invariant local polynomials in $a_\mu$ and ordinary derivatives thereof which do not vanish on the classical equation of motion of $a_\mu$.

The requirement of the validity of the WPC selects the Yang-Mills field strength squared and the Stieltjes mass term as the only possible physical operators admissible in the tree-level vertex functional [14].

A comparison with the linearly realized $SU(2)$ Yang-Mills theory can be useful. In the linearly realized framework the trace component $h$ of the $2 \times 2$ matrix

$$H = h + i\phi_a \tau_a, \quad h = v + \sigma$$

is an independent degree of freedom. The latter is parameterized by the Higgs field $\sigma$. $h$ acquires the vacuum expectation value $v$ via spontaneous symmetry breaking and correspondingly $\langle \sigma \rangle = 0$.

The construction of gauge-invariant variables out of $A_{a\mu}$ and $\sigma$ is easily performed via the field redefinitions

$$A_{a\mu} \rightarrow \tilde{h}_{a\mu} = T_r \left\{ iH^H H^\dagger D_{\mu}[A] H \tau_a \right\}, \quad \sigma \rightarrow \tilde{\sigma} = \sqrt{H^H H} - v,$$
where $D_\mu[A]$ is the covariant derivative

$$D_\mu[A] = \partial_\mu - iA_{\mu}^a \frac{\tau_a}{2}.$$  \hspace{1cm} (55)

Since $H$ transforms in the fundamental representation of SU(2)

$$H' = U_L H,$$ \hspace{1cm} (56)

the R.H.S. of eqs. (53), (54) are automatically gauge invariant. In fact eqs. (53), (54) can be understood as the result of an operatorial finite gauge transformation, generated by the matrix $H^\dagger / \sqrt{H^\dagger H} \in SU(2)$, acting on $A_\mu$ and $H$ respectively. At $\phi_a = 0$ $\tilde{h}_{a\mu}$ and $\tilde{\sigma}$ reduce to $A_{a\mu}$ and $\sigma$. In the linearly realized theory $\sigma$ is an ancestor field.

Any functional built out of $\tilde{h}_{a\mu}, \tilde{\sigma}$ and ordinary derivatives thereof is gauge-invariant. As a consequence, the following mass bilinears

$$m_{ab} \tilde{h}_{a\mu} \tilde{h}_{b\mu}, \quad m_{ab} = m_{ba}$$ \hspace{1cm} (57)

are admissible on symmetry grounds. However, upon expansion of $\tilde{h}_{a\mu}$ in components

$$\tilde{h}_{a\mu} = A_{a\mu} - \frac{2}{v} \left( \partial_\mu \phi_a + \epsilon_{abc} A_{b\mu} \phi_c \right) + \frac{2}{v^2} \left( \sigma \partial_\mu \phi_a + \phi_a \partial_\mu \sigma - A_{a\mu} \tilde{\phi}^2 - \epsilon_{abc} \phi_b \partial_\mu \phi_c \right) + O(1/v^3)$$ \hspace{1cm} (58)

one sees that eq. (57) contains vertices involving two $\sigma$'s, two $\phi$'s and two derivatives. Thus at one loop level diagrams like those in Fig. 1 arise. They are logarithmically divergent irrespective of the number of external $\sigma$-legs. A similar argument shows that the kinetic term $\partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma}$ contains a vertex $\sim \frac{1}{v^2} \sigma \partial_\mu \sigma \partial^\mu \phi_a$, which gives rise to the same diagrams as in Fig. 1. This implies that the WPC is maximally violated in the linear theory, unless one chooses the combination

$$T_r \left\{ H^\dagger H \frac{(-i)}{H^\dagger H} (D_\mu[A] H)^\dagger H - \frac{i}{H^\dagger H} H^\dagger D^\mu[A] H \right\} = T_r \{(D_\mu[A] H)^\dagger D^\mu[A] H\},$$ \hspace{1cm} (59)

i.e. the WPC in the linear theory is strong enough to select a single gauge-invariant combination which boils down to the usual covariant kinetic term (59) for the Higgs doublet $H$.

8 SU(2) $\otimes$ U(1)

For the EW group SU(2) $\otimes$ U(1) the SU(2) custodial symmetry is violated in the fermionic sector. In the nonlinearly realized theory this fact entails that two independent gauge bosons mass invariants can be introduced in a way compatible with the WPC. They can be parameterized as

$$M^2 Tr \left\{ \left( g A_\mu - \frac{g'}{2} \Omega_3 B_\mu \Omega^\dagger - F_\mu \right)^2 \right\} + M^2 \kappa \left( Tr \left\{ \left( g A_\mu - \frac{g'}{2} \Omega_3 B_\mu \Omega^\dagger - F_\mu \right) \tau_3 \right\} \right)^2.$$ \hspace{1cm} (60)
Figure 1: One-loop divergent graph with arbitrary number of external Higgs legs. Dashed lines denote Goldstone propagators (in the 't Hooft gauge).

g, g' are the SU(2) and U(1)$_Y$ coupling constants repectively, $B_\mu$ the U(1)$_Y$ connection. Notice that we have restored the coupling constants in front of the gauge fields in order to match with the conventions of [16]. The action of U(1)$_Y$ on $\Omega$ is on the right, i.e.

$$\Omega' = \Omega V^\dagger, \quad V = \exp \left( i\alpha \frac{\tau_3}{2} \right).$$

One can introduce the bleached (SU(2) invariant) combination [16]

$$w_\mu = w_{a\mu} \frac{\tau_a}{2} = g\Omega^\dagger A_\mu \Omega - g'B_\mu \frac{\tau_3}{2} + i\Omega^\dagger \partial_\mu \Omega$$

Under U(1)$_Y$ one gets

$$w'_\mu = V w_\mu V^\dagger. \quad (63)$$

Since $w_\mu$ is SU(2)-invariant, by the Gell-Mann-Nishijima formula one sees that on these variables the action of U(1)$_Y$ coincides with the action of U(1)$_{em}$. The two independent bilinears in eq.(60) correspond to independent mass terms for the two electrically neutral combinations

$$M_2^2 \left( w^+ w^- + \frac{1}{2} w_3^2 \right), \quad \frac{M_2^2 \kappa}{2} w_3^2. \quad (64)$$

On the other hand, for the linearly realized EW model the WPC condition by itself is sufficient in order to impose the validity of the tree-level Weinberg relation. I.e. relaxing power-counting renormalizability (in favour of the weaker WPC condition) does not allow to introduce a second independent mass parameter for the non-Abelian gauge bosons.

The argument closely parallels the one given in Sect. 6 interaction vertices with two $\sigma$’s, two $\phi$’s and two derivatives only disappear if the combination (59) is chosen, where now the covariant derivative must be replaced with its SU(2) $\otimes$ U(1) counterpart, i.e.

$$D_\mu H = \partial_\mu H - ig A_{\mu a} \frac{\tau_a}{2} H - ig' H B_\mu \frac{\tau_3}{2}.$$
The low-energy limit of the more fundamental theory endowed with the exact custodial symmetry can lead to a model where two independent mass parameters in the vector meson sector are allowed only if at low energies the EW symmetry is non-linearly realized. On the other hand, a stronger remnant (imposing the exact relation $\kappa = 0$) would be in place if the low-energy realization of the EW symmetry were linear.

If a global fit (including radiative corrections) to EW precision data [35] favours a solution where $\kappa \neq 0$, this might be an indirect evidence that at LEP energies the EW symmetry is in fact nonlinearly realized.

9 Conclusions

We have classified the physical observables in the nonlinearly realized massive SU(2) Yang-Mills theory within the mathematically consistent framework governed by the LFE and the WPC. This approach allows to take into account the non-trivial quantum deformations of the nonlinearly realized gauge symmetry to all orders in the loop expansion.

It has been shown that the bleached variables, introduced in [12, 14, 16] as solution of the linearized LFE, can be obtained through a canonical transformation w.r.t. the BV bracket associated with the BRST symmetry of the model. In this process the tangent space of the group SU(2) emerges naturally.

The role of the WPC in the linear vs. the nonlinear realization of the gauge symmetry has been clarified. We have found that the tree-level Weinberg relation in the EW theory holds even though power-counting renormalizability is dropped in favour of the weaker WPC.

From the modern point of view which considers the SM as a very accurate effective approximation to a more fundamental theory, the WPC can be used as a unified guiding tool for controlling the loop perturbative expansion both in the linearly and in the nonlinearly realized EW theory.

One can then compare the global fit to the existing LEP precision data [35]. Should the solution with a non-zero mass parameter $\kappa$ be preferred, this might represent a rather intriguing indication that the EW symmetry is in fact nonlinearly realized at the LEP scale. This would also point towards a scenario with no SM Higgs, an option which could be experimentally investigated at the LHC in the coming years [34].

The fit within the nonlinear EW model must face some non-trivial issues. Explicit computations [36] show in fact that the radiative corrections to pseudo-observables at the Z pole in the nonlinearly realized EW theory are not-oblique [37] and get some flavour-dependent non-SM-like corrections (via their top mass dependence). The comparison with the experimental data must be performed in such a way that the SM-dependent assumptions, controlling the experimental fit of Ref. [35], are properly taken into account in the non-linear setting, in particular in connection with the dependence on the second mass parameter of the hadronic contribution to the Z-$\gamma$ interference term [35]. This deserves further investigation.
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A \( H_0(S_{\hat{\Gamma}}) \sim H_0(S_0) \)

In this Appendix we prove that the mapping \( \mathcal{R} \) between \( H_0(S_{\hat{\Gamma}}) \) and \( H_0(S_0) \) is an isomorphism.

In order to show that \( \mathcal{R} \) is one-to-one we prove that \( \ker \mathcal{R} = \{ 0 \} \). If \( [I] \in \ker \mathcal{R} \), then its lowest order coefficient \( I^{(0)} \) is \( S_0 \)-exact, i.e.

\[
I^{(0)} = S_0 G_0
\]

for some local function \( G_0 \). Then one can write

\[
I = I^{(0)} + I - I^{(0)} = S_0 G_0 + I - I^{(0)} = S_{\hat{\Gamma}} G_0 + H_1
\]

where

\[
H_1 = -(S_{\hat{\Gamma}} - S_0) G_0 + I - I^{(0)}
\]

is of order at least one in the loop expansion. Let us now suppose that \( I \) is \( S_{\hat{\Gamma}} \)-exact up to order \( k \):

\[
I = S_{\hat{\Gamma}} G_{j-1} + H_j, \quad j = 1, 2, \ldots, k
\]

\( H_j \) is at least of order \( j \) in the loop expansion. By the nilpotency of \( S_{\hat{\Gamma}} \) one obtains from eq. (69)

\[
S_{\hat{\Gamma}} H_k = 0
\]

By projecting the above equation at the lowest non-vanishing order one gets

\[
S_0 H_k^{(k)} = 0
\]

Since we assume that the theory is non-anomalous, the cohomology of \( S_0 \) is empty in ghost number one, i.e. there exists a local function \( G^{(k)} \) such that

\[
H_k^{(k)} = S_0 G^{(k)}
\]

Then

\[
I = S_{\hat{\Gamma}} G_{k-1} + H_k^{(k)} + H_k^{(k+1)} + \ldots
\]

\[
= S_{\hat{\Gamma}} G_{k-1} + S_0 G_k^{(k)} + H_k^{(k+1)} + \ldots
\]

\[
= S_{\hat{\Gamma}} (G_{k-1} + G_k^{(k)}) - (S_{\hat{\Gamma}} - S_0) G_k^{(k)} + H_k - H_k^{(k)}
\]

\[
= S_{\hat{\Gamma}} G_k + H_{k+1}
\]
where $G_k = G_{k-1} + G_k^{(k)}$ and

$$H_{k+1} = -(S_\Gamma - S_0)G_k^{(k)} + H_k - H_k^{(k)}$$  \hspace{1cm} (74)

i.e. $I$ is $S_\Gamma$-exact up to order $k+1$.

Moreover if

$$S_0 I^{(0)} = 0 \hspace{1cm} (75)$$

one can recursively find coefficients $I^{(j)}$, $j \geq 1$ in such a way that

$$I = \sum_{j=0}^\infty I^{(j)} \hspace{1cm} (76)$$

is $S_\Gamma$-invariant. This can be proven as follows. Nilpotency of $S_\Gamma$ yields at order one

$$S_0 S_1 + S_1 S_0 = 0 \hspace{1cm} (77)$$

By using the above equation one obtains from eq.\((75)\)

$$S_0 S_1 I^{(0)} = 0 \hspace{1cm} (78)$$

Since the cohomology of $S_0$ is empty at ghost number one (no anomalies), there exists a local function $I^{(1)}$ such that

$$S_1 I^{(0)} = -S_0 I^{(1)} \hspace{1cm} (79)$$

Therefore

$$S_1 I^{(0)} + S_1 I^{(0)} = 0 \hspace{1cm} (80)$$

Suppose now that $S_\Gamma I = 0$ holds up to order $n - 1$

$$\sum_{j=0}^m S_j I^{(m-j)} = 0, \hspace{1cm} m = 0, 1, \ldots, n - 1 \hspace{1cm} (81)$$

Then

$$S_\Gamma \sum_{k=0}^{n-1} I^{(k)} = \Delta^{(n)} + \ldots \hspace{1cm} (82)$$

where

$$\Delta^{(n)} = \sum_{j=1}^n S_j I^{(n-j)} \hspace{1cm} (83)$$

Again by the nilpotency of $S_\Gamma$ one gets

$$S_\Gamma \Delta^{(n)} = 0, \hspace{1cm} (S_0 + S_1 + \ldots)(\Delta^{(n)} + \ldots) = 0 \hspace{1cm} (84)$$
The projection of the above equation at lowest order gives

$$S_0 \Delta^{(n)} = 0 \quad (85)$$

which, again under the assumption that no anomalies are present, implies

$$\Delta^{(n)} = -S_0 I^{(n)} \quad (86)$$

for a local function $I^{(n)}$ of order $n$ in the loop expansion. Then by eq. $(86)$ one has

$$S_0 I^{(n)} + \sum_{j=1}^{n} S_j I^{(n-j)} = 0 \quad (87)$$

i.e. eq. $(81)$ holds also at order $n$. This concludes the proof.

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