Refined Estimates for Simple Blow-ups of the Scalar Curvature Equation on $S^n$

Man Chun LEUNG
National University of Singapore*
matlmc@nus.edu.sg

Abstract

In their work [12] on a sharp compactness theorem for the Yamabe problem, Khuri, Marques and Schoen apply a refined blow-up analysis (what we call ‘second order blow-up argument’ in this article) to obtain highly accurate approximate solutions for the Yamabe equation. As for the conformal scalar curvature equation on $S^n$ with $n \geq 4$, we examine the second order blow-up argument and obtain refined estimate for a blow-up sequence near a simple blow-up point. The estimate involves local effect from the Taylor expansion of the scalar curvature function, global effect from other blow-up points, and the balance formula as expressed in the Pohozaev identity in an essential way.

Key Words: Scalar Curvature Equation; Blow-up; Balance Formula.

2010 AMS MS Classification: Primary 35J60; Secondary 53C21.

* Department of Mathematics, National University of Singapore, 10, Lower Kent Ridge Rd., Singapore 119076, Republic of Singapore.

** e-Appendix starts at page 44 onward.
§ 1. Introduction.

In this article, we expound local and global contributions to a refined ‘second order’ estimate for simple blow-ups (or simple isolated blow-ups as known in some literature) of the prescribed scalar curvature equation

\[(1.1) \quad \Delta_1 u - \tilde{c}_n n(n - 1) u + (\tilde{c}_n K) u \frac{n+2}{n-2} = 0 \quad \text{on} \quad S^n.\]

Here \( K \), fixed once it is given, is assumed to be smooth enough [say, in \( C^{n+4}(S^n) \)], \( \Delta_1 \) is the Laplacian on \( S^n \) with the standard metric \( g_1 \), and \( \tilde{c}_n = \frac{n-2}{4(n-1)} \) (\( n \geq 3 \)).

Via the stereographic projection \( \hat{P} : S^n \setminus \{N\} \to \mathbb{R}^n \), which sends the north pole \( N \in S^n \) to infinity, equation (1.1) can be expressed in the simple form

\[(1.2) \quad \Delta_0 v + (\tilde{c}_n K) v \frac{n+2}{n-2} = 0,\]

\[(1.3) \quad v(y) := u(\hat{P}^{-1}(y)) \cdot \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}} \quad \text{and} \quad K(y) := K(\hat{P}^{-1}(y)) \quad \text{for} \quad y \in \mathbb{R}^n.\]

In (1.2), \( \Delta_0 \) is the Laplacian on \( \mathbb{R}^n \) with the standard Euclidean metric \( g_0 \). Considered as a ‘dual’ to the Yamabe equation, the study on the non-compact behavior (or blow-up phenomenon) of equation (1.1) is found to be rich and vibrant. See for examples [7] [8] [15] [21] [22], and the references therein.

§ 1 a. Simple blow-up. When \( (\tilde{c}_n K) \) is equal to a constant, say, \( n(n - 2) \), equation (1.2) has a family of solutions:

\[(1.4) \quad A_{\epsilon_i, \zeta_i}(y) = \left( \frac{\epsilon_i}{\epsilon_i^2 + |y - \zeta_i|^2} \right)^{\frac{n-2}{2}}.\]

For non-constant \( K \), a sequence of positive solutions \( \{v_i\} \) of (1.2) which blows up at 0 is shown to be ‘close’ to a sequence found in (1.4). Precisely,

\[(1.5) \quad |v_i(y) - A_{\epsilon_i, \zeta_i}(y)| \leq \epsilon_i \cdot \epsilon_i^{-\frac{n+2}{n-2}} \quad \text{for} \quad |y - \zeta_i| \leq \epsilon_i R_i \quad \text{and} \quad i \gg 1,\]

with parameters \( \epsilon_i \to 0 \), \( |\zeta_i| \to 0 \) and \( R_i \to \infty \) specific to \( \{v_i\} \) [cf. (2.21) in § 2 d]. Here (via a rescaling), we assume throughout this article that

\[(1.6) \quad (\tilde{c}_n K)(0) = n(n - 2).\]

Estimate (1.5) is rather weak - its accuracy in general deteriorates when \( i \to \infty \). Moreover, (1.5) is valid (generally) in a sequence of shrinking balls \( B_{\zeta_i}(\epsilon_i R_i) \). [The
order of shrinkage $O(\epsilon_i)$ makes space for the bubbles described in (1.4) to be stacked up (developed vertically, cf. the Delaunay solution \[13\]), or be put in juxtaposition (developed horizontally). See \[15\] for a classification of blow-ups for equation (1.1).

One can characterize simple blow-up in a geometric manner - when the bound in (1.5) can be stabilized in terms of scale and accuracy, namely,

\begin{equation}
\frac{1}{C} \cdot A_{\epsilon_i, \zeta_i}(y) \leq v_i(y) \leq C \cdot A_{\epsilon_i, \zeta_i}(y) \text{ for all } |y - \zeta| \leq \rho_o \text{ and } i \gg 1.
\end{equation}

Here $\rho_o$ and $C$ are fixed positive numbers. See Proposition 2.24 for the precise statement. Cf. also the notion of quasi-isometry. Simple blow-ups is by far the most common non-compact behavior we encounter in equation (1.1). In \[16\] \[17\] \[19\], blow-up sequence with a fixed nonconstant $K$ (may not be symmetric) are constructed using the Lyapunov-Schmidt reduction method. See also \[26\].

§ 1 b. Description of the main result. In this article, we identify three factors affecting the fixed scale behavior of simple blow-ups.

I) The local behavior of $K$ in terms of the Taylor expansion

\begin{equation}
(\bar{c}_n K)(y) = n(n - 2) + [-P_\ell(y)] + R_{\ell+1}(y) \text{ for } y \in B_o(\rho_o).
\end{equation}

Here $P_\ell$ is a homogeneous polynomial of degree $\ell \in \mathbb{N}$, and $R_{\ell+1}$ the remainder in the Taylor expansion. [See (2.29) and (3.7) for the sign convention we use on $P_\ell$.] We know that if 0 is a blow-up point for equation (1.2), then $\ell \geq 2$ (that is, $\nabla K(0) = 0$; see Theorem 5.1 in \[15\] for the precise statement; cf. also \[7\]). Hence

\begin{equation}
\text{(1.9) number of critical points of } K \text{ is finite} \implies \text{equation (1.1) has at most finite number of blow-up points.}
\end{equation}

The leading polynomial term $P_\ell$ comes into the picture when we find the difference between $v_i$ and the standard solutions given in (1.4). See (3.11). The second order blow-up argument allows us to discern the central information enveloped in $P_\ell$. We discuss this point more in § 1 c and § 1 d.

II) ‘Flexibility’ of the simple blow-up as measured by $|\xi_i| = O(\lambda_i^a)$,

\begin{equation}
\text{(1.10) where } v_i(\xi_i) = \max \left\{ v_i(y) \mid y \in \overline{B_o(\rho_o)} \right\} \text{ for } i \gg 1,
\end{equation}

\begin{equation}
\text{(1.11) and } \lambda_i := [v_i(\xi_i)]^{-\frac{2}{n-2}}, \quad \xi_i \to 0 \text{ (the blow-up point).}
\end{equation}
Here \( \rho_o \) is a small fixed number (its size is related to other blow-up points). \( v_i \) could have other maximal points near \( \xi_i \), but their distances to \( \xi_i \) are at most \( o(\lambda_i) \) for \( i \gg 1 \). Refer to \( \S \) 2 g. The position parameter \( \xi_i \) appears in the expression for the difference \( [v_i - A_{\lambda_i}, \xi_i] \), see (3.11). Thanks to the work of Chen and Lin [7] [8], one can impose conditions, including the following main ones (see \( \S \) 2 g for the full detail):

\[
\| \nabla P_\ell(y) \| \geq C |y|^{\ell-1} \quad \text{for } y \in B_o(\rho_o)
\]

and \[
\int_{\mathbb{R}^n} \nabla P_\ell(y + \mathcal{X}) \cdot [A_1(y)]^{2n/\ell-2} \, dy \neq 0 \quad \text{for all } \mathcal{X} \in \mathbb{R}^n \setminus \{0\},
\]

resulting in

(1.12) \( |\xi_i| = o(\lambda_i) \) modulo a subsequence \( \{\lambda_i\} \) (that is, \( \lambda_i^{-1} \cdot \xi_i \to 0 \)).

III) Interaction with other blow-ups. This is expressed by a global harmonic function (or Green’s function)

(1.13) \[
\sum_{j=0}^k \frac{A_j}{|y - Y_j|^{n-2}} \quad \text{for } y \in \mathbb{R}^n \setminus \{\hat{Y}_0, \hat{Y}_1, \cdots, \hat{Y}_k\},
\]

‘effective’ outside a neighborhood containing all the blow-up points \( \{\hat{Y}_j\} \). \( A_j \) are positive numbers. A major challenge here is to match the information expressed in (1.13) (the ‘collapsed region’) with the one in (1.5) (the ‘blow-up’ region). See \( \S \) 2 e for a fuller discussion.

Main Theorem 1.14. For \( n \geq 4 \), let \( u_i \in C^{n+4}(S^n) \) be a sequence of positive solutions of equation (1.1), with \( K \in C^{n+4}(S^n) \), and \( v_i \) and \( K \) be associated to \( u_i \) and \( K \) via (1.3), respectively. Assume that \( \{u_i\} \) has a finite number of blow-up points – one of them is at the south pole, but none at the north pole. Take the following conditions (1.15) – (1.19) into account.

(1.15) \( 0 \) is a simple blow-up point for \( \{v_i\} \).

(1.16) \( K > 0 \) in \( S^n \), and \( K \) is given by the Taylor expansion in (1.8) in \( B_o(\rho_o) \).

(1.17) \( 2 \leq \ell \leq n - 2 \).

(1.18) The parameters \( \lambda_i \) and \( \xi_i \) corresponding to the simple blow-up point at 0

\[ \text{[via (1.10) and (1.11), respectively]} \]

satisfy (1.12), that is, \( |\xi_i| = o(\lambda_i) \).

(1.19) When \( \ell = n - 2 \) is even and there are more than one blow-up point

\[ \]
or when \( \ell \) is odd, we require that \( \Delta^{(h\ell)} \mathbf{P}_\ell(y) \equiv 0 \). Here \( h\ell \) is biggest
integer less than or equal to \( \ell / 2 \).

Then we can determine a polynomial \( \Gamma \) (constructible from \( \mathbf{P}_\ell \) via a fixed procedure),
so that the following estimate holds (modulo a subsequence)

\[
(1.20) \quad \left| v_i(y) - A_{\lambda_i, \xi_i}(y) - \left[ \lambda_i^{\ell+1} \times \Gamma(Y) \right] \cdot \mathbf{A}_{\lambda_i, \xi_i}(y) \right|^{n-2} - \mathcal{O}_H\left(\frac{n-2}{\lambda_i}\right) = o\left(\lambda_i^n\right)
\]

for \( y \in B_o(\rho_1) \) \( (\rho_1 \leq \rho_o \text{ is fixed}) \), where \( Y = \frac{y - \xi_i}{\lambda_i} \).

Here the term \( \mathcal{O}_H\left(\frac{n-2}{\lambda_i}\right) \) is defined via the global harmonic term \( (1.13) \), and its
precise expression is found in \( (6.58) \). (The precise construction of \( \Gamma \) is given in
Proposition \( 4.49 \)).

§ 1 c. Necessity of the condition \( \Delta^{(h\ell)} \mathbf{P}_\ell(y) \equiv 0 \). At first sight the condition

\[
(1.21) \quad ^{*} \Delta^{(h\ell)} \mathbf{P}_\ell(y) = \Delta_o \left( \cdots \left[ \Delta_o \left( \Delta_o \mathbf{P}_\ell \right) \right] \cdots \right)(y) \equiv 0
\]

\( h\ell \) for all \( y \in \mathbb{R}^n \) reveals that it is an
integrated part of the discussion. In fact, under the conditions in Main Theorem \( 1.14 \),
when \( \ell \) is even, we obtain \( \Delta^{(h\ell)} \mathbf{P}_\ell(y) \equiv 0 \) with the help of the Pohozaev identity (see
Proposition \( 6.1 \)). The vanishing of \( \Delta^{(h\ell)} \mathbf{P}_\ell \) allows us to construct the polynomial \( \Gamma \) in
Main Theorem \( 1.14 \) via a reduction method, which we begin to expound.
§ 1 d. Key features of the proof. In [12], Khuri, Marques and Schoen introduce refined blow-up estimates for the Yamabe equation. The method is based on a second order approximation coupled with a second order blow-up argument. We apply these methods to the scalar curvature equation (1.1), and highlight the following differences.

The second order inhomogeneous equation is given by

\[(1.22) \quad \Delta_o \Phi + n (n + 2) A_{1}^{\frac{1}{n+2}} \cdot \Phi = P_{\ell} \cdot A_{n+2}^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n.\]

Here \(A_1 = A_{1,0}\) as given in (1.4). We observe that the linear operator appeared on the left hand side of (1.22) – it is used extensively in the Lyapunov- Schmidt reduction method, see for examples [2] [4] [5] [16] [17] [19]. In [12], a solution of (1.22) is found by a linear algebra method. The method does not disclose the precise form of the solution, which is desirable when we construct sharper estimates for simple blow-ups. In this manuscript, we introduce a reduction method, which explores the recursive relations in equation (1.22) (expounded in § 4). The condition \(\Delta_o^{(b)l}P_{l} \equiv 0\) comes into the picture when we terminate the recursive process. As a consequence, we can determine in a step-by-step manner the exact form of the solution \(\Phi\). Although the detail is shown in § 4, we indicate here that we know precisely what is \(\Gamma\) in (1.23), once \(P_{\ell}\) is given.

Another unique feature here is that the global influence from other blow-up points has to be taken into account when we estimate in the accuracy of \(O(\lambda_i^{n-2})\) (see § 5). To do so, we have to extend the information given by the harmonic function in (1.13) to the whole neighborhood of the blow-up point at 0, in a manner so that the second blow-up argument still works. See (6.7), § 5 b and § 6.

§ 1 e. Applications: limitation on ‘flexibility’ of simple blow-up, and locations of the blow-up points. Consider the parameters \(\lambda_i\) and \(\xi_i\) corresponding to the simple blow-up point at 0 via (1.10) and (1.11). Suppose that

\[(1.23) \quad \xi_i = \lambda_i^\alpha \cdot \vec{X} \quad \text{for a fixed vector } \vec{X} \text{ and a fixed number } \alpha > 0,
\]

where \(1 < \alpha < 2\). Assume also that 0 is the only simple blow-up point and \(\ell = n - 2\). Then we have

\[(1.24) \quad P_{\ell}(\vec{X}) = 0 \quad (\text{here } \vec{X} \text{ is considered as a point in } \mathbb{R}^n).\]

See Theorem A.6.62 in the e-Appendix for the precise statement and full layers of information available, as well as the conditions for (1.24) to hold. In case of multiple simple blow-up points with Taylor expansions at each blow-up point given as in (1.8), where uniformly \(\ell = n - 2\), similar limitations exist, and they involve the locations of the simple blow-up points. See § A.7 in the e-Appendix for the exact formulas.
The information should be helpful when one seeks examples and investigates situations with multiple simple blow-up points. Cf. [20] on using the interaction of two close bubbles to find solutions of equation (1.1) for certain functions $K$. As a footnote, only recently blow-up sequence with a single simple blow-up point is constructed for a fixed and none identically constant $K$ [16] [17] [19]. Cf. also [14], and [4] [5] for the Yamabe equation.

§ 1f. General conditions, assumptions and conventions.

To keep the notation clean, and without losing sight of the technical details, we assume that

\begin{equation}
(1.25) \quad u_i \text{ and } K \text{ are in } C^{n+4}(S^n), \ u_i \text{ is a positive solution of (1.1)}.
\end{equation}

\begin{equation}
(1.26) \quad "v_i \text{ and } K \text{ descend from } u \text{ and } K \text{ via (1.3). Moreover, } K > 0 \text{ in } S^n, \text{ and } (\tilde{c}_n K)(0) = n(n - 2)."
\end{equation}

The degree of smoothness assumed on $u$ and $K$ can be reduced according to the content (especially in § 2).

\begin{itemize}
  \item[\bullet_1] Throughout this work, the dimension $n \geq 3$, except when otherwise is specifically mentioned, and $\tilde{c}_n = (n - 2)/[4(n - 1)]$. We observe the practice on using $C$, possibly with sub-indices, to denote various positive constants, which may be rendered differently from line to line according to the contents. Whilst we use $\bar{c}$ and $\bar{C}$, possibly with sub-index, to denote a fixed positive constant which always keeps the same value as it is first defined.
  
  \item[\bullet_2] Denote by $B_y(r)$ the open ball in $(\mathbb{R}^n, g_0)$ with center at $y$ and radius $r > 0$. Likewise, let $B_x(\rho)$ be the open ball in $(S^n, g_1)$ with center at $x \in S^n$ and radius $\rho \in (0, \pi]$. We also use the standard notation $\langle , \rangle$ to denote the inner product in $(\mathbb{R}^n, g_0)$.
  
  \item[\bullet_3] Given a sequence of positive numbers $\{\lambda_i\}$, and a positive number $m$, we say that a sequence of numbers $\{\gamma_i\}$ satisfies

\begin{equation}
(1.27) \quad \gamma_i = O_{\lambda_i}(m) \iff |\gamma_i| \leq C \lambda_i^m \quad \text{for } i \gg 1.
\end{equation}

Likewise,

\begin{equation}
\gamma_i = o_{\lambda_i}(m) \iff |\gamma_i| \leq c_i \lambda_i^m \quad \text{for } i \gg 1, \quad \text{where } c_i \geq 0 \text{ and } c_i \to 0 \quad \text{as } i \to \infty.
\end{equation}

The notations help to highlight the order and manage longer expressions inside the brackets.
A statement involving a sequence is said to hold “modulo a subsequence” if we can select a subsequence (from the original sequence in the statement) so that the statement is valid for this subsequence. As a rule, we assume that the statement is true for the original sequence so that the notations remain clean.

§ 2. Simple blow-up.

§ 2 a. Simple blow-up and its analytic definition. Intuitively, simple blow-up develops precisely one bubble in a neighborhood. Its analytic definition is given by R. Schoen in [25]. See also [12] and [21]. Via a rotation, we assume without loss of generality that the blow-up point is at the south pole. Let \( \{v_i\} \) be given as in (1.3). For a simple blow-up point, there exists a sequence \( \{\xi_{m_i}\} \rightarrow 0 \) such that

\[
\text{(2.1)}
\]

for each \( i \gg 1 \), \( \xi_{m_i} \) is a local maximum of \( v_i \), with \( \lim_{i \to \infty} v_i(\xi_{m_i}) = \infty \), and the rescaled average

\[
\text{(2.2)}
\]

has precisely one critical point in \((0, \rho_o)\). Here \( \rho_o > 0 \) is fixed (independent on \( v_i \) for \( i \gg 1 \)).

§ 2 b. Proportionality of simple blow-ups. The following estimate is essentially taken from Proposition 2.3 in [21]. We present it in the setting of this article.

Proportionality Proposition 2.3. Under the standard conditions (1.6), (1.25) and (1.26), let \( 0 \) be a simple blow-up point for \( \{v_i\} \), and the sequence \( \xi_{m_i} \rightarrow 0 \) carries the meaning as in (2.1) and (2.2). Then there exist positive constants \( \bar{C}_1 \) and \( \bar{\rho}_o \) such that

\[
\text{(2.4)}
\]

for \( 0 < |y - \xi_i| \leq \bar{\rho}_o \) and for all \( i \gg 1 \).

In addition, there is a number \( \bar{\rho}_1 \in (0, \bar{\rho}_o) \) such that (modulo a subsequence)

\[
\text{(2.5)}
\]

in \( C^2_{\text{loc}}(B_o(\bar{\rho}_1) \setminus \{0\}) \),
where $h$ is a harmonic function in $B_o(\tilde{\rho}_1)$. [Recall that $(\tilde{c}_n K)(0) = n(n - 2).$]

§ 2 c. Harmonic expression of the collapsed part. Consider a blow-up sequence of positive solutions $\{u_i\}$ of equation (1.1). Consider the situations where

(2.6) “the number of blow-up points is finite, say at $\beta_o = S, \cdots, \beta_k \in S^n \setminus \{N\}$, and at least one of them is a simple blow-up point (say, $\beta_o$).”

Take a point

(2.7) $x_c \notin \{\beta_o, \cdots, \beta_k, N\}$.

Under the general conditions (1.25), (1.26), and also (2.6), a subsequence of

(2.8) $\left\{ \frac{u_i}{u_i(x_c)} \right\}$

converges to a positive $C^2$-function $H$ defined on $S^n \setminus \{\beta_1, \cdots, \beta_k\}$. See [15]. With the stereographic projection $\tilde{P}$ onto $\mathbb{R}^n$, which sends $N$ to infinity, $H$ can be expressed as [cf. the transformation in (1.3)]

(2.9) $H(y) := [H \circ \tilde{P}^{-1}(y)] \cdot \left( \frac{2}{1 + |y|^2} \right)^{n-2},$

(2.10) $H(y) = \sum_{j=0}^{k} \frac{A_j}{|y - \hat{Y}_j|^{n-2}} \quad$ for $y \in \mathbb{R}^n \setminus \{\hat{Y}_o, \cdots, \hat{Y}_k\}$.

(2.11) Here $\hat{Y}_j := \mathcal{P}(\beta_j)$ for $0 \leq j \leq k$,

and $A_j$ are positive numbers. Refer to § 4 in [15].

The convergence can be quantified in the following manner. Given a sequence of positive numbers $\varepsilon_j \downarrow 0$, and a sequence of compact sets $\{C_j\}$ such that

(2.12) $C_1 \subset C_2 \subset \cdots$, \quad $\bigcup_{j=0}^{\infty} C_j = \mathbb{R}^n \setminus \{\hat{Y}_1, \cdots, \hat{Y}_k\}$,

there exists a sequence of natural numbers $N_i \uparrow \infty$ so that

(2.13) $\left| v_i(y) - [u_i(x_c)] \cdot H(y) \right| \leq \varepsilon_j \cdot [u_i(x_c)]$ \quad for $i \geq N_j$ and $y \in C_j$.

We point out when $i \to \infty$,

(2.14) “right hand side of (2.13)” $= \varepsilon_j \cdot [u_i(x_c)] \to 0$,

(2.15) the domain in which (2.13) holds is $C_j \rightarrow \mathbb{R}^n \setminus \{\hat{Y}_1, \cdots, \hat{Y}_k\}$.
Cf. (2.22) and (2.23) in the § 2 d.

§ 2 c.1. Change of the base point. We observe that, in (2.8), one can replace the base point \( u_i(x_c) \) by a sequence of numbers \( \{ \gamma_i \} \) so that

\[
(2.16) \quad C^{-1} \cdot \gamma_i \leq u_i(x_c) \leq C \gamma_i \quad \text{for } i \gg 1.
\]

A subsequence of \( \{ \gamma_i^{-1} \cdot u_i \} \) converges to a positive \( C^2 \)-function \( \tilde{H} \) defined on \( \mathbb{S}^n \setminus \{ \beta_0, \ldots, \beta_k \} \). With the stereographic projection \( \tilde{P} \) onto \( \mathbb{R}^n \), \( \tilde{H} \) can be expressed as in (2.10) and (2.11), with a scaling factor \( \lim_{i \to \infty} \gamma_i^{-1} \cdot u_i(x_c) \) inserted.

§ 2 d. Renormalization and first order approximation. Let 0 be a simple blow-up point for the sequence of positive solutions \( \{ v_i \} \) of equation (1.2). With the notations in (2.1) and (2.2), define

\[
(2.17) \quad V_i(\mathcal{Y}) := \frac{v_i(\xi_{m_i} + \lambda_{m_i} \cdot \mathcal{Y})}{v_i(\xi_{m_i})} \quad \text{for } \mathcal{Y} \in \mathbb{R}^n \text{ with } \lambda_{m_i} \cdot \mathcal{Y} \in B_0(\rho_0).
\]

where \( \lambda_{m_i} := \left[ v_i(\xi_{m_i})^{-\frac{2}{n-2}} \right] \). Here \( V_i \) satisfies the equation (extendable to \( \mathbb{R}^n \))

\[
(2.18) \quad \Delta_o V_i + \left[ (\tilde{c}_n K)(\xi_{m_i} + \lambda_{m_i} \cdot \mathcal{Y}) \right] V_i^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_0(\lambda_{m_i}^{-1}, \rho_0)
\]

Assuming (1.6), under the conditions (1.25) and (1.26), we invoke Proposition 2.1 in [21] (pp. 333) to conclude that, modulo a subsequence, \( \{ V_i \} \) converges to \( A_1 = A_{1,0} \) as given in (1.4). Cf. [6] and [10]. The convergence happens in \( C^1 \)-sense, uniformly in compact subsets in \( \mathbb{R}^n \) (for the variable \( \mathcal{Y} \)). This translates into a weak approximation of \( v_i \), which can be described in the following manner. Given sequences of positive numbers \( \{ \varepsilon_i \} \) and \( \{ R_i \} \) with \( \varepsilon_i \downarrow 0 \) and \( R_i \uparrow \infty \), via the Cantor diagonal argument on subsequences, we have

\[
(2.19) \quad |V_i(\mathcal{Y}) - A_1(\mathcal{Y})| \leq \varepsilon_i
\]

for all \( \mathcal{Y} \in B_0(R_i) \) and \( i \gg 1 \) (modulo a subsequence). Moreover, by choosing \( R_i \) to be smaller if necessary, we can take it that

\[
(2.20) \quad \varepsilon_i \cdot R_i^{2(n-1)} \to 0 \quad \text{and} \quad \lambda_i \cdot R_i \to 0 \quad \text{as } i \to \infty.
\]

[See also § 3 a in [15], and the proof of Proposition A.6.34 in the e-Appendix for the application of (2.20)] Via the change of variables

\[
y = \xi_{m_i} + \lambda_{m_i} \mathcal{Y}; \quad \mathcal{Y} \in B_0(R_i) \iff y \in B_{\xi_{m_i}}(\lambda_{m_i} \cdot R_i),
\]

\footnote{The job is made easier as we explain in § 2 f, for simple blow-up, we can take \( \xi_i \) to be a global maximum point of \( v_i \) in \( B_0(\rho_0) \).}
(.21), (.19) and (.4) yield

\[ |v_i(y) - A_{\lambda_m}, \xi_m(y)| \leq \frac{\varepsilon_i}{\lambda_m^{2}} \quad \text{for} \quad |y - \xi_m| \leq \lambda_m \cdot R_i \quad \text{and} \quad i \gg 1. \]

Cf. (.5). However, we do not know, a priori, how small we can take \( \varepsilon_i \) (relative to \( \lambda_i \)) and how large we can choose \( R_i \) (relative to \( \lambda_i^{-1} \)). In particular, the following scenario can occur.

\[ |v_i(y) - A_{\lambda_m}, \xi_m(y)| \leq \varepsilon_i \leq \lambda_m \cdot R_i \rightarrow 0. \]

Our goal is to introduce bubble estimates that are accurate up to \( O(\lambda_m^\tau) \) for \( \tau > 0 \) (as big as possible), and to “stabilize” the domain in which the estimates hold.

**§ 2.d.1. Joint between shrinking bubble estimate and the expanding global harmonic term.** As mentioned, there are diametric contrasts between bubble estimate (.21) and the global harmonic estimate presented in (.13). Adding to the list, observe that \( \Delta_o A_{\lambda_m, \xi_m} = -n(n-2) [A_{\lambda_m, \xi_m}]^{\frac{n+2}{n-2}} \) (\(< 0\)).

These two estimates do not immediately link to each other. We demonstrate their intricate relation when we present estimates that are accurate up to order \( O(\lambda_m^\tau) \).

**§ 2.e. An equivalent geometric expression.** Before we proceed to a closer relation between \( V_i \) and \( A_1 \), we examine a simpler estimate here. Not only the estimate is useful in later discussion, it is interesting in its own right. As for the proof, we present it in § A.1 in the e-Appendix.

**Proposition 2.24.** Under the standard conditions in (.6), (.25) and (.26), modulo a subsequence, \( 0 \) is a simple blow-up point for \( \{v_i\} \) if and only if there exist a sequence \( \zeta_i \in \mathbb{R}^n \), with

\[ \zeta_i \rightarrow 0 \quad \text{and} \quad \varepsilon_i := \frac{1}{|v_i(\zeta_i)|^{\frac{2}{n-2}}} \rightarrow 0, \quad \text{so that} \]

\[ \frac{1}{C} \cdot A_{\varepsilon_i, \zeta_i}(y) \leq v_i(y) \leq C \cdot A_{\varepsilon_i, \zeta_i}(y) \quad \text{for all} \quad |y - \zeta_i| \leq \rho_i. \]

Here \( C \geq 1 \) and \( \rho_i \) are positive constants independent on \( i \).

**§ 2.f. Shifting to the maximal point.** Let \( \xi_i \in B_o(\rho_o) \) be given in (.10). We can take \( \xi_m = \xi_i \) in (2.1) and (2.2). Moreover, suppose that there exists another
sequence of points \( \{ \tilde{\xi}_i \} \) which also satisfies (2.1) and (2.2) in the definition of simple\footnote{A simple blow-up point is defined as a point where the solution of the equation under consideration blows up in finite time.} blow-up points. We have (modulo a subsequence)

\[
\l| \tilde{\xi}_i - \xi_i \r| = o(\lambda_i) \quad \text{[\( \lambda_i \) as in (1.11)]}
\]

The proofs of the above statements, which require only standard techniques, can be found in §A.2 in the e-Appendix.

§2 g. Non-degenerate conditions and \( o(\lambda_i) \) restriction on flexibility.

Non-vanishing derivatives at the blow-up point tend to post restriction on the blow-up flexibility. One good example can be found in [8], which we highlight here, using the setting of the present article. Via Taylor’s expansion,

\[
(\tilde{\xi}_n K)(y) = n(n-2) + [-P_\ell(y)] + R_{\ell+1}(y) \quad \text{for } y \in B_o(\rho).
\]

Here we use multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \), and

\[
P_\ell(y) = \sum_{|\alpha| = \ell} \left[ D_\alpha^{(\ell)} (-\tilde{\xi}_n K) \bigg|_{y=0} \cdot \frac{y^\alpha}{\alpha!} \right],
\]

\[
R_{\ell+1}(y) = O\left( \max_{B_o(\rho_0 + \varepsilon')} |\nabla^{(\ell+1)} K| \times |y|^{\ell+1} \right).
\]

[Negative sign is introduced for later matching. See (3.7).] One can verify that

\[
\frac{|R_{\ell+1}(y)|}{|y|^{\ell}} \to 0 \quad \text{and} \quad \left\| \nabla R_{\ell+1}(y) \right\| \to 0 \quad \text{as } |y| \to 0.
\]

A more demanding condition is the lower bound

\[
C^{-1} |y|^{\ell-1} \leq \\sqrt{\left( \frac{\partial P_\ell(y)}{\partial y_1} \right)^2 + \cdots + \left( \frac{\partial P_\ell(y)}{\partial y_n} \right)^2} \quad \left( \leq C |y|^{\ell-1} \right)
\]

for \( y \in B_o(\rho) \). Cf. the example below. From (2.32), we have

\[
0 < c(\varepsilon) \leq \| \nabla K(y) \| \leq C \quad \text{for } \varepsilon \leq |y| \leq \rho, \text{ where } \rho \text{ is small enough}.
\]

The following is a direct application of Lemma 3.6 in [8], after checking (1.2), (1.6), and (3.2), and the conditions stated at the beginning of §3 in [8] (in particular,
\[ \alpha_i \leq n - 2, \text{ pp. 127, loc. cit.}, \] also verifying the conditions stated in Lemma 3.4 and Lemma 3.6 (loc. cit.), and taking \( p_i \equiv \frac{n+2}{n-2} \).

**Proposition 2.34.** Granted the general conditions in (1.6), (1.25) and (1.26), suppose that 0 is a simple blow-up point for \( \{v_i\} \). Assume also (2.28) and (2.32) for \( 2 \leq \ell \leq n - 2 \). If

\[ \int_{\mathbb{R}^n} \nabla P_\ell (y + \mathcal{X}) [A_1 (y)]^\frac{2n-\ell}{n-2} dy \neq 0 \quad \text{for all } \mathcal{X} \in \mathbb{R}^n \setminus \{0\}, \]

then, modulo a subsequence, we have \( |\xi_i| = o(\lambda_i) \) [Recall that \( A_1 = A_1, 0 \) is given in (1.4), \( \xi_i \) fulfills (1.10), \( \lambda_i \) is given in (1.11), and \( P_\ell \) in (2.29).]

§ 2 g.1 **Examples on \( K \) with local expansions fulfilling the conditions \( \Delta_0^h P_\ell \equiv 0 \), (2.32) and (2.35).** Recall that \( h_\ell \) is defined as the largest integer that is less than or equal to \( \ell/2 \). Consider \( n \) and \( \ell \geq 2 \), both even numbers, and

\[ (\tilde{c}_n K) (y) = n (n - 2) + \left[ (y_{i1}^\ell - y_{i2}^\ell) + \cdots + (y_{i,n-1}^\ell - y_{i,n}^\ell) \right] \]

for \( y \in B_{\rho_0} \). Using Hölder’s inequality, one can verify (2.32). Moreover,

\[ (y_{i1} + \mathcal{X}_1)^{\ell - 1} = y_{i1}^{\ell - 1} + C(\ell - 1, 2) \cdot y_{i1}^{\ell - 2} \mathcal{X}_1^2 + \cdots + C(\ell - 1, \ell - 2) \cdot y_{i1}^{\ell - 2} \mathcal{X}_1^{\ell - 2} + \mathcal{X}_1 \left[ C(\ell - 1, 2) \cdot y_{i1}^{\ell - 1 - 1} + \cdots + C(\ell - 1, \ell - 3) \cdot y_{i1}^2 \cdot \mathcal{X}_1^{\ell - 4} + \mathcal{X}_1^{\ell - 2} \right]. \]

Here \( \mathcal{X} = (\mathcal{X}_1, \cdots, \mathcal{X}_n) \), and \( C(j, k) = \frac{j!}{(j - k)! \cdot k!} \) \((j \geq k)\)

is the binomial coefficient. Note that all the powers in \( \mathcal{X}_1 \) inside the brackets are even numbers. As \[ \int_{\mathbb{R}^n} y_{i1}^{2j+1} [A_1 (y)]^\frac{2n-\ell}{n-2} dy = 0, \]

we have

\[ \int_{\mathbb{R}^n} (y_{i1} + \mathcal{X}_1)^{\ell - 1} [A_1 (y)]^\frac{2n-\ell}{n-2} dy = 0 \iff \mathcal{X}_1 = 0. \]

It follows that (2.35) is fulfilled with the form in (2.36). In addition, observe that

\[ \Delta_0^h \left[ (y_{i1}^\ell - y_{i2}^\ell) + \cdots + (y_{i,n-1}^\ell - y_{i,n}^\ell) \right] = 0 \quad (\ell \text{ being even}). \]

One can generalized (2.36) by introducing positive multipliers onto each \( (y_{i2j-1}^\ell - y_{i2j}^\ell) \).
3. Difference between the normalization $V_i$ and $A_1$.

After shifting from $\xi_{m_i}$ to $\xi_i$ as described in §2f, for the sake of simplicity, we continue to use the notation

\[ V_i(Y) := \frac{v_i(\xi_i + \lambda_i Y)}{M_i} \quad \text{for} \ Y \in \mathbb{R}^n. \]

Here

\[ M_i := v_i(\xi_i) \quad \text{and} \quad \lambda_i = M_i^{-\frac{2}{n-2}}, \quad \xi_i \text{ is given in (1.10)}. \]

Cf. (2.17) and §2f. As $A_1 = A_{1,0}$ satisfies the equation

\[ \Delta_o A_1 + n(n - 2)A_1^{n+2} = 0 \quad \text{in} \ \mathbb{R}^n, \]

together with equation (2.18), which holds after the changes $\xi_{m_i} \rightarrow \xi_i$ and $\lambda_{m_i} \rightarrow \lambda_i$, it can be seen that

\[ \Delta_o (V_i - A_1)(Y) = n(n - 2)\bigl\{ [A_1(Y)]^{n+2} - [V_i(Y)]^{n+2} \bigr\} \]
\[ + \bigl\{ n(n - 2) - \bar{c}_n K(\lambda_i Y + \xi_i) \bigr\} [A_1(Y)]^{\frac{n+2}{n-2}} \]
\[ + \bigl\{ n(n - 2) - \bar{c}_n K(\lambda_i Y + \xi_i) \bigr\} \bigl\{ [V_i(Y)]^{\frac{n+2}{n-2}} - [A_1(Y)]^{\frac{n+2}{n-2}} \bigr\} \]

for $Y \in \mathbb{R}^n$.

3a. Linear approximation to $\left(A_1^{\frac{n+2}{n-2}} - V_i^{\frac{n+2}{n-2}}\right)$ in case of simple blow-up.

It follows from Proposition 2.24 that (see §A.10 in the e-Appendix for details)

\[ \|[A_1(Y)]^{\frac{n+2}{n-2}} - [V_i(Y)]^{\frac{n+2}{n-2}} = \left( \frac{n + 2}{n - 2} \right) [A_1(Y)]^{\frac{4}{n-2}} \cdot [A_1(Y) - V_i(Y)] \]
\[ + O(1) [A_1(Y) - V_i(Y)]^2 \cdot [A_1(Y)]^{\frac{4}{n-2}-1} \]

for $|Y| \leq \rho_o \lambda_i^{-1}$. 

\( \| \nabla K(N) \| = 0 \), and by (1.6), \( (\tilde{c}_n K) (0) = n (n - 2) \), we assume that all the derivatives of \( K \) vanish at 0 up to (and equal to) order \( \ell - 1 \). Here \( \ell \geq 2 \) is an integer. Using multi-index \( \alpha = (\alpha_1, \cdots, \alpha_n) \), and the Taylor expansion of \( (\tilde{c}_n K) \), we obtain

\[
(3.6) \quad n (n - 2) - \tilde{c}_n K(\lambda_i \mathcal{Y} + \xi_i) = \sum_{|\alpha| = \ell} D^{(\ell)} (\tilde{c}_n K) \bigg|_0 \cdot \frac{\lambda_i \mathcal{Y} + \xi_i}{\alpha!} + O (1) \max_{|\lambda_i \mathcal{Y} + \xi_i| \leq \rho_o} \| \nabla^{(\ell+1)} K \| \cdot |\lambda_i \mathcal{Y} + \xi_i|^{\ell+1} 
\]

\[
= \lambda_i^\ell \cdot P_\ell (\mathcal{Y}) + \sum_{k=1}^\ell O \left( \max_{|\lambda_i \mathcal{Y} + \xi_i| \leq \rho_o^+} |\lambda_i \mathcal{Y} + \xi_i|^{\ell+1-k} \right) + R_3 (\mathcal{Y})
\]

for \( \lambda_i \cdot |\mathcal{Y}| \leq \rho_o, \ i \gg 1 \). In the above \( \rho_o^+ \) is slightly bigger than \( \rho_o \). Moreover,

\[
(3.7) \quad P_\ell (\mathcal{Y}) = \sum_{|\alpha| = \ell} D^{(\ell)} (-\tilde{c}_n K) \bigg|_0 \cdot \frac{\mathcal{Y}^\alpha}{\alpha!},
\]

\[
(3.8) \quad R_3 (\mathcal{Y}) = O \left( \max_{|\lambda_i \mathcal{Y} + \xi_i| \leq \rho_o^+} \| \nabla^{(\ell+1)} K \| \cdot |\lambda_i \mathcal{Y} + \xi_i|^{\ell+1} \right).
\]

§ 3 c. The mixed term. Consider the last term in (3.4). Using Taylor expansion as in (3.6), and the inequality

\[
(3.9) \quad a > b > 0 \quad \text{and} \quad p \geq 1 \implies a^p - b^p \leq \frac{1}{p} \cdot (a - b) \cdot a^{p-1},
\]

we obtain

\[
(3.10) \quad n (n - 2) - \tilde{c}_n K(\lambda_i \mathcal{Y} + \xi_i) \left\{ \left[ V_i (\mathcal{Y}) \right]^\frac{n+2}{n-2} - \left[ A_1 (\mathcal{Y}) \right]^\frac{n+2}{n-2} \right\} 
\]

\[
= O \left( \max_{|\lambda_i \mathcal{Y} + \xi_i| \leq \rho_o^+} \| \nabla^{(\ell)} K \| \cdot |\lambda_i \mathcal{Y} + \xi_i|^\ell \right) \times 
\]

\[
\times \left[ O (1) \cdot | V_i - V | \times \max \left\{ \left[ V_i (\mathcal{Y}) \right]^\frac{1}{n-2}, \left[ A_1 (\mathcal{Y}) \right]^\frac{1}{n-2} \right\} \right]
\]

for \( |\mathcal{Y}| \leq \lambda_i^{-1} \rho_o \).

§ 3 d. Isolating the key terms and the remainder. It follows from (3.4), (3.5), (3.6) and (3.10) that

\[
(3.11) \quad \Delta_o \left[ V_i - A_1 \right] (\mathcal{Y}) + n (n + 2) A_1^\frac{n+2}{n-2} \left[ V_i - A_1 \right] (\mathcal{Y}) = \lambda_i^\ell \cdot P_\ell (\mathcal{Y}) \cdot A_1^\frac{n+2}{n-2} + \text{RM} (\mathcal{Y})
\]
Here (refer to §6b and §6c)

\[ (3.12) \quad RM = RM_1 + RM_2 + RM_3 + RM_4, \]

\[
RM_1(Y) = \sum_{k=1}^{\ell} O \left( |\xi_i|^k \cdot (\lambda_i |Y|)^{\ell-k} \right),
\]

\[
RM_2(Y) = O \left( \max_{|\lambda_i y + \xi_i| \leq \rho_o} \| \nabla^{(\ell+1)} K \| \times |\lambda_i Y + \xi_i|^{\ell+1} \right),
\]

\[
RM_3(Y) = O \left( 1 \right) \left\{ \left[ A_1(Y) \right]^{\frac{4}{n-2}} \cdot \left[ V_i(Y) - A_1(Y) \right]^2 \right\},
\]

\[
RM_4(Y) = O \left( \max_{|\lambda_i y + \xi_i| \leq \rho_o} \| \nabla^{(\ell)} K \| \times |\lambda_i Y + \xi_i|^{\ell} \right) \times \left[ O \left( 1 \right) |V_i(Y) - A_1(Y)| \times \max \left\{ \left[ V_i(Y) \right]^{\frac{4}{n-2}}, \left[ A_1(Y) \right]^{\frac{4}{n-2}} \right\} \right]
\]

for \( |Y| \leq \lambda_i^{-1} \rho_o \). (We use " = " to tell us that the right hand side is the order of the term.)

\section{4. Cancelation of the \( O(\lambda_i^\ell) \) term in (3.11).}

We first ignore the order \( \lambda_i^\ell \) in equation (3.11) and consider the linear inhomogeneous equation

\[ (4.1) \quad \Delta_o \Pi + n(n+2) A_1^{\frac{4}{n-2}} \cdot \Pi = \mathcal{P}_\ell \cdot A_1^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n \]

(with unknown \( \Pi \)). Here \( \mathcal{P}_\ell \) is a homogeneous polynomial defined on \( \mathbb{R}^n \) of degree \( \ell \geq 1 \), and \( A_1 = A_{1,0} = \left( \frac{1}{1 + |Y|^2} \right)^{\frac{n-2}{2}} \). As in [12], potential solutions \( \Pi \) can be expressed in the following form

\[ (4.2) \quad \Pi(Y) = \frac{\Gamma(Y)}{(1 + R^2)^{\frac{n+2}{2}}} \quad \text{for } Y \in \mathbb{R}^n \quad (R = |Y|). \]

Putting (4.2) into (4.1), we obtain

\[ (4.3) \quad (1 + R^2) \cdot [\Delta_o \Gamma] - 2n [Y \cdot \nabla \Gamma] + 2n \Gamma = \mathcal{P}_\ell \quad \text{in } \mathbb{R}^n. \]

Our goal is to find polynomial solutions \( \Gamma \) of equation (4.3), and to keep the degree of \( \Gamma \) as close to \( \ell \) as possible (this is for the second order blow-up argument), and
to make explicit the dependence on $\mathcal{P}_\ell$. We first look at some selected examples.

**Example 4.4.** When $\mathcal{P}_\ell \equiv 0$. We can take

$$
\Gamma_1(\mathcal{Y}) := \sum_{j=1}^{n} c_j \mathcal{Y}_j, \quad \text{or} \quad \Gamma_2(\mathcal{Y}) := \mathcal{R}^2 - 1.
$$

Here $c_j$ are any constants, and $\mathcal{Y} = (\mathcal{Y}_1, \ldots, \mathcal{Y}_n) \in \mathbb{R}^n$. Interestingly, up to linear combinations of $\Gamma_1$ and $\Gamma_2$, these are the only possible solutions when $\mathcal{P}_\ell \equiv 0$ and when we restrict $\Gamma$ to be a polynomial of degree less than $n$. Cf. Theorems 4.16 and 4.21.

**Example 4.6.** When $\ell \geq 2$ and $\Delta_o \mathcal{P}_\ell = 0$. In this case we simply take $\Gamma_3 = c \mathcal{P}_\ell$:

$$
(1 + \mathcal{R}^2) \Delta_o \Gamma_3 - 2n [ (\mathcal{Y} \cdot \nabla \Gamma_3) - \Gamma_3 ] = -2n (\ell - 1) (c \mathcal{P}_\ell) = \mathcal{P}_\ell
$$

$$
\Rightarrow c = -\frac{1}{2n(\ell - 1)} \Rightarrow \Gamma_3 = -\frac{\mathcal{P}_\ell}{2n(\ell - 1)}.
$$

**Example 4.8.** When $\ell = 1$. As the left hand side of (4.3) is linear, we may assume that $\mathcal{P}_\ell(\mathcal{Y}) = \mathcal{Y}_1$. Consider

$$
\Gamma_4(\mathcal{Y}) := a \mathcal{R}^2 \mathcal{Y}_1 + b \mathcal{R}^4 \mathcal{Y}_1 \quad (\text{maximum degree } = 5).
$$

Direct calculation shows that when

$$
n = 4, \quad a = \frac{1}{2n + 4}, \quad \text{and} \quad b = \frac{2n - 4}{2n + 4} \cdot \frac{1}{4n + 16},
$$

$\Gamma_4$ is a solution of (4.3) with $\mathcal{P}_\ell(\mathcal{Y}) = \mathcal{Y}_1$. The example demonstrates that, in general, $\ell = 1$ make it harder to solve equation (4.3). Cf. an existence result for equation (1.1) obtained by Aubin in [1].

§ 4a. Solving (4.1) via the linear method ($\ell < n$). As in [12] (pp.152), we introduce the collection ($h_\ell$ is the biggest integer less than or equal to $\ell/2$)

$$
\mathcal{F}(\mathcal{P}_\ell) := \{ \text{linear combinations of } (\mathcal{R}^2)^j \Delta_o^{(k)} \mathcal{P}_\ell, \}
$$

$$
0 \leq j \leq k, \quad h_\ell = 0, 1, \cdots, h_\ell
$$

where $\mathcal{R} = |\mathcal{Y}|$. Note that $\Delta_o^{(k)} \mathcal{P}_\ell \equiv 0$ for $k \geq h_\ell + 1$. Comparing to the one introduced in [12], (4.10) has the index $j$ limited more strictly from above. Assuming that $\Delta_o^{(j)} \mathcal{P}_\ell \neq 0$ for $0 \leq j \leq h_\ell$, $\mathcal{F}(\mathcal{P}_\ell)$ is a vector space with dimension, in general, equal to $\frac{1}{2} (h_\ell + 1) (h_\ell + 2) \quad [= O(n^2) \text{ when } \ell \text{ is close to } n; \quad \text{an exceptional case is when } \mathcal{P}_\ell(\mathcal{Y}) = \mathcal{R}_\ell, \quad \text{where } \ell \text{ is an even number}]$. We list down some simple
properties concerning the operator in (4.3) and $F(\mathcal{P}_\ell)$. All these can be checked readily (see §A.3 in the e-Appendix for a proof).

**Lemma 4.11.** $\mathcal{P}_\ell \in F(\mathcal{P}_\ell)$. Moreover,

$$(4.12) \text{ if } \ell \text{ is odd and } \Delta_{o}^{(h_{\ell})} \mathcal{P}_\ell \neq 0 \implies \sum c_{j} \nabla_{j} \in F(\mathcal{P}_\ell);$$

$$(4.13) \text{ if } \ell \text{ is odd and } \Delta_{o}^{(h_{\ell})} \mathcal{P}_\ell \neq 0 \implies (\mathcal{R}^{2} - 1) \in F(\mathcal{P}_\ell).$$

Here at least one of the coefficient $c_{j} \neq 0$.

$$(4.14) \text{ The degree of each term in } F(\mathcal{P}_\ell) \text{ is at most } \ell.$$  

$$(4.15) (1 + \mathcal{R}^{2}) \Delta_{o} \bullet - 2n(\nabla \cdot \psi) + 2n \bullet : F(\mathcal{P}_\ell) \to F(\mathcal{P}_\ell) \text{ is linear.}$$

In principle, one can express the linear operator in (4.15) of Lemma 4.11 into a matrix by using the basis of $F(\mathcal{P}_\ell)$ as shown in (4.10), and determine whether there is a solution or not. However, for genuine cases, and when $\ell$ is close to $n$, the matrix is of the size (number of entries) in the order $O(n^{4})$. In [12], the authors observes that one can make use of the following Liouville-type theorem (shown in [8]; see also [3]) to demonstrate that a solution exists.

**Theorem 4.16.** Suppose $\psi$ is a smooth solution of the equation

$$(4.17) (1 + \mathcal{R}^{2}) \Delta_{o} \psi - 2n(\nabla \cdot \psi) + 2n \psi = 0 \text{ in } \mathbb{R}^{n}.$$  

$$(4.18) \text{ with } \lim_{r \to \infty} \frac{\psi(\nabla)}{\mathcal{R}^{n}} = 0 \quad (\mathcal{R} = |\nabla|).$$

$$(4.19) \text{ Then } \psi(\nabla) = c_{o}(\mathcal{R}^{2} - 1) + \sum_{j=1}^{n} c_{j} \nabla_{j} \psi \nabla \text{ for } \nabla \in \mathbb{R}^{n}.$$  

$$(4.20) \text{ Moreover, } \psi(0) = 0 \text{ and } \nabla \psi(0) = 0 \implies \phi \equiv 0 \text{ in } \mathbb{R}^{n}.$$  

We now describe the linear method use in [12] to find a polynomial solution to equation (4.3). The following result begins to reveal that the condition $\Delta_{o}^{(h_{\ell})} \mathcal{P}_\ell \equiv 0$ is tightly knitted together with the refined estimate we seek.

**Proposition 4.21.** Assume that $\mathcal{P}_\ell$ is a homogeneous polynomial of degree $\ell$, with $2 \leq \ell < n$. The linear operator which appears in (4.15) of Lemma 4.11 is a bijection if and only if $\Delta_{o}^{(h_{\ell})} \mathcal{P}_\ell \equiv 0$.

**Proof.** For the sufficient part, the proof goes in an essential manner as in the proof of Proposition 4.1 in [12], using Lemma 4.11 and Theorem 4.16 to show that the
linear operator is an injection, and hence a bijection. As for the necessary part, it follows from (4.12) and (4.13) of Lemma 4.11, and Example 4.4 [the kernel contains a non-identically zero element in \( \mathcal{F}(\mathcal{P}_\ell) \)].

\[ \square \]

**§ 4 b. The reduction method.** Concerning the solution we find via Proposition 4.21, besides being in \( \mathcal{F}(\mathcal{P}_\ell) \) [in particular, we have property (4.14) in Lemma 4.11], there is little we know about the solution itself. When we come to the bubble estimates, it is natural to ask for the precise form of the solution \( \Gamma \). In this section we introduce a constructive method which allows us to determine each coefficient in \( \Gamma \). We present the precise result.

**Lemma 4.22.** Let \( \mathcal{P}_\ell \) be a homogeneous polynomial of degree \( \ell \geq 2 \) (defined on \( \mathbb{R}^n \)). When \( n \) is even, assume also that \( \ell < n + 2 \) (no such condition when \( n \) is odd). Define a polynomial \( G \) via

\[
G(Y) = \sum_{0 \leq j \leq k} C_j^k \cdot (\mathcal{R}^2)^j [\Delta_o^{(k)} \mathcal{P}_\ell(Y)] \quad (\mathcal{R} = |Y|),
\]

where the coefficients \( C_j^k \) can be determined by using (4.48). Then \( G \) satisfies

\[
(1 + \mathcal{R}^2) \Delta_o G - 2n (Y \cdot \nabla G) + 2n G = \mathcal{P}_\ell + \Delta_o^{(h_\ell)} \mathcal{P}_\ell + \left\{ a_{h_\ell} \cdot (\mathcal{R}^2)^{h_\ell} + a_{h_{\ell-1}} \cdot (\mathcal{R}^2)^{h_{\ell-1}} + \cdots + a_1 \cdot (\mathcal{R}^2)^1 + a_o \right\}.
\]

Here the numbers \( a_k \) can be found by using (4.45).

The precise definitions of \( C_j^k \) and \( a_k \) are obtained on the way toward the proof of Lemma 4.22. The key property is that they depend only on \( n, \ell, j \) and \( k \) only, and are formed by an algebraic iteration process, which we start to describe.

**§ 4 b.1. First step in the proof of Lemma 4.22: the recurrent and reduction to powers of \( (\mathcal{R}^2) \).** Consider first the situation where

\[
\Delta_o^{(k)} \mathcal{P}_\ell \neq 0 \quad \text{for} \quad 1 \leq k \leq h_\ell - 1.
\]

As in Example 4.4, we take

\[
[C_o^o \cdot \mathcal{P}_\ell], \quad \text{where} \quad C_o^o = \frac{1}{2n(1 - \ell)},
\]

and obtain

\[
(1 + \mathcal{R}^2) \Delta_o [C_o^o \cdot \mathcal{P}_\ell] + 2n [1 - (Y \cdot \nabla)] [C_o^o \cdot \mathcal{P}_\ell] = \mathcal{P}_\ell + C_o^o \left[ (\mathcal{R}^2) \Delta_o \mathcal{P}_\ell + \Delta_o \mathcal{P}_\ell \right].
\]
That is, we obtain $P_\ell$ in the right hand side, but “pay the price” by introducing $[(\mathcal{R}^2) \Delta_o P_\ell]$ and $[\Delta_o P_\ell]$. Observe that the degree of $\Delta_o P_\ell$ is lowered to $\ell - 2$, while the degree of $[(\mathcal{R}^2) \Delta_o P_\ell]$ is still equal to $\ell$, but its structure appears simpler in the sense that 2 of the degree is taken over by $(\mathcal{R}^2)$.

If $\ell = 2$ or 3, we are done. Assume that $\ell \geq 4$. To proceed, we simplify the notations and highlight the change in order, and introduce

$$R = (\mathcal{R}^2) \quad \Rightarrow \quad R^j = (\mathcal{R}^2)^j, \quad D = [\Delta_o P_\ell] \quad \rightarrow \quad D_k := [\Delta_o^{(k)} P_\ell].$$

Using these notations, we have (cf. § A.3 in the e-Appendix)

$$\begin{align*}
(1 + R) \cdot \Delta_o [R^j D_k] &= (1 + \mathcal{R}^2) \Delta_o [(\mathcal{R}^2)^j \cdot \Delta_o^{(k)} P_\ell)] \\
&= A_{\ell,j,k} \cdot (R^j D_k) + R^{j+1} D_{k+1} \quad [\text{degree} = \ell + 2(j - k) \text{ on both terms}] \\
&\quad + A_{\ell,j,k} \cdot (R^{j-1} D_k) + R^j D_{k+1} \\
&\quad \quad [\uparrow \text{degree} = \ell + 2(j - k) - 2 \text{ on both terms}].
\end{align*}$$

Here

$$A_{\ell,j,k} = (2j) \cdot (2j + n - 2 + 2\ell - 4k)$$

We realize that the process produces one same term $[R^j D_k]$ (times a constant), one same order term, plus two lower order terms. We illustrate the procedure when the linear operator [the right hand side of (4.3)] acts on $[R^j D_k]$ via the following diagram $(j \geq 1)$.

```plaintext
Diagram 4.30. The four terms, their degrees, and the multipliers.
```
We represent schematically part of the reduction procedure in the following diagram, showing the terms produced (indicated by the arrows, including itself) when the term is acted upon by the operator \((1 + R) \cdot \Delta_o\).

\[
\begin{array}{cccc}
\mathcal{P}_\ell & \rightarrow & D & \rightarrow D_2 \ldots \rightarrow [\Delta_o^{(h_\ell)} \mathcal{P}_\ell] \\
\downarrow & \nearrow & \downarrow & \nearrow \\
(1^{\text{st}}) & R D & \rightarrow & R D_2 \rightarrow R D_3 \ldots \rightarrow (R^2) [\Delta_o^{(h_\ell)} \mathcal{P}_\ell] \\
\downarrow & \nearrow & \downarrow & \nearrow \\
(2^{\text{nd}}) & R^2 D_2 & \rightarrow & R^2 D_3 \rightarrow \ldots \rightarrow (R^2)^2 [\Delta_o^{(h_\ell)} \mathcal{P}_\ell] \\
\vdots & \downarrow & \vdots & \\
(\ell_1^{\text{th}}) & R^{h_\ell - 1} D_{h_\ell - 1} & \rightarrow & R^{h_\ell - 1} D_{h_\ell} = (R^2)^{h_\ell - 1} [\Delta_o^{(h_\ell)} \mathcal{P}_\ell] \\
\downarrow & \\
(\ell_2^{\text{th}}) & R^{h_\ell} D_{h_\ell} \\
\| \\
(R^2)^{h_\ell} [\Delta_o^{(h_\ell)} \mathcal{P}_\ell] \\
\end{array}
\]

Degree \((\leq)\) \(\ell\) \(\ell - 2\) \(\ldots\) \(1\) / \(0\) 
\(\ell\) is odd/even

Diagram 4.31. Showing the cancelation order (top \(\rightarrow\) down) on the first column.

Back to the case when \(\ell = 4\). In the second step, we seek to eliminate the term \(C_o \cdot [(R^2) \Delta_o \mathcal{P}_\ell] = C_o \cdot R D\), which appears in (4.27). From (4.29) and (4.27): we take

\[
C_1^1 = \frac{-C_o}{A_{\ell,1} - 2n(\ell - 1)} = \frac{-C_o}{2(n - 2)(2 - \ell)} \quad \text{(here } \ell \geq 4\text{)}.
\]

It follows that

\[
(1 + R^2) \Delta_o \{ [C_o \cdot \mathcal{P}_\ell] + [C_1^1 \cdot (R^2) \Delta_o \mathcal{P}_\ell] \} \\
+ 2n [1 - (\mathcal{Y} \cdot \nabla)] \{ [C_o \cdot \mathcal{P}_\ell] + [C_1^1 \cdot (R^2) \Delta_o \mathcal{P}_\ell] \}
\]

21
\[ = \mathcal{P}_\ell + \left[ C^o + A_{\ell,1,1} \cdot C^1 \right] \cdot \left[ \Delta^o \mathcal{P}_\ell \right] + C^1 \cdot \left[ (\mathcal{R}^2)^2 \Delta^2 \mathcal{P}_\ell + (\mathcal{R}^2) \Delta^2 \mathcal{P}_\ell \right]. \]

Inductively, we find \[ \text{refer to (4.23)}\]

\[
(4.34) \quad C^j_k = \frac{-C^j_{k-1}}{A_{\ell,j,j} - 2n(\ell - 1)} = \frac{-C^j_{k-1}}{(2j)[n - 2 + 2(\ell - j)] - 2n(\ell - 1)},
\]

where \(1 \leq j \leq h_\ell\). This enables us to cancel the terms in the first column in Diagram 4.31, ending with the term

\[
(4.35) \quad C_{h_\ell - 1}^{h_\ell - 1} \cdot \left[ (\mathcal{R}^2)^{h_\ell} \Delta^{(h_\ell)} \mathcal{P}_\ell \right],
\]

which is present on the right hand side of equation (4.24).

Next, we proceed to cancel the terms in the second column (Diagram 4.31), starting from top toward the bottom. Gradually we move right to the next column, always proceeding from top to bottom. We summarize the cancelation in the following two cases.

* Cancelation of terms in the top row in Diagram 4.31. From (4.33), and also from Diagram 4.31, we have the term

\[
\left[ C^o + C^1 \cdot A_{\ell,1,1} \right] \cdot \mathbf{D}
\]

to be canceled. This is done by adding the term

\[
- \frac{C^o + C^1 \cdot A_{\ell,1,1}}{-2n[(\ell - 2) - 1]} \cdot \mathbf{D} \implies C^o_k = - \frac{C^o + C^1 \cdot A_{\ell,1,1}}{-2n[(\ell - 2) - 1]}.
\]

With the help of the information depicted in Diagram 4.30, and via induction, we have

\[
(4.36) \quad C^o_k = - \frac{C^o_{k-1} + C^1_k \cdot A_{\ell,1,k}}{-2n[(\ell - 2k) - 1]} \quad \text{for} \quad 1 \leq k \leq h_\ell - 1.
\]

The numbers \(C^1_2, C^1_3, \ldots, C^1_{h_\ell - 1}\) are obtained below – see Remark 4.40. Note that

\[
\ell - 2k \geq \ell - 2(h_\ell - 1) \geq 2 \implies (\ell - 2k) - 1 \neq 0 \quad \text{[recall (4.25)].}
\]

This enables us to cancel the terms in the first row, ending with

\[
C^o_{h_\ell - 1} \cdot \Delta^{(h_\ell)} \mathcal{P}_\ell,
\]

22
which appears in the right hand side of (4.24).

* Cancelation of the ‘inside’ terms. Finally, consider any ‘inside’ term $R^j D_k$. We observe that $k > j$ ($k = j$ appears in the first column only).

\[
C_{k-1}^{j-1} R^{j-1} D_{k-1} \quad (j \geq 1)
\]

\[
(\times 1 \rightarrow) \downarrow
\]

\[
C_{k-1}^j R^j D_{k-1} (\times 1) \rightarrow R^j D_k
\]

\[
\nearrow (\leftarrow \times A_{\ell,j+1,k})
\]

\[
C_k^{j+1} R^{j+1} D_k
\]

Diagram 4.37. The three terms which give rise to an inside term (with multipliers).

Via induction and the discussion in (4.34) and (4.36), we may assume that the coefficients $C_{k-1}^{j-1}$, $C_{k-1}^j$ and $C_k^{j+1}$ are determined. The term $R^j D_k$ makes its presence on the right hand given by

\[
(4.38) \quad [C_{k-1}^{j-1} + C_k^j + A_{\ell,j+1,k}^{j+1}] \cdot R^j D_k
\]

To cancel it, we introduce the term

\[
-C_{k-1}^{j-1} + C_k^j + A_{\ell,j+1,k}^{j+1} \cdot R^j D_k
\]

to the left hand side,

\[
(4.39) \quad \ldots \ldots \quad \Rightarrow \quad C_k^j = -\frac{C_{k-1}^{j-1} + C_k^j + A_{\ell,j+1,k}^{j+1}}{A_{\ell,j+1,k} - 2n [\ell - 2k + 2j - 1]} \cdot R^j D_k
\]

Remark 4.40. Concerning the usage of $C_2^1$, $C_3^1$, $\ldots$, $C_{h-1}^1$ in (4.36), we remark that, based on (4.39), in order to determine $C_2^1$, we need only $C_1^0$, $C_1^1$ and $C_2^2$, which are known via (4.34) and (4.36). Afterward, we can determine $C_j^{j-1}$ for $3 \leq j \leq h_\ell$ (the other coefficient in the second column in Diagram 4.31). $C_3^2$, together with $C_2^0$ and $C_2^1$ help to determine $C_3^1$, and so on.

§ 4 b.2. Non-zero characters for $(R^2)j \Delta^{(k)}_\lambda = (R^2)j \Delta^{(j+\sqcup)}_\lambda$ ($j \geq 1$ and $\sqcup \geq 0$). In order to finish the proof for Lemma 4.22, we are required to show that the denominators in (4.34) and (4.39) are non-zero — under the conditions on $\ell$ as stated in Lemma 4.22. Note that

\[
\text{Degree } \{ (R^2)^j \cdot [\Delta^{(j+\sqcup)}_\lambda \mathcal{P}_\ell] \} = \ell + 2 [j - (j + \sqcup)] = \ell - 2 \sqcup \leq \ell \ (\sqcup \geq 0).
\]
Moreover, as the process stops when \( j + \sqcup = h_\ell \), we need only to consider the situation where \( j + \sqcup \leq h_\ell - 1 \). It follows that

\[
(4.41) \quad k - j = \sqcup \geq 0 \text{ and } k = j + \sqcup \leq h_\ell - 1 \implies j \leq h_\ell - 1; \\
j \geq 1 \implies \sqcup \leq h_\ell - 2.
\]

We investigate the characteristic equation, which is given by the denominator in (4.39) [note that the denominator in (4.34) corresponds to \( k = j \)]:

\[
A_{\ell,j,k} = 2n[\ell + 2(j - k) - 1] = 0
\]

\[
\iff (2j)[2j + (n - 2) + 2(\ell - 2k)] - 2n[\ell - 2(k - j) - 1] = 0
\]

\[
\iff (2j)^2 - (2j)[(n - 2 + 2\ell - 4\sqcup)] + n(2\ell - 4\sqcup - 2) = 0
\]

\[
\iff [(2j) - n] \cdot [(2j) - (2\ell - 4\sqcup - 2)] = 0
\]

\[
(4.42) \quad \cdots \iff j = \frac{n}{2} \text{ or } j = (\ell - 1) - 2\sqcup.
\]

*When \( n \) is even.* Here \( n/2 \) is an integer, (4.41) requires us to post the restriction

\[
(4.43) \quad j \leq h_\ell - 1 < \frac{n}{2} \iff h_\ell - 1 < \frac{n}{2} \iff 2 \cdot h_\ell < n + 2.
\]

That is, when \( \ell \) is even, we require

\[
(4.44) \quad \ell < n + 2.
\]

Similarly, when \( \ell \) is odd, we need

\[
2 \cdot \frac{\ell - 1}{2} < n + 2 \iff \ell < n + 3 \iff \ell < n + 2,
\]

as \( n \) is even \( \implies n + 3 \) is odd, and \( \ell \) is also odd in this case. For the second root in (4.42), since \( k \leq h_\ell - 1 \), we have

\[
j + \sqcup < \frac{\ell}{2} \implies j + \sqcup \leq \frac{\ell}{2} - 1 \implies \sqcup \leq \frac{\ell}{2} - 2 (j \geq 1) \implies j + 2\sqcup \leq \ell - 3.
\]

Thus the solution \( j = (\ell - 1) - 2\sqcup \iff j + 2\sqcup = \ell - 1 \) is too big to happen.

*When \( n \) is odd.* In this case, \( n/2 \) is not an integer. We need only to consider the second root in (4.42). As we want the term \( (R^2) \) to be present, and \( (\Delta_{j+\sqcup} P_\ell) \) is not yet reduced to first order, therefore

\[
j + \sqcup < \frac{\ell - 1}{2} \implies j + \sqcup \leq \frac{\ell - 1}{2} - 1 \implies \sqcup \leq \frac{\ell - 1}{2} - 2 (j \geq 1) \implies j + 2\sqcup \leq \ell - 3 \implies j < (\ell - 3) - 2\sqcup.
\]

24
Once again, the solution \( j = (\ell - 1) - 2 \sqcup \) is too big. This completes the showing that the denominators in (4.34) and (4.39) are non-zero under the conditions of Lemma 4.22.

**The residue.** As the ‘pure’ \((R^2)^1, (R^2)^2, \ldots, (R^2)^{h_\ell}\) terms are obtained as the by-products of the last cancelations in each column (see Diagram 4.31), except the last coefficient \( a_o \), all the others are combination the horizontal arrow and the downward arrow (refer to Diagram 4.31). Hence [together with Diagram 4.30; cf. also (4.35) and (4.23)] we have

(4.45)

\[
a_{h_\ell} = C_{h_\ell-1}^{h_\ell-1}, \quad a_{h_\ell-1} = \left[ C_{h_\ell-1}^{h_\ell-1} + C_{h_\ell-1}^{h_\ell-2} \right], \ldots, \quad a_{1} = \left[ C_{h_\ell-1}^{h_\ell-1} + C_{h_\ell-1}^{o} \right],
\]

and \( a_o = C_{h_\ell-1}^{o} \).

The argument is completed under condition (4.25). Finally, suppose

(4.46)

\[
\Delta_o^{(k_o)} \mathcal{P}_\ell \equiv 0 \quad \text{for an integer } k_o \in [1, h_\ell - 1].
\]

The process described in Diagram 4.31 ends earlier. In this case

(4.47)

\[
\mathcal{G} = \sum_{0 \leq j \leq k \leq k_o - 1} C_j^k \cdot (R^2)^j \Delta_o^{(k_o)} \mathcal{P}_\ell,
\]

where the coefficients \( C_j^k \) are the same as the above. Moreover, in this case [that is, with (4.46)] \( \mathcal{G} \) satisfies

\[
(1 + R^2) \Delta_o \mathcal{G} - 2n (Y \cdot \nabla \mathcal{G}) + 2n \mathcal{G} = \mathcal{P}_\ell.
\]

This completes the proof of Lemma 4.22. \( \square \)

We summarize the coefficient in the following.

\[
\uparrow \\
\quad C_o^o = \frac{-1}{2n (\ell - 1)}, \ldots, \quad C_j^j = \frac{-C_{j-1}^{j-1}}{A_{\ell,j,j} - 2n (\ell - 1)}, \ldots,
\]

(4.48)

\[
\quad C_1^o = -\frac{C_o^o + C_1^1 \cdot A_{\ell,1,1}}{-2n [(\ell - 2) - 1]}, \ldots, \quad C_k^o = -\frac{C_{k-1}^o + C_1^1 \cdot A_{\ell,1,k}}{-2n [(\ell - 2k) - 1]}, \ldots,
\]

\[
\downarrow \\
\quad C_k^j = -\frac{C_{k-1}^{j-1} + C_k^j + C_{j+1}^{j+1} \cdot A_{\ell,j+1,k}}{A_{\ell,j,k} - 2n \{ [\ell + 2 (j - k)] - 1 \}} \quad \text{for } 1 \leq j < k \leq h_\ell - 1.
\]
Proposition 4.49. Let $P_\ell$ be a homogeneous polynomial of degree $\ell$ defined on $\mathbb{R}^n$. Assume that

(i) when $n$ is even: $2 \leq \ell < n + 2$ and $\Delta \Omega^{(h_\ell)} P_\ell = 0$ ($\Delta \Omega^{(h_\ell)} P_\ell$ is degree zero);

(ii) when $n$ is odd: $2 \leq \ell$ and $\Delta \Omega^{(h_\ell)} P_\ell \equiv 0$ (here $\Delta \Omega^{(h_\ell)} P_\ell$ is degree one).

Then equation (4.3) has a polynomial solution $G_\Omega$ given by

\begin{equation}
G_\Omega = \sum_{0 \leq j \leq k \leq h_\ell - 1} C_j^k \cdot (R^2)^j \left[ \Delta \Omega^{(h_\ell)} P_\ell \right].
\end{equation}

The coefficients $C_j^k$ are presented in (4.48). In particular, the constant and the linear terms are not present in the solution, and the degree of each term in $G$ is at most $\ell$.

Refer to §A.4 in the e-Appendix for the case when $\Delta \Omega^{(h_\ell)} P_\ell \neq 0$.

§5. Mezzo-scale effect of the global harmonic term.

In this section we show that for estimates of $v_i$ with accuracy of order $O(\lambda^{n - 2})$ or better, the contribution from other blow-up points has to be taken into account. We continue to assume

\begin{equation}
\text{the standard conditions (1.6), (1.25) and (1.26), plus (2.16), with the notations in (1.10) and (1.11)}.
\end{equation}

§5a. Rescaled harmonic part. From Proposition 2.24, we can find small positive numbers $c_0$ and $c_1$ such that for $i \gg 1$,

\begin{equation}
C^{-1} \lambda_i^{n - 2} \leq v_i(y) \leq C \lambda_i^{n - 2} \quad \text{for } i \gg 1 \quad \text{and} \quad c_0 \leq |y| \leq c_1.
\end{equation}

Together with the Harnack inequality (cf. Theorem 8.20 and Corollary 8.21 in [11], p. 199), and the discussion in §2c.1, a subsequence of

\begin{equation}
\left\{ M_i^{-1} \cdot u_i \right\} = \left\{ \lambda_i^{-n/2} \cdot u_i \right\} \quad \text{[}M_i \text{ is given in (3.2)]}
\end{equation}

converges to a positive $C^2$-function $H_\lambda$ in $S^n \setminus \{\beta_o, \cdots, \beta_k\}$. (The convergence is uniform in every compact set in $S^n \setminus \{\beta_o, \cdots, \beta_k\}$.) With the stereographic projection $\tilde{P}$ onto $\mathbb{R}^n$, $H_\lambda$ can be expressed [cf. (1.3) and (2.9)] as

\begin{align}
H_\lambda(y) &:= \left[ H_\lambda \circ \tilde{P}^{-1}(y) \right] \cdot \left( \frac{2}{1 + |y|^2} \right)^{n/2}, \\
H_\lambda(y) &= \sum_{j=0}^k \frac{A_l}{|y - \hat{Y}_l|^n - 2}.
\end{align}
for \( y \in \mathbb{R}^n \setminus \{ \hat{Y}_o = 0, \ldots, \hat{Y}_k \} \). Here \( \hat{Y}_j := \mathcal{P}(\beta_j) \), \( 0 \leq j \leq k \), and \( \mathcal{A}_j \) are positive constants \([\text{a constant times } A_j \text{ which appears in (2.10) }]\). From the Proportionality Proposition 2.3, (5.3) and \( \S 2 \mathbf{f} \), together with (1.10) and (1.11), we obtain

\[
[v_i(\xi)] \cdot v_i(y) = \lambda_i^{-\frac{n-2}{2}} \cdot v_i(y) \to \frac{1}{|y|^{n-2}} + h(y) \quad \text{in } C^2_{\text{loc}}(\mathcal{B}_o(\hat{\rho}_i) \setminus \{0\}).
\]

Recall that we assume (without loss of generality) \((\tilde{c}_n K)(0) = n(n-2)\). Hence we know that

\[
(5.5) \quad \mathcal{A}_o = 1.
\]

\[
(5.6) \quad \text{Define } H_{\lambda \geq 1}(y) := \sum_{j=1}^{k} \frac{A_j}{|y - \hat{Y}_j|^{n-2}} \quad \text{for } y \in \mathbb{R}^n \setminus \{ \hat{Y}_1, \ldots, \hat{Y}_k \}.
\]

Note that \( H_{\lambda \geq 1}(y) = H_{\lambda}(y) - |y|^{-(n-2)} \) is well-defined and smooth on a neighborhood of 0.

\( \S 5 \mathbf{b} \). Estimating \( |\mathcal{V}_i(\mathcal{Y}) - A_1(\mathcal{Y})| \) on the mezzo-scale \( C_o \leq \lambda_1|\mathcal{Y}| \leq C_1 \).

We start with the convergence occurring in (5.2):

\[
\frac{u_i(x)}{\lambda_i^{n-2}} \to H_\lambda(x) \quad \text{for } x \in S^n \setminus \left[ \bigcup_{j=0}^{k} \mathcal{B}_\beta(x) \right], \quad \rho > 0 \text{ small but fixed}
\]

\[
\Rightarrow \left| \frac{u_i(x)}{\lambda_i^{n-2}} - H_\lambda(x) \right| \leq \varepsilon \quad \text{for all } i \gg 1 \text{ and } x \in S^n \setminus \left[ \bigcup_{j=0}^{k} \mathcal{B}_\beta(x) \right]
\]

\[
\Rightarrow \left| \frac{u_i(x)}{\lambda_i^{n-2}} \cdot \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}} - H_\lambda(x) \cdot \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}} \right| \leq \varepsilon \cdot \left( \frac{2}{1 + |y|^2} \right)^{\frac{n-2}{2}}
\]

for all \( i \gg 1 \) and \( x \in S^n \setminus \left[ N \cup \bigcup_{j=0}^{k} \mathcal{B}_\beta(x) \right]; \quad y = \hat{P}(x)
\]

\[
\Rightarrow \left| \frac{v_i(y)}{\lambda_i^{n-2}} - H_\lambda(y) \right| \leq \frac{C \varepsilon}{(1 + r)^n} \quad \text{for all } i \gg 1 \text{ and } y \in \mathbb{R}^n \setminus \left[ \bigcup_{l=0}^{k} \mathcal{B}_\hat{Y}_l(r_o) \right]
\]

\[
\Rightarrow \left| \frac{v_i(y)}{\lambda_i^{n-2}} - \frac{1}{|y|^{n-2}} - H_{\lambda \geq 1}(y) \right| \leq \frac{C \varepsilon}{(1 + r)^n} \quad \text{[using (5.5) and (5.6)]}
\]

(5.7) \quad \Rightarrow \left| \frac{\lambda_i^{n-2}}{y|^{n-2}} - [H_{\lambda \geq 1}(y)] \cdot \lambda_i^{n-2} \right| \leq \frac{C \varepsilon \cdot \lambda_i^{n-2}}{(1 + r)^n}

27
for all \( i \gg 1 \) and \( c_o \leq |y| \leq c_1 \). As usual, \( r = |y| \). Note that we may take

\[
(5.8) \quad c_1 \leq \frac{1}{2} \cdot \min_{1 \leq j \leq k} |\hat{y}_j|.
\]

In (5.7), we replace \( y \to \xi_i + \lambda_i \mathcal{Y} \). For \( i \gg 1 \) and \( c_o \leq |\xi_i + \lambda_i \mathcal{Y}| \leq c_1 \):

\[
(5.9) \quad \left| v_i (\xi_i + \lambda_i \mathcal{Y}) - \frac{\lambda_i^{n-2}}{|\xi_i + \lambda_i \mathcal{Y}|^{n-2}} - [H_{\lambda \geq 1} (\xi_i + \lambda_i \mathcal{Y})] \cdot \lambda_i^{n-2} \right| \leq \frac{C \varepsilon \cdot \lambda_i^{n-2}}{(1 + |\xi_i + \lambda_i \mathcal{Y}|)^n}
\]

\[
\Rightarrow \quad \left| \frac{v_i (\xi_i + \lambda_i \mathcal{Y})}{v_i (\xi_i)} - \frac{\lambda_i^{n-2}}{|\xi_i + \lambda_i \mathcal{Y}|^{n-2}} - [H_{\lambda \geq 1} (\xi_i + \lambda_i \mathcal{Y})] \cdot \lambda_i^{n-2} \right| \leq C \varepsilon \cdot \lambda_i^{n-2}
\]

\[
\Rightarrow \quad \left| \mathcal{V}_i (\mathcal{Y}) - \frac{1}{|[\lambda_i^{-1} \cdot \xi_i] + \mathcal{Y}|^{n-2}} - \lambda_i^{n-2} \cdot [H_{\lambda \geq 1} (\xi_i + \lambda_i \mathcal{Y})] \right| \leq C \varepsilon \cdot \lambda_i^{n-2}.
\]

Next, we seek to show that the second term in the last inequality above is “close” to \( A_1 \) in the mezzo-range, under the condition that \( |\lambda_i^{-1} \cdot \xi_i| = O(1) \). We first note that

\[
y = \xi_i + \lambda_i \mathcal{Y}, \quad \text{where } c_o \leq |y| \leq c_1
\]

\[
\Rightarrow \quad c_o \leq |\xi_i + \lambda_i \mathcal{Y}| \leq c_1 \quad \Rightarrow \quad c_o \cdot \lambda_i^{-1} \leq |(\lambda_i^{-1} \cdot \xi_i) + \mathcal{Y}| \leq c_1 \cdot \lambda_i^{-1}
\]

[assuming \( \lambda_i^{-1} \cdot |\xi_i| = O(1) \)]

\[
(5.10) \quad \Rightarrow \quad \left[ c_o - o(1) \right] \cdot \lambda_i^{-1} \leq |\mathcal{Y}| \leq \left[ c_1 + o(1) \right] \cdot \lambda_i^{-1},
\]

where \( o(1) \to 0 \) as \( i \to \infty \). Thus again, for \( i \gg 1 \) and \( c_o \leq |\xi_i + \lambda_i \mathcal{Y}| \leq c_1 \), we continue with

\[
(5.11) \quad \left| A_1 (\mathcal{Y}) - \frac{1}{|[\lambda_i^{-1} \cdot \xi_i] + \mathcal{Y}|^{n-2}} \right|
\]

\[
= \left| \left( \frac{1}{1 + |\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} - \left( \frac{1}{|\mathcal{Y}|^2 + 2 \mathcal{Y} \cdot [\lambda_i^{-1} \cdot \xi_i] + O(1)} \right)^{\frac{n-2}{2}} \right|
\]

\[
\leq \frac{2}{n-2} \left| \frac{1}{1 + |\mathcal{Y}|^2} - \frac{1}{|\mathcal{Y}|^2 + 2 \mathcal{Y} \cdot [\lambda_i^{-1} \cdot \xi_i] + O(1)} \right| \times
\]

\[
\times \max \left\{ \left( \frac{1}{1 + |\mathcal{Y}|^2} \right)^{\frac{n-2}{2} - 1}, \left( \frac{1}{|\mathcal{Y}|^2 + 2 \mathcal{Y} \cdot [\lambda_i^{-1} \cdot \xi_i] + O(1)} \right)^{\frac{n-2}{2} - 1} \right\}
\]

28
\[ C' \cdot \left| \frac{2 \langle \mathcal{Y}, [\lambda_i^{-1} \cdot \xi_j] \rangle + O(1) - 1}{[1 + |\mathcal{Y}|^2] \cdot [|\mathcal{Y}|^2 + 2 \langle \mathcal{Y}, [\lambda_i^{-1} \cdot \xi_j] \rangle + O(1)]} \right| \cdot \left( \frac{1}{|\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} \]

\[ \leq C' \cdot \left[ \frac{1}{|\mathcal{Y}|^4} + \frac{O(1)}{|\mathcal{Y}|^4} + \frac{|\mathcal{Y}| \cdot |\lambda_i^{-1} \cdot \xi_j|}{|\mathcal{Y}|^4} \right] \cdot \left( \frac{1}{|\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} \]

\[ \leq C' \cdot \left[ \frac{1}{|\mathcal{Y}|^4} + \frac{O(1)}{|\mathcal{Y}|^4} + \frac{O(1)}{|\mathcal{Y}|^3} \right] \cdot \left( \frac{1}{|\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} \]

\[ \leq C' \cdot \frac{O(1)}{|\mathcal{Y}|^3} \cdot \left( \frac{1}{|\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} \leq \frac{O(1)}{|\mathcal{Y}|^3} \cdot \left( \frac{1}{|\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} \leq O(1) \cdot \lambda_i^{n-1} . \]

In the above, we apply the inequality

\[ a > b > 0 \quad \text{and} \quad p \geq 1 \implies a^p - b^p \leq p^{-1} \cdot (a - b) \cdot a^{p-1} . \]

Note that when \( j \neq 0 \), from (5.8) and (5.10), we have

\[ (5.12) \quad \frac{A}{|\langle \xi_i + \lambda_i \mathcal{Y} \rangle - \hat{Y}_j |^{n-2}} = \frac{A}{|\langle \lambda_i \mathcal{Y} - \hat{Y}_j \rangle + \xi_i |^{n-2}} \]

\[ = \frac{A}{|\lambda_i \mathcal{Y} - \hat{Y}_j |^{n-2}} \cdot \left( \frac{1}{1 + \frac{\xi_i}{|\lambda_i \mathcal{Y} - \hat{Y}_j |}} \right)^{n-2} = \frac{A}{|\lambda_i \mathcal{Y} - \hat{Y}_j |^{n-2}} \cdot [1 + O(|\xi_i|)] \]

for \( |c_o - o(1)| \cdot \lambda_i^{-1} \leq |\mathcal{Y}| \leq |c_1 + o(1)| \cdot \lambda_i^{-1} \). Thus if we install the terms

\[ (5.13) \quad H_{\geq 1} (\mathcal{Y}) := \sum_{j \geq 1} \frac{A_j}{|\lambda_i \mathcal{Y} - \hat{Y}_j |^{n-2}} , \]

\( [H_{\geq 1} (\mathcal{Y}) \) is smooth and harmonic in \( B_o (c_1 \cdot \lambda_i^{-1}) \), and apply the triangle inequality, (5.8), (5.11) and (5.12) furnish us with the following mezzo-scale estimate.

**Lemma 5.14.** For \( n \geq 3 \), under the conditions and notations in (5.1) for \( \{v_i \} \), \( \lambda_i \) and \( \xi_i \), assume also that \( \lambda_i^{-1} \cdot |\xi_i| = O(1) \). For any \( \varepsilon > 0 \), we have

\[ (5.15) \quad |V_i (\mathcal{Y}) - A_1 (\mathcal{Y}) - \lambda_i^{n-2} \cdot H_{\geq 1} (\mathcal{Y})| \leq C \varepsilon \cdot \lambda_i^{n-2} + O(1) \cdot \lambda_i^{n-1} \]

for all \( i \gg 1 \) and \( \bar{c}_1 \cdot \lambda_i^{-1} \leq |\mathcal{Y}| \leq \bar{c}_2 \cdot \lambda_i^{-1} \). Here \( \bar{c}_1 > 0 \) can be taken to be any small (but fixed) constant as long as \( \bar{c}_1 < \bar{c}_2 \). [In (5.15), the order in the right hand side is \( O_{\lambda_i} (n - 2) \).]
§ 6. Second order blow-up argument and the proof of Main Theorem 1.14.

In this section, we take up all the assumptions stated in Main Theorem 1.14. To begin with, we observe the following.

Proposition 6.1. For \(n \geq 4\), under the general conditions listed in (5.1), we also take the following conditions (i) – (iii) into account.

(i) 0 is a simple blow-up point for \(\{v_i\}\).

(ii) \(K\) is given by (1.8) in \(B_\rho(\rho_o)\), where \(2 \leq \ell < n - 2\).

(iii) The parameters \(\lambda_i\) and \(\xi_i\) corresponding to the simple blow-up point at 0 [via (1.10) and (1.11)] satisfy (1.12), that is, \(|\xi_i| = o(\lambda_i)\).

(iv) \(\ell\) is even.

Then \(\Delta_0^{(h)} P_\ell(y) = 0\) [\(P_\ell\) is found in (1.8)]. When \(\ell\) is even, \(h_\ell = \ell/2\), and \(\Delta_0^{(h)} P_\ell\) is a number. The same conclusion also holds when \(\ell = n - 2\) with an additional assumption that 0 is the only blow-up point (\(\ell\) is still required to be even).

The key of the proof is to combine the change of center formula (see A.6.33 in the e-Appendix) with the condition \(|\xi_i| = o(\lambda_i)\). Other arguments actually proceed in similar fashion as those found in [15] and [21]. For the benefit of readers, we present the estimates in § A.6. d in the e-Appendix.

Together with condition (1.19) in the Main Theorem, Proposition 4.49 and Proposition 6.1, we can secure a solution \(\Pi_p\) of the equation

\[
\Delta_0 \Pi_p + n(n + 2) A_\frac{4}{n-2} \cdot \Pi_p = P_\ell \cdot A_\frac{n+2}{n-2} \quad \text{in } \mathbb{R}^n.
\]

Moreover, \(\Pi_p(\mathcal{Y}) = \frac{\Gamma_p(\mathcal{Y})}{(1 + \mathcal{R}^2)^{\frac{\ell}{2}}}\) for \(\mathcal{Y} \in \mathbb{R}^n\), where \(\mathcal{R} = |\mathcal{Y}|\).

Thanks to Proposition 4.49, the precise form of \(\Gamma_p\) is known once \(P_\ell\) is given. It follows from (3.11) that

\[
\Delta_0 (\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_p) + n(n + 2) A_\frac{2}{n-2} (\mathcal{V}_i - A_1 - \lambda_i^\ell \cdot \Pi_p) = \text{RM}
\]

in \(\mathbb{R}^n\), where the ‘remainder’ \(\text{RM}\) is given in (3.12), Cf. (3.1) – (3.3).

§ 6 a. Inclusion of the harmonic term via interpolation. Consider

\[
\mathcal{D}_{i}^{\Pi} (\mathcal{Y}) := \left[ \mathcal{V}_i (\mathcal{Y}) - A_1 (\mathcal{Y}) - \lambda_i^\ell \cdot \Pi_p(\mathcal{Y}) - \lambda_i^{n-2} \cdot H_{\geq 1}(\mathcal{Y}) \right] +
\]
in the region \(|\mathcal{Y}| \leq c\lambda_i^{-1}\). Here \(c \leq c_1\) is a positive constant to be fixed [(cf. (5.8)] , \(H_{\geq 1}\) is given in (5.13) , and

\[
(6.5) \quad h_o := H_{\geq 1}(0) \quad \text{[that is, setting \(\mathcal{Y} = 0\) in (5.13)]}.
\]

In addition, \(\tilde{R} \in C^\infty(\mathbb{R}^n)\) is a non-negative function which satisfies

\[
(6.6) \quad \tilde{R}(\mathcal{Y}) = |\mathcal{Y}| \quad \text{for} \quad |\mathcal{Y}| \geq 1, \quad \tilde{R}(0) = 0, \quad \nabla_{\mathcal{Y}} \tilde{R}(0) = \vec{0}, \quad \text{and} \quad |\Delta_o \tilde{R}| \leq C \quad \text{in} \ \mathbb{R}^n.
\]

§ 6 a.1. Joint between bubble estimate and the global harmonic term. From (1.4), (1.6) and (2.17), we know that \(V_i(0) = A_i(0) = 1\) for all \(i\). Here, we group together the terms in (6.4) which link to the harmonic function \(H_{\geq 1}\), and note that

\[
(6.7) \quad \lambda_i^{n-2} \cdot H_{\geq 1}(\mathcal{Y}) + [\lambda_i^{n-2} \cdot h_o - \frac{\tilde{R}(\mathcal{Y})}{c\lambda_i^{-1}}] = 0 \quad \text{if} \quad \mathcal{Y} = 0,
\]

\[
\lambda_i^{n-2} \cdot H_{\geq 1}(\mathcal{Y}) + [\lambda_i^{n-2} \cdot h_o - \frac{\tilde{R}(\mathcal{Y})}{c\lambda_i^{-1}}] = \lambda_i^{n-2} \cdot H_{\geq 1}(\mathcal{Y})
\]

if \(|\mathcal{Y}| = c\lambda_i^{-1}\). That is, via \(\tilde{R}\), the bubble estimate and the global harmonic term are joint. See the comments in § 2 d.1 , (2.14), (2.15), (2.22) and (2.23).

§ 6 b. Ingredients for the method to work. [12] provides the framework for what we call the “second level blow-up argument” (see also [8]) . It is an exquisite method which goes to the root of the blow-up phenomenon. Here we summarize the key steps and apply it to our situation.

§ 6 b.1. First order vanishing property. From (1.6), (3.1), (3.2), Proposition 4.49 (from there we know that \(\Gamma_p\) contains no constant term) , and (6.7), we obtain

\[
(6.8) \quad D_i^\Pi(0) = 0 \quad \text{for} \quad i \gg 1.
\]

Referring to (5.13), (observe that presence of \(\lambda_i\) in the right hand side below)

\[
(6.9) \quad \frac{\partial}{\partial \mathcal{Y}_i} \left[ \frac{A_i}{|\lambda_i \mathcal{Y} - \mathcal{Y}_j|^n} \right] \bigg|_{\mathcal{Y}=0} = (n - 2) A_i \cdot \left( \frac{\lambda_i \mathcal{Y}_j}{|\mathcal{Y}_j|^n} \right)^{\lambda_i} \left[ \mathcal{Y}_j = (\mathcal{Y}_{j_1}, \ldots, \mathcal{Y}_{j_m}) \right].
\]
From the definition of $A_1 = A_{1,0}$ [cf. (1.4)], (1.6), (3.1), (3.2), Proposition 4.49 (from there we recognize that $\Gamma_p$ contains no first order terms), (6.5), (6.6) and (6.9), we obtain

$$\| \nabla_Y D_i^\Pi (0) \| = 0 (\lambda_i^{n-1}) \quad \text{for} \quad i \gg 1.$$

**§ 6 b.2.** The maximum in $B_o(c \lambda_i^{-1})$. Because $\Delta_o [H \geq 1 - h_o] = 0$, we have

$$\Delta_o D_i^\Pi + n (n + 2) A_i^{\frac{4}{n-2}} \cdot D_i^\Pi = \text{R.H.S.}_i \quad \text{in} \quad B_o(c \lambda_i^{-1}),$$

where [by using (6.3) and (6.6)],

$$\text{R.H.S.}_i := \text{R.M.} - \lambda_i^{n-2} \left\{ \frac{h_o}{c \lambda_i^{-1}} \cdot \Delta_o \bar{R} + n (n + 2) \cdot \frac{h_o}{c \lambda_i^{-1}} \cdot A_i^{\frac{4}{n-2}} \cdot \bar{R} \right.$$

$$\left. + n (n + 2) A_i^{\frac{4}{n-2}} \cdot [H \geq 1 - h_o] \right\}.$$

From (2.26) in Proposition 2.24, Proposition 4.49 and the expression for $\Pi_p$ [that is, (6.2)], we have

$$\ell \leq n - 2 \implies \Lambda_i := \max_{|Y| \leq c \lambda_i^{-1}} |D_i^\Pi (Y)| < \infty \quad \text{for} \quad i \gg 1.$$

We assert that

$$\Lambda_i = o_{\lambda_i} (\ell).$$

The assertion is equivalent to

$$\Lambda_i = o (1) \chi_i^\ell \iff \frac{\chi_i^\ell}{\Lambda_i} = \frac{1}{o (1)} \iff \frac{\chi_i^\ell}{\Lambda_i} \to \infty.$$

Suppose that this is not the case. Then (modulo a subsequence)

$$\frac{\chi_i^\ell}{\Lambda_i} \leq C \quad \text{for all} \quad i \geq 1 \iff \frac{1}{\Lambda_i} \leq \frac{C}{\chi_i^\ell} \quad \text{for all} \quad i \geq 1.$$

In what follows, we seek to derive a contradiction from (6.16).

**§ 6 b.3.** Renormalization. Consider the function

$$W_i := \frac{D_i^\Pi}{\Lambda_i} \quad \text{defined in} \quad B_o(c \lambda_i^{-1}).$$

From (6.11), we have

$$\Delta_o W_i + n (n + 2) A_i^{\frac{4}{n-2}} \cdot W_i = \Lambda_i^{-1} \cdot \text{R.H.S.}_i \quad \text{in} \quad B_o(c \lambda_i^{-1}).$$
§ 6 b.4. Order of magnitude of the remainder. The key property we want to show about $\textbf{R.H.S}_i$ are the following (under the condition in the Main Theorem).

(6.19) Given any $R_o > 0$, $\Lambda_i^{-1} \cdot |\textbf{R.H.S}_i| \to 0$ uniformly in $B_o(R_o)$.

(6.20) $\frac{|\textbf{R.H.S}_i(\mathcal{Y})|}{\Lambda_i} \leq \frac{C}{(1 + R)^4} + O(\lambda_i) \cdot \chi_{B_o(1)}$ for $R \leq c\lambda_i^{-1}$.

Here $R = |\mathcal{Y}|$ and $\chi_{B_o(1)}$ is the characteristic function of the unit ball. These are demonstrated in § 6 c.

§ 6 b.5. Vanishing on the whole. From (6.8), (6.9), (6.10), (6.16) and (6.17), we know that

(6.21) $|W_i| \leq 1$ in $B_o(c\lambda_i^{-1})$.

(6.22) $W_i(0) = 0$ for $i \gg 1$, and $\nabla W_i(0) = O(\lambda_i) \to \vec{0}$ as $i \to \infty$.

It follows from (6.11), (6.18), (6.19), (6.21) and standard elliptic theory that (modulo a subsequence)

(6.23) $W_i \to W$ uniformly in every compact subset of $\mathbb{R}^n$.

Here $W$ is a $C^2$-function satisfying

(6.24) $\Delta o W + n(n + 2)A_i^{\frac{4}{n-2}} \cdot W = 0$ in $\mathbb{R}^n$.

In addition, from (6.22),

(6.25) $W(0) = 0$ \& $\nabla W(0) = \vec{0}$.

Moreover, in § 6 b.7, we show that

(6.26) $|W(\mathcal{Y})| \to 0$ when $|\mathcal{Y}| \to \infty$.

A standard boot-strap argument shows that $W$ is smooth in $\mathbb{R}^n$. It follows from the Liouville-type theorem for (6.24) (see Lemma 2.4 in [8], cf. also [3]) that

(6.27) $W \equiv 0$ in $\mathbb{R}^n \implies W_i \to 0$ uniformly in $B_o(R_o) \subset \mathbb{R}^n$.

Here $R_o$ can be any given positive number. On the other hand, by the definition of $\Lambda_i$, there is a point

(6.28) $\mathcal{Y}_{\mu_i} \in \overline{B_o(c\lambda_i^{-1})}$ such that $W_i(\mathcal{Y}_{\mu_i}) = 1$. 

33
We produce a contradiction with (6.27) by showing that we can find a positive number $R_o$ such that

$$|Y_{\mu_i}| \leq R_o \quad \text{for all} \quad i \gg 1 \quad \left(\iff \max_{B_o(R_o)} W_i = 1\right).$$

§ 6 b.6. Smallness of $|W_i|$ near the boundary $\partial B_o(c\lambda_i^{-1})$. Given (5.15) in Lemma 5.14, we turn our attention to $\lambda_i^\ell \cdot \Pi_p$ in (6.4). Based on Proposition 4.49 and condition (6.2), we have

(6.30) the degree of the polynomial $\Gamma_p$ in (6.2) is at most $n - 2$.

It follows that

(6.31) $|\lambda_i^\ell \cdot \Pi_p(Y)| = \lambda_i^\ell \cdot \frac{|\Gamma_p(Y)|}{(1 + R^2)^2}$

$$\leq C \cdot \frac{\lambda_i^\ell (1 + R)^{n - 2}}{(1 + R)^n} \leq \frac{C_1 \lambda_i^\ell}{(1 + R)^2} \leq C_2 \lambda_i^{\ell + 2}$$

for $R = |Y| \geq (1 - \delta) \cdot c\lambda_i^{-1}$

(6.32) $\rightarrow \Lambda_i^{-1} \cdot |\lambda_i^\ell \cdot \Pi(Y)| = O(\lambda_i^2)$ [via (6.16)]

for $R \geq (1 - \delta) \cdot c\lambda_i^{-1}$. Together with (6.4), (6.6), Lemma 5.15 and (6.31), we have

(6.33) $|W_i(Y)| = O(\epsilon) + O(\lambda_i^2)$ for $|Y| = c\lambda_i^{-1}$ and $i \gg 1$.

Moreover, for $(1 - \delta) \cdot [c\lambda_i^{-1}] \leq |Y| \leq [c\lambda_i^{-1}]$, we have

(6.34) $[\lambda_i^{n - 2} \cdot h_o] \cdot \left|1 - \frac{\tilde{R}(Y)}{c\lambda_i^{-1}}\right| = [\lambda_i^{n - 2} \cdot h_o] \cdot \left|1 - \frac{|Y|}{c\lambda_i^{-1}}\right| \leq [\delta \cdot h_o] \cdot \lambda_i^{n - 2}$.

Thus if we choose $\delta > 0$ to be small [relative to the constant $C$ in (6.16) and $h_o$ only], and combine (6.32) with (6.33), (6.34) and Lemma 5.15, we obtain

(6.35) $|W_i(Y)| \leq \frac{1}{2}$ for $(1 - \delta) \cdot [c\lambda_i^{-1}] \leq |Y| \leq [c\lambda_i^{-1}]$ and $i \gg 1$.

§ 6 b.7. The decay to 0 - proof of (6.26). To demonstrate (6.26), we show that for any positive number $\tilde{\epsilon}$ (small and given), we can find a positive number $R_{\tilde{\epsilon}}$ and a natural number $I_{\tilde{\epsilon}}$, such that

(6.36) $|W_i(Y)| \leq \tilde{\epsilon}$ for all $i \geq I_{\tilde{\epsilon}}$ and $R_{\tilde{\epsilon}} \leq |Y| \leq (1 - \delta) \cdot [c\lambda_i^{-1}]$. 

34
Here $\delta \in (0, 1)$ is fixed, and the integer $I_\varepsilon$ depends on $\varepsilon$ only. In particular, $(1 - \delta) \cdot [c \lambda_i^{-1}] \to \infty$ as $i \to \infty$. Once we have (6.36), together with the uniform convergence of $W$ to $W_i$ on any given compact subset of $\mathbb{R}^n$, we have

$$\frac{\varepsilon}{(1 - \delta) \cdot [c \lambda_i^{-1}] - 1} \to \infty$$

for all $|\mathcal{Y}| \geq \mathcal{R}_\varepsilon \implies (6.26)$.

To demonstrate the proof for (6.36), via (6.33), we already know that $|W_i|$ is ‘small’ along the boundary $\partial B_o(c \lambda_i^{-1})$. The value in $B_o(c \lambda_i^{-1})$ is governed by the equation describing $\Delta o W_i$ [that is, (6.18)], and the Green representation formula, which, in the present situation, is given by

$$W_i(Y) = \int_{B_o(c \lambda_i^{-1})} G_i(Y, \mathcal{Y}_{out}) \left\{ - n (n + 2)[A_1(Y)]^{\frac{4}{n-2}} \cdot W_i(Y) + \Lambda_i^{-1} \cdot \text{R.H.S}_i(Y) \right\} d\mathcal{Y}$$

$$+ \int_{\partial B_o(c \lambda_i^{-1})} [n \cdot \nabla G_i(Y, \mathcal{Y}_{out})] W_i(Y) d\mathcal{S}_Y$$

for $\mathcal{Y}_{out} \in B_o(c \lambda_i^{-1})$. Here $G_i$ is the Green function for $\Delta o$ in $B_o(c \lambda_i^{-1})$ with the Dirichlet boundary condition. See, for example, [23]. Note that

$$G_i(Y, Y') \approx \frac{1}{(n - 2) \|S^{n-1}\| \cdot |Y - Y'|^{n-2}}$$

when $Y$ is close to $Y'$. The sign is negative of the one used in [12]. Using (1.4) and (6.21), we obtain

$$|A_1(Y)|^{\frac{4}{n-2}} \cdot W_i(Y) \leq \left( \frac{1}{1 + \mathcal{R}^2} \right)^2 \leq \frac{C}{1 + \mathcal{R}^4}$$

for $Y \in \mathbb{R}^n$.

Consider points $\mathcal{Y}_{out}$ so that

$$|\mathcal{Y}_{out}| \leq (1 - \delta) \cdot [c \lambda_i^{-1}] .$$

Via proportional property on the Green function (see §A.5 in the e-Appendix; cf. also pp. 157 in [12]), we have

$$|G_i(Y, \mathcal{Y}_{out})| \leq \left[ C_1 + \frac{C_2}{\delta n - 2} \right] \cdot \frac{1}{|y - \mathcal{Y}_{out}|^{n-2}}$$

for $Y \in B_o(c \lambda_i^{-1}) \setminus \{Y_{out}\}$,

$$|n \cdot \nabla G_i(Y, \mathcal{Y}_{out})| \leq \frac{C_3}{\delta n \cdot \lambda_i^{n-1}}$$

for $Y \in \partial B_o(c \lambda_i^{-1})$. 

35
where $|Y_{out}| \leq (1 - \delta) \cdot [c \lambda_i^{-1}]$. Here $C_1$, $C_2$ and $C_3$ are positive constant independent on $i$ and $\delta$. It follows from (6.33) and (6.41) that

\[(6.42) \left| \int_{\partial B_0(c \lambda_i^{-1})} \left[ \mathbf{n} \cdot \nabla Y G_i(Y, Y_{out}) \right] W_i(Y) \, dS_Y \right| \leq \frac{C_2 \cdot \varepsilon}{\delta^n} \cdot \lambda_i^{n-1} \cdot \int_{\partial B_0(c \lambda_i^{-1})} dS_Y \leq \frac{C_2 \cdot \varepsilon}{\delta^n} \cdot \lambda_i^{n-1} \cdot \left[ \|S^{n-1}\| \cdot (c \lambda_i)^{n-1} \right] \leq \frac{C \cdot \varepsilon}{\delta^n}.\]

Here the consider $C$ is independent on $i$. Putting the information into (6.38), together with (6.20) and (6.40), we obtain

\[(6.43) \left| W_i(Y_{out}) \right| \leq \left[ C_1 + \frac{C_2}{\delta^{n-2}} \right] \cdot \int_{B_0(c \lambda_i^{-1})} \left( \frac{1}{|Y - Y_{out}|^{n-2}} \cdot \frac{1}{(1 + |Y|)^4} \right) dy + \int_{B_0(1)} O(\lambda_i) + \frac{C \cdot \varepsilon}{\delta^n} \quad \text{(from the harmonic term)} \]

\[\leq C_\delta \cdot \left[ \frac{1}{(1 + |Y_{out}|)} + O(\lambda_i) + O(\varepsilon) \right] \]

for $|Y_{out}| \leq (1 - \delta) \cdot [c \lambda_i^{-1}]$. Refer to [8] for the estimation of the first integral in (6.43). Thus we can find $R_\varepsilon > 0$ and $I_\varepsilon$ such that for all $i \geq I_\varepsilon$, we have

\[(6.44) \left| W_i(Y) \right| \leq \varepsilon \quad \text{for} \quad R_\varepsilon \leq |Y| \leq (1 - \delta) \cdot [c \lambda_i^{-1}].\]

§ 6 b.8. Further restriction on the location of the maximum – proof of (6.29). In view of (6.35), we actually have

\[Y_{\mu_i} \in B_o \left( (1 - \delta) \cdot [c \lambda_i^{-1}] \right) \quad \text{[cf. (6.28)]}.\]

Here $\delta > 0$ is chosen close to 0 (as explained in § 6 b.6). Argue as in (6.43), we arrive at a similar conclusion

\[(6.45) 1 = \left| W_i(Y_{\mu_i}) \right| \leq C \left[ \frac{1}{1 + |Y_{\mu_i}|} + O(\lambda_i) + O(\varepsilon) \right].\]

It follows that there is a fixed positive number $R_o$ such that

\[|Y_{\mu_i}| \leq R_o \quad \text{for} \quad i \gg 1.\]
Hence we establish (6.29), and obtaining a contradiction to (6.27). Thus (6.16) must be wrong. That is, (6.14) holds.

§ 6. Terms in the remainder R.H.S. in (6.12). Here we verify (6.19) and (6.20). Recall that RM is decomposed into four components as expressed in (3.12).

§ 6 c. First term. Under the condition $|\xi_i| = o_{\lambda_i}(1)$, we have

(6.46)

$$
\Lambda_i^{-1} \cdot \left[ |\xi_i|^k \cdot (\lambda_i |\mathcal{Y}|)^{\ell-k} \right] \times [A_1(\mathcal{Y})]^{\frac{n+2}{n-2}} \quad (1 \leq k \leq \ell; \quad |\xi_i| = o(1) \cdot \lambda_i)
$$

\[
\leq \lambda_i^{-\ell} \cdot o(1) \cdot \lambda_i^k 
\times R^{\ell-k} \cdot \left( \frac{1}{1 + R^2} \right)^{\frac{n+2}{n-2}} \quad \text{[using (6.16);}] \quad R = |\mathcal{Y}|
\]

\[
\leq o(1) \cdot (1 + R)^{\ell-k} \cdot \left( \frac{1}{1 + R} \right)^{n+2} \leq \frac{o(1)}{(1 + R)^5} \quad \text{for } R \leq c \lambda_i^{-1}
\]

\[
\rightarrow 0 \quad \text{uniformly in } B_o(R_o) \quad (i \gg 1; \quad \ell \leq n-2, \quad k \geq 1).
\]

§ 6 c.2. Second term.

(6.47)

$$
\Lambda_i^{-1} \cdot \left( \max_{B_o(\lambda_i \mathcal{Y}+\xi_i)} \| \nabla^{(\ell+1)} K \| \right) \cdot |\lambda_i \mathcal{Y} + \xi_i|^{\ell+1} \times [A_1(\mathcal{Y})]^{\frac{n+2}{n-2}}
$$

\[
\leq C \lambda_i^{-\ell} \left| \lambda_i \left( \mathcal{Y} + \frac{\xi_i}{\lambda_i} \right) \right|^{\ell+1} \times \frac{1}{(1 + R)^{n+2}} \leq C \lambda_i \cdot (1 + R)^{\ell+1} \cdot \frac{1}{(1 + R)^{n+2}}
\]

\[
[\text{via (6.16);}] \quad |\xi_i| = o(\lambda_i), \quad \ell \leq n]
\]

\[
\leq \frac{C \lambda_i}{(1 + R)^3} \leq \frac{C_1}{(1 + R)^2} \quad \text{for } R \leq c \lambda_i^{-1} \quad \Rightarrow \quad \lambda_i (1 + R) \leq C_2
\]

§ 6 c.3. Third term. Using $a^2 = (a - b)^2 + 2b(a - b) + b^2$, we obtain

(6.48)

$$
\Lambda_i^{-1} \cdot A_1^{\frac{4}{n-2}} (\mathcal{Y}_i - A_1)^2
$$

\[
\leq A_1^{\frac{4}{n-2}} \Lambda_i^{-1} \left\{ [\mathcal{Y}_i - A_1 - \lambda_i^\ell \cdot \Pi_p]^2 + 2(\lambda_i^\ell \cdot \Pi_p) |\mathcal{Y}_i - A_1 - \lambda_i^\ell \cdot \Pi_p| + (\lambda_i^\ell \cdot \Pi_p)^2 \right\}
\]

\[
\leq \left( \frac{1}{1 + R^2} \right)^{\frac{n+2}{n-2}} \left\{ |(\mathcal{Y}_i - A_1 - \lambda_i^\ell \cdot \Pi_p| \cdot |\mathcal{W}_i + O_{\lambda_i}((n-2)-\ell)|
\]

37
\[ + 2 |\lambda_\ell^\ell \cdot \Pi_p| \cdot |W_i + O_{\lambda_i((n - 2) - \ell)}| + (\Lambda_i^{-1} \lambda_\ell^\ell) \cdot [\lambda_\ell^\ell \cdot (\Pi_p)^2] \]

\[ \text{[using (6.4), (6.16) and (6.17)]} \]

\[ \leq \frac{C}{(1 + R)^{6-n}} \cdot \left\{ |(V_i(Y) - A_1(Y)) - \lambda_\ell^\ell \cdot \Pi_p| + \lambda_\ell^\ell |\Pi_p(Y)| + \lambda_\ell^\ell |\Pi_p(Y)|^2 \right\} \]

\[ \text{[as } |W_i + O_{\lambda_i((n - 2) - \ell)}| \text{]} \]

\[ \leq |W_i| + |O_{\lambda_i((n - 2) - \ell)}| \leq 1 + C \text{ when } \ell \leq (n - 2) \]

\[ \rightarrow 0 \text{ in } B_o(R_o) \text{ uniformly \ [from (2.59): } |V_i - A_1| \rightarrow 0 \text{ in } B_o(R_o)] \]

\[ \leq \frac{C_1}{(1 + R)^{6-n}} \cdot \left\{ |(V_i(Y) - A_1(Y)) + \lambda_\ell^\ell |\Pi_p(Y)| + \lambda_\ell^\ell |\Pi_p(Y)|^2 \right\} \]

\[ \leq \frac{C_2}{(1 + R)^{6-n}} \cdot \left\{ \frac{1}{(1 + R)^{n-2}} + \frac{\lambda_\ell^\ell \cdot (1 + R)^\ell}{(1 + R)^n} + \frac{\lambda_\ell^\ell \cdot (1 + R)^{2\ell}}{(1 + R)^{2n}} \right\} \]

\[ \text{[via (3.7) and Proposition 4.49]} \]

\[ \leq \frac{C}{(1 + R)^4} \text{ for } R \leq c\lambda_i^{-1} \quad [\Rightarrow \lambda_i(1 + R) \leq C_2]. \]

Here \( \ell \leq n - 2. \)

\section*{§ 6 c.4. Fourth term.} From (3.12), we obtain

(6.49)

\[ \text{RM}_4(Y) \]

\[ = \Lambda_i^{-1} \cdot \left\{ O \left( \max_{|\lambda_i Y + \xi| \leq \rho_o} \|\nabla^{(\ell)} K\| \times |\lambda_i Y + \xi|^\ell \right) \right\} \times \]

\[ \times \left[ O(1) |V_i - A_1| \times \max \{ \frac{V_i^{\frac{1}{n-2}}, \ A_1^{\frac{1}{n-2}} \right\} \right] \]

\[ \leq C \Lambda_i^{-1} \cdot |\lambda_i Y + \xi|^\ell \cdot \left[ \frac{4}{A_1^{\frac{n-2}{2}}}; \ |V_i - A_1| \right] \quad \text{[cf. \ (A.1.3)]} \]

for \( |Y| \leq c \cdot \lambda_i^{-1} \) \quad \text{[using Propositions 2.3 and 2.24]}

\[ \leq C_1 \Lambda_i^{-1} \lambda_\ell^\ell \cdot (1 + R)^\ell \left[ \frac{4}{A_1^{\frac{n-2}{2}}}; |V_i - A_1 - \lambda_\ell^\ell \Pi_p + \lambda_\ell^\ell |\Pi_p| \right] \]

\[ \leq C_2 \lambda_\ell^\ell \cdot (1 + R)^\ell \times \frac{1}{(1 + R)^4} \times |W_i + O_{\lambda_i((n - 2) - \ell)}| + \]

38
\[ + C_3 \left( \Lambda_{i-1}^\ell \right) \cdot (1 + \mathcal{R})^\ell \cdot \frac{1}{(1 + \mathcal{R})^4} \cdot \frac{\lambda_i^\ell (1 + \mathcal{R})^\ell}{(1 + \mathcal{R})^n} \]

\[ \leq \frac{C_4 \lambda_i^\ell \cdot (1 + \mathcal{R})^\ell}{(1 + \mathcal{R})^4} \cdot \left[ 1 + \frac{1}{(1 + \mathcal{R})^n - \ell} \right] \to 0 \text{ uniformly in } B_o(R_o) \]

\[ \leq \frac{C_5}{(1 + \mathcal{R})^4} \text{ for } \mathcal{R} = |\mathcal{Y}| \leq c \lambda_i^{-1} \text{ (when } \ell \leq n - 2). \]

§ 6 c. 5. The inserted harmonic term. The last couple of terms to be considered in \( \Lambda_{i-1}^{-1} \cdot \text{R.H.S.} \) [cf. (6.12), (6.19) and (6.20)] are

\[ \frac{1}{\Lambda_i} \cdot \lambda_i^{n-2} \cdot \frac{h_o}{c \lambda_i^{-1}} \cdot \Delta \mathcal{Y} \tilde{\mathcal{R}}(\mathcal{Y}) \leq C \lambda_i \cdot \chi_{B_o(1)} \quad [\text{via } (6.16) \text{ and } \ell \leq n - 2], \]

\[ \frac{\lambda_i^{n-2}}{\Lambda_i} \cdot n(n + 2) \cdot \frac{h_o}{c \lambda_i^{-2}} \cdot [A_1(\mathcal{Y})]^{\frac{n}{n-2}} \cdot \tilde{\mathcal{R}}(\mathcal{Y}) \]

\[ \leq C \lambda_i \cdot \chi_{B_o(1)} + C_1 \cdot \lambda_i \cdot \frac{\mathcal{R}}{(1 + \mathcal{R})^4} \to 0 \text{ uniformly in } B_o(R_o) \]

\[ \leq C \lambda_i \cdot \chi_{B_o(1)} + C_2 \cdot \frac{1}{(1 + \mathcal{R})^4} \text{ for } \mathcal{R} = |\mathcal{Y}| \leq c \lambda_i^{-1}, \]

\[ \frac{\lambda_i^{n-2}}{\Lambda_i} \cdot n(n + 2) [A_1(\mathcal{Y})]^{\frac{n}{n-2}} \cdot |H_{\geq 1}(\mathcal{Y}) | - h_o | \]

\[ \leq C \cdot \left( \frac{1}{1 + \mathcal{R}^2} \right)^4 \times \sum_{j \geq 1} \left| \frac{1}{|\lambda_i \mathcal{Y} - \hat{\mathcal{Y}}_j|^n} - \frac{1}{|\mathcal{Y}_j|^n} \right| \times \mathcal{A}_j \]

\[ \to 0 \text{ uniformly for } \mathcal{Y} \in B_o(R_o) \]

[recall (5.13) and (6.5), observe also that \( \lambda_i \mathcal{Y} \to \mathfrak{f} |\mathcal{Y}| \leq R_o \)]

\[ \leq \frac{C}{(1 + \mathcal{R})^4} \text{ for } \mathcal{R} \leq c \lambda_i^{-1}. \]

Thus we estimate each term in the remainder and confirm the right orders in (6.19) and (6.20). Combining the above discussion, we obtain the following.

**Theorem 6.53.** Under the conditions in Main Theorem 1.14, we have

\[ |\mathcal{D}_i^\Pi(\mathcal{Y})| = \alpha_i(\ell) \text{ for } |\mathcal{Y}| \leq c \lambda_i^{-1} \text{ and } i \gg 1 \text{ (modulo a subsequence).} \]
§ 6 d. Proof of Main Theorem 1.14 – zooming out to the original scale.

As in the transformation (2.17) (see also § 2 f), we note that, from the definitions of $V_i$, $A_1$, and (6.2), estimate (6.54) in Theorem 6.53 leads to

$$
(6.55) \quad \left| \frac{v_i(\xi_i + \lambda_i \mathcal{Y})}{v_i(\xi_i)} - \left( \frac{1}{1 + |\mathcal{Y}|^2} \right)^{\frac{n-2}{2}} - \lambda_i^{\ell} \cdot \frac{\Gamma_p (\mathcal{Y})}{(1 + |\mathcal{Y}|^2)^{\frac{n}{2}}} - \lambda_i^{n-2} \cdot H_{\geq 1} (\mathcal{Y}) + \lambda_i^{n-2} \cdot h_o \cdot \left( 1 - \frac{\tilde{R}}{c \lambda_i^{-1}} \right) \right| = o (\lambda_i^\ell)
$$

for $|\mathcal{Y}| \leq c \lambda_i^{-1}$. Via the transformation $y = \lambda_i \mathcal{Y} + \xi_i$ and the definition $M_i := v_i (\xi_i)$, (6.55) is rewritten as

$$
(6.56) \quad \left| v_i (y) - M_i \cdot \left( \frac{1}{1 + \lambda_i^{-2} |y - \xi_i|^2} \right)^{\frac{n-2}{2}} - M_i \cdot \lambda_i^{\ell} \cdot \frac{\Gamma_p (\lambda_i^{-1} (y - \xi_i))}{(1 + \lambda_i^{-2} |y - \xi_i|^2)^{\frac{n}{2}}} - M_i \cdot \lambda_i^{n-2} \cdot H_{\geq 1} (\lambda_i^{-1} (y - \xi_i)) + M_i \cdot \lambda_i^{n-2} \cdot h_o \cdot \left( 1 - \frac{\tilde{R} (\lambda_i^{-1} (y - \xi_i))}{c \lambda_i^{-1}} \right) \right| = M_i \cdot o (\lambda_i^\ell)
$$

for $|y| = |\lambda_i \mathcal{Y} + \xi_i| \leq c - o (1) \ (i \gg 1)$. Recall that $M_i = \lambda_i^{-\frac{n-2}{2}}$ [see (1.10), (1.11), (2.17) and § 2 f], and also the form of $H_{\geq 1}$ as expressed in (5.13). Hence we come to the conclusion that

$$
(6.57) \quad \left| v_i (y) - \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^{\frac{n-2}{2}} - \left[ \lambda_i^{\ell+1} \cdot \Gamma_p \left( \frac{y - \xi_i}{\lambda_i} \right) \right] \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^{\frac{n}{2}} - \lambda_i^{n-2} \left[ \sum_{j \geq 1} \left( \frac{A_j}{|y - \xi_i| - \hat{Y}_j |n-2} - \frac{A_j}{|\hat{Y}_j|^{n-2}} \right) + \lambda_i \cdot h_o \cdot \frac{\hat{R} (\mathcal{Y})}{c} \right] \right| = o_{\lambda_i} \left( \ell - \frac{n-2}{2} \right) \quad \text{for} \ |y| \leq \rho_2 \quad (i \gg 1).
$$

Here $\rho_2 > 0$ is a number slightly less than $c$. With (1.4), and
\( O_H \left( \lambda_i^{\frac{n-2}{2}} \right) := \lambda_i^{\frac{n-2}{2}} \left[ \sum_{j \geq 1} \left( \frac{A_j}{(y - \xi_i - Y_j^{n-2})} - \frac{A_j}{|Y_j|^{n-2}} \right) + \lambda_i \cdot \frac{h_0 \cdot \tilde{R}(\mathcal{V})}{c} \right], \)

we arrive at (1.20). [As usual, \( \mathcal{V} = \lambda_i^{-1} (y - \xi_i). \)]

e-Appendix is available at page 44 onward.

References

[1] T. Aubin, *Meilleures constantes dans le théorème d’inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32, 148 – 174 (1979).

[2] A. Ambrosetti, J. Garcia Azorero & I. Peral, *Perturbation of \( \Delta u + u^{\frac{N+2}{N-2}} = 0 \), the scalar curvature problem in \( \mathbb{R}^N \), and related topics*, J. Funct. Anal. 165 (1999), 117 – 149.

[3] A. Ambrosetti & A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on \( \mathbb{R}^n \)*, Progress in Mathematics 240, Birkhauser, Basel-Boston-Berlin, 2006.

[4] S. Brendle, *Blow-up phenomena for the Yamabe equation*, Journal of AMS 21 (2008), 951 – 979.

[5] S. Brendle & F. Marques, *Blow-up phenomena for the Yamabe equation. II.* J. Differential Geom. 81 (2009), 225 – 250.

[6] L. Caffarelli, B. Gidas & J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), 271 – 297.

[7] C.-C. Chen & C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes*, Comm. Pure Appl. Math. 50 (1997), 971 – 1019.

[8] C.-C. Chen & C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes. II.*, J. Differential Geometry 49 (1998), 115 – 178.

[9] X.-Z. Chen & X.-W. Xu, *The scalar curvature flow on \( S^n \) – perturbation theorem revisted*, Invent. Math. 187 (2012), 395 – 506.

[10] B. Gidas, W.-M. Ni & L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209 – 243.
[11] D. Gilbarg & N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Verlag, Berlin-Heidelberg-New York, 1998.

[12] M. Khuri, F. Marques & R. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. 81 (2009), 143–196.

[13] N. Korevaar, R. Mazzeo, F. Pacard & R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities*, Invent. Math. 135 (1999), 233–272.

[14] M.-C. Leung, *Blow-up solutions of nonlinear elliptic equations in $\mathbb{R}^n$ with critical exponent*, Math. Ann. 327 (2003), 723–744.

[15] M.-C. Leung, *Supported blow-up and prescribed scalar curvature on $S^n$*, Memoirs of the American Mathematical Society, 213 (2011), No. 1002.

[16] M.-C. Leung, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$. I. Uniform cancellation*, Comm. Contemporary Mathematics, 14 (2012), 1–31.

[17] M.-C. Leung, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$. II. Annular domains*, Calculus of Variations and PDE, 46 (2013), 1–29.

[18] M.-C. Leung & F. Zhou, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$. III. Aggregated and towering Blow-up*, Calculus of Variations and PDE, 54 (2015), 3009–3035.

[19] M.-C. Leung & F. Zhou, *Conformal scalar curvature equation on $S^n$: functions with two close critical points (twin pseudo-peaks)*. To Appear.

[20] Y.-Y. Li, *Prescribing scalar curvature on $S^n$ and related problems*. I, J. Differential Equations 120 (1995), 319–410.

[21] Y.-Y. Li, *Prescribing scalar curvature on $S^n$ and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. 49 (1996), 541–597.

[22] R. McOwen, *Partial Differential Equations, Methods and Applications*, Prentice-Hall, Upper Saddle River, 1996.

[23] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. 41 (1988), 317–392.

[24] R. Schoen, distributed notes for the courses at Stanford University (1988) and New York University (1989), unpublished.

[25] J. Wei & S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on $S^N$*, J. Funct. Anal. 258 (2010), 3048-3081.
DEPARTMENT OF MATHEMATICS,
NATIONAL UNIVERSITY OF SINGAPORE,
10, LOWER KENT RIDGE ROAD,
SINGAPRE 119076,
REPUBLIC OF SINGAPORE

matlmc@math.nus.edu.sg
In this appendix we follow the notations, conventions, equation numbers, section numbers, lemma, proposition and theorem numbers as used in the main article [18], unless otherwise is specifically mentioned (for instances, those equation numbers starting with ‘A’).

§ A.1 A proof of Proposition 2.24.

Proof of the necessary part ($\Rightarrow$). Here we can take $\zeta_i = \xi_{m_i}$ and $\epsilon_i = \lambda_{m_i}$, according to the analytic definition of simple blow-up point as in (2.1) and (2.2).

With the notations in (2.20), (2.26) is equivalent to

(A.1.1) \( C^{-1} \cdot A_1(\mathcal{Y}) \leq V_i(\mathcal{Y}) \leq CA_1(\mathcal{Y}) \) for \( |\mathcal{Y}| \leq \rho_o \lambda_{m_i}^{-1} \).

[Cf. (2.17) and (2.19).] It follows from (1.4) and \( A_1 = A_{1,0} \) that

(A.1.2) \( \frac{1}{2^{n-2}} \cdot \frac{1}{|\mathcal{Y}|^{n-2}} \leq A_1(\mathcal{Y}) \leq \frac{1}{|\mathcal{Y}|^{n-2}} \) for \( |\mathcal{Y}| \geq 1 \) [i.e. \( \mathcal{Y} \notin B_o(1) \)].

When \( i \gg 1, \ |\lambda_{m_i}\mathcal{Y}| \leq \rho_o \Rightarrow |\lambda_{m_i}\mathcal{Y} + \xi_i| \leq \bar{\rho}_o \) for \( 0 < \rho_o < \bar{\rho}_o \), from (2.19) and Proportionality Proposition 2.3 we obtain

(A.1.3)

\[ V_i(\mathcal{Y}) = \frac{v_i(\lambda_{m_i}\mathcal{Y} + \xi_i)}{M_i} \leq \frac{1}{M_i^2} \cdot \frac{C_1}{|\lambda_{m_i}\mathcal{Y} + \xi_i - \xi_i|^{n-2}} \]

\[ \leq \frac{1}{M_i^2} \cdot \frac{C_1}{\lambda_{m_i}^{-2} |\mathcal{Y}|^{n-2}} \leq \frac{C_1}{|\mathcal{Y}|^{n-2}} \] for \( 0 < |\lambda_{m_i}\mathcal{Y}| \leq \rho_o \) and \( i \gg 1 \).
Here we use (2.4). As we already know that \( \mathcal{V}_i(\mathcal{Y}) \to A_1(\mathcal{Y}) \) uniformly for \( \mathcal{Y} \in B_o(1) \), together with the above estimate and (A.1.2), we have

\[
(A.1.4) \quad \mathcal{V}_i(\mathcal{Y}) \leq C A_1(\mathcal{Y}) \quad \text{for } i \gg 1 \quad \text{and } |\mathcal{Y}| \leq \rho_o \lambda_m^{-1}.
\]

As for the lower bound in (A.1.1), from Propotionality Proposition 2.3, we have

\[
(A.1.5) \quad M_i \cdot v_i(y) \geq \frac{a}{|y|^{n-2}} + h(y) - o(1) \quad \text{for } 0 < \rho_1 \leq |y| \leq \rho_o.
\]

Here \( o(1) \to 0 \) when \( i \to \infty \). Choosing \( \rho_o \) to be small enough (correspondingly adjusting \( \rho_1 < \rho_o \)), we obtain

\[
(A.1.6) \quad M_i \cdot v_i(y) \geq \frac{2^{-1} \cdot a}{|y - \xi_m|^{n-2}} \quad \text{for } 0 < \rho_1 \leq |y - \xi_m| \leq \rho_o \quad \text{and } i \gg 1.
\]

Repeat the argument in (A.1.3), we have

\[
(A.1.7) \quad \mathcal{V}_i(\mathcal{Y}) \geq C_2 \frac{|\mathcal{Y}|^{n-2}}{|\mathcal{Y}|^{n-2}} \quad \text{for } \rho_1 \lambda_m^{-1} \leq |\mathcal{Y}| \leq \rho_o \lambda_m^{-1} \quad \text{and } i \gg 1.
\]

Again using the uniform convergence \( \mathcal{V}_i(\mathcal{Y}) \to A_1(\mathcal{Y}) \) for \( \mathcal{Y} \in B_o(1) \), and (A.1.2), we obtain

\[
(A.1.8) \quad \mathcal{V}_i(\mathcal{Y}) \geq \frac{C_3}{|\mathcal{Y}|^{n-2}} \quad \text{for } |\mathcal{Y}| = 1.
\]

Let us pay attention to the region \( 1 \leq |\mathcal{Y}| \leq \rho_o \lambda_m^{-1} \) and consider the function

\[
(A.1.9) \quad \left( \mathcal{V}_i(\mathcal{Y}) - \frac{C_4}{|\mathcal{Y}|^{n-2}} \right).
\]

Choose \( C_4 = \min \{ C_2, C_3 \} \). It follows from equation (2.18) that

\[
(A.1.10) \quad \Delta_o \left( \mathcal{V}_i(\mathcal{Y}) - \frac{C_4}{|\mathcal{Y}|^{n-2}} \right) = \Delta_o \mathcal{V}_i(\mathcal{Y}) < 0 \quad \text{for } 1 \leq |\mathcal{Y}| \leq \rho_o \lambda_m^{-1}.
\]

For the boundary, we apply (A.1.7) and (A.1.8), then we use the maximum principle to obtain

\[
(A.1.11) \quad \mathcal{V}_i(\mathcal{Y}) \geq \frac{C_4}{|\mathcal{Y}|^{n-2}} \quad \text{for } 1 \leq |\mathcal{Y}| \leq \rho_o \lambda_m^{-1}.
\]

As before, we already know that \( \mathcal{V}_i(\mathcal{Y}) \to A_1(\mathcal{Y}) \) for \( \mathcal{Y} \in B_o(1) \). Combining with the above estimate and (A.1.2), we obtain

\[
\mathcal{V}_i(\mathcal{Y}) \geq C^{-1} \cdot A_1(\mathcal{Y}) \quad \text{for } i \gg 1 \quad \text{and } |\mathcal{Y}| \leq \rho_o \lambda_m^{-1}.
\]
once we choose $C$ to be small enough. This complete the proof of \((\implies)\).

**Proof of the sufficient part \((\iff)\).** Assuming that we have (2.26), that is,

\[
{\frac{1}{C}} \cdot \left(\frac{\epsilon_i}{\epsilon_i^2 + |y - \zeta_i|^2}\right)^{\frac{n-2}{2}} \leq v_i(y) \leq C \cdot \left(\frac{\epsilon_i}{\epsilon_i^2 + |y - \zeta_i|^2}\right)^{\frac{n-2}{2}}
\]

for $|y - \zeta_i| \leq \rho_o$. It becomes clear that for $i \gg 1$, there is a point $\xi_i$ such that

\[
v_i(\xi_i) = \max \left\{ v_i(y) \mid y \in \overline{B_o(\rho_o)} \right\} \quad \text{and} \quad |\zeta_i - \xi_i| \leq B \cdot \epsilon_i.
\]

Here $B$ is a fixed positive number [one can take $B^2 = C^4 - 1$, where $C$ is the constant in (A.1.12)]. Moreover,

\[
\lambda_i := \frac{1}{v_i(\xi_i)} \implies c^{-1} \cdot \epsilon_i \leq \lambda_i \leq c \cdot \epsilon_i \quad \text{for} \quad i \gg 1.
\]

Here $c \geq 1$ is a constant. In addition, via the triangle inequality $|y - \xi_i| \leq |y - \zeta_i| + |\xi_i - \zeta_i|$, and vice versa, we have

\[
\frac{1}{B} \leq \frac{\epsilon_i^2 + |y - \zeta_i|^2}{\lambda_i^2 + |y - \zeta_i|^2} \leq B \quad \text{for} \quad y \in \mathbb{R}^n.
\]

Here we can take the constant $D = c^2 (B^2 + 1) = c^2 \cdot C^4$. Thus

\[
\frac{1}{C'} \cdot A_{\lambda_i, \xi_i}(y) \leq v_i(y) \leq C' \cdot A_{\lambda_i, \xi_i}(y) \quad \text{for} \quad |y - \zeta_i| \leq \rho_1,
\]

where $\rho_1 > 0$ is slightly less than $\rho_o$. Thus, without loss of generality, we may assume that

\[
\zeta_i = \xi_i \quad \text{and} \quad \epsilon_i = \lambda_i \quad \text{for} \quad i \gg 1.
\]

Directly,

\[
\left(\frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2}\right)^{\frac{n-2}{2}} \leq \frac{1}{|y - \xi_i|^{\frac{n-2}{2}}} \implies v_i(y) \leq \frac{C}{|y - \xi_i|^{\frac{n-2}{2}}}
\]

for $0 < |y - \xi_i| \leq \rho_1$. That is, 0 is an isolated blow-up point.

Next, consider the rescaled average

\[
r \mapsto \bar{w}_i(r) := r^{\frac{n-2}{2}} \cdot \frac{\int_{\partial B_{\xi_i}(r)} v_i dS}{\int_{\partial B_{\xi_i}(r)} 1 dS}
\]

46
Via the change of variables $r = e^{-t}$, and (A.1.16), we have

$$\frac{1}{2} C' \cdot \left[ \frac{1}{e^{t} + e^{-t}} \right]^{\frac{n-2}{2}} \leq \tilde{w}_i(t) \leq C' \cdot \left[ \frac{1}{e^{t} + e^{-t}} \right]^{\frac{n-2}{2}}$$

for $r = e^{-t} \leq \rho_o \iff t \geq T_1$, where $T_1 := -\ln \rho_1$.

In the above

$$e^{-t_i} = \lambda_i \iff t_i = -\ln \lambda_i \quad \text{for } i = 1, 2, \ldots \quad (t_i \to \infty \text{ as } i \to \infty).$$

By performing a blow-up analysis as in Theorem 4.2 in [?](cf. also §7c in [15]), and using (A.1.20), we obtain

$$\tilde{w}_i(t) = \tilde{w}_i(t + t_i) \to \left[ \frac{1}{e^{t} + e^{-t}} \right]^{\frac{n-2}{2}} = \frac{1}{2^{n-2}} \cdot \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}}.$$

The convergence is in $C^2$-sense, uniform on any given bounded interval in $\mathbb{R}$. Directly,

$$\frac{d}{dt} \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}} = 0 \quad \text{iff } t = 0, \quad \frac{d^2}{dt^2} \left( \frac{1}{\cosh t} \right)^{\frac{n-2}{2}} \leq -c^2 < 0 \quad \text{for } |t| \leq \delta.$$

Here $c$ is a constant which depends on the small number $\delta > 0$. It follows that $\tilde{w}_i$ has only one critical point in $[-\delta, \delta]$ for all $i \gg 1$. Likewise, the first statement in (A.1.23) shows that $\tilde{w}_i$ has no critical point in $[-T_2, T_2] \setminus [-\delta, \delta]$ for $i \gg 1$. Here $T_2 \in (-t_i - T_1, \infty)$ is a (fixed) large positive number. Any potential critical point in $[-(t_i - T_1), \infty) \setminus [-T_2, T_2]$ can be ruled out by using (A.1.20), together with Lemma 5.1 in [8] (cf. also Lemma 7.16 and Lemma 7.25 in [15], and the proof of Theorem 4.1 in [?]). Note that (A.1.20) implies

$$C^{-1} \cdot \left[ \frac{1}{\cosh t} \right]^{\frac{n-2}{2}} \leq \tilde{w}_i(t) \leq C \cdot \left[ \frac{1}{\cosh t} \right]^{\frac{n-2}{2}} \quad \text{for } t \in [-t_i - T_1, \infty).$$

Any “small” critical value for $\tilde{w}_i$ comes from a local minimum, and according to (5.3) in [8] and Lemma 5.1 (loc. cit.), $\tilde{w}_i$ has to increase in either direction, which eventually contradicts (A.1.24). Via a translation back to $\tilde{w}_i$ as defined in (A.1.19), it follows that, for each $i \gg 1$, $\tilde{w}_i(r)$ has only one critical point (around $r_i := e^{-t_i}$) for $r \in (0, \rho_i)$. This completes the checking that 0 is a simple blow-up point for \{v_i\}. \qed
§ A.2 Shifting to the maximal point.

From § A.1, in particular, (A.1.13) and (A.1.17), we can take $\xi_{m_i} = \xi_i$ in (2.1) and (2.2) in the definition of simple blow-up points. Suppose there exists another sequence of points $\{\tilde{\xi}_i\}$ which also satisfies (2.1) and (2.2). We show that (modulo a subsequence)

(A.2.1) \[ |\tilde{\xi}_i - \xi_i| = o(\lambda_i) \]

The proof, which requires only standard argument, can be readily recognized by people working on the area. For the benefit of general readers, we present the argument, and refer to available papers for selected technical details. Set

(A.2.2) \[ \tilde{\lambda}_i = \frac{1}{[v_i(\tilde{\xi}_i)]^{\frac{n-2}{2}}} \implies \tilde{\lambda}_i \geq \lambda_i \left( = \frac{1}{[v_i(\xi_i)]^{\frac{n-2}{2}}} \right). \]

To receive a contradiction, suppose

(A.2.3) \[ \tilde{\lambda}_i^{-1} \cdot |\tilde{\xi}_i - \xi_i| \to \infty \quad \text{(mod subsequence)}. \]

As in the proof of Lemma 3.10 in [15], both $\{\tilde{\xi}_i\}$ and $\{\xi_i\}$ can be used as in (2.17) to form (distinct) bubbling sequences (see [15]), contradicting that the blow-up is isolated (Proposition 3.32 in [15]). Hence there is a positive constant $B$ such that

(A.2.4) \[ \tilde{\lambda}_i^{-1} \cdot |\tilde{\xi}_i - \xi_i| \leq B \quad \text{for} \quad i \gg 1. \]

As in § 2d (see also the proof of Lemma 3.10 in [15]),

(A.2.5) \[ \tilde{V}_i(Y) := \frac{v_i(\tilde{\lambda}_i Y + \tilde{\xi}_i)}{v_i(\tilde{\xi}_i)} \to A_1(Y) \quad \text{in } C^1\text{-sense, uniformly for } Y \in B_0(R). \]

Once we take $R$ to be large enough, the point $Y_i$ defined by

(A.2.6) \[ \tilde{\lambda}_i Y_i + \tilde{\xi}_i = \xi_i \quad \text{satisfies } |Y_i| \leq B < R/2. \]

Hence $\tilde{V}_i$ has a critical point at $Y = Y_i$ for $i \gg 1$. In particular, $\nabla \tilde{V}_i(Y_i) = 0$ for $i \gg 1$. On the other hand,

(A.2.7) \[ \min \{ |\nabla A_1(Y)| : \delta \leq |Y| \leq R \} \geq c_{\delta, R}^2 > 0. \]

Here $c_{\delta, R}^2$ is a positive number depending on $\delta$ and $R$ ($c_{\delta, R}^2 \to 0$ as $\delta \to 0^+$). The convergence in (A.2.5) together with (A.2.6) and (A.2.7) shows that $|Y_i| \leq \delta$. Moreover, we can let $\delta \to 0$ as $i \to \infty$. Hence

(A.2.8) \[ |\tilde{\xi}_i - \xi_i| = o(\tilde{\lambda}_i). \]
Using again the convergence in (A.2.5) and (A.2.8), we have
\[ \lambda_i \leq \tilde{\lambda}_i \leq [1 + o(1)] \cdot \lambda_i \implies |\tilde{\xi}_i - \xi_i| = o(\lambda_i). \]

In the above, \( o(1) \to 0^+ \) as \( i \to \infty \).

§ A.3 Proof of Lemma 4.11.

The first conclusion in (i) follows directly from definition (4.10), and (ii) from the limitation \( j \leq k \). As for the second and third conclusions in (i), observe that when \( \ell \) is odd, \( \Delta^{(h \ell)}_o P_\ell \) is a degree one polynomial. If it is not equivalent to zero, then \( \Delta^{(h \ell)}_o P_\ell = \sum_j c_j Y_j \in F(P_\ell) \) as claimed.

Similarly, when \( \ell \) is even, \( \Delta^{(h \ell)}_o P_\ell = c \neq 0 \) is a number, then \( c \in F(P_\ell) \), and so does \( cR^2 \in F(P_\ell) \). Their difference is also in \( F(P_\ell) \).

As for (iii), we first observe that, via direct calculation, we have
\[ (A.3.1) \quad Y \cdot \nabla Q_l = l \cdot Q_l \quad \text{for any homogeneous polynomial with degree } l. \]

For \( j \geq 1 \), using the product formula
\[ \Delta_o (f \cdot g) = f \cdot (\Delta_o g) + 2 \left\langle \nabla f, \nabla g \right\rangle + g \cdot (\Delta_o f), \]
we obtain
\[ (A.3.2) \quad \Delta_o \left[(R^2)^j \Delta^k_o P_\ell \right] = (R^2)^j \Delta^{k+1}_o P_\ell + A_{\ell, j, k} \cdot (R^2)^{j-1} \Delta^k_o P_\ell, \]
\[ (A.3.3) \quad (R^2) \Delta_o \left[(R^2)^j \Delta^k_o P_\ell \right] = (R^2)^{j+1} \Delta^{k+1}_o P_\ell + A_{\ell, j, k} \cdot (R^2)^j \Delta^k_o P_\ell, \]
\[ (A.3.4) \quad A_{\ell, j, k} = (2j) \cdot (2j + n - 2 + 2 \ell - 4k). \]

As \( j \leq k \implies (j + 1) \leq (k + 1) \), the terms which appear on the right hand side above belong to \( F(P_\ell) \).
§ A.4 The case when $\Delta_o^{(h_i)} P_\ell \neq 0$.

For $\ell \leq n - 2$, where $n$ is even, we discuss how to eliminate the condition $\Delta_o^{(h_i)} P_\ell = 0$ by adding higher order (up to $n$-th order) terms. We start with

(A.4.1) \[(1 + R^2) \Delta_o (R^\ell) - 2n [\mathcal{Y} \cdot \nabla (R^\ell)] + 2n (R^\ell) \]

\[= \left( [1 + R^2] \Delta_o - 2n R \cdot \frac{\partial}{\partial R} + 2n \right) R^\ell = (\ell - 2) (\ell - n) R^\ell + \ell (\ell + n - 2) R^{\ell-2}. \]

In particular, when $\ell = 2$ or $n$, we have

(A.4.2) \[(1 + R^2) \Delta_o (R^n) - 2n [\mathcal{Y} \cdot \nabla (R^n)] + 2n (R^n) = 0 \cdot R^n + [2n (n - 1)] R^{n-2}, \]

(A.4.3) \[(1 + R^2) \Delta_o (R^2) - 2n [\mathcal{Y} \cdot \nabla (R^2)] + 2n (R^2) = 0 \cdot R^2 + [2n]. \]

Consider finding a radial function $F(r)$ so that

(A.4.4) \[(1 + R^2) \Delta_o F(r) - 2n [\mathcal{Y} \cdot \nabla F(r)] + 2n F(r) \]

\[= - [\Delta_o^{h_i} P_\ell] \cdot \{ a_{o} + a_{1} \cdot (R^2) + \cdots + a_{h_i-1} \cdot (R^2)^{h_i-1} + a_{h_i} \cdot (R^2)^{h_i} \}. \]

(i) \ We start with using a $(R^2)$ term to cancel the constant term. By (A.4.3) above, we won’t introduce any new $(R^2)$ term.

(ii) \ The $(R^2)$-term in the right hand side can be canceled by introducing an $(R^2)^2$ term [using (A.4.1), and $\ell (\ell + n - 2) \neq 0$]. By doing so, a new $(R^2)^2$-term is introduced to the right hand side.

(iii) \ The combined $(R^2)^2$-term in the right hand side can be canceled by introducing an $(R^2)^3$-term. By doing so, an $(R^2)^3$-term is introduced to the right hand side. The process goes on until we reach the $R^{n-2}$ term ($n \geq 4$ is even). Introducing a $R^n$-term cancels the $R^{n-2}$-term, and via (A.4.2), it does not re-introduce itself to the right hand side (that is, $R^n$ is not present).
Diagram A.4.5. The cancelation order from bottom upward (when $n$ is even).

In summary, when $n \geq 4$, $\ell$ even with $\ell \leq n - 2$, we can find a polynomial (A.4.6)

$$F(r) := \left[ \Delta_{\text{eq}}^{\text{P}} \right] \cdot \left\{ B_1 \cdot (R^2)^1 + B_2 \cdot (R^2)^2 + \cdots + B_k \cdot (R^2)^k + \cdots + B_n \cdot (R^2)^{\frac{n}{2}} \right\},$$

which satisfies (A.4.4), where

(A.4.7) $B_1 = -\frac{a_0}{2n}, \; B_2 = -\frac{a_1}{4(4 + n - 2)}, \; B_3 = -\frac{a_2 + (4 - 2)(4 - n) \cdot B_2}{6(6 + n - 2)},$

$$\cdots, \; B_k = -\frac{a_{k-1} + [2(k - 1) - 2] \cdot [2(k - 1) - n] \cdot B_{k-1}}{(2k)[(2k) + n - 2]} \quad \text{for} \quad 3 \leq k \leq \frac{n}{2}.$$  

In particular,

(A.4.8) $B_{\frac{n}{2}} = -\frac{a_{\frac{n}{2} - 1} + (n - 4)(-2) \cdot B_{\frac{n}{2} - 1}}{n \left[ 2(n - 1) \right]}.$
Proposition A.4.9. For \( n \geq 4 \) and \( \ell \leq n - 2 \), both being even, let \( \mathcal{P}_\ell \) be a homogeneous polynomial of degree \( \ell \). Then equation (4.3) has a solution given by

\[
\sum_{0 \leq j \leq k \leq h - 1} C^j_k \cdot (\mathcal{R}^2)^j [\Delta_{o}^{(k)} \mathcal{P}_\ell] + F(r)
\]

where \( F \) is given in (A.4.6), (A.4.7) and (A.4.8).

Although Proposition A.4.9 allows us to find a solution of equation (4.3) without the condition \( \Delta_{o}^{(h_\ell)} \mathcal{P}_\ell \equiv 0 \), the presence of an order \( n \) term (that is, \( \mathcal{R}^n \)) hinders the application of the second order blow-up argument, cf. §6b.6.

**Remark on uniqueness.** Let \( \Gamma_a \) and \( \Gamma_b \) be two polynomial solutions to equation (4.3), with maximum degrees < \( n \). Via Theorem 4.16, [mindful of condition (4.18) which requires the maximum degrees < \( n \)], we have

\[
(A.4.10) \quad [\Gamma_a - \Gamma_b](Y) = c_o (1 - \mathcal{R}^2) + \sum c_j Y_j \quad (\mathcal{R} = |Y|).
\]

Thus when \( \ell < n \), the solution found in Proposition 4.49 are “unique” in the case of (A.4.10). This cannot be extended to the solution found in Proposition A.4.9, even though \( \ell < n \), as the presence of \( \mathcal{R}^n \)-term voids the application of Theorem 4.16. (For a generic \( n \), \( B_4 \neq 0 \).)
§ A.5 Bounds on the Green function on \( B_o(a) \) with a point not too close to the boundary.

The Green’s function for \( \Delta_o \) on \( B_o(a) \) is given by
\[
G(y, \xi) = -\frac{1}{(n-2)\|S^{n-1}\|} \left[ \frac{1}{\|y - \xi\|^{n-2}} - \left( \frac{a}{\|\xi\|} \right)^{n-2} \frac{1}{\|y - \xi^*\|^{n-2}} \right],
\]
where \( \xi^* \) is the reflection of the point \( \xi \) upon the sphere \( \partial B_o(a) \), given by
\[
\xi^* = \frac{\frac{\|\xi\|}{\xi} \cdot \xi}{\|\xi\|} \quad \Rightarrow \quad \|\xi^*\| = \frac{\|\xi\|}{\gamma} > \|\xi\|, \quad \text{where} \quad \gamma \leq (1 - \delta).
\]

It follows that, for \( \|\xi\| \leq (1 - \delta) a \),
\[
\left( \frac{a}{\|\xi\|} \right)^{n-2} \frac{1}{\|y - \xi^*\|^{n-2}} \leq \left( \frac{1}{\gamma} \right)^{n-2} \frac{1}{\|a - \frac{\|\xi\|}{\gamma}\|^{n-2}} = \left( \frac{1}{1 - \gamma} \right)^{n-2} \frac{1}{a^{n-2}}
\]
\[
\leq \frac{1}{\delta^{n-2}} \cdot \frac{2^{n-2}}{\|y - \xi\|^{n-2}} \quad \text{(since} \quad |\xi - y| \leq 2a).\]

We obtain
\[
|G(y, \xi)| \leq \left[ C_1 + \frac{C_2}{\delta^{n-2}} \right] \frac{1}{\|y - \xi\|^{n-2}} \quad \text{for} \quad |\xi| \leq (1 - \delta) a \quad \text{and} \quad y \in B_o(a) \setminus \{\xi\}.
\]

As for the bound in (6.41), we note that the term
\[
n \cdot \nabla_y G_i(y, \xi)
\]
is indeed the Poisson kernel for \( \Delta_o \) on \( B_o(a) \), which is given by (e.g. [23] pp. 116)
\[
\frac{1}{a\|S^{n-1}\|} \cdot \frac{a^2 - |\xi|^2}{|y - \xi|^n} \quad \text{for} \quad y \in \partial B_o(a)
\]
\[
\leq \frac{a^2}{a\|S^{n-1}\| \cdot (\delta a)^n} \quad \text{for} \quad |\xi| \leq (1 - \delta) a \quad \text{and} \quad |y| = a
\]
\[
\leq \frac{C}{\delta^n} \cdot \frac{1}{a^{n-1}}.
\]
§ A.6 Balance and cancelation.

We first describe the classic balance formula for equation (1.2), due to Stanislav I. Pohozaev. See for examples [21] and [15].

**Theorem A.6.1.** Let $v$ and $K$ satisfy the general conditions in (1.27) and (1.28). We have the following.

(I) **Global Pohozaev’s identity.**

\[ \int_{\mathbb{R}^n} \langle y, \nabla K(y) \rangle [v(y)]^{\frac{2n}{n-2}} \, dy = 0. \]  

(II) **Mezzo-scale Pohozaev’s identity.** For a fixed number $\rho_0 > 0$, we have

\[ \int_{B_0(\rho_0)} \langle y, \nabla K(y) \rangle [v(y)]^{\frac{2n}{n-2}} \, dy = \frac{1}{C_n} \cdot \frac{2n}{n-2} \int_{\partial B_0(\rho_0)} \langle \tilde{V}, n \rangle \, dS, \]

\[ \tilde{V}(y) = \frac{n-2}{2} v(y) \nabla v(y) - \frac{\nabla v(y)}{2} y + \left[ \langle y, \nabla v(y) \rangle \right] \nabla v(y) + \frac{n-2}{2n} \cdot C_n \cdot \left\{ [v(y)]^{\frac{2n}{n-2}} K(y) \right\} y. \]

In (A.6.3) and (A.6.4), $y$ is treated as a vector, and $n$ is the unit outward normal on $\partial B_0(\rho_0)$.

**§ A.6.a. Order of vanishing outside the blow-up points.** Observe that, via (A.1.2) and gradient estimate [?], we obtain

\[ \max_{\partial B_0(\rho_0)} |\langle y, \nabla v_i(y) \rangle| \cdot v_i = O_{\lambda_i}(n-2), \quad \max_{\partial B_0(\rho_0)} |\nabla v_i|^2 = O_{\lambda_i}(n-2), \]

\[ \max_{\partial B_0(\rho_0)} |\nabla v_i| \cdot v_i = O_{\lambda_i}(n-2), \quad \max_{\partial B_0(\rho_0)} [v(y)]^{\frac{2n}{n-2}} |K(y)| \cdot |y| = O_{\lambda_i}(2n) \]

(A.6.5) \Rightarrow \int_{B_0(\rho_0)} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} \, dy = O_{\lambda_i}(n-2). \]

See also [15]. Whereas from (2.12) and (2.19), we have

\[ \left| \int_{\mathbb{R}^n \setminus \Omega} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} \, dy \right| = O_{\lambda_i}(n) \]

(A.6.6) \Rightarrow \int_{\Omega} \langle y, \nabla K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} \, dy = O_{\lambda_i}(n). \]
In the discussion,

\[(A.6.7) \quad \Omega = \bigcup_{j=0}^{k} B_{\hat{Y}_j}(\rho), \quad \left[ B_{\hat{Y}_j}(\rho) \cap B_{\hat{Y}_l}(\rho) = \emptyset \text{ for } j \neq l \right], \]

where (as usual) \( \{\hat{Y}_o, \cdots, \hat{Y}_k\} \) is the collection of all blow-up points.

\section*{A.6.b. Linking the Pohozaev integral to the condition \( \Delta_o^{(h_{\ell})} P_{\ell} \equiv 0 \).}

In the consideration of the integrals in the Pohozaev identities, we often encounter integral in the expression (A.6.9) below. We first record down the following observation.

\textbf{Lemma A.6.8.} For a homogeneous polynomial \( Q_{\ell} \) (defined on \( \mathbb{R}^n \)) of degree \( \ell \leq n - 1 \). If \( \ell \) is even, then the following equivalence holds.

\[(A.6.9) \quad \int_{\mathbb{R}^n} Q_{\ell}(y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = 0 \iff \Delta_o^{(h_{\ell})} Q_{\ell} = 0. \]

(Recall that \( h_{\ell} = \ell/2 \) when \( \ell \) is even.)

\textbf{Proof.} We observe that, as \( \ell \leq n - 1 \), the integral in (A.6.9) is absolutely convergent. Keeping the notation \( y = (y_1, \cdots, y_n) \in \mathbb{R}^n \), consider a typical term in \( Q_{\ell} : \)

\[(A.6.10) \quad y_{\alpha_1}^{\alpha_1} y_{\alpha_2}^{\alpha_2} \cdots y_{\alpha_n}^{\alpha_n}, \quad \text{where } \alpha_j \geq 0 \text{ and } \sum_{j=1}^{n} \alpha_j = \ell \leq n - 1. \]

If one of the indices (say, \( \alpha_j \)) is an odd natural number, via symmetry, we have

\[(A.6.11) \quad \int_{\mathbb{R}^n} \left[ y_{\alpha_1}^{\alpha_1} \cdot y_{\alpha_2}^{\alpha_2} \cdots y_{\alpha_j}^{\alpha_j} \cdots y_{\alpha_n}^{\alpha_n} \right] \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = 0 \quad (\alpha_j \text{ is odd}). \]

Direct calculation also shows that in this situation

\[\Delta_o^{(h_{\ell})} \left[ y_{\alpha_1}^{\alpha_1} \cdot y_{\alpha_2}^{\alpha_2} \cdots y_{\alpha_j}^{\alpha_j} \cdots y_{\alpha_n}^{\alpha_n} \right] = 0 \quad (\alpha_j \text{ is odd}). \]

(Recall that \( \Delta_o^{(h_{\ell})} Q_{\ell} \) is a number when \( \ell \) is even.) Thus we are left with the case where any one index in (A.6.10) is an even natural number or zero. Let us introduce the following notion: a multi-index

\[\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \quad \left( |\alpha| = \sum_{j=1}^{n} \alpha_j = \ell > 0 \right)\]
is even if each $\alpha_j$ ($1 \leq j \leq n$) is either an even natural number or zero. With respect to this, the simplest case that can happen to the integral in (A.6.11) is

$$J := \int_{\mathbb{R}^n} y_{1|k}^{2} \cdots y_{|n|}^{2} \left( \frac{1}{1 + r^2} \right)^n \, dy.$$  

We seek to reduce other even multi-index cases to that in (A.6.12). As $\ell \leq n - 1$, we arrange in this way

$$y_{1|k}^{k+2} \cdot y_{|2}^{\alpha_2} \cdots y_{|n|}^{\alpha_n - 1},$$

where $\alpha = (k + 2, \alpha_2, \cdots, \alpha_{n-1}, 0)$ is even.

Here $k \geq 2$ is an even number. Via symmetry, the ordering is not important when we compute the integral in (A.6.11). One obtains the following reduction formula.

$$\int_{\mathbb{R}^n} y_{1|k}^{k+2} \cdot \left[ y_{|2}^{\alpha_2} \cdots y_{|n|}^{\alpha_n - 1} \right] \left( \frac{1}{1 + |y|^2} \right)^n \, dy = (k + 1) \int_{\mathbb{R}^n} y_{1|k}^{k} \cdot \left[ y_{|2}^{\alpha_2} \cdots y_{|n|}^{\alpha_n - 1} \right] \left( \frac{1}{1 + |y|^2} \right)^n \, dy \quad \text{for } 2 \leq k \leq n - 3,$$

by using Fubini’s theorem and integration by parts formula. See §A.8 below.

In view of (A.6.11) and (A.6.13), we introduce the following notation. For an integer $m \geq 0$, define

$$m!_{-2} = 1 \quad \text{if } m = 0 \text{ or } 2; \quad m!_{-2} = 0 \quad \text{if } m \text{ is odd};$$

$$m!_{-2} = (m - 1)(m - 3)(m - 5) \cdots 3 \cdot 1 \quad \text{if } m \geq 4 \text{ is even}.$$

Via the vanishing formula (A.6.11) and the reduction formula (A.6.13), we have

$$\int_{\mathbb{R}^n} \left[ y_{1}^{\alpha_1} \cdots y_{n}^{\alpha_n} \right] \left( \frac{1}{1 + |y|^2} \right)^n \, dy = (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot J.$$  

On the other side, calculation shows that

$$B := \Delta^{(h_{\ell})}_o \left[ y_{1}^{2} \cdot y_{2}^{2} \cdots y_{|n|}^{2} \right] = \ell (\ell - 2) (\ell - 4) \cdots 2 \cdot 1.$$  

Claim. Let $\alpha_2, \cdots, \alpha_{n-1}$ be even natural numbers or zero, and

$$\ell = (k + 2) + \alpha_2 + \cdots + \alpha_{n-1}, \quad \text{where } k \geq 2 \text{ is an even integer}.$$  

Then

$$\Delta^{(h_{\ell})}_o \left\{ y_{1}^{k+2} \cdot \left[ y_{|2}^{\alpha_2} \cdots y_{|n|}^{\alpha_n - 1} \right] \right\} = (k + 1) \cdot \Delta^{(h_{\ell})}_o \left\{ y_{1}^{k} \cdot y_{|2}^{2} \cdots y_{|n|}^{\alpha_n - 1} \right\}.$$  

56
Refer to (A.6.17), set
\[(A.6.19) \quad \ell := \alpha_2 + \cdots + \alpha_{n-1} .\]

We demonstrate how to use induction on \(\ell\) to prove the assertion in §A.8 in this e-Appendix. Thus using (A.6.18) repeatedly, we are led to
\[(A.6.20) \quad \Delta^{(h_\ell)}_o \left[ y_{|1}^{\alpha_1} \cdots y_{|n}^{\alpha_n} \right] = (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot B .\]

Using the linearity of the operations and symmetry, (A.6.15) and (A.6.20) yield
\[(A.6.21) \quad \int_{\mathbb{R}^n} Q_\ell (y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = \frac{J}{B} \cdot [ \Delta^{(h_\ell)} Q_\ell ] .\]

In particular, we establish (A.6.9).

\[\Box\]

§ A.6. c. Change of center. With (6.57), let us write
\[(A.6.22) \quad v_i (y) = A (y) + B (y) + C (y) \quad \text{for} \quad |y| \leq \rho_1 ,\]

where
\[(A.6.23) \quad A (y) = \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^{n-2} ,\]
\[(A.6.24) \quad B (y) = \lambda_i \cdot \left[ \lambda_i^\ell \cdot \Gamma_p \left( \frac{y - \xi_i}{\lambda_i} \right) \right] \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right) ,\]
\[(A.6.25) \quad C (y) = [v_i - A - B] (y) = o_{\lambda_i} \left( \ell - \frac{n-2}{2} \right) + O_{\lambda_i} \left( \frac{n-2}{2} \right) .\]

For simplicity, we suppress the subindex \(i\) in \(A, B\) and \(C\). Because of the polynomial nature of \(\Gamma_p\), in general \(B\) is not rotationally symmetric, contrasting to \(A\). As this part of the discussion is used repeatedly in this article, we consider in general a homogeneous polynomial \(Q_\ell\) defined on \(\mathbb{R}^n\) with degree \(\ell \in [2, n-2]\). For \(\rho > 0\), consider the integral
\[
\int_{B_\rho (\rho)} Q_\ell (y) \cdot [A (y)]^{\frac{2n}{n-2}} dy .
\]

Here \(A\) is given in \((A.6.23)\). We first observe that
\[(A.6.26) \quad \left| \int_{B_\rho (\rho_2)} f (y) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n dy - \int_{B_{\xi_i} (\rho_2)} f (y) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y|^2} \right)^n dy \right| \leq \int_{B_\rho (\rho_2) \setminus B_{\rho_2} (\rho_2 - |\xi_i|)} |f (y)| \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n dy = O (|\xi_i| \cdot \lambda_i^n) .\]

57
for $i \gg 1$. Here $f$ is a bounded continuous function defined on a slightly bigger ball, and $\xi_i$ continues to find its meaning in (1.13). In particular, $|\xi_i| \to 0$. It follows that

$$
(\text{A.6.27}) \quad \int_{B_\rho(\rho)} Q_\ell(y) \cdot [A(y)]^{2n \over n+2} dy = \int_{B_{\xi_i}(\rho)} Q_\ell(y) \cdot [A(y)]^{2n \over n+2} dy + o_\rho(n).
$$

Let us arrange

$$
(\text{A.6.28}) \quad Q_\ell(y) = Q_\ell(\xi_i + |y - \xi_i|) = Q_\ell(\xi_i) + M^Q(y - \xi_i) + Q_\ell(y - \xi_i).
$$

Here the “intermediate” term $M^Q(y - \xi_i)$ can be further broken down into $\ell - 1$ terms based on the degree on $\xi_i$:

$$
(\text{A.6.29}) \quad M^Q(\xi_i; y - \xi_i)
$$

$$
= O(|\xi_i| \cdot |y - \xi_i|^{\ell-1}) + O(|\xi_i|^2 |y - \xi_i|^{\ell-2}) + \cdots + O(|\xi_i|^\ell \cdot |y - \xi_i|)
$$

$$
= \Xi_1^Q(\xi_i; y - \xi_i) + \Xi_2^Q(\xi_i, y - \xi_i) + \cdots + \Xi_{\ell-1}^Q(\xi_i, y - \xi_i),
$$

$$
(\text{A.6.30}) \quad \text{where (formally) } \Xi_h^Q(\xi_i, z) = \sum_{|\alpha| = h} {1 \over \alpha!} \cdot \xi_i^\alpha D_\alpha^{(h)} Q_\ell(z) \bigg|_{z = (y - \xi_i)}
$$

for $1 \leq h \leq \ell - 1$. We continue with

$$
(\text{A.6.31}) \quad \int_{B_{\xi_i}(\rho)} Q_\ell(y) \cdot [A(y)]^{2n \over n+2} dy = \int_{B_{\xi_i}(\rho)} Q_\ell(y) \left( {\lambda_i \over \lambda_i^2 + |y - \xi_i|^2} \right)^n dy
$$

$$
= \int_{B_{\xi_i}(\rho)} Q_\ell(y - \xi_i) \left( {\lambda_i \over \lambda_i^2 + |y - \xi_i|^2} \right)^n dy
$$

$$
+ \int_{B_{\xi_i}(\rho)} M^Q(\xi_i; y - \xi_i) \left( {\lambda_i \over \lambda_i^2 + |y - \xi_i|^2} \right)^n dy + \int_{B_{\xi_i}(\rho)} Q_\ell(\xi_i) \left( {\lambda_i \over \lambda_i^2 + |y - \xi_i|^2} \right)^n dy
$$

$$
= \int_{B_{\rho}(\rho)} Q_\ell(z) \left( {\lambda_i \over \lambda_i^2 + |z|^2} \right)^n dz + \int_{B_{\rho}(\rho)} M^Q(\xi_i; z) \left( {\lambda_i \over \lambda_i^2 + |z|^2} \right)^n dz +
$$

$$
+ Q_\ell(\xi_i) \int_{B_{\rho}(\rho)} \left( {\lambda_i \over \lambda_i^2 + |z|^2} \right)^n dz
$$

$$
= \lambda_i^{\ell} \int_{B_{\rho}(\lambda_i^{-1} \cdot \rho)} Q_\ell(y) \left( {1 \over 1 + |y|^2} \right)^n dy +
$$

58
+ \lambda^\ell_i \left[ \sum_{h=1}^{\ell - 1} \int_{B_{o}(\lambda^{-1}_i, \rho)} \Xi^Q_h \left( \frac{\xi_i}{\lambda_i}, y \right) \left( \frac{1}{1 + |y|^2} \right)^n dy \right] + \\
+ \lambda^\ell_i \cdot \left[ Q^\ell \left( \frac{\xi_i}{\lambda_i} \right) \right] \cdot \int_{B_{o}(\lambda^{-1}_i, \rho)} \left( \frac{1}{1 + |y|^2} \right)^n dy \quad (z \to \lambda_i \cdot y).

§ A.6. c.1 Expressing the integrals on \( \mathbb{R}^n \). Observe also that the above argument remains valid if we replace \( \rho \) by \( \infty \). Indeed, consider (in general) a homogeneous polynomial \( Q_k \) defined on \( \mathbb{R}^n \) with degree \( k \in [0, n - 2] \). For a sequence of positive numbers \( r_i \to \infty \):

\[ \left| \int_{\mathbb{R}^n \setminus B_{o}(r_i)} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n dy \right| \leq C \int_{r_i}^{\infty} \frac{r^k}{r^{2n}} \cdot r^{n-1} dr \leq C \int_{r_i}^{\infty} \frac{1}{r^3} dr \leq \frac{C_2}{r_i^2} \]  

\[ (A.6.32) \]

\[ \cdots \Rightarrow \int_{B_{o}(r_i)} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n dy = \int_{\mathbb{R}^n} Q_k(y) \left( \frac{1}{1 + |y|^2} \right)^n dy + O(r_i^{-2}), \]

for \( i \gg 1 \) and \( 0 \leq k \leq n - 2 \). Putting \( r_i = \lambda_i^{-1} \cdot \rho \), and combining (A.6.27), (A.6.31) and (A.6.32) with \( |\xi_i| \to 0 \), we obtain

\[ (A.6.33) \quad \int_{B_{o}(\rho)} Q^\ell \cdot [A(y)]^{\frac{2n}{n-2}} dy = \lambda^\ell_i \int_{\mathbb{R}^n} Q^\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n dy + \\
+ \lambda^\ell_i \left\{ \sum_{h=1}^{\ell - 1} \int_{\mathbb{R}^n} \left[ \Xi^Q_h \left( \frac{\xi_i}{\lambda_i}, y \right) \right] \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy \right\} + \\
+ \lambda^\ell_i \cdot \left[ Q^\ell \left( \frac{\xi_i}{\lambda_i} \right) \right] \cdot \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^n dy \quad + O_{\lambda_i}(\ell + 2) + o_{\lambda_i}(n). \]

§ A.6.d. \( \Delta^h_{o} \mathbf{P}_\ell \equiv 0 \) when \( \ell \) is even and \( \ell < n - 2 \), or when \( \ell = n - 2 \)  

(\( n \) being even) and with only one simple blow-up point.

Proposition A.6.34. For \( n \geq 4 \), under the general conditions (1.6), (1.25), (1.26), assume that \( \{u_i\} \) has finite number of blow-up points, one at the south pole, but none at the north pole. Take the following conditions (i) – (iii) into account.

(i) \( 0 \) is a simple blow-up point for \( \{v_i\} \).

(ii) \( K \) is given by (1.8) in \( B_{o}(\rho_o) \), where \( 2 \leq \ell < n - 2 \).
(iii) The parameters $\lambda_i$ and $\xi_i$ corresponding to the simple blow-up point at 0 [via (1.10) and (1.11)] satisfy (1.12), that is, $|\xi_i| = o(\lambda_i)$.

(iv) $\ell$ is even.

Then $\Delta_h^{(0)} P_\ell(y) = 0$ ($\Delta_0^{(h)} P_\ell$ is a number when $\ell$ is even). The same conclusion also holds when $\ell = n - 2$ with an additional assumption that 0 is the only blow-up point ($\ell$ is still required to be even).

**Proof.** The key is to combine the change of center formula (A.6.33) with the condition $|\xi_i| = o(\lambda_i)$, and observe the lower order terms. Other arguments actually proceed in similar fashion as those found in [15] and [21]. For the benefit of readers, we present the estimates in detail. From (A.6.5) and (A.6.6), we have

$$\int_{B_\rho(\rho_o)} \langle y, \nabla K(y) \rangle \left[ v_i(y) \right]^{2n} d\gamma = \begin{cases} O_{\lambda_i}(n - 2), & \text{in general;} \\ O_{\lambda_i}(n), & \text{one blow-up point.} \end{cases}$$

Throughout this proof we assume that the positive constant $\rho_o > 0$ is chosen to be small enough. Let us pay attention to (2.21) and the number $R_i$ satisfying (2.20), together with the remark in §2f (on shifting the center, see also §A.2). Note that

$$\int_{B_\rho(\rho_o)} \langle y, \nabla K(y) \rangle \left[ v_i(y) \right]^{2n} d\gamma = \int_{B_{\lambda_i R_i}} r \frac{\partial K}{\partial r} \left[ A(y) \right]^{2n} d\gamma \cdots (I)$$

$$+ \int_{B_{\lambda_i R_i}} r \frac{\partial K}{\partial r} \left\{ \left[ v_i(y) \right]^{2n} - \left[ A(y) \right]^{2n} \right\} d\gamma \cdots (II)$$

$$+ \int_{B_{\lambda_i R_i} \setminus B_{\lambda_i R_i}} r \frac{\partial K}{\partial r} \left[ v_i(y) \right]^{2n} d\gamma \cdots (III)$$

(i) We begin with the core term (I). From (1.8), we have

$$r \cdot \frac{\partial [\tilde{c}_n K]}{\partial r} = \langle y, \nabla [\tilde{c}_n K] \rangle = \ell \times [\mathbf{P}_\ell(y)] + O(\|y\|^{\ell+1})$$

for $y \in B_\rho(\rho_o)$. It follows that [recall that $A$ is given in (A.6.23)]

$$\int_{B_{\lambda_i R_i}} r \frac{\partial K}{\partial r} \left[ A(y) \right]^{2n} d\gamma = -\frac{\ell}{c_n} \cdot \int_{B_{\lambda_i R_i}} \mathbf{P}_\ell \cdot \left[ A(y) \right]^{2n} d\gamma +$$

$$+ \int_{B_{\lambda_i R_i}} O(\|y\|^{\ell+1}) \cdot \left[ A(y) \right]^{2n} d\gamma.$$
\[
- \frac{\ell}{c_n} \cdot \int_{B_o(\lambda_i R_i)} P_\ell \cdot [A(y)] \frac{2n}{n-2} dy \quad + \quad O(\lambda_i (\ell + 1)) \quad \text{for} \quad \ell \leq n - 2
\]
[see (A.6.41); this only requires \(|\xi_i| = O(\lambda_i)\)]

\[
= - \lambda_i^\ell \cdot \frac{\ell}{c_n} \cdot \int_{\mathbb{R}^n} P_\ell(y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy \quad + \quad o(\lambda_i (\ell))
\]

[estimating as in (A.6.31)–(A.6.33), \(\ell\) using \(|\xi_i| = o(1)\) and replacing \(\rho_o\) by \(\lambda_i \cdot R_i\) in (A.6.31), and take \(r_i = R_i\) in (A.6.32).]

Here \(2 \leq \ell \leq n - 2\).

(ii) Now we turn to the term marked (II). Via inequality (3.14) and (2.26) in Proposition 2.24, we have

\[
(A.6.39) \quad \left| \int_{B_{\ell_i}(\lambda_i R_i)} \langle y, \nabla K(y) \rangle \left\{ [v_i(y)] \frac{2n}{n-2} - [A(y)] \frac{2n}{n-2} \right\} dy \right|
\]

\[
\leq C \cdot \frac{\varepsilon}{\lambda_i^{n-2}} \cdot \frac{1}{\lambda_i^{n-2}} \cdot \int_{B_{\ell_i}(\lambda_i R_i)} r^\ell dy \quad \quad (r = |y|)
\]

\[
\leq C \cdot \frac{\varepsilon}{\lambda_i^n} \cdot \int_{B_o(\lambda_i (R_i + 1))} r^\ell dy \quad \quad (\lambda_i \cdot |\xi_i| \to 0)
\]

\[
\leq C_1 \cdot \varepsilon \cdot [\lambda_i (R_i + 1)]^{\ell + n} = C_1 \cdot \lambda_i^\ell \cdot [\varepsilon (R_i + 1)^{\ell + n}] = o(\lambda_i (\ell)).
\]

In the above we apply

\[
\varepsilon_i \cdot R_i^{2(n-1)} = o(1) \quad \quad \text{[see (2.20)].}
\]

(iii) As for the term marked (III), using the binomial expansion on \((|z| + |\xi_i|)^\ell\), together with (2.5) or (2.26) (similar to Lemma 2.4 in [21]), we have

\[
(A.6.40) \quad \left| \int_{B_o(\rho_o) \setminus B_o(\lambda_i R_i)} \langle y, \nabla K(y) \rangle [v_i(y)] \frac{2n}{n-2} dy \right|
\]

\[
\leq C_1 \cdot \int_{B_o(\rho_o) \setminus B_o(\lambda_i R_i)} |y|^\ell \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n dy
\]

\[
\leq C_2 \cdot \int_{B_{\ell_i}(\rho_o) \setminus B_{\ell_i}(\lambda_i (R_i - c))} |y|^\ell \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n dy
\]

61
here $\rho'_o$ is slightly bigger than $\rho_o$,  

\[ c \text{ is a big enough constant (require } |\xi_i| = O(\lambda_i)) \]

\[ \leq C_3 \int_{B_o(\rho'_o) \backslash B_o(\lambda_i (R_i - c))} \left( \sum_{j=0}^{\ell} |z|^{\ell-j} \cdot |\xi_i|^j \right) \cdot \left( \frac{\lambda_i}{\lambda_i^2 + |z|^2} \right)^n dz \quad (y = z + \xi) \]

\[ \leq C_4 \lambda_i^\ell \int_{\sum_{j=0}^{\ell} \int_{R_i - c}^{R_i} \left( \frac{1}{1 + r^2} \right)^n r^{\ell-j + (n-1)} dr } \text{ (polar coordinates and } r \to \lambda_i \cdot r) \]

\[ = o(\lambda_i^\ell) \quad \text{for } 2 \leq \ell \leq n - 2 \]

(needs only $|\xi| = O(\lambda_i); \quad R_i \to \infty \text{ and } \lambda_i \cdot R_i \to 0 \implies \lambda_i^{-1} \cdot \rho_o > R_i$).

Similarly,

\[(A.6.41) \int_{B_o(\lambda_i R_i)} |y|^{\ell+1} \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n dy \leq C_1 \int_{B_o(\lambda_i (R_i + c))} |y|^{\ell+1} \left( \frac{\lambda_i}{\lambda_i^2 + |y - \xi_i|^2} \right)^n dy \]

\[ \leq C_2 \sum_{j=0}^{\ell+1} \left( \sum_{j=0}^{\ell+1} |z|^{\ell+1-j} \cdot |\xi_i|^j \right) \text{ (polar coordinates and } r \to \lambda_i \cdot r) \]

which requires only $|\xi_i| = O(\lambda_i)$. Here (as before) $y = z + \xi$, and we apply the change of variables $z \to \lambda_i \cdot z$. Using (A.6.35), (A.6.36), (A.6.38), (A.6.39) and (A.6.40) we obtain

\[ \int_{B_o(\rho_o)} \langle y, \nabla K(y) \rangle \cdot [v_i(y)]^{2n} dy = \lambda_i^\ell \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n dy + o_{\lambda_i}(\ell) \]

\[(A.6.42) \quad \cdots \cdots \implies \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n dy = 0.\]

Here $2 \leq \ell \leq n - 2$. (A.6.42) together with Lemma A.6.8 ($\ell$ is even) imply that $\Delta_0^{(b_i)} P_\ell(y) \equiv 0$. Let us end the proof with the remark that the condition $|\xi_i| = o(\lambda_i)$ is only ‘fully’ used in the last step in (A.6.38).

We highlight that the smaller order term in (A.6.41) depends on convergence parameters $\varepsilon_i$ and $R_i$, as well as the condition $|\xi_i| = o(\lambda_i)$. In the next section, we apply the refined estimate (A.6.22) to discern out the layers of information hidden in $o_{\lambda_i}(\ell)$.
§ A.6.e. Isolating the key term with lowest order in $\lambda_i$. Suppose that estimate (6.57) holds for \{v_i\} inside $B_o(\rho_2)$, where $\rho_2 > 0$ is a constant. From (A.6.22)–(A.6.25), we obtain

$$\int_{B_o(\rho_2)} r \frac{\partial K}{\partial r} \cdot [v_i (y)]^{\frac{2n}{n-2}} \, dy = \int_{B_o(\rho_2)} r \frac{\partial K}{\partial r} [A (y)]^{\frac{2n}{n-2}} \, dy +$$

$$+ \int_{B_o(\rho_2)} r \frac{\partial K}{\partial r} \left( [(A + B + C) (y)]^{\frac{2n}{n-2}} - [A (y)]^{\frac{2n}{n-2}} \right) \, dy.$$

In order to estimate the last term in the above, we make use of the inequality

$$(A.6.44) \quad \int_{B_o(\rho_2)} \left| r \frac{\partial K}{\partial r} \right| \left| (A + B + C) (y) \right|^{\frac{2n}{n-2}} - \left| A (y) \right|^{\frac{2n}{n-2}} \, dy \leq \varepsilon \int_{B_o(\rho_2)} \left| r \frac{\partial K}{\partial r} \right| A^{\frac{2n}{n-2}} \, dy + \frac{C_n}{\varepsilon^\frac{n-2}{2}} \int_{B_o(\rho_2)} \left| r \frac{\partial K}{\partial r} \right| \left( |B|^{\frac{2n}{n-2}} + |C|^{\frac{2n}{n-2}} \right) \, dy.$$

Here $\varepsilon > 0$ is a given (small) number, and the dimensional constant $C_n$ is independent on $\varepsilon$. We demonstrate the argument toward (A.6.44) in § A.3 in the Appendix.

Remark A.6.45. Suppose that we seek to find $\varepsilon > 0$ so that

$$\varepsilon \cdot \lambda_i^\ell + \frac{C_n}{\varepsilon^{\frac{n-2}{2}}} \cdot \lambda_i^{\ell + a} = \varepsilon \cdot \lambda_i^\ell + \frac{C_n}{\varepsilon^\frac{n-2}{2}} \cdot \lambda_i^{a - t} \cdot \lambda_i^{\ell + t}.$$ 

That is, we want to re-distribute some order of $\lambda_i$ to the first term so that in the end the two terms have the same order $O_{\lambda_i} (\ell + t)$:

$$\varepsilon = \lambda_i^\ell \implies \frac{C_n}{(\lambda_i)^{\frac{2n}{n-2}}} = \lambda_i^{a - t} \implies \frac{2n}{n-2} \cdot t = (a - t) \implies t = \frac{n - 2}{3n - 2} \cdot a.$$

§ A.6.f Estimate on the leading order term. Recall (1.8) and (4.12).

$$\int_{B_o(\rho_2)} r \frac{\partial K}{\partial r} \cdot [A (y)]^{\frac{2n}{n-2}} \, dy = -\ell \cdot \int_{B_o(\rho_2)} P_\ell \cdot [A (y)]^{\frac{2n}{n-2}} \, dy +$$

$$+ \int_{B_o(\rho_2)} O \left( |y|^{\ell + 1} \right) \cdot [A (y)]^{\frac{2n}{n-2}} \, dy.$$

The first term in the right hand side of the last equation in (A.6.46) can be expanded by using (A.6.33). As for the second term, it can be estimated as in (A.6.41) (replac-
ing \(\lambda_i R_i\) by \(\rho_2\) showing that the term is of order \(O_{\lambda_i}(\ell + 1)\). Hence we obtain the following.

**Lemma A.6.47.** Under the conditions in (2.63), (A.6.23), \(\ell \in [2, n-2]\), suppose that, for \(i \gg 1\), \(\xi_i = \lambda_i^{1+\eta_0} \cdot \bar{X}\), where \(\bar{X} \in \mathbb{R}^n\) is a fixed vector. Then we have

\[
- \int_{B_\rho(\rho_2)} r \frac{\partial K}{\partial r} A^{\frac{2n}{n-2}} dy = \lambda_i^{\ell} \cdot \frac{\ell}{c_n} \int_{\mathbb{R}^n} P_\ell(y) \left(\frac{1}{1 + |y|^2}\right)^n dy + \\
+ \int_{\mathbb{R}^n} \left[ \lambda_i^{\ell+\eta_0} \cdot \Xi_i^P(\bar{X}, y) + \cdots + \lambda_i^{\ell+(\ell-1)\cdot\eta_0} \cdot \Xi_{\ell-1}^P(\bar{X}, y) \right] \left(\frac{1}{1 + |y|^2}\right)^n dy + \\
+ \lambda_i^{\ell+\cdot\eta_0} \cdot P_\ell(\bar{X}) \int_{\mathbb{R}^n} \left(\frac{1}{1 + |y|^2}\right)^n dy + O_{\lambda_i}(\ell + 1).
\]

Here \(\Xi_h^P\) is defined as in (A.6.29) and (A.6.30) by replacing \(Q_{\ell}\) by \(P_{\ell}\).

**§ A.6. g. Estimate on the term involving \(B\).** We first shift the center in the term

\[
\int_{B_\rho(\rho_2)} r \frac{\partial K}{\partial r} |B(y)|^{\frac{2n}{n-2}} dy \leq C \int_{B_\rho(\rho_2)} |y|^{\ell} \cdot |B(y)|^{\frac{2n}{n-2}} dy \\
\leq C \int_{B_\xi(\rho_2)} |y - \xi_i|^\ell + C_1 |\xi_i| \cdot |y - \xi_i|^{\ell-1} + \cdots + |\xi_i|^{\ell} \cdot |B(y)|^{\frac{2n}{n-2}} dy \\
+ o_{\lambda_i} \left(\frac{n^2}{n-2} + 1\right)
\]

[recall that \(|\xi_i| = o(\lambda_i)\)].

In the above, we apply the triangle inequality and the binomial expansion as in

\(|y|^{\ell} \leq (|y - \xi_i| + |\xi_i|)^\ell \quad (\ell \text{ a positive integer}).\]

Introduce the change of variables

\[
(A.6.50) \quad |y - \xi_i| = \rho = \lambda_i \tan \theta \quad \Rightarrow \quad \tan \theta = \frac{|y - \xi_i|}{\lambda_i} \quad \Rightarrow \quad |\gamma| = \tan \theta.
\]
Recall that in the expression (A.6.24) for $B$,

$$\left| B(y) \right|^{\frac{2n}{n-2}} = \lambda_i^{(\ell+1)-\frac{2n}{n-2}} |\Gamma_{\leq \ell} (\mathcal{Y})|^{\frac{2n}{n-2}} \left[ \frac{1}{\lambda_i (1 + \tan^2 \theta)} \right]^\frac{n^2}{n-2} \left( \mathcal{Y} = \frac{y - \xi_i}{\lambda_i} \right).$$

Moreover,

(A.6.51)

$$|\Gamma_{\leq \ell} (\mathcal{Y})| \leq C \left[ \mathcal{R}^2 + \cdots + \mathcal{R}^\ell \right] \implies \begin{cases} |\Gamma_{\leq \ell} (\mathcal{Y})| \leq C_1 \text{ for } \mathcal{R} = |\mathcal{Y}| \leq 1; \\ |\Gamma_{\leq \ell} (\mathcal{Y})| \leq C_2 \mathcal{R}^\ell \text{ for } |\mathcal{Y}| \geq 1. \end{cases}$$

For $0 \leq l \leq \ell$, we have

(A.6.52) \[ \int_{B_{\xi_i}(\rho_2)} \left| y - \xi_i \right|^l \cdot |B(y)|^{\frac{2n}{n-2}} \, dy \]

$$= \left( \int_{B_{\xi_i}(\lambda_i)} + \int_{B_{\xi_i}(\rho_2) \setminus B_{\xi_i}(\lambda_i)} \right) \left| y - \xi_i \right|^l \cdot |B(y)|^{\frac{2n}{n-2}} \, dy$$

$$\leq C_3 \int_0^\lambda r^l \cdot \left[ \lambda^{\ell+1 - \frac{n}{2}} \right]^{\frac{2n}{n-2}} \left[ r^{-n-1} \, d r \right]$$

[where $r = |y - \xi_i|$; using first half in (A.6.40)]

$$+ C_2 \int_{\arctan 1}^{\arctan \left( \frac{\rho_2}{\lambda_i} \right)} \left[ \lambda_i \tan \theta \right]^l \cdot \lambda_i^{(\ell+1)-\frac{2n}{n-2}} (\tan \theta)^{\frac{2n\ell}{n-2}} \left[ \frac{1}{\lambda_i (1 + \tan^2 \theta)} \right]^\frac{n^2}{n-2} \times$$

$$\times \left\{ \lambda_i^n [\tan \theta]^{n-1} \sec^2 \theta \right\} \, d \theta$$

$$\leq O_{\lambda_i} \left( l + \ell + \frac{2n}{n-2} \right) +$$

$$+ O_{\lambda_i} \left( l + [\ell + 1] \cdot \frac{2n}{n-2} + n - \frac{n^2}{n-2} \right) \int_0^{\arctan \left( \frac{\rho_2}{\lambda_i} \right)} \frac{\left[ \cos \theta \right]^{\frac{2n^2}{n-2}}}{\left[ \cos \theta \right]^{l+\frac{2n^2}{n-2} + (n-1)+2}} \, d \theta$$

$$\leq O_{\lambda_i} \left( l + \frac{2n \ell}{n-2} \right) + O_{\lambda_i} \left( l + \frac{2n \ell}{n-2} \right) \int_0^{\arctan \left( \frac{\rho_2}{\lambda_i} \right)} \frac{\left[ \cos \theta \right]^{\frac{2n^2}{n-2}}}{\left[ \cos \theta \right]^{l+(\frac{2n^2}{n-2} - 4)+4+(n+1)}} \, d \theta$$

$$\leq O_{\lambda_i} \left( l + \frac{2n \ell}{n-2} \right) + O_{\lambda_i} (l + 4) \int_0^{\arctan \left( \frac{\rho_2}{\lambda_i} \right)} \frac{\left[ \cos \theta \right]^{\frac{2n^2}{n-2}}}{\left[ \cos \theta \right]^{l+n+5}} \, d \theta \quad \text{[see § A.6 g.2]}$$
\[ = o_{\lambda_i} (l + 4) + o_{\lambda_i} (l + 4) \quad \text{(as } l \geq 2) \]

\[ \text{note that } l + n + 5 \leq (n - 2) + n + 6 = 2n + 3 \leq \frac{2n^2}{n - 2} . \]

Using (A.6.49), (A.6.52) and \(|\xi_i| = O(\lambda_i)\), we obtain the following.

**Lemma A.6.53.** Let \(B\) be given as in (A.6.24), \(|\xi_i| = O(\lambda_i)\), and \(K\) satisfies (1.8). Then for \(2 \leq \ell \leq n - 2\), we have

\[ (A.6.54) \quad \int_{B_{o}(p_2)} \left| r \frac{\partial K}{\partial r} \right| \cdot \left| B(y) \right|^{\frac{2n}{n - 2}} dy = o_{\lambda_i} (\ell + 4) . \]

§ A.6. h. *Estimate on the term involving \(C\).* As

\[ (A.6.55) \quad \ell \leq n - 2 \implies \ell + 1 - \frac{n}{2} \leq \frac{n - 2}{2} . \]

From (A.6.25) and \(|\langle y, \nabla K\rangle| \leq c\) in \(B_{o}(p_0)\), we obtain

\[ (A.6.56) \quad \int_{B_{o}(p_2)} \left| r \frac{\partial K}{\partial r} \right| \cdot \left| C(y) \right|^{\frac{2n}{n - 2}} dy = O_{\lambda_i} \left( \left[ \ell - \frac{n - 2}{2} \right] \cdot \frac{2n}{n - 2} \right) . \]

Observe that

\[ \left( \ell + 1 - \frac{n}{2} \right) \cdot \frac{2n}{n - 2} > \ell \]

\[ \iff \ell > \frac{n(n - 2)}{n + 2} \iff \ell = (n - 2) \text{ } \& \text{ } n \geq 4; \text{ or } \ell = (n - 3) \text{ } \& \text{ } n > 6 . \]

Moreover,

\[ (A.6.57) \quad \ell = n - 2 \implies \left( \ell + 1 - \frac{n}{2} \right) \cdot \frac{2n}{n - 2} = n , \]

and when \(\ell = (n - 3)\) and \(n > 6\),

\[ (A.6.58) \quad \left[ (n - 3) + 1 - \frac{n}{2} \right] \cdot \frac{2n}{n - 2} - (n - 3) = \frac{n - 6}{n - 2} > 0 . \]
§ A.6. i. Remaining estimates. Using \(|\langle y, \nabla K\rangle| \leq C |y|^{\ell}\) and \(|\xi_i| = o(\lambda_i)\) \[ actually we only need \(|\xi_i| = O(\lambda_i)\] , as in (A.6.33), we have

(A.6.59) \[ \varepsilon \int_{B_\rho(\rho_2)} \left| \frac{\partial K}{\partial r} \right| \cdot |A(y)|^{\frac{2n}{n-2}} dy = \varepsilon \cdot O_\lambda(\ell) \quad \text{for } 2 \leq \ell \leq n-2. \]

§ A.6. i.1. Estimate on the outside. Similar to (A.6.32), for \(l \leq n-1\), we have

(A.6.60) \[ \int_{\mathbb{R}^n \setminus B_\rho(\rho_2)} r^l \cdot \left( \frac{\lambda_i}{\lambda_i^2 + r^2} \right)^n dy \leq \int_{\mathbb{R}^n \setminus B_\rho(\rho_2)} r^l \cdot \left( \frac{\lambda_i}{r^2} \right)^n dy = \lambda_i^n \cdot \int_{\mathbb{R}^n \setminus B_\rho(\rho_2)} \frac{1}{r^{2n-l}} dy \leq C \lambda_i^n \cdot \int_{\rho_2}^{\infty} \frac{1}{r^{2n-l-(n-1)}} dr = O(\lambda_i^n). \]

§ A.6. i.2. The angle. Let

(A.6.61) \[ \theta_{\lambda_i} := \arctan \frac{\rho_2}{\lambda_i} \]

\[ \implies \cos \theta_{\lambda_i} = \frac{1}{\sec \theta_{\lambda_i}} = \frac{1}{\sqrt{\sec^2 \theta_{\lambda_i}}} = \frac{1}{\sqrt{1 + \tan^2 \theta_{\lambda_i}}} = \frac{1}{\sqrt{1 + \frac{\rho_2^2}{\lambda_i^2}}} = O(\lambda_i). \]

§ A.6. j. Limitation on flexibility.

Theorem A.6.62. Assume the conditions in Main Theorem 1.14 and restrict \(\ell\) to be either \(n-2\) or \(n-3\). Moreover, assume the following.

(i) When \(\ell = n-2\), we take it that \(S\) is the only blow-up point.

(ii) When \(\ell = n-3\), we take it that \(n > 6\), and it is possible to have other blow-up point(s).

Let \(\Xi^P_n\) be given in (A.6.29) and (A.6.30) with \(Q_\ell\) replaced by \(P_\ell\). Assume also that

(A.6.63) \[ \xi_i = \lambda_i^{1+\eta_o} \cdot \bar{X} \quad \text{for } i \gg 1 \text{ and a fixed } \bar{X} \in \mathbb{R}^n. \]

Here \(\eta_o \leq \begin{cases} \frac{2}{3n-2} & \text{when } \ell = n-2; \\ \frac{n-6}{(n-3)(3n-2)} & \text{when } \ell = n-3 \text{ and } n > 6. \end{cases} \)

Then it is necessary that \(P_\ell(\bar{X}) = 0\), and

\[ \int_{\mathbb{R}^n} \Xi^P_n (\bar{X}, y) \left( \frac{1}{1 + |y|^2} \right)^n dy = \cdots = \int_{\mathbb{R}^n} \Xi^P_{\ell-1} (\bar{X}, y) \left( \frac{1}{1 + |y|^2} \right)^n dy = 0. \]
Proof. Using estimate (A.6.43), (A.6.44), (A.6.48), (A.6.54) – (A.6.58), we obtain

\[(A.6.64) \quad \int_{B_o(\rho_2)} \langle y, \nabla K(y) \rangle [v_i(y)]^{2n-2} \, dy = \chi_i^\ell \cdot \frac{\ell}{c_n} \int_{\mathbb{R}^n} P_\ell(y) \left( \frac{1}{1 + |y|^2} \right)^n \, dy + \]

\[+ \int_{\mathbb{R}^n} [\lambda_i^{\ell+\eta_o} \cdot \Xi_1^p(\vec{X}, y) + \cdots + \lambda_i^{\ell+(\ell-1)\cdot\eta_o} \cdot \Xi_{\ell-1}^p(\vec{X}, y)] \cdot \left( \frac{1}{1 + |y|^2} \right)^n \, dy + \]

\[+ \lambda_i^{\ell+\eta_o} \cdot P_\ell(\vec{X}) \int_{\mathbb{R}^n} \left( \frac{1}{1 + |y|^2} \right)^n \, dy + O_{\lambda_i}(\ell + 1) + \]

\[+ \varepsilon \cdot O_{\lambda_i}(\ell) + \frac{C_n}{\varepsilon^{2n}} \cdot \begin{cases} O_{\lambda_i}(\ell + 2) & \text{for } \ell = n - 2; \\ O_{\lambda_i}(\ell + \frac{n-6}{n-2}) & \text{for } \ell = n - 3, \ n > 6. \end{cases} \]

Referring to Remark A.6.45:

\[(A.6.65) \quad \text{when } \ell = n - 2, \ \ell \cdot \eta_o < 2 \cdot \frac{n-2}{3n-2} \iff \eta_o < \frac{2}{3n-2} \]

\( \left( a = 2, \ t = \frac{2(n-2)}{3n-2} < 1 \right) \);

\[(A.6.66) \quad \text{when } \ell = n - 3, \ \ell \cdot \eta_o < \frac{n-6}{n-2} \cdot \frac{n-2}{3n-2} \iff \eta_o < \frac{n-6}{(n-3)(3n-2)} \]

\( \left( a = \frac{n-6}{n-2}, \ t = \frac{n-2}{3n-2} \cdot \frac{n-6}{n-2} < 1 \right) \).

Combining with (A.6.35), we come to the conclusion of the theorem. \( \square \)
§ A.7. \( \ell = n - 2 \) and multiple simple blow-up points – off-center cancelation.

We present the consideration on global cancelation/balance with finite number of blow-up points, say, at

\begin{equation}
\hat{Y}_o = 0, \; \hat{Y}_1, \ldots, \hat{Y}_k \quad (k \geq 1).
\end{equation}

[Cf. (2.7).] Throughout this section (§ A.7) we assume the general conditions (1.6), (1.25), (1.26), \( n > 6 \), and

\begin{equation}
\sum_{2 \leq l < n - 2} | \nabla (l) K (\hat{Y}_j) | = 0 \quad \text{for} \; 0 \leq j \leq k.
\end{equation}

\( \hat{Y}_j \) is a simple blow-up point and

\begin{equation}
\xi_{m_i} : \nu (\xi_{m_i}) = \max \left\{ \nu (y) \mid y \in \overline{B_{\hat{Y}_m}(\rho_3)} \right\}, \quad \xi_{m_i} \to \hat{Y}_m,
\end{equation}

\begin{equation}
\lambda_{m_i} := \frac{1}{\left[ \nu (\xi_{m_i}) \right]^{\frac{n-2}{2}}} \quad \text{for} \; 1 \leq m \leq k.
\end{equation}

Here \( \rho_3 > 0 \) is a constant (small enough) so that Proposition 2.3 and Proposition 2.24 hold after a translation to each individual blow-up point, and

\[ B_{\hat{Y}_m}(\rho_3) \cap B_{\hat{Y}_j}(\rho_3) = \emptyset \quad \text{for} \; j \neq m. \]

Via Proposition 2.24 and the Harnack inequality \[15\],

\begin{equation}
\frac{1}{C} \cdot \lambda_{m_i}^{\frac{n-2}{2}} \leq \min_{|y - \hat{Y}_m| = \rho_3} \nu (y) \leq \max_{|y - \hat{Y}_m| = \rho_3} \nu (y) \leq C \lambda_{m_i}^{\frac{n-2}{2}}
\end{equation}

for \( 1 \leq m \leq k \) and \( i \gg 1 \). As there is no blow-up point which appears at the north pole [cf. (2.11)], apply the Harnack inequality \[15\] again on

\[ S^n \setminus \hat{P}^{-1} \left( \bigcup_{j=0}^{k} B_{\hat{Y}_j}(\rho_3) \right) \]

and obtain

\begin{equation}
\frac{1}{C} \leq \frac{\lambda_{m_i}}{\lambda_i} \leq C \quad \text{for} \; 1 \leq m \leq k \; \text{and} \; i \gg 1.
\end{equation}
It follows that, modulo a subsequence,

\[(A.7.7)\quad S_m := \lim_{i \to \infty} \frac{\lambda_m}{\lambda_i} \text{ is well-defined for } 1 \leq m \leq k.\]

Using the global formula (A.6.6), together with (A.7.5) and (A.7.6), we have

\[(A.7.8)\quad \int_{\Omega} \langle y, \nabla_y K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} \, dy = O_{\lambda_i}(n),\]

where \(\Omega = B_o(\rho_3) \cup B_{\hat{Y}_1}(\rho_3) \cup \cdots \cup B_{\hat{Y}_k}(\rho_3).\)

\[\text{§ A.7. a. Off-origin blow-up point.}\]

Consider the simple blow-up at \(\hat{Y}_m,\) where \(0 < m \leq k.\) Via Taylor expansion,

\[(A.7.9)\quad \tilde{c}_n \cdot K(y) = (\tilde{c}_n \cdot K)(\hat{Y}_m) + \sum_{|\alpha| = n-2} \frac{1}{\alpha!} [D^{(n-2)}_{\alpha} (\tilde{c}_n K)(\hat{Y}_m)] \cdot (y - \hat{Y}_m)^\alpha + O(|y - \hat{Y}_m|^{n-1}) \quad \text{for} \quad |y - \hat{Y}_m| \leq \rho_3 \]

\[= (\tilde{c}_n \cdot K)(\hat{Y}_m) + [-P_{n-2, m}(y)] + O(|y - \hat{Y}_m|^{n-1}).\]

Here \(P_{n-2, m}(y)\) is defined by the equation above [see also (A.7.15)]. For the sake of continuity, we keep the sign convention on \('P',\) which we use in this article. Assume that

\[(A.7.10)\quad \Delta^{(hn-2)}_{\alpha} P_{n-2, m}(\hat{Y}_m) \equiv 0 \quad \text{for} \quad 0 \leq m \leq k,\]

and all the corresponding conditions as in Main Theorem (1.17) hold for each individual simple blow-up point, except \((\tilde{c}_n \cdot K)(\hat{Y}_m)\) may not be \(n(n-2)\). Thus the estimate contains a scaling factor

\[(A.7.11)\quad v_i(y) = \left[ \frac{n(n-2)}{(\tilde{c}_n K)(\hat{Y}_m)} \right]^{\frac{n-2}{4}} \cdot \left( \frac{\lambda_m}{\lambda_{m_1} + |y - \xi_{m_1}|^2} \right)^{\frac{n-2}{2}} + \]

\[+ \{ \text{expressions similar to those in (A.6.24) and (A.6.25)} \} \quad \text{for} \quad y \in B_{\hat{Y}_m}(\rho_3).\]

[Here \(\rho_3\) is made smaller if necessary.]

We find the first derivative by using change of variables \(y = z + \hat{Y}_m,\)

\[(A.7.12)\quad \tilde{c}_n \cdot \langle y, \nabla_y K(y) \rangle = \tilde{c}_n \cdot \langle (y - \hat{Y}_m), \nabla_y K(y) \rangle + \tilde{c}_n \cdot \langle \hat{Y}_m, \nabla_y K(y) \rangle,

\[(A.7.13)\quad \langle (y - \hat{Y}_m), \nabla_y K(y) \rangle = \langle z, (\nabla_y K)|_{y = z + \hat{Y}_m} \rangle

\[= \langle z, (\nabla_z K)|_{z = (y - \hat{Y}_m)} \rangle.\]
Here \( K \to (z) = K(z + \hat{Y}_m) \).

Consider the second expression in the right hand side of (A.7.12). As in (A.7.13), we have

\[
(A.7.14) \quad \tilde{c}_n \cdot \langle \hat{Y}_m, \ \nabla_y K(y) \rangle = \langle \hat{Y}_m, \ \nabla_y P_{n-2, m}(y) \rangle + O(|y - \hat{Y}_m|^{n-2})
\]

\[
= \langle \hat{Y}_m, \ \nabla_z P_{m \to (z)} \rangle |_{z = y - \hat{Y}_m} + O(|y - \hat{Y}_m|^{n-2}).
\]

Let \( L_m(z) := \langle \hat{Y}_m, \ \nabla_z P_{m \to (z)} \rangle \) for \( 1 \leq m \leq k \).

\( L_m \) is a homogeneous polynomial of degree \( n - 3 \). As in (A.7.13), from (A.7.9), we recognize

\[
\hat{P}_{m \to (z)} = \hat{P}_{n-2, m}(z + \hat{Y}_m) = - \sum_{|\alpha| = n-2} \frac{1}{\alpha!} \left[ D_{\alpha}^{(n-2)}(\tilde{c}_n K)(\hat{Y}_m) \right] \cdot z^\alpha.
\]

Observe that, via (A.7.10), \( \Delta^{h_n-2} P_{m \to (z)} \equiv 0 \).

In the following we assume that, for each \( m \) with \( 0 \leq m \leq k \), there is a positive number \( \eta_m \) such that

\[
\eta_m < \frac{n-6}{(n-3)(3n-2)} \quad \text{and} \quad \xi_{m_i} = \lambda_{m_i}^{1+\eta_m} \cdot \bar{X}_m \quad \text{for} \ i \gg 1,
\]

where \( \bar{X}_m \in \mathbb{R}^n \) is fixed. Consider the integral

\[
(A.7.17) \quad \tilde{c}_n \cdot \int_{B \hat{Y}_m(\rho_3)} \langle y, \ \nabla_y K(y) \rangle [v_i(y)]^{\frac{2n}{n-2}} dy
\]

[\( \downarrow \) via \( (A.7.12), (A.7.13) \) and \( (A.7.14) \)]

\[
= \tilde{c}_n \cdot \int_{B \hat{Y}_m(\rho_3)} \langle z, \ \nabla_z K \to (z) \rangle |_{z = (y - \hat{Y}_m)} [v_i(y)]^{\frac{2n}{n-2}} dy +
\]

\[
+ \int_{B \hat{Y}_m(\rho_3)} \langle \hat{Y}_m, \ \nabla_z P_{m \to (z)} \rangle |_{z = (y - \hat{Y}_m)} [v_i(y)]^{\frac{2n}{n-2}} dy + O_{\lambda_{m_i}}(n - 2)
\]

[using (2.26) & (2.27), as in (A.6.41) for the term \( O(|y - \hat{Y}_m|^{n-2}) \) \( \uparrow \)]

\[
= \tilde{c}_n \cdot \int_{B_0(\rho_3)} \langle z, \ \nabla_z K(z) \rangle [v_i(z + \hat{Y}_m)]^{\frac{2n}{n-2}} dz \quad (z = y - \hat{Y}_m)
\]

\[
+ \int_{B_0(\rho_3)} L_m(z) [v_i(z + \hat{Y}_m)]^{\frac{2n}{n-2}} dz + O_{\lambda_{m_i}}(n - 2)
\]

71
\[
\begin{align*}
\Delta_o^{(h_{n-2})} P_m \rightarrow 0 & \iff \Delta_o^{(n-3)} P_m \rightarrow 0 \\
& \iff \Delta_o^{(n-3)} L_m (z) = \Delta_o^{(n-3)} \langle \hat{Y}_m, \nabla_z P_m (z) \rangle \\
& = \langle \hat{Y}_m, \nabla_z \left[ \Delta_o^{(n-2)} P_{n-2, m} (z) \right] \rangle = 0.
\end{align*}
\]

Again we obtain (A.7.18) by using Lemma A.6.8.

When we add up the integrals and estimates from each simple blow-up point, for simplicity, we skip the terms with intermediate orders

\[
O_{\lambda_{m_i}} \left( [n - 3] + \eta_{m_i} \right) + \cdots + O_{\lambda_{m_i}} \left( [n - 3] + [n - 4] \cdot \eta_{m_i} \right),
\]

and draw a conclusion from (A.7.8) and (A.7.17) that
\( \mathcal{L}_m (\bar{X}_m) = \langle \hat{Y}_m, \nabla_z P_{n-2, m} (\bar{X}_m) \rangle = 0 \) (where \( \xi_m = \lambda_m^{1+\eta_m} \cdot \bar{X}_m \)),

(A.7.20) \quad \text{provided} \quad (n - 3) \cdot \eta_m < \frac{n - 6}{3n - 2} \quad \left( \text{observe that} \quad \frac{n - 6}{3n - 2} < 1 \right),

and there is no interference from other blow-up points, precisely:

(A.7.21)

\( (n - 3) \cdot \eta_m \neq h \cdot \eta_j \quad \text{for} \quad j \neq m, 1 \leq h \leq n - 3 \quad (h \text{ is a natural number}). \)

In case some of the \( \eta_h = \eta_m \), together with (A.7.7), we obtain

(A.7.22)

\[
\left[ \frac{n(n-2)}{(\bar{c}_n K)(\bar{Y}_m)} \right]^{\frac{n}{2}} \cdot S^{(n-3)+(n-3)\eta_m} \cdot \langle \hat{Y}_m, \nabla_z P_{n-2, m} (\bar{X}_m) \rangle
\]

\[
+ \sum_{0 < h \leq k, h \neq m} \left[ \frac{n(n-2)}{(\bar{c}_n K)(\bar{Y}_h)} \right]^{\frac{n}{2}} \cdot S^{(n-3)+(n-3)\eta_h} \cdot \langle \hat{Y}_h, \nabla_z P_{n-2, h} (\bar{X}_h) \rangle = 0 ,
\]

where we set the condition

(A.7.23) \quad \eta_j \neq \eta_m \implies \text{(A.7.21) holds}.

We summarize the conditions assumed in the balance and cancelation formulas (A.7.19) and (A.7.22): besides the ones mentioned next to them [they are (A.7.20), (A.7.21), and (A.7.23)], and the conditions found in Main Theorem 1.14 for each blow-up point, plus (A.7.1) – (A.7.4), (A.7.10) and (A.7.16).
§ A.8. Verification of (A.6.13) and (A.6.18).

Refer to (A.6.13) for the notation we use.

\[
\int_{\mathbb{R}^n} \left( \text{terms without } y_1 \text{ & } y_n \right) \cdot y_1^{k+2} \cdot \left( \frac{1}{1 + r^2} \right)^n dy \quad \text{(absolute convergence)}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \text{terms without } y_1 \text{ & } y_n \right) \times \quad \text{(Fubini’s Theorem)}
\]

\[
\times \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^{k+2} \cdot \left( \frac{1}{1 + [y_1^2 + \cdots + y_{n-1}^2] + y_n^2 + \rho^2} \right)^n dy_1 dy_n \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \text{terms without } y_1 \text{ & } y_n \right) \times
\]

\[
\times \left[ \int_{0}^{2\pi} \int_{0}^{\infty} \rho^{k+3} \cdot \left( \frac{1}{1 + [y_1^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \cdot \rho \, d\theta \, d\rho \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \text{terms without } y_1 \text{ & } y_n \right) \times
\]

\[
\times \left[ \int_{0}^{\infty} \rho^{k+3} \cdot \left( \frac{1}{1 + [y_1^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \cdot \left\{ \int_{0}^{2\pi} \sin^{k+2} \theta \, d\theta \right\} d\rho \right] dy_2 \cdots dy_{n-1}.
\]

Here “(terms without \( y_1 \) & \( y_n \))” is a polynomial on \( y_1, \cdots, y_{n-1}, \) having sufficiently low degree so that the integral is absolutely convergent. Likewise,

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \text{terms without } y_1 \text{ & } y_n \right) \times
\]

\[
\times \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^k y_n^2 \left( \frac{1}{1 + [y_1^2 + \cdots + y_{n-1}^2] + y_n^2 + \rho^2} \right)^n dy_1 dy_n \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \text{terms without } y_1 \text{ & } y_n \right) \times
\]

\[
\times \left[ \int_{0}^{\infty} \rho^{k+3} \cdot \left( \frac{1}{1 + [y_1^2 + \cdots + y_{n-1}^2] + \rho^2} \right)^n \cdot \left\{ \int_{0}^{2\pi} (\sin^{k} \theta) \, (\cos^2 \theta) \, d\theta \right\} d\rho \right] dy_2 \cdots dy_{n-1}.
\]

74
A direct calculation using integration by parts shows that
\[
\int_0^{2\pi} (\sin^{k+2} \theta) \, d\theta = -\int_0^{2\pi} (\sin^{k+1} \theta) \, d[\cos \theta] = (k + 1) \int_0^{2\pi} (\sin^k \theta) (\cos^2 \theta) \, d\theta.
\]
Hence we deduce (A.6.13).

To show (A.6.18), recall that
\[
\ell = \alpha_2 + \cdots + \alpha_{n-1}.
\]
We demonstrate how to use induction on \(\ell\) to prove the assertion. Recall that
\[
\ell \in [0, n-2) \text{ is even}.
\]

(I) When \(\ell = 0\), the term is a constant. We have
\[
\Delta_0^{(h_\ell)} y_{\ell+1}^{k+2} = (k + 2) (k + 1) \cdots 3 \cdot 2 \cdot 1;
\]
\[
\Delta_0\left\{(k + 1) y_{\ell+1}^k y_\ell^2\right\} = (k + 1) k (k - 1) y_{\ell+1}^{k-2} y_\ell^2 + 2 (k + 1) y_{\ell+1}^k;
\]
\[
\Delta_0^{(2)} \left\{(k + 1) y_{\ell+1}^k y_\ell^2\right\} = (k + 1) k (k - 1) (k - 2) (k - 3) y_{\ell+1}^{k-4} y_\ell^2 +
+ 2 \times 2 (k + 1) k (k - 1) y_{\ell+1}^{k-2};
\]
\[
\Delta_0^{(h_\ell-2)} \left\{(k + 1) y_{\ell+1}^k y_\ell^2\right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot y_{\ell+1}^2 y_\ell^2 +
+ 2 [h_\ell - 2] (k + 1) k (k - 1) \cdots 5 \cdot y_{\ell+1}^4;
\]
\[
\Delta_0^{(h_\ell-1)} \left\{(k + 1) y_{\ell+1}^k y_\ell^2\right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot [y_{\ell+1}^2 + y_\ell^2] +
+ 2 [h_\ell - 2] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 y_{\ell+1}^2;
\]
\[
\Delta_0^{h_\ell} \left\{(k + 1) y_{\ell+1}^k y_\ell^2\right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 \cdot 4 +
+ [\ell - 4] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1
= (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 [\ell - 4 + 4]
= (k + 2) (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1
\]
(as \(\ell = k + 2\) in this case).
Hence the case \( \ell = 0 \) is settled.

(II) As an induction hypothesis, suppose that

\[
\Delta^{(h \ell)}_o \left\{ y_{1i}^{k+2} \cdot [\ldots \text{degree} = \ell \ldots] \right\} = \Delta^{(h \ell)}_o \left\{ (k + 1) y_{1i}^k y_{1n}^2 \cdot [\ldots \text{degree} = \ell \ldots] \right\}
\]

holds for \( \ell = (k + 2) + \ell \), where \( k \geq 2 \) (variable), but \( \ell > 0 \) (fixed). We continue to use the notations above and there is no \( y_{1i} \) or \( y_{1n} \) inside the homogeneous polynomial denoted by \([\ldots \text{degree} = \ell \ldots] \). Let us go on to show

\[
(A.8.1) \quad \Delta^{(h \ell)}_o \left\{ y_{1i}^{k+2} \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = \Delta^{(h \ell)}_o \left\{ (k + 1) y_{1i}^k y_{1n}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\},
\]

where \( k \geq 2 \) is even. Let us find the first Laplacians:

\[
(A.8.2) \quad \Delta_o \left\{ y_{1i}^{k+2} \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = (k + 2) (k + 1) y_{1i}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots]
\]

+ \( y_{1i}^{k+2} \cdot \Delta_o \left\{ [\ldots \text{degree} = \ell + 2 \ldots] \right\}
\]

= \( k (k + 1) y_{1i}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots]
\]

+ 2 (k + 1) y_{1i}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots]

+ \( y_{1i}^{k+2} \cdot \Delta_o \left\{ [\ldots \text{degree} = \ell + 2 \ldots] \right\}
\]

\[
( \leftarrow \uparrow \text{degree} = \ell \rightarrow );
\]

\[
\Delta_o \left\{ (k + 1) y_{1i}^k y_{1n}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\} = (k + 1) k (k - 1) y_{1i}^{k-2} y_{1n}^2 \cdot [\ldots \text{degree} = \ell + 2 \ldots]
\]

+ 2 (k + 1) y_{1i}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots]

+ \( (k + 1) y_{1i}^k y_{1n}^2 \cdot \Delta_o \left\{ [\ldots \text{degree} = \ell + 2 \ldots] \right\}
\]

\[
( \leftarrow \uparrow \text{degree} = \ell \rightarrow );
\]

[ observe that \( (k + 2) (k + 1) - 2 (k + 1) = k (k + 1) \).]

Via the induction hypothesis, the last two terms in the respective expressions are equal. After simplification, to verify (A.8.1), it suffices to show that

\[
\Delta^{(h \ell - 1)}_o \left\{ y_{1i}^k \cdot [\ldots \text{degree} = \ell + 2 \ldots] \right\}
\]

76
Applying $\Delta_o$ on the terms

$\{y_i^k \cdot [\cdots \text{degree} = \ell + 2 \cdots]\}$ and $\{(k - 1) y_i^{k-2} y_i^2 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\}$,

using similar calculation and cancelation as in (A.8.2), we come down gradually to verify

(A.8.3) \[
\Delta_o \left( h_\ell \right) \left( y_i^4 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right)
= \Delta_o \left( h_\ell \right) \left( 3 y_i^2 y_i^2 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right).
\]

Note that

\[
\ell = (k + 2) + (\ell + 2) \implies \frac{\ell}{2} - \frac{k - 2}{2} = \frac{4 + (\ell + 2)}{2}.
\]

Apply the Laplacian on the two terms inside the brackets in (A.8.3) and obtain

\[
\Delta_o \left( y_i^4 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right) = 4 \cdot 3 \cdot y_i^2 \cdot [\cdots \text{degree} = \ell + 2 \cdots] + \\
+ y_i^4 \cdot \{ \Delta_o \cdot [\cdots \text{degree} = \ell + 2 \cdots]\};
\]

\[
( \leftarrow \uparrow \text{degree} = \ell \rightarrow )
\]

\[
\Delta_o \left\{ 3 y_i^2 y_i^2 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right\}
= 3 \cdot 2 \cdot 1 \cdot [y_i^2 + y_i^2] \cdot [\cdots \text{degree} = \ell + 2 \cdots] + \\
+ 3 y_i^2 y_i^2 \cdot \{ \Delta_o \cdot [\cdots \text{degree} = \ell + 2 \cdots]\};
\]

\[
( \leftarrow \uparrow \text{degree} = \ell \rightarrow ).
\]

Again we apply the induction hypothesis to cancel the last term in each expression above. Apply the Laplacian again and obtain

(A.8.4) \[
\Delta_o \left( 4 \cdot 3 \cdot y_i^2 \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right)
= 4 \cdot 3 \cdot 2 \cdot [\cdots \text{degree} = \ell + 2 \cdots] + 4 \cdot 3 \cdot y_i^2 \cdot \{\Delta_o \cdot [\cdots \text{degree} = \ell + 2 \cdots]\};
\]

(A.8.5) \[
\Delta_o \left\{ 3 \cdot 2 \cdot [y_i^2 + y_i^2] \cdot [\cdots \text{degree} = \ell + 2 \cdots]\right\}
= 3 \cdot 2 \cdot 4 \cdot [\cdots \text{degree} = \ell + 2 \cdots] + \\
+ 3 \cdot 2 \cdot [y_i^2 + y_i^2] \cdot \{\Delta_o \cdot [\cdots \text{degree} = \ell + 2 \cdots]\}.
\]

77
As $4 \cdot 3 \cdot y^2_{1l} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\}$

$= 2 \cdot 3 \cdot y^2_{1l} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\} +$

$\quad + 2 \cdot 3 \cdot y^2_{1l} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\}$,

$3 \cdot 2 \cdot \left[ y^2_{1l} + y^2_{nl} \right] \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\}$

$= 3 \cdot 2 \cdot y^2_{1l} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\} +$

$\quad + 3 \cdot 2 \cdot y^2_{nl} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\}$,

and $\Delta_o \left( \frac{2 + (l+2)}{2} \right) \left[ y^2_{1l} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\} \right]

= \Delta_o \left( \frac{2 + (l+2)}{2} \right) \left[ y^2_{nl} \cdot \{\Delta_o \cdots \text{degree } = \ell + 2 \cdots\} \right] (\leftarrow \text{equal to a number}),$

we apply the remaining order of Laplacian on (A.8.4) and (A.8.5), yielding the same numbers. Hence we verify (A.8.3), and so (A.8.2). This completes the induction step.

§ A.9. Verification of (A.6.44).

Let $A$, $B$ and $C$ be numbers such that

(A.9.1) $A > 0$, $A + B + C > 0$, and $\beta = \frac{2n}{n-2}$.

We show that for any number $\varepsilon > 0$ small enough, there exists a positive number $\bar{C}_\beta$ so that

(A.9.2) $| (A + B + C)^\beta - A^\beta | \leq \varepsilon A^\beta + \frac{\bar{C}_\beta}{\varepsilon^{\beta/2}} \left( |B|^\beta + |C|^\beta \right)$.

In this presentation, the proof of this statement relies on the following ($\beta$ being relaxed).

**Lemma A.9.3.** Let $\beta \geq 1$ be given. There is a positive number $c_\alpha$ such that for any number $\varepsilon \in (0, c_\alpha)$, we have

(A.9.4) $| (1 + t)^\beta - 1 | \leq \varepsilon + \frac{C_\beta}{\varepsilon^{\beta}} \cdot |t|^\beta \quad \text{for } t \in [-1, \infty)$.

Here $c_\alpha$ and $C_\beta$ do not depend on $\varepsilon$ or $t$. 78
Proof. Via Taylor expansion, there is a positive number $c_o (< 1)$ such that

(A.9.5) \[ |(1 + t)^\beta - 1| \leq (\beta + 1)|t| \quad \text{for} \quad |t| \leq \frac{c_o}{\beta + 1}. \]

For $\varepsilon \in (0, c_o)$, consider the case

\[
|t| \leq \frac{\varepsilon}{\beta + 1} \implies \frac{|(1 + t)^\beta - 1|}{\varepsilon} \leq \frac{(\beta + 1)|t|}{\varepsilon} \leq 1 \quad \text{[using (A.9.4)]}
\]

\[
\implies |(1 + t)^\beta - 1| \leq \varepsilon \implies (A.9.4) \text{ holds for } |t| \leq \frac{\varepsilon}{\beta + 1}.
\]

When

\[
1 \geq |t| \geq \frac{\varepsilon}{\beta + 1},
\]

we find $C_\varepsilon$ to be large enough so that

(A.9.6) \[
\frac{2^\beta + 1}{C_\varepsilon \cdot \left(\frac{\varepsilon}{\beta + 1}\right)^\beta} \leq 1, \quad \text{that is, } C_\varepsilon = \frac{1}{\varepsilon^{\beta}} \cdot (2^\beta + 1)^{(\beta + 1)^\beta}.
\]

It follows that

\[
|(1 + t)^\beta - 1| \leq (2^\beta + 1) \leq C_\varepsilon|t|^\beta \quad \text{for} \quad 1 \geq |t| \geq \frac{\varepsilon}{\beta + 1}.
\]

When $t > 1$, from (A.9.6)

\[
C_\varepsilon > 2^\beta \implies \frac{|(1 + t)^\beta - 1|}{C_\varepsilon t^\beta} \leq \frac{|(2t)^\beta|}{C_\varepsilon t^\beta} \leq 1
\]

\[
\implies |(1 + t)^\beta - 1| \leq C_\varepsilon t^\beta \quad \text{for} \quad t > 1.
\]

Combining the three cases, we have (A.9.4) with the choice of $C_\beta = (2^\beta + 1)(\beta + 1)^\beta$ as specified in (A.9.6). \qed

Proof of (A.9.2). Using (A.8.4), we obtain

\[
|(A + B + C)^\beta - A^\beta| = A^\beta \cdot \left(1 + \left[\frac{B + C}{A}\right]\right)^\beta - 1
\]

\[
\leq A^\beta \left(\varepsilon + \frac{C_\beta}{\varepsilon^{\beta}} \cdot \left|\frac{B + C}{A}\right|^{\beta}\right) \quad (A > 0 \& A + B + C > 0 \implies \frac{B + C}{A} \geq -1)
\]

\[
\leq A^\beta \left[\varepsilon + \frac{C_\beta}{\varepsilon^{\beta}} \cdot \left(|B| + |C|\right)^{\beta}\right] \leq A^\beta \cdot \left\{\varepsilon + \frac{C_\beta}{\varepsilon^{\beta}} \cdot \left[2^{\beta - 1} \left(|B|^{\beta} + |C|^{\beta}\right)\right]\right\}
\]

\[
\leq \varepsilon A^\beta + 2^{\beta - 1} \cdot \frac{C_\beta}{\varepsilon^{\beta}} \cdot \left(|B|^{\beta} + |C|^{\beta}\right) \quad \text{[using} \quad (|B| + |C|)^{\beta} \leq 2^{\beta - 1} (|B|^{\beta} + |C|^{\beta})\text{].}
\]

79
We can take $C_\beta = 2^{\beta-1} \cdot C_\beta$ to obtain (A.9.2), and take $\beta = \frac{2n}{n-2}$ to obtain 7.23.

§ A.10. Linear approximation to $\left( A_1^{\frac{n+2}{n-2}} - V_i^{\frac{n+2}{n-2}} \right)$ in case of simple blow-up.

The Taylor expansion

\[(1 + t)^p = 1 + pt + O(t^2) \cdot (1 + \tau)^{p-2} \quad \text{(here } 0 \leq \tau \leq t)\]

tells us that

\[a^p = [b + (a - b)]^p = b^p \left[ 1 + \frac{(a - b)}{b} \right]^p = b^p + p \cdot (a - b) b^{p-1} + O(1) (1 + \tau)^{p-2} (a - b)^2 \cdot b^{p-2}\]

for numbers with $a > b > 0$ and $p > 1$. It follows that

(A.10.1) $a^{\frac{n+2}{n-2}} - b^{\frac{n+2}{n-2}} = \left[ \frac{n+2}{n-2} \right] (a - b) b^{\frac{4}{n-2}} + O(1) (1 + \tau)^{\frac{4}{n-2} - 1} \cdot (a - b)^2 \cdot b^{\frac{n+2}{n-2} - 2}$.

Here

\[0 \leq \tau \leq \frac{a - b}{b}.
\]

For $A_1(\mathcal{Y}) \geq V_i(\mathcal{Y})$, (A.10.1) implies that

(A.10.2) $\left[ A_1(\mathcal{Y}) \right]^{\frac{n+2}{n-2}} - \left[ V_i(\mathcal{Y}) \right]^{\frac{n+2}{n-2}} = \left( \frac{n+2}{n-2} \right) [A_1(\mathcal{Y}) - V_i(\mathcal{Y})] \cdot [A_1(\mathcal{Y})]^{\frac{4}{n-2}} + O(1) (1 + \tau)^{\frac{4}{n-2} - 1} \cdot [A_1(\mathcal{Y}) - V_i(\mathcal{Y})]^2 \cdot [A_1(\mathcal{Y})]^{\frac{4}{n-2} - 1}$.

Since $n \geq 6 \implies 1 - \frac{4}{n-2} \geq 0 \implies (1 + \tau)^{\frac{4}{n-2} - 1} = \frac{1}{(1 + \tau)^{1 - \frac{4}{n-2}}} \leq 1,$

and

\[
\frac{A_1(\mathcal{Y}) - V_i(\mathcal{Y})}{V_i(\mathcal{Y})} \leq 1 + \frac{A_1(\mathcal{Y})}{V_i(\mathcal{Y})} \leq 1 + C, \quad 3 \leq n \leq 5 \implies (1 + \tau)^{\frac{4}{n-2} - 1} \leq C_1.
\]

Recalling (2.26) in Proposition 2.24, and also § 2f, we have

(A.10.3) $C^{-1} \cdot A_1(\mathcal{Y}) \leq V_i(\mathcal{Y}) \leq C A_1(\mathcal{Y})$ for $|\mathcal{Y}| \leq \rho_o \lambda_i^{-1}$.
It follows that

\[ A_1 (\mathcal{Y}) \geq V_i (\mathcal{Y}) \implies [A_1 (\mathcal{Y})]^{n+2 \over n-2} - [V_i (\mathcal{Y})]^{n+2 \over n-2} \]

\[ = \left( \frac{n+2}{n-2} \right) [A_1 (\mathcal{Y})]^{4 \over n-2} \cdot [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})] \]

\[ + O (1) [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})]^2 \cdot [A_1 (\mathcal{Y})]^{4 \over n-2} - 1. \]

For the opposite case

\[ V_i (\mathcal{Y}) > A_1 (\mathcal{Y}) \]

we obtain a similar expression as in (A.10.4), with the last term

\[ (A.10.5) \quad [A_1 (\mathcal{Y})]^{4 \over n-2} - 1 \quad \text{changed to} \quad [V_i (\mathcal{Y})]^{4 \over n-2} - 1. \]

We obtain

\[ \frac{1}{C_2 \cdot A_1 (\mathcal{Y})} \leq \frac{1}{V_i (\mathcal{Y})} \leq \frac{C_2}{A_1 (\mathcal{Y})} \quad \text{for} \quad |\mathcal{Y}| \leq \rho_o \lambda_i^{-1}. \]

Thus the two terms in (A.10.5) have the same order. We come to the conclusion that

\[ (A.10.6) \quad [A_1 (\mathcal{Y})]^{n+2 \over n-2} - [V_i (\mathcal{Y})]^{n+2 \over n-2} = \left( \frac{n+2}{n-2} \right) [A_1 (\mathcal{Y})]^{4 \over n-2} \cdot [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})] \]

\[ + O (1) [A_1 (\mathcal{Y}) - V_i (\mathcal{Y})]^2 \cdot [A_1 (\mathcal{Y})]^{4 \over n-2} - 1, \]

which holds for \(|\mathcal{Y}| \leq \rho_o \lambda_i^{-1}\). (A.10.6) is independent on which value is bigger, \(A_1 (\mathcal{Y})\) or \(V_i (\mathcal{Y})\).
References

[1] T. Aubin, *Meilleures constantes dans le théorème d’inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32, 148–174 (1979).

[2] A. Ambrosetti, J. García Azorero & I. Peral, *Perturbation of \( \Delta u + u^{\frac{N+2}{N-2}} = 0 \), the scalar curvature problem in \( \mathbb{R}^N \), and related topics*, J. Funct. Anal. 165 (1999), 117–149.

[3] A. Ambrosetti & A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on \( \mathbb{R}^n \)*, Progress in Mathematics 240, Birkhauser, Basel-Boston-Berlin, 2006.

[4] S. Brendle, *Blow-up phenomena for the Yamabe equation*, Journal of AMS 21 (2008), 951–979.

[5] S. Brendle & F. Marques, *Blow-up phenomena for the Yamabe equation. II.* J. Differential Geom. 81 (2009), 225–250.

[6] L. Caffarelli, B. Gidas & J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. 42 (1989), 271–297.

[7] C.-C. Chen & C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes*, Comm. Pure Appl. Math. 50 (1997), 971–1019.

[8] C.-C. Chen & C.-S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes. II*, J. Differential Geometry 49 (1998), 115–178.

[9] X.-Z. Chen & X.-W. Xu, *The scalar curvature flow on \( S^n \) – perturbation theorem revisited*, Invent. Math. 187 (2012), 395–506.

[10] B. Gidas, W.-M. Ni & L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 68 (1979), 209–243.

[11] D. Gilbarg & N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer-Verlag, Berlin-Heidelberg-New York, 1998.

[12] M. Khuri, F. Marques & R. Schoen, *A compactness theorem for the Yamabe problem*, J. Differential Geom. 81 (2009), 143–196.

[13] N. Korevaar, R. Mazzeo, F. Pacard & R. Schoen, *Refined asymptotics for constant scalar curvature metrics with isolated singularities*, Invent. Math. 135 (1999), 233–272.

[14] M.-C. Leung, *Blow-up solutions of nonlinear elliptic equations in \( \mathbb{R}^n \) with critical exponent*, Math. Ann. 327 (2003), 723–744.

82
[15] M.-C. Leung, *Supported blow-up and prescribed scalar curvature on $S^n$*, Memoirs of the American Mathematical Society, 213 (2011), No. 1002.

[16] M.-C. Leung, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$*. I. Uniform cancellation, Comm. Contemporary Mathematics, 14 (2012), 1–31.

[17] M.-C. Leung, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$*. II. Annular domains, Calculus of Variations and PDE, 46 (2013), 1–29.

[18] M.-C. Leung, *Refined estimates for simple blow-ups of the scalar curvature equation on $S^n$*. To appear in Transactions of the American Society.

[19] M.-C. Leung & F. Zhou, *Construction of blow-up sequences for the prescribed scalar curvature equation on $S^n$*. III. Aggregated and towering Blow-up, Calculus of Variations and PDE, 54 (2015), 3009–3035.

[20] M.-C. Leung & F. Zhou, *Conformal scalar curvature equation on $S^n$: functions with two close critical points (twin pseudo-peaks)*. Preprint.

[21] Y.-Y. Li, *Prescribing scalar curvature on $S^n$ and related problems*. I, J. Differential Equations 120 (1995), 319–410.

[22] Y.-Y. Li, *Prescribing scalar curvature on $S^n$ and related problems. II. Existence and compactness*, Comm. Pure Appl. Math. 49 (1996), 541–597.

[23] R. McOwen, *Partial Differential Equations, Methods and Applications*, Prentice-Hall, Upper Saddle River, 1996.

[24] R. Schoen, *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation*, Comm. Pure Appl. Math. 41 (1988), 317–392.

[25] R. Schoen, distributed notes for the courses at Stanford University (1988) and New York University (1989), unpublished.

[26] J. Wei & S. Yan, *Infinitely many solutions for the prescribed scalar curvature problem on $S^N$*, J. Funct. Anal. 258 (2010), 3048–3081.