Spin-Statistics connection and the gravity of the Universe: The Cosmic connection

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Abstract

I suggest that the Spin-Statistics connection is a consequence of the phase shifts on quantum scattering amplitudes due to the induced gravitomagnetic field of the whole Universe at critical density. This connection was recently brought out in the context of a new theory of relativity in flat space with matter, called Cosmic Relativity. This prediction of the correct gravitational phases is a consequence of any relativistic gravitation theory in the presence of the massive Universe. This can also be interpreted as related to the Mach’s principle applied to quantum phenomena. Perhaps this is the simplest valid proof of the Spin-Statistics Theorem, and it finally identifies the physical origin of the connection.

1 General remarks

The main result of this paper is the suggestion that the well known spin-statistics connection is a consequence of the gravitational interaction of the quantum system with the entire Universe. In other words, the Pauli exclusion is a consequence of the relativistic gravitational interaction with the critical Universe, which is always present. Motion relative to masses in the Universe

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generates gravitational vector potentials that interact with the spin. The curl of this potential is a magnetic analog in gravity, and the spin-gravitomagnetic interaction energy leads to phase changes in the quantum state. By considering the difference in phases induced by the gravitomagnetic interaction in two possible amplitudes for scattering of identical particles we are able to deduce the spin-statistics connection; identical integer spin particles obey Bose-Einstein statistics and identical half-integer spin particles obey Fermi-Dirac statistics.

This result might answer for the first time the long-standing query – what are the physical reasons behind the spin-statistics connection? It also answers why the connection is valid in non-relativistic, two-particle situations despite the general impression that it is a consequence of relativistic field theory.

The fundamental conceptual basis that underlies the physical effects discussed here is the theory of Cosmic Relativity, discussed in detail in gr-qc/0406023 [1]. There is also a discussion of the cosmic connection between gravity and spin dynamics in that paper. Here we focus on the spin-statistics connection exclusively, discuss the physical and mathematical arguments that links this to the gravity of the Universe, and also discuss briefly some aspects in the context of other proofs of the spin-statistics connection.

In Cosmic Relativity, a particle that is moving in the cosmic gravitational potential \( \phi \) of the Universe experiences a modified gravitational potential and a vector gravitational potential equal to

\[
\phi' = \phi (1 - V^2/c^2)^{-1/2} \\
A_i = \phi \frac{V_i}{c} (1 - V^2/c^2)^{-1/2}
\]

For Universe with critical density, the quantity \( \phi = 1 \) (in unit of \( c^2 \)). In the more rigorous and exact language of the metric for the massive Universe, this is equivalent to writing \( g_{0i} = \frac{V_i}{c} (1 - V^2/c^2)^{-1/2} \). Circular motion then gives a nonzero curl for the velocity field, and this is a gravitomagnetic field due to the entire Universe,

\[
\vec{B}_g = \nabla \times \vec{A} = \Omega (1 - v^2/c^2)^{-1/2}
\]

This gravitomagnetic field couples to the spin angular momentum. The modified potential modifies precession rate of spin, and also modifies quantum phases of states with spin. Though I have written the gravitomagnetic field in terms of the post-Newtonian potentials, to clarify the physical issues in
terms of familiar concepts like an interaction energy of the quantum spin in a gravitomagnetic field, the final expressions concerning ‘phase’ of quantum amplitudes are expected to be correct to all orders. This is because the phase is the product of the interaction energy and time duration of interaction, and these two variables are affected by gravity of the Universe in an exactly reciprocal way.

I have been convinced of a connection between quantum spin dynamics in the Universe and effects like Thomas precession, Spin-Statistics connection, and geometric phases of fundamental particles taken in closed trajectories in momentum space since about 1995. Around the same time, it also became clear to me that a consistent formulation of relativistic effects in a Universe with an operational way of precisely determining the ‘absolute velocity’ relative to a preferred rest frame (the CMBR), and also the possibility of an absolute time (temperature of the CMBR) can be done only by taking the gravity of the Universe as the major physical reason for all those effects usually attributed to kinematics [2]. But, it was a major psychological barrier to think that special relativity based on mere kinematics in empty, matter-free space then will have to be replaced by a new theory of relativity based only on gravitational effects of the matter-filled Universe. The two ideas go together, inseparably. Finally, such a new theory is proposed and discussed in detail in ref. [1]. I have previously discussed Thomas precession as an effect of spin-gravitomagnetic field coupling, and the spectral fine structure splitting as due to the gravitational interaction of the quantum spin with the Universe [3, 4].

2 The Spin-Statistics Connection

The spin-statistics connection in quantum mechanics is a very intriguing fact. It is one of the most basic invariant facts of the physical world, yet we understand it only in terms of certain consistency relations between mathematical constructs and not in terms of a deeper physical aspect related to properties of the quantum spin angular momentum and interaction between particles. This is evident in a quotation from Feynman [5], "Why is it that particles with half-integer spin are Fermi particles whose amplitudes add with the minus sign, whereas particles with integral spin are Bose particles whose amplitudes add with the positive sign? We apologize for the fact that we cannot give you an elementary explanation. ..has been worked out by Pauli from
complicated arguments of quantum field theory and relativity... It appears to be one of the few places in physics where there is a rule that can be stated very simply, but for which no one has found a simple and easy explanation... This probably means that we do not have a complete understanding of the fundamental principle involved.”

While there is no physical understanding of the connection, mathematical proofs invoking field theoretic reasoning exist. The proof by Pauli used the mathematical fact that the quantization of the field of integer spin particles is associated with commutator relations and the quantization of half-integer spin particles is associated with anti-commutator relations between operators, and these coupled with requirements from Lorentz invariance provided the proof [6]. Many rigorous proofs with assumptions on properties of quantum fields, and Lorentz invariance exist, and a proof by Schwinger relied on requirement of invariance under time reversal. See ref. [7] for details with very instructive and readable commentary. There have been many attempts to provide simple proofs, and the general assessment seems to be that there is no simple proof, let alone a physical understanding of the connection. There have been some discussion on whether there is any ‘simple proof’ and whether the claimed simple proofs are physically defendable, in the American Journal of Physics, initiated by a question by D. E. Neuenschwander [8]. This discussion motivated a thorough discussion by I. Duck and E. C. G. Sudarshan on many aspects of the simple proofs of the connection, and also a valuable discussion of many aspects of both the ‘hard proofs’ and the simple proofs in their recent book, Pauli and the Spin Statistics Theorem [7].

Before proceeding further I state the spin-statistics connection:

a) Particles with integer spin are bosons and they obey the Bose-Einstein statistics.

b) Particles with half-integer spin are fermions and they obey the Fermi-Dirac statistics.

Equivalently,

a) The amplitude for a scattering event between identical integer spin particles and the amplitude with an exchange of the particles add with a plus (+) sign. In other words, the phase difference between the scattering amplitude and the exchanged amplitude is an integer multiple of $2\pi$.

b) The amplitude for a scattering event between identical half-integer spin particles and the amplitude with an exchange of the particles add with a minus (−) sign. In other words, the phase difference between the scattering amplitude and the exchanged amplitude is an odd integer multiple of $\pi$. 
A geometric understanding of these statements was published by Berry and Robbins [9], and several authors have invoked the relation between rotation operators and exchange of particles in quantum mechanics to prove the spin-statistics theorem [10]. Sudarshan has been arguing for the existence of a simple proof that is free of arguments specific to relativistic quantum field theory [7]. While these attempts have clarified several issues regarding the connection, none provides a good physical understanding of the connection.

It may be noted that physically the connection is applicable for any two identical particles, in non-relativistic quantum mechanics. Thus we should expect that the physical proof need not depend on relativistic quantum field theory. In any case, there are certain minimal expectations for a physically valid proof of the spin-statistics connection: 1) It should of course derive the correct phase factors between the two possible scattering amplitudes, direct and exchanged, from a physical interaction, or from a geometro-physical reasoning affecting only the two particles, 2) the phase difference should be independent of the specific interaction between the two particles, 3) if there is any modification to the exactness of the phase difference (zero for bosons and $\pi$ for fermions), then it should be small enough not to affect observed facts like stability of atoms, black-body spectrum etc.

Consider the scattering of two identical particles, Fig. 1.

The upper event can happen by two quantum amplitudes, shown in the lower panel. The amplitudes for these two processes differ by only a phase
for identical particles in identical states, since the two processes are indistinguishable. The particles are assumed to be spin polarized in identical directions, perpendicular the plane containing the scattering event. Only then, the initial and final states and the two processes are indistinguishable. We note the important fact that the two amplitudes have to be considered always in the gravitational field of the entire Universe. The task is to calculate the phase difference between the two amplitudes due the gravitational interaction with the entire Universe. We can calculate the phase changes in any configuration, but at present we want to discuss only the phase difference for indistinguishable states.

First we note that the two processes are different in the angle through which the momentum vector of the particles turn in the scattering process. In fact that is the only difference between the two amplitudes. The difference in angles is just π. (This is why it is equivalent to an exchange - what is really exchanged is the momentum vector after the scattering). The calculation for each particle can be done by noting that the k-vector can be considered without deflection, but the entire Universe turned through an appropriate angle, with angular velocity of this turning decided by the rate of turning of the k-vector. It does not matter whether we are dealing with massless particles or massive particles since the only physical fact used is the change in the direction of the k-vector of the particle.

Figure 2 depicts the ‘motion of the Universe’ as seen from the rotating frame at the point where the scattering takes place. The part ‘A’ corresponds to the mass currents relative to each of the scattering particle when the scattering angle is as shown in the upper diagram. The other possible amplitude then happens with the scattering angle that is larger by π, and the mass currents act for a larger duration, as in ‘B’. In any relativistic theory of gravity such mass currents generate a Lense-Thirring field, or equivalently a gravito-magnetic field. This is equal to the curl of the vector potential generated by the mass currents, in the post-Newtonian language, and equivalently the ‘curl’ of the metric coefficient for the term $dx \, dt$ in the more rigorous metrical description.

The phase change in the state of each particle is given by the product of the duration of interaction and the spin-gravitomagnetic interaction energy,

$$
\varphi = \mathbf{s} \cdot \mathbf{\Omega} (1 - V^2/c^2)^{-1/2} \times t (1 - V^2/c^2)^{1/2} = s\theta
$$

where $\theta$ is the angle through which the k-vector has turned in the scattering process. Note that this is a gravitational phase shift due to the
Figure 2: The curved paths of distant galaxies relative to the turning momentum vector at the quantum scattering event. The large mass currents generate large gravitomagnetic field which couples to spin $s$ and affects the energy and phase of the quantum state. A and B show the real physical difference of the two amplitudes relative to the massive Universe. The change in phase in a critical Universe is simply $s\theta$, where $\theta$ is the angle through which the momentum vector of each particle turns during scattering.

relativistic gravitational interaction of the quantum spin with the entire causally connected Universe, and it involves all the fundamental constants $G$, $c$ and $\hbar$. These constants are hidden from the final expression because $\sum_i G m_i / c^2 r_i = 1$ in a critical Universe. When the density is not critical, these constants will appear explicitly in the expression for this phase change. Also note that this phase difference is an unambiguous prediction from any theory of relativistic gravity in a Universe with critical density.

The momentum vectors turn in the same sense for both the particles and therefore the total phase change is $\varphi_1 = 2s\theta_1$ where $\theta_1$ is the angle through which the $k$-vector turns for the first amplitude. For the second amplitude the phase change is

$$\varphi_2 = 2s\theta_2$$  (5)

The phase difference between the two amplitudes is

$$\Delta\varphi = 2s(\theta_1 - \theta_2) = 2s \times \pi$$  (6)

Now we can see how the spin-statistics connection is related to the phase changes due to the gravitational interaction of the spin with the Universe.
For zero-spin particles the proof is trivial since there is no spin coupling to the Universe,

$$\Delta \varphi = s(\theta_1 - \theta_2) = 0 \times 2\pi = 0$$

and therefore zero-spin particles are bosons and their scattering amplitudes add with a + sign.

It is sufficient to show the connection for spin-1/2 particles and the higher spin cases can be constructed from spin-half using the Schwinger construction [7]. The only case to be treated when dealing with identical, indistinguishable states of spin is the one in which the spins are identically pointing, perpendicular to the plane containing the \( k \)-vectors. The phase angle difference between the two amplitudes then is

$$\Delta \varphi = 2s(\theta_1 - \theta_2) = 2 \times \frac{1}{2} \times \pi = \pi$$

The relative phase is \( \exp(i\pi) = -1 \). The amplitudes add with a negative sign. Therefore, half-integer spin particles obey the Pauli exclusion and the Fermi-Dirac statistics.

Two features of this proof may be noted immediately; 1) scattering of any two particles in a multiparticle system of identical fermions introduces a minus \((-\) sign between the original amplitude and the exchanged amplitude for the total system, 2) the proof is valid for interacting particles since the phase changes due to interactions are identical for the two amplitudes, as all other dynamical phases in the relevant diagrams. The only phase difference is due to the rotation of the momentum vector in the presence of the Universe.

3 Concluding remarks

The remarkable connection between quantum physics and gravity of the Universe is indeed startling. But this cosmic connection is also a natural consequence in a critical Universe in which everything is gravitationally interacting with everything else. It is satisfying to see that a deep physical phenomenon is linked to a universal physical interaction and not just to mathematical structures and consistency.

Many of the measured geometric phases on particles with spin, when they are taken around trajectories in space with their momentum vector turning in space, are of the same nature. Those geometric phases are simply given
by

\[ \varphi = \oint s\,d\theta \]  \hspace{1cm} (9)

where I mean an integral over the entire trajectory with appropriate signs. This may suggest the relation between our proof and the discussion by Berry and Robbins [9]. It also clarifies why the proofs that depend on rotational properties of wavefunctions have some success.

By identifying the physical origin as the gravitational interaction, we can now calculate the phase change for the two amplitudes for any arbitrary initial and final states, and thus we have the most general statement about the relative phase between the amplitude of any such scattering event. It might be possible to extend this physical insight further and arrive at the spin-statistics connection without specifically referring to scattering amplitudes.

In discussions of the proofs of the spin-statistics connection it is argued that the phase factor is either +1 or −1, by requiring that the system comes back to the same state, including phase, after two exchanges. One exchange of the particles converts the two amplitudes into each other. However such assumptions have been criticized with good reasons [10]. We do not use such an assumption for establishing the spin-statistics connection from the gravitational interaction, and this is a great advantage. In fact, the correct phase factors emerge naturally for each particle independently as a dynamical phase that depends on the interaction of the spin of each particle with the Universe. The calculation is applicable to any spin, and does not by itself rule out other fractional spin particles and statistics, unless an additional assumption is made about the complete equivalence, including phase, of two-particle configuration after two exchanges. So, in our case, the standard spin-statistics connection can be shown without this assumption, and with this assumption it seems possible to show that there can only be half-integer and integer spin particles in this Universe. Duck and Sudarshan note that all proofs of the spin-statistics connection are ‘negative proofs’ showing the impossibility of the “wrong” statistics. The proof relating the gravity of the Universe and the statistics is a ‘positive proof’ explicitly calculating the relevant phase differences.

Finally, there is no doubt that the cosmic connection discussed here is physically simple and transparent, and the resulting proof is simple. The issue is only whether this exclusively is the cause of the spin-statistics connection. If it is, it certainly answers satisfactorily Neuenschwander’s [8] query I referred to earlier. Whether or not this is the only physical aspect involved in
the spin-statistics connection the phase changes we discussed are unavoidable consequence of quantum dynamics in this massive critical Universe. Therefore, they surely have a role to play in the phase differences of the scattering amplitudes we discussed, and hence in the spin-statistics connection.

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