Scaling and finite-size-scaling in the two dimensional random-coupling Ising ferromagnet

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1. INTRODUCTION

Random disorder plays important role in many interesting phenomena of condensed matter physics. Some examples are Kondo problem and metal insulator transition driven by random disorder. The two-dimensional (2D) randomly disordered Ising ferromagnet is the simplest nontrivial statistical model including the effect of another type of fluctuation in addition to usual thermal fluctuation. By the random disorder is meant either a random site dilution or random-valued positive coupling. The effect of the combined fluctuations of the thermal and quenched random disorder on the critical behavior of the system has been the main subject of the studies. The two dimensional random coupling (or random bond) Ising ferromagnet is defined by the Hamiltonian

\[ H = - \sum_{\langle ij \rangle} J_{ij} S_i S_j, \quad J_{ij} > 0, \quad S_i = \pm 1, \]

where the sum is over all the links of a lattice. To be more specific, for every link on a given lattice one assigns randomly to \( J_{ij} \) one of two different ferromagnetic coupling constants – one that of the pure system \( J \) and the other perturbed coupling \( J' \). In this way single realization of random distribution of coupling is made. Due to its random nature, obviously, the realization of the disorder is uncorrelated. We are interested in the thermodynamic behavior averaged over infinitely many different realizations of the distribution of the coupling constants.

McCoy and Wu\(^2\) made the first successful attempt for such a model, and proved that the specific heat \( C_v \) is non-divergent in the 2D Ising system with one-directional and correlated random bond disorder. According to McCoy and Wu, the divergence of \( C_v \) is caused by the coherence of all the bonds acting together, so destroying the coherence by introducing one or two directional bond disorders makes \( C_v \) finite.

Many authors\(^3\), however, regarded the non-diverging behavior of \( C_v \) in the McCoy-Wu model as a characteristic of the one-dimensional correlated disorder. Especially, Shalaev, Shankar, and Ludwig (SSL)\(^4\) based on their observation that the continuum limit of the 2D random bond disordered Ising model is a certain type of Gross-Neveu model were able to obtain analytical prediction on the critical behavior of the disordered system. In this theory the disorder is regarded as a weak perturbation to non-interacting free fermion (that is equivalent to the 2D pure Ising system), so that the theoretical prediction is supposed to hold for weak random disorder. The main theoretical prediction can be summerized as:

\[ \xi \sim t^{-\nu} |\ln t|^\varphi, \quad \nu = 1, \quad \varphi = 1/2 \]

\[ C_v \sim t^{-\alpha} |\ln t|, \quad \alpha = 0. \]

\[ \chi \sim \xi^{2-\eta}, \quad \eta = 1/4, \]

where \( t \) is the reduced temperature and \( \chi \) is thermodynamic magnetic susceptibility.

Note that in the context of SSL’s theory the critical behavior is modified by the disorder only logarithmically, with no difference in the values of the critical exponents from those of the corresponding pure system. In the presence of extremely weak disorder the system should be almost identical to the pure system, and the corresponding critical behavior must be almost indistinguishable from that of the pure system. This means that the modification of the critical behavior in this case must take place in extremely narrow scaling regime near criticality only, with the remaining scaling regime exhibiting the critical behavior of the pure system. The conventional belief is that Eqs. (2)–Eq. (4) remains to be valid for strongly disordered cases—only the range of scaling region where they hold becomes broader with the disorder. Hence the theory generally implies the presence of crossover from the critical behavior of the pure Ising system to the asymptotic behavior Eqs. (2)–Eq. (4).

The expressions reflecting the presence of the crossover thus read

\[ \xi \sim t^{-\nu} |\ln t|^\varphi, \quad \nu = 1, \quad \varphi = 1/2 \]

\[ C_v \sim t^{-\alpha} |\ln t| + C', \]

\[ \chi \sim \xi^{2-\eta}, \quad \eta = 1/4 \]
The coefficients $C$ and $C'$ as a function of the strength of disorder cannot be determined theoretically. Nevertheless, obviously, $C$ increases with the strength of disorder, and for a given value of $C$, Eq. (5) reduces to its asymptotic form only when $t$ is very small.

There is a more general argument, originally due to Harris, and later elaborated by semi-rigorous approaches, on the effect of random quenched disorder to critical behavior. The argument can be formulated in the context of the renormalization group theory, and it was obtained that weak, uncorrelated random disorder is irrelevant if

$$D\nu > 2,$$

(8)

where $\nu$ is the critical exponent of the correlation length of the pure system. In our case $D=2$, $\nu = 1$, the random disorder has been regarded to be marginal leading to a logarithmic correction to power-law critical singularity, being consistent with the prediction of SSL.

However, especially in 2D, a presence of marginal perturbation in certain case such as in Kosterlitz-Thouless type phase transition leads to continuously varying critical exponents. A rigorously known example of continuously varying critical exponents of $\gamma$ and $\nu$, but with the ratio $\gamma/\nu$ remains unchanged from the value of the pure Ising ferromagnet, takes place in some subspace of the coupling constants of the 2D Ashkin-Teller model. In this subspace, the model is equivalent to a certain annealed random coupling Ising ferromagnet. Also, many authors obtained qualitatively different expressions for the critical behavior of $C_v$ from that given by SSL. According to these expressions, $C_v$ is non-diverging at criticality as is the case in the McCoy-Weiss model.

It is now numerically established[8,19] that the value of $\eta$ does not depend on the strength of disorder in both 2D random-coupled and site diluted Ising ferromagnets. Another generic feature obtained from various numerical studies[8,19] is that other critical exponents such as $\gamma$ and $\nu$ increase with the strength of disorder at least effectively. In most cases this effective increase was interpreted as the multiplicative logarithmic correction. Whether these effectively varying critical exponents are due to crossover effect as claimed in Ref.[8,12–16] or are indication of genuine novel critical behavior induced by random disorder as claimed in Ref.[10,17] has been a controversial issue. Numerically it is very difficult to distinguish a pure power-law criticality with small critical exponent from a logarithmic singularity, unless thermodynamic data in extremely deep scaling regime are available. Previously claimed evidence for the predictions of SSL in most cases were nothing but the consistency with the logarithmic correction. This claimed consistency cannot rule out pure power-law singularity without the multiplicative logarithmic correction. Especially it was observed that the thermodynamic values of the specific heat develops non-diverging peak at a temperature larger than $\beta_c$, for very strongly disordered random site diluted Ising model. This contrasts clearly with Eq. (3), but agrees with other theoretical prediction[10,17].

In this paper we attempt to clarify the unresolved issue of the 2D random bond disordered Ising ferromagnet by numerically determining the functional form of the universal FSS function defined in the context of novel FSS scheme. As long as the FSS is valid, the form of the FSS function is unique for a given universality class. Accordingly, if the system is indeed in the same universality class irrespective of the strength of disorder then the form of each finite size scaling function must be identical for different strength of random disorder. We will also demonstrate that the thermodynamic data of $\xi$ in sufficiently deep scaling regime tend to scale steeper than the asymptotic expression of SSL, Eq. (2). In the sections to follow we elaborate on the FSS method employed here, give a detailed description of our Monte Carlo simulation, report our results, and finally conclude with some discussions.

II. GENERALIZED FINITE SIZE SCALING

The fundamental assumption of FSS theory[2] is that $A_L(t)$, the value of some physical variable $A$ on a finite lattice of linear size $L$, can be expressed as

$$A_L(t) = L^{\rho/\nu} f_A(s(L,t)), \quad s(L,t) \equiv L/\xi(t)$$

(9)

for a thermodynamic quantity $A$ which has a power-law critical singularity $A(t) \sim t^{-\rho}$ with the reduced temperature $t \equiv (\beta_c - \beta)/\beta_c$. Eq. (9) is valid for the values of $L$ and $\xi(t)$ that are sufficiently large; otherwise, there should be corrections to FSS.

Notice that using the critical form for $\xi$, $\xi(t) \sim t^{-\nu}$, one can rewrite the scaling variable $s(L,t)$ as

$$s(L,t) = (A(t)/L^{\rho/\nu})^{-\nu/\rho},$$

(10)

so that Eq. (9) may be rewritten as

$$A_L(t) = A(t) F_A(s(L,t)),$$

(11)

where the relation between the scaling functions $f_A$ and $F_A$ is given by

$$F_A(s) = s^{\rho/\nu} f_A(s).$$

(12)

For $A = \xi$, Eq. (11) shows that $\xi_L(t)/L$ is just a function of $\xi(t)/L$ and vice versa, and this leads to the relation

$$A_L(t) = A(t) Q_A(x(L,t)),$$

(13)

where $x(L,t) \equiv \xi_L(t)/L$ is the ratio of the correlation length on a finite lattice to the linear size of the lattice, and $Q_A(x)$ is given by

$$Q_A(x) = F_A(f_Q^{-1}(x)).$$

(14)

Using the same observation, it is trivial to obtain another equivalent form,
where $b$ is a scaling factor and $\mathcal{G}_A(x)$ is another scaling function.

It is evident that given $f_A$ one can determine the other two scaling functions from Eqs. (12) and (13). It is well-known that the standard scaling function $f_A$ is universal, so that all the other scaling functions, $\mathcal{F}_A$, $\mathcal{Q}_A$, and $\mathcal{G}_A$ should be universal as well. It has also been argued that a certain asymptotic form of $f_A(s)$ can be expressed in terms of the critical exponent $\delta$; by fitting this functional form one can extract an estimate for the critical exponent.

It is worth stressing that use of the scaling function $\mathcal{Q}$ rather than $\mathcal{F}$ or $f_A$ would be more convenient in many cases, particularly because one does not need the thermodynamic correlation length to define the former. Note that there is no explicit $t$ dependence of the scaling variables, so that knowledge of the critical temperature and $\nu$ is not required, and that $x$ becomes independent of $L$ at criticality. This $L$ independent value of $x$ at criticality, $x_c$, which characterizes a universality class for a given geometry, forms the upper bound of $x$. In other words, the scaling function $\mathcal{Q}$ is defined only over $0 \leq x \leq x_c$. A priori, the two limits of the scaling function $\mathcal{Q}$ are known for a continuous phase transition: $\lim_{x \to 0} \mathcal{Q}(x) \to 1$ and $\lim_{x \to x_c} \mathcal{Q} \to 0$, because $A_L$ converges to its thermodynamic value in the former case while $A(t)$ diverges in the latter case with $A_L(t)$ finite. $\mathcal{Q}_A(x)$ turns out to be a monotonically decreasing function of $x$ for $A = \chi$ and $\xi$.

It is important to realize that the knowledge of the scaling function $\mathcal{Q}$ near $x \approx 0$ plays as relevant role as that near $x \approx x_c$ to the extraction of necessary information of the critical behavior in (deep) scaling region. It can be easily seen by noting that $x(L,t)$ for a fixed temperature arbitrarily close to criticality can be made arbitrarily close to zero by simply choosing a value of $L$ sufficiently large.

Eqs. (11) and (13) do not include any critical exponents, so that one might conjecture that their validity can be extended to non-power-law singularities. Although a general proof of this conjecture is missing, for the two-dimensional ($2D$) $O(N)$ ($N > 2$) spin model which exhibits an exponential critical singularity, Lüscher [8] obtained an explicit expression for the inverse correlation length (mass gap),

$$\xi_L(t) \sim \xi_\infty(t) \left[ e^{-4\pi F(x)/x^2} \times (1 + O(\log L/L^2)) \right]^{1/2}$$

(16)

Eq. (16) is reduced to Eq. (13) for the sufficiently large values of $L$ as for $O(\log L/L^2)$ to be safely neglected. Note that $O(\log L/L^2)$ is comparable to statistical errors of typical Monte Carlo simulations at $L=20$ already. It is thus postulated [8] that Eq. (13) is a generalized FSS form that is valid irrespective of the functional form of critical singularity.

Many novel applications of FSS were obtained thanks to Eq. (13). The key observation is that the scaling variable $x$ has no explicit temperature dependence. First, it can be used to check the validity of FSS itself for a given physical system: one numerically calculates the scaling function $\mathcal{Q}_\xi(x)$ at two arbitrary temperatures close to criticality. If the two scaling function thus calculated “collapse” unto a single function then Eq. (13) must be valid. The demonstration of the validity of FSS for the $2D$ and $3D$ pure Ising models in this context can be found in Ref. (19). The numerical determination of the $\mathcal{Q}_\xi$ were carried out for a variety of other statistical models such as $2D$ Heisenberg and XY models. This check is much more unambiguous than the standard data collapse method of FSS on the model since it does not involve any adjustable parameters such as the critical temperature and the critical exponent of the correlation length $\nu$. Second, $\mathcal{Q}_\xi$ can be used to extrapolate infinite volume limit (thermodynamic) values of various physical variables as first demonstrated in Ref. (18).

III. THE SIMULATION

We consider binary distribution of $J_{ij}$. Namely the value of $J_{ij}$ at a link $< ij >$ is randomly distributed between two positive values $J$ and $J'$ with probability $p$ and $1-p$ respectively. For $p = 1/2$, the system is self-dual [24] with the self-dual point given by

$$\tanh(J) = \exp(-2J')$$

(17)

A self-dual point equals the critical point of a system, provided the system has only one critical point. We fix $J = 1$ and $p = 1/2$ without any loss of generality, and consider three different values of $J'$, i.e., $J' = 0.9$, 0.25, and 0.1. Accordingly the self-dual points (critical points) are given by $\beta = 0.4642819 \ldots$, 0.80705185\ldots, and 1.10389523 \ldots for $J' = 0.9$, 0.25, and 0.1 respectively.

Our raw-data for each $J'$ are obtained by choosing a realization of random $J'$, then running Monte Carlo simulations in single cluster algorithm with periodic boundary conditions; for each realization, measurements were taken over 10 000 configurations each of which was separated by 2-15 single cluster updatings according to auto-correlation times. The procedure is then repeated for different realizations of $J'$. The average over all the different realizations converges as the numbers of the random realization increase; basically this mean value of a physical quantity is something physically interesting. To achieve the necessary precision for our FSS scheme the numbers of the different realizations we used are approximately 20-40, 150-250, and 300-1000 for $J' = 0.9$, 0.25, and 0.1 respectively; yet, in general, the fluctuation among different realizations of the random disorder is more significant than the statistical error for a given realization. This was particularly the case for $J' = 0.1$. Nevertheless, the average over different realizations obviously converges to a
physically meaningful value. Our quoted error bars in our numerics are obtained by standard jack-knife method, taking into consideration the variation with different realization only. The smallest and largest values of L we used to extract our thermodynamic values are 20 and 400 respectively. For J′ = 0.05 it turns out that data of very large ξ are not necessary, yielding compelling results with the data for ξ ≲ 50 already. Thus the FSS extrapolation method is not used for this J′ and 250-500 different realizations of distribution of J′ turns out to be sufficient.

Determination of A∞ and the size dependence of A upon L, A_L, is essential to the computation of Q_A. The measurements of the correlation length must be taken into consideration for the computation of the scaling variable x. The correlation lengths are measured employing the standard formula of second moment correlation length [4]. For the simplicity of our presentation, here we mainly focus on the correlation length. For each J′, we chose three different inverse temperatures for the computations of the scaling variable and scaling function. It is important to select the values of the temperatures so that they are in the scaling regime. The criterion for this, which is known from the studies of well-known pure systems such as 2D Ising and three-state Potts models, is that the thermodynamic value of the correlation length at a temperature in the scaling regime should be sufficiently large, namely ξ ≳ 5.

IV. RESULT AND ANALYSIS

The Q_ξ(x) are calculated for J′ = 0.9, 0.25, and 0.1. For an illustration, the size dependence of the ξ and χ, and the results of the computations of x, Q_ξ(x), and Q_χ(x) obtained for J′ = 0.25 and β = 0.77 and for J′ = 0.1 and β = 1.04 are tabulated in Table [1]. It is worth noticing that both ξ_L and ξ_L are monotonically increasing function of L. (This is the case even in terms of the scaling variable s.) The L dependence becomes weaker with increasing L, and becomes vanishingly small for sufficiently large L. This L independent value within the statistical errors is the corresponding thermodynamic value. In Table [1] it is shown that one needs L/ξ_L ≥ 10 for the accurate measurements of the thermodynamic values. The value of the ratio (L to ξ_L) is larger than in the corresponding pure system that is approximately 10 [4]. It is worth stressing that the value of the ratio beyond which a measurement becomes thermodynamic is temperature independent due to Eq. [13]. One thus needs to simulate at least with L larger than 10 times of the correlation length to make the data be free of finite size effect.

The plots of Q_ξ(x) for the three values of β in the scaling regime are shown in Fig. [4] for J′ = 0.25. Clearly each data set belonging to different β collapses onto a single curve that is the universal FSS function for the value of J′. Thus the validity of the FSS is verified for the J′. We repeated similar procedure for the other two values of J′ and observed similar data-collapse. For comparison, the FSS function for each J′ is shown in our Fig. [3]. It is observed that the FSS function for J′ = 0.9 is indistinguishable from that of the corresponding pure system that was calculated in Ref. [13]. However, for stronger disorder the FSS scaling function clearly depends on J′.

The ansatz for our scaling function is

\[ \bar{Q}_\xi(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4. \]  (18)

The values of the coefficients c_i (i = 1, ..., 4) are calculated by fitting our data to the ansatz for each value of J′.

The thermodynamic correlation lengths are measured directly (without using any extrapolation method) for the data roughly over the range ξ ≲ 40. With the knowledge of the scaling function available, the thermodynamic values closer to criticality can now be easily estimated by the use of the single-step FSS extrapolation method [3]. For each value of J′ the thermodynamic values of the correlation length are evaluated over the ranges 5.7(1) ≤ ξ ≤ 204(2) ([5.7(1), 204(2)], [5.8(1), 217(3)], and [5.0(1), 203(5)]. For the J′ = 0.05 the thermodynamic data were obtained by direct measurement under the thermodynamic condition L/ξ_L ≥ 12, which is over the range 5.52(2) ≤ ξ ≤ 47.64(29). Our thermodynamic data are summerized in Table [1].

ln(ξ(t)) as a function of ln|t| is plotted in Fig. [3]. The slope of each straight line corresponds to the value of ν. It is evident that ν increases with decreasing J′ at least effectively. Fixing the critical points at the self-dual points in the χ^2 fits and assuming a pure power-law type critical behavior, we obtain ν = 1.01(1), 1.10(2), 1.20(3), 1.34(6) for J′ = 0.9, 0.25, 0.1, and 0.05 respectively. Assuming a scaling function with a nonconfluent correction term, e.g., \( ξ(t) \sim t^{-ν}(1+at) \), yields the estimate of the critical exponent, e.g., ν = 1.08(4) and ν = 1.17(5) for J′ = 0.25 and J′ = 0.1 respectively. Notice that for J′ = 0.9 the estimated value of ν is virtually the same as that in the pure system.

Although our thermodynamic data are extraordinarily broad compared to standard Monte Carlo or series expansion study can usually make available, it turns out that the data fit to a pure power-law as well as to the form with the multiplicative logarithmic correction. This can be easily seen by simple numerical experiment as well. Namely, generate some numbers according to a certain power-law with the value of ν slightly larger than 1, and fit them to a multiplicative correction of the form Eq. [3]. One can always find acceptable values of C and ˜ν over quite broad ranges, making it prohibitly difficult to get accurate estimate of the logarithmic exponent. With the use of the asymptotic form Eq. [3], one can get much shaper estimate of ˜ν than with Eq. [3]. The problem is that one does not know a priori whether all the data are beyond the crossover point. Nevertheless, we find that use of the asymptotic form rather than Eq. [3] is quite useful for our purpose.
A useful observation is that \( [1 + C|\ln(t)|]^\nu \) \((\nu > 0)\) is less singular than \(|\ln t|^\nu\) for any range of \( t > 0 \). Namely, note that
\[
(1 + C|\ln t|)^\nu \sim |\ln t|^\nu \left( C + 1/|\ln t|\right)^\nu, \tag{19}
\]
and that the second term on the right hand side of Eq.\( \text{(19)} \) suppresses the singularity for any physical value of \( t \). Thus the asymptotic singularity Eq.\( \text{(2)} \) is always more singular than the mixture of the singularities, Eq.\( \text{(3)} \). Accordingly, if a thermodynamic correlation data turn out to be more singular than the asymptotic singularity in an arbitrary portion of the scaling regime, then the prediction of SSL must be invalidated.

In Fig.\( \text{(4)} \) we plotted \( \xi(t)/(t^{-1}\ln t)^{0.5} \) for \( J' = 0.25, 0.1, \) and 0.05. For the \( J' = 0.25 \), it is observed that the value of the ratio decreases monotonically until temperature is very close to criticality, but starts to increase with further approaching to the criticality. This is surprising in view of the SSL’s picture, because the figure shows none of the data are either in the asymptotic regime or in the scaling regime of the pure system. For the \( J' = 0.1 \), we find that the data less close to the criticality are consistent with the prediction of SSL, but start to deviate from it as \( t \to 0 \). For the \( J' = 0.05 \), we observe that the data scale faster than the asymptotic form for all our data. We thus lead to the picture that seemingly consistency with the logarithmic correction for weak disorder starts to become invalidated in sufficiently deep scaling regime. For the very strong disorder, any portion of data are inconsistent with the scaling form.

Binder’s cumulant ratio at criticality, denoted by \( U_L^{(4)} \), is another universal quantity.\[4-6\] For each \( J' \) we measured it at the critical point with varying \( L \) (Table\( \text{III} \)). It is observed that given \( J' U_L^{(4)} \) is invariant with \( L \) within the statistical errors, and that it tend to increase uniformly with decreasing \( J' \). The value for \( J' = 0.9 \) is indistinguishable from the pure case, as is the case for the scaling function \( Q_\xi \).

V. CONCLUSION AND DISCUSSION

We have obtained clean numerical evidence that FSS holds for quenched random disordered Ising ferromagnet. The universal FSS function is found to be dependent upon the strength of disorder for strongly disordered cases. For very weak disorder, however, it appears that disorder does not induce new universality class to the resolution of our data. The behavior of Binder’s cumulant ratio is in complete agreement with the feature obtained from the FSS scaling function. We also show that the singularity of \( \xi \) is steeper than the theoretical prediction for the data sufficiently close to criticality, which almost certainly rules out the validity of SSL in the strong disordered system. Our result of varying exponent \( \nu \) combined with the established fact of the invariance of \( \gamma/\nu \) supports the scenario of weak universality.\[2\] The same numerical evidence was obtained for the random-coupling three-state Potts ferromagnet as well.\[2\]

As mentioned in the introduction most of previous numerical studies claiming for the evidence for the SSL came from the observation of the consistency of the logarithmic correction. The consistency, however, cannot disprove the pure power-law singularity that is essentially more steeper than the logarithmic correction. A very strong claim for the evidence for the SSL was made in a recent high temperature expansion study of the same model.\[7\] What the authors of the paper observe is the monotonic increase of the effective value of \( \gamma \) with the strength of disorder. On the other hand, they claimed that the same data fitted to the prediction of SSL, Eq.\( \text{(3)} \), give rise to the value of the logarithmic exponent independent of the disorder. This is unlikely to be mathematically correct because the effective increase of \( \gamma \) leads to the effective increase of the logarithmic exponent which can be easily checked by simple numerical experiment.

The study of the FSS behavior of the specific heat, instead of its thermodynamic behavior, should also be treated with caution. In Ref.\( \text{(4)} \) it was shown that the size dependence of the specific heat is different from that of the correlation length or the magnetic susceptibility. Namely, the specific heat increases with \( s \equiv L/\xi(t) \) for smaller \( s \) but starts to decreases with it for larger \( s \). In view of FSS, Eq.\( \text{(1)} \), what this implies is the existence of the upper bound value of the scaling variable \( s \) below which the specific heat increases with \( L \). Thus, for any value of \( L \), one is bound to observe monotonic increase of the specific heat at criticality where \( \xi \) diverges (in other words, \( s \) is smaller than the upper bound value for any finite \( L \) when \( \xi \to \infty \)), and this has nothing to do with the actual divergence of the specific heat at criticality. From the study of the FSS behavior of the pure system,\[4, 14, 15 \] we also note that FSS of the specific heat gives rise to less accurate result than that of \( \xi \) or \( \chi \).
TABLE I. Size dependence of various physical quantities and the computed scaling variable and scaling functions at $\beta = 0.77$ with $J' = 0.25$ (the upper part) and at $\beta = 1.04$ with $J' = 0.1$ (the lower part). The L independent value (within the statistical errors) is the corresponding thermodynamic value, which are $\xi = 18.9(2)$ and $\xi = 25.4(3)$ respectively for $\beta = 0.77$ and 1.04.

| L   | $\xi_L$ | $\chi_L$ | $x$  | $Q_\xi$ | $Q_\chi$ |
|-----|---------|----------|------|---------|----------|
| 20  | 11.51(9)| 142.0(8) | 0.576(5)| 0.609(11)| 0.311(5) |
| 24  | 12.69(9)| 178.0(1.2)| 0.529(4)| 0.671(12)| 0.389(4) |
| 30  | 14.21(12)| 231.1(1.7)| 0.474(4)| 0.752(14)| 0.506(6) |
| 34  | 15.02(10)| 262.4(1.7)| 0.442(3)| 0.794(14)| 0.574(6) |
| 40  | 15.77(8)| 301.5(1.8)| 0.394(2)| 0.834(13)| 0.660(7) |
| 50  | 16.94(10)| 356.3(2.9)| 0.339(2)| 0.896(15)| 0.780(10) |
| 60  | 17.42(10)| 390.1(3.3)| 0.290(2)| 0.922(15)| 0.854(11) |
| 70  | 17.78(13)| 408.6(4.0)| 0.254(2)| 0.941(17)| 0.894(13) |
| 80  | 18.20(18)| 428.4(5.6)| 0.228(2)| 0.963(20)| 0.937(16) |
| 150 | 18.71(23)| 456.0(5.0)| 0.125(1)| 0.992(18)| 0.998(15) |
| 200 | 18.86(11)| 456.3(2.5)| 0.094(1)| 0.998(16)| 0.998(10) |
| 250 | 18.85(17)| 456.2(3.8)| 0.075(1)| 0.997(21)| 0.998(14) |
| 20  | 12.61(9)| 154.0(8)| 0.630(5)| 0.498(7)| 0.198(2) |
| 30  | 16.17(14)| 267.1(2.1)| 0.539(5)| 0.639(10)| 0.344(5) |
| 50  | 20.41(23)| 467.7(5.8)| 0.408(5)| 0.807(15)| 0.603(11) |
| 80  | 23.01(19)| 647.1(7.2)| 0.288(2)| 0.909(15)| 0.834(15) |
| 120 | 24.16(17)| 720.1(7.5)| 0.201(1)| 0.955(14)| 0.928(16) |
| 160 | 24.55(15)| 746.5(6.4)| 0.153(1)| 0.970(14)| 0.962(14) |
| 200 | 24.86(14)| 759.0(5.9)| 0.124(1)| 0.983(13)| 0.978(14) |
| 240 | 25.17(14)| 777.7(4.8)| 0.105(1)| 0.995(13)| 1.002(13) |
| 300 | 25.35(16)| 776.2(4.5)| 0.084(1)| 1.002(14)| 1.000(12) |
| 360 | 25.29(20)| 775.3(7.8)| 0.070(1)| 1.000(19)| 0.999(16) |
TABLE II. Selection of thermodynamic $\xi$ for $J' = 0.9$, 0.25, 0.1 and 0.05. For the $J' = 0.05$ all the results are obtained by direct measurements (without the use of the FSS extrapolation method).

| $J'$ = 0.9 | $\beta$ | 0.42 | 0.44 | 0.45 | 0.455 | 0.46 | 0.462 | 0.463 |
|------------|--------|------|------|------|-------|------|-------|------|
| $\xi$      |        | 5.76(5) | 10.7(1) | 18.4(1) | 28.3(2) | 61.9(3) | 116(1) | 204(2) |
| $J'$ = 0.25 | $\beta$ | 0.70 | 0.74 | 0.77 | 0.78 | 0.79 | 0.80 | 0.803 |
| $\xi$      |        | 5.80(5) | 9.72(7) | 18.9(1) | 26.7(2) | 43.6(2) | 116(1) | 217(3) |
| $J'$ = 0.1  | $\beta$ | 0.87 | 0.92 | 1.0 | 1.04 | 1.06 | 1.08 | 1.093 |
| $\xi$      |        | 5.05(7) | 6.95(8) | 14.20(10) | 25.4(3) | 39.4(4) | 80.5(1.4) | 203(5) |
| $J'$ = 0.05 | $\beta$ | 1.00 | 1.10 | 1.20 | 1.25 | 1.28 | - | - |
| $\xi$      |        | 5.52(2) | 9.15(9) | 18.40(11) | 30.75(18) | 47.64(29) | - | - |
TABLE III. Binder’s cumulant ratio at the self-dual points, for the three values of $J'$. Note that $U_L(t = 0)$ for each $J'$ does not vary with $L$ within the statistical errors, thus showing that each self-dual point is indeed the critical point. Also, it is clear that $U_L(t = 0)$ increases with decreasing $J'$, although for $J' = 0.9$ it is hardly distinguishable from the value of the pure system. For $J' = 0.25$ we extended the measurements up to $L=400$, which does not show any sign of crossover.

| L   | $J' = 1.00$ | $J' = 0.90$ | $J' = 0.25$ | $J' = 0.10$ |
|-----|-------------|-------------|-------------|-------------|
| 20  | 1.8324(6)   | 1.834(1)    | 1.850(3)    | 1.862(3)    |
| 40  | 1.8321(6)   | 1.833(2)    | 1.846(3)    | 1.858(3)    |
| 60  | 1.8317(5)   | 1.832(1)    | 1.852(3)    | 1.854(4)    |
| 80  | 1.8318(5)   | 1.833(1)    | 1.847(3)    | 1.862(4)    |
| 100 | 1.8316(6)   | 1.832(2)    | 1.849(3)    | 1.858(4)    |
| 200 |             |             | 1.844(3)    |             |
| 400 |             |             |             |             |

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We generated numerical data from computer assuming pure power-law singularity $\chi \sim t^{-\gamma}$ with three arbitrary $\gamma = 2.1, 2.3, \text{ and } 2.5$ (corresponding to $J^2/J^1 \simeq 5, 8 \text{ and } 10$ in their paper), with the range of $t$ over $0.01 \leq t \leq 0.05$ and with relative errors one percent. It is found that the data are fitted to the asymptotic form of SSL, Eq.4 to yield the estimates of the logarithmic exponent 1.31, 2.06, and 2.44 respectively. The estimates of the logarithmic exponent depend on the choice of the range of $t$, being larger for $t$ smaller. Nevertheless, the feature of its monotonic increase with decreasing $J'$ is invariant with the choice of the range of $t$.

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FIG. 1. Numerical calculation of $Q_\xi$ for $J' = 0.25$, at three arbitrary inverse temperature $\beta = 0.74, 0.77, \text{ and } 0.79$ in the scaling regime. Our figure clearly indicates that the scaling function does not have explicit temperature dependence. For all the values of $\beta$, the smallest value of $L$ used is 20.

FIG. 2. The $Q_\xi(x)$ for the three values of $J'$. The dependence of the scaling function on $J'$ is obvious.

FIG. 3. $\ln \xi$ versus $|\ln t|$. The dotted lines represent the results of the best $\chi^2$ fits assuming pure power-law type singularity. The values of the slope, which correspond to the values of $\nu$, are 1.00, 1.10, 1.20, and 1.34 respectively for $J' = 0.90, 0.25, 0.10, \text{ and } 0.05$.

FIG. 4. The ratio $\xi(t)/(t^{-1}|\ln t|^{1/2})$ for $J' = 0.25, 0.1, \text{ and } 0.05$. The data for $J' = 0.1 \text{ and } 0.05$ are uniformly shifted so that the difference in the data points are clearly visible in a single figure. Here increasing value of the ratio as $t$ becomes smaller is an evidence that the data over the regime are inconsistent with the prediction of SSL. The tendency of the increasing ratio with $t$ becoming sufficiently small is observed for $J' = 0.25 \text{ and } 0.1$, showing that the seemingly consistency with the logarithmic correction of SSL for the larger values of $t$ becomes invalidated in sufficiently deep scaling regime. For the $J' = 0.05$, all the data scales steeper than the asymptotic scaling, showing that SSL cannot be correct for any data in the scaling regime for this $J'$. 

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Fig. 1

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\( \beta = 0.74 \)
\( \beta = 0.77 \)
\( \beta = 0.79 \)
Fig. 4

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\[ \xi(t)/(t^{-1} \ln t)^{0.5} \]

\( J = 0.25 \)
\( J = 0.1 \)
\( J = 0.05 \)