Frauchiger-Renner argument and quantum histories

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Abstract

In this article we reconstruct the Frauchiger and Renner argument, taking into account that the assertions of the argument are made at different times. To do this, we use a formalism of quantum histories, namely the Theory of Consistent Histories. We show that the supposedly contradictory conclusion of the argument requires computing probabilities in a family of histories that does not satisfy the consistency condition, i.e., an invalid family of histories for the theory.
I. INTRODUCTION

In April 2016, Frauchiger and Renner published an article online in which they introduced a *Gedankenexperiment* that led them to conclude that “no single-world interpretation can be logically consistent” [1]. In a new version of the paper, the authors moderated their original claim, concluding “that quantum theory cannot be extrapolated to complex systems, at least not in a straightforward manner” [2].

Since its first online publication, the Frauchiger and Renner (F-R) argument was extensively commented upon in the field of quantum foundations, since it was considered as a new no-go result for quantum mechanics whose strength relies on the fact that it is neutral regarding interpretation: on the basis of three seemingly reasonable assumptions that do not include interpretive premises, the argument leads to a contradiction. This fact has been conceived as pointing to a deep shortcoming of quantum mechanics itself, which contrasts with the extraordinary success of the theory.

In a previous article [3] a careful reconstruction of the F-R argument has been offered, which shows that the contradiction resulting from the F-R argument is inferred by making classical conjunctions between different and incompatible contexts, and, as a consequence, it is the result of a theoretically illegitimate inference. However, recently it has been suggested that the criticism does not take into account the fact that the inferences in the F-R argument are all carefully timed, and that this fact would circumvent the objection based on the contextuality of quantum mechanics. The purpose of this article is to analyze such a defense of the F-R argument.

If timing really matters in the F-R argument, it seems natural to reconstruct the argument using a formalism of *quantum histories*, which allows us to define logical operations between quantum properties at different times. The idea of quantum histories was mainly motivated by this limitation of quantum mechanics. In 1984, Robert Griffiths presented the first version of his Theory of Consistent Histories [4]; some years later, he introduced some modifications to that original version [5, 6]. Roland Omnès [7–11] also published a series of articles that contributed to the development of this theory. Simultaneously, Murray Gell-Mann and James Hartle developed a similar formalism [12, 14]. The Theory of Consistent Histories

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1 We thank Jeffrey Bub for pointing out this recent debate to us.
extends the formalism of quantum mechanics. It introduces the notion of history, which generalizes the notion of event: an elemental history is defined as a sequence of events at different times, where an event is the occurrence of a property. But since it is not possible to assign probabilities to the set of all histories, it is necessary to select a subset of histories that satisfies additional conditions.

In order to analyze the defense of the F-R argument on the basis of the fact that the assertions are made at different times, we will carefully reconstruct the argument in the framework of the Theory of Consistent Histories. This task will allow us to prove that the supposedly contradictory conclusion of the argument requires computing probabilities in a family of histories that is not consistent, i.e., an invalid family of histories for the theory.

II. THE F-R ARGUMENT

The Gedankenexperiment proposed in Frauchiger and Renner’s article is a sophisticated reformulation of Wigner’s friend experiment [15]. In that original thought experiment, Wigner considers the superposition state of a particle in a closed laboratory where his friend is confined. When Wigner’s friend measures the particle, the state collapses to one of its components. However, from the outside of the laboratory, Wigner still assigns a superposition state to the whole composite system: Particle + Friend + Laboratory.

The F-R argument relies on duplicating Wigner’s setup (Fig. 1). Let us consider two friends $F_1$ and $F_2$ located in separate and isolated laboratories $L_1$ and $L_2$. $F_1$ measures the observable $C$ of a biased “quantum coin” in the state $|\phi\rangle = \frac{1}{\sqrt{3}}|h\rangle + \sqrt{\frac{2}{3}}|t\rangle$, where $|h\rangle$ and $|t\rangle$ are the eigenstates of $C$, and $h$ and $t$ are its respective eigenvalues. $F_1$ prepares a qubit in the state $|\downarrow\rangle$ if the outcome is $h$, or in the state $|\rightarrow\rangle = \frac{|h\rangle + |\downarrow\rangle}{\sqrt{2}}$ if the outcome is $t$, and sends it to $F_2$. When $F_2$ receives the qubit, she measures its observable $S_z$. After these two measurements, the state of the whole system composed of the two laboratories is:

$$|\Psi\rangle = \frac{1}{\sqrt{3}}|H\rangle|\downarrow\rangle + \sqrt{\frac{2}{3}}|T\rangle|\Rightarrow\rangle,$$

(1)

where we have the following:

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2 We slightly modify the original terminology for clarity.
FIG. 1: Illustration of the Gedankenexperiment. Friend $F_1$ tosses a coin and measures its result. Depending on the outcome, she sends a qubit in a particular state. Then, Friend $F_2$ measures the spin of the qubit in the $z$ direction, obtaining $z = +\frac{1}{2}$ or $-\frac{1}{2}$. Finally, observers $W_1$ and $W_2$ measure the entire laboratories $L_1$ and $L_2$ obtaining outcomes $\text{fail}_X$ or $\text{ok}_X$ and $\text{fail}_Y$ or $\text{ok}_Y$, respectively.

- $|H\rangle$ and $|T\rangle$, eigenstates of an observable $A$ with eigenvalues $H$ and $T$, are the states of the entire laboratory $L_1$ when the outcome of $F_1$’s measurement is $h$ and $t$, respectively.

- $|\uparrow\rangle$ and $|\downarrow\rangle$, eigenstates of an observable $B$ with eigenvalues $\uparrow$ and $\downarrow$, are the states of the entire laboratory $L_2$ when the outcome of $F_2$’s measurement is $+1/2$ and $-1/2$, respectively.

- $|\Rightarrow\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\rangle$.

The Gedankenexperiment continues by considering two “Wigner” observers, $W_1$ and $W_2$, located outside the laboratories, who will respectively measure the observables $X$ and $Y$ of the laboratories $L_1$ and $L_2$, respectively:

- $X$ has the eigenvectors $|\text{fail}_X\rangle$ and $|\text{ok}_X\rangle$, such that:

$$
|\text{fail}_X\rangle = \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|T\rangle, \quad |\text{ok}_X\rangle = \frac{1}{\sqrt{2}}|H\rangle - \frac{1}{\sqrt{2}}|T\rangle,
$$

(2)
Y has the eigenvectors $|\text{fail}_Y\rangle$ and $|\text{ok}_Y\rangle$, such that:

$$|\text{fail}_Y\rangle = \frac{1}{\sqrt{2}} |\downarrow\rangle + \frac{1}{\sqrt{2}} |\uparrow\rangle, \quad |\text{ok}_Y\rangle = \frac{1}{\sqrt{2}} |\downarrow\rangle - \frac{1}{\sqrt{2}} |\uparrow\rangle. \quad (3)$$

Before analyzing the consequences of the experiment, Frauchiger and Renner point out that their argument can be conceived as a no-go theorem [2] that proves that three “naturalsounding” assumptions, $(Q)$, $(C)$, and $(S)$, cannot all be valid simultaneously.\(^3\)

$(Q)$ **Compliance with quantum theory:** Quantum mechanics is universally valid, that is, it applies to systems of any complexity, including observers. Moreover, an agent knows that a given proposition is true whenever the Born rule assigns probability 1 to it.

$(C)$ **Self-consistency:** Different agents’ predictions are not contradictory.

$(S)$ **Single-world:** From the viewpoint of an agent who carries out a particular measurement, this measurement has one single outcome.

On the basis of the above considerations – experimental setup and assumptions – the F-R argument proceeds as follows. First, in order to compute the probability that the measurements of $X$ and $Y$ yield the results $|\text{ok}_X\rangle$ and $|\text{ok}_Y\rangle$, respectively, the state described by equation (1) must be expressed as:

$$|\Psi\rangle = \frac{1}{\sqrt{12}} |\text{ok}_X\rangle|\text{ok}_Y\rangle - \frac{1}{\sqrt{12}} |\text{ok}_X\rangle|\text{fail}_Y\rangle + \frac{1}{\sqrt{12}} |\text{fail}_X\rangle|\text{ok}_Y\rangle + \sqrt{\frac{3}{4}} |\text{fail}_X\rangle|\text{fail}_Y\rangle. \quad (4)$$

From this equation, it is clear that the probability of obtaining $\text{ok}_X$ and $\text{ok}_Y$ is $1/12$.

The second part of the argument consists in showing that the observers involved in the experiment can draw a conclusion different from the above one on the basis of the following reasoning. Let us consider the probability that $F_2$ obtains $-1/2$ in her $S_z$ measurement and $W_1$ obtains $|\text{ok}_X\rangle$ in her $X$ measurement; in order to compute this probability, the state

\(^3\)In the 2016 paper, Frauchiger and Renner implicitly consider $(Q)$ and $(C)$ as unavoidable: as a consequence, they claim that their argument shows that “no single-world interpretation can be logically consistent” and, therefore, “we are forced to give up the view that there is one single reality” [1]. By contrast, in the 2018 paper, they stress that “the theorem itself is neutral in the sense that it does not tell us which of these three assumptions is wrong” [2]; as a consequence, they admit the possibility of different theoretical and interpretive viewpoints regarding their result, and include a table that shows which of the three assumptions each interpretation of quantum theory violates.
described by equation (4) must be expressed as:

\[ |\Psi\rangle = \sqrt{\frac{2}{3}}|\text{fail}_X\rangle |\downarrow\rangle + \frac{1}{\sqrt{6}}|\text{fail}_X\rangle |\uparrow\rangle - \frac{1}{\sqrt{6}}|\text{ok}_X\rangle |\uparrow\rangle. \tag{5} \]

From this equation it can be inferred that such a probability is zero. Then, if \(W_1\) obtains \(|\text{ok}_X\rangle\) in her \(X\) measurement on the laboratory \(L_1\), she can infer with certainty that the outcome of \(F_2\)’s \(S_z\) measurement on the qubit was +1/2. In turn, if \(F_2\) obtains +1/2 in her \(S_z\) measurement on the qubit, she can infer that the outcome of \(F_1\)’s \(C\) measurement on the quantum coin was \(t\), because otherwise \(F_1\) would send \(F_2\) the qubit in state \(|\downarrow\rangle\), see equation (1). And if \(F_1\) obtains \(t\) in her \(C\) measurement on the quantum coin, she can infer that the outcome of \(W_2\)’s \(Y\) measurement on the laboratory \(L_2\) will be \(|\text{fail}_Y\rangle\), because the outcome \(t\) is perfectly correlated with the state \(|\Rightarrow\rangle\) of the laboratory \(L_2\), and \(|\Rightarrow\rangle = |\text{fail}_Y\rangle\), see equation (3). Therefore, from a nested reasoning it can be concluded that, when \(W_1\) gets \(|\text{ok}_X\rangle\), she can infer that \(W_2\) certainly gets \(|\text{fail}_Y\rangle\). But this conclusion contradicts what was inferred from equation (4), that is, that there is a non-zero probability that \(W_1\) gets \(|\text{ok}_X\rangle\) and \(W_2\) gets \(|\text{ok}_Y\rangle\).

The reactions to the F-R argument have been multiple and varied (see for example [16–22], just to mention some of them). However, since the argument from which the contradiction is obtained involves quantum properties at different times, it seems natural to consider a description of the \textit{Gedankenexperiment} using the theory of quantum histories. This theory extends the formalism of quantum mechanics introducing the notion of quantum history: an elemental history is defined as a sequence of quantum properties at different times (see Sec. IV). As far as we know, there has not been a detailed reconstruction of the argument in terms of the Theory of Consistent Histories. In Section IV we will offer such a description and we will draw the conclusions that this formalism offers for this case.

Moreover, the vectors \(|H\rangle\) and \(|T\rangle\) of the previous discussion are states of the measurement instrument in the laboratory \(L_1\), while the vectors \(|\uparrow\rangle\) and \(|\downarrow\rangle\) are states of the measurement instrument in the laboratory \(L_2\). However, the states of the measurement instruments of observers \(W_1\) and \(W_2\) are not included. In the next section we give a complete description of the process including the Hilbert spaces corresponding to all measurement instruments.
III. THE DIACHRONIC DEVELOPMENT OF THE ARGUMENT

Let us recall that in the laboratory $L_1$ there is a quantum coin in the initial state

$$|\phi\rangle = \frac{1}{\sqrt{3}}|h\rangle + \sqrt{\frac{2}{3}}|t\rangle \in \mathcal{H}_C,$$

where $\mathcal{H}_C$ is the Hilbert space of the coin. The initial state of the rest of the laboratory $L_1$ (including observer $F_1$) is $|a_0\rangle \in \mathcal{H}_{F_1}$. Therefore, the Hilbert space of the entire laboratory $L_1$ is $\mathcal{H}_{L_1} = \mathcal{H}_C \otimes \mathcal{H}_{F_1}$. In turn, in the laboratory $L_2$ there is a qubit, which initially is in state $|q_0\rangle \in \mathcal{H}_q$, where $\mathcal{H}_q$ is the Hilbert space of the qubit. The initial state of the rest of laboratory $L_2$ (including observer $F_2$) is $|b_0\rangle \in \mathcal{H}_{F_2}$. Therefore, the Hilbert space of the entire laboratory $L_2$ is $\mathcal{H}_{L_2} = \mathcal{H}_q \otimes \mathcal{H}_{F_2}$.

Observer $W_1$ measures the observable $X$ of the laboratory $L_1$ with an apparatus, which is initially in a state $|w_{10}\rangle \in \mathcal{H}_{W_1}$, where $\mathcal{H}_{W_1}$ is the Hilbert space of the apparatus. In turn, observer $W_2$ measures the observable $Y$ of the laboratory $L_2$ with an apparatus initially in a state $|w_{20}\rangle \in \mathcal{H}_{W_2}$, where $\mathcal{H}_{W_2}$ is the Hilbert space of the corresponding apparatus.

Summing up, the Hilbert space of the entire process is $\mathcal{H} = \mathcal{H}_{L_1} \otimes \mathcal{H}_{L_2} \otimes \mathcal{H}_{W_1} \otimes \mathcal{H}_{W_2}$, and the initial state at time $t_0$ is

$$|\Psi_0\rangle = |\phi\rangle \otimes |a_0\rangle \otimes |q_0\rangle \otimes |b_0\rangle \otimes |w_{10}\rangle \otimes |w_{20}\rangle \in \mathcal{H}.$$  \hspace{1cm} (7)

In what follows, we describe the consecutive processes.

• Time interval $(t_0, t_1)$: Observer $F_1$ measures the quantum coin.

This process is represented by a unitary evolution $U_{10}$ in the Hilbert space $\mathcal{H}_{L_1} = \mathcal{H}_C \otimes \mathcal{H}_{F_1}$, satisfying

$$U_{10}(|h\rangle \otimes |a_0\rangle) = |h\rangle \otimes |a_h\rangle \equiv |H\rangle, \quad U_{10}(|t\rangle \otimes |a_0\rangle) = |t\rangle \otimes |a_t\rangle \equiv |T\rangle.$$ \hspace{1cm} (8)

• Time interval $(t_1, t_2)$: Observer $F_1$ prepares the qubit.

This process is represented by a unitary evolution $U_{21}$ in the Hilbert space $\mathcal{H}_{F_1} \otimes \mathcal{H}_q$, satisfying

$$U_{21}(|a_h\rangle \otimes |q_0\rangle) = |a_h\rangle \otimes |\downarrow\rangle, \quad U_{21}(|a_t\rangle \otimes |q_0\rangle) = |a_t\rangle \otimes |\rightarrow\rangle.$$ \hspace{1cm} (9)
• Time interval \((t_2, t_3)\): Observer \(F_2\) measures the qubit

This process is represented by a unitary evolution \(U_{32}\) in the Hilbert space \(\mathcal{H}_{L_2} = \mathcal{H}_q \otimes \mathcal{H}_{F_2}\), satisfying

\[
U_{32} (|\downarrow\rangle \otimes |b_0\rangle) = |\downarrow\rangle \otimes |b_\downarrow\rangle \equiv |\downarrow\rangle, \quad U_{32} (|\uparrow\rangle \otimes |b_0\rangle) = |\uparrow\rangle \otimes |b_\uparrow\rangle \equiv |\uparrow\rangle.
\] (10)

• Time interval \((t_3, t_4)\): Observer \(W_1\) measures the laboratory \(L_1\).

This process is represented by a unitary evolution \(U_{43}\) in the Hilbert space \(\mathcal{H}_{L_1} \otimes \mathcal{H}_{W_1}\), satisfying

\[
U_{43} (|\text{fail}_X\rangle \otimes |w_{10}\rangle) = |\text{fail}_X\rangle \otimes |w_{1\text{fail}}\rangle, \quad U_{43} (|\text{ok}_X\rangle \otimes |w_{10}\rangle) = |\text{ok}_X\rangle \otimes |w_{1\text{ok}}\rangle.
\]

• Time interval \((t_4, t_5)\): Observer \(W_2\) measures the laboratory \(L_2\).

This process is represented by a unitary evolution \(U_{54}\) in the Hilbert space \(\mathcal{H}_{L_2} \otimes \mathcal{H}_{W_2}\), satisfying

\[
U_{54} (|\text{ok}_Y\rangle \otimes |w_{20}\rangle) = |\text{ok}_Y\rangle \otimes |w_{2\text{ok}}\rangle, \quad U_{54} (|\text{fail}_Y\rangle \otimes |w_{20}\rangle) = |\text{fail}_Y\rangle \otimes |w_{2\text{fail}}\rangle.
\]

Once the steps for the time evolution are established, the argument leading to the contradictory result, reviewed in Section II, should be written in terms of probabilities involving properties at different times. For example, in Section II the value \(1/12\) was obtained for the probability for obtaining \(\text{ok}_X\) and \(\text{ok}_Y\). Considering the description of the time evolution given in this section, we should write

\[
\Pr \left( \{w_{2\text{ok}} \text{ at } t_5\} \land \{w_{1\text{ok}} \text{ at } t_4\} \right) = \frac{1}{12},
\] (11)

where \(\land\) represents the logical conjunction. This expression represents the probability for the measurement instrument of observer \(W_1\) to indicate \(w_{1\text{ok}}\) at time \(t_4\) and for the measurement instrument of observer \(W_2\) to indicate \(w_{2\text{ok}}\) at the later time \(t_5\).

The second part of the argument is based on the following conditional probabilities

\[
\Pr \left( \{b_\uparrow \text{ at } t_3\} \mid \{w_{1\text{ok}} \text{ at } t_4\} \right) = 1,
\] (12)

\[
\Pr \left( \{a_t \text{ at } t_1\} \mid \{b_\uparrow \text{ at } t_3\} \right) = 1,
\] (13)

\[
\Pr \left( \{w_{2\text{fail}} \text{ at } t_5\} \mid \{a_t \text{ at } t_1\} \right) = 1.
\] (14)
If the last three conditional probabilities could be considered simultaneously, then we could infer the following conditional probability:

$$\Pr(\{w_2 \text{ fail at } t_5\} | \{w_1 \text{ ok at } t_4\}) = 1.$$  \hfill (15)

Hence, $$\Pr(\{w_2 \text{ ok at } t_5\} \land \{w_1 \text{ ok at } t_4\}) = 0,$$ which is in contradiction with equation (11).

Since the previous argument involves logical operations between quantum properties at different times, it seems natural to analyze it using a formalism of quantum histories. In order to search for the possibility of obtaining equations (11), (12), (13) and (14) simultaneously, in the next section we will apply the Theory of Consistent Histories.

IV. THE F-R ARGUMENT IN TERMS OF QUANTUM HISTORIES

In what follows we present a brief summary of the Theory of Consistent Histories (TQH) \[4\]-\[14\]. In quantum mechanics, the properties of a system are represented by orthogonal projectors. Since an elementary history is a sequence of properties at consecutive times, the TQH represents each elementary history with a tensor product of orthogonal projectors. For example, a history of \(n\) times \(\Pi = \Pi_1 \otimes \ldots \otimes \Pi_n\) represents a sequence of properties \(\Pi_1, \ldots, \Pi_n\), at times \(t_1, \ldots, t_n\).

To define probabilities for quantum histories, it is necessary to define a family of histories. For this purpose, first we have to choose a context of properties at each time \(t_i\), i.e., a set of projectors that sum the identity of \(\mathcal{H}\) and that are mutually orthogonal:

$$\Pi_k \Pi_{k'} = \delta_{k,k'} \Pi_k, \quad \sum_{k_i} \Pi_{k_i} = I_{\mathcal{H}}, \quad k_i, k'_i \in \sigma_i, \quad i = 1, \ldots, n;$$

where \(I_{\mathcal{H}}\) is the identity of the Hilbert space \(\mathcal{H}\), and each \(\sigma_i\) is an index set.

Second, we define the atomic histories \(\Pi_{k_1, \ldots, k_n}\), choosing one projector \(\Pi_{k_i}\) at each time \(t_i\):

$$\Pi_{k_1, \ldots, k_n} = \Pi_{k_1} \otimes \ldots \otimes \Pi_{k_n}, \quad (k_1, \ldots, k_n) \in \tilde{\sigma}, \quad \tilde{\sigma} = \sigma_1 \times \ldots \times \sigma_n.$$

Then, we define the histories \(\Pi_{\Lambda}\) summing the histories \(\Pi_{k_1, \ldots, k_n}\) with \((k_1, \ldots, k_n) \in \Lambda \subseteq \tilde{\sigma}\), i.e., \(\Pi_{\Lambda} = \sum_{(k_1, \ldots, k_n) \in \Lambda} \Pi_{k_1, \ldots, k_n}\). These histories represent disjunctions of the histories \(\Pi_{k_1, \ldots, k_n}\).

Finally, the family of histories is the set obtained by making arbitrary disjunctions between product histories.
If $\rho_0$ is the initial state at time $t_0$, the probability of a general history $\Pi_{\Lambda}$ is defined in the following way:

$$\Pr_{\rho_0}(\Pi_{\Lambda}) = \text{Tr} \left[ C^\dagger(\Pi_{\Lambda}) \rho_0 C(\Pi_{\Lambda}) \right],$$

(16)

where we have introduced the chain operator $C(\Pi_{\Lambda}) = \sum_{(k_1, \ldots, k_n) \in \Lambda} C(k_1, \ldots, k_n)$, in which

$$C(k_1, \ldots, k_n) = U(t_0, t_1) \Pi_{k_1} U(t_1, t_2) \Pi_{k_2} \cdots U(t_{n-1}, t_n) \Pi_{k_n} U(t_n, t_0)$$

with $U(t_i, t_j) = e^{-iH(t_i - t_j)/\hbar}$.

In general, the probability definition given in equation (16) does not satisfy the axiom of additivity. Therefore, to have a well-defined probability, the atomic histories of a family of histories must satisfy an additional condition, called the consistency condition.

$$\text{Tr} \left[ C^\dagger(\Pi_{k_1, \ldots, k_n}) \rho_0 C(\Pi_{k'_1, \ldots, k'_n}) \right] = 0, \quad \forall (k_1, \ldots, k_n) \neq (k'_1, \ldots, k'_n).$$

(17)

Intuitively, the consistency condition measures the amount of interference between pairs of histories. When $n = 1$, this condition is automatically satisfied, and the probability expression of equation (16) reduces to the Born rule. However, in the general case, the consistency condition is not trivial, and when it is satisfied the probability expression provides a generalization of the Born rule.

In order to describe the F-R argument in terms of quantum histories, we first obtain the probability for the measurement instrument of the observer $W_1$ to indicate $w_{1\text{ ok}}$ at time $t_4$ and for the measurement instrument of the observer $W_2$ to indicate $w_{2\text{ ok}}$ at a later time $t_5$.

A suitable context of properties for time $t_4$ should include the properties $w_{1\text{ ok}}, w_{1\text{ fail}}$ and it has to be completed with the property $\neg (w_{1\text{ ok}} \lor w_{1\text{ fail}})$ (where $\lor$ is the disjunction and $\neg$ is the negation) in order to include all the degrees of freedom of the measurement instrument, for example the initial state $|w_{10}\rangle$ given in equation (7). These properties are represented by the following projectors:

$$\Pi_{w_{1\text{ ok}}} = I_{L_1} \otimes I_{L_2} \otimes |w_{1\text{ ok}}\rangle \langle w_{1\text{ ok}}| \otimes I_{W_2},$$

$$\Pi_{w_{1\text{ fail}}} = I_{L_1} \otimes I_{L_2} \otimes |w_{1\text{ fail}}\rangle \langle w_{1\text{ fail}}| \otimes I_{W_2},$$

$$\Pi_{\neg(w_{1\text{ ok}} \lor w_{1\text{ fail}})} = I_{\mathcal{H}} - \Pi_{w_{1\text{ fail}}} - \Pi_{w_{1\text{ ok}}},$$

(18)

where each $I_K$ is the identity of the corresponding Hilbert space $\mathcal{H}_K$. These three projectors provide a context of properties of the Hilbert space $\mathcal{H}$.
For time $t_5$, a suitable context of properties should include the properties of the measurement instrument of observer $W_2$, i.e., $w_1.ok$, $w_1.fail$, and it has to be completed with the property $\neg (w_1.ok \lor w_1.fail)$. These properties are represented by the following projectors:

$$\Pi_{w_2.ok} = I_L \otimes I_L \otimes I_W \otimes |w_2.ok\rangle\langle w_2.ok|,$$

$$\Pi_{w_2.fail} = I_L \otimes I_L \otimes I_W \otimes |w_2.fail\rangle\langle w_2.fail|,$$

$$\Pi_{-(w_2.ok \lor w_2.fail)} = I_H - \Pi_{w_2.fail} - \Pi_{w_2.ok}.$$ (19)

These three projectors also provide a context of properties of the Hilbert space $H$.

From the contexts of properties for times $t_4$ and $t_5$, we can generate a family of two-times histories, whose atomic histories are $\tilde{\Pi}_{k_4,k_5} = \Pi_{k_4} \otimes \Pi_{k_5}$, with $\Pi_{k_4}$ one of the projectors of equations (18) and $\Pi_{k_5}$ one of the projectors of equations (19). It is easy to verify that the family generated by these atomic histories satisfies the consistency conditions given in equation (17). Therefore, equation (16) can be used to compute the probability of quantum history $\tilde{\Pi}_{w_1.ok,w_2.ok} = \Pi_{w_1.ok} \otimes \Pi_{w_2.ok}$:

$$\Pr(\tilde{\Pi}_{w_1.ok,w_2.ok}) = \frac{1}{12}. \quad (20)$$

This shows that using the Theory of Consistent Histories, and explicitly considering the measurement instruments as quantum systems, we obtain the same result given in Section III for the first part of the argument.

In the same way, different consistent families of two-times histories can be defined to express equations (12), (13) and (14). However, if the three equations are going to be used together in the same argument, it is necessary to have a consistent family of four-times histories including the possible results of the instrument of the observer $F_1$ at time $t_1$, of the instrument of the observer $F_2$ at time $t_3$, of the instrument of observer $W_1$ at time $t_4$, and of the instrument of observer $W_2$ at time $t_5$.

For times $t_4$ and $t_5$, the contexts of properties given in equations (18) and (19) are adequate. For time $t_1$, a suitable context of properties should include the properties $a_h$, $a_t$, and it has to be completed with the property $\neg (a_h \lor a_t)$. These properties are represented by the following projectors:

$$\Pi_{a_h} = I_C \otimes |a_h\rangle\langle a_h| \otimes I_L \otimes I_W \otimes I_W,$$

$$\Pi_{a_t} = I_C \otimes |a_t\rangle\langle a_t| \otimes I_L \otimes I_W \otimes I_W,$$

$$\Pi_{-(a_h \lor a_t)} = I_H - \Pi_{a_h} - \Pi_{a_t}. \quad (21)$$
For time $t_3$, a suitable context of properties should include the properties of the measurement instrument of observer $F_2$, i.e., $b_\downarrow, b_\uparrow$, and it has to be completed with the property $\neg (b_\downarrow \lor b_\uparrow)$. These properties are represented by the following projectors:

\[
\Pi_{b_\downarrow} = I_{L_1} \otimes I_q \otimes |b_\downarrow\rangle \langle b_\downarrow| \otimes I_{W_1} \otimes I_{W_2},
\]
\[
\Pi_{b_\uparrow} = I_{L_1} \otimes I_q \otimes |b_\uparrow\rangle \langle b_\uparrow| \otimes I_{W_1} \otimes I_{W_2}, \tag{22}
\]
\[
\Pi_{\neg (b_\downarrow \lor b_\uparrow)} = I_{\mathcal{H}} - \Pi_{b_\downarrow} - \Pi_{b_\uparrow}.
\]

From the contexts of properties for times $t_1$, $t_3$, $t_4$ and $t_5$ we can generate a family of four-times histories, whose atomic histories are

\[
\tilde{\Pi}_{k_1,k_3,k_4,k_5} = \Pi_{k_1} \otimes \Pi_{k_3} \otimes \Pi_{k_4} \otimes \Pi_{k_5}, \tag{23}
\]

with $\Pi_{k_1}$, $\Pi_{k_3}$, $\Pi_{k_4}$ and $\Pi_{k_5}$ projectors chosen from equations (21), (22), (18) and (19), respectively.

The non-atomic histories that are involved in the F-R argument are the following:

\[
\tilde{\Pi}_{1t} = \Pi_{a_t} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}}, \tag{24}
\]
\[
\tilde{\Pi}_{3\uparrow} = I_{\mathcal{H}} \otimes \Pi_{b_\uparrow} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}}, \tag{25}
\]
\[
\tilde{\Pi}_{4\text{ok}} = I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \Pi_{w_{1\text{ok}}} \otimes I_{\mathcal{H}}, \tag{26}
\]
\[
\tilde{\Pi}_{5\text{fail}} = I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes I_{\mathcal{H}} \otimes \Pi_{w_{2\text{fail}}}. \tag{27}
\]

In terms of quantum histories, the F-R argument can be formulated as follows:

**First part**

\[
\text{Pr}(\tilde{\Pi}_{5\text{ok}} \land \tilde{\Pi}_{4\text{ok}}) = \frac{1}{12}. \tag{28}
\]

**Second part**

\[
\text{Pr}(\tilde{\Pi}_{3\uparrow}|\tilde{\Pi}_{4\text{ok}}) = 1, \text{ Pr}(\tilde{\Pi}_{1t}|\tilde{\Pi}_{3\uparrow}) = 1 \text{ and Pr}(\tilde{\Pi}_{5\text{fail}}|\tilde{\Pi}_{1t}) = 1. \tag{29}
\]

This implies $\text{Pr}(\tilde{\Pi}_{5\text{fail}}|\tilde{\Pi}_{4\text{ok}}) = 1$, and then $\text{Pr}(\tilde{\Pi}_{5\text{ok}}|\tilde{\Pi}_{4\text{ok}}) = 0$. Therefore,

\[
\text{Pr}(\tilde{\Pi}_{5\text{ok}} \land \tilde{\Pi}_{4\text{ok}}) = 0. \tag{30}
\]

The contradiction is obtained from equations (28) and (30).

In order to infer equation (30) from the equations (29), the quantum histories must belong to a single consistent family of histories, generated by the atomic histories of equations (23).
But such a family of histories does not satisfy the consistency conditions, given in equation (17).

To prove this statement, let us consider two atomic histories \( \tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}} \) and \( \tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}} \), representing different results for the four measurements,

\[
\tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}} = \Pi_{a_t} \otimes \Pi_{b_1} \otimes \Pi_{w_{1,ok}} \otimes \Pi_{w_{2,ok}},
\]

(31)

\[
\tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}} = \Pi_{a_h} \otimes \Pi_{b_1} \otimes \Pi_{w_{1,ok}} \otimes \Pi_{w_{2,ok}},
\]

(32)

and with the following chain operators

\[
C(\tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}}) = U(t_0,t_1)\Pi_{a_t}U(t_1,t_3)\Pi_{b_1}U(t_3,t_4)\Pi_{w_{1,ok}}U(t_4,t_5)\Pi_{w_{2,ok}}U(t_5,t_0),
\]

(33)

\[
C(\tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}}) = U(t_0,t_1)\Pi_{a_h}U(t_1,t_3)\Pi_{b_1}U(t_3,t_4)\Pi_{w_{1,ok}}U(t_4,t_5)\Pi_{w_{2,ok}}U(t_5,t_0).
\]

(34)

Considering the unitary time evolution of the complete quantum system and the initial state defined in equation (7), we obtain

\[
C^\dagger(\tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}})\mid \Psi_0 \rangle = C^\dagger(\tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}})\mid \Psi_0 \rangle = \frac{1}{\sqrt{12}}U(t_0,t_5)(\mid 0_{kX} \rangle \otimes \mid 0_{kY} \rangle \otimes \mid w_{1,ok} \rangle \otimes \mid w_{2,ok} \rangle),
\]

(35)

and therefore, according to equation (17), the consistency condition gives

\[
\text{Tr} \left[ C^\dagger(\tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}})\mid \Psi_0 \rangle \langle \Psi_0 | C(\tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}}) \right] = \langle \Psi_0 | C(\tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}}) C^\dagger(\tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}})\mid \Psi_0 \rangle = \frac{1}{12} \neq 0.
\]

(36)

This proves that the atomic histories \( \tilde{\Pi}_{a_t,b_1,w_{1,ok},w_{2,ok}} \) and \( \tilde{\Pi}_{a_h,b_1,w_{1,ok},w_{2,ok}} \) do not satisfy the consistency condition and, therefore, there is no family of consistent histories to describe the results of the four measurement instruments of the F-R experiment. For this reason, the conclusion of the second part of the F-R argument cannot be asserted.

Summing up, since the conclusion of the second part of the F-R argument is based on an illegitimate inference, the supposed contradiction of the F-R argument does not hold.

V. CONCLUSIONS

In a previous article \cite{3} one of us argued that the contradiction resulting from the F-R argument is inferred by making classical conjunctions between different and incompatible
contexts, and, as a consequence, it is the result of a theoretically illegitimate inference. However, it has been suggested that the criticism does not take into account the fact that the inferences in the F-R argument are all carefully timed, and this fact would circumvent the objection based on the contextuality of quantum mechanics.

If timing really matters in the F-R argument, it seems natural to reconstruct it using a theory of quantum histories, a formalism that allows us to deal with quantum properties at different times. We applied the Theory of Consistent Histories, and we showed that the contradiction resulting from the F-R argument is inferred by computing probabilities in a family of histories that is not consistent, i.e. an invalid family of histories for the theory.

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