QUANTUM DEFORMATION THEORY OF THE AIRY CURVE AND MIRROR SYMMETRY OF A POINT

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ABSTRACT. We present a quantum deformation theory of the Airy curve and use it to establish a version of mirror symmetry of a point.

1. INTRODUCTION

“To see a world in a grain of sand” is the first line of a famous poem by William Blake. For the author of this paper, the grain of sand is just a point, and the world he wants to see is mirror symmetry. More precisely, he wants to gain some more understanding on mirror symmetry by studying the mirror symmetry of a point.

It may sound absurd to consider mirror symmetry of a point, because mirror symmetry usually involves Calabi-Yau 3-folds [8], or Fano manifolds and their mirror geometries. As a space, a point is as simple as one can get (except for perhaps the empty space), and it seems that it does not have any interesting structure that be used to produce a mirror partner. Nevertheless, the Gromov-Witten theory of a point is very rich and does have a mirror theory encoded in the Airy curve:

\[ y = \frac{1}{2} x^2, \]

that is the subject of this paper. This simple curve that everyone sees in junior high school turns out to have a rich deformation theory that can be used as the mirror of the theory of 2D topological gravity, in more than one way.

For early history on mirror symmetry we refer to [29]. Let us recall some highlights that are are particularly relevant to this work. The original proposal of mirror symmetry relates the deformations of symplectic structure on one Calabi-Yau 3-fold with the deformations of the complex structure on its mirror Calabi-Yau 3-fold. Based on mirror symmetry, physicists made predictions on counting the number of holomorphic curves in quintic threefolds in genus zero [8]. For \( g > 0 \), physicists have developed holomorphic anomaly equation to make predictions in genus one [5], genus 2 [6], and in genus \( g \leq 51 \)
Mathematically, these predictions have been proved in genus zero [24, 33] and genus one [48]. A physical reformulation of the holomorphic anomaly equation was given in [44]. A mathematical formulation of quantum BCOV theory has been developed [11].

Parallel to the mirror symmetry of compact Calabi-Yau 3-folds, one can consider the local mirror symmetry of toric Calabi-Yau 3-folds [9]. Based on duality with Chern-Simons theory of link invariants [42, 45], a physical theory of topological vertex has been developed to compute local Gromov-Witten invariants in all genera [3, 2]. A mathematical theory of the topological vertex has been developed in [34, 35, 36].

Even though the theory of topological vertex solves the problem of computing local Gromov-Witten invariant of toric Calabi-Yau 3-folds in principle, its combinatorial nature does not match directly with the deformation theory aspects of mirror symmetry of compact Calabi-Yau 3-folds, therefore its success in all genera does not provide much insight for its compact counterpart, except for perhaps the Gopakumar-Vafa invariants [26, 40, 27, 28]. Eleven years ago, physicists [1] have proposed a mathematically mysterious approach based on deformation at infinities of local mirror curves. More recently local mirror curves have been used as the spectral curve to apply the appearance of Eynard-Orantin topological recursion [19] in the program of remodelling the local B-models [38, 7, 20]. Deformations of the local mirror curves do not play a role in this topological recursion formalism either. The renewed interest in local mirror symmetry and local mirror curves motivates the author to present a mathematical elaboration of the beautiful ideas in [1], with an emphasis on the deformation theory aspects. Through this one can gain more insights about the mirror symmetry in the compact Calabi-Yau case.

In physics Gromov-Witten theory of a point corresponds to the theory of 2D topological gravity originally studied using matrix models, and mathematically they correspond to intersection numbers of $\psi$-classes on Deligne-Mumford moduli spaces. Witten’s remarkable conjecture relates such numbers to KdV hierarchy and Virasoro constraints. Since its first proof by Kontsevich [31] it has served as a paradigm for researches in Gromov-Witten theory and its generalizations. Recent progresses on topological recursions related to the Witten-Kontsevich tau-function [18, 4, 46, 47] have made it clear that the Airy curve:

$$y = \frac{1}{2}x^2,$$

plays an important role in this theory. In this paper we will elaborate on the fact that all information about the correlation functions of the
2D topological gravity is encoded in the Airy curve, by suitable procedures of deformation and quantization, therefore, this curve serves as the mirror geometry in the B-theory for the point. This is an elaboration of some ideas in [1] where some technical details are missing. This will provide the prototype of quantum deformation theory of a class of curves which we will develop in subsequent work. Our ultimate goal will be to develop a version of quantum deformation to unite the two main driving forces in Gromov-Witten type theories: The Witten Conjecture/Kontsevich Theorem and the computation of Gromov-Witten invariants of quintic Calabi-Yau 3-folds.

Deformation theory was created by Riemann in his famous paper on Abelian functions where he introduced the notion of moduli spaces of Riemann surfaces. Deformation theory of compact complex manifolds was developed by Kodaira, Spencer and Kuranishi. There is also a deformation theory of isolated singularities. It is amazing that such theories all have found applications in string theory related to mirror symmetry. In the study of mirror symmetry of quintic Calabi-Yau 3-fold, deformation of its mirror has been applied to find genus zero free energy. We will somehow reverse this direction of applications: We will first use free energy in genus zero to construct the deformation of the Airy curve, then we will apply the canonical quantization derive the Virasoro constraints satisfied by free energies in higher genera. Generalizations to Type A and Type D singularities are straightforward and will be presented in future publications. Generalizations to other case are work in progress.

We now summarize the main results of this paper. We first consider the miniversal deformation $y = \frac{1}{2}x^2 + u_0$ of the Airy curve $y = \frac{1}{2}x^2$. Write $f^2 = 2y$, then we prove the following identity:

\[(2) \quad x = f(1 - \frac{2u_0}{f^2})^{1/2} = f - \frac{u_0}{f} - \sum_{n=0}^{\infty} \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots) \cdot f^{-(2n+3)},\]

where $F_0$ is the genus zero free energy of the theory of 2D topological gravity, in suitably chosen coordinates $u_n$’s. Next we construct a special deformation of the Airy curve of the following form:

\[(3) \quad x = f - \sum_{n\geq 0} (2n + 1)u_nf^{2n-1} - \sum_{n\geq 0} \frac{\partial F_0}{\partial u_n}(u) \cdot f^{-2n-3}.\]

We prove that it is uniquely characterized by the following property:

\[(4) \quad (x^2(f))_- = 0.\]
From (3) we obtain a special deformation of the equation of the Airy curve of the form:

$$2y = f^2 = c_0 + c_1x^2 + c_2x^4 + c_3x^6 + \cdots.$$ 

We regard $W = y$ as the deformed superpotential function and define

$$(5) \quad \phi_n := \frac{\partial W}{\partial u_n}.$$ 

They generate an algebra over the ring of formal series in $u_n$'s, closed under multiplications. We take this as a generalization of the Milnor ring. Furthermore, if one defines

$$(6) \quad \nabla_{i_0}^{\lambda} \phi_k := \partial_{i_0} \phi_k + \lambda \phi_i \cdot \phi_k,$$ 

then the operators $\nabla_{i_0}^{\lambda}$ define a family of flat connections, i.e.,

$$(7) \quad \nabla_{i_0}^{\lambda} \nabla_{i_1}^{\lambda} \phi_k = \nabla_{i_1}^{\lambda} \nabla_{i_0}^{\lambda} \phi_k,$$ 

for $i, j, k \geq 0$ and all $\lambda \in \mathbb{C}$. We then define $\langle \langle \phi_{j_1}, \cdots, \phi_{j_n} \rangle \rangle_0$ as in Landau-Ginzburg theory and show that

$$(8) \quad \langle \langle \phi_{j_1}, \cdots, \phi_{j_n} \rangle \rangle_0 = \frac{\partial^n F_0}{\partial t_{j_1} \cdots \partial t_{j_n}}.$$ 

To extend the picture to arbitrary genera, we quantize the above picture. We endow the space of series of the form

$$(9) \quad \sum_{n=0}^{\infty} (2n + 1) \tilde{u}_n z^{(2n-1)/2} + \sum_{n=0}^{\infty} \tilde{v}_n z^{-(2n+3)/2}$$ 

the following symplectic structure:

$$(10) \quad \omega = \sum_{n=0}^{\infty} d\tilde{u}_n \wedge d\tilde{v}_n.$$ 

Take the natural polarization that $\{q_n = \tilde{u}_n\}$ and $\{p_n = \tilde{v}_n\}$, one can consider the canonical quantization:

$$(11) \quad \hat{\tilde{u}}_n = \tilde{u}_n, \quad \hat{\tilde{v}}_n = \frac{\partial}{\partial \tilde{u}_n}.$$ 

Corresponding to the field $x$, we consider the following fields of operators on the Airy curve:

$$(12) \quad \hat{x}(z) = -\sum_{m \in \mathbb{Z}} \beta_{-(2m+1)} z^{m-1/2} = -\sum_{m \in \mathbb{Z}} \beta_{2m+1} z^{-m-3/2}$$
where \( f = z^{1/2} \) and the operators \( \beta_{2k+1} \) are defined by:

\[
\beta_{-(2k+1)} = (2k+1) \tilde{u}_k, \quad \beta_{2k+1} = \frac{\partial}{\partial \tilde{u}_k}.
\]

We define a notion of regularized products \( \hat{x}(z)^{\circ n} \) and show that they are related to the normally ordered products \( \hat{x}(z)^n \) by Bessel numbers. Then we show that the DVV Virasoro constraints satisfied by the Witten-Kontsevich tau-function is just the following equation:

\[
\hat{x}(z)^{\circ 2} Z_{WK} = 0.
\]

We conjecture that

\[
\hat{x}(z)^{\circ 2n} Z_{WK} = 0
\]

for \( n > 1 \). In forthcoming work, we will generalize such results to deformations of isolated singularities of Type A and Type D.

Roughly speaking, what we mean by quantum deformation theory is a deformation theory whose moduli space encodes the information of genus zero free energy on the big phase space. It is then necessary that the moduli space is infinite-dimensional. Furthermore, we require that the moduli space admits a natural quantization from which one can produce constraints that determines the free energy in all genera. This is the point of view that we will take to understand mirror symmetry in subsequent work, both for noncompact and compact Calabi-Yau manifolds.

The rest of the paper is arranged as follows. In §2 we fix some terminologies and recall some combinatorial preliminaries. We recall the Witten Conjecture/Kontsevich Theorem and DVV Virasoro constraints in §3 some explicit numerical computations based on formal series solution of the inviscid Burgers’ equation are also presented. We construct in §4 the special deformation of the Airy curve based on genus zero Gromov-Witten invariants of a point. In §5 we study the Landau-Ginzburg type theory associated to the miniversal deformation of the Airy curve and make extensions to the special deformation in §6. In §7 and §8 we present the quantum deformation of the Airy curve by quantization \( \hat{x} \) of the field \( x \), define and study its regularized products relating to Virasoro constraints and its generalizations.

2. Preliminaries

In this paper we will discuss the mirror symmetry between two “field theories”. We will use this section to make a provisional definition of what we mean by a field theory to make our framework as simple as
possible. We will also collect some combinatorial techniques crucial for our computations and proofs.

2.1. **Some terminologies of field theory.** The physical background for our definition is the topological matters coupled with 2D topological gravity. The definition that we will take below does not reflect all the interesting aspects of such theories, it only serves to suit for the discussions in this work. The reader may consult [32] for a more general definition.

First, we need a finite-dimensional complex vector space $V$. It will be called the *small phase space*, and we will call a vector $v \in V$ a *primary field*. It will also be referred to as a *matter field*. The small phase space is supposed to be graded by rational numbers:

$$V = \bigoplus_{i=1}^{k} V_{w_i},$$

where $w_1 < w_2 < \cdots < w_k$ are distinct rational numbers. If $v \in V_{w_j}$, then we say $v$ is a homogeneous field of degree $w_j$, and write

$$\deg v = w_j.$$

By the *big phase space* we mean $\mathbb{C}[z] \otimes_{\mathbb{C}} V$. We will write this space as $V[z]$. (Usually one reserves this notation for $V$ be a commutative ring so that $V[z]$ is also a ring. A field theory often naturally makes $V$ a ring.) An element of this space can be written as

$$v_0 + v_1 z + \cdots + v_n z^n,$$

where $v_0, \ldots, v_n \in V$. For $v \in V$, the element $v z^n$ will be called the $n$-th gravitational descendant (field) of $v$. Following Witten, we will also denote $v z^n$ by $\tau_n(v)$. The grading on $V$ induces a grading $V[z]$

$$V[z] = \bigoplus_{n \geq 0} \bigoplus_{i=1}^{k} V_{w_i} z^n,$$

i.e., if $v \in V_{w_j}$ has $\deg v = w_j$, then one sets

$$\deg \tau_n(v) = w_j + n.$$

By a field theory of the topological matters coupled to 2D topological gravity, we simply mean a collection of *correlators*: For $2g - 2 + n > 0$,

$$\langle \cdot, \cdots, \cdot \rangle_g : V[z] \otimes_{\mathbb{C}} \rightarrow \mathbb{C},$$

$$(\tau_{m_1}(v_1), \ldots, \tau_{m_n}(v_n)) \mapsto \langle \tau_{m_1}(v_1), \ldots, \tau_{m_n}(v_n) \rangle_g.$$
They are required to satisfy the following two conditions: The correlators are symmetric in its arguments, i.e.,

$$\langle \tau_{\sigma(1)}(v_{\sigma(1)}), \ldots, \tau_{\sigma(n)}(v_{\sigma(n)}) \rangle_g = \langle \tau_{\sigma(1)}(v_1), \ldots, \tau_{\sigma(n)}(v_n) \rangle_g,$$

for any permutation $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$; the correlators satisfy the following *selection rule*: Suppose that $v_1, \ldots, v_n$ are homogeneous fields, then

$$\langle \tau_{m_1}(v_1), \ldots, \tau_{m_n}(v_n) \rangle_g = 0$$

except for

$$\sum_{i=1}^n (m_i + w_i) = 3g - 3 + n.$$

Take a basis $\{e_1, \ldots, e_m\}$ of $V$ consisting of homogeneous elements, then one gets a basis $\{\tau_n(e_i)\}_{n \geq 1, 1 \leq i \leq m}$ of $V[z]$. The corresponding linear coordinates $\{t_{n,i}\}_{n \geq 1, 1 \leq i \leq m}$ will be called *coupling constants*. Define the *free energies* in genus $g$ by:

$$F_0(t) = \sum_{n \geq 3} \sum_{\substack{m_1, \ldots, m_n \geq 0 \atop 1 \leq i_1, \ldots, i_n \leq m}} t_{m_1,i_1} \cdots t_{m_n,i_n} \langle \tau_{m_1}(e_{i_1}), \ldots, \tau_{m_n}(e_{i_n}) \rangle_0,$$

and for $g \geq 1$,

$$F_g(t) = \sum_{n \geq 1} \sum_{\substack{m_1, \ldots, m_n \geq 0 \atop 1 \leq i_1, \ldots, i_n \leq m}} t_{m_1,i_1} \cdots t_{m_n,i_n} \langle \tau_{m_1}(e_{i_1}), \ldots, \tau_{m_n}(e_{i_n}) \rangle_g.$$

The *partition function* is defined by:

$$Z(t; \lambda) = \exp \sum_{g \geq 0} \lambda^{2g-2} F_g(t).$$

Define the *deformed $n$-point correlators* by:

$$\langle \langle \tau_{m_1}(e_{i_1}), \ldots, \tau_{m_n}(e_{i_n}) \rangle \rangle_g(t) = \frac{\partial^n}{\partial t_{m_1,i_1} \cdots \partial t_{m_n,i_n}} F_g(t).$$

From this definition it is clear that

$$\langle \langle \tau_{m_1}(e_{i_1}), \ldots, \tau_{m_n}(e_{i_n}), \tau_{m_{n+1}}(e_{i_{n+1}}) \rangle \rangle_g(t)$$

$$= \frac{\partial}{\partial t_{m_{n+1},i_{n+1}}} \langle \langle \tau_{m_1}(e_{i_1}), \ldots, \tau_{m_n}(e_{i_n}) \rangle \rangle_g(t).$$

Hence to determine the theory, it suffices to determine the deformed one-point functions $\langle \langle \tau_{m}(e) \rangle \rangle_g(t)$. 

2.2. **Lagrangian inversion.** Suppose that we have two formal power series:

\begin{align}
y &= x + a_2x^2 + a_3x^3 + \cdots, \\
x &= y + b_2y^2 + b_3y^3 + \cdots
\end{align}

are compositional inverse to each other. Then their coefficients are related to each other by the Lagrange inversion formula:

\begin{align}
a_n &= \frac{1}{n} \text{res} \left( \frac{dy}{x^n} \right) = \frac{1}{n} (y + b_2y^2 + \cdots)^{-n} \bigg|_{y = 1}, \\
b_n &= \frac{1}{n} \text{res} \left( \frac{dx}{y^n} \right) = \frac{1}{n} (x + a_2x^2 + \cdots)^{-n} \bigg|_{x = 1},
\end{align}

The following are some explicit examples:

\begin{align*}
a_2 &= -b_2, \\
a_3 &= -b_3 + 2b_2^2, \\
a_4 &= -b_4 + 5b_2b_3 - 5b_2^3.
\end{align*}

If one assigns

\begin{equation}
\deg b_n = 1 - n, \quad n \geq 2,
\end{equation}

then \(a_n\) is a weighted homogeneous polynomial in \(b_2, \ldots, b_n\) of degree \(1 - n\). We will refer to \(\{a_n\}_{n \geq 2}\) and \(\{b_n\}_{n \geq 2}\) as Lagrange dual of each other.

3. **The A-Theory: Intersection Theory on Moduli Spaces of Algebraic Curves**

In this section, we recall the A-theory of a point, i.e., the intersection theory of \(\psi\)-classes on Deligne-Mumford moduli spaces \(\overline{M}_{g,n}\). In the physics literature (see e.g. [43]), this corresponds to the 2D topological gravity. Main features of this theory are the Witten Conjecture/Kontsevich Theorem [43, 31] and the Virasoro constraints [15]. These results tie this theory to the KdV hierarchy, and hence the genus zero part to the dispersionless limit of the KdV hierarchy. One can reduce the genus zero part of the theory to inviscid Burger’s equation and its series solution by Lagrange inversion. We also derive formula (86) which is the starting point for mirror symmetry with quantum deformation theory of the Airy curve.
3.1. **Free energy and partition function of 2D topological gravity.** The moduli spaces for 2D topological gravity are the Deligne-Mumford spaces $\overline{M}_{g,n}$ of stable algebraic curves with $n$ marked points. Let $\psi_1, \ldots, \psi_n$ be the first Chern classes of the cotangent line bundles corresponding to the $n$ marked points. The correlators of the 2D topological gravity are defined as the following intersection numbers:

\[
\langle \tau_{a_1}, \ldots, \tau_{a_n} \rangle_g := \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.
\]

The correlator $\langle \tau_{a_1}, \ldots, \tau_{a_n} \rangle_g \neq 0$ only if

\[
a_1 + \cdots + a_n = 3g - 3 + n.
\]

This is due to the fact that

\[
\dim \overline{M}_{g,n} = 3g - 3 + n.
\]

The free energy of 2D topological gravity is defined by

\[
F(t; \lambda) = \sum_{g \geq 0} \lambda^{2g-2} F_g(t).
\]

where the genus $g$ part of the free energy is defined by:

\[
F_g(t) = \langle \exp \sum_{a \geq 0} t_a \tau_a \rangle_g = \sum_{n \geq 0} \frac{1}{n!} \sum_{a_1, \ldots, a_n \geq 0} t_{a_1} \cdots t_{a_n} \langle \tau_{a_1}, \ldots, \tau_{a_n} \rangle_g.
\]

We will assign the following grading:

\[
\deg t_i = 2 - 2i, \quad i = 0, 1, \ldots
\]

By (31), $F_g(t)$ is weighted homogeneous of degree

\[
\deg F_g = 6 - 6g.
\]

The partition function of 2D topological gravity, often referred to as the Witten-Kontsevich tau-function, is defined by

\[
Z_{WK}(t, \lambda) = \exp F(t, \lambda).
\]

We assign

\[
\deg \lambda = 3
\]

so that $Z_{WK}$ is weighted homogeneous of degree 0.
3.2. **Witten Conjecture/Kontsevich Theorem.** Witten [43] conjectured that $Z_{WK}$ is a tau-function of the KdV hierarchy. More precisely, define a sequence $\{R_n\}$ of differential polynomials in $u$ as follows:

\begin{equation}
R_1 = u, \quad \frac{\partial}{\partial x} R_{n+1} = \frac{1}{2n+1} \left( \partial_x u \cdot R_n + 2u \cdot \partial_x R_n + \frac{\lambda^2}{4} \partial_x^3 R_n \right).
\end{equation}

For example,
\begin{align*}
\partial_x R_2 &= u \cdot \partial_x u + \frac{1}{12} \partial_x^3 u, \\
R_2 &= \frac{1}{2} u^2 + \frac{1}{12} \partial_x^2 u, \\
\partial_x R_3 &= \frac{1}{2} u^2 \cdot \partial_x u + \frac{1}{12} u \cdot \partial_x^3 u + \frac{1}{6} \partial_x u \cdot \partial_x^2 u + \frac{1}{240} \partial_x^5 u, \\
R_3 &= \frac{1}{6} u^3 + \frac{1}{12} u \cdot \partial_x^2 u + \frac{1}{24} (\partial_x u)^2 + \frac{1}{240} \partial_x^4 u.
\end{align*}

Then
\begin{equation}
u = \lambda^2 \frac{\partial F}{\partial^2 t_0}
\end{equation}

satisfies the following sequence of equations:

\begin{equation}
\partial_{t_n} u = \partial_{t_0} R_{n+1},
\end{equation}

where $t_0 = x$.

He also pointed out that together with the string equation:
\begin{equation}
\langle \tau_0 \tau_{a_1} \cdots \tau_{a_n} \rangle_g = \sum_{i=1}^n \langle \tau_{a_1} \cdots \tau_{a_i-1} \cdots \tau_{a_n} \rangle_g,
\end{equation}

this hierarchy of nonlinear differential equations uniquely determines $Z_{WK}$ from the initial values:

\begin{equation}
\int_{\mathcal{M}_{0,3}} 1 = 1, \quad \int_{\mathcal{M}_{1,1}} \psi_1 = \frac{1}{24},
\end{equation}

After Kontsevich [31], there have appeared many proofs of this conjecture.

3.3. **Dispersionless limits and reduction to inviscid Burgers’ equation.** Write

\begin{equation}
u = u(0) + \lambda^2 u(1) + \cdots,
\end{equation}
Then from the KdV hierarchy one gets the following sequence of equations for \( u^{(0)} \):

\[
\partial_t u^{(0)} = \partial_x \frac{u^{n+1}(0)}{(n+1)!}.
\]

Since these flows commute with each other, one can solve them separately and inductively.

The case of \( n = 1 \) is the inviscid Burgers’ equation:

\[
\partial_t u = \partial_x \frac{u^2}{2}.
\]

The next result shows that all higher equations in the hierarchy (42) can be reduced to it.

**Proposition 3.1.** Suppose that \( u(x, t) \) satisfies the equation

\[
\partial_t u = \partial_x \frac{u^{n+1}}{(n+1)!}
\]

for some \( n > 1 \), then \( v = \frac{u^n}{n!} \) satisfies:

\[
\partial_t v = \partial_x \frac{v^2}{2}.
\]

**Proof.** Easy computations. \( \square \)

### 3.4. Series solutions of inviscid Burgers’ equation by Lagrangian inversion

Recall that the inviscid Burgers’ equation can be solved by the method of characteristics. Suppose that the initial value \( u(x, 0) = u_0(x) \) is given. If \( x(t) \) with \( x(0) = x_0 \) is a solution of the ordinary differential equation

\[
\frac{d}{dt} x(t) = -u(x(t), t),
\]

where \( u = u(x, t) \) satisfy \( u_t = uu_x \), then

\[
\frac{d}{dt} u(x(t), t) = u_x \cdot \frac{d}{dt} x(t) + u_t = -u_x u + u_t = 0,
\]

and so

\[
u(x(t), t) = u(x(0), 0) = u(x_0, 0) = u_0(x_0).
\]

Now from (46),

\[
\frac{d}{dt} x(t) = -u(x(t), t) = -u_0(x_0),
\]

and so

\[
x(t) = x_0 - tu_0(x_0).
\]
Plug this back into (47) and change $x_0$ into $x$:

(49) \[ u(x - tu_0(x), t) = u_0(x). \]

To see this provides formal series solutions, we set

(50) \[ A(x, t) = x + t \cdot u(x, t). \]

Then one has

\[ A(x - tu_0(x), t) = (x - tu_0(x)) + tu(x_0 - tu_0(x), t) = x - tu_0(x) + tu_0(x) = x. \]

So we get the following:

**Proposition 3.2.** Suppose that $A(y, t)$ is a formal series in $y$ which is the compositional inverse of the equation

(51) \[ y = x - tu_0(x), \]

i.e., $A(x - tu_0(x), t) = x$, then

(52) \[ u(x, t) = \frac{A(x, t) - x}{t} \]

is a formal series solution of the inviscid Burger’s equation with initial condition $u_0(x)$:

(53) \[ \partial_t u = \partial_x \frac{u^2}{2}, \quad u(x, 0) = u_0(x). \]

One can use Lagrange inversion to solve (51). Let

\[ A(x, t) = x + \sum_{k \geq 2} a_k(t)x^k, \]

then one has:

(54) \[ a_k(t) = \frac{1}{k} (x - tu(x, t))^{-k} \big|_{x^{-1}}. \]

Sometimes one can directly solve (51) by explicit expressions. The following are some examples.

**Example 3.3.** When $u_0(x) = x$, the equation

\[ y = x - tx \]

can be solved by

\[ x = \frac{y}{1 - t}, \]

i.e.,

\[ A(x, t) = \frac{x}{1 - t}. \]
It follows that
\[ u(x, t) = \frac{A(x, t) - x}{t} = \frac{x}{1 - t}. \]

One can easily checked that it is a solution.

*Example 3.4.* When \( u_0(x) = x^2 \), the equation
\[ y = x - tx^2 \]
can be solved by
\[ x = \frac{1 - (1 - 4ty)^{1/2}}{2t}, \]
i.e.,
\[ A(x, t) = \frac{1 - (1 - 4tx)^{1/2}}{2t}. \]

It follows that
\[ u(x, t) = \frac{A(x, t) - x}{t} = \frac{1 - 2tx - (1 - 4tx)^{1/2}}{2t^2}. \]

One can easily checked that it is a solution. The coefficients of the series expansion of \( u(x, t) \) are Catalan numbers:

\[ (55) \quad u(x, t) = \sum_{n=0}^{\infty} \frac{(2n + 2)!}{(n + 1)!(n + 2)!} t^n x^{n+2}. \]

In general we have:

**Lemma 3.5.** For \( m = 1, 2, \ldots \), the following equation
\[ \partial_t u = \partial_x \frac{u^2}{2}, \quad u(x, 0) = cx^m \]
is solved by the following generating series of generalized Catalan numbers:

\[ (56) \quad u(x, t) = x^m \sum_{k=1}^{\infty} \frac{c^k}{(m-1)k+1} \binom{mk}{k} (x^{m-1}t)^{k-1}. \]

**Proof.** One can use Lagrange inversion to solve
\[ y = x - ctx^m, \]
i.e., let \( x = A(y, t) = y + \sum_{n \geq 2} a_n y^n \),
\[ a_n = \frac{1}{2\pi \sqrt{-1}} \int \frac{x}{y^{n+1}} dy = -\frac{1}{n} \cdot \frac{1}{2\sqrt{-1}} \int xd\frac{1}{y^n} \]
\[ = \frac{1}{n} \cdot \frac{1}{2\sqrt{-1}} \int \frac{1}{y^n} dx = \frac{1}{n} \cdot \frac{1}{2\sqrt{-1}} \int \frac{1}{(x - cx^{m-1})^n} dx \]
\[ = \frac{1}{n} \cdot \frac{1}{2\sqrt{-1}} \int \frac{1}{x^n} \sum_{k=0}^{\infty} \frac{1}{(1 - cx^{m-1})^n} dx \]
\[ = \frac{1}{n} \cdot \frac{1}{2\sqrt{-1}} \int \frac{1}{x^n} \sum_{k=0}^{\infty} \left(-\frac{n}{k}\right) (-cx^{m-1})^k dx. \]

It is now clear that \( a_n \) only when \( n = k(m-1) + 1 \) for some \( k \geq 1 \),
\[ a_{k(m-1)+1} = \frac{1}{k(m-1) + 1} \left(-k(m-1) - 1\right) \left(-ct\right)^k \]
\[ = \frac{1}{k(m-1) + 1} \binom{km}{m} e^{k^{k^k}}. \]

hence
\[ A(x, t) = x + \sum_{k \geq 1} \frac{1}{k(m-1) + 1} \binom{km}{m} e^{k^{k^k}x^{k(m-1)+1}}. \]

The result then follows. \( \square \)

3.5. Some explicit calculations. Now we use the above method to carry out some explicit calculations. We will use the following notation when there is no danger of confusions: We keep only the relevant variables, e.g., \( u(t_0, t_2, t_5) \) means \( u(t) \) restricted to \( t_i = 0, i \neq 0, 2, 5 \).

First we solve
\[ \frac{\partial u(t_0)}{\partial t_n} = \frac{u_{n+1}(t_0)}{(n+1)!}, \quad u(t_0, 0, \ldots) = t_0. \]

This can be transformed to
\[ v = \frac{u_{n}^n}{n!}, \quad \frac{\partial v}{\partial t_n} = \frac{v_{n}^n}{n^2}, \quad v(t_0, 0, \ldots) = \frac{t_n^n}{n!}. \]

By Lemma 3.5
\[ v(t_0, t_n) = \frac{t_n^n}{n!} \sum_{k=1}^{\infty} \frac{1}{(n-1)k+1} \binom{nk}{k} \left(\frac{t_0^n - t_n^n}{n!}\right)^{k-1}. \]

Then we have
\[ u_0(t_0, t_n) = t_0 \left[ \sum_{k=1}^{\infty} \frac{1}{(n-1)k+1} \binom{nk}{k} \left(\frac{t_0^n - t_n^n}{n!}\right)^{k-1} \right]^{1/n}. \]
It turns out that we have

\[ u(0)(t_0, t_n) = t_0 \left[ 1 + \sum_{k=1}^{\infty} \frac{1}{(n-1)k+1} \binom{n^k}{k} \left( \frac{t_0 t_n}{n!} \right)^k \right] \]

This can be proved by using the following identity:

\[ \left[ \sum_{k=1}^{\infty} \frac{z^{k-1}}{(n-1)k+1} \binom{n^k}{k} \right]^{1/n} = 1 + \sum_{k=1}^{\infty} \frac{z^k}{(n-1)k+1} \binom{n^k}{k} \]

One can prove this identity by Lagrange inversion as follows. Let \( y = z + \sum_{k \geq 2} a_n z^k \) satisfy

\[ (y/z)^{1/n} = 1 + y, \]

i.e.

\[ z = \frac{y}{(1+y)^n}. \]

Then

\[ a_k = \frac{1}{2\pi i} \oint \frac{y}{y^{k+1} z^k} dy = \frac{1}{k} \frac{1}{2\pi i} \oint \frac{1}{z^k} dy = \frac{1}{k} \frac{1}{(1+y)^n} (1+y)^{kn} \]

\[ = \frac{1}{k} \binom{kn}{k} \binom{(k-1)n+1}{k}. \]

As straightforward consequence of string equation is that

\[ \partial_{t_n} F_0(t_0, 0, \ldots) = \frac{t_0^{n+2}}{(n+2)!}. \]

This matches with (59).

**Proposition 3.6.** The sequence \( u(0)(t_0), u(0)(t_0, t_1), \ldots \) satisfies the following relations:

\[ u(0)(x, t_1, \ldots, t_n) = u(0)(x + \frac{t_n}{n!} u(0)(x, t_1, \ldots, t_n), t_1, \ldots, t_{n-1}). \]

**Proof.** Set

\[ v(x, t_1, \ldots, t_n) = \frac{1}{n!} u(0)(t_0 = x, t_1, \ldots, t_n). \]

Then we have

\[ \partial_{t_n} v(x, t_1, \ldots, t_n) = \partial_x \left( \frac{v^2(x, t_1, \ldots, t_n)}{2} \right), \]

\[ v(x, t_1, \ldots, t_{n-1}) = \frac{1}{n!} u(0)(x, t_1, \ldots, t_{n-1}). \]
Then by Proposition, 

$$v(x,t_1,\ldots,t_n) = \frac{A(x,t_1,\ldots,t_n) - x}{t_n},$$

where $A(x,t_1,\ldots,t_n)$ satisfies: 

$$A(x,t_1,\ldots,t_n) - t_nv(A(x,t_1,\ldots,t_n),t_1,\ldots,t_{n-1}) = x.$$ 

The proof can be finished by noting:

$$A(x,t_1,\ldots,t_n) = x + t_n \cdot v(x,t_1,\ldots,t_n)$$

$$= x + \frac{t_n}{n!} u_{(0)}^n(x,t_1,\ldots,t_n),$$

$$v(A(x,t_1,\ldots,t_n),t_1,\ldots,t_{n-1})$$

$$= \frac{1}{n!} u_{(0)}^n(A(x,t_1,\ldots,t_n),t_1,\ldots,t_{n-1})$$

$$= \frac{1}{n!} u_{(0)}^n(x + \frac{t_n}{n!} u_{(0)}^n(x,t_1,\ldots,t_n),t_1,\ldots,t_{n-1}).$$

\[ \square \]

Using the above method, one finds $u_{(0)}(t_0,\ldots,t_n)$ and $F_0(t_0,\ldots,t_n)$ recursively. For example, first,

$$u_{(0)}(t_0) = t_0, \quad F_0(t_0) = \frac{t_0^3}{6};$$

next,

$$u_{(0)}(t_0,t_1) = u_{(0)}(t_0 + t_1 u_{(0)}(t_0,t_1)) = t_0 + t_1 u_{(0)}(t_0,t_1),$$

hence

$$u_{(0)}(t_0,t_1) = \frac{t_0}{1-t_1}, \quad F_0(t_0,t_1) = \frac{t_0^3}{6(1-t_1)}.$$

Going to the next level,

$$u_{(0)}(t_0,t_1,t_2) = u_{(0)}(t_0 + \frac{1}{2} t_2 u_{(0)}^2(t_0,t_1,t_2),t_1)$$

$$= \frac{t_0 + \frac{1}{2} t_2 u_{(0)}^2(t_0,t_1,t_2)}{1-t_1}.$$

It can be solved explicitly as follows:

$$u_{(0)}(t_0,t_1,t_2) = \frac{1 - t_1 - ((1-t_1)^2 - 2t_0t_2)^{1/2}}{t_2},$$

and so

$$F_0(t_0,t_1,t_2) = \frac{(1-t_1)^5}{15t_2^3} \left(1 - \frac{5t_0t_2}{(1-t_1)^2} + \frac{15t_0^2t_2^2}{2(1-t_1)^4} - \left(1 - \frac{2t_0t_2}{(1-t_1)^2}\right)^{5/2}\right).$$
The series expansion of \( u_0(t_0, t_1, t_2) \) is given by:

\[
u_0(t_0, t_1, t_2) = \frac{1 - t_1}{t_2} \left[ 1 - \left( 1 - \frac{2t_0 t_2}{(1 - t_1)^2} \right)^{1/2} \right] = \frac{t_0}{1 - t_1} + \frac{1}{2} \frac{t_0^2 t_2}{(1 - t_1)^3} + \frac{1}{2} \frac{t_0^3 t_2^2}{(1 - t_1)^5} + \frac{5}{8} \frac{t_0^4 t_2^3}{(1 - t_1)^7} + \cdots ,\]

In general, one can use Lagrange inversion recursively as follows:

\[
u_0(t_0 = x, t_1, \ldots, t_n) = \left( n! \frac{A(x, t_1, \ldots, t_n) - x}{t_n} \right)^{1/n} ,
\]

where

\[
A(x, t_1, \ldots, t_n) = x + \sum_{k \geq 2} a_k x^k ,
\]

with the coefficients \( a_k \) given by

\[
a_k = \frac{1}{k} \left( x - t_n \frac{n!}{t_n^n} u_0^n(x, t_1, \ldots, t_{n-1}) \right)^{-k} |_{x=1} .
\]

One can easily use a computer algebra system to automate the calculations using these formulas.

### 3.6. Virasoro constraints.

Dijkgraaf, E. Verlinde, H. Verlinde \[15\] and independently Fukuma, Kawai, Nakayama \[22\] showed that the Witten-Kontsevich tau-function can also be uniquely determined by a sequence of linear differential equations called the Virasoro constraints:

\[
\frac{\partial}{\partial u_{n+1}} Z_{WK} = \hat{L}_n Z_{WK}, \quad n \geq -1 ,
\]

where the operators \( \hat{L}_n \) are defined by:

\[
\hat{L}_n = \sum_{k=0}^{\infty} (2k+1) u_k \frac{\partial}{\partial u_{k+n}} + \frac{\lambda^2}{2} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial u_k \partial u_{n-k-1}} + \frac{u_0^2}{2\lambda^2} \delta_{n, -1} + \frac{\delta_{n, 0}}{8} .
\]

Here we have made the following change of coordinates:

\[
t_k = (2k + 1)!! u_k .
\]
In terms of the free energy, 

\[
\frac{\partial F}{\partial u_0} = \sum_{k=1}^{\infty} (2k + 1) u_k \frac{\partial F}{\partial u_{k-1}} + \frac{u_0^2}{2\lambda^2},
\]

\[
\frac{\partial F}{\partial u_1} = \sum_{k=0}^{\infty} (2k + 1) u_k \frac{\partial F}{\partial u_k} + \frac{1}{8},
\]

\[
\frac{\partial F}{\partial u_n} = \sum_{k=0}^{\infty} (2k + 1) u_k \frac{\partial F}{\partial u_{k+n-1}} + \lambda^2 \frac{\partial^2 F}{\partial u_k \partial u_{n-k-2}} + \frac{\lambda^2}{2} \sum_{k=0}^{n-2} \left( \frac{\partial^2 F}{\partial u_k \partial u_{n-k-2}} + \frac{\partial F}{\partial u_k} \frac{\partial F}{\partial u_{n-k-2}} \right), \quad n \geq 2.
\]

In particular, by comparing the coefficients of \(\lambda^{-2}\) on both sides of these equations:

\[
\frac{\partial F_0}{\partial u_0} = \sum_{k=1}^{\infty} (2k + 1) u_k \frac{\partial F_0}{\partial u_{k-1}} + \frac{u_0^2}{2},
\]

\[
\frac{\partial F_0}{\partial u_1} = \sum_{k=0}^{\infty} (2k + 1) u_k \frac{\partial F_0}{\partial u_k},
\]

\[
\frac{\partial F_0}{\partial u_n} = \sum_{k=0}^{\infty} (2k + 1) u_k \frac{\partial F_0}{\partial u_{k+n-1}} + \frac{1}{2} \sum_{k=0}^{n-2} \frac{\partial F_0}{\partial u_k} \frac{\partial F_0}{\partial u_{n-k-2}}, \quad n \geq 2.
\]

Now we take \(u_i = 0\) for \(i \geq 1\) in these equations, and set \(f_n = \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots)\), we get:

\[
f_0 = \frac{u_0^2}{2},
\]

\[
f_1 = u_0 f_0,
\]

\[
f_n = u_0 f_{n-1} + \frac{1}{2} \sum_{k=0}^{n-2} f_k f_{n-k-2}, \quad n \geq 2.
\]

After taking the generating series \(f(t) = f_0 + f_1 t + \cdots\), one can get from the above recursion relations the following equation:

\[
f(t) = \frac{u_0^2}{2} + u_0 t \cdot f(t) + \frac{1}{2} t^2 \cdot f(t)^2.
\]

One can easily find the following explicit expression for \(f(t)\):

\[
f(t) = \frac{(1 - u_0 t) - (1 - 2u_0 t)^{1/2}}{t^2} = \sum_{n \geq 0} \frac{(2n + 1)!!}{(n + 2)!!2^{n+3}} u_0^{n+2} t^n,
\]
and so

$$\frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots) = \frac{(2n + 1)!!}{(n + 2)!} u_0^n.$$  

We note the above result can be reformulated as follows:

$$z(1 - \frac{2u_0}{z^2})^{1/2} = z - \frac{u_0}{z} - \sum_{n=0}^{\infty} \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots) \cdot z^{-(2n+3)}.$$  

This will play an important role in the next section.

### 4. The Mirror Geometry: Special Deformation of the Airy Curve

From the point of view of Eynard-Orantin topological recursions, it has been clear that the Airy curve can serve as the mirror curve for the Gromov-Witten theory of a point [13, 14, 40, 47]. In this section we will study some special deformation of the Airy curve constructed from Gromov-Witten invariants of a point.

#### 4.1. Miniversal deformation of the Airy curve.

The Airy curve is the plane algebraic curve given by the equation:

$$y = \frac{1}{2}x^2$$

Consider the restriction of the projection $\pi : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto y$ to this curve, $(x, y) = (0, 0)$ is an isolated singularity of type $A_1$. Its miniversal deformation is given by

$$y = \frac{1}{2}x^2 + u_0.$$  

To unravel the information hidden in this simple formula, let $f$ be the Laurent series such that $f^2 = 2y = x^2 + 2u_0$, i.e.,

$$f = x(1 + \frac{2u_0}{x^2})^{1/2} = \sum_{j=0}^{\infty} \binom{1/2}{j} (2u_0)^j x^{1-2j}$$

$$= x + \frac{u_0}{x} - \frac{1}{2} u_0^2 x^{-1} + \frac{1}{2} u_0^3 x^{-3} - \frac{5}{8} u_0^4 x^{-5} + \frac{7}{8} u_0^5 x^{-7} - \frac{21}{16} u_0^6 x^{-9} + \cdots,$$
and let
\[ x = f(1 - \frac{2u_0}{f^2})^{1/2} = \sum_{j=0}^{\infty} \binom{1/2}{j} (-2u_0)^j x^{1-2j} \]
\[ = f - \frac{u_0}{f} - \frac{1}{2} \frac{u_0^2}{f^3} - \frac{1}{2} \frac{u_0^3}{f^5} - \frac{5}{8} \frac{u_0^4}{f^7} - \frac{7}{8} \frac{u_0^5}{f^9} - \frac{21}{16} \frac{u_0^6}{f^{11}} - \ldots \]
be its inverse series. One can now note the meaning of the coefficient \( s \) of \( f^{2n+3} \) in \( x \):

**Proposition 4.1.** The following equalities hold:

\[ x|_{f^{2n+3}} = -(2n+1)!! \frac{n+2}{(n+2)!} u_0^{n+2} = -\frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots). \]

This is not just a nice coincidence but a special case of the mirror symmetry between the intersection theory of \( \psi \)-classes on \( \overline{M}_{g,n} \) and the quantum deformation theory of the Airy curve to be presented below.

**4.2. Special deformation.** In last subsection we have seen that the miniversal deformation of the Airy curve
\[ y = \frac{1}{2} x^2 + u_0 \]
is given by the Puiseux series:
\[ x = f - \frac{u_0}{f} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n}(u_0, 0, \ldots) \cdot f^{-2n-3}. \]

This suggests to include the variables \( u_1, u_2, \ldots \) in the deformation, as in the following:

**Theorem 4.2.** Consider the following series:

\[ x = f - \sum_{n \geq 0} (2n+1)u_n f^{2n-1} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n}(u) \cdot f^{-2n-3}. \]

Then one has
\[ x^2 = 2y \left( 1 - \sum_{n \geq 1} (2n+1)u_n (2y)^{n-1} \right)^2 \]
\[ -2u_0 \left( 1 - \sum_{n \geq 1} (2n+1)u_n (2y)^{n-1} \right) \]
\[ + 2 \sum_{n \geq 0} \sum_{k \geq n+2} (2k+1)u_k \cdot \frac{\partial F_0}{\partial u_n} \cdot (2y)^{k-n-2}. \]
In particular,

\[(x^2)_2 = 0.\]

Here for a formal series \(\sum_{n \in \mathbb{Z}} a_n f^n,\)

\[(\sum_{n \in \mathbb{Z}} a_n f^n)_+ = \sum_{n \geq 0} a_n f^n, \quad (\sum_{n \in \mathbb{Z}} a_n f^n)_- = \sum_{n < 0} a_n f^n.\]

**Proof.** This is actually equivalent to the Virasoro constraints for \(F_0.\) Indeed,

\[x^2 = \left( f - \sum_{n \geq 1} (2n + 1) u_n f^{2n-1} - \frac{u_0}{f} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n} \cdot f^{-2n-3} \right)^2 \]

\[= \left( f - \sum_{n \geq 1} (2n + 1) u_n f^{2n-1} \right)^2 \]

\[= f^2 \left( 1 - \sum_{n \geq 1} (2n + 1) u_n f^{2n-2} \right)^2 - 2 u_0 \left( 1 - \sum_{n \geq 1} (2n + 1) u_n f^{2n-2} \right) \]

\[+ 2 \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n} \cdot f^{-2n-2} + 2 \sum_{n \geq 1} (2n + 1) u_n f^{2n-1} \cdot \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n} \cdot f^{-2n-3}\]

\[+ \left( \frac{u_0}{f} + \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n} \cdot f^{-2n-3} \right)^2.\]

It follows that

\[(x^2)_- = 2 \left( \frac{t^2}{2} \cdot f^{-2} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n} \cdot f^{-2n-2} \right) \]

\[+ \sum_{n \geq 0} \sum_{k} (2k + 1) u_k \cdot \frac{\partial F_0}{\partial u_{k+n-1}} \cdot f^{-2n-2} \]

\[+ \frac{1}{2} \sum_{n \geq 2} \sum_{j+k=n-2} \frac{\partial F_0}{\partial u_j} \cdot \frac{\partial F_0}{\partial u_k} \cdot f^{-2n-2} \).\]
It vanishes by Virasoro constraints for \( F_0 \). It follows that
\[
x^2 = (x^2)_+ = f^2 \left( 1 - \sum_{n \geq 1} (2n + 1) u_n f^{2n-2} \right)^2
- 2u_0 \left( 1 - \sum_{n \geq 1} (2n + 1) u_n f^{2n-2} \right)
+ 2 \sum_{n \geq 0} \sum_{k \geq n+2} (2k + 1) u_k \cdot \frac{\partial F_0}{\partial u_n} \cdot f^{2(k-n-2)}.
\]
The proof is completed by recalling \( 2y = f^2 \).

Recall we have assigned that \( \deg t_n = 2 - 2n \), and because \( t_n = (2n - 1)!! \cdot u_n \), we have \( \deg u_n = 2 - 2n \). Also because \( \deg F_0 = 6 \), so we have
\[
\deg \frac{\partial F_0}{\partial u_n} = 2n + 4.
\]
Therefore, if we assign
\[
(95) \quad \deg f = 1,
\]
then the right-hand side of (91) is weighted homogeneous of degree 1, it follows that we should take
\[
(96) \quad \deg x = 1.
\]

4.3. Uniqueness of special deformation of the Airy curve. Let us first prove a simple combinatorial result.

**Theorem 4.3.** There exists a unique series
\[
x = f - \sum_{n \geq 0} v_n f^{2n-1} - \sum_{n \geq 0} w_n f^{2n-3}
\]
such that each \( w_n \in \mathbb{C}[[v_0, v_1, \ldots]] \) and
\[
(98) \quad (x^2)_- = 0.
\]

**Proof.** We begin by rewriting (98) as a sequence of equations:
\[
(99) \quad w_0 = \frac{1}{2} v_0^2 + v_1 w_0 + v_2 w_1 + v_3 w_2 + \cdots,
\]
\[
(100) \quad w_1 = v_0 w_0 + v_1 w_1 + v_2 w_2 + \cdots,
\]
\[
(101) \quad w_2 = v_0 w_1 + v_1 w_2 + \cdots + \frac{1}{2} w_0^2,
\]
\[
(102) \quad w_3 = v_0 w_2 + \cdots + w_0 w_1,
\]
\[
(103) \quad \cdots \cdots \cdots
\]
Write
\[ w_n = w_n^{(0)} + w_n^{(1)} + \cdots, \]
where each \( w_n^{(k)} \) consists of monomials in \( v_0, v_1, \ldots \) of ordinary degree \( k \). Using such decompositions, one can deduce by induction from the above system of equations:

\[ w_n^{(j)} = 0, \quad n \geq 0, \quad j = 0, \ldots, n + 1, \]
and furthermore,

\[ w_0^{(2)} = \frac{1}{2} v_0^2, \]
\[ w_n^{(n)} = \sum_{j=1}^{n} v_j w_{j-1}^{(n-1)}, \quad n \geq 3, \]
\[ w_1^{(n)} = \sum_{j=0}^{\infty} v_j w_j^{(n-1)}, \quad n \geq 3, \]
\[ w_m^{(n)} = \sum_{j=0}^{\infty} v_j w_{j+m-1}^{n-1} + \frac{1}{2} \sum_{j=0}^{m-2} \sum_{k=j}^{m+j} w_j^{(k)} w_{m-2-j}^{(n-k)}, \quad m \geq 1, \quad n \geq m + 2. \]

It follows that one can recursively determine all \( w_m^{(n)} \) from the initial value \( w_0^{(2)} = \frac{1}{2} v_0^2 \).

By combining Theorem 4.2 with Theorem 4.3, we then get:

**Theorem 4.4.** The equation

\[ (x^2)_+ = 0 \]

for a series

\[ x = f - \sum_{n \geq 0} (2n + 1) u_n f^{2n-1} - \sum_{n \geq 0} w_n f^{-2n-3}, \]

where each \( w_n \in \mathbb{C}[[u_0, u_1, \ldots]] \) has a unique solution given by:

\[ x = f - \sum_{n \geq 0} (2n + 1) u_n f^{2n-1} - \sum_{n \geq 0} \frac{\partial F_0(u)}{\partial u_n} f^{2n-3}, \]

as a series in \( u = (u_0, u_1, \ldots) \), where \( F_0(u) \) is the free energy of the 2D topological gravity in genus zero.
4.4. Deformation of the superpotential function. From (92) one can also derive formula for deformation of the superpotential function.

By (92), one can formally write:

\begin{equation}
\frac{\delta F_0}{\delta u_n} = a_0 - \frac{2}{n+1} \cdot \left( \frac{\delta F_0}{\delta u_n} \right)^2 + \cdots,
\end{equation}

where

\begin{equation}
a_0 = -2u_0(1 - 3u_1) + 2 \sum_{n \geq 0} (2n + 5)u_{n+2} \frac{\delta F_0}{\delta u_n},
\end{equation}

\begin{equation}
a_1 = (1 - 3u_1)^2 + 2u_0 \cdot 5u_2 + 2 \sum_{n \geq 0} (2n + 7)u_{n+3} \cdot \frac{\delta F_0}{\delta u_n},
\end{equation}

and for \( m \geq 2 \),

\begin{equation}
a_m = -2(1 - 3u_1) \cdot (2m + 1)u_m + 2u_0 \cdot (2m + 3)u_{m+1}
+ \sum_{m_1, m_2 \geq 1, m_1 + m_2 = m+1} (2m_1 + 1)u_{m_1} \cdot (2m_2 + 1)u_{m_2}
+ 2 \sum_{n \geq 0} (2n + 2m + 1)u_{n+2+m} \cdot \frac{\delta F_0}{\delta u_n}.
\end{equation}

Note each \( a_m \) is weighted homogenous of degree

\begin{equation}
\deg a_m = 2 - 2m.
\end{equation}

For later use, we note the following specialization of these coefficients:

\begin{align*}
a_0(u_0, u_1, 0, \ldots) &= -2u_0(1 - 3u_1),
\end{align*}

\begin{align*}
a_1(u_0, u_1, 0, \ldots) &= (1 - 3u_1)^2,
\end{align*}

\begin{align*}
a_n(u_0, u_1, 0, \ldots) &= 0, \quad n \geq 2.
\end{align*}

We also note that by (84), for \( m \geq 0 \),

\begin{equation}
\frac{\partial a_m}{\partial u_n}(u_0) = 2(2n + 1) \cdot \frac{(2n - 2m - 3)!!}{(n - m)!} u_0^{n-m},
\end{equation}

where we have used the following convention:

\begin{align*}
(-1)!! &= 1, \quad (-3)!! = -1, \quad (-2n - 1)!! = 0, \quad n \geq 2.
\end{align*}

Now we have

\begin{equation}
\frac{x^2 - a_0}{a_1} = f^2 + \frac{a_2}{a_1} f^4 + \frac{a_3}{a_1} f^6 + \cdots,
\end{equation}

so we can apply the Lagrangian inversion to get:

\begin{align*}
f^2 &= \frac{x^2 - a_0}{a_1} + b_2 \cdot \left( \frac{x^2 - a_0}{a_1} \right)^2 + \cdots
\end{align*}

\begin{align*}
&= c_0 + c_1 x^2 + c_2 x^4 + c_3 x^6 + \cdots,
\end{align*}
where \( \{ \frac{a_2}{a_1}, \frac{a_3}{a_1}, \ldots \} \) and \( \{ b_2, b_3, \ldots \} \) are Lagrangian dual to each other, and

\[
c_0 = -\frac{a_0}{a_1} + \sum_{n=2}^{\infty} (-1)^n b_n \frac{a_0^n}{a_1^n},
\]

\[
c_1 = \frac{1}{a_1^2} + \sum_{n=2}^{\infty} (-1)^{n-1} n b_n \frac{a_0^{n-1}}{a_1^n},
\]

\[
c_m = \sum_{n=m}^{\infty} (-1)^m \binom{n}{m} b_n \frac{a_0^{n-m}}{a_1^n}, \quad m \geq 2.
\]

We have already seen that

\[
(121) \quad c_0(u_0, u_1, 0, \ldots) = \frac{2u_0}{1 - 3u_1},
\]

\[
(122) \quad c_1(u_0, u_1, 0, \ldots) = \frac{1}{(1 - 3u_1)^2},
\]

\[
(123) \quad c_m(u_0, u_1, 0, \ldots) = 0, \quad m \geq 2.
\]

The following are some examples: When \( u_j = 0 \) for \( j \geq 1 \), the equation of the Airy curve is deformed to:

\[
(124) \quad x^2 = 2y - 2u_0,
\]

and the superpotential function is deformed to

\[
(125) \quad y = \frac{1}{2} x^2 + u_0.
\]

When \( t_j = 0 \) for \( j \geq 2 \), the curve is deformed to

\[
(126) \quad x^2 = 2(1 - 3u_1)^2 y - 2u_0(1 - 3u_1),
\]

and the superpotential is deformed to

\[
(127) \quad y = \frac{1}{2(1 - 3u_1)^2} x^2 + \frac{u_0}{1 - 3u_1}.
\]

In other words, by making the following transformation:

\[
x \mapsto \frac{x}{1 - 3u_1}, \quad u_0 \mapsto \frac{u_0}{1 - 3u_1},
\]

one can obtain (127) from (125). When \( t_j = 0 \) for \( j \geq 3 \), the curve is deformed to:

\[
x^2 = -2u_0(1 - 3u_1) + 10u_2 \frac{\partial F_0}{\partial u_0}(u_0, u_1, u_2)
\]

\[
+ ((1 - 3u_1)^2 + 10u_0 u_2)(2y)
\]

\[
+ (30u_1 u_2 - 10u_2(1 - 3u_1))(2y)^2 + 25u_2^2(2y)^3,
\]
where $F_0$ is given by (67). By making the transformation
\[ x \mapsto \frac{x}{1 - 3u_1}, \quad u_0 \mapsto \frac{u_0}{1 - 3u_1}, \quad u_2 \mapsto \frac{u_2}{1 - 3u_1} \]
one can reduce to the case of $u_1 = 0$:
\[ x^2 = \frac{2}{135u_2}((1 - 30u_0u_2)^{3/2} - 1 - 90u_0u_2) \]
\[ + (1 + 10u_0u_2) \cdot f \]
\[ - 10u_2 \cdot f^2 + 25u_2^2 \cdot f^3. \]

One can solve this equation explicitly by writing $f$ as a function of $x^2$. To make the notations simpler, write $C = 5u_2 f$, $V = 10u_0u_2$, $U = (1 - 3V)^{1/2}$ and $X = 5u_2x^2$. Then one can rewrite the above equation as follows:
\[ C^3 - 2C^2 - \frac{1}{3}U(U + 2)(U - 2)C + \frac{2}{27}(U + 2)^2(U - 1) - X = 0. \]

An explicit solution is given by:
\[ C = \frac{1}{6}[108X - 8U^3 + 12(81X^2 - 12XU^3)^{1/2}]^{1/3} \]
\[ + \frac{2U^2}{3}[108X - 8U^3 + 12(81X^2 - 12XU^3)^{1/2}]^{-1/3} + \frac{2}{3}. \]

However, this is not very useful if we want to expand it as a series in $X$. Write $C = \sum_{n=0}^{\infty} C_n X^n = C_0 + Y$. Then one has
\[ C_0 = \frac{2}{3}(1 - U), \]
and
\[ Y^3 - 2UY^2 + U^2Y = X, \]

After dividing both sides by $U^3$:
\[ \left( \frac{Y}{U} \right)^3 - 2\left( \frac{Y}{U} \right)^2 + \frac{Y}{U} = \frac{X}{U^3}. \]

This can be solved explicitly by the following series of generalized Catalan numbers:
\[ \frac{Y}{U} = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{3n+1}{n} \left( \frac{X}{U^3} \right)^{n+1}. \]
Therefore,
\[
C = \frac{2}{3}(1 - U) + \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{3n+1}{n} \frac{X^{n+1}}{U^{3n+2}}
\]
\[
= \frac{2}{3}(1 - U) + \frac{X}{U^2} + \frac{2X^2}{U^5} + \frac{7X^3}{U^8} + \frac{30X^4}{U^{11}} + \frac{143X^5}{U^{14}} + \cdots.
\]
It follows that
\[
f = \frac{2}{15} \left(1 - (1 - 30u_0u_2)^{1/2}\right)
\]
(134)
\[
+ \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{3n+1}{n} \frac{(5u_2)^n x^{2n+2}}{(1 - 30u_0u_2)^{(3n+2)/2}}.
\]
More explicitly,
\[
2y = (2u_0 + 15u_2u_0^2 + 225u_2^2u_0^3 + \frac{16875}{4}u_2^3u_0^4 + \frac{354375}{4}u_2^4u_0^5 + \cdots)
\]
\[
+ (1 + 30u_2u_0 + 900u_2^2u_0^2 + 27000u_2^3u_0^3 + 810000u_2^4u_0^4 + \cdots)x^2
\]
\[
+ (10u_2 + 750u_2^2u_0 + 39375u_2^3u_0^2 + 1771875u_2^4u_0^3 + \cdots)x^4
\]
\[
+ (175u_2^2 + 21000u_2^3u_0 + 1575000u_2^4u_0^2 + \cdots)x^6
\]
\[
+ \cdots.
\]
By letting \(u_2 = 0\), one recovers the equation \(y = \frac{1}{2}x^2 + u_0\).

This example reveals some interesting features of the deformation (92). First, by keeping \(u_n = 0\) for \(n > 2\) and letting \(u_2 \neq 0\), the deformation changes the Airy curve from a rational curve to a hyperelliptic curve of degree 6, hence the genus is changed from \(g = 0\) to \(g = 3\). In general, one can do this recursively by taking one more of \(u_n\) to be nonzero, the effect will be including terms of higher degrees in \(y\), and so the curve will have larger genus. Secondly, the constant term of the polynomial in \(y\) on the right-hand side of (128) is no longer a polynomial in \(u_0, u_1, u_2\). Thirdly, the superpotential \(\frac{1}{2}f\) is deformed to become an infinite formal power series in \(x^2\). There does not seem to have motivations to study such deformations from a traditional mathematical point of view. So we have another nice example of how string theory enriches mathematical researches.

5. Landau-Ginzburg Theory Associated with the Miniversal Deformation of the Airy Curve

In last section we have seen that the miniversal deformation
\[
y = \frac{1}{2}x^2 + t_0
\]
of the Airy curve $y = \frac{1}{2}x^2$ encodes the information of $\frac{\partial F_0}{\partial u_n}(u_0)$. In this section we take $W = y = \frac{1}{2}x^2 + t_0$ as the superpotential and consider the corresponding Landau-Ginzburg theory by defining some correlation functions on the small phase space in this theory. It is clear that one does not get a field theory in the sense of Section 2, nevertheless, we will identify the correlation functions we define in this section with the corresponding correlation functions in the theory of the 2D topological gravity on the small phase space. The results in this section motivates the extension to the big phase space in the next section.

5.1. The primary field and its descendant fields. The primary field $\phi_0$ is defined by:

\begin{equation}
\phi_0(x) := \frac{\partial W}{\partial u_0} = (\partial_x L)_+ = 1,
\end{equation}

where $L = (2W)^{1/2}$ is the Puiseux series:

\begin{equation}
L = (x^2 + 2u_0)^{1/2} = x(1 + \frac{2u_0}{x^2})^{1/2} = x + \frac{u_0}{x} - \frac{1}{2} \frac{u_0^2}{x^3} + \frac{1}{2} \frac{u_0^3}{x^5} + \cdots.
\end{equation}

It gives a basis of the Jacobian ring

\begin{equation}
J_W := \mathbb{C}[x]/(\partial_x W).
\end{equation}

Inspired by [37], the authors of [16] constructed the $n$-th gravitational descendant field of $\phi_0(x)$ as follows:

\begin{equation}
\sigma_n(\phi_0) = \frac{1}{(2n-1)!!}(L^{2n}\partial_x L)_+ = \frac{1}{(2n+1)!!}(\partial_x L^{2n+1})_+.
\end{equation}

The following are some explicit examples:

$\phi_0 = 1$,

$\sigma_1(\phi_0) = x^2 + u_0$,

$\sigma_2(\phi_0) = \frac{1}{3!!}(x^4 + 3u_0x^2 + \frac{3}{2}u_0^2)$,

$\sigma_3(\phi_0) = \frac{1}{5!!}(x^6 + 5u_0x^4 + \frac{15}{2}u_0^2x^2 + \frac{5}{2}u_0^3)$,

$\sigma_4(\phi_0) = \frac{1}{7!!}(x^8 + 7u_0x^6 + \frac{35}{2}u_0^2x^4 + \frac{35}{2}u_0^3x^2 + \frac{35}{8}u_0^4)$,

$\sigma_5(\phi_0) = \frac{1}{9!!}(x^{10} + 9u_0x^8 + \frac{63}{2}u_0^2x^6 + \frac{105}{2}u_0^3x^4 + \frac{315}{8}u_0^4x^2 + \frac{63}{8}u_0^5)$.

Since they are polynomials in $x$, one can understand them as fields living on the Airy curve $y = \frac{1}{2}x^2$, or maybe more appropriately, on the deformed Airy curve $y = \frac{1}{2}x^2 + u_0$. 
5.2. Variations of the descendant fields with respect to \( u_0 \). It is clear that
\begin{align}
\partial_{u_0} W &= \phi_0(x) = 1, \\
\partial_{u_0} \phi_0(x) &= 0.
\end{align}

By differentiating both sides of \( L^2 = W \), one gets:
\begin{equation}
\partial_{u_0} L = \frac{1}{L}.
\end{equation}

**Lemma 5.1.** The variation of \( \sigma_n(\phi_0) \) is given by:
\begin{equation}
\partial_{u_0} \sigma_n(\phi_0) = \sigma_{n-1}(\phi_0),
\end{equation}
for \( n \geq 1 \).

**Proof.** Take \( \partial_{u_0} \) on both sides of (138) and apply (141):
\begin{align}
\partial_{u_0} \sigma_n(\phi_0) &= \frac{2n}{(2n-1)!!} (L^{2n-1} \cdot \partial_{u_0} L \cdot \partial_x L)_+ + \frac{1}{(2n-1)!!} (L^n \partial_x (\partial_{u_0} L))_+ \\
&= \frac{2n}{(2n-1)!!} (L^{2n-1} \cdot \frac{1}{L} \cdot \partial_x L)_+ + \frac{1}{(2n-1)!!} (L^n \partial_x (\frac{1}{L}))_+ \\
&= \frac{1}{(2n-3)!!} (L^{2n-2} \partial_x L)_+ = \sigma_{n-1}(\phi_0).
\end{align}

Because \( \sigma_n(\phi_0) \) is a degree \( n \) polynomial in \( x^2 \), with leading term \( x^{2n} \), as a corollary, we have

**Corollary 5.2.** The following recursion relation holds:
\begin{equation}
\sigma_n(\phi_0) = \frac{x^{2n}}{(2n-1)!!} + \int_0^{u_0} \sigma_{n-1}(\phi_0) du_0.
\end{equation}

5.3. Explicit expressions for descendant fields. By induction one then gets:

**Proposition 5.3.** An explicit solution of (143) is given by:
\begin{equation}
\sigma_n(\phi_0) = \sum_{j=0}^n \frac{1}{(2j-1)!!(n-j)!} x^{2j} u_0^{n-j}.
\end{equation}

One can use (144) to get the following formula for the generating series of \( \{\sigma_n(\phi_0)\}_{n \geq 0} \) (\( \sigma_0(\phi_0) = \phi_0 \)):
\begin{equation}
\sum_{n \geq 0} t^n \sigma_n(\phi_0) = e^{u_0 t} (1 + \sqrt{\pi x^2 t/2} e^{x^2 t/2} \text{erf}(\sqrt{x^2 t/2})),
\end{equation}
where \( \text{erf}(x) \) is the error function:

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt.
\]

As another application of (144), we have

**Proposition 5.4.** The following recursion relation holds for \( n \geq 0 \):

\[
\sigma_1(\phi_0) \cdot (2n-1)!! \sigma_n(\phi_0) = (2n+1)!! \sigma_{n+1}(\phi_0)
\]

\[
- u_0 \cdot (2n-1)!! \sigma_n(\phi) + \frac{(2n-1)!!}{(n+1)!} u_0^{n+1}.
\]

Using generating series, the recursion relations (147) can be explicitly solved as follows:

\[
\sum_{n \geq 0} (2n-1)!! \sigma_n(\phi_0) t^n = \frac{\sqrt{1 - 2u_0 t}}{1 - t(x^2 + 2u_0)}.
\]

It follows that

\[
(2n-1)!! \sigma_n(\phi_0) = \sum_{j=0}^{n} \frac{(2j-3)!!}{j!} \cdot u_0^j (x^2 + 2u_0)^{n-j}.
\]

5.4. **Descendant operator algebra and structure constants.** It is easy to see that

\[
(2j-1)!! \sigma_j(\phi_0) \cdot (2k-1)!! \sigma_k(\phi_0) = \sum_{j=0}^{j+k} c_{jk}^l u_0^{j+k-l} \cdot (2l-1)!! \sigma_l(\phi_0)
\]

for some constants \( c_{jk}^l \). In other words, \( \{\sigma_n(\phi_0)\}_{n \geq 0} \) generate an algebra over the ring \( \mathbb{C}[u_0] \). We will call this algebra the **descendant algebra on the small phase**.

To determine the structure constants \( c_{jk}^l \), let

\[
f_n(x) = \sum_{j=0}^{n} \frac{(2n-1)!!}{(2j-1)!!(n-j)!} x^{2j},
\]

then

\[
f_j(x) \cdot f_k(x) = \sum_{l=0}^{j+k} c_{jl}^l f_l(x).
\]
For example,
\[ f_0(x) = 1, \]
\[ f_1(x) = x^2 + 1, \]
\[ f_2(x) = x^4 + 3x^2 + \frac{3}{2}, \]
\[ f_3(x) = x^6 + 5x^4 + \frac{15}{2}x^2 + \frac{5}{2}, \]
\[ f_4(x) = x^8 + 7x^6 + \frac{35}{2}x^4 + \frac{35}{2}x^2 + \frac{35}{8}, \]
\[ f_5(x) = x^{10} + 9x^8 + \frac{63}{2}x^6 + \frac{105}{2}x^4 + \frac{315}{8}x^2 + \frac{63}{8}. \]

By (147), for \( n \geq 1 \),
\[ f_1 f_n = f_{n+1} - f_n + \frac{(2n-1)!!}{(n+1)!}. \]

One can check that
\[ f_2 f_2 = (f_4 - f_3 - \frac{1}{2} f_2) + \left( \frac{1}{2} f_1 + \frac{5}{8} f_0 \right), \]
\[ f_2 f_3 = (f_5 - f_4 - \frac{1}{2} f_3) + \left( \frac{5}{8} f_1 + \frac{7}{8} f_0 \right), \]
\[ f_2 f_4 = (f_6 - f_5 - \frac{1}{2} f_4) + \left( \frac{7}{8} f_2 + \frac{21}{16} f_0 \right), \]
\[ f_3 f_3 = (f_6 - f_5 - \frac{1}{2} f_4 - \frac{1}{2} f_3) + \left( \frac{5}{8} f_2 + \frac{7}{8} f_1 + \frac{21}{16} f_0 \right). \]

One recognizes the structure constants as the coefficients of the following series:
\[ \sqrt{1 - 2x} = 1 - \sum_{n=1}^{\infty} \frac{(2n-3)!!}{n!} x^n \]
\[ = 1 - x - \frac{1}{2} x^2 - \frac{1}{2} x^3 - \frac{5}{8} x^4 - \frac{7}{8} x^5 - \frac{21}{16} x^6 - \cdots, \]

**Proposition 5.5.** For \( k \geq j \geq 1 \),
\[ f_j f_k = f_{j+k} - \sum_{l=1}^{j} \frac{(2l-3)!!}{l!} \frac{f_{j+k-l}}{f_{j-l}} \]
\[ + \sum_{l=1}^{j} \frac{(2(k+l)-3)!!}{(k+l)!} \frac{f_{j}}{f_{j-l}}. \]
Proof. We prove (155) by induction on \( j \). For \( j = 1 \), (155) is just (153). Suppose that (155) holds for some \( j \geq 1 \). Then by (153),

\[
f_{j+1} = f_1 f_j + f_j - \frac{(2j - 1)!!}{(j+1)!}.
\]

And so for \( k \geq j + 1 \),

\[
f_{j+1} f_k = (f_1 f_j + f_j - \frac{(2j - 1)!!}{(j+1)!}) \cdot f_k
\]

\[
= f_1 \left( f_{j+k} - \sum_{l=1}^{j} \frac{(2l - 3)!!}{l!} f_{j+k-l} + \sum_{l=1}^{j} \frac{(2(k + l) - 3)!!}{(k + l)!} f_{j-l} \right)
\]

\[
+ \left( f_{j+k} - \sum_{l=1}^{j} \frac{(2l - 3)!!}{l!} f_{j+k-l} + \sum_{l=1}^{j} \frac{(2(k + l) - 3)!!}{(k + l)!} f_{j-l} \right)
\]

\[
- \frac{(2j - 1)!!}{(j+1)!} f_k
\]

\[
= \left( f_{j+k+1} - f_{j+k} + \frac{(2(j + k) - 1)!!}{(j + k + 1)!} \right)
\]

\[
- \sum_{l=1}^{j} \frac{(2l - 3)!!}{l!} \left( f_{j+k-l+1} - f_{j+k-l} + \frac{(2(j + k - l) - 1)!!}{(j + k - l + 1)!} \right)
\]

\[
+ \sum_{l=1}^{j} \frac{(2(k + l) - 3)!!}{(k + l)!} \left( f_{j-l+1} - f_{j-l} + \frac{(2(j - l) - 1)!!}{(j - l + 1)!} \right)
\]

\[
+ \left( f_{j+k} - \sum_{l=1}^{j} \frac{(2l - 3)!!}{l!} f_{j+k-l} + \sum_{l=1}^{j} \frac{(2(k + l) - 3)!!}{(k + l)!} f_{j-l} \right)
\]

\[
- \frac{(2j - 1)!!}{(j+1)!} f_k.
\]

Here in the last equality we have used (153). The proof is completed by obvious cancelations. \( \square \)

5.5. Some \( n \)-point correlation functions on the small phase space. In the physics literature \[13\], the authors first derived based on physical arguments a formula for three-point function on the small phase space in the Landau-Ginzburg model associated with \( W \), then obtained formulas for two-point functions and one-point functions by integrations. The free energy was obtained from the one-point functions in another work of the same authors \[14\] using the weighted homogeneity of the free energy. General \( n \)-point functions \( (n \geq 4) \) can
be obtained by differentiating the three-point function. For generalizations that include gravitational descendants and extension to big phase space, we use [16, 17, 39] as references. In this subsection we will present a mathematical reformulation of [13, 16] by reversing the above procedure. We will take the formula for one-point function as the definition of the one-point function, and define $n$-point functions ($n \geq 2$) by taking derivatives.

We define the one-point function on the small phase space by:

$$\langle \langle \phi_0(x) \rangle \rangle_0(u_0) = 1$$

(156)

and so for $n \geq 2$, the $n$-point function are defined by:

$$\langle \langle \phi_0(x)^n \rangle \rangle_0(u_0) = \frac{\partial}{\partial u_0} \langle \langle \phi_0(x)^{n-1} \rangle \rangle_0(u_0).$$

(157)

Using the equation (141), one can check that for $n \geq 2$,

$$\langle \langle \sigma_n(\phi_0) \rangle \rangle_0(u_0) = 1$$

(158)

In [16, (46)], the following generalization of (156) was given:

$$\langle \langle \sigma_n(\phi_0) \rangle \rangle_0(u_0) = \frac{1}{(2n+3)!} \text{res}(L^{2n+3} dx).$$

(159)

We will take this as the definition of $\langle \langle \sigma_n(\phi_0) \rangle \rangle_0(u_0)$, and so for $m \geq 1$, define

$$\langle \langle \sigma_n(\phi_0) \phi_0(x)^m \rangle \rangle_0(u_0) = \frac{\partial}{\partial u_0} \langle \langle \sigma_n(\phi_0) \phi_0(x)^{m-1} \rangle \rangle_0(u_0).$$

(160)

One can easily see that

$$\langle \langle \sigma_n(\phi_0) \phi_0(x)^m \rangle \rangle_0(u_0) = \frac{\prod_{j=0}^{m-1} (2n + 3 - 2j)}{(2n + 3)!!} \text{res}(L^{2n-2m+3} dx).$$

(161)

In particular, when $n = 0$,

$$\langle \langle \phi_0(x)^{m+1} \rangle \rangle_0(u_0) = \frac{\prod_{j=0}^{m-1} (3 - 2j)}{3} \text{res}(L^{-2m+3} dx).$$

(162)

This matches with (158). Also when $m = 3$,

$$\langle \sigma_n(\phi_0) \phi_0^2 \rangle_0(u_0) = \frac{1}{(2n-1)!!} \text{res}(L^{2n-1} dx) = \text{res} \left( \frac{\sigma_n(\phi_0) \phi_0 \phi_0}{\partial x W} dx \right).$$
Inspired by the last equality and \([13]\), we define
\[
\langle\langle \sigma_n(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0) \rangle\rangle_0(u_0) = \text{res} \left( \frac{\sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0)}{\partial_x W} \right) dx,
\]
and for \(m \geq 1\) recursively define:
\[
\langle\langle \sigma_n(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0)\phi_m \rangle\rangle_0(u_0) = \partial_{u_0} \langle\langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0)\phi_{m-1} \rangle\rangle_0(u_0).
\]

5.6. **Compatibility of the definitions.** In last subsection, we have given the definitions of the correlation functions \(\langle\langle \sigma_n(\phi_0)\phi_0(x)^m \rangle\rangle_0(u_0)\) and \(\langle\langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0)\phi_m^m \rangle\rangle_0(u_0)\). We now check their compatibility. First we need the following:

**Proposition 5.6.** The following formulas hold:

\[
\langle\langle \sigma_n(\phi_0) \rangle\rangle_0(u_0) = \frac{u_0^{n+2}}{(n+2)!},
\]
(165)
\[
\langle\langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0) \rangle\rangle_0(u_0) = \frac{u_0^{n_1+n_2+n_3}}{\prod_{j=1}^3 n_j!}.
\]
(166)

**Proof.** Formula (166) follows directly from the explicit formula for \(\sigma_n(\phi_0)\) given by (144) and the definition (163). For the proof of (165), we proceed as follows:

\[
\langle\langle \sigma_n(\phi_0) \rangle\rangle_0(u_0) = \frac{1}{(2n+3)!!} \text{res} (L^{2n+3} dx) = -\frac{1}{(2n+3)!!} \text{res} (xdL^{2n+3})
\]
\[
= -\frac{1}{(2n+1)!!} \text{res} (L^{2n+2} (L - \sum_{m=1}^{\infty} \frac{(2m-3)!!}{m!} L^{2m-1})) dL)
\]
\[
= \frac{u_0^{n+2}}{(n+2)!}.
\]

In the above we have used:

\[
x = (L^2 - 2u_0)^{1/2} = L(1 - \frac{2u_0}{L^2})^{1/2} = L - \sum_{m=1}^{\infty} \frac{(2m-3)!!}{m!} \frac{1}{L^{2m-1}}.
\]
By taking derivatives on both sides of (165),
\[ \langle \langle \sigma_n(\phi_0)\phi_0 \rangle \rangle_0(u_0) = \frac{u_0^{n+1}}{(n+1)!}, \]
\[ \langle \langle \sigma_n(\phi_0)\phi_0^2 \rangle \rangle_0(u_0) = \frac{u_0^n}{n!}. \]
On the other hand, taking \( n_1 = n \) and \( n_2 = n_3 = 0 \) in (166),
\[ \langle \langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\phi_0^2 \rangle \rangle_0(u_0) = \frac{u_0^{n_1+n_2}}{n_1!n_2!}, \]
It is a match. One also has for \( n_3 = 0 \) in (166):
\[ \langle \langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\phi_0 \rangle \rangle_0(u_0) = \frac{u_0^{n_1+n_2+1}}{(n_1+n_2+1) \cdot n_1!n_2!}, \]
and so after integration one would have obtained
\[ \langle \langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\rangle \rangle_0(u_0) = \frac{u_0^{n_1+n_2+1}}{(n_1+n_2+1) \cdot n_1!n_2!}, \]
if the left-hand side were already defined.

5.7. Identification with genus zero \( n \)-point functions in 2D topological gravity on the small phase space.

**Proposition 5.7.** For \( n_1, n_2, n_3, m \geq 0 \),
\[ \langle \langle \sigma_{n_1}(\phi_0)\sigma_{n_2}(\phi_0)\sigma_{n_3}(\phi_0)\phi_0^m \rangle \rangle_0(t_0) = \frac{\partial^{m+3} F_0}{\partial t_{n_1}\partial t_{n_2}\partial t_{n_3}\partial u_0}(t_0). \]

**Proof.** It suffices to prove the \( m = 0 \) case. Recall \( u(0) = \partial^2_{u_0} F_0 \) satisfies the dispersionless KdV hierarchy (42):
\[ \partial_{t_n} u(0) = \partial_{t_0} \frac{u^{n+1}(0)}{(n+1)!}. \]
Therefore, we have
\[
\partial_{t_{n_1}} \partial_{t_{n_2}} \partial_{t_{n_3}} u(0) = \partial_{t_{n_1}} \partial_{t_{n_2}} \partial_{t_0} \frac{u^{n_3+1}(0)}{(n_3+1)!} \\
= \partial_{t_{n_1}} \partial_{t_0} \left( \frac{u^{n_3}(0)}{n_3!} \partial_{t_{n_2}} u(0) \right) = \partial_{t_{n_1}} \partial_{t_0} \left( \frac{u^{n_3}(0)}{n_3!} \partial_{t_0} \left( \frac{u^{n_2+1}(0)}{(n_2+1)!} \right) \right) \\
= \partial_{t_{n_1}} \partial_{t_0}^2 \left( \frac{u^{n_2+n_3+1}(0)}{n_2!n_3!(n_2+n_3+1)!} \right) = \partial_{t_0}^2 \left( \frac{u^{n_1+n_2+n_3}(0)}{n_1!n_2!n_3!} \partial_{t_0} u(0) \right). \\
\]
After integration with respect to \( t_0 \) twice, one gets:
\[ \partial_{t_{n_1}} \partial_{t_{n_2}} \partial_{t_{n_3}} F_0 = \frac{u^{n_1+n_2+n_3}(0)}{n_1!n_2!n_3!} \partial_{t_0} u(0). \]
Now we restrict to the small phase space and use the fact that $u_{(0)}(t_0) = t_0 = u_0$ to get:

$$\partial_{t_1} \partial_{t_2} \partial_{t_3} F_0(t_0) = \frac{u_0^{n_1+n_2+n_3}}{n_1!n_2!n_3!}.$$  

The completed by comparing with (166). □

6. Field Theory Associated with Special Deformation of the Airy Curve

The discussion in last section clearly indicates that to obtain a field that is the mirror theory to the theory of 2D topological gravity, miniversal deformation of the Airy curve does not suffice. In IV we have constructed a general deformation

$$x = f - \sum_{n \geq 0} (2n + 1)u_n f^{2n-1} - \sum_{n \geq 0} \frac{\partial F_0}{\partial u_n}(u) \cdot f^{-2n-3}. \tag{170}$$

and

$$2y = f^2 = c_0 + c_1 x^2 + c_2 x^4 + c_3 x^6 + \cdots,$$

of the Airy curve $y = \frac{1}{2} x^2$ associated with the genus zero free energy $F_0(u)$. In this section we will take $W = y$ as the superpotential and construct a field theory in genus zero by generalizing the discussions in last section. In next section we will extend this field theory to arbitrary genus and prove it is mirror to the theory of 2D topological gravity.

6.1. The primary field and its descendant fields. As in last section, the primary field $\phi_0$ is defined by:

$$\phi_0 := \frac{\partial W}{\partial u_0}. \tag{171}$$

Since now the superpotential $W$ depends also on the coupling constants $u_n$, we define in the same fashion the following fields:

$$\phi_n := \frac{\partial W}{\partial u_n}. \tag{172}$$

This generalizes the treatment in [13] to the big phase space. Recall that $W \in \mathbb{C}[[u_0, u_1, \ldots]]_2$ and deg $u_n = 2 - 2n$. Therefore, $\phi_n$ is weighted homogenous of degree $2n$, i.e.

$$\phi_n \in \mathbb{C}[[x, u_0, u_1, \ldots]]. \tag{173}$$

In the following we will explain why $\phi_n$ can be regarded as gravitational descendants of $\phi_0$. 

We also define another sequence of fields $\sigma_n$ as in last section by

\begin{equation}
\sigma_n = (L^{2n} \partial x L)_+ = \frac{1}{2n+1} (\partial x L^{2n+1})_+.
\end{equation}

We now examine the relationship between $\{\phi_n\}$ and $\{\sigma_n\}$.

Recall the following relationship between $x^2$ and $W$:

\begin{equation}
x^2 = a_0 + a_1 \cdot (2W) + a_2 \cdot (2W)^2 + \cdots,
\end{equation}

where

\begin{equation}
a_0 = -2u_0(1 - 3u_1) + 2 \sum_{n \geq 0} (2n + 5) u_{n+2} \frac{\partial F_0}{\partial u_n},
\end{equation}

\begin{equation}
a_1 = (1 - 3u_1)^2 + 2u_0 \cdot 5u_2 + 2 \sum_{n \geq 0} (2n + 7) u_{n+3} \cdot \frac{\partial F_0}{\partial u_n},
\end{equation}

and for $m \geq 2$,

\begin{equation}
a_m = -2(1 - 3u_1) \cdot (2m + 1) u_m + 2u_0 \cdot (2m + 3) u_{m+1}
+ \sum_{m_1, m_2 \geq 1, m_1 + m_2 = m+1} (2m_1 + 1) u_{m_1} \cdot (2m_2 + 1) u_{m_2}
\end{equation}

\begin{equation}
\quad + 2 \sum_{n \geq 0} (2n + 2m + 1) u_{n+2+m} \cdot \frac{\partial F_0}{\partial u_n}.
\end{equation}

In particular, each $a_n \in \mathbb{C}[[u_0, u_1, \ldots]]_{2-2m}$, the degree $2-2m$ part of $\mathbb{C}[[u_0, u_1, \ldots]]$, while $W \in \mathbb{C}[[x, u_0, u_1, \ldots]]$.

**Lemma 6.1.** The following formula holds:

\begin{equation}
\phi_n = -\frac{\partial_{u_n} a_0 + 2\partial_{u_n} a_1 \cdot W + 4\partial_{u_n} a_2 \cdot W^2 + \cdots}{2x} \partial_x W.
\end{equation}

**Proof.** After taking $\partial_x$ on both sides of (175) we get:

\begin{equation}
\partial_x W = \frac{x}{a_1 + 4a_2 W + 12a_3 W^2 + \cdots}.
\end{equation}

Similarly, after taking $\partial_{u_n}$ on both sides of (175) we get:

\begin{equation}
\partial_{u_n} W = -\frac{\partial_{u_n} a_0 + 2\partial_{u_n} a_1 \cdot W + 4\partial_{u_n} a_2 \cdot W^2 + \cdots}{2(a_1 + 4a_2 W + 12a_3 W^2 + \cdots)},
\end{equation}

It follows that

\begin{equation}
\partial_{u_n} W = -\frac{\partial_{u_n} a_0 + 2\partial_{u_n} a_1 \cdot W + 4\partial_{u_n} a_2 \cdot W^2 + \cdots}{2x} \partial_x W.
\end{equation}

This completes the proof. □
Proposition 6.2. When restricted to the small phase space we have
\begin{equation}
\phi_n(x, u_0) = (2n + 1)!! \cdot \sigma_n(\phi_0)(x, u_0).
\end{equation}

Proof. When restricted to the small phase space, we have
\[ W = \frac{1}{2} x^2 + u_0, \]
and by (116),
\[ \frac{\partial a_m}{\partial u_n}(u_0) = 2(2n + 1) \cdot \frac{(2n - 2m - 3)!!}{(n-m)!} u_0^{n-m}, \]
therefore, by (179),
\[ \phi_n = -\frac{1}{2}(\frac{\partial a_0}{\partial u_n}(u_0) + \frac{\partial a_1}{\partial u_n}(u_0) \cdot (x^2 + 2u_0) + \cdots + \frac{\partial a_n}{\partial u_n}(u_0) \cdot (x^2 + 2u_0)^n) \]
\[ = -(2n + 1) \cdot \left( \frac{(2n - 3)!!}{n!} u_0^n + \frac{(2n - 5)!!}{(n-1)!} u_0^{n-1} \cdot (x^2 + 2u_0) + \cdots + u_0 \cdot (x^2 + 2u_0)^{n-1} - (x^2 + 2u_0)^n \right) \]
\[ = (2n + 1)!! \cdot \sigma_n(\phi_0). \]

In the last equality on the right-hand side we have used (149) \(\square\)

We have related \(\phi_n(x, u_0)\) to \(\sigma_n(\phi_0)(x, u_0)\) defined by (138). Unfortunately this relation do not hold anymore on the big phase space, as shown by the following example. Recall by (134), when only \(u_0\) and \(u_2\) are nonzero,
\begin{equation}
2W = L^2 = \frac{2}{15} \cdot \frac{1}{u_2} \left( 1 - (1 - 30u_0u_2)^{1/2} \right)
+ \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{3n+1}{n} \frac{(5u_2)^{n}x^{2n+2}}{(1 - 30u_0u_2)^{(3n+2)/2}}.
\end{equation}

It follows that
\[ \phi_0(x, u_0, u_2) = (1 - 30u_0u_2)^{-1/2} + \frac{3}{2} \sum_{n=0}^{\infty} \binom{3n+2}{n+1} \frac{(5u_2x^2)^{n+1}}{(1 - 30u_0u_2)^{(3n+4)/2}}. \]

Next we compute \(L\). First we rewrite the above equality as follows:
\begin{align*}
L^2 &= \frac{x^2}{1 - 30u_0u_2} \left( 1 + \frac{2}{15} \cdot \frac{(1 - (1 - 30u_0u_2)^{1/2})(1 - 30u_0u_2)}{u_2} x^{-2} \right)
+ \sum_{n=1}^{\infty} \frac{1}{n+1} \binom{3n+1}{n} \frac{(5u_2)^{n}x^{2n}}{(1 - 30u_0u_2)^{3n/2}}.
\end{align*}
Then we note

\[
\begin{align*}
( x(1 + x^{-2}(a_0 + a_2 x^4 + a_3 x^6 + \cdots))^{1/2})_+ \\
= x \left( 1 - \frac{1}{4} a_0 a_2 + \frac{3}{16} a_0^2 a_3 - \frac{5}{32} a_0^3 a_4 - \frac{15}{64} a_0^2 a_2^2 + \frac{35}{64} a_0^3 a_2 a_3 \\
- \frac{105}{256} a_0^3 a_2^3 - \frac{315}{512} a_0^4 a_2 a_4 + \frac{3465}{2048} a_0^4 a_2^2 a_3 - \frac{315}{1024} a_0^4 a_3^2 + \cdots \right) + \cdots,
\end{align*}
\]

it follows that

\[
L_+ = \frac{x}{(1 - 30 u_0 u_2)^{1/2}} (1 - 5 u_0 u_2 - 75 (u_0 u_2)^2 - \frac{2875}{2} (u_0 u_2)^3 + \cdots) + \cdots,
\]

and so

\[
\sigma_0 = \partial_x L_+ = \frac{1}{(1 - 30 u_0 u_2)^{1/2}} (1 - 5 u_0 u_2 - 75 (u_0 u_2)^2 - \frac{2875}{2} (u_0 u_2)^3 + \cdots) + \cdots,
\]

It is then clear that:

\[
(185) \quad \phi_0(u_0, u_2) \neq \sigma_0(u_0, u_2).
\]

6.2. **Family of flat connections.** Inspired by the theory of Frobenius manifolds [12], we introduce the following operators \((\lambda \in \mathbb{C}):\)

\[
(186) \quad \nabla^\lambda_{\frac{\partial}{\partial t_i}} \phi_k := \partial_{t_i} \phi_k + \lambda \phi_i \cdot \phi_k.
\]

**Proposition 6.3.** The operators \(\nabla^\lambda_{\frac{\partial}{\partial t_i}}\) define a family of flat connections, i.e.,

\[
(187) \quad \nabla^\lambda_{\frac{\partial}{\partial t_i}} \nabla^\lambda_{\frac{\partial}{\partial t_j}} \phi_k = \nabla^\lambda_{\frac{\partial}{\partial t_j}} \nabla^\lambda_{\frac{\partial}{\partial t_i}} \phi_k,
\]

for \(i, j, k \geq 0\) and all \(\lambda \in \mathbb{C}\).

**Proof.**

\[
\nabla^\lambda_{\frac{\partial}{\partial t_i}} \nabla^\lambda_{\frac{\partial}{\partial t_j}} \phi_k = (\partial_{t_i} + \lambda \phi_i \cdot) (\partial_{t_j} \phi_k + \lambda \phi_j \cdot \phi_k)
\]

\[
= \partial_{t_i} \partial_{t_j} \phi_k + \lambda (\phi_i \cdot \partial_{t_j} \phi_k + \partial_{t_i} (\phi_j \cdot \phi_k)) + \lambda^2 \phi_i \cdot \phi_j \cdot \phi_k
\]

\[
= \partial_{t_i} \partial_{t_j} \phi_k + \lambda (\phi_i \cdot \partial_{t_j} \phi_k + \phi_j \cdot \partial_{t_i} \phi_k + \partial_{t_i} \phi_j \cdot \phi_k)
\]

\[
+ \lambda^2 \phi_i \cdot \phi_j \cdot \phi_k
\]

\[
= \partial_{t_i} \partial_{t_j} \phi_k + \lambda (\phi_j \cdot \partial_{t_i} \phi_k + \phi_i \cdot \partial_{t_j} \phi_k + \partial_{t_j} \phi_i \cdot \phi_k)
\]

\[
+ \lambda^2 \phi_j \cdot \phi_i \cdot \phi_k
\]

\[
= \nabla^\lambda_{\frac{\partial}{\partial t_j}} \nabla^\lambda_{\frac{\partial}{\partial t_i}} \phi_k.
\]

\[\square\]
6.3. The n-point functions and the mirror symmetry in genus 0. We define the one-point function in genus zero on the big phase space by:

\[
\langle \langle \phi_j \rangle \rangle_0(u) = \frac{1}{(2j+3)!!} \text{res}(L^{2j+3}dx),
\]

and define n-point function \((n \geq 2)\) by taking derivatives:

\[
\langle \langle \phi_{j_1}, \ldots, \phi_{j_n} \rangle \rangle_0(u) = \frac{\partial}{\partial t_{j_1}} \langle \langle \phi_{j_2} \cdots \phi_{j_n} \rangle \rangle_0(u).
\]

The following result tells us how to recover the genus zero free energy of the 2D topological gravity from the special deformation of the Airy curve.

**Theorem 6.4.** We have the following identification of n-point functions:

\[
\langle \langle \phi_{j_1}, \ldots, \phi_{j_n} \rangle \rangle_0(u) = \frac{\partial^n F_0}{\partial t_{j_1} \cdots \partial t_{j_n}}(t).
\]

**Proof.** It suffices to prove the one-point case as follows:

\[
\langle \langle \phi_n \rangle \rangle_0(u) = \frac{1}{(2n+3)!!} \text{res}(L^{2n+3}dx) = -\frac{1}{(2n+3)!!} \text{res}(xdL^{2n+3})
\]

\[
= -\frac{1}{(2n+1)!!} \text{res} \left( L^{2n+2}(L - \sum_{m \geq 0} (2m + 1)u_mL^{2m-1}
\right.
\]

\[
- \sum_{m \geq 0} \frac{\partial F_0}{\partial u_m}(u) \cdot L^{-2m-3}dL \left. \right) \)

\[
= \frac{1}{(2n+1)!!} \frac{\partial F_0}{\partial u_n}(u) = \frac{\partial F_0}{\partial t_n}(t).
\]

\[\square\]

7. Quantum Deformation Theory of the Airy Curve

We have already shown that the free energy in genus zero of 2D topological quantum gravity can be used to produce special deformation of the Airy curve, and how to recover the free energy in genus zero of the 2D topological gravity from the deformed superpotential function. In this section we will see that this deformation lead to a quantization of the Airy curve that can be used to recover the free energy in all genera.
7.1. Symplectic reformulation of the special deformation. Rewrite (91) as follows:

\( x(z) = - \sum_{n \geq 0} (2n + 1)\tilde{u}_nz^{2n-1/2} - \sum_{n \geq 0} \frac{\partial F_0}{\partial \tilde{u}_n}(u) \cdot z^{-\frac{2n+3}{2}}, \)

where \( z = 2y = f^2 \), and

\( \tilde{u}_n = u_n - \frac{1}{3}\delta_{n,1}. \)

One can formally understand \( x \) as a field on the Airy curve.

Consider the space of

\( V = z^{1/2}\mathbb{C}[z]. \)

We write an element in \( V \) as

\( \sum_{n=0}^{\infty} (2n + 1)\tilde{u}_nz^{(2n-1)/2} + \sum_{n=0}^{\infty} \tilde{v}_nz^{-(2n+3)/2} \)

We regard \( \{\tilde{u}_n, \tilde{v}_n\} \) as linear coordinates on \( V \), and introduce the following symplectic structure on \( V \):

\( \omega = \sum_{n=0}^{\infty} d\tilde{u}_n \wedge d\tilde{v}_n. \)

It follows that

\( \tilde{v}_n = \frac{\partial F_0}{\partial u_n}(u) \)

defines a Lagrangian submanifold in \( V \), and so does

\( \tilde{v}_n = \frac{\partial (\lambda^2F)}{\partial u_n}(u). \)

In other words, free energy in all genera produces a deformation of a Lagrangian submanifold.

7.2. Canonical quantization of the special deformation of Airy curve. Take the natural polarization that \( \{q_n = \tilde{u}_n\} \) and \( \{p_n = \tilde{v}_n\} \), one can consider the canonical quantization:

\( \hat{\tilde{u}}_n = \tilde{u}_n, \quad \hat{\tilde{v}}_n = \frac{\partial}{\partial \tilde{u}_n}. \)

Corresponding to the field \( x \), consider the following fields of operators on the Airy curve:

\( \hat{x} = - \sum_{n=0}^{\infty} \beta_{-(2n+1)} f^{2n-1} - \sum_{n=0}^{\infty} \beta_{2n+1} f^{-2n-3}, \)
where the operators $\beta_{2k+1}$ are defined by:

\begin{equation}
\beta_{-(2k+1)} = (2k + 1)\hat{u}_k, \quad \beta_{2k+1} = \frac{\partial}{\partial \hat{u}_k}.
\end{equation}

It is better to write $\hat{x}$ in the $z$-coordinate:

\begin{equation}
\hat{x}(z) = -\sum_{m \in \mathbb{Z}} \beta_{-(2m+1)} z^{m-1/2} = -\sum_{m \in \mathbb{Z}} \beta_{2m+1} z^{-m-3/2}
\end{equation}

7.3. The 2-Reduced bosonic Fock space. As usual, the operators $\{\beta_{2n+1}\}_{n \geq 0}$ are called annihilators while the operators $\{\beta_{-(2n+1)}\}_{n \geq 0}$ are called creators. Given $\beta_{2n_1+1}, \ldots, \beta_{2n_k+1}$, their normally ordered products are defined:

\begin{equation}
: \beta_{2n_1+1} \cdots \beta_{2n_k+1} := \beta_{2n'_1+1} \cdots \beta_{2n'_k+1},
\end{equation}

where $n'_1 \geq \cdots \geq n'_k$ is a rearrangement of $n_1, \ldots, n_k$. Denote $|0\rangle$ the vector 1, and by $\Lambda^{(2)}$ the space spanned by elements of form $\beta_{-(2n_1+1)} \cdots \beta_{-(2n_k+1)}|0\rangle$. We will refer to $\Lambda^{(2)}$ as the 2-reduced bosonic Fock space. On this space one can define a Hermitian product by setting

\begin{equation}
\langle 0|0 \rangle = 1,
\end{equation}

\begin{equation}
\beta^{*}_{2n+1} = \beta_{-(2n+1)}.
\end{equation}

For a linear operator $A : \Lambda^{(2)} \to \Lambda^{(2)}$, its vacuum expectation value is defined by:

\begin{equation}
\langle A \rangle = \langle 0|A|0 \rangle.
\end{equation}

7.4. Regularized products of two fields. We now study the product of the fields $\hat{x}(z)$ with $\hat{x}(z)$. This cannot be defined directly, because for example,

\begin{equation}
\langle 0|\hat{x}(z)\hat{x}(z)|0 \rangle = \sum_{n \geq 0} (2n + 1) = +\infty.
\end{equation}

To fix this problem, we follow the common practice in the physics literature by using the normally ordered products of fields and regularization of the singular terms as follows. First note

\[ \hat{x}(z) \cdot \hat{x}(w) = \hat{x}(z)\hat{x}(w) : + \sum_{n=0}^{\infty} (2n + 1) z^{-(2n-3)/2} w^{(2n-1)/2} \]

\[ = \hat{x}(z)\hat{x}(w) : + \frac{z + w}{\sqrt{zw(z - w)}}. \]

It follows that

\begin{equation}
\langle \hat{x}(z) \cdot \hat{x}(w) \rangle = \frac{z + w}{\sqrt{zw(z - w)}}.
\end{equation}
hence

\[ \hat{x}(z) \cdot \hat{x}(w) =: \hat{x}(z) \cdot \hat{x}(w) : + (\hat{x}(z) \cdot \hat{x}(w)). \]

Now we have

\[ \hat{x}(z + \epsilon) \cdot \hat{x}(z) =: \hat{x}(z + \epsilon) \hat{x}(z) : + 2 \epsilon^2 z^2 - \epsilon^4 z^3 + \cdots. \]

We define the regularized product of \( \hat{x}(z) \) with itself by

\[ \hat{x}(z) \odot \hat{x}(z) = \hat{x}(z) \hat{x}(z) : + 1 - \frac{1}{4z^2}. \]

In other words, we simply remove the term that goes to infinity as \( \epsilon \to 0 \), and then take the limit.

7.5. Virasoro constraints and mirror symmetry for 2D topological gravity. By straightforward calculations one can get:

**Proposition 7.1.** Let \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \) be defined by:

\[ L(z) := \frac{1}{8} : \hat{x}(z)^2 := \frac{1}{8} \sum_{n \in \mathbb{Z}} \sum_{j+k=-n-1} : \beta_{-(2j+1)} \beta_{-(2k+1)} : z^{-n-2}. \]

Then one has the following commutation relations:

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{2m^3 + m}{48} \delta_{m,-n}. \]

Recall the special deformation of the Airy curve constructed in \[4\] using the genus zero free energy of the 2D topological gravity, is characterized by (Theorem \[4.2\] and Theorem \[4.3\]):

\[ (x^2)_- = 0. \]

We take the quantization of this equation to

\[ (\hat{x}(2)^{\odot 2})_- Z = 0. \]

The following result establish the mirror symmetry of the theory of 2D topological gravity and the quantum deformation theory of the Airy curve:

**Theorem 7.2.** The Witten-Kontsevich tau-function \( Z_{WK} \) satisfies the following equation:

\[ (\hat{x}(2)^{\odot 2})_- Z_{WK} = 0. \]
Proof. By the definition of $\hat{x}(x)$ and (209), one gets:

$$\hat{x}(z)^{\odot 2} = 2\left(\frac{1}{2}\beta_{-1}^2 + \sum_{n=0}^{\infty} \beta_{-(2n+3)}\beta_{2n+1}z^{-1}\right)$$

$$+ 2\sum_{n=0}^{\infty} (\beta_{-(2n+1)}\beta_{2n+1} + \frac{1}{8})z^{-2}$$

$$+ 2\sum_{m\geq 1}^{\infty} \beta_{-(2n+1)}\beta_{2n+2m+1} + \frac{1}{2}\sum_{j+k=m-1} \beta_j\beta_k z^{-m-2}.$$  

It is then straightforward to see that (214) is equivalent to the Virasoro constraints (73)-(75). □

8. Regularized Products of Quantum Fields on the Airy Curve

In last section we have defined the regularized product $\hat{x}(z)^{\odot 2}$, and identify its coefficients with operators of Virasoro constraints for 2D topological gravity discovered in [15]. In this section we will generalize it to $\hat{x}(z)^{\odot n}$ for $n > 2$ and conjecture the higher W-constraints

(215)  

$$(\hat{x}^{\odot 2n}) - Z_{WK} = 0$$

hold for all $n \geq 1$.

8.1. Definition of $\hat{x}(z)^{\odot n}$. One can inductively define $\hat{x}(z)^{\odot n}$:

(216)  

$$\hat{x}(z)^{\odot n} = \hat{x}(z) \odot \hat{x}(z)^{\odot n-1}.$$  

For example,

$$\hat{x}(z + \epsilon) \cdot \hat{x}(z)^{\odot 2} = \hat{x}(z + \epsilon) \cdot (\hat{x}(z)^2 + \frac{1}{4z^2})$$

$$= : \hat{x}(z + \epsilon)\hat{x}(z)^2 : + 2(\hat{x}(z + \epsilon)\hat{x}(z)) \cdot \hat{x}(z) + \frac{\hat{x}(z + \epsilon)}{4z^2}$$

$$= : \hat{x}(z + \epsilon)\hat{x}(z)^2 : + 2\left(\frac{2}{\epsilon^2} + \frac{1}{4z^2} - \frac{\epsilon}{4z^2} + \cdots\right) \cdot \hat{x}(z) + \frac{\hat{x}(z + \epsilon)}{4z^2},$$

the regularized product $\hat{x}(z) \cdot \hat{x}(z)^{\odot 2}$ is defined by:

$$\hat{x}(z) \odot \hat{x}(z)^{\odot 2} = \lim_{\epsilon \to 0} \left(\hat{x}(z + \epsilon) \cdot \hat{x}(z)^{\odot 2} - \frac{4}{\epsilon^2}\hat{x}(z)\right) = : \hat{x}(z)^3 : + \frac{3\hat{x}(z)}{4z^2}.$$  

By induction one can easily prove that
Theorem 8.1. For \( n \geq 1 \),
\[
\hat{x}(z)^\otimes n = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{j!} \binom{n}{2, \ldots, 2, n-2j} \frac{1}{(4z^2)^j} \hat{x}(z)^{n-2j}.
\]

Corollary 8.2. For \( n \geq 1 \),
\[
\hat{x}(z)^n := \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n}{j} \frac{1}{j!} \binom{n}{2, \ldots, 2, n-2j} \frac{1}{(4z^2)^j} \hat{x}(z)^{n-2j}.
\]

Proof. Consider the generating series:
\[
\sum_{n \geq 0} \hat{x}(z)^\otimes n \frac{t^n}{n!} = \sum_{n \geq 0} \frac{t^n}{n!} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{1}{j!} \binom{n}{2, \ldots, 2, n-2j} \frac{1}{(4z^2)^j} \hat{x}(z)^{n-2j} \\
= \sum_{j=0}^\infty \frac{t^{2j}}{j!(8z^2)^j} \sum_{k=0}^\infty \hat{x}(z)^k : \frac{t^k}{k!} \\
= \exp\left(\frac{t^2}{8z^2}\right) \sum_{k=0}^\infty \hat{x}(z)^k : \frac{t^k}{k!},
\]
and so
\[
\sum_{k=0}^\infty \hat{x}(z)^k : \frac{t^k}{k!} = \exp\left(-\frac{t^2}{8z^2}\right) \sum_{n \geq 0} \hat{x}(z)^\otimes n \frac{t^n}{n!} \\
= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j}{j!} \binom{n}{2, \ldots, 2, n-2j} \frac{1}{(4z^2)^j} \hat{x}(z)^{n-2j}.
\]

8.2. Relationship with Bessel polynomials. Using The On-Line Encyclopedia of Integer Sequences, one sees that the coefficients
\[
T(n, j) = \frac{1}{j!} \binom{n}{2, \ldots, 2, n-2j} = \frac{n!}{2^j j!(n-2j)!} = \binom{n}{2j}(2j-1)!!
\]
have the following exponential generating series:
\[
\sum_{n=0}^\infty \sum_{j=0}^{\lfloor n/2 \rfloor} T(n, j) \frac{z^n}{n!} t^j = \exp(z + \frac{1}{2}tz^2).
\]
These numbers are called Bessel numbers \([10]\) because if one sets
\[
y_n(x) = \sum_{k=0}^n T(2n-k, n-k) x^{n-k} = \sum_{k=0}^n \frac{(2n-k)!}{2^{n-k} k!(n-k)!} x^{n-k},
\]
then $y_n$ is the $n$-th Bessel polynomial that satisfies the Bessel equation:

$$x^2y''_n + (2x + 2)y'_n = n(n + 1)y_n.$$  

8.3. **Regularized products of** $\hat{x}^{\ominus m}$ **with** $\hat{x}^{\ominus n}$. Next we define $\hat{x}^{\ominus m} \odot \hat{x}^{\ominus n}$ for general $m, n$ in the same fashion. Let us first define $:\hat{x}(z)^m \odot \hat{x}(z)^n :$. We will first assume that $m \geq n$, then by Wick’s theorem,

$$\hat{x}(z + \epsilon)^m : \hat{x}(z)^n := \hat{x}(z + \epsilon)^m \hat{x}(z)^n : + mn : \hat{x}(z + \epsilon)^m-1 \hat{x}(z)^n-1 \cdot \langle \hat{x}(z + \epsilon) \hat{x}(z) \rangle + 2! \cdot \binom{m}{2} \cdot \binom{n}{2} : \hat{x}(z + \epsilon)^{m-2} \hat{x}(z)^{n-2} \cdot \langle \hat{x}(z + \epsilon) \hat{x}(z) \rangle^2 + \ldots \ldots .$$

We have a similar expression for $m < n$. Then after removing all the obvious singularities from $\langle \hat{x}(z + \epsilon) \hat{x}(z) \rangle^j$ for $j = 1, \ldots, n$, and taking $\epsilon \to 0$, one gets:

$$\hat{x}(z)^m : \hat{x}(z)^n := \sum_{j=0}^{\min\{m,n\}} \frac{j!}{(4z^2)^j} \cdot \binom{m}{j} \binom{n}{j} : \hat{x}(z)^{m+n-2j} : .$$

**Theorem 8.3.** For $m, n \geq 1$,

$$\hat{x}^{\ominus m} \odot \hat{x}^{\ominus n} = \hat{x}^{\ominus m+n}.$$ 

**Proof.** We have

$$\sum_{m \geq 0} : \hat{x}(z)^m : \frac{t_1^m}{m!} \odot \sum_{n \geq 0} : \hat{x}(z)^n : \frac{t_2^n}{n!} = \sum_{m,n \geq 0} \frac{t_1^m t_2^n}{m! n!} \sum_{j=0}^{\min\{m,n\}} \frac{j!}{(4z^2)^j} \cdot \binom{m}{j} \binom{n}{j} : \hat{x}(z)^{m+n-2j} : = \sum_{m,n,j \geq 0} \frac{(t_1 t_2)^j}{m! n!} \frac{(t_1 + t_2)^k}{j! (4z^2)^j} : \hat{x}(z)^{m+n} : = \exp\left(\frac{t_1 t_2}{4z^2}\right) \cdot \exp\left(-\frac{(t_1 + t_2)^2}{8z^2}\right) \cdot \sum_{n \geq 0} \hat{x}(z)^{\ominus n} \frac{(t_1 + t_2)^n}{n!} = \exp\left(-\frac{t_1^2 + t_2^2}{8z^2}\right) \sum_{n \geq 0} \hat{x}(z)^{\ominus n} \frac{(t_1 + t_2)^n}{n!}.$$
It follows that
\[ :\hat{x}(z)^m \otimes \hat{x}(z)^n : = \sum_{j=0}^{\min\{m,n\}} \frac{j!}{(4z^2)^j} \cdot \binom{m}{j} \binom{n}{j} :\hat{x}(z)^{m+n-2j} :. \]

It follows that
\[ \sum_{m\geq 0} \hat{x}(z)^{\otimes m} \frac{t^m}{m!} \otimes \sum_{n\geq 0} \hat{x}(z)^{\otimes n} \frac{t^n}{n!} = \sum_{n\geq 0} \hat{x}(z)^{\otimes n} \frac{(t_1 + t_2)^n}{n!}. \]

This completes the proof. □

Corollary 8.4. The regularized product \( \otimes \) is associative, i.e.,
\[ (\hat{x}(z)^{\otimes l} \otimes \hat{x}(z)^{\otimes m}) \otimes \hat{x}(z)^{\otimes n} = \hat{x}(z)^{\otimes l} \otimes (\hat{x}(z)^{\otimes m} \otimes \hat{x}(z)^{\otimes n}). \]

8.4. W-constraints. We make the following

Conjecture 8.5. The equalities
\[ (\hat{x}(z)^{\otimes 2n})_W Z_{WK} = 0 \]
hold for all \( n \geq 1. \)

Acknowledgements. This research is partially supported by NSFC grant 1171174.

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