Error estimates and a two grid scheme for approximating transmission eigenvalues

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Abstract. In this paper, using the linearization technique we write the Helmholtz transmission eigenvalue problem as an equivalent nonselfadjoint linear eigenvalue problem whose left-hand side term is a selfadjoint, continuous and coercive sesquilinear form. To solve the resulting nonselfadjoint eigenvalue problem, we give an $H^2$ conforming finite element discretization and establish a two grid discretization scheme. We present a complete error analysis for both discretization schemes, and theoretical analysis and numerical experiments show that the methods presented in this paper can efficiently compute real and complex transmission eigenvalues.

Key words. transmission eigenvalues, finite element method, two grid discretization scheme, error estimates

AMS subject classifications. 65N25, 65N30, 65N15

1. Introduction. Transmission eigenvalues not only have wide physical applications, for example, they can be used to obtain estimates for the material properties of the scattering object [6,7], but also have theoretical importance in the uniqueness and reconstruction in inverse scattering theory [13]. Many papers such as [7,13,23,28,29] study the existence of transmission eigenvalues, and [7,8,15] explore upper and lower bounds for the index of refraction $n(x)$ from knowledge of the transmission eigenvalues. The attention from the computational mathematics community is also increasing (see, e.g., [1,9,14,19,21,22,26,31,32]).

In this paper, we consider the Helmholtz transmission eigenvalue problem: Find $k \in \mathbb{C}$, $w, \sigma \in L^2(D)$, $w - \sigma \in H^2(D)$ such that

\begin{align}
\Delta w + k^2 n(x) w &= 0, \quad \text{in } D, \\
\Delta \sigma + k^2 \sigma &= 0, \quad \text{in } D, \\
w - \sigma &= 0, \quad \text{on } \partial D, \\
\frac{\partial w}{\partial \nu} - \frac{\partial \sigma}{\partial \nu} &= 0, \quad \text{on } \partial D,
\end{align}

where $D \subset \mathbb{R}^d$ ($d = 2,3$) is a bounded simply connected set containing an inhomogeneous medium, and $\nu$ is the unit outward normal to $\partial D$.

From [8,29] we know that for $u = w - \sigma \in H^2_0(D)$, the weak formulation for the transmission eigenvalue problem (1.1)-(1.4) can be stated as follows: Find $k^2 \in \mathbb{C}$, $k^2 \neq 0$, $u \in H^2_0(D) \setminus \{0\}$ such that

\begin{align}
\frac{1}{n(x) - 1}(\Delta u + k^2 u, \Delta v + k^2 n(x) v)_0 = 0, \quad \forall v \in H^2_0(D),
\end{align}

where $(\psi, \varphi)_0 = \int_D \psi \varphi dx$ denotes the $L^2(D)$ inner product. We denote $\lambda = k^2$ as usual, then (1.5) is a quadratic eigenvalue problem.

In recent years, based on this weak formulation, various numerical methods have

*This work was supported by the National Science Foundation of China (Grant Nos.11561014, 11201093).
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been presented to solve the transmission eigenvalue problem. The first numerical treatment appeared in [14] where three finite element methods were proposed for the Helmholtz transmission eigenvalues which have been further developed in [1, 9, 19, 21, 22, 31]. Among them [14, 22, 31] studied the $H^2$ conforming finite element method, [9, 14, 21] mixed finite element methods, [1] spectral-element method and [19] the Galerkin-type numerical method. Inspired by these works, this paper further studies the $H^2$ conforming finite element method and has three features as follows:

(1) A complete error analysis is presented. Due to the fact that the problem is neither elliptic nor self-adjoint, once its error analysis was viewed as a difficult task. Sun [31] first proved an error estimate for the $H^2$ conforming finite element approximation of (1.5). Following him Ji et al. [22] presented an accurate error estimate, but Theorems 1-2 therein are only valid for real eigenvalues and rely on the conditions of Lemma 3.2 in [31] which are not easy to verify in general, especially for multiple eigenvalues. In this paper, we use the linearization technique in [33] to write the weak formulation (1.5) as an equivalent nonselfadjoint linear eigenvalue problem (2.6). Then the $H^2$ conforming finite element approximation eigenpair $(\lambda_h, u_h)$ of (2.6) is exactly the one of (1.5). Fortunately, the left-hand side term of (2.6) is selfadjoint, continuous and coercive sesquilinear form, thus we can use Babuska-Osborn’s spectral approximation theory [2] to give a complete error analysis for the $H^2$ conforming finite element approximation of (2.6), namely the error analysis of (1.5). The difficulty of the error estimates in low norms lies in the nonsymmetry of the right-hand side term of (2.6) that involves derivatives. We introduce two auxiliary problems and overcome this difficulty by the Nitche technique in a subtle way. Our theoretical results are proved under general conditions and are valid for arbitrary real and complex eigenvalues.

(2) A finite element discretization with good algebraic properties is presented. We give an $H^2$ conforming finite element discretization (see (3.1) or (5.1)) that will lead to a positive definite Hermitian and block diagonal stiff matrix. Then we use this discretization to solve the transmission eigenvalue problem numerically and obtain both real and complex transmission eigenvalues of high accuracy as expected. Similar discretizations already exist in the literatures. Colton et al. [14] established the $H^2$ conforming finite element discretization for the formulation (1.5) (see (4.3) in [14]), by starting with discretizing (1.4) into a quadratic algebra eigenvalue problem and then linearizing it, which is in the opposite order of our treatment. Though our discrete form (5.1) is formally different from the one in [14] for the resulting mass matrix in [14] is positive definite Hermitian and block diagonal, the two discrete forms are equivalent from the linear algebra point of view. Gintides et al. [19] has used square root of matrices, which is a standard tool, and a Hilbert basis in $H^2_0(D)$ as test functions to discretize (1.5) into a linear eigenvalue problem (see (2.14) in [19]) and proved the convergence for the approximate eigenvalues.

(3) A two grid discretization scheme is proposed. As we know, it is difficult to solve nonselfadjoint eigenvalues problems in general. So, Sun [31] adopted iterative methods to work on a series of generalized Hermitian problems and finally to solve for the real transmission eigenvalues. Ji et al. [22] combined the iterative methods in [31] with the extended finite element method to establish a multigrid algorithm for solving the real transmission eigenvalues. This paper establishes a two grid discretization scheme directly for the nonselfadjoint eigenvalue problem (2.6) which is suitable for arbitrary real and complex eigenvalues. The two grid discretization scheme is an efficient ap-
approach which was first introduced by Xu [34] for nonsymmetric or indefinite problems, and successfully applied to eigenvalue problems later (see, e.g., [17, 24, 35, 36, 37] and the references therein). With the two grid discretization scheme, the solution of the transmission eigenvalue problem on a fine grid \( \pi_h \) is reduced to the solution of the primal and dual eigenvalue problem on a much coarser grid \( \pi_H \) and the solutions of two linear algebraic systems with the same positive definite Hermitian and block diagonal coefficient matrix on the fine grid \( \pi_h \), and the resulting solution still maintains an asymptotically optimal accuracy. The key to our theoretical analysis is to find a \((u^*_H, \omega^*_H)\) and prove that \(|B((u_H, \omega_H), (u_H^*, \omega_H^*))|\) (see Step 1 of Scheme 4.1) has a positive lower bound uniformly with respect to the mesh size \( H \), which is also critical for further establishing multigrid method and adaptive algorithm. We successfully apply spectral approximation theory to solve this problem (see Lemma 4.1 and Remark 4.1).

The \( H^2 \) conforming finite element discretization is easy to implement under the package of iFEM [11] with MATLAB, and the numerical results indicate that this discretization is efficient for computing transmission eigenvalues. Moreover, we can further improve the computational efficiency by using the two grid discretization scheme.

Throughout this paper, \( C \) denotes a positive constant independent of the mesh size \( h \), which may not be the same constant in different places. For simplicity, we use the symbol \( a \lesssim b \) to mean that \( a \leq Cb \).

2. A linear formulation of (1.5). In this paper, we suppose that the index of refraction \( n \in L^\infty(D) \) satisfies the following assumption

\[
1 + \delta \leq \inf_D n(x) \leq \sup_D n(x) < \infty,
\]

for some constant \( \delta > 0 \), although, with obvious changes, the theoretical analysis in this paper also holds for \( n \) strictly less than 1.

Inspired by the works in [9, 14, 21], we now apply the linearization technique in [33] to write the weak formulation (1.5) as an equivalent linear formulation.

From (1.5) we derive that

\[
\left( \frac{1}{n-1} \Delta u, \Delta v \right)_0 + k^2 \left( \frac{1}{n-1} u, \Delta v \right)_0 + k^2 (\Delta u, n \frac{n}{n-1} v)_0 + k^4 (\frac{n}{n-1} u, v)_0 = 0, \quad \forall v \in H^2_0(D).
\]

(2.1)

Introduce an auxiliary variable

\[
\omega = k^2 u,
\]

(2.2)

then

\[
(\omega, z)_0 = k^2 (u, z)_0, \quad \forall z \in L^2(D).
\]

(2.3)

Thus, combining (2.1) and (2.3), we arrive at a linear formulation: Find \( k^2 \in \mathbb{C} \) and nontrivial \((u, \omega) \in H^2_0(D) \times L^2(D)\) such that

\[
\left( \frac{1}{n-1} \Delta u, \Delta v \right)_0 = -k^2 \left( \frac{1}{n-1} u, \Delta v \right)_0
\]

(2.4)

\[
- k^2 (\Delta u, n \frac{n}{n-1} v)_0 - k^2 (\frac{n}{n-1} \omega, v)_0, \quad \forall v \in H^2_0(D),
\]

(2.5)

\[
(\omega, z)_0 = k^2 (u, z)_0, \quad \forall z \in L^2(D).
\]
This is a nonselfadjoint linear eigenvalue problem.

Suppose that \((k^2, u)\) is an eigenpair of (1.3), then it is obvious that \((k^2, u, \omega)\) is an eigenpair of (2.4), (2.5). On the other hand, suppose that \((k^2, u, \omega)\) satisfies (2.4)-(2.5), from (2.5) we get \(\omega = k^2 u\), and substituting it into (2.4) we get (2.1) (or (1.5)).

The above argument indicates that (2.4)-(2.5) and (1.5) are equivalent.

Let \(H^{-l}(D)\) be the “negative space”, with norm given by

\[
\|f\|_{-l} = \sup_{0 \neq v \in H^l_0(D)} \frac{|(f, v)|}{\|v\|_{l}}, \quad l = 1, 2.
\]

Define the Hilbert space \(H = H^1_0(D) \times H^2(D)\) with norm \(\|(u, \omega)\|_H = \|u\|_2 + \|\omega\|_0\),

and define \(H^r = H^s(D) \times H^{s-2}(D)\) with norm \(\|(u, \omega)\|_{H^r} = \|u\|_s + \|\omega\|_{s-2}, s = 0, 1, 2\).

\(H^2 = H\). It’s obvious that \(H \hookrightarrow H^r\) \((s = 0, 1)\) compactly (see[5, 35]).

Denote \(\lambda = k^2\). Let

\[
A((u, \omega), (v, z)) = \left(\frac{1}{n - 1} \Delta u, \Delta v\right)_0 + (\omega, z)_0,
\]

\[
B((u, \omega), (v, z)) = -\left(\frac{1}{n - 1} u, \Delta v\right)_0 - \left(\Delta u, \frac{n}{n - 1} v\right)_0 - \left(\frac{n}{n - 1} \omega, v\right)_0 + (u, z)_0,
\]

then (2.4)-(2.5) can be rewritten as: Find \(\lambda \in \mathbb{C}, (u, \omega) \in H \setminus \{0\}\) such that

\[
A((u, \omega), (v, z)) = \lambda B((u, \omega), (v, z)), \quad \forall (v, z) \in H.
\]

Thus we get the following theorem.

**Theorem 2.1.** If \((k^2, u)\) is an eigenpair of (1.3) and \(\omega = k^2 u\), then \((k^2, u, \omega)\) satisfies (2.6), and if \((k^2, u, \omega)\) satisfies (2.6), then \((k^2, u)\) is an eigenpair of (1.3) and \(\omega = k^2 u\).

By calculation we derive for any \((u, \omega), (v, z) \in H\),

\[
A((u, \omega), (v, z)) = \int_D \frac{1}{n - 1} \Delta u \Delta v - \omega \Delta v dx = \int_D \frac{1}{n - 1} \Delta v \Delta u + z \Delta v dx = A((v, z), (u, \omega)),
\]

\[
|A((u, \omega), (v, z))| \lesssim \|\Delta u\|_0 \|\Delta v\|_0 + \|\omega\|_0 \|z\|_0 \lesssim \|(u, \omega)\|_H \|(v, z)\|_H,
\]

\[
A((u, \omega), (u, \omega)) = \int_D \frac{1}{n - 1} \Delta u \Delta u + \omega \Delta u dx \gtrsim \|(u, \omega)\|_H^2.
\]

i.e., \(A(\cdot, \cdot)\) is a selfadjoint, continuous and coercive sesquilinear form on \(H \times H\).

We use \(A(\cdot, \cdot)\) and \(\|\cdot\|_A = A(\cdot, \cdot) \frac{1}{2}\) as an inner product and norm on \(H\), respectively. Then \(\|\cdot\|_A\) is equivalent to the norm \(\|\cdot\|_H\).

When \(n \in L^\infty(D), \forall (f, g), (v, z) \in H\), we deduce

\[
|B((f, g), (v, z))| = | - \frac{1}{n - 1} f \Delta v|_0 - (\Delta f, \frac{n}{n - 1} v)|_0 - \left(\frac{n}{n - 1} g, v\right)_0 + (f, z)_0| \lesssim \frac{1}{n - 1} f_0 \|v\|_2 + \frac{n}{n - 1} \Delta f \|v\|_1 + \|g\|_{-1} \|v\|_1 + \|f\|_0 \|z\|_0
\]

\[
\lesssim (f_0 + \frac{n}{n - 1} \Delta f \|v\|_1 + \|g\|_{-1}) ||(v, z)||_H. \tag{2.7}
\]

When \(n \in W^{1, \infty}(D), \forall (f, g) \in H^1, \forall (v, z) \in H\), we have

\[
|B((f, g), (v, z))| = |(\nabla \left(\frac{1}{n - 1} f\right), \nabla v)|_0 + (\nabla f, \nabla \left(\frac{n}{n - 1} v\right))_0 - \left(\frac{n}{n - 1} g, v\right)_0 + (f, z)_0| \lesssim \|f\|_1 \|v\|_1 + \|f\|_1 \|v\|_1 + \|g\|_{-1} \|v\|_1 + \|f\|_1 \|z\|_1
\]

\[
\lesssim (f_1 + \frac{n}{n - 1} g \|v\|_{-1}) ||(v, z)||_H. \tag{2.8}
\]
And when \( n \in W^{2,\infty}(D) \), \( \forall (f, g) \in H^0, \forall (v, z) \in H \), we have
\[
|B((f, g), (v, z))| = | - \left( \frac{1}{n-1} f, \Delta v \right) - \left( f, \Delta \left( \frac{n}{n-1} g \right) \right)_0 + \left( f, z \right)_0 |
\lesssim \|f\|_0 \|v\|_2 + \|f\|_0 \|v\|_2 + \|g\|_{-2}(v) + \|f\|_0 \|z\|_0
\]
(2.9)

Thus
\[
\|B((f, g), (v, z))\|_{H^0} \lesssim \|f\|_0 \|v\|_2 + \|f\|_0 \|v\|_2 + \|g\|_{-2}(v) + \|f\|_0 \|z\|_0
\]

We can see from (2.7), (2.8) that for any given \((f, g) \in H^s (s = 0, 1, 2)\), \(B((f, g), (v, z))\) is a continuous linear form on \(H\).

The source problem associated with (2.6) is given by: Find \((\psi, \varphi)\) such that
\[
A((\psi, \varphi), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in H.
\]
From the Lax-Milgram theorem we know that (2.10) admits a unique solution. Therefore, we define the corresponding solution operators \(T \colon H^s \rightarrow H (s=0,1,2)\) by
\[
A(T(f, g), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in H.
\]
Then (2.10) has the equivalent operator form:
\[
T(u, \omega) = \lambda^{-1}(u, \omega).
\]
Namely, if \((\lambda, u, \omega)\) is an eigenpair of (2.10), then \((\lambda, u, \omega)\) is an eigenpair of (2.12).

Conversely, if \((\lambda, u, \omega)\) is an eigenpair of (2.12), then \((\lambda, u, \omega)\) is an eigenpair of (2.10).

**Theorem 2.2.** Suppose \( n \in L^{\infty}(D) \), then \( T : H \rightarrow H \) is compact; suppose \( n \in W^{2-s,\infty}(D) \) \( (s = 0, 1) \), then \( T : H^s \rightarrow H^s \) is compact.

Proof. When \( n \in L^{\infty}(D) \), let \((v, z) = T(f, g)\) in (2.10), then from (2.7) we have
\[
\|T(f, g)\|_H \lesssim A(T(f, g), T(f, g)) = B((f, g), T(f, g))
\]
\[
\lesssim (\|f\|_0 + \|\frac{n}{n-1} \Delta f\|_{-1} + \|\frac{n}{n-1} g\|_{-1})\|T(f, g)\|_H,
\]
thus
\[
\|T(f, g)\|_H \lesssim \|f\|_0 + \|\frac{n}{n-1} \Delta f\|_{-1} + \|\frac{n}{n-1} g\|_{-1}, \quad \forall (f, g) \in H.
\]

Let \( S = S_1 \times S_2 \) be any bounded set in \(H\), because of the compact embedding \(H^0_0(D) \rightarrow L^2(D)\) and \(L^2(D) \rightarrow H^{-1}(D)\), then \(S_1\) is relatively compact in \(L^2(D)\), \(\{v : v = \frac{n}{n-1} \Delta f, f \in S_1\}\) is relatively compact in \(H^{-1}(D)\) and \(\{v : v = \frac{n}{n-1} g, g \in S_2\}\) is relatively compact in \(H^{-1}(D)\). And from (2.13) we conclude that \(T : H \rightarrow H\) is compact.

When \( n \in W^{2-s,\infty}(D) \) \( (s = 0, 1) \), in (2.10), let \((v, z) = T(f, g)\), then from (2.8) and (2.9) we have
\[
\|T(f, g)\|_H^2 \lesssim A(T(f, g), T(f, g)) = B((f, g), T(f, g)) \lesssim \|f\|_0 \|T(f, g)\|_H,
\]
thus
\[
\|T(f, g)\|_H \lesssim \|f\|_0, \quad \forall (f, g) \in H^s (s = 0, 1),
\]
which implies that \( T : H^s \rightarrow H \) is continuous. Due to the compact embedding \(H \rightarrow H^s, T : H^s \rightarrow H^s\) is compact.

Consider the dual problem of (2.10): Find \(\lambda^* \in \mathbb{C}, (u^*, \omega^*) \in H \setminus \{0\}\) such that
\[
A((v, z), (u^*, \omega^*)) = \bar{\lambda}^* B((v, z), (u^*, \omega^*)), \quad \forall (v, z) \in H.
\]
Define the corresponding solution operators \( T^* : \mathbf{H}^s \rightarrow \mathbf{H} \) by
\[
A((v, z), T^*(f, g)) = B((v, z), (f, g)), \quad \forall (v, z) \in \mathbf{H}.
\]
Then \((\ref{3.1})\) has the operator form:
\[
\lambda T^*(u^*, \omega^*) = \lambda^{s-1}(u^*, \omega^*).
\]
It can be proved that \( T^* \) is the adjoint operator of \( T \) in the sense of inner product \( A(\cdot, \cdot) \). In fact, from \((\ref{3.11})\) and \((\ref{2.16})\) we have
\[
\lambda = \lambda^* \rightarrow 0
\]
and
\[
A(T(f, g), (v, z)) = B((f, g), (v, z)) = A((f, g), T^*(v, z)), \quad \forall (f, g), (v, z) \in \mathbf{H}.
\]
Hence the primal and dual eigenvalues are connected via \( \lambda = \lambda^* \).

3. The \( H^2 \) conforming finite element approximation and its error estimates. Let \( \pi_h \) be a shape-regular grid of \( D \) with mesh size \( h \) and \( S^h \subset H^2_0(D) \) be a piecewise polynomial space defined on \( \pi_h \); for example, \( S^h \) is the finite element space associated with one of the Argyris element, the Bell element or the Bogner-Fox-Schmit element (BFS element) (see [12]). Thanks to \((\ref{2.2})\), we choose \( H_h \subset S^h \) with mesh size \( h \leadsto \mathbf{H} \), \( h \subset \mathbf{H} \). Thanks to \((\ref{2.2})\), we choose \( H_h = S^h \). Then \( \mathbf{H} \subset \mathbf{H} \) be a conforming finite element space. For the three finite element spaces mentioned above, since \( \bigcup_{h>0} S^h \) are dense in both \( L^2(D) \) and \( H^2_0(D) \), it is obvious that the following condition \((\ref{C1})\) holds:

\((\text{C1})\)  If \((\psi, \varphi) \in \mathbf{H} \), then as \( h \rightarrow 0 \),
\[
\inf_{(\psi, z) \in \mathbf{H}_h} \| (\psi, \varphi) - (v, z) \|_{\mathbf{H}} \rightarrow 0.
\]

From the operator interpolation theory (see e.g. [3]), the following \((\text{C2})\) is also valid:

\((\text{C2})\)  If \( \psi \in H^2_0(D) \cap H^{2+r}(D) \) \((r \in (0, 2))\), then
\[
\inf_{v \in S^h} \| \psi - v \|_s \lesssim h^{2+r-s} \| \psi \|_{2+r}, \quad s = 0, 1, 2.
\]

Throughout this paper, we suppose that \((\text{C1})\) is valid.

The \( H^2 \) conforming finite element approximation of \((\ref{2.0})\) is given by the following: Find \( \lambda_h \in \mathbf{C} \), \((u_h, \omega_h) \in \mathbf{H}_h \setminus \{0\} \) such that
\[
A(((u_h, \omega_h), (v, z)) = \lambda_h B((u_h, \omega_h), (v, z)), \quad \forall (v, z) \in \mathbf{H}_h.
\]

Consider the approximate source problem: Find \((\psi_h, \varphi_h) \in \mathbf{H}_h \setminus \{0\} \) such that
\[
A((\psi_h, \varphi_h), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}_h.
\]

We introduce the corresponding solution operator: \( T_h : \mathbf{H}^s \rightarrow \mathbf{H}_h \) \((s=0,1)\):
\[
A(T_h(f, g), (v, z)) = B((f, g), (v, z)), \quad \forall (v, z) \in \mathbf{H}_h.
\]

Then \((\ref{3.1})\) has the operator form:
\[
T_h(u_h, \omega_h) = \lambda^{s-1}_h(u_h, \omega_h).
\]

Define the projection operators \( P^1_h : H^2_0(D) \rightarrow S^h \) and \( P^2_h : L^2(D) \rightarrow S^h \) by
\[
\frac{1}{n-1} \Delta (u - P^1_h u), \Delta v)_0 = 0, \quad \forall v \in S^h,
\]
\[
(\omega - P^2_h \omega, z)_0 = 0, \quad \forall z \in S^h.
\]
Let
\[ P_h(u, \omega) = (P_h^1 u, P_h^2 \omega), \quad \forall (u, \omega) \in H. \]

Then \( P_h : H \to H_h \), and

\[ A((u, \omega) - P_h(u, \omega), (v, z)) = A((u, \omega) - (P_h^1 u, P_h^2 \omega), (v, z)) \]
\[ = \left( \frac{1}{n-1} \right) \Delta(u - P_h^1 u, \Delta v)_0 + (\omega - P_h^2 \omega, z)_0 = 0, \quad \forall (v, z) \in H_h, \]

i.e., \( P_h : H \to H_h \) is the Ritz projection.

Combining (3.7), (2.11) with (3.3), we deduce for any \((u, \omega) \in H\) that

\[ A(P_h T(u, \omega) - T_h(u, \omega), (v, z)) = A(P_h T(u, \omega) - T(u, \omega), (v, z)) \]
\[ + A(T(u, \omega) - T_h(u, \omega), (v, z)) = 0, \quad \forall (v, z) \in H_h, \]

thus we get
\[ (3.8) \quad T_h = P_h T. \]

**Theorem 3.1.** Let \( n \in L^\infty(D) \), then
\[ (3.9) \quad \| T - T_h \|_H \to 0, \]
and let \( n \in W^{2-s, \infty}(D) \), then
\[ (3.10) \quad \| T - T_h \|_{H^s} \to 0, \quad s = 0, 1. \]

*Proof.* When \( n \in L^\infty(D) \), for any \((f, g) \in H\), from (C1) we have

\[ \| (I - P_h)T(f, g) \|_H = \| (T(f, g) - P_h T(f, g) \|_H \lesssim \inf_{(v, z) \in H_h} \| (T(f, g) - (v, z) \|_H \to 0. \]

Since \( T : H \to H \) is compact, \( T\{ (f, g) \in H, \|(f, g)\|_H=1 \} \) is relatively compact. Thus by the definition of operator norm we have

\[ \| T - T_h \|_H = \sup_{(f, g) \in H, \| (f, g) \|_H=1} \| (T - T_h)(f, g) \|_H \]
\[ = \sup_{(f, g) \in H, \| (f, g) \|_H=1} \| (I - P_h)T(f, g) \|_H \to 0. \]

When \( n \in W^{2-s, \infty}(D) \), for \( s = 0, 1 \), we have

\[ \| (I - P_h)T(f, g) \|_{H^s} \lesssim \| (I - P_h)T(f, g) \|_H \to 0. \]

Since \( T : H^s \to H^s \) is compact, \( T\{ (f, g) \in H^s, \| (f, g)\|_{H^s}=1 \} \) is relatively compact, thus we have

\[ \| T - T_h \|_{H^s} = \sup_{(f, g) \in H^s, \| (f, g) \|_{H^s}=1} \| (T - T_h)(f, g) \|_{H^s} \]
\[ = \sup_{(f, g) \in H^s, \| (f, g) \|_{H^s}=1} \| (I - P_h)T(f, g) \|_{H^s} \to 0. \]

The proof is completed. \( \blacksquare \)
The conforming finite element approximation of (2.11) is given by: Find \( \lambda_h^* \in \mathbb{C} \), 
\((u_h^*, \omega_h^*) \in \mathbf{H}_h \setminus \{0\}\) such that

\[
(3.11) \quad A((v, z), (u_h^*, \omega_h^*)) = \overline{\lambda_h} B((v, z), (u_h^*, \omega_h^*)), \quad \forall (v, z) \in \mathbf{H}_h.
\]

Define the solution operator \( T_h^*: \mathbf{H}^* \to \mathbf{H}_h \) satisfying

\[
(3.12) \quad A((v, z), T_h^* (f, g)) = B((v, z), (f, g)), \quad \forall (v, z) \in \mathbf{H}_h.
\]

Lemma 3.1 has the following equivalent operator form

\[
(3.13) \quad T_h^*(u_h^*, \omega_h^*) = \lambda_h^{* - 1} (u_h^*, \omega_h^*).
\]

It can be proved that \( T_h^* \) is the adjoint operator of \( T_h \) in the sense of inner product \( A(\cdot, \cdot) \). In fact, from (3.11) and (3.12) we have

\[
(3.14) \quad A(T_h(f, g), (v, z)) = B((f, g), (v, z)) = A((f, g), T_h^*(v, z)), \quad \forall (f, g), (v, z) \in \mathbf{H}_h.
\]

Hence, the primal and dual eigenvalues are connected via \( \lambda_h = \overline{\lambda_h} \).

We need the following lemma (see Lemma 5 on page 1091 of [18]).

Lemma 3.2. Let \( \|T_h - T\|_H \to 0 \). Let \( \{\lambda_j\} \) be an enumeration of the eigenvalues of \( T \), each repeated according to its multiplicity. Then there exist enumerations \( \{\lambda_{j, h}\} \) of the eigenvalues of \( T_h \), with repetitions according to multiplicity, such that \( \lambda_{j, h} \to \lambda_j \) \( (j \geq 1) \).

In this paper we suppose that \( \{\lambda_j\} \) and \( \{\lambda_{j,h}\} \) satisfy the above lemma, and let \( \alpha = \lambda_k \) be the kth eigenvalue with the algebraic multiplicity \( q \) and the ascent \( \alpha \), \( \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1} \). Since \( \|T_h - T\|_H \to 0 \), \( q \) eigenvalues \( \lambda_{k,h}, \cdots, \lambda_{k+q-1,h} \) of (3.1) will converge to \( \lambda \).

Let \( E \) be the spectral projection associated with \( T \) and \( \lambda \), then \( R(E) = N((\lambda^{-1} - T)^\alpha) \) is the space of generalized eigenvectors associated with \( \lambda \) and \( T \), where \( R \) denotes the range and \( N \) denotes the null space. Let \( E_h \) be the spectral projection associated with \( T_h \) and the eigenvalues \( \lambda_{k,h}, \cdots, \lambda_{k+q-1,h} \); let \( M_h(\lambda_{k,h}) \) be the space of generalized eigenvectors associated with \( \lambda_{k,h} \) and \( T_h \), and let \( M_h(\lambda) = \sum_{i=k}^{k+q-1} M_h(\lambda_{i,h}) \), then \( R(E_h) = M_h(\lambda) \) if \( h \) is small enough. In view of the dual problem (2.11) and (3.1), the definitions of \( E^*, R(E^*) \), \( E_h^* \), \( M_h(\lambda_{k,h})^* \), \( M_h(\lambda^*) \) and \( R(E_h^*) \) are analogous to \( E^* \), \( R(E^*) \), \( E_h^* \), \( M_h(\lambda_{k,h}) \), \( M_h(\lambda) \) and \( R(E_h) \) (see [2]).

Given two closed subspaces \( R \) and \( U \), denote

\[
\delta(R, U) = \sup_{(u, \omega) \in R \atop \|(u, \omega)\|_H = 1} \inf_{(v, z) \in U} \|(u, \omega) - (v, z)\|_H,
\]

\[
\theta(R, U)_s = \sup_{(u, \omega) \in R \atop \|(u, \omega)\|_H = 1} \inf_{(v, z) \in U} \|(u, \omega) - (v, z)\|_{H^*}, \quad s = 0, 1.
\]

We define the gaps between \( R(E) \) and \( R(E_h) \) in \( \|\cdot\|_H \) as

\[
\tilde{\delta}(R(E), R(E_h)) = \max\{\delta(R(E), R(E_h)), \delta(R(E_h), R(E))\},
\]

and in \( \|\cdot\|_{H^*} \) as

\[
\tilde{\theta}(R(E), R(E_h))_s = \max\{\theta(R(E), R(E_h))_s, \theta(R(E_h), R(E))_s\}.
\]
Define
\[ \varepsilon_h(\lambda) = \sup_{(u, \omega) \in R(E)} \inf_{\|v, z\|_H = 1} \|(u, \omega) - (v, z)\|_H, \]
\[ \varepsilon_h^*(\lambda^*) = \sup_{(u^*, \omega^*) \in R(E^*)} \inf_{\|v, z\|_H = 1} \|(u^*, \omega^*) - (v, z)\|_H. \]

It follows directly from (C1) that
\[ \varepsilon_h(\lambda) \to 0 \quad (h \to 0), \quad \varepsilon_h^*(\lambda^*) \to 0 \quad (h \to 0). \]

Suppose that \( R(E), R(E^*) \subset H \cap (H^{2+r}(D))^2 \), then from (C2) we get
\[ \varepsilon_h(\lambda) \lesssim h^r, \quad \varepsilon_h^*(\lambda^*) \lesssim h^r. \]

Further suppose that \( R(E), R(E^*) \subset (H^6(D))^2 \) is the Argyris finite element space, then from the interpolation theory we have
\[ \varepsilon_h(\lambda) \lesssim h^4, \quad \varepsilon_h^*(\lambda^*) \lesssim h^4. \]

Note that when the functions in \( R(E) \) and \( R(E^*) \) are piecewise smooth \([3.15]\) and \([3.16]\) are also valid.

Thanks to \([2]\), we get the following Theorem 3.3.

**Theorem 3.3.** Suppose \( n \in L^\infty(D) \), then
\[ \hat{\delta}(R(E), R(E_h)) \lesssim \varepsilon_h(\lambda), \]
\[ |\lambda - \left( \frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h}^{-1} \right)| \lesssim \varepsilon_h(\lambda) \varepsilon_h^*(\lambda^*), \]
\[ |\lambda - \lambda_{j,h}| \lesssim [\varepsilon_h(\lambda) \varepsilon_h^*(\lambda^*)]^\frac{1}{2}, \quad j = k, k+1, \ldots, k+q-1. \]

Suppose \((u_h, \omega_h)\) with \( \|(u_h, \omega_h)\|_A = 1 \) is an eigenfunction corresponding to \( \lambda_{j,h} \) \((j = k, k+1, \ldots, k+q-1)\), then there exists an eigenfunction \((u, \omega)\) corresponding to \( \lambda \), such that
\[ \|(u_h, \omega_h) - (u, \omega)\|_H \lesssim \varepsilon_h(\lambda)^\frac{1}{2}. \]

**Proof.** From Theorem 3.1 we know \( \|T - T_h\|_H \to 0 \quad (h \to 0) \), thus from Theorem 8.1, Theorem 8.2, Theorem 8.3 and Theorem 8.4 of \([2]\) we get the desired results \([3.17], [3.18], [3.19] \) and \([3.20]\), respectively.

Next we discuss the error estimates in the \( \|\cdot\|_{H^s} \) \((s = 0, 1)\) norm by using the Aubin-Nitsche technique.

We need the following regularity assumption:
**R(D).** For any \( \xi \in H^{-s}(D) \) \((s = 0, 1)\), there exists \( \psi \in H^{2+r_s}(D) \) satisfying
\[ \Delta \left( \frac{1}{n-1} \Delta \psi \right) = \xi, \quad \text{in } D; \quad \psi = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \partial D, \]
and
\[ \|\psi\|_{2+r_s} \leq C_p \|\xi\|_{-s}, \quad s = 0, 1. \]
where \( r_1 \in (0, 1], r_0 \in (0, 2], C_p \) denotes the prior constant dependent on \( n(x) \) and \( D \) but independent of the right-hand side \( \xi \) of the equation.

It is easy to see that (3.21) is valid with \( r_s = 2 - s \) when \( n \) and \( \partial D \) are appropriately smooth. When \( n \) is a constant and \( D \subset \mathbb{R}^2 \) is a convex polygon, from Theorem 2 in [3] we can get \( r_1 = 1 \) and if the inner angle at each critical boundary point is smaller than 126.283696° then \( r_0 = 2 \).

Consider the auxiliary boundary value problem: Find \( \phi_1 \in H^2_0(D) \) such that

\[
\Delta \left( \frac{1}{n-1} \Delta \phi_1 \right) = -\Delta(u - P^1_h u), \quad \text{in } D,
\]

\[
\phi_1 = \frac{\partial \phi_1}{\partial \nu} = 0, \quad \text{on } \partial D.
\]

Let \( R(D) \) hold, then

\[
\|\phi_1\|_{2+r_1} \lesssim \|\Delta(u - P^1_h u)\|_{-1} \lesssim \|u - P^1_h u\|_1.
\]

Consider the auxiliary boundary value problem:

\[
\Delta \left( \frac{1}{n-1} \Delta \phi_2 \right) = u - P^1_h u, \quad \text{in } D,
\]

\[
\phi_2 = \frac{\partial \phi_2}{\partial \nu} = 0, \quad \text{on } \partial D.
\]

Let \( R(D) \) hold, then

\[
\|\phi_2\|_{2+r_0} \lesssim \|u - P^1_h u\|_0.
\]

**Lemma 3.4.** Suppose that \( R(D) \) and \((C2)\) are valid \((s = 0, 1)\), then for \((u, \omega) \in H\),

\[
\|(u, \omega) - P_h(u, \omega)\|_{H} \lesssim h^{r_s}\|(u, \omega) - P_h(u, \omega)\|_{H}, \quad s = 0, 1.
\]

**Proof.** The weak form of (3.22)–(3.23) is

\[
(\Delta v, \frac{1}{n-1} \Delta \phi_1)_0 = (\nabla v, \nabla (u - P^1_h u))_0, \quad \forall v \in H^2_0(D).
\]

Let \( v = u - P^1_h u \), then

\[
(\Delta(u - P^1_h u), \frac{1}{n-1} \Delta(\phi_1 - P^1_h \phi_1))_0 = (\nabla(u - P^1_h u), \nabla(u - P^1_h u))_0,
\]

which leads to

\[
\|\nabla(u - P^1_h u)\|_0^2 \lesssim \|\Delta(u - P^1_h u)\|_0 \|\Delta(\phi_1 - P^1_h \phi_1)\|_0
\]

\[
\lesssim \|\Delta(u - P^1_h u)\|_0 h^{r_1} \|u - P^1_h u\|_1,
\]

thus

\[
\|u - P^1_h u\|_1 \lesssim h^{r_1} \|u - P^1_h u\|_2.
\]

The weak form of (3.24)–(3.25) is

\[
(\Delta v, \frac{1}{n-1} \Delta \phi_2)_0 = (v, u - P^1_h u)_0, \quad \forall v \in H^2_0(D).
\]
Let $v = u - P_h^1 u$, then
\begin{equation}
(\Delta (u - P_h^1 u), \frac{1}{n - 1} \Delta (\phi_2 - P_h^1 \phi_2))_0 = (u - P_h^1 u, u - P_h^1 u)_0, \tag{3.29}
\end{equation}
thus
\begin{equation}
\|u - P_h^1 u\|_0 \lesssim \|\Delta (u - P_h^1 u)\|_0 \|\Delta (\phi_2 - P_h^1 \phi_2)\|_0 \lesssim \|u - P_h^1 u\|_2 h^{r_0} \|u - P_h^1 u\|_0.
\end{equation}
and
\begin{equation}
\|u - P_h^1 u\|_0 \lesssim h^{r_0} \|u - P_h^1 u\|_2. \tag{3.30}
\end{equation}
Combining (3.28) and (3.30) we get
\begin{equation}
\|u - P_h^1 u\|_s \lesssim h^{r_s} \|u - P_h^1 u\|_2, \quad s = 0, 1. \tag{3.31}
\end{equation}
By the definition of the negative norm,
\begin{equation}
\|\omega - P_h^2 \omega\|_{s-2} = \sup_{\xi \in H^{2-s}_0(D)} \frac{\|\omega - P_h^2 \omega, \xi\|_0}{\|\xi\|_{2-s}} \tag{3.32}
\end{equation}
Combining (3.31) with (3.32) we deduce that
\begin{equation}
\|(u, \omega) - P_h(u, \omega)\|_{H^s} = \|(u - P_h^1 u, \omega - P_h^2 \omega)\|_{H^s} \lesssim \|u - P_h^1 u\|_s + \|\omega - P_h^2 \omega\|_{s-2} \lesssim h^{r_s} \|u - P_h^1 u\|_2 + h^{2-s} \|\omega - P_h^2 \omega\|_0
\end{equation}
\begin{equation}
\lesssim (h^{r_s} + h^{2-s}) \|(u, \omega) - P_h(u, \omega)\|_{H^s}, \quad s = 0, 1,
\end{equation}
noting that $h^{2-s} \lesssim h^{r_s}$, we obtain (3.26).

**Theorem 3.5.** Suppose that $R(D)$ and (C2) are valid, and $n \in W^{2-s, \infty}(D)$ ($s = 0, 1$). Then
\begin{equation}
\hat{\theta}(R(E), R(E_h))_s \lesssim h^{r_s} \varepsilon_h(\lambda), \quad s = 0, 1. \tag{3.33}
\end{equation}
Let $(u_h, \omega_h)$ with $\|(u_h, \omega_h)\|_A = 1$ be an eigenfunction corresponding to $\lambda_h$, then there exists eigenfunction $(u, \omega)$ corresponding to $\lambda$, such that
\begin{equation}
\|(u_h, \omega_h) - (u, \omega)\|_{H^s} \lesssim (h^{r_s} \varepsilon_h(\lambda))^\frac{1}{s}, \quad s = 0, 1;
\end{equation}
further let $\alpha = 1$, then
\begin{equation}
\|(u_h, \omega_h) - (u, \omega)\|_{H^s} \lesssim h^{r_s} \|(u_h, \omega_h) - (u, \omega)\|_{H^s}, \quad s = 0, 1. \tag{3.35}
\end{equation}

**Proof.** Note that in $R(E)$ the norm $\|\cdot\|_{H^s}$ is equivalent to the norm $\|\cdot\|_{H^s}$, and $TR(E) \subset R(E)$, by (3.26) we deduce that
\begin{equation}
\|(T - T_h)|_{R(E)}\|_{H^s} = \sup_{(u, \omega) \in R(E)} \frac{\|T(u, \omega) - T_h(u, \omega)\|_{H^s}}{\|(u, \omega)\|_{H^s}} \tag{3.36}
\end{equation}
\begin{equation}
\lesssim \sup_{(u, \omega) \in R(E)} \frac{\|T(u, \omega) - P_h T(u, \omega)\|_{H^s}}{\|(u, \omega)\|_{H^s}} \lesssim h^{r_s} \sup_{(u, \omega) \in R(E)} \frac{\|T(u, \omega) - P_h T(u, \omega)\|_{H^s}}{\|(u, \omega)\|_{H^s}} \lesssim h^{r_s} \varepsilon_h(\lambda).
Thanks to Theorem 3.1, Theorem 7.1 in [2] and (3.30) we deduce
\[ \widehat{\theta}(R(E), R(E_h)) \leq \| (T - T_h) \|_{R(E)} \| H^\perp \| \leq h^{r_v} e_{h}(\lambda). \]

And from Theorem 7.4 in [2] we get
\[ \| (u_h, \omega_h) - (u, \omega) \|_{H^\perp} \leq \| (T - T_h)(u, \omega) \|_{H^\perp} \leq \| (u, \omega) - P_h(u, \omega) \|_{H^\perp}, \]

thus from (3.26) and \(\| (u, \omega) - P_h(u, \omega) \|_{H} \leq \| (u, \omega) - (u_h, \omega_h) \|_{H}\) we obtain (3.35). This completes the proof. \(\square\)

**Remark 3.1.** Using the same argument as in this section we can prove the error estimates of finite element approximation for the dual problem (2.15), for example, there hold the following two estimates
\[ (3.37) \quad \| (u_h^n, \omega_h^n) - (u^*, \omega^*) \|_{H} \leq e_{h}^n(\lambda^*)^{s/2}, \]
\[ (3.38) \quad \| (u_h^n, \omega_h^n) - (u^*, \omega^*) \|_{H^\perp} \leq (h^{r_v} e_{h}(\lambda^*))^{s/2}, \quad s = 0, 1. \]

4. **Two grid discretization scheme.** In this section we use the two grid discretization scheme to treat transmission eigenvalues problem.

**Definition 4.1.** \(\forall (v, z), (v^*, z^*) \in H, B((v, z), (v^*, z^*)) \neq 0\), define \(A((v, z), (v^*, z^*)) = \frac{B((v, z), (v^*, z^*))}{B((v^*, z^*),(v^*, z^*))}\) as the generalized Rayleigh quotient of \((v, z)\) and \((v^*, z^*)\).

We now outline the two grid discretization scheme.

**Scheme 4.1.** Two grid discretization scheme.

**Step 1.** Solve (4.1) on a coarse grid \(\pi_H\): Find \(\lambda_H \in \mathbb{C}, (u_H, \omega_H) \in H_H\) such that
\[ \| (u_H, \omega_H) \|_A = 1 \]
and find \((u_H^n, \omega_H^n) \in M_H(\lambda^*)\) according to Lemma 4.1 and Remark 4.1.

**Step 2.** Solve two linear boundary value problems on a fine grid \(\pi_h\): Find \((u_h, \omega_h) \in H_h\) such that
\[ A((u_h, \omega_h), (v, z)) = \lambda_H B((u_h, \omega_h), (v, z)), \quad \forall (v, z) \in H_h, \]
and find \((u_h^*, \omega_h^*) \in H_h\) such that
\[ A((v, z), (u_h^*, \omega_h^*)) = \lambda_H B((v, z), (u_h^*, \omega_h^*)), \quad \forall (v, z) \in H_h. \]

**Step 3.** Compute the generalized Rayleigh quotient \(\lambda^h = \frac{A((u_h^h, \omega_h^h), (u_h^*, \omega_h^*))}{B((u_h^h, \omega_h^h), (u_h^*, \omega_h^*))}\).

A basic condition to the two grid discretization scheme is that \(|B((u_H, \omega_H), (u_H^*, \omega_H^*))|\) has a positive lower bound uniformly with respect to \(H\) (see, e.g., Theorem 3.5 in [24]). The following Lemma 4.1 guarantees this condition. Using this condition, in the following Theorem 4.3 we can also prove \(|B((u_h, \omega_h), (u_h^*, \omega_h^*))|\) has a positive lower bound uniformly with respect to \(h\).

**Lemma 4.1.** Suppose that \(\lambda_H = \lambda_{j,H}\) \((j = k, k + 1, \ldots, k + q - 1)\), and \((u_H, \omega_H)\) is an eigenfunction corresponding to \(\lambda_H\). Let \((u_H^*, \omega_H^*)\) be the orthogonal projection of \((u_H, \omega_H)\) to \(M_H(\lambda^*)\) in the sense of inner product \(A(\cdot, \cdot)\), and let
\[ (4.2) \quad (u_H^*, \omega_H^*) = \frac{(u_H, \omega_H)}{\| (u_H, \omega_H) \|_A}. \]
Thus, there is a constant uniformly with respect to $v, z$ for any $(H)$ from (3.1), (4.3) and (4.4), deduce

From [2, 10], if

$$ f((v, z)) = A(E(v, z), (u', \omega')) \quad \text{where} \quad (u', \omega') = -\frac{E(u_H, \omega_H)}{\|E(u_H, \omega_H)\|_A}. $$

Since for any $(v, z) \in H$

$$ |f((v, z))| \leq \|E(v, z)\|_A \leq \|E\|_A \|A(v, z)\|_A, $$

$f$ is a linear and bounded functional on $H$ and $\|f\|_A \leq \|E\|_A$. Using Riesz Theorem, we know there exists $(\tilde{u}, \tilde{\omega}) \in H$ satisfying

$$ A((v, z), (\tilde{u}, \tilde{\omega})) = A(E(v, z), (u', \omega')), \quad \|\tilde{u}\|_A \leq \|E\|_A. $$

Now, for any $(v, z) \in H$,

$$ A((v, z), (\lambda^{-1} - T^*)^\alpha (\tilde{u}, \tilde{\omega})) = A((\lambda^{-1} - T)^\alpha (v, z), (\tilde{u}, \tilde{\omega})) $$

$$ = A(E(\lambda^{-1} - T)^\alpha (v, z), (u', \omega')) = A(\lambda^{-1} - T)^\alpha E(v, z), (u', \omega')) = 0, $$

i.e., $(\lambda^{-1} - T^*)^\alpha (\tilde{u}, \tilde{\omega}) = 0$. Hence $(\tilde{u}, \tilde{\omega}) \in R(E^*)$. Let $(u'_H, \omega'_H) = \frac{E_H(\tilde{u}, \tilde{\omega})}{A(u_H, \omega_H, E_H(\tilde{u}, \tilde{\omega}))},$

then $(u'_H, \omega'_H) \in R(E_H^*) = M_H(\lambda^*)$,

$$ 1 = A(u_H, \omega_H, (u'_H, \omega'_H)) $$

$$ = A(u_H, \omega_H) - (u'_H, \omega'_H) + (u_H, \omega_H) $$$$ (u'_H, \omega'_H) \leq \|u'_H, \omega'_H\|_A \|u'_H, \omega'_H\|_A. $$

From [2, 10], if $H \to 0$ we have

$$ \|(E_H^* - E^*)\|_{R(E^*)} \to 0, $$

$$ \|E(u_H, \omega_H)\|_A = \|E(u_H, \omega_H) - (u_H, \omega_H) + (u_H, \omega_H)\|_A \to 1, $$

and

$$ A((u_H, \omega_H), E_H^*(\tilde{u}, \tilde{\omega})) = A((u_H, \omega_H), (E_H^* - E^*)(\tilde{u}, \tilde{\omega})) + A((u_H, \omega_H), E^*(\tilde{u}, \tilde{\omega})) $$

$$ = A((u_H, \omega_H), (E_H^* - E^*)(\tilde{u}, \tilde{\omega})) + \|E(u_H, \omega_H)\|_A \to 1. $$

Thus, there is a constant $C_0$ independent of $H$ such that

$$ \|(u'_H, \omega'_H)\|_A = \frac{\|E_H(\tilde{u}, \tilde{\omega})\|_A}{A((u_H, \omega_H), E_H^*(\tilde{u}, \tilde{\omega}))} \leq C_0. $$

From [31, 143] and [14], deduce

$$ |B((u_H, \omega_H), (u'_H, \omega'_H))| = |\lambda^{-1}_H||A((u_H, \omega_H), (u'_H, \omega'_H))| $$

$$ = |\lambda^{-1}_H||A(u_H, \omega_H, (u'_H, \omega'_H))_A| = |\lambda^{-1}_H||A(u_H, \omega_H, (u'_H, \omega'_H))_A| $$

$$ = |\lambda^{-1}_H||(u'_H, \omega'_H))_A \geq \frac{1}{|\lambda_H||u'_H, \omega'_H)\|_A \geq C_0||\lambda_H|. $$

Then when $H$ is small enough $|B((u_H, \omega_H), (u'_H, \omega'_H))|$ has a positive lower bound uniformly with respect to $H$. $\Box$

The above lemma is fundamental for studying two grid method as well as multigrid method and adaptive algorithm for the transmission eigenvalue problem and many 2$\text{th}$ order nonselfadjoint elliptic eigenvalue problems.
Remark 4.1. Computational method for \((u_H^j, \omega_H^j)\).

Step 1. Find a basis \[\{(\psi_i, \varphi_i)\}_{i=k}^{k+q-1} \] in \(M_H(\lambda^*)\).

How to seek this basis efficiently is an important issue of linear algebra. When the ascent of \(A_i \) is equal to 1 \((i = k, \cdots, k+q-1)\), we can use the Arnoldi algorithm \cite{20, 23, 30} to solve the dual problem of (4.1) and obtain this basis and meanwhile MATLAB has provided implemented Arnoldi solvers “sptarn” and “eigs”; we can also use the two sided Arnoldi algorithm in \cite{16} to compute both left and right eigenvectors of \((\lambda, \omega)\) at the same time, and obtain a basis \[\{(u_i, \omega_i, \lambda_i)\}_{i=k}^{k+q-1} \] in \(M_H(\lambda)\) and a basis \[\{(\psi_i, \varphi_i)\}_{i=k}^{k+q-1} \] in \(M_H(\lambda^*)\).

Step 2. Solve the following equations: find \(\beta_i \in \mathbb{C}, i = k, \cdots, k+q-1\) such that

\[\sum_{i=k}^{k+q-1} \beta_i A((\psi_i, \varphi_i), (\psi, \varphi)) = A((u, \omega), (\psi, \varphi)), \quad l = k, \cdots, k+q-1.\]

Then \((u_H^l, \omega_H^l) = \sum_{i=k}^{k+q-1} \beta_i (\psi_i, \varphi_i)\) and \((u_H^l, \omega_H^l) = (u_H^l, \omega_H^l)/\|\|u_H^l, \omega_H^l\|\),

Remark 4.1. Computational method for \((u_H^j, \omega_H^j)\).

Step 1. Find a basis \[\{(\psi_i, \varphi_i)\}_{i=k}^{k+q-1} \] in \(M_H(\lambda^*)\).

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Step 2. Solve the following equations: find \(\beta_i \in \mathbb{C}, i = k, \cdots, k+q-1\) such that

\[\sum_{i=k}^{k+q-1} \beta_i A((\psi_i, \varphi_i), (\psi, \varphi)) = A((u, \omega), (\psi, \varphi)), \quad l = k, \cdots, k+q-1.\]

Then \((u_H^l, \omega_H^l) = \sum_{i=k}^{k+q-1} \beta_i (\psi_i, \varphi_i)\) and \((u_H^l, \omega_H^l) = (u_H^l, \omega_H^l)/\|\|u_H^l, \omega_H^l\|\),

Lemma 4.2. Let \((\lambda, u, \omega)\) and \((\lambda^*, u^*, \omega^*)\) be the eigenpair of (2.6) and (2.15), respectively. Then, \(\forall (v, z), (v^*, z^*) \in \mathbb{H}, B((v, z), (v^*, z^*)) \neq 0\), the generalized Rayleigh quotient satisfies

\[
\frac{A((v, z), (v^*, z^*))}{B((v, z), (v^*, z^*))} - \lambda = \frac{A((v, z) - (u, \omega), (v^*, z^*) - (u^*, \omega^*))}{B((v, z), (v^*, z^*))} - \lambda \frac{B((v, z) - (u, \omega), (v^*, z^*) - (u^*, \omega^*))}{B((v, z), (v^*, z^*))}.
\]

(4.5)

Proof. From (2.6) and (2.15) we have

\[
\begin{align*}
A((v, z) - (u, \omega), (v^*, z^*) - (u^*, \omega^*)) - \lambda B((v, z) - (u, \omega), (v^*, z^*) - (u^*, \omega^*))
= & \quad A((v, z), (v^*, z^*)) - A((v, z), (u^*, \omega^*)) \\
- & \quad A((u, \omega), (v^*, z^*)) + A((u, \omega), (u^*, \omega^*)) \\
- & \quad \lambda(B((v, z), (v^*, z^*)) - B((v, z), (u^*, \omega^*)) \\
- & \quad B((u, \omega), (v^*, z^*)) + B((u, \omega), (u^*, \omega^*)) \\
= & \quad A((v, z), (v^*, z^*)) - \lambda B((v, z), (u^*, \omega^*)) \\
- & \quad \lambda B((u, \omega), (v^*, z^*)) + \lambda B((u, \omega), (u^*, \omega^*)) \\
- & \quad \lambda(B((v, z), (v^*, z^*)) - B((v, z), (u^*, \omega^*)) \\
- & \quad B((u, \omega), (v^*, z^*)) + B((u, \omega), (u^*, \omega^*)) \\
= & \quad A((v, z), (v^*, z^*)) - \lambda B((v, z), (u^*, \omega^*)).
\end{align*}
\]

dividing both sides by \(B((v, z), (v^*, z^*))\) we obtain the desired conclusion. \(\blacksquare\)

Theorem 4.3. Let \(\lambda_H, (u_H^j, \omega_H^j), (u_H^j, \omega_H^j), \lambda^b, (u^b, \omega^b), (u^b, \omega^b)\) be the numerical eigenpairs obtained by Scheme 4.1, and \(\lambda_H = \lambda_j, H (j = k, k+1, \cdots, k+q-1)\).
Assume that the ascent of $\lambda$ are equal to 1, and that $R(D)$ and (C2) are valid, $n \in W^{2-s, \infty}(D)$ ($s = 0, 1$). Then there exists $(u, \omega) \in R(E)$ and $(u^*, \omega^*) \in R(E^*)$ such that $H$ is properly small there hold

\begin{align}
(4.6) \quad & ||(u^h, \omega^h) - (u, \omega)||_H \lesssim H^r \varepsilon_H(\lambda) + \varepsilon_h(\lambda), \\
(4.7) \quad & ||(u^{h^*}, \omega^{h^*}) - (u^*, \omega^*)||_H \lesssim H^r \varepsilon_H^*(\lambda^*) + \varepsilon_h^*(\lambda^*), \\
(4.8) \quad & |\lambda^h - \lambda| \lesssim \{H^r \varepsilon_H(\lambda) + \varepsilon_h(\lambda)\} \{H^r \varepsilon_H^*(\lambda^*) + \varepsilon_h^*(\lambda^*)\}, \quad s = 0, 1.
\end{align}

Proof. Let $(u, \omega) \in R(E)$ such that $(u, \omega) - (u_H, \omega_H)$ and $\lambda - \lambda_H$ both satisfy Theorem 3.3 and Theorem 3.5. From (2.12) we get $(u, \omega) = \lambda T(u, \omega)$, and from (3.3) and Step 2 of Scheme 4.1, we get $(u^h, \omega^h) = \lambda_H T_h(u_H, \omega_H)$. From (3.3), we have

\begin{align*}
\|\lambda_H T_h(u_H, \omega_H) - \lambda T_h(u, \omega)\|_H & \leq \|\lambda_H T_h(u_H, \omega_H) - \lambda T_h(u, \omega)\|_* + \|\lambda T_h(u, \omega) - T(u, \omega)\|_H \\
& \lesssim H^r \varepsilon_H(\lambda) + \varepsilon_h(\lambda),
\end{align*}

i.e., (4.6) holds. Similarly we can prove (4.7).

From (4.5), we have

\begin{align}
\lambda^h - \lambda & = \frac{A((u^h, \omega^h) - (u, \omega), (u^{h^*}, \omega^{h^*}) - (u^*, \omega^*))}{B((u^h, \omega^h), (u^{h^*}, \omega^{h^*}))} \\
& \quad - \lambda \frac{B((u^h, \omega^h) - (u, \omega), (u^{h^*}, \omega^{h^*}) - (u^*, \omega^*))}{B((u^h, \omega^h), (u^{h^*}, \omega^{h^*}))}.
\end{align}

Note that $(u_H, \omega_H)$ and $(u^h, \omega^h)$ just approximate the same eigenfunction $(u, \omega)$, $(u^{h^*}, \omega^{h^*})$ and $(u^*, \omega^*)$ approximate the same adjoint eigenfunction $(u^*, \omega^*)$, and $|B((u_H, \omega_H), (u_H, \omega_H))|$ has a positive lower bound uniformly with respect to $H$.

From

\begin{align*}
B((u^h, \omega^h), (u^{h^*}, \omega^{h^*})) & = B((u^h, \omega^h), (u^{h^*}, \omega^{h^*})) - B((u, \omega), (u^*, \omega^*)) \\
& \quad + B((u, \omega), (u^*, \omega^*)) - B((u_H, \omega_H), (u_H, \omega_H)) + B((u_H, \omega_H), (u_H, \omega_H)),
\end{align*}

we know that $|B((u^h, \omega^h), (u^{h^*}, \omega^{h^*}))|$ has a positive lower bound uniformly. Therefore from (4.5), we get

\begin{align}
|\lambda^h - \lambda| & \lesssim ||(u^h, \omega^h) - (u, \omega)||_H \|B((u^{h^*}, \omega^{h^*}) - (u^*, \omega^*))\|_H.
\end{align}

Substituting (4.6) and (4.7) into (4.10) yields (4.8).

5. Numerical experiment. In this section, we will report some numerical experiments for the finite element discretization (3.1) and two grid discretization scheme (Scheme 4.1) to validate our theoretical results.

We use MATLAB 2012a to solve (1.1)-(1.4) on a Lenovo G480 PC with 4G memory. Our program is implemented using the package iFEM [11].
Let \( \{ \xi_i \}_{i=1}^{N_h} \) be a basis of \( S_h \) and \( u_h = \sum_{i=1}^{N_h} u_i \xi_i, \omega_h = \sum_{i=1}^{N_h} \omega_i \xi_i \). A similar definition can be made for \( u_h^* \) and \( \omega_h^* \). Denote \( \overrightarrow{u} = (u_1, \ldots, u_{N_h})^T \) and \( \overrightarrow{\omega} = (\omega_1, \ldots, \omega_{N_h})^T \). Similarly \( \overrightarrow{u}^* \) and \( \overrightarrow{\omega}^* \) can be defined from \( u_h^* \) and \( \omega_h^* \). To describe our algorithm, we specify the following matrices in the discrete case.

| Matrix | Dimension | Definition |
|--------|-----------|------------|
| \( A_h \) | \( N_h \times N_h \) | \( a_{i,j} = \int_D \Delta \xi_i \Delta \xi_j dx \) |
| \( B_h \) | \( N_h \times N_h \) | \( b_{i,j} = -\int_D (\nabla \xi_i \cdot \nabla \xi_j + \Delta \xi_i \xi_j) dx \) |
| \( C_h \) | \( N_h \times N_h \) | \( c_{i,j} = -\int_D (\nabla \xi_i \cdot \nabla \xi_j) dx \) |
| \( D_h \) | \( N_h \times N_h \) | \( d_{i,j} = \int_D \xi_i \xi_j dx \) |

when \( n \in W^{1,\infty}(D) \), \( b_{i,j} = \int_D (\nabla \xi_i \cdot \nabla \xi_j + \nabla \xi_i \cdot \nabla (\nabla (\nabla^{-1} \xi_j))) dx \).

Then \( (5.1) \) and \( (5.11) \) can be written as the generalized eigenvalue problems

\[
(5.1) \quad \begin{pmatrix} A_h & 0 \\ 0 & D_h \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{\omega} \end{pmatrix} = \lambda_h \begin{pmatrix} B_h & C_h \\ D_h & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{\omega} \end{pmatrix},
\]

and

\[
(5.2) \quad \begin{pmatrix} A_h & 0 \\ 0 & D_h \end{pmatrix} \begin{pmatrix} \overrightarrow{u}^* \\ \overrightarrow{\omega}^* \end{pmatrix} = \lambda_h \begin{pmatrix} B_h & D_h \\ C_h & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{u}^* \\ \overrightarrow{\omega}^* \end{pmatrix}.
\]

Note that in \( (5.1) \), \( A_h \) is a positive definite Hermitian matrix and \( D_h \) can be equivalently replaced by the identity matrix \( I_h \), which will lead to two sparser coefficient matrices with a good properties. Thus the computation of the eigenpairs for \( (5.1) \) is efficient. Similarly, the matrices for Scheme 4.1 can be given but are omitted in this paper.

For convenience, we use the following notations in our tables and figures:
\( k_{j,h} = \sqrt{\lambda_{j,h}} \): The \( j \)th eigenvalue obtained by \( (5.1) \) on \( \pi_h \).
\( k_{j,H} = \sqrt{\lambda_{j,H}} \): The \( j \)th eigenvalue obtained by \( (5.1) \) on \( \pi_H \).
\( k_j^h = \sqrt{\lambda_j^h} \): The \( j \)th eigenvalue obtained by Scheme 4.1.

\( -\): Failure of computation due to running out of memory.

**5.1. Model problem on the unit square.** We first consider the case when \( D \) is the unit square \([0,1]^2\) and the index of refraction \( n = 16 \) or \( n = 8 + x_1 - x_2 \). We use BFS element to compute the problem, and the numerical results are shown in Tables 5.1-5.2. We also depict the error curves for the numerical eigenvalues (see Figure 5.1). According to regularity theory, we know \( r_0 = 2 \) and \( u, \omega \in H^4(D) \). When the ascent \( \alpha = 1 \): according to \( (3.19) \) and \( (3.15) \), the convergence order of the eigenvalue approximation \( k_{j,h} \) is four; according to \( (4.8) \), when \( h \geq H^2 \), the convergence order of the \( k_j^h \) is also four, i.e.,

\[
|k_j - k_{j,h}| \lesssim h^4, \quad |k_j - k_j^h| \lesssim h^4.
\]

It is seen from Figure 5.1 that the convergence order of the numerical eigenvalues on the unit square is four, which coincides with the theoretical result \( (5.3) \).

In addition, Tables 5.1-5.2 show that the numerical eigenvalues obtained by BFS elements give a good approximation; it is worthwhile noticing that two grid discretization scheme can achieve the same convergence order as solving the eigenvalue problem by BFS element directly. Moreover, we find that the two grid discretization scheme can be performed on finer meshes so that we can obtain more accurate numerical eigenvalues. Therefore, the two grid discretization scheme for solving transmission eigenvalue problem is more efficient.
Table 5.1
The eigenvalues obtained by BFS element on the unit square, \( n = 16 \).

| \( j \) | \( H \) | \( k_{j,H} \) | \( k_{j}^h \) | \( k_{j,h} \) |
|------|------|----------|----------|----------|
| 1    | \( \sqrt{2} \) | 1.8800518272 | 1.8795932933 | 1.8795931085 |
| 1    | \( \sqrt{2} \) | 1.8796216444 | 1.8795911813 | 1.8795911812 |
| 1    | \( \sqrt{2} \) | 1.8795931085 | 1.8795911697 | 1.8795911747 |
| 2    | \( 2 \) | 2.4462555154 | 2.4442475976 | 2.444247101 |
| 2    | \( 2 \) | 2.4443713201 | 2.4442361446 | 2.4442361333 |
| 2    | \( 2 \) | 2.4442447101 | 2.4442361002 | — |
| 3    | \( \sqrt{2} \) | 2.8681931483 | 2.8664515120 | 2.8664469634 |
| 3    | \( \sqrt{2} \) | 2.8665606968 | 2.8664391607 | 2.8664391408 |
| 3    | \( \sqrt{2} \) | 2.8664469634 | 2.8664391111 | — |
| 4    | \( 2 \) | 3.5424522436 | 3.5387180040 | 3.5387126105 |
| 4    | \( 2 \) | 3.5389469722 | 3.5386967795 | 3.5386967579 |
| 4    | \( 2 \) | 3.5387126105 | 3.5386967008 | — |
| 5,6  | \( 3 \) | 4.4971031374 | 4.4964665266 | 4.4965591247 |
| 5,6  | \( 3 \) | 4.4965591247 | 4.4965519816 | 4.4965519832 |
| 5,6  | \( 3 \) | 4.4965591247 | 4.4965519531 | — |

Table 5.2
The eigenvalues obtained by BFS element on the unit square, \( n = 8 + x_1 - x_2 \).

| \( j \) | \( H \) | \( k_{j,H} \) | \( k_{j}^h \) | \( k_{j,h} \) |
|------|------|----------|----------|----------|
| 1    | \( \sqrt{2} \) | 2.8234457937 | 2.8221946996 | 2.8221945051 |
| 1    | \( \sqrt{2} \) | 2.8222709846 | 2.8221893629 | 2.8221893619 |
| 1    | \( \sqrt{2} \) | 2.8221945051 | 2.8221893480 | — |
| 2    | \( \sqrt{2} \) | 3.5424522436 | 3.5387180040 | 3.5387126105 |
| 2    | \( \sqrt{2} \) | 3.5389469722 | 3.5386967795 | 3.5386967579 |
| 2    | \( \sqrt{2} \) | 3.5387126105 | 3.5386967008 | — |
| 5,6  | \( 2 \) | 4.4971031374 | 4.4964665266 | 4.4965591247 |
| 5,6  | \( 2 \) | 4.4965591247 | 4.4965519816 | 4.4965519832 |
| 5,6  | \( 2 \) | 4.4965591247 | 4.4965519531 | — |

5.2. Model problem on the L-shaped domain. We consider the case when \( D = (-1, 1)^2 \backslash ([0, 1] \times (-1, 0)) \) is the L-shaped domain and the index of refraction \( n = 16 \) or \( n = 8 + x_1 - x_2 \). The numerical results obtained by BFS element are shown in Tables 5.3-5.4 and Figure 5.2.

Figure 5.2 indicates that on the L-shaped domain, the convergence order of \( k_{1,h}, k_{2,h}, k_{5,h} \) is around 1.3 and 2.3 respectively for \( n = 16 \), and the convergence order of \( k_{1,h}, k_{5,h} \) is around 1.4 for \( n = 8 + x_1 - x_2 \). This fact suggests that the four eigenfunctions on the non-convex domain do have singularities to different degrees.

5.3. Model problem on the circular domain. We also investigate the case of \( n \) being piecewise constant for a disk \( D \) of the radius 1. Let \( n(x) = n_1 \) for \( x \in D_1 \) and \( n(x) = n_2 \) for \( x \in D \backslash D_1 \), \( D_1 \) being an inner disk of the radius \( r_1 < 1 \). For this disk domain we generate a triangular mesh with \( h \approx 1/40 \) and number of degrees of freedom 101040 and move the nodes outside and nearest the inner disk onto the inner circle. And we use the Argyris element to solve (1.1)-(1.4) on the mesh and the numerical eigenvalues associated with different \( n \) and \( r_1 \) are shown in Table 5.5. The analytic eigenvalues can be referred to Tables 2 and 3 in [19].
Table 5.3
The eigenvalues obtained by BFS element on the L-shaped domain, n = 16.

| j | H  | h       | k_{1,H} | k_{j,h} | k_{j,h} |
|---|-----|---------|---------|---------|---------|
| 1 | 8   | 1       | 1.4850653844 | 1.4781249432 | 1.4780403370 |
| 1 | 16  | 1       | 1.480242297  | 1.4770298927  | 1.4770116105 |
| 1 | 32  | 1       | 1.5705634174 | 1.5697720130  | 1.569735335  |
| 2 | 8   | 1       | 1.5699010557 | 1.5697293878  | —          |
| 2 | 16  | 1       | 1.5699010557 | 1.5697293878  | —          |
| 2 | 32  | 1       | 1.7078128918 | 1.7055170229  | 1.7055794458 |
| 3 | 8   | 1       | 1.7061981331 | 1.7051443044  | —          |
| 3 | 16  | 1       | 1.7061981331 | 1.7051443044  | —          |
| 3 | 32  | 1       | 1.7834680505 | 1.7831208381  | 1.7831208523 |
| 4 | 8   | 1       | 1.7831489498 | 1.7831163182  | —          |
| 4 | 16  | 1       | 1.7831489498 | 1.7831163182  | —          |
| 4 | 32  | 1       | 1.7831489498 | 1.7831163182  | —          |

Table 5.4
The eigenvalues obtained by BFS element on the L-shaped domain, n = 8 + x₁ - x₂.

| j | H  | h       | k_{1,H} | k_{j,h} | k_{j,h} |
|---|-----|---------|---------|---------|---------|
| 1 | 8/8  | 2       | 2.3127184233 | 2.3045393229 | 2.3043808707 |
| 1 | 16/16| 2       | 2.3069612717 | 2.3032153953 | 2.3031811119 |
| 1 | 32/32| 2       | 2.3069612717 | 2.3026641395 | —         |
| 2 | 8/8  | 2       | 2.3974892428 | 2.3957871378 | 2.3957863810 |
| 2 | 16/16| 2       | 2.3960567236 | 2.3957182671 | 2.3957182148 |
| 2 | 32/32| 2       | 2.3960567236 | 2.3956994585 | —         |
| 3,6| 8/8  | ±0.5743458101 | ±0.5664545248i | ±0.5664307881i | —         |
| 3,6| 16/16| ±0.5686045702 | ±0.5654537885i | ±0.5654487200i | —         |
| 3,6| 32/32| ±0.5686045702 | ±0.5654994478i | —         | —         |

![Error curves on the unit square for k₁, k₂ with n = 16 (left), and for k₁, k₅ with n = 8 + x₁ - x₂ (right).](image-url)
The eigenvalues obtained by Argyris element on the circle of radius 1 for piecewise constant $n$. 

| $n_1$, $n_2$, $r_1$ | $\{k_{1,1}, h_{1,1}, k_{2,1}, h_{2,1}, k_{3,1}, h_{3,1}, k_{4,1}, h_{4,1}, k_{5,1}, h_{5,1}, k_{6,1}, h_{6,1}, k_{7,1}, h_{7,1}, k_{8,1}, h_{8,1}, k_{9,1}, h_{9,1}, k_{10,1}, h_{10,1}, k_{11,1}\}$ |
|---|---|
| 1 3 0.5 | 1.4975, 1.7354, 1.7357, 2.1709, 2.1721, 2.3570, 0.4881, 2.6995, 2.6999 |
| 2 4 0.5 | 2.4349 $\pm$ 0.6969, 3.5812 $\pm$ 0.5731, 3.5824 $\pm$ 0.5730, 3.9137, 3.9196, 4.1160 |
| 5 8 0.6 | 1.7891, 2.2512, 2.520, 2.5574 $\pm$ 0.3780, 2.6681, 2.6717, 3.0376, 3.0388 |
| 5 3 0.6 | 2.2996 $\pm$ 0.7437, 2.8876, 2.8885, 3.1861, 3.2953, 3.2982 |
| 10 8 0.7 | 1.3732, 1.7322, 1.7328, 2.1101, 2.1121, 2.3665 $\pm$ 0.4401, 2.4958, 2.4964 |
| 2 4 0.7 | 2.4280 $\pm$ 0.6299, 3.7582 $\pm$ 0.6135, 3.7593 $\pm$ 0.6136, 4.9689 $\pm$ 0.4384, 4.9751 $\pm$ 0.4395, 5.0515 |
| 13 11 0.8 | 1.1499, 1.4899, 1.4903, 1.8261, 1.8278, 2.1570, 2.1575 |
| 3 6 0.8 | 2.2225 $\pm$ 0.4796, 3.5482, 3.5485, 3.6711, 3.6748, 3.8012 |
| 6 13 0.9 | 1.5929, 2.0271, 2.0280, 2.487 $\pm$ 0.2643, 2.5142, 2.5171, 3.0034, 3.0043 |
| 6 2 0.9 | 2.0222, 2.4190, 2.4192, 2.7401 $\pm$ 0.4468, 2.9309, 2.9317 |

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