On the Thermodynamics and Geodesic Stability of Charged Symmetric Black Holes in $f(R)$ Gravity

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In a recent paper, Nashed and Capozziello presented a new class of charged, spherically symmetric black hole solutions of $f(R)$ gravity with an asymptotic flat or (anti-)de Sitter behavior. These metrics depend on a dimensional parameter $\alpha$ and are interesting because they cannot reduce to general relativity solutions. However, there are some issues in the study of the physical and thermodynamical properties of these black holes that we correct in this paper. Some of our results show that the entropy is always positive within the allowed values of $\alpha$ and due to the well behaved quantities such as Gibbs free energy, we conclude that there is no such a phase transition as discussed in the previous work. We also study the geodesics in these space-times and particularly, the stability of the circular orbits to obtain the radius of the Innermost Stable Circular Orbit.

I. INTRODUCTION

Recently [1], Nashed and Capozziello found a new class of charged spherically symmetric black hole solutions with a flat or (anti-)de Sitter asymptotic behavior in the context of the $f(R)$ gravitational scenario with the particular function $f(R) = R - 2\alpha\sqrt{R - 2\Lambda}$, where $\Lambda$ is the cosmological constant. Although the reported metrics are indeed solutions of the Maxwell-$f(R)$ field equations, the physical properties, thermodynamics and stability study presented by the authors in [1] have some errors that we will correct in this paper.

The work is organized as follows: in Section II we present the charged, spherically symmetric black hole solutions found by Nashed and Capozziello in [1] and calculate the correct radius of the event horizon as well as the values of two of the curvature invariants. In section III we introduce some thermodynamical quantities such as Hawking temperature, entropy and Gibbs free energy to show their behavior for the balck hole solutions. The results indicate that there are no phase transitions for these solutions. Section IV is dedicated to study the geodesics and geodesic deviation stability for circular orbits. The obtained equations permit us to find the radius of the Innermost Stable Circular Orbit (ISCO) for these black holes. Finally, Section V presents some conclusions.

II. THE CHARGED AND SPHERICAL BLACK HOLE SOLUTIONS

The model of gravity used to obtain the black hole solutions is given by the action

$$S = S_g + S_{EM}$$

where $S_{EM}$ is the electromagnetic field action and

$$S_g = \frac{1}{2\kappa} \int \sqrt{-g} [f(R) - \Lambda]$$

is the gravitational action with $\Lambda$ the cosmological constant, $R$ the Ricci scalar, $g$ the determinant of the metric and $\kappa$ the gravitational constant. The resulting Maxwell-$f(R)$ field equations are

$$R_{\mu\nu}f'(R) - \frac{1}{2} g_{\mu\nu} f(R) - 2g_{\mu\nu}\Lambda + g_{\mu\nu}\Box f'(R) - \nabla_\mu \nabla_\nu f'(R) = 8\pi T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci tensor, $F^{\mu\nu}$ is the electromagnetic field strength tensor, $f'(R) = \frac{df}{dR}$ and

$$T^{\mu\nu} = \frac{1}{4\pi} \left[ g_{\alpha\beta} F_\mu^\alpha F_\nu^\beta - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \right].$$

The particular model considered to obtain the solutions is given by the function

$$f(R) = R - 2\alpha\sqrt{R - 8\Lambda},$$

where $\alpha$ is a dimensional parameter with positive values.
A. The Asymptotically Flat Black Hole

The first solution presented in [1] represents a charged, spherically symmetric black hole in the absence of cosmological constant and therefore, the space-time is asymptotically flat. The line element is

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (7)

with

$$B(r) = \frac{1}{2} - \frac{1}{3\alpha r} + \frac{1}{3\alpha r^2}.$$  \hspace{1cm} (8)

The gauge potential in this case is given by

$$A = \frac{1}{\sqrt{3\alpha r}}.$$  \hspace{1cm} (9)

From these equations it is clear that the parameter $\alpha$ cannot be zero and therefore this solution cannot reduce to a metric in general relativity, e.g. Schwarzschild’s or Reissner-Nordstrom’s solutions. Ricci’s curvature scalar $R = g_{\mu\nu}R^{\mu\nu}$ and Kretchmann scalar $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ for this new solution are

$$R = \frac{1}{r^2}$$  \hspace{1cm} (10)

$$K = \frac{9\alpha^2 r^4 + 12\alpha r^3 - 12(\alpha - 1)r^2 - 48r + 56}{9\alpha^2 r^8}.$$  \hspace{1cm} (11)

These invariants show that the point $r = 0$ is an essential singularity and that $\alpha \neq 0$.

The event horizon of a spherically symmetric black hole with a line element as that defined in Eq. (7) is located at the radius $r = r_H$ defined by the largest positive root of the equation $B(r) = 0$.

For the asymptotically flat black hole of Eq. (8), the horizon radius is

$$r_H(\alpha) = \frac{1}{3\alpha} \left[ 1 + \sqrt{1 - 6\alpha} \right]$$  \hspace{1cm} (12)

which differs from that reported in Eq. (26) in [1]. It is clear that the existence of the horizon implies that $0 < \alpha \leq \frac{1}{6}$. The value $\alpha = \frac{1}{6}$ corresponds to an extremal black hole with a degenerate horizon radius $r_{H}^{ext} = 2$ (see the complete behavior in Fig. 1).

B. The (Anti-)de Sitter Black Holes

Considering a non-vanishing cosmological constant $\Lambda$, Nashed and Capozziello obtained a solution representing a charged, spherically symmetric black hole which asymptotically behaves as (anti-)de Sitter space-time. The line element has the same structure given in Eq. (7) but this time

$$B(r) = \frac{1}{2} - \frac{2\Lambda}{3} r^2 - \frac{1}{3\alpha r} + \frac{1}{3\alpha r^2},$$  \hspace{1cm} (13)

while the gauge potential is given again by

$$A = \frac{1}{\sqrt{3\alpha r}}.$$  \hspace{1cm} (14)

This metric gives the Ricci’s and Kretchmann’s curvature invariants

$$R = g^{\mu\nu} R_{\mu\nu} = \frac{1 + 8\Lambda r^2}{r^2},$$  \hspace{1cm} (15)

$$K = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{96\Lambda^2 \alpha^2 r^8 + 24\Lambda\alpha^2 r^6 + 9\alpha^2 r^4 + 12\alpha r^3 - 12(\alpha - 1)r^2 - 48r + 56}{9\alpha^2 r^8},$$  \hspace{1cm} (16)

Concerning the event horizon, the condition $B(r_H) = 0$ for the asymptotically (anti-)de Sitter black hole gives the
horizon radius as the largest root of the polynomial

\[4A\alpha r_H^4 - 3A\alpha r_H^2 + 2r_H - 2 = 0.\]  \hspace{1cm} (17)

This equation also differs from that reported in Eq. (26) in [1] by a sign in the last term. A numerical analysis
let us plot the horizon radius as function of \(\alpha\) as shown in Fig. 2.

\[\begin{align*}
\text{\textbf{FIG. 2.}} & \quad \text{Radius of the event horizon as function of the dimensional parameter } \alpha \text{ for some values of the cosmological constant in the asymptotically (anti-)de Sitter black hole.}
\end{align*}\]

III. BLACK HOLE THERMODYNAMICS

In this section we will study the black hole thermodynamic properties such as Hawking temperature, entropy, quasi-local energy and Gibbs free energy.

As usual, the Hawking temperature for these spherically symmetric black holes is given by the expression

\[T_H = \frac{1}{4\pi} \left. \frac{dB}{dr} \right|_{r=r_H}, \hspace{1cm} (18)\]

while the Bekenstein-Hawking entropy and the quasi-local energy in the \(f(R)\) model of gravity are [2-5]

\[S = \frac{1}{4} A_H f'(R(r_H)) \hspace{1cm} (19)\]

and

\[E_H = \frac{1}{4} \int \left[ 2f'(R(r_H)) + r_H^2 f(R(r_H)) - r_H^2 R(r_H)f'(R(r_H)) \right] dr_H, \hspace{1cm} (20)\]

where \(A_H = 4\pi r_H^2\) is the area of the event horizon.

A. Asymptotically Flat Black Hole

The temperature associated with the horizon of the asymptotically flat black hole, see Eq. [5], gives the function

\[T_H(\alpha) = \frac{3\alpha(1 + \sqrt{1 - 6\alpha - 6\alpha})}{4\pi(1 + \sqrt{1 - 6\alpha})^3}. \hspace{1cm} (21)\]

The complete behavior of this temperature as function of the parameter \(\alpha\) is significantly different from that reported in [1] and can be seen in Fig. 3. The value \(\alpha = \frac{1}{6}\), corresponding to the extremal black hole, gives a zero temperature, just as in black holes such as Reissner-Nördstrom or Kerr solutions in general relativity. Hence, the process of evaporation of these black holes will terminate in a frozen remnant with a final horizon radius \(r_H^\text{ext} = 2\). It is also interesting to note that the temperature function presents a maximum of \(T_H^{\text{max}} \approx 0.00682667\) at the parameter \(\alpha_m = \frac{1}{6} (\sqrt{2} - 1) \approx 0.138071\). This behavior is a unique characteristic of these black holes, with no similar situations in general relativity solutions.

\[\begin{align*}
\text{\textbf{FIG. 3.}} & \quad \text{Hawking temperature as function of the dimensional parameter } \alpha \text{ for the asymptotically flat black hole.}
\end{align*}\]

On the other hand, the entropy for the asymptotically flat black hole is the function

\[S(\alpha) = \frac{\pi}{27\alpha^2} \left[ 1 + \sqrt{1 - 6\alpha} \right]^2 \left[ 2 - \sqrt{1 - 6\alpha} \right], \hspace{1cm} (22)\]

from which it is clear that the entropy is always positive for \(0 \leq \alpha < \frac{1}{6}\) and has a smooth behavior in this range of \(\alpha\), as shown in Fig. 3. Therefore, there is no such a phase transition as the one reported in the analysis of Nashed and Capozziello [1].

To complete the thermodynamic analysis, we calculate the quasi-local energy for the asymptotically flat black hole, obtaining

\[E(\alpha) = \frac{1 + \sqrt{1 - 6\alpha} + 3\alpha}{12\alpha}. \hspace{1cm} (23)\]
and then, the Gibbs free energy gives

\[ G(\alpha) = \frac{5}{36\alpha} - \frac{(1 - \sqrt{1 - 6\alpha})}{12(1 + \sqrt{1 - 6\alpha})} \]  

(24)

The behavior of these function is shown in Figures 5 and 6. Both of them have a positive value and smooth behavior in the allowed range of the parameter \( \alpha \). This is a clear indication that no phase transitions occur for the asymptotically flat black hole [7].

Concerning the temperature, note that the cosmological constant produces an interesting behavior for small \( \alpha \), making \( T \) to diverge. Hence, the temperature function has a minimum and a maximum, depending on the value of \( \Lambda \). However, there is a value of \( \alpha \) for which the black hole freezes (\( T \to 0 \)).

On the other hand, the entropy function does not change significantly with the introduction of the cosmological constant, it has positive values and a smooth behavior in the allowed range of \( \alpha \).

Finally, the quasi-local energy for the asymptotically (anti-)de Sitter black hole gives the function

\[ E(r_H) = \frac{r_H}{2} - \frac{3\alpha}{8} r_H^2 + \frac{\Lambda \alpha}{2} r_H^4 \]  

(27)

and therefore, the Gibbs energy takes the form

\[ G(r_H) = \frac{1}{2} \alpha \Lambda r^4 - \frac{3\alpha r^2}{8} + \frac{r}{2} - (1 - \alpha r) \left( -4\alpha \Lambda r^4 + r - 2 \right) \]  

\[ \frac{12\alpha r}{12\alpha r} \]  

(28)

In Figures 9 and 10 we plot these functions for some particular values of the cosmological constant. Both of them have a smooth behavior with no indication of phase transitions for the asymptotically (A)dS black holes.
FIG. 7. Hawking temperature $T_H$ as function of the dimensional parameter $\alpha$ for some values of the cosmological constant in the asymptotically (anti-)de Sitter black hole.

FIG. 8. Bekenstein-Hawking entropy $S$ as function of the dimensional parameter $\alpha$ for some values of the cosmological constant in the asymptotically (anti-)de Sitter black hole.

FIG. 9. Quasi-local energy as function of the parameter $\alpha$ for the asymptotically (anti-)de Sitter black hole.

FIG. 10. Gibb’s free energy as function of the parameter $\alpha$ for the asymptotically (anti-)de Sitter black hole.

IV. GEODESICS STABILITY

A. Circular Geodesics

The study of the geodesics describing a test particle moving along a trajectory $x^\alpha(\lambda)$ with 4-velocity $\dot{x}^\alpha = dx^\alpha/d\lambda$ around one of the described black holes, gives the conservation of the specific energy, $E$, and the specific axial component of the angular momentum, $l$,

$$E = B(r)\dot{t} \quad , \quad l = r^2 \dot{\phi} ,$$

(29)

where the dot represents derivative with respect to the particle’s proper time. From the conservation of the rest-mass $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1$, we obtain the equation of motion for the radial coordinate in the equatorial plane ($\theta = \pi/2$),

$$\ddot{r}^2 = E^2 - V_{\text{eff}}^2(r)$$

(30)

where the effective potential function is

$$V_{\text{eff}}^2(r) = B(r) \left( 1 + \frac{l^2}{r^2} \right) .$$

(31)
The typical behavior of the effective potential for the asymptotically flat black hole (8) is shown in Figure 11. Note that the form of the curve indicates the existence of bound and unbound orbits, as well as the possibility of circular trajectories. These are defined by the conditions

\[ \pm \frac{\pi}{2} \rightarrow \frac{d\theta}{d\lambda} = 0, \quad \frac{dr}{d\lambda} = 0. \] (32)

The typical behavior of the effective potential for the (anti-)de Sitter solution in (13) we get the angular velocities for different values of angular momentum with the particular choice \( \alpha = 1/12 \) and \( \Lambda = 0 \). There is an ISCO for \( l_{\text{ISCO}} \approx 12.7848 \) at \( r_{\text{ISCO}} \approx 20.6658 \) and we notice that as \( l \) increases from this value, there are both stable and unstable circular orbits.

In Fig. 11 we show how the effective potential changes for different values of angular momentum with the particular values \( \alpha = 1/12 \) and \( \Lambda = 0 \). There is an ISCO for \( l_{\text{ISCO}} \approx 12.7848 \) at \( r_{\text{ISCO}} \approx 20.6658 \) and we notice that as \( l \) increases from this value, there are both stable and unstable circular orbits.

By numerically solving equation (37) using the function \( B(r) \) in equation (13) we obtain the radius of the ISCO as function of the parameter \( \alpha \) for some values of the cosmological constant, as shown in Figure 13.

For the asymptotically flat solution in equation (8) and for the (anti-)de Sitter solution in (13) we get the angular velocities

\[ \Omega_{\text{flat}} = \sqrt{\frac{r - 2}{6\alpha r^4}}, \quad \Omega_{\text{AdS}} = \sqrt{\frac{r - 2 - 4\Lambda \alpha r^4}{6\alpha r^4}}, \] (35)

which correspond to Kepler’s third law for particles moving around these two black holes. The typical behavior of these angular velocities are shown in Figure 12.

Due to the characteristics of the black hole metrics, it is interesting to study the Innermost Stable Circular Orbit (ISCO), which is determined by the conditions

\[ \frac{dV_{\text{eff}}^2}{dr} \bigg|_{r=r_{\text{ISCO}}} = 0 \quad \text{and} \quad \frac{d^2V_{\text{eff}}^2}{dr^2} \bigg|_{r=r_{\text{ISCO}}} = 0. \] (36)

These equations combine to give the radius of the ISCO through the relation

\[ 3B(r) - 2rB'(r) + \frac{rB''(r)B}{B'} = 0. \] (37)

FIG. 11. Effective potential as function of the radial coordinate for different values of angular momentum with the particular choice \( \alpha = 1/12 \) and \( \Lambda = 0 \).

FIG. 12. Angular velocity of a particle moving around the \( f(R) \) black holes as function of the radius of the circular orbit.

FIG. 13. Radius of the ISCO as function of the parameter \( \alpha \) for some particular values of the cosmological constant.
For a circular orbit satisfying conditions (32), the geodesic deviation equations reduce to

\[
\begin{align*}
\frac{d^2 \xi^0}{d\lambda^2} + \frac{B'}{B} \frac{dt}{d\lambda} \frac{d\xi^0}{d\lambda} + \frac{B'}{B} \frac{dr}{d\lambda} \frac{d\xi^0}{d\lambda} + \frac{dt}{d\lambda} \frac{dr}{d\lambda} \frac{d\xi^0}{d\lambda} + \frac{dt}{d\lambda} \frac{d\xi^1}{d\lambda} \left( \frac{B'}{B} \right) &= 0 \\
\frac{d^2 \xi^1}{d\lambda^2} + B'B' \frac{dt}{d\lambda} \frac{d\xi^0}{d\lambda} - \frac{B'}{B} \frac{dr}{d\lambda} \frac{d\xi^1}{d\lambda} - 2rB \frac{d\theta}{d\lambda} \frac{d\xi^2}{d\lambda} - 2rB \sin^2 \theta \frac{d\phi}{d\lambda} \frac{d\xi^3}{d\lambda} + \frac{1}{2} \frac{dt}{d\lambda} \frac{2}{(dr/d\lambda)} \xi^3 \partial_\beta (BB') \\
- \frac{1}{2} \left( \frac{dr}{d\lambda} \right) \frac{2}{(dr/d\lambda)} \xi^3 \partial_\beta \left( \frac{B'}{B} \right) - \left( \frac{d\theta}{d\lambda} \right) \frac{2}{(dr/d\lambda)} \xi^3 \partial_\beta (rB) - \left( \frac{d\phi}{d\lambda} \right) \frac{2}{(dr/d\lambda)} \xi^3 \partial_\beta (rB \sin^2 \theta) &= 0 \\
\frac{d^2 \xi^2}{d\lambda^2} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{d\xi^2}{d\lambda} - \frac{2}{r} \frac{d\theta}{d\lambda} \frac{d\xi^1}{d\lambda} - 2 \sin \theta \cos \theta \frac{d\phi}{d\lambda} \frac{d\xi^3}{d\lambda} + \frac{2}{r} \frac{d\theta}{d\lambda} \frac{d\xi^1}{d\lambda} \left( \frac{1}{r} \right) - \left( \frac{d\phi}{d\lambda} \right) \frac{2}{(dr/d\lambda)} \xi^3 \partial_\beta (\sin \theta \cos \theta) &= 0 \\
\frac{d^2 \xi^3}{d\lambda^2} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{d\xi^3}{d\lambda} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{d\xi^1}{d\lambda} + \frac{2 \cos \theta \sin \theta}{\sin \theta} \frac{d\phi}{d\lambda} \frac{d\xi^2}{d\lambda} + \frac{2 \cos \theta \sin \theta}{\sin \theta} \frac{d\phi}{d\lambda} \frac{d\xi^3}{d\lambda} + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{d\xi^2}{d\lambda} \xi^3 \partial_\beta \left( \frac{1}{r} \right) + \frac{2}{r} \frac{d\phi}{d\lambda} \frac{d\xi^3}{d\lambda} \xi^3 \partial_\beta (\sin \theta) &= 0.
\end{align*}
\]
\[
\omega^2 = \frac{48\alpha^2 \Lambda r^6 - 60\alpha \Lambda r^5 + 96\alpha \Lambda r^4 - 3\alpha r^3 + 6r^2 - 18r + 16}{6\alpha r^2 (4\alpha \Lambda r^4 - r + 2)} > 0
\]  
(45)

which is plotted in Fig. 15 for the particular value \( \Lambda = -0.1 \) and for some values of the parameter \( \alpha \).

C. Geodesic motion in the equatorial plane

To study the geodesic motion of a test particle in the equatorial plane, we obtain the equation of the orbit by introducing a new coordinate \( u = 1/r(\phi) \) and rewriting (30) in terms of the azimuthal angle \( \phi \) and the constant of motion \( l = r^2\dot{\phi} \). This gives the first order differential equation

\[
\left( \frac{du}{d\phi} \right)^2 = \frac{1}{l} \left[ E^2 - B(u)(1 + u^2l^2) \right]
\]  
(46)

which, after differentiating with respect to \( \phi \) and dividing by \( 2du/d\phi \), gives the second order differential equation of motion

\[
\frac{d^2u}{d\phi^2} = -\left[ uB(u) + \frac{1}{2} \left( \frac{1}{l^2} + u^2 \right) \frac{dB}{du} \right].
\]  
(47)

Using (13) for \( B(u) \), we get

\[
\frac{d^2u}{d\phi^2} = -\left( \frac{1}{2} + \frac{1}{3\alpha l^2} \right) u(\phi) + \frac{1}{2}\alpha u^2(\phi)
\]

\[-\frac{2}{3\alpha} u^3(\phi) - \frac{2\Lambda}{3l^2 u^3(\phi)} + \frac{1}{6\alpha l^2}. \]

(48)

This equation looks like the orbital equation obtained in general relativity for Schwarzschild’s or Reissner-Nordstrom solutions in the presence of a cosmological constant. However, it is important to remember that it is not possible to obtain a general relativity limit of this relation because the parameter \( \alpha \) can not be zero.

Numerical solutions of (48) for the asymptotically flat and (anti-)de Sitter metrics when \( \alpha = 0.15 \) give the bound orbits shown in Figures 16 and 17.

V. CONCLUSION

The charged, spherically symmetric solutions depending on a dimensional parameter \( 0 < \alpha < \frac{1}{6} \) reported by Nashed and Capozziello in [1] satisfy the Maxwell-\( f(R) \) field equations in the presence of a cosmological constant. In this paper we have obtained the correct description of the physical and thermodynamical properties associated with these black hole type solutions, including radius of the horizon, Hawking temperature, entropy, quasi-local energy and Gibbs free energy. The behavior of these quantities show that the black holes evaporate similarly to the Reissner-Nordstrom process. In particular, we show that the evaporation process is well-behaved and terminates in a frozen remnant with a final horizon radius \( r_{\text{H}}^\text{free} = 2 \). Accordingly, the entropy and the Gibbs
free energy both have a smooth behavior, showing no phase transitions for the black holes in the allowed range of the parameter $\alpha$. This result is significantly different from that obtained in [1], where the authors reported a non-existing phase transition due to a wrong calculation of the thermodynamic properties.

We also studied the geodesic deviation equation and the existence of circular geodesic trajectories to show that these orbits are stable in a range of radii depending on the parameter $\alpha$. Then, similarly to Schwarzschild and Reissner-Nordstrom black holes, there exist an Innermost Stable Circular Orbit (ISCO) with a radius that we calculated numerically from the geodesic deviation equation.

Finally, we want to conclude this work emphasizing that the solutions presented here were obtained in the context of $f(R)$ gravity and due to the allowed range for the parameter $\alpha$, they cannot be reduced to general relativity. Hence, there are many features that can be studied about these solutions which may be used to distinguish them from general relativity metrics, as we will present in future works.

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