Brill-Noether Loci and the Gonality Stratification of $\mathcal{M}_g$

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1 Introduction

For an irreducible smooth projective complex curve $C$ of genus $g$, the gonality defined as $\text{gon}(C) = \min\{d \in \mathbb{Z}_{\geq 1} : \text{there exists a } g_d \text{ on } C\}$ is perhaps the second most natural invariant: it gives an indication of how far $C$ is from being rational, in a way different from what the genus does. For $g \geq 3$ we consider the stratification of the moduli space $\mathcal{M}_g$ of smooth curves of genus $g$ given by gonality:

$$\mathcal{M}_{g,1}^1 \subseteq \mathcal{M}_{g,2}^1 \subseteq \ldots \subseteq \mathcal{M}_{g,k}^1 \subseteq \ldots \subseteq \mathcal{M}_g,$$

where $\mathcal{M}_{g,k}^1 := \{[C] \in \mathcal{M}_g : C \text{ has a } g_k\}$. It is well-known that the $k$-gonal locus $\mathcal{M}_{g,k}^1$ is an irreducible variety of dimension $2g + 2k - 5$ when $k \leq [(g+3)/2]$; when $k \geq [(g+3)/2]$ one has that $\mathcal{M}_{g,k}^1 = \mathcal{M}_g$ (see for instance [AC]). The number $[(g+3)/2]$ is thus the generic gonality for curves of genus $g$.

For positive integers $g, r$ and $d$, we introduce the Brill-Noether locus

$$\mathcal{M}_{g,d}^r = \{[C] \in \mathcal{M}_g : C \text{ carries a } g_d^r\}.$$ The Brill-Noether Theorem (cf. [ACGH]) asserts that when the Brill-Noether number $\rho(g, r, d) = g - (r+1)(g-d+r)$ is negative, the general curve of genus $g$ has no $g_d^r$'s, hence in this case the locus $\mathcal{M}_{g,d}^r$ is a proper subvariety of $\mathcal{M}_g$. We study the relative position of the loci $\mathcal{M}_{g,d}^r$ when $r \geq 3$ and $\rho(g, r, d) < 0$ with respect to the gonality stratification of $\mathcal{M}_g$. Typically, we would like to know the gonality of a ‘general’ point $[C] \in \mathcal{M}_{g,d}^r$, or equivalently the gonality of a ‘general’ smooth curve $C \subseteq \mathbb{P}^r$ of genus $g$ and degree $d$. Since the geometry of the loci $\mathcal{M}_{g,d}^r$ is very messy (existence of many components, some nonreduced and/or not of expected dimension), we will content ourselves with computing $\text{gon}(C)$ when $[C]$ is a general point of a ‘nice’ component of $\mathcal{M}_{g,d}^r$ (i.e. a component which is generically smooth, of the expected dimension and with general point corresponding to a curve with a very ample $g_d^r$).

Our main result is the following:

**Theorem 1.** Let $g \geq 15$ and $d \geq 14$ be integers with $g$ odd and $d$ even, such that $d^2 > 8g$, $4d < 3g + 12$, $d^2 - 8g + 8$ is not a square and either $d \leq 18$ or $g < 4d - 31$. If

$$(d', g') \in \{(d, g), (d+1, g+1), (d+1, g+2), (d+2, g+3)\},$$

then there exists a regular component of the Hilbert scheme $\text{Hilb}_{d', g';3}$ whose general point $[C]$ is a smooth curve such that $\text{gon}(C) = \min(d' - 4, [(g'+3)/2])$. 

Here by $\text{Hilb}_{d,g,r}$ we denote the Hilbert scheme of curves $C \subseteq \mathbb{P}^r$ with $p_a(C) = g$ and $\text{deg}(C) = d$. A component of $\text{Hilb}_{d,g,r}$ is said to be regular if its general point corresponds to a smooth irreducible curve $C \subseteq \mathbb{P}^r$ such that the normal bundle $N_{C/\mathbb{P}^r}$ satisfies $H^1(C, N_{C/\mathbb{P}^r}) = 0$. By standard deformation theory (cf. [Mod] or [Se]), a regular component of $\text{Hilb}_{d,g,r}$ is generically smooth of the expected dimension $\chi(C, N_{C/\mathbb{P}^r}) = (r + 1)d - (r - 3)(g - 1)$. Note that for $r = 3$ the expected dimension of the Hilbert scheme is just $4d$. We refer to Section 4 for a natural extension of Theorem 1 for curves in higher dimensional projective spaces.

As for the numerical conditions entering Theorem 1, we note that the inequality $d^2 > 8g$ ensures the existence of smooth curves $C \subseteq \mathbb{P}^3$ with $g(C) = g$ and $\text{deg}(C) = d$ (see Section 2), $4d < 3g + 12 \Leftrightarrow \rho(g, 3, d) < 0$ is just the condition that $\mathcal{M}_{g,d}^3$ is a proper subvariety of $\mathcal{M}_g$, while the remaining requirements are mild technical conditions.

A remarkable application of Theorem 1 is a new proof of our result (cf. [Fa]):

**Theorem 2** The Kodaira dimension of the moduli space of curves of genus 23 is $\geq 2$.

We recall that for $g \geq 24$ Harris, Mumford and Eisenbud proved (cf. [HM], [EH]) that $\mathcal{M}_g$ is of general type whereas for $g \leq 16$, $g \neq 14$ we have that $\kappa(\mathcal{M}_g) = -\infty$. The famous Slope Conjecture of Harris and Morrison predicts that $\mathcal{M}_g$ is uniruled for all $g \leq 22$ (see [Mod]). Therefore the moduli space $\mathcal{M}_{23}$ appears as an intriguing transition case between two extremes: uniruledness and being of general type.

To put our Theorem 1 into perspective, let us note that for $r = 2$ we have the following result of M. Coppens (cf. [Co]): let $\nu : C \rightarrow \Gamma$ be the normalization of a general, irreducible plane curve of degree $d$ with $\delta = g - \binom{d-1}{2}$ nodes. Assume that $0 < \delta < (d^2 - 7d + 18)/2$. Then $\text{gon}(C) = d - 2$.

This theorem says that there are no $g_{d-3}^1$’s on $C$. On the other hand a $g_{d-2}^1$ is given by the lines through a node of $\Gamma$. The condition $\delta < (d^2 - 7d + 18)/2$ from the statement is equivalent with $\rho(g, 1, d - 3) < 0$. This is the range in which the problem is non-trivial: if $\rho(g, 1, d - 3) \geq 0$, the Brill-Noether Theorem provides $g_{d-3}^1$’s on $C$.

For $r \geq 3$ we might hope for a similar result. Let $C \subseteq \mathbb{P}^r$ be a suitably general smooth curve of genus $g$ and degree $d$, with $\rho(g, r, d) < 0$. We can always assume that $d \leq g - 1$ (by duality $g_d^r \mapsto |K_C - g_d^r|$ we can always land in this range). One can expect that a $g_k^1$ computing $\text{gon}(C)$ is of the form $g_d^r(-D) = \{E - D : E \in g_d^r, E \geq D\}$ for some effective divisor $D$ on $C$. Since the expected dimension of the variety of $e$-secant $(r - 2)$-plane divisors

$$V_{e-1}^{r-1}(g_d^r) := \{D \in C_e : \text{dim } g_d^r(-D) \geq 1\}$$

is $2r - 2 - e$ (cf. [ACGH]), we may ask whether $C$ has finitely many $(2r - 2)$-secant $(r - 2)$-planes (and no $(2r - 1)$-secant $(r - 2)$-planes at all). This is known to be true for curves with general moduli, that is, when $\rho(g, r, d) \geq 0$ (cf. [Hir]): for instance a smooth curve $C \subseteq \mathbb{P}^3$ with general moduli has only finitely many 4-secant lines and no 5-secant lines. No such principle appears to be known for curves with special moduli.

**Definition:** We call the number $\min(d - 2r + 2, [(g + 3)/2])$ the expected gonality of a smooth nondegenerate curve $C \subseteq \mathbb{P}^r$ of degree $d$ and genus $g$.

One can approach such problems from a different angle: find recipes to compute the
gonality of various classes of curves $C \subseteq \mathbb{P}^r$. Our knowledge in this respect is very scant: we know how to compute the gonality of extremal curves $C \subseteq \mathbb{P}^r$ (that is, curves attaining the Castelnuovo bound, see [ACGH]) and the gonality of complete intersections in $\mathbb{P}^3$ (cf. [Ba]): If $C \subseteq \mathbb{P}^3$ is a smooth complete intersection of type $(a, b)$ then $\text{gon}(C) = ab - l$, where $l$ is the degree of a maximal linear divisor on $C$. Hence an effective divisor $D \subseteq C$ computing $\text{gon}(C)$ is residual to a linear divisor of degree $l$ in a plane section of $C$.

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## 2 Linear systems on $K3$ surfaces in $\mathbb{P}^r$

We will construct smooth curves $C \subseteq \mathbb{P}^r$ having the expected gonality starting with sections of smooth $K3$ surfaces. We recall a few basic facts about linear systems on $K3$ surfaces (cf. [SD]).

Let $S$ be a smooth $K3$ surface. For an effective divisor $D \subseteq S$, we have $h^1(S, D) = h^0(D, \mathcal{O}_D) - 1$. If $C \subseteq S$ is an irreducible curve then $H^1(S, C) = 0$, and by Riemann-Roch we have that $\dim |C| = 1 + C^2/2 = p_a(C)$. In particular $C^2 \geq -2$ for every irreducible curve $C$. Moreover we have equivalences

- $C^2 = -2 \iff \dim |C| = 0 \iff C$ is a smooth rational curve and
- $C^2 = 0 \iff \dim |C| = 1 \iff p_a(C) = 1$.

For a $K3$ surface one also has a ‘strong Bertini’ Theorem (cf. [SD]):

**Proposition 2.1** Let $\mathcal{L}$ be a line bundle on a $K3$ surface $S$ such that $|\mathcal{L}| \neq \emptyset$. Then $|\mathcal{L}|$ has no base points outside its fixed components. Moreover, if $\text{bs} |\mathcal{L}| = \emptyset$ then either

- $\mathcal{L}^2 > 0$, $h^1(S, \mathcal{L}) = 0$ and the general member of $|\mathcal{L}|$ is a smooth, irreducible curve of genus $\mathcal{L}^2/2 + 1$, or
- $\mathcal{L}^2 = 0$ and $\mathcal{L} = \mathcal{O}_S(kE)$, where $k \in \mathbb{Z}_{\geq 1}$, $E \subseteq S$ is an irreducible curve with $p_a(E) = 1$. We have that $h^0(S, \mathcal{L}) = k + 1$, $h^1(S, \mathcal{L}) = k - 1$ and all divisors in $|\mathcal{L}|$ are of the form $E_1 + \cdots + E_k$ with $E_i \sim E$.

We are interested in space curves sitting on $K3$ surfaces and the starting point is Mori’s Theorem (cf. [Mo]): if $d > 0$, $g \geq 0$, there is a smooth curve $C \subseteq \mathbb{P}^3$ of degree $d$ and genus $g$, lying on a smooth quartic surface $S$, if and only if (1) $g = d^2/8 + 1$, or (2) $g < d^2/8$ and $(d, g) \neq (5, 3)$. Moreover, we can choose $S$ such that $\text{Pic}(S) = \mathbb{Z}H = \mathbb{Z}(4/d)C$ in case (1) and such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, with $H^2 = 4, C^2 = 2g - 2$ and $H \cdot C = d$, in case (2).

In each case $H$ denotes a plane section of $S$. Note that from the Hodge Index Theorem one has the necessary condition $(C \cdot H)^2 - H^2C^2 = d^2 - 8(g - 1) \geq 0$.

Mori’s result has been extended by Rathmann to curves in higher dimensional projective spaces (cf. [Ra], see also [Kn]): For integers $d > 0, g > 0$ and $r \geq 3$ such that $d^2 \geq 4g(r - 1) + (r - 1)^2$, there exists a smooth $K3$ surface $S \subseteq \mathbb{P}^r$ of degree $2r - 2$ and a smooth curve $C \subseteq S$ of genus $g$ and degree $d$ such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$, where $H$ is a hyperplane section of $S$.

We will repeatedly use the following simple observation:
Proposition 2.2 Let \( S \subseteq \mathbb{P}^r \) be a smooth K3 surface of degree \( 2r - 2 \) with a smooth curve \( C \subseteq S \) such that \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \) and assume that \( S \) has no \((-2)\) curves. A divisor class \( D \) on \( S \) is effective if and only if \( D^2 \geq 0 \) and \( D \cdot H > 2 \).

Remark: If \( S \subseteq \mathbb{P}^r \) is a smooth K3 surface of degree \( 2r - 2 \) with Picard number 2 as above, \( S \) has no \((-2)\) curves when the equation
\[
(r - 1)m^2 + mnd + (g - 1)n^2 = -1
\]
has no solutions \( m, n \in \mathbb{Z} \). This is the case for instance when \( d \) is even and \( g \) and \( r \) are odd. Furthermore, a necessary condition for \( S \) to have genus 1 curves is that \( d^2 - 4(g - 1)(r - 1) \) is a square.

3 Brill-Noether special linear series on curves on K3 surfaces

The first important result in the study of special linear series on curves lying on K3 surfaces was Lazarsfeld’s proof of the Brill-Noether-Petri Theorem (cf. [Laz]). He noticed that there is no Brill-Noether type obstruction to embed a curve in a K3 surface: if \( C_0 \subseteq S \) is a smooth curve of genus \( g \geq 2 \) on a K3 surface such that \( \text{Pic}(S) = \mathbb{Z}C_0 \), then the general curve \( C \in |C_0| \) satisfies the Brill-Noether-Petri Theorem, that is, for any line bundle \( A \) on \( C \), the Petri map \( \mu_0(C, A) : H^0(C, A) \otimes H^0(C, K_C \otimes A^\vee) \rightarrow H^0(C, K_C) \) is injective. We mention that Petri’s Theorem implies (trivially) the Brill-Noether Theorem.

The general philosophy when studying linear series on a K3-section \( C \subseteq S \) of genus \( g \geq 2 \), is that the type of a Brill-Noether special \( g^r_d \) often does not depend on \( C \) but only on its linear equivalence class in \( S \), i.e. a \( g^r_d \) on \( C \) with \( \rho(g, r, d) < 0 \) is expected to propagate to all smooth curves \( C' \in |C| \). This expectation, in such generality, is perhaps a bit too optimistic, but it was proved to be true for the Clifford index of a curve (see [GL]): for \( C \subseteq S \) a smooth K3-section of genus \( g \geq 2 \), one has that \( \text{Cliff}(C') = \text{Cliff}(C) \) for every smooth curve \( C' \in |C| \). Furthermore, if \( \text{Cliff}(C) \) is odd, then there exists a line bundle \( L \) on \( S \) such that for all smooth \( C' \in |C| \) the restriction \( L_{|C'} \) computes \( \text{Cliff}(C') \). Recall that the Clifford index of a curve \( C \) of genus \( g \) is defined as
\[
\text{Cliff}(C) := \min \{ \text{Cliff}(D) : D \in \text{Div}(C), h^0(D) \geq 2, h^1(D) \geq 2 \},
\]
where for an effective divisor \( D \) on \( C \), we have \( \text{Cliff}(D) = \deg(D) - 2(h^0(D) - 1) \). Note that in the definition of \( \text{Cliff}(C) \) the condition \( h^1(D) \geq 2 \) can be replaced with \( \deg(D) \leq g - 1 \). Another invariant of a curve is the Clifford dimension of \( C \) defined as
\[
\text{Cliff-dim}(C) := \min \{ r \geq 1 : \exists g^r_d \text{ on } C \text{ with } d \leq g - 1, \text{ such that } d - 2r = \text{Cliff}(C) \}.
\]
Curves with Clifford dimension \( \geq 2 \) are rare: smooth plane curves are precisely the curves of Clifford dimension 2, while curves of Clifford dimension 3 occur only in genus 10 as complete intersections of two cubic surfaces in \( \mathbb{P}^3 \).
Harris and Mumford during their work in [HM] conjectured that the gonality of a K3-section should stay constant in a linear system: if $C \subseteq S$ carries an exceptional $g_d^1$ then every smooth $C' \in |C|$ carries an equally exceptional $g_d^1$. This conjecture was later disproved by Donagi and Morrison (cf. [DMo]). They came up with the following counterexample: let $\pi : S \to \mathbb{P}^2$ be a K3 surface, double cover of $\mathbb{P}^2$ branched along a smooth sextic and let $\mathcal{L} = \pi^*O_{\mathbb{P}^2}(3)$. The genus of a smooth $C \in |\mathcal{L}|$ is 10. The general $C \in |\mathcal{L}|$ carries a very ample $g_5^1$, hence $\text{gon}(C) = 5$. On the other hand, any curve in the codimension 1 linear system $|\pi^*H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(3))|$ is bielliptic, therefore has gonality 4. Under reasonable assumptions this turns out to be the only counterexample to the Harris-Mumford conjecture. Ciliberto and Pareschi proved that if $4. Let $(d, g)$ and every $\text{Cliff}(C', D_{IC'}) \leq \text{Cliff}(C, Z)$, for any smooth $C' \in |C|$.

- There is $Z_0 \in |Z|$, consisting of distinct points such that $Z_0 \subseteq D \cap C$.

Throughout this paper, for a smooth curve $C$ we denote, as usual, by $W_d^r(C)$ the scheme whose points are line bundles $A \in \text{Pic}^d(C)$ with $h^0(C, A) \geq r + 1$, and by $G_d^r(C)$ the scheme parametrizing $g_d^r$'s on $C$.

4. The gonality of curves in $\mathbb{P}^r$

For a wide range of $d, g$ and $r$ we construct curves $C \subseteq \mathbb{P}^r$ of degree $d$ and genus $g$ having the expected gonality. We start with a case when we can realize our curves as sections of K3 surfaces.

Theorem 3 Let $r \geq 3, d \geq r^2 + r$ and $g \geq 0$ be integers such that $\rho(g, r, d) < 0$ and with $d^2 > 4(r-1)(g + r - 2)$ when $r \geq 4$ while $d^2 > 8g$ when $r = 3$. Let us assume moreover that 0 and $-1$ are not represented by the quadratic form

$$Q(m, n) = (r - 1)m^2 + mnd + (g - 1)n^2, \quad m, n \in \mathbb{Z}.$$  

Then there exists a smooth curve $C \subseteq \mathbb{P}^r$ of degree $d$ and genus $g$ such that $\text{gon}(C) = \min(d-2r+2, [(g+3)/2])$. If $\text{gon}(C) = d - 2r + 2 < [(g+3)/2]$ then $\dim W_{d-2r+2}^1(C) = 0$ and every $g_{d-2r+2}$ is given by the hyperplanes through a $(2r-2)$-secant $(r-2)$-plane.

Proof: By Rathmann’s Theorem there exists a smooth K3 surface $S \subseteq \mathbb{P}^r$ with $\deg(S) = 2r - 2$ and $C \subseteq S$ a smooth curve of degree $d$ and genus $g$ such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$,
where $H$ is a hyperplane section. The conditions $d, g$ and $r$ are subject to, ensure that $S$ does not contain $(-2)$ curves or genus $1$ curves.

We prove first that Cliff-$\dim(C) = 1$. It suffices to show that $C \subseteq S$ is an ample divisor, because then by using Prop.3.3 from [CilP] we obtain that either Cliff-$\dim(C) = 1$ or $C$ is a smooth plane sextic, $g = 10$ and $(S, C)$ are as in Donagi-Morrison’s example (then Cliff-$\dim(C) = 2$). The latter case obviously does not happen.

We prove that $C \cdot D > 0$ for any effective divisor $D \subseteq S$. Let $D \sim mH + nC$, with $m, n \in \mathbb{Z}$, such a divisor. Then $D^2 = (2r - 2)m^2 + 2mnt + n^2(2g - 2) \geq 0$ and $D \cdot H = (2r - 2)m + dn > 2$. The case $m \leq 0, n \leq 0$ is impossible, while the case $m \geq 0, n \geq 0$ is trivial. Let us assume $m > 0, n < 0$. Then $D \cdot C = md + n(2g - 2) > -n(d^2/(2r - 2) + 2g - 2) + d/(r - 1) > 0$, because $d^2/(2r - 2) > 2g$. In the remaining case $m < 0, n > 0$ we have that $nD \cdot C \geq -mD \cdot H > 0$, so $C$ is ample by Nakai-Moishezon.

Our assumptions imply that $d \leq g - 1$, so $O_{C}(1)$ is among the line bundles from which Cliff$(C)$ is computed. We get thus the following estimate on the gonality of $C$:

$$\text{gon}(C) = \text{Cliff}(C) + 2 \leq \text{Cliff}(C, H_{C}) + 2 = d - 2r + 2,$$

which yields $\text{gon}(C) \leq \min(d - 2r + 2, [(g + 3)/2]).$

For the rest of the proof let us assume that $\text{gon}(C) < [(g + 3)/2]$. We will then show that $\text{gon}(C) = d - 2r + 2$. Let $|Z|$ be a complete, base point free pencil computing $\text{gon}(C)$. By applying Prop.3.1, there exists an effective divisor $D \subseteq S$ satisfying

$$h^0(S, D) \geq 2, h^0(S, C - D) \geq 2, \deg(D_{|C}) \leq g - 1, \text{ gon}(C) = \text{Cliff}(D_{|C}) + 2 \text{ and } Z \subseteq D \cap C.$$

We consider the exact cohomology sequence:

$$0 \rightarrow H^0(S, D - C) \rightarrow H^0(S, D) \rightarrow H^0(C, D_{|C}) \rightarrow H^1(S, D - C).$$

Since $C - D$ is effective and $\sim 0$, one sees that $D - C$ cannot be effective, so $H^0(S, D - C) = 0$. The surface $S$ does not contain $(-2)$ curves, so $|C - D|$ has no fixed components; the equation $(C - D)^2 = 0$ has no solutions, therefore $(C - D)^2 > 0$ and the general element of $|C - D|$ is smooth and irreducible. Then it follows that $H^1(S, D - C) = H^1(S, C - D)^{\vee} = 0$. Thus $H^0(S, D) = H^0(C, D_{|C})$ and

$$\text{gon}(C) = 2 + \text{Cliff}(D_{|C}) = 2 + D \cdot C - 2 \dim|D| = D \cdot C - D^2.$$

We consider the following family of effective divisors

$$\mathcal{A} := \{D \in \text{Div}(S) : h^0(S, D) \geq 2, h^0(S, C - D) \geq 2, C \cdot D \leq g - 1\}.$$

Since we already know that $d - 2r + 2 \geq \text{gon}(C) \geq \alpha$, where $\alpha = \min\{D \cdot C - D^2 : D \in \mathcal{A}\}$, we are done if we prove that $\alpha \geq d - 2r + 2$. Take $D \in \mathcal{A}$ such that $D \sim mh + nc$, $m, n \in \mathbb{Z}$. The conditions $D^2 > 0, D \cdot C \leq g - 1$ and $2 < D \cdot H < d - 2$ (use Prop.2.2 for the last inequality) can be rewritten as

$$(r - 1)m^2 + mnd + n^2(g - 1) > 0 \text{ (i)}, 2 < (2r - 2)m + nd < d - 2 \text{ (ii)}, md + (2n - 1)(g - 1) \leq 0 \text{ (iii)}.$$

We have to prove that for any $D \in \mathcal{A}$ the following inequality holds

$$f(m, n) = D \cdot C - D^2 = -(2r - 2)m^2 + m(d - 2nd) + (n - n^2)(2g - 2) \geq f(1, 0) = d - 2r + 2.$$
We solve this standard calculus problem. Denote by
\[ a := \frac{d + \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2} \text{ and } b := \frac{d - \sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}. \]

We dispose first of the case \( n < 0 \). Assuming \( n < 0 \), from (i) we have that either \( m < -bn \) or \( m > -an \). If \( m < -bn \) from (ii) we obtain that \( 2 < n(d - (2r-2)b) < 0 \), because \( n < 0 \) and \( d - (2r-2)b = \sqrt{d^2 - 4(r-1)(g-1)} > 0 \), so we have reached a contradiction.

We assume now that \( n < 0 \) and \( m > -an \). From (iii) we get that \( m \leq (g-1)(1-2n)/d \). If \( -an > (g-1)(1-2n)/d \) we are done because there are no \( m, n \in \mathbb{Z} \) satisfying (i), (ii) and (iii), while in the other case for any \( D \in A \) with \( D \sim mH + nC \), one has the inequality
\[ f(m, n) > f(-an, n) = \frac{(d^2 - 4(r-1)(g-1)) + d\sqrt{d^2 - 4(r-1)(g-1)}}{2r-2}(-n). \]

When \( r \geq 4 \) since we assume that \( \sqrt{d^2 - 4(r-1)(g-1)} \geq 2r - 2 \), it immediately follows that \( f(m, n) \geq d > d - 2r + 2 \). In the case \( r = 3 \) when we only have the weaker assumption \( d^2 > 8g \), we still get that \( f(-an, n) > d - 4 \) unless \( n = -1 \) and \( d^2 - 8g < 8 \). In this last situation we obtain \( m \geq (d + 4)/4 \) so \( f(m, -1) \geq f((d + 4)/4, -1) > d - 4 \).

The case \( n > 0 \) can be treated rather similarly. From (i) we get that either \( m < -an \) or \( m > -bn \). The first case can be dismissed immediately. When \( m > -bn \) we use that for any \( D \in A \) with \( D \sim mH + nC \),
\[ f(m, n) \geq \min\{f(-(g-1)(2n-1)/d, n), \max\{f(-bn, n), f((2 - nd)/(2r - 2), n)\}\}. \]

Elementary manipulations give that
\[ f(-(g-1)(2n-1)/d, n) = (g-1)/2 [(2n-1)^2(d^2 - 4(r-1)(g-1))/d^2 + 1] \geq d - 2r + 2 \]
(use only that \( d \leq g - 1 \) and \( d^2 > 4(r-1)g \), so we cover both cases \( r = 3 \) and \( r \geq 4 \) at once). Note that in the case \( n > 0 \) we have equality if and only if \( n = 1, m = -1 \) and \( d = g - 1 \).

Moreover, \( f(-bn, n) = n(2g - 2 - bd) \geq 2g - 2 - bd \) and \( 2g - 2 - bd > d - 2r + 2 \iff 2r - 2 < \sqrt{d^2 - 4(r-1)(g-1)} < d - 2r + 2 \). When this does not happen we proceed as follows: if \( \sqrt{d^2 - 4(r-1)(g-1)} \geq d - 2r + 2 \) then if \( n = 1 \) we have that \( m > -b \geq -1 \), that is \( m \geq 0 \), but this contradicts (ii). When \( n \geq 2 \), we have \( f((2 - nd)/(2r - 2), n) = [(d^2 - 4(r-1)(g-1))(n^2 - n) + (2d - 4)]/(2r - 2) > d - 2r + 2 \). Finally, the remaining possibility \( 2r - 2 \geq \sqrt{d^2 - 4(r-1)(g-1)} \) does not occur when \( r \geq 4 \) while in the case \( r = 3 \) we either have \( f(-bn, n) > d - 4 \) or else \( n = 1 \) and then \( m > -(d + 4)/4 \) hence \( f(m, 1) > f(-(d + 4)/4, 1) = d - 4 \).

All this leaves us with the case \( n = 0 \), when \( f(m, 0) = -(2r - 2)m^2 + md \). Clearly \( f(m, 0) \geq f(1, 0) \) for all \( m \) complying with (i), (ii) and (iii).

Thus we proved that \( \text{gon}(C) = d - 2r + 2 \). We have equality \( D \cdot C - D^2 = d - 2r + 2 \) where \( D \in A \), if and only if \( D = H \) or in the case \( d = g - 1 \) also when \( D = C - H \). The latter possibility can be ruled out since \( d = g - 1 \) is not compatible with the assumptions \( d \geq r^2 + r \) and \( d - 2r + 2 < [(g + 3)/2] \). Therefore we can always assume that the divisor
on $S$ cutting a $g_{d-2r+2}^1$ on $C$ is the hyperplane plane section of $S$. Since $Z \subseteq H \cap C$, if we denote by $\Delta$ the residual divisor of $Z$ in $H \cap C$, we have that $h^0(C, H|_{C - \Delta}) = 2$, so $\Delta$ spans a $\mathbb{P}^{r-2}$ hence $|Z|$ is given by the hyperplanes through the $(2r-2)$-secant $(r-2)$-plane $(\Delta)$. This shows that every pencil computing $\text{gon}(C)$ is given by the hyperplanes through a $(2r-2)$-secant $(r-2)$-plane.

There are a few ways to see that $C$ has only finitely many $(2r-2)$-secant $(r-2)$-planes. The shortest is to invoke Theorem 3.1 from [CilP]: since $\text{gon}(C') = d - 2r + 2$ is constant as $C'$ varies in $|C|$, for the general smooth curve $C' \in |C|$ one has $W^1_{d-2r+2}(C') = 0$.

Remarks: 1. Keeping the assumptions and the notations of Theorem 3 we note that when $d - 2r + 2 < [(g + 3)/2]$ the linear system $|C|$ is $(d - 2r - 1)$-very ample, i.e. for any 0-dimensional subscheme $Z \subseteq S$ of length $\leq d - 2r$ the map $H^0(S, C) \to H^0(S, C \otimes O_Z)$ is surjective. Indeed, by applying Theorem 2.1 from [BS] if $|C|$ is not $(d - 2r - 1)$-very ample, there exists an effective divisor $D$ on $S$ such that $C - 2D$ is $\mathbb{Q}$-effective and

$$C \cdot D - (d - 2r) \leq D^2 \leq C \cdot D/2 < d - 2r,$$

hence $C \cdot D - D^2 \leq d - 2r$. On the other hand clearly $D \in A$, thus $C \cdot D - D^2 \geq d - 2r + 2$, a contradiction.

2. One can find quartic surfaces $S \subseteq \mathbb{P}^3$ containing a smooth curve $C$ of degree $d$ and genus $g$ in the case $g = d^2/8 + 1$ (which is outside the range Theorem 3 deals with). Then $d = 4m, g = 2m^2 + 1$ with $m \geq 1$ and $C$ is a complete intersection of type $(4, m)$. For such a curve, $\text{gon}(C) = d - l$, where $l$ is the degree of a maximal linear divisor on $C$ (cf. [Ba]). If $S$ is sufficiently general so that it contains no lines, by Bezout, $C$ cannot have 5-secant lines so $\text{gon}(C) = d - 4$ in this case too.

When $r = 3$ we want to find out when the curves constructed in Theorem 3 correspond to smooth points of $\text{Hilb}_{d, 9, 3}$. We have the following:

**Proposition 4.1** Let $C \subseteq S \subseteq \mathbb{P}^3$ be a smooth curve sitting on a quartic surface such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C$ with $H$ being a plane section and assume furthermore that $S$ contains no $(-2)$ curves. Then $H^1(C, N_{C/\mathbb{P}^3}) = 0$ if and only if $d \leq 18$ or $g < 4d - 31$.

**Proof:** We use the exact sequence

$$0 \longrightarrow N_{C/S} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow N_{S/\mathbb{P}^3} \otimes O_C \longrightarrow 0,$$

where $N_{S/\mathbb{P}^3} \otimes O_C = O_C(4)$ and $N_{C/S} = K_C$. We claim that there is an isomorphism $H^1(C, N_{C/\mathbb{P}^3}) = H^1(C, O_C(4))$. Suppose this is not the case. Then the injective map $H^1(C, K_C) \to H^1(C, N_{C/\mathbb{P}^3})$ provides a section $\sigma \in H^0(N_{C/\mathbb{P}^3} \otimes K_C)$ which yields a splitting of the dual of the exact sequence (2), hence (2) is split as well. Using a result from [GH, p.252] we obtain that $C$ is a complete intersection with $S$. This is clearly a contradiction. Therefore one has $H^1(C, N_{C/\mathbb{P}^3}) = H^1(C, O_C(4))$.

We have isomorphisms $H^1(C, 4H|_C) = H^2(S, 4H - C) = H^0(S, C - 4H)^\vee$. According to Prop.2.2 the divisor $C - 4H$ is effective if and only if $(C - 4H)^2 \geq 0$ and $(C - 4H) \cdot H > 2$, from which the conclusion follows. \qed
We need to determine the gonality of nodal curves not of compact type and which consist of two components meeting at a number of points. We have the following result:

**Proposition 4.2** Let $C = C_1 \cup_\Delta C_2$ be a quasi-transversal union of two smooth curves $C_1$ and $C_2$ meeting at a finite set $\Delta$. Denote by $g_1 = g(C_1), g_2 = g(C_2), \delta = \text{card}(\Delta)$. Let us assume that $C_1$ has only finitely many pencils $g_1^d$, where $\delta \leq d$ and that the points of $\Delta$ do not occur in the same fibre of one of these pencils. Then $\text{gon}(C) \geq d + 1$. Moreover if $\text{gon}(C) = d + 1$ then either (1) $C_2$ is rational and there is a degree $d$ map $f_1 : C_1 \to \mathbb{P}^1$ and a degree $1$ map $f_2 : C_2 \to \mathbb{P}^1$ such that $f_1|_\Delta = f_2|_\Delta$, or (2) there is a $g_{d+1}^1$ on $C_1$ containing $\Delta$ in a fibre.

**Proof:** Let us assume that $C$ is $k$-gonal, that is, a limit of smooth $k$-gonal curves. If $g = g_1 + g_2 + \delta - 1$, we consider the space $\overline{\mathcal{H}}_{g,k}$ of Harris-Mumford admissible coverings of degree $k$ and we denote by $\pi : \overline{\mathcal{H}}_{g,k} \to \mathcal{M}_g$ the proper map sending a covering to the stable model of its domain (see [HM]). Since $[C] \in \overline{\mathcal{M}}_{g,k}^1 = \text{Im}(\pi)$, it follows that there exists a semistable curve $C'$ whose stable model is $C$ and a degree $k$ admissible covering $f : C' \to Y$, where $Y$ is a semistable curve of arithmetic genus $0$. We thus have that $f^{-1}(Y_{\text{sing}}) = C'_{\text{sing}}$ and if $p \in C_1' \cap C_2'$ with $C_1'$ and $C_2'$ components of $C'$, then $f(C_1')$ and $f(C_2')$ are distinct components of $Y$ and the ramification indices at the point $p$ of the restrictions $f|_{C_1'}$ and $f|_{C_2'}$ are the same.

We have that $C' = C_1 \cup C_2 \cup R_1 \cup \ldots \cup R_\delta$, where for $1 \leq i \leq \delta$ the curve $R_i$ is a (possibly empty) destabilizing chain of $\mathbb{P}^1$'s inserted at the nodes of $C$. Let us denote $\{p_i\} = C_1 \cap R_i$ and $\{q_i\} = C_2 \cap R_i$; if $R_i = \emptyset$ then we take $p_i = q_i \in \Delta \subseteq C$.

We first show that $k \geq d + 1$. Suppose $k \leq d$. Since $C_1$ has no $g_{d-1}^1$'s it follows that $k = d$ and that $f^{-1}f(C_1) = C_1$. If there were distinct points $p_i$ and $p_j$ such that $f(p_i) \neq f(p_j)$, then $f(R_i) \neq f(R_j)$ and the image curve $Y$ would no longer have genus $0$. Therefore $f(p_i) = f(p_j)$ for all $i, j \in \{1, \ldots, \delta\}$, that is $\Delta$ appears in the fibre of a $g_1^1$ on $C_1$, a contradiction.

Assume now that $k = d + 1$. Then either $\deg(f|_{C_1}) = d$ or $\deg(f|_{C_1}) = d + 1$. If $\deg(f|_{C_1}) = d$ and $\deg(f|_{C_1}) = d + 1$, then again $f^{-1}f(C_1) = C_1$ and by the same reasoning $f$ maps all the $p_i$'s to the same point and this yields case (2) from the statement of the Proposition. If $\deg(f|_{C_1}) = d$ then $f^{-1}f(C_1) = C_1 \cup D$, where $D$ is a smooth rational curve mapped isomorphically to its image via $f$. If $D = C_2$ then the condition that the dual graph of $Y$ is a tree implies that $f(p_i) = f(q_i)$ for all $i$ and this yields case (1) from the statement. Finally, if $D \neq C_2$ then $f(C_1) \neq f(C_2)$. We know that there are $1 \leq i < j \leq \delta$ such that $f(p_i) \neq f(p_j)$. The image $f(C_2)$ belongs to a chain $R$ of $\mathbb{P}^1$'s such that either $R \cap f(C_1) = \{f(p_i)\}$ or $R \cap f(C_1) = \{f(p_j)\}$. In the former case $f(p) = f(p_i)$ for all $p \in \Delta - \{p_j\}$ while in the latter case $f(p) = f(p_j)$ for all $p \in \Delta - \{p_i\}$. In each case by adding a base point we obtain a $g_{d+1}^1$ on $C_1$ containing $\Delta$ in a fibre. \qed

Theorem 3 provides curves $C \subseteq \mathbb{P}^3$ of expected gonality when $d$ is even and $g$ is odd (equation (1) has no solutions in this case). Naturally, we would like to have such curves when $d$ and $g$ have other parities as well. We will achieve this by attaching to a ‘good’ curve of expected gonality either a 2 or 3-secant line or a 4-secant conic.
Theorem 1 Let \( g \geq 15 \) and \( d \geq 14 \) be integers with \( g \) odd and \( d \) even, such that \( d^2 > 8g, 4d < 3g + 12, d^2 - 8g + 8 \) is not a square and either \( d \leq 18 \) or \( g < 4d - 31 \). If

\[
(d', g') \in \{(d, g), (d + 1, g + 1), (d + 1, g + 2), (d + 2, g + 3)\},
\]

then there exists a regular component of \( \text{Hilb}_{d', g', 3} \) with general point \([C']\) a smooth curve such that \( \text{gon}(C') = \min(d' - 4, [(g' + 3)/2]) \).

Proof: For \( d \) and \( g \) as in the statement we know by Theorem 3 and by Prop.4.1 that there exists a smooth nondegenerate curve \( C \subseteq \mathbb{P}^3 \) of degree \( d \) and genus \( g \), with \( \text{gon}(C) = \min(d - 4, [(g + 3)/2]) \) and \( H^1(C, N_{C/\mathbb{P}^3}) = 0 \). We can also assume that \( C \) sits on a smooth quartic surface \( S \) and \( \text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \). Moreover, in the case \( d - 4 < [(g + 3)/2] \) the curve \( C \) has only finitely many \( g_{d-4}' \)'s, all given by planes through a 4-secant line.

i) Let us settle first the case \((d', g') = (d + 1, g + 1)\). Take \( p, q \in C \) general points, \( L = \overline{pq} \subseteq \mathbb{P}^3 \) and \( X := C \cup L \). By applying Lemma 1.2 from [BE], we know that

\[
H^1(X, N_X) = 0 \quad \text{and the curve } X \text{ is smoothable in } \mathbb{P}^3,
\]

that is, there exists a flat family of curves \( \{X_t\} \) in \( \mathbb{P}^3 \) over a smooth and irreducible base, with the general fibre \( X_t \) smooth while the special fibre \( X_0 \) is \( X \). If \( d - 4 < [(g + 3)/2] \), then since \( C \) has only finitely many \( g_{d-4}' \)'s, by applying Prop.4.2 we get that \( \text{gon}(X) = d - 3 \). In the case \( d - 4 \geq [(g + 3)/2] \) we just notice that \( \text{gon}(X) \geq \text{gon}(C) = [(g' + 3)/2] \).

ii) Next we tackle the case \((d', g') = (d + 1, g + 2)\). Assume first that \( d - 4 < [(g + 3)/2] \Leftrightarrow d' - 4 < [(g' + 3)/2] \). We apply Lemma 1.2 from [BE] to a curve \( X := C \cup L \), where \( L \) is a suitable trisecant line to \( C \). In order to conclude that \( X \) is smoothable in \( \mathbb{P}^3 \) and that \( H^1(X, N_X) = 0 \), we have to make sure that the trisecant line \( L = \overline{pqq'} \) with \( p, q, q' \in C \) can be chosen in such a way that

\[
L, T_p(C), T_q(C) \text{ and } T_{qq'}(C) \text{ do not all lie in the same plane}. \tag{3}
\]

We claim that when \( C \in |C| \) is general in its linear system, at least one of its trisecants satisfies (3). Suppose not. Then for every smooth curve \( C \in |C| \) and for every trisecant line \( L \) to \( C \) condition (3) fails.

We consider a 0-dimensional subscheme \( Z \subseteq S \) where \( Z = p + q + q' + u + u' \), with \( p, q, q' \in S \) being collinear points while \( u \) and \( u' \) are general infinitely near points to \( q \) and \( q' \) respectively. The linear system \( |C| \) is at least 5-very ample (cf. Remark 1), hence a general curve \( C \in |C - Z| \) is smooth and possesses a trisecant line for which (3) holds, a contradiction.

Since the scheme of trisecants to a space curve is of pure dimension 1, it follows that for a general curve \( C \in |C| \), through a general point \( p \in C \) there passes a trisecant line \( L \) for which (3) holds. We have that \( X := C \cup L \) is smoothable in \( \mathbb{P}^3 \) and \( H^1(X, N_X) = 0 \). We conclude that \( \text{gon}(X) = d - 3 \) by proving that there is no \( g_{d-4}' \) on \( C \) containing \( L \cap C \) in a fibre.

If \( C \in |C| \) is general, any line in \( \mathbb{P}^3 \) (hence also a 4-secant line to \( C \)) can meet only finitely many trisecants. Indeed, assuming that \( m \subseteq \mathbb{P}^3 \) is a line meeting infinitely many trisecants, we consider the correspondence

\[
T = \{(p, t) \in C \times m : pt \text{ is a trisecant to } C\}
\]
and the projections $\pi_1 : T \to C$ and $\pi_2 : T \to m$. If $\pi_2$ is surjective, then $\Nm_{\pi_1}(\pi_2)$ yields a $g^1_d$ on $C$, a contradiction. If $\pi_2$ is not surjective then there exists a point $t \in \PP^3$ such that $t \overline{\pi}$ is a trisecant to $C$ for each $p \in C$. This possibility cannot occur for a general $C \in |C|$: Otherwise we take general points $t \in \PP^3$ and $p, p' \in S$ and if we denote

$$B := \{ C \in |C| : p, p' \in C \text{ and } \overline{tx} \text{ is a trisecant to } C \text{ for each } x \in C \},$$

we have that $\dim B \geq g - 5$. On the other hand since $t \overline{pp}$ and $t \overline{pp'}$ are trisecants for all curves $C \in B$, there must be a 0-dimensional subscheme $Z \subseteq (t \overline{pp} \cup t \overline{pp'}) \cap S$ of length 6 such that $B \subseteq |C - Z|$, hence $\dim B \leq \dim |C - Z| = g - 6$ (use again that $|C|$ is 5-very ample), a contradiction. In this way the case $d - 4 < [(g + 3)/2]$ is settled.

When $d - 4 \geq [(g + 3)/2]$ we apply Theorem 3 to obtain a smooth curve $C_1 \subseteq \PP^3$ of degree $d$ and genus $g + 2$ such that $\text{gon}(C_1) = (g + 5)/2$ and $H^1(C_1, N_{C_1}) = 0$. We take $X_1 := C_1 \cup L_1$ with $L_1$ being a general 1-secant line to $C_1$. Then $X_1$ is smoothable and $\text{gon}(X_1) = \text{gon}(C_1) = (g + 5)/2$.

iii) Finally, we turn to the case $(d', g') = (d + 2, g + 3)$. Take $H \subseteq \PP^3$ a general plane meeting $C$ in $d$ distinct points in general linear position and pick 4 of them: $p_1, p_2, p_3, p_4 \in C \cap H$. Choose $Q \subseteq H$ a general conic such that $Q \cap C = \{ p_1, p_2, p_3, p_4 \}$. Then $\text{gon}(C_1) = (g + 5)/2$ and $H^1(X, N_X) = 0$.

Assume first that $d' - 4 \leq [(g' + 3)/2]$. We claim that $\text{gon}(X) \geq \text{gon}(C) + 2$. According to Prop.4.2 the opposite could happen only in 2 cases: a) There exists a $g^1_{d-3}$ on $C$, say $\overline{Z}$, such that $|Z|(-p_1 - p_2 - p_3 - p_4) \neq \emptyset$. b) There exists a degree $d - 4$ map $f : C \to \PP^1$ and a degree 1 map $f' : Q \to \PP^1$ such that $f(p_i) = f'(p_i)$, for $i = 1, \ldots, 4$.

Assume that a) does happen. We denote by $U = \{ D \in C_4 : |O_{C}(1)|(-D) \neq \emptyset \}$ the irreducible 3-fold of divisors of degree 4 spanning a plane and also consider the correspondence

$$\Sigma = \{(L, D) \in W^1_{d-3}(C) \times U : |L|(-D) \neq \emptyset \},$$

with the projections $\pi_1 : \Sigma \to W^1_{d-3}(C)$ and $\pi_2 : \Sigma \to U$. We know that $\pi_2$ is dominant, hence $\dim \Sigma \geq 3$ and therefore $\dim W^1_{d-3}(C) \geq 2$.

If $\rho(g, 1, d - 3) < 0$ by Prop.3.1 we get that every base-point-free $g^1_{d-3}$ on $C$ is cut out by a divisor $D$ on $S$ such that $D \in A$ (see the proof of Theorem 3 for this notation) and $C \cdot D - D^2 = \Cliff(C, D_C) + 2 \leq d - 3$, hence $C \cdot D - D^2 \leq d - 4$ for parity reasons. As pointed out at the end of the proof of Theorem 3 this forces $D \sim H$, that is, all base-point-free $g^1_{d-3}$’s on $C$ are given by planes through a trisecant line. Thus $C$ has $\infty^2$ trisecants, a contradiction.

If $\rho(g, 1, d - 3) \geq 0$, then $g = 2d - 9$ and we can assume that there is $L \in \pi_1(\Sigma)$ such that $|O_C(1) - L| \neq \emptyset$. The map $\pi_1$ is either generically finite hence $\dim W^1_{d-4}(C) \geq \dim W^1_{d-3}(C) - 2 \geq 1$ (cf. [FHL]), a contradiction, or otherwise $\pi_1$ has fibre dimension 1. This is possible only when there is a component $A$ of $W^1_{d-4}(C)$ with $\dim(A) \geq 2$ and such that the general $L \in A$ satisfies $|O_C(1) - L| \neq \emptyset$ and every $L \in A$ has non-ordinary ramification so that the monodromy of each $g^1_{d-3}$ is not the full symmetric group. Applying again [FHL] there is $L \in W^1_{d-4}(C)$ such that $\{ L \} + W^0_1(C) \subseteq A$, in particular $L$ has non-ordinary ramification too. It is easy to see that this contradicts the $(d - 7)$-very ampleness of $|C|$ asserted by Remark 1.
We now rule out case b). Suppose that b) does happen and denote by $L \subseteq \mathbb{P}^3$ the 4-
secant line corresponding to $f$. Let \{p\} = $L \cap H$, and pick $l \subseteq H$ a general line. As $Q$ was
a general conic through $p_1, \ldots, p_4$ we may assume that $p \notin Q$. The map $f' : Q \to l$ is (up
to a projective isomorphism of $l$) the projection from a point $q \in Q$, while $f(p_i) = \overline{pq} \cap l$,
for $i = 1, \ldots, 4$. By Steiner’s Theorem from classical projective geometry, the condition
\((f(p_1)f(p_2)f(p_3)f(p_4)) = (f'(p_1)f'(p_2)f'(p_3)f'(p_4))\) is equivalent with $p_1, p_2, p_3, p_4, p$ and
$q$ being on a conic, a contradiction since $p \notin Q$.

Finally, when $d' - 4 > [(g' + 3)/2]$, we have to show that $\text{gon}(X) \geq \text{gon}(C) + 1$. We
note that $\dim G^1_{(g+3)/2}(C) = 1$ (for any curve one has the inequality $\dim G^1_{\text{gon}} \leq 1$).
By taking $H \in (\mathbb{P}^3)^V$ general enough, we obtain that $p_1, \ldots, p_4$ do not occur in the same
fibre of a $\mathfrak{g}^1_{(g+3)/2}$.

\textbf{Remark:} Theorem 1 can be viewed as a non-containment relation $\mathcal{M}^3_{g',d'} \not\subseteq \mathcal{M}^1_{g',d'-5}$
between different Brill-Noether loci when $d'$ and $g'$ are as in Theorem 1 and moreover
$d' - 4 \leq [(g' + 3)/2]$. We can turn this problem on its head and ask the following question:
given $g$ and $k$ such that $k < (g + 2)/2$, when is it true that the general $k$-gonal curve of
 genus $g$ has no other linear series $\mathfrak{g}^r_d$ with $\rho(g, r, d) < 0$, that is, the pencil computing the
gonality is the only Brill-Noether exceptional linear series?

In [Fa2] we prove using limit linear series the following result: fix $g$ and $k$ positive
integers such that $-3 \leq \rho(g, 1, k) < 0$. If $\rho(g, 1, k) = -3$ assume furthermore that $k \geq 6$.
Then the general $k$-gonal curve $C$ of genus $g$ has no $\mathfrak{g}^r_d$ with $\rho(g, r, d) < 0$ except $\mathfrak{g}^1_k$ and
$|K_C - \mathfrak{g}^1_k|$. In other words the $k$-gonal locus $\mathcal{M}^1_{g,k}$ is not contained in any other proper
Brill-Noether locus $\mathcal{M}^r_{g,d}$ with $r \geq 2, d \leq g - 1$ and $\rho(g, r, d) < 0$.

In seems that other methods are needed to extend this result for more negative values
of $\rho(g, 1, k)$.

5 The Kodaira dimension of $\mathcal{M}_{23}$

In this section we explain how Theorem 1 gives a new proof of our result $\kappa(\mathcal{M}_{23}) \geq 2$ (cf.
[Fa]). We refer to [Fa] for a detailed analysis of the geometry of $\mathcal{M}_{23}$ ; in that paper we
also conjecture that $\kappa(\mathcal{M}_{23}) = 2$ and we present evidence for such a possibility.

Let us denote by $\overline{\mathcal{M}}_g$ the moduli space of Deligne-Mumford stable curves of genus $g$.
We study the multicanonical linear systems on $\overline{\mathcal{M}}_{23}$ by exhibiting three explicit multi-
canonical divisors on $\overline{\mathcal{M}}_{23}$ which are (modulo a positive combination of boundary classes
coming from $\overline{\mathcal{M}}_{23} - \mathcal{M}_{23}$) of Brill-Noether type, that is, loci of curves having a $\mathfrak{g}^r_d$
when $\rho(23, r, d) = -1$.

On $\mathcal{M}_{23}$ there are three Brill-Noether divisors corresponding to the solutions of the
equation $\rho(23, r, d) = -1$: the 12-gonal divisor $\mathcal{M}^1_{23,12}$, the divisor $\mathcal{M}^2_{23,17}$ of curves having
a $\mathfrak{g}^2_{17}$ and finally the divisor $\mathcal{M}^3_{23,20}$ of curves possessing a $\mathfrak{g}^3_{20}$. If we denote by $\overline{\mathcal{M}}_{g,d}$
the closure of $\mathcal{M}^r_{g,d}$ inside $\overline{\mathcal{M}}_g$, the classes $[\overline{\mathcal{M}}_{g,d}] \in \text{Pic}_Q(\overline{\mathcal{M}}_g)$ when $\rho(g, r, d) = -1$ have been
computed (see [EH],[Fa]). It is quite remarkable that for fixed $g$ all classes $[\overline{\mathcal{M}}_{g,d}]$ are
proportional. One also knows the canonical divisor class (cf. [HM]):

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]},$$
and by comparing for \( g = 23 \) this formula with the expression of the classes \( [\overline{M}_{23,d}] \), we find that there are constants \( m, m_1, m_2, m_3 \in \mathbb{Z}_{>0} \) such that

\[
mK_{\overline{M}_{23}} = m_1[\overline{M}_{23,12}] + E = m_2[\overline{M}_{23,17}] + E = m_3[\overline{M}_{23,20}] + E,
\]

where \( E \) is the same positive combination of the boundary classes \( \delta_1, \ldots, \delta_{11} \).

As explained in [Fa], since \( \overline{M}_{23,12}, \overline{M}_{23,17}, \) and \( \overline{M}_{23,20} \) are mutually distinct irreducible divisors, we can show that the multicanonical image of \( \overline{M}_{23} \) cannot be a curve once we construct a smooth curve of genus 23 lying in the support of exactly two of the divisors \( M_{23,12}, M_{23,17}, \) and \( M_{23,20} \). In this way we rule out the possibility of all three intersections of two Brill-Noether divisors being equal to base-locus \( (mK_{\overline{M}_{23}}) \cap \overline{M}_{23} \).

In [Fa] we found such genus 23 curves using an intricate construction involving limit linear series (cf. Proposition 5.4 in [Fa]). Here we can construct such curves in a much simpler way by applying Theorem 1 when \((d, g) = (18, 23)\): there exists a smooth curve \( C \subseteq \mathbb{P}^3 \) of genus 23 and degree 18 such that \( \operatorname{gon}(C) = 13 \); moreover \( C \) sits on a smooth quartic surface \( S \subseteq \mathbb{P}^3 \) such that \( \operatorname{Pic}(S) = \mathbb{Z}C \oplus \mathbb{Z}H \).

Since \( C \) has a very ample \( g^3_{18} \), by adding 2 base points it will also have plenty of \( g^3_{20} \)'s and also \( g^2_{17} \)'s of the form \( g^3_{18}(-p) = \{ D \in g^3_{18} : D \geq p \} \), for any \( p \in C \). Therefore \( [C] \in (M_{23,20} \cap M_{23,17}) - M_{23,12} \), and Theorem 2 now follows.

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