A simplified He’s frequency–amplitude formulation for nonlinear oscillators

Zi-Yin Ren

Abstract
Derived from an ancient Chinese algorithm, He’s frequency–amplitude formulation is an effective approach to finding an approximate solution of a nonlinear oscillator. In this article, based on He’s formulation, a simplified formulation is proposed. Some nonlinear oscillators are adopted as examples to demonstrate the solving process using this simplified formulation. Through the demonstration, it can be seen that the solving process is simplified.

Keywords
He’s frequency–amplitude formulation, ancient Chinese mathematics, nonlinear oscillators

Introduction
Nonlinear behaviours can be observed in plenty of real-world phenomena. And nonlinear oscillation has been a hot research topic for many years and involved in applications of different areas, such as automotive, sensing, fluid–solid interaction, aerospace, micro- and nano-scale and bioengineering. The significance of nonlinear oscillation theory has many reasons, such as developing new devices in micro- and nano-scales, analysing real-world cases with the consideration of non-linearity to better insight into the oscillatory devices, having uncertainties in the model parameters and improving the design of nonlinear microelectromechanical systems. Therefore, researchers from different fields have explored the nature of nonlinear oscillators for decades.

Currently, more and more analytical and approximate methods have been developed to find approximate solutions of nonlinear oscillators, for instance, He’s variational iteration method, the homotopy perturbation method, the variational method, the parameter-expanding method and other methods. Researchers worldwide have sought a concise and practical approach to solve various nonlinear oscillators for a long time.

In 2006, a Chinese mathematician, Prof. Ji-Huan He, suggested a simple approach to finding an approximate solution of a nonlinear oscillator in his review article. This method has been further developed in Refs. and is now named after Prof. He, called He’s frequency–amplitude formulation.

This formulation is famous for its accuracy and convenience: only a few calculations can result in an accurate solution for the whole solution domain. He’s frequency–amplitude formulation is derived from an ancient Chinese algorithm. In this article, we will review the ancient Chinese algorithm and He’s frequency–amplitude formulation. Furthermore, a simplified formulation will be proposed and demonstrated.

He’s Frequency–Amplitude Formulation
The oldest method of approximating the real roots of a nonlinear equation is suggested by chapter 7 of Nine Chapters (九章数学) in China. To illustrate the idea of this algorithm, we can start by considering the following equation
Let $x_1$ and $x_2$ be the approximate solutions of equation (1), which lead to the remainders $f(x_1)$ and $f(x_2)$ respectively, and the following result will be acquired through the ancient Chinese algorithm

$$x = x_2 f(x_1) - x_1 f(x_2) / f(x_1) - f(x_2)$$

Some application of this algorithm can be found in Refs. 31–35. The modern development of this algorithm led to the widely used He’s frequency–amplitude formulation.28–30

He’s frequency–amplitude formulation can be derived by considering a generalized nonlinear oscillator with initial conditions in the form

$$\mu'' + f(\mu) = 0, \quad \mu(0) = A, \quad \mu'(0) = 0$$

Then we can use two trial functions

$$\mu_1(t) = A \cos \omega_1 t$$

$$\mu_2(t) = A \cos \omega_2 t$$

which are respectively the solutions of the following equations

$$\mu'' + \omega_1^2 \mu = 0, \quad \mu(0) = A, \quad \mu'(0) = 0$$

$$\mu'' + \omega_2^2 \mu = 0, \quad \mu(0) = A, \quad \mu'(0) = 0$$

where $\omega_1$ and $\omega_2$ are the frequencies of the linear oscillators. And we will have the following remainders

$$R_1(t) = f(\mu_1) - \omega_1^2 A \cos \omega_1 t$$

$$R_2(t) = f(\mu_2) - \omega_2^2 A \cos \omega_2 t$$

Like the Chinese ancient algorithm, the approximate solution $\mu = A \cos \omega t$ can be denoted as the following equation

$$\mu = \frac{\mu_2 R_1(t) - \mu_1 R_2(t)}{R_1(t) - R_2(t)}$$

Expand the approximate solution $\mu = A \cos \omega t$ in a power series

$$\mu = A \cos \omega t = A \left(1 - \frac{1}{2!} \omega^2 t^2 + \frac{1}{4!} \omega^4 t^4 - \ldots\right)$$

In the same way, we will have

$$\mu_1 = A \cos \omega_1 t = A \left(1 - \frac{1}{2!} \omega_1^2 t^2 + \frac{1}{4!} \omega_1^4 t^4 - \ldots\right)$$

$$\mu_2 = A \cos \omega_2 t = A \left(1 - \frac{1}{2!} \omega_2^2 t^2 + \frac{1}{4!} \omega_2^4 t^4 - \ldots\right)$$

We can take the first three terms in the power series

$$\mu = A \cos \omega t = A \left(1 - \frac{1}{2!} \omega^2 t^2 + \frac{1}{4!} \omega^4 t^4\right)$$

and similarly

$$\mu_1 = A \cos \omega_1 t = A \left(1 - \frac{1}{2!} \omega_1^2 t^2 + \frac{1}{4!} \omega_1^4 t^4\right)$$
\[
\mu_2 = A \cos \omega_2 t = A \left(1 - \frac{1}{2!} \omega_2^2 t^2 + \frac{1}{4!} \omega_2^4 t^4 \right)
\] (16)

Then substituting equations (14)–(16) in equation (10)

\[
A - \frac{1}{2!} A^2 \omega^2 + \frac{1}{4!} A^4 \omega^4
\]

\[
= A - \frac{1}{2!} A^2 \left[ \frac{\omega_2^2 R_1(t_1) - \omega_2^2 R_2(t_2)}{R_1(t_1) - R_2(t_2)} \right] + \frac{1}{4!} A^4 \left[ \frac{\omega_4^2 R_1(t_1) - \omega_4^2 R_2(t_2)}{R_1(t_1) - R_2(t_2)} \right]
\] (17)

In the light of the terms of \( t^2 \), we will have

\[
\omega^2 = \frac{\omega_2^2 R_1(t_1) - \omega_2^2 R_2(t_2)}{R_1(t_1) - R_2(t_2)}
\] (18)

which has been used in many previous works.\(^{36-48}\)

In the light of the \( t^4 \) terms, we will have

\[
\omega^4 = \frac{\omega_4^2 R_1(t_1) - \omega_4^2 R_2(t_2)}{R_1(t_1) - R_2(t_2)}
\] (19)

which is also feasible in solving nonlinear oscillators.\(^{49}\) Besides, an alternative modification of He’s formulation has been proposed in Ref.\(^{50}\).

**A Simplified Frequency–Amplitude Formulation**

Inspired by the ancient Chinese algorithm (2), a simplified frequency–amplitude formulation can be envisaged. If we replace all \( \omega^2 \) in equation (18) with \( \omega \), the following formulation can be obtained

\[
\omega = \frac{\omega_2^2 R_1(t_1) - \omega_2^2 R_2(t_2)}{R_1(t_1) - R_2(t_2)}
\] (20)

which is very similar to equation (2).

In the following part of this article, we will use some simple examples to demonstrate the calculation process of using this formulation. Through the demonstration, hopefully, its efficiency and convenience can be shown.

**Examples**

We can consider the Duffing equation as an example

\[
\mu'' + \mu + \varepsilon \mu^3 = 0, \quad \mu(0) = A, \quad \mu'(0) = 0
\] (21)

Two trial functions can be chosen

\[
\mu_1(t) = A \cos t
\] (22)

\[
\mu_2(t) = A \cos \omega t
\] (23)

By substituting equations (22) and (23) in (21), the following two residuals can be obtained

\[
R_1(t_1) = \varepsilon A^3 \cos^3 t_1
\] (24)

\[
R_2(t_2) = \varepsilon A^3 \cos^3 \omega t_2 + (1 - \omega^2) A \cos \omega t_2
\] (25)

According to equation (20), the following approximate frequency can be obtained
\[
\omega = \frac{\omega R_1(t_1) - R_2(t_2)}{R_1(t_1) - R_2(t_2)} = \frac{\omega A^3 \cos^3 t_1 - \varepsilon A^3 \cos^3 \omega t_2 - (1 - \omega^2)A \cos \omega t_2}{\varepsilon A^3 \cos^3 t_1 - \varepsilon A^3 \cos^3 \omega t_2 - (1 - \omega^2)A \cos \omega t_2}
\]

Choosing the location where \( t_1 = t_2 = 0 \), we will have

\[
\omega = \frac{\omega A^3 - \varepsilon A^3 - (1 - \omega^2)A}{\varepsilon A^3 - \varepsilon A^3 - (1 - \omega^2)A} = \frac{(\omega - 1)\varepsilon A^3 + (\omega^2 - 1)A}{(\omega^2 - 1)A} = 1 + \frac{\varepsilon A^2}{1 + \omega}
\]

By rearranging equation (27), the following frequency–amplitude relationship can be obtained

\[
\omega^2 = 1 + \varepsilon A^2
\]

which is the same as the relationship in Ref. 28.

We can also consider the following nonlinear oscillator

\[
\frac{d^2 \mu}{dw^2} + \frac{1}{1 + \varepsilon A^2} = 0, \quad \mu(0) = A, \quad \mu'(0) = 0
\]

At last, a golden mean location where \( t_1 = T_1/12 \) and \( t_2 = T_2/12 \) can also be considered. According to equation (20), the following equation can be obtained

\[
\omega = \frac{\omega A^3 \cos^3 (\pi/6) - \varepsilon A^3 \cos^3 (\pi/6) - (1 - \omega^2)A \cos(\pi/6)}{\varepsilon A^3 \cos^3 (\pi/6) - \varepsilon A^3 \cos^3 (\pi/6) - (1 - \omega^2)A \cos(\pi/6)} = 1 + \frac{3\varepsilon A^2}{4(\omega + 1)}
\]

The result can be simplified as

\[
\omega^2 = \frac{3}{4}\varepsilon A^2 + 1
\]

which is the same as the result in Ref. 36.

Two trial solutions can be the same as equations (22) and (23). By substituting equations (22) and (23) in (33), we can get the following residuals

\[
R_1(t) = \frac{A \cos t}{1 + \varepsilon A^2 \cos^2 t} - A \cos t
\]

\[
R_2(t) = \frac{A \cos \omega t}{1 + \varepsilon A^2 \cos^2 \omega t} - A \omega^2 \cos \omega t
\]
When } t_1 = t_2 = 0 \), we will have
\[ \omega^2 = \frac{1}{1 + \varepsilon A^2} \] (37)

When } t_1 = T_1/12 \) and } t_2 = T_2/12 \), we will have
\[ \omega^2 = \frac{1}{1 + (3/4)\varepsilon A^2} \] (38)

When } t_1 = 0.0955T_1 \) and } t_2 = 0.0955T_2 \), we will have
\[ \omega^2 = \frac{1}{1 + 0.6811\varepsilon A^2} \] (39)

These results are precisely the same as the results obtained in Ref. 37.

**Conclusion**

In this article, a simplified frequency–amplitude formulation has been proposed based on He’s frequency–amplitude formulation. This simplified formulation is more convenient and concise than He’s formulation and is feasible in practical applications. By applying the simplified formulation, the calculation process can be simplified without the loss of accuracy.

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**ORCID iD**

Zi-Yin Ren © https://orcid.org/0000-0001-7268-8914

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