Market Ecology of Active and Passive Investors

Andrea Capocci and Yi-Cheng Zhang

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Abstract

We study the role of active and passive investors in an investment market with uncertainties. Active investors concentrate on a single or a few stocks with a given probability of determining the quality of them. Passive investors spread their investment uniformly, resembling buying the market index. In this toy market stocks are introduced as good and bad. If a stock receives sufficient investment it will survive, otherwise die. Active players exert a selective pressure since they can determine to an extent the investment quality. We show that the active players provide the driving force whereas the passive ones act as free riders. While their gains do not differ too much, we show that the active players enjoy an edge. Their presence also provides better gains to the passive players and stocks themselves.

1 Introduction

In the standard finance literature, it is generally stated that competitive financial markets are efficient and an arbitrage opportunity would instantly disappear once smart investors act upon it. This goes under the name of Efficient Market Hypothesis (EMH) [1], [2]. The proponents never specified how much a probabilistic edge would be reduced and how many smart investors there should be to bring the market to efficiency. The general conclusion they draw from such doctrine is that active investors just waste their time, since nobody can earn above-market gains. Practitioners, on the other hand, do not heed to
such advice and it seems to be impossible to reconcile the academics and practice.

Recently an alternative theory is put forward [3] [4] which maintains that probabilistic edge can never really disappear, but can only be reduced. There is a quantitative relationship between the reduction and the amount of investment the active players commit. This is called Marginally Efficient Market (MEM) theory. Under this general framework in this paper we study the interplay of various players in a toy stock market. Active investors hold relative focus; passive ones just buy everything in the market; as well as the stocks which can survive or die. In this market ecology we aim to understand the interaction among various players, given the market uncertainties. The salient features coming out from our investigation are: the active players provide the driving force such that the market index can be sustained to a higher level than random chance would warrant, the passive players enjoy a free ride. In general we find that under quite general conditions active players enjoy an edge over the market index on which the passive players ride free, this provides incentives to the active players to stay active. On the investment side the stocks are not guaranteed of survival. The more the active players, the more of the good stocks survive. So in this ecology all players benefit from there being active players.

2 The model

In a market, investors can choose between two main strategies: either they focus on some stock, study it in depth and then decide whether to invest on it or not, or they can choose to be passive, and invest on all the available stocks to safely enjoy the average gain (if any) thanks to the central limit theorem. Stocks, on the other hand, can provide positive or negative return, and we assume that their survival depends on their capability to attract new investments.

We assume that there are $I$ investors and $S$ stocks in which they can invest. Time is described by the integer variable $t$. Each stocks $i$ is associated with a spin variable $\sigma_i = \pm 1$, that describes its quality: if $\sigma_i = +1$, the stock $i$ is positive and it is good to invest on it, while if $\sigma_i = -1$ the stock $i$ is negative and one should not invest on it. An investor can be active or passive. Each agent can invest at most a unit at each time step. At each time step, he chooses the stocks to
be included in his portfolio according to his active or passive strategy, then invests an equal amount in each of them. We define the gain $g^\alpha$ of a given investor $\alpha$ by

$$g^\alpha = \frac{\sum_{i=1}^{S} f^{\alpha}_i \sigma_i}{P},$$

(1)

where $f_i = 1$ if stock $i$ has been included in the portfolio of investor $\alpha$ and 0 otherwise, and $P = \sum_i^{S} f^{\alpha}_i$ is the number of stocks included in the portfolio. In other words, a portfolio gives a positive gain if it includes more good stocks than bad ones.

Let us now define the two strategies. Passive investors want to take advantage of the average return of the market. Therefore, they invest a capital $1/S$ on each stock ($f^{\text{pass}}_i = 1$ for all $i = 1, 2, \ldots$) and $P = S$. On the other hand, each active investor examines a single randomly drawn stock and judges it. Once he has examined a given stock, he invests all his unit on it if he thinks it is a good stock, otherwise he does not invest at all. As we did in [4], we want to represent active investors with bounded rationality, who may have a wrong perception of stocks. We then assume that an active investor who examines a given stock has a wrong perception of it with the probability $D$.

When all the investors have made their choice, stocks are selected by the amount of the investment they manage to receive. Stocks that fail to attract sufficient investment go bankrupt. Then, a stocks that receives less investment than a fixed threshold $T$ is replaced by a new stock. The new stock is good with the probability $p_0$ and bad with probability $1 - p_0$. This process is repeated many times, starting from a random initial condition. We denote by $p(t)$ the fraction of good stocks at time $t$. The market performance is measured by the average quality of the market, or index, $2p(t) - 1$. The system, after a many time steps, falls in a stationary state where the fraction of good stocks fluctuate around $p^*$. We assume that the fraction of active investors is $\rho$. Let us suppose, for the time being, that $\rho$ is fixed. We can compute the average gains of active and passive investors. An active investor gets a gain of +1 if he examines a good stock and if his perception of the stock is right. The probability to examine a good stock drawn at random is $p(t)$, and the probability to have a right perception of it is $1 - D$, then the gain is equal to 1 with the probability $p(t)(1 - D)$. The same investor has a negative return, that is, he loses 1, if he examines a bad stock, but has a wrong perception of it and decides nevertheless to invest.
Reasoning as above, one can see that this happens with probability 
\((1 - p(t))D\). Otherwise, he receives no return, and this case does not
give contribution to the average. Thus, the average gain of the active
investors at time \(t\) is \(g^{\text{act}} = p(t) - D\). On the other hand, the passive
investors enjoy the average behavior of the whole market and their
average gain at time \(t\) is \(g^{\text{pass}} = 2p(t) - 1\). If \(p(t) < 1 - D\), which
implies \(g^{\text{act}} > g^{\text{pass}}\), the active investors enjoy a greater gain than the
passive investors.

3 The stationary state

Let \(q^{(+)}(t) (q^{(-)}(t))\) be the probability to go bankrupt for a good
(bad) stock at time \(t\). In the following, we will drop the explicit time
dependence (if not necessary) to avoid a too heavy notation. At the
stationary state, where the number of good stock replaced by bad ones
equals the number of bad stocks replaced by good ones, we find

\[
p^*q^{(+)}(1 - p_0) = (1 - p^*)q^{(-)}p_0; \quad (2)
\]

From this, we obtain

\[
p^* = p_0q^{(-)}/[(1 - p_0)q^{(+)} + p_0q^{(-)}]. \quad (3)
\]

Let us consider a given bad stock and check its total investment re-
ceived to determine its fate. \(q^{(-)}\) is the probability that a given bad
stock receives less than \(T\) units of investment. All stocks, bad or good,
enjoy the same investment coming from the passive investors, spread
out over all the market. This amount per stock is a constant equal to
\(T_0 = I(1 - \rho)/S\).

To fill the gap between \(T_0\) and the threshold \(T\), the bad stock
should receive sufficient investment from the active investors who mis-
take the bad stock for good. The probability that \(k\) active investors
invest on a bad stock obeys a binomial distribution. Each active in-
vestor examines a stock with probability \(1/S\). If it is a bad stock,
the investor invests on it if he mistakes it, which happens with the
probability \(D\). Then, the probability of receiving a unit of investment
from each active investor is \(D/S\). The probability to receive \(k\) units of
investment from all active investors is \(b(k, \rho I, D/S)\), where \(b(n, M, z)\)
is the usual binomial probability of observing \(n\) events on \(M\) trials, if
the probability of the single event is equal to \(z\), i.e.

\[
b(n, M, z) \equiv \binom{M}{n} z^n(1 - z)^{M-n}. \quad (4)
\]
Then, the probability for a bad stock to go bankrupt is

\[ q^(-) = \sum_{k=0}^{T-T_0} b(k, \rho I, D/S). \]  \hspace{1cm} (5)

Likewise, the computation of the bankrupt probability for a good stock \( q^+ \) gives

\[ q^+ = \sum_{k=0}^{T-T_0} b(k, \rho I, (1 - D)/S). \]  \hspace{1cm} (6)

By means of the binomial distribution computed above, we can evaluate the average investment received by each stock. From active agents, each good stock receives on average \((1 - D)\rho I/S\) and each bad stock \(D\rho I/S\). By adding \(T_0\) to both quantities, we obtain that each good stock receives on average \((1 - D\rho)I/S\) units of investment and each bad stock receives \([1 - D(1 - \rho)]I/S\). By replacing eqs. (5) and (6) in equation (3), we obtain

\[ p^* = p^*(\rho) = \frac{p_0 \sum_{k=0}^{T-T_0} b(k, \rho I, D/S)}{\sum_{k=0}^{T-T_0} [(1 - p_0)b(k, \rho I, D/S) + p_0b(k, \rho I, (1 - D)/S)]}. \]  \hspace{1cm} (7)

The above formula can be easily verified by simulations. For different values of \(T\) we obtain qualitatively different behaviors. If \(T > I/S\), i.e. if the market is under severe selective pressure, the market index increases monotonically as a function of the number of active investors. For low values of \(\rho\), i.e. when active investors are rare, all stocks receive about the same amount of investment, since the passive agents invest the same quantity \(1/S\) on each stock regardless its return. Thus, for \(\rho = 0\), each stock receives \(I/S < T\) investment and no one survives:

\[ q^+ = 1, \]  \hspace{1cm} (8)

\[ q^- = 1. \]  \hspace{1cm} (9)

and

\[ p^* = p_0. \]  \hspace{1cm} (10)

As \(\rho\) (the fraction of active investors) increases, investment is allocated in a more selective way, since the active investors can distinguish to a certain extent good from bad stocks. Thus, if there are more active investors, good stocks have a higher probability to receive the minimal \(T\) investment and then survive. The presence of the active investors then gives an edge to the good stocks, and \(p\) increased with \(\rho\) as
shown in figure 1. When $D \to 0$ and $T = I/S$, the active investor exercise the strongest selective pressure on the market (provided that $T > I/S$), since they invest with great accuracy and the threshold is exactly the average investment received by all stocks. In this situation, the number of good stocks that receive less than $T$ investment is the minimal. We now ask whether the selective pressure on the market can so strong that passive investor enjoy a greater gain than the active ones. This happens if $p^* > 1 - R$ in a market where active agents invest with a very high precision, that is, $D \to 0$.

Since $p^*$ is an increasing function of $\rho$, we check if the condition is verified for $\rho = 1$. Replacing $\rho = 1$, $D \simeq 0$ and $T = I/S$ in equation (7), we obtain

$$p^* \simeq \frac{p_0 \sum_{k=0}^{I/S} \delta(k)}{(1 - p_0) \sum_{k=0}^{I/S} \delta(k) + p_0 \sum_{k=0}^{I/S} b(k, I, 1/S)}.$$  \hspace{1cm} (11)

The $\delta$ functions in equation (11) come from the limit

$$\lim_{z \to 0} b(n, M, z) = \delta(n).$$  \hspace{1cm} (12)

Since

$$\sum_{k=0}^{I/S} \delta(k) = 1,$$  \hspace{1cm} (13)

and

$$\sum_{k=0}^{I/S} b(k, I, 1/S) \simeq \frac{1}{2},$$  \hspace{1cm} (14)

we have, from equation (11),

$$p^*(\rho = 1) \simeq \frac{2p_0}{2 - p_0}.$$  \hspace{1cm} (15)

Thus, the condition $p^* > 1 - D$ can be satisfied only if $p_0 > 2/3 + O(\epsilon)$.

If $T < I/S$, we observe a radically different behavior. If there is no active investors ($\rho = 0$), each stock receives $I/S > T$ investments, thus all stocks survive and we have

$$q^{(+)} = q^{(-)} = 0.$$  \hspace{1cm} (16)

This remains true until the fraction of active investors is such that

$$(1 - \rho)I/S \geq T.$$  \hspace{1cm} (17)
Then, there is a critical fraction of investors $\rho_c$ such that if $\rho < \rho_c$ all stock receive more than the threshold investment. Equation (17) implies $\rho_c = 1 - ST/I$. Beyond this value, the investment coming from the passive agents is no more sufficient to keep all the investors above the threshold $T$. As the number of active investors increases over the value $\rho_c$, investment move from bad to good stocks, and the bankrupt probability for a bad stocks grows much more than for the good stocks, as one can see in figure 1. For higher values of $\rho$, however, the fraction of good stocks $p$ starts to decrease. Indeed, when there are many active players bad stocks can take advantage of their more frequent errors and the survival probability for a bad stock slightly increases, as it is shown in figure 1.

4 Effort

Let us now define the effort $E$. We use the same notation of ref. [4] and assume that the error probability $D$ is a decreasing function of the effort; we choose for simplicity

$$D = 1/(E + 2). \quad (18)$$

Thus, $E$ is positive definite and gives a measure of the skill of the active investors, which is assumed to be constant for all the active investors. If the active investors provide no effort ($E = 0$) they would invest at random. Error would then be the largest ($D = \frac{1}{2}$). On the other hand, a perfect perception of the market cannot be reached by a finite effort, since $D = 0$ corresponds to $E = \infty$. We want to investigate the role of the effort in a market, where the fraction of active investors is fixed and the system is in the stationary state. The active investors’ ability to distinguish good stocks from bad ones is a function of the effort they provide.

Let us consider the gain of an investor as a function of his effort with $\rho$ fixed. By replacing $p(\rho)$ in the expressions of the gains $g^{act}$ and $g^{pass}$, one can show that both the gains of the active and passive investors grow monotonically with $E$. This behavior is shown in figure 2.

We conclude that the effort by the active investors exercises a selective pressure on the market performance. We notice that if the active investors provide the minimal effort $E = 2$, which corresponds to an error probability $D = 1/2$, the number of bad and good stocks
weeded out of the market are equal, since both kinds of investors are indifferent to the return of the stock on which they are investing.

We then generalize, by numerical simulation, our study of the market assuming that the active agents invest on more than only one stock. As we did in ref. [4], we assume that the missing information, measured by the error probability $D$, grows with the number of stocks examined by the active investors, $N$ and that a residual ignorance $D_0$ remains also for $N \to 0$, i.e.

$$D(N) = D_0 + A \frac{N}{E},$$  \hspace{1cm} (19)$$

where $A$ is a constant determined by imposing $D(S) = 1/2$. This means that an investor who examines all stocks actually behaves like a random investor.

We see that the market performance of an active investor $2p - 1$ has a maximum for a finite value $0 < N^{opt} < S$, as it is shown in figure 3.

Indeed, the of information needed to include more stock in a portfolio then a tradeoff has to take place between the need of diversification and the need of including good stocks in a portfolio. This confirms the conclusions of [4] and [3], where the diversification of a portfolio is found to be bounded by a finite cost of information about the stocks, which acts as a source of inefficiency in a market.

5 Complex behavior of self organized investors

Let us now suppose that the passive investors do not invest in all stocks, but they choose at random the stocks to be included in their portfolio. This is the easiest strategy to follow: such a noise trader chooses its portfolio at random and invests on each stock with probability $1/2$, regardless the stock is good or bad. Once he has chosen the stocks, he invests an equal amount on each of them. As above, the passive investors have a unit to invest, and they invest their entire capital. An investor who follows this strategy enjoys the average return of the market, after many time steps.

On the other hand, an active investor follows the same strategy as in the previous model: he examines one single stock drawn at random and, if he thinks that it is a good stock, he invests on it.
We now allow investors to adapt and change strategy when their last investment gives a non positive gain, so that the density of active investors $\rho(t)$ changes with time: if an investor enjoys a null or negative gain, he switches to the other strategy with the probability $\rho_0$. As above, we study the properties of the system in the stationary state, where $\rho(t)$ and $p(t)$ start oscillating around their stationary values $\rho^*$ and $p^*$ respectively.

By numerical simulation shown in figure ??, we observe that the fraction of passive investors $1 - \rho^*$ at the stationary state increases with the effort of the active investors. Likewise, the market performance increases with the effort of the active investors, as do the average gains of active and passive investors.

This can be qualitatively interpreted by a simple argument. The active investors have a clear advantage if their effort is increased, since they invest in the market with a higher precision. But the passive investors as well take advantage of the effort: the better the active investors work, the easier is the life for the passive investors, so that their population increases. If there are many active investors, the market becomes more selective and the fraction of good stock increases. This allows more passive strategies to reach a better results and grow in number. As a non trivial results, it appears that the more the active investors are successful, i.e. invest with a greater accuracy, the more the passive investors are favored.

6 Approximated analysis of the self-organized model

The properties at the stationary state can be approximately computed. We study the stationary state of the system as a function of the effort provided by the active investors, whose expression, from equation (18) can be written as $E = D^{-1} - 2$. We note by $\tau_a$ the probability that an active strategy obtains a negative or null gain, and with $\tau_f$ the same probability for an average strategy. At the stationary state, the numbers of agents who switch from active to passive strategy and from passive to active strategy must be the same.

The probability that an active investor switches to a passive strategy is equal to $\tau^{act}(p) = D(1 - \rho_0)(1 - p)$, i.e. the product of the probability $(1 - p)$ of examining a bad stock, the probability $D$ of having a wrong perception of that stock and the probability $1 - \rho_0$ of
replacing an active strategy with a passive one.

On the other hand, a passive investor does not include a fixed number of stocks in his portfolio. Let us assume that he invests on \( n \) stocks. In such a case, the probability to include \( m \) bad stocks is a binomial one, \( b(m, n, p) \). Thus, for a given \( n \), the probability to obtain a negative gain is \( \tau_{\text{pass}}(p) = \sum_{m<n/2} b(m, n, p) \). The probability to invest over \( n \) stocks drawn at random obeys a binomial distribution, \( P(n) = b(n, S, 1/2) \). Then, we can average \( \tau_{\text{pass}} \) over all values of \( n \), and we obtain

\[
\tau_{\text{pass}}(p) = \sum_{n} P(n) \sum_{m<n/2} b(m, n, p). \tag{20}
\]

By imposing the detailed balance condition

\[
(1 - \rho)\tau_{\text{pass}}(p) = \rho \tau_{\text{act}}(p), \tag{21}
\]

we can compute the active investors fraction at the stationary state, \( \rho^* \) as a function of the fraction of good stocks at the stationary state, \( p^* \). Indeed, the stationary state condition implies

\[
\rho^* = \frac{\rho_0 \tau_{\text{pass}}(p^*)}{(1 - \rho_0)\tau_{\text{act}}(p^*) + \rho_0 \tau_{\text{pass}}(p^*)}. \tag{22}
\]

One can proceed as above to write a detailed balance equation for the stocks. A given stock is replaced if it receives investment less than \( T \). As seen above, the number of stocks included in its portfolio obeys a binomial distribution with mean value \( S/2 \). Let us assume that all the passive investors invest on \( S/2 \) stocks, investing \( 2/S \) on each stock. Under this assumption, a stock included in the portfolio of \( Sx/2 \) passive investors receives from them an investment equal to \( x \). Thus, the probability \( Q_<(m) \) that a given stock receives from the passive agents an investment less than \( m \) is equal to the probability to be included in the portfolio of less than \( Sm/2 \) passive agents over \( (1 - \rho)I \). The probability to be included in the portfolio of \( k \) passive investors is \( b(k, (1 - \rho)I, 1/2) \). Thus, the probability that a given stock receives less than \( m \) investment from the passive investors is equal to

\[
Q_<(m) = \sum_{k=0}^{Sm/2} b(k, (1 - \rho)I, 1/2). \tag{23}
\]

The probability \( Q_<(m) \) does not depend on the quality of the stock, since this investment comes from the agents who invest at random on
good and bad stocks. The active investors, conversely, provide an effort $E$ in examining the quality of stocks. The probability that a given good stock receives an investment from an active agent is $(1-D)/S$, where $1/S$ is the probability to be examined by the active investor and $(1-D)$ is the probability to be exactly judged, i.e. that the investor has a correct perception of the good stock and invests on it. Therefore, the probability that a good stock receives $l$ investments from the $\rho I$ active investors is $R^{(+)}(l) = b(l, \rho I, D/S)$. Likewise, for a bad stock we obtain $R^{(-)}(l) = b(l, \rho I, (1-D)/S)$. Then, the probability that a stock receives less than $T$ is

$$q^{(+)} = \sum_{l=0}^{T} R^{(+)}(l)Q_{<}(T-l),$$

and

$$q^{(-)} = \sum_{l=0}^{T} R^{(-)}(l)Q_{<}(T-l),$$

for good and bad stocks. We notice that these two probabilities are functions of $\rho$. They allow us to write

$$p^* = p^*(\rho^*) = \frac{p_0 q^{(-)}(\rho^*)}{p_0 q^{(+)}(\rho^*) + (1-p_0)q^{(-)}(\rho^*)}$$

and, by replacing $\rho^* = \rho^*(p^*)$ from equation (22), we obtain a self-consistent equation for $p^*$, that can be solved numerically but exactly. As a consequence, we can plot $p^*$ and $\rho^*$ as functions of the effort $E$. The results are verified by numerical simulation.

### 7 Conclusion

We describe the statistical properties of a simplified market model composed by investors and stocks. Stocks can only be good or bad, according to the sign of their return. Investors can be active, i.e. study the stocks and invest on the positive ones, or passive, that is, behave as free riders and take advantage of the market performance, without any selective effort. We investigate the stationary state of this model. First we consider a fixed population of active investors, and we select stocks according to the investment they manage to attract. We observe by numerical and analytical means that increasing the fraction of active investors improve the market performance, which
can be exploited as well by passive investors. Second, we consider a model in which investors can switch from active to passive strategy and *vice versa*. We observe that as the accuracy of active investors is increased, the population of passive investors grows. We can then conclude that active and passive investors live in a sort of *symbiosis*, although the two kinds of investors are in competition (selective force *vs* diversification). Indeed, the gain of the active agents, due to an increased accuracy, can result in a greater gain for the free riders as well, due to a better performance of the whole market.

**References**

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[2] E.F. Fama, *Journal of Financial Economics* **49** (1998) 283.

[3] Y.-C. Zhang, *Toward a Theory of Marginally Efficient Markets, Physica A*, **269** 30 (1999).

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Figure 1: Market Performance at the stationary state as a function of the active investor fraction in a model with $I = 1000$, $S = 100$, $p_0 = 0.5$, $T = 11$ (circles) and $T = 9$ (plus), according to equation 11.
Figure 2: Average Gain of active (solid line) and passive (dashed line) investors as a function of the effort $E$ of the active investors, in a model with $\rho = 0.1$, $I = 500$, $S = 100$, $T = 5$, $p_0 = 0.5$
Figure 3: Market performance as a function of the diversification of active strategies with $\rho = 0.2$, $I = 500$, $S = 100$, $T = 5$, $p_0 = 0.5$
Figure 4: Passive investors fraction (solid line: theory, circles simulation), positive stock fraction (short dashed line, squares), active (dashed line, diamonds) and passive (long-dashed line, triangles) investor fraction as a function of the effort $E$ in the self organized model, with $I = 500, S = 100, T = 5, p_0 = 0.5$