EXPlicit fÖllmer–SCHweizer decomPosition and discretization with jump correction in exponential lévy Models

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Abstract. We investigate two hedging problems in exponential Lévy models. First, we provide an explicit representation for the Föllmer–Schweizer decomposition of European type options under mild conditions, which implies a closed-form expression of the corresponding local risk-minimizing strategies. Secondly, we discretize stochastic integrals driven by an exponential Lévy process using a jump correction method. The convergence rate of the resulting discretization error as the expected number of discretization times increases is measured in weighted BMO spaces, implying also $L^p$-estimates, $p \in (2, \infty)$. Moreover, the effect of a change of measure satisfying a reverse Hölder inequality is addressed. As an application, the error caused by discretizing the local risk-minimizing strategies is investigated in dependence of properties of the Lévy measure, the regularity of the payoff function and the chosen random discretization times.

1. Introduction

This article is concerned with hedging problems in semimartingale financial markets driven by exponential Lévy processes. We investigate two problems corresponding to two typical types of risks for hedging an option. The first one comes from the incompleteness of the market. We consider the semimartingale setting and aim to determine an explicit form for the Föllmer–Schweizer decomposition of European type options which provides directly a closed form for the local risk-minimizing strategies (a similar closed form expression in the martingale setting has been established in [8, 22, 39, 40]). The second type of risk is due to the impossibility of continuously rebalancing a hedging portfolio which leads to the discrete-time hedging. We use an approximation scheme based on tracking jumps of the driving process, the so-called discretization with jump correction, and measure the discretization error in weighted bounded mean oscillation (BMO) spaces. This approach enables to achieve good distributional tail estimates for the error such as a $p$th-order polynomial decay, $p \in (2, \infty)$.

Let us introduce some notations to state the main results. Let $T \in (0, \infty)$ be a fixed time horizon and $X = (X_t)_{t \in [0,T]}$ a Lévy process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the augmented natural filtration of $X$ which satisfies the usual conditions (right continuity and completeness). Assume that $\mathcal{F} = \mathcal{F}_T$. Let $\sigma \geq 0$ be the coefficient of the standard Brownian component and $\nu$ the Lévy measure of $X$, see (2.1). We assume that the underlying discounted price process is modelled by the exponential $S = e^X$.

1.1. Explicit Föllmer–Schweizer (FS) decomposition. Because models with jumps typically correspond to incomplete markets, in general there is no hedging strategy which is self-financing and replicates an option at maturity. Hence, one has to look for certain strategies that minimize some types of risk. In the current work, we choose the quadratic hedging approach which is a popular method to deal with the problem in models with jumps. We refer
the reader to the survey article [37] for this approach. Two typical types of quadratic hedging strategies are the local risk-minimizing (LRM) strategies and the mean-variance hedging (MVH) strategies. Roughly speaking, the LRM strategy is mean-self-financing, replicates an option at maturity and minimizes the riskiness of the cost process locally in time, while the MVH strategy is self-financing and minimizes the global hedging error in the mean square sense. Both types of those strategies are intimately related to the so-called FS decomposition. Namely, in our (exponential Lévy) setting, the FS decomposition gives directly the LRM strategy, and the MVH strategy then can be determined based on this decomposition. This article discusses the FS decomposition and focuses on the LRM strategies only.

Assume that \( S \) is square integrable so that it is a semimartingale satisfying the structure condition, and that the mean-variance trade-off process of \( S \) is deterministic and bounded, see Remark 4.2. Then, the FS decomposition of an \( H \in L_2(\mathbb{P}) \) is of the form

\[
H = H_0 + \int_0^T \vartheta^H_t dS_t + L^H_T,
\]

where \( H_0 \in \mathbb{R}, \vartheta^H \) is an admissible integrand specified in (4.2), and \( L^H \) is an \( L_2(\mathbb{P}) \)-martingale starting at zero which is orthogonal to the martingale part of \( S \). The integrand \( \vartheta^H \) is called the LRM strategy of \( H \), and it is unique up to a \( \mathbb{P} \otimes \lambda \)-null set where \( \lambda \) is the Lebesgue measure. A key tool to study the FS decomposition is the minimal (signed) local martingale measure for \( S \) (see [36]), and we denote this signed measure by \( \mathbb{P}^* \) from now on. Recently, [6, Theorem 4.3] indicated that under a regularity condition for \( \mathbb{P}^* \), we can determine the LRM strategy \( \vartheta^H \) based on the martingale representation of \( H \) with respect to \( \mathbb{P}^* \).

There are many works interested in finding an explicit representation for the FS decomposition and the LRM strategy in the semimartingale framework (see, e.g., [2, 18, 19, 23, 39]). In the exponential Lévy setting and in the case of a European type option \( H = g(S_T) \), Hubalek et al. [19] assumed that the function \( g \) can be represented as an integral transform of a finite complex measure from which one can determine a closed form for the LRM strategy. The key idea of this approach is the separation of the function \( g \) and the underlying price process \( S \) by using a kind of inverse Fourier transform. An advantage of this method is that one gains much flexibility for choosing the underlying Lévy process where there is no extra regularity required for the driving process \( S \) except some mild integrability.

As our first main result, Theorem 1.1 below provides a closed form for the LRM strategy \( \vartheta^H \) of a European type option \( H = g(S_T) \). To obtain this result, except of some mild integrability conditions, we neither assume any regularity for the payoff function \( g \) nor require any extra condition for the small jump behavior of \( X \) and nor need the presence of the diffusion component. Instead of those regularities, we require the condition that \( \mathbb{P}^* \) exists as a true probability measure (see Assumption 4.4) which leads to a constraint for the characteristics of \( X \). This result might be regarded as a counterpart of [19, Proposition 3.1] in which only the square integrability is required for \( S \) while the function \( g \) is supposed to be the integral transform of finite complex measures. The notation \( \mathbb{E}^* \) below means the expectation with respect to \( \mathbb{P}^* \).

**Theorem 1.1.** Assume that \( X \) is not a.s. deterministic and \( S = e^X \) is square \( \mathbb{P} \)-integrable. If \( \mathbb{P}^* \) is a probability measure (i.e., Assumption 4.4 holds), then for any Borel function \( g: (0, \infty) \to \mathbb{R} \) with \( \mathbb{E}^*[g(yS_t)] < \infty, \forall (t, y) \in [0, T] \times (0, \infty) \) and \( g(S_T) \in L_2(\mathbb{P}) \cap L_2(\mathbb{P}^*) \) the following assertions hold:

1. The LRM strategy \( \vartheta^H \) corresponding to \( H = g(S_T) \) is of the form

\[
\vartheta^H_t = \frac{1}{\kappa(\sigma, \nu)} \left( \sigma^2 \partial_y G^*(t, S_t) + \int_{\mathbb{R}} \frac{G^*(t, e^x S_t) - G^*(t, S_t)}{S_t} (e^x - 1) \nu(dx) \right)
\]

for \( \mathbb{P} \otimes \lambda \)-a.e. \( (\omega, t) \in \mathbb{R} \times [0, T] \), where \( \kappa(\sigma, \nu) := \sigma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \in (0, \infty) \), \( G^*(t, y) := \mathbb{E}^*[g(yS_{T-t})] \), and we set \( \partial_y G^* := 0 \) when \( \sigma = 0 \) by convention.

2. There exists a process \( \tilde{\vartheta}^H \) which is adapted and càdlàg on \([0, T]\) such that
Remark 4.5. The main tool for the proof is a martingale representation theorem. Furthermore, the càdlàg property of \( \tilde{\vartheta} \), which is useful to design some Riemann-type approximations for \( \int_0^T \tilde{\vartheta}_t^\alpha dS_t \). For example, an approximation scheme based on tracking jumps of \( \tilde{\vartheta} \) has been constructed in [34].

We also employ this càdlàg version of the LRM strategy for the discrete-time hedging problem in Section 5. Such a path regularity for integrands in the martingale setting was studied in [27].

Several formulas resembling (1.2) have been established in [22, Formula (2.12)], [8, Formula (4.1)], [39, Formula (45)], or in [40, Formula (4.2)]. However, in fact, (1.2) is different from those. The formulas in [8, 22, 39, 40] were obtained by projecting \( H \) orthogonally down to the space of stochastic integrals driven by a (local) martingale, while (1.2) is derived from the FS decomposition which is a different orthogonal decomposition in the semimartingale framework.

Theorem 1.1 is proved in Section 4. The main tool for the proof is a martingale representation for functionals of \( X_T \) in which the integrands with respect to the Brownian part and the jump part are determined explicitly. Such a martingale representation is established in Proposition 3.4 by using Malliavin calculus.

1.2. Discretization with jump correction method via the weighted BMO-approach.

We continue to investigate the discrete-time approximation problem for stochastic integrals driven by the exponential Lévy process \( S \). Let \( E = (E_t)_{t \in [0,T]} \) be the error process given by

\[
E_t := \int_0^t \tilde{\vartheta}_u dS_u - A_t, \quad t \in [0,T],
\]

where \( \tilde{\vartheta} \) is an admissible integrand and \( A = (A_t)_{t \in [0,T]} \) is a discretization approximation for the stochastic integral. In mathematical finance, the stochastic integral can be interpreted as the theoretical hedging portfolio which is continuously readjusted. However, in practice one can only rebalance the portfolio finitely many times, and this fact leads to a discretization of the stochastic integral, represented by \( A \).

In the case that \( A = A^{\text{Rm}} \) is the Riemann approximation, the corresponding error \( E = E^{\text{Rm}} \) and its convergence rate have been investigated in the \( L_2 \)-sense in several works. When \( S \) is assumed to be a martingale, the discretization error along deterministic time-nets was examined by, among others, Brodén and Tankov [5] and Geiss, Geiss and Laukkarinen [12]. Although the approaches in [5] and in [12] are different, both arrived at a result stating that, if the stochastic integral is sufficiently regular, then the convergence rate of the error measured in \( L_2 \) is of order \( n^{-\frac{1}{2}} \) when \( n \), which represents the cardinality of used time-nets, tends to infinity. This obtained rate is shown to be asymptotically optimal in their settings. Later, Rosenbaum and Tankov in [34] considered a model where the driving process is a purely discontinuous local martingale. The authors shown that if the small jump activity of the semimartingale integrand behaves like an \( \alpha \)-stable process with \( \alpha \in (1,2) \) and if the employed discretization times-nets are (random) hitting times of a suitably chosen space grid, then one can achieve the convergence rate \( n^{-\frac{1}{2}} \), which is significantly better than \( n^{-\frac{1}{2}} \) above. Moreover, this rate was obtained under the \( L_2 \)-norm and was proved to be asymptotically optimal in that framework.

In fact, the martingale property of the driving process is technically convenient for handling the discretization errors in \( L_2 \) and it also simplifies the selection of quadratic hedging strategies. However, the problem of discrete-time hedging is not necessarily considered under the martingale setting. Therefore, the second part of this article is twofold: We examine the discretization errors under the setting of semimartingale with jumps and investigate the convergence rates for the error in \( L_p \) for \( p \in (2, \infty) \). Those are natural extensions for the martingale- and for the \( L_2 \)-setting and they seem to be still missing in the literature.
To address those goals, we use the weighted BMO-approach which has been recently exploited in [40]. In addition, we employ the discretization method introduced in [40], the so-called jump correction method which was constructed by tracking jumps of the driving process $S$. Moreover, we show how the weighted BMO-approach can be used to obtain $L_p$-estimates, $p \in (2, \infty)$, for the corresponding error. This approach also allows a change of the underlying measure which leaves the error estimates unchanged provided the change of measure satisfies a reverse Hölder inequality, see Proposition 5.3. The latter is frequently encountered in mathematical finance, and it is particularly useful here to switch the discretization problem between the martingale setting and the semimartingale setting.

The main results of this part are Theorems 5.8 and 5.15. We provide in Theorem 5.8 several estimates for the discretization error measured under weighted BMO-norms and describe a situation so that $L_p$-estimates can be achieved for $p \in (2, \infty)$. Theorem 5.15 serves as an application of Theorem 5.8 where we consider the approximation problem for the stochastic integral term in (1.1) and the chosen integrand is the LRM strategy of a European type option. The results show how the interplay between the regularity of payoff functions and the small jumps intensity of the underlying Lévy process affects the convergence rate.

Let us list some illustrative examples, which are direct consequences of Theorem 5.15, showing the convergence rates for $E_{g}^{corr}$. Here, $E_{g}^{corr}$ denotes the global discretization error resulted from the approximation with jump correction method using the LRM strategy of a European payoff $H = g(S_T)$. To reveal the effect of the small jump activity of $X$, we assume that $X$ does not have a Brownian component and the small jump intensity of $X$ behaves like an $\alpha$-stable process with $\alpha \in (0, 2)$. Then, for the European call/put option (or any Lipschitz $g = g^{\text{Lip}}$) and for the binary option (or any bounded and Borel function $g = g^{\text{bdd}}$), under $\mathbb{E}S_T^p < \infty$ with $p > 3$ and some mild conditions we have

$$
\|E_{g}^{corr}\|_{L_p} \leq c \begin{cases}
\frac{1}{n^\frac{1}{p}} & \text{if } \alpha \in (0, 1), \\
\frac{1 + \log n}{n} & \text{if } \alpha = 1, \\
\frac{1}{n^\frac{1}{\alpha}} & \text{if } \alpha \in (1, 2),
\end{cases}$$

where the latter case holds for any $0 < \delta < \frac{1}{2}(1 - \frac{1}{\alpha})(\frac{2}{\alpha} - 1)$. The parameter $n$ in those estimates represents the expected cardinality of the used time-nets. Our results show the convergence rate $n^{-\frac{1}{\alpha}}$ not only for Lipschitz $g^{\text{Lip}}$ but also for certain Hölder functionals $g$. We remark here that the obtained rate $n^{-\frac{1}{\alpha}}$ is consistent with that in [34, Remark 5 with $\beta = 0$], and moreover, our approach enables the case $\alpha \in (0, 1]$ and the $L_p$-setting which are not treated there.

1.3. Structure of the article. In Section 2, we introduce the notation, recall Malliavin–Sobolev spaces and exponential Lévy processes. Section 3 aims to establish a martingale representation with explicit integrands for functionals of a Lévy process. Section 4 is devoted to prove Theorem 1.1 above. Section 5 presents the discrete-time hedging problem with the weighted BMO-approach for exponential Lévy models. Appendix A provides the proof for main results of Section 5, and some technical results are given in Appendix B.

2. Preliminaries

2.1. General notations. Denote $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. For $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ as usual. For $A, B \geq 0$ and $c \geq 1$, by $A \sim_{c} B$ we mean $A/c \leq B \leq cA$. Subindexing a symbol by a label indicates the place where that symbol appears (e.g., $c_{(5.1)}$ refers to formula (5.1)).

Let $\mathcal{B}(\mathbb{R})$ be the Borel $\sigma$-algebra on $\mathbb{R}$. The Lebesgue measure on $\mathcal{B}(\mathbb{R})$ is denoted by $\lambda$, and we also write $dx$ instead of $\lambda(dx)$ for simplicity. For $p \in [1, \infty]$ and $A \in \mathcal{B}(\mathbb{R})$, the space $L_p(A)$ consists of all $p$-order integrable Borel functions on $A$ with respect to $\lambda$, where the essential supremum is taken when $p = \infty$. For a measure $\mu$ on $\mathcal{B}(\mathbb{R})$, its support is given by

$$
\text{supp } \mu := \{x \in \mathbb{R} : \mu((x - \varepsilon, x + \varepsilon)) > 0, \forall \varepsilon > 0\}.
$$
Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\xi : \Omega \to \mathbb{R}\) a random variable. Denote by \(\mathbb{P}_\xi\) the push-forward measure of \(\mathbb{P}\) with respect to \(\xi\). If \(\xi\) is \(\mathbb{P}\)-integrable (non-negative), then the (generalized) conditional expectation of \(\xi\) given a sub-\(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\) is denoted by \(\mathbb{E}_{\mathbb{P}}^\mathcal{G}[\xi]\) for which we usually omit the reference measure \(\mathbb{P}\) if there is no risk of confusion. Set \(L_p(\mathbb{P}) := L_p(\Omega, \mathcal{F}, \mathbb{P})\).

For a non-empty and open interval \(U \subseteq \mathbb{R}\), let \(C^\infty(\Omega)\) denote the family of all functions \(f\) which have derivatives of all orders on \(U\).

### 2.2. Notation for stochastic processes.

Let \(T > 0\) be a fixed finite time horizon, and let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space equipped with a right continuous filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0,T]}\). Assume that \(\mathcal{F}_0\) is generated by \(\mathbb{P}\)-null sets only. The conditions imposed on \(\mathcal{F}\) allow us to assume that every martingale adapted to this filtration is càdlàg (right-continuous with left limits). We use the following notations and conventions where

\[
\mathbb{I} = [0, T) \quad \text{or} \quad \mathbb{I} = [0, T].
\]

- For processes \(X = (X_t)_{t \in \mathbb{I}}\) and \(Y = (Y_t)_{t \in \mathbb{I}}\), we write \(X = Y\) to indicate that \(X_t = Y_t\) for all \(t \in \mathbb{I}\) a.s., and analogously when the relation "\(=\)" is replaced by some other standard relations such as "\(\leq\)", "\(\geq\)", etc.
- For a càdlàg process \(X = (X_t)_{t \in \mathbb{I}}\), the process \(X_\cdot = (X_{t-})_{t \in \mathbb{I}}\) is defined by setting \(X_{0-} := X_0\) and \(X_{t-} := \lim_{s \downarrow t} X_s\) for \(t \in \mathbb{I}\setminus\{0\}\). We set \(\Delta X := X - X_\cdot\).
- \(\mathcal{CL}(\mathbb{I})\) denotes the family of all càdlàg and \(\mathbb{F}\)-adapted processes.
- \(\mathcal{CL}_0(\mathbb{I})\) (resp. \(\mathcal{CL}^+(\mathbb{I})\)) consists of all \(X \in \mathcal{CL}(\mathbb{I})\) with \(X_0 = 0\) a.s. (resp. \(X \geq 0\)).
- For \(p \in [1, \infty)\) and \(X \in \mathcal{CL}([0, T])\), we set \(\|X\|_{S_p(\mathbb{P})} := \sup_{t \in [0, T]} |X_t|L_p(\mathbb{P})\).
- \(\mathcal{P}\) is the predictable \(\sigma\)-algebra\(^1\) on \(\Omega \times [0, T]\) and \(\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R})\).

We recall some notions regarding semimartingales on the finite time interval \([0, T]\).

- An \(M \in \mathcal{CL}([0, T])\) is called a local (resp. locally square integrable) \(\mathbb{P}\)-martingale if there is a sequence of non-decreasing stopping times \((\rho_n)_{n \geq 1}\) taking values in \([0, T]\) such that \(\mathbb{P}(\rho_n < T) \to 0\) as \(n \to \infty\) and the stopped process \(M_{\rho_n} = (M_{t \wedge \rho_n})_{t \in [0, T]}\) is a \(\mathbb{P}\)-martingale (resp. square integrable \(\mathbb{P}\)-martingale) for all \(n \geq 1\). Let \(\mathcal{M}_0^2(\mathbb{P})\) be the space of all square integrable \(\mathbb{P}\)-martingales \(M = (M_t)_{t \in [0, T]}\) with \(M_0 = 0\) a.s.
- An \(S \in \mathcal{CL}([0, T])\) is called a \(\mathbb{P}\)-semimartingale if \(S\) can be written as a sum of a local \(\mathbb{P}\)-martingale and a process of finite variation a.s. The quadratic covariation of two semimartingales \(S\) and \(R\) is denoted by \([S, R]\). The predictable \(\mathbb{P}\)-compensator of \([S, R]\), if it exists, is denoted by \([S, R]_\mathbb{P}\).
- Let \(M, N\) be locally square integrable \(\mathbb{P}\)-martingales. Then, \(M\) and \(N\) are said to be \(\mathbb{P}\)-orthogonal if \([M, N]_\mathbb{P}\) is a local \(\mathbb{P}\)-martingale, or equivalently, \([M, N]_\mathbb{P}\) = 0.

### 2.3. Lévy process and Itô’s chaos expansion.

Let \(X = (X_t)_{t \in [0, T]}\) be a real-valued Lévy process on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), i.e. \(X_0 = 0\), \(X\) has independent and stationary increments and \(X\) has càdlàg paths. Let \(\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}\) denote the augmented natural filtration generated by \(X\). Throughout this article, we assume \(\mathcal{F} = \mathcal{F}_T\). According to the Lévy–Khintchine formula (see, e.g., [35, Theorem 8.1]), the characteristic exponent \(\psi\) of \(X\), which is defined by

\[
\mathbb{E}e^{iuX_1} = e^{-t\psi(u)}, \quad u \in \mathbb{R}, t \in [0, T],
\]

is of the form

\[
\psi(u) = -i\gamma u + \frac{\sigma^2 u^2}{2} - \int_{\mathbb{R}} \left( e^{iux} - 1 - iux \mathbb{1}_{\{|x| \leq 1\}} \right) \nu(dx), \quad u \in \mathbb{R}. \tag{2.1}
\]

\(^1\)\(\mathcal{P}\) is the \(\sigma\)-algebra generated by \(\{A \times \{0\} : A \in \mathcal{F}_0\} \cup \{A \times (s, t] : 0 \leq s < t \leq T, A \in \mathcal{F}_s\}\).
Here, $\gamma \in \mathbb{R}$, while $\sigma \geq 0$ is the coefficient of the Brownian component, and $\nu : B(\mathbb{R}) \to [0, \infty]$ is a Lévy measure, i.e., $\nu(\{0\}) := 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty$. The triplet $(\gamma, \sigma, \nu)$ is called the characteristics of $X$. To indicate explicitly the characteristics of $X$ under $\mathbb{P}$, we write
\[ (X|\mathbb{P}) \sim (\gamma, \sigma, \nu) \text{ or } (X|\mathbb{P}) \sim \psi. \]

We present briefly the Malliavin calculus for Lévy processes by means of Itô’s chaos expansion which is the main tool to prove Proposition 3.4. For further details, we refer to [1, 30, 31, 38] and the references therein. Define the $\sigma$-finite measures $\mu$ on $B(\mathbb{R})$ and $\mathfrak{m}$ on $B([0, T] \times \mathbb{R})$ by setting
\[ \mu(dx) := \sigma^2 \delta_{0}(dx) + x^2 \nu(dx) \quad \text{and} \quad \mathfrak{m} := \lambda \otimes \mu, \]
where $\delta_{0}$ is the Dirac measure at zero. For $B \in B([0, T] \times \mathbb{R})$ with $\mathfrak{m}(B) < \infty$, the random measure $M$ is defined by
\[ M(B) := \sigma \int_{\{t \in [0, T] : (t, 0) \in B\}} dW_t + L_2(\mathbb{P}) \cdot \lim_{n \to \infty} \int_{B \cap ([0, T] \times \{\frac{1}{n} < |x| < n\})} x \widetilde{N}(dt, dx), \]
where $W$ is the standard Brownian motion and $\widetilde{N}(dt, dx) := N(dt, dx) - dt \nu(dx)$ is the compensated Poisson random measure appearing in the Lévy–Itô decomposition of $X$ (see, e.g., [1, Theorem 2.4.16]). Set $L_2(\mu^0) = L_2(\mathfrak{m}^0) := \mathbb{R}$. For $n \geq 1$, we denote
\[ L_2(\mu^{\otimes n}) := L_2(\mathbb{R}^n, B(\mathbb{R}^n), \mu^{\otimes n}), \]
\[ L_2(\mathfrak{m}^{\otimes n}) := L_2(([0, T] \times \mathbb{R})^n, B(([0, T] \times \mathbb{R})^n), \mathfrak{m}^{\otimes n}). \]

The multiple integral $I_n : L_2(\mathfrak{m}^{\otimes n}) \to L_2(\mathbb{P})$ is defined in the sense of Itô [21] by a standard way using approximation, where it is given for simple functions as follows: for
\[ \xi_{n}^{m} := \sum_{k=1}^{m} a_k \mathbb{1}_{B_1^k \times \cdots \times B_n^k}, \]
where $a_k \in \mathbb{R}$, $B_i^k \in B([0, T] \times \mathbb{R})$ with $\mathfrak{m}(B_i^k) < \infty$ and $B_i^k \cap B_j^k = \emptyset$ for $k = 1, \ldots, m$, $i, j = 1, \ldots, n$, $i \neq j$ and $m \geq 1$, we define
\[ I_n(\xi_{n}^{m}) := \sum_{k=1}^{m} a_k M(B_1^k) \cdots M(B_n^k). \]

Then, [21, Theorem 2] asserts the following Itô chaos expansion
\[ L_2(\mathbb{P}) = \bigoplus_{n=0}^{\infty} \{I_n(\xi_{n}) : \xi_{n} \in L_2(\mathfrak{m}^{\otimes n})\}, \]
where $I_0(\xi_0) := \xi_0 \in \mathbb{R}$. For $n \geq 1$, the symmetrization $\tilde{\xi}_{n}$ of a $\xi_{n} \in L_2(\mathfrak{m}^{\otimes n})$ is
\[ \tilde{\xi}_{n}((t_1, x_1), \ldots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi} \xi_{n}((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)})), \]
where the sum is taken over all permutations $\pi$ of $\{1, \ldots, n\}$, so that $I_n(\xi_n) = I_n(\tilde{\xi}_n)$ a.s. The Itô chaos decomposition verifies that $\xi \in L_2(\mathbb{P})$ if and only if there are $\xi_n \in L_2(\mathfrak{m}^{\otimes n})$ such that $\xi = \sum_{n=0}^{\infty} I_n(\xi_n)$ a.s., and this expansion is unique if every $\xi_n$ is symmetric, i.e. $\xi_n = \xi_{\tilde{n}}$. Furthermore, $||\xi||^2_{L_2(\mathbb{P})} = \sum_{n=0}^{\infty} n! ||\xi_n||^2_{L_2(\mathfrak{m}^{\otimes n})}$. 

**Definition 2.1.** Let $D_{1,2}$ be the Malliavin–Sobolev space of all $\xi = \sum_{n=0}^{\infty} I_n(\xi_n) \in L_2(\mathbb{P})$ such that
\[ \|\xi\|_{D_{1,2}}^2 := \sum_{n=0}^{\infty} (n + 1)! ||\xi_n||^2_{L_2(\mathfrak{m}^{\otimes n})} < \infty. \]
The Malliavin derivative operator $D: \mathbb{D}_{1,2} \to L_2(\mathbb{P} \otimes \mathbb{m})$, where $L_2(\mathbb{P} \otimes \mathbb{m}) := L_2(\Omega \times [0,T] \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}([0,T] \times \mathbb{R}), \mathbb{P} \otimes \mathbb{m})$, is defined for $\xi = \sum_{n=0}^{\infty} I_n(\xi_n) \in \mathbb{D}_{1,2}$ by
\[
D_{t,x} \xi := \sum_{n=1}^{\infty} n I_{n-1}(\xi_n((t,x),\cdot)), \quad (\omega, t, x) \in \Omega \times [0,T] \times \mathbb{R}.
\]

2.4. Exponential Lévy processes. Let $X$ be a Lévy process with $(X|\mathbb{P}) \sim (\gamma, \sigma, \nu)$. The stochastic exponential of $X$, denoted by $\mathcal{E}(X)$, is the càdlàg process that satisfies the stochastic differential equation (SDE)
\[
d\mathcal{E}(X) = \mathcal{E}(X) - dX, \quad \mathcal{E}(X)_0 = 1.
\]
We apply [1, Theorem 5.1.6] with the truncation function $x \mathbb{1}_{\{|x| \leq 1\}}$ instead of $x \mathbb{1}_{\{|x| < 1\}}$ to obtain that if $\mathcal{E}(X) > 0$, then there is a Lévy process $Y$ with $(Y|\mathbb{P}) \sim (\gamma_Y, \sigma_Y, \nu_Y)$ such that $\mathcal{E}(X) = e^Y$, where $\sigma_Y = \sigma$ and
\[
\nu_Y = \nu \circ h^{-1} \quad \text{for } h(x) := \ln(1 + x),
\]
\[
\gamma_Y = \gamma - \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left( \mathbb{1}_{\{|h(x)| \leq 1\}} h(x) - x \mathbb{1}_{\{|x| \leq 1\}} \right) \nu(dx).
\]
Conversely, there is a Lévy process $Z$ with $(Z|\mathbb{P}) \sim (\gamma_Z, \sigma_Z, \nu_Z)$ such that $e^X = \mathcal{E}(Z)$. Moreover, one has $\sigma_Z = \sigma$ and
\[
\nu_Z = \nu \circ \tilde{h}^{-1} \quad \text{for } \tilde{h}(x) := e^x - 1,
\]
\[
\gamma_Z = \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left( \mathbb{1}_{\{|\tilde{h}(x)| \leq 1\}} \tilde{h}(x) - x \mathbb{1}_{\{|x| \leq 1\}} \right) \nu(dx).
\]

3. Martingale representation with explicit integrands

This section is to prove a martingale representation for $f(X_T)$ by using Malliavin calculus. There are two key observations: first, the kernels in the chaos expansion of $f(X_T) \in L_2(\mathbb{P})$ do not depend on the time variables which implies the Malliavin differentiability of $\mathbb{E}_f, \mathbb{F}_t[f(X_T)]$ for any $t \in [0,T)$, see Lemma 3.3; secondly, the Malliavin derivative of a functional of $X_t$, provided it is Malliavin differentiable, can be expressed in an explicit form, see Lemma 3.2.

Assume $(X|\mathbb{P}) \sim (\gamma, \sigma, \nu)$ in this section. We first need the following:

**Lemma 3.1** ([16], Theorem 9.13(1)). Assume $\sigma > 0$. Let $f: \mathbb{R} \to \mathbb{R}$ be a Borel function with $\mathbb{E}|f(X_T)|^q < \infty$ for some $q > 1$. Then, $\mathbb{E}|f(x + X_{T-t})| < \infty$ for all $(t,x) \in [0,T] \times \mathbb{R}$, and the function $x \mapsto f(t,x) := \mathbb{E}f(x + X_{T-t})$ belongs to $C^\infty(\mathbb{R})$ for any $t \in [0,T)$. Furthermore,
\[
\mathbb{E}_{\mathcal{F}_s}[\partial_x F(t,X_t)] = \partial_x F(s,X_s) \quad \text{a.s.}
\]
for any $0 \leq s < t < T$.

**Lemma 3.2** ([24]). Let $t \in (0,T]$ and a Borel function $f: \mathbb{R} \to \mathbb{R}$ with $f(X_t) \in L_2(\mathbb{P})$. Then, $f(X_t) \in \mathbb{D}_{1,2}$ if and only if the following two assertions hold:
\begin{itemize}
  \item[(a)] when $\sigma > 0$, $f$ has a weak derivative\(^2\) $f'_w(x)$ on $\mathbb{R}$ with $f'_w(X_t) \in L_2(\mathbb{P})$;
  \item[(b)] the map $(s, x) \mapsto \frac{f(X_t + x) - f(X_t)}{x} I_{[0,t]}(s, x) \in L_2(\mathbb{P} \otimes \mathbb{m})$ belongs to $L_2(\mathbb{P} \otimes \mathbb{m})$.
\end{itemize}

\(^2\)A locally integrable $h$ is called a weak derivative of a locally integrable $f$ on $\mathbb{R}$ if $\int_{\mathbb{R}} f(x)\phi'(x)dx = -\int_{\mathbb{R}} f(x)\phi(x)dx$ for all smooth functions $\phi$ with compact support in $\mathbb{R}$. If such an $h$ exists (unique up to a $\lambda$-null set), then we denote $f'_w := h$. 
Lemma 3.2. Let us turn to the case $\sigma > 0$ that (3.1) is implied by (3.3). Moreover, in the case $\sigma = 0$ we let $F(t, \cdot)$ be a Borel function such that $F(t, X_t) = \mathbb{E}_{\mathcal{F}_t}[f(X_T)]$ a.s. and set $\partial_{x} F := 0$.

Proof. Items (1) and (2) are due to [16, Lemma D.1]. For Item (3), it is clear for the case $\sigma = 0$ that (3.1) is implied by Lemma 3.2. Let us turn to the case $\sigma > 0$. According to Lemma 3.1, one has $F(t, \cdot) \in C^\infty(\mathbb{R})$, and hence $\partial_{x} F(t, \cdot) \in L_2(\mathbb{R})$ a.e. with respect to the Lebesgue measure $\lambda$. Since the law of $X_t$ is absolutely continuous with respect to $\lambda$, it holds that $(F(t, \cdot))'_{\sigma}(X_t) = \partial_{x} F(t, X_t)$ a.s. Then, (3.1) follows from Lemma 3.2. \hfill $\Box$

The following result provides a martingale representation for $f(X_T)$ with explicit integrands.

Proposition 3.4. Let $f: \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\mathbb{E}|f(x + X_t)| < \infty$ for all $(t, x) \in [0, T] \times \mathbb{R}$. If $f(X_t) \in L_2(\mathbb{P})$, then

$$
\mathbb{E} \int_0^T \sigma \partial_{x} F(t, X_{t-})^2 dt + \mathbb{E} \int_0^T \int_{\mathbb{R}} |F(t, X_{t-} + x) - F(t, X_{t-})|^2 \nu(dx) dt < \infty
$$

and, a.s.,

$$
f(X_T) = \mathbb{E} f(X_T) + \int_0^T \sigma \partial_{x} F(t, X_{t-}) dW_t + \int_0^T \int_{\mathbb{R} \setminus \{0\}} (F(t, X_{t-} + x) - F(t, X_{t-})) \tilde{N}(dt, dx),
$$

where $F(t, x) := \mathbb{E} f(x + X_{t-})$ for $(t, x) \in [0, T] \times \mathbb{R}$, and we set $\partial_{x} F := 0$ if $\sigma = 0$.

Proof. For $(t, x) \in [0, T] \times \mathbb{R}$, denote

$$
\mathcal{D} F(t, x) := \partial_{x} F(t, X_{t-}) 1_{\{x=0\}} + \frac{F(t, X_{t-} + x) - F(t, X_{t-})}{x} 1_{\{x \neq 0\}},
$$

where we recall that $\partial_{x} F := 0$ if $\sigma = 0$ by convention. The assumption $\mathbb{E}|f(x + X_t)| < \infty$ for all $(t, x) \in [0, T] \times \mathbb{R}$ implies that $(F(t, X_{t} + x) - F(t, X_{t}))_{t \in [0, T]}$ is a martingale for each $x \in \mathbb{R}$. Moreover, in the case $\sigma > 0$, the assumption $f(X_t) \in L_2(\mathbb{P})$ and Lemma 3.1 imply that $F(t, \cdot) \in C^\infty(\mathbb{R})$ for all $t \in [0, T]$ and $(\partial_{x} F(t, X_{t}))_{t \in [0, T]}$ is a martingale.

Step 1. We show that for any $t \in (0, T)$,

$$
C(t) := \mathbb{E} \int_0^t \int_{\mathbb{R}} |\mathcal{D} F(s, x)|^2 \mathfrak{m}(ds, dx) < \infty.
$$

Note that $(t, x) \mapsto F(t, x)$ is Borel measurable by Fubini’s theorem. Since $X_{t-}$ is predictable, it implies that $(\omega, t, x) \mapsto F(t, X_{t-} + x)$ is $\mathcal{P}$-measurable. Hence, $\mathcal{D} F$ given in (3.3) is $\mathcal{P}$-measurable. Since $X_s = X_{s-}$ a.s. for each $s \in [0, T]$, using Fubini’s theorem and the martingale property, together with (3.1), we obtain for any $t \in (0, T)$ that

$$
C(t) = \mathbb{E} \int_0^t |\sigma \partial_{x} F(s, X_s)|^2 ds + \mathbb{E} \int_0^t \int_{\mathbb{R}} |F(s, X_s + x) - F(s, X_s)|^2 \nu(dx) ds
$$
Lemma 3.2 \[(3.4) \text{we have for} \ \Omega /BD \text{where the second equality comes from the fact that}\]

\[X(\sigma) \text{this formula seems to be considered either when the Lévy process} \]

\[\text{functions} \tilde{\sigma} \text{for all} \ m \ \text{satisfy}\]

\[t \leq \sum_{i} - F(t, X_t) = E \sum_{i} F(t, X_t) + \int_{t}^{t} D F(s, x)(ds, dx). \]

\[\text{Hence, the stochastic integral} \int_{0}^{t} D F(s, x)(ds, dx) \text{exists as an element in} \ L_2(\mathbb{P}).\]

**Step 2.** Fix \( t \in (0, T). \) We prove that, a.s.,

\[F(t, X_t) = E f(X_T) + \int_{t}^{t} D F(s, x)(ds, dx). \tag{3.4} \]

The representation (3.4) can be regarded as a consequence of the Clark–Ocone formula. However, this formula seems to be considered either when the Lévy process \( X \) is square integrable or when \( X \) has no Brownian component (i.e., \( \sigma = 0 \)), see, e.g., [3, 26, 30, 31, 38]. So, for the reader’s convenience, we present here a complete proof for (3.4) where neither square integrability nor \( \sigma = 0 \) is assumed. Due to the denseness of the simple multiple stochastic integrals in \( L_2(\mathbb{P}) \) (see [11, Lemma 2.1]), in order to obtain (3.4) it is sufficient to show that

\[E[I_m(k_m)F(t, X_t)] = E[I_m(k_m) \int_{0}^{t} D F(s, x)(ds, dx)] \tag{3.5} \]

for all \( m \geq 1 \) and all functions \( k_m \) of the form

\[k_m = \mathbb{1}_{B_1 \times \cdots \times B_m}, \tag{3.6} \]

where \( B_i = (s_i, t_i) \times (a_i, b_i) \) in which \( (a_i, b_i) \) are finite intervals and the time intervals \( (s_i, t_i) \subset [0, t] \)

satisfy \( t_{i-1} \leq s_i, i = 2, \ldots, m. \)

Since \( F(t, X_t) \in D_{1,2} \) by Lemma 3.3(2), applying Lemma 3.2 we have for \( \mathbb{P} \otimes \mathbb{m} \text{-a.e.} \) \((\omega, s, x) \in \Omega \times [0, T] \times \mathbb{R},\)

\[D_{s,x} F(t, X_t) = \partial_t F(t, X_t) \mathbb{1}_{[0,t] \times \mathbb{R}}(s, x) + \frac{F(t, X_t + x) - F(t, X_t)}{x} \mathbb{1}_{(x \neq 0)}(s, x). \tag{3.7} \]

Moreover, for each \((s, x) \in [0, t] \times \mathbb{R}, \) the martingale property implies that, a.s.,

\[E(F_{s,x} \partial_t F(t, X_t) \mathbb{1}_{\{x=0\}} + \frac{F(t, X_t + x) - F(t, X_t)}{x} \mathbb{1}_{\{x \neq 0\}}) = D F(s, x), \]

where the second equality comes from the fact that \( X_s = X_{s-} \) a.s.

We let \( f(X_T) = \sum_{n=0}^{\infty} I_n(f_n \mathbb{1}_{[0,T]}^o) \) and \( F(t, X_t) = \sum_{n=0}^{\infty} I_n(f_n \mathbb{1}_{[0,T]}^o) \) as in Lemma 3.3(1) and (2) respectively, where \( f_n \in L_2(\mu^o) \) are symmetric. Let \( k_m \) be of the form as in (3.6). Since functions \( f_n \) are symmetric, the left-hand side of (3.5) is computed as follows

\[LHS_{(3.5)} = m! \int_{B_1 \times \cdots \times B_m} \check{f}_m(x_1, \ldots, x_m) m(dx_1) \cdots m(dx_m). \tag{3.8} \]

For the right-hand side of (3.5), writing \( I_m(k_m) = \int_{B_m} I_{m-1}(k_{m-1}) M(ds, dx), \) where \( k_{m-1} := \mathbb{1}_{B_1 \times \cdots \times B_{m-1}}, \) and using Fubini’s theorem we obtain

\[RHS_{(3.5)} = \int_{B_m} I_{m-1}(k_{m-1}) D F(s, x)(ds, dx) \]

\[= \int_{B_m} \mathbb{E} \left[ I_{m-1}(k_{m-1}) E_{F_s} \left[ \partial_t F(t, X_t) \mathbb{1}_{\{x=0\}} + \frac{F(t, X_t + x) - F(t, X_t)}{x} \mathbb{1}_{\{x \neq 0\}} \right] \right] m(ds, dx) \]

\[= \mathbb{E} \int_{B_m} I_{m-1}(k_{m-1}) D_{s,x} F(t, X_t) m(ds, dx) \tag{3.9} \]

\[= \mathbb{E} \int_{B_m} I_{m-1}(k_{m-1}) \left( L_2(\mathbb{P} \otimes \mathbb{m}) \lim_{j \to \infty} \sum_{i=1}^{j} I_{i-1} \left( \check{f}_i(t, x) \mathbb{1}_{[0,T]}^o \mathbb{1}_{[0,T]}^o(s) \right) \right) m(ds, dx) \]
\begin{equation}
\begin{aligned}
&= m \int_{B_m} \mathbb{E} \left[ I_{m-1}(k_{m-1})I_{m-1} \left( \tilde{f}_m(\cdot, x) \mathbb{1}_{[0,t]}(s) \right) \right] \mathbb{m}(ds, dx) \\
&= m! \int_{B_m} \int_{B_{m-1} \times \cdots \times B_{m-1}} \tilde{f}_m(x_1, \ldots, x_{m-1}, x) \mathbb{m}(ds_1, dx_1) \cdots \mathbb{m}(ds_{m-1}, dx_{m-1}) \mathbb{m}(ds, dx).
\end{aligned}
\tag{3.10}
\end{equation}

Here, one uses (3.7) and the fact that \( I_{m-1}(k_{m-1}) \) is \( \mathcal{F}_s \)-measurable for all \( s \in (s_m, t_m] \) to obtain (3.9). Combining (3.8) with (3.10) yields (3.5).

Step 3. For any \( t \in (0, T) \), Jensen’s inequality implies that \( \mathbb{E}|f(X_T)|^2 \geq \mathbb{E}|f(X_t)|^2 \). Then, we apply Step 2 and Itô’s isometry to obtain
\[ 
\mathbb{E}|f(X_T)|^2 \geq \mathbb{E}|f(X_t)|^2 + \mathbb{E} \int_0^t \int \mathbb{D}F(s, x)^2 \mathbb{m}(ds, dx).
\]

Letting \( t \uparrow T \), we infer that the stochastic integral \( \int_0^T \int \mathbb{D}F(s, x)M(ds, dx) \) exists as an element in \( L_2(\mathbb{P}) \) and equals to \( L_2(\mathbb{P}) \)-limit to \( \int_0^t \int \mathbb{D}F(s, x)M(ds, dx) \). On the other hand, due to the martingale convergence theorem, \( F(t, X_t) = \mathbb{E}_{\mathcal{F}_t}[f(X_T)] \rightarrow \mathbb{E}_{\mathcal{F}_\infty}[f(X_T)] \) a.s. and in \( L_2(\mathbb{P}) \) as \( t \uparrow T \), where \( \mathcal{F}_\infty := \sigma(\cup_{t \in [0, T]} \mathcal{F}_t) \). Since \( \mathcal{F}_t \) is the augmented natural filtration of the Lévy process \( X \), it holds that \( \mathcal{F}_\infty = \mathcal{F}_T \), and hence the desired conclusion follows.

Remark 3.5. Proposition 3.4 extends [8, Proposition 7] in which the function \( f \) has a polynomial growth and \( X \) satisfies certain conditions. A similar representation to (3.2) in a general framework (with different assumptions from ours) can be found in the proof of [22, Theorem 2.4]. On the other hand, when \( f(X_T) \) is Malliavin differentiable then one can use the Clark–Ocone formula (see, e.g., [2, 3, 26]) to obtain its explicit martingale representation. However, the Malliavin differentiability of \( f(X_T) \) fails to hold in many contexts. For example, if \( f(x) = \mathbb{1}_{[K, \infty)}(x) \) for some \( K \in \mathbb{R} \), and if \( X \) is of infinite variation and \( X_T \) has a density satisfying a mild condition, then \( f(X_T) \) is not Malliavin differentiable, see [25, Theorem 6(b)].

The representation (3.2) is a Clark–Ocone type formula but \( f(X_T) \) is not necessarily differentiable in the Malliavin sense. The idea exploiting Malliavin calculus to obtain explicit integrands in the martingale representation of non-differentiable \( f(X_T) \) was used in [32] for more general underlying process \( X \). However, their results require the presence of the diffusion component of \( X \), which in our context means \( \sigma > 0 \).

4. Closed form for the local risk-minimizing strategy

This section gives the proof of Theorem 1.1. First, let us fix the setting of this section.

4.1. Setting. Let \( S = e^X \) be the exponential of a Lévy process \( X \) with \( (X|\mathbb{P}) \sim (\gamma, \sigma, \nu) \). Assume that \( \sigma^2 + \nu(\mathbb{R}) \in (0, \infty) \) and \( \int_{x>1} e^{2x} \nu(dx) < \infty \).

Condition \( \sigma^2 + \nu(\mathbb{R}) > 0 \) is simply to exclude the degenerate case that \( X \) is a.s. deterministic. Since \( \int_{x>1} e^{2x} \nu(dx) < \infty \Leftrightarrow \int_{|x|>1} e^{2x} \nu(dx) < \infty \), it follows from [35, Theorem 25.3] that \( \int_{x>1} e^{2x} \nu(dx) < \infty \) is equivalent to the square integrability of \( S \). Moreover, Itô’s formula yields
\[ 
S = 1 + \left( \int_0^T \sigma S_{t-}dW_t + \int_0^T \int_{\mathbb{R}_0} S_{t-}(e^x - 1)N(dt, dx) \right) + \gamma(4.1) \int_0^T S_{t-}dt =: 1 + S^m + S^{fv},
\]
where \( S^m \) and \( S^{fv} \) respectively denote the martingale part and the predictable finite variation part in the representation of \( S \), and where
\[ 
\gamma(4.1) := \gamma + \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{\{|x|\leq 1\}}) \nu(dx).
\]
Recall from Theorem 1.1 the notation
\[ 
\kappa(\sigma, \nu) = \sigma^2 + \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \in (0, \infty).
\]
4.2. Föllmer–Schweizer (FS) decomposition. We briefly present the FS decomposition of a random variable and the notion of the minimal local martingale measure which is the key tool to determine the FS decomposition. We refer the reader to [37] for a survey about these objects.

In this article, we follow [19, p.863] and use the family of admissible strategies as

\[ A(S|\mathbb{P}) := \left\{ \vartheta \text{ predictable} : \mathbb{E} \int_0^T \vartheta_t^2 S_t^2 \, dt < \infty \right\}. \tag{4.2} \]

It turns out that if \( \vartheta \in A(S|\mathbb{P}) \), then

\[ \mathbb{E} \int_0^T \vartheta_t^2 d[S,S]_t = \mathbb{E} \int_0^T \vartheta_t^2 d[S^m,S^m]_t = \mathbb{E} \int_0^T \vartheta_t^2 d\langle S^m, S^m \rangle_t = \kappa(\sigma, \nu) \mathbb{E} \int_0^T \vartheta_t^2 S_t^2 \, dt < \infty. \tag{4.3} \]

**Definition 4.1 ([37]).** (1) An \( H \in L_2(\mathbb{P}) \) admits an FS decomposition if one can express

\[ H = H_0 + \int_0^T \vartheta_t^H \, dS_t + L_t^H, \]

where \( H_0 \in \mathbb{R}, \vartheta_t^H \in A(S|\mathbb{P}) \) and \( L_t^H \in M_2^0(\mathbb{P}) \) is \( \mathbb{P} \)-orthogonal to \( S^m \).

(2) The integrand \( \vartheta_t^H \) is called the local risk-minimizing strategy of \( H \).

**Remark 4.2.** In our context, \( S \) satisfies the structure condition, and the mean-variance trade-off process \( \hat{K} \) of \( S \) in the sense of [37, p.553] is

\[ \hat{K}_t = \frac{\gamma_t^2}{\kappa(\sigma, \nu)} \]

which is uniformly bounded in \( (\omega, t) \in \Omega \times [0, T] \). Hence, [29, Theorem 3.4] implies that any \( H \in L_2(\mathbb{P}) \) admits a unique FS decomposition.

We continue with the notion of minimal martingale measure.

**Definition 4.3 ([36], Section 2).** Let \( \mathcal{E}(U) \in \text{CL}([0, T]) \) be the stochastic exponential of \( U \), i.e. \( d\mathcal{E}(U) = \mathcal{E}(U) \, dU \) with \( \mathcal{E}(U)_0 = 1 \), where

\[ U = -\frac{\gamma_t^{(4.1)}}{\kappa(\sigma, \nu)} \left( \sigma W + \int_0^t \int_{\mathbb{R}_0} (e^x - 1) \tilde{N}(ds, dx) \right). \tag{4.4} \]

If \( \mathcal{E}(U) > 0 \), then the probability measure \( \mathbb{P}^* \) defined by

\[ d\mathbb{P}^* := \mathcal{E}(U)^{-1} \, d\mathbb{P} \]

is called the minimal martingale measure for \( S \).

Since \( U \) given in (4.4) is a Lévy process and belongs to \( M_2^0(\mathbb{P}) \), it is known that \( \mathcal{E}(U) \) is also an \( L_2(\mathbb{P}) \)-martingale (see, e.g., [33, Ch.V, Theorem 67] or [12, Lemma 1]).

We now provide a condition imposed on the characteristics of \( X \) such that \( \mathbb{P}^* \) exists as a probability measure. Let \( (U|\mathbb{P}) \sim (\gamma_U, \sigma_U, \nu_U) \) and denote

\[ \alpha_U(x) := -\frac{\gamma^{(4.1)}(e^x - 1)}{\kappa(\sigma, \nu)}, \quad x \in \mathbb{R}. \]

Then, it follows from (4.4) that

\[ \gamma_U = -\int_{|\alpha_U(x)| > 1} \alpha_U(x) \nu(dx), \quad \sigma_U = \frac{\gamma^{(4.1)}(e^x - 1)}{\kappa(\sigma, \nu)}, \quad \nu_U = \nu \circ \alpha_U^{-1}. \tag{4.5} \]

Because

\[ \mathcal{E}(U) > 0 \Leftrightarrow \Delta U > -1 \Leftrightarrow \nu_U((-\infty, -1]) = 0 \Leftrightarrow \nu\left( \left\{ x \in \mathbb{R} : 1 - \frac{\gamma^{(4.1)}(e^x - 1)}{\kappa(\sigma, \nu)} \leq 0 \right\} \right) = 0, \]

the following assumption ensures the existence of \( \mathbb{P}^* \) as a probability measure:
Assumption 4.4. $\nu(\mathbb{R}\setminus A) = 0$ where $A := \left\{ x \in \mathbb{R} : 1 - \frac{\gamma(4.1)(e^x - 1)}{\kappa(\sigma, \nu)} > 0 \right\}$.

A sufficient condition for Assumption 4.4 is

$$\gamma(4.1)(e^x - 1) < \kappa(\sigma, \nu), \quad \forall x \in \text{supp } \nu,$$

and (4.6) is particularly satisfied when

$$0 \geq \gamma(4.1) \geq -\kappa(\sigma, \nu).$$

Assume that Assumption 4.4 holds. Then, an application of Girsanov’s theorem (see, e.g., [9, Propositions 2 and 3]) asserts that $X$ is a Lévy process under $\mathbb{P}^*$ with $(X|\mathbb{P}^*) \sim (\gamma^*, \sigma^*, \nu^*)$, where

$$\gamma^* = \gamma - \frac{\gamma(4.1)}{\kappa(\sigma, \nu)} \left( \sigma^2 + \int_{|x| \leq 1} x(e^x - 1)\nu(dx) \right),$$

$$\sigma^* = \sigma \quad \text{and} \quad \nu^*(dx) = \left( 1 - \frac{\gamma(4.1)(e^x - 1)}{\kappa(\sigma, \nu)} \right) \nu(dx).$$

Moreover, a calculation shows $\int_{|x| > 1} e^x\nu^*(dx) < \infty$ and $\gamma^* + \frac{\sigma^2}{2} + \int_{|x| \leq 1} (e^x - 1 - x^2)1_{\{|x| \leq 1\}}\nu^*(dx) = 0$ which implies that $S = e^X$ is a $\mathbb{P}^*$-martingale via Itô’s formula. Let $W^*$ and $\tilde{N}$ respectively denote the Brownian motion and the compensated Poisson random measure of $X$ under $\mathbb{P}^*$, then

$$W^*_t = W_t + \frac{\gamma(4.1)\sigma}{\kappa(\sigma, \nu)} t, \quad \tilde{N}(dt, dx) = \tilde{N}(dt, dx) + \frac{\gamma(4.1)}{\kappa(\sigma, \nu)} (e^x - 1)\nu(dx)dt.$$

In the sequel, let $\mathbb{E}^*$ (resp. $\mathbb{E}_G^*$) denote the expectation (resp. conditional expectation given a $\sigma$-algebra $G \subseteq \mathcal{F}$) with respect to $\mathbb{P}^*$.

4.3. Proof of Theorem 1.1. Let $f(x) := g(e^{x})$ and $F^*(t, x) := \mathbb{E}^*f(x + X_{T-t})$ so that $G^*(t, e^{x}) = F^*(t, x)$ for $(t, x) \in [0, T) \times \mathbb{R}$. We define

$$D_jG^*(t, x) := G^*(t, e^{x}S_{t-j}) - G^*(t, S_{t-j}), \quad (t, x) \in [0, T) \times \mathbb{R}.$$

(1) We present here a direct proof for this assertion, an alternative argument in an abstract setting can be found in [6, Proof of Theorem 4.3]. By assumption, $f(X_T) = g(S_T) \in L_2(\mathbb{P}^*)$ and $\mathbb{E}^*|f(x + X_t)| = \mathbb{E}^*|g(e^{x}S_t)| < \infty$ for any $(t, x) \in [0, T) \times \mathbb{R}$, we apply Proposition 3.4 to obtain

$$K^* = \mathbb{E}^*g(S_T) + \int_0^T \mathbb{E}S_t - \partial_y G^*(t, S_{t-j})dW_j^* + \int_0^T \int_{R_0} D_jG^*(t, x)\tilde{N}(dt, dx),$$

where $K^* = (K^*_t)_{t \in [0, T]}$ denotes a càdlàg version of the $L_2(\mathbb{P}^*)$-martingale $(\mathbb{E}^*[g(S_T)])_{t \in [0, T]}$, and where $W^*$ and $\tilde{N}$ are introduced in (4.8). Then, $\mathcal{E}(U)K^*$ is a martingale under $\mathbb{P}$. Since the $\mathbb{P}$-martingale $U$ given in (4.4) satisfies

$$\|\langle U, U \rangle_T \|_{L_\infty(\mathbb{P})} = \frac{\gamma^2(4.1)T}{\kappa(\sigma, \nu)^2} \left( \sigma^2 + \int_\mathbb{R} (e^x - 1)^2\nu(dx) \right) < \infty,$$

it implies that $\mathcal{E}(U)$ is regular and satisfies $(R_2)$ in the sense of [7, Proposition 3.7]. Since $K^*_T = g(S_T) \in L_2(\mathbb{P})$ by assumption, we apply [7, Theorem 4.9((i))$\Rightarrow$(ii))] to obtain

$$\mathbb{E}[K^*, K^*_T] < \infty.$$

Combining this with (4.9) yields

$$\mathbb{E} \int_0^T \sigma^2|S_{t-j}\partial_y G^*(t, S_{t-j})|^2dt + \mathbb{E} \int_0^T \int_{R_0} |D_jG^*(t, x)|^2N(dt, dx) = \mathbb{E}[K^*, K^*_T] < \infty.$$

Since $d\nu(dx)$ is the predictable $\mathbb{P}$-compensator of $N(dt, dx)$, it implies that

$$\mathbb{E} \int_0^T \sigma^2|S_{t-j}\partial_y G^*(t, S_{t-j})|^2dt + \mathbb{E} \int_0^T \int_{\mathbb{R}} |D_jG^*(t, x)|^2\nu(dx)dt < \infty.$$
Using Cauchy–Schwarz’s inequality yields

\[
\mathbb{E} \int_0^T \sigma^2 S_{t-}^2 |\partial_y G^*(t, S_{t-})| \, dt + \mathbb{E} \int_0^T \int_\mathbb{R} |\mathcal{D}_f G^*(t, x)S_{t-}(e^x - 1)| \nu(dx) \, dt \\
\leq \sqrt{\mathbb{E} \int_0^T S_{t-}^2 \, dt} \sqrt{\mathbb{E} \int_0^T |\sigma^2 S_{t-} \partial_y G^*(t, S_{t-})|^2 \, dt} \\
+ \sqrt{\int_\mathbb{R} (e^x - 1)^2 \nu(dx)} \sqrt{\mathbb{E} \int_0^T S_{t-}^2 \, dt} \sqrt{\mathbb{E} \int_0^T |\mathcal{D}_f G^*(t, x)|^2 \nu(dx) \, dt} \\
< \infty.
\]

On the other hand, the FS decomposition of \( H = g(S_T) \) is

\[
g(S_T) = H_0 + \int_0^T \vartheta_t^H \, dS_t + L_t^H
\]

where \( H_0 \in \mathbb{R}, \vartheta_t^H \in \mathcal{A}(S|\mathbb{P}) \) and \( L_t^H \in \mathcal{M}_0^c(\mathbb{P}) \) is \( \mathbb{P} \)-orthogonal to the martingale component \( S^m \) of \( S \). According to \([37, \text{Eq. (3.10)}]\), it holds that \( L_t^H \) is a local \( \mathbb{P}^* \)-martingale. Note that \( \int_0^T \vartheta_t^H \, dS_t \) is also a local \( \mathbb{P} \)-martingale. Using Cauchy–Schwarz’s inequality and (4.3) yields

\[
\mathbb{E}^* \left[ |L_t^H|_T \right] \leq \| \mathcal{E}(U)_T \|_{L_2(\mathbb{P})} \sqrt{\mathbb{E}[L_t^H, L_t^H]_T} < \infty,
\]

\[
\mathbb{E}^* \int_0^T |\vartheta_t^H|^2 \, d[S, S]_t \leq \| \mathcal{E}(U)_T \|_{L_2(\mathbb{P})} \sqrt{\mathbb{E} \int_0^T |\vartheta_t^H|^2 \, d[S, S]_t} < \infty.
\]

Hence, the Burkholder–Davis–Gundy inequality verifies that both \( L_t^H \) and \( \int_0^T \vartheta_t^H \, dS_t \) are \( \mathbb{P}^* \)-martingales. Combining (4.9) with (4.12), we derive \( H_0 = \mathbb{E}^* g(S_T) \) and

\[
\int_0^T \vartheta_t^H \, dS_t + L_t^H = \int_0^T \sigma S_{t-} \partial_y G^*(t, S_{t-}) \, dW_t^\ast + \int_0^T \int_{\mathbb{R}^2} \mathcal{D}_f G^*(t, x) \tilde{N}^\ast(dt, dx).
\]

Recall that the martingale part of \( S \) is \( S^m = \int_0^T \sigma S_{t-} \, dW_t + \int_0^T \int_{\mathbb{R}_0} S_{t-}(e^x - 1) \tilde{N}(dt, dx) \). Since \( \langle L^H, S^m \rangle_\mathbb{P} = 0 \) by the definition of the FS decomposition, we take the predictable quadratic covariation on both sides of (4.13) with \( S^m \) under \( \mathbb{P} \) and notice that the integrability condition (4.11) holds to obtain

\[
\kappa(\sigma, \nu) \int_0^T \vartheta_t^H S_{t-}^2 \, dt = \int_0^T \sigma^2 S_{t-}^2 \partial_y G^*(t, S_{t-}) \, dt + \int_0^T \int_{\mathbb{R}_0} \mathcal{D}_f G^*(t, x) S_{t-}(e^x - 1) \nu(dx) \, dt,
\]

which yields (1.2) as desired.

(2) It follows from Cauchy–Schwarz’s inequality and (4.10) that

\[
\mathbb{E}^* \int_0^T |\sigma^2 S_{t-} \partial_y G^*(t, S_{t-})| \, dt + \mathbb{E}^* \int_0^T \int_{\mathbb{R}} |\mathcal{D}_f G^*(t, x)(e^x - 1)| \nu(dx) \, dt \\
\leq \sqrt{T} \| \mathcal{E}(U)_T \|_{L_2(\mathbb{P})} \sqrt{\mathbb{E} \int_0^T |\sigma^2 S_{t-} \partial_y G^*(t, S_{t-})|^2 \, dt} \\
+ \| \mathcal{E}(U)_T \|_{L_2(\mathbb{P})} \sqrt{T} \int_{\mathbb{R}} (e^x - 1)^2 \nu(dx) \sqrt{\mathbb{E} \int_0^T |\mathcal{D}_f G^*(t, x)|^2 \nu(dx) \, dt} \\
< \infty.
\]
By assumption, it is clear that \((G^*(t, e^x S_t) - G^*(t, S_t))_{t \in [0,T]}\) is a \(\PP^*\)-martingale for each \(x \in \RR\). In the case \(\sigma > 0\), due to \(g(S_T) \in L_2(\PP^*)\) and Lemma 3.1, \((S_t \partial_y G^*(t, S_t))_{t \in [0,T]}\) is also a \(\PP^*\)-martingale. Hence, the function
\[
[0, T) \ni t \mapsto \mathbb{E}^\ast[\sigma^2 S_t \partial_y G^*(t, S_t)] + \mathbb{E}^\ast \int_\RR |G^*(t, e^x S_t) - G^*(t, S_t)||e^x - 1|\nu(dx)
\]
is non-decreasing by the martingale property. In addition, noting that \(S_t = S_t\) a.s. for each \(t \in [0, T]\), we infer from (4.14) and Fubini's theorem that
\[
\mathbb{E}^\ast[\sigma^2 S_t \partial_y G^*(t, S_t)] + \mathbb{E}^\ast \int_\RR |G^*(t, e^x S_t) - G^*(t, S_t)||e^x - 1|\nu(dx) < \infty
\]
for all \(t \in [0, T]\). Therefore,
\[
\left(\frac{1}{\kappa(\sigma, \nu)} \left(\sigma^2 S_t \partial_y G^*(t, S_t) + \int_\RR (G^*(t, e^x S_t) - G^*(t, S_t))(e^x - 1)\nu(dx)\right)\right)_{t \in [0,T]}
\]
is a \(\PP^*\)-martingale for which one can find a càdlàg modification, denoted by \(\varphi^y\). Then, the process \(\tilde{\varphi}^y\) defined by
\[
\tilde{\varphi}^y := \varphi^y / S
\]
satisfies the desired requirements.

\(\blacksquare\)

**Remark 4.5.** Let \(\tilde{\varphi} \in \text{CL}([0, T))\) such that \(\tilde{\varphi} = \tilde{\varphi}^y\) for \(\PP \otimes \lambda\)-a.e. \((\omega, t) \in \Omega \times [0, T), \) where \(\tilde{\varphi}^y\) given in (4.15). Then, \(\PP(\tilde{\varphi}_t = \tilde{\varphi}^y_t, \ \forall t \in [0, T]) = 1\) due to the càdlàg property. Hence, \(\tilde{\varphi}_-\) is also a LRM strategy of \(H = g(S_T)\). Moreover, for any \(t \in [0, T]\),
\[
\tilde{\varphi}_t = \frac{1}{\kappa(\sigma, \nu)} \left(\sigma^2 S_t \partial_y G^*(t, S_t) + \int_\RR (G^*(t, e^x S_t) - G^*(t, S_t))(e^x - 1)\nu(dx)\right) \text{ a.s.}
\]

5. **Discrete-time hedging via weighted BMO-approach**

This section is a continuation of the work in [40] in the exponential Lévy models. First, we use the discrete-time approximation for stochastic integrals introduced in [40] and investigate the resulting error in weighted BMO spaces. Consequently, the \(L_p\)-estimates, \(p \in (2, \infty)\), for the error are provided. Secondly, to illustrate the obtained results, the approximated stochastic integrals we consider is the integral term, which can be interpreted as the hedgeable part, in the FS decomposition of a European type option.

### 5.1. Weighted bounded mean oscillation (BMO)

Let \(S([0, T])\) denote the family of all stopping times \(\rho : \Omega \rightarrow [0, T]\). Set \(\inf \emptyset := \infty\).

#### Definition 5.1 ([15, 16])

1. Let \(\text{BMO}_p^\ast(\PP)\) consist of all \(Y \in \text{CL}_0([0, T])\) with \(\|Y\|_{\text{BMO}_p^\ast(\PP)} < \infty\), where
\[
\|Y\|_{\text{BMO}_p^\ast(\PP)} := \inf \left\{ c \geq 0 : \mathbb{E}_{\mathcal{F}_p}[|Y_T - Y_0| - c^p] \leq c^p \Phi_p \text{ a.s., } \forall \rho \in S([0, T]) \right\}.
\]

2. (Weight regularity) We let \(\Phi \in \text{SM}_p(\mathcal{P})\) if \(\|\Phi\|_{\text{SM}_p(\mathcal{P})} < \infty\), where
\[
\|\Phi\|_{\text{SM}_p(\mathcal{P})} := \inf \left\{ c \geq 0 : \mathbb{E}_{\mathcal{F}_p}[\sup_{t \in [0,T]}|\Phi(t)|^p] \leq c^p \Phi_p \text{ a.s., } \forall a \in [0, T] \right\}.
\]

The theory of non-weighted BMO-martingales, i.e., when \(\Phi \equiv 1\) and \(Y\) is a martingale, can be found in [33, Ch.IV]. Remark that the weighted BMO spaces above were introduced and discussed in [15] for general càdlàg processes which are not necessarily martingales.

#### Definition 5.2 ([15])

For \(s \in (1, \infty)\), we denote by \(\mathcal{R}H_s(\mathcal{P})\) the family of all probability measures \(\mathcal{Q}\) equivalent to \(\mathcal{P}\) such that \(d\mathcal{Q}/d\mathcal{P} =: U \in L_s(\mathcal{P})\) and there exists a constant \(c_{(5.1)} > 0\) such that \(U\) satisfies the following reverse Hölder inequality
\[
\mathbb{E}_{\mathcal{F}_p}[U^s] \leq c_{(5.1)}^s \left(\mathbb{E}_{\mathcal{F}_p}[U]^s\right) \text{ a.s., } \forall \rho \in S([0, T]).
\]
We refer the reader to [15, 16] for further properties of those quantities. Proposition 5.3 below recalls some features of weighted BMO which are crucial for our applications, and their proofs can be found in [16, Proposition A.6] and [40, Proposition 2.5].

**Proposition 5.3** ([16, 40]). Let \( p \in (0, \infty) \) and \( \Phi \in \text{CL}^+([0,T]) \).

1. For any \( r \in (0, \infty) \), there is a \( c_1 = c_1(r, p) > 0 \) such that \( \| \cdot \|_{S_p(\mathbb{P})} \leq c_1 \| \Phi \|_{S_p(\mathbb{P})} \| \cdot \|_{\text{BMO}^p(\mathbb{P})} \).
2. If \( \Phi \in SM_p(\mathbb{P}) \), then for any \( r \in (0, p] \) there exists a \( c_2 = c_2(r, p, \| \Phi \|_{SM_p(\mathbb{P})}) > 0 \) such that \( \| \cdot \|_{\text{BMO}^p(\mathbb{P})} \leq c_2 \| \cdot \|_{\text{BMO}^p(\mathbb{P})} \).
3. If \( \mathbb{Q} \in \mathcal{RH}_s(\mathbb{P}) \) for some \( s \in (1, \infty) \) and \( \Phi \in SM_p(\mathbb{P}) \), then there exists a \( c_3 = c_3(s, p) > 0 \) such that \( \| \cdot \|_{\text{BMO}^p(\mathbb{Q})} \leq c_3 \| \cdot \|_{\text{BMO}^p(\mathbb{P})} \).

### 5.2. Setting.

We assume **Subsection 4.1** from now until the end of **Section 5**, i.e., \( S = e^X \) is a square integrable exponential Lévy process under \( \mathbb{P} \) with \( (X|\mathbb{P}) \sim (\gamma, \sigma, \nu) \) and \( \sigma^2 + \nu(\mathbb{R}) \in (0, \infty) \).

Then, as aforementioned in **Subsection 2.4**, there is a Lévy process \( Z \) such that \( S \) is the stochastic exponential of \( Z \), i.e.,

\[
dS_t = S_t - dZ_t, \quad t \in [0, T].
\]

(5.2)

Let \( (Z|\mathbb{P}) \sim (\gamma_Z, \sigma, \nu_Z) \). Then, the square integrability of \( S \) implies \( \int_{\mathbb{R}} z^2 \nu_Z(dz) < \infty \).

The stochastic integral we are going to discretize is of the form

\[
\int_0^T \tilde{\vartheta}_t - dS_t = \int_0^T \tilde{\vartheta}_u - S_t - dZ_t
\]

for suitable \( \tilde{\vartheta} \in \text{CL}([0, T]) \).\(^3\) introduced in **Assumption 5.4** below.

### 5.3. Discrete-time approximation with jump correction scheme.

Although the martingale framework is convenient for handling the discrete-time hedging problem as one can simplify the selection of quadratic hedging strategies and reduce some technicalities, it is more natural to consider the problem in the semimartingale framework. In principle, one can switch the semimartingale setting to the martingale setting by an appropriate change of measure. However, in general, the convergence rates of the discretization error might be slower if one use Hölder’s inequality to switch distributional estimates back to the original semimartingale setting. The approach we use here is beneficial from the change of measure feature of weighted BMO where the convergence rates remain the same after a (suitable) change of measure. Here, we do not assume the (local) martingale property under \( \mathbb{P} \) for the underlying price process \( S \). The idea is as follows: For \( \tilde{\vartheta} \) as in (5.3), one looks for a suitable equivalent probability measure \( \mathbb{P} \sim \mathbb{P} \) such that the process \( M := \tilde{\vartheta}S \) is a square integrable martingale (not necessarily uniformly integrable) under \( \mathbb{P} \). We then exploit the \( \mathbb{P} \)-orthogonality of increments of \( M \), which naturally appear in the discretization of the stochastic integral, to achieve favorable estimates for the approximation error which possibly yields considerably good convergence rates. Here, \( S \) is not necessarily a (local) \( \mathbb{P} \)-martingale but a \( \mathbb{P} \)-semimartingale. One also notices that the obtained convergence rates are still measured under \( \mathbb{P} \). Thanks to Proposition 5.3(3), under satisfiable conditions, one can switch from \( \mathbb{P} \) back to the original measure \( \mathbb{P} \) so that the convergence rates under \( \mathbb{P} \) remain unchanged. Such a change of measure is in general not unique, and hence, we have a flexibility to choose an appropriate \( \mathbb{P} \) so that the problem is convenient to handle. In our situation considered later, one can choose \( \mathbb{P} \) such that \( S \) is also a (local) \( \mathbb{P} \)-martingale.

Let us introduce the main assumption.

\(^3\)We use the tilde sign to indicate the càdlàg property of the applied process.
Assumption 5.4. For \( \hat{\vartheta} \in \mathcal{CL}([0, T)), \) \( \theta \in (0, 1], \) an a.s. non-decreasing \( \Theta \in \mathcal{CL}^+([0, T]) \) and for a probability measure \( \hat{\mathbb{P}} \) equivalent to \( \mathbb{P}, \) we let

\[
\hat{\vartheta} \in \mathcal{A}(\theta, \Theta|\hat{\mathbb{P}})
\]

if the following conditions hold:

(a) \( \mathbb{E}\int_0^T \hat{\vartheta}_t^2 S_t^2 dt < \infty. \)
(b) \( M := \hat{\vartheta} S \in \mathcal{CL}([0, T)) \) is an \( L_2(\hat{\mathbb{P}})\)-martingale.
(c) (Growth condition) There are constants \( c(5.4), c(5.5) > 0, \) which might depend on \( \theta, \) such that

(i) Case \( \theta = 1: \) One has

\[
|\hat{\vartheta}_a| \leq c_{(5.4)} \Theta_a \ a.s., \forall a \in [0, T).
\]

(ii) Case \( \theta \in (0, 1): \) One has

\[
\mathbb{E}_{\hat{\mathbb{P}}} \left[ \int_{(a,T)} (T - t)^{-\theta} \hat{\vartheta}_t^2 S_t^2 dt \right] \leq c_{(5.5)}^2 \Theta_a^2 \ a.s., \forall a \in [0, T).
\]

(d) \( Z \) is an \( L_2(\hat{\mathbb{P}})\)-semimartingale with the canonical representation

\[
Z_t = \tilde{Z}_0 + \int_0^t b_t^Z \, ds + \sigma W_t^\hat{\mathbb{P}} + \int_0^t \int_{\mathbb{R}} z(N_Z - \pi^Z_t)(dz, ds), \quad t \in [0, T],
\]

where \( \tilde{Z}_0 \in \mathbb{R}, W_t^\hat{\mathbb{P}} \) is a Brownian motion under \( \hat{\mathbb{P}}, \) \( \pi^Z_t \) is the predictable \( \hat{\mathbb{P}}\)-compensator of the jump measure \( N_Z \) of \( Z \) such that

\[
\pi^Z_t(\omega, dz, dt) = \nu^Z_t(t, dz) dt, \quad (\omega, t, z) \in \Omega \times [0, T] \times \mathbb{R}
\]

where \( \nu^Z_t(t, \cdot), t \in [0, T], \) are Lévy measures, and that

\[
\nu((5.6)) := \left\| t \mapsto \int_{\mathbb{R}} z^2 \nu^Z_t(t, dz) \right\|_{L_\infty([0, T], \lambda)} < \infty,
\]

and \( \tilde{b}^Z \) is a measurable and deterministic function with

\[
\tilde{b}((5.7)) := \left\| \tilde{b}^Z \right\|_{L_\infty([0, T], \lambda)} < \infty.
\]

Let us briefly comment on the conditions (a)–(c) above:

- Since càdlàg functions have at most countable many discontinuities, condition (a) ensures that \( \hat{\vartheta}_- \) is an admissible integrand, i.e., \( \hat{\vartheta}_- \in \mathcal{A}(S|\mathbb{P}) \) for \( \mathcal{A}(S|\mathbb{P}) \) given in (4.2).
- Apparently, condition (b) looks unnatural, however, this is typically satisfied in applications where \( \hat{\vartheta}_- \) is the hedging strategies of European options derived from the quadratic hedging approach. We clarify this fact in Example 5.5 below.
- The parameter \( \theta \in (0, 1] \) in (c) is related to the growth property of \( \hat{\vartheta} \) (relatively to the corresponding weights and measures) when the time variable approaches \( T. \) Remark that the bigger \( \theta \) is, the less singular \( \hat{\vartheta} \) has at \( T, \) which leads the approximated stochastic integral to be more regular. In particular, in the Black–Scholes model where \( \hat{\vartheta} \) is the delta-hedging strategy of a European payoff, \( \theta \) can be interpreted as the fractional smoothness of the payoff in the sense of [17] in which \( \theta = 1 \) corresponds to the smoothness of order 1. Therefore, in this article we call the situation where (c) holds for \( \theta = 1 \) as regular regime. Further discussion on (c) is given in Proposition 5.6.

Example 5.5 (Exponential Lévy model). Consider a European payoff \( g(S_T). \)

1. ( MVH and delta hedging in the martingale framework) Assume that \( S \) is a \( \mathbb{P}\)-martingale. Analogously to the Black–Scholes model, the delta hedging strategy in the exponential Lévy model is defined by \( \vartheta_t = \frac{\partial G}{\partial S}(t, S_t) \) provided that the derivative exists, where \( G(t, y) := \mathbb{E}_\mathbb{P}(g(y S_{T-t}) \) for \( (t, y) \in [0, T] \times (0, \infty). \) Although being not optimal in general, the delta hedging is often used by financial market practitioners. Then, it is shown in [40, Theorem
4.2] that under an integrability condition, both delta hedging and MVH strategies satisfy Assumption 5.4(b) with $\mathbb{P} = \mathbb{P}$.

(2) *(LRM in the semimartingale framework)* Under the assumptions of Theorem 1.1, if $\tilde{\vartheta}$ is a càdlàg modification for the LRM strategy of $g(S_T)$, then Theorem 1.1(2) verifies that $M := \tilde{\vartheta}S$ is a càdlàg $\mathbb{P}^*$-martingale. In particular, in the application when $g$ is bounded or Hölder/Lipschitz continuous, then $M$ is square integrable under $\mathbb{P}^*$, see Theorem 5.15(3) below. Hence, Assumption 5.4(b) holds for $\tilde{\vartheta}$ and $\mathbb{P} = \mathbb{P}^*$.

**Proposition 5.6.** (1) For $0 < \theta' < \theta < 1$, one has $A(\theta, \Theta|\mathbb{P}) \subseteq A(\theta', \Theta|\mathbb{P})$.

(2) If $M = \tilde{\vartheta}S$ is an $L_2(\mathbb{P})$-martingale and $\theta \in (0, 1)$, then (5.5) is equivalent to the following system of two inequalities:

\[
|\tilde{\vartheta}_a| \leq c(5.8)(T - a)^{\theta - 1} \Theta_a \quad \text{a.s. for all } a \in [0, T); \quad (5.8)
\]

\[
\mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - t)^{-\theta} d(M^\mathbb{P}_t) \right] \leq c(5.9) \Theta^2_a S^2_a \quad \text{a.s. for all } a \in [0, T). \quad (5.9)
\]

Here, $c(5.8), c(5.9) > 0$ are constants independent of $a$.

(3) Assume Items (a), (b), (d) in Assumption 5.4. If $\Theta S \in \mathcal{S}M_2(\mathbb{P})$ and if there exist a constant $c(5.10) > 0$ and a measurable function $w : [0, T) \to [0, \infty)$ such that

\[
|\tilde{\vartheta}_t| \leq c(5.10)w(t)\Theta_t \quad \text{a.s., } \forall t \in [0, T) \quad (5.10)
\]

and that $I_w := \{\delta \in (0, 1) : \int_0^T (T - t)^{-\delta} w(t)^2 dt < \infty\} \neq \emptyset$. Then $\tilde{\vartheta} \in \cap_{0 < \theta' < \theta} A(\theta', \Theta|\mathbb{P})$ for $\theta := \sup I_w$.

**Proof.** (1) is obvious. (2) Let $\theta \in (0, 1)$ and assume that $M = \tilde{\vartheta}S$ is an $L_2(\mathbb{P})$-martingale. For $a \in [0, T)$, by the orthogonality and Fubini’s theorem, we get, a.s.,

\[
\mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - t)^{-\theta} \tilde{\vartheta}_t^2 S_t^2 dt \right] = \mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - t)^{-\theta} (M_t - M_a)^2 dt \right] + M_a^2 \int_a^T (T - t)^{-\theta} dt
\]

\[
= \mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - t)^{-\theta} \int_{(a,t]} d(M_u^\mathbb{P}_t) dt \right] + M_a^2 \frac{(T - a)^{1-\theta}}{1-\theta}
\]

\[
= \frac{1}{1-\theta} \left[ \mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - u)^{1-\theta} d(M_u^\mathbb{P}_t) \right] + (T - a)^{1-\theta} M_a^2 \right],
\]

which yields the equivalence as desired.

(3) For any $0 < \theta' < \theta = \sup I_w$, there exists $\delta \in (\theta', \theta]$ such that $\int_0^T (T - t)^{-\delta} w(t)^2 dt < \infty$. Since $\Theta S \in \mathcal{S}M_2(\mathbb{P})$, it holds that, for $a \in [0, T)$, a.s.,

\[
\mathbb{E}^{\mathbb{P}}_{\mathcal{F}_a} \left[ \int_{(a,T)} (T - t)^{-\theta} \tilde{\vartheta}_t^2 S_t^2 dt \right] \leq \frac{c(5.10)}{\mathcal{S}M_2(\mathbb{P})} \Theta_S^2 S_a \int_a^T (T - t)^{-\delta} w(t)^2 dt
\]

\[
\leq \frac{c(5.10)}{\mathcal{S}M_2(\mathbb{P})} \Theta_S^2 \int_0^T (T - t)^{-\delta} w(t)^2 dt \Theta_a^2 S_a^2,
\]

which then implies $\tilde{\vartheta} \in A(\theta', \Theta|\mathbb{P})$.

\[\square\]

5.3.1. *Deterministic discretization times.* Let $\mathcal{T}_{\text{det}}$ denote the family of all deterministic time-nets $\tau = (t_i)_{i=0}^n$, $0 = t_0 < t_1 < \cdots < t_n = T$, $n \geq 1$. The mesh size of $\tau = (t_i)_{i=0}^n \in \mathcal{T}_{\text{det}}$ associated with a parameter $\theta \in (0, 1]$ is defined by

\[
\|\tau\|_\theta := \max_{i=1,\ldots,n} \frac{t_i - t_{i-1}}{(T - t_{i-1})^{1-\theta}} = \max_{i=1,\ldots,n} \sup_{a \in [t_{i-1}, t_i]} \frac{t_i - a}{(T - a)^{1-\theta}}.
\]
Let \( \tau_n \in \mathcal{T}_{\text{det}} \) with \( \# \tau_n = n + 1 \). By a short calculation we find that \( \| \tau_n \|_\theta \geq T^\theta / n \). Minimizing \( \| \tau_n \|_\theta \) over \( \tau_n \in \mathcal{T}_{\text{det}} \) with \( \# \tau = n + 1 \) leads us to the following time-nets, which were exploited in [12, 14, 15, 16, 40]: For \( \theta \in (0,1] \) and \( n \geq 1 \), the adapted time-net \( \tau_n^\theta = (t_i^\theta)_{i=0}^n \) is given by
\[
 t_i^\theta := T - T \left( 1 - \frac{i}{n} \right)^{1/\theta}, \quad i = 1, \ldots, n, \quad \text{and satisfies} \quad \frac{T^\theta}{n} \leq \| \tau_n \|_\theta \leq \frac{T^\theta}{\det i_1}, \quad n \geq 1. \quad (5.11)
\]

5.3.2. Approximation with jump correction. We recall from [40] the discrete-time approximation with the jump correction method. Let \( \tilde{\vartheta} \in \text{CL}([0,T]) \) such that \( \mathbb{E} \int_0^T \tilde{\vartheta}_t^2 S_t^2 \, dt < \infty \). The Riemann approximation \( A_{\text{Rm}}(\tilde{\vartheta}, \tau) \) of \( \int_0^T \tilde{\vartheta}_t \, dS_t \) along with the time-net \( \tau = (t_i)_{i=0}^n \in \mathcal{T}_{\text{det}} \) is given by
\[
 A_{\text{Rm}}(\tilde{\vartheta}, \tau) := \sum_{i=1}^n \tilde{\vartheta}_{t_{i-1}} (S_{t_{i-1}} - S_{t_{i-1} \land t}), \quad t \in [0,T].
\]

The following random times are employed to capture the relatively large jumps of \( S \): for \( \varepsilon > 0 \) and \( \kappa \geq 0 \), we define \( \rho(\varepsilon, \kappa) = (\rho_i(\varepsilon, \kappa))_{i \geq 0} \) by setting \( \rho_0(\varepsilon, \kappa) := 0 \) and
\[
 \rho_i(\varepsilon, \kappa) := \inf\{ T > t > \rho_{i-1}(\varepsilon, \kappa) : |\Delta S_t| > \varepsilon(T - t)^\kappa S_t \} \land T, \quad i \geq 1.
\]
Since \( Z \) is càdlàg and the considered filtration satisfies the usual conditions, it holds that \( \rho_i(\varepsilon, \kappa) \) are stopping times. The quantity \( \varepsilon(T - t)^\kappa \) is the threshold at the time \( t \) from which we decide which jumps of \( S \) are (relatively) large. If we normalize \( T = 1 \) and let \( t = 0 \), then \( \varepsilon > 0 \) can be regarded as the initial jump size threshold. Moreover, for \( \kappa > 0 \), this threshold continuously shrinks as \( t \uparrow T \), and thus the parameter \( \kappa \) indices the jump size decay rate. The reason for using the decay function \( \varepsilon(T - t)^\kappa \) is to compensate the growth of integrands.

**Definition 5.7 ([40]).** Let \( \varepsilon > 0 \), \( \kappa \in [0, \frac{1}{2}] \) and \( \tau = (t_i)_{i=0}^n \in \mathcal{T}_{\text{det}} \).

1. Let \( \tau \cup \rho(\varepsilon, \kappa) \) denote the combined time-net derived from combining \( \tau \) with \( \rho(\varepsilon, \kappa) \) and re-ordering their time-knots.
2. For \( t \in [0,T] \), we define
\[
 \tilde{\vartheta}_t^\rho := \sum_{i=1}^n \tilde{\vartheta}_{t_{i-1}} 1_{(t_{i-1}, t]}(t),
\]
\[
 A_{\text{corr}}^\rho(\tilde{\vartheta}, \tau|\varepsilon, \kappa) := A_{\text{Rm}}^\rho(\tilde{\vartheta}, \tau) + \sum_{\rho_i(\varepsilon, \kappa) \in [0,T] \cap [0,T]} \left( \tilde{\vartheta}_{\rho_i(\varepsilon, \kappa)} - \tilde{\vartheta}_{\rho_i(\varepsilon, \kappa)} \right) \Delta S_{\rho_i(\varepsilon, \kappa)}.
\]
\[
 E_{\text{corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) := \int_0^T \tilde{\vartheta}_u - dS_u - A_{\text{corr}}^\rho(\tilde{\vartheta}, \tau|\varepsilon, \kappa).
\]

The number of random jump times is denoted by
\[
 \mathcal{N}(\varepsilon, \kappa) := \inf\{ i \geq 1 : \rho_i(\varepsilon, \kappa) = T \}.
\]
Under the condition (5.6), we apply Proposition 5.10 with \( \alpha = 2 \) and \( \varepsilon_n = \varepsilon \) to conclude that \( \mathcal{N}(\varepsilon, \kappa) < \infty \) a.s. for any \( \varepsilon > 0 \) and \( \kappa \in [0, \frac{1}{2}] \). Hence, the sum in the definition of \( A_{\text{corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) \) is a finite sum a.s. Then, we adjust this sum on a set of probability zero so that one may assume that \( A_{\text{corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) \) and \( E_{\text{corr}}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) \) belong to \( \text{CL}_0([0,T]) \).

5.3.3. A general approximation result. There are three factors which affect the approximation rates: the presence of the diffusion component of \( Z \), the small jump activity of the underlying Lévy process, and the growth property of the integrand which is described by \( \theta \).

**Theorem 5.8.** Let \( \theta \in (0,1] \), \( \kappa := \frac{1-\alpha}{2} \) and \( \tilde{\vartheta} \in \text{A}(\theta, \Theta)[\mathbb{P}] \). Let \( \Phi \in \text{CL}^+(0,T] \) such that
\[
 \max \{ \Theta_{t} S_t, \Theta_{t-} S_{t-} \} \leq \Phi_t, \quad \forall t \in [0,T].
\]
Assume that \( \Phi \in \mathcal{S} M_2(\mathbb{P}) \) and that
\[
 c_{(5.12)}(\alpha) := \sup_{t \in (0,1]} r^\alpha \left\| \int_{|\nu| < |\zeta| < 1} v_{\nu}^\vartheta(t, dz) \right\|_{L_{\infty}(0,T], \lambda)} < \infty \quad \text{for some} \ \alpha \in (0,2]. \quad (5.12)
\]
Then the following assertions hold:
Theorem 5.8

The idea for the proof of Item (3)

Corollary 5.9

If \( \Phi \in S_p(\mathbb{P}) \) for some \( p \in (2, \infty) \), then (5.13) holds for \( \mathcal{X} = S_p(\mathbb{P}) \).

(2) (Switch back to \( \mathbb{P} \)) If \( \mathbb{P} \in \mathcal{RH}_s(\mathbb{P}) \) and \( \Phi \in SM_r(\mathbb{P}) \) for some \( s \in (1, \infty) \), \( r \in (0, 2] \), then (5.13) holds for \( \mathcal{X} = BMO^p_2(\mathbb{P}) \).

(4) (Switch back to \( \mathbb{P} \)) If \( \mathbb{P} \in \mathcal{RH}_s(\mathbb{P}) \) and \( \Phi \in SM_r(\mathbb{P}) \cap S_p(\mathbb{P}) \) for some \( s \in (1, \infty) \), \( r \in (0, 2] \), \( p \in [r, \infty) \), then (5.13) holds for \( \mathcal{X} = S_p(\mathbb{P}) \).

Proof. The idea for the proof of Item (1) follows closely [40, Proofs of Theorems 3.11 and 3.15]. Hence, we provide in Appendix A the essential steps of the proof. Items (2), (3) and (4) follow from applying Proposition 5.3 (Items 1 and 3).

A minimization procedure for the function \( H(\theta, \alpha, \varepsilon, \|\tau\|_\theta) \) over \( \varepsilon > 0 \) leads to the selection

\[
\varepsilon = \varepsilon(\theta, \alpha, \|\tau\|_\theta) := \begin{cases} 
\|\tau\|_\theta^{2(1-\kappa(\alpha-1))} & \text{if } \alpha \in [1, 2] \\
\|\tau\|_\theta & \text{if } \alpha \in (0, 1). 
\end{cases}
\]

Continuing to minimize \( H(\theta, \alpha, \varepsilon(\theta, \alpha, \|\tau\|_\theta), \|\tau\|_\theta) \) over \( \tau \in \mathcal{T}_\det \) with \( \#\tau = n + 1 \) yields to the adapted time-net \( \tau = \tau^\theta_n \). Therefore, we arrive at the following corollary.

Corollary 5.9 (Convergence rates in terms of discretization cardinality). Assume the assumptions of Theorem 5.8.

(1) For \( \mathcal{X} = BMO^p_2(\mathbb{P}) \) one has

\[
\sup_{n \geq 1} R(n) \| E^{\text{corr}}(\tilde{\theta}, \tau^\theta_n | \varepsilon_n, 1 - \theta) \|_{\mathcal{X}} < \infty, 
\]

where the rate of convergence \( R(n) \) and the initial jump size threshold \( \varepsilon_n \) are provided depending on \( \sigma \) as follows:

when \( \sigma = 0 \) one has

\[
R(n) = \begin{cases} 
1/\varepsilon_n = n \frac{1}{\alpha-1} & \text{if } \alpha \in (1, 2) \\
n/(1 + \log n), \varepsilon_n = 1/n & \text{if } \alpha = 1 \\
1/\varepsilon_n & \text{if } \alpha \in (0, 1),
\end{cases}
\]

when \( \sigma > 0 \) one has

\[
R(n) = \begin{cases} 
1/\varepsilon_n = n \frac{1}{\alpha-1} & \text{if } \alpha \in (0, 1) \\
1/\varepsilon_n = \frac{1}{\alpha} & \text{if } \alpha \in (0, \frac{3-\theta}{2}).
\end{cases}
\]

(2) (Switch back to \( \mathbb{P} \)) If \( \Phi \in S_p(\mathbb{P}) \) for some \( p \in (2, \infty) \), then (5.14) holds for \( \mathcal{X} = S_p(\mathbb{P}) \).

(3) (Switch back to \( \mathbb{P} \)) If \( \mathbb{P} \in \mathcal{RH}_s(\mathbb{P}) \) and \( \Phi \in SM_r(\mathbb{P}) \) for some \( s \in (1, \infty) \), \( r \in (0, 2] \), then (5.14) holds for \( \mathcal{X} = BMO^p_2(\mathbb{P}) \).

(4) (Switch back to \( \mathbb{P} \)) If \( \mathbb{P} \in \mathcal{RH}_s(\mathbb{P}) \) and \( \Phi \in SM_r(\mathbb{P}) \cap S_p(\mathbb{P}) \) for some \( s \in (1, \infty) \), \( r \in (0, 2] \), \( p \in [r, \infty) \), then (5.14) holds for \( \mathcal{X} = S_p(\mathbb{P}) \).

Proof. (1) The conclusions for \( R(n) \) and \( \varepsilon_n \) when \( \sigma = 0 \) are straightforward from the minimization argument for \( H(\theta, \alpha, \varepsilon, \|\tau\|_\theta) \) above, together with (5.11). For \( \sigma > 0 \), the minimization procedure for the denominator in (5.13) is given as follows in which the constants \( c_1, c_2 > 0 \) do not depend on \( n \).
For \( \alpha \in (0, \frac{3-\theta}{2-\theta}] \supset (0, 1] \), choose \( \varepsilon_n = \frac{1}{\sqrt{n}} \) to get that \( \sup_{n \geq 1} \sqrt{n}H(\theta, 0, \frac{1}{\sqrt{n}}, ||\tau_n^0||_\theta) < \infty \). Then, \( \max \{ \sigma \sqrt{||\tau_n^0||_\theta}, H(\theta, 0, \frac{1}{\sqrt{n}}, ||\tau_n^0||_\theta) \} \sim c_1 \frac{1}{\sqrt{n}} \), and this yields \( R(n) = \sqrt{n} \).

For \( \alpha \in (\frac{3-\theta}{2-\theta}, 2) \subset (1, 2] \), it is easy to check that \( \frac{1}{\alpha}(1 - \frac{1}{2}(1-\theta)(\alpha - 1)) \leq \frac{1}{2} \). Then, we choose \( \varepsilon_n = \frac{n^{-1/2}(1-\frac{1}{2}(1-\theta)(\alpha - 1))}{\alpha} \) to obtain that \( \max \{ \sigma \sqrt{||\tau_n^0||_\theta}, H(\theta, 0, \varepsilon_n, ||\tau_n^0||_\theta) \} \leq c_2 n^{-\frac{1}{2}(1-\frac{1}{2}(1-\theta)(\alpha - 1))} \).

Hence, \( R(n) = n^{\frac{1}{2}(1-\frac{1}{2}(1-\theta)(\alpha - 1))} \).

Items (2), (3) and (4) follow directly from the respective ones in Theorem 5.8. \( \square \)

In (5.14), to disclose the role of the parameter \( n \) in \( R(n) \) we need the following:

**Proposition 5.10.** If (5.12) holds for some \( \alpha \in (0, 2] \), then for any \( \theta \in (0, 1] \) and \( (\varepsilon_n)_{n \geq 1} \subset (0, \infty) \) with \( \inf_{n \geq 1} n\varepsilon_n > 0 \) there exists a constant \( c_{5.17} > 0 \) such that for any \( n \geq 1 \), \( \tau_n \in T_{\text{det}} \) with \( \#\tau_n = n + 1 \) one has

\[
\|\#\tau_n^0 \cup \rho(\varepsilon_n, \kappa)\|_{L_2(\mathbb{P})} \sim c_{5.17} n. \tag{5.17}
\]

**Proof.** The idea of the proof is similar to that in [40, Proposition 3.16], and hence, we omit the presentation here. \( \square \)

**Remark 5.11** (Role of the parameter \( n \) in (5.14)). The discretization time-nets employed in the approximation in (5.14) are \( \tau_n^0 \cup \rho(\varepsilon_n, \kappa) \) for any \( n \geq 1 \), with \( \kappa = \frac{1-\theta}{2} \in [0, \frac{1}{2}] \). Because of the randomness, a natural way to quantify their cardinality is to evaluate its expectation. This approach has been considered, for example, in [10] or [34, Eq. (10)] with \( \beta = 0 \).

1. In Proposition 5.10, it is shown that the \( L_2(\mathbb{P}) \)-norm of \( \tau_n^0 \cup \rho(\varepsilon_n, \kappa) \) is comparable to the cardinality of the deterministic time-net \( \tau_n^0 \) for any \( \varepsilon_n \) taken case-wise from (5.15, 5.16). It means that the argument \( n \) in the convergence rate \( R(n) \) can be regarded under \( \mathbb{P} \) as the 2-norm, and consequently, as any \( q \)-norm, \( q \in [1, 2) \), of the employed time-nets. Moreover, the relation (5.17) indicates that the approximation with jump correction scheme using the time-nets \( \tau_n^0 \cup \rho(\varepsilon_n, \kappa) \) does, in average, not increase the time complexity compared to the classical Riemann approximation with the respective deterministic discretization times \( \tau_n^0 \).

2. To quantify \( \tau_n^0 \cup \rho(\varepsilon_n, \kappa) \) under the original measure \( \mathbb{P} \), we have the following: If the density \( d\mathbb{P}/d\mathbb{P} \in L_r(\mathbb{P}) \) for some \( r \in (1, \infty) \), then

\[
\|\#\tau_n^0 \cup \rho(\varepsilon_n, \kappa)\|_{L_2(1/2,L_2(\mathbb{P}))} \sim c n, \quad n \geq 1,
\]

for some \( c \geq 1 \) not depending on \( n \). This relation can be derived from using directly Hölder’s inequality together with (5.17). In particular, when \( r = 2 \), then the expected number of \( \tau_n^0 \cup \rho(\varepsilon_n, \kappa) \) under \( \mathbb{P} \) is up to a multiplicative positive constant, comparable to \( n \).

### 5.4. Discretization for LRM strategies with jump correction method.

#### 5.4.1. Hölder spaces and \( \alpha \)-stable-like measures.

Before tending to applications of Theorem 5.8, let us introduce in this part Hölder spaces where the payoff functions belong to and \( \alpha \)-stable-like measures which describes the small jump intensity of the underlying process.

**Definition 5.12.** (1) **(Hölder spaces)** For \( \eta \in [0, 1] \), let \( C^{0,\eta} \) be the family of all Borel functions \( g: (0, \infty) \to \mathbb{R} \) with \( |g|_{C^{0,\eta}} < \infty \), where

\[
|g|_{C^{0,\eta}} := \inf \{ c \geq 0 : |g(x) - g(y)| \leq c|x - y|^{\eta}, \forall x, y > 0, x \neq y \}.
\]

(2) For \( \alpha \in (0, 2] \), we let \( \ell \in \mathcal{US}(\alpha) \) if \( \ell \) is a Lévy measure with

\[
\sup_{r \in (0, 1)} r^\alpha \int_{r < |x| \leq 1} \ell(dx) < \infty.
\]

(3) **(\( \alpha \)-stable-like measures)** For \( \alpha \in (0, 2] \) and a Lévy measure \( \ell \), we let \( \ell \in \mathcal{S}(\alpha) \) if \( \ell = \ell_1 + \ell_2 \) for some Lévy measures \( \ell_1, \ell_2 \) which satisfy that

\[
\ell_1(dx) = \frac{k(x)}{|x|^{\alpha+1}} \mathbb{1}_{\{x \neq 0\}} dx \quad \text{and} \quad \ell_2 \in \mathcal{US}(\alpha),
\]
where
- \( k \geq 0 \) is right-continuous,
- \( \limsup_{x \to 0} k(x) < \infty \) and \( \liminf_{x \to 0} k(x) + \liminf_{x \to 0^+} k(x) > 0 \),
- \( x \mapsto |x|^{-\alpha} k(x) \) is non-decreasing on \((-\infty, 0)\) and non-increasing on \((0, \infty)\).

**Remark 5.13.** (1) It is obvious that \( C^{0,1} \) consists of Lipschitz continuous functions on \((0, \infty)\) and \( C^{0,0} \) contains bounded Borel functions which are *not* necessarily continuous.
(2) Evidently, any Lévy measure belongs to \( \mathcal{U}S(2) \).
(3) Observe that the function \( k \) in (5.18) is bounded on compact intervals. Therefore, for any \( \alpha \in (0, 2) \), one has \( \ell_1 \in \mathcal{U}S(\alpha) \), and then \( S(\alpha) \subset \mathcal{U}S(\alpha) \). Some further properties of \( \mathcal{U}S(\alpha) \) and \( S(\alpha) \) are given in Lemma B.1.

We now provide some examples in the context of financial modelling using Hölder functions and \( \alpha \)-stable-like measures above.

**Example 5.14.** (1) The European call and put payoff functions are Lipschitz, hence they belong to \( C^{0,1} \). For \( \eta \in (0, 1) \), the *power call* \( g_\eta(y) := ((y - K) \vee 0)\eta \) with strike \( K > 0 \) belongs to \( C^{0,\eta} \). The binary payoff function \( g_0(y) := 1_{(K, \infty)}(y) \) belongs to \( C^{0,0} \) obviously.

(2) The following examples are taken from [20]:
- A *CGMY process* with parameters \( C, G, M > 0 \) and \( Y \in (0, 2) \) has the Lévy measure
  \[
  \nu_{\text{CGMY}}(dx) = C e^{Gx} \mathbf{1}_{\{x < 0\}} + e^{-Mx} \mathbf{1}_{\{x > 0\}} \mathbf{1}_{\{x \neq 0\}} dx.
  \]
  According to Lemma B.1(3), \( \nu_{\text{CGMY}} \in \mathcal{S}(Y) \).
- A *Normal Inverse Gaussian* (NIG) process has a Lévy density \( p_{\text{NIG}}(x) := \nu_{\text{NIG}}(dx)/dx \) that satisfies
  \[
  0 < \liminf_{x \to 0} x^2 p_{\text{NIG}}(x) \leq \limsup_{x \to 0} x^2 p_{\text{NIG}}(x) < \infty.
  \]
  Hence, Lemma B.1(3) verifies that \( \nu_{\text{NIG}} \in \mathcal{S}(1) \).
- The Lévy measure \( \nu_- \) of a *spectrally negative Lévy process* is
  \[
  \nu_-(dx) = c \mathbf{1}_{\{x < 0\}} |x|^{-1-\alpha} dx
  \]
  for some \( c > 0 \) and \( \alpha \in (0, 2) \). Then Lemma B.1(3) shows that \( \nu_- \in \mathcal{S}(\alpha) \).

**5.4.2. Applications to LRM strategies.** This part provides a realization of Theorem 5.8 where the approximated stochastic integral is the integral term in the FS decomposition of \( g(S_T) \) (see Definition 4.1). Furthermore, we choose the càdlàg version \( \tilde{\gamma}^g \) of the LRM strategy, which is feasible due to Theorem 1.1(2), so that the integral we are going to approximate is of the form

\[
\int_0^T \tilde{\gamma}^g_t dS_t.
\]

Then, under the assumptions of Theorem 1.1, it follows from Remark 4.5 that, for \( t \in [0, T) \),

\[
\tilde{\gamma}^g_t = \frac{1}{\kappa(\sigma, \nu)} \left( \sigma^2 \partial_y G^*(t, S_t) + \int \frac{G^*(t, e^x S_t) - G^*(t, S_t)}{S_t} (e^x - 1) \nu(dx) \right) \quad \text{a.s.} \quad (5.19)
\]

The weight processes in this context are given by

\[
\Theta(\eta) := \sup_{u \in [0, \cdot]} |S_u^{-1}|, \quad \Psi(\eta) := \Theta(\eta) S, \quad \Phi(\eta) := \Psi(\eta) + \sup_{u \in [0, \cdot]} |\Delta \Psi(\eta)| u|.
\]

Regarding the coefficient \( \gamma_{(4, 1)} \), here we focus on the case \( \gamma_{(4, 1)} \neq 0 \) since the case \( \gamma_{(4, 1)} = 0 \), which corresponds to the martingale setting, has been investigated in [40, Section 4]. Recall from Definition 4.3 the minimal martingale measure \( \mathbb{P}^* \) of \( S \).

**Theorem 5.15.** Assume Assumption 4.4, \( \gamma_{(4, 1)} \neq 0 \) and \( \int_{x > 1} e^{2x} \nu(dx) < \infty \). Let \( \eta \in [0, 1] \) and \( g \in C^{0,\eta} \). Then the following assertions hold:
(1) Both \( \Psi(\eta) \) and \( \Phi(\eta) \) belong to \( \mathcal{SM}_3(\mathbb{P}) \cap \mathcal{SM}_2(\mathbb{P}^*) \).
(2) \( \mathbb{P}^* \in \mathcal{R}_3(\mathbb{P}) \) and \( \| \cdot \|_{\text{BMO}^2(x)(\mathbb{P}^*)} \leq c \| \cdot \|_{\text{BMO}^2(x)(\mathbb{P})} \) for some constant \( c > 0 \).

(3) In the notations of Assumption 5.4, one has \( \tilde{\vartheta}^0 \in \mathcal{A}(\theta, \Theta(\eta))\|\mathbb{P}^* \) for \( \vartheta \) given in Theorem 1.1 and for the parameter \( \theta \) case-wise provided in Table 1.

(4) For the adapted time-nets \( x_n^\theta \) given in (5.11) and for \( \mathcal{X} = \text{BMO}^2_{(p)}(\mathbb{P}^*) \), one has
\[
\sup_{n \geq 1} R(n) \| E^x_{\mathbb{P}^*}((\tilde{\vartheta}^0, \tau_n^\theta | x_n, \frac{1-x}{2})\|_\mathcal{X} < \infty,
\]
where \( \theta, R(n), \varepsilon_n \) are case-wise provided in Table 1.

(5) Let \( s \in (1, +\infty) \). Assume in addition that
\[
\begin{align*}
\int_{x < 1} e^{(1-s)x} \nu(dx) &< \infty \text{ and } \ln(1 + \frac{\kappa(\sigma, \nu)}{\gamma(\alpha)}) = x_0 \notin \supp \nu \text{ when } \frac{\kappa(\sigma, \nu)}{\gamma(\alpha)} \in (-1, \infty); \\
\int_{x < 1} e^{(1-s)x} \nu(dx) &< \infty \text{ when } \frac{\kappa(\sigma, \nu)}{\gamma(\alpha)} = -1.
\end{align*}
\]
Then, one has \( \mathbb{P} \in \mathcal{R}_a(\mathbb{P}^*) \), and there is a constant \( c' \geq 1 \) such that
\[
\| \cdot \|_{\text{BMO}^2(x)(\mathbb{P}^*)} \sim c' \| \cdot \|_{\text{BMO}^2(x)(\mathbb{P})}.
\]
Consequently, (5.20) holds for \( \mathcal{X} = \text{BMO}^2_{(p)}(\mathbb{P}) \) and for \( \mathcal{X} = L_3(\mathbb{P}) \).

(6) Under the assumptions of Item (5), if \( \int_{x > 1} e^{\theta x} \nu(dx) < \infty \) for some \( p \in (3, \infty) \), then (5.20) holds for any \( \mathcal{X} \in \{ \text{BMO}^2_{(p)}(\mathbb{P}^*), S_{p-1}(\mathbb{P}^*), \text{BMO}^1_{(p)}(\mathbb{P}), S_{p}(\mathbb{P}) \} \).

Table 1: Parameter \( \theta \), convergence rate \( R(n) \) and initial jump size threshold \( \varepsilon_n \)

| Interplay between \( g \) and \( X \) | \( \sigma = 0 \) and \( \nu \in \mathbb{L}(\alpha) \) for some \( (\eta, \alpha) \in (0, 1) \times (0, 1+\eta) \) \( \cup \{1\} \times (0, 2) \) | Values of \( \theta \) | \( R(n) \) and \( \varepsilon_n \) |
|---|---|---|---|
| C1 | \( \theta = 1 \) | \( R(n) = 1/\varepsilon_n = \sqrt{n} \) if \( \alpha \in (1, 2] \), \( R(n) = n/(1 + \log n) \) if \( \alpha = 1 \), \( R(n) = 1/\varepsilon_n = n \) if \( \alpha \in (0, 1) \) | |
| C2 | \( \forall \theta \in \left(0, \frac{2(1+n)}{\alpha} - 1\right) \) | \( R(n) = 1/\varepsilon_n = n^{\alpha/(1-\theta(1-\alpha))} \) if \( (\eta, \alpha) \neq (0, 1) \), \( R(n) = n/(1 + \log n) \) if \( (\eta, \alpha) = (0, 1) \) | |
| C3 | \( \sigma > 0 \) and \( \eta = 1 \) | \( \theta = 1 \) | \( R(n) = 1/\varepsilon_n = \sqrt{n} \) if \( \alpha \in (0, 1-\frac{\theta}{2\theta}) \), \( R(n) = 1/\varepsilon_n = n^{\alpha/(1-\theta(1-\alpha))} \) if \( \alpha \in (\frac{\theta}{2\theta}, 2] \) | |
| C4 | \( \sigma > 0, \eta \in (0, 1) \) and \( \nu \in \mathbb{L}(\alpha) \) \( \forall \theta \in (0, \eta) \) | \( R(n) = 1/\varepsilon_n = \sqrt{n} \) \( \text{ if } \alpha \in (0, 1-\frac{\theta}{2\theta}) \), \( R(n) = 1/\varepsilon_n = n^{\alpha/(1-\theta(1-\alpha))} \) \( \text{ if } \alpha \in (\frac{\theta}{2\theta}, 2] \) | |

Remark 5.16. The parameter \( \theta \), which describes the growth property of \( \tilde{\vartheta}^0 \), is the outcome of the interplay between the small jump activity of \( X \) and the Hölder regularity of the payoff function.

For the proof of Theorem 5.15, we need the following lemma.

Lemma 5.17. Under Assumption 4.4, the following assertions hold:

(1) For \( \alpha \in (0, 2] \), one has \( \nu \in \mathbb{L}(\alpha) \) if and only if \( \nu^* \in \mathbb{L}(\alpha) \).

(2) For \( \alpha \in (0, 2) \), one has \( \nu \in \mathbb{L}(\alpha) \) if and only if \( \tilde{\nu}^* \in \mathbb{L}(\alpha) \).

(3) Assume \( \gamma(\alpha) \neq 0 \). Then, for \( r \in [1, \infty) \) one has
\[
E e^{rx} \nu(dx) < \infty \forall t > 0 \Leftrightarrow \int_{|x| > 1} e^{rx} \nu(dx) < \infty 
\]
\[ \Leftrightarrow \int_{x>1} e^{(r-1)x} \nu^*(dx) < \infty \Leftrightarrow \int_{|x|>1} e^{(r-1)x} \nu^*(dx) < \infty \Leftrightarrow E^* e^{(r-1)X_t} < \infty, \forall t > 0. \]

**Proof.** We recall \(\nu^*(dx) = \left(1 - \frac{\gamma(4,1)}{\kappa(\sigma, \nu)}(e^x - 1)\right)\nu(dx) =: \left(\frac{d\nu^*}{d\nu}\right)(x) \nu(dx)\) from (4.7) and the set \(A\) from Assumption 4.4.

(1) follows directly from the fact that \(\lim_{x \to 0} \frac{d\nu^*}{d\nu}(x) = 1\) and \(\sup_{x \in (0,1)} \left(\frac{d\nu^*}{d\nu}\right)(x) < \infty\).

(2) Assume that \(\nu = \nu_1 + \nu_2 \in S(\alpha)\), where \(\nu_1, \nu_2\) are Lévy measures which satisfy (5.18) for \(\ell_i = \nu_i, i = 1, 2\). We define
\[
\nu_1^*(dx) := \nu^*(dx) - \nu_i^*(dx) = \mathbb{1}_{\{x \in A\}} \nu^*(dx) - \nu_i^*(dx)
\]
and set
\[
\nu_2^*(dx) := \nu^*(dx) - \nu_1^*(dx) = \mathbb{1}_{\{x \in A\}} \nu^*(dx) - \nu_i^*(dx)
\]
\[
\nu_2^*(dx) := \nu^*(dx) - \nu_i^*(dx) = \mathbb{1}_{\{x \in A\}} \nu^*(dx) - \nu_i^*(dx)
\]
\[
\text{and set } \nu_2^*(dx) := \nu^*(dx) - \nu_i^*(dx), \text{ an analogous argument as in the first implication yields } \nu = \nu_1 + \nu_2 \in S(\alpha).
\]

(3) follows from combining [35, Theorem 25.3] with the relation between \(\nu\) and \(\nu^*\). \(\Box\)

**Proof of Theorem 5.15.** Recall \((X|P^*) \sim (\gamma^*, \sigma, \nu^*)\) from (4.7) and \(A\) from Assumption 4.4.

(1) Combining Lemma 5.17(3) with Lemma B.3, we obtain that \(\Psi(\eta) \in SM_3(F) \cap SM_2(F^*)\). Thanks to Lemma B.2, one has \(\Psi(\eta) \in SM_3(F) \cap SM_2(F^*)\).

(2) We recall \(\mathcal{E}(U)\) from Definition 4.3 and notice that \(\mathcal{E}(U) > 0\) by Assumption 4.4. According to Subsection 2.4, there is a Lévy process \(V\) with \((V|F) \sim (\gamma_V, \sigma, \nu_V)\) such that \(\mathcal{E}(U) = e^V\).

Due to (4.5), by letting \(h(x) := \log(1 + x)\) for \(x > -1\) one has
\[
\nu_V = \nu_U \circ h^{-1} = (\nu \circ \alpha_U^{-1}) \circ h^{-1} = \nu \circ (h \circ \alpha_U)^{-1}.
\]

Since \(h(\alpha_U(x)) = \log(1 + \gamma(4,1)e^x)\) for \(x \in A\), there is an \(\varepsilon(5,22) > 0\) such that
\[
\{x \in A : |h(\alpha_U(x))| > 1\} \subseteq A \setminus (-\varepsilon(5,22), \varepsilon(5,22)).
\]

Then, the assumption \(\int_{|x|>1} e^{3x} \nu(dx) < \infty\) implies that
\[
\int_{|x|>1} e^{3x} \nu_V(dx) = \int_{|h(\alpha_U(x))|>1} e^{h(\alpha_U(x))} \nu(dx)\]
\[
\leq \int_{A \setminus (-\varepsilon(5,22), \varepsilon(5,22))} \left(1 - \frac{\gamma(4,1)}{\kappa(\sigma, \nu)}(e^x - 1)\right)^3 \nu(dx) < \infty.
\]
Let \((V|\mathbb{F}) \sim \psi_V\). Since \((e^{3V_t+t\psi_V(-3)t})_{t \in [0,T]}\) is a càdlàg martingale, it follows from the optional stopping theorem that for any stopping time \(\rho: \Omega \to [0,T]\), a.s.,
\[
\mathbb{E}_{\mathbb{F}_\rho}[e^{3V_T}] = e^{-T\psi_V(-3)}\mathbb{E}_{\mathbb{F}_\rho}[e^{3V_T+T\psi_V(-3)}] = e^{-T\psi_V(-3)}e^{3V_\rho+\rho\psi_V(-3)} \\
\leq e^{T\psi_V(-3)}e^{3V_\rho} = e^{T\psi_V(-3)}\left|\mathbb{E}_{\mathbb{F}_\rho}[e^{V_T}]\right|^3,
\]
where we use the martingale property of \(e^V\) for the last equality. According to Definition 5.2, \(d\mathbb{P}^* = e^{V_T}d\mathbb{P} \in \mathcal{RH}_3(\mathbb{F})\). Hence, Proposition 5.3(3) yields \(\|\cdot\|_{\text{BMO}^2(\nu^*)} \leq c\|\cdot\|_{\text{BMO}^2(\eta^*)}\).

(3) Since the function \(g \in C^{0,\eta}\) has at most linear growth at infinity and \(\int_{|x|>1} e^{3x}\nu(dx) < \infty\), which is equivalent to \(\int_{|x|>1} e^{2x}\nu^*(dx) < \infty\) by Lemma 5.17(3), the assumptions of Theorem 1.1 are satisfied so that (5.19) is applicable. We now verify Assumption 5.4 with \(\mathbb{P} = \mathbb{P}^*\).

- For Item (d), we notice that \(S\) is an \(L_2(\mathbb{P}^*)\)-martingale. Hence, the stochastic logarithm \(Z\) appeared in (5.2) is a Lévy process under \(\mathbb{P}^*\) and is also an \(L_2(\mathbb{P}^*)\)-martingale. Denote
\[
(Z|\mathbb{P}^*) \sim (\gamma_Z^*, \sigma, \nu_Z^*).
\]
Then, Item (d) is satisfied for \(Z^0 = 0, b^Z = 0, W^\mathbb{P}^* = W^*\) and \(\pi^* = \lambda \otimes \nu_Z^*\).

- For Item (e), we aim to apply Proposition B.4 with the selections
\[
\mathcal{Q} = \mathbb{P}^* \quad \text{and} \quad \ell = \nu
\]
so that, for any \(t \in [0,T]\), (5.19) gives
\[
|\tilde{\nu}_\ell^t| = (\kappa(\sigma, \nu))^\cdot |I^\nu^*(T-t, S_t)| \quad \text{a.s.}
\]
Let us examine each case in Table 1. One recalls that, thanks to Lemma 5.17, \(\nu \in \mathfrak{US}(\alpha) \Leftrightarrow \nu^* \in \mathfrak{US}(\alpha)\) and \(\nu \in \mathfrak{S}(\alpha) \Leftrightarrow \nu^* \in \mathfrak{S}(\alpha)\).

For C1, the given range of \((\eta, \alpha)\) yields \(\int_{|x|<1} |x|^{1+\eta}\nu(dx) < \infty\). Thus, Proposition B.4(2) gives
\[
|\tilde{\nu}_\ell^t| \leq (\kappa(\sigma, \nu))^{-1}c_{B.2}S_{\ell,t}^{-\gamma} \leq (\kappa(\sigma, \nu))^{-1}c_{B.2}\Theta(t)^{-} \quad \text{a.s.},
\]
which shows that (c) holds with \(\theta = 1\) and \(\Theta = \Theta(\eta)\).

For C2, when \((\eta, \alpha) \in [0,1) \times \{1+\eta, 2\}\) (resp. \((\eta, \alpha) \in [0,1) \times \{1+\eta\}\)) we apply Proposition B.4(3) to find that (5.10) is satisfied for \(\Theta = \Theta(\eta)\) and \(w(t) = (T-t)^{\frac{1}{1+\eta}}\) (resp. \(w(t) = \max\{1, \log \frac{1}{t}\}\)). Hence, applying Proposition 5.6(3) yields \(\tilde{\nu}^g \in \mathfrak{g} \cap \{0 < \theta \log_{\frac{1}{2}} \gamma \mathfrak{A}(\theta, \Theta(\eta))|\mathbb{P}^*\}\) for any \((\eta, \alpha) \in [0,1) \times \{1+\eta\}\).

For C3, we use Proposition B.4(1) to get \(\theta = 1\) and \(\Theta = \Theta(\eta)\).

For C4, we apply Proposition B.4(1) again to find that (5.10) is satisfied for \(\Theta = \Theta(\eta)\) and \(w(t) = (T-t)^{\frac{1}{2}}\). Then, Proposition 5.6(3) shows \(\tilde{\nu}^g \in \mathfrak{g} \cap \{0 < \theta \mathfrak{A}(\theta, \Theta(\eta))|\mathbb{P}^*\}\).

- For Item (b), \(\tilde{\nu}^S\) is a \(\mathbb{P}^*\)-martingale because of Theorem 1.1(2). The square \(\mathbb{P}^*\)-integrability of \(\tilde{\nu}^S\) follows from the growth property of \(\tilde{\nu}\) in Item (c) and \(\Phi(\eta) \in \mathfrak{SM}_2(\mathbb{P}^*)\).

- For Item (a) follows from combining the growth property of \(\tilde{\nu}\) with \(\Phi(\eta) \in \mathfrak{SM}_2(\mathbb{F}) \subset \mathfrak{SM}_2(\mathbb{P})\).

(4) Our purpose is to apply Corollary 5.9 with \(\mathbb{P} = \mathbb{P}^*\), \(\Theta = \Theta(\eta)\) and \(\Phi = \Phi(\eta)\). We check the assumptions of Theorem 5.8. By definition, it is clear that
\[
\max\{\Theta(\eta)^{-}, \Theta(\eta)S\} \leq \Phi(\eta)
\]
and that \(\Phi(\eta) \in \mathfrak{SM}_2(\mathbb{P}^*)\) according to Item (1). Condition (5.12) in this context simply means that \(\nu_Z^* \in \mathfrak{US}(\alpha)\). Since \(\nu_Z^* = \nu^* \circ h^{-1}\) for \(h(x) := e^x - 1\), it is straightforward that
\[
\nu_Z^* \in \mathfrak{US}(\alpha) \Leftrightarrow \nu^* \in \mathfrak{US}(\alpha) \Leftrightarrow \nu \in \mathfrak{US}(\alpha),
\]
where the second equivalence is due to Lemma 5.17. In Table 1, the conclusions for \(R(n)\) and \(\varepsilon_n\) in the cases C1 and C2 (resp. cases C3 and C4) follow directly from (5.14) and (5.15) (resp. (5.16)).
(5) Step 1. For $\nu_V$ given in (5.21), we first show that $\int_{|x|>1} e^{(1-s)x} \nu_V(dx) < \infty$. Indeed,

$$\int_{|x|>1} e^{(1-s)x} \nu_V(dx) = \int_{|\nu_V(x)|>1} \left(1 - \frac{\gamma(4.1)(e^x-1)}{\kappa(\sigma,\nu)}\right)^{1-s} \nu(dx) \leq \int_{A}\left(1 - \frac{\gamma(4.1)(e^x-1)}{\kappa(\sigma,\nu)}\right)^{1-s} \nu(dx) =: I(5.23). \quad (5.23)$$

We consider three cases for $\kappa(\sigma,\nu)$ as follows:

- Case 1: $\kappa(\sigma,\nu) > 0$. Since $x \notin \text{supp } \nu$ by assumption, it implies that $\nu((x_0 - \varepsilon_0, x_0 + \varepsilon_0)) = 0$ for some $\varepsilon_0 > 0$. Moreover, one has $1 - \frac{\gamma(4.1)(e^x-1)}{\kappa(\sigma,\nu)} \geq |x - x_0|^{-\frac{\gamma(4.1)}{\kappa(\sigma,\nu)}}$ for all $x \in A$ by the mean value theorem. Hence,

$$I(5.23) = \int_{|x-x_0| \geq \varepsilon_0, x \in \Omega(-\varepsilon, \varepsilon)} \left(1 - \frac{\gamma(4.1)(e^x-1)}{\kappa(\sigma,\nu)}\right)^{1-s} \nu(dx),$$

$$\leq \varepsilon_0^{-1-s} \frac{\gamma(4.1)}{\kappa(\sigma,\nu)} \int_{|x-x_0| \geq \varepsilon_0, x \in \Omega(-\varepsilon, \varepsilon)} e^{(1-s)x} \nu(dx) < \infty,$$

where the finiteness is due to the assumption $\int_{x<-1} e^{(1-s)x} \nu(dx) < \infty$.

- Case 2: $\kappa(\sigma,\nu) = 0$. We have $I(5.23) = \int_{\Omega(-\varepsilon, \varepsilon)} e^{(1-s)x} \nu(dx) < \infty$.

- Case 3: $\kappa(\sigma,\nu) < 0$. In this case one has $\gamma(4.1) < 0$, which implies that $\inf_{x \in \mathbb{R}} \left(1 - \frac{\gamma(4.1)(e^x-1)}{\kappa(\sigma,\nu)}\right) = 1 + \frac{\gamma(4.1)}{\kappa(\sigma,\nu)} > 0$. Hence,

$$I(5.23) \leq \left(1 + \frac{\gamma(4.1)}{\kappa(\sigma,\nu)}\right)^{1-s} \int_{\Omega(-\varepsilon, \varepsilon)} \nu(dx) < \infty.$$

We conclude from three cases above that $\int_{|x|>1} e^{(1-s)x} \nu_V(dx) < \infty$, or equivalently,

$$e^{-t\psi_V((s-1)i)} = Ee^{(1-s)V_t} < \infty, \quad t > 0.$$

Step 2. We show $\mathbb{P} \in \mathcal{RH}_s(\mathbb{P})$. Indeed, since $d\mathbb{P} = e^{-V_T} d\mathbb{P}^e$ and $e^V = E(U)$ is a $\mathbb{P}$-martingale, it holds that $e^{-V}$ is a $\mathbb{P}^e$-martingale. We have for any $t \in [0, T]$ that, a.s.,

$$E_{\mathbb{P}^e}[e^{s(-V_T)}] = e^{-V_t} E_{\mathbb{P}}[e^{-sV_T} e^{V_T}] = e^{-V_t} E_{\mathbb{P}}[e^{(1-s)V_T}] \leq e^{T\psi_V((s-1)i)} e^{-sV_t}.$$

By a standard approximation argument, we infer that

$$E_{\mathbb{P}^e}[e^{s(-V_T)}] \leq e^{T\psi_V((s-1)i)} e^{-sV_t} = e^{T\psi_V((s-1)i)} \left|E_{\mathbb{P}}[e^{-V_T}]\right|^{s} \quad \text{a.s.}$$

for any stopping times $\rho: \Omega \to [0, T]$. Therefore, $\mathbb{P} \in \mathcal{RH}_s(\mathbb{P}^e)$.

Step 3. Thanks to Step 2 and Items (1), (2), we apply Proposition 5.3(3) with $\mathbb{Q} = \mathbb{P}^e$ and $p = 2$ to obtain

$$\| \cdot \|_{\text{BMO}^2(\mathbb{Q})} \sim_c \| \cdot \|_{\text{BMO}^2(\mathbb{P})},$$

The “Consequently” part is straightforward because of $\Phi(\eta) \in \mathcal{SM}(\mathbb{P})$ and Proposition 5.3(1).

(6) An analogous argument as in the proof of Item (1) shows that both $\Psi(e)$ and $\Phi(e)$ belong to $\mathcal{SM}_p(\mathbb{P}) \cap \mathcal{SM}_{p-1}(\mathbb{P}^e)$. We now apply Proposition 5.3(2) with $r = 2$ and then combine this with Item (5) to derive the assertion. \qed
APPENDIX A. PROOF OF THEOREM 5.8, ITEM (1)

In this proof, we often use the fact that a càdlàg (left-continuous with right limits) function has countably many discontinuities which then implies that when integrating such a function with respect to the Lebesgue measure we may use its càdlàg version without changing the value of the integral. The proof is divided in the following steps.

**Step 1:** Growth in time and no fixed-time discontinuities of \( \tilde{\vartheta} \). For \( 0 \leq a < T \), a.s.,

\[
|\tilde{\vartheta}_a| \leq \begin{cases} 
  c(A.4) \Theta_a & \text{if } \theta = 1 \\
  c(5.8)(T - a)^{\frac{\theta - 1}{\theta}} \Theta_a & \text{if } \theta \in (0, 1),
\end{cases}
\]  

(A.1)

where the case \( \theta \in (0, 1) \) holds because of Proposition 5.6(2) (here, by assumption, \( M = \tilde{\vartheta}S \) is an \( L_2(\mathbb{P}) \)-martingale), and where \( c(A.1) := c(5.4) \lor c(5.8) \). Moreover, it follows from the monotonicity of \( \Theta \) and the càdlàg property of \( \tilde{\vartheta}, \Theta \) that

\[
|\tilde{\vartheta}_a - \tilde{\vartheta}_a'| \leq 2c(A.1)(T - t)^{\frac{\theta - 1}{\theta}} \Theta_t \quad \text{for all } 0 \leq a < t < T \text{ a.s.} 
\]  

(A.2)

On the other hand, since \( \tilde{\vartheta}S \) is a martingale adapted to the natural completed filtration of a Lévy process, which is a quasi-left continuous filtration (see, e.g., [33, p.150, Exercise 9]), it follows from [33, p.190] that \( \tilde{\vartheta}S \) has no fixed-time of discontinuities, i.e., \( \tilde{\vartheta}_tS_t = \tilde{\vartheta}_{t-}S_{t-} \) a.s. for \( t \in [0, T) \). Since \( \tilde{S}_t \) is a semimartingale and \( \tilde{\vartheta}_t = \tilde{\vartheta}_{t-} \) a.s. for \( t \in [0, T) \). Hence, \( \tilde{\vartheta} \) has no fixed-time of discontinuities. As a consequence, for any \( \tau = (t_n)_{n=0}^n \in \mathcal{T}_{\text{det}} \) one has, a.s.,

\[
\tilde{\vartheta}_a \equiv \sum_{i=1}^n (\tilde{\vartheta}_{t_{i-}} - \tilde{\vartheta}_{t_i})1_{(t_{i-}, t_i]}(u), \quad \forall u \in [0, T].
\]

**Step 2:** One-step approximation. Let \( 0 \leq a < t < T \) arbitrarily. Since \( S \) is a \( \mathbb{F} \)-semimartingale which satisfies the SDE \( dS_t = S_t \, dZ_t \), using a Gronwall argument as in [40, Lemma 5.1] yields constants \( c(A.3), d(A.3) > 0 \) independent of \( a, t \) such that, a.s.,

\[
\mathbb{E}_F^a\left[ \int_{(a,t]} |\tilde{\vartheta}_u - \tilde{\vartheta}_a|^2S_u^2 \, du \right] \leq 2\mathbb{E}_F^a\left[ \int_{(a,t]} |M_u - M_a|^2 \, du \right] + 2\mathbb{E}_F^a\left[ \int_{(a,t]} |S_u - S_a|^2 \, du \right]
\]

\[
= 2\mathbb{E}_F^a\left[ \int_{(a,t]} (t - u)d(M)_u^2 \right] + 2d(A.3)(t - a)^2 \tilde{\vartheta}_a^2 S_a^2 
\]

\[
\leq 2\mathbb{E}_F^a\left[ \int_{(a,t]} (t - u)^{1-\theta} d(M)_u^2 \right] + c(A.4)(t - a)\Theta_a^2 S_a^2,
\]

(A.4)

where \( c(A.4) := c(A.1)d(A.3) \).

**Step 3:** Multi-step approximation. For \( \tau = (t_n)_{n=0}^n \in \mathcal{T}_{\text{det}} \) and \( t \in [0, T] \), we set

\[
\langle \tilde{\vartheta}, \tau \rangle_t := \int_0^t \left[ \tilde{\vartheta}_u - \sum_{i=1}^n \tilde{\vartheta}_{t_{i-}}1_{(t_{i-}, t_i]}(u) \right]^2 S_u^2 \, du \overset{a.s.}{=} \int_0^t |\tilde{\vartheta}_u - \tilde{\vartheta}_a|^2 S_u^2 \, du.
\]

Then there exists a constant \( c(A.5) > 0 \) such that for any \( \tau \in \mathcal{T}_{\text{det}} \) and any \( a \in [0, T] \),

\[
\mathbb{E}_F^a\langle \tilde{\vartheta}, \tau \rangle_T - \langle \tilde{\vartheta}, \tau \rangle_a \leq c^2(A.5) \| \tau \| \Phi_a^2 \quad \text{a.s.}
\]  

(A.5)

Indeed, for \( \tau = (t_n)_{n=0}^n \in \mathcal{T}_{\text{det}} \) and \( a \in [t_{k-1}, t_k) \), \( k = 1, \ldots, n \), we let \( s_i := a \lor t_i, i = k - 1, \ldots, n \). Then, applying (A.4) with the aid of Proposition 5.6(2) yields, a.s.,

\[
\mathbb{E}_F^a\langle \tilde{\vartheta}, \tau \rangle_T - \langle \tilde{\vartheta}, \tau \rangle_a \leq 2\mathbb{E}_F^a\left[ |\tilde{\vartheta}_a - \tilde{\vartheta}_{t_{k-1}}|^2 \int_{(a,t_k]} S_u^2 \, du \right] + 2\sum_{i=k}^n \mathbb{E}_F^a\left[ \int_{(s_{i-1}, s_i]} |\tilde{\vartheta}_u - \tilde{\vartheta}_a|^2 S_u^2 \, du \right]
\]
\[
\leq 8c^2_{(A,1)}c^2_{(A,3)} \frac{t_k - a}{(T - a)^{\theta - 1}} \Phi_{s_i}^2 + 4\|\sigma\|_{\mathbb{F}_a} \int_{(a,T]} (T - u)^{1 - \theta} d\langle M \rangle_u^p + c^2_{(A,4)} \sum_{i=k}^n (s_i - s_{i-1}) \Phi_{s_{i-1}}^2 \]
\[
\leq c^2_{(A,5)} \|\sigma\|_{\mathbb{F}_a},
\]
where \(c_{(A,5)} > 0\) is determined by \(c^2_{(A,5)} = 8c^2_{(A,1)}c^2_{(A,3)} + 4c^2_{(5,9)} + 4Tc_{(A,4)}\|\Phi\|_{S_{M2}(\mathbb{P})}^2\).

**Step 4:** Examine \(E^{corr}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\). Since
\[
\int_0^T \int_{|z| > (T - u)^{-\kappa}} |z| \pi_{Z}^u(\text{d}z, \text{d}u) \leq \frac{1}{\varepsilon} \int_0^T (T - u)^{-\kappa} \int_{|z| \leq \varepsilon (T - u)^{-\kappa}} z^2 \nu_{Z}(u, \text{d}u) \, \text{d}u \leq \frac{\nu_{(5,6)} T^{1-\kappa}}{\varepsilon (1 - \kappa)} < \infty,
\]
the jump part of \(Z\) can be decomposed under \(\mathbb{P}\) as
\[
z \cdot (N_Z - \pi_{Z}^u) = (1 \{0 < |z| \leq \varepsilon (T - u)^{-\kappa}\} \cdot z) \cdot (N_Z - \pi_{Z}^u) + (1 \{|z| > \varepsilon (T - u)^{-\kappa}\} \cdot z) \cdot (N_Z - (1 \{|z| > \varepsilon (T - u)^{-\kappa}\} \cdot z) \cdot \pi_{Z}^u.
\]
so that
\[
Z = \tilde{Z}_0 + \bar{b}^Z \cdot \lambda + \sigma W^\mathbb{P} + Z^{1,\varepsilon,\kappa} + Z^{2,\varepsilon,\kappa} - \tau^{\varepsilon,\kappa}.
\]

Then we have the following decomposition
\[
(\tilde{\vartheta} - \tilde{\vartheta}^T) \cdot S = ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot Z
\]
\[
= ((\tilde{\vartheta} - \tilde{\vartheta}^T) S \cdot \tilde{b}^Z) \cdot \lambda + (\sigma (\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot W^\mathbb{P}
\]
\[
+ ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot Z^{1,\varepsilon,\kappa} + ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot Z^{2,\varepsilon,\kappa} - ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot \tau^{\varepsilon,\kappa}.
\]

We remark that the integral processes on the right-hand side of the equation above exist in \(L_2(\mathbb{P})\) as a by-product of estimates in Step 5-Step 8 below. The correction term of \(A^{corr}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\) given in Definition 5.7 satisfies that
\[
\sum_{\rho_1(\varepsilon, \kappa) \in [0, T]} \left( \tilde{\vartheta}_{\rho_1(\varepsilon, \kappa)} - \tilde{\vartheta}_{\rho_1(\varepsilon, \kappa)}^T \right) \Delta S_{\rho_1(\varepsilon, \kappa)} = ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot Z^{2,\varepsilon,\kappa}.
\]

Hence, we arrive at the following decomposition
\[
E^{corr}(\tilde{\vartheta}, \tau|\varepsilon, \kappa) = ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot \tilde{b}^Z \cdot \lambda + (\sigma (\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot W^\mathbb{P}
\]
\[
= : E^B(\tilde{\vartheta}, \tau)
\]
\[
+ ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot Z^{1,\varepsilon,\kappa} - ((\tilde{\vartheta} - \tilde{\vartheta}^T) S) \cdot \tau^{\varepsilon,\kappa},
\]

where \(E^B\) denotes the error involving with the process \(\tilde{b}^Z\), \(E^C\) (resp. \(E^S, E^D\)) is the error related to the continuous martingale (resp. small jump, drift) part of \(Z\). By the triangle inequality,
\[
\|E^{corr}(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\|_{BMO^p_{\mathbb{F}}(\mathbb{P})} \leq \sum_{i \in \{B,C\}} \|E^i(\tilde{\vartheta}, \tau)\|_{BMO^p_{\mathbb{F}}(\mathbb{P})} + \sum_{i \in \{S,D\}} \|E^i(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\|_{BMO^p_{\mathbb{F}}(\mathbb{P})}.
\]

Before investigating the right-hand side of (A.6), let us introduce a variant for the \(BMO^p_{\mathbb{F}}(\mathbb{P})\)-norm: for \(Y \in CL_0([0, T])\), we let \(Y \in bmo_{\mathbb{F}}(\mathbb{P})\) if
\[
\|Y\|_{bmo^p_{\mathbb{F}}(\mathbb{P})} := \inf \left\{ c > 0 : \|E^p_{\mathbb{F}}\left[ |Y_T - Y_a|^2 \right]\|_{\mathbb{F}_a} \leq c^2 \Phi_{s_i}^2, \forall a \in [0, T] \right\} < \infty.
\]

---

4For a semimartingale \(Y\), a random measure \(\Pi\), and suitable integrands \(\vartheta\) and \(f\), we denote by \(\vartheta \cdot Y = ((\vartheta \cdot Y)_t)_{t \in [0, T]}\) and by \(f \cdot \Pi = ((f \cdot \Pi)_t)_{t \in [0, T]}\) the integral processes which are càdlàg and start at zero with
\[
(\vartheta \cdot Y)_t := \int_{(0,t]} \vartheta_u \text{d}Y_u, \quad (f \cdot \Pi)_t := \int_{(0,t] \times \mathbb{R}_+} f(z, s) \Pi(\text{d}z, \text{d}s), \quad t \in (0, T].
\]
In particular, if the integrator is induced by the Lebesgue measure, then \(\vartheta \cdot \lambda = ((\vartheta \cdot \lambda)_t)_{t \in [0, T]}\) denotes a continuous process given by \(\vartheta \cdot \lambda_t := \int_0^t \vartheta_u \text{d}u\).
Step 5: Estimate \( \| E_b^b(\bar{\vartheta}, \tau) \|_{\text{BMO}^2(\mathbb{F})} \). One first observes that \( E_b^b(\bar{\vartheta}, \tau) \) is a continuous process which then implies that \( E_b^b(\bar{\vartheta}(\tau), \tau) = E_b^b(\bar{\vartheta}, \tau) \) for any stopping times \( \rho \). Hence,

\[
\| E_b^b(\bar{\vartheta}, \tau) \|_{\text{BMO}^2(\mathbb{F})} = \inf \left\{ c \geq 0 : E^F_{\mathbb{F}} \left[ | E_b^b(\bar{\vartheta}, \tau) - E_b^b(\bar{\vartheta}(\tau), \tau)|^2 \right] \leq c^2 \Phi^2 \right. \quad \text{a.s., } \forall \rho \in \mathcal{S}([0, T]) \right\}
\]

\[
= \inf \left\{ c \geq 0 : E^F_{\mathbb{F}} \left[ | E_b^b(\bar{\vartheta}, \tau) - E_b^b(\bar{\vartheta}(\tau), \tau)|^2 \right] \leq c^2 \Phi^2 \right. \quad \text{a.s., } \forall \rho \in [0, T] \right\}
\]

\[
= \| E_b^b(\bar{\vartheta}, \tau) \|_{\text{BMO}^2(\mathbb{F})},
\]

where in order to get the second equality we may replace the stopping times \( \rho \) by the deterministic times \( a \) by using a standard approximation argument as for instance in [16, Proposition A.4]. Since \( S = 1 + S_{-} \cdot Z \), we write \( S = 1 + S^m_{\mathbb{F}} + S^{fv}_{\mathbb{F}} \) as the canonical semimartingale decomposition of \( S \) under \( \mathbb{F} \) where \( S^m_{\mathbb{F}} := \sigma S_{-} \cdot W^\mathbb{F} + (S_{-} - \pi^Z) \) and \( S^{fv}_{\mathbb{F}} := (S_{-} - \bar{Z}) \cdot \lambda \). Now, for \( a \in [t_{k-1}, t_k), k = 1, \ldots, n \), by setting \( s_i := a \vee t_i, i = k - 1, \ldots, n \) we obtain

\[
\frac{1}{4} E^F_{\mathbb{F}} \left[ | E_b^b(\bar{\vartheta}(\tau), \tau) - E_b^b(\bar{\vartheta}(\tau), \tau)|^2 \right] = \frac{1}{4} E^F_{\mathbb{F}} \left[ \left| \int_a^T (\bar{\vartheta}_u - \bar{\vartheta}^\tau_u) S_u \bar{b}_u^Z \, du \right|^2 \right]
\]

\[
= \frac{1}{4} E^F_{\mathbb{F}} \left[ (\bar{\vartheta}_a - \bar{\vartheta}_{t_{k-1}}) \int_a^{t_{k-1}} S_u \bar{b}_u^Z \, du + \sum_{i=k}^n \int_{s_{i-1}}^{s_i} (M_{u} - M_{s_{i-1}}) \bar{b}_u^Z \, du \right]^2
\]

\[
\leq E^F_{\mathbb{F}} \left[ (\bar{\vartheta}_a - \bar{\vartheta}_{t_{k-1}}) \int_a^{t_{k-1}} S_u \bar{b}_u^Z \, du \right]^2 + E^F_{\mathbb{F}} \left[ \sum_{i=k}^n \int_{s_{i-1}}^{s_i} (M_{u} - M_{s_{i-1}}) \bar{b}_u^Z \, du \right]^2
\]

\[
= I^{(1)}_{(A.7)} + I^{(2)}_{(A.7)} + I^{(3)}_{(A.7)} + I^{(4)}_{(A.7)}. \quad (A.7)
\]

Recall \( \kappa = \frac{1-\theta}{2} \) and denote

\[
B_{(A.8)} := \sup_{i=k, \ldots, n} \sup_{u \in (s_{i-1}, s_i) \cap (0, T)} \left[ \frac{1}{(T-u)^\kappa} \int_u^{s_i} | \bar{b}_u^Z | \, dr \right]. \quad (A.8)
\]

- For \( I^{(1)}_{(A.7)} \), using the non-decreasing property of \( \Theta \) we get, a.s.,

\[
I^{(1)}_{(A.7)} \leq 4 c_{(A.1)}^2 (T-a)^{\theta-1} \Theta^2 \sup_{a \in [a, T]} \left[ \int_a^{t_k} S_u \bar{b}_u^Z \, du \right]^2
\]

\[
\leq 4 c_{(A.1)}^2 (T-a)^{1-\theta} B_{(A.8)}^2 (T-a)^{1-\theta} E^F_{\mathbb{F}} \left[ \sup_{u \in (a, T]} \Phi^2_u \right]
\]

\[
\leq \left[ 4 c_{(A.1)}^2 \right] E^F_{\mathbb{F}} \left[ \sup_{u \in [a, T]} \Phi^2_u \right].
\]

- For \( I^{(2)}_{(A.7)} \), we use the orthogonality of martingale increments and notice that \( \bar{b}^Z \) is deterministic to find that the mixed terms in the square expansion vanish under the respective conditional expectation. Then, using the stochastic Fubini theorem and applying the conditional Itô isometry yield, a.s.,

\[
I^{(2)}_{(A.7)} = \sum_{i=k}^n \left[ \int_{s_{i-1}}^{s_i} (M_{u} - M_{s_{i-1}}) \bar{b}_u^Z \, du \right]^2 = \sum_{i=k}^n \left[ \int_{s_{i-1}}^{s_i} \int_{u}^{s_i} \bar{b}_u^Z \, dr \, dM_u \right]^2
\]

\[
\leq \sum_{i=k}^n \int_{s_{i-1}}^{s_i} \left[ \int_{u}^{s_i} \bar{b}_u^Z \, dr \right]^2 \left[ \int_{a}^{t_k} (T-u)^{1-\theta} \, dM_u \right]^2
\]

\[
\leq c_{(5.9)}^2 B_{(A.8)}^2 \Phi^2_a.
\]
• For $I_{(A.7)}^{(3)}$, we also proceed in the same way as for $I_{(A.7)}^{(2)}$ to get, a.s.,

$$I_{(A.7)}^{(3)} = \sum_{i=k}^{n} \mathbb{E}_{F_u} \left[ \left( S_{u_i} - S_{u_i-1} \right)^2 \right] = \sum_{i=k}^{n} \mathbb{E}_{F_u} \left[ \left( \tilde{b}_{u_i} - \tilde{b}_{u_i-1} \right)^2 \right] \leq B_{(A.8)}^2 \sum_{i=k}^{n} \mathbb{E}_{F_u} \left[ \left( (T - s_{u_i})^{\theta - 1} \right) \right] \leq \left( \sigma^2 + \nu_{(5.6)}B_{(A.8)}^2 \right) \mathbb{E}_{F_u} \left[ \left( \sigma^2 \right) \right] \leq \left( \sigma^2 + \nu_{(5.6)} \right) \mathbb{E}_{F_u} \left[ \left( \sigma^2 \right) \right].$$

• For $I_{(A.7)}^{(4)}$, a standard argument using Fubini’s theorem yields, a.s.,

$$I_{(A.7)}^{(4)} \leq \mathbb{E}_{F_u} \left[ \left( \sum_{i=k}^{n} c(1.1) \left( T - s_{u_i} \right)^{\theta - 1} \right) \sum_{i=k}^{n} \mathbb{E}_{F_u} \left[ \left( \tilde{b}_{u_i} - \tilde{b}_{u_i-1} \right)^2 \right] \right].$$

Plugging those estimates for $I_{(A.7)}^{(1)}$, $I_{(A.7)}^{(4)}$ into (A.7), we get a constant $c(1.9) > 0$ not depending on $\varepsilon, \tau$ such that

$$\mathbb{E}_{F_u} \left[ \left( E_{(A.7)}^{\tilde{b}}(\tilde{\vartheta}, \tau) - E_{(A.7)}^a(\tilde{\vartheta}, \tau) \right)^2 \right] \leq c(1.9) B_{(A.8)}^2 \mathbb{E}_{F_u} \left[ \left( \sigma^2 \right) \right].$$

A simple computation shows $B_{(A.8)} \leq T^{4/\theta} b(5.7) ||\tau|| \theta$, and hence,

$$\left\| E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) \right\|_{\text{BMO}_2^2(\mathbb{F})} \leq \left[ c(1.9) T^{4/\theta} b(5.7) \right] \left\| \tau \right\| \theta.$$

**Step 6:** Estimate $\left\| E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) \right\|_{\text{BMO}_2^2(\mathbb{F})}$. By the same reason as in Step 5, the pathwise continuity of $E_{(A.7)}^{C}(\tilde{\vartheta}, \tau)$ gives

$$\left\| E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) \right\|_{\text{BMO}_2^2(\mathbb{F})} = \inf \left\{ \varepsilon \geq 0 : \mathbb{E}_{F_u} \left[ \left| E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) - E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) \right|^2 \right] \leq \varepsilon^2 \left\| \tau \right\| \theta \}, \forall a \in [0, T].$$

For $a \in [0, T]$, it follows from the conditional Itô isometry that, a.s.,

$$\mathbb{E}_{F_u} \left[ \left( E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) - E_{(A.7)}^{C}(\tilde{\vartheta}, \tau) \right)^2 \right] = \sigma^2 \mathbb{E}_{F_u} \left[ \left( \tilde{\vartheta}_u - \tilde{\vartheta}_{u-1} \right)^2 S_{u}^2 du \right] \leq c(1.5) \sigma^2 \left\| \tau \right\| \theta.$$
where $|D E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)|_\Phi := \inf\{c \geq 0 : |D E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)|_\Phi \leq c\Phi_t \text{ for all } t \in [0, T] \text{ a.s.}\}$. For the first term on the right-hand side of (A.12) and for $a \in [0, T]$, using the conditional Itô isometry gives, a.s.,

$$\mathbb{E}_{F_a}^F[|E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa) - E^S_a(\tilde{\vartheta}, \tau|\varepsilon, \kappa)|^2] = \mathbb{E}_{F_a}^F\left[\int_a^T |\tilde{\vartheta}_u - \tilde{\vartheta}_u|^2\|S_u\|^2 \int_{0<|z|\leq \varepsilon(T-u)^\alpha} z^2 \nu_z^F(u, dz) du\right]\leq \left\|u \mapsto \int_{0<|z|\leq \varepsilon T^n} z^2 \nu_z^F(u, dz) \right\|_{L^\infty([0,T],\lambda)} \mathbb{E}_{F_a}^F[(\tilde{\vartheta}, \tau)\tau - (\tilde{\vartheta}, \tau)_a].$$

By an argument using Fubini’s theorem as in (B.1), we obtain

$$\left\|u \mapsto \int_{0<|z|\leq \varepsilon T^n} z^2 \nu_z^F(u, dz) \right\|_{L^\infty([0,T],\lambda)} \leq c(A.13)(\varepsilon \land 1)^{2-\alpha}$$

for some constant $c(A.13) > 0$ depending at most on $\nu(5.6), c(5.12)(\alpha), \alpha, T, \kappa$. Thus,

$$\left\|E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})} \leq c(A.5)\sqrt{c(A.13)}(\varepsilon \land 1)^{1-\frac{2}{\alpha}} \sqrt{\|\tau\|_\theta}.$$  

Moreover, due to (A.2) we can find an $\Omega_0$ with probability one such that on $\Omega_0$ one has

$$\forall t \in [0, T] : |D E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)| \leq \frac{2c(A.1)(T-t)^{\frac{\kappa-1}{\alpha}}}{\alpha} \Theta_{t-S_t-\varepsilon(T-t)\frac{1}{\varepsilon} < 1} \leq 2c(A.1)(T-t)^{\frac{\kappa-1}{\alpha}} \Theta_{t-S_t-\varepsilon(T-t)\frac{1}{\varepsilon} < 1} \leq 2c(A.1)\varepsilon\Phi_t.$$  

Hence, $|D E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)|_\Phi \leq 2c(A.1)\varepsilon$. Plugging those estimates into (A.12) yields

$$\left\|E^S(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})} \leq c(A.14)(\varepsilon \land 1)^{1-\frac{2}{\alpha}} \sqrt{\|\tau\|_\theta}$$

for $c(A.14) := \max\{c(A.5)\sqrt{c(A.13)}, \sqrt{2c(A.1)}\}$.

**Step 8**: Estimate $\left\|E^D(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})}$. We first define

$$D_{\varepsilon, \kappa}(u) := \int_{|z|>\varepsilon(T-u)^\alpha} z^2 \nu_z^F(u, dz), \quad u \in [0, T].$$

Then, by the same estimation as given in [40, Proof of Theorem 3.15, Step 1] we get a constant $c(A.15) > 0$ independent of $\tau = (t_i)^n_{i=0} \in \mathcal{T}_{\text{det}}$ and $\varepsilon > 0$ such that

$$D_{\varepsilon, \kappa}(u) := \sup_{i=1, \ldots, n} \sup_{u \in (t_{i-1}, t_i]} \left[\frac{1}{(T-u)^\kappa} \int_{t_i}^{t_{i-1}} |D\varepsilon_{\varepsilon, \kappa}(r)| dr\right] \leq c(A.15) \left\|\tau\|_\theta + \varepsilon^{1-\alpha} \|\tau\|_\theta^{1-\kappa(\alpha-1)} \right\|\tau\|_\theta$$

if (5.12) holds with $\alpha \in (1, 2]$

$$\left\|\tau\|_\theta \right\|\tau\|_\theta$$

if (5.12) holds with $\alpha = 1$

$$\left\|\tau\|_\theta \right\|\tau\|_\theta$$

if (5.12) holds with $\alpha \in (0, 1)$.  

We now estimate $\left\|E^D(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})}$ analogously as for $\left\|E^S(\tilde{\vartheta}, \tau)\right\|_{\text{bmo}_2(\mathbb{F})}$ in Step 5, where $D_{\varepsilon, \kappa}(u)$ and $D_{A.15}$ play the role of $\tilde{b}_0^\varepsilon$ and $B(A.8)$ respectively. This then leads us to

$$\mathbb{E}_{F_a}^F\left[|E^D(\tilde{\vartheta}, \tau|\varepsilon, \kappa) - E^D_a(\tilde{\vartheta}, \tau|\varepsilon, \kappa)|^2\right] \leq c(A.9)^2D_{A.15}(\kappa)^2 \Phi_a^2 \text{ a.s. for all } a \in [0, T]$$

which means that

$$\left\|E^D(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})} = \left\|E^D(\tilde{\vartheta}, \tau|\varepsilon, \kappa)\right\|_{\text{bmo}_2(\mathbb{F})} \leq c(A.9)D_{A.15}.  \quad (A.16)$$

Combining (A.10), (A.11), (A.14) and (A.16) with (A.6) yields (5.13).
APPENDIX B. SOME TECHNICAL RESULTS

B.1. Some properties of stable-like-measures. We recall $\mathcal{US}(\alpha), \mathcal{S}(\alpha)$ from Definition 5.12.

**Lemma B.1** (See also [40], Remark 4.5). Let $\ell$ be a Lévy measure.

1. For $\alpha \in (0, 2]$ one has
   \[
   \sup_{r \in (0,1)} r^\alpha \int_{\frac{2}{3} \leq |x| \leq r} \ell(dx) < \infty \iff \ell \in \mathcal{US}(\alpha) \iff \limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) < \infty.
   \]
   
2. If $\ell \in \mathcal{S}(\alpha)$ for some $\alpha \in (0, 2)$, then $\alpha$ is equal to the Blumenthal–Getoor index of $\ell$, i.e.,
   \[\alpha = \inf \{ q \in [0, 2] : \int_{|x| \leq 1} |x|^q \ell(dx) < \infty \} \]

3. If $\ell$ has a density $p(x) := \ell(dx)/dx$ which satisfies
   \[
   \liminf_{x \to 0^+} (-x)^{1+\alpha} p(x) + \lim sup_{x \to 0^+} x^{1+\alpha} p(x) > 0 \quad \text{and} \quad \lim sup_{x \to 0^+} |x|^{1+\alpha} p(x) < \infty
   \]
   for some $\alpha \in (0, 2)$, then $\ell \in \mathcal{S}(\alpha)$.

**Proof.** (1) We show $A(\alpha) \Rightarrow \ell \in \mathcal{US}(\alpha) \Rightarrow B(\alpha) \Rightarrow A(\alpha)$. Denote
   \[
c_1(\alpha) := \sup_{r \in (0,1)} r^\alpha \int_{r < |x| \leq 1} \ell(dx) \quad \text{and} \quad c_2(\alpha) := \sup_{r \in (0,1)} r^\alpha \int_{0 < |x| < r} \ell(dx).
   \]

- $A(\alpha) \Rightarrow \ell \in \mathcal{US}(\alpha)$: Let $r \in (0,1)$ and $m_r \in \mathbb{N} \cup \{0\}$ with $2^{-m_r - 1} < r \leq 2^{-m_r}$. Then
   \[
r^\alpha \int_{r < |x| \leq 1} \ell(dx) \leq 2^{-\alpha m_r} \sum_{k=0}^{m_r} \sum_{0 < |x| < 2^{-k}} \ell(dx) \leq c_2(\alpha) 2^{-\alpha m_r} \sum_{k=0}^{m_r} 2^\alpha \leq c_2(\alpha) 2^{-\alpha m_r} \frac{2^{\alpha(m_r+1)}}{2^\alpha - 1} = c_2(\alpha) \frac{2^\alpha}{2^\alpha - 1}.
   \]

- $\ell \in \mathcal{US}(\alpha) \Rightarrow B(\alpha)$: For $|u| > 1$, one has
   \[
   \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) = \frac{1}{|u|^\alpha} \left( \int_{|x| < |u|} + \int_{|ux| \leq |u|} + \int_{|ux| > |u|} \right) (1 - \cos(ux)) \ell(dx)
   \leq \frac{2}{|u|^\alpha} \int_{|x| < |u|} \ell(dx) + |u|^{-\alpha} \int_{|ux| \leq |u|} x^2 \ell(dx) + \frac{2}{|u|^\alpha} \int_{|ux| > |u|} \ell(dx)
   \leq 2c_1(\alpha) + |u|^{-2\alpha} \int_{|x| < |u|} x^2 \ell(dx) + \frac{2}{|u|^\alpha} \int_{|x| > |u|} \ell(dx).
   \]

If $\alpha = 2$, then it is obvious from the estimate above that $\ell \in \mathcal{US}(2) \Rightarrow B(2)$. For $\alpha \in (0, 2)$, we use Fubini’s theorem to get that, for $r \in (0,1]$,
\[
\int_{|x| \leq r} x^2 \ell(dx) = \int_{|x| \leq r} \int_0^r dy \ell(dx) = \int_0^r \int_{\sqrt{y} \leq |x| \leq r} \ell(dx)dy
   \leq c_1(\alpha) \int_0^r y^{-\frac{2}{2\alpha}} dy = \frac{2c_1(\alpha)}{2 - \alpha} r^{2 - \alpha}.
   \]

Hence,
\[
\limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) \leq 2c_1(\alpha) + \frac{2c_1(\alpha)}{2 - \alpha} < \infty.
   \]

- $B(\alpha) \Rightarrow A(\alpha)$: Using the inequality $1 - \cos y \geq \frac{y^2}{4}$, $y \in [0, 2]$, we infer that, for $|u| \geq 2$,
   \[
   \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) \geq \frac{1}{|u|^\alpha} \int_{2|ux| > 1} (1 - \cos(ux)) \ell(dx)
   \]
   and
   \[
   \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) \leq \frac{1}{|u|^\alpha} \int_{|ux| \leq 1} (1 - \cos(ux)) \ell(dx)
   \]
   Hence, taking $r \to \infty$, we see that
   \[
   \limsup_{|u| \to \infty} \frac{1}{|u|^\alpha} \int_{\mathbb{R}} (1 - \cos(ux)) \ell(dx) \leq 2c_1(\alpha) + \frac{2c_1(\alpha)}{2 - \alpha} < \infty
   \]
We set
\[ \text{SM} \]
where we set \( \ell \).

Then, \( \ell \). For \( Q \), we define \( \Phi(\eta) \) by setting
\[ \Phi(\eta) := e_{X_t} \sup_{\alpha \in [0, t]} e^{(\alpha - 1) X_t}, \quad t \in [0, T]. \]

Lemma B.2 ([40], Proposition A.1). If \( \Psi \in SM_q(Q) \) for some \( q \in (0, \infty) \), then \( \Phi \in SM_q(Q) \).

Assume that \( X = (X_t)_{t \in [0, T]} \) is a Lévy process with \( (X_t) \sim (\gamma, \sigma^2, \nu^2) \). For \( \eta \in [0, 1] \), we define \( \Psi(\eta) \) by
\[ \Psi(\eta) := e_{X_t} \sup_{\alpha \in [0, t]} e^{(\alpha - 1) X_t}, \quad t \in [0, T]. \]

Lemma B.3 ([40], Proposition A.2). If \( \int_{|x| > 1} e^{\eta x} \nu^2(dx) < \infty \) for some \( q \in (1, \infty) \), then \( \Psi(\eta) \) is a Lévy semigroup on \( H \) spaces. In this part we recall \( C^{0, \eta} \) and \( S(\alpha) \) from Definition 5.12.

Assume that \( X = (X_t)_{t \geq 0} \) is a Lévy process with respect to a probability measure \( Q \) with \( (X_t) \sim (\gamma, \sigma^2, \nu^2) \). Let \( \eta \in [0, 1], g \in C^{0, \eta} \), and assume that \( \int_{|x| > 1} e^{\eta x} \nu^2(dx) < \infty \). Then, for all \( t > 0 \) and \( y, z > 0 \) one has \( \mathbb{E}^Q[g(ye^{X_t})] < \infty \) and
\[ |\mathbb{E}^Q[g(ye^{X_t})] - \mathbb{E}^Q[g(ze^{X_t})]| \leq |g|_{C^{0, \eta}} \mathbb{E}^Q[e^{\eta X_t}] |y - z|^\eta. \]

This leads us to define the map \( Q_t : C^{0, \eta} \to C^{0, \eta} \) by setting
\[ Q_t g(y) := \mathbb{E}^Q[g(ye^{X_t})], \quad y > 0, t \geq 0. \]

It is clear that \( Q_{t+s} = Q_t \circ Q_s \) for all \( s, t \geq 0 \) which means that \( (Q_t)_{t \geq 0} \) is a semigroup on \( C^{0, \eta} \).

For a Lévy measure \( \ell \) on \( B(\mathbb{R}) \) and a Borel function \( g \), we write symbolically
\[ I_\ell(t, y) := |\sigma^2| \partial_y Q_t g(y) + \int_{\mathbb{R}} Q_t g(e^x y) - Q_t g(y) \left( \frac{e^x - 1}{x} \right) \ell(dx), \quad y > 0, t \geq 0, \]
where we set \( \partial_y Q_t g := 0 \) if \( \sigma^2 = 0 \).
Proposition B.4 ([40], Proposition B.5). Let \( g \in C^{0,0} \) with \( \eta \in [0,1] \). Assume that \( \ell \) is a Lévy measure with \( \int_{|x|>1} e^{(\eta+1)x} \ell(dx) < \infty \). Then, for any \( T \in (0, \infty) \) there is a \( c(B.2) > 0 \) such that
\[
|I^Q_\ell(t,y)| \leq c(B.2) V(t) y^{\eta-1}, \quad \forall (t,y) \in (0,T] \times (0, \infty), \tag{B.2}
\]
where the cases for \( V(t) \) are provided as follows:

1. If \( \sigma^Q > 0 \) and \( \int_{|x|>1} e^{x\sigma^Q} \ell(dx) \) is finite, then \( V(t) = t^{\frac{2-\eta}{\eta}} \).
2. If \( \sigma^Q = 0 \), \( \int_{|x|>1} e^{x\sigma^Q} \ell(dx) < \infty \) and \( \int_{|x|\leq 1} |x|^{\eta+1} \ell(dx) \) is finite, then \( V(t) = 1 \).
3. If \( \sigma^Q = 0 \), \( \eta \in [0,1] \) and if the following two conditions hold:
   a. \( \nu^Q \in S(\alpha) \) for some \( \alpha \in (0,2) \) and \( \int_{|x|>1} e^{r \nu^Q} \ell(dx) \) is finite,
   b. there is a \( \beta \in [0,2] \) such that

\[
0 < c(B.3) := \sup_{r \in (0,1)} r^\beta \int_{r<|x|\leq 1} \ell(dx) < \infty, \tag{B.3}
\]

then
\[
V(t) = \begin{cases} 
   t^{\frac{n+1-\beta}{\beta}} & \text{if } \beta \in (1+\eta,2] \\
   \max\{1, \log(1/t)\} & \text{if } \beta = 1+\eta \\
   1 & \text{if } \beta \in [0,1+\eta).
\end{cases}
\]

Here, the constant \( c(B.2) \) may depend on \( \beta \) in Item (3).

One remarks that the definition of \( S(\alpha) \) in Proposition B.4 is slightly more general than that in [40, Proposition B.5], however, the proof remains the same.

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