Classification of the stable solution to the fractional \((2 < s < 3)\) Lane-Emden equation

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Abstract

We classify the stable solutions (positive or sign-changing, radial or not) to the following nonlocal Lane-Emden equation:

\[-(\Delta)^s u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^n\]

for \(2 < s < 3\).

1 Introduction and Main results

Consider the stable solution of the following equation

\[-(\Delta)^s u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}\]

where \(-(\Delta)^s\) is the fractional Laplacian operator for \(2 < s < 3\).

The motivation of studying such an equation is originated from the classical Lane-Emden equation

\[-\Delta u = |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^n \tag{1.2}\]

and its parabolic counterpart, which have played a crucial role in the development of nonlinear PDEs in the last decades. These arise in astrophysics and Riemannian geometry. The pioneering works on Eq. (1.2) were contributed by R. Fowler \([12, 13]\). Later, the ground-breaking result on equation (1.2) is the fundamental Liouville-type theorems established by Gidas and Spruck \([14]\), they claimed that the Eq. (1.2) has no positive solution whenever \(p \in (1, 2^* - 1)\); where \(2^* = 2n/(n - 2)\) if \(n \geq 3\) and \(2^* = \infty\) if \(n \leq 2\). The critical case \(p = 2^* - 1\), Eq. (1.2) has a unique positive solution

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up to translation and rescaling which is radial and explicitly formulated, see Caffarelli- Gidas-Spruck [1]. Since then many experts in partial differential equations devote to the above equations for various parameters \( s \) and \( p \).

For the nonlocal case of \( 0 < s < 1 \), a counterpart of the classification results of Gidas and Spruck [14], and Caffarelli- Gidas-Spruck [1] holds for the fractional Lane-Emden equation (1.1), see the works due to Li [19] and Chen-Li-Ou [5]. In these cases, the Sobolev exponent is given by \( P_S(n, s) = \frac{n + 2s}{n - 2s} \) if \( n > 2s \), and otherwise \( P_S(n, s) = \infty \).

Recently, for the nonlocal case of \( 0 < s < 1 \), Davila, Dupaigne and Wei in [6] gave a complete classification of finite Morse index solution of (1.1); for the nonlocal case of \( 1 < s < 2 \), Fazly and Wei in [17] gave a complete classification of finite Morse index solution of (1.1). For the local cases \( s = 1 \) and \( s = 2 \), such kind of classification is proved by Farina in [10] and Davila, Dupaigne, Wang and Wei in [7], respectively. For the case \( s = 3 \), the Joseph-Lundgren exponent (for the triharmonic Lane-Emden equation) is obtained and classification is proved by in [21].

However, when \( 2 < s < 3 \), the equation (1.1) has not been considered so far. In this paper we classify the stable solution of (1.1).

There are many ways of defining the fractional Laplacian \( (-\Delta)^s \), where \( s \) is any positive, noninteger number. Caffarelli and Silvestre in [2] gave a characterization of the fractional Laplacian when \( 0 < s < 1 \) as the Dirichlet-to-Neumann map for a function \( u \) satisfying a higher order elliptic equation in the upper half space with one extra spatial dimension. This idea was later generalized by Yang in [27] when the \( s \) is being any positive, noninteger number. See also Chang-Gonzales [4] and Case-Chang [3] for general manifolds.

To introduce the fractional operator \( (-\Delta)^s \) for \( 2 < s < 3 \), just like the case of \( 1 < s < 2 \), via the Fourier transform, we can define
\[
(-\Delta)^s u(\xi) = |\xi|^{2s} \hat{u}(\xi)
\]
or equivalently define this operator inductively by \( (-\Delta)^s = (-\Delta)^{s-2} \circ (-\Delta)^2 \).

**Definition 1.1.** We say a solution \( u \) of (1.1) is stable outside a compact set if there exists \( R_0 > 0 \) such that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n + 2s}} \, dx \, dy - p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 \, dx \geq 0
\]
for any \( \varphi \in C^\infty_c(\mathbb{R}^n \setminus B_{R_0}) \).

Set
\[
p_s(n) = \begin{cases} 
\infty & \text{if } n \leq 2s, \\
n + 2s & \text{if } n > 2s.
\end{cases}
\]

The first main result of the present paper is the following
Theorem 1.1. Suppose that $n > 2s$ and $2 < s < \delta < 3$. Let $u \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |z|)^{n+2s}dz)$ be a solution of (1.1) which is stable outside a compact set. Assume

(1) $1 < p < p_s(n)$ or
(2) $p_s(n) < p$ and

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

then the solution $u \equiv 0$.

(3) $p = p_s(n)$, then $u$ has finite energy, i.e.,

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2s}} dxdy = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition $u$ is stable, then $u \equiv 0$.

Remark 1.1. In the Theorem 1.1 the condition (1.4) is optimal. In fact, the radial singular solution $u = |x|^{-\frac{s}{p-1}}$ is stable if

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} \leq \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

See [22].

Remark 1.2. The hypothesis (2) of Theorem 1.1 is equivalent to

$$p < p_c(n) := \begin{cases} +\infty & \text{if } n \leq n_0(s), \\ \frac{n+2s-2-2\sqrt{n}}{n-2s-2-2\sqrt{n}} & \text{if } n > n_0(s), \end{cases}$$

where $n_0(s)$ is the largest root of $n - 2s - 2 - 2\sqrt{n} = 0$, see [22]. More details and further sharp results about $a_{n,s}$ and $n_0(s)$ see [23].

Remark 1.3. In this remark, we further analyze the hypothesis (2) in Theorem 1.1. Recall that when $s = 1$ the condition (1.4) gives an upper bounded of $p$ (originated from Joseph and Lundgren [18]), it is

$$p < p_c(n) := \begin{cases} +\infty & \text{if } n \leq 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11. \end{cases}$$

For the case $s = 2$, (1.4) induce the upper bound of $p$ which is given by the following formula (cf. Gazzola and Grunau [16]):

$$p < p_c(n) = \begin{cases} +\infty & \text{if } n \leq 12, \\ \frac{n+2+\sqrt{n^2+4n^2-8n+32} \sqrt{n^2+8n+32}}{n-6+\sqrt{n^2+4n^2-8n+32}} & \text{if } n \geq 13. \end{cases}$$
In the triharmonic case, the corresponding exponent given by see (21) is the following

\[ p < p_c(n) = \begin{cases} 
\infty & \text{if } n \leq 14, \\
n + 4 - 2D(n) & \text{if } n \geq 15,
\end{cases} \]

where

\[ D(n) := \frac{1}{6} \left( 9n^2 + 96 - \frac{1536 + 1152n^2}{D_0(n)} - \frac{3}{2}D_0(n) \right)^{1/2}; \]

\[ D_0(n) := \frac{-1}{6} \left( 9n^2 + 96 - \frac{1536 + 1152n^2}{D_1(n)} - \frac{3}{2}D_1(n) \right)^{1/2}; \]

\[ D_1(n) := -94976 + 20736n + 103104n^2 - 10368n^3 + 3072n^4 - 108n^6; \]

\[ D_2(n) := 6131712 - 16644096n^2 + 1936384n^4 - 4818944n^6 + 4818944n^8 - 4320n^9 + 1800n^{10} - 216n^{11} + 9n^{12}. \]

2 Preliminary

Throughout this paper we denote \( b := 5 - 2s \) and define the operator

\[ \Delta_b w := \Delta w + \frac{b}{y}w_y = y^{-b} \text{div}(y^b \nabla w) \]

for a function \( w \in W^{3,2}(\mathbb{R}^{n+1}; y^b \text{d}x \text{d}y) \). We firstly quote the following result.

**Theorem 2.1.** (See [27]) Assume \( 2 < s < 3 \). Let \( u_e \in W^{3,2}(\mathbb{R}^{n+1}; y^b \text{d}x \text{d}y) \) satisfy the equation

\[ \Delta^3_b u_e = 0 \quad (2.1) \]

on the upper half space for \( (x, y) \in \mathbb{R}^n \times \mathbb{R}_+ \) (where \( y \) is the spacial direction) and the boundary conditions:

\[ u_e(x, 0) = f(x), \]

\[ \lim_{y \to 0} y^b \frac{\partial u_e}{\partial y} = 0, \]

\[ \frac{\partial^2 u_e}{\partial y^2} \big|_{y=0} = \frac{1}{2s} \Delta_x u_e \big|_{y=0}, \]

\[ \lim_{y \to 0} C_{n,s} y^b \frac{\partial}{\partial y} \Delta^2_b u_e = (-\Delta)^s f(x), \quad (2.2) \]

where \( f(x) \) is some function defined on \( H^s(\mathbb{R}^n) \). Then we have

\[ \int_{\mathbb{R}^n} |\xi|^2 |\hat{u}(\xi)|^2 \text{d}\xi = C_{n,s} \int_{\mathbb{R}^{n+1}_+} y^b |\nabla \Delta_b u_e(x, y)|^2 \text{d}x \text{d}y. \quad (2.3) \]
Applying the above theorem to solutions of (1.1), we conclude that the extended function \( u_e(x, y) \), where \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( y \in \mathbb{R}^+ \), satisfies

\[
\begin{align*}
\Delta_b^k u_e &= 0 \text{ in } \mathbb{R}^{n+1}_+, \\
\lim_{y \to 0} y^b \frac{\partial u_e}{\partial y} &= 0 \text{ on } \partial \mathbb{R}^{n+1}_+, \\
\frac{\partial^2 u_e}{\partial y^2} \bigg|_{y=0} &= \frac{1}{2} \Delta_x u_e \bigg|_{y=0} \text{ on } \partial \mathbb{R}^{n+1}_+, \\
\lim_{y \to 0} y^b \frac{\partial}{\partial y} \Delta_b^2 u_e &= -C_{n,s} |u_e|^{p-1} u_e \text{ in } \mathbb{R}^{n+1}_+.
\end{align*}
\] (2.4)

Moreover,

\[
\int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi = C_{n,s} \int_{\mathbb{R}^{n+1}_+} y^b |\nabla \Delta_b u_e(x, y)|^2 \, dx dy
\]

and \( u(x) = u_e(x, 0) \).

Define

\[
E(\lambda, x, u_e) = \int_{\mathbb{R}^{n+1}_+ \setminus \partial B_1} \frac{1}{2} \theta_1 \left| \nabla \Delta_b^1 u_e^\lambda \right|^2 - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u_e^\lambda|^{p+1} \\
+ \sum_{0 \leq i,j \leq 4, i+j \leq 5} C_{i,j} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^i \lambda^{i+j} \frac{d^i u_e^\lambda}{d\lambda^i} \frac{d^j u_e^\lambda}{d\lambda^j} \\
+ \sum_{0 \leq i,s \leq 2, i+s \leq 3} C_{i,s} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^i \lambda^{i+s} \nabla S^n \frac{d^i u_e^\lambda}{d\lambda^i} \nabla S^n \frac{d^s u_e^\lambda}{d\lambda^s} \\
+ \sum_{0 \leq i,k \leq 1, i+k \leq 1} C_{i,k} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^i \lambda^{i+k} \Delta S^n \frac{d^i u_e^\lambda}{d\lambda^i} \Delta S^n \frac{d^k u_e^\lambda}{d\lambda^k} \\
+ \left( \frac{s}{p-1} + 1 \right) \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^s (\Delta_b u_e^\lambda)^2.
\] (2.5)

The following is the monotonicity formula which will play an important role.

**Theorem 2.2.** Let \( u_e \) satisfy the equation (2.1) with the boundary conditions (2.2), we have the following

\[
\frac{dE(\lambda, x, u_e)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^b \left( 3\lambda^5 \left( \frac{d^3 u_e^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left( \frac{d^2 u_e^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left( \frac{du_e^\lambda}{d\lambda} \right)^2 \right) \\
+ \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^b \left( 2\lambda^3 |\nabla S^n \frac{d^2 u_e^\lambda}{d\lambda^2}|^2 + B_1 \lambda |\nabla S^n \frac{du_e^\lambda}{d\lambda}| \right) \\
+ \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^b \lambda (\Delta S^n \frac{du_e^\lambda}{d\lambda})^2.
\] (2.6)
where $\theta_1 = \frac{p}{2}$ and

$$
A_1 := 10\delta_1 - 2\delta_2 - 56 + \alpha_0^2 - 2\alpha_0 - 2\beta_0 - 4,
A_2 := -18\delta_1 + 6\delta_2 - 4\delta_3 + 2\delta_4 + 72 - \alpha_0^2 + \beta_0^2 + 2\alpha_0 + 2\beta_0,
B_1 := 8\alpha - 4\beta - 2\beta_0 + 4(n + b) - 14,
$$

$$
\alpha := n + b - \frac{4s}{p - 1}, \beta := \frac{2s}{p - 1}(3 + \frac{2s}{p - 1} - n - b),
\alpha_0 := n + b - \frac{4s}{p - 1}, \beta_0 := \frac{2s}{p - 1}(1 + \frac{2s}{p - 1} - n - b)
$$

and

$$
\begin{align*}
\delta_1 &= 2(n + b) - \frac{8s}{p - 1}, \\
\delta_2 &= (n + b)(n + b - 2) - (n + b)\frac{12s}{p - 1} + \frac{12s}{p - 1}(1 + \frac{2s}{p - 1}), \\
\delta_3 &= - \frac{8s}{p - 1}(1 + \frac{2s}{p - 1})(2 + \frac{2s}{p - 1}) + 2(n + b)\frac{6s}{p - 1}(1 + \frac{2s}{p - 1}) \\
&\quad - (n + b)(n + b - 2)(1 + \frac{4s}{p - 1}), \\
\delta_4 &= (3 + \frac{2s}{p - 1})(2 + \frac{2s}{p - 1})(1 + \frac{2s}{p - 1})\frac{2s}{p - 1} \\
&\quad - 2(n + b)(1 + \frac{2s}{p - 1})(2 + \frac{2s}{p - 1})\frac{2s}{p - 1} \\
&\quad + (n + b)(n + b - 2)(2 + \frac{2s}{p - 1})\frac{2s}{p - 1}.
\end{align*}
\tag{2.7}
$$

We will give the proof of Theorem 2.2 in the next section. Now we would like to state a consequent result of Theorem 2.2. Recall that $E(\lambda, x, u_\epsilon)$, defined in (2.5), can be divided into two parts: the integral over the ball $B_\lambda$ and the terms on the boundary $\partial B_\lambda$. We note that in our blow-down analysis, the coefficients (including positive or negative, big or small) of the boundary terms can be estimated in a unified way, therefore we may change some coefficients of the boundary terms in $E(\lambda, x, u_\epsilon)$. After such a change, we denote the new functional by $E^c(\lambda, x, u_\epsilon)$.

Define

$$
p_m(n) := \begin{cases} 
+\infty & \text{if } n < 2s + 6 + \sqrt{73}, \\
\frac{5n + 10 - \sqrt{15(n - 2s)^2 + 120(n - 2s) + 370}}{5n - 10 - \sqrt{15(n - 2s)^2 + 120(n - 2s) + 370}} & \text{if } n \geq 2s + 6 + \sqrt{73}.
\end{cases}
\tag{2.8}
$$

We have the following

**Theorem 2.3.** Assume that $\frac{n + 2s}{2} < p < p_m(n)$, then $E^c(\lambda, x, u_\epsilon)$ is a nondecreasing function of $\lambda > 0$. Furthermore,

$$
\frac{dE^c(\lambda, x, u_\epsilon)}{d\lambda} \geq C(n, s, p)\lambda^{2s + \frac{2s}{p - 1} - 6 - n} \int_{\mathbb{R}^{n+1} \cap \partial B_\lambda(x_0)} y^b\left(\frac{2s}{p - 1}u_\epsilon + \lambda \partial_\nu u_\epsilon\right)^2,
$$

$$
6
$$
where $C(n, s, p)$ is a constant independent of $\lambda$.

By carefully comparing \( \frac{n+2s}{n-2s} < p < p_m(n) \) with $p \geq \frac{n+2s}{n-2s}$ and (1.4), we get the following (see the last section of the current paper) monotonicity formula for our blow down analysis.

**Theorem 2.4.** Assume that $p > \frac{n+2s}{n-2s}$ and (1.4), then $E^c(\lambda, x, u_e)$ is a nondecreasing function of $\lambda > 0$. Furthermore,

$$
\frac{dE^c(\lambda, x, u_e)}{d\lambda} \geq C(n, s, p) \lambda^{\frac{n+2s}{p-1}} - 6^2 - n \int_{\mathbb{R}^{n+1} \cap \partial B_\lambda(x_0)} y^b \left( \frac{2s}{p-1} u_e + \lambda \partial \tau u_e \right)^2,
$$

where $C(n, s, p)$ is a constant independent of $\lambda$.

### 3 Monotonicity formula and the proof of Theorem 2.2

The derivation of the monotonicity for the (1.1) when $2 < s < 3$ is complicated in its process, we divide it into several subsections. In subsection 3.1, we derive $\frac{d}{d\lambda} \bar{E}(u_e, \lambda)$. In subsection 3.2, we calculate $\frac{\partial}{\partial y_i} u_e$ and $\frac{\partial^2}{\partial y_i^2} u_e$, $i, j = 1, 2, 3, 4$. In subsection 3.3, the operator $\Delta_2$ and its representation will be given. In subsection 3.4, we decompose $\frac{d}{d\lambda} \bar{E}(u_e, 1)$. Finally, combine with the above four subsections, we can obtain the monotonicity formula, hence get the proof of Theorem 2.2.

Suppose that $x_0 = 0$ and denote by $B_\lambda$ the balls centered at zero with radius $\lambda$. Set

$$
\bar{E}(u_e, \lambda) := \lambda^{\frac{n+2s}{p-1}} - n \left( \int_{\mathbb{R}^{n+1} \cap \partial B_\lambda} \frac{1}{2} y^b |\nabla \Delta_b u_e|^2 - \frac{C_{n,s}}{p+1} \int_{\partial B^{n+1} \cap \partial B_\lambda} |u_e|^{p+1} \right).
$$

#### 3.1 The derivation of $\frac{d}{d\lambda} \bar{E}(u_e, \lambda)$

Define

$$
v_e := \Delta_b u_e, \quad v^\lambda_e(X) := \lambda^{\frac{2s}{p-1}} u_e(\lambda X), \quad w_e(X) := \Delta_b v_e, \quad w^\lambda_e(X) := \lambda^{\frac{2s}{p-1}+2} v_e(\lambda X),
$$

where $X = (x, y) \in \mathbb{R}^{n+1}$. Therefore,

$$
\Delta_b u^\lambda_e(X) = v^\lambda_e(X), \quad \Delta_b v^\lambda_e(X) = w^\lambda_e(X).
$$

Hence

$$
\Delta_b w^\lambda_e = 0, \\
\lim_{y \to 0} y^b \frac{\partial u_e}{\partial y} = 0, \\
\frac{\partial^2 u_e}{\partial y^2} \bigg|_{y=0} = \frac{1}{2s} \Delta_x u_e \bigg|_{y=0}, \\
\lim_{y \to 0} C_{n,s} y^b \frac{\partial}{\partial y} w^\lambda_e = -C_{n,s} |u_e|^{p-1} u_e.
$$
In addition, differentiating (3.2) with respect to $\lambda$ we have

$$\Delta_b \frac{du^\lambda}{d\lambda} = \frac{dv^\lambda}{d\lambda}, \quad \Delta_b \frac{dv^\lambda}{d\lambda} = \frac{dw^\lambda}{d\lambda}.$$  

Note that

$$\overline{E}(u^\lambda, \lambda) = \overline{E}(u^\lambda, 1) = \int_{\mathbb{R}^{n+1}_+ \cap B_1} \frac{1}{2} y^b |\nabla v^\lambda|^2 - \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u^\lambda|^p + 1.$$  

Taking derivative of the energy $\overline{E}(u^\lambda, 1)$ with respect to $\lambda$ and integrating by part we have:

$$\frac{d\overline{E}(u^\lambda, 1)}{d\lambda} = \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \nabla v^\lambda \nabla \frac{dv^\lambda}{d\lambda} - C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u^\lambda|^p - 1 u^\lambda \frac{du^\lambda}{d\lambda}$$

$$= \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v^\lambda}{\partial n} \frac{dv^\lambda}{d\lambda} - \int_{\mathbb{R}^{n+1}_+ \cap B_1} (y^b \Delta v^\lambda + y^b \frac{\partial v^\lambda}{\partial y} \frac{dv^\lambda}{d\lambda}) - C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u^\lambda|^p - 1 u^\lambda \frac{du^\lambda}{d\lambda}$$

$$= - \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v^\lambda}{\partial y} \frac{dv^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} y^b \frac{\partial v^\lambda}{\partial r}$$

$$- \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta v^\lambda \frac{dv^\lambda}{d\lambda} + y^b \frac{\partial v^\lambda}{\partial y} \frac{dv^\lambda}{d\lambda} - \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda}.$$  

Now note that from the definition of $v^\lambda$ and by differentiating it with respect to $\lambda$, we get the following identity for $X \in \mathbb{R}^{n+1}_+$,

$$r \frac{\partial v^\lambda}{\partial r} = \lambda \frac{\partial v^\lambda}{\partial r} - \left( \frac{2s}{p - 1} + 2 \right) v^\lambda.$$  

Hence,

$$\int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v^\lambda}{\partial r} \frac{dv^\lambda}{d\lambda} = \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \left( \lambda \frac{dv^\lambda}{d\lambda} - \left( \frac{2s}{p - 1} + 2 \right) v^\lambda \frac{dv^\lambda}{d\lambda} \right)$$

$$= \lambda \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \left( \frac{dv^\lambda}{d\lambda} \right)^2 \left( \frac{s}{p - 1} + 1 \right) \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} y^b (v^\lambda)^2.$$  

Note that

$$- \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta v^\lambda \frac{dv^\lambda}{d\lambda} = \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v^\lambda}{\partial y} \frac{v^\lambda}{d\lambda} - \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} y^b \frac{\partial v^\lambda}{\partial r} \frac{v^\lambda}{d\lambda}$$

$$+ \int_{\mathbb{R}^{n+1}_+ \cap B_1} \nabla v^\lambda \nabla (y^b \frac{dv^\lambda}{d\lambda}).$$
Integration by part we have

\[
\int_{\mathbb{R}^{n+1} \cap B_1} y^b \nabla v^\lambda \frac{d\nu^\lambda}{d\lambda} = - \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial v^\lambda}{\partial y} \frac{d\nu^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1} \cap \partial B_1} y^b \frac{\partial v^\lambda}{\partial r} \frac{d\nu^\lambda}{d\lambda}
\]

\[- \int_{\mathbb{R}^{n+1} \cap B_1} \nabla \cdot (y^b \nabla v^\lambda) \frac{d\nu^\lambda}{d\lambda} = - \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial v^\lambda}{\partial y} \frac{d\nu^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1} \cap \partial B_1} y^b \frac{\partial v^\lambda}{\partial r} \frac{d\nu^\lambda}{d\lambda}
\]

Now

\[- \int_{\mathbb{R}^{n+1} \cap B_1} y^b \Delta v^\lambda \Delta_b \frac{d\nu^\lambda}{d\lambda} = - \int_{\mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda (\Delta \frac{d\nu^\lambda}{d\lambda} + y \nabla \frac{d\nu^\lambda}{d\lambda})
\]

\[- \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda \frac{\partial \nu^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1} \cap \partial B_1} y^b \nabla \Delta_b v^\lambda \nabla \frac{d\nu^\lambda}{d\lambda}
\]

\[- \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda \frac{\partial \nu^\lambda}{\partial y} \frac{d\nu^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1} \cap \partial B_1} y^b \nabla \Delta_b v^\lambda \nabla \frac{d\nu^\lambda}{d\lambda}
\]

\[- \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda \frac{\partial \nu^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda} + \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial \Delta_b v^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda}
\]

\[- \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda \frac{\partial \nu^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda} + \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial \Delta_b v^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda}
\]

\[- \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \Delta_b v^\lambda \frac{\partial \nu^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda} + \int_{\partial \mathbb{R}^{n+1} \cap B_1} y^b \frac{\partial \Delta_b v^\lambda}{\partial n} \frac{d\nu^\lambda}{d\lambda}
\]
Here we have used that $\Delta^2 v_c^\lambda = \Delta^2 u_c^\lambda = 0$. Therefore, combine with the above arguments we get that

\[
\int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \nabla v_c^\lambda \nabla \frac{du_c^\lambda}{d\lambda} = -\int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v_c^\lambda}{\partial y} \frac{du_c^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v_c^\lambda}{\partial r} \frac{du_c^\lambda}{d\lambda} \\
+ \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta v_c^\lambda \frac{\partial}{\partial y} \frac{du_c^\lambda}{d\lambda} - \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \Delta v_c^\lambda \frac{\partial}{\partial r} \frac{du_c^\lambda}{d\lambda} \\
- \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial}{\partial y} \Delta v_c^\lambda \frac{du_c^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial}{\partial r} \Delta v_c^\lambda \frac{du_c^\lambda}{d\lambda}
\]

(3.4)

Here, we have used that $\frac{\partial \Delta v_c^\lambda (x, 0)}{\partial y} = 0$, $\frac{\partial}{\partial y} \frac{du_c^\lambda}{d\lambda} = 0$ on $\partial \mathbb{R}^{n+1}_+$ and $\lim_{y \to 0} y^b \frac{\partial}{\partial y} \Delta v_c^\lambda = -C_{n,s} |u_c^\lambda|^{p-1} u_c^\lambda$. By (3.3) and (3.4) we obtain that

\[
\frac{d}{d\lambda} E(u_c^\lambda, 1) = \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial v_c^\lambda}{\partial r} \frac{du_c^\lambda}{d\lambda} + \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b \frac{\partial w_c^\lambda}{\partial r} \frac{du_c^\lambda}{d\lambda} \\
- \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b w_c^\lambda \frac{\partial}{\partial r} \frac{du_c^\lambda}{d\lambda}
\]

(3.5)

Recall (3.1) and differentiate it with respect to $\lambda$, we have

\[
\frac{du_c^\lambda (X)}{d\lambda} = \frac{1}{\lambda} \left( \frac{2s}{p-1} u_c^\lambda (X) + r \partial_r u_c^\lambda (X) \right),
\]

\[
\frac{dv_c^\lambda (X)}{d\lambda} = \frac{1}{\lambda} \left( \left( \frac{2s}{p-1} + 2 \right) v_c^\lambda (X) + r \partial_r v_c^\lambda (X) \right),
\]

\[
\frac{dw_c^\lambda (X)}{d\lambda} = \frac{1}{\lambda} \left( \left( \frac{2s}{p-1} + 4 \right) w_c^\lambda (X) + r \partial_r w_c^\lambda (X) \right).
\]

Differentiate the above equations with respect to $\lambda$ again we get

\[
\lambda \frac{d^2 u_c^\lambda (X)}{d\lambda^2} + \frac{du_c^\lambda (x)}{d\lambda} = \frac{2s}{p-1} \frac{du_c^\lambda (X)}{d\lambda} + r \partial_r \frac{du_c^\lambda}{d\lambda}.
\]
Hence, for $X \in \mathbb{R}^{n+1}_+ \cap B_1$, we have

\[
\partial_r(u^\lambda_e(X)) = \lambda \frac{du^\lambda_e}{d\lambda} - \frac{2s}{p-1} u_e,
\]

\[
\partial_r(\frac{d\lambda_e^\lambda}{d\lambda}) = \lambda \frac{d^2u^\lambda_e}{d\lambda^2} + (1 - \frac{2s}{p-1}) \frac{d\lambda_e^\lambda}{d\lambda},
\]

\[
\partial_r(v^\lambda_e(X)) = \lambda \frac{dv^\lambda_e}{d\lambda} - (\frac{2s}{p-1} + 2) v_e^\lambda,
\]

\[
\partial_r(w^\lambda_e(X)) = \lambda \frac{dw^\lambda_e}{d\lambda} - (\frac{2s}{p-1} + 4) w_e^\lambda.
\]

Plugging these equations into (3.5), we get that

\[
\frac{d}{d\lambda} E(u^\lambda_e, 1) = \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} y^b(\lambda \frac{dv^\lambda_e}{d\lambda} \frac{d^2\lambda_e^\lambda}{d\lambda^2} - \frac{2s}{p-1} + 2) \frac{dv^\lambda_e}{d\lambda}
\]

\[
+ y^b(\lambda \frac{dw^\lambda_e}{d\lambda} \frac{d\lambda_e^\lambda}{d\lambda} - \frac{2s}{p-1} + 4) \frac{du^\lambda_e}{d\lambda}
\]

\[
- y^b(\lambda \frac{dw^\lambda_e}{d\lambda} \frac{d^2u^\lambda_e}{d\lambda^2} + (1 - \frac{2s}{p-1}) \frac{d\lambda_e^\lambda}{d\lambda})
\]

\[
= \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} y^b[\lambda \frac{dv^\lambda_e}{d\lambda} \frac{d^2\lambda_e^\lambda}{d\lambda^2} - \frac{2s}{p-1} + 2] \frac{dv^\lambda_e}{d\lambda}
\]

\[
+ y^b[\lambda \frac{dw^\lambda_e}{d\lambda} \frac{d\lambda_e^\lambda}{d\lambda} - \lambda \frac{dw^\lambda_e}{d\lambda} \frac{d^2u^\lambda_e}{d\lambda^2} - 5y^b \frac{dw^\lambda_e}{d\lambda} \frac{dv^\lambda_e}{d\lambda}]
\]

\[
:= E_d1(u^\lambda_e, 1) + E_d2(u^\lambda_e, 2).
\]

### 3.2 The calculations of $\frac{\partial}{\partial r} u^\lambda_e$ and $\frac{\partial}{\partial \lambda} u^\lambda_e$, $i, j = 1, 2, 3, 4$

Note

\[
\lambda \frac{du^\lambda_e}{d\lambda} = \frac{2s}{p-1} u_e^\lambda + r \frac{\partial}{\partial r} u_e^\lambda.
\]

Differentiating (3.7) once, twice and thrice with respect to $\lambda$ respectively, we have

\[
\lambda \frac{d^2u^\lambda_e}{d\lambda^2} + \frac{du^\lambda_e}{d\lambda} = \frac{2s}{p-1} \frac{du^\lambda_e}{d\lambda} + r \frac{\partial}{\partial r} \frac{du^\lambda_e}{d\lambda},
\]

\[
\lambda \frac{d^3u^\lambda_e}{d\lambda^3} + 2 \frac{d^2u^\lambda_e}{d\lambda^2} = \frac{2s}{p-1} \frac{d^2u^\lambda_e}{d\lambda^2} + r \frac{\partial}{\partial r} \frac{d^2u^\lambda_e}{d\lambda^2},
\]

\[
\lambda \frac{d^4u^\lambda_e}{d\lambda^4} + 3 \frac{d^3u^\lambda_e}{d\lambda^3} = \frac{2s}{p-1} \frac{d^3u^\lambda_e}{d\lambda^3} + r \frac{\partial}{\partial r} \frac{d^3u^\lambda_e}{d\lambda^3}.
\]

Similarly, differentiating (3.7) once, twice and thrice with respect to $r$ respectively we have

\[
\lambda \frac{\partial}{\partial r} \frac{du^\lambda_e}{d\lambda} = (\frac{2s}{p-1} + 1) \frac{\partial}{\partial r} \frac{u^\lambda_e}{d\lambda} + r \frac{\partial^2}{\partial r^2} \frac{u^\lambda_e}{d\lambda}.
\]

\[
\lambda \frac{\partial}{\partial r} \frac{d^2u^\lambda_e}{d\lambda^2} = \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{d^2u^\lambda_e}{d\lambda^2} + r \frac{\partial}{\partial r} \frac{d^2u^\lambda_e}{d\lambda^2},
\]

\[
\lambda \frac{\partial}{\partial r} \frac{d^3u^\lambda_e}{d\lambda^3} = \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{d^3u^\lambda_e}{d\lambda^3} + r \frac{\partial}{\partial r} \frac{d^3u^\lambda_e}{d\lambda^3}.
\]
\[ \lambda \frac{\partial^2}{\partial r^2} \frac{du^\lambda}{d\lambda} = \left( \frac{2s}{p-1} + 2 \right) \frac{\partial^2}{\partial r^2} u^\lambda_e + r \frac{\partial^3}{\partial r^3} u^\lambda_e, \quad (3.12) \]

\[ \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda}{d\lambda} = \left( \frac{2s}{p-1} + 3 \right) \frac{\partial^3}{\partial r^3} u^\lambda_e + r \frac{\partial^4}{\partial r^4} u^\lambda_e. \quad (3.13) \]

From (3.7), on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), we have

\[ \frac{\partial u^\lambda_e}{\partial r} = \lambda \frac{du^\lambda_e}{d\lambda} - \frac{2s}{p-1} u^\lambda_e. \]

Next from (3.8), on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), we derive that

\[ \frac{\partial}{\partial r} \frac{du^\lambda_e}{d\lambda} = \lambda \frac{d^2 u^\lambda_e}{d\lambda^2} + \left( 1 - \frac{2s}{p-1} \right) \frac{du^\lambda_e}{d\lambda}. \]

From (3.11), combine with the two equations above, on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), we get

\[ \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} = \lambda \frac{\partial^2}{\partial r^2} d \frac{u^\lambda_e}{d\lambda} - \left( 1 + \frac{2s}{p-1} \right) \frac{du^\lambda_e}{d\lambda} - \left( 1 - \frac{2s}{p-1} \right) \frac{du^\lambda_e}{d\lambda}. \]

(3.14)

Differentiating (3.8) with respect to \( r \), and combine with (3.8) and (3.9), we get that

\[ \frac{\partial^2}{\partial r^2} d \frac{u^\lambda_e}{d\lambda} = \lambda \frac{\partial^2}{\partial r^2} d \frac{u^\lambda_e}{d\lambda} = \left( \frac{2s}{p-1} + 1 \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} + r \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda}, \]

\[ \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} = \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} - \left( 2 + \frac{2s}{p-1} \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} + \left( \frac{6s}{p-1} \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} \]

(3.15)

From (3.12), on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), combine with (3.14) and (3.15), we have

\[ \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} = \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} - \left( 2 + \frac{2s}{p-1} \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} - \left( \frac{6s}{p-1} \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} \]

(3.16)

Now differentiating (3.8) once with respect to \( r \), we get

\[ \lambda \frac{\partial^2}{\partial r^2} \frac{d^2 u^\lambda_e}{d\lambda^2} = \left( \frac{2s}{p-1} + 1 \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda} + r \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda}, \]

then on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), we have

\[ \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} = \lambda \frac{\partial^3}{\partial r^3} \frac{du^\lambda_e}{d\lambda} - \left( \frac{2s}{p-1} + 1 \right) \frac{\partial^2}{\partial r^2} \frac{du^\lambda_e}{d\lambda}. \]

(3.17)
Now differentiating (3.9) twice with respect to \( r \), we get
\[
\lambda \frac{\partial}{\partial r} \frac{d^3u_\lambda}{d\lambda^3} = \left( \frac{2s}{p-1} - 1 \right) \frac{\partial}{\partial r} \frac{d^2u_\lambda}{d\lambda^2} + r \frac{\partial^2}{\partial r^2} \frac{d^2u_\lambda}{d\lambda^2},
\]
hence on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), combine with (3.9) and (3.10) there holds
\[
\frac{\partial^2}{\partial r^2} \frac{d^2u_\lambda}{d\lambda^2} = \lambda \frac{\partial}{\partial r} \frac{d^3u_\lambda}{d\lambda^3} + (1 - \frac{2s}{p-1}) \frac{\partial}{\partial r} \frac{d^2u_\lambda}{d\lambda^2}
\]
\[
= \lambda^2 \frac{d^4u_\lambda}{d\lambda^4} + \lambda \left( 4 - \frac{4s}{p-1} \right) \frac{d^3u_\lambda}{d\lambda^3} + \lambda^2 \left( 2 - \frac{2s}{p-1} \right) \frac{d^2u_\lambda}{d\lambda^2}.
\]
Now differentiating (3.8) with respect to \( r \), we have
\[
\lambda \frac{\partial}{\partial r} \frac{d^2u_\lambda}{d\lambda^2} = \frac{2s}{p-1} \frac{\partial}{\partial r} \frac{du_\lambda}{d\lambda} + r \frac{\partial^2}{\partial r^2} \frac{du_\lambda}{d\lambda}.
\]
This combine with (3.8) and (3.9), on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), we have
\[
\frac{\partial^2}{\partial r^2} \frac{du_\lambda}{d\lambda} = \lambda \frac{\partial}{\partial r} \frac{d^2u_\lambda}{d\lambda^2} - 2 \frac{\partial}{\partial r} \frac{du_\lambda}{d\lambda}
\]
\[
= \lambda^2 \frac{d^4u_\lambda}{d\lambda^4} + 2s \frac{d^3u_\lambda}{d\lambda^3} - \frac{2s}{p-1} \frac{d^2u_\lambda}{d\lambda^2}.
\]
Now from (3.17), combine with (3.18) and (3.19), we get
\[
\frac{\partial^3}{\partial r^3} \frac{du_\lambda}{d\lambda} = \lambda^3 \frac{d^4u_\lambda}{d\lambda^4} + \lambda^2 \left( 3 - \frac{6s}{p-1} \right) \frac{d^3u_\lambda}{d\lambda^3} - \lambda \left( 1 - \frac{2s}{p-1} \right) 6s \frac{d^2u_\lambda}{d\lambda^2}
\]
\[
+ (1 - \frac{2s}{p-1}) \left( 1 + \frac{2s}{p-1} \right) \frac{2s}{p-1} \frac{du_\lambda}{d\lambda}.
\]
From (3.13), on \( \mathbb{R}^{n+1}_+ \cap \partial B_1 \), combine with (3.20) then
\[
\frac{\partial^4}{\partial r^4} u_\lambda \frac{du_\lambda}{d\lambda} = \lambda^4 \frac{d^4u_\lambda}{d\lambda^4} - \lambda^3 \frac{8s}{p-1} \frac{d^3u_\lambda}{d\lambda^3} + \lambda^2 \left( 2 + \frac{4s}{p-1} \right) \frac{6s}{p-1} \frac{d^2u_\lambda}{d\lambda^2}
\]
\[
- \lambda \left( 1 + \frac{2s}{p-1} \right) \left( 1 + \frac{s}{p-1} \right) \frac{16s}{p-1} \frac{du_\lambda}{d\lambda}
\]
\[
+ (3 + \frac{2s}{p-1}) (2 + \frac{2s}{p-1}) \frac{2s}{p-1} \frac{du_\lambda}{d\lambda}.
\]
In summary, we have that
\[
\frac{\partial^3}{\partial r^3} u_\lambda \frac{du_\lambda}{d\lambda} = \lambda^3 \frac{d^3u_\lambda}{d\lambda^3} - \lambda^2 \frac{6s}{p-1} \frac{d^2u_\lambda}{d\lambda^2} + \lambda \left( \frac{6s}{p-1} + \frac{12s^2}{(p-1)^2} \right) \frac{du_\lambda}{d\lambda}
\]
\[
- (2 + \frac{2s}{p-1}) \left( 1 + \frac{2s}{p-1} \right) \frac{2s}{p-1} u_\lambda.
\]
and
\[
\frac{\partial^2 u^\lambda}{\partial r^2} = \lambda^2 \frac{d^2 u^\lambda}{d\lambda^2} - \lambda \frac{4s}{p-1} \frac{d u^\lambda}{d\lambda} + (1 + \frac{2s}{p-1}) \frac{2s}{p-1} u^\lambda \frac{d u^\lambda}{d\lambda} - \frac{2s}{p-1} u^\lambda.
\]

### 3.3 On the operator $\Delta^2_b$ and its representation

Note that
\[
\Delta_b u = y^{-b} \nabla \cdot (y^b \nabla u) = u_{rr} + \frac{n+b}{r} u_r + \frac{1}{r^2} \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} u),
\]
where $\theta_1 = \frac{x}{r}, r = \sqrt{|x|^2 + y^2}$. Set $v = \Delta_b u$ and $\Delta_b^2 u := w$. Then
\[
w = \Delta_b v = v_{rr} + \frac{n+b}{r} v_r + \frac{1}{r^2} \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} v)
= \partial_{rrrr} u + \frac{2(n+b)}{r} \partial_{rr} u + \frac{(n+b)(n+b-2)}{r^2} \partial_{rr} u - \frac{(n+b)(n+b-2)}{r^3} \partial_r u
+ r^{-4} \theta_1^{-b} \text{div}_{S^n} (\theta_1^{b-1} \nabla_{S^n} (\theta_1^b \nabla_{S^n} u))
+ 2r^{-4} \theta_1^{-b} \text{div}_{S^n} (\theta_1^{b-1} \nabla_{S^n} (u_{rr} + \frac{n+b-2}{r} u_r))
- 2(n+b-3) r^{-4} \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} u).
\]

On $\mathbb{R}^{n+1}_+ \cap \partial B_1$, we have
\[
w = \partial_{rrrr} u + 2(n+b) \partial_{rr} u + (n+b)(n+b-2) \partial_{rr} u - (n+b)(n+b-2) \partial_r u
+ \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} (\theta_1^{b-1} \nabla_{S^n} (\theta_1^b \nabla_{S^n} u)))
+ 2 \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} (u_{rr} + \frac{n+b-2}{r} u_r))
- 2(n+b-3) \theta_1^{-b} \text{div}_{S^n} (\theta_1^b \nabla_{S^n} u)
:= I(u) + J(u) + K(u) + L(u).
\]
By these notations, we can rewrite the term $E_{d2}(u^\lambda_e, 1)$ appear in (3.6) as following

$$E_{d2}(u^\lambda_e, 1)$$

$$= \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^h_1 \left( \frac{d}{d\lambda} \frac{d}{d\lambda^2} \lambda^2 u^\lambda_e \right) - \lambda \theta^h_1 I(u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} - 5\theta^h_1 I(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda}$$

$$+ \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda \theta^h_1 \left( \frac{d}{d\lambda} I(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} - \lambda \theta^h_1 I(u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} - 5\theta^h_1 I(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} \right)$$

$$+ \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda \theta^h_1 \left( \frac{d}{d\lambda} K(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} - \lambda \theta^h_1 K(u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} - 5\theta^h_1 K(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} \right)$$

$$+ \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda \theta^h_1 \left( \frac{d}{d\lambda} L(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} - \lambda \theta^h_1 L(u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} - 5\theta^h_1 L(u^\lambda_e) \frac{d u^\lambda_e}{d\lambda} \right) \tag{3.21}$$

we define as

$$E_{d2}(u^\lambda_e, 1) := I + J + K + L$$

$$:= I_1 + I_2 + I_3 + J_1 + J_2 + J_3 + K_1 + K_2 + K_3 + L_1 + L_2 + L_3,$$

where $I_1, I_2, I_3, J_1, J_2, J_3, K_1, K_2, K_3, L_1, L_2, L_3$ are corresponding successively to the 12 terms in (3.21). By the conclusions of subsection 2.2, we have

$$I(u^\lambda_e) = \frac{\partial}{\partial r} u^\lambda_e + 2(n + b)\frac{\partial}{\partial r} u^\lambda_e - (n + b)(n + b - 2)\frac{\partial}{\partial r} u^\lambda_e$$

$$= \lambda^2 \left( \frac{12}{p - 1} \frac{2s}{p - 1} - (n + b) \frac{12}{p - 1} \frac{2s}{p - 1} + (n + b)(n + b - 2) \right) \frac{d^2 u^\lambda_e}{d\lambda^2}$$

$$+ \lambda^2 \left( \frac{12}{p - 1} \frac{2s}{p - 1} (2 + \frac{2s}{p - 1}) + (n + b) \frac{6}{p - 1} \frac{2s}{p - 1} \right) \frac{d^2 u^\lambda_e}{d\lambda^2}$$

$$+ \frac{12}{p - 1} \frac{2s}{p - 1} \frac{d^2 u^\lambda_e}{d\lambda^2}$$

$$+ \lambda^2 \left( \frac{12}{p - 1} \frac{2s}{p - 1} (2 + \frac{2s}{p - 1}) \frac{2s}{p - 1} \right) \frac{d^2 u^\lambda_e}{d\lambda^2}$$

$$+ \lambda^2 \left( \frac{12}{p - 1} \frac{2s}{p - 1} (2 + \frac{2s}{p - 1}) \frac{2s}{p - 1} \right) \frac{d^2 u^\lambda_e}{d\lambda^2}$$

$$+ \lambda^2 \left( \frac{12}{p - 1} \frac{2s}{p - 1} (2 + \frac{2s}{p - 1}) \frac{2s}{p - 1} \right) \frac{d^2 u^\lambda_e}{d\lambda^2} \tag{3.22}$$
For convenience, we denote that
\[ I(u_\lambda^e) = \lambda^4 \frac{d^4 u_\lambda^e}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u_\lambda^e}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u_\lambda^e}{d\lambda^2} + \lambda \delta_3 \frac{du_\lambda^e}{d\lambda} + \delta_4 u_\lambda^e, \tag{3.23} \]

where \( \delta_i \) are the corresponding coefficients of \( \lambda^i \frac{d^i u_\lambda^e}{d\lambda^i} \) appeared in (3.22) for \( i = 1, 2, 3, 4 \). Now taking the derivative of (3.23) with respect to \( \lambda \), we get
\[ \frac{d}{d\lambda} I(u_\lambda^e) = \lambda^4 \frac{d^5 u_\lambda^e}{d\lambda^5} + \lambda^3 (\delta_1 + 4) \frac{d^4 u_\lambda^e}{d\lambda^4} + \lambda^2 (3\delta_1 + \delta_2) \frac{d^3 u_\lambda^e}{d\lambda^3} + \lambda (2\delta_2 + \delta_3) \frac{d^2 u_\lambda^e}{d\lambda^2} + (\delta_3 + \delta_4) \frac{du_\lambda^e}{d\lambda} \tag{3.24} \]

and
\[ \partial_{rr} u_\lambda^e + (n + b - 2) \partial_r u_\lambda^e = \lambda^2 \frac{d^2 u_\lambda^e}{d\lambda^2} + \lambda (n + b - 2 - \frac{4s}{p-1}) \frac{du_\lambda^e}{d\lambda} + \frac{2s}{p-1} \left( 3 + \frac{2s}{p-1} - n - b \right) u_\lambda^e \tag{3.25} \]

\[ := \lambda^2 \frac{d^2 u_\lambda^e}{d\lambda^2} + \lambda \alpha \frac{du_\lambda^e}{d\lambda} + \beta u_\lambda^e. \]

Hence,
\[ \frac{d}{d\lambda} \left[ \partial_{rr} u_\lambda^e + (n + b - 2) \partial_r u_\lambda^e \right] = \lambda^2 \frac{d^3 u_\lambda^e}{d\lambda^3} + \lambda (\alpha + 2) \frac{d^2 u_\lambda^e}{d\lambda^2} + (\alpha + \beta) \frac{du_\lambda^e}{d\lambda}. \tag{3.26} \]

here \( \alpha = n + b - 2 - \frac{4s}{p-1} \) and \( \beta = \frac{2s}{p-1} \left( 3 + \frac{2s}{p-1} - n - b \right) \).
3.4 The computations of $I_1$, $I_2$, $I_3$ and $I$

$$I_1 := \int_{\mathbb{R}^n_{\geq 1} \cap \partial B_1} \lambda \theta^1 \frac{d}{d\lambda} I(\lambda) \frac{du_1^\lambda}{d\lambda}$$

$$= \int_{\mathbb{R}^n_{\geq 1} \cap \partial B_1} \theta^1 \left( \lambda^5 \frac{d^5 u_1^\lambda}{d\lambda^5} + \lambda^4 (4 + \delta_1) \frac{d^4 u_1^\lambda}{d\lambda^4} + \lambda^3 (3 \delta_1 + \delta_2) \frac{d^3 u_1^\lambda}{d\lambda^3} ight. $$

$$+ \lambda^2 (2 \delta_2 + \delta_3) \frac{d^2 u_1^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du_1^\lambda}{d\lambda} + \left. \left( \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du_1^\lambda}{d\lambda} \right) \right)$$

$$= \frac{d}{d\lambda} \int_{\mathbb{R}^n_{\geq 1} \cap \partial B_1} \theta^1 \left[ \lambda^5 \frac{d^4 u_1^\lambda}{d\lambda^4} \frac{du_1^\lambda}{d\lambda} - \lambda^5 \frac{d^3 u_1^\lambda}{d\lambda^3} \frac{du_1^\lambda}{d\lambda} + \left( \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + 3 \delta_1 - \delta_2 + \delta_3 - 12 \lambda^2 \frac{du_1^\lambda}{d\lambda} \right) \right]$$

$$+ \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du_1^\lambda}{d\lambda} \right) \right)$$

$$+ \left. \left( \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + 3 \delta_1 - \delta_2 + \delta_3 - 12 \lambda^2 \frac{du_1^\lambda}{d\lambda} \right) \right) \right]$$

$$+ \left. \left( \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + 3 \delta_1 - \delta_2 + \delta_3 - 12 \lambda^2 \frac{du_1^\lambda}{d\lambda} \right) \right) \right]$$

$$+ \left. \frac{d}{d\lambda} \int_{\mathbb{R}^n_{\geq 1} \cap \partial B_1} \theta^1 \left[ (12 - 3 \delta_1 + \delta_2 + \delta_4) \lambda (\frac{du_1^\lambda}{d\lambda})^2 \right] $$

$$+ (\delta_1 - 4 - ？ \delta_2) \lambda^3 \frac{d^2 u_1^\lambda}{d\lambda^2} + \lambda^5 (\frac{du_1^\lambda}{d\lambda})^2 \right) \right]$$

$$+ \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + \lambda (\delta_3 + \delta_4) \frac{du_1^\lambda}{d\lambda} \right) \right)$$

$$+ \left. \left( \lambda^2 \frac{d^2 u_1^\lambda}{d\lambda^2} + 3 \delta_1 - \delta_2 + \delta_3 - 12 \lambda^2 \frac{du_1^\lambda}{d\lambda} \right) \right) \right]$$

where $\delta_i (i = 1, 2, 3, 4)$ are defined in (3.22) and (3.23). Denote $f = u_1^\lambda, f' := \frac{du_1^\lambda}{d\lambda}$, we have used the following differential identities:

$$\lambda^5 f'''' f' = [\lambda^5 f'''' f' - 5 \lambda^4 f''' f' - 5 \lambda^2 f'' f' + 20 \lambda^3 f'' f' + 30 \lambda^2 f'' f']'$$

$$+ 60 \lambda (f')^2 - 20 \lambda^3 (f'')^2 + \lambda^5 (f''')^2 + 10 \lambda^4 f''' f'',$$

$$\lambda^4 f''''' f' = [\lambda^4 f''''' f' - 4 \lambda^4 f'' f' - 6 \lambda^2 f f']' - 12 \lambda (f')^2 + 4 \lambda^3 (f'')^2 - \lambda^4 f''' f'',$$

$$\lambda^3 f'''' f' = [\lambda^3 f'''' f' - \frac{3 \lambda^2}{2} f' f']' + 3 \lambda (f')^2 - \lambda^3 (f'')^2,$$

and

$$\lambda^2 f'' f' = \left[ \frac{\lambda^2}{2} f' f' \right]' - \lambda (f')^2.$$
\[ I_2 := -\lambda \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b I(u^b_c) \frac{d^2 u^\lambda_c}{d\lambda^2} \]
\[ = -\lambda \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left( \lambda^4 \frac{d^4 u^\lambda_c}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u^\lambda_c}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \delta_3 \frac{du^\lambda_c}{d\lambda} + \delta_4 u^\lambda_c \right) \frac{d^2 u^\lambda_c}{d\lambda^2} \]
\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ -\lambda^5 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{d^2 u^\lambda_c}{d\lambda^2} - \delta_4 \lambda \frac{du^\lambda_c}{d\lambda} u^\lambda_c \right] \]
\[ + \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ \lambda^5 \left( \frac{d^3 u^\lambda_c}{d\lambda^3} \right)^2 - \delta_2 \lambda^5 \left( \frac{d^2 u^\lambda_c}{d\lambda^2} \right)^2 + \delta_4 \lambda \left( \frac{du^\lambda_c}{d\lambda} \right)^2 \right] \]
\[ + \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ (5 - \delta_1) \lambda^3 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{d^2 u^\lambda_c}{d\lambda^2} - \delta_3 \lambda^2 \frac{d^2 u^\lambda_c}{d\lambda^2} + \delta_4 \frac{du^\lambda_c}{d\lambda} u^\lambda_c \right]. \]

(3.28)

Here we have used that
\[-\lambda^5 f'''' f''' = \left[ -\lambda^5 f'''' f''' \right]' + 5\lambda^4 f''' f'' + \lambda^5 (f''')^2\]
and
\[-\lambda f''' f' = [-\lambda f'']' + f'' f + \lambda (f'')^2.\]

\[ I_3 := -5 \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b I(u^b_c) \frac{du^\lambda_c}{d\lambda} \]
\[ = -5 \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ \lambda^4 \frac{d^4 u^\lambda_c}{d\lambda^4} + \lambda^3 \delta_1 \frac{d^3 u^\lambda_c}{d\lambda^3} + \lambda^2 \delta_2 \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \delta_3 \frac{du^\lambda_c}{d\lambda} + \delta_4 u^\lambda_c \right] \frac{du^\lambda_c}{d\lambda} \]
\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ -5 \lambda^4 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{du^\lambda_c}{d\lambda} + (20 - 5\delta_1) \lambda^3 \frac{d^2 u^\lambda_c}{d\lambda^2} \frac{du^\lambda_c}{d\lambda} \right] \]
\[ + \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ (5\delta_1 - 20) \lambda^3 \frac{d^2 u^\lambda_c}{d\lambda^2} \frac{du^\lambda_c}{d\lambda} - 5\delta_3 \lambda \frac{du^\lambda_c}{d\lambda} \right] \]
\[ + \int_{\mathbb{R}^n_+ \cap \partial B_1} \theta^b \left[ 5\lambda^4 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{d^2 u^\lambda_c}{d\lambda^2} + (15\delta_1 - 60 - 5\delta_2) \lambda^3 \frac{d^2 u^\lambda_c}{d\lambda^2} \frac{du^\lambda_c}{d\lambda} - 5\delta_4 \lambda \frac{du^\lambda_c}{d\lambda} u^\lambda_c \right]. \]

(3.29)

Here we have use that
\[-\lambda^4 f''''' f''' = \left[ -5\lambda^4 f''''' f''' + 20\lambda^3 f'''' f'' \right]' - 20\lambda^3 (f''')^2 - 60\lambda^2 f''' f' + 5\lambda^4 f'''' f''\]
and
\[-\lambda^3 f''' f' = \left[ -\lambda^3 f'' f' \right]' + 3\lambda^2 f'' f' + \lambda^3 (f'')^2.\]
Now we add up $I_1, I_2, I_3$ and further integrate by part, we can get the term $\mathcal{I}$.

$$\mathcal{I} := I_1 + I_2 + I_3$$

$$= \frac{d}{d\lambda} \int_{\mathbb{B}^{n+1} \cap \partial B_1} \theta^i_0 \left[ \lambda^5 \frac{d^4 u^\lambda_c}{d\lambda^4} \frac{du^\lambda_c}{d\lambda} - 2\lambda^5 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{d^2 u^\lambda_c}{d\lambda^2} \right]$$

$$+ (\delta_1 - 6)\lambda^3 \frac{d^3 u^\lambda_c}{d\lambda^3} \frac{du^\lambda_c}{d\lambda} + (24 - 6\delta_1 + \delta_2)\lambda^3 \frac{d^2 u^\lambda_c}{d\lambda^2} \frac{du^\lambda_c}{d\lambda}$$

$$+ (9\delta_1 - 3\delta_2 - 36)\lambda^2 \frac{du^\lambda_c}{d\lambda} \frac{du^\lambda_c}{d\lambda}$$

$$+ (8 - \delta_1)\lambda^4 \left( \frac{d^2 u^\lambda_c}{d\lambda^2} \right)^2 - \delta_4 \lambda \frac{du^\lambda_c}{d\lambda} u_c^\lambda - 2\delta_4 (u_c^\lambda)^2$$

$$+ \int_{\mathbb{B}^{n+1} \cap \partial B_1} \theta^i_1 \left( 2\lambda^5 \frac{d^3 u^\lambda_c}{d\lambda^3} \right)^2 + (10\delta_1 - 2\delta_2 - 56)\lambda^3 \frac{d^2 u^\lambda_c}{d\lambda^2} \right)^2$$

$$+ (-18\delta_1 + \delta_2 + 4\delta_3 + 2\delta_4 + 72)\lambda \left( \frac{du^\lambda_c}{d\lambda} \right)^2 \right).$$

Since $u_c^\lambda(X) = \lambda^{\frac{2s}{p-1}} u_c(\lambda X)$, we have the following

$$\lambda^4 \frac{d^4 u^\lambda_c}{d\lambda^4} = \lambda^{\frac{4s}{p-1}} \left[ \frac{2s}{p-1} - \frac{2s}{p-1} - 1 \right] \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) (\frac{2s}{p-1} - 3) u_c(\lambda X)$$

$$+ \frac{8s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \left( \frac{2s}{p-1} - 2 \right) \lambda \partial_r u_c(\lambda X)$$

$$+ \frac{12s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \lambda^2 \partial_{rr} u_c(\lambda X)$$

$$+ \frac{8s}{p-1} \lambda^3 \partial_{rrr} u_c(\lambda X) + \lambda^4 \partial_{rrrr} u_c(\lambda X) \right],$$

and

$$\lambda^3 \frac{d^3 u^\lambda_c}{d\lambda^3} = \lambda^{\frac{3s}{p-1}} \left[ \frac{2s}{p-1} - \frac{2s}{p-1} - 1 \right] \left( \frac{2s}{p-1} - 2 \right) u_c(\lambda X)$$

$$+ \frac{6s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \lambda \partial_r u_c(\lambda X)$$

$$+ \frac{6s}{p-1} \lambda^2 \partial_{rr} u_c(\lambda X) + \lambda^3 \partial_{rrr} u_c(\lambda X) \right],$$

$$\lambda^2 \frac{d^2 u^\lambda_c}{d\lambda^2}$$

$$= \lambda^{\frac{2s}{p-1}} \left[ \frac{2s}{p-1} - \frac{2s}{p-1} - 1 \right] u_c(\lambda X) + \frac{4s}{p-1} \lambda \partial_r u_c(\lambda X) + \lambda^2 \partial_{rr} u_c(\lambda X) \right],$$

and

$$\lambda \frac{du^\lambda_c}{d\lambda} = \lambda^{\frac{s}{p-1}} \left[ \frac{2s}{p-1} u_c(\lambda X) + \lambda \partial_r u_c(\lambda X) \right].$$
Hence, by scaling we have the following derivatives:

\[
\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \theta_1^b \lambda^5 \frac{d^4 u_e^*}{d\lambda^4} \frac{d u_e}{d\lambda} = \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \lambda^{2s+\frac{n-5}{2}} - \frac{2s}{2s+1}(\frac{2s}{p-1} - 1)(\frac{2s}{p-1} - 2)(\frac{2s}{p-1} - 3)u_e
\]
\[
+ \frac{8s}{p-1}(\frac{2s}{p-1} - 1)\lambda \partial_r u_e + \frac{12s}{p-1}(\frac{2s}{p-1} - 1)\lambda^2 \partial_{rr} u_e
\]
\[
+ \frac{8s}{p-1} \lambda^3 \partial_{rrr} u_e + \lambda^4 \partial_{rrrr} u_e [\frac{2s}{p-1} u_e + r \lambda \partial_r u_e];
\]

\[
\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \theta_1^b \lambda^3 \frac{d^3 u_e^*}{d\lambda^3} \frac{d^2 u_e}{d\lambda^2} \frac{d u_e}{d\lambda} = \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \lambda^{2s+\frac{n-5}{2}} - \frac{2s}{2s+1}(\frac{2s}{p-1} - 1)(\frac{2s}{p-1} - 2)u_e
\]
\[
+ \frac{6s}{p-1}(\frac{2s}{p-1} - 1)\lambda \partial_r u_e + \frac{6s}{p-1} \lambda^2 \partial_{rr} u_e + \lambda^3 \partial_{rrr} u_e
\]
\[
[\frac{2s}{p-1} u_e + \lambda \partial_r u_e];
\]

\[
\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \theta_1^b \lambda^3 \frac{d^2 u_e^*}{d\lambda^2} \frac{d u_e}{d\lambda} \frac{d u_e}{d\lambda} = \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \lambda^{2s+\frac{n-5}{2}} - \frac{2s}{2s+1}(\frac{2s}{p-1} - 1)u_e
\]
\[
+ \frac{4s}{p-1} \lambda \partial_r u_e + \lambda^2 \partial_{rr} u_e [\frac{2s}{p-1} u_e + \lambda \partial_r u_e]
\]

and

\[
\frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \theta_1^b \lambda^2 \frac{d u_e^*}{d\lambda} \frac{d u_e}{d\lambda} \frac{d u_e}{d\lambda} = \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1}\cap \partial B_1} \lambda^{2s+\frac{n-5}{2}} - \frac{2s}{2s+1}(\frac{2s}{p-1} - 1)u_e + \lambda \partial_r u_e]^2.
\]
Further,
\[
\frac{d}{d\lambda} \int_{\mathbb{R}^{n+1} \cap \partial B_1} \theta_1^\lambda \frac{d^2 u_e}{d\lambda^2} \frac{d^2 u_e}{d\lambda^2} = d \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1} \cap \partial B_1} \lambda^{\frac{2s+1}{p-1} - n - 5} y^b \left[ \frac{2s}{p-1}(\frac{2s}{p-1} - 1) u_e \right] \\
+ \frac{4s}{p-1} \lambda \partial_r u_e + \lambda^2 \partial_{rr} u_e \right]^2,
\]

and
\[
\frac{d}{d\lambda} \int_{\mathbb{R}^{n+1} \cap \partial B_1} \theta_1^\lambda \frac{d u_e}{d\lambda} u_e^2 = d \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1} \cap \partial B_1} \lambda^{\frac{2s+1}{p-1} - n - 5} y^b u_e^2.
\]

### 3.5 The computations of $J_i, K_i, L_i$ ($i = 1, 2, 3$) and $\mathcal{J}, \mathcal{K}, \mathcal{L}$

Firstly,
\[
J_1 := \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \frac{d}{d\lambda} \frac{d u_e}{d\lambda} = \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \lambda \theta_1^\lambda \left( \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \right) \frac{d u_e}{d\lambda}
\]

\[
= - \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \nabla_{\mathcal{S}_n} \frac{d u_e}{d\lambda}
\]

\[
= \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \theta_1^\lambda \left( \Delta_{\mathcal{S}_n} \frac{d u_e}{d\lambda} \right)^2,
\]

3.5 The computations of $J_i, K_i, L_i$ ($i = 1, 2, 3$) and $\mathcal{J}, \mathcal{K}, \mathcal{L}$

Firstly,
\[
J_1 := \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \frac{d}{d\lambda} \frac{d u_e}{d\lambda} = \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \lambda \theta_1^\lambda \left( \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \right) \frac{d u_e}{d\lambda}
\]

\[
= - \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \nabla_{\mathcal{S}_n} \frac{d u_e}{d\lambda}
\]

\[
= \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \theta_1^\lambda \left( \Delta_{\mathcal{S}_n} \frac{d u_e}{d\lambda} \right)^2,
\]

3.5 The computations of $J_i, K_i, L_i$ ($i = 1, 2, 3$) and $\mathcal{J}, \mathcal{K}, \mathcal{L}$

Firstly,
\[
J_1 := \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \frac{d}{d\lambda} \frac{d u_e}{d\lambda} = \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \lambda \theta_1^\lambda \left( \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \right) \frac{d u_e}{d\lambda}
\]

\[
= - \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \nabla_{\mathcal{S}_n} \left( \theta_1^\lambda \frac{d u_e}{d\lambda} \right) \nabla_{\mathcal{S}_n} \frac{d u_e}{d\lambda}
\]

\[
= \lambda \int_{\mathbb{R}^{n+1} \cap \partial B_1} \theta_1^\lambda \left( \Delta_{\mathcal{S}_n} \frac{d u_e}{d\lambda} \right)^2,
\]
here we have used integrate by part formula on the unit sphere $S^n$.

\[
J_2 := - \lambda \int_{B^+_n \cap \partial B_1} \theta^b J(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\
= - \lambda \int_{B^+_n \cap \partial B_1} \text{div}_{S^n}(\theta^b \nabla_{S^n}(\theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda))) \frac{d^2 u^\lambda}{d\lambda^2} \\
= \lambda \int_{B^+_n \cap \partial B_1} \theta^b \nabla_{S^n}(\theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda)) \nabla_{S^n} \frac{d^2 u^\lambda}{d\lambda^2} \\
= - \lambda \int_{B^+_n \cap \partial B_1} \theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda) \\
= \frac{d}{d\lambda} \int_{B^+_n \cap \partial B_1} \lambda \theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} \\
+ \lambda \int_{B^+_n \cap \partial B_1} \theta_{-b} \left[ \frac{d}{d\lambda} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda) \right]^2 \\
= \frac{d}{d\lambda} \left( \int_{B^+_n \cap \partial B_1} \theta^b \left( - \frac{1}{2} \lambda \frac{d}{d\lambda} (\Delta_{S^n} u^\lambda)^2 + \frac{1}{2} (\Delta_{S^n} u^\lambda)^2 \right) \right) \\
+ \int_{B^+_n \cap \partial B_1} \theta^b \lambda (\Delta_{S^n} \frac{d u^\lambda}{d\lambda})^2,
\]

here we denote that $g = \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda)$, $g' = \frac{d}{d\lambda} \text{div}_{S^n}(\theta^b \nabla_{S^n} u^\lambda)$ and we have used that

\[-\lambda g g' = [-gg']' + gg' + \lambda (g')^2 = [-gg' + \frac{1}{2} g^2]' + \lambda (g')^2.\]

Further,

\[
J_3 := - 5 \int_{B^+_n \cap \partial B_1} \theta^b J(u^\lambda) \frac{du^\lambda}{d\lambda} \\
= - 5 \int_{B^+_n \cap \partial B_1} \text{div}_{S^n}(\theta^b \nabla_{S^n}(\theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u))) \frac{du^\lambda}{d\lambda} \\
= 5 \int_{B^+_n \cap \partial B_1} \theta^b \nabla_{S^n}(\theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u)) \nabla_{S^n} \frac{du^\lambda}{d\lambda} (3.34) \\
= - 5 \int_{B^+_n \cap \partial B_1} \theta_{-b} \text{div}_{S^n}(\theta^b \nabla_{S^n} u) \frac{d}{d\lambda} \text{div}_{S^n}(\theta^b \nabla_{S^n} u) \\
= - \frac{5}{2} \frac{d}{d\lambda} \int_{B^+_n \cap \partial B_1} \theta^b (\Delta_{S^n} u^\lambda)^2.
\]
Therefore, combine with (3.32), (3.33) and (3.34), we get that

\[ J := J_1 + J_2 + J_3 \]

\[ = 2\lambda \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b-} \left[ \frac{d}{d\lambda} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right]^2 \]

\[ - 4 \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{-b} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u \right) \frac{d}{d\lambda} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u \right) \]

\[ + \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} - \lambda \theta_1^{b} \left[ \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right] \frac{d}{d\lambda} \left[ \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right] \]

\[ = 2\lambda \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b-} \left[ \frac{d}{d\lambda} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right]^2 \]

\[ - 2 \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{-b} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \]

\[ + \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} - \lambda \theta_1^{-b} \left[ \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right] \frac{d}{d\lambda} \left[ \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right] \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b} \left( - 2 (\Delta_{S^n} u_e^\lambda)^2 - \frac{1}{2} \lambda \frac{d}{d\lambda} (\Delta_{S^n} u_e^\lambda)^2 \right) \]

\[ + 2\lambda \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b} (\Delta_{S^n} u_e^\lambda)^2 \].

Note that

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b} \left[ \frac{d}{d\lambda} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right]^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\frac{n+4}{n+2}} \lambda^{-5} \left( \lambda^2 \Delta_b u_e - \lambda^2 \partial_{rr} u_e - (n + b) \lambda \partial_r u_e \right)^2, \]

and

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta_1^{b} \lambda \frac{d}{d\lambda} \left[ \frac{d}{d\lambda} \text{div}_{S^n} \left( \theta_1^{b} \nabla_{S^n} u_e^\lambda \right) \right]^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\frac{n+4}{n+2}} \lambda^{-4} \frac{d}{d\lambda} \left( \lambda^2 \Delta_b u_e - \lambda^2 \partial_{rr} u_e - (n + b) \lambda \partial_r u_e \right)^2. \]
Next we compute $K_1, K_2, K_3$ and $K$.

$$K_1 = \lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \frac{d}{d\lambda} K(u^\lambda_c) \frac{d u^\lambda_c}{d\lambda}$$

$$= 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \text{div}_{S^n} \left( \theta^b \nabla_{S^n} \left( \frac{d}{d\lambda} \left( (\partial_{rr} + (a + b - 2)\partial_r)u^\lambda_c \right) \right) \right) \frac{d u^\lambda_c}{d\lambda}$$

$$= 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \text{div}_{S^n} \left( \theta^b \nabla_{S^n} \left( \lambda^3 \frac{d^3 u^\lambda_c}{d\lambda^3} + \lambda^2 (\alpha + 2) \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \left( \alpha + \beta \right) \frac{d u^\lambda_c}{d\lambda} \right) \right) \frac{d u^\lambda_c}{d\lambda}$$

$$= -2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \nabla_{S^n} \left( \lambda^3 \frac{d^3 u^\lambda_c}{d\lambda^3} + \lambda^2 (\alpha + 2) \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \left( \alpha + \beta \right) \frac{d u^\lambda_c}{d\lambda} \right) \nabla_{S^n} \frac{d u^\lambda_c}{d\lambda}$$

$$+ \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} -\lambda^3 \theta^b \frac{d}{d\lambda} \left( \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c \right)^2 + (2 - 2\alpha)\lambda^2 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c \right)$$

$$\cdot \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u^\lambda_c \right)^2 + 2\lambda^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u^\lambda_c \right)^2$$

$$- (2\alpha + 2\beta)\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c \right)^2. \quad (3.37)$$

Here we denote that $h = \nabla_{S^n} u^\lambda_c, h' = \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c$, and have used that

$$- \lambda^3 h'h'' = \left[ - \frac{\lambda^3}{2} \frac{d}{d\lambda} (h')^2 \right]' + 3\lambda^2 h'h'' + \lambda^3 (h'')^2.$$  

Next,

$$K_2 := -\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b K(u^\lambda_c) \frac{d^2 u^\lambda_c}{d\lambda^2}$$

$$= -2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \text{div}_{S^n} \left( \theta^b \nabla_{S^n} \left( \lambda^2 \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \left( \alpha + \beta \right) \frac{d u^\lambda_c}{d\lambda} \right) \right) \frac{d^2 u^\lambda_c}{d\lambda^2}$$

$$= 2\lambda \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \nabla_{S^n} \left( \lambda^2 \frac{d^2 u^\lambda_c}{d\lambda^2} + \lambda \left( \alpha + \beta \right) \frac{d u^\lambda_c}{d\lambda} \right) \nabla_{S^n} \frac{d^2 u^\lambda_c}{d\lambda^2}$$

$$= \frac{d}{d\lambda} \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left[ 2\beta \lambda \nabla_{S^n} u^\lambda_c \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c - \beta \nabla_{S^n} (u^\lambda_c)^2 \right]$$

$$+ 2\lambda^3 \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d^2}{d\lambda^2} \nabla_{S^n} u^\lambda_c \right)^2 - 2\lambda \beta \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c \right)^2$$

$$+ 2\lambda^2 \alpha \int_{\mathbb{R}_+^{n+1} \cap \partial B_1} \theta^b \left( \frac{d}{d\lambda} \nabla_{S^n} u^\lambda_c \right)^2 \frac{d^2}{d\lambda^2} \nabla_{S^n} u^\lambda_c. \quad (3.38)$$

Here we have used that

$$2\lambda hh'' = [2\lambda hh' - h^2]' - 2\lambda (h')^2.$$
Further,

\[ K_3 := -5 \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 K(u^i_0) \frac{du^i}{d\lambda} \]

\[ = -10 \int_{\mathbb{R}^n_{+} \cap \partial B_1} \text{div}_{\mathbb{R}^n} (\theta^i_1 \nabla S^u (\lambda^2 \frac{d^2 u^i}{d\lambda^2} + \lambda \frac{du^i}{d\lambda} + \beta u^i \frac{d\lambda}{d\lambda})) \frac{du^i}{d\lambda} \]

\[ = 10 \int_{\mathbb{R}^n_{+} \cap \partial B_1} \frac{d\lambda}{d\lambda} \left[ \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 \nabla S^u (\lambda^2 \frac{du^i}{d\lambda}) \nabla S^u u^i \right] + 10 \alpha \int_{\mathbb{R}^n_{+} \cap \partial B_1} \frac{d\lambda}{d\lambda} \left( \nabla S^u u^i \right)^2 \]

\[ + 10 \lambda^2 \int_{\mathbb{R}^n_{+} \cap \partial B_1} \frac{d\lambda}{d\lambda} \left[ \theta^i_1 \frac{d^2}{d\lambda^2} \nabla S^u u^i \right]^2 \]

Now combine with (3.37), (3.38) and (3.39), we get that

\[ K := K_1 + K_2 + K_3 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 \left[ - \lambda^3 \frac{d}{d\lambda} (\nabla S^u u^i)^2 \right] \]

\[ + 2 \beta \lambda \frac{d}{d\lambda} \left( \nabla S^u u^i \right)^2 + 4 \beta \left( \nabla S^u u^i \right)^2 + 6 \lambda^2 (\nabla S^u \frac{du^i}{d\lambda})^2 \]

\[ + 4 \lambda^3 \frac{d}{d\lambda} \left( \nabla S^u u^i \right)^2 \]

\[ + (8 \alpha - 4 \beta - 12) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 (\frac{d}{d\lambda}) \left( \nabla S^u u^i \right)^2. \]

Notice that by scaling we have

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 (\nabla S^u u^i)^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \lambda^2 \frac{d^{s+1}}{d\lambda^{s+1}} - n - 4 \frac{d^2}{d\lambda^2} \left[ \lambda^2 (\nabla u)^2 - \lambda^2 |\partial_r u|^2 \right]. \]

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 \lambda \frac{d}{d\lambda} (\nabla S^u u^i)^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \lambda^{2s+1} \frac{d^{s+1}}{d\lambda^{s+1}} - n - 4 \frac{d^2}{d\lambda^2} \left[ \lambda^2 (\nabla u)^2 - \lambda^2 |\partial_r u|^2 \right] \]

and

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^i_1 \lambda^3 \frac{d}{d\lambda} (\nabla S^u u^i)^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \lambda^{2s+1} \frac{d^{s+1}}{d\lambda^{s+1}} - n - 4 \frac{d^2}{d\lambda^2} \left[ \lambda^2 (\nabla u)^2 - \lambda^2 |\partial_r u|^2 \right]. \]
Finally, we compute \( L \).

\[
L_1 := \int_{\mathbb{R}^n_{+} \cap \partial B_1} \lambda \frac{d}{d\lambda} L(u^\lambda_e) \frac{du^\lambda_e}{d\lambda} \\
= -2(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \text{div}_{S^n}(\theta^h \nabla_{S^n} u^\lambda_e) \frac{du^\lambda_e}{d\lambda} \\
= 2(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^h (\nabla_{S^n} u^\lambda_e)^2;
\]

\[
L_2 := \int_{\mathbb{R}^n_{+} \cap \partial B_1} -\lambda \theta^h L(u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} \\
= 2(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \text{div}_{S^n}(\theta^h \nabla_{S^n} u^\lambda_e) \frac{d^2 u^\lambda_e}{d\lambda^2} \\
= -2(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \lambda \theta^h \nabla_{S^n} u^\lambda_e \frac{d^2 u^\lambda_e}{d\lambda^2} \nabla_{S^n} u^\lambda_e \\
= -(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^h \frac{d}{d\lambda} \left[ 2 \lambda \nabla_{S^n} u^\lambda_e \nabla_{S^n} u^\lambda_e \frac{du^\lambda_e}{d\lambda} - (\nabla_{S^n} u^\lambda_e)^2 \right] \\
+ 2(n + b - 3) \lambda \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^h \left| \nabla_{S^n} u^\lambda_e \right|^2;
\]

\[
L_3 := \int_{\mathbb{R}^n_{+} \cap \partial B_1} -5 \theta^h L(u^\lambda_e) \frac{du^\lambda_e}{d\lambda} \\
= 10(n + b - 3) \int_{\mathbb{R}^n_{+} \cap \partial B_1} \text{div}_{S^n}(\theta^h \nabla_{S^n} u^\lambda_e) \frac{du^\lambda_e}{d\lambda} \\
= -10(n + b - 3) \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^h \nabla_{S^n} u^\lambda_e \nabla_{S^n} u^\lambda_e \frac{du^\lambda_e}{d\lambda} \\
= -5(n + b - 3) \frac{d}{d\lambda} \int_{\mathbb{R}^n_{+} \cap \partial B_1} \theta^h \left| \nabla_{S^n} u^\lambda_e \right|^2.
\]
Hence,

\[ \mathcal{L} := L_1 + L_2 + L_3 \]

\[ = -(n + b - 3) \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^b_1 \left[ \frac{d}{d\lambda} (\nabla S_n u^\lambda_e)^2 \right] \]

\[ - 4(n + b - 3) \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^b_1 (\nabla S_n u^\lambda_e)^2 \]

\[ + 4(n + b - 4) \lambda \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^b_1 |\nabla S_n u^\lambda_e|^2 \]

\[ = -(n + b - 3) \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\epsilon - 1 - n - 4} y^b \frac{d}{d\lambda} \left[ \lambda^2 |\nabla u^\lambda_e|^2 - \lambda^2 |\partial_r u^\lambda_e|^2 \right] \]

\[ - 4(n + b - 3) \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\epsilon - 1 - n - 5} y^b \left[ \lambda^2 |\nabla u^\lambda_e|^2 - \lambda^2 |\partial_r u^\lambda_e|^2 \right]. \]

By rescaling, we have

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^b_1 \left[ \frac{d}{d\lambda} (\nabla S_n u^\lambda_e)^2 \right] \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\epsilon - 1 - n - 4} y^b \frac{d}{d\lambda} \left[ \lambda^2 |\nabla u^\lambda_e|^2 - \lambda^2 |\partial_r u^\lambda_e|^2 \right]; \]

\[ \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \theta^b_1 |\nabla S_n u^\lambda_e|^2 \]

\[ = \frac{d}{d\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_1} \lambda^{2\epsilon - 1 - n - 5} y^b \left[ \lambda^2 |\nabla u^\lambda_e|^2 - \lambda^2 |\partial_r u^\lambda_e|^2 \right]. \]

### 3.6 The term $\overline{E}_d t_1$

Notice that on the boundary $\partial B_1$,

\[ v^\lambda_e = \Delta_b u^\lambda_e \]

\[ = \lambda^2 \frac{d^2 u^\lambda_e}{d\lambda^2} + (n + b - \frac{4s}{p - 1}) \lambda \frac{du^\lambda_e}{d\lambda} + \frac{2s}{p - 1} (1 + \frac{2s}{p - 1} - n - b) u^\lambda_e + \Delta_\theta u^\lambda_e \]

\[ := \lambda^2 \frac{d^2 u^\lambda_e}{d\lambda^2} + \alpha_0 \lambda \frac{du^\lambda_e}{d\lambda} + \beta_0 u^\lambda_e + \Delta_\theta u^\lambda_e. \]

Integrate by part, it follows that
\[
\int_{\partial B_1} y^b \lambda \left( \frac{dv_\lambda^e}{d\lambda} \right)^2 = \int_{\partial B_1} \left( \lambda^5 \left( \frac{d^3 u_\lambda^e}{d\lambda^3} \right)^2 + (\alpha_0^2 - 2\alpha_0 - 2\beta_0) \right) \lambda \left( \frac{d^2 u_\lambda^e}{d\lambda^2} \right)^2 \\
+ (-\alpha_0^2 + \beta_0^2 + 2\alpha_0 + 2\beta_0) \lambda \left( \frac{du_\lambda^e}{d\lambda} \right)^2 \\
+ \int_{\partial B_1} \left( -2\lambda^3 (\nabla \theta \frac{d^2 u_\lambda^e}{d\lambda^2})^2 + (10 - 2\beta_0) \lambda (\nabla \theta \frac{du_\lambda^e}{d\lambda})^2 \right) \\
+ \lambda (\Delta \theta \frac{du_\lambda^e}{d\lambda})^2 \\
+ \frac{d}{d\lambda} \left( \int_{\partial B_1} \sum_{0 \leq i, j \leq 4, i+j \leq 2} c_{i,j}^1 \lambda^{i+j} \frac{d^i u_\lambda^e}{d\lambda^i} \frac{d^j u_\lambda^e}{d\lambda^j} \\
+ \sum_{0 \leq s, t \leq 4, s+t \leq 2} c_{s,t}^2 \lambda^{s+t} \frac{d^s u_\lambda^e}{d\lambda^s} \frac{d^t u_\lambda^e}{d\lambda^t} \right), 
\]

(3.43)

where \(c_{i,j}^1, c_{s,t}^2\) depending on \(a, b\) hence on \(p, n\).

**Proof of Theorem 2.2** Notice that the equation (3.6), combine with the estimates on \(I, J, K, L\) and (3.43), we obtain Theorem 2.2. \(\square\)

### 4 Energy estimates and Blow down analysis

In this section, we do some energy estimates for the solutions of (1.1), which are important when we perform a blow-down analysis in the next section.

#### 4.1 Energy estimates

**Lemma 4.1.** Let \(u\) be a solution of (1.1) and \(u_\varepsilon\) satisfy (2.4), then there exists a positive constant \(C\) such that

\[
\int_{\partial R^{n+1}_+} |u_\varepsilon|^{p+1} \eta^6 + \int_{R^{n+1}_+} y^b |\nabla \Delta_b u_\varepsilon|^2 \eta^6 \\
\leq C \left[ \int_{R^{n+1}_+} y^b |\Delta_b u_\varepsilon|^2 \eta^4 |\nabla \eta|^2 + \int_{R^{n+1}_+} y^b |\nabla u_\varepsilon|^2 \frac{|\Delta \eta|^6}{\eta^6} \right] + 3 \int_{R^{n+1}_+} y^b u_\varepsilon^2 |\nabla \Delta_b u_\varepsilon|^2 \eta^6 \\
+ \int_{R^{n+1}_+} y^b |\nabla u_\varepsilon|^2 \eta^2 |\nabla \eta|^4 + \int_{R^{n+1}_+} y^b |\nabla^2 u_\varepsilon|^2 \eta^4 |\nabla \eta|^2 \\
+ \int_{R^{n+1}_+} y^b |\nabla u_\varepsilon|^2 \eta^4 |\nabla^2 \eta|^2 \right].
\]

(4.1)
Proof. Multiply the equation (2.4) with \( y^b u_e \eta^6 \), where \( \eta \) is a test function, we get that

\[
0 = \int_{\mathbb{R}^{n+1}_+} y^b u_e \eta^6 \Delta_b^3 u_e = \int_{\mathbb{R}^{n+1}_+} u_e \eta^6 \text{div}(y^b \nabla \Delta_b^2 u_e)
\]

\[
= - \int_{\partial \mathbb{R}^{n+1}_+} u_e \eta^6 \frac{\partial}{\partial y} \Delta_b^2 u_e - \int_{\mathbb{R}^{n+1}_+} y^b \nabla (u_e \eta^6) \nabla \Delta_b^2 u_e
\]

\[
= C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+} |u_e|^{p+1} \eta^6 - \int_{\partial \mathbb{R}^{n+1}_+} y^b \frac{\partial (u_e \eta^6)}{\partial y} \Delta_b^2 u_e + \int_{\mathbb{R}^{n+1}_+} y^b \Delta_b (u_e \eta^6) \Delta_b^2 u_e
\]

\[
= C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+} |u_e|^{p+1} \eta^6 + \int_{\mathbb{R}^{n+1}_+} y^b \Delta_b (u_e \eta^6) \Delta_b^2 u_e
\]

\[
= C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+} |u_e|^{p+1} \eta^6 - \int_{\partial \mathbb{R}^{n+1}_+} \Delta_b (u_e \eta^6) y^b \frac{\partial \Delta_b u_e}{\partial y} - \int_{\mathbb{R}^{n+1}_+} y^b \nabla (\Delta_b (u_e \eta^6)) \nabla \Delta_b u_e
\]

\[
= C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+} |u_e|^{p+1} \eta^6 - \int_{\mathbb{R}^{n+1}_+} y^b \nabla (\Delta_b (u_e \eta^6)) \nabla \Delta_b u_e.
\]

Hence, we have

\[
C_{n,s} \int_{\partial \mathbb{R}^{n+1}_+} |u_e|^{p+1} \eta^6 = \int_{\mathbb{R}^{n+1}_+} y^b \nabla (\Delta_b (u_e \eta^6)) \nabla \Delta_b u_e.
\]  

(4.2)

Since \( \Delta_b (\xi \eta) = \eta \Delta_b \xi + \xi \Delta_b \eta + 2 \nabla \xi \nabla \eta \), we have

\[
\Delta_b (u_e \eta^6) = \eta^6 \Delta_b u_e + u_e \Delta_b \eta^6 + 12 \eta^5 \nabla u_e \nabla \eta,
\]

therefore,

\[
\nabla \Delta_b (u_e \eta^6) \nabla \Delta_b u_e = 6 \eta^5 \Delta_b u_e \nabla \eta \nabla \Delta_b u_e + (\eta^6 (\nabla \Delta_b u_e))^2 + \Delta_b \eta^6 \nabla u_e \nabla \Delta_b u_e + u_e \nabla \Delta_b \eta^6 \nabla \Delta_b u_e + 6 \eta^4 (\nabla \eta \nabla \Delta_b u_e) (\nabla u_e \nabla \eta) + 12 \eta^5 \sum_{i,j} \partial_i u_e \partial_i \eta \partial_j \Delta_b u_e + 12 \eta^5 \sum_{i,j} \partial_i u_e \partial_i \eta \partial_j \Delta_b u_e.
\]

(4.4)

Here \( \partial_j (j = 1, ..., n, n + 1) \) denote the derivatives with respect to \( x_1, ..., x_n, y \) respectively. A similar way can be applied to deal with the following term \( |\nabla \Delta_b (u_e \eta^3)|^2 \).

On the other hand, by the stability condition, we have

\[
p \int_{\mathbb{R}^n} |u|^{p+1} \eta^6 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{u(x) \eta^3(x) - u(y) \eta^3(y)}{|x - y|^{n+2s}} \right)^2 = \frac{1}{C_{n,s}} \int_{\mathbb{R}^{n+1}_+} y^b |\nabla \Delta_b (u_e \eta^3)|^2.
\]

(4.5)

(Here we notice that \( u_e (x, 0) = u(x) \), see Theorem 2.1 (2.3))
Combining with (4.3), (4.4) and (4.5), we have
\[
\int_{R^{n+1}} y^b |\nabla \Delta_b u_c|^2 \eta^6 \\
\leq C \int_{R^{n+1}} y^b (\nabla \Delta_b u_c)^2 \eta^6 + C(\varepsilon) \int_{R^{n+1}} y^b |\nabla \Delta_b u_c|^2 \eta^4 |\nabla \eta|^2 \\
+ \int_{R^{n+1}} y^b |\nabla u_c|^2 \left( \frac{|\Delta_b u_c|^2}{\eta^6} + \eta^4 |\nabla^2 \eta|^2 \right) \\
+ \int_{R^{n+1}} y^b u_c^2 \frac{|\nabla \Delta_b \eta|^2}{\eta^6} + \int_{R^{n+1}} y^b |\nabla u_c|^2 \eta^4 |\nabla \eta|^2 + \int_{R^{n+1}} y^b |\nabla^2 u_c|^2 \eta^4 |\nabla \eta|^2,
\]
we can select \( \varepsilon \) so small that \( C \varepsilon \leq \frac{\frac{1}{2}}{2} \). Combine with (4.3) and (4.4), we obtain our conclusion.

\[ \]

**Corollary 4.1.** Let \( u \) be a solution of (1.1) and \( u_c \) satisfy (2.4), then
\[
\int_{\partial R^{n+1} \cap B_{R/2}} |u_c|^{p+1} + \int_{R^{n+1} \cap B_{R/2}} y^b (\nabla \Delta_b u_c)^2 \\
\leq C \left[ R^{-6} \int_{R^{n+1} \cap B_{R}} y^b u_c^2 + R^{-4} \int_{R^{n+1} \cap B_{R}} y^b |\nabla u_c|^2 \\
+ R^{-2} \int_{R^{n+1} \cap B_{R}} y^b (|\Delta_b u_c|^2 + |\nabla^2 u_c|^2) \right].
\]

**Proof.** We let \( \eta = \xi^m \) where \( m > 0 \) in the estimate (4.1). We have
\[
\int_{\partial R^{n+1}} |u_c|^{p+1} \xi^m + \int_{R^{n+1}} y^b |\nabla \Delta_b u_c|^2 \xi^6m \\
\leq C \left[ \int_{R^{n+1}} y^b (|\Delta_b u_c|^2 + |\nabla^2 u_c|^2) \xi^6m^2 |\nabla \xi|^2 \\
+ \int_{R^{n+1}} y^b |\nabla u_c|^2 \xi^6m^4 (|\nabla^2 \xi|^2 + |\nabla \xi|^4) + \int_{R^{n+1}} y^b u_c^2 \xi^6m^6 - |\nabla \xi|^2 \right].
\]
Let \( \xi = 1 \) in \( B_{R/2} \) and \( \xi = 0 \) in \( B_R \), satisfying \( |\nabla \xi| \leq \frac{C}{\pi} \), then we have the desired estimates.

**Lemma 4.2.** Suppose that \( u \) is a solution of (1.1) which is stable outside some ball \( B_R \subset \mathbb{R}^n \). For \( \eta \in C_c^\infty (\mathbb{R}^n \setminus \overline{B_R}) \) and \( x \in \mathbb{R}^n \), define
\[
\rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy. 
\]
Then
\[
\int_{\mathbb{R}^n} y^b u_c^2 \rho dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) \eta(x) - u(y) \eta(y)|^2 |x - y|^{n+2s} dxdy \leq C \int_{\mathbb{R}^n} u^2 \rho dx. 
\]
Lemma 4.3. Let $m > n/2$ and $x \in \mathbb{R}^n$. Set
\[
\rho(x) := \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} \, dy \quad \text{where} \quad \eta(x) = (1 + |x|^2)^{-m/2}.
\] (4.8)

Then there is a constant $C = C(n, s, m) > 0$ such that
\[
C^{-1}(1 + |x|^2)^{-n/2-s} \leq \rho(x) \leq C(1 + |x|^2)^{-n/2-s}.
\] (4.9)

Corollary 4.2. Suppose that $m > n/2$, $\eta$ is given by (4.8) and $R > R_0 > 1$. Define
\[
\rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x - y|^{n+2s}} \, dy, \quad \text{where} \quad \eta_R(x) = \eta(x/R)\psi(x/R)
\] (4.10)

for a standard test function $\psi$ that $\psi \in C^\infty(\mathbb{R}^n)$, $0 \leq \psi \leq 1$, $\psi = 0$ on $B_1$ and $\psi = 1$ on $\mathbb{R}^n \setminus B_2$. Then there exists a constant $C > 0$ such that
\[
\rho_R(x) \leq C \eta^2(x/R)|x|^{-(n+2s)} + R^{-2s} \rho(x/R).
\]

Lemma 4.4. Suppose that $u$ is a solution of (1.1) which is stable outside a ball $B_{R_0}$. Consider $\rho_R$ which is defined in (4.10) for $n/2 < m < n/2 + s(p + 1)/2$. Then there exists a constant $C > 0$ such that
\[
\int_{\mathbb{R}^n} u^2 \rho_R \leq C \left( \int_{B_{3R_0}} u^2 + R^{n-2s} \rho_R^{n+1} \right)
\] for any $R > 3R_0$.

The proofs of Lemma 4.2, Corollary 4.2, Lemma 4.3 and Lemma 4.4 are similar to that of Lemmas 2.1, 2.2, 2.4 in [6] and we omit the details here.

Lemma 4.5. Suppose that $p \neq \frac{n+2s}{2}$. Let $u$ be a solution of (1.1) which is stable outside a ball $B_{R_0}$ and $u_c$ satisfy (2.4). Then there exists a constant $C > 0$ such that
\[
\int_{B_R} y^b |u_c|^2 \leq CR^{n+6-2s} \psi^b,
\]
\[
\int_{B_R} y^b |\nabla u_c|^2 \leq CR^{n+4-2s} \psi^b,
\]
\[
\int_{B_R} y^b (|\nabla^2 u_c|^2 + |\Delta u_c|^2) \leq CR^{n+2-2s} \psi^b.
\]

Proof. Recall that the Possion formula for the fractional equation for the case $0 < s < 1$ (see [2]), we can generalize the expression formula to the general case with non-integer positive real number. Therefore,
\[
u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \frac{y^{2s}}{(|x - z|^2 + y^2)^{\frac{n+2s}{2}}} \, dz.
\]

Then we have
\[
|u_c(x, y)|^2 \leq C \int_{\mathbb{R}^n} u^2(z) \frac{y^{2s}}{(|x - z|^2 + y^2)^{\frac{n+2s}{2}}} \, dz, \quad (4.11)
\]
Now we turn to estimate the following integration, which provides a unify way to deal with our desired estimates.

By a straightforward calculation we have

\[
\partial_y u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{2sy^{2s-1}}{|x-z|^2 + y^2} \frac{n+2s}{2} - \frac{(n+2s)y^{2s+1}}{|x-z|^2 + y^2} \right] dz,
\]

also

\[
\partial_x u_c(x, y) = -C_{n,s} \int_{\mathbb{R}^n} u(z) \frac{(n+2s)(x_j - z_j)y^{2s}}{|x-z|^2 + y^2} \frac{n+2s+2}{2} dz,
\]

for \( j = 1, 2, ..., n \). Hence by Hölder’s inequality we have

\[
|\nabla u_c(x, y)|^2 \leq C \int_{\mathbb{R}^n} \frac{u^2(z)y^{2s-2}}{|x-z|^2 + y^2} \frac{n+2s}{2} dz. \tag{4.12}
\]

By a straightforward calculation we have

\[
\partial_{x_j} u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{(n+2s)(n+2s+2)(x_j - z_j)^2 y^{2s}}{|x-z|^2 + y^2} \right. \left. \frac{n+2s+1}{2} \right] dz,
\]

\[
\partial_{x_j} u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{(n+2s)(n+2s+2)(x_j - z_j)^2 y^{2s+1}}{|x-z|^2 + y^2} \right. \left. \frac{n+2s+1}{2} \right] dz,
\]

and

\[
\partial_{y^j} u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{2s(2s-1)y^{2s-2}}{|x-z|^2 + y^2} \right. \left. \frac{n+2s}{2} \right] - \frac{(n+2s)(4s+1)y^{2s}}{|x-z|^2 + y^2} \frac{n+2s+1}{2} dz,
\]

\[
\partial_{y^j} u_c(x, y) = C_{n,s} \int_{\mathbb{R}^n} u(z) \left[ \frac{2s(2s-1)y^{2s-2}}{|x-z|^2 + y^2} \right. \left. \frac{n+2s}{2} \right] - \frac{(n+2s)(4s+1)y^{2s}}{|x-z|^2 + y^2} \frac{n+2s+1}{2} dz + \frac{(n+2s)(n+2s+2)y^{2s+2}}{|x-z|^2 + y^2} \frac{n+2s+1}{2} dz.
\]

Therefore, we have

\[
|\nabla^2 u_c(x, y)| + |\Delta_{y} u_c(x, y)| \leq C \int_{\mathbb{R}^n} |u(z)| \frac{y^{2s-2}}{|x-z|^2 + y^2} \frac{n+2s}{2} dz.
\]

Hence,

\[
|\nabla^2 u_c(x, y)|^2 + |\Delta_{y} u_c(x, y)|^2 \leq C \int_{\mathbb{R}^n} u^2(z) \frac{y^{2s-4}}{|x-z|^2 + y^2} \frac{n+2s}{2} dz. \tag{4.13}
\]

Now we turn to estimate the following integration, which provides a unify way to deal with our desired estimates.
Define
\[
A_k := \int_{|x| \leq R} u^2(z) \left[ \int_{|z| = 2} \frac{y^{2k+1}}{(|x - z|^2 + y^2)^{\frac{n+2s}{2}}} dy \right] dz dx
\]
\[
= \int_{|x| \leq R} u^2(z) \left[ \int_{|z| = 2} \alpha^{\frac{1}{2}} \frac{d\alpha}{|x - z|^2 + \alpha + k \alpha^{\frac{1}{2}}} \right] dz dx
\]
\[
\leq \frac{1}{2} \int_{|x| \leq R} u^2(z) \left[ \int_{0}^{R^2} \frac{d\alpha}{|x - z|^2 + \alpha + k \alpha^{\frac{1}{2}}} \right] dz dx
\]
\[
= \frac{1}{2} \left( n + 2s - k \right) \int_{|x| \leq R} u^2(z) \left[ (|x - z|^2)^{k - \frac{n+2s}{2}} + 1 \right]
\]
\[- \left( (|x - z|^2 + R^2)^{k - \frac{n+2s}{2} + 1} \right],
\]
where \( k = 0, 1, 2 \). Split the integral to \(|x - z| \leq 2R\) and \(|x - z| > 2R\), for the case of \(|x - z| \leq 2R\), we see that
\[
\int_{|x| \leq R, |x - z| \leq 2R} u^2(z) \left[ (|x - z|^2)^{k - \frac{n+2s}{2}} + 1 \right] 
\]
\[- (|x - z|^2 + R^2)^{k - \frac{n+2s}{2} + 1} \right]
\]
\[
\leq \int_{|x| \leq R, |x - z| \leq 2R} u^2(z) \left[ (|x - z|^2)^{k - \frac{n+2s}{2}} + 1 \right]
\]
\[
\leq CR^{2k-2s+2} \int_{|z| \leq 3R} u^2(z) dz
\]
\[
\leq R^{2k-2s+2} \left( \int_{B_{3R}} |u|^{p+1} \eta_{R}^{2} \right)^{2/(p+1)} \left( \int_{B_{3R}} \eta_{R}^{4/(p-1)} \right)^{(p-1)/(p+1)}
\]
\[
\leq CR^{2k-2s+2} \left( \int_{B_{3R}} u^2(z) \rho_{R}(z) \right)^{2/(p+1)}
\]
\[
\leq CR^{n+2k+2-2s+2 + \frac{n+4}{p+1}}.
\]
Here we have used Lemma 4.2 and 4.4. For the case of \(|x - z| > 2R\), by the mean value theorem, we have
\[
\int_{|x| \leq R, |x - z| > 2R} u^2(z) \left[ (|x - z|^2)^{k - \frac{n+2s}{2}} + 1 \right] 
\]
\[- (|x - z|^2 + R^2)^{k - \frac{n+2s}{2} + 1} \right]
\]
\[
\leq R^2 \int_{|x| \leq R, |x - z| > 2R} u^2(z) \left[ (|x - z|^2)^{k - \frac{n+2s}{2}} \right]
\]
\[
\leq CR^{n+2} \int_{|z| \geq R} u^2(z) |z|^{2k-n-2s} dz
\]
\[
\leq CR^{n+2} \left[ \int_{|z| \geq R} (u^{p+1}(z)) \right]^{2/(p+1)} \left( \int_{|z| \geq R} |z|^{(2k-n-2s) \frac{p+1}{p+1}} \right)^{(p-1)/(p+1)}
\]
\[
\leq CR^{n+2k+2-2s+2 + \frac{n+4}{p+1}}.
\]
Here we have used Lemma 4.2. Hence, we obtain that
\[
A_k \leq CR^{n+2k+2-2s+2 + \frac{n+4}{p+1}},
\]
where $C = C(n, s, p)$ independent of $R$. Now, combine with (4.11), (4.12) and (4.13), recall that $b = 5 - 2s$, we have

$$
\int_{B_R} y^b u_e^2 dxdy \leq A_2, \quad \int_{B_R} y^b |\nabla u_e|^2 dxdy \leq A_1,
$$

$$
\int_{B_R} y^b \left( |\nabla^2 u_e|^2 + |\Delta_b u_e|^2 \right) dxdy \leq A_0.
$$

Apply (4.15), we finish our proof.

Combine Corollary 4.1 and Lemma 4.4, we have the following lemma.

**Lemma 4.6.** Let $u$ be a solution of (1.1) which is stable outside a ball $B_{R_0}$ and $u_e$ satisfy (2.4). Then there exists a positive constant $C$ such that

\[
\int_{\partial \mathbb{R}^{n+1}_+ \cap B_R} |u_e|^{p+1} + R^{-6} \int_{\mathbb{R}^{n+1}_+ \cap B_R} y^b |u_e|^2 + R^{-4} \int_{\mathbb{R}^{n+1}_+ \cap B_R} y^b |\nabla u_e|^2
\]

\[
+ R^{-2} \int_{\mathbb{R}^{n+1}_+ \cap B_R} y^b \left( |\Delta_b u_e|^2 + |\nabla^2 u_e|^2 \right) + \int_{\mathbb{R}^{n+1}_+ \cap B_R} y^b |\nabla \Delta_b u_e|^2 \leq CR^{n-2s+\frac{p+1}{p-1}}.
\]
4.2 Blow down analysis and the proof of Theorem 1.1

The proof of Theorem 1.1. Suppose that \( u \) is a solution of (1.1) which is stable outside the ball of radius \( R_0 \) and suppose that \( u_e \) satisfies (2.4). In the subcritical case, i.e., \( 1 < p < p_s(n) \), Lemma 4.2 implies that \( u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n) \). Multiplying (1.1) with \( u \) and integrate, we obtain that

\[
\int_{\mathbb{R}^n} |u|^{p+1} = \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}.
\]  

(4.16)

Multiplying (1.1) with \( u^\lambda(x) = u(\lambda x) \) yields

\[
\int_{\mathbb{R}^n} |u|^{p-1} u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} u^\lambda = \lambda^s \int_{\mathbb{R}^n} w w_\lambda,
\]

where \( w = (-\Delta)^{s/2} u \). Following the ideas provided in [24, 25] and using the change of variable \( z = \sqrt{\lambda} x \), we can get the following Pohozaev identity

\[
-\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} = \frac{2s-n}{2} \int_{\mathbb{R}^n} |w|^2 + \frac{d}{d\lambda} \int_{\mathbb{R}^n} w^{\frac{s}{n-2}} u_1^{1/\sqrt{s}} dz \bigg|_{\lambda=1} = \frac{2s-n}{2} \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}.
\]

Hence, we have the following Pohozaev identity

\[
\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} = \frac{n-2s}{2} \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}.
\]

For \( p < p_s(n) \), this equality above together with (4.16) proves that \( u \equiv 0 \). For \( p = p_s(n) \), this equality above means that the energy is finite. Further, since \( u \in \dot{H}^s(\mathbb{R}^n) \), apply the stability inequality with test function \( \psi = u \eta^2(y) \), and let \( R \to +\infty \) (where \( \eta \) is cutoff function), then we get that

\[
\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} \leq \|u\|^2_{\dot{H}^s(\mathbb{R}^n)}.
\]

This together with (4.16) gives that \( u \equiv 0 \).

Now we consider the supercritical case, i.e., \( p > \frac{n+2s}{n-2s} \), we perform the proof via a few steps.

**Step 1.** \( \lim_{\lambda \to \infty} E(u_e, 0, \lambda) < \infty \).

From Theorem 2.4 we know that \( E \) is nondecreasing w.r.t. \( \lambda \), so we only need to show that \( E(u_e, 0, \lambda) \) is bounded. Note that

\[
E(u_e, 0, \lambda) = \frac{1}{\lambda} \int_\lambda^{2\lambda} E(u_e, 0, t) dt \leq \frac{1}{\lambda^\gamma} \int_\lambda^{2\lambda} \int_t^{t+\lambda} E(u_e, 0, \gamma) d\gamma dt.
\]
From Lemma 4.6, we have that

\[
\frac{1}{\lambda^2} \int_{\lambda}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap B_{\gamma}} \gamma^{2s - \alpha - n} \left[ \int_{\mathbb{R}^{n+1}_+ \cap B_{\gamma}} \frac{1}{2} y^b |\nabla \Delta_b u_e|^2 dy dx \right. \\
\left. - \frac{C_{n,s}}{p+1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_{\gamma}} |u_e|^{p+1} dx \right] d\gamma dt \leq C,
\]

where \( C > 0 \) is independent of \( \gamma \).

\[
\frac{1}{\lambda^2} \int_{\lambda}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_{\gamma}} \gamma^{2s - \alpha - n} y^b \left[ \right. \\
\left. \frac{2s}{p-1} \left( \frac{2s}{p-1} - 1 \right) \gamma \partial_r u_e + \frac{6s}{p-1} \gamma^2 \partial_{rr} u_e + \gamma^3 \partial_{rrr} u_e \right]
\]

\[
\leq C \frac{1}{\lambda^2} \int_{\lambda}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_{\gamma}} \gamma^{2s - \alpha - n} y^b \left[ u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2 \right]
\]

\[
\leq C \frac{1}{\lambda^2} \int_{\lambda}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_{\lambda}} \gamma^{2s - \alpha - n} y^b \left[ u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2 \right]
\]

\[
\leq C \lambda^{n-2s+\alpha+6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_{\lambda}} \gamma^{2s - \alpha - n} y^b \left[ u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2 \right]
\]

\[
\leq C \lambda^{n-2s+\alpha+6} \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_{\lambda}} \gamma^{2s - \alpha - n} y^b \left[u_e^2 + \gamma^2 (\partial_r u_e)^2 + \gamma^4 (\partial_{rr} u_e)^2 + \gamma^6 (\partial_{rrr} u_e)^2 \right]
\]

\[
\leq C
\]

and
\[ \left| \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_{t}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\gamma} \gamma^{2s+\frac{p+1}{p-1}-n-4} y^b \frac{d}{d\gamma} \left( \gamma^2 \Delta_e u_e - \gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e \right)^2 \right| \leq \left| \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_{t}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\gamma} y^b \left[ 2\gamma^2 \Delta_e u_e - 2\gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e \right] \right| \\
\leq \left| \frac{1}{\lambda^2} \int_{\lambda}^{2\lambda} \int_{t}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\gamma} y^b \left[ 2\gamma^2 \Delta_e u_e - 2\gamma^2 \partial_{rr} u_e - (n+b)\gamma \partial_r u_e \right] \right| \\
\leq C \lambda^{n-2s+\frac{p+1}{p-1}+6} \int_{\lambda}^{2\lambda} \int_{t}^{t+\lambda} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\gamma} y^b \left[ \gamma^4 \left( \Delta_e u_e \right)^2 + \gamma^4 \left( \partial_{rr} u_e \right)^2 + \gamma^2 \partial_r u_e \right] \\
\leq C. \quad (4.18) \]

Integrate by part, by the scaling identity of section 3, for example (3.31), (3.36), (3.41), and (3.42), we can treat the remaining terms by a similar way as the estimates (4.16) and (4.18).

**Step 2.** There exists a sequence \( \lambda_i \to \infty \) such that \( (u^\lambda_i) \) converges weakly to a function \( u^\infty \) in \( H_{\text{loc}}^2(\mathbb{R}^n; y^b dx dy) \), this is a direct consequence of Lemma 4.6.

**Step 3. The function \( u^\infty \) is homogeneous.** Due to the scaling invariance of \( E \) (i.e., \( E(u_e, 0, R\lambda) = E(u^\lambda_e, 0, R) \)) and the monotonicity formula, for any given \( R_2 > R_1 > 0 \), we see that

\[
0 = \lim_{i \to \infty} \left( E(u_e, 0, R_2\lambda_i) - E(u_e, 0, R_1\lambda_i) \right) \\
= \lim_{i \to \infty} \left( E(u^\lambda_e, 0, R_2) - E(u^\lambda_e, 0, R_1) \right) \\
\geq \liminf_{i \to \infty} \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}^{n+1}_+} y^b r^{2s+\frac{p+1}{p-1}-n-6} \left( \frac{2s}{p-1} u^\lambda_e + r \frac{\partial u^\lambda_e}{\partial r} \right)^2 dy dx \\
\geq \int_{(B_{R_2} \setminus B_{R_1}) \cap \mathbb{R}^{n+1}_+} y^b r^{2s+\frac{p+1}{p-1}-n-6} \left( \frac{2s}{p-1} u^\infty + r \frac{\partial u^\infty}{\partial r} \right)^2 dy dx.
\]

In the last inequality we have used the weak convergence of the sequence \( (u^\lambda_e) \) to the function \( u^\infty \) in \( H_{\text{loc}}^2(\mathbb{R}^n; y^b dx dy) \). This implies that

\[
\frac{2s}{p-1} u^\infty + \frac{\partial u^\infty}{\partial r} = 0 \quad \text{a.e. in } \mathbb{R}^{n+1}_+.
\]

Therefore, \( u^\infty \) is homogeneous.
Step 4. \( u_c^\infty = 0 \). This is a direct consequence of Theorem 3.1 in [17].

Step 5. \( (u_c^\lambda) \) converges strongly to zero in \( H^3(B_R \setminus B_\varepsilon; y^b dx dy) \) and \( (u_c^\lambda) \) converges strongly in \( L^{p+1}(\partial \mathbb{R}^{n+1}_+ \cap (B_R \setminus B_\varepsilon)) \) for all \( R > \varepsilon > 0 \). These are consequent results of Lemma 4.6 and Theorem 1.5 in [9].

Step 6. \( u_c = 0 \). Note that

\[
\overline{E}(u_c, \lambda) = \overline{E}(u_c, 1, 1)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b |\nabla \Delta_b u_c^\lambda|^2 dx dy - \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u_c^\lambda|^{p+1} dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap B_1} y^b |\nabla \Delta_b u_c^\lambda|^2 dx dy - \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_+ \cap B_1} |u_c^\lambda|^{p+1} dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap (B_1 \setminus B_\varepsilon)} y^b |\nabla \Delta_b u_c^\lambda|^2 dx dy - \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_+ \cap (B_1 \setminus B_\varepsilon)} |u_c^\lambda|^{p+1} dx
\]

\[
= \varepsilon^{-2} \left[ \frac{E(u_c, \lambda) - E(u_c, 1)}{\lambda} \right] + \frac{1}{2} \int_{\mathbb{R}^{n+1}_+ \cap (B_1 \setminus B_\varepsilon)} y^b |\nabla \Delta_b u_c^\lambda|^2 dx dy
\]

\[
- \frac{C_{n,s}}{p + 1} \int_{\partial \mathbb{R}^{n+1}_+ \cap (B_1 \setminus B_\varepsilon)} |u_c^\lambda|^{p+1} dx.
\]

Letting \( \lambda \to +\infty \) and then \( \varepsilon \to 0 \), we deduce that \( \lim_{\lambda \to +\infty} \overline{E}(u_c, \lambda) = 0 \). Using the monotonicity of \( E \),

\[
E(u_c, \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(t) dt \leq \sup_{[\lambda, 2\lambda]} \overline{E} + \frac{1}{\lambda} \int_{\lambda}^{2\lambda} [E - \overline{E}]
\]

\[
\leq \sup_{[\lambda, 2\lambda]} \overline{E} + \frac{1}{\lambda} \int_{\lambda}^{2\lambda} \lambda^{2\lambda + 1 - n - 5} \int_{\mathbb{R}^{n+1}_+ \cap \partial B_\lambda} y^b [(u_c)^2 + \lambda^2 |\nabla u_c|^2 + \lambda^4 (|\Delta_b u_c|^2 + |\nabla^2 u_c|^2)]
\]

\[
+ \frac{1}{\lambda} \int_{\lambda}^{2\lambda} \lambda^{2\lambda + 1 - n - 5} \int_{\mathbb{R}^{n+1}_+ \cap (B_2 \setminus B_\lambda)} y^b [(u_c)^2 + \lambda^2 |\nabla u_c|^2 + \lambda^4 (|\Delta_b u_c|^2 + |\nabla^2 u_c|^2)]
\]

\[
= \sup_{[\lambda, 2\lambda]} \overline{E} + \frac{1}{\lambda} \int_{\lambda}^{2\lambda} \int_{\mathbb{R}^{n+1}_+ \cap (B_2 \setminus B_1)} y^b [(u_c)^2 + |\nabla u_c|^2 + |\Delta_b u_c|^2 + |\nabla^2 u_c|^2]
\]

\[
(4.19)
\]

and so \( \lim_{\lambda \to +\infty} E(u_c, \lambda) = 0 \). Since \( u \) is smooth, we also have \( E(u_c, 0) = 0 \). Since \( E \) is monotone, \( E \equiv 0 \) and so \( \overline{E} \) must be homogenous, a contradiction unless \( u_c \equiv 0 \).
Let $k := \frac{2s}{n-1}$ and $m := n - 2s$. By a direct calculation, we obtain that

$$A_1 = -10k^2 + 10mk - m^2 + 12m + 25,$$

$$A_2 = 3k^4 - 6mk^3 + (3m^2 - 12m - 30)k^2 + (12m^2 + 30m)k + 9m^2 + 36m + 27,$$

$$B_1 = -6k^2 + 6mk + 12m + 30.$$

(5.1)

Notice that our supercritical condition $p > \frac{n+2s}{n-2s}$ is equivalent to $0 < k < \frac{n-2s}{2}$. Next, we have the following lemma which yields the sign of $A_2$ and $B_1$.

**Lemma 5.1.** If $p > \frac{n+2s}{n-2s}$, then $A_2 > 0$ and $B_1 > 0$.

**Proof.** From (5.1), we derive that

$$A_2 = 3(k + 1)(k + 3)(k - (m + 1))(k - (m + 3)),$$

(5.2)

and the roots of $B_1 = 0$ are

$$\frac{1}{2}m - \frac{1}{2}\sqrt{m^2 + 8m + 20}, \quad \frac{1}{2}m + \frac{1}{2}\sqrt{m^2 + 8m + 20}.$$ 

Recall that $p > \frac{n+2s}{n-2s}$ is equivalent to $0 < k < \frac{n}{2}$, we get the conclusion.

To show monotonicity formula, we proceed to prove the following inequality. That is, there exist real numbers $c_{i,j}$ and positive real number $\epsilon$ such that

$$3\lambda^5 \left( \frac{d^3 u^\lambda}{d\lambda^3} \right)^2 + A_1 \lambda^3 \left( \frac{d^2 u^\lambda}{d\lambda^2} \right)^2 + A_2 \lambda \left( \frac{du^\lambda}{d\lambda} \right)^2 \geq \epsilon \lambda \left( \frac{du^\lambda}{d\lambda} \right)^2 + \frac{d}{d\lambda} \left( \sum_{0 \leq i,j \leq 2} c_{i,j} \lambda^{i+j} \left( d \frac{d u^\lambda}{d\lambda} \right)^2 \right).$$

(5.3)

To deal with the rest of the dimensions, we employ the second idea: we find nonnegative constants $d_1, d_2$ and constants $c_1, c_2$ such that we have the following Jordan form decomposition:

$$3\lambda^5 (f''')^2 + A_1 \lambda^3 (f'')^2 + A_2 \lambda (f')^2 = 3\lambda (\lambda^2 f''' + c_1 \lambda f'')^2 + d_1 \lambda (\lambda f'' + c_2 f')^2 + d \lambda (\sum_{i,j} e_{i,j} \lambda^{i+j} f^{(i)} f^{(j)}),$$

(5.4)

where the unknown constants are to be determined.

**Lemma 5.2.** Let $p > \frac{n+2s}{n-2s}$ and $A_1$ satisfy

$$A_1 + 12 > 0,$$

(5.5)

then there exist nonnegative numbers $d_1, d_2$, and real numbers $c_1, c_2, e_{i,j}$ such that the differential inequality (5.4) holds.
Proof. Since
\[ 4\lambda^4 f'''f'' = \frac{d}{d\lambda} (2\lambda^4 (f'')^2) - 8\lambda^3 (f'')^2 \]
and
\[ 2\lambda^2 f'' f' = \frac{d}{d\lambda} (\lambda^2 (f')^2) - 2\lambda (f')^2, \]
by comparing the coefficients of \(\lambda^3 (f'')^2\) and \(\lambda (f')^2\), we have that
\[ d_1 = A_1 - 3c_1^2 + 12c_1, \quad d_2 = A_2 - (c_2^2 - 2c_2)(A_1 - 3c_1^2 + 12c_1). \]
In particular,
\[ \max_{c_1} d_1(c_1) = A_1 + 12 \text{ and the critical point is } c_1 = 2. \]
Since \(A_2 > 0\), we select that \(c_1 = 2, c_2 = 0\). Hence, in this case, by a direct calculation we see that \(d_1 = A_1 + 12 > 0\). Then we get the conclusion.

We conclude from Lemma 5.2 that if \(A_1 + 12 > 0\) then (5.3) holds. This implies that when \(m < 6 + \sqrt{73}, p > \frac{n+2s}{m+2s} \) or \(m \geq 6 + \sqrt{73}\) and
\[ \frac{n+2s}{n-2s} < p < \frac{5m+20s-\sqrt{15m^2+120m+370}}{5m-\sqrt{15m^2+120m+370}}, \]
then (5.3) holds.

Let
\[ p_m(n) := \begin{cases} +\infty & \text{if } n < 2s + 6 + \sqrt{73}, \\ \frac{5n+10s-\sqrt{15(n-2s)^2+120(n-2s)+370}}{5n-10s+\sqrt{15(n-2s)^2+120(n-2s)+370}} & \text{if } n \geq 2s + 6 + \sqrt{73}, \end{cases} \]
(5.7)

Combining all the lemmas of this section, we obtain the Theorem 2.3.

Now we proceed to prove Theorem 2.4. From Corollary 1.1 of [22], we know that if \(n > 2s, s > 0, p > \frac{n+2s}{n+2s} \), then there exists \(n_0(s)\), where \(\frac{1}{\sqrt{n}} < a_{n,s} < \frac{1}{2} \frac{n-2s}{\sqrt{n}} + \frac{1}{\sqrt{n}}\), such that the inequality (1.4) always holds whenever \(n \leq n_0(s)\); while when \(n > n_0(s)\), then the inequality (1.4) is true if and only if
\[ p < p_2 := \frac{n+2s}{n-2s} - 2a_{n,s}\sqrt{n}, \]
where \(n_0(s)\) is in fact the largest \(n\) satisfying \(n-2s - 2a_{n,s}\sqrt{n} < 0\). In particular,
\[ \frac{n+2s}{n-2s} < \frac{n+2s-4}{n-2s-4} < p_2 < +\infty. \] Therefore, we introduce
\[ p_c(n) := \begin{cases} +\infty & \text{if } n \leq n_0(s), \\ \frac{n+2s-2-2a_{n,s}\sqrt{n}}{n-2s-2-2a_{n,s}\sqrt{n}} & \text{if } n > n_0(s). \end{cases} \]
(5.8)
From [23], we use the sharp estimate \( n_0(s) < 2s + 8.998 \) for \( 2 < s < 3 \), then
\[
n_0(s) \leq 2s + 8.998 < 2s + 6 + \sqrt{73} \simeq 2s + 14.544. \tag{5.9}
\]

On the other hand, via the sharp estimate \( a_{n,s} < 1 \) from [23]
\[
\frac{5n + 10s - \sqrt{15(n - 2s)^2 + 120(n - 2s) + 370}}{5n - 10s - \sqrt{15(n - 2s)^2 + 120(n - 2s) + 370}} > \frac{n + 2s - 2 - 2a_{n,s}\sqrt{n}}{n - 2s - 2 - 2a_{n,s}\sqrt{n}} \tag{5.10}
\]
provided that \( s \in (2, 3) \) and that
\[
225m^4 - 720m^3 - 17244m^2 - 29088m + 7236 > 0, \text{ where } m = n - 2s. \tag{5.11}
\]
The (5.11) holds whenever \( m > 11.12 \), that is \( n > 2s + 11.12 \). This combine with (5.9) we obtain that \( p_c(n) < p_m(n) \). Therefore we get Theorem 2.4. \( \square \)
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