ASYMPTOTIC FORMULAE FOR EULERIAN SERIES

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Abstract. Let \((a; q)_{\infty}\) be the \(q\)-Pochhammer symbol and \(\text{Li}_2(x)\) be the dilogarithm function. Let \(\prod_{\alpha, \beta, \gamma}\) be a finite product with every triple \((\alpha, \beta, \gamma) \in (\mathbb{R}_{>0})^3\) and \(S_{\alpha \beta \gamma} \in \mathbb{R}\). Also let the triple \((A, B, v) \in (\mathbb{R}_{>0} \times \mathbb{R}^2) \cup \{(0)^2 \times \mathbb{R}_{>0}\} \cup \{(0) \times \mathbb{R}_{<0} \times \mathbb{R}\}\). In this work, we let \(z = e^v\), denote by \(H_{-1}(u) = vu - Au^2 + \sum_{\alpha} \text{Li}_2(e^{-\alpha u}) \sum_{\beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma}\) and consider the Eulerian series

\[H(z; q) = \sum_{m=0}^{\infty} \frac{q^{Am^2 + Bm}z^m}{\prod_{\alpha, \beta, \gamma} (q^{am+\gamma}; q^\beta)_\infty^{S_{\alpha \beta \gamma}}}.

We prove that if there exist an \(\varepsilon > 0\) such that \(H_{-1}(u)\) is an increasing function on \([0, \varepsilon)\), then as \(q \to 1^-\),

\[H(z; q) = (1 + o(|\log q|^p)) \int_0^\infty \frac{q^{Ax^2 + Bx}z^x}{\prod_{\alpha, \beta, \gamma} (q^{ax+\gamma}; q^\beta)_\infty^{S_{\alpha \beta \gamma}}} \, dx
\]

holds for each \(p \geq 0\). We also obtain full asymptotic expansions for \(H(z; q)\) which satisfy above condition as \(q \to 1^-\). The complete asymptotic expansions for related basic hypergeometric series could be derived as special cases.

1. Introduction and Statement of Results

We begin with the definition of Eulerian series, which could be found in [1].

Definition 1. Eulerian series are combinatorial formal power series which are constructed from basic hypergeometric series.

In his last letter to Hardy, Ramanujan listed 17 examples of functions in Eulerian series that he called mock theta functions. The first three pages in which Ramanujan explained what he meant by a ”mock theta function” are very obscure. Hardy comments that a mock theta function is a function defined by a \(q\)-series convergent when \(|q| < 1\), for which we can calculate asymptotic formulæ, when \(q\) tends to a ”rational point” \(e^{2\pi is/r}\) of the unit circle, of the same degree of precision as those furnished for the ordinary theta functions by the theory of linear transformation (see [2]).

In the same letter, Ramanujan also noted that for other Eulerian series, approximations analogous to mock theta function may not exist. He claimed that

\[\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{(q; q)_m^2} = \sqrt{\frac{t}{2\pi \sqrt{5}}} \exp \left( \frac{\pi^2}{5t} + c_1 t + \cdots + c_p t^p + O(t^{p+1}) \right)
\]

holds for each integer \(p \geq 1\), \(q = e^{-t}, t \to 0^+\) with infinitely many \(c_j \neq 0\). Here we use the \(q\)-Pochhammer symbol \((a; q)_m = \prod_{k=0}^{m-1} (1 - aq^k)\) for \(a, q \in \mathbb{C}, |q| < 1\) and \(m \in \mathbb{N} \cup \{\infty\}\).

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Although this example have been discussed by Watson [2] and McIntosh [3], we can’t prove this claim until today. In same paper [3], McIntosh also provided the complete asymptotics: Let \( a, b \in \mathbb{R}_{>0}, c, t \in \mathbb{R}, q = e^{-t} \) with \( t \to 0^+ \), then

\[
\sum_{m=0}^{\infty} \frac{q^{bm^2+cn}}{(q; q)_n} q^m \sim \exp \left( \frac{C_0}{t} + \sum_{k=0}^{\infty} C_k t^k \right),
\]

where \( C_k \) are constants depends only on \( a, b \) and \( c \). He note that his method could applicable to wide variety unimodal series with some limit conditions which required, see Theorem 2 of [4].

Moreover, let \( A \) be a positive definite symmetric \( r \times r \) matrix, \( B \) a vector of length \( r \), and \( C \) a scalar, all three with rational coefficients. Zagier [5, Section 3, Chapter II] define a function \( f_{A, B, C}(z) \) by the \( r \)-fold \( q \)-hypergeometric series

\[
f_{A, B, C}(z) = \sum_{x=(x_1, \ldots, x_r) \in \mathbb{N}^r} \frac{q^{\frac{1}{2}x^TAx + x^TB + C}}{(q; q)_{x_1} \cdots (q; q)_{x_r}},
\]

with \( z \in \mathbb{C}(\Re z > 0) \) and ask when \( f_{A, B, C}(z) \) is a modular function. Zagier give a method involving the asymptotic expansion of (1.2), with \( q = e^{-t} \) for \( t \to 0^+ \). In [5], Zagier also outline some methods to computing the asymptotics and solve the question for \( r = 1 \). In [6], Vlasenko and Zwegers use the ideal comes from Zagier to give the asymptotics for \( r \geq 2 \).

Finally, the \( q \)-hypergeometric series are \( q \)-analogue generalizations of generalized hypergeometric series. In 1940, Wright [7, 8] has been established six theorems on the asymptotic expansion of the generalized hypergeometric function. Zhang [9, 10] has investigated Plancherel-Rotach type asymptotics for certain \( q \)-hypergeometric series. However, nothing of the asymptotics as (1.1) is known for more general \( q \)-hypergeometric series.

The purposes of this paper is establish a complete asymptotic expansion for more general Eulerian series or \( q \)-hypergeometric series. We first fixed the following \( q \)-notation:

\[
(a; q)_z = \frac{(a; q)_\infty}{(aq^z; q)_\infty}, \quad (a_1, a_2, \ldots, a_n; q)_z = \prod_{j=1}^{n} (a_j; q)_z
\]

for \( a, a_1, \ldots a_n, q \in \mathbb{C}, |q| < 1 \) and \( z \in \mathbb{C} \). We will focus on the formal Eulerian series

\[
\mathcal{H}(z; q) = \sum_{m=0}^{\infty} \prod_{a,b,c,d} (q^a; q^b)^{S(a,b,c,d)} q^{Am^2+Bm} z^m,
\]

where \( q \in (0, 1) \), \( z = e^{\nu} \); the triple \((A, v, B) \in (\mathbb{R}_{>0} \times \mathbb{R}^2) \cup (\{0\}^2 \times \mathbb{R}_{>0}) \cup (\{0\} \times \mathbb{R}_{<0} \times \mathbb{R}) \); the product \( \prod_{a,b,c,d} \) is a finite product, every quadruple \((a, b, c, d) \in \mathbb{R}^4 \) with \( b, c, a + bd > 0 \) and \( S(a, b, c, d) \in \mathbb{R} \) is a function in \( a, b, c, d \). Clearly,

\[
\mathcal{H}(z; q) = \mathcal{H}(z; q) \prod_{a,b,c,d} (q^a; q^b)^{-S(a,b,c,d)},
\]

where

\[
\mathcal{H}(z; q) = \sum_{m=0}^{\infty} \left( \prod_{a,b,c,d} (q^{bcm+a+bd}; q^b)^{S(a,b,c,d)} \right) q^{Am^2+Bm} z^m.
\]
The asymptotics of the general term of the product in (1.4) has been understood well by McIntosh [11]. Thus we just need consider $\mathcal{H}(z; q)$, which could be rewritten as the following simple form

$$
\mathcal{H}(z; q) = \sum_{m=0}^{\infty} \frac{q^{Am^2 + Bm} z^m}{\prod_{\alpha, \beta, \gamma} (q^{\alpha m + \gamma}; q^b)_\infty^{S_{\alpha \beta \gamma}}}.
$$

(1.6)

In order to formulate the main result of this paper, we first denote by

$$
H_{-1}(u) = u \log z - Au^2 + \sum_{a} \text{Li}_2(e^{-\alpha u}) \sum_{\beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma}
$$

$$
:= u \log z - Au^2 - \sum_{1 \leq j \leq H} \text{Li}_2(e^{-\alpha_j u}) f(\alpha_j)
$$

(1.7)

with $\text{Li}_2(\cdot)$ is the dilogarithm function be defined by (4.6), $f(\alpha_j) \neq 0$ for $1 \leq j \leq H$ and $\alpha_1 < \alpha_2 < \cdots < \alpha_H$. Comparing (1.5) and (1.6) we also have

$$
H_{-1}(u) = vu - Au^2 - \sum_{b, c} \text{Li}_2(e^{-bcu}) \sum_{a, d} S(a, b, c, d) \frac{S(a, b, c, d)}{b}.
$$

(1.8)

Then, our main results as follows.

**Theorem 1.** Let $\mathcal{H}(z; q)$, $\mathcal{H}(z; q)$ and $H_{-1}(u)$ be defined as above. If there exist an $\varepsilon > 0$ such that $H_{-1}(u)$ is an increasing function on $[0, \varepsilon)$, then as $q \to 1^-$,

$$
\mathcal{H}(z; q) = (1 + o(|\log q|^p)) \int_0^{\infty} \frac{q^{Ax^2 + Bx z^x}}{\prod_{\alpha, \beta, \gamma}(q^{\alpha x + \gamma}; q^b)_\infty^{S_{\alpha \beta \gamma}}} dx
$$

holds for each $p \geq 0$. Furthermore, we have

$$
\mathcal{H}(z; q) = (1 + o(|\log q|^p)) \int_0^{\infty} \frac{q^{Ax^2 + Bx z^x}}{\prod_{a, b, c, d}(q^a; q^b)_{c+d}^{S(a, b, c, d)}} dx
$$

holds for each $p \geq 0$ as $q \to 1^-$.

We shall first prove the following more general results and then Theorem 1 could be derived as special cases.

**Lemma 1.1.** Let $F(x, t)$ and $E_\ell(u)(\ell \in \mathbb{Z}_{\geq -1})$ be some real analytic functions in $(0, \infty)$. For each $x \gg 1/t$ and $m \in \mathbb{N}$, suppose that $F(x, t)$ satisfies the complete asymptotics

$$
\frac{\partial^m F(x, t)}{\partial x^m} \sim t^m \int_{-\infty}^{\infty} E^{(m)}_{\ell}(xt) t^{\ell}
$$

for $t \to 0^+$. Also suppose that the set of all local maximum points of $E_{-1}(u)$ on $(0, \infty)$ is a nonempty finite set $\mathcal{E}_E$. Then, for each $p \geq 0$, as $t \to 0^+$

$$
\sum_{u_m/t < m \leq u_M/t} \exp(F(m, t)) = (1 + o(t^p)) \int_{u_m/t}^{u_M/t} \exp(F(x, t)) dx,
$$

where $u_m = \min_{u \in \mathcal{E}_E} u - 1/|\log t|$ and $u_M = \max_{u \in \mathcal{E}_E} u + 1/|\log t|$. 

We have the complete asymptotic expansion for above Lemma 1.1

**Lemma 1.2.** Let $F(x, t)$, $E_{\ell}(u)(\ell \in \mathbb{Z}_{\geq 1})$, $u_m$ and $u_M$ be defined as Lemma 1.1. Let $\kappa_{2\ell}(u, t)$ be defined by (3.16). Then we roughly have for each $\ell \in \mathbb{N}_1$,

$$\kappa_{2\ell}(u, t) \ll t^{\frac{e}{4u(2u+1)}}.$$

Further more, we have the asymptotic expansion in $\kappa_{2\ell}(u, t)$ of the form

$$\sum_{u_m/t < m \leq u_M/t} e^{F(m, t)} \sim \sum_{u \in \mathcal{F}_E} \frac{\exp(F(u/t, t))}{V(u, t)} \sum_{\ell=0}^{\infty} \frac{\Gamma(2\ell + 1)}{2k_u^{2\ell}} \kappa_{2\ell}(u, t) \frac{1}{k_u^{2\ell}}.$$

where $k_u$ is the minimum positive integer such that $E_{2k_u}^{(2)}(u) \neq 0$, $\kappa_0(u, t) = 1$ and $V(u, t)$ be defined by (3.14). In particular, we have the leading asymptotics

$$\sum_{u_m/t < m \leq u_M/t} e^{F(m, t)} \sim \sum_{u \in \mathcal{F}_E} e^{F_0(u)} \frac{1}{k_u} \Gamma(\frac{1}{2k_u}) \left(\frac{-(2k_u)!}{E_{2k_u}^{(2)}(u)}\right)^{\frac{1}{2k_u}} t^{-1 + \frac{1}{2k_u}} e^{E_{-1}(u)/t}.$$

**Remark 1.3.** From the assumption of Lemma 1.1, (3.14) and (3.16) it is not difficult to see that $V(u, t)$ and $\kappa_{2\ell}(u, t)$ have asymptotic expansions in powers of $t^{1/k_u}$, this implies that the asymptotic expansion of Lemma 1.2 could be rewritten as an asymptotic expansion in powers of $t^{1/k_u}$ of the form

$$\sum_{u_m/t < m \leq u_M/t} e^{F(m, t)} \sim \sum_{u \in \mathcal{F}_E} C_F(u) t^{-1 + \frac{1}{2k_u}} e^{E_{-1}(u)/t} \left(1 + \sum_{\ell=1}^{\infty} C_{F_j}(u) t^{\ell/k_u}\right)$$

with

$$C_F(u) = e^{F_0(u)} \frac{1}{k_u} \Gamma(\frac{1}{2k_u}) \left(\frac{-(2k_u)!}{E_{2k_u}^{(2)}(u)}\right)^{\frac{1}{2k_u}}$$

and for each $j \in \mathbb{N}_1$, $C_{F_j}(u) \in \mathbb{R}$ depends only on $F(\cdot)$ and $u$.

Finally, we obtain the full asymptotic behavior of Eulerian series (1.3) and (1.6).

**Theorem 2.** Let $H_{-1}(u)$ be defined by (1.7) and $H_{2l}(u)(\ell \in \mathbb{Z}_{\geq 0})$ be defined by Lemma 4.3. Also let $\mathcal{S}_H$ be the set of all local maximum points of $H_{-1}(u)$ on $(0, \infty)$. Then, under the assumption of Theorem 1 we have

$$\mathcal{H}(z; q) = N_H(t) + I_H(t),$$

where $N_H(t) = 0$ for $\mathcal{S}_H$ is an empty set and if $\mathcal{S}_H$ nonempty then

$$N_H(t) \sim \sum_{u \in \mathcal{S}_H} C_u t^{-1 + \frac{1}{2m_u}} e^{H_{-1}(u)/t} \left(1 + \sum_{j \geq 1} C_j(u) t^{j/m_u}\right)$$

with the coefficients $C_j(u) \in \mathbb{R}$ are constant depends only on $\mathcal{H}(z; \cdot)$ and $u$ be determined by (5.15), $m_u$ is the minimum positive integer such that $H_{-1}^{(2m_u)}(u) \neq 0$ and

$$C_u = \frac{e^{H_0(u)} \Gamma(\frac{1}{2m_u})}{m_u} \frac{-(2m_u)!}{H_{-1}^{(2m_u)}(u)}.$$

\[ I_{H}(t) = 0 \text{ for } A > 0 \text{ or } \log z < 0 \text{ or } f(\alpha_1) < 0 \text{ and if } f(\alpha_1) > 0 \text{ then} \]
\[
I_{H}(t) \sim \frac{\Gamma(B/\alpha_1)}{\alpha_1[f(\alpha_1)]^{B/\alpha_1}} t^{B/\alpha_1 - 1} \left(1 + \sum_{\lambda \in \Lambda(H)} C_\lambda(H)t^\lambda\right),
\]
where the set \( \Lambda(H) \subset \mathbb{Q}_{>0} \) satisfy \( \inf_{\lambda \neq \mu, \mu \in \Lambda(H)} |\lambda - \mu| > 0 \) be defined by (5.17), and \( C_\lambda(H) \in \mathbb{R} \) for \( \lambda \in \Lambda(H) \) be determined by (5.19) depends only on \( H(1; \cdot) \).

**Remark 1.4.** We can obtain the complete asymptotic expansion for \( H(z; q) \). For each \( \ell \in \mathbb{N} \), let the Bernoulli polynomials \( B_\ell(x) \) and the Bernoulli number \( B_\ell \) be defined by (4.1) and (4.2), respectively. Then, we have
\[
H(z; q) \sim H(z; q)C_H t^{B_H} \exp \left(\frac{A_H}{t} + \sum_{\ell=1}^{\infty} A_\ell t^\ell\right)
\]
with
\[
A_H = \sum_{a,b,c,d} \frac{\pi^2 S(a, b, c, d)}{6b}, A_\ell = \sum_{a,b,c,d} \frac{B_\ell S(a, b, c, d) b^\ell}{\ell(\ell + 1)!} B_{\ell+1} \left(\frac{a}{b}\right) \ell \in \mathbb{N},
\]
\[
B_H = \sum_{a,b,c,d} \left(\frac{a}{b} - \frac{1}{2}\right) S(a, b, c, d) \text{ and } C_H = \prod_{a,b,c,d} \left(\frac{\Gamma(a/b)/\sqrt{2\pi}}{S(a,b,c,d)}\right).
\]

**Remark 1.5.** It is clear that the asymptotic results of [3] and [4] of McIntosh could be derived as special cases of the Eulerian series (1.3) with the conditions which satisfied. Moreover, our theorem is more general, the asymptotic expansion is similar with the definition of the mock theta function of Gordon and McIntosh [12]. Furthermore, some special \( H(1; q) \) are generating functions in many partition problems, for example, our results can give the complete asymptotic expansion for all partition generating functions in Bringmann and Mahlburg [13]. Using the Tauberian theorem of Ingham (see [13, Theorem 3.1] and [14]), which allows us to describe the asymptotic behavior of the coefficients of a power series using the analytic nature of the partition generating functions \( H(1; q) \).

This paper is organized as follows. In Section 2, we first determine the stationary point of \( F(x, t) \) and consider it’s Taylor expansion at the stationary point. In Section 3, we prove Lemma 1.1 and Lemma 1.2. In Section 4, we first collect properties on Bernoulli numbers, Bernoulli polynomials and the polylogarithm function. Then, we give the asymptotic facts for the general term of the product in (1.4) and prove that the logarithm of general term of \( H(z; q) \) satisfies the assumption of \( F(m, t) \) in Lemma 1.1. In Section 5 we prove Theorem 1 and Theorem 2. We will prove Theorem 1 in Subsection 5.1–5.3 and prove Theorem 2 in last subsection of this section. In Section 6, we apply our main theorem to the some confluent basic hypergeometric series, some simple Eulerian series and some mock theta functions.

### 2. The stationary point of \( F(x, t) \)

**Lemma 2.1.** Let \( t > 0 \) sufficiently small be fixed, let \( F(x, t), E_l(u) \) and \( \mathcal{S}_E \) be defined as Lemma 1.1. Also let \( \mathcal{S}_F \) be the set of all maximum point of \( F(x, t) \) for all \( x > 1/t \). Then for each \( X \in \mathcal{S}_F \) there exist an \( u \in \mathcal{S}_E \) such that
\[
X = (u + r_u(t))/t,
\]
with \( r_u(t) \ll t^{1/(2k_u - 1)} \) and where \( k_u \) is the minimum positive integer such that \( E_{k_u - 1}^{(2k_u)}(u) \neq 0. \)
Proof. Letting \( X \in \mathcal{F} \) then
\[
0 = \frac{\partial F}{\partial x}(X, t) \sim E_{-1}(Xt) + \sum_{\ell=1}^{\infty} E'_{\ell-1}(Xt)t^\ell.
\]
(2.1)

Thus we have \( \lim_{t \to 0^+} (Xt) \in \mathcal{F} \), namely, \( X \sim u/t \) for some \( u \in \mathcal{F} \). On the other hand, it is clear that for each \( u \in \mathcal{F} \) there exist an \( r_u(t) \in \mathbb{R} \) with \( r_u(t) = o(t) \) for \( t \to 0^+ \) such that \( X = (u + r_u(t))/t \) satisfies (2.1). Further, the using of Taylor theorem yields
\[
\sum_{k=1}^{\infty} \frac{E_{-1}^{(k+1)}(u)}{k!} r_u(t)^k + \sum_{\ell=1}^{\infty} t^\ell \sum_{k=0}^{\infty} \frac{E'_{\ell-1}^{(k+1)}(u)}{k!} r_u(t)^k \sim 0.
\]

Since \( u \) is the maximum point of \( E_{-1}(x) \) in \( x \in (0, \infty) \), so the minimum positive integer \( k \) such that \( E_{-1}^{(k+1)}(u) \neq 0 \) must be an odd integer \( 2k_u - 1 \). Moreover, we have \( E_{-1}^{(2k_u)}(u) < 0 \) and
\[
\sum_{k=2k_u-1}^{\infty} \frac{E_{-1}^{(k+1)}(u)}{k!} r_u(t)^k + \sum_{\ell=1}^{\infty} t^\ell \sum_{k=0}^{\infty} \frac{E'_{\ell-1}^{(k+1)}(u)}{k!} r_u(t)^k \sim 0.
\]

This implies that \( r_u(t) \ll t^{1/(2k_u-1)} \). Which completes the proof of the lemma. \( \square \)

Next, we obtain the Taylor series for \( F(m, t) \) at \( u/t \) with \( u \in \mathcal{F} \).

**Lemma 2.2.** Let \( F(x, t), E_\ell(u) \) and \( \mathcal{F} \) be defined as Lemma 1.1. Also let \( t > 0 \) sufficiently small. Then there exist a constant \( \theta_E > 0 \) depends only on \( E_{-1}(\cdot) \) such that for \( m \in [(u - \theta_E)/t, (u + \theta_E)/t] \) we have the Taylor expansion
\[
F(m, t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{\partial^\ell F}{\partial x^\ell}(u/t, t)(m - u/t)^\ell.
\]

**Proof.** First, we have for each \( N \in \mathbb{N} \),
\[
\partial^N F(x, t)/\partial x^N = O \left( t^{N-1} \left| E_{-1}^{(N)}(xt) \right| \right)
\]
holds for each \( x \gg 1/t \). Thus for \( m \in [(u-\theta)/t, (u+\theta)/t] \) with \( \theta \in (0, u/2) \), we find that
\[
\int_{u/t}^{m} \frac{\partial^{N+1} F(x, t) (m - x)^N}{\partial x^{N+1} N!} dx \ll \frac{t^N |m - u|^{N+1}}{(N + 1)!} \max_{|x - u|/t \leq \theta/t} \left| E_{-1}^{(N+1)}(xt) \right|.
\]

We note that \( E_{-1}(u) \) is real analytic function, then there exist a \( C_u > 0 \) depends only on \( u \) and \( E_{-1}(\cdot) \) such that
\[
\sup_{0 < \theta < u/2} \left( \frac{1}{(N + 1)!} \max_{|x - u|/t \leq \theta/t} \left| E_{-1}^{(N+1)}(xt) \right| \right) \leq C_u^{N+1}.
\]

Thus
\[
\int_{u/t}^{m} \frac{\partial^{N+1} F(x, t) (m - x)^N}{\partial x^{N+1} N!} dx \ll \frac{t^{N+1}}{\theta^{N+1}} C_u^{N+1}.
\]

Then, by setting
\[
\theta_E = \min_{u \in \mathcal{F}} \left( \frac{u}{3 + 2uC_u} \right)
\]

and recalling the Taylor theorem

\[ F(m, t) = \sum_{\ell=0}^{N} \frac{\partial^\ell F(u/t, t)}{\partial x^\ell} (m - u/t)^\ell + \int_{u/t}^{m} \frac{\partial^{N+1} F(u, t)(m - x)^N}{\partial x^{N+1}} \, dx, \]

we immediately obtain the proof of this lemma. \( \square \)

3. The proof of the Lemma 1.1 and Lemma 1.2

3.1. The proof of the Lemma 1.1. Let \( t > 0 \) sufficiently small be fixed. We have first

\[ \sum_{u_m/t < m \leq u_M/t} e^{F(m,t)} = \sum_{m \in M_F} \exp[F(m,t)] + \sum_{u_m/t < m \leq u_M/t} \exp[F(m,t)] \]

\[ := M_F + R_F, \quad (3.1) \]

where the number set

\[ M_F = \bigcup_{u \in S_E} M_u \text{ with } M_u = \{ m \in \mathbb{N} : |m - u/t| \leq t^{\theta_u - 1} \} \]

and \( \theta_u \in (1/(2k_u + 1), 1/(2k_u)) \) for each \( u \in S_E \), where \( k_u \) be defined as Lemma 2.1. It is easily seen that the above is a disjoint union as \( t \to 0^+ \).

3.1.1. The estimate of \( R_F \). For \( u \in S_E \) and \( m \in M_u \) we note that

\[ \frac{\partial^\ell F}{\partial x^\ell}(u/t, t) \sim t^{\ell-1} \left( E_{-1}^{(\ell)}(u) + t E_0^{(\ell)}(u) \right) \leq \begin{cases} t^\ell & \ell \in [1, 2k_u - 1] \\ t^{\ell-1} & \ell \geq 2k_u \end{cases} \quad (3.2) \]

holds for each \( \ell \in \mathbb{N}_1 \), thus the using of Lemma 2.2 yields

\[ F(m, t) = \sum_{\ell \geq 0} \frac{1}{\ell!} \frac{\partial^\ell F}{\partial x^\ell}(u/t, t)(m - u/t)^\ell \]

\[ = F(u/t, t) + \frac{1}{(2k_u)!} \frac{\partial^{2k_u} F}{\partial x^{2k_u}}(u/t, t)(m - u/t)^{2k_u} + o(1) \]

\[ = F(u/t, t) + \frac{1}{(2k_u)!} E_{-1}^{(2k_u)}(u)(m - u/t)^{2k_u} t^{2k_u - 1} + o(1) \quad (3.3) \]

holds for \( t \to 0^+ \). Therefore from the facts that \( E_{-1}^{(2k_u)}(u) < 0 \) for each \( u \in S_E \), the monotonicity of \( F(m, t) \) and (3.3) we obtain that

\[ R_F \ll \frac{1}{t} \sum_{u \in S_E} e^{F(u/t, t)} \exp \left( \frac{E_{-1}^{(2k_u)}(u)}{(2k_u)!} t^{-1+2\theta_u k_u} + o(1) \right) \]

\[ \ll \sum_{u \in S_E} \exp \left( F(u/t, t) - \delta_u t^{-1+2\theta_u k_u} \right) \quad (3.4) \]

for some \( \delta_u > 0 \) depends only on \( E_{-1}(\cdot) \) and \( u \).
3.1.2. The estimate of $M_F$. We write

$$T_F(u, t) = \sum_{|m-u/t| \leq t^{\theta u-1}} \exp [F(m, t) - F(u/t, t)]$$

$$= \sum_{|m-u/t| \leq t^{\theta u-1}} \exp \left[ \sum_{\ell \geq 1} \frac{1}{\ell!} \frac{\partial^\ell F}{\partial x^\ell} (u/t, t)(m - u/t)^\ell \right].$$

We denote by

$$f(m, t) = \sum_{\ell \geq 1} \frac{1}{\ell!} \frac{\partial^\ell F}{\partial x^\ell} (u/t, t)(m - u/t)^\ell,$$

then for $x \in M_u$ and each $N \in \mathbb{N}_1$, it is not hard to prove that

$$\partial^N f(x, t) / \partial x^N \ll_N \begin{cases} t^{N-1+\theta u} & N \leq 2k_u - 1 \\ t^{N-1} & N \geq 2k_u \end{cases} \ll t^{\theta u} \quad (3.5)$$

by Lemma 2.2. The Euler–Maclaurin formula (see for example [15, Theorem D.2.1]) gives that for each $N \in \mathbb{N}$,

$$T_F(u, t) = \int_{[u-t^{\theta u}/t]}^{[(t^{\theta u}+u)/t]} \left( e^{f(x, t)} - \frac{(-1)^N}{N!} B_N(x - \lfloor x \rfloor) \frac{\partial^N e^{f(x, t)}}{\partial x^N} \right) dx$$

$$+ \sum_{\ell=0}^N (-1)^{\ell+1} B_{\ell+1} \frac{\partial^\ell e^{f(x, t)}}{\partial x^\ell} \bigg|_{[(u-t^{\theta u}/t]}

\quad \sum_{m \in \mathbb{N}} \prod_{j=1}^N \frac{N!}{m_j! j^m_j} \prod_{j=1}^N \left( \frac{\partial^j f(x, t)}{\partial x^j} \right)^{m_j}

\quad \equiv e^{(2k_u)}(u) t^{-1+2\theta u k_u} + o(1) \ll e^{-\delta_u t^{-1+2\theta u k_u}} \quad (3.6)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the greatest integer function and the least integer function, respectively. Using (3.3) it is not hard to see that

$$e^{f(x, t)} \bigg|_{[(u-t^{\theta u}/t)]} \ll \exp \left( \frac{E_{-1}^{(2k_u)}(u)}{(2k_u)!} t^{-1+2\theta u k_u} + o(1) \right) \ll e^{-\delta_u t^{-1+2\theta u k_u}} \quad (3.7)$$

where $\delta_u$ be defined as (3.4). From Faà di Bruno’s formula

$$\frac{\partial^N e^{f(x, t)}}{\partial x^N} = e^{f(x, t)} \sum_{m_1, m_2, \ldots, m_N \in \mathbb{N}} \prod_{m_1+2m_2+\ldots+N m_N = N} \frac{N!}{m_1! m_2! \ldots m_N!} \prod_{j=1}^N \left( \frac{\partial^j f(x, t)}{\partial x^j} \right)^{m_j}

\quad \equiv e^{(2k_u)}(u) t^{-1+2\theta u k_u} + o(1) \ll e^{-\delta_u t^{-1+2\theta u k_u}} \quad (3.8)$$

and the estimate (3.5), we obtain that

$$\frac{\partial^N e^{f(x, t)}}{\partial x^N} \bigg|_{[(u-t^{\theta u}/t)]} \ll e^{-\delta_u t^{2\theta u k_u-1}} \sum_{m_1, m_2, \ldots, m_N \in \mathbb{N}} \prod_{m_1+2m_2+\ldots+N m_N = N} \left( p_j^{\theta_u} \right)^{m_j} \ll e^{-\delta_u t^{2\theta u k_u-1}} \quad (3.8)$$
for each $N \in \mathbb{N}_1$, where $\delta_u$ also be defined as (3.4). Further more,

$$
\int_{[(u-t^\theta_u)/t]}^{[(t^\theta_u+u)/t]} \left| \frac{\partial^N e^{f(x,t)}}{\partial x^N} \right| \, dx \ll \int_{|x-u/t| \leq t^\theta_u-1} \sum_{m_1, m_2, \ldots, m_N \in \mathbb{N}} \prod_{j=1}^{N} (t^\theta_u)^{m_j} \\
\ll t^\theta_u N \int_{|x-u/t| \leq t^\theta_u-1} e^{f(x,t)} \, dx
$$

(3.9)

holds for each $N \in \mathbb{N}_1$. On the other hand, by (3.3) we see

$$
\int_{|x-u/t| \leq t^\theta_u-1} e^{f(x,t)} \, dx \gg \int_{|y| \leq t^\theta_u-1} \exp \left( \frac{y^{2k_u} u^{-2k_u-1}}{(2k_u)!} E^{(2k_u)}_{-1}(u) \right) \, dy \gg 1/t.
$$

(3.10)

Thus from (3.6)–(3.10), it is clear that

$$
T_F(u, t) = (1 + o(t^p)) \int_{|x-u/t| \leq t^\theta_u-1} \exp (F(x,t) - F(u/t, t)) \, dx,
$$

(3.11)

holds for each $p \geq 0$ as $t \to 0^+$.  

3.1.3. The final estimate. From (3.4), (3.10) and (3.11), it is obvious that for each $p \geq 0$,

$$
\sum_{u_m/t < m \leq u_M/t} e^{F(m,t)} = \sum_{u \in \mathcal{F}_E} \left(1 + o(t^p)\right) \int_{|x-u/t| \leq t^\theta_u-1} e^{F(x,t)} \, dx + O \left(e^{F(u/t,t) - \delta_u u^{-1+2\theta_u k_u}}\right)
$$

$$
= \left(1 + o(t^p)\right) \sum_{u \in \mathcal{F}_E} \int_{|x-u/t| \leq t^\theta_u-1} e^{F(x,t)} \, dx
$$

(3.12)

holds for $t \to 0^+$. Further more, if we denote by

$$
\mathcal{m} = (u_m/t, u_M/t) \setminus \left( \cup_{u \in \mathcal{F}_E} \{x \in \mathbb{R} : |x - u/t| \leq t^\theta_u-1\} \right).
$$

Then by the monotonicity of $F(m, t)$, we obtain

$$
\int_{\mathcal{m}} e^{F(x,t)} \, dx \ll \frac{u_M - u_m}{t} \sum_{u \in \mathcal{F}_E} e^{F(u/t,t) - \delta_u u^{-1+2\theta_u k_u}}.
$$

(3.13)

Combining (3.12), (3.13) and similar with (3.12), we obtain that for each $p \geq 0$, as $t \to 0^+$

$$
\sum_{u_m/t < m \leq u_M/t} e^{F(m,t)} = \left(1 + o(t^p)\right) \int_{u_m/t}^{u_M/t} e^{F(x,t)} \, dx.
$$

Which completes the proof of the Lemma 1.1.
3.2. **The proof of the Lemma 1.2.** It is not difficult to compute that the integral in (3.11) equals to

\[
\int_{|x| \leq \theta u - 1} \exp \left( \sum_{\ell \geq 1} \frac{1}{\ell!} \frac{\partial^\ell F}{\partial x^\ell} (u/t, t) x^\ell \right) \, dx
\]

\[
= \frac{1}{V(u, t)} \int_{|y| \leq \theta u - 1/V(u, t)} \exp \left( -y^{2k_u} + \sum_{k \geq 1}^* \lambda_k(u, t) y^k \right) \, dy
\]

\[
= \frac{1}{V(u, t)} \int_{|y| \leq \theta u - 1/(2k_u)} e^{-y^{2k_u}} \exp \left( \sum_{k \geq 1}^* \lambda_k(u, t) y^k \right) \, dy + O \left( e^{-\delta_u t^{-1+2\theta u k_u}} \right),
\]

where * means that \( k \neq 2k_u \) and

\[
V(u, t) = \left( -1 \frac{\partial^{2k_u} F}{\partial x^{2k_u}} (u/t, t) \right)^{\frac{1}{2k_u}}
\]

(3.14)

and

\[
\lambda_\ell(u, t) = \frac{1}{\ell! V(u, t)^\ell} \frac{\partial^\ell F}{\partial x^\ell} (u/t, t), \ell \in (\mathbb{N} \setminus \{2k_u\}).
\]

Further more, we have the estimates

\[
\frac{1}{V(u, t)} = \left( \frac{(2k_u)!}{E_0^{(2k_u)}(u)} \right)^{\frac{1}{2k_u}} t^{1/(2k_u) - 1} \left( 1 - \frac{1}{2k_u} E_0^{(2k_u)}(u) t + O(t^2) \right)
\]

and

\[
\lambda_\ell(u, t) \sim \left( \frac{(2k_u)!}{E_0^{(2k_u)}(u)} \right)^{\frac{1}{2k_u}} t^{\ell/(2k_u)} \times \begin{cases} E_0^{(\ell)}(u) & \ell \in [1, 2k_u) \\ t^{-1} E_0^{(\ell)}(u) & \ell \in (2k_u, \infty). \end{cases}
\]

(3.15)

by Newton’s generalized binomial theorem. If we write

\[
\exp \left( \sum_{k \geq 1}^* \lambda_k(u, t) x^k \right) = \sum_{\ell=0}^\infty \kappa_\ell(u, t) x^\ell,
\]

then

\[
\kappa_\ell(u, t) = \sum_{\ell_\ell, r_\rho, r \neq 2k_u} \prod_{r \geq 1} \frac{[\lambda_r(u, t)]^{\ell_r}}{r_\rho!} \left( \frac{(2k_u)!}{E_0^{(2k_u)}(u)} \right)^{\frac{r}{2k_u}} \frac{\partial^r F}{\partial x^r} (u/t, t)
\]

\[
\times \left( \frac{\partial^{2k_u} F}{\partial x^{2k_u}} (u/t, t) \right)^{\frac{r}{(2k_u)}} \right)^{\ell_r}. \quad (3.16)
\]
Hence the using of (3.15) and (3.16) implies that
\[
\kappa_{\ell}(u, t) \ll \sum_{\ell_r \in \mathbb{N}} \prod_{r=1}^{2k_u-1} t^{r_{2k_u}} \prod_{r=2k_u+1}^{\ell} t^{-r_{2k_u+1}} \ll t^{2k_u} \sum_{\ell_r \in \mathbb{N}} \prod_{r=2k_u+1}^{\ell} t^{-r_{2k_u+1}} \ll t^{2k_u(2k_u+1)}
\]
for each \( \ell \in \mathbb{N} \). Thus it is not hard to show that
\[
T_F(u, t) = 1 + o(t^N) \left( \sum_{\ell=0}^{2k_u(2k_u+1)N} \kappa_{\ell}(u, t) \int_{\mathbb{R}} e^{-y^{2k_u}} y^{\ell} dy + o(t^N) \right)
\]
\[
= \frac{1}{V(u, t)} \left( \sum_{\ell=0}^{(2k_u+1)N} \Gamma \left( \frac{2\ell + 1}{2k_u} \right) \frac{k_{2\ell}(u, t)}{k_u} + o(t^N) \right)
\]
holds for each \( N \in \mathbb{N} \). Then combining (3.12) we finish the proof of Lemma 1.2.

4. Preliminary results of \( \mathcal{H}(z; q) \)

To prove Theorem 1, we first need the following concepts and lemmas.

4.1. some special functions. Recall that the Bernoulli polynomials \( B_{\ell}(x) \) are involved in the generating function is
\[
\frac{ze^{zx}}{e^z - 1} = \sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell!} B_{\ell}(x),
\]
where \( |z| < 2\pi \). The Bernoulli numbers are given by
\[
B_{\ell} = B_{\ell}(0) \text{ for } \ell \in \mathbb{N}.
\]
It is a well known fact that
\[
B_0 = 1, B_1 = -\frac{1}{2}, B_{2\ell+1} = 0 \text{ and } B_{2\ell} \sim (-1)^{\ell+1} \frac{2(2\ell)!}{(2\pi)^{2\ell}} \text{ as } \ell \to \infty \text{ for } \ell \in \mathbb{N}_1.
\]
The polylogarithm function \( \text{Li}_s(z) \) is
\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.
\]
If \( s = 2 \), then it is called that dilogarithm function, say
\[
\text{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2},
\]
and converges for \( |z| \leq 1 \). If \( s = 1 \), then
\[
\text{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1 - z),
\]
which converges for $|z| < 1$. If $s$ is a non-positive integer, say $s = -r$, then the polylogarithm function is defined recursively by

$$\text{Li}_{-r}(z) = z \frac{d}{dz} \text{Li}_{1-r}(z) \text{ for } r \in \mathbb{N}$$ (4.6)

and $|z| < 1$. Thus we have the following lemma.

**Lemma 4.1.** *(See [11, Lemma 2])* Let integer $n \leq 2$, $a$ and $x$ be real number. Then, we have

$$\text{Li}_n(ae^x) = \sum_{k=0}^{\infty} \frac{\text{Li}_{n-k}(a)}{k!} x^k$$

holds for $|a| < 1$ and $|x| < \min(-\log |a|, \pi)$.

We need the following asymptotic of $q$-shifted factorials which proof follows immediately from Theorem 2 of [11] or Theorem 2 of [16] with change of variables.

**Lemma 4.2.** Let $b > 0$ and real $a/b \notin \mathbb{Z}_{\leq 0}$. Then, we have

$$(e^{-at}, e^{-bt}) \sim \frac{\sqrt{2\pi t}^{\frac{a}{b} - \frac{b}{a}}}{\Gamma(a/b)} \exp \left( -\frac{\pi^2}{6bt} - \sum_{\ell=1}^{\infty} \frac{b^\ell B_{\ell+1}(a/b)}{\ell(\ell+1)!} t^\ell \right).$$

### 4.2. Some asymptotic properties of $H(z; q)$.

#### 4.2.1. The asymptotics for the product term of $H(z; q)$.

From Lemma 4.2, we have

$$\prod_{a,b,c,d} (q^a; q^b)_{\infty}^{-S(a,b,c,d)}$$

$$\sim \prod_{a,b,c,d} \left( \frac{\Gamma(a/b)t^{\frac{a}{b} - \frac{b}{a}}}{\sqrt{2\pi}} \exp \left( \frac{\pi^2}{6bt} + \sum_{\ell=1}^{\infty} \frac{b^\ell B_{\ell+1}(a/b)}{\ell(\ell+1)!} t^\ell \right) \right)^{S(a,b,c,d)}$$

$$= t^{\sum_{a,b,c,d} \left( \frac{a}{b} - \frac{b}{a} \right) S(a,b,c,d)} \prod_{a,b,c,d} \left( \Gamma(a/b)/\sqrt{2\pi} \right)^{S(a,b,c,d)}$$

$$\times \exp \left( \frac{1}{t} \sum_{a,b,c,d} \frac{\pi^2 S(a,b,c,d)}{6b} + \sum_{\ell=1}^{\infty} t^\ell \sum_{a,b,c,d} \frac{B_{\ell} S(a,b,c,d) b^\ell}{\ell(\ell+1)!} B_{\ell+1} \left( \frac{a}{b} \right) \right).$$

Namely, we have

$$\prod_{a,b,c,d} (q^a; q^b)_{\infty}^{-S(a,b,c,d)} \sim C_H t^{B_H} \exp \left( \frac{A_H}{t} + \sum_{\ell=1}^{\infty} A_{\ell} t^\ell \right)$$

with

$$A_H = \sum_{a,b,c,d} \frac{\pi^2 S(a,b,c,d)}{6b}, \quad A_{\ell} = \sum_{a,b,c,d} \frac{B_{\ell} S(a,b,c,d) b^\ell}{\ell(\ell+1)!} B_{\ell+1} \left( \frac{a}{b} \right) \quad \ell \in \mathbb{N}_1,$$

$$B_H = \sum_{a,b,c,d} \left( \frac{a}{b} - \frac{1}{2} \right) S(a,b,c,d)$$

and

$$C_H = \prod_{a,b,c,d} \left( \Gamma(a/b)/\sqrt{2\pi} \right)^{S(a,b,c,d)}.$$
4.2.2. Some asymptotic properties of $\mathcal{H}(z; q)$. We first denote by

$$\mathcal{F}(m, t) = \log \left( \prod_{a,b,c,d} \left( q^{bcm+a+bd}; q^b \right)^{S(a,b,c,d)} q^{Am^2 + Bm^2 z^m} \right),$$

then

$$\mathcal{H}(z; q) = \sum_{m \in \mathbb{N}} \exp (\mathcal{F}(m, t)). \quad \text{(4.7)}$$

It is not hard to compute that

$$\mathcal{F}(m, t) = \sum_{a,b,c,d} S(a, b, c, d) \sum_{\ell=0}^\infty \log \left( 1 - q^{bcm+a+bd+\ell} \right) - Am^2 t - Bmt + mv$$

$$= mv - Am^2 t - Bmt - \sum_{a,b,c,d} S(a, b, c, d) \sum_{\ell=0}^\infty \sum_{k=1}^\infty \frac{q^{k(bcm+a+bd)+\ell}}{k}$$

$$= mv - Am^2 t - Bmt - \sum_{a,b,c,d} S(a, b, c, d) \sum_{k=1}^\infty \frac{e^{-k(1-a+bd)}t}{k(1 - e^{-kbt})} \quad \text{(4.8)}$$

or, we have

$$\mathcal{F}(m, t) = mv - Am^2 t - Bmt + \sum_{a,\beta,\gamma} S_{a\beta\gamma} \sum_{k=1}^\infty \frac{e^{-k(1-a+bd)}t}{k(1 - e^{-kbt})} \quad \text{(4.9)}$$

by (1.6). The series $\mathcal{F}(m, t)$ converges uniformly on compact subsets of $t > 0$ and $m \geq 0$. Moreover, it is easily seen that the function $\mathcal{F}(x, t)$ with $(x, t) \in (\mathbb{R}_{>0})^2$ is a real analytic function. Under the following lemma, we can apply Lemma 1.1 to prove Theorem 1.

**Lemma 4.3.** Let $\mathcal{F}(x, t)$ be defined as (4.8) and let the real number $\delta > 0$. Then, for all $X \geq t^{-\delta}$, we have

$$\frac{\partial^N \mathcal{F}(X, t)}{\partial X^N} = t^N \left( \sum_{\ell=-1}^M H_{\ell}^{(N)}(Xt)t^\ell + o(t^{M+2}) \right)$$

as $t \to 0^+$ for each $N, M \in \mathbb{N}$, where $H_{-1}(u)$ be defined by (1.7) and

$$H_m(u) = \begin{cases} \sum_{a,b,c,d} \text{Li}_1(e^{-bcu}) \left( d + \frac{a}{b} - \frac{1}{2} \right) S(a, b, c, d) - Bu & m = 0 \\ (-1)^m \sum_{a,b,c,d} b^m S(a,b,c,d) \frac{B_{m+1}(a+bd)}{(m+1)!} \text{Li}_{1-m}(e^{-bcu}) & m \in \mathbb{N}_1, \end{cases}$$

or

$$H_m(u) = \begin{cases} - \sum_{a,\beta,\gamma} \text{Li}_1(e^{-\alpha u}) \left( \frac{a}{\beta} - \frac{1}{2} \right) S_{a\beta\gamma} - Bu & m = 0 \\ (-1)^{m-1} \sum_{a,\beta,\gamma} b^m S_{a\beta\gamma} \frac{B_{m+1}(\gamma/\beta)}{(m+1)!} \text{Li}_{1-m}(e^{-\alpha u}) & m \in \mathbb{N}_1 \end{cases}$$

by comparing (4.8) and (4.9). In particular, we have for each $X \gg 1/t$

$$\frac{\partial^N \mathcal{F}(X, t)}{\partial X^N} = t^N \left( \sum_{\ell=-1}^M H_{\ell}^{(N)}(Xt)t^\ell + o(t^{M+1}) \right)$$

as $t \to 0^+$ for each $N \in \mathbb{N}$. 

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Proof. We shall prove the case \( N = 0 \), the proof of the cases \( N \geq 1 \) is similar. It is not hard to see that
\[
\mathcal{F}(X, t) = X v - AX^2 t - BX t
- \sum_{a, b, c, d} \frac{S(a, b, c, d)}{bt} \sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^2} \frac{-kb t}{e^{-kt} - 1} + R_1(X, t),
\]
where
\[
R_1(X, t) \ll \sum_{a, b, c, d} \sum_{k > t^{2\delta/3 - 1}} \frac{e^{-ktbcX}}{kt} \ll e^{-t^{-\delta/4}}
\]
by using the condition \( X \geq t^{-\delta} \). By (4.2) we have
\[
\sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^2} \frac{-kb t}{e^{-kt} - 1} = \sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^2} \sum_{\ell = 0}^{\infty} B_{\ell}(-kt)^{\ell}
\]
Then use the asymptotic formula of \( B_{\ell} \) (see (4.3)), for each \( M \in \mathbb{N} \)
\[
\sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^2} \frac{-kb t}{e^{-kt} - 1} = \sum_{\ell = 0}^{M} B_{\ell}(-bt)^{\ell} \sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^{2-\ell}} + R_2(X, t),
\]
where as \( t \to 0^{+} \),
\[
R_2(X, t) \ll \sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-ktbcX}}{k^2} \sum_{\ell > M} (kt)^{\ell} \ll |bt^{2\delta/3}|^{M+1} \ll t^{2(M+1)}.
\]
Using (4.4), it is not difficult seen that
\[
\sum_{k \leq t^{2\delta/3 - 1}} \frac{e^{-k(bX + a + bd)t}}{k^2} \frac{-kb t}{e^{-kt} - 1} = \sum_{\ell = 0}^{M} B_{\ell}(-bt)^{\ell} \sum_{\ell = 0}^{M} \frac{Li_{2-\ell}(e^{-k(bX + a + bd)t})}{\ell!} + O(t^{4}(M+1)).
\]
By Lemma 4.1
\[
\frac{d^{M}Li_{2-\ell}(e^{-bXt}e^{x})}{dx^{M}} = Li_{2-\ell-M}(e^{-bXt}e^{x}).
\]
Note the fact that
\[
Li_{1-m}(e^{-bXt}) \ll \left( \sum_{k \leq t^{2\delta/3 - 1}} + \sum_{k > t^{2\delta/3 - 1}} \right) k^{m-1} e^{-kbXt} \ll t^{(2\delta/3-1)m}
\]
holds for \( m \leq |\log t| \), then Taylor Theorem implies that
\[
Li_{2-\ell}(e^{-k(bX + a + bd)t}) = \sum_{k=0}^{M} \frac{Li_{2-\ell-k}(e^{-bXt})}{k!}(-a + bd)^{k} + R_{3}(X, t),
\]
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where

\[
R_3(X, t) = \int_0^{-(a+bd)t} \left( e^{-beXt} e^x \right) \frac{(-(a+bd)t-x)^M}{M!} dx
\]

\[
\ll \frac{(a+bd)t)^{M+1}}{(M+1)!} Li_{1-M} \left( e^{-beXt} \right) \ll t^{2M\delta/3} t^{(2\delta/3-1)\ell}.
\]

for \( \ell \leq M \) as \( t \to 0^+ \). Therefore, we obtain that

\[
\sum_{k \leq t^{2\delta/3-1}} e^{-k(bcX+a+bd)t} \frac{-kbt}{k^2} \frac{e^{-kbt}}{k!} = \sum_{0 \leq k \leq M} \sum_{\ell \leq M} B_{\ell}(-bt)^\ell \frac{Li_{2-\ell-k} \left( e^{-beXt} \right)}{k!} \left( -(a+bd)t \right)^k + O \left( t^{\delta(M+1)} \right).
\]

Thus we obtain that as \( t \to 0^+ \),

\[
\mathcal{F}(X, t) = Xv - AX^2 t - BX t + O \left( t^{\delta(M+1)} + \sum_{M<k \leq 2M} t^{k-1+(2\delta/3-1)(k-1)} \right)
\]

\[
- \sum_{m=0}^{M} (-t)^m \sum_{a,b,c,d} Li_{2-m} \left( e^{-beXt} \right) \frac{B_{\ell}b^\ell (a+bd)^m S(a,b,c,d)}{\ell! (m-\ell)!}.
\]

Namely,

\[
\mathcal{F}(X, t) = \frac{1}{t} \left( v(Xt) - A(Xt)^2 - \sum_{a,b,c,d} Li_{2} \left( e^{-beXt} \right) b^{-1} S(a,b,c,d) \right)
\]

\[
+ \sum_{a,b,c,d} Li_{1} \left( e^{-beXt} \right) (B_0(a+bd) + B_1 b) \frac{S(a,b,c,d)}{b} - B(Xt)
\]

\[
+ \sum_{m=1}^{M} (-1)^m \sum_{a,b,c,d} Li_{1-m} \left( e^{-beXt} \right) \frac{b^m S(a,b,c,d)}{(m+1)!} B_{m+1} \left( \frac{a+bd}{b} \right) + o \left( t^{\delta M/2} \right)
\]

for each \( M \in \mathbb{N} \). Finally, note that if \( X \gg 1/t \) then \( Li_{1-m} \left( e^{-beXt} \right) \ll 1 \), thus we immediately get the proof of (4.10). Which completes the proof of the lemma.

\[ \square \]

5. The proof of the main theorem

Let us denote by \( \mathcal{S}_H \) the set of all local maximum points of \( H_{-1}(x) \) with \( x \in \mathbb{R}_{>0} \) and let \( t > 0 \) sufficiently small. We first have \( \mathcal{S}_H \) is a finite point set or an empty set by the definition of \( H_{-1}(u) \) in Lemma 4.3. We rewritted (4.7) as

\[
\mathcal{H}(z; q) = \left( \sum_{m \leq f_m/t} + \sum_{f_m/t < m \leq f_M/t} + \sum_{m > f_M/t} \right) e^{\mathcal{F}(m,t)} := \Sigma_1 + \Sigma_2 + \Sigma_3,
\]

where \( f_m \) and \( f_M \) be defined as follows:
1.1. Lemma 4.3. We also write by

\[ I_H(z; q) = \int_0^\infty e^{F(x, t)} \, dx = \left( \int_0^{f_m/t} + \int_{f_m/t}^{f_M/t} + \int_{f_M/t}^\infty \right) e^{F(x, t)} \, dx := I_1 + I_2 + I_3. \]

Note that if \( I_H \) is nonempty, then by Lemma 4.3 and Lemma 1.1, as \( t \to 0^+ \)

\[ \Sigma_2 = (1 + o(t^p))I_2 \quad (5.1) \]

holds for each \( p \geq 0 \). Thus we just need consider the estimates of \( \Sigma_1, I_1, \Sigma_3 \) and \( I_3 \).

5.1. The treat of \( \Sigma_1 \) and \( I_1 \).

Lemma 5.1. Let \( F(m, t) \) be defined as (4.9) and \( f_m \) be defined as above. Then, we have as \( t \to 0^+ \)

\[ \Sigma_1 \ll \exp (F(f_m/t, t) + |\log t|^3) \quad \text{and} \quad I_1 \ll \exp (F(f_m/t, t) + |\log t|^3). \]

Proof. First of all, let \( t \to 0^+ \) and \( x \geq 0 \). It is not difficult to compute that

\[
\frac{\partial F(x, t)}{\partial x} = v - 2Axt - Bt - t \sum_{\alpha, \beta, \gamma} S_{\alpha \beta \gamma} \sum_{k=1}^\infty \frac{e^{-k(\alpha x + \gamma \beta t)}}{e^k - 1}.
\]

\[
= v - 2Axt - t \sum_{\alpha, \beta, \gamma} S_{\alpha \beta \gamma} \frac{\left[ \frac{1}{k} \right]}{\frac{1}{e^k} - 1} + O \left( \exp \left( - \frac{\alpha}{2\beta} x \right) \right)
\]

\[
= v - 2Axt - t \sum_{\alpha, \beta, \gamma} S_{\alpha \beta \gamma} \frac{\left[ \frac{1}{k} \right]}{k^\beta} + O \left( \frac{1}{1 + x + t} \right)
\]

\[
= v - 2Axt - \sum_{\alpha, \beta, \gamma} S_{\alpha \beta \gamma} \frac{\left[ \frac{1}{k} \right]}{k} + O \left( \frac{1}{1 + x + t} \right).
\]

Therefore it is not hard seen that

\[
\frac{\partial F(x, t)}{\partial x} = \begin{cases} H_{-1}'(xt) + O(1/x + t) & x \geq 1 \\ O(|\log t|) & x \in [0, 2]. \end{cases} \quad (5.2)
\]

Thus for each \( X \in [1, f_m/t] \) and \( Y \gg 1/t \), we have

\[
F(X, t) = F(Y, t) - \int_X^Y \left( H_{-1}'(xt) + O(1/x + t) \right) \, dx
\]

\[
= F(Y, t) - H_{-1}(Yt)/t + H_{-1}(Xt)/t + O (|\log t|). \]

Using Lemma 4.3 we obtain that

\[
F(X, t) = H_{-1}(Xt)/t + O (|\log t|).
\]
Therefore as $t \to 0^+$, we have
\[ \exp (\mathcal{F}(x, t)) \ll \exp (H_{-1}(xt)/t + |\log t|^2) \]
Moreover, from the assumption of Theorem 1 and Lemma 4.3 we find that
\[ H_{-1}(xt) \leq H_{-1}(f_m) = \mathcal{F}(f_m/t, t) + O(1) \]
for each $x \in [0, f_m/t]$. By (5.4) we have $\mathcal{F}(X, t) \ll \mathcal{F}(1, t) + |\log t|$ for $X \in [0, 1]$. Hence
\[ \Sigma_1 \ll \sum_{m \leq f_m/t} \exp (H_{-1}(mt)/t + O(|\log t|)) \ll \exp (\mathcal{F}(f_m/t, t) + |\log t|^3) \]
and
\[ I_1 \ll \int_0^{f_m/t} \exp (H_{-1}(xt)/t + O(|\log t|)) \, dx \ll \exp (\mathcal{F}(f_m/t, t) + |\log t|^3) . \]
Which completes the proof of the lemma. \[\square\]

5.2. The estimate of $\Sigma_3$ and $I_3$. If $A = v = 0$ then
\[ \mathcal{H}(z; q) = \mathcal{H}(1; q) = \sum_{m \in \mathbb{N}} q^{B_m} \prod_{\alpha, \beta, \gamma} (q^{\alpha m + \gamma}; q^3)_{\infty} \]
and for $x \gg 1/t$,
\[ \mathcal{F}(x, t) = \frac{1}{t} \sum_{\alpha} \text{Li}_2(e^{-\alpha x}) \sum_{\alpha, \beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma} - Btx + O \left( \sum_{\alpha} e^{-\alpha xt} \right) + O(t) \quad \text{(5.3)} \]
and
\[ \frac{\partial \mathcal{F}(x, t)}{\partial x} = -\sum_{\alpha} \alpha \text{Li}_1(e^{-\alpha x}) \sum_{\alpha, \beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma} - Bt + O \left( \sum_{\alpha} te^{-\alpha xt} \right) + O(t^2) \quad \text{(5.4)} \]
by Lemma 4.3. The assumption of Theorem 1 implies that
\[ H_{-1}(u) = \sum_{\alpha} \text{Li}_2(e^{-\alpha u}) \sum_{\beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma} \neq 0. \]
and hence $\sum_{\beta, \gamma} \beta^{-1} S_{\alpha \beta \gamma} \neq 0$ for some $\alpha$. Therefore we can rewritten (5.4) as
\[ \frac{\partial \mathcal{F}(x, t)}{\partial x} = -\sum_{k=1}^{H} \alpha_k \log(1 - e^{-\alpha_k xt}) f(\alpha_k) - Bt + O \left( te^{-\alpha_1 xt} + t^2 \right). \]
Furthermore,
\[ \frac{\partial \mathcal{F}(x, t)}{\partial x} = e^{-\alpha_1 xt} \alpha_1 f(\alpha_1) \left( 1 + O \left( e^{-\alpha_1 xt} + e^{-(\alpha_2 - \alpha_1) xt} \right) \right) - Bt(1 + O(t)). \quad \text{(5.5)} \]
This yields if $f(\alpha_1) > 0$ then $\partial \mathcal{F}(x, t)/\partial x = 0$ has only one solution $X_\mathcal{H}$ for $x > C/t$ with $C > 0$ be sufficiently large depends only on $\mathcal{H}(1; \cdot)$. Moreover, it is clear to compute that
\[ X_\mathcal{H} = \frac{1}{\alpha_1 t} \left( \log \left( \frac{1}{t} \right) + O(1) \right). \]
In this case, we have the following lemma.
Lemma 5.2. If \( f(\alpha_1) > 0 \) then as \( t \to 0^+ \)
\[
\sum_3 = (1 + o(t^p)) I_3 + O(\exp(\mathcal{F}(f_M/t, t) + 2|\log t|))
\]
holds for each \( p \geq 0 \).

Proof. The proof of this lemma is similar with the proof of (3.11). We first give the estimates of the derivatives of \( \mathcal{F}(x, t) \) for \( x > A_h|\log t|/t \) with \( A_h = 1/(2 + 2\alpha_1) \). From (5.3) we have
\[
\mathcal{F}(x, t) = -f(\alpha_1) e^{-\alpha_1 xt} (1 + O(e^{-\alpha_1 xt} + e^{-(\alpha_2 - \alpha_1) xt})) - Bxt + O(e^{-\alpha_1 xt})
\]
\[
= -f(\alpha_1) e^{-\alpha_1 xt} (1 + o(1)) - Bxt.
\]
From (5.5) we have
\[
\frac{\partial \mathcal{F}(x, t)}{\partial x} \ll e^{-\alpha_1 xt} + t \ll t^{A_h}.
\]
From Lemma 4.3 we have for each \( N \geq 2 \),
\[
\frac{\partial^N \mathcal{F}(x, t)}{\partial x^N} \ll t^{N-1} \ll t^{A_h N}.
\]
Thus similar with (3.7) and (3.8), for each \( N \in \mathbb{N} \) we have
\[
\frac{\partial^N e^{\mathcal{F}(x, t)}}{\partial x^N} \bigg|_{[A_h|\log t|/t]}^\infty \ll e^{\mathcal{F}([A_h|\log t|/t], t)} \ll \exp\left(-\frac{f(\alpha_1)}{t} e^{-\alpha_1 [A_h|\log t|/t]t} (1 + o(1))\right)
\]
\[
\ll \exp\left(-f(\alpha_1) t^{-1+\alpha_1 A_h} (1 + o(1))\right) \ll \exp\left(-1/\sqrt{t}\right). \quad (5.6)
\]
Similar with (3.9), we have for each \( N \in \mathbb{N}_1 \),
\[
\frac{\partial^N e^{\mathcal{F}(x, t)}}{\partial x^N} \ll e^{\mathcal{F}(x, t)} \sum_{m_1, m_2, \ldots, m_N \in \mathbb{N}} \prod_{j=1}^N (t^{j A_h})^{m_j} \ll t^{A_h N} e^{\mathcal{F}(x, t)}.
\]
Further more, it is not not difficult compute that
\[
\int_{[A_h|\log t|/t]}^{\infty} e^{\mathcal{F}(x, t)} \, dx = \int_{[A_h|\log t|/t]}^{\infty} \exp\left(-Bxt - \frac{f(\alpha_1)}{t} e^{-\alpha_1 xt} (1 + o(1))\right) \, dx
\]
\[
= \frac{1}{Bt} \int_0^{t^{A_h B (1+o(1))}} \exp\left(-\frac{f(\alpha_1)}{t} y^{\alpha_1/B} (1 + o(1))\right) \, dy \approx t^{\alpha_1/B - 1}. \quad (5.7)
\]
Thus similar with (3.11) and the using of Euler–Maclaurin formula yields that
\[
\sum_{m > [A_h|\log t|/t]} e^{\mathcal{F}(m, t)} = (1 + o(t^p)) \int_{[A_h|\log t|/t]}^{\infty} e^{\mathcal{F}(x, t)} \, dx \quad (5.8)
\]
holds for each \( p \geq 0 \). On the other hand, it is not difficult seen that \( \mathcal{F}(x, t) \) has an unique minimal point on \( x \in [f_M/t, A_h|\log t|/t] \) if \( \mathcal{S}_H \) is nonempty or \( \mathcal{F}(x, t) \) is an increasing
function if $\mathcal{S}_H$ is empty. Thus
\[
\sum_{f_M/t < m \leq [A_h | \log t]/t} e^{\mathcal{F}(m,t)} \ll t^{-2} \exp (\mathcal{F}(f_M/t,t)) + t^{-2} \exp (\mathcal{F}([A_h | \log t]/t,t))
\]
\[
\ll \exp(\mathcal{F}(f_M/t,t) + 2|\log t| + \exp (-1/\sqrt{t}) \tag{5.9}
\]
and
\[
\mathcal{I}_3 = \int_{[A_h | \log t]/t}^{\infty} e^{\mathcal{F}(x,t)} \, dx \ll t^{-2} (\exp (\mathcal{F}(f_M/t,t)) + \exp (\mathcal{F}([A_h | \log t]/t,t)))
\]
\[
\ll \exp(\mathcal{F}(f_M/t,t) + 2|\log t| + \exp (-1/\sqrt{t}) \tag{5.10}
\]
by (5.6). Combining (5.8), (5.7), (5.9) and (5.10) we immediately obtain the proof of the lemma. \qed

We have the following estimate for $\Sigma_3$ and $\mathcal{I}_3$.

**Lemma 5.3.** Let $\mathcal{F}(m,t)$ be defined as (4.8) and $f_M$ be defined as above. If $A > 0$ or $v < 0$ or $f(\alpha_1) < 0$, then as $t \to 0^+$,
\[
\Sigma_3 \ll \exp (\mathcal{F}(f_M/t, t) + 4|\log t|) \text{ and } \mathcal{I}_3 \ll \exp (\mathcal{F}(f_M/t, t) + 4|\log t|).
\]

*Proof.* If $x \geq 1/t^4$ then as $t \to 0^+$,
\[
\mathcal{F}(x,t) = xv - Ax^2t - Bxt - \sum_{\alpha, \beta, \gamma} S_{\alpha\beta\gamma} \sum_{k=1}^{\infty} \frac{e^{-k(\alpha x + \gamma) t}}{k (e^{kx^2 t} - 1)}
\]
\[
< xv - Ax^2t - Bxt + \sum_{\alpha, \beta, \gamma} |S_{\alpha\beta\gamma}| \sum_{k=1}^{\infty} \frac{e^{-k\alpha x t}}{k^2 \beta t}
\]
\[
= xv - Ax^2t - Bxt + O(e^{-1/t^2}). \tag{5.11}
\]
If $f(\alpha_1) < 0$, then from (5.5) we have $\partial \mathcal{F}(x,t)/\partial x < 0$ for $x > C_1/t$ with $C_1 > 0$ be sufficiently large depends only on $\mathcal{H}(1; \cdot)$. On the other hand, by (5.4) if $x \gg 1/t$ then
\[
\frac{\partial \mathcal{F}(x,t)}{\partial x} = v - 2Ax^2 + \sum_{\alpha} \alpha \log(1 - e^{-\alpha x t}) \sum_{\beta, \gamma} S_{\alpha\beta\gamma} \beta^{-1} \gamma^{-1} + O(1/x + t).
\]
Thus $A > 0$ or $v < 0$ implies that $\partial \mathcal{F}(x,t)/\partial x < 0$ holds for $x > C_2/t$ for some $C_2 > 0$ sufficiently large depends only on $\mathcal{H}(z; \cdot)$. Therefore under the condition of the lemma, the definition of $f_M$ and the estimate of $r_u(t)$ in Lemma 2.1 for $\mathcal{F}(x,t)$ implies that $\partial \mathcal{F}(x,t)/\partial x < 0$ for all $x > f_M/t$. Hence
\[
\Sigma_3 = \sum_{f_M/t < m \leq 1/t^4} e^{\mathcal{F}(m,t)} + \sum_{m > 1/t^4} e^{\mathcal{F}(m,t)}
\]
\[
\ll \exp (\mathcal{F}(f_M/t, t) + 4|\log t|) + \sum_{m \geq 1/t^4} e^{mv - Am^2t - Bmt}
\]
\[
\ll \exp (\mathcal{F}(f_M/t, t) + 4|\log t|) + e^{-1/t^2}.
\]
Moreover, Lemma 4.3 implies that $\mathcal{F}(x, t) \ll 1/t$ for all $x \geq 1/t$, hence

$$
\Sigma_3 \ll \exp (\mathcal{F}(f_M/t, t) + 4|\log t|).
$$

Now, the estimate for $\mathcal{I}_3$ is easy to establish. Thus we complete the proof of the lemma. □

5.3. The final estimate for $\mathcal{H}(z; q)$. From Lemma 2.2, we have for each $u \in \mathcal{S}_H$, as $t \to 0^+$,

$$
\mathcal{F}(u/t \pm 1/(t|\log t|), t) = \mathcal{F}(u/t, t) + \frac{1}{(2k_u)!} \frac{\partial^{2k_u} \mathcal{F}}{\partial x^{2k_u}} (u/t, t) \frac{1}{t|\log t|^{2k_u}} (1 + o(1))
$$

$$
< \mathcal{F}(u/t, t) - 2/\sqrt{t}.
$$

Therefore if $\mathcal{S}_H$ is nonempty then

$$
\mathcal{I}_2 \gg \sum_{u \in \mathcal{S}_H} e^{\mathcal{F}(u/t, t)}/t
$$

by Lemma 1.2. Therefore from Lemma 5.1 we have for each $p \geq 0$, as $t \to 0^+$,

$$
\Sigma_1 \ll t^p \mathcal{I}_2, \ \mathcal{I}_1 \ll t^p \mathcal{I}_2.
$$

From Lemma 5.2 and Lemma 5.3 we have: Let $f(\alpha_1)$ be defined as (??). If $A = v = 0$ and $f(\alpha_1) > 0$ then

$$
\Sigma_3 = (1 + o(t^p)) \mathcal{I}_3 + o(t^p \mathcal{I}_2).
$$

If $A > 0$ or $v < 0$ or $f(\alpha_1) < 0$, then

$$
\Sigma_3 \ll t^p \mathcal{I}_2 \text{ and } \mathcal{I}_3 \ll t^p \mathcal{I}_2.
$$

Thus together with (5.1), we obtain that for each $p \geq 0$, as $t \to 0^+$,

$$
\mathcal{H}(z; q) = (1 + o(t^p)) \mathcal{I}_H(z; q).
$$

If $\mathcal{S}_H$ is empty then $f(\alpha_1) > 0$, thus from Lemma 5.2 we have as $t \to 0^+$,

$$
\mathcal{H}(z; q) = \Sigma_1 + \Sigma_3 + \mathcal{I}_1 - \mathcal{I}_1
$$

$$
= O \left( \exp \left( \mathcal{F}(1/t, t) + |\log t|^3 \right) \right) + (1 + o(t^p)) \mathcal{I}_3 + \mathcal{I}_1
$$

$$
= (1 + o(t^p)) \mathcal{I}_H(z; q) + o(t^p \mathcal{I}_3) = (1 + o(t^p)) \mathcal{I}_H(z; q)
$$

holds for each $p \geq 0$.

Which completes the proof of the Theorem 1.

5.4. The proof of Theorem 2. From above discussion, it is not difficult seen that for each $p \geq 0$, as $t \to 0^+$

$$
\mathcal{H}(z; q) = (1 + o(t^p)) \left( \Sigma_2 + \int_{\log |\log t|/t}^{\infty} e^{\mathcal{F}(x, t)} \, dx \right), \quad (5.12)
$$

where if $\mathcal{S}_H$ is empty then $\Sigma_2 = 0$. If $\mathcal{S}_H$ is nonempty then we can obtain the full asymptotics by Lemma 1.2. Thus we just need to consider the integration part of above. On the other hand, combining Subsection 4.2.1, the full asymptotic expansion for $\mathcal{H}(z; q)$ immediately follows the full asymptotic expansion of $\mathcal{H}(z; q)$. We first give the results of $\Sigma_2$. 
5.4.1. Asymptotic expansion for $\Sigma_2$. Let denote by $m_u$ the minimum positive integer such that $H^{(2m_u)}(u) \neq 0$.

\[
V_H(u, t) = \left( -1 \frac{\partial^{2m_u} F}{\partial x^{2m_u}}(u/t, t) \right) \frac{1}{2m_u}, \tag{5.13}
\]

\[
\mu_\ell(u, t) = \sum_{\ell_r \in \mathbb{N}} \prod_{r \geq 1} \frac{1}{\ell_r!} \left( \frac{[(2m_u)!]^{r/(2m_u)}}{r!} \frac{\partial^r F}{\partial x^r}(u/t, t) \right) \frac{1}{\ell_r} \tag{5.14}
\]

where $\ast$ means that $r \neq 2m_u$ and from Subsection 3.2 the estimate

\[
\mu_\ell(u, t) \ll t^{m_u(2m_u + 1)}
\]

holds for each $\ell \in \mathbb{N}$. From Lemma 1.2 we have the full asymptotic expansion in $\mu_{2\ell}(u, t)$ of the form

\[
\Sigma_2 \sim \sum_{u \in \Sigma_H} \frac{\exp(F(u/t, t))}{V_H(u, t)} \sum_{\ell = 0}^{\infty} \frac{\Gamma((2\ell + 1)/2m_u)}{2m_u} \mu_{2\ell}(u, t) \frac{\mu_{2\ell}(u, t)}{m_u}, \tag{5.15}
\]

5.4.2. Asymptotic expansion for the integration part. Now we consider the integral of (5.12). In fact, we shall consider the case is when $A = v = 0$ with $f(\alpha_1) > 0$. We have first

\[
I_H(t) := \int_{\log |\log t/t|}^{\infty} e^{F(x/t)} dx
\]

\[
= \int_{\log |\log t/t|}^{\infty} \exp \left( -Bxt + \sum_{k=1}^{H} \sum_{\ell = 1}^{\alpha_\ell} e^{-k\alpha_\ell x} \sum_{\beta, \gamma} S_{\alpha_\ell, \beta, \gamma} e^{-k\beta t} \right) dx
\]

by (4.9). We denote by

\[
h_{k, \alpha}(t) := -\sum_{\beta, \gamma} S_{\alpha_\ell, \beta, \gamma} e^{-k\beta t} k(1 - e^{-\beta t}).
\]

Then for each $k \in \mathbb{N}_1$ and $\alpha_\ell$, it is obvious that

\[
h_{k, \alpha}(t) = \frac{1}{k^{2t}} \left( f(\alpha_\ell) - \sum_{r \geq 1} \frac{(-kt)^r}{r!} \sum_{s=0}^{r} \frac{r!}{s!} \beta^{s \gamma^{r-s}} S_{\alpha_\ell, \beta, \gamma} \right).
\]

Further more, we obtain that

\[
I_H(t) = \frac{1}{Bt} \int_{0}^{1/(\log t)^B} \exp \left( -\sum_{k=1}^{H} \sum_{\ell = 1}^{\alpha_\ell} y^{k\alpha_\ell/B} h_{k, \alpha}(t) \right) dy
\]

\[
= \frac{1 + o(t^p)}{Bt|h_1, \alpha_1(t)|^{\alpha_1}} \int_{0}^{1/(t \log t)^{B}} \frac{\partial}{\partial u} \left| u \right|^{-1} e^{-u} \exp \left( -\sum_{(\ell, k) \in \mathbb{N}_1^2 \setminus \{(1, 1)\}} \sum_{\ell \leq H} \frac{u^{k\alpha_\ell}}{|h_{1, \alpha_1}(t)|^{\alpha_1}} \right) dy
\]

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for each \( p \geq 0 \). Moreover, for each \( k \in \mathbb{N} \), we have

\[
\frac{h_{k,\alpha}(t)}{[h_{1,\alpha_1}(t)]^{k_{\alpha_1}}} = \frac{f(\alpha)}{k^2[f(\alpha)]^{|\alpha_1|}} \left( 1 + \sum_{r \geq 1} b_r(\alpha_1, \alpha_\ell, k)t^r \right)^{k_{\alpha_1} - 1} (5.16)
\]

holds for some \( b_r(\alpha_1, \alpha_\ell, k) \in \mathbb{R}(r \geq 1) \) as \( t \to 0^+ \). In order to give the asymptotic expansion, we shall define the countable set

\[
\Lambda(\mathcal{H}) := \left\{ \lambda_1 + \sum_{\ell = 2}^{H} \sum_{k = 1}^{\infty} \lambda_{\ell k}(k\alpha_\ell/\alpha_1 - 1) : \lambda_1, \lambda_{\ell k} \in \mathbb{N}, 2 \leq \ell \leq H, k \geq 1 \right\}. \quad (5.17)
\]

Then, we formally have

\[
\exp \left( - \sum_{(\ell, k) \in \mathbb{N}_0^2 \setminus \{(1, 1)\}} \frac{u^{k\alpha_\ell/\alpha_1}h_{k,\alpha_\ell}(t)}{[h_{1,\alpha_1}(t)]^{k\alpha_1/\alpha_1}} \right) = \sum_{\lambda \in \Lambda(\mathcal{H})} \kappa_\lambda(t)u^\lambda \quad (5.18)
\]

with some \( \kappa_\lambda(t) \in \mathbb{R} \). Moreover, for each \( \lambda \in \Lambda(\mathcal{H}) \), by (5.16) and (5.18)

\[
\kappa_\lambda(t) \ll \sum_{\ell_1 + \cdots + \ell_H = \lambda} \left( \sum_{\ell_1 \geq 2} t^{\sum_{\ell_r \geq 2}(r-1)\ell_1} \right) \prod_{j=2}^{H} \left( \sum_{\ell_{j_1} + 2\ell_{j_2} + \cdots = \ell_j} t^{\sum_{r \geq 1}(\alpha_j/\alpha_1 - 1)\ell_{j_r}} \right) \ll \sum_{\ell_1 + \cdots + \ell_H = \lambda} t^{\ell_1/2} \prod_{j=2}^{H} t^{(\alpha_j/\alpha_1 - 1)\ell_j} = \sum_{\ell_1 + \cdots + \ell_H = \lambda} t^{\ell_1/2 + \sum_{j=2}^{H}(\alpha_j/\alpha_1 - 3/2)\ell_j}.
\]

Thus it is not hard to see that

\[
\kappa_\lambda(t) \ll t^{\lambda/2} + t^{(\alpha_2/\alpha_1 - 1)\lambda}.
\]

We choose

\[
\alpha(h) = \frac{1}{4 + 2/(\alpha_2/\alpha_1 - 1)},
\]

then it not difficult seen that

\[
I_\mathcal{H}(t) = \frac{1 + o(t^p)}{\alpha_1 t[h_{1,\alpha_1}(t)]^{\frac{\alpha_1}{\alpha_1}}} \int_0^{t^{-\alpha(h)}} u^{\alpha_1 - 1} e^{-u} \exp \left( - \sum_{(\ell, k) \in \mathbb{N}_0^2 \setminus \{(1, 1)\}} \frac{u^{k\alpha_\ell/\alpha_1}h_{k,\alpha_\ell}(t)}{[h_{1,\alpha_1}(t)]^{k\alpha_1/\alpha_1}} \right) du
\]

and furthermore,

\[
I_\mathcal{H}(t) = \frac{1 + o(t^p)}{V(t)} \int_0^{t^{-\alpha(h)}} y^{\frac{\alpha_1}{\alpha_1} - 1} e^{-y} \sum_{\lambda \in \Lambda(\mathcal{H})} \kappa_\lambda(t)y^\lambda dy \sim \sum_{\lambda \in \Lambda(\mathcal{H})} \kappa_\lambda(t) \frac{\kappa_\lambda(t)}{V(t)} \Gamma \left( \frac{B}{\alpha_1} + \lambda \right). \quad (5.19)
\]

where

\[
V(t) = \alpha_1 t[h_{1,\alpha_1}(t)]^{\frac{\alpha_1}{\alpha_1}}.
\]
6. Some applications of the main results

6.1. Asymptotics of the basic hypergeometric series. We write \( a = (a_1, a_2, \ldots, a_r) \in \mathbb{R}_{>0}^r, \ b = (b_1, b_2, \ldots, b_s) \in \mathbb{R}_{>0}^s \) with \( \ell := s - r > 0 \). Assuming \( v \in \mathbb{R} \) and \( t > 0 \) and considering the following basic hypergeometric series:

\[
r \phi_s(a, b; t, v) := \sum_{k=0}^{\infty} \left( \frac{e^{-ta_1}, \ldots, e^{-ta_r}, e^{-t}}{e^{-tb_1}, \ldots, e^{-tb_s}, e^{-t}} \right)_k e^{-t(z)} e^{v k}.
\]

Applying Theorem (1), we define

\[
\Phi_m(u) = \begin{cases} 
v u - \ell (u^2/2 + \text{Li}_2(e^{-u})) & m = -1 \\
\sum_{j=1}^{s} (b_j - \frac{1}{2}) - \sum_{\mu=1}^{r} (a_\mu - \frac{1}{2}) \text{Li}_1(e^{-u}) + \frac{s - r}{2} u & m = 0.
\end{cases}
\]

Then

\[
\Phi'_{-1}(u) = v - \ell \log(e^u - 1),
\]

then \( \lim_{u \to 0^+} \Phi'_{-1}(u) = +\infty \) and hence \( r \phi_s \) satisfy the assumption of Theorem 1. Therefore we obtain that for each \( p \geq 0 \), as \( t \to 0^+ \),

\[
r \phi_s(a, b; t, v) = (1 + o(t^p)) \int_0^{\infty} \left( \frac{e^{-ta_1}, \ldots, e^{-ta_r}, e^{-t}}{e^{-tb_1}, \ldots, e^{-tb_s}, e^{-t}} \right)_x e^{-t(z)} e^{v x} dx.
\]

Furthermore, \( u = \log(1 + e^{v/\ell}) \) is only one solution of \( \Phi'_{-1}(u) = 0 \) and

\[
\Phi''_{-1}(\log(1 + e^{v/\ell})) = -\ell(1 + e^{-v/\ell}) < 0.
\]

Thus, by Theorem 2 we have

\[
r \phi_s(a, b; t, v) \sim C_\phi t^{B_\phi} \exp \left( \frac{A_\phi}{t} \right) \left( 1 + \sum_{j=1}^{\infty} C_j t^j \right)
\]

with

\[
A_\phi = \frac{\ell}{2} \left( \frac{2v}{\ell} \log(1 + e^{v/\ell}) - \log^2(1 + e^{v/\ell}) + \frac{\pi^2}{3} - 2 \text{Li}_2 \left( \frac{1}{1 + e^{v/\ell}} \right) \right),
\]

\[
B_\phi = \sum_{\nu=1}^{s} b_\nu - \sum_{\mu=1}^{r} a_\mu - \frac{\ell}{2} + 1,
\]

\[
C_\phi = \frac{(2\pi)^{\ell/2}}{\sqrt{\ell}} \frac{(1 + e^{v/\ell})^{\ell/2}}{\prod_{\nu=1}^{s} \Gamma(b_\nu)} \prod_{\mu=1}^{r} \Gamma(a_\mu) \left( 1 + e^{-v/\ell} \right)^{\sum_{\nu=1}^{s} b_\nu - \sum_{\mu=1}^{r} a_\mu - \frac{1}{2}}.
\]

and for \( j \in \mathbb{N}_1 \), the coefficients \( C_j \in \mathbb{R} \) are constant depends only on \( r \phi_s(a, b; \cdot; v) \). In particular,

\[
r \phi_s(a, b; t, 0) \sim \frac{1}{\pi^{\ell/2}} \sqrt{\frac{2\pi}{\ell}} \prod_{\nu=1}^{s} \Gamma(b_\nu) \prod_{\mu=1}^{r} \Gamma(a_\mu) (2t)^{\sum_{\nu=1}^{s} b_\nu - \sum_{\mu=1}^{r} a_\mu - \frac{e^{-t}}{2}} \exp \left( \frac{\ell \pi^2}{12t} \right).
\]

6.2. Asymptotics of some simple Eulerian Series.
6.2.1. A simple Eulerian Series. Let \( A, C, D, E > 0, \ F \geq 0, \ G \in \mathbb{N}_1 \) and \( B \in \mathbb{R} \). We consider the following Eulerian series.

\[
\mathcal{R}(q) = \sum_{m \in \mathbb{N}} \frac{q^{Am^2+Bm}}{(q^C; q^D)_{Em+F}^G} = \frac{1}{(q^C; q^D)_\infty^G} \sum_{m \in \mathbb{N}} q^{Am^2+Bm} (q^{DEm+C+DF}; q^D)_\infty^G.
\]

By (1.8), Lemma 4.3, Theorem 1 and Theorem 2, we define

\[
R_{-1}(u) = -Au^2 - \frac{G}{D} \text{Li}_2(e^{-DEu})
\]

and

\[
R_0(u) = -Bu - G \text{Li}_1(e^{-DEu}) \left( \frac{1}{2} - F - \frac{C}{D} \right)
\]

to replace \( H_{-1}(u) \) and \( H_0(u) \), respectively. Then we have

\[
R'_{-1}(u) = -2Au - GE \log(1 - e^{-DEu})
\]

and

\[
R''_{-1}(u) = -2A - \frac{DGE^2}{e^{DEu} - 1}.
\]

Thus \( \lim_{u \to 0^+} R'_{-1}(u) = +\infty \), Therefore for each \( p \geq 0 \), as \( q \to 1^- \),

\[
\mathcal{R}(q) = \frac{1 + o(|\log q|^p)}{(q^C; q^D)_\infty^G} \int_0^\infty q^{Ax^2+Bx}(q^{DEx+C+DF}; q^D)_\infty^G dx.
\]

Furthermore, \( R'_{-1}(u) = 0 \) with \( u > 0 \) equivalent to

\[
(e^{-u})^{A/(EG)} + (e^{-u})^{DE} - 1 = 0.
\]

This equation just have one solution \( \zeta_R \) on \((0, \infty)\) and \( R''_{-1}(<R) < 0 \). Thus by Theorem 2 it is easy to prove that the leading asymptotics

\[
\mathcal{R}(e^{-t}) \sim C_R t^{B_R} \exp \left( \frac{A_R}{t} \right),
\]

where

\[
A_R = -A\zeta_R^2 + \frac{G}{D} \left( \frac{\pi^2}{6} - \text{Li}_2(e^{-DE\zeta_R}) \right), \quad B_R = \frac{CG}{D} - \frac{G + 1}{2}
\]

and

\[
C_R = \left( \frac{1}{2\pi} \right)^{G/2} \left( \frac{C}{D} \right)^G \exp \left( \frac{A(F+C/D-1)}{E} - B \right) \frac{\zeta_R}{\sqrt{2A + DGE^2e^{(DE-2A/(EG))\zeta_R}}}
\]

[5]
6.2.2. *Some examples on mock theta functions.* We now apply our main result to some mock theta functions. It is easy check that there are about a half mock theta functions of the website [17] can directly use Theorem 1 and Theorem 2 to obtain the complete asymptotic expansion. Here just gives the illustration of the following two examples.

**Example 1.** The following example is a mock theta function of Ramanujan, which we re proved in Andrews [18] and Hickerson[19].

\[ F_0(q) = \sum_{m \in \mathbb{N}} \frac{q^{m^2}}{(q^{m+1}; q)_m} = \sum_{m \in \mathbb{N}} \frac{(q^{2m+1}; q)_\infty}{(q^m; q)_\infty} q^{m^2}. \]

By (1.7), Lemma 4.3, Theorem 1 and Theorem 2, we define

\[ F(u) = -u^2 - \text{Li}_2(e^{-2u}) + \text{Li}_2(e^{-u}) \]

and

\[ F_00(u) = -\frac{1}{2} \log(1 + e^{-u}) \]

to replace \( H^{-1}(u) \) and \( H_0(u) \), respectively. Then we have

\[ F'(u) = -2u - \log(1 - e^{-u}) - 2 \log(1 + e^{-u}) \]

and

\[ F''(u) = 1 - \frac{1}{1 - e^{-u}} - \frac{2}{1 + e^{-u}}. \]

Then \( \lim_{u \to 0^+} F'_{-1}(u) = +\infty \), thus as \( q \to 1^- \)

\[ F_0(q) = (1 + o(|\log q|^p)) \int_0^\infty \frac{(q^{2x+1}; q)_\infty}{(q^{x+1}; q)_\infty} q x^2 \, dx \]

holds for each \( p \geq 0 \). Moreover, \( F'_{-1}(u) = 0 \) with \( u > 0 \) equivalent to

\( (e^{-u})^3 + 2(e^{-u})^2 - e^{-u} - 1 = 0. \)

We have

\[ \zeta_F := -\log \left( \frac{2}{3} \sqrt{7} \cos \left( \frac{1}{3} \cos^{-1} \left( -\frac{1}{2} \sqrt{7} \right) \right) - \frac{2}{3} \right) = 0.2206 \ldots \]

is the solution and \( F''(\zeta_F) < 0 \). Thus we have the leading asymptotics

\[ F_0(e^{-t}) \sim e^{F_00(\zeta_F)} \sqrt{\frac{-2}{F''(\zeta_F)}} \Gamma \left( \frac{1}{2} \right) \sqrt{\frac{1}{t}} e^{F(\zeta_F)/t} \]

\[ = \left( \frac{1 - e^{-\zeta_F}}{2 - e^{-\zeta_F} + e^{-2\zeta_F}} \right)^{\frac{1}{2}} \sqrt{\frac{2\pi}{t}} \exp \left( \frac{1}{t} \left( \text{Li}_2(e^{-\zeta_F}) - \zeta_F^2 - \text{Li}_2(e^{-2\zeta_F}) \right) \right). \]
Example 2. The following mock theta function were proved in Berndt and Chan [20].

\[ \phi_m(q) = \sum_{m \in \mathbb{N}_1} \frac{q^m(-q; q)_{2m-1}}{(q; q^2)_m} \]

\[ = (-q; q)_\infty \sum_{m \in \mathbb{N}_1} \frac{(q^{2m+1}; q^2)_\infty q^m}{(-q^{2m}; q)_\infty}. \]

Clearly,

\[ \sum_{m \in \mathbb{N}_1} \frac{(q^{2m+1}; q^2)_\infty q^m}{(-q^{2m}; q)_\infty} = q \sum_{m \in \mathbb{N}} \frac{(q^{2m+3}; q^2)_\infty (q^{2m+2}; q)_\infty q^m}{(q^{4m+4}; q^2)_\infty}. \]

By (1.7) and Theorem 2, we define

\[ P(u) = (\text{Li}_2(e^{-4u}) - 3\text{Li}_2(e^{-2u}))/2 \]

to replace \( H_{-1}(u) \). Then

\[ \lim_{u \to 0^+} P'_{-1}(u) = \lim_{u \to 0^+} (-\log(1 - e^{-2u}) + 2 \log(1 + e^{2u})) = +\infty, \]

thus Theorem 1 implies that for each \( p \geq 0 \), as \( q \to 1^- \)

\[ \phi_m(q) = (1 + o(|\log q|^p)) \frac{(-q; q)_\infty}{(q; q^2)_\infty} \int_0^\infty \frac{(q^{2x+3}; q^2)_\infty (q^{2x+2}; q)_\infty}{(q^{4x+4}; q^2)_\infty} q^x dx. \]

Furthermore, it is not difficult to prove that the above

\[ \phi_m(q) = (1 + o(|\log q|^p)) \frac{(-q; q)_\infty}{(q; q^2)_\infty} \int_0^1 \frac{(qy; q^2)_\infty}{(-y; q)_\infty} \frac{dy}{2\sqrt{y}} \]

holds for each \( p \geq 0 \) as \( q \to 1^- \). Moreover, from Theorem 2 it is easy to obtain the leading asymptotics:

\[ \phi_m(e^{-t})(t) \sim \frac{\exp(\pi^2/(6t))}{2\sqrt{3\pi t}}. \]

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