Family Gromov-Witten Invariants for Kähler Surfaces

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Abstract

We use analytic methods to define Family Gromov-Witten Invariants for Kähler surfaces. We prove that these are well-defined invariants of the deformation class of the Kähler structure.

Gromov-Witten invariants are counts of holomorphic curves in a symplectic manifold $X$. To define them using the analytic approach one chooses an almost complex structure $J$ compatible with the symplectic structure and considers the set of maps $f : \Sigma \to X$ from Riemann surfaces $\Sigma$ which satisfy the (nonlinear elliptic) $J$-holomorphic map equation

$$\overline{\partial}_J f = 0.$$  \hfill (0.1)

After compactifying the moduli space of such maps, one imposes constraints, requiring, for example, that the image of the map passes through specified points. With the right number of constraints and a generic $J$, the number of such maps is finite. That number is a GW invariant; it depends only on the symplectic structure of $X$.

There are some beautiful conjectures about what the counts of holomorphic curves on Kähler surfaces ought to be ([V],[KP],[YZ],[G]). However, as currently defined, the GW invariants with those many point constraints corresponding to the dimension of holomorphic curves on Kähler surfaces with $p_g \geq 1$ are all zero! The reason is that the dimension of the GW moduli space for generic almost complex structures is strictly less than the dimension of the space of relevant holomorphic curves on Kähler surfaces. This discrepancy in dimension occurs because Kähler structures are very special, namely the linearizations obtained from (0.1) are not onto for Kähler structures when $p_g \geq 1$ — Donaldson details this in [D](see also [BL3]).

Clearly a new version of the invariants is needed — one which counts the relevant holomorphic curves. Work in that direction is just beginning. Bryan and Leung ([BL1],[BL2]) defined such invariants for K3 and Abelian surfaces by using the Twistor family; they were also able to calculate their invariants in important cases. Behrend-Fantechi [BF] have defined invariants for a more general class of algebraic surfaces using algebraic geometry. We approach the same issues using the geometric analysis approach to GW invariants.
Given a Kähler manifold \((X, \omega, J, g)\) we construct a \(2p_g\)-dimensional family of elements \(K_J(f, \alpha)\) in \(\Omega^{0,1}(f^*TX)\), where \(\alpha\) is a real part of a holomorphic 2 form. We then modify the \(J\)-holomorphic map equation (1) by considering the pairs \((f, \alpha)\) satisfying
\[
\overline{\partial}_J f = K_J(f, \alpha).
\] (0.2)

The solutions of this equation form a moduli space whose dimension is \(2p_g\) larger than the dimension of the usual GW moduli space.

Because \(\alpha\) ranges over a vector space compactness is an issue. Here things get interesting because there are instances when the moduli space for (0.2) is not compact. In fact, when the map represents a component of a canonical divisor the moduli space is never compact. Nevertheless, there is a simple analytic criterion — the uniform boundedness of the energy of the map and the \(L^2\) norm of \(\alpha\) — that ensures that the moduli space is compact.

**Theorem 0.1** Let \((X, J)\) be a Kähler surface and fix a genus \(g\) and a class \(A \in H_2(X, \mathbb{Z})\). Denote by \(C(J)\) the supremum of \(E(f) + ||\alpha||_{L^2}\) over all \((J, \alpha)\)-holomorphic maps from genus \(g\) curves into \(X\) which represent \(A\). If \(C(J)\) is finite, then the family GW invariants
\[GW^{J,H}_{g,k}(X, A)\]
are well-defined. They are invariant under deformations \(\{J_t\}\) of the Kähler structure with \(C(J_t)\) bounded. Furthermore, if \(A\) is a \((1,1)\) class then all the maps which contribute to these invariants are in fact \(J\)-holomorphic.

The last sentence of Theorem 0.1 means that the invariants for \((1,1)\) classes are counts of holomorphic curves in \((X, J)\). That is not the same as saying the invariants are enumerative, since there is no claim that each curve is counted with multiplicity one. But it does mean that the family GW invariants, which \textit{a priori} are counts of maps which are holomorphic with respect to families of almost complex structures on \(X\), are in fact calculable from the complex geometry of \((X, J)\) alone.

Theorem 0.1 yields well-defined family GW invariants provided there is a finite energy bound \(C(J)\). Following the Kodaira classification of surfaces, we verify the energy bound case-by-case using geometric arguments. That yields the following cases where the family GW invariants are well-defined.

**Proposition 0.2** The moduli space for a class \(A\) is compact, and hence the family GW invariants \(GW^{J,H}_{g,k}(X, A)\) are well-defined, when \((X, J)\) is

\(a\) a K3 or Abelian surface with \(A \neq 0\),
\(b\) a minimal elliptic surface \(\pi: E \to C\) with Kodaira dimension 1 with \(-A\cdot (\text{fiber class}) \neq \deg \pi_*(A)\), and
\(c\) a minimal surface of general type and \(A\) is of type \((1,1)\) such that \(A\) is not a linear combination of components of the canonical class \(K\).

Next consider a minimal surface \((X, J)\) of general type. As mentioned above, Proposition 0.2 does not apply to classes \(A\) which are linear combinations of components of the canonical class.
(such as $A = mK$ for $m \geq 1$) because, for those $A$, the family moduli space is not compact (see Example 3.5). However, we can still define invariants for such classes by adapting the approach used by Ionel and Parker ([IP1,IP2]) to define GW invariants. That gives the following extension of Proposition 0.3.

**Proposition 0.3** Let $(X, J)$ be a minimal surface of general type. Suppose $A$ is of type $(1,1)$ and the class $A$ and the genus $g$ satisfy

$$-K \cdot A + g - 1 \geq 0.$$  

Then there are well-defined invariants for the class $A$ and the genus $g$ which coincide with the family invariants whenever $A$ also satisfies Condition (c) of Proposition 0.2.

Section 1 gives the definition of a $(J, \alpha)$-holomorphic map and some of the analytic consequences of that definition, most notably an expression for the energy in terms of pullback of the symplectic form and the form $\alpha$. Section 2 begins by describing the relation between a complete linear system $|C|$ — or more generally a Severi variety — and the moduli space of $(J, \alpha)$-holomorphic maps. That leads us to consider the family of $(J, \alpha)$-holomorphic maps in which $\alpha$ is the real part of holomorphic 2-form; the corresponding family moduli space should be an analytic version of the Severi variety. As partial justification of that view, we prove the last statement of Theorem 0.1: any $(J, \alpha)$-holomorphic map which represents a $(1,1)$ class is in fact holomorphic (theorem 2.4).

Section 3 summarizes the analytic results which lead to the definition of the family GW-invariants. That involves constructing the virtual moduli cycle by adapting the method of Li and Tian [LT]. Thus defined, the family invariants satisfy a Composition Law analogous to those of ordinary GW-invariants.

Section 4 contains examples of Kähler surfaces with $p_g \geq 1$ with well-defined family invariants. We focus on minimal surfaces and establish the results summarized in Propositions 0.2 and 0.3 above. For the case of K3 and Abelian surfaces we prove that our family GW-invariants agree with the invariants defined by Bryan and Leung. That is done in the course of the proof of Theorem 4.3 by relating the holomorphic 2-forms to the Twistor family.

The appendix contains a brief discussion of how the family GW invariants defined here relate to those defined by Behrend and Fantachi in [BF].

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1 $(J, \alpha)$-holomorphic maps

A $J$-holomorphic map into an almost complex manifold $(X, J)$ is a map $f : \Sigma \to X$ from a complex curve $\Sigma$ (a closed Riemann surface with complex structure $j$) whose differential is
complex linear. Equivalently, $f$ is a solution of the $J$-holomorphic map equation

$$\bar{\partial} Jf = 0 \quad \text{where} \quad \bar{\partial} Jf = \frac{1}{2} (df + J df).$$

In this section we will show that when $X$ is four-dimensional there is natural infinite-dimensional family of almost complex structures parameterized the $J$-anti-invariant 2-forms on $X$.

Let $(X, J)$ be a 4-dimensional almost Kähler manifold with the hermitian triple $(\omega, J, g)$. Using $J$, we can decompose $\Omega^2(X) \otimes \mathbb{C}$ as $\Omega^2(X) \otimes \mathbb{C} = \Omega^1_+ \oplus (\Omega^0_+ \oplus \Omega^0_-)$. This leads to the following decomposition

$$\Omega(X) = \Omega^+_J(X) \oplus \Omega^-_J(X)$$

where $\Omega^+_J(X) = (\Omega^1_+)_R$ and $\Omega^-_J(X) = (\Omega^2_0)_R (=(\Omega^0_0 \oplus \Omega^0_2)_R)$. Note that $\alpha \in \Omega^-_J(X)$ iff $\alpha(Ju, Jv) = -\alpha(u, v)$.

**Definition 1.1** A 2-form $\alpha$ is called $J$-anti-invariant if $\alpha$ is in $\Omega^-_J(X)$. Each $\alpha$ in $\Omega^-_J(X)$ defines an endomorphism $K_\alpha$ of $TX$ by the equation

$$\langle u, K_\alpha v \rangle = \alpha(u, v). \quad (1.1)$$

It follows that

$$\langle K_\alpha u, v \rangle = -\langle u, K_\alpha v \rangle, \quad JK_\alpha = -K_\alpha J, \quad \text{and} \quad \langle Ju, K_\alpha u \rangle = 0. \quad (1.2)$$

**Definition 1.2** For $\alpha \in \Omega^-_J(X)$, a map $f : \Sigma \to X$ is called $(J, \alpha)$-holomorphic if

$$\bar{\partial} Jf = K_J(f, \alpha) \quad (1.3)$$

where $K_J(f, \alpha) = K_\alpha(\partial f \circ j) = \frac{1}{2} K_\alpha(df - J df \circ j)$.

The next proposition and its corollary list some pointwise relations involving the quantities that appear in the $(J, \alpha)$-holomorphic equation. We state these first for general $C^1$ maps, then specialize to $(J, \alpha)$-holomorphic maps.

**Proposition 1.3** Fix a metric within the conformal class $j$ and let $dv$ be the associated volume form. Then for any $C^1$ map $f$ we have the pointwise equalities

(a) $|\bar{\partial} Jf|^2 \ dv = \frac{1}{2} |df|^2 \ dv - f^* \omega$,

(b) $\langle \bar{\partial} Jf, K_J(f, \alpha) \rangle \ dv = f^* \alpha$,

(c) $K^2_\alpha = -|\alpha|^2 \ Id$,

(d) $|K_J(f, \alpha)|^2 \ dv = f^*(|\alpha|^2) \left(\frac{1}{2} |df|^2 \ dv + f^* \omega\right)$. 

4
Proof. Fix a point \( p \in \Sigma \) and an orthonormal basis \( \{e_1, e_2 = je_1\} \) of \( T_p\Sigma \). Setting \( v_1 = df(e_1) \) and \( v_2 = df(e_2) \), we have \( 2\overline{\partial}_J f(e_1) = v_1 + Jv_2 \) and \( 2K_J(f, \alpha)(e_1) = K_{\alpha} v_2 - JK_{\alpha} v_1 \), and similarly \( 2\overline{\partial}_J f(e_2) = v_2 - Jv_1 \) and \( 2K_J(f, \alpha)(e_2) = -K_{\alpha} v_1 - JK_{\alpha} v_2 \). Therefore,

\[
4|\overline{\partial}_J f|^2 = |v_1 + Jv_2|^2 + |v_2 - Jv_1|^2 = 2(|v_1|^2 + |v_2|^2) + 4\langle v_1, Jv_2 \rangle
= 2|df|^2 - 4f^*\omega(e_1, e_2).
\]

That gives (a), and (b) follows from the similar computation

\[
4\langle \overline{\partial}_J f, K_J(f, \alpha) \rangle = \langle v_1 + Jv_2, K_{\alpha} v_2 - JK_{\alpha} v_1 \rangle + \langle v_2 - Jv_1, -K_{\alpha} v_1 - JK_{\alpha} v_2 \rangle
= \langle v_1, K_{\alpha} v_2 \rangle - \langle v_1, JK_{\alpha} v_1 \rangle + \langle Jv_2, K_{\alpha} v_2 \rangle - \langle Jv_2, JK_{\alpha} v_1 \rangle
- \langle v_2, K_{\alpha} v_1 \rangle - \langle v_2, JK_{\alpha} v_2 \rangle + \langle Jv_1, K_{\alpha} v_1 \rangle + \langle Jv_1, JK_{\alpha} v_2 \rangle
= 4\langle v_1, K_{\alpha} v_2 \rangle
= 4f^*\alpha(e_1, e_2).
\]

Next fix \( x \in X \) and an orthonormal basis \( \{w^1, w^2, w^3, w^4\} \) of \( T_x^*X \) with \( w^2 = -Jw_1 \) and \( w^4 = -Jw^3 \). Then the six forms

\[
w^1 \wedge w^2 \pm w^3 \wedge w^4, \quad w^1 \wedge w^3 \pm w^2 \wedge w^4, \quad w^1 \wedge w^4 \pm w^2 \wedge w^3
\]

give an orthonormal basis of \( \Lambda^2(T_x^*X) \), and two of these span the subspace of \( J \) anti-invariant forms. Hence

\[
\alpha = a(w^1 \wedge w^3 - w^2 \wedge w^4) + b(w^1 \wedge w^4 + w^2 \wedge w^3)
\]

for some \( a \) and \( b \), and in this basis \( K_{\alpha} \) is the matrix

\[
\begin{pmatrix}
0 & 0 & a & b \\
0 & 0 & b & -a \\
-a & -b & 0 & 0 \\
-b & a & 0 & 0
\end{pmatrix}
\]

Consequently, \( K_{\alpha}^2 = -(a^2 + b^2)Id = -|\alpha|^2Id \). Lastly, since \( K_{\alpha} \) is skew-adjoint, (c) shows that

\[
|K_J(f, \alpha)|^2 = -\langle \partial_J f \circ j, K_{\alpha}^2(\partial_J f \circ j) \rangle = f^*(|\alpha|^2)|\partial_J f|^2.
\]

Equation (d) then follows from (a) because \( |df|^2 = |\partial_J f|^2 + |\overline{\partial}_J f|^2 \). \( \square \)

Corollary 1.4 Suppose the map \( f : \Sigma \to X \) is \( (J, \alpha) \)-holomorphic. Then

(a) \( |\overline{\partial}_J f|^2 \ dv = f^*\alpha \),

(b) \( (1 - f^*(|\alpha|^2)) \ |df|^2 \ dv = 2(1 + f^*(|\alpha|^2)) \ f^*\omega \), and

(c) \( f^*(|\alpha|^2) |df|^2 = (1 + f^*(|\alpha|^2)) \ |\overline{\partial}_J f|^2 \).
Proof. Since $f$ is $(J, \alpha)$-holomorphic, $|\overline{\partial} J f|^2 = \langle \overline{\partial} J f, K_J(f, \alpha) \rangle = |K_J(f, \alpha)|^2$, so (a) follows from Proposition 1.3b while (b) and (c) follow from Proposition 1.3 (a) and (d).

There is a second way of writing the $(J, \alpha)$-holomorphic equation (1.3). For each $\alpha \in \Omega_J(X)$, $\text{Id} + JK_\alpha$ is invertible since $JK_\alpha$ is skew-adjoint. Hence

$$J_\alpha = (\text{Id} + JK_\alpha)^{-1}J(\text{Id} + JK_\alpha)$$

(1.4)

is an almost complex structure. A map $f : \Sigma \to X$ is $(J, \alpha)$-holomorphic if and only if $f$ is $J_\alpha$-holomorphic, i.e. satisfies

$$\overline{\partial} J_\alpha f = \frac{1}{2} (df + J_\alpha df) = 0.$$ 

(1.5)

From this perspective, a solution of the $(J, \alpha)$-holomorphic equation is a $J_\alpha$ holomorphic map with $J_\alpha$ lying in the family (1.4) parameterized by $\alpha \in \Omega_J(X)$. In particular, we see from (1.5) that the $(J, \alpha)$-holomorphic equation is elliptic.

**Proposition 1.5** For any $\alpha \in \Omega_J(X)$, the almost complex structure $J_\alpha$ on $X$ satisfies

$$\langle J_\alpha u, J_\alpha v \rangle = \langle u, v \rangle \quad \text{and} \quad J_\alpha = \frac{1 - |\alpha|^2}{1 + |\alpha|^2} J - \frac{2}{1 + |\alpha|^2} K_\alpha$$

(1.6)

**Proof.** From (1.2), the endomorphisms $A_+ = \text{Id} + JK_\alpha$ and $A_- = \text{Id} - JK_\alpha$ are transposes, and $A_+ J = JA_-$ and $A_+ K_\alpha = K_\alpha A_-$. Consequently, $A_+^{-1}$ and $A_-^{-1}$ are transposes, with $A_-^{-1} J = JA_+^{-1}$ and $A_-^{-1} K_\alpha = K_\alpha A_+^{-1}$ and therefore $A_+^{-1} A_+ = A_+ A_-^{-1}$. Consequently,

$$\langle J_\alpha u, J_\alpha v \rangle = \langle A_+^{-1} J A_+ u, A_+^{-1} J A_+ v \rangle = \langle J A_+^{-1} A_+ u, JA_+^{-1} A_+ v \rangle$$

$$= \langle A_+^{-1} A_+ u, A_+^{-1} A_+ v \rangle = \langle u, A_+ A_+^{-1} A_+ v \rangle$$

$$= \langle u, v \rangle.$$

On the other hand, noting that $K_\alpha^2 = -|\alpha|^2 \text{Id}$, it is easy to verify that

$$(\text{Id} + JK_\alpha)^{-1} = \frac{1}{1 + |\alpha|^2} \text{Id} - \frac{1}{1 + |\alpha|^2} JK_\alpha.$$ 

(1.7)

With that, the second part of (1.6) follows from the definition of $J_\alpha$. \qed

In summary, $(J, \alpha)$-holomorphic maps can be regarded as solutions of the $J_\alpha$-holomorphic map equation $\overline{\partial} J_\alpha f = 0$ for a family of almost complex structures parameterized by $\alpha$ as in (1.5). We will frequently move between these two viewpoints.

## 2 Curves and Canonical Families of $(J, \alpha)$ Maps

Given a Kähler surface $X$, we would like to use $(J, \alpha)$-holomorphic curves to solve the following problem in enumerative geometry:
Enumerative Problem  Give a (1, 1) homology class $A$, count the curves in $X$ that represent $A$, have a specified genus $g$, and pass through the appropriate number of generic points.

We begin this section with some dimension counts which show that in order to interpret this problem in terms of holomorphic maps we need to consider families of maps of dimension $p_g$. We then show that there is a very natural family of $(J, \alpha)$-holomorphic maps with exactly that many parameters. We conclude the section with a theorem showing that such maps do indeed represent holomorphic curves in $X$.

One can phrase the above enumerative problem in terms of the Severi variety $V_g(C) \subset |C|$, which is defined to be the closure of the set of all curves with geometric genus $g$. Assuming that $C - K$ is ample, it follows from the Riemann-Roch theorem that the dimension of the complete linear system $|C|$ is given in terms of $p_g = \dim \mathcal{C} H^0.2(X)$ and $q = \dim \mathcal{C} H^0.1(X)$ by

$$\dim \mathcal{C}|C| = \frac{C^2 - C \cdot K}{2} + p_g - q$$

and the formal dimension of the Severi variety is

$$\dim \mathcal{C}V_g(C) = -K \cdot C + g - 1 + p_g - q. \quad (2.1)$$

The right-hand side of (2.1) is the ‘appropriate number’ of point constraints to impose; the set of curves in $V_g(C)$ through that many generic points should be finite, making the enumerative problem well-defined.

Now let $\mathcal{M}_g(X, A)$ be the moduli space of holomorphic maps from Riemann surfaces of genus $g$, which represent homology class $A$. Then its virtual dimension is given by

$$\dim \mathcal{C}\mathcal{M}_g(X, A) = -K \cdot A + g - 1. \quad (2.2)$$

The image of a map in $\overline{\mathcal{M}}_g(X, |C|)$ might be not a divisor in $|C|$, instead it is a divisor in some other complete linear system $|C'|$ with $|C'| = |C|$. As in [BL3], we define the parameterized Severi variety

$$V_g(|C|) = \coprod_{|C'|=|C|} V_g(C')$$

Its expected dimension is now given by

$$\dim \mathcal{C}V_g(|C|) = -K \cdot C + g - 1 + p_g. \quad (2.3)$$

We still have $p_g$ dimensional difference between (2.3) and (2.2). Therefore, the cut-down moduli space by (2.3) many point constraints is empty when $p_g \geq 1$. This implies that the GW invariants with (2.3) many point constraints are zero, whenever $p_g \geq 1$.

We show that there is a natural — in fact obvious — $p_g$-dimensional family of $(J, \alpha)$-holomorphic maps associated with every Kähler surface.
Definition 2.1 Given a Kähler surface $X$, define the parameter space $\mathcal{H}$ by

$$\mathcal{H} = \{ \alpha + i\bar{\alpha} \mid \alpha \in H^{2,0}(X) \} \quad (2.4)$$

Here $H^{2,0}(X)$ means the set of holomorphic $(2,0)$ forms on $X$. Note that all forms $\alpha \in H^{2,0}(X)$ are closed since $d\alpha = \partial\alpha + \bar{\partial}\alpha = \partial\alpha$ is a $(3,0)$ form and hence vanishes because $X$ is a complex surface. Thus $\mathcal{H} \subset \Omega^2(X)$ is a $2p_g$-dimensional real vector space of closed forms. We give it the (real) inner product defined by the $L^2$ inner product of forms:

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta. \quad (2.5)$$

We can use the forms $\alpha \in \mathcal{H}$ to parameterize the right-hand side of the $(J, \alpha)$-holomorphic map equation (1.2).

Definition 2.2 Henceforth the term ‘$(J, \alpha)$-holomorphic map’ means a map satisfying (1.2) for $\alpha$ in the above family $\mathcal{H}$.

Lemma 2.3 The zero divisor $Z(\alpha)$ of each nonzero $\alpha \in \mathcal{H}$ represents the canonical class.

Proof. Write $\alpha = \beta + \bar{\beta}$ with $\beta \in H^{2,0}(X)$. Since $\beta$ is a section of the canonical bundle, this means that $Z(\alpha) = Z(\beta)$ represents the canonical divisor with appropriate multiplicities. \hfill \Box

Next, using this $2p_g$ dimensional parameter space $\mathcal{H}$, we define the family moduli space

$$\overline{\mathcal{M}}_g^H(X, [C]) = \{ (f, \alpha) \mid \overline{\partial}_J f = 0, [\text{Im } f] = [C], \text{ and } \alpha \in \mathcal{H} \}$$

Since we just parameterize the $\overline{\partial}$-operator by $2p_g$ dimensional parameter space, the formal dimension of the family moduli space is given by

$$(\text{Formal}) \dim \overline{\mathcal{M}}_g^H(X, [C]) = -K \cdot C + g - 1 + p_g$$

On the other hand, we define a component of the canonical class to be a homology class of a component of some canonical divisor.

Theorem 2.4 If $f$ is a $(J, \alpha)$-holomorphic map which represents a class $A \in H^{1,1}(X)$. Then $f$ is, in fact, holomorphic. Furthermore, if $A$ is not a linear combination of components of the canonical class, then $\alpha = 0$.

Proof. Since $\alpha \in H^{2,0}(X) \oplus H^{2,0}(X)$ is closed and $A \in H^{1,1}(X)$, it follows from Corollary 1.4a that

$$\int_X |\overline{\partial}_J f|^2 = \alpha(A) = 0.$$ 

Thus $f$ is holomorphic, that is, $\overline{\partial}_J f \equiv 0$. Consequently, $|\alpha|^2|\text{df}|^2 \equiv 0$ by Corollary 1.4c. Since $\text{df}$ has at most finitely many zeros, we can conclude that $\alpha = 0$ along the image of $f$. Hence $\alpha = 0$, otherwise it contradicts to the assumption on $A$ by Lemma 2.3. \hfill \Box
3 Family GW-Invariants

Let $X$ be a complex surface with a Kähler structure $(\omega, J, g)$. In this section we will define the Family Gromov-Witten Invariants associated to $(X, J)$ and the parameter space $\mathcal{H}$ of (2.4). We also state some properties of these invariants.

Our approach is the same analytic arguments as that of Li and Tian [LT] to show that the moduli space of $(J, \alpha)$-holomorphic maps carries a virtual fundamental class whenever it is compact. While compactness is automatic for the usual Gromov-Witten invariants, it must be verified case-by-case for the family GW invariants (see Example 3.5). Thus compactness appears as a hypothesis in the results of this section.

First, we recall the notion of $C^l$ stable maps as defined in [LT]. Fix an integer $l \geq 0$ and consider pairs $(f; \Sigma; x_1, \ldots, x_k)$ consisting of

1. a connected nodal curve $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$ of arithmetic genus $g$ with distinct smooth marked points $x_1, \ldots, x_k$, and
2. a continuous map $f : \Sigma \to X$ so that each restriction $f_i = f|_{\Sigma_i}$ lifts to a $C^l$-map from the normalization $\tilde{\Sigma}_i$ of $\Sigma$ into $X$.

**Definition 3.1** A stable $C^l$ map of genus $g$ with $k$ marked points is a pair $(f; \Sigma; x_1, \ldots, x_k)$ as above which satisfies the stability condition:

- If the homology class $[f_i] \in H_2(X, \mathbb{Q})$ is trivial, then the number of marked points in $\Sigma_i$ plus the arithmetic genus of $\Sigma_i$ is at least three.

Two stable maps $(f; \Sigma; x_1, \ldots, x_k)$ and $(f'; \Sigma'; x'_1, \ldots, x'_k)$ are equivalent if there is a biholomorphic map $\sigma : \Sigma \mapsto \Sigma'$ such that $\sigma(x_i) = x'_i$ for $1 \leq i \leq k$ and $f' = f \circ \sigma$. We denote by

$$F_{g,k}^l(X, A)$$

the space of all equivalence classes $[f; \Sigma; x_1, \ldots, x_k]$ of $C^l$-stable maps of genus $g$ with $k$ marked points and with total homology class $A$. The topology of $F_{g,k}^l(X, A)$ is defined by sequential convergence as in section 2 of [LT]. There are two continuous maps from $F^l$. First, there is an evaluation map

$$ev : F_{g,k}^l(X, A) \to X^k$$ (3.1)

which records the images of the marked points. Second, for $2g + k \geq 3$, collapsing the unstable components of the domain gives a stabilization map

$$st : F_{g,k}^l(X, A) \to \mathcal{M}_{g,k}$$ (3.2)

to the compactified Deligne-Mumford space of genus $g$ curves with $k$ marked points. For $2g+k < 3$ we define $\mathcal{M}_{g,k}$ to be the topological space of consisting of a single point and define (3.2) to be the map to that point.
We next construct a ‘generalized bundle’ \(E\) over \(\mathcal{F}_{g,k}(X,A) \times \mathcal{H}\), again following [LT]. Recall that each \(\alpha \in \mathcal{H}\) defines an almost complex structure \(J_\alpha\) on \(X\) by (1.4). Denote by \(\text{Reg}(\Sigma)\) the set of all smooth points of \(\Sigma\). For each \(([f; \Sigma, x_1, \ldots, x_k], \alpha)\), define
\[
\Lambda_j^\Sigma (f^*TX)
\]
to be the set of all continuous sections \(\nu\) of \(\text{Hom}(T\text{Reg}(\Sigma), f^*TX)\) with \(\nu \circ j_\Sigma = -J_\alpha \circ \nu\) such that \(\nu\) extends continuously across the nodes of \(\Sigma\). We take \(E\) to be the bundle whose fiber over \(([f; \Sigma, x_1, \ldots, x_k], \alpha)\) is \(\Lambda_j^\Sigma (f^*TX)\) and give \(E\) the continuous topology as in section 2 of [LT].

We then define a section \(\Phi : \mathcal{F}_{g,k}(X,A) \times \mathcal{H} \to E\) by
\[
\Phi([f; \Sigma, x_1, \ldots, x_k], \alpha) = df + J_\alpha df j_\Sigma.
\]
(3.3)

The right-hand side of (3.3) vanishes for \(J_\alpha\)-holomorphic maps. Thus \(\Phi^{-1}(0)\) is the moduli space of \((J, \alpha)\)-holomorphic maps. The following is a family version of Proposition 2.2 in [LT].

**Proposition 3.2** Suppose that the set \(\Phi^{-1}(0)\) is compact. Then the section \(\Phi\) gives rise to a generalized Fredholm orbifold bundle with a natural orientation and with index
\[
r = 2c_1(X)[A] + 2(g - 1) + 2k + \dim \mathcal{H}.
\]
(3.4)

By Theorem 1.2 of [LT], the bundle \(E\) has a rational homology “Euler class” in \(\mathcal{F}_{g,k}(X,A) \times \mathcal{H}\); in fact, since \(\mathcal{H}\) is contractible this Euler class lies in \(H_r(\mathcal{F}_{g,k}(X,A); \mathbb{Q})\) where \(r\) is the index (3.4). We call this class the **virtual fundamental cycle** of the moduli space of family holomorphic maps parameterized by \(\mathcal{H}\) and denote it by \([\overline{\mathcal{M}}_{g,k}^{I,\mathcal{H}}(X,A)]_{\text{vir}}\).

(3.5)

In particular,
\[
\dim [\overline{\mathcal{M}}_{g,k}^{I,\mathcal{H}}(X,A)]_{\text{vir}} = 2c_1(X)[A] + 2(g - 1) + 2k + 2p_g.
\]
(3.6)

The next issue is whether the virtual fundamental cycle is independent of the Kähler structure on \(X\). The next proposition is analogous to the Proposition 2.3 in [LT]. It shows that the virtual fundamental cycle depends only on certain deformation class of the Kähler structure.

**Proposition 3.3** Let \((\omega_t, J_t, g_t), 0 \leq t \leq 1,\) be a continuous family of Kähler structures on \(X\). Let \(\mathcal{H}_t\) be the corresponding continuous family of finite subspaces defined by (2.4) and let \(\Phi_t\) be the corresponding family of sections of \(E_t\) over \(\mathcal{F}_{g,k}(X,A) \times \mathcal{H}_t\). If \(\Phi_t^{-1}(0)\) is compact for all \(0 \leq t \leq 1,\) then
\[
[\overline{\mathcal{M}}_{g,k}^{J_0,\mathcal{H}_0}(X,A)]_{\text{vir}} = [\overline{\mathcal{M}}_{g,k}^{J_1,\mathcal{H}_1}(X,A)]_{\text{vir}}.
\]

The family GW invariants can now be defined by pulling back cohomology classes by the evaluation and stabilization maps and integrating over the virtual fundamental cycle. That of course requires that the virtual fundamental cycle exists, so we must assume that we are in a situation where \(\Phi_t^{-1}(0)\) is compact.
Definition 3.4 Whenever the virtual fundamental cycle $[\mathcal{M}^{J,\mathcal{G}}_{g,k}(X,A)]^{\text{vir}}$ exists, we define the family $GW$ invariants of $(X,J)$ to be the map

$$GW_{g,k}^{J,\mathcal{G}}(X,A) : \{H^*(X;\mathbb{Q})\}^k \times H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q}) \rightarrow \mathbb{Q}$$

defined on $\alpha \in \{H^*(X;\mathbb{Q})\}^k$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,k};\mathbb{Q})$ by

$$GW_{g,k}^{J,\mathcal{G}}(X,A)(\beta; \alpha) = [\mathcal{M}^{J,\mathcal{G}}_{g,k}(X,A)]^{\text{vir}} \cap (\text{st}^* \beta \cup \text{ev}^* \alpha).$$

The condition that $\Phi^{-1}(0)$ is compact must be checked “by hand”. In general, $\Phi^{-1}(0)$ is compact for some choices of $A$, but not for others.

Example 3.5 Let $(X,J)$ be a Kähler surface with $p_g > 1$. Then there is a non-zero element $\beta \in H^{2,0}$ whose zero set $Z(\beta)$ is non-empty, represents the canonical class $K$, and whose irreducible components can be parameterized by holomorphic maps. Fix a parameterization $f : \Sigma \rightarrow X$ of one such component; this represents a non-zero class $A \in H_2(X,\mathbb{Z})$. Then $\alpha = \beta + \beta$ lies in the space $\mathcal{H}$ of (2.1) and $\Phi(f, \lambda \alpha) = 0$ for all real $\lambda$. Thus on any Kähler surface with $p_g > 1$, the set $\Phi^{-1}(0)$ is not compact for an component of the canonical class $A$.

On the other hand, in the next section we will give examples of classes $A$ in Kähler surfaces with $p_g > 1$ for which $\Phi^{-1}(0)$ is compact.

Recall that the energy of a map $f : (\Sigma,j) \rightarrow X$ is defined by

$$E(f) = \frac{1}{2} \int_{\Sigma} |df|^2 d\mu$$

where $\mu$ is the metric in the conformal class $j$.

Theorem 3.6 If there is a constant $C$, depending only on $(X,\omega,J,g)$ such that $E(f) + ||\alpha|| < C$ for all $(J,\alpha)$-holomorphic maps into $(X,J)$, then $\Phi^{-1}(0)$ is compact and hence the family $GW$ invariants are well-defined.

Proof. Consider a sequence $(f_n, \alpha_n)$ of $J_\alpha$-holomorphic maps. The uniform bound on $||\alpha_n||$ implies that the $J_\alpha$ lie in a compact family. Since $E(f_n) < C$ the proof of Gromov’s Compactness Theorem (see [PW] and [IS]) shows that $(f_n, \alpha_n)$ has a convergent subsequence. Consequently, $\Phi^{-1}(0)$ is compact as in the hypothesis of Proposition 3.2. That means that the virtual fundamental cycle (3.5) is well-defined. The family $GW$ invariants are then given by Definition 3.4.

We conclude this section by an important property of the family $GW$ invariants. This property generalizes the composition law of ordinary Gromov-Witten invariants. For that we consider maps from a domain $\Sigma$ with node $p$ and relate them to maps whose domain is the normalization of $\Sigma$ at $p$. When the node is separating the genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and is a natural map

$$\sigma : \mathcal{N}_{g_1,k_1+1} \times \mathcal{N}_{g_2,k_2+1} \rightarrow \mathcal{N}_{g,k}$$

(3.7)
defined by gluing \((k_1 + 1)\)-th marked point of the first component to the first marked point of the second component. We denote by \(PD(\sigma)\) the Poincaré dual of the image of this map \(\sigma\).

We denote by \(E_1 \oplus \tilde{E}_2\) (resp. \(E_1 \oplus E_2\)) the generalized bundle over
\[
\mathcal{F}_{g_1,k_1+1}(X, A_1) \times \mathcal{F}_{g_2,k_2+1}(X, A_2) \times \mathcal{H} \times [0, 1]
\]
whose fiber over \([(f_1, \Sigma_1; \{x_i\}), [f_2, \Sigma_2; \{y_j\}], \alpha, t]\) is \(\Lambda_{g_1,1}^1, J_{t_0}^1 \oplus \Lambda_{g_2,1}^1, J_{t_0}^1\) (resp. \(\Lambda_{g_1,1}^0, J_{t_0}^1 \oplus \Lambda_{g_2,1}^0, J_{t_0}^1\)).

The formula
\[
\Psi_1([f_1, \Sigma_1; \{x_i\}, [f_2, \Sigma_2; \{y_j\}], \alpha]) = (df_1 + J_\alpha df_1 j_{\Sigma_1}, df_2 + J_\alpha df_2 j_{\Sigma_2})
\]
(resp. \(\Psi_2([f_1, \Sigma_1; \{x_i\}, [f_2, \Sigma_2; \{y_j\}], \alpha]) = (df_1 + J_\alpha df_1 j_{\Sigma_1}, df_2 + J_\alpha df_2 j_{\Sigma_2})\))
then defines a section of \(E_1 \oplus \tilde{E}_2\) (resp. \(E_1 \oplus E_2\)).

On the other hand, for non-separating nodes there is another natural map
\[
\theta : \overline{\mathcal{M}}_{g-1,k+2} \mapsto \overline{\mathcal{M}}_{g,k}
\]
defined by gluing the last two marked points. We also write \(PD(\theta)\) for the Poincaré dual of the image of \(\theta\). The composition law is then the following two formulas.

**Proposition 3.7 (Composition Law)** Let \(\{H_\gamma\}\) be any basis of \(H^*(X; Z)\) and \(\{H^\gamma\}\) be its dual basis and suppose that \(GW_{g,k}^{J,\mathcal{H}}(X, A)\) is defined.

(a) For any decomposition \(A = A_1 + A_2\) if either \(\Psi_1^{-1}(0)\) or \(\Psi_2^{-1}(0)\) is compact, then
\[
GW_{g,k}^{J,\mathcal{H}}(X, A)(PD(\sigma); \alpha) = \sum_{\alpha = \alpha_1 \oplus \alpha_2} I_{A_1, A_2}(\alpha) \text{ where }
\]
\[
I_{A_1, A_2}(\alpha) = \sum_\gamma GW_{g_1,k_1+1}(X, A_1)(\alpha_1, H_\gamma)GW_{g_2,k_2+1}(X, A_2)(H^\gamma, \alpha_2)\text{ if }\Psi_1^{-1}(0)\text{ is compact,}
\]
\[
I_{A_1, A_2}(\alpha) = \sum_\gamma GW_{g_1,k_1+1}(X, A_1)(\alpha_1, H_\gamma)GW_{g_2,k_2+1}(X, A_2)(H^\gamma, \alpha_2)\text{ if }\Psi_2^{-1}(0)\text{ is compact,}
\]

\(GW\) denotes the ordinary GW invariant, and \(\alpha = \alpha_1 \oplus \alpha_2\) in \([H^*(X)]^{k_1} \otimes [H^*(X)]^{k_2}\).

(b) \(GW_{g,k}^{J,\mathcal{H}}(X, A)(PD(\theta); \alpha) = \sum_\gamma GW_{g-1,k+2}^{J,\mathcal{H}}(X, A)(\alpha, H_\gamma, H^\gamma)\).

**Sketch of proof**

(a) We denote by \(E_1 \oplus E_2\) the generalized bundle over
\[
\mathcal{F}_{g_1,k_1+1}(X, A_1) \times \mathcal{F}_{g_2,k_2+1}(X, A_2) \times \mathcal{H}
\]
whose fiber over \([(f_1, \Sigma_1; \{x_i\}), [f_2, \Sigma_2; \{y_j\}], \alpha]\) is \(\Lambda_{g_1,1}^1, J_{t_0}^1 \oplus \Lambda_{g_2,1}^1, J_{t_0}^1\). The formula
\[
\Psi_1([f_1, \Sigma_1; \{x_i\}, [f_2, \Sigma_2; \{y_j\}], \alpha]) = (df_1 + J_\alpha df_1 j_{\Sigma_1}, df_2 + J_\alpha df_2 j_{\Sigma_2})
\]
also defines a section of \(E_1 \oplus E_2\). Similarly to Proposition 3.2, this bundle is a generalized Fredholm orbifold bundle. We denote its virtual fundamental cycle by \(J_{A_1, A_2}\).
On the other hand, there is a natural map
\[ p : \mathcal{F}_{g_1,k_1+1}(X, A_1) \times \mathcal{F}_{g_2,k_2+1}(X, A_1) \times \mathcal{H} \to X \times X \]
defined by \( ([f_1, \Sigma_1; \{x_i\}], [f_2, \Sigma_2; \{y_j\}], \alpha) \to (f_1(x_{k_1+1}), f_2(y_1)) \). Thus we have a homology class
\[ \sum \sum J_{A_1, A_2} \cap (ev_{k_1+1}^*H_\gamma \cup ev_1^*H_\gamma) \in H_*(\cup p^{-1}(\Delta); \mathbb{Q}) \]
where the sum and the union are over all decompositions of \( A \) and \( \Delta \) is the diagonal in \( X \times X \).

We set
\[ I_{A_1, A_2} = \sum \sum J_{A_1, A_2} \cap (ev_{k_1+1}^*H_\gamma \cup ev_1^*H_\gamma) \quad (3.9) \]

There is also a surjective map
\[ \pi : \cup p^{-1}(\Delta) \to st^{-1}(\text{Im } \sigma) \]
obtained by identifying \( x_{k_1+1} \) and \( y_1 \), where \( st \) is the stabilization map of \( \mathcal{F}_{g,k}(X, A) \times \mathcal{H} \). Note that we can consider
\[ [\overline{\mathcal{M}}_{g,k}(X, A)]^\text{vir} \cap \text{PD}(\sigma) \]
as a homology class in \( H_*(st^{-1}(\text{Im } \sigma); \mathbb{Q}) \). Similarly to the case of GW invariants, we have
\[ [\overline{\mathcal{M}}_{g,k}(X, A)]^\text{vir} \cap \text{PD}(\sigma) = \pi_* (\sum I_{A_1, A_2}) \quad (3.10) \]

Now, suppose \( \Psi^{-1}_1(0) \) is compact. For each \( 0 \leq t \leq 1 \), consider the restriction of \( E_1 \times \tilde{E}_2 \) and its section \( \Psi_1 \) to
\[ \mathcal{F}_{g_1,k_1+1}(X, A_1) \times \mathcal{F}_{g_2,k_2+1}(X, A_2) \times \mathcal{H} \times \{t\} \]
This is also a generalized Fredholm orbifold bundle with a virtual fundamental cycle, denoted by
\[ [\overline{\mathcal{M}}_t]^\text{vir} = [\overline{\mathcal{M}}_{(g_1,k_1+1),(g_2,k_2+1)}(X, A_1, A_2, t)]^\text{vir} \]
Note that by definition \([\overline{\mathcal{M}}_1]^\text{vir} = J_{A_1, A_2} \) and
\[ [\overline{\mathcal{M}}_0]^\text{vir} = [\overline{\mathcal{M}}_{g_1,k_1+1}(X, A_1)]^\text{vir} \otimes [\overline{\mathcal{M}}_{g_2,k_2+1}(X, A_2)]^\text{vir} \]
where \([\overline{\mathcal{M}}_{g_2,k_2+1}(X, A_2)]^\text{vir} \) is a class which defines GW invariants. By the same argument as in Proposition 3.3, we finally have
\[ J_{A_1, A_2} = [\overline{\mathcal{M}}_1]^\text{vir} = [\overline{\mathcal{M}}_0]^\text{vir} = [\overline{\mathcal{M}}_{g_1,k_1+1}(X, A_1)]^\text{vir} \otimes [\overline{\mathcal{M}}_{g_2,k_2+1}(X, A_2)]^\text{vir} \]
as homology classes in \( H_*(\mathcal{F}_{g_1,k_1+1}(X, A_1) \times \mathcal{F}_{g_2,k_2+1}(X, A_1); \mathbb{Q}) \). Together with (3.9) and (3.10) this implies the first composition law.
(b) Similarly as above, we have an evaluation map of last two marked points
\[ p : \mathcal{F}_{g-1,k+2}(X,A) \times \mathcal{H} \to X \times X \]
\[ ([f, \Sigma; \{ x_i \}], \alpha) \to (f(x_{k+1}), f(x_{k+2})). \]

There is also a surjective map \( \pi : p^{-1}(\Delta) \to st^{-1}(\text{Im } \theta) \). It follows that
\[ \left[ \mathcal{M}^{g,1,\mathcal{H}}_{g,k}(X,A) \right]^{\text{vir}} \cap PD(\theta) = \pi_* \left( \sum \left[ \mathcal{M}^{g,1,\mathcal{H}}_{g-1,k+2}(X,A) \right]^{\text{vir}} \cap (ev^*_k H \wedge ev^*_k H^\gamma) \right) \]
which implies the second Composition Law. \( \square \)

That completes our overview of the family GW invariants. We next look at some examples, namely the various types of minimal Kähler surfaces. There we can use the specific geometry of the space to verify that the moduli space is compact and hence the family GW invariants are well-defined.

4 Kähler surfaces with \( p_g \geq 1 \)

In this section we will focus on the family GW-invariants for minimal Kähler surfaces \( X \) with \( p_g \geq 1 \). The Enriques-Kodaira Classification [BPV] separates such surfaces into the following three types.

1. \( X \) is K3 or Abelian surface with canonical class \( K = 0 \). In this case, \( p_g = 1 \).

2. \( X \) is an elliptic surface \( \pi : X \to C \) with Kodaira dimension 1. If the multiple fibers \( B_i \) have multiplicity \( m_i \), then a canonical divisor is
\[ K = \pi^* D + \sum (m_i - 1) B_i \quad \text{where} \quad \deg D = 2g(C) - 2 + \chi(O_X) \quad (4.1) \]

3. \( X \) is a surface of general type with \( K^2 > 0 \).

We will examine these cases one at a time. For each we will show that the family invariants \( GW^{g,1,\mathcal{H}}_{g,k}(X,A) \) are well-defined. By Theorem 3.6 the key issue is bounding the energy \( E(f) \) and the pointwise norm \( |\alpha| \) uniformly for all \((J, \alpha)\)-holomorphic maps into \( X \).

K3 and Abelian Surfaces

Let \( (X,J) \) be a K3 or Abelian surface. Since the canonical class is trivial, Yau’s proof of the Calabi conjecture implies that \( (X,J) \) has a Kähler structure \( (\omega, J, g) \) whose metric \( g \) is Ricci flat. For such a structure all holomorphic \((0,2)\) forms are parallel, and hence have pointwise constant norm (see [B]). Thus \( \mathcal{H} \cong \mathbb{C} \) consists of closed forms \( \alpha \) with \( |\alpha| \) constant. Furthermore, the structure is also hyperkähler, meaning that there is a three-dimensional space of Kähler structures which is isomorphic as an algebra to the imaginary quaternions. The unit two-sphere in that space is the so-called Twistor Family of complex structures.

Consider the set \( T_0 = \{ J_\alpha \mid \alpha \in \mathcal{H} \} \). Since \( \alpha \) has no zeros, equation (1.6) shows that \( J_\alpha \to -J \) uniformly as \( |\alpha| \to \infty \). We can therefore compactify \( T_0 \) to \( T \cong \mathbb{P}^1 \) by adding \(-J\) at infinity.
Proposition 4.1 \( T \) is the Twistor Family induced from the hyperkähler metric \( g \).

Proof. Let \( \alpha \in \mathcal{H} \) with \(|\alpha| = 1\). It then follows from Proposition 1.5 that \( J_\alpha = -K_\alpha \) and \( (\alpha, J_\alpha, g) \) is a Kähler structure on \( X \). On the other hand, we define \( \alpha' \) by \( \alpha'(u, v) = \alpha(u, Jv) \). Then \( |\alpha'| = 1 \) and \( \alpha' \in \mathcal{H} \) since \( \beta' \) is holomorphic for each holomorphic 2-form \( \beta \). Moreover, by definition we have

\[
J_{\alpha'} = -K_{\alpha'} = -JK_\alpha = JJ_\alpha.
\]

Since \( (\alpha', J_{\alpha'}, g) \) is also Kähler and \( JJ_\alpha J_{\alpha'} = -Id \), the Kähler structures \( \{ J, J_\alpha, J_{\alpha'} \} \) multiply as unit imaginary quaternions. It follows that \( T \) is the Twistor Family induced from the hyperkahler metric \( g \). \( \Box \)

Lemma 4.2 Let \( A \) be a nontrivial homology class with \( \omega(A) \geq 0 \). Then there exists a constant \( C_A \) such that every \((J, \alpha)\)-holomorphic map \( f : C \to X \) representing \( A \) with \( \alpha \in \mathcal{H} \) satisfies

\[
E(f) = \frac{1}{2} \int_{\Sigma} |df|^2 < \omega(A) + C_A \quad \text{and} \quad |\alpha| \leq 1.
\]

Proof. Since \(|\alpha| \) is a constant, we can integrate Corollary 1.4b to conclude that \(|\alpha| \leq 1\). Let \( C_A \) be an upper bound for the function \( \alpha \mapsto |\alpha(A)| \) on the set of \( \alpha \in \mathcal{H} \) with \(|\alpha| \leq 1\). Because \( \alpha \) is closed, Proposition 1.3a and Corollary 1.4a imply that

\[
E(f) = \frac{1}{2} \int_{C} |df|^2 = \frac{1}{2} \int_{\Sigma} f^*(\omega + \alpha) = \omega(A) + \alpha(A) \leq \omega(A) + C_A. \quad \Box
\]

Theorem 4.3 Let \((X, J)\) be a K3 or Abelian surface. For each non-trivial \( A \in H_2(X, \mathbb{Z}) \), the invariants \( GW_{g,k}^{J,\mathcal{H}}(X, A) \) are well-defined and independent of \( J \). Furthermore, if \( A = mB \) and \( A' = mB' \) where \( B \) and \( B' \) are primitive with the same square, then

\[
GW_{g,k}^{J,\mathcal{H}}(X, A) = GW_{g,k}^{J,\mathcal{H}}(X, A').
\]

Proof. For any nontrivial homology class \( A \), we can choose a Ricci flat Kähler structure \((\omega, J, g)\) such that \( \omega(A) \geq 0 \) (if \( \omega(A) < 0 \), then we choose \((-\omega, -J, g))\). It then follows from Lemma 4.2 and Theorem 3.6 that \( GW_{g,k}^{J,\mathcal{H}}(X, A) \) is well-defined.

Bryan and Leung have applied the machinery of Li and Tian to define family GW invariants associated to the Twistor Family \( T \) [BL1, BL2]. Their invariants, which we denote by

\[
\Phi_{g,k}^T(X, A),
\]

are actually independent of the Twistor Family since the moduli space of complex structures on \( X \) is connected. On the other hand, if \( A = mB \) and \( A' = mB' \) where \( B \) and \( B' \) are primitive with the same square, then there is an orientation preserving diffeomorphism of \( X \) which sends the class \( B \) to the class \( B' \). That implies that \( \Phi_{g,k}^T(X, A) = \Phi_{g,k}^T(X, A') \).

To complete the proof it suffices to show that

\[
GW_{g,k}^{J,\mathcal{H}}(X, A) = \Phi_{g,k}^T(X, A).
\]
For that, recall from Theorem 1.2 of [LT] that the moduli cycle is defined from a section $s$ of a generalized Fredholm orbifold bundle $E \to B$ and is represented by a cycle that lies in an arbitrarily small neighborhood of $s^{-1}(0)$. Both sides of (4.2) are defined in that way using the same Fredholm bundle $E$ over the space of Kähler structures. In the first case $B = \{ J_\alpha \mid \alpha \in \mathcal{H} \}$ and $s^{-1}(0)$ is the set of all $(f, \alpha)$ where $f$ is a $J_\alpha$-holomorphic map, and in the second case $B = T$ is the Twistor Family and $s^{-1}(0)$ is the set of $J_\alpha$-holomorphic maps for $J_\alpha \in T$. By Proposition 4.1 $\{ J_\alpha \mid \alpha \in \mathcal{H} \}$ parameterizes the Twistor Family after adding a point at infinity to $\mathcal{H}$. But since $\omega(A) \geq 0$, Lemma 4.2 shows that $|\alpha| \leq 1$ for all $J_\alpha$ holomorphic maps representing the homology class $A$ with $\alpha \in \mathcal{H}$. Thus the moduli cycle is bounded away from the point at infinity, so the two definitions of the moduli cycle are exactly equal. That gives (4.2) \quad \Box

Elliptic Surfaces

First, we recall the well-known facts about minimal elliptic surfaces $X$ with Kodaira dimension 1 [FM].

1. $X$ is elliptic in a unique way.

2. Every deformation equivalence is through elliptic surfaces.

Therefore, there is a unique elliptic structure $\pi : (X, J) \to C$. Moreover, for the fiber class $F$ and any homology class $A \in H_2(X; \mathbb{Z})$, the integer

$$F \cdot A + \deg(\pi_* A)$$

is well-defined for each complex structure $J$ and it is invariant under the deformation of complex structure $J$.

Let $(\omega, J, g)$ be a Kähler structure on $X$ and $\mathcal{H}$ be as in (2.4). For $\alpha \in \mathcal{H}$, let $||\alpha||$ denote the $L^2$ norm as in (2.5).

**Lemma 4.4** Let $A \in H_2(X; \mathbb{Z})$ such that the integer (4.3) is positive. Then, there exit uniform constants $E_0$ and $N$ such that for any $J_\alpha$-holomorphic map $f : \Sigma \to X$, representing homology class $A$, with $\alpha \in \mathcal{H}$, we have

$$E(f) = \frac{1}{2} \int_\Sigma |df|^2 \leq E_A, \quad ||\alpha|| \leq N.$$

**Proof.** It follows from (4.1) and Lemma 2.3 that for any nonzero $\alpha \in \mathcal{H}$, the zero set of $\alpha$ lies in the union of fibers $F_i$. Let $N(\alpha)$ be a (non-empty) union of $\varepsilon$-tubular neighborhoods of the $F_i$. Denote by $S$ the unit sphere in $\mathcal{H}$ and set

$$m(J) = \min_{\alpha \in S} \min_{x \in X \setminus N(\alpha)} |\alpha| \quad \text{and} \quad N = \frac{2}{m(J)}.$$

We can always choose a smooth fiber $F \subset X \setminus N(\alpha)$ such that $f$ is transversal to $F$. Let $f^{-1}(F) = \{ p_1, \ldots, p_n \}$ and for each $i$ fix a small holomorphic disk $D_i$ normal to $F$ at $f(p_i)$. We can further assume that $f$ is transversal to each $D_i$ at $f(p_i)$.
Define \( \text{sgn}(r) \) to be the sign of a real number \( r \) if \( r \neq 0 \), and 0 if \( r = 0 \). Denote by \( I(S, f)_p \) the local intersection number of the map \( f \) and a submanifold \( S \hookrightarrow X \) at \( f(p) \). In terms of bases \( \{ e_1, e_2 = j e_1 \} \) of \( T_p \Sigma \), \( \{ v_1, v_2 = j v_1 \} \) of \( T_{f(p_i)} F \), and \( \{ v_3, v_4 = j v_3 \} \) of \( T_{f(p_i)} D_i \) we have

\[
I(F, f)_{p_i} = \text{sgn} \left( (v^1 \wedge v^2 \wedge v^3 \wedge v^4)(v_1, v_2, f_* e_1, f_* e_2) \right) = \text{sgn} \left( (v^3 \wedge v^4)(f_* e_1, f_* e_2) \right),
\]

\[
I(D_i, f)_{p_i} = \text{sgn} \left( (v^1 \wedge v^2 \wedge v^3 \wedge v^4)(f_* e_1, f_* e_2, v_3, v_4) \right) = \text{sgn} \left( (v^1 \wedge v^2)(f_* e_1, f_* e_2) \right).
\]

Comparing with \( \text{sgn} f^* \omega(e_1, e_2) = \text{sgn} \left( (v^1 \wedge v^2)(f_* e_1, f_* e_2) + (v^3 \wedge v^4)(f_* e_1, f_* e_2) \right) \) shows that

\[
I(F, f)_{p_i} + I(D_i, f)_{p_i} = \text{sgn} (f^* \omega)(e_1, e_2).
\]

Now suppose \( m(J)||\alpha|| \geq 2 \). Then \( ||\alpha|| \geq 2 \) along each \( F_i \), so by (4.4) and Corollary 1.4b

\[
\sum_i \left( I(f, F)_{p_i} + I(f, D)_{p_i} \right) < 0.
\]

This contradicts to our assumption \( A \cdot f + \deg(\pi_* A) > 0 \) since by definition \( \sum_i I(f, F)_{p_i} = A \cdot f \) and \( \sum_i I(f, D_i)_{p_i} = \deg(\pi_* A) \). Therefore \( ||\alpha|| < N \) with \( N \) as above. The energy bound follows exactly same arguments as in the proof of Lemma 4.2. \( \square \)

**Proposition 4.5** For any homology class \( A \) with (4.3) positive, the invariants \( GW_{g,k}^{J, H}(X, A) \) are well-defined and depend only on the deformation class of \( (X, J) \).

**Proof.** It follows from Lemma 4.4 and Theorem 3.6 that the invariants \( GW_{g,k}^{J, H}(X, A) \) are well-defined. On the other hand, (4.3) is invariant under the deformation of \( J \). Therefore, applying Proposition 3.3, we can conclude that the invariants only depend on the deformation equivalence class of \( J \). \( \square \)

**Surfaces of General Type**

Let \( (X, J) \) be a minimal surface of general type.

**Proposition 4.6** If \( A \) is of type \((1,1)\) and is not a linear combination of components of the canonical class, then we can define the invariant \( GW_{g,k}^{J, H}(X, A) \). They are invariant under the deformations of complex structures which preserve \((1,1)\)-type of \( A \).

**Proof.** Lemma 2.4 and Theorem 3.6 imply that the invariants \( GW_{g,k}^{J, H}(X, A) \) are well-defined under the assumption that \( A \) is type \((1,1)\). On the other hand, Proposition 3.3 also implies that the invariants \( GW_{g,k}^{J, H}(X, A) \) are invariant under deformations of the complex structure which preserve the \((1,1)\) type of \( A \). \( \square \)

**A Linear Combination of Components of the Canonical Class**

There is an alternative way of defining invariants: As in [IP1, IP2], we define an invariant from a rational homology class in \( \overline{M}_{g,k} \times X^k \) that is induced from the family moduli space. This
invariant is same as the family invariant whenever both invariants are well-defined. In particular, this invariant is also well-defined for a class $A$ which is a linear combination of components of canonical class and satisfies (4.5) below.

Let $(X, J)$ be a minimal K"ahler surface and $A$ be a class of type $(1, 1)$ with

$$-A \cdot K + g - 1 \geq 0 \quad (4.5)$$

where $K$ is the canonical class. As in section 3, denote by $E$ the generalized bundle over $\mathcal{T}_{g,k}(X, A) \times \mathcal{H}$ with a section $\Phi$ defined by $\Phi(f, \alpha) = df + J_\alpha df$.

**Proposition 4.7** The section $\Phi$ as above gives rise to a well-defined homology class $[\overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(X, A)]$ in $H_{2r}(\overline{\mathcal{M}}_{g,k} \times X^k; \mathbb{Q})$, where $r = -A \cdot K + g - 1 + p_g + k$.

**Proof.** It follows from Lemma 2.3 and Theorem 2.4 that

$$\overline{\mathcal{M}}_{g,k}(X, A) \subset \Phi^{-1}(0) \subset \overline{\mathcal{M}}_{g,k}(X, A) \cup (F \times \mathcal{H}) \quad (4.6)$$

where $F$ is the set of all $(f, 0)$ in $\Phi^{-1}(0)$ such that the image of $f$ lies entirely in some canonical divisor. Since $\dim_K [K] = \dim_{\mathbb{C}} H^{2,0}(X) - 1 = p_g - 1$, under the stabilization and evaluation maps $st \times ev$ the image of $F \times \mathcal{H}$ lies in (real) dimension less than $2(p_g - 1 + k)$. Hence, the assumption (4.5) implies that the image of $F \times \mathcal{H}$ lies in dimension at most $2r - 2$.

Let $d$ be the dimension of $\overline{\mathcal{M}}_{g,k} \times X^k$. Then there exists a neighborhood $V$ of the image of $F \times \mathcal{H}$ such that (i) the basis for $H_{d-2r}(\overline{\mathcal{M}}_{g,k} \times X^k; \mathbb{Q})$ are represented by cycles $D_i$ for $1 \leq i \leq m$ and (ii) each cycle $D_i$ does not intersect with $V$. Let $U$ be the preimage of $V$ under the map $st \times ev$. It then follows from the proof of Proposition 2.2 in [LT] that $\overline{\mathcal{M}}_{g,k}(X, A)$ can be covered by finitely many smooth approximations $U_i$ for $1 \leq i \leq n$ such that $F \subset \bigcup_{1 \leq i \leq n} U_i \subset U$ for some $l \leq n$. Then using the arguments in the proof of Theorem 1.2 in [LT] we can construct a cocycle $Z$ with its boundary lying in $U$.

As in the proof of Proposition 4.2 of [KM], we now define a homology class $[\overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(X, A)]$ as a homology class in $H_{2r}(\overline{\mathcal{M}}_{g,k} \times X^k; \mathbb{Q})$ determined by the intersection numbers between $(st \times ev)(Z)$ and the cycles $D_i$ for $1 \leq i \leq m$. Since the image of $F$ lies in dimension at most $2r - 2$, this homology class is well-defined. $\square$

**Definition 4.8** As in Definition 3.4, we define the invariant by

$$GW_{g,k}^{J,\mathcal{H}}(X, A)(\beta; \alpha) = [\overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(X, A)] \cap (\beta \cup \alpha).$$

where $\alpha$ in $[H^*(X; \mathbb{Q})]^k$ and $\beta$ in $H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{Q})$.

These invariants are unchanged under deformations of complex structures which preserve the $(1,1)$-type of $A$. In particular, if $A = mK$ for $m \geq 1$, then $GW_{g,k}^{J,\mathcal{H}}(X, A)$ is invariant under all deformations of the complex structure since $mK$ is always of type $(1,1)$.

On the other hand, if $A$ is not a linear combination of components of canonical class, then the set $F$ which appears in (4.6) is empty, and hence

$$\overline{\mathcal{M}}_{g,k}^{J,\mathcal{H}}(X, A) = \Phi^{-1}(0) = \overline{\mathcal{M}}_{g,k}(X, A).$$
Therefore the invariants defined in Definitions 3.4 and 4.8 coincide. That completes the proof of Proposition 0.3 stated in the introduction.

These invariants differ from the more familiar GW invariants.

**Example 4.9** The generic element of the linear system $|K|$ of the canonical class is an embedded curve of genus $g = K^2 + 1$. For that genus, C. Taubes proved that the (standard) GW invariant $GW_g(X,K)$ is the same as the Seiberg-Witten invariant and is equal to 1 for all surfaces of general type.

On the other hand, the family invariant for that genus vanishes. That can be seen from a dimension count: the linear system $|K|$ has (real) dimension $2(p_g - 1)$, while the family GW invariant lies in dimension $2(-K \cdot K + (g - 1) + p_g) = 2p_g$ (cf. Proposition 4.7).

## 5 Appendix – Relations with the Behrend-Fantechi Approach

Behrend and Fantechi [BF] have defined modified GW invariants for Kähler surfaces using algebraic geometry. While their techniques are completely different from ours, the definitions seem to be, at their core, equivalent. In this appendix we make several observations which relate their approach to ours. This is necessarily tentative because the paper [BF] is not yet available; we are relying on the terse description given in [BL3].

In algebraic geometry, the virtual fundamental class $[\overline{M}_{g,k}(X,A)]^{vir}$ is obtained from the relative tangent-obstruction spaces together with the tangent-obstruction spaces of Deligne-Mumford space $\overline{M}_{g,k}$. Behrend and Fantechi modified their machinery, intrinsic normal cone and obstruction complex, by replacing the relative obstruction space $H^1(f^*TX)$ by the kernel of the map

$$H^1(f^*TX) \rightarrow H^2(X,\mathcal{O})$$

defined by dualizing of the composition

$$H^0(X,\Omega^2) \rightarrow H^0(f^*\Omega^2) \rightarrow H^0(f^*\Omega^1 \otimes f^*\Omega^1) \rightarrow H^0(f^*\Omega^1 \otimes \Omega^1).$$

(A.1)

In order for their machinery to work, the map (A.1) is of constant rank — in particular surjective — for every $f$ in $\overline{M}_{g,k}(X,A)$ [BL3]. Composing (A.2) with the Kodaira-Serre dual map, we have

$$H^0(X,\Omega) \rightarrow H^0(f^*\Omega^1 \otimes \Omega^1) \rightarrow H^1(f^*TX).$$

(A.3)

This map is given by $\beta \rightarrow K_\beta df j$.

**Proposition A.1** Let $(X,J)$ be a Kähler surface and $A \in H^{1,1}(X,\mathbb{Z})$. Then the family moduli space $\overline{M}^H_{g,k}(X,A)$ is compact if and only if the map (A.1) is surjective for every $f$ in $\overline{M}_{g,k}(X,A)$.
Proof. By Theorem 2.4, $M_{g,k}(X, A)$ consists of pairs $(f, \alpha)$ with $f \in \overline{M}_{g,k}(X, A)$ and with the image of $f$ contained in the zero set of $\alpha$; the latter condition means that $K_\alpha = 0$ along the image, so $K_\alpha df = 0$ for all $(f, \alpha)$. As usual, $M_{g,k}(X, A)$ is compact by the Gromov Compactness Theorem.

Now, suppose (A.1) is surjective. Then by duality (A.3) is injective. This implies $\alpha = 0$ and hence $M_{g,k}(X, A) = \overline{M}_{g,k}(X, A)$ is compact. Conversely, suppose for some $f \in M_{g,k}(X, A)$ there is a $\beta$ in the kernel of (A.3). Then setting $\alpha = \beta + \bar{\beta}$ we have $\partial_\beta f = tK_\alpha df = 0$ — and hence $(f, t\alpha) \in \overline{M}_{g,k}(X, A)$ — for all real $t$. That means that $M_{g,k}(X, A)$ is compact only when (A.3) is injective or equivalently when (A.1) is surjective. □

The map (A.3) is directly related to the linearization of the $(J, \alpha)$-holomorphic map equation.

Suppose that $A$ is $(1, 1)$ and that the family moduli space $\overline{M}_{g,k}(X, A)$ is compact as in Proposition A.1. Consider the linearization of the $(J, \alpha)$-holomorphic map equation at $(f, j, \alpha)$. Since $J$ is Kähler, the linearization reduces to

$$L_f \oplus Jdf \oplus L_0 : \Omega^0(f^*TX) \oplus T_j\overline{M}_{g,n} \oplus \mathcal{H} \to \Omega^{0,1}(f^*TX)$$

where

$$\begin{align*}
L_f(\xi) &= \nabla \xi + J\nabla \xi j \\
L_0(\beta) &= -2K_\beta df j
\end{align*}$$

In fact, this $L_f$ is exactly (twice) the Dolbeault derivative $\overline{\partial}$. Therefore, $\text{Ker}(L_f)$ are $\text{Coker}(L_f)$ are identified with the Dolbeault cohomology groups $H^0(f^*TX)$ and $H^{0,1}(f^*TX)$, respectively.

**Proposition A.2** Under either of the two equivalent conditions of Proposition A.1 there are natural identifications $H^1(f^*TX) \simeq H^{0,1}(f^*TX)$ and $H^0(X, \Omega^2) \simeq \mathcal{H}$ under which identification the map is identified with (A.3) with

$$L_0 : \mathcal{H} \to \text{Coker}(L_f \oplus Jdf).$$

By Proposition A.1 this map is injective if and only if the family moduli space $\overline{M}_{g,k}(X, A)$ is compact.

Proof. It follows by comparing the formulas for $L_0$ and (A.3) that $L_0$ maps $\mathcal{H}$ into $\text{Coker}(L_f)$. On the other hand, given $h \in T_j\overline{M}_{g,n}$, there is a family $j_t$ with $j_0 = j$ and $\frac{dj_t}{dt} \big|_{t=0} = h$. It follows from Proposition 1.3b and $\langle \beta, A \rangle = 0$ that

$$0 = \frac{d}{dt} \big|_{t=0} \int_{\Sigma, j_t} f^*(\beta) = \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma, j_t} (df + Jdf j_t, K_\beta f_s j_t) = \int_{\Sigma} (Jdf(h), K_\beta f_s j).$$

This implies that $L_0$ maps $\mathcal{H}$ into $\text{Coker}(L_f \oplus Jdf)$. □

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