CONNECTED FAIR DETACHMENTS OF HYPERGRAPHS

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Abstract. Let $G$ be a hypergraph whose edges are colored. An $(\alpha, n)$-detachment of $G$ is a hypergraph obtained by splitting a vertex $\alpha$ into $n$ vertices, say $\alpha_1, \ldots, \alpha_n$, and sharing the incident hinges and edges among the subvertices. A detachment is fair if the degree of vertices and multiplicity of edges are shared as evenly as possible among the subvertices within the whole hypergraph as well as within each color class. In this paper we solve an open problem from 70s by finding necessary and sufficient conditions under which a $k$-edge-colored hypergraph $G$ has a fair detachment in which each color class is connected. Previously, this was not even known for the case when $G$ is an arbitrary graph (i.e. 2-uniform hypergraph). We exhibit the usefulness of our theorem by proving a variety of new results on hypergraph decompositions, and completing partial regular combinatorial structures.

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1. INTRODUCTION AND STATEMENT OF MAIN RESULT

All hypergraphs under consideration are finite. We allow edges to occur multiple times, and we also allow each vertex to have multiple occurrences within an edge. In other words, the edge set and every element of the edge set is a multiset. By an edge of the form $\{u_1^{m_1}, \ldots, u_s^{m_s}\}$, we mean an edge in which vertex $u_i$ occurs $m_i$ times for $1 \leq i \leq s$.

Let $n$ be a positive integer. Given a hypergraph $H := (V, E)$ and a vertex $\alpha$ of $H$, we obtain a hypergraph $F$ on $|V| + n - 1$ vertices and $|E|$ edges in the following way.

(i) Replace $\alpha$ in $G$ by $n$ new vertices $\alpha_1, \ldots, \alpha_n$ in $F$;
(ii) Replace each edge \( \{\alpha^p\} \cup U \) in \( G \) by an edge \( \{\alpha^1_p, \ldots, \alpha^n_p\} \cup U \) in \( F \), where \( p \geq 1, \alpha \notin U \subset V \) and \( p_1 + \cdots + p_n = p \).

(iii) Leave the remaining vertices and edges of \( G \) intact and add them to \( F \).

The resulting hypergraph \( F \) is called an \((\alpha, n)\)-detachment of \( G \), and \( G \) is an amalgamation of \( F \) (Amalgamation here is different from that of Nešetřil [34]). It is clear that \( F \) is in general far from being uniquely determined.

There are two active and somewhat independent lines of research on detachments of (hyper)graphs. In the first line of research which was initiated by Nash-Williams [29, 30, 31, 32, 33], an arbitrary graph is given, the aim is to find conditions under which the detachment maintains certain structural properties such as edge-connectivity properties or the detached vertices have prescribed degrees. This area has connections with submodular functions, combinatorial optimization and matroid theory [15, 18, 26, 28]. For an overview of this line of work together with several other related results on variations of “splitting off” operations, we refer the reader to [19].

In the second line of research which was initiated by Hilton [21], a specific graph (typically an amalgamation of a complete multigraph, possibly multipartite) is given whose edges are colored. The aim is to find conditions under which the detachment satisfies various global fairness properties (such as degree and multiplicity conditions) as well as fairness properties within each color. For a recent survey, we refer the reader to [10]. This approach is quite powerful in solving problems in two particular areas: graph decompositions, and completing partial combinatorial structures such as latin squares, edge-colored graphs, etc. [1, 2, 17, 21, 22, 23, 25, 27, 35].

The results of the first line of work are typically interesting from a pure mathematical point of view, while the second line of research is more appealing to those with an applied viewpoint. The goal of this paper is to study detachments by combining these two approaches. Unlike most results in the literature where the focus is graphs, here we work with arbitrary hypergraphs whose edges are colored. Our goal is to find a detachment ensuring connectivity of each color class is maintained and at the same time several fairness properties are satisfied in the resulting hypergraph as a whole as well as within each color class.

In order to state our main result, we need a few pieces of notation. More precise definitions are provided in Section 2.

Here \( x \approx y \) means \( [y] \leq x \leq [y] \). For a hypergraph \( G := (V, E) \) whose edges are colored with \( k \) colors (the set of colors is always \( [k] := \{1, \ldots, k\} \)), \( c(G) \) is the number of components of \( G \); for \( j \in [k] \), \( G(j) \) is the color class \( j \) of \( G \); for \( \alpha \in V \), \( \deg_G(\alpha) \) is the total number of occurrences of \( \alpha \) among all edges of \( E \); \( \omega(G) \) is the number of “pieces” of \( G \) after removal of the vertex \( \alpha \in V \); and for \( t \geq 0, U, X \subset V \), \( \text{mult}_G(\alpha^t, X) \) and \( \text{mult}_G(U, X) \) are the number of repetitions of \( \{\alpha^t\} \cup X \) and \( U \cup X \) in \( E \), respectively. Finally, \( G \) is called \( \alpha \)-simple if \( \text{mult}_G(\alpha^2, X) = 0 \) for every \( X \subset V \).

Here is our main result.

**Theorem 1.1.** Let \( k \) and \( n \) be positive integers. Let \( H := (V, E) \) be a hypergraph whose edges are colored with \( k \) colors, and let \( \alpha \in V \) such that

\[
\text{mult}_H(\alpha^t, Y) = 0 \quad \text{for each } t > n \text{ and each } Y \subseteq V.
\]

Then there exists an \((\alpha, n)\)-detachment \( F \) of \( H \) such that \( F \) is \( \alpha \)-simple and the following conditions hold.
(F1) For each $i \in [n]$,
\[ \deg_F(\alpha_i) \approx \frac{\deg_H(\alpha)}{n}; \]

(F2) For each $i \in [n]$ and each $j \in [k]$,
\[ \deg_{F(j)}(\alpha_i) \approx \frac{\deg_{H(j)}(\alpha)}{n}; \]

(F3) For each $U \subseteq \{\alpha_1, \ldots, \alpha_n\}$ and each $X \subseteq V\{\alpha_1, \ldots, \alpha_n\}$,
\[ \text{mult}_F(U, X) \approx \frac{\text{mult}_H(\alpha[U], X)}{\binom{n}{|U|}}; \]

(F4) For each $U \subseteq \{\alpha_1, \ldots, \alpha_n\}$, each $X \subseteq V\{\alpha_1, \ldots, \alpha_n\}$, and each $j \in [k]$,
\[ \text{mult}_{F(j)}(U, X) \approx \frac{\text{mult}_{H(j)}(\alpha[U], X)}{\binom{n}{|U|}}; \]

(C1) For each $j \in [k]$,
\[ c(F(j)) = c(H(j)) \text{ if and only if } \deg_{H(j)}(\alpha) - \omega_{\alpha}(H(j)) \geq n - 1. \]

We refer to conditions (F1)–(F4), and (C1) as fairness, and connectivity condition(s), respectively.

A special case of Theorem 1.1 was known for the case when $H$ is an amalgamation of the complete $n$-vertex graph [24] (Hilton, Johnson, and Rodger [24, p. 268] noted that the case of connectivity is “a tough nut to crack”). For other results on connected detachments of graphs, we refer the reader to [9, 11, 27, 32, 35]. For detachments of 3-uniform hypergraphs and arbitrary hypergraphs, see [3] and [4], respectively (none of these two papers address connectivity). We also note that previously, Theorem 1.1 was not even known for the case when $H$ is an arbitrary graph. Moreover, necessary and sufficient conditions under which a hypergraph has a connected $(\alpha, n)$-detachment were not known (condition (C1) by itself settles this).

The paper is organized as follows. More precise definitions are given in Section 2. In Section 3, we discuss all the necessary tools needed to deal with connectivity in Theorem 1.1. In Section 4, we construct two laminar families; these are crucial to deal with fairness conditions in Theorem 1.1. Before we delve into proving our main result, in Section 5, we prove a generalization of the famous Baranyai’s theorem [13]. We prove our main result in Section 6. The rest of the paper is devoted to various applications of our main result. In Sections 7 and 8, we are concerned with the problem of extending edge-colorings of hypergraphs. Finally, in Section 9, we decompose complete (not necessarily uniform) hypergraphs into (almost) regular connected spanning sub-hypergraphs. The results of this section extend the Berge-Johnson theorem [16].

2. More Precise Notation

In this paper, $\mathbb{R}$, $\mathbb{N}$, and $[k]$ denote the set of reals, positive integers, and $\{1, \ldots, k\}$, respectively. For $x, y \in \mathbb{R}$, $[x]$ and $\lfloor x \rfloor$ denote the integers such that $x - 1 < [x] \leq x \leq [x] < x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leq x \leq \lfloor y \rfloor$. The relation $\approx$ is transitive, and for $x, y \in \mathbb{R}, a, b, c \in \mathbb{Z}, n \in \mathbb{N}$: (i) $x \approx y$ implies $x/n \approx y/n$; (ii) $a = b - c$ and $c \approx x$, implies
A hypergraph $G$ is a pair $(V(G), E(G))$ where $V(G)$ is a finite set called the vertex set, $E(G)$ is the edge multisets, and every edge is itself a multi-subset of $V(G)$. This means that not only can an edge occur multiple times in $E(G)$, but also each vertex can have multiple occurrences within an edge. A hypergraph is said to be non-trivial if it has at least one edge. By an edge of the form $\{u_1^{m_1}, u_2^{m_2}, \ldots, u_s^{m_s}\}$ (or for short a $u_1^{m_1} \ldots u_s^{m_s}$-edge), we mean an edge in which vertex $u_i$ occurs $m_i$ times for $1 \leq i \leq r$. The total number of occurrences of an vertex $v$ among all edges of $E(G)$ is called the degree, $\deg_G(v)$ of $v$ in $G$, and $\Delta(G)$ denotes the maximum degree of $G$. For an edge $e$ in $G$, $|e|$ is the number of distinct vertices in $e$, and the multiplicity of $e$, written $\text{mult}_G(e)$, is the number of repetitions of $e$ in $E(G)$ (note that $E(G)$ is a multisets, so an edge may appear multiple times). If $\{u_1^{m_1}, \ldots, u_s^{m_s}\}$ is an edge in $G$, then we abbreviate $\text{mult}_G(\{u_1^{m_1}, \ldots, u_s^{m_s}\})$ to $\text{mult}_G(u_1^{m_1} \ldots u_s^{m_s})$. If $U_1, \ldots, U_s$ are multi-subsets of $V(G)$, then $\text{mult}_G(U_1, \ldots, U_s)$ means $\text{mult}_G(\bigcup_{i \in [s]} U_i)$, where the union of $U_i$s is the usual union of multiset. Whenever it is not ambiguous, we drop the subscripts; for example we write $\deg(v)$ and $\text{mult}(e)$ instead of $\deg_G(v)$ and $\text{mult}_G(e)$, respectively.

If the multiplicity of a vertex $\alpha$ in an edge $e$ is $p$, we say that $\alpha$ is incident with $p$ distinct objects, say $h_1(\alpha, e), \ldots, h_p(\alpha, e)$. We call these objects hinges, and we say that $e$ is incident with $h_1(\alpha, e), \ldots, h_p(\alpha, e)$.

A $k$-edge-coloring of $G$ is a mapping $f : E(G) \rightarrow [k]$ and color class $i$ of $G$, written $G(i)$, is the spanning sub-hypergraph of $G$ induced by the edges of color $i$.

Let $G$ be a hypergraph, let $U$ be some finite set, and let $\Psi : V(G) \rightarrow U$ be a surjective mapping. The map $\Psi$ extends naturally to $E(G)$. For $A \in E(G)$ we define $\Psi(A) = \{\Psi(x) : x \in A\}$. Note that $\Psi$ need not be injective, and $A$ may be a multisets. Then we define the hypergraph $F$ by taking $V(F) = U$ and $E(F) = \{\Psi(A) : A \in E(G)\}$. We say that $F$ is an amalgamation of $G$, and that $G$ is a detachment of $F$. Associated with $\Psi$ is a (number) function $g$ defined by $g(u) = |\Psi^{-1}(u)|$; to be more specific we will say that $G$ is a $g$-detachment of $F$. Then $G$ has $\sum_{u \in V(F)} g(u)$ vertices. Note that $\Psi$ induces a bijection between the edges of $F$ and the edges of $G$, and that this bijection preserves the size of an edge. We adopt the convention that it preserves the color also, so that if we amalgamate or detach an edge-colored hypergraph the amalgamation or detachment preserves the same coloring on the edges.

Intuitively speaking, an $(\alpha, n)$-detachment of $G$ is a hypergraph obtained by splitting a vertex $\alpha$ into $n$ vertices, say $\alpha_1, \ldots, \alpha_n$, and sharing the incident hinges and edges among the subvertices. In an $\alpha$-detachment $G'$ of $G$ in which we split $\alpha$ into $\alpha$ and $\beta$, an edge of the form $\{\alpha^p, u_1, \ldots, u_z\}$ in $G$ will be of the form $\{\alpha^{p-i}, \beta^i, u_1, \ldots, u_z\}$ in $G'$ for some $i$, $0 \leq i \leq p$. Note that a hypergraph and its detachments have the same hinges.

If we replace every edge $e$ of a hypergraph $G$ by $\lambda$ copies of $e$, then we denote the new hypergraph by $\lambda G$. For hypergraphs $G_1, \ldots, G_t$ with the same vertex set $V$, we define their union, written $\bigcup_{i \in [t]} G_i$, to be the hypergraph with vertex set $V$ and edge set $\bigcup_{i \in [t]} E(G_i)$. For a hypergraph $G$ and $V \subseteq V(G)$, let $G - V$ be the hypergraph whose vertex set is $V(G) - V$ and whose edge set is $\{e \in V(G) : e \notin V\}$, and let $G \setminus V$ be the hypergraph whose vertex set is $V(G)$ and whose edge set is $\{e \in E(G) : e \notin V\}$.

A hypergraph $G$ is said to be $k$-regular if every vertex has degree $k$. A $k$-factor of a hypergraph $G$ is a $k$-regular spanning sub-hypergraph of $G$. 
3. Wings

A vertex $\alpha$ in a connected hypergraph $G$ is a cut vertex if there exist two non-trivial sub-hypergraphs $I, J$ of $G$ such that

1. $I \cup J = G$,
2. $V(I \cap J) = \{\alpha\}$, and
3. $E(I \cap J) = \emptyset$.

A sub-hypergraph $W$ of a hypergraph $G$ is an $\alpha$-wing of $G$ if

1. $W$ is non-trivial and connected,
2. $\alpha$ is not a cut vertex of $W$, and
3. no edge in $E(G) \setminus E(W)$ is incident with a vertex in $V(W) \setminus \{\alpha\}$.

The set of all $\alpha$-wings of $G$ is denoted by $\mathcal{W}_\alpha(G)$, and $\omega_\alpha(G) := |\mathcal{W}_\alpha(G)|$. If $G$ is non-trivial and connected and $\alpha$ is not a cut vertex of $G$, then $\mathcal{W}_\alpha(G) = \{G\}$. Moreover, if $G$ is disconnected, then $\mathcal{W}_\alpha(G) = \mathcal{W}_\alpha(H)$ where $H$ is the component of $G$ that contains $\alpha$.

An $\alpha$-wing $W$ is (i) slim if $\deg_W(\alpha) = 1$, (ii) wide if $\deg_W(\alpha) \geq 2$, (iii) small if $V(W) = \{\alpha\}$, and (iv) large if $V(W) \neq \{\alpha\}$. Let $\omega_{\alpha}^{s}(G)$ be the number of large $\alpha$-wings in $G$.

Let $G'$ be an $(\alpha, 2)$-detachment of $G$, and let $W$ be an $\alpha$-wing in $G$. Let $W'$ denote the sub-hypergraph of $G'$ whose hinges are the same as those in $W$. With respect to this detachment, $W$ is (i) attached if $\deg_{W'}(\alpha) = \deg_{W}(\alpha)$, (ii) semi-detached if $1 \leq \deg_{W'}(\alpha) < \deg_{W}(\alpha)$, and (iii) detached if $\deg_{W'}(\alpha) = 0$. Note that semi-detached wings are always wide.

**Lemma 3.1.** Let $G, H$ be two connected hypergraphs such that $\alpha \in V(G)$, $\alpha \notin V(H)$, and $V(G) \cap V(H) \neq \emptyset$. If $G$ is an $\alpha$-wing, then so is $G \cup H$.

**Proof.** Since $G$ is an $\alpha$-wing, $G \cup H$ is non-trivial. Since both $G$ and $H$ are connected, and $V(G) \cap V(H) \neq \emptyset$, $G \cup H$ is also connected. Moreover, $\alpha$ is not-cut vertex of $G$, so it is also a non-cut vertex of $G \cup H$. Therefore, $G \cup H$ is an $\alpha$-wing. [\qed]

**Lemma 3.2.** Let $G'$ be an $(\alpha, 2)$-detachment of a hypergraph $G$. Let $X_1, \ldots, X_r$ be the list of all detached $\alpha$-wings, and let $W_1, \ldots, W_s$ be the nonempty list of all semi-detached $\alpha$-wings. Then

$$Z := (\bigcup_{i \in [r]} X'_i) \bigcup (\bigcup_{i \in [s]} W'_i)$$

is an $\alpha$-wing in $G'$.

**Proof.** Let $X = \{X_1, \ldots, X_r\}$. There are two cases to consider.

1. $X = \emptyset$: Since $s \geq 1$, $Z$ is non-trivial. For all $i \in [s]$, $W_i$ is connected and $\alpha \in V(W_i)$, and so $Z$ is connected. Let $\{\beta\} = V(G') \setminus V(G)$. Since $W_i$ is semi-detached, $\{\alpha, \beta\} \subseteq V(W'_i)$ for all $i \in [s]$. First we show that no edge in $E(G') \setminus E(Z)$ is incident with a vertex in $V(Z) \setminus \{\alpha\}$. Suppose by contrary that there is some edge $e \in E(G') \setminus E(Z)$ with $e \cap (V(Z) \setminus \{\alpha\}) \neq \emptyset$. Since $\beta \notin e$, we have

$$e \in E(G) \setminus E(W_i) \text{ and } e \cap (V(W_i) \setminus \{\alpha\}) \neq \emptyset \text{ for some } i \in [s],$$

contradicting the fact that $W_i$ is an $\alpha$-wing in $G$.

To complete the proof, we need to show that $\alpha$ is not a cut vertex of $Z$. Suppose by contrary that $\alpha$ is a cut vertex of $W$. There must exist two non-trivial sub-hypergraphs $I', J'$ of $G'$ such that

$$I' \cup J' = G', V(I' \cap J') = \{\alpha\}, E(I' \cap J') = \emptyset.$$
Without loss of generality, let us assume that $\beta \in V(I')$ (so $\beta \notin V(J')$). For each $i \in [s]$, define $I'_i = W'_i \cap I'$, $J'_i = W'_i \cap J'$. We have
\[
I'_i \cup J'_i = (W'_i \cap I') \cup (W'_i \cap J') = W'_i \cap (I' \cup J') = W'_i \cap G' = W'_i,
\]
\[
V(I'_i \cap J'_i) = V(I' \cap J' \cap W'_i) = \{\alpha\},
\]
\[
E(I'_i \cap J'_i) \subseteq E(I' \cap J') = \emptyset.
\]
If for some $i \in [s]$, $I'_i$ is trivial, then $J'_i = W'_i$. But $\beta \notin V(J')$ and $\beta \in V(W'_i)$, so this is clearly a contradiction. Thus all $I'_i$'s are non-trivial. Moreover, since $J'$ is non-trivial and $J' = \cup_{i=1}^s J'_i$, there is some $k \in [s]$ so that $J'_k$ is non-trivial. Now, let $I_k$ and $J_k$ be the sub-hypergraphs of $G$ obtained by amalgamating $\alpha$ and $\beta$ in $I_k$ and $J_k$, respectively. Both $I_k$ and $J_k$ are non-trivial sub-hypergraphs of $W_k$, and
\[
I_k \cup J_k = W_k, V(I_k \cap J_k) = \{\alpha\}, E(I_k \cap J_k) = \emptyset.
\]
So, $\alpha$ is a cut vertex of $W_k$, which contradicts with $W_k$ being an $\alpha$-wing.

(2) $X \neq \emptyset$: Combining the first case with Lemma 3.2 settles this case.

**Lemma 3.3.** Let $G'$ be an $(\alpha, 2)$-detachment of $G$ with $V(G') = V(G) \cup \{\beta\}$, and let $A$ and $B$ be the set of those $\alpha$-wings of $G$ that are detached and semi-detached in $G'$, respectively. Then

(i) $\omega_\alpha(G') = \omega_\alpha(G) - |A| - |B| + 1$;
(ii) $\deg_{G'}(\alpha) = \deg_G(\alpha) - \sum_{W \in A} \deg_W(\alpha) - \sum_{W \in B} \deg_W(\beta)$;
(iii) $\deg_{G'}(\alpha) - \omega_\alpha(G') \leq \deg_G(\alpha) - \omega_\alpha(G) - 1$.

**Proof.** The proof of (i) is an immediate consequence of the previous lemma, and (ii) is sufficiently obvious. For every $W \in A$, $d_W(\alpha) \geq 1$ and for every $W \in B$, $d_W(\beta) \geq 1$. By combining (i) and (ii), we have
\[
\deg_{G'}(\alpha) - \omega_\alpha(G') = \deg_G(\alpha) - \sum_{W \in A} \deg_W(\alpha) - \sum_{W \in B} \deg_W(\beta) - (\omega_\alpha(G) - |A| - |B| + 1)
\]
\[
= \deg_G(\alpha) - \omega_\alpha(G) - \sum_{W \in A} (\deg_W(\alpha) - 1) - \sum_{W \in B} (\deg_W(\beta) - 1) - 1
\]
\[
\leq \deg_G(\alpha) - \omega_\alpha(G) - 1.
\]
This proves (iii). □

**Lemma 3.4.** Let $G'$ be an $(\alpha, 2)$-detachment of a hypergraph $G$ with $V(G') = V(G) \cup \{\beta\}$. Then $c(G') = c(G)$ if and only if some wide $\alpha$-wing in $G$ is semi-detached in $G'$.

**Proof.** This is immediate from definitions. □

**Lemma 3.5.** Let $F$ be an $(\alpha, n)$-detachment of a hypergraph $G$ such that $c(F) = c(G)$. Then
\[
\deg_G(\alpha) - \omega_\alpha(G) \geq n - 1.
\]

**Proof.** The proof is by induction on $n$. The case $n = 1$ is trivial, because for each $W \in \mathcal{W}_\alpha(G)$, $d_W(\alpha) \geq 1$. The case $n = 2$ is immediate from Lemma 3.4.

To prove the inductive step, let $F$ be an $(\alpha, n)$-detachment of $G$ obtained by splitting $\alpha$ into $\alpha_1, \ldots, \alpha_n$ such that $c(F) = c(G)$. Let $H$ be the hypergraph obtained by amalgamating
Then both $\alpha_2, \ldots, \alpha_n$ in $\mathcal{F}$ into $\alpha_2$ in $\mathcal{H}$. Since $\mathcal{H}$ is an $(\alpha, 2)$-detachment of $\mathcal{G}$, by Lemma 3.3(iii) we have

$$ \deg_\mathcal{G}(\alpha) - \omega_\alpha(\mathcal{G}) \geq \deg_\mathcal{H}(\alpha) - \omega_\alpha(H) + 1. $$

Since $\mathcal{H}$ is an $(\alpha, n - 1)$-detachment of $\mathcal{F}$ and $c(\mathcal{H}) = c(\mathcal{F})$, by the induction hypothesis we have

$$ \deg_\mathcal{H}(\alpha) - \omega_\alpha(H) \geq n - 2. $$

Therefore, $\deg_\mathcal{G}(\alpha) - \omega_\alpha(\mathcal{G}) \geq n - 1$. \hfill $\square$

4. Laminar Families of Hinges

A family $\mathcal{A}$ of sets is laminar if, for every pair $A, B$ of sets belonging to $\mathcal{A}$, either $A \subseteq B$, or $B \subseteq A$, or $A \cap B = \emptyset$.

For the rest of this section, $\mathcal{G} = (V, E)$ is a $k$-edge-colored hypergraph, and $\alpha \in V$. The set of hinges incident with $\alpha$ in $\mathcal{G}$ will be denoted by $\mathbb{H}_\alpha(\mathcal{G})$. In this section, we construct two laminar families of subsets of $\mathbb{H}_\alpha(\mathcal{G})$. These families will be crucial in the proof of our main result. For an edge $e$ in $\mathcal{G}$, $\mathbb{H}_e(\alpha)$ will be the set of hinges incident with both $\alpha$ and $e$ in $\mathcal{G}$; abbreviate $\mathbb{H}_\mathcal{G}^{(\alpha \cap U)}(\alpha)$ to $\mathbb{H}_\mathcal{G}^{r, U}(\alpha)$ (for $r \geq 1$ and $U \subseteq V \setminus \{\alpha\}$). For $i \in [k]$, let

$$ i\mathbb{H}_\mathcal{G}(\alpha) = \bigcup_{W \in \mathcal{W}(\mathcal{G}(i)), \deg_W(\alpha) \geq 2} \mathbb{H}_W(\alpha). $$

In other words, $i\mathbb{H}_\mathcal{G}(\alpha)$ is the set of all hinges incident with $\alpha$ in some wide $\alpha$-wing of $\mathcal{G}(i)$. Observe that

- $|\mathbb{H}_\mathcal{G}(\alpha)| = \deg_\mathcal{G}(\alpha)$,
- $|\mathbb{H}_\mathcal{G}(i)(\alpha)| = \deg_\mathcal{G}(i)(\alpha)$ for $i \in [k]$,
- $\mathbb{H}_e(\alpha) \subseteq \mathbb{H}_\mathcal{G}(i)(\alpha)$ for $i \in [k]$ and $e \in E(\mathcal{G}(i))$,
- $|\mathbb{H}_\mathcal{G}^{r, U}(\alpha)| = r \cdot \mult_\mathcal{G}(\alpha \setminus U)$ for $r \geq 1$ and $U \subseteq V \setminus \{\alpha\}$.
- $|\mathbb{H}_\mathcal{G}^{r, U}(\alpha)| = r \cdot \mult_\mathcal{G}(\alpha \setminus U)$ for $i \in [k], r \geq 1$ and $U \subseteq V \setminus \{\alpha\}$.

The proof of the following lemma is immediate from definitions.

**Lemma 4.1.** Let

$$ \mathcal{A} = \bigcup_{i \in [k]} \left\{ \mathbb{H}_\mathcal{G}(i)(\alpha), \ldots, \mathbb{H}_\mathcal{G}(k)(\alpha) \right\}, $$

$$ \bigcup_{i \in [k]} \left\{ i\mathbb{H}_\mathcal{G}(\alpha), \ldots, k\mathbb{H}_\mathcal{G}(\alpha) \right\}, $$

$$ \bigcup \left\{ \mathbb{H}_\mathcal{G}(\alpha) : e \in E \right\}, $$

$$ \bigcup \left\{ \mathbb{H}_W(\alpha) : W \in \mathcal{W}(\mathcal{G}(i)), \deg_W(\alpha) \geq 2, i \in [k] \right\}, $$

and

$$ \mathcal{B} = \bigcup_{i \in [k]} \left\{ \mathbb{H}_\mathcal{G}^{r, U}(\alpha) : r \geq 1, U \subseteq V \setminus \{\alpha\} \right\}, $$

$$ \bigcup \left\{ \mathbb{H}_\mathcal{G}^{r, U}(\alpha) : r \geq 1, U \subseteq V \setminus \{\alpha\}, i \in [k] \right\}. $$

Then both $\mathcal{A}$ and $\mathcal{B}$ are laminar families of subsets of $\mathbb{H}_\mathcal{G}(\alpha)$. 
5. Connected Baranyai’s Theorem

A hypergraph is almost regular if there is an integer $d$ such that the degree of each vertex is $d$ or $d + 1$. We exhibit the usefulness of our main result by providing the following applications.

**Theorem 5.1.** Let $a_1, \ldots, a_k \in \mathbb{N}$ and $\sum_{j \in [k]} a_j = \binom{n}{h}$. Then the complete $\lambda$-fold $n$-vertex $h$-uniform hypergraph $\lambda K_h^n$ can be decomposed into almost regular spanning sub-hypergraphs $\mathcal{F}(1), \ldots, \mathcal{F}(k)$ so that for all $j \in [k]$ the following hold.

(i) $\mathcal{F}(j)$ has $a_j$ edges;

(ii) $\deg_{\mathcal{F}(j)}(v) \approx h a_j/n$ for each $v \in V(\mathcal{G}_j)$;

(iii) $\mathcal{F}(j)$ is connected if and only if $a_j \geq (n-1)/(h-1)$.

**Proof.** Given $a_1, \ldots, a_k$ with $\sum_{j \in [k]} a_j = \binom{n}{h}$, let $\mathcal{H}$ be a hypergraph with $V(\mathcal{H}) = \{\alpha\}$, $E(\mathcal{H}) = \{\{\alpha^h\}^\binom{n}{h}\}$.

We color the edges of $\mathcal{H}$ with $k$ colors so that $\text{mult}_{\mathcal{H}(j)}(\alpha^h) = a_j$ for $j \in [k]$. It is clear that for every $j \in [k]$, $\deg_{\mathcal{H}(j)}(\alpha) = h a_j$, $c(\mathcal{H}(j)) = 1$, and $\omega_\alpha(\mathcal{H}(j)) = \text{mult}_{\mathcal{H}(j)}(\alpha^h) = a_j$. By Theorem 1.1, there exists an $(\alpha, n)$-detachment $\mathcal{F}$ of $\mathcal{H}$ such that the following conditions hold.

(a) By (F2), for each $i \in [n]$ and each $j \in [k]$,

$$\deg_{\mathcal{F}(j)}(\alpha_i) \approx \frac{h a_j}{n};$$

(b) By (F3), for each $U \subseteq \{\alpha_1, \ldots, \alpha_n\}$ with $|U| = h$,

$$\text{mult}_{\mathcal{F}}(U) \approx \frac{\text{mult}_{\mathcal{H}}(\alpha^h)}{\binom{n}{h}} = \frac{\lambda_h^n}{\binom{n}{h}} = \lambda;$$

(c) By (C1), for each $j \in [k]$,

$\mathcal{F}(j)$ is connected if and only if $ha_j - a_j \geq n - 1.$

$\square$

**Theorem 5.1** without condition (iii) is known as the Baranyai’s theorem [13]. A special case of Theorem 5.1 where almost regularity is replaced by regularity is proven in [5].

6. **Proof of Theorem 1.1**

Let $k, n \in \mathbb{N}$. Let $\mathcal{H} = (V, E)$ be a $k$-edge-colored hypergraph with $\alpha \in V$ such that $\text{mult}_{\mathcal{H}}(\alpha^t, Y) = 0$ for each $t > n$ and each $Y \subseteq V$.

We show that there exists an $(\alpha, n)$-detachment $\mathcal{F}$ of $\mathcal{H}$ such that $\mathcal{F}$ is $\alpha$-simple, and that conditions (F1)–(F4) and (C1) hold.

Let $I \subseteq [k]$ be the set of all colors so that for each $i \in I$, $\deg_{\mathcal{H}(i)}(\alpha) - \omega_\alpha(\mathcal{H}(i)) \geq n - 1$.

Given an arbitrary $(\alpha, n)$-detachment $\mathcal{F}$ of $\mathcal{H}$, if for some $i \in [k]$, $c(\mathcal{H}(i)) = c(\mathcal{F}(i))$, then by Lemma 3.5 we must have $i \in I$. Since an $\alpha$-detachment only affects the component containing $\alpha$, to prove (C1) it suffices to show that if $\mathcal{H}(i)$ is connected, then so is $\mathcal{F}(i)$, for all $i \in I$. For the rest of this section we will assume that $\mathcal{H}(i)$ is connected for all $i \in I$. 
We shall construct a sequence of hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_n$ with $\mathcal{H}_1 := \mathcal{H}, \mathcal{H}_n := \mathcal{F}, \alpha_1 := \alpha$, and $V(\mathcal{H}_{\ell}) = V(\mathcal{H}_{\ell-1}) \cup \{\alpha_\ell\}$ for $2 \leq \ell \leq n$, where $\mathcal{H}_{\ell}$ is a detachment of $\mathcal{H}_{\ell-1}$ obtained by splitting off the new vertex $\alpha_\ell$ from $\alpha$ so that we end up with $\alpha$ and $\alpha_\ell$ in $\mathcal{H}_{\ell}$, such that for all $\ell \in [n]$, $\mathcal{H}_{\ell}$ satisfies the following conditions.

For all $i \in [\ell], j \in [k], 0 \leq r \leq n - \ell + 1, t > n - \ell + 1, s \in I, Y \subseteq V, U \subseteq \{\alpha_2, \ldots, \alpha_\ell\}, X \subseteq V \setminus \{\alpha_1, \ldots, \alpha_\ell\}, 1 \leq r + |U| \leq n$:

1. $\text{mult}_{\mathcal{H}_{\ell}}(\alpha^i, Y) = 0$;

2. $\text{deg}_{\mathcal{H}_{\ell}}(\alpha_i) \approx \begin{cases} \frac{(n - \ell + 1) \text{deg}_{\mathcal{H}}(\alpha)}{\text{deg}_{\mathcal{H}}(\alpha)} & \text{for } i = 1 \\ \frac{n}{\text{deg}_{\mathcal{H}}(\alpha)} & \text{for } 2 \leq i \leq \ell \end{cases}$;

3. $\text{deg}_{\mathcal{H}_{\ell(j)}}(\alpha_i) \approx \begin{cases} \frac{(n - \ell + 1) \text{deg}_{\mathcal{H}(j)}(\alpha)}{\text{deg}_{\mathcal{H}(j)}(\alpha)} & \text{for } i = 1 \\ \frac{n}{\text{deg}_{\mathcal{H}(j)}(\alpha)} & \text{for } 2 \leq i \leq \ell \end{cases}$;

4. $\frac{\text{mult}_{\mathcal{H}_{\ell}}(\alpha^r, U, X)}{\binom{n-\ell+1}{r}} \approx \frac{\text{mult}_{\mathcal{H}}(\alpha^{r+|U|}, X)}{\binom{n}{r+|U|}}$;

5. $\frac{\text{mult}_{\mathcal{H}_{\ell(j)}}(\alpha^r, U, X)}{\binom{n-\ell+1}{r}} \approx \frac{\text{mult}_{\mathcal{H}(j)}(\alpha^{r+|U|}, X)}{\binom{n}{r+|U|}}$;

6. $\mathcal{H}_{\ell}(s))$ is connected;

7. $\text{deg}_{\mathcal{H}_{\ell}(s)}(\alpha) - \omega_{\mathcal{H}}(\mathcal{H}_{\ell}(s)) \geq n - \ell$.

We prove this by induction on $\ell$. For $\ell = 1$, $\mathcal{H}$ clearly satisfies all the conditions (1)–(7), so we can let $\mathcal{H}_1$ be $\mathcal{H}$.

Before we proceed with the inductive step, we verify that when $\ell = n$ (and so $n - \ell + 1 = 1$), we can let $\mathcal{F} = \mathcal{H}_n$: Condition (1) implies that $\mathcal{H}_n$ is $\alpha$-simple, and conditions (2) and (3) are equivalent to (F1) and (F2) respectively. To see the equivalence of (4) and (F3), notice that the set $U$ in (F3) either contains $\alpha_1 = \alpha$ or it does not. According to condition (4), for $0 \leq r \leq 1, U \subseteq \{\alpha_2, \ldots, \alpha_n\}, X \subseteq V \setminus \{\alpha_1, \ldots, \alpha_n\}, 1 \leq r + |U| \leq n$ we have

$$\frac{\text{mult}_{\mathcal{H}_n}(\alpha^r, U, X)}{\binom{1}{r}} \approx \frac{\text{mult}_{\mathcal{H}}(\alpha^{r+|U|}, X)}{\binom{n}{r+|U|}}.$$

For $r = 0$, this is equivalent to

$$\text{mult}_{\mathcal{H}_n}(U, X) \approx \frac{\text{mult}_{\mathcal{H}}(\alpha^{|U|}, X)}{\binom{n}{|U|}},$$

and for $r = 1$, it is equivalent to

$$\text{mult}_{\mathcal{H}_n}(\alpha, U, X) \approx \frac{\text{mult}_{\mathcal{H}}(\alpha^{|U|+1}, X)}{\binom{n}{|U|+1}}.$$

The proof of the equivalence of conditions (5) and (F4) is very similar. Condition (6) together with the discussion at the beginning of this section implies (C1).
To prove the inductive step, let us assume that $G := H_{\ell}$ has been constructed for some $\ell < n$, and $H_{\ell}$ satisfies (1)--(7). We denote the corresponding conditions for $G' := H_{\ell+1}$ by (1)'--(7)'.

In order to decide how to share the edges incident with $\alpha$ in $G$ between $\alpha$ and the new vertex $\alpha_{\ell+1}$ in $G'$, let $\mathcal{A}$ and $\mathcal{B}$ be the laminar families of subsets of $H_G(\alpha)$ defined in Section 4. By [32, Lemma 2], there exist a subset $Z$ of $H_G(\alpha)$ such that

\[
|Z \cap P| \approx \frac{|P|}{n - \ell + 1} \quad \text{for every } P \in \mathcal{A} \cup \mathcal{B}.
\]

Let $G'$ be the $(\alpha, 2)$-detachment of $G$ with $V(G') \setminus V(G) = \{\beta\} := \{\alpha_{\ell+1}\}$ such that the hinges which were incident with $\alpha$ in $G$ are incident in $G'$ with $\alpha$ or $\beta$ according to whether they do not or do belong to $Z$, respectively. More precisely,

\[
H_{G'}(\beta) = Z, \quad H_{G'}(\alpha) = H_G(\alpha) \setminus Z.
\]

Note that (9) provides a full description of $G'$. It remains to show that $G'$ indeed satisfies (1)'--(7)'. Throughout the rest of our proof, we shall repeatedly use properties of $\approx$ and (1)--(9) without further explanation.

By the induction hypothesis, for $t > n - \ell + 1$ and $Y \subseteq V$ we have $\text{mult}_G(\alpha^t, Y) = 0$, or equivalently, for $e \in E(G)$ with $\alpha \in e$, we have $|H_G^e(\alpha)| \leq n - \ell + 1$. Since $H_G^e(\alpha) \in \mathcal{A}$, we have

\[
|Z \cap H_G^e(\alpha)| \approx \frac{|H_G^e(\alpha)|}{n - \ell + 1} \leq 1.
\]

Therefore, $|Z \cap H_G^e(\alpha)| \in \{0, 1\}$, and so $\text{mult}_{G'}(\beta^q, U) = 0$ for $q \geq 2$ and $U \subseteq V(G')$. If $|H_G^e(\alpha)| = n - \ell + 1$, then $|Z \cap H_G^e(\alpha)| = 1$, so $|H_G^e(\alpha)| = n - \ell$, and if $|H_G^e(\alpha)| < n - \ell + 1$, then $|H_G^e(\alpha)| \leq |H_G^e(\alpha)| \leq n - \ell$. Therefore, in both cases (1)' is satisfied.

Since $H_G(\alpha) \in \mathcal{B}$, we have

\[
\deg_{G'}(\beta) = |Z| = |Z \cap H_G(\alpha)| \approx \frac{|H_G(\alpha)|}{n - \ell + 1} = \frac{\deg_G(\alpha)}{n - \ell + 1} \approx \frac{\deg_H(\alpha)}{n},
\]

\[
\deg_{G'}(\alpha) = |H_{G'}(\alpha)| = |H_G(\alpha) \setminus Z| \approx \frac{(n - \ell + 1) \deg_H(\alpha)}{n} - \frac{\deg_H(\alpha)}{n} = \frac{(n - \ell) \deg_H(\alpha)}{n}.
\]

Moreover, for $2 \leq i \leq \ell$ we have $\deg_{G'}(\alpha_i) = \deg_G(\alpha_i) = \deg_H(\alpha)/n$. Therefore $G'$ satisfies (2)'.
For $j \in [k]$, we have $\mathbb{H}_{G(j)}(\alpha) \in \mathcal{A}$, and so,

\[
\deg_{G(j)}(\beta) = |Z \cap \mathbb{H}_{G(j)}(\alpha)|
\approx \frac{|\mathbb{H}_{G(j)}(\alpha)|}{n - \ell + 1} = \frac{\deg_{G(j)}(\alpha)}{n - \ell + 1}
\approx \frac{d_{H(j)}(\alpha)}{n},
\]

\[
\deg_{G(j)}(\alpha) = \deg_{G(j)}(\alpha) - \deg_{G(j)}(\beta)
\approx \frac{(n - \ell + 1) \deg_{H(j)}(\alpha)}{n} - \frac{\deg_{H(j)}(\alpha)}{n}
= \frac{(n - \ell) \deg_{H(j)}(\alpha)}{n}.
\]

This proves (3)'.

For $r \geq 1, U \subseteq V(G) \setminus \{\alpha\}$, we have $\mathbb{H}_{G}^{rU}(\alpha) \in \mathcal{B}$. Therefore,

\[
\mult_{G}(\alpha^{r-1}, \beta, U) = |Z \cap \mathbb{H}_{G}^{rU}(\alpha)|
\approx \frac{|\mathbb{H}_{G}^{rU}(\alpha)|}{n - \ell + 1}
= \frac{r}{n - \ell + 1} \mult_{G}(\alpha^{r}, U),
\]

\[
\mult_{G}(\alpha^{r}, U) = \mult_{G}(\alpha^{r}, U) - \mult_{G^{r}}(\alpha^{r-1}, \beta, U)
\approx \mult_{G}(\alpha^{r}, U) - \frac{r}{n - \ell + 1} \mult_{G}(\alpha^{r}, U)
= \frac{(n - \ell - r + 1)}{n - \ell + 1} \mult_{G}(\alpha^{r}, U).
\]

To prove (4)', let $U \subseteq \{\alpha_{2}, \ldots, \alpha_{\ell+1}\}$, and let $X \subseteq V(G^{r}) \setminus \{\alpha_{1}, \ldots, \alpha_{\ell+1}\}$. There are two cases to consider.

(i) If $\beta \in U$, then

\[
\frac{\mult_{G}(\alpha^{r}, U, X)}{\binom{n-\ell}{r}} \approx \frac{(r + 1) \mult_{G}(\alpha^{r+1}, U \setminus \{\beta\}, X)}{(n - \ell + 1) \binom{n-\ell}{r}}
= \frac{\mult(\alpha^{r+1}, U \setminus \{\beta\}, X)}{\binom{n-\ell+1}{r+1}}
\approx \frac{\mult_{H}(\alpha^{r+1+|U\setminus\{\beta\}|}, X)}{\binom{n}{r+1+|U\setminus\{\beta\}|}}
= \frac{\mult_{H}(\alpha^{r+|U|}, X)}{\binom{n}{r+|U|}}.
\]
(ii) If $\beta \notin U$, then

$$\frac{\text{mult}_{G'}(\alpha^r, U, X)}{\binom{n-\ell}{r}} \approx \frac{(n - \ell - r + 1) \text{mult}_{G}(\alpha^r, U, X)}{(n - \ell + 1)\binom{n-\ell}{r}} = \frac{\text{mult}_{G}(\alpha^r, U, X)}{(n-\ell+1)\binom{n-\ell}{r}} \approx \frac{\text{mult}_{H}(\alpha^{r+|U|}, X)}{(r+|U|)\binom{n}{r+|U|}}.$$ 

This proves (4').

For $j \in [k], r \geq 1, U \subseteq V(G)\setminus\{\alpha\}$, we have $\mathbb{H}^r_{G(j)}(\alpha) \in \mathcal{B}$. Therefore,

$$\text{mult}_{G(j)}(\alpha^{r-1}, \beta, U) = |Z \cap \mathbb{H}^r_{G(j)}(\alpha)| \approx \frac{|\mathbb{H}^r_{G(j)}(\alpha)|}{n - \ell + 1} = \frac{r}{n - \ell + 1} \text{mult}_{G(j)}(\alpha^r, U),$$

$$\text{mult}_{G(j)}(\alpha^r, U) = \text{mult}_{G(j)}(\alpha^r, U) - \text{mult}_{G(j)}(\alpha^{r-1}, \beta, U) \approx \frac{n - \ell - r + 1}{n - \ell + 1} \text{mult}_{G(j)}(\alpha^r, U).$$

To prove (5)', let $j \in [k], U \subseteq \{\alpha_2, \ldots, \alpha_{\ell+1}\}$, and let $X \subseteq V(G')\setminus\{\alpha_1, \ldots, \alpha_{\ell+1}\}$. There are two cases to consider.

(i) If $\beta \in U$, then

$$\frac{\text{mult}_{G(j)}(\alpha^r, U, X)}{\binom{n-\ell}{r}} \approx \frac{(r + 1) \text{mult}_{G(j)}(\alpha^{r+1}, U\setminus\{\beta\}, X)}{(n - \ell + 1)\binom{n-\ell}{r}} = \frac{\text{mult}_{G(j)}(\alpha^{r+1}, U\setminus\{\beta\}, X)}{(n-\ell+1)\binom{n-\ell}{r+1}} \approx \frac{\text{mult}_{H(j)}(\alpha^{r+1+|U\setminus\{\beta\}|}, X)}{(r+1+|U\setminus\{\beta\}|)\binom{n}{r+1+|U\setminus\{\beta\}|}} = \frac{\text{mult}_{H(j)}(\alpha^{r+|U|}, X)}{(r+|U|)\binom{n}{r+|U|}}.$$
(ii) If $\beta \notin U$, then 
\[
\frac{\text{mult}_{G^\prime}(\alpha^r, U, X)}{\binom{n-\ell}{r}} \approx \frac{(n - \ell - r + 1) \text{mult}_{G^\prime}(\alpha^r, U, X)}{(n - \ell + 1)\binom{n-\ell}{r}} \approx \frac{\text{mult}_{G^\prime}(\alpha^r, U, X)}{\binom{n-\ell+1}{r}} = \frac{\text{mult}_{H^\prime}(\alpha^{r+|U|}, X)}{\binom{n}{r+|U|}}.
\]

This proves (5)'.

To prove the connectivity conditions (6)' and (7)', let us fix a color $i \in I$ (If $I = \emptyset$, there is nothing to prove). By Lemma 3.4, in order to establish that $G^\prime$ is connected, it is enough to show that

(a) There exists an $\alpha$-wing $W$ in $G(i)$ such that $\deg_W(\alpha) \geq 2$;

(b) For every $\alpha$-wing $W$ in $G(i)$ with $\deg_W(\alpha) \geq 2$, 
\[|Z \cap \mathbb{H}_W(\alpha)| < \deg_W(\alpha);\]

(c) 
\[|Z \cap i\mathbb{H}_G(\alpha)| \geq 1.\]

To prove (a), suppose by contrary that for every $W \in \mathcal{W}_\alpha(G(i))$, $\deg_W(\alpha) = 1$. Then, 
\[\deg_{G(i)}(\alpha) - \omega_\alpha(G(i)) = \sum_{W \in \mathcal{W}_\alpha(G(i))} (\deg_W(\alpha) - 1) = 0.\]

But this contradicts (7), which says $\deg_{G(i)}(\alpha) - \omega_\alpha(G(i)) \geq n - \ell \geq 1$.

To prove (b), let $W \in \mathcal{W}_\alpha(G(i))$ with $\deg_W(\alpha) \geq 2$. Since $\mathbb{H}_W(\alpha) \in \mathcal{A}$, we have 
\[|Z \cap \mathbb{H}_W(\alpha)| \approx \frac{|\mathbb{H}_W(\alpha)|}{n - \ell + 1} = \frac{\deg_W(\alpha)}{n - \ell + 1}.\]

Since $\deg_W(\alpha) \geq 2$ and $n - \ell + 1 \geq 2$, we have 
\[|Z \cap \mathbb{H}_W(\alpha)| < \deg_W(\alpha).\]

To prove (c), let $A_i = \{W \in \mathcal{W}_\alpha(G(i)) : \deg_W(\alpha) \geq 2\}$. By (a), $|A_i| \geq 1$. We have 
\[n - \ell \leq \deg_{G(i)}(\alpha) - \omega_\alpha(G(i)) = \sum_{W \in \mathcal{W}_\alpha(G(i))} (\deg_W(\alpha) - 1) = \sum_{W \in A_i} \deg_W(\alpha) - |A_i| \leq |i\mathbb{H}_G(\alpha)| - 1.\]

Since $i\mathbb{H}_G(\alpha) \in \mathcal{A}$, 
\[|Z \cap i\mathbb{H}_G(\alpha)| \approx \frac{|i\mathbb{H}_G(\alpha)|}{n - \ell + 1} \geq 1.\]

This completes the proof of (6)', and we conclude that $G^\prime(i)$ is connected for all $i \in I$.

Now, we need to show that 
\[\deg_{G^\prime(i)}(\alpha) - \omega_\alpha(G^\prime(i)) \geq n - \ell - 1.\]
For \( W \in \mathcal{W}_\alpha(\mathcal{G}(i)) \) with \( \deg_W(\alpha) \geq 2 \), we have \( H_W(\alpha) \in \mathcal{A} \), and so

\[
\deg_{W'}(\beta) = |Z \cap \mathcal{H}_W(\alpha)| \\
\approx \frac{|\mathcal{H}_W(\alpha)|}{n - \ell + 1} = \frac{\deg_W(\alpha)}{n - \ell + 1},
\]

\[
\deg_{W'}(\alpha) = \deg_W(\alpha) - \deg_{W'}(\beta) \\
\approx \deg_W(\alpha) - \frac{\deg_W(\alpha)}{n - \ell + 1} \\
= \frac{(n - \ell) \deg_W(\alpha)}{n - \ell + 1}.
\]

Let \( S_i \) be the set of detached slim \( \alpha \)-wings of \( G(i) \), and let \( L_i \) be the set of semi-detached (wide) \( \alpha \)-wings of \( G(i) \). Note that \( L_i \neq \emptyset \). We have

\[
\deg_{\mathcal{G}'(i)}(\alpha) = \deg_{\mathcal{G}(i)}(\alpha) - |S_i| - \sum_{W \in L_i} \deg_{W'}(\beta).
\]

Moreover, by Lemma 3.3(i),

\[
\omega_\alpha(\mathcal{G}'(i)) = \omega_\alpha(\mathcal{G}(i)) - |S_i| - |L_i| + 1.
\]

Therefore,

\[
\deg_{\mathcal{G}'(i)}(\alpha) - \omega_\alpha(\mathcal{G}'(i)) + 1 = \deg_{\mathcal{G}(i)}(\alpha) - |S_i| - \sum_{W \in L_i} \deg_{W'}(\beta) \\
- (\omega_\alpha(\mathcal{G}(i)) - |S_i| - |L_i| + 1) + 1 \\
= (\deg_{\mathcal{G}(i)}(\alpha) - \omega_\alpha(\mathcal{G}(i))) - \left( \sum_{W \in L_i} \deg_{W'}(\beta) - |L_i| \right) \\
= (\deg_{\mathcal{G}(i)}(\alpha) - \omega_\alpha(\mathcal{G}(i))) - \sum_{W \in L_i} (\deg_{W'}(\beta) - 1).
\]

Let \( \overline{L}_i = \{ W \in L_i : \deg_{W'}(\beta) \geq 2 \} \). There are two cases to consider.

(I) \( \overline{L}_i = \emptyset \): In this case, for each \( W \in L_i \), \( \deg_{W'}(\beta) = 1 \), and so \( \sum_{W \in L_i} (\deg_{W'}(\beta) - 1) = 0 \). Thus,

\[
\deg_{\mathcal{G}'(i)}(\alpha) - \omega_\alpha(\mathcal{G}'(i)) + 1 = \deg_{\mathcal{G}(i)}(\alpha) - \omega_\alpha(\mathcal{G}(i)) \geq n - \ell,
\]

and the proof is complete.

(II) \( \overline{L}_i \neq \emptyset \): In this case, there is some \( W \in L_i \) with \( \deg_{W'}(\beta) \geq 2 \). Since \( W \) is a semi-detached wide \( \alpha \)-wing, we have \( \deg_W(\alpha) \geq 3 \). In fact, since \( \deg_{W'}(\beta) \geq 2 \) and

\[
\deg_{W'}(\beta) \approx \frac{\deg_W(\alpha)}{n - \ell + 1},
\]


we have $\deg_W(\alpha) \geq n - \ell + 2$. The following completes the proof.

$$
\deg_{G^\prime(i)}(\alpha) - \omega^\prime(G'(i)) + 1 = \sum_{w \in \mathcal{W}_i(G^\prime(i))} (\deg_W(\alpha) - 1) - \sum_{w \in \mathcal{L}} (\deg_W(\alpha) - 1)
$$

$$
\geq \sum_{w \in \mathcal{L}_i} (\deg_W(\alpha) - 1) - \sum_{w \in \mathcal{L}} (\deg_W(\alpha) - 1)
$$

$$
= \sum_{w \in \mathcal{L}_i} (\deg_W(\alpha) - \deg_W(\alpha))
$$

$$
= \sum_{w \in \mathcal{L}_i} \deg_W(\alpha)
$$

$$
\geq \sum_{w \in \mathcal{L}_i} \deg_W(\alpha) \frac{n - \ell}{n - \ell + 1} - \frac{n - \ell}{n - \ell + 1}
$$

$$
\geq \sum_{w \in \mathcal{L}_i} \frac{(n - \ell)(\deg_W(\alpha) - 1)}{n - \ell + 1}
$$

$$
= \frac{n - \ell}{n - \ell + 1} \sum_{w \in \mathcal{L}_i} (\deg_W(\alpha) - 1))
$$

$$
\geq \frac{n - \ell}{n - \ell + 1} |\mathcal{L}_i|(n - \ell + 1)
$$

$$
= |\mathcal{L}_i|(n - \ell)
$$

$$
\geq n - \ell.
$$

□

7. Embedding Arbitrary Colorings into Connected Regular Colorings

The main theme of this section is the following problem.

**Problem 1.** Suppose that the edges of $G := \lambda K^h_m$ are arbitrarily colored with $q$ colors. Find all values of $r$ and $n$ such that the given $q$-edge-coloring of $G$ can be extended to a connected $r$-factorization of $\lambda K^h_n$.

Assuming that $q$ is the number of non-empty color classes, we must have $1 \leq q \leq \lambda \binom{m}{h}$. For $i \in [q]$, let $\Delta_i = \Delta(G(i))$. It is clear that $1 \leq \Delta_i \leq \lambda \binom{m-1}{h-1}$ for each $i \in [q]$. Let $\Delta = \max_{i \in [q]} \Delta_i$. In order to extend the $q$-edge-coloring of $G$ to an $r$-factorization of $\lambda K^h_n$, it is necessary that $r \geq \Delta$, and $n \geq N_r$ where $N_r := \min\{a \in \mathbb{N} \mid q \leq \binom{a-1}{h-1}/r\}$.

Throughout this section, we assume that

(10) \hspace{1cm} n, m, h, r, \lambda \in \mathbb{N}, \quad n > m > h, \quad r \geq 1.

A partial $r$-factorization of a set $S \subseteq \binom{[n]}{h}$ is a coloring of $S$ with $\binom{n-1}{h-1}/r$ colors so that the number of times each element of $[n]$ appears in each color class is at most $r$. Each color class is called a partial $r$-factor. Note that a color class may be empty.

Embedding connected factorizations have been only studied for graphs [24, 27, 32, 35]. For recent progress on embedding factorizations of hypergraphs, we refer the reader to [6, 7, 8, 12, 14, 20]. In this section, we prove connected analogues of various hypergraph embedding
results. For the sake of brevity, we shall skip some of the algebraic manipulations that are similar to those in [6, 7].

First, we prove a simple observation.

**Lemma 7.1.** Let $\mathcal{G} = (V, E)$ be a hypergraph. We have the following.

(a) Adding an edge $e$ to $\mathcal{G}$ decreases the number of components by some $t \in [0, |e|]$.
(b) Deleting an edge $e$ to $\mathcal{G}$ increases the number of components by some $t \in [0, |e|]$.
(c) $c(\mathcal{G}) \geq |V| - \sum_{e \in E} (|e| - 1)$.
(d) If $\mathcal{G}$ is $h$-uniform, then $c(\mathcal{G}) \geq |V| - (h - 1)|E|$.
(e) If $\mathcal{G}$ is $h$-uniform and connected, then $|E| \geq \frac{|V|-1}{h-1}$.

*Proof.* Parts (a) and (b) are sufficiently obvious. Adding one-by-one the edges of $\mathcal{G}$ to a trivial hypergraph with $|V|$ vertices and applying (a) each time, completes the proof of (c). Part (d) is a special case of (c), and (e) is obtained by letting $c(\mathcal{G}) = 1$ in (d). \qed

For convenience, let us refer to the vertices in $V(K^n_h) \setminus V(K^n_m)$ as the new vertices, the edges in $E(K^n_h) \setminus E(K^n_m)$ as the new edges, and the colors in $\{q+1, \ldots, k\}$ as the new colors if $k > q$. A triple $(n, r, \lambda)$ is said to be $h$-admissible if $h|rn$ and $r|\lambda(n^{(n-1)}_{h-1})$.

**Lemma 7.2.** If a partial $r$-factorization of $\mathcal{G} := \lambda K^n_m$ can be extended to a connected $r$-factorization of $\lambda K^n_m$, then

(N1) The triple $(n, r, \lambda)$ is $h$-admissible;
(N2) No component of any color class of $\mathcal{G}$ is $r$-regular;
(N3) For all $j \in [k]$,

\[ rm - rn + \frac{rn}{h} \leq |E(\mathcal{G}(j))| \leq \frac{rn}{h} - \frac{n - m + c(\mathcal{G}(j)) - 1}{h - 1}, \]

where $k := \frac{\lambda(n^{(n-1)}_{h-1})}{r}$.

*Proof.* Suppose that a partial $r$-factorization of $\mathcal{G} := \lambda K^n_m$ is extended to a connected $r$-factorization of $\lambda K^n_m$. For $\lambda K^n_m$ to be $r$-factorable, the degree of each vertex should be divisible by $r$, and the existence of an $r$-factor in $\lambda K^n_m$ implies that $h|rn$. Therefore, $(n, r, \lambda)$ is $h$-admissible, which also implies that $k := \lambda(n^{(n-1)}_{h-1})/r$ is an integer. Since each partial $r$-factor $F$ is extended to a connected $r$-factor, there must be (at least) an edge connecting the new vertices to each component of $F$, and therefore no component of any color class of $\lambda K^n_m$ can be $r$-regular.

In an $r$-factorization of $\lambda K^n_m$, each color class has $rn/h$ edges, and each of the $n - m$ new vertices is incident with exactly $r$ edges of each color class, and so all the $n - m$ new vertices are incident with at most $r(n - m)$ edges of each color class. Therefore, for $j \in [k]$

\[ |E(\mathcal{G}(j))| + r(n - m) \geq \frac{rn}{h}. \]

For each $j \in [k]$, $\lambda K^n_m(j)$ is obtained by adding $\frac{rn}{h} - |E(\mathcal{G}(j))|$ new edges to $n - m$ new vertices and $c(\mathcal{G}(j))$ components. By lemma 7.1(e), in order to ensure that $\lambda K^n_m(j)$ is connected for each $j \in [k]$, we must have

\[ \frac{rn}{h} - |E(\mathcal{G}(j))| \geq \frac{n - m + c(\mathcal{G}(j)) - 1}{h - 1}. \]
Remark 7.3.

(a) For a connected $r$-factor to exist, we must have that $r \geq 2$. This is also implied from (N2); if by contrary $r = 1$, then each edge in the partial 1-factorization of $\lambda K_m^h$ is 1-regular, which is a contradiction.

(b) If $n \geq \frac{h}{k-1} m$, then $rm - rn + rn/h \leq 0$, so the left hand side inequality in (N3) will be trivial.

(c) Condition (N3) implies that for all $j \in [k]$, $rm - rn + rn/h \leq (n - m + c(G(j)) - 1)/(h - 1)$, or equivalently, $c(G(j)) \leq (n - m)(rh - r - 1) + 1$.

Let $\lambda K_m^h$ be the hypergraph obtained by adding a new vertex $\alpha$ and new edges to $\lambda K_m^h$ so that

$$\text{mult}(\alpha^i, X) = \lambda \left( \frac{n - m}{i} \right)$$

for each $i \in [h]$, and $X \subseteq V(\lambda K_m^h)$ with $|X| = h - i$.

In other words, $\lambda K_m^h$ is an amalgamation of $\lambda K_n^h$, obtained by identifying a set of $n - m$ vertices in $\lambda K_n^h$.

An application of Theorem 1.1 is the following corollary that reduces Problem 1 to a much simpler problem.

Corollary 7.4. A partial $r$-factorization of $\lambda K_m^h$ can be extended to a connected $r$-factorization of $\lambda K_n^h$ if and only if the new edges of $\mathcal{H} := \lambda K_m^h$ can be colored such that

$$\deg_{\mathcal{H}(i)}(v) = \begin{cases} r & \text{if } v \neq \alpha, \\ r(n - m) & \text{if } v = \alpha, \end{cases} \quad \forall i \in [k],$$

and

$$\omega_{\alpha}(\mathcal{H}(j)) \leq (r - 1)(n - m) + 1 \quad \forall i \in [k],$$

where $k := \lambda \left( \frac{n - 1}{h - 1} \right)/r \in \mathbb{N}$.

Proof. First, suppose that a partial $r$-factorization of $\lambda K_m^h$ can be extended to a connected $r$-factorization of $\lambda K_n^h$. By amalgamating the new $n - m$ vertices of $\lambda K_m^h$ into a single vertex $\alpha$, we clearly obtain $\mathcal{H}$. The $k$-edge-coloring of $\lambda K_n^h$ (in which each color class is a connected $r$-factor) induces a $k$-edge-coloring in $\mathcal{H}$ that satisfies (11). Since each color class of $\lambda K_n^h$ is connected, so is each color class of $\mathcal{H}$. Therefore by Lemma 3.5, we must have that $r(n - m) - \omega_{\alpha}(\mathcal{H}(j)) \geq n - m - 1$, and so (12) is satisfied.

Conversely, suppose that the edges of $\mathcal{H}$ are colored so that (11) and (12) are satisfied. By Theorem 1.1, there exists an $(\alpha, n - m)$-detachment $\mathcal{F}$ of $\mathcal{H}$ with $V(\mathcal{F}) = V(\mathcal{H}) \cup \{\alpha_2, \ldots, \alpha_{n-m}\}$ (here, $\alpha_1 := \alpha$) such that the following conditions hold.

(a) By (F1), for each $i \in [n - m]$ and each $j \in [k],$

$$\deg_{\mathcal{F}(j)}(\alpha_i) \approx \frac{\deg_{\mathcal{H}(j)}(\alpha)}{n - m} = r;$$

(b) By (F3), for each $U \subseteq \{\alpha_1, \ldots, \alpha_{n-m}\}$ and each $X \subseteq V(\lambda K_m^h)$ with $|U| + |X| = h,$

$$\text{mult}_{\mathcal{F}}(U, X) \approx \frac{\text{mult}_H(\alpha^{[U]}, X)}{\binom{n-m}{|U|}} = \frac{\lambda \binom{n-m}{|U|}}{\binom{n-m}{|U|}} = \lambda;$$
(c) Since \((12)\) is satisfied, by (C1), for each \(j \in [k]\),
\[
c(F(j)) = c(H(j)) = 1.
\]
Combining (a) and (c) implies that each color class of \(F\) is a connected \(r\)-factor, and (b) implies that \(F \cong \lambda K_n^r\).

The case \(\lambda = 1, r \geq 3\) of the following is proved by Hilton et al. [24, Theorem 3.1].

**Theorem 7.5.** A partial \(r\)-factorization of \(G := \lambda K_m\) can be extended to a connected \(r\)-factorization of \(\lambda K_n\) if and only if

(i) \((n, r, \lambda)\) is \(2\)-admissible;

(ii) No component of any color class of \(G\) is \(r\)-regular;

(iii) \[
rm - \frac{rn}{2} \leq |E(G(j))| \leq \frac{rn}{2} - n + m - c(G(j)) + 1 \quad \forall j \in [k],
\]
where \(k := \lambda(n - 1)/r\).

**Proof.** The necessity is implied by Lemma 7.2. To prove the sufficiency, we need to show that the edges of \(H := \lambda K_m\) can be colored with \(k := \lambda(n - 1)/r\) colors so that (11) and (12) are satisfied. Let \(G := (V, E)\). For \(v \in V\), we color the \(\alpha v\)-edges so that for \(j \in [k]\),
\[
\text{mult}_j(\alpha v) = r - \deg_{G(j)}(v).
\]
This is possible, because
\[
\sum_{j \in [k]} \text{mult}_j(\alpha v) = rk - \deg_G(v) = \lambda(n - 1) - \lambda(m - 1) = \lambda(n - m).
\]
Then we color the \(\alpha^2\)-edges so that for \(j \in [k]\),
\[
\text{mult}_j(\alpha^2) = \frac{rn}{2} - rkm + |E|
\]
\[
= \lambda \left(\frac{n}{2}\right) - \lambda m(n - 1) + \lambda \left(\frac{m}{2}\right) = \lambda \left(n - m\right).
\]
So, coloring the \(\alpha^2\)-edges is possible.

For the rest of the proof, let us fix \(j \in [k]\). It is clear that \(\deg_j(v) = r\) for \(v \in V\). Moreover,
\[
\deg_j(\alpha) = 2 \text{mult}_j(\alpha^2) + \sum_{v \in V} \text{mult}_j(\alpha v)
\]
\[
= rn - 2rm + 2|E(G(j))| + \sum_{v \in V} (r - \deg_{G(j)}(v))
\]
\[
= rn - 2rm + 2|E(G(j))| + rm - 2|E(G(j))|
\]
\[
= r(n - m).
\]
So (11) is satisfied. By the necessary condition (ii), there is at least an edge colored \(j\) between \(\alpha\) and each component of \(G(j)\), therefore \(H(j)\) is connected. The \(\alpha\)-wings of \(H(j)\) are the \(\alpha^2\)-edges, and \(\alpha\) and the components of \(G(j)\) and the edges joining \(\alpha\) to vertices of that component. We have
\[
\omega_\alpha(H(j)) = \text{mult}_j(\alpha^2) + c(G(j))
\]
\[
= \frac{rn}{2} - rm + |E(G(j))| + c(G(j))
\]
\[
\leq rn - rm - n + m + 1 = (r - 1)(n - m) + 1,
\]
where the last inequality holds by (iii). Thus, (12) is satisfied and the proof is complete. \(\square\)
The next theorem is a generalization of a result of the author and Rodger [12].

**Theorem 7.6.** Let \( n > 2m + (\gamma - 1)/2 \) where \( \gamma = \sqrt{8m^2 - 16m - 7} \). Suppose that a partial \( r \)-factorization of \( G := \lambda K^3_m \) is given. Moreover, assume that if \( r = 2 \), then either \( n \geq 3.5m - 3 \), or the number of components of each color class of \( G \) is at most \((2m + \gamma)/5 + 1\). Then the given partial \( r \)-factorization of \( G \) can be extended to a connected \( r \)-factorization of \( \lambda K^3_m \) if and only if \((n, r, \lambda)\) is \( 3 \)-admissible and no component of any color class of \( \lambda K^3_m \) is \( r \)-regular.

**Proof.** The necessity is implied by Lemma 7.2. To prove the sufficiency, we need to show that the edges of \( H := \lambda K^3_m \) can be colored with \( k := \lambda(n - 1)/r \) colors so that (11) and (12) are satisfied.

Let \( G := (V, E) \) be the given partially \( r \)-factored \( \lambda K^3_m \). First we color the \( \alpha uv \)-edges in \( H \) where \( u, v \in V \) and \( u \neq v \). We color these edges greedily so that \( \deg_j(x) \leq r \) for each \( x \in V \) and \( j \in [k] \). We claim that this coloring can be done in such a way that all edges of this type are colored. Suppose by contrary that there is an edge \( \alpha uv \) in \( H \) that can not be colored. This implies that for each \( j \in [k] \) either \( \deg_j(u) = r \) or \( \deg_j(v) = r \). Thus for each \( j \in [k] \), \( \deg_j(u) + \deg_j(v) \geq r \). We have

\[
\lambda \left( \frac{n - 1}{2} \right) = rk \leq \sum_{i \in [k]} (\deg_i(u) + \deg_i(v)) \leq 2\lambda \left[ \left( \frac{m - 1}{2} \right) + (n - m)(m - 1) - 1 \right]
\]

It is a routine exercise to verify that the above inequality contradicts \( n > 2m + (\gamma - 1)/2 \).

We color the \( \alpha^2 u \)-edges so that \( \text{mult}_1(\alpha^2 u) = r - \deg_j(u) \) for \( u \in V, j \in [k] \). The following shows that we can indeed color all such edges.

\[
\sum_{j \in [k]} (r - \deg_j(u)) = \lambda \left( \frac{n - 1}{2} \right) - \lambda \left( \frac{m - 1}{2} \right) - \lambda(n - m)(m - 1) = \lambda \left( \frac{n - m}{2} \right).
\]

For \( j \in [k] \), let \( a_j, b_j, c_j \) be the number of \( uvw \)-edges, \( u\alpha \)-edges, and \( u\alpha^2 \)-edges colored \( j \), respectively (for distinct \( u, v, w \in V \) ). We color the \( \alpha^3 \)-edges so that \( d_j := \text{mult}_1(\alpha^3) = \frac{rn}{3} - rm + 2a_j + b_j \) for \( j \in [k] \). Since \( (n, r, \lambda) \) is \( 3 \)-admissible and \( n \geq 3m \), \( d_j \) is a non-negative integer for all \( j \in [k] \). The following confirms that all \( \alpha^3 \)-edge can be colored.

\[
\sum_{j \in [k]} \left( \frac{rn}{3} - rm + 2a_j + b_j \right) = \lambda \left( \frac{n}{3} \right) - \lambda \left( \frac{m}{2} \right) + 2\lambda \left( \frac{m}{3} \right) + \lambda(n - m) \left( \frac{m}{3} \right) = \lambda \left( \frac{n - m}{3} \right).
\]

Now that we are done with the edge-coloring of \( H \), we show that (11) and (12) hold. For the rest of the proof, we fix \( j \in [k] \). It is clear that \( \deg_j(v) = r \) for \( v \in V \). Since \( rm = \sum_{v \in V} \deg_j(v) = 3a_j + 2b_j + c_j \), we have

\[
\deg_j(\alpha) = b_j + 2c_j + 3d_j = 3(a_j + b_j + c_j + d_j) - (3a_j + 2b_j + c_j) = rn - rm = r(n - m).
\]

To complete the proof, we need to show that \( \omega_\alpha(H(j)) \leq (r - 1)(n - m) + 1 \). Since no component of \( G(j) \) is \( r \)-regular, and we were able to color the \( uvw \)-edges and \( u\alpha^2 \) edges (for distinct \( u, v \in V \) ) such that \( \deg_j(u) = r \) for each \( u \in V \), there must be an edge between each component of \( G(j) \) and \( \alpha \) in \( H(j) \). Hence, \( H(j) \) is connected. For distinct \( u, v \in V \), \( u\alpha \)-edges can potentially connect \( \alpha \) to two different components of \( G(j) \), and so

\[
2b_j + c_j \geq c(G(j)).
\]
Therefore, \( b_j + 2c_j \geq b_j + c_j/2 \geq c(G(j))/2 \). There are \( d_j \) small \( \alpha \)-wings in \( \mathcal{H}(j) \). Moreover, every component of \( G(j) \) corresponds to at most one large \( \alpha \)-wing in \( \mathcal{H}(j) \). Thus, if 
\[
\omega(\mathcal{H}(j)) \leq d_j + c(G(j)).
\]
So, it suffices to show that \( rn/3 - rm + 2a_j + b_j + c(G(j)) \leq (r - 1)(n - m) + 1 \) or equivalently,
\[
\frac{2rn}{3} - n + m + 1 \geq 2a_j + b_j + c(G(j)).
\]
Since \( rm = 3a_j + 2b_j + c_j \), we have
\[
2a_j + b_j + c(G(j)) \leq \frac{2rm}{3} - \frac{1}{3}(b_j + 2c_j) + c(G(j)) \\
\leq \frac{2rm}{3} + \frac{5}{6}c(G(j))
\]
Thus, to prove (12), it is enough to show that
\[
\frac{2rn}{3} - n + m + 1 \geq \frac{2rm}{3} + \frac{5}{6}c(G(j)),
\]
or equivalently,
\[
5c(G(j)) \leq (4r - 6)(n - m) + 6.
\]
It is clear that \( c(G(j)) \leq m \). If \( r \geq 3 \), we have
\[
5c(G(j)) \leq 5m < 6(m + \gamma - 1) < 6(n - m) \leq (4r - 6)(n - m) + 6.
\]
If \( r = 2 \) and \( n \geq 3.5m - 3 \), we have
\[
5c(G(j)) \leq 5m = 2(2.5m - 3) + 6 \leq 2(n - m) + 6 = (4r - 6)(n - m) + 6.
\]
If \( r = 2 \) and \( c(G(j)) \leq (2m + \gamma)/5 + 1 \), we have
\[
5c(G(j)) \leq 2m + \gamma + 5 = 2(m + \gamma - 1) + 6 < 2(n - m) + 6 = (4r - 6)(n - m) + 6.
\]
\[\square\]

**Remark 7.7.** The necessary condition (N3) of Lemma 7.2 did not appear in Theorem 7.6. This is because (i) we assumed that \( n > 3m \), and so the left-hand side inequality of (N3) is trivial, and (ii) the right hand side in (N3) is equivalent to \( 2rn/3 - n + m + 1 \geq 2|E(G(j))| + c(G(j)) \), but we showed that a stronger statement, namely (13), holds.

The next three results without the connectivity condition were previously obtained in [6].

**Theorem 7.8.** For \( n \geq 4.847323m \), any partial \( r \)-factorization of \( \lambda K^4_m \) can be extended to a connected \( r \)-factorization of \( \lambda K^4_n \) if and only if \( (n, r, \lambda) \) is \( 4 \)-admissible, and no component of any color class of \( \lambda K^4_n \) is \( r \)-regular.

**Proof.** The necessity is implied by Lemma 7.2. To prove the sufficiency, we need to show that the edges of \( \mathcal{H} := \lambda K^4_m \) can be colored with \( k := \lambda(n - 1)/r \) colors so that (11) and (12) are satisfied.

Let \( G := (V, E) \) be the given partially \( r \)-factored \( \lambda K^4_n \). First we color all the \( uva \)-edges of \( \mathcal{H} \) (for distinct \( u, v, w \in V \)) greedily so that \( \deg_i(x) \leq r \) for each \( x \in V \) and \( i \in [k] \). Suppose
by contrary that there is an edge $uvw\alpha$ that can not be colored (for distinct $u,v,w \in V$). This implies that
\[
\lambda\left(\frac{n-1}{3}\right) \leq \sum_{i=1}^{k} (\deg_{i}(u) + \deg_{i}(v) + \deg_{i}(w)) \leq 3\lambda\left[\left(\frac{m-1}{3}\right) + (n-m)\left(\frac{m-1}{2}\right) - 1\right].
\]
which contradicts $n > 4m, m \geq 5$.

We greedily color all the $u\alpha^{2}$-edges of $\mathcal{H}$ (for distinct $u,v \in V$) so that $\deg_{i}(x) \leq r$ for each $x \in V$ and $i \in [k]$. If by contrary, some edge $uvw\alpha$ remains uncolored (for distinct $u,v \in V$), then
\[
\lambda\left(\frac{n-1}{3}\right) \leq \sum_{i=1}^{k} (\deg_{i}(u) + \deg_{i}(v)) \\
\leq 2\lambda\left[\left(\frac{m-1}{3}\right) + (n-m)\left(\frac{m-1}{2}\right) + (m-1)\left(\frac{n-m}{2}\right) - 1\right].
\]
Using Mathematica it can be shown that this inequality does not have any real solution under the constraints $m \geq 5, n \geq 4.847323m$.

We color all the $u\alpha^{3}$-edges so that $\text{mult}_{j}(u\alpha^{3}) = r - \deg_{s\in(j)}(u)$ for $u \in V, j \in [k]$. Since for each $u \in V$,
\[
\sum_{i \in [k]} (r - \deg_{i}(u)) = \lambda\left(\frac{n-1}{3}\right) - \lambda\left[\left(\frac{m-1}{3}\right) + (n-m)\left(\frac{m-1}{2}\right) + (m-1)\left(\frac{n-m}{2}\right)\right] \\
= \lambda\left(\frac{n-m}{3}\right);
\]
coloring the $u\alpha^{3}$-edges is possible.

For $j \in [k]$, let $a_{j}, b_{j}, c_{j}, d_{j}$ be the number of $uvw\alpha$-edges, $uww\alpha$-edges, $uvw\alpha$-edges, and $u\alpha^{3}$-edges colored $j$, respectively (for distinct $u,v,w,x \in V$). We color the $\alpha^{4}$-edges so that $e_{j} := \text{mult}_{j}(\alpha^{4}) = rm/4 - rm + 3a_{j} + 2b_{j} + c_{j}$ for $j \in [k]$. Since $(n,r,\lambda)$ is 4-admissible and $n > 4m$, $e_{j}$ is a non-negative integer for all $j \in [k]$. The following confirms that all $\alpha^{4}$-edge can be colored.
\[
\sum_{j \in [k]} (\frac{rm}{4} - rm + 3a_{j} + 2b_{j} + c_{j}) = \lambda\left(\frac{n}{4}\right) - \lambda m\left(\frac{n-1}{3}\right) + 3\lambda\left(\frac{m}{4}\right) \\
+ 2\lambda(n-m)\left(\frac{m}{3}\right) + \lambda\left(\frac{n-m}{2}\right)\left(\frac{m}{2}\right) = \lambda\left(\frac{n-m}{4}\right).
\]

Now that the edge-coloring of $\mathcal{H}$ is completed, we shall show that this coloring satisfies (11) and (12). For the rest of the proof, we fix $j \in [k]$. It is clear that $\deg_{j}(v) = r$ for $v \in V$. Since $rm = \sum_{v \in V} \deg_{j}(v) = 4a_{j} + 3b_{j} + 2c_{j} + d_{j}$, we have
\[
\deg_{j}(\alpha) = b_{j} + 2c_{j} + 3d_{j} + 4e_{j} = 4(a_{j} + b_{j} + c_{j} + d_{j} + e_{j}) - (4a_{j} + 3b_{j} + 2c_{j} + d_{j}) \\
= rn - rm = r(n-m).
\]
To complete the proof, we need to show that $\omega_{\alpha}(\mathcal{H}(j)) \leq (r-1)(n-m) + 1$. Observe that $\mathcal{H}(j)$ is connected, and
\[3b_{j} + 2c_{j} + d_{j} \geq c(\mathcal{G}(j)).\]
Therefore, \( b_j + 2c_j + 3d_j \geq b_j + 2c_j/3 + d_j/3 \geq c(G(j))/3 \). There are \( e_j \) small alpha-wings in \( H(j) \). Moreover, every component of \( G(j) \) corresponds to no more than one large alpha-wing in \( H(j) \). Therefore,

\[
\omega_\alpha(H(j)) \leq e_j + c(G(j)).
\]

So, it suffices to show that \( rn/4 - rm + 3a_j + 2b_j + c_j + c(G(j)) \leq (r - 1)(n - m) + 1 \) or equivalently,

\[
(15) \quad \frac{3rn}{4} - n + m + 1 \geq 3a_j + 2b_j + c_j + c(G(j)).
\]

Recall that \( n > 4m, r \geq 2 \). It is clear that \( c(G(j)) \leq m \). Since \( rm = 4a_j + 3b_j + 2c_j + d_j \), we have \( 3a_j + 2b_j + c_j = 3rm/4 - (b_j + 2c_j + 3d_j)/4 \).

The following completes the proof.

\[
\frac{3rn}{4} - n + m + 1 > \left( \frac{3r}{4} - 1 \right)(4m) + m + 1
\]

\[
= 3rm - 3m + 1
\]

\[
\geq \frac{3rm}{4} + \frac{11m}{12}
\]

\[
\geq \frac{3rm}{4} + \frac{11}{12}c(G(j))
\]

\[
\geq \frac{3rm}{4} - \frac{1}{4}(b_j + 2c_j + 3d_j) + c(G(j))
\]

\[
= 3a_j + 2b_j + c_j + c(G(j)).
\]

\[\square\]

**Remark 7.9.** Since \( n > 4m \) and (15) holds, condition (N3) did not appear in the statement of Theorem 7.8.

**Theorem 7.10.** For \( n \geq 6.285214m \), any partial r-factorization of \( \lambda K_m^5 \) can be extended to a connected r-factorization of \( \lambda K_n^5 \) if and only if \( (n, r, \lambda) \) is 5-admissible, and no component of any color class of \( \lambda K_m^5 \) is r-regular.

**Proof.** The necessity is implied by Lemma 7.2. To prove the sufficiency, we need to show that the edges of \( H := \lambda K_m^5 \) can be colored with \( k := \lambda \left( \frac{n - 1}{4} \right) \) colors so that (11) and (12) are satisfied.

Let \( G := (V, E) \) be the given partially r-factored \( \lambda K_m^5 \). For distinct \( u, v, w, x \in V \), we claim that we can greedily color all the \( uvwx \)-edges, \( uvw \alpha^2 \)-edges, and \( uv \alpha^3 \)-edges of \( H \) (in that particular order) so that \( \deg_i(x) \leq r \) for each \( x \in V \) and \( i \in [k] \).

If there is an edge \( uvwx \alpha \) that can not be colored, then by obtaining a lower bound and an upper bound for \( \sum_{i \in [k]} \left( \deg_i(u) + \deg_i(v) + \deg_i(w) + \deg_i(x) \right) \) we have the following which contradicts \( n > 4m \) and \( m \geq 5 \).

\[
\lambda \left( \frac{n - 1}{4} \right) \leq 4\lambda \left[ \left( \frac{m - 1}{4} \right) + (n - m) \left( \frac{m - 1}{3} \right) - 1 \right].
\]

If there is an edge \( uvwx \alpha^2 \) that can not be colored, then by obtaining a lower bound and an upper bound for \( \sum_{i \in [k]} \left( \deg_i(u) + \deg_i(v) + \deg_i(w) \right) \) we have the following which contradicts
\(n > 6m\) and \(m \geq 6\).

\[
\lambda \left( \frac{n-1}{4} \right) \leq 3\lambda \left[ \frac{(m-1)}{4} + (n-m) \frac{(m-1)}{3} + \frac{(m-1)}{2} \frac{(n-m)}{2} - 1 \right].
\]

Finally, if there is an edge \(uv\alpha^3\) that cannot be colored, then by obtaining a lower bound and an upper bound for \(\sum_{i \in [k]} (\deg_i(u) + \deg_i(v))\) we have the following, which according to Mathematica, contradicts \(m \geq 6, n \geq 6.285214m\).

\[
\lambda \left( \frac{n-1}{4} \right) \leq 2\lambda \left[ \frac{(m-1)}{4} + (n-m) \frac{(m-1)}{3} + \frac{(m-1)}{2} \frac{(n-m)}{2} + (m-1) \frac{(n-m)}{3} - 1 \right].
\]

We color all the \(uv\alpha^4\)-edges so that \(\text{mult}_j(uv\alpha^4) = r - \deg_{g(j)}(u)\) for \(u \in V, j \in [k]\). Since for each \(u \in V\),

\[
\sum_{j \in [k]} (r - \deg_j(u)) = \lambda \left( \frac{n-1}{4} \right) - \lambda \left( \frac{m-1}{4} \right) - \lambda(n-m) \frac{(m-1)}{3} - \lambda \left( \frac{m-1}{2} \right) \frac{(n-m)}{2} - \lambda(m-1) \frac{(n-m)}{3} = \lambda \left( \frac{n-m}{4} \right),
\]

coloring the \(uv\alpha^4\)-edges is possible.

For \(j \in [k]\), let \(a_j, b_j, c_j, d_j, e_j\) be the number of \(uvwx\)-edges, \(uvwx\alpha\)-edges, \(uv\alpha^2\)-edges, \(uv\alpha^3\)-edges, and \(uv\alpha^4\)-edges colored \(j\), respectively (for distinct \(u, v, w, x, y \in V\)). We color the \(\alpha^5\)-edges so that \(f_j := \text{mult}_j(\alpha^5) = rn/5 - rm + 4a_j + 3b_j + 2c_j + d_j\) for \(j \in [k]\). Since \((n, r, \lambda)\) is 5-admissible and \(n > 5m\), \(f_j\) is a non-negative integer for all \(j \in [k]\). The following confirms that all \(\alpha^5\)-edge can be colored.

\[
\sum_{j \in [k]} \frac{rn}{5} - rm + 4a_j + 3b_j + 2c_j + d_j = \lambda \left( \frac{n}{5} \right) - \lambda m \left( \frac{n-1}{4} \right) + 4\lambda \left( \frac{m}{5} \right) + 3\lambda(n-m) \left( \frac{m}{4} \right) + 2\lambda \left( \frac{m-1}{2} \right) \left( \frac{n-m}{3} \right) \left( \frac{m}{2} \right) = \lambda \left( \frac{n-m}{5} \right).
\]

The edge-coloring of \(\mathcal{H}\) is completed, so now we show that this coloring satisfies (11) and (12). For the rest of the proof, we fix \(j \in [k]\). It is clear that \(\deg_j(v) = r\) for \(v \in V\). Since \(rm = \sum_{v \in V} \deg_j(v) = 5a_j + 4b_j + 3c_j + 2d_j + e_j\), we have

\[
\deg_j(\alpha) = b_j + 2c_j + 3d_j + 4e_j + 5f_j
\]

\[
= 5(a_j + b_j + c_j + d_j + e_j + f_j) - (5a_j + 4b_j + 3c_j + 2d_j + e_j)
\]

\[
= rn - rm = r(n-m).
\]

To complete the proof, we need to show that \(\omega_\alpha(\mathcal{H}(j)) \leq (r-1)(n-m) + 1\). Observe that \(\mathcal{H}(j)\) is connected, and

\[
4b_j + 3c_j + 2d_j + e_j \geq c(\mathcal{G}(j)).
\]

Therefore, \(b_j + 2c_j + 3d_j + 4e_j \geq b_j + 3c_j/4 + d_j/2 + e_j/4 \geq c(\mathcal{G}(j))/4\). There are \(f_j\) small \(\alpha\)-wings in \(\mathcal{H}(j)\). Moreover, every component of \(\mathcal{G}(j)\) corresponds to at most one large \(\alpha\)-wing.
in $\mathcal{H}(j)$. Therefore,
\[
\omega_a(\mathcal{H}(j)) \leq f_j + c(\mathcal{G}(j)).
\]
So, it suffices to show that $rn/5 - rm + 4a_j + 3b_j + 2c_j + d_j + c(\mathcal{G}(j)) \leq (r - 1)(n - m) + 1$ or equivalently,
\[
(16) \quad \frac{4rn}{5} - n + m + 1 \geq 4a_j + 3b_j + 2c_j + d_j + c(\mathcal{G}(j)).
\]
Recall that $n > 6.5m, r \geq 2$. It is clear that $c(\mathcal{G}(j)) \leq m$. Since $rm = 5a_j + 4b_j + 3c_j + 2d_j + e_j$, we have $4a_j + 3b_j + 2c_j + d_j = 4rm/5 - (b_j + 2c_j + 3d_j + 4e_j)/5$.

The following completes the proof.
\[
\frac{4rn}{5} - n + m + 1 \geq \left(\frac{4r}{5} - 1\right)(6m) + m + 1
\]
\[
= \frac{24rm}{5} - 5m + 1
\]
\[
> \frac{4rm}{5} + \frac{19m}{20}
\]
\[
\geq \frac{4rm}{5} + \frac{19}{20}c(\mathcal{G}(j))
\]
\[
\geq \frac{4rm}{5} - \frac{1}{5}(b_j + 2c_j + 3d_j + 4e_j) + c(\mathcal{G}(j))
\]
\[
= 4a_j + 3b_j + 2c_j + d_j + c(\mathcal{G}(j)).
\]

\[\square\]

Observe that the necessary condition (N3) did not appear in the statement of Theorem 7.10, for $n > 5m$ and (16) is satisfied.

Let $V \subset V(\lambda K_n^h)$ with $|V| = m < n$. Then $\mathcal{H} := \lambda K_n^h - V \cong \bigcup_{i=0}^{h-1} \lambda^{m-i} K_{n-i}^{h-i}$. A partial $r$-factorization of $\mathcal{H}$ is a coloring of the edges of $\mathcal{H}$ with at most $\lambda^{(n-1)\over r}$ colors so that for each color $i$, $\text{deg}_{\mathcal{H}(i)}(v) \leq r$ for each vertex of $\mathcal{H}$. In the next result, we settle the problem of extending a partial $r$-factorization of $\mathcal{H}$ to a connected $r$-factorization of $\lambda K_n^h$. Note that here we are not only extending the coloring, but also the edges of size less than $h$ to edges of size $h$.

**Theorem 7.11.** For $V \subset V(\lambda K_n^h)$ with $|V| = m < n$, any partial $r$-factorization of $\mathcal{H} := \lambda K_n^h - V$ can be extended to a connected $r$-factorization of $\lambda K_n^h$ if and only if

(i) $(n, r, \lambda)$ is $h$-admissible;
(ii) $r \geq 2$;
(iii) Each color class of $\mathcal{H}$ is $r$-regular;
(iv) No component of any color class of $\mathcal{H}$ is $h$-uniform;
(v) For $j = 1, \ldots, \lambda^{(n-1)\over r}$,
\[
|E(\mathcal{H}(j))| \leq \frac{rn}{h};
\]
(vi) For $j = 1, \ldots, \lambda^{(n-1)\over r}$,
\[
c(\mathcal{H}(j)) \leq rn\left(1 - \frac{1}{h}\right) - \sum_{i \in [h-1]} i e_j^{i+1} - m + 1,
\]
where $e_j^t$ is the number of edges of size $t$ in $\mathcal{H}(j)$ for $t \in [h]$. 
So, to prove (vi), we need to show that the edges of size $\text{mult}_k$ hold. Let $h$ be the maximum $G$ of (v) is implied. (iv) is necessary. The number of edges in each color class of $H$ is $h$-regular, hence (iv) is necessary. The number of edges in each color class of $\lambda K_n^h$ is $rn/h$, thus the necessity of (v) is implied.

To prove the necessity of (vi), let us fix a color $j$, and let $G$ be the hypergraph containing the $c(H(j))$ components of $H(j)$ together with the $m$ vertices from $V$. We need to extend the edges of size $i$ (for $i < h$) in $H(j)$ to edge of size $h$ in $G$, and add edges of size $h$, so that the resulting hypergraph $\lambda K_n^h(j)$ is connected. Adding each edge of size $h$ decreases the number of components of $G$ by at most $h - 1$. Moreover, extending each edge of size $i$ (for $1 \leq i < h$) in $H(j)$ to edge of size $h$ in $G$ decreases the number of components of $G$ by at most $h - i$. Therefore,

$$1 = c(G) \geq c(H(j)) + m - (h - 1) \text{mult}_j(\alpha^h) - \sum_{i \in [h-1]} (h - i) e^i_j.$$ 

So, to prove (vi), we need to show that

$$(h - 1) \frac{rn}{h} = \sum_{i \in [h-1]} i e^{i+1}_j + (h - 1) \text{mult}_j(\alpha^h) + \sum_{i \in [h-1]} (h - i) e^i_j$$

which is true, because

$$\frac{rn}{h} = \text{mult}_j(\alpha^h) + \sum_{i \in [h]} e^i_j.$$ 

To prove the sufficiency, suppose that a partial $r$-factorization of $H$ is given and (i)–(vi) hold. Let $k = \frac{\lambda}{r} \left( \binom{n-1}{h-1} \right)$, and let $F = \lambda K_n^{h-m}$. For $0 \leq i \leq h$, an edge of type $\alpha^i$ in $F$ is an edge in $F$ containing $\alpha^i$ but not containing $\alpha^{i+1}$. There are $\binom{m}{h-1} \binom{n-m}{h-i}$ edges of type $\alpha^i$ in $F$. Due to the one-to-one correspondence between the edges of size $i$ in $H$ and the edges of type $\alpha^{h-i}$ in $F$ (for each $i \in [h]$), we can color the edges of type $\alpha^i$ in $F$ with the same color as the corresponding edge in $H$ for $0 \leq i \leq h - 1$. We claim that we color the remaining edges of $F$ (edges of type $\alpha^h$) so that the two conditions of Corollary 7.4 are satisfied.

For $i \in [h], j \in [k]$, let $\text{mult}_j(\alpha^i, \cdot)$ be the number of edges of type $\alpha^i$ in $F(j)$. For $j \in [k], \text{mult}_j(\alpha^h, \cdot) = \text{mult}_{F(j)}(\alpha^h)$. We color the edges of type $\alpha^h$ so that

$$\text{mult}_j(\alpha^h, \cdot) = \frac{rn}{h} - r(n - m) + \sum_{i \in [h-1]} i \text{mult}_j(\alpha^{h-i-1}, \cdot) \quad \text{for } j \in [k].$$

Since $(n, r, \lambda)$ is $h$-admissible, $\text{mult}_j(\alpha^h, \cdot)$ is an integer for $j \in [k]$. The following shows that $\text{mult}_j(\alpha^h, \cdot) \geq 0$ for $j \in [k]$.

$$\frac{rn}{h} \geq |E(H(j))| = \sum_{i \in [h-1]} \text{mult}_j(\alpha^i, \cdot)$$

$$= \sum_{i \in [h]} i \text{mult}_j(\alpha^{h-i}, \cdot) - \sum_{i \in [h-1]} i \text{mult}_j(\alpha^{h-i-1}, \cdot)$$

$$= r(n - m) - \sum_{i \in [h-1]} i \text{mult}_j(\alpha^{h-i-1}, \cdot).$$
The following confirms that all the $\alpha^h$-edges will be colored.

$$\sum_{j \in [k]} \text{mult}_j(\alpha^h, \cdot) = \binom{n}{h} - (n-m)\binom{n-1}{h-1} + \sum_{i=2}^{h} (i-1) \binom{m}{h-i} \binom{n-m}{i}$$

$$= \sum_{i=0}^{h} \binom{m}{i} \binom{n-m}{h-i} - \sum_{i=1}^{h-1} \binom{n-m}{i} \binom{m}{h-i}$$

$$- (n-m)\binom{n-m-1}{h-1} + \sum_{i=2}^{h} (i-1) \binom{m}{h-i} \binom{n-m}{i}$$

$$= \binom{m}{h} - (n-m)\binom{n-m-1}{h-1} + h\binom{n-m}{h} = \binom{m}{h}.$$

For $j \in [k]$, we have

$$\deg_{F(j)}(\alpha) = \sum_{i \in [h]} i \text{mult}_j(\alpha^i, \cdot) = h \sum_{0 \leq i \leq h} \text{mult}_j(\alpha^{h-i}, \cdot) - \sum_{i \in [h]} i \text{mult}_j(\alpha^{h-i}, \cdot)$$

$$= rn - r(n-m) = rm.$$

Let us fix $j \in [k]$. To complete the proof we need to show that

$$\omega_{\alpha}(F(j)) \leq m(r-1) + 1.$$  

Since no component of any color class of $H(j)$ is $h$-uniform, $F(j)$ is connected, and $\omega_{\alpha}(F(j)) = \text{mult}_j(\alpha^h, \cdot) + c(H(j))$. Thus, to prove (17), we need to show that

$$c(H(j)) \leq rm - \text{mult}_j(\alpha^h, \cdot) - m + 1,$$

or equivalently,

$$c(H(j)) \leq rn(1 - \frac{1}{h}) - \sum_{i \in [h-1]} i \text{mult}_j(\alpha^{h-i-1}, \cdot) - m + 1,$$

which is true by (vi).  

Let $V \subset V(\lambda K_n^h)$ with $|V| = m < n$, and let $H := \lambda K_n^h \setminus V$. We are interested in finding the conditions under which a partial $r$-factorization of $H$ can be extended to a connected $r$-factorization of $\lambda K_n^h$. Let us first look at some necessary conditions.

**Lemma 7.12.** For $V \subset V(\lambda K_n^h)$ with $|V| = m < n$, if a partial $r$-factorization of $H := \lambda K_n^h \setminus V$ can be extended to an $r$-factorization of $\lambda K_n^h$, then

(i) $(n, r, \lambda)$ is $h$-admissible;

(ii) $r \geq 2$;

(iii) $\deg_j(v) = r$ for each $v \in V(\lambda K_n^h) \setminus V$ and $j \in [k]$;

(iv) No component of any color class of $H - V$ is $h$-uniform;

(v) For each $j \in [k],$

$$|E(H(j))| \leq \frac{rn}{h};$$

(vi) For each $j \in [k],$

$$c(H(j)) \leq rn(1 - \frac{1}{h}) - \sum_{i \in [h-1]} ie^{i+1}_j - m + 1,$$

where $e^{i}_j$ is the number of edges $e \in E(H(j))$ with $|e \cap (V(\lambda K_n^h) \setminus V)| = t$ for $t \in [h].$
The proof is very similar to that of Theorem 7.11 and we shall skip it here.

An edge \( e \) in \( \mathcal{H} \) is of type \( i \), if \( |e \cap V| = i \) (for \( 0 \leq i \leq h - 1 \)). Let \( P \) be a partial \( r \)-factorization of \( \mathcal{H} \). Then a partial \( r \)-factorization \( Q \) of \( \mathcal{H} \) is said to be \( P \)-friendly if}

(1) the color of each edge of type 0 is the same in \( P \) and \( Q \), and
(2) the number of edges of type \( i \) and color \( j \) is the same in \( P \) and \( Q \) for each \( i \in [h - 1] \) and each color \( j \).

**Corollary 7.13.** Let \( V \subseteq V(\lambda K^h_n) \) with \( |V| = m < n \), and let \( P \) be a partial \( r \)-factorization of \( \mathcal{H} := \lambda K^h_n \setminus V \), and assume that conditions (i)–(vi) of Lemma 7.12 are satisfied. Then there exists a \( P \)-friendly partial \( r \)-factorization of \( \mathcal{H} \) that can be extended to a connected \( r \)-factorization of \( \lambda K^h_n \).

**Proof.** From the given partial \( r \)-factorization \( P \) of \( \mathcal{H} \), we obtain a partial \( r \)-factorization \( Q \) of \( \mathcal{H} - V \). Using Theorem 7.11, we extend \( Q \) to a connected \( r \)-factorization of \( G \cong \lambda K^h_n \). The partial \( r \)-factorization of \( G \setminus V \) is \( P \)-friendly. \( \square \)

### 8. Embedding Regular Colorings into Connected Regular Colorings

Throughout this section, \( m, n, h, r, s \in \mathbb{N} \) and \( n > m > h \). In this section, we investigate the following problem.

**Problem 2.** Given an arbitrary \( r \)-factorization of \( G := \lambda K^h_m \), find all values of \( s \) and \( n \) such that the given \( r \)-factorization of \( G \) can be extended to a connected \( s \)-factorization of \( \lambda K^h_n \).

We completely solve this problem for \( h = 2, 3 \), and nearly solve it for \( h = 4, 5 \).

**Lemma 8.1.** If an \( r \)-factorization of \( G := \lambda K^h_m \) can be embedded into a connected \( s \)-factorization of \( \lambda K^h_n \), then

(M1) \((m, r, \lambda) \) and \((n, s, \lambda) \) are \( h \)-admissible;
(M2) \( 1 < s/r \leq (n-1)/(m-1) \);
(M3) \( n \geq h_{h-1} m \) if \( s/r < (n-1)/(m-1) \);
(M4) \( n \geq h_{h-1} m \) if \( s/r = (n-1)/(m-1) \).

**Proof.** We apply Lemma 7.2 with the simple observation in mind that an \( r \)-factorization of \( G \) is a partial \( s \)-factorization of \( G \). Since \( G \) is \( r \)-factorable, \((m, r, \lambda) \) is \( h \)-admissible, so (M1) is immediate from (N1). The number of colors in an \( r \)-factorization of \( G \) is \( q := \lambda(h_{h-1})/r \), and the number of colors in an \( s \)-factorization of \( \lambda K^h_n \) is \( k := \lambda(n_{h-1})/s \), and so we must have that \( q \leq k \), or equivalently, \( s/r \leq (n_{h-1})/(h_{h-1}) \). By (N2), no component of any color class of \( G \) is \( s \)-regular, so we must have that \( s > r \). This proves (M2).

Since for \( j \in [q] \), \( |E(G(j))| = rm/h \), and for \( j \in (q, k] \), \( |E(G(j))| = 0 \), (N3) implies that \( rm/h \geq sm - sn - sn/h \), or equivalently, \( n \geq h_{h-1} m \). This proves (M4). Moreover, if \( k > q \), then \( sm - sn + sn/h \leq 0 \), or \( n \geq h_{h-1} m \). This proves (M3). \( \square \)

**Remark 8.2.**

(a) Since for \( j \in (q, k] \), \( c(G(j)) = m \), (N3) also implies that \( sn/h - (n-m+m-1)/(h-1) \geq 0 \). Therefore, \( n(sh - h - s) \geq -h \) which holds because \( s \geq 2, h \geq 2 \).

(b) It seems that condition (M4) can be eliminated. Here, we prove this for \( h = 2, 3 \). Since \( n \geq m + 1 \), we have \((n-m-1)(n-m) \geq 0 \), or equivalently, \( n \geq (2 - \frac{m-1}{n-1}) m \). If \( k = q \), this implies that \( n \geq (2 - r/s)m \). Similarly, \( 2(n-m)(n-m-1)(2n+m-4) \geq 0 \).
0, or equivalently, \( n \geq \frac{m}{2} \left( 3 - \frac{(m-1)}{(n-1)} \right) \). Again, if \( k = q \), this implies that \( n \geq \frac{m}{2} (3 - r/s) \).

(c) By (N3), for \( j \in [q] \), \( sn/h - (n - m + c(G(j)))/(h - 1) \geq rm/h \), or equivalently, 
\( c(G(j)) \leq (h-1)(sn-rm)/h - n + m + 1 \). Although this is a necessary condition, it is in fact trivial. To see this, it is enough to show that \( \frac{h-1}{h} (sn-rm) - n + m + 1 \geq m/h \), or equivalently,
\[
n \geq \frac{(h-1)r-h+1}{(h-1)s-h} - \frac{h}{(h-1)s-h},
\]
which is true because \( n > m \).

The following result settles Problem 2 for graphs. Although this result is a special case of Theorem 7.5, it may shed some light to the hypergraph analogue.

**Theorem 8.3.** An \( r \)-factorization of \( \lambda K_m \) can be extended to a connected \( s \)-factorization of \( \lambda K_n \) if and only if the following conditions are satisfied.

1. \((m, r, \lambda) \) and \((n, s, \lambda) \) are \( 2 \)-admissible;
2. \( 1 < s/r \leq (n-1)/(m-1) \);
3. \( n \geq 2m \) if \( s/r < (n-1)/(m-1) \);

**Proof.** The proof is obtained by combining Theorem 7.5 and Lemma 8.1.

The next result completely settles Problem 2 for \( h = 3 \). For a similar result without the connectivity condition, we refer the reader to [7].

**Theorem 8.4.** An \( r \)-factorization of \( \lambda K^3_m \) can be embedded into a connected \( s \)-factorization of \( \lambda K^3_n \) if and only if the following conditions are satisfied.

1. \((m, r, \lambda) \) and \((n, s, \lambda) \) are \( 3 \)-admissible;
2. \( 1 < s/r \leq (n-1)/(m-1) \);
3. If \( r (n-1) > s (m-1) \), then \( n \geq 3m/2 \);
4. \( (n-m) (m/2) \geq (m-n/3) \left( \frac{(n-1)}{2} - \frac{s (m-1)}{r} \right) \);

**Proof of Necessity.** Suppose that an \( r \)-factorization of \( \lambda K^3_m \) can be extended to a connected \( s \)-factorization of \( \lambda K^3_n \). By Lemma 8.1, conditions (i)–(iii) are necessary. Condition (iv) is equivalent to \( (n-m) (m/2) \geq (k-q) (sm - sn/3) \) where \( q := \frac{1}{3} (m-1) \) and \( k := \frac{1}{2} (n-1) \), and is trivial if \( k = q \). So let us assume that \( k > q \). For \( j \in \{0, k\} \), let \( b_j, c_j, \) and \( d_j \) be the number of edges colored \( j \) in \( \lambda K^3_n \) with 2, 1, and 0 vertices in \( \lambda K^3_m \), respectively. In color class \( j \) of \( \lambda K^3_n \) with \( j \in (q, k] \), the degree sum of all the vertices in \( \lambda K^3_m \) is \( sm \) and
\[
sm = 2b_j + c_j.
\]
Moreover, in color class \( j \) of \( \lambda K^3_n \) with \( j \in (q, k] \), the total number of edges is \( sn/3 \) and we have
\[
\frac{sn}{3} = b_j + c_j + d_j.
\]
Therefore,
\[
sm - \frac{sn}{3} = b_j - d_j \leq b_j.
\]
We have the following which proves (iv).
\[
(k - q) (sm - \frac{sn}{3}) \leq \sum_{j \in (q, k]} b_j \leq \sum_{j \in (0, k]} b_j = (n-m) \binom{m}{2}.
\]
Proof of Sufficiency. Suppose that an \( r \)-factorization of \( G := \lambda K_m^3 \) is given and that (i)–(v) hold. Let \( H \) be a hypergraph with vertex set \( \{u, \alpha\} \) such that \( \text{mult}_H(u^i \alpha^{3-i}) = \lambda \left( \binom{m}{i} \binom{n-m}{3-i} \right) \) for \( i \in [0, 2] \). We define \( t_1, t_2, \rho_1, \rho_2 \) as follows.

\[
\begin{align*}
t_1 & := sm - \frac{sn}{3} - \frac{2rm}{3}, & \rho_1 & := \frac{sm - rm}{2}, \\
t_2 & := sm - \frac{sn}{3}, & \rho_2 & := \frac{sm}{2}.
\end{align*}
\]

By (i), \( t_1, t_2 \in \mathbb{Z} \). Since by (iii) for \( k > q, n \geq 3m/2 \) and by Remark 8.2 for \( k = q, n \geq \frac{m}{2} (3 - r/s) \), we have \( t_1 \leq \rho_1 \). Moreover, (iii) implies that \( t_2 \leq \rho_2 \).

Claim 1. We can color the \( u^2\alpha \)-edges of \( H \) so that

\[
\begin{aligned}
\frac{t_1}{\rho_1} & \leq \text{mult}_H(j)(u^2\alpha) \leq \begin{cases} \\
\lceil \rho_1 \rceil & \text{for } j \in (0, q], \\
\lfloor \rho_2 \rfloor & \text{for } j \in (q, k].
\end{cases}
\end{aligned}
\]

First, we show that

\[
q[\rho_1] + (k - q)[\rho_2] \geq \lambda(n - m) \binom{m}{2},
\]

or equivalently,

\[
\lambda m \binom{n - m}{2} \geq 2q \text{frac}(\rho_1) + 2(k - q) \text{frac}(\rho_2).
\]

If \( \text{frac}(\rho_1) = 0 \), and either \( \text{frac}(\rho_2) = 0 \) or \( k = q \), then (19) is trivial. To prove (19), there are four remaining cases to consider.

1. If \( k = q, \text{frac}(\rho_1) = 0.5 \), then \( m(s - r) \) is odd. We claim that \( n \geq m + 2 \). If by contrary \( n = m + 1 \), then \( r(m) = s(m-1) \) and so \( 2s = m(s - r) \). Since \( m(s - r) \) is odd, this is a contradiction.

Now we show that either \( s - r \geq 2 \) or \( n - m \geq 3 \). If by contrary, \( s = r + 1, n = m + 2 \), then \( rm(m + 1) = (r + 1)(m - 1)(m - 2) \) or equivalently, \( m^2 - (4r + 3)m + 2r + 2 = 0 \). Therefore, \( m = \frac{1}{2}(4r + 3 + \sqrt{16r^2 + 16r + 1}) \). Since \( (4r + 1)^2 < 16r^2 + 16r + 1 < (4r + 2)^2 \), \( 16r^2 + 16r + 1 \) is never a perfect square, but \( m \) is an integer, and so we have a contradiction.

Since \( k = q \), we have

\[
\frac{s}{r} \binom{m - 1}{2} = \binom{n - 1}{2} = \binom{m - 1}{2} + \binom{n - m}{2} + (m - 1)(n - m).
\]

Therefore,

\[
\frac{s - 1}{r} \binom{m - 1}{2} = \binom{n - m}{2} + (m - 1)(n - m).
\]

To prove (19), we have

\[
rm \binom{n - m}{2} \geq \binom{m - 1}{2} = \left( \frac{r}{s - r} \right) \left( \binom{n - m}{2} + (m - 1)(n - m) \right) \Longleftrightarrow
\]

\[
[(s - r)m - 1] \binom{n - m}{2} \geq (m - 1)(n - m) \Longleftrightarrow
\]

\[
[(s - r)m - 1] (n - m - 1) \geq 2(m - 1)
\]

which is true because either \( s - r \geq 2 \) or \( n - m \geq 3 \).
If $\frac{\rho_1}{\rho_2} = 0$, $\frac{\rho_2}{\rho_2} = 0.5$, $k > q$, then (19) is equivalent to
\[
\frac{sm}{2} - 1 + (s - r) \frac{m - 1}{2} \geq r(n - m)(m - 1),
\]
which holds because $n - m \geq 3$ (Note that $m$ is odd and so $m \geq 5$).

(3) If $\frac{\rho_1}{\rho_2} = 0$, $\frac{\rho_2}{\rho_2} = 0.5$, $k > q$, then (19) is equivalent to
\[
rm(n - m)(n - m - 1) \geq (m - 1)(n - m - 1)
\]
which holds because $n - m \geq (m + 1)/2$ (Note that $m$ is odd and by (iii) $m \geq 3m/2$).

(4) If $\frac{\rho_1}{\rho_2} = 0$, $\frac{\rho_2}{\rho_2} = 0.5$, $k > q$, then (19) is equivalent to
\[
\frac{sm}{2} - 1 + (s - r) \frac{m - 1}{2} \geq r(n - m)(m - 1)
\]
which holds because $n - m \geq 2$ and $(n - m)(n - m - 1) \geq n - m - 2 - 1$ (Note that $n - m \geq (m + 1)/2$).

To complete the proof of Claim 1, observe that if $\nu_1 \leq 0$, $\nu_2 \leq 0$, then the left hand sides of (18) are clearly satisfied. It remains to consider two cases.

(1) If $\nu_1 \geq 0$, $\nu_2 \geq 0$, we have
\[
\lambda(n - m) \left( \frac{m}{2} \right) \geq \lambda m \left( \frac{n - 1}{2} \right) - \lambda \left( \frac{n}{3} \right) - 2\lambda \left( \frac{m}{3} \right) = q\nu_1 + (k - q)\nu_2.
\]

(2) If $\nu_1 \leq 0$, $\nu_2 \geq 0$, by the necessary condition
\[
\lambda(n - m) \left( \frac{m}{2} \right) \geq (k - q)\nu_2.
\]

Claim 2. We can color the remaining edges of $\mathcal{H}$ such that
\[
\deg_{\mathcal{H}(j)}(x) = \begin{cases} 
  m(s - r) & \text{if } x = u, j \in (0, q], \\
  sm & \text{if } x = u, j \in (q, k], \\
  s(n - m) & \text{if } x = \alpha, j \in (0, k].
\end{cases}
\]

We color the $u\alpha^2$-edges so that
\[
\text{mult}_{\mathcal{H}(j)}(u\alpha^2) = \begin{cases} 
  2(\rho_1 - \text{mult}_{\mathcal{H}(j)}(u^2\alpha)) & \text{for } j \in (0, q], \\
  2(\rho_2 - \text{mult}_{\mathcal{H}(j)}(u^2\alpha)) & \text{for } j \in (q, k].
\end{cases}
\]

By (18) and the following, coloring of $u\alpha^2$-edges is possible.
\[
2\rho_1 q + 2\rho_2 (k - q) - 2\lambda(n - m) \left( \frac{m}{2} \right) = \lambda m \left( \frac{n - m}{2} \right).
\]

Then we color the $\alpha^3$-edges so that
\[
\text{mult}_{\mathcal{H}(j)}(\alpha^3) = \begin{cases} 
  \text{mult}_{\mathcal{H}(j)}(u^2\alpha) - \nu_1 & \text{for } j \in (0, q], \\
  \text{mult}_{\mathcal{H}(j)}(u^2\alpha) - \nu_2 & \text{for } j \in (q, k].
\end{cases}
\]

By (18) and the following, coloring of $\alpha^3$-edges is possible.
\[
\lambda(n - m) \left( \frac{m}{2} \right) - \nu_1 q - \nu_2 (k - q) = \lambda \left( \frac{n - m}{3} \right).
\]
Combining the following with (21) and (22) shows that $H$ satisfies (20), thus completing the proof of Claim 2.

\[
\deg_{H(j)}(x) = \begin{cases} 
2 \mult_{H(j)}(u^2 \alpha) + \mult_{H(j)}(u \alpha^2) & \text{if } x = u, \\
2 \mult_{H(j)}(u \alpha^2) + \mult_{H(j)}(u^2 \alpha) + 3 \mult_{H(j)}(\alpha^3) & \text{if } x = \alpha.
\end{cases}
\]

Since each edge of $H$ contains $\alpha$, $c(H(j)) = 1$ for each $j \in (0, k]$. By Theorem 1.1, there exists a $(u, m)$-detachment $F$ of $H$ obtained by splitting off $u$ in $H$ into $u_1, \ldots, u_m$ in $F$ such that the following conditions hold.

1. By (F1), for each $i \in [m]$,

\[
\deg_{F(j)}(u_i) \approx \frac{\deg_{H(j)}(u)}{m} = \begin{cases} 
\frac{s - r}{m} & \text{for } j \in (0, q], \\
\frac{s}{m} & \text{for } j \in (q, k].
\end{cases}
\]

2. By (F3), for each $U \subseteq \{u_1, \ldots, u_m\}$ with $0 \leq |U| \leq 2$,

\[
\mult_F(U, \alpha^{3-|U|}) \approx \frac{\mult_{H}(u^{|U|}, \alpha^{3-|U|})}{\binom{m}{|U|}} = \frac{\lambda \binom{m}{|U|}}{\binom{\alpha}{n-|U|}} = \lambda \left( \frac{n-m}{3-|U|} \right).
\]

Since all edges of $F$ contain $\alpha$, $c(F(j)) = 1$ for $j \in (0, k]$.

Without loss of generality we may assume that $V(\lambda K^h_m) = \{u_1, \ldots, u_m\}$, and we think of the given $r$-factorization of $\lambda K^3_m$ as a $q$-edge-coloring of $\lambda K^3_m$ so that each color class induces an $r$-factor. Let $G$ be a hypergraph whose vertex set is $V(F) = \{\alpha, u_1, \ldots, u_m\}$, whose edges are $E(\lambda K^3_m) \cup E(F)$, and its edges are colored according to the colors of edges of $\lambda K^3_m$ and $F$. Clearly, $G \cong \lambda K^3_m$, and $G$ contains an $r$-factorization of $\lambda K^3_m$.

**Claim 3.** The edge-coloring of $G$ satisfies

\[
\deg_{G(j)}(v) = \begin{cases} 
\frac{s}{s(n-m)} & \text{if } v \neq \alpha, \\
\frac{s}{s(n-m)} & \text{if } v = \alpha, \\
& \forall j \in [k],
\end{cases}
\]

and

\[
\omega(\alpha, G(j)) \leq (s - 1)(n - m) + 1 \quad \forall j \in [k],
\]

If we prove this claim, then by Corollary 7.4 the proof of Theorem 8.4 will be complete.

It is straightforward to check that (23) holds. Recall that $\omega_{\alpha}(G(j))$ is the number of large $\alpha$-wings in $G(j)$. We have

\[
\omega(\alpha, G(j)) = \omega_{\alpha}(G(j)) + \mult_{G(j)}(\alpha^3)
\]

\[
= \omega_{\alpha}(G(j)) + \mult_{H(j)}(u^2 \alpha) - \begin{cases} 
\lambda \frac{\alpha}{h} & \text{for } j \in (0, q], \\
0 & \text{for } j \in (q, k].
\end{cases}
\]

Therefore, (24) is equivalent to

\[
\mult_{H(j)}(u^2 \alpha) + \omega_{\alpha}(G(j)) \leq \frac{2sn}{3} - n + m + 1 - \begin{cases} 
\frac{2rm}{3} & \text{for } j \in (0, q], \\
0 & \text{for } j \in (q, k].
\end{cases}
\]
By (18) it suffices to show that
\[
\omega^L_n(\mathcal{G}(j)) \leq \frac{2sn}{3} - n + m + 1 - \begin{cases} 
\frac{2rm}{3} + \rho_1 & \text{for } j \in (0, q], \\
\rho_2 & \text{for } j \in (q, k]. 
\end{cases}
\]

\[
= \begin{cases} 
\left\{ n\left(\frac{2s}{3} - 1\right) + m\left(1 - \frac{s}{2} - \frac{r}{6}\right) + 1 \right. & \text{for } j \in (0, q], \\
\left\{ n\left(\frac{2s}{3} - 1\right) + m\left(1 - \frac{s}{2}\right) + 1 \right. & \text{for } j \in (q, k]. 
\end{cases}
\]

For \( j \in (0, q] \), we have \( c(\lambda K_m^3(j)) \leq \frac{m}{3} \), and for \( j \in (q, k] \), we have \( c(\lambda K_m^3(j)) = m \). Since \( \omega^L_n(\mathcal{G}(j)) \leq c(\lambda K_m^3(j)) \), we have

\[
(26) \quad \omega^L_n(\mathcal{G}(j)) \leq \begin{cases} 
m/3 & \text{for } j \in (0, q], \\
m & \text{for } j \in (q, k]. 
\end{cases}
\]

Therefore, to prove (24) it suffices to show that
\[
m/3 \leq n\left(\frac{2s}{3} - 1\right) + m\left(1 - \frac{s}{2} - \frac{r}{6}\right) + 1 \quad \text{for } j \in (0, q],
\]

and
\[
m \leq n\left(\frac{2s}{3} - 1\right) + m\left(1 - \frac{s}{2}\right) + 1 \quad \text{for } j \in (q, k].
\]

Equivalently, it suffices to show that
\[
n\left(\frac{2s}{3} - 1\right) \geq \begin{cases} 
m\left(\frac{s}{2} + \frac{r}{6} - \frac{2}{3}\right) - 1 & \text{for } j \in (0, q], \\
m\left(\frac{s}{2s - 3}\right) - 1 & \text{for } j \in (q, k]. 
\end{cases}
\]

This can be simplified to

\[
(27) \quad n \geq -\frac{3}{2s - 3} + \left\{ \left(\frac{3s + r - 4}{2(2s - 3)}\right)m \quad \text{for } j \in (0, q], \\
\left(\frac{3s}{2(2s - 3)}\right)m \quad \text{for } j \in (q, k]. 
\end{cases}
\]

Since \( r \geq 1 \) and \( s \geq r + 1, \frac{3s + r - 4}{2(2s - 3)} \leq 3/2 \). Moreover, if \( s \geq 3 \), \( \frac{3s}{2(2s - 3)} \leq 3/2 \).

Recall that for \( k > q \), we have that \( n \geq 3m/2 \). Thus, to complete the proof of (24) for \( k > q \), it remains to consider the case \( j \in (q, k], r = 1, s = 2 \). Let us fix \( j \in (q, k] \). We need to show that

\[
(28) \quad \omega^L_n(\mathcal{G}(j)) + \text{mult}_{\mathcal{H}(j)}(u^2\alpha) \leq \frac{n}{3} + m + 1.
\]

Define a graph \( G = (V(\mathcal{G})\setminus\{\alpha\}, E) \) where \( uv \) is an edge in \( G \) if \( uv\alpha \) is an edge in \( \mathcal{G}(j) \). We have
\[
|V(G)| = m, \quad \Delta(G) \leq 2, \quad |E| = \text{mult}_{\mathcal{H}(j)}(u^2\alpha), \quad c(G) = \omega^L_n(\mathcal{G}(j)).
\]

Thus, (28) is equivalent to

\[
(29) \quad c(G) + |E| \leq \frac{n}{3} + |V(G)| + 1.
\]
Using the fair detachment we can guarantee that \( G \) is simple: by Theorem 1.1 (F4), for \( u, v \in V(G) \setminus \{\alpha\} \),
\[
\text{mult}_{g(j)}(uv\alpha) = \text{mult}_{f(j)}(uv\alpha) \approx \frac{\text{mult}_{H(j)}(u^2\alpha)}{\binom{m}{2}} \leq \frac{m}{\binom{m}{2}} < 1.
\]

Since \( \Delta(G) \leq 2 \), each component of \( G \) is a path (possibly of length zero) or a cycle. Therefore, \( c(G) = p + q \), where \( p \) and \( q \) are the number of paths and cycles in \( G \), respectively. Since \( G \) is simple, \( q \leq |E|/3 = \text{mult}_{H(j)}(u^2\alpha)/3 \leq m/3 < n/3 + 1 \). Since a cycle has the same number of vertices and edges, \(|V(G)| - |E| = p\). We have
\[
c(G) - |V(G)| + |E| = q \leq \frac{n}{3} + 1,
\]
This proves (29), and the proof of (24) for \( k > q \) is complete.

Now, we consider the case when \( k = q \). If \( n = m + 1 \), then (24) simplifies to \( \omega_\alpha(H(j)) \leq s \) which is trivial because \( \text{deg}_{H(i)}(\alpha) = s \) and \( \text{deg}_{H(i)}(\alpha) \geq \omega_\alpha(H(j)) \). Earlier, we showed that if \( k = q \) and \( n \geq m + 2 \), then either \( s - r \geq 2 \) or \( n - m \geq 3 \). Since \( \frac{3s+4}{2(2s-3)} \leq 3/2 \), and for \( s \geq r + 2 \), we have \( \frac{3s+r-4}{2(2s-3)} \leq 1 \), (27) holds when either \( n \geq 3m/2 \) or \( s \geq r + 2 \).

Therefore, to prove (24), it remains to solve the case when \( k = q, s = r + 1, m + 3 \leq n < 3m/2 \). By (27), it suffices to show that
\[
n \geq \left( \frac{4r - 1}{2(2r - 1)} \right) m - \frac{3}{2r - 1},
\]
or, equivalently,
\[
r \geq \frac{m + 2a - 6}{4a},
\]
where \( a := n - m \).

First, we show that \( r \geq 2 \). If by contrary \( r = 1 \), we have \( 3|m, 3|n, (n-1) = 2(m-1) \).
Therefore, \( (n-1)(n-2) = 2(m-1)(m-2) \), and so \( 2 \equiv 4 \pmod{3} \), which is a contradiction.

Since \( k = q \) and \( s = r + 1 \), we have \( r^{(m+\alpha-1)} = (r + 1)^{(m-1)} \), and so
\[
r = \frac{m^2 - 3m + 2}{(n - m)(n - m - 3)}.
\]
Combining \( r \geq 2 \) and (32) we have
\[
m \geq \frac{1}{2}(\sqrt{24a^2 + 1} + 4a + 3) > (2 + \sqrt{6})a.
\]
If we show that (31) holds, then we are done. Suppose by contrary that \( r < \frac{m + 2a - 6}{4a} \).
Combining this with (32) we have
\[
m < \frac{1}{4}(\sqrt{41a^2 - 126a + 89} + 5a - 3) < \frac{1}{4}(5 + \sqrt{11})a.
\]
Thus, \( (2 + \sqrt{6})a < \frac{1}{4}(5 + \sqrt{11})a \) which is a contradiction.

This completes the proof of (24) for \( k = q \).

\[\square\]

**Remark 8.5.** Let \( \lambda = 1, m = 5, n = 9, r = 6, s = 7 \). Then conditions (i)-(iii) of Theorem 8.3 are satisfied, but condition (iv) does not hold. This shows that we can not eliminate the necessary condition (iv) from Theorem 8.3.
Theorem 8.6. For \( n \geq 4m \), any \( r \)-factorization of \( \lambda K^4_m \) can be extended to a connected \( s \)-factorization of \( \lambda K^4_n \) if and only if

(i) \((m, r, \lambda)\) and \((n, s, \lambda)\) are 4-admissible;
(ii) \( 1 < s/r \leq \binom{n-1}{3}/\binom{m-1}{3} \).

Proof. The necessity of (i) and (ii) is obvious by Lemma 8.1. To prove the sufficiency, suppose that an \( r \)-factorization of \( G := \lambda K^4_m \) is given, and that (i) and (ii) hold. Let \( H \) be a hypergraph with vertex set \( \{u, \alpha\} \) such that \( \text{mult}_H(u^4\alpha^i) = \lambda\binom{m}{i}\binom{n-m}{4-i} \) for \( i \in [0, 3] \). Let \( q = \lambda\binom{m-1}{3}/r, k = \lambda\binom{n-1}{3}/s \). By (i), both \( q \) and \( k \) are positive integers, and by (ii), \( k \geq q \).

Claim 1. We can color the \( u^3\alpha \)-edges and \( u^2\alpha^2 \)-edges (in that particular order) such that

\[
\text{mult}_{H(j)}(u^3\alpha) \leq \begin{cases} \rho_1 & \text{for } j \in (0, q], \\ \rho_2 & \text{for } j \in (q, k], \end{cases}
\]

and

\[
\text{mult}_{H(j)}(u^2\alpha^2) \leq \begin{cases} \rho_{1j} & \text{for } j \in (0, q], \\ \rho_{2j} & \text{for } j \in (q, k], \end{cases}
\]

where

\[
\begin{align*}
\rho_1 &:= \frac{sm}{3} - \frac{rm}{3} \\
\rho_2 &:= \frac{sm}{3} \\
\rho_{1j} &:= \frac{sm}{2} - \frac{3}{2} \text{mult}_{H(j)}(u^3\alpha) - \frac{rm}{2}, \quad j \in (0, q] \\
\rho_{2j} &:= \frac{sm}{2} - \frac{3}{2} \text{mult}_{H(j)}(u^3\alpha), \quad j \in (q, k].
\end{align*}
\]

Observe that \( \rho_{1j} = \frac{3}{2}(\rho_1 - \text{mult}_{H(j)}(u^3\alpha)) \) for \( j \in (0, q] \), and \( \rho_{2j} = \frac{3}{2}(\rho_2 - \text{mult}_{H(j)}(u^3\alpha)) \) for \( j \in (q, k] \). To prove that such a coloring is possible, we need to show that

\[
\lambda(n-m)\binom{m}{3} \leq q[\rho_1] + (k-q)[\rho_2],
\]

and

\[
\lambda\binom{m}{2}\binom{n-m}{2} \leq \sum_{j\in(0,q]}[\rho_{1j}] + \sum_{j\in(q,k]}[\rho_{2j}].
\]

We have

\[
\frac{3}{\lambda} (q[\rho_1] + (k-q)[\rho_2]) \geq \frac{q}{\lambda} (m(s-r) - 2) + \frac{k-q}{\lambda} (sm-2) \\
\geq m \left[ \left( \binom{n-1}{3} - \binom{m-1}{3} \right) - 2 \binom{n-1}{3} \right].
\]

Therefore, to prove (35), it suffices to show that

\[
(m-2)\binom{n-1}{3} - m\binom{m-1}{3} \geq 3(n-m)\binom{m}{3}.
\]
Since \( m \geq 5, n \geq 4m \), we have
\[
\frac{6}{m - 2} \left[ (m - 2) \binom{n - 1}{3} - m \binom{m - 1}{3} - 3(n - m) \binom{m}{3} \right] \\
= n^3 - 6m^2 - 3m^2n + 3mn + 11n + 2m^3 + m^2 - 3m - 6 \\
= n\left(n(n - 6) - 3m^2 + 3m + 11\right) + m(m - 3) + (2m^3 - 6) \\
\geq n(13m^2 - 21m + 11) > 0.
\]
We have
\[
\frac{2}{\lambda} \left( \sum_{j \in \{0, q\}} |\rho_{1j}| + \sum_{j \in \{q, k\}} |\rho_{2j}| \right) \\
= \frac{q}{\lambda}(m(s - r) - 1) + \frac{k - q}{\lambda}(ms - 1) - 3(n - m) \binom{m}{3} \\
\geq (m - 1) \binom{n - 1}{3} - m \binom{m - 1}{3} - 3(n - m) \binom{m}{3}.
\]
So to prove (36), it is enough to show that
\[
(m - 1) \binom{n - 1}{3} - m \binom{m - 1}{3} - 3(n - m) \binom{m}{3} \geq 2 \binom{m}{2} \binom{n - m}{2}.
\]
We have
\[
\frac{6}{m - 1} \left[ (m - 1) \binom{n - 1}{3} - m \binom{m - 1}{3} - 3(n - m) \binom{m}{3} - 2 \binom{m}{2} \binom{n - m}{2} \right] \\
= n^3 - 3mn^2 - 6m^2 + 3m^2n - n^2 - 4m^2 + 9mn - 6m + 11n - 6 \\
> n^2(n - 3m - 6) + m(3mn - m^2 + 9n - 4m - 6) \\
\geq n^2(m - 6) + m(11m^2 + 32m - 6) > 0.
\]
This completes the proof of Claim 1.

**Claim 2.** We can color the remaining edges of \( \mathcal{H} \) such that
\[
\deg_{\mathcal{H}(\alpha)}(x) = \begin{cases} 
  m(s - r) & \text{if } x = u, j \in (0, q], \\
  sm & \text{if } x = u, j \in (q, k], \\
  s(n - m) & \text{if } x = \alpha, j \in (0, k].
\end{cases}
\]
We color the \( u\alpha^3 \)-edges such that
\[
\lambda(\mathcal{H}(\alpha^3)) = \begin{cases} 
  2\rho_{1j} - 2 \lambda(\mathcal{H}(u^2\alpha^2)) & \text{for } j \in (0, q], \\
  2\rho_{2j} - 2 \lambda(\mathcal{H}(u^2\alpha^2)) & \text{for } j \in (q, k].
\end{cases}
\]
The following shows that this is possible.
\[
\sum_{j \in \{0, q\}} \left( 2\rho_{1j} - 2 \lambda(\mathcal{H}(u^2\alpha^2)) \right) + \sum_{j \in \{q, k\}} \left( 2\rho_{2j} - 2 \lambda(\mathcal{H}(u^2\alpha^2)) \right) = \lambda \binom{n - m}{3}.
\]
We color the \( \alpha^4 \)-edges such that
\[
\lambda(\mathcal{H}(\alpha^4)) = \begin{cases} 
  2 \lambda(\mathcal{H}(u^3\alpha)) + \lambda(\mathcal{H}(u^2\alpha^2)) + sn/4 - sm + 3rm/4 & \text{for } j \in (0, q], \\
  2 \lambda(\mathcal{H}(u^3\alpha)) + \lambda(\mathcal{H}(u^2\alpha^2)) + sn/4 - sm & \text{for } j \in (q, k].
\end{cases}
\]
The following confirms that this is possible.

\[
\sum_{j \in [0,q]} \left( 2 \mult_{H(j)}(u^3\alpha) + \mult_{H(j)}(u^2\alpha^2) + sn/4 - sm + 3rm/4 \right) \\
+ \sum_{j \in (q,k]} \left( 2 \mult_{H(j)}(u^3\alpha) + \mult_{H(j)}(u^2\alpha^2) + sn/4 - sm \right) = \lambda \left( \frac{n-m}{4} \right).
\]

Combining the following with (38) and (39) shows that $H$ satisfies (37), thus completing the proof of Claim 2.

\[
\begin{aligned}
\deg_{H(j)}(u) &= 3 \mult_{H(j)}(u^3\alpha) + 2 \mult_{H(j)}(u^2\alpha^2) + \mult_{H(j)}(u\alpha^3), \\
\deg_{H(j)}(\alpha) &= \mult_{H(j)}(u^3\alpha) + 2 \mult_{H(j)}(u^2\alpha^2) + 3 \mult_{H(j)}(u\alpha^3) + 4 \mult_{H(j)}(\alpha^4).
\end{aligned}
\]

All edges of $H$ contain $\alpha$, and so $c(H(j)) = 1$ for all $j \in (0,k]$. By Theorem 1.1, there exists a $(u,m)$-detachment $F$ of $H$ obtained by splitting off $u$ in $H$ into $u_1, \ldots, u_m$ in $F$ such that the following conditions hold.

1. By (F1), for each $i \in [m],$

\[
\deg_{F(j)}(u_i) = \deg_{H(j)}(u) \approx \frac{\deg_{H(j)}(u)}{m} = \begin{cases} s - r & \text{for } j \in (0,q), \\
 s & \text{for } j \in (q,k]. \end{cases}
\]

2. By (F3), for each $U \subseteq \{u_1, \ldots, u_m\}$ with $0 \leq |U| \leq 3$

\[
\mult_{F}(U, \alpha^{4-|U|}) \approx \mult_{H}(u^{\binom{|U|}{2}}, \alpha^{4-|U|}) = \lambda \left( \frac{m}{|U|} \right)^{\binom{n-m}{4-|U|}} = \lambda \left( \frac{n-m}{4 - |U|} \right).
\]

Since all edges of $F$ contain $\alpha$, $c(F(j)) = 1$ for $j \in (0,k]$.

Let $V(\lambda K^4_m) = \{u_1, \ldots, u_m\}$, and think of the given $r$-factorization of $\lambda K^4_m$ as a $q$-edge-coloring of $\lambda K^4_m$ so that each color class induces an $r$-factor. Let $G$ be a hypergraph whose vertex set is $V(F) = \{\alpha, u_1, \ldots, u_m\}$, whose edges are $E(\lambda K^4_m) \cup E(F)$, and its edges are colored according to the colors of edges of $\lambda K^4_m$ and $F$. Clearly, $G \cong \lambda K^4_m$, and $G$ contains an $r$-factorization of $\lambda K^4_m$.

**Claim 3.** The edge-coloring of $G$ satisfies

\[
\deg_{G(j)}(v) = \begin{cases} s & \text{if } v \neq \alpha, \\
 s(n-m) & \text{if } v = \alpha, \end{cases} \quad \forall j \in [k],
\]

and

\[
\omega_{\alpha}(G(j)) \leq (s-1)(n-m) + 1 \quad \forall j \in [k],
\]

Once we prove this claim, applying Corollary 7.4 will complete the proof of Theorem 8.6. The proof of (40) is straightforward.
By (39), (33), and (34), for $j \in (0, q]$, 
\[
\omega_{\alpha}(G(j)) = \text{mult}_{G(j)}(\alpha^4) + \omega_{\alpha}^L(G(j)) \\
\leq \text{mult}_{G(j)}(\alpha^4) + c(\lambda K_m^4(j)) \\
\leq \text{mult}_{G(j)}(\alpha^4) + \frac{m}{4} \\
= 2 \text{mult}_{H(j)}(u^3 \alpha) + \text{mult}_{H(j)}(u^2 \alpha^2) + \frac{sn}{4} - sm + \frac{3rm}{4} + \frac{m}{4} \\
\leq 2 \text{mult}_{H(j)}(u^3 \alpha) + \rho_1j + \frac{sn}{4} + \frac{3sm}{2} + \frac{rm}{2} + \frac{m}{4} \\
= \frac{\rho_1}{2} + \frac{sn}{4} - \frac{sm}{2} + \frac{rm}{2} + \frac{m}{4} \\
= \frac{sn}{4} - \frac{sm}{3} - \frac{rm}{12} + \frac{m}{4},
\]
and for $j \in (q, k]$, 
\[
\omega_{\alpha}(G(j)) = \text{mult}_{G(j)}(\alpha^4) + \omega_{\alpha}^L(G(j)) \\
\leq \text{mult}_{G(j)}(\alpha^4) + c(\lambda K_m^4(j)) \\
\leq \text{mult}_{G(j)}(\alpha^4) + m \\
= 2 \text{mult}_{H(j)}(u^3 \alpha) + \text{mult}_{H(j)}(u^2 \alpha^2) + \frac{sn}{4} - sm + m \\
\leq 2 \text{mult}_{H(j)}(u^3 \alpha) + \rho_2j + \frac{sn}{4} - sm + m \\
= \frac{\rho_2}{2} + \frac{sn}{4} - \frac{sm}{2} + m \\
= \frac{sn}{4} - \frac{sm}{3} + m.
\]
Since $sn/4 - sm/3 - rm/12 + m/4 \leq sn/4 - sm/3 + m$, to prove (41), it is enough to show that $sn/4 - sm/3 + m \leq (s - 1)(n - m)$, or equivalently, that $3n(3s - 4) - 8sm \geq 0$. Since $n \geq 4m, s \geq 2$, we have $3n(3s - 4) - 8sm \geq 4m(7s - 12) > 0$. $\square$

**Theorem 8.7.** For $n \geq 5m$, any $r$-factorization of $\lambda K_m^5$ can be extended to a connected $s$-factorization of $\lambda K_n^5$ if and only if

(i) $(m, r, \lambda)$ and $(n, s, \lambda)$ are 5-admissible;

(ii) $1 < s/r \leq \binom{n-1}{m-4}/\binom{m-1}{4}$.

**Proof.** The necessity of (i) and (ii) is obvious by Lemma 8.1. To prove the sufficiency, suppose that an $r$-factorization of $G := \lambda K_m^5$ is given, and that (i) and (ii) hold. Let $H$ be a hypergraph with vertex set $\{u, \alpha\}$ such that $\text{mult}_H(u^i \alpha^{5-i}) = \lambda(n_m^i)/\binom{m}{5-i}$ for $i \in [0, 4]$. Let $q = \lambda(m-1)/r, k = \lambda(n-1)/s$. By (i), both $q$ and $k$ are positive integers, and by (ii), $k \geq q$. 
Claim 1. We can color the the $u^4\alpha$-edges, $u^3\alpha^2$-edges, and $u^2\alpha^3$-edges (in that particular order) such that

\begin{equation}
\operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha) \leq \begin{cases} 
\rho_1 & \text{for } j \in (0, q], \\
\rho_2 & \text{for } j \in (q, k], 
\end{cases}
\end{equation}

\begin{equation}
\operatorname{mult}_{\mathcal{H}(j)}(u^3\alpha^2) \leq \begin{cases} 
\rho_{1j} & \text{for } j \in (0, q], \\
\rho_{2j} & \text{for } j \in (q, k], 
\end{cases}
\end{equation}

and

\begin{equation}
\operatorname{mult}_{\mathcal{H}(j)}(u^2\alpha^3) \leq \begin{cases} 
\rho'_{1j} & \text{for } j \in (0, q], \\
\rho'_{2j} & \text{for } j \in (q, k], 
\end{cases}
\end{equation}

where

\begin{align*}
\rho_1 &= \frac{sm}{4} - \frac{rm}{4}, \\
\rho_2 &= \frac{sm}{4}, \\
\rho_{1j} &= \frac{sm}{3} - \frac{4}{3}\operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha) - \frac{rm}{3}, & j \in (0, q] \\
\rho_{2j} &= \frac{sm}{3} - \frac{4}{3}\operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha), & j \in (q, k] \\
\rho'_{1j} &= \frac{sm}{2} - 2\operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha) - \frac{3}{2}\operatorname{mult}_{\mathcal{H}(j)}(u^3\alpha^2) - \frac{rm}{2}, & j \in (0, q] \\
\rho'_{2j} &= \frac{sm}{2} - 2\operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha) - \frac{3}{2}\operatorname{mult}_{\mathcal{H}(j)}(u^3\alpha^2), & j \in (q, k]
\end{align*}

Observe that $\rho_{1j} = \frac{4}{3}(\rho_1 - \operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha))$, $\rho'_{1j} = \frac{3}{2}(\rho_{1j} - \operatorname{mult}_{\mathcal{H}(j)}(u^3\alpha^2))$ for $j \in (0, q]$, and $\rho_{2j} = \frac{4}{3}(\rho_2 - \operatorname{mult}_{\mathcal{H}(j)}(u^4\alpha))$, $\rho'_{2j} = \frac{3}{2}(\rho_{2j} - \operatorname{mult}_{\mathcal{H}(j)}(u^3\alpha^2))$ for $j \in (q, k]$. To prove that such a coloring is possible, we need to show that

\begin{equation}
\lambda(n - m) \left(\binom{m}{4}\right) \leq q[\rho_1] + (k - q)[\rho_2],
\end{equation}

\begin{equation}
\lambda \left(\binom{m}{3}\right) \left(\binom{n - m}{2}\right) \leq \sum_{j \in (0, q]} [\rho_{1j}] + \sum_{j \in (q, k]} [\rho_{2j}],
\end{equation}

and

\begin{equation}
\lambda \left(\binom{m}{2}\right) \left(\binom{n - m}{3}\right) \leq \sum_{j \in (0, q]} [\rho'_{1j}] + \sum_{j \in (q, k]} [\rho'_{2j}].
\end{equation}

We have

\[
\frac{4}{\lambda} (q[\rho_1] + (k - q)[\rho_2]) \geq q \lambda (m(s - r) - 3) + \frac{k - q}{\lambda} (sm - 3)
\]

\[
\geq m \left[ \left( \binom{n - 1}{4} - \binom{m - 1}{4} \right) - 3 \left( \binom{n - 1}{3} \right) \right].
\]

Therefore, to prove (45), it suffices to show that

\[
(m - 3) \left( \binom{n - 1}{4} - m \binom{m - 1}{4} \right) \geq 4(n - m) \left( \binom{m}{4} \right).
\]
Since \( m \geq 6, n \geq 5m \), we have
\[
\frac{24}{m-3} \left[ (m-3) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) \right]
= n^4 - 10n^3 + 35n^2 - 4m^3n + 12m^2n - 8mn - 50n + 3m^4 - 5m^3 - 6m^2 + 8m + 24
= n \left( n^2(n-10) + 35n - 4m^3 + 12m^2 - 8m - 50 \right) + m(3m^3 - 5m^2 - 6m + 8) + 24
> 25m^2(5m - 10) + 35(5m) - 4m^3 + 12m^2 - 8m - 50
= m^2(121m - 238) + 167m - 50 > 0.
\]

We have
\[
\frac{3}{\lambda} \left( \sum_{j \in \{0,q\}} |\rho_{1j}| + \sum_{j \in \{q,k\}} |\rho_{2j}| \right)
\geq \frac{q}{\lambda} \left( m(s-r) - 2 \right) + \frac{k-q}{\lambda} (ms - 2) - 4(n-m) \left( \frac{m}{4} \right)
\geq (m-2) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right).
\]

So to prove (46), it is enough to show that
\[
(m-2) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) \geq 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right).
\]

We have
\[
\frac{24}{m-2} \left[ (m-2) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) - 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right) \right]
= n^4 - 10n^3 + 6m^2n^2 + 35n^2 - 50n + 6mn^2 + 8m^3n
- 18mn + 10m^2n - 3m^4 - 8m^3 - m^2 + 12m + 24
> n^2 \left( n(n-10) - 6m^2 \right) + n(35n - 50 + 6mn + 8m^3 - 18m + 10m^2) - 3m^4 - 8m^3 - m^2
> (19m^2 - 50m) + 5m(8m^3 + 40m^2 + 157m - 50) - 3m^4 - 8m^3 - m^2 > 0.
\]

We have
\[
\frac{2}{\lambda} \left( \sum_{j \in \{0,q\}} |\rho_{1j}'| + \sum_{j \in \{q,k\}} |\rho_{2j}'| \right)
\geq \frac{q}{\lambda} \left( m(s-r) - 1 \right) + \frac{k-q}{\lambda} (ms - 1) - 4(n-m) \left( \frac{m}{4} \right) - 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right)
\geq (m-1) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) - 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right).
\]

So to prove (47), it is enough to show that
\[
(m-1) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) - 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right) \geq 2 \left( \frac{m}{2} \right) \left( \frac{n-m}{3} \right).
\]
For $m \geq 10$, we have
\[
\frac{24}{m-1} \left[ (m-1) \left( \frac{n-1}{4} \right) - m \left( \frac{m-1}{4} \right) - 4(n-m) \left( \frac{m}{4} \right) - 3 \left( \frac{m}{3} \right) \left( \frac{n-m}{2} \right) - 2 \left( \frac{m}{2} \right) \left( \frac{n-m}{3} \right) \right]
\]
\[
= n^3(n-4m-10) + n(n(6m^2+24m+35) - 4m^3 - 22m^2 - 44m - 50)
\]
\[
+ m^4 + 7m^3 + 18m^2 + 24m + 24
\]
\[
> n^3(m-10) + n(26m^3 + 98m^2 + 121m - 50) > 0.
\]

Using elementary calculus, the inequality can be verified for $m \in \{6,7,8,9\}$, and we shall skip the details here. This completes the proof of Claim 1.

**Claim 2.** We can color the remaining edges of $\mathcal{H}$ such that
\[
\deg_{\mathcal{H}(j)}(x) = \begin{cases} 
 m(s-r) & \text{if } x = u,j \in (0,q], \\
 sm & \text{if } x = u,j \in (q,k], \\
 s(n-m) & \text{if } x = \alpha,j \in (0,k]. 
\end{cases}
\]

We color the $u\alpha^4$-edges such that
\[
\text{mult}_{\mathcal{H}(j)}(u\alpha^4) = \begin{cases} 
 2p'_{1j} - 2\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) & \text{for } j \in (0,q], \\
 2p'_{2j} - 2\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) & \text{for } j \in (q,k]. 
\end{cases}
\]

The following shows that this is possible.
\[
\sum_{j \in (0,q]} \left( 2p'_{1j} - 2\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) \right) + \sum_{j \in (q,k]} \left( 2p'_{2j} - 2\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) \right) = \lambda m \left( \frac{n-m}{4} \right).
\]

We color the $\alpha^5$-edges such that for $j \in [k]$
\[
\text{mult}_{\mathcal{H}(j)}(\alpha^5) = 3\text{mult}_{\mathcal{H}(j)}(u^4\alpha) + 2\text{mult}_{\mathcal{H}(j)}(u^3\alpha^2) + \text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) + \frac{sn}{5} - sm + \gamma_j,
\]
where $\gamma_j = 4rm/5$ if $j \in (0,q], \gamma_j = 0$ if $j \in (q,k]$. The following confirms that this is possible.
\[
\sum_{j \in (0,q]} \left( 3\text{mult}_{\mathcal{H}(j)}(u^4\alpha) + 2\text{mult}_{\mathcal{H}(j)}(u^3\alpha^2) + \text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) + \frac{sn}{5} - sm + \frac{4rm}{5} \right)
\]
\[
+ \sum_{j \in (q,k]} \left( 3\text{mult}_{\mathcal{H}(j)}(u^4\alpha) + 2\text{mult}_{\mathcal{H}(j)}(u^3\alpha^2) + \text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) + \frac{sn}{5} - sm \right) = \lambda \left( \frac{n-m}{5} \right).
\]

Combining the following with (49) and (50) shows that $\mathcal{H}$ satisfies (48), thus completing the proof of Claim 2.

\[
\deg_{\mathcal{H}(j)}(u) = 4\text{mult}_{\mathcal{H}(j)}(u^4\alpha) + 3\text{mult}_{\mathcal{H}(j)}(u^3\alpha^2) + 2\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) + \text{mult}_{\mathcal{H}(j)}(u\alpha^4),
\]
\[
\deg_{\mathcal{H}(j)}(\alpha) = \text{mult}_{\mathcal{H}(j)}(u^4\alpha) + 2\text{mult}_{\mathcal{H}(j)}(u^3\alpha^2) + 3\text{mult}_{\mathcal{H}(j)}(u^2\alpha^3) + 4\text{mult}_{\mathcal{H}(j)}(u\alpha^4)
\]
\[
+ 5\text{mult}_{\mathcal{H}(j)}(\alpha^5).
\]

All edges of $\mathcal{H}$ contain $\alpha$, and so $c(\mathcal{H}(j)) = 1$ for all $j \in (0,k]$. By Theorem 1.1, there exists a $(u,m)$-detachment $\mathcal{F}$ of $\mathcal{H}$ obtained by splitting off $u$ in $\mathcal{H}$ into $u_1, \ldots, u_m$ in $\mathcal{F}$ such that the following conditions hold.

(1) By (F1), for each $i \in [m],$
\[
\deg_{\mathcal{F}(j)}(u_i) \approx \frac{\deg_{\mathcal{H}(j)}(u)}{m} = \begin{cases} 
 s-r & \text{for } j \in (0,q], \\
 s & \text{for } j \in (q,k]. 
\end{cases}
\]
(2) By (F3), for each $U \subseteq \{u_1, \ldots, u_m\}$ with $0 \leq |U| \leq 4$
\[
\mult_F(U, \alpha^{5-|U|}) \approx \frac{\mult_H(u^{|U|}, \alpha^{5-|U|})}{\binom{m}{|U|}} = \lambda\left(\frac{m}{|U|}\right)^{n-m} = \lambda\left(\frac{n-m}{5-|U|}\right).
\]
Since all edges of $F$ contain $\alpha$, $c(F(j)) = 1$ for $j \in (0, k]$.
Let $V(\lambda K^5_m) = \{u_1, \ldots, u_m\}$, and think of the given $r$-factorization of $\lambda K^5_m$ as a $q$-edge-coloring of $\lambda K^5_m$ so that each color class induces an $r$-factor. Let $G$ be a hypergraph whose vertex set is $V(F) = \{\alpha, u_1, \ldots, u_m\}$, whose edges are $E(\lambda K^5_m) \cup E(F)$, and its edges are colored according to the colors of edges of $\lambda K^5_m$ and $F$. Clearly, $G \cong \lambda K^5_m$, and $G$ contains an $r$-factorization of $\lambda K^5_m$.

**Claim 3.** The edge-coloring of $G$ satisfies
\[
\deg_{G(j)}(v) = \begin{cases} 
\frac{s}{s(n-m)} & \text{if } v \neq \alpha, \\
\frac{s}{s(n-m)} & \text{if } v = \alpha, \
\end{cases} \quad \forall j \in [k],
\]
and
\[
\omega_{\alpha}(G(j)) \leq (s-1)(n-m) + 1 \quad \forall j \in [k],
\]
Once we prove this claim, applying Corollary 7.4 will complete the proof of Theorem 8.7.

The proof of (51) is straightforward.
By (50), (42)–(44), for $j \in (0, q]$,
\[
\omega_{\alpha}(G(j)) = \mult_{G(j)}(\alpha^5) + \omega_{\alpha}^{F}(G(j))
\leq \mult_{G(j)}(\alpha^5) + c(\lambda K^5_m(j))
\leq \mult_{G(j)}(\alpha^5) + \frac{m}{5}
= 3 \mult_{H(j)}(u^4\alpha) + 2 \mult_{H(j)}(u^3\alpha^2) + \mult_{H(j)}(u^2\alpha^3) + \frac{sn}{5} - sm + \frac{4rm}{5} + \frac{m}{5}
\leq \mult_{H(j)}(u^4\alpha) + \frac{1}{2} \mult_{H(j)}(u^3\alpha^2) + \frac{sn}{5} - sm - \frac{3rm}{10} + \frac{m}{5}
\leq \frac{1}{3} \mult_{H(j)}(u^4\alpha) + \frac{sn}{5} - sm - \frac{7rm}{15} + \frac{m}{5}
\leq \frac{sn}{5} - sm - \frac{11rm}{20} + \frac{m}{5},
\]
and for $j \in (q, k]$,
\[
\omega_{\alpha}(G(j)) = \mult_{G(j)}(\alpha^5) + \omega_{\alpha}^{F}(G(j))
\leq \mult_{G(j)}(\alpha^4) + c(\lambda K^5_m(j))
\leq \mult_{G(j)}(\alpha^5) + m
= 3 \mult_{H(j)}(u^4\alpha) + 2 \mult_{H(j)}(u^3\alpha^2) + \mult_{H(j)}(u^2\alpha^3) + \frac{sn}{5} - sm + m
\leq \mult_{H(j)}(u^4\alpha) + \frac{1}{2} \mult_{H(j)}(u^3\alpha^2) + \frac{sn}{5} - sm - m
\leq \frac{1}{3} \mult_{H(j)}(u^4\alpha) + \frac{sn}{5} - sm + m
\leq \frac{sn}{5} - sm + \frac{m}{3} + m
\leq \frac{sn}{5} - sm + \frac{m}{3} + m.
Since \( sn/5 - sm/4 - 11rm/20 + m/5 \leq sn/5 - sm/4 + m \), to prove (52), it is enough to show that \( sn/5 - sm/4 + m \leq (s - 1)(n - m) \), or equivalently, that \( 4n(4s - 5) - 15sm \geq 0 \). Since \( n \geq 5m, s \geq 2 \), we have \( 4n(4s - 5) - 15sm \geq 5m(13s - 20) > 0 \). \( \square \)

9. Connected Factorizations of (Non-uniform) Complete Hypergraphs

For positive integer column vectors \( \Lambda := [\lambda_1 \ldots \lambda_m]^T \) and \( H := [h_1 \ldots h_m]^T \) with \( 1 \leq h_1 < h_2 < \cdots < h_m \leq n \), let \( \Lambda K_n^H \) denote an \( n \)-vertex hypergraph such that there are \( \lambda_i \) edges of size \( h_i \) incident with every \( h_i \) vertices for \( i \in [m] \). In other words,

\[
\Lambda K_n^H = \bigcup_{i \in [m]} \lambda_i K_n^{h_i}.
\]

For a hypergraph \( G \) and a positive integer column vector \( R := [r_1 \ldots r_k]^T \), an \( R \)-factorization of \( G \) is a partition (decomposition) \( \{ F_1, \ldots, F_k \} \) of \( E(G) \) in which \( F_i \) is an \( r_i \)-factor for \( i \in [k] \).

**Theorem 9.1.** \( \Lambda K_n^H \) is \( R \)-factorable if and only if

\[
\sum_{i \in [m]} \lambda_i \left( \frac{n - 1}{h_i - 1} \right) = \sum_{i \in [k]} r_i,
\]

and there exists a non-negative integer matrix \( A = [a_{ij}]_{k \times m} \) such that

\[
AH = nR, \quad \sum_{i \in [k]} a_{ij} = \lambda_j \left( \frac{n}{h_j} \right) \text{ for } j \in [m].
\]

Moreover, for each \( i \in [k] \), an \( r_i \)-factor is connected if and only if

\[
\sum_{j \in [m]} a_{ij} \leq n(r_i - 1) + 1.
\]

**Proof.** To prove the necessity, suppose that \( \Lambda K_n^H \) is \( R \)-factorable where \( F_i \)'s are the corresponding \( r_i \)-factors for \( i \in [k] \). Since \( F_i \) is \( r_i \)-regular and \( \Lambda K_n^H \) is \( \sum_{i \in [m]} \lambda_i \binom{n-1}{h_i-1} \)-regular, (53) must hold. For \( i \in [k], j \in [m], \) let \( a_{ij} \) be the number of edges of size \( h_j \) (counting multiplicities) in \( F_i \). Since \( \{ F_1, \ldots, F_k \} \) is a partition of the edges of \( \Lambda K_n^H \), we have

\[
\sum_{i \in [k]} a_{ij} = \lambda_j \left( \frac{n}{h_j} \right) \text{ for } j \in [m].
\]

Double counting the degree sum in \( F_i \), we obtain

\[
\sum_{j \in [m]} a_{ij} h_j = nr_i \quad \text{for } i \in [k].
\]

Let us fix an \( i \in [k] \). If \( F_i \) is connected, then by Lemma 7.1 we have

\[
1 \geq n - \sum_{e \in F_i} (|e| - 1) = n - \sum_{j \in [m]} a_{ij} (h_j - 1) = n - r_i n + \sum_{j \in [m]} a_{ij}.
\]

Therefore, (55) holds.

To prove the sufficiency, let \( \mathcal{H} \) be a hypergraph with a single vertex \( \alpha \) such that

\[
\text{mult}_{\mathcal{H}}(\alpha^{h_j}) = \lambda_j \left( \frac{n}{h_j} \right) \text{ for } j \in [m].
\]

Since for \( j \in [m], \sum_{i \in [k]} a_{ij} = \lambda_j \binom{n}{h_j} \), we can color the edges of \( \mathcal{H} \) so that

\[
\text{mult}_{\mathcal{H}(i,j)}(\alpha^{h_j}) = a_{ij} \text{ for } i \in [k], j \in [m].
\]
Let us fix an \( i \in [k] \). We have
\[
\deg_{\mathcal{H}(i)}(\alpha) = \sum_{j \in [m]} a_{ij} h_j = nr_i,
\]
and
\[
\omega_\alpha(\mathcal{H}(i)) = \sum_{j \in [m]} a_{ij}.
\]
Therefore, if (55) holds,
\[
\deg_{\mathcal{H}(i)}(\alpha) - \omega_\alpha(\mathcal{H}(i)) = nr_i - \sum_{j \in [m]} a_{ij} \geq n - 1.
\]
By Theorem 1.1, there exists an \((\alpha, n)\)-detachment \( \mathcal{F} \) of \( \mathcal{H} \) such that the following conditions hold.

(a) For each \( i \in [n] \) and each \( j \in [k] \),
\[
\deg_{\mathcal{F}(j)}(\alpha_i) \approx \frac{\deg_{\mathcal{H}(j)}(\alpha)}{n} = \frac{nr_j}{n} = r_j;
\]
(b) For each \( U \subseteq \{\alpha_1, \ldots, \alpha_n\} \) with \( |U| = h_j, j \in [m] \),
\[
\text{mult}_\mathcal{F}(U) \approx \frac{\text{mult}_{\mathcal{H}}(\alpha[U])}{\binom{n}{h_j}} = \frac{\lambda_j \binom{n}{h_j}}{\binom{n}{h_j}} = \lambda_j;
\]
(c) For each \( i \in [k] \), if (55) holds, then \( c(\mathcal{F}(i)) = 1 \).

By (b), \( \mathcal{F} \cong \Lambda K_n^H \), by (a) \( \mathcal{F}(i) \) is an \( r_i \)-factor, and by (c), if (55) holds for some \( i \), then \( \mathcal{F}(i) \) is connected.

If we forbid edges of size 1, then condition (55) will become simpler.

**Corollary 9.2.** \( \Lambda K_n^H \) is R-factorable if and only if (53) and (54) are satisfied. Moreover, if \( \min H \geq 2 \), then for each \( i \in [k] \), an \( r_i \)-factor is connected if and only if one of the following hold.

(i) \( r_i = 1, h_m = n, a_{im} = 1, a_{ij} = 0 \) for \( j \in [m - 1] \),
(ii) \( r_i \geq 2 \).

**Proof.** Since \( \min H \geq 2 \), we have \( 2 \leq h_1 < h_2 < \cdots < h_m \leq n \). Fix \( i \in [k] \).

Let us assume that (55) holds. If \( r_i \geq 2 \), there is nothing to prove. If \( r_i = 1 \), then by (55), \( \sum_{j \in [m]} a_{ij} \leq 1 \). Thus, there exists an \( \ell \in [m] \) such that \( a_{i\ell} = 1, a_{ij} = 0 \) for \( j \neq \ell \). Moreover, by (54), \( h_\ell = \sum_{j \in [m]} a_{ij} h_j = n \), and so \( \ell = m \).

Now, we show that if (i) or (ii) holds, then (55) is satisfied. If (i) holds, then clearly (55) holds (It is also easy to see that if \( h_m = n, a_{im} = 1 \), then the corresponding \( r_i \)-factor is connected). Let us assume that \( r_i \geq 2 \). By (54),
\[
\frac{r_i n}{h_1} = \frac{1}{h_1} \sum_{j \in [m]} a_{ij} h_j = a_{i1} + \sum_{j \in [m]} \frac{h_j}{h_1} a_{ij} > \sum_{j \in [m]} a_{ij}.
\]
Therefore, it suffices to show that \( r_i n / h_1 \leq n(r_i - 1) + 1 \), or equivalently, \( r_i \geq \frac{h_1(n - 1)}{n(h_i - 1)} \). Since \( r_i \geq 2 \), it is enough to show that \( 2n(h_1 - 1) \geq h_1(n - 1) \), which is true because \( h_1 \geq 2 \).

For two column vectors \( Q = [q_1 \ldots q_k]^T \), \( R = [r_1 \ldots r_k]^T \), we say that \( Q \leq R \) if \( q_i \leq r_i \) for \( i \in [k] \). A \((q, r)\)-factor in a hypergraph \( \mathcal{G} \) is a spanning sub-hypergraph in which
\[
q \leq \deg(v) \leq r \text{ for each } v \in V(\mathcal{G}).
\]
A \((Q, R)\)-factorization of \(G\) is a partition \(\{F_1, \ldots, F_k\}\) of \(E(G)\) in which \(F_i\) is a \((q_i, r_i)\)-factor for \(i \in [k]\).

The proof of the following result is very similar to that of Theorem 9.1, and we shall skip it here.

**Theorem 9.3.** \(\Lambda K^H_n\) is \((Q, R)\)-factorable if and only if

\[
\sum_{i \in [k]} q_i \leq \sum_{i \in [m]} \lambda_i \left( \frac{n - 1}{h_i - 1} \right) \leq \sum_{i \in [k]} r_i,
\]

and there exists a non-negative integer matrix \(A = [a_{ij}]_{k \times m}\) such that

\[
nQ \leq AH \leq nR, \quad \sum_{i \in [k]} a_{ij} = \lambda_j \left( \frac{n}{h_j} \right) \text{ for } j \in [m].
\]

Moreover, for each \(i \in [k]\), an \(r_i\)-factor is connected if and only if

\[
\sum_{j \in [m]} a_{ij}(h_j - 1) \geq n - 1.
\]

An almost \(k\)-factor of a hypergraph is \((k - 1, k)\)-factor. Theorem 9.3 can be used to construct connected almost factorizations of \(\Lambda K^H_n\).

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