LINEAR EXTENSIONS OF PARTIAL ORDERS 
ON ABELIAN GROUPS

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ABSTRACT. Partially ordered groups, also known as po-groups, are groups 
with a compatible partial order. Results from M. I. Zajceva and H.-H. Teh are 
combined in order to provide a full characterisation of linear order extensions of 
a given order on a group. In contrast to Teh this approach provides a method 
to discuss linear orders of different abelian rank in a uniform manner. This will 
be achieved by modelling the linear orders using hyperplanes in a real vector 
spaces. Among some additional remarks a construction of an archimedian 
directed order is given for every torsion free abelian group.

INTRODUCTION

Linear and lattice ordered groups are well-studied algebraic structures. The 
structure of linearly ordered abelian groups has mainly been discovered by the 
work of Otto Hölder [12] and Hans Hahn [11]. Later, Friedrich Levi [15] provided 
a first characterisation of lattice ordered groups and showed that an abelian group 
can be linearly ordered if it is torsion free. It must be noticed that at the time of 
these early works the theory of modules and vector spaces had not settled down. 
First works on these topics from the middle of the 19th century have rarely been 
noticed in the community. It took nearly 100 years until today’s definitions for 
vector spaces and modules had been fixed.

Анатолий Иванович Мальцев [31] (A. I. Maľcev) investigated necessary and 
sufficient conditions for linear orders on abelian groups. Later, М. И. Зайцева [28] 
(M. I. Zajceva) published a characterisation of finitely generated archimedian 
linearly ordered abelian groups. With a corresponding decomposition into archime-
dian subgroups, in that work she discusses also the main properties of a description 
of linear orders on finitely generated groups. The cardinality of such orders on a 
given group has been determined by Shin-Ichi [19] to ℵ. Eventually, Hoon-Heng Teh 
[25] provided another characterisation of linearly ordered groups based on Hahn’s 
theorem.

The current work provides a characterisation of linear order extensions of abelian 
partially ordered groups, which has been developed from scratch, based on the 
usage and an example from Charles Holland and other authors (cf. e.g., [13]). Levi 
discussed the characteristics of lattice ordered groups starting with abelian groups 
embedded into one and two dimensional vector spaces and analysed them with 
rising dimension (bottom-up approach). He provides tools to describe the image of 
a positive cone of a lattice ordered group in a vector space by means of rays and

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edges. We will discuss the same phenomenon by construction of hyperplanes in a vector space that are constructed using independent sets of a given group (top-down approach). Though, this method considers the positive cone in less granularity, it provides a better bridge to methods from convex geometry than Levi’s approach.

Zajceva’s equations for archimedian orders on $n$-generated abelian groups can be interpreted as certain hyperplanes in $n$-dimensional real vector spaces. Thus, this work can also be considered an extension of her results onto arbitrary abelian groups and allows to discuss existing orders on them. A further advantage of this view consists in the possibility to investigate archimedian and several non-archimedian orders together based on a common mathematical structure. This facet and the discussion of linear order extensions of given order relations are also an enhancement to Teh’s work, who discusses the archimedian rank of linear orders.

The constructive nature of the given article is dedicated to be more accessible for non-mathematicians and mathematicians with a background different from group theory. This is achieved by using injective abelian groups, which need different tools and provide slightly different insights than the usual approach using free abelian $\ell$-groups or free vector lattices and the duality theorems provided by W. M. Beynon [2].

To achieve this, after a short clearing of notions, mappings between orders on different structures will be investigated. This discussion has the aim to enable us to represent a given order on a torsion free abelian group in a convenient way in the vector space of the direct sum $\oplus_{E} \mathbb{R}$, where the set of indices $E$ corresponds to a maximal independent set in the given group. There we will use a characterisation of linearly ordered groups that is a little bit different from those given by Zajceva and Teh (cf. Theorem 24). Finally, with Theorem 28 we will provide a characterisation of linear order extensions on partially ordered abelian groups, with the use of half spaces and linearly ordered hyperplanes. Thus, we can use methods from convex geometry to discuss linearly ordered groups. This will be demonstrated in Section 7, where some known results have been resembled. This section also shows how to construct an archimedian directed order on an arbitrary torsion free abelian group.

1. Preliminaries

In this section we repeat the basics used in this article. The main facts can be found in the usual textbooks about lattice ordered groups (e.g. [1, 3, 5, 32, 10, 14, 29, 30]).

A group $G = (G, \cdot, -1, e)$ is called partially ordered iff there exists a partial order $\leq \subseteq G \times G$ such that the following condition holds for all elements $x, y, a, b \in G$:

$$a \leq b \Rightarrow xay \leq xby.$$  

(1)

It is called linearly ordered iff $\leq$ is a linear order. A group homomorphism $\varphi$ from a partially ordered group $G$ into another one $G'$ is called o-homomorphism iff it is also an order homomorphism. Furthermore it is called an o-isomorphism if it is additionally an order isomorphism.

A very fundamental theorem allows us to characterise each partial order $\leq$ on a group $G$ by means of its positive cone $G_+ := \{ g \in G \mid 0 \leq g \}$:

\footnote{The given condition is sufficient to assure that the partial order $\leq$ is also compatible with the other group operations (inverse element and neutral element).}
Theorem 1. The positive cone \( G_+ \) of a partially ordered group \((G, \cdot, -1, e, \leq)\) fulfils the following conditions:

1. \( G_+ \cdot G_+ \subseteq G_+ \) \( G_+ \) is a semigroup
2. \( G_+ \cap (G_+)^{-1} = \{ e \} \) \( G_+ \) is a pure subset
3. \( \forall g \in G : g^{-1} G g \subseteq G_+ \) \( G_+ \) is invariant subsemigroup.

If \( G \) is a directed set, it further fulfils,

4. \( G = G_+ \cdot (G_+)^{-1} \).

and if \( G \) is linearly ordered

5. \( G = G_+ \cup (G_+)^{-1} \).

Conversely, if in \( G \) a set \( P \subseteq G \) exists, that fulfils (2) to (4), there exists an order on \( G \) such that \( P \) is the set of positive elements of \( G \). If furthermore the Condition (5) is met, the order is directed and, if (6) is true, it is a linear order.

Proof. Cf. [14], Theorem 2.1.1 or [30], II.2.1. \( \square \)

An Element \( a \in G_+ \) is called infinitesimal with respect to another element \( b \) iff the relation \( a^n \leq b \) holds for every integer \( n \in \mathbb{Z} \). Consequently 0 is infinitesimal with respect to every other element. If there is no other element infinitesimal with respect to any other element then the group is called archimedian.

It is a well-known fact that each finite group can only be discretely ordered. This implies that only torsion free groups can be linearly ordered. Thus, throughout this paper we will refer to torsion free groups when ever we talk about groups, unlike otherwise stated.

An independent set \( E \subseteq G \) of an abelian group \( G = (G, +, -1, 0) \) is defined as a set of its elements, which fulfills the equation

\[
\sum_{a \in E'} a(a)a = 0,
\]

for each finite subset \( E' \subseteq E \) and a mapping \( a \in \oplus_{E} \mathbb{Z} \) iff for all \( a \in E' \) the condition \( a(a) = 0 \) is met. Zorn’s lemma assures the existence of a maximal independent set. A subset \( E \subseteq G \) is independent, iff the subgroup \( (E) \), which is generated by \( E \), is isomorphic to the direct sum of the cyclic groups \( \langle a \rangle \) of all of its elements \( a \in E \). A subgroup \( S \subseteq G \) is called essential subgroup iff for each subgroup \( S' \subseteq G \) the intersection \( S \cap S' \neq \{ 0 \} \) is non-trivial. An independent set \( E \) is maximal iff its generated subgroup \( (E) \) is an essential subgroup of \( G \). For further information we refer to [8] or any other text book on the theory of abelian groups.

Given an arbitrary set \( E \) we define the direct sum \( \oplus_{E} G \) according to

\[
\oplus_{E} G := \{ \mathbf{r} : E \to G, |\text{supp}\mathbf{r}| < \infty \},
\]

where \( \text{supp}\mathbf{r} := \{ a \in E \mid \mathbf{r}(a) \neq 0 \} \) is the support of the mapping \( \mathbf{r} \). The unit vectors will be denoted by

\[
e_a(b) := \begin{cases} 1 & \text{iff } a = b, \\ 0 & \text{otherwise.}
\end{cases}
\]
If the generating set $E$ is not uniquely determined by the context (e.g. when we are using different bases of a vector space) we will write $x_E(e)$ instead of $x(e)$ for any element $e \in E$.

A vector space $\mathcal{V}$ over an ordered field $K = (K, +, -, 0, \cdot, -1, 1)$ is called ordered vector space iff $\mathcal{V}$ is an ordered group w.r.t. addition and for all vectors $v \in \mathcal{V}+$, and all positive elements $\alpha \in K_+$ of the field the condition $\alpha v \in \mathcal{V}+$ is met. Consequently, the set $\mathcal{V}+$ is also a positive cone in the geometrical sense (cf. Fig. 1).

If $A$ is a set of vectors of a vector space, $L_A$ denotes their linear closure. The set of linear combinations of vectors from $A$ with exclusively non-negative coefficients will be symbolised by $L_A^+\mathcal{S}$. As usual a subset $A \subseteq \mathcal{V}$ is called linear subspace $A \leq \mathcal{V}$ if $L_A = A$ is true. A subspace $\mathcal{H}$ of a vector space $\mathcal{V}$ is called hyperplane if there exists a vector $a \in \mathcal{V}$ such that the conditions $L_\mathcal{H} = \mathcal{H} \preceq \mathcal{V}$ and $L(\mathcal{H} \cup \{a\}) = \mathcal{V}$ hold. For some base $B$ of the vector space $\mathcal{V}$ let $S \subseteq B$ be a subbase. Then for every vector $x$ the sum

$$P_{S, B} := \sum_{s \in S} x_B(s)$$

is called (coordinate) projection of $x$ into $L_S$ along $B$.

In Section 3 we need the notion of a neighbourhood. Since vectors in the direct sum have finite support, we can use the standard scalar product in the resulting vector space for any ordered field $K$. To be consistent with the following definitions, we use a generalised version: Let $B$ be a base of the vector space $\oplus_E K$, $x, y \in \oplus_E K$ two vectors, and $r : E \rightarrow K_+=\{0\}$ a positive mapping. Then

$$\langle x, y \rangle_{r, B} := \sum_{b \in B} r(b) \cdot x_B(b) y_B(b)$$

denotes a scalar product, as $x$ and $y$ have a finite support, and have values that differ from zero only at finitely many entries. For the same reason given any basis $B$ and any positive integer $n \in \mathbb{N}$

$$(12) \quad \|x\|_{n, r, B} := \sqrt{\sum_{b \in B} r(b) \cdot |x_B(b)|^n} \quad \text{and} \quad \|x\|_{\infty, r, B} := \max_{b \in B} r(b) \cdot |x_B(b)|$$

are norms, and

$$(13) \quad \Delta_{n, r, B}(x, y) := \|y - x\|_{n, r, B} \quad \text{and} \quad \Delta_{\infty, r, B}(x, y) := \|y - x\|_{\infty, r, B}$$

The additive group $(K, +, -, 0)$ is an ordered group such that $0 \leq 1$, as well as the multiplicative group of the positive elements $(K_+ \setminus \{0\}, \cdot, -1, 1)$.

3 The complete argumentation be found in the proof of Theorem 2.
are metrics on $\oplus_E \mathbb{K}$. For the special case $n = 2$ we have the usual identity
\[ \sqrt{\langle x, x \rangle} = \|x\|_{2, \mathbb{R}}. \]
For $x \in \oplus_E \mathbb{K}$ and $r \in \mathbb{K}$ we use the following definition for an open ball $B_{r, n, \mathbb{R}}(x)$ (including $n = \infty$):
\[ B_{r, n, \mathbb{R}}(x) := \{ y \in \oplus_E \mathbb{K} | \Delta_{n, \mathbb{R}}(x, y) < 1 \}. \]
As the generated topologies of these open balls are not necessarily the same, we define a finer topology that includes all of them:

**Definition 1.** A subset $M \subseteq \mathcal{V}$ of a vector space $\mathcal{V}$ is called open, iff for each pair of vectors $a \in M, b \in \mathcal{V}$ the set
\[ \{ \lambda \in \mathbb{R} | \lambda a + (1 - \lambda)b \in M \} \]
is open in the set of real numbers $\mathbb{R}$ with the standard topology.

As all vectors have finite support, this defines a topology on $\mathcal{V}$ where all open balls are open sets. The boundary of a set $M$ with respect to this topology is denoted by $\partial M$, and its interior by $\text{int} M$.

For two subsets $A, B \subseteq \mathcal{G}$ of a group $\mathcal{G}$ we use the usual definition of a complex sum
\[ A + B := \{ a + b | a \in A, b \in B \}. \]
If $A$ and $B$ are linear subspaces of a vector space whose intersection $A \cap B = \{ 0 \}$ is the singleton containing the zero vector $0$, then we denote this by $A \oplus B$. Thus, for a hyperplane $\mathcal{H} \leq \mathcal{V}$ there exists a vector $a$ such that $\mathcal{V} = \mathcal{H} \oplus \mathbb{K}a$.

### 2. Preparation of injective groups

In this section we mainly resemble Walker’s theorem [27] for ordered groups providing a construction of the embedding into the injective abelian Groups with the formalism used throughout the current article.

We will characterise all linear order extensions of an abelian group $\mathcal{G}$, in this article using the vector space $\oplus_E \mathbb{R}$ for a set $E$, that is constructed depending on $\mathcal{G}$. In order to achieve this we will use properties from the integer module $\oplus_E \mathbb{Z}$ and its divisible extension, the rational vector space $\oplus_E \mathbb{Q}$ as the latter is the smallest injective group containing $\oplus_E \mathbb{Z}$. While the first one is isomorphic to a subgroup of $\mathcal{G}$, the latter one is isomorphic to a supergroup of the divisible hull of $\mathcal{G}$. As it is useful to have a well-defined base in the considered vector spaces, we shortly discuss the necessary steps for the construction of the mentioned algebraic structures on the basis of the given torsion free abelian group. Furthermore, in this section we discuss a method to transfer orders between these structures [4] Let’s start with the following well-known fact:

**Lemma 2 (Folklore).** Let $\mathcal{G} = (G, \cdot, -1, e, \leq)$ be a po-group, $\mathcal{G}' = (G', \cdot', -1', e')$ a group and $\varphi : \mathcal{G}' \to \mathcal{G}$ an injective homomorphism. Then $\mathcal{G}'$ forms a po-group together with the relation $\subseteq$ defined by
\[ g \subseteq h :\Leftrightarrow \varphi(g) \leq \varphi(h). \]

[4] It proved to be useful, to favour monomorphisms over subgroup relations, here.
The mapping \( \varphi \) is a group isomorphism from \( G' \) onto \( \varphi[G'] \) and according to equation (16) it is also an order isomorphism. So it remains to show that the group \( G' \) is a partially ordered group together with the relation \( \preceq \). Let \( a, b, x, y \in G' \) elements of the group \( G' \). Then the following equivalences hold:

\[
\begin{align*}
  a \preceq b & \iff \varphi(a) \leq \varphi(b) \\
  & \iff \varphi(x) \cdot \varphi(a) \cdot \varphi(y) \leq \varphi(x) \cdot \varphi(b) \cdot \varphi(y) \\
  & \iff \varphi(x \cdot a \cdot y) \leq \varphi(x \cdot b \cdot y) \\
  & \iff x \cdot a \cdot y \preceq x \cdot b \cdot y.
\end{align*}
\]

So \( \varphi \) is an \( o \)-homomorphism from \( G' \) into \( G \). \( \square \)

This lemma allows us to relate possible orders on the three structures \( \oplus E \mathbb{Z}, \oplus E \mathbb{Q} \) and \( \oplus E \mathbb{R} \). Figure 2 shows these relationships with respect to the identical embeddings. In the left hand Figure 2(a) the diagram of the embeddings is shown, while the arrows on the right hand side 2(b) show the possible ways of induction or transfer of the orders between the structures.

In order to add an arbitrary torsion free abelian group into this system of monomorphisms and induced orders, suppose \( E \) is an independent set of the group \( G \). Thus, considering cyclic groups, a positive integer \( n \in \mathbb{N} \setminus \{0\} \) exists for each element \( g \in G \) such that \( ng \in \langle E \rangle \). Thereby, we found a more or less unique representation for each Element of \( G \) by elements of a maximal independent set \( E \):

**Lemma 3.** Let \( G \) be a torsion free abelian group and \( E \) a maximal independent set in \( G \). Then for each element \( g \in G \) a positive integer \( q_g \in \mathbb{N} \setminus \{0\} \) and a mapping \( p_g : E \to \mathbb{Z} \) exist such that the following equation holds:

\[
q_g g = \sum_{a \in E} p_g(a)a.
\]

The factors \( q_g \) and \( p_g(a) \) are uniquely defined up to multiplication by a common rational number.

**Proof.** As \( \langle E \rangle \) is an essential subgroup, for each element \( g \in G \) the intersection \( \langle E \rangle \cap \langle g \rangle \neq \emptyset \) is non-empty. This implies that a positive integer \( q_g \) exists such that \( q_g g \in \langle E \rangle \). This proves the existence of a representation of the form (17).
Let
\[ q_g g = \sum_{a \in E} p_g(a) a \quad \text{and} \quad q_g' g = \sum_{a \in E} p_g'(a) a \]
two such representations according to (17). Considering their difference we get
\[ 0 = (q_g - q_g') g = \sum_{a \in E} (p_g - p_g')(a) a. \]

Since \( E \) is independent, we deduce from this equation that for each \( a \in E \) the condition \( p_g(a) - p_g'(a) = 0 \) holds. So we proved the identity \( p_g = p_g' \) for any fixed number \( q_g \).

Let’s assume that we have different representations of the form (17). Since the mapping \( p_g \) is unique for each \( q_g \), there must be different numbers on the left hand side too. Let \( q_g \in \mathbb{N} \setminus \{0\} \) the smallest integer for which a representation in the form (17) exists. Furthermore, let us assume there exists a number \( q_g' \in \mathbb{N} \setminus \{0,1\} \), which is coprime to \( q_g \) such that:
\[ q_g' g = \sum_{a \in E} p_g'(a) a. \]

Then the following two equations hold:
\[ (q_g - q_g') g = \sum_{a \in E} (p_g - p_g')(a) a \quad \text{and} \quad (q_g + q_g') g = \sum_{a \in E} (p_g + p_g')(a) a. \]

This allows us to use the Euclidian algorithm to find two integers \( c, d \) such that \( cq_g + dq_g' = \gcd(q_g, q_g') \) holds. Using these the equation
\[ \gcd(q_g, q_g') g = (cq_g + dq_g') g = \sum_{a \in E} (cp_g + dp_g')(a) a. \]
holds. Since \( 0 < \gcd(q_g, q_g') \leq q_g \) and \( q_g \) was chosen minimal and \( q_g' \) coprime to \( q_g \), there exists a contradiction. So \( q_g' \) is a multiple of \( q_g \).

Let \( q_g' = bq_g \) for some positive integer \( b \in \mathbb{N} \setminus \{0\} \). Then we have
\[ \sum_{a \in E} bp_g(a) a = bq_g g = q_g' g = \sum_{a \in E} p_g'(a) a. \]

This proves that both \( q_g' \) and \( p_g' \) are multiples with the same factor of \( q_g \) and \( p_g \), respectively. \( \square \)

Using this Lemma 3 we can find an embedding of \( G \) with a maximal independent set \( E \) into the direct sum \( \oplus_E \mathbb{Q} \) (cf. Walker’s theorem [27]):

**Theorem 4.** Let \( G = (G, +, -, 0) \) be a torsion free abelian group and \( E \subseteq G \) a maximal independent set in \( G \). Let for each \( g \in G \) a mapping \( p_g : E \to \mathbb{Z} \) and a positive integer \( q_g \in \mathbb{N} \setminus \{0\} \) defined according to equation (17). Then the mapping
\[ (18) \quad \varphi : G \to \oplus_E \mathbb{Q} : g \mapsto \frac{p_g}{q_g} \]
is a monomorphism from \( G \) into \( \oplus_E \mathbb{Q} \).
Proof. Lemma 3 ensures for each \( g \in G \) the existence of integers \( q_g, p_g(a) \) (for all \( a \in E \)) that fulfil the following equation:

\[
q_g g = \sum_{a \in E} p_g(a)a
\]

Let \( \varphi : G \to \oplus_E \mathbb{Q} : g \mapsto \frac{p_g}{q_g} \) be the mapping as defined in equation (18). Then for each \( c \in \mathbb{Z} \) the equation \( \varphi(g) = \frac{p_g}{q_g} \) holds. From this we infer together with Lemma 3 that \( \varphi \) is well-defined.

Furthermore, having \( \varphi(\,cg\,) = c \varphi(\,g\,) \) we can combine the two representations

\[
q_g g = \sum_{a \in E} p_g(a)a, \quad q_h h = \sum_{a \in E} p_h(a)a
\]

of elements \( g, h \in G \), into one common formula:

\[
q_g q_h (g + h) = q_h \sum_{a \in E} p_g(a)a + q_g \sum_{a \in E} p_h(a)a.
\]

Thus, the mapping \( \varphi \) is a group homomorphism from \( \mathbb{Z} \) into \( \oplus_E \mathbb{Q} \) with respect to the addition. Consequently,

\[
\varphi(g + h) = \frac{q_h}{q_g} p_g + \frac{q_g}{q_h} p_h = \varphi(g) + \varphi(h)
\]

Let \( \varphi(g) = 0 \). Then for each \( a \in E \) the element \( p_g(a) = 0 \) is zero, because \( E \) is an independent set. Thus, for any \( q_g \in \mathbb{N} \setminus \{0\} \) the equality \( q_g g = 0 \) holds, which implies \( g = 0 \). So \( \varphi \) is injective, which means it is a monomorphism from \( \mathbb{Z} \) into \( \oplus_E \mathbb{Q} \).

\[
\square
\]

**Corollary 5.** The homomorphism \( \varphi \) maps each element \( a \in E \) of the independent set onto a unit vector, i.e., \( \varphi(a) = e_a \).

With Theorem 4 each order in the real (rational) vector space \( \oplus_E \mathbb{R} \) defines an order on the group \( \mathbb{G} \).

**Theorem 6.** Let \( \mathbb{G} \) be a torsion free abelian group and \( E \) a maximal independent set in \( \mathbb{G} \). Furthermore, let \( \varphi : \mathbb{G} \to \oplus_E \mathbb{Q} \) be defined as in equation (18). Then the mapping

\[
(19) \quad \psi : \oplus_E \mathbb{Z} \to \mathbb{G} : \mathbf{x} \mapsto \sum_{a \in E} x(a)a
\]

is a monomorphism such that for all \( g \in \langle E \rangle \) and all \( \mathbf{x} \in \oplus_E \mathbb{Z} \) the conditions

\[
(20) \quad \psi(\varphi(g)) = g \quad \text{and} \quad \varphi(\psi(\mathbf{x})) = \mathbf{x}
\]

hold.

Proof. Firstly we show that \( \psi \) is a homomorphism. Since \( \mathbb{G} \) is commutative, we can rewrite the sum in the following way:

\[
\psi(\mathbf{x} + \mathbf{y}) = \sum_{a \in E} (\mathbf{x} + \mathbf{y})(a)a = \sum_{a \in E} \mathbf{x}(a)a + \sum_{a \in E} \mathbf{y}(a)a = \psi(\mathbf{x}) + \psi(\mathbf{y}).
\]

The last identity holds, since \( \mathbb{G} \) is torsion free. Furthermore, \( E \) is an independent set in \( \mathbb{G} \) and the equation \( e_a(1) = 1 \) holds. Thus, \( \sum_{a \in E} f(a)a = 0 \) implies \( f = 0 \), which proves that \( \psi \) is an injective homomorphism.
Since \( \varphi \) and \( \psi \) are monomorphisms and \( \langle E \rangle \) is a subgroup in \( \mathbb{G} \), it suffices to show the equations (20) for the elements of the generating sets \( E \) and \( \{ e_a \mid a \in E \} \). For those it is easy to show the following identities:

\[
\psi(\varphi(a)) = \psi(e_a) = a, \quad \text{and} \\
\varphi(\psi(e_a)) = \varphi(a) = e_a.
\]

Since these identities hold for any element of the corresponding set they are satisfied for any element of the corresponding (sub)group, too. \( \square \)

The preceding two theorems prove that the diagram in Figure 3(a) commutes. Thus, we can transfer orders along the arrows in Figure 3(b) between the different groups. Together with Theorem 24 this would be sufficient to rework the theorem of M. I. Zajceva (M. I. Zajceva) \[28\] in the language of vector spaces. Since we want to discuss order extensions, we must firstly transfer the order from a given group into the corresponding real vector space. The following section will provide this link.

3. Copying the order into the vector space

Any order in the vector space constructed in the last section implies an order in our group (cf. Figure 2(b)). So far the vector space has no idea about the existing order in the group \( \mathbb{G} \). This section closes this gap. There are some well-known theorems such as Hahn’s theorem \[11\] or the Conrad-Harvey-Holland-Theorem (cf. \[9\], chap. 4.6) which cover the embedding of ordered abelian groups into real vector spaces. These theorems provide embeddings for each order, but they do not assure that it is possible to use the same embedding for all orders.

We can transfer the order from the group onto an integer module (see Figure 3). However, going this path we loose information about group elements. On the other hand, the integer module \( \oplus_E \mathbb{Z} \) is a set of size nearly zero in the real vector space \( \oplus_E \mathbb{R} \), so it is not easy to tell which linear vector space orders correspond to one linear order in the integer module.

As the notion convex is already order-theoretically defined in the language of partially ordered groups, we choose a different wording, here:

**Definition 2** (cf. \[5\], Def. 3.2). Let \( \mathbb{G} = (G, +, −, 0, \leq) \) a partially ordered abelian group. The order \( \leq \) is called **semiclosed** iff for all \( a \in G \) and \( n \in \mathbb{N} \setminus \{0\} \) the
following implication holds:

\[(21) \quad 0 \leq na \Rightarrow 0 \leq a.\]

We show in this section that semiclosed orders on any abelian torsion free group can be easily extended to vector space orders. This will be done by forming the convex hull. Doing so, we prove that in this particular case the induced order on the group will be the same as the original one. Fortunately, the class of semiclosed ordered torsion free abelian groups is the relevant class of groups to consider, here. Each partially ordered abelian group has a canonical semiclosed order extension, which itself is a suborder of any linear order extension of the group. So the restriction to semiclosed groups does not influence the generality of this construction.

As shown in Figure 4 the concept of a semiclosed order can be considered as an order on the group, whose positive cone is convex in a geometrical sense (we will discuss this later). In geometric discussions about convex sets the convex hull plays an important role. As a semiclosed order has a “convex” positive cone, we can ask, whether the positive cone of any partial order has a well defined “convex hull”. The following lemma discusses this fact:

**Lemma 7** (cf. [5], Cor. 29.10). Let \( G = (G, +, -, 0, \leq) \) be a partially ordered abelian group and the order relation \( \leq' \) defined by

\[(22) \quad 0 \leq' g :\iff \exists m \in \mathbb{N} \setminus \{0\} : 0 \leq mg.\]

Then \( \leq' \) is a semiclosed order relation and \((G, +, -, 0, \leq')\) is a partially ordered group. Each linear or lattice order on \( G \) is an extension of \( \leq' \).

**Proof.** Firstly, we have to show that \( P := \{ g \in G \mid 0 \leq' g \} \) fulfils the conditions (2) to (4). Let \( g, h \in P \) be two elements of this set. Then there exist two positive integers \( m, n \in \mathbb{N} \setminus \{0\} \) such that \( 0 \leq mg \) and \( 0 \leq nh \). This implies \( 0 \leq mng \) and \( 0 \leq mnh \). Consequently, we have \( 0 \leq mng + mnh = mn(g + h) \) and \( 0 \leq' g + h \). Since \( G \) is a commutative group, the set \( P \) is a semigroup and invariant. If for any nonzero element \( g \neq 0 \) the inequality \( 0 \leq mg \) holds, there exists no positive number \( n \in \mathbb{N} \) such that \( ng \leq 0 \). Otherwise from \( ng \leq 0 \) follows \( mng \leq 0 \) and from \( 0 \leq mg \) follows \( 0 \leq mng \). This would imply \( g = 0 \), which has been excluded.

Thus, with \( P \cap -P = \{0\} \) the set \( P \) is a pure subset of \( G \). This shows that \((G, +, -, 0, \leq')\) is a partially ordered group. Since the condition (22) is always true in lattice ordered groups (cf. [5], Prop. 3.6) and the positive cones of \( \leq \) and \( \leq' \) differ by exactly those elements, which contradict this condition, the order \( \leq' \) is a suborder of any lattice order extension of \( \leq \).
Up to now, the transitive closure of the arrows in Figure 3(b) is a partial order. Our aim is to consider the orders on the different groups to be more or less equivalent. The following theorem extends a given order on a subgroup to a semiclosed order on the containing vector space. As a result of this theorem, we will be able to invert the arrows in the lower right triangle of this figure.

**Theorem 8.** Let $\oplus E Z$ an integer module and $\oplus E R$ a vector space of equal dimension. Then there exists an injective mapping, which maps each semiclosed order on $\oplus E Z$ to a vector space order on $\oplus E R$. The same is true for the combinations $\oplus E Z / \oplus E Q$ and $\oplus E Q / \oplus E R$.

**Proof.** It is sufficient to prove that the positive cone $(\oplus E Z)_+$ can be embedded into a convex invariant subset $P$ of the vector space $\oplus E R$ that fulfills the Conditions (2) to (4). We will show the injectivity afterwards. As candidate for the set $P$ we choose the convex hull of $\varphi[\psi((\oplus E Z)_+)]$ in the real vector space $\oplus E R$ using the monomorphisms $\varphi$ and $\psi$ defined in (18) and (19). Let

$$P := \text{conv}(\varphi[\psi((\oplus E Z)_+)])$$

$$P' := \mathcal{L}^+(\varphi[\psi((\oplus E Z)_+)])$$

As $P'$ contains any positive linear combinations of its elements it also contains the special case of convex combinations. This implies $P \subseteq P'$. Since the zero vector $0 \in \oplus E Z$ lies in the positive cone and for each positive vector $x \in (\oplus E Z)_+$ its multiple $nx \in (\oplus E Z)_+$ is positive for every $n \in \mathbb{N}$, for each $\alpha \in \mathbb{R}_+$ we also have $\alpha x \in P$. This can be deduced from the equation

$$\alpha \sum_{i=0}^{n} \beta_i x_i = \sum_{i=0}^{n} \alpha \beta_i x_i.$$

Thus, for any two vectors $x, y \in P$ and any two positive numbers $\alpha, \beta \in \mathbb{R}_+$ we get

$$\alpha x + \beta y = \frac{\alpha}{\alpha + \beta} ((\alpha + \beta) x) + \left( 1 - \frac{\alpha}{\alpha + \beta} \right) (\alpha + \beta) y \in P,$$

which proves $P' \subseteq P$ and together with the preceding results $P = P'$. Furthermore, the set $P$ is a subsemigroup of $\oplus E R$. As this vector space is commutative, $P$ is already an invariant semigroup.

The set $\mathcal{B} := \{e_a \mid a \in E\}$ is a generating set of the module $\oplus E Z$. Thus, $\mathcal{B}$ is a basis of $\oplus E R$. Since $P \subseteq \oplus E R$, the following equation holds:

$$\mathcal{L} P = \mathcal{L} \left( \mathcal{L}^+((\oplus E Z)_+) \right).$$

Thus, for any vector $x \in \oplus E Z \cap P$ there exists a linear combination of elements of a linear independent set $B \subseteq (\oplus E Z)_+$ and a positive integer $n \in \mathbb{N}$ such that

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

where $\alpha_i \in \mathbb{R}_+$ and $a_i \in B$. Since the range of $x$ is a set of integers, the coefficients $\alpha_i$ cannot be irrational, because they are solutions of the system of linear equations

$$\forall m \in E : \sum_{i=1}^{n} \alpha_i a_i(m) = x(m) \quad \text{for } \alpha = \beta = 0 \text{ we already proved } 0 \in \mathbb{Z} \subseteq P.$$
As \( a_i(m) \in \mathbb{Z} \) and \( x(m) \in \mathbb{Z} \) are integers, the coefficients \( \alpha_i \) are rational numbers. Let \( k \) be the least common denominator of all those numbers \( \alpha_i \). Then we can multiply the system with \( k \) and get the modified system of linear equations

\[
\forall m \in E : \sum_{i=1}^{n} (k\alpha_i)a_i(m) = x'(m),
\]

where \( x' \) is defined as \( x' := kx \). Thus, the vector \( x' \) is positive in \( \oplus E\mathbb{Z} \) and so the vector \( x \) is non-negative. Since the order on \( \oplus E\mathbb{Z} \) is semiclosed, this also implies \( x \in (\oplus E\mathbb{Z})_+ \). Consequently, we showed that \( x \in P \cap \oplus E\mathbb{Z} \) holds iff \( x \in P \cap \oplus E\mathbb{Z} \).

Finally, we mapped each order \( \leq \) on \( \oplus E\mathbb{Z} \) onto an order \( \leq' \) on \( \oplus E\mathbb{R} \) such that the positive cone of \( \leq' \) reproduces the positive cone of \( \leq \) by the intersection with \( \oplus E\mathbb{Z} \). From that follows that the mapping from the set of partial orders on the group \( \oplus E\mathbb{Z} \) into the set of vector space orders on \( \oplus E\mathbb{R} \) is injective.

The proof of the other two combinations follows the same path.  

As already stated, this theorem ensures that the paths in the lower right triangle in Figure 5(b) are equivalent. Due to the additional arrow from \( (\oplus E\mathbb{Z}, \leq|\mathbb{Z}) \) to \( (\oplus E\mathbb{Q}, \leq|\mathbb{Q}) \) this is not clear for the upper left triangle in this figure. The following lemma closes this gap.

**Lemma 9.** If the group \( G \) is semiclosed ordered, \( E \) is a maximal independent set in \( G \) and the monomorphisms \( \varphi \) and \( \psi \) are defined according to the Equations (18) and (19) the following equation holds:

\[
(23) \quad \text{conv}_Q \left[ \psi^{-1}[G_+ \cap (E)] \right] \cap \varphi[G] = \varphi[G_+]
\]

**Proof.** Let \( x \in \varphi[G_+] \). Then there exists an element \( g \in G_+ \) such that \( \varphi(g) = x \). Since \( E \) is a maximal independent set there exists a natural number \( n \in \mathbb{N} \setminus \{0\} \) such that \( ng \in (E) \) is an element of the subgroup which is generated by \( E \). Thus \( ng \) is contained in the image of the group homomorphism \( \psi \). Considering \( \oplus E\mathbb{Q} \) the vector \( \frac{1}{n} \psi^{-1}(ng) \) is an element of the convex hull \( \text{conv}_Q \left( \left\{ 0, \psi^{-1}(ng) \right\} \right) \). Since...
the neutral element is always an element of the positive cone, we have shown one direction of (23):

$$\text{conv}_Q \left[ \psi^{-1}(G_+ \cap \langle E \rangle) \right] \cap \varphi[G] \supseteq \varphi[G_+]$$.

Let $x \in \text{conv}_Q \left[ \psi^{-1}(G_+ \cap \langle E \rangle) \right] \cap \varphi[G]$. Then there exists elements $g \in G$ and $g_1, \ldots, g_n \in G_+ \cap \langle E \rangle$, positive rational numbers $x_1, \ldots, x_n \in \mathbb{Q}_+$ and a positive integer $n \in \mathbb{N} \setminus \{0\}$ such that the following condition holds:

$$x = \varphi(g) = \sum_{i=1}^{n} x_i \psi^{-1}(g_i), \text{ where } \sum_{i=1}^{n} x_i = 1.$$  

Let $q \in \mathbb{N}$ be the smallest common denominator of $\{x_1, \ldots, x_n\}$. Then the products $\{qx_1, \ldots, qx_n\} \subseteq \mathbb{N}$ are positive integers. From Theorem 6 and Equation (20) we infer that for each $i \in \{1, \ldots, n\}$ we can use the identity $\psi^{-1}(g_i) = \varphi(g_i)$. Using these two modifications we get the equation

$$q \varphi(g) = \sum_{i=1}^{n} qx_i \varphi(g_i).$$

Since $\varphi$ is an injective group homomorphism, we can rewrite this equation as

$$\varphi(qg) = \varphi \left( \sum_{i=1}^{n} qx_i g_i \right)$$

and get $qg = \sum_{i=1}^{n} qx_i g_i$.

Thus, $qg \in G_+$, which implies $g \in G_+$, since the order of $G$ is semiclosed. This proves the other inclusion

$$\text{conv}_Q \left[ \psi^{-1}(G_+ \cap \langle E \rangle) \right] \cap \varphi[G] \subseteq \varphi[G_+]$$

and, thus, the lemma is proved. \(\square\)

4. Linear orders in real vector spaces

Here, we will discuss linear vector space orderings. As already mentioned in the last section this touches the area of fundamental theorems of the theory of $\ell$-groups with the additional insight how different orders can be represented in the same vector space.

We will prove that the positive cone in the generated subspace of a class of archimedian equivalent elements of a linearly ordered vector space is defined by linear half spaces which are bounded by linearly ordered hyperplanes. Using Zorn’s lemma, it can be shown that this is equivalent to the description by linearly ordered bases and an extension to Teh’s result [25]. In contrast to that work, we use properties of normed vector spaces to prove this result for all torsion free abelian groups.

At first we discuss possible linear orders on the vector space $\oplus E \mathbb{R}$ for a given set $E$. In each of these orders a hyperplane will contain all infinitesimal elements, and the space will be filled with copies of this hyperplane (c.f., Figure 6).

**Lemma 10.** Let $\mathfrak{H}$ be a linearly ordered hyperplane in the vector space $\oplus E \mathbb{R}$ and $a \in \oplus E \mathbb{R} \setminus \mathfrak{H}$ a vector which is linearly independent from $\mathfrak{H}$. Then the set

$$(24) \quad \mathfrak{H}_+ \cup (\mathfrak{H} + (\mathbb{R}_+ \setminus \{0\})a)$$

defines a linear order on the vector space $(\oplus E \mathbb{R}, +, -, o)$. 

Furthermore, since $H \oplus I$ in short:

$P \in \mathbb{R}$ where $H$ above follows.

Let $P := H_+ \cup (H + (\mathbb{R}_+ \setminus \{0\}))a$ be the half space, which is defined by the positive cone of the hyperplane $H_+$ and the vector $a$. Since $\oplus_{\mathbb{R}}$ is a group considering the addition, We have to prove the conditions (2) to (4) considering $P$ as positive cone of $\oplus_{\mathbb{R}}$. This will prove that $P$ is an invariant semigroup in $\oplus_{\mathbb{R}}$ and for each vector $x \in \oplus_{\mathbb{R}}$ either $x \in P$ or $-x \in P$ holds.

Let $x \in P$ be a vector in $P$. Then either the condition $x \in H_+$ or $x \in H + (\mathbb{R}_+ \setminus \{0\})a$ holds.

1. The condition $x \in H_+$ is true iff $-x \in H_-$, since $H$ is a vector space itself and linearly ordered by precondition.
2. If $x \in H + (\mathbb{R}_+ \setminus \{0\})a$, then a unique vector $x' \in H$ and a unique positive real number $x \in \mathbb{R}_+ \setminus \{0\}$ exist such that $x = x' + xa$. This is equivalent to $-x = -x' + (-x)a \in H + (\mathbb{R}_- \setminus \{0\})a$

Furthermore, since $H$ is as hyperplane in $\oplus_{\mathbb{R}}$ and $a \notin H$, the following holds:

$\oplus_{\mathbb{R}} = H \oplus Ra = H + Ra$

$= H + (\mathbb{R}_+ \setminus \{0\})a \cup (H_+ \setminus \{0\}) \cup \{0\} \cup (H_- \setminus \{0\}) \cup H + (\mathbb{R}_- \setminus \{0\})a$

$= P \cup \{0\} \cup \{0\} \cup -P \cup \{0\}$

In short: $\oplus_{\mathbb{R}} = P \cup -P$, so the Condition (4) is fulfilled. From the two points above follows $P \cap -P = \{0\}$, which is the Condition (3).

Let’s consider the remaining Conditions (2) and (4). Let $x, \eta \in P$ be two vectors in $P$. Then there exist unique decompositions

$x = x' + xa$ and

$\eta = \eta' + ya$

where $x', \eta' \in H$ and $x, \eta \in \mathbb{R}_+$. For two non-negative real numbers $\alpha, \beta \in \mathbb{R}_+$ this leads to the equation

$\alpha x + \beta \eta = (\alpha x' + \beta \eta') + (\alpha x + \beta y)a$.

Since the values $\alpha, \beta, x, y \geq 0$ are non-negative, the inequality $\alpha x + \beta y \geq 0$ is true.

Equally, the sum $\alpha x' + \beta \eta' \in H$ belongs to the hyperplane. Consequently, for a non-negative sum $\alpha x + \beta y \neq 0$ also the sum of the vectors $\alpha x + \beta \eta \in P$ is a member of the set $P$. On the other hand, the equation $\alpha x + \beta y = 0$ induces $\alpha x = \beta y = 0$ as both values are non-negative. This means that $\alpha x \in H_+$, as well as $\beta \eta \in H_+$. From the semigroup properties of the positive cone of the hyperplane $H$ follows that also the linear combination $\alpha x + \beta \eta \in H_+$ is an element of this subspace. So we have shown for all vectors $x, \eta \in P$ from the half space $P$ and all non-negative real numbers $\alpha, \beta \in \mathbb{R}$ that the linear combination $\alpha x + \beta \eta \in P$ belongs to $P$,
too. Thus, \( P \) is a subsemigroup of \( \oplus E \mathbb{R} \), so the Condition (2) holds, and since \( \oplus E \mathbb{R} \) is commutative \( -P \) is also an invariant subsemigroup of the vector space, which proves condition (3). From Theorem 1 follows that \( P \) defines a linear order on the vector space \( \oplus E \mathbb{R} \) considered as a group. Using the unique decomposition \( \mathbf{x} = \mathbf{x}' + \alpha \mathbf{a} \) for any vector \( \mathbf{x} \in P \) with \( \mathbf{x}' \in \mathcal{H} \) we infer that \( \alpha \mathbf{x}' \in \mathcal{H} \) and \( \alpha \mathbf{x}' + \alpha \mathbf{a} \in P \). Thus the vector space \( \oplus E \mathbb{R} \) is a linearly ordered vector space. \( \square \)

In the opposite direction we can only limit the codimension of the set of infinitesimal elements.

**Lemma 11.** Let \( \oplus E \mathbb{R} = (\oplus \mathbb{R}, +, -, \circ, \leq) \) be an ordered vector space. Then the boundary \( \partial \left( (\oplus E \mathbb{R})_+ \right) \) of the positive cone is a linear subspace of \( \oplus E \mathbb{R} \).

*Proof.* Let \( \mathbf{x} \in \partial \left( (\oplus E \mathbb{R})_+ \right) \) a boundary vector of the positive cone. Then for all \( r \in \mathbb{R} \setminus \{ 0 \} \) the intersections \( B_r(\mathbf{x}) \cap (\oplus E \mathbb{R})_+ \) and \( B_r(\mathbf{x}) \cap (\oplus E \mathbb{R})_- \) are non-empty. Furthermore, for \( r \neq 0 \) either \( \mathbf{x} \in (\oplus E \mathbb{R})_+ \) or \( -\mathbf{x} \in (\oplus E \mathbb{R})_+ \) holds.

In fact, \( \mathbf{x} \in \partial \left( (\oplus E \mathbb{R})_+ \right) \) is satisfied iff \( -\mathbf{x} \in \partial \left( (\oplus E \mathbb{R})_+ \right) \). This follows as \( \mathbf{x}' \in B_r(\mathbf{x}) \cap (\oplus E \mathbb{R})_+ \) is a member of the subset of the positive elements of a neigh-bourhood of the vector \( \mathbf{x} \) iff \( -\mathbf{x}' \in B_r(-\mathbf{x}) \setminus (\oplus E \mathbb{R})_+ \) is located in a corresponding subset of negative elements. An analogue result is true for \((\oplus E \mathbb{R})_-)\). Thus, the neighbourhood \( B_r(-\mathbf{x}) \) contains as well positive as well as negative elements.

In a next step we show that \( \partial \left( (\oplus E \mathbb{R})_+ \right) \) is a linear subspace in \( \oplus E \mathbb{R} \). To do that, we choose two vectors \( \mathbf{x}, \mathbf{y} \in \partial \left( (\oplus E \mathbb{R})_+ \right) \). Then for arbitrary positive real numbers \( r, \alpha, \beta \in \mathbb{R}_+ \setminus \{ 0 \} \) four vectors \( \mathbf{x}', \mathbf{x}'', \mathbf{y}' \) and \( \mathbf{y}'' \) exist in the following way:

\[
\mathbf{x}' \in B_{\frac{r}{\alpha}}(\mathbf{x}) \cap (\oplus E \mathbb{R})_+ \quad \mathbf{x}'' \in B_{\frac{r}{\alpha}}(\mathbf{x}) \setminus (\oplus E \mathbb{R})_+ \\
\mathbf{y}' \in B_{\frac{r}{\beta}}(\mathbf{y}) \cap (\oplus E \mathbb{R})_+ \quad \mathbf{y}'' \in B_{\frac{r}{\beta}}(\mathbf{y}) \setminus (\oplus E \mathbb{R})_+
\]

Depending on the values of \( \alpha \) and \( \beta \) this leads to the following six cases:

\( \alpha > 0 \) and \( \beta > 0 \): Here, we get the equations

\[
\alpha \mathbf{x}' + \beta \mathbf{y}' = \alpha \mathbf{x} + \alpha (\mathbf{x}' - \mathbf{x}) + \beta \mathbf{y} + \beta (\mathbf{y}' - \mathbf{y}) \tag{\text{and}}
\]

\[
\alpha \mathbf{x}'' + \beta \mathbf{y}'' = \alpha \mathbf{x} + \alpha (\mathbf{x}'' - \mathbf{x}) + \beta \mathbf{y} + \beta (\mathbf{y}'' - \mathbf{y}) \nonumber
\]

Thus, in \( B_r(\alpha \mathbf{x} + \beta \mathbf{y}) \) are positive as well as negative elements of the vector space \( \oplus E \mathbb{R} \), so \( \alpha \mathbf{x} + \beta \mathbf{y} \) is also an element of \( \partial \left( (\oplus E \mathbb{R})_+ \right) \).

\( \alpha > 0 \) and \( \beta < 0 \): This case an be deduced to the last case using the representation

\[
\alpha \mathbf{x} + \beta \mathbf{y} = \alpha \mathbf{x} + (-\beta)(-\mathbf{y}),
\]

since \( \mathbf{y} \) is an element of the boundary \( -\mathbf{y} \) is within the boundary, too.

\( \alpha < 0 \) and \( \beta > 0 \): As the addition is commutative this is identical to the pre-decreasing case.
\( \alpha < 0 \) and \( \beta < 0 \): With the same idea as in the predeceasing cases, the result follows from the first statement.

\( \alpha \neq 0 \) and \( \beta = 0 \): Here, we have \( \beta \eta = a \). So analogue to the first case follows:

\[
\alpha \eta' + \beta \eta'' = \alpha \eta + \alpha (\eta' - \eta) \in B_\gamma(\alpha \eta) \cap (\oplus E \mathbb{R})_+
\]

\[
\in B_\gamma(\alpha \eta) \backslash (\oplus E \mathbb{R})_+
\]

\( \alpha = 0 \): For \( \beta \neq 0 \) this case is analogue to the predeceasing case. Otherwise we know \( \alpha \eta + \beta \eta = a \in \partial (\oplus E \mathbb{R})_+ \).

So far, we have shown that the relevant operations of the vector space, addition of vectors, and multiplication with scalars do not exit \( \partial (\oplus E \mathbb{R})_+ \). Thus, this set is a linear subspace of \( \oplus E \mathbb{R} \).

The next step is to determine the dimension of the boundary

**Lemma 12.** Let \( \oplus E \mathbb{R} = (\oplus_M \mathbb{R}, +, -, o, \leq) \) be an ordered vector space. Then for the codimension of the boundary \( \partial (\oplus E \mathbb{R})_+ \) the following equation holds:

\[
\text{codim} \partial (\oplus E \mathbb{R})_+ = \begin{cases} 
0, & \text{iff } (\oplus E \mathbb{R})_+ = \emptyset \\
1, & \text{else}
\end{cases}
\]

**Proof.** Let us assume that the interior of \( (\oplus E \mathbb{R})_+ \) is non-empty. By Lemma 11 the boundary \( \partial (\oplus E \mathbb{R})_+ \) is a linear subspace of \( \oplus E \mathbb{R} \). To be a hyperplane, this set has to be a subspace with codimension 1. To show this, we assume that we have at least codimension 2 and deduce a contradiction from that assumption. At the beginning we have to find a vector, which is not in \( \partial (\oplus E \mathbb{R})_+ \). Let \( \mathbf{r}, \eta \in (\oplus E \mathbb{R})_+ \) be two arbitrary positive elements of \( \oplus E \mathbb{R} \). Then either \( \mathbf{r} \leq \eta \) or \( \eta \leq \mathbf{r} \) follows from the linear order. Let w.l.o.g. \( \mathbf{r} \leq \eta \). This leads to the inequality \( o \leq \eta - \mathbf{r} \) and we get for \( 0 \leq \lambda \leq 1 \):

\[
\lambda \mathbf{r} + (1 - \lambda) \eta = \lambda \mathbf{r} + (1 - \lambda)(\eta - \mathbf{r}) + (1 - \lambda)\mathbf{r} \\
= \mathbf{r} + (1 - \lambda)(\eta - \mathbf{r}) \geq \mathbf{r},
\]

since \( 0 \leq 1 - \lambda \) holds, and the positive cone \( (\oplus E \mathbb{R})_+ \) is convex in the sense of ordered groups. Consequently, \( (\oplus E \mathbb{R})_+ \) is convex in the sense of vector space and with the positive cone, the negative cone \( (\oplus E \mathbb{R})_- = -(\oplus E \mathbb{R})_+ \) has this property, too. So we have the identity \( \mathcal{L}(\oplus E \mathbb{R})_+ = \mathcal{L}(\oplus E \mathbb{R})_- = \oplus E \mathbb{R} \), since \( \oplus E \mathbb{R} = (\oplus E \mathbb{R})_- \cup (\oplus E \mathbb{R})_+ \). As for each base vector \( \mathbf{e}_a \) with \( a \in E \) either \( \mathbf{e}_a \) or the vector \( -\mathbf{e}_a \) is positive, and we can replace in the basis each negative basis vector with its inverse we can assume w.l.o.g. that \( \{ \mathbf{e}_a | a \in E \} \subseteq (\oplus E \mathbb{R})_+ \). This is also true for its convex hull, which leads to \( \text{conv}\{ \mathbf{e}_a | a \in E \} \cap \text{int}(\oplus E \mathbb{R})_+ \neq \emptyset \).

Let \( a \in \text{conv}\{ \mathbf{e}_a | a \in E \} \cap \text{int}(\oplus E \mathbb{R})_+ \), and let us assume that the codimension of \( \partial (\oplus E \mathbb{R})_+ \) is greater than 1. Then besides \( a \) another vector \( \mathbf{r} \in (\oplus E \mathbb{R})_+ \) exists which is linearly independent from \( \{ a \} \cup \partial (\oplus E \mathbb{R})_+ \). Consequently, there exists an isomorphism \( \xi \), which maps the linear closure of \( a \) and \( \mathbf{r} \) onto \( \mathbb{R}^2 \) such that \( \xi(a) = (1, 0) \) and \( \xi(\mathbf{r}) = (0, 1) \) are true. Obviously, the natural order on \( \mathbb{R}^2 \)}
is compatible to the order, which is transferred from $\oplus_E \mathbb{R}$ to $\mathbb{R}^2$ by $\xi$. So we can extend the order on $\mathbb{R}^2$ linearly in such a way, that $\xi$ is an o-isomorphism.

Using this order extension, we know that $(-\frac{1}{2}, -1) < (0, 0) < (1, \frac{1}{2})$ holds.

Now, we will define a mapping

$$f : \mathbb{R} \to \{-1, 0, 1\} : \lambda \mapsto \begin{cases} 1 & (1, \frac{1}{2}) - \lambda(\frac{3}{2}, \frac{3}{2}) > (0, 0) \\ -1 & (1, \frac{1}{2}) - \lambda(\frac{3}{2}, \frac{3}{2}) < (0, 0) \\ 0 & \text{else} \end{cases}$$

Obviously, $f(0) = 1$ and $f(1) = -1$. Thus, $f$ is discontinuous according the natural topology on $\mathbb{R}$. Let $\lambda_0$ be a place of discontinuity of $f$. Since $\mathbb{R}^2$ is linearly ordered and

$$(0, 0) \not\in \left(1, \frac{1}{2}\right) - \mathbb{R} \left(\frac{3}{2}, \frac{3}{2}\right),$$

one of the identities $f(\lambda_0) = 1$ or $f(\lambda_0) = -1$ is met. Thus, for each $\varepsilon \in \mathbb{R}$ there exists a real number $\lambda_1 \in \mathbb{R}$ such that $|\lambda_0 - \lambda_1| < \varepsilon$ and $f(\lambda_1) = -f(\lambda_0)$. This implies that in $B_1(\varphi^{-1}((1, \frac{1}{2}) - \lambda_0(\frac{3}{2}, \frac{3}{2})))$ a positive vector $\eta' \in (\oplus_E \mathbb{R})_+$ and a negative vector $\eta'' \not\in (\oplus_E \mathbb{R})_+$ exist. So $\varphi^{-1}((1, \frac{1}{2}) - \lambda_0(\frac{3}{2}, \frac{3}{2})) \in \partial ((\oplus_E \mathbb{R})_+ \setminus \{0\})$, this is a contradiction to the assumption that this set has no non-zero representatives in the linear closure of $r$ and $a$. Thus, the boundary $\partial ((\oplus_E \mathbb{R})_+)$ is a hyperplane in $\oplus_E \mathbb{R}$ with $\partial ((\oplus_E \mathbb{R})_+) \oplus \mathbb{R}a = \oplus_E \mathbb{R}$.

As already shown $\text{conv}\{e_a | a \in E\}$ can be considered to be a subset of the positive cone. It also generates the vector space, so the set $\text{conv}\{e_a | a \in E\} \cap \text{int}((\oplus_E \mathbb{R})_+)$ is empty iff $\text{int}((\oplus_E \mathbb{R})_+)$ is empty. In that case the codimension is zero.

Consequently the boundary of the positive cone in any ordered vector space is either a hyperplane or it is the space itself. With this knowledge we can define some additional notations for a vector $a$ of an ordered vector space:

(26) $\mathcal{U}_a := \mathcal{L}\{r \in \oplus_E \mathbb{R} | o \leq r \leq |a|\}$  and

(27) $\mathcal{H}_a := \mathcal{L}\{r \in \oplus_E \mathbb{R} | o \leq r \ll |a|\}$

In general the subspace $\mathcal{U}_a$ exists only for such vectors $a$ that have assigned a unique absolute element $|a| \geq o$. For that it is sufficient that there exists a lattice ordered subspace containing $a$. If the vector space is linearly ordered $\mathcal{H}_a$ is a hyperplane and Equation (27) can be extended to

$\mathcal{H}_a = \partial ((\mathcal{U}_a)_+)$

After we have shown that all linearly ordered hyperplanes define linear orders, we can prove that all linear orders on the vector space $\oplus_E \mathbb{R}$ have such a representation.

**Theorem 13.** Let $\oplus_E \mathbb{R} = (\oplus_E \mathbb{R}, +, -, o, \leq)$ be an ordered vector space. The partial order $\leq$ is linear iff one of the following conditions is true:

1. The positive cone $(\oplus_E \mathbb{R})_+$ has the form

(28) $$(\oplus_E \mathbb{R})_+ = \mathcal{H}_a \cup (\mathcal{H} + (\mathbb{R}_+ \setminus \{0\})a),$$

where $\mathcal{H}$ is a linearly ordered hyperplane in $\oplus_E \mathbb{R}$ and $a \in \oplus_E \mathbb{R} \setminus \mathcal{H}$ is a positive vector from the interior of the positive cone.

2. The vector space $\oplus_E \mathbb{R}$ is the union of an infinite chain of proper subspaces of the form given in (27).
(3) $E = \emptyset$.

Proof. As $\emptyset$ is trivial we have to prove that for every non-trivial vector space either 1 or 2 is true. By Lemma 10 the vector space is linearly ordered if the condition 1 holds. By Lemma 11 and Lemma 12 the remaining part consists of two questions:

(1) Is it sufficient that a vector space is the union of an infinite chain of linearly ordered subspaces in order to get a linear order?

(2) Can every linear order with empty interior of the positive cone be described by such a chain?

In order to answer the first question, suppose there are two incomparable elements $a, b \in \oplus E$. Then there exist linearly ordered subspaces $U_a$ and $U_b$ in the chain of subspaces that generates the order such that $a \in U_a$ and $b \in U_b$. As both vector spaces belong to the same chain either $U_a \leq U_b$ or $U_b \leq U_a$ holds. As $a$ and $b$ are incomparable and the vector spaces are linearly ordered we get the additional condition $a \not\in U_b$ and $b \not\in U_a$. Thus both vector spaces do not belong to the same chain in the subspace lattice, which is a contradiction to the assumption. Consequently, we can answer the first question with “Yes”.

To answer the other question let $a, b \in \oplus E$ be two vectors with $a < b$. Then either $a \in H_b$ or $H_a \oplus Ra = H_b \oplus Rb$ holds. Thus, the set

$$\mathcal{M}_b := \{ H_a \oplus Ra \mid a \leq b \}$$

is a chain in the subspace lattice of $\oplus E$. Thus, each order ideal in the ordered set $(\mathcal{M}, \leq)$ with

$$\mathcal{M} = \bigcup_{b \in \oplus E} \mathcal{M}_b$$

is a chain. With a similar argumentation we can prove that each order filter in this ordered set is a chain, too. Thus, $(\mathcal{M}, \leq)$ is a chain itself. As it describes the linear order as discussed in the previous question, also this question can be answered with “Yes”.

Corollary 14. Considering the hyperplane $\mathcal{H}$ from Lemma 10, Condition 7 and one of the linear orders, which arise from this hyperplane, in Theorem 13 the following equation holds:

$$\mathcal{H}_+ = (\oplus E)_+ \cap \partial \left( (\oplus E)_+ \right).$$

Proof. The positive cone of the hyperplane is uniquely defined. \qed

The previous lemma shows that linear vector space orders on $\oplus E$ can be characterised by hyperplanes in sufficiently chosen subspaces. Thus, each linear order on a given abelian group can be represented by one or an infinite chain of linearly ordered hyperplanes in this space. Obviously, for each linearly ordered hyperplane $\mathcal{H}$ two linear orders on $\oplus E$ exist: A first one, where the open half-space marked by a given vector $a \in \oplus E \setminus \mathcal{H}$ is positive and a second one where it is negative. Consequently, a given basis of the vector space can be a subset of the positive cone only in one of these two cases.

As the lattice of linear subspaces is inductively ordered and since all subspaces of a given subspace are subspaces of hyperplanes in the bigger subspace, we can apply Zorn’s lemma and receive the result that each linear order on a vector space $\mathcal{W}$ is uniquely defined by a given linearly ordered base $(\mathcal{B}, \leq)$ according to the following construction:
Let \( a \in B \) be a base vector. Furthermore, let \( \mathcal{V}_a = L\{ a' \mid a' \leq a \} \) and \( \mathcal{H}_a := L\{ a' \mid a' < a \} \) a linearly ordered hyperplane. Then the subspace \( \mathcal{V}_a \) can be linearly ordered by Lemma 10 such that \( a > 0 \) is positive and \( \mathcal{V}_a \) is itself ordered as described above for the vectors \( a' < a \) in \( B \).

Moreover, the vector spaces \( \mathcal{V}_a \) form a chain in the lattice of subspaces of \( \oplus E \mathbb{R} \) that fulfills the following condition:

\[
\oplus E \mathbb{R} = \bigcup_{a \in B} \mathcal{V}_a.
\]

Thus, \( \oplus E \mathbb{R} \) can be linearly ordered, since the order on the hyperplane above could be chosen arbitrarily. Obviously, for a given linear order the basis is not unique as the subspaces and their order is not influenced by the multiplication of basis vectors with scalars.

We will discuss this construction not in further details\(^6\) as some aspects of this topic have been used already in different flavours in other literature e.g., the works of The\[25\] and Zajceva\[28\].

5. Linear orders on groups

The characterisations of linear orders on groups in The’s\[25\] and Zajceva’s\[28\] articles both have drawbacks. The first description needs different vector spaces to describe all linear orders on a given group, while the inductive nature of the latter restricts it to mainly countable dimensional vector spaces. Nevertheless, several authors have discussed the existence of linear orders on infinitely generated groups by considering finitely generated subgroups (cf.\[7\]\[18\]\[22\]). So it is convenient to expect also characterisations of linear orders on groups based on those which have been found for finitely generated groups.

The approach we use here has been developed independently from the mentioned articles. With the results of the preceding sections, we are able to describe the linear orders on an abelian group with the help of linearly ordered hyperplanes. Given an abelian group \( G = (G,+,-,0) \) and a maximal independent set \( E \subseteq G \), each linearly ordered base in \( \oplus E \mathbb{R} \) defines an order on \( G \), and each linear order on the vector space \( \oplus E \mathbb{R} \) can be reached that way. Linearly ordered bases define systems of hyperplanes that are linearly ordered by set inclusion. Each of this systems defines a different order on \( \oplus E \mathbb{R} \). This is assured by Lemma\[10\] Theorem\[13\] and the monomorphisms in the Diagram\[5(a)\] Theorem\[8\] and Lemma\[9\] assure that each linear order on the group can be described as linear order on the real vector space \( \oplus E \mathbb{R} \).

So there is (among others) one question remaining: Which linearly ordered bases define the same linear order on the Group \( G \)? We will address this question in the current section.

A subspace \( \mathcal{U} \) of \( \oplus E \mathbb{R} \) may be multidimensional, though it contains no rational vectors \( (\mathcal{U} \cap \oplus E \mathbb{Q} = \{ 0 \}) \). On the other hand we know that the linear closure of the rational vectors is the complete vector space. Thus, omitting a vector from a basis leads to loss of rational vectors. This means, that every vector in a linearly ordered base is necessary in order to define a linear order on the rational “subspace“. In order to describe this we will use the following notions.

\(^6\)The complete discussion will be made available via\[23\]
Corollary 17. For every basis vector $y \in \mathcal{B}$ we consider two linearly ordered bases equivalent iff they generate the same order, i.e. their primary ideals generate the same subspaces.

\begin{equation}
L(U \setminus \{ y \}) \cap \oplus E \mathbb{Q} \neq L(U \cap \oplus E \mathbb{Q}).
\end{equation}

Otherwise $y$ is called (rationally) passive.

An element $a \in \mathcal{B}$ is called activator of another element $y \in \mathcal{B}$ iff both vectors $a$ and $y$ are active in the principal ideal $\downarrow a$. If $y$ is its own activator it is called self-activator. The set of activators of an element $y$ is denoted by $\text{Act}_{\mathcal{B}}$.

Furthermore let $\mathcal{V}$ be a subspace of $\oplus E \mathbb{R}$. Then the set $\text{Act}_{\mathcal{B}} \mathcal{V} := L(\oplus E \mathbb{Q} \cap \mathcal{V})$ is called the active subspace of $\mathcal{V}$.

Corollary 15. Let $(\mathcal{B}, \leq)$ be a linearly ordered basis of $\oplus E \mathbb{R}$ and $U \subseteq \mathcal{B}$ a corresponding subset. Then the active subspace of $LU$ is the linear hull of the active vectors from $U$ in $LU$.

Lemma 16. Let $(\mathcal{B}, \leq)$ the linearly ordered basis of the linearly ordered vector space $\oplus E \mathbb{R}$. Then the set of activators of an element $y \in \mathcal{B}$ is an order filter in the set of self-activators of $\mathcal{B}$.

Proof. By definition, a vector $a$ is an activator of $y$ iff it is a self-activator and the condition

$$L(\downarrow a \setminus \{ y \}) \cap \oplus E \mathbb{Q} \neq L(\downarrow a \setminus \{ y \}) \cap \oplus E \mathbb{Q}.$$

holds. The latter condition is equivalent to the existence of a vector $b \in \mathcal{V}_a \cap \oplus E \mathbb{Q}$ whose projection along $\mathcal{B}$ into $L_y$ is non-zero. Such a vector exists, iff there exists a basis vector $c$ such that $\mathcal{V}_a = \mathcal{V}_b$ and $c$ is an activator of $y$. Since in linearly ordered vector spaces $b \in \mathcal{V}_a$ is the same as $\mathcal{V}_b \subseteq \mathcal{V}_a$, we know that $c \leq a$. The vector $b$ also proves that this basis vector $c$ is also an activator of $y$ as $b \not\in H_b = H_y$. On the other hand for each activator $c \leq a$ of $y$ we can find a vector $b$ whose projection into $L_y$ is non-zero. Thus we have proved so far: A self-activator $a \in \mathcal{B}$ is an activator of $y$ iff there exists a self-activator $c \leq a$ that is also an activator of $y$. This implies that for each activator $c \in \text{Act}_y$ of $y$ each self-activator $a \geq c$ is also an activator of $y$. \hfill \Box

Corollary 17. For every basis vector $x$ and every activator $\eta \in \text{Act}_x$ the relation $\eta \in \text{Act}_x$ holds.

We consider two linearly ordered bases equivalent iff they generate the same order, i.e. their primary ideals generate the same subspaces $\oplus E \mathbb{Q}$.

Lemma 18. For every linearly ordered basis $\mathcal{B}$ there exists an equivalent basis $\mathcal{B}'$ such that for each basis vector $x \in \mathcal{B}'$ the condition

\begin{equation}
x \in \text{Act}_x \iff x \in \oplus E \mathbb{Q}
\end{equation}

holds.

Proof. Let $x \in \mathcal{B}$ be a self-activator ($x \in \text{Act}_x$). We know by definition that $L(L(\downarrow x) \cap \oplus E \mathbb{Q}) = L(L_x \cap \oplus E \mathbb{Q})$ and $L(L(\downarrow x \setminus \{ x \}) \cap \oplus E \mathbb{Q}) = H_x \cap \oplus E \mathbb{Q}$ hold. By the presumption we know $(L_x \setminus H_x) \cap \oplus E \mathbb{Q} \neq \emptyset$. Let us choose a vector $f(y) \in (L_x \setminus H_x) \cap \oplus E \mathbb{Q}$ for every self-activator $x \in \mathcal{B}$. In that case we know $\mathcal{V}_f(y) = L_y$. Otherwise choose $f(y) = y$. Then $f$ is a mapping from $\mathcal{B}$ into the vector space $\oplus E \mathbb{R}$. And $\mathcal{B}' := \{ f(y) \mid y \in \mathcal{B} \}$ fulfills the condition if $f(y) \leq f(\eta) \iff x \leq y$. \hfill \Box
Thus, we can develop a standard form for linearly ordered bases.

**Definition 4.** A linearly ordered base \( \mathcal{B} \) is called **clarified**, iff it fulfills the condition

\[
\forall x \in \text{Act'} x \Leftrightarrow x \in \oplus_{E} \mathbb{Q}.
\]

As we can describe every linear order by a clarified base, in the remaining part of this section, we will consider all bases to be clarified if it is not stated otherwise.

**Lemma 19.** Let \( \mathcal{B} \) be a base of \( \oplus_{E} \mathbb{R} \) and \( a, b \in \mathcal{B} \) be two different basis vectors. Then the projection of \( \oplus_{E} \mathbb{Q} \) along \( \mathcal{B} \setminus \{ a, b \} \) into \( \mathcal{L}(a, b) \) is 2-dimensional.

**Proof.** Since both vectors \( a \) and \( b \) are linear combinations of the standard basis, the projection of the standard base vectors contains a nontrivial vector \( \mathfrak{z} \). Let \( P \) be the projection along \( \mathcal{B} \setminus \{ a, b \} \). Then from \( \mathfrak{z} \in P[\oplus_{E} \mathbb{Q}] \) follows the set inclusion \( \mathcal{L}\{\mathfrak{z}\} \subseteq \mathcal{L}(P[\oplus_{E} \mathbb{Q}]) \subseteq \mathcal{L}\{a, b\} \).

Suppose \( \mathcal{L}\{\mathfrak{z}\} = \mathcal{L}(P[\oplus_{E} \mathbb{Q}]) \). Then we can express every rational vector as a linear combination of \( \mathfrak{z} \) and \( \mathcal{B} \setminus \{ a, b \} \). This leads to the set inclusion

\[
\oplus_{E} \mathbb{Q} \subseteq \mathcal{L}((\mathcal{B} \setminus \{ a, b \}) \cup \{ \mathfrak{z} \}) \subseteq \mathcal{L}_{\mathcal{B}} = \mathcal{L}(\oplus_{E} \mathbb{Q}).
\]

As the linear hull is a closure operator, we get immediately the set inclusion

\[
\mathcal{L}(\oplus_{E} \mathbb{Q}) \subseteq \mathcal{L}((\mathcal{B} \setminus \{ a, b \}) \cup \{ \mathfrak{z} \}) \subseteq \mathcal{L}_{\mathcal{B}} = \mathcal{L}(\oplus_{E} \mathbb{Q}).
\]

As \( \mathfrak{z} \) is linear dependend from the vectors \( a \) and \( b \), at least one of the equations \( \mathcal{L}\{a, b\} = \mathcal{L}\{a, \mathfrak{z}\} \) and \( \mathcal{L}\{a, b\} = \mathcal{L}\{\mathfrak{z}, b\} \) holds. W.l.o.g. we can assume \( \mathcal{L}\{a, b\} = \mathcal{L}\{a, \mathfrak{z}\} \). Then we can exchange \( b \) in \( \mathcal{B} \) with \( \mathfrak{z} \), too. We get

\[
\mathcal{L}(\oplus_{E} \mathbb{Q}) = \mathcal{L}(\oplus_{E} \mathbb{R}) = \mathcal{L}_{\mathcal{B}} = \mathcal{L}((\mathcal{B} \setminus \{ b \}) \cup \{ \mathfrak{z} \})
\]

As we have seen above, from this follows that \( a \) is linear dependent from the set \( (\mathcal{B} \setminus \{ a, b \}) \cup \{ \mathfrak{z} \} \). Thus, we can express \( a \) as a linear combination of the vectors from \( (\mathcal{B} \setminus \{ a, b \}) \cup \{ \mathfrak{z} \} \). As \( \mathcal{B} \) is a basis this representation has a nonzero \( \mathfrak{z} \) component and can express \( \mathfrak{z} \) as a linear combination of the vectors from \( \mathcal{B} \setminus \{ b \} \). Consequently we get the following equations

\[
\mathcal{L}((\mathcal{B} \setminus \{ a, b \}) \cup \{ \mathfrak{z} \}) = \mathcal{L}(\mathcal{B} \setminus \{ b \}) = \mathcal{L}(\oplus_{E} \mathbb{Q}) = \mathcal{L}(\mathcal{B}).
\]

This shows that either \( \mathcal{B} \) is not a basis, that \( a = b \), or that the assumption \( \mathcal{L}\{\mathfrak{z}\} = \mathcal{L}(P[\oplus_{E} \mathbb{Q}]) \) is wrong. \( \square \)

Using this lemma, we can find a first condition, which explains when two different orders on \( \oplus_{E} \mathbb{R} \) imply the same order on \( \oplus_{E} \mathbb{Q} \).

**Lemma 20.** Let the vector space \( \oplus_{E} \mathbb{R} \) be ordered twice with the linear order relations \( \leq \) and \( \leq' \). Furthermore let \( \mathcal{B} \) a Basis that describes both \( \leq \) and \( \leq' \), and Act respectively Act’ the corresponding operators that provide the set of activators of \( \leq \) respectively \( \leq' \). Then the induced orders on the rational vector space \( \oplus_{E} \mathbb{Q} \) coincide iff for every element \( x \in \mathcal{B} \) the sets of Activators Act \( x \) and Act’ \( x \) are equal.

**Proof.** We have to consider four cases:

1. The two bases are equally ordered. In that case the orders also share the same sets of activators.
2. There exist two self-activators \( \mathfrak{x} \) and \( \mathfrak{y} \) with \( \mathfrak{x} \leq \mathfrak{y} \) and \( \mathfrak{y} \leq' \mathfrak{x} \).
3. The sets of self-activators are different and the orders are the same on those elements which are self-activators according to both orders.
Lemma 19 we can find a vector \( H \) with \( H \times a \) to Act is smaller than the one of a non-zero \( z \) proves the assertion.

Proof. Let us choose a rational vector \( a \subseteq (V_f \setminus H_f) \cap \oplus E Q \)
such that the component of \( a \) in \( f \) direction and the component of \( b \) in \( \eta \) direction are non-zero. Then \( a < b <' a \) holds.

In case 3 a vector \( f \subseteq B \) exists such that \( f \in \text{Act} f \) and \( f \notin \text{Act}' f \) hold. Then we choose a rational vector \( a \subseteq (V_f \setminus H_f) \cap \oplus E Q \) such that the component in \( f \) direction is different from zero. Let us choose \( \eta \subseteq B \) such that \( V'_\eta = V'_\eta \). As \( a \) has a non-zero \( \eta \) component, according to B, we have the inequality \( V_\eta \subseteq V_f \) leading to \( \text{Act} f \subseteq \text{Act} \eta \). As it has a non-zero \( f \) component, too, we get \( \text{Act} \eta \subseteq \text{Act}' f \). By Lemma 19 we can find a vector \( b \subseteq V'_\eta \cap \oplus E Q \) whose \( f \) component is larger than the one of \( a \) and whose \( \eta \) component is smaller than the one of \( a \). Then \( a < b <' a \) proves the assertion.

For the remaining part 4 we prove that two inactive basis vectors \( f, \eta \subseteq B \) do not influence the result. Suppose the relations \( f \notin \text{Act}' f, f \notin \text{Act} f, \) and \( f < \eta <' f \). Suppose there are two vectors \( a, b \subseteq \oplus E V \) with \( a < b \). As all vectors have finite support with respect to the standard base, their support with respect to \( B \) is finite, too. Thus, there exists a largest basis vector \( z \subseteq B \) such that the \( z \) component of \( a \) is smaller than the one of \( b \). Then both vectors \( a, b \subseteq V_z \) belong to its vector space, while at least one of them is not in the corresponding hyperplane \( H_z \) leading to \( z \in \text{Act} z \). Thus, \( f \neq z \). Suppose \( f < z <' f \) and \( a \) has a non-zero \( f \) component. Then there would be a self-activator \( z' \subseteq \text{Act} z' \cap \text{Act} f \) such that \( a \) has a non-zero \( z' \) component. As \( a \subseteq V_z \) this implies \( z' < z <' z' \) which has been excluded by the presumption. Consequently, either \( a \) has a zero \( f \) component or \( f < z \Leftrightarrow f <' z \) holds. The same is true for any other non-self-activator. As the orders on the self-activators coincide between the relations \( \leq \) and \( \leq' \), the \( z \) component remains the main component to determine the order of \( a \) and \( b \) leading to \( a <' b \).

In Theorem 13 we have shown that in \( \oplus E R \) each linearly ordered hyperplane defines a different linear order. On the other hand it is not obvious, which linearly ordered hyperplanes define different linear orders on \( \oplus E Q \).

Lemma 21. Let \( \leq \) and \( \leq' \) be two linear orders on \( \oplus E R \) and \( a \subseteq \oplus E Q \) a rational vector. Furthermore we denote the vector spaces from Equation (22) with \( V_a \) and \( V'_a \) and the corresponding linearly ordered hyperplanes according to Equation (27) with \( H_a \) and \( H'_a \). If the active subspaces \( \text{Act} V_a \) and \( \text{Act} V'_a \) are not equal or the restrictions of the hyperplanes \( H_a \cap \text{Act} V_a \) and \( H'_a \cap \text{Act} V'_a \) to the active subspaces \( V_a \) and \( V'_a \) differ, then the induced orders w.r.t. Lemma 3 differ on \( \oplus E Q \).

Proof. Suppose at first \( \text{Act} V_a = \text{Act} V'_a \). According to the presumption the set

\[
    U := (\text{Act} V_a)_+ \setminus V'_a_+ = (\text{Act} V_a)_+ \cap V'_a_- \setminus H'_a_+ = (\text{Act} V_a)_+ \cap (V'_a)_- \setminus \{0\}
\]

is non-empty. Let \( B \) the linearly ordered base that describes the order relation \( \leq \) and \( B' \) the one of \( \leq' \). As the hyperplanes \( H_a \) and \( H'_a \) differ, there exists a base vector \( b \subseteq B \cap H_a \cap \text{Act} V_a \setminus H'_a \) and a base vector \( b' \subseteq B \cap H'_a \cap \text{Act} V'_a \setminus H_a \). Both vectors are linear independent. As \( b \notin H_a \), for the base vector \( \eta \subseteq B \) with \( V_\eta = V_a \) the set \( B = B \setminus \{\eta\} \cup \{b'\} \) is another basis that describes \( \leq \). As \( a \subseteq \oplus E Q \) the
vector \( b' \) is active in \( V_n \), by Lemma 19 the projection of \( \oplus E Q \cap V_n \) along \( B \) into \( L\{ b, b' \} \) is two-dimensional. Thus it contains for all 4 combinations of positive or negative (non-zero) scalars with respect to \( b \) and \( b' \) at least one element. This implies that at least one element exists that is positive with respect to both orders, one that is positive with respect to \( \leq \) and negative with respect to \( \leq' \), one that is negative with respect to both orders and at least one that is negative with respect to \( \leq \) and positive with respect to \( \leq' \). The latter one is sufficient to prove that the orders differ.

The remaining part of the proof considers \( \text{Act } V_n \neq \text{Act } V'_n \). W.l.o.g. considering the definition of \( \text{Act} \) there exists a vector \( b \in \text{Act } V_n \cap \oplus E Q \setminus \text{Act } V'_n \). Furthermore we can choose an integer \( n \in \mathbb{Z} \) such that \( b < a^n \) and as \( b \notin V'_n \) we get the other inequality \( a \ll b \).

After having shown that different hyperplanes imply different orders, the next question we have to address, considers the effect of having different orders on \( \oplus E R \), which share a common dividing hyperplane.

**Lemma 22.** Let \( \leq \) and \( \leq' \) be two differing linear orders on \( \oplus E R \), which have the same hyperplane \( S_n = S'_n \) for some rational vector \( a \in \oplus E Q \). If the vector \( a \) is positive w.r.t. each of the orders (\( 0 \leq a \) and \( 0 \leq' a \)), then the induced orders on \( V_n \cap \oplus E Q \) are identical iff the induced orders on \( S_n \cap \oplus E Q \) and \( S'_n \cap \oplus E Q \) are equal.

**Proof.** According to Lemma 2 all elements of \( \oplus E Q \setminus S \) and all elements of \( S \cap \oplus E Q \) are either positive with respect to both orders on \( \oplus E Q \) or negative. Thus, the order on \( \oplus E R \setminus \oplus E Q \) does not influence the order on \( \oplus E Q \).

**Lemma 23.** Let \( \leq \) and \( \leq' \) be two linear orders on \( \oplus E R \). They are equal on \( \oplus E Z \) iff the coincide on \( \oplus E Q \).

**Proof.** As \( \oplus E Z \subseteq \oplus E Q \) the orders are equal on \( \oplus E Z \) if they are equal on \( \oplus E Q \). For the other direction let \( a \in \oplus E Q \) such that \( a < o < a' \), i.e. it is positive with respect to one and negative with respect to the other order. Then it has finitely many rational coordinates with respect to the standard base of \( \oplus E Q \). Let \( a \in \mathbb{N} \) be the least common denominator of the coordinates of \( a \). Then \( aa \in \oplus E Z \) is an integer vector. Thus we get the inequality \( aa < o < aa \).

With Corollary 23 and the monomorphisms \( \varphi \) and \( \psi \) as defined in Theorems 4 and 9 we also have a characterisation of linear orders on an arbitrary abelian group:

**Theorem 24** (Characterisation of linear orders on abelian groups).

Let \( G = (G, +, -, 0) \) be an abelian group, \( E \subseteq G \) a maximal independent set in \( G \), and \( \varphi \) the canonical embedding of \( G \) into \( \oplus E R \) as described in Lemma 4. Let further \( U \subseteq \oplus E Q \) be a set that generates \( \oplus E R \) in the following way:

\[
\oplus E R = \bigcup_{a \in U} V_n.
\]

Two linear orders \( \leq \) and \( \leq' \) on \( G \) are different iff some vector \( a \in U \) fulfills one of the following conditions:

1. \( S_n \cap \text{Act } V_n \neq S'_n \cap \text{Act } V'_n \)
2. \( \text{Act } V_n \neq \text{Act } V'_n \)
3. \( (S_n)_+ \cap \oplus E Q \neq (S'_n)_+ \cap \oplus E Q \).
Proof. As shown in the first part of this article Diagram 5(a) commutes and with Corollary 23 each linear order on $\oplus_E \mathbb{Q}$ uniquely defines an order on the group $G$. Lemma 9 finally assures that there are no more linear orders on $G$, since each linear order is semiclosed.

Lemma 2 tells us that we can transfer the order from $\oplus_E \mathbb{R}$ onto $\oplus_E \mathbb{Q}$ and $\oplus_E \mathbb{Z}$. The Theorems 4 and 6 assure that these orders can also be considered as orders on the group $G$. Let $a \in U$ one of the selected vectors. As $\mathcal{V}_a = \mathcal{V}_a \oplus a$ and similar for $\leq'$, Theorem 13 describes the positive cone of $\mathcal{V}_a$ using a hyperplane. Lemma 21 shows that Conditions 1 and 2 indicate a difference between the orders $\leq$ and $\leq'$. If these conditions do not hold, Lemma 22 proves Condition 3. Thus the restrictions of the orders $\leq$ and $\leq'$ to $\mathcal{V}_a \cup \mathcal{V}_a'$ are equal iff none of these three conditions holds.

From $\oplus_E \mathbb{R} = \bigcup_{a \in U} \mathcal{V}_a = \text{Act} \oplus_E \mathbb{R}$ also the restricted equation $\oplus_E \mathbb{Q} = \bigcup_{a \in U} \mathcal{V}_a \cap \oplus_E \mathbb{Q}$ follows as well as the same for the other order relation $\leq$. Thus, the orders $\leq$ and $\leq'$ are equal on $\oplus_E \mathbb{Q}$ iff their restrictions to $\text{Act} \mathcal{V}_a$ are equal for each vector $a \in U$. $\square$

**Corollary 25.** Linear orders on groups can be completely described by linearly ordered hyperplanes and their intersections in certain subspaces of a real vector space $\oplus_E \mathbb{R}$.

Theorem 24 provides a characterisation of linearly ordered groups independent of the ideas described in The’s 25 and Zajceva’s 28 papers. Teh describes linear orders by embedding them into vector spaces of the form $\oplus_E \mathbb{R}$ where the cardinality $|E|$ is smaller or equal to that of the continuum $c$. For the cardinality of these subsets he defines a prototype of an order and considers all other orderings to be homomorphic images of the prototypic vector spaces. The dimensions of these spaces correspond to the archimedian rank of the generated order in $G$. Our characterisation does not touch the notion of archimedian orders. The description using linearly ordered bases provides additional details about the case of infinite archimedian rank. The work of Zajceva describes archimedian orders by certain equations. It is not evident that finite linear equations can be used in general for infinitely generated groups. Nevertheless, these equations can be used in several cases since they are descriptions of the hyperplanes, which we used in Theorem 24.

So far, we didn’t use the properties of $\oplus_E \mathbb{R}$ as a scalar product space. This will be necessary for discussing the plane equations.

Beforehand we head over to linear order extensions, we can shortly bridge to Zajceva’s results. Two elements $g, h \in G$ of an abelian linearly ordered group $G = (G, +, -, 0, \leq)$ with $g \leq h$ are considered archimedian equivalent iff there exist two integers $m, n \in \mathbb{Z}$ such that $h^m \leq g \leq h^n$ holds. The group $G$ is archimedian iff all of its elements are archimedian equivalent to each other. In other words: For each two elements $g, h \in G$ with $0 \leq h - g$ there exists a number $n \in \mathbb{Z}$ such that $h - ng \leq 0$. When we describe the linear order on $G$ by a hyperplane as in Theorem 24 this is equivalent to: for all vectors $\mathfrak{r}, \mathfrak{s} \in \varphi[G]$ with $\mathfrak{r} - \mathfrak{s} \in \mathcal{H}_+ \cup \mathcal{H} + (\mathbb{R}_+ \setminus \{0\})a$ there exists a number $n \in \mathbb{Z}$ such that $\mathfrak{s} - ng \in \mathcal{H}_- \cup \mathcal{H} + (\mathbb{R}_- \setminus \{0\})a$. If the intersection $\mathcal{H} \cap \varphi[G]$ of the dividing hyperplane of the order and the image of the group is empty, then the vector $\mathfrak{r}$ is linearly independent from $\mathcal{H}$ and, thus, the linear closure $L(\mathcal{H} \cup \{\mathfrak{r}\}) = \oplus_E \mathbb{R}$ is the whole vector space. Thus, there exists an integer $n$ such that $\mathfrak{s} - ng$ is an
element of the negative half space. On the other hand if \( r \in \mathcal{H} \) is an element of the hyperplane and \( \eta \not\in \mathcal{H} \) is not, then the straight line \( \eta + r \mathbb{R} \) is parallel to \( \mathcal{H} \) and thus \( \eta - nr \) will never be negative. Thus, the group is archimedian iff there is no element \( g \in G \) such that \( \varphi(g) \in \mathcal{H} \). According to Corollary \ref{cor:archimedian} this is equivalent to the condition that no vector in \( \mathcal{H} \) has only rational coordinates with respect to the standard base defined by the independent set \( E \) as described in the preceding sections. This implies that the quotient of any two coordinates of the normal vector of \( \mathcal{H} \) must be irrational if it exists. Otherwise we can scale the normal vector by a real number such that it has two rational coordinates. On the other hand if all such quotients are irrational, all vectors containing only rational coordinates are never orthogonal to the normal vector.

In \cite{Zajceva} Zajceva discusses linear orders on abelian groups with a maximal independent set \( E \) by assigning a set of rationally independent numbers \( E' \) with a bijective mapping \( \alpha : E \to E' : e \mapsto \alpha(e) \) to them. So each element \( g \in G \) with a rational representation \( g = \sum_{a \in E} \xi_a(g) a \) (\( \xi_a : G \to \mathbb{Q} \)) is positive iff the real number \( \sum_{a \in E} \xi_a(g) \alpha(a) \) is positive. These equations and inequalities can be considered as the scalar product of the normal vector to the dividing hyperplane \( \mathcal{H} \) in the case of abelian groups with a finite maximal independent set. However if the maximal independent set \( E \) is infinite, the normal vector must have infinite support and thus is not an element of the vector space. In that case using our approach to find such a representation is less suitable than Zajceva’s method. It is again the hierarchy of subspaces generated by the order, that extends this description.

6. Linear Order Extensions

Before we discuss linear order extensions on arbitrary partially ordered torsion free abelian groups, we start with the special case of ordered vector spaces. So far we have discussed linearly ordered hyperplanes as representations of linear orders on vector spaces. Now we can relate them to other orders. Figure \ref{fig:linear-extensions} illustrates of the following theorem.

**Lemma 26.** Let \((\mathcal{R}, \leq)\) be an ordered real vector space, which has the positive cone \(\mathcal{R}_+\) with non-empty interior. A linearly ordered hyperplane \(\mathcal{H}\) defines a linear
order extension \( \leq' \) of \( \leq \) iff for its positive cone \( \mathcal{N}_+ \) the following condition is true:

\[
\mathcal{R}_+ \cap \mathcal{H} \subseteq \mathcal{H}_+ \cap \partial \mathcal{R}_+
\]

(34)

Proof. Let \( \leq' \) be a linear order on the vector space \( \mathcal{R} \) defined by a hyperplane \( \mathcal{H} \), which is an order extension of the order \( \leq \). Moreover, \( \leq' \) is also a linear order on \( \mathcal{H} \). Then for the positive cone \( \mathcal{R}_+ := \{ \mathbf{r} \in \mathcal{R} \mid 0 \leq' \mathbf{r} \} \) the following relation is satisfied:

\[
\mathcal{R}_+ \subseteq \mathcal{R}_+
\]

(35)

Suppose there exists an element \( \mathbf{r} \in \text{int}(\mathcal{R}_+) \cap \mathcal{H} \). Hence, neighbourhood \( U(\mathbf{r}) \) exists such that \( U(\mathbf{r}) \subseteq \mathcal{R}_+ \). On the other hand with \( \mathcal{H} = \partial \mathcal{R}_+ \) the inequality \( U(\mathbf{r}) \setminus \mathcal{R}_+ \neq \emptyset \) holds. So we deduce \( (U(\mathbf{r}) \cap \mathcal{R}_+) \setminus \mathcal{R}_+ \neq \emptyset \), which implies \( \mathcal{R}_+ \setminus \mathcal{R}_+ \neq \emptyset \) (\( \dagger \)). This is a contradiction to (35). Consequently, the intersection \( \text{int} \mathcal{R}_+ \cap \mathcal{H} = \emptyset \) is empty. Hence, \( \mathcal{R}_+ \cap \mathcal{H} \subseteq \partial \mathcal{R}_+ \), which implies \( \mathcal{R}_+ \cap \mathcal{H} \subseteq \partial \mathcal{R}_+ \cap \mathcal{H} \).

Let now \( \mathbf{r} \in \mathcal{R}_+ \cap \mathcal{H}_- \setminus \{ \mathbf{0} \} \). In this case the vector \( \mathbf{r} \) would be positive with respect to the given order \( \leq \) and with respect to the order \( \leq' \) that is generated by the hyperplane \( \mathcal{H} \) it would be negative. Hence, the orders would not be compatible(\( \dagger \)). Since \( \mathcal{H} \) is linearly ordered, the remaining case is \( \mathbf{r} \in \mathcal{H}_+ \) and thus,

\[
\mathcal{R}_+ \cap \mathcal{H}_+ \subseteq \mathcal{H}_+.
\]

So far we have shown that each linear order extension of \( \leq \) fulfills condition (34).

If the hyperplane \( \mathcal{H} \) fulfills the Condition (34), on one side of the hyperplane there are no elements of \( \mathcal{R}_+ \). In the contrary case two vectors \( \mathbf{r} \in \text{int} \mathcal{R}_+ \) and \( \mathbf{y} \in \mathcal{H} \) and a positive real number \( \alpha \in \mathcal{R}_+ \setminus \{ \mathbf{0} \} \) exist such that \( \mathbf{r} \) is on one side of \( \mathcal{H} \) and \( \mathbf{y} := \mathbf{y} - \alpha (\mathbf{r} - \mathbf{y}) \) is on the other side of \( \mathcal{H} \). Then \( \mathbf{y} - \mathbf{r} = (1 + \alpha) (\mathbf{y} - \mathbf{r}) \). Let \( U \) be an open environment of \( \mathbf{r} \). Then the set

\[
U' := \{ \mathbf{z}' \in \oplus E \mathcal{R} \mid z' = z + \frac{1}{1 + \alpha} (z - y'), z \in U \}
\]

is neighbourhood of \( \mathbf{y} \) and as \( \mathcal{R}_+ \) is convex \( U' \subseteq \mathcal{R}_+ \) is one of its subsets.

Thus, we have \( \mathbf{y} \in \text{int} \mathcal{R}_- \cap \mathcal{H} \) (\( \dagger \)) in contradiction to (34). Thus only on one side of \( \mathcal{H} \) can be positive elements with respect to the order \( \leq \).

If on both sides of \( \mathcal{H} \) there are no elements of \( \mathcal{R}_+ \) i.e., \( \mathcal{R}_+ \subseteq \mathcal{H} \), we can freely choose the side of the positive cone of \( \leq' \). In this case we choose an arbitrary vector \( \mathbf{a} \in \mathcal{R} \setminus \mathcal{H} \) and receive in dependence from \( \mathbf{a} \) one of the two possible orders.

If the set \( \mathcal{R}_+ \setminus \mathcal{H} \) is non-empty, we choose a vector \( \mathbf{a} \in \mathcal{R}_+ \setminus \mathcal{H} \). In this case \( P := \mathcal{H}_+ \cup (\mathcal{R}_+ \setminus \{ \mathbf{0} \}) \mathbf{a} \) is the set of positive elements of a linearly ordered group. The independence from the chosen representative \( \mathbf{a} \) has been shown for \( \mathcal{R}_+ \setminus \mathcal{H} \neq \emptyset \) in the proof of Theorem 13.

As for each vector \( \mathbf{a} \in \oplus E \mathcal{R} \) the subspace \( \mathcal{V}_a \) has a hyperplane that divides its positive cone according to the order relation \( \leq' \) from the negative one, we can immediately conclude the following corollary:

**Corollary 27.** Let \( U \subseteq \oplus E \mathbb{Q} \) be a set that generates \( \oplus E \mathcal{R} \) in the following way:

(36)

\[
\oplus E \mathcal{R} = \bigcup_{\mathbf{a} \in U} \mathcal{V}_a.
\]

Then \( \leq' \) is a linear extension of \( \leq \) iff for each element \( \mathbf{a} \in U \) the following relation holds:

\[
\mathcal{R}_+ \cap \mathcal{H}'_+ \subseteq (\mathcal{H}'_a)_+ \cap \partial \mathcal{R}_+
\]

(37)
The main theorem of this article is illustrated in Figure 8. It describes the set of linear order extensions of a given order on an abelian partially ordered group:

**Theorem 28** (Characterisation of linear order extensions on abelian po-groups). Let $\mathcal{G} = (G, +, - , 0 , \leq)$ a po-group, $E$ a maximal independent set in $\mathcal{G} = \bigoplus E \mathbb{R}$ the corresponding vector space with the embeddings $\varphi : G \to \bigoplus E \mathbb{R}$ and $\psi : \bigoplus E \mathbb{Z} \to G$ as described in (18) and (19). Then each linear order extension of $\leq$ can be characterised by a set $U \subseteq \bigoplus E \mathbb{Q}$ and for each element $a \in U$ a linearly ordered hyperplane $\mathcal{H}_a$ which fulfills the condition

$$\text{conv}(\varphi[G_+]) \cap \mathcal{H}_a \subseteq (\mathcal{H}_a)_+ \cap \partial \left(\text{conv}(\varphi[G_+])\right).$$

For each element $a \in U$ the corresponding hyperplane $\mathcal{H}_a$ defines a unique linear order extension in the subspace $\mathcal{V}_a$ if $\mathcal{V}_a \cap \varphi[G_+] \setminus \mathcal{H}_a$ is non-empty, otherwise it can be used to define exactly 2 linear order extensions in this subspace.

**Proof.** The existence of the monomorphisms has been proved in the Lemmata 4 and 6. With Theorem 8 and Lemma 9 we get the description of the positive cone $G_+$ as $\text{conv}(\varphi[G_+])$ in the vector space $\bigoplus E \mathbb{R}$. For non-semiclosed orders this construction adds only these elements to the positive cone of $G$, which are positive in every linear order extension on $G$, as Lemma 4 shows. Thus this restriction can be loosened. Theorem 24 gives us a description of linear orders on $G$ in the vector space $\bigoplus E \mathbb{R}$ as well as the number of orders defined by the hyperplane. Thus all parts of this theorem have been proved elsewhere in this article. \qed

With Theorem 28 we have described a geometrical characterisation of linear order extensions of partially ordered torsion free abelian groups. This allows us, to use methods from the convex geometry for the construction and investigation of such linear order extensions. The following section illustrates this.

### 7. Further results

As an illustration of the method provided above, in this section some other proofs and additional conclusions are added.

So far we have only described existing order extensions. The question has not been touched whether such an order extension or hyperplane does exist. The following theorem fills this gap. It has been proved several times (cf. [6, 17, 32, 33]).

**Theorem 29.** Each partially ordered torsion free group $\mathcal{G}$ has a compatible linear order extension.
Proof. Consider the complete sublattice of the subspace lattice of \( \mathfrak{V} \) that is generated by the set \( W := \{ \mathfrak{W}_a \mid a \in \mathfrak{V} \} \). This sublattice has a maximal chain \( \mathcal{K} \). As each vector \( a \in \mathfrak{V} \) is contained in an element of \( \mathcal{K} \), the union \( \bigcup \mathcal{K} = \mathfrak{V} \) is the complete vector space \( \mathfrak{V} \). We define

\[
\mathcal{S}_a := \bigcup \{ \mathfrak{W} \in \mathcal{K} \mid a \notin \mathfrak{W} \}.
\]

As the subspace \( \mathfrak{W}_a := \mathfrak{W}_a + \mathcal{S}_a \) is the join of \( \mathfrak{W}_a \) and \( \mathcal{S}_a \) in the subspace lattice, \( \mathfrak{W}_a \) is the upper neighbour of \( \mathcal{S}_a \) in \( \mathcal{K} \). Then there exists a maximal linear independent set \( B \subseteq \mathfrak{W}_a \) that contains \( a \) and is linearly independent from \( \mathcal{S}_a \). Then for each vector \( b \in B \) the vector space \( \mathfrak{W}_b \) is the same as \( \mathfrak{W}_a \) (otherwise \( b \in \mathcal{S}_a \)).

For each vector \( c \in \mathfrak{W}_a \cap \mathfrak{W}_b \) there exists a real number \( c \in \mathbb{R} \) such that \( cc \leq a \) which implies \( c \leq a - cc \). Thus there exists a basis \( \mathfrak{B}_a \) containing \( a \) and a basis of \( \mathcal{S}_a \) such that for all Elements \( r \in \mathfrak{W}_a \cap \mathfrak{W}_b \) the distance \( \|a - r\|_{\mathfrak{V}} \geq d \in \mathbb{R} \setminus \{0\} \) is strictly positive (choose all base vectors to be positive between \( a \) and \( c \)) and such that \( a \) is the largest element in the basis \( \mathfrak{B}_a \). Then \( \text{conv}( \mathfrak{B}_a \cap \mathfrak{W}_b, (a) \cup \mathfrak{W}_a) \) has non-empty interior and does not hit \( \mathfrak{W}_a \). Then there exists a hyperplane \( \mathcal{H}_a \) such that \( \mathcal{S}_a \subseteq \mathcal{H}_a \) and \( \mathcal{S}_a \) does not hit \( \mathfrak{W}_a \cap \mathfrak{W}_b \setminus \mathcal{S}_a \).

Let us chose a base \( \mathfrak{B}_a \) such that it contains a base of \( \mathcal{S}_a \) together with \( a \) with the additional condition that \( \mathfrak{B}_a \setminus \{a\} \cup \mathcal{S}_a \subseteq \mathcal{H}_a \). Then we define a linear order relation \( \leq' \) on \( \mathfrak{B}_a \) that has \( a \) as largest element and is defined in a way such that for any two elements \( r \in \mathfrak{B}_a \cap \mathfrak{H} \setminus \mathcal{S}_a \) and \( \eta \in \mathcal{S}_a \) the inequality \( \eta \leq' r \) holds. Then \( \leq' \) defines a linear order on \( \mathfrak{W}_a \) such that the positive cone of \( \mathfrak{W}_a \) with respect to the original order \( \leq \) is a subset of the positive cone that is defined by \( \leq' \). Furthermore we can use \( \mathfrak{B}_a \setminus \mathcal{S}_a \) to extend any linearly ordered base of \( \mathcal{S}_a \) to a linearly ordered base of \( \mathfrak{W}_a \) that fulfills the same condition.

As the vector \( a \) has been chosen arbitrarily, we can find such a subbase \( \mathfrak{B}_a \) for every subspace \( \mathfrak{W}_a \in \mathcal{K} \), and the union \( \mathfrak{B} \) of all these subbases is a basis of the original vector space \( \mathfrak{V} \). The order on the chain \( \mathcal{K} \) defines a linear preorder on \( \mathfrak{B} \), which can be turned into a linear order by using the orders of the subbases \( \mathfrak{B}_a \) as developed above. Thus we can define a linear order whose positive cone is a superset of the one of the original order \( \leq \), which means that it is a linear order extension.

As every torsion free ordered abelian group \( \mathfrak{G} \) has a closed order extension, we can use this construction to construct a linear order extension on \( \mathfrak{G} \).

Looking at the proof and the preceding sections it is obvious that every linear order extension can be found with this construction: Every linear order can be expressed by a maximal chain in the subspace lattice, and it is an order extension iff it provides an extension of the positive cone.

In general, an order relation on a given set is the intersection of its linear order extensions (cf. [24]). If the order must be compatible to a given algebraic structure, the set of choices for the linear extensions is much more restricted. For abelian groups the following theorem has been proved a long time ago (cf. [7], Corollary 6). Also in this case our method provides an alternative proof.

**Theorem 30.** Let \( \mathfrak{G} = (G, +, -, 0, \leq) \) be a partially ordered torsion free abelian group. Then the order relation \( \leq \) is the intersection of all of its linear order extensions iff it is semiclosed.
Proof. One direction is obvious: Each lattice ordered group is semiclosed, thus also each linearly ordered group. So, the positive cone of an arbitrary order on a given group $G$ contains for each $n \in \mathbb{N} \setminus \{0\}$ together with $na$ also the element $a$ and, thus, the intersection is also semiclosed (cf. Lemma 7).

Considering the other direction, let $\leq$ be semiclosed. Furthermore, let $O$ the set of all linear order extensions of $\leq$ which are compatible with $\mathcal{B}$ and $E$ a maximal independent set in $\mathcal{B}$. Then for each monomorphism $\varphi: \mathcal{B} \to \bigoplus E$ with $a \mapsto e_a$ for all $a \in E$ the set $P := \text{conv} \{\varphi[\mathcal{B}]\}$ is the unique representation of the positive cone in $\bigoplus E$ according to Theorem 8 and Lemma 9. Thus it is sufficient to show that $P$ is the intersection of the corresponding positive half spaces of elements of $O$. Let $P'$ the intersection of all of these half spaces of the linear order extensions. Then $P'$ is non-empty because of Theorem 29. Furthermore, $P \subseteq P'$. Now, we will show the inclusion $\bigoplus E \setminus P \subseteq \bigoplus E \setminus P'$.

Let $r \in \bigoplus E \setminus P$. We infer from the convexity of $P$ the equality $\mathbb{R}r \cap P = \{0\}$. Let $\eta \in \text{conv}(\mathbb{R}_+r \cup -P) \cap \text{conv}(\mathbb{R}_-r \cup P) = \mathcal{L}^+(\mathbb{R}_+r \cup -P) \cap \mathcal{L}^-(\mathbb{R}_-r \cup P)$. The two sets are equal as both are convex and $0 \in \mathbb{R}_+r \cap P$. Then there exist $\alpha, \beta \in \mathbb{R}_+$ and vectors $p, q \in P$ such that $\alpha r - p = \eta = -\beta r + q$. Consequently, $(\alpha+\beta)r = p+q \in P$, which implies with the properties of being a cone for $\alpha \neq 0$ or $\beta \neq 0$ that also $r$ is in $P$ ($\dagger$). Consequently, $\alpha = \beta = 0$ and it follows $\eta = p = q = 0$. This implies $\text{conv}(\mathbb{R}_-r \cup P) \cap \text{conv}(\mathbb{R}_+r \cup -P) = \{0\}$.

As proven in Theorem 29 a linear order extension of $\leq$ exists, for which $r$ is negative. Consequently, $r \notin P'$. Thus, $\bigoplus E \setminus P \subseteq \bigoplus E \setminus P'$ follows and so $P' \subseteq P$, from which we get $P = P'$. This means that a semiclosed order $\leq$ is the intersection of all of its order extensions. □

Hölder's theorem\(^\text{7}\) has not been used so far, but can be proved easily, now. The main idea will be formulated as a separate lemma:

Lemma 31. A linearly ordered base of $\bigoplus E \mathbb{R}$ defines an archimedean order on $\bigoplus E \mathbb{Q}$ iff it fulfills the following conditions:

1. There exists a largest basis vector $b$.
2. The largest vector $b$ is the only self-activator in $\mathcal{B}$.

\(\footnote{In fact, Hölder proved the theorem not for archimedean orders, but for such ordered semigroups that provide a Dedekind cut. For a link to the current theorem cf. \cite{9}, XIII.12.} \)
Proof. At first we show that an archimedian order has only one self-activator. Suppose there are two self-activators \( a \) and \( b \). W.l.o.g. we can assume \( a, b \in \oplus E \mathbb{Q} \). As the basis is linearly ordered either \( a < b \) or \( b < a \) is true. Let’s assume the first one holds. This implies \( a \in H^b \), which means \( a \) is infinitesimal smaller than \( b \) (\( \preceq \)). Thus there is at most one self-activator in \( B \). As \( \oplus E \mathbb{Q} \subseteq \mathcal{L} B \) holds there is at least one self-activator in \( B \).

The projection along \( B \) into the subspace that is generated by two basis vectors is 2-dimensional as proofed in Lemma 19. This implies that every non-self-activator has a self-activator above it. Thus the unique self-activator of an archimedian linear order must be the maximal element.

On the other hand if these two conditions are fulfilled, we can choose the maximal element \( b \). Then the corresponding hyperplane \( H^b \) is disjoint from \( \oplus E \mathbb{Q} \). Thus the order is completely defined by the projection along \( B \) into \( \mathbb{R}^b \). Suppose there are two vectors \( a_1 \) and \( a_2 \) in \( \oplus E \mathbb{Q} \) with the same projection into \( \mathbb{R}^b \). Then the difference \( a_1 - a_2 \in H^b \cap \oplus E \mathbb{Q} \) is both infinitesimal smaller than \( b \) and rational (\( \preceq \)). □

Geometrically, this implies that every archimedian linear order can be represented by a linearly ordered hyperplane in the complete vector space \( \oplus \mathbb{Q} \mathbb{R} \) that does not intersect with the rational vectors from \( \oplus \mathbb{Q} \mathbb{R} \) as shown in Figure 9.

It has been proved many times that an archimedian \( o \)-group is abelian. This shall not be repeated, here. Thus, we prove the following theorem only for abelian groups.

**Theorem 32** ([3], XIII.12; Hölder, [12]; ). *Every archimedian linearly ordered abelian group is isomorphic to a subgroup of the real numbers.*

Proof. Let \( E \) be a maximal independent set in the group and \( B \) a linearly ordered base of \( \oplus E \mathbb{R} \). Then, by the preceding lemma the order on the group is defined by the projection along \( B \) into the subspace \( \mathbb{R}^b \) of the maximal basis vector. □

**Corollary 33.** *The abelian group \( \mathbb{Z}^\mathbb{R} \) with respect to elementwise addition and subtraction does not permit an archimedian linear order.*

Proof. For the cardinalities we get the inequality

\[
|\mathbb{Z}^\mathbb{R}| \geq 2^{|\mathbb{R}|} = |\mathcal{P}(\mathbb{R})| > |\mathbb{R}|.
\]

Thus, there is no isomorphism from \( \mathbb{Z}^\mathbb{R} \) into a subset of the real numbers. □

The following theorem touches a question that has been published in [4] as Problem 1.7. As it is weaker, it can be considered only as a step into the right direction.

**Theorem 34.** *Every torsion free abelian group permits an archimedian directed order.*

Proof. Let \( G \) be an abelian group. And \( E \) a maximal independent set. Then \( G \) can be embedded into \( \oplus E \mathbb{Q} \). Let \( \varphi : G \rightarrow \oplus E \mathbb{Q} \) be the corresponding embedding. The product order \( (a \leq b \iff \forall e \in E : a(e) \leq b(e)) \) on \( \oplus E \mathbb{Q} \) is an archimedian order: For any two vectors \( a \) and \( b \) the set \( \text{supp } a \cup \text{supp } b \) is finite. If both \( \text{supp } a \cap \text{supp } b \neq \emptyset \) and \( \text{supp } b \setminus \text{supp } a \neq \emptyset \) are non-empty either \( a \parallel b \) or \( a \parallel -b \) holds, thus the two vectors fulfil the archimedian property.

For the remaining case we may assume \( \text{supp } b \subseteq \text{supp } a \). Then there exist an element \( e \in E \) and integers \( a, b \in \mathbb{Z} \) such that \( a a(e) = b b(e) \neq 0 \). If \( a(a) \) is strictly
positive, then $a$ can be chosen strictly positive, too. This leads to $(a + 1)a > bb(e) > b(e)$ and $(a + 1)a < b$ on the one hand, and $2bb(e) = 2aa(e) > a(e)$ and $2bb < a$ on the other hand, the archimedian property. For a negative coordinate $a(e)$ and a negative scalar $a$ the inequalities read $(a - 1)a(e) > bb(e) > b(e), (a - 1)a < b, 2bb(e) = 2aa(e) > a(e)$ and $2bb < a$. As $\varphi$ is an embedding, this proves also that the given order is also archimedian on $G$.

Finally the order is a directed order. Let $a$ and $b$ the vectors that correspond to two arbitrary group elements. Then we define a mapping $\epsilon : E \rightarrow \mathbb{Z}$ such that $\epsilon(e)$ is the least integer that fulfills the conditions

$$a(e) \leq \epsilon(e) \quad \text{and} \quad b(e) \leq \epsilon(e).$$

With the relation $\text{supp } \epsilon \subseteq \text{supp } a \cup \text{supp } b$ the vector $\epsilon$ is a well-defined element of $\oplus_E \mathbb{Z}$ and thus it represents an element of $G$ such that $a \leq \epsilon$ and $b \leq \epsilon$.

8. Conclusion and further topics

In Theorem 24 we have provided a method to describe linearly ordered abelian groups by means of linearly ordered hyperplanes in a the vector space, arising from the set of mappings with finite support from a maximal independent set into the real numbers. The vector space has been defined as the direct sum $\oplus_E \mathbb{R}$, where $E$ denotes a maximal independent set in the given abelian torsion free group $G$. This description enables us to investigate all linear orders on $G$ in one common vector space. Thus, we can use it to describe all linear order extensions of a partially ordered abelian group. We concentrated on torsion free groups, as only those can be ordered linearly. The description used for linear orders can also be used for semiclosed partially ordered groups, which have been shown to be compatible with the attempt to catalogue all compatible linear order extensions of a given partially ordered group. Finally such a characterisation has been given in Theorem 28.

Some additional examples provided an insight, how this method can be applied to mathematical problems arising from the work with partially ordered abelian groups.

Though abelian groups have interesting applications, a general description for arbitrary groups would be interesting. The groups considered here are torsion free abelian groups and have been embedded into some vector space $\oplus_E \mathbb{Q}$ which is always possible as the latter is an injective group. Another idea would be to consider those groups, which can be embedded into vector spaces over skew fields.

Linear order extensions play an important role in the modelling of tone systems (cf. [21]) and ordered generalised interval systems [16] in mathematical music theory. In this topic it is necessary to describe factorisations of $\ell$-groups by non-convex subgroups based on linear order extensions. This leads to further interesting questions in the direction of cylindrically and cyclically ordered groups. This theory has its application in software development e.g., for understanding just intonation logics based on the Tonnetz as described by Martin Vogel [20] and provided by Mutabor ([20]).

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