A NOTE ON THE BANACH LATTICE $c_0(\ell^n_2)$ AND ITS DUAL

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Abstract

The main purpose of this paper is to study some geometric properties on $c_0$ sum of the finite dimensional Banach lattice, $\ell^n_2$ and its dual. Among other results, we show that the Banach lattices $c_0(\ell^n_2)$ has the strong Gelfand-Phillips property, but does not have the positive Grothendieck property. We also prove that the closed unit ball of $l_\infty(\ell^n_2)$ is an almost limited set.

Keywords: Banach lattices, Dunford-Pettis property, Dunford-Pettis* property, Gelfand-Phillips property, weak Dunford-Pettis property, weak Dunford, weak Grothendieck property, positive Grothendieck property, strong Gelfand-Phillips property.

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1 Introduction

Throughout this paper $X$ and $Y$ will denote Banach spaces, $E$ and $F$ will denote Banach lattices. We denote by $B_X$ the closed unit ball of $X$. In a Banach lattice the additional lattice structure provides a large number of tools that are not available in more general Banach spaces. This fact facilitates the study of geometric properties of Banach lattices. We will start by recalling concepts of specific sets in Banach spaces and their consequences on their geometric properties.
A bounded set $A \subset X$ is Dunford-Pettis (resp. limited) if every weakly null sequence in $X'$ converges uniformly to zero on $A$ (resp. if every weak* null sequence in $X'$ converges uniformly to zero on $A$). Concerning these sets, we can consider a few properties in the class of Banach spaces. A Banach space $X$ has the $DP$ property if every relatively weakly compact subset of $X$ is Dunford-Pettis. Or equivalently, $x'_n(x_n) \to 0$ for every $x_n \xrightarrow{w} 0$ in $X$ and $x'_n \xrightarrow{w^*} 0$ in $X'$, $X$ has the $DP^*$ property if every relatively weakly compact subset of $X$ is limited. Or, equivalently, $x'_n(x_n) \to 0$ for every $x_n \xrightarrow{w} 0$ in $X$ and $x'_n \xrightarrow{w^*} 0$ in $X'$ and $X$ has Gelfand-Phillips property (in short GP property), if every limited subset of $X$ is relatively compact.

Of course the DP* property implies the DP. On the other hand, $L_1[0,1]$ and $c_0$ are Banach spaces with the DP property without the DP*. Schur spaces have all three properties listed above. Separable and reflexive spaces are examples of Banach spaces with the GP property. For more information concerning those properties we refer [1, 4, 6, 8].

In the class of Banach lattices, the lattice structure allows us to consider disjoint sequences. A sequence $(x_n) \subset E$ is disjoint if $|x_n| \wedge |x_m| = 0$ for every $n \neq m$. A bounded subset $A \subset E$ is almost Dunford-Pettis (resp. almost limited) if every disjoint weakly null sequence in $E'$ converges uniformly to zero on $A$ (resp. if every disjoint weak* null sequence in $E'$ converges uniformly to zero on $A$). Next, we will present the geometric properties in Banach lattices that the definitions given for the sets above appear naturally. A Banach lattices $E$ has the weak $DP$ property (in short wDP) if every relatively weakly compact subset of $E$ is almost Dunford-Pettis, $E$ has weak $DP^*$ property (wDP*) if every relatively weakly compact subset of $E$ is almost limited and $E$ has strong GP property (sGP) if every almost limited subset of $E$ is relatively compact.

Of course the DP and the DP* properties imply, respectively, the wDP and the wDP*. In [9], Leung gave the first example of a Banach lattice with the wDP and without the DP. In [7], the authors showed that $L_1[0,1]$ have the wDP* property even though it does not have the DP*. Note that the sGP property is stronger than the GP. For instance, $L_1[0,1]$ does not have the GP property. We refer [2, 3, 7] for more details concerning those properties.
Recall that a Banach space $X$ has the *Grothendieck property* if every weak* null sequence in $X'$ is weakly null. For example, $\ell_\infty$ has the Grothendieck property. For a Banach lattice, we can consider the weak Grothendieck property and the positive Grothendieck property. From [15], $E$ is said to have the *positive Grothendieck property* if every positive weak* null sequence in $E'$ is weakly null. Following [11], $E$ has the *weak Grothendieck property* if every disjoint weak* null sequence in $E'$ is weakly null. Clearly, the Grothendieck property implies both the positive Grothendieck and the weak Grothendieck. For instance, $\ell_1$ has the weak Grothendieck property, but it fails to have the positive Grothendieck, and $c$ is a Banach lattice with the positive Grothendieck property without the weak Grothendieck property.

Our goal is to add examples of Banach lattices that have the properties described here, for that we will study the Banach lattices $(\bigoplus_{n=1}^\infty \ell^n_2)_0$, $(\bigoplus_{n=1}^\infty \ell^n_2)_1$ and $(\bigoplus_{n=1}^\infty \ell^n_2)_\infty$ and describe their properties. The importance of such spaces appears in [14], when Stegall showed that $(\bigoplus_{n=1}^\infty \ell^n_2)_\infty$ does not have DP property, but its predual, $(\bigoplus_{n=1}^\infty \ell^n_2)_1$, has it.

2 Results

First, we are going to fix some notations. Denote by $\ell^n_2$ the Banach lattice $\mathbb{R}^n$ with Euclidean norm and let

$$E = \left( \bigoplus_{n=1}^\infty \ell^n_2 \right)_0 = c_0(\ell^n_2), \quad E' = \left( \bigoplus_{n=1}^\infty \ell^n_2 \right)_1 = l_1(\ell^n_2) \quad \text{and} \quad E'' = \left( \bigoplus_{n=1}^\infty \ell^n_2 \right)_\infty = l_\infty(\ell^n_2).$$

(1)

It is well known that if $X'$ has the DP property, then $X$ has the DP property. In [14], Stegall showed that $E'$ is a Schur space. Consequently, $E'$ has the DP property. However, its dual $E''$ does not have it. This was the first example of a Banach space with the DP property whose dual space does not have it.

We can consider in $E$ a natural structure of Banach lattice induced by its unconditional basis $(e^n_{i,j})_{i,j}$ where $e^n_j = (0, \ldots, 0, \underbrace{e_j, 0, \ldots}_{i})$ with $e_j = (0, \ldots, 0, 1(j), 0, \ldots)$. Thus $E'$ and $E''$ also are
Banach lattices with their dual structures. In the following, \( E, E' \) and \( E'' \) will be fixed as in [1]. Our goal in this section is to study the geometric properties of such Banach lattices.

Since \( E' \) has the Schur property, it has the DP, the DP* and the sGP properties. Then, \( E \) has the DP and (hence) wDP properties. We begin this section showing that \( E \) does not have the wDP* property.

**Proposition 2.1** The Banach lattice \( E \) does not have the wDP* property.

**Proof:** Let \( T : c_0 \to E \) be the positive diagonal operator given by

\[
T(\alpha_j) = \begin{pmatrix}
\alpha_1 & 0 & 0 & \ldots \\
0 & \alpha_2 & 0 & \ldots \\
& 0 & \alpha_3 & \ldots \\
& & & \ddots
\end{pmatrix}.
\]

If \((e_n)\) is the canonical sequence in \( c_0 \) then \( Te_n \overset{w^*}{\to} 0 \) in \( E \). On the other hand, the sequence \( e'_{n,n} = (0, \ldots, 0, e_n, 0, \ldots) \in E' \) is disjoint and weak* null with \( e'_{n,n}(Te_n) = 1 \) for every \( n \). So \( E \) does not have the wDP* property. \( \square \)

In [14] Stegall showed that \( E'' \) does not have the DP property. It is natural to ask if \( E'' \) has the wDP property. We will show that \( E'' \) does not have it. To do this, we need the next two Lemmas. Recall that a Banach lattice \( F \) has the wDP property if and only if for all Banach space \( Y \) and weakly compact operator \( T : F \to Y, T \) is almost Dunford-Pettis operator, that means, \( T \) maps disjoint weakly null sequences of \( F \) onto norm null sequences in \( Y \).

**Lemma 2.2** Let \( F \) and \( G \) be Banach lattices such that \( F \) has the wDP property. If \( T : F \to G \) is a surjective lattice isomorphism, then \( G \) also has the wDP property.

**Proof:** Let \( X \) be a Banach space and \( S : G \to X \) be a weakly compact operator. So \( S \circ T : F \to X \) is a weakly compact operator. As a consequence we get \( S \circ T \) is an almost DP operator. Let \((y_n) \subset G\) be a disjoint weakly null sequence, then there exists \((x_n) \subset F\) a disjoint weakly null sequence such that \( Tx_n = y_n \) for all \( n \). As \( S(y_n) = S(T(x_n)) \) we have that \( S(y_n) \to 0 \) in \( X \). \( \square \)
Lemma 2.3 If $F$ has the wDP property, then every complemented sublattice of $F$ also has it.

Proof: Let $G$ be a complemented sublattice of $F$ and let $P : F \to G$ be the bounded projection. Consider $X$ be a Banach space and $T : G \to X$ be a weakly compact operator, so $T \circ P : F \to X$ is also a weakly compact operator. Thus, $T \circ P$ is an almost DP operator. If $(z_n)$ is a disjoint weakly null sequence in $G$, since $G$ is a complemented sublattice of $F$, it follows that $(z_n)$ is a disjoint weakly null sequence in $F$. As $T(z_n) = T(P(z_n))$, and we get that $T(z_n) \to 0$ and the result is true. \qed

Now we can prove that $E''$ does not have the wDP property.

Theorem 2.4 The Banach lattice $E''$ does not have the wDP property.

Proof: Consider the bounded linear operator $R : E' \to \ell_2$ given by

$$R(x) = (x_{1,1} + x_{2,1} + \cdots, x_{2,2} + x_{3,2} + \cdots, \cdots).$$

So, $R' : \ell_2 \to E''$ is an isomorphism between Banach spaces and $R'(:\ell_2)$ is a complemented subspace of $E'$. Moreover, we have that

$$R'(a) = \begin{pmatrix} \alpha_1 & \alpha_1 & \alpha_1 & \alpha_1 & \cdots \\ \alpha_2 & \alpha_2 & \alpha_2 & \cdots \\ \alpha_3 & \alpha_3 & \cdots \\ \alpha_4 & \cdots \end{pmatrix}.$$ 

And as

$$R'(|a|) = \begin{pmatrix} |\alpha_1| & |\alpha_1| & |\alpha_1| & |\alpha_1| & \cdots \\ |\alpha_2| & |\alpha_2| & |\alpha_2| & \cdots \\ |\alpha_3| & |\alpha_3| & \cdots \\ |\alpha_4| & \cdots \end{pmatrix} = |R'(a)|,$$

then $R'$ is a lattice isomorphism and $R'(:\ell_2)$ is a complemented sublattice of $E''$. Since $\ell_2$ does not have the wDP property, it follows from Lemma 2.2 that $R'(:\ell_2)$ cannot have the wDP property. By Lemma 2.3 follows that $E''$ cannot have the wDP property. \qed
As a consequence of Theorem 2.4, $E''$ cannot have the wDP* property. Now, we will prove that $E$ has the sGP property. Observe that $E$ has the GP property, because it is separable. We begin showing that $E$ is Dedekind complete.

**Lemma 2.5** The Banach lattice $E$ is a Dedekind complete Banach lattice.

**Proof:** Let $A \subset E$ such that $a \leq x$ for every $a \in A$ and some $x \in E^+$. In particular,

$$a = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ a_{2,2} & a_{3,2} & \cdots \\ a_{3,3} & \cdots \\ \vdots & \ddots & \ddots \\ \end{pmatrix} \leq \begin{pmatrix} x_{1,1} & x_{2,1} & x_{3,1} & \cdots \\ x_{2,2} & x_{3,2} & \cdots \\ x_{3,3} & \cdots \\ \vdots & \ddots & \ddots \\ \end{pmatrix} = x$$

holds for every $a \in A$. Consequently, $a_{i,j} \leq x_{i,j}$ in $\mathbb{R}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$. As $\mathbb{R}$ is Dedekind complete, let $z_{i,j} = \sup \{a_{i,j} : a = (a_{k,l})_{l \leq k} \in A\}$. Now, let $z = (z_{i,j})_{j \leq i}$. Since $x_{i,j} \leq z_{i,j}$ holds for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$, it follows that $z \in E$. Now we prove that $z = \sup A$. In fact, if $y \in E$ is such that $a \leq y$ for every $a \in A$, then $a_{i,j} \leq y_{i,j}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$. Thus $z_{i,j} \leq y_{i,j}$ for every $i \in \mathbb{N}$ and $j = 1, \ldots, i$. Hence $z$ is the supremum of $A$ in $E$. \hfill \Box

As a consequence of above Lemma, we get that $E$ has order continuous norm. Using this fact we have the following Theorem:

**Theorem 2.6** The Banach lattice $E$ has the sGP property.

**Proof:** By Theorem 2.1 of [2], it suffices to prove that $E$ is a discrete Banach lattice. Indeed, let $[0, y]$ be an order interval in $E$. Write

$$y = \begin{pmatrix} y_{1,1} & y_{2,1} & y_{3,1} & \cdots \\ y_{2,2} & y_{3,2} & \cdots \\ y_{3,3} & \cdots \\ \vdots & \ddots & \ddots \\ \end{pmatrix}.$$
Let
\[
x = \begin{pmatrix}
y_{1,1} & 0 & 0 & \cdots \\
0 & 0 & \cdots \\
0 & \cdots \\
\vdots & \ddots & \ddots 
\end{pmatrix}.
\]
So \( x \in [0, y] \). If \( |z| \leq x \) in \( E \), it follows that
\[
z = \begin{pmatrix}
z_{1,1} & 0 & 0 & \cdots \\
0 & 0 & \cdots \\
0 & \cdots \\
\vdots & \ddots & \ddots 
\end{pmatrix}
\]
with \( |z_{1,1}| \leq y_{1,1} \) in \( \mathbb{R} \), what implies that there exists a real number \( t \) such that \( z_{1,1} = ty_{1,1} \). Thus \( z = tx \).

Note that \( E'' \) does not have the GP property. Hence it does not have the sGP. Indeed, consider the positive operator \( S : \ell_\infty \to E'' \) given by
\[
S(\alpha_j) = \begin{pmatrix}
\alpha_1 & 0 & 0 & \cdots \\
\alpha_2 & 0 & \cdots \\
\alpha_3 & \cdots 
\end{pmatrix}.
\]
We have \( (e_n) \subset \ell_\infty \) is a weakly null limited sequence. This implies that \( (Se_n) \) is a weakly null limited sequence in \( E'' \) such that \( \|Se_n\|_\infty = 1 \) for all \( n \). Thus \( E \) cannot have the Gelfand-Phillips property.

Next, we study the Grothendieck type properties in \( E \) and \( E' \). We show that \( E \) does not have neither the weak nor the positive Grothendieck properties.

**Proposition 2.7** The Banach lattice \( E \) does not have neither the weak Grothendieck nor the positive Grothendieck properties.
Proof: Let \((e'_{n,n}) \subset E'\) as in the proof of Proposition 2.11. This sequence is positive, disjoint and weak* null in \(E'\). However, \((e_{n,n})'\) is not weakly null. Indeed, if \(x'' = (e_1, e_2, e_3, \ldots)\), then \(x''(e'_{n,n}) = 1\) for all \(n\).

The following Proposition shows that \(E'\) has the weak Grothendieck property, but it fails to have the positive Grothendieck property. Recall that the positive element

\[
e = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & \cdots \\ 1 & \cdots \\ \vdots \\
\end{pmatrix}
\]

is an order unit of \(E''\), i.e. \(B_{E''} = [-e, e]\).

**Proposition 2.8** The Banach lattice \(E'\) has the weak Grothendieck property, however it does not have the positive Grothendieck property.

Proof: Let \((x''_n) \subset E''\) be a disjoint weak* null sequence. Since \((x''_n)\) is bounded, there exists \(M > 0\) such that \(x''_n \in [-Me, Me]\) for every \(n \in \mathbb{N}\). Consequently, \((x''_n)\) is a disjoint order bounded sequence in \(E''\). Hence \(x'' \stackrel{\ast}{\rightharpoonup} 0\) in \(E''\) (pg. 192, [1]). As the diagonal operator \(T : \ell_1 \to E'\),

\[
T(\alpha_j) = \begin{pmatrix} \alpha_1 & 0 & 0 & \cdots \\ \alpha_2 & 0 & \cdots \\ \alpha_3 & \cdots \\
\end{pmatrix},
\]

is a lattice isometry, it follows that \(T\) is a lattice embedding. Proposition 2.3.11 of [12] yields that \(\ell_1\) is positively completemented in \(E'\), then \(E'\) cannot have the positive Grothendieck property.

**Corollary 2.9** The norm in \(E''\) is not order continuous.

Proof: By Proposition 4.9 from [11], if \(F\) has the weak Grothendieck property and \(F'\) has order continuous norm, then \(F\) also has the positive Grothendieck property. From this fact and Proposition 2.8 it follows that \(E''\) does not have order continuous norm.
Now we want to study if $E''$ has the Grothendieck property. We begin with the following Lemma:

**Lemma 2.10** Consider $E'_{\perp} = \{ f \in E'' : f(x) = 0, \forall x \in E \}$. Then $E'' = E' \oplus E'_{\perp}$ and $E'_{\perp}$ is an ideal in $E''$.

**Proof:** Let $f \in E''$ and put $a_{i,j} = f(e_{i,j})$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, i$. Then

$$a' = \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & \cdots \\ a_{2,2} & a_{3,2} & \cdots \\ a_{3,3} & \cdots \\ \vdots \end{pmatrix} \in E'.$$  

Indeed, since

$$\sum_{i=1}^{\infty} \| a_i \|_2 \leq \sum_{i=1}^{\infty} \sum_{j=1}^{i} |a_{i,j}| = \sum_{i=1}^{\infty} \sum_{j=1}^{i} f(e_{i,j}e_{i,j}) = f(\sum_{i=1}^{\infty} \sum_{j=1}^{i} e_{i,j}e_{i,j}) \leq \| f \|.$$  

On the other hand, if $x = (x_{i,j})_{1 \leq i \leq j} \in E$, then

$$f(x) - a'(x) = \lim_{n \to \infty} \left[ \sum_{i=1}^{n} \sum_{j=1}^{i} f_{i,j}(x_{i,j}e_{i,j}) - \sum_{i=1}^{n} \sum_{j=1}^{i} a_{i,j}(x_{i,j}) \right] = 0.$$  

As $E' \cap E'_{\perp} = \{0\}$, we get $E''' = E' \oplus E'_{\perp}$.

We observe that $E$ has order continuous norm, so it is an ideal in $E''$. Using this fact we claim that $E'_{\perp}$ is an ideal in $E'''$ as well. Indeed, let $x''$, $y'' \in E''$ with $|y''| \leq |x''|$ and $x'' \in E'_{\perp}$. If $x \in E^+$ and $|y''| \leq x$ in $E''$, then $|y''| \in E$, what implies that $|x''|(x) = \sup \{|x''(y)| : |y| \leq x\} = 0$ for all $x \in E^+$. Finally, if $x \in E$,

$$|y''(x)| \leq |y''|(|x|) \leq |x''|(|x|) = 0.$$  

Therefore $y'' \in E'_{\perp}$.  

Since $E'$ has order continuous norm, $E'$ is an ideal in $E''$. On the other hand, since $E'_{\perp}$ also is an ideal in $E''$, by Theorem 1.41 in [1], we have that $E'$ and $E'_{\perp}$ are projection bands in $E''$. Now we can prove a version of Phillip's Lemma for $E''$. The Proposition follows the same idea used in Theorem 4.67 of [1].
Proposition 2.11 Every weak* null sequence in $E''$ converges uniformly to zero on $B_E$. Consequently, $B_E$ is a limited set in $E''$.

Proof: Let $(f_n) \subset E'''$ be a weak* null sequence and write $f_n = x_n + g_n$ with $(x_n) \subset E'$ and $(g_n) \subset E^\perp$. As $E'$ is a projection band, $x_n \overset{\omega}{\rightharpoonup} 0$ in $E'''$. Then $x_n \overset{\omega}{\rightharpoonup} 0$ in $E'$, and since $E'$ has the Schur property, $x_n \to 0$ in $E'$. As a consequence, $\|f_n\|_{B_E} = \sup_{x \in B_E} |x_n(x)| \leq \|x_n\| \to 0$. \(\blacksquare\)

In [5] the authors has showed that $B_E$ is a limited set in $E''$, but they used another technique in the another context.

The next result we classify the closed unit balls of $E$, $E'$ and bounded subset $E''$ concerning if they are (or not) almost Dunford-Pettis or almost limited. As $E'$ has the Schur property, $B_E$ is a Dunford-Pettis set.

Proposition 2.12 1. The closed unit ball of $E$ is not almost limited.

2. The closed unit ball of $E'$ is not almost Dunford-Pettis.

3. Every norm bounded subset in $E''$ is almost limited.

Proof: (1) Consider $T : c_0 \to E$ and $(e_{n,n}') \subset E'$ as in the proof of Proposition 2.1. Since $\|e_{n,n}'\|_{B_E} = \sup_{x \in B_E} |e_{n,n}'(x)| \geq e_{n,n}'(Te_n) = 1$ for all $n$, we have that $B_E$ is not almost limited.

(2) The unit diagonal sequence $e_{n,n}'' = (0, \ldots, 0, e_n, 0, \ldots)$ for all $n$ is weakly null and disjoint in $E''$. Since $\sup_{x \in B_{E'}} |e_{n,n}''(x)| \geq \sup_{x \in B_{E'}} |e_{n,n}''(e_{n,n}')| = 1$ for all $n$, we have that $B_{E'}$ is not almost Dunford-Pettis. As a consequence $B_{E''}$ it is not almost limited.

(3) Consider $A \subset E''$ a norm bounded subset, then there exists $M > 0$ such that $A \subset M \cdot B_{E''} = [-Me, Me] = \text{sol}(Me)$. By Lemma 2.1 of [10], we have that $M \cdot B_{E''}$ is almost limited. Consequently, $A$ is almost limited as well. \(\blacksquare\)

Recall that $A \subset E'$ is said to be $L$-almost limited if every almost limited weakly null sequence $(x_n) \subset E$ converges uniformly to zero on $A$, i.e. $\|J_A(x_n)\|_A \to 0$. Since $E$ and $E'$ have the sGP property, we have that $B_{E'}$ and $B_{E''}$ are L-almost limited sets (see Proposition 3.3 in [10]).
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