ON THE IRREDUCIBLE ACTION OF PSL(2, R) ON THE 3-DIMENSIONAL EINSTEIN UNIVERSE.

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Abstract. We describe the orbits of the irreducible action of PSL(2, R) on the 3-dimensional Einstein universe Ein^{1,2}. This work completes the study in [2], and is one element of the classification of cohomogeneity one actions on Ein^{1,2} ([7]).

1. Introduction

1.1. Einstein universe. Let \( \mathbb{R}^{2,n+1} \) denote a \((n+3)\)-dimensional real vector space equipped with a non-degenerate symmetric bilinear form \( q \) with signature \((2,n+1)\). The nullcone of \( \mathbb{R}^{2,n+1} \) is

\[ \mathcal{N}^{2,n+1} = \{ v \in \mathbb{R}^{2,n+1} \setminus \{0\} : q(v) = 0 \} . \]

The \((n+1)\)-dimensional Einstein universe \( \text{Ein}^{1,n} \) is the image of the nullcone \( \mathcal{N}^{2,n+1} \) under the projectivization:

\[ \mathbb{P} : \mathbb{R}^{2,n+1} \setminus \{0\} \to \mathbb{P}^{n+2} . \]

The degenerate metric on \( \mathcal{N}^{2,n+1} \) induces a \( O(2,n+1) \)-invariant conformal Lorentzian structure on Einstein universe. The group of conformal transformations on \( \text{Ein}^{1,n} \) is \( O(2,n+1) \) [4].

A lightlike geodesic in Einstein universe is a photon. A photon is the projectivisation of an isotropic 2-plane in \( \mathbb{R}^{2,n+1} \). The set of photons through a point \( p \in \text{Ein}^{1,n} \) denoted by \( L(p) \) is the lightcone at \( p \). The complement of a lightcone \( L(p) \) in Einstein universe is the Minkowski patch at \( p \) and we denote it by \( \text{Mink}(p) \). A Minkowski patch is conformally equivalent to the \((n+1)\)-dimensional Minkowski space \( \mathbb{E}^{1,n} \) [1].

The complement of the Einstein universe in \( \mathbb{P}^{n+2} \) has two connected components: the \((n+2)\)-dimensional Anti de-Sitter space \( \text{AdS}^{1,n+1} \) and the generalized hyperbolic space \( \mathbb{H}^{2,n} \): the first (respectively the second) is the projection of the domain \( \mathbb{R}^{2,n+1} \) defined by \( \{ q < 0 \} \) (respectively \( \{ q > 0 \} \)).

An immersed submanifold \( S \) of \( \text{AdS}^{1,n+1} \) or \( \mathbb{H}^{2,n} \) is of signature \((p,q,r)\) (respectively \( \text{Ein}^{1,n} \)) if the restriction of the ambient pseudo-Riemmanian metric (respectively the conformal Lorentzian metric) is of signature \((p,q,r)\), meaning that the radical has dimension \( r \), and that maximal definite negative and positive subspaces have dimensions \( p \) and \( q \), respectively. If \( S \) is nondegenerate, we forgot \( r \) and simply denote its signature by \((p,q)\).

1.2. The irreducible representation of PSL(2, R). A subgroup of \( O(2,n+1) \) is irreducible if it preserves no proper subspace of \( \mathbb{R}^{2,n+1} \). By [3, Theorem 1], up to conjugacy, \( SO_c(1,2) \simeq PSL(2,\mathbb{R}) \) is the only irreducible connected Lie subgroup of \( O(2,3) \).

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On the other hand, for every integer $n$, it is well known that, up to isomorphism, there is only one $n$-dimensional irreducible representation of $\text{PSL}(2, \mathbb{R})$. For $n = 5$, this representation is the natural action of $\text{PSL}(2, \mathbb{R})$ on the vector space $V = \mathbb{R}[X,Y]$ of homogeneous polynomials of degree 4 in two variables $X$ and $Y$. This action preserves the following quadratic form

$$q(a_4X^4 + a_3X^3Y + a_2X^2Y^2 + a_1XY^3 + a_0Y^4) = 2a_4a_0 - \frac{1}{2}a_1a_3 + \frac{1}{6}a_2^2.$$

The quadratic form $q$ is nondegenerate and has signature $(2, 3)$. This induces an irreducible representation $\text{PSL}(2, \mathbb{R}) \to O(2, 3)$.

**Theorem 1.1.** The irreducible action of $\text{PSL}(2, \mathbb{R})$ on the 3-dimensional Einstein universe $\text{Ein}^{1,2}$ admits three orbits:

- An 1-dimensional lightlike orbit, i.e. of signature $(0, 0, 1)$
- A 2-dimensional orbit of signature $(0, 1, 1)$.
- An open orbit (hence of signature $(1, 2)$) on which the action is free.

The 1-dimensional orbit is lightlike, homeomorphic to $\mathbb{RP}^1$, but not a photon. The union of the 1-dimensional orbit and the 2-dimensional orbit is an algebraic surface, whose singular locus is precisely the 1-dimensional orbit. It is the union of all projective lines tangent to the 1-dimensional orbit. Figure 1 describes a part of the 1 and 2-dimensional orbits in the Minkowski patch $\text{Mink}(Y^4)$.

![Figure 1. Two partial views of the intersection of the 1 and 2-dimensional orbits in Einstein universe with Mink(Y^4). Red: Part of the 1-dimensional orbit in Minkowski patch. Green: Part of the 2-dimensional orbit in Minkowski patch.](image)

We will also describe the actions on Anti de-Sitter space and the generalized hyperbolic space $\mathbb{H}^{2,2}$:

**Theorem 1.2.** The orbits of $\text{PSL}(2, \mathbb{R})$ in the Anti de-Sitter component $\text{AdS}^{1,3}$ are Lorentzian, i.e. of signature $(1, 2)$. They are the leaves of a codimension 1 foliation. In addition, $\text{PSL}(2, \mathbb{R})$ induces three types of orbits in $\mathbb{H}^{2,2}$: a 2-dimensional spacelike orbit (of signature $(2, 0)$) homeomorphic to the hyperbolic plane $\mathbb{H}^2$, a 2-dimensional Lorentzian orbit (i.e., of signature $(1, 1)$) homeomorphic to the de-Sitter space $\text{dS}^{1,1}$, and four kinds of 3-dimensional orbits where the action is free:

- one-parameter family of orbits of signature $(2, 1)$ consisting of elements with four distinct non-real roots,
• one-parameter family of Lorentzian (i.e. of signature (1, 2)) orbits consisting of elements with four distinct real roots,
• two orbits of signature (1, 1, 1),
• one-parameter family of Lorentzian (i.e. of signature (1, 2)) orbits consisting of elements with two distinct real roots, and a complex root \( z \) in \( \mathbb{H}^2 \) making an angle \( \theta \) smaller than \( 5\pi/6 \) with the two real roots.

Remark 1.3. F. Fillastre indicated to us an alternative description for the last case stated in Theorem 1.2 these orbits correspond to polynomials whose roots in \( \mathbb{CP}^1 \) are ideal vertexes of regular ideal tetraedra in \( \mathbb{H}^3 \).

2. Proofs of the Theorems

Let \( f \) be an element in \( \mathbb{V} \). We consider it as a polynomial function from \( \mathbb{C}^2 \) into \( \mathbb{C} \). Actually, by specifying \( Y = 1 \), we consider \( f \) as a polynomial of degree at most 4. Such a polynomial is determined, up to a scalar, by its roots \( z_1, z_2, z_3, z_4 \) in \( \mathbb{CP}^1 \) (some of these roots can be \( \infty \) if \( f \) can be divided by \( Y \)).

It provides a natural identification between \( \mathbb{P}(\mathbb{V}) \) and the set \( \mathbb{CP}^1 \) made of 4-tuples (up to permutation) \((z_1, z_2, z_3, z_4)\) of \( \mathbb{CP}^1 \) such that if some \( z_i \) is not in \( \mathbb{RP}^1 \), then its conjugate \( \bar{z}_i \) is one of the \( z_j \)’s. This identification is \( \text{PSL}(2, \mathbb{R}) \)-equivariant, where the action of \( \text{PSL}(2, \mathbb{R}) \) on \( \mathbb{CP}^1 \) is simply the one induced by the diagonal action on \((\mathbb{CP}^1)^4\).

Actually, the complement of \( \mathbb{RP}^1 \) in \( \mathbb{CP}^1 \) is the union of the upper half-plane model \( \mathbb{H}^2 \) of the hyperbolic plane, and the lower half-plane. We can represent every element of \( \mathbb{CP}^1 \) by a 4-tuple (up to permutation) \((z_1, z_2, z_3, z_4)\) such that:
- either every \( z_i \) lies in \( \mathbb{RP}^1 \),
- or \( z_1, z_2 \) lies in \( \mathbb{RP}^1 \), \( z_3 \) lies in \( \mathbb{H}^2 \) and \( z_4 = \bar{z}_3 \),
- or \( z_1, z_2 \) lies in \( \mathbb{H}^2 \) and \( z_3 = \bar{z}_1, z_4 = \bar{z}_2 \).

Theorems 1.1 and 1.2 will follow from the following proposition:

Proposition 2.1. Let \( [f] \) be an element of \( \mathbb{P}(\mathbb{V}) \). Then:
- it lies in \( \text{Ein}^{1,2} \) if and only if it has a root of multiplicity at least 3, or two distinct real roots \( z_1, z_2 \), and two complex roots \( z_3, z_4 = \bar{z}_3 \) with \( z_3 \) in \( \mathbb{H}^2 \) and such that the angle at \( z_3 \) between the hyperbolic geodesic rays \([z_3, z_1]\) and \([z_3, z_2]\) is \( 5\pi/6 \),
- it lies in \( \text{AdS}^{1,3} \) if and only if it has two distinct real roots \( z_1, z_2 \), and two complex roots \( z_3, z_4 = \bar{z}_3 \), with \( z_3 \) in \( \mathbb{H}^2 \) and such that the angle at \( z_3 \) between the hyperbolic geodesic rays \([z_3, z_1]\) and \([z_3, z_2]\) is \( > 5\pi/6 \),
- it lies in \( \mathbb{H}^{2,2} \) if and only if it has no real roots, or four distinct real roots, or a root of multiplicity exactly 2, or it has two distinct real roots \( z_1, z_2 \), and two complex roots \( z_3, z_4 = \bar{z}_3 \), with \( z_3 \) in \( \mathbb{H}^2 \) and such that the angle at \( z_3 \) between the hyperbolic geodesic rays \([z_3, z_1]\) and \([z_3, z_2]\) is \( < 5\pi/6 \).

Proof of Proposition 2.1. Assume that \( f \) has no real root. Hence we are in the situation where \( z_1, z_2 \) lie in \( \mathbb{H}^2 \) and \( z_3 = \bar{z}_1, z_4 = \bar{z}_2 \). By applying a suitable element of \( \text{PSL}(2, \mathbb{R}) \), we can assume \( z_1 = i, \) and \( z_2 = ri \) for some \( r > 0 \). In other words, \( f \) is in the \( \text{PSL}(2, \mathbb{R}) \)-orbit of \((X^2 + Y^2)(X^2 + r^2Y^2)\). The value of \( q \) on this polynomial is \( 2 \times 1 \times r^2 + 1/(1 + r^2)^2 > 0 \), hence \([f]\) lies in \( \mathbb{H}^{2,2} \).

Hence we can assume that \( f \) admits at least one root in \( \mathbb{RP}^1 \), and by applying a suitable element of \( \text{PSL}(2, \mathbb{R}) \), one can assume that this root is \( \infty \), i.e. that \( f \) is a multiple of \( Y \).
We first consider the case where this real root has multiplicity at least 2:

\[ f = Y^2(aX^2 + bXY + cY^2) \]

Then, \( q(f) = \frac{1}{6}a^2 \): it follows that if \( f \) has a root of multiplicity at least 3, it lies in \( \mathbb{E}^{1,2} \), and if it has a real root of multiplicity 2, it lies in \( \mathbb{H}^{2,2} \).

We assume from now that the real roots of \( f \) have multiplicity 1. Assume that all roots are real. Up to \( \text{PSL}(2, \mathbb{R}) \), one can assume that these roots are 0, 1, \( r \) and \( \infty \) with \( 0 < r < 1 \).

\[ f(X, Y) = XY(X - Y)(X - rY) = X^3Y - (r + 1)XY^2 + rXY^3 \]

Then, \( q(f) = -\frac{1}{2}r + \frac{1}{2}(r + 1)^2 = \frac{1}{2}(r^2 - r + 1) > 0 \). Therefore \( f \) lies in \( \mathbb{H}^{2,2} \) once more.

The only remaining case is the case where \( f \) has two distinct real roots, and two complex conjugate roots \( z, \bar{z} \) with \( z \in \mathbb{H}^2 \). Up to \( \text{PSL}(2, \mathbb{R}) \), one can assume that the real roots are 0, \( \infty \), hence:

\[ f(X, Y) = XY(X - zY)(X - \bar{z}Y) = XY(X^2 - 2|z| \cos \theta XY + |z|^2Y^2) \]

where \( z = |z|e^{i\theta} \). Then:

\[ q(f) = \frac{2|z|^2}{3}(\cos^2 \theta - \frac{3}{4}) \]

Hence \( f \) lies in \( \mathbb{E}^{1,2} \) if and only if \( \theta = \pi/6 \) or \( 5\pi/6 \). The proposition follows easily. \( \square \)

**Remark 2.2.** In order to determine the signature of the orbits induced by \( \text{PSL}(2, \mathbb{R}) \) in \( \mathbb{P}(\mathbb{V}) \), we consider the tangent vectors induced by the action of 1-parameter subgroups of \( \text{PSL}(2, \mathbb{R}) \). We denote by \( E \), \( P \) and \( H \), the 1-parameter elliptic, parabolic and hyperbolic subgroups stabilizing \( i \), \( \infty \) and \( \{0, \infty\} \), respectively.

**Proof of Theorem 1.1** It follows from Proposition 2.1 that there are precisely three \( \text{PSL}(2, \mathbb{R}) \)-orbits in \( \mathbb{E}^{1,2} \):

- one orbit \( \mathcal{N} \) comprising polynomials with a root of multiplicity 4, i.e. of the form \( [(sY - tX)^4] \) with \( s, t \in \mathbb{R} \). It is clearly 1-dimensional, and equivariantly homeomorphic to \( \mathbb{RP}^1 \) with the usual projective action of \( \text{PSL}(2, \mathbb{R}) \). Since \( \frac{d}{dt}|_{t=0}(Y - tX)^4 = -4XY^3 \) is a q-null vector, this orbit is lightlike (but cannot be a photon since the action is irreducible),

- one orbit \( \mathcal{L} \) comprising polynomials with a real root of multiplicity 3, and another real root. These are the polynomials of the form \( [(sY - tX)^3(s'Y - t'X)] \) with \( s, t, s', t' \in \mathbb{R} \). It is 2-dimensional, and it is easy to see that it is the union of the projective lines tangent to \( \mathcal{N} \). The vectors tangent to \( \mathcal{L} \) induced by the 1-parameter subgroups \( P \) and \( E \) at \( [XY^3] \in \mathcal{L} \) are \( v_P = -Y^4 \) and \( v_E = 3X^2Y^2 + Y^4 \). Obviously, \( v_P \) is orthogonal to \( v_E \) and \( v_E + v_P \) is spacelike. Hence \( \mathcal{L} \) is of signature \((0, 1, 1)\).

- one open orbit comprising polynomials admitting two distinct real roots and a root \( z \) in \( \mathbb{H}^2 \) making an angle \( 5\pi/6 \) with the two real roots in \( \partial\mathbb{H}^2 \). The stabilizers of points in this orbit are trivial since an isometry of \( \mathbb{H}^2 \) preserving a point in \( \mathbb{H}^2 \) and one point in \( \partial\mathbb{H}^2 \) is necessarily the identity. \( \square \)

**Proof of Theorem 1.2** According to Proposition 2.1, the polynomials in \( \text{AdS}^{1,3} \) have two distinct real roots, and a complex root \( z \) in \( \mathbb{H}^2 \) making an angle \( \theta \) greater than \( 5\pi/6 \) with the two real roots. It follows that the action in \( \text{AdS}^{1,3} \) is free, and that the orbits are the level sets of the function \( \theta \). Suppose that \( M \) is a \( \text{PSL}(2, \mathbb{R}) \)-orbit in \( \text{AdS}^{1,3} \). There exists \( r \in \mathbb{R} \) such that \( [f] = [Y(X^2 + Y^2)(X - rY)] \in M \). The orbit induced by the 1-parameter elliptic subgroup \( E \) at \([f] \) is \( \gamma(t) = [(X^2 + Y^2)((\sin t \cos t - r \sin^2 t)X^2 - (\sin t \cos t + r \cos^2 t)Y^2 + (\cos^2 t - \sin^2 t + 2r \sin t \cos t)XY)] \).

Then \( q(\frac{d}{dt}|_{t=0}X^2 + Y^2) = -2 - 2r^2 < 0 \). This implies, as for any submanifold of a Lorentzian manifold admitting a timelike vector, that \( M \) is Lorentzian, i.e., of signature \((1, 2)\).

The case of \( \mathbb{H}^{2,2} \) is the richest one. According to Proposition 2.1, there are four cases to consider:
• No real roots. Then $f$ has two complex roots $z_1$, $z_2$ in $\mathbb{H}^2$ (and their conjugates). It corresponds to two orbits: one orbit corresponding to the case $z_1 = z_2$: it is spacelike and has dimension 2. It is the only maximal $\text{PSL}(2, \mathbb{R})$-invariant surface in $\mathbb{H}^{2, 2}$ described in [2] Section 5.3. The case $z_1 \neq z_2$ provides a one-parameter family of 3-dimensional orbits on which the action is free (the parameter being the hyperbolic distance between $z_1$ and $z_2$). One may assume that $z_1 = i$ and $z_2 = ri$ for some $r > 0$. Denote by $M$ the orbit induced by $\text{PSL}(2, \mathbb{R})$ at $[f] = [(X^2 + Y^2)(X^2 + r^2Y^2)]$. The vectors tangent to $M$ at $[f]$ induced by the 1-parameter subgroups $H$, $P$ and $E$ are:

$$v_H = -4X^4 + 4r^2Y^4, \quad v_P = -4X^3Y - 2(r^2 + 1)XY^3, \quad v_E = 2(r^2 - 1)X^3Y + 2(r^2 - 1)XY^3,$$

respectively. The timelike vector $v_H$ is orthogonal to both $v_P$ and $v_E$. It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1, 1)$. Therefore, the tangent space $T_{[f]}M$ is of signature $(2, 1)$.

• Four distinct real roots. This case provides a one-parameter family of 3-dimensional orbits on which the action is free - the parameter being the cross-ratio between the roots in $\mathbb{R}P^1$. Denote by $M$ the $\text{PSL}(2, \mathbb{R})$-orbit at $[f] = [XY(X - Y)(X - rY)]$ (here as explained in the proof of Proposition 2.1 we can restrict ourselves to the case $0 < r < 1$). The vectors tangent to $M$ at $[f]$ induced by the 1-parameter subgroups $H$, $P$, and $E$ are:

$$v_H = -rY^4 + 2(r + 1)XY^3 - 3X^2Y^2, \quad v_P = -2X^3Y + 2rXY^3, \quad v_E = X^4 - rY^4 + 3(r - 1)X^2Y^2 + 2(r + 1)XY^3 - 2(r + 1)X^3Y,$$

respectively. A vector $x = av_H + bv_P + cv_E$ is orthogonal to $v_P$ if and only if $2ra + b(r + 1) + c(r + 1)^2 = 0$. Set $a = (b(r + 1) + c(r + 1)^2)/-2r$ in

$$q(x) = 2ra^2 + \frac{3}{2}b^2 + \left(\frac{7}{2}(r^2 + 1) - r \right)c^2 + 2(r + 1)ab + 2(r + 1)^2 + ac(2r^2 - 2r + 5).$$

Consider $q(x) = 0$ as a quadratic polynomial $F$ in $b$. Since $0 < r < 1$, the discriminant of $F$ is non-negative and it is positive when $c \neq 0$. Thus, the intersection of the orthogonal complement of the spacelike vector $v_P$ with the tangent space $T_{[f]}M$ is a 2-plane of signature $(1, 1)$. This implies that $M$ is Lorentzian, i.e., of signature $(1, 2)$.

• A root of multiplicity 2. Observe that if there is a non-real root of multiplicity 2, when we are in the first "no real root" case. Hence we consider here only the case where the root of multiplicity 2 lies in $\mathbb{R}P^1$. Then, we have three subcases to consider:

- two distinct real roots of multiplicity 2: The orbit induced at $X^2Y^2$ is the image of the $\text{PSL}(2, \mathbb{R})$-equivariant map

$$dS^{1, 1} \subset \mathbb{P}(\mathbb{R}_2[X, Y]) \rightarrow H^{2, 2}, \quad [L] \mapsto [L^2],$$

where $\mathbb{R}_2[X, Y]$ is the vector space of homogeneous polynomials of degree 2 in two variables $X$ and $Y$, endowed with discriminant as a $\text{PSL}(2, \mathbb{R})$-invariant bilinear form of signature $(1, 2)$ [2] Section 5.3. (Here, $L$ is the projective class of a Lorentzian bilinear form on $\mathbb{R}_2$).

The vectors tangent to the orbit at $X^2Y^2$ induced by the 1-parameter subgroups $P$ and $E$ are $v_P = -2XY^3$ and $v_E = 2X^3Y - 2XY^3$, respectively. It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1, 1)$. Hence, the orbit induced at $X^2Y^2$ is Lorentzian.
– three distinct real roots, one of them being of multiplicity 2: Denote by $M$ the orbit induced by $\text{PSL}(2, \mathbb{R})$ at $[f] = [XY^2(X - Y)]$. The vectors tangent to $M$ at $[f]$ induced by the 1-parameter subgroups $H$, $P$ and $E$ are:

$$v_H = -2XY^3, \quad v_P = Y^4 - 2XY^3, \quad v_E = Y^4 - X^4 - 2X^2Y^2 + X^3Y - XY^3,$$

respectively. Obviously, the lightlike vector $v_H + v_P$ is orthogonal to $T[f]M$. Therefore, the restriction of the metric on $T[f]M$ is degenerate. It is easy to see that the quotient of $T[f]M$ by the action of the isotropic line $\mathbb{R}(v_H + v_P)$ is of signature $(1, 1)$. Thus, $M$ is of signature $(1, 1, 1)$.

– one real root of multiplicity 2, and one root in $\mathbb{H}^2$: Denote by $M$ the orbit induced by $\text{PSL}(2, \mathbb{R})$ at $[f] = [Y^2(X^2 + Y^2)]$. The vectors tangent to $M$ at $[f]$ induced by the 1-parameter subgroups $H$, $P$ and $E$ are:

$$v_H = 4Y^4, \quad v_P = -2XY^3, \quad v_E = 2X^3Y + 2XY^3,$$

respectively. Obviously, the lightlike vector $v_H$ is orthogonal $T[f]M$. Therefore, the restriction of the metric on $T[f]M$ is degenerate. It is easy to see that the quotient of $T[f]M$ by the action of the isotropic line $\mathbb{R}(v_H)$ is of signature $(1, 1)$. Thus $M$ is of signature $(1, 1, 1)$.

- Two distinct real roots, and a complex root $z$ in $\mathbb{H}^2$ making an angle $\theta$ smaller than $5\pi/6$ with the two real roots. Denote by $M$ the orbit induced by $\text{PSL}(2, \mathbb{R})$ at $[f] = [Y(X^2 + Y^2)(X - rY)]$. The vectors tangent to $M$ at $[f]$ induced by the 1-parameter subgroups $H$, $P$ and $E$ are:

$$v_H = -4rY^4 - 2X^3Y + 2XY^3, \quad v_P = -3X^2Y^2 + 2rXY^3 - Y^4,$$

$$v_E = X^4 - Y^4 - 2rX^3Y - 2rXY^3,$$

respectively. The following set of vectors is an orthogonal basis for $T[f]M$ where the first vector is timelike and the two others are spacelike.

$$\{(7r + 3r^3)v_H + (6 - 2r^2)v_P + (5 + r^2)v_E, 4v_P + v_E, v_H\}.$$ 

Therefore, $M$ is Lorentzian, i.e., of signature $(1, 2)$.

\[ \Box \]

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