Recent results on a non-minimal coupling between curvature and matter

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Abstract. This work presents a review of recent findings from the consideration of a non-minimal coupling between matter and geometry, namely the possibility of mimicking dark matter in clusters and the description of gravitational collapse — thus adding to the wide range of phenomena already covered by the theory.

1. Introduction
Dark gravity is the collective name under which a growing number of researchers are tackling several outstanding issues in contemporary Cosmology, namely dark matter and dark energy: instead of assuming that Einstein’s General Relativity (GR) is valid at the relevant energy scales and focusing on the search for the missing matter or energy contribution, this alternative posits instead that the dark sector of our Universe merely reflect deviations of gravity from GR, so that no real dark matter or dark energy species are in principle required.

Several phenomenological attempts to generalize GR have surfaced, positing putative low-energy modifications of Einstein’s theory that should be derived from a yet unknown high-energy, fundamental theory of gravitation: in particular, \( f(R) \) theories substitute the linear scalar curvature \( R \) term in the Einstein-Hilbert action with a function \( f(R) \) [1]. This may be generalized by considering a non-minimal coupling with matter [2],

\[
S = \int \left[ \kappa f_1(R) + f_2(R) \mathcal{L} \right] \sqrt{-g} \, d^4x , \quad \kappa = \frac{c^4}{16\pi G} .
\]  

(1)

This work presents a view of recent results obtained with this framework [3, 4], developed in collaboration with O. Bertolami, C. Bastos and P. Frazão. Other results [5, 6, 7, 8, 9, 10, 11] were discussed in Ref. [12].

2. The model
Variation with respect to the action Eq. (1) yields the modified Einstein field equations,

\[
2(\kappa F_1 + F_2 \mathcal{L}) G_{\mu\nu} = 2(\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) (\kappa F_1 + F_2 \mathcal{L}) - [\kappa (F_1 R - f_1) + F_2 \mathcal{L} R] g_{\mu\nu} + f_2 T_{\mu\nu} ,
\]

(2)

where \( F_i(R) \equiv f_i'(R) \). One recovers GR by setting \( f_1(R) = R \) and \( f_2(R) = 1 \). Its trace is

\[
\kappa (F_1 R - 2 f_1) + F_2 \mathcal{L} R = \frac{1}{2} f_2 T - 3 \Box (\kappa F_1 + F_2 \mathcal{L}) ,
\]

(3)
where $T$ is the trace of the energy-momentum tensor. The Bianchi identities may be used to derive the non-(covariant) conservation of the energy-momentum tensor,

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2} (\gamma^{\mu\nu} L - T^{\mu\nu}) \nabla_\mu R .$$ (4)

This may be rewritten as an extra force imparted on test particles, so that its trajectory will deviate from geodesical motion. Thus, the Equivalence Principle may be broken if the r.h.s. of the last equation varies significantly for different matter distributions [2, 13].

### 3. Cluster dark matter mimicking

One first reviews a mechanism that mimics the dark matter at cluster scales [3]. In order to isolate the effect of the non-minimal coupling, one sets $f_1(R) = 1$ and a power-law $f_2(R) = 1 + (R/R_n)^n$. Inserting this into Eq. (3), together with the Lagrangian density $\mathcal{L} = -\rho$ and the energy-momentum tensor $T_{\mu\nu} = \rho U_\mu U_\nu$ for non-relativistic dust yields

$$R = (1 - 2n) \left( \frac{R}{R_n} \right)^n \rho \frac{2}{\kappa R_n} - 3n \Box \left[ \left( \frac{R}{R_n} \right)^n \rho \frac{\kappa}{R_n} \right] .$$ (5)

If the terms stemming from the non-minimal coupling dominate, an exact solution is found when the last term vanishes; interpreting the additional curvature obtained as a mimicked “dark matter”, one obtains a direct translation between visible and mimicked dark matter,

$$\rho_{dm} \equiv 2\kappa R = \rho_n \left[ (1 - 2n) \frac{\rho}{\rho_n} \right]^{1/(1-n)} , \quad \rho_n \equiv 2\kappa R_n .$$ (6)

A more evolved study of Eq. (5) shows that its most general solution exhibits small oscillations around Eq. (6): this makes the gradient term actually dominate Eq. (5), and yields a perturbative $f_2(R) \ll 1$, which guarantees that the weak, strong, null and dominant energy conditions are satisfied and prevents the appearance of Dolgov-Kawasaki instabilities [14].

In order to fit the dark matter density profiles of several clusters (A133, A262, A383, A478, A907, A1413, A1795, A1991, A2029, A2390, RX J1159+5531, MKW 4, USGC S152 and A586, which are almost virialized and spherical clusters [15]), one numerically solves Eq. (5) with $\rho$ given by available curves for the visible matter density, and fit the resulting $\rho_{dm}$ to the reported dark matter profiles. A best fit is obtained for a value $n = 0.2$, as depicted in Fig. 1 for the Abell A907 cluster. The other clusters also yield an adequate fit [3], showing that the outlined mechanism, first proposed in the context of galactic dark matter [8], is also useful at this scale.

This fitting procedure yields an upper bound for $r_n \equiv R_n^{-1/2}$, based upon the two following considerations: firstly, it is found that the dark matter density profiles reported in Ref. [15] are impossible to mimic if the non-minimal coupling is non-perturbative, so that $f_2(R) \sim 1$ — which numerically translates into $r_{0.2} \ll 1$ Mpc. A second, more stringent bound is derived from the physical constraint that the extra force arising from the non-conservation of the energy-momentum tensor Eq. (4) is much smaller than the Newtonian gravitational force: this leads to the stronger condition $r_{0.2} \ll 51$ pc. With the latter constraint and the positive exponent $n = 0.2$, the non-minimal coupling is much smaller than unity, $f_2(R) \sim 1 + 10^{-8}$.

### 4. Gravitational Collapse

One now discusses the features of the gravitational collapse of a spherical body of homogeneous dust linearly coupled to gravity [4], thus generalizing the familiar Oppenheimer-Snyder scenario (OS). One considers $f_2(R) = 1 + \epsilon R/\kappa$ and a trivial $f_1(R) = R$, in order to highlight the
effect of the former. Compatibility between a non-minimally coupled preheating mechanism and Starobinsky inflation [11] dictates that the coupling strength is of the order $10^9 < \epsilon < 10^{13}$.

The Lagrangean density of matter appears explicitly in the modified field Eq. (2): this has prompted the question as to what is the right form for this quantity, as discussed in Ref. [7] — where it was argued that $\mathcal{L} = -\rho$, not $\mathcal{L} = p$, is the correct one. Nevertheless, one asserts the physical impact of the two forms by writing $\mathcal{L} = -\alpha p$: the choice $\mathcal{L} = -\rho$ corresponds to $\alpha = 1$, while $\mathcal{L} = p$ is obtained by setting $\alpha = 0$ (as the pressure vanishes in the dust distribution). The parameter $\alpha$ is thus a binary variable, considerably simplifying several calculations.

Introducing the above expression, together with the energy-momentum tensor for dust and the adopted forms for $f_1(R)$ and $f_2(R)$ into the modified field Eq. (2), one eventually obtains

$$\frac{1}{6} \kappa a^2 \rho = \left( \kappa^2 - \epsilon \rho \right) k + \left[ \kappa^2 + (3 \alpha - 1) \epsilon \rho \right] \left( a \right)^2 + \epsilon (\alpha - 1) a \dot{a} \rho ,$$

where $a(t)$ is scale factor and $k$ the spatial curvature of the FRW describing the homogeneous collapse: the latter is given by $k = k_0/[1 - (1 + \alpha)\epsilon_0/2]$, with $k_0$ the value found in GR and $\epsilon_0 \equiv \epsilon \rho_0/k^2$. Since the initial density is much smaller than the typical density of neutron stars, $\rho_0 \ll \rho_N \sim 10^{18}$ kg/m$^3$, this parameter is vanishingly small, $\epsilon_0 \ll 10^{-62}$.

Using Eq. (3) and recalling that $\alpha$ is a binary variable, one integrates Eq. (4) to find

$$\rho(t) = \rho_0 \left( \frac{\kappa + \epsilon R}{\kappa + \epsilon R_0} \right)^{\alpha - 1} \left( \frac{a_0}{a} \right)^3 = \frac{\rho_0 a^{-3}}{1 + \epsilon (1 - \alpha) \frac{\epsilon_0}{2 \kappa^2} (a^{-3} - 1)} .$$

Combining Eq. (7) and (8), one obtains the differential equation

$$\dot{a} = -\sqrt{\frac{k(1 - a)}{a^3 + (a + 1) \alpha \epsilon_0}} \frac{a^2 + (a + 1) \alpha \epsilon_0}{a^3 + 2 \alpha \epsilon_0} , \quad a(t_0) = 1 .$$

Analysis of Eq. (9) and Eq. (8) shows that, in the $\mathcal{L} = -\rho \rightarrow \alpha = 1$ scenario, the gravitational collapse deviates (very weakly, as $\epsilon_0 \ll 1$) from GR due to the more evolved dynamics; this is depicted in Fig. 2 for larger values of $\epsilon_0$, for clarity. The usual dependence of the density on the scale factor $\rho \sim a^{-3}$ is maintained, and a point-like singularity with infinite density reached.

The $\mathcal{L} = p \rightarrow \alpha = 0$ scenario is much more interesting: although the dynamics of the scale factor are qualitatively the same as in OS collapse, the modified dependence for the density yields a geometric point-like singularity with a final finite density $\rho \rightarrow \rho_f = 2k^3/\epsilon$. Since $\epsilon$ is very large, this falls well below the Planckian domain, $\rho \rightarrow \rho_f \ll M_P^2$, although still many orders of magnitude above the typical density of neutron stars.

To match the interior FRW metric with the outer Schwarzschild metric, one inspects the behaviour of the extrinsic curvature $K_{ab}$ across the surface of the body, which reads

$$K^+_{ab} = \left( 1 - \frac{\epsilon \alpha}{k^2} \rho \right) K^-_{ab} + \frac{\epsilon \alpha}{k^2} \rho K^- h_{ab} .$$

In the $\mathcal{L} = p \rightarrow \alpha = 1$ scenario, the extrinsic curvature is continuous, and this matching shows that the mass $M$ of the spherical body (as measured from the gravitational potential away from it) according to $M = M_0/(1 - \epsilon_0/2)$, where $M_0$ is the usual gravitational mass obtained in GR. As this depends on the initial density $\rho_0$, different event horizons arise from the collapse of stars with the same mass, but distinct radius — thus breaking the no-hair theorem.

The alternative description $\mathcal{L} = -\rho \rightarrow \alpha = 1$ is problematic, as the matching of the inner and outer spacetimes is unfeasible unless unnatural extra terms are added to the boundary action. This could be related to the non-vanishing effective pressure that arises due to the non-minimal coupling — i.e. a non-vanishing $p_{eff} \sim G_{rr} \neq 0$ — reminiscent of the similar matching problem found in the gravitational collapse of a homogeneous sphere with pressure in GR.
5. Conclusions and Outlook
A non-minimal coupling between geometry and matter covers a broad spectrum of applications in an elegant, natural way, as these and other results have shown. The different forms adopted may be regarded as an approximation to a more evolved forms for $f_2(R)$, each valid in a particular regime: early vs. late time, central vs. long range, etc. — hinting at a Laurent series expansion for the latter. Yet untested terms of the above series may be evaluated by studying other phenomena and environments, where distinct curvatures and densities are at play.

Acknowledgments
This work was developed in the context of the First International Conference on Mathematical Modeling in Physical Sciences, 3–7 September 2012, Budapest, Hungary. I wish to thank the organization, specially Elins Vagenas, for the invitation and hospitality.

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