Hypergraph LSS-ideals and coordinate sections of symmetric tensors

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Joint work with Volkmar Welker

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\begin{itemize}
    \item $\mathbb{K}$ field;
    \item $H = ([n], E)$ cluster hypergraph;
    \item $d \geq 1$ an integer;
    \item $S = \mathbb{K}[y_{ik} : i \in [n], k \in [d]]$;
    \item $e \in E, \quad f_e^{(d)} = \sum_{k=1}^{d} \prod_{i \in e} y_{ik}$;
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\[ \mathbb{K} = \mathbb{R}, H \text{ graph } \implies V(L^\mathbb{K}_H(d)) = OR_H(\mathbb{R}^d) \]
An orthogonal representation of $H$ in $\mathbb{R}^d$ assigns to each $i \in [n]$ a vector $u_i \in \mathbb{R}^d$ such that $u_i^T u_j = 0$, for $\{i, j\} \in \bar{E}$. 
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- 1979 Lovász
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- 1979 Lovász

- 1984 Grötschel, Lovász and Schrijver
\( L^K_H(d) \) Lovász-Saks-Schrijver ideal (LSS-ideal)
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- 2015 Herzog, Machia, Saeedi Madani and Welker
• 2018 Conca, Welker
Definition (Conca-Welker)

Given a hypergraph $H = (V, E)$, a positive matching of $H$ is a matching $M$ of $H$ such that there exists a weight function $w: V \to \mathbb{R}$ satisfying:

$$
\sum_{v \in e} w(v) > 0, \quad \text{if } e \in M, \\
\sum_{v \in e} w(v) < 0, \quad \text{if } e \notin M.
$$
Definition (Conca-Welker)

Let $H = (V, E)$ be a hypergraph. A positive matching decomposition (or pmd) of $H$ is a partition $E = \bigcup_{i=1}^{p} E_i$ of $E$ into pairwise disjoint subsets such that $E_i$ is a positive matching of $(V, E \setminus \bigcup_{j=1}^{i-1} E_j)$, for $i = 1, \ldots, p$. The $E_i$s are called the parts of the pmd. The smallest $p$ for which $H$ admits a pmd with $p$ parts will be denoted by $\text{pmd}(H)$. 

The following theorem establishes a nice connection between the pmd of a hypergraph with the algebraic properties of the corresponding LSS-ideal.

Theorem (Conca-Welker)

Let $H = (V, E)$ be a hypergraph. Then for $d \geq \text{pmd}(H)$ the ideal $L_{K[H]}(d)$ is a radical complete intersection. In particular, $L_{K[H]}(d)$ is prime if $d \geq \text{pmd}(H) + 1$. 

Gharakhloo-Welker

LSS-ideals and Tensors
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LSS-ideal $L^K_\Gamma(d)$ and determinantal ideal of the $(d + 1)$-minors of a generic symmetric matrix $I_{d+1}^K(Y_{\Gamma}^{\text{Sym}})$:
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The ideal $I_{d+1}^{K}(Y_{\Gamma}^{\text{Sym}})$ is radical (resp. is prime, has the expected height) if the ideal $L_{\Gamma}^{K}(d)$ is radical (resp. is prime, is a complete intersection).
LSS-ideal $L^{k}_{\Gamma}(d)$ and determinantal ideal of the $(d + 1)$-minors of a generic symmetric matrix $I^{K}_{d+1}(Y^{\text{Sym}}_{\Gamma})$:

The ideal $I^{K}_{d+1}(Y^{\text{Sym}}_{\Gamma})$ is radical (resp. is prime, has the expected height) if the ideal $L^{k}_{\Gamma}(d)$ is radical (resp. is prime, is a complete intersection).

Therefore the ideal $I^{K}_{d+1}(Y^{\text{Sym}}_{\Gamma})$ is radical complete intersection for $d \geq \text{pmd}(\Gamma)$ and is prime for $d \geq \text{pmd}(\Gamma) + 1$. 
Connection of LSS-ideals of $k$-uniform hypergraphs and coordinate sections of the variety which is the closure of the set of symmetric tensors of bounded rank ($S^d_{n,k}$):
Connection of LSS-ideals of $k$-uniform hypergraphs and coordinate sections of the variety which is the closure of the set of symmetric tensors of bounded rank ($S_{n,k}^d$):

Let $\mathbb{K}$ be an algebraically closed field. Consider the map

$$
\phi : (\mathbb{K}^n)^d \longrightarrow \underbrace{\mathbb{K}^n \otimes \ldots \otimes \mathbb{K}^n}_k
$$

$$(v_1, \ldots, v_d) \mapsto \sum_{j=1}^{d} v_j \otimes \cdots \otimes v_j$$

$$
= \sum_{j=1}^{d} \sum_{1 \leq i_1, \ldots, i_k \leq n} (v_j)_{i_1} \cdots (v_j)_{i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \in \underbrace{\mathbb{K}^n \otimes \ldots \otimes \mathbb{K}^n}_k
$$
The Zariski closure of the image of \( \phi \) is the variety \( S_{n,k}^d \) of symmetric tensors of (symmetric) rank \( \leq d \). The coefficient of \( e_{i_1} \otimes \cdots \otimes e_{i_k} \) in \( \phi(v_1, \ldots, v_d) \) is

\[
\sum_{j=1}^{d} (v_j)_{i_1} \cdots (v_j)_{i_k} = f_{\{i_1 < \cdots < i_k\}}^{(d)}(v_1, \ldots, v_d).
\]
The Zariski closure of the image of $\phi$ is the variety $S_{n,k}^d$ of symmetric tensors of (symmetric) rank $\leq d$. The coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_k}$ in $\phi(v_1, \ldots, v_d)$ is

$$\sum_{j=1}^{d} (v_j)_{i_1} \cdots (v_j)_{i_k} = f^{(d)}_{\{i_1<\cdots<i_k\}}(v_1, \ldots, v_d).$$

Therefore if we restrict the map $\phi$ to $V(L^K_H(d))$, then we have a parameterization of the coordinate section of $S_{n,k}^d$ with 0 coefficient at $e_{i_1} \otimes \cdots \otimes e_{i_k}$ for $\{i_1, \ldots, i_k\} \in E$. 
The Zariski closure of the image of $\phi$ is the variety $S^{d}_{n,k}$ of symmetric tensors of (symmetric) rank $\leq d$. The coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_k}$ in $\phi(v_1, \ldots, v_d)$ is

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Theorem 1 (Gharakhloo-Welker)

Let $H = ([n], E)$ be a $k$-uniform hypergraph and $d \geq 2$. If $L^K_H(d)$ is prime, then $L^K_H(d)$ is a complete intersection.
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Theorem 2 (Gharakhloo-Welker)

Let $H = ([n], E)$ be a $k$-uniform hypergraph and $d \geq 2$. If $L^K_H(d - 1)$ is a complete intersection, then $L^K_H(d)$ is prime.
Theorem (Conca-Welker)

Let $\mathbb{K}$ be a field. Then for a hypergraph $H = (V, E)$:

$$\text{pmd}(H) \leq d \Rightarrow L^\mathbb{K}_H(d) \text{ is a complete intersection.}$$

(1)
Theorem (Avramov-Huneke)

Let $R$ be a complete intersection and $M$ be an $R$-module presented by the matrix $A \in R^{m \times n}$. Then

1. $\text{Sym}_R(M)$ is a complete intersection $\iff \text{height } (I_t(A)) \geq m - t + 1$ for all $t = 1, \ldots, m$.

2. $\text{Sym}_R(M)$ is a domain and $I_m(A) \neq 0 \iff R$ is a domain and \( \text{height } (I_t(A)) \geq m - t + 2 \) for all $t = 1, \ldots, m$. 




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For a $k$-uniform hypergraph $H = ([n], E)$ and a number $d \geq 1$ we fix

\[ S = \mathbb{K}[y_{ij} : i \in [n], j \in [d]], \quad S' = \mathbb{K}[y_{ij} : i \in [n-1], j \in [d]], \]

\[ H' = H \setminus \{n\}, \quad R = S'/L_{H'}^\mathbb{K}(d), \]

\[ U = \left\{ \{i_1, \ldots, i_{k-1}\} \subseteq [n-1] \mid \{i_1, \ldots, i_{k-1}, n\} \in E \right\}, \quad u = \left| U \right|. \]
Remark

Let $H = ([n], E)$ be a $k$-uniform hypergraph. Then $S/L^K_H(d)$ is the symmetric algebra of the cokernel of the linear map $R^u \xrightarrow{A^T} R^d$ defined by the $u \times d$ matrix $A$ where

$$A = \left( y_{i_1j}y_{i_2j} \cdots y_{i_{k-1}j} \right)_{\{i_1, \ldots, i_{k-1}\} \in U, j \in [d]} \in S' u \times d.$$
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Lemma (Gharakhloo-Welker)

Let $H = ([n], E)$ be a $k$-uniform hypergraph. Then for every $2 \leq t \leq u$ the $t$-minor $f_t$ of

$$A = \left( y_{i_1j}y_{i_2j} \cdots y_{i_{k-1}j} \right) \{i_1, \ldots, i_{k-1}\} \in U, j \in [d] \in S' u \times d$$

corresponding to the first $t$ rows and columns is non-zero in $S/L^K_H(d)$. 
**Definition (Gharakhloo-Welker)**

For integers $k, c > 0$ such that $0 < k - 1 \leq n - c$ let $W$ be a set of $(k - 1)$-subsets of $[n - c]$ with $|W| > 0$.

By $H_{W,c}$ we denote the hypergraph

$$
([n], \left\{ \{i_1, \ldots, i_k\} \mid \{i_1, \ldots, i_{k-1}\} \in W, i_k \in \{n - c + 1, \ldots, n\}\right\}).
$$
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$$( [n], \{ \{i_1, \ldots, i_k\} \mid \{i_1, \ldots, i_{k-1}\} \in W, \ i_k \in \{n - c + 1, \ldots, n\} \} ).$$

Proposition (Gharakhloo-Welker)

If $L_{H}^{IK}(d)$ is prime, then $H$ does not contain $H_{W,c}$ for any set $W$ of $(k - 1)$-subsets of $[n - c]$ with $|W| + c > d$. 
Lemma (Gharakhloo-Welker)

For integers \( n, u > 0 \) and \( k \geq 2 \) assume \( U = \{A_1, \ldots, A_u\} \) for distinct \((k - 1)\)-subsets \( A_1, \ldots, A_u \) of \([n]\). For elements \((y_{ij})_{i \in [n], j \in [d]}\) from the Noetherian ring \( R \) and variables \((y_{i d + 1})_{i \in [n]}\) define \( m_{ij} = \prod_{\ell \in A_i} y_{\ell j} \) for \( i \in [u] \) and \( j \in [d] \). Consider the matrix

\[
M = \begin{bmatrix}
m_{11} & m_{12} & \cdots & m_{1d} \\
m_{21} & m_{22} & \cdots & m_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
m_{u1} & m_{u2} & \cdots & m_{ud}
\end{bmatrix}
\]
with entries in $R$ and the matrix $M'$ arising from $M$ by adding the new column

$$[m_{1d+1}, \cdots, m_{ud+1}]^T$$

with entries in $T = R[Y] = R[y_{1d+1}, \cdots, y_{nd+1}]$. Then for all $1 < t \leq u$ we have

$$\text{height } I_t(M') \geq \min\{\text{height } I_{t-1}(M), \text{height } I_t(M) + 1\}. \quad (2)$$
Definition (Gharakhloo-Welker)

Let $k \geq 2$. A $k$-uniform hypergraph $H = (V, E)$ is called a ($k$-uniform) tree if

- For each pair of edges $e, e' \in E$ we have $|e \setminus e'| = 1$.
- For each pair of vertices $v, v' \in V$, for which there is no edge in $E$ containing both, there exists a unique sequence $e_1, \ldots, e_r \in E$ such that:
  - $(a)$ $v \in e_1$ and $v' \in e_r$ and $v, v' \not\in e_2, \ldots, e_{r-1}$,
  - $(b)$ for each $1 \leq i \leq r$ we have $|e_i \setminus e_{i+1}| = 1$,
  - $(c)$ for each $1 \leq i \neq j \leq r$ we have $e_i \setminus e_j = \emptyset$. 

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(T1) For each pair of edges $e, e' \in E$ we have $|e \cap e'| \leq 1$.

(T2) For each pair of vertices $v, v' \in V$, for which there is no edge in $E$ containing both, there exists a unique sequence $e_1, \ldots, e_r \in E$ such that:

(a) $v \in e_1$ and $v' \in e_r$ and $v, v' \notin e_2, \ldots, e_{r-1}$,
(b) For each $1 \leq i \leq r - 1$, we have $|e_i \cap e_{i+1}| = 1$,
(c) For each $1 \leq i \neq j \leq r$ where $|i - j| \geq 2$ we have $e_i \cap e_j = \emptyset$. 

Positive matching decomposition for hypergraph
Definition (Gharakhloo-Welker)

Let \( k \geq 2 \). A \( k \)-uniform hypergraph \( H = (V, E) \) is called a \((k\text{-uniform})\) tree if

\[\begin{align*}
\text{(T1)} & \quad \text{For each pair of edges } e, e' \in E \text{ we have } |e \cap e'| \leq 1. \\
\text{(T2)} & \quad \text{For each pair of vertices } v, v' \in V, \text{ for which there is no edge in } E \\
& \quad \text{containing both, there exists a unique sequence } e_1, \ldots, e_r \in E \text{ such that:}
\end{align*}\]

\[\begin{align*}
& \quad (a) \quad v \in e_1 \text{ and } v' \in e_r \text{ and } v, v' \not\in e_2, \ldots, e_{r-1}, \\
& \quad (b) \quad \text{For each } 1 \leq i \leq r - 1, \text{ we have } |e_i \cap e_{i+1}| = 1, \\
& \quad (c) \quad \text{For each } 1 \leq i \neq j \leq r \text{ where } |i - j| \geq 2 \text{ we have } e_i \cap e_j = \emptyset.
\end{align*}\]
Theorem (Gharakhloo-Welker)

Let \( H = (V, E) \) be a \( k \)-uniform tree. Then \( \text{pmd}(H) = \Delta(H) \).
Theorem (Gharakhloo-Welker)

Let $H = (V, E)$ be a $k$-uniform tree. Then $\text{pmd}(H) = \Delta(H)$.

Proof strategy:

- Using induction on $\Delta(H)$. 

Gharakhloo-Welker  

LSS-ideals and Tensors
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- Using induction on $\Delta(H)$.

- Let $k \geq 2$ and $H = (V, E)$ be a $k$-uniform hypergraph which is a tree. Then there exists at least one vertex $v \in V$, with $\deg_H(v) = 1$. 
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$\triangleright$ Using induction on $\Delta(H)$.

$\triangleright$ Let $k \geq 2$ and $H = (V, E)$ be a $k$-uniform hypergraph which is a tree. Then there exists at least one vertex $v \in V$, with $\deg_H(v) = 1$.

$\triangleright$ Let $k \geq 2$ and let $H = (V, E)$ be a $k$-uniform hypergraph which is a tree with $\Delta(H) \geq 2$. Then there is a positive matching $M \subseteq E$ such that $V(M)$ contains all vertices $v$ of degree $\deg(v) \geq 2$. 
Corollary

Let $H = ([n], E)$ be a $k$-uniform tree. Then the coordinate sections of the variety $S^{d}_{n,k}$ with respect to $H$ for $\Delta(H) + 1 \leq d \leq \binom{n+k-1}{k} - n$, are irreducible.
Proposition (Gharakhloo-Welker)

Let $H = (V, E)$ be the complete 3-uniform hypergraph on $n$ vertices and with $\binom{n}{3}$ edges. Then for every $3 \leq l_1 \leq 2n - 3$ and $5 \leq l_2 \leq 2n - 1$, the set $E_{l_1, l_2} = \{\{a, b, c\} \in E \mid a < b < c, \ a + b = l_1, b + c = l_2\}$ is a matching and $E = \bigcup_{l_1, l_2} E_{l_1, l_2}$.

In addition, the cardinality of the set

$$E_n := \{(l_1, l_2) \mid \text{there exist } 1 \leq a < b < c \leq n, \ l_1 = a + b, l_2 = b + c\}$$

is $\frac{3}{2}n^2 - \frac{15}{2}n + 10$. 


Conjecture (Gharakhloo-Welker)

Let $H = (V, E)$ be a 3-uniform hypergraph with $n$ vertices. Then
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\text{pmd}(H) \leq \frac{3}{2}n^2 - \frac{15}{2}n + 10.
\]
Conjecture (Gharakhloo-Welker)

Let $H = (V, E)$ be a 3-uniform hypergraph with $n$ vertices. Then
\[ \text{pmd}(H) \leq \frac{3}{2}n^2 - \frac{15}{2}n + 10. \]

\(<\text{If the conjecture holds, then for } 3\left(\frac{n}{2}\right) - 6n + 10 \leq d \leq \left(\frac{n+2}{3}\right) - n + 1 \text{ every coordinate section of } S_{n,k}^d \text{ is irreducible.}\)
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Thanks for your attention