The hyperholomorophic line bundle

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Dedicated to Klaus Hulek on the occasion of his 60th birthday

1 Introduction

In a recent paper [10], A.Haydys introduced a natural line bundle with connection on a hyperkähler manifold with an $S^1$-action of a certain type. The curvature is of type $(1,1)$ with respect to all complex structures in the hyperkähler family and for this reason is called hyperholomorphic. In [12] a description of this line bundle via a holomorphic bundle on the twistor space was given and in this format calculated for a number of examples of interest to physicists. These are mostly moduli spaces of solutions to gauge-theoretic equations.

In this article we give examples with a more geometrical flavour, in particular on minimal resolutions of Kleinian singularities and cotangent bundles of coadjoint orbits of a compact Lie group. We first approach the subject from the differential-geometric point of view, giving some explicit formulae, and then from the twistor viewpoint, where, as in [12], the holomorphic point of view demonstrates a naturality which is not apparent from the explicit expressions.

In a more general result, which contributes to the examples, we show how the hyperholomorphic bundle descends naturally in a hyperkähler quotient, and for the quotient by a linear action on flat space can be identified with a canonical hyperholomorphic line bundle.

2 The differential geometric viewpoint

2.1 The hyperholomorphic connection

Let $M$ be a hyperkähler manifold with Kähler forms $\omega_1, \omega_2, \omega_3$ relative to complex structures $I, J, K$. If the de Rham cohomology class $[\omega_1/2\pi] \in H^2(M, \mathbb{R})$ is in the image of the integral cohomology then there exists a line bundle $L$ and hermitian connection $\nabla$ with curvature
\( \omega_1 \), unique up to tensoring with a flat \( U(1) \) bundle. Since \( \omega_1 \) is of type \((1, 1)\) with respect to the complex structure \( I \), \( L \) also has a holomorphic structure defined by the \( \bar{\partial} \)-operator \( \nabla^0 \). Given a local holomorphic section \( s \) of \( L \), then \( \omega_1 = dd^c \log \|s\|^2 / 2 \). Hence, if we multiply the hermitian metric by \( e^{2f} \) the curvature of the connection on \( L \) compatible with this new structure is

\[
F = \omega_1 + dd^c f.
\]

Suppose now we have a circle action which fixes \( \omega_1 \) but acts on the other forms by the transformation \((\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3) \). The manifold \( M \) must necessarily be noncompact for this. Suppose further that we have chosen a lift of the action to \( L \). This implies in particular the existence of a moment map – a function \( \mu \) such that \( i_X \omega_1 = d\mu \) where \( X \) is the vector field generated by the action. Then the result of Haydys [10] (see also [12]) is:

**Theorem 1** The 2-form \( \omega_1 + dd^c \mu \) is of type \((1, 1)\) with respect to complex structures \( I, J, K \).

Thus rescaling the natural metric by \( e^{2\mu} \) gives a new connection which defines a holomorphic structure on \( L \) relative to all complex structures in the quaternionic family. This is a hyperholomorphic connection, and \( L \) is the hyperholomorphic bundle of the title.

There are relatively few hyperkähler metrics which one can write down explicitly but it is instructive to find the line bundle in these cases.

**Example:** Flat quaternionic space \( H^n \). Writing \( H^n = C^n \oplus jC^n \) we have

\[
\omega_1 = \frac{i}{2} \sum_i (dz_i \wedge d\bar{z}_i + dw_i \wedge d\bar{w}_i), \quad \omega_2 + i\omega_3 = \sum_i dz_i \wedge dw_i
\]

and the action \((z, w) \mapsto (z, e^{i\theta}w)\) is of the required form. Then

\[
F = \omega_1 + dd^c \mu = \frac{i}{2} \sum_i (dz_i \wedge d\bar{z}_i - dw_i \wedge d\bar{w}_i).
\]

In the complex structure \( I \) this is the trivial holomorphic line bundle with hermitian metric \( h = (\|z\|^2 - \|w\|^2) / 2 \).

In the above we have specified a particular action of the circle on the three Kähler forms \( \omega_1, \omega_2, \omega_3 \). More generally, if an irreducible hyperkähler manifold \( M \) has a circle symmetry then it acts on the three-dimensional space of covariant constant 2-forms preserving the inner product. The action is either trivial, in which case it is called triholomorphic, or it leaves fixed a one-dimensional subspace with an orthogonal complement on which the action is rotation by \( n\theta \). The case above is \( n = 1 \). This occurs for example on the cotangent bundle of a complex manifold where the action is scalar multiplication in a fibre and the symplectic form is the canonical one. In the general case, \( \mathbb{Z}_n \subset S^1 \) preserves the three Kähler forms.
and so the quotient $M/Z_n$ is a hyperkähler orbifold with a circle action as above. The local geometry of the hyperholomorphic bundle is then the same, but the curvature form on $M$ is $F = \omega_1 + ndd^c \mu$.

In what follows we shall also consider flat space as above but with the action $(z, w) \mapsto e^{i\theta}(z, w)$. Then $n = 2$ since $(\omega_2 + i\omega_3) \mapsto e^{2i\theta}(\omega_2 + i\omega_3)$. The moment map $\mu = -((\|z\|^2 + \|w\|^2))/2$ and so $F = \omega_1 + 2dd^c \mu = 0$ and the hyperholomorphic line bundle is trivial as a line bundle with connection. This may seem uninteresting, but in Theorem 4 we shall see how it defines the bundle for a hyperkähler quotient of $H^n$.

2.2 Hermitian symmetric spaces

Biquard and Gauduchon gave in [1] an explicit formula for a hyperkähler metric which, in the complex structure $I$, is defined on the total space of the cotangent bundle of a hermitian symmetric space $G/H$. A circle action is given by multiplication of a cotangent vector by a unit complex number and the form $\omega_2 + i\omega_3$ is the canonical symplectic form on the cotangent bundle.

If $p : T^*(G/H) \to G/H$ is the projection and $\omega$ is the Kähler form of the symmetric space $G/H$ then the hyperkähler metric is defined by $\omega_1 = p^* \omega + dd^c h$ where, for a cotangent vector $v$, $h$ is the quartic function on the fibres defined by $h(v) = (f(IR(Iv, v)v, v))$. Here $R(u, v)$ is the curvature tensor of $G/H$ and $f$ is the analytic function

$$f(u) = \frac{1}{u} \left( \sqrt{1 + u} - 1 - \log \frac{1 + \sqrt{1 + u}}{2} \right).$$

This function is applied to $IR(Iv, v)$ which is a non-negative hermitian transformation. In fact since the curvature of a symmetric space is constant we can also view the quadratic map $R(Iv, v)$ from $(g/h)^*$ to $h \subset g$ as a multiple of the moment map for the isotropy action of $H$. The strange function $f(u)$ has the property that

$$(uf(u))' = \frac{1}{2u}(\sqrt{1 + u} - 1) \quad (1)$$

We first calculate the moment map $\mu$ for the circle action. Since the action is purely in the fibres of the cotangent bundle we have

$$i_X \omega_1 = i_X (p^* \omega + dd^c h) = i_X dd^c h.$$

Now the action preserves both $h$ and the complex structure so $(di_X + iXd)d^c h = L_X d^c h = d^c (L_X h) = 0$, which means that $i_X \omega_1 = -d(i_X d^c h)$ and we can take $\mu = -i_X d^c h = (IX)(h)$. The vector field $X$ was generated by $v \mapsto e^{i\theta} v$ so $IX$ is generated by $v \mapsto \lambda^{-1} v$ for $\lambda \in \mathbb{R}^+$. Hence

$$\mu(v) = \frac{\partial}{\partial \lambda} h(\lambda^{-1} v)|_{\lambda = 1}.$$
But $h(v) = (f(u)v, v)$ where $u = IR(Iv, v)$ is homogeneous of degree 2 in $v$ and so $\mu(v) = -2(u^f(u)v, v) - 2(f(u)v, v) = -2((uf(u))v, v)$. Using (11) we see that

\[
F = \omega_1 + dd^c \mu = p^\ast \omega + dd^c k
\]

where $k(v) = (g(IR(Iv, v))v, v)$ for the function

\[
g(u) = -\frac{1}{u} \left( \log \frac{1 + \sqrt{1 + u^2}}{2} \right).
\]

This is an explicit formula for the curvature of the hyperholomorphic line bundle (assuming $\omega$ is normalized so that $[\omega/2\pi]$ is an integral class).

Note that on the zero-section $v = 0$, $F$ restricts to $\omega$ and is $S^1$-invariant. From [5],[6] we can say that this is the unique hyperholomorphic extension to $T^\ast(G/H)$ of this line bundle with connection on $G/H$. Later we shall view this in a more natural setting.

### 2.3 Multi-instanton metrics

The most concrete examples of hyperkähler metrics are the gravitational multi-instantons of Gibbons and Hawking [7]. These are four-dimensional and in this dimension a hyperholomorphic connection is the same thing as an anti-self-dual one. The general Ansatz for this family of metrics consists of taking a harmonic function $V$ on an open set in $\mathbb{R}^3$, with its flat metric. Writing locally $*dV = d\alpha$ the metric has the form

\[
g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}(d\theta + \alpha)^2.
\]

Then $\omega_1 = Vdx_2 \wedge dx_3 + dx_1 \wedge (d\theta + \alpha)$ is a Kähler form and similarly for $\omega_2, \omega_3$.

An example is flat space $\mathbb{C}^2$ with a circle action $(z_1, z_2) \mapsto (e^{i\theta}z_1, e^{-i\theta}z_2)$. (Note that this action is triholomorphic, and so is not of the type we have been considering). The quotient space is $\mathbb{R}^3$ with Euclidean coordinates $x_1 = (|z_1|^2 - |z_2|^2)/2, x_2 + ix_3 = z_1z_2$ and then the metric has the above form if $V = 1/2r$. The flat space $\mathbb{C}^2\backslash\{0\}$ is here expressed as a principal circle bundle over $\mathbb{R}^3\backslash\{0\}$ and $d\theta + \alpha$ is the connection form for the horizontal distribution defined by metric orthogonality. The curvature of the connection is $d\alpha = *dV$ and the function $V^{-1/2}$ is the length of the vector field $Y$ generated by the action.

The general case has the same principal bundle structure but the flat example shows that a $1/r$ singularity for $V$ does not produce a singularity in the metric: it is simply a fixed point of the circle action on the four-manifold. With this in mind, setting

\[
V = \sum_{i=1}^{k+1} \frac{1}{|x - a_i|}
\]

for distinct points $a_i \in \mathbb{R}^3$ defines a nonsingular, complete hyperkähler manifold $M$. 
If the points \( a_i \) lie on the \( x_1 \)-axis then rotation about that axis induces an isometric circle action generating a vector field \( X \). This involves lifting the action on \( \mathbb{R}^3 \) to the \( S^1 \)-bundle with connection form \( \alpha \), commuting with the circle action. Such a lifting is defined by a vector field of the form \( X = X_H + fY \), where \( X_H \) is the horizontal lift of

\[
x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}
\]

and, since \( *dV \) is the curvature of the connection, \( i_X *dV = df \). Since \( L_X V = 0 \) the local existence of such an \( f \) is assured. This means

\[
df = (x_2 V_2 + x_3 V_3)dx_1 - x_2 V_1 dx_2 - x_3 V_1 dx_3.
\]

It follows that, with \( a_i = (a_i, 0, 0) \),

\[
f = \sum_{i=1}^{k+1} \frac{x_1 - a_i}{|x - a_i|} + c. \tag{2}
\]

The Kähler form \( \omega_1 \) is given by \( \omega_1 = V dx_2 \wedge dx_3 + dx_1 \wedge (d\theta + \alpha) \). This is the curvature of a connection if its periods lie in \( 2\pi \mathbb{Z} \). Now the segment \([a_i, a_{i+1}]\) defines a one-parameter family of \( S^1 \)-orbits which become single points over the end-points and therefore form a 2-sphere in \( M \). The manifold retracts onto a neighbourhood of a chain \([a_1, a_2], [a_2, a_3], \ldots, [a_k, a_{k+1}]\) of \( k \) such spheres which therefore generate \( H_2(M, \mathbb{Z}) \). Integrating \( \omega_1 \) over the \( i \)th sphere gives \( 2\pi(a_{i+1} - a_i) \) and so for integrality we require \( a_{i+1} - a_i \) to be an integer. With these conditions we have, from Haydys’s theorem, a hyperholomorphic line bundle which, since the two actions commute, is invariant under the triholomorphic circle action on \( M \).

Kronheimer [13] showed that \( S^1 \)-invariant instantons on the multi-instanton space became monopoles on \( \mathbb{R}^3 \) with Dirac singularities at the marked points \( a_i \). Since the hyperholomorphic bundle is invariant we can define it this way by a \( U(1) \) monopole: a harmonic function \( \phi \) on \( \mathbb{R}^3 \) and a connection \( A \) such that \( F = dA = *d\phi \). The Ansatz is

\[
\hat{A} = A - \phi V^{-1}(d\theta + \alpha) \tag{3}
\]

where \( \hat{A} \) is a local connection form on \( M \). Thus

\[
\omega_1 + dd^c \mu = dA - d(\phi V^{-1}) \wedge (d\theta + \alpha) - \phi V^{-1} *dV
\]

and taking the interior product with \( Y \) we obtain

\[
i_Y (\omega_1 + dd^c \mu) = -dx_1 + i_Y dd^c \mu = d(\phi V^{-1}).
\]

Since \( Y \) is triholomorphic, it preserves \( I \) and since it commutes with \( X \) it preserves \( \mu \) so as in the previous section \( d(\phi V^{-1}) = -dx_1 - d(i_Y d^c \mu) \) and up to an additive constant,

\[
\phi V^{-1} = -x_1 - i_Y d^c \mu = -(x_1 + i_Y (Ii_X \omega_1)) = -(x_1 - g(X, Y))
\]

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Now \( g(X, Y) = V^{-1} f \). therefore

\[
\phi = -x_1 V + f = -\sum_{i=1}^{k+1} \frac{a_i}{|x - a_i|} + c.
\]

Note however that \( A \mapsto A + c\alpha, \phi \mapsto \phi + cV \) takes \( \hat{A} \) to \( A + c\alpha - (\phi + cV)V^{-1}(d\theta + \alpha) = \hat{A} - cd\theta \) and so preserves the anti-self-dual curvature form \( d\hat{A} \) (this absorbs the constant ambiguity too). We can therefore also take

\[
\phi = \sum_{i=1}^{k} \frac{a_{k+1} - a_i}{|x - a_i|} + c
\]

and since the coefficients \( a_{k+1} - a_i \) are integers, this is a genuine \( U(1) \)-monopole which satisfies the Dirac quantization condition.

There remains the question of the constant \( c \). This is not in general zero since \( k = 1 \) is flat space and we have seen the non-zero curvature of the connection in the previous section. Here we have by construction also a circle action which preserves all three Kähler forms so given one lifting of the rotation action on \( \mathbb{R}^3 \) to \( M \) we can compose with a homomorphism to the triholomorphic circle to obtain another. The constant \( c \) will then change by \( 2\pi n, n \in \mathbb{Z} \).

Remark: When \( c = 0 \) the curvature \( F \) is a linear combination of \( \mathcal{L}^2 \) harmonic forms \([15],[9]\). In this case if \( k = 2m \) and \( x \) lies on the \( x_1 \)-axis with \( a_m \leq x_1 \leq a_{m+1} \) then \([2]\) shows that \( f = 0 \). Note for future reference that this means that the middle 2-sphere in the chain is point-wise fixed by the circle action.

The complex structure \( I \) for the metrics above is the minimal resolution of the Kleinian singularity \( xy = z^{k+1} \). There are, thanks to Kronheimer \([14]\), hyperkähler metrics on all such resolutions. These are produced by a finite-dimensional hyperkähler quotient construction and this is semi-explicit – the quotient metric of a subspace of flat space defined by a finite number of quadratic equations – but the hyperholomorphic line bundle is well adapted to the quotient construction.

### 2.4 Hyperkähler quotients

The hyperkähler quotient construction of \([11]\) proceeds as follows. Given a hyperkähler manifold with a triholomorphic action of a Lie group \( G \) we have, under appropriate conditions, three moment maps \( \nu_1, \nu_2, \nu_3 \) corresponding to the three Kähler forms \( \omega_1, \omega_2, \omega_3 \) and hence a vector-valued moment map \( \nu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3 \). Then, assuming \( G \) acts freely on \( \nu^{-1}(0) \), the manifold \( M = \nu^{-1}(0)/G \) with its quotient metric is hyperkähler.

In our situation we have a distinguished complex structure \( I \) preserved by a circle action. The construction can then be viewed in a slightly different way. Firstly \( \nu_c = \nu_2 + i\nu_3 \) is
holomorphic with respect to $I$ and so the zero set $M_0 = \nu^{-1}_c(0)$ is a complex submanifold of $M$ and hence $\omega_1$ restricts to it as a Kähler form. The group $G$ preserves $M_0$ and $\nu_3$ is the moment map for the restriction of $\omega_1$. Hence the hyperkähler quotient is the symplectic quotient of $M_0$ by this action.

**Theorem 2** Suppose $M$ has a circle action as in Section 2.1 commuting with $G$, so that the hyperkähler quotient $\bar{M}$ has an induced action. Then the hyperholomorphic line bundle on $M$ descends naturally to the hyperholomorphic line bundle of $\bar{M}$.

**Proof:** First recall that for a symplectic manifold $(N, \omega)$ with $\frac{\omega}{2\pi}$ integral there is a line bundle—the prequantum line bundle—with a unitary connection whose curvature is $\omega$. Given a lift of the action of a group $G$, the invariant sections on the zero set of the moment map define the prequantum line bundle on the symplectic quotient.

To see this more concretely, let $Y$ be the vertical vector field of the principal $U(1)$-bundle $P$, $X_a$ the vector field on $N$ given by $a \in \mathfrak{g}$ and $\mu : N \to \mathfrak{g}^*$ the moment map. Then a lift commuting with the $U(1)$-action is defined by $(X_a)_H + \langle \mu, a \rangle Y$ where $(X_a)_H$ is the horizontal lift. An arbitrary section of the line bundle is defined by a function $f$ on $P$, equivariant under the vertical action, and an invariant section satisfies $((X_a)_H + \langle \mu, a \rangle Y)f = 0$. Thus on $\mu^{-1}(0)$ we have $(X_a)_H f = 0$ which means that the section is covariant constant along the $G$-orbits. Hence the connection is pulled back from the symplectic quotient $\mu^{-1}(0)/G$.

This is the construction for a symplectic manifold. Now suppose we take our hyperkähler manifold with circle action and commuting triholomorphic $G$-action with hyperkähler moment map $\nu$. We want to apply the above to $N = M_0 = \nu^{-1}_c(0)$ for the symplectic quotient of $M_0$ is the hyperkähler quotient of $M$. Now the circle action does not preserve $\omega_2 + i\omega_3$ but it acts on $d\nu_c = d(\nu_2 + iv_3)$ by multiplication by $e^{i\theta}$. If we make a choice of moment map so that the action on $\nu_c$ is the same scalar multiplication, then the action will preserve $M_0 = \nu^{-1}_c(0)$. Moreover, $\mu$ restricted to $M_0$, is the moment map for $\omega_1$ restricted to $M_0$.

The line bundle with hyperholomorphic connection on $M$, and hence its restriction to $M_0$, was obtained from the prequantum line bundle by rescaling the hermitian metric by $e^{2\mu}$. By what we have just seen, this descends to $\bar{M}$, the symplectic quotient of $N = M_0$. However, $G$ commutes with the circle action and so $\mu$ is $G$-invariant. It is also the moment map for the induced action on the quotient, and it follows that rescaling the prequantum hermitian metric on $\bar{M}$ gives the hyperholomorphic bundle.

One other aspect of the quotient is that it comes equipped with a canonical principal $G$-bundle with a hyperholomorphic connection. Indeed $\nu^{-1}(0)/G = \bar{M}$ and $\nu^{-1}(0)$ is the total space of the principal $G$-bundle. The induced metric defines an orthogonal subspace in the tangent space to the orbit directions and this is the horizontal space of a connection, which is hyperholomorphic. A differential-geometric proof of this was give in [8] but it can be seen very naturally from the twistor space point of view which we carry out in the next
section. In fact, with fewer formulae and more geometry, the hyperholomorphic bundle appears much more naturally using holomorphic techniques.

3 The twistor viewpoint

3.1 The holomorphic bundle

This section is essentially a review of the construction in [12]. The twistor space $Z$ of a hyperkähler manifold $M$ is the product $Z = M \times S^2$ given the complex structure $(I_u, I)$ where $I_u = u_1 I + u_2 J + u_2 K$ for a unit vector $u \in \mathbb{R}^3$ and where the second factor is the complex structure of $S^2 = \mathbb{P}^1$. The projection $\pi : Z \to \mathbb{P}^1$ is holomorphic and for each $x \in M$, $(x, S^2)$ is a holomorphic section, a twistor line.

The fibre over $u \in S^2$ is the hyperkähler manifold $M$ with complex structure defined by $u$ but it also has a holomorphic symplectic form relative to this complex structure. Using an affine coordinate $\zeta$ on $\mathbb{P}^1$ where $u_2 + i u_3 = 2 \zeta/(1 + |\zeta|^2)$ the complex structures $I, -I$ are defined by $\zeta = 0, \infty$ and the holomorphic symplectic form is $(\omega_2 + i \omega_3) + 2 i \omega_1 \zeta + (\omega_2 - i \omega_3) \zeta^2$.

Globally, this is a twisted relative 2-form $\omega$ a holomorphic section of $\Lambda^2 T_Z/\mathbb{P}^1(2)$ where the $(2)$ denotes the tensor product with the line bundle $\pi^* \mathcal{O}(2)$, reflecting the quadratic dependence on $\zeta$.

Example: The twistor space for flat $H^n$ is the total space of the vector bundle $\mathbb{C}^{2n}(1)$ over $\mathbb{P}^1$. This is given by holomorphic coordinates $(v, \xi, \zeta) \in \mathbb{C}^{2n+1}$ on the open set $U$ defined by $\zeta \neq \infty$ and $(\bar{v}, \bar{\xi}, \bar{\zeta})$ for $V$ by $\zeta \neq 0$, with identification $(\bar{v}, \bar{\xi}, \bar{\zeta}) = (v/\zeta, \xi/\zeta, 1/\zeta)$ over $\zeta \in \mathbb{C}^*$. In these coordinates $Z$ is expressed as a $C^\infty$-product by the map $(z, w, \zeta) \mapsto (z + \zeta \bar{w}, w - \zeta \bar{z}, \zeta)$.

If a bundle on $M$ has a hyperholomorphic connection its curvature is of type $(1,1)$ with respect to all complex structures parametrized by $\zeta$ and it follows that its pull-back to $Z = M \times S^2$ has a holomorphic structure. Conversely any holomorphic vector bundle on $Z$ which is trivial on the twistor lines $(x, S^2)$ defines a hyperholomorphic connection on a vector bundle over $M$. This is the hyperkähler version of the Atiyah-Ward result for anti-self-dual connections. For a line bundle the triviality on twistor lines is simply the vanishing of the first Chern class. To get a unitary connection we impose a reality condition. It follows that to describe a hyperholomorphic line bundle on $M$ we simply look for a holomorphic line bundle $L_Z$ on $Z$ determined by the circle action.

Example: In flat space with the action $(z, w) \mapsto (z, e^{i\theta} w)$ one can calculate the line bundle directly. The $(1,0)$-forms on $Z$ for $\zeta \neq \infty$ are spanned by $dz_i + \zeta d\bar{w}_i, dw_i - \zeta \bar{d}z_i, d\zeta$ and then with

$$\log h_U = \frac{1}{2} \sum_i z_i \bar{z}_i - w_i \bar{w}_i + \zeta \bar{z}_i \bar{w}_i + \bar{\zeta} z_i w_i$$

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we find
\[ \bar{\partial}_Z \log h_U = \frac{1}{2} \sum z_i w_i d\bar{\zeta} + z_i d\bar{z}_i - w_i d\bar{w}_i + \bar{\zeta} d(z_i w_i) \]
and hence \( \bar{\partial}_Z \partial_Z \log h_U = (\sum -dz_id\bar{z}_i + dw_id\bar{w}_i)/2 \), the curvature of the hyperholomorphic line bundle, on the open set \( U \). Defining \( \log h_V = -\log h_U(-1/\bar{\zeta}) \) on \( V \), the pair \((h_U, h_V)\) defines a hermitian metric on the line bundle with holomorphic transition function on \( U \cap V \)

\[ \exp(-\sum v_i \xi_i/2\zeta). \]

The link between the differential geometric and holomorphic points of view is proved in \cite{[12]}. In fact the line bundle \( L_Z \) is essentially the prequantum line bundle for the family of holomorphic symplectic manifolds defined by \( Z \).

To understand this, and to see where the circle action enters in the construction, first note that since \( \omega_2 + i\omega_3 \) transforms as \( (\omega_2 + i\omega_3) \mapsto e^{i\theta}(\omega_2 + i\omega_3) \), differentiating with respect to \( \theta \) we have \( \omega_2 = L_X \omega_3 = d_X \omega_3 \) and so \( \omega_2 \) and similarly \( \omega_3 \) are exact. Thus the 2-form \( (\omega_2 + i\omega_3)/2i\zeta + \omega_1 + (\omega_2 - i\omega_3)\zeta/2i \) has the same cohomology class as \( \omega_1 \) for any \( \zeta \) and is therefore, given the integrality condition on \( \omega_1/2\pi \), the curvature of a line bundle on \( M \).

In the complex structure at \( \zeta \), \( (\omega_2 + i\omega_3)/2i\zeta + \omega_1 + (\omega_2 - i\omega_3)\zeta/2i \) is of type \((2,0)\) therefore has no \((0,2)\) part: hence the bundle has a holomorphic structure.

Now observe that the induced circle action on the twistor space generates a holomorphic vector field \( V \) on \( Z \). Since the action fixes \( \pm I \), \( V \) projects to the vector field \( i\zeta d/d\zeta \) on \( \mathbb{P}^1 \) vanishing at \( \zeta = 0, \infty \). This is a holomorphic section \( s \) of \( O(2) \) and so the 2-form we wrote above, \( (\omega_2 + i\omega_3)/2i\zeta + \omega_1 + (\omega_2 - i\omega_3)\zeta/2i \) is, on a specific fibre, the restriction of the meromorphic relative differential form \( \omega/2is \in \Omega^2_{Z/\mathbb{P}^1} \). It turns out that this relative form is the restriction of a closed meromorphic 2-form \( F_Z \) on \( Z \), which is the curvature of a meromorphic connection on the holomorphic line bundle.

**Theorem 3** [12] The line bundle \( L_Z \) on the twistor space \( Z \) admits a meromorphic connection such that

- there are simple poles at \( \zeta = 0, \infty \)
- the curvature \( F_Z \) restricts to
  \[ \frac{1}{2i\zeta}(\omega_2 + i\omega_3) + \omega_1 + \frac{1}{2i}(\omega_2 - i\omega_3)\zeta \]
  on each fibre over \( \mathbb{C}^* \subset \mathbb{P}^1 \)
- \( iV F_Z = 0 \) where \( V \) is the vector field generated by the circle action.

**Remark:** Suppose the holomorphic vector field \( V \) integrates to a \( \mathbb{C}^* \)-action. Then as \( F_Z \) is closed, the last property tells us that this action gives a symplectic isomorphism between any of the holomorphic symplectic manifolds over \( \zeta \in \mathbb{C}^* \).
Given that such a connection exists, the line bundle is essentially uniquely determined by the residue of the connection, for any two such bundles \( L, L' \) with the connections as above and with the same residue at \( \zeta = 0, \infty \), the resulting holomorphic connection on \( L' L^* \) would have a curvature which is a holomorphic 2-form. But the normal bundle of a twistor line is \( \mathbb{C}^{2n}(1) \) and so \( T^*_Z \cong \mathbb{C}^{2n}(-1) \oplus \mathcal{O}(-2) \) on such a line. It follows that there are no holomorphic forms of positive degree on a twistor space since there is a twistor line through each point. Hence the connection on \( L' L^* \) is flat and this is in any case the ambiguity in choosing a prequantum connection.

The residue is canonically determined by the data of the action as follows (see [12] for details). Since the connection has a singularity on a divisor of \( \mathcal{O}(2) \), its residue will be a section of \( T^*_Z(2) \) on that divisor. Now since \( TP_1 \cong \mathcal{O}(2) \) the projection \( \pi : Z \to P^1 \) gives an exact sequence of bundles:

\[
0 \to \mathcal{O} \to T^*_Z(2) \to T^*_Z/P^1(2) \to 0
\]

and the twisted relative form \( \omega \) identifies \( T^*_Z/P^1 \) with \( T^*_Z(2) \). The resulting extension

\[
0 \to \mathcal{O} \to E \to T^*_Z/P^1 \to 0
\]

can be identified with \( TP/C^* \) where \( P \) is the holomorphic principal bundle of the prequantum line bundle for the real symplectic form \( \omega_1 \). The vector field \( V \) on \( Z \) is tangential to the fibres at \( \zeta = 0, \infty \) and is \( X \) itself. The moment map defines an invariant lift to \( P \) and hence a section of \( TP/C^* \). Under the isomorphism above, this is the residue of the connection. If we restrict it as a form to the fibre \( \zeta = 0 \) it is \( i_V(\omega_2 + i\omega_3)/2i \)

**Examples:**

i) For flat space with the action \((z, w) \mapsto e^{i\theta}(z, w)\) the line bundle \( L_Z \) is trivial and the connection with the trivial action is just the meromorphic one-form

\[
\frac{1}{2\zeta} \sum_i \xi_i d\bar{v}_i - \bar{v}_i d\xi_i = \frac{\zeta}{2} \frac{\sum_i \xi_i d\bar{v}_i - \bar{v}_i d\xi_i}{\zeta} = \frac{1}{2\zeta} \sum_i \xi_i d\bar{v}_i - \bar{v}_i d\xi_i.
\]

With the action \( u \mapsto e^{i\theta}u \) it is

\[
2\pi i n \frac{d\zeta}{\zeta} + \frac{1}{2\zeta} \sum_i \xi_i d\bar{v}_i - \bar{v}_i d\xi_i
\]

ii) Flat space with the other action \((z, w) \mapsto (z, e^{i\theta}w)\) requires local connection forms \( A_U, A_V \) such that \( A_V = A_U + g_U^1 d\bar{g}_U \). Define

\[
A_U = \frac{1}{2\zeta} \sum_i v_i d\xi_i \quad A_V = -\frac{1}{2\zeta} \sum_i \bar{v}_i d\xi_i
\]

then on \( U \cap V \)

\[
A_V - A_U = -\frac{\zeta}{2} \sum_i \xi_i d\bar{v}_i - \frac{1}{2\zeta} \sum_i v_i d\xi_i = -d \left( \frac{1}{2\zeta} \sum_i v_i \xi_i \right).
\]
3.2 Hyperkähler quotients

In the twistor formalism the hyperkähler quotient is a very natural operation: it is just the fibrewise holomorphic symplectic quotient as long as the holomorphic vector fields generated by $G$ integrate to an action of the complexification $G^c$. Each $a \in \mathfrak{g}$ gives a holomorphic vector field $V_a$ tangential to the fibres of $\pi : Z \to \mathbb{P}^1$ and the three moment maps for $V_a$, $a \in \mathfrak{g}$ give a complex section

$$\nu = (\nu_2 + i\nu_3) + 2i\nu_1 \zeta + (\nu_2 - i\nu_3)\zeta^2$$

of $\mathfrak{g}^c(2)$. The twistor space $\bar{Z}$ of the hyperkähler quotient is then simply $\nu^{-1}(0)/G^c$ where the metric plays a role in determining the stable points for this quotient by a complex group. With this viewpoint the descent of the hyperholomorphic bundle through a quotient is, given Theorem 3, the descent of the prequantum line bundle in a symplectic quotient (it is straightforward to check that the residue descends appropriately).

As we saw in the previous section, a hyperkähler quotient brings with it a canonical hyperholomorphic $G$-bundle. In fact, in the twistor interpretation, $\nu^{-1}(0)$ is a principal $G^c$-bundle over the twistor space $\bar{Z} = \nu^{-1}(0)/G^c$ and it satisfies the reality condition to define a hyperholomorphic principal $G$-bundle over $\bar{M}$. A homomorphism $G \to U(1)$ then defines a hyperholomorphic line bundle and this raises the obvious question about whether, given a circle action, this is the hyperholomorphic bundle of the title.

In fact for a manifold to be a smooth hyperkähler quotient of flat space such homomorphisms must exist. The standard moment map for a linear action is quadratic and the origin lies in $\nu^{-1}(0)$, so for smoothness we must change this by a constant. Equivariance however demands that the constant is an invariant in $\mathfrak{g}^*$: a homomorphism from $\mathfrak{g}$ to $\mathbb{R}$.

Consider flat space $\mathbb{H}^n$ as a right $\mathbb{H}$-module, then $U(n) \subset Sp(n)$ is the subgroup commuting with left multiplication by $e^{i\theta}$: this is a distinguished complex structure $I$. Let $G \subset U(n)$ and $c \in \mathfrak{g}^*$ be a $G$-invariant element. If $c$ is integral it corresponds to a homomorphism $\chi : G \to U(1)$. Let $\nu$ be the standard quadratic hyperkähler moment map for the linear action, then taking the reduction at $\nu = (c,0,0)$, the cohomology class of the Kähler form $\omega_1$ lies in $2\pi H^2(\bar{M},\mathbb{Z})$. Indeed the integrality for $c$ gives a lift of the $G$-action to the prequantum line bundle on $\nu_{c}^{-1}(0)$ which descends.

**Theorem 4** If the hyperkähler quotient $\bar{M}$ of $\mathbb{H}^n$ by $G$ with $\nu = (c,0,0)$ is smooth, then the hyperholomorphic line bundle is $\nu^{-1}(c,0,0) \times_G \mathbb{C}$ endowed with the canonical connection, where $G$ acts via $\chi : G \to U(1)$.

**Proof:** From the twistor point of view the line bundle $L_Z$ on the quotient is defined by the property that local sections are the same as local $G^c$-invariant sections of the holomorphic line bundle on $\nu^{-1}(0)$. For flat space and the circle action above the latter, as we observed in Section 2.4, is a trivial holomorphic bundle but has a non-trivial action defined by $\chi$. Thus on $Z$ the line bundle is associated to the principal $G^c$-bundle $\nu^{-1}(0)$ by $\chi$. $\square$
Examples:

i) The simplest example is the cotangent bundle of a complex Grassmannian, one of the Hermitian symmetric spaces of Section 2.2. In this case the flat space is $M = V \oplus jV$ for $V = \text{Hom}(C^k, C^n)$ and $G = U(n)$ acting in the obvious way. There is just a one-dimensional space of invariant elements in $\mathfrak{g}^*$ and $H^2(\bar{M}, Z) \cong \mathbb{Z}$. Notice that $-1$ acting on the vector space is represented by $-1 \in U(n)$ and hence acts trivially on the quotient. It is therefore $e^{2i\theta}$ which acts effectively on the quotient. Since $e^{i\theta}$ acts on $\omega_2 + i\omega_3$ in flat space by multiplication by $e^{2i\theta}$, on the quotient the induced action is the standard one: in fact the fibre action on the cotangent bundle.

ii) Taking $M = V \oplus jV$ where $V = \text{End}(C^k) \oplus \text{Hom}(C^k, C^n)$ and $G = U(k)$ one obtains the moduli space of $U(k)$-instantons on $\mathbb{R}^4$ of charge $k$ or, with a non-zero moment map, the moduli space of noncommutative instantons. For $n = 1$ this is the Hilbert scheme $(C^2)^{[k]}$ of $k$ points on $C^2$ and the hyperholomorphic line bundle with complex structure $I$ is defined by the exceptional divisor. The circle action is induced from scalar multiplication on $C^2$ and so the action on $\omega_2 + i\omega_3$ is multiplication by $e^{2i\theta}$, since on the open set of $(C^2)^{[k]}$ consisting of the configuration space of $C^2$ the symplectic form is the sum of $k$ copies of $dz \wedge dw$.

iii) In [14] Kronheimer constructed asymptotically locally Euclidean hyperk"{a}hler metrics on minimal resolutions of Kleinian singularities $(C^2/\Gamma$ for $\Gamma \subset SU(2)$ a finite group) by the quotient construction. The construction is as follows. Let $R = L^2(\Gamma)$ be the regular representation, $C^2$ the basic representation from $\Gamma \subset SU(2)$ and put $M = (C^2 \otimes \text{End}(R))^\Gamma$. Since $\text{End}(R)$ has real structure $A \mapsto A^*$ and $SU(2) \cong Sp(1)$ this is a quaternionic vector space and the group $G = U(R)^\Gamma$ acts as quaternionic unitary transformations. The ALE space appears as a hyperk"{a}hler quotient of $M$ by the action of $G$. If $R_0, \ldots, R_k$ are the irreducible representations of $\Gamma$, of dimension $d_i$ then

$$R = \bigoplus_i C^{d_i} \otimes R_i$$

and so $U(R)^\Gamma \cong U(d_0) \times \cdots \times U(d_k)$. From the McKay correspondence each $R_i$ labels a vertex of an extended Dynkin diagram of type $A, D, E$ and then

$$M = \bigoplus_{i \leftrightarrow j} \text{Hom}(C^{d_i}, C^{d_j})$$

(5)

the sum taken over the edges of the diagram, once with each orientation. As shown in [14], the invariant subspace of $\mathfrak{g}^*$ can be identified with the Cartan subalgebra of the Lie algebra of type $A, D, E$ as can the cohomology $H^2(M, \mathbb{R})$, with $H_2(M, \mathbb{Z})$ the root lattice. The case of $A_k$ is the multi-instanton metric of Section 2.3 where the chain of 2-spheres constructed explicitly realizes the Dynkin diagram of type $A_k$.

Here the circle action on the symplectic form of the quotient will be the standard one if there is an element in $G$ which acts as $-1$. For this, from (5) we need to show that there exist $c_i = \pm 1$, $0 \leq i \leq k$, such that if $i, j$ are joined by an edge of the extended Dynkin diagram...
then $c_ic_j = -1$. For $A_1$ this is trivial. Consider the extended Dynkin diagram (for $k > 1$) as a simplicial complex, then this is the same as asking that the $\mathbb{Z}_2$-cocycle associating $-1$ to each 1-simplex is a coboundary. The diagrams of type $D_k, E_6, E_7, E_8$ are contractible and so have zero first cohomology so this is true. For $A_k$ the diagram is homeomorphic to a circle and the cohomology class in $H^1$ vanishes if there is an even number of edges, which is when $k$ is odd.

Now the $D, E$ Dynkin diagrams have a trivalent vertex which, in the presence of our circle action, corresponds to a rational curve of self-intersection $-2$ which is pointwise fixed, since there cannot be just three fixed points. And, as pointed out in Section 2.3 when $k$ is odd, the central curve in the $A_k$ case is fixed.

In these cases, with respect to the complex structure $I$, we have a rational curve of self-intersection $-2$ and a neighbourhood of such a curve is biholomorphic to a neighbourhood of the zero section of the cotangent bundle. Moreover the circle action is the standard scalar multiplication in the fibre. Applying [5] this means that the Kronheimer metric with circular symmetry is the unique hyperkähler extension of the induced metric on the distinguished 2-sphere.

### 3.3 Coadjoint orbits

The Hermitian symmetric spaces which we considered in Section 2.2 are special cases of coadjoint orbits of compact semi-simple Lie groups with their canonical Kähler structure. There is a very natural description of the twistor space of a hyperkähler metric on the cotangent bundle of such a space due originally to Burns [3]. In fact, that paper only asserts the existence of such a metric in a neighbourhood of the zero section, but it was written before hyperkähler quotients, and in particular the infinite-dimensional gauge-theoretic versions, were discovered. Much later, armed with a knowledge of existence theorems for Nahm’s equations, Biquard [2] revisited this description being assured of global existence. The action of scalar multiplication in the cotangent fibres by $e^{i\theta}$ gives a circle action and we shall now seek a concrete description of the line bundle $L_Z$ using Burns’s approach.

Let $G$ be a semisimple compact Lie group with a bi-invariant metric and $z \in \mathfrak{g}$ be an element with centralizer $H$. Then in the complex group $G^c$ there are parabolic subgroups $P_+, P_-$ with $P_+ \cap P_- = H^c$. The real (co)adjoint orbit $G/H \cong G^c/P_+ \cong G^c/P_-$, and the complex coadjoint orbit is $G^c/H^c$.

The Lie algebra $\mathfrak{p}_+ = \mathfrak{h} + \mathfrak{n}_+$ where $\mathfrak{n}_+$ is nilpotent, $z \in \mathfrak{h}$ by definition and we define two complex manifolds

$$Z_0 = G^c \times_{P_+} \{C \cdot z + \mathfrak{n}_+\} \quad Z_\infty = G^c \times_{P_-} \{C \cdot z + \mathfrak{n}_-\}.$$  

Since $P_\pm$ fixes $z$ modulo $\mathfrak{n}_\pm$, the coefficient of $z$ defines a projection $\pi_0 : Z_0 \to \mathbb{C}$ and similarly for $Z_\infty$. The fibre over 0 is the cotangent bundle $T^* G^c/P_+ \cong G^c \times_{P_+} \mathfrak{n}_+$ and for $\zeta \neq 0$, $G^c \times_{P_+} \{\zeta z + \mathfrak{n}_+\}$ is an affine bundle over $G^c/P_+$.  

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There is another description, however, for $(g, \zeta z + x_+) \mapsto (\text{Ad}(\zeta z + x_+), \zeta)$ identifies the fibre at $\zeta \neq 0$ with the $G^c$-orbit of $\zeta z$. Note that the map $z \mapsto \zeta z$ defines an isomorphism with the orbit of $z$ which is not symplectic for the canonical Kostant-Kirillov form $\omega_{\text{can}}$ but is for its rescaling $\omega_{\text{can}}/\zeta$.

The twistor space is obtained by identifying $Z_0, Z_\infty$ over $\zeta \in \mathbb{C}^*$ by $(x, \zeta) \mapsto (\zeta^{-2}x, \zeta^{-1})$. Then the two projections define $\pi : Z \to \mathbb{P}^1$ and $\omega_{\text{can}}$ defines the twisted relative symplectic form.

We define line bundles $L_+, L_-$ over $Z_0, Z_\infty$ by pulling back the prequantum line bundle on $G/H = G^c/P_\pm$ using the projections $p_0 : Z_0 \to G^c/P_+, p_\infty : Z_\infty \to G^c/P_-$. Then to define a line bundle $L_Z$ on $Z$ we need an isomorphism between $L_+$ and $L_-$ over $\mathbb{C}^* \subset \mathbb{P}^1$. But the prequantum line bundle is homogeneous, defined by representations $\chi_\pm : P_\pm \to \mathbb{C}^*$, and these agree on $H^c = P_+ \cap P_-$. This therefore gives an isomorphism $p_*^+ L_+ \cong p_*^- L_-$ on $Z_0 \cap Z_\infty \cong G^c/H^c \times \mathbb{C}^*$.

To show that this truly is the twistor version of the hyperholomorphic bundle we may simply note that it does generate a hyperholomorphic line bundle but by the invariance of the construction it is homogeneous on the zero section $G/H$ and hence agrees with a hyperholomorphic bundle there. Invoking [5],[6] once more we see that they are isomorphic everywhere.

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