Simulating all multipartite non-signalling channels via quasiprobabilistic mixtures of local channels in generalised probabilistic theories

Paulo J. Cavalcanti¹, John H. Selby¹, Jamie Sikora², and Ana Belén Sainz¹

¹International Centre for Theory of Quantum Technologies, University of Gdańsk, 80-309 Gdańsk, Poland
²Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

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Non-signalling quantum channels – relevant in, e.g., the study of Bell and Einstein-Podolsky-Rosen scenarios – may be simulated via affine combinations of local operations in bipartite scenarios. Moreover, when these channels correspond to stochastic maps between classical variables, such simulation is possible even in multipartite scenarios. These two results have proven useful when studying the properties of these channels, such as their communication and information processing power, and even when defining measures of the non-classicality of physical phenomena (such as Bell non-classicality and steering). In this paper we show that such useful quasi-stochastic characterizations of channels may be unified and applied to the broader class of multipartite non-signalling channels. Moreover, we show that this holds for non-signalling channels in quantum theory, as well as in a larger family of generalised probabilistic theories. More precisely, we prove that non-signalling channels can always be simulated by affine combinations of corresponding local operations, provided that the underlying physical theory is locally tomographic – a property that quantum theory satisfies. Our results then can be viewed as a generalisation of Refs. [Phys. Rev. Lett. 111, 170403] and [Phys. Rev. A 88, 022318 (2013)] to the multipartite scenario for arbitrary tomographically local generalised probabilistic theories (including quantum theory). Our proof technique leverages Hardy’s duotensor formalism, highlighting its utility in this line of research.

1 Introduction

Quantum operations are at the core of communication and information processing tasks, and how well we can perform at the latter may depend on the properties of the quantum operations that we have at hand. One particular set of operations of interest is that of non-signalling quantum channels [1], i.e., those that cannot be used by two distant parties to exchange information in a way that is against the laws of relativity theory. Bipartite non-signalling quantum operations have been extensively studied, specially since they play a central role in Bell [2] and Einstein-Podolsky-Rosen ‘steering’ [3, 4] scenarios, which in
turn underpin cryptographic protocols [5, 6]. In addition, the simulation of bipartite non-signalling quantum channels via affine combinations of local operations has provided valuable insight on the exploration of the advantage they provide for communication and information processing tasks [7, 8].

In recent years it has become fruitful to study quantum theory from the ‘outside’, that is, by placing it as one theory within a broad landscape of logically consistent theories. This allows one to understand why quantum theory has particular features, and also its possibilities and limitations for various applications. The framework of generalised probabilistic theories [9, 10] (GPTs) has become the preferred tool for such studies, for example, shedding light on matters pertaining to: cryptography [11–16]; computation [17–23]; interference [24–30]; thermodynamics [31–36]; contextuality [37–40]; nonlocality [41–47]; steering [48–51]; decoherence [52–55]; information processing [56–62]; incompatibility [63–67]; uncertainty [68–71]; as well as providing a foundational view of the primitive structures of physical theories [72–88]. For a comprehensive introduction to the field see Refs. [47, 89, 90].

In this work we investigate no-signalling channels in GPTs. In particular, we prove a useful technical result, namely that multipartite non-signalling channels in locally-tomographic GPTs [9] can be simulated by affine combinations of product (local) channels (Theorem 5.1). Our results can be viewed as a generalisation of those of Ref. [7] and of Ref. [8, Lem. 1] to arbitrary tomographically local GPTs: the former applies only to multipartite non-signalling stochastic maps on classical variables, while the latter applies to bipartite non-signalling quantum channels.

Our proofs leverage the convenient duotensor formalism of Ref. [91] with a slight twist based on Ref. [92] which allows us to directly lift the result of Ref. [7] (using a generalisation of Lem. 2 in Ref. [8]) to this more general setting. We believe that this way of lifting structural properties of stochastic maps to properties of channels in arbitrary tomographically local GPTs via the duotensor formalism [91] may be a useful tool in future research.

2 Generalised Probabilistic Theories: the basics

The framework of Generalised Probabilistic Theories (GPTs) can be used to define arbitrary physical theories. The simplicity of the framework enables various alternative theories to be formulated and explored while allowing at the same time a deep study of the probabilistic and compositional aspects of such theories. It is based on the tenet that a minimal requirement of any physical theory is that it must make probabilistic predictions about the outcomes of experiments. Whilst this is conceptually extremely minimal, the mathematical consequences of this lead to a rich formal structure known as a GPT.

Because physical theories describe predictions about measurement outcomes in experiments, a few elements are necessarily present in all of them. Namely, these theories need to talk about types of systems, possible states for each of them, possible measurement outcomes, transformations, and the operation of discarding a system (see Table 1). In quantum theory, these elements are, respectively, the Hilbert spaces, the density operators on them, positive operators upper-bounded by the identity, completely positive trace-non-increasing (CPTNI) linear maps, and the (partial) trace operation.

Having those elements present, although necessary, is not sufficient to express the full form of a physical theory. Some structure relating them are implied by the way that experiments are performed. Abstractly speaking, a notion of connectivity between those elements must also be present because, in experiments, we perform actions on systems,
that is, we subject them to processes, and these processes can happen in parallel (independently) or in sequence. This motivates a notion of compositionality of processes.

From this notion of how the experimental processes connect, or compose, a convenient diagrammatic notation can be defined so as to capture the entire structure of the GPTs. We can represent any process by a box, and encode the type of system on which it happens as a labelled input wire at the bottom of it. (Hence, we have also implied that systems are represented by wires.) Additionally, since the type of a system can change after a process, we denote the output type of a process by a labelled wire on top of its box. In this notation, then, a system type $S$, and a transformation $T$ from a system type $A$ to a system type $B$, respectively, appear as $\boxed{S}$ and $\boxed{T}$. (1)

Note that a state of a system can be conceptualised as some preparation procedure, which, abstractly speaking, is also a process. Hence we can represent it as a box that has no input wire but has as output the wire corresponding to the type of that system. Similarly, an effect, or measurement outcome, is a box with input wire corresponding to the system where it can be observed, and no output wire. We follow the convention that states and effects are represented by triangular boxes, so a state $\sigma$ and an effect $e$ of a system $S$ appear as $\boxed{\sigma}$ and $\boxed{e}$. (2)

respectively. Because of this, the discarding operation, since it has an input but no output, appears as a special effect in the theory. This effect is sometimes called the deterministic effect and is unique for each system type$^1$. In this notation, it is represented by $\boxed{S}$. (3)

These diagrammatic pieces can be connected when the input/output wire types match. This represents the sequential composition of processes. When processes are instead drawn side by side, we are representing their parallel composition. By connecting boxes, therefore,

$^1$The uniqueness of this discarding effect means that we are dealing with so-called causal GPTs [93].
we can then construct more complex diagrams, i.e. complex processes, such as

\[
\begin{array}{c}
\hspace{1cm} e \\
\hline
\hspace{1cm} h \\
\hline
\hspace{1cm} f \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array}
\]

where we omit the wire labels for simplicity, but it should be clear that only matching types can be connected.

When a diagram has no loose wires, they are interpreted as numbers, which in the case of GPTs are the probabilities generated by the theory. For instance,

\[
\begin{array}{c}
\hspace{1cm} e \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array} = \text{Prob}(e|g, \sigma)
\]

(5)

denotes the probability that the outcome associated to effect \( e \) is observed when the system is prepared in state \( \sigma \) and a transformation \( g \) is applied to it.

Of course, we might need to describe systems that are composed by simpler parts – multipartite systems – so we can emphasize that some system is composite by drawing the wires of its parts side by side

\[
\begin{array}{c}
\hspace{1cm} A \otimes B \\
\hline
\end{array} = A \mid B
\]

(6)

When we represent bipartite composite systems by the two wires together, its deterministic effect is represented by the composition of the deterministic effects of its parts:

\[
\begin{array}{c}
\hspace{1cm} \overline{A \otimes B} \\
\hline
\end{array} = \overline{A} \overline{B}
\]

(7)

With what we have, we can represent simple experimental processes, composite processes, and probabilities of outcomes in those experiments. To reason about them, we need now a notion of equality of processes, or, in other words, a notion of tomography. We say that two processes are equal if they give the same probabilities in all situations, so

\[
\begin{array}{c}
\hspace{1cm} f \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array} = \begin{array}{c}
\hspace{1cm} f \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array} \quad \forall \sigma, e.
\]

(8)

In fact, in this paper we will work with a special class of GPTs which satisfy the principle of tomographic locality \([9, 91]\). This means that:

\[
\begin{array}{c}
\hspace{1cm} f \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array} = \begin{array}{c}
\hspace{1cm} f \\
\hline
\hspace{1cm} g \\
\hline
\hspace{1cm} \sigma
\end{array} \quad \forall \sigma, e,
\]

(9)
that is, in a tomographically local theory we can do process tomography without a side channel.

An important type of processes is that of those that are *discard-preserving*, which means they satisfy the following:

\[ \mathcal{F} = \mathcal{I}. \]  

(10)

In the case of quantum theory, these correspond to the trace-preserving maps. Physically-realisable discard-preserving processes in a GPT are known as *channels*. Discard-preservation also defines a notion of causality for processes [93, 94]: a process is said to be causal if it is discarding preserving. This is so because this condition ensures compatibility with relativistic causal structure [95].

A final ingredient in the GPT formalism is the possibility to represent convex mixtures of processes. This stems from the requirement that in an experiment we can always decide to perform \( f \) with probability \( p \) or \( g \) with probability \( 1 - p \), at least, provided that \( f \) and \( g \) have the same input and output systems. This is introduced through the definition of a sum of processes that distributes over diagrams:

\[ \sum_i p_i f_i \mathcal{A} B = \sum_i p_i f_i \mathcal{A} B. \]  

(11)

Note that this definition implies that we can only sum processes with the same input/output types. From this, since a probability \( p \) can be a number (diagram without loose wires) of the theory, we can write

\[ \mathcal{A} B = p f \mathcal{A} B + (1 - p) g \mathcal{A} B = \sum p_i f_i \mathcal{A} B \]  

(12)

to describe convex mixtures of processes.

At this point a notion of order can be defined for processes:

\[ f \mathcal{A} B \leq g \mathcal{A} B \iff \exists h \mathcal{A} B \text{ s.t. } f \mathcal{A} B + h \mathcal{A} B = g \mathcal{A} B. \]  

(13)

This order allows us to define discard-non-increasing processes. A process \( f \) is said to be discard-non-increasing if and only if

\[ f \mathcal{A} B \leq g \mathcal{A} B. \]  

(14)

Note that in any GPT all (physically-realisable) processes must be discard-nonincreasing; this corresponds to the constraint in quantum theory that processes are trace-nonincreasing. In particular, this means that for any effect in the theory, there must be another effect such that they sum to the deterministic effect. Note that this is important for the definition
of measurements. In quantum theory, for example, the deterministic effect is the trace operation, or multiplication by identity followed by the trace, and it is required that the POVM elements forming a measurement sum to identity, so each of them is less than or equal to the deterministic effect.

Since in this work we focus on the class of GPTs that are tomographically local, we can moreover use the particular duotensor notation of Ref. [91]. Next we will present the basics of this notation.

3 Duotensor basics

Here we present an adaptation to the duotensor formalism where, in addition to the GPT systems of the previous section, we also have classical systems representing measurement outcomes and control systems. In order to distinguish these two kinds of systems, the classical ones will be drawn horizontally. We will also label them by finite sets, \( \Lambda \):

\[
\Lambda.
\]  

(15)

The physical processes transforming between these classical systems are (sub)stochastic maps between these finite sets. We draw these as white boxes, such as:

\[
\Lambda \rightarrow \Lambda'.
\]  

(16)

A particularly useful example which we will make use of in this work is the copy map, which we draw as a white dot and is defined by:

\[
\lambda \ni \Lambda \rightarrow \Lambda', \quad \forall \lambda \in \Lambda.
\]  

(17)

Note that the copy map satisfies:

\[
\Lambda \rightarrow \Lambda' \rightarrow \Lambda' \rightarrow \Lambda \times \Lambda' \rightarrow \Lambda \times \Lambda',
\]  

(18)

that is, copying the components of a system is the same as copying the composite system.

In contrast to the physical (sub)stochastic maps, we will draw mathematically well defined but unphysical processes as black boxes such as:

\[
\Lambda \rightarrow \Lambda'.
\]  

(19)

which, in this case, would be a linear map from \( \Lambda \) to \( \Lambda' \) which is not (sub)stochastic, e.g., it may have negative coefficients.

Note that in contrast to the approach of Ref. [91], rather than labelling horizontal systems by black and white dots, we instead label the processes as being either black or white. This is equivalent but more convenient for us as, on the one hand, we can interpret the color as representing whether or not a process is physical, and, on the other hand, it takes us to a more standard category-theoretic notation. Indeed, categorically there is
no distinction between the horizontal and vertical wires, it is simply a convenient way to label the different objects, at which point it is clear that all of the processes that we draw below live inside the category of real linear maps.

For each system $S$ in the GPT we define a particular minimal informationally-complete state preparation and measurement. We call these the fiducial preparation and fiducial measurement. A state preparation is a box which has a classical input and a GPT output where the classical input controls which state is prepared, whilst a measurement is a box which has a GPT input and a classical output where the classical output encodes the result of the measurement. We can therefore denote the fiducial preparation and fiducial measurement for a system $S$ as:

$$S \xrightarrow[\Lambda_S]{\Lambda_S} S$$

(20)

where without loss of generality we take $\Lambda_S$ to index both the fiducial set of states and the fiducial set of effects. Moreover, all of the fiducial states are normalised and the fiducial effects sum to the unit effect, such that:

$$\Lambda_S = \Lambda_S |I\rangle \text{ and } \Lambda_S^* |I\rangle = \Lambda_S^* |S\rangle.$$  

Note that here we follow the convention of Ref. [92] rather than Ref. [91], as the former demands that the fiducial effects form a measurement whilst the latter does not. Note that this does not constitute a loss of generality as a minimal informationally-complete measurement can be shown to exist for any GPT.

Now, for each system $S$, define the fiducial transition matrix by

$$\Lambda_S \Lambda_S := S \xrightarrow[\Lambda_S]{\Lambda_S} S$$

(22)

and note that eq. (21) implies that these fiducial transition matrices are stochastic maps, such that:

$$\Lambda_S \Lambda_S |I\rangle = \Lambda_S |I\rangle.$$

(23)

Now, the fact that the fiducial preparation and measurement are informationally-complete means that they are invertible linear maps. Importantly, however, these inverses are not typically physical transformations. We therefore denote them as:

$$\Lambda_S^S \Lambda_S^S := \Lambda_S^S \Lambda_S^S S$$

(24)

such that:

$$\Lambda_S^S \Lambda_S^S = \Lambda_S = \Lambda_S \Lambda_S \text{ and } \Lambda_S S = \Lambda_S S \Lambda_S^S = S = \Lambda_S I S S.$$  

(25)

We can then moreover define

$$\Lambda_S \Lambda_S := \Lambda_S \Lambda_S S.$$  

(26)
which can easily be seen using Eq. (25) to be the inverse of the fiducial transition matrix. Hence:

$$\Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \Lambda S = \begin{array}{c} \Lambda S \\ \Lambda S \end{array} \Lambda S .$$

(27)

The fiducial transition matrix and its inverse (the white and black squares respectively) are known as hopping metrics in the terminology of Hardy.

It is also easy to see from these conditions that:

$$\Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} ,$$

(28)

which, in particular, means that:

$$\begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} .$$

(29)

Moreover, it is also easy to show that:

$$\Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \Lambda S \begin{array}{c} \Lambda S \\ \Lambda S \end{array} , \quad \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \begin{array}{c} \Lambda S \\ \Lambda S \end{array} , \quad \begin{array}{c} \Lambda S \\ \Lambda S \end{array} = \begin{array}{c} \Lambda S \\ \Lambda S \end{array} .$$

(30)

The key use of all of this for us, is that it allows us to map any GPT channel to a stochastic map and back again as follows: a GPT channel is mapped to a stochastic map via

$$T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} \mapsto T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} ,$$

(31)

and the stochastic map associated to the GPT channel can be mapped back to the GPT channel via

$$T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} \mapsto T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} ,$$

(32)

It is clear that the RHS of Eq. (31) is indeed stochastic as it is positive (since it is composed out of physically realisable GPT transformations) and satisfies:

$$T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} = T \begin{array}{c} T \cdots T \end{array} C \begin{array}{c} S \cdots S \end{array} = \begin{array}{c} S \cdots S \end{array} = \begin{array}{c} S \cdots S \end{array} = \begin{array}{c} S \cdots S \end{array} .$$

(33)
where the second equality holds because $C$ is a GPT channel rather than a generic GPT process. Similar arguments imply that if $C$ satisfies certain no-signalling conditions then so to will the associated stochastic map.

For example, a bipartite channel $B$ is said to be non-signalling if:

$$\begin{align*}
T_1 & \longrightarrow T_2 \\
B & \quad S_1 \quad S_2
\end{align*} \quad \text{and} \quad \begin{align*}
T_3 & \longrightarrow T_2 \\
B & \quad S_1 \quad S_2
\end{align*}
$$

from which it is easy to show that the associated stochastic map will also be non-signalling, for example:

$$\begin{align*}
\begin{array}{c}
T_1 \\
B \\
S_1 \quad S_2
\end{array} & = \begin{array}{c}
T_1 \\
S_1 \quad S_2
\end{array} = \begin{array}{c}
T_2 \\
S_1 \quad S_2
\end{array} = \begin{array}{c}
T_2 \\
S_1 \quad S_2
\end{array} \quad (34)
\end{align*}
$$

This straightforwardly generalises to multipartite GPT channels, and also to the case where only some of the no-signalling conditions hold. That is, the non-signalling structure of the channel and of the associated stochastic map are the same.

4 Geometry of transformations

In this section we present a geometric perspective on some of the processes discussed above, as well as on particular types of channels.

4.1 States

First let us start by discussing the geometry of the state space for some system $S$. Schematically this looks like:

Formally, we have some real vector space $V_S$ which contains a convex cone of states, $\mathcal{T}^S$, which is closed, pointed, and full dimensional, with an intersecting hyperplane which defines the normalised vectors. The intersection of this hyperplane and the state cone defines the normalised state space, $\Omega_S$. A subnormalized state $s$ is a vector in the cone such that there exists $\alpha \geq 1$ such that $\alpha s$ is normalized. In particular, the convex set of subnormalized states spans the vector space and, moreover, there exists at least one normalized state which is interior to the cone.
4.2 Effects

Next let us consider the geometry of the effect space for some system $S$. Schematically this looks like:

Formally, the effect space of $S$ lives inside the dual of the vector space of states, $V_S^*$, and consists of a convex cone of effects, $\mathcal{T}_S$, which is closed, pointed and full dimensional. The unique “normalised” effect (the discarding effect) $\varphi$, is the unique linear functional that evaluates to 1 on the intersecting hyperplane defining the normalised states. This must be in the interior of the effect cone such that it is an order unit for the cone. That is, we have that every effect $e$ in the cone can be rescaled to an effect $\alpha e$, for some $\alpha > 0$, such that there exists some $e'$ in the cone which satisfies $\alpha e + e' = \varphi$. The set of subnormalised effects, $\mathcal{E}_S$, can be defined as those that satisfy this condition for some $\alpha \geq 1$. In particular, this ensures that the convex set of subnormalised effects spans the dual space and that $\varphi$ is in the interior of the effect cone.

4.3 Physical transformations

Finally, we turn to our main focus which is the geometry of transformations within a tomographically local GPT. Schematically this looks like:

As we are assuming tomographic locality, the transformations from $S$ to $T$ live inside the vector space of linear maps from $V_S$ to $V_T$, which we denote as $\mathcal{L}(V_S, V_T)$. The geometric picture that we present here is not as standard in the literature as it is for the state and effect cases, and so we now explain how this structure arises.

In this picture we have a convex set of normalised transformations which are defined by the intersection of an affine set (namely, the discard-preserving linear maps) and a convex cone (namely, the cone of transformations, $\mathcal{T}_S^T$). We can then view this as a positive cone such that the discard-nonincreasing transformations are those that are “underneath” the discard-preserving transformations in the associated partial order.
As there exists a set of states which span $V_T$ and a set of effects which span $V_S^*$, then, using the fact that $\mathcal{L}(V_S, V_T) \cong V_S^* \otimes V_T$, we have that

$$\mathcal{L}(V_S, V_T) \cong \text{span} \left\{ \begin{array}{c} s \\ s \in \Omega_T, e \in T_S \end{array} \right\}. \quad (39)$$

This means that any linear map in $\mathcal{L}(V_S, V_T)$ can be written as

$$\left( \sum_{i=1}^{k} \frac{\Pi}{\Delta_{i}^{S}} \right) - \left( \sum_{i'=1}^{k'} \frac{\Pi}{\Delta_{i'}^{S}} \right) \quad (40)$$

for some finite values of $k$ and $k'$, where $s_1, \ldots, s_k, s'_1, \ldots, s'_{k'}$ are normalised states, and $e_1, \ldots, e_k, e'_1, \ldots, e'_{k'}$ are in the effect cone.

With this in mind, we define the following convex subcone $\mathcal{K}$ of the cone of transformations $T^T_S$ which is useful in our analysis:

$$\mathcal{K} := \left\{ \sum_{i=1}^{k} \frac{\Pi}{\Delta_{i}^{S}} \bigg| s_i \in \Omega_T, e_i \in T_S, k \text{ finite} \right\} \subseteq T^T_S \subset \mathcal{L}(V_S, V_T). \quad (41)$$

Equipped with this definition, we can express $\mathcal{L}(V_S, V_T)$ neatly as

$$\mathcal{L}(V_S, V_T) = \mathcal{K} - \mathcal{K} := \{ \phi_1 - \phi_2 | \phi_1, \phi_2 \in \mathcal{K} \}, \quad (42)$$

which means that $\mathcal{K}$ spans $\mathcal{L}(V_S, V_T)$.

4.4 Measure-and-prepare transformations and discard-preserving channels

Of particular interest is the set of physical transformations referred to as measure-and-prepare, which we denote as MP:

$$\text{MP} := \left\{ \sum_{i=1}^{k} \frac{\Pi}{\Delta_{i}^{S}} \bigg| s_1, \ldots, s_k, e_1, \ldots, e_k \in \Omega_T, e_1, \ldots, e_k \in T_S, \sum_{i=1}^{k} e_i = \Phi, k \text{ finite} \right\}, \quad (43)$$

recalling that $\Phi$ is the discarding effect.

Note that since these are physically possible in any GPT, they are a subset\(^2\) of the valid transformations, that is, they live inside the convex cone $T^T_S$ and, in fact, MP $\subseteq \mathcal{K}$.

A measure-and-prepare transformation $\phi \in \text{MP}$ has the additional property of being discard-preserving:

$$\Phi = \sum_{i=1}^{k} \frac{\Pi}{\Delta_{i}^{S}} = \sum_{i=1}^{k} \frac{\Delta_{i}^{S}}{1} = \frac{\Pi}{1}. \quad (44)$$

\(^2\)In quantum theory, for example, these are a proper subset of all quantum channels and are known as entanglement-breaking channels.
We denote the set of discard-preserving linear maps as $\text{DP}$ and define it formally as
\[
\text{DP} := \left\{ f \mid \tilde{f} = \tilde{f} \right\}. \tag{45}
\]
Note that the set of discard-preserving maps forms an affine space (see Appendix A for definitions relating to “affine” concepts), which is easily proven given the definition above.

Note that $\text{DP}$ may contain non-physical transformations. However, from the above discussion, it does contain the measure-and-prepare transformations. Neatly, we have that $\text{MP} \subseteq \text{DP} \cap \mathcal{K}$. Perhaps surprisingly, this containment is not strict, as shown in Lemma 1 (see the Appendix).

The sets $\text{DP}$, $\text{MP}$, and $\mathcal{K}$ allow us to get a useful characterization of the discard-preserving linear maps, which we now discuss.

### 4.5 A useful characterization of discard-preserving linear maps

The following theorem characterizes the set of discard-preserving maps in terms of those that are also measure-and-prepare.

**Theorem 4.1.** Any discard-preserving linear map can be written as an affine combination of measure-and-prepare transformations and any affine combination of measure-and-prepare transformations is a discard-preserving linear map. More formally,
\[
\text{DP} = \text{Aff}(\text{MP}), \tag{46}
\]
where Aff denotes the affine hull operation.

We provide a proof of Theorem 4.1 preceded by a background on convex geometry in Appendix A.

### 5 A characterisation of no-signalling GPT channels

Critical to our result is that of Ref. [7]. In the duotensor formalism presented in the previous section, the result of Ref. [7] is: if we have some non-signalling stochastic map $S$ it can be written as an affine combination of product stochastic maps:
\[
\vdots S \vdots = \sum_{\alpha \in A} q_{\alpha} \vdots s_{\alpha}^{1} \vdots \ldots \vdots s_{\alpha}^{n} \vdots \tag{47}
\]
where $n$ is the number of input/output system, $q_{\alpha} \in \mathbb{R}$, $\sum_{\alpha \in A} q_{\alpha} = 1$, and the $s_{\alpha}^{i}$ are stochastic maps. That is the $q_{\alpha}$ define a quasiprobability distribution $q$ over the set $A$. We can therefore equivalently draw this as:
\[
\vdots S \vdots = \sum_{\alpha \in A} q_{\alpha} \vdots s_{\alpha}^{1} \vdots \ldots \vdots s_{\alpha}^{n} \vdots, \tag{48}
\]
where the white dot is the copy operation, the quasiprobability distribution $q$ is a black triangle because it is not physically realisable as it can have negative coefficients, and the $S_{i}$ are stochastic maps controlled by the variable $A$. 

[12]
The duotensor formalism of Ref. [91], together with the above understanding of the geometry of GPT transformations, allow us to easily lift this result to arbitrary no-signalling channels in arbitrary tomographically local GPTs.

**Theorem 5.1.** Any non-signalling GPT channel $C$ in a tomographically local GPT $G$, can be written as an affine combination of product channels.

**Proof.** Consider some $n$-partite non-signalling channel $C$ in a tomographically local GPT:

\[
\begin{array}{c}
T_1 \cdots T_n \\
\hline
C \\
S_1 \cdots S_n
\end{array}
\]  

(49)

By decomposing the input and output identities using Eq. (29), we obtain:

\[
\begin{array}{c}
T_1 \cdots T_n \\
\hline
C \\
S_1 \cdots S_n
\end{array} = \sum_{\alpha \in A} q_\alpha s_\alpha^1 \cdots s_\alpha^n
\]  

(50)

We can then note that:

\[
\begin{array}{c}
\vdots
\hline
S \\
\vdots
\end{array} := \sum_{\alpha \in A} q_\alpha s_\alpha^1 \cdots s_\alpha^n
\]  

(51)

is a non-signalling stochastic map, and hence we can apply the result of Ref. [7] to obtain:

\[
\begin{array}{c}
\vdots
\hline
S \\
\vdots
\end{array} = \sum_{\alpha \in A} q_\alpha s_\alpha^1 \cdots s_\alpha^n
\]  

(52)

where $s_\alpha^i$ are stochastic maps, $q_\alpha \in \mathbb{R}$, and $\sum_{\alpha \in A} q_\alpha = 1$. By substituting this back in Eq. (50), we obtain:

\[
\begin{array}{c}
T_1 \cdots T_n \\
\hline
C \\
S_1 \cdots S_n
\end{array} = \sum_{\alpha \in A} q_\alpha s_\alpha^1 \cdots s_\alpha^n
\]  

(53)

where

\[
x_\alpha^i := \begin{array}{c}
\triangle
\hline
s_\alpha^i
\end{array}
\]  

(54)
It is then easy to check that the $x_i^\alpha$ are discard preserving:

\[
\begin{align*}
    x_i^\alpha &= s^{\alpha i} = s^{\alpha i} = 55 \\
    &= s^{\alpha i} = 56 \\
    &= . 57
\end{align*}
\]

We can then use Thm. 4.1 to write each $x_i^\alpha$ as an affine combination of GPT channels:

\[
\begin{align*}
    x_i^\alpha &= \sum_\beta x_i^{\alpha \beta} c_i^{\alpha \beta} \\
    &= 58
\end{align*}
\]

We can write this instead as:

\[
\begin{align*}
    x_i^\alpha &= C_i^{\alpha} \cdot R_i^\alpha \\
    &= 59
\end{align*}
\]

where $C_i^{\alpha}$ is a classically controlled channel and $R_i^\alpha$ is a quasidistribution.

Now, let us define:

\[
\begin{align*}
    A &\quad R_i &\quad C_i^{\alpha} \\
    &= 60
\end{align*}
\]

such that

\[
\begin{align*}
    \alpha A R_i C_i^{\alpha} &= R_i C_i^{\alpha} \\
    &= 61
\end{align*}
\]

where $C_i^{\alpha}$ is a classically controlled channel and $R_i$ is a quasistochastic map.

Putting this together with Eq. (52) we find that:

\[
\begin{align*}
    T_{i_1} \cdots T_{i_n} s_{i_1} \cdots s_{i_n} &= \sum_{\alpha} q_\alpha x_1^\alpha \cdots x_n^\alpha \\
    &= \sum_{\alpha} q_\alpha R_1 C_1^{\alpha} \cdots R_n C_n^{\alpha} \\
    &= Q' \\
    &= 62 \\
    &= 63 \\
    &= 64
\end{align*}
\]

where $Q'$ is the quasidistribution defined by the $q_\alpha$, i.e.:

\[
\begin{align*}
    Q' &\quad := q_\alpha \\
    &= 65
\end{align*}
\]
for all \( \alpha \).

Next, let us define \( Q'' \) by

\[
Q'' := \begin{array}{c}
\vdots \\
\vdots \\
R_1 \\
\vdots \\
R_n \\
\vdots \\
\end{array}
\] (66)

such that we can now combine this with Eq. (64) to write our channel as

\[
\begin{array}{c}
T_1 \cdots T_n \\
C \\
S_1 \cdots S_n \\
\end{array}
= \begin{array}{c}
\vdots \\
\vdots \\
C'_{1} \\
\vdots \\
\vdots \\
\end{array}
\] (67)

Eq. 67 gives us a quasiprobability distribution over a set of variables, one for each \( C'_i \). In the remainder of this proof we show that this can be rewritten as a quasidistribution over a single variable, which is then copied to each of the \( C''_i \)'s. Diagrammatically, this means that the copy operation should be the last operation prior to the \( C'_i \)'s. It is then this quasidistribution over a single variable which defines our affine combination of product channels.

Now, define “all but system \( i \)” marginalisation maps, \( D_i \) as:

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
D_i \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\] (68)

where the case \( i \neq 1 \) follows similarly.

We can then write:

\[
\begin{array}{c}
T_1 \cdots T_n \\
C \\
S_1 \cdots S_n \\
\end{array}
= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
C'_{1} \\
\vdots \\
\vdots \\
\end{array}
\] (69)

\[
= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
D_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
C''_1 \\
\vdots \\
\end{array}
\] (70)

\[
= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
C'_n \\
\vdots \\
\end{array}
\] (71)

\[
= \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\] (72)
where in the last step we have simply merged together parallel wires into a single composite wire, whilst using Eq. (18) to write the composite of copies as a copy of the composite.

By decomposing the quasidistribution \( Q'' \) we can equivalently write this as:

\[
\begin{align*}
&\sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta} = \sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta} \\
&= \sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta}
\end{align*}
\]

where \( c_{i}^{\beta} \) are GPT channels and \( q''_{\beta} \) is a quasidistribution. That is, any no-signalling GPT channel can be written as an affine combination of product GPT channels.

Note that, if we have a GPT, such as quantum theory, in which one can always reversibly encode classical data into a GPT system, then we can rewrite this as:

\[
\begin{align*}
&\sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta} = \sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta} \\
&= \sum_{\beta} q''_{\beta} C_{1}^{\beta} \cdots C_{n}^{\beta}
\end{align*}
\]

where \( E \) is the encoding map, \( D \) the decoding map, \( s_{Q} \) is some vector which is not necessarily a physical GPT state, and where the \( C_{i}^{\prime} \) are GPT channels.

6 Outlook

In this work we have provided a characterisation of multipartite non-signalling channels in arbitrary locally-tomographic theories: these channels can always be represented as affine combinations of local channels. In the case where the input and output system types are classical, i.e., where the channel is a multipartite non-signalling stochastic map, we recover the result of Ref. [7]. In the case of bipartite non-signalling channels whose inputs and outputs are quantum systems, we in turn recover the results of Ref. [8].
Our proof technique highlights the usefulness of the duotensor formalism, and we hope this will motivate its use throughout the quantum community. In particular, we show how it can be used to lift properties of multipartite stochastic maps, to arbitrary tomographically local GPTs. This motivates the question as to which other properties of stochastic maps can be similarly lifted?

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Note added

We are not aware of this result appearing elsewhere in the literature, but suspect that this might be the case. We are happy to give credit if such a result is brought to our attention and will update the manuscript accordingly. We do expect, however, that our approach is novel and may find other applications in future works.

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A bit of convex geometry and a proof of Theorem 4.1

This appendix aims to arrive at the proof of Theorem 4.1. In order to do so, a background on convex geometry is provided, and the concepts presented are used to prove Lemmas 1-4. Then, Lemmas 1, 3 and 4 are directly used to prove Theorem 4.1, while Lemma 2 is used to prove Lemma 3. All the sets DP, MP, and $K$ that are referred to here are defined in Section 4.

A.1 Convex Geometry

Given a real vector space $V$ and a (finite) set of vectors $v_1, \ldots, v_k \in V$, we define an affine combination of those vectors to be of the form

$$\sum_{i=1}^{k} q_i v_i$$

(78)

where $q_i \in \mathbb{R}$ and $\sum_{i=1}^{k} q_i = 1$. Note the distinction between an affine combination and a convex combination, where the latter also requires that each $q_i$ is nonnegative. Given a set of vectors $C$, its affine hull is the set of all affine combinations of vectors in $C$ and is denoted $\text{Aff}(C)$. Lastly, if a set is equal to its affine hull, that is, it contains all of its affine combinations, then we say that the set is affine (or an affine space). Geometrically, one can view an affine space as a subspace translated by a fixed single vector.

Conceptually, a convex set contains all the line segments between all pairs of points in the set. An affine set contains all the lines that extend beyond the endpoints of the line segments. This brings us to the definition of the core of a set. Thinking of line segments, $x \in S$ is in the core of a set $S$ if for all $z \in V$, there exists a $t_z > 0$ such that $x + t_z z \in S$ for all $t \in [0, t_z]$. Conceptually, this means that given $x$, you can start drawing a line in any direction and stay within the set $S$. Formally,

$$\text{core}(S) := \{x \in S | \forall z \in V, \exists t_z > 0, \text{ such that } x + t z \in S, \text{ for all } t \in [0, t_z]\}.$$  

(79)

---

3This difference is analogous to the difference between quasiprobability distributions and (proper) probability distributions.
A.2 Lemmas and proof of Theorem 4.1

**Lemma 1.** $\text{MP} = \text{DP} \cap \mathcal{K}$.

**Proof.** Since $\text{MP} \subseteq \text{DP} \cap \mathcal{K}$, all that remains to show is the opposite containment. Let $\phi \in \text{DP} \cap \mathcal{K}$ be a fixed, arbitrary vector. Since $\phi \in \mathcal{K}$, we can write it as

$$\phi = \sum_{i=1}^{k} \alpha_i s_i e_i,$$

where $k$ is finite, $s_1, \ldots, s_k \in \Omega_T$ are normalized states, and $e_1, \ldots, e_k \in T_S$ are in the effect cone.

It remains to show that $e_1, \ldots, e_k$ sum to $\Phi$. Since $\phi \in \text{DP}$, we have that

$$\sum_{i=1}^{k} \alpha_i s_i e_i = \phi,$$

(80)

since the states $s_i$ are normalised. This is the desired equality we seek. Finally, note that by the partial order given in Section 4.2, for all $e_i$ in the sum above, we have $e_i \in \mathcal{E}_S$, so $\phi$ satisfies all requirements for membership in MP.

**Lemma 2.** Suppose $S$ is a set and

$$K = \left\{ \sum_{i=1}^{n} \alpha_i s_i \bigg| \alpha_i \geq 0, s_i \in S, n \text{ finite} \right\},$$

(82)

is a full-dimensional cone, i.e., $V = K - K$. Suppose for $x \in K$, we have that for all $s \in S$, there exists $t_s > 0$ such that $x - ts \in K$ for all $t \in [0, t_s]$. Then $x \in \text{core}(K)$.

Note that the only difference between the definition of $\text{core}(K)$ and the condition above is that the vectors in the statement above are not arbitrary but rather belong to a set which generates a full-dimensional cone.

**Proof of Lemma 2.** Since $V = K - K$, for any arbitrary $v \in V$ we can write $v = y - z$ where $y, z \in K$. Then for $x \in K, t \geq 0$, we can write the following

$$x + tv = x + tyr - tz.$$

(83)

So, for $x$ to be in $\text{core}(K)$, it suffices to find a $t_v > 0$ such that $x + tv \in K$ for all $t \in [0, t_v]$. To do that, we have two cases to analyse, $z = 0$ and $z \neq 0$. Note that if $z = 0$, $x + tv = x + ty \in K$ for all $t \geq 0$ since $x, y \in K$, and $K$ is cone, so this case is trivial. Suppose $z \in K$ is nonzero, then we can write it as $\sum_i \alpha_i s_i$ where $\alpha_i > 0$ and $s_i \in S$ and the sum is finite. Then we have

$$x + tv = x + ty - \sum_i \alpha_i ts_i.$$

(84)

For brevity, define $a = \sum_i \alpha_i > 0$. By hypothesis, let $t_i > 0$ be such that

$$x - ats_i \in K$$

(85)
for all \( t \in [0,t_i] \). This exists since \( a \) is positive and by assumption there is some \( t'_i = at_i \) such that \( x - ts_i \in K \) for all \( t \in [0,t'_i] \).

We now have

\[
x + tv = x + ty - tz
= ty + x - \sum_i \alpha_i ts_i
= ty + \frac{1}{a} \left( \sum_i \alpha_i x - \sum_i a \cdot \alpha_i ts_i \right)
= ty + \frac{1}{a} \sum_i \alpha_i (x - ats_i)
\]

which is in \( K \) for all \( t \in [0,t_v] \) where \( t_v := \min_i \{ t_i \} \) (which is positive since there are finitely many indices \( i \)). This concludes the proof. \( \square \)

We use this particular case characterization of a core element to prove the following lemma which is helpful in our proof of Theorem 4.1.

**Lemma 3.** Let \( \mu \in \text{int}(T^T) \) be a normalised state. Then

\[
\frac{\mu^T}{\mu} \in \text{DP} \cap \text{core}(K),
\]

where, recall, \( \phi \in \text{int}(E_S) \) is the discarding effect.

**Proof.** Clearly

\[
\frac{\mu^T}{\mu} \in \text{DP},
\]

as \( \mu \) is normalised, so all that remains to show is that it is in \( \text{core}(K) \).

Define

\[
S := \left\{ \frac{\mu^T}{\mu} \left|\begin{array}{c}
s \in T^T, e \in T_S
d\end{array}\right. \right\}.
\]

Since \( K \) (as defined in Equation (41)) is the convex hull of \( S \) (as defined in Equation (82)) and \( L(V_S, V_T) = K - K \), by Lemma 2, it suffices to show that for a fixed \( s \in T^T \) and \( e \in T_S \), there exists \( \tilde{t} > 0 \) such that

\[
\frac{\mu^T}{\mu} - t \frac{s^T}{s} \in K
\]

for all \( t \in [0,\tilde{t}] \). For \( t > 0 \), we can write

\[
\frac{\mu^T}{\mu} - t \frac{s^T}{s} = \frac{\mu^T}{\mu} - \sqrt{t} \frac{s^T}{s} + \sqrt{t} \frac{s^T}{s} - t \frac{s^T}{s}
\]

(94)
\[
  \left( \sqrt{\mu_{TT}} - \sqrt{t} \sqrt{\mu_{TT}} \right) \frac{\mu_{TT}}{1S} + \sqrt{t} \sqrt{\mu_{TT}} \left( \frac{\mu_{TT}}{1S} - \sqrt{t} \sqrt{\mu_{TT}} \right). \tag{95}
\]

Note that
\[
  \sqrt{\mu_{TT}} - \sqrt{t} \sqrt{\mu_{TT}} \in T_T \quad \text{and} \quad \frac{\mu_{TT}}{1S} - \sqrt{t} \sqrt{\mu_{TT}} \in T_S \tag{96}
\]
for all sufficiently small \( t > 0 \) as \( \mu \) and \( \Phi \) are interior in their respective cones. Therefore,
\[
  \frac{\mu_{TT}}{1S} - t \frac{\mu_{TT}}{1S} \in K \tag{97}
\]
is in \( K \) for all \( t > 0 \) sufficiently small. This concludes the proof. \( \square \)

**Lemma 4.** Given a real vector space \( V \), let \( S \subseteq V \) be a subset and let \( A \subseteq V \) be an affine space. If \( \text{core}(S) \cap A \neq \emptyset \), then \( \text{Aff}(S \cap A) = A \).

Before diving into the proof, we explain the idea first since the proof is actually quite simple to picture geometrically, but the proof we give here is algebraic. We start with a point \( x \in \text{core}(S) \cap A \) and draw the line segment between that point to some other arbitrary fixed point \( z \in A \). Since \( x \in \text{core}(S) \), there is a point on that line segment, call it \( y \), such that it is still in \( S \). And since \( A \) is affine, \( y \) is in \( A \) as well. The proof concludes by noting that \( z \) is an affine combination of \( x \) and \( y \).

\[
  \text{Proof of Lemma 4.} \quad \text{Since} \quad S \cap A \subseteq A, \text{we have that} \quad \text{Aff}(S \cap A) \subseteq \text{Aff}(A) = A. \text{Therefore, all that remains to show is the reverse containment. To this end, let} \quad z \in A \text{ be a fixed, arbitrary vector and let} \quad x \in \text{core}(S) \cap A \text{ (which exists by hypothesis). Define}
\]
\[
  y = (1 - t)x + tz, \tag{99}
\]
for some \( t > 0 \) which we define momentarily. Notice that \( y \) is an affine combination of \( x \) and \( z \), both of which are in \( A \), and thus \( y \in A \) as well. We now want to show that \( y \in S \). Note that \( y \) can be rewritten as
\[
  y = x + t(z - x). \tag{100}
\]
Since $x \in \text{core}(S)$, there exists a $t \in (0,1)$ such that $y \in S$. Thus, we have $x$ and $y$ both belonging to $S \cap A$. Notice that

$$z = \left(\frac{1}{t}\right) y + \left(\frac{t - 1}{t}\right) x.$$  \hfill (101)

Since $z$ is an affine combination of $x$ and $y$, both belonging to $S \cap A$, the result follows. \hfill \Box

Lemmas 1-4 together allow us to write a short proof for Theorem 4.1.

**Proof of Theorem 4.1.** We want to show that $\text{DP} = \text{Aff}(\text{MP})$.

By Lemma 1, we know

$$\text{MP} = \text{DP} \cap \mathcal{K}.$$  \hfill (102)

Now, Lemma 3 tells us that

$$\text{DP} \cap \text{core}(\mathcal{K}) \neq \emptyset.$$  \hfill (103)

If we set $A := \text{DP}$ and $S := \mathcal{K}$, the assumptions in Lemma 4 are satisfied and we can use it to conclude that

$$\text{DP} = \text{Aff}(\mathcal{K} \cap \text{DP}) = \text{Aff}(\text{DP} \cap \mathcal{K}) = \text{Aff}(\text{MP}).$$  \hfill (104)

\hfill \Box