Complex Random Vectors and ICA Models: Identifiability, Uniqueness and Separability

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Abstract—In this paper the conditions for identifiability, separability and uniqueness of linear complex valued independent component analysis (ICA) models are established. These results extend the well-known conditions for solving real-valued ICA problems to complex-valued models. Relevant properties of complex random vectors are described in order to extend the Darmois-Skitovich theorem for complex-valued models. This theorem is used to construct a proof of a theorem for each of the above ICA model concepts. Both circular and noncircular complex random vectors are covered. Examples clarifying the above concepts are presented.

Index Terms—Blind methods, circularity, complex linear models, complex Darmois-Skitovich theorem, differential entropy, independent component analysis (ICA), noncircular complex random vectors, properness.

I. INTRODUCTION

Independent component analysis (ICA) [1] is a relatively new signal processing and data analysis technique. It may be used, for example, in blind source separation (BSS) and identifying or equalizing instantaneous multiple-input multiple-output (I-MIMO) models. It has found applications, e.g., in wireless communications, biomedical signal processing and data mining (see [2] for references). In instantaneous complex-valued ICA problem

\[ \tilde{x} = A\tilde{\alpha}, \]

the goal is to recover the original source signal vectors \( \tilde{\alpha} \) from the observation vectors \( \tilde{x} \) blindly without explicit knowledge of the sources or the linear mixing system \( A \). ICA is based on the crucial assumption that the underlying unknown source signals are statistically independent. Recent textbooks provide an interesting tutorial material and a partial review on ICA [2], [3].

The theorems for linear combinations of real-valued random vectors and theoretical conditions on separation for real-valued signals are now well-known [1], [4], [5]. Even though algorithms for separation of complex-valued signals have been developed, for example [1], [6], the conditions when the separation is possible have not been established. Also recent papers, e.g., [7]–[10], proposing ICA algorithms for complex-valued data ignore this important issue.

In this paper we construct theorems stating the conditions for identifiability, separability, and uniqueness of complex-valued linear ICA models. These results extend the theorems proved for the real-valued instantaneous ICA model [1], [5] to the complex case. Both circular (proper) and noncircular complex random vectors are covered by the theorems. These conditions depend not only on the probabilistic structure of the sources but also the linear space structure of the mixing. In order to prove the theorems, the celebrated Darmois-Skitovich theorem [4] needs to be extended to linear combinations of complex random variables. A good number of statistical properties of circular and noncircular complex vectors have to be considered in the process of constructing the proof. This is due to the special operator structure that may be used for complex random vectors. In addition, the second order statistical properties of noncircular complex vectors may not be defined using the covariance matrix alone [11]–[13]. General complex Gaussian random vectors is an important class of random vectors that need to be addressed in detail. There are relatively few papers where noncircular complex random vectors are studied [11]–[16]. Hence, many of the key results needed in proving the theorems are included in this paper and presented in a unified manner. This also allows a direct derivation of some fundamental information-theoretic quantities like the entropy of a complex normal random vector.

The paper is organized as follows. In Section II relevant properties that distinguish complex random vectors from real random vectors are described in detail. Especially, the correlation structure is used to study complex normal random vectors. These properties are needed in proving the Darmois-Skitovich theorem for the complex case. This theorem plays a key role in establishing the conditions for identifiability, separability and uniqueness of complex linear ICA models in Section III. Finally, some concluding remarks are given. Most of the proofs are presented in appendices.

II. RELEVANT PROPERTIES OF COMPLEX RANDOM VECTORS

The traditional probability theory is concerned with real-valued random variables (r.v.s) and random vectors (r.v.c.s). The theory has been generalized to various algebraic structures. Main studies are in the frameworks of locally compact spaces and complete separable metric spaces (see, e.g., [17]–[20] and references therein). However, the most natural extension from the engineering point of view is the complex Hilbert space. It seems to have gained relatively little attention. Some results on complex normal r.v.c.s can be found in [21], [22]. The second-order structure of complex r.v.c.s has been studied in [11]–[13], [15], and a general framework for higher-order statistics can be found from [23]. Some research has been conducted on complex elliptically symmetric distributions [24].
and on complex stable distributions [25]. Polya’s theorem to complex case is presented in [26]. The only systematic Hilbert space approach known to the authors is [14]. This may be due to the fact that the additive structure of the complex Hilbert space is the same as that of the real Hilbert space. However, the multiplicative structure and the operator structure are different giving r.v.c.s in a complex Hilbert space distinct properties. Even though many results from the general abstract theory apply directly to the complex Hilbert space case, the systematic treatment considering both the additive and the multiplicative structure seems to be missing.

In Section II-B the finite dimensional Hilbert space is reviewed by constructing an isomorphism into a real-valued Hilbert space. This isomorphism shows essentially the difference between the real and complex Hilbert spaces. In Section II-C some basic properties of r.v.c.s in the complex Hilbert space are stated, the second-order structure of complex r.v.c.s is studied in Section II-D. Complex normal r.v.c.s are studied in Section II-E and, finally, the complex Darmois-Skitovich theorem is proved in Section II-F.

A. Notation

Let us begin with some definitions and notations. We have used typewriter font for all random objects, e.g. \( x \), in order to distinguish them from deterministic ones, e.g. \( x \). For random vectors, e.g. \( \vec{x} \), we have used the vec symbol in order to separate them from scalar random variables. For deterministic objects, the bold face lower case letters are used for vectors, e.g. \( \vec{z} \), and the bold face upper case letters are used for matrices, e.g. \( W \).

The modulus of a complex number \( z = z_R + jz_I \in \mathbb{C} \) is denoted \( |z| = \sqrt{z^*z} = \sqrt{z_R^2 + z_I^2} \), where the superscript \( * \) denotes the complex conjugate, \( z^* = z_R - jz_I \), and \( j = \sqrt{-1} \) is the imaginary unit. Recall that any nonzero complex number \( z \) can be given in polar form \( z = \alpha e^{j\theta} \), where \( \alpha > 0, \theta \in \mathbb{R} \). The number \( \theta \) is called an argument of the complex number \( z \), and the argument \( \theta = \text{Arg}(z) \) such that \( -\pi \leq \theta < \pi \) is called the principal argument. The real part of a \( p \)-dimensional complex vector \( (z_1, z_2, \cdots, z_p)^T = \vec{z} \in \mathbb{C}^p \), where \( T \) is the ordinary transpose, is denoted by \( z_R \) and the imaginary part by \( z_I \). The Euclidean norm of a vector \( \vec{z} \) is denoted \( \|z\|^2 = \langle z, z \rangle = z^H z \), where \( \langle \cdot, \cdot \rangle \) is the inner product and the superscript \( H \) denotes the conjugate transpose, i.e., the Hermitian adjoint. A complex matrix \( C \in \mathbb{C}^{p \times p} \) is termed [27] symmetric if \( C^T = C \) and Hermitian if \( C^H = C \). Furthermore, the matrix \( C \) is orthogonal if \( C^T C = C C^T = I_p \) and unitary if \( C^H C = C C^H = I_p \), where \( I_p \) denotes the \( p \times p \) identity matrix.

B. Complex Hilbert space isomorphism

Let \( C = C_R + jC_I \in \mathbb{C}^{m \times p} \) and \( \vec{z} = z_R + jz_I \in \mathbb{C}^p \). We use the following notations

\[
C_R = \begin{pmatrix} C_R & -C_I \\ C_I & C_R \end{pmatrix}, \quad z_R = \begin{pmatrix} z_R \\ z_I \end{pmatrix}
\]

for the associated \( 2m \times 2p \) real matrix and \( 2p \)-variate real vector, respectively. The mapping \( \vec{z} \mapsto z_R \) gives naturally a group isomorphism between the additive Abelian groups \( \mathbb{C}^p \) and \( \mathbb{R}^{2p} \). In the case \( m = p = 1 \), the mapping given by \( C \mapsto C_R \) defines a field isomorphism (e.g., [14], [22]) between the complex numbers and a subset of real two dimensional matrices. Therefore, one can construct real structures where the role of complex multiplication is played by the special matrices.

Now consider the mapping

\[
C\vec{z} \mapsto (C\vec{z})_R = C_R z_R.
\]

It is continuous and therefore preserves the topological properties, i.e., it is a homeomorphism [19]. Let \( \text{diag}(z) \) (as in Matlab) denote the diagonal matrix with components of \( z \) in its main diagonal and zeros elsewhere. Since \( \mathbb{C}^p \) is a vector space, where the scalar multiplication for \( c \in \mathbb{C} \) is given by

\[
\vec{cz} \triangleq \begin{pmatrix} cz_1 \\ \vdots \\ cz_p \end{pmatrix} = \text{diag}(c \cdots c)\vec{z},
\]

the mapping \( \vec{cz} \) defines a vector space isomorphism between the standard \( p \)-dimensional complex vector space and a \( 2p \)-dimensional real-valued vector space given by the mapping. It is important to realize that this associated real-valued vector space is not isomorphic to the standard real vector space \( \mathbb{R}^{2p} \). Furthermore, by equating \( z^R \) with \( C \) in (3) it is easily verified that the mapping \( \vec{z} \mapsto z^R = (z^R)^R((z^R)_R) \) associates a (complex) inner product for \( \mathbb{R}^{2p} \). Therefore, the mapping \( (3) \) is also a Hilbert space isomorphism. Again, it should be emphasized that the inner product given by the mapping is not the standard Euclidean inner product in \( \mathbb{R}^{2p} \). However, the vector norms, and hence metrics, are equivalent in both.

The following properties are easily established.

**Lemma 1:** Let \( C \in \mathbb{C}^{p \times p} \) and \( \vec{z} \in \mathbb{C}^p \).

(i) \( |\det(C)|^2 = |\det(C_R)| \).

(ii) \( C \) is Hermitian iff \( C_R \) is symmetric. Then \( \det(C) = \det(C_R) \) and \( 2 \times \text{rank}(C) = \text{rank}(C_R) \).

(iii) \( C \) is nonsingular iff \( C_R \) is nonsingular.

(iv) \( C \) is unitary iff \( C_R \) is orthogonal.

(v) \( z^H z = z^H_R z_R \).

(vi) \( C \) is Hermitian positive definite iff \( C_R \) is symmetric positive definite.

(vii) Any polynomial with complex coefficients in variables \( z_R \) can be equivalently given in variables \( (z, z^*) \).

**Proof:** These properties are direct consequences of the isomorphism, see, e.g., [22], [24]. The last property follows from the identities \( z_R = \frac{1}{2}(z + z^*) \) and \( z_I = \frac{j}{2}(z - z^*) \).

Since the variables \( (z, z^*) \) in Lemma 1.(vii) are dependent, we call such complex polynomials wide sense polynomials. The idea of using also the complex conjugate variable has turned out to be highly useful in, e.g., complex parameter estimation [28] and blind channel equalization [16].

C. Complex random vectors

A \( p \)-variate complex random vector (r.v.c.) \( \vec{z} \) is defined as an r.v. of the form

\[
\vec{x} = \vec{z}_R + j\vec{z}_I,
\]
where \( \tilde{x}_k \) and \( \tilde{z}_k \) are \( p \)-variate real r.v.c.s, i.e., \( \tilde{x}_k \) and \( \tilde{z}_k \) are measurable functions from a probability space to \( \mathbb{R}^p \). This is equivalent to \( \tilde{x} \) to be measurable from the probability space into \( \mathbb{C}^p \) due to the separability of the complex space. Therefore, the probabilistic structure of the r.v.c.s in \( \mathbb{C}^p \) and the probabilistic structure of the r.v.c.s in \( \mathbb{R}^{2p} \) is the same. However, the operator structure is different as it is evident from the previous section. This gives distinct properties to the r.v.c.s with complex values, and justifies studying them separately. Throughout this paper all complex r.v.c.s are assumed to be full. This means that the induced measure of a \( p \)-dimensional r.v.c. is not contained in any lower dimensional complex subspace.

Since the probabilistic structures of r.v.c.s in \( \mathbb{C}^p \) and in \( \mathbb{R}^{2p} \) are the same, the operator structure of r.v.c.s in \( \mathbb{C}^p \) can be studied by first using the isomorphism \( \tilde{\nu} \) and then applying the concepts associated with the real r.v.c.s. However, we define these associated concepts directly on \( \mathbb{C}^p \), since this approach is notationally more convenient.

The expectation \( E[\cdot] \) of a complex r.v. \( \tilde{x} \) is defined as
\[
E_{\tilde{x}}[\tilde{x}] = E_{\tilde{x}_R}[\tilde{x}_R] + j E_{\tilde{x}_I}[\tilde{x}_I],
\]
and the distribution function \( F_{\tilde{x}} \) is given as \( F_{\tilde{x}}(z) \triangleq F_{\tilde{x}_R}(z)_R \), where \( z = (z_1, \ldots, z_p) \in \mathbb{C}^p \) and \( F_{\tilde{x}_R} \) denotes the distribution function of real-valued r.v. \( \tilde{x}_R \). Then for independent r.v.s \( (a_1, \ldots, a_p)^T = \tilde{s} \), we have
\[
F_{\tilde{x}}(z) = F_{\tilde{x}_R}(z)_R = \prod_{k=1}^{p} F_{\tilde{a}_k}(z_k)_R = \prod_{k=1}^{p} F_{\tilde{a}_k}(z_k). \tag{7}
\]

The same way we define the probability density function \( f_{\tilde{x}} \) (if it exists) of a \( p \)-dimensional complex r.v. \( \tilde{x} \) as \( f_{\tilde{x}}(z) \triangleq f_{\tilde{x}_R}(z)_R \), and the characteristic function (c.f.) \( \{14\} \) as
\[
\varphi_{\tilde{x}}(z) \triangleq \varphi_{\tilde{x}_R}(z)_R = E_{\tilde{x}_R}[\exp(j \langle z, \tilde{x} \rangle_R)] = E_{\tilde{x}_R}[\exp(j \text{Re}(z \tilde{x}^*)_R)].
\]

It follows directly from Eq. (7) that for independent complex r.v.s \( (a_1, \ldots, a_p)^T = \tilde{s} \),
\[
\varphi_{\tilde{x}}(z) = \prod_{k=1}^{p} \varphi_{\tilde{a}_k}(z_k). \tag{9}
\]

Using a standard property of real c.f.s and the properties of the isomorphism \( \tilde{\nu} \), we have a useful relation for the c.f. of an r.v. \( \tilde{x} \) and the c.f. of the linearly transformed r.v. \( C\tilde{x} \). Namely, for any complex matrix \( C \), we have
\[
\varphi_{C\tilde{x}}(z) = \varphi_{C\tilde{x}_R}(z)_R = \varphi_{\tilde{x}_R}((C_R^T)z_R)_R = \varphi_{\tilde{x}_R}((C_R^T)z_R)_R = \varphi_{\tilde{x}}(C^* z). \tag{10}
\]

Finally, a c.f. \( \varphi_{\tilde{x}}(z) \) is called analytic if \( \varphi_{\tilde{x}_R}(z)_R \) is an analytic real c.f. \( \{29\} \), i.e., the real c.f. \( \varphi_{\tilde{x}_R}(z)_R \) has a regular extension defined on \( \mathbb{C}^{2p} \) in some neighborhood of \( \mathbb{C}^p \).

**D. Second-order statistics of complex random vectors**

An r.v. \( \tilde{x} \) has finite second order statistics if \( E_{\tilde{x}}[(z, \tilde{x})]^2 < \infty \) for all \( z \in \mathbb{C}^p \). This is clearly equivalent to the existence of finite second order statistics for both real r.v.c.s \( \tilde{x}_R \) and \( \tilde{x}_I \). All r.v.c.s in this section are assumed to have finite second order statistics. Such r.v.c.s are in general called second-order complex r.v.c.s.

The second-order statistics between two real r.v.c.s may be described by the covariance matrix. The complex covariance matrix \( \text{cov}[\tilde{x}_1, \tilde{x}_2] \) of two complex r.v.c.s \( \tilde{x}_1 \) and \( \tilde{x}_2 \) may be defined as
\[
\text{cov}[\tilde{x}_1, \tilde{x}_2] \triangleq E_{\tilde{x}_1, \tilde{x}_2}[(\tilde{x}_1 - E_{\tilde{x}_1}[\tilde{x}_1])(\tilde{x}_2 - E_{\tilde{x}_2}[\tilde{x}_2])^H]. \tag{11}
\]
However, considering the real representations of the complex r.v.c.s, it can be seen that the complex covariance matrix does not give complete second order description. For that we define the pseudo-covariance matrix \( \text{pcov}[\tilde{x}_1, \tilde{x}_2] \) \{11\] as
\[
\text{pcov}[\tilde{x}_1, \tilde{x}_2] \triangleq E_{\tilde{x}_1, \tilde{x}_2}[(\tilde{x}_1 - E_{\tilde{x}_1}[\tilde{x}_1])(\tilde{x}_2 - E_{\tilde{x}_2}[\tilde{x}_2])^T] = \text{cov}[\tilde{x}_1, \tilde{x}_2]. \tag{12}
\]

Two complex r.v.c.s \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are uncorrelated if real r.v.c.s \( (\tilde{x}_1)_R \) and \( (\tilde{x}_2)_R \) are uncorrelated, i.e., \( \text{cov}[(\tilde{x}_1)_R, (\tilde{x}_2)_R] = 0_{2p \times 2p} \), where \( 0_{2p \times 2p} \) denotes the \( 2p \times 2p \) matrix of zeros. Then, by using the properties from the previous section, the following lemma \{11\] follows directly.

**Lemma 2:** Complex r.v.c.s \( \tilde{x}_1 \) and \( \tilde{x}_2 \) are uncorrelated if and only if \( \text{cov}[\tilde{x}_1, \tilde{x}_2] = \text{pcov}[\tilde{x}_1, \tilde{x}_2] = 0_{p \times p} \).

As it is the case with real r.v.c.s, the internal correlation structure of a single r.v. \( \tilde{x} \) may be of interest in addition to correlation between two r.v.c.s. Then we define \( \text{cov}[\tilde{x}] \triangleq \text{cov}[\tilde{x}, \tilde{x}] \) and \( \text{pcov}[\tilde{x}] \triangleq \text{pcov}[\tilde{x}, \tilde{x}] \), and call them the covariance matrix and the pseudo-covariance matrix of an r.v. \( \tilde{x} \), respectively. It is easily seen that the covariance matrix \( \text{cov}[\tilde{x}] \) is Hermitian and the pseudo-covariance matrix is symmetric. Since all r.v.c.s are assumed to be full, the covariance matrix \( \text{cov}[\tilde{x}] \) is also positive definite. R.v.c. \( \tilde{x} \) is said to have uncorrelated components if all its marginal r.v.s \( x_k \) and \( \tilde{x}_k \), \( k \neq l \), are uncorrelated. The following lemma is a simple consequence of Lemma 2.

**Lemma 3:** A complex r.v. \( \tilde{x} \) has uncorrelated components if and only if its covariance matrix and pseudo-covariance matrix are diagonal.

An r.v. \( \tilde{x} \) is said to be spatially white, if \( \text{cov}[\tilde{x}] = \sigma^2 I_p \) for some \( \sigma^2 > 0 \). If \( \text{pcov}[\tilde{x}] = 0_{p \times p} \), then the r.v.c. is called second order circular (or circularly symmetric). Some authors prefer the term proper \{11\], \{14\]. Circular r.v.c.s have gained most of the attention in the literature of complex r.v.c.s. This is likely due to the fact that all the second order information of circular r.v.c.s is contained in the covariance matrix, which, on the other hand, behaves like the covariance matrix for the real r.v.c.s. However, in this paper we need the complete second-order description of circular r.v.c.s to be derived next. Our approach is to use our best knowledge novel, mainly based on the following theorem. For alternative characterizations, see \{12\]–\{14\].

**Theorem 1:** Any full complex \( p \)-dimensional r.v. \( \tilde{x} \) with finite second order statistics can be transformed by using a nonsingular square matrix \( C \) such that the r.v. \( \tilde{s} = (s_1, \ldots, s_p)^T = C\tilde{x} \) has the following properties:

\footnote{The pseudo-covariance matrix is called the relation matrix in \{12\] and the complementary covariance matrix in \{13\].}
(i) \( \text{cov}[\vec{g}] = I_p \)

(ii) \( \text{pcov}[\vec{g}] = \text{diag}(\lambda[\vec{g}]) \), where \( \lambda[\vec{g}] = (\lambda_1, \ldots, \lambda_p)^T \)

denotes a vector such that \( \lambda_1 \geq \cdots \geq \lambda_p \).

Proof: It is easily verified that \( \text{cov}[C\vec{x}] = C \text{cov}[\vec{x}]C^H \)
and \( \text{pcov}[C\vec{x}] = C \text{pcov}[\vec{x}]C^T \). By Corollary 4.6.12(b) in [27], if a matrix \( A \) is Hermitian and positive definite and a matrix \( B \) is symmetric, then there exists a nonsingular matrix \( C \) such that \(CAC^H = I_p \) and \( CBC^T \) is a diagonal matrix with nonnegative diagonal entries. Since the covariance matrix is Hermitian and positive definite and the pseudo-covariance matrix is symmetric, the proof is completed by noticing that the diagonal entries can be ordered by permuting the rows of \( C \).

Since \( \text{cov}[x] = \text{cov}[x_R] + \text{cov}[x_I] \) and \( \text{pcov}[x] = \text{cov}[x_R] - \text{cov}[x_I] + 2j \text{cov}[x_R, x_I] \) for any complex r.v. \( x = x_R + jx_I \), it follows that in Theorem 1, \( \Re\{\Lambda_k\}, \Im\{\Lambda_k\} \) are 0 and \( 1 \geq \lambda_k = \Re\{\Lambda_k\} - \text{cov}[\Im\{\Lambda_k\}] \geq 0 \), \( k = 1, \ldots, p \). The r.v.s satisfying the properties of Theorem 1 have a special structure, and they are here called strongly uncorrelated. Any strongly uncorrelated r.v. is white with \( \text{cov}[\vec{x}] = I_p \), but the converse is not true. In general, for a given r.v. \( \vec{x} \), the strongly uncorrelated r.v. \( \vec{y} \) and the strong-uncorrelating transform \( C \) given by Theorem 1 are not unique. However, we have the following.

Theorem 2: For a given r.v. \( \vec{x} \), the vector \( \lambda[\vec{y}] \) in Theorem 1 is unique.

Proof: Suppose there exist two nonsingular transformations \( C_1 \) and \( C_2 \) such that r.v.s \( \vec{y}_1 = C_1 \vec{x} \) and \( \vec{y}_2 = C_2 \vec{x} \) satisfy the properties in Theorem 1. Let \( C_1 = U_1 \Lambda_1 V_1^H \) and \( C_2 = U_2 \Lambda_2 V_2^H \), be the singular value decompositions (SVD) of the transform matrices. Now \( I_p = C_1 \text{cov}[\vec{x}] C_1^H = C_2 \text{cov}[\vec{x}] C_2^H \), and therefore \( \text{cov}[\vec{x}] = V_1 \Lambda_1^{-2} V_1^H = V_2 \Lambda_2^{-2} V_2^H \). Since \( \text{cov}[\vec{x}] \) is positive definite, it follows \( V_1 \Lambda_1^{-2} V_1^H = V_2 \Lambda_2^{-2} V_2^H \). Now

\[
\text{pcov}[\vec{y}_1] = U_1 \Lambda_1 V_1^H \text{pcov}[\vec{x}] V_1 \Lambda_1 V_1^H U_1^T
\]

\[
= U_1 V_1^H (V_1^H V_1) \Lambda_1 V_1^H \text{pcov}[\vec{x}] V_1 \Lambda_1 V_1^H U_1^T
\]

\[
= U_1 V_1^H (V_1^H V_1) \text{pcov}[\vec{x}] V_1 \Lambda_1^{-2} V_1^H U_1^T
\]

\[
= U_1 V_1^H V_1 (U_2^H U_2) \text{pcov}[\vec{x}] V_2 \Lambda_2^{-2} V_2^H U_2^T
\]

\[
= U_1 V_1^H V_1 (U_2^H U_2) \text{pcov}[\vec{x}] V_2 \Lambda_2^{-2} V_2^H U_2^T
\]

\[
= U_1 V_1^H V_1 (U_2^H U_2) \text{pcov}[\vec{x}] V_2 \Lambda_2^{-2} V_2^H U_2^T
\]

\[
= U_1 V_1^H V_1 (U_2^H U_2) \text{pcov}[\vec{y}_2] U_2^T
\]

and since \( U_1 V_1^H V_2 U_2^T \) is unitary, \( \text{pcov}[\vec{y}_1] \) and \( \text{pcov}[\vec{y}_2] \) have the same singular values. Since by the assumption \( \text{pcov}[\vec{y}_1] \) and \( \text{pcov}[\vec{y}_2] \) are diagonal with sorted entries, it follows \( \text{pcov}[\vec{y}_1] = \text{pcov}[\vec{y}_2] \).

Remark 1: The proof of Theorem 2 gives a way to construct a strong-uncorrelating transform \( C \) as follows:

(i) Find the usual whitening transform \( D = \text{cov}[\vec{x}]^{-\frac{1}{2}} \), i.e., the inverse of the matrix square root of \( \text{cov}[\vec{x}] \).

(ii) Any symmetric matrix \( B \) has a special form of SVD known as Takagi’s factorization (see [27]). The factorization is given as \( B = U \Lambda U^T \), where \( U \) is unitary and \( \Lambda \) is a diagonal matrix with real nondecreasing nonnegative main diagonal entries. An example of the factorization is given in Eq. (13). Hence, find \( \text{pcov}[D\vec{x}] = U \Lambda U^T \).

(iii) Set \( C = U^H D \).

Notice also that the vector \( \lambda[\vec{y}] \) contains the singular values of the pseudo-covariance matrix of a white r.v. with unit variances.

The previous theorems lead to a useful characterization of second-order complex r.v.s.

**Definition 1:** The vector \( \lambda[\vec{x}] \equiv \lambda[\vec{y}] = (\lambda_1, \ldots, \lambda_p)^T \) in Theorem 1 is called the circularity spectrum of an r.v. \( \vec{x} \). An element of the circularity spectrum corresponding to an r.v. is called a circularity coefficient.

Any r.v. \( \vec{x} \) is clearly second order circular if and only if its circularity spectrum is a zero vector, i.e., \( \lambda[\vec{x}] = 0_{p \times 1} \).

**Corollary 1:** If the circularity spectrum of an r.v. has distinct elements, all rows corresponding to nonzero circularity coefficients of the strong-uncorrelating transform are unique up to multiplication of the row by \(-1\). A row corresponding to the zero coefficient is unique up to multiplication of the row by \(e^{j\theta}, \theta \in \mathbb{R} \).

Proof: The left unitary factor in the SVD of a block matrix with distinct singular values is determined up to right multiplication by the matrix \( \Lambda = \text{diag}(e^{j\theta_1}, \ldots, e^{j\theta_p}) \) and the right unitary factor is determined by the left unitary factor [27]. In the special form for a symmetric matrix (Takagi’s factorization), \( \theta_k = 0 \) or \( \theta_k = \pi \) for the values of \( k \) corresponding to nonzero singular values. Therefore, \( U_1 V_1^H V_2 U_2^T = \Lambda C_2 \) (14)
by the proof of Theorem 2.

Some properties of the circularity coefficient are listed in the following lemma, whose proof is given in Appendix D.

**Lemma 4:** Let \( x \) and \( y \) be uncorrelated second-order complex r.v.s. Then

(i) \( 0 \leq \lambda[x] \leq \frac{\text{pcov}[x]}{\text{cov}[x]} \leq 1 \) for any nonzero constant \( c \in \mathbb{C} \),

(ii) \( \lambda[x] = 1 \) if and only if \( x = c(\vec{z} R + \alpha) \) for some unit variance real r.v. \( \vec{z} R \) and deterministic constants \( 0 \neq c \in \mathbb{C} \), \( \alpha \in \mathbb{R} \),

(iii) \( \lambda[x + y] = \frac{\text{pcov}[x + y]}{\text{cov}[x] + \text{cov}[y]} \leq \max\{\lambda[x], \lambda[y]\} \)
with the equality if and only if \( \lambda[x] = \lambda[y] \) and \( \text{Arg}(\text{pcov}[x]) = \text{Arg}(\text{pcov}[y]) \) if \( \lambda[x] \neq 0 \).

**E. Complex normal random vectors**

There are no commonly agreed definitions of what is meant by complex normal r.v.s. It is natural to require that a r.v. \( \vec{x} \) is normal (Gaussian) if the real r.v. \( \vec{x} R \) is multivariate normal. Such r.v.s are generally called wide sense normal r.v.s [14]. Since the real complex normal r.v. is completely characterized by its mean vector and covariance, the results from the previous section show that a wide sense complex
normal r.v.c. is completely specified by its mean, covariance matrix, and pseudo-covariance matrix.

However, all wide sense normal r.v.c.s do not possess all the properties that real normal r.v.c.s do. Only a special subclass of wide sense normal r.v.c.s has a density function similar to the real r.v.c.s [21], [22], maximizes the entropy [11], or has the 2-stability property (Polya’s characterization) [26]. Such r.v.c.s are called narrow sense normal r.v.c.s [14]. They are wide sense normal r.v.c.s such that the real and imaginary parts of any linear projection of the r.v.c. are independent and have equal variances. This condition is equivalent to the requirement that a wide sense normal r.v.c. is second order circular (see, e.g., [11]).

In order to establish the properties of the complex ICA model of Eq. (1), neither wide sense normal in its full generality nor narrow sense normal is adequate, and a more specific characterization of complex normal r.v.c.s is needed. This is done next. From now on, we will use the term “complex normal” to mean wide sense complex normal r.v.c.

The main result is the following decomposition theorem for complex normal random vectors.

**Theorem 3:** An r.v.c. \( \mathbf{n} \) is complex normal with circularity spectrum \( \lambda \) if and only if

\[
\mathbf{n} = C(\mathbf{i} \eta_R + j \mathbf{i} \eta_I) + \mu
\]  

(15)

for some nonsingular matrix \( C \), a complex constant vector \( \mu \), and multinormal real independent r.v.c.s \( \eta_R \sim N(0_{p \times 1}, \frac{1}{2} \text{diag}(\lambda)) \) and \( \eta_I \sim N(0_{p \times 1}, \frac{1}{2} \text{diag}(\lambda)) \). Also \( \text{cov}[\mathbf{n}] = C \text{cov}[\eta_R] C^T \), \( \text{pcov}[\mathbf{n}] = C \text{diag}(\lambda) C^T \), and \( \text{E}_z[\mathbf{n}] = \mu \).

**Proof:** It is obvious that the r.v.c. \( \mathbf{n} \) in Eq. (15) is complex normal, \( \text{cov}[\mathbf{n}] = C \text{cov}[\eta_R] C^T \), \( \text{pcov}[\mathbf{n}] = C \text{diag}(\lambda) C^T \), and \( \text{E}_z[\mathbf{n}] = \mu \). Thus, it remains to show that any complex normal r.v.c. can be given the form (15).

Let \( \mathbf{n} \) be a complex normal r.v.c. Without loss of generality assume it is zero mean. By Theorem [1] there exists a nonsingular matrix \( D \) such that \( \text{cov}[DN] = I_p \) and \( \text{pcov}[DN] = \text{diag}(\lambda) \). Let \( \mathbf{i} \eta_R \sim N(0_{p \times 1}, \frac{1}{2} D \text{diag}(\lambda)) \) and \( \mathbf{i} \eta_I \sim N(0_{p \times 1}, \frac{1}{2} D \text{diag}(\lambda)) \) be real independent r.v.c.s. Now \( \text{cov}[\mathbf{i} \eta_R + j \mathbf{i} \eta_I] = D(I_p + \frac{1}{2} \text{diag}(\lambda)) + \frac{1}{2} D(I_p - \frac{1}{2} \text{diag}(\lambda) = I_p \) and \( \text{pcov}[\mathbf{i} \eta_R + j \mathbf{i} \eta_I] = \text{diag}(\lambda) \). Hence \( DN \) and \( \mathbf{i} \eta_R + j \mathbf{i} \eta_I \) have the same second order structure. Since a zero mean complex normal r.v.c. is completely characterized by the covariance and the pseudo-covariance matrices, it follows \( DN = \mathbf{i} \eta_R + j \mathbf{i} \eta_I \), and the claim follows by setting \( C = D^{-1} \).

A complex normal r.v.c. \( \mathbf{n} \) such that \( C = I_p \) and \( \mu = 0_{p \times 1} \) in the representation (15), i.e., \( \mathbf{n} = \mathbf{i} \eta_R + j \mathbf{i} \eta_I \), is called standard complex normal with the circularity spectrum \( \lambda \). Clearly any centered and strongly uncorrelated complex normal r.v.c. is standard. Also, it is seen that any complex normal r.v.c. may be alternatively specified by the mean, the circularity spectrum, and the (inverse of) strong-uncorrelation matrix \( C \).

The previous decomposition allows the derivation of differential entropy of a complex normal r.v.c. in a closed form. Entropy \( h(\mathbf{n}) \) of an r.v.c. \( \mathbf{n} \) is defined as the entropy [30] of the real r.v.c. \( \mathbf{x} \). The following result has been implicitly derived in [31] without reference to circularity coefficients.

**Corollary 2:** The differential entropy \( h(\mathbf{n}) \) of a zero-mean complex normal r.v.c. \( \mathbf{n} \) with the circularity coefficients \( \lambda_k \neq 0 \) is given by

\[
h(\mathbf{n}) = \log(\text{det}(\pi e \text{cov}[\mathbf{n}]]) + \frac{1}{2} \sum_{k=1}^{p} \log(1 - \lambda_k^2). \tag{16}
\]

**Proof:** Let \( \mathbf{n} = C \mathbf{i} \eta \) be the decomposition given by Theorem 3. Now \( \text{det}(2 \text{cov}[\mathbf{i} \eta]) = \prod_{k=1}^{p} (1 - \lambda_k^2) \), and the differential entropy of real-valued normal r.v.c. [30] simplifies as

\[
h(\mathbf{n}) = \frac{1}{2} \log(\text{det}(2 \pi e \text{cov}[\mathbf{i} \eta])) = \frac{1}{2} \log(\text{det}(2 \pi e C \text{cov}[\mathbf{i} \eta] C^T)) = \frac{1}{2} \log(\text{det}(\pi e C \text{cov}[\mathbf{i} \eta] C^T)) + \frac{1}{2} \log(\text{det}(2 \text{cov}[\mathbf{i} \eta])) = \frac{1}{2} \log((\pi e)^{2p} \text{det}(CCC^T)) + \frac{1}{2} \log(\prod_{k=1}^{p} (1 - \lambda_k^2)) = \frac{1}{2} \log((\pi e)^{2p} \text{det}(\text{cov}[\mathbf{i} \eta])) + \frac{1}{2} \sum_{k=1}^{p} \log(1 - \lambda_k^2) = \frac{1}{2} \log((\pi e)^{2p} \text{det}(\text{cov}[\mathbf{i} \eta])) + \frac{1}{2} \sum_{k=1}^{p} \log(1 - \lambda_k^2) = \log(\text{det}(\pi e \text{cov}[\mathbf{i} \eta])) + \frac{1}{2} \sum_{k=1}^{p} \log(1 - \lambda_k^2)
\]

by the properties of Lemma 1.

Since the summation term on the right of Eq. (16) is always nonpositive and the entropy of real r.v.c.s with the given covariance is maximized for Gaussian r.v.c.s [30], it may be seen that the entropy of complex r.v.c.s with the given covariance is maximized for a narrow sense complex normal r.v.c. [11], i.e., for a complex normal r.v.c. with zero pseudo-covariance. Theorem 3 allows also an easy derivation of the c.f. of a complex normal r.v.c. [12], [14].

**Corollary 3:** The c.f. of a complex normal r.v.c. \( \mathbf{n} \) is given by

\[
\varphi_{\mathbf{n}}(z) = \exp\left(-\frac{1}{4} z^H \text{cov}[\mathbf{n}] z - \frac{1}{4} \text{Re}\{z^H \text{pcov}[\mathbf{n}] z^*\} + j \text{Re}\{z^H \text{E}_z[\mathbf{n}] z\}\right)
\]

\[
= \exp\left(-\frac{1}{4} \text{Re}\{z^H \text{cov}[\mathbf{n}] z + \text{pcov}[\mathbf{n}] z^*\} + j \text{Re}\{z^H \text{E}_z[\mathbf{n}] z\}\right).
\]

(18)

**Proof:** By Theorem 3 \( \mathbf{n} = C(\mathbf{i} \eta_R + j \mathbf{i} \eta_I) + \mu \). Let \( z = \mathbf{z}_R + j \mathbf{z}_I \in C^p \), and \( \mathbf{i} \eta = \mathbf{i} \eta_R + j \mathbf{i} \eta_I \). Now

\[
\varphi_{\mathbf{n}}(z) = \varphi_{\mathbf{n}}(z_R) = \exp\left(-\frac{1}{4} z_R^H (I_p + \text{diag}(\lambda)) z_R + z_I^T (I_p - \text{diag}(\lambda)) z_I \right)
\]

\[
= \exp\left(-\frac{1}{4} (z_R^H \mathbf{z}_R + z_I^T \mathbf{z}_I + z_R^H \text{diag}(\lambda) z_R - z_I^T \text{diag}(\lambda) z_I)\right)
\]

(19)
and by Eq. (10)
\[
\varphi(z) = \varphi_{C\bar{\eta}+\mu}(z) = \varphi_{C\bar{\eta}}(z) \exp(j \text{Re}\{\langle z, \mu \rangle\})
\]
\[
= \varphi_{C\bar{\eta}}(\mathbf{C}^H \mathbf{z}) \exp(j \text{Re}\{\langle z, \mu \rangle\})
\]
\[
= \exp(-\frac{1}{4}(z^H \mathbf{C} \mathbf{C}^H z + \text{Re}\{\mathbf{z}^T \mathbf{C}^* \text{diag}(\mathbf{\Lambda}) \mathbf{C}^H \mathbf{z}\}))
\]
\[
= \exp(-\frac{1}{4}(z^H \mathbf{C} \mathbf{C}^H z + \text{Re}\{\mathbf{z}^H \mathbf{C} \text{diag}(\mathbf{\Lambda}) \mathbf{C}^H \mathbf{z}\} + j \text{Re}\{z^H \mu\})
\]
(20)

Corollary B shows in particular that the second characteristic function \(\psi_2 = \log \varphi(z)\) of a complex r.v. \(\mathbf{x}\) is a second-order wide sense polynomial in variables \((z, z^*)\). Theorem 5 can be also used to derive the density function of a complex normal r.v. However, unlike the c.f., the density function of a wide sense normal r.v. does not appear to have a simple form. See [12] for expressions for the density function in terms of the covariance and the pseudo-covariance matrices. The following example essentially shows that in some cases the distribution of a standard complex normal r.v. is invariant to orthogonal transformations.

**Example 1:** Let the components of \(\bar{\mathbf{z}}\) be uncorrelated complex normal r.v. with the same circularity coefficient \(\lambda\). Now for a diagonal matrix \(\mathbf{A}\) the r.v.c. \(\mathbf{A}\bar{\mathbf{z}}\) is standard complex normal with the circularity spectrum \((\lambda, \ldots, \lambda)^T\), and for any (real-valued) orthonormal matrix \(\mathbf{O}\), \(\text{cov}[\mathbf{O}\mathbf{A}\bar{\mathbf{z}}] = \mathbf{O}\text{cov}[\mathbf{A}\bar{\mathbf{z}}] \mathbf{O}^H = \mathbf{O}\text{cov}[\mathbf{A}\bar{\mathbf{z}}] \mathbf{O}^H = \mathbf{O}I_p \mathbf{O}^T = I_p\) and \(\text{cov}[\mathbf{O}\mathbf{A}\bar{\mathbf{z}}] = \mathbf{O}\text{cov}[\mathbf{A}\bar{\mathbf{z}}] \mathbf{O}^T = \mathbf{O}\text{cov}[\mathbf{A}\bar{\mathbf{z}}] \mathbf{O}^T = \mathbf{O}(\lambda I_p) \mathbf{O}^T = \lambda I_p\). Therefore, the r.v.c. \(\mathbf{O}\mathbf{A}\bar{\mathbf{z}}\) is also standard complex normal.

**F. Darmois-Skitovich theorem for complex random variables**

One of the main characterization theorems for real r.v.s is the well-known Darmois-Skitovich theorem (see [4]). The theorem is fundamental for proving the identifiability of real ICA models [1], [5]. From here we extend the theorem to complex r.v.s.

The proofs of the complex Darmois-Skitovich theorem and the proof of a closely related characterization theorem (Theorem B in Section 5) are both based on a complex functional equation (Lemma 5 in Appendix III). The functional equation is an extension of the corresponding equation for real variables (see, e.g., Lemma 1.5.1 in [4]) to complex variables. Using the mapping \(\mathbf{z} \rightarrow \mathbf{z}^H \mathbf{C} \mathbf{C}^H \mathbf{z}\) may be easily seen to be a direct consequence of the real multivariate theorem [32] (see also [4], [33]). A direct proof is given in Appendix III for the sake of completeness.

The complex extension of Darmois-Skitovich theorem has exactly the same form as the real theorem with the wide sense complex normal r.v.s taking the role of real normal r.v.s. Hence, this theorem is an example where the analogy [22] between theories of narrow sense complex normal r.v.s and real normal r.v.s is broken.

**Theorem 4 (Complex Darmois-Skitovich):** Let \(s_1, \ldots, s_n\) be mutually independent complex r.v.s. If the linear forms (the r.v.s)
\[
x_1 = \sum_{k=1}^{n} \alpha_k s_k \text{ and } x_2 = \sum_{k=1}^{n} \beta_k s_k,
\]
(21)

where \(\alpha_k, \beta_k \in \mathbb{C}, k = 1, \ldots, n,\) are independent, then r.v.s \(s_k\) for which \(\alpha_k \beta_k \neq 0\) are complex normal.

**Sketch of the proof:** The complete proof is given in Appendix III and it follows the proof of the real-valued Darmois-Skitovich theorem (see [4]) with appropriate extensions to complex field. The idea is to consider two forms of the logarithm of the joint c.f. of \(x_1\) and \(x_2\) following from independence. This functional equation is only satisfied for wide sense polynomials showing that the r.v. \(x_1\) is complex normal. This is only possible if r.v.s \(s_k\) are complex normal.

Although narrow sense complex normal r.v.s had to be admitted to the complex Darmois-Skitovich theorem, it may still appear in the view of Corollary III that complex normal r.v.s appearing in the theorem cannot be completely arbitrary. That is, it may appear that some of the circularity coefficients of normal r.v.s should be equal. It is true if \(n = 2\). However, it is not generally true as it is shown in the next example.

**Example 2:** Let \(\bar{\mathbf{r}}_1 = (r_1, r_2, r_3)^T\) be standard complex normal r.v. with the circularity spectrum \(\lambda_1 = (\frac{5}{2}, \frac{1}{2}, \frac{1}{2})^T\). Then \(\bar{\mathbf{r}}_2 = \frac{1}{\sqrt{2}}(\frac{3}{2}, -\frac{1}{2}, \frac{1}{2})\bar{\mathbf{r}}_1\) is also standard complex normal r.v. with the circularity spectrum \(\lambda_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T\). Thus marginals of \(\bar{\mathbf{r}}_2\) are independent, and the Darmois-Skitovich theorem applies. However, the circularity spectrum of \(\bar{\mathbf{r}}_1\) is distinct. Notice also that by Example III the r.v. obtained from \(\bar{\mathbf{r}}_2\) by multiplying with any orthogonal matrix is also standard complex normal r.v. with the same circularity spectrum.

**III. COMPLEX ICA MODELS**

In this section, we show that complex ICA is actually a well-defined concept, and we establish theoretical conditions similar to the real-valued case [5]. In Section III-A the main definitions along with some illustrative examples are given. Also a crucial characterization theorem giving a connection between vector coefficients and complex normal r.v.s is proved. Finally, in sections III-B and III-C the conditions for separability, identifiability, and uniqueness of complex ICA models, respectively, are derived.

**A. Definitions and problem statement**

A general linear instantaneous complex-valued ICA model may be described by the equation
\[
\mathbf{z} = \mathbf{A}\bar{\mathbf{a}},
\]
(22)

where \((s_1, \ldots, s_m)^T = \bar{\mathbf{a}}\) are unknown complex-valued independent non-degenerate r.v.s, i.e., sources. \(\mathbf{A}\) is a complex constant \(p \times m\) unknown mixing matrix, \(p \geq 2\), and \(\mathbf{z} = (x_1, \ldots, x_p)^T\) are mixtures, i.e., the observed complex r.v. (sensor array output). The couple \((\mathbf{A}, \bar{\mathbf{a}})\) is called a representation of r.v. \(\mathbf{z}\). If no column in the mixing matrix \(\mathbf{A}\) is collinear with another column in the matrix, i.e., all columns are pairwise linearly independent, the representation is called reduced. All representations are assumed to be reduced throughout.
this paper. Furthermore, a reduced representation for the r.v.c. \( \bar{x} \) in the model (22) is called proper, if it satisfies all the assumptions made about the model.

The model of Eq. (22) is defined to be

(i) identifiable, or the mixing matrix is (essentially) unique, if in every proper representation \((A, \bar{s})\) and \((B, \bar{x})\) of \( \bar{x} \), every column of complex matrix \( A \) is collinear with a column of complex matrix \( B \) and vice versa,

(ii) unique if the model is identifiable and furthermore the source r.v.c.s \( \bar{s} \) and \( \bar{x} \) in different proper representations have the same distribution for some permutation up to changes of location and complex scale, and

(iii) separable, if for every complex matrix \( W \) such that \( W \bar{x} \) has \( m \) independent components, we have \( AP\bar{s} = W\bar{x} \) for some diagonal matrix \( A \) with nonzero diagonals and permutation matrix \( P \). Moreover, such a matrix \( W \) has to always exist.

It is completely possible for the model (22) to be identifiable but not unique nor separable as it is shown in the next example.

**Example 3:** As an example of a model which is identifiable but is not separable nor unique, consider independent non-normal r.v.c. s, \( k = 1, \ldots, 4 \). Let \( \eta_1, \eta_2, \) and \( \eta_3 \) be independent standard normal r.v.c.s with the same circularity coefficient. Then also r.v.c. \( \eta_1 + \eta_2 \) and \( \eta_1 - \eta_2 \) are independent. Now

\[
\begin{pmatrix}
  s_1 + s_3 + s_4 + \eta_1 + \eta_2 \\
  s_2 + s_3 - s_4 + \eta_1 - \eta_2
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 + \eta_1 \\ s_4 + \eta_2
\end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} s_1 + \eta_1 + \eta_2 \\ s_2 + \eta_1 - \eta_2 \\ s_3 \\ s_4
\end{pmatrix},
\]

which shows that the corresponding model can not be unique. However, it is identifiable. R.v.c.s of the form \( s + n \), where \( n \) is a normal r.v. independent of \( s \), are said to have a normal component.

It follows from the reduction assumption that the number of columns, i.e., the number of sources or the model order, is the same in every proper representation of \( \bar{x} \) in identifiable models. If \( W \) is a separating matrix, then linear manifolds of \( AP \) and \( W \) must coincide, and therefore \( p \geq \text{rank}(W) = \text{rank}(AP) = m \), i.e., there has to be at least as many mixtures as sources in a separable model. This fact also emphasizes that identifiability of the model (22) depends also on the linear operator structure, and since the linear operators defined on \( \mathbb{R}^{2p} \) and \( \mathbb{C}^p \) are not isomorphic, one can not simply consider real-valued model with twice the observation dimension when studying the complex ICA model (22). This is illustrated in the following example.

**Example 4:** By simply considering real-valued models with twice the dimension, it may actually seem that the complex separation is possible only under very strict conditions. Indeed, let \( x_k, k = 1, \ldots, 4 \), be independent real-valued r.v.c.s, and let \( A_1, A_2, B_1, \) and \( B_2 \) be \( 2 \times 2 \) nonsingular real matrices. Define \( \bar{s}_1 = A_1 (r_1, r_2)^T \) and \( \bar{s}_2 = A_2 (r_3, r_4)^T \). Now \( \bar{s}_1 \) and \( \bar{s}_2 \) are independent, but so are also \( \bar{y}_1 \) and \( \bar{y}_2 \),

\[
\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} = \begin{pmatrix} B_1 & 0_{2 \times 2} \\ 0_{2 \times 2} & B_2 \end{pmatrix} \begin{pmatrix} A_1^{-1} & 0_{2 \times 2} \\ 0_{2 \times 2} & A_2^{-1} \end{pmatrix} \begin{pmatrix} \bar{s}_1 \\ \bar{s}_2 \end{pmatrix},
\]

for any permutation matrix \( P \). However, \( \bar{y}_1 \) and \( \bar{y}_2 \) are mixtures of \( \bar{s}_1 \) and \( \bar{s}_2 \) for many permutations \( P \).

The previous example is easily generalized to the ICA models that have multidimensional independent sources, i.e., one is looking for independent multidimensional subspaces. The example shows that models can not be identified or separated without additional constraints on the internal dependency structure of the sources or the allowed mixing matrices.

Since linear operators in complex and real spaces are not isomorphic, the classes of separable source r.v.c.s are not the same. That is, some source r.v.c.s considered in complex mixtures can be separated although their real-valued representations in real mixtures can not. This is shown in the next example.

**Example 5:** Let \( \eta_1, \ldots, \eta_{2m} \) be independent standard zero mean unit variance real Gaussian r.v.c.s. Define

\[
\tilde{\eta} = \left( \frac{1}{\sqrt{m+1}}(\sqrt{m}\eta_1 + j\eta_{m+1}), \frac{1}{\sqrt{m}}(\sqrt{m-1}\eta_2 + j\eta_{m+2}), \ldots, \frac{1}{\sqrt{2}}(\eta_m + j\eta_{2m}) \right).
\]

Now it is easily seen that \( \tilde{\eta} \) is a standard normal r.v.c with the distinct circularity spectrum \( \lambda[\tilde{\eta}] = (\frac{m-1}{m+1}, \frac{m-2}{m+1}, \ldots, 0)^T \). If \( \tilde{\eta}_k \) is taken as the source r.v.c. in the real-valued ICA model, i.e., \( \tilde{y} = B\tilde{\eta}_k \) and \( B \) is a \( 2p \times 2m \) real-valued matrix, \( p \geq m \), the model is not separable [5]. However, the complex model involving \( \tilde{\eta} \) itself, i.e., \( \bar{x} = A\tilde{\eta} \) and \( A \) is a \( p \times m \) complex-valued matrix, is separable by Corollary 1.

The following characterization theorem is the base of the identifiability and uniqueness theorems. It is an extension of a real theorem [4, Theorem 10.3.1] to the complex case. The idea of the proof is similar to the proof of Darmois-Skitovich theorem, and the proof given follows loosely that of the real counterpart with appropriate complex extensions.

**Theorem 5:** Let \((A, \bar{s})\) and \((B, \bar{x})\) be two reduced representations of a \( p \)-dimensional complex r.v.c. \( \bar{x} \), where \( A \) and \( B \) are constant complex matrices of dimensions \( p \times m \) and \( p \times n \), respectively, and \( \bar{s} = (s_1, \ldots, s_m)^T \) and \( \bar{x} = (x_1, \ldots, x_n)^T \) are complex r.v.c.s with independent components. Then the following properties hold.

(i) If the \( k \)th column of \( A \) is not collinear with any column of \( B \), then the r.v. \( s_k \) is complex normal.

(ii) If the \( k \)th column of \( A \) is collinear with the \( l \)th column of \( B \), then the logarithms of the c.f.s of r.v.c.s \( s_k \) and \( x_l \) differ by a wide sense polynomial in a neighborhood of the origin.

**Proof:**

(i) By Lemma 4 (see Appendix III), there exists a \( 2 \times p \) matrix \( C \) such that the \( k \)th column of \( D_1 = CA \) is not collinear with any other column of \( D_1 \), or with any column of \( D_2 = CB \). Then \( C\bar{x} = D_1\bar{s} = D_2\bar{x} \), and
applying Lemma 8(iii) (see Appendix III), it is seen that the r.v. $s_k$ is complex normal.

(ii) By definitions the $k$th column of $A$, say $\alpha$, is collinear only with the $l$th column of $B$, say $\beta$. Therefore by Lemma 7(see Appendix III), there exists a $2 \times p$ matrix $C$ such that the $k$th column of $D_1 = CA$ is not collinear with any other columns of $D_1$, or with any column of $D_2 = CB$ except possibly the $l$th. Furthermore, since $C\alpha = C(c\beta) = c(C\beta)$ for some $c \in \mathbb{C}$, it is seen that $(D_1, \bar{s})$ and $(D_2, \bar{x})$ are reduced representations of $C\bar{x}$ such that Lemma 8(iii) gives the claim.

B. Separability

ICA is commonly used as a Blind Source Separation method, where the problem is to extract the original signals from the observed linear mixture. Therefore, separability of the ICA model is an important issue. The separability theorem for the complex ICA model below may be surprising, since it allows also separation of some complex normal mixtures.

Theorem 6 (Separability): The model of Eq. (22) is separable if and only if the complex mixing matrix $A$ is of full column rank and there are no two complex normal source r.v.s with the same circularity coefficient.

Proof: Suppose the model is separable. Since $m = \text{rank}(WA) \leq \text{rank}(A) \leq m$, the mixing matrix $A$ is of full column rank $m$. If there were two complex normal source r.v.s with the same circularity coefficient, by Example 1 in Section II-E, there would exist matrices that produce $m$ independent components but which are not diagonal matrices for any permutation of the columns.

To the other direction, suppose the mixing matrix $A$ is of full column rank and there are no two complex normal source r.v.s with the same circularity coefficient. Now $A^\#$, where the superscript $\#$ denotes the Moore-Penrose generalized inverse [27], is a separating matrix. Suppose $W$ is a matrix such that $W\bar{x}$ has $m$ independent components. If $WA$ is not of the form $AP$, then there exist at least two columns such that they both contain at least two nonzero elements. By Lemma 10 (see Appendix III) there can not exist only one such column since the sources are nondegenerate. Assume without loss of generality that the first $l$ columns $\beta_k, k = 1, \ldots, l \leq m$, of $WA$ are columns with at least two nonzero elements, and denote the corresponding matrix of rank $l$ by $B = (\beta_1, \ldots, \beta_l)$. By Theorem 4 the r.v. $s_k$ corresponding to the column $\beta_k, k = 1, \ldots, l$, is complex normal, and we assume, without loss of generality, that the r.v. $\eta_1 = (s_1, \ldots, s_l)^T$ is standard complex normal. By Theorem 11 (see Appendix III) all components of $\bar{s}_2 = B\eta_2$ are complex normal, and by Lemma 3 (see Appendix III) all components of $\bar{s}_2$ are independent. Choose any $l$ rows of $B$ such that the corresponding submatrix $\bar{B}$ is of rank $l$, and $\bar{B}$ contains a row with two nonzero elements. Since $\bar{B}$ is not diagonal for any permutation by construction, $\eta_1$ is standard, and $\bar{s}_2$ has independent components, it follows from Corollary 1 that $\eta_1$ cannot have a distinct circularity spectrum, which is a contradiction. Therefore, $WA$ is of the form $AP$, and the model is separable.

Remark 2: If the source $\bar{s}$ has finite second order statistics and the circularity spectrum $\lambda[\bar{s}]$ is distinct, then the separation can be achieved by simply performing the strong-uncorrelating transform by Corollary 1. In this case, there is no additional restrictions on the distribution of the source r.v.s, and therefore some normal r.v.s can be also separated. An example of such a mixture is seen in Example 5.

C. Identifiability

Identifiability considers reconstruction of the mixing matrix.

- This is useful in some problems, where the immediate interest may not be in the sources themselves but in how they were mixed (e.g., channel matrix in MIMO communications).

Theorem 7 (Identifiability): The model of eq. (22) is identifiable, if

(i) no source r.v. is complex normal, or

(ii) $A$ is full column rank and there are no two complex normal source r.v.s with the same circularity coefficient.

Proof:

(i) Since there are no complex normal r.v.s, by Theorem 5, every column has to be collinear with exactly a column in another proper representation, i.e., the model is identifiable.

(ii) Let $(A, \bar{s})$ and $(B, \bar{x})$ be proper representations of $\bar{x}$. Since the model is separable by Theorem 6 and $A^\#$ is a separating matrix, $A^# B = PA$ for a permutation matrix $P$ and a diagonal matrix $\Lambda$. By the uniqueness of the generalized inverse, it follows $APA = B$.

There is a striking contrast between the two cases in Theorem 7. Namely, if there are more sources than mixtures not a single normal r.v. is allowed whereas in the other case all source r.v.s can be normal. The following example shows the reason why we can not allow a single normal r.v. for identifiability when there are more sources than sensors.

Example 6: Consider independent non-normal r.v.s $s_1, s_2$, and standard normal r.v.s $\eta_1$ and $\eta_2$ with the same circularity coefficient. Now

$$\bar{x} = \begin{pmatrix} s_1 + s_2 + 2\eta_1 \\ s_1 + 2\eta_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 + 2\eta_1 \\ 2\eta_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} s_1 + \eta_1 + \eta_2 \\ s_2 \\ \eta_1 - \eta_2 \end{pmatrix}$$

and the last column shows that the model is not identifiable.

It is evident from the previous example and from the separation theorem that another identifiability condition could be formulated by essentially allowing a single normal r.v. and not allowing other source r.v.s to have normal components with the same circularity coefficient. However, this condition is unnecessarily complicated. Therefore, it is not stated in a formal manner.
D. Uniqueness

Uniqueness considers the case where one is interested not only in the mixing matrix but also in the distribution of the sources.

Theorem 8 (Uniqueness): The model of Eq. (22) is unique if either of the following properties hold.

(i) The model is separable.
(ii) All c.f.s of source r.v.s are analytic (or all c.f.s are non-vanishing), and none of the c.f.s has an exponential factor with a wide sense polynomial of degree at least two, i.e., no source r.v. has the c.f. \( \varphi(z) = \varphi_1(z) \exp(\mathcal{P}(z, z^*)) \) for a c.f. \( \varphi_1(z) \) and for some wide sense polynomial \( \mathcal{P}(z, z^*) \) of degree at least two.

Proof:

(i) Let \((A, \tilde{s})\) and \((B, \tilde{x})\) be proper representations of \( \tilde{x} \). By Theorem 7, the model is identifiable, and therefore \( APA = B \) for a permutation matrix \( P \) and a diagonal matrix \( A \). Now \( \tilde{s} = A^\# \tilde{x} = A^\# B \tilde{x} = P \Lambda \tilde{x} \).

(ii) There cannot be any complex normal r.v.s, and therefore the model is identifiable by Theorem 7. Now the logarithms of the c.f.s of the source variables in two proper representations differ by a wide sense polynomial by Theorem 5. However, by the assumption this wide sense polynomial can be at most of degree 2, i.e., the source variables have the same distribution up to changes of location and complex scale.

A nonunique but identifiable mixture was described in Example 3. By slightly restricting the allowed mixing matrices, it is possible in the real case to obtain more classes of unique models [5]. Further work is needed to determine if those theorems can be extended to the complex case.

IV. CONCLUSION

In this paper conditions for separability, identifiability, and uniqueness of complex-valued linear ICA models are established. Both circular and noncircular complex random vectors are covered by the results. So far these conditions have been known for real random vectors only. The conditions for identifiability, and uniqueness are sufficient and the separability condition is also found to be necessary. In order to show these results, a proof of complex extension of the Darmois-Skitovitch Theorem is constructed. Some second-order properties and characterizations of linear forms of complex random vectors are reviewed and new results found in the process of proving the theorem. As a by-product of establishing the conditions, a theorem on differential entropy for complex normal random vectors is proved and a slightly surprising result about separating complex Gaussian sources is found.

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APPENDIX I

Proof of Lemma 4

Proof of Lemma 4: By Theorem 1 there exist nonzero constants \( a, b \in \mathbb{C} \) such that r.v.s \( s = ax \) and \( r = by \) are strongly uncorrelated.

(i) Since \( \text{cov}[s] + \text{cov}[r] = 0 \), \( 0 \leq \lambda[x] = \lambda[s] = \text{cov}[s] - \text{cov}[r] = 1 - 2 \text{cov}[r] \leq 1 \). Also \( \phi'(x) = 2 \), and thus by uniqueness \( \lambda[|x|] = \lambda[s] = \lambda[x] \).

Furthermore

\[
\lambda[x] = \text{pcov}[s] = \frac{\text{pcov}[|s|]}{\text{cov}[s]} = \frac{\text{pcov}[|ax|]}{\text{cov}[ax]}
\]

\[
= \frac{|a|^2 \text{pcov}|x|}{|a|^2 \text{cov}|x|} = \frac{|a|^2 \text{pcov}[x]}{|a|^2 \text{cov}[x]} = \frac{\text{pcov}[x]}{\text{cov}[x]}
\]

(27)

(ii) \( \lambda[x] = 1 - 2 \text{cov}[r] = 1 \) if and only if \( \text{cov}[s] = 0 \).

(iii) Suppose \( \lambda[x] \geq \lambda[y] \). Using the first part of the lemma for an r.v. \( x + y \), uncorrelateness, and the triangle inequality, we have

\[
\lambda[x + y] = \frac{\text{pcov}[x + y]}{\text{cov}[x + y]} = \frac{\text{pcov}[x] + \text{pcov}[y]}{\text{cov}[x] + \text{cov}[y]}
\]

\[
= \frac{\text{pcov}[\frac{s}{a}] + \text{pcov}[\frac{r}{b}]}{\text{cov}[s] + \text{cov}[r]} = \frac{\frac{1}{a} |a|^2 \text{pcov}[s] + \frac{1}{b} |b|^2 \text{pcov}[r]}{|a|^2 |a|^2 + |b|^2 |b|^2}
\]

\[
= \frac{\frac{1}{a} |a|^2 \lambda[x] + \frac{1}{b} |b|^2 \lambda[y]}{|a|^2 |a|^2 + |b|^2 |b|^2} \leq \lambda[x],
\]

(28)

which proves the inequality.

If both r.v.s \( x \) and \( y \) are second order circular, then clearly the equality holds in (28). Now suppose the condition for the equality holds in the noncircular case, and let \( \lambda = \lambda[x] = \lambda[y] \) and \( \theta = \text{Arg}(\text{pcov}[x]) = \text{Arg}(\text{pcov}[y]) \).

Then

\[
\lambda[x + y] = \frac{\text{pcov}[x] + \text{pcov}[y]}{\text{cov}[x] + \text{cov}[y]}
\]

\[
= \frac{\lambda \text{cov}[x] e^{\theta} + \lambda \text{cov}[y] e^{\theta}}{\text{cov}[x] + \text{cov}[y]}
\]

\[
= \frac{\lambda |e^{\theta}| \text{cov}[x] + \lambda |e^{\theta}| \text{cov}[y]}{\text{cov}[x] + \text{cov}[y]} = \lambda.
\]

(29)

To the other direction, the last inequality in (28) holds with the equality iff \( \lambda[x] = \lambda[y] \). If now \( \lambda[x] \neq 0 \), then the triangle inequality in (28) holds with the equality iff

\[
0 < \frac{b^2}{a^2} = \frac{b^2 \text{pcov}[s]}{a^2 \text{pcov}[r]} = \frac{\text{pcov}[x]}{\text{pcov}[y]}
\]

(30)

Hence \( \text{Arg}(\text{pcov}[x]) = \text{Arg}(\text{pcov}[y]) \) by the polar forms of \( \text{pcov}[x] \) and \( \text{pcov}[y] \).
APPENDIX II
PROOF OF THE COMPLEX DARMOIS-SKITOVICH THEOREM AND RELATED THEOREMS

The following theorem is a direct consequence of the multivariate version of the real Marcinkiewicz theorem. The theorem shows essentially that a complex normal r.v. is the only r.v. whose second c.f. is a wide sense polynomial.

**Theorem 9 (Complex Marcinkiewicz):** If in some neighborhood of zero the c.f. \( \varphi_x \) of a complex r.v. \( x \) admits the representation
\[
\varphi_x(z) = \exp(\mathcal{P}(z, z^*)),
\]
where \( \mathcal{P} \) is a wide sense polynomial, then the r.v. \( x \) is complex normal.

**Proof:** Fix \( z_0 \in \mathbb{C} \), and define a c.f. \( \varphi_0(t) \equiv \varphi_x(tz_0) = \exp(\mathcal{P}(tz_0, tz_0^*)) \) for \( t \in \mathbb{R} \). Then for some \( \varepsilon > 0 \), \( \log \varphi_0(t) \) is a polynomial in \( t \), \( |t| < \varepsilon \). Therefore, by a version of α-decomposition theorem (see [34, Theorem 7.4.2]) the relation is valid for all \( t \) and \( \varphi_0(t) \) is normal. Since \( z_0 \) is assumed to be arbitrary, it follows that the equation (31) is valid for all \( z \). By the last property of Lemma 1 \( \mathcal{P}(z, z^*) \) is a polynomial in \( z_r \), and the claim follows from the multivariate (bivariate) Marcinkiewicz’s theorem (e.g., [29, Theorem 3.4.3]).

Also the well-known Cramer’s theorem has a direct complex counterpart.

**Theorem 10 (Complex Cramer):** If \( s_1 \) and \( s_2 \) are independent r.v.s such that \( s_1 + s_2 \) is a complex normal r.v., then each of the r.v.s \( s_1 \) and \( s_2 \) is complex normal.

**Proof:** This is a direct corollary to the real multivariate Cramer’s theorem (e.g., [34, Theorem 6.3.2]).

**Lemma 5:** Consider the equation, assumed valid for \( |s_1|, |s_2| < \varepsilon \),
\[
\sum_{k=1}^{p} \psi_k(z_1 + c_k s_2) = h_1(z_1) + h_2(z_2),
\]
where \( \psi_k \), \( k = 1, \ldots, p \), \( h_1 \), and \( h_2 \) are continuous complex-valued functions of complex variables and the nonzero complex numbers \( c_k \), \( k = 1, \ldots, p \), are distinct. Then all the functions in (32) are wide sense polynomials in \( (z, z^*) \) of degree not exceeding \( p \).

**Proof:** Let \( d_k^{(1)} = (1 - \frac{c_k}{cp})b_1 \). Now, for small enough \( b_1 \), we have
\[
\sum_{k=1}^{p} \psi_k(z_1 + b_1 + c_k s_2 - \frac{b_1}{cp}) = \sum_{k=1}^{p} \psi_k(z_1 + d_k^{(1)} + c_k s_2) = h_1(z_1 + b_1) + h_2(z_2 - \frac{b_1}{cp})
\]
by substituting \( (z_1 + b_1) \) for \( z_1 \) and \((z_2 - \frac{b_1}{cp}) \) for \( z_2 \) in (32). Subtracting (32) from (33), we obtain
\[
\sum_{k=1}^{p-1} \Delta_k^{(1)} \psi_k(z_1 + c_k s_2) = \frac{1}{b_1} h_1(z_1) + \frac{1}{cp} h_2(z_2),
\]
where \( \Delta_k \) is the general difference operator defined by
\[
\Delta_k f(z) = f(z + a_k) - f(z)
\]
and
\[
\sum_{k=0}^{n+1} a_k f(z) = \sum_{k=0}^{n} \Delta_k [f(z) + f(z)],
\]
for any constants \( a_k \). Equation (34) is of the same form as (32) except the number of the terms in the sum is lower. Let \( d_k^{(2)} = (1 - \frac{c_k}{cp})b_2 \). Again by substituting and subtracting, we obtain from (34) the equation
\[
\sum_{k=1}^{p-2} \Delta_k^{(2)} \psi_k(z_1 + c_k s_2) = \frac{2}{b_1 b_2} h_1(z_1) + \frac{2}{cp} \frac{a_1}{a_2} h_2(z_2),
\]
Continuing the process, we end up with the equation
\[
\frac{1}{b_1 b_2 \cdots b_{p-1}} h_1(z_1) + \frac{1}{cp} \frac{a_1}{a_2} \cdots \frac{a_{p-1}}{a_p} h_2(z_2).
\]
This is the generalized Cauchy’s equation for complex variables [35] showing that \( \Delta_k = \frac{1}{b_1 b_2 \cdots b_{p-1}} \psi_k(z_1 + c_k s_2) \) for some constants \( a_k \in \mathbb{C} \). Since coefficients \( b_k \) are arbitrary in the neighborhood of zero, and by continuity, the difference operator structure [36] shows that \( \psi_k(z) \) is a wide sense polynomial in \((z, z^*)\) of degree not exceeding \( p \). By renumbering, the same is obtained for \( \psi_k(z) \), \( k = 1, \ldots, p \), and thus also for \( h_1(z) \) and \( h_2(z) \).

**Proof of Theorem 11** The joint c.f. of \( (x_1, x_2)^T \) is given as
\[
\varphi_{x_1, x_2}(z_1, z_2) = E_{x_1, x_2} \exp(j \mathcal{P}(z_1, z_2)^T (x_1, x_2)^T)
\]
\[
= E_{x_1, x_2} \exp(j \mathcal{P}(z_1, z_2)^T, \sum_{k=1}^{n} (\alpha_k s_1^k, \beta_k s_2^k)^T)
\]
\[
= E_{x_1, x_2} \exp(j \sum_{k=1}^{n} \mathcal{P}(\alpha_k s_1^k + \beta_k s_2^k)^T)
\]
\[
= \prod_{k=1}^{n} E_{s_k} \exp(j \mathcal{P}(\alpha_k s_1^k + \beta_k s_2^k))
\]
\[
= \prod_{k=1}^{n} \varphi_{s_k}(\alpha_k s_1^k + \beta_k s_2^k),
\]
Thus by combining equations (38) and (39), we get
\[
\prod_{k=1}^{n} \varphi_{s_k}(\alpha_k s_1^k + \beta_k s_2^k) = \prod_{k=1}^{n} \varphi_{s_k}(\alpha_k s_1^k) \prod_{k=1}^{n} \varphi_{s_k}(\beta_k s_2^k).
\]
As always, there exists a neighborhood of zero such that all c.f.s in Eq. (40) are nonzero. Let \( x_k = \alpha_k s_k \) and \( c_k = \beta_k / \alpha_k \) for \( \alpha_k \neq 0 \), and \( c_k = \beta_k \) for \( \alpha_k = 0 \). Then, by Eq. (40), we can rewrite Eq. (40) for some positive \( \varepsilon > |z_1|, |z_2| \) by setting
\[
\psi_k = \log \varphi_{x_k}
\] as
\[
\sum_{k=1}^{l} \psi_k(z_1 + c_k z_2) = \sum_{k=1}^{l} \psi_k(z_1) + \sum_{k=1}^{l} \psi_k(c_k z_2),
\] (41)
where it is assumed without loss of generality that \( l \) first r.v.s \( x_k, k = 1, \ldots, l \), are such that \( \alpha_k \beta_k \neq 0 \), and therefore components \( \psi_k, k > l \), cancel out. By combining functions \( \psi_k \) with the equal arguments to a single function \( \psi \) and renumbering, Eq. (41) may be rewritten as
\[
\sum_{k=1}^{l} \psi_k(z_1 + c_k z_2) = \sum_{k=1}^{l} \psi_k(z_1) + \sum_{k=1}^{l} \psi_k(c_k z_2)
\] (42)
such that numbers \( c_k, k = 1, \ldots, q \leq l \), are distinct. Therefore, \( \sum_{k=1}^{l} \psi_k(z_1) \) is a wide sense polynomial by Lemma 5. By Theorem 9 the r.v. \( \sum_{k=1}^{l} x_k \) is complex normal. Thus by Theorem 10 each r.v. \( x_k \), and hence each r.v. \( s_k, k = 1, \ldots, l \), is complex normal.

**APPENDIX III**

**ADDITIONAL CHARACTERIZATION LEMMAS**

**Lemma 6:** Let \( \alpha_1, \ldots, \alpha_m \) be given nonzero vectors of an inner product space. Then there exist a vector \( \beta \), which is not orthogonal to any of the given vectors.

**Proof:** Suppose \( \beta \) is not orthogonal to any \( \alpha_l, l = 1, \ldots, k-1 \), but is orthogonal to \( \alpha_k \). Then a scalar \( c \in \mathbb{C} \) can be chosen such that \( \langle \beta, \alpha_l \rangle = -c \langle \alpha_k, \alpha_l \rangle \) for all \( l \leq k \). Now the vector \( \beta = \beta + c \alpha_k \) is not orthogonal to any \( \alpha_l, l \leq k \).

Since \( \alpha_1 \) is nonzero, \( \beta_1 = \alpha_1 \) is not orthogonal to \( \alpha_1 \). Choose \( \beta_2 = \beta_1 + c_2 \alpha_2 \), where \( c_2 \) is a scalar as above if \( \beta_1 \) is orthogonal to \( \alpha_2 \), and \( c_2 = 0 \) otherwise. By iterating the procedure \( m-1 \) times, it is seen that \( \beta_m \) is a required type of vector.

**Lemma 7:** Let \( \alpha_1, \ldots, \alpha_m \) be given \( p \)-dimensional nonzero complex vectors such that \( \alpha_1 \) is not collinear with any \( \alpha_k \), \( k \neq 1 \). Then there exists a \( 2 \times p \) matrix \( C \) such that \( C \alpha_1 \) is not collinear with any \( C \alpha_k \), \( k \neq 1 \).

**Proof:** Denote \( \alpha_k = (\alpha_k)_1, \ldots, \alpha_k)_p \), \( k = 1, \ldots, m \). Without loss of generality we assume that the coefficients \( \alpha_k \), \( k = 1, \ldots, m \), are either zero or one. Furthermore, we may take \( \alpha_1 = 1 \) by permuting the indices.

Suppose \( \alpha_1 \) is not collinear with \( \alpha_k \), i.e., \( \alpha_1 \neq \alpha_k \), for any \( k \neq 1 \). Define
\[
C = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
\beta_1 & \beta_2 & \cdots & \beta_p
\end{pmatrix},
\] (43)
where \( \beta = (\beta_1, \ldots, \beta_p) \) is a vector such that
\[
\langle \beta, \alpha_1 - \alpha_k \rangle \neq 0, \quad k = 2, \ldots, m.
\] (44)
By Lemma 5 such a vector \( \beta \) exists. Now vectors \( C \alpha_k \) are again such that the first component is either zero or one. Thus \( C \alpha_1 \) can be collinear with another vector \( C \alpha_k \) only if \( \alpha_{k1} = 1 \). But then the difference
\[
C \alpha_1 - C \alpha_k = \begin{pmatrix}
1 \\
\beta_1^T \alpha_1
\end{pmatrix} - \begin{pmatrix}
1 \\
\beta_1^T \alpha_k
\end{pmatrix} = \begin{pmatrix}
0 \\
\langle \beta, (\alpha_1 - \alpha_k) \rangle
\end{pmatrix}
\] (45)
is not zero by construction. Thus \( C \alpha_1 \) is not collinear with any \( C \alpha_k \), \( k \neq 1 \), and \( C \) is a required type of matrix.

**Lemma 8:** Let \( A \) and \( k \) be two reduced representations of a 2-dimensional complex r.v. \( \bar{x} \), where \( A \) and \( B \) are constant complex matrices of dimensions \( 2 \times m \) and \( 2 \times n \) respectively, and \( \bar{z} = (z_1, \ldots, z_m)^T \) and \( \bar{x} = (x_1, \ldots, x_n)^T \) are complex r.v.s with independent components. Then the following properties hold.

(i) If the \( k \)th column of \( A \) is not collinear with any column of \( B \), then the r.v. \( s_k \) is complex normal.

(ii) If the \( k \)th column of \( A \) is collinear with the \( l \)th column of \( B \), then the logarithms of the c.f.s of \( s_k \) and \( x_l \) differ by a wide sense polynomial in a neighborhood of the origin.

**Proof:**

(i) Without loss of generality we assume that matrices \( A \) and \( B \) are scaled such that the first rows consist only of zeros and ones. This amounts only to the scale of r.v.s \( z_1 \) and r.v.s \( x_1 \). Furthermore, since the components of \( \bar{x} \) can be interchanged if necessary, the first entry of the \( k \)th column of \( A \) can be taken to be one.

As always, there exists a neighborhood \( \varepsilon > 0 \) of zero such that all c.f.s are nonzero, and the logarithms of c.f.s are well-defined. Therefore for \( z = (z_1, z_2)^T \in \mathbb{C}^2 \), \( |z_1| < \varepsilon, |z_2| < \varepsilon \), we have the properties (10) and (11) that
\[
\log \varphi_{z}(x) = \log \varphi_{z}(A^T z) = \log \varphi_{\tilde{z}}(B^T z)
\] (46)
\[
= \sum_{i=1}^{m} \log \varphi_{z_i}(\alpha_{i1} z_1 + \alpha_{i2} z_2)
\] (47)
where \( A = (\alpha_{i1}), B = (\beta_{i1}, \ldots, \gamma_{i}) \). Let \( q \) be the number of different noncollinear columns with nonzero coefficients in \( A \) and \( B \) other than the \( k \)th column of \( A \). Now substituting (47) from (46), and combining the terms with equal nonzero coefficient arguments to functions \( h_l \), and with one zero coefficient to \( f \) and \( g \), respectively, we get an equation of the form
\[
\log \varphi_{z_k}(z_1 + \alpha_{2k} z_2) + \sum_{l=1}^{q} h_l(z_1 + \gamma_l z_2) = f(z_1) + g(z_2)
\] (48)
if \( \alpha_{2k} \neq 0 \), and of the form
\[
\sum_{l=1}^{q} h_l(z_1 + \gamma_l z_2) = \log \varphi_{z_k}(z_1) + g(z_2)
\] (49)
if \( \alpha_{2k} = 0 \). Numbers \( \alpha_{2k}, \gamma_1, \ldots, \gamma_q \) are now distinct, and then by Lemma 5, \( \log \varphi_{z_k} \) must be a wide sense polynomial in \( (z, z^*) \) of degree not exceeding \( q \). Thus by Theorem 9 the r.v. \( s_k \) is complex normal.
(ii) By definitions of representations, $k$th column of $A$ is collinear only with the $l$th column of $B$. Thus one of the $h$’s in the proof of part (i) is the difference the logarithms of the c.f.s of $s_2$ and $x_1$, and the claim follows from Lemma 8.

**Lemma 9:** Suppose independent complex r.v.s $s_1$ and $s_2$ are independent of complex normal r.v.s $n_1$ and $n_2$. If $s_1 + n_1$ is independent of $s_2 + n_2$, then also $n_1$ and $n_2$ are independent.

**Proof:** Since the r.v. $(s_1, s_2)^T$ is independent of the r.v. $(n_1, n_2)^T$, the joint c.f. can be written as

$$
\varphi_{s_1+n_1, s_2+n_2}(z_1, z_2) = \varphi_{s_1, s_2}(z_1, z_2) \varphi_{n_1, n_2}(z_1, z_2) \quad (50)
$$

On the other hand, using the independence of $s_1+n_1$ and $s_2+n_2$, we have

$$
\varphi_{s_1+n_1, s_2+n_2}(z_1, z_2) = \varphi_{s_1+n_1}(z_1) \varphi_{s_2+n_2}(z_2) \quad (51)
$$

and therefore

$$
\varphi_{s_1}(z_1) \varphi_{s_2}(z_2) \varphi_{n_1, n_2}(z_1, z_2) = \varphi_{s_1}(z_1) \varphi_{s_2}(z_2) \varphi_{s_1}(z_1) \varphi_{s_2}(z_2) \varphi_{n_1, n_2}(z_1, z_2) \quad (52)
$$

Then, in some neighborhood of zero, all c.f.s in \ref{52} are nonzero, and we have

$$
\varphi_{s_1, s_2}(z_1, z_2) = \varphi_{s_1}(z_1) \varphi_{s_2}(z_2) \quad (53)
$$

in the neighborhood. By the $\alpha$-decomposition theorem [34, Theorem 7.4.2], the equation valid for all $z_1$ and $z_2$, i.e., $n_1$ and $n_2$ are independent.

**Lemma 10:** If complex r.v.s $n$ and $s$ are independent and $n+s$ is independent of $n$, then $n$ is degenerate (i.e., a constant).

**Proof:** By Theorem 8, the r.v. $n$ is complex normal. As in the proof of Lemma 9 it follows that the equation

$$
\varphi_n(z_1 + z_2) = \varphi_n(z_1) \varphi_n(z_2) \quad (54)
$$

is satisfied in a neighborhood of zero. This is only possible if $n$ is a degenerate complex normal r.v., i.e., a complex normal r.v. with zero variance.
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