Riemann-Cartan space time $U_4$ is considered here. It has been shown that when we link topological Nieh-Yan density with the gravitational constant then we get Einstein-Hilbert Lagrangian as a consequence.

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1. Introduction

A generic Riemann manifold is endowed with two fundamental and independent entities: a metric and an affine connection. If the affine connection is not assumed to be a function of the metric, then the local geometry is endowed with two independent tensors, curvature and torsion. It was Einstein’s point of view - that torsion would be an unnecessary addition which was not required for the most economical and successful theory of space time. Cartan on the other hand refused to accept it on the ground that the two notions are logically independent and therefore Einstein’s proposal was a particular case. For more details on this controversy one can see the letters exchanged between them between 1929 and 1932 [1]. In recent years, much more abstract geometrical frameworks are modelled to handle many pertinent issues of particle physics especially for a consistent quantum theory of gravitation. In this setup torsion does not look a strange idea.

Curvature plays an important role in the characterization of the topological structure of the manifold. It is a remarkable result of differential geometry that certain global features of a manifold are determined by some local functionals of its intrinsic geometry. Pontryagin and Euler classes are two well known examples in four dimension. Values of these invariants depend on the global properties of the manifold. These invariants are expected to be related to the global values of some physical observables.

The topological features associated with the gauge orbit space of a non-Abelian gauge theory when the topological $\theta$-term is introduced in the Lagrangian corresponds to a vortex line, and the gauge orbit space appears to be multiply connected in nature. This has an implication in the loop space formalism, in the sense that the latter involves nonlocality and there is no way we could arrive at a corresponding continuum
limit. In the gravity with out the metric formalism, it has been observed that the \( \theta \)-term in the Lagrangian effectively corresponds to the introduction of torsion and it is found to be associated with the vortex line and the cosmological constant. In the following section we shall review some topological aspects of Riemann-Cartan space time. In section 3 we shall try to derive the metric considering the connection as the primary object and then, after interpreting the gravitational constant as connected to a diffeomorphism invariant density of the manifold, will get the Einstein-Hilbert Lagrangian as a consequence. Section 4 is the section of discussion.

2. Axial vector torsion and topological invariants

Cartan’s structural equations for a Riemann-Cartan space-time \( U_4 \) are given by

\[
T^a = de^a + \omega^a_b \wedge e^b, \tag{1}
\]
\[
R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \tag{2}
\]

here \( \omega^a_b \) and \( e^a \) represent the spin connection and the local frames respectively.

From these two expressions, curvature seems to be more fundamental than torsion. Definition (2) requires only connection whereas definition (1) requires both connection and local frame. But, since on any smooth metric manifold a local frame is necessarily always defined, torsion can exist even if the connection vanishes. This implies torsion and curvature should be treated on a similar footing. Torsion appears rather naturally in the commutator of two covariant derivatives for the group of diffeomorphisms of a manifold in a coordinate basis,

\[
[\nabla_\mu, \nabla_\nu]V^A = -T^A_{\mu\nu}V^A + R^A_{B\mu\nu}V^B \tag{3}
\]

where \( V^A \) represents any tensor (or spinor) under diffeomorphisms (or under the group of tangent rotations), and \( R^A_{B\mu\nu} \) is the curvature tensor in the corresponding representation. Here curvature and torsion play quite different roles: \( T^A_{\mu\nu} \) is the structure function for the diffeomorphism group and \( R^A_{B\mu\nu} \) is central charge. This indicates that the gauge approach of gravity can be achieved through the local version of the Lorentz-Poincaré symmetry where \( R^A_{B\mu\nu} \) and \( T^A_{B\mu\nu} \), respectively, represent rotational symmetry and translational symmetry in the tangent space.

In \( U_4 \) there exists two invariant four forms. One is the well known Pontryagin density \( P \) and the other is the less known Nieh-Yan density \( \bar{N} \) given by

\[
P = R^{ab} \wedge R_{ab}, \tag{4}
\]
\[
and \quad \bar{N} = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b. \tag{5}
\]
P and N being four forms, \( \mathcal{P} = \int P d^4x \) and \( \mathcal{N} = \int N d^4x \) are diffeomorphism invariant quantities representing certain global features of the manifold. \( \mathcal{P} \) is dimensionless whereas \( \mathcal{N} \) has the dimension of \( L^2 \). This naturally suggests a fundamental (length)\(^2\) constant like Newton’s gravitational constant \( G \) or the inverse cosmological constant \( \Lambda^{-1} \).

Chandia and Zanelli\(^{16}\) have shown that if we combine the spin connection and the vierbein together in a connection for \( SO(5) \), in the tangent space, in the form

\[
W^{AB} = \begin{bmatrix}
\omega^{ab} & \frac{1}{l} e^a \\
-\frac{1}{l} e^b & 0
\end{bmatrix},
\]

where \( a, b = 1, 2, \ldots, 4 \); \( A, B = 1, 2, \ldots, 5 \) and \( l \) is a fundamental length constant, then we obtain the \( SO(5) \) curvature 2-form,

\[
F^{AB} = dW^{AB} + W^{AC} \wedge W^{CB}
\]

\[
= \begin{bmatrix}
R^{ab} & \frac{1}{l} e^a \wedge e^b & \frac{1}{l} T^a \\
-\frac{1}{l} T^b & 0
\end{bmatrix},
\]

and the \( SO(5) \) Pontryagin density,

\[
F^{AB} \wedge F_{AB} = R^{ab} \wedge R_{ab} + \frac{2}{l^2} [T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b].
\]

The first term of the right hand side of (9) is the \( SO(4) \) Pontryagin density and hence we can write,

\[
\frac{2}{l^2} \int_{M_4} N = P_4[SO(5)] - P_4[SO(4)]
\]

\[
= \text{const.} \times \frac{l^2}{2} (z_1 + z_2 + z_3), \quad z_i \in \mathbb{Z}.
\]

Here \( P_4[G] \) is the Pontryagin class for a compact group \( G \) on a four dimensional manifold.

From (11) and (12) we see that, when the vierbein is well defined,

\[
N = d(e^a \wedge T_a)
\]

\[
= T^a \wedge T_a - e^a \wedge DT_a
\]

where \( DT_a = dT_a + \omega^b_{a} \wedge T_b \)

\[
= R_{ab} \wedge e^b.
\]

From (12) we see that the Nieh-Yan density is the total derivative of a Chern-Simon like term,

\[
e^a \wedge T_a \sim e^{\mu\alpha\beta} T_{\mu\alpha\beta}
\]

\[
= S^\mu
\]
where $S^\mu$, the axial vector part of the torsion, only contributes to the three form.

We can decompose the torsion tensor $T_{\alpha\beta\gamma}$ into its three irreducible components, given by,

\begin{align}
T_{\beta} &= T^\alpha_{\beta\alpha} \\
S^\mu &= \epsilon^{\mu\alpha\beta} T_{\nu\alpha\beta} \\
\text{and } q_{\alpha\beta\mu} &= T_{\alpha\beta\mu} - \frac{1}{3} (T_{\beta} g_{\alpha\mu} - T_{\mu} g_{\alpha\beta}) + \frac{1}{6} \epsilon_{\alpha\beta\mu\nu} S^\nu
\end{align}

The minimal action of a spin $\frac{1}{2}$ field $\psi$ with an external gravitational field with torsion is given by 17,

\begin{align}
S_{\frac{1}{2}, \text{min}} &= i \int d^4 x \sqrt{g} \bar{\psi} (\gamma^\alpha \nabla_\alpha - \frac{i}{8} \gamma^5 \gamma^\alpha S_\alpha - i m) \psi,
\end{align}

where the covariant derivative $\nabla_\alpha$ is torsionless. Here we see that only the axial vector part of the torsion interacts with the spin $\frac{1}{2}$ field and the other irreducible components of the torsion tensor, $T_{\alpha}$ and $q_{\alpha\beta\gamma}$, completely decouple from the action. Therefore having no source in the matter lagrangian we can simply assume $T_{\alpha} = 0$ and $q_{\alpha\beta\gamma} = 0$. With this assumption the Nieh-Yan density (13) becomes

\begin{align}
N &= -e^a \wedge DT_a.
\end{align}

In terms of vierbeins we can write the spin connection in $U_4$ as 17,

\begin{align}
\omega_{\mu ab} &= \bar{\omega}_{\mu ab} + \frac{1}{4} K^\alpha_{\lambda\mu}(e^\lambda_a e_{ba} - e^\lambda_b e_{aa}), \\
\text{where } \bar{\omega}_{\mu ab} &= \frac{1}{4} (e_{ba} \partial_\mu e^\alpha_a - e_{aa} \partial_\mu e^\alpha_b) + \frac{1}{4} \Gamma^\alpha_{\lambda\mu}(e^\lambda_a e_{ba} - e^\lambda_b e_{aa})
\end{align}

Here $\Gamma^\alpha_{\lambda\mu}$ is the ordinary Christoffel symbol and $K^\alpha_{\lambda\mu} = \frac{1}{2} (T^\alpha_{\lambda\mu} - K_{\lambda\mu}^\alpha - K_{\lambda\mu}^\alpha)$ is the contorsion tensor. By $\nabla_\alpha$ here we mean the torsionless covariant derivative with respect to the connection $\bar{\omega}_{\mu ab}$ whereas the full-connection $\omega_{\mu ab}$ gives the $U_4$ covariant derivative $D_\alpha$. With this nomenclature we see that, from 13, the Nieh-Yan density takes the following form,

\begin{align}
N &= -e^a \wedge DT_a \\
\sim & \nabla_\mu S^\mu \\
= & \partial_\mu S^\mu.
\end{align}
From (25) it is clear that $N$ is an invariant under Lorentz rotation in tangent space but it is not in the case of (A)dS boost there i.e. when $\frac{1}{l} e^a$ itself transforms as a gauge field. There is a known lemma\textsuperscript{18} which states that:

**Lemma:** For $d = 4k$, the only parity-odd $d$-forms built from $e^a, R^{ab}$ and $T^a$, invariant under AdS group, are the Chern characters for $SO(d+1)$.

This lemma is equally valid in case of Lorentz group $SO(d-1,1)$ when the (A)dS group is either $SO(d,1), SO(d-1,2)$ or $SO(d+1)$, because the analysis that follows is insensitive of the signature. For $d = 4$ there is only one such AdS invariant, the second Chern character of the AdS group, given by\textsuperscript{18}

$$R^A_B \wedge R^B_A = R^a_b \wedge R^b_a + \frac{2}{l^2} (T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b).\quad (28)$$

Hence none of $P$ and $N$ is AdS invariant only their combination

$$P_4(SO(5-i,i)), i=1 \text{ or } 2 = P + \frac{2}{l^2} N,\quad (29)$$

is (A)dS invariant. And in particular in the case of axial vector torsion, i.e. when $T^a \wedge T_a = 0$ in (28), this invariant has no explicit dependence on torsion.

3. Connection, Curvature, Metric and Gravitational Constant

From the definition of curvature in (2) we see that only the knowledge of connection is required. The role of local frames are only implicit here. From (28) we see that, in the case of axial vector torsion, the $SO(4,1)$ Pontryagin density reduces to $SO(3,1)$ Pontryagin density unless the torsion contributing density $R^{ab} \wedge e_a \wedge e_b \neq 0$. Here we heuristically try to define vierbeins $e_{\mu}^a$ from $SO(3,1)$ curvatures $R_{\mu\nu ab}$ to get the curvature $R_{\mu\nu\alpha\beta}$ having only external indices. To include torsion here we forgo just one property of Einstein’s curvature tensor, viz, $\epsilon^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\mu} = 0$. So we define vierbeins $e_{\mu}^a$ and the external curvature $R_{\mu\nu\alpha\beta}$ simultaneously given by

$$R_{\mu\nu\alpha\beta} = R_{\mu\nu ab} e^a_{\alpha} e^b_{\beta},\quad (30)$$

such that

$$R_{\mu\nu\alpha\beta} = R_{\alpha\beta\mu\nu}.\quad (31)$$

Equation (31) gives fifteen equations to determine 16 vierbeins, hence another constraint is required to completely specify the vierbeins. The natural answer comes from the Nieh-Yan invariant, if we impose the condition,

$$R_{\mu\nu ab} e^a_{\alpha} e^b_{\beta} e^{\nu\alpha\beta} = e \hat{g},\quad (32)$$
where $e = det(e^a_\mu)$ and $\dot{g}$ is a scalar (constant!) of dimension $L^{-2}$. Therefore in one hand we have $36 \, R_{\mu\nu a b}$ and $1 \, \dot{g}$, i.e. altogether $37$ independent components. On the other hand we have $16$ vierbeins $e^a_\mu$ and $21$ external curvature components $R_{\mu\nu a \beta}$, i.e. altogether $37$ components. Hence vierbeins and external curvatures are completely specified in terms of the internal $SO(3,1)$ curvatures and one scalar $\dot{g}$ representing torsion. Here, it is to be noted that, (31) together with (32) give minimal extension to the Einstein’s theory to incorporate torsion. Both the constraints can be derived from an action principle using standard techniques of Lagrange multipliers such that the vierbeins can be varied independently and then the Lagrange multipliers can be set to take zero values. Also when we treat the vierbeins as independent fields (31) and (32) impart constraint on the torsion part of the $SO(3,1)$ connection. Thus degrees of freedom of this theory is greater than that of Einstein’s theory with torsion contributing to the additional degree.

With respect to the completely antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ we can decompose $R_{\mu\nu a \beta}$ into two independent parts, given by,

$$R_{\mu\nu a \beta} = \bar{R}_{\mu\nu a \beta} + \frac{1}{24}e\dot{g}\epsilon_{\mu\nu a \beta}, \quad (33)$$

such that

$$\bar{R}_{\mu\nu a \beta}\epsilon^{\mu\nu a \gamma} = 0 \quad (34)$$

and

$$R_{\mu\nu a \beta}\epsilon^{\mu\nu a \gamma} = \frac{1}{4}e\dot{g}\delta^\gamma_\beta. \quad (35)$$

After this decomposition the $SO(3,1)$ Pontryagin density can be written as

$$\epsilon^{\mu\nu a \beta}R_{\mu\nu a b}R_{\alpha \beta}^{ab} = \epsilon^{\mu\nu a \beta}\bar{R}_{\mu\nu \gamma \delta}\bar{R}_{\alpha \beta}^{\gamma \delta} + \frac{1}{6}e\dot{g}R \quad (36)$$

where

$$R = \bar{R}_{\alpha \beta}^{\alpha \beta} \quad (37)$$

$$= R_{\alpha \beta}^{\alpha \beta}. \quad (38)$$

Here, as vierbeins are already introduced, use of metric has been done for raising and lowering the external indices. As long as (31) and (32) are valid, $SO(4,1)$ Pontryagin density can be written as

$$R^A_B \wedge R^B_A \sim \epsilon^{\mu\nu a \beta}\bar{R}_{\mu\nu \gamma \delta}\bar{R}_{\alpha \beta}^{\gamma \delta} + \frac{1}{6}e\dot{g}R - \frac{2}{l^2}e\dot{g}. \quad (39)$$

It is well known that, if we treat any Pontryagin density as a Lagrangian, it will contribute nothing locally, at least in the classical level, but globally it signifies some global property of the gauge field and the manifold. Hence to have an effective field theory we can consider either of the first two terms of (39), but not both, to produce locally
nontrivial action. Therefore we heuristically propose the gravitational Lagrangian as,

$$\mathcal{L}_G = \epsilon^{\mu\alpha\beta} \bar{R}_{\mu\nu\gamma\delta} \bar{R}_{\alpha\beta} \gamma^\delta + (a + \frac{1}{6})e\hat{g}R - \frac{2}{l^2}e\hat{g}, \quad (40)$$

where $a$ is a pure number. For $a = 0$ this Lagrangian is locally trivial. In particular taking $a = 1$, locally this Lagrangian is equivalent to the Einstein-Hilbert Lagrangian,

$$\mathcal{L}_{\mathcal{E}H} = \frac{1}{\kappa} eR, \quad (41)$$

with $\hat{g} = \frac{1}{\kappa}$, here $\kappa$ is Einstein’s gravitational constant and then Newton’s constant of gravitation is given by,

$$G = \frac{c^2}{8\pi \hat{g}}. \quad (42)$$

From (42) there is a possibility that $G$ is a variable when $\hat{g}$, the torsional part of the curvature, is a variable. Following Dirac’s large number hypothesis one can argue that, strictly speaking, $G$ is not a constant like the fine structure constant $\alpha = \frac{e^2}{\hbar c}$. 

4. Discussion

One of the most interesting problems of elementary particle physics is to understand the gravitation or the gravitational constant. According to Brans-Dicke theory, the value of $G$ is determined by the value of the Brans-Dicke scalar field $\phi$. The Brans-Dicke version of Einstein-Cartan theory, with nonzero torsion and vanishing non-metricity, was discussed by many authors. In these approaches $\phi$ acts as a source of torsion. In our approach $\hat{g}$ has topological origin where in one hand $e\hat{g}$ is from the torsional curvature of $SO(3, 1)$ gauge group and on the other hand it is the Nieh-Yan density. In a recent paper it has been shown that torsion is a natural consequence in a non-commutative $U(1)$ Yang-Mills theory where gauge symmetries give very natural and explicit realizations of the mixing of spacetime and internal symmetries. Here torsion measures the noncommutativity of displacement of points in the flat spacetime in the teleparallel theory and the noncommutativity scale is given by the Planck length. So this approach is very much akin to our present approach if we consider that torsion is connected to internal space which makes the spacetime noncommutative such that the gravitational constant fixes the scale of noncommutativity. This supports the findings of some other works, when the gauge group is
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$SL(2, C)$ which is the covering group of $SO(3, 1)$, where torsion has been shown to be linked with CP-violation$^{27}$ and fermion mass$^{28}$.

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