On quantum-mechanical equations of motion in representation dependent of external sources

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Abstract

In the present paper, we consider in detail the aspects of the Heisenberg’s equations of motion, related to their transformation to the representation dependent of external sources. We provide with a closed solution as to the variation-derivative motion equations in the general case of a normal form (symbol) chosen. We show that the action in the path integral does depend actually on a particular choice of a normal symbol. We have determined both the aspects of the latter dependence: the specific boundary conditions for virtual trajectories, and the specific boundary terms in the action.

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1 Introduction

In the present paper, we consider in detail the general scenario of quantum-mechanical dynamical description as originated by Heisenberg, Schroedinger, Dirac, Schwinger, Dyson, DeWitt [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], and then developed essentially by Fradkin, Faddeev [13, 14, 15], and members of their scientific schools [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31].

We begin with the standard Heisenberg’s equations of motion as formulated for an autonomous dynamical system, and then we define the transformation operator generating a new representation where dynamical equations look as formulated in the presence of external sources. We show that the generating operator introduced appears to be, at the same time, a generating functional (of the external sources) for all chronologic - ordered products of operators in the new representation. On the ground of that observation, we reformulate the Heisenberg’s equation of motion in terms of chronologic - ordered operator products via the respective time delocalization as for the normal symbol of the Hamiltonian operator [32, 33, 34, 35, 36]. In this way, we derive closed variation - derivative equations as for the generating operator of chronologic - ordered operator products.

We show that the variation-derivative equations derived are integrable in the sense of the compatibility criterion, due to the canonical commutation relations as for the equal - time operators. In this way, we find ourselves allowed to integrate the variation derivative -equations along the straight line, as to obtain their closed solution when using the Weyl’s normal form (symbol). Then, by making use of the Baker - Campbell - Hausdorff formula, one can extend the solution obtained as to the cases of the $QP$ or the $PQ$ normal symbols.

By applying the functional Fourier transformation as to the generating operator in its dependence on the external sources, one derives the path - integral form of the solution to the variation-derivative equations of motion. It appears that the action in the path integral depends actually on the particular normal form (symbol) chosen [35, 36]. We consider the two types of the path integral. The first kind, called ”general”, does not assume the phase variables to be split explicitly into co-ordinates and momenta. Vice versa, the second kind of the path integral, called ”split”, does. For both the kinds, we provide with convenient ways as to how to parameterize the dependence of the action on the particular choice of normal symbol. The latter dependence has the two aspects: the specific boundary conditions for virtual trajectories, and the specific form of the boundary terms in the action. We have determined both these aspects. It is also a remarkable feature that a restricted virtual trajectory is expressed in terms of an unrestricted velocity, being the latter an actual path integration variable. In the latter sense, our path integrals represent a quantization of the field of velocities.

Finally, we extend the above results as to their form symmetric in the chronologic and the anti-chronologic ordering. With this purpose, we introduce the second kind of the exter-
nal source, and then modify appropriately the equation defining the generating operator. We provide with a closed variation-derivative solution symmetric in the chronologic and the anti-chronologic operator ordering.

2 Heisenberg’s equations of motion and generating operator for chronologic products

Let $Z^A(t)$, $\varepsilon(Z^A) =: \varepsilon_A$, be canonical phase-variable operators, and $H(Z)$ be an original Hamiltonian, so that the Heisenberg’s equations of motion read

$$i\hbar \frac{\partial Z^A}{\partial t} = [Z^A, H], \quad [Z^A, Z^B] = i\hbar \omega^{AB} = \text{const.}$$ (2.1)

Let $J_A(t)$, $\varepsilon(J_A) =: \varepsilon_A$, be an external source, and let us define a transformation operator $U(t)$ as to satisfy the equation

$$i\hbar \frac{\partial U}{\partial t} = -J_A Z^A U, \quad U(-\infty) = 1.$$ (2.2)

Now, we define canonical phase-variable operators in the source-dependent representation, $Z'^A := U^{-1} Z^A U$, as to satisfy the modified equation of motion

$$i\hbar \frac{\partial Z'^A}{\partial t} = [Z'^A, H' - J_B Z'^B].$$ (2.3)

It follows from (2.2) that

$$i\hbar U^{-1}(t) \frac{\partial}{\partial t} \frac{\delta}{\delta J_A(t')} U(t) = -\delta(t - t') Z'^A(t) - J_B(t) Z'^B(t) U^{-1}(t) \frac{\delta}{\delta J_A(t')} U(t).$$ (2.5)

On the other hand, the left-hand side in (2.5) rewrites as

$$i\hbar \frac{\partial}{\partial t} U^{-1}(t) \frac{\delta}{\delta J_A(t')} U(t) - \left( i\hbar \frac{\partial}{\partial t} U^{-1}(t) \right) \frac{\delta}{\delta J_A(t')} U(t).$$ (2.6)

Here in (2.6), in the second term, the first parentheses rewrite as

$$i\hbar \frac{\partial}{\partial t} U^{-1}(t) = -U^{-1}(t) \left( i\hbar \frac{\partial}{\partial t} U(t) \right) U^{-1}(t) = J_B(t) Z'^B(t) U^{-1}(t).$$ (2.7)

In this way, the second term in (2.6) cancels exactly the second term in the right-hand side in (2.5), and the latter (2.5) takes the form

$$i\hbar \frac{\partial}{\partial t} U^{-1}(t) \frac{\delta}{\delta J_A(t')} U(t) = -\delta(t - t') Z'^A(t'),$$ (2.8)
whose causal solution reads
\[
\frac{\hbar}{i} U^{-1}(t) \frac{\delta}{\delta J_A(t')} U(t) = \theta(t - t') Z'^A(t'),
\]
so that
\[
\frac{\hbar}{i} \frac{\delta}{\delta J_B(t')} Z'^A(t) = [Z'^A(t), \theta(t - t') Z'^B(t')(-1)^{\varepsilon_{A\varepsilon_B}}].
\]
In the limit \( t \to \infty \), it follows from (2.9), (2.10) that
\[
\left( \frac{\hbar}{i} \right)^2 \frac{\delta^2}{\delta J_B(t'') \delta J_A(t')} U(\infty) = U(\infty) T(Z'^B(t'') Z'^A(t')),
\]
where the binary chronologic-ordered product is defined by
\[
T(Z'^B(t'') Z'^A(t')) =: \left( \theta(t'' - t') Z'^B(t'') Z'^A(t') + \theta(t' - t'') Z'^A(t') Z'^B(t'')(-1)^{\varepsilon_{B\varepsilon_A}} \right),
\]
see also (2.18) below for the general case.

Let \( F[Z] \) be a functional of classical functions \( Z^A(t) \), and
\[
F[0] = 0, \quad F_A(t|Z) =: F[Z] \frac{\delta}{\delta Z^A(t)}.
\]
By applying the variation-derivative operator,
\[
F \left[ \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right],
\]
as to the equation (2.2), we get
\[
\left( i\hbar \frac{\partial}{\partial t} + J_A Z^A \right) F \left[ \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right] U = -\frac{\hbar}{i} F_A \left( t \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right) Z^A U.
\]
It follows from (2.15) that
\[
F \left[ \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right] U(t) = \int_{-\infty}^{t} dt' U(t) U^{-1}(t') F_A \left( t' \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right) Z^A(t') U(t') = U(t) \int_{-\infty}^{t} dt' F_A \left( t' |\theta(t' - (\cdot)) Z' (\cdot) + \frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)} \right) Z'^A(t').
\]
Here in (2.16), in the second equation, we have used the above eq.(2.9). If one chooses a functional \( F[Z] \) in the monomial form,
\[
F[Z] = Z^{A_1}(t_1) \cdots Z^{A_n}(t_n),
\]
then, in the limit \( t \to \infty \), the right-hand side in (2.16) does generate the corresponding \( T \)-product as to the primed operators (2.3) in the external source-dependent representation,
\[
\frac{\hbar}{i} \frac{\delta}{\delta J_{A_1}(t_1)} \cdots \frac{\hbar}{i} \frac{\delta}{\delta J_{A_n}(t_n)} U(\infty) = U(\infty) T(Z'^{A_1}(t_1) \cdots Z'^{A_n}(t_n)).
\]
3 Variation - derivative equations of motion for generating operator and their closed solution

Now, let us represent the primed Hamiltonian $H'$ in the form of its chronologic-ordered delocalized symbol $H_R$,

$$H' = H(Z'(t)) = T(H_R(t|Z'(\cdot)))$$  \hspace{1cm} (3.1)

and the same for the commutator

$$[Z'^A(t), H(Z'(t))] = i\hbar \omega^{AB} T\left(\frac{\delta}{\delta Z^B(t)} \int_{-\infty}^{\infty} dt' H_R(t'|Z'(\cdot))\right).$$  \hspace{1cm} (3.2)

Then, due to the eq.(2.18), it follows from the primed Heisenberg’s equation (2.4) that the variation-derivative equation of motion holds as for the operator $U(\infty)$,

$$\left[\dot{Z}^A(t) - \omega^{AB} \left(\frac{\delta}{\delta Z^B(t)} \int_{-\infty}^{\infty} dt' H_R(t'|Z(\cdot))\right) - J_B(t) \omega^{BA}\right] \left(Z = \frac{\hbar}{i} \frac{\delta}{\delta J}\right) U(\infty) = 0.$$  \hspace{1cm} (3.3)

A variation-derivative solution to that equation has the form \[37, 35\]

$$U(\infty) = \exp \left\{ -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt H_R(t|\frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)}) \right\} \times \exp \left\{ -\frac{i}{\hbar} \frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J_A(t) D^{AB}(t-t') J_B(t') (-1)^{\varepsilon_B} \right\} \times N \left( \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt J_A(t) Z'^A_{in} \right\} \right),$$  \hspace{1cm} (3.4)

$$D^{AB}(t-t') = \frac{1}{i\hbar} \left( Z'^A_{in} Z'^B_{in} \theta(t-t') + Z'^B_{in} Z'^A_{in} \theta(t'-t) (-1)^{\varepsilon_A\varepsilon_B} - N(Z'^A_{in} Z'^B_{in}) \right),$$  \hspace{1cm} (3.5)

$$\frac{\partial}{\partial t} D^{AB}(t-t') = \omega^{AB} \delta(t-t'),$$  \hspace{1cm} (3.6)

$$Z'^A_{in} = \frac{\hbar}{i} U^{-1}(\infty) \frac{\delta U(\infty)}{\delta J_A(-\infty)}.$$  \hspace{1cm} (3.7)

In order to explain as to how to derive the solution \[3.4\], let us seek for a solution to \[3.3\] in the natural form

$$U(\infty) = \exp \left\{ -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt H_R(t|\frac{\hbar}{i} \frac{\delta}{\delta J(\cdot)}) \right\} U_{ext}(\infty),$$  \hspace{1cm} (3.8)

so that the operator $U_{ext}(\infty)$ does satisfy the equation

$$\left[\frac{\partial}{\partial t} \frac{\hbar}{i} \frac{\delta}{\partial J_A(t)} + \omega^{AB} J_B(t) (-1)^{\varepsilon_B}\right] U_{ext}(\infty) = 0,$$  \hspace{1cm} (3.9)
or in its integral form,
\[ \frac{\hbar}{i} \frac{\delta U_{\text{ext}}(\infty)}{\delta J_A(t)} = U_{\text{ext}}(\infty) \left( Z_{A}^{tA} - \int_{-\infty}^{t} dt' \omega^{AB} J_B(t')(-1)^{\varepsilon_B} \right), \] (3.10)

with
\[ Z_{in}^{tA} =: Z^{tA}(-\infty), \] (3.11)

being \textit{in} - operators independent of the external sources \( J_A(t) \). The equations (3.10) are integrable due to the canonical commutation relations as for the operators (3.11); we have from (3.10)
\[ X^{BA}(t', t) =: \left( \frac{\hbar}{i} \right)^2 \frac{\delta^2 U_{\text{ext}}(\infty)}{\delta J_B(t')\delta J_A(t)} = U_{\text{ext}}(\infty) \left[ \left( Z_{in}^{tB} - \int_{-\infty}^{t'} ds' \omega^{BD} J_D(s')(-1)^{\varepsilon_D} \right) \right] \times \left( Z_{in}^{tA} - \int_{-\infty}^{t} ds \omega^{AC} J_C(s)(-1)^{\varepsilon_C} \right) + \frac{\hbar}{i} \omega^{BA} \theta(t - t'). \] (3.12)

By anti - symmetrizing the operators (3.12), we get
\[ X^{BA}(t', t) - X^{AB}(t, t')(-1)^{\varepsilon_B\varepsilon_A} = U_{\text{ext}}(\infty) \left[ [Z_{in}^{tB}, Z_{in}^{tA}] + \frac{\hbar}{i} \left( \omega^{BA} \theta(t - t') - \omega^{AB} \theta(t' - t)(-1)^{\varepsilon_B\varepsilon_A} \right) \right]. \] (3.13)

As the \textit{in} - operators do satisfy the canonical commutation relations, the commutator in the right-hand side in (3.13) equals to \( i\hbar \omega^{BA} \), while the intrinsic parentheses equal to
\[ \omega^{BA}(\theta(t - t') + \theta(t' - t)) = \omega^{BA}. \] (3.14)

Thus, we have shown that the anti - symmetric part of the operators (3.12) is zero,
\[ X^{BA}(t', t) - X^{AB}(t, t')(-1)^{\varepsilon_B\varepsilon_A} = 0, \] (3.15)

which means that the integrability holds (3.15) as to the equation (3.10). By multiplying the latter (3.10) by \( \delta J_A(t) \) from the left, and integrating then over \( dt \), we get the complete variation as for the operator \( U_{\text{ext}}(\infty) \),
\[ \frac{\hbar}{i} \delta U_{\text{ext}}(\infty) = U_{\text{ext}}(\infty) \int_{-\infty}^{\infty} dt \delta J_A(t) \left( Z_{in}^{tA} - \int_{-\infty}^{t} dt' \omega^{AB} J_B(t')(-1)^{\varepsilon_B} \right). \] (3.16)

By making a formal rescaling
\[ J_A(t) \to \lambda J_A(t), \] (3.17)

with \( \lambda \) being a Boson parameter, due to the integrability (3.15) shown above, we are allowed to choose the variation of the rescaled source along the straight line,
\[ \delta(\lambda J_A(t)) = d\lambda J_A(t). \] (3.18)
Then we get the differential equation in $\lambda$,

\[
\frac{\hbar}{i} \frac{\partial U_{\text{ext}}(\infty)}{\partial \lambda} = U_{\text{ext}}(\infty) \left[ \int_{-\infty}^{\infty} dt J_A(t) Z_{in}^A - \lambda \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J_A(t) \omega^{AB} \frac{1}{2} \text{sign}(t - t') J_B(t) (-1)^{\varepsilon_B} \right].
\] (3.19)

By integrating that equation together with the condition $U_{\text{ext}}(\infty)|_{\lambda=0} = 1$, and taking then $\lambda = 1$, we arrive finally at the solution for the $U_{\text{ext}}(\infty)$ as given by the product of the second and the third exponential in (3.4), with $N$ being the Weyl normal form; other types of normal form for $N$ do follow via the Baker - Campbell - Hausdorff invariant formula,

\[
\exp \left\{ i \frac{\hbar}{\mathcal{D} \int_{-\infty}^{\infty} dt J_A(t) Z_{in}^A} \right\} = \exp \left\{ -i \frac{\hbar}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' J_A(t) \frac{1}{i\hbar} (N_W - N)(Z_{in}^A Z_{in}^B) J_B(t') (-1)^{\varepsilon_B} \right\} \times N \exp \left\{ i \frac{\hbar}{\mathcal{D} \int_{-\infty}^{\infty} dt J_A(t) Z_{in}^A} \right\}. \] (3.20)

4 Path-integral solution for generating operator

The path integral solution can easily be derived from (3.4) by making use of a functional Fourier transformation as for the second exponential. Let us split temporarily the complete set of canonical phase-variable operators $Z^A$ into the sectors of co-ordinates $Q^i$ and momenta $P_i$,

\[
Z^A =: (Q^i; P_i). \] (4.1)

Denote via $N_{(a,b)}$, $a + b = 1$, the $(a, b)$ - type normal form:

\[
a = 0, \ b = 1 : \quad QP, \] (4.2)
\[
a = 1, \ b = 0 : \quad PQ, \] (4.3)
\[
a = \frac{1}{2}, \ b = \frac{1}{2} : \quad W_{\text{cyl}}. \] (4.4)

Then, we have

\[
H_R(t|Z(\cdot)) = H_{(a,b)}(Z_R(t)), \] (4.5)

where the delocalized set of phase variables is given explicitly by

\[
Z_R^A(t) =: (Q^i(t + \text{sign}(b-a)0); P_i(t)). \] (4.6)

As for the cases (4.2) - (4.4), the $\mathcal{D}$ function (3.5) rewrites in the form [39]

\[
\mathcal{D}^{AB}(t - t') = \frac{1}{2} \text{sign}(t - t') \omega^{AB} + \frac{1}{i\hbar} \left( N_{(\frac{1}{2}, \frac{1}{2})} - N_{(a,b)} \right) (Z_{in}^A Z_{in}^B). \] (4.7)

Thus, the $(a, b)$ - type in' - symbol $U_{(a,b)}(\infty)$ of the operator $U(\infty)$ is given by the path integral

\[
U_{(a,b)}(\infty) = \int [DV] \exp \left\{ i \frac{\hbar}{W} \right\}, \] (4.8)
where the measure \([DV]\) contains the required normalization factor as to maintain that (4.8) equals to one at \(J = 0\), while the action \(W\) has the form

\[
W =: \int_{-\infty}^{\infty} dt \left[ \frac{1}{2} Z^A(t) \omega_{AB} \dot{Z}^B(t) - H_{(a,b)}(Z_R(t)) + J_A(t) Z^A(t) \right] +
+ \frac{1}{2} Z^A(\infty) \omega_{AB} Z^B(-\infty) + \frac{1}{2} (P_i(\infty) + P_i(-\infty) - 2 P'_{i in})(Q^i(\infty) - Q^i(-\infty)),
\]

with \(Z^A(t)\) being a restricted virtual trajectory,

\[
\frac{1}{2} \left( Z^A(\infty) + Z^A(-\infty) \right) -
- \frac{1}{i\hbar} \left( N_{(a,b)} \right) \left( Z'^{A}_{in} Z'^{B}_{in} \right) \omega_{BC} \left( Z^C(\infty) - Z^C(-\infty) \right) = Z'^{A}_{in},
\]

as expressed via an unrestricted velocity \(V^A(t)\),

\[
Z^A(t) =: Z'^{A}_{in} + \int_{-\infty}^{\infty} dt' D^{AB}(t - t') \omega_{BC} V^C(t').
\]

In this way, the above path integral (4.8) appears to be, in fact, an actual quantization of the field of velocities. It follows from the general boundary conditions (4.10) that there holds their split version in the form

\[
b Q^i(\infty) + a Q^i(-\infty) = Q'^i_{in},
\]

\[
a P_i(\infty) + b P_i(-\infty) = P'_{i in}.
\]

In order to derive in a natural way the path integral (4.8) from the variation-derivative solution (3.4), one may notice the path-integral Fourier representation for the \((a, b)\)-type \(in'\)-symbol of the operator \(U_{ext}(\infty)\),

\[
U_{ext(a,b)}(\infty) = \int [DV] \exp \left\{ \frac{i}{\hbar} W_{ext} \right\},
\]

\[
W_{ext} =: \int_{-\infty}^{\infty} dt \left[ \frac{1}{2} Z^A(t) \omega_{AB} \dot{Z}^B(t) + J_A(t) Z^A(t) \right] +
+ \frac{1}{2} Z^A(\infty) \omega_{AB} Z^B(-\infty) + \frac{1}{2} (P_i(\infty) + P_i(-\infty) - 2 P'_{i in})(Q^i(\infty) - Q^i(-\infty)),
\]

where the formulae (4.7), (4.10), (4.11) must be taken into account. As the action (4.15) is quadratic in \(Z^A(t)\), the path integral (4.14) is a Gaussian one. Therefore, its value is given by taking the Boltzmann factor, \(\exp \left\{ \frac{i}{\hbar} W_{ext} \right\}\), at the extremum of the action (4.15), which is given by

\[
Z^A(t) = Z'^{A}_{in} - \int_{-\infty}^{\infty} dt' D^{AB}(t - t') J_B(t')(-1)^{\varepsilon_B}.
\]
In this way, one reproduces immediately the product of the second and the third exponential in (3.4).

Now, let us split the set of external sources in accordance with the splitting (4.1) as for the phase variables

\[ J_A =: (I_i; K^i). \] (4.17)

Then, the variation-derivative solution (3.4) takes the split form,

\[
U(\infty) = \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \frac{\hbar}{i} \frac{\delta}{\delta I(\cdot)} \left[ \frac{\hbar}{i} \frac{\delta}{\delta K(\cdot)} \right] \right\} \times \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' I_i(t) \mathcal{D}^j(t - t') K^j(t') (-1)^{\varepsilon_j} \right\} \times \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt (I_i(t) Q^i_{in} + K^i(t) P^i_{in}) \right\}, \] (4.18)

where

\[
\mathcal{D}^j(t - t') = \frac{1}{i\hbar} \left( Q^i_{in} P^i_{in} \theta(t - t') + P^i_{in} Q^i_{in} \theta(t' - t) (-1)^{\varepsilon_i \varepsilon_j} - N(Q^i_{in} P^i_{in}) \right) = \delta^j_i \left( a \theta(t - t') - b \theta(t' - t) \right). \] (4.19)

In a similar way we get for the transposed \( \mathcal{D} \)-function,

\[
\mathcal{D}^i_j(t - t') = -(-1)^{\varepsilon_i \varepsilon_j} \delta^i_j \left( b \theta(t - t') - a \theta(t' - t) \right). \] (4.20)

The non-split function (3.5) / (4.7) rewrites in terms of the blocks (4.19), (4.20)

\[
\mathcal{D}^{AB}(t - t') = \begin{pmatrix} 0 & \mathcal{D}^j_i(t - t') \\ \mathcal{D}^i_j(t - t') & 0 \end{pmatrix}. \] (4.21)

Introduce the following path-integral Fourier representation

\[
\exp \left\{ -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' I_i(t) \mathcal{D}^j(t - t') K^j(t') (-1)^{\varepsilon_j} \right\} = \int [DP][DV] \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left[ P^i(t) \left( V^i(t) + K^i(t) (-1)^{\varepsilon_i} \right) + I_i(t) \int_{-\infty}^{\infty} dt' \mathcal{D}^i_j(t - t') V^j(t') \right] \right\}. \] (4.22)

By making use of that representation, we arrive finally at the following split path-integral formula as for the \((a, b)\)-type \( in' \)-symbol of the operator \( U(\infty) \),

\[
U_{(a, b)}(\infty) = \int [DP][DV] \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left[ P^i(t) \dot{Q}^i(t) - H_{(a, b)}(Q_R(t), P_R(t)) + I_i(t) Q^i(t) + K^i(t) P^i(t) \right] - \frac{i}{\hbar} P^i_{in}(Q^i(\infty) - Q^i(-\infty)) \right\}, \] (4.23)
where
\[
bQ^i(\infty) + aQ^i(-\infty) = Q^i_{\text{in}},
\]
(4.24)

\[
Q^i(t) = Q^i_{\text{in}} + \int_{-\infty}^{\infty} dt' D^i_j(t-t')V^j(t').
\]
(4.25)

Here in (4.23), the \(Q_R\) and \(P_R\) are just given explicitly in components in the right-hand side in (4.6). In the split path integral (4.23), the momenta \(P_i(t)\) are non-restricted integration variables, so that there are no boundary conditions to them. However, consider the classical equations of motions for velocities \(V^i(t)\), as they follow from the action in (4.23),

\[
P_i(t) = P'_{i \text{in}} + \left(a \int_{-\infty}^{\infty} -b \int_{-\infty}^{t} dt' \frac{\partial H_{(a,b)}}{\partial Q^i(t')}\right).
\]
(4.26)

It follows from (4.26)

\[
\dot{P}_i(t) = -\frac{\partial H_{(a,b)}}{\partial Q^i(t)},
\]
(4.27)

\[
aP_i(\infty) + bP_i(-\infty) = P'_{i \text{in}},
\]
(4.28)

Thus, in the split path integral (4.23), we have reproduced the boundary conditions (4.13) for momenta \(P_i(t)\) at the classical level.

As to compare our proposed path integrals for the ones of Berezin and Shubin, the situation is the following. In the book of these authors [36], the general (non-split) path integral, similar to our (4.8), is given as applied only to the case of the Weyl symbol. In the cases of the \(PQ\) and the \(QP\) symbol, these authors have given only split path integrals, similar to our (4.23). It should also be mentioned that all our path integrals have just the respective unrestricted velocities as their actual integration variables, rather than restricted trajectories.

5 Formulation symmetric in chronologic and anti-chronologic products

So far, we did proceed from the basic idea of the chronologic ordering of operators. However, it is a remarkable feature that there also exists an approach [40, 41] based on symmetric use of both the chronologic and the anti-chronologic product. In that approach, one proceeds with extended version of the equation (2.2),

\[
\frac{i\hbar}{\partial t} = -J_A Z^A U + U \tilde{J}_{\tilde{A}} Z^A, \quad U(-\infty) = 1
\]
(5.1)

where \(Z^A\) are the Heisenberg’s operators as defined by (2.1), and \(\tilde{J}_A(t)\) is a new external source. The equation (5.1) is satisfied naturally with the factorized ansatz

\[
U = U_J(U_J)^{-1},
\]
(5.2)
where the operator \( U_J \) is defined by (2.2),
\[
\frac{i\hbar}{\partial t} \frac{\partial U_J}{\partial t} = -J_A Z^A U_J, \quad U_J(-\infty) = 1,
\]
(5.3)
and the operator \( U_J \) is defined by (5.3) with the replacement \( J_A \to \tilde{J}_A \).

Then, by making use of the same method as the one applied as to the eq. (2.2), one can
derive easily a variation - derivative solution as to the equation (5.1),
\[
H \frac{\partial}{\partial t} \frac{\partial U}{\partial t} - \frac{i}{\hbar} \left[ H, U \right] + \frac{1}{2} J(t) D^{AB}(t - t') J(t') + \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left\{ \frac{\partial}{\partial t} \left( D^{AB} \right) \right\} = 0,
\]
(5.10)
where \( H \) is defined by (4.9); the \( \tilde{U} \) is defined by (5.3) with the replacement \( J_A \to \tilde{J}_A \).

By applying the respective Fourier representations in \( J \) and \( \tilde{J} \) as to the (5.1), one can obtain
the corresponding path-integral representation for the \((a, b)\)-type \( i \hbar \) - symbol of the operator \( U(\infty) \),
\[
U(\infty) = \exp \left\{ -i \frac{1}{\hbar} \int_{-\infty}^{\infty} dt \left[ H_R \left( t \frac{\hbar}{i} \frac{\delta}{\delta J_A(\cdot)} \right) - H_R \left( t \frac{\hbar}{i} \frac{\delta}{\delta J_A(\cdot)} \right) \right] \right\} \times
\exp \left\{ -i \frac{1}{\hbar} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' \left[ J_A(t) D^{AB}(t - t') \tilde{J}_B(t') + \tilde{J}_A(t) \tilde{D}^{AB}(t - t') \tilde{J}_B(t') \right] \right\} \times
\exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left( J_A(t) - \tilde{J}_A(t) \right) Z^{IA}(t) \right\},
\]
(5.4)
where the delocalization \( H_R(t|Z(\cdot)) \) was defined in (4.5), (4.6), while its anti-chronologic counterpart reads
\[
H_R(t|Z(\cdot)) = H(Z_R(t)),
\]
(5.5)
\[
Z^A_R(t) = \left( Q^i (t + \text{sign}(a - b)0); P_i(t) \right),
\]
(5.6)
and the kernels \( D \) are defined by
\[
D^{AB}(t - t') = \frac{1}{i \hbar} \left( Z^{IA}_{in} Z^{IB}_{in} \theta(t - t') + Z^{IB}_{in} Z^{IA}_{in} \theta(t' - t)(-1)^{\varepsilon_A \varepsilon_B} - N(Z^{IA}_{in} Z^{IB}_{in}) \right),
\]
(5.7)
\[
\tilde{D}^{AB}(t - t') = \frac{1}{i \hbar} \left( Z^{IA}_{in} Z^{IB}_{in} \theta(t' - t) + Z^{IB}_{in} Z^{IA}_{in} \theta(t - t')(1)^{\varepsilon_A \varepsilon_B} - N(Z^{IA}_{in} Z^{IB}_{in}) \right),
\]
(5.8)
\[
\tilde{D}^{AB}(t - t') = \frac{1}{i \hbar} \left( N(Z^{IA}_{in} Z^{IB}_{in}) - Z^{IA}_{in} Z^{IB}_{in} \right).
\]
(5.9)
By applying the respective Fourier representations in \( J \) and \( \tilde{J} \) as to the (5.1), one can obtain
the corresponding path-integral representation for the \((a, b)\)-type \( i \hbar \) - symbol of the operator \( U(\infty) \),
\[
U_{(a,b)}(\infty) = \int \left[ D V \right] \left[ \tilde{D} V \right] \exp \left\{ \frac{i}{\hbar} \tilde{W} \right\} \ast \exp \left\{ -\frac{i}{\hbar} \tilde{W} \right\},
\]
(5.10)
Here in (5.10): the \( \ast \) is the \( N_{(a,b)} \) - symbol multiplication; the \( W \) is given by (4.9); the \( \tilde{W} \) is
given by (4.9) with the replacements:
\[
Z^A(t) \to \tilde{Z}^A(t) =: Z^{IA}_{in} + \int_{-\infty}^{\infty} dt' \tilde{V}^C(t') \omega_{CB} \tilde{D}^{BA}(t' - t),
\]
(5.11)
\[ \tilde{D}^{AB}(t - t') = D^{AB}(t' - t), \quad (5.12) \]
\[ Z^A_R \rightarrow \tilde{Z}^A_R, \quad J_A(t) \rightarrow \tilde{J}_A(t). \quad (5.13) \]

The relation
\[ \star = \exp \left\{ -\frac{i}{\hbar} \frac{\partial}{\partial Z_{in}^A} i\hbar \tilde{D}^{AB} \frac{\partial}{\partial Z_{in}^B} \right\} \quad (5.14) \]

has been used when deriving (5.10), with \( \tilde{D}^{AB} \) given by (5.9) where \( t \) and \( t' \) dependence omitted [39].

6 Conclusion

In the present paper, we have considered the general aspects of quantum-mechanical description of dynamical evolution. It is the general feature that the typical path-integral solutions of the quantum dynamics depend actually on the particular choice of the normal form (symbol) of the operators used. The latter dependence has the two aspects: the specific form of the boundary conditions as for the virtual trajectory, and the specific form of the boundary terms in the action. In the present paper, we have determined both the mentioned aspects, as demonstrated explicitly in the formulae (4.9) - (4.11) and (4.23) - (4.25).

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Appendix A. Antisources in gauge theories

Here in this Appendix A, we extend the above consideration naturally, as to the general case of gauge theories, whose original Hamiltonian \( H \) does commute with a Fermion nilpotent BFV-BRST charge operator \( Q \),
\[ [H, Q] = 0, \quad Q^2 = \frac{1}{2} [Q, Q] = 0. \quad (A.1) \]

In this case, we modify the equation (2.2) as
\[ i\hbar \frac{\partial U}{\partial t} = (-J_A Z^A - J^*_A (i\hbar)^{-1} [Z^A, Q]) U, \quad (A.2) \]
by introducing new antisources,

\[ J_A^*(t), \quad \varepsilon(J_A^*) = \varepsilon_A + 1. \]  

(A.3)

Then, the equation (2.4) becomes

\[ i\hbar \frac{\partial Z'^A}{\partial t} = [Z'^A, H' - J_B Z'^B - J_B^*(i\hbar)^{-1}[Z'^B, Q']], \]  

(A.4)

so that the equation of motion holds as for the \( Q' \),

\[ i\hbar \frac{\partial Q'}{\partial t} = [J A Z'^A, Q'], \]  

(A.5)

where we have used (A.1). The equation (2.9) remains valid with the new \( U \) and \( Z'^A \). There is also the new equation as to hold,

\[ \frac{\hbar}{i} \frac{U^{-1}(t)}{\delta J_A^*(t')} \frac{\delta}{\delta J_A^*(t')} U(t) = \theta(t - t')(i\hbar)^{-1}[Z'^A, Q'](t'). \]  

(A.6)

It follows then from the (A.5), (A.6) that the dynamical change of the BFV-BRST charge \( Q' \) caused with the sources and antisources equals to

\[ Q'(t) - Q'(-\infty) = U^{-1}(t)[Q, U(t)] = -i\hbar U^{-1}(\infty) \int_{-\infty}^{t} dt' J_A(t') \frac{\delta}{\delta J_A^*(t')} U(\infty). \]  

(A.7)

which is an operator valued "ancestor" as to the well-known quantum master equation. Indeed, by making use of the \( \text{in}' \)-normal Fourier representation as for the generating operator \( U(\infty) \),

\[ U(\infty) =: \int [DV] \exp \left\{ \frac{i}{\hbar} \int_{-\infty}^{\infty} dt J_A(t) (Z^A(t) - Z'^A_{\text{in}}) \right\} \tilde{U}(\infty), \]  

(A.8)

where \( Z^A(t) \) is defined by (4.11), we rewrite the (A.7) (at \( t = \infty \)) in the form directly related as to the "naive" master equation (see also the footnote as for the formula (A.8)),

\[ [\tilde{U}(\infty), Q] = \left( \frac{\hbar}{i} \right)^2 \Delta \tilde{U}(\infty), \]  

(A.9)

with the \( \Delta \) being a functional "odd Laplacian",

\[ \Delta =: -\int_{-\infty}^{\infty} dt (-1)^{\varepsilon_A} \left( \frac{\partial}{\partial t} \frac{\delta}{\delta V^A(t)} \right) \frac{\delta}{\delta J_A^*(t)}. \]  

(A.10)

\[ ^{3}\text{In fact, these objects are the same as the antifields introduced in [42, 43].} \]

\[ ^{4}\text{By the way, the } \tilde{U}(\infty) \text{ entering (A.8) is expressed in terms of the respective spectral density } \tilde{U}(\infty) \text{ as} \]

\[ \tilde{U}(\infty) = N \left( \exp \left\{ \int_{-\infty}^{\infty} dt \left( \frac{\delta}{\delta V^A(t)} - \frac{\delta}{\delta V^A(-\infty)} \right) \right\} \right) \tilde{U}(\infty). \]
Let $|\Phi\rangle, |\Phi'\rangle$ be two physical states annihilated by the Hermitian operator $Q$,

$$Q|\Phi\rangle = 0, \quad Q|\Phi'\rangle = 0, \quad Q = Q^\dagger. \quad \text{(A.11)}$$

It follows then from (A.9) that the physical matrix element of the operator $\tilde{U}(\infty)$ between the two states is annihilated by the $\Delta$, (A.10),

$$\Delta(\Phi'|\tilde{U}(\infty)|\Phi) = 0. \quad \text{(A.12)}$$

Now, consider in short an $Sp(2)$ extension of the main construction above. In the latter case, we have an $Sp(2)$ vector valued Fermion BFV-BRST charge operator \[^{11,13,16}\]

$$[H, Q^a] = 0, \quad Q^aQ^b + (a \leftrightarrow b) = [Q^a, Q^b] = 0. \quad \text{(A.13)}$$

The equation (A.2) now modifies as to become

$$i\hbar \frac{\partial U}{\partial t} = (- J_A Z^A - J^*_a(i\hbar)(i\hbar)^{-1}[Z^A, Q^a] - J^*_A(i\hbar)^{-1}[[Z^A, Q^a], Q^b]) \frac{1}{2} \varepsilon_{ab} U, \quad \text{(A.14)}$$

where we have introduced the antisources, $J^*_a(t), J^*_A(t)$,

$$\varepsilon(J^*_a) = \varepsilon_A + 1, \quad \varepsilon(J^*_A) = \varepsilon_A. \quad \text{(A.15)}$$

The equation (A.4) now becomes

$$i\hbar \frac{\partial Z^A}{\partial t} = [Z^A, H'] - J^*_B Z^B - J^*_b(i\hbar)^{-1}[Z^B, Q^b] - J^*_B(i\hbar)^{-1}[[Z^B, Q^b], Q^a] \frac{1}{2} \varepsilon_{ab}. \quad \text{(A.16)}$$

In turn the equation (A.7) rewrites in the form

$$Q^a(t) - Q^a(-\infty) = U^{-1}(t)[Q^a, U(t)] = -i\hbar U^{-1}(\infty) \int_{-\infty}^t dt' J_A(t') \frac{\delta}{\delta J^*_a(t')} - \varepsilon_{ab} J^*_A(t') \frac{\delta}{\delta J^*_A(t')} U(\infty), \quad \text{(A.17)}$$

where the normalization $\varepsilon^{12} = -\varepsilon_{12} = 1$ is used. The respective counterpart to the formula (A.9) reads

$$[\tilde{U}(\infty), Q^a] = \left(\frac{\hbar}{i}\right)^2 \Delta^a_+ \tilde{U}(\infty), \quad \text{(A.18)}$$

where

$$\Delta^a_+ =: \Delta^a + \frac{i}{\hbar} V^a, \quad \text{(A.19)}$$

$$\Delta^a =: - \int_{-\infty}^{\infty} dt (-1)^{\varepsilon^A} \left(\frac{\partial}{\partial t} \delta V^A(t) \frac{\delta}{\delta J^*_a(t)} \right), \quad V^a =: \varepsilon^{ab} \int_{-\infty}^{\infty} dt J^*_b(t) \frac{\delta}{\delta J^*_A(t)}. \quad \text{(A.20)}$$

Let $|\Phi\rangle, |\Phi'\rangle$ be two physical states annihilated by the Hermitian operators $Q^a$,

$$Q^a|\Phi\rangle = 0, \quad Q^a|\Phi'\rangle = 0, \quad Q^a = (Q^a)^\dagger. \quad \text{(A.21)}$$

It follows then from (A.18) that the physical matrix element of the operator $\tilde{U}(\infty)$ between the two states is annihilated by the $\Delta^a_+$, (A.19),

$$\Delta^a_+ (\Phi'|\tilde{U}(\infty)|\Phi) = 0. \quad \text{(A.22)}$$
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