The Geometry of Billiards in Ellipses and their Poncelet Grids

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Abstract. The goal of this paper is an analysis of the geometry of billiards in ellipses, based on properties of confocal central conics. The extended sides of the billiards meet at points which are located on confocal ellipses and hyperbolas. They define the associated Poncelet grid. If a billiard is periodic then it closes for any choice of the initial vertex on the ellipse. This gives rise to a continuous variation of billiards which is called billiard’s motion though it is neither a Euclidean nor a projective motion. The extension of this motion to the associated Poncelet grid leads to new insights and invariants.

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1. Introduction

A billiard is the trajectory of a mass point within a domain with ideal physical reflections in the boundary. Already for two centuries, billiards in ellipses have attracted the attention of mathematicians, beginning with J.-V. Poncelet and A. Cayley. One basis for the investigations was the theory of confocal conics. In 2005 S. Tabachnikov published a book on billiards, based on the theory of completely integrable systems [26]. In several publications and in the book [13], V. Dragović and M. Radnović studied billiards, also in higher dimensions, from the viewpoint of dynamical systems.

Computer animations of billiards in ellipses, which were carried out by D. Reznik [21], stimulated a new vivid interest on this well studied topic, where algebraic and analytic methods are meeting (see, e.g., [2, 3, 11, 19, 20] and many further references in [21]). These papers focus on invariants of periodic billiards when one vertex varies on the ellipse while the caustic remains fixed. This variation is called billiard motion though neither angles
nor side lengths remain fixed and this is also not a projective motion preserving the circumscribed ellipse.

The goal of this paper is a geometric analysis of billiards in ellipses and their associated Poncelet grid, starting from properties of confocal conics. We concentrate on a certain symmetry between the vertices of any billiard and the contact points with the caustic, which can be an ellipse or hyperbola. Billiard motions induce motions of associated billiards with the same caustic and circumscribed confocal ellipses.

2. Metric properties of confocal conics

A family of confocal central conics (Figure 1) is given by
\[
\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} = 1, \quad k \in \mathbb{R} \setminus \{ -a^2, -b^2 \}
\] (2.1)
serves as a parameter in the family. All these conics share the focal points
\[ F_{1,2} = (\pm d, 0), \quad d := a^2 - b^2. \] (2.2)

The confocal family sends through each point \( P \) outside the common axes of symmetry two orthogonally intersecting conics, one ellipse and one hyperbola [15, p. 38]. The parameters \((k_e, k_h)\) of these two conics define the elliptic coordinates of \( P \) with
\[-a^2 < k_h < -b^2 < k_e.\]

If \((x, y)\) are the cartesian coordinates of \( P \), then \((k_e, k_h)\) are the roots of the quadratic equation
\[
k^2 + (a^2 + b^2 - x^2 - y^2)k + (a^2b^2 - b^2x^2 - a^2y^2) = 0, \]
while conversely
\[
x^2 = \frac{(a^2 + k_e)(a^2 + k_h)}{d^2}, \quad y^2 = -\frac{(b^2 + k_e)(b^2 + k_h)}{d^2}. \] (2.4)

Let \((a, b) = (a_c, b_c)\) be the semiaxes of the ellipse \( e \) with \( k = 0 \). Then, for points \( P \) on a confocal ellipse \( e \) with semiaxes \((a_e, b_e)\) and \( k = k_e > 0 \), i.e., exterior to \( c \), the standard parametrization yields
\[
P = (x, y) = (a_e \cos t, b_e \sin t), \quad 0 \leq t < 2\pi,
\]
with \( a_e^2 = a_c^2 + k_e, \quad b_e^2 = b_c^2 + k_e \). \( \) (2.5)

For the elliptic coordinates \((k_e, k_h)\) of \( P \) follows from (2.3) that
\[
k_e + k_h = a_e^2 \cos^2 t + b_e^2 \sin^2 t - a_c^2 - b_c^2.
\]

After introducing the respective tangent vectors of \( e \) and \( c \), namely
\[
\mathbf{t}_e(t) := (-a_e \sin t, b_e \cos t), \quad \mathbf{t}_c(t) := (-a_c \sin t, b_c \cos t), \quad \text{where} \quad \|\mathbf{t}_e\|^2 = \|\mathbf{t}_c\|^2 + k_e,
\] (2.6)
we obtain\footnote{The norm \(|\mathbf{t}_c|\) equals half length of the diameter of \(e\) which is parallel to \(\mathbf{t}_c\).}
\[
k_h = k_h(t) = -(a_c^2 \sin^2 t + b_c^2 \cos^2 t) = -||\mathbf{t}_c(t)||^2 = -||\mathbf{t}_e(t)||^2 + k_e \tag{2.7}
\]
and
\[
||\mathbf{t}_e(t)||^2 = k_e - k_h(t). \tag{2.8}
\]
Note that points on the confocal ellipses \(e\) and \(c\) with the same parameter \(t\) have the same coordinate \(k_h\). Consequently, they belong to the same confocal hyperbola (Figure 5). Conversely, points of \(e\) or \(c\) on this hyperbola have a parameter out of \(\{t, -t, \pi + t, \pi - t\}\).

\[\text{Figure 1.} \quad \text{The tangents from the point} \ P \ \text{to the conics of a confocal family are symmetric w.r.t. the tangents at} \ P \ \text{to the confocal conics passing through} \ P.\]

Normal vectors of \(e\) and \(c\) can be defined respectively as
\[
\mathbf{n}_e(t) := \left( \frac{\cos t}{a_e}, \frac{\sin t}{b_e} \right), \quad \text{where} \ ||\mathbf{n}_c(t)|| = \frac{||\mathbf{t}_c(t)||}{a_c b_c}. \tag{2.9}
\]

We complete with two useful relations between the parameter \(t\) and the second elliptic coordinate \(k_h(t)\):
\[
\tan^2 t = -\frac{b_c^2 + k_h(t)}{a_c^2 + k_h(t)} \quad \text{and} \quad \sin t \cos t = \frac{a_h b_h}{d^2} \tag{2.10}
\]
with \(a_h\) and \(b_h\) as semiaxes of the hyperbola corresponding to the parameter \(t\) i.e., \(a_h^2 = a_c^2 + k_h\) and \(b_h^2 = -(b_c^2 + k_h)\).

\textbf{Proof.} From \[\text{2.7}\] follows
\[
k_h = -\frac{a_c^2 \tan^2 t + b_c^2}{1 + \tan^2 t}, \quad \text{hence} \quad \tan^2 t (a_c^2 + k_h) = -b_c^2 - k_h
\]
and
\[
\sin t \cos t = \frac{\tan t}{1 + \tan^2 t} = \frac{\sqrt{-(b_c^2 + k_h)(a_c^2 + k_h)}}{a_c^2 - b_c^2} = \frac{a_h b_h}{d^2}. \quad \square
\]
Referring to Figure 1, the following lemma addresses an important property of confocal conics (note, e.g., [15, p. 38, 309]).

Lemma 2.1. The tangents drawn from any fixed point $P$ to the conics of a confocal family share the axes of symmetry, which are tangent to the two conics passing through $P$.

This means, if a ray is reflected at $P$ in one of the conics passing through, then the incoming and the outgoing ray contact the same confocal ellipse or hyperbola.

Below, we report about results concerning a pair of confocal conics. Due to their meaning for billiards in ellipses, we restrict ourselves to pairs $(e,c)$ of confocal ellipses with $c$ in the interior $e$, and we call $c$ the caustic (Figure 2).

Lemma 2.2. Let $P = (a_e \cos t, b_e \sin t)$ with elliptic coordinates $(k_e, k_h)$ be a point on the ellipse $e$ with $k_e > 0$ and $c$ be the confocal ellipse with $k = 0$. Then, the angle $\theta(t)/2$ between the tangent at $P$ to $e$ and any tangent from $P$ to $c$ satisfies

$$\sin^2 \frac{\theta}{2} = \frac{k_e}{\|t_e(t)\|^2} = \frac{k_e}{k_e - k_h}, \quad \tan \frac{\theta}{2} = \pm \sqrt{-\frac{k_e}{k_h}},$$

$$\cos \theta = 1 - \frac{2k_e}{\|t_e(t)\|^2} = \frac{k_h + k_e}{k_h - k_e}, \quad \sin \theta = \pm \frac{2\sqrt{-k_e k_h}}{k_e - k_h}.$$  

(2.11) (2.12)

Proof. The tangent $t_P$ to $e$ at $P = (a_e \cos t, b_e \sin t)$ in direction of $t_e$ has the slope

$$f := \tan \alpha_1 = \frac{-b_e \cos t}{a_e \sin t}.$$
If $s_1$ and $s_2$ denote the slopes of the tangents from $P$ to $c$, then they satisfy
\[ y - b_e \sin t = s_i(x - a_e \cos t), \quad i = 1, 2. \]

As tangents of $c$, their homogeneous line coordinates
\[ (u_0: u_1: u_2) = ((b_e \sin t - s_i a_e \cos t): s_i : -1) \]
must satisfy the tangential equation $-u_0^2 + a_e^2 u_1^2 + b_e^2 u_2^2 = 0$ of $c$. This results in a quadratic equation for the unknown $s$, namely
\[ (a_e^2 \sin^2 t - k_e)s^2 + 2a_e b_e s \sin t \cos t + (b_e^2 \cos^2 t - k_e) = 0. \]

We conclude
\[ s_1 + s_2 = \frac{-2a_e b_e \sin t \cos t}{a_e^2 \sin^2 t - k_e} \quad \text{and} \quad s_1 s_2 = \frac{b_e^2 \cos^2 t - k_e}{a_e^2 \sin^2 t - k_e}. \]

The slopes $f = \tan \alpha_1$ of $t_P$ and $\tan \alpha_2 = s_1$ or $s_2$ of the tangents to $c$ imply for the enclosed signed angle $\theta(t)/2$ (for brevity, we often suppress the parameter $t$)
\[ \tan \frac{\theta}{2} = \tan(\alpha_1 - \alpha_2) = \frac{s_1 - f}{1 + s_1 f} = \frac{f - s_2}{1 + s_2 f}, \]

hence
\[ \tan^2 \frac{\theta}{2} = \frac{(s_1 - f)(f - s_2)}{(1 + s_1 f)(1 + s_2 f)} = \frac{f(s_1 + s_2) - s_1 s_2 - f^2}{f(s_1 + s_2) + 1 + f^2 s_1 s_2}. \]

After some computations, we obtain
\[ \tan^2 \frac{\theta}{2} = \frac{k_e}{a_e^2 \sin^2 t + b_e^2 \cos^2 t - k_e} = \frac{k_e}{\|t_e\|^2 - k_e} = \frac{k_e}{\|t_e\|^2}, \]

therefore
\[ \cot^2 \frac{\theta}{2} = \frac{\|t_e\|^2}{k_e} - 1 \quad \text{and} \quad \sin^2 \frac{\theta}{2} = \frac{1}{1 + \cot^2 \frac{\theta}{2}} = \frac{k_e}{\|t_e\|^2}, \]

where $k_e = a_e^2 - b_e^2 = b_e^2 - b_e^2$, and finally
\[ \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} = 1 - \frac{2k_e}{\|t_e\|^2 + k_e} = \frac{\|t_e\|^2 - k_e}{\|t_e\|^2 + k_e}. \]

**Remark 2.3.** A change of the origin $k = 0$ for the elliptic coordinates in a family of confocal conics corresponds to a shift of the coordinates. Hence, if in Lemma 2.2 the ellipse $c$ is replaced by another confocal conic with the coordinate $k$, then the formulas (2.11) and (2.12) remain valid under the condition that we replace $k_e$ by $k_e - k$ and $k_h$ by $k_h - k$.

**Lemma 2.4.** Let $P_1P_2$ be a chord of the ellipse $e$, which contacts the caustic $c$ at the point $Q_1$. Then the signed distances of the line $[P_1, P_2]$ to the center $O$ and to the pole $R_1$ w.r.t. $e$ have the constant product $-k_e$. The lines $[P_1, P_2]$ and $[Q_1, R_1]$ are orthogonal (Figure 2).
Proof. Let the side $P_1P_2$ touch the caustic $c$ at the point $Q_1 = (a_c \cos t'_1, b_c \sin t'_1)$. Then the Hessian normal form of the spanned line $t_Q = [P_1, P_2]$ reads

$$t_Q: \frac{b_c \cos t'_1 x + a_c \sin t'_1 y - a_c b_c}{\sqrt{b_c^2 \cos^2 t'_1 + a_c^2 \sin^2 t'_1}} = 0,$$

Its pole w.r.t. $e$ has the coordinates

$$R_1 = \left(\frac{a_c^2 \cos t'_1}{a_c}, \frac{b_c^2 \sin t'_1}{b_c}\right). \quad (2.13)$$

This yields for the signed distances to the line $t_Q$

$$\bar{O}t_Q = -\frac{a_c b_c}{\|t_c(t'_1)\|} \quad (2.14)$$

and

$$\frac{\bar{R}_1 t_Q}{\bar{R}_1 Q_1} = \frac{k_c(b_c^2 \cos^2 t'_1 + a_c^2 \sin^2 t'_1)}{a_c b_c \|t_c(t'_1)\|} = \frac{k_c \|t_c(t'_1)\|}{a_c b_c}. \quad (2.15)$$

Thus, we obtain a constant product $\bar{O}t_Q \cdot \bar{R}_1 t_Q = -k_c$, as stated above.

The last statement holds since $R_1$ and $Q_1$ are the poles of $[P_1, P_2]$ w.r.t. the confocal conics $e$ and $c$. It is wellknown that the poles of any line $\ell$ w.r.t. a family of confocal conics lie on a line $\ell^*$ orthogonal to $\ell$ (see, e.g., [15, p. 340]).

3. Confocal conics and billiards

By virtue of Lemma 2.1 all sides of a billiard inscribed to the ellipse $e$ with parameter $k = k_c$ are tangent to a fixed conic $c$ confocal with $e$ (Figure 2). This conic $c$ with parameter $k_c$ is called caustic. It can be a smaller ellipse with $-b^2 < k_c < k_e$ or a hyperbola with $-a^2 < k_c < -b^2$ or, in the limiting case with $k_c = -b^2$, consist of the pencils of lines with the focal points $F_1, F_2$ of $c$ as carriers. At the beginning, we confine ourselves to an ellipse as caustic with $k_c = 0$ (Figure 2), and we speak of elliptic billiards. Only the Figures 10, 11 and 12 show examples of (periodic) billiards in $e$ with a hyperbola as caustic, called hyperbolic billiards.

For billiards $\ldots P_1P_2P_3 \ldots$ in the ellipse $e$ and with the ellipse $c$ as caustic, we assume from now on a counter-clockwise order and signed exterior angles $\theta_1, \theta_2, \theta_3, \ldots$ (see Figure 2). The tangency points $Q_1, Q_2, \ldots$ of the billiard’s sides $P_1P_2, P_2P_3, \ldots$ with the caustic $c$ subdivide the sides into two segments. We denote the lengths of the segments adjacent to $P_i$ as

$$l_i := \overline{P_iQ_i} \quad \text{and} \quad r_i := \overline{P_iQ_{i-1}}. \quad (3.1)$$

Based on the parametrizations $(a_c \cos t, b_c \sin t)$ of $e$ and $(a_c \cos t', b_c \sin t')$ of $c$, we denote the respective parameters of $P_1, Q_1, P_2, Q_2, P_3, \ldots$ with $t_1, t'_1, t_2, t'_2, t_3, \ldots$ in strictly increasing order.

The following two lemmas deal with sides of billiards in the ellipse $e$. 


Lemma 3.1. The connecting line \([P_i, P_{i+1}]\) of the vertices with respective parameters \(t_1, t_2\) on \(e\) contacts the caustic \(c\) if and only if
\[
\frac{a^2_e}{a^2_c} \cos^2 \frac{t_1 + t_2}{2} + \frac{b^2_e}{b^2_c} \sin^2 \frac{t_1 + t_2}{2} = \cos^2 \frac{t_1 - t_2}{2}.
\]
This is equivalent to
\[
\sin^2 \frac{t_1 - t_2}{2} = \frac{k_e}{a_e b_e} \left\| t_e \left( \frac{t_1 + t_2}{2} \right) \right\|^2.
\]
Proof. The line connecting the points \((a_e \cos t_i, b_e \sin t_i), i = 1, 2\), has homogeneous line coordinates \((u_0 : u_1 : u_2)\) equal to
\[
(a_e b_e (\cos t_1 \sin t_2 - \sin t_1 \cos t_2) : b_e (\sin t_1 - \sin t_2) : a_e (\cos t_2 - \cos t_1)).
\]
It contacts the caustic \(c\) if \(-u^2_0 + a^2_e u^2_1 + b^2_e u^2_2 = 0\), i.e.,
\[
\frac{a^2_e}{a^2_c} \frac{b^2_e}{b^2_c} \sin^2 \frac{t_1 - t_2}{2} \cos^2 \frac{t_1 + t_2}{2} + \frac{b^2_e}{b^2_c} \sin^2 \frac{t_1 + t_2}{2} \sin^2 \frac{t_1 - t_2}{2} = \frac{a^2_e}{a^2_c} \frac{b^2_e}{b^2_c} \sin^2 \frac{t_2 - t_1}{2} \cos^2 \frac{t_2 + t_1}{2}.
\]
Under the condition \(\sin[(t_1 - t_2)/2] \neq 0\) we obtain the first claimed equation. The second follows after the substitutions \(a^2_c = a^2_e - k_e\) and \(b^2_c = b^2_e - k_e\) from
\[
1 - \frac{k_e}{a^2_e b^2_e} \left( \frac{b^2_e \cos^2 \frac{t_1 + t_2}{2} + a^2_e \sin^2 \frac{t_1 + t_2}{2}}{2} \right) = \cos^2 \frac{t_2 - t_1}{2}.
\]
by \((2.9)\) .

Lemma 3.2. Referring to the notation in Lemma 3.1, if the side \(P_i P_{i+1}\) contacts the caustic \(c\) at \(Q_i\) with parameter \(t_i\), then
\[
\sin t_i' = \frac{b_c}{b_e} \frac{\sin \frac{t_i + t_{i+1}}{2}}{\cos \frac{t_i + t_{i+1}}{2}}, \quad \cos t_i' = \frac{a_c}{a_e} \frac{\cos \frac{t_i + t_{i+1}}{2}}{\cos \frac{t_i + t_{i+1}}{2}}, \quad \tan t_i' = \frac{b_c a_c}{a_e b_e} \tan \frac{t_i + t_{i+1}}{2}.
\]
Proof. The tangent to \(c\) at \(Q_i\) has the line coordinates
\[
(u_0 : u_1 : u_2) = (-a_c b_c : b_c \cos t'_1 : a_c \sin t'_1),
\]
which must be proportional to those in the proof of Lemma 3.1.

Remark 3.3. The half-angle substitution
\[
\tau_i := \tan \frac{t_i}{2} \quad \text{for } i = 1, 2
\]
allows to express the equation in Lemma 3.1 (for \(i = 1\)) in projective coordinates on \(e\). We obtain a symmetric biquadratic condition
\[
b^2_e k_e \tau_1^2 \tau_2^2 - b^2_e a^2_e (\tau_1^2 + \tau_2^2) + 2(a^2_e k_e + a^2_e b^2_e) \tau_1 \tau_2 + b^2_e k_e = 0,
\]
which defines a 2-2-correspondence on \(e\) between the endpoints \(P_1, P_2\) of a chord which contacts \(c\). This remains valid after iteration, i.e., between the initial point \(P_i\) and the endpoint \(P_{N+1}\) of a billiard after \(N\) reflections in \(e\). Now, we recall a classical argument for the underlying Poncelet porism (see also \cite{17} and the references there): A 2-2-correspondence different from the identity keeps fixed at most four points. However, four fixed points on \(e\)
are already known as contact points between \(e\) and the common complex conjugate (isotropic) tangents \(c\) with the caustic \(c\), since tangents of \(e\) remain fixed under the reflection in \(e\). If therefore one \(N\)-sided billiard in \(e\) with caustic \(c\) closes, then the correspondence is the identity and all these billiards close.

Given any billiard \(P_1P_2 \ldots\) in the ellipse \(e\), let \(p_i = (x_i, y_i)\) denote the position vector of \(P_i\) for \(i = 1, 2, \ldots\), while \(u_1, u_2, \ldots\) denote the unit vectors of the oriented sides \(P_1P_2, P_2P_3, \ldots\). By (2.9), the vector \(n_{e|i} := (x_i/a_e^2, y_i/b_e^2)\) is orthogonal to \(e\) at \(P_i\). According to [2, Proposition 2.1], the scalar product

\[
J_e := -\langle u_i, n_{e|i} \rangle
\]

is invariant along the billiard in \(e\) and called Joachimsthal integral (note also [26, p. 54]).

The invariance of the Joachimsthal integral, which also holds in higher dimensions for billiards in quadrics, is the key result for the integrability of billiards, i.e., in the planar case for the existence of a caustic [2, p. 3]. In our approach, the invariance of \(J_e\) follows from Lemma 2.2.

**Lemma 3.4.** The Joachimsthal integral \(J_e := -\langle u_i, n_{e|i} \rangle\) equals

\[
J_e = \frac{\sqrt{k_e}}{a_e b_e}
\]

with \(k_e\) as elliptic coordinate of \(e\) w.r.t. \(c\), i.e., \(k_e = a_e^2 - a_c^2 = b_e^2 - b_c^2\).

**Proof.** From (2.11) follows for the points \((a_e \cos t, b_e \sin t)\) of \(e\)

\[
J_e = -\langle u, n_e \rangle = -\cos \left( \frac{\pi}{2} + \frac{\theta}{2} \right) \|n_e\| = \sin \frac{\theta}{2} \|n_e\| = \sin \left( \frac{\theta}{2} \frac{\|t_e\|}{\|t_e\|^2} \right),
\]

hence by (2.11), (2.7), (2.6), and (2.9)

\[
J_e^2 = \sin^2 \frac{\theta}{2} \|n_e\|^2 = \frac{k_e}{\|t_e\|^2} \|n_e\|^2 = \frac{k_e}{a_e^2 b_e^2}.
\]

\(\text{They follow from the second equation in Lemma 3.1 from } t_1 = t_2.\)
This confirms the claim. \[\Box\]

3.1. Poncelet grid

The following theorem is the basis for the Poncelet grid associated to each billiard. We formulate and prove a projective version. The special case dealing with confocal conics, has already been published by Chasles [9, p. 841] and later by Böhmer in [8, p. 221]. The same theorem was studied in [22] and in [1]. In [18], the authors proved it in a differential-geometric way.

In the theorem and proof below, the term conic stands for regular dual conics, i.e., conics seen as the set of tangent lines, but also for pairs of line pencils and for single line pencils with multiplicity two. Expressed in terms of homogeneous line coordinates, the corresponding quadratic forms have rank 3, 2 or 1, respectively. Moreover, we use the term range for a pencil of dual conics. The term net denotes a 2-parametric linear system of dual curves of degree 2. Obviously, conics and ranges included in a net play the role of points and lines of a projective plane within the 5-dimensional projective space of dual conics. Any two ranges in a net share a conic (compare with [10 Théorèmes I–IV]).

**Theorem 3.5.** Let \( c \) be a regular conic and \( A_1, B_1 \) two points such that the tangents \( t_1, \ldots, t_4 \) drawn from \( A_1 \) and \( B_1 \) to \( c \) form a quadrilateral. Its remaining pairs of opposite vertices are denoted by \((A_i, B_i), i = 2, 3\). Then,

1. for each conic \( c_1 \) passing through \( A_1 \) and \( B_1 \), the range \( R_c \) spanned by \( c \) and \( c_1 \) contains conics \( c_i \) passing through \( A_i \) and \( B_i \), simultaneously. The tangents at \( A_j \) and \( B_j \) to \( c_j \) for \( j = 1, 2, 3 \) meet at a common point \( R \). If \( c_i \) has rank 2, then we obtain, as the limit of \( c_i \), the diagonal \([A_i, B_i]\) of the quadrilateral \( t_1, \ldots, t_4 \).
2. This result holds also in the limiting case $t_1 = t_2$, where the chord $A_1B_1$ of $c_1$ contacts $c$ at $B_2$.

In Figure 4 the particular case is displayed where $c$ and $c_1 = e$ span a range $\mathcal{R}_e$ of confocal conics (note also [9] Fig. 19). Then by Lemma 2.1 the tangents at $A_j$ and $B_j$ to $c_j$ are angle bisectors of the quadrilateral. In case of a rank deficiency of $c_i$, either one axis of symmetry of the confocal family or the line at infinity shows up as $c_i$.

Proof. The conics tangent to $t_1, \ldots, t_4$ define a range $\mathcal{R}_t$, which includes for $j = 1, 2, 3$ the pairs of line pencils $(A_j, B_j)$ as well as the initial conic $c$. On the other hand, $c$ and $c_1$ span a range $\mathcal{R}_c$. Since both ranges share the conic $c$, they span a net $\mathcal{N}$ of conics.

The pair $(A_1, B_1)$ of line pencils spans together with $c_1$ the range of conics sharing the points $A_1, B_1$ and the tangents there, which meet at a point $R$. This range, which also belongs to $\mathcal{N}$, contains the rank-1 conic with carrier $R$. Each pair of line pencils $(A_i, B_i)$, $i = 1, 2$, spans with the pencil $R$ again a range within $\mathcal{N}$. This range shares with the range $\mathcal{R}_c$ a conic $c_i$ passing through $A_i$ and $B_i$ with respective tangent lines through $R$.

All these conclusions remain valid in the case, when $\mathcal{R}_t$ consists of conics which touch $c$ at $B_2$ and are tangent to $t_3$ and $t_4$.

As already indicated by the notation, we are interested in the particular case of Theorem 3.5 where the conics $c$ and $e$ in the range $\mathcal{R}_c$ are confocal. The following result follows directly from Theorem 3.5 and summarizes properties of the Poncelet grid. For the points of intersection between extended sides of a billiard $\ldots P_0P_1P_2 \ldots$ we use the notation

$$S_i^{(j)} := \begin{cases} [P_{i-k-1}, P_{i-k}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k, \\ [P_{i-k}, P_{i-k+1}] \cap [P_{i+k}, P_{i+k+1}] & \text{for } j = 2k - 1 \end{cases}$$

(3.3)

where $i = \ldots, 0, 1, 2, \ldots$ and $j = 1, 2, \ldots$ Note that there are $j$ sides between those which intersect at $S_i^{(j)}$, and ‘in the middle’ of these $j$ sides there is for even $j$ the vertex $P_i$ and otherwise the point of contact $Q_i$. At the same token, the point $S_i^{(j)}$ is the pole of the diagonal $[Q_{i-k-1}, Q_{i+k}]$ or $[Q_{i-k}, Q_{i+k}]$ of the polygon $\ldots Q_1Q_2Q_3 \ldots$ of contact points w.r.t. the caustic.

Theorem 3.6. Let $\ldots P_0P_1P_2 \ldots$ be a billiard in the ellipse $e$ with sides $P_iP_{i+1}$ which contact the ellipse $c$ at the respective points $Q_i$ for all $i \in \mathbb{Z}$. Then the vertices $S_i^{(j)}$ of the associated Poncelet grid are distributed on the following conics.

1. The points $S_i^{(1)}, S_i^{(3)}, \ldots$ are located on the confocal hyperbola through $Q_i$, while the points $S_i^{(2)}, S_i^{(4)}, \ldots$ are located on the confocal hyperbola through $P_i$.

\footnote{An extended version of this theorem in [23] addresses the symmetry between the ranges $\mathcal{R}_e$ and $\mathcal{R}_t$. This generalizes the statement that in the case of confocal conics $c$ and $c_1$ the quadrilateral $A_1A_2B_1B_2$ has an incircle $d$ (Figures 4, 6 and 14 compare with [11, 13]).}
2. For each \( j \in \{1, 2, \ldots\} \), the points \( S_i^{(j)} S_{i+(j+1)}^{(j)} \) are vertices of another billiard with the caustic \( c \) inscribed in a confocal ellipse \( e^{(j)} \), provided that \( e^{(j)} \) is regular. Otherwise \( e^{(j)} \) coincides with an axis of symmetry or with the line at infinity. The locus \( e^{(j)} \) is independent of the position of the initial vertex \( P_0 \in e \).

![Figure 5. Periodic billiard \((N = 7)\) with extended sides.](image)

**Proof.** 1. The side lines \([P_0, P_1]\) (\(P_0 = P_7\) in Figure 5) and \([P_2, P_3]\) meet at \( S_1^{(1)} \), while \([P_1, P_2]\) contacts \( c \) at \( Q_1 \). By Theorem 3.6, the points \( Q_1 \) and \( S_1^{(1)} \) belong to the same confocal hyperbola.

Now we go one step away from \( Q_1 \): the tangents from \( P_0 \) and \( P_3 \) to \( c \) intersect at \( S_1^{(1)} \) and \( S_1^{(3)} = [P_1, P_0] \cap [P_3, P_4] \). The confocal conic through \( S_1^{(1)} \) and \( S_1^{(3)} \) must again be the hyperbola through \( Q_1 \). This follows by continuity after choosing \( Q_1 \) on one axis of symmetry. Iteration confirms the first claim in Theorem 3.6.

The tangents to \( c \) from \( P_0 \) and \( P_2 \) form a quadrilateral with \( P_1 \) and \( S_1^{(2)} \) as opposite vertices. Therefore, there exists a confocal conic passing through both points. This conic must be a hyperbola, as can be concluded by continuity: If \( P_1 \) is specified at a vertex of \( c \), then due to symmetry the points \( P_1 \) and \( S_1^{(2)} \) are located on an axis of symmetry.

The tangents to \( c \) from \( P_{-1} \) and \( P_3 \) form a quadrilateral with \( S_1^{(2)} \) and \( S_1^{(4)} \) as opposite vertices. Theorem 3.6 and continuity guarantee that this is again the confocal hyperbola through \( P_1 \). Iteration shows the same of \( S_1^{(6)} \) etc. However, the points \( P_1, S_1^{(2)}, S_1^{(4)}, \ldots \) need not belong to the same branch of the hyperbola.

2. The tangents through \( P_2 \) and \( S_2^{(2)} \) (note Figure 5) form a quadrilateral with \( S_2^{(1)} \) and \( S_2^{(1)} \) as opposite vertices. This time, continuity shows that the
two points belong to the same confocal ellipse \( e^{(1)} \). The same holds for the tangents through \( P_3 \) and \( S_3^{(2)} \) etc.

Similarly, starting with the points \( P_0 \) and \( P_3 \), we find the ellipse \( e^{(2)} \) through \( S_1^{(2)} \) and \( S_2^{(2)} \), and so on.

In order to prove that these ellipses \( e^{(1)}, e^{(2)}, \ldots \) are independent of the choice of the initial point \( P_1 \in e \), we follow an argument from [2] proof of Corollary 2.2: The claim holds for all confocal ellipses \( e \) where billiards with the same caustic \( c \) are aperiodic and traverse \( e \) infinitely often. Since these ellipses form a dense set, the claim holds also for those with periodic billiards. The invariance of the ellipses \( e^{(1)}, e^{(2)}, \ldots \) is already mentioned in [2] Theorem 7.

An alternative proof consists in demonstrating that the elliptic coordinate \( k_e^{(j)} \) of \( e^{(j)} \) for all \( j \) does not depend on the parameter \( t \). As one example, we present below in (3.7), (3.6) and (3.8) formulas for the semiaxes \( a_{e|1}, b_{e|1} \) and the elliptic coordinate \( k_{e|1} \) of \( e^{(1)} \).

**Remark 3.7.** Figure 5 reveals, that the polygons \( P_1 S_1^{(1)} P_2 S_2^{(1)} \ldots \) as well as \( P_1 S_1^{(2)} P_3 S_3^{(2)} \ldots \) and \( S_1^{(2)} S_2^{(1)} S_2^{(2)} S_1^{(1)} \ldots \) are zigzag billiards in rings bounded by two confocal ellipses. However, we find also zigzag billiards between two confocal hyperbolas, e.g., \( S_1^{(2)} P_2 P_1 S_2^{(2)} \ldots \) or the twofold covered \( \ldots S_1^{(2)} S_1^{(1)} P_1 Q_1 P_1 S_1^{(1)} S_1^{(2)} \ldots \). Billiards between other combinations of confocal conics can be found in [12].

The coming lemma addresses invariants related to the incircles of quadrilaterals built from the tangents to \( c \) from any two vertices \( P_i \) and \( P_j \) of a billiard in \( e \) (note circle \( d \) in Figure 4 and [1] 5 [18]).

**Lemma 3.8.** Referring to Figure 6, the power \( w^2 \) of the point \( S_i^{(1)} \) w.r.t. the incircle of the triangle \( P_i P_{i+1} S_i^{(1)} \) is the same for all \( i \). Similarly, the power \( w_i^2 \) of \( S_i^{(2)} \) w.r.t. the incircle of the quadrangle \( P_i S_i^{(1)} S_{i+1}^{(2)} S_i^{(1)} \) is constant.

**Proof.** According to Graves’s construction [15, p. 47], an ellipse \( e \) can be constructed from a smaller ellipse \( c \) in the following way: Let a closed piece of string strictly longer than the perimeter of \( c \) be posed around \( c \). If point \( P \) is used to pull the string taut, then \( P \) traces a confocal ellipse \( e \). Consequently, for each vertex \( P_i \) and neighboring tangency points \( Q_{i-1} \) and \( Q_i \) of a billiard in \( e \) with caustic \( c \), the sum of the lengths \( Q_{i-1} P_i \) and \( P_i Q_i \) minus the length of the elliptic arc between \( Q_{i-1} \) and \( Q_i \), i.e.,

\[
D_e := Q_{i-1} P_i + P_i Q_i - Q_{i-1} Q_i \quad (3.4)
\]

is constant (Figure 6).

The incircle of \( P_2 P_3 S_2^{(1)} \) has the center \( R_2 \) and the radius \( Q_2 R_2 \) by (2.15). The power of \( P_2 \) w.r.t. this circle is \( l^2_2 \), that of \( P_3 \) is \( r^2_3 \). From Graves’ construction follows for the ellipse \( e \) that

\[
D_e = r_2 + l_2 - Q_1 Q_2 = r_3 + l_3 - Q_3 Q_3 = \text{const}.
\]
is the same for all $P_i$, provided that $\overline{Q_iQ_j}$ denotes the length of the (shorter)
arc along $c$ between $Q_i$ and $Q_j$. For the analogue invariant at $e^{(1)}$ follows (Figure 6)
\[
D_{e^{(1)}} := \frac{Q_1S_2^{(1)}}{S_2^{(1)}Q_3} - \frac{Q_1Q_3}{2D_e + 2w} = \text{const.,}
\]
hence $w = \text{const.}$, where $w^2$ is the power of $S_2^{(1)}$ w.r.t. the said incircle.

Since the incircle of the quadrangle $P_2S_1^{(1)}S_2^{(2)}S_2^{(1)}$ is an excircle of the triangle $P_2P_3S_2^{(1)}$ (Figure 6), the power of $P_2$ w.r.t. the excircle equals $w^2$, too. This
follows from elementary geometry.

As an alternative, the constancy of $w$ can also be concluded from the fact,
that neighboring circles with centers $R_1^1$, $R_i$ or $R_i^1$, $R_{i+1}$ share three tangents,
and one circle is an incircle, the other an excircle of the triangle. Therefore,
on the common side the same lengths $w$ shows up twice and also at the
adjacent pairs of neighboring circles. Since the distance $w$ is constant for all
aperiodic billiards, it reveals also for periodic billiards that $w$ is independent
of the choice of the initial vertex. In a similar way follows the invariance of
the lengths $w_1$, as shown in Figure 6.

\[\text{Figure 6. The power } w^2 \text{ of } S_2^{(1)} \text{ w.r.t. the incircle of the}
\text{triangle } S_2^{(1)}P_2P_3 \text{ shows up at all } S_i^{(1)} \text{ and equals the power}
of } P_i \text{ w.r.t. the incircle of the quadrangle } P_iS_i^{(1)}S_i^{(2)}S_i^{(1)}.\]
It needs to be noted that $S_2^{(1)}$ or $S_2^{(2)}$ can be located on the other branch of the related hyperbola. Then the said ‘incircle’ of the triangle $P_2P_3S_2^{(1)}$ has to be replaced by the excircle which contacts $c$ at $Q_2$ and is tangent to the side lines $[P_1,P_2]$ and $[P_3,P_4]$. Similarly, the said ‘incircle’ of the quadrangle becomes an excircle. In all these cases, Lemma 3.8 and the proof given above have to be adapted.

Remark 3.9. The Theorems 3.5 and 3.6 as well as the constancy of the length $w$ according to Lemma 3.8 are also valid in spherical geometry (note [23]) and in hyperbolic geometry. On the sphere (see Figure 7), the caustic consists of a pair of opposite components, and for $N$-periodic billiards the confocal spherical ellipse $e^{(j)}$ coincides with $e^{(N-2-j)}$ w.r.t. the opposite caustic. We obtain other families of incircles when we focus on pairs of consecutive sides of the billiards in $e^{(1)}$, $e^{(2)}$ and so on. However, these circles are not mutually disjoint. By the way, the centers $R_i^{(j)}$ of all these circles are the poles of diagonals of $\ldots P_1P_2P_3\ldots$ w.r.t. the ellipse $e$.

For the sake of completeness, we express below in (3.9) the distance $w$ in terms of the semiaxes of $e$ and $c$. For this purpose, we compute first
the semiaxes \(a_{e|1}\) an \(b_{e|1}\) of \(e^{(1)}\), since we need the coordinates of \(S_2^{(1)}\). From (2.13) and (2.15) follows
\[
R_2 = \left( \frac{a_e^2 \cos t_2'}{a_c}, \frac{b_e^2 \sin t_2'}{b_c} \right), \quad Q_2R_2 = \frac{k_e \| t_e(t_2') \|}{a_c b_c}.
\]
By virtue of Theorem 3.5 the tangents from \(Q_2\) to the confocal hyperbola through \(Q_2\) contact at
\[
Q_2 = (a_c \cos t_2', b_c \sin t_2') \in c \quad \text{and} \quad S_2^{(1)} = (a_{e|1} \cos t_2', b_{e|1} \sin t_2') \in e^{(1)}.
\]
Both points lie on the polar of \(R_2\) w.r.t. the hyperbola in balance with the elliptic coordinate \(k_h = -\|t_e(t_2')\|^2\). This yields the condition
\[
\frac{a_e^2 \cos t_2'}{a_c(a_e^2 + k_h(t_2'))} a_{e|1} \cos t_2' + \frac{b_e^2 \sin t_2'}{b_c(b_e^2 + k_h(t_2'))} b_{e|1} \sin t_2' = 1
\]
or, by virtue of (2.7),
\[
a_e^2 b_c a_{e|1} - b_e^2 a_e b_{e|1} = a_c b_c d^2, \quad \text{where} \quad a_{e|1}^2 - b_{e|1}^2 = d^2.
\]
We eliminate \(a_{e|1}\) and obtain after some computation the quadratic equation
\[
(b_e^4 - 2b_e^2 b_e^4 - b_e^2 d^2) b_{e|1}^2 + 2a_e^2 b_c b_e b_{e|1} + b_e^2 (a_e^2 d^2 - a_e^4) = 0.
\]
The second solution besides \(b_{e|1} = b_c\) is
\[
b_{e|1} = \frac{b_c(a_e^2 d^2 - a_e^4)}{b_e^4 - 2b_e^2 b_e^4 - b_e^2 d^2} = \frac{b_c(a_e^2 b_e^2 + d^2 k_e)}{a_e^2 b_e^2 - k_e^2}.
\]
This implies
\[
a_{e|1} = \frac{a_c}{a_e^2 b_c} (b_c d^2 + b_e^2 b_{e|1}) = \frac{a_c(a_e^2 b_e^2 - d^2 k_e)}{a_e^2 b_e^2 - k_e^2}
\]
and
\[
k_{e|1} = b_{e|1}^2 - b_c^2 = k_e \left(\frac{2a_c b_c a_{e|1} b_{e|1}}{a_e^2 b_c^2 - k_e^2}\right)^2.
\]
This leads finally to
\[
w := \frac{2a_c b_c \sqrt{k_e}}{a_e^2 b_c^2 - k_e^2}.
\]

Negative semiaxes \(a_{e|1}, b_{e|1}\) and a negative \(w\) in the formulas above mean that the points \(S_2^{(1)}\) are located on the respective second branches of the hyperbolas and the incircles of the triangles \(P_{i+1}P_i\) become excircles. In the case of a vanishing denominator for \(k_e = a_c b_c\) (periodic four-sided billiard) the ellipse \(e^{(1)}\) is the line at infinity.

If on the right-hand side of the formulas (3.7), (3.6) and (3.8) we replace \(a_e, b_e, k_e\) respectively by \(a_{e|1}, b_{e|1}, k_{e|1}\), then we obtain expressions for \(a_{e|3}, b_{e|3}, k_{e|3}\), i.e.,
\[
k_{e|3} = k_{e|1} \left(\frac{2a_c b_c a_{e|1} b_{e|1}}{a_e^2 b_c^2 - k_e^2}\right)^2,
\]
and this can be iterated.
3.2. Conjugate billiards

For two confocal ellipses $c$ and $e$, there exists an axial scaling

$$\alpha: (x, y) \mapsto \left(\frac{a_c}{a_e} x, \frac{b_c}{b_e} y\right) \quad \text{with} \quad c \to e . \quad (3.11)$$

Corresponding points share the parameter $t$. Hence, they belong to the same confocal hyperbola (Figure 8). The affine transformation $\alpha$ maps the tangency point $Q_i \in c$ of the side $P_i P_{i+1}$ to a point $P'_i \in e$, while $\alpha^{-1}$ maps $P_i$ to the tangency point $Q'_{i-1}$ of $P_{i-1} P'_i$, i.e.,

$$\alpha: Q_i \mapsto P'_i, \quad Q'_{i-1} \mapsto P_i .$$

This results from the symmetry between $t_i$ and $t'_i$ in the equation

$$b_c a_e \cos t_i \cos t'_i + a_c b_e \sin t_i \sin t'_i = a_e b_c \quad (3.12)$$

which expresses that $P_i \in e$ with parameter $t_i$ lies on the tangent to $c$ at $Q_i$ with parameter $t'_i$. Referring to Figure 8, $\alpha$ sends the tangent $[P_{i-1}', P'_i]$ to $c$ at $Q'_{i-1}$ to the tangent $[R_{i-1}, R_i]$ to $e$ at $P_i$. Hence, by $\alpha$ the polygon $Q_1 Q_2 \ldots$ is mapped to $P'_1 P'_2 \ldots$ and furthermore to that of the poles $R_1 R_2 \ldots$ of the billiard’s sides $P_1 P_2$, $P_2 P_3$, $\ldots$.

![Figure 8. The periodic billiard $P_1 P_2 \ldots P_5$ in $e$ with the caustic $c$ and the conjugate billiard $P'_1 P'_2 \ldots P'_5$.](image)

**Definition 3.10.** Referring to Figure 8, the billiard $\ldots P'_0 P'_1 P'_2 \ldots$ is called *conjugate* to the billiard $\ldots P_0 P_1 P_2 \ldots$ in the ellipse $e$ with the ellipse $c$ as caustic, when the axial scaling $\alpha: c \to e$ defined in (3.11) maps the tangency point $Q_i$ of the side $P_i P_{i+1}$ to the vertex $P'_i$.

**Lemma 3.11.** For each billiard $\ldots P_0 P_1 P_2 \ldots$ in the ellipse $e$ with the ellipse $c$ as caustic, there exists a unique conjugate billiard $\ldots P'_0 P'_1 P'_2 \ldots$, and the relation between the two billiards in $e$ is symmetric. Moreover,

$$l_i = P_i Q_i = P'_i Q'_{i-1} = r'_i \quad \text{and} \quad r_i = P_i Q_{i-1} = P'_{i-1} Q'_{i-1} = l'_{i-1}. \quad (3.13)$$
Proof. From the symmetry in (3.12) follows for \( \alpha : c \to e \) that \( P_i \) is the preimage of \( P_i \) is the tangency point \( Q_{i-1} \) of \( P'_{i-1}P'_i \). The congruences stated in (3.13) follow from Ivory’s Theorem for the two diagonals in the curvilinear quadrangle \( P_iP_i'Q_iQ'_{i-1} \). In view of the sequence of parameters \( t_1, t'_1, t_2, t'_2, t_3, \ldots \) of the vertices \( P_1, P'_1, P_2, P'_2, P_3, \ldots \) on \( e \), the switch between the original billiard and its conjugate corresponds the interchange of \( t_i \) with \( t'_i \) for \( i = 1, 2, \ldots \). \( \square \)

Finally we recall that, based on the Arnold-Liouville theorem from the theory of completely integrable systems, it is proved in [18] that there exist canonical coordinates \( u \) on the ellipses \( e \) and \( c \) such that for any billiard the transitions from \( P_i \to P_{i+1} \) and \( Q_i \to Q_{i+1} \) correspond to shifts of the respective canonical coordinates \( u_i \) and \( u_{i+1} \) by \( 2\Delta u \). Explicit formulas for the parameter transformation \( t \mapsto u \) are provided in [24].

![Figure 9. An example of canonical coordinates on c, e and e(1), this time with origin Q1 and unit point Q3](image)

Figure 9 shows how on \( c \) such coordinates can be constructed by iterated subdivision, provided that \( Q_N \) and \( Q_2 \) get the respective canonical coordinates \( u = 0 \) and \( 1 \). A comparison with Figure 8 reveals that, in the sense of a canonical parametrization, the contact point \( Q_i \) is exactly halfway from the \( P_i \) to \( P_{i+1} \), i.e.,

\[
\Delta u_i = u_i + \Delta u, \quad u_{i+1} = u_i + 2\Delta u.
\]

Hence, the transition from a billiard to its conjugate is equivalent to a shift of canonical coordinates by \( \Delta u \).

### 3.3. Billiards with a hyperbola as caustic

As illustrated in Figure 10 billiards in ellipses \( e \) with a confocal hyperbola \( c \) as caustic are zig-zags between an upper and lower subarc of \( e \). If the initial point \( P_1 \) is chosen at any point of intersection between \( e \) and the hyperbola \( c \), then the billiard is twofold covered, and the first side \( P_1P_2 \) is tangent to \( c \) at \( P_1 \).
Here we report briefly, in which way these billiard differ from those with an elliptic caustic. Proofs are left to the readers. In view of the associated Poncelet grid, we start with the analogue to Theorem 3.6 (see Figures 10 and 12).

**Figure 10.** Periodic billiard $P_1 P_2 \ldots P_{12}$ in the ellipse $e$ with the hyperbola $c$ as caustic, together with the hyperbolas $e^{(1)}$, $e^{(3)}$ and the ellipse $e^{(2)}$ with the inscribed billiard consisting of three quadrangles $S_i^{(2)} S_{i+3}^{(2)} S_{i+6}^{(2)} S_{i+9}^{(2)}$.

**Theorem 3.12.** Let $\ldots P_0 P_1 P_2 \ldots$ be a billiard in the ellipse $e$ with the hyperbola $c$ as caustic.

1. Then the points $S_i^{(1)}$, $S_i^{(3)}$, $\ldots$ are located on confocal ellipses through the contact point $Q_i$ of $[P_i, P_{i+1}]$ with $c$, while the points $S_i^{(2)}$, $S_i^{(4)}$, $\ldots$ are located on the confocal hyperbola through $P_i$.

2. For even $j$, the points $\ldots S_i^{(j)} S_{i+(j+1)}^{(j)} S_{i+2(j+1)}^{(j)} \ldots$ are vertices of another billiard with the caustic $c$ inscribed in a confocal ellipse $e^{(j)}$, provided that the $S_i^{(j)}$ are finite.

For odd $j$, the points $S_i^{(j)}$ are located on confocal hyperbolas $e^{(j)}$ or an axis of symmetry. At each vertex of $\ldots S_i^{(j)} S_{i+(j+1)}^{(j)} S_{i+2(j+1)}^{(j)} \ldots$, one angle bisector is tangent to $e^{(j)}$.

All conics $e^{(j)}$ are independent of the position of the initial vertex $P_1 \in e$.

At the 12-periodic billiard depicted in Figure 10 the billiard inscribed to $e^{(2)}$ splits into three quadrangles (dashed). Note that for odd $j$ there are
some points $S_{i}^{(j)}$ where the tangent to $e^{(j)}$ is the interior bisector of the angle $\angle S_{i-(j+1)}^{(j)} S_{i}^{(j)} S_{i+(j+1)}^{(j)}$. Hence, we obtain no billiards inscribed to hyperbolas $e^{(j)}$ with the hyperbola $c$ as caustic. In Figure 12 the 6-periodic billiard $P_1 P_2 \ldots P_6$ yields two triangles $S_1^{(1)} S_{i+2}^{(1)} S_{i+4}^{(1)}$ inscribed to $e^{(1)}$; one of them is shaded in green.

Lemma 3.8 is also valid for hyperbolas as caustic. An example is depicted in Figure 11. The power $w^2$ of $P_9$ w.r.t. the circle tangent to the four consecutive sides $[P_7, P_8], [P_8, P_9], [P_9, P_{10}]$, and $[P_{10}, P_{11}]$ equals that of $P_{12}$ w.r.t. the incircle of the quadrilateral with sides $[P_{10}, P_{11}], [P_{11}, P_{12}], [P_{12}, P_1]$, and $[P_1, P_2]$.

Also for billiards $P_1 P_2 \ldots$ in $e$ with a hyperbola $c$ as caustic, there exists a conjugate billiard $P'_{1} P'_{2} \ldots$, and the relation is symmetric. However, the definition is different. It uses the singular affine transformation

\[ \alpha_h : e \rightarrow F_1 F_2 \text{ with } P_i \mapsto T_i = [P_i, P_{i+1}] \cap [F_1, F_2], \] (3.15)

with $F_1$ and $F_2$ as the focal points of $e$ and $c$.

**Definition 3.13.** Referring to Figure 12 the billiard $\ldots P'_0 P'_1 P'_2 \ldots$ in the ellipse $e$ with the hyperbola $c$ as caustic is called *conjugate* to the billiard $\ldots P_0 P_1 P_2 \ldots$ in $e$ with the same caustic $c$ if the axial scaling $\alpha_h$ defined in (3.15) maps the point $P'_i$ to the intersection $T_i$ of $P_i P_{i+1}$ with the principal axis.
is symmetric. Moreover, if $P_0P_1P_2\ldots$ is a billiard in the ellipse $e$ with the hyperbola $c$ as caustic, together with the conjugate billiard $P_0'P_1'P_2'\ldots$. The associated polygon with vertices on $e^{(1)}$ splits into two triangles $S_1^{(1)}S_3^{(1)}S_5^{(1)}$ (green shaded) and $S_2^{(1)}S_4^{(1)}S_6^{(1)}$.

**Lemma 3.14.** Let $\ldots P_0P_1P_2\ldots$ be a billiard in the ellipse $e$ with the hyperbola $c$ as caustic. Then to this billiard and to its mirror w.r.t. the principal axis exists a conjugate billiard $\ldots P_0'P_1'P_2'\ldots$, and it is unique up to a reflection in the principal axis. The relation between the two conjugate billiards in $e$ is symmetric. Moreover, if $T_i'$ denotes the intersection of $P_i'P_{i+1}'$ with the principal axis, then

$$P_iT_i = P_i'T_{i-1}' \quad \text{and} \quad P_iT_{i-1} = P_{i-1}'T_{i-1}' .$$

**Proof.** The singular affine transformation $\alpha_h$ maps $P_1'$ to $T_1$ and $P_1$ to a point $T'$ ($= T_6'$ in Figure 12). We assume that $P_1$ and $P_1'$ lie on the same side of the principal axis, since otherwise we apply a reflection in the axis. Then we obtain a curvilinear Ivory quadrangle $P_1'T_1T'T'P_1$ with diagonals of equal lengths. On the other hand, the lines $[P_i',T_i']$ and $[P_1,T_1]$ must contact the same confocal conic (see, e.g., [7] p. 153 or [23] Lemma 1)). Hence, the billiard through the point $P_1' \in e$ with caustic $c$ contains one side on the line $[P_1',T']$. Iteration confirms the claim.

Let $P_1'$ and $P_2'$ be the images of $P_1$ and $P_2$ under reflection in the principal axis of $e$ (Figure 12). Then, a comparison with Figure 9 reveals that the confocal hyperbola through the intersection $T_1 = [P_1,P_2] \cap [P_1',P_1]$ lies ‘in the middle’ between the hyperbolas through $P_1$ and $P_2$. 

\[\text{Figure 12. Periodic billiard } P_1P_2\ldots P_6 \text{ in the ellipse } e \text{ with the hyperbola } c \text{ as caustic, together with the conjugate billiard } P_0'P_1'P_2'\ldots. \text{The associated polygon with vertices on } e^{(1)} \text{splits into two triangles } S_1^{(1)}S_3^{(1)}S_5^{(1)} \text{ (green shaded) and } S_2^{(1)}S_4^{(1)}S_6^{(1)}.\]
Remark 3.15. If \( \ldots P_0 P_1 P_2 \ldots \) and \( \ldots P'_0 P'_1 P'_2 \ldots \) is a pair of conjugate billiards in an ellipse \( e \), then the points of intersection \([P_i, P_{i+1}] \cap [P'_i, P'_{i-1}]\) are located on a confocal ellipse \( e' \) inside \( e \). This holds for ellipses and hyperbolas as caustics. If the billiards are \( N \)-periodic, then in the elliptical case, the restriction of the two billiards to the interior of \( e' \) is \( 2N \)-periodic; conversely, \( e \) plays the role of \( e^{(1)} \) w.r.t. \( e' \) (Figure 3). In the hyperbolic case, the restriction to the interior of \( e' \) gives two symmetric \( 2N \)-periodic billiards, provided that also the reflected billiards are involved (Figure 12).

We conclude with citing a result from [25] about billiards in ellipses. It states that for each billiard \( \ldots P_1 P_2 P_3 \ldots \) in an ellipse \( e \) with a hyperbola as caustic there exists a billiard \( \ldots P^*_1 P^*_2 P^*_3 \ldots \) in \( e^* \) with an ellipse as caustic such that corresponding sides \( P_i P_{i+1} \) and \( P^*_i P^*_{i+1} \) are congruent.

4. Periodic \( N \)-sided billiards

Let the billiard \( P_1 P_2 \ldots P_N \) in the ellipse \( e \) be periodic with an ellipse \( c \) as caustic. Then, the sequence of parameters \( t_1, t'_1, t_2, \ldots, t_N, t'_N \) of the vertices \( P_i \) and the intermediate contact points \( Q_1, \ldots, Q_N \) with \( c \) is cyclic. Each side line intersects only a finite number of other side lines. Hence, the corresponding Poncelet grid contains a finite number of confocal ellipses \( e^{(j)} \) through the points \( S^{(j)} \), namely \( \lfloor \frac{N-2}{2} \rfloor \) (including possibly the line at infinity), provided that \( N \geq 5 \) (Figure 5). The sequence of ellipses \( e, e^{(1)}, e^{(2)}, \ldots \) is cyclic, and

\[
e^{(j)} = e^{(N-2-j)}.
\]

(4.1)

For example, in the case \( N = 7 \) (Figure 6), the ellipse \( e^{(2)} \) coincides with \( e^{(3)} \).

Definition 4.1. The sum of exterior angles \( \theta_i \) of a periodic billiard in an ellipse \( e \) is an integer multiple of \( 2\pi \), namely \( 2\tau \pi \). We call \( \tau \in \mathbb{N} \) the turning number of the billiard. It counts the loops of the billiard around the center \( O \) of \( e \), anti-clockwise or clockwise.

If the periodic billiard \( P_1 P_2 \ldots P_N \) has the turning number \( \tau = 1 \) (Figure 13), then the billiard \( S_1^{(1)} S_3^{(1)} S_5^{(1)} \ldots \) in \( e^{(1)} \) has \( \tau = 2 \), that of \( S_1^{(2)} S_4^{(2)} \ldots \) in \( e^{(2)} \) the turning number \( \tau = 3 \), and so on. In cases with \( g = \gcd(N, \tau) > 1 \) the corresponding billiard splits into \( g \frac{N}{g} \)-sided billiards, each with turning number \( \tau/g \) (note [22, Theorem 1.1]).

4.1. Symmetries of periodic billiards

The following is a corollary to Theorem 3.6

Corollary 4.2. Let \( P_1 P_2 \ldots P_N \) be an \( N \)-sided periodic billiard in the ellipse \( e \) with the ellipse \( c \) as caustic.

(i) For even \( N \) and odd \( \tau \), the billiard is centrally symmetric.

(ii) For odd \( N = 2n + 1 \) and odd \( \tau \), the billiard is centrally symmetric to the conjugate billiard, where \( P_i \) corresponds to \( P'_{i+n} \).
(iii) If $N$ is odd and $\tau$ is even, then the conjugate billiard coincides with the original one, and $P_i = P'_i$.

\textbf{Proof.} By virtue of Theorem 3.6 the lines $[P_{i-j-1}, P_{i-j}]$ and $[P_{i+j}, P_{i+j+1}]$ for $j = 1, 2, \ldots$ meet at the point $S_{i}(j)$ on the confocal hyperbola through $P_{i}$.

(i) This means for even $N = 2n$, odd $\tau$ and $j = n - 1$, that also the opposite vertex $S_{i}^{(n)} = P_{i-n} = P_{i+n}$ belongs to this hyperbola. If $P_{i}$ is specified at a vertex on the minor axis of the ellipse $e$, then $P_{i+n}$ is the opposite vertex. Continuity implies that the two points belong to different branches of the hyperbola and are symmetric w.r.t. the center $O$ of $e$.

(ii), (iii): If $N$ is odd, say $N = 2n + 1$, then for $j = n - 1$ the sides $[P_{i-n+1}, P_{i-n}]$ and $[P_{i+n-1}, P_{i+n}]$ intersect at a point on the hyperbola through $Q_{i+n}$ and $P'_{i+n}$. For odd $\tau$ (Figures 8 and 13), the same continuity argument as before proves that $P_{i}$ and $P'_{i+n}$ are opposite w.r.t. $O$.

If $\tau$ is even (Figure 14), then the choice of $P_{i} \in e$ on an axis of symmetry shows the coincidence with $P'_{i+n} \in e$, and this must be preserved, when $P_{i}$ varies continuously on $e$. In the case of even $\tau$ and $N$ the billiard splits. \hfill \Box

**Figure 13.** Periodic billiard $P_{1}P_{2}\ldots P_{9}$ with $\tau = 1$. Note that $l_{2} = P_{2}Q_{2} = Q_{6}P_{7} = r_{7}$ and $S_{2}^{(1)}P_{3} = P_{7}S_{7}^{(1)}$. The associated billiard in $e^{(2)}$ splits into three triangles.

The billiards with a hyperbola $c$ as caustic (see Figures 10 and 12) oscillate between the upper and lower section of $e$. Therefore, only billiards with an even $N$ can be periodic. Also for billiards of this type, it possible to define

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3All subscripts in this section are understood modulo $N$.
a turning number \( \tau \) which counts how often the points \( P_1, P_2, P_3, \ldots, P_N \) (Figure 12) run to and fro along the upper component of \( e \).

The symmetry properties of these periodic \( N \)-sided billiards differ from those in Corollary 4.2. They follow from Theorem 3.6 since opposite vertices \( P_i \) and \( P_{i+N/2} \) belong to the same confocal hyperbola.

**Corollary 4.3.** Let \( P_1P_2 \ldots P_N \) be an \( N \)-sided periodic billiard in the ellipse \( e \) with the hyperbola \( c \) as caustic.

(i) For \( N \equiv 0 \pmod{4} \), the billiard is symmetric w.r.t. the secondary axis of \( e \) and \( c \).

(ii) For \( N \equiv 2 \pmod{4} \) and odd turning number \( \tau \), the billiards are centrally symmetric. For even \( \tau \), each billiard is symmetric w.r.t. the principal axis of \( e \) and \( c \).

### 4.2. Some invariants

As a direct consequence of the results so far, we present new proofs for the invariants \( k_{101}, k_{118} \) and \( k_{119} \) listed in [21, Table 2], though this table refers already to proofs for some of them in [2] and [11].

We begin with a result that has first been proved for a much more general setting in [26, p. 103].

**Lemma 4.4.** The length \( L_e \) of a periodic \( N \)-sided billiard in the ellipse \( e \) with the ellipse \( c \) as caustic is independent of the position of the initial vertex \( P_1 \in e \).

**Proof.** We refer to Graves’s construction [15, p. 47]. According to (3.4) holds

\[
D_e := Q_{i-1}P_i + \overrightarrow{P_iQ_i} - \overrightarrow{Q_iQ_{i-1}}.
\]

This yields for an \( N \)-sided billiard with turning number \( \tau \) the total length

\[
L_e = N \cdot D_e - \tau \cdot P_c, \tag{4.2}
\]

where \( P_c \) denotes the perimeter of the caustic \( c \). Thus, \( L_e \) does not depend of the choice of the initial vertex \( P_1 \in e \). \( \square \)

If the billiard in \( e \) has the turning number \( \tau \), then its extension in \( e^{(1)} \) has the turning number \( 2\tau \), and from (3.5) and (3.9) follows

\[
L_{e|1} = ND_{e|1} - 2\tau P_c = 2N(D_e + w) - 2\tau P_c = 2L_e + 2N \frac{2a_e b_e \sqrt{k_e^3}}{a_e^2 b_e^2 - k_e^2}, \tag{4.3}
\]

The following theorem on the invariant \( k_{118} \) in [21] deals with the lengths \( r_i \) and \( l_i \) of the segments \( Q_{i-1}P_i \) and \( P_iQ_i \), as defined in (3.1).

\footnote{The turning number of hyperbolic billiards becomes more intuitive when the billiard is seen as the limit of a focal billiard in the sense of [25, Theorem 2].}
Theorem 4.5. In each $N$-sided periodic billiard opposite segments are congruent, i.e., if $N = 2n$, then $r_{i+n} = r_i$ and $l_{i+n} = l_i$, and if $N = 2n + 1$, then $r_{i+n} = l_{i-1}$ and $l_{i+n} = r_i$. Thus, for odd $N$ holds

$$\sum_{i=1}^{N} l_i = \sum_{i=1}^{N} r_i = \frac{L_e}{2}.$$ 

Proof. By Corollary 4.2 for even $N = 2n$ the central symmetry implies for opposite segments $r_i = r_{i+n}$ and $l_i = l_{i+n}$.

If $N = 2n + 1$, then $l_i = \overline{P_iQ_i}$ shows up as $l'_{i+n} = \overline{P_{i+n}Q_{i+n}}$ at the conjugate billiard and, by virtue of (3.13), this equals $r_{i+n+1} = \overline{P_{i+n+1}Q_{i+n}}$ (Figure 13). Similarly follows $r_i = l_{i+n}$. \hfill $\Box$

Remark 4.6. In the particular case $N = 3$ the two segments adjacent to any side are congruent (note in Figure 13 the triangular billiards in $e^{(2)}$). Therefore, the Cevians $[P_i, Q_{i+1}]$ are concurrent and meet at the Nagel point of the triangle. This has already been proved in [20] and agrees with the circles through $Q_i$ and centered at $R_i$ (see Figures 6 and 14), which for $N = 3$ are excircles of the triangle $P_1P_2P_3$.

The following theorem has first been proved in [21] p. 4]. Another proof can be found in [3] Cor. 3.2]. We give below a new proof.

Theorem 4.7. For the exterior angles $\theta_1, \ldots, \theta_N$ of the periodic $N$-sided elliptic billiard in the ellipse $e$, the sum of cosines is independent of the initial vertex, namely

$$\sum_{i=1}^{N} \cos \theta_i = N - J_e L_e = N - \frac{\sqrt{k_e}}{a_e b_e} L_e,$$

where $L_e$ is the common perimeter of these billiards in $e$. 

![Figure 14. Periodic billiard with N = 7 and τ = 2.](image-url)
Proof. The pedal points $F_i$ and $F_{i-1}$ on the sides $P_i P_{i+1}$ and $P_{i-1} P_i$ w.r.t. the center $O$ have the position vectors
\[
\mathbf{f}_{i,i-1} = \mathbf{p} + \frac{\lambda_{i,i-1}}{||\mathbf{t}_e||} \left( \cos \frac{\theta_i}{2} \mathbf{t} \pm \sin \frac{\theta_i}{2} \mathbf{t}^\perp \right),
\]
where \( 0 = \langle \mathbf{f}_{i,i-1}, \left( \cos \frac{\theta_i}{2} \mathbf{t} \pm \sin \frac{\theta_i}{2} \mathbf{t}^\perp \right) \rangle. \]

Here, $\lambda_i$ and $\lambda_{i+1}$ denote the signed distances from the vertex $P_i$ in one case towards $P_{i+1}$, in the other opposite to $P_{i-1}$. From
\[
\langle \mathbf{p}_i, \mathbf{t} \rangle = (-a_e^2 + b_e^2) \cos t \sin t \quad \text{and} \quad \langle \mathbf{p}_i, \mathbf{t}^\perp \rangle = -a_e b_e
\]
follows
\[
\lambda_{i,i-1} = \frac{1}{||\mathbf{t}_e||} \left( (-a_e^2 + b_e^2) \cos t \sin t \cos \frac{\theta_i}{2} \pm a_e b_e \sin \frac{\theta_i}{2} \right).
\]

This implies by virtue of (2.11) and after reversing the orientation for $\lambda_{i-1}$,
\[
\lambda_i - \lambda_{i-1} = \frac{2a_e b_e}{||\mathbf{t}_e||^2} \sqrt{k_e}.
\]

Since the sum over all signed lengths between $P_i$ and the adjacent pedal points gives the total perimeter $L_e$ of the billiard, we obtain by (2.12)
\[
L_e = \sum_{i=1}^{N} \left( \frac{\mathbf{P}_i \mathbf{F}_i + \mathbf{P}_{i-1} \mathbf{F}_{i-1}}{k_e} \right) = \frac{a_e b_e}{\sqrt{k_e}} \sum_{i=1}^{N} \frac{2k_e}{||\mathbf{t}_e||^2} = \frac{a_e b_e}{\sqrt{k_e}} \sum_{i=1}^{N} (1 - \cos \theta_i),
\]

hence
\[
\sum_{i=1}^{N} \frac{1}{||\mathbf{t}_e||^2} = \frac{L_e}{2a_e b_e \sqrt{k_e}} \quad \text{and also} \quad \sum_{i=1}^{N} \cos \theta_i = N - \frac{\sqrt{k_e}}{a_e b_e} L_e,
\]
as stated.

Remark 4.8. Note that the result in [2] relates to the interior angles of the billiard. As already mentioned in [2] Theorem 7, the constant sum of cosines holds also for the ‘extended’ billiards in $(j)$, where the exterior angles are $\theta_i + \theta_{i+1} + \cdots + \theta_{i+j}$ (note Figure 5).
Similar to (2.14), the distance of $O$ to the tangent $t_P$ to $e$ at $P$ equals
\[ \overline{Ot_P} = \frac{a_e b_e}{\|t_e\|}, \quad (4.6) \]
From (4.5) follows for the vertices of a billiard a result of [3, Cor. 3.2]. The invariant $k_{119}$, first proved by P. Roitmann, deals with the curvature of $e$, namely
\[ \kappa_e(t) := \frac{a_e b_e}{\|t_e(t)\|^3} \]
by [15, p. 79]. The result referring to this and given below is again a consequence of (4.5).

**Corollary 4.9.** The squared distances from the center $O$ to the tangents $t_{P_i}$ at the vertices $P_i$ of the periodic $N$-sided elliptic billiard in the ellipse $e$ have a constant sum, independent of the initial vertex, namely
\[ \sum_{i=1}^N \overline{Ot_{P_i}}^2 = \frac{a_e b_e}{2 \sqrt{k_e}} L_e. \]
The curvatures $\kappa_i$ of $e$ at the vertices $P_i$ give rise to an invariant sum
\[ \sum_{i=1}^N \kappa_i^{2/3} = \frac{L_e}{2 \sqrt{k_e}} (a_e b_e)^{-1/3}. \]

We conclude this section with a comment on (4.1) in connection with (3.8) and (3.10). For example, we obtain $k_{e|1} = k_e$ for $N = 3$, $k_e = a_e b_e$ for $N = 4$, $k_{e|2} = k_{e|1}$ for $N = 5$, and $k_{e|2} = \infty$ for $N = 6$. This yields algebraic conditions for the dimension of the ellipse $e$ with an inscribed $N$-periodic billiard when the ellipse $c$ is given as caustic. However, we need to recall that A. Cayley gave already an explicit solution for this problem in a projective setting ([16] or [15, Theorem 9.5.4]). Other approaches are provided in [4, Sect. VI] and [14, Sect. 11.2.3.9]. Equivalent conditions in terms of elliptic functions can be deduced from [24, Corollary 3].

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