LARGE DEVIATIONS FOR
DENOMINATORS OF CONTINUED FRACTIONS

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ABSTRACT. We give an exponential upper bound on the probability with which
the denominator of the $n$th convergent in the regular continued fraction expan-
sion stays away from the mean $n\pi^2/12\log 2$. The exponential rate is best possible,
given by an analytic function related to the dimension spectrum of Lyapunov
exponents for the Gauss transformation.

1. Introduction

Each irrational number $x \in (0, 1)$ has the continued fraction expansion

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

where each $a_i$ is a positive integer. Let $p_n, q_n$ be relatively prime positive integers satisfying

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_n}}}$$

Then $p_n/q_n$ converges to $x$ as $n \to \infty$, and the rate of this convergence is deter-
mined by the growth rate of the denominator $q_n$:

$$\frac{1}{2q_{n+1}^2} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2}.$$ 

One important problem in the metric theory of continued fractions is to investigate
the limit behavior of $q_n$ for typical irrationals. It was Khinchin [9] who proved the
existence of an absolute constant $\gamma$ such that $(1/n) \log q_n \to \gamma$ as $n \to \infty$ Lebesgue-
a.e. Lévy [10] showed $\gamma = \pi^2/12\log 2$. Hence, for any closed interval $K$ not containing
$\gamma$, the Lebesgue measure of the event $\{\log q_n - \gamma n \in K\}$ converges to 0 as $n \to \infty$.

Of interest to know is the rate of this convergence. If it is exponential, namely
there is an upper bound of the form $Ce^{-\delta n}$ for some $C > 0$ and $\delta > 0$, then the
smallest such $\delta$ would have some intrinsic meaning.

\textit{Key words:} Diophantine approximation, continued fraction, large deviations.
Some rates in this convergence are available from central limit theorems. Denote by $\lambda$ the Lebesgue measure restricted to $(0, 1)$. Misevičius [11] showed that
\[
\sup_{\alpha \in \mathbb{R}} \left| \lambda \left\{ \frac{\log q_n - \gamma n}{\sigma \sqrt{n}} \leq \alpha \right\} - \int_{-\infty}^{\alpha} e^{-\frac{x^2}{2}} dx \right| = O \left( \frac{\log n}{\sqrt{n}} \right),
\]
where $\sigma > 0$. The results of Morita [12] and Vallée [16] improve the order to $O(1/\sqrt{n})$. It follows that for every $\alpha < 2\gamma$,
\[
\lambda \left\{ \frac{2}{n} \log q_n \leq \alpha \right\} = O \left( \frac{1}{\sqrt{n}} \right),
\]
This estimate is far from optimal. From the result of Araújo and Bufetov [1, Theorem B],
\[
\limsup_{n \to \infty} \frac{1}{n} \log \lambda \left\{ \frac{2}{n} \log q_n \leq \alpha \right\} \leq I(\alpha),
\]
where the number $I(\alpha) > 0$ is defined below. Hence, the convergence takes place at an exponential rate, and the rate can be chosen arbitrarily close to $I(\alpha)$. The aim of this paper is to show that $I(\alpha)$ is the best exponential rate.

We now define $I(\alpha)$ and state our main result. For each $\alpha \in [0, \infty]$ define
\[
L(\alpha) = \left\{ x \in (0, 1) : \liminf_{n \to \infty} \frac{2}{n} \log q_n(x) = \limsup_{n \to \infty} \frac{2}{n} \log q_n(x) = \alpha \right\}.
\]
Put $b(\alpha) = \dim_H L(\alpha)$, where $\dim_H$ denotes the Hausdorff dimension. Define
\[
I(\alpha) = \alpha(1 - b(\alpha)).
\]
Put $\alpha_{\min} = \log \frac{\sqrt{5} + 1}{2}$.

**Main Theorem.** The following holds:

- for every $n \geq 1$ and every $\alpha \in (2\gamma + \frac{16}{n}, \infty)$,
\[
\lambda \left\{ \frac{2}{n} \log q_n \geq \alpha \right\} \leq C_{\alpha} e^{-I(\alpha)n};
\]
- for every $n \geq 1$ with $\alpha_{\min} < 2\gamma - \frac{16}{n}$ and every $\alpha \in (\alpha_{\min}, 2\gamma - \frac{16}{n})$,
\[
\lambda \left\{ \frac{2}{n} \log q_n \leq \alpha \right\} \leq C_{\alpha} e^{-I(\alpha)n},
\]
where $C_\alpha := e^{16(|I'(\alpha)|+1)}$.

The Main Theorem follows from a combination of the multifractal analysis [8, 13] and the thermodynamic formalism [3, 15] associated with the Gauss transformation $T: (0, 1) \to (0, 1)$ given by $Tx = 1/x - \lfloor 1/x \rfloor$. It is well-known (see e.g., [7] or Lemmas 2.1 and 2.2) that there exists a constant $C > 1$ such that for any irrational $x \in (0, 1)$ and $n \geq 1$,
\[
C^{-1} q_n^2(x) \leq |DT^n(x)| \leq C q_n^2(x).
\]
These double inequalities permit to translate the analysis of $\log q_n$ to that of the Birkhoff sum of the function $\log |DT|$ under the iteration of $T$. The $L(\alpha)$ is the set of irrationals in $(0, 1)$ for which the Lyapunov exponent for $T$ is equal to $\alpha$. 

Then \( L(\alpha) \neq \emptyset \) holds if and only if \( \alpha \in [\alpha_{\min}, \infty) \), see [8, 13]. The function \( \alpha \in [\alpha_{\min}, \infty) \mapsto b(\alpha) \) is known as the dimension spectrum of Lyapunov exponents. It is a non-convex function, analytic on \((\alpha_{\min}, \infty) \) [8, Theorem 1.3], and \( b(\alpha) = 1 \) holds if and only if \( \alpha = 2\gamma \). Hence, \( \alpha \in (\alpha_{\min}, \infty) \mapsto I(\alpha) \) is analytic and \( I(\alpha) = 0 \) holds if and only if \( \alpha = 2\gamma \).

The graph of the function \( \alpha \in [\alpha_{\min}, \infty) \mapsto I(\alpha) \) is shown in FIGURE 1. Since \( b(\alpha_{\min}) = 0 \) by [8, Theorem 1.3], \( I(\alpha_{\min}) = \alpha_{\min} \) holds. Since \( I \) is convex by Lemma 2.3, \( I'(\alpha) \) increases for \( \alpha > 2\gamma \). Since \( b(\alpha) \to 1/2 \) as \( \alpha \to \infty \) by [8, Theorem 1.3], the asymptote exists with slope \( 1/2 \).

Since \( T \) has infinitely many branches and \( \log |DT| \) is unbounded, some finite approximations are necessary for a proof of the Main Theorem. We take finite subsystems, and estimate the exponent of the deviation probabilities in terms of entropy and Lyapunov exponents of invariant probability measures of \( T \) supported on the subsystems (Lemma 3.1). Then, using the variational formula for the dimension spectrum [13] we relate the exponent to the function \( I \). At the very end we use the convexity and the smoothness of \( I \) (in fact, \( C^2 \) is sufficient) to bound error terms arising from the nonlinearity of \( T \) and deduce the desired upper bounds.

From [1, Theorem B] the following asymptotic lower bounds hold:

- for every \( \alpha \in (2\gamma, \infty) \),
  \[
  \liminf_{n \to \infty} \frac{1}{n} \log \lambda \left\{ \frac{2}{n} \log q_n \geq \alpha \right\} \geq -I(\alpha);
  \]

- for every \( \alpha \in (\alpha_{\min}, 2\gamma) \),
  \[
  \liminf_{n \to \infty} \frac{1}{n} \log \lambda \left\{ \frac{2}{n} \log q_n \leq \alpha \right\} \geq -I(\alpha).
  \]

This means that the exponent \( I(\alpha) \) in the Main Theorem is the best possible one. However, the result below on sample means of independent and identically distributed (i.i.d.) random variables leaves the possibility that the upper bounds in the Main Theorem can be improved.

**Theorem 1.** [2, Theorem 1] Let \((X_n)_{n \geq 1}\) be a sequence of i.i.d. random variables with positive variance with mean 0. Assume the moment generating function \( c(t) = \)
\[ \log E(e^{tX_1}) \text{ is finite on some interval } U. \] Let \( \alpha > 0 \) and \( t_\alpha \in U \) be such that \( J(\alpha) := \sup_{t \in U} e^{t_\alpha - t} = e^{t_0 - t}. \) Then for every \( \alpha > 0, \)
\[ P(S_n \geq \alpha n) = \frac{b_n(1 + o(1))}{\sqrt{2\pi n}} e^{-J(\alpha)n}, \]
where \( S_n = X_1 + \cdots + X_n \), \((b_n)_{n \geq 1}\) is a sequence of constants and \( \inf_n b_n > 0, \sup_n b_n < \infty. \)

In [2] it was shown that \( \frac{b_n(1 + o(1))}{\sqrt{2\pi n}} \leq 1, \) and so \( P(S_n \geq \alpha n) \leq e^{-J(\alpha)n} \) holds. Such an exponential upper bound was obtained in [5], and follows from Cramér’s theorem on the LDP, see [4, pp.26-27]. In the non-i.i.d. case, results, for uniformly hyperbolic systems on compact metric spaces with Hölder continuous functions are available [4 Lemma A.1], [17 Theorem 1], which provide upper and lower bounds in agreement with the i.i.d. case in Theorem [1]. The bounds in [4 Lemma A.1] are valid only for those \( \alpha \) close to the mean.

2. Preliminary lemmas

Before entering the proof of the Main Theorem we need some preliminary lemmas. For each integer \( n \geq 1 \) denote by \( \mathcal{A}^n \) the collection of maximal open intervals on which \( T^n \) is well-defined and continuous. Notice that \( q_n \) is constant on each element \( A \in \mathcal{A}^n \). This constant value is denoted by \( q_n(A) \). For a finite set \( \mathcal{B} \) of \( \mathcal{A}^n \) denote by \([\mathcal{B}]\) the union of all its elements.

**Lemma 2.1.** For every integer \( n \geq 1 \) and every \( A \in \mathcal{A}^n \),
\[ \frac{1}{2} \leq \frac{\lambda(A)}{q_n(A)^2} < 1. \]

**Proof.** Assume \( n = 1 \). Each \( A \in \mathcal{A}^1 \) has the form \( A = (1/(k + 1), 1/k) \) for some \( k \geq 1 \). Then \( q_1(A) = k \) and so the double inequalities hold. Assume \( n \geq 2 \) and let \( A \in \mathcal{A}^n \). For each \( i = 1, \ldots, n \), \( p_i \), \( q_i \) are constant on \( A \). Denote these constant values by \( p_i(A) \) and \( q_i(A) \). By [7 p.18], the endpoints of the interval \( A \) are \( p_n(A)/q_n(A) \) and \( (p_n(A) + p_{n-1}(A))/(q_n(A) + q_{n-1}(A)) \). As a consequence,
\[ \lambda(A) = \frac{1}{q_n(A)(q_n(A) + q_{n-1}(A))}. \]
Since \( q_n(A) > q_{n-1}(A) \) we obtain the desired double inequalities. \( \square \)

The next lemma used to control the nonlinearity of \( T \) can be proved by elementary calculations and hence omitted. See e.g., [6 p.253 Claim] for details.

**Lemma 2.2.** For every integer \( n \geq 1 \) and every \( A \in \mathcal{A}^n \),
\[ \sup_{x,y \in A} \frac{DT^n(x)}{DT^n(y)} \leq e^{16}. \]

Write \( \phi = -\log |DT| \) and denote by \( \mathcal{M}_\phi(T) \) the set of \( T \)-invariant Borel probability measures on \((0,1)\) for which \( \phi \) is integrable. For each \( \mu \in \mathcal{M}_\phi(T) \) denote by \( h(\mu) \) the Kolmogorov-Sinaï entropy of \( \mu \) with respect to \( T \), and define \( \chi(\mu) = -\int \phi d\mu \). Put \( F(\mu) = h(\mu) - \chi(\mu) \). It is known [18] that \( \chi(\mu) \geq \alpha_{\min} \) and \( F(\mu) \leq 0. \)
Lemma 2.3. For every $\alpha \in [\alpha_{\min}, \infty)$,
\[
I(\alpha) = \inf\{-F(\mu) : \mu \in M_\phi(T), \chi(\mu) = \alpha\}.
\]
In particular, $I$ is convex.

Proof. Denote the infimum by $\tilde{I}(\alpha)$. Choose a sequence $\{\nu_n\}$ in $M_\phi(T)$ with $\chi(\nu_n) = \alpha$ and $\lim h(\nu_n)/\chi(\nu_n) = b(\alpha)$. Then $\tilde{I}(\alpha) \leq -\lim F(\nu_n) = I(\alpha)$. To show the reverse inequality, choose a sequence $\{\mu_n\}$ in $M_\phi(T)$ with $\chi(\mu_n) = \alpha$ and $-F(\mu_n) \to \tilde{I}(\alpha)$ as $n \to \infty$. Fix a measure $\nu \in M_\phi(T)$ with $\chi(\nu) < \alpha$. For each $n$ large enough fix $p_n \in (0, 1]$ with $\chi(p_n\mu_n + (1 - p_n)\nu) = \alpha$. Then $\lim p_n \to 1$ and hence $\lim h(p_n\mu_n + (1 - p_n)\nu) = \alpha - \tilde{I}(\alpha)$. The variational formula in [13] gives
\[
b(\alpha) = \sup \left\{ \frac{h(\mu)}{\chi(\mu)} : \mu \in M_\phi(T), \chi(\mu) = \alpha \right\},
\]
and therefore $\alpha^{-1}(\alpha - \tilde{I}(\alpha)) \leq b(\alpha)$, namely $I(\alpha) \leq \tilde{I}(\alpha)$ as required. The convexity of $I$ is a consequence of the affinity of entropy and Lyapunov exponent on measures.

3. Upper bound with best exponential rate

We are in position to prove the Main Theorem.

Lemma 3.1. Let $n \geq 1$ be an integer and let $\alpha > 0$. Let $B^n(\alpha)$ be a non-empty finite subset of $\{A \in F^n : (2/n)\log q_a(A) \geq \alpha\}$. There exists a measure $\mu \in M_\phi(T)$ such that
\[
\chi[B^n(\alpha)] \leq e^{16e^{F(\mu)n}} \quad \text{and} \quad \chi(\mu) \geq \alpha - \frac{16}{n}.
\]

Proof. Put $\hat{T} = T^n$ and $\Lambda = \bigcap_{m=0}^\infty \hat{T}^{-m}[B^n(\alpha)]$. Then $\Lambda$ is a compact set and $\hat{T}|_\Lambda : \Lambda \to \Lambda$ is continuous. Put $\hat{\phi} = -\log |D\hat{T}|$ and fix $y_0 \in \Lambda$. Lemma 2.2 implies $\sum_{i=0}^{m-1} (\hat{\phi}(\hat{T}^i(x)) - \hat{\phi}(\hat{T}^i(y))) \leq 16$ for every $m \geq 1$, every $x, y \in \Lambda$ such that $\hat{T}^i(x), \hat{T}^i(y)$ belong to the same element of $B^n(\alpha)$ for each $i = 0, \ldots, m - 1$. The variational principle [3, Lemma 1.20] gives
\[
\sup_{\nu \in M(\hat{T}|_\Lambda)} \left( h_{\hat{T}|_\Lambda}(\hat{\nu}) + \int \hat{\phi} d\hat{\nu} \right) = \lim_{m \to \infty} \frac{1}{m} \log \left( \sum_{x \in (\hat{T}|_\Lambda)^{-m}(y_0)} \exp \sum_{i=0}^{m-1} \hat{\phi}(\hat{T}^i(x)) \right),
\]
with $M(\hat{T}|_\Lambda)$ the space of $\hat{T}|_\Lambda$-invariant Borel probability measures endowed with the weak*-topology and $h_{\hat{T}|_\Lambda}(\hat{\nu})$ the entropy of $\hat{\nu} \in M(\hat{T}|_\Lambda)$ with respect to $\hat{T}|_\Lambda$.

By Lemma 2.2 $\inf_{\nu} e^{\hat{\phi}} \geq e^{-16} \chi[\nu]$ holds for every $\nu \in D^n$. Hence
\[
\sum_{x \in (\hat{T}|_\Lambda)^{-m}(y_0)} \exp \left( \sum_{i=0}^{m-1} \hat{\phi}(\hat{T}^i(x)) \right) \geq \left( \inf_{y' \in \Lambda} \sum_{x \in (\hat{T}|_\Lambda)^{-1}(y')} e^{\hat{\phi}(x)} \right)^m \geq \left( e^{-16} \chi[B^n(\alpha)] \right)^m.
\]
Taking logs of both sides, dividing by \( m \) and plugging the result into the previous inequality gives

\[
\lim_{m \to \infty} \frac{1}{m} \log \left( \sum_{x \in \mathcal{B}(\alpha)_{-m} \cap \mathcal{B}(\alpha)} \exp \left( \sum_{i=0}^{m-1} \phi (\mathcal{T}^i(x)) \right) \right) \geq \log \lambda [\mathcal{B}^n(\alpha)] - 16. 
\]

Plugging this into the previous inequality yields

\[
\sup_{\tilde{\nu} \in \mathcal{M}(\mathcal{T}_\alpha)} \left( h_{\mathcal{T}_\alpha}(\tilde{\nu}) + \int \hat{\phi} d\tilde{\nu} \right) \geq \log \lambda [\mathcal{B}^n(\alpha)] - 16.
\]

Since \( \mathcal{M}(\mathcal{T}_\alpha) \) is compact and \( \mathcal{M}(\mathcal{T}_\alpha) \ni \tilde{\nu} \mapsto h_{\mathcal{T}_\alpha}(\tilde{\nu}) + \int \hat{\phi} d\tilde{\nu} \) is upper semi-continuous, there exists a measure \( \tilde{\mu} \in \mathcal{M}(\mathcal{T}_\alpha) \) which attains this supremum. The measure \( \mu = (1/n) \sum_{i=0}^{n-1} \tilde{\mu} \circ \mathcal{T}^{-i} \) is in \( \mathcal{M}_\phi(T) \). From the second inequality in Lemma 2.1 and Lemma 2.2, \( \inf \lambda [\mathcal{B}(\alpha, c)] \log |D \mathcal{T}| \geq an - 16 \) holds. Hence \( \chi(n) = (1/n) \int \log |D \mathcal{T}| d\tilde{\mu} \geq \alpha - 16/n \) as required. \( \square \)

**Proof of the Main Theorem.** Let \( n \geq 1 \) be an integer. We concentrate on the case \( \alpha \in (2\gamma + \frac{16}{n}, \infty) \) since the case \( \alpha \in (\alpha_{\min}, 2\gamma - \frac{16}{n}) \) is identical with the obvious modifications of statements. Denote by \( \lambda_n \) the distribution of \( (2/n) \log q_n \).

For each \( c > 1 \) there is a finite subset \( \mathcal{B}(\alpha, c) \) of \( \mathcal{B}^n \) such that \( \lambda_n([\alpha, \infty)) \leq c\lambda [\mathcal{B}^n(\alpha, c)] \). By Lemma 3.1 there exists \( \mu \in \mathcal{M}_\phi(T) \) which satisfies \( \chi(\mu) \leq e^{16} \exp(F(\mu)n) \) and \( \chi(\mu) \geq \alpha - 16/n \). Therefore

\[
\lambda_n([\alpha, \infty)) \leq ce^{16} \exp(F(\mu)n)
\]

\[
\leq ce^{16} \exp \left( \sup \left\{ F(\mu) : \mu \in \mathcal{M}_\phi(T), \chi(\mu) \geq \alpha - \frac{16}{n} \right\} \right)
\]

\[
\leq ce^{16} e^{-\inf_{\beta \in [\alpha-16/n, \infty)} I(\beta)\} \}
\]

\[
= ce^{16} e^{-I(\alpha-16/n)n}
\]

\[
\leq ce^{16} e^{-I(\alpha)n+1} e^{-I(\alpha)n}.
\]

For the last inequality we have used the convexity and the smoothness of \( I \). Since \( c > 1 \) is arbitrary, we obtain \( \lambda_n([\alpha, \infty)) \leq C_\alpha e^{-I(\alpha)n} \) as required. \( \square \)

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