ARC SPACES AND EQUIVARIANT COHOMOLOGY

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Abstract. We present a new geometric interpretation of equivariant cohomology in which one replaces a smooth, complex $G$-variety $X$ by its associated arc space $J_\infty X$, with its induced $G$-action. This not only allows us to obtain geometric classes in equivariant cohomology of arbitrarily high degree, but also provides more flexibility for equivariantly deforming classes and geometrically interpreting multiplication in the equivariant cohomology ring. Under appropriate hypotheses, we obtain explicit bijections between $\mathbb{Z}$-bases for the equivariant cohomology rings of smooth varieties related by an equivariant, proper birational map.

As applications, we present geometric $\mathbb{Z}$-bases for the equivariant cohomology rings of a smooth toric variety (with respect to the dense torus) and a partial flag variety (with respect to the general linear group).

1. Introduction

Let $X$ be a smooth complex algebraic variety equipped with an action of a linear algebraic group $G$. In this article, we consider two constructions associated to this situation. The equivariant cohomology ring $H^*_G X$ is an interesting and useful object encoding information about the topology of $X$ as it interacts with the group action; for example, fixed points and orbits are relevant, as are representations of $G$ on tangent spaces. The arc space of $X$ is the scheme $J_\infty X$ parametrizing morphisms $\text{Spec}\mathbb{C}[[t]] \to X$; this construction is functorial, so $J_\infty G$ is a group acting on $J_\infty X$. Except when $X$ is zero-dimensional, $J_\infty X$ is not of finite type over $\mathbb{C}$, but it is a pro-variety, topologized as a certain inverse limit. Due to their connections with singularity theory [16, 18, 33] and their central role in motivic integration [13, 29], arc spaces have recently proved increasingly useful in birational geometry.
The present work stems from a simple observation: The projection \( J^\infty X \to X \) is a homotopy equivalence, and is equivariant with respect to \( J^\infty G \to G \), so there is a canonical isomorphism \( H^*_G X = H^*_G J^\infty X \) (Lemma 2.1). Very broadly, our view is that interesting classes in \( H^*_G X \) arise from the \( J^\infty G \)-equivariant geometry of \( J^\infty X \).

The philosophy we wish to emphasize is motivated by analogy with two notions from ordinary cohomology of (smooth) algebraic varieties. First, classes in \( H^{2k} X \) come from subvarieties of codimension \( k \). We seek invariant subvarieties of codimension \( k \) to correspond to classes in \( H^*_{J^\infty G} X \). Since \( H^*_G X \) typically has nonzero classes in arbitrarily large degrees, however, \( X \) must be replaced with a larger—in fact, infinite-dimensional—space. The traditional approach to equivariant cohomology, going back to Borel, replaces \( X \) with the mixing space \( \mathbb{E} G \times^G X \), which does not have a \( G \)-action; we will instead study \( J^\infty X \), which is intrinsic to \( X \) and on which \( G \) acts naturally.

The second general notion is that cup product in \( H^* X \) should correspond to transverse intersection of subvarieties. In the \( C^\infty \) category, of course, this is precise: any two subvarieties can be deformed to intersect transversely, and the cup product is represented by the intersection. On the other hand, \( X \) often has only finitely many \( G \)-invariant subvarieties, so in the equivariant setting, no such moving is possible within \( X \) itself. Replacing \( X \) with \( J^\infty X \), one gains much greater flexibility to move invariant cycles.

Our main theorem addresses the first notion, and says that under appropriate hypotheses, \( J^\infty G \)-orbits in \( J^\infty X \) determine a basis over the integers:

**Theorem 5.7.** Let \( G \) be a connected linear algebraic group acting on a smooth complex variety \( X \), with \( D \subseteq X \) a \( G \)-invariant closed subset such that \( G \) acts on \( X \setminus D \) with unipotent stabilizers. Suppose \( J^\infty X \setminus J^\infty D = \bigcup_j U_j \) is an equivariant affine paving, in the sense of Definition 5.6. Then

\[
H^*_G X = \bigoplus_j \mathbb{Z} \cdot [U_j].
\]

Since arc spaces are well-suited to the study of the birational geometry of \( X \), one should also consider proper equivariant birational maps \( f : Y \to X \). When \( X \) satisfies the conditions of Theorem 5.7 and the paving is compatible with \( f \), we establish a geometric bijection between \( \mathbb{Z} \)-bases of \( H^*_G X \) and \( H^*_G Y \) (Corollary 5.10).

In Sections 6 and 7, we address the second notion, and relate the cup product in \( H^*_G X \) to intersections in \( J^\infty X \). Let \( V_1, \ldots, V_s \) be \( G \)-invariant subvarieties of a smooth
variety $X$. The main results of these sections say that under certain restrictions on the singularities of the $V_i$'s, products of their equivariant cohomology classes are represented by multi-contact loci in the arc space of $X$. (Basic facts about contact loci are reviewed in §3; the notation used in the following theorems is also introduced there.) Our first theorem about multiplication requires the subvarieties to be equivariant local complete intersections (see §6):

**Theorem 6.1.** Assume $G$ is a connected reductive group, $X^G$ is finite, and the natural map $i^*: H^*_G X \to H^*_G X^G$ is injective. Consider a chain of equivariant local complete intersection subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X.$$ If $m = (m_1, \ldots, m_s)$ is a tuple of non-negative integers, then let $\lambda(m)$ be the partition defined by $\lambda_i = m_i + \cdots + m_s$, and let $\text{Cont}^{\leq \lambda(m)}(V_\bullet) \subseteq J_\infty X$ denote the locus of arcs with contact order $\lambda_i$ with $V_i$ for $1 \leq i \leq s$. If $\text{codim} \text{Cont}^{\leq \lambda(m)}(V_\bullet) = \sum_{i=1}^s m_i \text{codim} V_i$, then

$$[V_1]^{m_1} \cdots [V_s]^{m_s} = [\text{Cont}^{\leq \lambda(m)}(V_\bullet)].$$

Replacing the e.l.c.i. hypothesis with a requirement that the singular locus be small, we obtain a similar result:

**Theorem 7.3.** Consider a chain of $G$-invariant subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X,$$

and a tuple $m = (m_1, \ldots, m_s)$ of non-negative integers. We have

$$[V_1]^{m_1} \cdots [V_s]^{m_s} = [\text{Cont}^{\leq \lambda(m)}(V_\bullet)]$$

whenever $\min\{\text{codim}(\text{Sing}(V_i), X)\} > \sum_i m_i \text{codim}(V_i, X)$.

Our initial motivation for this work came from the theory of toric varieties. By a theorem of Ishii, orbits of generic arcs in a toric variety $X$ are parametrized by the same set which naturally indexes a $\mathbb{Z}$-basis for $H^*_T X$, namely, points in the lattice $N$ of one-parameter subgroups of $T$. In Section 8, we apply our results to extend this bijection to a natural isomorphism of rings (Corollary 8.3).

As another application, we consider the action of $GL_n$ on $n \times n$ matrices (by left multiplication) in Section 9. We show that Theorem 5.7 applies, using a paving defined in terms of contact loci with certain determinantal varieties (Corollary 9.5).
The hypotheses for Theorems 6.1 and 7.3 fail for these subvarieties—these determinantal varieties are generally not l.c.i., and they have large singular sets—but the conclusions appear to hold (Conjecture 9.6). (It would be interesting to have a more general framework which explains this.) Finally, we apply our result concerning the behavior of equivariant cohomology under birational maps (Corollary 5.10) to deduce a geometric basis for the $GL_n$-equivariant cohomology of a partial flag variety (Corollary 9.10).

In the theory of equivariant cohomology, one often chooses finite-dimensional algebraic approximations to the mixing space (see, e.g., [24, §2]). In this context, one may attempt to find representatives for classes in $H^*_G X$ via subvarieties of the approximation space (cf. [9, §2.2]) or deform to transverse position to compute products (cf. [3]). Our approach uses the jet schemes $J_m X = \text{Hom}(\text{Spec } \mathbb{C}[t]/(t^{m+1}), X)$ as finite-dimensional approximations to $J_\infty X$. This seems to be unrelated to the mixing space approximations; as mentioned before, it has the advantage of being canonical to $X$ and carrying large group actions.

Equivariant classes in jet schemes have also been studied by Bérczi and Szenes [5], from a somewhat different point of view; our results overlap in a simple special case. They consider the space

$$J_d(n, k) = \text{Hom}(\text{Spec } \mathbb{C}[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^{d+1}, \mathbb{A}^k),$$

and compute the classes of contact loci $\text{Cont}^d(\{0\})$. In general, this is quite complicated, but in our case, when $n = 1$, the class in question is $c^d_k \in H^*_{GL_k}J_d(1, k) = \mathbb{Z}[c_1, \ldots, c_k]$. This is also an easy case of Conjecture 9.6 (see Remark 9.9(3)).

Arc spaces have also been used by Arkhipov and Kapranov to study the quantum cohomology of toric varieties [2]. There may be an interesting relation between their point of view and ours, but we do not know a direct connection. In Section 10, we suggest a related correspondence between equivariant orbifold cohomology and twisted arcs in the sense of Yasuda [42].

For the convenience of the reader, we include brief summaries of basic facts about equivariant cohomology (Section 2) and jet schemes (Section 3), together with references. In Section 4, we prove a technical fact about stabilizers (Proposition 4.5) which is used in the proof of Theorem 5.7. Sections 5–9 contain the main results and applications described above. We conclude the paper with a short discussion of questions and projects suggested by the ideas presented here.
Notation and conventions. All schemes are over the complex numbers. For us, a **variety** is a separated reduced scheme of finite type over \( \mathbb{C} \), assumed to be pure-dimensional but not necessarily irreducible. Throughout, \( G \) will be a connected linear algebraic group over \( \mathbb{C} \), and \( X \) will be a \( G \)-variety.

Unless otherwise indicated, cohomology will be taken with \( \mathbb{Z} \) coefficients, with respect to the usual (complex) topology.

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2. **Equivariant cohomology**

We refer the reader to [24] or [10] for an introduction to equivariant cohomology, as well as proofs and details. Here we collect the basic properties we will need, and give a few illustrative examples. As always, \( G \) is a connected linear algebraic group acting on the left on \( X \).

A map \( f : X \to X' \) is **equivariant** with respect to a homomorphism \( \varphi : G \to G' \) if \( f(g \cdot x) = \varphi(g) \cdot f(x) \) for all \( g \in G, x \in X \). Equivariant cohomology is contravariant for equivariant maps: one has \( f^* : H^*_G X' \to H^*_G X \).

The following two facts play a key role in our arguments:

**Lemma 2.1.** Suppose \( X \to X' \) is equivariant with respect to \( G \to G' \), and both are locally trivial fiber bundles with contractible fibers. Then the induced map \( H^*_G X' \to H^*_G X \) is an isomorphism.

**Lemma 2.2.** The equivariant cohomology of an orbit is described as follows: for a closed subgroup \( G' \subseteq G \), one has \( H^*_G (G/G') = H^*_{G'}(pt) \).

**Example 2.3.** For a representation \( V \) of \( G \), one has \( H^*_G V = H^*_{G'}(pt) \).

**Example 2.4.** If \( G \) is contractible, then \( H^*_G(pt) = \mathbb{Z} \).

When \( X \) is smooth, a closed \( G \)-invariant subvariety \( Z \subseteq X \) of codimension \( c \) defines a class \([Z]\) in \( H^c_G X \). If \( Z_1, \ldots, Z_k \) denote the irreducible components of \( Z \), then \([Z] = [Z_1] + \cdots + [Z_k]\).

An equivariant vector bundle \( V \to X \) has **equivariant Chern classes** \( c^G_i(V) \) in \( H^i_G X \), with the usual functorial properties of Chern classes.
Example 2.5. A representation $V$ of $G$ is simply an equivariant vector bundle on a point, so one has corresponding Chern classes $c_i^G(V) \in H^*_G(pt)$. For $V = \mathbb{C}^n$, with $GL_n$ acting by the standard representation, the Chern classes $c_i = c_i^G(V)$ freely generate $H^*_{GL_n}(pt)$.

A key feature of equivariant cohomology is that $H^*_G X$ is canonically an algebra over $H^*_G(pt)$, via the constant map $X \to pt$. In contrast to the non-equivariant situation, $H^*_G(pt)$ is typically not trivial.

Example 2.6. If $T \cong (\mathbb{C}^*)^n$ is a torus with character group $M \cong \mathbb{Z}^n$, then $H^*_T(pt) = \text{Sym}^* M \cong \mathbb{Z}[t_1, \ldots, t_n]$. The inclusion $(\mathbb{C}^*)^n \hookrightarrow GL_n$ induces an inclusion

$$H^*_{GL_n}(pt) = \mathbb{Z}[c_1, \ldots, c_n] \hookrightarrow \mathbb{Z}[t_1, \ldots, t_n],$$

sending $c_i$ to the $i$th elementary symmetric function in $t$.

We will use equivariant Borel-Moore homology $\overline{H}^*_G X$ as a technical tool; see [15, p.605] or [11, Section 1] for some details. The main facts are analogous to the non-equivariant case, for which a good reference is [22, Appendix B]; we summarize them here.

If $X$ has (pure) dimension $d$, then $\overline{H}^*_G X = 0$ for $i > 2d$ and $\overline{H}^*_{2d} X = \bigoplus \mathbb{Z}$, with one summand for each irreducible component of $X$. In contrast to the non-equivariant case, $\overline{H}^*_G X$ may be nonzero for arbitrarily negative $i$. If $X$ is smooth of dimension $d$, then $\overline{H}^*_G X = H^{2d-i}_G X$.

Borel-Moore homology is covariant for equivariant proper maps and contravariant for equivariant open inclusions. For $Z \subseteq X$ a $G$-invariant closed subvariety of codimension $c$, there is a fundamental class $[Z]$ in $\overline{H}^{2d-2c}_G X$. More generally, if $Z \subseteq X$ is any $G$-invariant closed subset, with $U = X \setminus Z$ the open complement, there is a long exact sequence

$$\cdots \to \overline{H}^*_i Z \to \overline{H}^*_i X \to \overline{H}^*_i U \to \overline{H}^{*+1}_i Z \to \cdots.$$ 

Definition 2.7. A $d$-dimensional variety $X$ has trivial equivariant Borel-Moore homology if

$$\overline{H}^*_i X = \begin{cases} 
\mathbb{Z} & \text{if } i = 2d; \\
0 & \text{otherwise.}
\end{cases}$$

Example 2.8. For us, the main examples of such varieties arise as follows. An affine family of $G$-orbits is a smooth map $S \to \mathbb{A}^n$ of $G$-varieties, with $G$ acting trivially on $\mathbb{A}^n$, such that there is a section $s : \mathbb{A}^n \to S$, and the map $G \times \mathbb{A}^n \to S$, $(g, x) \mapsto g \cdot s(x)$ is smooth and surjective. In other words, as a smooth scheme
over $\mathbb{A}^n$, $S$ is the geometric quotient of the group scheme $\mathcal{G} = G \times \mathbb{A}^n$ by a closed subgroup scheme $\mathcal{H}$ over $\mathbb{A}^n$, so we may write $S = \mathcal{G}/\mathcal{H}$.

When $\mathcal{H} \to \mathbb{A}^n$ has contractible fibers—i.e., the stabilizers (in $G$) of points in $S$ are contractible subgroups—the projection $S \to \mathbb{A}^n$ is a (Serre) fibration, by [31, Corollary 15(ii)]. It follows that $H^*_G(S) = H^*_G(G/H_0) = \mathbb{Z}$, where $G/H_0 \subseteq S$ is the fiber over $0 \in \mathbb{A}^n$. Since $S$ is smooth, we conclude that $S$ has trivial Borel-Moore homology.

The following is an equivariant analogue of [22, Appendix B, Lemma 6]:

**Lemma 2.9.** Suppose $X$ has a filtration by $G$-invariant closed subvarieties $X_s \subseteq X_{s-1} \subseteq \cdots \subseteq X_0 = X$ such that each complement $U_i = X_i \setminus X_{i+1}$ has trivial equivariant Borel-Moore homology. Then, for $0 \leq k < \text{codim}(X_s, X)$, we have

$$\Omega^G_{2d-2k}X = \bigoplus_{\text{codim} U_i = k} \mathbb{Z} \cdot [U_i]$$

and $\Omega^G_{2d-2k+1}X = 0$. Consequently, if $X$ is smooth we have

$$H^G_{2k}X = \bigoplus_{\text{codim} U_i = k} \mathbb{Z} \cdot [U_i]$$

and $H^G_{2k-1}X = 0$, for $0 \leq k < \text{codim}(X_s, X)$.

We omit the proof, which proceeds exactly as in the non-equivariant case (using induction and the long exact sequence).

We will also need a slight refinement, whose proof is immediate from the long exact sequence:

**Lemma 2.10.** Let $X_0 \subseteq X$ be a $G$-invariant open subset. Then the induced map $\Omega^G_kX \to \Omega^G_kX_0$ is an isomorphism for $2d \geq k > 2\dim(X \setminus X_0) + 1$.

### 3. Arc spaces and jet schemes

In this section, we review some aspects of the theory of arc spaces and jet schemes, and set notation for the rest of the paper. We refer the reader to [33] and [17] for more details.

Let $X$ be a scheme over $\mathbb{C}$ of finite type. The $m$th jet scheme of $X$ is a scheme $J_mX$ over $\mathbb{C}$ whose $\mathbb{C}$-valued points parameterise all morphisms $\text{Spec} \mathbb{C}[t]/(t^{m+1}) \to X$. For example, $J_0X = X$ and $J_1X = TX$ is the total tangent space of $X$. In what follows, we will often identify schemes with their $\mathbb{C}$-valued points.
For $m \geq n$, the natural ring homomorphism $\mathbb{C}[t]/(t^{m+1}) \to \mathbb{C}[t]/(t^{n+1})$ induces truncation morphisms

$$\pi_{m,n} : J_m X \to J_n X,$$

and we write

$$\pi_m = \pi_{m,0} : J_m X \to X.$$

The inclusion $\mathbb{C} \hookrightarrow \mathbb{C}[t]/(t^{m+1})$ induces a morphism $\text{Spec} \mathbb{C}[t]/(t^{m+1}) \to \text{Spec} \mathbb{C}$, and hence a morphism

$$s_m : X \to J_m X,$$

called the \textbf{zero section}, with the property that $\pi_m \circ s_m = \text{id}$.

The truncation morphisms $\pi_{m,m-1} : J_m X \to J_{m-1} X$ form a projective system whose projective limit is a scheme $J_\infty X$ over $\mathbb{C}$, which is typically not of finite type. The scheme $J_\infty X$ is called the \textbf{arc space} of $X$, and the $\mathbb{C}$-valued points of $J_\infty X$ parameterise all morphisms $\text{Spec} \mathbb{C}[[t]] \to X$. For each $m$, there is a truncation morphism

$$\psi_m : J_\infty X \to J_m X,$$

induced by the natural ring homomorphism $\mathbb{C}[[t]] \to \mathbb{C}[[t]]/(t^{m+1}) = \mathbb{C}[t]/(t^{m+1})$.

Both $J_m$ and $J_\infty$ are functors from the category of schemes of finite type over $\mathbb{C}$ to the category of schemes over $\mathbb{C}$, and both preserve fiber squares (cf. [17, Remark 2.8]). For a morphism $f : X \to Y$, we write $f_m : J_m X \to J_m Y$ for the corresponding morphism of jet schemes. The following lemma should be compared with Theorem 3.11.

\textbf{Lemma 3.1.} \cite[Proposition 5.12]{17} \textit{If $X$ is a smooth variety and $V$ is a closed subscheme of $X$ with $\dim V < \dim X$, then}

$$\lim_{m \to \infty} \text{codim}(J_m V, J_m X) = \infty.$$  

\textbf{Remark 3.2.} If $X$ is a smooth variety of dimension $d$, then $J_m X$ is a smooth variety of dimension $(m+1)d$, and the truncation morphisms $\pi_{m,m-1} : J_m X \to J_{m-1} X$ are Zariski-locally trivial fibrations with fiber $\mathbb{A}^d$ \cite[Corollary 2.11]{17}. Moreover, the projection $\psi_0 : J_\infty X \to X$ is a Zariski-locally trivial fibration with contractible fibers.

When $X$ is singular, $J_m X$ may not be reduced or irreducible, and may not be pure-dimensional. However, if $\text{Sm}(X)$ denotes the smooth locus of $X$, then the closure of $\pi^{-1}_m \text{Sm}(X) \subseteq J_m X$ is an irreducible component of dimension $(m+1)d$. 
Example 3.3. Let $X = \mathbb{A}^n = \text{Spec } \mathbb{C}[x_1, \ldots, x_n]$. An $m$-jet $\text{Spec } \mathbb{C}[t]/(t^{m+1}) \rightarrow \mathbb{A}^n$ corresponds to a ring homomorphism $\mathbb{C}[x_1, \ldots, x_n] \rightarrow \mathbb{C}[t]/(t^{m+1})$, and hence to an $n$-tuple of polynomials in $t$ of degree at most $m$. We conclude that $J_m \mathbb{A}^n \cong \mathbb{A}^{(m+1)n}$, and we write $\{x_i^{(j)} \mid 1 \leq i \leq r, 0 \leq j \leq m\}$ for the corresponding coordinates.

Similarly, an arc is determined by an $n$-tuple of power series over $\mathbb{C}$, and $J_\infty \mathbb{A}^n$ is an infinite-dimensional affine space.

Example 3.4. With the notation of the previous example, if $X \subseteq \mathbb{A}^n$ is defined by equations $\{f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0\}$, then an $m$-jet $\text{Spec } \mathbb{C}[t]/(t^{m+1}) \rightarrow X$ corresponds to a ring homomorphism

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \rightarrow \mathbb{C}[t]/(t^{m+1}).$$

The closed subscheme $J_m X \subseteq J_m \mathbb{A}^n \cong \mathbb{A}^{(m+1)n}$ is therefore defined by the equations

$$f_i \left( \sum_{j=0}^{m} x_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} x_n^{(j)} t^j \right) \equiv 0 \mod t^{m+1} \quad \text{for } 1 \leq i \leq r.$$

In other words, let $f_i^{(k)}$ be the the coefficient of $t^k$ in $f_i(\sum_{j=0}^{m} x_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} x_n^{(j)} t^j)$, so it is a polynomial in the variables $\{x_i^{(j)} \mid 1 \leq i \leq r, \ 0 \leq j \leq m\}$. Then $J_m X$ is defined by the $(m+1)r$ equations $\{f_i^{(k)} = 0 \mid 1 \leq i \leq r, \ 0 \leq k \leq m\}$.

In fact, if $R = \mathbb{C}[x_i^{(k)} \mid 1 \leq i \leq n, \ k \geq 0]$ and $D : R \rightarrow R$ is the unique derivation over $\mathbb{C}$ satisfying $D(x_i^{(k)}) = x_i^{(k+1)}$, then $f_i^{(k)} = D^k(f_i)$ [33, pg. 5].

Assume $X$ is smooth of dimension $d$. A cylinder $C$ in $J_\infty X$ is a subset of the arc space of $X$ of the form $C = \psi_m^{-1}(S)$, for some $m \geq 0$ and some constructible subset $S \subseteq J_m X$. The cylinder $C$ is called open, closed, locally closed, or irreducible if the corresponding property holds for $S$, and the codimension of $C$ is defined to be the codimension of $S$ in $J_m X$. That these notions are well-defined follows from the fact that $\pi_m, m-1$ is a Zariski-locally trivial fibration with fiber $\mathbb{A}^d$ (Remark 3.2). A subset of $J_\infty X$ is called thin if it is contained in $J_\infty V$ for some proper, closed subset $V \subseteq X$.

Lemma 3.5. [17, Proposition 5.11] Let $X$ be a smooth variety and let $C \subseteq J_\infty X$ be a cylinder. If the complement of a disjoint union of cylinders $\bigsqcup_j C_j \subseteq C$ is thin, then $\lim_{j \rightarrow \infty} \text{codim } C_j = \infty$ and $\text{codim } C = \min_j \text{codim } C_j$.

Interesting examples of cylinders arise as follows. Let $V$ be a proper, closed subscheme of $X$ defined by an ideal sheaf $\mathcal{I}_V \subseteq \mathcal{O}_X$, and let $\gamma : \text{Spec } \mathbb{C}[t] \rightarrow X$ be an arc. The pullback of $\mathcal{I}_V$ via $\gamma$ is either an ideal of the form $(t^\alpha)$, for some non-negative integer $\alpha$, or the zero ideal. In the former case, the contact order
ordγ(V) of V along γ is defined to be α; in the latter case, ordγ(V) is infinite by convention, and γ lies in J∞V ⊆ J∞X. For each non-negative integer e, set

\[ \text{Cont}^{\geq e}(V) = \{ \gamma \in J_\infty X \mid \text{ord}_\gamma(V) \geq e \}, \]

so \( \text{Cont}^{\geq 0}(V) = J_\infty X \) and \( \text{Cont}^{\geq e}(V) = \psi_{e-1}^{-1}(J_{e-1}V) \) for \( e > 0 \). We see that \( \text{Cont}^{\geq e}(V) \) is a closed cylinder and

\[ \text{Cont}^e(V) = \{ \gamma \in J_\infty X \mid \text{ord}_\gamma(V) = e \} = \text{Cont}^{\geq e}(V) \setminus \text{Cont}^{\geq e+1}(V) \]

is a locally closed cylinder.

Cylinders of this form are called contact loci. For each \( m \geq e \), we let

\[ \text{Cont}^{\geq e}(V)_m = \psi_m(\text{Cont}^{\geq e}(V)) \quad \text{and} \quad \text{Cont}^e(V)_m = \psi_m(\text{Cont}^e(V)), \]

denote the loci of \( m \)-jets with contact order with \( V \) at least \( e \) and precisely \( e \), respectively.

If subvarieties \( V_1, \ldots, V_s \) of \( X \) are specified, along with an \( s \)-tuple of nonnegative integers \( e = (e_1, \ldots, e_s) \), we write

\[ \text{Cont}^{\geq e}(V) = \bigcap_{i=1}^s \text{Cont}^{\geq e_i}(V_i) \]

and

\[ \text{Cont}^e(V) = \bigcap_{i=1}^s \text{Cont}^{e_i}(V_i) \]

for the corresponding multi-contact loci.

**Remark 3.6.** Ein, Lazarsfeld and Mustaţă [16] gave a correspondence between closed, irreducible cylinders of \( J_\infty X \) and divisorial valuations of the function field of \( X \).

We recall some results relating arc spaces and singularities [33, 34, 18]. Let \( X \) be a \( \mathbb{Q} \)-Gorenstein variety, and let \( f : Y \to X \) be a resolution of singularities such that the exceptional locus \( E = E_1 \cup \cdots \cup E_r \) is a simple normal crossings divisor. The relative canonical divisor has the form \( K_Y/X = \sum_{i=1}^r a_i E_i \), for some integers \( a_i \), and \( X \) has terminal, canonical, or log canonical singularities if \( a_i > 0 \), \( a_i \geq 0 \), or \( a_i \geq -1 \), respectively, for all \( i \).

**Theorem 3.7** ([18, Theorem 1.3]). If \( X \) is a normal, local complete intersection (l.c.i.) variety, then it has log canonical (canonical, terminal) singularities if and only if \( J_m X \) is pure dimensional (irreducible, normal) for all \( m \geq 0 \).
Remark 3.8. In general, the closure of $J_m \text{Sm}(X)$ (the jet scheme of the smooth locus) in $J_m X$ is an irreducible component of dimension $d(m + 1)$. Thus when $J_m X$ is pure-dimensional, its dimension is $d(m + 1)$.

Remark 3.9. A result of Elkik [19] and Flenner [20] states that a Gorenstein variety has canonical singularities if and only if it has rational singularities.

Remark 3.10. In fact, Mustaţă proves that if $X$ is a normal, l.c.i. variety with canonical (equivalently, rational) singularities, then $J_m X$ is l.c.i., reduced, and irreducible for all $m \geq 0$.

Let $X$ be a smooth variety and let $V$ be a proper, closed subscheme. An important invariant measuring the singularities of $V$ is the log canonical threshold $\text{lct}(X, V)$. We refer the reader to [34] for details.

Theorem 3.11 ([34, Corollary 0.2]). If $X$ is a smooth variety and $V$ is a proper, closed subscheme, then

$$\text{lct}(X, V) = \dim X - \max_m \frac{\dim J_m V}{m + 1}.$$ 

Moreover, the maximum is achieved for $m$ sufficiently divisible.

The main ingredient in the proofs of the above results is the following theorem which is motivated by Kontsevich’s theory of motivic integration.

Theorem 3.12 ([13]). Let $f : Y \to X$ be a proper, birational morphism between smooth varieties $Y$ and $X$. If $m \geq 2e$ are non-negative integers and $\text{Cont}^e(K_{Y/X})_m \subseteq J_m Y$ denotes the locus of $m$-jets with contact order $e$ with the relative canonical divisor, then the restriction of the induced map $f_m : J_m Y \to J_m X$ to $\text{Cont}^e(K_{Y/X})_m$,

$$f_m : \text{Cont}^e(K_{Y/X})_m \to \text{f}_m(\text{Cont}^e(K_{Y/X})_m),$$

is a Zariski-locally trivial fibration with fiber $\mathbb{A}^e$.

4. Equivariant geometry of jet schemes

Let $G$ be a linear algebraic group acting on a smooth complex variety $X$. Functoriality of $J_m$ (for $m$ in $\mathbb{N} \cup \{\infty\}$) implies that $J_m G$ is an algebraic group with an induced action on $J_m X$ (cf. [26, Proposition 2.6]). The main result of this section is Proposition 4.5, which gives a sufficient condition for the stabilizer of a point in $J_m X$ to be contractible.
Example 4.1. Let a torus $T$ act on $\mathbb{A}^n$ via the characters $\chi_1, \ldots, \chi_n$, i.e., $t \cdot (z_1, \ldots, z_n) = (\chi_1(t)z_1, \ldots, \chi_n(t)z_n)$. Recall that $J_m\mathbb{A}^n$ is identified with $n$-tuples of truncated polynomials (i.e., elements of $\mathbb{C}[t]/(t^{m+1})$). The characters also define homomorphisms $J_mT \to J_m\mathbb{C}^*$. Identifying $J_m\mathbb{C}^*$ with truncated polynomials with nonzero constant term, $J_mT$ acts on $J_m\mathbb{A}^n$ by

$$\gamma \cdot (\xi_1, \ldots, \xi_n) = (\chi_1(\gamma)\xi_1, \ldots, \chi_n(\gamma)\xi_n),$$

where the multiplication on the RHS is multiplication of truncated polynomials.

It is also convenient to identify $J_m\mathbb{A}^n$ with $n \times (m+1)$ matrices, with the entries in the $k$th column corresponding to the coefficients of $t^{k-1}$. Under this identification, the zero section $T_0 \subseteq J_mT$ acts simply by scaling the $i$th row by $\chi_i(t) \in \mathbb{C}^*$. The fixed subspace $(J_m\mathbb{A}^n)^{T_0}$ is identified with the rows where the corresponding character is zero; note that $(J_m\mathbb{A}^n)^{T_0}$ is the $m$th jet scheme $J_m(\mathbb{A}^n)^T$ of the fixed locus $(\mathbb{A}^n)^T$.

The same discussion holds for any (possibly disconnected) diagonalizable group $H$; for finite groups, of course, there is no difference between $J_mH$ and the zero section.

We refer the reader to [7] and [39] for basic properties of linear algebraic groups. In particular, we will need the following fact.

Lemma 4.2. Let $U$ be a complex unipotent group, and let $\mathfrak{u}$ denote its Lie algebra. The exponential map $\exp : \mathfrak{u} \to U$ is an isomorphism of complex varieties. In particular, $U$ is contractible. □

Let $e$ denote the identity element of $G$ and, for any $m \geq 0$, consider the projection $\pi_m : J_mG \to G$ and the associated exact sequence of algebraic groups

$$1 \to \pi_m^{-1}(e) \to J_mG \xrightarrow{\pi_m} G \to 1.$$  (1)

The zero section $s_m : G \to J_mG$ (see Section 3) identifies $J_mG$ with the semidirect product $\pi_m^{-1}(e) \rtimes G$.

The following lemma is stated in the Appendix in [33].

Lemma 4.3. For any $m \geq 0$, the kernel $\pi_m^{-1}(e)$ of the projection $\pi_m : J_mG \to G$ is a unipotent group.

The easy proof was related to us by Mustaţă; one uses induction on $m$, the exact sequence

$$1 \to T_eG \to \pi_m^{-1}(e) \to \pi_{m-1}^{-1}(e) \to 1,$$

and the fact that an extension of a unipotent group by another unipotent group is unipotent.
Lemma 4.4. Let $G$ be a linear algebraic group, with maximal torus $T$. Then the zero section $T_0 \subseteq J_mT \subseteq J_mG$ is a maximal torus of $J_mG$.

Proof. This is a general fact about unipotent extensions: Suppose $G = G'/U$, with $U \subseteq G'$ unipotent; then a torus in $G'$ is maximal if and only if its image in $G$ is maximal. Since every torus in $G'$ intersects $U$ trivially, and hence maps isomorphically to $G$, one implication is obvious. For the other, let $T' \subseteq G'$ be a maximal torus, let $T \subseteq G$ be a maximal torus containing the image of $T'$, and let $H' \subseteq G'$ be the preimage of $T$, so $H'$ is solvable. Then $H'/U = T$, so a maximal torus of $H'$ has the same dimension as $T$. It follows that $T$ is the image of $T'$. (To obtain the statement of the lemma, put $G' = J_mG$ and $T' = T_0$.) \hfill \Box

Proposition 4.5. Let $G$ be a connected linear algebraic group acting on a smooth variety $X$, and let $D \subseteq X$ be a $G$-invariant closed subset, with irreducible components $\{D_i\}$. Assume that the action of $G$ on $X \setminus D$ has unipotent stabilizers. Then $J_mG$ acts on $J_mX \setminus \bigcup J_mD_i$ with unipotent stabilizers.

Proof. First we reduce to the case where $G$ is a torus. Suppose the stabilizer $\Gamma \subseteq J_mG$ of $x_m \in J_mX$ is not unipotent, and let $\gamma_m \in \Gamma$ be a nontrivial semisimple element fixing $x_m$. We wish to show that $x_m$ lies in $J_mD_i$, for some irreducible component $D_i \subseteq D$.

Choose a maximal torus $T \subseteq G$, so the zero section $T_0 \subseteq J_mT \subseteq J_mG$ is a maximal torus. Since $\gamma_m$ is semisimple, it lies in a maximal torus of $J_mG$ ([39, Theorem 6.4.5]). Since all maximal tori are conjugate ([39, Theorem 6.4.1]), there is an element $c \in J_mG$ such that $c\gamma_mc^{-1} \in T_0$. This fixes $c \cdot x_m$, and since each irreducible component $D_i$ is $G$-invariant, $x_m$ lies in $J_mD_i$ if and only if $c \cdot x_m$ does. Therefore we may assume $\gamma_m$ lies in the torus $T_0$. Let $H_0 = \Gamma \cap T_0$ be the subgroup of $T_0$ fixing $x_m$; this is a diagonalizable group containing $\gamma_m$. Write $H \subseteq T$ for its isomorphic image in $G$.

Let $x = \pi_m(x_m) \in X$. By assumption, $\pi_m(\gamma_m)$ fixes $x$, so $x$ lies in $D$. Let $K \subseteq H$ be the maximal compact subgroup. Using the slice theorem (see [4, I.2.1]), we may replace $X$ with a $K$-invariant analytic neighborhood of $x$, and assume $X = \mathbb{A}^n$ with $H$ acting linearly by characters $\chi_1, \ldots, \chi_n$. Moreover, since $H$ is reductive, we have an equality of fixed point sets $X^H = X^K$. Since the fixed locus $(\mathbb{A}^n)^H$ is irreducible and contained in $D$, we have $(\mathbb{A}^n)^H \subseteq D_i$, for some $i$. Using Example 4.1, we conclude that $x_m \in (J_m\mathbb{A}^n)^{H_0} = J_m(\mathbb{A}^n)^H \subseteq J_mD_i$. \hfill \Box
5. Jet schemes and equivariant cohomology

In this section, we relate the equivariant cohomology ring $H^*_G X$ of a connected linear algebraic group $G$ acting on a smooth complex variety $X$ of dimension $d$, with the geometry of the jet schemes $J_m X$ of $X$, and prove a criterion for producing a geometric $\mathbb{Z}$-basis for $H^*_G X$.

We will use the following lemma freely throughout the rest of the paper; its proof is immediate from Lemma 2.1 and the fact that when $X$ is smooth, the morphisms $\pi_m : J_m X \to X$ and $\pi_m : J_m G \to G$ are fiber bundles with contractible fibers (Remark 3.2). When $m = \infty$, we may and will define $H^*_{J_m G} J_\infty X$ to be $H^*_{J_m G} X$.

**Lemma 5.1.** Let $X$ be a smooth $G$-variety. For any $m \in \mathbb{N} \cup \{\infty\}$, we have isomorphisms

$$H^*_G X \cong H^*_G J_m X \cong H^*_G J_m G.$$

For a $G$-invariant (or $J_m G$-invariant) closed subvariety $Z \subseteq J_m X$, we let $[Z]$ denote the corresponding class in $H^*_G X$ under the isomorphism of Lemma 5.1. Observe that a closed cylinder $C = \psi^{-1}_m(S)$, for some $S \subseteq J_m X$, is $G$-invariant (or $J_\infty G$-invariant) if and only if $S$ is $G$-invariant (respectively, $J_m G$-invariant). In this case, it follows from Lemma 5.1 that there is a well-defined class $[C] = [S] \in H^*_G X$.

**The following lemma is a direct application of Lemma 2.9 and Lemma 2.10.**

**Lemma 5.2.** Let $G$ be a connected linear algebraic group acting on a smooth complex variety $X$, with $D \subseteq X$ a $G$-invariant closed subset with irreducible components $D_1, \ldots, D_t$. Suppose there exists a filtration by $J_m G$-invariant closed subvarieties $Z_s \subseteq \cdots \subseteq Z_0 = J_m X \setminus \bigcup_i J_m D_i$, such that each $U_j = Z_j \setminus Z_{j+1}$ has trivial equivariant Borel-Moore homology (see Definition 2.7). If $k = \min\{\text{codim}(Z_s, J_m X), \min\{\text{codim}(J_m D_i, J_m X)\}\} - 1$, then

$$H^{\leq 2k}_G X = \bigoplus_{\text{codim } U_j \leq k} \mathbb{Z} \cdot [U_j].$$

**Remark 5.3.** Lemma 3.1 implies that $\lim_{m \to \infty} \text{codim}(J_m D_i, J_m X) = \infty$. In fact, Theorem 3.11 implies that $\text{codim}(J_m D_i, J_m X) \geq (m + 1) \text{let}(X, D_i)$, and equality is achieved for $m$ sufficiently divisible.

In order to state our results, we introduce the following notation. Recall from Example 2.8 that a $G$-variety $S$ is an **affine family of $G$-orbits** if there is a smooth map $S \to \mathbb{A}^n$, and $S$ is identified with a geometric quotient of $G \times \mathbb{A}^n$ by some closed subgroup scheme over $\mathbb{A}^n$. 
Definition 5.4. Let $D \subseteq X$ be a $G$-invariant closed subset with irreducible components $D_1, \ldots, D_t$. A locally closed cylinder $C \subseteq J_\infty X$ is an \textbf{affine family of orbits} (with respect to $D$) if $C = \psi^{-1}_m(S)$ for some $S \subseteq J_m X$, such that $S \cap J_m D_i = \emptyset$ for $1 \leq i \leq t$, and $S$ is an affine family of $J_m G$-orbits.

Remark 5.5. With the notation above, suppose that $G$ acts on $X \setminus D$ with unipotent stabilizers. By Proposition 4.5 and Lemma 4.2, the stabilizer of $x \in S \subseteq J_m X$ is contractible, so Example 2.8 shows that $S$ has trivial equivariant Borel-Moore homology. Moreover, for any $m' \geq m$, $\pi^{-1}_{m', m}(S) \subseteq J_{m'} X \setminus \bigcup_i J_{m'} D_i$ is smooth and hence has trivial equivariant Borel-Moore homology by Remark 3.2 and Lemma 2.1.

Definition 5.6. With the notation of Lemma 5.2, a decomposition $J_\infty X \setminus J_\infty D = \bigcup_j U_j$ into a non-empty, disjoint union of cylinders is an \textbf{equivariant affine paving} if there exists a filtration $J_\infty D \subseteq \cdots \subseteq Z_{s+1} \subseteq Z_s \subseteq \cdots \subseteq Z_0 = J_\infty X$ by $J_\infty G$-invariant closed cylinders in $J_\infty X$ containing $J_\infty D$ such that $U_j = Z_j \setminus Z_{j+1}$ is an affine family of orbits.

We are now ready to present our first main theorem.

Theorem 5.7. Let $G$ be a connected linear algebraic group acting on a smooth complex variety $X$, with $D \subseteq X$ a $G$-invariant closed subset such that $G$ acts on $X \setminus D$ with unipotent stabilizers. If $J_\infty X \setminus J_\infty D = \bigcup_j U_j$ is an equivariant affine paving, then

$$H^*_G X = \bigoplus_j \mathbb{Z} \cdot [U_j].$$

Proof. We assume the notation of Definition 5.6. Fix a degree $k$, and note that the filtration is either finite or satisfies $\lim_{j \to \infty} \codim U_j = \infty$ by Lemma 3.5; therefore the set $\{j \mid \codim U_j \leq k\}$ is always finite. Let $s - 1$ be the largest index in this finite set (so $\codim Z_s > k$ by Lemma 3.5). Now choose $m$ large enough so that $Z_j = \psi^{-1}_m(\psi_m(Z_j))$ for $j \leq s$, and $U_j = \psi^{-1}_m(S_j)$ for $j < s$, where $S_j \subseteq J_m X \setminus \bigcup_i J_m D_i$ is an affine family of $J_m G$-orbits. Also choose $m$ large enough so that $2 \min\{\codim(J_m D_i, J_m X)\} > k$ (see Remark 5.3), where the $D_i$ are the irreducible components of $D$. Setting $Z'_j = \psi_m(Z_j) \setminus \bigcup_i J_m D_i$, we have a filtration of $J_m G$-invariant closed subvarieties

$$Z'_s \subseteq \cdots \subseteq Z'_0 = J_m X \setminus \bigcup_i J_m D_i,$$
such that each $\psi_m(U_j) = Z'_j \setminus Z'_{j+1}$ has trivial equivariant Borel-Moore homology by Remark 5.5. The result now follows from Lemma 5.2. □

**Remark 5.8.** If $G$ acts on a smooth variety $X$ with a free, dense open orbit $U$, then $G$ acts on $U$ with trivial, and hence unipotent, stabilizers. Applications of Theorem 5.7 of this type are given in Section 8 and Section 9.

**Remark 5.9.** The simplest type of cylinder which is an affine family of orbits consists of a single $J\infty G$-orbit in $J\infty X$. The existence of an equivariant affine paving involving only cells of this type is quite restrictive, however. Indeed, suppose $X$ is compact and $G$ acts freely on $X \setminus D$. The valuative criterion for properness [25, Theorem II.4.7] implies that there is a bijection between $J\infty G$-orbits of $J\infty X \setminus \bigcup_j J\infty D_i$ and elements of the affine Grassmannian $G((t))/G[[t]]$, and the latter is uncountable unless $G$ is diagonalizable. Since our notion of paving assumes countably many orbits—in fact, finitely many in any given codimension—essentially the only examples of this type are compactifications of tori, i.e. toric varieties (see Section 8).

For the remainder of the section, we will consider a proper, equivariant birational map $f : Y \to X$ between smooth $G$-varieties $Y$ and $X$, for some connected linear algebraic group $G$. We will apply our results above to describe a method for comparing the $G$-equivariant cohomology of $X$ and $Y$.

Suppose that $D \subseteq X$ is a $G$-invariant closed subset such that $G$ acts on $X \setminus D$ with unipotent stabilizers, and, with the notation of Definition 5.6, consider an equivariant affine paving $J\infty X \setminus J\infty D = \bigcup_j U_j$. Recall that the relative canonical divisor $K_{Y/X}$ on $Y$ is the divisor defined by the vanishing of the Jacobian of $f : Y \to X$, and that $f_\infty : J\infty Y \to J\infty X$ denotes the morphism of arc spaces corresponding to $f$. We say that the paving is compatible with $f$ if $f^{-1}_\infty(U_j) \subseteq \text{Cont}^{e_j}(K_{Y/X})$ for some non-negative integer $e_j$ and for all $j$. In this case, we will write $e_j = \text{ord}_{f^{-1}_\infty(U_j)}(K_{Y/X})$.

**Corollary 5.10.** Let $G$ be a connected linear algebraic group and let $f : Y \to X$ be a proper, equivariant birational map between smooth $G$-varieties $Y$ and $X$. Let $D \subseteq X$ be a $G$-invariant closed subset such that $G$ acts on $X \setminus D$ with unipotent stabilizers, and let $J\infty X \setminus J\infty D = \bigcup_j U_j$ be an equivariant affine paving which is compatible with $f$. These data determine a bijection between $\mathbb{Z}$-bases of $H^*_G Y$ and $H^*_G X$, explicitly given by

$$H^*_G Y = \bigoplus_j \mathbb{Z} \cdot [f^{-1}_\infty(U_j)], \quad H^*_G X = \bigoplus_j \mathbb{Z} \cdot [U_j].$$

Moreover, $\text{codim} U_j = \text{codim} f^{-1}_\infty(U_j) + e_j$, where $e_j = \text{ord}_{f^{-1}_\infty(U_j)}(K_{Y/X})$.
The paving of Definition 5.6,

\[ J_\infty D \subseteq \cdots \subseteq Z_{j+1} \subseteq Z_j \subseteq \cdots \subseteq Z_0 = J_\infty X, \]

lifts to a chain of \( J_\infty G \)-invariant closed cylinders containing \( J_\infty(f^{-1}(D)) \):

\[ f_\infty^{-1}(J_\infty D) = J_\infty(f^{-1}(D)) \subseteq \cdots \subseteq f_\infty^{-1}(Z_{j+1}) \subseteq f_\infty^{-1}(Z_j) \subseteq \cdots \subseteq f_\infty^{-1}(Z_0) = J_\infty Y. \]

Here \( f_\infty^{-1}(Z_j) \setminus f_\infty^{-1}(Z_{j+1}) = f_\infty^{-1}(U_j) \), and \( f^{-1}(D) \) denotes the scheme-theoretic inverse image of \( D \).

Fix a degree \( k \) and note that the set \( \{ j \mid \operatorname{codim} f_\infty^{-1}(U_j) \leq k \} \) is finite by Lemma 3.5. Let \( s-1 \) be an index greater than \( \max(\{ j \mid \operatorname{codim} f_\infty^{-1}(U_j) \leq k \}) \) and \( \max(\{ j \mid \operatorname{codim} U_j \leq k \}) \). By the proof of Theorem 5.7 and Lemma 5.2, we may choose \( m \) sufficiently large such that we have a filtration of \( J_m G \)-invariant closed subvarieties

\[ Z'_s \subseteq \cdots \subseteq Z'_0 = J_m X \setminus \bigcup_i J_m D_i, \]

such that each \( \psi_m^X(U_j) = Z'_j \setminus Z'_{j+1} \) has trivial Borel-Moore homology and \( U_j = \psi_m^X(\psi_m^X(U_j)) \). Moreover, if \( D_1, \ldots, D_t \) denote the irreducible components of \( D \), then we may choose \( m \) large enough so that \( 2 \min\{ \operatorname{codim}(J_m f^{-1}(D_i), J_m Y) \} > k \) (by Lemma 3.1) and \( m \geq 2e_j \) for \( 0 \leq j \leq s-1 \).

Consider the filtration

\[ f_m^{-1}(Z'_s) \subseteq \cdots \subseteq f_m^{-1}(Z'_0) = J_m Y \setminus \bigcup_i J_m f^{-1}(D_i), \]

with \( f_m^{-1}(Z'_j) \setminus f_m^{-1}(Z'_{j+1}) = f_m^{-1}(\psi_m^X(U_j)) = \psi_m^Y(f_m^{-1}(U_j)) \). By Theorem 3.12, the restriction \( f_m : f_m^{-1}(U_j) \to U_j \) is a \( J_m G \)-equivariant, Zariski-locally trivial fibration with fiber \( \mathbb{A}^{e_j} \). We conclude that \( f_m^{-1}(U_j) \) has trivial equivariant Borel-Moore homology and \( \operatorname{codim} U_j = \operatorname{codim} f_\infty^{-1}(U_j) + e_j \). Using Lemma 2.9 and Lemma 2.10, we conclude that \( H^{2k-1}_G Y = 0 \) and

\[ H^{2k}_G Y = \bigoplus_{\operatorname{codim} U_j + e_j = k} \mathbb{Z} \cdot [f_\infty^{-1}(U_j)]. \]

The result now follows from Theorem 5.7.

In the succeeding two sections, we will give criteria to interpret multiplication in the equivariant cohomology ring geometrically. An answer to the following question may be very useful in proving Conjecture 9.6 (cf. Example 9.12 and Remark 9.7):

**Question 5.11.** Under suitable assumptions, can one compare the multiplication of classes in the \( \mathbb{Z} \)-bases of \( H^*_G Y \) and \( H^*_G X \) determined by Corollary 5.10?
Remark 5.12. The relationship between the graded dimensions of $H^*_G(Y; \mathbb{C})$ and $H^*_G(X; \mathbb{C})$, and the relative canonical divisor $K_{Y/X}$, would be predicted by an equivariant version of motivic integration. We say that two smooth $G$-varieties $X$ and $Y$ are **equivariantly K-equivalent** if there is a smooth $G$-variety $Z$ and $G$-equivariant, proper birational maps $Z \to X$ and $Z \to Y$ such that $K_{Z/X} = K_{Z/Y}$. For example, one may consider equivariantly $K$-equivalent toric varieties with respect to the torus action (cf. Section 8). As in the non-equivariant case, one expects that if $X$ and $Y$ are equivariantly $K$-equivalent, then $\dim \mathbb{C} H^i_G(X; \mathbb{C}) = \dim \mathbb{C} H^i_G(Y; \mathbb{C})$ for all $i \geq 0$.

Question 5.13. Do there exist interesting examples of $G$-equivariantly $K$-equivalent varieties, where $G$ is non-trivial, other than $K$-equivalent toric varieties?

6. Multiplication of classes I

In this section and the next, we use jet schemes to give a geometric interpretation of multiplication in the equivariant cohomology ring $H^*_G X$ of a smooth variety $X$ acted on by a connected linear algebraic group $G$. We present two sets of results, with different assumptions on the singularities of subvarieties: the first concerns local complete intersection varieties (treated in this section), and the second requires the singular locus to be sufficiently small (discussed in the following section).

It will be convenient to introduce some terminology for this section. A subvariety $V \subseteq X$ is an **equivariant complete intersection** if it has codimension $r$ and is the scheme-theoretic intersection of $r$ $G$-invariant hypersurfaces in $X$. Similarly, $V \subseteq X$ is an **equivariant local complete intersection (e.l.c.i.)** if it is a local complete intersection variety locally cut out by $G$-invariant hypersurfaces. Of course, a $G$-invariant l.c.i. subvariety need not be e.l.c.i.: for example, the origin in $\mathbb{C}^n$ is not cut out by $GL_n$-invariant hypersurfaces (since there are no such hypersurfaces).

For a tuple of non-negative integers $m = (m_1, \ldots, m_s)$, let $\lambda(m) = (\lambda_1, \ldots, \lambda_s)$ be the partition defined by $\lambda_i = m_1 + \cdots + m_s$. The main theorem of this section is the following:

**Theorem 6.1.** Assume the following:

(*) $G$ is a connected reductive group, $X^G$ is finite, and the natural map $\iota^* : H_G^* X \to H_G^* X^G$ is injective.

Consider a chain of e.l.c.i. subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X,$$
and a tuple $m = (m_1, \ldots, m_s)$ of non-negative integers. If $\text{codim} \text{Cont}^\lambda(m)(V_\bullet) = \sum_{i=1}^s m_i \text{codim} V_i$, then

$$[V_1]^{m_1} \cdots [V_s]^{m_s} = [\text{Cont}^\lambda(m)(V_\bullet)]$$

**Remark 6.2.** In the statement of the above theorem, observe that if the hypothesis $\text{codim} \text{Cont}^\lambda(m)(V_\bullet) = \sum_{i=1}^s m_i \text{codim} V_i$ holds for all tuples $m = (m_1, \ldots, m_s)$ of non-negative integers, then $[\text{Cont}^\lambda(m)(V_\bullet)] = [\text{Cont}^\lambda(m)(V_\bullet)]$.

We will prove Theorem 6.1 by reducing to the case of $\mathbb{A}^d$. (The assumption $(\ast)$ is needed only for the reduction, so we do not require it in what follows, when $X = \mathbb{A}^d$.)

Let $G$ act on $\mathbb{A}^d$ and let $V \subseteq \mathbb{A}^d$ be a $G$-invariant hypersurface, defined by $f \in \mathbb{C}[x_1, \ldots, x_n]$. Recall from Example 3.4 that $J_m V \subseteq J_m \mathbb{A}^d$ is defined by equations $\{f(k) \mid 0 \leq k \leq m\}$ in the variables $\{x_i(k) \mid 1 \leq i \leq n, 0 \leq k \leq m\}$.

**Lemma 6.3.** For $0 \leq k \leq m$, the hypersurface $V^{(k)} := \{f + f^{(k)} = 0\} \subseteq J_m \mathbb{A}^d$ is $G$-invariant, and under the isomorphism of Lemma 5.1, $[V^{(k)}] = [V] \in H^*_c \mathbb{A}^d$.

**Proof.** The lemma is trivial when $k = 0$, so assume $k \geq 1$. Since $V$ is invariant, $g \cdot f = \lambda(g)f$ for some character $\lambda : G \to \mathbb{C}^\times$. With the notation of Example 3.4, it follows from the definition of the action of $G$ on $J_m \mathbb{A}^d$ that

$$(g \cdot f) \left( \sum_{k=0}^m x_1^{(k)} t^k, \ldots, \sum_{k=0}^m x_d^{(k)} t^k \right) = \lambda(g)f \left( \sum_{k=0}^m x_1^{(k)} t^k, \ldots, \sum_{k=0}^m x_d^{(k)} t^k \right).$$

In particular, considering coefficients of $t^k$ on both sides gives $g \cdot f^{(k)} = \lambda(g)f^{(k)}$ for $0 \leq k \leq m$, and we conclude that $V^{(k)}$ is $G$-invariant.

Let $V \subseteq J_m \mathbb{A}^d \times \mathbb{A}^1$ be defined by the equation $f + \zeta f^{(k)} = 0$ (where $\zeta$ is the parameter on $\mathbb{A}^1$). Thus $V \to \mathbb{A}^1$ is an equivariant family of hypersurfaces in $J_m \mathbb{A}^d$, whose fibers at $\zeta = 0$ and $\zeta = 1$ are $V$ and $V^{(k)}$, respectively. (The polynomials $f$ and $f^{(k)}$ involve different variables, so $f + \zeta f^{(k)}$ is never identically zero; hence each fiber has the same dimension.) Since $V$ is a hypersurface in an affine space, it follows that the projection $V \to \mathbb{A}^1$ is flat; indeed, one easily checks that $\mathbb{C}[V]$ is torsion free and hence free over $\mathbb{C}[\zeta]$. We conclude that $[V^{(k)}] = [V]$.

**Remark 6.4.** If $G \cong (\mathbb{C}^\times)^r$ is a torus, then the equivariant cohomology class of a torus-invariant subvariety $V \subseteq \mathbb{A}^d$ is equal to its multi-degree [32, Chapter 8]. In this case, it follows from the description of $f^{(k)}$ as an iterated derivation of $f$ in Example 3.4 that $V$ and $V^{(k)}$ have the same multi-degree, implying the above lemma.
Recall that for a tuple of non-negative integers \( \mathbf{m} = (m_1, \ldots, m_s) \), we let \( \lambda(\mathbf{m}) = (\lambda_1, \ldots, \lambda_s) \) be the partition defined by \( \lambda_i = m_i + \cdots + m_s \).

**Proposition 6.5.** Consider a chain of equivariant complete intersection subvarieties

\[
V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq \mathbb{A}^d,
\]
and a tuple \( \mathbf{m} = (m_1, \ldots, m_s) \) of non-negative integers. If \( \text{codim} \text{Cont}^{\geq \lambda(\mathbf{m})}(V_s) = \sum_{i=1}^s m_i \text{codim} V_i \), then

\[
[V_1]^{m_1} \cdots [V_s]^{m_s} = [\text{Cont}^{\geq \lambda(\mathbf{m})}(V_\bullet)].
\]

**Proof.** We will show that \( \text{Cont}^{\geq \lambda(\mathbf{m})}(V_\bullet) \) is an equivariant complete intersection. Fix \( m \geq \lambda_1 - 1 \), so the equations defining \( \text{Cont}^{\geq \lambda(\mathbf{m})}(V_\bullet) \) are the same as those defining \( \bigcap \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_1-1}V_i) \) in \( J_m \mathbb{A}^d \). It will suffice to prove the claimed equation in \( H^*_G J_m \mathbb{A}^d \).

For each \( i \), let \( r_i = \text{codim} V_j \) and let \( f_{i,1}, \ldots, f_{i,r_i} \) be (semi-invariant) polynomials defining \( V_i \). Thus

\[
\{ f_{i,j}^{(k)} \mid 1 \leq j \leq r_i, \ 0 \leq k \leq \lambda_i - 1 \}
\]
defines \( J_{\lambda_i-1}V_i \) in \( J_{\lambda_i-1} \mathbb{A}^d \), as well as \( \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_i-1}V_i) \).

Now consider \( V_s \subseteq V_{s-1} \). Since \( J_{\lambda_s-1}V_s \subseteq J_{\lambda_i-1}V_{s-1} \), we have a containment of ideals

\[
(f_{s,j}^{(k)} \mid 1 \leq j \leq r_s, \ 0 \leq k \leq \lambda_s - 1) \supseteq (f_{s-1,j}^{(k)} \mid 1 \leq j \leq r_{s-1}, \ 0 \leq k \leq \lambda_{s-1} - 1).
\]

To cut out \( \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_i-1}V_s) \cap \pi_{m,\lambda_s-1}^{-1}(J_{\lambda_{s-1}-1}V_{s-1}) \), then, we need the \( m_s \cdot r_s \) equations

\[
\{ f_{s,j}^{(k)} \mid 1 \leq j \leq r_s, \ 0 \leq k \leq \lambda_s - 1 \}
\]
together with the \( m_{s-1} \cdot r_{s-1} \) equations

\[
\{ f_{s,j}^{(k)} \mid 1 \leq j \leq r_{s-1}, \ \lambda_s \leq k \leq \lambda_{s-1} - 1 \}.
\]

Continuing in this way, we obtain \( \sum_{i=1}^s m_i \cdot r_i \) equations defining \( \bigcap \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_i-1}V_i) \); by hypothesis, this is the codimension of \( \bigcap \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_i-1}V_i) \), so it is a complete intersection. It follows that

\[
[\bigcap \pi_{m,\lambda_1-1}^{-1}(J_{\lambda_i-1}V_i)] = \prod_{i=1}^s \prod_{j=1}^{r_i} \prod_{k=\lambda_i-1}^{\lambda_i-1} [V_{i,j}^{(k)}],
\]

where \( V_{i,j}^{(k)} \subseteq J_m \mathbb{A}^d \) is the \( G \)-invariant hypersurface defined by \( f_{i,j}^{(k)} \). By Lemma 6.3, the class \( [V_{i,j}^{(k)}] \) is independent of \( k \), and since \( V_i \) is a complete intersection, we have

\[
\prod_{j=1}^{r_i} [V_{i,j}^{(0)}] = [V_i].
\]

The proposition follows. \( \Box \)
In practice, the codimension condition in the above proposition may be difficult to check. It would be very interesting to have a nice answer to the following question.

**Question 6.6.** Can one give a geometric criterion for the codimension condition in Proposition 6.5 to be satisfied for all tuples \( m = (m_1, \ldots, m_s) \) of non-negative integers?

In the case when \( V_s = V_1 = V \subseteq \mathbb{A}^d \), we have the following answer.

**Corollary 6.7.** Suppose \( V \subseteq \mathbb{A}^d \) is an equivariant complete intersection. Then \( [J_mV] = [V]^{m+1} \) whenever \( J_mV \) is pure-dimensional. In particular, if \( V \) is normal and \( [V] \) is a non-torsion element in \( H^*_G X \), then this equation holds for all \( m \geq 0 \) if and only if \( V \) has log canonical singularities.

**Proof.** The first statement follows from Proposition 6.5, and the second is immediate from Theorem 3.7. □

**Proof of Theorem 6.1.** By hypothesis (\( * \)), \( H^*_G X \) embeds in \( H^*_G X^G \), so it suffices to establish the formula (2) after restriction to a fixed point \( p \in X^G \). Since \( G \) is reductive, the slice theorem gives a \( G \)-invariant (étale or analytic) neighborhood of \( p \) equivariantly isomorphic to \( \mathbb{A}^d \). Now apply Proposition 6.5. □

**Remark 6.8.** If one uses \( \mathbb{Q} \) coefficients for cohomology, the hypothesis (\( * \)) in Theorem 6.1 can be replaced by the following:

(\( *' \)) \( G \) is connected, and for a maximal torus \( T \subseteq G \), \( X^T \) is finite and the map \( H^*_T X \to H^*_T X^T \) is injective.

Moreover, we may assume that our subvarieties \( \{V_i\} \) are e.l.c.i with respect to \( T \). Indeed, (\( * \)) applies to \( T \), and \( H^*_G X \) embeds in \( H^*_T X \) as the subring of Weyl invariants, by a theorem of Borel.

**Remark 6.9.** Corollary 6.7 also extends to e.l.c.i. subvarieties, using either hypothesis (\( * \)) or (\( *' \)).

The following variant is useful in practice and follows immediately from Theorem 6.1.

**Corollary 6.10.** Assume hypothesis (\( * \)), and let \( Y_1, \ldots, Y_s \) be invariant subvarieties of \( X \) such that each intersection \( V_i = Y_1 \cap \cdots \cap Y_i \) is proper and e.l.c.i. For a tuple \( m = (m_1, \ldots, m_s) \) of non-negative integers, if \( \text{codim} \ Cont_{\geq m}(Y_\bullet) = \sum_{i=1}^s m_i \text{codim} Y_i \), then

\[
[Y_1]^{m_1} \cdot [Y_s]^{m_s} = [\text{Cont}_{\geq m}(Y_\bullet)].
\]
Example 6.11. Suppose \( G \) and \( X \) satisfy (\( \ast \)), and let \( D = D_1 + \cdots + D_s \) be a \( G \)-invariant normal crossings divisor in \( X \). Corollary 6.10 and Remark 6.2 apply, so

\[
[D_1]^{m_1} \cdots [D_s]^{m_s} = [\text{Cont}_m(D)] = [\text{Cont}^m(D)].
\]

7. Multiplication of classes II

Requiring that a subvariety be an equivariant local complete intersections is quite restrictive. In the general case, we have a partial result, which instead places a restriction on the dimension of the singular locus. Let \( \text{Sm}(V) \) and \( \text{Sing}(V) \) denote the smooth and singular loci of a variety \( V \), respectively. Throughout this section, \( G \) is assumed to be reductive.

In what follows, we will embed \( X \) as a smooth subvariety of \( J_mX \) via the zero section, and write \( \Delta_{m+1} : X \hookrightarrow X \times \cdots \times X \) for the diagonal embedding of \( X \) in the \((m+1)\)-fold product.

Lemma 7.1. If \( X \) is a smooth \( G \)-variety of dimension \( d \), then there are canonical isomorphisms \( N_{X/J_mX} \cong N_{\Delta_{m+1}/X \times \cdots \times X} \cong T_X \oplus m \).

Proof. By the functorial definition of jet schemes,

\[
T(J_mX) = \text{Hom}(\text{Spec} \mathbb{C}[s]/(s^2), J_mX) = \text{Hom}(\text{Spec} \mathbb{C}[s,t]/(s^2, t^{m+1}), X).
\]

Hence, for a closed point \( x \) in \( X \), an element of \( T_xJ_mX \) corresponds to a \( \mathbb{C} \)-algebra homomorphism

\[
\theta : \mathcal{O}_{X,x} \to \mathbb{C}[s,t]/(s^2, t^{m+1}), \quad \theta(y) = \theta_0(y) + \sum_{i=0}^{m} \varphi_i(y)s^it^i,
\]

where \( \theta_0 : \mathcal{O}_{X,x} \to \mathbb{C} \) is the \( \mathbb{C} \)-algebra homomorphism corresponding to \( x \). That \( \theta \) is a \( \mathbb{C} \)-algebra homomorphism is equivalent to requiring that \( \theta_0(y) + s\varphi_i(y) \) is a closed point in \( T_xX \) for \( 0 \leq i \leq m \). Hence, we have a natural isomorphism

\[
T_xJ_mX \cong T_xX \times \cdots \times T_xX.
\]

Moreover, identifying \( X \) with the zero section, we have an embedding of \( T_xX \) in \( T_xJ_mX \) whose image corresponds to the subspace where \( \varphi_1 = \cdots = \varphi_m = 0 \). On the other hand,

\[
T_x(X \times \cdots \times X) \cong T_xX \times \cdots \times T_xX,
\]

and \( T_x\Delta_{m+1} \) is the image of \( T_xX \) under the diagonal embedding in \( T_xX \times \cdots \times T_xX \). Hence (with a slight abuse of notation)

\[
N_{X/J_mX,x} = \{(\varphi_0, \varphi_1, \ldots, \varphi_m) \mid \varphi_i \in T_xX \}/\{(\varphi, 0, \ldots, 0) \mid \varphi \in T_xX\},
\]
$N_{\Delta_{m+1}/X\times\cdots\times X,x} = \{(\varphi'_0, \varphi'_1, \ldots, \varphi'_{m}) \mid \varphi'_i \in T_xX\}/\{(\varphi', \varphi', \ldots, \varphi') \mid \varphi' \in T_xX\}$

and there is a natural isomorphism sending

$$(\varphi_0, \varphi_1, \ldots, \varphi_m) \mapsto (\varphi_0, \varphi_0 - \varphi_1, \ldots, \varphi_0 - \varphi_m).$$

One easily verifies that this extends to a canonical global isomorphism. □

For the remainder of the section we consider a chain of invariant subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X,$$

and a tuple $m = (m_1, \ldots, m_s)$ of non-negative integers with $m_s > 0$. Recall that $\lambda(m) = (\lambda_1, \ldots, \lambda_s)$ denotes the partition defined by $\lambda_i = m_i + \cdots + m_s$, and $\text{Cont}^{\geq \lambda(m)}(V_s)$ denotes the associated multi-contact locus. If $U = X \setminus \bigcup_i \text{Sing}(V_i)$, then $\text{Cont}^{\geq \lambda(m)}(V_s)$ restricts to a smooth, irreducible cylinder in $J_{\infty}U$. The closure of this restricted cylinder in $J_{\infty}X$ is an irreducible cylinder of codimension $\sum_i m_i \text{codim} V_i$ which we denote by $\text{Cont}^{\geq \lambda(m)} \text{Sm}(V_s)$.

**Remark 7.2.** Consider a chain of invariant smooth subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X,$$

and a tuple $m = (m_1, \ldots, m_s)$ of non-negative integers with $m_s > 0$. Fix $m \geq \lambda_1 - 1$, so that $\text{Cont}^{\geq \lambda(m)}(V_s)_m = \bigcap \pi_{m,\lambda_i-1}^{-1}(J_{\lambda_i-1}V_i)$ in $J_mX$. The proof of Lemma 7.1 gives a canonical isomorphism between the normal bundle of $V_s$, embedded via the diagonal embedding in $\underbrace{V_s \times \cdots \times V_s}_{m_s \text{ times}} \times \underbrace{V_1 \times \cdots \times V_1 \times X \times \cdots \times X}_{m_1 \text{ times} \times \text{m+1-\lambda_1 times}}$, and the normal bundle of $V_\bullet$, embedded via the zero section in $\text{Cont}^{\geq \lambda(m)}(V_s)_m$, and the normal bundle of $V_s$, embedded via the diagonal embedding in $\underbrace{V_s \times \cdots \times V_s}_{m_s \text{ times}} \times \underbrace{V_1 \times \cdots \times V_1 \times X \times \cdots \times X}_{m_1 \text{ times} \times \text{m+1-\lambda_1 times}}$.

**Theorem 7.3.** Let $X$ be a smooth $G$-variety of dimension $d$, and consider a chain of invariant subvarieties

$$V_s \subseteq V_{s-1} \subseteq \cdots \subseteq V_1 \subseteq X,$$

and a tuple $m = (m_1, \ldots, m_s)$ of non-negative integers. We have

$$[V_1]^{m_1} \cdots [V_s]^{m_s} = [\text{Cont}^{\geq \lambda(m)}(V_s)]$$

whenever $\min\{\text{codim}(\text{Sing}(V_\tau), X)\} > \sum_i m_i \text{codim}(V_i, X)$. (By convention, $\dim \emptyset = -\infty$.)

**Proof.** Clearly we may assume that $m_s > 0$. Let $c_i$ denote the codimension of $V_i$. Let $Z = \bigcup\text{Sing}(V_\tau)$ and let $U = X \setminus Z$, so we have an exact sequence

$$\cdots \rightarrow H^{G}_{2(d-\sum m_i c_i)}Z \rightarrow H^{2m_i c_i}_G X \rightarrow H^{2m_i c_i}_G U \rightarrow H^{G}_{2(d-\sum m_i c_i)-1}Z \rightarrow \cdots.$$
By the assumption on \( \dim Z \), the left and right terms are zero, so the restriction map \( H^G_{\sum 2m_i c_i} X \to H^G_{\sum 2m_i c_i} U \) is an isomorphism. Replacing \( X \) with \( U \) and \( V_s \) with \( V_s \cap U \), we reduce to the case when each \( V_s \) is smooth.

Fix \( m \geq \lambda_1 - 1 \), so that \( \text{Cont}^{\geq \lambda(m)}(V_\bullet)_m = \bigcap \pi^{-1}_{m, \lambda_1 - 1}(J_{\lambda_1 - 1} V_i) \) in \( J_m X \). Let \( K \subseteq G \) be the maximal compact subgroup; since a reductive group retracts onto its maximal compact subgroup, \( G \)- and \( K \)-equivariant cohomology are naturally isomorphic, and we identify the two for the rest of this argument. The slice theorem (see [4, I.2.1]) gives a \( K \)-invariant neighborhood \( U_X \subseteq J_m X \) of \( X \) which is \( K \)-equivariantly isomorphic to a neighborhood of the zero section in \( N_{X/J_m X} \). Note that restriction to the zero section \( H^*_G J_m X \to H^*_G U_X \to H^*_G X \) is an isomorphism by Lemma 5.1. Since \( U_X \) retracts onto \( X \), the map \( H^*_G U_X \to H^*_G X \) is also an isomorphism, and hence the restriction \( H^*_G J_m X \to H^*_G U_X \) is an isomorphism.

By the canonical isomorphism \( N_{X/J_m X} \cong N_{\Delta_{m+1}(X)/X \times \cdots \times X} \) of Lemma 7.1, \( U_X \) is \((K\text{-equivariantly})\) isomorphic to an open neighborhood of the diagonal \( \Delta_{m+1}(X) \) in \( X \times \cdots \times X \). Moreover, Remark 7.2 implies that the class of \( \text{Cont}^{\geq \lambda(m)}(V_\bullet)_m \) in \( J_m X \) restricts to the class of the intersection \( U_{V_\bullet, m_i} \) of \( U_X \) with

\[ V_{\bullet, m_i} := V_1 \times \cdots \times V_s \times \cdots \times V_i \times \cdots \times V_1, \]

where \( \sum_{m_i} = m \). Consider the commutative diagram:

\[
\begin{array}{ccc}
H^G_{\sum 2m_i c_i} (X \times \cdots \times X) & \to & H^G_{\sum 2m_i c_i} (U_X) \\
\downarrow & & \downarrow \\
H^G_{\sum 2m_i c_i} (X \times \cdots \times X) & \to & H^G_{\sum 2m_i c_i} (X) \\
\end{array}
\]

Here the horizontal arrows are restriction to the tubular neighborhoods; the composition in the bottom row is \( \Delta^* \). Going counter-clockwise around the diagram, we have \( \Delta^*[V_{\bullet, m_i}] = [V_1]^{m_1} \cdots [V_s]^{m_s} \). Going clockwise, we have \([\text{Cont}^{\geq \lambda(m)}(V_\bullet)]\), completing the proof.

**Remark 7.4.** The condition on singular loci in Theorem 7.3 is quite restrictive. In particular, it implies that \( \text{codim}(\bigcup_i \text{Sing}(V_i), X) > \sum_i m_i \text{codim} V_i \), and hence that \( \text{codim} \text{Cont}^{\geq \lambda(m)}(V_\bullet) = \text{codim} \text{Cont}^{\geq \lambda(m)} \text{Sm}(V_\bullet) = \sum_i m_i \text{codim} V_i \). Hence this condition is stronger than the codimension condition in Theorem 6.1.

In the case when \( V_s = V_1 = V \), Theorem 7.3 reduces to the following corollary.

**Corollary 7.5.** Let \( X \) be a smooth \( G \)-variety of dimension \( d \), and let \( V \subseteq X \) be a \( G \)-invariant connected subvariety of codimension \( c \). We have

\[ [V]^{m+1} = [J_m \text{Sm}(V)] \]
whenever codim(Sing(V), X) > (m + 1)c. (By convention, dim \emptyset = -\infty.)

**Example 7.6.** Some condition on the singular locus is necessary. For example, let $V \subseteq \mathbb{A}^2$ be the cuspidal cubic defined by $x^3 - y^2 = 0$; this is invariant for the action of $T = \mathbb{C}^*$ by $z \cdot (x, y) = (z^2 x, z^3 y)$. Since $V$ has degree 6 with respect to the grading corresponding to the $\mathbb{C}^*$-action, we have $[V] = 6t$ in $H^*_T \mathbb{A}^2 \cong \mathbb{Z}[t]$. The tangent bundle $TV = J^1 V$ is defined by the two equations $x^3 - y^2 = 0$ and $3x^2 x_1 - 2yy_1 = 0$, each of which has degree 6, so $[TV] = 36t^2 = [V]^2$. On the other hand, $TV$ has two irreducible components, and one can check that $[TSm(V)] = 18t^2$.

**Question 7.7.** For an arbitrary $G$-invariant subvariety $V \subseteq X$ of codimension $c$, let $[J_m V]_{\exp}$ denote the sum of all components of $J_m V$ which have “expected codimension” $c(m + 1)$ in $J_m X$. When does $[V]^{m+1} = [J_m V]_{\exp}$ hold?

8. **Example: smooth toric varieties**

The goal of this section is to apply our results to give a geometric interpretation of the equivariant cohomology ring of a smooth toric variety. We refer the reader to [21] for an introduction to toric varieties.

Let $X = X(\Sigma)$ be a smooth $d$-dimensional toric variety corresponding to a fan $\Sigma$ in a lattice $N$ of rank $d$. Let $v_1, \ldots, v_r$ denote the primitive integer vectors of the rays in $\Sigma$ and let $D_1, \ldots, D_r$ denote the corresponding torus-invariant prime divisors of $X$. The **Stanley-Reisner ring** $\text{SR}(\Sigma)$ is the quotient of $\mathbb{Z}[x_1, \ldots, x_r]$ by the ideal generated by monomials of the form $x_{i_1} \cdots x_{i_d}$, such that $v_{i_1}, \ldots, v_{i_d}$ do not span a cone in $\Sigma$. If $T$ denotes the torus associated to $X$, then we have the following description of the equivariant cohomology ring $H^*_T X$:

**Theorem 8.1.** [6, Theorem 8] With the notation above, there is a natural isomorphism $H^*_T X \cong \text{SR}(\Sigma)$, sending $[D_i]$ to $x_i$, for $1 \leq i \leq r$.

Our goal is to apply our results to give a new geometric proof of this result. Observe that $\text{SR}(\Sigma)$ has a $\mathbb{Z}$-basis indexed by lattice points in $N$ which lie in the support $|\Sigma|$ of $\Sigma$: if $\sigma$ is a maximal cone with primitive integer vectors $v_{i_1}, \ldots, v_{i_d}$, then a lattice point $v = \sum_{j=1}^d a_j v_j$ corresponds to the monomial $x^v := x_{i_1}^{a_1} \cdots x_{i_d}^{a_d}$ in $\text{SR}(\Sigma)$. In fact, $\text{SR}(\Sigma)$ is isomorphic to the **deformed group ring** $\mathbb{Z}[N]^\Sigma$ of $\Sigma$: $\mathbb{Z}[N]^\Sigma$ is the $\mathbb{Z}$-algebra with $\mathbb{Z}$-basis $\{ y^v \mid v \in |\Sigma| \cap N \}$ and multiplication defined by

$$y^u \cdot y^v = \begin{cases} y^{u+v} & \text{if } u, v \in \sigma \text{ for some } \sigma \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$
On the other hand, with the notation above, for each \( v = \sum_{j=1}^{d} a_j v_i \in |\Sigma| \cap N \) consider the cylinder \( \text{Cont}^v(D) := \bigcap_{1 \leq j \leq d} \text{Cont}^{a_j}(D_i) \) in \( J_\infty X \). One verifies the decomposition

\[
J_\infty X \setminus \bigcup_i J_\infty D_i = \prod_{v \in |\Sigma| \cap N} \text{Cont}^v(D).
\]

We will let \( \text{Cont}^{\geq v}(D) \) denote the closure of \( \text{Cont}^v(D) \) in \( J_\infty X \), and define a partial order \( \leq \) on \( |\Sigma| \cap N \) by setting \( w \leq v \) if \( w - v \) lies in some maximal cone in \( \Sigma \) containing \( v \) and \( w \). The following lemma may be deduced from the case when \( X = \mathbb{A}^d \) (see Example 4.1), and also follows from a more general result of Ishii.

**Lemma 8.2.** [26] The cylinders \( \{ \text{Cont}^v(D) \mid v \in |\Sigma| \cap N \} \) are precisely the \( J_\infty T \)-orbits of \( J_\infty X \setminus \bigcup_i J_\infty D_i \), and \( \text{Cont}^{\geq v}(D) \) is an equivariant affine paving, and \( \text{Cont}^v(E) \subseteq \bigcup_{w \leq v} \text{Cont}^w(D) \).

We are now ready to state our geometric interpretation of the equivariant cohomology ring of \( X \).

**Corollary 8.3.** There is a natural isomorphism \( H^*_T X \cong \text{SR}(\Sigma) \) such that the class \( [\text{Cont}^{\geq v}(D)] \in H^*_T X \) corresponds to the monomial \( x^v \in \text{SR}(\Sigma) \), for each lattice point \( v \in |\Sigma| \cap N \).

**Proof.** It follows from Lemma 8.2 that \( J_\infty X \setminus \bigcup_i J_\infty D_i = \prod_{v \in |\Sigma| \cap N} \text{Cont}^v(D) \) is an equivariant affine paving, and hence Theorem 5.7 implies that the classes \( \{ [\text{Cont}^{\geq v}(D)] \mid v \in |\Sigma| \cap N \} \) form a \( \mathbb{Z} \)-basis of \( H^*_T X \). Moreover, it follows from Example 6.11 that these classes satisfy the multiplication rule (3):

\[
[\text{Cont}^{\geq u}(D)] \cdot [\text{Cont}^{\geq v}(D)] = \begin{cases} 
[\text{Cont}^{\geq u+v}(D)] & \text{if } u, v \in \sigma \text{ for some } \sigma \in \Sigma, \\
0 & \text{otherwise}.
\end{cases}
\]

\[ \square \]

**Remark 8.4.** In Section 5, we described how one can compare equivariant cohomology rings under proper, birational morphisms. In the toric setting we have the following application: a proper, birational morphism \( f : Y(\Delta) \to X(\Sigma) \) between smooth toric varieties corresponds to a refinement \( \Delta \) of a fan \( \Sigma \) in a lattice \( N \). Let \( \psi \) and \( \varphi \) denote the piecewise linear functions on \( |\Delta| = |\Sigma| \) with value 1 on the primitive integer vectors of \( \Delta \) and \( \Sigma \), respectively, and let \( E \) and \( D \) denote the union of the torus-invariant divisors of \( Y \) and \( X \), respectively. We have a bijection between \( \mathbb{Z} \)-bases of \( H^*_T Y \) and \( H^*_T X \) such that, for each \( v \in |\Delta| \cap N \), \( \text{Cont}^v(E) \subseteq \text{Cont}^{\varphi(v)}(\psi(v))(K_{Y/X}) \) and

\[
[\text{Cont}^{\geq v}(E)] \in H^{2\varphi(v)}_T Y, \quad [\text{Cont}^{\geq v}(D)] \in H^{2\psi(v)} T X.
\]
Remark 8.5. Toric prevarieties are not necessarily separated analogues of toric varieties which first arose in Wlodarczyk’s work on embeddings of varieties [41]. The geometry of toric prevarieties is controlled by an associated multi-fan (roughly speaking, a multi-fan is a fan where one does not require two cones to intersect along a common face), and we refer the reader to Section 4 in [35] for an introduction to the subject. The analogue of Corollary 8.3 holds in this case: if $X = X(\Sigma)$ is a smooth $d$-dimensional toric prevariety associated to a multi-fan $\Sigma$ in a lattice $N$ of rank $d$ and $T$ denotes the corresponding torus, then the equivariant cohomology ring $H_T^*X$ is isomorphic to the Stanley-Reisner ring $\text{SR}(\Sigma)$ of $\Sigma$ [35]. On the other hand, if $D_1, \ldots, D_r$ denote the $T$-invariant divisors of $X$, then the classes

$$\{ \text{Cont}^a(D_\bullet) \in H_T^*X \mid a = (a_1, \ldots, a_r) \in \mathbb{N}^r, \bigcap_{a_i > 0} D_i \neq \emptyset \}$$

form a $\mathbb{Z}$-basis of $H_T^*X$, corresponding to a monomial basis of $\text{SR}(\Sigma)$.

Remark 8.6. Hypertoric varieties may be viewed as a complex-symplectic analogue of toric varieties; their geometry is related to the combinatorics of matroids and hyperplane arrangements. (We refer the reader to [28] and [37] for an introduction to the subject.) A smooth $2d$-dimensional hypertoric variety $Y$ comes with the action of a $d$-dimensional torus $T$, and Proudfoot and Webster [38] observed that there is an associated smooth pretoric variety $X = X(\Sigma)$ with torus $T$, and a natural $T$-equivariant affine bundle $p : Y \to X$. In particular, $H_T^*Y \cong H_T^*X$. With the notation of Remark 8.5, the classes

$$\{ \text{Cont}^a(p^{-1}(D_\bullet)) \in H_T^*Y \mid (a_1, \ldots, a_r) \in \mathbb{N}^r, \bigcap_{a_i > 0} p^{-1}(D_i) \neq \emptyset \}$$

therefore form a $\mathbb{Z}$-basis of $H_T^*Y$, corresponding to a monomial basis of $\text{SR}(\Sigma)$.

9. Example: the general linear group

In this section, we apply our results to give a geometric interpretation of the $GL_n$-equivariant cohomology ring of a partial flag variety.

Consider $G = GL_n(\mathbb{C})$ acting by left multiplication on the variety of $n \times n$ matrices $M_{n,n} = M_{n,n}(\mathbb{C})$. Since $M_{n,n}$ is contractible, Lemma 2.1 implies that $H_G^* M_{n,n} \cong H_G^*(\text{pt}) = \Lambda_G$. Our first aim is to present a natural, geometric $\mathbb{Z}$-basis for $\Lambda_G$. Consider the chain of closed subvarieties

$$V_n \subseteq \cdots \subseteq V_1 \subseteq V_0 = M_{n,n},$$

where

$$V_r = \{ A = (a_{i,j}) \in M_{n,n} \mid \text{rk}(a_{i,j})_{1 \leq j \leq n+1-r} < n + 1 - r \}.$$
That is, $V_r$ is the subvariety of $M_{n,n}$ defined by setting all $(n+1-r) \times (n+1-r)$ minors involving the first $n+1-r$ columns equal to zero. It is well known that $V_r$ is a normal, irreducible variety of codimension $r$ in $M_{n,n}$.

Remark 9.1. Note that $V_n \cong \mathbb{A}^{n(n-1)}$ is a smooth ($T$-equivariant) complete intersection, and $V_1$ is a singular hypersurface provided $n \geq 2$. On the other hand, $V_r$ is not a local complete intersection variety for $1 \leq r < n$.

The jet schemes of determinantal varieties have been studied by Mustaţă [33], Yuen [43], Košir and Sethuraman [30], and recently by Docampo [14]. We will use the following result:

Theorem 9.2 ([30, Theorem 3.1]). The jet schemes $J_m V_r$ are irreducible for all $m \geq 0$ and $1 \leq r \leq n$.

Remark 9.3. The cases $r = n$ and $r = 1$ are easy: $V_n$ is smooth and the result is immediate, while $V_1$ is a normal hypersurface (hence a local complete intersection) with canonical singularities, so the theorem follows from Theorem 3.7. The case $r = n - 1$ is due to Mustaţă [33, Example 4.7].

Recall from Section 6 and Section 7 that for a tuple of non-negative integers $\mathbf{m} = (m_1, \ldots, m_n)$, we let $\lambda(\mathbf{m}) = (\lambda_1, \ldots, \lambda_n)$ be the partition defined by $\lambda_i = m_i + \cdots + m_n$. Moreover, we may consider the $J_\infty G$-invariant cylinders

$$\text{Cont}^\lambda(V_\bullet) = \text{Cont}^{\lambda(\mathbf{m})}(V_\bullet) := \bigcap_{i=1}^n \text{Cont}^{\lambda_i}(V_i) \subseteq J_\infty M_{n,n}$$

and

$$\text{Cont}^{\geq \lambda}(V_\bullet) = \text{Cont}^{\geq \lambda(\mathbf{m})}(V_\bullet) := \bigcap_{i=1}^n \text{Cont}^{\geq \lambda_i}(V_i) \subseteq J_\infty M_{n,n},$$

and observe that

$$J_\infty M_{n,n} \setminus J_\infty V_1 = \coprod_{\lambda} \text{Cont}^\lambda(V_\bullet),$$

where $\lambda$ varies over all partitions of length at most $n$.

Lemma 9.4. The contact locus $\text{Cont}^\lambda(V_\bullet)$ is an affine family of orbits.

Proof. Identify $J_\infty M_{n,n}$ with $n \times n$ matrices whose entries are power series in $\mathbb{C}[[t]]$, and set $m_i = \lambda_i - \lambda_{i+1}$ for $1 \leq i \leq n$, so that $\lambda = \lambda(\mathbf{m})$. For brevity, we will use the notation

$$C = \text{Cont}^\lambda(V_\bullet) \quad \text{and} \quad C_m = \text{Cont}^\lambda(V_\bullet)_m$$

in this proof.
Let $L \subseteq C$ be the set of $n \times n$ upper triangular matrices with $(i,i)^{th}$ entry equal to $t^{m_{n+1-i}}$ and $(i,j)^{th}$ entry equal to a polynomial in $t$ of degree strictly less than $m_{n+1-j}$ for $i > j$; this is an affine space $\mathbb{A}^N$, for $N = n(m_1 + \cdots + m_n) = n\lambda_1$. Let $L_m \subseteq C_m \subseteq J_m M_{n,n}$ be defined similarly. Take $m > \lambda_1$, so that $L_m \cong L \cong \mathbb{A}^N$ and $L_m$ is not contained in $J_m V_1$.

Using row operations, one sees that every $J_m G$-orbit in $C_m$ has a unique representative in $L_m$. We claim that the map $p : C_m \to L_m$ given by

$$p(x) = (J_m G \cdot x) \cap L_m$$

is a smooth, algebraic morphism of varieties. To see this, consider $x$ as a matrix, and assume it lies in the open subset $U \subseteq C_m$ where the top-left minor of size $i$ has order $m_{n+1-i}$ (in $t$). (By definition, $C_m$ is covered by $n!$ such open sets $U_w$, one for each permutation, since some minor on the first $i$ columns has order $m_{n+1-i}$.) Thus the entry in position $(1,1)$ has the form $x_{1,1} = t^{m_n} \cdot q(t)$, where $q(t)$ is an invertible element of $\mathbb{C}[t]/(t^{m+1})$. Scale the $n^{th}$ row by $q(t)^{-1}$, and use row operations to set the entries below $x_{1,1}$ to zero. Note that the entries of the resulting matrix $x'$ are rational functions of the coordinates of $x$. Repeat this process for $x'$, starting with $x'_{2,2}$, with the additional step of using row operations to ensure the entry $x'_{1,2}$ is a polynomial of degree strictly less than $m_{n-1}$. Continuing in this way, one obtains a matrix in $L_m$ whose entries are rational functions of the coordinates of $x$; that is, we have described a morphism $U \to L_m$. Here is an example, for $n = 2$, $\lambda = (2,1)$, and $m = 3$:

$$x = \begin{bmatrix} t + t^2 & 1 + 2t \\ t & 1 + t^2 \end{bmatrix} \sim \begin{bmatrix} t & (1 + 2t)(1 - t + t^2) \\ t & 1 + t^2 \end{bmatrix} \sim \begin{bmatrix} t & 1 + t - t^2 + 2t^3 \\ 0 & -t + 2t^2 - 2t^3 \end{bmatrix} \sim \begin{bmatrix} t & 1 + t - t^2 + 2t^3 \\ 0 & t \end{bmatrix} = p(x).$$

The map is defined similarly on the other open sets $U_w$, by composing with an appropriate permutation of the rows. Since $p(x)$ is the unique element of $L_m$ in the orbit $J_m G \cdot x$, it follows that these maps patch to give a morphism $C_m \to L_m$. (In fact, we have described morphisms $s_w : U_w \to J_m G$, with $s_w(x) \cdot x = p(x)$. These maps to $J_m G$ do not glue, however—only the composition with the action map is well defined on the overlaps of the $U_w$'s.)

Finally, consider $\mathcal{G} = J_m G \times L_m$ as a group scheme over $L_m$, and let $\mathcal{H} \subseteq \mathcal{G}$ be the flat subgroup scheme defined by $\mathcal{H} = \{(g,x) \mid g \cdot x = x\}$. Since the quotient $\mathcal{G}/\mathcal{H} = C_m$ exists as a scheme (in fact, a variety), general facts about quotients
imply that the maps $G \to C_m$ and $C_m \to L_m$ are smooth (see, e.g., [27, §I.5]). The lemma follows.

Our geometric description of $\Lambda_G$ now follows immediately from Theorem 5.7:

**Corollary 9.5.** With the notation above, the classes $\overline{\text{Cont}^\lambda(V_\bullet)}$ form a $\mathbb{Z}$-basis of $\Lambda_G$, as $\lambda$ varies over all partitions of length at most $n$.

Recall from Example 2.5 that $\Lambda_G = \mathbb{Z}[c_1, \ldots, c_n]$, where $c_i$ is the $i^{th}$ equivariant Chern class of the standard representation of $G = GL_n$. We offer the following conjecture.

**Conjecture 9.6.** If $m = (m_1, \ldots, m_n)$ is a tuple of non-negative integers and $\lambda = \lambda(m)$, then $[\text{Cont}^{\geq \lambda}(V_\bullet)] = [\overline{\text{Cont}^\lambda(V_\bullet)}] = c_1^{m_1} \cdots c_n^{m_n}$.

**Remark 9.7.** It follows from Lemma 9.4 that $\text{Cont}^\lambda(V_\bullet)$ is a smooth cylinder of codimension $\sum_i \lambda_i$, and hence $\text{Cont}^{\geq \lambda}(V_\bullet)$ and $\overline{\text{Cont}^\lambda(V_\bullet)}$ are closed cylinders of codimension $\sum_i \lambda_i$. In particular, the classes in Conjecture 9.6 all have the correct degree.

**Remark 9.8.** If $\lambda_1 = \cdots = \lambda_r = m + 1$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$, then, using Theorem 9.2, $\text{Cont}^{\geq \lambda}(V_\bullet) = \overline{\text{Cont}^\lambda(V_\bullet)} = \psi_m^{-1}(J_m V_r)$.

**Remark 9.9.** We can establish Conjecture 9.6 in several cases:

1. The fact that $[V_r] = c_r$ is well known; for example, it follows from the Giambelli-Thom-Porteous formula for cohomology classes of degeneracy loci [23, §14].
2. Since $V_1$ is a normal e.l.c.i. with rational singularities, it follows from Corollary 6.7 that $[V_1]^{m+1} = [J_m V_1] = c_1^{m+1}$.
3. Since $V_n$ is smooth, Corollary 7.5 says $[J_m V_n] = [V_n]^{m+1} = c_n^{m+1}$. (This is also easy to see directly.)
4. For $m = 1$ and any $r$, Corollary 7.5 implies $[J_1 V_r] = [V_r]^2$. Indeed, for $1 \leq r < n$, the singular locus of $V_r$ has codimension $2(r+1)$, so the hypothesis of Corollary 7.5 is satisfied when $(m-1)r < 2$.
5. When $n = 2$, the conjecture follows from Theorem 6.1, Remark 6.2 and Theorem 3.7.
6. When $n = 3$, we have verified that $[J_m V_2] = [V_2]^{m+1} = c_2^{m+1}$ for $m \leq 5$ using Macaulay 2.
Now we use Corollary 5.10 to relate the discussion above with partial flag varieties. Fix integers $0 = r_0 < r_1 < r_2 < \cdots < r_k < r_{k+1} = n$, and consider the partial flag variety

$$\text{Fl}(\mathbf{r}) = \text{Fl}(r_1, \ldots, r_k; n) = \{(V_{r_1} \subseteq \cdots \subseteq V_{r_k} \subseteq \mathbb{C}^n) \mid \dim V_{r_i} = r_i\}.$$  

Let $F_\bullet \in \text{Fl}(\mathbf{r})$ be the standard (partial) flag, and let $P$ be the parabolic subgroup of $G$ which fixes $F_\bullet$. That is, $P$ is the group of invertible block upper-triangular matrices, with diagonal blocks of sizes $r_1, r_2 - r_1, \ldots, r_k - r_{k-1}, n - r_k$:

\[
\begin{array}{c|c|c|c}
 r_1 & r_2 - r_1 & n - r_k \\
\hline
 * & * & \cdots & * \\
 0 & * & \cdots & * \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & 0 & \cdots & * \\
\end{array}
\]

Let $\mathfrak{p}$ be the Lie algebra of $P$; it consists of all matrices with the same block form as $P$. Note that $P$ acts on $\mathfrak{p}$ by left matrix multiplication, and that $\text{Fl}(\mathbf{r})$ is naturally identified with $G/P$. Consider

$$Y = G \times^P \mathfrak{p},$$  

the quotient of $G \times \mathfrak{p}$ by the relation $(g \cdot p, x) \sim (g, p \cdot x)$ for $p \in P$. This comes with a $G$-equivariant map $\varphi : Y \to M_{n,n}$, induced by the multiplication map $G \times \mathfrak{p} \to M_{n,n}$ sending $(g, x)$ to $g \cdot x$. It is also a vector bundle over $G/P = \text{Fl}(\mathbf{r})$ via the first projection, and hence $H^*_G Y \cong H^*_G \text{Fl}(\mathbf{r})$ by Lemma 2.1. Moreover, we have an identification

$$Y = S_{r_1}^\oplus \oplus S_{r_2-r_1}^\oplus \oplus \cdots \oplus S_{n-r_k}^\oplus \subseteq S_n^\oplus \cong \text{Fl}(\mathbf{r}) \times M_{n,n},$$  

where $S_r$ is the tautological rank $r$ bundle on $\text{Fl}(\mathbf{r})$. (So $S_n = \mathbb{C}^n$ is the trivial bundle.) From this perspective, the map $\varphi$ is simply projection on the second factor; in particular, $\varphi$ is proper.

Recall that $V_{n+1-r} \subseteq M_{n,n}$ is the locus of matrices where the first $r$ columns have rank strictly less than $r$. One sees that $\varphi$ is an isomorphism over the open set $M_{n,n} \setminus V_{n+1-r_k}$. Moreover, $E = \varphi^{-1}(V_{n+1-r_k})$ is a reduced divisor with $k$ irreducible components $E_{n+1-r_1}, \ldots, E_{n+1-r_k}$. (To see this, lift $\varphi$ to the multiplication map $\bar{\varphi} : G \times \mathfrak{p} \to M_{n,n}$, and observe that $\bar{\varphi}^{-1}(V_{n+1-r_k})$ is defined by the vanishing of the principal $r_k \times r_k$ minor in $\mathfrak{p}$.) This determinant factors into $k$ block determinants, of sizes $r_1, r_2 - r_1, \ldots, r_k - r_{k-1}$. In fact, for $1 \leq i \leq k$, $\varphi^{-1}(V_{n+1-r_i}) = E_{n+1-r_1} + \cdots + E_{n+1-r_i}$.
To apply Corollary 5.10, we compute $K_{Y/M,n,n}$. This is equivalent to $K_Y$, since $K_{M,n,n} = 0$. In fact, we have

$$K_{Y/M,n,n} = K_Y = \sum_{i=1}^{k} (n - r_i)E_{n+1-r_i}.$$

We leave the details of this calculation to the reader; it can be done by considering the vector bundle projection $Y \to Fl(r)$, and using standard formulas for $K_{Fl(r)}$ and the relative canonical divisor of a vector bundle.

For $r$ not among the $r_i$’s, let $\tilde{V}_{n+1-r}$ be the “proper transform” of $V_{n+1-r}$, that is, the closure of $\varphi^{-1}(V_{n+1-r} \setminus V_{n+1-(r-1)})$; let $\tilde{V}_{n+1-r} = E_{n+1-r}$. If $r_{i-1} < r < r_i$, then $\tilde{V}_{n+1-r} \subseteq E_{n+1-r}$ and $\varphi^{-1}(V_{n+1-r}) = \tilde{V}_{n+1-r} + E_{n+1-r_1} + \cdots + E_{n+1-r_{i-1}}$.

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ of length at most $n$, we let $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, where $\tilde{\lambda}_{n+1-r} = \lambda_{n+1-r} - \lambda_{n+1-r_{i-1}}$ for $r_{i-1} < r < r_i$, and $\tilde{\lambda}_{n+1-r} = \lambda_{n+1-r}$ for $r \leq r_1$. Alternatively, if $\lambda'$ is the subpartition of $\lambda$ given by

$$\begin{align*}
\lambda_{n+1-r_k} &\geq \cdots \geq \lambda_{n+1-r_{k-1}} \\
&\qquad \begin{array}{c}
\text{times} \kern 10pt n-r_k \end{array} \\
&\geq \lambda_{n+1-r_{k-1}} \geq \cdots \geq \lambda_{n+1-r_{k-2}} \\
&\qquad \begin{array}{c}
\text{times} \kern 10pt r_{k-1}-r_{k-2} \end{array} \\
&\qquad \begin{array}{c}
\text{times} \kern 10pt r_2-r_1 \end{array} \\
&\geq \cdots \geq \lambda_{n+1-r_1} \\
&\geq \cdots \geq \lambda_{n+1-r_1},
\end{align*}$$

then $\tilde{\lambda} = \lambda - \lambda'$.

Observing that $\varphi_\infty^\lambda(\text{Cont}^{\lambda}(V_*)) = \text{Cont}^{\tilde{\lambda}}(\tilde{V}_*) \subseteq \text{Cont}^{\lambda'}(K_{Y/M,n,n})$, where

$$e_\lambda = \sum_{i=1}^{k} (n - r_i)\tilde{\lambda}_{n+1-r_i} = \sum_{i=1}^{k} (n - r_i)(\lambda_{n+1-r_i} - \lambda_{n+1-r_{i-1}}) = \sum_{i=1}^{n} \lambda'_i,$$

we have the following application of Corollary 5.10 and Remark 9.7.

**Corollary 9.10.** With the notation above, the classes $[\text{Cont}^{\tilde{\lambda}}(V_*)]$ form a $\mathbb{Z}$-basis of $H^*_G Fl(r)$, as $\lambda$ varies over all partitions of length at most $n$. Moreover, the degree of $[\text{Cont}^{\lambda'}(\tilde{V}_*)]$ in $H^*_G Fl(r)$ equals $\sum_i \lambda'_i$.

Note that $H^*_G Y = H^*_G(G/P) = H^*_P(pt) = \Lambda_P$ is isomorphic to the ring of “multiply-symmetric functions”

$$\Lambda_P = \mathbb{Z}[t_1, \ldots, t_n]^{S_{d_1} \times \cdots \times S_{d_k}},$$

where $d_i = r_{i+1} - r_i$. One may view this isomorphism as induced from the inclusion of $p$ into $G \times P p$ sending $A$ to $(1, A)$, which is equivariant with respect to the composition of the inverse map $P \to P$, $g \mapsto g^{-1}$, and the inclusion $P \hookrightarrow G$. The corollary therefore describes a $\mathbb{Z}$-linear isomorphism

$$\Lambda_G = \mathbb{Z}[t_1, \ldots, t_n]^{S_n} \to \mathbb{Z}[t_1, \ldots, t_n]^{S_{d_1} \times \cdots \times S_{d_k}} = \Lambda_P.$$
For \( r_i + 1 \leq n + 1 - j \leq r_{i+1} \), let \( c_{j,r} \in \mathbb{Z}[t_1, \ldots, t_n]^{S_d \times \cdots \times S_d} \) denote the \((r_{i+1} - n + j)\)th elementary symmetric function in the variables \( t_{r_i+1}, \ldots, t_{r_{i+1}} \). We offer the following conjecture.

**Conjecture 9.11.** If \( \mathbf{m} = (m_1, \ldots, m_n) \) is a tuple of non-negative integers and \( \tilde{\lambda} = \tilde{\lambda}(\mathbf{m}) \), then

\[
[\text{Cont}^{\geq \tilde{\lambda}}(\check{V}_*)] = [\text{Cont}^{\tilde{\lambda}}(\check{V}_*)] = (-1)^{\Sigma_i c_{1,r}^1 \cdots c_{n,r}^m}.
\]

**Example 9.12.** We will show that the conjecture holds in the case of the full flag variety \( \text{Fl}(n) = G/B \). Indeed, in this case \( E = \varphi^{-1}(V_1) = E_n + \cdots + E_1 \) is a simple normal crossings divisor with \( n \) irreducible components \( E_i = \check{V}_i \). (To see this, lift \( \varphi \) to the multiplication map \( \tilde{\varphi} : G \times b \to M_{n,n} \), and observe that \( \tilde{\varphi}^{-1}(V_{n+1-r}) \) is defined by the vanishing of the product of the diagonal entries in \( b \).) Moreover, \( \tilde{\lambda} = \mathbf{m} \) and \([E_i] = -t_{n+1-i}\) under the isomorphism \( H^*_G Y \cong H^*_B(b) \cong H^*_T(pt) = \mathbb{Z}[t_1, \ldots, t_n] \). The conjecture now follows from Corollary 9.10 and Example 6.11.

### 10. Final remarks

It would be interesting to extend the ideas of this paper to the case when \( X \) has singularities. In the case when \( X \) has orbifold singularities, we suggest that, on the one hand, one should replace the equivariant cohomology \( H^*_G X \) with the **equivariant orbifold cohomology ring** \( H^*_{G, \text{orb}}(X; \mathbb{Q}) \). Orbifold cohomology was introduced by Chen and Ruan [12] and an algebraic version was developed by Abramovich, Graber and Vistoli [1]. One may extend their definitions to define the equivariant version \( H^*_{G, \text{orb}}(X; \mathbb{Q}) \). On the other hand, for \( m \in \mathbb{N} \cup \{ \infty \} \), we suggest replacing the jet schemes \( J_m X \) with the stack of **twisted jets** \( J_m X \), as defined by Yasuda [42].

In the case when \( X = X(\Sigma) \) is a simplicial toric variety corresponding to a fan \( \Sigma \) in a lattice \( N \), one can extend the ideas of Borisov, Chen and Smith [8] to show that \( H^*_{T, \text{orb}}(X; \mathbb{Q}) \) is isomorphic to the deformed group \( \mathbb{Q}[N]^\Sigma \) [36]: the ring \( \mathbb{Q}[N]^\Sigma \) has basis \( \{ y^v \mid v \in \Sigma \cap N \} \), where \( |\Sigma| \) denotes the support of \( \Sigma \), and multiplication

\[
y^u \cdot y^v = \begin{cases} 
y^{u+v} & \text{if } u, v \in \sigma \text{ for some } \sigma \in \Sigma, \\
0 & \text{otherwise.}
\end{cases}
\]

On the other hand, an explicit description of the stacks \( J_m X \) was given by the second author in [40]: roughly speaking, away from a closed substack of infinite codimension, the \( J_\infty T \)-orbits of \( J_\infty X \) consist of cylinders \( \{ C_v \mid v \in \Sigma \cap N \} \). One expects that under the isomorphism \( H^*_{T, \text{orb}}(X; \mathbb{Q}) \cong \mathbb{Q}[N]^\Sigma \), the class \([C_v]\) in \( H^*_{T, \text{orb}}(X; \mathbb{Q}) \) corresponds to \( y^v \) in \( \mathbb{Q}[N]^\Sigma \) for all \( v \in \Sigma \cap N \).
More generally, we expect that our main results should extend to other situations. For example, the evidence for Conjecture 9.6 suggests that the hypotheses in Theorems 6.1 and 7.3 can be relaxed. It would also be interesting to study spherical varieties in the spirit of Theorem 5.7, generalizing the example of toric varieties.

Finally, it would be useful to develop a version of this theory for varieties over an arbitrary field, using equivariant Chow groups. The statements of our results make sense in this context, so we expect this should be possible; however, there are a few technical obstacles, since several of our proofs use analytic neighborhoods and the long exact sequence for Borel-Moore homology.

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