Chiral Symmetry Breaking in Three Dimensional QED

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Abstract. Over the past few years three dimensional Quantum Electrodynamics (QED\textsuperscript{3}) has attracted a lot of attention, because it may be an effective theory for the underdoped and non-superconducting region of the phase diagram of high \(T_c\) cuprate compounds. We present results from lattice simulations of the non-compact version of the theory in order to address the issue of whether chiral symmetry is spontaneously broken when the number of fermion flavours \(N_f\) is less than a critical value \(N_{f_c}\). Our results provide strong evidence that QED\textsuperscript{3} is chirally symmetric for \(N_f \geq 1\), implying that a pseudogap phase separates the superconducting phase from the antiferromagnetic phase.

1. Introduction

Interest in QED\textsuperscript{3} has recently been revived by the suggestion that the model may be an effective theory for the underdoped and non-superconducting region of the phase diagram of high-\(T_c\) superconducting cuprate compounds \cite{1,2}. In brief, superconductivity in these substances is confined to planes defined by CuO\textsubscript{2} layers, thus motivating a \((2 + 1)d\) description. The superconducting order parameter has a \(d\)-wave symmetry, implying that there are four nodes in the gap function as the Fermi surface (which in \((2+1)d\) is a curve) is circumnavigated. At each node the low-energy quasiparticle excitations obey an approximately linear dispersion relation with the result that it is possible to rewrite the action for eight distinct low energy species (spin up and spin down at each of four nodes) in a relativistically invariant form in terms of \(N_f = 2\) species of four-component Dirac spinors. For phenomenologically relevant models the action for each individual flavor exhibits a spatial anisotropy, a feature ignored in this paper. If QED\textsuperscript{3} is a relevant effective theory for cuprates, then the abstract theoretical problem of the value of \(N_{f_c}\) assumes concrete phenomenological reference. If \(N_{f_c} > 2\), then the theory is chirally broken at zero temperature. On retranslating from the Dirac spinor basis to the original electron degrees of freedom, the chiral order parameter is reinterpreted as an order parameter for spin density waves, whose wavevector gets shorter and shorter as doping is decreased, until at zero doping the Néel antiferromagnetic state is recovered \cite{2}. This picture therefore predicts the existence of a phase boundary between superconducting (dSC) and antiferromagnetic (AFM) phases at some non-zero doping in the zero temperature limit. If, on the other hand, \(N_{f_c} < 2\), the chirally symmetric ground state manifests itself as a tongue of “pseudogap” phase separating dSC from AFM, in which normal Fermi liquid properties may be modified as a result of a
non-perturbative anomalous dimension for the fermion field \[1\]. In addition, QED\(_3\) has found interesting applications in the unconventional quantum Hall effect in graphene \[3\].

The study of quantum field theories in which the ground state shows a sensitivity to the number of fermion flavours \(N_f\) is intrinsically interesting. Apparently, for \(N_f > N_{fc}\), the attractive interaction between a fermion and an antifermion due to photon exchange is overwhelmed by the fermion screening of the theory’s electric charge. Initial studies of QED\(_3\) based on Schwinger Dyson equations (SDEs) using the photon propagator derived from the leading order \(1/N_f\) expansion suggested that for \(N_f\) less than \(N_{fc} \approx 3.2\) chiral symmetry is broken \[4\]. Other studies taking non-trivial vertex corrections into account predicted chiral symmetry breaking for arbitrary \(N_f\) \[5\]. Studies which treat the vertex consistently in both numerator and denominator of the SDEs have found \(N_{fc} < \infty\), with a value either in agreement with the original study \[6\], or slightly higher \(N_{fc} \approx 4.3\) \[7\]. An argument based on a thermodynamic inequality predicted \(N_{fc} \leq \frac{3}{2}\) \[8\], a result that was later challenged in \[9\].

Progress in the direction of gauge covariant solutions for the propagators of QED\(_3\) showed that in the Landau gauge a chiral phase transition exists at \(N_{fc} \approx 4\) \[10\]. A gauge invariant determination of \(N_{fc}\) based on the divergence of the chiral susceptibility gives \(N_{fc} \approx 2.16\) \[11\]. It has also been shown that the issue of the gauge dependence of \(N_{fc}\) extracted from SDEs becomes irrelevant if Landau-Khalatnikov-Fradkin transformations are taken into account \[12\].

Recent lattice simulations showed that chiral symmetry is broken for \(N_f = 1\), whereas \(N_f = 2\) appeared chirally symmetric with an upper bound of \(10^{-4}\) on the dimensionless condensate \[13\].

The principal obstruction to a definitive answer has been large finite volume effects resulting from the presence of a massless photon in the spectrum, which prevent a reliable extrapolation to the thermodynamic limit. Recent lattice simulations of the three-dimensional Thirring model, which may have the same universal properties as QED\(_3\), predicted \(N_{fc} = 6.6(1)\) \[14\]. In this paper we present preliminary results from lattice simulations of QED\(_3\) on large lattices in an effort to detect chiral symmetry breaking for \(N_f = 0.5, \ldots, 2\).

### 2. Lattice Model and Simulations

We are considering the four-component formulation of QED\(_3\) where the Dirac algebra is represented by the \(4 \times 4\) matrices \(\gamma_0, \gamma_1\) and \(\gamma_2\). This formulation preserves parity and gives each spinor a global \(U(2)\) symmetry generated by \(\gamma_3, \gamma_5\) and \(i\gamma_3\gamma_5\); the full symmetry is then \(U(2N_f)\). If the fermions acquire dynamical mass the \(U(2N_f)\) symmetry is broken spontaneously to \(U(N_f) \times U(N_f)\) and \(2N_f^2\) Goldstone bosons appear in the particle spectrum.

The action of the lattice model we study is

\[
S = \frac{\beta}{2} \sum_{x,\mu<\nu} \Theta_{\mu\nu}(x)\bar{\Theta}_{\mu\nu}(x) + \sum_{i=1}^{N} \sum_{x,x'} \bar{\chi}_i(x)M(x,x')\chi_i(x')
\]

(1)

where

\[
\Theta_{\mu\nu}(x) = \theta_{x\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x\nu}.
\]

\[
M(x,x') = m\delta_{x,x'} + \frac{1}{2} \sum_{\mu} \eta_{\mu}(x)[\delta_{x',x+\hat{\mu}}U_{x\mu} - \delta_{x',x-\hat{\mu}}U_{x\mu}^*].
\]

This describes interactions between \(N\) flavours of Grassmann-valued staggered fermion fields \(\chi, \bar{\chi}\) defined on the sites \(x\) of a three-dimensional cubic lattice, and real photon fields \(\theta_{x\mu}\) defined on the link between nearest neighbour sites \(x, x + \hat{\mu}\). Since \(\Theta^2\) is unbounded from above, eq.(1) defines a non-compact formulation of QED; note however that to ensure local gauge invariance the fermion-photon interaction is encoded via the compact connection

\[
U_{x\mu} \equiv \exp(i\theta_{x\mu}), \text{ with } U_{x+\hat{\mu},-\mu} = U_{x\mu}^*.
\]

In the fermion kinetic matrix \(M\) the Kawamoto-Smit phases \(\eta_{\mu}(x) = (-1)^{x_1 + \cdots + x_{\mu-1}}\) are designed to ensure relativistic covariance in the continuum limit, and \(m\) is the bare fermion mass.
If the physical lattice spacing is denoted $a$, then in the continuum limit $a\partial \to 0$, eq.\(1\) can be shown to be equivalent up to terms of $O(a^2)$ to

$$S = \sum_{j=1}^{N_f} \bar{\psi}^j [\gamma_\mu (\partial_\mu + igA_\mu) + m] \psi^j + \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$$ \(2\)

ie. to continuum QED in 2+1 euclidean dimensions, with $\psi, \bar{\psi}$ describing $N_f$ flavours of four-component Dirac spinor acted on by $4 \times 4$ matrices $\gamma_\mu$, and $N_f \equiv 2N$. The continuum photon field is related to the lattice field via $\theta_{x\mu} = agA_\mu(x)$, with dimensional coupling strength $g$ given by $g^2 = (a\beta)^{-1}$, and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The continuum limit is thus taken when the dimensionless inverse coupling $\beta \to \infty$.

Our numerical simulations were performed using the standard Hybrid Molecular Dynamics algorithm. We checked the effects of lattice discretization on the values of the chiral condensate $\langle \bar{\psi}\psi \rangle$ that we extracted from eq.\(3\) by comparing data extracted from simulations at fixed physical volume $(L/\beta)^3$ and fixed physical mass $\beta m$. The results show that the lattice discretization effects are small for $N_f > 0.5$ at $\beta = 0.90, m = 0.005$ on 54$^3$ lattices, whereas for $N_f = 0.5$ there is an $\sim 8\%$ discrepancy between the values of the dimensionless condensate $\beta^2 \langle \bar{\psi}\psi \rangle$ at $\beta = 0.90$ and $\beta = 1.20$.

- **Figure 1.** $\beta^2 \langle \bar{\psi}\psi \rangle$ vs $\beta m$ for $N_f = 1.5$.
- **Figure 2.** $\beta^2 \langle \bar{\psi}\psi \rangle$ vs $\beta m$ for $N_f = 2$.

We fitted $\beta^2 \langle \bar{\psi}\psi \rangle$ at different values of $\beta m$ for $N_f = 1.5, N_f = 2$ and fixed $\beta = 0.90$ to

$$\beta^2 \langle \bar{\psi}\psi \rangle = a_0 + a_1 \cdot (\beta m).$$ \(3\)

The finite size effects are small especially for $N_f = 2$ [see Figs. 1 and 2]. For $N_f = 1.5, 80^3$ we extracted from eq.\(4\) $a_0 = -6 \times 10^{-7}$ with a statistical error $8 \times 10^{-7}$, whereas for $N_f = 2, 80^3$ we extracted $a_0 = 1.5 \times 10^{-6}$ with a statistical error $10^{-6}$. These results provide evidence that QED$_3$ with $N_f = 1.5$ and $N_f = 2$ is chirally symmetric with an accuracy $10^{-6}$.

Next, we fitted the values of the dimensionless condensate at different $N_f, m$, and lattice sizes to a renormalization group inspired equation of state that includes a finite size scaling term $\beta^2 \langle \bar{\psi}\psi \rangle$

$$m = A((\beta - \beta_c) + CL^{-\frac{1}{\nu}})\beta^2 \langle \bar{\psi}\psi \rangle^p + B(\beta^2 \langle \bar{\psi}\psi \rangle)^\delta,$$ \(4\)

where $p = \delta - 1/\beta_m$. The results extracted from this fit are: $A = 0.0477(38), B = 0.79(2), C = 10.7(8), N_f c = 1.52(6), \delta = 1.177(7), p = 0.73(2)$. The data and the fitting functions are shown in Fig. 3. These results are consistent with a second order phase transition.
Figure 3. Fits of $\beta^2\langle\bar{\psi}\psi\rangle$ vs. $N_f$ to a finite volume scaling form of the equation of state.

3. Summary
The extrapolations of $\beta^2\langle\bar{\psi}\psi\rangle$ vs $\beta m$ to the chiral limit on lattices with small finite size effects show that QED$_3$ with $N_f \geq 1.5$ is chirally symmetric with an accuracy of $O(10^{-6})$, which is in agreement with the theoretical prediction of [8]. This may imply that the $N_f = 2$ chirally symmetric ground state manifests itself in the phase diagram of the cuprate compounds as a tongue of “pseudogap” phase separating the superconducting from antiferromagnetic phases, in which normal Fermi liquid properties may be modified as a result of a non-perturbative anomalous dimension for the fermion field [1]. The preliminary results extracted from fits to a finite volume equation of state are consistent with a second order phase transition scenario at $N_{fc} \approx 1.5$. However, as we mentioned in the previous section lattice discretization artifacts are not negligible for $N_f \leq 0.5$. This implies that $N_{fc}$ could be even smaller than 1.5. We are performing simulations closer to the continuum limit to clarify this issue.

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