Stability of the electron cyclotron resonance

Joachim Asch ∗† Olivier Bourget‡ Cédric Meresse§

30.12.2014

Abstract

We consider the magnetic AC Stark effect for the quantum dynamics of a single particle in the plane under the influence of an oscillating homogeneous electric and a constant perpendicular magnetic field. We prove that the electron cyclotron resonance is insensitive to impurity potentials.

1 Introduction

For $T > 0$ let $E \in C^0(\mathbb{R}, \mathbb{R}^2)$ be a $T$ periodic function and $V$ be a $T$ periodic multiplication operator by a real valued bounded function in $L^2(\mathbb{R}^2, dq)$. The time dependent family of Hamiltonians under investigation is

$$H(t) = H_0(t) + V(t) \quad (t \in \mathbb{R})$$

with

$$H_0(t) = H_{La} - \langle E(t), q \rangle \quad \text{and} \quad H_{La} = \frac{1}{2} \left( -i \nabla - \frac{q^\perp}{2} \right)^2,$$

where $a^\perp$ denotes the direct perpendicular of a vector $a$. $H_{La}, H_0(t), H(t)$ are essentially selfadjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$.

$H(\cdot)$ describes the dynamics of a quantum particle of mass $m$ and charge $e$ for a magnetic field of strength $B$ with $eB > 0$, in units of magnetic length $\sqrt{\frac{\hbar}{eB}}$, gyration time $\frac{m}{eB}$, and energy $\frac{\hbar eB}{m}$. $E$ defines the strength of the electric field and $V$ the impurity potential.

∗Aix Marseille Université, CNRS, CPT UMR 7332, F–13288 Marseille cedex 9, France, e-mail : asch@cpt.univ-mrs.fr
†Université de Toulon, CNRS, CPT UMR 7332, B.P. 20132, F–83957 La Garde, France
‡Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Av. Vicuña Mackenna 4860, C.P. 690 44 11, Macul, Santiago, Chile
§Aix Marseille Université, CNRS, CPT UMR 7332, F–13288 Marseille cedex 9, France
Electron cyclotron resonance means the growth of the system’s kinetic energy if the frequency of the electric field is in resonance with the cyclotron frequency defined by the magnetic field and the particle: the spectrum of $H_{La}$ is pure point and equals $N + \frac{1}{2}$, thus $e^{-i2\pi H_{La}} = e^{-i\pi I}$, all orbits generated by $H_{La}$ are periodic with cyclotron frequency equal to 1; in case of resonance, i.e. if $T$ is an integer multiple of $2\pi$, the Floquet operator of $H_0$ is a product of phase space translation operators which may accelerate the system.

To state this more precisely denote

$$R(t) := \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

the rotation matrix of angle $t$, $U$ the propagator defined by $H$, and the (vector) operator of asymptotic velocity

$$v_{asy} := s - \lim_{n \to \infty} \frac{1}{nT} (U^*(nT)qU(nT) - q)$$
on its maximal domain.

Denote the Fréchet space

$$S(1) := \{ f \in C^\infty(\mathbb{R}^2; \mathbb{R}); \|\partial^\alpha f\|_\infty < \infty, \quad \alpha \in \mathbb{N}_0^2 \}.$$ 

We always make the assumption

(A): Let $T > 0$. Suppose that $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $V \in C^0(\mathbb{R}; S(1))$ are $T$ periodic functions.

The family $H$ then defines a unitary propagator $U$, see theorem 2.1. Denote by $U(T) := U(T, 0)$ its Floquet operator.

We summarize our main results in two theorems:

**Theorem 1.1** Assume (A) and suppose $|\nabla V(t, q)| \to_{|q| \to \infty} 0$, uniformly in $t$.

If $T \in 2\pi \mathbb{N}$ and $\left| \int_0^T E(t) \, dt \right| + \left| \int_0^T R(t)E(t) \, dt \right| > 0$ then

1. the spectrum of $U(T)$ is purely absolutely continuous except possibly a finite number of eigenvalues;
2. $v_{asy}$ has discrete spectrum independent of $V$:

$$v_{asy} = \frac{1}{T} \int_0^T \left( R(t)E^\perp(t) - E^\perp(t)dt \right) P_{ac}(U(T))$$

$P_{ac}$ denoting the projector on the absolutely continuous subspace;
3. for $\psi$ in the domain of $H_{La}$ it holds

$$\frac{1}{(NT)^2} \langle U(nT)\psi, H_{La}U(nT)\psi \rangle \sim_{n \to \infty} \rho \| P_{ac}(U(T))\psi\|^2$$
with the $V$-independent rate

$$
\rho = \left| \frac{1}{T} \int_0^T R(t) E(t) \, dt \right|^2.
$$

If $T \notin 2\pi \mathbb{N}$ then

1. if $\int_0^T E(t) \, dt \neq 0$
   
   (a) the spectrum of $U(T)$ is purely absolutely continuous except possibly a finite number of eigenvalues;
   
   (b) $v_{\text{asy}}$ has discrete spectrum independent of $V$:

   $$
v_{\text{asy}} = \left( \frac{1}{T} \int_0^T E^\perp(t) \, dt \right) P_{\text{ac}}(U(T));
   $$

2. if $\int_0^T E(t) \, dt = 0$ then $v_{\text{asy}} = 0$.

**Theorem 1.2** Assume (A).

1. Suppose $\left| \int_0^T E(t) \, dt \right| > 0$ and $|\nabla V| < \left| \frac{1}{T} \int_0^T E \right|$ or $\left| \int_0^T R(t) E(t) \, dt \right| > 0$ and $|\nabla V| < \left| \frac{1}{T} \int_0^T R E \right|$ then the spectrum of $U(T)$ is purely absolutely continuous.

2. Suppose $V(t, q) \to q \to \infty 0$ uniformly in $t$. If $T \in 2\pi \mathbb{Q}$ and $\int_0^T E = 0 = \int_0^T R(s)E(s) \, ds$ then the spectrum of $U(T)$ is pure point.

**Remarks 1.3**

1. In 1.1 no decay of $V$ is assumed, on the other hand, due to the decay assumption on $\nabla V$ we do not say anything for random or periodic $V$.

2. While $|\nabla V| < \left| \frac{1}{T} \int_0^T E \right|$ implies that the spectrum of $U(T)$ is purely absolutely continuous, the physical interesting situation is the linear response regime of small electric or large magnetic field. See [G] and references therein.

3. $v_{\text{asy}}$ may be considered as proportional to the Floquet Hall-current. For the static Quantum Hall Effect, i.e. $E = \text{const.}$, we have in particular $v_{\text{asy}} = -E^\perp P_{\text{ac}}(U(T))$. Remark that the time dependency of the electric field can suppress this current in the resonant case; as an example : if $E(t) := E_0 + R(-t)E_0$, for an $E_0 \in \mathbb{R}^2$, then $v_{\text{asy}} = 0$.

4. In 1.2.2 the spectrum of $U(T)$ is pure point in the resonant case. So local potentials are not likely to accelerate the system. We guess that this remains true in the non resonant case; however, no method or proof seems to be available.
In [GY] the AC Stark effect was discussed for a system with \( H_{La} \) replaced by the isotropic harmonic oscillator. The situation is similar as the eigenvalues of the oscillator are also equidistant, they are only finitely degenerated though. [GY] proved for a sinusoidal electric field that the spectrum of the Floquet hamiltonian is purely absolutely continuous if \( \| \nabla V \|_\infty \) is small.

[BF] proved 1.2.1 for the case \( \| \nabla V \| < \frac{1}{T} \int_0^T E \). The problem of stability, remark 1.3.4, under non-resonant time periodic local perturbations was raised in [EV]. Progress on this difficult problem has recently been achieved for the case of the harmonic oscillator in one dimension [W, GT].

We shall discuss general properties of the propagator in section 2. Then in section 3 we first prove a dynamical compactness result which is our main technical contribution and allows us to apply a unitary Mourre theorem. We finish the paper with the proof of theorems 1.1, 1.2.

## 2 The propagator

\( H_0(\cdot) \) is quadratic in the canonical operators; as in the case of the Stark effect [AH, KY] its propagator can be determined explicitly. We present it here in terms of center and velocity operators. Although this is folklore we provide an explicit proof. See [Fo] for the metaplectic representation in general.

If not otherwise specified the following operators and identities are to be understood as acting on Schwartz-space \( \mathcal{S}(\mathbb{R}^2) \). Denote \( D := -i \nabla \). The (vector-) operators of center \( c \) and velocity \( v \) are

\[
v = (v_1, v_2) := D - \frac{1}{2} q^\perp; \quad c = (c_1, c_2) := \frac{1}{2} q - D^\perp.
\]

Relevant commutation relations are

\[
\begin{align*}
[v_1, v_2] &= i, \quad [c_2, c_1] = i, \quad [c_j, v_k] = 0 \quad \forall j, k \\
[c, H_{La}] &= 0, \quad [v^\perp, H_{La}] = iv.
\end{align*}
\]

To relate to the usual position and momentum operators note that

\[
q = c + v^\perp, \quad D = \frac{1}{2} \left( e^\perp + v \right).
\]

While the operator of kinetic energy is \( H_{La} = \frac{1}{2} v^2 \) the name center operator is motivated by the relation

\[
e^{itH_{La}} q e^{-itH_{La}} = c + R(-t) v^\perp
\]

which follows from (1).
Denote for \( a \in \mathbb{R}^2 \) and a vector operator \( w \):
\[
\langle a, w \rangle := a_1 w_1 + a_2 w_2.
\]

Note that this an abuse of notation as we also use \( \langle \cdot, \cdot \rangle \) for the scalar product in \( L^2(\mathbb{R}^2) \) in the statement of theorem 1.1.

Note that the domain of \( H(t) \) is time dependent in general. We have

**Theorem 2.1** Assume (A). For \( t \in \mathbb{R} \) consider the operators
\[
H_{La} := \frac{1}{2} \left( D - \frac{1}{2} q^\perp \right)^2, \quad H_0(t) := H_{La} - \langle E(t), q \rangle, \quad H(t) = H_0(t) + V(t)
\]
which are all essentially selfadjoint on \( S(\mathbb{R}^2) \).

Then the family \( H(\cdot) \) generates a unitary propagator \( U \) which leaves \( S(\mathbb{R}^2) \) invariant. Furthermore for \( \psi \in S(\mathbb{R}^2) \), \( t \mapsto U(t,t_0)\psi \) is the solution of
\[
i \partial_t \psi(t) = H(t)\psi(t), \quad \psi(t_0) = \psi.
\]

Denote
\[
S(t,t_0) := e^{i\left(\int_{t_0}^t E, E\right)} e^{i\left(\int_{t_0}^t RE, v^\perp\right)} e^{-i\varphi(t,t_0)}
\]
with
\[
\varphi(t,t_0) := \int_{t_0}^t \left( \frac{1}{2} \left( E(s), \int_0^s E^\perp \right) - \frac{1}{2} \left( R(s) E(s), \int_0^s R E^\perp \right) \right) ds.
\]

\( S \) is a unitary propagator. For the propagators \( U_0 \) generated by \( H_0(\cdot) \) and \( U \) it holds:
\[
U_0(t,t_0) = e^{-itH_{La}} S(t,t_0) e^{it_0H_{La}}, \quad U(t,t_0) = U_0(t,0) \Omega(t,t_0) U_0(0,t_0) \tag{3}
\]
where \( \Omega \) is the propagator generated by the family of bounded operators
\[
V(t,x(t,q,D))
\]
which is the Weyl quantization of the symbol \( V(t,x(t,q,p)) \) with
\[
x(t,q,p) := \frac{1}{2} q - p^\perp - \int_0^t E^\perp + R(-t) \left( \frac{1}{2} q + p^\perp + \int_0^t RE^\perp \right) \tag{4}
\]
for \( q, p \in \mathbb{R}^2 \). In particular it holds for \( T = n2\pi, \quad n \in \mathbb{Z} \)
\[
U_0(T) := U_0(T,0) = e^{i\alpha_n} e^{i\left(\int_0^T E, E\right)} e^{i\left(\int_0^T RE, v^\perp\right)} \tag{5}
\]
with \( i\alpha_n := -i\pi n - i\varphi(T), i\beta_n := -i\pi n - i\varphi(T) + \frac{1}{4} \left( \int_0^T (I - R) E^\perp, \int_0^T (I + R) E \right) \), \( I \) the \( 2 \times 2 \) identity matrix.
\textbf{Proof.} We use the commutation relations (1). Note that for observables \( w_f \) which are linear polynomials in \( D_j, q_k \) it holds
\[
e^{-i(w_1+w_2)} = e^{-iw_1}e^{-iw_2}e^{\frac{1}{2}[w_1,w_2]} \quad (6)
\]
which for the case \([w_1, w_2] = i \) and \( a \in C^1(\mathbb{R}; \mathbb{R}^2) \) implies for \( \psi \in \mathcal{S}(\mathbb{R}^2) \)
\[
i\partial_t e^{-i(a,w)}\psi = \left( \langle \hat{a}, w \rangle + \frac{1}{2} \langle \hat{a}, a^\perp \rangle \right) e^{-i(a,w)}\psi. \quad (7)
\]
Furthermore \( e^{iH_la}v^\perp e^{-iH_la} = R(-t)v^\perp \) because the derivatives of both functions coincide as well as their values at \( t = 0 \).

To apply formula (7) we write
\[
S = e^{-i(a,c)}e^{-i(b,v^\perp)}e^{-i\varphi}
\]
with \( a(t) := -\int_0^t E, b(t) := -\int_0^t RE \) and conclude for \( \psi \in \mathcal{S}(\mathbb{R}^2) \):
\[
i(\partial_t S)S^{-1}\psi = \left( \langle \hat{a}, c \rangle + \langle b, v^\perp \rangle - \frac{1}{2} \langle \hat{a}, a^\perp \rangle + \frac{1}{2} \langle b, b^\perp \rangle + \varphi \right) \psi \\
= \left( -\langle E, c \rangle - \langle RE, v^\perp \rangle \right) \psi
\]
by definition of \( \varphi \).

Also:
\[
i\partial_t e^{-itH_la}S(t,0)\psi = \left( H_{la} + e^{-itH_la}i\partial_t SS^{-1}(t)e^{itH_la} \right) e^{-itH_la}S(t,0)\psi \\
= \left( H_{la} - \langle E(t), c \rangle - \langle R(t)E(t), R(t)v^\perp \rangle \right) e^{-itH_la}S(t,0)\psi \\
= H_0(t)e^{-itH_la}S(t,0)\psi.
\]

It follows that \( U_0(t,0) = e^{-itH_la}S(t,0) \); (3) results from \( U_0(t,t_0) = U_0(t,0)(U_0(t,0))^{-1} \).

The formula for \( U_0(T) \) in terms of \( q, D \) follows from the identity
\[
\langle a, c \rangle + \langle b, v^\perp \rangle = \left\langle a + \frac{b}{2}, q \right\rangle + \langle (a - b)^\perp, D \rangle
\]
and the identity (6).

Now define \( \Omega(t,0) := U_0^*(t,0)U(t,0) \) then
\[
i\partial_t \Omega(t,0)\psi = U_0^*(t,0)V(t)U_0(t,0)\Omega(t,0)\psi;
\]
to prove the equality \( U_0^*VU_0(t) = V(t, x(t, q, D)) \) note that for \( a \in \mathbb{R}^2 \) it follows from (1):
\[
e^{i\langle a, v^\perp \rangle v^\perp}e^{-i\langle a, v^\perp \rangle} = v^\perp - a^\perp, \quad e^{i(a,c)}ce^{-i(a,c)} = c + a^\perp.
\]
Together with (2) this yields
\[ U_0^*(t,0)qU_0(t,0) = c - \int_0^t E^\perp + R(-t) \left( v^\perp + \int_0^t RE^\perp \right) \]
\[ = x(t,q,D) = \frac{1}{2} q - D^\perp - \int_0^t E^\perp + R(-t) \left( \frac{1}{2} q + D^\perp + \int_0^t RE^\perp \right). \]

The equality \( U_0^*VU_0(t) = V(t,x(t,q,D)) \) now follows from Egoroff’s theorem because \( H_0 \) is an operator quadratic in \( q \) and \( D \), cf [Fo]. By the Calderon Vaillancourt theorem \( V(t,x(t,q,D)) \) is bounded thus the propagator \( \Omega \) is well defined. The invariance of Schwartz space now follows from \( i\partial_t \psi(t) = V(t,x(t,q,D)) \psi(t) \) and the fact that the pseudodifferential operator \( V(t,x,t,q,D)) \) leaves \( S(\mathbb{R}^2) \) invariant.

Note the free evolution of the observables \( c, v^\perp, H_{La} \):

**Corollary 2.2** For \( \psi \in S(\mathbb{R}^2) \)
\[ U_0^*(t,0)cU_0(t,0)\psi = c\psi - \int_0^t E^\perp \psi \]
\[ U_0^*(t,0)v^\perp U_0(t,0)\psi = R(-t) \left( v^\perp \psi + \int_0^t RE^\perp \psi \right) \]
\[ U_0^*(t,0)H_{La}U_0(t,0)\psi = \frac{1}{2} \left( v^\perp + \int_0^t RE^\perp \right)^2 \psi. \]

\( H(t) \) being a \( T \) periodic family of operators one has by Floquet’s theorem
\[ U(t) = M(t) (U(T))^\frac{1}{T} \]
for a unitary \( T \)-periodic family \( M \) and \( U(T) \) the Floquet operator. Eigenvectors of \( U(T) \) give rise to periodic orbits whereas for \( \psi \) in the absolutely continuous spectrum \( n \mapsto \langle \psi, U(nT)\psi \rangle \) decays. Define
\[ a_c(t) := - \int_0^t E^\perp, \quad a_v(t) := \int_0^t RE^\perp \]
and the commuting operators
\[ A_c := \langle a_c(T), c \rangle, \quad A_v := \langle a_v(T), v^\perp \rangle. \]

For \( T \in 2\pi \mathbb{N} \) the operator \( U_0(T) \) is, apart from a phase, the product of the two commuting phase space translation operators; its spectrum is absolutely continuous if \( a_c(T) \) or \( a_v(T) \) is non zero.

In order to prove spectral properties in the general case we shall use the following result of unitary Mourre theory which was proven in [ABCF], Theorem 3.3.
Theorem 2.3 Let $U$ be unitary and $A$ selfadjoint such that
$$U^*AU - A \quad \text{and} \quad [U^*AU, A]$$
are densely defined and can be extended as bounded operators. Suppose that
$$U^*AU - A \geq cl + K$$
for a $c > 0$ and $K$ a compact operator. Then the spectrum of $U$ is absolutely continuous with possibly a finite number of eigenvalues. If $K = 0$ the spectrum is purely absolutely continuous.

For the free case we have

**Proposition 2.4** Let $T > 0$ and $\sigma$ the spectrum of $U_0(T)$. It holds:
$$U_0^*(T)A_vU_0(T) - A_v = |a_v(T)|^2.$$  \hspace{1cm} (12)

1. If $a_v(T) \neq 0$ then $\sigma$ is purely absolutely continuous, $\sigma = \sigma_{ac}$.
2. If $a_v(T) = 0$ then
   (a) if $T \notin 2\pi \mathbb{N}$ then $\sigma = \sigma_{pp}$
   (b) if $T \in 2\pi \mathbb{N}$ then
$$U_0^*(T)A_vU_0(T) - A_v = |a_v(T)|^2$$ \hspace{1cm} (13)
   if $a_v(T) \neq 0$ then $\sigma = \sigma_{ac}$ if $a_v(T) = 0$ then $\sigma = \sigma_{pp}$

**Proof.** The identities (12), (13) follow from corollary 2.2. The second order commutators, which are densely defined on $S(\mathbb{R}^2)$, vanish:
$$[(U_0^*(T)A_cU_0(T) - A_c), A_c] = [(U_0^*(T)A_vU_0(T) - A_v), A_v] = 0.$$

It follows from Theorem 2.3 that the spectrum of $U_0(T)$ is purely absolutely continuous if $a_v(T) \neq 0$ for any $T$ or $a_v(T) \neq 0$ for $T \in 2\pi \mathbb{N}$.

Concerning the remaining cases, remark that for $\psi \in S(\mathbb{R}^2)$ and $t \in \mathbb{R}$ it holds
$$\sup_{t \in \mathbb{R}} \| (I + c^2 + v^2) U_0(t) \psi \| = \sup_{t \in \mathbb{R}} \| \left( I + (c - a_v(t))^2 + (\nu^+ + a_v(t))^2 \right) \psi \|.$$

This supremum is finite iff $a_c$ and $a_v$ are bounded functions of time. On the other hand $(1 + c^2 + v^2)^{-1}$ is a compact operator and thus the trajectory \{ $U_0(t)\psi, t \in \mathbb{R}$ \} is relatively compact iff $a_c$ and $a_v$ are bounded. If all trajectories are relatively compact one knows, see [EV], that $\sigma = \sigma_{pp}$.

$E$ being $T$-periodic $a_v(T) = 0$ implies that $a_v(\cdot)$ is a bounded function. $R(\cdot)E(\cdot)$ is an almost periodic function whose associated Fourier series is $\sum_{m \in \mathbb{Z}} a_n \beta_m e^{i(n + m \frac{T}{2})}$ with $(\beta_m) \in \ell^2(\mathbb{Z}, \mathbb{C}^2), a_n \in \mathbb{C}$. $\lim_{r \to \infty} \frac{1}{r} \int_0^r RE(\cdot) = 0$ implies boundedness of $a_v(\cdot)$, see [F]; this mean is zero if $T \notin 2\pi \mathbb{N}$ or $T \in 2\pi \mathbb{N}$ and $a_v(T) = 0$. Remark that in the latter case $U_0$ is just multiplication by a phase factor. \[\Box\]
1. The case $a_v(T) \neq 0$ and $a_v(T) = 0$ is essentially the dynamics of the usual quantum Hall effect, it is the case $a_v(T) \neq 0$ which we coin “resonant”.

3 Compactness and the proof of Theorem 1.1

In order to go from the unperturbed to the full propagator, we shall prove the following dynamical compactness result.

**Theorem 3.1** Let $T_1 < T_2$ be real, let $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $V \in C^0(\mathbb{R}; S(1))$.

Denote $U$ the propagator defined by $H$ in Theorem 2.1.

1. For $f \in C^0([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ such that $f(t, q) \to_{|q| \to \infty} 0$ uniformly in $t$, it holds for the multiplication operator by $f$ and $U$ of theorem 2.1

$$\int_{T_1}^{T_2} U^*(t)f(t)U(t)dt$$

is compact.

2. For $f \in C^\infty([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ such that for an $\varepsilon > 0$ and all $\alpha \in \mathbb{N}_0^2$

$$\sup_t \sup_{q \in \mathbb{R}^2} |\langle q \rangle^{(1+\varepsilon)} \partial_q^\alpha f(t, q)| < \infty$$

we have

$$\int_{T_1}^{T_2} U_0^*(t)f(t)U_0(t)dt$$

belongs to the $\alpha$-th Schatten class for $\alpha > 4$.

The proof is build on the following lemma

**Lemma 3.2** $T_1 < T_2$ be real, let $E \in C^0(\mathbb{R}; \mathbb{R}^2)$ and $x(t, q, p)$ defined in equation (4).

If $f \in C^\infty([T_1, T_2] \times \mathbb{R}^2, \mathbb{R})$ such that for an $\varepsilon > 0$ and all $\alpha \in \mathbb{N}_0^2$

$$\sup_t \sup_{q \in \mathbb{R}^2} |\langle q \rangle^{(1+\varepsilon)} \partial_q^\alpha f(t, q)| < \infty,$$

then it holds for

$$f_{av}(q, p) := \int_{T_1}^{T_2} f(t, x(t, q, p))dt$$

$$\sup_{(q, p) \in \mathbb{R}^4} |(1 + q^2 + p^2)^{1/2} \partial_q^\alpha \partial_p^\beta f_{av}(q, p)| < \infty \quad \forall \alpha, \beta \in \mathbb{N}_0^2$$

**Proof** of lemma 3.2.

Let $h_{La}, h_0$ be the hamiltonian symbols associated to the partial differential operators $H_{La}, H_0$

$$h_{La}(q, p) := \frac{1}{2} \left(p - \frac{q^1}{2}\right)^2, \quad h_0(q, p) := h_{La}(q, p) - \langle E(t), q \rangle.$$
Denote $\Phi_{h_0}$ the hamiltonian flow generated by $h_0$. $x(t, q, p)$ is the projection of $\Phi_{h_0}(t, q, p)$ to the $q$-component, i.e. the configuration space part of the trajectory, originating in $q, p$.

$$x(t, q, p) = \chi_{La}(t, q, p) + b(t)$$

for a function $b \in C^0(\mathbb{R}, \mathbb{R}^2)$ and $\chi_{La}(t, q, p)$ the projection to the $q$-component of the trajectory generated by $h_{La}$. $\chi_{La}(t, q, p)$ is linear in $q, p$ and, with the symbols of the center and velocity operators

$$v(q, p) := p - \frac{q}{2}, \quad c(q, p) := q - v^\perp,$$

it holds

$$\chi_{La}(t, q, p) = c(q, p) + R(-t)v^\perp(q, p)$$

describing the cyclotron orbit with center $c$ and radius $|v|$.

Denote $z = (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2$, $\langle z \rangle := (1 + q^2 + p^2)^{1/2}$. $b$ being bounded on $[T_1, T_2]$ the boundedness of $\langle q \rangle^{1+\varepsilon} f(q)$ implies

$$|f(t, x(t, z))| \leq \frac{cte}{(1 + (\chi_{La}(t, z))^2)^{\frac{1+\varepsilon}{2}}}$$

for a $cte > 0$, all $z$ and $t \in [T_1, T_2]$.

Denote $\hat{z}(q, p) := (c(q, p), v^\perp(q, p)) \in \mathbb{R}^2 \times \mathbb{R}^2$ and remark that

$$\frac{1}{\sqrt{2}} |z| \leq |\hat{z}| \leq \sqrt{2} |z|.$$

Because of the time periodicity of $\chi_{La}$ we may assume $[T_1, T_2] \subset [0, 2\pi]$. In order to estimate the decay of $\int_{T_1}^{T_2} |f(\chi(t, z))| \, dt$ we separate the time events “orbit is close to the origin” and “orbit is far from the origin” as follows:

For a fixed number $d \in \left(0, \frac{1}{\sqrt{2}}\right)$ define for $z \in \mathbb{R}^4$

$$I_>(z) := \{t \in [0, 2\pi], |\chi_{La}(t, z)| \geq d|z|\}, \quad I_<(z) := [0, 2\pi] \setminus I_>$$

and estimate the contributions to $f_{av}(z)$ of these sets. By (14):

$$\sup_z \langle z \rangle \int_{[T_1, T_2] \cap I_>(z)} |f(t, x(t, z))| \, dt \leq \sup_z \frac{\langle z \rangle cte |T_2 - T_1|}{(1 + (d|z|)^2)^{\frac{1+\varepsilon}{4}}} < \infty.$$

Now for $z$ fixed let $t_0 \in [0, 2\pi]$ be the point for which the distance to the origin $t \mapsto |\chi_{La}(t, z)|$ is minimal. If $I_<(z) \neq \emptyset$ then $t_0 \in I_<(z)$. It holds

$$\partial^2_t |\chi_{La}(t, z)|^2 = -2\langle c, R(-t)v^\perp \rangle$$

10
and thus for $t \in I_c(z)$:

$$|\chi_{La}(t, z)|^2 = c^2 + v^2 + 2(c, R(-t)v^\perp) \leq 2d^2|z|^2 =: \hat{d}^2|z|^2,$$

which implies

$$\partial_t^2|\chi_{La}(t, z)|^2 \geq (1 - \hat{d}^2)|z|^2.$$ It follows that for a $\tau_t \in [t_0, t]$

$$|\chi_{La}(t, z)|^2 = |\chi_{La}(t_0, z)|^2 + \partial_t^2|\chi_{La}(\tau_t)|^2 \frac{(t-t_0)^2}{2} \geq \frac{(1 - \hat{d}^2)}{2} |z|^2 (t-t_0)^2$$

and

$$\sup_{z} \int_{T_1}^{T_2} \int_{I_c(z)} |f(t, x(t, .))| dt \leq \sup_{z} \int_{T_1}^{T_2} \frac{cte(z)}{1 + \left(1 - \hat{d}^2 \right) |z|^2 (t-t_0)^2} dt < \infty$$

and the claim is proven for $f$. Analogously the claim for the derivatives follows from the assumptions on the derivatives of $f$ and the affine character of $z \mapsto x(t, z)$.

**Remark 3.3** Intuitively lemma 3.2 means that the sojourn time of a classical particle in the “support” of $f$ decays as $1/ \left(1 + q^2 + p^2 \right)^{1/2}$ in the initial data because the center respectively the radius of the cyclotron orbit grows linearly in these data. Remarks that analogous results are likely to hold for similar systems.

**Proof** of theorem 3.1.

We use pseudodifferential Weyl Calculus, see [Fo] and references therein. We first deduce the second point from lemma 3.2

By Egoroff’s theorem $U_0^*(t)fU_0(t)$ (respectively its integral) is the Weyl quantization of the symbol $z \mapsto f(x(t, z))$ (respectively $\int_{T_1}^{T_2} f(x(t, z)) dt$).

This is due to the fact that $h_0$ is a quadratic polynomial. Lemma 3.2 means that

$$\int_{T_1}^{T_2} f(t, x(t, .)) dt \in S(m, g),$$

the Hörmander class with respect to the euclidean metric

$$g = \sum_i dq_i^2 + dp_i^2$$

and weight function

$$m(z) = \langle z \rangle = (1 + q^2 + p^2)^{1/2}.$$ $m$ and $\int_{T_1}^{T_2} f(t, x(t, .)) dt$ are both in $L^p(\mathbb{R}^4)$ for $p > 4$ thus by theorem 2.1 of [BT], see also their proposition 4.2, the second claim follows.

Now $C^\infty_0(\mathbb{R})$ is uniformly dense in the continuous functions vanishing at infinity and the space of compact operators in the bounded operators is norm closed. Thus we conclude that
Now use that for \( t, s \).

Since

\[ \text{Assume Theorem 3.4} \]

It follows

\[ \| \Omega(t) - \Omega(s) \| \leq |t - s||V|. \]

Now use that for \( t, s \in I_j, \tau \in (t, s) \) it holds \( \int_s^t \| B \| = O \left( \frac{1}{N} \right) \) and \( \| \Omega(t) - \Omega(s) \| = O \left( \frac{1}{N^2} \right) \).

It follows

\[ \left\| \int_0^{T_j} U^* (T) \sum_{j=0}^{N-1} \Omega^*(jT/N) \left( \int I_j U^*_0 B U_0 \right) \Omega(jT/N) \right\| = O \left( \frac{1}{N} \right). \]

Since \( \int I_j U^*_0 B U_0 \) is compact \( \forall j \), we conclude that \( \int_0^T U^* B U \) is compact as the space of compact operators is norm closed.

In particular \( \int_0^T U^*(t) K(t) U(t) \) is compact so the theorem is proven.

We now prove the Mourre estimate needed to apply theorem 2.3.

**Theorem 3.4** Assume (A). Denote \( U \) the propagator defined by \( H(t) = H_0(t) + V \) on \( L^2(\mathbb{R}^2) \) and the quantities \( a_c, A_c, a_v, A_v \) defined in (11).

1. Suppose \( \int_0^T E \neq 0 \) and \( \left\{ \int_0^T E \mathbf{1}_- \nabla V(q) \right\} \rightarrow |q| \rightarrow \infty 0 \). Then there exists a compact operator \( C \) on \( L^2(\mathbb{R}^2) \) such that

\[ U^*(T) A_c U^*(T) - A_c = |a_c(T)|^2 \mathbb{1} + C, \]

\[ [U^*(T) A_c U(T) - A_c, A_c] \text{ is bounded.} \]

2. Suppose \( T \in 2\pi \mathbb{N} \) and \( |\nabla V(q)| \rightarrow |q| \rightarrow \infty 0 \). If \( \int_0^T R(s) E(s) ds \neq 0 \) then there exists a compact operator \( C \) on \( L^2(\mathbb{R}^2) \) such that

\[ U^*(T) A_v U^*(T) - A_v = |a_v(T)|^2 \mathbb{1} + C, \]

\[ [U^*(T) A_v U(T) - A_v, A_v] \text{ is bounded.} \]
Proof. Throughout this proof the symbols $a, A$ without subscript denote one of the $a_v, A_v$ or $a_c, A_c$ defined in (11). To prove the results concerning the first commutator we first determine a continuous family of bounded operators $K$ such that

$$U^*(T)A_U(T) - A - |a|^2 = -i \int_0^T U^*(t)K(t)U(t)dt,$$

and such that $\int_0^T U^*(t)K(t)U(t)dt$ is compact.

Each of the following computations is to be understood first on $S(\mathbb{R}^2)$ then by extension to $L^2(\mathbb{R}^2)$; recall that by Theorem 2.1 all the involved operators leave $S(\mathbb{R}^2)$ invariant.

By (12, 13) : $U_0^*(T)A_U(0) - A = |a|^2$. It follows

$$U^*(T)A_U(T) - |a|^2 - A = U^*(0)AU_0^*(T) - A$$

$$= -i \int_0^T U^*(t)((U_0(t)AU_0^*(t), H(t)) + [H_0(t), U_0(t)AU_0^*(t)])U(t) dt$$

$$= -i \int_0^T U^*(t)\underbrace{[U_0(t)AU_0^*(t), V]}_{=:K(t)} U(t) dt. \quad (15)$$

Explicitly by Corollary 2.2

$$K_c(t) = [U_0(t)A_cU_0^*(t), V] = \langle a_c(T), -D^\perp \rangle, V \rangle = i \langle a_c(T), \nabla V^\perp \rangle$$

independently of $t$ and

$$K_v(t) = [U_0(t)A_vU_0^*(t), V] = \langle a_v(T), R(-t)D^\perp \rangle, V \rangle = -i \langle R(t)a_v(T), \nabla V^\perp \rangle.$$

By the decay assumption of $\nabla V$ theorem 3.1 is applicable which in both cases implies the compactness of $\int_0^T U^*(t)K(t)U(t)dt$.

Concerning the double commutator observe :

$$[U^*(T)A_U(T) - A, A] =_{(15)} [\Omega^*(T)A\Omega(T) - A, A] =$$

$$[\Omega^*(T)A, \Omega(T)], A = \Omega^*(T)\underbrace{[[A, \Omega(T)], A + \Omega^*(T)[A, \Omega(T)]] \Omega^*(T)[A, \Omega(T)]}.\quad \Omega(T)$$

Thus it is sufficient to prove boundedness of $[A, \Omega(t)]$ and $[A, [A, \Omega(t)]]$ to infer boundedness of $[U^*(T)A_U(T) - A, A]$. Or

$$-i\partial_t\Omega(t) = -U_0^*(t)VU_0(t)\Omega(t), \quad \Omega(0) = \mathbb{I}.$$
Thus the triple of operators $\Omega(t)$, $[A, \Omega(t)]$, $[A, [A, \Omega(t)]]$ is the solution of the system

$$
\Omega(t) = 1 + i \int_0^t L \Omega \\
[A, \Omega(t)] = i \int_0^t ([A, L] \Omega + L [A, \Omega]) \\
[A, [A, \Omega(t)]] = i \int_0^t ([A, L] [A, \Omega] + L [A, [A, \Omega]])
$$

in the Banach space $K^3$ where $K := C^0([0, T], \mathbb{B}(L^2(\mathbb{R}^2)))$ with the norm $\sup_t \| \cdot \|_\ast$ provided that $L, [A, L]$ and $[A, [A, L]]$ belong to $K$.

$V$ is bounded thus $L \in K$. Now $[A, L] = -U_0^\ast [U_0 A U_0^\ast, V] U_0 = (15) - U_0^\ast K(t) U_0$ and so we have shown that $[A, L] \in K$. Finally

$$
[A, [A, L]] = -U_0^\ast [U_0 A U_0^\ast, K] U_0.
$$

Using the explicit expressions for $K$ we get

$$
[U_0 A_c U_0^\ast, K_c] = \left\langle a_c(T), D^\bot \right\rangle, \left\langle a_c(T), D^\bot \right\rangle, V \right\rangle = \left\langle a_c(T)^\bot, Hess(V)a_c(T)^\bot \right\rangle
$$

$$
[U_0 A_v U_0^\ast, K_v](t) = \left\langle R(t) a_v(T), D^\bot \right\rangle, \left\langle R(t) a_v(T), D^\bot \right\rangle, V \right\rangle = \left\langle R(t) a_v(T)^\bot, Hess(V) R(t) a_v(T)^\bot \right\rangle.
$$

Thus from our assumptions of boundedness of the second derivatives of $V$ the results on the double commutator follow and end the proof of Theorem 3.4.

\[\]

We now have all elements to provide the proofs of theorems 1.1 and 1.2

**Proof.** of Theorem 1.1

Recall equations (8,9,10) for the free evolution of observables. As in equation (15) one calculates for $\psi \in S(\mathbb{R}^2)$:

$$
U^\ast(t) v^\bot U(t) \psi = U^* U_0(t) \left( R(-t) v^\bot + \int_0^t R(s-t) E^\bot(s) ds \right) U_0^\ast U(t) \psi
$$

$$
= R(-t) \left( v^\bot \psi + i \int_0^t U^\ast(s) \left( R(s) D^\bot, \right) U(s) \psi \ ds \right) + \int_0^t R(s-t) E^\bot(s) ds \psi
$$

$$
= U_0^\ast(t) v^\bot U_0(t) \psi + \int_0^t U^\ast(s) R(s) \nabla V^\bot U(s) \psi \ ds
$$

and

$$
U^\ast(t) c U(t) \psi = U_0^\ast(t) c U_0(t) \psi + \int_0^t U^\ast(s) \left( -\nabla V^\bot \right) U(s) \psi \ ds,
$$
\[
U^*(t)qU(t)\psi = U_0^*(t)qU_0(t)\psi + \int_0^t U^*(s) \left( R(s-t) - \mathbb{I} \right) \nabla V^\perp U(s)\psi \, ds.
\]

(17)

Note that from Floquet’s theorem, i.e.: \( U(t+T,t_0+T) = U(t,t_0) \), it follows that
\[
C_q(nT) = \sum_{j=0}^{n-1} U^*(jT)C_q(T)U(jT) \quad (n \in \mathbb{N}).
\]

By the decay hypothesis on \( \nabla V \) and theorem 3.1 \( C_q(T) \) is compact.

Case \( T \in 2\pi\mathbb{N} \):
1) follows from theorem 2.3 and theorem 3.4.
2)
\[
\frac{1}{nT} (U^*(nT)qU(nT) - q) = \frac{1}{T} \int_0^T (R - \mathbb{I})E^\perp + \frac{1}{nT} \sum_{j=0}^{n-1} U^*(jT)C_q(T)U(jT).
\]

(18)

By Wiener’s theorem
\[
\lim_{n \to \infty} \frac{1}{nT} (U^*(nT)qU(nT) - q) P_{ac}(U(T))\psi = \frac{1}{T} \int_0^T (R - \mathbb{I})E^\perp P_{ac}(U(T))\psi.
\]

Applied to an eigenvector \( \psi_\lambda \) of \( U(T) \) with eigenprojection \( P_\lambda \) the right hand side of equation (18) converges to
\[
\frac{1}{T} \int_0^T (R - \mathbb{I})E^\perp \psi_\lambda + \frac{1}{T} P_\lambda C_q(T)\psi_\lambda.
\]

In order to prove
\[
P_\lambda \left( \int_0^T (R - \mathbb{I})E^\perp + C_q(T) \right) P_\lambda = P_\lambda (U^*(T)qU(T) - q) P_\lambda = 0 \quad (19)
\]

we adapt an argument of [PSS] to prove the virial theorem. For each component \( q_j \) \((j \in 1, 2)\) define the bounded operators \( R_x := \frac{x}{x + i q_j} \) \((x > 0)\). Then \( q_j R_x \) are bounded and \( s - \lim_{x \to \infty} R_x = \mathbb{I} \). Furthermore
\[
[U(T), q_j R_x] = R_x [U(T), q_j] R_x.
\]

Now \( P_\lambda [U(T), q_j R_x] P_\lambda = 0 \) implies \( P_\lambda R_x [U(T), q_j] R_x P_\lambda = 0 \) and this implies in the limit \( x \to \infty \)
\[
P_\lambda (q_j - U^*(T)q_j U(T)) P_\lambda = 0
\]
as \( [U(T), q_j] = U(T) (q_j - U^*(T)q_j U(T)) \). Thus (19) is proven and we conclude that
\[
v_{asy} = \frac{1}{T} \int_0^T (R - \mathbb{I})E^\perp P_{ac}(U(T)).
\]

15
3) By an analogous argument as in point 2 it follows

\[ s - \lim_{n \to \infty} \frac{1}{nT} \left( U^*(nT)v^\perp U(nT) - v^\perp \right) = \frac{1}{T} \int_0^T RE^\perp P_{ac}(U(T)). \]

Now for \( \psi \in S(\mathbb{R}^2) \):

\[
\langle U^*(nT)\psi, H_{La} U(nT)\psi \rangle - \langle \psi, H_{La} \psi \rangle \\
= \left\langle \left( v^\perp(nT) - v^\perp \right) \psi, \left( v^\perp(nT) - v^\perp \right) \psi \right\rangle \\
+ \left\langle \left( v^\perp(nT) - v^\perp \right) \psi, v^\perp \psi \right\rangle + \left\langle v^\perp \psi, \left( v^\perp(nT) - v^\perp \right) \psi \right\rangle \\
\sim_{n \to \infty} (nT)^2 \left| \frac{1}{T} \int_0^T RE^\perp \right|^2 \| P_{ac}(U(T)) \psi \|^2
\]

because the last to terms on the right hand side are \( \mathcal{O}(n^2 \langle \psi, H_{La} \psi \rangle \frac{1}{2}) \).

Case \( T \not\in 2\pi \mathbb{Z} \).

1) Follows from theorem 2.3 and theorem 3.4 because \( a_c(T) \neq 0 \).

2) By (8) and (17) we have for \( \psi \in S(\mathbb{R}^2) \)

\[
\frac{1}{nT}(U^*(nT)qU(nT) - q) \psi \right\rangle = \frac{1}{nT} \left( R(-nT) - \mathbb{I} \right) v^\perp \psi \\
+ \frac{1}{nT} \int_0^{nT} \left( R(s - nT) - \mathbb{I} \right) E^\perp(s) \psi + \frac{1}{nT} \sum_{j=0}^{n-1} U^*(jT)C_q(jT)U(jT) \psi \\
\to_{n \to \infty} \lim \frac{1}{n} \int_0^{nT} \left( R(s - nT) - \mathbb{I} \right) E^\perp(s) P_{ac}(U(T)) \psi \\
= - \frac{1}{T} \int_0^T E^\perp P_{ac}(U(T)) \psi
\]

because \( \lim_{\tau \to \infty} \int_0^\tau RE^\perp = 0 \) as shown in proposition 2.4.

**Proof.** of Theorem 1.2

1) From the proof of Theorem 3.4 we have

\[
U^*(T)A_c U^*(T) - A_c = |a_c(T)|^2 \mathbb{I} + \int_0^T U^*(t) \left( a_c(T), \nabla V^\perp \right) U(t) dt.
\]

\[
U^*(T)A_v U^*(T) - A_v = |a_v(T)|^2 \mathbb{I} - \int_0^T U^*(t) \left( R(t)a_v(T), \nabla V^\perp \right) U(t) dt.
\]

The assumption of smallness of \( \nabla V \) implies that one of right hand sides is a positive operator and the result follows from theorem 2.3.
2) Theorem 3.1 and the decay of $V$ imply compactness of
\[
\int_0^T U_0^*(t)VU_0(t) \, dt.
\]
By the last argument in the proof of theorem 3.1 this implies compactness of
\[
\int_0^T U_0^*(t)VU(t) \, dt = U_0^*(T)U(T) - \mathbb{I}.
\]
Thus $U(T) - U_0(T)$ is compact and the essential spectra of $U(T)$ and $U_0(T)$ coincide. By formula (3) $U_0(T) = e^{-i\varphi(T)}e^{-iTH_LA}$ so its spectrum is the discrete set of points $\left(e^{-i\varphi(T)}e^{-i(n+\frac{1}{2})T}\right)_{n\in\mathbb{N}_0}$. By Weyl's theorem the spectrum of $U(T)$ is pure point and the theorem 1.2 is proven.

Acknowledgements
We acknowledge gratefully support from the grants Fondecyt Grant 1120786; Scientific Nucleus Milenio ICM PROY-P07-027-F, ECOS-CONICYT C10E01. OB thanks CPT, JA and CM thank Facultad de Matemáticas of PUC for hospitality.

References

[ABCF] Astaburuaga, M A and Bourget, O and Cortés, V H and Fernandez, C., Floquet operators without singular continuous spectrum, Journal of Functional Analysis, (238),489–517, (2006).

[AH] Avron, J. and Herbst, I.: Spectral and scattering theory of Schrödinger operators related to the Stark effect. Commun. Math. Phys., 52, 239–254, 1977.

[BF] Bourget, O. and Fernandez, C.: Absence of singular spectrum for some time-periodic magnetic systems. Spectral and scattering theory for quantum magnetic systems, 25?31, Contemp. Math., 500, Amer. Math. Soc., Providence, RI, 2009.

[BT] E. Buzano and J. Toft. Schatten-von Neumann properties in the Weyl calculus. J. Funct. Anal., 259(12):3080–3114, 2010.

[EV] V. Enss and K. Veselić. Bound states and propagating states for time-dependent hamiltonians. Ann.Inst.H.Poincaré, sect. A, 39:159–191, 1983.

[F] Fink, A. M.: Almost periodic differential equations Lecture Notes in Mathematics, Vol. 377, Springer, Berlin, 1974

[Fo] Folland, G.B.: Harmonic Analysis in Phase Space Princeton, 1989
[G] Graf, Gian Michele Aspects of the integer quantum Hall effect. Proc. Sympos. Pure Math., 76, Part 1, Amer. Math. Soc., Providence, 429–442 (2007)

[GT] Grébert, B., and Thomann, L.: KAM for the Quantum Harmonic Oscillator Comm. Math. Phys., 307, 383?427 (2011)

[GY] Graffi, A. and Yajima, K.: Absolute Continuity of the Floquet Spectrum for a Nonlinearly Forced Harmonic Oscillator Comm. Math. Phys., 215, 245–250 (2000)

[KY] Kitada, H. and Yajima, K.: A scattering theory of time-dependent long range potentials. Duke Math. Journ., 49, 341–376, 1982.

[PSS] Perry, P., Sigal, I. M., Simon, B. Spectral analysis of N-body Schrödinger operators. Ann. of Math., 114 no. 3, 519–567, 1981.

[W] Wang, W.-M. Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations. Comm. Math. Phys., 277 no. 2, 459–496, 2008.