THE MINIMAL RESOLUTIONS OF DOUBLE POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$
WITH ACM SUPPORT

ELENA GUARDO AND ADAM VAN TUYL*

Abstract. Let $Z$ be a finite set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ and suppose further that $X$, the support of $Z$, is arithmetically Cohen-Macaulay (ACM). We present an algorithm, which depends only upon a combinatorial description of $X$, for the bigraded Betti numbers of $I_Z$, the defining ideal of $Z$. We then relate the total Betti numbers of $I_Z$ to the shifts in the graded resolution, thus answering a special case of a question of Römer.

Introduction

Given a set of fat points $Z$ in $\mathbb{P}^n$, it has been the goal of many authors to describe the homological invariants encoded in the graded minimal free resolution of $I_Z$, the defining ideal of $Z$. A non-exhaustive list of references includes [1, 5, 6, 7, 8, 16, 21]. Many interesting questions about these numerical characters remain open; Harbourne’s survey [15] on these problems in $\mathbb{P}^2$ provides a good entry point to this material.

Recently, many authors have extended this circle of problems to include fat points in multiprojective spaces. The Hilbert function ([11, 12, 13]) and the Castelnuovo-Mumford regularity ([14, 20]) are two such topics that have been investigated. Besides their intrinsic interest, motivation to study such points arises from a paper of Catalisano, Geramita, and Gimigliano [3] which exhibited a connection between specific values of the Hilbert function of a set of fat points in a multiprojective space and the dimensions of certain secant varieties of the Segre varieties. We contribute to this ongoing research program by providing an algorithm to compute the bigraded minimal free resolution of the ideal of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support is arithmetically Cohen-Macaulay.

The $\mathbb{N}^2$-graded polynomial ring $S = k[x_0, x_1, y_0, y_1]$ with deg $x_i = (1, 0)$ and deg $y_i = (0, 1)$ is the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. If $P = R \times Q \in \mathbb{P}^1 \times \mathbb{P}^1$ is a point in $\mathbb{P}^1 \times \mathbb{P}^1$, then the defining ideal of $P$ is $I_P = (L_R, L_Q)$ with deg $L_R = (1, 0)$ and deg $L_Q = (0, 1)$. If $X = \{P_1, \ldots, P_s\}$ is a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and $m_1, \ldots, m_s$ are positive integers, then the ideal $I_Z = I_{P_1}^{m_1} \cap \cdots \cap I_{P_s}^{m_s}$ is an $\mathbb{N}^2$-homogeneous ideal that defines a scheme of fat points $Z = \{(P_1; m_1), \ldots, (P_s; m_s)\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The set of points $X$ is called the support of $Z$, while the integer $m_i$ is called the multiplicity of $P_i$. When all the $m_i$s equal two, we call $Z$ a set of double of points. A set of (reduced or non-reduced) points $Z$ is said to be arithmetically Cohen-Macaulay (ACM) if its associated coordinate
ring $S/I_Z$ is Cohen-Macaulay. While it is always true that $Z$ is ACM if $Z \subseteq \mathbb{P}^n$, if $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $r \geq 2$, then $Z$ may or may not be ACM (e.g., see [22]).

We shall focus on sets of double points $Z$ in $\mathbb{P}^1 \times \mathbb{P}^1$ whose support $X$ is ACM. Such schemes were studied by the first author [11] who used combinatorial information about $X$ to determine both the minimal generators of $I_Z$ and its associated Hilbert function. As shown in [11, 13], these schemes are rarely ACM. However, because the support $X$ is ACM, we can associate to $Z$ a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of the integer $s = |X|$ which is related to the relative positions of the points of $X$, i.e., the number of points which share the same first coordinate, and so on. We extend the results of [11] by constructing an algorithm to obtain the bigraded minimal resolution of $I_Z$ from $\lambda$.

Our algorithm (see Algorithm 5.1) is based upon the following steps:

- Using $\lambda$ we construct a scheme $Y$ of reduced and double points, which we call the **completion of $Z$**, such that $Z \subseteq Y$ and $Y$ is ACM (see Theorem 2.2). Applying a theorem of [13], we compute the bigraded minimal free resolution of $I_Y$ from $\lambda$.
- Using $\lambda$ we use $F_1, \ldots, F_p$ to construct bihomogeneous forms such that $I_Z = I_Y + (F_1, \ldots, F_p)$ and where $\deg F_i$ is a function of $\lambda$ (see Theorem 3.6).
- For $j = 0, \ldots, p$, we set $I_0 = I_Y$ and $I_j = (I_{j-1}, F_j)$. For each $j = 1, \ldots, p$, we show (see Lemma 4.3) that $(I_{j-1} : F_j)$ is the defining ideal of a complete intersection of points whose type (and hence minimal resolution) can be computed from $\lambda$.
- For each $j = 1, \ldots, p$, we have a short exact sequence

$$0 \to S/(I_{j-1} : F_j)(-\deg F_j)\to S/I_{j-1} \to S/I_j \to 0.$$  

We prove (cf. Theorem 4.4) that the mapping cone construction gives the bigraded minimal free resolution of $S/I_j$ for each $j$.
- Because the minimal resolution of $I_Y = I_0$ depends only upon $\lambda$, we can reiteratively use the mapping cone construction and the fact that $(I_{j-1} : F_j)$ is a complete intersection to compute the minimal resolution $I_Z = I_p$.

Römer [19] recently asked if the total graded Betti numbers of an ideal $I$ are bounded by the shifts that appear within the minimal graded free resolution of $I$. As an application of Algorithm 5.1, we show (see Theorem 6.1) that the ideals $I_Z$ satisfy this bound, thus extending work of both Römer [19] and Miró-Roig [17].

Some final observations are in order. First, our approach to computing the bigraded minimal free resolution is similar to the approach taken by Catalisano [4]. Catalisano showed that the Hilbert function and resolution of fat points on a nonsingular conic in $\mathbb{P}^2$ can be computed via an algorithm that depends only upon the multiplicities of the points, and without reference to the coordinates of the points. Second, by viewing $I_Z$ as a graded ideal of $S = k[x_0, x_1, y_0, y_1]$, then the ideal $I_Z$ defines a set of "fat lines" in $\mathbb{P}^3$, and our algorithm describes their graded minimal free resolutions. We are not of aware of any other such result about the resolutions of "fat lines". Finally, the ideals $I_Z$ give a new family of examples of codimension two non-perfect ideals whose resolution can be described (see [18] for another such class arising from lattice ideals).
In this paper \( k \) is an algebraically closed field of characteristic zero and \( \mathbb{N} := \{0, 1, 2, \ldots \} \).

1. Preliminaries

1.1. Points and fat points in \( \mathbb{P}^1 \times \mathbb{P}^1 \). We continue to use the notation and definitions from the introduction. Suppose that \( P = [a_0 : a_1] \times [b_0 : b_1] \) is a point of \( \mathbb{P}^1 \times \mathbb{P}^1 \). The bihomogeneous ideal associated to \( P \) is the ideal \( I_P = (a_1x_0 - a_0x_1, b_1y_0 - b_0y_1) \). The ideal \( I_P \) is a prime ideal of height two that is generated by an element of degree \((1, 0)\) and an element of degree \((0, 1)\). If \( P = R \times Q \), then we shall usually write \( I_P = (L_R, L_Q) \) where \( L_R \) is the form of degree \((1, 0)\) and \( L_Q \) is the form of degree \((0, 1)\). Because \( \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{Q} \), the quadric surface in \( \mathbb{P}^3 \), it is useful to note that \( L_R \) defines a line in one ruling of \( \mathbb{Q} \), \( L_Q \) defines a line in the other ruling, and \( P \) is the point of intersection of these two lines.

Let \( X \) be any set of \( s \) points in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( \pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) denote the projection morphism defined by \( P = R \times Q \mapsto R \). Similarly, let \( \pi_2 \) denote the other projection morphism. The set \( \pi_1(X) = \{ R_1, \ldots, R_r \} \) is the set of \( r \leq s \) distinct first coordinates that appear in \( X \), while \( \pi_2(X) = \{ Q_1, \ldots, Q_t \} \) is the set of \( t \leq s \) distinct second coordinates. The set \( X \) is therefore a subset of \( \{ R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X) \} \). When \( P \in X \), we write \( P = P_{i,j} \) to mean that \( P = R_i \times Q_j \).

For \( i = 1, \ldots, r \), let \( L_{R_i} \) denote the degree \((1, 0)\) form that vanishes at all the points of \( X \) which have first coordinate \( R_i \). Similarly, for \( j = 1, \ldots, t \), let \( L_{Q_j} \) denote the degree \((0, 1)\) form that vanishes at all the points whose second coordinate is \( Q_j \). The defining ideal of \( I_X \) is then the ideal

\[
I_X = \bigcap_{P_{i,j} \in X} I_{P_{i,j}} = \bigcap_{P_{i,j} \in X} (L_{R_i}, L_{Q_j}).
\]

As noted above, \( X \) is a subset of \( \{ R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X) \} \). When we have equality, then \( X \) is called a complete intersection of type \((r, t)\), denoted \( X = CI(r, t) \), where \( r = |\pi_1(X)| \) and \( t = |\pi_2(X)| \). The name follows from the fact that

\[
I_X = \bigcap_{P_{i,j} \in X} I_{P_{i,j}} = (L_{R_1} \cdots L_{R_r}, L_{Q_1} \cdots L_{Q_t}) = (F, G)
\]

where \( \deg F = (r, 0) \) and \( \deg G = (0, t) \), and furthermore, \( F \) and \( G \) form a regular sequence on \( S \). When \( X = CI(r, t) \), then the bigraded resolution of \( I_X \) is

\[
0 \longrightarrow S(-r, -t) \longrightarrow S(-r, 0) \oplus S(0, -t) \longrightarrow I_X \longrightarrow 0
\]

which follows from the Koszul resolution, but also taking into account that \( I_X \) is bigraded.

If \( X \) is a finite set of \( s \) points in \( \mathbb{P}^1 \times \mathbb{P}^1 \), and \( m_{i,j}, \ldots, m_{s,j} \) are \( s \) positive integers, then \( Z \) denotes the subscheme of \( \mathbb{P}^1 \times \mathbb{P}^1 \) defined by the saturated bihomogeneous ideal

\[
I_Z = \bigcap_{P_{i,j} \in X} I_{P_{i,j}}^{m_{i,j}} = \bigcap_{P_{i,j} \in X} (L_{R_i}, L_{Q_j})^{m_{i,j}}.
\]

We call \( Z \) a fat point scheme (or sometimes, a set of fat points) of \( \mathbb{P}^1 \times \mathbb{P}^1 \). When all the \( m_{i,j} \) equal one, then \( Z = X \), and \( X \) is called a reduced set of points.

From time to time, we will wish to represent our fat point schemes pictorially. Because \( \mathbb{P}^1 \times \mathbb{P}^1 \) is isomorphic to the quadric surface \( \mathcal{Q} \subseteq \mathbb{P}^3 \), we can draw fat point schemes...
on \( Q \) as subschemes whose support is contained in the intersection of lines of the two rulings of \( Q \). For example, if \( P_{i,j} = R_i \times Q_j \in \mathbb{P}^1 \times \mathbb{P}^1 \), then the fat point scheme \( Z = \{(P_{1,1}; 4), (P_{1,2}; 2), (P_{2,2}; 3)\} \) can be visualized as

\[
\begin{array}{c|c|c}
| & Q_1 & Q_2 \\
\hline
R_1 & 1 & 2 \\
R_2 & \underline{3} & \\
\end{array}
\]

where a dot represents a point in the support and the number its multiplicity.

1.2. ACM points and fat points. As noted in the introduction, a set of (fat) points in \( \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \) with \( r \geq 2 \) may or may not be arithmetically Cohen-Macaulay (ACM). Currently, only ACM sets of (fat) points in \( \mathbb{P}^1 \times \mathbb{P}^1 \) have been classified. ACM sets of points in \( \mathbb{P}^1 \times \mathbb{P}^1 \) were first classified via their Hilbert function in [9]. An alternative classification was provided by the second author [22], which we recall here.

We associate to a set of points \( X \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) two tuples \( \alpha_X \) and \( \beta_X \) as follows. Let \( \pi_1(X) = \{R_1, \ldots, R_s\} \) be the \( r \) distinct first coordinates in \( X \). Then, for each \( R_i \in \pi_1(X) \), let \( \alpha_i := |\pi_1^{-1}(R_i)| \), i.e., the number of points in \( X \) which have \( R_i \) as its first coordinate. After relabeling the \( \alpha_i \) so that \( \alpha_i \geq \alpha_{i+1} \) for \( i = 1, \ldots, r-1 \), we set \( \alpha_X = (\alpha_1, \ldots, \alpha_r) \).

Analogously, for each \( Q_i \in \pi_2(X) = \{Q_1, \ldots, Q_t\} \), we let \( \beta_i := |\pi_2^{-1}(Q_i)| \). After relabeling so that \( \beta_i \geq \beta_{i+1} \) for \( i = 1, \ldots, t-1 \), we set \( \beta_X = (\beta_1, \ldots, \beta_t) \).

Recall that a tuple \( \lambda = (\lambda_1, \ldots, \lambda_r) \) with \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \) is a partition of an integer \( s \) if \( \sum \lambda_j = s \). So, by construction, \( \alpha_X \) and \( \beta_X \) are partitions of \( s = |X| \). The conjugate of a partition \( \lambda \), denoted \( \lambda^* \), is the tuple \( \lambda^* = (\lambda_1^*, \ldots, \lambda_t^*) \) where \( \lambda_i^* = \#\{ \lambda_j \in \lambda | \lambda_j \geq i \} \).

With this notation, we can state Theorem 4.8 of [22]:

**Theorem 1.1.** A set of reduced points \( X \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \) is ACM if and only if \( \alpha_X^* = \beta_X \).

**Example 1.2.** Let \( P_1 = [1 : 0] \) and \( P_2 = [0 : 1] \) in \( \mathbb{P}^1 \), and consider \( X = \{P_1 \times P_1, P_2 \times P_2\} \) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). In this example \( \alpha_X = (1, 1) \) and \( \beta_X = (1, 1) \), but \( \alpha_X^* = (2) \neq \beta_X \), so \( X \) is not ACM. The set \( X \) is the simplest example of a non-ACM set of points.

**Example 1.3.** Consider the following set of points in \( \mathbb{P}^1 \times \mathbb{P}^1 \):

\[
X = \begin{array}{cccccc}
& Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
R_1 & & & & & & \\
R_2 & & & & & & \\
R_3 & & & & & & \\
R_4 & & & & & & \\
R_5 & & & & & & \\
\end{array}
\]

For this set of points, \( \pi_1(X) = \{R_1, R_2, R_3, R_4, R_5\} \). Then

\[
|\pi_1^{-1}(R_1)| = 3, \quad |\pi_1^{-1}(R_2)| = 5, \quad |\pi_1^{-1}(R_3)| = 6, \quad |\pi_1^{-1}(R_4)| = 1, \quad \text{and} \quad |\pi_1^{-1}(R_5)| = 1.
\]

So, \( \alpha_X = (6, 5, 3, 1, 1) \). Now counting the number of points whose second coordinate is \( Q_i \) for \( i = 1, \ldots, 6 \), we have \( \beta_X = (5, 3, 3, 2, 2, 1) \). So \( X \) is ACM because \( \alpha_X^* = \beta_X \).
Remark 1.4. Suppose that $X$ is ACM with $\alpha_X = (\alpha_1, \ldots, \alpha_r)$ and $\beta_X = (\beta_1, \ldots, \beta_t)$. Because $\alpha_X = \beta_X$, we can assume after relabeling that $\alpha_i = |\pi_1^{-1}(R_i)|$ for each $i = 1, \ldots, r$, and $\beta_j = |\pi_2^{-1}(Q_j)|$ for each $j = 1, \ldots, t$. So, when $X$ is ACM, the points of $X$ can be represented by a Ferrers diagram for the partition $\alpha_X$.

The two authors [13] found a similar combinatorial description for classifying ACM fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. We recall this procedure. Let $X$ denote the support of a fat point scheme $Z$, and suppose that $|X| = s$. For each $R_i \in \pi_1(X)$, set

$$Z_{1,R_i} := \{(P_{i,j_1}, m_{i,j_1}), (P_{i,j_2}, m_{i,j_2}), \ldots, (P_{i,j_n}, m_{i,j_n})\}$$

where $P_{i,j_k} = R_i \times Q_{j_k}$ for some $Q_{j_k} \in \pi_2(X)$. Thus $\pi_1(\text{Supp}(Z_{1,R_i})) = \{R_i\}$, and $I_Z = \bigcap_{i=1}^{r} I_{Z_{1,R_i}}$. For each $R_i \in \pi_1(X)$ define $i := \max\{m_{i,j_1}, \ldots, m_{i,j_n}\}$. Then, for $k = 0, \ldots, \ell_i - 1$, we set

$$a_{i,k} := \sum_{j=1}^{\alpha_i} (m_{i,j} - k)_{+} \text{ where } (n)_{+} := \max\{n, 0\}.$$ 

We then put all the numbers $a_{i,k}$ into a tuple; that is, let

$$\alpha_Z := (a_{1,0}, \ldots, a_{1,\ell_1-1}, a_{2,0}, \ldots, a_{2,\ell_2-1}, \ldots, a_{r,0}, \ldots, a_{r,\ell_r-1}).$$

Similarly, for each $Q_j \in \pi_2(X)$, define

$$Z_{2,Q_j} := \{(P_{1,j_1}, m_{1,j_1}), (P_{1,j_2}, m_{1,j_2}), \ldots, (P_{1,j_n}, m_{1,j_n})\}$$

where $P_{1,j_k} = R_{i_k} \times Q_j$ are those points of $\text{Supp}(Z)$ whose projection onto its second coordinate is $Q_j$. Thus $\pi_2(\text{Supp}(Z_{2,Q_j})) = \{Q_j\}$. For $Q_j \in \pi_2(X)$ define $l'_j := \max\{m_{i,j_1}, \ldots, m_{i,j_n}\}$. Then, for each integer $0 \leq k \leq l'_j - 1$, we define

$$b_{j,k} := \sum_{i=1}^{\beta_j} (m_{i,j} - k)_{+} \text{ where } (n)_{+} := \max\{n, 0\}.$$ 

As in the case of $\alpha_Z$, we place all the values $b_{j,k}$ into a tuple:

$$\beta_Z := (b_{1,0}, \ldots, b_{1,l'_1-1}, b_{2,0}, \ldots, b_{2,l'_2-1}, \ldots, b_{r,0}, \ldots, b_{r,l'_r-1}).$$

If we reorder the entries of $\alpha_Z$ and $\beta_Z$ in non-increasing ordering, i.e., $\alpha_i \geq \alpha_{i+1}$ and $\beta_i \geq \beta_{i+1}$ for all $i$, then $\alpha_Z$ and $\beta_Z$ are partitions of $\deg Z$. The following result of the authors [13, Theorem 4.8] then extends Theorem [1.1]. Note that when $Z = X$, then $\alpha_Z = \alpha_X$ and $\beta_Z = \beta_X$, so Theorem [1.1] is a special case of the following theorem.

Theorem 1.5. A set of fat points $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is ACM if and only if $\alpha^*_Z = \beta_Z$.

When $Z$ is ACM, we can in fact describe the entire resolution of $I_Z$ using only the tuple $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$. Define the following two sets from $\alpha_Z$:

$$SZ_0 := \{(m, 0), (0, \alpha_1)\} \cup \{(i-1, \alpha_i) \mid \alpha_i - \alpha_{i-1} < 0\}$$

$$SZ_1 := \{(m, \alpha_m)\} \cup \{(i-1, \alpha_{i-1}) \mid \alpha_i - \alpha_{i-1} < 0\}.$$ 

We take $\alpha_{-1} = 0$. With this notation, we have
Theorem 1.6. Suppose that $Z$ is an ACM set of fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $\alpha_Z = (\alpha_1, \ldots, \alpha_m)$. Then the bigraded minimal free resolution of $I_Z$ is given by
\[ 0 \rightarrow \bigoplus_{(i,j) \in S\mathbb{Z}_1} S(-i, -j) \rightarrow \bigoplus_{(i,j) \in S\mathbb{Z}_0} S(-i, -j) \rightarrow I_Z \rightarrow 0 \]
where $S\mathbb{Z}_0$ and $S\mathbb{Z}_1$ are constructed from $\alpha_Z$ as above.

Our goal is to describe the resolution of the following special class of fat points.

Convention 1.7. For the remainder of this paper, $Z$ will denote a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with the property that $\text{Supp}(Z) = X$ is an ACM scheme and the partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ will denote the partition $\alpha_X$.

Example 1.8. Let $X$ be as in Example 1.3. The scheme $Z$ defined by $I_Z = \bigcap_{P_{i,j} \in X} I_{P_{i,j}}^2$ is an example of a set of points that satisfies Convention 1.7. For this set of points, $\lambda = \alpha_X = (6, 5, 3, 1, 1)$. In light of Remark 1.4 we can visualize this set as

\[ Z = \begin{array}{ccccc}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
R_1 & 2 & 2 & 2 & 2 & 2 \\
R_2 & 2 & 2 & 2 & 2 & 2 \\
R_3 & 2 & 2 & 2 & 2 & 2 \\
R_4 & 2 & 2 & 2 & 2 & 2 \\
R_5 & 2 & 2 & 2 & 2 & 2 \\
\end{array} \]

For this set of fat points, we have
\[ \alpha_Z = (12, 10, 6, 6, 5, 3, 2, 2, 1, 1) \text{ and } \beta_Z = (10, 6, 6, 5, 4, 4, 3, 3, 2, 2, 1). \]

It then follows that $Z$ is not ACM because $\alpha_Z^* = (10, 8, 6, 5, 5, 4, 4, 2, 2, 2, 1, 1) \neq \beta_Z$.

2. The completion of $Z$

Let $Z$ be a set of double points that satisfies Convention 1.7 and let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition that describes the ACM support $X$. In this section we build a scheme $Y$, which we call the completion of $Z$, that contains $Z$. The scheme $Y$ will be an ACM set of fat points that will form the base step in our recursive formula to compute the bigraded resolution of $I_Z$. The notion of a completion was originally introduced by the first author in [11] to describe the minimal generators and Hilbert function of $I_Z$.

Geometrically, the completion of $Z$ is formed by adding a number of simple (reduced) points to $Z$ so that the support of the new scheme becomes a complete intersection. If $X$ is the support of $Z$, and if $\pi_1(X) = \{R_1, \ldots, R_r\}$ and $\pi_2(X) = \{Q_1, \ldots, Q_t\}$, then
\[ X \subseteq W = \{ R_i \times Q_j \mid R_i \in \pi_1(X) \text{ and } Q_j \in \pi_2(X) \}. \]

Note that $W$ is a complete intersection of reduced points.

Definition 2.1. Suppose that $Z$ is set of double points that satisfies Convention 1.7. With the notation as above, the completion of $Z$ is the scheme
\[ Y := Z \cup (W \setminus X). \]
Note that the support of the completion is the complete intersection $CI(r, t)$. (Because of Convention[1,2], we have $t = \lambda_1$.) As first proved in[11], the completion of $Z$ is ACM. In fact, the bigraded minimal free resolution of $I_Y$ is a function of $\lambda$.

**Theorem 2.2.** Let $Y$ be the completion of the scheme $Z$. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the tuple describing $X = \text{Supp}(Z)$, then

(i) $\alpha_Y = (\lambda_1 + \lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \lambda_r, \lambda_1, \lambda_2, \ldots, \lambda_r)$.

(ii) $Y$ is ACM.

(iii) the bigraded minimal free resolution of $I_Y$ has the form

$$0 \to \bigoplus_{(i,j) \in S_0} S(-i, -j) \to \bigoplus_{(i,j) \in S_1} S(-i, -j) \to I_Y \to 0$$

where

$S_0 = \{(2r, 0), (r, \lambda_1), (0, 2\lambda_1)\} \cup \{(i - 1, \lambda_1 + \lambda_i) | \lambda_i - \lambda_{i-1} < 0\}$

$S_1 = \{(2r, \lambda_r), (r, \lambda_1 + \lambda_r)\} \cup \{(i - 1, \lambda_1 + \lambda_{i-1}) | \lambda_i - \lambda_{i-1} < 0\}$

Proof. Statement (i) follows directly from the construction of $Y$. For statement (ii), it suffices to note that if $\lambda^* = (\lambda_1^*, \ldots, \lambda_r^*)$, then $\beta_Y = (\lambda_1^* + \lambda_1^*, \ldots, \lambda_r^* + \lambda_1^*, \lambda_1^*, \ldots, \lambda_r^*)$. Moreover, one can check that $\alpha_Y^* = \beta_Y$, so that by Theorem[1,5] it follows that $Y$ is ACM. The bigraded resolution of (iii) follows from Theorem[1,6].

**Example 2.3.** Let $Z$ be the scheme of Example[1,8]. The completion of $Z$ is the scheme

$Y = \begin{array}{ccccccc}
Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\
R_1 & & & & & & \\
& R_2 & & & & & \\
& & R_3 & & & & \\
& & & R_4 & & & \\
& & & & R_5 & & \\
\end{array}$

where $\bullet$ means a double point and $\circ$ means a simple point (we have suppressed the multiplicities). Because $\lambda = (6, 5, 3, 1, 1)$, it follows that $\alpha_Y = (12, 11, 9, 7, 7, 6, 5, 3, 1, 1)$. Then the shifts in the bigraded minimal free resolution of $I_Y$ are given by

$S_0 = \{(10, 0), (8, 1), (7, 3), (6, 5), (5, 6), (3, 7), (2, 9), (1, 11), (0, 12)\}$

$S_1 = \{(10, 1), (8, 3), (7, 5), (6, 6), (5, 7), (3, 9), (2, 11), (1, 12)\}$.

3. The Generators of $I_Z$ and $I_Y$

Using the tuple $\lambda$, we construct a matrix whose entries are either two or one. We then extract information from this matrix to describe the minimal generators of $I_Z$ and $I_Y$. This technique originated with the first author[11] to describe the minimal generators and the Hilbert function of $I_Z$; this method can also describe the generators of $I_Y$.

Because $I_Y \subseteq I_Z$, we will identify a family of bigraded forms $\{F_1, \ldots, F_p\}$ such that $F_i \notin I_Y + (F_1, \ldots, F_{i-1})$ for $i = 1, \ldots, p$ and $I_Z = I_Y + (F_1, \ldots, F_p)$.
Definition 3.1. If $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the partition associated to $Z$, then the degree matrix of $Z$ is the $r \times \lambda_1$ matrix $M_\lambda$ where

$$(M_\lambda)_{i,j} = \begin{cases} 2 & j \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$$

Remark 3.2. If the points in the support of $Z$ have been relabeled according to Remark 1.4, then $(M_\lambda)_{a,b}$ is the multiplicity of the point $P_{a,b}$ in $Y$, the completion of $Z$.

We now recall some definitions given in [11] using the degree matrix of $Z$.

Definition 3.3. The base corners of $Z$ is the set:

$$C_0 := \{(i, j) \mid (M_\lambda)_{i,j} = 1 \text{ but } (M_\lambda)_{i-1,j} = (M_\lambda)_{i,j-1} = 2\}.$$ 

Given the base corners of $Z$, we then set

$$C_1 := \{(i, l) \mid (i, j), (k, l) \in C_0 \text{ and } i > k\}.$$

The corners of $Z$ is then the set $C := C_0 \cup C_1$. We shall assume that the elements of $C$ have been ordered from largest to smallest with respect to the lex order.

Remark 3.4. The set of base corners $C_0$ can be computed directly from the partition $\lambda$ associated to $Z$. Precisely, $C_0 := \{(i, \lambda_i + 1) \mid \lambda_i - \lambda_{i-1} < 0\}$.

Definition 3.5. For each $(i,j) \in C$, set

$$u_{i,j} := m_{1,j} + m_{2,j} + \cdots + m_{i-1,j} \quad \text{and} \quad v_{i,j} := m_{i,1} + m_{i,2} + \cdots + m_{i,j-1}$$

were $m_{a,b} = (M_\lambda)_{a,b}$. That is, $u_{i,j}$, respectively $v_{i,j}$, is the sum of the entries in $M_\lambda$ in the column above, respectively in the row to the left, of the position $(i,j)$. If $(i,j) = (i_\ell,j_\ell)$ is the $\ell$th largest element of $C$ with respect to the lexicographical order, the form

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{\ell-1}}^{m_{\ell-1,j}} L_{Q_1}^{m_{1,1}} \cdots L_{Q_{\ell-1}}^{m_{\ell-1,1}}$$

were $m_{a,b} = (M_\lambda)_{a,b}$ is called the form relative to the corner $(i,j)$.

Theorem 3.6. Let $Z$ be a fat point scheme that satisfies Convention 1.7, and furthermore, assume that the points in the support have been relabeled using Remark 1.4. If $(i,j) = (i_\ell,j_\ell)$ is the $\ell$th largest element of $C$ with respect to the lex order, then let

$$F_\ell = L_{R_1}^{m_{1,j}} \cdots L_{R_{\ell-1}}^{m_{\ell-1,j}} L_{Q_1}^{m_{1,1}} \cdots L_{Q_{\ell-1}}^{m_{\ell-1,1}}$$

be the form relative to the corner $(i,j)$. Set $I_0 := I_Y$, and $I_\ell := (I_{\ell-1}, F_\ell)$ for $\ell = 0, \ldots, |C|$. Then

1. $\deg F_\ell = (u_{i,j}, v_{i,j})$.
2. $F_\ell \notin I_{\ell-1}$.
3. $I_Z = I_Y + (F_1, \ldots, F_p)$ where $p = |C|$.
4. $I_\ell$ is generated by the generators of $I_Y$, and all the forms relative to corners $(a,b)$ with $(a,b)$ bigger than or equal to $(i_\ell,j_\ell)$.

Proof. Statement (i) is immediate from the definition of $F_\ell$. For statement (ii), note that after relabeling, $P_{i_\ell,j_\ell} = R_{i_\ell} \times Q_{j_\ell}$ is a reduced point of $Y$. Furthermore, every element of $I_{\ell-1}$ vanishes at the point $P_{i_\ell,j_\ell}$, i.e., $I_{\ell-1} \subseteq I_{P_{i_\ell,j_\ell}} = (L_{R_{i_\ell}}, L_{Q_{j_\ell}})$, but the form $F_\ell \notin I_{P_{i_\ell,j_\ell}}$.  

Statements (iii) and (iv) are Theorem 3.15 of [11].
A slight variation of the above technique enables us to describe the generators of $I_Y$.

**Definition 3.7.** Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the partition associated to $Z$, and suppose $M_\lambda$ is the degree matrix of $Z$. The degree matrix of $Y$ is the $(r + 1) \times (\lambda_1 + 1)$ matrix

$$M_Y = \begin{bmatrix} M_\lambda & 1 \\ 1 & 1 \end{bmatrix}$$

where $1$ denotes the appropriately sized matrix consisting only of ones.

**Definition 3.8.** Let $C_0$ be the base corners of $Z$ constructed from $\lambda = (\lambda_1, \ldots, \lambda_r)$. The outside corners of $Z$ is the set

$$OC = \{(r + 1, 1), (1, \lambda_1 + 1), (r + 1, \lambda_1 + 1)\} \cup \{(r + 1, j), (i, \lambda_1 + 1) \mid (i, j) \in C_0\}.$$ 

**Theorem 3.9.** Let $Z$ be a fat point scheme that satisfies Convention 1.7 and furthermore, assume that the points in the support have been relabeled using Remark 1.4. If $(i, j) = (i_\ell, j_\ell) \in OC$, then set

$$G_\ell = L_{R_{i_\ell}}^{m_{i_\ell,j_\ell}} \cdots L_{R_{i_{\ell-1}}}^{m_{i_{\ell-1},j_{\ell-1}}} L_{Q_{j_\ell}}^{m_{i,j_\ell}} \cdots L_{Q_{j_{\ell-1}}}^{m_{i,j_{\ell-1}}}$$

were $m_{a,b} = (M_Y)_{a,b}$. Then $\{G_1, \ldots, G_q\}$ where $q = |OC|$ is a minimal set of generators of $I_Y$.

**Proof.** For each $\ell = 1, \ldots, q$, one can show that $G_\ell$ passes through all the points of $Y$ to the correct multiplicity. By comparing the degrees of each $G_\ell$ with the degrees of the minimal generators of $I_Y$ from the bigraded minimal free resolution in Theorem 2.2, we then see that the $G_\ell$’s form a minimal set of generators of $I_Y$.

We end this section with an example illustrating these ideas.

**Example 3.10.** Let $\lambda = (6, 5, 3, 1, 1)$ be the $\lambda$ associated to the fat point scheme $Z$ of Example 1.8. Then the degree matrices of $Z$ and $Y$ are given by

$$M_\lambda = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{bmatrix} \quad M_Y = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$ 

Then $C_0 = \{(4, 2), (3, 4), (2, 6)\}$, ordered lexicographically. The corners of $Z$ is the set

$$C := C_0 \cup \{(4, 4), (4, 6), (3, 6)\} = \{(4, 6), (4, 4), (4, 2), (3, 6), (3, 4), (2, 6)\}.$$ 

The positions of the underlined 1’s in $M_\lambda$ correspond to the elements of $C$.

The outside corners, which correspond to the positions of the underlined 1’s in the matrix $M_Y$, is the set $OC = \{(6, 1), (6, 2), (6, 4), (6, 6), (6, 7), (1, 7), (2, 7), (3, 7), (4, 7)\}$. As an example of Theorem 3.9 consider $(6, 6) \in OC$. Associated to this tuple is the form

$$G = L_{R_1}^2 L_{R_2}^1 L_{R_3}^1 L_{R_4}^1 L_{R_5}^1 L_{Q_1}^1 L_{Q_2}^1 L_{Q_3}^1 L_{Q_4}^1 L_{Q_5}^1.$$ 

We see from the picture of Example 2.8 that $G$ passes through all the points (with correct multiplicity) of $Y$. Also, $\deg G = (6, 5)$ is one of the degrees of the minimal generators.
Observation 3.11. The following fact will be used implicitly in the next section. For each \((i, j) \in \mathcal{C}\) there exists non-negative integers \(c\) and \(d\) such that \((i + c + 1, j)\), \((i, j + d + 1)\) and \((i + c + 1, j + d + 1)\) are either elements of \(\mathcal{C}\) or \(\mathcal{C}'\). Although we leave the proof of this fact to the reader, we can illustrate this observation using the above example. Note that \((4, 2)\) is a corner of \(Z\). There exists two integers \(c = 1\) and \(d = 1\) such that \((4 + 1 + 1, 2)\), \((4, 2 + 1 + 1)\) and \((4 + 1 + 1, 2 + 1 + 1)\) are also corners or outside corners.

4. The resolution of \(I_Z\)

Let \(F_1, \ldots, F_p\) be the \(p\) forms of Theorem 3.6 where \(F_\ell\) is the form relative to the corner \((i_\ell, j_\ell) \in \mathcal{C}\). As in Theorem 3.6, we set \(I_0 = I_Y\) and \(I_\ell = (I_{\ell - 1}, F_\ell)\) for \(\ell = 1, \ldots, p\). Then, for each \(1 \leq \ell \leq p\), we have a short exact sequence

\[
0 \to S/(I_{\ell - 1} : F_\ell)(-u_{i_\ell,j_\ell}, -v_{i_\ell,j_\ell}) \to S/I_{\ell - 1} \to S/I_\ell = S/(I_{\ell - 1}, F_\ell) \to 0
\]

where \(\deg F_\ell = (u_{i_\ell,j_\ell}, v_{i_\ell,j_\ell})\). Using the short exact sequence and the mapping cone construction, we will reiteratively describe the bigraded minimal free resolution of \(I_Z\).

To use the mapping cone construction in conjunction with (4.1), we will prove that \((I_{\ell - 1} : F_\ell)\) is a complete intersection for each \(\ell = 1, \ldots, p\) whose type can be determined through the following family of matrices. Let \(C = \{(i_1, j_1), \ldots, (i_p, j_p)\}\) be the corners of \(Z\) ordered from largest to smallest with respect to the lex order. Then set \(\mathcal{M}_0 = \mathcal{M}_\lambda\), and for \(\ell = 1, \ldots, p\), let \(\mathcal{M}_\ell\) be the \(r \times \lambda\) matrix where

\[
(M_\ell)_{i,j} = \begin{cases} 0 & \text{if } (i, j) \succeq (i_\ell, j_\ell) \\ (M_{\ell - 1})_{i,j} & \text{otherwise.} \end{cases}
\]

Here \(\succeq\) denotes the partial order where \((i_1, j_1) \succeq (i_2, j_2)\) if and only if \(i_1 \geq i_2\) and \(j_1 \geq j_2\).

Example 4.1. Before preceding to the main results of this paper, we describe in more detail what our algorithm does geometrically, and how we shall use the matrices \(\mathcal{M}_\ell\). Let \(Z_\ell\) denote the scheme of fat points defined by the ideal \(I_\ell\), where \(Z_0 = Y\) is the completion of \(Z\). Roughly speaking, at each step in our algorithm, we are removing a set of points from \(Z_{\ell - 1}\) to form the set of points \(Z_\ell\). In particular, at each step we are removing a complete intersection whose type can be ascertained from the matrix \(\mathcal{M}_{\ell - 1}\).

We illustrate some of these ideas by using our running example (Example 1.8) of \(\lambda = (6, 5, 3, 1, 1)\). The matrix \(\mathcal{M}_0 = \mathcal{M}_\lambda\) of Example 3.10 describes the multiplicities of the fat points \(Z_0 = Y\). By Example 3.10, the largest corner of \(Z\) is \((4, 6)\). The element

\[
F_1 = L_{R_1}^2L_{R_2}L_{R_3}L_{Q_1}^2L_{Q_2}L_{Q_3}L_{Q_4}L_{Q_5}
\]

is the form relative to the corner \((4, 6)\). The form \(F_1\) passes through all the points of \(Z_0 = Y\) with correct multiplicity, except the points \(P_{a,b} = R_a \times Q_b\) with \((4, 6) \preceq (a, b) \preceq (5, 6)\). These points are \(C = \{R_4 \times Q_6, R_5 \times Q_6\}\), a complete intersection of points of type \((2, 1)\) defined by \(I_C = (L_{R_4}L_{R_5}, L_{Q_6})\). The type can be found by starting at the location of the first corner \((4, 6)\) in \(\mathcal{M}_0\), and summing the entry in position \((4, 6)\) and all those below it (in this case, \(1 + 1 = 2\)), to get the first coordinate of the type, and summing the entry in position \((4, 6)\) and all those to right (in this case, only 1) to get the second coordinate.
The ideal \( I_1 = (I_0, F_1) \) is then the defining ideal of \( Z_1 \), where

\[
Z_1 = Y \setminus CI(2, 1) = Y \setminus \{P_{4,6}, P_{5,6}\}.
\]

Observe now that the matrix

\[
M_1 = \begin{bmatrix}
2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 1 \\
2 & 2 & 2 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 0 \\
2 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

describes the multiplicities of the fat point scheme \( Z_1 \):

where \( \bullet \) means a double point and \( \circ \) means a simple point.

The next largest corner of \( Z \) is \((4, 4)\), and the form

\[
F_2 = L_{R_1}^2 L_{R_2}^2 L_{R_3} L_{Q_1}^2 L_{Q_2} L_{Q_3}
\]

is the form relative to the second corner \((4, 4)\). The form \( F_2 \) now passes through all the points of the scheme \( Z_1 \) with correct multiplicity, except the points \( P_{a,b} \) with \((4, 4) \preceq (a, b) \preceq (5, 5)\). These points are \( C = \{R_4 \times Q_4, R_4 \times Q_5, R_5 \times Q_4, R_5 \times Q_5\} \), a complete intersection of type \((2, 2)\) defined by \( I_C = (L_{R_4} L_{R_5}, L_{Q_4} L_{Q_5}) \). The type can be found by starting at the location of the second corner \((4, 4)\) in \( M_1 \), and summing the entry in position \((4, 4)\) and all those below it (in this case, \(1 + 1 = 2\)), to get the first coordinate of the type, and summing the entry in position \((4, 4)\) and all those to right (in this case, \(1 + 1 + 0 = 2\)) to get the second coordinate.

The ideal \( I_2 = (I_1, F_1) \) now defines the scheme

\[
Z_2 = Z_1 \setminus CI(2, 2) = Z_1 \setminus \{P_{4,4}, P_{4,5}, P_{5,4}, P_{5,5}\} = Y \setminus \{P_{4,4}, P_{4,5}, P_{4,6}, P_{5,4}, P_{5,5}, P_{5,6}\},
\]

and analogously, the matrix \( M_2 \) describes the multiplicities of the fat point scheme \( Z_2 \):
Continuing in this fashion, we remove all the simple points from $Y$ by removing a suitably sized complete intersection at each step until we get $Z_6 = Z$. In general, the matrices $M_\ell$ allow us to keep track of the size of the complete intersection we are cutting out from $Z_\ell$ at each step.

**Remark 4.2.** Let $\{(i_1, j_1), \ldots, (i_p, j_p)\}$ be the corners of $Z$ starting from the largest corner of $Z$; the complete intersection $C$ that we remove at each step from $Y$ is formed from the points $P_{a,b}$ with $(i_\ell, j_\ell) \preceq (a, b) \preceq (i_\ell + c, j_\ell + d)$ and such that $(i_\ell, j_\ell)$, $(i_\ell, j_\ell + c + 1)$ and $(i_\ell + d + 1, j_\ell)$ are either corners or outside corners of $Z$.

In the next lemma we show $(I_{\ell-1} : F_{\ell})$ is a complete intersection of points.

**Lemma 4.3.** With the notation as above, let $(i, j) = (i_\ell, j_\ell)$ be the $\ell$th corner of $C$. Then $(I_{\ell-1} : F_{\ell}) = I_{CI(a_i,j_i)}$ where $a_{i,j} = m_{i,j} + \cdots + m_{r,j}$, $b_{i,j} = m_{i,j} + \cdots + m_{i,\lambda_1}$ and $m_{a,b} = (M_{\ell-1})_{a,b}$.

**Proof.** Without loss of generality, assume that the points of $Z$ have been relabeled in accordance to Remark 1.4. From the construction of $M_{\ell-1}$ there exists integers $c$ and $d$ such that $m_{i,j} = m_{i+1,j} = \cdots = m_{i+c,j} = 1$, but $m_{i+c+1,j} = \cdots = m_{r,j} = 0$, and similarly, $m_{i,j} = \cdots = m_{i,j+d} = 1$, but $m_{i,j+d+1} = \cdots = m_{i,\lambda_1} = 0$. Set

$$A = L_{R_1}^{m_{i,j}} \cdots L_{R_{i+c}}^{m_{i+c,j}} = L_{R_1} \cdots L_{R_{i+c}}$$

and $B = L_{Q_1}^{m_{i,j}} \cdots L_{Q_{j+d}}^{m_{i,j+d}} = L_{Q_1} \cdots L_{Q_{j+d}}$.

It will now suffice to show that $(I_{\ell-1} : F_{\ell}) = (A, B)$.

Note that $(A, B)$ defines a complete intersection $C = CI(a_{i,j}, b_{i,j})$. Because the points have been rearranged in accordance to Remark 1.4, $P_{a,b} = R_a \times Q_b \in C$ if and only if $(i, j) \preceq (a, b) \preceq (i + c, j + d)$. The points of $C$ form a subset of the reduced points of $Y$.

By Theorems 3.6 and 3.9, $I_{\ell-1} = (G_1, \ldots, G_q, F_1, \ldots, F_{\ell-1})$. The forms $G_i$ vanish at all the points of $C \subseteq Y$. By Theorem 3.6 we have $F_i \in I_C$ for $1 \leq i \leq \ell - 1$. However,

$$F_{\ell} = L_{R_1}^{m_{i,j}} \cdots L_{R_{i+c}}^{m_{i+c,j}} L_{Q_1}^{m_{i,j}} \cdots L_{Q_{j+d}}^{m_{i,j+d}}$$

from which it follows that for every $P_{a,b} \in C$, $F_{\ell}(P_{a,b}) \neq 0$. So, if $HF_{\ell} \in I_{\ell-1} \subseteq I_C$, then $H \in I_C$. That is, $(I_{\ell-1} : F_{\ell}) \subseteq I_C = (A, B)$.

From the construction of $M_{\ell-1}$, $(i + c + 1, j)$ is either a corner or outside corner of $Z$. In either case, set

$$F = L_{R_1}^{n_{a,b}} \cdots L_{R_{i+c}}^{n_{a,b}+j} L_{R_{i+c}}^{n_{a,b}+j} L_{Q_1}^{n_{a,b}+j} \cdots L_{Q_{j+d}}^{n_{a,b}+j}$$

where $n_{a,b}$ refers to the entries in $M_Y = (n_{a,b})$, the degree matrix of $Y$. If $(i+c+1, j) \in C$, then $F \in I_{\ell-1}$ by Theorem 3.6 if $(i+c+1, j) \in OC$, then $F \in I_{\ell-1}$ by Theorem 3.9. Now set

$$F_{\ell}A = L_{R_1}^{m_{i,j}} \cdots L_{R_{i+c}}^{m_{i+c,j}} L_{R_{i+c}}^{m_{i+c,j}} L_{Q_1}^{m_{i,j}} \cdots L_{Q_{j+d}}^{m_{i,j+d}}$$

We claim that $F$ divides $F_{\ell}A$, and hence $F_{\ell}A \in I_{\ell-1}$. To see this we compare the matrices $M_Y$ and $M_{\ell-1}$. By construction $(M_Y)_{a,b} = (M_{\ell-1})_{a,b}$ for all $(a, b) \preceq (i + c, j)$. So, the exponents of the $L_{R_1}$’s in $F_{\ell}A$ and $F$ are actually the same.

On the other hand, note that $n_{a,b} \geq n_{b,a}$ if $a \geq b$ in $M_Y$, i.e., the columns are non-increasing. Since $m_{i,t} = n_{i,t}$ for $t = 1, \ldots, j - 1$, we have that the exponents of the
Let $(i, j) = (i_\ell, j_\ell)$ be the $\ell$th largest corner of $Z$, and assume that the points of $Z$ have been rearranged in accordance to Remark 1.4. Let

\[ F_\ell = L_{Q_{1}}^{m_{i,j}} \cdots L_{Q_{i-1}}^{m_{i-1,j}} L_{R_{i}}^{m_{i,j}} \cdots L_{R_{j-1}}^{m_{j,i-1}} \]

be the form relative to the corner $(i, j)$ with $\deg F_\ell = (u_{i,j}, v_{i,j})$. Note that for all $(a, b)$ with $(a, b) \preceq (i, j)$, we have $(M_{\ell-1})_{a,b} = (M_{1})_{a,b}$. So, the integers $u_{i,j}$ and $v_{i,j}$ as defined above are the same as those of Theorem 3.6.

By Lemma 4.3, we know that $(I_{\ell-1} : F_\ell) = I_{CI(a_{i,j}, b_{i,j})}$. By using (1.1), a minimal bigraded free resolution of $(I_{\ell-1} : F_\ell)$ is:

\[ 0 \to S(-a_{i,j}, -b_{i,j}) \to S(-a_{i,j}, 0) \oplus S(0, -b_{i,j}) \to (I_{\ell-1} : F_\ell) \to 0. \]

When we apply the mapping cone construction to the short exact sequence (4.1), we get that (4.2) is a bigraded free resolution of $I_{\ell}$. It therefore suffices to verify that this resolution is minimal.

The map in (4.1)

\[ S/(I_{\ell-1} : F_\ell)(-u_{i,j}, -v_{i,j}) \xrightarrow{\phi_{\ell}} S/I_{\ell-1} \]

lifts to a map from the minimal resolution of $S/(I_{\ell-1} : F_\ell)$ to that of $S/I_{\ell-1}$:

\[ 0 \to S \xrightarrow{\delta_1} S^2 \xrightarrow{\phi_{\ell}} S \xrightarrow{\epsilon} S/(I_{\ell-1} : F_\ell) \to 0 \]

\[ \downarrow \delta_2 \quad \downarrow \delta_1 \quad \downarrow \times F_\ell \quad \downarrow \times F_\ell \]

\[ 0 \to F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} S \xrightarrow{\epsilon} S/I_{\ell-1} \to 0. \]
We have suppressed all the shifts in the resolutions. The maps in each square commute. Again suppressing the shifts, the resolution of \( S/I_\ell \) given by the mapping cone construction has the form

\[
0 \to S \oplus F_2 \xrightarrow{\Phi_2} S^2 \oplus F_1 \xrightarrow{\Phi_1} S \oplus F_0 \xrightarrow{\Phi_0} S \to S/I_\ell \to 0
\]

where the maps are

\[
\Phi_2 = \begin{bmatrix} -\phi_1 & 0 \\ \delta_2 & \varphi_2 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} -\phi_0 & 0 \\ \delta_1 & \varphi_1 \end{bmatrix}, \quad \text{and} \quad \Phi_0 = [F_\ell \varphi_0].
\]

After fixing a basis, each map \( \phi_i, \varphi_i, \) and \( \delta_i \) can be represented by a matrix with entries in \( S \). It will therefore suffice to show that all the nonzero entries of the matrix corresponding to the map \( \Phi_i \) for \( i = 0, 1, 2 \) belong to the maximal ideal \((x_0, x_1, y_0, y_1)\) of \( S \). The matrices corresponding to \( \phi_i \) and \( \varphi_i \) already have this property because they are the maps in the maximal resolution of \( S/(I_{\ell-1} : F_\ell) \) and \( S/I_{\ell-1} \), respectively. So, we need to show that there exists maps \( \delta_1 \) and \( \delta_2 \) that make each square commute, and when these maps are represented as matrices, all the nonzero entries belong to \((x_0, x_1, y_0, y_1)\).

From Observation 3.11 because \((i,j) \in C\), there exist integers \( c \) and \( d \) such that \((i+c+1,j), (i,j+d+1)\), and \((i+c+1,j+d+1)\) are either corners or outside corners of \( Z \); in particular, we choose \( c \) and \( d \) as in the proof of Lemma 4.3 that is, \( m_{i,j} = m_{i+1,j} = \cdots = m_{i+c,j} = 1 \), but \( m_{i+c+1,j} = \cdots = m_{r,j} = 0 \), and similarly, \( m_{i,j} = \cdots = m_{i,j+d} = 1 \), but \( m_{i,j+d+1} = \cdots = m_{r,j+d+1} = 0 \) with \( m_{a,b} = (M_{\ell-1})_{a,b} \). Set

\[
A = L_{R_i}^{m_{i,j}} \cdots L_{R_i+c}^{m_{i+c,j}} = L_{R_i} \cdots L_{R_i+c} \quad \text{and} \quad B = L_{Q_j}^{m_{i,j}} \cdots L_{Q_j+d}^{m_{i,j+d}} = L_{Q_j} \cdots L_{Q_j+d}.
\]

Because \((I_{\ell-1} : F_\ell) = (A,B)\) is a complete intersection, the maps \( \phi_0 \) and \( \phi_1 \) are simply the Koszul maps. As matrices, these maps are

\[
\phi_1 = \begin{bmatrix} B \\ -A \end{bmatrix} \quad \text{and} \quad \phi_0 = \begin{bmatrix} A & B \end{bmatrix}.
\]

We also let

\[
H_1 = L_{R_i}^{n_{i,j}} \cdots L_{R_i+c}^{n_{i+c,j}} L_{Q_j}^{n_{i+c+1,j}}, \quad H_2 = L_{R_i}^{n_{i,j+d+1}} \cdots L_{R_i-1}^{n_{i-1,j+d+1}} L_{Q_j}^{n_{i,j}}, \quad H_3 = L_{R_i}^{n_{i,j+d+1}} \cdots L_{R_i+c}^{n_{i+c,j+d+1}} L_{Q_j}^{n_{i+c+1,j+d}}.
\]

where \( n_{a,b} = (M_Y)_{a,b} \).

Now \((i+c+1,j), (i,j+d+1)\), and \((i+c+1,j+d+1)\) are either corners or outside corners of \( Z \). In the case that they are corners of \( Z \), then they are larger than the corner \((i,j)\). So by Theorems 3.6 and 3.9 we have that the forms \( H_1, H_2, H_3 \) are minimal generators of \( I_{\ell-1} \).

After a suitable change of basis, we can then write \( \varphi_0 \) as

\[
\varphi_0 = [H_1 \ H_2 \ H_3 \ K_1 \ \cdots \ K_s]
\]

where \( K_1, \ldots, K_s \) denote the other minimal generators of \( I_{\ell-1} \).
Let
\[ C = \frac{F_\ell A}{H_1} = \frac{L_{m_1} \cdots L_{m_{i,j}} \cdots L_{m_{i,j}}} {L_{R_i} \cdots L_{R_{i+c}} L_{Q_1} \cdots L_{Q_{i,j}}} L_{R_i} \cdots L_{R_{i+c}}. \]

Now, by the construction of \( \mathcal{M}_Y \) and \( \mathcal{M}_{\ell-1} \), we also have \( (\mathcal{M}_Y)_{a,b} = (\mathcal{M}_{\ell-1})_{a,b} \) for all \( (a,b) \leq (i + c, j + d) \). The exponents of the \( L_{R_i} \)’s in the above expression are then the same on the top and bottom, and thus they cancel out, i.e.,
\[ C = \frac{F_\ell A}{H_1} = \frac{L_{m_{i,j}} \cdots L_{m_{i,j}}} {L_{Q_1} \cdots L_{Q_{i,j}}} + \frac{L_{m_{i,j}} \cdots L_{m_{i,j}}} {L_{Q_1} \cdots L_{Q_{i,j}}} + 1. \]

Because \( (i, j) \) is a corner and \( (i + c + 1, j) \) is either a corner or outside corner of \( Z \), by construction of the \( \mathcal{M}_Y \), there exist some \( j' \leq j - 1 \) such that \( n_{i+c+1,j'} < n_{i,j'} = m_{i,j} \).

(The columns of \( \mathcal{M}_Y \) are non-increasing, so if \( n_{i+c+1,j'} = n_{i,j'} \) for all \( j' \leq j - 1 \), then the first \( j - 1 \) entries of rows \( i \) through \( i + c + 1 \) are the same, and thus there would not be a corner (or outside corner) in position \( (i + c + 1, j) \).) Because of this fact, we have \( \deg C > 0 \). A similar argument implies that if \( D = \frac{F_\ell B}{H_2} \), then \( \deg D > 0 \).

Because \( F_\ell H_3 = H_1 H_2 \), we have the following two syzygies:
\[ BH_1 - DH_3 = 0 \quad \text{and} \quad AH_2 - CH_3 = 0. \]

That is, \((B, 0, -D, 0, \ldots, 0)^T \) and \((0, A, -C, 0, \ldots, 0)^T \) are two elements of \( \mathbb{F}_0 \), written as vectors, in \( \ker \varphi_0 = \text{Im} \varphi_1 \). Let \( \underline{a} = (a_1, \ldots, a_m)^T \), respectively, \( \underline{b} = (b_1, \ldots, b_m)^T \) denote an element of \( \mathbb{F}_1 \) with \( \varphi_1(\underline{a}) = (B, 0, -D, 0, \ldots, 0)^T \), respectively, \( \varphi_1(\underline{b}) = (0, A, -C, 0, \ldots, 0)^T \).

With this notation, we can now prove:

**Claim.** The maps \( \delta_1 \) and \( \delta_2 \) are given by
\[
\delta_2 = \begin{bmatrix}
Ca_1 - Db_1 \\
\vdots \\
Ca_m - Db_m
\end{bmatrix}
\quad \text{and} \quad
\delta_1 = \begin{bmatrix}
C & 0 \\
0 & D \\
\vdots & \vdots \\
0 & 0
\end{bmatrix}.
\]

**Proof.** We just need to show that each square containing a \( \delta_i \) commutes. Now \( \varphi_0 \delta_1 = [H_1 C \quad H_2 D] = [F_\ell A \quad F_\ell B] \). This map is the same as composing the map \( \phi_0 \) with the map defined by multiplication by \( F_\ell \). For the second square,
\[
\varphi_1 \delta_2 = C \varphi_1(\underline{a}) - D \varphi_1(\underline{b}) = C(B, 0, -D, 0, \ldots, 0)^T - D(0, A, -C, 0, \ldots, 0)^T
\]
\[
= (CB, -DA, 0, \ldots, 0)^T = \delta_1 \phi_1.
\]

This completes the proof of the claim. \( \square \)

Because \( C \) and \( D \) are nonconstant bihomogeneous forms, every nonzero entry of \( \delta_1 \) and \( \delta_2 \) belongs to \( (x_0, x_1, y_0, y_1) \subseteq S \). Therefore, the resolution of \( I_\ell \) is minimal, as desired. \( \square \)

**Remark 4.5.** As observed in Example 4.1, the ideal \( I_\ell \) corresponds to a subscheme of \( Y \) formed by removing a number of complete intersections of reduced points. The above theorem allows us to calculate the bigraded minimal free resolution for each such subscheme “between” \( Y \) and \( Z \), that is, those schemes we called \( Z_\ell \) in Example 4.1.
The resolution of $I_0 = I_Y$ depends only upon $\lambda$. By repeatedly applying Theorem 4.4 we obtain the minimal resolution of $I_p = I_Z$. Furthermore, the shifts that appear at each step only depend upon $\mathcal{M}_{\ell-1}$ which is constructed from $\lambda$. Thus, there is an algorithm to compute the bigraded minimal free resolution of a fat point scheme $Z$ which satisfies Convention 1.7. For the convenience of the reader, we explicitly write out this algorithm.

**Algorithm 5.1.** *(Computing bigraded resolution)*

**Input:** $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ where $\lambda$ describes the ACM support of $Z$.

**Output:** The shifts in the bigraded minimal free resolution of $I_Z$.

**Step 1:** Compute the shifts in the bigraded resolution of $I_Y$ where $Y$ is the completion of $Z$.

- $\mathcal{SY}_0 := \{(2r,0),(r,\lambda_1),(0,2\lambda_1)\} \cup \{(i-1,\lambda_1+\lambda_i),(i+r-1,\lambda_i) \mid \lambda_i - \lambda_{i-1} < 0\}$
- $\mathcal{SY}_1 := \{(2r,\lambda_r),(r,\lambda_1+\lambda_r)\} \cup \{(i-1,\lambda_1+\lambda_{i-1}),(i+r-1,\lambda_{i-1}) \mid \lambda_i - \lambda_{i-1} < 0\}$

**Step 2:** Locate the corners

- $C_0 := \{(\lambda_i + 1, i) \mid \lambda_i - \lambda_{i-1} < 0\} = \{(i_1,j_1),\ldots,(i_s,j_s)\}$ (lex ordered from largest to smallest)
- $C_1 := \{(i_a,j_b) \mid (i_a,j_a),(i_b,j_b) \in C_0 \text{ and } a > b\}$
- $C := C_0 \cup C_1$ and order $C$ in lexicographical order (largest to smallest)

**Step 3:** Calculate the shifts in the resolution of $I_Z$.

- Let $\mathcal{M}_\lambda$ be the $r \times \lambda_1$ matrix where $(\mathcal{M}_\lambda)_{i,j} = \begin{cases} 2 & \text{if } j \leq \lambda_i \\ 1 & \text{otherwise} \end{cases}$
- Set $\mathcal{SZ}_0 := \mathcal{SY}_0$, $\mathcal{SZ}_1 := \mathcal{SY}_1$, and $\mathcal{SZ}_2 := \{\}$
- For each $(i,j) \in C$ (working largest to smallest) do
  
  \[ u_{i,j} := (\mathcal{M}_\lambda)_{1,j} + \cdots + (\mathcal{M}_\lambda)_{i-1,j} \]
  \[ v_{i,j} := (\mathcal{M}_\lambda)_{i,j} + \cdots + (\mathcal{M}_\lambda)_{i-1,j} \]
  \[ a_{i,j} := (\mathcal{M}_\lambda)_{i,j} \]
  \[ b_{i,j} := (\mathcal{M}_\lambda)_{i,j} \]
  \[ \mathcal{SZ}_0 := \mathcal{SZ}_0 \cup \{u_{i,j},v_{i,j}\} \]
  \[ \mathcal{SZ}_1 := \mathcal{SZ}_1 \cup \{(u_{i,j} + a_{i,j},v_{i,j}), (u_{i,j} + v_{i,j} + b_{i,j})\} \]
  \[ \mathcal{SZ}_2 := \mathcal{SZ}_2 \cup \{(u_{i,j} + a_{i,j},v_{i,j} + b_{i,j})\} \]
  \[ (\mathcal{M}_\lambda)_{i,j} := \begin{cases} 0 & \text{if } (i',j') \geq (i,j) \\ (\mathcal{M}_\lambda)_{i,j} & \text{otherwise} \end{cases} \]

**Step 4:** Return $\mathcal{SZ}_0$, $\mathcal{SZ}_1$, and $\mathcal{SZ}_2$ (the shifts at the 0th, 1st, and 2nd step of the resolution, respectively).

**Remark 5.2.** The above algorithm has been implemented in CoCoA [2] and Macaulay 2 [10], and can be downloaded from the second author’s web page.

**Example 5.3.** We use Algorithm 5.1 to compute the bigraded resolution of the fat points of Example 1.8. We have already computed $\mathcal{SY}_0$ and $\mathcal{SY}_1$ in Example 2.3. To calculate the remaining elements of $\mathcal{SZ}_0$, $\mathcal{SZ}_1$, and $\mathcal{SZ}_2$, where $\mathcal{SZ}_i$ is the set of shifts in $i$th free module appearing the resolution of $I_Z$, we need the numbers $u_{i,j},v_{i,j},a_{i,j},b_{i,j}$ for each...
corner \((i, j) \in \mathcal{C}\). We have presented these numbers in the table below:

| \((i, j) \in \mathcal{C}\) | \(u_{i,j}\) | \(v_{i,j}\) | \(a_{i,j}\) | \(b_{i,j}\) |
|----------------|--------|--------|--------|--------|
| (4,6)          | 4      | 6      | 2      | 1      |
| (4,4)          | 5      | 4      | 2      | 2      |
| (4,2)          | 6      | 2      | 2      | 2      |
| (3,6)          | 3      | 8      | 1      | 1      |
| (3,4)          | 4      | 6      | 1      | 2      |
| (2,6)          | 2      | 10     | 1      | 1      |

By using Theorem 4.4 and the above information, we have

\[
SZ_0 = \{(6,2), (5,4), (4,6), (4,6), (3,8), (2,10)\} \cup SY_0
\]

\[
SZ_1 = \{(8,2), (7,4), (6,6), (6,4), (5,6), (5,6), (4,8), (4,8), (4,7), (3,10), (3,9), (2,11)\} \cup SY_1
\]

\[
SZ_2 = \{(8,4), (7,6), (6,7), (5,8), (4,9), (3,11)\}.
\]

**Remark 5.4.** From Algorithm 5.1 we see that \(Z\) is ACM if and only if \(\mathcal{C} = \emptyset\) if and only if \(\lambda = (\lambda_1, \ldots, \lambda_1)\), that is, if the support of \(Z\) is a complete intersection and \(Z = Y\).

6. An Application: A Question of R"omer

Let \(I\) be a homogeneous ideal of \(R = k[x_1, \ldots, x_n]\) and consider the minimal graded free resolution of \(R/I\)

\[
0 \to \mathbb{F}_p \to \mathbb{F}_{p-1} \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0
\]

where \(\mathbb{F}_i = \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{i,j}(R/I)}\). The number \(p = \text{projdim}(R/I)\) is the **projective dimension**, while the numbers \(\beta_{i,j}(R/I)\) are the \(i, j\)-th **graded Betti numbers** of \(R/I\). R"omer [19] recently initiated an investigation into the relationship between the **ith Betti number** of \(R/I\), i.e., \(\beta_i(R/I) = \sum_{j \in \mathbb{Z}} \beta_{i,j}(R/I)\), and the shifts that appear with the minimal free resolution. Among other things, R"omer asked what ideals satisfy the bound

\[
\beta_i(R/I) \leq \frac{1}{(i-1)!(p-i)!} \prod_{j \neq i} M_j
\]

where \(M_i = \max\{j \mid \beta_{i,j}(R/I_Z) \neq 0\}\) denotes the maximum shift that appears in \(\mathbb{F}_i\). In this section, we show the ideals \(I_Z\) studied in this paper satisfy (6.1). Precisely,

**Theorem 6.1.** Let \(Z\) be a set of double points in \(\mathbb{P}^1 \times \mathbb{P}^1\) with ACM support. Then all the \(i\)th Betti numbers of \(S/I_Z\) satisfy the upper bound (6.1).

Although we have viewed \(S/I_Z\) as a bigraded ring up to this point, the ring \(S/I_Z\) also can be given a graded structure by defining the \(i\)th graded piece to be \((S/I_Z)_i = \bigoplus_{a+b=i}(S/I_Z)_{a,b}\). As noted, \(S/I_Z\) is rarely Cohen-Macaulay, so this family provides further evidence that (6.1) holds for all codimension 2 ideals (R"omer showed (6.1) is true for all codimension 2 Cohen-Macaulay ideals).

We continue to use the notation we developed in previous sections. In particular, we continue to assume \(Z\) satisfies Convention 1.7. We first show how to obtain precise formulas for \(\beta_i(R/I_Z)\) for \(i = 1, 2\) and 3, and lower bounds for \(M_1, M_2, M_3\) in terms of \(\lambda\). With this information, the verification of the bound (6.1) is a straightforward exercise.
Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be any partition, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1$. We set
\[ d(\lambda) = \# \{i \mid \lambda_i - \lambda_{i-1} < 0 \}. \]
Also, let $i^* = \min \{i \mid \lambda_i - \lambda_{i-1} < 0 \}$. This means $\lambda_1 = \lambda_2 = \cdots = \lambda_{i^* - 1} > \lambda_{i^*}$.

**Lemma 6.2.** Let $Z$ be a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support with associated tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$. Let $d = d(\lambda)$. Then
\[
\begin{align*}
(i) & \quad \beta_1(S/I_Z) = 2d + 3 + \binom{d+1}{2}, \\
(ii) & \quad \beta_2(S/I_Z) = 2d + 2 + \binom{d+1}{2}, \\
(iii) & \quad \beta_3(S/I_Z) = \binom{d+1}{2}.
\end{align*}
\]

**Proof.** Let $Y$ be the completion of $Z$. By Theorem 2.2, $R/I_Y$ is ACM, and $\beta_1(R/I_Y) = 3 + 2d$ and $\beta_2(R/I_Y) = 2 + 2d$. By Theorem 3.6 there exist $p$ forms $F_1, \ldots, F_p$ such that $I_Z = I_Y + (F_1, \ldots, F_p)$. Here, $p$ is the number of corners which is $p = \binom{d+1}{2}$. So $\beta_1(R/I_Z) = 2d + 3 + \binom{d+1}{2}$. By Theorem 4.4, each generator $F_i$ contributes two first syzygies and one second syzygy. Hence $\beta_2(R/I_Z) = 2d + 2 + \binom{d+1}{2}$ and $\beta_3(R/I_Z) = \binom{d+1}{2}$. \hfill $\square$

**Lemma 6.3.** Let $Z$ be a set of double points in $\mathbb{P}^1 \times \mathbb{P}^1$ with ACM support with associated tuple $\lambda = (\lambda_1, \ldots, \lambda_r)$, and $d(\lambda) > 0$. Then
\[
\begin{align*}
(i) & \quad 2\lambda_1 \leq M_1, \\
(ii) & \quad 2\lambda_1 + 1 \leq M_2, \\
(iii) & \quad \lambda_1 + \lambda_{i^*} + 3 \leq M_3.
\end{align*}
\]

**Proof.** Let $Y$ be the completion of $Z$. By Theorem 2.2 there is a generator of $I_Y$ of bidegree $(0, 2\lambda_1)$ and a first syzygy of $I_Y$ of bidegree $(i^* - 1, \lambda_1 + \lambda_{i^* - 1})$. By Algorithm 5.1 we thus have that the bigraded shift $(0, -2\lambda_1)$ appears in $F_1$ and $(-i^* + 1, -\lambda_1 - \lambda_{i^* - 1})$ appears as a shift in $F_2$. So, if we only consider the graded resolution of $S/I_Z$, we have that there must be a shift of $-2\lambda_1$ in $F_1$, and a shift of $-i^* + 1 - \lambda_1 - \lambda_{i^* - 1} \leq -1 - \lambda_1 - \lambda_1$ in $F_2$. So $M_1 \geq 2\lambda_1$ and $M_2 \geq 2\lambda_1 + 1$.

Note that $(i^*, \lambda_{i^*} + 1)$ is a base corner of $Z$, and is in fact the smallest corner of $Z$ with respect to the lexicographical ordering. Consider the matrix $(M_p)$ as defined before Lemma 4.3. It must have the following form:
\[
\begin{bmatrix}
2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\
\vdots & & & & & & \\
2 & 2 & \cdots & 2 & 1 & \cdots & 1 \\
\vdots & & & & & & 
\end{bmatrix}.
\]

That is, the first row contains $\lambda_1$ twos, and row $i^*$ contains $\lambda_{i^*}$ twos and $\lambda_1 - \lambda_{i^*}$ ones. By Theorem 4.4 there is a second syzygy of $I_Z$ whose bidegree is $(u, v)$ where $u$ is the sum of the entries in column $\lambda_{i^*} + 1$ and $v$ is the sum of the entries in row $i^*$ of the above matrix. Hence $u \geq 2 + 1$ and $v = 2\lambda_{i^*} + (\lambda_1 - \lambda_{i^*}) = \lambda_1 + \lambda_{i^*}$. So, in the graded resolution of $R/I_Z$, there is a shift of $-u - v \leq -3 - \lambda_1 - \lambda_{i^*}$, from which we deduce $M_3 \geq \lambda_1 + \lambda_{i^*} + 3$. \hfill $\square$

With the above lemmas, we now prove Theorem 6.1.
Proof. (of Theorem 6.1) Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be the tuple associated to the support \( Z \), and set \( d = d(\lambda) \). If \( d = 0 \), then \( \lambda = (\lambda_1, \ldots, \lambda_1) \), and in this case \( S/I_Z \) is Cohen-Macaulay of codimension 2, and thus satisfies the bound (6.1) by [19, Corollary 4.2].

So, we can assume that \( d \geq 1 \). In this case \( S/I_Z \) is not ACM because \( \beta_3(S/I_Z) = \binom{d+1}{2} > 0 \). Before proceeding, we note that \( \lambda_1 - 1 \geq d \) and \( \lambda_i \geq d \). We need to verify (6.1) for \( i = 1, 2 \) and 3 where \( p = 3 \) in this case. We consider each case separately.

Case: \( i = 1 \). In this case, we have

\[
\beta_1(S/I_Z) = 2d + 3 + \binom{d+1}{2} = \frac{1}{2}(d+2)(d+3).
\]

But \( d + 2 \leq (2d + 3) \) and \( d + 3 \leq (2d + 3) \) for all \( d \geq 1 \), so

\[
\beta_1(S/I_Z) \leq \frac{1}{2}(2d+3)(2d+3) \leq \frac{1}{2}(2(d+1)+1)((d+1)+d+2) \leq \frac{1}{2}(2\lambda_1+1)(\lambda_1+\lambda_i+2) \leq \frac{1}{(1-1)!(3-1)!}M_2M_3.
\]

Case: \( i = 2 \). For this case

\[
\beta_2(S/I_Z) = 2d + 2 + \binom{d+1}{2} = 2d + 2 + (d+1)d = (d+1)(d+2) \leq 2(d+1)(d+3) = (2(d+1))(2(d+2)) = (2(d+1))((d+1)+d+3) \leq (2\lambda_1)(\lambda_1+\lambda_i+3) \leq \frac{1}{(2-1)!(3-2)!}M_1M_3.
\]

Case: \( i = 3 \). In our final case we have

\[
\beta_3(S/I_Z) = \binom{d+1}{2} \leq \binom{\lambda_1+1}{2} \leq \lambda_1(\lambda_1+1) \leq \lambda_1(2\lambda_1+1) = \frac{1}{2}2\lambda_1(2\lambda_1+1) \leq \frac{1}{(3-1)!(3-3)!}M_1M_2.
\]

So, the bound (6.1) is satisfied for all \( i \). \( \square \)

References

[1] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points. Invent. Math. 140 (2000) 303–325.
[2] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it
[3] M.V. Catalisano, A. V. Geramita, A. Gimigliano, Ranks of tensors, secant varieties of Segre varieties and fat points. Linear Algebra Appl. 355 (2002) 263–285.
[4] M. Catalisano, “Fat” points on a conic. Comm. Algebra 19 (1991) 2153–2168.
[5] G. Fatabbi, A. Lorenzini, On a sharp bound for the regularity index of any set of fat points. J. Pure Appl. Algebra 161 (2001) 91–111.
[6] S. Fitchett, B. Harbourne, S. Holay, Resolutions of fat point ideals involving eight general points of \( \mathbb{P}^2 \). J. Algebra 244 (2001) 684–705.
[7] A.V. Geramita, J. Migliore, L. Sabourin, On the first infinitesimal neighborhood of a linear configuration of points in \( \mathbb{P}^2 \). J. Algebra 298 (2006) 563-611.
[8] A. Gimigliano, M. Id`a, The ideal resolution for generic 3-fat points in \( \mathbb{P}^2 \). J. Pure Appl. Algebra 187 (2004) 99-128.
S. Giuffrida, R. Maggioni, A. Ragusa, On the postulation of 0-dimensional subschemes on a smooth quadric. Pacific J. Math. 155 (1992) 251–282.

D. R. Grayson, M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry. http://www.math.uiuc.edu/Macaulay2/

E. Guardo, Fat point schemes on a smooth quadric. J. Pure Appl. Algebra 162 (2001) 183–208.

E. Guardo, A survey on fat points on a smooth quadric. Algebraic structures and their representations, 61-87, Contemp. Math., 376, Amer. Math. Soc., Providence, RI, 2005.

E. Guardo, A. Van Tuyl, Fat points in $\mathbb{P}^1 \times \mathbb{P}^1$ and their Hilbert functions. Canad. J. Math. 56 (2004) 716–741.

H. T. Hà, A. Van Tuyl, The regularity of points in multi-projective spaces. J. Pure Appl. Algebra 187 (2004) 153-167.

B. Harbourne, Problems and Progress: A survey on fat points in $\mathbb{P}^2$. Queen’s Papers in Pure and Appl. Math. 123 (2002) 85–132.

B. Harbourne, Free resolutions of fat point ideals on $\mathbb{P}^2$. J. Pure Appl. Algebra 125 (1998) 213–234.

R.M. Miró-Roig, Betti numbers of determinantal ideals. (2007) Preprint math.AC/0701435

I. Peeva, B. Sturmfels, Syzygies of codimension 2 lattice ideals. Math. Z. 229 (1998) 163-194.

T. Römer, Betti numbers and shifts in minimal graded free resolutions. (2007) Preprint math.AC/0701190

J. Sidman, A. Van Tuyl, Multigraded regularity: syzygies and fat points. Beiträge Algebra Geom. 47 (2006) 67-87.

N.V. Trung, G. Valla, Upper bounds for the regularity index of fat points. J. Algebra 176 (1995) 182-209.

A. Van Tuyl, The Hilbert functions of ACM sets of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. J. Algebra 264 (2003) 420-441.

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIALE A. DORIA, 6 - 95100 - CATANIA, ITALY
E-mail address: guardo@dmi.unict.it

DEPARTMENT OF MATHEMATICS, LAKEHEAD UNIVERSITY, THUNDER BAY, ON P7B 5E1, CANADA
E-mail address: avantuyl@sleet.lakeheadu.ca