A Beable Interpretation of the Continuous Spontaneous Localization Model

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Abstract

We extend the beable interpretation, due to Bell, to the continuous spontaneous localization model (CSL). Results obtained by Vink are generalized to the modified Schrödinger equation of Ghirardi, Pearle and Rimini (GPR), which allows a beable interpretation for both position and momentum.

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1. Introduction
The usual interpretation of quantum mechanics deals fundamentally with results of measurements and therefore presupposes, besides a system, an apparatus to perform the measurements. However what the apparatus is and how to distinguish it from the system are questions with vague answers. In face of this problem, Bell\cite{bell} proposed an interpretation in terms of ‘beables’ instead of observables. Beables correspond to things that exist independently of the observation, therefore they can be assigned well defined values. In this way we avoid a cut between the microscopic (quantum) world and the macroscopic (classical) world.

Vink\cite{vink} showed that two other well known interpretations of quantum mechanics - the causal interpretation associated with Bohm\cite{bohm} and the stochastic interpretation due to Nelson\cite{nelson} - are particular cases of the beable interpretation as developed by Bell. Moreover, he proposed that all observables, even those that do not commute, can attain beable status simultaneously.

In this paper we investigate the model proposed by GPR for the free particle\cite{gpr} from the beable point of view. We treat position and momentum as beables despite the fact that they are non-commuting and show that in the continuum limit (following Vink) they satisfy two stochastic differential equations, reproducing the results as obtained by Ghirardi, Rimini and Weber (GRW) for the average values and dispersions\cite{grw}.

We start the paper by presenting Vink’s description of causal and stochastic interpretations as particular cases of Bell’s beables. After a brief exposition of the CSL model we obtain the main results of this work.

2. Causal and Stochastic Interpretations from the Beable Point of View
In Bohm’s causal interpretation the wave function $\psi$, which is a solution of the Schrödinger equation, is a field that guides the particle, whose trajectory is obtained by solving the equation

$$\dot{x} = \frac{\nabla S(x,t)}{M},$$

where $S/\hbar$ is the phase of the wave function written in the polar form

$$\psi(x,t) = R(x,t) \exp\left[\frac{iS(x,t)}{\hbar}\right].$$

and $M$ is the particle mass.

In Nelson’s stochastic approach particles play a preponderant role and are subjected to a stochastic process given by

$$dx = \left[2\nu\frac{\nabla R(x,t)}{R(x,t)} + \frac{\nabla S(x,t)}{M}\right]dt + (2\nu)^{\frac{1}{2}}dw(t),$$

where $w(t)$ is a white noise: $<dw(t)> = 0$ and $<dw(t)^2> = dt$ with $\nu$ the diffusion constant given by $\nu = \hbar/2M$. Note that when $\nu = 0$ we recover the causal interpretation.

Both approaches deal with trajectories, the particles have a definite position even if not observed, making position a beable in Bell’s sense.
Vink [2] showed a connection between the two approaches above and Bell’s beable interpretation. Unlike Bell’s approach, which is done in terms of fermion number, a discrete quantity, Vink shows that the beable concept can be extended to any observable in case it takes discrete values on small scales.

In case we want to find the trajectories of a set of commuting dynamical variables $O^i$, each one with $m$ discrete eigenvalues, we write the continuity equation in the $O$ representation as

$$\partial_t P_m = \sum_n J_{mn}, \quad (4)$$

where the probability density $P_m$ and the source matrix $J_{mn}$ are defined by

$$P_m(t) = |<O_m|\psi(t)>|^2, \quad (5)$$

$$J_{mn}(t) = 2Im\{<\psi(t)|O_m><O_m|H|O_n><O_n|\psi(t)>\}. \quad (6)$$

We are using GPR notation $H = p^2/2m\hbar + V(x)/\hbar$.

Following Bell, the probability distribution of $O_m$ values, $P_m(t)$, satisfies the master equation

$$\partial_t P_m = \sum_n (T_{mn}P_n - T_{nm}P_m), \quad (7)$$

where $T_{mn}dt$ is the transition probability expressing mathematically the probability for jumps from state $n$ to state $m$. To reconcile the quantum and stochastic views we equate (4) and (7):

$$J_{mn} = (T_{mn}P_n - T_{nm}P_m), \quad (8)$$

with $T_{mn} \geq 0$ and $J_{mn} = -J_{nm}$.

There is great freedom to find solutions of equation (8). Bell chooses a particular one for $n \neq m$,

$$T_{mn} = \begin{cases} J_{mn}/P_n & J_{mn} > 0 \\ 0 & J_{mn} \leq 0 \end{cases}$$

Restricting the position of a particle in one dimension to the sites of a lattice, $x = an$, with $n = 1, ..., N$ and $a$ the lattice distance, it follows from the discrete version of the Schrödinger equation that $J_{mn}$ is given by

$$J_{mn} = \frac{1}{Ma}\{[S(an)]'P_n\delta_{n,m-1} - [S(an)]'P_n\delta_{n,m+1}\}, \quad (9)$$

where use was made of the polar form of the wave function and $\psi(x+a)$ is expanded up to first order in $a$. In the expression above $[S(an)]' = [S(an + a) - S(an)]/a$.

For forward movement, Bell’s choice becomes

$$T_{mn} = \frac{[S(an)]'}{Ma}\delta_{n,m-1} \quad (10)$$
which gives the average displacement $dx = S(x)'dt/M$. In the continuum limit, as $a \to 0$, the particle has a velocity given by

$$\dot{x} = \frac{\nabla S(x)}{M}$$

which corresponds to the result of the causal approach.

However, we could also add to $T_{mn}$ the solution of the homogeneous equation, $T_{mn}^0$

$$T_{mn}^0 P_n - T_{nm}^0 P_m = 0$$

for which we can choose a Gaussian with width $\sigma$

$$T_{mn}^0 \propto \exp \left\{ - \left[ m - n - \frac{2\sigma \ln(P_m/P_n)}{4(m - n)} \right]^2 / 2\sigma \right\}.$$  

Assuming $\sigma$ sufficiently small, we can approximate $[\ln(P_m/P_n)]/(m - n)$ by $2a[R(\alpha)]' / R(\alpha)$ arriving at the following Langevin equation for the particle position in the continuum limit

$$dx = \left[ (\beta \sigma a^2) \frac{\nabla R(x)}{R(x)} + \frac{\nabla S(x)}{M} \right] dt + (\beta \sigma a^2)^{1/2}dw,$$

where $\beta$ is a free parameter.

This equation coincides with Nelson’s stochastic equation with $\beta \sigma a^2 = 2\nu$.

### 3. CSL Model

In this model the wave function is subjected to a stochastic process in Hilbert space. In one dimension the evolution equation in the Stratonovich form is

$$d\psi(x, t) = \left\{ [-iH - \lambda]dt + \int dzdB(z, t)G(x - z) \right\} \psi(x, t)$$

which does not preserve the norm of the wave function. A norm preserving evolution is described by the following Stratonovich equation

$$d\phi(x, t) = \left\{ [-iH - \gamma \int dzK(x, z)] dt + \left[ \int dzdB(z, t)L(x, z) \right] \right\} \phi(x, t).$$

where

$$K(x, z) = (G(x - z))^2[1 - 3||\psi||^2 + 2||\psi||^4]$$

and

$$L(x, z) = G(x - z)[1 - ||\psi||^2]$$

In the equations above $dB$ is a white noise ($<dB>=0$ and $<dB^2>=\gamma dt$) and
\[ G(x-z) = \sqrt{\frac{\alpha}{2\pi}} \exp \left[ -\alpha \frac{(x-z)^2}{2} \right] \]  

(19)

is an indication of the localization of the wave function. The length parameter \( \alpha \) and the frequency parameter \( \lambda \) are related to \( \gamma \) according to \( \gamma = \lambda (4\pi/\alpha)^{1/2} \). They are chosen in such a way that the new evolution equations do not give different results from the usual Schrödinger unitary evolution for microscopic systems with few degrees of freedom, but when a macroscopic system is described there is a fast decay of the macroscopic linear superpositions which are quickly transformed into statistical mixtures \[5,6\].

We now formulate this model in terms of beables. The reader may be worried that Vink’s approach does not hold for a non-linear evolution such as in equation (16), however the non-linear terms only involve expectation values in the given state and this equation gives the increment \(d\phi\) for a given \(dt\). All we have to do in order to adapt Vink’s approach to our case is to compute (4) in the following way: \(dP_{mn}/dt = \sum J_{mn}\). Taken this precaution into account a solution of equation (8) for the CSL model can be given as

\[ T_{mn}^x dt = \frac{J_{mn}^x}{P_n} dt + \beta T_{mn}^{ox} dt \]  

(20)

where, for the modified Schrödinger equation (16) \[\mathbb{1}\] and forward movement, \(J_{mn}^x\) \[\mathbb{2}\] is given by

\[ J_{mn}^x = \frac{1}{Ma}[S(\alpha n)] P_n \delta_{n,m-1} + 2 \left( -\gamma \int dz K(x,z) + \int dz \frac{\partial B(z,t)}{\partial t} L(x,z) \right) P_n \delta_{m,n} \]  

(21)

The last term in equation (21) comes from the non-unitary part of the evolution of the wave function (16), but does not contribute to the displacement \(dx\) and consequently \(dx\) obeys an equation of the same form as equation (14). However, the wave function is now different from the one unitarily evolved by the action of \(H, \psi_S(x,t)\). In order to obtain the normalized wave function \(\phi(x,t)\) it is easier to solve equation (15) than the non-linear equation (16) and then use \[\mathbb{3,4}\]: \(\phi(x,t) = \psi(x,t)/||\psi||\), which gives

\[ \psi(x,t) = \frac{\psi_S(x,t)}{||\psi||} \exp \left[ -\lambda t \right] \exp \left\{ \int_0^t \left[ \int dz G(x-z) dB(z,t') \right] \right\}, \]  

(22)

Notice that the norm \(||\psi||\) does not depend on \(x\), therefore the stochastic differential equation for the free particle is

\[ dx = \frac{p_o}{M} dt + 2\nu \left\{ \int_0^t \left[ \int dz dB(z,t') \frac{\partial G(x-z)}{\partial x} \right] \right\} dt + (2\nu)^{1/2} dw. \]  

(23)

where \(dw\) and \(dB\) are two independent white noises.

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\[1\] Notice that as the beable approach deals with transition probabilities we use the norm preserving evolution equation (16).

\[2\] The superscript \(x\) is used to remind that position is the beable.
In equation (23) the first term on the right hand side describes a single free particle
deterministic evolution. The two other terms describe the stochastic processes, with the
last one being a standard diffusion and the second term, a non standard diffusion which
exhibits the non-locality of the localization process. This second term indicates that the
particle position tracks the wave function. The position increment induced by this term
drives the particle to where the wave function is increasing according to the fluctuating
term in equation (22).

Considering now momentum as a beable, we repeat the procedure for position and restrict
it to the sites of a lattice, \( p = bn \), with \( n = 1, ..., N \) and \( b \) the lattice distance. The norm
preserving evolution equation in the \( p \) representation
\[
\frac{\partial \tilde{\phi}(p,t)}{\partial t} = -iH(p)\tilde{\phi}(p,t) - \sqrt{\frac{1}{2\pi\hbar}} \int dp' dx dz \gamma K(x,z) \exp \left[ -i\frac{(p-p')x}{\hbar} \right] \tilde{\phi}(p',t) + \sqrt{\frac{1}{2\pi\hbar}} \int dp' dx dz \frac{\partial B(z,t)}{\partial t} L(x,z) \exp \left[ -i\frac{(p-p')x}{\hbar} \right] \tilde{\phi}(p',t)
\]
after being discretized gives
\[
J^p_{mn} = \sqrt{\frac{1}{2\pi\hbar}} \sum_{m'} \int dx dz \left[ -\gamma K(x,z) + \frac{\partial B(z,t)}{\partial t} L(x,z) \right] \exp \left[ -ib\frac{(m-m')x}{\hbar} \right] \delta_{m,n}
\]

The transition probability for jumps in momentum is
\[
T^p_{mn} dt = \frac{J^p_{mn}}{P_n} dt + \xi T^{op}_{mn} dt
\]
where
\[
T^{op}_{mn} \propto \exp \left\{ -\left[ m - n - \frac{2\Omega \ln(P_m/P_n)}{4(m-n)} \right]^2 / 2\Omega \right\}
\]

Assuming \( \Omega \) small, we can approximate \( \ln(P_m/P_n) \) by \( m - n \) and obtain for \( dp \)
\[
dp = \left[ (\xi \Omega b^2) \nabla_p \tilde{\phi}(p) \tilde{\phi}(p) \phi^*(p) \phi(p) \right] dt + (\xi \Omega b^2)^{1/2} dw
\]

The localization process in configuration space leads to a spreading of the wave-packet in
momentum space, which in turn, removes the momentum dependence on \( |\phi(p)|^2 \), eliminating
the first term in equation (27).

In order to obtain the same mean values and dispersions for position and momentum in
the GRW model for a free particle, we need to use \( \xi \Omega b^2 = \hbar^2 \alpha \lambda / 2 \) and can thus write
\[
dp = \hbar \sqrt{\frac{\alpha \lambda}{2}} dt
\]
Notice that the stochastic process for momentum is a consequence of the collapse of the wave function, which vanishes when GRW parameters ($\alpha, \lambda$) go to zero.

The two stochastic differential equations for position and momentum (eqs. 23 and 29) lead to the same Fokker-Planck equation for the phase-space density as obtained by GRW.

4. Concluding Remarks

We have given a beable interpretation for the CSL model which leads quite naturally to simultaneous beable status for both position and momentum.

The localization of the wave function, in addition to inducing fluctuation in momentum (eq. 29), introduces a new stochastic process for the displacement (eq. 23), which drives the particle to where the wave function localizes.

We have treated in this paper the free particle case and intend to extend this treatment to other cases in a future publication.

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