Entanglement in joint $\Lambda \bar{\Lambda}$ decay

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Received: 9 December 2014 / Revised: 12 May 2015
Published online: 2 July 2015 – © Societ à Italiana di Fisica / Springer-Verlag 2015
Communicated by H. Wittig

Abstract. We investigate the joint $\Lambda \bar{\Lambda}$ decay in the reaction $e^+ e^- \rightarrow \gamma \Lambda(\rightarrow p\pi^-) \bar{\Lambda}(\rightarrow \bar{p}\pi^+)$. This reaction may provide information on the electromagnetic form factors of the Lambda baryon, in the time-like region. We present a conventional diagram-based calculation where production and decay steps are coherent and summations over final-state proton and anti-proton spins are performed. The resulting cross-section distribution is explicitly covariant as it is expressed in scalar products of the four-momentum vectors of the participating particles. We compare this calculation with that of the folding method which we extend and make explicitly covariant. In the folding method production and decay distributions, not amplitudes, are folded together. Of particular importance is then a correct counting of the number of possible intermediate-hyperon-spin states.

1 Introduction

The BABAR Collaboration \cite{1} has studied a number of $e^+ e^-$ annihilation reactions. One of them is the initial-state radiation reaction, $e^+ e^- \rightarrow \gamma \Lambda(\rightarrow p\pi^-) \bar{\Lambda}(\rightarrow \bar{p}\pi^+)$, which offers opportunities to determine the electromagnetic form factors of the $\Lambda$ hyperon in the time-like region. This determination is achieved by varying the energy of the radiated photon.

A theoretical analysis of the above reaction is presented in refs. \cite{2} and \cite{3}. It is based on a recipe here referred to as folding. The cross-section distribution is obtained by multiplying distributions functions for the $\Lambda \bar{\Lambda}$ production with the decay-distribution functions for the Lambda and anti-Lambda hyperons, all for fixed hyperon-spin directions. This product is then averaged over the hyperon-spin directions. The disadvantage of this method, as used, is that several coordinate systems are employed in the calculation, and there seems to be a problem of properly counting the number of intermediate hyperon-spin states.

We propose a calculation from first principles, the conventional perturbation method, calculating directly the relevant matrix element as

\[ j_\mu(p_1, p_2) = -ie\bar{u}(p_1)O_\mu(p_1, p_2)v(p_2) \]

\[ O_\mu(p_1, p_2) = G_1(P^2)\gamma_\mu - \frac{1}{2M}G_2(P^2)Q_\mu, \]

with $P = p_1 + p_2$ and $Q = p_1 - p_2$.

The form factors $G_1$ and $G_2$ are related to the more commonly used form factors $F_1$ and $F_2$, and the electric

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written as \( \Gamma \), and our notation follows Pilkuhn [5].

The cross-section distribution for the reaction \( e^+e^- \to \gamma \Lambda \to p\pi^- \), \( A \to \bar{p}\pi^+ \) is written as

\[
\sigma = \frac{1}{2\sqrt{s}(s,m^*_e,m^*_\gamma)^2} |M|^2 \ d\text{Lips}(k_1+k_2;q,l_1,l_2,q_1,q_2),
\]

where the average over the squared matrix element indicates summation over final proton and anti-proton spins and average over initial electron and positron spins. The definitions of the particle momenta are explained in fig. 1.

We would like to remove some trivial factors from the squared matrix element, namely the powers of the electron charge, the squares of the intermediate Lambda and anti-Lambda denominators, and the intermediate-photon denominator. Together they build the compounded factor

\[
K = \frac{e^6}{(P^2)^2} \left( \frac{1}{(s_1-M^2)^2+M^2\Gamma^2(\sqrt{s})} \right) \left( \frac{1}{(s_2-M^2)^2+M^2\Gamma^2(\sqrt{s})} \right),
\]

with \( s_1 = p_1^2 \) and \( s_2 = p_2^2 \). The hyperon width \( \Gamma(\sqrt{s}) = \Gamma(\sqrt{s}; \Lambda \to \text{all}) \) is the total Lambda (anti-Lambda) decay width, and since it is narrow, \( \Gamma \ll M \), it may be evaluated at \( \sqrt{s} = M \). Furthermore, due to the smallness of the width we may approximate

\[
\frac{1}{(s-M^2)^2+M^2\Gamma^2(\sqrt{s})} \approx \frac{\pi}{MT(M)} \delta(s-M^2).
\]

As a consequence of the above factorization we can write

\[
|M|^2 = K |M_{\text{red}}|^2,
\]

with

\[
K = \frac{(4\pi\alpha)^3}{(P^2)^2} \frac{\pi^2}{M^2\Gamma^2(M)} \delta(s_1-M^2)\delta(s_2-M^2).
\]

Since the intermediate-hyperon states are states whose masses in the narrow-width approximation may be considered fixed, it is useful to rewrite the phase-space expression making this explicit by using the following nesting formula:

\[
d\text{Lips}(k_1+k_2;q,l_1,l_2,q_1,q_2) = \frac{1}{(2\pi)^2} ds_1 ds_2
\]

\[
\times d\text{Lips}(k_1+k_2;q,p_1,p_2)
\]

\[
\times d\text{Lips}(p_1;1,q_1) d\text{Lips}(p_2;l_2,q_2),
\]

with \( p_1^2 = s_1 \) and \( p_2^2 = s_2 \). Multiplication by \( K \) puts the hyperons on their mass shells.

### 4 Lepton tensor

The leptonic four-current is defined as

\[
L_\mu(k_1,k_2,q) = \bar{v}(k_2)\gamma_\mu \left( \frac{k_1-q}{k_1-q} \right) + m_e \bar{u}(k_1)
\]

\[
+ \bar{v}(k_2)\left( \frac{-k_2+q}{k_1-q} \right) + m_e \gamma_\mu u(k_1),
\]

where index \( \mu \) is tied to the lepton-intermediate-photon vertex. For the cross-section distribution we need the corresponding leptonic tensor,

\[
L_{\nu\mu}(k_1,k_2,q) = \frac{1}{4} \sum L_\nu(k_1,k_2,q) L_\mu(k_1,k_2,q),
\]

where the sum runs over initial lepton spins and final photon polarizations. We neglect the electron mass \( m_e \) compared with other masses and energies. Furthermore, the lepton tensor enters the cross-section distribution contracted with the hadron tensor. The hadron tensor is gauge invariant, which means that when contracted with four vectors \( P^\mu \) or \( P^\nu \) zero result is obtained. Hence, dependencies \( P_\mu \) or \( P_\nu \) in the lepton tensor may be ignored. As a consequence, the relevant part of the lepton tensor becomes symmetric in its indices and equal to

\[
L_{\nu\mu} = L_{\mu\nu}
\]

\[
= \frac{1}{y_1 y_2} \left[ -4(s-y_1-y_2)(k_{1\nu}k_{1\mu}+k_{2\nu}k_{2\mu}) 
\right.
\]

\[
- (2s(s-y_1-y_2)+y_1^2+y_2^2)g_{\nu\mu} \right],
\]

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We remark that eq. (13), agrees with that of Czyż.

The definition of the hadronic tensor is

\[ H_{\nu\mu} = \frac{\alpha^2}{4\pi} \rho_{\nu\mu} \]

with

\[ s = (k_1 + k_2)^2, \quad y_1 = -(k_1 - q)^2 + m_c^2 = 2k_1 \cdot q, \]
\[ y_2 = -(k_2 - q)^2 + m_c^2 = 2k_2 \cdot q. \]

We remark that

\[ s - y_1 - y_2 = (k_1 + k_2 - q)^2 = P^2, \]

and \( P = p_1 + p_2 \). Our expression for the lepton tensor, eq. (13), agrees with that of Czyż et al. [3].

### 5 Hadron tensor

The hadronic four-current \( H_{\nu\mu}(p_1, p_2, l_1, l_2) \) describes, in addition to the coupling of the intermediate photon to the hyperons, also their decays. The two parts are coherent. The denominators of the hyperon propagators have already been extracted into the \( K \) factor of eq. (6) so we are left with

\[ H_{\nu\mu} = \bar{u}(l_1)[A + B\gamma_5][\bar{\psi}_2 + M]O_{\nu\mu}(p_1, p_2) \]
\[ (\bar{\psi}_2 - M)[A' + B'\gamma_5]v(l_2), \]

where the Lambda vertex function \( O_{\nu\mu}(p_1, p_2) \) is defined in eq. (2),

\[ O_{\nu\mu}(p_1, p_2) = G_1(P^2)\gamma_\mu - \frac{1}{2M}G_2(P^2)Q_{\nu\mu}. \]

The definition of the hadronic tensor is

\[ H_{\nu\mu} = \sum H_{\nu\mu} \]

with the sum running over the lepton flavor and anti-lepton flavors.

The calculation of the hadronic tensor is simplified by noting that

\[ H_{\nu\mu} = Sp[Y_A\bar{O}_\nu X_A O_{\nu\mu}], \]

with \( \bar{O} = \gamma_0 O^\dagger \gamma_0 \), and

\[ X_A = (\bar{\psi}_2 + M) [R_A - S_A\gamma_5(l_1 \cdot p_1 + M_1l_1)] \]
\[ Y_A = (\bar{\psi}_2 - M) [\bar{R}_A + S_A\gamma_5(l_2 \cdot p_2 - M_2l_2)]. \]

The \( R \) and \( S \) parameters govern the Lambda-hyperon decays and are defined in appendix A. We also note that the scalar products \( l_1 \cdot p_1 = l_2 \cdot p_2 = \) constants.

We decompose the hadron tensor into powers of \( R \) and \( S \), writing

\[ H_{\nu\mu} = \bar{R}_A R_A H_{\nu\mu}^{RR} + \bar{R}_A S_A H_{\nu\mu}^{RS} + \bar{S}_A R_A H_{\nu\mu}^{SR} + \bar{S}_A S_A H_{\nu\mu}^{SS}. \]

The explicit expression for the first partial-hadron tensor is the following:

\[ H_{\nu\mu}^{RR} = 2(p_1 \cdot p_\mu - P^2 y_{\nu\mu} - Q_{\nu\mu} |G_1|^2 \]
\[ + 2Q_{\nu\mu} \left( 2 \Re(G_1^2) - \frac{Q^2}{4M^2} |G_2|^2 \right). \]

The argument of the form factors, which is \( P^2 \), is here omitted, and \( P^2 = 4M^2 - Q^2 \). We also remark that the two contributing terms are separately gauge invariant, i.e., they vanish upon contraction with \( P^\mu \) or \( P^\nu \).

The terms involving spin contributions look like

\[ H_{\nu\mu}^{RS} = -4i|G_1|^2 \left[ l_1 \cdot p_1 \epsilon(p_2 p_1)_{\nu\mu} - M^2 \epsilon(p_1 + p_2, l_1)_{\nu\mu} \right] \]
\[ + 2G_1 G_2 Q_{\nu\mu} \epsilon(p_2 p_1 l_1)_{\nu\mu} \]
\[ - 2G_1 G_2 Q_{\nu\mu} \epsilon(p_2 p_1 l_1)_{\nu\mu}, \]

with \( \epsilon_{0123} = 1 \) and

\[ \epsilon(p_2 p_1 l_1)_{\nu} = \epsilon_{\alpha\beta\gamma\nu} p_2^\alpha p_1^\beta l_1^\gamma, \]
\[ \epsilon(p_2 p_1 l_1)_{\nu} = \epsilon_{\alpha\beta\gamma\nu} p_2^\alpha p_1^\beta l_1^\gamma. \]

Now, we observe that the imaginary part of the tensor \( H_{\nu\mu}^{RS} \) is anti-symmetric in its indices, whereas the real part is symmetric. Since the hadron tensor is to be contracted with a lepton tensor, eq. (13), which is symmetric in its indices, the contribution to the cross-section distribution, effectively, comes only from the real part of \( H_{\nu\mu}^{RS} \). Keeping the symmetry of the lepton tensor in mind, we write

\[ H_{\nu\mu}^{RS} = -4 \Im(G_1 G_2^* Q_{\nu\mu} \epsilon(p_2 p_1 l_1)_{\nu}). \]

The same reasoning leads to the formula

\[ H_{\nu\mu}^{SR} = -4 \Im(G_1 G_2^* Q_{\nu\mu} \epsilon(p_2 p_1 l_1)_{\nu}). \]

Expressions (29) and (30) are related by the substitutions \((G_1, p_1, l_1) = (G_2, p_2, l_2)\).

The tensors discussed so far have a simple form. The double-spin part is rather more complicated so we write it as a sum of several terms,

\[ H_{\nu\mu}^{SS} = |G_1|^2 A_{\nu\mu}^{11} + G_1 G_2 A_{\nu\mu}^{12} + G_1 G_2 A_{\nu\mu}^{22}, \]

with

\[ A_{\nu\mu}^{11} = -2 \left( p_1 \cdot l_1 p_2 \cdot l_2 + M^2 l_1 \cdot l_2 \right) \]
\[ \times \left[ P_\nu P_\mu - P^2 q_{\nu\mu} - Q_{\nu\mu} \right] \]
\[ - 4M^2 \frac{1}{2} P^2 (l_1 l_2)_{\nu\mu} + l_1 l_2 g_{\nu\mu} \]
\[ - P \cdot l_1 (l_2 p_1 p_2 + l_2 p_1 l_1) \]
\[ - P \cdot l_1 (l_2 p_1 p_2 + l_1 p_2), \]

\[ A_{\nu\mu}^{12} = -2M^2 \left[ Q_{\nu\mu} l_1 \cdot l_2 - Q_{\nu\mu} l_1 \cdot l_2 P \cdot l_2 + Q_{\nu\mu} l_2 \cdot P \cdot l_1 \right] \]
\[ + 2Q_{\nu\mu} \left[ p_2 l_1 p_2 l_2 p_1 l_1 - p_2 p_1 l_1 p_2 l_2 \right] \]
\[ - \frac{1}{2} l_1 l_2 P^2 + \frac{1}{2} l_2 p_1 l_1 P^2 \]
\[ = A_{\nu\mu}^{12}, \]

and

\[ A_{\nu\mu}^{22} = -Q_{\nu\mu} \left[ \frac{Q^2}{2M^2} (p_1 \cdot l_1)(p_2 \cdot l_2) - M^2 l_1 \cdot l_2 \right] \]
\[ + Q \cdot l_1 Q \cdot l_2. \]
There are alternative ways of formulating the above expressions. Equation (33) could have been written as

$$A_{\nu\mu}^{11} = -2M^2 [Q_p Q_p l_1 \cdot l_2 - Q_p l_1 Q \cdot l_2 - Q_p l_2 Q \cdot l_1] + 2Q_p [p_{1p} p_{2p} \cdot l_2 p_{2l} p_{1l} \cdot l_1 + p_{2p} p_{1p} \cdot l_2 p_{1l} p_{2l} \cdot l_1] + \frac{1}{2} l_{1p} p_{2l} \cdot l_2 Q^2 - \frac{1}{2} l_{2p} p_{1l} \cdot l_1 Q^2, \quad (35)$$

Now, from the symmetry expressed in eq. (33) it follows that the imaginary part of the hadronic tensor of eq. (31) vanishes. Furthermore, we take into consideration that the hadronic tensor is contracted with a symmetric lepton tensor we may write eq. (31) as

$$H_{\nu\mu}^{11} = |G||^2 A_{\nu\mu}^{11} + 2 \text{Re}(G_1 G_2^*) A_{\nu\mu}^{21} + |G^2||^2 A_{\nu\mu}^{22}. \quad (36)$$

### 6 Cross-section distribution

The next step in the calculation is the contraction of hadronic and leptonic tensors. The reduced cross-section distribution is defined as

$$|\mathcal{M}_{\text{red}}|^2 = L^\mu\nu H_{\mu\nu}, \quad (37)$$

and we decompose the right-hand side as

$$|\mathcal{M}_{\text{red}}|^2 = \bar{R}_A R_A M^{RR} + \bar{R}_A S_A M^{RS} + S_A R_A M^{SS}. \quad (38)$$

From the structure of the lepton tensor of eq. (13), we conclude that each of the $M$ functions has two parts,

$$M = \frac{1}{y_1 y_2} \left[ -4P^2 A - (2s P^2 + y_1^2 + y_2^2)B \right]. \quad (39)$$

The $A$ factor is obtained by contracting the hadron tensor with the symmetric tensor $k_1 k_1 k_1 + k_2 k_2 k_2$, and the $B$ factor by contracting the hadron tensor with the tensor $g_{\mu\nu}$. Remember that terms in the lepton tensor containing $P_\mu$ or $P_\nu$ do not give any contribution due to the gauge invariance of the hadronic tensor.

The leading term of eq. (38) is $M^{RR}$ and it is independent of variables that relate to spin dependences in the hyperon decay distributions. We have

$$A^{RR} = 2|G_1|^2 \left[ (k_1 \cdot P)^2 + (k_2 \cdot P)^2 - (k_1 \cdot Q)^2 - (k_2 \cdot Q)^2 \right] + 4 \text{Re}(G_1 G_2^*) \left[ (k_1 \cdot Q)^2 + (k_2 \cdot Q)^2 \right] - |G|^2 \left( \frac{Q^2}{2M^2} \right) \left[ (k_1 \cdot Q)^2 + (k_2 \cdot Q)^2 \right], \quad (40)$$

and similarly

$$B^{RR} = -4|G_1|^2 (P^2 + 2M^2) + 4 \text{Re}(G_1 G_2^*) Q^2 - |G|^2 \left( \frac{Q^2}{2M^2} \right)^2. \quad (41)$$

Thus, the distribution function $M^{RR}$ does not depend on the decay momenta $l$ and $q$ of the Lambda hyperons.

Next in order are terms linear in the spin variables,

$$A^{RS} = -4\text{Im}(G_1 G_2^*) \left[ (k_1 \cdot Q) \text{det}(p_2 p_1 l_1 k_1) + k_2 \cdot Q \text{det}(p_2 p_1 l_2 k_2) \right], \quad (42)$$

$$A^{SR} = -4\text{Im}(G_1 G_2^*) \left[ (k_1 \cdot Q) \text{det}(p_2 p_1 l_2 k_1) + k_2 \cdot Q \text{det}(p_2 p_1 l_1 k_2) \right], \quad (43)$$

with $\text{det}(abcd) = \epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$ and

$$B^{RS} = 0, \quad (44)$$

$$B^{SR} = 0. \quad (45)$$

The expressions for the spin-spin contributions are more complicated. We have for the $A$ term

$$A^{SS} = -2|G_1|^2 \left[ ((k_1 \cdot P)^2 + (k_2 \cdot P)^2 - (k_1 \cdot Q)^2 - (k_2 \cdot Q)^2 \right] - (k_2 \cdot Q)^2 \right) \times (P_{1l} + P_{1p} l_{1p} l_{1l} + P_{2l} + P_{2p} l_{2p} l_{2l}) + 2M^2 \left( (k_1 \cdot Q)^2 + (k_2 \cdot Q)^2 \right) - M^2 \left( P \cdot l_2 (k_1 \cdot Q k_1 \cdot l_1 + k_2 \cdot Q k_2 \cdot l_1) - P \cdot l_1 (k_1 \cdot Q k_2 \cdot l_2 + k_2 \cdot Q k_2 \cdot l_1) \right) - k_2 \cdot Q \left( k_2 \cdot p_{1p} l_{1l} p_{2l} - k_2 \cdot p_{2p} l_{1p} l_{1l} + \frac{1}{2} P^2 k_1 \cdot l_{1p} l_{1l} + \frac{1}{2} P^2 k_1 \cdot l_{1p} l_{1l} \right) - k_2 \cdot Q \left( k_2 \cdot p_{1p} l_{1p} l_{1l} - k_2 \cdot p_{2p} l_{1p} l_{1l} + \frac{1}{2} P^2 k_2 \cdot l_{1p} l_{1l} + \frac{1}{2} P^2 k_2 \cdot l_{1p} l_{1l} \right) - |G|^2 \left( \frac{Q^2}{2M^2} \right) \left[ (k_1 \cdot Q)^2 + (k_2 \cdot Q)^2 \right] \times \left( Q^2 \cdot l_{1p} l_{1l} - M^2 l_{1l} \right) + 2M^2 Q \cdot l_1 Q \cdot l_1, \quad (46)$$

and for the $B$ term

$$B^{SS} = +4|G_1|^2 \left[ (P^2 + 2M^2) (P_{1l} + l_{1p} l_{1l} + P_{2l} + l_{2p} l_{2l}) - M^2 (P_{2l} l_{2l} + 2P \cdot l_{1l} p_{1l}) - 4 \text{Re}(G_1 G_2^*) \left[ Q^2 M^2 l_{1l} \cdot l_2 - M^2 (Q \cdot l_1 P \cdot l_2 - Q \cdot l_2 P \cdot l_1) - (P_{1l} + P_{1p} l_{1l} + P_{2l} + P_{2p} l_{2l}) \right) \right] - |G|^2 \left( \frac{Q^2}{2M^2} \right) \left[ Q^2 \cdot l_{1p} l_{1l} - M^2 l_{1l} \right) + 2M^2 Q \cdot l_1 Q \cdot l_1. \quad (47)$$
7 Discussion

The distributions presented so far refer to distributions in the momenta of the hyperon-decay products. It might be of interest to integrate, say over the proton and pion momenta of the Lambda hyperon. To do this we need to perform the integral over $l_{1\mu}$. This is done with recourse to the formula

$$l_{\mu} \, dlips(p; l, q) = \frac{p \cdot l}{M^2} \, p_{\mu} \, dlips(p; l, q), \quad (48)$$

where $p^2 = M^2$, and $p \cdot l$ constant. Thus, the effect of the integration is equivalent to making the substitution

$$l_{\mu} \rightarrow \left[ \frac{M^2 + m^2 - \mu^2}{2M^2} \right] p_{\mu}. \quad (49)$$

The phase-space volume is $l_A/4\pi M$, with $l_A$ the decay momentum in the Lambda rest system.

The functions $A^{RR}$ and $B^{RR}$ of eqs. (40) and (41) do not depend on the variable $l_1$, but $A^{RS}$ of eq. (42) does. Evidently, making the substitution (49) one obtains

$$A^{RS}(l_1 \rightarrow p_1) = 0. \quad (50)$$

The other terms that depend on $l_1$ are $A^{SS}$ of eq. (46) and $B^{SS}$ of eq. (47). Similarly, also here a substitution gives

$$A^{SS}(l_1 \rightarrow p_1) = 0, \quad (51)$$

$$B^{SS}(l_1 \rightarrow p_1) = 0. \quad (52)$$

This result is important since it shows that the lifetime of the Lambda hyperon does not depend on the decay-parameter $S_A$, and hence is independent of the production mechanism.

Integration over the decay distributions of both hyperons gives

$$d\sigma(e^+e^- \rightarrow \gamma A\bar{A}) = \frac{(4\pi\alpha)^3}{28} \frac{1}{(P^2)^2} \frac{\Gamma_A^2}{T_2} \frac{M^{RR}}{T_2} \times dlips(k_1 + k_2; q, p_1, p_2), \quad (53)$$

with widths $\Gamma = \Gamma(A \rightarrow all)$ and $\Gamma_A = \Gamma(A \rightarrow p\pi^-)$. Consequently, the last factor describes the branching-ratio product of the hyperon-decay channels. Since $P$ is the four-momentum of the virtual photon, the second factor originates with its propagator. Finally, the algebraic expression for the cross-section distribution function is

$$M^{RR} = \frac{1}{y_1 y_2} \left[ -4P^2 A^{RR} - (2sP^2 + y_1^2 + y_2^2) B^{RR} \right], \quad (54)$$

with $A^{RR}$ and $B^{RR}$ as in eqs. (40) and (41). The defining expression for $M^{RR}$ is given in eq. (21) as

$$M^{RR} = L^{\mu \nu} \text{Sp}[(\gamma_2 - M)\tilde{O}_\nu(\gamma_1 + M)O_\mu], \quad (55)$$

supporting the interpretation of eq. (53) as the cross-section distribution for $A\bar{A}$ production, with summation over final hyperon-spin states.

8 Folding method

We shall now demonstrate that the folding method used in refs. [2] and [3] for calculating cross-section distributions can give the same result as the present, conventional method. To this end we need some properties of the Lambda–four-spin vector $s(p, n)$ of appendix A,

$$s(p, n) = \left( \frac{\bf{n} \cdot \bf{p}}{M}, \frac{\bf{E} \cdot \bf{n}}{M} \hat{\bf{p}} + \bf{n} - \hat{\bf{p}}(\bf{n} \cdot \hat{\bf{p}}) \right), \quad (56)$$

where the three-vector $\bf{n}$ identifies the quantization direction of the spin in the Lambda rest system. For each $\bf{n}$ there are two spin states, represented by $s(p, n)$ and $-s(p, n)$.

We assume all quantization directions $\bf{n}$ equally likely and define averages such that

$$\langle 1 \rangle = 1, \quad \langle n^k \rangle = 0, \quad \langle n^k n^l \rangle = \delta^{kl}. \quad (57)$$

It is not difficult to show that these relations imply both $\langle s^a(p, n) \rangle = 0$, and

$$\langle s^a(p, n) s^b(p, n) \rangle = \frac{1}{M^2} \delta^{\mu \nu} p^\mu - g^{\mu \nu}, \quad (58)$$

conditions which are explicitly covariant.

In the present investigation cross-section distributions are obtained by squaring the sum of the two matrix elements corresponding to the diagrams of fig. 1, i.e., by calculating

$$|\mathcal{M}(e^+e^- \rightarrow \gamma A(\rightarrow p\pi^-)\bar{A}(\rightarrow \bar{p}\pi^+))|^2. \quad (59)$$

The matrix element of a diagram is a product of a hyperon-production step and subsequent hyperon-decay steps, with sums over the intermediate hyperon-spin states. This is embodied in the hadron tensor of eq. (21).

In the folding method of refs. [2] and [3] one first calculates cross-section and decay distributions for given hyperon spins, and then averages their product over spin-quantization directions according to eq. (58). Thus, the prescription is to form

$$\left\langle |\mathcal{M}(e^+e^- \rightarrow \gamma A_n\bar{A}_{n'})|^2 \times |\mathcal{M}(A_n \rightarrow p\pi^-)|^2 \, |\mathcal{M}(\bar{A}_{n'} \rightarrow \bar{p}\pi^+)|^2 \right\rangle_{nn'}. \quad (60)$$
In addition, we should multiply by a factor of four, since for each quantization direction there are two spin possibilities, spin up and spin down. Details of the calculation are given in appendix B.

The squared matrix element for Lambda decay when summed over final proton spin states is, as in appendix A,

$$|\mathcal{M}(A_n \rightarrow p\pi^-)|^2 = R_A + M S_A l_1 \cdot s_1, \quad (61)$$

and the squared matrix element for hyperon production $$|\mathcal{M}(e^+e^- \rightarrow \gamma A_n\Lambda_n)|^2$$ contains the projector

$$u(p_1, s_1)\bar{u}(p_1, s_1) = (\bar{p}_1 + M) \frac{1}{2}(1 + \gamma_5 \slashed{s}_1), \quad (62)$$

with $$s_1 = s(p_1, n)$$. Multiplying the product of these two expressions by the factor of two, for the two spin possibilities, and taking the average according to eq. (58), it follows that

$$2 \left( (\bar{p}_1 + M) \frac{1}{2}(1 + \gamma_5 \slashed{s}_1) [R_A + M S_A l_1 \cdot s_1] \right) = (\bar{p}_1 + M) [R_A - S_A \gamma_5 (p_1 \cdot l_1 + M l_1)] . \quad (63)$$

This result is immediately recognized as the Lambda-hyperon factor $$X_\Lambda$$ of eq. (22) in the trace form of the hadronic tensor, eq. (21).

For the anti-Lambda hyperon the projector is

$$v(p_2, s_2)\bar{v}(p_2, s_2) = (\bar{p}_2 - M) \frac{1}{2}(1 + \gamma_5 \slashed{s}_2), \quad (64)$$

and combined with the anti-Lambda-decay distribution

$$\bar{R}_A + M S_A l_2 \cdot s_2, \quad (65)$$

it leads to the average

$$2 \left( (\bar{p}_2 - M) \frac{1}{2}(1 + \gamma_5 \slashed{s}_2) [\bar{R}_A + M S_A l_2 \cdot s_2] \right) = (\bar{p}_2 - M) [\bar{R}_A - \bar{S}_A \gamma_5 (p_2 \cdot l_2 - M l_2)] , \quad (66)$$

a result identical to the $$Y_A$$ factor of eq. (23) which describes the anti-Lambda-hyperon factor of the hadronic tensor, eq. (21).

We conclude that the folding method as described here leads to the same result as a conventional evaluation of Feynman diagrams, provided the number of spin states is correctly counted.

In the conventional calculation there are correlations between the hyperon-decay products already in the matrix element. In the cross-section distribution, e.g., this is manifested by the term $$l_1 \cdot l_2$$. Moreover, the matrix element involves a sum over intermediate hyperon polarizations, but once we have chosen the spin-quantization direction, there are only two contributions, spin up and spin down, as is clear from the decomposition

$$\bar{p} + M = \sum_{s = \pm} u(p, s)\bar{u}(p, s)$$

$$= (\bar{p} + M) \frac{1}{2}(1 + \gamma_5 \slashed{s}) + (\bar{p} + M) \frac{1}{2}(1 - \gamma_5 \slashed{s}), \quad (67)$$

where $$s = s(p, n)$$ and $$n$$ an arbitrary but fixed direction. The sum is independent of the spin-quantization direction.

In the folding calculation of eq. (60) one starts with a product of distribution functions for fixed quantization directions, $$n$$ and $$n'$$. As a consequence, the cross-section-distribution function factorizes into a product of distribution functions. This implies vanishing correlation between the decay products of the two hyperons. However, taking the average of a product distribution over the quantization directions $$n$$ and $$n'$$ does not necessarily yield a product distribution. Instead correlations between the various factors are created, in such a way as to reproduce the correct result.

9 Model form factors

We have derived expressions for cross-section distributions and polarizations in the hyperon-production reaction $$e^+e^- \rightarrow \gamma A \rightarrow p\pi^- \Lambda$$ (eq. 21). The two methods have been used; a direct evaluation of the Feynman diagrams, and an application of our folding method. The two methods give the same result. The folding method is similar to the calculational methods employed in refs. [2] and [3].

The BABAR Collaboration [1, 6] has employed initial-state radiation in $$e^+e^-$$ annihilation as a tool for mapping out time-like hyperon form factors. For the Lambda hyperon the region from threshold to $$M_{AA} = 3 \text{GeV}/c^2$$ has been covered. Results are consistent with the ratio $$|G_E/G_M| \approx 1$$. The Lambda polarization has also been measured.

Other experiments have investigated the Lambda form factors in direct $$e^+e^-$$ annihilation into hyperon pairs [7, 8], experiments which however yield form factors at a single time-like momentum, corresponding to the c.m. energy in the reaction.

We have not investigated the structure of the hyperon form factors themselves. A model based on generalized vector-meson dominance and Regge recurrences of the vector mesons has been proposed by Körner and Kuroda [9], The model has no free parameters. The DM2 Collaboration [8] claims agreement with this model, although based on poor statistics. The experiment using the CLEO-c detector [7] claims disagreement.

A modified vector-meson dominance model is also described in ref. [10], and a QCD-based asymptotic expression for the form factors in ref. [11].

I thank Bengt Karlsson, Stefan Leupold, and Karin Schöning for discussions.

Appendix A. Lambda-decay parameters

The matrix element for $$A \rightarrow p\pi^-$$ decay is commonly written as

$$\mathcal{M}(A \rightarrow p\pi^-) = \bar{u}_p(l)[A + B\gamma_5]u_A(p). \quad (A.1)$$
For a Lambda hyperon, of polarization $+\mathbf{n}$ in its rest system, the square of this matrix element, after summation over final-state-proton polarizations, becomes

$$\sum |\mathcal{M}|^2 = \text{Sp}[(A^* - B^*\gamma_5)(I + m)|A + B\gamma_5(\not{p} + M)\frac{1}{2}(1 + \gamma_5\not{\nu})|].$$ (A.2)

The spin four-vector $s = s(p, n)$ satisfies $s \cdot s = -1$ and $s \cdot p = 0$, where $p$ is the hyperon four-momentum. The mass of the hyperon is $M$ and that of the proton $m$. The spin vector for polarization $-\mathbf{n}$ is $-s(p, n)$.

In the rest system of the Lambda $s(p, n) = (0, \mathbf{n})$ and $p = (M, 0)$. In a coordinate system where the Lambda has three-momentum $\mathbf{p}$, the spin vector is

$$s(p, n) = \frac{n_\parallel}{M}(\mathbf{p}, E\mathbf{p}) + (0, n_\perp),$$ (A.3)

with $n_\parallel = \mathbf{n} \cdot \mathbf{p}$ and

$$n_\perp = \mathbf{n} - \mathbf{p}(\mathbf{n} \cdot \mathbf{p}).$$ (A.4)

In the first part of expression (A.3) we notice the helicity vector $h(p) = (|\mathbf{p}|, E\mathbf{p})/M$.

Evaluation of the trace gives the distribution function

$$\sum |\mathcal{M}|^2 = R_A + MS_A l \cdot s,$$ (A.5)

where

$$R_A = |A|^2((M + m)^2 - \mu^2) + |B|^2((M - m)^2 - \mu^2),$$ (A.6)

$$S_A = 4\text{Re}(A^*B),$$ (A.7)

and $\mu$ the pion mass. In the rest system of the Lambda the decay distribution is

$$\sum |\mathcal{M}|^2 = R_A(1 + \alpha_A \hat{A} \cdot \mathbf{n}),$$ (A.8)

$$\alpha_A = -\frac{l_A MS_A}{R_A},$$ (A.9)

$$l_A = \frac{1}{2M} \left[\frac{(M + m)^2 - \mu^2}{(M - m)^2 - \mu^2}\right]^{1/2},$$ (A.10)

with $l_A$ the decay momentum in the Lambda rest system. For unpolarized decay we average over the two spin vectors $s(p, n)$ and $-s(p, n)$, and get

$$|\mathcal{M}|^2_{\text{unpol}} = R_A.$$ (A.11)

The decay width is

$$\Gamma_A(A \to p\pi^-) = \frac{l_A}{8\pi M^2}R_A,$$ (A.12)

with $l_A$ as in eq. (A.10).

The matrix element for the charge conjugate decay, $\bar{A} \to \bar{p}\pi^+$, is

$$\mathcal{M}(\bar{A} \to \bar{p}\pi^+) = \bar{v}_A(p)[A' + B'\gamma_5]\nu_\mu(l),$$ (A.13)

and the corresponding decay-distribution function

$$\sum |\mathcal{M}|^2 = \bar{R}_A + MS_{\bar{A}} l \cdot s,$$ (A.14)

where

$$\bar{R}_A = |A'|^2((M + m)^2 - \mu^2) + |B'|^2((M - m)^2 - \mu^2),$$ (A.15)

$$\bar{S}_A = 4\text{Re}(A'^*B').$$ (A.16)

For unpolarized decay of anti-Lambda, $|\mathcal{M}|^2_{\text{unpol}} = \bar{R}_A$.

### Appendix B. Hadronic folding tensor

In order to facilitate comparison with Czyż et al. [3], we have calculated the hadronic tensor $K_{\nu\mu}(s_1, s_2)$ for the reaction $e^+e^- \to \gamma\Lambda\bar{A}$. Here, the spin vector for the final state Lambda is denoted $s_1$ and for the anti-Lambda $s_2$, both referring to spin up. The hadronic tensor is defined by

$$K_{\nu\mu}(s_1, s_2) = \text{Sp}\left[\hat{O}_\nu(\not{p}^1_1 + M)\frac{1}{2}(1 + \gamma_5\not{\nu}_1) \times O_\mu(\not{p}^2_2 - M)\frac{1}{2}(1 + \gamma_5\not{\nu}_2)\right].$$ (B.1)

with the matrix $O_{\mu}$ is in eq. (19). This hadronic tensor is gauge invariant, and can be decomposed as

$$K_{\nu\mu}(s_1, s_2) = K_{\nu\mu}^{00}(0, 0) + K_{\nu\mu}^{05}(s_1, 0) + K_{\nu\mu}^{50}(0, s_2) + K_{\nu\mu}^{55}(s_1, s_2).$$ (B.2)

The functional arguments indicate the spin vectors involved.

The first term on the right hand side has been calculated before, and

$$K_{\nu\mu}^{00}(0, 0) = \frac{1}{4}H_{\nu\mu}^{RR},$$ (B.3)

with $H_{\nu\mu}^{RR}$ defined in eq. (25). Since the leptonic tensor is symmetric in its indices we need only retain the symmetric part of the hadronic tensor. It follows that

$$K_{\nu\mu}^{05}(s_1, 0) = -\frac{1}{2M}\text{Im}(G_1G_2^*\mu)[Q_{\nu\mu}(p_1, p_2, s_1)_\mu + Q_{\nu\mu}(p_1, p_2, s_1)_\nu],$$ (B.4)

$$K_{\nu\mu}^{50}(0, s_2) = -\frac{1}{2M}\text{Im}(G_1G_2^*\mu)[Q_{\nu\mu}(p_1, p_2, s_2)_\mu + Q_{\nu\mu}(p_1, p_2, s_2)_\nu],$$ (B.5)

with the epsilon function of eq. (27).

The contribution depending on both spin vectors is

$$K_{\nu\mu}^{55}(s_1, s_2) = |G_1|^2B_{\nu\mu}^{s_1} + |G_2|^2B_{\nu\mu}^{s_2},$$ (B.6)
with

\[
B^1_{\nu\mu} = -s_1 \cdot s_2 \left( p_{1\nu} p_{2\mu} + p_{1\mu} p_{2\nu} - \frac{1}{2} g_{\nu\mu} p^2 \right) - \frac{1}{2} p^2 (s_{1\nu} s_{2\mu} + s_{1\mu} s_{2\nu}) - g_{\nu\mu} p_1 \cdot s_2 p_2 \cdot s_1 + p_1 \cdot s_1 (s_{1\nu} p_{2\mu} + s_{1\mu} p_{2\nu}) + p_2 \cdot s_1 (s_{2\nu} p_{1\mu} + s_{2\mu} p_{1\nu}).
\]

\[
B^2_{\nu\mu} = \frac{1}{4 M^2} Q_\nu Q_\mu \left( \frac{1}{2} [s_1^2 + s_1 p_1 s_2 + p_1 \cdot s_2 p_2 \cdot s_1] \right).
\]

\[
B^3_{\nu\mu} = -Q_\nu Q_\mu s_1 \cdot s_2 + \frac{1}{2} [s_1 (Q_\nu s_{1\mu} + Q_\mu s_{1\nu}) - p_2 \cdot s_1 (Q_\nu s_{2\mu} + Q_\mu s_{2\nu})].
\]

This hadronic tensor corresponds to the one of Czyż et al., eq. (7) of ref. [3], but written on a covariant form.

Next, we fold the \( \Lambda \Lambda \) tensor with the decay distributions of the hyperons, i.e., we first multiply

\[
4 [R_A + M S_A l_1 \cdot s_1] [R_A + M S_A l_2 \cdot s_2]
\]

and then average over spin decays according to eq. (58). Remember that the factor of four is needed to properly include the total number of spin states. The directional average involves

\[
\langle s_1 (l_1 \cdot s_1) \rangle = \frac{p_1 \cdot l_1}{M^2} p_1 - l_1.
\]

Thus, the hadronic tensor \( K_{\nu\mu}(s_1, s_2) \) of eq. (B.1) is replaced by the folded tensor

\[
H_{\nu\mu} = 4 \left[ R_A R_A K_{\nu\mu}^{00}(0,0) - M R_A S_A K_{\nu\mu}^{05}(l_1,0) - M S_A R_A K_{\nu\mu}^{50}(0,l_2) + M^2 S_A S_A \left\{ K_{\nu\mu}^{55}(l_1,l_2) - \frac{p_1 \cdot l_1}{M^2} K_{\nu\mu}^{55}(l_1,l_2) - \frac{p_2 \cdot l_2}{M^2} K_{\nu\mu}^{55}(l_1,l_2) + \frac{p_1 \cdot l_1 p_2 \cdot l_2}{M^2} K_{\nu\mu}^{55}(l_1,l_2) \right\} \right],
\]

which agrees, term by term, with the tensor of eq. (24). For the calculation remember that \( B_{\nu\mu}^0(p_1, p_2) = 0 \). This is the form of the hadronic tensor which is most suitable for comparison with Czyż et al. [3].

**Appendix C. Polarization**

Suppose we integrate over the anti-\( \Lambda \Lambda \) decay distribution. Then the reduced cross-section distribution of eq. (38) is reduced to

\[
|\mathcal{M}_{\text{red}}|^2 = \bar{R}_A [R_A M_{RR} + S_A M_{RS}] .
\]

The function \( M_{RS} \) has only a contribution from \( A_{RS} \) of eq. (42), which has the form of a four-spin vector \( S_1 \) contracted with the proton four-momentum vector \( l_1 \). We define \( S_1 \) from

\[
k_1 \cdot Q \epsilon \langle p_2 p_1 l_1 k_1 \rangle + k_2 \cdot Q \epsilon \langle p_2 p_1 l_2 k_2 \rangle = S_1 \cdot l_1 ,
\]

From \( p_1 \cdot S_1 = 0 \) it follows that \( S_1 \) is a typical spin vector, except for the fact it is not properly normalized. This is easily arranged for, by defining

\[
s_{\nu\mu}(p_1) = \frac{1}{D} S_{\nu\mu} = \frac{1}{D} \left[ k_1 \cdot Q \epsilon \langle p_1 p_2 k_1 \rangle + k_2 \cdot Q \epsilon \langle p_1 p_2 k_2 \rangle \right],
\]

with \( S_1 \cdot S_1 = -D^2 \), and

\[
D^2 = (k_1 \cdot Q)^2 \left[ \begin{array}{c} p_1 \cdot p_1 + p_2 \cdot p_1 + \frac{k_1 \cdot k_1}{D} \\ p_2 \cdot p_2 + p_1 \cdot p_2 + \frac{k_2 \cdot k_2}{D} \end{array} \right] .
\]

We can now summarize the Lambda decay distribution as

\[
\left| \mathcal{M}_{\text{red}} \right|^2 = \bar{R}_A R_A M_{RR} \left[ 1 - P_A A_{\Lambda l_1 \cdot s_1 / l_\Lambda} \right] ,
\]

with the Lambda polarization

\[
P_A = \frac{16 \text{Im}(G_1 G_2^*) P^2 D / M}{4 P^2 A_{RR} + (2 P F_2 + y_1^2 + y_2^2) B_{RR}^2} ,
\]

and \( l_\Lambda \) the decay momentum in the Lambda rest system, eq. (A.10). If we go to the Lorentz system where the Lambda hyperon is at rest, \( p_1 = (M, 0, 0, 0) \), then the normalized spin vector reads

\[
s_{\nu\mu}(p_1) = -\frac{M}{D} \left[ k_1 \cdot Q p_2 \times k_1 + k_2 \cdot Q p_2 \times k_2 \right] .
\]

From this expression we easily derive alternative expressions for the determinants of eq. (C.4).

**References**

1. BABAR Collaboration (B. Aubert et al.), Phys. Rev. D 76, 092006 (2007).
2. L.V. Kardapoltsev, Bachelor's thesis, Novosibirsk State University, 2007, unpublished.
3. H. Czyż, A. Grzeliliska, J.H. Kühn, Phys. Rev. D 75, 074026 (2007).
4. O.D. Dalkarov, P.A. Khakhulin, A.Yu. Voronin, Nucl. Phys. A 833, 104 (2010).
5. H. Pilkuhn, Relativistic Particle Physics (Springer-Verlag, Berlin, 1979).
6. F. Anulli, Nucl. Phys. B (Proc. Suppl.) 225-227, 205 (2012).
7. S. Dobbs, A. Tomaradze, T. Xiao, Kamal K. Seth, G. Bonvicini, Phys. Lett. B 739, 90 (2014).
8. DM2 Collaboration (D. Bisello et al.), Z. Phys. C 48, 23 (1990).
9. J.G. Körner, M. Kuroda, Phys. Rev. D 16, 2165 (1977).
10. A.Z. Dubnicková, S. Dubnicka, M.E. Biagini, A. Castro, Czech. J. Phys. 41, 1177 (1991).
11. V.L. Chernyak, A.R. Zhitnitskii, V.G. Serbo, JETP Lett. 26, 594 (1977).