GEOMETRIC STRUCTURES ON ORBIFOLDS AND HOLonomy REPRESENTATIONS

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ABSTRACT. An orbifold is a topological space modeled on quotient spaces of a finite group actions. We can define the universal cover of an orbifold and the fundamental group as the deck transformation group. Let $G$ be a Lie group acting on a space $X$. We show that the space of isotopy-equivalence classes of $(G, X)$-structures on a compact orbifold $\Sigma$ is locally homeomorphic to the space of representations of the orbifold fundamental group of $\Sigma$ to $G$ following the work of Thurston, Morgan, and Lok. This implies that the deformation space of $(G, X)$-structures on $\Sigma$ is locally homeomorphic to the space of representations of the orbifold fundamental group to $G$ when restricted to the region of proper conjugation action by $G$.

1. Introduction

An orbifold is a topological space with neighborhoods modeled on the orbit-spaces of finite group actions on open subsets of euclidean spaces. Often orbifolds arise as quotient spaces of manifolds by proper actions of discrete groups. They are so-called good orbifolds. The manifold itself could be chosen to be a simply connected one. In this case, the manifold is said to be the universal covering space of the orbifold and the deck transformation group the (orbifold) fundamental group of the orbifold.

For example, the quotient spaces of the hyperbolic spaces by discrete subgroups of isometries are orbifolds (especially, when the group has torsion elements). Very good orbifolds are orbifolds which are quotient spaces of manifolds by finite group actions. The good orbifolds are as good as manifolds since they admit universal covering space as manifolds.

Looking at quotients of manifolds by group actions as orbifolds sometimes gives us useful methods such as decomposition or putting geometric structures by cut-and-paste methods. This is one of the reasons why we study orbifolds instead of just manifolds.

Let $G$ be a Lie group acting on a space $X$ transitively and effectively. Then $(G, X)$ is said to be a geometry. A $(G, X)$-structure on an orbifold $M$ is given by a maximal atlas of charts to orbit spaces of finite subgroups of $G$ acting on open subsets of $X$. A $(G, X)$-structure on an orbifold implies that the orbifold is good.
as first observed by Thurston. This generalizes the notion of \((G, X)\)-structures on manifolds introduced by Ehresmann.

We will define in this paper, the space \(S(\Sigma)\) of isotopy-equivalence classes of \((G, X)\)-structures on a given orbifold \(\Sigma\). Given a \((G, X)\)-structure on \(\Sigma\), we can define an immersion \(D\) from its universal cover to \(X\) and a homomorphism \(h: \pi_1(\Sigma) \rightarrow G\) for the orbifold fundamental group \(\pi_1(\Sigma)\) of \(\Sigma\). \(D\) is said to be a developing map, and \(h\) a holonomy homomorphism. \((D, h)\) determines the \((G, X)\)-structure but given a \((G, X)\)-structure \((D, h)\) is determined up to the following action

\[
(D, h(\cdot)) \mapsto (\vartheta \circ D, \vartheta \circ h(\cdot) \circ \vartheta^{-1})
\]
for \(\vartheta \in G\). The so-called development pair \((D, h)\) is essentially defined by an analytic continuation of charts as in the manifold cases. (See Goldman \[9\] for more details on \((G, X)\)-structures on manifolds.) The space \(S(\Sigma)\) can be considered as the space of equivalence classes of development pairs of \((G, X)\)-structures on \(\Sigma\) under the isotopy action of \(\Sigma\) commuting with the deck-transformation group. The deformation space \(D(\Sigma)\) of \((G, X)\)-structures on \(\Sigma\) is obtained from \(S(\Sigma)\) as a quotient space by the above action of \(G\).

We need to assume that \(\Sigma\) is compact and \(\pi_1(\Sigma)\) is finitely-presented. One can define a map, so-called pre-holonomy map,

\[\mathcal{PH}: S(\Sigma) \rightarrow \text{Hom}(\pi_1(\Sigma), G)\]

induced by an isotopy-invariant function assigning a \((G, X)\)-structure with a developing map \(D\) to its holonomy homomorphism associated with \(D\). \(\text{Hom}(\pi_1(\Sigma), G)\) is naturally a real algebraic subset of \(G^n\) where \(n\) is the number of generators of \(\pi_1(\Sigma)\) defined by relations and hence is a topological space.

**Theorem 1.** Let \((X, G)\) be a geometry, and \(\Sigma\) be a compact orbifold with a finitely-presented orbifold fundamental group \(\pi_1(\Sigma)\). Then

\[\mathcal{PH}: S(\Sigma) \rightarrow \text{Hom}(\pi_1(\Sigma), G)\]

is a local homeomorphism.

If the orbifold is given an additional “cellular structure” in some sense, then the compactness should imply that the fundamental group is finitely generated. (However, we do not prove it here.)

The proof of the above theorem for manifolds was first given by Thurston (perhaps much earlier by Ehresmann), again by Canary-Epstein-Green \[3\], and simultaneously by Lok (following Morgan). Our proof generalizes that in the manifold case by Lok \[13\] following Morgan’s lectures (see Weil \[24\] and Canary-Epstein-Green \[3\] also). (We mention that this can be also done using Goldman’s idea in \[9\].) (There are related works by Kapovich \[12\] and Gallo-Kapovich-Marden \[8\] where some results were proved for 2-orbifolds partially.)

Let us denote by \(\text{Hom}(\pi_1(\Sigma), G)^*\) the part where the conjugation action of \(G\) given by

\[h(\cdot) \mapsto gh(\cdot)g^{-1}, \quad g \in G\]

is stable (see \[14\]). Let \(D^*(\Sigma)\) the inverse image of the above set by \(\mathcal{PH}\). (We assume here that \(G\) is a group of \(\mathbb{R}\)-points of an algebraic group defined over the real number fields.)
Corollary 1. Let $\Sigma$ be a compact orbifold with a finitely-presented orbifold fundamental group $\pi_1(\Sigma)$. Then the map

$$\mathcal{H} : \mathcal{D}^*(\Sigma)/G \to \text{Hom}(\pi_1(\Sigma), G^*)/G$$

induced by $\mathcal{P}\mathcal{H}|\mathcal{D}^*(\Sigma)$ is a local homeomorphism to its image.

This result will be used in Choi-Goldman [5], which is the main reason why we wrote this paper.

This paper intentionally is technical and gives many details since excellent intuitive writings on the subjects are already available in Chapter 5 of Thurston’s notes [23] and the paper by Scott [22]. (See also Kato [13] and Matsumoto and Montesinos-Amilibia [16].) Although this material has already been exposed well, we believe that it should be a glad duty of mathematical community to continuously reinterpret old ideas and make precise and refine what initial attempts to understand have left to the next generation. Hopefully, this writing will convey the idea of orbifolds to readers in a more rigorous way.

In fact some of the preliminary materials in this paper have been also exposed in Haefliger [10], Bridson-Haefliger [1], and Ratcliffe [19] but they study only orbifolds with geometric structures where $G$ acts as an isometry group on a space $X$. For our purposes with geometric structures without metrics, their results are not enough and hence the need for writing them down arose. For geometric structures such as projectively flat and conformally flat structures on $n$-dimensional orbifolds, the results in this paper are new except for the work of Kapovich described above.

In Section two, we introduce orbifolds, orbifold-maps, isotopies of orbifold-maps, and so on. We also explain the Riemannian metric on orbifolds and coverings by normal neighborhoods.

In Section three, we first review fiber-products of topological covering spaces. We discuss covering orbifolds of orbifolds, and discuss its simple properties. We define fiber-products of orbifold-covering maps by first doing so for orbifold-coverings of elementary neighborhoods, and then extending the definitions to any collection of orbifold-coverings. We prove a theorem of Thurston that there exists a so-called universal covering orbifold for any orbifold by fiber-products. We provide a proof of the fact that the deck transformation group acts transitively on the universal covering orbifolds. From these results, we obtain most properties of orbifold-coverings similar to topological coverings. Finally, we show that a good orbifold has a manifold as the universal covering orbifold. The author faithfully follows and gives selective details of Chapter 5 of Thurston [23]. (In fact, this material should be published by Thurston in his next book. This part has grown unintentionally large and is only there to provide a technical background to this paper.)

In Section four, we discuss the geometric structures on orbifolds. We show that orbifolds with geometric structures are good, and find the developing maps and the holonomy homomorphisms for orbifolds. We define the deformation space of $(G, X)$-structures on an orbifold, which is the space of equivalence classes of $(G, X)$-structures under isotopies and $(G, X)$-diffeomorphisms. The so-called isotopy-equivalence space of $(G, X)$-structures on an orbifold $\Sigma$ is defined to be the space of equivalence classes of a pair $(D, f)$ where $D$ is a developing map for a $(G, X)$-orbifold $M$, and $f$ is a lift of an orbifold-diffeomorphism defined on the universal cover of $\Sigma$. The equivalence relation is given by an isotopy action on $f$. We show
that $G$ acts on the isotopy-equivalence space so that the quotient space is the deformation space of $(G, X)$-structures here. We define a pre-holonomy map from the isotopy-equivalence space of $(G, X)$-structures on an orbifold $M$ to the space of representations $\text{Hom}(\pi_1(\Sigma), G)$ given by sending $(D, f)$ to the holonomy homomorphism composed with the isomorphism $\pi_1(\Sigma) \to \pi_1(M)$ induced by $f$. Here $\pi_1(\Sigma)$ denotes the deck transformation group of $\Sigma$.

In Section five, we prove Theorem 1 that the space of isotopy classes of $(G, X)$-structures on an orbifold is locally homeomorphic to the space of representations of the fundamental group to $G$ by the pre-holonomy map. The proof is essentially the same as that of Morgan and Lok [15] but we modify slightly for clarity and completeness. The basic idea is to deform small neighborhoods first and patch them together using “bump” functions as we change the representation by a small amount in a cone-neighborhood of the representation space as described in Canary-Epstein-Green [3].

The local finite group actions complicate the proof somewhat but not greatly if we use the old ideas of Palais-Stewart [18] which yield three necessary lemmas on conjugating finite group actions by diffeomorphisms in the beginning of Section five. For technical purposes, we explain the Riemannian metric structures on orbifolds here. The proof of Theorem 1 is as follows:

(I) We choose three model-neighborhood coverings $\{U_i\}$, $\{W_i\}$, and $\{V_i\}$ of the orbifold so that they are nested, i.e.,

$$\text{Cl}(U_i) \subset V_i, \text{Cl}(V_i) \subset W_i$$

for each $i$. Then we lift each open set to a connected open set in the universal cover and choose deck transformations so that patching the lifted sets by a selection of deck transformations gives us back the orbifold.

(II) We show that there is a local section of the pre-holonomy map: we show that as we deform the holonomy representations, we can deform the model neighborhoods by conjugating with respect to finite group action deformations. We patch the deformations together to form a deformation of $M$.

(III) We finally show that the pre-holonomy map is a local homeomorphism; that is, if the holonomy homomorphisms of two $(G, X)$-structures are equal and they are close with one another, then we will find a $(G, X)$-orbifold-diffeomorphism between them.

We would like to thank Yves Benoist, William Goldman, Karsten Grove, Silvio Levi, Misha Kapovich, Hyuk Kim, Inkang Kim, John Millson, and Shmuel Weinberger for their helpful comments and encouragements. We especially thank Bill Thurston for discovering the ideas in this paper. We thank greatly the referee for his detailed comments to polish this paper up.

2. Topology of orbifolds

In this paper, we assume that the action of a group on a topological space is locally faithful; that is, for each nonidentity element $g$ restricts to nonidentity on each open subset of the space where the group acts on. By this requirement, the set of fixed points of any nontrivial subgroup is always nowhere dense. Also, if two elements agree locally, then they are equal. For finite groups, this is always true by M.H.A. Newman [17].
In this paper, we will consider only differentiable maps and sets with differentiable structures. Most of the difficulty of topological group actions disappear in this case. For example, Newman’s result is trivial in differentiable cases.

There are extensive literature on the finite group actions on manifolds using many interesting methods for which this author has no expertise on: For example, many actions on an n-cell are not conjugate to linear actions. In fact there are finite group actions without fixed points (see Floyd and Richardson \[7\] and Buchdhal, Kwasik, and Schultz \[2\]). However, if one chooses a sufficiently small ball around a fixed point, then the action of the finite group fixing that point is conjugate to a linear action. Also, every smooth action of a compact Lie group on a 3-dimensional Euclidean space is differentiably conjugate to a linear action as claimed by Thurston and shown by Kwasik and Schultz \[14\]. (See Davis \[6\] for a survey.)

Most of the material of this section is in Chapter 5 of Thurston (91/12/19 version) \[23\]: We will repeat it here for reader’s convenience and the difficulty of the writing there and some omissions. See also Satake \[20\], \[21\], Kato \[13\], and Matsumoto and Montesinos-Amilibia \[16\].

An n-dimensional orbifold is a Hausdorff, second-countable space Y so that each point has a neighborhood homeomorphic to the quotient space \(U\) of an open set in \(\mathbb{R}^n\) by an action of a finite group. Moreover, if such a neighborhood V of y, modeled on a pair (\(\tilde{V}, G_V\)) is a subset of another such neighborhood U, modeled on a pair (\(\tilde{U}, G_U\)), then the inclusion map \(\phi_{V,U} : V \rightarrow U\) lifts to an imbedding \(\tilde{\phi}_{V,U} : \tilde{V} \rightarrow \tilde{U}\) equivariant with respect to a homomorphism \(\psi_{V,U} : G_V \rightarrow G_U\) so that the following diagram is commutative.

\[
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\tilde{\phi}_{V,U}} & \tilde{U} \\
\downarrow & & \downarrow \\
\tilde{V}/G_V & \rightarrow & \tilde{U}/\psi_{V,U}(G_V) \\
\downarrow & & \downarrow \\
V & \xrightarrow{\phi_{V,U}} & U
\end{array}
\] (1)

Note that the pair \((\tilde{\phi}_{V,U}, \psi_{V,U})\) can be chosen differently; i.e., the pair \(\vartheta \circ \tilde{\phi}_{V,U}\) and \(\vartheta \circ \psi_{V,U}(\cdot) \circ \vartheta^{-1}\) for \(\vartheta \in G_U\) satisfies the above equation as well. Thus, \((\tilde{\phi}_{V,U}, \psi_{V,U})\) associated to the map \(\phi_{V,U}\) is unique up to an element of \(G_U\). If \(\phi_{V,U} : \tilde{V} \rightarrow \tilde{U}\) and \(\phi_{U,W} : \tilde{U} \rightarrow \tilde{W}\) are inclusion maps, then we are forced to have

\[
\begin{align*}
\tilde{\phi}_{V,W} &= \vartheta \circ \tilde{\phi}_{U,W} \circ \tilde{\phi}_{V,U} \\
\psi_{V,W}(\cdot) &= \vartheta \circ \psi_{U,W} \circ \psi_{V,U}(\cdot) \circ \vartheta^{-1}, \text{ for } \vartheta \in G_3
\end{align*}
\] (2)

where \(G_3\) is the finite group associated with W. (V is said to be a model neighborhood and \((\tilde{V}, G_V)\) the model pair where \(G_V\) is a finite group acting on an open subset \(V\) of \(\mathbb{R}^n\).)

A maximal family of coverings \(\mathcal{O}\) with models satisfying the above conditions is said to be an orbifold structure on Y. (That is, an orbifold structure is a maximal collection of model pairs with inclusion equivalence classes satisfying the above properties.) Y is said to be the underlying space of \((Y, \mathcal{O})\) where \(\mathcal{O}\) has models as above. Given an orbifold \(M\), we denote by \(X_M\) the underlying space in this paper.
(We won’t often distinguish between the underlying space and the orbifold itself, particularly later on.)

Given two orbifolds $M$ and $N$, an orbifold-map is a map $f : X_M \to X_N$ so that for each point $x$ of $X_N$, a neighborhood of $x$ modeled on $(U, G)$, and an inverse image of $y$, there is a neighborhood of $y$ modeled on $(V, G')$ and a smooth map $\tilde{f} : V \to U$ inducing $f$ equivariant with respect to a homomorphism $\psi : G' \to G$. (That is, we record the lifting $\tilde{f}$ but $\tilde{f}$ is determined only up to $G, G'$, i.e., the map $g \circ \tilde{f} \circ g'$, $g \in G, g' \in G'$ and homomorphism $\psi(\cdot)$ changed to $g \circ \psi(g'(\cdot))g''-1 \circ g^{-1}$.

Moreover, such liftings have to be consistent in a way that one can take two copies of equation (1) for $M$ and $N$ and write $\tilde{f}$ and induced maps between corresponding elements.)

An orbifold-diffeomorphism is an orbifold-map which has an inverse function with lifts that again forms an orbifold-map.

An orbifold with boundary is a Hausdorff, second-countable space so that each point has a neighborhood modeled on an open set intersected with the upper-half space and a finite group acting on it. The interior is a set of points with neighborhoods modeled on open balls. The boundary is the complement of the interior. (The boundary is a boundaryless orbifold of codimension one.)

A singular point $x$ of an orbifold is a point of the underlying space which has a neighborhood whose model neighborhood has a nontrivial element of the group fixing a point corresponding to $x$. A nonsingular point, so-called regular point, of an orbifold always has a neighborhood homeomorphic to a ball. The set of regular points is an open dense subset of the underlying space since the set of fixed points of a finite group in a model pair is a nowhere dense closed set. The set of singular points is nowhere dense since so is the set of fixed points of a differentiable group action. In this paper, often a point in the open subset of the model pair is said to be regular or singular depending on what the image in the quotient is.

A suborbifold of an orbifold $N$ is an imbedded subset $Y$ of $X_N$ with an orbifold structure so that for each point $x$ of $Y$, and a neighborhood $V$ modeled on $(V', G)$, the neighborhood $V \cap Y$ is modeled on $(V' \cap F, G|F)$ where $F$ is a submanifold of $\mathbb{R}^n$ where $G$ acts, and $G|F$ denotes the image subgroup of the restriction homomorphism to groups acting on $F$.

The boundary of an orbifold is a suborbifold clearly.

Example 1. A class of examples are given as follows: Let $M$ be a manifold and $\Gamma$ a discrete group acting on $M$ properly but not necessarily freely. Then $M/\Gamma$ has an orbifold structure: Let $x$ be a point of $M/\Gamma$ and $\bar{x}$ a point of $M$ corresponding to $x$. Then a subgroup $I_{\bar{x}}$ of $\Gamma$ fixes $\bar{x}$. There is a ball-neighborhood $U$ of $\bar{x}$ on which $I_{\bar{x}}$ acts and for any $g \in \Gamma - I_{\bar{x}}, g(U) \cap U$ is empty. Then $U/I_{\bar{x}}$ is a neighborhood of $x$ modeled on $(U, I_{\bar{x}})$. If $V$ is another such neighborhood in $U/I_{\bar{x}}$ containing a point $y$, then a component $V'$ of its inverse image in $U$ is acted upon by a subgroup $I'$ of $I_{\bar{x}}$. Also, for any $g \in \Gamma - I'$, $g(V') \cap V'$ is empty. Therefore, the inclusion $V \to U/I_{\bar{x}}$ satisfies the conditions for equations (1).

Given two orbifolds $M$ and $N$, the product space $X_M \times X_N$ obviously has an orbifold structure; i.e., we model on $(U \times V, G_U \times G_V)$ if $(U, G_U)$ and $(V, G_V)$ are model pairs for neighborhoods of $M$ and $N$ respectively. The product space with this orbifold structure is denote by $M \times N$.

A homotopy of two orbifold-maps $f_1, f_2 : M \to N$ from an orbifold $M$ to another one $N$ is an orbifold-map $F : M \times [0, 1] \to N$ where $[0, 1]$ is the unit interval and
that the support of $\phi R$ is a Riemannian metric on orbifolds $\phi R$.

Given an orbifold $M$, an \textit{isotopy} $f: M \to M$ is a self-orbifold-diffeomorphism, i.e., an automorphism, so that there is a homotopy $F : M \times [0, 1] \to M$ so that $F_0$ is the identity map and $F_1 = f$, and $F_t$ is an orbifold-diffeomorphism for each $t$.

Two orbifold-diffeomorphisms $f_1, f_2 : M \to M'$ are isotopic if there is a homotopy $F : M \times [0, 1] \to M'$ so that $F_0 = f_1$ and $F_1 = f_2$ and $F_t$ are orbifold-diffeomorphisms.

A \textit{Riemannian} metric on an orbifold is a Riemannian metric on each model open set invariant under the associated finite group action and where inclusion induced maps for model pairs are isometries. (See Satake \[20\] and \[21\] for more details.) A \textit{partition of unity} for an open cover of an orbifold is a collection of functions whose supports are in compact subsets of the elements of the open cover, which sum to 1 and correspond to finite-group-invariant smooth functions on model pairs. An orbifold with an open cover has a partition of unity. We can always put a Riemannian metric on a compact orbifold: Cover the orbifold by the modeled neighborhoods and choose a locally finite subcover $\{V_i\}$ and a partition of unity. Let $(U_i, G_i)$ be the model pairs. Choose a Riemannian metric on $U_i$ and by taking an average over the finite group action $G_i$, we obtain an invariant metric on each modeled neighborhood $V_i$. Let $\{\phi_i\}$ be the partition of unity on the orbifold such that the support of $\phi_i$ is in a compact subset of $V_i$ for each $i$. Then $\phi_i$ pulls back to a smooth function $\tilde{\phi}_i$ which is $G_i$ equivariant. Let us choose a Riemannian metric $\mu_i$ on each $U_i$ so that $G_i$ acts by isometries. Then on each $U_i$, we may form a smooth pseudo metric $\tilde{\phi}_i^* \mu_j^*$ which is induced from the inclusion map to $U_j$. Clearly, $\tilde{\phi}_i^* \mu_j^*$ is a smooth metric on $U_i$ where $G_i$ acts as isometries and the inclusion maps induce isometries. This defines a global metric on the orbifold.

Furthermore, for any model neighborhood, its model pair has a well-defined induced Riemannian metric invariant under the group action. We may consider such metrics Riemannian metrics on orbifolds.

We make a quotient space of the tangent bundle $T(U_i)$ over $U_i$ by $G_i$ to obtain $2n$-dimensional orbifold $O_i$. We can easily patch $O_i$s together to obtain a $2n$-orbifold $T(M)$ with a map $p : T(M) \to M$ so that the inverse image of a point is a vector space modulo a finite group action. Let $T_{x_0}(M)$ denote the fiber over $x_0 \in M$.

If $x_0 \in M$ is a singular point in $V_i$, then we can choose an open ball $U_{x_0}$ in $U_i$ so that the subgroup $G_{x_0}$ of $G_i$ fixing the point $\tilde{x}_0$ corresponding to $x_0$ acts on it. Then there is a neighborhood of $V_{x_0}$ of $x_0$ which is modeled on $(U_{x_0}, G_{x_0})$.

An exponential map from $T_{x_0}(M)$ to $V_{x_0}$ is locally defined by the exponential map on the model open set $U_{x_0}$ which is clearly invariant under the finite group action if $x_0$ is singular. If $x_0$ is regular, we can use the ordinary exponential map. We can obviously patch these maps to obtain a global map $\exp_{x_0} : T_{x_0}(M) \to M$.

We can find $r > 0$ so that $\exp_{x_0}$ imbeds the ball $B_r(0) \subset T_{x_0}U_{x_0}$ of positive radius $< r$ to a strictly convex ball in $U_{x_0}$. (They have smooth convex boundary.) Thus, the exponential map from each $x_0 \in M$ sends a quotient space of a ball positive radius $< r$ to a quotient space of a strictly convex ball in $M$. The images are said to be \textit{normal neighborhoods}. 

$F(x, 0) = f_1(x)$ and $F(x, 1) = f_2(x)$ for every $x \in M$. We define an orbifold-map $F_t : M \to N$ to be given by $F_t(x) = F(x, t)$ with appropriate liftings in model pairs of $M$ and $N$.

Given an orbifold $M$, an \textit{isotopy} $f : M \to M$ is a self-orbifold-diffeomorphism, i.e., an automorphism, so that there is a homotopy $F : M \times [0, 1] \to M$ so that $F_0$ is the identity map and $F_1 = f$, and $F_t$ is an orbifold-diffeomorphism for each $t$.
Lemma 1. Let \( x \) be a point of an orbifold \( M \). Then \( x \) has a model pair \((\tilde{V}, G_V)\) so that \( \tilde{V} \) is simply-connected.

Proof. We may choose \( \tilde{V} \) sufficiently small so that \( G_V \) action is conjugate to a linear action. For example, we can choose the normal neighborhood. This is Proposition 5.4.1 of [23]. \( \square \)

We will need to find a covering of an orbifold by model sets of certain forms. A covering \( \{O_i\} \) of an orbifold \( M \) is said to be a nice covering if it satisfies

- Each \( O_i \) is connected and open.
- \( O_i \) has a model pair \((\tilde{O}_i, G_i)\) so that \( \tilde{O}_i \) is simply-connected.
- The intersection of any finite collection of \( O_i \) has the above two properties.

Proposition 1. An orbifold \( M \) has a nice locally finite covering.

Proof. We will assume that \( M \) is compact. If \( M \) is only locally compact, the proof is similar. First cover \( M \) by a finite collection of normal neighborhoods. For a point \( x \) of a model pair \((U, G_V)\) of a normal neighborhood, there exists a radius \( r > 0 \) such that the ball \( B_r(x) \) of radius \( r \) in \( U \) has the property that for each pair of points \( y \) and \( z \) of \( B_r(x) \), there exists a unique geodesic segment connecting \( y \) and \( z \). Thus, using the Lebesgue number, we can find a real number \( r_0 > 0 \) so that for each point \( x \) of \( M \), the ball \( B_r(x) \) of radius \( r \), \( 0 < r < r_0 \) in \( M \) has the generic convexity property; i.e., for each generic pair of points \( y \) and \( z \) of \( B_r(x) \), there exists a unique geodesic connecting \( y \) and \( z \). Given two balls \( B_{r/8}(x) \) and \( B_{r/8}(y) \), their intersection \( B_{r/8}(x) \cap B_{r/8}(y) \), which is open, can have only one component. By an induction, we can show that any finite intersection of balls \( \bigcap_{i=1}^n B_{r/8}(x_i) \) is connected. The collection consisting of \( B_{r/8}(x_i), i = 1, \ldots, n \), covering \( M \) is a nice covering. \( \square \)

3. Fiber products and the universal covering orbifolds

In some cases, we will allow covering spaces to have many components for convenience, which does not cause too much confusion. In this case, a morphism of topological covering spaces \((X_1, p_1)\) and \((X_2, p_2)\) is a map \( f : X_1 \to X_2 \) such that \( p_1 = p_2 \circ f \) so that \( f \) induces injective correspondence between the components of \( X_1 \) and components of \( X_2 \), and if \( X_1 \) is connected, and \( X_1 \) and \( X_2 \) have base points, we require that \( f \) sends \( X_1 \) to the component of \( X_2 \) containing the base point. Note that \( f \) need not be surjective. However, if \( X_1 \) and \( X_2 \) are connected, morphisms are surjective.

Let us briefly review the notion of (topological) fiber-products in ordinary covering space theory so that we can generalize the notion to that for orbifold-covering spaces. Given a sequence of covering maps \( p_i : Y_i \to Y \) for \( i \) in some index set \( I \), in the ordinary sense, one can form a fiber-product \( p^f : Y^f \to Y^c \) by setting \( Y^f \) to be the subspace of \( \prod_{i \in I} Y_i \) such that for \( (x_i)_{i \in I} \in Y^f \)

\[ p_j \circ \pi_j((x_i)_{i \in I}) = p_k \circ \pi_k((x_i)_{i \in I}) \]

for all \( j, k \) and \( \pi_i : \prod_{i \in I} Y_i \to Y_i \) the projection to the \( i \)-th factor. The covering map \( p^f : Y^f \to Y \) is given by \( p^f((x_i)) = p_1(x_1) \), and \( Y^f \) covers \( Y_1 \) by a morphism \( p^f_1 : Y^f \to Y_1 \), so that \( p_1 \circ p^f_1 = p^f \), which is given by the projection to the \( i \)-th factor.

Given base-points \( y_i \) of \( Y_i \) mapping to a base point \( y \) of \( Y \), we decide that the corresponding point \( y^f = (y_i)_{i \in I} \) of \( Y^f \) be a base point of \( Y^f \).
The fiber product is not necessarily connected. For example, consider the fiber-product of two-fold covering of and a four-fold covering of $S^1$. The fiber-product maps to $S^1$ by 8 to 1, but it has two components mapping 4 to 1.

Let $Y_0^f$ be the component containing the base point. Then $Y_0^f$ is the covering of $Y$ corresponding to $\bigcap_{i \in I} p_{i,*}(\pi_1(Y_i, y_i))$. This follows by considering which loops of $Y$ based at $y$ can be lifted to loops in the fiber-product; A loop lifts if and only if it is in the above intersection. We also see that each of the other components is a covering corresponding to

$$\bigcap_{i \in I} p_{i,*}(\pi_1(Y_i, y_i))$$

where $(y'_i)_{i \in I}$ is a point of the component so that $p_i(y'_i) = y$.

We can also verify that $Y_0^f$ has a universal property that if $(Y'', p'')$ is a connected covering space of $Y$ with a base point $y''$ and $q_i : Y'' \to Y_i$ is a covering morphism sending $y''$ to $y_i$ for each $i$, then there exists a covering morphism $q' : Y'' \to Y_0^f$ sending $y''$ to $y'$. (Of course, we can change $y_i$ to any $y'_i$ with $p_i(y'_i) = y$.)

Also, the universal property characterizes $Y_0^f$ up to covering isomorphisms. That is, if $(Z, p_Z)$ is a connected covering space of $Y$ so that there is a covering morphism $q_{Z,i} : Z \to Y_i$, so that $z \mapsto y_i$ for a base point $z$ of $Z$ and each $i$, and $Z$ satisfies the universal property of $Y_0^f$ above, then there exists a covering isomorphism $L : Z \to Y_0^f$ such that

$$q_{Z,i} = p'_i \circ L.$$  \hspace{1cm} (3)

Finally, if the collection $\{(Y_i, p_i)\}$ contains all the covering spaces of $Y$ up to isomorphisms, we see that components of $Y^f$ are universal covers of $Y$.

**Example 2.** Since we need the group action descriptions to define orbifold-fiber products, we view above example more in terms of group actions: Let a connected manifold $Y$ have a connected regular covering space $\tilde{Y}$ with the covering map $\tilde{p}$ and subgroups $\Gamma_i$ of the deck transformation group $\Gamma$. Then let $Y_i$ be the quotient space $\tilde{Y}/\Gamma_i$, and $p_i : \tilde{Y} \to Y_i$ the covering map for each $i$. Let $\Gamma_i \setminus \Gamma$ denote the right coset space of $\Gamma_i$ in $\Gamma$. The projection map

$$\tilde{p}_i : \tilde{Y} \times \Gamma_i \setminus \Gamma \to Y_i$$

induces a map

$$(\tilde{Y} \times \Gamma_i \setminus \Gamma)/\Gamma \to \tilde{Y}/\Gamma$$

where $\Gamma$ acts on the first space by

$$\gamma((\tilde{x}, \Gamma_i \gamma)) = (\gamma(\tilde{x}), \Gamma_i \gamma \gamma^{-1})$$

for $\gamma \in \Gamma$ and on the second space in the usual way. Since $\Gamma$ acts transitively on each sheets of

$$\tilde{Y} \times \Gamma_i \setminus \Gamma$$

and $\Gamma_i$ acts on the sheet $\tilde{Y} \times \Gamma_i \setminus 1$, we see that this map is obviously the same map as $p_i$. The fiber product of $\tilde{p}_i$'s is clearly equal to the projection

$$\tilde{Y} \times \prod_{i \in I} \Gamma_i \setminus \Gamma \to \tilde{Y}.$$  

Define the left action of $\Gamma$ on the first space by

$$\gamma((\tilde{x}, (\Gamma_i \gamma)_{i \in I})) = ((\gamma(\tilde{x}), (\Gamma_i \gamma \gamma^{-1})_{i \in I})$$

for $\gamma \in \Gamma$. 


Since $\Gamma$-equivalence classes of the first space correspond exactly to the fiber products of the $\Gamma$-equivalence classes of $\tilde{Y}$, the fiber product of $p_i$s equals
\[
(\tilde{Y} \times \prod_{i \in I} (\Gamma_i \setminus \Gamma))/\Gamma \to \tilde{Y}/\Gamma
\]
induced by the projection. As before, the fiber product may have many components. Using path-considerations again, we see that each component is isomorphic to a cover
\[
\tilde{Y}/\bigcap_{i \in I} \gamma_i \Gamma_i \gamma_i^{-1}
\]
of $\tilde{Y}/\Gamma$ where $\gamma_i$, $i \in I$, is a sequence of a coset-representative of $\Gamma_i \setminus \Gamma$ for each $i$. Another way to see this is that given a component
\[
\tilde{Y} \times \prod_{i \in I} \Gamma_i \gamma_i
\]
for a sequence $\gamma_i$, the group acting on it equals $\bigcap_{i \in I} \gamma_i \Gamma_i \gamma_i^{-1}$. Thus, the quotient of the component corresponds to the component described above.

If $\{\Gamma_i\}_{i \in I}$ are all of the subgroups of $\Gamma$, then each component equals $\tilde{Y}$.

We note that the covering map from the fiber-product to $Y$ is given by sending $[\tilde{x}, (\Gamma_i \gamma_i)_{i \in I}]$ to $[p(\tilde{x})]$, and the covering morphism from the fiber product to $Y_i$ for each $i$ is given by sending $[\tilde{x}, (\Gamma_i \gamma_i)_{i \in I}]$ to $[\gamma_i(\tilde{x})]$ in $\tilde{Y}/\Gamma_i$.

A covering orbifold of an orbifold $M$ is an orbifold $\tilde{M}$ with a surjective orbifold-map $p : X_{\tilde{M}} \to X_M$ such that each point $x \in X_M$ has a neighborhood $U$, so-called an elementary neighborhood, with a homeomorphism $\phi : U/G_U \to U$ and an open subset of $\tilde{U}$ in $\mathbb{R}^n$ or $\mathbb{R}^{n-1}$ with a group $G_U$ acting on it, so that each component $V_i$ of $p^{-1}(U)$ has a homeomorphism $\phi_i : U/G_i \to V_i$ (in the orbifold structure) where $G_i$ is a subgroup of $G_U$. We require the quotient map $U \to V_i$ induced by $\phi_i$ composed with $p$ should be the quotient map $\tilde{U} \to U$ induced by $\phi$. We often don’t assume $X_M$ is connected. In this case, only components of $M$ need to be orbifolds and $M$ itself does not. See also Bridson-Haefliger [11].

A fiber of a point of $M$ is the inverse image $p^{-1}(x)$.

Given an orbifold-map $f : Y \to Z$ and a covering $(Z_1, p_1)$ of $Z$, if an orbifold-map $\tilde{f} : Y \to Z_1$ satisfies $p_1 \circ \tilde{f} = f$ and $\tilde{f}$ lifts for every model pair of points of $Z_1$ in the consistent way for $Z$, $\tilde{f}$ is said to be a lifting of $f$.

Two covering orbifolds $(Y_1, p_1)$ and $(Y_2, p_2)$ of an orbifold $Y$ are isomorphic if there is an orbifold-diffeomorphism $f : Y_1 \to Y_2$ so that $p_2 \circ f = p_1$. A covering automorphism or deck transformation $Y_1 \to Y_1$ is a covering isomorphism of $Y_1$ to itself. (Thus, $f$ is a lifting of $p_1$.) More generally, a morphism $(Y_1, p_1) \to (Y_2, p_2)$ is an orbifold-map $f : Y_1 \to Y_2$ so that $p_2 \circ f = p_1$ where distinct components go to distinct components (see Proposition 5). If $Y_1$ and $Y_2$ are connected, then $f$ obviously is surjective by an open and closedness argument. For orbifolds with base points, we require that a morphism should preserve base points. A covering $(Y_1, p_1)$ is regular if the automorphism group acts transitively on fibers over regular points.

Given coverings $(Y_1, p_1)$ over $\tilde{Y}$ and $(Y_2, p_2)$ over $Z$, an orbifold-map $f : Y_1 \to Y_2$
covers an orbifold-map \( g : Y \to Z \) if the following diagram is commutative:

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{f} & Y_2 \\
p_1 \downarrow & & \downarrow p_2 \\
Y & \xrightarrow{g} & Z.
\end{array}
\]

In this paper, if \( Z \) is a cover of an orbifold \( Y \), then by \( Z' \) we mean the inverse image of the regular part \( Y' \) of \( Y \). The inverse image consists of regular points of \( Z' \); however, the converse is not true in general. (It will be clear from the context whether one means just a regular part or the part over the regular part.)

We now go over some preliminary results on orbifold-covering maps:

**Proposition 2.** Let \((Y_1,p_1)\) and \((Y_2,p_2)\) be coverings over an orbifold \( Y \). Let \( f : Y_1 \to Y_2 \) be a covering morphism so that \( f : Y'_1 \to Y'_2 \) is a covering isomorphism where \( Y'_1 \) and \( Y'_2 \) are inverse images of the regular part \( Y' \) of \( Y \). Then \( f \) itself is a covering isomorphism.

**Proof.** For the model pairs, the groups have to be isomorphic. The rest is straightforward.

**Proposition 3.** Let \((Y_1,p_1)\), \((Y_2,p_2)\), and \((Y_3,p_3)\) be coverings over \( Y \). Let \( f_1 : Y_1 \to Y_3 \), \( f_2 : Y_2 \to Y_3 \), and \( f_3 : Y_1 \to Y_2 \) be covering morphisms so that \( f_1|Y'_1 = f_2 \circ f_3|Y'_1 \). Then \( f_1 = f_2 \circ f_3 \).

**Proof.** Again a local consideration proves this in a straightforward manner.

By a *path*, we mean a smooth orbifold-map from an interval to an orbifold. The two central properties of covering space theory survive in the orbifold-covering space theory:

**Proposition 4.** Let \( Y \) be an orbifold and \( p : Y' \to Y \) an orbifold-covering map.

- Let \( x \) be a point of \( Y \) and \( x' \) a point in \( p^{-1}(x) \). A path \( f : I \to Y \) such that \( f(0) = x \) lifts to a unique path \( f' : I \to Y' \) in \( Y' \) such that \( f'(0) = x' \).
- Let \( f_1 : Z \to Y' \) and \( f_2 : Z \to Y' \) be orbifold-maps lifting \( f : Z \to Y \). If \( f_1(x) = f_2(x) \) for a regular point \( x \in Z \), then \( f_1 = f_2 \).

**Proof.** The first one has the same proof as the ordinary topological theory since liftings are determined up to the action of local groups and we only need to match them. We use the open and closeness for the second one.

We discuss somewhat about so-called doubling. A *mirror point* is a singular point of an orbifold with a model pair \((U,G)\) where \( G \) is an order-two group acting on the open subset \( U \) of \( \mathbb{R}^n \) fixing a hyperplane meeting \( U \). The set of mirror points is said to be the *mirror set*.

We can form an orbifold \( M^d \) covering \( M \) so that there are two points in the inverse image of each regular point of \( M \). The so-called 2-fold cover \( M^d \) has no mirror points: Let \( V_i, i = 1, 2, \ldots \), be model neighborhoods covering \( M \) and \( V_i \) has model pairs \((U_i,G_i)\). The model open set \( U_i \) has an induced orientation from \( \mathbb{R}^n \). For each \( i \), we define a new pair of form \((U_i \times \{-1,1\},G_i)\) where \( G_i \) acts by \( g((x,l)) = (g(x), \text{sign}(g)l), l = \pm 1 \) where \( \text{sign}(g) \) is defined to be 1 if \( g \) is orientation-preserving and \(-1 \) if not. For each morphism \( V_i \cap V_j \to V_i \), we simply define the lift

\[
\tilde{\phi}_{V_i \cap V_j, V_i} : V_i \cap V_j \times \{-1,1\} \to V_i \times \{-1,1\}
\]
to be \( \tilde{\phi}_{V_1 \cap V_2} \times \text{id}_{(-1,1)} \) and the homomorphism \( \tilde{\phi}_{V_1 \cap V_2} \) to be the old one \( \phi_{V_1 \cap V_2} \). If we paste together these sets with thus-defined morphisms, we obtain a new orbifold \( M^d \). Also, projections \( U_i \times \{-1, 1\} \to U_i \) define an orbifold-covering map \( M^d \to M \) which is two-fold. (Note that \( M^d \) is not the “doubled orbifold” defined by Thurston in general. For example consider the orbifold which is the quotient-orbifold of \( \mathbb{R}^3 \) by the group of order-two generated by \(-1\) times the identity map.)

**Lemma 2.** Let \( f : M_1 \to M_2 \) be an orbifold-map, and \( p_1 : M_1^d \to M_1 \) and \( p_2 : M_2^d \to M_2 \) be the two-fold orbifold covering maps as defined above. Then there exists an orbifold-map \( f^d : M_1^d \to M_2^d \) covering \( f \).

**Proof.** For each model pair \((U_i \times \{-1, 1\}, G_i)\) of \( M_i^d \), we define \( f^d \) to be \( f \times \text{id}_{(-1,1)} \).

**Proposition 5.** Let \( p : N \to M \) be an orbifold-covering map where \( N \) and \( M \) are connected orbifolds. Let \( f : N \to N \) be a morphism. Then \( f \) is a covering-isomorphism.

**Proof.** First, we consider the case when \( M \) has no mirror points. Then we claim that \( N' \) is connected: In the model pair of a singular point of \( N \), if there are no element fixing a hypersurface, then the set of regular points in the model open set \( U \) is connected since the actions are conjugate to linear actions on sufficiently small open sets. Thus, each point of \( U \) can be a boundary point of only one component of \( N' \). Therefore, \( N' \) can have only one component. (By same reason, \( M' \) is connected since the identity map is an orbifold-covering map.)

Now, \( f|N' : N' \to N' \) is a topological covering automorphism. Thus, \( f|N' \) is one-to-one and onto. By Proposition 2, \( f \) is a covering isomorphism.

If \( M \) has mirror-points, then we form the two-fold orbifold-covering map \( p : M^d \to M \). Then an orbifold-map \( f^d : M^d \to M^d \) covering \( f \) is also a covering morphism of \( p \) from \( M^d \) to itself. Since \( M^d \) has no mirror-points, \( f^d \) is a covering isomorphism. Therefore, so is \( f \) obviously.

**Proposition 6.** A model neighborhood of a point of an orbifold \( Y \) is elementary for any orbifold-covering map if the open subset of the model pair is simply connected.

**Proof.** Let \( V \) be a model neighborhood of \( x \in Y \), and \((\tilde{V}, G_V)\) the model pair, and \( p : Y' \to Y \) an orbifold-covering map. For a path \( f \) in \( \tilde{V} \) with the base point \( \tilde{x} \), we can lift \( q \circ f \) to a path in \( Y' \) easily by using the elementary neighborhoods. Two homotopic paths \( f \) and \( f' \) lift to homotopic path-classes again using elementary neighborhoods. By taking a base point in \( \tilde{V} \) and path-classes from the base point to all points of \( \tilde{V} \), we can lift \( q \) to \( \tilde{q} : \tilde{V} \to V' \) so that \( p \circ \tilde{q} = q \). Since any path in \( V' \) can be lifted, \( \tilde{q} \) is obviously a surjective orbifold-map. (This works in the same manner as in the covering space theory.) From here, it is straightforward to verify that \( V' \) is of form \( \tilde{V} \) quotient out by a finite subgroup \( G_V' \) of \( G_V \). That is, we show that the inverse image of every point of \( V' \) is an orbit of \( G_V' \) by an open and closedness argument. Thus, \( V \) is elementary.

**Proposition 7.** Let \( V \) be an \( n \)-orbifold which is a quotient space of an \( n \)-ball \( \tilde{V} \) by a finite group \( G_V \) acting on it. Then the following statements hold:

(i) A connected covering orbifold \( V_1 \) of \( V \) is isomorphic to \( \tilde{V}/G_{V'} \) for a subgroup \( G_{V'} \subset G_V \) with a covering map \( p : \tilde{V}/G_{V'} \to V = \tilde{V}/G \) induced by
the identity map \( \tilde{V} \to \tilde{V} \); i.e., the set of the isomorphism classes of connected covering orbifolds is in one-to-one correspondence with the conjugacy classes of subgroups of \( G \).

(ii) Given two covering orbifolds \( \tilde{V}/G_1 \) and \( \tilde{V}/G_2 \), a covering morphism \( \tilde{V}/G_1 \to \tilde{V}/G_2 \) is induced by an element \( g \in G : \tilde{V} \to \tilde{V} \) so that \( gG_1g^{-1} \subset G_2 \). The covering morphisms are in one-to-one correspondence with double cosets of form \( G_2gG_1 \) with \( g \) satisfying \( gG_1g^{-1} \subset G_2 \).

(iii) The covering automorphism group of a covering orbifold \( V' \) is given by \( N(G_v')/G_v' \) where \( G_v' \) is a subgroup corresponding to \( V' \) and \( N(G_v') \) is the normalizer of \( G_v' \) in \( G_v \).

Proof. (i) The first part follows from Proposition 6. The second part is a consequence of (ii).

(ii) The morphism lifts to an orbit-preserving map \( f : \tilde{V} \to \tilde{V} \), which sends each \( G_1 \)-orbit to a \( G_2 \)-orbit. Again \( f \) is an element of \( G \) covering the identity map of \( V \). Thus \( gG_1(x) \subset G_2(g(x)) \) for each regular point \( x \) of \( \tilde{V} \). Thus, \( gG_1g^{-1} \subset G_2 \).

The elements \( g \) and \( g' \) induce a same map \( \tilde{V}/G_1 \to \tilde{V}/G_2 \) if and only if \( g' = gG_1g_1 \) for \( g_1 \in G_1 \) and \( g_2 \in G_2 \).

(iii) follows from (ii). \( \square \)

We first define orbifold-fiber products of orbifold-covering spaces of a model pairs: Let \( G_i, i \in I \), be a collection of subgroups of a finite group \( G \) acting on an open subset \( V \) of \( \mathbb{R}^n \). Then \( p_i : V/G_i \to V/G \) form a collection of orbifold-covering maps.

\[
p : V \times \prod_{i \in I} G_i \to V
\]

is a covering map. We let \( G \) act on it by

\[
\gamma(v, G_i \gamma_i) = (\gamma(v), G_1 \gamma \gamma_i^{-1})_{i \in I}.
\]

Define the orbifold-fiber-product to be

\[
p^f : V^f = (V \times \prod_{i \in I} G_i \) \to V/G \]

where \( p^f \) is induced by \( p \); i.e., the orbit \( [v, G_i \gamma_i] \) of \( (v, G_i \gamma_i) \) is sent to the orbit \( [v] \) of \( v \).

As in Example 2, each component of \( V^f \) is of form \( V/\bigcap_{i \in I} G_i \gamma_i^{-1} \) for a sequence of representatives \( \gamma_i, i \in I \), of cosets \( G_i \) \( G \). \( p^f \) is obviously given by projection, and \( G \) acts transitively on the components of \( V^f \).

There are covering morphisms \( q_i : V^f \to V/G_i \) given by \( q_i : [v, G_i \gamma_i] \mapsto [\gamma_i v] \).

If we replace \( \tilde{V} \) by \( \tilde{V}^r \), we obtain an ordinary fiber product \( V^r \) of the maps \( (p_i|V^r_i)_{i \in I} \). We can easily see that \( V^r \) is identifiable with the inverse image of \( p^{f-1}(V^r) \) by construction of \( V^f \). (This is nicely explained in Chapter 5 of Thurston 23). Also, the discussion over the regular part reduces to Example 2.

Let \( y \in V/G \) be a base point and a regular point. Given a base point \( y_i \) of \( V/G_i \) for each \( i \) mapping to the base point \( y \) of \( V/G \), we can form a base point \( y^f \) of \( V^f \): Choose a base point \( y_0 \) in \( V \) so that \( y_i = [\gamma_i(y_0)] \) for some \( \gamma_i \in G \). Then let \( y = (y_0, (G_i \gamma_i)_{i \in I}) \) and \( y^f \) be the equivalence class of \( y \) under the action of \( G \) in \( V^f \).

Again \( V^f \) has a universal property that if \( (V'', p'') \) is a connected covering space of \( V/G \) with a base point \( y'' \) and \( q_i : V'' \to V/G_i \) is a covering morphism sending
$y''$ to base points $y_i$ for each $i$, then there exists a covering morphism $q' : V'' \to V^f$ sending $y''$ to $y^f_i$ so that $q_i \circ q' = q''_i$: We regard $V''$ as $\tilde{V}/G''_V$ for a subgroup $G''_V$ of $G_V$. We can lift $q''_i$ to an orbifold-covering $\tilde{q}_i : \tilde{V} \to V/G_i$ for each $i$, and hence to a covering map $\tilde{q}_i : \tilde{V} \to \tilde{V}$, which is given by an element $g_i$ of $G_V$ by Proposition 7 satisfying

$$q_i G''_V g_i^{-1} \subset G_i.$$  

We map $\tilde{V}$ to

$$V^f = (\tilde{V} \times \prod_{i \in I} (G_i \backslash G_V))/G_V$$

by sending $u$ to the class of $(u, (G_i g_i)_{i \in I})$. This map induces a well-defined (diagonal) morphism $q'$ from $V''$ to $V^f$ by equation (4).

$$\begin{array}{ccc}
\tilde{V} & \longrightarrow & V^f = (\tilde{V} \times \prod_{i \in I} (G_i \backslash G_V))/G_V \\
\downarrow & & \downarrow q_i \\
V'' = \tilde{V}/G''_V & \longrightarrow & \tilde{V}/G_i \\
\downarrow p_i & & \downarrow \\
V = \tilde{V}/G_V
\end{array}$$

(5)

Clearly, such a diagonal map is unique. Moreover, we may verify that $y''$ goes to $y^f$. Thus the component $V'_0$ of $V^f$ containing $y^f$ has a universal property. Also, since we can change components, each component of $V^f$ or $V^f$ itself has a universal property without base point conditions.

Also, the universal property characterizes components of $V^f$ or $V^f$ itself up to covering isomorphisms. That is, if $(Z, p_Z)$ is a connected covering space of $V$ so that there is a covering morphism $q_{Z,i} : Z \to V_i$, so that $z \mapsto y_i$ for each $i$ and $Z$ satisfies the universal properties of $Y^f$ above, then there exists a covering isomorphism $L : Z \to V^f_i$ such that $q_{Z,i} = p'_i \circ L^{-(*)}$: The proof is obvious from the universal properties of $Z$ and $V^f$.

**Example 3.** Let $V$ be a closed interval and $\mathbb{Z}_2$ the group of order two fixing a point of $V$. Let $p_1 : V \to V/\mathbb{Z}_2$ and $p_2 : V \to V/\mathbb{Z}_2$ be two orbifold-covering maps. Then the orbifold-fiber-product is obviously given by two copies of $V$ mapping to $V/\mathbb{Z}_2$.

Let us list a collection of orbifold-coverings $(Y_i, p_i)$, $i \in I$, of a connected orbifold $Y$ for some index set $I$. Let $y_i$ be base-points of $Y_i$ so that $p_i(y_i) = y$ for a regular base point $y$ of $Y$. ($y_i$ are all regular.) We now define orbifold-fiber-product of orbifold-covering maps $(Y_i, p_i)$: Let us cover $Y$ by a collection $Z$ of connected model neighborhoods so that the open subsets of their model pairs are simply connected. We also require that finite intersections of model neighborhoods are always connected and the open subset of their model pairs are simply connected. Such a covering of $Y$ exists by Proposition 11.

Let $V \in Z$ be a model-neighborhood of $Y$ with a model pair $(\tilde{V}, G_V)$ where $\tilde{V}$ is simply-connected. Take a component $V'_i$, of $p_i^{-1}(V)$ for each $Y_i$. Let $G'_V$ denote the subgroup of $G_V$ so that $\tilde{V} \to V^i$ is the covering map given as a quotient map for $G'_V$ acting on $\tilde{V}$. For each choice of $j$ for $i$, we define a map $J$ defined on $I$ to the set of components of $p_i^{-1}(V)$ for $i \in I$ so that $J(i)$, also denoted by $j(i)$, is
a component of \( p_i^{-1}(V) \). With a fixed function \( J \), we form a fiber product \( V^J \) of \( V_i^{j(i)}, i \in I \). We define

\[
V^J = (\tilde{V} \times \prod_{i \in I}(G_i^{j(i)} \backslash G_V))/G_V
\]

where the \( G_V \)-action is given by

\[
\gamma(u, (G_i^{j(i)} \gamma_i)_{i \in I}) = (\gamma u, (G_i^{j(i)} \gamma_i \gamma^{-1})_{i \in I}).
\]

We define \( q_j : V^J \to V \) to be the obvious covering map sending \((u, (G_i^{j(i)} \gamma_i)_{i \in I})\) to the class of \( u \) in \( V \), i.e., the orbit \([u] \) of \( u \). From \( V^J \), we can define a covering map \( q_i : V^J \to V_i^{j(i)} \) by sending \([u, (G_i^{j(i)} \gamma_i)_{i \in I}] \) to \([\gamma_i u] \) in the equivalence class of \( \tilde{V}/G_i^{j(i)} \). This clearly is a well-defined orbifold-morphism.

We note the universal property of \( V^J \) that given a sequence of morphisms \( q_i'' : V'' \to V_i^{j(i)} \) for all \( i \), there is a covering morphism \( q' : V'' \to V^J \) so that \( q_i \circ q' = q_i'' \).

Now, we define \( \tilde{V} \) as the disjoint union \( \coprod_j V^J \) for all functions \( J \). It has an obvious covering map \( \hat{p} : \tilde{V} \to V \). We can define a morphism \( q_i : \tilde{V} \to \bigcup_j V^J = p_i^{-1}(V) \) by defining \( q_i \) as above for each of \( V^J \).

Since \( \coprod_j V^J \) contains the fiber products of

\[
p_i|p_i^{-1}(V^J) : p_i^{-1}(V^J) \to V^J,
\]

we see that if the base point \( y \) of \( Y \) is in \( V \), then there exists a regular point \( y^J \) mapping to \( y \) under \( q_i \) for each \( i \). Thus, we construct \( y^J \) in this manner, later to be identified.

Also, \( \tilde{V} \) has a following universal property: given a sequence of morphisms

\[
q_i'' : V'' \to \bigprod_j V^J = p_i^{-1}(V)
\]

for all \( i \) so that \( p_i \circ q_i'' \) is a fixed covering map \( V'' \to V \), there exists a unique morphism \( q' : V'' \to V \) so that \( q_i \circ q' = q_i'' \). This follows from considering where each component of \( V'' \) maps to. The covering \( \tilde{V} \) is said to be a fiber product of \( p_i^{-1}(V), i \in I \).

Let \( U \) be a connected open subset of \( V \), such as \( U = V \cap V' \) for another model neighborhood \( V' \) in the covering \( Z \). We assume that \( U \) is modeled on a pair \( (\hat{U}, G_U) \).

Then components of \( q^{-1}(U) \) in \( \tilde{V} \) are homeomorphic to \( \hat{U} \) and the subgroup of \( G_V \) acting on a component is isomorphic to \( G_U \).

Let \( U_i^{j,k} \) denote the components of \( p^{-1}(U) \) in \( V_i^j \) for each \( i, j \). Let \( G_{U,i}^{j,k} \) denote a subgroup of \( G_U \) so that \( \hat{U}/G_{U,i}^{j,k} \) equals the covering \( U_i^{j,k} \to U \).

Let \( K \) be a function defined on \( I \) by sending \( i \) to an index \( k(i) \) among the indices \( k \) of components of form \( U_i^{j(i),k} \). Let the fiber product

\[
U^{J,K} = (\hat{U} \times \prod_{i \in I}(G_{U,i}^{j(i),k(i)} \backslash G_U))/G_U
\]

be defined where \( G_U \) acts by

\[
(u, (G_{U,i}^{j(i),k(i)} \gamma_i)_{i \in I}) \mapsto (\gamma u, (G_{U,i}^{j(i),k(i)} \gamma_i \gamma^{-1})_{i \in I}), \quad \text{for } \gamma \in G_U.
\]
Let
\[ q_{i}^{J,K} : U^{J,K} \rightarrow U_{i}^{j(i),k(i)} \subset p_{i}^{-1}(U) \]
be the morphism defined by sending \((u, (G_{i}^{j(i),k(i)}))_{i \in I}\) to the equivalence class of \(\gamma_{i}u\). Let \(p_{U}^{J,K} : U^{J,K} \rightarrow U\) denote the covering map.

Define \(U^{J}\) by taking the disjoint union \(\bigsqcup_{K} U^{J,K}\), and \(\hat{U}\) by \(\bigsqcup_{J} U^{J}\). We define morphisms
\[ q_{U,i}^{J} : U^{J} \rightarrow \bigsqcup_{j,k} U_{i}^{j,k} = p_{i}^{-1}(U) \]
by restricting it to be \(q_{i}^{J,K}\) for appropriate components, and define morphisms
\[ q_{U,i}^{J} : \hat{U} \rightarrow \bigsqcup_{j,k} U_{i}^{j,k} = p_{i}^{-1}(U) \]
similarly. We let \(p_{U}^{J} : U^{J} \rightarrow U\) and \(\hat{p}_{U} : \hat{U} \rightarrow U\) denote the covering maps. We note that \(\hat{U}\) has the appropriate universal property also: i.e., if
\[ q_{i}''^{J} : U'' \rightarrow \bigsqcup_{j,k} U_{i}^{j,k} = p_{i}^{-1}(U) \]
is a morphism for each \(i\), then there exists a unique morphism \(q_{U,i}'' : U'' \rightarrow \hat{U}\), so that
\[ (6) \quad q_{U,i} \circ q_{U,i}'' = q_{i}'' . \]

We will now identify \(\hat{p}^{-1}(U)\) in \(V\) with \(\hat{U}\): Since \(q_{i} : \hat{U} \rightarrow p_{i}^{-1}(V)\) is a morphism, \(\hat{p}^{-1}(U)\) is mapped to \(p_{i}^{-1}(U)\) by \(q_{i}\). Thus, there is a morphism \(f : \hat{p}^{-1}(U) \rightarrow \hat{U}\) by the universal property of \(\hat{U}\). By construction, \(f\) sends \((u, (G_{i}^{j(i)}))_{i \in I}\) for \(p(u) \in U\) to a point of \(\hat{U}\) mapping to \(q_{i}(\gamma_{i}(u))\) under \(q_{U,i}\) for each \(i\). We obtain a morphism
\[ f|\hat{p}^{-1}(U') : \hat{p}^{-1}(U') \rightarrow \hat{p}_{U}^{-1}(U') , \]
which is an ordinary covering-isomorphism between fiber products of ordinary covering spaces \(\hat{p}^{-1}(U')\) and \(\hat{p}_{U}^{-1}(U')\) of \(U'\). (This follows since the two sets are obviously topological fiber-products over \(U'\) and \(f\) is the natural identification with a commutative diagram:
\[
\begin{array}{ccc}
\hat{p}^{-1}(U') & \xrightarrow{f} & \hat{p}_{U}^{-1}(U') \\
\downarrow q_{i} & & \downarrow q_{U,i} \\
p_{i}^{-1}(U') & \xrightarrow{id} & p_{i}^{-1}(U') ,
\end{array}
\]
By Proposition \(2\) we can identify \(\hat{U}\) as a suborbifold of \(\hat{V}\). Since \(q_{U,i} \circ f = q_{i}\) by the uniqueness part of equation \(6\) the orbifold-map \(q_{i}\) on \(\hat{V}\) extends \(q_{U,i}\) on the suborbifold \(\hat{U}\). Also, since \(\hat{p}_{U} \circ f = p\) while \(f\) is a morphism, \(\hat{p}_{U}\) is extended to \(\hat{p} : \hat{V} \rightarrow V\).

We let \(Yf\) be the quotient space of union of all \(\hat{U}s\) as \(U\) ranges over all open model subsets in the covering \(Z\) with identification given as above. We call \(Yf\) the orbifold-fiber-product of \(Ys\). We obviously have an orbifold-map \(p^f : Yf \rightarrow Y\) extended from \(p\) defined over the model neighborhoods. Obviously, base points \(y's\) in \(Ys\) correspond to a unique point in \(Yf\), to be denoted by \(y^f\) again.

We choose a set \(Y\) as a component of \(Yf\) containing the base point \(y^f\). Let \(\hat{p} : Yf \rightarrow Y\) be the restriction of \(p^f\).
The topology of $\hat{Y}$ is given by the basis which are sets of form of components of $V^J$ as $V$ ranges over the elementary neighborhoods of $Y$. There are well defined morphisms $\hat{p} : \hat{Y} \to Y$ and $q_i : \hat{Y} \to Y_i$ extending $p$ and $q_i$ on each sets of form $\hat{V}$. The extension exists by equation 6. We say that $\hat{Y}$ is a component of a fiber product of $Y_i$s.

We show that $\hat{Y}$ is Hausdorff and second-countable: Let $x$ and $y$ be two points of $\hat{Y}$. If $\hat{p}(x)$ and $\hat{p}(y)$ are distinct, then choose disjoint elementary neighborhoods in $Y$ and their inverse images are disjoint open sets. If $\hat{p}(x)$ equals $\hat{p}(y)$, then $x$ and $y$ are in different sets of form of a component of $V^J$ for some elementary open set $V$, then as components of $V^J$ are basis elements, $x$ and $y$ are contained in disjoint open sets. If $x$ and $y$ are in the same component $U$ of some $V^J$, then the inverse image in $\hat{V}$ of $x$ and that of $y$ meet up to an action of a finite group $G_V$ where $(V,G_V)$ is the model pair for $V$ considering $U$ as a quotient space of $\hat{V}$. Since $x$ and $y$ are distinct, there are certainly disjoint neighborhoods in $U$ as $U$ is a quotient space of $\hat{V}$ by a proper subgroup of $G_V$ where $x$ and $y$ are in different orbits.

Since each open set of form $V^J$ is locally compact, $\hat{Y}$ is locally compact, and $\hat{Y}$ is metrizable. Since a component of the topological fiber product $Y^r$ of $Y^r$ is separable, so is $\hat{Y}$; thus, $\hat{Y}$ is a Hausdorff second countable set. Since $\hat{Y}$ is covered by model neighborhoods, $\hat{Y}$ is an orbifold, and $\hat{p} : \hat{Y} \to Y$ is an orbifold-covering map.

Similarly, components of $\hat{Y}^J$ enjoy the same properties, i.e., they are Hausdorff second countable and hence are orbifolds. Thus, the fiber-product $\hat{Y}^J$ is a disjoint union of orbifolds. Therefore, $p^J : \hat{Y}^J \to Y$ is an orbifold-covering map, i.e., the aim of our construction.

We verify a universal property: Let $(Z,p_2)$ be a covering orbifold of $X$ so that there are covering morphisms $q_i^Z : Z \to Y_i$ sending the base point $z$ to $y_i$s where $p_2(z) = p_i(y_i) = y$. We show that there exists a covering morphism $Z \to \hat{Y}^J$ sending $z$ to $y^J$. Define $Z^r$ to be the inverse image of $Y^r$ in $Y$. Then $Z^r$ is an open dense set, and it has a morphism to $\hat{Y}^J$ over $X^r$ sending $z$ to $y^J$. Since $\hat{Y}^J$ is a fiber-product of $Y_i$s. Now, there is a unique extension of this map from $Z$ to $\hat{Y}^J$ since the extensions are given by the universal properties of fiber-products of the inverse images of model neighborhoods.

**Example 4.** For reader’s convenience, let us explain Thurston’s example in Chapter 5 of [23]. We consider two orbifold-coverings of an interval $I = [-1,1]$ with two mirror points at the end. Then $I$ is covered by a circle $\mathbb{S}^1$ where the covering map is given by a projection to the $x$-axis. This is a regular covering with $\mathbb{Z}_2$ acting by reflections in the $x$-axis. Let $p_1 : \mathbb{S}^1 \to I$ be this covering. The next covering is an orbifold-covering $p_2 : J \to I$ where $J$ is an interval $[-1,3]$ with two mirror points at the end. Define $p_2 : J \to I$ by $p_2(t) = t$ if $-1 \leq t \leq 1$ and $p(t) = -t + 2$ if $1 \leq t \leq 3$. Again, this is a regular covering.

Cover $I$ by three open sets $I_1 = [-1,-\epsilon), I_2 = (-2\epsilon, 2\epsilon), I_3 = (\epsilon, 1]$ where $0 < \epsilon < 1/2$. The above construction tells us that the fiber product of inverse images of $I_2$ is a union of four arcs. Over $I_1$, it is a union of two arcs. Over $I_3$, it is a union of two arcs. Since the fiber products over $(-2\epsilon, -\epsilon)$ and $(\epsilon, 2\epsilon)$ are unions of four arcs obviously identifiable as subsets, we clearly see that the fiber product is isomorphic to a four-fold covering $\mathbb{S}^1 \to I$ with four open arcs mapping to $I_1$ or $I_3$ as two-fold coverings.
In the note of Thurston \[23\], he proved that each orbifold \( Y \) has a so-called universal covering orbifold \( \tilde{Y} \) with an orbifold-map \( p_Y : \tilde{Y} \to Y \) so that given an orbifold-covering map \( p_Z : Z \to Y \) where \( Z \) has a connected underlying space, there is an orbifold-map \( q : \tilde{Y} \to Z \) so that \( p \circ q \) equals \( p_Y \).

We give a more precise definition: Let \( y \) be a regular base point of \( Y \). A universal covering orbifold is a connected covering \((\tilde{Y}, p_Y)\) with a regular base point \( \tilde{y} \) mapping to \( y \) so that for any covering \( p_Z : Z \to Y \) where \( Z \) is a connected orbifold with a regular base point \( z \) over \( y \) there is an orbifold-covering-morphism \( q : \tilde{Y} \to Z \) so that \( q(\tilde{y}) = z \) and \( p_Z \circ q = p_Y \). We require that this should hold for any choices of regular base points \( y, \tilde{y} \in p_Y^{-1}(y) \), and \( z \in p_Z^{-1}(y) \).

The uniqueness of the universal covering of an orbifold up to covering isomorphisms (preserving base points) is obvious from the definition.

The group of automorphisms of a universal cover of \( Y \) is said to be the fundamental group, and we denote it by \( \pi_1(Y) \).

**Proposition 8** (Thurston). Let \( Y \) be a connected orbifold. Then there exists a universal covering orbifold \( \tilde{Y} \) unique up to covering isomorphism. Moreover, the fundamental group of \( \tilde{Y} \) acts transitively on the inverse image of the base point \( y \).

**Proof.** Let \( \tilde{Y} \) be obtained by a list of coverings \((Y_i, p_i)\) with base point \( y_i \) over a base point \( y \) which lists one-element from each isomorphism class of covering maps of \( Y \) preserving base-points. Let \( \tilde{y} \) be the base point from the above construction.

Given any covering \( p : Z \to Y \) with a base point \( z \), since we obviously have our \( Z \) isomorphic to say \( Y_i \), \( i \in I \), there exists a covering morphism \( q_Y : \tilde{Y} \to Z \) where \( q_Y(\tilde{y}) = \tilde{y} \).

Let \( y' \) be a point of \( p^{f^{-1}}(y) \) different from \( y \). We show that there exists a deck transformation sending \( y' \) to \( y' \): Clearly, \((\tilde{Y}, \tilde{p})\) with \( \tilde{y} \) as a base point is in the list of all covering maps of \( Y \). Thus, there exists a morphism \( g : \tilde{Y} \to \tilde{Y} \) sending \( \tilde{y} \) to \( y' \). By Proposition 8 \( g \) is a deck transformation.

Now, let \((Z, p_Z)\) be a cover of \( Y \) with a base point \( z \) mapping to a base point \( x \) of \( Y \), perhaps different from above \( y \). Let \( y' \) be a point of \( p^{f^{-1}}(x) \) of \( \tilde{Y} \) (deemed to be our new base point). Find a path \( \alpha \) from \( y' \) to \( \tilde{y} \), which maps to a path \( \alpha' \) from \( x \) to \( y \) on \( Y \), and find a path \( \alpha'' \) on \( Z \) from \( z \) ending at a point \( z' \) lifting \( \alpha' \). Then \( p_Z(z') = y = p^{f}\(\tilde{y}\) \). By above construction of fiber-products, there exists a morphism \( g : \tilde{Y} \to Z \) so that \( g(\tilde{y}) = \tilde{y} \) for some \( y'' \in p^{-1}(y) \) since \( \tilde{Y} \) and \( Z \) are connected. By precomposing with \( g \) a deck transformation, we may assume that \( g(\tilde{y}) = z' \). Since \( \alpha \) goes to \( \alpha'' \), we see that \( g(y') \) maps to \( z \). Therefore \( \tilde{Y} \) satisfies the definition of a universal cover. \( \Box \)

**Proposition 9.** Let \( p_1 : \tilde{Y} \to Y_1 \) and \( p_2 : \tilde{Y} \to Y_2 \) be universal covering orbifold-maps. Then any orbifold-map \( f : Y_1 \to Y_2 \) covering an orbifold-diffeomorphism \( g : Y \to Y \) lifts to an orbifold-diffeomorphism \( \tilde{f} : \tilde{Y} \to \tilde{Y} \). The lift is unique if we decide the value \( f(y_0) \) among \( p_2^{-1}(f(p_1(y_0))) \) for a base-point \( y_0 \) of \( Y \), where \( \tilde{f}(y_0) \) can be chosen to be any such point. Finally, if \( g \) is the identity, then \( \tilde{f} \) is an automorphism.

**Proof.** Since \( f : Y_1 \to Y_2 \) is a covering map, \( f \circ p_1 \) is the universal covering of \( Y_2 \). Since \( p_2 \) is also a universal covering of \( Y_2 \), there is a morphism \( \tilde{f} : \tilde{Y} \to \tilde{Y} \) so that \( p_2 \circ \tilde{f} = f \circ p_1 \) by the uniqueness of the isomorphism class of universal covering orbifolds. The uniqueness follows since \( \tilde{f} \) restricts to \( \tilde{Y}^r \) a lift of \( f \) restricted \( Y_1^r \)
and the ordinary covering space theory. The freedom of choice follows by the transitivity of automorphism group in the fiber of $Y \to Y_2$. The final statement follows trivially.

The fundamental group $\pi_1(Y)$ acts on the universal cover $\tilde{Y}$ as a group of orbifold-diffeomorphisms. The action is proper since given a point of a model neighborhood of $Y$ which is a component of a preimage of a model neighborhood of $Y$, it may return to the neighborhood only finitely many times since the only equivalent points are in the orbits. Therefore, $\tilde{Y}/\pi_1(Y)$ is again an orbifold.

We obtain most of the useful results of the covering space theory for topological spaces in the orbifold case.

**Corollary 2.** (i) The fundamental group acts transitively on each fiber of a universal cover $\tilde{Y}$, i.e., on the inverse image of a regular point $x$ of $Y$. Moreover, $\tilde{Y}/\pi_1(Y) = Y$.

(ii) Each covering space $p_1 : Y_1 \to Y$ is isomorphic to a covering map $p' : \tilde{Y}/\Gamma \to Y$ where $p'$ is induced from the universal covering map $p : \tilde{Y} \to Y$ and $\Gamma$ is a subgroup of $\pi_1(Y)$.

(iii) The isomorphism classes of covering spaces of $Y$ are in one-to-one correspondence with the conjugacy classes of subgroups of $\pi_1(Y)$.

(iv) The group of automorphisms of a covering space $\tilde{Y}/\Gamma$ is isomorphic to $N(\Gamma)/\Gamma$ where $N(\Gamma)$ is the normalizer of $\Gamma$ in $\pi_1(Y)$.

(v) A covering $Y' \to Y$ is regular if and only if $Y'$ is isomorphic to $\tilde{Y}/\Gamma$ for a normal subgroup $\Gamma$ of $\pi_1(Y)$.

(vi) Let $Y_1$ and $Y_2$ be orbifolds with universal covering orbifolds $\tilde{Y}_1$ and $\tilde{Y}_2$ respectively. A lift $f : \tilde{Y}_1 \to \tilde{Y}_2$ of an orbifold-diffeomorphism $g : Y_1 \to Y_2$ is an isomorphism.

*Proof.* (i) This follows from definition, i.e., the condition on base-points. The covering map $p_Y : \tilde{Y} \to Y$ induces an orbifold-map

$$\tilde{Y}/\pi_1(Y) \to Y$$

which is one-to-one over the regular part. $p_Y$ restricts to a homeomorphism over the regular part since it is proper and is a local homeomorphism. Therefore, $p_Y$ is an orbifold-diffeomorphism by Proposition 2.

(ii) There is a morphism $q : \tilde{Y} \to Y_1$. Since $\tilde{Y}$ covers any cover of $Y_1$ also, $\tilde{Y}$ is again a universal cover of $Y_1$. Since $Y_1$ is a quotient of its universal cover, (ii) follows.

(iii) If two covering spaces $p_1 : \tilde{Y}/\Gamma_1 \to Y$ and $p_2 : \tilde{Y}/\Gamma_2 \to Y$ are isomorphic, then there exists a morphism $f : \tilde{Y}/\Gamma_1 \to \tilde{Y}/\Gamma_2$. $f$ lifts to a morphism $\gamma : \tilde{Y} \to \tilde{Y}$ by Proposition 2. Since $f$ is a morphism, $f$ satisfies $p_2 \circ f = p_1$, and so $f$ covers identity on $Y$. Thus, the lift $\gamma$ of $f$ is a deck transformation of $p_Y$. In order that $\gamma$ descends to a map $f$, we need that for each $\alpha \in \Gamma_1$, there exists $\alpha' \in \Gamma_2$ so that $\gamma \alpha = \alpha' \gamma$. Thus, $\gamma \Gamma_1 \gamma^{-1} \subset \Gamma_2$. A converse argument shows that a conjugate of $\Gamma_2$ is in $\Gamma_1$. Hence, $\Gamma_1$ and $\Gamma_2$ are conjugate.

(iv) This follows from (iii).

(v) Given a base point $y'$ of $\tilde{Y}$ mapping to a base point $y$ of $Y$, each point of $p_Y^{-1}(y)$ is of form $\gamma(y')$ for $\gamma \in \pi_1(Y)$. Given $\tilde{Y}/\Gamma$ for a subgroup $\Gamma$ of $\pi_1(Y)$, and two points in the fiber of $y$, we see that there exists a deck transformation $\gamma$ sending
Proposition 10. Let $Y$ be an orbifold and $f : Y \to Y$ and $g : Y \to Y$ orbifold-diffeomorphisms with homotopy $H : Y \times I \to Y$. Then for any choice of lift $\tilde{f} : \tilde{Y} \to \tilde{Y}$ of $f$, there is a unique lift $\tilde{H} : \tilde{Y} \times I \to \tilde{Y}$ which becomes a homotopy between $\tilde{f}$ and a lift $\tilde{g} : \tilde{Y} \to \tilde{Y}$ of $g$.

Proof. Clearly, the lift $H' : \tilde{Y}' \times I \to \tilde{Y}'$ of $H|_{Y' \times I} : Y' \times I \to Y'$, which is a homotopy between $\tilde{f}|_{Y'} : Y' \to Y'$ and a map $g' : \tilde{Y}' \to \tilde{Y}'$, exists. We give a local description.

Let $(x,t)$ be a point of $Y \times I$. There is a model open subset $(N \times J, G_N)$ of $(x,t)$ where $G_N$ acts so that $(y,s) \mapsto (g(y), s)$ for $g \in G_N$ where $J$ is a small open interval containing $t$. $H$ lifts to $H^1 : N \times J \to M$ where $M$ is a model open subset where a finite group $G_M$ also acts on, and there is a homomorphism $k : G_N \to G_M$ so that $H^1(g(y), s) = k(g)(H^1(y, s))$. We also have $f(g(y)) = k(g)(f(y))$ for $g \in G_N$ and $k(g) \in G_M$.

Let $q(N \times J)$ be the neighborhood of $(x,t)$ in $Y \times I$ corresponding to $N \times J$. Take a component $N' \times J$ of the inverse image of $q(N \times J)$ in $Y \times I$. Then $\tilde{f}$ maps it into a component $M'$ of the inverse image of $M$ in $\tilde{Y}$. Moreover, we may assume without loss of generality that $\tilde{f}$ has the same model as above $N$ and $N'$ and an associated homomorphism $k : G'' \to G''$ which becomes a homotopy $H' : \tilde{Y} \times I \to \tilde{Y}$. Obviously, a lift $\tilde{g}$ is obtained by restricting $\tilde{H}$. □

Given orbifolds $M$ and $N$, and an orbifold-diffeomorphism $f : M \to N$ which lifts to a diffeomorphism $\tilde{f} : \tilde{M} \to \tilde{N}$, we obtain an induced homomorphism $\tilde{f}_* : \pi_1(M) \to \pi_1(N)$. For each deck-transformation $\vartheta$ of $M$, let $\tilde{f}_*(\vartheta)$ be the deck-transformation $\tilde{f} \circ \vartheta \circ \tilde{f}^{-1}$.

By Proposition 11 given orbifold-diffeomorphisms $f_1, f_2 : M \to N$ with a homotopy $H$, a lift $\tilde{f}_1 : \tilde{M} \to \tilde{N}$ of $f_1$ is homotopic to a lift $\tilde{f}_2 : \tilde{M} \to \tilde{N}$ of $f_2$ by a lift of a homotopy $H : \tilde{M} \times [0, 1] \to \tilde{N}$.

Proposition 11. If $\tilde{f}_2 : \tilde{M} \to \tilde{N}$ is a diffeomorphism homotopic to $\tilde{f}_1$ by a homotopy $h : M \times [0, 1] \to \tilde{N}$ equivariant with respect to $f_1 : \pi_1(M) \to \pi_1(N)$, then $\tilde{f}_2 \circ \tilde{f}_1^{-1}$ is homotopic to $\gamma' = \tilde{f}_2 \circ \gamma \circ \tilde{f}_1^{-1}$, and let $H$ be the homotopy between them given by $H_t = h_t \circ \gamma \circ h_t^{-1}$ for $t \in [0, 1]$. Then $H_t : \tilde{N} \to \tilde{N}$ is a deck transformation for
each $t \in [0, 1]$. (To see this simply post-compose $H_t$ with the covering map of $N$.) Since the group of deck transformations is discrete in $C^\omega$-topology, $\gamma'$ and $\gamma''$ are equal.

**Remark 1.** We are not aware of the full theory of liftings of maps of orbifold-covering spaces using fundamental groups. But it might be desirable to have one for other purposes than required in this paper. At any rate, the inclusion of such a theory will fully complete the orbifold-covering-space theory, which might be an interesting project.

**Remark 2.** For two-dimensional orbifolds, the constructions of the universal covers are considerably simpler, and are exposed in Scott [22].

**Remark 3.** A good orbifold is an orbifold with a covering orbifold that is a manifold. Since its universal covering orbifold covers a manifold, each of the model pairs of the universal covering orbifold has a trivial group action. Thus, the universal covering orbifold is a manifold.

A very good orbifold is an orbifold with a finite regular cover that is a manifold.

A good orbifold $Y$ is always orbifold-diffeomorphic to $M/\Gamma$ where $M$ is a simply-connected manifold and $\Gamma$ is a discrete group acting on $M$ properly.

A good orbifold $M$ has a covering that is a simply-connected manifold $\tilde{M}$. Then it is a universal covering orbifold. Finally, if $Y = M/\Gamma$ and $M$ is simply connected, then $\pi_1(Y)$ equals $\Gamma$.

4. *(G,X)-structures on orbifolds*

A *(G,X)-structure* on an orbifold $M$ is a collection of charts $\phi_U : U \to X$ for each model pair $(U,H_U)$ so that $\phi_U$ conjugates the action of $G_U$ with that of a finite subgroup $G_U$ of $G$ on $\phi(U)$ by an isomorphism $i_U : H_U \to G_U$, and the inclusion map induced map $U \to \tilde{V}$ is always realized by an element $\vartheta$ of $G$ and the homomorphism $G_U \to G_V$ is given by a conjugation by $\vartheta$; i.e., $g \mapsto \vartheta \circ g \circ \vartheta^{-1}$. A different choice equals $\varphi \vartheta$ for $\varphi \in G_U$ and the homomorphism change by conjugation by $\varphi$. (Again, we need the assumption that the action of $G$ is locally faithful)

A maximal such family of collections $(\phi_U, i_U)$ is said to be a *(G,X)-structure* of $M$. A *(G,X)-structure* on $M$ induces *(G,X)-structures* on its covering orbifolds.

If an orbifold $M$ has a *(G,X)-structure*, then we can choose a model pair $(U,G_U)$ for each model set where $U \subset X$ and a subgroup $G_U$ of $G$ using the charts.

A *(G,X)-map* $f$ between two *(G,X)-orbifolds* $M$ and $N$ is a map so that for each point $x$ of $M$ and a point $y$ of $N$ so that $x = f(y)$, and a neighborhood $U$ of $x$ modeled on a pair $(\tilde{U},H_U)$ with a chart $\phi_U$ and an isomorphism $i_U : H_U \to G_U \subset G$, there is a neighborhood $\tilde{V}$ of $y$ modeled on a pair $(\tilde{V},H_V)$ with a chart $\phi_V$ and an isomorphism $i_V : H_V \to G_V$ so that $f$ lifts to a map $\tilde{f} : \tilde{V} \to \tilde{U}$ equivariant with respect to a homomorphism $H_V \to H_U$ induced by a homomorphism $G_V \to G_U$ given by a conjugation $g \mapsto \vartheta g \vartheta^{-1}$ by some $\vartheta \in G$.

**Theorem 2** (Thurston). A *(G,X)-orbifold* $M$ is a good orbifold. There exists an immersion $D$ from the universal covering manifold $\tilde{M}$ to $X$ so that

$$D \circ \vartheta = h(\vartheta) \circ D, \vartheta \in \pi_1(M)$$
holds for a homomorphism \( h : \pi_1(M) \to G \), where \( D \) is a local \((G,X)\)-map. Moreover, any such immersion equals \( g \circ D \) for \( g \in G \), with the associated homomorphism \( g \circ h(\cdot) \circ g^{-1} \).

**Proof.** This is found in Chapter 5 of Thurston [23] (see also Bridson-Haefliger [11] and Matsumoto and Montesinos-Amilibia [16]). We rewrite it here for the reader’s convenience: Let \( N \) be a neighborhood of \( x \in \Sigma \), and \((\tilde{N}, H_N)\) the model pair for \( \tilde{N} \) an open set in \( X \) and \( H_N \) the associated finite group acting on \( \tilde{N} \). We form \( G \times \tilde{N} \) and give an action of \( H_N \) by \( \gamma(g,y) = (\gamma g, \gamma y) \). Then \( G(N) = (G \times \tilde{N})/H_N \) is a manifold and has a projection \( p_N : G(N) \to N \) induced by the projection to the second factor. (Here, \( p_N \) is an orbifold-map.)

We find a nice locally finite cover of \( M \) by such neighborhoods \( \{N_1, N_2, \ldots\} \). If \( N_i \) and \( N_j \) meet, then \( N_i \cap N_j \) has an inclusion map \( i : N_i \cap N_j \to N_i \). Then there is a connected open subset \( A \) of \( \tilde{N}_i \) and a subgroup \( H_A \) acting on it being a model for \( N_i \cap N_j \). We form \( G(N_i \cap N_j)_A \) where \( A \) denotes the fact we used \( A \) as a model and find a map \( \tilde{i} : G(N_i \cap N_j)_A \to G(N_i) \) induced by \( (g,x) \mapsto (g,x) \). The map \( i \) is a well-defined imbedding since a different choice \( i \) gives us \( \vartheta \circ i, \vartheta \in G \), and so \( i \) is replaced by a map \( (g,x) \mapsto (\vartheta g, \vartheta x) \).

We find an open subset \( B \) of \( \tilde{N}_j \) corresponding to \( N_i \cap N_j \), and form \( G(N_i \cap N_j)_B \) similarly, and find an imbedding \( G(N_i \cap N_j)_B \to G(N_j) \). Since \( G(N_i \cap N_j)_A \) and \( G(N_i \cap N_j)_B \) can be identified by the identification of the model pairs, we see that \( G(N_i) \) and \( G(N_j) \) can be pasted by this relation. We can easily show that such identifications of \( G(N_1), G(N_2), \ldots \) are possible, and obtain a manifold \( G(M) \) from the identifications.

The foliation of \( G(N_i) \) with leaves that are images of \( g \times \tilde{N}_i \) for \( g \in G \) gives rise to a foliation on \( G(M) \) whose leaves meet the fibers at unique points. Take a leaf \( L \) in \( G(M) \). Then \( p_N|L : L \to M \) is an orbifold-covering map and \( L \) is a manifold. Take a universal cover \( \tilde{L} \) of \( L \) with covering map \( p_L \). Then \( p_N \circ p_L \) is a universal covering map of \( M \). \( L \) has a \((G,X)\)-structure since it covers \( M \): one can induce charts. Then \( \tilde{L} \) has a \((G,X)\)-structure.

Since by Remark 4 \( \tilde{L} \) is a universal cover of \( M \), Corollary 2 implies that \( \tilde{L}/\Gamma \) for the fundamental group \( \Gamma \) is \((G,X)\)-diffeomorphic to \( M \) by a map induced by \( p_N \circ p_L \). As \( \tilde{L} \) is a \((G,X)\)-manifold, \( M \) has a developing map \( D : \tilde{L} \to X \) (which follows from the geometric structure theory for manifolds). For a deck transformation \( \gamma, D \circ \gamma \) is also a \((G,X)\)-map, and this means that \( D \circ \gamma = h(\gamma) \circ D \) for some \( h(\gamma) \in G \). We can clearly verify that \( h : \Gamma \to G \) is a homomorphism. The rest of the conclusion follows in the same way as the geometric structure theory for manifolds.

**Remark 4.** In most cases, geometric orbifolds are also very good due to Selberg’s lemma since our Lie groups are often subgroups of linear groups.

We now assume that \( M \) is a compact \((G,X)\)-orbifold with a universal cover \( \tilde{M} \). A pair \((D,h)\) of immersions \( D : \tilde{M} \to X \) equivariant with respect to a homomorphism \( h : \pi_1(M) \to G \) is said to be a development pair of \( M \). \( D \) is called a developing map and \( h \) a holonomy homomorphism. Conversely, given such a pair \((D,h)\), they give charts to \( \tilde{M} \), and hence induces a \((G,X)\)-structure on \( \tilde{M} \). Since a deck-transformation is a \((G,X)\)-map \( \tilde{M} \to \tilde{M} \), we see that \( M = \tilde{M}/\pi_1(M) \) has an induced \((G,X)\)-structure from \( \tilde{M} \).
We say that two such pairs \((D, h)\) and \((D', h')\) are \(G\)-equivalent if \(D' = \vartheta \circ D\) and \(h'(z) = \vartheta \circ h(z) \circ \vartheta^{-1}\) for \(\vartheta \in G\).

Let us look at the set \(\mathcal{M}(M)\) of all \((G, X)\)-structures on \(M\) and introduce an equivalence relation that two \((G, X)\)-structures \(\mu_1\) and \(\mu_2\) are equivalent if there is an isotopy \(\phi : M \to M\) so that the induced \((G, X)\)-structure \(\phi^*(\mu_1)\) obtained by pulling back charts equals \(\mu_2\). The deformation space of \((G, X)\)-structures on \(M\) (without topology) is defined to be this set \(\mathcal{M}(M)/\sim\).

We reinterpret this space as follows: consider the set of diffeomorphisms \(f : M \to M'\) where \(M'\) is a \((G, X)\)-manifold. We introduce an equivalence relation that \(f\) and \(f' : M \to M''\) are equivalent if there is a \((G, X)\)-diffeomorphism \(\phi : M' \to M''\) so that \(\phi \circ f\) is isotopic to \(f'\). The set of equivalence classes corresponds in one-to-one manner with the above space by sending \(f : M \to M'\) to \(f_*\) for the \((G, X)\)-structure \(\mu\) on \(M'\).

We present yet another version of this set: Clearly, \(\tilde{M} \times I\) is a universal cover of \(M \times I\) and the group of deck transformation group is isomorphic to \(\pi_1(M)\) with an obvious action. We identify \(\pi_1(M \times I)\) with \(\pi_1(M)\). Consider the set of diffeomorphisms \(\tilde{f} : \tilde{M} \to \tilde{M}'\) equivariant with respect to an isomorphism \(\pi_1(M) \to \pi_1(M')\) where \(M'\) is a \((G, X)\)-manifold. Introduce an equivalence relation that \(\tilde{f} : \tilde{M} \to \tilde{M}'\) and \(\tilde{f}' : \tilde{M} \to \tilde{M}''\) are equivalent if there are a \((G, X)\)-diffeomorphism \(\tilde{\phi} : \tilde{M}' \to \tilde{M}''\) and an isotopy \(H : \tilde{M} \times [0, 1] \to \tilde{M}''\) equivariant with respect to the isomorphisms

\[\tilde{\phi}_* : \pi_1(M') \to \pi_1(M'')\]

respectively so that

\[H_0 = \tilde{\phi} \circ \tilde{f}\]

Let us denote this space by \(D_1(M)\). The set of the equivalence classes is certainly in one-to-one correspondence with the above set since two different choices of lifts of \(f : M \to M'\) differ by a deck transformation of \(M'\) which is a \((G, X)\)-diffeomorphism.

We now give the final version in order to introduce topology: Following Lok’s thesis [13], we define the isotopy-equivalence space \(S(M)\) of \((G, X)\)-structures for a compact orbifold \(M_0\) to be the space of equivalence classes of pairs \((D, \tilde{f} : M_0 \to \tilde{M})\) where \(\tilde{f}\) is a diffeomorphism equivariant with respect to an isomorphism \(\pi_1(M_0) \to \pi_1(M)\) and \(D : \tilde{M} \to X\) is an immersion equivariant with respect to a homomorphism \(\pi_1(M) \to G\). Two such pairs \((D, \tilde{f})\) and \((D', \tilde{f}')\) are isotopy-equivalent if and only if there are a diffeomorphism \(\tilde{\phi} : \tilde{M} \to \tilde{M}'\) lifting a diffeomorphism \(\phi : M \to M'\) with \(D' \circ \tilde{\phi} = D\) and an isotopy \(H : \tilde{M} \times [0, 1] \to \tilde{M}'\) equivariant with respect to the isomorphism \(\tilde{\phi}_* : \pi_1(M) \to \pi_1(M')\) so that \(H_0 = \tilde{\phi} \circ \tilde{f}\) and \(H_1 = \tilde{f}'\).

The topology is given on the set of pairs by \(C^\infty\)-topology on \(D \circ \tilde{f}\), i.e., a sequence of functions converges if it does on every compact subset of \(M_0\) uniformly in \(C^\infty\)-sense. \((s \geq 1\) is sufficient for all purposes.\) We give the quotient topology on \(S(M_0)\).

There is a natural \(G\)-action on \(S(M_0)\) given by

\[\gamma(D, \tilde{f}) = (\gamma \circ D, \tilde{f}), \gamma \in G.\]

Let \(D(M_0)\) be the quotient space under this action. Then \(D_1(M_0)\) and \(D(M_0)\) are also in one-to-one correspondence given by sending \(\tilde{f} : M_0 \to M'\) to the equivalence class of \((D, \tilde{f})\) where \(D : M' \to X\) is a developing map of \(M'\). Therefore, we call \(D(M_0)\) the deformation space of \((G, X)\)-structures on \(M_0\).
The set of all homomorphisms $h : \pi_1(M) \to G$ is denoted by $\text{Hom}(\pi_1(M), G)$. We restrict our attention to the case that $\pi_1(M)$ is finitely presented. Let $g_1, \ldots, g_n$ denote the generators of $\pi_1(M)$, and $R_1, \ldots, R_m$ the relations. Then $H = \text{Hom}(\pi_1(M), G)$ can be injectively mapped into $G^n$ by sending a homomorphism $h$ to the element $(h(g_1), \ldots, h(g_n))$ corresponding to generators. The relations give us the subset of $G^n$ where $H$ can lie. Actually, the subset defined by the relations gives us precisely the image. Thus, we identify $H$ with this subset. The subset has a subspace topology of a real algebraic set, which we give to $H$. Obviously, it is a metric space if $G$ has a metric.

There is an action by conjugations on $H$ sending a homomorphism $h(\cdot)$ to $\vartheta \circ h(\cdot) \circ \vartheta^{-1}$ for $\vartheta \in G$. $H/G$ may not be a Hausdorff space. There is a subset $H^s$ of $H$, where $G$ acts properly, consisting of points lying in stable orbits when $G$ is the group of $\mathbb{R}$-points of an algebraic group $\bar{G}$ defined over $\mathbb{R}$. $H^s/G$ is a Hausdorff real analytic space.

We define a pre-holonomy map

$$\mathcal{P}\mathcal{H} : \mathcal{S}(M_0) \to \text{Hom}(\pi_1(M_0), G)$$

by sending $(D, \vec{f} : M_0 \to \hat{M})$ to the holonomy representation $h \circ \vec{f}_*$ where $\vec{f}_*$ is the induced homomorphism $h(\pi_1(M_0)) \to \pi_1(M)$.

First of all, this is well-defined: Let $(D', \vec{f}' : M_0 \to \hat{M})$ be an equivalent pair. Then $D = D' \circ \vec{f}$ for a lift $\vec{f}$ of an isotopy $\phi : M \to M'$. For a deck transformation $\vartheta$ of $\hat{M}_0$, we obtain

$$h(\vec{f}_*(\vartheta)) \circ D = D \circ \vec{f} \circ \vartheta \circ \vec{f}^{-1} = D' \circ \vec{f}' \circ \vartheta \circ \vec{f}'^{-1} \circ \vec{f}^{-1} \circ \vec{f} \circ \vartheta = D' \circ \vec{f}' \circ \vartheta \circ \vec{f}'^{-1} \circ \vec{f} \circ \vartheta$$

by Proposition 11

$$= h'(\vec{f}'_* (\vartheta)) \circ D' \circ \vec{f}'$$

(7)

Therefore, we obtain $h \circ \vec{f}_* = h' \circ \vec{f}'_*$.

Also $\mathcal{P}\mathcal{H}$ is continuous: Let $C(M_0)$ denote the space of pairs $(D, \vec{f} : M_0 \to \hat{M})$ with $C^*$-topology. Then sending $(D, \vec{f} : M_0 \to M)$ to $h \circ \vec{f}_*$ is a continuous map, which we denote by

$$\mathcal{P}\mathcal{P}\mathcal{H} : C(M_0) \to \text{Hom}(\pi_1(M), G)$$

for later purposes: the $C^*$-convergence of sequence of $D_i \circ \vec{f}_i$ restricted on compact subsets for a sequence of pairs $(D_i, \vec{f}_i)$ implies the uniform $C^\infty$-convergence of $h_i(\vec{f}_*(\vartheta))$ for each deck transformation $\vartheta$. (The sequence of locally defined maps $D_i \circ \vec{f}_i \circ \vartheta \circ \vec{f}_i^{-1} \circ D_i^{-1}$ converges in $C^\infty$-topology in sufficiently small compact domains in $X$ and hence in $C^\infty$-topology as $G$ acts smoothly on $X$. $M_0$ needs to be compact here.)

5. The proof of Theorem 1

Again, let $G$ be a Lie group acting on a space $X$ smoothly with the local properties mentioned above. Let us now present three lemmas 9, 14 and 15 on the perturbation of the finite group actions and conjugation by diffeomorphisms.

Lemma 3. Let $G_B$ be a finite subgroup of $G$ acting on an $n$-ball $B$ in $X$. Let $h_t : G_B \to G$, $t \in [0, \epsilon]$, $\epsilon > 0$, be an analytic parameter of representations of $G_B$ so
that $h_0$ is the inclusion map. Then for $0 \leq t \leq \epsilon$, there exists a continuous family of diffeomorphisms $f_t : B \to B_t$ to open balls $B_t$ in $X$ so that $f_t$ conjugates $h(G_B)$-action to $h_i(G_B)$-action; i.e., $f_t^{-1} h_t(g) f_t = h(g)$ for each $g \in G_B$ and $t \in [0, \epsilon]$.

Proof. We take a product $X \times [0, 1]$ and let $v$ be a vector field in the positive $[0, 1]$-direction in the product space. Then $G_B$ acts smoothly on $X \times [0, 1]$ by sending $(x, t)$ to $(h_t(g)(x), t)$ for $g \in G_B$. We average $g^*(v)$ for $g \in G_B$ to obtain a smooth $G_B$-invariant vector field $V$. The integral curve $l$ of $V$ starting from $(x, 0)$ is mapped to an integral curve $m$ of $V$ starting from $(g(x), 0)$ by the $G_B$-action. Thus, the endpoint $l(1)$ is sent to $m(1)$, and so $g(l(1)) = m(1)$. Hence, let $f'_t(x)$ equal the point of the path from $(x, 0)$ at time $t$. Then $f_t = p \circ f'_t : X \to X$ for the projection $p : X \times [0, 1] \to X$ is a desired diffeomorphism and $f_t(B)$ is the desired open ball.

A point $x$ of a real algebraic set has a neighborhood with a semi-algebraic homeomorphism to a cone over a semi-algebraic set $S$ in the boundary of a small ball with a cone-point at the origin corresponding to $x$.

Lemma 4. Let $G_B$ be a finite subgroup of $G$ acting on an $n$-ball $B$. Suppose that $h$ is a point of an algebraic set $V = \text{Hom}(G_B, G)$ for a finite group, and let $C$ be a cone-neighborhood of $h$. Then for each $h' \in C$, there is a corresponding diffeomorphism

$$f_{h'} : B \to B_{h'}, B_{h'} = f_{h'}(B)$$

so that $f_{h'}$ conjugates the $h(G_B)$-action on $B$ to the $h'(G_B)$-action on $B_{h'}$; i.e., $f_{h'}^{-1} h'(g) f_{h'} = h(g)$ for each $g \in G$. Moreover, the map $h' \mapsto f_{h'}$ is continuous from $C$ to the space $C^\infty(B, X)$ of smooth functions from $B$ to $X$.

Proof. Parameterize $C$ by $[0, \epsilon] \times S$ for a semi-algebraic set $S$ with $\{0\} \times S$ corresponding to $h$ and, for each $x \in S$, there is a map $[0, \epsilon] \times x \to C^\infty(B, X)$ from the above lemma. Again, we obtain a smooth $G_B$-invariant vector field $V_x$ on $X \times [0, \epsilon]$ as above, and $V_x$ depends continuously on $x$. From this, we see that $f_{x,t}$ corresponding to a representation corresponding to $(x, t)$ depends continuously on $(x, t)$.

An isotopy of an embedded submanifold extends to one of the ambient manifold in a continuous manner, which is the following version of Cerf’s “first isotopy and extension theorem” (see Lok [13]):

Theorem 3. Let $F$ be a closed smooth submanifold of $X$ with corners. Let $\mathcal{E}(X)$ denote the space of isotopies $X \times [0, 1] \to X$ with the $C^s$-topology. Let $\mathcal{E}(F, X)$ denote the space of embeddings of $F$ in $X$ with $C^s$-topology. Consider the map

$$\Phi : \mathcal{E}(X) \to \mathcal{E}(F, X)$$

given by sending an isotopy $f_t$ to $f_t|F$. Then there is a neighborhood of the inclusion $i : F \to X$ of $\mathcal{E}(F, X)$ on which there is a continuous section $s$ of $\Phi$ and $s(i) = e$ where $e$ is the identity isotopy in $\mathcal{E}(X)$.

Continuing to use the notation of Lemma 4 we define a parameterization $l : S \times [0, \epsilon] \to C$ which is injective except at $S \times \{0\}$ which maps to $h$. (We fix $l$ although $C$ may become smaller and smaller). For $h' \in S$, we denote by $l(h')(t) = h$ be a ray in $S$ so that $l(h')(0) = h$ and $l(h')(\epsilon) = h'$. Let the finite group $G_B$ act on a submanifold $F$ of $B$. A $G_B$-equivariant isotopy $H : F \times [0, \epsilon] \to X$ is a map so that
$H_t$ is an imbedding for each $t \in [0, \epsilon')$, with $0 < \epsilon' \leq \epsilon$, conjugating the $G_B$-action on $F$ to the $l(h')t(G_B)$-action on $X$, where $H_0$ is an inclusion map $F \to X$. The above lemma 4 says that for each $h' \in C$, there exists a $G_B$-equivariant isotopy $H : B \times [0, \epsilon) \to X$. We will denote by $H(h')_t : B \to X$ the map obtained from $H$ for $h'$ and $t = \epsilon'$. Note also by the similar proof, for each $h' \in S$, there exists a $G_B$-equivariant isotopy $H : F \times [0, \epsilon) \to X$.

**Lemma 5.** Let $H : F \times [0, \epsilon') \times S \to X$ be a map so that $H(h') : F \times [0, \epsilon') \to X$ is a $G_B$-equivariant isotopy of $F$ for each $h' \in S$ where $0 < \epsilon' \leq \epsilon$ for some $\epsilon > 0$. Then $H$ can be extended to $\hat{H} : B \times [0, \epsilon'') \times S \to X$ so that $\hat{H}(h') : B \times [0, \epsilon'') \to X$ is a $G_B$-equivariant isotopy of $B$ for each $h' \in S$ where $0 < \epsilon'' \leq \epsilon'$.

**Proof.** Let $\mathcal{E}(C, F, X)$ denote the space of all $G_B$-equivariant isotopies of $F \to X$ with the above parametrization $l$. The maps $H(h')_t : F \to X$ are in $\mathcal{E}(F, X)$. There exists a section $s' : W \subset \mathcal{E}(C, F, X) \to \mathcal{E}(X)$ where $W$ is a neighborhood of $(h, l[F])$ by Theorem 5. Hence, there exists $\epsilon''$, $0 < \epsilon'' \leq \epsilon$, so that $H(h')_{\delta} \in W$ for $0 < \delta < \epsilon'$ for all $h' \in S$. We define $H'(x, \delta, h')$ for $x \in B$ to be $s(H(h')_{\delta})_1(x)$. This defines a function $H' : X \times [0, \epsilon'') \times S \to X$ so that $H'(h') : X \times [0, \epsilon'') \to X$ is an isotopy for each $h'$.

We modify $H'(h')$ to be $G_B$-equivariant. We define

$$H''(h') : X \times [0, \epsilon'') \times X \times [0, \epsilon'']$$

to be given by $H''(h')(x, t) = (H'(h')(x), t)$. Then $H''(h')(x, t)$ for a given $x$ is an integral curve of a vector field $V$ on $X \times [0, \epsilon'']$ with the component in the $[0, \epsilon'']$-direction equal to 1. Since $H(h')$ is an $G_B$-equivariant isotopy, we see that $V$ restricted to the image of $H''(h')(F \times [0, \epsilon'')]$ is $G_B$-invariant. We may now average the image vector fields of $V$ under $G_B$, and call $V'$ the resulting $G_B$-invariant vector field on $X \times [0, \epsilon'']$. Again $V'$ restricted to the image equals $V$ on the image and the second component equals 1. The integral curves of $V'$ give us a $G_B$-equivariant isotopy $\hat{H}(h') : X \times [0, \epsilon'') \to X$ extending $H(h') : F \times [0, \epsilon'') \to X$. Since the section $s$ is continuous, and we do averaging and integration, it follows that $\hat{H} : X \times [0, \epsilon'') \times S \to X$ is continuous. Now restrict $\hat{H}$ to $B \times [0, \epsilon'') \times S$. □

**Remark 5.** We choose some arbitrary Riemannian metric on a neighborhood of $B$, and can assume that the images of $f_h$s are all in this neighborhoods (see below). By our construction, given any $\epsilon > 0$, we can make sure that the $C^\infty$-norm of $f_h$, constructed in above lemmas, minus the inclusion map of $B$ is less than $\epsilon$ in some coordinate systems if we choose the neighborhoods $C$ sufficiently small near $h$. In particular, we can assume that for each $\epsilon > 0$, there is a neighborhood $C$ of $h$ so that $d(f_h(x), x) \leq \epsilon$ for $x \in B$ and $h' \in C$ where $f_h$ is obtained from above three lemmas.

If $B$ was strictly convex with smooth boundary, we see that $f_h(B)$ is also strictly convex with smooth boundary as the boundary convexity is given by a $C^2$-condition.

We can trivially generalize Lemma 4 so that $B$ could be a union of disjoint balls with some finite groups acting on each.

Now, we begin the step (I) of the proof of Theorem 1 as stated in the outline in the introduction:

Let $M$ be a compact $(G, X)$-orbifold. We choose a nice finite cover $U_1, \ldots, U_k$ of $M$ so that $\text{Cl}(U_i) \subset W_i$ and $\text{Cl}(W_i) \subset V_i$ for nice finite covers $W_1, \ldots, W_k$ and $V_1, \ldots, V_k$. (This can be done by change radii by small amounts. See the proof
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of Proposition [11] We assume that $U_i, W_i, V_i$ all have the open balls as the open subsets in the model pairs.

Give $M$ a Riemannian metric, and let $\tilde{M}$ be the universal cover of $M$. Since $\tilde{M}$ is a manifold, it has an induced Riemannian metric in the ordinary sense. The components of the inverse images of the balls $V_i$ above, are strictly convex balls which are images of exponential maps. By their strict convexity, any two of them meet in a strictly convex ball, i.e., in a contractible subset.

For each $V_i$, choose an arbitrary component $L_i$ in $\tilde{M}$ of its inverse image. $L_i$ is homeomorphic to an $n$-ball, and there exists a finite subgroup $\Gamma_i$ of the fundamental group $\Gamma$ of $M$ acting on $L_i$, and $(\Gamma_i, \Gamma_i)$ is a model pair for $V_i$. We choose $M_i$ and $N_i$ in $L_i$ corresponding to $U_i$ and $W_i$ respectively.

Given $i, j$, if $V_i$ and $V_j$ meet, then there exists a deck-transformation $\gamma_{ij}$ so that $L_i \cap \gamma_{ij} L_j \neq \emptyset$. The choice of $\gamma_{ij}$ is not unique if $\Gamma_i$ and $\Gamma_j$ are not trivial since one can always multiply $\gamma_{ij}$ in the left by an element of $\Gamma_i$ in the right by an element of $\Gamma_j$. Let $\Gamma_{ij}$ denote the all such possibilities for $L_i$ and $L_j$. Clearly, we have

$$\gamma^{-1} \in \Gamma_{ji} \text{ if and only if } \gamma \in \Gamma_{ij}.$$  

If $L_i \cap \gamma(L_i) \neq \emptyset$, then $\gamma$ acts on $L_i$ since $L_i$ is a normal neighborhood. Hence, we have

$$\Gamma_{ij} = \Gamma_i.$$  

Clearly, every element of $\Gamma_{ij}$ can be written $\gamma_1 \gamma_2$ where $\gamma_1 \in \Gamma_i$, $\gamma_2 \in \Gamma_j$, and $\gamma$ is a fixed element of $\Gamma_{ij}$. Thus, one can make sense of the statement that the coset space $\Gamma_{ij}/\Gamma_j$ is in one-to-one correspondence with $\Gamma_i$.

We note that $L_i \cap \gamma L_j$ for $\gamma \in \Gamma_{ij}$ is a convex ball, hence contractible. The same can be said for $M_i \cap \gamma M_j$ and $N_i \cap \gamma N_j$. We assume that $L_i \cap \gamma L_j \neq \emptyset$ if and only if $M_i \cap \gamma M_j \neq \emptyset$ if and only if $N_i \cap \gamma N_j \neq \emptyset$.

We claim that $\bigcup_{i,j} \Gamma_{ij}$ is a set of generators of $\pi_1(M)$. Let $\gamma \in \pi_1(M)$. Since $\tilde{M}$ is connected, there is a path from $L_1$ to $\gamma(L_1)$. There exists a collection $A_1, A_2, \ldots, A_n$ of open sets so that $A_1 = L_1$, $A_n = \gamma(L_1)$, $A_j \cap A_{j+1} \neq \emptyset$ for $j = 1, \ldots, n-1$, and $A_j$ is of form $\gamma_j(L_{k_j})$ for some $k_j$ and $j = 1, \ldots, n$. Since $A_j$ and $A_{j+1}$ meet, and so $\gamma_j(L_{k_j})$ and $\gamma_{j+1}(L_{k_{j+1}})$ meet, it follows that $\gamma^{-1}_j \gamma_{j+1}$ lies in $\Gamma_{k_j, k_{j+1}}$. We have

$$A_1 = L_1$$
$$A_2 = \gamma_{1k_2} L_{k_2}, \text{ for } \gamma_{1k_2} \in \Gamma_{1k_2}$$
$$A_3 = \gamma_{1k_2} \gamma_{2k_3} L_{k_3}, \text{ for } \gamma_{2k_3} \in \Gamma_{2k_3}$$
$$\vdots$$

$$A_n = \gamma(L_1) = \gamma_{1k_2} \gamma_{2k_3} \gamma_{3k_4} \cdots \gamma_{n-1, n} L_1, \text{ for } \gamma_{n-1} \in \Gamma_{k_{n-1}}$$

Thus, we see that

$$\gamma = \gamma' \gamma_{1k_2} \gamma_{2k_3} \gamma_{3k_4} \cdots \gamma_{n-1} \text{ for some } \gamma' \in \Gamma_1$$

We can write any element of $\Gamma$ as a product of elements in $\bigcup_{i,j} \Gamma_{ij}$.

Also, we see that

$$\gamma \circ \gamma' \in \Gamma_{ik}$$

if $\gamma \in \Gamma_{ij}$ and $\gamma' \in \Gamma_{jk}$ and $\gamma \circ \gamma'(x) \in L_i$ for some $x \in L_k$.

Next, we do the step (II) of the outline. More precisely, we will find a neighborhood $\Omega$ of $h \circ \tilde{f}_x$ in $\text{Hom}(\pi_1(M_0), G)$ so that there is a continuous map $s : \Omega \rightarrow C(M_0)$
where $\mathcal{P}\mathcal{H} \circ s$ is the identity map and $s(h \circ f_s) = (D, \tilde{f})$ where $D$ is a developing map $\tilde{M} \to X$ where $\tilde{M}$ is a universal cover of an orbifold $M$, and $\tilde{f}$ is a lift of a diffeomorphism $M_0 \to M$. The map $s$ induces a continuous map

$$\tilde{s} : \Omega \to \mathcal{S}(M_0),$$

which is a local section of $\mathcal{P}\mathcal{H}$.

This will be accomplished by the following steps: (i) First we specify $\Omega$. Perturbations in $\Omega$ induce deformations of the model neighborhoods, and we construct the orbifold using the deformations in $\Omega$ of the holonomy of the patching deck transformations. (ii) We show that the constructed orbifold is diffeomorphic to $M_0$. This will be done by patching together the deformation maps of model neighborhoods. (iii) We show that the diffeomorphism lifts to the diffeomorphisms of the universal cover. Using this fact, we can show that the deformed orbifold indeed has the desired deformed holonomy homomorphism.

(i) Let $(D, \tilde{f} : M_0 \to \tilde{M})$ be an element of $C(M_0)$. Let $h$ be the associated holonomy homomorphism $\pi_1(M_0) \to G$.

One can construct the underlying space of $X_M$ from $V_i$s. That is, we introduce an equivalence relation on the disjoint union $\bigsqcup_{i=1}^n L_i$ given by letting $x \sim y$ if $x = \gamma_{ij}(y)$ for $x \in L_i, y \in L_j$. Obviously, the orbifold structure is encoded in this construction; thus, we can construct $M$ back from these.

We can also construct $M$ from $\bigsqcup_{i=1}^n D(L_i)$ from the equivalence relation that $x \sim_M y$ if $x \in h(\gamma_{ij})(y)$ for $x \in D(L_i), y \in D(L_j)$. This is easily shown to be an equivalence relation (see equations $\mathbb{E}$, $\mathbb{F}$, and $\mathbb{G}$). Let $Q : \bigsqcup D(L_i) \to M$ denote the quotient map. The components of the inverse images of $Q(D(L_i))$ under the universal covering map $\tilde{M} \to M$ form a covering of $M$.

Remark 6. The open sets of form $\gamma(L_i)$ constitute a cover of the universal covering orbifold $\tilde{M}$. For given three sets $\gamma_i(L_{i1}), \gamma_i(L_{i2}),$ and $\gamma_i(L_{i3})$ so that $\gamma_i(L_{i1}) \cap \gamma_i(L_{i2}) \cap \gamma_i(L_{i3}) \neq \emptyset$ with $l = 1, 2, 3$ in cyclic sense, we require that $D$ restricted to their union should be an imbedding and their intersection should be of generic type in $C^\infty$ deformations of $L_i$s. We require the same pattern for $M_i$ and $N_i$ as well. (We don’t want a sudden change in the intersection pattern of these three sets, i.e., we need the stability.)

For convenience, we identify $M$ with $M_0$ and $\pi_1(M)$ with $\pi_1(M_0)$ by $\tilde{f} : M_0 \to \tilde{M}$: We choose a cone-neighborhood $\Omega$ of $h$ in $\text{Hom}(\pi_1(M), G)$ so that for each finite group $\Gamma_i$ associated with $L_i$ are in a neighborhood $\hat{C}_i$ of $\text{Hom}(\Gamma_i, G)$ satisfying Remark $\mathbb{H}$ for $B$ equal to $D(N_i)$ or $D(M_i)$, and we choose Riemannian metrics on $D(L_i)$ from $M$ pushed to $X$ by the map $D|L_i$. (Also, we fix a parameterization of $\hat{C}_i$ by $[0, \epsilon] \times S_i$ for some semi-algebraic set $S_i$.) That is, we assume that for $h'$ in $\hat{C}_i$s, the closures of $f_{h'}(B)$ are subsets of $D(L_i)$ or $D(N_i)$ for $f_{h'}$ obtained in the lemmas $\mathbb{H}$ and $\mathbb{I}$ respectively. (In the following, $\hat{C}_i$ will be modified further in various steps; $\Omega$ will be modified correspondingly.)

From now on, we will denote by the same symbol $f_{h'}$ these functions for $D(N_i)$ and $D(M_i)$. Also, we denote by $D'$ the maps $f_{h'} \circ D$ restricted on $N_i$ and $M_i$ respectively.

Given $h'$ in $\Omega$, we will construct a real projective manifold $M'$ which is homeomorphic to $M_0$.
We define $D'$ on sets of form $\gamma(N_i)$ or $\gamma(M_i)$ for a deck transformation $\gamma$ to be $h'(\gamma) \circ D' \circ \gamma^{-1}$ on these sets. (They are not yet consistently defined.) We need to choose $\Omega$ sufficiently small so that for sets $\gamma_i(N_{i1}), \gamma_i(N_{i2}), \gamma_i(N_{i3})$ so that $\gamma_i(N_{i1}) \cap \gamma_i(N_{i+1}) \neq \emptyset$ for $i = 1, 2, 3$ in cyclic sense, their intersection pattern does not change under $D'$ (as well as under $D$). Such $\Omega$ exists by Remark 6.

We define a topological space

$$\coprod_{j \in J} D'(N_j) / \sim_M,$$

where $\sim_M$ is defined as follows: $x \in D'(N_i)$ and $y \in D'(N_j)$ are equivalent if $x = h'(\gamma)(y)$ for $y \in \Gamma_{ij}$. This obviously is reflexive, symmetric, and transitive by equations (8), (9), and (11) and the stability. Let $Q' : \coprod_{j \in J} D'(N_j) / \sim_M \to M'$ be the quotient map.

We claim that $M'$ is an orbifold: We show that $M'$ is Hausdorff. Let $x \in D'(N_i)$ and $y \in D'(N_j)$, and suppose that they are not equivalent. If $i \neq j$ and $\Gamma_{ij} = \emptyset$, then $Q'(D'(N_i))$ and $Q'(D'(N_j))$ are disjoint neighborhoods of $Q'(x)$ and $Q'(y)$ respectively. If $i \neq j$ and $\Gamma_{ij} \neq \emptyset$, then define a map

$$D'' : D'(N_i) \coprod_{[\gamma] \in \Gamma_{ij}/\Gamma_j} D'(N_j)^{[\gamma]} \to X$$

by letting $D''|D'(N_i)$ be the inclusion map, and $D''|D'(N_j)^{[\gamma]}$ be the map $h'(\gamma)|D'(N_j)$ where $\gamma$ is a representative of $[\gamma]$ and $D'(N_j)^{[\gamma]}$ is a copy of $D'(N_j)$ for each $[\gamma] \in \Gamma_{ij}/\Gamma_j$. (By equation (12), $\Gamma_i = \Gamma_{ij}/\Gamma_j$.) Since $x$ and $y$ are not equivalent, $D''(x)$ and $D''(y^{[\gamma]})$ for a copy $y^{[\gamma]}$ of $y$, every $\gamma \in \Gamma_j$ and $[\gamma] \in \Gamma_{ij}/\Gamma_j$ are not equal. We assume without loss of generality that the above map $D''$ is an imbedding by choosing our neighborhoods sufficiently small. Thus, there exist disjoint neighborhoods of $D''(x)$ and the set $\{D''(y^{[\gamma]})\}$ in $D''(N_i)$ which is $\Gamma_i$-invariant. Then the component of the neighborhood containing $y$ and that containing $x$ have no equivalent points since every equivalence between $D'(L_i)$ and $D'(L_j)$ arises from $\Gamma_{ij}$ (see equation (14)). The disjoint neighborhoods clearly map to disjoint neighborhoods in $M'$ by $Q'$. If $i = j$, a similar argument applies. Since we need to consider only finitely many $j$ for each $i$, the quotient space $M'$ is a Hausdorff space.

Since $D''(\text{Cl}(N_i))$ are compact, and we can easily define a map from $\coprod_{j \in J} D'(N_j)$ to $M'$ by extending the quotient map $\coprod_{i \in I} D'(\text{Cl}(N_i)) \to M'$, we see that $M'$ is compact. Since $M'$ contains a countable dense subset clearly, $M'$ is second countable.

Also, $M'$ is obviously a $(G, X)$-orbifold since we obtained $M'$ by patching together the finite subgroup orbits in open subsets of $X$: $Q'(D'(N_j))$ form an open cover of $M'$ modeled on the pairs $(D'(N_j), \Gamma_j)$.

(i) We will construct an orbifold-diffeomorphism $\phi : M \to M'$: Define an imbedding $I_i : Q(D(\text{Cl}(M_i))) \to Q'(D'(N_i))$ by

$$I_i = Q' \circ f_{h'|\Gamma_i} \circ (Q|D((\text{Cl}(N_i))))^{-1}$$

obtained by Lemma 6 if $\Gamma_i$ is not trivial, or

$$I_i = Q' \circ (Q|D(\text{Cl}(N_i)))^{-1}|D(\text{Cl}(M_i))$$

if $\Gamma_i$ is trivial. (We define $I_i : D(\text{Cl}(M_i)) \to D'(N_i)$ to be $f_{h'|\Gamma_i}$, which covers the above map.) The problem is that $I_i$'s are not consistently defined over the overlaps of $Q(D(\text{Cl}(M_i)))$ and hence, we need to modify the map. We have an ordering
\( M_1, M_2, \ldots, M_n \) for some \( n \). We look at the sets of form
\[
Q(D(Cl(M_{i_1}))) \cap Q(D(Cl(M_{i_2}))) \cap \cdots \cap Q(D(Cl(M_{i_t}))),
\]
with indices satisfying \( i_1 < i_2 < \cdots < i_t \) for some \( t \). There is an upper bound \( t_0 \) on \( t \). (Note that for given \( t_0 \), the collection of the sets of above forms is composed of quotients of disjoint contractible compact submanifolds since our covering is nice.) We define a map \( \phi : M \to M' \) by defining it to be \( I_{i_t} \) on each set of the above form for \( t = t_0 \) and the lowest index \( i_1 \). Note that \( I_{i_t} \) defined on the inverse image of the above set in \( D(Cl(M_{i_t})) \) is a \( \Gamma_{i_t} \)-equivariant isotopy. Also, since \( \phi \) is well-defined, \( \phi \) lifts to a \( \Gamma_j \)-equivariant isotopy defined on the inverse image of the set in \( D(Cl(M_{i_t})) \) for each \( j = 2, \ldots, t \) mapping to \( D'(N_{i_t}) \).

We begin an inductive definition: Suppose that we defined an immersion \( \phi \) from the union of sets of form
\[
Q(D(Cl(M_{i_1}))) \cap Q(D(Cl(M_{i_2}))) \cap \cdots \cap Q(D(Cl(M_{i_t})))
\]
for indices satisfying \( i_1 < i_2 < \cdots < i_t \) so that \( \phi \) lifts to a smooth \( \Gamma_j \)-equivariant isotopy on the inverse image under \( \phi \) in \( D(Cl(M_{i_t})) \) to \( D'(N_{i_t}) \), \( j = 1, \ldots, t \).

Then we define a map from the union of sets of form
\[
Q(D(Cl(M_{i_1}))) \cap Q(D(Cl(M_{i_2}))) \cap \cdots \cap Q(D(Cl(M_{i_{t-1}})))
\]
for indices satisfying \( i_1 < i_2 < \cdots < i_{t-1} \). Take one of them say \( A \) of form
\[
Q(Cl(D(M_{i_1}))) \cap Q(Cl(D(M_{i_2}))) \cap \cdots \cap Q(Cl(D(M_{i_{t-1}})));
\]
with indices satisfying \( i_1 < i_2 < \cdots < i_{t-1} \). The subset \( \tilde{A} = Q^{-1}(A) \cap D(Cl(M_{i_t})) \) is an imbedded submanifold on which \( \Gamma_{i_t} \) acts. Let \( A' \) be the subset
\[
\bigcup_{t_{i_t} = 1}^n Cl(Q(D(M_{i_1}))) \cap Cl(Q(D(M_{i_2}))) \cap \cdots \cap Cl(Q(D(M_{i_{t-1}}))) \cap Cl(Q(D(M_{i_t}))),
\]
of \( A \) where \( i_1 < i_2 < \cdots < i_{t-1}, i_t \neq i_1, \ldots, i_{t-1}, \) and \( \phi \) is already defined with above properties on \( A' \). The subset \( A' = Q^{-1}(A') \cap D(Cl(M_{i_t})) \) is an imbedded submanifold of \( \tilde{A} \) on which \( \Gamma_{i_t} \) acts. \( \phi \) lifts to \( \tilde{\phi} \) on \( \tilde{A} \) and using Lemma 5 we obtain a \( \Gamma_{i_t} \)-equivariant isotopy \( \tilde{\phi} : A \to D'(N_{i_t}) \). This induces an imbedding \( \phi : A \to M' \) extended from \( A' \). Since \( \phi \) is well-defined, \( \phi \) lifts to a \( \Gamma_j \)-equivariant isotopy from the inverse image of \( A \) in \( D(Cl(M_{i_t})) \) for \( j = 1, \ldots, t-1 \) to \( D'(N_{i_t}) \). (Note here that the neighborhoods \( \tilde{C}_i \) are taken to be smaller and smaller because of Lemma 5 in this induction process. Also, an ambiguity of choice of the lift is taken care of by the fact that \( \tilde{\phi} \) should continuously deform to an identity map; i.e., \( \tilde{\phi} \) is an isotopy.) Therefore, the map \( \phi \) on \( A' \) extends to a smooth map \( \phi' : A \to M' \). We can do this for sets of form \( A \) consistently since they overlap in sets of form \( A' \) where \( \phi \) is already defined. By induction, we obtain a map \( \phi : M \to M' \).

Therefore, we defined for each \( M_i \) a map \( \tilde{\phi}_i : D(M_i) \to D'(L_i) \) which is a lifting of \( \phi \) from the model of neighborhoods \( Q(D(M_i)) \) to that of \( Q'(D'(L_i)) \). For \( \eta \in \Gamma_{ij} \),
\[
(14) \quad h'(\eta) \circ \tilde{\phi}_j \circ h(\eta^{-1})|D(M_i) \cap \eta D(M_j) = \tilde{\phi}_j|D(M_i) \cap \eta D(M_j).
\]
are diffeomorphic equivariantly with respect to an isomorphism ˜γM. We construct ˜γM quotient space of M and hence, φ which are covered by the open sets only once, we see that the local degree of x if h = y when ˜φ and φ are the inclusion maps.

By construction, the map φ : M → M′ induces a smooth map φ|M′ : M′ → M′. By taking a finite open cover of M initially, so that there are some points which are covered by the open sets only once, we see that the local degree of φ|M′ : M′ → M′ is equal to one. This map is proper and locally diffeomorphic, and hence, φ|M′ : M′ → M′ is a diffeomorphism. Therefore, φ : M → M′ is an orbifold-diffeomorphism (see the proof of Proposition 2).

(iii) Since M and M′ are orbifold-diffeomorphic, their universal covers ˜M and ˜M′ are diffeomorphic equivariantly with respect to an isomorphism π1(M) → π1(M′).

We construct ˜M′ explicitly from ˜M as follows: ˜M is covered by open sets of form γMi for γ ∈ π1(M), i = 1, ..., n. The universal cover ˜M can be considered as a quotient space of

\[ \prod_{\gamma \in \pi_1(M)} h(\gamma)(D(M_i)) \]

under the equivalence relation that

\[ x \in h(\gamma)(D(M_i)) \sim y \in h(\gamma')(D(M_j)) \]

if x = y and γ−1γ′ ∈ Γij (or γ(Mi) and γ′(Mj) meet). Let ˜Q : \[ \prod h(\gamma)(D(M_i)) \] → ˜M denote the quotient map. (We take distinct copies in the disjoint union of h(γ)(D(Mi)) for each γ unless γ−1γ′ belongs to Γi, in which case, we consider h(γ)(D(Mi)), same as h(γ′)(D(Mi)).)

We define ˜M′ as the quotient space of \[ \prod h'(\gamma')D'(N_i) \] again with the equivalence relation

\[ x \in h'(\gamma')(D'(N_i)) \sim y \in h'(\gamma')(D'(N_j)) \]

if x = y and γ−1γ′ ∈ Γij. (Again, we use the above copying rule.) Let ˜Q′ : \[ \prod h'(\gamma')(D'(N_i)) \] → ˜M′ denote the quotient map. ˜M′ is shown to be a manifold just as M′ is shown to be an orbifold. Since ˜M is good, ˜M is a simply-connected manifold. Also, from a nerve consideration, ˜M′ has the same nerve of covering as ˜M as sufficiently implied by Remark 9. Thus, ˜M′ is a simply-connected manifold.

We define a map pM′ : ˜M′ → M′ by defining

\[ p_{M'}(\tilde{Q}'(h'(\gamma')(D'(N_i)))) : \tilde{Q}'(h'(\gamma')(D'(N_i))) \rightarrow Q'(D'(N_i)) \]

by sending a point corresponding to h'(γ)(x) to Q′(x) for x ∈ \[ \prod D'(N_i) \]. The map pM′ is clearly an orbifold-covering map. Moreover, π1(M) acts on ˜M′ by letting ˜φ ∈ π1(M) act by sending x ∈ ˜Q′(h′(γ))D′(N_i) to a point in ˜Q′(h′(\vartheta)h′(γ))D′(N_i)) by a map ˜Q′ ◦ h′(\vartheta) ◦ ˜Q′−1. This is a well-defined automorphism of ˜M′, and ˜M′/π1(M) is orbifold-diffeomorphic to M′. (Of course, the covering map pM : ˜M → M and the action of π1(M) on ˜M can be defined the same way.)

The above diffeomorphism ˜φ lifts to a diffeomorphism ˜φ : ˜M → ˜M′; we first recall the lift ˜φi : D(Mi) → D′(N_i) of φ : Q(D(Mi)) → Q′(D′(N_i)), and for h(γ)(D(Mi)) where γ ∈ π1(M), we define

\[ \tilde{\phi} : h(\gamma)(D(M_i)) \rightarrow h'(\gamma)(D'(N_i)) \]
by letting \( \tilde{o}(x) \) to be \( h'(\gamma) \circ \tilde{o}_i \circ h(\gamma)^{-1}(x) \). This is well-defined: Let \( y \) be a point of \( h(\gamma')(D(M_j)) \) for some \( j, \gamma' \in \pi_1(M) \) so that \( x = y \) and \( \gamma^{-1} \gamma' \in \Gamma_{ij} \). Then
\[
h'(\gamma') \circ \tilde{o}_j \circ h(\gamma)^{-1}(y) = h'(\gamma)h'(\gamma^{-1} \gamma') \circ \tilde{o}_j \circ h(\gamma^{-1} \gamma)(h^{-1}(x)).
\]
By equation (4), the right-hand side of the above equation is now \( h'(\gamma) \circ \tilde{o}_i \circ h(\gamma^{-1})(x) \). This defines a smooth map \( \tilde{o} : \tilde{M} \rightarrow \tilde{M}' \), which is an immersion.

Since \( \tilde{M} \) and \( \tilde{M}' \) have the same nerve of open coverings by open balls, we see that \( \tilde{M}' \) is a simply connected manifold. Therefore, \( \tilde{M}' \) is a universal covering orbifold of \( M' \) by Remark 4. Since \( p_{M'} \circ \tilde{o} = \phi \circ p_{\tilde{M}} \) clearly, \( \tilde{o} \) is a lift of an orbifold-diffeomorphism \( \phi \) and hence is an isomorphism by Corollary 2. The above map \( \tilde{o} \) is equivariant, i.e.,
\[
\tilde{o} \circ \gamma = \gamma \circ \tilde{o}, \quad \text{for each } \gamma \in \pi_1(M).
\]
Thus, we see that \( \tilde{M}' \) is the universal covering space of \( M' \) and \( \pi_1(M) \) and \( \pi_1(M') \) are isomorphic by \( \tilde{o}_* \) induced from \( \tilde{o} \).

We define a developing map \( D' : \tilde{M}' \rightarrow X \) by defining
\[
D'(\tilde{Q}'(h(\gamma)D'(N_i))) = \tilde{Q}'^{-1}h'(\gamma)D'(N_i).
\]
This defines a smooth immersion over \( \tilde{M}' \) in a consistent manner. We consider \( D' \circ \vartheta \) for \( \vartheta \in \pi_1(M) \). Then on \( \tilde{Q}'(D'(N_i)) \), it equals
\[
\tilde{Q}'^{-1} \circ \tilde{Q}' \circ h'(\vartheta) \circ \tilde{Q}'^{-1}
\]
which equals \( h'(\vartheta) \circ \tilde{Q}'^{-1} \). We obtain \( D' \circ \vartheta = h'(\vartheta) \circ D' \). Therefore, the holonomy homomorphism is \( h' : \pi_1(M) \rightarrow G \) under the identification \( \pi_1(M') = \pi_1(M) \). Or equivalently, \( h'' \circ \phi_\ast = h' \) where \( h'' \) is the holonomy homomorphism of \( M' \).

To summarize, for each \( h' \in \Omega \), we defined \( M'(h') \) with a development pair \( (D',h'') \) and a diffeomorphism \( \phi_{h'} : M \rightarrow M' \) lifting to a diffeomorphism \( \tilde{\phi}_{h'} : \tilde{M} \rightarrow \tilde{M}'(h') \) so that \( h'' \circ \tilde{\phi}_{h'} = h' \). (For objects we defined above, we attach a suffix \( h' \) to indicate that they are constructed for \( h' \).) In fact, we constructed a map \( s' : \Omega \rightarrow C(M) \) where
\[
s'(h') = (D',\tilde{\phi}_{h'} : \tilde{M} \rightarrow \tilde{M}'(h')).
\]
By Lemma 5 and our inductive construction, we can verify that
\[
\tilde{\phi}_{i,h'} : D(M_i) \rightarrow D'(N_i)
\]
depends continuously on \( h' \), and hence, so does
\[
\tilde{\phi}_{h'} | h(\gamma)(D(M_i)) : h(\gamma)(D(M_i)) \rightarrow h'(\gamma)(D'(N_i)).
\]
This proves the continuity of section \( s \) completing the step (II).

We will show that
\[
\mathcal{P}H : S(M_0) \rightarrow \text{Hom}(\pi_1(M_0), G)
\]
is locally injective; i.e., for each \( (D, \tilde{f} : \tilde{M}_0 \rightarrow \tilde{M}) \) there is a neighborhood where \( \mathcal{P}H \) is injective. This will be the step (III) of the outline.

Again, we identify \( \tilde{M} \) with \( \tilde{M}_0 \) by \( \tilde{f} \). Let us give \( M \) a Riemannian metric with covering by neighborhoods modeled on \( (\tilde{L}_i, \Gamma_i) \), \( i = 1, \ldots, n \), in \( \tilde{M} \) as above. We choose \( M_i, N_i \) as above in \( L_i \).
Let $\iota:\tilde{M} \rightarrow \tilde{M}$ denote the identity map. We choose a neighborhood $\mathcal{O}$ of $(D_{1},\iota):\tilde{M} \rightarrow M$ in $C(M)$ so that any two elements $(D_{1},\tilde{f}_{1}:\tilde{M} \rightarrow \tilde{Y}_{1})$ and $(D_{2},\tilde{f}_{2}:\tilde{M} \rightarrow \tilde{Y}_{2})$ satisfy that $D_{1} \circ \tilde{f}_{1}$ is sufficiently $C^{\infty}$-close to $D_{2} \circ \tilde{f}_{2}$ so that

$$D_{1} \circ \tilde{f}_{1}(\text{Cl}(M_{i})) \subset D_{2} \circ f_{2}(N_{i}) \subset D(L_{i})$$

and

$$D_{2} \circ \tilde{f}_{2}(\text{Cl}(M_{i})) \subset D_{1} \circ \tilde{f}_{1}(N_{i}) \subset D(L_{i}),$$

and the corresponding holonomy homomorphisms $h_{1}$ and $h_{2}$ belong to $\Omega$ for $\Omega$ defined above. (We will add two more conditions on $\mathcal{O}$ making it smaller.)

Let $q:C(M) \rightarrow S(M)$ be the quotient map defined above. $q$ is an open map since $S(M)$ is the space of orbits in $C(M)$ under the action of the group of isotopies of $M$.

We may assume that

$$\text{PH}(q(\mathcal{O})) = \text{PH}(\mathcal{O}) \subset \Omega$$

by choosing $\mathcal{O}$ sufficiently small.

We claim that on $q(\mathcal{O})$, which is a neighborhood of the equivalence class of $(D_{1},\iota):\tilde{M} \rightarrow \tilde{Y}_{1}$ and $(D_{2},\iota):\tilde{M} \rightarrow \tilde{Y}_{2})$ in $S(M)$, $PH(q(\mathcal{O}))$ is injective.

This will prove Theorem 1 since $PH(q(\mathcal{O}))$ has an inverse $s$ restricted to the image in $\Omega$. The image of $PH(q(\mathcal{O}))$ is open since that of $PH(\mathcal{O})$ is open. The latter image is open since for each point of its image, we can find a small neighborhood $\Omega'$ in $\Omega$ so that a section $s'$ defined on $\Omega'$ has images in $\mathcal{O}$ as we can control the $C^{\infty}$-norm of conjugating diffeomorphisms of model sets by the size of the holonomy perturbations (see Remark 5). Thus, $PH(q(\mathcal{O}))$ is a homeomorphism to an open subset of $\text{Hom}(\pi_{1}(M),G)$.

Given $(D_{1},\tilde{f}_{1}:M \rightarrow \tilde{Y}_{1})$ and $(D_{2},\tilde{f}_{2}:\tilde{M} \rightarrow \tilde{Y}_{2})$ in $\mathcal{O}$ with the holonomy homomorphisms $h_{1} \circ \tilde{f}_{1}$ and $h_{2} \circ \tilde{f}_{2}$ which are equal, we show that $(D_{1},\tilde{f}_{1})$ and $(D_{2},\tilde{f}_{2})$ are isotopy equivalent. We assumed that $D_{1} \circ \tilde{f}_{1}(M_{i}) \subset D_{2} \circ \tilde{f}_{2}(N_{i})$ for each $i$. We start from $\tilde{f}_{1}(M_{1})$, and lift the map $D_{1}\tilde{f}_{1}(M_{1})$ by $D_{2}^{-1}$ to $\tilde{f}_{2}(N_{1})$.

We identify $\pi_{1}(M)$ with $\pi_{1}(Y_{1})$ and $\pi_{1}(Y_{2})$ by $f_{1*}$ and $f_{2*}$ respectively. If $\gamma(M_{j})$ meets $M_{1}$ for $\gamma \in \Gamma_{1}$, then

$$D_{1}(\tilde{f}_{1}(\gamma(M_{j}))) = h_{1}(\gamma)(D_{1}(\tilde{f}_{1}(M_{j}))) \subset h_{2}(\gamma)(D_{2}(\tilde{f}_{2}(N_{j}))) = D_{2}(\tilde{f}_{2}(\gamma(N_{j})))$$

since $h_{1}(\gamma) = h_{2}(\gamma)$. We lift $D_{1}\tilde{f}_{1}(\gamma(M_{j}))$ by $(D_{2}\tilde{f}_{2}(\gamma(N_{j})))^{-1}$ into $D_{2}(\tilde{f}_{2}(\gamma(N_{j})))$. By an induction in this manner, we see that we can lift an immersion $D_{1}:\tilde{Y}_{1} \rightarrow X$ to an immersion $f_{12}:\tilde{Y}_{1} \rightarrow \tilde{Y}_{2}$ by $D_{2}$ so that $D_{2} \circ f_{12} = D_{1}$.

Since $h_{1} = h_{2}$, considering $\tilde{Y}_{1}$ and $\tilde{Y}_{2}$ as quotient spaces of the sets of form $h_{1}(\gamma)D_{1}(\tilde{f}_{1}(M_{j}))$ and $h_{2}(\gamma)D_{2}(\tilde{f}_{2}(N_{j}))$, this map is also seen to be $\pi_{1}(M)$-equivariant; i.e.,

$$f_{12} \circ \tilde{f}_{1} \circ \gamma = \gamma \circ f_{12} \circ \tilde{f}_{1}, \gamma \in \pi_{1}(M);$$

or in other words,

$$f_{12} \circ \tilde{f}_{1}(\gamma) = \tilde{f}_{2}(\gamma) \circ f_{12} \text{ for } \gamma \in \pi_{1}(M).$$

Thus,

$$f_{12*} \circ \tilde{f}_{1}(\gamma) = \tilde{f}_{2}(\gamma).$$

We now show that $f_{12} \circ \tilde{f}_{1}$ is isotopic to $\tilde{f}_{2}$ by an isotopy $H:\tilde{M} \times [0,1] \rightarrow \tilde{Y}_{2}$ equivariant with respect to the homomorphism $f_{2*}:\pi_{1}(M) \rightarrow \pi_{1}(Y_{2})$. 

Let $Y_2$ have the Riemannian metric pushed by $\tilde{f}_2$ with distance metric $d_{Y_2}$. Then $\tilde{f}_2$ is an isometry. Since $M$ is compact, $f_{12} \circ \tilde{f}_1 : M \to \tilde{Y}_2$ is a map so that

$$d_{Y_2}(\tilde{f}_2(x), f_{12} \circ \tilde{f}_1(x)) \leq \epsilon$$

for some small $\epsilon > 0$.

We may choose our neighborhood $O$ in the beginning so that $\epsilon$ may be chosen to be smaller than the minimum radius of the normal neighborhoods for every point of $M$. Thus, one can find a unique geodesic from $f_{12}(x)$ to $f_{12} \circ \tilde{f}_1(x)$ for each $x \in M$. For each point $y$ of $\tilde{Y}_2$, let $v$ be an equivalence class of a vector in $T_y \tilde{Y}_2$ so that $\exp_y(v) = f_{12} \circ \tilde{f}_1 \circ \tilde{f}_2^{-1}(y)$. Since $f_{12} \circ \tilde{f}_1 \circ \tilde{f}_2^{-1}$ is a $\pi_1(Y_2)$-equivariant diffeomorphism by equation 15, $v$ is a $\pi_1(Y_2)$-invariant vector field.

Let us denote by $E : T(Y_2) \to \tilde{Y}_2 \times \tilde{Y}_2$ the map given by sending $(z, w)$ to $(z, \exp_y(w))$ for $z \in \tilde{Y}_2$ and $w \in T_z(\tilde{Y}_2)$. Then $E$ is a differentiable map invertible near the diagonal $\Delta$ in $\tilde{Y}_2 \times \tilde{Y}_2$. Let us call $E^{-1}$ the inverse in a neighborhood of $\Delta$. Since $E^{-1}$ is a smooth map, $v$ is a smooth vector field on $\tilde{Y}_2$.

If we choose $O$ sufficiently near $(D, \epsilon_n)$, then $v$ is very small so that the map $g_t : \tilde{Y}_2 \to \tilde{Y}_2$ defined by $g_t(x) = \exp_x(tv)$ are immersions for $t \in [0,1]$: We look at the variation of the Jacobian from the Jacobian of the identity map. Each $g_t$ descends to an immersions $g'_t : Y_2 \to Y_2$ with local degree 1 isotopic to the identity map. Since $g'_t$ is a proper-map and of local-degree 1, $g'_t$ restricts to a diffeomorphism $Y_2^r \to Y_2^r$. As we showed above, $g'_t$ are orbifold-diffeomorphisms by Proposition 1 and so are $g_t$ by Corollary 1. Thus, we require this to hold for $O$.

Let us denote by $H(y, t)$ the point $\exp_y(tv)$ for $t \in [0,1]$. Then $H$ is a smooth function $\tilde{Y}_2 \times [0,1] \to \tilde{Y}_2$ so that $H(y, 0) = y$ for $y \in \tilde{Y}_2$ and $H(f_{12}(x), 1) = f_{12} \circ \tilde{f}_1(x)$ for every $x \in \tilde{Y}_1$. Therefore, $H$ is $\pi_1(Y_2)$-equivariant isotopy between $\tilde{f}_2$ and $f_{12} \circ \tilde{f}_1$, since so are $v$ and $v_t$. Thus, $\tilde{f}_2$ and $f_{12} \circ \tilde{f}_1$ are isotopic, and $(D_1, \tilde{f}_1)$ and $(D_2, \tilde{f}_2)$ are isotopy-equivalent. \hfill $\square$

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