Abstract

We show that Martin’s Maximum ++ implies Woodin’s $P_{max}$ axiom ($\ast$).

1 Introduction.

Cantor’s Continuum Problem, which later became Hilbert’s first Problem (see [7]), asks how many real numbers there are. This question got a non-answer through the discovery of the method of forcing by Paul Cohen: CH, Cantor’s Continuum Hypothesis, is independent from ZFC (see [3]), the standard axiom system for set theory which had been isolated by Zermelo and Fraenkel. CH states that every uncountable set of reals has the same size as $\mathbb{R}$.

Ever since Cohen, set theorists have been searching for natural new axioms which extend ZFC and which settle the Continuum Problem. See e.g. [21], [22], [9], and the discussion in [4]. There are two prominent such axioms which decide CH in the negative and which in fact both prove that there are $\aleph_2$ reals: Martin’s Maximum (MM, for short) or variants thereof on the one hand (see [6]), and Woodin’s axiom ($\ast$) on the other hand (see [20]). See e.g. [12], [19], and [14].

Both of these axioms may be construed as maximality principles for the theory of the structure $(H_{\omega_2}; \in)$, but up to this point the relationship between MM and ($\ast$) was a bit of a mystery, which led M. Magidor to call ($\ast$) a “competitor” of MM ([12, p. 18]). Both MM and ($\ast$) are inspired by and formulated in the language of forcing.
and they both have “the same intuitive motivation: Namely, the universe of sets is rich” ([12, p. 18]).

This paper resolves the tension between MM and (⋆) by proving that MM⁺⁺, a strengthening of MM, actually implies (⋆), see Theorem 2.1 below, so that MM and (⋆) are actually compatible with each other. This answers [20, Question (18) a) on p. 924], see also [20, p. 846], [12, Conjecture 6.8 on p. 19], and [14, Problem 14.7].

2 Preliminaries.

Martin’s Maximum⁺⁺, abbreviated by MM⁺⁺, see [6] (cf. also [20, Definition 2.45 (2)]), is the statement that if P is a forcing which preserves stationary subsets of ω₁, if \{Dᵢ: i < ω₁\} is a collection of dense subsets of P, and if \{τᵢ: i < ω₁\} is a collection of P-names for stationary subsets of ω₁, then there is a filter g ⊂ P such that for every i < ω₁,

(i) g ∩ Dᵢ ≠ ∅ and

(ii) \((τᵢ)^g = \{ξ < ω₁: ∃p ∈ g_≤^P ⪰ P_≤^P ξ ∈ τᵢ\}\) is stationary.

Woodin’s Pmax axiom (⋆), see [20, Definition 5.1], is the statement that

(i) AD holds in L(ℝ) and

(ii) there is some g which is Pmax-generic over L(ℝ) such that P(ω₁) ∩ V ⊂ L(ℝ)[g].

Already PFA, the Proper Forcing Axiom, which is weaker than MM⁺⁺, implies ADL(ℝ) and much more, see [17], [8], and [15, Chapter 12]. This paper produces a proof of the following result.

**Theorem 2.1** Assume Martin’s Maximum⁺⁺. Then Woodin’s Pmax-axiom (⋆) holds true.

Our key new idea is (Σ.8) on page 9 below.

P. Larson, see [10] and [11], has shown that MM⁺ω is consistent with ¬(⋆) relative to a supercompact limit of supercompact cardinals.

Throughout our entire paper, “ω₁” will always denote ω₁^V, the ω₁ of V.

Let us fix throughout this paper some A ⊂ ω₁ such that ω₁^{L[A]} = ω₁. Let us define gₐ as the set of all Pmax conditions p = (N; ∈, I, a) such that there is a generic iteration

\((Nᵢ, σᵢǧ: i ≤ j ≤ ω₁)\)
of \( p = N_0 \) of length \( \omega_1 + 1 \) such that if we write \( N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*) \), then 
\[ I^* = (\text{NS}_{\omega_1})^V \cap N_{\omega_1} \text{ and } a^* = A. \]

**Lemma 2.2 (Woodin)** Assume that \( \text{NS}_{\omega_1} \) is saturated and that \( \mathcal{P}(\omega_1)^\# \) exists.

1. \( g_A \) is a filter.
2. If \( g_A \) is \( \mathbb{P}_{\text{max}} \)-generic over \( L(\mathbb{R}) \), then \( \mathcal{P}(\omega_1) \subset L(\mathbb{R})[g] \).

**Proof.** This routinely follows from the proof of [20, Lemma 3.12 and Corollary 3.13] and from [20, Lemma 3.10]. \( \square \)

One may also use \( \text{BMM} \) plus \( \text{"NS}_{\omega_1} \) is precipitous” to show that \( g_A \) is a filter, this is by the proof from [2].

Let \( \Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \). We say that \( \Gamma \) is *productive* iff for all \( k < \omega \) and all \( D \in \Gamma \cap \mathcal{P}(\mathbb{R}^{k+2}) \), if \( D \) is universally Baire (see [5]) as being witnessed by the trees \( T \) and \( U \) on \( k^+ \omega \times \text{OR} \), i.e., \( D = p[T] \) and for all posets \( \mathbb{P} \),

\[
\models_{\mathbb{P}} p[U] = \mathbb{R}^{k+2} \setminus p[T],
\]

and if

\[
\hat{U} = \{(s \upharpoonright (k + 1), (s(k + 1), t)) : (s, t) \in U\}.
\]

so that \( (x_0, \ldots, x_k) \in p[\hat{U}] \) iff there is some \( y \) such that \( (x_0, \ldots, x_k, y) \in p[U] \), then there is a tree \( \hat{T} \) on \( k^+ \omega \times \text{OR} \) such that for all posets \( \mathbb{P} \),

\[
\models_{\mathbb{P}} p[\hat{U}] = \mathbb{R}^{k+1} \setminus p[\hat{T}].
\]

Let us denote by \( \Gamma^\infty \) the collection of all \( D \in \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \) which are universally Baire. If \( D \in \Gamma^\infty \), then there is an unambiguous version of \( D \) in any forcing extension of \( V \), which as usual we denote by \( D^* \). (2) then means that if \( D = p[U] \) and \( E = p[\hat{U}] \), then in any forcing extension of \( V \), \( E^* = \mathbb{R} D^* \).

If \( \Gamma \subset \Gamma^\infty \) is productive and if \( D \in \Gamma \), then any projective statement about \( D \) is absolute between \( V \) and any forcing extension of \( V \), \( \text{2} \) i.e., if \( \varphi \) is projective, \( x_1, \ldots, x_k \in \mathbb{R} \), and \( \mathbb{P} \) is any poset, then

\[
V \models \varphi(x_1, \ldots, x_k, D) \iff \models_{\mathbb{P}} \varphi(\hat{x}_1, \ldots, \hat{x}_k, D^*).
\]

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1. Here and elsewhere we often confuse a model with its underlying universe.
2. This seems to be wrong if we just assume \( \Gamma \subset \Gamma^\infty \), but the hypothesis that \( \Gamma \) be productive is crossed out.
By a theorem of Woodin, see e.g. [18, Theorem 1.2], combined with the key result of Martin and Steel in [13], the pointclass $\Gamma^\infty$ is productive under the hypothesis that there is a proper class of Woodin cardinals.

The proof from [17] produces the result that under PFA, the universe is closed under the operation $X \mapsto M^\#(X)$, which implies that every set of reals in $L(\mathbb{R})$ is universally Baire and that $\bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\mathbb{R})$ is productive. See e.g. [16, Section 3, pp. 187f.] on the relevant argument. Therefore, in the light of Lemma 2.2, Theorem 2.1 follows from the following more general statement.

**Theorem 2.3** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

(iv) Martin’s Maximum++ holds true.

Then $g_A$ is $\mathbb{P}_{\text{max}}$-generic over $L(\Gamma, \mathbb{R})$.\(^3\)

In the light of Lemma 2.2 and [6, Corollary 17], Theorem 2.3 readily follows from the following via a standard application of MM++.

**Lemma 2.4** Let $\Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k)$. Assume that

(i) $\Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R})$,

(ii) $\Gamma \subset \Gamma^\infty$,

(iii) $\Gamma$ is productive, and

(iv) $\text{NS}_{\omega_1}$ is saturated.\(^4\)

Let $D \subset \mathbb{P}_{\text{max}}$ be open dense, $D \in L(\Gamma, \mathbb{R})$.\(^5\) There is then a stationary set preserving forcing $\mathbb{P}$ such that in $V^\mathbb{P}$ there is some $p = (N; \in, I^*, a^*) \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$$

of $p = N_0$ of length $\omega_1 + 1$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})_{V^\mathbb{P}} \cap N_{\omega_1}$ and $a^* = A$.

\(^3\)In the presence of a proper class of Woodin cardinals, hypotheses (i), (ii), and (iv) then give $(*)_{\Gamma}$, see [16, Definition 4.1].

\(^4\)We could weaken this hypothesis to “$\text{NS}_{\omega_1}$ is precipitous.”

\(^5\)By hypothesis, $D$ is then universally Baire in the codes, so that there is an unambiguous version of $D$ in any forcing extension of $V$, which again we denote by $D^*$.
The attentive reader will notice that we don’t need the full power of \( \text{MM}^{++} \) in order to derive Theorem 2.3 from Lemma 2.4, the hypothesis that \( D\text{-BMM}^{++} \) holds true for all \( D \in \Gamma^\infty \) would suffice, see [20, Definition 10.123]. By the proof of [1, Theorem 2.7], (*) is then actually equivalent to a version of BMM; we state this as follows.

**Theorem 2.5** Let \( \Gamma \subset \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \). Assume that
1. \( \Gamma = \bigcup_{k<\omega} \mathcal{P}(\mathbb{R}^k) \cap L(\Gamma, \mathbb{R}) \),
2. \( \Gamma \subset \Gamma^\infty \),
3. \( \Gamma \) is productive, and
4. \( \text{NS}_{\omega_1} \) is saturated.\(^6\)

The following statements are then equivalent.

1. \( D\text{-BMM}^{++} \) holds true for all \( D \in \Gamma \).
2. \( g_A \) is \( \mathbb{P}_{\text{max}} \)-generic over \( L(\Gamma, \mathbb{R}) \).

**Theorem 2.6** Assume that there is a proper class of Woodin cardinals. The following statements are then equivalent.

1. \( D\text{-BMM}^{++} \) holds true for all \( D \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}) \).
2. \( (*) \).

Our next section is entirely devoted to a proof of Lemma 2.4.

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## 3 The forcing.

Let us assume throughout the hypotheses of Lemma 2.4. We aim to verify its conclusion.

Let us fix \( D \subset \mathbb{P}_{\text{max}} \), an open dense set in \( L(\Gamma, \mathbb{R}) \). By hypotheses (ii) and (iii) in the statement of Lemma 2.4 we will have that

\(^6\)Again, we could weaken this hypothesis to “\( \text{NS}_{\omega_1} \) is precipitous.”
(D.1) \( V^{\text{Col}(\omega, \omega_2)} \models "D^* \text{ is an open dense subset of } \mathbb{P}_{\text{max}}." \)

Let us identify \( D \) with a canonical set of reals coding the elements of \( D \),\(^7\) and let \( T \in V \) be a tree on \( \omega \times 2^{\aleph_2} \) such that

(D.2) \( V^{\text{Col}(\omega, \omega_2)} \models D^* = p[T] \).

Let us write \( \kappa = (2^{\aleph_2})^+ \), (4)

so that \( T \in H_\kappa \). Let \( d \) be \( \text{Col}(\kappa, \kappa) \)-generic over \( V \). In \( V[d] \), let \( (\bar{A}_\lambda : \lambda < \kappa) \) be a \( \Diamond_\kappa \)-sequence, i.e., for all \( \bar{A} \subset \kappa \), \( \{ \lambda < \kappa : \bar{A} \cap \lambda = \bar{A}_\lambda \} \) is stationary. Also, let \( c: \kappa \to H^V_\kappa = H^V_{\kappa[d]} \), \( c \in V[d] \), be bijective. For \( \lambda < \kappa \), let

\[
Q_\lambda = c^{"\lambda} \text{ and } A_\lambda = c^{"\bar{A}_\lambda}.
\]

(5)

Let \( C \subset \kappa \) be club such that for all \( \lambda \in C \),

(i) \( Q_\lambda \) is transitive,

(ii) \( \{ T, ((H_\omega)^V; \in, (\text{NS}_{\omega_1})^V, A) \} \cup 2^{\aleph_2} \subset Q_\lambda \),

(iii) \( Q_\lambda \cap \text{OR} = \lambda \) (so that \( c \upharpoonright \lambda: \lambda \to Q_\lambda \) is bijective), and

(iv) \( (Q_\lambda; \in) \prec (H_\kappa; \in) \).

In \( V[d] \), for all \( P, B \subset H_\kappa \), the set of all \( \lambda \in C \) such that

\[
(Q_\lambda; \in, P \cap Q_\lambda, B \cap Q_\lambda) \prec (H_\kappa; \in, P, B)
\]

is club, and the set of all \( \lambda \in C \) such that \( B \cap Q_\lambda = A_\lambda \) is stationary, so that

(\( \Diamond \)) For all \( P, B \subset H_\kappa \) the set

\[
\{ \lambda \in C : (Q_\lambda; \in, P \cap Q_\lambda, A_\lambda) \prec (H_\kappa; \in, P, B) \}
\]

is stationary.

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\(^7\text{We will have to spell out a bit more precisely below in which way we aim to have the elements of } p[T] \text{ code the elements of } D, \text{ see } (\Sigma.5) \text{ below.} \)
We shall sometimes also write $Q_\kappa = H_\kappa$.

We shall now go ahead and produce a stationary set preserving forcing $\mathbb{P} \in V[d]$ which adds some $p \in D^*$ and some generic iteration

$$(N_i, \sigma_{ij}; i \leq j \leq \omega_1)$$

of $p = N_0$ such that if we write $N_{\omega_1} = (N_{\omega_1}; \in, I^*, a^*)$, then $I^* = (\text{NS}_{\omega_1})^{V[d]^p} \cap N_{\omega_1}$ and $a^* = A$. As the forcing $\text{Col}(\kappa, \kappa)$ which added $d$ is certainly stationary set preserving, this will verify Lemma 2.4.

$\text{NS}_{\omega_1}$ is still saturated in $V[d]$ and (D.1) and (D.2) are still true in $V[d]$, so that in order to simplify our notation, we shall in what follows confuse $V[d]$ with $V$, i.e., pretend that in addition to “$\text{NS}_{\omega_1}$ is saturated” plus (D.1) and (D.2), $(\bigcirc)$ is also true in $V$.

Working under these hypotheses, we shall now recursively define a $\subset$-increasing and continuous chain of forcings $\mathbb{P}_\lambda$ for all $\lambda \in C \cup \{\kappa\}$. The forcing $\mathbb{P}$ will be $\mathbb{P}_\kappa$.

Assume that $\lambda \in C \cup \{\kappa\}$ and $\mathbb{P}_\mu$ has already been defined in such a way that $\mathbb{P}_\mu \subset Q_\mu$ for all $\mu \in C \cap \lambda$.

We shall be interested in objects $\mathfrak{C}$ which exist in some outer model and which have the following properties.

$$\mathfrak{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij}; i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n); n < \omega \rangle, \langle \lambda_\delta, X_\delta; \delta \in K \rangle, \quad (6)$$

and

(C.1) $M_0, N_0 \in \mathbb{P}_{\text{max}}$,

(C.2) $x = \langle k_n; n < \omega \rangle$ is a real code for $N_0$ and $\langle (k_n, \alpha_n); n < \omega \rangle \in [T]$,

(C.3) $\langle M_i, \pi_{ij}; i \leq j \leq \omega_1^{N_0} \rangle \in N_0$ is a generic iteration of $M_0$ which witnesses that $N_0 < M_0$ in $\mathbb{P}_{\text{max}}$,

(C.4) $\langle N_i, \sigma_{ij}; i \leq j \leq \omega_1 \rangle$ is a generic iteration of $N_0$ such that if $N_{\omega_1} = (N_{\omega_1}; \in, I^*, A^*)$,

then $A^* = A$.

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8$W$ is an outer model iff $W$ is a transitive model of $\text{ZFC}$ with $W \supset V$ and which has the same ordinals as $V$; in other words, $W$ is an outer model iff $V$ is an inner model of $W$.

9There is no requirement on $I^*$ matching the non-stationary ideal of some model in which $\mathfrak{C}$ exists.
\begin{equation}
\langle M_i, \pi_{ij} : i \leq j \leq \omega_1 \rangle = \sigma_{\omega_1}(\langle M_i, \pi_{ij} : i \leq j \leq \omega_1^{N_0} \rangle) \text{ and } \\
M_{\omega_1} = ((H_{\omega_2})^V; \in, (\text{NS}_{\omega_1})^V, A), \tag{7}
\end{equation}

(C.6) \( K \subset \omega_1 \)

and for all \( \delta \in K \),

(C.7) \( \lambda_\delta < \lambda \), and if \( \gamma < \delta \) is in \( K \), then \( \lambda_\gamma < \lambda_\delta \) and \( X_\gamma \cup \{ \lambda_\gamma \} \subset X_\delta \), and

(C.8) \( X_\delta \prec (Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \) and \( X_\delta \cap \omega_1 = \delta \).

We need to define a language \( \mathcal{L} \) (independently from \( \lambda \)) whose formulae will be able to describe \( \mathcal{C} \) with the above properties by producing the models \( M_i \) and \( N_i \), \( i < \omega_1 \), as term models out of equivalence classes of terms of the form \( \dot{n} \), \( n < \omega \). The language \( \mathcal{L} \) will have the following constants.

\begin{itemize}
    \item \( \dot{T} \) intended to denote \( T \)
    \item \( \dot{x} \) for every \( x \in H_{\kappa} \)
    \item \( \dot{n} \) for every \( n < \omega \)
    \item \( \dot{M}_i \) for \( i < \omega \)
    \item \( \dot{\pi}_{ij} \) for \( i \leq j \leq \omega_1 \)
    \item \( \dot{\vec{M}} \) intended to denote \( (M_j, \pi_{jj'}: j \leq j' \leq \omega_1^{N_i}) \) for \( i < \omega_1 \)
    \item \( \dot{N}_i \) for \( i < \omega_1 \)
    \item \( \dot{\sigma}_{ij} \) for \( i \leq j < \omega_1 \)
    \item \( \dot{\sigma}_i \) intended to denote the distinguished \( a \)-predicate of \( M_i, N_i, i < \omega_1 \)
    \item \( \dot{\mathcal{I}} \) intended to denote the distinguished ideal of \( N_i, i < \omega_1 \)
    \item \( \dot{X}_\delta \) for \( \delta < \omega_1 \)
\end{itemize}

Formulae of \( \mathcal{L} \) will be of the following form.

\[ \Gamma \vdash \varphi(\xi_1, \ldots, \xi_k, \dot{n}_1, \ldots, \dot{n}_\ell, \dot{\alpha}, \dot{\mathcal{I}}, \dot{M}_j, \ldots, \dot{M}_{j_m}, \dot{\pi}_{q_{1r_1}}, \ldots, \dot{\pi}_{q_{sr_s}}, \dot{\vec{M}})^\gamma \]

for \( i < \omega_1, \xi_1, \ldots, \xi_k < \omega_1, n_1, \ldots, n_\ell < \omega, j_1, \ldots, j_m < \omega, q_1 \leq r_1 < \omega_1, \ldots, q_s \leq r_s < \omega_1 \)

\[ \Gamma \vdash \dot{\pi}_{i \omega_1}(\dot{n}) = x^\gamma \] for \( i < \omega_1 \) and \( x \in H_{\omega_2} \)

\[ \Gamma \vdash \dot{\pi}_{\omega_1 \omega_1}(x) = x^\gamma \] for \( x \in H_{\omega_2} \)

\[ \Gamma \vdash \dot{\sigma}_{ij}(\dot{n}) = \dot{m}^\gamma \] for \( i \leq j < \omega_1, n, m < \omega \)
\[ \begin{align*}
\varphi(\vec{u}, \vec{a}) & \in \bar{T} \iff \vec{u} \in {}^{<\omega}\omega \text{ and } \vec{a} \in {}^{<\omega}(2^{\mathbb{N}}) \\
\varphi(\delta, \lambda) & \in \bar{T} \iff \delta < \omega_1, \lambda < \kappa \\
\varphi(x, \bar{X}_\delta) & \in \bar{T} \iff \delta < \omega_1, x \in H_\kappa
\end{align*} \]

Let us write \( \mathcal{L}^\lambda \) for the collection of all \( \mathcal{L} \)-formulae except for the formulae which mention elements outside of \( Q_\lambda \), i.e., except for the formulae of the form \( \varphi(\lambda) \) for \( \delta < \omega_1 \) and \( \lambda \leq \delta < \kappa \). We may and shall assume that \( \mathcal{L} \) is built in a canonical way so that \( \mathcal{L}^\lambda \subseteq Q_\lambda \).

We say that the objects \( C \) as in (6) are pre-certified by a collection \( \Sigma \) of \( \mathcal{L}^\lambda \)-formulae if and only if (C.1) through (C.8) are satisfied by \( C \) and there are surjections \( e_i : \omega \to N_i \) for \( i < \omega_1 \) such that the following hold true.

(S.1) \( \varphi(\xi_1, \ldots, \xi_k, n_\ell, \dot{a}, \dot{I}, \dot{M}_j, \ldots, \dot{M}_{\dot{m}}, \dot{\pi}_{q_{r_1}}, \ldots, \dot{\pi}_{q_{r_{\dot{s}}}}, \dot{\lambda}) \in \Sigma \) iff
\( i < \omega_1, \xi_1, \ldots, \xi_k \leq N_i, n_\ell < \omega, j_1, \ldots, j_m \leq N_i, q_1 \leq r_1 \leq \omega_1, \ldots, q_{\dot{s}} \leq r_{\dot{s}} \leq \omega_1 \) and
\( \dot{N}_i \models \varphi(\xi_1, \ldots, \xi_k, e_i(n_\ell), A \cap \omega_i, I, M_j, \ldots, M_{\dot{m}}, \pi_{q_{r_1}}, \ldots, \pi_{q_{r_{\dot{s}}}}, \dot{\lambda}^i) \),
where \( I^{N_i} \) is the distinguished ideal of \( N_i \) and \( \dot{\lambda} = (M_j, \pi_{jj'} : j \leq j' \leq \omega_1) \).

(S.2) \( \varphi(\pi_{\omega_1}(\dot{a})) = x \in \Sigma \) iff \( i < \omega_1, n < \omega, \) and \( \pi_{\omega_1}(e_i(n)) = x \),

(S.3) \( \varphi(\pi_{\omega_1}(x)) = x \in \Sigma \) iff \( x \in H_\omega \),

(S.4) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \),

(S.5) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \),

(S.6) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \),

(S.7) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \),

Letting \( F \) with \( \text{dom}(F) = \omega \) be the monotone enumeration of the Gödel numbers of all \( \varphi(\dot{u}, \dot{a}) \in \bar{T} \subseteq \Sigma \) with \( \varphi(\dot{u}, \dot{a}) \in \bar{T} \subseteq \Sigma \) iff there is some \( n < \omega \) such that \( \langle \dot{u}, \dot{a} \rangle = \langle (F(m), \alpha_m) : m < n \rangle \) and \( F(m) = k_m \) for all \( m < n \),

(S.6) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \) for all \( m < n \),

(S.7) \( \varphi(\sigma_{ij}(\dot{a})) = \dot{a} \in \Sigma \) iff \( i < j < \omega_1, n, m < \omega, \) and \( \sigma_{ij}(e_i(n)) = e_j(m) \) for all \( m < n \),

We say that the objects \( C \) as in (6) are certified by a collection \( \Sigma \) of formulae if and only if \( C \) is pre-certified by \( \Sigma \) and in addition,

(S.8) if \( \delta \in K \), then \( |\Sigma|^{<\omega} \cap X_\delta \cap E \neq \emptyset \) for every \( E \subseteq \mathbb{P}_{\lambda_\delta} \) which is dense in \( \mathbb{P}_{\lambda_\delta} \) and definable over the structure
\[
\langle Q_{\lambda_\delta}; \in, \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta} \rangle
\]
By way of definition, we call $C$ as in (6) a **semantic certificate** iff there is a collection $\Sigma$ of formulae such that $C$ is certified by $\Sigma$. We call $\Sigma$ a **syntactic certificate** iff there is a semantic certificate $C$ such that $C$ is certified by $\Sigma$. Given a syntactic certificate $\Sigma$, there is a unique semantic certificate $C$ such that $C$ is certified by $\Sigma$. Even though it is obvious how to construct $C$ from $\Sigma$, in the proof of Lemma 3.3 below we will provide details on how to derive a semantic certificate from a given $\Sigma$.

Let $\Sigma \cup p$ be a set of formulae, where $p$ is finite. We say that $p$ is **certified by** $\Sigma$ if and only if there is some (unique) $C$ as in (6) such that $C$ is certified by $\Sigma$ and

$$(\Sigma.9) \quad p \in [\Sigma]<\omega.$$  

We may also say that $p$ is certified by $C$ as in (6) iff there is some $\Sigma$ such that $C$ and $p$ are both certified by $\Sigma$ – and we will then also refer to $\Sigma$ as a syntactical certificate for $p$ and to $C$ as the associated semantical certificate.

We are then ready to define the forcing $\mathbb{P}_\lambda$. We say that $p \in \mathbb{P}_\lambda$ if and only if $V^{\text{Col}(\omega, \lambda)} \models "There is a set $\Sigma$ of $L^\lambda$-formulae such that $p$ is certified by $\Sigma."$ (8)

Let $p$ be a finite set of formulae of $L^\lambda$. By the homogeneity of $\text{Col}(\omega, \lambda)$, if there is some $h$ which is $\text{Col}(\omega, \lambda)$-generic over $V$ and there is some $\Sigma \in V[h]$ such that $p$ is certified by $\Sigma$, then for all $h$ which are $\text{Col}(\omega, \lambda)$-generic over $V$ there is some $\Sigma \in V[h]$ such that $p$ is certified by $\Sigma$. It is then easy to see that $\langle \mathbb{P}_\lambda : \lambda \in C \cup \{\kappa}\rangle$ is definable over $V$ from $\langle A_\lambda : \lambda < \kappa\rangle$ and $C$, and is hence an element of $V$.

Again let $p$ be a finite set of formulae of $L^\lambda$. By $\Sigma_1^1$ absoluteness, if there is any outer model in which there is some $\Sigma$ which certifies $p$, then there is some $\Sigma \in V^{\text{Col}(\omega, \lambda)}$ which certifies $p$.\cite{11} This simple observation is important in the verification that $\mathbb{P}_\lambda$ is actually non-empty, cf. Lemma 3.2, and in the proof of Lemma 3.8.

It is easy to see that

1. $\mathbb{P} = \mathbb{P}_\kappa \subset H_\kappa$,
2. If $\tilde{\lambda} \leq \lambda$ are both in $C \cup \{\kappa\}$, then $\mathbb{P}_{\tilde{\lambda}} \subset \mathbb{P}_\lambda$, and

\cite{10}Equivalently, $[\Sigma]<\omega \cap E \neq \emptyset$ for every $E \subset \mathbb{P}_\lambda \cap X_\delta$ which is dense in $\mathbb{P}_\lambda \cap X_\delta$ and definable over the structure $$(X_\delta; e, \mathbb{P}_\lambda \cap X_\delta, A_\lambda \cap X_\delta)$$

from parameters in $X_\delta$.

\cite{11}In fact, if $P$ is a transitive model of $\text{KP}$ plus the axiom Beta with $(Q_\lambda; \langle A_\lambda : \tilde{\lambda} < \lambda\rangle) \in P$ and if $p \in \mathbb{P}_\lambda$, then there is some $\Sigma \in P^{\text{Col}(\omega, \lambda)}$ which certifies $p$.  

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(iii) if \( \lambda \in C \cup \{ \kappa \} \) is a limit point of \( C \cup \{ \kappa \} \), then \( \mathbb{P}_\lambda = \bigcup_{\lambda \in C \cap \lambda} \mathbb{P}_\lambda \), so that there is some club \( D \subset C \) such that for all \( \lambda \in D \),

\[ \mathbb{P}_\lambda = \mathbb{P} \cap Q_\lambda. \]

Hence (\( \diamond \)) gives us the following.

(\( \diamond (\mathbb{P}) \)) For all \( B \subset H_\kappa \) the set

\[ \{ \lambda \in C : (Q_\lambda ; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa ; \in, \mathbb{P}, B) \} \]

is stationary.

The first one of the following lemmas is entirely trivial.

**Lemma 3.1** Let \( \Sigma \) be a syntactic certificate, and let \( p, q \in [\Sigma]^{\omega} \). Then \( p \) and \( q \) are compatible conditions in \( \mathbb{P} \).

**Lemma 3.2** Let \( \lambda \in C \cup \{ \kappa \} \). Then \( \emptyset \in \mathbb{P}_\lambda \).

**Proof.** See the proof of [1, Theorem 2.8], or the proof of [16, Theorem 4.2]. Notice that for all \( \lambda \in C \cup \{ \kappa \} \), \( \emptyset \in \mathbb{P}_\lambda \) iff \( \emptyset \in \mathbb{P} \).

Let \( h \) be \( \text{Col}(\omega, \omega_2) \)-generic over \( V \), and write \( \rho = \omega_3^V = \omega_1^{V[h]} \). Inside \( V[h] \),

\[ ((H_\omega^V ; \in, (\text{NS}_\omega^V) ; V, A) \]

is a \( \mathbb{P}_{\text{max}} \) condition, call it \( p \). Let \( q \in (\mathbb{P}_{\text{max}})^V[h], q < p, q \in D^* \), cf. (D.1). Let \( q = N_0 = (N_0 ; \in, I, a) \). Let \( (M_i, \pi_{ij} : i \leq j \leq \omega^{N_0}) \in N_0 \) be the unique generic iteration of \( p \) which witnesses \( q < p \).

Let \( (N_i, \sigma_{ij} : i \leq j \leq \rho) \in V[h] \) be a generic iteration of \( N_0 \) such that \( \rho = \omega_1^{N_0} \).\(^{12}\)

Let

\[ (M_i, \pi_{ij} : i \leq j \leq \rho) = \sigma_{0\rho}((M_i, \pi_{ij} : i \leq j \leq \omega^{N_0})) \]

We may lift (10) to a generic iteration

\[ (M_i^+, \pi_{ij}^+ : i \leq j \leq \rho) \]

of \( V \). Let us write \( M = M_{\rho}^+ \) and \( \pi = \pi_{0\rho}^+ \).

\(^{12}\)If we wished, then we could even arrange that writing \( N_{\rho} = (N_{\rho} ; \in, I^*, a^*) \), we have that \( I^* = (\text{NS}_{\rho})^{V[h]} \cap N_{\rho} \), but this is not relevant here; cf. footnote 9.
Let \( \langle k_n, \alpha_n : n < \omega \rangle \) be such that \( x = \langle k_n : n < \omega \rangle \) is a real code for \( N_0 \) à la (\( \Sigma, 5 \)) and \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T] \). We then clearly have that \( \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle \in [\pi(T)] \).

It is now easy to see that

\[
\mathcal{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \rho \rangle, \langle (k_n, \pi(\alpha_n)) : n < \omega \rangle, \rangle
\]  

(11)
certifies \( \emptyset \), construed as the empty set of \( \pi(\mathcal{L}^\kappa) \) formulae: as the third component \( \langle \rangle \) of \( \mathcal{C} \) in (11) is empty, any set of surjections \( e_i : \omega \rightarrow N_i, i < \omega_1 \), will induce a syntactic certificate for \( \emptyset \) whose associated semantic certificate is \( \mathcal{C} \). By \( \Sigma_1^1 \) absoluteness, there is then some \( \mathcal{C} \in M^{\text{Col}(\omega_1)} \) as in (11) which certifies \( \emptyset \) so that \( \emptyset \in \pi(P) \). By the elementarity of \( \pi \), then, there is some \( \mathcal{C} \in V^{\text{Col}(\omega_1)} \) which certifies \( \emptyset \), construed as the empty set of \( \mathcal{L}^\kappa \) formulae. Hence \( \emptyset \in P \).

\[\Box\]

**Lemma 3.3** Let \( \lambda \in C \cup \{\kappa\} \). Let \( g \subset P_\lambda \) be a filter such that \( g \cap E \neq \emptyset \) for all dense \( E \subset P_\lambda \) which are definable over \( (Q_\lambda ; e, P_\lambda) \) from elements of \( Q_\lambda \). Then \( \bigcup g \) is a syntactic certificate.

**Proof.** It is obvious how to read off from \( \bigcup g \) a candidate

\[
\mathcal{C} = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle \lambda_\delta, X_\delta : \delta \in K \rangle
\]

like in (6) for a semantical certificate for \( \bigcup g \). Let us be somewhat explicit, though. A variant of what is to come shows how to derive \( \mathcal{C} \) from a given syntactic certificate \( \Sigma \), where \( \mathcal{C} \) is unique such that \( \Sigma \) certifies \( \mathcal{C} \), cf. the remark on p. 10.

For \( i, j < \omega_1 \) and \( \tau, \sigma \in \{ \hat{n} : n < \omega \} \cup \omega_1 \) define

\( \tau \sim_i \sigma \) if \( \tau \) \( \rangle \) and \( \sigma \rangle \in \bigcup g \)

\( (i, \tau) \sim_{\omega_1} (j, \sigma) \) if \( i \leq j \lor \exists \rho \{ \tau \rangle = \sigma \rangle, \tau \rangle \in \bigcup g \}

\( \{ (i, \tau) \sim_{\omega_1} (j, \sigma) \} \in \bigcup g \)

\( \tau \rangle \in (H_{\omega_2})^V \)

\( M_i = \{ (\tau) : \tau \in \{ \hat{n} : n < \omega \} \cup \omega_1 \land \tau \rangle \in \bigcup g \} \)

\( M_{\omega_1} = (H_{\omega_2})^V \)

\( N_i = \{ (\tau) : \tau \in \{ \hat{n} : n < \omega \} \cup \omega_1 \} \)

\( N_{\omega_1} = \{ (i, \tau) : i \in \omega_1 \land \tau \rangle \in \bigcup g \} \)
\[ [\tau_i, \bar{c}_i, [\sigma_i] \text{ iff } \forall \tilde{N}_i \models \tau \in \sigma \in \bigcup g \]
\[ [i, \tau] \in I^{N_i} \text{ iff } \forall \tilde{N}_i \models \tau \in \check{I} \in \bigcup g \]
\[ [\tau_i]_i \in a^{N_i} \text{ iff } \forall \tilde{N}_i \models \tau \in \check{a} \in \bigcup g \]
\[ \pi_{ij}([\tau_i]) = [\sigma]_j \text{ iff } \forall \tilde{N}_j \models \pi_{ij}(\tau) = \sigma \in \bigcup g \]
\[ \pi_{i\omega_1}([\tau_i]) = x \text{ iff } \forall \tilde{N}_i \models \pi_{i\omega_1}(\tau) = x \in (H_{\omega_1})^V \]
\[ \sigma_{ij}([\tau_i]) = [\sigma]_j \text{ iff } \forall \tilde{N}_j \models \sigma_{ij}(\tau) = \sigma \in \bigcup g \]
\[ (k, \alpha) = (k^n, \alpha^n) \text{ iff } \exists \bar{u} \exists \bar{\alpha}(\bar{u}, \bar{\alpha}) \in \hat{T} \in \bigcup g \wedge k = \bar{u}(n) \wedge \alpha = \bar{\alpha}(n) \]
\[ \delta \in K^g \text{ iff } \exists \bar{\lambda} \bar{\delta} \mapsto \bar{\lambda} \in \bigcup g \]
\[ \bar{\lambda} = \lambda^g \text{ iff } \delta \in K^g \wedge \bar{\delta} \mapsto \bar{\lambda} \in \bigcup g \]
\[ x \in X^g \text{ iff } \delta \in K^g \wedge \bar{\delta} \mapsto x \in \hat{X} \in \bigcup g \]

We will first have that \( \bar{c}_0 \) is wellfounded and that in fact (the transitive collapse of) the structure \( N_0 = (N_0; \bar{c}_0, a^{N_0}, I^{N_0}) \) is an iterable \( \mathbb{P}_{\text{max}} \) condition. This is true because straightforward density arguments give (C.2), i.e., that \( \langle (k_n, \alpha_n) : n < \omega \rangle \in [T] \) and \( \langle k_n : n < \omega \rangle \) will code the theory of \( N_0 \) à la \( \Sigma_3 \).

Another set of easy density arguments will give that \( (N_i; \sigma_{ij} : i \leq j) \) is a generic iteration of \( N_0 \), were we identify \( N_i \) with the structure \( (N_i; \bar{c}_i, a^{N_i}, I^{N_i}) \). To verify this, let us first show:

**Claim 3.4** For each \( i < \omega_1 \) and for each \( \xi \leq \omega_1 \), \( [\xi]_i \) represents \( \xi \) in (the transitive collapse of the well-founded part of) the term model for \( N_i \); moreover, \( a^{N_i} = A \cap \omega_1^{N_i} \). Hence \( a^{N_{\omega_1}} = A \).

**Proof of Claim 3.4.** Straightforward arguments using (\( \Sigma_1 \)) show that \( [\xi]_i \) must always represent \( \xi \) in \( N_i \) as given by any certificate. Claim 3.4 then follows by straightforward density arguments. \( \square \) (Claim 3.4)
Similarly:

**Claim 3.5** Let \( i < \omega_1 \). \( N_{i+1} \) is generated from \( \text{ran}(\sigma_{i+1}) \cup \{\omega_1^N_i\} \) in the sense that for every \( x \in N_{i+1} \) there is some function \( f \in \omega_1^N_i (N_i) \cap N_i \) such that \( x = \sigma_{i+1}(f)(\omega_1^N_i) \).

**Claim 3.6** Let \( i < \omega_1 \). \( \{X \in \mathcal{P}(\omega_1^N_i) \cap N_i : \omega_1^N_i \in \sigma_{i+1}(X)\} \) is generic over \( N_i \) for the forcing given by the \( P_i \)-positive sets.

**Claim 3.7** Let \( i \leq \omega_1 \) be a limit ordinal. For every \( x \in N_i \) there is some \( j < i \) and some \( \bar{x} \in N_j \) such that \( x = \sigma_j(\bar{x}) \).

\((N_i, \sigma_{ij} : i \leq j \leq \omega_1)\) is then indeed a generic iteration of \( N_0 \). As \( N_0 \) is iterable, we may and shall identify \( N_i \) with its transitive collapse, so that (C.4) holds true.

Another round of density arguments will show that \( \mathcal{C} \) satisfies (C.1), (C.3), (C.5), (C.6), and (C.7), where we identify \( M_i \) with the structure \((M_i; \in, (\text{NS}_{\omega_1^M_i})^{M_i}, A \cap \omega_1^{M_i})\). Let us now verify (C.8) and (C.9).

As for (C.8), \( X_\delta \cap N_1 = \delta \) for \( \delta \in K \) is easy. Let \( x_1, \ldots, x_k \in X_\delta, \delta \in K \). Suppose that

\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \exists v \varphi(v, x_1, \ldots, x_k).
\]

Let \( p \in g \) be such that \( \{\gamma x_1 \in X_\delta^{-1}, \ldots, \gamma x_k \in X_\delta^{-1}, \gamma \delta \mapsto \lambda_\delta^{-1}\} \subset p \). Let \( q \leq p \), and let \( \Sigma \) be a syntactical certificate for \( q \) whose associated semantical certificate is

\[
\mathcal{C}' = \langle M_i', \pi_{ij}', N_i', \sigma_{ij}' : i \leq j \leq \omega_1 \rangle, (k_n', \alpha_n') : n < \omega, \lambda'_\delta, X_\delta' : \delta \in K' \rangle.
\]

Then \( \delta \in K' \) and

\[
\{x_1, \ldots, x_k\} \subset X_\delta' < (Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}),
\]

so that by (12) we may choose some \( x \in X_\delta' \) with

\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(x, x_1, \ldots, x_k).
\]

Let \( r = q \cup \{\gamma x \in X_\delta^{-1}\} \).

By density, there is then some \( y \in X_\delta \) such that

\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(y, x_1, \ldots, x_k).
\]

The proof of (C.9) is similar. Let again \( \delta \in K \). Let \( E \subset \mathbb{P}_{\lambda_\delta} \cap X_\delta^g \) be dense in \( \mathbb{P}_{\lambda_\delta} \cap X_\delta \), and \( r \in E \) iff \( r \in \mathbb{P}_{\lambda_\delta} \cap X_\delta \) and

\[
(Q_{\lambda_\delta}; \in \mathbb{P}_{\lambda_\delta}, A_{\lambda_\delta}) \models \varphi(r, x_1, \ldots, x_k).
\]

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Let $p \in g$ be such that $\{\tau x_1 \in \dot{X}_\delta, \ldots, \tau x_k \in \dot{X}_\delta, \tau \delta \mapsto \lambda_\delta\} \subseteq p$. Let $q \leq p$, and again let $\Sigma$ be a syntactical certificate for $q$ whose associated semantical certificate is

$$C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle X_\delta' : \delta \in K' \rangle.$$  

Then $[\Sigma]^{<\omega} \cap X_\delta'$ has an element, say $r$, such that (13) holds true. Let $s = q \cup r \cup \{\tau r \in \dot{X}_\delta\}$.

By density, then, $g \cap X_\delta \cap E \neq \emptyset$. □

**Lemma 3.8** Let $g$ be $\mathbb{P}$-generic over $V$. Let

$$C = \langle M_i, \pi_{ij}, N_i, \sigma_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k_n, \alpha_n) : n < \omega \rangle, \langle X_\delta' : \delta \in K \rangle$$

be the semantic certificate associated with the syntactic certificate $\bigcup g$. Let

$$N_{\omega_1} = (N_{\omega_1}; \in, A, I^*)$$

and let $T \in (\mathcal{P}(\omega_1) \cap N_{\omega_1}) \setminus I^*$. Then $T$ is stationary in $V[g]$.

If $C$, $I^*$, and $T$ are as in the statement of Lemma 3.8, then by Lemma 3.3 and (C.3) and (C.4) we will have that $(\mathbf{NS}_{\omega_1})^V = I^* \cap V$, so that the conclusion of Lemma 3.8 also gives that $\mathbb{P}$ preserves the stationarity of $T$. In other words:

**Corollary 3.9** $\mathbb{P}$ preserves stationary subsets of $\omega_1$.

**Proof** of Lemma 3.8. Let $\dot{N}_{\omega_1} \in V^\mathbb{P}$ be a canonical name for $N_{\omega_1}$, and let $\dot{I}^* \in V^\mathbb{P}$ be a canonical name for $I^*$. Let $\bar{p} \in g$, $\dot{C}$, $\dot{S} \in V^\mathbb{P}$, and $i < \omega_1$ and $n < \omega$ be such that

(i) $T = \dot{S}^\bar{p}$,

(ii) $\bar{p} \Vdash \dot{C} \subseteq \omega_1$ is club,

(iii) $\bar{p} \Vdash \dot{S} \in (\mathcal{P}(\omega_1) \cap \dot{N}_{\omega_1}) \setminus \dot{I}^*$,

(iv) $\bar{p} \Vdash \dot{S}$ is represented by $[i, n]$ in the term model producing $\dot{N}_{\omega_1}$.

We may and shall also assume that

$$\tau \dot{N}_i \Vdash \dot{n} \text{ is a subset of the first uncountable cardinal, yet } \dot{n} \notin \dot{I} \in \bar{p}. \quad (14)$$

Let $p \leq \bar{p}$ be arbitrary. We aim to produce some $q \leq p$ and some $\delta < \omega_1$ such that $q \Vdash \delta \in \dot{C} \cap \dot{S}$, see Claim 3.11 below.
For $\xi < \omega_1$, let
\[ D_\xi = \{ q \leq p : \exists \eta \geq \xi (\eta < \omega_1 \land q \Vdash \check{\eta} \in \check{C}) \}, \]
so that $D_\xi$ is open dense below $p$. Let
\[ E = \{ (q, \eta) \in \mathbb{P} \times \omega_1 : q \Vdash \check{\eta} \in \check{C} \}. \]
Let us write
\[ \tau = ((D_\xi : \xi < \omega_1), E). \]
We may and shall identify $\tau$ with some subset of $H_\kappa$ which codes $\tau$.

By $(\bigcirc (\mathbb{P}))$, we may pick some $\lambda \in C$ such that $p \in \mathbb{P}_\lambda$ and
\[(Q_\lambda; \in, \mathbb{P}_\lambda, A_\lambda) \prec (H_\kappa; \in, \mathbb{P}, \tau). \quad (15)\]

Let $h$ be $\text{Col}(\omega, 2^{\aleph_2})$-generic over $V$, and let $g' \in V[h]$ be a filter on $\mathbb{P}_\lambda$ such that $p \in g'$ and $g'$ meets every dense set which is definable over $(N_\lambda; \in, \mathbb{P}_\lambda, A_\lambda)$ from parameters in $N_\lambda$. By Lemma 3.3, $\bigcup g'$ is a syntactic certificate for $p$, and we may let
\[ \langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_i, X'_i : \delta \in K' \rangle \]
be the associated semantic certificate. In particular, $K' \subset \lambda$.

Let $S$ denote the subset of $\omega_1$ which is represented by $[i, \check{n}]$ in the term model giving $N'_{\omega_1}$, so that if $N'_{\omega_1} = (N'_{\omega_1}, \in, A, I')$, then by (14),
\[ S \in (\mathcal{P}(\omega_1) \cap N'_{\omega_1}) \setminus I'. \quad (16) \]

Let us also write $\rho = \omega_{1}^{V[h]} = (2^{\aleph_2})^{+V}$. Inside $V[h]$, we may extend $\langle N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle$ to a generic iteration
\[ \langle N'_i, \sigma'_{ij} : i \leq j \leq \rho \rangle \]
such that
\[ \omega_1 \in \sigma'_{\omega_1, \omega_1+1}(S). \quad (17) \]
This is possible as $\omega_{1}^{N'_{\omega_1}} = \sup\{\omega_{1}^{N_j} : j < \omega_1\} = \omega_1$ and by (16). Let
\[ \langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle = \sigma_{0,\rho}(\langle M'_i, \pi'_{ij} : i \leq j \leq \omega_{1}^{N_0} \rangle), \]
so that $\langle M'_i, \pi'_{ij} : i \leq j \leq \rho \rangle$ is an extension of $\langle M'_i, \pi'_{ij} : i \leq j \leq \omega_1 \rangle$. 

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Recalling (7), we may lift \((M^+_i, \pi^+_ij : \omega_1 \leq i \leq j \leq \rho)\) to a generic iteration \((M^+_i, \pi^+_ij : \omega_1 \leq i \leq j \leq \rho)\) of \(V\). Let us write \(M = M^+_\rho\) and \(\pi = \pi^+_\rho\).

The key point is now that \((M^+_i, \pi^+_ij, \sigma^+_ij : i \leq j \leq \rho)\) may be used to extend \(\pi'' \bigcup g'\) to a syntactic certificate

\[\Sigma \supset \pi'' \bigcup g'\]  

for \(\pi(p)\) in the following manner. Let \(K^* = K' \cup \{\omega_1\}\). For \(\delta \in K'\), let \(\lambda^*_\delta = \pi(\lambda'_\delta)\) and \(X^*_\delta = \pi''X'_\delta\). Also, write \(\lambda^*_\omega_1 = \pi(\lambda)\) and \(X^*_\omega_1 = \pi''Q\). Let

\[
\mathcal{C}^* = \langle M'_i, \pi'_ij, N'_i, \sigma'_ij : i \leq j \leq \rho \rangle, \langle (k'_n, \pi(\alpha'_n)) : n < \omega \rangle, \langle \lambda^*_\delta, X^*_\delta : \delta \in K^* \rangle.
\]

It is then straightforward to verify that \(\mathcal{C}^*\) is a semantic certificate for \(\pi(p)\), and that in fact there is some syntactic certificate \(\Sigma\) as in (18) such that \(\mathcal{C}^*\) is certified by \(\Sigma\).

Now let \([\dot{m}]_{\omega_1+1}\) represent \(\sigma'_{\omega_1\omega_1+1}(S)\) in the term model for \(N'_{\omega_1+1}\) provided by \(\Sigma\), so that\(^{13}\)

\[
\{\Gamma \dot{\sigma}_{\omega_1+1}(\dot{n}) = \dot{m}, \Gamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}\} \subseteq \Sigma,
\]

in other words,

\[
\pi(p) \cup \{\Gamma \dot{\sigma}_{\omega_1+1}(\dot{n}) = \dot{m}, \Gamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}\} \text{ is certified by } \Sigma.
\]

(19)

Let us now define

\[
q^* = \pi(p) \cup \{\Gamma \dot{\sigma}_{\omega_1+1}(\dot{n}) = \dot{m}, \Gamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}, \Gamma \omega_1 \mapsto \pi(\lambda)^c\}.
\]

(20)

We thus established the following.

Claim 3.10 \(q^* \in \pi(\mathbb{P})\), as being certified by \(\Sigma\).

The elementarity of \(\pi : V \rightarrow M^+_\rho\) then gives some \(\delta < \omega_1\) and some \(\mu < \kappa\) such that

\[
q = p \cup \{\Gamma \dot{\sigma}_{\delta+1}(\dot{n}) = \dot{m}, \Gamma \dot{N}_{\delta+1} \models \delta \in \dot{m}, \Gamma \delta \mapsto \lambda^c\} \in \mathbb{P}.
\]

(21)

Claim 3.11 \(q \models \delta \in \dot{C} \cap \dot{S}\).

\(^{13}\)Here, \(\dot{\sigma}_{\omega_1+1}\) and \(\dot{N}_{\omega_1+1}\) are terms of the language associated with \(\pi(\mathbb{P}_\lambda)\) and \(\Gamma \dot{\sigma}_{\omega_1+1}(\dot{n}) = \dot{m}\) and \(\Gamma \dot{N}_{\omega_1+1} \models \omega_1 \in \dot{m}\) are formulae of that language.
Proof of Claim 3.11. \( q \models \delta \in \hat{S} \) readily follows from \( \{ \gamma \sigma_{\delta+1}(\hat{n}) = \hat{m} \gamma, \gamma \hat{N}_{\delta+1} \models \delta \in \hat{m} \gamma \} \subset q \), the fact that \( \bar{p} \geq p \) forces that \( \hat{S} \) is represented by \([i, \hat{n}]\) in the term model giving \( \hat{N}_{\omega_1} \), and the fact that by Claim 3.4, \([\delta]_{\delta+1} \) represents \( \delta \) in the model \( N_{\delta+1} \) of any semantic certificate for \( q \).

Let us now show that \( q \models \check{\delta} \in \check{C} \). We will in fact show that \( q \) forces that \( \check{\delta} \) is a limit point of \( \check{C} \). Otherwise there is some \( r \leq q \) and some \( \eta < \delta \) such that

\[
\left( 22 \right)
\]

\[
r \models \check{C} \cap \check{\delta} \subset \check{\eta}.
\]

Let

\[
\langle M'_i, \pi'_{ij}, N'_i, \sigma'_{ij} : i \leq j \leq \omega_1 \rangle, \langle (k'_n, \alpha'_n) : n < \omega \rangle, \langle \lambda'_\delta, X'_\delta : \delta \in K' \rangle
\]

-certify \( r \). We must have that

(a) \( \delta \in K' \),

(b) \( X'_\delta \prec (Q_\lambda; \in, P_\lambda, A_\lambda) \),

(c) \( X'_\delta \cap \omega_1 = \delta \), and

(d) for some \( \Sigma \) such that the objects from (23) are certified by \( \Sigma \), \( [\Sigma]^{<\omega} \cap X'_\delta \cap E \neq \emptyset \) for every \( E \subset P_\lambda \) which is dense in \( P_\lambda \cap X'_\delta \) and definable over the structure \((Q_\lambda; \in, P_\lambda, A_\lambda)\) from parameters in \( X'_\delta \).

Notice that \( A_\lambda = \tau \cap Q_\lambda \), and hence \( A_\lambda \) may be identified with \((D_\xi \cap Q_\lambda : \xi < \omega_1), E \cap Q_\lambda\). As \( \eta < \delta \subset X'_\delta \), \( D_\eta \) is definable over the structure \((Q_\lambda; \in, P_\lambda, A_\lambda)\) from a parameter in \( X'_\delta \). By (15), \( D_\eta \cap Q_\lambda \) is dense in \( P_\lambda \). By (d) above, there is then some \( s \in [\Sigma]^{<\omega} \cap X'_\delta \cap D_\eta \cap Q_\lambda \).

By (15) again, the unique smallest \( \eta' \geq \eta \) with \( s \models \check{\eta'} \in \check{C} \) must be in \( X'_\delta \), hence \( \eta' < \delta \) by (c) above. By Lemma 3.1, \( s \) is compatible with \( r \). We have reached a contradiction with (22). \( \square \)
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