On Jacobi quasi-Nijenhuis algebroids and Courant–Jacobi algebroid morphisms

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ABSTRACT

We propose a definition of a Jacobi quasi-Nijenhuis algebroid and show that any such Jacobi algebroid has an associated quasi-Jacobi bialgebroid. Therefore, an associated Courant–Jacobi algebroid is also obtained. We introduce the notions of quasi-Jacobi bialgebroid morphism and Courant–Jacobi algebroid morphism, also providing some examples of Courant–Jacobi algebroid morphisms.

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0. Introduction

It is well known that one can consider, on a Lie algebroid, some additional geometric tools that provide richer structures such as, among others, Poisson Lie algebroids, Poisson–Nijenhuis Lie algebroids [1,2], Lie bialgebroids [3] and quasi-Lie bialgebroids [4]. This procedure can be repeated if we start with a Jacobi algebroid $(A, \phi)$, i.e., a Lie algebroid $A$ together with a 1-cocycle $\phi \in \Gamma(A^*)$. The notion of a Jacobi algebroid was introduced in [5,6] and it is the one that fits in the Jacobi framework. As in the Poisson case, one can define Jacobi bialgebroids [5,6], quasi-Jacobi bialgebroids [7], Jacobi–Nijenhuis algebroids [8], and so on. In the case of a Poisson–Nijenhuis Lie algebroid $(A, \pi, N)$, $\pi$ is a Poisson bivector on $A$ which is compatible, in a certain sense, with the Nijenhuis operator $N$, i.e., a vector bundle map $N : A \to A$ with vanishing torsion. Relaxing the condition which imposes the vanishing of the torsion, and admitting that its nonzero value depends on a certain closed 3-form, one obtains what is called a Poisson quasi-Nijenhuis Lie algebroid, a notion recently introduced in [9] for the case of manifolds and then extended to general Lie algebroids in [10]. It is worthwhile mentioning that the notion considered in [10] is even more general since the compatibility between the Poisson bivector and the operator $N$ is less restrictive, leading to the so-called Poisson quasi-Nijenhuis Lie algebroid with background. Either in the Poisson quasi-Nijenhuis case or in the Poisson quasi-Nijenhuis with background case, there is an associated quasi-Lie bialgebroid. Since the double of a quasi-Lie bialgebroid is a Courant algebroid, one has a Courant algebroid associated to each Poisson quasi-Nijenhuis Lie algebroid (with background).

In the Jacobi setting, the notion of a Jacobi–Nijenhuis algebroid was introduced in [8] and it is a quadruple $(A, \phi, \pi, N)$, where $(A, \phi)$ is a Jacobi algebroid and $\pi$ is a Jacobi bivector (i.e., $[\pi, \pi]^\phi = 0$) that is compatible with the operator $N$ which
has vanishing torsion with respect to the bracket $[\cdot, \cdot]^\phi$. Admitting a nonzero torsion that depends on a certain $d^\phi$-closed 3-form, we define a Jacobi quasi-Nijenhuis algebroid and show that it has an associated quasi-Jacobi bialgebroid and therefore also an associated Courant–Jacobi algebroid [11,12]. Another concept we introduce, having a relevant role in the paper, is that of a quasi-Jacobi bialgebroid morphism.

An important notion related to Courant and Courant–Jacobi algebroids $A \to M$ is that of Dirac structure — a special vector subbundle $L \to M$ of $A \to M$ which inherits, by restriction, a Lie algebroid structure. In [13,14], given a Courant algebroid $A \to M$, the authors considered a vector subbundle $L \to \Pi$ over a submanifold $\Pi$ of $M$ and introduced the notion of a Dirac structure supported on a submanifold, which is obviously a generalization of a Dirac structure. In a natural way, we extend the definition to the case of Dirac structures ofCourant–Jacobi algebroids supported on submanifolds of the base manifold and, using this notion, we introduce the concept of Courant–Jacobi algebroid morphism. We prove that a morphism of quasi-Jacobi bialgebroids produces a Courant–Jacobi algebroid morphism as well as some Jacobi quasi-Nijenhuis algebroids do.

Very recently, while we were finishing this paper, the notion of a Jacobian quasi-Nijenhuis algebroid was also introduced in [15], where the author claims the equivalence between a Jacobian quasi-Nijenhuis structure on a Courant algebroid and a quasi-Jacobi bialgebroid structure on its dual. The author proves his result just on functions and exact 1-forms, but these in general do not generate the space of 1-forms in Jacobi (or Lie) algebroids, so the proof is not correct. Only in the case of the tangent bundle to a manifold does the equivalence hold.

This paper is divided into two sections. In Section 1, we introduce the notion of quasi-Jacobi bialgebroid morphism, and we give a simpler definition of Courant–Jacobi algebroids and we extend to Courant–Jacobi algebroids the notion of a Dirac structure supported on a submanifold of the base manifold, which was initially given in [13,14] for Courant algebroids morphisms. Finally, we define Courant–Jacobi morphism and show that, in the case where the Courant–Jacobi algebroids are doubles of quasi-Jacobi bialgebroids, a morphism $(\Phi, \phi)$ of Courant–Jacobi algebroids provides an example of a Dirac structure supported on graph $\phi$.

In Section 2, after introducing the concept of a Jacobian quasi-Nijenhuis algebroid, we show that it has an associated Poisson quasi-Nijenhuis Lie algebroid. The main result of this section states that to each Jacobi quasi-Nijenhuis algebroid one can associate a quasi-Jacobi bialgebroid, and in the case where the Lie algebroid is the tangent bundle to a manifold, one has a one-to-one correspondence. For a special kind of Jacobi quasi-Nijenhuis algebroid, we show that the graph of the quasi-Nijenhuis operator determines a Courant–Jacobi algebroid morphism.

1. Quasi-Jacobi bialgebroid morphisms

1.1. Quasi-Jacobi bialgebroids

Recall that a Jacobi algebroid [5] or generalized Lie algebroid [6] is a pair $(A, \phi)$, where $A = (A, [\cdot, \cdot], \rho)$ is a Lie algebroid over a manifold $M$ and $\phi \in \Gamma(A^*)$ is a 1-cocycle, i.e., $d\phi = 0$. A Jacobi algebroid has an associated Schouten–Jacobi bracket on the graded algebra $\Gamma(\wedge A)$ of multivector fields on $A$ given by

$$[P, Q]^\phi = [P, Q] + (p - 1)\rho \wedge i_Q P - (-1)^{p-1}(q - 1)i_p P \wedge Q,$$

for $P \in \Gamma(\wedge P A), Q \in \Gamma(\wedge A), p \geq 1, q \geq 1$.

In a Jacobi algebroid the anchor $\rho$ is replaced by $\rho^\phi$, which is the representation of the Lie algebra $\Gamma(A)$ on $C^\infty(M)$ given by

$$\rho^\phi(X)f = \rho(X)f + f(\phi, X).$$

The cohomology operator $d^\phi$ associated with this representation is the $\phi$-differential of $A$ and is given by

$$d^\phi \omega = d\omega + \phi \wedge \omega, \quad \omega \in \Gamma(\wedge A^*).$$

Any vector bundle map $\Psi : A \to B$ induces a map $\Psi^* : \Gamma(B^*) \to \Gamma(A^*)$ which assigns to each section $\alpha \in \Gamma(B^*)$ the section $\Psi^* \alpha$ given by

$$\Psi^* \alpha(X)(m) = \langle \alpha(\Psi(m)), \Psi_m X(m) \rangle, \quad \forall m \in M, X \in \Gamma(A),$$

where $\Psi : M \to N$ is the map induced by $\Psi$ on the base manifolds. We denote by the same symbol $\Psi^*$ the extension of this map to the multisectons of $B^*$, where we set $\Psi^* f = f \circ \Psi$, for $f \in C^\infty(N)$.

Now, let $A \to M$ and $B \to N$ be two Lie algebroids and $\phi_A \in \Gamma(A^*), \phi_B \in \Gamma(B^*)$ 1-cocycles, so that $(A, \phi_A)$ and $(B, \phi_B)$ are Jacobi algebroids. Then it is natural to give the following definition.

Definition 1.1. A Jacobi algebroid morphism from $(A, \phi_A)$ to $(B, \phi_B)$ is a vector bundle map $\Psi : A \to B$ such that $\Psi^* : (\Gamma(\wedge B^*), d_B^{\phi_B}) \to (\Gamma(\wedge A^*), d_A^{\phi_A})$ is a chain map.

The definition of a Jacobi algebroid morphism has already been given in [16] in terms of Lie algebroid morphisms. The two definitions are equivalent, as is stated in the following proposition.

Proposition 1.2. Let $(A, \phi_A)$ and $(B, \phi_B)$ be Jacobi algebroids and $\Psi : A \to B$ a vector bundle map. Then $\Psi$ is a Jacobi algebroid morphism if and only if the following conditions are satisfied:

(i) $\Psi^*: (\Gamma(\wedge B^*), d_B^{\phi_B}) \to (\Gamma(\wedge A^*), d_A^{\phi_A})$ is a chain map, i.e. $\Psi$ is a Lie algebroid morphism;

(ii) $\Psi^* \phi_B = \phi_A$. 

If conditions (i) and (ii) hold, then $\Psi$ is clearly a Jacobi algebroid morphism. Conversely, if $\Psi$ is a Jacobi algebroid morphism, then by evaluating the condition
\[ \Psi^* \circ d_B^\psi = d_A^\phi \circ \Psi^* \]
over the constant function $1: N \to \mathbb{R}$ one obtains condition (ii). Then, condition (i) follows as well. \qed

In the Jacobi framework, the structure analogous to that of quasi-Lie bialgebroid is given by the following definitions.

**Definition 1.3.** Let $(A, \phi)$ be a Jacobi algebroid and $W \in \Gamma(A)$. Then a (degree-one) $W$-quasi differential on $(A, \phi)$ is a linear operator $\tilde{d}_*: \Gamma(\wedge^* A) \to \Gamma(\wedge^{*+1} A)$ such that
\begin{enumerate}[(i)]  
  \item $\tilde{d}_*(P \wedge Q) = \tilde{d}_* P \wedge Q + (-1)^p P \wedge \tilde{d}_* Q - W \wedge P \wedge Q,$  
  \item $d_p [P, Q]_A^\phi = [d_P, Q]_A^\phi + (-1)^p [P, d_Q]_A^\phi,$  
\end{enumerate}
for any $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^{q} A)$.

If $\tilde{d}_*$ is a $W$-quasi differential on $(A, \phi)$, then considering the constant function $1: M \to \mathbb{R}$ from (i) we get $\tilde{d}_* 1 = W$. So $d_p = \tilde{d}_* - W \wedge P$ is the associated degree-one derivation of the Gerstenhaber algebra $(\Gamma(\wedge^* A), \wedge, [\cdot, \cdot]_A)$.

**Definition 1.4.** A quasi-Jacobi bialgebroid is a Jacobi algebroid $(A, \phi)$ equipped with a $W$–quasi differential $\tilde{d}_*$ and a 3-section of $A$, $X_A \in \Gamma(\wedge^3 A)$ such that
\[ \tilde{d}_* X_A = 0 \quad \text{and} \quad \tilde{d}_*^2 = [X_A, -]_A^\phi. \]

**Remark 1.5.** This definition is equivalent to that of [17]. Given a quasi-Jacobi bialgebroid $(A, \phi_A, \tilde{d}_*, X_A)$, by using $d_*$ one can define a map $\rho_{\phi_A}: A^* \to TM$ by
\[ \rho_{\phi_A}(\alpha)(f) = \alpha(d_f) \]
and a bracket $[\cdot, \cdot]_A^\phi$ on the sections of $A^*$ by
\[ [\alpha, \beta]_A^\phi = \rho_{\phi_A}(\alpha)(\beta(X)) - \rho_{\phi_A}(\beta)(\alpha(X)) - d_* X(\alpha, \beta), \]
for $\alpha, \beta \in \Gamma(A)^*$, $f \in C^\infty(M)$ and $X \in \Gamma(A)$. Then, the Jacobi bracket $[\cdot, \cdot]_A^W$ on $\Gamma(A)^*$ is defined in the same way as in (1).

In particular, when $\phi = 0$ and $W = 0$, then $(A, \tilde{d}_*, X_A)$ is a quasi-Lie bialgebroid [4]. On the other hand, when $X_A = 0$ then a Jacobi bialgebroid $(A, \phi, \tilde{d}_*)$ is defined.

Examples of quasi-Jacobi bialgebroids are given by the quasi-Jacobi bialgebroid associated to a twisted-Jacobi manifold [17,7].

We propose the following definition of morphism between quasi-Jacobi bialgebroids:

**Definition 1.6.** Let $(A, \phi_A, \tilde{d}_A, X_A)$ and $(B, \phi_B, \tilde{d}_B, X_B)$ be quasi-Jacobi bialgebroids over $M$ and $N$, respectively. A vector bundle map $\Psi: A \to B$ is a quasi-Jacobi bialgebroid morphism if
\begin{enumerate}[(1)]  
  \item $\Psi$ is a Jacobi algebroid morphism;  
  \item $\Psi^*$ is compatible with the brackets on the sections of $A^*$ and $B^*$:  
    \[ [\Psi^* \alpha, \Psi^* \beta]_B^\psi = \Psi^* [\alpha, \beta]_A^\phi; \]
  \item the vector fields $\rho_{\phi_B}(\alpha)$ and $\rho_{\phi_A}(\Psi^* \alpha)$ are $\Psi$-related:  
    \[ T\psi \cdot \rho_{\phi_A}(\Psi^* \alpha) = \rho_{\phi_B}(\alpha) \circ \psi; \]
  \item $\Psi X_A = X_B \circ \psi$;  
  \item $\Psi W_A = W_B \circ \psi$;  
\end{enumerate}
where $\alpha, \beta \in \Gamma(B^*)$ and $\psi: M \to N$ is the smooth map induced by $\Psi$ on the base and $W_A = \tilde{d}_A 1, W_B = \tilde{d}_B 1$ are the associated sections of $A^*, B^*$, respectively.

**Example 1.7.** When $X_A = X_B = 0$, we have a Jacobi bialgebroid morphism, i.e., a Jacobi algebroid morphism $\Psi: A \to B$ such that conditions (2) and (3) above are satisfied and $\Psi W_A = W_B \circ \psi$.

**Example 1.8.** Let $(A, \phi_A, \tilde{d}_A, X_A)$ and $(B, \phi_B, \tilde{d}_B, X_B)$ be quasi-Jacobi bialgebroids over the same base manifold $M$. We can see that a base-preserving quasi-Jacobi bialgebroid morphism (i.e., such that $\psi = \text{id}$) is a vector bundle map $\Psi: A \to B$ such that $\Psi^* \circ d_B^\psi = d_A^\phi \circ \Psi^*$, $\Psi \circ \tilde{d}_B = \tilde{d}_A \circ \Psi$ and $\Psi X_A = X_B$.

In Definition 1.6, when $\phi = 0$ and $W = 0$, then $(A, \tilde{d}_A, X_A)$ and $(B, \tilde{d}_B, X_B)$ are quasi-Lie bialgebroids and we obtain as a special case the definition of a quasi-Lie bialgebroid morphism as a vector bundle map $\Psi: A \to B$ such that $\Psi$ is a Lie algebroid morphism and satisfies conditions (2)–(4).
1.2. Courant–Jacobi algebroids

Next, following the ideas of [18], we will present a definition of a Courant–Jacobi algebroid which is simpler than the original one in [11].

**Definition 1.9.** A Courant–Jacobi algebroid is a vector bundle $E \to M$ equipped with a fiberwise nondegenerate symmetric bilinear form $(\cdot, \cdot)$, a vector bundle map $\rho : E \to TM \oplus \mathbb{R}$ and a bilinear bracket $\circ$ on $\Gamma(E)$ satisfying

- \((CJ1)\) \(e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)\);
- \((CJ2)\) \(\mathcal{E}_{\rho(e)}(e_2, e_3) = \langle e_1, e_2 \circ e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle\);
- \((CJ3)\) \(\mathcal{E}_{\rho(e)}(e, e) = 2 \langle e_1 \circ e, e \rangle\);

for all \(e_1, e_2, e_3 \in \Gamma(E)\).

Recall that in the definition of Courant–Jacobi algebroid in [11], instead of (CJ1)–(CJ3), one has the following conditions:

- \((CJ4)\) \(e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)\);
- \((CJ5)\) \(\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]\);
- \((CJ6)\) \(\langle e_1 \circ e, e \rangle = \langle e_1, e \circ e \rangle\);
- \((CJ7)\) \(\mathcal{E}_{\rho(e)}(e, e) = 2 \langle e_1 \circ e, e \rangle\);

for all \(e, e_1, e_2, e_3 \in \Gamma(E)\).

The next proposition proves that **Definition 1.9** is equivalent to the one in [11].

**Proposition 1.10.** Let \(E \to M\) be a vector bundle equipped with a fiberwise nondegenerate symmetric bilinear form $(\cdot, \cdot)$, a vector bundle map $\rho : E \to TM \oplus \mathbb{R}$ and a bilinear bracket $\circ$ on $\Gamma(E)$. Then, the conditions (CJ1)–(CJ3) are equivalent to (CJ4)–(CJ7).

**Proof.** Conditions (CJ1) and (CJ4) are the same. Let us first show that conditions (CJ2)–(CJ4) imply conditions (CJ2) and (CJ3). For any \(e_1, e_2, e_3 \in \Gamma(E)\), we have from condition (CJ4) and the bilinearity of $\circ$ and $(\cdot, \cdot)$

\[
\mathcal{E}_{\rho(e)}(e_2 + e_1, e_1 + e_3, e_1 + e_3) = 2 \langle e_1 \circ e_2, e_1 \rangle + 2 \langle e_1 \circ e_3, e_1 \rangle + 2 \langle e_2 \circ e_3, e_1 \rangle + 2 \langle e_2 \circ e_3, e_2 \rangle + 2 \langle e_2 \circ e_3, e_3 \rangle.
\]

On the other hand, we also have

\[
\mathcal{E}_{\rho(e)}(e_2 + e_1, e_1 + e_3, e_1 + e_3) = \mathcal{E}_{\rho(e)}(e_2, e_2) + 2\mathcal{E}_{\rho(e)}(e_2, e_3) + \mathcal{E}_{\rho(e)}(e_3, e_3).
\]

From (3) and (4), and using again condition (CJ4), we obtain (CJ3). Condition (CJ3) allows us to write

\[
\langle e_1 \circ (e_2 \circ e_3), e_3 \rangle = \langle e_1, e_2 \circ e_3 \rangle + \langle e_1, e_2 \circ e_3 \rangle,
\]

or, equivalently,

\[
\langle e_1 \circ e_3, e_2 \rangle + \langle e_1, e_2 \circ e_3 \rangle = \langle e_1, (e_2 \circ e_3 \rangle + \langle e_3, e_2 \rangle),
\]

which expresses the equality of the right-hand sides of conditions (CJ2) and (CJ3) and therefore (CJ2) holds.

Now let us prove that, conversely, (CJ2) and (CJ3) imply (CJ4)–(CJ7). If we take \(e_1 = e_2 \in (CJ3)\), we obtain

\[
\mathcal{E}_{\rho(e)}(e_2, e_2) = 2 \langle e_1 \circ e_2, e_2 \rangle.
\]

which is exactly (CJ4). Moreover, if we take \(e_3 = e_2 \in (CJ2)\), we get

\[
\mathcal{E}_{\rho(e)}(e_2, e_2) = 2 \langle e_1 \circ e_2, e_2 \rangle.
\]

From (6) and (5), we conclude that (CJ3) holds.

It remains to show that \(\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]\), for all \(e_1, e_2 \in \Gamma(E)\). For any \(e_1 \in \Gamma(E)\), \(\rho(e_1)\) is a first-order differential operator on \(M\). Then, \(\mathcal{E}_{\rho(e_1)}(1) = \theta(e_1)\) with $\theta$ a 1-cocycle and, using (CJ3), we have

\[
e_1 \circ f e_2 = f e_1 \circ e_2 + \mathcal{E}_{\rho(e_1)} f - \theta(e_1) e_2,
\]

for any \(f \in C^\infty(M)\). Starting with the Jacobi identity and applying (7), we get

\[
e_1 \circ (e_2 \circ f e_3) = (e_1 \circ e_2) \circ f e_3 + e_2 \circ (e_1 \circ f e_3) = f (e_1 \circ e_2) \circ e_3 + (\mathcal{E}_{\rho(e_1) \circ f e_3}) e_3
\]

\[
- \theta(e_1 \circ e_2) f e_3 + f e_2 \circ (e_1 \circ e_3) + (\mathcal{E}_{\rho(e_2) f e_3}) (e_1 \circ e_2) - \theta(e_2) f e_1 \circ e_3
\]

\[
+ (\mathcal{E}_{\rho(e_1) f e_3}) (e_2 \circ e_3) + \mathcal{E}_{\rho(e_2)} (\mathcal{E}_{\rho(e_1) f e_3}) (e_3 - \theta(e_2) (\mathcal{E}_{\rho(e_1) f e_3}) e_3 - \theta(e_1) f e_2 \circ e_3
\]

\[
- \mathcal{E}_{\rho(e_2)} (\theta(e_1) f e_3) e_3 + \theta(e_2) \theta(e_1) f e_3.
\]
On the other hand,
\[
\begin{align*}
e_1 \circ (e_2 \circ f_3) &= e_1 \circ (f_2 \circ e_3 + (\mathcal{E}_{p(e_2)f}e_3 - \theta(e_2)fe_3) \\
&= f_1 \circ (e_2 \circ e_3) + (\mathcal{E}_{p(e_1)f}e_2) e_3 - \theta(e_1)fe_2 \circ e_3 + (\mathcal{E}_{p(e_3)f}e_2) e_3 - \theta(e_3)f_1 \circ e_3 - \mathcal{E}_{p(e_1)f}(\theta(e_2)f) e_3 + \theta(e_1)\theta(e_2)fe_3.
\end{align*}
\]
From (8) and (9), we obtain
\[
(\mathcal{E}_{p(e_1)e_2}f) - \mathcal{E}_{p(e_1)e_2}f + \mathcal{E}_{p(e_2)}f = 0,
\]
and so,
\[
\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]. \quad \square
\]
An alternate definition which was also proved to be equivalent to that of a Courant–Jacobi algebroid was given in [12]
under the name of a generalized Courant algebroid.

Associated with the bracket \( \circ \), we can define a skew-symmetric bracket on the sections of \( E \) by
\[
\ll e_1, e_2 \rr = \frac{1}{2} (e_1 \circ e_2 - e_2 \circ e_1),
\]
and the properties (CJ1)–(CJ3) can be expressed in terms of this bracket.

**Remark 1.11.** Note that in Definition 1.9 when \( \rho^*(0,1) = 0 \) (where \( (0,1) \in \Gamma(T^*M \times \mathbb{R}) = \Omega^1(M) \times C^\infty(M) \)), then by replacing \( \rho \)
with its natural projection \( \rho^! := \pi \circ \rho : E \to TM \) we obtain a Courant algebroid structure on the vector bundle \( E \to M \).

**Example 1.12** (Double of a Quasi-Jacobi Bialgebroid [17]). Let \((A, \phi, \tilde{\alpha}_s, X_A)\) be a quasi-Jacobi bialgebroid over \( M \) and let \( W \in \Gamma(A) \) be the associated section. Its double \( E = A \oplus A^* \) is a Courant–Jacobi algebroid when it is equipped with the pairing \( (X + \alpha, Y + \beta) = \frac{1}{2} (\alpha(Y) + \beta(X)) \), the anchor \( \rho = \rho_A^\phi + \rho_A^W \), and the bracket
\[
(X + \alpha) \circ (Y + \beta) = \left( [X, Y]_A + \tilde{\mathcal{E}}_{\alpha\beta} Y - i_\beta \tilde{\alpha}_s X + X_A(\alpha, \beta, -) + \left( [\alpha, \beta]_A^W + \mathcal{E}_A^\phi + \mathcal{E}_A^\beta i_\alpha d\beta - i_\beta d\alpha \right) \right),
\]
where \( \mathcal{E}_A^\phi \) and \( \tilde{\mathcal{E}}_A \) are the Lie derivative and the quasi-Lie derivative operators defined, respectively, by \( d\phi \) and \( \tilde{\alpha}_s \).

Taking \( X_A = 0 \), we have the Courant–Jacobi algebroid structure on the double of a Jacobi bialgebroid. In particular, when the Jacobi bialgebroid is \( (TM \times \mathbb{R}, (0,1), \tilde{\alpha}_s = 0) \), its double is the standard Courant–Jacobi algebroid associated to the manifold \( M, \mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \). The bilinear form reads
\[
((X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)) = \frac{1}{2} (i_{X_2} \alpha_1 + i_{X_1} \alpha_2 + f_1 g_2 + f_2 g_1),
\]
the bilinear bracket is given by
\[
((X_1, f_1) + (\alpha_1, g_1)) \circ ((X_2, f_2) + (\alpha_2, g_2)) = ([X_1, X_2], X_1(f_2) - X_2(f_1)) + \left( \mathcal{E}_{X_1} \alpha_2 - i_{X_2} d\alpha_1 \\
+ f_1 \alpha_2 - f_2 \alpha_1 + g_2 df_1 + f_2 dg_1, X_1(g_2) - X_2(g_1) + i_{X_2} \alpha_1 + f_1 g_2 \right),
\]
and the anchor is
\[
\rho((X_1, f_1) + (\alpha_1, g_1)) = (X_1, f_1),
\]
for \((X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))\), with \( i = 1, 2 \).

### 1.3. Dirac structures supported on a submanifold

A Dirac subbundle or Dirac structure of a Courant–Jacobi algebroid \( E \) is a vector subbundle \( A \subset E \), which is maximal isotropic with respect to the pairing \( \langle \cdot, \cdot \rangle \) and is integrable in the sense that the space of the sections of \( A \) is closed under the bracket on \( \Gamma(E) \). Restricting the skew-symmetric bracket of \( E \) and the natural projection \( pr_1 \circ \rho : E \to TM \) of the anchor \( \rho \) to \( A \), we obtain a Lie algebroid structure on \( A \).

As a way to generalize Dirac structures we have the concept of generalized Dirac structures or Dirac structures supported on a submanifold of the base manifold. In the case of Courant algebroids, they were dealt with in [13,14]. That definition can be generalized to the Courant–Jacobi case as follows.

**Definition 1.13.** On a Courant–Jacobi algebroid \( E \to M \), a Dirac structure supported on a submanifold \( P \) of \( M \) or a generalized Dirac structure is a vector subbundle \( F \) of \( E|_p \) such that

D1) for each \( x \in P, F_x \) is maximal isotropic;

D2) \( F \) is compatible with the anchor, i.e., \( \rho_{|p}(F) \subset TP \times \mathbb{R} \);

D3) for each \( e_1, e_2 \in \Gamma(E) \), such that \( e_{1|p}, e_{2|p} \in \Gamma(F) \), we have \( (e_1 \circ e_2)_p \in \Gamma(F) \).
Obviously, a Dirac structure supported on the whole base manifold $M$ is a Dirac structure in the usual sense.

Let $E \to M$ be a Courant–Jacobi algebroid and $L \to P$ a vector subbundle of $A$ over a submanifold $P$ of $M$. We denote by $L^\perp \to P$ the vector subbundle of $A$ over $P$ whose fiber over any $x \in P$ is $L^\perp_x := \{ e_x \in E_x \mid \forall e'_x \in L_x, \langle e_x, e'_x \rangle = 0 \}$. The conormal bundle $\nu^*(P) \to P$ is defined by $\nu^*(P) := \{ e_x \in E_x^* \mid \forall e_x \in L_x, \alpha_x(e_x) = 0 \}$.  

**Theorem 1.14.** Let $E = A \oplus A^*$ be the double of a quasi-Jacobi bialgebroid $(\mathfrak{A}, \mathfrak{d}, \tilde{d}_e, \mathcal{A}_\tau)$ over the manifold $M$, $L \to P$ a vector subbundle of $A$ over a submanifold $P$ of $M$ and $F = L \oplus L^\perp$. Then $F$ is a Dirac structure supported on $P$ if and only if the following conditions hold:

1. $L$ is a Lie subalgebroid of $A$;
2. $L^\perp$ is closed for the bracket on $A^*$ defined by $d_e$;
3. $L^\perp$ is compatible with the anchor, i.e., $\rho_{ip}(L^\perp) \subset TP \times \mathbb{R}$;
4. $X_{Alp} \in \Gamma(\wedge^3 L)$.

**Proof.** Since $F = L \oplus L^\perp$, this is a Lagrangian subbundle of $E$. If $F$ is a Dirac structure supported on $P$, then we immediately see that the stated four conditions are satisfied.

Conversely, suppose that $L$ is a Lie subalgebroid of $A$, $L^\perp \subset A^*$ is closed for $[\cdot, \cdot]_{A^*}$, $\rho_{ip}(L^\perp) \subset TP \times \mathbb{R}$ and $X_{Alp} \in \Gamma(\wedge^3 L)$. Obviously $F$ is compatible with the anchor, i.e., $\rho_{ip}(L \oplus L^\perp) = \rho_{ip}(L) + \rho_{ip}(L^\perp) \subset TP \times \mathbb{R}$. We are left to prove that $F$ is closed with respect to the bracket on $E$. Let $X, Y \in \Gamma(A)$ and $\alpha, \beta \in \Gamma(A^*)$ such that $X + \alpha$ and $Y + \beta$ restricted to $P$ are sections of $F$. By definition, the bracket on $E$ is given by

$$(X + \alpha) \circ (Y + \beta) = [X, Y]_A + i_X d_Y - i_Y d_X + d_A (\alpha(Y)) + X_A(\alpha, \beta, -) + (\alpha, W) Y - \langle (\beta, W), X \rangle + \langle (\beta, X), W \rangle + [\alpha, \beta]_A + F_X \beta - i_Y d\alpha + \langle X, \phi \rangle \beta - \langle Y, \phi \rangle \alpha + \langle \alpha, Y \rangle \phi,$$

where $W = \tilde{d}_e 1$.

By hypothesis, we immediately have that

$$(X, Y)_A = [X_A, Y_A]_L \in \Gamma(L),$$
$$(\alpha, \beta)_A = [\alpha_A, \beta_A]_L \in \Gamma(L^\perp),$$
$$(\alpha, Y)_A, \phi |_p + \langle (\alpha, X), Y \rangle |_p W_p = 0,$$
$$(\alpha, W)_A |_p - \langle (\beta, W), X \rangle |_p + \langle (\beta, X), W \rangle |_p + \langle (\alpha, \beta) |_p \alpha |_p F,$$

and

$X_A(\alpha, \beta, -)_A |_p \in \Gamma(L).$

Now, notice that $\alpha(Y)_A |_p = 0$, so $d\alpha(Y)_A |_p \in \nu^*(P) = \nu^*(TP)$. Since $\rho_{ip}(L^\perp) \subset TP$, we have that $d_e \alpha(Y)_A |_p = \rho_{ip}^* \nu^*(P)$, $\alpha(Y)_A |_p \in \Gamma(L)$.

Analogously, $d\beta(X)_A |_p = \rho_{ip}^* \nu^*(P)$, $\beta(X)_A |_p \in \Gamma(L^\perp)$.

Also,

$$d_Y (\alpha, \beta)_A |_p = (\rho_{ip} \alpha_{A^*} \cdot \beta(Y) - \rho_{ip} \beta_{A^*} \cdot \alpha(Y) - [\alpha, \beta]_{A^*} |_p (Y)) |_p = 0,$$

so $i_X d_y \in \Gamma(L)$ and $i_X X = \Gamma(L^\perp)$.

All these conditions allow us to say that $(X + \alpha) \circ (Y + \beta) \in \Gamma(L \oplus L^\perp)$ and, consequently, $F$ is a Dirac structure supported on $P$. \hfill $\Box$

**Corollary 1.15.** Let $E = A \oplus A^*$ be the double of a Jacobi bialgebroid; then $F = L \oplus L^\perp$ is a Dirac structure supported on $P$ if and only if $L$ and $L^\perp$ are Lie subalgebroids of $A$ and $A^*$.

Notice that, when $\phi = 0$ and $W = 0$, then $E = A \oplus A^*$ is the double of a Lie bialgebroid. Then, if we have also $P = M$, i.e., the Dirac structure is global, we obtain Proposition 7.1 of [3].

**Corollary 1.16.** Let $E^\perp_A(M) = (TM \times \mathbb{R}) \oplus (TM^* \times \mathbb{R})$ be the standard Courant–Jacobi algebroid twisted by the $3$-form $(d\omega, \omega) \in \Gamma(\wedge^3 (TM \times \mathbb{R})^\perp)$ and $i : P \hookrightarrow M$ a submanifold of $M$. Then $F = (TP \times \mathbb{R}) \oplus (TP \times \mathbb{R})^\perp$ is a Dirac structure of $E^\perp_A(M)$ supported on $P$ if and only if $i^* \omega = 0$.

**Proof.** Since the Jacobi algebroid structure on $TM \times \mathbb{R}$ is the null structure, we immediately have the three first conditions of **Theorem 1.14**. Condition (4) is ensured by $i^* \omega = 0$ because, in this case, we also have $i^* d\omega = 0$, and these two conditions are equivalent to $(d\omega, \omega) \in \Gamma(\wedge^3 (TP \times \mathbb{R})^\perp)$. \hfill $\Box$
Consider the following vector bundles over $E \times \tilde{E}$ supported on graph $\phi$, where $\phi : E \to M'$ is a smooth map and $\tilde{E}$ denotes the Courant–Jacobi algebroid obtained from $E'$ by changing the sign of the bilinear form.

If $E$ and $E'$ are in particular Courant algebroids, then a Dirac structure in $E \times \tilde{E}$ supported on graph $\phi$ defines a Courant algebroid morphism $[13, 14]$. Several examples of Courant–Jacobi morphisms, extending the known ones for Courant algebroids, appear naturally (see for instance [19]).

**Example 1.21.** A Courant–Jacobi morphism from $E$ to the zero Courant algebroid over a point is the same as a Dirac structure of the Courant–Jacobi algebroid $E$.

**Theorem 1.22.** Let $E_1 = A \oplus A^*$ and $E_2 = B \oplus B^*$ be doubles of quasi-Jacobi bialgebroids $(A, \phi_A, \tilde{A}, A_\lambda, X_A)$ and $(B, \phi_B, \tilde{A}, A_\lambda, X_B)$ and $(\Phi, \phi) : A \to B$ a quasi-jacobi bialgebroid morphism; then

$$F = \left\{ (a + \Phi b^*, \Phi a + b^*) \mid a \in A, b^* \in B^*_{\phi(x)}, x \in M \right\} \subset E_1 \times E_2$$

is a Dirac structure supported on graph $\phi$, i.e., $F$ is a Courant–Jacobi algebroid morphism.

**Proof.** Consider the following vector bundles over $M$ (or graph $\phi \simeq M$):

$$L_\lambda = \text{graph } \Phi_\lambda = \{(a, \Phi_\lambda a) \mid a \in A_x \} \subset A_x \times B_{\phi(x)}$$

and

$$L_\lambda^+ = \{(\Phi^* b^*, -b^*) \mid b^* \in B^*_{\phi(x)} \} \subset A_\lambda^* \times B^*_{\phi(x)}.$$

Since $\Phi$ is a Jacobi algebroid morphism, $L$ is clearly a Lie subalgebroid of $A \times B$ (when seen as a vector bundle over graph $\phi \simeq M$). Analogously, we can also conclude that $L^+$ is closed for the bracket on $A^* \times B^*$ (where $B^*$ denotes the vector bundle $B^*$ with bracket $\{-, -\}_B$) and it is compatible with the anchor $\rho_{A^* \times B^*}^\Phi = (\rho_{A^*}^\Phi, -\rho_{B^*}^\Phi)$. Since $\Phi X_A = X_B$, we conclude that $L \oplus L^+$ is a Dirac structure supported on graph $\phi$ of the double $(A \times B) \oplus (A^* \times B^*)$ which is the Courant–Jacobi algebroid $(A \oplus A^*) \times (B \oplus B^*)$. Finally, observe that the vector bundle morphism $b + b^* \mapsto b - b^*$ induces a canonical isomorphism between $F$ and $L \oplus L^+$, and the result follows. □

**Corollary 1.23.** Let $E_1 = A \oplus A^*$ and $E_2 = B \oplus B^*$ be doubles of quasi-Lie bialgebroids $(A, d_{a\lambda}, A_\lambda, X_A)$ and $(B, d_{\lambda B}, X_B)$ and $(\Phi, \phi) : A \to B$ a quasi-Lie bialgebroid morphism; then

$$F = \left\{ (a + \Phi b^*, \Phi a + b^*) \mid a \in A, b^* \in B^*_{\phi(x)}, x \in M \right\} \subset E_1 \times E_2$$

is a Courant algebroid morphism.

2. **Jacobi quasi-Nijenhuis algebroids**

Let $(A, \phi)$ be a Jacobi algebroid. Recall that the induced Lie algebroid structure from $A$ by $\phi$ on the vector bundle $\hat{A} = A \times \mathbb{R}$ over $M \times \mathbb{R}$ is defined by the anchor

$$\hat{\rho}(X) = \rho(X) + \langle \phi, X \rangle \frac{\partial}{\partial t}, \quad X \in \Gamma(A),$$

and the bracket induced by $[,]$ for time-independent multivectors

$$[P, Q]_\hat{\lambda} = [P, Q], \quad P, Q \in \Gamma(\wedge^* A).$$

The differential in $\hat{A}$ induced by $(A, \phi)$ is denoted by $\hat{\partial}$. The differential obtained in $\hat{A}$ when $\phi = 0$ is denoted by $\hat{\partial}$. 

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Let \((A, \phi)\) be a Jacobi algebroid and suppose that a Jacobi bivector is given on \(A\), i.e., a bivector \(\pi \in \Gamma(\wedge^2 A)\) such that \([\pi, \pi]^0 = 0\). It follows that \(\tilde{\pi} = e^{-\pi}\) is a Poisson bivector on \(\hat{A}\) and, consequently, it defines a Lie algebroid structure on \(\hat{A}^*\) (over \(M \times \mathbb{R}\)) given by

\[
[\alpha, \beta]_\tilde{\pi} = \hat{\mathcal{L}}_{\tilde{\pi}\alpha}\beta - \hat{\mathcal{L}}_{\tilde{\pi}\beta}\alpha - \hat{d}\tilde{\pi}(\alpha, \beta),
\]

(12)

where \(\alpha, \beta \in \Gamma(\hat{A}^*)\) and \(\hat{\mathcal{L}}\) is the Lie derivative in \(\hat{A}\). In particular, for \(\alpha, \beta \in \Gamma(A^*)\), we have

\[
[\epsilon^1\alpha, \epsilon^1\beta]_{\tilde{\pi}} = \epsilon^1(\mathcal{L}_{\pi\alpha}\beta - \mathcal{L}_{\pi\beta}\alpha - d\pi(\alpha, \beta)),
\]

(14)

The Lie bracket

\[
[\alpha, \beta]_\pi = \mathcal{L}_{\pi\alpha}\beta - \mathcal{L}_{\pi\beta}\alpha - d\pi(\alpha, \beta),
\]

(15)

together with the anchor

\[
\rho_\pi = \rho \circ \pi^*,
\]

(16)

ends \(A^*\) with a Lie algebroid structure over \(M\).

In order to introduce Jacobian quasi-Nijenhuis algebroids we recall now the notion of torsion of a vector bundle map in a Jacobi algebroid.

Let \((A, \phi)\), with \(A = (A, [\cdot, \cdot], \rho)\), be a Jacobi algebroid over a manifold \(M\). The torsion of a vector bundle map \(N : A \to A\) (over the identity) is defined by

\[
T_N(X, Y) := [NX, NY] - N[X, Y]_N, \quad X, Y \in \Gamma(A),
\]

(17)

where \([\cdot, \cdot]_N\) is given by

\[
[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \Gamma(A).
\]

When \(T_N = 0\), the vector bundle map \(N\) is called a Nijenhuis operator, the triple \(A_N = (A, [\cdot, \cdot]_N, \rho_N = \rho \circ N)\) is a new Lie algebroid and \(\phi_i := N^*\phi\) is a \(1\)-cocycle of \(A_N\) so that \((A_N, \phi_1)\) is a new Jacobi algebroid. Finally, \(N : (A_N, \phi_1) \to (A, \phi)\) is an example of Jacobi algebroid morphism.

Now assume also that a Jacobi bivector \(\pi \in \Gamma(\wedge^2 A)\) is given on \((A, \phi)\).

**Definition 2.1.** On a Jacobi algebroid \((A, \phi)\) with a Jacobi bivector \(\pi \in \Gamma(\wedge^2 A)\), we say that a vector bundle map \(N : A \to A\) is compatible with \(\pi\) if \(N\pi^* = \pi^*N^*\) and the Magri–Morosi concomitant vanishes:

\[
e(\pi, N)(\alpha, \beta) = [\alpha, \beta]_N - [\alpha, \beta]_N^* = 0,
\]

where \([\cdot, \cdot]_N\) is the bracket defined by the bivector field \(N\pi \in \Gamma(\wedge^2 A)\), and \([\cdot, \cdot]_N^*\) is the Lie bracket obtained from the Lie bracket \([\cdot, \cdot]_\pi\) by deformation along the tensor \(N^*\).

**Definition 2.2.** A **Jacobi quasi-Nijenhuis algebroid** \((A, \phi, \pi, N, \varphi)\) is a Jacobi algebroid \((A, \phi)\) equipped with a Jacobi bivector \(\pi\), a vector bundle map \(N : A \to A\) compatible with \(\pi\) and a \(3\)-form \(\varphi \in \Gamma(\wedge^3 A^*)\) such that \(d^\varphi = 0\) (i.e., \(\varphi\) is \(d^\pi\)-closed), and

\[T_N(X, Y) = -\pi^* (i_{X \wedge Y} \varphi) \quad \text{and} \quad d^\varphi (i_{N} \varphi) = 0,
\]

where \(i_{N}\varphi(X, Y, Z) = \varphi(NX, Y, Z) + \varphi(X, NY, Z) + \varphi(X, Y, NZ)\).

In particular, when \(\phi = 0\), we have a **Poisson quasi-Nijenhuis Lie algebroid** \((A, \pi, N, \varphi)\) [10,9].

**Proposition 2.3.** If \((A, \phi, \pi, N, \varphi)\) is a Jacobi quasi-Nijenhuis algebroid, then its Poissonization \((\hat{A}, e^{-\pi}, N, e^\varphi)\) is a Poisson quasi-Nijenhuis Lie algebroid.

**Proof.** It is well known that \((\hat{A}, e^{-\pi})\) is a Poisson Lie algebroid over \(M \times \mathbb{R}\). Also we have that

\[
\hat{d}(e^\varphi) = e^\varphi d^\varphi = 0,
\]

and

\[
\hat{d}(i_N e^\varphi) = \hat{d}(e^\varphi i_N) = e^\varphi d^\varphi (i_N \varphi) = 0
\]

and

\[
T_N(X, Y) = -\pi^* (i_{X \wedge Y} \varphi) = -(e^{-\pi})^* (i_{X \wedge Y} e^\varphi),
\]

and we conclude that \((\hat{A}, e^{-\pi}, N, e^\varphi)\) is a Poisson quasi-Nijenhuis Lie algebroid. \(\Box\)
When \((A, \phi) = (TM \times \mathbb{R}, (0, 1))\) and \(\mathcal{N} : TM \times \mathbb{R} \to TM \times \mathbb{R}\) is the \(C^\infty(M)\)-linear map given, for any section \((X, f)\) of \(\Gamma(TM \times \mathbb{R})\), by \(\mathcal{N}(X, f) = (NX + fY, i_X \gamma + fg)\), where \(N\) is a \((1, 1)\)-tensor field on \(M\), \(Y \in \mathfrak{X}(M)\), \(\gamma \in \Omega^1(M)\) and \(g \in C^\infty(M)\), we may introduce the notion of Jacobi quasi-Nijenhuis manifold.

**Definition 2.4**. A Jacobi quasi-Nijenhuis manifold is a Jacobi manifold \((M, \Lambda, E)\) together with a \(C^\infty(M)\)-linear map \(\mathcal{N} := (N, Y, \gamma, g) : TM \times \mathbb{R} \to TM \times \mathbb{R}\) compatible with \((\Lambda, E)\) and a 2-form \(\omega \in \Omega^2(M)\) such that

\[
\mathcal{T}_\mathcal{N} = - (\Lambda, E)^2 \circ (d\omega, \omega) \quad \text{and} \quad \hat{\omega}^{(0,1)}(i_X (d\omega, \omega)) = 0.
\]

In the case where \(\omega = 0\), \((M, (\Lambda, E), \mathcal{N})\) is a Jacobi–Nijenhuis manifold [20].

Let us now consider the Poissonization \((\hat{M}, \hat{A})\) of the Jacobi manifold \((M, \Lambda, E)\), i.e., \(\hat{M} = M \times \mathbb{R}\) and \(\hat{A} = e^{-\tau} (A + \frac{\partial}{\partial t} \Lambda)\), and the tensor field \(\hat{N}\) of type \((1, 1)\) on \(\hat{M}\) given by

\[
\hat{N} = N + Y \otimes dt + \frac{\partial}{\partial t} \otimes \gamma + \gamma \frac{\partial}{\partial t} \otimes dt.
\]

**Proposition 2.5**. The quadruple \((M, (\Lambda, E), \mathcal{N}, \omega)\) is a Jacobi quasi-Nijenhuis manifold if and only if \((\hat{M}, \hat{A}, \hat{N}, \hat{\omega}(e^t \omega))\) is a Poisson quasi-Nijenhuis manifold.

**Proof.** It is known [20] that \(\mathcal{N}\) is compatible with \((\Lambda, E)\) if and only if \(\hat{N}\) is compatible with \(\hat{A}\). Moreover, a direct computation shows that

\[
\mathcal{T}_\mathcal{N}((X_1, f_1), (X_2, f_2)) = -(\Lambda, E)^2((d\omega, \omega)((X_1, f_1), (X_2, f_2), -))
\]

is equivalent to

\[
\mathcal{T}_\hat{N} \left( X_1 + f_1 \frac{\partial}{\partial t}, X_2 + f_2 \frac{\partial}{\partial t} \right) = - \hat{A}^2 \left( \hat{\omega}^{(0,1)} \left( X_1 + f_1 \frac{\partial}{\partial t}, X_2 + f_2 \frac{\partial}{\partial t} \right) \right),
\]

for all \(X_i \in \mathfrak{X}^1(M), f_i \in C^\infty(M), i = 1, 2\).

Finally,

\[
i_\mathcal{N}(d\omega, \omega) = (i_N d\omega, i_\gamma d\omega) + (\gamma \wedge \omega, g\omega) + (0, i_\Lambda \omega),
\]

and since

\[
\hat{\omega}^{(0,1)}(i_X (d\omega, \omega)) = e^t (d\omega, d\omega) + (d_\omega (\omega, \omega) - (0, 1) \wedge (d_\omega \omega, -d_\omega \omega))((X_1, f_1), (X_2, f_2), (X_3, f_3)),
\]

the proof is completed. \(\square\)

The main aim of this section is to prove that for any Jacobi quasi-Nijenhuis algebroid there is an associated quasi-Jacobi bialgebroid structure on its dual.

**Theorem 2.6**. If \((A, \phi, \pi, N, \psi)\) is a Jacobi quasi-Nijenhuis algebroid, then \((A^*, \psi, W, \hat{d}^N_\pi \phi, \psi)\) is a quasi-Jacobi bialgebroid, where

\[
W = - \pi^{-1}(\phi).
\]

**Proof.** From **Proposition 2.3**, we know that \((\hat{A}, \pi, N, e^t \phi)\), with \(\pi = e^{-t} \pi\), is a Poisson quasi-Nijenhuis algebroid. By [10, Theorem 3.2] it follows that \((\hat{A}_N^*, \hat{d}_N, e^t \phi)\) is a quasi-Lie bialgebroid. Notice that \(\hat{d}_N\) is the derivation associated to \(\hat{A}_N\), i.e., to the deformation by the tensor \(N\) of the Lie algebroid \(\hat{A}\). Consider now \(A_N\), i.e., the deformation by the tensor \(N\) of the Lie algebroid \(A\). Since \(N\) is not Nijenhuis, then \(A_N\) is not necessarily a Lie algebroid. Nevertheless, the procedure described in (10) and (11) can be applied to \((A_N, N^* \phi)\) obtaining a new bracket \([\cdot, \cdot]_{\hat{A}_N}\) and a new anchor \(\hat{\rho}_N\) on \(\hat{A}\) or, equivalently, a differential \(\hat{d}_N\). This differential coincides with \(\hat{d}_N\) because, for \(X, Y \in \Gamma(A)\), we have

\[
[X, Y]_{\hat{A}_N} := [NX, Y]_{\hat{A}} + [X, NY]_{\hat{A}} - N[X, Y]_{\hat{A}}
\]

and

\[
\hat{\rho}_N(X) := \hat{\rho} \circ N(X) = \rho(NX) + \langle \phi, NX \rangle \frac{\partial}{\partial t}
\]

and

\[
\hat{\rho}_N(X) = \hat{\rho} \circ N(X) + \langle N^* \phi, X \rangle \frac{\partial}{\partial t}
\]

and

\[
\hat{\rho}_N(X).
\]
Thus \((\hat{A}_N^\alpha \circ \hat{d}_E, e' \varphi) = (\hat{A}_N^\alpha, \psi)\), and this is known to induce a quasi-Jacobi bialgebroid structure on \(A^*\): \((\hat{A}_N^\alpha, W, d_N^{\circ \varphi}, \varphi)\) (see [7, Theorem 4.1]). □

For the case of Jacobi quasi-Nijenhuis manifolds we have an equivalence:

**Proposition 2.7.** The quadruple \((M, (\Lambda, E), N, \omega)\) is a Jacobi quasi-Nijenhuis manifold if and only if \(((T^*M \times \mathbb{R})_{(\Lambda, E)}, (E, 0), d_N^{\circ \varphi}, (\omega, \omega))\) is a quasi-Jacobi bialgebroid.

**Proof.** From **Proposition 2.5.**, \((M, (\Lambda, E), N, \omega)\) is a Jacobi quasi-Nijenhuis manifold if and only if \((\hat{M}, \hat{\Lambda}, \hat{N}, e' \omega)\) is a Poisson quasi-Nijenhuis manifold, which is equivalent to \(((T^*M \times \mathbb{R})_{(\Lambda, E)}, (E, 0), d_N^{\circ \varphi}, (\omega, \omega))\) being a quasi-Jacobi bialgebroid over \(M\) [9]. This, in turn, is equivalent to \(((T^*M \times \mathbb{R})_{(\Lambda, E)}, (E, 0), d_N^{\circ \varphi}, (\omega, \omega))\) being a quasi-Jacobi bialgebroid over \(M\) [7]. Notice that \((A, E)^\gamma(0, 1) = (E, 0)\) and \(N^*(0, 1) = (\gamma, g)\). □

Suppose that \((\Lambda, \omega, \pi, N, \varphi)\) is a Jacobi quasi-Nijenhuis bialgebroid. The double of the quasi-Jacobi bialgebroid \((A^*_N, W, d_N^{\circ \varphi}, \varphi)\) is a Courant–Jacobi algebroid (see **Example 1.12**) that we denote by \(E^\varphi_N\).

An interesting case is when the 3-form \(\varphi\) is the image by \(N^*\) of another \(d^\varphi\)-closed 3-form \(\psi\): \(\varphi = N^* \psi\) and \(d^\varphi \psi = 0\).

In this case, \((\Lambda, \omega, \pi, \varphi)\) is a twisted Jacobian algebroid because

\[
[N_\pi, N_\pi^\psi] = 2\pi^\varphi(\varphi) = 2\pi^\varphi(N^* \psi) = 2N\pi^\varphi(\psi)
\]

and \(A^*\) has a structure of Lie algebroid: \(A^*_N = (A^*, \{\cdot, \cdot\}_N^N, N_\pi^\varphi)\) with 1-cocycle \(W_1 = -N\pi^\varphi(\phi)\) (see [7]).

Equipping \(A\) with the differential \(d'\) given by

\[
d'f = df \quad \text{and} \quad d'\alpha = d\alpha - N_{\pi^\varphi} \alpha,
\]

for \(f \in C^\infty(M)\) and \(\alpha \in \Gamma(A^*)\), we obtain a quasi-Jacobi bialgebroid: \((A^*_N, W, d'^\varphi, \pi, \varphi)\) [7]. Its double is a Courant–Jacobi algebroid and we denote it by \(E^\varphi_N\).

**Theorem 2.8.** Let \((\Lambda, \omega, \pi, N, \varphi)\) be a Jacobi quasi-Nijenhuis algebroid and suppose that \(\varphi = N^* \psi\), for some \(d^\varphi\)-closed 3-form \(\psi\), then

\[
F = \{ (a + N^* \alpha, Na + \alpha) \mid a \in A \text{ and } \alpha \in A^* \} \subset E^\psi_N \times E^\varphi_N
\]

defines a Courant–Jacobi algebroid morphism between \(E^\psi_N\) and \(E^\varphi_N\).

In order to prove the theorem, we need to prove the following property.

**Lemma 2.9.** Let \((\Lambda, \omega, \pi, N, \varphi)\) be a Jacobi quasi-Nijenhuis Lie algebroid; then

\[
\langle T_{N^*}(\alpha, \beta), X \rangle = \varphi(\pi^\varphi \alpha, \pi^\varphi \beta, X),
\]

for all \(X \in \Gamma(A)\) and \(\alpha, \beta \in \Gamma(A^*)\).

**Proof.** The compatibility between \(N\) and \(\pi\) implies that (see [1])

\[
\langle T_{N^*}(\alpha, \beta), X \rangle = \langle \alpha, T_N(X, \pi^\varphi \beta) \rangle,
\]

so

\[
\langle T_{N^*}(\alpha, \beta), X \rangle = \langle \alpha, -\pi^\varphi (i_{\pi^\varphi \beta} \varphi) \rangle = -\varphi(X, \pi^\varphi \beta, \pi^\varphi \alpha) = \varphi(\pi^\varphi \alpha, \pi^\varphi \beta, X). \quad \square
\]

**Proof of Theorem.** First, notice that \(N^*: A^*_N \rightarrow A^*_\pi\) is a Jacobi algebroid morphism because it is obviously compatible with the anchors, \(NW = -N\pi^\varphi \phi = W_1\) and

\[
\begin{align*}
N^* & [\alpha, \beta]_N^\psi = N^* [\alpha, \beta]_N + N^* \psi(\pi^\varphi \alpha, \pi^\varphi \beta, -) \\
& = [N^* \alpha, N^* \beta]_\pi - T_{N^*}(\alpha, \beta) + \varphi(\pi^\varphi \alpha, \pi^\varphi \beta, -) = [N^* \alpha, N^* \beta]_\pi.
\end{align*}
\]

Let \([\cdot, \cdot]\) be the bracket on the sections of \(A\) induced by the differential \(d'\). First notice that \(N^*\) preserves all the cocycles. Also

\[
[X, f'] = [df, X] = (df, X),
\]

so

\[
N^* d'f = d_N f, \quad \text{for all } f \in C^\infty(M) \text{ and } X \in \Gamma(A).
\]

And since

\[
[X, Y'] = [X, Y] - (N \pi)^\varphi(\psi(X, Y, -))
\]


we have
\[ N [X, Y]_N = [NX, NY] - \mathcal{T}_N(X, Y) = [NX, NY] + \pi^*(i_{X,Y}N^*\psi) \]
\[ = [NX, NY] + \psi(NX, NY, N\pi^*\nu) = [NX, NY]' \]
for all \( X, Y \in \Gamma(A) \).

In this way we may conclude that \( N^*: A^*_{\pi,N} \rightarrow A^*_\pi \) is a quasi-Jacobi bialgebroid morphism (see Definition 1.6), and the result follows from Theorem 1.22. □

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