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An optimal fourth-order family of modified Cauchy methods for finding solutions of nonlinear equations and their dynamical behavior

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Abstract: In this paper, we derive and analyze a new one-parameter family of modified Cauchy method free from second derivative for obtaining simple roots of nonlinear equations by using Padé approximant. The convergence analysis of the family is also considered, and the methods have convergence order three. Based on the family of third-order method, in order to increase the order of the convergence, a new optimal fourth-order family of modified Cauchy methods is obtained by using weight function. We also perform some numerical tests and the comparison with existing optimal fourth-order methods to show the high computational efficiency of the proposed scheme, which confirm our theoretical results. The basins of attraction of this optimal fourth-order family and existing fourth-order methods are presented and compared to illustrate some elements of the proposed family have equal or better stable behavior in many aspects. Furthermore, from the fractal graphics, with the increase of the value m of the series in iterative methods, the chaotic behaviors of the methods become more and more complex, which also reflected in some existing fourth-order methods.

Keywords: iterative methods, Newton’s method, Cauchy’s method, order of convergence, Padé approximant

MSC 2010: 65H05, 37F10

1 Introduction

In this paper, we consider iterative methods to find a simple root \( \alpha \), i.e., \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \), of a nonlinear equation

\[
f(x) = 0,
\]

where \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) for an open interval \( I \) is a scalar function.

Finding the simple root of the nonlinear equation (1) is a common and important problems in numerical analysis of science and engineering, and iterative methods are usually used to approximate a solution of these equations. We know that Newton’s method is an important and basic approach for solving nonlinear
equations [1, 2], and its formulation is given by
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \]  
(2)
this method converges quadratically.

The classical Cauchy's method [2] is expressed as
\[ x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f(x_n)}} \frac{f(x_n)}{f'(x_n)}, \]  
(3)
where
\[ L_f(x_n) = \frac{f''(x_n)f(x_n)}{f'(x_n)^2}. \]  
(4)
This family methods given by (3) is a well-known third-order method. However, the method depends on the second derivatives in computing process, and therefore their practical applications are restricted rigorously.

In recent years, several methods with free second derivatives have been developed, see [3–10] and references therein. In this paper, we will improve the family defined by (3) and obtain third and optimal fourth order family of second-derivative-free variants of Cauchy's methods by using Padé approximant. The rest of the paper is organized as follows: In Section 2, we present a new third order family of modified Cauchy method and show the order of convergence of this family; In Section 3, different numerical tests confirm the theoretical results, and the new methods are comparable with other known methods and give better results in many cases; In Section 4, based on the family of third-order method, a new optimal fourth-order family of iterative methods is obtained by using weight function; In Section 5, numerical tests and the comparison with the existing optimal fourth-order methods are included to confirm our theoretical results; In Section 6, the basins of attraction of the existing optimal fourth-order methods and our methods are presented and compared to illustrate their performances. Finally, we infer some conclusions.

### 2 Development of the third order method and its convergence analysis

In order to avoid the evaluation of the second derivatives \( f''(x_n) \) of Cauchy’s method (3), we consider approximating it by the derivative \( y''(x_n) \) of the following second degree Padé approximant:
\[ y(t) = \frac{a_1 + a_2(t - w_n) + a_3(t - w_n)^2}{1 + a_4(t - w_n)}. \]  
(5)
where \( a_1, a_2, a_3 \) and \( a_4 \) are real parameters. We impose the tangency conditions
\[ y(x_n) = f(x_n), \quad y'(x_n) = f'(x_n), \quad y(w_n) = f(w_n), \]  
(6)
where \( x_n \) is \( n \)th iterate and
\[ w_n = x_n - \frac{f(x_n)}{f'(x_n)}. \]  
(7)

By using the tangency conditions from (6), we obtain the value of \( a_1, a_2, a_4, \) and \( a_3 \) is determined in terms of \( a_3 \) in the following
\[ a_1 = f(w_n), \quad a_2 = f'(x_n) - \frac{2f'(x_n)f(w_n)}{f(x_n)}, \quad a_4 = \frac{a_3}{f'(x_n)} - \frac{f'(x_n)f(w_n)}{f'(x_n)^2}. \]  
(8)

From (5), we also have
\[ y''(t) = \frac{2[a_3 - a_2a_4 + a_1a_4^2]}{[1 + a_4(t - w_n)]^3}. \]  
(9)
Substituting (8) into (9) yields
\[ f''(x_n) = y''(x_n) = \frac{2f^4(x_n)f(w_n)}{f(x_n)[f^2(x_n)f(x_n) + \alpha_x f^2(x_n) - f^2(x_n)f(x_n)]}. \] \hspace{1cm} (10)

Using (10) we can approximate
\[ L_f(x_n) = \frac{f''(x_n)f(x_n)}{f^2(x_n)} = \frac{2f^2(x_n)f(w_n)}{f^2(x_n)f(x_n) + \alpha_x f^2(x_n) - f^2(x_n)f(x_n)}. \] \hspace{1cm} (11)

We define
\[ L_f, \mu(x_n, w_n) = \frac{2f^2(x_n)f(w_n)}{f^2(x_n)f(x_n) + \mu f^2(x_n) - f^2(x_n)f(x_n)}. \] \hspace{1cm} (12)

Using \( L_f, \mu(x_n, w_n) \) instead of \( L_f(x_n) \), we obtain a new one-parameter family of modified Cauchy method free from second derivative
\[ x_{n+1} = x_n - \frac{2}{1 + \sqrt{1 - 2L_f, \mu(x_n, w_n)}} f(x_n), \] \hspace{1cm} (13)

where \( \mu \in R \). Similar to the classical Cauchy’s method, a square root is required in (13). However, this may cost expensively, even fail in the case \( 1 - 2L_f, \mu(x_n, w_n) < 0 \). In order to avoid the calculation of the square roots, we will derive some forms free from square roots by Taylor approximation [4].

It is easy to know that Taylor approximation of \( \sqrt{1 - 2L_f, \mu(x_n, w_n)} \) is
\[ \sqrt{1 - 2L_f, \mu(x_n, w_n)} = \sum_{k=0}^{m} \left( \frac{k}{2} \right)^k (-2L_f, \mu(x_n, w_n))^k, \] \hspace{1cm} (14)

where \( m > 0 \).

Using (14) in (13), we can obtain the following form
\[ x_{n+1} = x_n - \frac{2}{1 + \sum_{k=0}^{m} \left( \frac{k}{2} \right)^k (-2L_f, \mu(x_n, w_n))^k} f(x_n), \] \hspace{1cm} (15)

where \( \mu \in R \).

On the other hand, it is clear that
\[ \frac{2}{1 + \sqrt{1 - 2L_f, \mu(x_n, w_n)}} = \frac{1 - \sqrt{1 - 2L_f, \mu(x_n, w_n)}}{L_f, \mu(x_n, w_n)} = \sum_{k=0}^{m} \left( \frac{1}{2} \right)^{k+1} (-1)^k 2^{k+1} L_f, \mu(x_n, w_n)^k \] \hspace{1cm} (16)

Then, Using (16) in (13), we also can construct a new family of iterative methods as follows:
\[ x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{1}{2} \right)^{k+1} (-1)^k 2^{k+1} L_f, \mu(x_n, w_n)^k \right) f(x_n), \] \hspace{1cm} (17)

where \( \mu \in R, m > 0 \).

We have the convergence analysis of the methods by (17).

**Theorem 2.1.** Let \( \alpha \in I \) be a simple zero of sufficiently differentiable function \( f : I \subset R \to R \) for an open interval \( I \). If \( x_0 \) is sufficiently close to \( \alpha \), then the order of convergence of the methods defined by (17) is three, and the error equation
\[ e_{n+1} = \left[ \frac{c_2 \mu}{f'(\alpha) - c_2^2} + c_2 e_n^3 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5) \right], \] \hspace{1cm} (18)

**Proof.** Let \( e_n = x_n - \alpha \), we use the following Taylor expansions:
\[ f(x_n) = f'(\alpha)[e_n + c_2 e_n^3 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \] \hspace{1cm} (19)
where \( c_k = \frac{f^{(k)}(a)}{k!} \). Furthermore, we have

\[
f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)].
\] (20)

Dividing (19) by (20),

\[
f(x_n)\frac{f'\left(\frac{x_n}{f'(x_n)}\right)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_3^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^2 - 3c_4)e_n^4 + O(e_n^5).
\] (21)

From (21), we get

\[
w_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 - 2(c_3^2 - c_3)e_n^3 - (7c_2c_3 - 4c_2^2 - 3c_4)e_n^4 + O(e_n^5).
\] (22)

Expanding \( f(w_n) \) in Taylor’s Series about \( \alpha \) and using (22), we get

\[
f(w_n) = f'(\alpha)[w_n - \alpha + c_2(w_n - \alpha)^2 + c_3(w_n - \alpha)^3 + c_4(w_n - \alpha)^4 + \cdots]
\]

\[
= f'(\alpha)c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (5c_4^2 + 3c_4 - 7c_2c_3)e_n^4 + O(e_n^5).
\] (23)

Since (20), we obtain

\[
f'^2(x_n) = f'^2(\alpha)[1 + 4c_2e_n + (6c_3^3 + 4c_2^3)e_n^2 + (8c_4 + 12c_2c_3)e_n^3
\]

\[
+ (10c_5 + 16c_2c_4 + 9c_3^2)e_n^4 + O(e_n^5)].
\] (24)

Because of (19), we get

\[
f^2(x_n) = f'^2(\alpha)[e_n^2 + 2c_2e_n^3 + (2c_3 + c_2^2)e_n^4 + O(e_n^5)].
\] (25)

From (23) and (24), we get

\[
f'^2(x_n)f(w_n) = f'^3(\alpha)c_2e_n^2 + 2(c_3 + c_2^2)e_n^3 + (7c_2c_3 + 3c_4 + c_2^3)e_n^4
\]

\[
+ (4c_5 + 4c_3c_2^2 + 10c_2c_4 + 6c_2^3)e_n^5 + O(e_n^6)].
\] (26)

From (20), (24), (25) and (26), we obtain

\[
L_{f, \mu}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{f'^2(x_n)f(x_n) + \mu f^2(x_n) - f'^2(x_n)f(w_n)} = 2c_2e_n + (4c_3 - 4c_2^2 - \frac{2\mu c_2}{f'(a)}e_n^2
\]

\[
+ (6c_4 - 12c_2c_3 + 6c_3^2 + \frac{8c_4^2 - 4c_3c_2}{f'(a)} + \frac{2\mu^2 c_2}{f'^2(a)}e_n^3 + (8c_5 + 22c_3c_2^2 - 16c_2c_4
\]

\[- 8c_3^2 - 6c_2^4 + \frac{26c_2c_3\mu - 20\mu c_2}{f'(a)} - 2\mu c_2}{f'^2(a)} + \frac{4c_3\mu^2 - 12\mu^2 c_2^2}{f'^3(a)} + \frac{2\mu^3 c_2}{f'^3(a)}e_n^4 + O(e_n^5).
\] (27)

Furthermore, from (27) we have

\[
\sum_{k=0}^{m} \left( \frac{1}{2} \right)^k (k+1) (-1)^k 2^{k+1} L_{f, \mu}(x_n, w_n)^k = 1 + \frac{1}{2} L_{f, \mu}(x_n, w_n)^2 + \frac{5}{8} L_{f, \mu}(x_n, w_n)^3
\]

\[
+ \frac{7}{8} L_{f, \mu}(x_n, w_n)^4 + \cdots
\]

\[= 1 + c_2e_n + \left( 2c_3 - \frac{\mu c_2}{f'(a)} \right)e_n^2 + \left( 3c_4 + 2c_2c_3 - \frac{2\mu c_2}{f'(a)} + \frac{\mu^2 c_2}{f'^2(a)} \right)e_n^3
\]

\[+ \left( 8c_5 + 22c_3c_2^2 - \frac{16\mu c_2}{f'(a)} - 10c_3^2 + \frac{9\mu c_2}{f'(a)} + \frac{6\mu^2 c_2^2}{f'^2(a)} + 12c_2c_4 \right)e_n^4
\]

\[+ O(e_n^5).
\] (28)

Since (17) and (28), we have

\[
x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{1}{2} \right)^k (k+1) (-1)^k 2^{k+1} L_{f, \mu}(x_n, w_n)^k \right) \frac{f(x_n)}{f'(x_n)}
\]
from $e_{n+1} = x_{n+1} - \alpha$, we have

$$e_{n+1} = -\left(c_2^2 - \frac{\mu c_2}{f'(\alpha)}\right) e_n^3 + O(e_n^4).$$

Then the methods defined by (17) is shown to converge of the order three.

Similar to the proof of Theorem 2.1, we can prove that the methods defined by (12) and (15) are third-order methods.

**Some special cases**

1°: If $\mu = 0$, from (12) and (17) we obtain

$$L_{f,0}(x_n, w_n) = \frac{2f(w_n)}{f(x_n) - f(w_n)},$$

$$x_{n+1} = x_n - \left(\sum_{k=0}^{m} \left(\frac{1}{k+1}\right)(-1)^k 2^{k+1} L_{f,0}(x_n, w_n)^k\right) \frac{f(x_n)}{f'(x_n)},$$

where $m > 0$. For $m = 2$, we obtain a third-order method (LM1)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f,0}(x_n, w_n) + \frac{1}{2} L_{f,0}(x_n, w_n)^2\right) \frac{f(x_n)}{f'(x_n)},$$

For $m = 3$, we obtain from (17) a third-order method (LM2)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f,0}(x_n, w_n) + \frac{1}{2} L_{f,0}(x_n, w_n)^2 + \frac{5}{8} L_{f,0}(x_n, w_n)^3\right) \frac{f(x_n)}{f'(x_n)}.$$

2°: If $\mu = 1$, from (12) we obtain

$$L_{f,1}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{f'^2(x_n)f(x_n) + f^2(x_n) - f'^2(x_n)f(w_n)},$$

For $m = 2$, we obtain from (17) a third-order method (LM3)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f,1}(x_n, w_n) + \frac{1}{2} L_{f,1}(x_n, w_n)^2\right) \frac{f(x_n)}{f'(x_n)}.$$

3°: If $\mu = -\frac{1}{2}$, from (12) we obtain

$$L_{f,-\frac{1}{2}}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{f'^2(x_n)f(x_n) - \frac{1}{2} f^2(x_n) - f'^2(x_n)f(w_n)},$$

For $m = 2$, we obtain from (17) a third-order method (LM4)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f,-\frac{1}{2}}(x_n, w_n) + \frac{1}{2} L_{f,-\frac{1}{2}}(x_n, w_n)^2\right) \frac{f(x_n)}{f'(x_n)}.$$

4°: If $\mu = -1$, for $m = 2$, we obtain a third-order method (LM5) from (12) and (17)

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f,-1}(x_n, w_n) + \frac{1}{2} L_{f,-1}(x_n, w_n)^2\right) \frac{f(x_n)}{f'(x_n)}.$$

5°: If $\mu = \frac{1}{2}$, from (12) and (15) we obtain some iterative methods as follows:

For $m = 1$, we obtain a third-order method

$$x_{n+1} = x_n - \frac{2}{2 - L_{f,-\frac{1}{2}}(x_n, w_n) f'(x_n)} f(x_n).$$
For $m = 2$, we obtain a third-order method (LM$_6$)

$$x_{n+1} = x_n - \frac{4}{f(x_n)} f(x_n)$$

For $m = 3$, we obtain a third-order method

$$x_{n+1} = x_n - \frac{4}{f(x_n)} f(x_n)$$

3 Numerical examples of the third order methods

In this section, we present the results of numerical simulations in Table 2 to compare the efficiencies of the methods. The considered methods are Newton method (NM), the method of Weerakoon and Fernando [8] (WF), the method of Potra and Pták (PP) [9], Chebyshev’s method (CMH) [11–12], Halley’s method (HM) [11], and our new methods (33) (LM$_1$), (34) (LM$_2$), (36) (LM$_3$), (38) (LM$_4$), (39) (LM$_5$), and (41) (LM$_6$). Displayed in Table 2 are the number of iterations (IT), the number of function evaluations (NFE) counted as the sum of the number of evaluations of the function itself plus the number of evaluations of the derivative, the absolute residual error of the corresponding function value ($|f(x_n)|$), the computing time (TIME, the unit of time is one second) and the distance of two consecutive approximations $\delta = |x_n - x_{n-1}|$. All computations were done using Matlab 7.1 environment with a ADM athlon(tm) II X2 250-3.01 GHz based PC. We accept an approximate solution rather than the exact root, depending on the precision $\epsilon$ of the computer. We use the following stopping criteria for computer programs: $|f(x_n)| < \epsilon$, we used the fixed stopping criterion $\epsilon = 10^{-15}$. “-” is divergence. We used the following test functions and display the computed approximate zero $x^\ast$ in Table 1 [13].

4 Development of the optimal fourth order method and its convergence analysis

Corresponding to the well known Traub’s method (see [14]), this scheme (17) with order of convergence three, is not optimal in the sense of Kung-Traub conjecture [14]. In this section, we introduce parametric weight functions and the well-known technique of undetermined coefficients to the family of iterative methods (17) to increase the order of convergence to four.

| Test functions | $x^\ast$ |
|----------------|---------|
| $f_1(x) = x^4 + 4x^2 - 10$ | 1.3652300134140969 |
| $f_2(x) = x^2 - e^x - 3x + 2$ | 0.25753028543986076 |
| $f_3(x) = \sin(x)e^x + \ln(1 + x^2)$ | 0 |
| $f_4(x) = (x - 1)^3 - 1$ | 2 |
| $f_5(x) = \cos(x) - x$ | 0.73908513321516067 |
| $f_6(x) = \sin^2(x) - x^2 + 1$ | 1.4044916482153411 |
| $f_7(x) = e^{x^2 + 7x - 30} - 1$ | 3 |
### Table 2: Comparison of various third-order methods and Newton’s method.

| Method | IT | NFE | \(|f(x_n)| \) | Time | \(\delta \) |
|--------|----|-----|-------------|------|---------|
| \(f_1: x_0 = 1\) |
| NM     | 5  | 10  | 0.046897    | 2.126987475037367e-011 |
| WF     | 3  | 9   | 0.016892    | 2.284722713019605e-006 |
| PP     | 4  | 12  | 0.033053    | 1.558753126573720e-013 |
| CHM    | 4  | 12  | 0.033136    | 1.643130076445232e-014 |
| HM     | 3  | 9   | 0.018876    | 3.698649917449615e-007 |
| LM₁    | 3  | 9   | 0.017555    | 7.65677843550274e-006 |
| LM₂    | 3  | 9   | 0.017310    | 6.519974116159233e-009 |
| LM₃    | 3  | 9   | 0.016915    | 6.545923781452599e-006 |
| LM₄    | 4  | 12  | 0.033269    | 2.220446049250313e-016 |
| LM₅    | 4  | 12  | 0.032977    | 2.220446049250313e-016 |
| LM₆    | 3  | 9   | 0.016579    | 1.582211815787105e-006 |
| \(f_1: x_0 = 2\) |
| NM     | 5  | 10  | 0.048387    | 5.020497351182485e-010 |
| WF     | 4  | 12  | 0.021366    | 4.440892098506626e-016 |
| PP     | 4  | 12  | 0.033571    | 7.949196856316121e-014 |
| CHM    | 4  | 12  | 0.033689    | 2.065014825802791e-014 |
| HM     | 3  | 9   | 0.016968    | 3.107350415199051e-006 |
| LM₁    | 3  | 9   | 0.017158    | 1.87020466026076e-007 |
| LM₂    | 4  | 12  | 0.033365    | 2.220446049250313e-016 |
| LM₃    | 3  | 9   | 0.016895    | 3.063923381674272e-008 |
| LM₄    | 3  | 9   | 0.016990    | 3.636202978718472e-007 |
| LM₅    | 3  | 9   | 0.017591    | 6.449765435052159e-007 |
| LM₆    | 4  | 12  | 0.033007    | 2.220446049250313e-016 |
| \(f_2: x_0 = 0\) |
| NM     | 4  | 8   | 0.026519    | 2.665312415217613e-012 |
| WF     | 3  | 9   | 0.016351    | 7.80181479797131e-012 |
| PP     | 3  | 9   | 0.016522    | 1.219191414492116e-012 |
| CHM    | 3  | 9   | 0.018237    | 8.906764215055318e-013 |
| HM     | 3  | 9   | 0.017423    | 7.374600929921371e-012 |
| LM₁    | 3  | 9   | 0.017213    | 1.014743844507393e-013 |
| LM₂    | 3  | 9   | 0.017442    | 1.497690860219336e-013 |
| LM₃    | 3  | 9   | 0.017380    | 3.035904860837491e-013 |
| LM₄    | 3  | 9   | 0.016920    | 2.620348382720295e-012 |
| LM₅    | 3  | 9   | 0.016538    | 1.656591530618812e-011 |
| LM₆    | 2  | 6   | 0.000403    | 1.015871229748111e-005 |
| \(f_2: x_0 = 0.5\) |
| NM     | 4  | 8   | 0.040693    | 1.791899961745003e-013 |
| WF     | 3  | 9   | 0.017298    | 6.424749621203318e-012 |
| PP     | 3  | 9   | 0.016290    | 4.607425552194400e-014 |
| CHM    | 3  | 9   | 0.017653    | 3.087480271446452e-011 |
| HM     | 3  | 9   | 0.018192    | 4.208039472430869e-011 |
| LM₁    | 3  | 9   | 0.017986    | 1.054711873393899e-015 |
| LM₂    | 3  | 9   | 0.017973    | 7.21644966063518e-016 |
| LM₃    | 3  | 9   | 0.017978    | 1.497135748707024e-013 |
| LM₄    | 3  | 9   | 0.017530    | 9.942047185518277e-014 |
| LM₅    | 3  | 9   | 0.016987    | 7.234768339969833e-013 |
| LM₆    | 3  | 9   | 0.016680    | 1.887379141862766e-015 |
| $f_3 : x_0 = 1$ |
|-----------------|
| NM 7 14 3.537126081266182e-024 | 0.076967 | 1.085848323840232e-012 |
| WF 4 12 2.621304391538411e-016 | 0.031335 | 4.33031069188726e-006 |
| PP 5 15 8.196910187379942e-033 | 0.050273 | 8.806888499109001e-012 |
| CHM 5 15 6.352230116524407e-022 | 0.051221 | 2.520356663650445e-011 |
| HM 5 15 7.257520328033399e-029 | 0.054265 | 8.459855063117184e-015 |
| LM$_1$ 4 12 8.271806125530277e-025 | 0.035949 | 3.479185746483636e-009 |
| LM$_2$ 4 12 0 | 0.034977 | 1.804501237019987e-013 |
| LM$_3$ 4 12 0 | 0.033510 | 3.732361177500448e-009 |
| LM$_4$ 4 12 0 | 0.033691 | 6.861191797005728e-010 |
| LM$_5$ 4 12 0 | 0.032995 | 3.650246134545045e-015 |
| LM$_6$ 4 12 0 | 0.033798 | 2.539428973634579e-010 |

| $f_3 : x_0 = 0.5$ |
|-----------------|
| NM 6 12 5.905159674954809e-020 | 0.060337 | 1.402992074360412e-010 |
| WF 4 12 1.764824578467612e-017 | 0.032156 | 4.200981459101664e-009 |
| PP 4 12 1.694834561079519e-019 | 0.034667 | 2.767209186089879e-007 |
| CHM 4 12 8.798671206634291e-017 | 0.033659 | 3.05965058808672e-007 |
| HM 4 12 3.044907255908736e-017 | 0.035225 | 5.518067735074518e-009 |
| LM$_1$ 4 12 0 | 0.034032 | 4.539201791677570e-014 |
| LM$_2$ 4 12 0 | 0.035989 | 5.35324260457416e-014 |
| LM$_3$ 4 12 0 | 0.032770 | 2.221451036901571e-013 |
| LM$_4$ 3 9 1.163082115698561e-019 | 0.016919 | 2.852561908633870e-007 |
| LM$_5$ 4 12 0 | 0.034210 | 6.743878120329082e-014 |
| LM$_6$ 3 9 0 | 0.017372 | 1.053838990356347e-011 |

| $f_4 : x_0 = 2.5$ |
|-----------------|
| NM 6 12 0 | 0.055351 | 1.154631945610163e-014 |
| WF 4 12 0 | 0.031820 | 7.314593375440381e-012 |
| PP 4 12 0 | 0.032233 | 4.221685223262537e-010 |
| CHM 4 12 0 | 0.033641 | 9.853584614916144e-011 |
| HM 4 12 0 | 0.033511 | 4.662936703425658e-014 |
| LM$_1$ 3 9 6.661338147750939e-016 | 0.017800 | 1.544537542308433e-008 |
| LM$_2$ 4 12 0 | 0.033585 | 2.244870955792067e-013 |
| LM$_3$ 3 9 0 | 0.016430 | 6.254471388249676e-007 |
| LM$_4$ 3 9 0 | 0.016473 | 1.067152234579538e-006 |
| LM$_5$ 4 12 0 | 0.032690 | 4.440892908500626e-016 |
| LM$_6$ 4 12 0 | 0.032958 | 2.042810365310288e-014 |

| $f_4 : x_0 = 3.5$ |
|-----------------|
| NM 7 14 0 | 0.086241 | 2.877564853065451e-011 |
| WF 5 15 0 | 0.049605 | 6.550315845288424e-013 |
| PP 5 15 0 | 0.048335 | 4.512221707388743e-010 |
| CHM 5 15 0 | 0.049367 | 4.188738245147761e-011 |
| HM 4 12 0 | 0.033641 | 4.485352507632712e-006 |
| LM$_1$ 4 12 0 | 0.035118 | 1.079692646399622e-008 |
| LM$_2$ 4 12 0 | 0.034134 | 7.838174553853605e-013 |
| LM$_3$ 4 12 0 | 0.035020 | 3.379705404427114e-010 |
| LM$_4$ 4 12 0 | 0.033277 | 1.283696526854783e-008 |
| LM$_5$ 4 12 0 | 0.033268 | 8.547096808086963e-009 |
| LM$_6$ 4 12 0 | 0.032565 | 8.725183908708800e-009 |
### An optimal fourth-order family of modified Cauchy methods for finding solutions

#### $f_5 : x_0 = 0$

| Method | Order | Coefficients | Error |
|--------|-------|--------------|-------|
| NM     | 5     | 10           | 0.049831 1.701233598438989e-010 |
| WF     | 3     | 9            | 0.017922 7.792236328407753e-007 |
| PP     | 4     | 12           | 0.032661 1.50058566291943e-010 |
| CHM    | 4     | 12           | 0.033834 5.32797279989847e-009 |
| HM     | 4     | 12           | 0.032642 1.121325254871408e-014 |
| LM_1   | 4     | 12           | 0.033695 3.819167204710539e-014 |
| LM_2   | 3     | 9            | 0.017802 8.247395144600489e-008 |
| LM_3   | 4     | 12           | 0.037679 1.818811767861917e-011 |
| LM_4   | 4     | 12           | 0.033281 8.344436253082677e-013 |
| LM_5   | 4     | 12           | 0.033370 8.471505719143124e-010 |
| LM_6   | 4     | 12           | 0.032211 2.348121697082206e-013 |

#### $f_5 : x_0 = 1$

| Method | Order | Coefficients | Error |
|--------|-------|--------------|-------|
| NM     | 4     | 8            | 0.030217 1.701233598438989e-010 |
| WF     | 2     | 6            | 4.440892098500626e-16 2.67427701733964e-005 |
| PP     | 3     | 9            | 0.016561 9.809075773858922e-011 |
| CHM    | 3     | 9            | 0.018428 1.600380383770528e-009 |
| HM     | 3     | 9            | 0.018336 6.624212289807474e-010 |
| LM_1   | 3     | 9            | 0.017117 2.252753539266905e-012 |
| LM_2   | 3     | 9            | 0.016831 4.671929509925121e-012 |
| LM_3   | 3     | 9            | 0.018593 2.668459120336308e-009 |
| LM_4   | 3     | 9            | 0.016778 1.749711486809247e-012 |
| LM_5   | 3     | 9            | 0.016682 1.375262126401822e-010 |
| LM_6   | 3     | 9            | 0.017519 3.14879344824041e-010 |

#### $f_6 : x_0 = 1$

| Method | Order | Coefficients | Error |
|--------|-------|--------------|-------|
| NM     | 6     | 12           | 3.330669073875470e-016 3.059774655866931e-013 |
| WF     | 4     | 12           | 4.440892098500626e-16 1.793023507445923e-010 |
| PP     | 16    | 48           | 4.440892098500626e-16 1.531728257564424e-007 |
| CHM    | 5     | 15           | 4.440892098500626e-16 6.883094094689568e-010 |
| HM     | 4     | 12           | 4.440892098500626e-16 2.686739719592879e-013 |
| LM_1   | 4     | 12           | 3.330669073875470e-016 1.042735342515755e-008 |
| LM_2   | 4     | 12           | 3.330669073875470e-016 7.038286398142191e-009 |
| LM_3   | 4     | 12           | 4.440892098500626e-016 3.42001305036852e-008 |
| LM_4   | 4     | 12           | 3.330669073875470e-016 1.918714076509787e-010 |
| LM_5   | 4     | 12           | 4.440892098500626e-016 1.002852445530778e-007 |
| LM_6   | 4     | 12           | 3.330669073875470e-016 6.483733550055604e-010 |

#### $f_6 : x_0 = 2.5$

| Method | Order | Coefficients | Error |
|--------|-------|--------------|-------|
| NM     | 6     | 12           | 3.330669073875470e-016 1.404654170755748e-012 |
| WF     | 4     | 12           | 3.330669073875470e-016 4.229505634611996e-012 |
| PP     | 4     | 12           | 3.330669073875470e-016 1.030850205196998e-008 |
| CHM    | 4     | 12           | 3.330669073875470e-016 4.624686628221753e-009 |
| HM     | 4     | 12           | 4.440892098500626e-016 9.462626682221753e-009 |
| LM_1   | 4     | 12           | 4.440892098500626e-016 1.265654248072679e-014 |
| LM_2   | 3     | 9            | 3.330669073875470e-016 1.7161538492606e-008 |
| LM_3   | 4     | 12           | 4.440892098500626e-016 4.662936703425658e-015 |
| LM_4   | 4     | 12           | 3.330669073875470e-016 1.501021529293212e-013 |
| LM_5   | 4     | 12           | 4.440892098500626e-016 8.837375276016246e-014 |
| LM_6   | 4     | 12           | 3.330669073875470e-016 2.375877272697835e-014 |
We consider using a weight function $H(\mu(x_n, w_n, \gamma_i))$ instead of $\mu$ in the operator (12), and consider the well-known technique of undetermined coefficients to design a new operator $L_{f,H,\tilde{\mu}}(x_n, w_n)$ as follows

$$L_{f,H,\tilde{\mu}}(x_n, w_n) = \frac{2\mu f^2(x_n)f(w_n)}{\mu_2 f^2(x_n)f(x_n) + \mu_3 H(\mu(x_n, w_n, \gamma_i)) f^2(x_n) - \mu_4 f^2(x_n)f(w_n)},$$

(43)

where $H(\mu(x_n, w_n, \gamma_i))$ is a function of real variable

$$\mu(x_n, w_n, \gamma_i) = \frac{\gamma_1 f(x_n) + \gamma_2 f(w_n)}{1 + \gamma_3 f(x_n) + \gamma_4 f(w_n)},$$

(44)

$\gamma_i(i = 1, \ldots, 4)$ and $\mu_j(j = 1, \ldots, 4)$ are real parameters. Then, using (43) in (17), we also can construct two new optimal fourth-order family of modified Cauchy methods as follows:

$$x_{n+1} = x_n - \frac{2f(x_n)}{1 + \sum_{k=0}^{m-1} \left(\frac{1}{k}\right) (-2L_{f,H,\tilde{\mu}}(x_n, w_n))^k \frac{f(x_n)}{f'(x_n)}},$$

(45)

and

$$x_{n+1} = x_n - \frac{\sum_{k=0}^{m} \left(\frac{1}{k+1}\right) (-1)^k 2^{k+1} L_{f,H,\tilde{\mu}}(x_n, w_n)^k \frac{f(x_n)}{f'(x_n)}},$$

(46)

where $m > 0$.

In the following result, we present the conditions that the weight function $H(\mu(x_n, w_n, \gamma_i))$ and the parameters must satisfy for obtaining two families of iterative methods with fourth-order of convergence, which becoming optimal schemes by Kung-Traub conjecture.
Theorem 3.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval $I$, such that $a \in I$ is a simple solution of the nonlinear equation $f(x) = 0$. Let $H : \mathbb{R} \to \mathbb{R}$ be any sufficiently differentiable function satisfying $H(0) = 0$, $|H'(0)| < \infty$, $|H''(0)| < \infty$. If $x_0$ is close enough to $a$, $\mu_1 = \mu_2 \neq 0$, and $\mu_\alpha = 0$, then the method defined by (46) has fourth-order of convergence and its error equation is:

$$e_{n+1} = \frac{1}{\mu_2} c_2 \left( \mu_2 H'(0) \gamma_1 - \mu_2 c_3 \right) e_n^3 + O(e_n^4). \quad (47)$$

Proof. Let $e_n = x_n - a$, because of the Taylor series expansions of $f(x_n)$ and $f(w_n)$, we have

$$\mu(x_n, w_n, \gamma_i) = \frac{\gamma_1 f(x_n) + \gamma_2 f(w_n)}{1 + \gamma_3 f(x_n) + \gamma_4 f(w_n)}$$

$$= f'(a) \gamma_1 e_n + \left( f''(a)(\gamma_2 c_2 + \gamma_2 c_2) - f'^2(a) \gamma_1 \gamma_3 \right) e_n^2 + \left( \frac{1}{2} f'(a)(2\gamma_1 c_3 - 4\gamma_2 c_2 + 4\gamma_2 c_3) \right) e_n^3$$

$$- f'^2(a) \gamma_1 \gamma_3 c_3 + f'^3(a) \gamma_1 \gamma_3 - \frac{1}{2} f'(a) \gamma_1 (2\gamma_3 f'(a)c_2 + 2\gamma_4 f'(a)c_2) e_n^3$$

$$+ \left( - \frac{1}{6} f''(a)(\gamma_1 (6\gamma_3 f'(a)c_3 + 12\gamma_4 f'(a)c_3) - 12\gamma_6 f'(a)c_3^2) + \frac{1}{6} f''(a)(3\gamma_2 c_3^2 + 6\gamma_1 c_4 - 42\gamma_4 c_3) + \frac{1}{2} f''(a)(2\gamma_1 c_3 - 4\gamma_2 c_2 + 4\gamma_2 c_3) \gamma_3 + f''(a)(\gamma_1 c_2 + 2\gamma_2 c_2) c_3 \right)$$

$$- f''(a) \gamma_1 \gamma_3 c_3^2 + f''(a) \gamma_1 \gamma_3 (2\gamma_3 f'(a)c_2 + 2\gamma_4 f'(a)c_2) - \frac{1}{2} f''(a)(\gamma_1 c_2 + 2\gamma_2 c_2)(2\gamma_3 f'(a)c_2$$

$$+ 2\gamma_4 f'(a)c_2) e_n^3 + O(e_n^4). \quad (48)$$

Taking into account the expansion of $\mu(x_n, w_n, \gamma_i)$, and using Taylor series expansion of $H(\mu(x_n, w_n, \gamma_i))$ around 0, we obtain

$$H(\mu(x_n, w_n, \gamma_i)) = H(0) + H'(0)\mu(x_n, w_n, \gamma_i) + \frac{H''(0)}{2!} \mu^2(x_n, w_n, \gamma_i) + \frac{H'''(0)}{3!} \mu^3(x_n, w_n, \gamma_i) + O(\mu^4(x_n, w_n, \gamma_i))$$

$$= H(0) + f'(a)H'(0)\gamma_1 e_n + \left( f''(a)\gamma_1 c_2 + \frac{1}{2} f''(a)\gamma_1 \gamma_3 \right) e_n^2 + \left( f''(a)\gamma_1 \gamma_3 c_3 - 2 f''(a)\gamma_1 c_3 \right)$$

$$+ f''(a)\gamma_1 \gamma_3 c_3^2 + f''(a)\gamma_1 \gamma_3 (2\gamma_3 f'(a)c_2 + 2\gamma_4 f'(a)c_2) - \frac{1}{2} f''(a)(\gamma_1 c_2 + 2\gamma_2 c_2)(2\gamma_3 f'(a)c_2$$

$$+ 2\gamma_4 f'(a)c_2) e_n^3 + O(e_n^4). \quad (49)$$

From (20), (24), (25), (26) and (43), we obtain

$$L_{f,H,p}(x_n, w_n) = \frac{2\mu_0 f^2(x_n) f(w_n)}{[\mu_2 f(x_n) f(w_n) + \mu_3 f(x_n) f(w_n) - \mu_4 f^2(x_n) f(w_n)]}$$

$$= 2 \frac{\mu_1 \mu_2 c_2 e_n + \left( 8 \frac{\mu_1}{\mu_2} c_2 + 2 \frac{\mu_1}{\mu_2} (2c_3 - 2c_2^2) + \frac{1}{3} \frac{\mu_1}{f'(a)\mu_2} c_2 (6f'(a)\mu_4 c_2 - 30f_2 f'(a)c_2$$

$$- 6 \mu_3 c_4) e_n^3 + \left( 8 \frac{\mu_1}{\mu_2} c_2 + 8 \frac{\mu_1}{\mu_2} (2c_3 - 2c_2^2) c_2 + \frac{4}{3} \frac{\mu_1}{f'(a)\mu_2} c_2 (6f'(a)\mu_4 c_2 - 30f_2 f'(a)c_2$$

$$- 6 \mu_3 c_4) + 12 \frac{\mu_1}{\mu_2} c_2 c_3 + \frac{\mu_1}{\mu_2} (10c_3^2 - 14c_2 c_3 + 6c_4) + \frac{4}{3} \frac{\mu_1}{f'(a)\mu_2} c_2 c_2 (2c - 2c_2^2)(6f'(a)\mu_4 c_2$$

$$- 30f_2 f'(a)c_2 - 6 \mu_3 c_4) + \frac{1}{18} f'(a)\mu_2^2 c_2 (6f'(a)\mu_4 c_2 - 30f_2 f'(a)c_2 - 6 \mu_3 c_4)$$

$$+ \frac{1}{6} f'(a)\mu_2^2 c_2 (12f'(a)\mu_1 H(0) - 84f(\mu_2)\mu_4 c_3 - 24\mu_3 H(c_2) + 24f'(a)\mu_4 c_2$$

$$- 96f'(a)\mu_2 c_2 + 24f'(a)\mu_4 c_3) e_n^3 + O(e_n^4). \quad (50)$$

Substituting (50) into (46), we have

$$x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{1}{k+1} (-1)^k 2^{k+1} L_{f,H,p}(x_n, w_n)^k \right) \frac{f(x_n)}{f'(x_n)} \right)^k$$
\[ x_n = \left(1 + \frac{1}{2} L_{f,H,H}(x_n, w_n) + \frac{1}{2} L_{f,H,H}(x_n, w_n)^2 + \frac{5}{8} L_{f,H,H}(x_n, w_n)^3 + \frac{7}{8} L_{f,H,H}(x_n, w_n)^4 + \cdots \right) f(x_n) \]

\[ = x_n - \left( \frac{\mu_1}{\mu_2} c_2 - c_2 \right) e_n - \left( 3 \frac{\mu_1}{\mu_2} + \frac{1}{2} (2 c_3 - 2 c_2^2) \right) e_n^2 - \left( \frac{\mu_1}{\mu_2} \right)^2 c_2 (3 f''(a)(a) + 30 f''(a) \mu_2 c_2 - 6 \mu_3 H(0)) + 2 \frac{\mu_2}{\mu_3} c_2^2 - 2 c_3 + 2 c_3^2 \right) e_n^3 - \left( \frac{1}{6} x \frac{\mu_1}{\mu_2} \right)^2 (3 f''(a)(a) + 30 f''(a) \mu_2 c_2 - 6 \mu_3 H(0)) + 2 \frac{\mu_2}{\mu_3} c_2^2 - 2 c_3 + 2 c_3^2 \right) e_n^4 - \left( \frac{1}{6} x \frac{\mu_1}{\mu_2} \right)^2 (3 f''(a)(a) + 30 f''(a) \mu_2 c_2 - 6 \mu_3 H(0)) + 2 \frac{\mu_2}{\mu_3} c_2^2 - 2 c_3 + 2 c_3^2 \right) e_n^5. \]  

(51)

From \( e_{n+1} = x_{n+1} - \alpha \), we consider that if \( H(0) = 0, \mu_1 = \mu_2, \mu_4 = 0 \), Then, we obtain the error equation of (46) in the form:

\[ e_{n+1} = \frac{1}{\mu_2} c_2 (\mu_3 H'(0) \gamma_1 - \mu_2 c_3) e_n^6 + O(e_n^5). \]  

(52)

Then the methods defined by (46) is shown to converge of the order four. □

Similar to the proof of Theorem 3.1, we can prove that the methods defined by (43) and (45) are fourth-order methods.

When \( H(0) = 0, \mu_1 = \mu_2, \mu_4 = 0 \), we obtain

\[ L_{f,H,H}(x_n, w_n) = \frac{2 \mu f''(x_n)f(w_n)}{\mu_1 f''(x_n)f(x_n) + \mu_3 H(\mu(x_n, w_n, \gamma_1))f'(x_n)}. \]  

(53)

Let \( \lambda = \frac{\mu_2}{\mu_1} \) in (53), we have

\[ L_{f,H,H}(x_n, w_n) = \frac{2 f''(x_n)f(w_n)}{f''(x_n)f(x_n) + \lambda H(\mu(x_n, w_n, \gamma_1))f'(x_n)}. \]  

(54)

From the expansion of \( L_{f,H,H}(x_n, w_n) \) in (54), we can obtain following members of family (45) and (46).

**Some special cases**

1°: If we consider the following weight function \( H_1 = H(\mu(x_n, w_n, \gamma_1)) = 0 \), from (46) and (54) we obtain

\[ L_{f,H_1,H}(x_n, w_n) = \frac{2 f(w_n)}{f_1(x_n)}. \]  

(55)

\[ x_{n+1} = x_n - \left( \sum_{k=0}^{m} \left( \frac{1}{2} \right)^k \right) f''(x_n) f(x_n) \]  

(56)

where \( m > 0 \). For \( m = 2 \), we obtain a recently developed fourth-order method by Khattri et al. (KM1) [13]

\[ x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_{f,H_1,H}(x_n, w_n) + \frac{1}{2} L_{f,H_1,H}(x_n, w_n)^2 \right) f(x_n) \]

(57)

For \( m = 3 \), we also get the existing optimal fourth-order method by Khattri et al. (KM2) [13]

\[ x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_{f,H_1,H}(x_n, w_n) + \frac{1}{2} L_{f,H_1,H}(x_n, w_n)^2 + \frac{5}{8} L_{f,H_1,H}(x_n, w_n)^3 \right) f(x_n) \]

(58)

For \( m = 4 \), we obtain the developed fourth-order method by Khattri et al. (KM3) [13], which is given by

\[ x_{n+1} = x_n - \left( 1 + \frac{1}{2} L_{f,H_1,H}(x_n, w_n) + \frac{1}{2} L_{f,H_1,H}(x_n, w_n)^2 + \frac{5}{8} L_{f,H_1,H}(x_n, w_n)^3 + \frac{7}{8} L_{f,H_1,H}(x_n, w_n)^4 \right) f(x_n) \]
2°: Now, we consider the following weight function, which also satisfies all the conditions of Theorem 3.1. If 
\[ H_2 = H(\mu(x_n, w_n, \gamma_i)) \]
we have
\[ f(x_n) = \frac{1}{h^2(x_n)} + 2h^2(x_n) + \frac{5}{h^3(x_n)} + \frac{14h^4(x_n)}{h^5(x_n)}. \]  \hspace{1cm} (59)

For \( m = 2 \), from (46) and (60) we obtain a new fourth-order method (LTM1)
\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f, H_i, 1}(x_n, w_n) + \frac{1}{2} L_{f, H_i, 1}(x_n, w_n)^2\right) f(x_n) \] \hspace{1cm} (61)

For \( m = 3 \), we obtain a new fourth-order method (LTM2)
\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f, H_i, 1}(x_n, w_n) + \frac{5}{8} L_{f, H_i, 1}(x_n, w_n)^3 + \frac{7}{8} L_{f, H_i, 1}(x_n, w_n)^4\right) f(x_n) \] \hspace{1cm} (62)

For \( m = 4 \), we obtain a new fourth-order method
\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f, H_i, 1}(x_n, w_n) + \frac{5}{8} L_{f, H_i, 1}(x_n, w_n)^3 + \frac{7}{8} L_{f, H_i, 1}(x_n, w_n)^4\right) f(x_n) \] \hspace{1cm} (63)

For \( m = 2 \), from (44) and (60) we obtain a new fourth-order method (LTM3)
\[ x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_i, 1}(x_n, w_n) - L_{f, H_i, 1}(x_n, w_n)^2} f(x_n) \] \hspace{1cm} (64)

For \( m = 3 \), we obtain a new fourth-order method (LTM4)
\[ x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_i, 1}(x_n, w_n) - L_{f, H_i, 1}(x_n, w_n)^2} f(x_n) \] \hspace{1cm} (65)

3°: If \( H_3 = H(\mu(x_n, w_n, \gamma_i)) = \mu(x_n, w_n, \gamma_i), \gamma_1 = -1, \gamma_2 = 1, \gamma_3 = -1, \gamma_4 = 1 \) and \( \lambda = -\frac{1}{4} \), from (43) and (54), we have
\[ L_{f, H_i, \frac{1}{4}}(x_n, w_n) = \frac{2f^2(x_n) f(w_n)}{f^2(x_n) - \frac{7}{8} H_3 f^2(x_n)}. \] \hspace{1cm} (66)

For \( m = 2 \), from (45) and (66) we obtain a new fourth-order method (LTM5)
\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f, H_i, -\frac{1}{4}}(x_n, w_n) + \frac{1}{2} L_{f, H_i, -\frac{1}{4}}(x_n, w_n)^2\right) f(x_n) \] \hspace{1cm} (67)

For \( m = 3 \), we obtain a new fourth-order method (LTM6)
\[ x_{n+1} = x_n - \left(1 + \frac{1}{2} L_{f, H_i, -\frac{1}{4}}(x_n, w_n) + \frac{5}{8} L_{f, H_i, -\frac{1}{4}}(x_n, w_n)^3\right) f(x_n) \] \hspace{1cm} (68)

For \( m = 2 \), from (44) and (66) we obtain a new fourth-order method
\[ x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_i, -\frac{1}{4}}(x_n, w_n) - L_{f, H_i, -\frac{1}{4}}(x_n, w_n)^2} f(x_n) \] \hspace{1cm} (69)

For \( m = 3 \), we obtain a new fourth-order method
\[ x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_i, -\frac{1}{4}}(x_n, w_n) - L_{f, H_i, -\frac{1}{4}}(x_n, w_n)^2} f(x_n) \] \hspace{1cm} (70)
If \( H_n = H(\mu(x_n, w_n, \gamma)) = \frac{\mu(x_n, w_n, \gamma)}{1-2\mu(x_n, w_n, \gamma)} \), \( \gamma_1 = -1, \gamma_2 = 1, \gamma_3 = -\frac{q}{2}, \ \gamma_6 = 2 \) and \( \lambda = \frac{1}{2} \), from (43) and (54), we have

\[
L_{f, H_n, \gamma}(x_n, w_n) = \frac{2f'^2(x_n)f(w_n)}{f''(x_n)f(x_n)^2 + \frac{1}{2}H_n f^2(x_n)}, \tag{71}
\]

For \( m = 2 \), from (46) and (71) we obtain a new fourth-order method

\[
x_{n+1} = x_n + \left(1 + \frac{1}{2}L_{f, H_n, \gamma}(x_n, w_n) + \frac{1}{2}L_{f, H_n, \gamma}(x_n, w_n)^2 \right) \frac{f(x_n)}{f'(x_n)}. \tag{72}
\]

For \( m = 3 \), we obtain a new fourth-order method (LTM7)

\[
x_{n+1} = x_n - \left(1 + \frac{1}{2}L_{f, H_n, \gamma}(x_n, w_n) + \frac{1}{2}L_{f, H_n, \gamma}(x_n, w_n)^2 + \frac{5}{8}L_{f, H_n, \gamma}(x_n, w_n)^3 \right) \frac{f(x_n)}{f'(x_n)}. \tag{73}
\]

For \( m = 2 \), from (45) and (71) we obtain a new fourth-order method (LTM9)

\[
x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_n, \gamma}(x_n, w_n) - L_{f, H_n, \gamma}(x_n, w_n)^2} \frac{f(x_n)}{f'(x_n)}. \tag{74}
\]

For \( m = 3 \), we obtain a new fourth-order method

\[
x_{n+1} = x_n - \frac{4}{4 - 2L_{f, H_n, \gamma}(x_n, w_n) - L_{f, H_n, \gamma}(x_n, w_n)^2 - L_{f, H_n, \gamma}(x_n, w_n)^3} \frac{f(x_n)}{f'(x_n)}. \tag{75}
\]

### 5 Numerical examples of the optimal fourth order methods

This section is devoted to verify the validity and effectiveness of our theoretical results which we have proposed earlier. We present the numerical results obtained by applying the proposed methods on some scalar equations. We are going to compare LTM1-LTM9 with some fourth-order known methods as Kung-Traub scheme [14]

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)},
\]

\[
x_{n+1} = y_n - \frac{f^2(x_n)}{f(x_n) - f(y_n)^2} f'(x_n), \tag{76}
\]

denoted by KTM; three methods of Khattat et al. [13], denoted by KM1, KM2 and KM3 (see (57), (58) and (59) in special cases of fourth order methods); the fourth-order method by Chun [15] denoted by CM1; two fourth-order methods by Chun and Ham [16] denoted by CM2, CM3; the fourth-order method by Kou et al. [17] denoted by NSPP; the fourth-order method by Sharma and Bahl [18] denoted by SBM, which the method are applied to solve systems of nonlinear equations by Sharma et al. [19]. The CM1 is given as

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)},
\]

\[
x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)[f(x_n) - f(y_n)]} f(x_n) + \frac{f'(x_n)f(y_n)}{f^2(x_n) + f'^2(x_n)}. \tag{77}
\]

The CM2 is defined as

\[
y_n = x_n - \frac{f(x_n)f'(x_n)}{f'^2(x_n) - f(x_n)},
\]

\[
L_n = f'^2(x_n) - f(x_n),
\]
methods LTM performs the worst, while the method CM performs slightly worse, and SBM fails in convergence for the case \( f_2(x_0) \). In test function \( f_2(x) \), the methods KM, CM, and our method LTM have better performances. The numerical results also show that KM has smaller residual error in the corresponding function \( |f_2(x_0)| \) compared with LTM.

Regarding the results of test function \( f_3(x) \), we claim that our methods and the existing fourth-order methods have almost similar performance.

From the results of the test function \( f_3(x) \), our methods LTM, LTM, LTM, LTM, and the existing method KM, KM, CM, CM, CM require less number of iterations (IT) and function evaluations (NFE) than other methods, which demonstrate that several of our methods converge faster than some existing ones.

In test function \( f_3(x) \), the methods KM, CM, CM, and our method LTM have better performances in terms of the speed of convergence. The existing methods KM and SBM fail in convergence for the case \( f_3(x) \). The results show that our fourth-order method LTM can compete with KTm, KM, CM, NSPP, SBM and KM.

In test function \( f_3(x) \), in terms of convergence, the methods KM, SBM, LTM, LTM, LTM, LTM perform slightly worse, while the method CM performs the worst.

In test function \( f_3(x) \), we also check the effectiveness of our methods when we consider the same nonlinear equation with same initial approximation. Then, we find that the methods KM, CM and our methods LTM, LTM, LTM, LTM, LTM, LTM perform better than KTm, KM, KM, CM, CM, NSPP and
Table 3: Comparison of some different fourth-order methods.

| Method | IT | NFE | $|f(x_n)|$ | $\delta$ | $\rho$ |
|--------|----|-----|----------|--------|------|
| $f_1(x) = 0$ | | | | | |
| KTM | 19 | 57 | 0 | 1.007297327770829e-006 | 3.57 |
| $x_0 = -0.1$ | | | | | |
| KM$_1$ | 64 | 192 | 0 | 1.477826749862743e-009 | 3.76 |
| KM$_2$ | 45 | 135 | 0 | 4.900428897136600e-005 | 3.72 |
| KM$_3$ | 102 | 306 | 0 | 1.091067414571434e-005 | 4.42 |
| CM$_1$ | 6 | 18 | 0 | 9.497031339122941e-007 | 3.15 |
| CM$_2$ | 9 | 27 | 0 | 1.145895600629387e-009 | 3.80 |
| CM$_3$ | 45 | 135 | 0 | 2.819966482547898e-014 | 3.49 |
| NSPP | 14 | 42 | 0 | 2.220446049250313e-016 | 1.98 |
| SBM | 12 | 36 | 0 | 2.60283572479857e-010 | 3.81 |
| LTM$_1$ | 49 | 147 | 0 | 1.015936601511669e-008 | 3.71 |
| LTM$_2$ | 21 | 63 | 0 | 1.554312234475219e-015 | 4.09 |
| LTM$_3$ | 17 | 51 | 0 | 1.74305148661496e-013 | 3.90 |
| LTM$_4$ | 11 | 33 | 0 | 3.606588179216885e-007 | 4.10 |
| LTM$_5$ | 21 | 63 | 0 | 4.322075137341841e-006 | 4.31 |
| LTM$_6$ | 26 | 78 | 0 | 2.886579864025407e-015 | 3.62 |
| LTM$_7$ | 16 | 48 | 0 | 5.212047367475492e-006 | 4.00 |
| LTM$_8$ | 14 | 42 | 0 | 6.794564910705958e-014 | 4.18 |
| $f_2(x) = 0$ | | | | | |
| KTM | 5 | 15 | 0 | 7.216449660063518e-016 | 3.89 |
| $x_0 = 5$ | | | | | |
| KM$_1$ | 5 | 15 | 0 | 1.105560087921731e-011 | 4.13 |
| KM$_2$ | 5 | 15 | 0 | 1.15296611073225e-013 | 3.95 |
| KM$_3$ | 4 | 12 | | 9.358037653284246e-006 | 4.12 |
| CM$_1$ | 4 | 12 | 8.8817849701252e-016 | 1.99775433807696e-004 | 6.61 |
| CM$_2$ | 5 | 15 | 0 | 2.425837308805967e-014 | 3.91 |
| CM$_3$ | - | - | - | - | - |
| NSPP | 5 | 15 | 0 | 2.473671267821942e-011 | 4.20 |
| SBM | 5 | 15 | 4.440892098500626e-016 | 9.436895709313831e-016 | 3.78 |
| LTM$_1$ | 5 | 15 | 0 | 5.860528728973691e-011 | 4.24 |
| LTM$_2$ | 5 | 15 | 0 | 6.93334278784103e-014 | 3.94 |
| LTM$_3$ | 4 | 12 | 4.440892098500626e-016 | 1.094604859936954e-004 | 5.29 |
| LTM$_4$ | 5 | 15 | 0 | 2.942091015256665e-015 | 3.96 |
| LTM$_5$ | 5 | 15 | 0 | 1.32602056683184e-007 | 3.54 |
| LTM$_6$ | 5 | 15 | 0 | 6.67910716144942e-013 | 3.51 |
| LTM$_7$ | 5 | 15 | 0 | 5.632716515437532e-013 | 3.54 |
| LTM$_8$ | 5 | 15 | 0 | 7.216449660063518e-016 | 3.64 |
| Method | IT | NFE | \(|f(x_n)|\) | \(\delta\) | \(\rho\) |
|--------|----|-----|-------------|---------|--------|
| \(f_3(x) = 0\) |
| KTM    | 4  | 12  | 2.698906001294071e-024 | 1.217853159082307e-008 | 3.62   |
| KM_1   | 4  | 12  | 0             | 2.148368599356326e-007 | 3.45   |
| KM_2   | 4  | 12  | 0             | 9.592177833278592e-011 | 4.65   |
| KM_3   | 4  | 12  | 0             | 5.846342610051233e-000 | 4.20   |
| CM_1   | 4  | 18  | 0             | 1.19123721644855e-000 | 3.75   |
| CM_2   | 5  | 15  | 0             | 1.656923030498442e-000 | 3.50   |
| CM_3   | 5  | 15  | 5.048709793414476e-029 | 2.684470484445867e-013 | 3.61   |
| NSPP   | 4  | 12  | 1.985233470127266e-023 | 4.289648665705555e-008 | 3.55   |
| SBM    | 4  | 12  | 2.418605025271162e-021 | 4.917930580759401e-011 | 3.78   |
| LTM_1  | 4  | 12  | 5.293955920339377e-023 | 2.170159571681513e-007 | 3.45   |
| LTM_2  | 4  | 12  | 0             | 9.881239316361645e-011 | 4.65   |
| LTM_3  | 4  | 12  | 6.617444900424221e-024 | 9.954034684185604e-009 | 3.61   |
| LTM_4  | 4  | 12  | 1.615587133892632e-027 | 1.42851676682764e-011 | 4.27   |
| LTM_5  | 4  | 12  | 3.308722450212111e-024 | 2.045966094113560e-008 | 3.59   |
| LTM_6  | 4  | 12  | 3.231174267785264e-027 | 2.309911940542863e-011 | 4.33   |
| LTM_7  | 4  | 12  | 0             | 7.43207119110223e-011 | 4.26   |
| LTM_8  | 4  | 12  | 4.13590362765138e-025 | 2.239552597536676e-009 | 3.65   |

| \(f_4(x) = 0\) |
|----------------|
| KTM    | 5  | 15  | 0             | 1.776356839400251e-015 | 3.95   |
| KM_1   | 5  | 15  | 0             | 6.295657328792004e-010 | 3.78   |
| KM_2   | 4  | 12  | 0             | 5.425764586330928e-005 | 3.72   |
| KM_3   | 4  | 12  | 0             | 1.13548533082044e-006  | 3.82   |
| CM_1   | 4  | 12  | 0             | 1.661871202074394e-005 | 3.34   |
| CM_2   | 4  | 12  | 0             | 4.525455072901252e-006 | 3.51   |
| CM_3   | 4  | 12  | 0             | 1.644155478430776e-009 | 2.87   |
| NSPP   | 5  | 15  | 0             | 7.429612480791548e-013 | 3.89   |
| SBM    | 5  | 15  | 0             | 1.02240518265144e-014  | 3.93   |
| LTM_1  | 5  | 15  | 0             | 6.829425913679188e-010 | 3.78   |
| LTM_2  | 4  | 12  | 6.661338147750939e-016 | 5.805743723130696e-005 | 3.71   |
| LTM_3  | 5  | 15  | 0             | 4.440892098500626e-016 | 3.86   |
| LTM_4  | 4  | 12  | 6.661338147750939e-016 | 1.499435946517025e-007 | 3.89   |
| LTM_5  | 5  | 15  | 0             | 9.895031460871451e-010 | 3.86   |
| LTM_6  | 4  | 12  | 0             | 4.299091316228854e-005 | 3.24   |
| LTM_7  | 4  | 12  | 0             | 1.04592253612575e-005  | 4.31   |
| LTM_8  | 4  | 12  | 0             | 5.39998182232676e-005  | 3.37   |
| Method | IT | NFE | $|f(x_0)| \varepsilon$ | $\rho$ |
|--------|----|-----|------------------|------|
| $f_5(x) = 0$ | KTM | 9 | 27 | 0 | 1.288968931589807e-013 | 3.80 |
| $x_0 = 5.8$ | KM$_1$ | 5 | 15 | 0 | 5.195843755245733e-014 | 3.30 |
| | KM$_2$ | 4 | 12 | 0 | 1.78989045807431e-011 | 3.60 |
| | KM$_3$ | - | - | - | - |
| | CM$_1$ | 5 | 15 | 1.110223024625157e-016 | 6.661338147750939e-016 | 3.97 |
| | CM$_2$ | 4 | 12 | 0 | 4.142503895465666e-009 | 3.84 |
| | CM$_3$ | 4 | 12 | 1.110223024625157e-016 | 8.643999676372083e-006 | 2.93 |
| | NSPP | 18 | 54 | 0 | 1.818157153921085e-005 | 4.93 |
| | SBM | - | - | - | - |
| | LTM$_1$ | 5 | 15 | 0 | 1.394815624942147e-004 | 5.37 |
| | LTM$_2$ | 4 | 12 | 1.110223024625157e-016 | 1.033639332229663e-004 | 2.87 |
| | LTM$_3$ | 9 | 27 | 0 | 1.776356839400251e-015 | 4.10 |
| | LTM$_4$ | 16 | 48 | 0 | 1.953010203822325e-004 | 5.98 |
| | LTM$_5$ | 6 | 18 | 0 | 9.492295838242626e-011 | 3.68 |
| | LTM$_6$ | 5 | 15 | 1.110223024625157e-016 | 2.07982643160738e-005 | 4.89 |
| | LTM$_7$ | 6 | 18 | 0 | 7.094251031070087e-006 | 3.25 |
| | LTM$_8$ | 6 | 18 | 1.110223024625157e-016 | 4.067230241489028e-007 | 4.21 |

| Method | IT | NFE | $|f(x_0)| \varepsilon$ | $\rho$ |
|--------|----|-----|------------------|------|
| $f_6(x) = 0$ | KTM | 5 | 15 | 4.440892098500626e-016 | 2.725704772088555e-005 | 3.05 |
| $x_0 = 30$ | KM$_1$ | 6 | 18 | 4.440892098500626e-016 | 2.819966482547898e-014 | 3.90 |
| | KM$_2$ | 5 | 15 | 3.330669073875470e-016 | 3.551058632700332e-006 | 3.61 |
| | KM$_3$ | 5 | 15 | 3.330669073875470e-016 | 2.469841886565405e-009 | 4.11 |
| | CM$_1$ | 5 | 15 | 4.440892098500626e-016 | 2.90072567032454e-008 | 4.28 |
| | CM$_2$ | 5 | 15 | 3.330669073875470e-016 | 5.619937435419331e-006 | 3.47 |
| | CM$_3$ | - | - | - | - |
| | NSPP | 5 | 15 | 8.881784197001252e-016 | 1.200137685715141e-004 | 2.70 |
| | SBM | 6 | 18 | 4.440892098500626e-016 | 4.440892098500626e-016 | 3.97 |
| | LTM$_1$ | 6 | 18 | 4.440892098500626e-016 | 2.287059430727823e-014 | 3.91 |
| | LTM$_2$ | 5 | 15 | 3.330669073875470e-016 | 1.494727950301922e-006 | 3.74 |
| | LTM$_3$ | 5 | 15 | 4.440892098500626e-016 | 7.169287327135621e-006 | 3.21 |
| | LTM$_4$ | 5 | 15 | 4.440892098500626e-016 | 1.95256579032674e-011 | 4.06 |
| | LTM$_5$ | 6 | 18 | 3.330669073875470e-016 | 2.093081263865315e-011 | 3.82 |
| | LTM$_6$ | 6 | 18 | 4.440892098500626e-016 | 3.9960288650564e-015 | 4.03 |
| | LTM$_7$ | 5 | 15 | 3.330669073875470e-016 | 6.713351008989520e-005 | 3.12 |
| | LTM$_8$ | 5 | 15 | 4.440892098500626e-016 | 1.567170138416785e-005 | 3.77 |
The function \( \Phi(\bar{\lambda}) = \frac{8\pi c P \bar{\lambda}^{-5}}{e^{\frac{\bar{\lambda}}{c T}} - 1} \), which calculates the energy density within an isothermal blackbody. In the expression of formula (83), \( \bar{\lambda} \) is the wavelength of the radiation, \( T \) is the absolute temperature of the blackbody, \( B \) is Boltzmann’s constant, \( P \) is the Planck’s constant and \( c \) is the speed of light. In some cases, due to the needs of the application, it is often necessary to determine wavelength \( \bar{\lambda} \) which corresponds to maximum energy density \( \Phi(\bar{\lambda}) \). To find the critical points, we use the Chain Rule to differentiate the function of equation (83), and obtain

\[
\Phi'(\bar{\lambda}) = \left( \frac{8\pi c P \bar{\lambda}^{-6}}{e^{\frac{\bar{\lambda}}{c T}} - 1} \right) \left( \frac{c P}{e^{\frac{\bar{\lambda}}{c T}}} \frac{e^{\frac{\bar{\lambda}}{c T}}}{e^{\frac{\bar{\lambda}}{c T}} - 1} - 5 \right),
\]

so to find the critical number of \( \Phi \) for the maxima, we solve the equation

\[
\frac{c P}{e^{\frac{\bar{\lambda}}{c T}}} \frac{e^{\frac{\bar{\lambda}}{c T}}}{e^{\frac{\bar{\lambda}}{c T}} - 1} - 5 = 0.
\]

Consider the relationship between variables, if \( X = \frac{c P}{A B T} \), then the equation (85) is converted into the following nonlinear equation

\[
F(X) = e^{-X} + \frac{X}{\frac{5}{2}} - 1 = 0.
\]

The function (86) is continuous, and it has a solution \( X = 0 \), which is what we do not interest. We want to obtain positive roots of the nonlinear function, so that requires us to apply iterative method to get
approximate solution of this equation. Here, our desired root is \( X^* = 4.96511423174428 \). Keeping in view this fact, we apply KTM, KM, KM_2, KM_3, CM_1, CM_2, CM_3, NSPP, SBM, and our methods LTM_1-LTM_8 to the nonlinear equation (86) and compare. Displayed in Table 4 are the number of iterations (IT), the number of function evaluations (NFE), the absolute residual error of the corresponding function value, \( |F(X_n)| \), and the distance of two consecutive approximations \( |X_n - X_{n-1}| \), where “−” is divergence. We use the following stopping criteria for computer programs: \( |F(X_n)| < \epsilon = 10^{-15} \). Note that in Table 4, in terms of iterations number (IT) and function evaluations (NFE), the fourth-order methods have the same performance. KTM, KM_1, KM_2, CM_1, NSPP, SBM, and our methods LTM_1, LTM_2, LTM_3, LTM_4, LTM_5, LTM_6, LTM_7, and LTM_8 have smaller residual error in the nonlinear function as compared to the other methods of fourth-order. Our methods is slightly better at computing time. Consequently, the roots of \( F(X) = 0 \) give the maximum wavelength of radiation \( \lambda \) by means of the following relation:

\[
\tilde{\lambda} = \frac{cP}{X^*BT} = \frac{cP}{4.96511423174428BT}.
\]

**6 Basin of attractions**

In this section, we study some dynamical properties of the family of iterative methods (45) and (46) based on their basins of attraction when they are applied to the complex polynomial \( P(z) \). We investigate the structure of the basins of attraction for comparing convergence and stability of the family of iterative methods. Here we briefly introduce some necessary dynamical concepts and basic results to be used later. Most of them can be found in the classic works such as [20, 23–30] and references therein. Let \( R : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a rational map on the Riemann sphere. The origin of a point \( z_0 \in \hat{\mathbb{C}} \) is defined as the set \( \{z_0, R(z_0), R^2(z_0), \ldots, R^m(z_0), \ldots\} \). A point \( z_0 \in \hat{\mathbb{C}} \) is a fixed point of a rational map \( R \) if satisfy \( R(z_0) = z_0 \). A periodic point \( z_0 \) of period \( m > 1 \) is a point such that \( R^m(z_0) = z_0 \), where \( m \) is the smallest such integer. A point \( z_0 \) is called attracting if satisfy \( |R'(z_0)| < 1 \), repelling if satisfy \( |R'(z_0)| > 1 \), and neutral if satisfy \( |R'(z_0)| = 0 \). Moreover, if satisfy \( |R'(z_0)| = 0 \), the fixed point is super attracting.
Let $z_f^*$ be an attracting fixed point of the rational function $R$. The basin of attraction of the fixed point $z_f^*$ is defined
\[
\mathcal{A}(z_f^*) = \{ z_0 \in \mathbb{C} : R^n(z_0) \to z_f^*, \ n \to \infty \}.
\] (88)

The set of points whose orbits tends to an attracting fixed point $z_f^*$ is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set, the Julia $\beta(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

Some known and existing fourth-order methods and our fourth-order methods are considered, they are KTM (76), KM1 (57), KM2 (58), KM3 (59), CM1 (77), CM2 (78), CM3 (79), NSPP (80), SBM (81), LTM1 (61), LTM2 (62), LTM3 (64), LTM4 (65), LTM5 (67), LTM6 (68), LTM7 (73) and LTM8 (74). In our experiments, we take a square region $D = [-2, 2] \times [-2, 2]$ of the complex plane, with $400 \times 400$ points, and we apply the iterative methods starting in very $z_0$ in the square. The iterative methods can converge to the root or, eventually, diverges. As an illustration, we consider the stopping criterium for convergence to be less than a tolerance $e = 10^{-7}$ and a maximum of 200 iterations. If a sequence $\{z_n\}$ with the residual $|P(z_n)| < e$, generated by the iterative method for the initial guess $z_0$ within the maximum iteration, then we decide the iterative method converges for $z_0$, otherwise we consider the method to be divergent. We take black color for denoting lack of convergence to any of the roots or convergence to the infinity.

**Test problem 1.** Let $P_1(z) = z^3 - 1$ having three simple zeros $\{-\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, 1\}$. Based on the Figure 1-3, we observe that the method CM3 is the best method in terms of less chaotic behavior on the boundary points, the methods KTM, CM2, NSPP, SBM, LTM2 and LTM8 are better. From the three Figures, we find that, with the increase of the value $m$ of (56), the chaotic behaviors of the methods KM1, KM2 and KM3 become more and more complex, which the feature of attraction basins is also reflected in our methods LTM1, LTM2, LTM3 and LTM4. In the next we have taken polynomials of increasing degree.

**Test problem 2.** Let $P_2(z) = z^5 + z$ having five simple zeros $\{-0.7071067812 \pm 0.7071067812i, 0, 0.7071067812 \pm 0.7071067812i\}$. We conclude based on Figure 4-6 that the methods CM2 and LTM3 outperform all the others, and the methods LTM8, KTM and SBM are better in terms of less chaotic behavior than other methods. However, the fractal picture of the method LTM8 has some non convergent points. The method CM3 has the most divergence points in Figure 5, so it performs worst in this test problem. Since the value of $m$ is bigger, the method KM3 has the most complex behavior on the boundary points.

**Test problem 3.** Let $P_3(z) = z^6 + z - 1$ having six simple zeros $\{0.7780895987, -1.1347241384, -0.4510551586 \pm 1.0023645716i, 0.6293724285 \pm 0.7357559530i\}$. In the fractal pictures from Figure 7-9, it is clear that the methods KTM, CM3, NSPP, SBM and our methods LTM1, LTM2 have the largest basins of attraction as compared to the other methods. In addition, although LTM3, LTM8 have a small amount of no convergence points, the two methods have less chaotic behavior on the boundary points than other methods, including the known and existing fourth-order methods KTM, KM1, KM2, KM3, CM1, NSPP and SBM. In terms of the dynamical behavior on the boundary points, the method KM1 is most complex, followed by the methods LTM6 and KM2. From Figures 6(a), the method LTM4 has more non-convergence point regions than other methods.

### 7 Conclusions

In this paper, we have designed and studied a new one-parameter family of modified Cauchy methods for finding solutions of nonlinear equations by using Padé approximant. The convergence analysis of the methods was also considered, and the methods have convergence order three. Based on the family of third-order method, a new optimal fourth-order family of iterative methods (in the sense of Kung-Traub’s conjecture) is obtained by using weight function. We observed from numerical study that the proposed methods are efficient and demonstrate equal or better performance as compared with other well-known fourth-order methods. Finally, the dynamical analysis of this optimal fourth-order family and existing fourth-order methods have been made on some different polynomials, showing some elements of...
the proposed family have equal or better stable behavior in many aspects. Furthermore, the fractal graphics show the chaotic behaviors of our methods become more and more complex with the increase of the value of the series in iterative methods, which also reflected in the existing fourth-order methods KM₁, KM₂ and KM₃.

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Figure 1: Basins of attraction of the methods KTM, KM, KM, CM, CM respectively for $P_1(z) = z^3 - 1$. 
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Figure 2: Basins of attraction of the methods CM, NSPP, SBM, LTM, LTM, and LTM respectively for $P_1(z) = z^3 - 1$. 
Figure 3: Basins of attraction of the methods LTM$_4$, LTM$_5$, LTM$_6$, LTM$_7$, and LTM$_8$ respectively for $P_1(z) = z^3 - 1$. 
Figure 4: Basins of attraction of the methods KTM, KM, KM, KM, CM, and CM respectively for \( P_2(z) = z^5 + z \).
Figure 5: Basins of attraction of the methods CM, NSPP, SBM, LTM, LTM and LTM respectively for \( P_2(z) = z^5 + z \).
Figure 6: Basins of attraction of the methods LTM₄, LTM₅, LTM₆, LTM₇ and LTM₈ respectively for \( P(x) = x^5 + x \).
Figure 7: Basins of attraction of the methods KTM, KM, KM, KM, CM, and CM respectively for $P_3(z) = z^5 + z - 1$. 
Figure 8: Basins of attraction of the methods CM₃, NSPP, SBM, LTM₁, LTM₂ and LTM₃ respectively for \( P₃(z) = z^6 + z - 1 \).
Figure 9: Basins of attraction of the methods LTM$_4$, LTM$_5$, LTM$_6$, LTM$_7$, and LTM$_8$ respectively for $P_3(z) = z^6 + z - 1$. 