Generating the cosmological constant from a conformal transformation

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ABSTRACT: The homogeneous Friedman-Lemaître-Robertson-Walker (FLRW) cosmology of a free scalar field with vanishing cosmological constant was recently shown to be invariant under the one-dimensional conformal group $\text{SL}(2, \mathbb{R})$ acting as Mobius transformations on the proper time. Here we generalize this analysis to arbitrary transformations of the proper time, $\tau \mapsto \tilde{\tau} = f(\tau)$, which are not to be confused with reparametrizations of the time coordinate. First, we show that the FLRW cosmology with a non-vanishing cosmological constant $\Lambda \neq 0$ is also invariant under a $\text{SL}(2, \mathbb{R})$ group of conformal transformations. Second, we show that a cosmological constant can be generated from the $\Lambda = 0$ case through particular conformal transformations, realizing a compactification or de-compactification of the proper time depending on the sign of $\Lambda$. Finally, we propose an extended FLRW cosmological action invariant the full group $\text{Diff}(S^1)$ of conformal transformations on the proper time, by promoting the cosmological constant to a gauge field for conformal transformations or by modifying the scalar field action to a Schwarzian action.
Introduction

Symmetry is an essential concept in theoretical physics. A theory is defined as a representation of a symmetry group, which gives the constants of motion, allows for the integration of the equations of motion, constrains the quantization and the correlation functions. This role becomes even more crucial in the quest of quantum gravity, since the physical content of classical general relativity is entirely encoded in its gauge invariance under space-time diffeomorphisms and that, from this perspective, quantum gravity can be considered as the search for a (possibly anomalous) quantum representation of a regularization or extension of the group of diffeomorphisms.

In this work, we apply this logic to the homogeneous and isotropic sector of general relativity (GR). This is indeed an perfect arena to test quantum gravity methods, not only because its highly symmetric setting is simpler to handle than the generic setting of full general relativity, but also because it is the only setting (as for now) where we can reasonably hope to identify signatures of the quantum regime of gravity through the early universe dynamics.

In this context, a deeper analysis of the Friedman-Lemaître-Robertson-Walker (FLRW) cosmology of a homogeneous free scalar field coupled to general relativity with vanishing cosmological constant $\Lambda = 0$ has recently identified a hidden conformal symmetry on top of the gauge invariance under standard time reparametrizations (or 1D diffeomorphisms) [1]. We underline that those conformal transformations, which change the proper time, are to be distinguished from the usual time reparametrizations, which change the time coordinate while keeping the proper time invariant. More precisely, it was shown that FLRW cosmology
at $\Lambda = 0$ with flat three-dimensional slices is invariant under 1D conformal group $\text{SL}(2, \mathbb{R})$ acting as Mobius transformations on the proper time,
\begin{align}
\tau \quad \mapsto \quad \tilde{\tau} = \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \quad \text{with} \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.
\end{align}

(0.1)

associated to a suitable time-dependent rescaling of the scale factor. The corresponding Noether charges form a $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra, previously studied in [2] and named the CVH algebra. This symmetry and algebraic structure allow one to map the FLRW cosmology with a free scalar field onto the conformal quantum mechanics introduced by de Alfaro, Fubini and Furlan [3] and to discuss it in terms of 1D conformal field theory (CFT$_1$) [1].

The initial goal of the present work is to extend this analysis to the case of a non-vanishing cosmological constant $\Lambda \neq 0$ and to understand whether de Sitter (dS) and anti de Sitter (AdS) cosmologies are also invariant under a similar one-dimensional conformal symmetry. We show that this is indeed the case: the FLRW cosmology of a free scalar field at $\Lambda \neq 0$ is invariant under Mobius transformation of the proper time composed with a map with constant Schwarzian derivative. This now makes possible to discuss the quantization of FLRW cosmology with an arbitrary value of $\Lambda$, both for dS and AdS cosmology, in terms of SL$(2, \mathbb{R})$ representations and CFT$_1$ as done for the $\Lambda = 0$ in [1].

A very interesting side-product of this analysis is that those constant Schwarzian functions map the FLRW cosmology with vanishing cosmological constant onto FLRW cosmology with a non-vanishing cosmological constant. Not only this gives a new twist to the interpretation of the Schwarzian derivative as curvature, but more importantly it shows that a conformal transformation of the proper time allows to generate a cosmological constant from the free theory, at least in the homogeneous setting of FLRW cosmology. More precisely, the action of FLRW cosmology at $\Lambda = 0$ is not invariant under arbitrary conformal transformations $\tau \mapsto \tilde{\tau} = f(\tau)$. Such transformations generate a potential term proportional to the Schwarzian derivative of $f$, as already noticed in [1]. When the Schwarzian vanishes, this shows that FLRW cosmology at $\Lambda = 0$ is invariant under Mobius transformations. In the case of conformal transformation of the proper time $f(\tau)$ has a constant Schwarzian, this generates a constant potential, i.e. a cosmological constant. This seems to be in the same line of thought as the application of Korteweg-de Vries solitons to cosmology as proposed in [4].

We push the logic of conformal invariance further and propose how to modify and extend the FLRW action principle in order to make it invariant not merely under the SL$(2, \mathbb{R})$ group of Mobius transformations but more generally under the whole Diff$(S^1)$ group of arbitrary conformal transformations in proper time $\tau \mapsto \tilde{\tau} = f(\tau)$. We envision two different approaches. First, in section IV, we propose to turn the cosmological constant into a cosmological field $\Lambda(\tau)$ which plays the role of a gauge field for the conformal transformations, similarly to the proposal for implementing the equivalence principle in quantum mechanics by Faraggi and Matone [5]. It remains to be seen if such a scenario can be derived from general relativity or conformal gravity. Second, in section V, we propose to modify the action for the free scalar field into a Schwarzian action. The resulting conformally-invariant FLRW action can be identified with the Schwarzian boundary action for two-dimensional
dilatonic gravity à la Jackiw-Teitelboim [6, 7]. The differences lay in the physical interpretation of the model. Indeed, the AdS$_2$ space-time of dilatonic gravity becomes the phase space of FLRW cosmology as explained in [1]. Moreover, it is not yet clear how the FLRW cosmology action could be understood as a boundary theory.

At the end of the day, we hope that this investigation of the conformal symmetry of FLRW cosmology can open the door to a systematic study of the quantization of FLRW cosmology in CFT terms, but also that the fully conformal invariant versions of the FLRW action could lead to an interesting dynamics for the early universe.

1 FLRW cosmology

Let us consider the Friedman-Lemaître-Robertson-Walker (FLRW) cosmology of an homogeneous massless free scalar field $\phi$ coupled to the space-time geometry foliated by flat spatial slices. The 4d Lorentzian metric is parametrized by the lapse function $N(t)$ and the scale factor $a(t)$, and reads:

$$d{s}^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j,$$

(1.1)

The FLRW cosmological action is derived as the reduced Einstein-Hilbert action integrated on a fiducial spatial cell of volume $V_\circ$. Noting the cosmological constant $\Lambda$, one obtains, up to total derivatives:

$$S[N,a,\phi] = V_\circ \int dt \left[ a^3 \phi'^2 - \frac{3}{8\pi G} \frac{aa'^2}{N} - \frac{\Lambda}{8\pi G} Na^3 \right].$$

(1.2)

with the primes denoting the derivation with respect to the time coordinate, $\phi' = d_t \phi$ for the scalar field and $a' = d_t a$ for the scale factor. One can complete re-absorb the volume of the cosmological fiducial cell by introduce the volume variable $v = V_\circ a^3$:

$$S[N,v,\phi] = \int dt \left[ v \phi'^2 - \frac{1}{8\pi G} \frac{v'^2}{3vN} - \frac{\Lambda}{8\pi G} Nv \right].$$

(1.3)

We proceed to the canonical analysis by defining the conjugate momenta:

$$\pi_\phi = \frac{\partial L}{\partial \phi'} = v \frac{\phi'}{N}, \quad b = - \frac{\partial L}{\partial v'} = \frac{1}{12\pi G} \frac{v'}{Nv}.$$

(1.4)

Notice the opposite convention for the volume variable. Performing the Legendre transform, we obtain the Hamiltonian form of the action:

$$S[N,v,\phi] = \int dt \left[ \pi_\phi \phi' - bv' - N \mathcal{H} \right] \quad \text{with} \quad \mathcal{H} = \frac{1}{2} \left[ \frac{\pi_\phi^2}{v} - \kappa^2 vb^2 + \frac{3\Lambda}{\kappa^2} v \right],$$

(1.5)

where we have introduced the notation $\kappa = \sqrt{12 \pi G}$ for the Planck length (up to a numerical factor). The lapse $N$ is a non-dynamical variable and plays the role of a Lagrange multiplier enforcing the Hamiltonian constraint $\mathcal{H} = 0$.

The volume is a length cube, $[v] = L^3$, while its conjugate momentum is an inverse volume, $[b] = L^{-3}$. The scalar field is an inverse length, $[\phi] = L^{-1}$ and $[\pi_\phi] = L$. Since the
cosmological constant is a curvature, \( [\Lambda] = L^{-2} \), and the lapse is dimensionless, \( [N] = 1 \), one easily checks that the Hamiltonian constraint \( \mathcal{H} \) is an inverse length as expected, \( [\mathcal{H}] = L^{-1} \).

The Poisson bracket with the Hamiltonian constraint gives the evolution in the proper time \( \tau \), defined by absorbing the lapse factor \( d\tau = N dt \) in the time coordinate:

\[
d\tau v = \frac{1}{N} dv = \{v, \mathcal{H}\} = \kappa^2 vb, \quad d\tau \phi = \frac{\pi_\phi}{v}, \quad d\tau \pi_\phi = 0. \tag{1.6}
\]

In order to close the system of differential equations for the evolution, instead of computing the equation of motion for the conjugate momentum \( b \), it is more convenient to introduce the observable \( C = vb \). This observable generates phase space dilatation in the geometric sector, on the volume \( v \) and the extrinsic curvature \( b \). Its evolution gives:

\[
d\tau v = \kappa^2 C, \quad d\tau C = \{C, \mathcal{H}\} = -\mathcal{H} + \frac{3\Lambda}{\kappa^2} v. \tag{1.7}
\]

Taking into account that the Hamiltonian constraint must vanish on-shell, \( \mathcal{H} = 0 = d\tau \mathcal{H} \), this gives a closed set of differential equations, which are straightforward to integrate. This reflects that the three observables, \( v, C \) and \( \mathcal{H} \), provided with the canonical Poisson bracket, form a closed \( \mathfrak{sl}_2 \) Lie algebra, named the CVH algebra in [1, 2, 8, 9].

Imposing that the Hamiltonian constraint vanishes, \( \mathcal{H} = 0 \), the equation of motion for the volume is a straightforward second order equation:

\[
d^2 \tau v = 3\Lambda v. \tag{1.8}
\]

The behavior of the cosmological trajectories clearly depend on the sign of the cosmological constant:

- The vanishing cosmological constant case \( \Lambda = 0 \):

  The volume evolve linearly in proper time:

  \[
  C = C_0 = \frac{\pi_\phi}{\kappa}, \quad v = \epsilon \kappa \pi_\phi (\tau - \tau_0), \quad \phi = \frac{\epsilon}{\kappa} \ln \left[ \epsilon \kappa^{-2} \pi_\phi (\tau - \tau_0) \right], \tag{1.9}
  \]

  with \( \epsilon e^{\kappa \phi} \) is constant during the cosmological evolution. There is a single constant of integration, which can be interpreted as the choice of origin in proper time \( \tau_0 \). The arbitrary sign \( \epsilon = \pm \) signals the contraction or expansion phase of the universe. Since the volume of the universe is meant to always remain positive, the expanding solution \( \epsilon = + \) is valid for positive times \( \tau \geq \tau_0 \), while the contracting solution \( \epsilon = - \) is valid for negative times \( \tau \leq \tau_0 \). The origin \( \tau_0 \) is the cosmological singularity.

- The positive cosmological constant case \( \Lambda > 0 \):

  The evolution of the volume becomes exponential, with two independent solutions:

  \[
  v = v_+ e^{+\omega \tau} + v_- e^{-\omega \tau}, \quad C = \frac{\omega}{\kappa^2} \left[ v_+ e^{+\omega \tau} - v_- e^{-\omega \tau} \right] \quad \text{with} \quad \omega = \sqrt{3\Lambda}, \tag{1.10}
  \]
where \( v_{\pm} \) are the two constants of integration. So a generic solution is a superposition of contracting and expanding solutions. The scalar field evolution can be exactly integrated:

\[
\phi = \phi_0 + \frac{\pi \phi}{\omega \sqrt{v_+ v_-}} \arctan \left( \frac{v_+}{v_-} e^{\omega \tau} \right). \tag{1.11}
\]

We have regular bouncing solutions such that the volume always stays positive,

\[
\forall \tau \in \mathbb{R}, \quad v = v_0 \cosh \omega (\tau - \tau_0), \quad \phi = \phi_0 + \frac{2\pi \phi}{\omega v_0} \arctan e^{\omega (\tau - \tau_0)}, \tag{1.12}
\]

and singular solutions such that the volume vanishes for a finite value of the proper time,

\[
\forall \tau > \tau_0, \quad v = v_0 \sinh \omega (\tau - \tau_0), \quad \phi = \phi_0 - \frac{2\pi \phi}{\omega v_0} \operatorname{artanh} e^{-\omega (\tau - \tau_0)}. \tag{1.13}
\]

Both types of solutions are illustrated on the plots in fig.1.

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**Figure 1.** FLRW cosmological evolution for a positive cosmological constant \( \Lambda > 0 \) in terms of the proper time \( \tau \) with parameters \( \omega = 1, \pi \phi = \frac{1}{2}, v_0 = 1, \phi_0 = \tau_0 = 0 \); the volume \( v \) is in blue and the scalar field \( \phi \) is in orange. On the left hand side, a bouncing solution with \( v = v_0 \cosh \omega (\tau - \tau_0) \): the universe follows a contracting then a expanding phase while the scalar field evolves monotonically and remains bounded. On the right hand side, a singular solution with \( v = v_0 \sinh \omega (\tau - \tau_0) \): the universe starts with an initial singularity with vanishing volume and divergent scalar field \( \phi \to +\infty \), then the volume grows exponentially while the scalar field decreases toward 0.

- The negative cosmological constant case \( \Lambda < 0 \):

The evolution of the volume becomes oscillatory with real frequency \( \omega = \sqrt{-3\Lambda} > 0 \):

\[
v = v_0 \cos \omega (\tau - \tau_0), \quad C = -\frac{\omega v_0}{\kappa^2} \sin \omega (\tau - \tau_0), \tag{1.14}
\]

with the two constants of integration \( v_0 \) and \( \tau_0 \). Since the volume \( v \) is meant to stay positive, these solutions are valid only the proper time intervals \( |\tau_0 - \frac{\pi}{2}, \tau_0 + \frac{\pi}{2}| \) of duration \( \pi/\omega \). The trajectory for the scalar field can also be integrated:

\[
\phi = \phi_0 + \frac{2\pi \phi}{\omega v_0} \operatorname{artanh} \left( \tan \left( \frac{\omega}{2} (\tau - \tau_0) \right) \right), \quad v \cosh \left[ \frac{\omega v_0}{\pi \phi} (\phi - \phi_0) \right] = v_0, \tag{1.15}
\]

which is again valid only on the interval \( \tau \in \left[ \tau_0 - \frac{\pi}{2}, \tau_0 + \frac{\pi}{2} \right] \) around the time origin \( \tau_0 \), with the scalar field monotonically growing from \(-\infty\) to \(+\infty\). As we can see on the plots on fig.2, there are initial and final singularities.

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**Figure 2.** FLRW cosmological evolution for a negative cosmological constant $\Lambda < 0$ in terms of the proper time $\tau$ with parameters $\omega = 1$, $\pi_\phi = \frac{1}{2}$, $v_0 = 1$, $\phi_0 = \tau_0 = 0$: the dynamics starts with the initial singularity has vanishing volume $v \to 0$ and divergent scalar field $\phi \to -\infty$ and drives the evolution towards a final singularity with $v \to 0$ and $\phi \to +\infty$. The volume $v$ is in blue and the scalar field $\phi$ is drawn in orange.

### 2 Conformal Transformations \\ Schwarzian action

After this short review of FLRW cosmology with a non-vanishing cosmological constant, we would like to study its behavior under conformal transformations defined as local scale transformations in proper time, as introduced in [1]:

$$
\begin{align*}
\tau &\mapsto \tilde{\tau} = f(\tau) \\
v &\mapsto \tilde{v}(\tilde{\tau}) = h(\tau) v(\tau) \\
\phi &\mapsto \tilde{\phi}(\tilde{\tau}) = \phi(\tau)
\end{align*}
$$

(2.1)

where $h(\tau) = \partial_\tau f$ is the Jacobian factor for infinitesimal time variations, $d\tilde{\tau} = h(\tau) d\tau$. The 3d volume $v$ has a weight +1 with respect to those scale transformations, while the scalar field is assumed to have a trivial transformation with vanishing weight.

It is crucial not to confuse these conformal transformations with time reparametrizations, which change the coordinate time and the lapse without affecting the proper time. Indeed, time reparametrizations act as:

$$
\begin{align*}
t &\mapsto \tilde{t} = \theta(t) \\
N(t) &\mapsto \tilde{N}(\tilde{t}) = \dot{\theta}(t)^{-1} N(t) \\
d\tau &\mapsto d\tilde{\tau} = d\tau \\
v &\mapsto \tilde{v}(\tilde{t}) = v(t) \\
\phi &\mapsto \tilde{\phi}(\tilde{t}) = \phi(t)
\end{align*}
$$

(2.2)

and do not change the proper time $\tau$. The conformal transformations described above are thus different from the time diffeomorphisms.
In order to study the action of conformal transformations, it is convenient to write the FLRW action directly in proper time, by absorbing the lapse field in the time coordinate:

\[ S[v, \phi] = \int d\tau \left [ v \frac{\dot{\phi}^2}{2} - \frac{1}{2\kappa^2} v^2 - \frac{3\Lambda}{2\kappa^2} v \right ], \]  

(2.3)

with the Planck length \( \kappa = \sqrt{12\pi G} \) plays the role of a unit of time. The dot denote the total variation with respect to the proper time, \( \dot{v} = d\tau v \) for the volume and \( \dot{\phi} = d\tau \phi \) for the scalar field. Let us compute the variation of action under a conformal transformation:

\[ \tilde{S}[	ilde{v}, \tilde{\phi}] = \int d\tilde{\tau} \left [ \frac{\dot{\tilde{v}}^2}{2} - \frac{1}{2\kappa^2} \dot{\tilde{v}}^2 - \frac{3\Lambda}{2\kappa^2} \tilde{v} \right ] \]

(2.4)

where the Schwarzian derivative of \( f \) appears as a volume term:

\[ \text{Sch}[f] = \frac{f^{(3)}}{f} - 3 \left ( \frac{f'}{f} \right )^2 = \frac{h'}{\hat{h}} - 3 \frac{1}{2} \left ( \frac{h'}{\hat{h}} \right )^2 = \frac{d^2}{d\tau} (\ln h) - \frac{1}{2} (d\tau \ln h)^2. \]  

(2.5)

We first consider the vanishing cosmological constant action, \( S_0 \) for \( \Lambda = 0 \). As one can see from the variation computed above, the action \( S_0 \) is invariant (up to a total derivative term) under conformal transformation with vanishing Schwarzian, i.e. Mobius transformations in proper time, as noticed earlier in [1]:

\[ f(\tau) = M_{(\alpha, \beta, \gamma, \delta)}(\tau) = \frac{\alpha \tau + \beta}{\gamma \tau + \delta}, \quad (\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4, \quad \text{Sch}[M] = 0. \]  

(2.6)

As shown in [1], this \( \text{SL}(2, \mathbb{R}) \) conformal invariance leads to conformal Noether charges, which have been identified as (the initial conditions) for the three observables \( v, C \) and \( H \). These Noether charges form a closed \( \mathfrak{sl}_2 \) Lie algebra called the CVH algebra.

In this paper, we would like to show that it is possible to extend this logic to an arbitrary non-vanishing cosmological constant, \( \Lambda \neq 0 \). Indeed, discarding the total derivative term, the variation of the action reads:

\[ \Delta f S = \int d\tau \frac{v}{\kappa^4} \left [ \text{Sch}[f] - \frac{3\Lambda}{2} \left ( f'^2 - 1 \right ) \right ]. \]  

(2.7)

Thus the action is invariant if the conformal transformation satisfies:

\[ \text{Sch}[f] - \frac{3\Lambda}{2} f'^2 = -\frac{3\Lambda}{2}. \]  

(2.8)

Using the composition law for the Schwarzian derivative, \( \text{Sch}[f_1 \circ f_2] = (f_2)^2 \text{Sch}[f_1] \circ f_2 + \text{Sch}[f_2] \) (which is the cocycle equation for the Schwarzian), we can re-absorb the \( f'^2 \) term by a suitable composition. Choosing a function \( F \) with constant Schwarzian \( \text{Sch}[F] = -3\Lambda/2 \), the condition above reduces to:

\[ \text{Sch}[F \circ f] = -\frac{3\Lambda}{2} = \text{Sch}[M \circ F]. \]  

(2.9)
This means that if $f$ is a Mobius transformation $M$ up to conjugation (by composition) with the function $F$ of constant Schwarzian, more explicitly $F \circ f = M \circ F$, then the FLRW action $S$ is invariant under the conformal transformation $f$. We are left with determining the existence of a map $F$ with constant Schwarzian derivative. This will be shown in the next section.

Moreover, pushing this logic further, it appears that one can generate an arbitrary cosmological constant out of the vanishing cosmological constant case. Indeed, as computed above, starting from the theory at $\Lambda = 0$, let us do a conformal transformation and ignore the total derivative term, we get:

$$\tilde{S}_f = \int \! \! \! d\tau \left[ v \ddot{\phi}^2 - \frac{1}{2\kappa^2} \dot{v}^2 + \frac{1}{\kappa^2} v \text{Sch}[f] + (\text{total derivative}) \right].$$

The action is invariant under Mobius transformation, but a conformal transformation with a constant Schwarzian derivative creates an effective cosmological constant term. Thus, choosing a function $F$ with constant Schwarzian and calling $\Lambda_{\text{eff}} = -\frac{2}{3} \text{Sch}[F]$, this leads to:

$$\tilde{S}_F = \int \! \! \! d\tau \left[ v \ddot{\phi}^2 - \frac{1}{2\kappa^2} \dot{v}^2 - \frac{3\Lambda_{\text{eff}}}{2\kappa^2} v + (\text{total derivative}) \right],$$

and we have created a cosmological constant by a mere conformal transformation in proper time.

In short, this means that the cosmological constant amounts to a non-trivial conformal transformation. A consequence is that the cosmological trajectories for arbitrary $\Lambda$ should be obtained from the cosmological trajectories at $\Lambda = 0$ by this conformal map. We check this explicitly below in the next section.

3 Cosmological Constant as a Conformal Soliton

In the previous section, we showed that a non-vanishing cosmological constant corresponds to a conformal transformation with constant Schwarzian derivative for the theory at $\Lambda = 0$. We refer to this as a conformal soliton, along the lines of [4, 10, 11] which proposed to use Korteweg-de Vries (KdV) solitons in cosmology. This is an elegant way to generate a non-trivial four-dimensional curvature through conformal transformations in the homogeneous context of FLRW cosmology. This is also an actual implementation of the Schwarzian derivative as curvature (see [12] for a synthetic overview of the Schwarzian derivative).

We underline that it realizes a non-trivial transformation of the proper time, very different from time reparametrizations. As we explain below, this corresponds to a compactification or de-compactification of the proper time depending on the sign of the cosmological constant. The two maps $\tau \mapsto \tanh(\Omega \tau)$ and $\tau \mapsto \tan(\Omega \tau)$ actually already appear in the framework of conformal quantum mechanics to switch between null $\mathfrak{sl}(2, \mathbb{R})$-generators and space-like or time-like generators [3], and were also recently used to study the temperature of causal diamonds [13].
3.1 $\Lambda > 0$ from proper time compactification

For a positive cosmological constant, we need a map with constant negative Schwarzian derivative. The condition $\text{Sch}[F] = -3\Lambda/2 < 0$ is solved by:

$$F(\tau) = \tanh(\Omega \tau) \quad \Rightarrow \quad \text{Sch}[F] = -2\Omega^2,$$

(3.1)

with the proper frequency given by $\Omega = \sqrt{3\Lambda}/2$. We can compose this map with an arbitrary translation in proper time (since this is a special case of a Mobius transformation, with vanishing Schwarzian) in order to off-set the origin in $\tau$.

So we start with the FLRW action with vanishing cosmological constant, perform the non-trivial conformal change of proper time $\tilde{\tau} = F(\tau)$, and get the theory with positive cosmological constant:

$$S_0[\tau, v, \phi] \rightarrow \tilde{S}[\tilde{\tau}, \tilde{v}, \tilde{\phi}] = S_0[\tilde{\tau}, \tilde{v}, \tilde{\phi}] = S_\Lambda[\tau, v, \phi].$$

(3.2)

The proper time $\tilde{\tau}$ for the theory with vanishing cosmological constant corresponds to the proper time $\tau$ for the theory with $\Lambda > 0$. This implies that a $\Lambda = 0$ trajectory in $\tilde{\tau}$ should define a $\Lambda > 0$ trajectory in $\tau$. Let us check this explicitly, starting with an expanding trajectory:

$$\tilde{v}(\tilde{\tau}) = \kappa \pi \tilde{\tau} + \tilde{v}_0 = \kappa \pi (\tilde{\tau} - \tilde{\tau}_0), \quad \tilde{\phi}(\tilde{\tau}) = \kappa^{-1} \ln \left[ \kappa \pi (\tilde{\tau} - \tilde{\tau}_0) \right],$$

(3.3)

we perform the conformal transformation $F$ properly off-set by the initial proper times, $F(\tau - \tau_0) = \tilde{\tau} - \tilde{\tau}_0$, and get:

$$v(\tau) = \frac{\tilde{v}(\tilde{\tau})}{F(\tau - \tau_0)} = \frac{\kappa \pi \tanh \Omega (\tau - \tau_0)}{\Omega \cosh^{-2} \Omega (\tau - \tau_0)}, \quad \phi(\tau) = \tilde{\phi}(\tilde{\tau}) = \kappa^{-1} \ln \left[ \kappa \pi \tanh \Omega (\tau - \tau_0) \right].$$

(3.4)

This exactly matches the cosmological solutions for $\Lambda > 0$, with an initial singularity, given in (1.10) and (1.11) with $2\Omega = \sqrt{3\Lambda} = \omega$:

$$v(\tau) = \frac{\kappa \pi \phi}{\omega} \sinh \omega (\tau - \tau_0)$$

(3.5)

If we play with the off-set in proper time $\tau_0$ and glue the conformal transformations of collapsing and expanding trajectories, we will obtain arbitrary linear combination $v = v_+ e^{\omega \tau} + v_- e^{-\omega \tau}$, and similarly for the scalar field, including the regular bouncing solutions.

Let us underline that the conformal map $F$ realizes a compactification: the proper time $\tau$ for the cosmological evolution with a positive cosmological constant $\Lambda > 0$ gets compactified to the proper time $\tilde{\tau} = F(\tau) = \tanh \frac{\omega}{2} \tau$ describing the evolution for a vanishing cosmological constant.

3.2 $\Lambda < 0$ from proper time decompactification

For a negative cosmological constant, we need a map with constant positive Schwarzian derivative. The condition $\text{Sch}[F] = -3\Lambda/2 > 0$ is solved by:

$$F(\tau) = \tan \frac{\omega}{2} \tau \quad \Rightarrow \quad \text{Sch}[F] = +\frac{1}{2} \omega^2,$$

(3.6)
with the proper frequency given by \( \omega = \sqrt{-3\Lambda} \), where we have anticipated the factor 2 rescaling of the frequency for the cosmological trajectories.

Off-set transformations \( F(\tau - \tau_0) = \tilde{\tau} - \tilde{\tau}_0 \) map the \( \Lambda = 0 \) trajectories in \( \tilde{\tau} \) into \( \Lambda < 0 \) trajectories in \( \tau \). Starting with an expanding trajectory \( \tilde{\nu}(\tilde{\tau}) = \kappa \pi \phi (\tilde{\tau} - \tilde{\tau}_0) \), we get:

\[
v(\tau) = \frac{\tilde{\nu}(\tilde{\tau})}{F(\tau - \tau_0)} = \frac{2\kappa \pi \phi \tan \frac{\omega}{2}(\tau - \tau_0)}{\omega \cos^{-2} \frac{\omega}{2}(\tau - \tau_0)} = \frac{\kappa \pi \phi}{\omega} \sin \omega(\tau - \tau_0),
\]

and similarly for the scalar field \( \phi \). This exactly matches solutions for \( \Lambda < 0 \) given in (1.14) and (1.15). Depending on the initial off-set, we indeed obtain arbitrary oscillations \( v = v_0 \cos \omega \tau + w_0 \sin \omega \tau \) and similarly for the scalar field.

For a negative constant cosmological, the conformal map \( F \) now realizes a de-compactification: the proper time \( |\tau| \leq \frac{\pi}{\omega} \) with finite range for the cosmological evolution with a positive cosmological constant \( \Lambda < 0 \) gets de-compactified to the proper time \( \tilde{\tau} = F(\tau) = \tan \frac{\omega}{2} \tau \in \mathbb{R} \) describing the evolution for a vanishing cosmological constant.

### 4 Cosmological potential as conformal gauge field

Up to now, we have focused on obtaining a constant volume term interpreted as a (effective) cosmological constant. Coming back to the action of conformal transformations on the original FLRW action, as derived earlier in (2.4),

\[
S[v, \phi] = \int d\tau \left[ \frac{\dot{v}^2}{2} - \frac{1}{2\kappa^2} v^2 - \frac{3\Lambda}{2\kappa^2} v \right]
\]

\[
\tilde{S}[\tilde{v}, \tilde{\phi}] = S[\tilde{\tau}, \tilde{v}, \tilde{\phi}] = \int d\tau \left[ \frac{\dot{\tilde{v}}^2}{2} - \frac{1}{2\kappa^2} \tilde{v}^2 - \frac{3}{2\kappa^2} (h^2 \Lambda - \frac{2}{3} \text{Sch}[f]) \right]
\]

where we have discarded total derivatives, we see that an arbitrary conformal transformation creates a potential term \( Q(\tau) \propto h^2 \Lambda - \frac{2}{3} \text{Sch}[f] \). It is important to stress that this cosmological potential depends on the proper time and is not a self-interaction potential for the scalar field \(^1\). This means that we can derive cosmological trajectories with a (proper) time dependent potential from the bare cosmological trajectories (with or without cosmological constant) by a suitable conformal transformation.

We would like to go further and propose a way to make the FLRW action fully invariant under conformal transformations. We can turn the cosmological constant \( \Lambda \) into a cosmological field \( \Lambda(\tau) \), which plays the role of a gauge field for the conformal transformation in proper time. More precisely, we assume that this new cosmological field transforms as:

\[
\tau \mapsto \tilde{\tau} = f(\tau)
\]

\[
\Lambda(\tau) \mapsto \tilde{\Lambda}(\tilde{\tau}) = h(\tau)^{-2} \left[ \Lambda(\tau) + \frac{2}{3} \text{Sch}[f] \right]
\]

\(^1\) It might be possible to exchange a time-dependent potential \( Q(t) \) with a self-interaction potential \( V[\phi] \) by choosing the scalar field \( \phi \) as time coordinate. Using such a dynamical choice of time coordinate is consistent with the logic of using an internal clock in general relativity and would implement the “de-parametrization” of the theory (i.e. getting rid of the arbitrary time coordinate). Studying this possible mapping between \( Q(t) \) and \( V[\phi] \) is out of the scope of the present paper but is nevertheless a very interesting question.
This cosmological field transforms with a $-2$ weight plus a derivative term, similarly to a connection. Then the extended FLRW action is invariant under all conformal transformations $\tilde{\tau} = f(\tau)$. The action

$$S^{\text{ext}}[v, \phi, \Lambda] = \int d\tau \left[ v\frac{\dot{\phi}^2}{2} - \frac{1}{2\kappa^2} \frac{\dot{\phi}^2}{v} - \frac{3\Lambda}{2\kappa^2} v \right]$$

transforms as

$$\tilde{S}^{\text{ext}}[\tilde{v}, \tilde{\phi}] = \int d\tilde{\tau} \left[ \tilde{v}\frac{(d\tau\dot{\phi})^2}{2} - \frac{1}{2\kappa^2} \frac{(d\tau\dot{\phi})^2}{\tilde{v}} - \frac{3\Lambda}{2\kappa^2} \tilde{v} \right] = \int d\tau \left[ v\frac{\dot{\phi}^2}{2} - \frac{1}{2\kappa^2} \frac{\dot{\phi}^2}{v} - \frac{3\Lambda}{2\kappa^2} v - \frac{1}{\kappa^2} \frac{d(\tilde{v}^{-1} h)}{d\tau} \right].$$

Up to the total derivative, turning the cosmological constant into a gauge field for the conformal transformations, or “conformal connection” in short, makes the FLRW action invariant under the conformal re-parametrizations of the proper time $\tau$. Let us stress that this is an extra symmetry on top of the usual time coordinate re-parametrization corresponding the one-dimensional diffeomorphisms in the time direction.

This procedure to make FLRW cosmology fully invariant under conformal transformation is along the lines of the equivalence postulate for quantum mechanics pushed forward in e.g. [5, 14–18]. It would be interesting to investigate further if this extension of FLRW cosmology, 1. has relevant phenomenological implications (e.g. for the cosmological constant problem); 2. is comparable to quintessence models or other scenarii with a varying cosmological constant (e.g. [19–22]); 3. could be derived from a symmetry reduction of Weyl gravity or any other conformally-invariant version of general relativity.

A key question is to classify the equivalence classes of cosmological fields $\Lambda(\tau)$ under conformal transformations, i.e. the conformal orbits, in order to study the resulting cosmological dynamics and evolution of each class.

### 5 Schwarzian Action for the Scalar Field

The extension of FLRW cosmology above turning the cosmological constant into a conformal connection implicitly implies a modification of gravity and/or the introduction of a new field. Another way to make FLRW cosmology conformally-invariant is to modify the action for the scalar field $\phi$. Again along the lines of [5], we propose to use the Schwarzian action instead of the minimal coupling of a free and massless scalar field:

$$S^{\text{conf}}[v, \phi] = \frac{1}{\kappa^2} \int d\tau \left[ v\text{Sch}[\phi] - \frac{\dot{\phi}^2}{2v} \right],$$

where the cosmological constant is set to 0.

A straightforward calculation, equivalent to the composition law for the Schwarzian derivative, yields the behavior of the $\text{Sch}[\phi]$ factor under conformal transformations

$$\text{Sch}[\phi](\tau) \xrightarrow{\tilde{\tau} = f(\tau)} \tilde{\text{Sch}}[\phi] = \frac{d^3 \tilde{\phi}}{d\tilde{\tau}^3} \phi - \frac{3}{2} \left( \frac{d^2 \tilde{\phi}}{d\tilde{\tau}^2} \right)^2 = h(\tau)^{-2} \left[ \text{Sch}[\phi] - \text{Sch}[f] \right],$$
with the Jacobian factor $h = \frac{d\tau}{f}$. The term $\text{Sch}[f]$ compensates the one resulting from the conformal transformation of the kinetic term of the volume and makes this new action invariant under arbitrary conformal transformation in proper time:

$$S_{\text{conf}}^{\text{f}}[v, \phi] \overset{\tau = \frac{f(\tau)}{\sqrt{\phi}}}\longrightarrow \tilde{S}_{\text{conf}}^{\text{f}}[\tilde{v}, \tilde{\phi}] = S_{\text{conf}}^{\text{f}}[v, \phi],$$

up to a total derivative term.

It is clear that such a scalar field is not standard, it is not a mere addition of a self-interaction potential, it involves a third derivative of the field and it does not look like any usual inflationary ansatz (e.g. see [23]). In this context, it would be very interesting to, 1. solve the conformally-invariant FLRW evolution and see how it differs from the standard FLRW cosmology in the semi-classical regime and close to the singularity; 2. check if it leads to an inflationary phase; 3. if there exists a regime where the scalar field behaves as standard scalar field with mass and potential; 4. if this Schwarzian action can be derived from a suitable dimensional reduction or boundary term of general relativity à la Jackiw-Teitelboim (see e.g. [24]).

Let us conclude with a couple of remarks:

- One can add to the Schwarzian action for the scalar field (5.1) the usual kinetic term in $v\dot{\phi}^2$, which is conformally invariant on its own and does not affect the properties of the action under conformal transformation. The action acquires a new term and coupling:

$$S_{\beta}^{\text{conf}}[v, \phi] = \frac{1}{\kappa^2} \int d\tau \left[ v\text{Sch}[\phi] + \beta \frac{1}{2} v\dot{\phi}^2 - \dot{\phi}^2 \right].$$

In fact, it was well-known that such a term can be re-absorbed directly in the Schwarzian derivative by the composition of $\phi$ with a $\tan$ or $\tanh$ map as the compactification and de-compactification used to generate a cosmological constant:

for $\beta > 0$, \quad $\Phi = \tan \frac{\sqrt{\beta}}{2} \phi \Rightarrow \text{Sch}[\Phi] = \text{Sch}[\phi] + \frac{\beta}{2} \phi^2$,

(5.5)

for $\beta < 0$, \quad $\Phi = \tanh \frac{\sqrt{-\beta}}{2} \phi \Rightarrow \text{Sch}[\Phi] = \text{Sch}[\phi] + \frac{\beta}{2} \phi^2$.

(5.6)

For instance, for $\beta > 0$, one would use the conformally-invariant action:

$$S_{\beta}^{\text{conf}}[v, \phi] = \frac{1}{\kappa^2} \int d\tau \left[ v\text{Sch} \left[ \tan \frac{\sqrt{\beta}}{2} \phi \right] - \frac{\beta}{2} \phi^2 \right].$$

(5.7)

- Although the Schwarzian action involves third derivatives of the scalar field, it is nevertheless possible to understand it as a second order action principle. Indeed the Schwarzian derivative of the scalar field is defined as:

$$\text{Sch}[\phi] = \frac{\phi^{(3)}}{\phi} - 3 \left( \frac{\phi'}{\phi} \right)^2 = d_\tau^2 \ln \phi - \frac{1}{2} (d_\tau \ln \phi)^2.$$
Although the scalar field \( \phi \) has a trivial conformal weight, the secondary field \( \dot{\phi} \) behave as a field of weight \(-1\) and the field \( \psi \equiv \ln \dot{\phi} \) simply gets shifted by a conformal transformation:

\[
\begin{align*}
\phi \mapsto \tilde{\phi}(\tilde{\tau}) &= \phi(\tau) \\
d_\tau \phi \mapsto d_{\tilde{\tau}} \tilde{\phi} &= h(\tau)^{-1} d_\tau \phi \\
\psi \mapsto \tilde{\psi}(\tilde{\tau}) &= \psi(\tau) - \ln h(\tau)
\end{align*}
\]

The conformally-invariant FLRW action can then be written as a usual second-order scalar field coupled to the cosmological metric \( \mathcal{V}[\psi] \propto e^{+2\psi} \) similar to power-law potentials used in inflationary scenarios (see [23] for an overview of inflationary potentials). It would be enlightening to investigate the cosmological trajectories resulting from this action principle and understand their early universe behavior.

6 Discussion

We have proposed a new way to understand the cosmological constant, in the context of FLRW cosmology, as a conformal change in proper time. Such transformations, \( \tau \mapsto \tilde{\tau} = f(\tau) \), with a rescaling of the scale factor, are not to be confused with changes in coordinate time, a.k.a. time reparametrizations \( t \mapsto \tilde{t}(t) \), which leave the proper time \(^3\) invariant and are a gauge symmetry of cosmology and more generally of general relativity. We have shown how conformal transformations with constant Schwarzian derivative, that effectively compactify or decompactify the proper time, maps the FLRW cosmology of a free scalar field at \( \Lambda = 0 \) onto the FLRW cosmology with an arbitrary cosmological constant (given in terms of the value of the Schwarzian derivative). This means that, after conformal transformation, the cosmological dynamics of the scale factor and scalar matter, and the propagation of (test) particles, happen as if there were a non-vanishing cosmological constant.

\(^2\) We nevertheless point out that the field equations taking \( \phi \) or \( \psi \) as primary field are slightly different. This is the difference between a third-order action and a second-order action. Taking the pure Schwarzian action as example, we underline the difference between the following two action principles:

\[
\begin{align*}
S[\psi] &= \int d\tau \frac{1}{2} \dot{\psi}^2 & \neq & S[\phi, \psi] &= \int d\tau \left[ \frac{1}{2} \dot{\psi}^2 - \lambda(\dot{\phi} - e^{\psi}) \right],
\end{align*}
\]

where \( \lambda \) is a Lagrange multiplier. The equation of motion for the former action is simply the second-order equation \( \ddot{\psi} = 0 \), while the equation of motion for the latter action is the third-order field equation for the Schwarzian action (or the dimensionally-reduced 2d Liouville theory), \( \ddot{\psi} = \lambda e^\psi \) with \( \lambda(\tau) = \lambda_0 \) constant. While the solutions of \( \ddot{\psi} = 0 \) are \( \psi(\tau) = A\tau + B \), the solutions \( \ddot{\psi} = \lambda_0 e^\psi \) with non-vanishing \( \lambda_0 \) are \( \psi(\tau) = -2 \ln \tau + \psi_0 \) with \( \lambda_0 e^{\psi_0} = 2 \). These are extra solutions of the third order theory compared to the second order theory and can be understood as ghost modes. See e.g. [6, 7] for details on the classical and quantum Schwarzian theory.

\(^3\) While the time coordinate \( t \) is unphysical and can be considered as pure gauge (up to time-like boundary conditions), the proper time \( \tau \) is physical in the sense that it is the natural time associated to a physical process, the propagation of test particles (i.e. particles in the weak coupling regime).
Moreover, we recall that FLRW cosmology at $\Lambda = 0$ is invariant under conformal transformations with vanishing Schwarzian derivative, i.e. invariant under the $\text{SL}(2, \mathbb{R})$ group of Mobius transformations of the proper time, as shown in [1]. Here, we have extended this result to FLRW cosmology at $\Lambda \neq 0$, showing that it also admits a $\text{SL}(2, \mathbb{R})$ symmetry, consisting in Mobius transformations conjugated by a compactification or decompactification map depending on the sign of $\Lambda$. This opens the door to discussing the quantization of the FLRW cosmology of a scalar field in terms of $\text{SL}(2, \mathbb{R})$ and 1D conformal field theory whatever the sign and value of the cosmological constant $\Lambda$. In particular, it would be interesting the investigate how the presence of the cosmological constant modifies the CFT two point correlation function of flat FLRW quantum cosmology presented in [1].

To go further, there are two natural directions. On the one hand, in the context of cosmology, adding a potential for the scalar field or adding inhomogeneities would break the $\text{SL}(2, \mathbb{R})$ invariance and one would like to understand if the $\text{SL}(2, \mathbb{R})$ algebraic structure could still be helpful to quantize the theory and compute the correlation functions. On the other hand, we would like to investigate how our results extend to full general relativity, i.e. understand how conformal transformations in proper time for a space-like foliation could generate a cosmological constant, or more generally other polynomial terms in the curvature. This could potentially map some classes of $f(R)$ gravity back to Einstein gravity through such conformal transformations.

To conclude on a less speculative note, we have proposed a modified FLRW action principle, fully invariant under the whole $\text{Diff}(S^1)$ group of conformal transformations in proper time (and not only under the $\text{SL}(2, \mathbb{R})$ subgroup of Mobius transformations). Assuming a vanishing cosmological constant, it consists in switching the free scalar field for a Schwarzian field:

$$S[v, \phi] = \frac{1}{\kappa^2} \int d\tau \left[ v \frac{\dot{\phi}^2}{2} - \frac{\dot{\phi}^2}{2v} \right] \longrightarrow S^{\text{conf}}[v, \phi] = \frac{1}{\kappa^2} \int d\tau \left[ v \text{Sch}[\phi] - \frac{\dot{\phi}^2}{2v} \right]. \quad (6.1)$$

It is of course essential to check whether this new conformally-invariant version leads to a plausible dynamics of the early universe and realistic cosmology. Nevertheless, another intriguing remark is that this extended Schwarzian action looks very similar to the Schwarzian action arising on the boundary of 2D dilatonic gravity, as used in the study of an $\text{AdS}_2$/CFT$_1$ correspondence [6, 7, 24],

$$S_{\text{Schw}}[\varphi, u] = \int d\tau \varphi(\tau) \text{Sch}[u](\tau), \quad (6.2)$$

where $\varphi$ is the dilaton field and $u$ parametrizes the boundary conditions at infinity. Identifying $(v, \phi)$ to $(\varphi, u)$, we see that the difference is that FLRW cosmology retains a kinetic term for the volume of the universe, leading to an evolving background for the scalar field $\phi$. In fact, it is this kinetic term for the volume that makes this version of FLRW cosmology invariant under arbitrary conformal transformation while the Schwarzian action $S_{\text{Schw}}$ is only invariant under the $\text{SL}(2, \mathbb{R})$ subgroup. On the other hand, in the context of $\text{AdS}_2$ gravity, the dilaton field is usually set to a constant, either through by boundary conditions and equations of motion or by a time redefinition. This means that FLRW conformal cosmology that we propose might be obtained through a slightly different choice of boundary.
conditions for Jackiw-Teitelboim gravity. In light of this relation, a natural outlook is to investigate this interplay between FLRW cosmology and the AdS$_2$/CFT$_1$ correspondence at the level of the quantum theory and correlation functions.

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