An Analytical Study of the Diffraction of Light by a Circular Aperture Using Spherical Harmonics for $n \leq 1$

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Based on the importance of spherical harmonics and their applicability in many physical problems, this research aimed to study the diffraction pattern of light by a circular aperture starting from the first Rayleigh–Sommerfeld diffraction equation and to expand the polar radius of a point on the surface of the circular aperture based on spherical harmonics. We depended on this theoretical framework in our paper. We calculated the optical intensity compounds $C_{00}^0$, $C_{01}^1$, $C_{11}^1$, $C_{-11}^1$ for $(n = 0, 1, m = -1, 0, 1)$. We studied the intensity distributions in three special cases (along the optical axis, at the geometrical focal plane, and along the boundary of the geometrical shadow). We presented numerical comparative examples to illustrate the variation of the intensity versus a ratio $(Z/A)$ is the ratio of the distance between the circular aperture and the observation plane to a radius of the circular aperture, and we used Maple program to represent these results. We noticed that the expansion we made using spherical harmonic analysis led to an increase in the number of fringes bright enough to be visible to the naked eye. We then concluded with a brief discussion of the results.

1. Introduction

In the study of light, it is very important to understand the manner of diffraction of light passing through aperture, slit, or obstruction. Fresnel interpreted [1] the diffraction phenomenon according to the Huygens–Fresnel principle [2–5] with a wave-based approach, and the Huygens–Fresnel principle states that every point of a wavefront is a new secondary source that emits spherical wavelets. The Huygens–Fresnel principle has a close relationship with the diffraction by a circular aperture. The appropriate choice of the shape, size, and type of material from which the aperture is made has great scientific and practical importance; Basyigit et al. [6] studied various forms of apertures (rectangular, square, circular, hexagonal, and equilateral triangle shape apertures), and they ranked it from best performance to the worst with respect to electrical shielding effectiveness (ESE). They also studied polarization type of applied electromagnetic field. It is important to note that we neglected polarization in our study because the study is scalar. Basyigit and Dogan [7] investigated the effect of aperture size of metallic enclosure on ESE by using both Robinson’s analytical model and ANN methods (ANN is a universal approximation method used in electromagnetic research studies).

The calculation of optical intensity is an interesting subject among many features of studying the diffraction by a circular aperture. This topic has been a matter of interest to many researchers [8–15]. Sheppard and Cooper [9] studied Fresnel diffraction by a circular aperture for off-axis illumination and various different levels of approximation. They obtained a new approximate equation for the shape of contours of equal intensity in the focal plane describing a distortion of the contours in comparison with those of elliptical shape predicted by Murty [10]. Acosta and Reino [14] presented a method for studying the intensity distribution of a beam diffracted by a circular aperture in terms of Bessel functions.

In this research, we studied the intensity distributions of spherical waves diffracted by a circular aperture in three cases as follows:
(i) Along the optical axis
(ii) In the geometrical focal plane
(iii) Along the boundary of the geometrical shadow

We calculated the optical intensity compounds $C_0, C_1, C_{-1}$ based on the spherical harmonic analysis for $(n = 0, 1, m = -1, 0, 1)$. In addition, we discussed the special cases of intensity on graphs for a Rayleigh distance $d_0$. We presented numerical comparative examples to illustrate our analytical results, which greatly improve the design of optical devices that have a same circular aperture.

### 2. Theoretical Formulation

If we assume that the light waves impinging on a screen contain a circular aperture, the diffused light diffracts only from the circular section of the aperture. It is clear that the same process occurs in the eye, microscope, telescope, and the camera lens. In order to calculate the resulting optical intensity, we started from the following diffraction equation [8]:

$$ U(\rho, R) = \frac{A}{R} \left( \frac{Z}{R} \right) (ik + \frac{1}{R}) \int_0^1 e^{i(2\pi \rho') J_0(\rho')d\rho}, $$

where $A$ is the radius of the circular aperture, $k$ is the wave number, $J_0(\rho)$ is the zero order of Bessel function, $(u, v, w)$ are the optical coordinates, and $\rho, \rho'$ are cylindrical radial coordinates in the observation and aperture planes, respectively (Figure 1).

The geometry of the experiment has a great importance in our study. Figure 1 shows the geometry of diffraction for a uniform, monochromatic spherical wave diffracted by a circular aperture. There is a close relationship between the parameters of this figure and the diffraction phenomenon. The most important parameter is the size of circular aperture. The maximum limits of intensity become sharper and narrower with the increase in aperture size. Diffraction is the bending of waves around corners that occurs when a portion of a wavefront is cut off by a barrier or obstacle, and as a result of the differences in the diffraction angles, fringes form a specific pattern of intensity on the observation screen.

A "Z" distance between circular aperture and observation screen is an important parameter in the diffraction phenomenon. It determines the conditions of transmission between the different patterns of diffraction (Rayleigh–Sommerfeld, Fresnel, and Fraunhofer), and this is what we explained later in the results.

### 3. Spherical Harmonics

Spherical harmonics are a frequency-space basis for representing a series of special functions defined over the sphere. They have been applied in various fields ranging from the representation of gravitational and magnetic fields of planetary bodies in geodesy to atomic electron configurations in physical chemistry. They also appear in quantum mechanics as the solutions of the Schrödinger equation in spherical coordinates and in computer graphics [16].

In our research, each point of the wavefront diffracted by a circular aperture is a new secondary source that emits spherical wavelets according to the Huygens–Fresnel Principle [2–5]. This makes the spherical harmonics a natural basis for our calculations. The main goals of the spherical harmonic analysis are to expand the polar radius of a point on the surface of the circular aperture that takes equation (2) and to calculate the associated coefficients of the spherical harmonic series [17]:

$$ \hat{\rho}(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_n^{m} Y_n^{m}(\theta, \phi), $$

where $\hat{\rho}(\theta, \phi)$ is the polar radius from the circular aperture center, $\phi, \theta$ are the corresponding spherical coordinates, $\theta \in [0, \pi]$ is polar (colatitudinal) coordinate and represents the angle of diffraction, $\phi \in [0, 2\pi]$ is azimuthal (longitudinal) coordinate, $C_n^{m}$ is the associated spherical harmonic coefficient, and $Y_n^{m}(\theta, \phi)$ are the spherical harmonic functions that are a series of special functions defined on the surface of a sphere and are given by the following relationship [18, 19]:

$$ Y_n^{m}(\theta, \phi) = \sqrt{\frac{(2n + 1)(n - m)!}{4\pi(n + m)!}} P_n^{m}(\cos \theta)e^{im\phi}, $$

where $n$ is the degree of the associated Legendre function, $m$ is the order of the associated Legendre function, and $P_n^{m}(\cos \theta)$ are the associated Legendre functions which are expressed by the following formula [18–21]:

$$ P_n^{m}(x) = \left(1 - x^2\right)^{m/2} \frac{d^{[m]}_n}{dx^{[m]}} \left[\frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n\right]. $$

For $(n = 0, m = 0)$, we get an equation means that the zero degree spherical harmonic representation of all point on the surface of the circular aperture is a sphere, whose radius is determined by $C_0^{0}$:

$$ \hat{\rho}_0 = C_0^{0} Y_0^{0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} C_0^{0}, $$

where $Y_0^{0}(\theta, \phi)$ is spherically symmetric and represents the monopole moment of the function on a sphere and $Y_n^{m}(\theta, \phi)$ represents the dipole moments of the function on a sphere. To simplify the calculations and based on the mathematical properties of the spherical harmonic function [19],
we express the coefficients $C_0^0$, $C_1^{-1}$, $C_1^1$, which represent the optical intensity compounds for $n = 1, m = -1, 0, 1$ with two real and imaginary parts as follows:

\[ C_0^0 = c, \]
\[ C_1^1 = a + ib, \]
\[ C_1^{-1} = -a + ib. \]

Note that $C_n^m$ is always a real number.

$Y_n^m(\theta, \phi)$ are expressed from the standard spherical harmonic function by the following formulas [18, 19]:

\[ Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \]
\[ Y_1^0(\theta, \phi) = \frac{3}{8\pi} \left(\sin \theta \cos \phi + i \sin \theta \sin \phi\right), \]
\[ Y_1^{-1}(\theta, \phi) = \frac{3}{8\pi} \left(\sin \theta \cos \phi - i \sin \theta \sin \phi\right). \]

Relationship (2) is expanded for $n = 0, 1, m = -1, 0, 1$ in the following formula:

\[ \rho(\theta, \phi) = C_0^0 Y_0^0 + C_1^1 Y_1^{-1} + C_1^{-1} Y_1^1. \]

By deriving relationship (11), we obtain:

\[ d\rho = \left[ \frac{3}{2\pi} \sqrt{a^2 + b^2} \cos \theta - \frac{1}{2} \frac{3}{\pi} c \sin \theta \right] d\theta. \]

We calculated the Bessel function $J_0(\nu \rho)$ in relationship (1) based on the following rule (assuming $x = \nu \rho$):

\[ J_0(\nu \rho) = \frac{1}{2\pi} \int_0^{2\pi} e^{j\nu \rho \sin \theta} d\theta. \]

We expanded the exponential function $e^{j\nu \rho \sin \theta}$ in relationship (13) according to the following power series (assuming that $(a = j

\[ e^{a} = \sum_{n=0}^{\infty} a^n. \]

By solving the necessary integral, we obtain:

\[ J_0(\nu \rho) = 1 + i \nu \sqrt{\frac{3}{2\pi}} \sqrt{a^2 + b^2}. \]

We also expanded the exponential function $e^{(i/2)\nu \rho \sin \theta}$ in relationship (11) according to power series (14) (assuming that $(a = (i/2)\nu \rho)$):

\[ e^{(i/2)\nu \rho \sin \theta} \approx 1 + i \frac{\nu}{8\pi} (C_0^0)^2 + i \frac{3\nu}{8\pi} a^2 \cos^2 \theta + i \frac{3\nu}{4\pi} (a^2 + b^2) \sin^2 \theta. \]

Substituting relationships (11), (12), (15), and (16) into relationship (1), after solving the resulting integral, we obtain the following optical wave amplitude $U(\rho, R)$:

\[ U(\rho, R) = A \left( \frac{Z}{R} \right) \left[ ik + \frac{1}{R} \right] \rho \left[ \frac{-\sqrt{3}}{2\pi} c \rho \right] C_0^0 + \frac{3\sqrt{a^2 + b^2}}{16\sqrt{2\pi} \pi} uv \rho \rho (C_0^0)^3 + \frac{3\sqrt{a^2 + b^2}}{16\sqrt{2\pi} \pi} \mu \rho \rho (C_0^0)^3 - \frac{9\sqrt{3}}{128\sqrt{\pi} \pi} \mu \rho \rho (C_0^0)^3 \right]

\[ + \frac{3(a^2 + b^2)}{4\sqrt{2\pi} \pi^2} \mu v \rho \rho C_0^0 + \frac{9\sqrt{3} (a^2 + b^2)^2}{64\sqrt{\pi} \pi} uv \rho \rho - i \left[ \frac{3\sqrt{a^2 + b^2}}{2\sqrt{2\pi} \pi} v \rho \rho (C_0^0)^3 + \frac{3\sqrt{3}}{16\pi} \mu \rho \rho (C_0^0)^3 \right]

\[ + \frac{\sqrt{3}}{16\pi} \mu \rho \rho (C_0^0)^3 + \frac{3\sqrt{a^2 + b^2}}{4\pi^2} \mu v \rho \rho C_0^0 + \frac{9(a^2 + b^2)\sqrt{a^2 + b^2}}{32\sqrt{2\pi}} \mu u \rho + \frac{9\sqrt{3}(a^2 + b^2)}{64\sqrt{2\pi} \pi} \mu \rho \rho (C_0^0)^3 \right]. \]
Relationship (17) expresses the optical field amplitude at any point behind the circular aperture, which represents the sum of spherical wavelets emitted from secondary sources and has the same primary wave frequency. After squaring the previous result (optical wave amplitude), we obtain the following optical intensity:

\[
I(u, v) = A^4 \left(\frac{Z^2}{R^2}\right) \left(k^2 + \frac{1}{R^2}\right) \left\{ \frac{3}{4\pi^2} C^0_c^2 (C^0_0)^2 + \frac{9(a^2 + b^2)}{8\pi^3} v^2 c^2 (C^0_0)^2 + \frac{3\sqrt{a^2 + b^2}}{2\pi^2} u v c^2 C^0_0 \left(\frac{\sqrt{3}}{4\pi\sqrt{2\pi}} (C^0_0)^3\right) \right. \\
+ \frac{\sqrt{3}}{4\pi\sqrt{2\pi}} C^0_0 \left(\frac{9\sqrt{a^2 + b^2}}{32\sqrt{\pi}} c^2 + \frac{\sqrt{3}}{\pi\sqrt{2\pi}} (a^2 + b^2) C^0_0 + \frac{9(a^2 + b^2)\sqrt{a^2 + b^2}}{16\sqrt{\pi}} + \frac{9(a^2 + b^2)}{16\pi^2} u^2 v^2 c^2 \right. \\
+ \left. \left(\frac{C^0_0}{4\pi\sqrt{2\pi}} + \frac{c^2 C^0_0}{4\pi\sqrt{2\pi}} - \frac{3\sqrt{3}}{32\sqrt{\pi}} c^2 + \left(a^2 + b^2\right) C^0_0 + \frac{3\sqrt{3}}{16\sqrt{\pi}} \right)^2 \right. \\
+ \frac{u^2 c^2}{16\pi^2} \left(\frac{\sqrt{3} C^0_0}{4\pi} + \frac{\sqrt{3} c^2 C^0_0}{4\pi} + \frac{\sqrt{3}}{\pi} \left(a^2 + b^2\right) C^0_0 + \frac{9(a^2 + b^2)\sqrt{a^2 + b^2}}{8\sqrt{2}} - \frac{9\sqrt{a^2 + b^2}}{16\sqrt{2}} c^2 \right)^2 \left\}.
\]

(18)

4. Particular Cases

There are three cases of intensity study [3, 4]:

(i) The intensity along the axis (v = 0)

(ii) The intensity in the geometrical focal plane (u = 0)

\[
I(u, 0) = A^4 \left(\frac{Z^2}{R^2}\right) \left(k^2 + \frac{1}{R^2}\right) \left\{ \frac{3}{4\pi^2} C^0_c^2 (C^0_0)^2 + \frac{u^2 c^2}{16\pi^2} \left(\frac{\sqrt{3} C^0_0}{4\pi} + \frac{\sqrt{3} c^2 C^0_0}{4\pi} + \frac{\sqrt{3}}{\pi} \left(a^2 + b^2\right) C^0_0 + \frac{9(a^2 + b^2)\sqrt{a^2 + b^2}}{8\sqrt{2}} - \frac{9\sqrt{a^2 + b^2}}{16\sqrt{2}} c^2 \right)^2 \right\}. 
\]

(19)

We obtain “R” from Figure 1:

\[
R = \frac{Z}{\sqrt{1 - \sin^2(\theta)}} = \frac{Z}{\cos(\theta)} 
\]

(20)

The relationship of the optical coordinate u [8] is given as follows:

\[
u = \frac{2kA^2}{Z \left[1 + (A^2/Z^2)^{(1/2)}\right]} + 1.
\]

(21)

We calculated the values of the coefficients C^m_n from the following renormalized spherical harmonic relationship [17, 22]:

\[
C_n^m (\theta, \varphi) = \sqrt{\frac{4\pi}{2n + 1}} Y_n^m (\theta, \varphi), 
\]

(22)

\[
\begin{align*}
C_0^0 &= 1, \\
C_1^0 &= \cos \theta, \\
C_1^1 &= \frac{1}{\sqrt{2}} (\sin \theta \cos \varphi + i \sin \theta \sin \varphi), \\
C_1^{-1} &= \frac{1}{\sqrt{2}} (\sin \theta \cos \varphi - i \sin \theta \sin \varphi).
\end{align*}
\]

We studied a special case (\varphi = 0) where the studying point from the aperture is located on the x-axis. We obtained the following coefficients:
\[ C_0 = 1, \]
\[ C_1^0 = \cos \theta, \]
\[ C_1^1 = -\frac{1}{\sqrt{2}} \sin \theta, \]
\[ C_1^1 = \frac{1}{\sqrt{2}} \sin \theta. \]  

Substituting relationships (20), (21), and (23) into relationship (18), we obtain the optical intensity along the optical axis:

\[
I(u, 0) = \left( \frac{2d_{RL}^2 \pi^2 A^2}{\lambda Z^2} \cos^4 (\theta) + \frac{A^4 \cos^6 (\theta)}{Z^4} \right) \cdot \left[ \frac{3}{4\pi^2} \cos^2 (\theta) + \frac{9 \sin^2 (\theta) \cos^2 (\theta)}{16\pi^2} \right] \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^2 
\]

where \( d_{RL} \) is a Rayleigh distance (which is an important parameter used in the theory of laser beams; it represents the axial distance from the radiating aperture to the observation plane) and is given by the following formula (see [23], pp.658):

\[ d_{RL} = \frac{2(2A)^2}{\lambda}. \]  

Substituting \( u = 0 \) and relationships (20) and (23) into relationship (18), we obtain the optical intensity in the geometrical focal plane:

\[
I(0, v) = \left( \frac{2d_{RL}^2 \pi^2 A^2}{\lambda Z^2} \cos^4 (\theta) + \frac{A^4 \cos^6 (\theta)}{Z^4} \right) \cdot \left[ \frac{3}{4\pi^2} \cos^2 (\theta) + \frac{9 \sin^2 (\theta) \cos^2 (\theta)}{16\pi^2} \right] \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^2 
\]

Substituting \( (u = \pm v) \):

The third case: the intensity along the boundary of the geometrical shadow \( (u = 0) \): Substituting \( (u = u) \) and relationships (20), (21), and (23) into relationship (18), we obtain the optical intensity along the boundary of the geometrical shadow:

\[
I(u, u) = \left( \frac{2d_{RL}^2 \pi^2 A^2}{\lambda Z^2} \cos^4 (\theta) + \frac{A^4 \cos^6 (\theta)}{Z^4} \right) \cdot \left[ \frac{3}{4\pi^2} \cos^2 (\theta) + \frac{9 \sin^2 (\theta) \cos^2 (\theta)}{16\pi^2} \right] \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^2 
\]

\[ - \frac{3 \sin (\theta) \cos^2 (\theta)}{2\sqrt{2} \pi^2} \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^2 \left( \frac{\sqrt{3}}{4\pi \sqrt{2\pi}} + \frac{\sqrt{3}}{4\pi \sqrt{2\pi}} \cos^2 (\theta) - \frac{9 \sin (\theta) \cos^2 (\theta)}{32\sqrt{2\pi}} + \frac{9 \sin (\theta) \cos^2 (\theta)}{2\pi \sqrt{2\pi}} + \frac{9 \sin (\theta)}{32\sqrt{2\pi}} \right) 
\]

\[ + \frac{9 \sin^2 (\theta) \cos^2 (\theta)}{32\pi^2} \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^4 \left( \frac{1}{4\pi \sqrt{2\pi}} + \frac{\cos^2 (\theta)}{4\pi \sqrt{2\pi}} - \frac{3\sqrt{3}}{32\sqrt{2\pi}} \sin^2 (\theta) + \frac{\sin^2 (\theta)}{2\pi \sqrt{2\pi}} + \frac{3\sqrt{3} \sin (\theta)}{32\sqrt{2\pi}} \right)^2 
\]

\[ + \frac{\cos^2 (\theta)}{16\pi^2} \left( \frac{2\pi d_{RL}}{Z(1 + \sqrt{1 + (A^2/Z^2)})} \right)^2 \left( \frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3} \cos^2 (\theta)}{4\pi} + \frac{\sqrt{3} \sin^2 (\theta)}{2\pi} + \frac{9 \sin (\theta) \cos^2 (\theta)}{32} - \frac{9 \sin (\theta) \cos^2 (\theta)}{32} \right)^2 
\]
5. Numerical Examples and Results

In this paper, we studied the diffraction pattern of light by a circular aperture starting from the first Rayleigh–Sommerfeld diffraction equation. We computed the optical intensity along the optical axis (25), intensity in the geometrical focal plane (26), and intensity along the boundary of the geometrical shadow (27). We obtained the graphs using Maple program [24] for the following numerical values: \( A = (1 \times 10^{-4}) \text{m} \) (radius of the circular aperture), \( \lambda = (6.328 \times 10^{-7}) \text{m} \) wavelength, \( d_R = (12.6 \times 10^{-2}) \text{m} \) Rayleigh distance, \( Z = (135, 115, 197) \times 10^{-4} \text{m} \) distance between the circular aperture and the observation plane, and the optical coordinate value \( \psi = 3.832 \), which represents the first zero in intensity [9]. We got the following graphs.

Since curve of the optical intensity distribution is a visual representation of the diffracted light by a circular aperture, we can explain the resulting curves with a set of the following points:

(i) In Figures 2(a), 2(c), and 2(e), we observe that the presence of bands of frequencies appears clearly in specific and repeated regularly domains. This is due to the introduction of spherical harmonics in calculation of the optical intensity, where spherical harmonics represent a complete set of orthogonal and regular functions defined on the surface of a sphere. They are very similar to Fourier series where the frequencies appear intermittently on the observation screen.

(ii) By truncating a portion of the curves in Figures 2(a), 2(c), and 2(e), we obtain Figures 2(b), 2(d), and 2(f), where the intensity levels appear close to each other at the beginning, and then these levels spaced from each other with an increase in the value of \( Z \). It is worthy to note that the obtained curves in Figures 2(b), 2(d), and 2(f) are identical to those obtained by Sheppard and Hrynevych (see [8], figure 2).

(iii) We also observed that the intensity in relationships (24), (26), and (27) is a sinusoidal change. The resulting relationship contains the functions \( \sin(\theta) \) and \( \cos(\theta) \), which are important for studying periodic phenomena such as waves. The graphs show that the intensity oscillates to reach maximum and minimum values appear as successive bright and dark fringes because of constructive and destructive interference of waves.

(iv) Figures 2 and 3 illustrate the intensity variation versus the ratio \( \left( \frac{Z}{A} \right) \) (the ratio of the distance between the circular aperture and the observation plane to the radius of the circular aperture).

(v) Figure 4 represents the diffraction of a monochromatic light wave by a circular aperture, whose illumination phase is almost constant [23] (where \( D = 2A \) is diameter of the circular aperture and its unit is meter, \( \lambda \) is the wavelength and its unit is meter, \( Z \) is the distance between the circular aperture and the observation plane and its unit is meter, \( d_R \) is Rayleigh distance and its unit is meter, and \( \theta \) is the diffraction angle and its unit is radian). We can distinguish between Fresnel and Fraunhofer diffraction according to the relationship between a Rayleigh distance and the distance \( Z \), if:

\[
\begin{align*}
Z < d_R & \quad \text{Fresnel diffraction} \\
Z > d_R & \quad \text{Fraunhofer diffraction}
\end{align*}
\]

Accordingly, Figures 2(a), 2(c), and 2(e) express Fresnel diffraction at the following distances \( Z = (135, 115, 197) \times 10^{-4} \text{m} \) and Figures 3(a), 3(c), and 3(e) express Fraunhofer diffraction at a distance \( Z = (655 \times 10^{-3}) \text{m} \), which achieves condition (28). We also mention that each band in Figures 2(a), 2(c), 2(e), 3(a), 3(c), and 3(e) is a set of the successive bright and dark fringes. By truncating a portion of the drawing field, we obtain Figures 3(b), 3(d), and 3(f) which show the shape of any band in curves of Figures 3(a), 3(c), and 3(e).

(i) Since the expansion coefficients are the analogs of Fourier coefficients (see [25], pp. 277–278) and as a result of the expansion of the optical intensity compounds, we obtained more complicated shapes, where the plots in Figures 2 and 3 show that the number of resulting intensity peaks increase in comparison with the intensity in the previously studied case [8].

(ii) As shown in Figure 5, the curves in Figures 2(a), 2(c), 2(e), 3(a), 3(c), and 3(e) are succession of diffraction patterns at increasing distance from aperture (Fresnel diffraction pattern is shown in Figure 2; Fraunhofer diffraction pattern is shown in Figure 3).

(iii) In Figures 2(c), 2(d), 3(c), and 3(d) (which represent the optical intensity resulting in the geometrical focal plane), there is a division in the maximum peaks of intensity. This may be due to the interference or diffraction of the higher orders in the optical intensity expansion.

(iv) We started our study from the first Rayleigh–Sommerfeld diffraction formula that describes the Huygens–Fresnel principle. This formula is used to represent the spread of optical fields and it gives a physically realistic prediction for the axial intensity close to a circular aperture, whereas the second Rayleigh–Sommerfeld and Kirchhoff diffraction formulas do not [27]. We also chose the first Rayleigh–Sommerfeld diffraction formula because it is accurate in our study, and importantly, it can be solved readily analytically, whereas Kirchhoff theory is not mathematically consistent, despite its success [28]. In addition, we expanded our study to include the Fresnel and Fraunhofer diffraction regions based on a Rayleigh distance. We obtained a set of results explained above.

(v) Lommel succeeded in expressing the complex disturbance in terms of convergent series of Bessel functions based on the Huygens–Fresnel integral.
Figure 2: Optical intensity resulting from a plane wave diffraction by a circular aperture using spherical harmonics for $n \leq 1$. (a, b) Optical intensity resulting along the optical axis for $Z = (135 \times 10^{-4})$m. (c, d) Optical intensity resulting in the geometrical focal plane for $Z = (115 \times 10^{-4})$m. (e, f) Optical intensity resulting along the boundary of the geometrical shadow for $Z = (197 \times 10^{-4})$m.
Born and Wolf (see [4], pp. 435–442) studied Fraunhofer diffraction by a circular aperture. They discussed the three special cases of light intensity (the intensity along the axis, the intensity in the geometrical focal plane, and the intensity along the boundary of the geometrical shadow). They obtained a set of results as follows.

Intensity distribution along the optical axis is characterized by mathematical Sinc function square \( \left( \frac{\sin}{c} \right)^2 \). Intensity distribution in the geometrical focal plane is characterized by square of Bessel function of first order \( J_1^2 \), while the intensity along the boundary of the geometrical shadow is characterized by the Bessel function of zero order \( J_0 \) and its square \( (J_0)^2 \) and cosine

Figure 3: (a, b) Optical intensity resulting along the optical axis for \( Z = (655 \times 10^{-3}) \) m. (c, d) Optical intensity resulting in the geometrical focal plane for \( Z = (655 \times 10^{-3}) \) m. (e, f) Optical intensity resulting along the boundary of the geometrical shadow for \( Z = (655 \times 10^{-3}) \) m.
function. From these results, they obtained the curves of intensity that have a principal maximum in the center, and then the intensity oscillates and gradually amplitude diminishes with increasing distance \((z)\). In other words, they obtained a bright disk surrounded by concentric bright and dark rings. Then, the intensity of the bright rings decreases rapidly with their radius. Normally, only the first or second rings being bright enough are visible to the naked eye. So, the total incident energy is contained within the central core of the diffraction pattern. In this paper, we expanded the polar radius of a point on the surface of the circular aperture based on spherical harmonics. We obtained intensity containing sine and cosine functions of higher orders \((\text{relationships (24), (26) and (27)})\). Because of this expansion, we obtained intensity distribution curves versus the ratio \(Z/A\) (Figures 2 and 3), in which the successive bright and dark fringes are distributed larger and more clearly, which provides a better view of the image to the naked eye. In case of studying without taking into account the mathematical expansion process of the polar radius (relationship (2)) which we did, the relationship of intensity will be reduced to a simpler relationship (see Ref. \([6]\), equation (21)).

6. Discussion

This paper provides an insight into the effect of expansion coefficients on optical intensity distribution. It has been observed that there is a good agreement between our results and the previous results of Born and Wolf \([4]\) and Sheppard and Hrynevych \([8]\), if we neglect the mathematical expansion process in relationship (2) and calculate the integral in relationship (1) in terms of Lommel functions. In addition, we studied the special intensity cases. We provided numerical comparison examples, and we used Maple program to represent these results. We noticed that the expansion we made using spherical harmonic analysis led to an increase in the number of fringes bright enough to be visible to the naked eye. Bands of frequencies also appeared in the curves of Figures 2(a), 2(c), 2(e), 3(a), 3(c), and 3(e) due to the introduction of spherical harmonics in the calculations. Finally, as practical applications on optical devices such as the camera or telescope, we can obtain more accurate and clear images, and this is what modern science seeks to accomplish.

7. Conclusion

In this work, we studied the intensity distributions of a monochromatic light beam diffracted by a circular aperture in a homogeneous medium. The relationship of optical intensity was obtained using expansion of the spherical harmonics series in three-dimensional space for \(n \leq 1\), where spherical harmonics form an orthonormal basis on the unit sphere. They can be considered as signal-processing tools on the unit sphere, analogous to the Fourier series or sines and cosines on the line or circle, while the Zernike polynomials (which are closely related to the general spherical harmonics) are especially useful in numerical calculations, but any wavefront aberration can be decomposed in terms of them, and advantages of orthogonality are lost in noncircular cases \([29]\).

We also calculated the optical wave amplitudes \(C_0, C_1, C_2, C_1^{-1}\). We discussed these results for Rayleigh distance \(d_{\text{RL}}\). We used the correction \([8]\) of the optical coordinate \(u\) in order to calculate the optical intensity along the optical axis and along the boundary of the geometrical shadow. We also used a specific value for the optical coordinate \(v\) \([9]\) in order to calculate the optical intensity in the geometrical focal plane. We performed a set of calculations
using spherical harmonics expansion and applied this method in the illuminated region from the near field to the far field.

Finally, we recommend extending spherical harmonics to higher orders for \( n > 1 \) to obtain more accurate results and changing the geometry shape of the used aperture and recalculating again.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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