Generalized Euler-Lagrange equations for variational problems with scale derivatives

Ricardo Almeida  Delfim F. M. Torres
ricardo.almeida@ua.pt  delfim@ua.pt
Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal

Abstract
We obtain several Euler-Lagrange equations for variational functionals defined on a set of Hölder curves. The cases when the Lagrangian contains multiple scale derivatives, depends on a parameter, or contains higher-order scale derivatives are considered.

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1 Introduction
In 1992 L. Nottale introduced the theory of scale-relativity without the hypothesis of space-time differentiability [6, 7]. A rigorous mathematical foundation to Nottale’s scale-relativity theory was then given by J. Cresson in 2005 [3, 4]. Roughly speaking, the calculus of variations developed in [3] cover sets of non differentiable curves, by substituting the classical derivative by a new complex operator, known as the scale derivative. Here we proceed with the theory started in [3] and continued in [1, 5, 6], by presenting several Euler-Lagrange equations in the class of Hölderian curves and generalizing the previous results. For the necessary terminology and motivation to the study of such non-differentiable calculus of variations we refer the reader to [1, 2, 3, 6].

The paper is organized as follows. In Section 2 we review the necessary notions of scale calculus. Our results are then given in Section 3 (i) in §3.1 we prove the Euler-Lagrange equation for functionals defined by multiple scale derivatives; (ii) in §3.2 we deduce the Euler-Lagrange equation with a dependence on a complex parameter ξ; (iii) finally, in §3.3 we characterize the extremals of variational functionals containing higher-order scale derivatives.

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2 Preliminaries

In what follows, function $f \in C^0$ and $\epsilon > 0$ is a real number. The $\epsilon$-left and $\epsilon$-right quantum derivatives are defined by the formulas

$$\Delta^- f(x) = \frac{f(x - \epsilon) - f(x)}{\epsilon} \quad \text{and} \quad \Delta^+ f(x) = \frac{f(x + \epsilon) - f(x)}{\epsilon},$$

respectively. The $\epsilon$ scale derivative of $f$ at $x$ is given by

$$\Box_{\epsilon} f(x) = \frac{1}{2}(\Delta^+ f(x) + \Delta^- f(x)) - i \frac{1}{2}(\Delta^+ f(x) - \Delta^- f(x)).$$

**Theorem 1** ([3]). Let $f, g \in C^0$ and $\epsilon > 0$. Then,

$$\Box_{\epsilon} (f \cdot g) = \Box_{\epsilon} f \cdot g + f \cdot \Box_{\epsilon} g + i \frac{\epsilon}{2}(\Box_{\epsilon} f \Box_{\epsilon} g - \Box_{\epsilon} f \Box_{\epsilon} g - \Box_{\epsilon} f \Box_{\epsilon} g),$$

where $\Box_{\epsilon} (\cdot)$ is the complex conjugate of $\Box_{\epsilon} (\cdot)$.

The space of curves under consideration is the space of Hölderian curves with Hölder exponent $\alpha$. More precisely, given a real number $\alpha \in (0, 1)$ and a sufficiently small real parameter $\epsilon$, $0 < \epsilon < 1$, we define

$$C^\alpha_\epsilon(a,b) = \{y : [a - \epsilon, b + \epsilon] \to \mathbb{R} \mid y \in H^\alpha\}.$$

In [3] variational functionals of the type $\Phi(y) = \int_a^b f(x,y(x),\Box_{\epsilon}y(x)) \, dx$ are studied in the class $y \in C^\alpha_\epsilon(a,b)$. As variation curves, it is considered those of the following type: let $\beta$ be a positive real satisfying condition

$$\beta \geq (1-\alpha)1_{[0,1/2]} + \alpha 1_{[1/2,1]} \cdot$$

and $h \in C^\beta_\epsilon(a,b)$ be such that $h(a) = 0 = h(b)$. A curve of the form $y + h$ is called a variation of $y$. One says that $\Phi$ is differentiable on $C^\alpha_\epsilon(a,b)$ if, for all curves $y \in C^\alpha_\epsilon(a,b)$ and for all variations $y + h$,

$$\Phi(y + h) - \Phi(y) = F_y(h) + R_y(h),$$

where $F_y$ is a linear operator on the space $C^\beta_\epsilon(a,b)$ and $R_y(h) = O(h^2)$. A curve $y$ is an extremal for $\Phi$ on $C^\beta_\epsilon(a,b)$ if $[F_y(h)]_\epsilon = 0$ for all $\epsilon > 0$ and all $h \in C^\beta_\epsilon(a,b)$.

**Theorem 2** ([3]). The curve $y$ is an extremal for $\Phi$ on $C^\beta_\epsilon(a,b)$ if and only if

$$[\partial_2 f(x, y(x), \Box_{\epsilon}y(x)) - \Box_{\epsilon} (\partial_3 f)(x, y(x), \Box_{\epsilon}y(x))]_\epsilon = 0$$

for every $\epsilon > 0$.

3 Main Results

We obtain several generalizations of Theorem 2. Theorem 4 of [3.1] coincides with Theorem 2 in the particular case $n = 1$; Theorem 6 of [3.2] coincides with Theorem 2 in the particular case when the Lagrangian $L$ does not depend on the parameter $\xi$; Theorem 8 of [3.3] coincides with Theorem 2 in the particular case when the Lagrangian $L$ does not depend on $\Box^2_\epsilon y(x)$ and $\xi$. **
3.1 Euler-Lagrange equation for multiple scale derivatives

Let $\epsilon := (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$ be a vector such that $0 < \epsilon_k \ll 1$ for all $k \in \{1, \ldots, n\}$. For continuous functions $f$ let

$$\Delta^\epsilon_k f(x) = \frac{f(x + \epsilon_k) - f(x)}{\epsilon_k}, \quad \sigma = \pm.$$

We define the $\epsilon_k$ scale derivative of $f$ at $x$ by the rule

$$\Box \epsilon_k f(x) = \frac{1}{2} \left( \Delta^+\epsilon_k f(x) + \Delta^-\epsilon_k f(x) \right) - i \frac{1}{2} \left( \Delta^\epsilon_k f(x) - \Delta^-\epsilon_k f(x) \right).$$

Let $\alpha \in (0, 1)$ be a real, $\epsilon_M := \max\{\epsilon_1, \ldots, \epsilon_n\}$, and

$$C^\alpha_{\epsilon_M}(a, b) = \{ y : [a - \epsilon_M, b + \epsilon_M] \to \mathbb{R} | y \in H^\alpha \}.$$

We consider the following type of functionals: $\Phi : C^\alpha_{\epsilon_M}(a, b) \to \mathbb{C}$ of form

$$\Phi(y) = \int_a^b L(x, y(x), \Box \epsilon_i y(x), \ldots, \Box \epsilon_n y(x)) \, dx,$$

where $L : \mathbb{R}^2 \times \mathbb{C}^n \to \mathbb{C}$ is a given $C^1$ function, called the Lagrangian. As in [3], we assume the Lagrangian $L$ to satisfy

$$\|DL(x, y(x), \Box \epsilon_i y(x), \ldots, \Box \epsilon_n y(x))\| \leq C$$

for all $x$ and $\epsilon_1 > 0, \ldots, \epsilon_n > 0$, where $C$ is a positive constant, $D$ denotes the differential, and $\| \cdot \|$ is a norm for matrices.

**Definition 1.** We say that $\Phi$, as in [1], is differentiable on $C^\alpha_{\epsilon_M}(a, b)$ if for all curves $y \in C^\alpha_{\epsilon_M}(a, b)$ and for all variations $y + h$, $h \in C^3_{\epsilon_M}(a, b)$, the equality

$$\Phi(y + h) - \Phi(y) = F_y(h) + R_y(h)$$

holds with $F_y$ a linear operator on the space $C^3_{\epsilon_M}(a, b)$ and $R_y(h) = O(h^2)$.

Let us determine the expression of $F_y(h)$. To simplify notation, let $u := (x, y, \Box \epsilon_1 y, \ldots, \Box \epsilon_n y)$. Then,

$$\Phi(y + h) - \Phi(y) = \int_a^b \left[ L(x, y + h, \Box \epsilon_i y + \Box \epsilon_i h, \ldots, \Box \epsilon_n y + \Box \epsilon_n h) - L(x, y, \Box \epsilon_i y, \ldots, \Box \epsilon_n y) \right] \, dx$$

$$= \int_a^b \left[ \partial_2 L(u) \cdot h + \partial_3 L(u) \cdot \Box \epsilon_i h + \cdots + \partial_{n+2} L(u) \cdot \Box \epsilon_n h \right] \, dx + O(h^2).$$

Integrating by parts (see Theorem [1]), we obtain:

$$F_y(h) = \int_a^b \left[ \partial_2 L(u) \cdot h + \Box \epsilon_i (\partial_3 L(u) \cdot h) - \Box \epsilon_i (\partial_3 L(u) \cdot h) - i \frac{\epsilon_1}{2} \sum \Box \epsilon_i (\partial_3 L(u), h)$$

$$+ \cdots + \Box \epsilon_n (\partial_{n+2} L(u) \cdot h) - \Box \epsilon_n (\partial_{n+2} L(u) \cdot h) - i \frac{\epsilon_n}{2} \sum \Box \epsilon_n (\partial_{n+2} L(u), h) \right] \, dx,$$
We denote by $$\Sigma_{\epsilon_k}(p,q) = \Box_{\epsilon_k}p\Box_{\epsilon_k}q - \Box_{\epsilon_k}p\Box_{\epsilon_k}q - \Box_{\epsilon_k}p\Box_{\epsilon_k}q.$$ Therefore,

$$F_{y}(h) = \int_{a}^{b} \left[ \partial_{3}L(u) - \Box_{\epsilon_{1}}(\partial_{3}L(u)) - \cdots - \Box_{\epsilon_{n}}(\partial_{n+2}L(u)) \right] \cdot h \, dx + \int_{a}^{b} \Box_{\epsilon_{1}}(\partial_{3}L(u) \cdot h) + \cdots + \Box_{\epsilon_{n}}(\partial_{n+2}L(u) \cdot h) \, dx - i \int_{a}^{b} \left[ \frac{\epsilon_{1}}{2} \Sigma_{\epsilon_{1}}(\partial_{3}L(u),h) + \cdots + \frac{\epsilon_{n}}{2} \Sigma_{\epsilon_{n}}(\partial_{n+2}L(u),h) \right] \, dx.$$

We just proved the following result:

**Theorem 3.** For all $$\epsilon_{1} > 0, \ldots, \epsilon_{n} > 0$$, the functional $$\Phi$$ defined by (1) is differentiable, and

$$F_{y}(h) = \int_{a}^{b} \left[ \partial_{3}L(u) - \Box_{\epsilon_{1}}(\partial_{3}L(u)) - \cdots - \Box_{\epsilon_{n}}(\partial_{n+2}L(u)) \right] \cdot h \, dx + \int_{a}^{b} \Box_{\epsilon_{1}}(\partial_{3}L(u) \cdot h) + \cdots + \Box_{\epsilon_{n}}(\partial_{n+2}L(u) \cdot h) \, dx - i \int_{a}^{b} \left[ \frac{\epsilon_{1}}{2} \Sigma_{\epsilon_{1}}(\partial_{3}L(u),h) + \cdots + \frac{\epsilon_{n}}{2} \Sigma_{\epsilon_{n}}(\partial_{n+2}L(u),h) \right] \, dx.$$

**Definition 2.** Let $$a(\epsilon), \epsilon = (\epsilon_{1}, \ldots, \epsilon_{n})$$, be a real or complex valued function. We denote by $$[\cdot]_{\epsilon}$$ the linear operator such that

$$a(\epsilon) - [a(\epsilon)]_{\epsilon} \xrightarrow{\epsilon \to 0} 0 \quad \text{and} \quad [a(\epsilon)]_{\epsilon} = 0 \text{ if } \lim_{\epsilon \to 0} a(\epsilon) = 0.$$

**Definition 3.** A curve $$y$$ is an extremal for $$\Phi$$ on $$C_{M\alpha}^{b}(a,b)$$ if $$[F_{y}(h)]_{\epsilon} = 0$$ for all $$\epsilon_{1} > 0, \ldots, \epsilon_{n} > 0$$ and $$h \in C_{M\alpha}^{b}(a,b).$$

We now prove an Euler-Lagrange type equation for functionals of type (1).

**Theorem 4.** A curve $$y$$ is an extremal for $$\Phi$$ defined by (1) on $$C_{M\alpha}^{b}(a,b)$$ if and only if

$$\left[ \partial_{3}L(u) - \sum_{k=1}^{n} \Box_{\epsilon_{k}}(\partial_{k+2}L)(u) \right]_{\epsilon} = 0 \quad (2)$$

for all $$\epsilon_{1} > 0, \ldots, \epsilon_{n} > 0$$, where $$u = (x, y(x), \Box_{\epsilon_{1}}y(x), \ldots, \Box_{\epsilon_{n}}y(x)).$$

**Proof.** Let $$k \in \{1, \ldots, n\}$$. Then, for all $$t \in [a,b]$$

$$\sup_{x \in [t,t+\epsilon\kappa] } |\partial_{k+2}L(x, y(x), \Box_{\epsilon_{1}}y(x), \ldots, \Box_{\epsilon_{n}}y(x))| \leq C_{k}\epsilon_{m}^{-1}.$$
for some constant $C_k$, where $\epsilon_m := \min \{\epsilon_1, \ldots, \epsilon_n\}$. Therefore, as $\epsilon$ goes to zero, the quantities

$$\int_a^b \Box_\epsilon (\partial_{k+2}L(u) \cdot h) \, dx \quad \text{and} \quad \epsilon_k \int_a^b \Sigma_\epsilon (\partial_{k+2}L(u), h) \, dx$$

also go to zero (cf. [3, Lemma 3.2]). Applying the bracket operator $[\cdot]_\epsilon$ to $F_y(h)$, the formula (2) follows from the fundamental lemma of the calculus of variations. \hfill \Box

In [1] we study the isoperimetric problem in the scale calculus context, considering integral constraints containing scale derivatives. There, with the help of an appropriate auxiliary function, we prove a necessary condition for extremals. We now extend the main result of [1] to functionals containing multiple scale derivatives.

**Definition 4.** Let $\Phi(y) = \int_a^b L(x, y(x), \Box_{\epsilon_1} y(x), \ldots, \Box_{\epsilon_n} y(x)) \, dx$ and $\Psi(y) = \int_a^b g(x, y(x), \Box_{\epsilon_1} y(x), \ldots, \Box_{\epsilon_n} y(x)) \, dx$ be two functionals on $C^\alpha_{\epsilon M} (a, b)$. A curve $y$ is called an extremal of $\Phi$ subject to the constraint $\Psi(y) = c$, $c \in \mathbb{C}$, if for all $m \in \mathbb{N}$ and all variations $\hat{y} = y + \sum_{k=1}^m h_k$, where $(h_k)_{1 \leq k \leq m} \in C^\beta_{\epsilon M} (a, b)$ are such that $\Psi(\hat{y}) = c$, one has $[F_y(h_k)]_\epsilon = 0$ for all $\epsilon_1 > 0, \ldots, \epsilon_n > 0$ and all $k \in \{1, \ldots, m\}$.

Using the techniques in the proof of Theorem 4 and in [1, Theorem 4], the following result can be easily obtained.

**Theorem 5.** Suppose that $y \in C^\alpha_{\epsilon M} (a, b)$ is an extremal for the functional $\Phi$ on $C^\beta_{\epsilon M} (a, b)$ subject to the constraint $\Psi(y) = c$, $c \in \mathbb{C}$. If

1. $y$ is not an extremal for $\Psi$;

2. both limits

$$\lim_{\epsilon \to 0} \max_{x \in [a, b]} |\partial_2 L - \Box_{\epsilon_1} (\partial_3 L) - \ldots - \Box_{\epsilon_n} (\partial_{n+2} L)|$$

and

$$\lim_{\epsilon \to 0} \max_{x \in [a, b]} |\partial_2 g - \Box_{\epsilon_1} (\partial_3 g) - \ldots - \Box_{\epsilon_n} (\partial_{n+2} g)|$$

are finite along $y$;

then there exists $\lambda \in \mathbb{C}$ such that

$$[\partial_2 K - \Box_{\epsilon_1} (\partial_3 K) - \ldots - \Box_{\epsilon_n} (\partial_{n+2} K)]_\epsilon = 0$$

holds along the curve $y$, where $K = L - \lambda g$. 

5
3.2 Dependence on a parameter

We now characterize the extremals in the case when the variational functional depends on a complex parameter $\xi$. We consider functionals of the form

$$\Phi(y, \xi) = \int_a^b L(x, y(x), \square_x y(x), \xi) \, dx,$$

where $(y, \xi) \in C^\alpha(a, b) \times \mathbb{C}$. We say that $\Phi$ is differentiable if for all $(y, \xi) \in C^\alpha(a, b) \times \mathbb{C}$ and for all $(h, \delta) \in C^\beta_c(a, b) \times \mathbb{C}$ one has

$$\Phi(y + h, \xi + \delta) - \Phi(y, \xi) = F(y, \xi)(h, \delta) + R(y, \xi)(h, \delta),$$

where $F(y, \xi)$ is a linear operator on the space $C^\alpha(a, b) \times \mathbb{C}$ and $R(y, \xi)(h, \delta) = O(|(h, \delta)|^2)$. With similar calculations as done before, we deduce:

$$\int_a^b \left[ L(x, y + h, \square_x y + \square_x h, \xi + \delta) - L(x, y, \square_x y, \xi) \right] \, dx$$

$$\begin{aligned}
&= \int_a^b \left[ \partial_2 L(u) \cdot h + \partial_3 L(u) \cdot \square_x h + \partial_4 L(u) \cdot \delta \right] \, dx \\
&= \int_a^b \left[ \partial_2 L(u) - \square_x (\partial_3 L(u)) \right] \cdot h \, dx + \int_a^b \left[ \partial_4 L(u) \right] \cdot \delta \, dx,
\end{aligned}$$

where $u := (x, y(x), \square_x y(x), \xi)$. For $\delta = 0$ we obtain the Euler-Lagrange equation $[\partial_2 L(u) - \square_x (\partial_3 L(u))] = 0$ for all $\epsilon > 0$. For $h = 0$ we get $\int_a^b [\partial_4 L(u)] \, dx = 0$. In summary, we have:

**Theorem 6.** The pair $(y, \xi)$ is an extremal for $\Phi$ given by (3), i.e., $[F(y, \xi)(h, \delta)] = 0$, if and only if

$$\begin{cases}
[\partial_2 L(u) - \square_x (\partial_3 L(u))] = 0 \\
\int_a^b [\partial_4 L(u)] \, dx = 0
\end{cases}$$

for all $\epsilon > 0$, where $u = (x, y(x), \square_x y(x), \xi)$.

**Example 1.** Let $\Phi$ be given by the expression

$$\Phi(y, \xi) = \int_{-1}^1 (\square_x y - \square_x |x|)^2 + (\xi x)^2 \, dx.$$

Then, $(y, \xi) = (|x|, 0)$ is an extremal of $\Phi$:

$$[\partial_2 L(u) - \square_x (\partial_3 L(u))] = [\square_x (2(\square_x y - \square_x |x|))] = 0$$

and

$$\int_{-1}^1 [\partial_4 L(u)] \, dx = \int_{-1}^1 [2\xi x^2] \, dx = 0.$$

**Example 2.** Consider now the functional $\Phi(y, \xi) = \int_{-1}^1 (\xi \cdot \square_x y - \square_x |x|)^2 \, dx$, $\xi \in \mathbb{C}$. Similarly as in Example 1, it can be proved that $(y, \xi) = (|x|, 1)$ is an extremal of $\Phi$. Observe that if we substitute $y$ by $|x|$ and $\xi \in \mathbb{R}$ in $\Phi$, simple calculations show that $\Phi(|x|, \xi) = 2(\xi - 1)^2(1 - e)$. This function has a global minimizer for $\xi = 1$.  

6
3.3 Higher-order Euler-Lagrange equation

Let $\alpha \in (0, 1)$, $\epsilon > 0$, $n \in \mathbb{N}$, and $f \in C^{n-1}[a, b]$ be a real valued function.

**Definition 5.** For $k = 1, \ldots, n$ let

$$\Delta^k_\epsilon f(x) = \frac{f^{(|k|-1)}(x + \epsilon) - f^{(|k|-1)}(x)}{\epsilon}, \quad \sigma = \pm.$$

We define the $k$th $\epsilon$-scale derivative of $f$ at $x$ by

$$\square^k_\epsilon f(x) = \frac{1}{2} (\Delta^k_\epsilon + f(x) + \Delta^k_\epsilon - f(x)) - \frac{1}{2} (\Delta^k_\epsilon + f(x) - \Delta^k_\epsilon - f(x)).$$

Consider variational functionals of the form

$$\Phi(y) = \int_a^b L(x, y(x), \square_\epsilon^1 y(x), \ldots, \square_\epsilon^n y(x)) \, dx,$$

for curves $y$ of class $C^{n-1}$ and $y, \square_\epsilon^1 y, \ldots, \square_\epsilon^n y \in C^\alpha_c(a, b)$. Observe that, as $\epsilon \to 0$, we obtain the standard functional of the calculus of variations with higher-order derivatives:

$$\Phi(y) = \int_a^b L(x, y(x), y'(x), \ldots, y^{(n)}(x)) \, dx.$$ We study the case $n = 2$:

$$\Phi(y) = \int_a^b L(x, y(x), \square_\epsilon^1 y(x), \square_\epsilon^2 y(x)) \, dx.$$ (4)

Results for the general case are easily proved by induction. Let $h$ be a function of class $C^1$ such that $h, \square_\epsilon^1 h \in C^\alpha_c(a, b)$, $h(a) = 0 = h(b)$ and $h'(a) = 0 = h'(b)$. Observe that $\square_\epsilon^2 y = \square_\epsilon^1 y'$ and $\square_\epsilon^2 h = \square_\epsilon^1 h'$. Thus, by Theorem [1] and the standard integration by parts formula (here we are assuming that $L$ and $y$ are at least of class $C^2$),

$$[\Phi(y + h) - \Phi(y)]_\epsilon$$

$$= \int_a^b \left[ L(x, y + h, \square_\epsilon^2 y + \square_\epsilon^1 h, \square_\epsilon^2 y + \square_\epsilon^2 h) - L(x, y, \square_\epsilon^2 y, \square_\epsilon^2 y) \right]_\epsilon \, dx$$

$$= \int_a^b \left[ \partial_2 L \cdot h + \partial_3 L \cdot \square_\epsilon^1 h + \partial_4 L \cdot \square_\epsilon^2 h \right]_\epsilon \, dx + O(h^2)$$

$$= \int_a^b \left[ \partial_2 L \cdot h + \partial_3 L \cdot \square_\epsilon^1 h + \partial_4 L \cdot \square_\epsilon^1 h' \right]_\epsilon \, dx + O(h^2)$$

$$= \int_a^b \left[ \partial_2 L \cdot h - \square_\epsilon^1 (\partial_3 L) \cdot h - \square_\epsilon^1 (\partial_4 L) \cdot h' \right]_\epsilon \, dx + O(h^2)$$

$$= \int_a^b \left[ \partial_2 L - \square_\epsilon^1 (\partial_3 L) + (\square_\epsilon^1 (\partial_4 L))' \right]_\epsilon \, h \, dx + O(h^2).$$

We just deduced the Euler-Lagrange equation for (4):
**Theorem 7.** Let $L$ be a Lagrangian of class $C^2$, and $\Phi$ as in (4) be defined on the class $C^2$ of curves such that $y, \Box^1 y \in C^2(a,b)$. Function $y$ is an extremal for $\Phi$ if and only if

$$\left[ \partial_2^2 L - \Box^1 (\partial_3 L) + (\Box^1 (\partial_4 L))^1 \right]_\epsilon = 0$$

for all $\epsilon > 0$.

**Remark 1.** In contrast with the classical theory of the calculus of variations for functionals containing second-order derivatives, where typically admissible functions are of class $C^4[a,b]$, here it is enough to work with $C^2$ curves.

We can easily include the case when the functional depends on a complex parameter $\xi$, as was done in Section 3.2.

**Theorem 8.** Let $\Phi$ be the functional defined by

$$\Phi(y, \xi) = \int_a^b L(x, y(x), \Box^1 y(x), \Box^2 y(x), \xi) \, dx.$$ 

The pair $(y, \xi)$ is an extremal for $\Phi$ if and only if

$$\left\{ \int_a^b \left[ \partial_2^2 L - \Box^1 (\partial_3 L) + (\Box^1 (\partial_4 L))^1 \right]_\epsilon \right\} = 0$$

for all $\epsilon > 0$, where $u = (x, y(x), \Box^1 y(x), \Box^2 y(x), \xi)$.

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