K₀ OF INVARIANT RINGS AND NONABELIAN H¹

MARTIN LORENZ

Department of Mathematics
Temple University
Philadelphia, PA 19122-6094
lorenz@math.temple.edu

Abstract. We give a description of the kernel of the induction map K₀(R) → K₀(S), where S is a commutative ring and R = S^G is the ring of invariants of the action of a finite group G on S. The description is in terms of H¹(G, GL(S)).

Introduction

This article is concerned with the relationship between K₀(S) and K₀(R), where S is a commutative ring and R = S^G denotes the subring of invariants under the action of a finite group G on S.

Specifically, working under the assumption that the trace tr : S → R, s ↦ ∑_{g ∈ G} s^g, is surjective, we shall study the kernel of the induction map

\[ \text{Ind}_R^S = K₀(f) : K₀(R) → K₀(S) \]

that is associated with the inclusion f : R ↪ S. We will describe an embedding of Ker(Ind_­R^S) into the cohomology set H¹(G, GL(S)). Moreover, we will endow H¹(G, GL(S)) with a natural commutative monoid structure, essentially coming from the “block diagonal” maps GL_n × GL_m → GL_{n+m} ↪ GL, such that our embedding identifies Ker(Ind_­R^S) with the group of units U(H¹(G, GL(S))). To further describe this unit group, we define S_H, for any subgroup H of G, to be the factor of S modulo the intersection of all maximal ideals of S whose inertia group contains H. Letting ρ_H : H¹(G, GL(S)) → H¹(H, GL(S_H)) denote the map that is given by restriction from G to H and the canonical map S → S_H, our main result reads as follows.

Theorem. Ker(Ind_­R^S) ∼= U(H¹(G, GL(S))) = ∩_H Ker ρ_H, where H ranges over all cyclic (or, equivalently, all) subgroups of G.

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The theorem follows via direct limits from a corresponding result, with \( \text{GL}_n(\cdot) \) in place of \( \text{GL}(\cdot) \), for the kernel of the maps \( \mathbf{P}_n(f) : \mathbf{P}_n(R) \to \mathbf{P}_n(S) \), where \( \mathbf{P}_n(\cdot) \) denotes the set of isomorphism classes of f.g. projective modules of constant rank \( n \).

As applications, we present a version of Hilbert’s Theorem 90 for Galois actions on commutative rings and quickly derive the (known) structure of the Picard groups of linear and multiplicative invariants \((\mathbb{K}, \mathbb{L})\). Some open problems are also discussed.

Notations and Conventions. Throughout this note, 
- \( S \) will be a ring, assumed commutative from \( \S 2.3 \) onwards,
- \( G \) will be a finite group acting by automorphisms on \( S \); the action will be written \( s \mapsto s^g \), and
- \( R = S^G \) will denote the ring of \( G \)-invariants in \( S \).

We make the standing hypothesis that the trace map \( \text{tr} : S \to R, \ s \mapsto \sum_{g \in G} s^g \) is surjective or, equivalently:

\[ (*) \quad \text{There exists an element } x \in S \text{ with } \text{tr}(x) = 1. \]

1. Nonabelian \( H^1 \)

1.1. Definition. We recall the definition of nonabelian \( H^1 \) following \[ SE \] \( \S 1.5 \) (or \[ BS \], \[ Se2 \] p. 123ff), except that our group actions are on the right.

Let \( X \) be a \( G \)-group, with \( G \)-action written as \( x \mapsto x^g \). A \((1-)\)cocycle is a map \( d : G \to X \) satisfying

\[ d(gg') = d(g)^g d(g') \quad (g, g' \in G). \]

The set of 1-cocycles of \( G \) in \( X \) will be denoted by \( Z^1(G, X) \). Two cocycles \( d, e \in Z^1(G, X) \) are called cohomologous if there exists an element \( x \in X \) satisfying \( d(g) = e(g) x^g x^{-1} \) for all \( g \in G \). This defines an equivalence relation on \( Z^1(G, X) \). The set of equivalence classes is

\[ H^1(G, X), \]

a pointed set with distinguished element the class of the unit cocycle \( 1(g) = 1 \) for all \( g \in G \).

Remark. \( H^1(G, X) \) parametrizes the conjugacy classes of complements of \( X \) in the split extension \( X \rtimes G \), exactly as in the familiar special case where \( X \) is abelian (cf. \[ KG \], 11.1.2, 11.1.3).

1.2. Examples. (1) Suppose \( G \) acts trivially on \( X \). Then \( Z^1(G, X) = \text{Hom}(G, X) \) and

\[ H^1(G, X) = \text{Hom}(G, X)/X, \]

with \( X \) acting by conjugation on \( \text{Hom}(G, X) \). The distinguished class consists of \( 1 \) alone.

(2) If \( G = \langle g \rangle \) is cyclic of order \( m \), then each cocycle \( d \in Z^1(G, X) \) is determined by the element \( x = d(g) \in X \), and the eligible elements of \( X \) are precisely those satisfying
the condition \( x^{q_m - 1}x^{q_m - 2}\ldots x^q = 1 \). Moreover, if \( d, e \in Z^1(G, X) \) correspond in this manner to \( x, y \in X \), respectively, then \( d \) and \( e \) are cohomologous precisely if there exists \( z \in X \) with \( x = z^qyz^{-1} \). Thus, writing \( \sim \) for the equivalence relation on \( X \) determined by this condition, we have

\[
H^1(G, X) \cong \{ x \in X : x^{q_m - 1}x^{q_m - 2}\ldots x^q = 1 \} / \sim.
\]

(3) If the order of \( X \) is finite and coprime to \( |G| \) then \( H^1(G, X) \) is trivial, by the uniqueness part of the Schur-Zassenhaus Theorem (cf. [Ro], 9.1.2).

(4) If \( X \) is a linear algebraic group over an algebraically closed field whose characteristic does not divide \( |G| \) and \( G \) acts algebraically on \( X \) then \( H^1(G, X) \) is finite. This is essentially due to A. Weil who explicitly dealt with the case of a trivial \( G \)-action on \( X \) ([Bou3], 9.1.2, cf. also [Bou2]). The general case is an easy consequence.

1.3. Functoriality and direct limits. Suppose that we are given a homomorphism of groups \( \alpha : G \to G' \) and a \( G' \)-group \( X' \). Then \( X' \) can be viewed as a \( G \)-group via \( \alpha \).

Any map of \( G \)-groups (i.e., any group homomorphism compatible with the \( G \)-actions) \( f : X' \to X \) gives rise to a map \( Z^1(G', X') \to Z^1(G, X) \), \( d \mapsto f \circ d \circ \alpha \), and this map passes down to a map of pointed sets

\[
(\alpha, f)^1_* : H^1(G', X') \to H^1(G, X).
\]

In particular, we have the restriction maps \( H^1(G, X) \to H^1(H, X) \) for subgroups \( H \) of \( G \) and the inflation maps \( H^1(G/N, X^N) \to H^1(G, X) \) for normal subgroups \( N \). Here, \( X^N \) denotes the \( N \)-invariants in \( X \). We will write \( f^1_* \) for \( (\text{Id}_G, f)^1_* \).

The following easy lemma is surely well-known, but I am not aware of a reference. (For commutative cohomology, see [Br], Prop. (4.6) on p. 195.)

**Lemma.** Let \( (X_n, f_{mn}) \) be a direct system of \( G \)-groups and let \( X = \varinjlim X_n \) be the direct limit with its induced \( G \)-action (cf. [Bou3], p. A I.117). If all \( f_{mn} : X_n \to X_m \) \((m \geq n)\) are injective then \( H^1(G, X) \cong \varprojlim H^1(G, X_n) \) (as pointed sets).

**Proof.** Put \( \varphi_{mn} = (f_{mn})^1_* : H^1(G, X_n) \to H^1(G, X_m) \) and \( \varphi_n = (f_n)^1_* \), where \( f_n : X_n \to X \) is the canonical map. Then the relations \( f_m \circ f_{mn} = f_n \) entail \( \varphi_m \circ \varphi_{mn} = \varphi_n \), and so there is a unique map \( \varphi : \varinjlim H^1(G, X_n) \to H^1(G, X) \) with \( \varphi \circ \psi_n = \varphi_n \), where \( \psi_n : H^1(G, X_n) \to \varprojlim H^1(G, X_n) \) is the canonical map. We show that \( \varphi \) is bijective; the fact that \( \varphi \) respects distinguished elements is clear.

For surjectivity, let \( d \in Z^1(G, X) \) be given. Since \( G \) is finite, there is an \( n \) with \( d(G) \subseteq f_n(X_n) \), and so the class of \( d \) in \( H^1(G, X) \) belongs to \( \text{Im} \varphi_n \subseteq \text{Im} \varphi \).

As to injectivity, we must show that if \( a, b \in H^1(G, X_n) \) satisfy \( \varphi_n(a) = \varphi_n(b) \) then there exists \( m \geq n \) with \( \varphi_{mn}(a) = \varphi_{mn}(b) \) (cf. [Bou2], Prop. 6 on p. E III.62). Say \( a \) and \( b \) are the classes of \( d, e \in Z^1(G, X_n) \), respectively. Then \( f_n \circ d \) and \( f_n \circ e \) are cohomologous in \( Z^1(G, X) \), i.e., there exists \( x \in X \) with \( (f_n \circ d)(g) = x^q(f_n \circ e)(g)x^{-1} \) for all \( g \in G \). Now \( x \in f_m(X_m) \) for some \( m \geq n \), say \( x = f_m(y) \). Then

\[
(f_m \circ f_{mn} \circ d)(g) = f_m(y^q)(f_m \circ f_{mn} \circ e)(g)f_m(y^{-1}).
\]
Since all $f_{mn}$ are injective, so are the maps $f_n$ ([Bou2], Remarque 1 on p. E III.63). Hence, $(f_{mn} \circ d)(g) = y^g(f_{mn} \circ e)(g)y^{-1}$ holds for all $g \in G$ which shows that $f_{mn} \circ d$ and $f_{mn} \circ e$ are cohomologous in $Z^1(G, X_m)$, as required.

1.4. The monoid structure of $H^1(G, \text{GL}(S))$. Lemma [L3] implies in particular that in $H^1(G, \text{GL}(S)) \cong \operatorname{lim}_{\rightarrow} H^1(G, \text{GL}_n(S))$, where $G$ acts on $\text{GL}(S)$ and on $\text{GL}_n(S)$ via its action on $S$. Our goal here is to endow $H^1(G, \text{GL}(S))$ with the structure of a commutative monoid.

More generally, let $X$ be any $G$-stable subgroup of $\text{GL}(S)$ containing the matrices

$$s_{m,n} = (-1)^{(m+1)n} \begin{pmatrix} 0_{n \times m} & -1_{n \times n} \\ 1_{m \times m} & 0_{m \times n} \\ & & \ddots \end{pmatrix}.$$

Note that $\det s_{m,n} = 1$. (In fact, it is not hard to show that the subgroup of $\text{GL}(S)$ that is generated by the matrices $s_{m,n}$ consists of all monomial matrices with entries $\pm 1$ and with determinant 1.) Hence $s_{m,n} \in \text{E}(S)$, the group generated by the elementary matrices ([Ba], Prop. 1.6 on p. 226). Furthermore,

$$s_{m,n}^{-1} \begin{pmatrix} a_{m \times n} & b_{m \times m} \\ & 1 \\ & & \ddots \end{pmatrix} = \begin{pmatrix} b_{m \times m} & a_{n \times n} \\ & 1 \\ & & \ddots \end{pmatrix}.$$ 

Thus, if $x = \begin{pmatrix} x_{n \times n} \\ & 1 \\ & & \ddots \end{pmatrix} \in X_n = X \cap \text{GL}_n(S)$ then, for any $m$,

$$x[m] = s_{m,n}^{-1} x s_{m,n} \in X_{m+n}.$$ 

For $a, b \in H^1(G, X)$, define $a + b \in H^1(G, X)$ as follows. Choose cocycles $d, e \in Z^1(G, X)$ representing $a$ and $b$, respectively. Since $G$ is finite, we have $d \in Z^1(G, X_m)$ and $e \in Z^1(G, X_n)$ for suitable $m$ and $n$. For $g \in G$, put

$$(d \oplus_m e)(g) = \left( \begin{array}{c} d(g)_{m \times m} \\ & e(g)_{n \times n} \\ & \ddots \end{array} \right) \in X.$$ 

It is trivial to verify that $d \oplus_m e$ is a cocycle, and we define $a + b$ to be its class in $H^1(G, X)$. To show that this is well-defined, we first note that the class of $d \oplus_m e$ is
independent of \( m \), as long as \( d(G) \subseteq X_m \). Indeed, if \( t \geq 0 \) then
\[
(d \oplus_{m+t} e)(g) = (s_{t,n}[m])^{-1} (d \oplus_m e)(g)s_{t,n}[m],
\]
and \( s_{t,n}[m] = s_{t,n}[m]^g \in X \). Thus, \( d \oplus_{m+t} e \) and \( d \oplus_m e \) are cohomologous. Now suppose that \( a \) and \( b \) are also represented by the cocycles \( d' \in Z^1(G, X) \) and \( e' \in Z^1(G, X) \), respectively. So there are matrices \( x, y \in X \) with
\[
d'(g) = x^g d(g) x^{-1} \quad \text{and} \quad e'(g) = y^g e(g) y^{-1}.
\]
Fix \( r \) so that \( x, y, d(G), e(G) \) are all contained in \( X_r \), and hence so are \( d'(G) \) and \( e'(G) \). By the foregoing, it suffices to show that \( d \oplus_r e \) and \( d' \oplus_r e' \) are cohomologous.

But, putting \( z = x \cdot y[r] \in X \), we have \( (d' \oplus_r e')(g) = z^g (d \oplus_r e)(g) z^{-1} \). Thus, \( a + b \in H^1(G, X) \) is indeed well-defined. Commutativity and associativity of + follow along similar lines, thereby making \( H^1(G, X) \) a commutative monoid with neutral element the class of the unit cocycle. — For a different description of the monoid structure of \( H^1(G, X) \) for \( X = \text{GL}(S) \), see Lemma \([27]\).

1.5. Monoid maps. Let \( S' \) be another ring that is acted on by a group \( G' \) and suppose that we are given a group homomorphism \( \alpha : G \to G' \) and a \( G' \)-equivariant ring map \( \phi : S' \to S \), where \( G \) acts on \( S' \) via \( \alpha \). Then we obtain a map of \( G' \)-groups \( \text{GL}(\phi) : \text{GL}(S') \to \text{GL}(S) \). Thus, if \( X' \subseteq \text{GL}(S') \) is a \( G' \)-stable subgroup containing the matrices \( s_{m,n} \in \text{GL}(S') \) of \([1.4]\) and if \( X \subseteq \text{GL}(S) \) is \( G \)-stable containing the image of \( X' \) then we have a map of \( G \)-groups \( f : X' \to X \). The map induced on cocycles \( Z^1(G', X') \to Z^1(G, X) \) (cf. \([36]\)) is easily seen to respect the operations \( \oplus_m \) of \([1.4]\), and hence the map \( (\alpha, f)_* : H^1(G', X') \to H^1(G, X) \) of \([36]\) is actually a monoid map. This applies in particular to restriction and induction maps (with \( \phi = \text{Id}_S \)). Moreover, we have the following

**Lemma.** Let \( X \subseteq \text{GL}(S) \) be a \( G \)-stable subgroup with \( X \supseteq E(S) \) and let \( f : X \to X^{ab} = X/[X, X] \) denote the canonical map. Then \( f_*^1 : H^1(G, X) \to H^1(G, X^{ab}) \) is a monoid map, where \( H^1(G, X^{ab}) \) has its usual group structure.

**Proof.** Consider \( d, e \in Z^1(G, X) \), say \( d, e \in Z^1(G, X_m) \). Then, by \([34]\), Prop. (1.7) on p. 226,
\[
(d \oplus_m e)(g) \equiv d(g) \cdot e(g) = \begin{pmatrix} (d(g))_{m \times m} & e(g)_{m \times m} \\ 1 & \ddots \end{pmatrix} \pmod{E(S)}.
\]
Inasmuch as \( X^{ab} = X / E(S) \) (cf. \([34]\), Thm. (2.1) on p. 228), our assertion follows.

1.6. Units. The kernel of \( \text{Ind}_R^S : K_0(R) \to K_0(S) \) will turn out to be isomorphic to the group of units \( U(H^1(G, \text{GL}(S))) \) of the monoid \( H^1(G, \text{GL}(S)) \) (see \([28]\)). Here, we make some preliminary observations on the unit group
\[
U(H^1(G, X)),
\]
where \( X \) is any \( G \)-invariant subgroup of \( \text{GL}(S) \) containing the matrices \( s_{m,n} \), as in §1.4.

**Lemma.** Let \( N = \text{Ker}_G(X) \) denote the kernel of the action of \( G \) on \( X \). Then \( U(H^1(G,X)) \cong U(H^1(G/N,X)) \) via inflation. In particular, if \( G \) acts trivially on \( X \), then \( U(H^1(G,X)) \) contains only the unit class.

**Proof.** Use the inflation-restriction sequence \( H^1(G/N,X) \rightarrow H^1(G,X) \rightarrow H^1(N,X) \) ([56], §I.5.8). This sequence is exact, the first map (inflation) is injective, and both maps are monoid maps. Thus it induces an exact sequence of groups

\[
1 \rightarrow U(H^1(G/N,X)) \rightarrow U(H^1(G,X)) \rightarrow U(H^1(N,X)) .
\]

Part (ii) therefore reduces to the claim that \( U(H^1(N,X)) \) is trivial. To verify this, recall that \( H^1(N,X) = \text{Hom}(N,X)/X \), with \( X \) acting by conjugation on \( \text{Hom}(N,X) \), and the unit class consists of the unit map \( 1 \) alone (cf. §1.2). Thus, letting \( \langle . \rangle \) denote \( X \)-conjugacy classes, the equation \( \langle d \rangle + \langle e \rangle = \langle 1 \rangle \) for \( d,e \in \text{Hom}(N,X) \) is equivalent with \( (d \oplus_m e)(g) = 1 \) for all \( g \in N \), where \( m \) is chosen as above. But the latter condition says that \( d = e = 1 \).

\[\Box\]

2. **The Kernel of Induction**

2.1. **The skew group ring.** We will let

\[ T = S \ast G \]

denote the skew group ring that is associated with the given \( G \)-action on \( S \). Thus \( T \) is an associative ring containing \( S \) as a subring and \( G \) is a subgroup of \( \text{U}(T) \), the group of units of \( T \). The elements of \( G \) form a free basis of \( T \) as right \( S \)-module. Multiplication in \( T \) is based on the rule \( ga \cdot hb = gh a^g b \) for \( a,b \in S, g,h \in G \). The ring \( S \) is an \( R \)-\( T \)-bimodule with action

\[ r \cdot a \cdot gb = ra^g b \quad (r \in R, a,b \in S, g \in G) . \]

Hypothesis (*) entails that \( txt = t \), where we have put \( t = \sum_{g \in G} g \in T \). So \( e = tx \) is an idempotent element of \( T \) with \( eT = tT = tS \cong S_T \). In particular, \( S_T \) is projective and the ideal \( I = TeT \) of \( T \) satisfies \( I^2 = I \) and \( S_T \cdot I = S_T \).

2.2. **Some module categories.** Let \( \text{proj} \ R \) denote the category of finitely generated projective (right) \( R \)-modules, and similarly for \( T \), and let \( \text{add} \ S_T \) denote the full subcategory of \( \text{proj} \ T \) whose objects are the direct summands of the modules \( S^*_n \) for \( n \geq 0 \). The following lemma is well-known but we include the proof for the reader’s convenience.

**Lemma.** (i) The functors \( E: \text{proj} \ R \rightarrow \text{add} \ S_T \), \( Q \mapsto Q \otimes_R S_T \) and \( F: \text{add} \ S_T \rightarrow \text{proj} \ R \), \( P \mapsto P^G \) yield an equivalence of categories \( \text{proj} \ R \cong \text{add} \ S_T \).

(ii) A module \( P \) in \( \text{proj} \ T \) belongs to \( \text{add} \ S_T \) precisely if \( PI = P \).
Proof. (i) For \( Q \in \text{proj} R \), let \( \varphi_Q : Q \to (F \circ E)(Q) = (Q \otimes_R S_T)^G \) denote the \( R \)-linear map given by \( \varphi_Q(q) = q \otimes 1 \). Then \( \varphi_R : R \to (R \otimes_R S_T)^G \cong (S_T)^G = R \) is an isomorphism, and hence so is \( \varphi_{R^n} \) for every \( n \) and \( \varphi_Q \) for every \( Q \). Thus \( \varphi \) is a natural equivalence of functors \( \text{Id}_{\text{proj} R} \cong F \circ E \). Similarly, defining \( \psi_P : (E \circ F)(P) = P^G \otimes_R S_T \to P \) in \( \text{add} S_T \) by \( \psi_P(p \otimes s) = ps \), we obtain a natural equivalence of functors \( E \circ F \cong \text{Id}_{\text{add} S_T} \).

(ii) All modules \( P \) in \( \text{add} S_T \) satisfy \( PI = P \), because \( S \cdot I = S \). Conversely, if \( P \in \text{proj} T \) satisfies \( P = PI = PeT \) then, for some \( n \), \( (eT)^n \cong S_T^n \) maps onto \( P \), and so \( P \) is a direct summand of \( S_T^n \).

2.3. Another description of \( \text{add} S_T \). From now on, \( S \) is assumed commutative. We let \( \text{Max} S \) denote the set of maximal ideals of \( S \). For each \( \mathfrak{M} \in \text{Max} S \), we put \( G^Z(\mathfrak{M}) = \{ g \in G : \mathfrak{M}^g = \mathfrak{M} \} \), the decomposition group of \( \mathfrak{M} \), and

\[
T(\mathfrak{M}) = (S/\mathfrak{M})^*G^Z(\mathfrak{M})
\]

the skew group ring that is associated with the action of \( G^Z(\mathfrak{M}) \) on \( S/\mathfrak{M} \). As in §2.1, \( S/\mathfrak{M} \) is a right module over \( T(\mathfrak{M}) \); this module structure can be viewed as coming from \( S_T \) by restriction to \( S^*G^Z(\mathfrak{M}) \) and reduction mod \( \mathfrak{M} \). The following description of \( \text{add}(S_T) \) is adapted from [Kr], Proposition 3.

Lemma. A module \( P \) in \( \text{proj} T \) belongs to \( \text{add} S_T \) if and only if, for all \( \mathfrak{M} \in \text{Max} S \), there is an isomorphism of \( T(\mathfrak{M}) \)-modules \( P/P\mathfrak{M} \cong (S/\mathfrak{M})^r \) (r = rank \( P_{\mathfrak{M}} \)).

Proof. The condition is surely necessary. For, if \( P \) is a direct summand of \( S_T^n \), then \( P/P\mathfrak{M} \) is a direct summand of the homogeneous \( T(\mathfrak{M}) \)-module \( (S/\mathfrak{M})^n \).

For the converse, consider some \( \mathfrak{M} \in \text{Max} S \) and put \( \mathfrak{M}^0 = \bigcap_{g \in G} \mathfrak{M}^g \), a \( G \)-stable ideal of \( S \). Then

\[
S/\mathfrak{M}^0 \cong \bigoplus_{g \in G^Z(\mathfrak{M}) \backslash G} S/\mathfrak{M}^g \cong (S/\mathfrak{M}) \otimes_{T(\mathfrak{M})} T
\]

as \( T \)-modules. Similarly, \( P/P\mathfrak{M}^0 \cong P \otimes_S S/\mathfrak{M}^0 \cong (P/P\mathfrak{M}) \otimes_{T(\mathfrak{M})} T \) as \( T \)-modules, with \( G \) acting “diagonally” on \( P \otimes_S S/\mathfrak{M}^0 \): \( (p \otimes s)g = pg \otimes s^g \). By hypothesis, \( P/P\mathfrak{M} \) is isomorphic to \( (S/\mathfrak{M})^r \) as \( T(\mathfrak{M}) \)-modules, and so

\[
P/P\mathfrak{M}^0 \cong (S/\mathfrak{M}^0)^r
\]

as \( T \)-modules. Since \( S = S \cdot I \) (cf. §2.1), this isomorphism implies \( P = PI + P\mathfrak{M}^0 \), and since \( \mathfrak{M} \) was arbitrary, we further conclude that \( P = PI \) (cf. [Bou], Prop. 11 on p. 113). In view of Lemma 2.2(ii), this shows that \( P \) belongs to \( \text{add} S_T \).

2.4. The induction triangle. For each \( n \geq 0 \), we let \( \mathbf{P}_n(R) \) denote the set of isomorphism classes of f.g. projective \( R \)-modules of constant rank \( n \), and similarly for \( \mathbf{P}_n(S) \). These are pointed sets with distinguished elements \( (R^n \rangle \) and \( \langle S^n \rangle \), respectively, where \( \langle . \rangle \) denotes isomorphism classes. Furthermore, \( \mathbf{P}_{S,n}(T) \) will denote the set of isomorphism classes of f.g. projective right \( T \)-modules having constant rank \( n \).
as $S$-modules, with distinguished element $(S^n_T)$. We have a commutative diagram of pointed sets (cf. [Ba], Prop. (7.3) on p. 130)

$$
\begin{array}{ccc}
\mathbf{P}_n(R) & \xrightarrow{\Phi_n} & \mathbf{P}_n(S) \\
\Phi_n = (\cdot) \otimes R & \downarrow & \downarrow \\
\mathbf{P}_{S,n}(T) & \xrightarrow{\text{Res}_{S,n}^T} & \\
\end{array}
$$

By Lemma $2.2(i)$, $\Phi_n$ is injective. The kernels of the other two maps will be described in §2.3 and 2.6 below. Recall that the kernel of a map of pointed sets is defined to be the preimage of the distinguished element of the target set.

2.5. The kernel of $\text{Res}_{S,n}^T$. We now consider the restriction map $\text{Res}_{S,n}^T : \mathbf{P}_{S,n}(T) \rightarrow \mathbf{P}_n(S)$, as in §2.4.

**Lemma.** $\text{Ker}(\text{Res}_{S,n}^T) \cong H^1(G, \text{GL}_n(S))$ as pointed sets.

**Proof.** Each cocycle $d \in Z^1(G, \text{GL}_n(S))$ gives rise to a $T$-module structure $(S^n)_d$ on $S^n$ via

$$x \cdot gs = x^g d(g)s \quad (x \in S^n, g \in G, s \in S).$$

This action extends the regular $S$-module structure on $S^n$. Conversely, if $\cdot$ is any right $T$-module operation on $S^n$ extending the regular $S$-module structure then write, for $g \in G$,

$$e_i \cdot g = \sum_{j=1}^n e_j d_{i,j}(g),$$

where $e_i \in S^n$ is the basis element with 1 in the $i$-th position and 0s elsewhere and $d_{i,j}(g) \in S$. A routine verification shows that $d = (d_{i,j})$ is a cocycle of $G$ in $\text{GL}_n(S)$ and the given $T$-module structure on $S^n$ is identical with $(S^n)_d$.

Since $(S^n)_d$ for the unit cocycle $d = 1$ is just $S^n_T$, we obtain a surjective map of pointed sets

$$Z^1(G, \text{GL}_n(S)) \rightarrow \text{Ker}(\text{Res}_{S,n}^T), \quad d \mapsto \langle (S^n)_d \rangle.$$

Finally, for $d, e \in Z^1(G, \text{GL}_n(S))$, we have $(S^n)_d \cong (S^n)_e$ as $T$-modules precisely if there is an $S$-module isomorphism $(S^n)_d \cong (S^n)_e$ that commutes with the $G$-actions, that is, for some matrix $a \in \text{GL}_n(S)$, $(x \cdot g)a = (xa) \cdot g$ holds for all $x \in S^n, g \in G$. The latter condition is equivalent with $x^g d(g) = x^g a^g e(g) a^{-1}$ which in turn just says that $d$ and $e$ are cohomologous. This completes the proof of the lemma. \hfill $\Box$

**Remark.** With $T(\mathfrak{M})$ as in §2.3, we have a map of pointed sets $\mathbf{P}_{S,n}(T) \rightarrow \mathbf{P}_{S/\mathfrak{M},n}(T(\mathfrak{M}))$, $P \mapsto P/P \mathfrak{M}$, which restricts to a map $\text{Ker}(\text{Res}_{S,n}^T) \rightarrow \text{Ker}(\text{Res}_{S/\mathfrak{M},n}^T)$. In terms of the identification provided by the above Lemma, the latter becomes the map

$$\rho_{\mathfrak{M},n} : H^1(G, \text{GL}_n(S)) \rightarrow H^1(G^Z(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M}))$$

that is given by restriction from $G$ to $G^Z(\mathfrak{M})$ and reduction modulo $\mathfrak{M}$. 

2.6. The kernel of $P_n(f)$. For each subgroup $H \leq G$, we put

$$J(H) = \bigcap_{\mathfrak{M} \in \text{Max } S} \mathfrak{M} \quad \text{and} \quad S_H = S/J(H),$$

where $G^T(\mathfrak{M}) = \{g \in G : s^g - s \in \mathfrak{M} \text{ for all } s \in S\}$ is the inertia group of $\mathfrak{M}$. In addition to the maps $\rho_{\mathfrak{M},n}'$ introduced in Remark 2.5, we will consider the analogous restriction-reduction maps $\rho_{\mathfrak{M},n}$:

$$\rho_{\mathfrak{M},n} : H^1(G, \text{GL}_n(S)) \to H^1(G^T(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M}))$$

and

$$\rho_{H,n} : H^1(G, \text{GL}_n(S)) \to H^1(H, \text{GL}_n(S_H)).$$

Since the actions of $G^T(\mathfrak{M})$ on $\text{GL}_n(S/\mathfrak{M})$ and of $H$ on $\text{GL}_n(S_H)$ are trivial, we have $H^1(G^T(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M})) = \text{Hom}(G^T(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M}))/\text{GL}_n(S/\mathfrak{M})$ and similarly for $H^1(H, \text{GL}_n(S_H))$ (cf. §1.2). We let $C$ denote the set of cyclic subgroups of $G$.

**Proposition.** As pointed sets,

$$\text{Ker } P_n(f) \cong \bigcap_{\mathfrak{M} \in \text{Max } S} \text{Ker } \rho_{\mathfrak{M},n}' = \bigcap_{\mathfrak{M} \in \text{Max } S} \text{Ker } \rho_{\mathfrak{M},n} = \bigcap_{C \in C} \text{Ker } \rho_{C,n} = \bigcap_{H \leq G} \text{Ker } \rho_{H,n}.$$

**Proof.** In view of the induction triangle in §2.4, $\text{Ker } P_n(f) \cong \text{Im } \Phi_n \cap \text{Ker } (\text{Res}^T_{\mathfrak{M},n})$. Furthermore, by virtue of Lemmas 2.2 and 2.3, if $\langle P \rangle \in P_{S,n}(T)$ then $\langle P \rangle \in \text{Im } \Phi_n$ iff $\langle P/P\mathfrak{M} \rangle$ is the distinguished element of $P_{S/\mathfrak{M},n}(S(T\mathfrak{M}))$ for all $\mathfrak{M} \in \text{Max } S$. Therefore, by Remark 2.3,

$$\text{Im } \Phi_n \cap \text{Ker } (\text{Res}^T_{\mathfrak{M},n}) \cong \bigcap_{\mathfrak{M} \in \text{Max } S} \text{Ker } \rho_{\mathfrak{M},n'},$$

which establishes the $\cong$ in the proposition. Now $\rho_{\mathfrak{M},n} = \text{Res}^{G^Z(\mathfrak{M})}_{G^T(\mathfrak{M})} \circ \rho_{\mathfrak{M},n}'$, where

$$\text{Res}^{G^Z(\mathfrak{M})}_{G^T(\mathfrak{M})} : H^1(G^Z(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M})) \to H^1(G^T(\mathfrak{M}), \text{GL}_n(S/\mathfrak{M}))$$

is the restriction map. Since this map has trivial kernel, by the generalized “Theorem 90” ([Se], Lemme 1 on p. 129 and §5.8(a)), we conclude that $\text{Ker } \rho_{\mathfrak{M},n}' = \text{Ker } \rho_{\mathfrak{M},n}$ which proves the first equality.
Consider \( a \in H^1(G, \text{GL}_n(S)) \), say \( a \) is the class of \( d \in Z^1(G, \text{GL}_n(S)) \). Then, in view of \( \text{II}.2(1) \),

\[
\begin{align*}
\ker \rho_{\mathfrak{m},n} & \iff \forall \mathfrak{m} \in \text{Max } S \ \forall g \in G^T(\mathfrak{M}) : \ d(g) \equiv 1_{n \times n} \mod \mathfrak{M} \\
& \iff \forall g \in G \ \forall \mathfrak{M} \ni J(\langle g \rangle) : \ d(g) \equiv 1_{n \times n} \mod J(\langle g \rangle) \\
& \iff a \in \bigcap_{\mathfrak{m} \in \text{Max } S} \ker \rho_{\mathfrak{m},n},
\end{align*}
\]

proving the second equality. Finally, the proof of \( \bigcap \ker \rho_{\mathfrak{m},n} = \bigcap H \ker \rho_{H,n} \) is completely analogous, based on the observation that \( H \subseteq G^T(\mathfrak{M}) \) if and only if \( \mathfrak{M} \ni J(H) \). This completes the proof of the proposition.

2.7. **Stabilization.** For \( m \geq n \), we now consider the stabilization maps \( P_n(R) \rightarrow P_m(R) \), \( \langle P \rangle \mapsto \langle P \oplus R^{m-n} \rangle \), the analogous map for \( S \), and the map \( P_{S,n}(T) \rightarrow P_{S,m}(T) \), \( \langle Q \rangle \mapsto \langle Q \oplus S_T^{m-n} \rangle \). These are maps of pointed sets which are compatible with the maps \( \Phi_n \), \( \text{Res}^S_{S,n} \) and \( P_n(f) \) in the induction triangle (\( \text{II}.4 \)). Thus we obtain a commutative diagram

\[
\begin{array}{ccc}
\lim P_n(R) & \xrightarrow{\varphi = \lim \Phi_n} & \lim P_n(S) \\
\xrightarrow{r = \lim \text{Res}^S_{S,n}} & & \\
\lim P_{S,n}(T) & \xrightarrow{=} & \lim P_n(f)
\end{array}
\]

with \( \varphi \) injective, by \( \text{II}.2 \), Prop. 7 on p. E III.64. Explicitly (cf. \( \text{II}.2 \), p. E III.61),

\[
\lim P_n(R) = \bigcup_{n \geq 0} P_n(R) / \sim,
\]

where \( \bigcup \) denotes the disjoint union and \( x \sim y \) for \( x = \langle P \rangle \in P_n(R) \), \( y = \langle Q \rangle \in P_m(R) \) iff \( P \oplus R^{t-n} \cong Q \oplus R^{t-m} \) for some \( t \geq \max(m,n) \). In other words, \( \langle P \rangle \sim \langle Q \rangle \) iff \( P \) and \( Q \) are stably isomorphic. Now, \( \bigcup_{n \geq 0} P_n(R) \) is a commutative monoid under \( \oplus \) and the equivalence relation \( \sim \) respects this structure. Thus, \( \lim P_n(R) \) becomes a commutative monoid with identity element the stable isomorphism class of \( \{0\} \), that is, the f.g. free modules. Actually, \( \lim P_n(R) \) is a group: If \( \langle P \rangle \in P_n(R) \) is given then \( P \oplus Q \cong R^r \) for suitable \( Q \) and \( r \), and hence \( \langle P \rangle \oplus \langle Q \rangle = \langle R^r \rangle \sim \langle 0 \rangle \). In fact, letting \( K_0(R) \) denote the kernel of

\[
\text{rank} : K_0(R) \rightarrow H_0(R) = \{\text{continuous maps Spec } R \rightarrow \mathbb{Z}\},
\]
as usual (cf. [Ba], p. 459), the map sending \( \langle P \rangle \in \mathbf{P}_n(R) \) to \([P] - [R^n] \in \widetilde{K}_0(R)\) passes down to a homomorphism of groups

\[
\lim_{\leftarrow} \mathbf{P}_n(R) \to \widetilde{K}_0(R)
\]

which is easily seen to be an isomorphism (cf. [W], Chap. II, Lemma 2.3.1).

The foregoing is valid for any commutative ring \( R \), and so also applies to \( \lim_{\leftarrow} \mathbf{P}_n(S) \).

Under the identification \( \lim_{\leftarrow} \mathbf{P}_n \simeq \widetilde{K}_0 \), the top map \( \lim_{\leftarrow} \mathbf{P}_n(f) \) of the above triangle becomes

\[
\widetilde{K}_0(f) = \text{Ind}_R^S \bigg|_{\widetilde{K}_0(R)} : \widetilde{K}_0(R) \to \widetilde{K}_0(S).
\]

Things are similar for \( \lim_{\leftarrow} \mathbf{P}_{S,n}(T) \): Isomorphism classes \( \langle X \rangle \in \mathbf{P}_{S,n}(T) \) and \( \langle Y \rangle \in \mathbf{P}_{S,m}(T) \) become identified precisely if \( X \oplus S_{t-n} \cong Y \oplus S_{t-m} \) for some \( t \geq \max(m, n) \).

However, since the submonoid of \( \big( \bigcup_{n \geq 0} \mathbf{P}_{S,n}(T), \oplus \big) \) that is generated by \( \langle S_T \rangle \) need no longer be cofinal, \( \lim_{\leftarrow} \mathbf{P}_{S,n}(T) \) is merely a commutative monoid. The maps \( \varphi \) and \( r \) of the above triangle are monoid maps and \( \varphi \) is mono, as was pointed out earlier.

Thus, summarizing, we have the following stabilized induction triangle of commutative monoids and groups

\[
\begin{array}{ccc}
\widetilde{K}_0(R) & \xrightarrow{\widetilde{K}_0(f)} & \widetilde{K}_0(S) \\
\varphi = (.) \otimes_R S_T & & r \\
\lim_{\leftarrow} \mathbf{P}_{S,n}(T) & &
\end{array}
\]

**Lemma.** \( \text{Ker } r \cong H^1(G, \text{GL}(S)) \) as commutative monoids.

**Proof.** Since direct limits commute with kernels ([Bou2], Cor. (ii) on p. E III.65),

\[
\text{Ker } r = \lim_{\leftarrow} \left( \text{Ker } \text{Res}_{S,n}^T \right).
\]

Next, we infer from Lemma 2.3 (and its proof) that the stabilization map \( \text{Ker } \text{Res}_{S,n}^T \rightarrow \text{Ker } \text{Res}_{S,n}^T \) \( (m \geq n) \) translates into the map \( H^1(G, \text{GL}_n(S)) \rightarrow H^1(G, \text{GL}_m(S)) \) that is induced by \( \text{GL}_n(S) \rightarrow \text{GL}_m(S), \ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \). Thus, Lemma 2.3 implies that \( \text{Ker } r \cong H^1(G, \text{GL}(S)) \), at least as pointed sets. The fact that this \( \cong \) does respect the additive structures is a consequence of the obvious isomorphism (using the notations of §3 1.4 2.3)

\[
(S^m)_d \oplus (S^n)_e \cong (S^{m+n})_{d \oplus m e}
\]

for \( d \in Z^1(G, \text{GL}_m(S)) \) and \( e \in Z^1(G, \text{GL}_n(S)) \). \( \square \)
2.8. **Proof of the main result.** For each subgroup $H \leq G$, let
\[
\rho_H : H^1(G, \text{GL}(S)) \to H^1(H, \text{GL}(S_H)) = \text{Hom}(H, \text{GL}(S_H))/\text{GL}(S_H)
\]
be given by restriction from $G$ to $H$ and reduction modulo $J(H)$; so $\rho_H = \lim_{\rightarrow} \rho_{H,n}$ (cf. §2.6). Similarly, for $\mathfrak{M} \in S$, put
\[
\rho_{\mathfrak{M}} = \lim_{\rightarrow} \rho_{\mathfrak{M},n} : H^1(G, \text{GL}(S)) \to H^1(G^T(\mathfrak{M}), \text{GL}(S/\mathfrak{M})).
\]
By §2.6, these maps are monoid maps, and hence their kernels are submonoids of $H^1(G, \text{GL}(S))$. It is now a simple matter to prove the Theorem stated in the Introduction. Recall that $\mathcal{C}$ denotes the set of cyclic subgroups of $G$.

**Theorem.** $Ker K_0(f) \cong U(H^1(G, \text{GL}(S)))$, an isomorphism of groups, and
\[
U(H^1(G, \text{GL}(S))) = \bigcap_{C \in \mathcal{C}} Ker \rho_C = \bigcap_{H \leq G} Ker \rho_H = \bigcap_{\mathfrak{M} \in \text{Max } S} Ker \rho_{\mathfrak{M}}.
\]

If $S$ is Noetherian of Krull dimension $d$ and $n > d$ then $Ker K_0(f) \cong Ker P_n(f)$.

**Proof.** Recall that $K_0(.)$ decomposes naturally as $K_0(.) = H_0(.) \oplus \tilde{K}_0(.)$, with $\tilde{K}_0(.) \cong \lim_{\rightarrow} P_n(.)$. If the ring in question is Noetherian of Krull dimension $d$, then $\lim_{\rightarrow} P_n(.) \cong P_m(.)$ holds for any $m > d$ ([Ba2]).

Applying this to the inclusion $f : R \hookrightarrow S$, we obtain that $K_0(f) = H_0(f) \oplus \tilde{K}_0(f)$. Here, $H_0(f)$ is injective (cf. [Ba], Lemma (3.1) on p. 459), and hence
\[
Ker K_0(f) = Ker \tilde{K}_0(f) \cong \lim_{\rightarrow} Ker P_n(f)
\]
(again using the fact that direct limits commute with kernels). Moreover, if $S$ is Noetherian of Krull dimension $d$ (and hence so is $R$, by virtue of hypothesis (*) ), then $Ker K_0(f) \cong Ker P_n(f)$ for $n > d$ which establishes the last assertion of the theorem.

Writing the formula for $Ker P_n(f)$ in Proposition 2.6 as
\[
Ker P_n(f) \cong Ker \rho_n,
\]
where $\rho_n = \{ \rho_{C,n} \} : H^1(G, \text{GL}_n(S)) \to \prod_{C \in \mathcal{C}} H^1(C, \text{GL}_n(S_C))$, the stabilization map $Ker P_n(f) \to Ker P_m(f)$ ($m \geq n$) becomes the map on $H^1$ that is induced by the maps $\text{GL}_n \to \text{GL}_m$, $a \mapsto (\alpha^\top \beta^\top \gamma^\top \delta^\top)$, for $S$ and $S_C$. Thus, using Lemma 1.3 and the fact that direct limits commute with kernels and with finite direct products (cf. [Bou2], loc. cit. and Prop. 10/Cor. on p. E III.67/8), we deduce that
\[
Ker K_0(f) \cong Ker \rho,
\]
where $\rho = \{ \rho_C \} : H^1(G, \text{GL}(S)) \to \prod_{C \in \mathcal{C}} H^1(C, \text{GL}(S_C))$. The $\cong$ is additive, by Lemma 2.7. Since $Ker K_0(f)$ is a group, its image in $H^1(G, \text{GL}(S))$ must be contained in the unit group $U(H^1(G, \text{GL}(S)))$. On the other hand, we infer from Lemma 1.6 that
Goldman \( \text{[AG]} \), if and only if 
\[ \text{G} \]
Thus, by Proposition 2.6 and Theorem 2.8, 
\[ 1.3 \text{ on p. 4). In this case, hypothesis (*) is satisfied (\text{[CHR]}, Lemma 1.6 on p. 7).} \]

3. Applications and Problems

3.1. Galois actions. The \( G \)-action on \( S \) is Galois, in the sense of Auslander and Goldman \([\text{AG}]\), if and only if \( G^T(\mathfrak{M}) \) is trivial for all \( \mathfrak{M} \in \text{Max} \ S \) (cf. \text{[CHR]}, Theorem 1.3 on p. 4). In this case, hypothesis (*) is satisfied (\text{[CHR]}, Lemma 1.6 on p. 7). Thus, by Proposition 2.6 and Theorem 2.8,
\[
\text{Ker } \mathfrak{P}_n(f) \cong H^1(G, \text{GL}_n(S)) \quad \text{and} \quad \text{Ker } K_0(f) \cong H^1(G, \text{GL}(S)).
\]

In particular, \( H^1(G, \text{GL}(S)) \) is a group with the operation of \( \text{[L4]} \). In fact:

**Proposition.** If the action of \( G \) on \( S \) is Galois then \( \text{Ker } K_0(f) \cong H^1(G, \text{GL}(S)) \) is annihilated by a power of \( |G| \). For \( S \) Noetherian of Krull dimension \( d \), \( |G|^d \) will do.

**Proof.** By \text{[CHR]}, Lemma 4.1 on p. 13, \( S \) is f.g. projective of constant rank equal to \( |G| \) as \( R \)-module. Therefore,
\[
[S_R] - |G|[R] = [S_R] - |G|1_{K_0(R)} \in \widetilde{K}_0(R)
\]
and, moreover, the restriction map \( \text{Res}_R^S : K_0(S) \to K_0(R) \) is defined. The composite \( \text{Res}_R^S \circ K_0(f) : K_0(R) \to K_0(R) \) is clearly multiplication with \( S_R \). Hence,
\[
\text{Ker } K_0(f) \subseteq \text{ann}_{K_0(R)}([S_R])
\]
Recall that \( \widetilde{K}_0(R) \) is a nil ideal of \( K_0(R) \), and if \( S \) (or, equivalently, \( R \)) is Noetherian of Krull dimension \( d \), then \( \widetilde{K}_0(R)^{d+1} = \{0\} \) (\text{[B4]}, pp. 477 and 473). Furthermore, \( \text{Ker } K_0(f) \subseteq \widetilde{K}_0(R) \) (cf. the proof of Theorem 2.8). Hence, \( \text{Ker } K_0(f) \cdot ([S_R] - |G|)^t = \{0\} \) for some \( t \), with \( t = d \) a possible choice for \( S \) Noetherian of Krull dimension \( d \). Consequently, \( \text{Ker } K_0(f) \cdot |G|^t = \{0\} \).

The proposition can be viewed as an extension of Hilbert’s “Theorem 90” (cf. \text{[Sc]}, Lemme 1 on p. 129) to commutative rings.

3.2. I don’t know if Proposition 3.1 generalizes to arbitrary \( G \)-actions:

**Question.** Is \( \text{Ker } K_0(f) \cong U(H^1(G, \text{GL}(S))) \) always annihilated by \( |G|^d \) if \( S \) is Noetherian of Krull dimension \( d \)?

This is indeed so for \( d = 0 \), in which case \( K_0(R) = H_0(R) \) and \( K_0(f) = H_0(f) \) is mono, and for \( d = 1 \), where \( K_0(R) = H_0(R) \oplus \text{Pic}(R) \) and \( \text{Ker } K_0(f) = \text{Ker } \text{Pic}(f) \) is
isomorphic to a subgroup of $H^1(G, U(S))$ (cf. §3.3 below). By a routine direct limit argument, a positive answer to the above question would imply that the kernel of the induction map $K_0(f) : K_0(R) \to K_0(S)$ is always $|G|$-primary (i.e., every element is annihilated by a power of $|G|$), for any commutative ring $S$. Finally, I note that the dual statement for $G_0$ is known to hold ([BrL1] or [BrL2]): The cokernel of the restriction map $G_0(S) \to G_0(R)$ is annihilated by $|G|^{d+1}$. The proof given in [BrL1] results from an analysis of the so-called coniveau filtration of $G_0$. The key to the above problem might very well be the Grothendieck $\gamma$-filtration ([SGA6], [FL])

$$K_0(R) = F^0_\gamma K_0 \supseteq \widetilde{K}_0(R) = F^1_\gamma K_0 \supseteq \cdots \supseteq F^{d+1}_\gamma K_0 = 0.$$  

The first two slices are $F^0_\gamma / F^1_\gamma = H_0(R)$ and $F^1_\gamma / F^2_\gamma = \text{Pic}(R)$. Not much appears to be known about the higher slices.

3.3. Picard groups. For any commutative ring $\mathcal{R}$, the set $P_1(\mathcal{R})$ of isomorphism classes of f.g. projective $\mathcal{R}$-modules of constant rank 1 forms a group under $\otimes_\mathcal{R}$, with identity element the distinguished element $\langle \mathcal{R} \rangle$. This group is the Picard group of $\mathcal{R}$, usually denoted $\text{Pic}(\mathcal{R})$ (cf. [Ba], p. 131ff).

Specializing Proposition 2.6 to the case $n = 1$ and letting $U(\cdot) = \text{GL}_1(\cdot)$ denote groups of units, we obtain the following result (cf. [Kr], [DMV], [L]):

**Proposition.** There is an isomorphism of groups

$$\text{Ker Pic}(f) \cong \bigcap_{C \in \mathcal{C}} \text{Ker} \left( H^1(G, U(S)) \to \text{Hom}(C, U(S_C)) \right)$$

$$\cong \bigcap_{H \leq G} \text{Ker} \left( H^1(G, U(S)) \to \text{Hom}(H, U(S_H)) \right)$$

and an exact sequence of commutative monoids

$$1 \to \text{Ker } \sigma \to \text{Ker } K_0(f) \to \text{Ker Pic}(f) \to 1,$$

where $\sigma : H^1(G, \text{SL}(S)) \to \prod_{C \in \mathcal{C}} H^1(C, \text{SL}(S_C))$.

**Proof.** The fact that the isomorphism of Proposition 2.6 is an isomorphism of groups, not just of pointed sets, for $n = 1$ is a consequence of the identity of $T$-modules $S_d \otimes_S S_e \cong S_{de}$ for $d, e \in Z^1(G, U(S))$. The exact sequence is a consequence of the exact sequence $1 \to \text{SL}(\cdot) \to \text{GL}(\cdot) \xrightarrow{\text{det}} U(\cdot) \to 1$ which is split by the canonical embedding $U(\cdot) = \text{GL}_1(\cdot) \hookrightarrow \text{GL}(\cdot)$. Indeed, this sequence leads to a commutative
diagram of pointed sets ([S], §§5.4, 5.5)

\[
\begin{array}{ccc}
1 & 1 \\
\downarrow & \downarrow \\
H^1(G, \text{SL}(S)) & \prod_{C \in C} H^1(C, \text{SL}(S_C)) \\
\downarrow & \downarrow \\
H^1(G, \text{GL}(S)) & \prod_{C \in C} H^1(C, \text{GL}(S_C)) \\
\mu_1 \left( \begin{array}{c}
\pi_1 \\
\Pi \pi_{1,C} \\
\Pi \mu_{1,C}
\end{array} \right) & \\
\downarrow & \downarrow \\
H^1(G, \text{U}(S)) & \prod_{C \in C} H^1(C, \text{U}(S_C)) \\
\rho_1 & \\
1 & 1
\end{array}
\]

with \( \pi_1 \circ \mu_1 = \text{Id} \) and \( \pi_{1,C} \circ \mu_{1,C} = \text{Id} \). Here, \( \rho \) and \( \rho_1 \) are the usual restriction-reduction maps, as in the proof of Theorem 2.8. The diagram yields the exact sequence

\[ 1 \to \ker \sigma \to \ker \rho \to \ker \rho_1 \to 1 \]

which is in fact a sequence of commutative monoids, by §1.3. Finally, \( \ker K_0(f) \cong \ker \rho \) and \( \ker \text{Pic}(f) \cong \ker \rho_1 \).

3.4. Linear actions. Here, \( S = S(V) \) is the symmetric algebra of a finite dimensional \( k \)-vector space \( V \) and \( G \) is a subgroup of \( \text{GL}(V) = \text{Aut}_k(V) \). The \( G \)-action on \( V \) extends uniquely to an action of \( G \) on \( S \) by \( k \)-algebra automorphisms. Hypothesis (*) amounts to the requirement that \( |G|^{-1} \in k \), which will be assumed, and linear actions are never Galois. Both assertions follow from the existence of an augmentation \( \varepsilon : S \to k \) which is \( G \)-invariant (i.e., \( \varepsilon(s^g) = \varepsilon(s) \) holds for all \( s \in S \), \( g \in G \)); it is given by \( \varepsilon(V) = \{0\} \).

3.4.1. The factors \( S_H \). We now describe the factors \( S_H = S/J(H) \) of \( S \) that were introduced in §2.8. Fix a subgroup \( H \) of \( G \) and let \( V(H) \) denote the subspace of \( V \) that is generated by the elements \( v - v^h \) (\( v \in V, h \in H \)). Then \( H \subseteq G^T(\mathcal{M}) \) iff \( V(H) \subseteq \mathcal{M} \). Thus, \( J(H) \) is the intersection of all \( \mathcal{M} \in \text{Max } S \) with \( V(H) \subseteq \mathcal{M} \). Now \( V(H)S \) is a prime ideal of \( S \); in fact, \( S/V(H)S \cong S(V_H) \), where \( V_H = V/V(H) \) is the vector space of \( H \)-coinvariants of \( V \). Moreover, since \( kH \) is semisimple, we have \( V = V^H \oplus V(H) \). Therefore, \( J(H) = V(H)S \) and

\[ S_H \cong S(V_H) \cong S(V^H) \]

The canonical map \( S \to S_H \) is \( H \)-equivariantly split by \( S_H \cong S(V^H) \hookrightarrow S \).
3.4.2. Picard group \([\mathbb{K}]\). Here, \(\text{Pic}(S) = 1\) and so \(\text{Ker} \, \text{Pic}(f) = \text{Pic}(R)\). Also, \(U(S) = k^*\) and all \(U(S_H) = k^*\), and so the intersection in Proposition 3.3 reduces to the intersection of the kernels of the restriction maps \(\text{Hom}(G, k^*) \rightarrow \text{Hom}(H, k^*)\) which is obviously trivial. Thus,

\[ \text{Pic}(R) = 1. \]

3.4.3. A problem of Kraft. Recall that \(\text{Pic}(R)\) is an image of \(\tilde{K}_0(R)\) (cf. 3.2). It is an open question, raised by Kraft ([Kr], Problem 5.1), whether in fact \(\tilde{K}_0(R)\) is trivial or, equivalently, \(\text{Ker} \, K_0(f) = \{0\}\). A positive answer to Question 3.2 would easily entail this in characteristics \(> 0\), and for fixed-point-free actions in characteristic 0. In the latter case, Kraft’s problem has already been settled by a different method by Holland [Ho], at least for algebraically closed base fields \(k\). The following Proposition gives a cohomological reformulation of Holland’s result, and of a result of Gubeladze [Gu]. We put \(S_+ = VS\) and, as usual, \(\text{GL}(S, S_+)\) denotes the kernel of the reduction map \(\text{GL}(S) \rightarrow \text{GL}(S/S_+) = \text{GL}(k)\), and similarly for \(\text{GL}_n(S, S_+)\).

**Proposition.** Assume that \(k\) is algebraically closed of characteristic 0. If \(G\) acts fixed-point-freely on \(V\) then \(H^1(G, \text{GL}(S, S_+))\) is trivial. If, in addition, \(G\) is abelian (and hence actually cyclic) then \(H^1(G, \text{GL}_n(S, S_+))\) is trivial for all \(n\).

**Proof.** By fixed-point-freeness, \(V(H) = V\) holds for all non-identity subgroups \(H\) of \(G\), and hence \(J(H) = S_+\). Thus, in the notation of Theorem 2.8, \(\text{Ker} \, \rho_G \subseteq \text{Ker} \, \rho_H\) and so \(U(H^1(G, \text{GL}(S))) = \text{Ker} \, \rho_G\). Finally, by [Se], Prop. 38 on p. 49, the split exact sequence of \(G\)-groups \(1 \rightarrow \text{GL}(S, S_+) \rightarrow \text{GL}(S) \rightarrow \text{GL}(S/S_+) = \text{GL}(k) \rightarrow 1\) gives rise to an exact sequence of pointed sets

\[ 1 \rightarrow H^1(G, \text{GL}(S, S_+)) \longrightarrow H^1(G, \text{GL}(S)) \xrightarrow{\rho_G} H^1(G, \text{GL}(S/S_+)) \rightarrow 1. \]

Inasmuch as \(\text{Ker} \, \rho_G \cong \text{Ker} \, K_0(f)\) is trivial, by [Ho], triviality of \(H^1(G, \text{GL}(S, S_+))\) follows (and conversely).

If \(G\) is abelian then \(R\) is an affine normal semigroup algebra (cf. [H]), and hence all f.g. projectives over \(R\) are free, by Gubeladze’s Theorem [Gu]. In other words, all maps \(P_n(f)\) have trivial kernel which in turn says that \(H^1(G, \text{GL}_n(S, S_+))\) is trivial, exactly as above, using Proposition 2.6 instead of Theorem 2.8.

3.5. Multiplicative actions. In this case, \(S = kA\) is the group algebra of a f.g. free abelian group \(A\) and \(G\) is a subgroup of \(\text{GL}(A) = \text{Aut}_Z(A)\) acting on \(S\) by means of the unique extension of the natural \(\text{GL}(A)\)-action on \(A\). Again, hypothesis (\(\ast\)) is equivalent to \(|G|^{-1} \in k\) and multiplicative actions are never Galois, because of the \(G\)-invariant augmentation \(\varepsilon : S \rightarrow k\) given by \(\varepsilon(A) = \{1\}\).

3.5.1. The factors \(S_H\). Fix a subgroup \(H\) of \(G\), let \([A, H]\) denote the subgroup of \(A\) that is generated by the elements \(a^{-1}a^H\) \((a \in A, h \in H)\), and let \(\omega([A, H])S\) denote the ideal of \(S\) that is generated by the elements \(a - 1\) with \(a \in [A, H]\). So \(\omega([A, H])S\) is the kernel of the canonical map of group algebras \(S = kA \rightarrow kA_H\),
where $A_H = A/[A,H]$ denotes the $H$-coinvariants of $A$. Clearly, $H \subseteq G^T(M)$ iff $\omega([A,H])S \subseteq M$. We claim that $\omega([A,H])S$ is a semiprime ideal of $S$. Since $[H]$ is nonzero in $k$, this will follow if we can show that the torsion group of $A_H$ is annihilated by a power of $[H]$. To this end, write $(.)' = (.) \otimes_{\mathbb{Z}} \mathbb{Z}[1/[H]]$ and view $A'$ as a module over the group ring $\mathbb{Z}'H$. Putting $e = |H|^{-1} \sum_{h \in H} h \in \mathbb{Z}'H$, an idempotent of $\mathbb{Z}'H$, we have $A \subseteq A' = (A')^e \oplus (A')^{1-e} = (A')^H \oplus [A',H]$. Thus $A'/[A',H]$ is torsion-free, and hence so is $A/(A \cap [A',H])$. Since every element of $[A',H]/[A,H]$ is annihilated by a power of $[H]$, our claim follows. We conclude that $J(H) = \omega([A,H])S$ and

$$S_H \cong kA_H.$$  

The canonical map $S \to S_H$ is not split, in general, as $S_H$ need not be a domain.

3.5.2. Picard group ($\text{Pic}$). Again, Pic($S$) = 1 and so Pic($R$) can be determined from Proposition 3.3. Here, $U(S) = k^* \times A$ and $U(S_C) = k^* \times U_1$ with $A_C \subseteq U_1$, the group of normalized (augmentation 1) units of $S_C = kA_C$ (a strict inclusion, in general). The maps $H^1(G, U(S)) = \text{Hom}(G, k^*) \times \text{Hom}(G, A) \to \text{Hom}(C, U(S)) = \text{Hom}(C, k^*) \times \text{Hom}(C, U_1)$ decompose as the direct product of the restriction maps $\text{Hom}(G, k^*) \to \text{Hom}(C, k^*)$ times the maps $\text{Hom}(G, A) \to \text{Hom}(C, A) = \text{Hom}(C, A_C) \hookrightarrow \text{Hom}(C, U_1)$. The first factor contributes nothing to the kernel, as for linear actions. Since the map $H^1(C, A) \to H^1(C, A_C)$ is mono for cyclic $C$ (e.g., [3], p. 79), the contribution from the second factor is the kernel of the restriction map $H^1(G, A) \to H^1(C, A)$. Therefore,

$$\text{Pic}(R) = \bigcap_{C \in C} \text{Ker} \left( \text{Res}_C^G : H^1(G, A) \to H^1(C, A) \right).$$

This group need not be trivial; the results of a systematic computer aided investigation of all cases with rank $A \leq 4$ are reported in [4].

3.6. Moding out the radical. Returning to the general situation where $S$ is an arbitrary commutative ring satisfying (*), we briefly consider the reduction maps $p : \text{GL}(S) \to \text{GL}(S/\text{rad}S)$ and $p_n : \text{GL}_n(S) \to \text{GL}_n(S/\text{rad}S)$. Here, rad $S$ denotes the Jacobson radical of $S$. The following lemma is an application of Lemma 2.3.

**Lemma.** The maps $(p_n)_*^1 : H^1(G, \text{GL}_n(S)) \to H^1(G, \text{GL}_n(S/\text{rad}S))$ have trivial kernel, and so does $p_*^1 : H^1(G, \text{GL}(S)) \to H^1(G, \text{GL}(S/\text{rad}S))$.

**Proof.** It suffices to consider the maps $(p_n)_*^1$; the case of $p_*^1$ then follows by taking lim.

Using the identification $\text{Ker}(\text{Res}_S^G) \cong H^1(G, \text{GL}_n(S))$ of Lemma 2.3 and writing $S = S/\text{rad}S$ and $T = \mathbb{S}_*G$ (cf. [2]), the map $(p_n)_*^1$ becomes the map $\text{Ker}(\text{Res}_S^G) \to \text{Ker}(\text{Res}_{S_n}^\mathbb{S})$ that is given by $(P) \mapsto \langle P = P/\text{rad}S \rangle$. Say $\langle P \rangle$ belongs to the kernel of this map, that is, $\langle P \rangle \cong \mathbb{S}_{\mathbb{T}}^n$ or, equivalently, $P/\text{rad}S \cong S_{\mathbb{T}}^n/S_{\mathbb{T}}^n \text{rad}S$. Since $\text{rad}S \subseteq \text{rad}T$ (cf. [2], Theorem 7.2.5), the Nakayama Lemma implies that $P \cong S_{\mathbb{T}}^n$.
(cf. [Ba], Prop. (2.12) on p. 90). Thus $\langle P \rangle$ is the distinguished element of $\text{Ker} (\text{Res}_{S,n}^T)$, as required.

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