GAP FUNCTIONS AND HAUSDORFF CONTINUITY OF SOLUTION MAPPINGS TO PARAMETRIC STRONG VECTOR QUASIEQUILIBRIUM PROBLEMS

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Abstract. In this paper, we consider parametric strong vector quasiequilibrium problems in Hausdorff topological vector spaces. Firstly, we introduce parametric gap functions for these problems, and study the continuity property of these functions. Next, we present two key hypotheses related to the gap functions for the considered problems and also study characterizations of these hypotheses. Then, afterwards, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to these problems. Finally, as applications, we derive several results on Hausdorff (lower) continuity properties of the solution mappings in the special cases of variational inequalities of the Minty type and the Stampacchia type.

1. Introduction. One of the classes of problems in optimization, which has attracted attention of mathematicians all over the world, is the class of equilibrium problem. The equilibrium problem was named by Blum and Oettli [19] as a generalization of the variational inequality and optimization problems. This model has been proved to contain also other important problems related to optimization, namely, minimax problems, complementarity problems, Nash equilibrium, fixed-point and coincidence-point problems, traffic network problems, etc. During the last two decades, there have been many papers devoted to equilibrium and related problems. The first and the most important topic is the existence conditions for this class of problems (see, e.g., [24, 27, 29, 30, 36, 40, 49], and the references therein). Another important topic is the stability and sensitivity analysis, including semi-continuity, continuity, Hölder/Lipschitz continuity and differentiability properties.

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of the solution mappings to equilibrium and related problems. In fact, differentiability of the solution mappings is a rather high level of regularity and is somehow close to the Lipschitz continuous property (due to the Rademacher theorem). For generalized differentiation properties and applications to mathematical programs with equilibrium constraints, we would like to refer the reader to the two volume book by Mordukhovich [45, 46]. For the efficient use of the solution mappings, of course the higher level of regularity such as Hölder/Lipschitz continuity (see, e.g., [1, 2, 4, 5, 7, 8, 10, 11, 15, 16, 17, 38, 39]) they possess, the better they are. However, to have a certain property of the solution mapping, usually the problem data needs to possess the same level of the corresponding property, and this assumption about the data is often not satisfied in practice. In addition, in a number of practical situations such as mathematical models for competitive economies, the semicontinuity of the solution mapping is enough for the efficient use of the models. Hence, the study of the semicontinuity and continuity properties of solution mappings in the sense of Berge and Hausdorff is among the most interesting and important topic in the stability and sensitivity analysis of equilibrium problem, see, e.g., [3, 6, 9, 12, 28, 33, 34, 42, 43, 47, 48, 52].

It is well known that gap functions is an efficient approach to study existence conditions for problems related to optimization. Recently, gap functions have been used in stability conditions for such problems. In 1997, Zhao [50] introduced the key hypothesis \((H_1)\) related to a gap function of optimization problem and showed that \((H_1)\) was a sufficient condition for the Hausdorff lower semicontinuity of the solution mapping to the parametric nonlinear optimization problem. Then, by giving some characterizations of the hypothesis \((H_1)\), Kien [32] sharpened the main results of Zhao. Motivated by [32, 50], Li and Chen [41] and Chen and Li [20] presented hypothesis \((H_g)\) and employed it to obtain sufficient conditions for the solution mapping to the parametric weak vector variational inequality in Banach spaces to be Hausdorff lower semicontinuous. By applying an assumption similar to \((H_g)\), Chen et al. [21] improved and extended the results in [20] to the generalized vector quasivariational inequality. Answering an open question put forward in [21], Zhong and Huang [51] proved that the hypothesis \((H_g)\) was a sufficient and necessary condition for Hausdorff lower semicontinuity of solution mapping to the set-valued weak vector variational inequality in Banach spaces. Recently, the gap function method and the hypothesis \((H_g)\) were employed in [37] to derive the Hausdorff continuity property of the solution mapping of the scalar quasivariational inequality of the Minty type in finite dimensional spaces. Because of technical difficulties, as far as we know, there has been a single paper Zhong and Huang [52] devoted to the Hausdorff lower semicontinuity property of the solution map to equilibrium problems (in fact, the authors introduced hypothesis \((H_g)'\), similar to the one in [51], then the authors proved that the hypothesis \((H_g)'\) was not only sufficient but also necessary for the Hausdorff lower semicontinuity of the solution mapping to the parametric weak vector quasiequilibrium problem in Banach spaces), and there have not been any works on Hausdorff lower semicontinuity and Hausdorff continuity of the solution mappings to strong equilibrium and related problems using above mentioned approaches.

Motivated and inspired by the above observations, in this paper, we use the gap function method to study the continuity property of solution mappings to parametric strong vector quasiequilibrium problems in Hausdorff topological vector spaces.
We introduce gap functions of such problems and investigate their properties. Similar to the above mentioned papers, we also present two key hypotheses related to the gap functions of the problems and study some characterizations of these hypotheses. Then, we prove that these hypotheses are not only sufficient but also necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of solution mappings to the considered problems. As applications, we derive several results on the special cases of variational inequalities of the Minty type and the Stampacchia type. Our results are new and improve some key results in the literatures (e.g., [20, 32, 37, 41, 50]).

The paper is organized as follows. Sect. 2 is devoted to the setting of parametric quasiequilibrium problems and some preliminary results which are needed in the sequel. In Sect. 3, we introduce the gap functions for these problems and give some preliminary results which are needed in the sequel. In Sect. 4, we propose two key hypotheses \((H_p(\gamma_0))\) and \((H_0(\gamma_0))\) and obtain some characterizations of these hypotheses. Then we show that these conditions are sufficient and necessary for the solution mappings to the considered problems to be Hausdorff lower semicontinuous and Hausdorff continuous. As applications, we derive some results for the special cases of quasivariational inequalities of the Minty type and the Stampacchia type in the last section.

2. Preliminaries. Let \(X, Y, Z, P\) be Hausdorff topological vector spaces, \(A \subseteq X, B \subseteq Y\) and \(\Gamma \subseteq P\) be nonempty subsets, and let \(C\) be a closed convex cone in \(Z\) with \(\text{int} C \neq \emptyset\). Let \(K : A \times \Gamma \rightrightarrows A, T : A \times \Gamma \rightrightarrows B\) be multifunctions and \(f : A \times B \times A \times \Gamma \rightarrow Z\) be an equilibrium function, i.e., \(f(x, t, x, \gamma) = 0\) for all \(x \in A, t \in B, \gamma \in \Gamma\). Motivated and inspired by variational inequalities in the sense of Minty and Stampacchia, we consider the following two parametric strong vector quasiequilibrium problems.

\((\text{QEP}_1)\) Find \(x \in K(x, \gamma)\) such that
\[
f(x, t, y, \gamma) \in C, \forall y \in K(x, \gamma), \forall t \in T(y, \gamma).
\]

\((\text{QEP}_2)\) Find \(x \in K(x, \gamma)\) and \(t \in T(x, \gamma)\) such that
\[
f(x, t, y, \gamma) \in C, \forall y \in K(x, \gamma).
\]

For each \(\gamma \in \Gamma\), we denote the solution sets of \((\text{QEP}_1)\) and \((\text{QEP}_2)\) by \(S_1(\gamma)\) and \(S_2(\gamma)\), respectively. In general, \(S_1(\gamma)\) and \(S_2(\gamma)\) are distinct (as shown below in special cases (c) and (d)). If \(T(z, \gamma) = T(\gamma)\) then \(S_1(\gamma) \subset S_2(\gamma)\) for all \(\gamma \in \Gamma\), the following example shows that the converse is not true.

Example 2.1. Let \(X = Y = Z = P = \mathbb{R}, A = B = [0, 2], \Gamma = [0, 1], C = \mathbb{R}_+\), \(K(x, \gamma) = [\gamma, 2], T(x, \gamma) = [0, 1]\) and \(f(x, t, y, \gamma) = t(x - y)e^7\).

By direct calculations, we have \(S_1(\gamma) = \{2\}\) and \(S_2(\gamma) = [\gamma, 2]\) for each \(\gamma \in \Gamma\).

Since the existence of solutions has been well studied, throughout the article, we assume that \(S_1(\gamma) \neq \emptyset\) and \(S_2(\gamma) \neq \emptyset\), for each \(\gamma\) in a neighborhood of the reference point. To provide our motivations for these settings, we discuss some special cases of the problems.

(a) If \(X = \mathbb{R}^n, Y = \mathbb{R}^n, C = \mathbb{R}_+, \ f(x, z, y, \gamma) = \langle z, y - x \rangle\), then the problem \((\text{QEP}_1)\) reduces to the parametric scalar quasivariational inequality of the Minty type (in short, (MVI(\gamma))) studied in [37].

(b) If \(X = \mathbb{R}^n, Y = \mathbb{R}^n, C = \mathbb{R}_+, K(x, \gamma) = A, T(x, \gamma) = T(x), f(x, z, y, \gamma) = \langle z, y - x \rangle\), then the problem \((\text{QEP}_2)\) reduces to the Stampacchia variational inequalities (in short, (VI(T,A))) studied in [14].
Let $x$ be a multifunction.

$G$ and usc at $x$.

$d$) If $C = \mathbb{R}_+, K(x, \gamma) = A, f(x, z, y, \gamma) = \langle z, y - x \rangle$, then the problems (QEP$_1$) and (QEP$_2$) reduce to the variational inequalities of the Minty type and Stampacchia type (in short, (MVIP) and (SVIP)), respectively, studied in [25, 31, 35].

Firstly, we recall some basic definitions and some of their properties, see [9, 13, 18]. Let $X, Y$ be topological vector spaces, $A \subset X$ be a nonempty compact subset, and $G : X \rightrightarrows Y$ be a multifunction. $G$ is said to be lower semicontinuous (H-lsc) at $x_0$ if $G(x_0) \cap U \neq \emptyset$ for some open set $U \subseteq Y$. $G$ is said to be upper semicontinuous (H-usc) at $x_0$ if for each open set $U \supseteq G(x_0)$, there is a neighborhood $N$ of $x_0$ such that, $U \supseteq G(x)$, $\forall x \in N$. $G$ is said to be Hausdorff upper semicontinuous (H-usc) at $x_0$ if for each neighborhood $B$ of the origin in $Y$, there exists a neighborhood $N$ of $x_0$ such that, $G(x) \subseteq G(x_0) + B$, $\forall x \in N$. $G$ is said to be Hausdorff lower semicontinuous (H-lsc) at $x_0$ if for each neighborhood $B$ of the origin in $Y$, there exists a neighborhood $N$ of $x_0$ such that $G(x_0) \subseteq G(x) + B$, $\forall x \in N$. $G$ is said to be continuous at $x_0$ if it is both H-lsc and H-usc at $x_0$.

If $G$ has compact values, then $G$ is usc at $x_0$ if and only if, for each net $\{x_n\} \subset X$ which converges to $x_0$ and for each $y_0 \in G(x_0)$, there is $y_0 \in G(x_n)$ such that $y_0 \rightarrow y_0$.

Lemma 2.1. ([13, 18]). Let $X$ and $Y$ be topological vector spaces and $G : X \rightrightarrows Y$ be a multifunction.

(i) If $G$ is usc at $x_0$, then $G$ is H-usc at $x_0$. The converse is true if $G(x_0)$ is compact.

(ii) If $G$ is H-lsc at $x_0$, then $G$ is lsc at $x_0$. The converse is true if $G(x_0)$ is compact.

(iii) $G$ is lsc at $x_0$ if and only if for each net $\{x_n\} \subset X$ which converges to $x_0$ and for each $y_0 \in G(x_0)$, there is $y_0 \in G(x_n)$ such that $y_0 \rightarrow y_0$.

Lemma 2.2. ([26, 44]) For any fixed $e \in \text{int}C$, $y \in Y$ and the nonlinear scalarization function $\xi_e : Y \rightarrow \mathbb{R}$ defined by $\xi_e(y) := \min \{r \in \mathbb{R} : y \in re - C\}$, we have

(i) $\xi_e$ is a continuous and convex function on $Y$;

(ii) $\xi_e(y) \leq r \Leftrightarrow y \in re - C$;

(iii) $\xi_e(y) > r \Leftrightarrow y \not\in re - C$.

3. Gap functions for (QEP$_1$) and (QEP$_2$). In this section, we introduce the parametric gap functions for (QEP$_1$) and (QEP$_2$). Then we study some properties of these functions which will be needed in Sect. 4. In the rest of this section, we assume that $f$ is continuous on $A \times B \times A \times \Gamma$.

Definition 3.1. A function $g : A \times \Gamma \rightarrow \mathbb{R}$ is said to be a parametric gap function for problem (QEP$_1$) ((QEP$_2$, respectively), if:
(a) \( g(x, \gamma) \geq 0 \), for all \( x \in K(x, \gamma) \);
(b) \( g(x, \gamma) = 0 \) if and only if \( x \in S_1(\gamma) \).

Now we suppose that \( K \) and \( T \) have compact valued in a neighborhood of the reference point. We define two functions \( p : A \times \Gamma \rightarrow R \) and \( h : A \times \Gamma \rightarrow R \) as follows

\[
p(x, \gamma) = \max_{t \in T(y, \gamma)} \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)),
\]

and

\[
h(x, \gamma) = \min_{t \in T(x, \gamma)} \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)).
\]

Since \( K(x, \gamma) \) and \( T(x, \gamma) \) are compact sets for any \((x, \gamma) \in A \times \Gamma\), \( \xi_e \) and \( f \) are continuous, \( p \) and \( h \) are well-defined.

**Theorem 3.2.**  
(i) The function \( p(x, \gamma) \) defined by (1) is a parametric gap function for problem \((QEP_1)\).  
(ii) The function \( h(x, \gamma) \) defined by (2) is a parametric gap function for problem \((QEP_2)\).

**Proof.**  
(i) We define a function \( \varphi : E(\Gamma) \times B \times \Gamma \rightarrow R \), where \( E(\Gamma) = \cup_{\gamma \in \Gamma} E(\gamma) = \cup_{\gamma \in \Gamma} \{ x \in A : x \in K(x, \gamma) \} \), as follows

\[
\varphi(x, t, \gamma) = \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)), x \in E(\gamma), t \in B, \gamma \in \Gamma.
\]

(a) It is easy to see that \( \varphi(x, t, \gamma) \geq 0 \). Indeed, suppose to the contrary that there is \((x_0, t_0, \gamma_0) \in E(\gamma_0) \times B \times \Gamma \), such that \( \varphi(x_0, t_0, \gamma_0) < 0 \), then

\[
0 > \varphi(x_0, t_0, \gamma_0) = \max_{y \in K(x_0, \gamma_0)} \xi_e(-f(x_0, t_0, y, \gamma_0)) \geq \xi_e(-f(x_0, t_0, y, \gamma_0)), \forall y \in K(x_0, \gamma_0).
\]

When \( y = x_0 \), we have

\[
\xi_e(-f(x_0, t_0, x_0, \gamma_0)) = \xi_e(0) = 0,
\]

which is a contradiction. Hence,

\[
p(x, \gamma) = \max_{t \in T(y, \gamma)} \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)) \geq 0.
\]

(b) By definition, \( p(x_0, \gamma_0) = 0 \) if and only if, for any \( y \in K(x_0, \gamma_0) \) and \( t \in T(y, \gamma_0) \),

\[
\xi_e(-f(x_0, t, y, \gamma_0)) \leq 0.
\]

By Lemma 2.2(ii), this inequality holds if and only if, for any \( y \in K(x_0, \gamma_0), t \in T(y, \gamma_0) \),

\[
-f(x_0, t, y, \gamma_0) \in -C,
\]

or

\[
f(x_0, t, y, \gamma_0) \in C,
\]

i.e., \( x_0 \in S_1(\gamma_0) \). Hence, \( p \) is a parametric gap function for problem \((QEP_1)\).

(ii) We define a function \( \psi : E(\Gamma) \times B \times \Gamma \rightarrow R \) as follows

\[
\psi(x, t, \gamma) = \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)), x \in E(\gamma), t \in B, \gamma \in \Gamma.
\]

Using the same arguments as above, we also have \( \psi(x, t, \gamma) \geq 0 \), (i.e., \( h(x, \gamma) \geq 0 \)), and \( h(x_0, \gamma_0) = 0 \) if and only if \( x_0 \in S_2(\gamma_0) \). \(\square\)
The following result gives sufficient conditions for the parametric gap function $p$ to be continuous.

**Theorem 3.3.** Assume that $K$ and $T$ are continuous with compact values in $A \times \Gamma$. Then, $p$ and $h$ are continuous on $A \times \Gamma$.

**Proof.** Since the proof techniques are similar, we discuss only the continuity of $h$. Firstly, we prove that $h$ is lower semicontinuous on $A \times \Gamma$. Let $a \in \mathbb{R}$, and $\{(x_\alpha, \gamma_\alpha)\} \subseteq A \times \Gamma$, $(x_\alpha, \gamma_\alpha) \to (x_0, \gamma_0)$ and $h(x_\alpha, \gamma_\alpha) \leq a$ for all $\alpha$, we need to show that $h(x_0, \gamma_0) \leq a$. We have,

$$h(x_\alpha, \gamma_\alpha) = \min_{t \in T(x_\alpha, \gamma_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t, y, \gamma_\alpha)) \leq a,$$

and hence,

$$h(x_\alpha, \gamma_\alpha) = \min_{t \in T(x_\alpha, \gamma_\alpha)} \psi(x_\alpha, t, \gamma_\alpha),$$

where $\psi$ is defined as in Theorem 3.2. Since $\xi_e$ is continuous and $K$ is continuous with compact values on $A \times \Gamma$, Proposition 23, in Section 1 of Chapter 3 [13], deduces that $\psi$ is continuous. By the compactness of $T$, there exists $t_\alpha \in T(x_\alpha, \gamma_\alpha)$ such that

$$h(x_\alpha, \gamma_\alpha) = \min_{t \in T(x_\alpha, \gamma_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t, y, \gamma_\alpha)) = \psi(x_\alpha, t_\alpha, \gamma_\alpha).$$

Let $y_0 \in K(x_0, \gamma_0)$ be arbitrary. As $K$ is lower semicontinuous on $A \times \Gamma$, there exists $y_0 \in K(x_\alpha, \gamma_\alpha)$ such that $y_0 \to y_0$. Since $y_0 \in K(x_\alpha, \gamma_\alpha)$, we have

$$\xi_e(-f(x_\alpha, t_\alpha, y_0, \gamma_\alpha)) \leq a. \tag{3}$$

Since $T$ is upper semicontinuous with compact values on $A \times \Gamma$, we can assume that, there exists $t_0 \in T(x_0, \gamma_0)$ such that $t_\alpha \to t_0$. From the continuity of $f$ and $\xi_e$, taking the limit in (3), we have

$$\xi_e(-f(x_0, t_0, y_0, \gamma_0)) \leq a. \tag{4}$$

By the arbitrariness of $y_0$, it follows from (4) that

$$\psi(x_0, t_0, \gamma_0) = \max_{y \in K(x_0, \gamma_0)} \xi_e(-f(x_0, t_0, y, \gamma_0)) \leq a,$$

for some $t_0 \in T(x_0, \gamma_0)$. Hence,

$$h(x_0, \gamma_0) = \min_{t \in T(x_0, \gamma_0)} \max_{y \in K(x_0, \gamma_0)} \xi_e(-f(x_0, t, y, \gamma_0)) \leq a.$$

This proves that, for $a \in \mathbb{R}$, the lower level set $\{(x, \gamma) : h(x, \gamma) \leq a\}$ is closed. Therefore, $h$ is lower semicontinuous on $A \times \Gamma$.

Next, we will show that $h$ is upper semicontinuous on $A \times \Gamma$. Let $a \in \mathbb{R}$, $\{(x_\alpha, \gamma_\alpha)\} \subseteq A \times \Gamma$, $(x_\alpha, \gamma_\alpha) \to (x_0, \gamma_0)$ and $h(x_\alpha, \gamma_\alpha) \geq a$ for all $\alpha$. We need to show that $h(x_0, \gamma_0) \geq a$. We have,

$$h(x_\alpha, \gamma_\alpha) = \min_{t \in T(x_\alpha, \gamma_\alpha)} \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t, y, \gamma_\alpha)) \geq a,$$

and so,

$$\max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t, y, \gamma_\alpha)) \geq a, \forall t \in T(x_\alpha, \gamma_\alpha). \tag{5}$$
Let \( t_0 \in T(x_0, \gamma_0) \) be arbitrary. As \( T \) is lower semicontinuous on \( A \times \Gamma \), there exists \( t_\alpha \in T(x_\alpha, \gamma_\alpha) \) such that \( t_\alpha \to t_0 \). Since \( t_\alpha \in T(x_\alpha, \gamma_\alpha) \), it follows from (5) that
\[
\max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t_\alpha, y, \gamma_\alpha)) \geq a.
\]
Combining the compactness of \( K \) and the continuity of \( f \) and \( \xi_e \), there exists \( y_\alpha \in K(x_\alpha, \gamma_\alpha) \) such that
\[
\xi_e(-f(x_\alpha, t_\alpha, y_\alpha, \gamma_\alpha)) = \max_{y \in K(x_\alpha, \gamma_\alpha)} \xi_e(-f(x_\alpha, t_\alpha, y, \gamma_\alpha)) \geq a. \tag{6}
\]
Since \( K \) is upper semicontinuous with compact values, we can assume that \( y_\alpha \to y_0 \), for some \( y_0 \in K(x_0, \gamma_0) \) (take a subnet of \( \{y_\alpha\} \) if necessary). Taking the limit in (6), we obtain
\[
\xi_e(-f(x_\alpha, t_\alpha, y_0, \gamma_\alpha)) \geq a,
\]
and hence,
\[
\max_{y \in K(x_\alpha, \gamma_0)} \xi_e(-f(x_\alpha, t_\alpha, y, \gamma_0)) \geq a. \tag{7}
\]
Since \( t_0 \in T(x_0, \gamma_0) \) is arbitrary, it follows from (7) that
\[
h(x_0, \gamma_0) = \min_{t \in T(x_0, \gamma_0)} \max_{y \in K(x_0, \gamma_0)} \xi_e(-f(x_0, t, y, \gamma_0)) \geq a,
\]
i.e., for each \( a \in \mathbb{R} \), the upper level set \( \{(x, \gamma) : h(x, \gamma) \geq a\} \) is closed. Hence, \( h \) is upper semicontinuous on \( A \times \Gamma \).

\textbf{Remark 1.} As far as we know, there have not been any works on gap functions for parametric strong vector quasiequilibrium problems, and hence our parametric gap functions \( p \) and \( h \) are new and cannot be compared with the existing ones in the literatures (see e.g., parametric gap functions in [21, 41, 51, 52]).

4. \textbf{Hausdorff continuity of solution mappings.} In this section, we establish that conditions \((H_p(\gamma_0))\) and \((H_h(\gamma_0))\) are sufficient and necessary for the Hausdorff lower semicontinuity and Hausdorff continuity of the solution mappings to \((QEP_1)\) and \((QEP_2)\).

\textbf{Theorem 4.1.} Assume that \( A \) is compact, \( K \) is continuous with compact values on \( A \), and \( \text{lev}_{\geq c} f \) is closed. Then,

(i) \( S_1 \) is both upper semicontinuous and closed with compact values in \( \Gamma \) if \( T \) is lower semicontinuous on \( A \),

(ii) \( S_2 \) is both upper semicontinuous and closed with compact values in \( \Gamma \) if \( T \) is upper semicontinuous with compact values on \( A \),

where \( \text{lev}_{\geq c} f = \{ (x, t, y, \gamma) \in X \times Z \times X \times \Gamma : f(x, t, y, \gamma) \in C \} \).

\textbf{Proof.} As an example we present only the proof for (ii). Suppose, on the contrary, that \( S_2 \) is not usc at \( \gamma_0 \), for some \( \gamma_0 \in \Gamma \). Then there are an open subset \( U \), \( S_2(\gamma_0) \subset U \), and a net \( \{\gamma_\alpha\} \subset \Gamma \), \( \gamma_\alpha \to \gamma_0 \), such that there exists \( x_\alpha \in S_2(\gamma_\alpha) \setminus U \), for all \( \alpha \). By the compactness of \( A \), we can assume that \( x_\alpha \) tends to \( x_0 \), for some \( x_0 \in A \). Since \( A \) is compact and \( K \) is continuous with compact values, we conclude that \( K \) is closed at \( (x_0, \gamma_0) \), and hence \( x_0 \in K(x_0, \gamma_0) \). Now we show that \( x_0 \) belongs to \( S_2(\gamma_0) \). Suppose, to establish a contradiction, that \( x_0 \notin S_2(\gamma_0) \). As \( x_\alpha \in S_2(\gamma_\alpha) \), there is \( t_\alpha \in T(x_\alpha, \gamma_\alpha) \) such that
\[
f(x_\alpha, t_\alpha, y, \gamma_0) \in C, \forall y \in K(x_\alpha, \gamma_\alpha).
\]
Since $T$ is upper semicontinuous and with compact values at $(x_0, \gamma_0)$, we can assume that $t_\alpha \to t_0$ for some $t_0 \in T(x_0, \gamma_0)$. By above contradiction assumption, there must be $y_0 \in K(x_0, \gamma_0)$, such that

$$f(x_0, t_0, y_0, \gamma_0) \notin C. \quad (8)$$

Using the lower semicontinuity of $K$ at $(x_0, \gamma_0)$, we pick $y_\alpha \in K(x_\alpha, \gamma_\alpha)$ such that $y_\alpha \to y_0$. Due to $y_\alpha \in K(x_\alpha, \gamma_\alpha)$,

$$f(x_\alpha, t_\alpha, y_\alpha, \gamma_\alpha) \in C. \quad (9)$$

Combining (9) and the closedness of lev$_{\gamma \geq 0} f$, we conclude that

$$f(x_0, t_0, y_0, \gamma_0) \in C.$$  

This contradicts (8). Hence, $x_0 \in S_2(\gamma_0) \subseteq U$, which is again a contradiction, since $x_\alpha \notin U$ for all $\alpha$. Thus, $S_2$ is usc on $\Gamma$.

Next, we show that $S_2$ is closed on $\Gamma$. Let $\gamma_0 \in \Gamma$ be arbitrary, and suppose that $\{\gamma_\alpha\} \subset \Gamma$, $x_\alpha \in S_2(\gamma_\alpha)$, $(x_\alpha, \gamma_\alpha) \to (x_0, \gamma_0)$. By using the same arguments as above, we also achieve that $x_0 \in S_2(\gamma_0)$, and hence $S_2$ is closed on $\Gamma$.

Since $S_2$ is closed on $\Gamma$, $S_2(\gamma)$ is a closed subset of $A$ for all $\gamma \in \Gamma$. Hence, $S_2$ is compact-valued as $A$ is compact.

As examples, we provide the following examples to illustrate the essentialness of the assumptions of Theorem 4.1(i).

**Example 4.1.** Let $X = Y = Z = P = \mathbb{R}$, $A = B = [-1, 1], \Gamma = [0, 1], C = \mathbb{R}_+$, $K(x, \gamma) = (-\gamma, 1], T(x, \gamma) = [0, 1]$ and $f(x, t, y, \gamma) = x^2 - yx$. It is easy to see that $K$ is lsc, $T$ and $f$ are continuous, but $S_1$ is not upper semicontinuous at 0 (in fact, $S_1(0) = \{1\}$, and $S_1(\gamma) = \{0, 1\}, \forall \gamma \neq 0$). The reason is that $K$ is not use at $(x, 0)$.

**Example 4.2.** Let $X = Y = Z = P = \mathbb{R}$, $A = B = [-1, 1], \Gamma = [0, 1], C = \mathbb{R}_+$, $K(x, \gamma) \equiv [0, 1], f(x, t, y, \gamma) = t(y - x)e^\gamma$, and

$$T(x, \gamma) = \begin{cases} [-1, 0], & \text{if } \gamma = 0, \\ [0, 1], & \text{otherwise}. \end{cases}$$

Then, all the assumptions of Theorem 4.1(i) are fulfilled except the lower semicontinuity of $T$. Direct computations give us the solution map

$$S_1(\gamma) = \begin{cases} \{1\}, & \text{if } \gamma = 0, \\ \{0\}, & \text{if } \gamma \neq 0, \end{cases}$$

which is not use at $\gamma = 0$.

**Example 4.3.** Let $X = Y = Z = P = \mathbb{R}$, $A = B = \Gamma = [0, 1], C = \mathbb{R}_+$, $K(x, \gamma) \equiv [0, 1], T(x, \gamma) \equiv [0, 3]$, and

$$f(x, t, y, \gamma) = \begin{cases} (x - y)e^\gamma, & \text{if } \gamma = 0, \\ (y - x)e^\gamma, & \text{otherwise.} \end{cases}$$

It is clear that all the assumptions of Theorem 4.1(i) are satisfied except the closedness of lev$_{\gamma \geq 0} f$, (in fact, taking $x_n = 0, y_n = 1 + \frac{1}{n}, \gamma_n = \frac{1}{n}$, then $(x_n, y_n, \gamma_n) \to (0, 1, 0)$ and $f(x_n, y_n, \gamma_n) = (1 + \frac{1}{n})e^{1/n} > 0$ but $f(0, 1, 0) = -1 < 0$). By direct calculations, we obtain that $S_1(0) = \{1\}$ and $S_1(\gamma) = \{0\}, \forall \gamma \neq 0$, and thus $S_1$ is not use at 0.
Motivated by the hypotheses \((H_1)\) in [32, 50], \((H_g)\) in [21, 41] and \((H'_g)\) in [52], we introduce the following key assumptions.

\((H_p(\gamma_0))\) : Given \(\gamma_0 \in \Gamma\). For any open neighborhood \(U\) of the origin in \(X\), there exist \(\rho > 0\) and a neighborhood \(V(\gamma_0)\) of \(\gamma_0\) such that for all \(\gamma \in V(\gamma_0)\) and \(x \in E(\gamma) \setminus (S_1(\gamma) + U)\), one has \(p(x, \gamma) \geq \rho\).

\((H_h(\gamma_0))\) : Given \(\gamma_0 \in \Gamma\). For any open neighborhood \(U\) of the origin in \(X\), there exist \(\rho > 0\) and a neighborhood \(V(\gamma_0)\) of \(\gamma_0\) such that for all \(\gamma \in V(\gamma_0)\) and \(x \in E(\gamma) \setminus (S_2(\gamma) + U)\), one has \(h(x, \gamma) \geq \rho\).

Encouraged by [32, 52], we also study the characterizations of the hypotheses \((H_p)\) and \((H_g)\) as follows.

**Proposition 1.** Suppose that all the conditions in Theorem 3.3 are satisfied. For any open neighborhood \(U\) of the origin in \(X\), let

\[
\Xi_U(\gamma) := \inf_{x \in E(\gamma) \setminus (S_1(\gamma) + U)} p(x, \gamma),
\]

\[
\Theta_U(\gamma) := \inf_{x \in E(\gamma) \setminus (S_2(\gamma) + U)} h(x, \gamma).
\]

Then,

(i) \((H_p(\gamma_0))\) holds if and only if for any open neighborhood \(U\) of the origin in \(X\),

\[
\liminf_{\gamma \to \gamma_0} \Xi_U(\gamma) > 0.
\]

(ii) \((H_h(\gamma_0))\) holds if and only if for any open neighborhood \(U\) of the origin in \(X\),

\[
\liminf_{\gamma \to \gamma_0} \Theta_U(\gamma) > 0.
\]

**Proof.** Similar proof techniques can be employed to prove the two cases. We demonstrate only (i). If \((H_p(\gamma_0))\) holds, then for any open neighborhood \(U\) of the origin in \(X\), there exist \(\rho > 0\) and a neighborhood \(V(\gamma_0)\) of \(\gamma_0\) such that for all \(\gamma \in V(\gamma_0)\) and \(x \in E(\gamma) \setminus (S_1(\gamma) + U)\), one has \(p(x, \gamma) \geq \rho\). This concludes that \(\Xi_U(\gamma) \geq \rho\), for each \(\gamma \in V(\gamma_0)\), and hence

\[
\liminf_{\gamma \to \gamma_0} \Xi_U(\gamma) \geq \rho > 0.
\]

Conversely, for any open neighborhood \(U\) of the origin in \(X\),

\[
\omega = \liminf_{\gamma \to \gamma_0} \Xi_U(\gamma) > 0,
\]

then there exist a neighborhood \(V(\gamma_0)\) of \(\gamma_0\) such that

\[
\Xi_U(\gamma) \geq \rho := \frac{1}{2} \omega > 0,
\]

for all \(\gamma \in V(\gamma_0)\). Hence, for any \(x \in E(\gamma) \setminus (S_1(\gamma) + U)\), we have

\[
p(x, \gamma) \geq \rho > 0,
\]

i.e., \((H_p(\gamma_0))\) is satisfied. \(\square\)

Now, we show that the hypotheses \((H_p(\gamma_0))\) and \((H_h(\gamma_0))\) are not only sufficient but also necessary for the Hausdorff lower semicontinuity of the solution mappings to \((QEP_1)\) and \((QEP_2)\), respectively.

**Theorem 4.2.** Suppose that \(A\) is compact, \(K\) and \(T\) are continuous with compact values in \(A \times \Gamma\), and \(f\) is continuous in \(A \times B \times A \times \Lambda\). Then,

(i) \(S_1\) is Hausdorff lower semicontinuous on \(\Gamma\) if and only if \((H_p(\gamma_0))\) is satisfied for all \(\gamma_0 \in \Gamma\),
Proof. We present only the proof for (i). Assume, by contradiction, that \((S \cup \{a\})\) is compact, and hence we can assume that \(x \in X\) is a neighborhood of the origin in \(X\), and so we have

\[
\delta \in \xi \in H \in 0.
\]

Employing the hypothesis \((H_p(\gamma_0))\), there exists \(\rho > 0\) such that \(p(\xi_\beta, \gamma_\beta) \geq \rho\). By Theorem 3.3, \(p\) is continuous on \(\lambda \times \gamma\), and hence,

\[
p(x_0, \gamma_0) \geq \rho > 0,
\]

i.e.,

\[
p(x_0, \gamma_0) = \max_{t \in T(\gamma, \gamma_0)} \max_{y \in K(x_0, \gamma_0)} \xi_t(\alpha x_0, t, y, 0, \gamma_0)) > 0.
\]

Thus, there exist \(y_0 \in K(x_0, \gamma_0)\) and \(t_0 \in T(y_0, \gamma_0)\) such that

\[
\xi_t(\alpha x_0, t_0, y_0, \gamma_0) > 0.
\]

By Lemma 2.2(iii), we have

\[
f(x_0, t_0, y_0, \gamma_0) \not\in -C,
\]

or

\[
f(x_0, t_0, y_0, \gamma_0) \not\in C,
\]

which is in contradiction with \(x_0 \in S_1(\gamma_0)\). Therefore, \(S_1\) is Hausdorff lower semicontinuous on \(\Gamma\).

Conversely, suppose that \(S_1\) is Hausdorff lower semicontinuous at \(\gamma_0\), but \((H_p(\gamma_0))\) is not satisfied, for some \(\gamma_0 \in \Gamma\). By Proposition 1, there exists an open neighborhood \(U\) of the origin in \(X\), such that

\[
\liminf_{\gamma \to \gamma_0} \Xi_U(\gamma) = 0.
\]

Then there exists a net \(\{\gamma_\alpha\} \subset \Gamma\) with \(\gamma_\alpha \to \gamma_0\) such that

\[
\lim_{\gamma \to \gamma_0} \Xi_U(\gamma) = \lim_{\gamma \to \gamma_0} \inf_{\gamma \to \gamma_0} \{x \in E(\gamma) \setminus (S_1(\gamma) + U)\} p(x, \gamma_\alpha) = 0. \quad (11)
\]
Since $K$ is usc with compact values and $A$ is compact, we conclude that $K$ is closed. So, for each $\gamma \in \Gamma$, $E(\lambda)$ is a closed subset of $A$, and hence it is compact. Therefore, for each $\alpha$, $E(\gamma(\alpha)) \setminus \{S(\gamma(\alpha)) + U\}$ is compact. Since $p$ is continuous, there exists $x_0 \in E(\gamma(\alpha)) \setminus \{S(\gamma(\alpha)) + U\}$ satisfying $\Xi_U(\gamma(\alpha)) = p(x_0, \gamma(\alpha))$. Clearly, (11) implies the fact that

$$\lim_{\gamma(\alpha) \to \gamma_0} p(x_0, \gamma(\alpha)) = 0.$$  

Combining the compactness of $A$ and assumption (i), we can assume that $x_\alpha \to x_0$ for some $x_0 \in K(x_0, \gamma_0)$. By the continuity of $p$, we have $p(x_0, \gamma_0) = 0$ and so $x_0 \in S_1(\gamma_0)$. For any $\delta \in S_1(\gamma_0)$, since $S_1$ is Hausdorff lower semicontinuous on $\Gamma$, we can find a net $\{\delta_\alpha\}$, $\delta_\alpha \in S_1(\gamma_\alpha)$, such that $\delta_\alpha \to \delta$. As $x_\alpha \in E(\gamma(\alpha)) \setminus \{S(\gamma(\alpha)) + U\}$, we conclude that $x_\alpha - \delta_\alpha \notin U$, i.e., $x_0 - \delta \notin U$, for all $\delta \in S_1(\gamma_0)$. This in turn contradicts the fact that $x_0 \in S_1(\gamma_0)$. Thus, the proof is completed. \hfill $\square$

The following examples show that $(H_p(\gamma_0))$ in Theorem 4.2 is essential.

**Example 4.4.** Let $X = Y = Z = \mathbb{R}$, $A = B = [-1, 1]$, $\Gamma = [0, 1]$, $C = \mathbb{R}^+$, $K(x, \gamma) = [-1, 1]$, $T(x, \gamma) = [1, 3]$ and $f(x, t, y, \gamma) = \gamma y - \gamma x + yx^2 - x^3$.

It follows from the direct computations that

$$S_1(\gamma) = \begin{cases} \{-1, 0\}, & \text{if } \gamma = 0, \\ \{-1\}, & \text{otherwise}. \end{cases}$$

Hence, $S_1$ is not Hausdorff lower semicontinuous on $\Gamma$. Theorem 4.2(i) implies the fact that $(H_p(\gamma_0))$ does not hold at 0. Indeed, taking $e = 1 \in \text{int}\mathbb{R}^+$, we have

$$p(x, \gamma) = \max_{t \in T(y, \gamma)} \max_{y \in K(x, \gamma)} \xi_e(-f(x, t, y, \gamma)) = \max_{y \in K(x, \gamma)} (x^3 - yx^2 + \gamma x - \gamma y) = x^3 + x^2 + \gamma x + \gamma.$$  

We have $p$ is a parametric gap function of $(\text{QEP}_1)$. For given $\gamma_0 \in \Gamma$, for any open neighborhood $U_\varepsilon(0) = (-\varepsilon, \varepsilon)$, choose $\varepsilon$ such that $0 < \varepsilon < 1$. For any $\alpha > 0$ take $\{\gamma_\alpha\}, \gamma_\alpha \to 0$ with $0 < \gamma_\alpha < \beta$ and $x_\alpha = 0 \in E(\gamma_\alpha) \setminus \{S(\gamma_\alpha) + U_\varepsilon(0)\}$. We have $p(x_\alpha, \gamma_\alpha) = \gamma_\alpha < \beta$. Hence, $(H_p(\gamma_0))$ does not hold at 0.

**Example 4.5.** Let $X = C[0, 1]$, where $C[0, 1]$ is the space of all continuous functions of $[0, 1]$ into $\mathbb{R}$, $Y = Z = \mathbb{R}$, $A = \{x \in C[0, 1] : \int_0^1 x(u)du \leq 1\}$, $B = \Gamma = [0, 1]$, $C = \mathbb{R}^+$, $K(x, \gamma) = A$, $T(x, \gamma) = \{\gamma\}$ and $f(x, t, y, \gamma) = e^x \int_0^1 t|y(u) - y(u)|du$. In $C[0, 1]$, we consider a norm given by

$$\|x\| = \int_0^1 |x(u)|du, \forall x \in C[0, 1].$$

Then, $(C[0, 1], \|\cdot\|)$ is not a Banach space. Indeed, let $\{x_n\} \subset C[0, 1]$,

$$x_n(u) = \begin{cases} 0, & \text{if } u \in [0, \frac{1}{2}], \\ n(u - \frac{1}{2}), & \text{if } u \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}], \\ 1, & \text{if } u \in [\frac{1}{2} + \frac{1}{n}, 1]. \end{cases}$$

Then $\{x_n\}$ is a Cauchy sequence in $C[0, 1]$, but $x_n \not\to x$ for any $x \in C[0, 1]$. Let $\gamma_0 = 0$ and $\gamma_n = \frac{1}{n}$ and $e = 1 \in \text{int}\mathbb{R}_+$. It is clear that $h(x, \gamma_n) \to 0$ as $n \to \infty$, and hence $H_{\gamma_n}(\gamma_0)$ is not satisfied. Applying Theorem 4.2(ii), we conclude that $S_2$ is not Hausdorff lower semicontinuity on $\Gamma$. In fact, by direct computations,
we have $x_0 = 0 \in S_2(0)$. Suppose that there is $x_n \in S_2(\gamma_n)$, $x_n \to x_0$, i.e.,
$$f_0 \int_0^1 |x_n(u) - x_0(u)| \, du \to 0.$$ Let $y = 1 \in K(x, \gamma)$, we have
$$f(x, \gamma_n, y, \gamma_n) = \frac{1}{n} e^{\frac{1}{n}} \int_0^1 |x_n(u) - 1| \, du < 0,$$
for $n$ sufficiently large, which is impossible as $x_n \in S_2(\gamma_n)$, and hence $S_2$ is not Hausdorff lower semicontinuous at 0.

Now, we apply the results of Theorems 4.1, 4.2 in deriving the sufficient and necessary conditions for the solution mappings to (QEP$_1$) and (QEP$_2$) to be Hausdorff continuous.

**Theorem 4.3.** Suppose that all the conditions in Theorem 4.2 are satisfied. Then,
(i) $S_1$ is Hausdorff continuous with compact values in $\Gamma$ if and only if $(H_p(\gamma_0))$
holds for all $\gamma_0 \in \Gamma$,
(ii) $S_2$ is Hausdorff continuous with compact values in $\Gamma$ if and only if $(H_h(\gamma_0))$
holds for all $\gamma_0 \in \Gamma$.

**Remark 2.** As mentioned in the Introduction, until now there have not been any papers devoted to the continuity properties in the sense of Berge and Hausdorff for the solution mappings to the strong vector quasiequilibrium problems by gap function method, and hence our results, Theorems 4.2, 4.3 are new and cannot be compared with the existing ones in the literature. Moreover, by using the imposed hypotheses $(H_p)$ and $(H_h)$, we can study the sufficient and necessary conditions for the continuity properties of solution mappings to these problems in Hausdorff topological vector spaces, while most of the corresponding existing results only apply to problems in Banach spaces.

5. **Applications.** Since the equilibrium problems contain optimization problems, variational inequalities, fixed-point and coincidence-point problems, the complementarity problems, the network traffic problems, etc, the results of the previous sections can be employed to derive the corresponding results for such special cases. In this section, as an application, we consider the special case of quasivariational inequalities of the types of Minty and Stampacchia, and apply some typical obtained results to this special setting.

Let $X, Y, Z, A, B, C, K, T$ be as in Sect. 2, $L(X; Y)$ be the space of all linear continuous operators from $X$ into $Y$ and $g : A \times \Lambda \to A$ be a vector function. $\langle t, x \rangle$ denotes the value of a linear operator $t \in L(X; Y)$ at $x \in X$.

For each $\gamma \in \Gamma$, we consider the following two parametric strong vector quasivariational inequalities of the types of Minty and Stampacchia (in short, (MQVI) and (SQVI), respectively).

(MQVI) Find $x \in K(x, \gamma)$ such that
$$\langle t, y - g(x, \gamma) \rangle \in C, \forall y \in K(x, \gamma), \forall t \in T(y, \gamma).$$

(SQVI) Find $x \in K(x, \gamma)$ and $t \in T(x, \gamma)$ such that
$$\langle t, y - g(x, \gamma) \rangle \in C, \forall y \in K(x, \gamma).$$

By setting
$$f(x, t, y, \gamma) = \langle t, y - g(x, \gamma) \rangle,$$
the problems (MQVI) and (SQVI) become special cases of (QEP$_1$) and (QEP$_2$), respectively.
For each $\gamma \in \Gamma$, we denote the solution sets of the problems (MQVI) and (SQVI) by $\Phi(\gamma)$ and $\Psi(\gamma)$, respectively. We assume that these solution sets are nonempty in a neighborhood of a reference point.

The following results are derived from the main results of Sect. 4.

**Corollary 1.** Assume that $A$ is compact, $K$ and $T$ are continuous with compact values in $A \times \Gamma$, and $g$ is continuous in $A \times \Gamma$. Then,

(i) $\Phi$ is Hausdorff lower semicontinuous on $\Gamma$ if and only if $(H_p(\gamma_0))$ holds for all $\gamma_0 \in \Gamma$,

(ii) $\Psi$ is Hausdorff lower semicontinuous on $\Gamma$ if and only if $(H_h(\gamma_0))$ holds for all $\gamma_0 \in \Gamma$.

**Proof.** Combining the continuity of $g$ and (12), we conclude that $f$ is continuous. It is easy to see that all the assumptions of Theorem 4.2 are satisfied, and hence by applying this theorem we obtain the conclusions of Corollary 1.

**Remark 3.** (a) In special case, $X = \mathbb{R}^n, Y = \mathbb{R}^m, C = \mathbb{R}_+$, $g(x, \gamma) = x$, the problem (MQVI) reduces to the parametric scalar quasivariational inequality of the Minty type which was studied in [37]. As our parametric gap function $p(x, \gamma)$ is new and different from the existing ones in the literature, Corollary 1(i) is new and extends Theorem 4.4 in [37]. Moreover, our hypothesis $(H_p)$ is not only sufficient but also necessary for the Hausdorff lower semicontinuity of the solution map to (MQVI), while the assumption $(H_g)$ in [37] is only sufficient for the corresponding property of the solution map to the considered problem.

(b) As far as we know, there have not been any works on Hausdorff lower semicontinuity of the solution map to parametric strong quasivariational inequalities of the Stampacchia type, and hence, even for the special case, Corollary 1(ii) is new.

For winding up this section, we present the sufficient and necessary conditions for Hausdorff continuity of the solution mappings to the problems (MQVI) and (SQVI).

**Corollary 2.** Suppose that all the conditions in Corollary 1 are satisfied. Then,

(i) $\Phi$ is Hausdorff continuous with compact values in $\Gamma$ if and only if $(H_p(\gamma_0))$ holds for all $\gamma_0 \in \Gamma$,

(ii) $\Psi$ is Hausdorff continuous with compact values in $\Gamma$ if and only if $(H_h(\gamma_0))$ holds for all $\gamma_0 \in \Gamma$.

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