THE EXISTENCE OF LINEAR SELECTION AND THE QUOTIENT LIFTING PROPERTY

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Abstract. Lifting properties for Banach spaces are studied. An alternate version of the lifting property due to Lindenstrass and Tzafriri is proposed and a characterization, up to isomorphism, is given. The quotient lifting property for pairs of Banach spaces \((X, J)\), with \(J\) proximinal in \(X\), is considered and several conditions for the property to hold are given.

1. Introduction

Following Lindenstrass and Tzafriri [8], we say a Banach space \(Y\) has the lifting property (LP) if for every bounded operator \(\psi\) from a Banach space \(X\) onto a Banach space \(W\) and for every \(S \in L(Y,W)\), there is \(\hat{S} \in L(Y,X)\) such that \(S = \psi \circ \hat{S}\). We note that in [8] it is shown that \(\ell_1\) has the LP, but that can easily be extended to show that any space isomorphic to \(\ell_1\) has the LP. The converse for infinite dimensional separable spaces is given in [8]. If we put the attention on \(W\) rather than \(Y\), we say \(W\) has the alternate lifting property (ALP), if for an operator \(\psi\) from \(X\) onto \(W\) and \(S \in L(Y,W)\), there is \(\hat{S} \in L(Y,X)\) such that \(S = \psi \circ \hat{S}\). From the ideas in [8] we have the following theorem.

**Theorem.** Let \(W\) be a Banach space. Then \(W\) satisfies the ALP if and only if \(W\) is isomorphic to \(\ell_1(\Gamma)\), for a suitable index set \(\Gamma\).

We consider the case where \(W\) is the quotient of \(X\) by a closed subspace \(J\) with \(\psi = \pi\), where \(\pi\) is the quotient map. We are interested in the existence of norm preserving lifts of a bounded operator \(S : Y \to X/J\). If, for every \(Y\) and \(S\), such a lift exists then we say that the pair \((X, J)\) has quotient lifting property (QLP).
First, we give an alternate version of the QLP. We say that $Y$ satisfies the alternate quotient lifting property (AQLP) if given a Banach space $X$ with closed subspace $J$ and bounded operator $S$ from $Y$ to $X/J$, there exists $\hat{S} \in \mathcal{L}(Y,X)$ such that $S = \pi \circ \hat{S}$. We obtain a similar characterization for Banach spaces with the AQLP.

**Theorem.** If $Y$ is Banach space then $Y$ satisfies the AQLP if and only if $Y$ is isomorphic to $\ell_1(\Gamma)$, for a suitable index set $\Gamma$.

We remark that we have not required $\|\hat{S}\| = \|S\|$ in the alternate versions as we do for the QLP.

In this paper we first study conditions on a Banach space that ensure the existence of liftings for operators either into the space (ALP) or on the space (AQLP), and for which a diagram commutes. We also investigate conditions for the existence of norm preserving lifts of operators (QLP) and its interconnections with metric projections, linear selections and proximinality.

We start by recalling some definitions to be used throughout. For a closed subspace $J$ of $X$, we denote by $\pi$ the canonical quotient map from $X$ to $X/J$, given by $\pi(x) = x + J$ ($\|\pi\| = 1$ unless $J = X$). A subspace $J$ of a normed linear space $X$ is called proximinal (resp. Chebyshev) if for each $x$ in $X$, the set of best approximations to $x$ in $J$,

$$P_J(x) := \{j \in J : \|x - j\| = \text{dist}(x,J)\}$$

is nonempty (resp. a singleton). The set valued map $P_J$ is called the metric projection onto $J$. For a proximinal subspace $J$, a selection for $P_J$ is a function $p : X \to J$ with $p(x) \in P_J(x)$, for every $x$. If $p$ is linear we call it a linear selection. The metric complement of $J$ is defined to be $J_0 = \{x \in X : \|x\| = \text{dist}(x,J)\}$. We also recall that an M-ideal in a Banach space is a subspace for which the annihilator is an L-summand of the dual space. For details we refer the reader to Chapter 1 in [6]. It is a known fact that M-ideals are proximinal.
In section 2 we prove the theorems stated above. These theorems characterize spaces with the ALP and spaces with the AQLP, up to isomorphism. We draw some conclusions concerning subspaces of Banach spaces with the property.

In section 3 we show that proximinality is a necessary condition for the QLP and the metric complement being a subspace is sufficient for QLP.

In section 4 we study the relation between properties of the metric complement of \( J \) and the QLP for \((X, J)\). It is worth to mention that the existence of a linear selection is an invariant condition under isometric isomorphisms. From this fact we derive that the QLP holds for several pairs of Banach spaces. Moreover, for proximinal subspaces, we show that the QLP is equivalent to the existence of a linear selection.

In Section 5 we consider the QLP for subspaces that are M-ideals. If \( X \) is reflexive and \( J \) is an \( M \)-ideal in \( X \), then \((X, J)\) has the QLP. The same holds for \( C([0, 1]^n) \) and \( J \) an \( M \)-ideal in \( C([0, 1]^n) \). The QLP does not hold, in general, for \((C(Ω), J)\), with \( Ω \) a compact Hausdorff space and \( J \) an \( M \)-ideal of \( C(Ω) \). Nevertheless, the QLP holds when \( Ω \) is compact and metrizable.

### 2. The Alternate Lifting Properties

We define the following alternate lifting properties for Banach spaces.

**Definition 1.** Let \( W \) be a Banach space. Then,

- \( W \) has the ALP if and only if given Banach spaces \( X \) and \( Y \), an operator \( ψ \) from \( X \) onto \( W \) and \( S ∈ L(Y, W) \), there is \( ˆS ∈ L(Y, X) \) such that \( S = ψ ◦ ˆS \).

- \( Y \) has the AQLP if and only if for every Banach space \( X \), a closed subspace \( J \) and a bounded operator \( S \) from \( Y \) to \( X/J \), there exists \( ˆS ∈ L(Y, X) \) such that \( S = π ◦ ˆS \), with \( π \) denoting the quotient map from \( X \) onto \( X/J \).

**Theorem 2.** Let \( W \) be a Banach space. Then \( W \) satisfies the ALP if and only if \( W \) is isomorphic to \( ℓ_1(Γ) \), for some suitable index set \( Γ \).
Proof. The statement is straightforward for finite dimensional spaces. We present the proof for the infinite dimensional case. We recall that any Banach space is the quotient of $\ell_1(\Gamma)$ for some suitable index set $\Gamma$ (see [10] page 21), and therefore is the range of a bounded operator on $\ell_1(\Gamma)$. Let $W$ satisfy the ALP and in that definition, let $X = \ell_1(\Gamma)$, $\psi$ a bounded operator from $X$ onto $W$, $Y = W$, and $S$ the identity operator on $W$, denoted by $Id$. Then by the ALP, there exists $\hat{S}$ from $W$ to $\ell_1(\Gamma)$ such that

$$Id = S = \psi \circ \hat{S}.$$  

It is easy to see that $\hat{S} \circ \psi$ is a bounded projection onto a subspace of $\ell_1(\Gamma)$. It follows from the Koethe extension of Theorem 2.a.3 in [8, p. 108] that $W$ is isomorphic to $\ell_1(\Gamma)$.

On the other hand, suppose $T$ is an isomorphism from $\ell_1(\Gamma)$ onto $W$. Let $w_\gamma = Te_\gamma$, where $\{e_\gamma\}$ denotes the standard family of functions from $\Gamma$ into the scalar field such that $e_\gamma(\alpha) = 1$ for $\gamma = \alpha$, and 0 otherwise. Thus if $w \in W$, there exists an absolutely summable family $\{\alpha_\gamma\}$ of scalars such that $w = \sum_{\gamma \in \Gamma} \alpha_\gamma w_\gamma$. There exists $x_\gamma \in X$ such that $\psi(x_\gamma) = w_\gamma$ and since $\psi$ is surjective and open, the family $\{x_\gamma\}$ must be bounded. For $y \in Y$ there is an absolutely summable family of scalars $\{\alpha_\gamma\}$ such that $S(y) = \sum_{\gamma \in \Gamma} \alpha_\gamma w_\gamma$. Since the series $\sum_{\gamma \in \Gamma} \alpha_\gamma x_\gamma$ is absolutely summable, it must converge to some $x \in X$. We define $\hat{S}(y) = x$ and it follows that $\hat{S}$ is a bounded linear operator from $Y$ to $X$ such that $S = \psi \circ \hat{S}$.  

Similar considerations apply to the AQLP (see Definition [1]) to prove the characterization for Banach spaces with the AQLP.

**Theorem 3.** Let $Y$ be a Banach space. Then $Y$ satisfies the AQLP if and only if $Y$ is isomorphic to $\ell_1(\Gamma)$.

**Proof.** Suppose $Y$ has AQLP. As above, $Y$ is isometric with a quotient space $\ell_1(\Gamma)/J$ for some index set $\Gamma$. Let $X = \ell_1(\Gamma)$ and $J$ its subspace as given. Let $S$ be the isometry between $Y$ and $\ell_1(\Gamma)$. For $\hat{S}$ guaranteed by AQLP, the operator $\hat{S} \circ S^{-1} \circ \pi$ is a projection onto a complemented
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subspace of $\ell_1(\Gamma)$. The remainder of the proof follows as in the proof of Theorem 2.

Therefore the class of Banach spaces with the ALP coincides with that of the AQLP.

**Theorem 4.** Let $W$ be a Banach space and let $W_1$ be a complemented subspace of $W$. If $W$ has the AQLP then $W_1$ has the AQLP.

**Proof.** Let $X$ be a Banach space and $J$ be a closed subspace of $X$. Since $W_1$ is complemented in $W$, there exists a projection $P$ from $W$ onto $W_1$. Given $S \in L(W_1, X/J)$ and since $W$ has the AQLP, there exists $\tilde{S} \circ P$, a lift of $S \circ P$, such that $\pi \circ \tilde{S} \circ P = S \circ P$. Then $\tilde{S} \circ P|_{W_1}$ is a lift for $S$. This concludes the proof.

Theorems 3 and 4 imply that every separable infinite dimensional complemented subspace of a space with the AQLP (or ALP) is isomorphic to $\ell_1$. We invoke the main theorem in [3] to conclude that $C(\Omega, E)$ (with $\Omega$ an infinite compact Hausdorff space and $E$ an infinite dimensional Banach space) does not satisfy the AQLP. In particular, $C(\Omega, C(\Omega)); C(K_1 \times K_2)$, where $K_1$ and $K_2$ are infinite compact Hausdorff spaces, do not satisfy the AQLP, since each of these spaces contain a complemented copy of $c_0$. See [3].

3. Proximinality and the Quotient Lifting Property

In this section we start with the definition for a pair of Banach spaces to have the QLP. The authors are grateful to T.S.S.R.K.Rao for mentioning a property considered in [6] that lead to this definition.

**Definition 5.** Let $X$ be a Banach space and $J$ be a closed subspace of $X$. The pair $(X, J)$ has the QLP if and only if for every Banach space $Y$ and every bounded operator $S : Y \to X/J$ there exists a bounded operator $T$ from $Y$ to $X$ lifting $S$ while preserving the norm, i.e. $\|T\| = \|S\|$ and $\pi \circ T = S$.

Given $S$, as in the Definition 5, there may exist several liftings, i.e. bounded operators $T$ such that $S = \pi \circ T$. For each such lifting, $T$, we
have $\|S\| \leq \|T\|$. It is easy to construct examples where the inequality is strict.

**Theorem 6.** Let $X$ be a Banach space and $J$ a closed subspace of $X$. Then

(i) If $(X, J)$ has the QLP then $J$ is proximinal in $X$.

(ii) If $J$ is proximinal in $X$ and $P_J$ has a linear selection, then $(X, J)$ has the QLP.

**Proof.** If $J$ is not proximinal in $X$, then there must exist a norm one element $x \in X$ such that $\text{dist}(x, J) < \|x - j\|$, for every $j \in J$. We define $Y = \text{span}\{x\}$. Let $S : Y \to X/J$ be such that $S(x) = x + J$ then $\|S\| = \text{dist}(x, J)$. Every bounded operator $T : Y \to X$ such that $\pi \circ T = S$, satisfies $T(x) = x + j$, for some $j \in J$. Then $\|S\| \leq \|x + j\| \leq \|T\|$. This proves (i).

For (ii), suppose $P_J$ admits a linear selection $p : X \to J$ and let $\psi : X/J \to X$ be defined by $x + J \mapsto x - p(x)$. To see that $\psi$ is well defined, we suppose that $x_1 + J = x_2 + J$. Then $x_2 = x_1 + j$ for some $j \in J$, and by the linearity of $p$, we can write

$$\psi(x_2 + J) = x_2 - p(x_2) = x_1 + j - p(x_1) - j = x_1 - p(x_1) = \psi(x_1 + J).$$

Thus $\psi$ is a linear isometry. For a bounded operator $S : Y \to X/J$, $\tilde{S} = \psi \circ S$ provides the lifting as desired. □

**Corollary 7.** If $J$ is an $M$-summand in $X$, then $(X, J)$ has QLP.

**Proof.** If $J$ is an $M$-summand in $X$, then $J$ is proximinal and has a linear (continuous) selection, namely, the $M$-projection $P$. Therefore the statement. □

David Yost in [11] proved that a Banach space is reflexive if and only if every closed subspace is proximinal. Hence every nonreflexive space must contain a nonproximinal closed subspace and therefore this pair does not have the QLP.

We now give an example of a pair of spaces for which the proximinality condition holds but the QLP does not hold. First, we recall the
notion of total subset of a Banach space. A subset $F$ of the dual space $X^*$ of a Banach space $X$ is said to be total if $f(x) = 0$ for each $f \in F$ implies that $x = 0$.

**Example 8.** We consider the pair $(\ell_\infty, c_0)$. From [6] we know $c_0$ is an $M$-ideal, hence a proximinal subspace of $\ell_\infty$. We show that the pair $(\ell_\infty, c_0)$ does not have the QLP. To see this, we observe that there is no injective bounded linear map from $\ell_\infty/c_0$ to $\ell_\infty$. We assume the existence of an injective bounded linear map $\phi$ from $\ell_\infty/c_0$ to $\ell_\infty$. Since $(\ell_\infty)^*$ contains a countable total set, $\{\tau_i : \tau_i(e_j) = \delta_{ij}\}$, then $\phi^* \tau_i$ is a total set for $(\ell_\infty/c_0)^*$. Indeed, for $z \in \ell_\infty$ such that $\phi^* \tau_i(z) = 0$ for all $i$, then $\phi z = 0$. Since $\phi$ is injective, then $z = 0$. Arterbaun and Whitley in [1] showed that $(\ell_\infty/c_0)^*$ has no countable total subset. Therefore, the identity map from $\ell_\infty/c_0$ to $\ell_\infty/c_0$ has no lift. If such a lift existed, it would be an injective bounded linear map. This is impossible and it shows that the pair $(\ell_\infty, c_0)$ does not have the QLP.

We note that the quotient space $\ell_\infty/c_0$ does not have the ALP. To establish this take $S$ to be the identity map from $\ell_\infty/c_0$ to $\ell_\infty/c_0$ and $\psi$ to be the quotient map from $\ell_\infty$ to $\ell_\infty/c_0$. If there exists $\hat{S}$ from $\ell_\infty/c_0$ to $\ell_\infty$ such that $S = \psi \circ \hat{S}$, then $\hat{S}$ is an injective map but no such map exists. Theorem 2 implies that $\ell_\infty/c_0$ is not isomorphic to a $\ell_1(\Gamma)$, for any index set $\Gamma$.

For completeness of exposition we include the following theorems from [5] and from [4] to be used later.

**Theorem 9.** (See Deutsch, [5]) Let $J$ be a proximinal subspace of a normed linear space $X$. Then the following are equivalent

1. $P_J$ has a linear selection.
2. $J_0$, the metric complement of $J$, contains a closed subspace $J_1$ such that $X = J \oplus J_1$.

**Theorem 10.** (See Cheney and Wulbert [4]) Let $J$ be a subspace of $X$. Then

1. $J$ is a proximinal subspace if and only if $X = J + J_0$. 

     

(2) J is Chebyshev if and only if \( X = J + J_0 \) and the representation of each \( x \in X \) as \( x = j + j_0 \), where \( j \in J \) and \( j_0 \in J_0 \), is unique.

Thus we have the following proposition.

**Proposition 11.** Let J be a proximinal subspace of X. Then

(i) If \( J_0 \) is a subspace of X then \((X, J)\) has the QLP.

(ii) (cf. [7]) \( J_0 \) is a subspace if and only if J is Chebyshev and \( P_J \) is linear.

**Proof.** From Theorem 10, we have \( X = J + J_0 \). By assumption \( J_0 \) is a subspace of X, so Theorem 9 asserts that \( P_J \) has a linear selection. So (i) follows from Theorem 6-(ii).

Since J is proximinal in X and \( J_0 \) is a subspace, Theorem 10-(1) implies that \( X = J + J_0 \). It is clear that \( J \cap J_0 = \{0\} \), then \( X = J \oplus J_0 \). The existence of a linear selection of the metric projection \( P_J \) follows from Theorem 9. The uniqueness of the representation of any element in X implies that J is Chebyshev by Theorem 10-(2). Conversely, since \( J_0 = \{x - P_J(x) : x \in X\} \), the linearity of the metric projection implies \( J_0 \) is a subspace of X. \( \square \)

Next example shows that \( J_0 \) being a subspace is not a necessary condition for the QLP.

**Example 12.** (See example 2.7 of [5]) Let \( X = \ell^2_\infty(\mathbb{R}) \) and \( J = \text{span}\{e_1\} \). It is easy to see that \( J_0 \) is not a subspace. Just consider \((0, 2)\) and \((1, -2)\), both in \( J_0 \) with sum \((1, 0) \notin J_0 \). We observe that \( p : X \to J \) given by \( p(x, y) = (x, 0) \) is a linear selection for the metric projection, Theorem 2-(ii) implies that \((\ell^2_\infty(\mathbb{R}), J)\) has the QLP.

4. **On metric complement of subspaces for low dimensional Banach spaces**

In this section we start with a characterization of the metric complements of subspaces of \( \ell^2_\infty(\mathbb{R}) \). First, we notice that for \( J = \{0\} \) or \( J = X \) then \( J_0 = X \) or \( J_0 = \{0\} \) respectively.
Proposition 13. Let $X = \ell^{(2)}_\infty(\mathbb{R})$ and let $J$ be a non-trivial subspace of $X$. Then $J_0$ is a subspace of $X$ if and only if $J$ is generated by $(u,v)$ with $uv \neq 0$.

Proof. We claim that $J_0$ is homogeneous, i.e. given $(a,b) \in J_0$ and $\lambda$ a nonzero scalar, then $(\lambda a, \lambda b) \in J_0$. Towards this claim we just observe that

$$\inf_{j \in J} \|(\lambda a, \lambda b) - j\| = |\lambda| \inf_{j \in J} \|(a, b) - \lambda^{-1} j\| = |\lambda| \|(a, b)\| = \|(\lambda a, \lambda b)\|.$$

Let $J$ be a non-trivial subspace of $\ell^{(2)}_\infty(\mathbb{R})$ generated by $(u,v)$. If $uv = 0$, WLOG assume that $u \neq 0$ and $v = 0$, then $J = \text{span}(1, 0)$. If $J_0$ is a subspace we have $w_0 = (0, 1)$ and $w_1 = (1, 1)$ belong to $J_0$ but $w_1 - w_0 = (1, 0)$ does not belong to $J_0$. This shows that $J_0$ is not a subspace. If $uv \neq 0$, let $a = \frac{v}{u}$ then $J$ is generated by $(1, a)$. For $x_1 = (1, c)$

$$\text{dist}(x_1, J) = \inf_{t \in \mathbb{R}} \{\max\{|1 - t|, |c - at|\}\}$$

The graphs of $|1 - t|$ and $|c - at|$ plotted in the figure above illustrate the cases to be considered in the computation of $\text{dist}(x_1, J)$. In each case, the distance is equal to the second coordinate of $P$. We now proceed with the computations.

Case I: $a > 0$

If $c > a$, $\text{dist}(x_1, J) = \frac{c - a}{1 + a}$ which is not equal to $\|x_1\|_\infty$ for any choice of $c$. If $c < a$, $\text{dist}(x_1, J) = \frac{a - c}{1 + a}$ which is equal to $\|x_1\|_\infty$ only if $c = -1$. So $J_0$ is generated by $(1, -1)$ and is a subspace of $X$.

Case II: $a < 0$

If $c > a$, $\text{dist}(x_1, J) = \frac{c - a}{1 - a}$ which is equal to $\|x_1\|_\infty$ only if $c = 1$. If $c < a$, $\text{dist}(x_1, J) = \frac{a - c}{1 - a}$ which is not equal to $\|x_1\|_\infty$ for any choice of $c$. So $J_0$ is generated by $(1, 1)$ and is a subspace of $X$. 

\[\square\]
Remark 14. (1) Let $X$ be a Banach space and $J$ a proximinal subspace of codimension one. Then $(X, J)$ has the QLP. This follows from Corollary 2.8 in [5].

(2) If $J$ is a closed subspace of a Hilbert space, $H$, then the pair $(H, J)$ has the QLP since the metric projection on a subspace of a Hilbert space is linear.

The next result gives a sufficient condition for the existence of a linear selection of the metric projection onto a proximinal subspace.

Proposition 15. Let $X$ be a Banach space and let $J$ be a closed subspace of $X$. If $(X, J)$ has the QLP then the metric projection $P_J$ has a linear selection.

Proof. Since $(X, J)$ has the quotient lifting property then there exists a bounded operator $\tilde{I}d : X/J \to X$ such that $Id = \pi \circ \tilde{I}d$ and $\|Id\| = 1$. It is easy to check that $\tilde{I}d \circ \pi$ is a projection on $X$. For simplicity we denote this composition by $P$. Further, $X = \text{Range}(P) \oplus \ker(P)$ and $\ker(P) = J$. It is clear that $J \subset \ker(P)$, if $x \in \ker(P)$ then $\tilde{I}d(x + J) = 0$ and $x + J = \pi \circ \tilde{I}d(x + J) = \pi(0) = J$. Hence $x \in J$. We claim that $\text{Range}(P) \subset J_0$, the metric complement of $J$. Towards this claim, we observe that, for $x \in \text{Range}(P)$, $P(x) = \tilde{I}d(x + J) = x$ and $\|x\| \leq \|x + J\| = \text{dist}(x, J) \leq \|x\|$. Therefore $x \in J_0$. An application of the Theorem [6] implies that the metric projection onto $J$ has a linear selection. □

Remark 16. Given $J$, a proximinal subspace of a Banach space $X$, and let $P_J$ denote the metric projection onto $J$. Then the theorems [6] and [15] imply the equivalence of the following statements:

- $(X, J)$ has the QLP.
- $P_J$ has a linear selection.

Example 17. Let $\Omega$ be a compact Hausdorff space and let $J$ be the subspace of all constant functions in $C(\Omega)$. The pair $(C(\Omega), J)$ has the QLP if and only if the cardinality of $\Omega$ is less or equal to 2. It is clear that $J$ is Chebyshev.
If $\Omega$ has 3 distinct points, $a, b, c$, we consider three pairwise disjoint open neighborhoods, $U_a$, $U_b$ and $U_c$ of $a$, $b$ and $c$, respectively. Then there exist continuous functions $f_a$ and $f_b$ on $\Omega$ satisfying the conditions: $f_a : \Omega \to [0, 1]$, $f_a(a) = 1$ and $f_a(x) = 0$, for all $x \notin U_a$; $f_b : \Omega \to [-1, 0]$, $f_b(b) = -1$ and $f_b(x) = 0$, for all $x \notin U_b$. We set $f = f_a + f_b$. Similarly we define $g = g_b + g_c$, with $g_b : \Omega \to [-1, 0]$, $g_b(b) = -1$ and $g_b(x) = 0$, for all $x \notin U_b$; $g_c : \Omega \to [0, 1]$, $g_c(c) = 1$ and $g_c(x) = 0$, for all $x \notin U_c$. The constant function equal to 0 is closest to $f$ and $g$ but $-1/2$ is closest to $f + g$. This implies that $P_J$ is not linear and then $(C(\Omega), J)$ does not have the QLP by Remark 16. The other implication follows from Proposition 13.

This example shows that $X/J$ having the ALP does not imply that the metric projection onto $J$ has a linear selection. The lift of the identity operator on $X/J$ may not have norm 1, in which case the argument given in the proof for the Proposition 15 does not hold.

We add a few remarks that follow straightforwardly from previous results and theorems in [5].

- If $\Omega$ contains $n$ isolated points, there is an $n$-dimensional subspace $J$ of $C_0(\Omega)$ for which $P_J$ admits a linear selection. Therefore $(C_0(\Omega), J)$ has the QLP.
- If $J = \text{span}\{f\}$, then $(C_0(\Omega), J)$ has the QLP iff the support of $f$ contains at most 2 points.
- If $f$ is in $L^1(\Omega, \Sigma, \mu) = X$, then for $J = \text{span}\{f\}$, $(X, J)$ has the QLP if support of $f$ is purely atomic and contains at most two atoms. A similar statement holds for $L^p$, with $1 < p < \infty$ and $p \neq 2$.

The next result addresses the problem of whether the existence of a linear selection for a pair of spaces transfers to other pairs of spaces defined from the given one. We consider sequence spaces, spaces of vector valued continuous functions on a compact Hausdorff space $\Omega$, Lebesgue-Bochner integrable function spaces $(L_1(\mu, X))$ and space of $\mu$-measurable Pettis integrable functions ($\hat{P}_1(\mu, X)$). For the definitions and properties of these spaces we refer the reader to [10]. Given a
Banach space $X$ we denote by $S(X)$ any one of the following sequence spaces: $\ell_p(X)$ ($1 \leq p \leq \infty$), $c_0(X)$ or $c(X)$.

We denote by $F(X)$ any one of the following function spaces: $C(\Omega, X)$, with a compact Hausdorff space, or $\text{Lip}(\Omega, X)$, with $\Omega$ a compact metric space endowed with one of the standard norms. We denote by $I(X)$ any one of the following function spaces: $L_1(\mu, X)$ or $\hat{P}_1(\mu, X)$. Further, $L(Y, X)$ denotes the space of bounded operators from $Y$ into $X$.

**Proposition 18.** Let $X$ and $Y$ be Banach spaces and let $J$ be a proximinal subspace of $X$ such that the metric projection $P_J$ has a linear selection $p : X \to J$. Then the map $f \mapsto p \circ f$ from $S(X)$ ($F(X)$, $I(X)$ or $L(Y, X)$) onto $S(J)$ (resp. $F(J)$, $I(J)$, $L(Y, J)$) is a linear selection.

**Proof.** The proof follows easily from $\|p(x_n)\| \leq 2\|(x_n)\|$.

Let $J$ be a proximinal subspace of $X$ and let $p$ be a linear selection. We observe that if $f : Y \to X$ is an isometric isomorphism, then $f^{-1}(J)$ is proximinal in $Y$, and $f^{-1} \circ p \circ f$ defines a linear selection from $Y$ to $f^{-1}(J)$.

We denote by $X \hat{\otimes}_\pi Y$ the tensor product space of $X$ and $Y$ endowed with the projective norm and by $X \hat{\otimes}_\epsilon Y$ the tensor product space of $X$ and $Y$ endowed with the injective norm. We refer the readers to [10] for more details. If $X$ and $Y$ are isometrically isomorphic we denote this by $X \cong Y$. The following identifications of tensor products are known: $\ell_1 \hat{\otimes}_\pi X \cong \ell_1(X)$; $\ell_1 \hat{\otimes}_\epsilon X \cong \ell_1[X]$, the space of unconditionally summable sequences in $X$; $c_0 \hat{\otimes}_\epsilon X \cong c_0(X)$; $C(\Omega) \hat{\otimes}_\epsilon X \cong C(\Omega, X)$, the space of continuous functions from $\Omega$ to $X$; $L_1(\mu) \hat{\otimes}_\pi X \cong L_1(\mu, X)$ and $L_1(\mu) \hat{\otimes}_\epsilon X \cong \hat{P}_1(\mu, X)$. These facts, together with the above observation and Proposition [18] establish the next corollary.

**Corollary 19.** If $X$ is a Banach space, $J$ is a closed proximinal subspace of $X$, and $p : X \to J$ is a linear selection, then there is a linear selection from $\ell_1 \hat{\otimes}_\pi X$ onto $\ell_1 \hat{\otimes}_\pi J$, $\ell_1 \hat{\otimes}_\epsilon X$ onto $\ell_1 \hat{\otimes}_\epsilon J$, $c_0 \hat{\otimes}_\epsilon X$ onto $c_0 \hat{\otimes}_\epsilon J$, and $\hat{P}_1(\mu, X)$ onto $\hat{P}_1(\mu, J)$.
onto $c_0 \hat{\otimes} J$, $C(\Omega) \hat{\otimes} X$ onto $C(\Omega) \hat{\otimes} J$, $L_1(\mu) \hat{\otimes} X$ onto $L_1(\mu) \hat{\otimes} J$, and $L_1(\mu) \hat{\otimes} X$ onto $L_1(\mu) \hat{\otimes} J$.

Remark 20. Using Theorem 6 we conclude that all the pairs listed in Proposition 18 have the QLP.

5. M-ideals and the quotient lifting property

In this section we explore the QLP for pairs where the subspace is an M-ideal. Towards this we establish the existence of a linear selection for the metric projection.

In [6] (see p.59) it is mentioned that Ando, Choi and Effros proved that if $J$ is an $L_1$-predual M-ideal in Banach space $X$ and $Y$ is a separable Banach space, any bounded linear operator $T \in L(Y, X/J)$ has a bounded linear lifting $\tilde{T} \in L(Y, X)$ such that $\|T\| = \|\tilde{T}\|$ and $T = \pi \circ \tilde{T}$. Our next result shows that this statement holds for $X = C(\Omega)$, where $\Omega$ is compact metrizable, any M-ideal $J$ and all Banach spaces $Y$. Therefore the pair $(C(\Omega), J)$ has the QLP. Let us recall a definition from [2].

Definition 21. A Banach space $X$ is called a $\pi$-space if there is a sequence $F_n$ of finite dimensional subspaces such that $F_1 \subset F_2 \subset F_3 \ldots$ with $\bigcup_{n=1}^{\infty} F_n = X$ and each $F_n$ is the range of a projection of norm one.

Remark 22. Separable $L_p(1 \leq p < \infty)$ and $C(\Omega)$ on compact metrizable $\Omega$ are $\pi$-spaces (See [2].)

Proposition 23. The pair $(C(\Omega), J)$, where $\Omega$ is a compact metrizable space and $J$ is an M-ideal of $C(\Omega)$, has the QLP.

Proof. Since $J$ is an M-ideal of $C(\Omega)$, then $J = \{f \in C(\Omega) : f|_D \equiv 0\}$ for some closed subset $D$ of $\Omega$. By Remark 22, $C(\Omega)$ is a $\pi$-space and $C(D)$ is isometrically isomorphic to $C(\Omega)/J$. Therefore $C(\Omega)/J$ is a $\pi$-space. By Theorem 5 of [2] there exists a norm one linear map $\phi$ from $C(\Omega)/J$ to $C(\Omega)$ such that $\pi \circ \phi = Id$. For a Banach space $Y$ and bounded linear map $S : Y \to C(\Omega)/J$, $\tilde{S} : Y \to C(\Omega)$ defined
by \( \tilde{S} = \phi \circ S \) is a lifting such that \( S = \pi \circ \tilde{S} \) and \( \|S\| = \|\tilde{S}\| \). This concludes the proof. \hfill \Box

Our next result shows on how to construct a linear selection onto an arbitrary M-ideal of the space of continuous functions on a particular class of Euclidean domains.

**Proposition 24.** Let \( J \) be an M-ideal of \( C([0,1]^n) \), where \( n \) is a positive integer. Then the metric projection \( P_J \) has a linear selection.

**Proof.** Since \( J \) is an M-ideal of \( C([0,1]^n) \) there exists a closed subset \( D \) of \([0,1]^n\) such that \( J = \{ f \in C([0,1]^n) : f|_D \equiv 0 \} \) (See [6] p. 4.)

We observe that \( g \in J_0 \) if and only if \( \|g\|_\infty = \text{dist}(g,J) = \|g|_D\|_\infty \).

We present the proof for the cases: \( n = 1 \) and \( n = 2 \); other cases follow similarly.

For \( n = 1 \), we first consider \( D = [t_1, t_2] \cup [t_3, t_4] \). Let \( f \in C[0,1] \), then we define \( f_1 \) as

\[
(\star) \quad f_1(x) = \begin{cases} 
    f(x) - f(t_1 \chi(-\infty,t_1) + t_4 \chi(t_4,\infty)) & \text{for } x \not\in [t_2, t_3] \\
    0 & \text{for } x \in D \\
    f(x) - f(t_2) + \frac{x - t_2}{t_3 - t_2} [f(t_2) - f(t_3)] & \text{for } x \in [t_2, t_3]
\end{cases}
\]

The map \( p : C([0,1]) \rightarrow J \) defined by \( f \mapsto f_1 \) is a linear selection and \( f - f_1 \in J_0 \). For an arbitrary closed subset of \([0,1]\) we assign the value 0 on \( D \) and extend over the open intervals in \([0,1] \setminus D \) as in (\( \star \)).

\[\text{Figure 1. } D \text{ is a closed subset of } [0,1]^2 \text{ and } D' = D \cap OP\]

Let \( n = 2 \) and \( f \in C([0,1]^2) \). For \( x \in [0,1]^2 \) we will define \( f_1 = f_{i,m} \) along the line of slope \( m \) passing through 0 and \( x \). Let \( D' \) be the intersection of \( D \) with the line passing through 0 and \( x \), see Figure
1. For simplicity of notation, we assume that $D'$ has two connected components, i.e. $D' = [t_1, t_2] \cup [t_3, t_4]$. Then,

$$f_{1m}(x) = \begin{cases} f_1(x) & \text{for } \|x\| \not\in [\|t_2\|, \|t_3\|] \\ f(x) - f(t_2) + \frac{\|x-t_2\|}{\|t_3-t_2\|} [f(t_2) - f(t_3)] & \text{for } \|x\| \in [\|t_2\|, \|t_3\|] \end{cases}$$

The map $p : C([0,1]^2) \to J$ defined by $f \mapsto f_1$ is a linear selection. □

The same techniques are unsuitable for general Hausdorff spaces.

It is well known that M-ideals are proximinal (see [6], p.50). It is not the case that for every Banach space $X$ and an M-ideal $J$ in $X$, the pair $(X, J)$ has the QLP, as we have seen with the pair $(\ell_\infty, c_0)$. However the conclusion is different for reflexive Banach spaces.

Given a Banach space $B$, let $J_B$ denote the canonical isometric embedding of $B$ into its double dual, $B^{**}$.

**Proposition 25.** If $X$ is reflexive and $J$ is an M-ideal of $X$ then $(X, J)$ has the QLP.

**Proof.** Let $Y$ be a Banach space and $S : Y \to X/J$, a bounded operator. We denote by $\pi : X \to X/J$, the quotient map, we assume that $J \neq X$. Since the dual of $X/J$ is identified with the annihilator of $J$, $J^\perp$, we have $S^* : J^\perp \to Y^*$. Let $P$ denote the projection $X^* = J^\perp \oplus_1 J^\perp \to J^\perp$. The composition $S^* \circ P : X^* \to Y^*$, is such that $\|S^* \circ P\| = \|S\|$, and $(S^* \circ P)^* : Y^{**} \to X^{**} = X$ also has the same norm as $S$. We now show that $\pi \circ (J_X^{-1} \circ (S^* \circ P)^* \circ J_Y) = S$. Let $y \in Y$, $(S^* \circ P)^*(y^{**}) = J_X(x)$, and $\pi(x) = x + J$. Given $y \in Y$ and $\eta \in X^*$ then

$$(S^* \circ P)^* \circ J_Y(y)(\eta) = P(\eta)(S(y)).$$

On the other hand, $S(y) = x + J$, identified with its image in $(X/J)^{**}$, $(x + J)^{(**}(P(\eta)) = P(\eta)(x + J) = P(\eta)(S(y))$. This shows that $\pi \circ (J_X^{-1} \circ (S^* \circ P)^* \circ J_Y) = S$ and completes the proof. □

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