Whitehead and Ganea constructions for fibrewise sectional category

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Abstract

We introduce the notion of fibrewise sectional category via a Whitehead-Ganea construction. Fibrewise sectional category is the analogue of the ordinary sectional category in the fibrewise setting and also the natural generalization of the fibrewise unpointed LS category in the sense of Iwase-Sakai. On the other hand the fibrewise pointed version is the generalization of the fibrewise pointed LS category in the sense of James-Morris. After giving their main properties we also establish some comparisons between such two versions.

Introduction

The sectional category $\text{secat}(p)$ of any map $p : E \to B$ is the least non negative integer $k$ such that there exists a cover of $B$ constituted by $k + 1$ open subsets on each of which $p$ has a local homotopy section. When $p$ is a fibration, then we can consider local strict sections in the definition, retrieving the usual notion of sectional category (or Schwarz genus) of $p$ (see [22]). This is a lower bound of the Lusternik-Schnirelmann category of the base space and also a generalization as $\text{secat}(p) = \text{cat}(B)$ when $E$ is contractible. Apart from its usefulness in mathematical problems such as the computation of the roots of a complex polynomial, the embedding problem or the classification of bundles, the sectional category is also crucial for the notion of topological complexity of a space. The topological complexity of a space $X$, denoted as $\text{TC}(X)$, is the sectional category of the evaluation fibration $\pi : X^I \to X \times X$, $\alpha \mapsto (\alpha(0), \alpha(1))$. It was established by Farber

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in order to face the motion planning problem in robotics from a topological perspective. The topological complexity is also interesting in algebraic topology itself, independently of its original purpose in robotics, as it is closely related to difficult tasks such as the immersion problem for real projective spaces \cite{12}. Since its apparition, this relatively new numerical homotopy invariant has been of great interest for many researchers working on applied algebraic topology. It is remarkable the work of Iwase-Sakai \cite{14} who related the topological complexity to what they call fibrewise unpointed LS category $\text{cat}_B^*(\cdot)$ in the fibrewise setting. Namely, if $\Delta_X$ denotes the diagonal map, $pr_2$ the projection onto the second factor, and $d(X) \equiv X \xrightarrow{\Delta_X} X \times X \xrightarrow{pr_2} X$ the induced fibrewise pointed space over $X$, then they proved that $\text{TC}(X) = \text{cat}_X^X(d(X))$. They also work with a pointed version in the fibrewise context and consider $\text{TC}^M(X)$ the monoidal topological complexity and $\text{cat}_B^B(\cdot)$ the fibrewise pointed LS category in the sense of James-Morris \cite{17}. In this case the equality $\text{TC}^M(X) = \text{cat}_X^X(d(X))$ holds.

The notion of fibrewise unpointed LS category of Iwase-Sakai, and the notion of fibrewise pointed LS category of James-Morris suggest a natural generalization, an analogue notion of sectional category in the category of fibrewise (pointed) spaces over a space $B$. The aim of this paper is to establish such a generalization, the fibrewise sectional category, denoted by $\text{secat}_B(\cdot)$, as well as a Whitehead-Ganea approach of it. We also present its pointed version, $\text{secat}_B^B(\cdot)$, and some of their most important properties. In order to present our work we have divided the paper into four sections. In the first section we establish a Strøm-type model category and give some background about fibrewise homotopy such as fibrewise homotopy pullbacks, pushouts and joins. In the second section we introduce the main notion of the paper, the one of fibrewise sectional category, and give its main properties. Among them its Whitehead-Ganea approach (so it can be considered as an abstract sectional category in the sense of \cite{18} or \cite{5}) in which we will strongly use the results about fibrewise joins given in the previous section. This can be summarized in the following theorem. For details about its statement the reader is referred to the first sections of the paper:

**Theorem 0.1.** Let $f : E \to X$ be any fibrewise map between normal spaces, or a closed fibrewise cofibration with $X$ normal. Then the following statements are equivalent:

(i) $\text{secat}_B(f) \leq n$

(ii) The diagonal map $\Delta_{n+1} : X \to \prod_{B}^{n+1} X$ factors, up to fibrewise homotopy,
through the fibrewise sectional n-fat wedge

\[
\begin{array}{c}
X \\
\Delta_{n+1} \\
\prod_{n+1} X
\end{array}
\xymatrix{
X \ar[r]^{T_B^n(f)} \ar[d]_{\Delta_{n+1}} & \Delta_n \\
\prod_{n+1} X
}
\]

(iii) The \(n\)-th fibrewise Ganea map \(p_n : G^n_B(f) \to X\) admits a fibrewise homotopy section.

It is also important the relationship of fibrewise sectional category with the ordinary sectional category. In this sense we have obtained the following result, which is not true in general. Here by a fibrant space over \(B\) we mean a fibrewise space over \(B\) in which the projection is a Hurewicz fibration.

**Theorem 0.2.** Let \(f : E \to X\) be a fibrewise map between fibrant spaces over \(B\). Then \(\text{secat}_B(f) = \text{secat}(f)\).

The third section is dedicated to the fibrewise pointed case, where similar results are displayed and proved. Finally, in the last section we compare the unpointed and the pointed versions. As we will see, they are not so different as one might think. We present two results, being quite general the first of them.

**Theorem 0.3.** Let \(f : E \to X\) be a fibrewise pointed map in \(\text{Top}_w(B)\) between normal spaces, or a closed fibrewise cofibration with \(X\) normal. Then

\[\text{secat}_B(f) \leq \text{secat}_B^P(f) \leq \text{secat}_B(f) + 1\]

The second one, which closes our paper, is more restrictive but also interesting. It is a generalization of a result given by Dranishnikov when he compares topological complexity and monoidal topological complexity.

**Theorem 0.4.** Let \(f : E \to X\) a fibrewise pointed map between pointed fibrant and cofibrant spaces over \(B\). Suppose that \(B\) is a CW-complex and \(X\) a paracompact Hausdorff space satisfying the following conditions:

(i) \(f : E \to X\) is a \(k\)-equivalence \((k \geq 0)\);

(ii) \(\dim(B) < (\text{secat}_B(f) + 1)(k + 1) - 1\).

Then \(\text{secat}_B(f) = \text{secat}_B^P(f)\).
Of course, in all these theorems we also obtain corollaries replacing in the corresponding statements fibrewise sectional category by fibrewise unpointed LS category or fibrewise pointed sectional category by fibrewise pointed LS category. Therefore, by Iwase-Sakai, we also retrieve known results about topological complexity and monoidal topological complexity.

1 Fibrewise homotopy theory.

We begin by giving some preliminary definitions and results on fibrewise homotopy theory that will be important throughout all the paper. For basic notation and terminology theory we have considered the reference [3].

Let $B$ be a fixed topological space. A fibrewise space over $B$ consists of a pair $(X, p_X)$, where $X$ is a topological space and $p_X : X \to B$ a map from $X$ to $B$. The map $p_X$ is usually called the projection. If there is not ambiguity we will denote $X$ the fibrewise space $(X, p_X)$. If $X$ and $Y$ are fibrewise spaces, then a fibrewise map (over $B$) $f : X \to Y$ is just a map $f : X \to Y$ satisfying $p_Y f = p_X$

\[ \begin{array}{ccc} X & \xrightarrow{f} & Y \\ p_X \searrow & & \nwarrow p_Y \\ & B & \\
\end{array} \]

The corresponding category of fibrewise spaces and fibrewise maps over $B$ will be denoted as $\text{Top}_B$.

If $X$ and $Y$ are fibrewise spaces, then the binary fibrewise product of $X$ and $Y$ will be denoted by $X \times_B Y = \{(x, y) \in X \times Y : p_X(x) = p_Y(y)\}$. We will denote by $\prod^n_B X$ the product of $n$ copies of a given fibrewise space $X$.

The fibrewise cylinder of a fibrewise space $X$ is just the usual product space $X \times I$ (where $I$ denotes the closed unit interval $[0, 1]$) together with the composite $X \times I \xrightarrow{\text{pr}} X \xrightarrow{p_X} B$ as the projection $p_{X \times I}$. We will denote by $I_B(X)$ the fibrewise cylinder of $X$. The definition of fibrewise homotopy $\simeq_B$ between fibrewise maps comes naturally as well as the notion of fibrewise homotopy equivalence.

1.1 Axiomatic homotopy for fibrewise spaces

Now, for any fibrewise space $X$ take the following pullback in the category $\text{Top}$ of spaces and maps. Here $X^I$ (and $B^I$) denotes the free path-space provided with the cocompact topology, $p_X^I$ is the obvious map induced by $p_X$ and $c : B \to B^I$.
is the map that carries any \( b \in B \) to the natural constant path \( c_b \) in \( B^I \).

\[
\begin{array}{ccc}
P_B(X) & \xrightarrow{I} & X^I \\
\downarrow & & \downarrow p_X \\
B & \xrightarrow{c} & B^I
\end{array}
\]

Thus, \( P_B(X) = B \times_B X^I = \{(b, \alpha) \in B \times X^I : c_b = p_X \alpha \} \) being \( P_B(X) \to B \)
the obvious projection onto \( B \). \( P_B(X) \) is called the \textit{fibrewise cocylinder} of \( X \)
(or \textit{fibrewise free path space} of \( X \)). The fibrewise cylinder and fibrewise cocylinder
constructions give rise to functors

\[
I_B, P_B : \text{Top}_B \to \text{Top}_B
\]

Associated to the functor \( I_B \) there are defined natural transformations \( i_0, i_1 : X \to I_B(X) \)
and \( \rho : I_B(X) \to X \) (given by \( i_*(x) = (x, \epsilon) \) and \( \rho(x, t) = x \)).
Analogously, associated to \( P_B \) there are defined natural transformations \( d_0, d_1 : P_B(X) \to X \)
and \( c : X \to P_B(X) \) (given by \( d_*(b, \alpha) = \alpha(\epsilon) \) and \( c(x) = (p_X(x), c_x) \)).
Moreover, we have that \( (I_B, P_B) \) is an \textit{adjoint pair} in the
sense of Baues (see [1, p.29]).

A fibrewise map \( j : A \to X \) is said to be a \textit{fibrewise cofibration} (over \( B \))
if it satisfies the Homotopy Extension Property, that is, for any fibrewise map \( f : X \to Y \)
and any fibrewise homotopy \( H : I_B(A) \to Y \) such that \( Hi_0 = f_j \),
there exists a fibrewise homotopy \( \tilde{H} : I_B(X) \to Y \) such that \( \tilde{H}i_0 = f \) and \( \tilde{H}I_B(j) = H \)

As known, the fibrewise cofibrations are cofibrations in the usual sense. One
has just to take into account that, for a given space \( Z \), the product \( B \times Z \) is
a fibrewise space considering the canonical projection \( B \times Z \to B \). Therefore,
\( j : A \to X \) is a \textit{fibrewise embedding}, that is, \( j : A \to j(A) \) is a fibrewise
homeomorphism. Then we can consider, without loss of generality, that
the fibrewise cofibrations are pairs of the form \((X, A)\). Such pairs are also called
\textit{fibrewise cofibred pairs}.  

5
An important characterization of fibrewise cofibred pair, slightly changed in [20] Prop 5.2.4 and also proved in [3] in the closed case, is given by what is called a fibrewise Strøm structure.

**Proposition 1.1.** Let \((X, A)\) be a fibrewise pair. Then \((X, A)\) is fibrewise cofibred if and only if \((X, A)\) admits a fibrewise Strøm structure, that is, a pair \((\varphi, H)\) consisting of:

1. A map \(\varphi : X \to I\) satisfying \(A \subseteq \varphi^{-1}(\{0\})\);
2. A fibrewise homotopy \(H : I_B(X) \to X\) satisfying \(H(x, 0) = x, H(a, t) = a\) for all \(x \in X, a \in A, t \in I\), and \(H(x, t) \in A\) whenever \(t > \varphi(x)\).

If \(A\) is closed the \(\varphi\) can be taken so that \(A = \varphi^{-1}(\{0\})\).

An interesting consequence of Proposition 1.1 is the following fact. Recall that given \(A, U\) subspaces of a topological space \(X\), \(U\) is said to be a halo of \(A\) in \(X\) if there exists a map \(\varphi : X \to I\) such that \(A \subseteq \varphi^{-1}(\{0\})\) and \(\varphi^{-1}([0, 1)) \subseteq U\).

**Corollary 1.2.** Given \((X, A)\) any fibrewise cofibred pair, then \(A\) is a fibrewise strong deformation retract of an open neighborhood \(U\) in \(X\). Such open subset \(U\) is a halo of \(A\).

We will be particularly interested in closed fibrewise cofibred pairs (or closed fibrewise cofibrations), which are closed fibrewise pairs \((X, A)\) (i.e., \(A\) is a closed subspace of \(X\)) such that \((X, A)\) is fibrewise cofibred. As in the classical topological case, this is not a very restrictive condition; for instance, if \((X, A)\) is any fibrewise cofibred pair with \(X\) Hausdorff, then necessarily \(A\) is a closed subspace of \(X\).

A fibrewise fibration is a fibrewise map \(p : E \to Y\) such that it verifies Homotopy Lifting Property with respect to any fibrewise space

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & E \\
\downarrow{i_0} & & \downarrow{p} \\
I_B(Z) & \xrightarrow{H} & Y
\end{array}
\]

If \(p : E \to Y\) is any fibrewise map such that it is a Hurewicz fibration, then \(p\) is a fibrewise fibration. However, in general, the converse is not true. For instance, if \(X\) is a fibrewise space, then \(p_X : X \to B\) is always a fibrewise fibration, but \(p_X\) need not be a Hurewicz fibration.
Denote by $\text{fib}_B$, $\text{cof}_B$ and $\text{he}_B$ the classes of fibrewise fibrations, closed fibrewise cofibrations (equivalently, closed fibrewise cofibred pairs) and fibrewise homotopy equivalences, respectively. It is not hard to check that $\text{Top}_B$ is an $I$-category and a $P$-category in the sense of Baues (see [1] p.31 for definitions). More is true, one can also check that the Relative Homotopy Lifting Property holds, i.e. if $(X, A)$ is any closed fibrewise cofibred pair and $p : E \to Y$ any fibrewise fibration, then any commutative diagram in $\text{Top}_B$ of the form

\[
\begin{array}{ccc}
X \times \{0\} \cup A \times I & \to & E \\
\uparrow & & \downarrow p \\
I_B(X) & \to & Y
\end{array}
\]

admits a lift. All these facts are summarized in the following proposition

**Proposition 1.3.** The category $\text{Top}_B$ together with the classes of $\text{cof}_B$, $\text{fib}_B$ and $\text{he}_B$ has an IP category structure in the sense of Baues. In particular, $\text{Top}_B$ is a cofibration category and a fibration category of Baues.

Following the reasonings given by Strøm in [21] we also have the following theorem, which should be compared with [20, Th 5.2.8].

**Theorem 1.4.** The category $\text{Top}_B$ together with the classes of $\text{cof}_B$, $\text{fib}_B$ and $\text{he}_B$ has a proper closed model category structure in the sense of Quillen.

### 1.2 Homotopy pushouts, pullbacks and joins in the fibrewise setting.

In this subsection we will establish the Cube Lemma in $\text{Top}_B$. This result will be crucial in order to deal with a Whitehead-Ganea type characterization for fibrewise sectional category. First we need to present the notions of *fibrewise homotopy pullback* and *fibrewise homotopy pushout*, which are simply the corresponding homotopy limits in the fibrewise axiomatic setting. Given a fibrewise homotopy commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow h \\
Z & \xrightarrow{k} & K
\end{array}
\]
with a fibrewise homotopy $H : hf \simeq_B kg$, there is another homotopy commutative diagram

\[
\begin{array}{ccc}
E_{h,k} & \xrightarrow{p} & Y \\
\downarrow q & & \downarrow h \\
Z & \xrightarrow{G} & K \\
\downarrow k & & \\
K & & \\
\end{array}
\]

in which $E_{h,k} = \{(y, (b, \theta), z) \in Y \times P_B(K) \times Z : h(y) = \theta(0), k(z) = \theta(1)\}$ with the natural projection $E_{h,k} \to B$, given by $(y, (b, \theta), z) \mapsto b$. Here $p$ and $q$ are the obvious restrictions of the projections and $G$ is the fibrewise homotopy defined as $G(y, (b, \theta), z, t) = \theta(t)$. There is a fibrewise whisker map $w : X \to E_{h,k}$ given by $w(x) = (f(x), (p_X(x), H(x, -)), g(x))$ satisfying $pw = f$, $qw = g$ and $G(w \times id) = H$. The homotopy commutative square (*) is said to be a fibrewise homotopy pullback whenever $w$ is a fibrewise homotopy equivalence. Given $h$ and $k$ fibrewise maps there always exist their fibrewise homotopy pullback. The square (**) is called the standard fibrewise homotopy pullback.

There is the dual notion in the sense of Eckmann-Hilton. Given $f : X \to Y$ and $g : X \to Z$ fibrewise maps we can consider the quotient fibrewise space $C_{f, g} := (Y \sqcup I_B(X) \sqcup Z)/\sim$ where $\sim$ is the equivalent relation generated by the elemental relations $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$, for all $x \in X$. The induced fibrewise map $w' : C_{f, g} \to K$ is called the fibrewise co-whisker map. If $w'$ is a fibrewise homotopy equivalence then the square of (**) is called fibrewise homotopy pushout.

**Remark 1.5.** As in the classical case, fibrewise homotopy pullbacks and fibrewise homotopy pushouts can be also characterized by the weak universal property of fibrewise homotopy limits and colimits or through factorization properties. The reader is referred to [19], [6] or [2].

A combination of fibrewise homotopy pullbacks and fibrewise homotopy pushouts is the fibrewise join of two fibrewise maps $f : X \to Z$ and $g : Y \to Z$. Namely, the fibrewise join of $f$ and $g$, $X \ast_Z Y$, is the fibrewise homotopy pushout.
of the fibrewise homotopy pullback of \( f \) and \( g \),

\[
\begin{array}{ccc}
\bullet & \rightarrow & Y \\
\downarrow & & \downarrow \\
X \ast_Z Y & \rightarrow & g \\
\downarrow & & \downarrow \\
X & \rightarrow & Z
\end{array}
\]

being the dotted arrow the corresponding co-whisker map, induced by the weak universal property of fibrewise homotopy pushouts.

The next result relates fibrewise cofibrations and fibrewise fibrations and is one of the key tools in the proof of the Cube Lemma.

**Lemma 1.6.** Consider a fibrewise pullback of the following form, where \( p \) is a fibrewise fibration over \( B \)

\[
\begin{array}{ccc}
P & \xrightarrow{j'} & E \\
p' \downarrow & & \downarrow p \\
A & \xrightarrow{j} & X
\end{array}
\]

If \( j : A \to X \) is a closed fibrewise cofibration, then so is its base change \( j' : P \to E \).

**Proof.** We can suppose that \((X, A)\) is a closed fibrewise cofibred pair, and \( j \) the natural inclusion, so that there exists \((\varphi, H)\) a fibrewise Strom structure. Moreover, \( P = A \times_X E = p^{-1}(A) \) with \( j' \) the inclusion and \( p' \) the corresponding restriction of \( p \). Now take a lift in the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{id} & E \\
\downarrow \iota & & \downarrow p \\
I_B(E) & \xrightarrow{\overline{\Pi}} & X
\end{array}
\]

Defining \( \overline{H}(e, t) = \overline{\Pi}(e, \min\{t, \varphi p(e)\}) \), then one can easily check that \((\overline{H}, \varphi p)\) is a fibrewise Strom structure for the pair defined by the inclusion \( j' \). \qed
Theorem 1.7 (Cube Lemma). Given a fibrewise homotopy commutative cube

in which the bottom face is a fibrewise homotopy pushout and all sides are fibrewise homotopy pullbacks, then the top face is also a fibrewise homotopy pushout.

Proof. Following an analogous reasoning to the one given in [19] we can suppose without loss of generality that the cube is strictly commutative in which:

- The arrows $A \to B$ and $A \to C$ are closed fibrewise cofibrations, and the bottom face is a fibrewise pushout;
- $Z \to D$ is a fibrewise fibration and all side squares are fibrewise pullbacks.

This way the top square is of the form

\[
\begin{array}{ccc}
A \times_D Z & \longrightarrow & B \times_D Z \\
\downarrow & & \downarrow \\
C \times_D Z & \longrightarrow & Z
\end{array}
\]

where, by Lemma 1.6 above, all the arrows are closed fibrewise cofibrations. Now consider $P$ the fibrewise pushout of $C \times_D Z \leftarrow A \times_D Z \longrightarrow B \times_D Z$ and $\theta : P \to Z$ the fibrewise map induced by the pushout property. A simple inspection proves that $\theta$ is a fibrewise isomorphism (compare with [7, 6.1] ) concluding that this square is a fibrewise (homotopy) pushout.

Remark 1.8. Note that, even satisfying the Cube Lemma, $\textbf{Top}_B$ is not a $J$-category in the sense of Doeraene [6] as it has no zero object.

2 Fibrewise sectional category.

And now we will establish the main notion of the paper in the fibrewise context, the one of fibrewise sectional category. First we give some background on fibrewise LS category and its unpointed version.
Let $X$ be a fibrewise space. The \textit{fibrewise L.-S. category} of $X$, $\text{cat}_B(X)$, is the minimal number $n \geq 0$ such that there exists a cover $\{U_i\}_{i=0}^n$ of $X$ by $n+1$ open subsets, each of them admitting a fibrewise homotopy commutative diagram in $\textbf{Top}_B$ of the form

$$
\begin{array}{ccc}
U_i & \rightarrow & X \\
\downarrow_{p_X|U_i} & & \downarrow_{s_i} \\
B & & 
\end{array}
$$

If there is no such $n$, then we say $\text{cat}_B(X) = \infty$. This notion was given by James-Morris in $[17]$ (see also $[16]$ and $[3]$). Unfortunately, this invariant is not very manageable from the axiomatic point of view. Instead we will consider a certain variant of $\text{cat}_B(-)$, given by Iwase-Sakai in $[14]$. For this, we deal with fibrewise pointed spaces. By a \textit{fibrewise pointed space} over $B$ we mean a fibrewise space $X$ together with a fibrewise map $s_X : B \to X$ (i.e., $s_X : B \to X$ is a section of $p_X$).

If $X$ is a fibrewise pointed space, then the \textit{fibrewise unpointed L.-S. category} of $X$, $\text{cat}_B^*(X)$, is the minimal number $n \geq 0$ such that there exists a cover $\{U_i\}_{i=0}^n$ of $X$ by $n+1$ open subsets, each of them \textit{fibrewise categorical}, that is, admitting a fibrewise homotopy commutative diagram in $\textbf{Top}_B$ of the form

$$
\begin{array}{ccc}
U_i & \rightarrow & X \\
\downarrow_{p_X|U_i} & & \downarrow_{s_X} \\
B & & 
\end{array}
$$

If there is no such $n$, then we say $\text{cat}_B^*(X) = \infty$.

Obviously, $\text{cat}_B(X) \leq \text{cat}_B^*(X)$ and the equality holds when $X$ is \textit{vertically connected}, that is, all possible sections $s_i : B \to X$ are fibrewise homotopic to $s_X$ (see $[17]$). As James-Morris assert, when $B$ is a CW-complex a fibre bundle over $B$ with fibre $F$ is vertically connected if $\dim(B)$ does not exceed the connectivity of $F$.

### 2.1 Open-like definition of fibrewise sectional category

We want to give the natural generalization of fibrewise unpointed LS category by considering the analogous notion of sectional category in the fibrewise setting.

**Definition 2.1.** Let $f : E \to X$ be a fibrewise map over $B$ and consider an open subset $U$ of $X$. Then $U$ is said to be \textit{fibrewise sectional} if there exists a
morphism \( s : U \rightarrow E \) in \( \textbf{Top}_B \) such that the following triangle commutes up to fibrewise homotopy

\[
\begin{array}{ccc}
U & \xrightarrow{\text{in}} & X \\
\downarrow s & & \downarrow f \\
E & \xrightarrow{f} & \end{array}
\]

The fibrewise sectional category of \( f \), denoted as \( \text{secat}_B(f) \), is the minimal number \( n \) such that \( X \) admits a cover \( \{U_i\}_{i=0}^n \) constituted by fibrewise sectional open subsets. If there is no such \( n \), then \( \text{secat}_B(f) = \infty \).

When \( f : E \rightarrow X \) is a fibrewise fibration, then we may suppose in the definition of fibrewise sectional that the triangles are strictly commutative. On the other hand, from the definition of fibrewise sectional category it is clear the identity

\[ \text{cat}_B^*(X) = \text{secat}_B(s_X). \]

We can generalize this fact. Observe that given a fibrewise homotopy commutative diagram in \( \textbf{Top}_B \) of the form

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & E' \\
\downarrow f & & \downarrow f' \\
X & \xrightarrow{X} & \end{array}
\]

one has the inequality \( \text{secat}_B(f') \leq \text{secat}_B(f) \). Indeed, if \( U \) is an open fibrewise sectional categorical subset of \( X \) with local fibrewise homotopy section \( s : U \rightarrow E \) of \( f \), then \( s' = \lambda s : U \rightarrow E' \) is a local fibrewise homotopy section of \( f' \).

A fibrewise contractible space (or just a shrinkable space) is any fibrewise space having the fibrewise homotopy type of \( B \). On the other hand if \( X \) and \( Y \) are fibrewise pointed spaces over \( B \), by a fibrewise pointed map \( f : X \rightarrow Y \) we mean a fibrewise map such that \( fs_X = sy \). Applying the above comments to the commutative triangle \( fs_E = s_X \) we obtain

**Proposition 2.2.** Let \( f : E \rightarrow X \) be any fibrewise pointed map. Then

\[ \text{secat}_B(f) \leq \text{cat}_B^*(X) \]

If \( E \) is fibrewise contractible, then \( \text{secat}_B(f) = \text{cat}_B^*(X) \).

One can also check that the fibrewise unpointed LS category \( \text{cat}_B^*(-) \) is invariant by fibrewise pointed maps which are fibrewise homotopy equivalences.
Proposition 2.3. If \( f : X \to Y \) is a fibrewise pointed map such that it is a fibrewise homotopy equivalence, then

\[
\text{cat}^*_B(X) = \text{cat}^*_B(Y).
\]

In particular, if \( X \) is a fibrewise pointed space over \( B \), then \( \text{cat}^*_B(X) = 0 \) if and only if \( X \) is fibrewise contractible.

2.2 The axiomatic approach of fibrewise sectional category

Now we study the fibrewise sectional category \( \text{secat}_B(-) \) from a Whitehead-Ganea approach. Let \( f : E \to X \) be a fibrewise map. For each \( n \geq 0 \) we consider the fibrewise sectional \( n \)-fat wedge as the fibrewise map \( j_n : T^n_B(f) \to \prod^{n+1}_B X \) inductively defined as follows: set \( T^0_B(f) = E \), \( j_0 = f : E \to X \), and define \( j_n : T^n_B(f) \to \prod^{n+1}_B X \) as the join in \( \text{Top}_B \)

On the other hand, the \( n \)-th fibrewise Ganea map of \( X \), \( G^n_B(X) \xrightarrow{p_n} X \), is inductively defined as follows:

Set \( p_0 := f : E \to X \) (so \( G^0_B(f) = E \)). If \( p_{n-1} \) is already constructed, then \( G^n_B(f) \) is the fibrewise join of \( G^{n-1}_B(f) \xrightarrow{p_{n-1}} X \leftarrow^f E \), and \( p_n \) is the induced whisker map:

The following result is a direct consequence of Theorem 1.7, the Cube Lemma in the fibrewise context. Its proof is similar to the classical case (see, for instance, [1]) and therefore is omitted and left to the reader.
**Lemma 2.4.** Let \( f : E \to X \) be a fibrewise map. Then, for each \( n \geq 0 \), there is a fibrewise homotopy pullback

\[
\begin{array}{ccc}
G_B^n(f) & \xrightarrow{j_n} & T_B^n(f) \\
\downarrow{p_n} & & \downarrow{j_n} \\
X & \xrightarrow{\Delta_{n+1}} & \Pi_B^{n+1} X
\end{array}
\]

where \( \Delta_{n+1} \) denotes the \( n+1 \)-st diagonal map.

**Lemma 2.5.** Let \( f : E \to X \) be a closed fibrewise cofibration. Then, the fibrewise sectional \( n \)-fat wedge is, up to fibrewise homotopy,

\[
T_B^n(f) = \{(x_0, x_1, \ldots, x_n) \in \prod_B^{n+1} X \mid x_i \in E \text{ for some } i \in \{0, 1, \ldots, n\}\}
\]

being \( j_n : T_B^n(f) \to \prod_B^{n+1} X \) the canonical inclusion.

**Proof.** It is evident for \( n = 0 \). Next, we check that the following square is a fibrewise homotopy pullback:

\[
\begin{array}{ccc}
E \times_B T_B^{n-1}(f) & \xrightarrow{f \times_B \text{id}} & X \times_B T_B^{n-1}(f) \\
\downarrow{\text{id} \times_B j_{n-1}} & & \downarrow{\text{id} \times_B j_{n-1}} \\
E \times_B \Pi_B^n X & \xrightarrow{f \times_B \text{id}} & \Pi_B^{n+1} X
\end{array}
\]

The standard fibrewise homotopy pullback of \( f \times_B \text{id} \) and \( \text{id} \times_B j_{n-1} \), call it \( L \), is given by the elements

\[
(e, \bar{x}, \bar{b}, \gamma, x, \bar{y}) \in (E \times_B \Pi_B^n X) \times P_B(\Pi_B^{n+1} X) \times (X \times_B T_B^{n-1}(f))
\]

for which \( \gamma(0) = (f(e), \bar{x}) \) and \( \gamma(1) = (x, \bar{y}) \). Define \( \omega : E \times_B T_B^{n-1}(f) \to L \) by \( \omega(e, \bar{y}) = (e, \bar{y}, (b, C(f(e), \bar{y})), f(e), \bar{y}) \) (where \( C(f(e), \bar{y}) \) denotes the constant path in \( (f(e), \bar{y}) \)) and \( \omega' : L \to E \times_B T_B^{n-1}(f) \) by \( \omega'(e, \bar{x}, (b, \gamma), x, \bar{y}) = (e, \bar{y}) \). Then, \( \omega' \omega = \text{id} \) and \( \omega \omega' \simeq_B \text{id} \) through the fibrewise homotopy

\[
H(e, \bar{x}, (b, \gamma), x, \bar{y}; t) = (e, \gamma_2(t), (b, \delta(t)), \gamma_1(1 - t), \bar{y}),
\]

being \( \gamma = (\gamma_1, \gamma_2) \) and \( \delta(t)(s) = (\gamma_1(s(1-t)), \gamma_2((1-s)t + s)) \).

Finally, taking into account that \( f \times_B \text{id} : E \times_B T_B^{n-1}(f) \to X \times_B T_B^{n-1}(X) \) is a closed fibrewise cofibration, the fibrewise homotopy pushout of \( f \times_B \text{id} \) and the inclusion \( \text{id} \times_B j_{n-1} \) is just its honest fibrewise pushout. The result follows by induction.
Remark 2.6. If $f$ is any fibrewise map, then we can factor it through a closed fibrewise cofibration followed by a fibrewise homotopy equivalence

\[ E \xrightarrow{f} X \xrightarrow{\simeq} X' \]

Therefore any fibrewise map can be considered, up to fibrewise homotopy, as a closed fibrewise cofibration. Moreover, as this factorization can be taken through the fibrewise mapping cylinder, if $E$ and $X$ are normal, then $X'$ is also normal.

Theorem 2.7. Let $f : E \to X$ be any fibrewise map between normal spaces, or a closed fibrewise cofibration with $X$ normal. Then the following statements are equivalent:

(i) $\text{secat}_B(f) \leq n$

(ii) The diagonal map $\Delta_{n+1} : X \to \prod_{B}^{n+1} X$ factors, up to fibrewise homotopy, through the fibrewise sectional $n$-fat wedge

\[ X \xrightarrow{T_B^n(f)} \prod_{B}^{n+1} X \]

(iii) The $n$-th fibrewise Ganea map $p_n : G^n_B(f) \to X$ admits a fibrewise homotopy section.

Proof. Statements (ii) and (iii) are equivalent as a consequence of Lemma 2.4 and the weak universal property for fibrewise homotopy pullbacks. Now we check that statements (i) and (ii) are equivalent. Taking into account the above remark and the fact that statements (i)-(iii) are invariant by fibrewise homotopy equivalences, we can suppose without loss of generality that $f$ is a fibrewise closed cofibration with $X$ normal.

Assume that $\text{secat}_B(f) \leq n$ and consider $\{U_i\}_{i=0}^n$ an open cover of $X$ and $H_i : I_B(U_i) \to X$ a fibrewise homotopy satisfying $H_i(x,0) = x$ and $H_i(x,1) = f s_i(x)$, for all $x \in U_i$, where $s_i : U_i \to E$ is a fibrewise map. As $X$ is a normal space there exist, for each $i$, closed subsets $A_i, B_i$ and an open subset $\Theta_i$ such that $A_i \subseteq \Theta_i \subseteq B_i \subseteq U_i$ and $\{A_i\}_{i=0}^n$ covers $X$. Now, by Urysohn
characterization of normality, take $h_i : X \to I$ a map such that $h_i(A) = \{1\}$ and $h_i(X \setminus \Theta_i) = \{0\}$. Then we obtain a fibrewise homotopy $L = (L_0, ..., L_n) : I_B(X) \to \prod_{B}^{n+1} X$, where $L_i : I_B(X) \to X$ is the fibrewise map defined as follows:

$$L_i(x, t) = \begin{cases} x & x \in X \setminus B, \\ H_i(x, th_i(x)) & x \in U_i \end{cases}$$

Taking into account Lemma 2.5 and that $\{A_i\}_{i=0}^n$ covers $X$ it is straightforward to check that $H : \Delta_{n+1} \cong_B j_n \varphi$, where $\varphi : X \to T^n_B(f)$ is defined as $\varphi(x) := L(x, 1)$.

Conversely, suppose a fibrewise map $\varphi : X \to T^n_B(f)$ and a fibrewise homotopy $L : \Delta_{n+1} \cong_B j_n \varphi$. Then, $L = (L_0, ..., L_n)$ and $j_n \varphi = (\varphi_0, ..., \varphi_n)$ where $L_i : I_B(X) \to X$ and $\varphi_i : X \to X$ for each $i$. Since $f : E \hookrightarrow X$ is fibrewise closed cofibration, by Remark 2.2 we have that $E$ is a fibrewise strong deformation retract of an open neighborhood $U$ in $X$. Take a fibrewise retraction $r : U \to E$ and a fibrewise homotopy $H : I_B(U) \to X$ such that $H(x, 0) = x$, $H(x, 1) = fr(x)$ for all $x \in U$, and $H(e, t) = e = f(e)$, for all $e \in E$ and $t \in I$. Defining $U_i = \varphi_i^{-1}(U)$ we obtain $\{U_i\}_{i=0}^n$ an open cover of $X$ and fibrewise homotopies $G_i : I_B(U_i) \to X$

$$G_i(x, t) = \begin{cases} L_i(x, 2t) & , 0 \leq t \leq \frac{1}{2} \\ H(\varphi_i(x), 2t - 1) & , \frac{1}{2} \leq t \leq 1 \end{cases}$$

If $s_i : U_i \to E$ denotes the composite $U_i \xrightarrow{\varphi_i} U \xrightarrow{r} E$, then $G_i(x, 0) = x$ and $G_i(x, 1) = fs_i(x)$, for all $x \in U_i$.

**Remark 2.8.** Observe that taking $f = s_X$ in the above theorem we obtain the corresponding Whitehead-Ganea characterization of $\text{cat}^*_B(\cdot)$. Compare with [22].

### 2.3 Fibrewise sectional category and ordinary sectional category.

Recall that if $f : X \to Y$ is any map, then the sectional category $\text{secat}(f)$ can be defined as the least non-negative integer $n$ such that $Y$ admits an open cover $\{U_i\}_{i=0}^n$ where each $U_i$ has a local homotopy section $s_i : U_i \to X$ of $f$. When $f$ is a Hurewicz fibration, then in this definition we can take local strict sections, recovering the usual notion of sectional category (or Schwarz genus [22]) for fibrations.

On the other hand, one of the main motivations of the invariant $\text{cat}^*_B(\cdot)$ is the fact that it retrieves the notion of topological complexity of a space in
the sense of Farber [10, 11]. The topological complexity of a space \( X \), denoted \( \text{TC}(X) \), is defined as the sectional category of \( \pi : X^I \to X \times X, \alpha \to (\alpha(0), \alpha(1)) \).

Of course, since the fibration \( \pi : X^I \to X \times X \) is homotopy equivalent to the diagonal map \( \Delta_X : X \to X \times X \), then we can redefine \( \text{TC}(X) \) as \( \text{secat}(\Delta_X) \).

The product space \( X \times X \) can be seen as a fibrewise pointed space over \( X \) with \( \Delta_X \), the diagonal map as the section and \( \text{pr}_2 : X \times X \to X \) the projection. Denoting by \( d(X) \) such a fibrewise pointed space we have the following known equality (see [14]):

\[
\text{TC}(X) = \text{cat}^*_X(d(X))
\]

This fact can be generalized for the unpointed LS category. Under a not very restrictive condition on the fibrewise pointed space \( X \) we will be able to prove that \( \text{cat}^*_B(X) = \text{secat}(s_X) \).

By a \textit{fibrant space} over \( B \) we mean a fibrewise space over \( B, X \), such that the projection \( \text{pr}_X : X \to B \) is a Hurewicz fibration. A \textit{pointed fibrant space} over \( B \) is a fibrant space over \( B \) which is also pointed over \( B \).

**Theorem 2.9.** Let \( f : E \to X \) be a fibrewise map between fibrant spaces over \( B \). Then

\[
\text{secat}_B(f) = \text{secat}(f)
\]

**Proof.** The inequality \( \text{secat}(f) \leq \text{secat}_B(f) \) is obvious so it only remains to check \( \text{secat}_B(f) \leq \text{secat}(f) \). First consider a factorization of \( f \) in \( \text{Top} \)

\[
\begin{tikzcd}
E \ar{r}{f} \ar{dr}[swap]{\lambda} & X \\
\hat{E} \ar{ur}{p}
\end{tikzcd}
\]

by a homotopy equivalence \( \lambda \) followed by a Hurewicz fibration \( p \). We observe that this factorization also has sense in \( \text{Top}_B \) considering in \( \hat{E} \) the composite \( \hat{E} \xrightarrow{p} X \xrightarrow{\text{pr}_X} B \) as the natural projection, which is a Hurewicz fibration. Being \( E \) and \( \hat{E} \) fibrant spaces over \( B \) we conclude that actually \( \lambda : E \to \hat{E} \) is a fibrewise homotopy equivalence. Furthermore, \( \text{secat}(f) = \text{secat}(p) \) and \( \text{secat}_B(f) = \text{secat}_B(p) \).

If \( U \) is any open subset of \( X \) with a strict local section \( s : U \to \hat{E} \) of \( p \), then \( s \) is necessarily a fibrewise map over \( B \) and therefore \( U \) is also an open fibrewise sectional subset of \( X \). The result follows taking open coverings. \( \square \)

**Corollary 2.10.** Let \( X \) be a pointed fibrant space over \( B \). Then

\[
\text{cat}^*_B(X) = \text{secat}(s_X).
\]
As an interesting and surprising consequence of Theorem 2.9 and its corollary we have that, in order to deal with \( \text{secat}_B(f) \) or \( \text{cat}^*_B(X) \) we may forget the projections and work in the non-fibrewise world, at least for weak fibrant spaces.

**Remark 2.11.** Observe that, in general, the above equality is not true. For example, consider \( X = ([0, 1] \times \{0, 1\}) \cup (\{0\} \times [0, 1]) \) and \( B = [0, 1] \) with \( s_X(t) = (t, 0) \) and \( p_X(t, s) = t \). If \( X \) were fibrewise contractible, then the fibre \( p_X^{-1}(\{t\}) \) would have the same ordinary homotopy type of \( \{t\} \), for all \( t \in [0, 1] \); but by connectivity reasons this is not the case. Therefore, by Corollary 2.3 we have that \( \text{cat}^*_B(X) \geq 1 \). But \( \text{secat}(s_X) = 0 \), since \( X \) is contractible in the ordinary sense.

The hypothesis of Theorem 2.9 can be relaxed. Recall that a Dold fibration (or weak fibration) is a map \( p : X \to B \) which satisfies the weak covering homotopy property (see [8] for details). A Dold fibration is completely characterized by the fact that it is fibrewise homotopy equivalent to a Hurewicz fibration. Then we can consider weak fibrant spaces over \( B \), i.e., fibrewise spaces over \( B \) in which the projection is a Dold fibration. It is immediate to check the equality \( \text{secat}_B(f) = \text{secat}(f) \) when \( E \) and \( X \) are just weak fibrant spaces over \( B \), and the equality \( \text{cat}^*_B(X) = \text{secat}(s_X) \) when \( X \) is a pointed weak fibrant space over \( B \).

### 3 The fibrewise pointed case.

The category of fibrewise pointed spaces and fibrewise pointed maps will be denoted by \( \text{Top}(B) \). Observe that the space \( B \) together with the identity is the zero object so that \( \text{Top}(B) \) is a pointed category. Any subspace \( A \subseteq X \) containing the section (i.e., \( s_X(B) \subseteq A \)) is a fibrewise pointed space. This way, the inclusion \( A \hookrightarrow X \) is a fibrewise pointed map. These class of subspaces will be called fibrewise pointed subsets of \( X \).

For any fibrewise pointed space \( X \) we can consider its fibrewise pointed cylinder as the following pushout

\[
\begin{array}{ccc}
B \times I & \xrightarrow{pr} & B \\
\downarrow{s_X \times id} & & \downarrow{} \\
X \times I & \longrightarrow & I^B_B(X)
\end{array}
\]

with the obvious projection induced by the pushout property. This pointed cylinder functor gives the notion of fibrewise pointed homotopy between fibrewise
pointed maps, that will be denoted by $\simeq^B_B$. We point out that giving a fibrewise pointed homotopy $F : I^B_B(X) \to Y$ is the same as giving a fibrewise homotopy $F' : I_B(X) \to Y$ such that $F'(s_X(b), t) = s_X(b)$, for all $b \in B$ and $t \in I$. The notion of fibrewise pointed homotopy equivalence comes naturally. On the other hand we can consider

$$P^B_B(X) = B \times_B X^I = \{(b, \alpha) \in B \times X^I : c_b = p_X \alpha\}$$

which is simply the fibrewise space $P_B(X)$ together with the section $(\text{id}_B, c_{s_X}) : B \to P^B_B(X)$ induced by the pullback property. There are functors

$I^B_B, P^B_B : \text{Top}(B) \to \text{Top}(B)$

as well as induced natural transformations $i_0, i_1 : X \to I^B_B(X)$, $\rho : I^B_B(X) \to X$ and $d_0, d_1 : P^B_B(X) \to X$, $c : X \to P^B_B$. Moreover, $(I^B_B, P^B_B)$ is an adjoint pair in the sense of Baues.

Associated to these functors, there are defined the notions of (closed) fibre-wise pointed cofibration and of fibrewise pointed fibration, which are characterized by the natural Homotopy Extension Property and the Homotopy Lifting Property in $\text{Top}(B)$, respectively. Moreover, $\text{Top}(B)$ has an I-category and a $P$-category structure in the sense of Baues ([1, p.31]).

Any fibrewise pointed map which is a fibrewise cofibration is a fibrewise pointed cofibration. Also, any fibrewise fibration is a fibrewise pointed fibration. Unfortunately, as P. May and Sigurdsson assert in [20, p.82], on each case the converse is not true even for the simplest case, in which $B$ is a point. Despite this fact, in order to deal with LS-type invariants in the homotopy category of $\text{Top}(B)$, we still can achieve a reasonable structure on the category of fibrewise pointed spaces based one the fibrewise structure.

A fibrewise well-pointed space is a fibrewise pointed space $X$ in which the section $s_X : B \to X$ is a closed fibrewise cofibration. Let $\text{Top}_w(B)$ denote the full subcategory of $\text{Top}(B)$ consisting of fibrewise well-pointed spaces. One cannot expect $\text{Top}_w(B)$ to be a model category as it is not closed under finite limits and colimits. However the following proposition is enough for our purposes. First we need a technical lemma, whose proof is analogous to the ordinary case and therefore is omitted (see [21]).

**Lemma 3.1.** Let $j : A \to X$ and $i : X \to Y$ be fibrewise maps such that $i$ and $ij$ are closed fibrewise cofibrations. Then $j$ is also a closed fibrewise cofibration.

**Proposition 3.2.** $\text{Top}_w(B)$ is closed under the pullbacks of fibrewise pointed
maps which are fibrewise fibrations. Similarly, $\textbf{Top}_w(B)$ is closed under the pushouts of fibrewise pointed maps which are closed fibrewise cofibrations.

**Proof.** As the cobase change of a fibrewise cofibration is a fibrewise cofibration, the second statement of the lemma is trivially true. Now suppose $p : E \to X$ and $f : X' \to X$ fibrewise pointed maps between fibrewise well-pointed spaces where $p$ is a fibrewise fibration. Consider the following diagram of pullbacks

$$
\begin{array}{ccc}
F & \xrightarrow{i} & E' & \xrightarrow{f'} & E \\
\downarrow & & \downarrow & & \downarrow \\
B & \xrightarrow{s_{X'}} & X' & \xrightarrow{f} & X
\end{array}
$$

As $X'$ and $X$ are well-pointed and $p$ is a fibrewise fibration, by Lemma 1.6 $i$ and $f'i$ are closed fibrewise cofibrations. Considering Proposition 3.1 and the fact that $(f'i)s_F = s_E$ is a closed fibrewise cofibration we have that $F$ is well-pointed and therefore $s_{E'} = is_F$ is a closed fibrewise cofibration. \qed

A certain version of the following result also appears in [20, Prop 5.2.2, Prop 5.2.3].

**Proposition 3.3.** Let $f : X \to Y$ be a fibrewise pointed map between fibrewise well-pointed spaces over $B$. Then,

(i) $f$ is a fibrewise pointed fibration if and only if $f$ is a fibrewise fibration;

(ii) If $f$ is a closed map, then $f$ is a fibrewise pointed cofibration if and only if $f$ is a fibrewise cofibration;

(iii) $f$ is a fibrewise pointed homotopy equivalence if and only if $f$ is a fibrewise homotopy equivalence.

**Proof.** Part (i) is given in [3, Prop 16.3], while part (iii) is consequence of the abstract Dold’s theorem in the cofibration category $\textbf{Top}_B$ ([11, p.96]). A simple inspection reveals that the reasonings given in [21] can be straightforwardly extrapolated to the fibrewise setting for the proof of part (ii). \qed

These facts lead to a great simplification in this setting and allow us to prove directly the following result.

**Theorem 3.4.** The category $\textbf{Top}_w(B)$ with the structure inherited by $\textbf{Top}_B$ is a J-category in the sense of Doeraene [7].
Proof. Considering the above two propositions combined with Proposition 1.3 we have that $\textbf{Top}_B$ induces in $\textbf{Top}_w(B)$ a cofibration and fibration category structures in the sense of Baues. Observe that the factorizations of a fibrewise pointed map are taken through the fibrewise mapping cylinder or the fibrewise mapping track. Again, propositions 3.3 and 3.2 together with Theorem 1.7 show that $\textbf{Top}_w(B)$ satisfy the Cube Lemma.

**Definition 3.5.** Let $f: E \to X$ be a fibrewise map over $B$ and consider an open subset $U$ of $X$ containing $s_X(B)$. Then $U$ is said to be fibrewise pointed sectional if there exists a morphism $s: U \to E$ in $\textbf{Top}(B)$ such that the following triangle commutes up to fibrewise pointed homotopy:

$$
\begin{array}{ccc}
U & \xrightarrow{in} & X \\
\downarrow{s} & \nearrow{f} & \\
E & & \\
\end{array}
$$

The fibrewise pointed sectional category of $f$, denoted as $\text{secat}^B_B(f)$ is the minimal number $n$ such that $X$ admits an open cover $\{U_i\}_{i=0}^n$ constituted by fibrewise pointed sectional subsets. When such $n$ does not exist then $\text{secat}^B_B(f) = \infty$.

When $f$ is a fibrewise pointed fibration, having a local strict section of $f$ is equivalent to having a local fibrewise pointed homotopy section of $f$. On the other hand it is immediate to check the identity

$$\text{secat}^B_B(s_X) = \text{cat}^B_B(X),$$

that is, the pointed fibrewise sectional category of $s_X$ is exactly the fibrewise pointed LS category in the sense of James-Morris [17].

A fibrewise pointed contractible space is any fibrewise pointed space having the same fibrewise pointed homotopy type of $B$. The proof of the following result is analogous to the fibrewise unpointed case.

**Proposition 3.6.** Let $f: E \to X$ be any fibrewise pointed map. Then

$$\text{secat}^B_B(f) \leq \text{cat}^B_B(X)$$

If $E$ is fibrewise pointed contractible, then $\text{secat}^B_B(f) = \text{cat}^B_B(X)$.

As a consequence of Theorem 3.4 by [15] or [18] we can define a manageable axiomatic notion of fibrewise pointed sectional category from two equivalent approaches: Whitehead’s and Ganea’s. Moreover, the fibrewise pointed $n$-fat
wedge and the \( n \)-th fibrewise pointed Ganea fibration can be chosen as \( j_n : T^n_B(f) \to \prod^{n+1}_B X \) and \( p_n : G^n_B(f) \to X \), the ones given in the unpointed case. The proof of the following theorem is completely analogous to the one given in Theorem 2.7 and therefore is omitted. For the particular case \( f = s_X \) compare the equivalence of (i) and (ii) in our theorem with [17] Prop 6.1 and Prop 6.2.

**Theorem 3.7.** Let \( f : E \to X \) be a fibrewise pointed map in \( \text{Top}_w(B) \) between normal spaces, or a closed fibrewise pointed cofibration with \( X \) normal. Then the following statements are equivalent:

(i) \( \text{secat}^B_B(f) \leq n \)

(ii) The diagonal map \( \Delta_{n+1} : \prod^{n+1}_B X \to \prod^{n+1}_B X \) factors, up to fibrewise pointed homotopy, through the fibrewise \( n \)-fat wedge

\[
\begin{array}{ccc}
X & \xrightarrow{T^n_B(f)} & \prod^{n+1}_B X \\
\downarrow \Delta_n & & \uparrow j_n \\
\prod^{n+1}_B X & & \\
\end{array}
\]

(iii) The \( n \)-th fibrewise Ganea map \( p_n : G^n_B(f) \to X \) admits a fibrewise pointed homotopy section.

**Remark 3.8.** As in the unpointed case, observe that taking \( f = s_X \) in the above theorem we obtain a Whitehead-Ganea characterization of \( \text{cat}^B_B(\cdot) \). Compare with [14].

We conclude this section with some comments and remarks. Recall from [14] that the monoidal topological complexity of a space \( X \), denoted \( \text{TC}^M(X) \), is the least integer \( n \) such that \( X \times X \) can be covered by \( n + 1 \) open sets \( \{U_i\}_{i=0} \) on each of which there is a local section \( s_i \) of the free path fibration

\[
\pi_X : X^I \to X \times X, \quad \pi_X(\gamma) = (\gamma(0), \gamma(1))
\]

satisfying that \( \Delta_X(X) \subset U_i \) and \( s_i(x, x) = c_x \), for all \( x \in X \).

As we have previously commented the product space \( X \times X \) can be seen as a fibrewise pointed space over \( X, d(X) \), with \( \Delta_X \) the diagonal map as the section and \( pr_2 : X \times X \to X \) the projection. We have the equality

\[
\text{TC}^M(X) = \text{cat}^X_X(d(X))
\]
for any space $X$ (see [14] for details). In particular $\text{TC}^M(X) = \text{secat}_X^X(\Delta_X)$. We assert that the monoidal topological complexity can also be seen as

$$\text{TC}^M(X) = \text{secat}_X^X(\pi_X).$$

Indeed, considering $X^I$ as a fibrewise pointed space over $X$ with section $c : X \to X^I$ sending $x \mapsto c_x$ ($c_x$ is the constant path at $x$) and projection $\text{ev}_1 : X^I \to X$ the evaluation map $\alpha \mapsto \alpha(1)$, we have that in the following commutative diagram in $\text{Top}(X)$

$$
\begin{array}{ccc}
X & \xrightarrow{c} & X^I \\
\downarrow{\Delta_X} && \downarrow{\pi_X} \\
X \times X & \xrightarrow{\text{ev}_1} & X
\end{array}
$$

the homotopy equivalence $c : X \to X^I$ is actually a fibrewise pointed homotopy equivalence over $X$. Trivially $\text{ev}_1 c = \text{id}_X$, and the formula $H(\alpha, t)(s) = \alpha((1 - t)s + t)$ defines a fibrewise pointed homotopy over $X$ between $\text{id}_X^I$ and the composite $c \text{ev}_1$.

### 4 Unpointed versus pointed fibrewise sectional category.

In this last section we will check how close $\text{secat}_B(-)$ and $\text{secat}_B^B(-)$ are. Obviously, the inequality $\text{secat}_B(-) \leq \text{secat}_B^B(-)$ always holds.

**Theorem 4.1.** Let $f : E \to X$ be a fibrewise pointed map in $\text{Top}_w(B)$ between normal spaces, or a closed fibrewise cofibration with $X$ normal. Then

$$\text{secat}_B(f) \leq \text{secat}_B^B(f) \leq \text{secat}_B(f) + 1$$

**Proof.** We can suppose without loss of generality that $f : E \to X$ is a closed fibrewise cofibration with $X$ normal. Therefore, by Corollary [1,2] $E$ is a fibrewise strong deformation retract of an open neighborhood $N$ in $X$. Take $r : N \to E$ a fibrewise retraction and a fibrewise homotopy $G : I_B N \to X$ with $G(x, 0) = x$, $G(x, 1) = fr(x)$, for all $x \in N$ and $G(e, t) = e$, for all $e \in E$ and $t \in I$.

Suppose that $\text{secat}_B(f) = n$ and take $\{U_i\}_{i=0}^n$ an open cover of $X$, where each $U_i$ is fibrewise sectional with $s_i : U_i \to E$ the local fibrewise homotopy section of $f$. For each $i \in \{0, ..., n\}$ we choose $H_i : I_B U_i \to X$ a fibrewise homotopy satisfying $H_i(x, 0) = x$ and $H_i(x, 1) = fs_i(x)$, for all $x \in U_i$. 

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If we consider $V_i = U_i \setminus E$ for $i \in \{0, ..., n\}$ and $V_{n+1} = N$, then we have that $\{V_i\}_{i=0}^{n+1}$ is an open cover of $X$, and by normality one can find $\{W_i\}_{i=0}^{n+1}$ a refined open cover of $X$ such that $W_i \subseteq \overline{W_i} \subseteq V_i$, for all $i \in \{0, ..., n+1\}$.

Now we define the open subset

$$\mathcal{N} = N \cap (X \setminus \overline{W_0}) \cap ... \cap (X \setminus \overline{W_n})$$

Obviously, $\mathcal{N} \cap W_i = \emptyset$ and $E \subseteq \mathcal{N}$, for all $i \in \{0, ..., n\}$.

If $O_i = W_i \cup \mathcal{N}$, for $i \in \{0, ..., n\}$ and $O_{n+1} = N$, then $\{O_i\}_{i=0}^{n+1}$ is a cover of $X$ constituted by $n + 2$ open fibrewise pointed categorical subsets (observe that $W_{n+1} \subseteq V_{n+1} = N$). Indeed, it only remains to check that $O_i$ is fibrewise pointed sectional, for $i \neq n + 1$. In this case the following fibrewise pointed homotopy $L_i : I_B(O_i) \to X$ proves this fact

$$L_i(x, t) = \begin{cases} H_i(x, t), & x \in W_i \\ G(x, t), & x \in \mathcal{N} \end{cases}$$

Remark 4.2. Observe that Theorem [4.1] is also true when we consider fibrewise pointed embeddings $f : E \hookrightarrow X$ with $X$ normal and such that $f$ admits an open fibrewise pointed sectional subset $U \subseteq X$. For instance, when $X$ is an ENR (Euclidean Neighborhood Retract) then $\Delta_X : X \to X \times X$ is known to satisfy this condition.

As a corollary of the above result we have that if $X$ is any normal fibrewise well-pointed space over $B$, then

$$\text{cat}_B^*(X) \leq \text{cat}_B^o(X) \leq \text{cat}_B^*(X) + 1$$

In particular, if $X$ is a locally finite simplicial complex (or more generally, an ENR), then

$$\text{TC}(X) \leq \text{TC}_M^h(X) \leq \text{TC}(X) + 1.$$ 

These two latter results have been already proved in [15] and [9].

In order to establish the statement of our second theorem we need the following lemma whose proof can be found in [20 Th 24.1.2]. Here the bracket $[Z, K]_B$ denotes the set of fibrewise homotopy classes of fibrewise maps $Z \to K$ over $B$. 

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Lemma 4.3. Let $X$ be a CW-complex over $B$ and $e : Y \to Z$ an $n$-equivalence over $B$ between fibrant spaces. Then
\[ e_* : [X, Y]_B \to [X, Z]_B; \quad [\alpha] \mapsto [e\alpha] \]
is a bijection if $\dim(X) < n$ and a surjection if $\dim(X) = n$.

If $X$ is a fibrewise pointed space over $B$, then we say that $X$ is cofibrant when the section $s_X$ is a closed cofibration in $\textbf{Top}$. Observe the difference with the notion of being well-pointed, in which $s_X$ is a closed fibrewise cofibration. Every well-pointed fibrewise space is cofibrant but, in general, the converse is not true.

Now we are ready for our result.

Theorem 4.4. Let $f : E \to X$ be a fibrewise pointed map between pointed fibrant and cofibrant spaces over $B$. Suppose, in addition, that $B$ is a CW-complex, $X$ is a paracompact Hausdorff space and the following conditions are satisfied:

(i) $f : E \to X$ is a $k$-equivalence ($k \geq 0$);

(ii) $\dim(B) < (\text{secat}_B(f) + 1)(k + 1) - 1$.

Then $\text{secat}_B(f) = \text{secat}_B^B(f)$.

Proof. Take a factorization of $f$ in $\textbf{Top}$

by a closed cofibration and homotopy equivalence $\lambda$ followed by a Hurewicz fibration $p$. Then $\hat{E}$ is a fibrewise pointed space over $B$ considering $\lambda s_E$ as section and the composite $p_X p$ as a projection. Moreover $\lambda$ is a fibrewise pointed homotopy equivalence as it is a homotopy equivalence between fibrant and cofibrant spaces. Therefore, we have the equalities

$\text{secat}_B(f) = \text{secat}_B(p) = \text{secat}(p)$ and $\text{secat}_B^B(f) = \text{secat}_B^B(p)$.

Suppose $\text{secat}(p) = n$ and take the iterated classical join of $n + 1$ copies of $p$, $j^n_p : *_X \hat{E} \to X$. Observe that this is a Hurewicz fibration which is a fibrewise pointed map over $B$. Then $j^n_p$ admits a strict section $\sigma : *_X \hat{E} \to X$ which,
necessarily, must be a fibrewise map over $B$. However, $\sigma$ need not be a fibrewise pointed map over $B$. In order to solve this problem we proceed as follows:

Taking into account that $j^n_p$ is an $((n+1)(k+1) - 1)$-equivalence over $X$ between fibrant spaces over $X$, and that $\dim(B) < (n+1)(k+1) - 1$, we have by Lemma 4.3 that

$$(j^n_p)_* : [B, \ast_X \hat{E}]_X \to [B, X]_X$$

is a bijection and, in particular, is injective. Now, if $s_* : B \to \ast_X \hat{E}$ denotes the corresponding section for $\ast_X \hat{B}$, then

$$(j^n_p)_*(\ast s_X) = [j^n_p \ast s_X] = [s_X] = (j^n_p)_*([s_* \hat{E}])$$

Therefore there exists a fibrewise homotopy over $X$, $H : \ast s_X \simeq_X \ast s_* \hat{B}$.

As $s_X : B \hookrightarrow X$ is a closed cofibration and $j^n_p : \ast_X \hat{E} \to X$ a Hurewicz fibration, by the ordinary Relative Homotopy Lifting Property we can consider a lift in Top

$$X \times \{0\} \cup B \times I \xrightarrow{(\sigma, H)} \ast_X \hat{E}$$

We define $\sigma' := \tilde{H} \iota_1 : X \to \ast_X \hat{E}$. Then one can straightforwardly check that $\sigma'$ is a fibrewise pointed map over $B$ such that $j^n_p \sigma' = id_X$. From this fact one can easily check that there exists an open cover $\{U_i\}_{i=0}^n$ of $X$ in which $U_i$ contains $s_X(B)$ and there is $s_i : U_i \to \hat{E}$ a strict local section of $p$, satisfying $s_is_X = \lambda s_E$, for all $i$. But this implies that $\secat^B_B(p) \leq n$.

\[\square\]

**Corollary 4.5.** Let $B$ be a CW-complex over $X$. Suppose, in addition, that $X$ is a paracompact Hausdorff pointed fibrant and cofibrant space over $B$ satisfying the following conditions:

(i) $s_X : B \to X$ is a $k$-equivalence ($k \geq 0$);

(ii) $\dim(B) < (\cat_B^p(X) + 1)(k+1) - 1$.

Then the equality $\cat_B^p(X) = cat_B^p(X)$ holds.

Specialising to topological complexity we obtain the following known result \[9\].

**Corollary 4.6.** \[9\] Let $X$ be a $k$-connected CW-complex such that $\dim(X) < (TC(X) + 1)(k+1) - 1$. Then $TC(X) = TC^M(X)$. 26
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