THE INITIAL-BOUNDARY VALUE PROBLEM FOR SOME QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS ON THE HALF-LINE

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Abstract. We prove local well-posedness for the initial-boundary value problem associated to some quadratic nonlinear Schrödinger equations on the half-line. The results are obtained in the low regularity setting by introducing an analytic family of boundary forcing operators, following the ideas developed in [14].

1. Introduction

This paper is concerned with the initial-boundary value problem (IBVP) on the right half-line for some quadratic nonlinear Schrödinger equations, namely

\[
\begin{align*}
  i\partial_t u - \partial_x^2 u &= N_i(u, \overline{u}), \quad (x, t) \in (0, +\infty) \times (0, T), \quad i = 1, 2, 3, \\
  u(x, 0) &= u_0(x), \quad x \in (0, +\infty), \\
  u(0, t) &= f(t), \quad t \in (0, T),
\end{align*}
\]

where \( N_1(u, \overline{u}) = u^2 \), \( N_2(u, \overline{u}) = |u|^2 \) or \( N_3(\overline{u}) = \overline{u}^2 \).

The appropriate spaces for the initial and boundary data is motivated by the behavior of the solutions for linear Schrödinger equation in \( \mathbb{R} \). Let \( e^{-it\partial_x^2} \) the linear homogeneous solution group in \( \mathbb{R} \) for the Schrödinger equation. The following smoothing effect

\[
\|e^{-it\partial_x^2}\phi\|_{L_\infty^2 \dot{H}^{2s+1}((\mathbb{R}_t))} \leq c\|\phi\|_{\dot{H}^s(\mathbb{R})},
\]

can be found in [16], where this inequality is sharp in the sense that \( \frac{2s+1}{4} \) cannot be replaced by any higher number. We are thus motivated to consider the IBVP (1.1) in the setting

\[
u_0 \in H^s(\mathbb{R}_+) \quad \text{and} \quad f(t) \in H^{\frac{2s+1}{4}}(\mathbb{R}_+). \tag{1.2}\]

Our goal in studying (1.1)-(1.2) is to obtain low regularity results. Then we consider \( s \leq 0 \), where compatibility conditions between the values \( u(0) \) and \( f(0) \) are not required.
The motivation to study the IBVP (1.1)-(1.2) come from the associated initial-value problem (IVP) in $\mathbb{R}$, that is,

$$
\begin{aligned}
  i\partial_t u - \partial_x^2 u &= N(u, \overline{u}), & (x, t) \in \mathbb{R} \times (0, T), \\
  u(x, 0) &= u_0(x), & x \in \mathbb{R}.
\end{aligned}
$$

(1.3)

The IVP (1.3) has been studied by several authors of the last decades (see e.g. the monograph by Linares and Ponce [19]). Here we mainly discuss the principal results for quadratic nonlinearities. Cazenave and Weissler [6] and Tsutsumi [20] obtained local well-posedness (LWP), i.e., existence, uniqueness and continuity of the data-to-solution map, in $H^s(\mathbb{R})$ for $s \geq 0$, for any quadratic nonlinear term, i.e., $|N(u, \overline{u})| \leq c|u|^2$. The proof is based on the version of the Strichartz estimate for the free Schrödinger group $e^{-it\partial_x^2}$ found in [1]. Kenig, Ponce and Vega [17] showed LWP in $H^s(\mathbb{R})$ for $s \in (-\frac{3}{4}, 0]$ for the nonlinear terms $u^2$ or $\overline{u}^2$, and $s \in (-\frac{1}{2}, 0]$ for nonlinearity $|u|^2$, by using Fourier restriction norm method introduced by Bourgain in [3]. Bejenaru and Tao [1] showed LWP in $H^s(\mathbb{R})$ for $s \geq -1$ when $N(u, \overline{u}) = u^2$, by an iteration using a modification of the standard Bourgain spaces. These authors also proved local ill-posedness (LIP) for $s < -1$, in the sense that the data-to-solution map fails to be continuous. In [18] Kishimoto obtained LWP in $H^s(\mathbb{R})$ for $s \geq -1$ and LIP for $s < -1$ when $N(u, \overline{u}) = \overline{u}^2$.

The solutions we construct to the IBVP (1.1)-(1.2) shall have the following properties.

Definition 1.1. A function $u(x, t)$ will be called a distributional solution of (1.1) on $[0, T]$ if

(a) **Well-defined nonlinearity**: $u$ belongs to some space $X$ with the property that $u \in X$ implies $N_i(u, \overline{u})$, $i = 1, 2, 3$ is a well-defined distribution;

(b) $u(x, t)$ satisfies the equation (1.1) in the sense of distributions on the set $(x, t) \in (0, +\infty) \times (0, T)$;

(c) **Space traces**: $u \in C([0, T]; H^s(\mathbb{R}_+^+))$ and in this sense $u(\cdot, 0) = u_0$;

(d) **Time traces**: $u \in C(\mathbb{R}_+^+; H^s(\mathbb{R}_+^+; H^{3/4}))$, and in this sense $u(0, \cdot) = f$.

In our case, $X$ shall be the restriction on $\mathbb{R}^+ \times (0, T)$ of the functions belonging to the Space $X^{s,b} \cap C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C(\mathbb{R}_x; H^{s+1} (\mathbb{R}_t))$, where $X^{s,b}$ is the Bourgain space, where

$$
\|u\|_{X^{s,b}} = \left( \int \int \left| \xi \right|^{2s} \left| \tau - \xi^2 \right|^{2b} |\hat{u}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}.
$$

The space $X^{s,b}$, with $b > \frac{1}{2}$, is typically employed in the study of the IVP (1.3). As in the work of Colliander and Kenig [7], to handle with the Duhamel boundary forcing operator in our study of the IBVP, we need to take $b < \frac{1}{2}$, force us to intercept $X^{s,b}$ with the space $C(\mathbb{R}_t; H^s(\mathbb{R}_x))$, because for $b < \frac{1}{2}$, $X^{s,b}$ is not embedded into $C(\mathbb{R}_t; H^s(\mathbb{R}_x))$. An important point here is that for $b > \frac{1}{2}$ the bilinear estimates in Bourgain Spaces $X^{s,b}$ were already obtained in [17], while here we need to prove these estimates in the case $b < \frac{1}{2}$.

Our main results are summarized in the following theorem.

**Theorem 1.1.** Let $s_i = -3/4$ for $i = 1, 3$ and $s_2 = -1/4$. For any initial-boundary data $(u_0, f) \in H^s(\mathbb{R}^+) \times H^{\frac{3}{4} + s} (\mathbb{R}^+)$, with $s \in (s_i, 0]$, there exists a positive time $T_i$, depending only
of \( \|u_0\|_{H^s(\mathbb{R}^+)} \) and \( \|f\|_{H^{-\frac{1}{2}+}(\mathbb{R}^+)} \), and a distributional solution \( \tilde{u}_i(x,t) \) of the IBVP (1.1)–(1.2) with nonlinearity \( N_i(u,\overline{u}) \), which is defined in \( C([0,T]; H^s(\mathbb{R}^+)) \cap C([0, T]; H^{2\frac{1}{4}+1}(0,T)) \). Moreover, the map \( (u_0,f) \mapsto u_i \) from \( H^s(\mathbb{R}^+) \times H^{-\frac{1}{2}+}(\mathbb{R}^+) \) into \( C\left([0,T]; H^{s}(\mathbb{R}_+^+)\right) \cap C\left([0,T]; H^{2\frac{1}{4}+1}(0,T)\right) \) is analytic.

As regards the question of regularity this result is similar of the classical results obtained by Kenig, Ponce and Vega in [17] for the pure IVP (1.3) on \( \mathbb{R} \).

The proof of Theorem 1.1 involves the approach of Colliander and Kenig [7], in their treatment of the generalized Korteweg-de Vries on the half-line. In this work they introduced a new method to solve IBVP for nonlinear dispersive partial differential equations by recasting these problems as IVP with an appropriate forcing term. We also use the ideas contained in Holmer’s works [13] and [14], in his study of nonlinear Schrödinger and Korteweg-de Vries (KdV) equations on the half-line, by adapting the method of [7].

The method consists in solving a forced IVP in \( \mathbb{R} \), analogous to the (1.1),

\[
\begin{aligned}
\frac{\partial\tilde{u}}{\partial t} - \partial_x^2 \tilde{u} &= N(\tilde{u},\overline{u}) + D(x)h(t); \quad (x,t) \in \mathbb{R} \times (0,T), \\
\tilde{u}(x,0) &= \tilde{u}_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\]

where \( \tilde{u}_0 \) is a nice extension of \( u_0 \) and \( D \) is a distribution supported in \( \mathbb{R}^- \). The boundary forcing function \( h(t) \) is selected to ensure that

\[
\tilde{u}(0,t) = f(t), \quad t \in (0,T).
\]

Upon constructing the solution \( \tilde{u} \) of (1.4), we obtain a distributional solution of (1.1), by restriction, as \( u = \tilde{u}|_{(\mathbb{R}^+ \times (0,T))} \).

The solution of (1.4) satisfying (1.5) is constructed using the classical restricted norm method of Bourgain (see [3] and [17]) and the inversion of a Riemann-Liouville fractional integration operator.

The crucial point in this work is the study of the Duhamel boundary forcing operator of [13],

\[
\mathcal{L}f(x,t) = 2e^{-i\frac{\pi}{4}} \int_0^t e^{i(t-t')\partial_x^2} \delta_0(x) \mathcal{I}_{-\frac{1}{2}} f(t') dt',
\]

where \( \mathcal{I}_{-\frac{1}{2}} \) is the the Riemann-Liouville fractional integral of order \(-1/2\) given by

\[
\mathcal{I}_{-\frac{1}{2}} f(t') = \frac{1}{\Gamma(1/2)} \int_0^t (t-s)^{-1/2} f'(s) ds, \quad f \in C_0^\infty(\mathbb{R}^+),
\]

in the context of Bourgain Spaces. In [13] the operator \( \mathcal{L} \) was studied in the context of Strichartz estimates, to solve the IBVP associated to the classical nonlinear Schrödinger equation, with nonlinearity \( \lambda|u|^{\alpha-1}u \). We will obtain the Bourgain spaces estimates,

\[
\|\psi(t) \mathcal{L} f(x,t)\|_{X^{s,b}} \leq c\|f\|_{H_0^{2\frac{1}{4}+1}(\mathbb{R}^+)}
\]

for \(-\frac{1}{2} < s < \frac{1}{2}\), where \( \psi(t) \) is a smooth cutoff function. To extend this estimate for others values of \( s \) we define an analytic family of boundary forcing operators, analogous to the one in [13] in the study of the IBVP for the KdV equation on the half-line, in the low regularity setting. This family of operator generalizes the single operator \( \mathcal{L} \) of [13].
We have applied the same technical to study the IBVP associated to the Schrödinger-Korteweg-de Vries system on the half-line \[5\].

We do not explore here the uniqueness of solutions obtained in Theorem 1.1. Actually we have uniqueness in the sense of Kato (see \[15\]) only for a reformulation of the IBVP \((1.1)\) as an integral equation posed in \(\mathbb{R}\), such that solves \((1.1)\). As there are many ways to transform the IBVP \((1.1)\) into an integral equation, we do not have uniqueness in the strong sense. We believe that the method given by Bona, Sun and Zhang in \[4\] can be applied here. These authors introduced the concept of mild solutions, to treat the question of uniqueness for the Korteweg-de-Vries equation on the half-line. In \[5\] the same authors applied this method to get uniqueness of the solutions for the classical nonlinear Schrödinger equation, with nonlinearity \(|u|^{\lambda-1}u\) on the half-line and on the bounded interval.

This paper is organized as follows: in the next section, we discuss some notation, introduce function spaces and recall some needed properties of these function spaces, and review the definition and basic properties of the Riemann-Liouville fractional integral. Sections 3, 4 and 5 respectively, are devoted to the needed estimates for the linear group, the Duhamel boundary forcing operators and the Duhamel inhomogeneous solution operator. In Section 6 we define the Duhamel boundary forcing operators class, adapted from \[14\], and prove the needed estimates for it. In Section 7 we prove the bilinear estimates. Finally, Section 8 is devoted to prove Theorem 1.1.

2. Notations, Function Spaces, Riemann-Liouville Fractional Integral

2.1. Notations. For \(\phi = \phi(x) \in S(\mathbb{R})\), \(\hat{\phi}(\xi) = \int e^{-i\xi x} \phi(x) dx\) will denote the Fourier transform of \(\phi\). For \(u = u(x,t) \in S(\mathbb{R}^2)\), \(\hat{u}(\xi,\tau) = \int e^{-i(\xi x + \tau t)} u(x,t) dx dt\) denotes its space-time Fourier transform, \(\mathcal{F}_x u(\xi,t)\) denotes its space Fourier transform and \(\mathcal{F}_t u(x,\tau)\) its time Fourier transform. Let \(|\xi| = 1 + |\xi|\). Define \((\tau - i\gamma)^\alpha\) as the limit, in the sense of distributions, of \((\tau - \gamma i)^\alpha\), when \(\gamma \to 0^-\). \(\chi_A\) denotes the characteristic function of an arbitrary set \(A\).

2.2. Function Spaces. For \(s \geq 0\) define \(\hat{\phi} \in H^s(\mathbb{R}^+)\) if exists \(\tilde{\phi}\) such that \(\phi(x) = \tilde{\phi}(x)\) for \(x > 0\), in this case we set \(\|\phi\|_{H^s(\mathbb{R}^+)} = \inf_{\tilde{\phi} \in \mathcal{S}} \|\tilde{\phi}\|_{H^s(\mathbb{R})}\). For \(s \geq 0\) define

\[ H_0^s(\mathbb{R}^+) = \{\phi \in H^s(\mathbb{R}^+); \text{ supp } \phi \subset [0,\infty)\}. \]

For \(s < 0\), define \(H^s(\mathbb{R}^+)\) as the dual space to \(H_0^{-s}(\mathbb{R}^+)\), and define \(H_0^s(\mathbb{R}^+)\) as the dual space to \(H^{-s}(\mathbb{R}^+)\).

Also define the \(C_0^\infty(\mathbb{R}^+) = \{\phi \in C^\infty(\mathbb{R}); \text{ supp } \phi \subset [0,\infty)\}\) and \(C_0^\infty(\mathbb{R}^+)\) as those members of \(C_0^\infty(\mathbb{R}^+)\) with compact support. We remark that \(C_0^\infty(\mathbb{R}^+)\) is dense in \(H_0^s(\mathbb{R}^+)\) for all \(s \in \mathbb{R}\).

Throughout the paper, we fix a cutoff function \(\psi \in C_0^\infty(\mathbb{R})\) such that \(\psi(t) = 1\) if \(t \in [-1,1]\) and \(\text{supp } \psi \subset [-2,2]\).

The following lemmas state elementary properties of the Sobolev spaces. For the proofs we refer the reader \[7\].

**Lemma 2.1.** For \(-\frac{1}{2} < s < \frac{1}{2}\) and \(f \in H^s(\mathbb{R})\), we have \(\|\chi_{(0,\infty)} f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}\).

**Lemma 2.2.** If \(0 \leq s < \frac{1}{2}\), then \(\|\psi f\|_{H^s(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}\) and \(\|\psi f\|_{H^{-s}(\mathbb{R})} \leq c\|f\|_{H^{-s}(\mathbb{R})}\), where \(c\) only depends of \(s\) and \(\psi\).
Lemma 2.3. Let $f \in H^s_0(\mathbb{R}^+)$, with $-\infty < s < +\infty$. Then $\|\psi f\|_{H^s_0(\mathbb{R}^+)} \leq c \|f\|_{H^s_0(\mathbb{R}^+)}$.

For $s, b \in \mathbb{R}$, we introduce the classical Bourgain spaces $X^{s,b}$ and the modified Bourgain spaces $W^{s,b}$ related to the Schrödinger equation as the completion of $S'(\mathbb{R}^2)$ under the norms

$$
\|u\|_{X^{s,b}} = \left( \int \int |\xi|^{2s}(\tau - \xi^2)^{2b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2},
$$

$$
\|u\|_{W^{s,b}} = \left( \int \int |\tau|^{s}(\tau - \xi^2)^{2b}|\hat{u}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2}.
$$

2.3. Riemann-Liouville fractional integral. The tempered distribution $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ is defined as a locally integrable function for Re $\alpha > 0$ by

$$
\left\langle \frac{t^{\alpha-1}}{\Gamma(\alpha)}, f \right\rangle = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} f(t) dt.
$$

For Re $\alpha > 0$, we have that

$$
\frac{t^{\alpha-1}}{\Gamma(\alpha)} = \partial_k \left( \frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} \right),
$$

for all $k \in \mathbb{N}$. This expression can be used to extend the definition, in the sense of distributions, of $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ to all $\alpha \in \mathbb{C}$.

A change of contour calculation shows that

$$
\left( \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) = e^{-\frac{1}{2} \pi \alpha (\tau - i0)^{-\alpha}},
$$

where $(\tau - i0)^{-\alpha}$ is the distributional limit. If $f \in C^\infty_0(\mathbb{R}^+)$, we define

$$
\mathcal{I}_\alpha f = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f.
$$

Thus, when Re $\alpha > 0$,

$$
\mathcal{I}_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
$$

and $\mathcal{I}_0 f = f$, $\mathcal{I}_1 f(t) = \int_0^t f(s) ds$, and $\mathcal{I}_{-1} f = f'$. Also $\mathcal{I}_\alpha \mathcal{I}_\beta = \mathcal{I}_{\alpha+\beta}$.

The following lemmas, whose proofs can be found in [14], state some useful properties of the Riemann-Liouville fractional integral operator.

Lemma 2.4. If $f \in C^\infty_0(\mathbb{R}^+)$, then $\mathcal{I}_\alpha f \in C^\infty_0(\mathbb{R}^+)$, for all $\alpha \in \mathbb{C}$.

Lemma 2.5. If $0 \leq \alpha < \infty$ and $s \in \mathbb{R}$, then $\|\mathcal{I}_{-\alpha} h\|_{H^s_0(\mathbb{R}^+)} \leq c \|h\|_{H^{s+\alpha}_0(\mathbb{R}^+)}$.

Lemma 2.6. If $0 \leq \alpha < \infty$, $s \in \mathbb{R}$ and $\mu \in C^\infty_0(\mathbb{R})$, then $\|\mu \mathcal{I}_\alpha h\|_{H^s_0(\mathbb{R}^+)} \leq c \|h\|_{H^{s-\alpha}_0(\mathbb{R}^+)}$, where $c = c(\mu)$.

For more details on the distribution $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ see [10].
3. Linear Version

We define the unitary group associated to the linear Schrödinger equation as

\[ e^{-it\partial_x^2} \phi(x) = \frac{1}{2\pi} \int e^{ix\xi} e^{it\xi^2} \hat{\phi}(\xi)d\xi, \]

it follows that

\[
\begin{align*}
(i\partial_t - \partial^2_x)e^{-it\partial_x^2}\phi(x) &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \\
\left. e^{-it\partial_x^2}\phi(x) \right|_{t=0} &= \phi(x), \quad x \in \mathbb{R}.
\end{align*}
\]

\[ (3.1) \]

**Lemma 3.1.** Let \( s \in \mathbb{R}, 0 < b < 1 \). If \( \phi \in H^s(\mathbb{R}) \), then

(a) (Space traces) \( \|e^{-it\partial_x^2}\phi(x)\|_{C(\mathbb{R}; H^s(\mathbb{R}))} \leq c\|\phi\|_{H^s(\mathbb{R})}; \)

(b) (Time traces) \( \|\psi(t)e^{-it\partial_x^2}\phi(x)\|_{C(\mathbb{R}; H^{s+\frac{1}{2}}(\mathbb{R}))} \leq c\|\phi\|_{H^s(\mathbb{R})}; \)

(c) (Bourgain spaces) \( \|\psi(t)e^{-it\partial_x^2}\phi(x)\|_{X^{s,b}} \leq c\|\psi\|_{H^1(\mathbb{R})}\|\phi\|_{H^s(\mathbb{R})}. \)

**Proof.** The assertion in (a) follows from properties of group \( e^{-it\partial_x^2} \), (b) was obtained in [16] and the proof of (c) can be found in [12]. \( \square \)

4. The Duhamel Boundary Forcing Operator

We now introduce the Duhamel boundary forcing operator of Holmer [13]. For \( f \in C_0^\infty(\mathbb{R}^+) \), define the boundary forcing operator

\[
\mathcal{L}f(x,t) = 2e^{it\frac{a}{4}} \int_0^t e^{-i(t-t')\partial_x^2} \delta_0(x)\mathcal{I}_{-\frac{1}{2}} f(t')dt'.
\]

The equivalence of the two definitions is clear from the formula

\[
\mathcal{F}_x \left( \frac{e^{-i\frac{a}{4}\text{sgn } a}}{2|a|^{1/2}\sqrt{\pi}} e^{\frac{a^2}{4t}} \right)(\xi) = e^{-ia\xi^2}, \quad \forall \ a \in \mathbb{R}.
\]

From this definition, we see that

\[
\begin{align*}
(i\partial_t - \partial^2_x)\mathcal{L}f(x,t) &= 2e^{it\frac{a}{4}} \delta_0(x)\mathcal{I}_{-\frac{1}{2}} f(t), \quad x, t \in \mathbb{R}, \\
\mathcal{L}f(x,0) &= 0, \quad x \in \mathbb{R}.
\end{align*}
\]

The following lemma, whose proof can be found in [13], states some continuity properties of \( \mathcal{L}f(x,t) \).

**Lemma 4.1.** Let \( f \in C_0^{\infty}(\mathbb{R}^+) \).

(a) For fixed \( t \), \( \mathcal{L}f(x,t) \) is continuous in \( x \) for all \( x \in \mathbb{R} \) and \( \partial_x \mathcal{L}f(x,t) \) is continuous in \( x \) for \( x \neq 0 \) with

\[
\lim_{x \to 0^-} \partial_x \mathcal{L}f(x,t) = -e^{i\frac{a}{2}}\mathcal{I}_{-1/2} f(t), \quad \lim_{x \to 0^+} \partial_x \mathcal{L}f(x,t) = e^{i\frac{a}{2}}\mathcal{I}_{-1/2} f(t).
\]
(b) For $N, k$ nonnegative integers and fixed $x$, $\partial_t^k \mathcal{L} f(x, t)$ is continuous in $t$ for all $t \in \mathbb{R}^+$. We also have the pointwise estimates, on $[0, T]$,

$$|\partial_t^k \mathcal{L} f(x, t)| + |\partial_x \mathcal{L} f(x, t)| \leq c(x)^{-N},$$

where $c = c(f, N, k, T)$.

Let $f \in C_0^\infty(\mathbb{R}^+)$, set $u(x, t) = e^{-it\partial_x^2} \phi(x) + \mathcal{L}(f - e^{-it\partial_x^2} \phi(x))|_{x=0}$). Then, by Lemma 4.1 (a) $u(x, t)$ is continuous in $x$. Thus $u(0, t) = f(t)$ and $u(x, t)$ solves the problem

$$\begin{cases}
(i\partial_t - \partial_x^2)u(x, t) = 2e^{it\frac{\pi}{2}} \delta(x)\mathcal{I}_{-1/2}(f - e^{-it\partial_x^2} \phi(x)|_{x=0}), & (x, t) \in \mathbb{R}, \\
u(x, 0) = \phi(x), & x \in \mathbb{R}, \\
u(0, t) = f(t), & t \in \mathbb{R}. 
\end{cases}$$

This would suffice to solve the linear analogue of the half-line problem.

5. Nonlinear versions

We define the Duhamel inhomogeneous solution operator $D$ as

$$Dw(x, t) = -i \int_0^t e^{-i(t-t')\partial_x^2} w(x, t') dt',$$

it follows that

$$\begin{cases}
(i\partial_t - \partial_x^2)Dw(x, t) = w(x, t), & (x, t) \in \mathbb{R} \times \mathbb{R}, \\
Dw(x, 0) = 0, & x \in \mathbb{R}. 
\end{cases}$$

**Lemma 5.1.** Let $s \in \mathbb{R}$. Then:

(a) (Space traces) If $-\frac{1}{2} < c < 0$, then

$$\|\psi(t)Dw(x, t)\|_{C(\mathbb{R}; H^s(\mathbb{R}^+))} \leq c\|w\|_{X^{s,c}};$$

(b) (Time traces) If $-\frac{1}{2} < c < 0$, then

$$\|\psi(t)Dw(x, t)\|_{C(\mathbb{R}^+; H^{s+1}(\mathbb{R}))} \leq \begin{cases}
\|w\|_{X^{s,c}}, & \text{if } \frac{-1}{2} \leq s \leq \frac{1}{2}, \\
c\|w\|_{H^s} + \|w\|_{X^{s,c}}, & \text{for all } s \in \mathbb{R};
\end{cases}$$

(c) (Bourgain spaces estimates) If $-\frac{1}{2} < c \leq 0 \leq b \leq c + 1$, then

$$\|\psi(t)Dw(x, t)\|_{X^{s,b}} \leq \|w\|_{X^{s,c}}.$$

**Remark 5.1.** We note that the $W^{s,b}$ (time-adapted) Bourgain spaces, used in Lemma 5.1 (b), are introduced in order to cover the full values of $s$.

**Proof.** To prove (a), we use $2\chi_{(0, t)}(t') = \text{sgn}(t') + \text{sgn}(t - t')$, then

$$\begin{align*}
\psi(t) \int_0^t e^{i(t-t')\partial_x^2} w(., t') dt' &= \psi(t) \int_\xi e^{it\xi} \int_\tau \frac{e^{it\tau} - e^{it\xi^2}}{\tau - \xi^2} \hat{w}(\xi, \tau) d\tau d\xi \\
&= \psi(t) \int_\xi e^{it\xi} \int_{|\tau - \xi^2| \leq 1} \frac{e^{it\tau} - e^{it\xi^2}}{\tau - \xi^2} \hat{w}(\xi, \tau) d\tau d\xi + \psi(t) \int_\xi e^{it\xi} \int_{|\tau - \xi^2| > 1} \frac{e^{it\tau} - e^{it\xi^2}}{\tau - \xi^2} \hat{w}(\xi, \tau) d\tau d\xi \\
&:= w_1 + w_2.
\end{align*}$$
To estimate $w_1$, we use $|\psi(t)\frac{e^{it\tau}-e^{it\xi^2}}{\tau-\xi^2}| \leq c$, when $|\tau - \xi^2| \leq 1$, more the Cauchy-Schwarz inequality.

To estimate $w_2$, we use $|\psi(t)\frac{e^{it\tau}-e^{it\xi^2}}{\tau-\xi^2}| \leq c(\tau - \xi^2)^{-1}$, when $|\tau - \xi^2| > 1$, and the Cauchy-Schwarz inequality. To prove (b), we take $\theta(t) \in C^\infty(\mathbb{R})$ such that $\theta(t) = 1$ for $|t| < \frac{1}{3}$ and $\text{supp } \theta \subset [-\frac{2}{3}, \frac{2}{3}]$, then

$$
\mathcal{F}_x \left( \psi(t) \int_0^t e^{i(t-t')\partial_x^2} w(x,t') \right)(\xi) = \psi(t) \int_\tau e^{it\tau} - \xi^2 \hat{w}(\xi,\tau) \, d\tau
$$

$$
= \psi(t)e^{it\xi^2} \int_\tau e^{i(t-\xi^2)}(-1)\theta(\tau - \xi^2)\hat{w}(\xi,\tau) \, d\tau + \psi(t) \int_\tau e^{i\tau}(-1)\theta(\tau - \xi^2)\hat{w}(\xi,\tau) \, d\tau
$$

$$
- \psi(t)e^{it\xi^2} \int_\tau \frac{1}{\tau - \xi^2} \hat{w}(\xi,\tau) \, d\tau := \mathcal{F}_x w_1 + \mathcal{F}_x w_2 - \mathcal{F}_x w_3.
$$

By the power series expansion for $e^{it(\tau-\xi^2)}$, $w_1(x,t) = \sum_{k=1}^{\infty} \frac{\psi_k(t)}{k!} e^{-it\partial_x^2} \phi_k(x)$, where $\psi_k(t) = \frac{1}{k!} \frac{\phi_k(x)}{\tau-\xi^2}$ is the Cauchy-Schwarz inequality,

$$
|\phi_k|^2_{H^s(\mathbb{R})} = c \int_\xi \langle \xi \rangle^{2s} \left( \int_{\{\tau-\xi^2\leq \frac{\xi}{2}\}} \sum_{k=1}^{\infty} (\tau - \xi^2)^{k-1} \theta(\tau - \xi^2)\hat{w}(\xi,\tau) \right)^2 \, d\xi
$$

$$
\leq c \int_\xi \langle \xi \rangle^{2s} \int_\tau \langle \tau - \xi^2 \rangle^{2c} |\hat{w}(\xi,\tau)|^2 \, d\tau d\xi.
$$

This completes the estimate for $w_1$. To treat $w_2$, we change variables $\eta = \xi^2$, then

$$
||w_2||_{C(\mathbb{R} \times H^{2s+1}(\mathbb{R}))} \leq c \int_\tau \langle \tau \rangle^{2s+1} 2b \int_{|\eta|<1} \langle \tau - \eta \rangle^{-\frac{1}{2}} |\hat{w}(\pm \eta^{\frac{1}{2}},\tau)| \, d\eta \, d\tau.
$$

By Cauchy-Schwarz (5.2) is bounded by

$$
c \int_\tau \langle \tau \rangle^{2s+1} G(\tau) \int_{|\eta|<1} \langle \tau - \eta \rangle^{-2b} \langle |\eta|^{-\frac{1}{2}} |\hat{w}(\pm \eta^{\frac{1}{2}},\tau)| \rangle \, d\eta \, d\tau,
$$

where $G(\tau) = \int_\eta \langle \tau - \eta \rangle^{-2+2b} |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-s} \, d\eta$. Separating $G(\tau)$ in the regions $|\eta| \leq 1$, $2|\eta| \leq |\tau|$, $|\eta| \leq 2|\eta|$, and using that $-\frac{1}{2} < s \leq \frac{1}{2}$,

Now suppose $-\frac{1}{2} < s \leq \frac{1}{2}$. Applying the Cauchy-Schwarz inequality we obtain

$$
||w_2||_{C(\mathbb{R} \times H^{2s+1}(\mathbb{R}))} \leq c \int_\tau \langle \tau \rangle^{2s+1} G(\tau) \int_{|\eta|<1} \langle \tau + \xi^2 \rangle^{2c} \langle \xi \rangle^{2s} |\hat{w}(\xi,\tau)|^2 \, d\xi \, d\tau,
$$

where $G(\tau) = c \int_\eta \langle \tau - \eta \rangle^{-2-2c} |\eta|^{-\frac{1}{2}} \langle \eta \rangle^{-s} \, d\eta$.

Separating in the regions $|\eta| \leq 1$, $2|\eta| \leq |\tau|$, $|\tau| \leq 2|\eta|$ and using that $-\frac{1}{2} < s \leq \frac{1}{2}$, we can obtain $G(\tau) \leq c(\tau)^{-2s+2}$. This completes the estimate for $w_2$. To obtain $w_3$, we
write \( w_3 = \psi(t)e^{-it\partial^2_x}\phi(x) \), where \( \tilde{\phi}(\xi) = \int \frac{1 - e^{\xi^2}}{\tau - \xi^2} \tilde{w}(\xi, \tau) d\tau \). Using Lemma 3.1 (b) and the Cauchy-Schwarz inequality, we obtain
\[
\|w_3\|_{C(\mathbb{R}_t; H^{2s+\frac{1}{3}}(\mathbb{R}))} = c\|\psi(t)e^{-it\partial^2_x}\phi(x)\|_{C(\mathbb{R}_t; H^{2s+\frac{1}{3}}(\mathbb{R}))} \leq c\|\phi\|_{H^s(\mathbb{R})}
\leq c\int \langle \xi \rangle^{2s} \left( \int_{\tau} |\tilde{w}(\xi, \tau)|^2 (\tau - \xi^2)^{2c} d\tau \int \frac{d\tau}{(\tau - \xi^2)^{2-2c}} \right) d\xi.
\]

Since \( c > -\frac{1}{2} \), we have \( \int \frac{1}{(\tau - \xi^2)^{2-2c}} d\tau \leq c \). This completes the estimate for \( w_3 \).

The statement (c) is a standard estimate and can be found in [12].

Let
\[
\Lambda(u)(t) = e^{-it\partial^2_x}\tilde{u}_0 + \mathcal{D}(N(u, \overline{u}))(t) + \mathcal{L}h(t),
\]
where, \( h(t) = \chi_{(0, +\infty)}(f(t) - e^{-it\partial^2_x}\tilde{u}_0|_{x=0} - \mathcal{D}(N(u, \overline{u}))(t)|_{x=0})\big|_{(0, +\infty)} \) and \( \tilde{u}_0 \) is a good extension of \( u_0 \) in \( \mathbb{R} \).

Using (4.2) and (5.1), we see that
\[
(i\partial_t - \partial^2_x)\Lambda(u)(t) = N(u, \overline{u}) + 2e^{i\overline{\lambda}}\delta_0(x)\mathcal{I}_{-\frac{1}{2}}h(t)
\]
and \( \Lambda(u)(0, t) = f(t) \).

Thus, a way to solve the IBVP (1.1) is to prove that \( \Lambda \) (or more precisely its time truncated versions) defines a contraction in a suitable Banach space. However, as pointed out before, since we are interested in low regularity results, we will use the auxiliary Bourgain spaces. Because of the estimate for the Duhamel forcing operator in our study, we need to take \( b < \frac{1}{2} \), in order to Lemma 6.2 (c) to be valid. However, the space \( X^{s, b} \), with \( b < \frac{1}{2} \), fails to be embedded into \( C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \). For this reason, we choose to work with the Banach space \( Z \) given by
\[
Z = C(\mathbb{R}_t; H^s(\mathbb{R}_x)) \cap C(\mathbb{R}_t; H^{2s+\frac{1}{3}}(\mathbb{R}_t)) \cap X^{s, b}.
\]

In Section 7 we will obtain the bilinear estimates associated to nonlinearities \( N_1(u, \overline{u}) = u^2 \) and \( N_2(u, \overline{u}) = \overline{u}^2 \), in Bourgain spaces \( X^{s, b} \), for \( -\frac{3}{4} < s \leq 0 \).

Unfortunately, we will able to prove the estimate for the Duhamel boundary forcing operator in Bourgain spaces
\[
\|\psi(t)\mathcal{L}f(x, t)\|_{X^{s, b}} \leq \|f\|_{H_0^{2s+\frac{1}{3}}(\mathbb{R}_t)}
\]
if \( -\frac{1}{2} < s < \frac{1}{2} \). In the next section, inspired by [14], to obtain this estimate on the full interval \((-\frac{3}{4}, 0]\), we will define an analytic family of boundary operators \( \mathcal{L}^\lambda \), for \( \lambda \in \mathbb{C} \), such that \( \mathcal{L}^0 = \mathcal{L} \) and
\[
\left\{
\begin{array}{ll}
(i\partial_t - \partial^2_x)\mathcal{L}^\lambda f(x, t) = 2e^{i\overline{\lambda}}\mathcal{I}_{\frac{1}{\lambda}}\mathcal{L}_{-\frac{1}{2}}\mathcal{L}^\lambda f(t), & (x, t) \in \mathbb{R}^2, \\
\mathcal{L}^\lambda f(0, t) = e^{i\lambda\overline{\lambda}}f(t), & t \in \mathbb{R}.
\end{array}
\right.
\]

Due to the support properties of \( \frac{x^{\lambda-1}}{1(\lambda)} \), \( (i\partial_t - \partial^2_x)\mathcal{L}^\lambda f(x, t) = 0 \) for \( x > 0 \), in the sense of distributions. For any \( -\frac{3}{4} < s \leq 0 \), we will be able to address the right half-line problem (1.1) by replacing \( \mathcal{L} \) to \( \mathcal{L}^\lambda \) for suitable \( \lambda = \lambda(s) \).
6. The Duhamel Boundary Forcing Operator Class

Define, for \( \lambda \in \mathbb{C} \), such that \( \text{Re} \, \lambda > -2 \), and \( f \in C_0^\infty(\mathbb{R}^+) \),

\[
\mathcal{L}^\lambda f(x, t) = \left[ \frac{x^{\lambda-1}}{\Gamma(\lambda)} * \mathcal{L}(\mathcal{I}_{-\frac{i}{2}} f)(\cdot, t) \right](x).
\]

It follows that

\[
\mathcal{L}^\lambda f(x, t) = \frac{1}{\Gamma(\lambda + 2)} \int_{x}^{+\infty} (y-x)^{\lambda-1} \mathcal{L}(\mathcal{I}_{-\frac{i}{2}} f)(y, t)dy,
\]

for \( \text{Re} \, \lambda > 0 \). (6.1)

For \( \text{Re} \, \lambda > -2 \), by using (4.1), we see that

\[
\mathcal{L}^\lambda f(x, t) = \frac{1}{\Gamma(\lambda + 2)} \int_{x}^{+\infty} (y-x)^{\lambda+1} \partial_y^2 \mathcal{L}(\mathcal{I}_{-\frac{i}{2}} f)(y, t)dy
\]

\[
= \frac{1}{\Gamma(\lambda + 2)} \int_{x}^{+\infty} (y-x)^{\lambda+1} (i\partial_y \mathcal{I}_{-\frac{i}{2}} f)(y, t)dy - \frac{2ie^{i\pi/4}x^{\lambda+1}}{\Gamma(\lambda + 2)} \mathcal{I}_{-1/2-\lambda/2} f(t).
\]

From (4.1) we see that

\[
(i\partial_t - \partial_y^2) \mathcal{L}^\lambda f(x, t) = 2e^{i\pi/4} \frac{x^{\lambda-1}}{\Gamma(\lambda)} \mathcal{I}_{-\frac{i}{2}-\frac{\lambda}{2}} f(t),
\]

in the sense of distributions.

Using Lemma 4.1 we see that \( \mathcal{L}^\lambda f(x, t) \) is well defined for \( \text{Re} \, \lambda > -2 \) and \( t \in [0, 1] \).

As \( \frac{x^{\lambda-1}}{\Gamma(\lambda)} \bigg|_{\lambda=0} = \delta_0 \), then \( \mathcal{L}^0 f(x, t) = \mathcal{L} f(x, t) \) and (6.2) implies \( \mathcal{L}^{-1} f(x, t) = \partial_x \mathcal{L}(\mathcal{I}_{1/2}) f(x, t) \).

The dominated convergence Theorem and Lemma 4.1 imply that, for fixed \( t \in [0, 1] \) and \( \text{Re} \, \lambda > -1 \), \( \mathcal{L}^\lambda f(x, t) \) is continuous in \( x \) for \( x \in \mathbb{R} \).

The next Lemma states the values of \( \mathcal{L}^\lambda f(x, t) \) at \( x = 0 \).

**Lemma 6.1.** Let \( f \in C_0^\infty(\mathbb{R}^+) \). If \( \text{Re} \, \lambda > -1 \), then

\[
\mathcal{L}^\lambda f(0, t) = e^{i\lambda t} f(t).
\]

**Proof.** From (6.2), we have

\[
\mathcal{L}^\lambda f(0, t) = \int_0^{+\infty} \frac{y^{\lambda+1}}{\Gamma(\lambda + 2)} \partial_y^2 \mathcal{L}(\mathcal{I}_{-\frac{i}{2}} f)(y, t)dy.
\]

By complex differentiation under the integral sign, (6.2) proves that \( \mathcal{L}^\lambda f(0, t) \) is analytic in \( \lambda \), for \( \text{Re} \, \lambda > -1 \). By analyticity, we shall only compute (6.3) for \( 0 < \lambda < 2 \).

For the computation in the range \( 0 < \lambda < 2 \), we use the representation (6.1), to obtain

\[
\mathcal{L}^\lambda f(0, t) = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} (y)^{\lambda-1} \mathcal{L}(\mathcal{I}_{-\frac{i}{2}} f)(y, t)dy
\]

\[
= \frac{1}{\Gamma(\lambda)} \int_0^t (t-t')^{-\frac{1}{2}} \mathcal{I}_{-\frac{i}{2}-\frac{\lambda}{2}} f(t') \int_0^{+\infty} y^{\lambda-1} \frac{1}{\sqrt{\pi}} e^{-y^2} dydt'.
\]
Set \( I = \int_0^{+\infty} y^{\lambda-1} e^{-\frac{y^2}{4(t-t')^2}} dy \). Changing variables \( r = \frac{y^2}{4(t-t')^2} \), then \( y = r^{\frac{1}{2}}(t-t')^{\frac{1}{2}} \) and \( dy = r^{-\frac{1}{2}}(t-t')^{\frac{1}{2}} dr \), it follows that
\[
I = \int_0^{+\infty} \left[ r^{\frac{1}{2}} 2(t-t')^{\frac{1}{2}} \right]^{\lambda-1} r^{-\frac{1}{2}} e^{-ir(t-t')^{\frac{1}{2}}} dr = 2^{\lambda-1} (t-t')^{\frac{\lambda}{2}} \int_0^{+\infty} r^{\frac{\lambda}{2}-1} e^{-ir} dr.
\]
By a change of contour,
\[
I = 2^{\lambda-1} (t-t')^{\frac{\lambda}{2}} \int_0^{+\infty} r^{\frac{\lambda}{2}-1} e^{-r} dr = 2^{\lambda-1} (t-t')^{\frac{\lambda}{2}} \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}, \quad \text{for } \lambda \in (0,2).
\]
Using the formula \( \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)} = \frac{2^{1-\lambda} \sqrt{\pi}}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)} \), we obtain
\[
L^\lambda f(0,t) = 2^{\lambda-1} \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)} \int_0^t (t-t')^{\frac{\lambda}{2}} I_{-\frac{\lambda}{2}} f(t') dt' = \frac{1}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)} \int_0^t (t-t')^{\frac{\lambda}{2}} I_{-\frac{\lambda}{2}} f(t') dt' = \frac{\Gamma\left(\frac{\lambda}{2}\right)}{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)} f(t) = e^{i\lambda t} f(t).
\]

Now we state the needed estimates for the Duhamel boundary forcing operators class.

**Lemma 6.2.** Let \( s \in \mathbb{R} \). Then
(a) (Space traces) If \( \frac{s}{2} < \lambda < s + \frac{1}{2} \), \( \lambda > -1 \) and \( \text{supp } f \subset [0,1] \), then
\[
\| L^\lambda f(x,t) \|_{C(\mathbb{R};\mathcal{H}^s(\mathbb{R}^n))} \leq c \| f \|_{H^{\frac{2s+1}{4}}_0(\mathbb{R}^+)};
\]
(b) (Time traces) If \( -1 < \lambda < 1 \), then
\[
\| \psi(t)L^\lambda f(x,t) \|_{C(\mathbb{R}^1;H^{\frac{2s+1}{4}}_0(\mathbb{R}^n))} \leq c \| f \|_{H^{\frac{2s+1}{4}}_0(\mathbb{R}^+)};
\]
(c) (Bourgain spaces) If \( b < \frac{1}{2} \), \( s - \frac{3}{4} < \lambda < s + \frac{1}{4} \) and \( -1 < \lambda < \frac{1}{2} \), then
\[
\| \psi(t)L^\lambda f(x,t) \|_{X^{s,b}_0} \leq c \| f \|_{H^{\frac{2s+1}{4}}_0(\mathbb{R}^+)}.
\]

**Remark 6.1.** As in the treatment for the IBVP associated to KdV equation [7], the assumption \( b < \frac{1}{2} \) is crucial in the proof of Lemma 6.2 (c). This fact forced us to work with the Bourgain spaces with \( b < \frac{1}{2} \). In [9] was derived regularity properties for the cubic nonlinear Schrödinger equation on the half-line by using Laplace transform method, being also necessary to work with Bourgain spaces \( X^{s,b}_0 \) with \( b < 1/2 \).

**Proof.** We follow closely the argument in [14]. By density, we can assume that \( f \in C_{0,c}(\mathbb{R}^+). \)
Using \( \mathcal{F}_x \left( \frac{\lambda}{\Gamma(\lambda)} \right) (\xi) = c(\xi-i0)^{-\lambda} \), we see that \( \mathcal{F}_x(L^\lambda f)(\xi,t) = c(\xi-i0)^{-\lambda} \int_0^t e^{i(t-t')^2} I_{-\frac{\lambda}{2}} f(t') dt'. \)
As \( \lambda > -1 \), the distribution \((\xi - i0)^{-\lambda}\) is a locally integrable function. The changing of variables \( \eta = \xi^2 \) gives that, for fixed \( t \),

\[
\|L^{\lambda}f(x,t)\|_{H^s(\mathbb{R})}^2 \leq c \int_{\eta} |\eta|^{-\frac{s}{2}} \left| \int_0^t e^{i(t-t')\eta}\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f}(t')dt' \right|^2 d\eta
\]

\[
= c \int_{\eta} |\eta|^{-\frac{s}{2}} \left| (\chi(-\infty,t)\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(\eta) \right|^2 d\eta.
\]

Since \(-1 < s - \lambda - \frac{1}{2} < 1\), Lemma 2.2 implies

\[
\int_{\eta} |\eta|^{-\frac{s}{2}} \left| (\chi(-\infty,t)\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(\eta) \right|^2 d\eta \leq c \int_{\eta} \langle \eta \rangle^{s - \lambda - \frac{1}{2}} \left| (\chi(-\infty,t)\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(\eta) \right|^2 d\eta.
\]

By Lemmas 2.1 (to remove \( \chi(-\infty,t) \)) and 2.5 (to estimate \( \mathcal{I}_{\frac{1}{2} - \frac{1}{2} f} \)), we obtain

\[
c \int_{\eta} \langle \eta \rangle^{s - \lambda - \frac{1}{2}} \left| (\chi(-\infty,t)\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(\eta) \right|^2 d\eta \leq c \|f\|_{H^0_{\frac{2s+1}{2}}(\mathbb{R}^+)}^2,
\]

which proves (a). Now we prove (b). By Lemma 2.3 we can ignore the test function.

Changing variables \( t \to t - t' \), we get

\[
(I - \partial^2_t)^{\frac{2s+1}{4}} \left( \frac{x^{\lambda-1}}{\Gamma(\lambda)} * \int_{-\infty}^t e^{-i(t-t')\partial^2_x} \delta(x) h(t') dt' \right)
\]

\[
= \left( \frac{x^{\lambda-1}}{\Gamma(\lambda)} * \int_{-\infty}^t e^{-i(t-t')\partial^2_x} \delta(x)(I - \partial^2_t)^{\frac{2s+1}{4}} h(t') dt' \right).
\]

It suffices to prove

\[
\left\| \int_{\xi} e^{ix(\xi - i0)^{-\lambda}} \int_{-\infty}^t e^{i(t-t')\xi^2 (\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(t')} dt' d\xi \right\|_{L^2_x L^2_t(\mathbb{R})} \leq c \|f\|_{L^2_{\mathbb{R}}(\mathbb{R}^+)}.
\]

Using \( \chi(-\infty,t) = \frac{1}{2} \text{sgn}(t - t') + \frac{1}{2} \), we obtain

\[
\int_{\xi} e^{ix(\xi - i0)^{-\lambda}} \int_{-\infty}^t e^{i(t-t')\xi^2 (\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(t')} dt' d\xi
\]

\[
= \int_{\xi} e^{ix(\xi - i0)^{-\lambda}} \int_{-\infty}^{+\infty} \chi(-\infty,t)e^{i(t-t')\xi^2 (\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(t')} dt' d\xi
\]

\[
+ \int_{\xi} e^{ix(\xi - i0)^{-\lambda}} \int_{-\infty}^{+\infty} \frac{1}{2} e^{i(t-t')\xi^2 (\mathcal{I}_{\frac{1}{2} - \frac{1}{2} f})(t')} dt' d\xi := I + II.
\]
We treat $I$ and $II$ separately. The change of variables $\eta = \xi^2$ implies

$$II = \int_{\xi} e^{ix\xi} (I - I_{\lambda - \frac{1}{2} + \frac{1}{2}} f) (\xi^2)(\xi - i0)^{-\lambda} e^{it\xi^2} d\xi = \int_{\xi} e^{ix\xi} (\xi^2 - i0)^{-\lambda} e^{it\xi^2} d\xi$$

$$= \int_{\eta=0}^{+\infty} e^{ix\eta^{\frac{1}{2}}} (\eta - i0)^{\frac{\lambda+1}{2}} \hat{f}(\eta)(\frac{1}{2} - \eta)^{-\lambda} e^{it\eta^{\frac{1}{2}}} \frac{1}{2} d\eta$$

$$+ \int_{\eta=0}^{+\infty} e^{-ix\eta^{\frac{1}{2}}} (\eta - i0)^{\frac{\lambda+1}{2}} \hat{f}(\eta)(-\frac{1}{2} - \eta)^{-\lambda} e^{it\eta^{\frac{1}{2}}} \frac{1}{2} d\eta$$

$$= c_1 \int_{0}^{+\infty} e^{ix\eta^{\frac{1}{2}}} \hat{f}(\eta)e^{it\eta} d\eta + c_2 e^{-ix\eta^{\frac{1}{2}}} \hat{f}(\eta)e^{it\eta} d\eta,$$

which implies $\|II(x,t)\|_{L^2_t} \leq c\|f\|_{L^2_t}$. To estimate $I$, we write $I = \left(\frac{1}{2} e^{ix^2} sgn(\cdot) \ast I_{\lambda - \frac{1}{2} - \frac{1}{2}} f\right)(t)$. Since

$$\mathcal{F}_t \left(\frac{1}{2} e^{ix^2} sgn(\cdot) \ast I_{\lambda - \frac{1}{2} - \frac{1}{2}} f\right)(\tau) = \frac{(\tau - i0)^{\frac{\lambda+1}{2}} \hat{f}(\tau)}{\tau - \xi^2},$$

it follows that

$$I = \int_{\tau} e^{ix\tau} \lim_{\epsilon \to 0} \int_{|\tau - \xi^2| > \epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{\lambda+1}{2}} (\xi - i0)^{-\lambda} d\xi}{(\tau - \xi^2)} \hat{f}(\tau) d\tau.$$

Therefore, it suffices to show that the function

$$g(\tau) := \lim_{\epsilon \to 0} \int_{|\tau - \xi^2| > \epsilon} \frac{e^{ix\xi}(\tau - i0)^{\frac{\lambda+1}{2}} (\xi - i0)^{-\lambda} d\xi}{(\tau - \xi^2)}$$

is limited.

Changing of variables $\xi \to |\tau|^\frac{1}{2} \xi$ and using $(\xi|\tau|^\frac{1}{2} - i0)^{-\lambda} = |\tau|^{-\frac{1}{2}} (c_1 \xi_+^\lambda + c_2 \xi_-^\lambda)$, we obtain

$$g(\tau) = \int_{\xi} \frac{e^{i|\tau|^\frac{1}{2} \xi \phi} (\xi_+^\lambda + c_2 \xi_-^\lambda)}{\tau - |\tau|^2} d\xi$$

$$= c_1 \chi_{\{\tau > 0\}} \int_{\xi} \frac{e^{i|\tau|^\frac{1}{2} \xi \phi} c_1 \xi_+^\lambda + c_2 \xi_-^\lambda}{1 - \xi^2} d\xi + c_2 \chi_{\{\tau < 0\}} \int_{\xi} \frac{e^{i|\tau|^\frac{1}{2} \xi \phi} c_1 \xi_+^\lambda + c_2 \xi_-^\lambda}{1 + \xi^2} d\xi$$

$$:= g_1 + g_2.$$

The second integral is uniformly limited in $\tau$ if $\lambda < 1$, this can be obtained by considering the cases $|\xi| < 1$ and $|\xi| \geq 1$. Now we estimate $g_1$. Let $\theta \in C^\infty(\mathbb{R})$ such that $\theta(\xi) = 1$ in $[\frac{1}{2}, \frac{3}{2}]$ and $\theta(t) = 0$ outside $(\frac{1}{2}, \frac{3}{2})$. Then we obtain

$$g_1 = c_1 \chi_{\{\tau > 0\}} \int_{\xi} e^{i|\tau|^\frac{1}{2} \xi \phi} \frac{c_1 \xi_+^\lambda \theta(\xi)}{1 - \xi^2} d\xi + c_1 \chi_{\{\tau > 0\}} \int_{\xi} e^{i|\tau|^\frac{1}{2} \xi \phi} \frac{1 - \theta(\xi)(c_1 \xi_+^\lambda + c_2 \xi_-^\lambda)}{1 - \xi^2} d\xi$$

$$= g_{11} + g_{12}.$$
The second integral is clearly limited when \( \lambda > -1 \). To estimate \( g_{11} \), we write
\[
g_{11} = c_1 \chi(\tau > 0) \int e^{i\tau \frac{\hat{\phi}}{2}} x \xi - \frac{c_1 \xi_+^{\lambda} \theta(\xi)}{(1-\xi)(1+\xi)} d\xi = c_1 \chi(\tau > 0) F_{\xi}^{-1} \left( \frac{c_1 \xi_+^{\lambda} \theta(\xi)}{(1-\xi)(1+\xi)} \right) (x|\tau|^{\frac{1}{2}})
\]
\[
= \left[ (\text{sgn}(1-\xi) \ast F_{\xi}^{-1} \left( \frac{c_1 \xi_+^{\lambda} \theta(\xi)}{(1+\xi)} \right) \right] (x|\tau|^{\frac{1}{2}}).
\]
This becomes convolution of Schwartz class function with a phase shifted \( \text{sgn}(x) \) function, completing the proof of (b).

To prove (c), we first note that
\[
F_x(L^\lambda f)(\xi,t) = c(\xi-i0)^{-\lambda} \int \frac{e^{it\tau} - e^{it\xi^2}}{\tau - \xi^2} (\tau - i0)^{\frac{1}{2} + \frac{1}{2}} f(\tau) d\tau.
\]
Now let \( \theta(\tau) \in C^\infty \) such that \( \theta(\tau) = 1 \) for \( |\tau| \leq 1 \) and \( \theta(\tau) = 0 \) for \( |\tau| \geq 2 \). Define \( u_1, u_2, u_3 \) by
\[
F_x u_1(\xi,t) = (\xi-i0)^{-\lambda} \int \frac{e^{it\tau} - e^{it\xi^2}}{\tau - \xi^2} \theta(\tau - \xi^2)(\tau - i0)^{\frac{1}{2} + \frac{1}{2}} f(\tau) d\tau,
\]
\[
F_x u_2(\xi,t) = (\xi-i0)^{-\lambda} \int \frac{e^{it\tau}}{\tau - \xi^2} (1 - \theta(\tau - \xi^2))(\tau - i0)^{\frac{1}{2} + \frac{1}{2}} f(\tau) d\tau,
\]
\[
F_x u_3(\xi,t) = (\xi-i0)^{-\lambda} \int \frac{e^{it\xi^2}}{\tau - \xi^2} (1 - \theta(\tau - \xi^2))(\tau - i0)^{\frac{1}{2} + \frac{1}{2}} f(\tau) d\tau,
\]
it follows that \( L^\lambda f = u_1 + u_2 - u_3 \).

We start by estimating \( u_2 \). It’s immediate that
\[
\|u_2\|_{X^{s,\theta}}^2 \leq c \int \int |\xi|^{2s} \frac{(\tau - \xi^2)^{2b}}{|\tau - \xi^2|^2} (1 - \theta(\tau - \xi^2))^2 |\xi|^{-2\lambda} |\tau|^{\lambda + 1} |\hat{f}(\tau)|^2 d\tau d\xi
\]
\[
= c \int |\tau|^{\lambda + 1} \left( \int \frac{|\xi|^{-2\lambda} |\xi|^{2s} d\xi}{|\tau - \xi^2|^{2-2b}} \right) |\hat{f}(\tau)|^2 d\tau.
\]
Thus, it suffices to obtain
\[
I(\tau) = \int \frac{|\eta|^{-\frac{1}{2} - \lambda} |\eta|^s d\eta}{|\tau - \eta|^{2b}} \leq c(\tau)^{s - \lambda - \frac{1}{2}}.
\]
This will be obtained by separating some cases. For \( |\tau| \leq \frac{\tau}{2} \), we have that \( \langle \tau - \eta \rangle \sim \langle \eta \rangle \). It follows that
\[
I(\tau) \leq c \int \frac{|\eta|^{-\frac{1}{2} - \lambda} |\eta|^s d\eta}{|\tau - \eta|^{2b}} \leq c \int |\eta|^{-\frac{1}{2} - \lambda} + c \int \frac{d\eta}{|\eta|^{2 - 2b} - s + \frac{1}{2} + \lambda} \leq c,
\]
where we have used that \( \lambda < -\frac{1}{2} \) and \( \lambda > -\frac{1}{2} + 2b + s \).

Now suppose that \( |\tau| > \frac{\tau}{2} \) and \( |\eta| \geq \frac{|\tau|}{2} \). In this case we use \( \lambda \geq s - \frac{1}{2} \) and \( b < \frac{1}{2} \) to obtain
\[
I(\tau) \leq c \int \frac{\langle \eta \rangle^{s - \frac{1}{2} - \lambda}}{|\tau - \eta|^{2b}} \leq c(\tau)^{s - \frac{1}{2} - \lambda} \int \frac{d\eta}{|\tau - \eta|^{2b}} \leq c(\tau)^{s - \frac{1}{2} - \lambda}.
\]
On the case $|\tau| \geq \frac{1}{2}$ and $|\eta| \leq \frac{|\tau|}{2}$ we have $\langle \tau - \eta \rangle \geq \frac{1}{2}(\tau)$. Then

$$I(\tau) \leq c(\tau)^{2b-2} \int_{|\eta| \leq \frac{|\tau|}{2}} \langle \eta \rangle^s |\eta|^{-\frac{1}{2} - \lambda} \leq c(\tau)^{2b-2} \langle \tau \rangle^{s+\frac{1}{2} - \lambda} \leq c(\tau)^{s-\frac{1}{2} - \lambda}.$$  

This completes the estimate for $u_2$. To estimate $u_3$, we write $u_3(x,t) = \theta(t)e^{-it\partial_x^2}\phi(x)$, where

$$\hat{\phi}(\xi) = (\xi - i0)^{-\lambda} \int \frac{1 - \psi(\tau - \xi^2)}{\tau - \xi^2} (I_{-\frac{\lambda}{2} - \frac{1}{2}})(\tau) d\tau.$$ 

Then, by Lemma 3.1, it suffices to show that

$$\|\phi\|_{H^s(R)} \leq c\|f\|_{H^0_{\alpha+1}(R^+)}.$$  

Arguing as in the proof of Lemma 5.8 in [14], there exists a function $\beta \in S(R)$ such that

$$\frac{1 - \psi(\tau - \xi^2)}{\tau - \xi^2} (I_{-\frac{\lambda}{2} - \frac{1}{2}})(\tau) d\tau = \int (I_{-\frac{\lambda}{2} - \frac{1}{2}})(\tau) \beta(\tau - \xi^2) d\tau.$$  

By using Cauchy-Schwarz and $|\beta(\tau - \xi^2)| \leq c(\tau - \xi^2)^{-N}$, for $N >> 1$, we have that

$$\|\phi\|^2_{H^s(R)} \leq \int \langle \xi \rangle^{2s} |\xi|^{-2\lambda} \left( \int_{\tau} \beta(\tau - \xi^2)|\tau|^{\lambda+1} \hat{f}(\tau) d\tau \right)^2 d\xi \leq c \int_{\tau} \left( \int_{|\eta| \leq \frac{1}{2}} \langle \eta \rangle^{s} |\tau - \eta|^{-2N+2} d\eta \right) |\tau|^{\lambda+1} |\hat{f}(\tau)|^2 d\tau.$$  

Using (6.5), we see that

$$\int_{|\eta| \leq \frac{1}{2}} \langle \eta \rangle^{s} |\tau - \eta|^{-2N+2} d\eta \leq c(\tau)^{s-\lambda - \frac{1}{2}}.$$  

Substituting (6.6) in (6.7) we obtain (6.6).

Finally we estimate $u_1$. By the power series expansion for $e^{\mu(\tau - \xi^2)}$, we write $u_1(x,t) = \sum_{k=1}^{\infty} \psi_k(t)e^{-it\partial_x^2}\phi_k(x)$, where $\psi_k(t) = i^kk^k\psi(t)$ and $\psi_k(t) = i^kk^k\psi(t)$ and

$$\hat{\phi}_k(\xi) = (\xi - i0)^{-\lambda} \int_{\tau} (\tau - \xi^2)^{k-1} \theta(\tau - \xi^2)(\tau)^{\frac{1}{2} + \frac{1}{2}} \hat{f}(\tau) d\tau.$$ 

Using (3.1), we need to show that $\|\phi_k\|_{H^s(R)} \leq c\|f\|_{H^0_{\alpha+1}(R^+)}$.

By Cauchy-Schwarz inequality,

$$\|\phi_k\|^2_{H^s(R)} \leq c \int \langle \xi \rangle^{2s} |\xi|^{-2\lambda} \int_{|\tau - \xi^2| \leq 1} |\tau|^{\lambda+1} |\hat{f}(\tau)|^2 d\tau d\xi.$$  

The changing of variables $\eta = \xi^2$ implies

$$\int_{|\tau - \xi^2| \leq 1} \langle \xi \rangle^{2s} |\xi|^{-2\lambda} d\xi \leq c \int_{|\tau - \eta| \leq 1} |\eta|^{-\lambda - \frac{1}{2}} |\eta|^s d\eta \leq c(\tau)^{s-\lambda - \frac{1}{2}}.$$  

Combining (6.9) and (6.10), we obtain $\|\phi_k\|_{H^s(R)} \leq c\|f\|_{H^0_{\alpha+1}(R^+)}$. 

\[\square\]
7. Bilinear Estimates

Lemma 7.1. (Estimates for nonlinear term $N_1$)

(a) Given $-\frac{3}{4} < s \leq 0$, there exists $b = b(s) < \frac{1}{2}$ such that

$$\|uv\|_{X^{s,-b}} \leq c\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}. \quad (7.1)$$

(b) Given $-\frac{3}{4} < s \leq -\frac{1}{2}$, there exists $b = b(s) < \frac{1}{2}$ such that

$$\|uv\|_{W^{s,-b}} \leq c\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}. \quad (7.2)$$

Lemma 7.2. (Estimate for nonlinear term $N_2$) Given $-\frac{1}{4} < s \leq 0$, there exists $b = b(s) < \frac{1}{2}$ such that

$$\|\overline{uv}\|_{X^{s,-b}} \leq c\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

Lemma 7.3. (Estimates for nonlinear term $N_3$)

(a) Given $-\frac{3}{4} < s \leq 0$, there exists $b = b(s) < \frac{1}{2}$ such that

$$\|\overline{uv}\|_{X^{s,-b}} \leq c\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

(b) Given $-\frac{3}{4} < s \leq -\frac{1}{2}$, there exists $b = b(s) < \frac{1}{2}$ such that

$$\|\overline{uv}\|_{W^{s,-b}} \leq c\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}.$$

We shall prove these lemmas by the calculus techniques of [17]. We begin with some elementary integral estimates, whose proofs can be found in [12], [2] and [14], respectively.

Lemma 7.4. Let $b_1, b_2$, such that $b_1 + b_2 > \frac{1}{2}$ and $b_1, b_2 < \frac{1}{2}$. Then

$$\int \frac{dy}{(y - \alpha)^{2b_1}(y - \beta)^{2b_2}} \leq \frac{c}{(\alpha - \beta)^{2b_1 + 2b_2 - 1}}.$$

Lemma 7.5. If $b > \frac{1}{2}$, then

$$\int_{-\infty}^{\infty} \frac{dx}{(\alpha_0 + \alpha_1 x + x^2)^b} \leq c.$$

Lemma 7.6. If $b < \frac{1}{2}$, then

$$\int_{|x| < \beta} \frac{dx}{(x)^{4b - 1}|\alpha - x|^b} \leq \frac{(1 + \beta)^{2-4b}}{(\alpha)^{\frac{b}{2}}}. $$

7.1. Proof of Lemma 7.1. Let $\rho = -s \in [0, \frac{3}{4})$. For $u, v \in X^{s,b} = X^{-\rho,b}$ define

$$g(\xi, \tau) = (\tau - \xi^2)^b(\xi)^{-\rho}\hat{u}(\xi, \tau), \quad h(\xi, \tau) = (\tau - \xi^2)^b(\xi)^{-\rho}\hat{\psi}(\xi, \tau).$$

Then $\|g\|_{L^2} = \|u\|_{X^{s,b}}$ and $\|h\|_{L^2} = \|v\|_{X^{s,b}}$. We write the left hand side of (7.1) in terms of $f$ and $g$, i.e.,

$$\|uv\|_{X^{s,-b}} := \sup_{\|\phi\|_{L^2} \leq 1} W(g, h, \phi),$$

where,

$$W(g, h, \phi) = \int_{\mathbb{R}^4} \frac{(\xi)^s}{(\tau - \xi^2)^b(\xi_1)^s(\tau_1 - \xi_1^2)^b(\xi - \xi_1)^s(\tau - \tau_1 - (\xi - \xi_1)^2)^b} \phi(\xi, \tau) d\xi d\tau_1 d\xi d\tau.$$
Initially, we treat the case $\rho = 0$. We integrate over $\xi_1$ and $\tau_1$ first, and then we use the Cauchy-Schwarz and H"older inequalities to obtain

$$W^2(g, h, \phi) \leq \|\phi\|_{L^2}^2 \times \left\| \frac{1}{(\tau - \xi_1^2)b} \int \int \frac{g(\xi_1, \tau_1)}{\langle \tau_1 - \xi_1^2 \rangle^b} \frac{h(\xi - \xi_1, \tau - \tau_1)}{\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^b} d\xi_1 d\tau_1 \right\|_{L^2_{\xi_1}L^2_{\tau_1}}^2$$

$$\leq \|\phi\|_{L^2}^2 \|g\|_{L^2}^2 \|h\|_{L^2}^2 \times \left\| \frac{1}{(\tau - \xi_1^2)^{2b}} \int \int \frac{d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^2 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b}} \right\|_{L^\infty_{\xi_1 \tau_1}}. \tag{7.3}$$

Using Lemmas 7.3 and 7.5, we obtain

$$\frac{1}{(\tau - \xi_1^2)^{2b}} \int \int \frac{d\tau_1 d\xi_1}{\langle \tau_1 - \xi_1^2 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b}} \leq c \int_{-\infty}^{\infty} \frac{d\xi_1}{\langle \tau - \xi_1^2 + 2\xi_1(\xi - \xi_1) \rangle^{4b-1}}, \tag{7.4}$$

that is bounded provided $\frac{3}{4} < b < \frac{1}{2}$. This proves the case $\rho = 0$.

To prove the case $\rho \in (1/2, 3/4)$, we write

$$W(g, h, \phi) = \int_{\mathbb{R}^4} \frac{(\xi_1)^s}{\langle \tau - \xi_1^2 \rangle^b} \frac{g(\xi_1, \tau_1)}{\langle \tau_1 - \xi_1^2 \rangle^b} \frac{h(\xi - \xi_1, \tau - \tau_1)}{\langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^b} \phi(\xi, \tau) d\xi_1 d\tau_1 d\xi d\tau$$

$$= \int_{\mathcal{R}_1} + \int_{\mathcal{R}_2} + \int_{\mathcal{R}_3} + \int_{\mathcal{R}_4} := W_1 + W_2 + W_3 + W_4,$$

where

$$\mathcal{R}_1 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \leq 1\},$$

$$\mathcal{R}_2 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \geq 1, \max\{|\tau - \xi_1^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 - (\xi - \xi_1)^2|\} = |\tau - \xi_1^2|\},$$

$$\mathcal{R}_3 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \geq 1, \max\{|\tau - \xi_1^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 - (\xi - \xi_1)^2|\} = |\tau_1 - \xi_1^2|\},$$

$$\mathcal{R}_4 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \geq 1, \max\{|\tau - \xi_1^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 - (\xi - \xi_1)^2|\} = |\tau - \tau_1 - (\xi - \xi_1)^2|\}.$$

The estimate for $W_1$ can be obtained as in the case $\rho = 0$, since the frequencies cancel each other out. To estimate $W_2$ we integrate over $\xi_1$ and $\tau_1$ first, and then we use the Cauchy-Schwarz and H"older inequalities to obtain

$$W_2^2 \leq c \|\phi\|_{L^2}^2 \|g\|_{L^2}^2 \|h\|_{L^2}^2 \times \left\| \frac{|\xi_1|^{2s}}{(\tau - \xi_1^2)^{2b}} \int \int \frac{\chi_A(\xi, \tau)}{\langle \xi_1 \rangle^{2b} \langle \tau_1 - \xi_1^2 \rangle^{2b} \langle \xi - \xi_1 \rangle^{2s} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b}} d\tau_1 d\xi_1 \right\|_{L^\infty_{\xi_1 \tau_1}}. \tag{7.5}$$

where $A(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbb{R}^2; \max\{|\tau - \xi_1^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 + (\xi - \xi_1)^2|\} = |\tau - \xi_1^2|\}.$

For $W_3$ we integrate over $\xi$ and $\tau_1$ first, and then we use the Cauchy-Schwarz and H"older inequalities to obtain
Thus, in \( A \) we have
\[
\langle | | \rangle \leq \frac{3}{2} | \tau - \xi^2 | \text{ and } \langle | | \rangle \leq | \tau - \xi^2 | + | (\tau - (\xi - \xi_1)^2)| \leq 2 | \tau - \xi^2 | .
\]
By symmetry, the estimate for \( W \) is similar to the \( W_3 \) estimate. From the estimates (7.5) and (7.6) it suffices to show that
\[
\left\| \frac{\xi}{(\xi - \xi_1)^2} \int \frac{2}{(\xi - \xi_1)^2} \right\| \leq c.
\]
To estimate (7.7) and (7.8), we will use the algebraic relation
\[
(\tau - \xi^2 - (\tau - \xi^2)) = -2 \xi_1 (\xi - \xi_1).
\]
Thus, in \( A(\xi, \tau) \) we have
\[
| \xi_1 (\xi - \xi_1) | \leq \frac{3}{2} | \tau - \xi^2 | \text{ and } | \tau - \xi^2 + 2 \xi (\xi - \xi_1) | \leq | \tau - \xi^2 | + | (\tau - (\xi - \xi_1)^2)| \leq 2 | \tau - \xi^2 | .
\]
We can assume that \( | \xi - \xi_1 | \geq 1 \), since otherwise, the bound reduces to the case \( \rho = 0 \). Thus, we have \( | \xi_1 |^\rho | \xi - \xi_1 |^\rho \leq c| \xi_1 |^\rho | \xi - \xi_1 |^\rho \).

Using Lemma 7.6, we see that
\[
\langle | | \rangle \leq \frac{c}{(\tau - \xi^2)^2 (\xi - \xi_1)^2} \int \langle \xi_1 | \xi - \xi_1 \rangle^2 | \tau - (\xi - \xi_1)^2 |^2 \hat{d}_f \xi_1 ,
\]
where we have used that \( \frac{1}{2} < b < \frac{1}{2} \).

Set \( \eta = \tau - \xi^2 + 2 \xi_1 (\xi - \xi_1) \). Then \( \xi_1 = \frac{1}{2} (\xi + (2 \tau - \xi^2 - 2 \eta)^{\frac{1}{2}}) \), \( | 2 \xi_1 - \xi | = | 2 \tau - \xi^2 - 2 \eta |^{\frac{1}{2}} \) and \( \hat{d}_f \xi_1 = \frac{c}{| \tau - \eta - \xi_1 |^{\frac{1}{2}}} d\eta \). Substituting into right hand side (7.12) and using Lemma 7.6 we estimate the right hand side of (7.12) by
\[
\langle | | \rangle \leq \frac{c}{(\tau - \xi^2)^2} \int \langle | | \rangle^2 | \tau - \xi^2 |^{\frac{1}{2}}
\]
This expression is bounded, provided \( b \geq \frac{1}{4} + \frac{c}{4} \). Note that as \( \rho < \frac{3}{4} \), we can choice \( b(\rho) \) depending of \( \rho \) such that \( b(\rho) < \frac{1}{2} \).
Now we prove (7.8) in the case $\rho \in (\frac{1}{2}, \frac{3}{4})$. In $B(\xi_1, \tau_1)$, using (7.9), we have that
\[
|\xi_1(\xi - \xi_1)| \leq \frac{3}{2}|\tau_1 - \xi_1^2| \quad \text{and} \quad |\tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1)| \leq |\tau - \xi^2| + |\tau - \tau_1 - (\xi - \xi_1)^2| \leq 2|\tau_1 - \xi^2_1|. \tag{7.13}
\]
From Lemma 7.4 we obtain
\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{B} \frac{(|\xi_1||\xi - \xi_1|)^{2\rho}d\xi d\tau}{\langle \xi \rangle^{2\rho} \langle \tau - \xi_1^2 \rangle^{2b} \langle \tau - \tau_1 - (\xi - \xi_1)^2 \rangle^{2b}} \leq \frac{1}{\langle \tau - \xi_1^2 \rangle^{2b-2\rho}} \int_{D} \frac{d\xi}{\langle \xi \rangle^{2\rho} \langle \tau - \xi_1^2 + 2\xi_1(\xi - \xi_1) \rangle^{4b-1}},
\]
where $D = D(\xi_1, \tau_1) = \{\xi \in \mathbb{R}; |\tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1)| \leq 2|\tau_1 - \xi_1^2| \}$ and $\frac{1}{2} < b < \frac{3}{4}$. Now we divide $D$ into two parts $D_1$ and $D_2$, where, $D_1 = \{\xi \in D; |2\xi_1(\xi - \xi_1)| \leq \frac{|\tau_1 - \xi_1^2|}{2} \}$, $D_2 = \{\xi \in D; |\tau_1 - \xi_1^2| \leq |\xi_1(\xi - \xi_1)| \leq \frac{3|\tau_1 - \xi_1^2|}{2} \}$.
In $D_1$ we have $|\tau_1 - \xi_1^2| \leq 2|\tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1)|$, it follows that
\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b-2\rho}} \int_{D_1} \frac{d\xi}{\langle \xi \rangle^{2\rho} \langle \tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1) \rangle^{4b-1}} \leq \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b-2\rho + 4b-1}} \int_{D_2} \frac{d\xi}{\langle \xi \rangle^{2\rho}}. \tag{7.15}
\]
This expression is bounded, provided $\rho > \frac{1}{2}$ and $b \geq \frac{1}{6} + \frac{\rho}{2}$. Note that as $\rho < \frac{3}{4}$, we can choose $b(\rho) < \frac{1}{2}$.
We subdivide $D_2$, into three pieces, i.e., $D_{21} \cup D_{22} \cup D_{23}$, where
\[
D_{21} = \{\xi \in D_2; \frac{|\xi|}{4} \leq |\xi_1| \leq 100|\xi_1| \}, \quad D_{22} = \{\xi \in D_2; 1 \leq |\xi_1| \leq \frac{|\xi|}{4} \} \text{ and } \quad D_{23} = \{\xi \in D_2; 100|\xi| \leq |\xi_1| \}.
\]
In $D_{21}$ we have $|\xi|^2 \sim |\xi_1|^2 \geq c(\tau_1 - \xi_1^2)$. Set $\eta = \tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1)$. Then $d\eta = -2\xi_1 d\xi$.
Using Lemma 7.6 we obtain
\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b-2\rho}} \int_{D_{21}} \frac{d\xi}{\langle \xi \rangle^{2\rho} \langle \tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1) \rangle^{4b-1}} \leq \frac{c}{\langle \tau_1 - \xi_1^2 \rangle^{2b-\rho}} \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{4b-2}},
\]
which is bounded provided $b \geq \frac{1}{3} + \frac{\rho}{6}$.
In $D_{22}$, we have $\langle \tau_1 - \xi_1^2 \rangle \geq \frac{1}{2} |\xi_1| - |\xi_1| \geq \frac{1}{2} |\xi_1| (|\xi| - |\xi_1|) \geq \frac{1}{2} |\xi_1| (4|\xi_1| - \xi_1) = 2|\xi_1|^2$.
Then, by Lemma 7.6 we obtain
\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{D_{22}} \frac{(|\xi_1||\xi - \xi_1|)^{2\rho}d\xi}{\langle \xi \rangle^{2\rho} \langle \tau_1 - \xi_1^2 + 2\xi_1(\xi - \xi_1) \rangle^{4b-1}} \leq \frac{c}{\langle \tau_1 - \xi_1^2 \rangle^{2b}} \int_{|\eta| \leq |\tau_1 - \xi_1^2|} \frac{|\xi_1|^2 d\eta}{\langle \xi_1 \rangle^{4b-1}} \leq \frac{c}{\langle \tau_1 - \xi_1^2 \rangle^{6b-\frac{\rho}{2} - \rho}}\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{4b-1}},
\]
which is bounded provided $b \geq \frac{1}{4} + \frac{\rho}{6}$. In $D_{23}$ we use $|\xi_1|^2 \sim |\tau_1 - \xi_1^2|$ to obtain
\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b-2\rho}} \int_{D_{23}} \frac{d\xi}{\langle \xi_1 \rangle^{2\rho}} \langle \tau_1 - \xi_1^2 + 2\xi_1 (\xi_1 - \xi) \rangle^{4b-1} \leq \frac{c}{\langle \tau_1 - \xi_1^2 \rangle^{2b-2\rho}} \int_{|\eta| \leq 2|\tau_1 - \xi_1^2|} \frac{d\eta}{\langle \xi_1 \rangle^{4b-1}}
\]

which is bounded provided \(b \geq \frac{\rho}{2} + \frac{1}{4}\). Since \(\rho < \frac{1}{2}\), we can choose \(b = b(\rho)\) such that \(b < \frac{1}{4}\).

The inequality (7.8) in the case \(\rho \in (0, \frac{1}{4})\) follows interpolate the cases \(\rho = 0\) and \(\rho \in \left(\frac{1}{2}, \frac{3}{4}\right)\). This finish the proof of the part (a) of the lemma.

Arguing as in the part (a) of proof, to obtain Lemma 7.1 (b), it suffices to prove

\[
\left\| \frac{1}{\langle \tau - \xi_1^2 \rangle^{2b(\tau)}} \int \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi d\tau_1}{\langle \tau - \xi_1^2 \rangle^{2b(\tau)} - (\xi - \xi_1)^2} \right\|_{L_{\xi,\tau}^\infty} \leq c, \tag{7.16}
\]

where \(A(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbb{R}^2; \max \{|\tau - \xi_2^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 - (\xi - \xi_2)^2|\} = |\tau - \xi_2|\}, \) and

\[
\left\| \frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b_1}} \int \int_{B} \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi d\tau}{\langle \tau - \xi_1^2 \rangle^{2b_2(\tau)} - (\xi - \xi_1)^2} \right\|_{L_{\xi_1,\tau_1}^\infty} \leq c, \tag{7.17}
\]

where \(B = B(\xi_1, \tau_1) = \{(\xi, \tau) \in \mathbb{R}^2; \max \{|\tau - \xi_2^2|, |\tau_1 - \xi_1^2|, |\tau - \tau_1 - (\xi - \xi_2)^2|\} = |\tau_1 - \xi_1^2|\} \).

The inequality (7.16) can be obtained in the same way of the estimate (7.7). To estimate (7.17), we can assume \(10|\tau| < |\xi_2|\), since otherwise, the bound reduces to the inequality (7.8).

In this case we have \(|\tau - \xi_2^2| \sim |\xi_2^2|\). Thus, we need to show

\[
\frac{1}{\langle \tau_1 - \xi_1^2 \rangle^{2b_1}} \int \int_{B} \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi d\tau}{\langle \tau - \xi_1^2 \rangle^{2b_2(\tau)} - (\xi - \xi_1)^2} \leq c. \tag{7.18}
\]

Using the definition of \(B\) and (7.9), we see that \(|\xi_1| |\xi - \xi_1| \leq |\tau_1 - \xi_1^2|\), it follows that

\[
\frac{(|\xi_1| |\xi - \xi_1|)^{2\rho}}{|\tau_1 - \xi_1^2|} \leq \frac{1}{2|\tau_1 - \xi_1^2|^{2b_2 - \frac{1}{2}}} \leq c, \tag{7.19}
\]

where we have used that \(\rho < \frac{1}{2} b\). Using (7.19) and Lemma 7.4 we see that the left hand side of (7.18) is bounded by

\[
c \int \int_{B} \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi d\tau}{\langle \tau - \xi_1^2 \rangle^{2b}(\tau - (\xi - \xi_1)^2)}} \leq c \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi d\tau}{\langle \tau_1 + (\xi - \xi_1)^2 \rangle^{2b+2b_2 - 1}(\xi_1)^{2b}}.
\]

To estimate the last integral, we divide the integration domain in two parts, i.e., \(D_1 = \{\xi \in \mathbb{R}; |\xi_1| > 2|\xi - \xi_1| \text{ or } |\xi - \xi_1| > 2|\xi_1| \}\) and \(D_2 = \{\xi \in \mathbb{R}; \frac{1}{2}|\xi - \xi_1| \leq |\xi_1| \leq 2|\xi - \xi_1| \}\).

In \(D_1\) we have

\[
\frac{(|\xi_1| |\xi - \xi_1|)^{2\rho}}{|\xi_1|} \leq \frac{c}{|\xi_1|^{4\rho}} \leq c, \text{ for } \rho < 3b.
\]

Thus, Lemma 7.5 implies

\[
\int_{D_1} \frac{(|\xi_1| |\xi - \xi_1|)^{2\rho} d\xi}{\langle \tau_1 + (\xi - \xi_1)^2 \rangle^{2b+2b_2 - 1}(\xi_1)^{2b}} \leq c \int \frac{d\xi}{\langle \tau_1 + (\xi - \xi_1)^2 \rangle^{2b+2b_2 - 1}} \leq c,
\]

where we have used that \(b > \frac{3}{4} - \frac{\rho}{2}\).
Now we subdivide $D_2$ in two pieces, i.e., $D_{21} \cup D_{22}$, where $D_{21} = \{ \xi \in D_2; \ 10|\xi_1||\xi - \xi_1| \leq |\tau_1 - \xi_1^2| \}$ and $D_{22} = \{ \xi \in D_2; \ |\xi_1||\xi - \xi_1| \sim |\tau_1 - \xi_1^2| \}$.

In $D_{21}$ we have $|\tau_1 - \xi_1^2 - 2\xi_1(\xi - \xi_1)| \sim |\tau_1 - \xi_1^2|$. Then, by Lemma 7.1,

$$\frac{c}{(\tau_1 - \xi_1^2)^{2b}} \int_{B \cap D_{21} \times R} (\xi_1||\xi - \xi_1||)^{2d} d\xi d\tau \leq \frac{c}{(\tau_1 - \xi_1^2)^{2b-2\rho}} \int_{B \cap D_{21} \times R} (\xi + \xi_1^2)^{2b} (\tau - \tau_1 - (\xi - \xi_1)^2)^{2b} \xi d\xi d\tau \leq c \int_{\xi_1 \leq |\xi|} (\xi_1 - \xi_1^2)^{6b-1-2\rho} \leq c \int_{\xi_1 \leq |\xi|} (\xi_1 - \xi_1^2)^{6b-1-2\rho} \leq c \int_{\xi_1 \leq |\xi|} (\xi_1 - \xi_1^2)^{6b-1-2\rho - \frac{3}{2}},$$

which is bounded provided $\rho \leq 3b - \frac{3}{4}$. Since $\rho < \frac{3}{4}$ we can choose $b$ depending of $\rho$, such that $b < \frac{1}{2}$. In $D_{22}$ we have $|\xi_1|^2 \sim |\tau_1 - \xi_1^2|$, and consequently

$$\frac{1}{(\tau_1 - \xi_1^2)^{2b}} \int_{D_{22}} (\xi_1||\xi - \xi_1||)^{2d} d\xi \leq \frac{c}{(\tau_1 - \xi_1^2)^{2d}} \int_{D_{22}} (\tau_1 - \xi_1^2)^{4b-1} \xi d\xi d\tau \leq c \int_{|\eta| \leq 2|\xi_1 - \xi_1^2|} (\xi_1||\eta||)^{4b-1} \leq c \int_{|\eta| \leq 2|\xi_1 - \xi_1^2|} (\xi_1||\eta||)^{4b-1},$$

which is bounded provided $\rho \leq 3b - \frac{3}{4}$, and hence the proof of Lemma 7.1 is completed.

### 7.2. Proof of Lemma 7.2

Let $\rho = -s \in [0, \frac{3}{4}]$. For $u, v \in X^{s,b}$, define

$$g(\xi, \tau) = (\tau - \xi_1^2)^{\rho} \hat{u}(\xi, \tau), \ h(\xi, \tau) = (\tau - \xi_1^2)^{\rho} \hat{v}(\xi, \tau).$$

Then $\|g\|_{L_{1/2}^s L_2^b} = \|u\|_{X^{s,b}}, \ \|h\|_{L_{1/2}^s L_2^b} = \|v\|_{X^{s,b}}$. Using $\hat{\tau}(\xi, \tau) = \frac{\xi_{\rho}}{(\tau + \xi_{\rho})^{1/2}} \tau(-\xi, -\tau)$, we have

$$\|N_2(u, v)\|_{X^{s,b}} = \sup_{\|\phi\|_{L_{1/2}^s} \leq 1} W(g, h, \phi),$$

where

$$W(g, h, \phi) = \int_{\mathbb{R}^2} \frac{\xi_{\rho}}{(\tau - \xi_1^2)^{2b}} \int_{\mathbb{R}^2} \frac{\xi_{\rho}}{(\tau - \xi_1^2)^{2b}} g(\xi - \xi_1, \tau - \tau_1) \phi(\xi_1, \tau) d\xi_1 d\tau_1 d\xi d\tau.$$

Initially, we treat the case $\rho = 0$. Integrating first in $\xi, \tau$, changing of variables $\tau_2 = \tau - \tau_1$, $\xi_2 = \xi - \xi_1$, and using Cauchy-Schwarz and Hölder inequalities, we obtain

$$W^2(g, h, \phi) \leq c \phi \frac{1}{L_{1/2}^s} \frac{2}{L_2^b} \frac{1}{L_{1/2}^s} \leq c.$$

Thus we need to show

$$\sup_{(\xi_2, \tau_2) \in \mathbb{R}^2} \frac{1}{(\tau_2 - \xi_2^2)^{2b}} \int_{\mathbb{R}^2} \frac{d\xi d\tau}{(\tau - \xi_1^2)^{2b}(\tau - \tau_2 + (\xi - \xi_2)^2)^{2b}} \leq c. \quad (7.20)$$
Applying Lemmas 7.4 and 7.3 we see that the left hand side of (7.20) is bounded by
\[ c \int \frac{d\xi}{(2\xi^2 - 2\xi \xi_2 - \tau_2 \xi_2^2)^{4b-1}} \leq c, \]
where we have used that \( \frac{3}{8} < b < \frac{1}{2} \).

Now we treat the case \( \rho \in (0, \frac{1}{4}) \). In this case we shall assume that \( |\xi_1| \geq 10 \) and \( |\xi - \xi_1| \geq 10 \), since otherwise, the bound reduces to the case \( \rho = 0 \). We write
\[
W(g, h, \phi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s}{\langle \tau - \xi^2 \rangle^c} \frac{\langle \xi_1 \rangle^s}{\langle \tau_1 + \xi_1^2 \rangle^b} \frac{g(\xi - \xi_1, \tau - \tau_1)}{(\xi - \xi_1)^b (\tau - \tau_1 - (\xi - \xi_1)^2)^b} \phi(\xi, \tau) d\xi_1 d\tau_1 d\xi d\tau
\]
\[
= \int_{\mathcal{R}_1} + \int_{\mathcal{R}_2} + \int_{\mathcal{R}_3} + \int_{\mathcal{R}_4} + \int_{\mathcal{R}_5} := W_1 + W_2 + W_3 + W_4 + W_5,
\]
where
\[
\mathcal{R}_1 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \geq 10, |\xi| \leq 1\},
\]
\[
\mathcal{R}_2 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi| \geq 10, |\xi| \geq 1,
max\{||\tau - \xi_1^2|, |\tau_1 + \xi_1^2|, |\tau_1 - (|\xi_1| - |\xi|)^2|\} = |\tau_1 + \xi_1^2|\}.
\]
\[
\mathcal{R}_3 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi_1| \geq 10, 1 < |\xi| < |\xi_1|,
max\{||\tau - \xi_1^2|, |\tau_1 + \xi_1^2|, |\tau_1 - (|\xi_1| - |\xi|)^2|\} = |\tau_1 - \xi_1^2|\}.
\]
\[
\mathcal{R}_4 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi| \geq 10, |\xi| \geq |\xi_1|,
max\{||\tau - \xi_1^2|, |\tau_1 + \xi_1^2|, |\tau_1 - (|\xi_1| - |\xi|)^2|\} = |\tau_1 - \xi_1^2|\}.
\]
\[
\mathcal{R}_5 = \{(\xi, \xi_1, \tau, \tau_1) \in \mathbb{R}^4; |\xi| \leq 1, |\xi_1| \geq 10,
max\{||\tau - \xi_1^2|, |\tau_1 + \xi_1^2|, |\tau_1 - (|\xi_1| - |\xi|)^2|\} = |\tau_1 - (|\xi_1| - |\xi|)^2|\}.
\]
Thus, we need to show the following estimates
\[
\sup_{|\xi_1| > 10, \tau_1 \in \mathbb{R}} \left( \frac{1}{\tau_1 + \xi_1^2} \right)^{2b} \int \frac{|\xi_1|^2 |\xi - \xi_1|^{2\rho} d\xi d\tau}{(|\xi_1|^2 - |\tau - \tau_1 - (\xi - \xi_1)^2|)^{2b} (|\tau - \xi_1^2|)^{2b}} \leq c, \quad (7.21)
\]
\[
\sup_{|\xi| > 10, \tau \in \mathbb{R}} \left( \frac{1}{\tau + \xi^2} \right)^{2b} \int \frac{\chi_A(\xi_1, \tau_1) |\xi_1|^2 |\xi - \xi_1|^{2\rho} d\xi_1 d\tau}{(|\tau + \xi_1^2| + |\tau - \tau_1 - (\xi - \xi_1)^2|)^{2b} (|\tau - \xi_1^2|)^{2b}} \leq c, \quad (7.22)
\]
\[
\sup_{|\xi| > 1, \tau_1 \in \mathbb{R}} \left( \frac{1}{\tau + \xi_1^2} \right)^{2b} \int \frac{\chi_B(\xi_1, \tau_1) |\xi_1|^2 |\xi - \xi_1|^{2\rho} d\xi_1 d\tau_1}{(|\tau_1 + \xi_1^2| + |\tau - \tau_1 - (\xi - \xi_1)^2|)^{2b} (|\tau - \xi_1^2|)^{2b}} \leq c, \quad (7.23)
\]
\[
\sup_{|\xi| > 1, \tau_1 \in \mathbb{R}} \left( \frac{1}{\tau + \xi_1^2} \right)^{2b} \int \frac{\chi_C(\xi_1, \tau_1) |\xi_1|^2 |\xi - \xi_1|^{2\rho} d\xi_1 d\tau_1}{(|\tau_1 + \xi_1^2| + |\tau - \tau_1 - (\xi - \xi_1)^2|)^{2b} (|\tau - \xi_1^2|)^{2b}} \leq c, \quad (7.24)
\]
\[
\sup_{|\xi_1| > 1, \tau_1 \in \mathbb{R}} \left( \frac{1}{\tau_1 + \xi_1^2} \right)^{2b} \int \frac{\chi_D(\xi_1, \tau_1) |\xi_1|^2 |\xi - \xi_1|^{2\rho} d\tau_1 d\xi}{(|\tau_1 + \xi_1^2| + |\tau - \tau_1 - (\xi - \xi_1)^2|)^{2b} (|\tau - \xi_1^2|)^{2b}} \leq c, \quad (7.25)
\]
where
\[ A(\xi, \tau) = \{(\xi, \tau) \in \mathbb{R}^2; |\xi| \geq 1, \max\{|\tau - \xi^2|, |\tau + \xi^2|, |\tau - (\xi - \xi_1)^2|\} = |\tau + \xi^2| \}, \]
\[ B(\xi, \tau) = \{(\xi, \tau) \in \mathbb{R}^2; |\xi| > 10, |\xi| \leq |\xi_1|, \max\{|\tau - \xi^2|, |\tau + \xi^2|, |\tau - (\xi - \xi_1)^2|\} = |\tau - \xi^2| \}, \]
\[ C(\xi, \tau) = \{(\xi, \tau) \in \mathbb{R}^2; |\xi| > 10, |\xi| \geq |\xi_1|, \max\{|\tau - \xi^2|, |\tau + \xi^2|, |\tau - (\xi - \xi_1)^2|\} = |\tau - \xi^2| \}, \]
\[ D(\tau_1, \xi_1) = \{(\xi, \tau) \in \mathbb{R}^2; |\xi| \geq 1, |\xi - \xi_1| \geq 10, \max\{|\tau - \xi^2|, |\tau - (\xi - \xi_1)^2|\} = |\tau - \xi^2| \}. \]

First, we obtain (7.21). Using Lemma 7.4 we see that the left hand side of (7.24) is bounded by
\[
\sup_{|\xi_1| > 10, \tau_1 \in \mathbb{R}} \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{d\xi d\tau}{(\tau - \xi_1 - (\xi - \xi_1)^2)^{2b}(\tau - \xi_1^2)^{2b}} \leq \sup_{|\xi_1| > 10, \tau_1 \in \mathbb{R}} \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{d\xi}{(\tau_1 + \xi_1^2 - 2\xi \xi_1)^{4b-1}}. \tag{7.26} \]

Changing variables \( \eta = \tau_1 + \xi_1^2 - 2\xi \xi_1 \), then \( d\eta = -2\xi_1 d\xi \) and \( |\eta| \leq |\tau_1 + \xi_1^2| + |\xi_1| \). Substituting into (7.26), we see that (7.26) is controlled by
\[
\sup_{|\xi_1| > 10, \tau_1 \in \mathbb{R}} \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{d\eta}{(\eta)^{4b-1}} \leq \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{d\eta}{(\eta)^{4b-1}} \leq \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{|\tau_1 + \xi_1^2|^{2-4b}}{(|\tau_1 + \xi_1|^{2-4b} + |\xi_1|^{2-4b})}, \]

which is bounded provided \( b > \frac{1}{3} \) and \( \rho < b - \frac{1}{3} \).

To obtain (7.22), we use the following algebraic relation
\[ \tau - \tau_1 - (\xi - \xi_1)^2 + (\tau_1 + \xi_1^2) - (\tau - \xi_1^2) = 2\xi \xi_1. \tag{7.27} \]

By Lemma 7.4 we have that the left hand side of (7.22) is bounded by
\[
\sup_{|\xi_1| > 10, \tau_1 \in \mathbb{R}} \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{2b}} \int \frac{d\xi}{(\tau_1 + \xi_1^2 - 2\xi \xi_1)^{4b-1}}. \tag{7.28} \]

Set \( \eta = \tau + \xi_1^2 - 2\xi \xi_1 \). Then \( d\eta = -2\xi_1 d\xi \). By (7.27), we have \( |\eta| \leq c|\tau_1 + \xi_1^2| \) in \( A(\xi_1, \tau_1) \). Substituting into (7.28), we obtain that (7.28) is bounded by \( \frac{c|\xi_1|^{4\rho}}{(\tau_1 + \xi_1^2)^{4b-1}} \), which is bounded provided \( b > \frac{1}{3} \) and \( \rho \leq \frac{1}{4} \). Now we obtain (7.23). By Lemma 7.4 we see that left hand side of (7.23) is bounded by
\[
\sup_{|\xi_1| > 1, \tau \in \mathbb{R}} \frac{cX(|\xi_1|)}{(\tau - \xi_1^2)^{2b}} \int \frac{X(|\xi_1|; \eta_1 \in B(\xi, \tau)) (\xi_1)^{4\rho} d\xi_1}{(\tau - \xi_1^2 - 2\xi \xi_1)^{4b-1}} \leq \sup_{|\xi_1| > 1, \tau \in \mathbb{R}} \frac{cX(|\xi_1|)}{(\tau - \xi_1^2)^{2b-4\rho}} \int \frac{X(|\xi_1|; \eta_1 \in B(\xi, \tau)) d\xi_1}{(\tau - \xi_1^2 - 2\xi \xi_1)^{4b-1}}. \tag{7.29} \]
Set $\eta = \tau - \xi^2 + 2\xi \xi_1$, then $d\eta = 2\xi d\xi_1$ and $|\eta| \leq 2|\tau - \xi^2|$. Substituting into (7.29), we obtain that (7.29) is bounded by

$$\sup_{|\xi| > 1, \tau \in \mathbb{R}} \frac{c \chi(|\xi| > 1)}{|\tau - \xi^2|^{2b-4\rho} (\xi)^{2\rho} |\xi|} \int_{|\eta| \leq 2|\tau - \xi^2|} \frac{d\eta}{|\eta|^{4b-1}} \leq \sup_{|\xi| > 1, \tau \in \mathbb{R}} \frac{c \chi(|\xi| > 1)}{|\tau - \xi^2|^{2b-4\rho} (\tau - \xi^2)^{4b-2}}. \quad (7.30)$$

This expression is bounded, provided $\rho \leq \frac{3}{2} b - \frac{1}{2}$.

Now we estimate (7.24). By Lemma 7.4 we see that left hand side of (7.24) is bounded by

$$\sup_{|\xi| > 1, \tau \in \mathbb{R}} \frac{\chi(|\xi| > 1)}{|\tau - \xi^2|^{2b-\rho} (\xi)^{2\rho} (\tau - \xi^2)^{2b}} \int_{|\eta| \leq 2|\tau - \xi^2|} \frac{\chi_{C(\xi, \tau)}(|\xi| - |\xi - \xi_1|)^\rho}{(\tau + \xi^2)^{2b} (\tau - \tau_1 - (\xi - \xi_1)^2)^{2b}} \frac{d\xi_1 d\tau_1}{(\tau - \xi^2)^{4b-1}}. \quad (7.31)$$

Set $\eta = \tau - \xi^2 + 2\xi \xi_1$, then $d\eta = 2\xi d\xi_1$ and

$$|\eta| = |\tau_1 + \xi_1^2 + \tau - \tau_1 - (\xi - \xi_1)^2| \leq 2|\tau - \xi^2|, \text{ in } B(\xi, \tau).$$

Substituting into (7.31), we see that (7.31) is bounded by

$$\sup_{|\xi| > 1, \tau \in \mathbb{R}} \frac{\chi(|\xi| > 1)}{|\tau - \xi^2|^{2b-\rho} (\xi)^{2\rho} (\tau - \xi^2)^{2b}} \int_{|\eta| \leq 2|\tau - \xi^2|} \frac{d\eta}{|\eta|^{4b-1}} \leq \sup_{|\xi| > 1, \tau \in \mathbb{R}} \frac{c}{|\tau - \xi^2|^{6b-\rho-2}},$$

which is bounded provided and $\rho \leq 6b - 2$.

Finally, we shall obtain (7.25). By Lemma 7.4 we see that left hand side of (7.25) is bounded by

$$\sup_{|\xi| > 1, \tau_1, \tau_2 \in \mathbb{R}} \frac{\chi(|\xi| > 10)}{(\tau_1 - \xi_1^2)^{2b}} \int_{D(\xi_1, \tau_1)} \frac{(|\xi| - |\xi - \xi_1|)^{2\rho}}{(\xi)^{2\rho} (\tau_1 - \xi_1^2 - 2\xi_2 \xi_1)^{4b-1}},$$

where $D(\xi_1, \tau_1) = \{ \xi ; \exists \tau ; (\xi, \tau) \in D(\xi_1, \tau_1) \}$.

Now we divide $D(\xi_1, \tau_1)$ into two pieces, i.e., $D_1(\xi_1, \tau_1) = \{ \xi \in D(\xi_1, \tau_1) ; |\xi| \leq 100|\xi_1| \}$ and $D_2(\xi_1, \tau_1) = \{ \xi \in D(\xi_1, \tau_1) ; |\xi| > 100|\xi_1|, |\xi| \leq 500|\tau_1 - \xi_1^2| \}$.

In $D_1(\xi_1, \tau_1)$ we use the algebraic relation

$$(\tau - \xi^2) - (\tau_1 - \xi_1^2) - (\tau - \tau_1 + (\xi - \xi_1)^2) = -2\xi(\xi - \xi_1)$$

and Lemma 7.4 to obtain

$$\sup_{|\xi_1| > 10, \tau_1, \tau_2 \in \mathbb{R}} \frac{\chi(|\xi_1| > 10)}{(\tau_1 - \xi_1^2)^{2b}} \int_{D(\xi_2, \tau_2)} \frac{\chi_{D_1(|\xi_1| > 10)}(|\xi_1| - |\xi_1 - \xi_2|)^{2\rho}}{(\xi_2)^{2\rho} (\tau_1 - \xi_1^2 - 2\xi_2 \xi_1)^{4b-1}} \leq \sup_{|\xi_1| > 10, \tau_1, \tau_2 \in \mathbb{R}} \frac{c \chi_{D_1} d\xi_1}{(\tau_1 - \xi_1^2 - 2\xi_2 \xi_1)^{4b-1}},$$

which is bounded provided $b > \frac{3}{8}$ and $\rho \leq b$.

In $D_2$ we have $|\xi_1| - |\xi - \xi_1| \leq c|\xi_1|^2 \leq c|\tau_1 - \xi_1^2|^2$, it follows that the integral in this region can be controlled by

$$\sup_{|\xi_1| > 10, \tau_1, \tau_2 \in \mathbb{R}} \frac{c |\xi_1|^{4\rho} \chi(|\xi_1| > 10)}{(\tau_1 - \xi_1^2)^{2b}} \int_{D(\xi_2, \tau_2)} \frac{\chi_{D_2} d\xi_1}{(\xi_2)^{2\rho} (\tau_1 - \xi_1^2 - 2\xi_2 \xi_1)^{4b-1}}. \quad (7.32)$$
By Lemma 7.3, the expression (7.32) is bounded if \( b > \frac{3}{8} \) and \( \rho < \frac{b}{2} \), and hence the proof of Lemma 7.2 is completed.

7.3. Proof of Lemma 7.3. To get the part (a) it suffices to show that

\[
\left\|\frac{1}{(\tau - \xi^2)^{2b}} \int \int \frac{d\xi_1 d\tau_1}{(\tau_1 + \xi_1^2)^{2b}(\tau - \tau_1 + (\xi - \xi_1)^2)^{2b}}\right\|_{L^\infty_{\xi,\tau}} \leq c, \tag{7.33}
\]

\[
\left\|\frac{1}{(\tau - \xi^2)^{2b}(\xi^2)^{2\rho}} \int \int \frac{(|\xi_1||\xi - \xi_1|)^{2\rho} d\xi_1 d\tau_1}{A_3 \langle \tau_1 + \xi_1^2 \rangle^{2b}(\tau - \tau_1 + (\xi - \xi_1)^2)^{2b}}\right\|_{L^\infty_{\xi,\tau}} \leq c, \tag{7.34}
\]

\[
\left\|\frac{\chi_{\{|\xi_1|>\delta\}}}{(\tau_1 + \xi_1^2)^{2b}} \int \int \frac{(|\xi_1||\xi - \xi_1|)^{2\rho} d\tau d\xi}{B_3 \langle \tau - \xi_1^2 \rangle^{2b}(\tau - \tau_1 + (\xi - \xi_1)^2)^{2b}}\right\|_{L^\infty_{\xi_1,\tau_1}} \leq c, \tag{7.35}
\]

for \( \rho \in \left(\frac{1}{2}, \frac{3}{4}\right) \), where

\[ A_3 = A_3(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbb{R}^2; \max\{|\tau - \tau_1 + (\xi - \xi_1)^2|, |\tau_1 + \xi_1^2|, |\tau - \xi^2|\} = |\tau - \xi^2|\}, \]

\[ B_3 = B_3(\xi_1, \tau_1) = \{(\xi, \tau) \in \mathbb{R}^2; \max\{|\tau - \tau_1 + (\xi - \xi_1)^2|, |\tau_1 + \xi_1^2|, |\tau - \xi^2|\} = |\tau_1 + \xi_1^2|\}. \]

The estimate (7.33) can be obtained by using the same argument given in the estimate (7.4) in the proof of Lemma 7.1 (part (a)). The estimates (7.34) and (7.35) can be obtained by following the ideas used in the estimates (7.7) and (7.8) in the proof of Lemma 7.1 (part (a)) and by using the following algebraic relation

\[
\tau - \tau_1 + (\xi - \xi_1)^2 + \tau_1 + \xi_1^2 - (\tau - \xi^2) = (\xi - \xi_1)^2 + \xi_1^2 + \xi^2. \tag{7.36}
\]

To obtain the part (b) it suffices to prove that

\[
\left\|\frac{\chi_{\{|10|\,|\xi_1|\leq \delta^2\}}}{(\tau - \xi_1^2)^{2b}(\tau)^{\rho}} \int \int \frac{(|\xi_1||\xi - \xi_1|)^{2\rho} d\tau d\xi}{A_3 \langle \tau_1 + \xi_1^2 \rangle^{2b}(\tau - \tau_1 + (\xi - \xi_1)^2)^{2b}}\right\|_{L^\infty_{\xi,\tau}} \leq c, \tag{7.37}
\]

\[
\left\|\frac{1}{(\tau + \xi_1^2)^{2b}} \int \int \frac{\chi_{\{|10|\,|\xi_1|\leq \delta^2\}}(\xi_1\,|\xi - \xi_1|)^{2\rho} d\tau d\xi}{B_3 \langle \tau - \xi_1^2 \rangle^{2b}(\tau - \tau_1 + (\xi - \xi_1)^2)^{2b}}\right\|_{L^\infty_{\xi_1,\tau_1}} \leq c, \tag{7.38}
\]

where \( A_3 = A_3(\xi, \tau) = \{(\xi_1, \tau_1) \in \mathbb{R}^2; \max\{|\tau - \tau_1 + (\xi - \xi_1)^2|, |\tau_1 + \xi_1^2|, |\tau - \xi^2|\} = |\tau - \xi^2|\} \) and \( B_3 = B_3(\xi_1, \tau_1) = \{(\xi, \tau) \in \mathbb{R}^2; \max\{|\tau - \tau_1 + (\xi - \xi_1)^2|, |\tau_1 + \xi_1^2|, |\tau - \xi^2|\} = |\tau_1 + \xi_1^2|\} \).

These estimates can be obtained by using a similar argument to that described in the proof of Lemma 7.1 (part (b)). The detailed computation is omitted here.

8. Proof of Theorem 1.1.

We will prove Theorem 1.1 for the nonlinearity \( N_1(u, \overline{u}) = u^2 \), the proofs for the nonlinearities \( N_2(u, \overline{u}) = |u|^2 \) and \( N_3(u, \overline{u}) = \overline{u}^2 \) follow the same ideas, by using Lemmas 7.2 and 7.3.

Using a scaling argument, we can assume

\[
\|u_0\|_{H^s(\mathbb{R}^+)} + \|f\|_{H^s(\mathbb{R}^+)} = \delta,
\]
for $\delta$ sufficiently small. Indeed, $u$ solves the problem (1.1) if and only if for $\lambda > 0$, $u_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^2 t)$ solves the problem (1.1) with initial-boundary conditions $u_\lambda(x, 0) = \lambda^2 u_0(\lambda x)$ and $u_\lambda(0, t) = \lambda^2 f(\lambda^2 t)$. Thus for $s \leq 0$, $\|u_\lambda(\cdot, 0)\|_{H^s(\mathbb{R})} \leq \lambda^{3/2 + s}\|u(\cdot, 0)\|_{H^s(\mathbb{R})}$ and $\|u_\lambda(0, \cdot)\|_{H^{2+s,1}(\mathbb{R}^+)} \leq \lambda^{3/2 + s}\|u(0, \cdot)\|_{H^{2+s,1}(\mathbb{R}^+)}$.

Select an extension $\tilde{u}_0 \in H^s(\mathbb{R})$ of $u_0$ such that $\|\tilde{u}_0\|_{H^s(\mathbb{R})} \leq c\|u_0\|_{H^s(\mathbb{R})}$. Let $b = b(s) < \frac{1}{2}$ such that the estimates (7.1) and (7.2) are valid.

Let

$$\Lambda(u)(t) = \psi(t)e^{-it\partial_x^2} \tilde{u}_0 + \psi(t)D(u^2)(t) + \psi(t)\mathcal{L}^\lambda h(t),$$

where $h(t) = e^{-it\partial_x^2}(\chi_0, +\infty)\psi(t)f(t) - \psi(t)e^{-it\partial_x^2} \tilde{u}_0|_{x=0} - \psi(t)D(\psi^2)(t)|_{x=0}\big|_{(0, +\infty)}$ and $\lambda = \lambda(s)$ is such that $-1 < \lambda < \frac{1}{2}$ and $s - \frac{1}{2} < \lambda < s + \frac{1}{2}$.

Initially, we show that $\mathcal{L}^\lambda h(t)$ is well defined. By Lemmas 2.1, 2.2, 3.1, 5.1, 7.1 and 7.2 we have

$$\|h\|_{H^{2+s,1}(\mathbb{R}^+)} \leq \|\chi_0, +\infty(\psi(t)f(t) - \psi(t)e^{-it\partial_x^2} \tilde{u}_0|_{x=0} - \psi(t)D(\psi^2)(t)|_{x=0})\|_{H^{2+s,1}(\mathbb{R})} \leq c\|f\|_{H^{2+s,1}(\mathbb{R}^+)} + \|\tilde{u}_0\|_{H^s(\mathbb{R})} + \|u^2(x, t)\|_{X^{s-b}} + c_1(s)\|u^2(x, t)\|_{W^{s-b}} \leq c\|f\|_{H^{2+s,1}(\mathbb{R})} + \|\tilde{u}_0\|_{H^s(\mathbb{R})} + \|u\|_{X^{s,b}},$$

where $c_1(s) = 0$, if $s \in (-1/2, 0)$ and $c_1(s) = 1$, if $s \in (-3/4, -1/2]$. Since $u \in Z$ we obtain $h \in H^{2+s,1}(\mathbb{R}^+)$. If $-\frac{3}{4} < s \leq 0$, then $-\frac{1}{8} \leq \frac{2s+1}{4} \leq \frac{1}{4}$, and Lemma 2.1 shows that $h \in H^{2+s,1}_0(\mathbb{R}^+)$. Thus, $\mathcal{L}^\lambda h(t)$ is well defined and $\|h\|_{H^{2+s,1}(\mathbb{R}^+)} \sim\|h\|_{H^{2+s,1}_0(\mathbb{R}^+)}$.

Our goal is to show that $\Lambda$ defines a contraction map on any ball of $Z$.

Using Lemmas 3.1, 5.1 and 7.1 we see that

$$\|\psi(t)e^{-it\partial_x^2} \tilde{u}_0\|_Z \leq c\|\tilde{u}_0\|_{H^s(\mathbb{R})} \leq c\|u_0\|_{H^s(\mathbb{R})} \text{ and } \|\psi(t)D(u^2)(t)\|_Z \leq c\|u\|_{X^{s,b}}^2.$$

Combining Lemma 5.2 and expression (8.1) we obtain

$$\|\psi(t)\mathcal{L}^\lambda h(t)\|_Z \leq c\|h\|_{H^{2+s,1}_0(\mathbb{R}^+)} \leq c\|f\|_{H^{2+s,1}_0(\mathbb{R}^+)} + \|u_0\|_{H^s(\mathbb{R})} + \|u\|_Z^2,$$

for $-1 < \lambda < \frac{1}{2}$ and $s - \frac{1}{2} < \lambda < s + \frac{1}{2}$. Note that for $s > -\frac{1}{2}$, we can take $\lambda = 0$ and as $s > -\frac{4}{5}$ we can choose adequate $\lambda = \lambda(s)$. Thus, we obtain

$$\|\Lambda u\|_Z \leq c\|f\|_{H^{2+s,1}_0(\mathbb{R}^+)} + \|u_0\|_{H^s(\mathbb{R})} + \|u\|_Z^2.$$

Similarly, we obtain for $u$ and $v$ in $Z$,

$$\|\Lambda u - \Lambda v\|_Z \leq c^2(\|u\|_Z + \|v\|_Z)\|u - v\|_Z.$$

Let $B_{a,b}(c\delta) = \{u \in Z; \|u\|_Z \leq 2c\delta\}$. If $u \in B_{a,b}(c\delta)$, then (8.2) implies

$$\|\Lambda u\|_Z \leq c\delta + c(2\delta)^2 \leq 2c\delta.$$
Now we choose \( \delta \), such that \( 4c^2\delta \leq \frac{1}{2} \). By (8.3) if \( u \) and \( v \) in \( B_{s,b}(c\delta) \) we have
\[
\|\Lambda u - \Lambda v\|_Z \leq 4c^2\delta\|u - v\|_Z \leq \frac{1}{2}\|u - v\|_Z.
\]
Thus, \( \Lambda \) defines a contraction on \( B_{s,b}(c\delta) \) and consequently there exists a unique function \( \tilde{u} \in B_{s,b}(c\delta) \) such that \( \tilde{u} = \Lambda(\tilde{u}) \). Therefore \( u(x,t) := \tilde{u}(x,t)|_{\mathbb{R}^+ \times [0,1]} \) solves the IBVP (1.1) with nonlinearity \( N_1(u,\overline{u}) = u^2 \), in the time interval \([0,1]\).

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