Cosmological Time Crystals From Einstein-Cubic Gravities

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ABSTRACT

By including appropriate Riemman cubic invariants, we find that the dynamics of classical time crystals can be straightforwardly realized in Einstein gravity on the FLRW metric. The time reflection symmetry is spontaneously broken in the two vacua with the same scale factor $a$, but opposite $\dot{a}$. The tunneling from one vacuum to the other provides a robust mechanism for bounce universes; it always occurs for systems with positive energy density. For suitable matter energy-momentum tensor we also construct cyclic universes. Cosmological solutions that resemble the classical time crystals can be constructed in massive gravity.

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1 Introduction

Recently, a fascinating concept of time crystal was proposed \[1,2\], and subsequently realized in experiments \[3,5\]. A time crystal refers to a ground state that breaks the time-translational invariance, such that it is periodic in both space and time. The subject has attracted considerable attention, e.g. \[6,11\]. From the view of effective dynamics \[12\], a minimal classical time crystal can be realized mathematically by including a potential with a well-defined regulator; it is characterized by a non-smooth reversal of the velocity at the boundary. What is unusual is that the sudden velocity reversal is not caused by a “brick wall” potential; it is a consequence of spontaneous symmetry breaking analogous to the Higgs mechanism, but in the momentum space.

It is interesting and important to study this idea in the context of gravity. There are many important time-dependent systems in General Relativity (GR) such as the evolution of our Universe, dynamical black holes, gravitational waves and so on. The time crystal behavior of an oscillating scalar field in the expanding Friedmann-Lemaître-Robertson-Walker (FLRW) universe was constructed in \[13,14\]. The aim of this paper is to study the possibility of treating the universe itself as a time crystal. Such a cosmological model naturally depicts a cyclic universe, which would necessarily violate the null-energy condition (NEC) in GR. Nevertheless, the idea of our Universe being cyclic still attracts much attention, and the most famous model is the ekpyrotic universe, based on string theory \[15\]. The time crystal mechanism provides an alternative realization.

Constructing cosmological time crystals was attempted in gravity on non-commutativity geometry \[16\]. In fact, the dynamics of classical time crystals \[1,12\] can be easily realized in gravities, when we include higher-order curvature invariants. In this paper we consider Einstein gravity extended with appropriate Riemann cubic invariants, coupled to some homogeneous and isotropic perfect fluid. A new feature arising is that there is now a Hamiltonian constraint owing to the general diffeomorphism whilst in a classical mechanical system, the Hamiltonian yields an arbitrary conserved energy that should be bounded below.

In section 2, we construct the cubic invariants that facilitate the time crystal mechanism of \[1,12\]. We study the properties of the resulting cosmological solutions. In section 3, we present three explicit examples based on different types of the perfect fluid models. In the last example, we consider massive gravity instead of Einstein gravity. We discuss and conclude the paper in section 4.
2 The theory and the cosmological model

A crucial ingredient in our construction involves the Riemann cubic invariants, which have in general eight terms:

\[
L^{(3)} = \sqrt{-g} \left( e_1 R^3 + e_2 R_{\mu\nu} R^{\mu\nu} + e_3 R_{\mu}^{\nu\rho} R_{\rho}^{\mu\nu} + e_4 R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + e_5 R R^{\mu\nu\rho\sigma} R_{\mu\rho\nu\sigma} + e_6 R^{\mu\nu\rho \gamma} R_{\rho \gamma \nu \mu} + e_7 R^{\mu \nu \rho \sigma} R_{\rho \sigma \alpha \beta} R^{\alpha \beta} R^{\gamma} R_{\mu \gamma} \right). \quad (2.1)
\]

In de Sitter (dS) or anti-de Sitter (AdS) spacetimes, these generate additional linear massive scalar and spin-2 modes. Decoupling of the ghost-like spin-2 mode requires (e.g. [17])

\[
12 e_2 + 9 e_3 + 5 e_4 + 48 e_5 + 16 e_6 + 24 e_7 - 3 e_8 = 0. \quad (2.2)
\]

In this paper, we study cosmology in the FLRW metric

\[
ds^2 = -dt^2 + a(t)^2 (dx_1^2 + dx_2^2 + dx_3^2). \quad (2.3)
\]

In the effective Lagrangian, the absence of \(\dddot{a}^2\) and \(\dddot{a}^3\) requires

\[
216 e_1 + 60 e_2 + 18 e_3 + 16 e_4 + 48 e_5 + 14 e_6 + 12 e_7 + 3 e_8 = 0,

36 e_1 + 12 e_2 + 5 e_3 + 3 e_4 + 12 e_5 + 4 e_6 + 4 e_7 = 0. \quad (2.4)
\]

The first equation decouples the massive scalar mode. Intriguingly these are precisely the same conditions for the holographic \(a\)-theorem [18]. With these conditions, we find that the effective Lagrangian is given by

\[
L_{(3)} = \frac{2}{5} \lambda \frac{a^6}{a^3}, \quad \lambda = 2(36 e_1 + 6 e_2 + e_4 + 4 e_5). \quad (2.5)
\]

We may restrict further to Ricci polynomials, namely \(7 R^3 - 36 R_{\mu\nu} R^{\mu\nu} + 36 R_{\nu}^{\mu\nu} R_{\rho}^{\nu} R_{\mu}^{\rho} [19]\). The analogous construction for quadratic invariants yields the Weyl-squared term, which gives no contribution to equations of motion for the FLRW metric.

We focus on Einstein gravity coupled to some perfect fluid with energy-momentum tensor \(T^a_b = \text{diag}(\rho_m, p_m, p_m, p_m)\), and the Riemann cubics. The Einstein equations yield

\[
\frac{3a^2}{a^2} - \lambda \frac{\dot{a}^6}{a^6} = \rho_m, \quad -\frac{2\dot{a}}{a} - \frac{\ddot{a}}{a^2} + \lambda \left( \frac{2\dot{a}\ddot{a}}{a^5} - \frac{\dot{a}^6}{a^6} \right) = p_m. \quad (2.6)
\]
For simplicity, we consider an effective theory where the energy density is a function of the scale factor \(a\). For example, the vacuum energy density is given by \(\rho_m = \Lambda_0\), the bare cosmological constant. In radiation or matter dominated universes, we have \(\rho_m \sim a^{-4}\) and \(a^{-3}\) respectively. For a free massless scalar \(\phi\), we have instead \(\rho_m \sim a^{-6}\). If we express \(\rho_m = V(a)/(2a^3)\), the energy-momentum conservation requires \(p_m = V'(a)/(6a^2)\), giving rise to the equation of state \(w = -aV'/(3V)\).

The equations of motion can now be derived from the effective Lagrangian

\[
L = -6a\dot{a}^2 + \frac{2}{3}\lambda \frac{\dot{a}^6}{a^3} - V. \tag{2.7}
\]

The negative kinetic energy \(-\dot{y}^2/2\) proposed in [1,12] is hard to justify in classical mechanics, it arises naturally in gravity. This term should not be viewed as ghost-like owing to the general diffeomorphism, which imposes the Hamiltonian constraint

\[
H = -6a\dot{a}^2 + 2\lambda \frac{\dot{a}^6}{a^3} + V = 0. \tag{2.8}
\]

In fact this constraint is equivalent to the first equation in (2.6). By contrast, although \(H\) is conserved in classical mechanics, it does not necessarily vanish.

In this paper, we shall consider only \(\lambda > 0\), such that the gravitational part of the Hamiltonian, \(H_0 = -6a\dot{a}^2 + 2\lambda \dot{a}^6/a^3\), is bounded below. A key property is that in terms of the canonical momentum

\[
p = \frac{\partial L}{\partial \dot{a}} = -12a\dot{a}(1 - \frac{\lambda \dot{a}^4}{5a^4}), \tag{2.9}
\]

\(H_0(p,a)\) can be multi-valued and the minimum of \(H_0\) occurs not when \(p = 0\), but when

\[
\frac{\partial H_0}{\partial \dot{a}} = 0, \quad \implies \quad \dot{a} = \pm \lambda^{-\frac{1}{4}} a, \quad H_0^{\text{min}} = -\frac{4a^3}{\sqrt{\lambda}}. \tag{2.10}
\]

The \(p = 0 = \dot{a}\) point is instead a local maximum. This is analogous to the Higgs mechanism, but in the momentum space. \(H_0\) as function of \(p\) is depicted in Fig. 1.

The evolution of \(a(t)\) near \(H_0^{\text{min}}\) depends on the potential \(V(a)\). In order to study this behavior, we also examine \(H_0\) as a function of \(\dot{a}\), depicted also in Fig. 1. The Hamiltonian constraint (2.8) at \(H_0^{\text{min}}\) implies that

\[
\tilde{V}(a) \equiv V(a) - \frac{4a^3}{\sqrt{\lambda}} = 0. \tag{2.11}
\]
Figure 1: The sparrow tail in the left plot is characteristic when we include a higher-order kinetic term. The cusp singularities imply that $\dot{a}$ cannot be continuous at the tip. Instead the system tunnels from one vacuum to the other, keeping $\dot{a}^2$ continuous. The right plot shows that $H_0(\dot{a})$ shapes like a two-dimensional Mexican hat. In both plots we set $\lambda = 1$ and $a = 1$. For general quantities, the true vacua are at $\dot{a} = \pm \lambda^{1/4} a$, corresponding to $H_0^{\text{min}} = -4a^3/\sqrt{\lambda}$.

If it has no solution, (e.g. $V = -3a^3$, for a negative cosmological constant,) then $H_0^{\text{min}}$ can never be reached. If equation (2.11) has a solution at $a = a_0$, the system may reach $a_0$, but cannot stay there since $\dot{a} \neq 0$. Thus we must require that the cubic equation (2.8) for $\dot{z} = \dot{a}^2$ has a positive root, and it does if and only if $\bar{V}(a) \leq 0$ in the connected region. In fact, there are two positive roots, given by

$$\dot{a}^2 = \frac{2a^2}{\sqrt{\lambda}} \cos \frac{1}{3} \left( \arccos \left[ \frac{a_0^3 V(a)}{a^3 V(a_0)} \right] - k\pi \right), \quad k = \pm 1. \quad (2.12)$$

The third root, corresponding to $k = 3$, is negative and should be ignored. In the vicinity of $a = a_0$, we have

$$\dot{a}^2 = \frac{a_0^2}{\sqrt{\lambda}} \left( 1 + k \sqrt{\frac{2}{a_0} (1 + w(a_0)) (a - a_0)} + \cdots \right). \quad (2.13)$$

For matter satisfying the NEC ($w \geq -1$), we must have $a \geq a_0$, thus a bounce occurs at $a = a_0$. Furthermore, at $a = a_0$, we have two vacua with $\dot{a}_0^\pm = \pm \lambda^{-1/4} a_0$ respectively, and they are energy degenerate, but break the time reflection symmetry. The tunneling from one vacuum to the other keeps $\dot{a}^2$ continuous, but causes $\dot{a}$ to jump from the negative to the positive, or vice versa, analogous to the situation when a pingpong hits a brick wall.

It is instructive to determine whether there should be an external “brick wall” source, since $\ddot{a}$, and hence the curvature, has a $\delta$-function singularity of the comoving time. Assuming that the turning point $a = a_0$ occurs at $t = 0$, we can solve the function $a(t)$ at small $t$. The solution is smooth except at $t = 0$, and the extra source required for the bouncing
behavior is formally given by

\[ \rho_{\text{ext}} = 0, \quad p_{\text{ext}} = 4k\lambda^{-1/4}\sqrt{2\lambda^{-1/4}(1 + w(a_0))}\sqrt{|t|} \delta(t), \quad k = -1, 1. \]  

(2.14)

Since the energy-momentum conservation does not involve a time derivative of \( p \), we can effectively treat \( p_{\text{ext}} = 0 \). It was demonstrated [12] that this matter source can be replaced by some well-defined regulator in classical time crystals.

The physical picture is clear. Owing to the spontaneous symmetry breaking, the cosmology splits into two energy-degenerated vacua, with \( \dot{a}^\pm \) respectively. As the universe shrinks to \( a_0 \), it tunnels from the \( \dot{a}^- \) vacuum to the \( \dot{a}^+ \) one and starts to expand, creating a bounce at \( a = a_0 \). It follows from (2.11) that this bounce mechanism is robust and will always occur for positive energy density.

3 Explicit examples

I. Bounce universes: The simplest example is perhaps when \( w \) is a constant, for which \( V = 2q^2a^{-3w} \) where \( q \) is a constant. For \( \lambda = 0 \), the universe is expanding with \( a = (\frac{3}{4}(1 + w)^2q^2t^2)^{1/(3(1+w))} \), with an initial spacetime singularity at \( t = 0 \). For non-vanishing \( \lambda \), a bounce must occur, at \( a_0^{3(1+w)} = \sqrt{\lambda q^2}/2 \). The \( k = -1 \) solution is governed by

\[ a^2 = \frac{2}{\sqrt{\lambda}} a^2 \cos \frac{1}{3} \left( \arccos \left[ \left( \frac{a_0}{a} \right)^{3(1+w)} \right] + \pi \right). \]  

(3.1)

Thus we see that for the standard cosmology with positive energy density, the introduction the Riemann cubics generates a bounce universe. In Fig. 2 we plot \( a(t) \) for various \( w \).

![Figure 2](image)

Figure 2: The left plot shows bounce universes, where \( \dot{a} \) reverses its sign at \( t = 0, a_0 = 1 \). We have chosen \( \lambda = 1 \) and \( w = -1, -2/3, -1/3, 1/3 \) for the lines from the top to the bottom. The right plot is the cosmological time crystal where \( \dot{a} \) reverses signs at both \( a_\pm = 1, 3 \). The theory involves massive gravity, discussed in the third example.

II. Cosmological time crystals: If the potential \( V \) has a zero at finite \( a = A > a_0 \);
furthermore, $\tilde{V}(a) > 0$ for $a \in (a_0, A)$, then $a(t)$ shrinks smoothly from $a = A$, until at $a = a_0$ where it bounces, creating a cyclic universe. As a concrete example, we consider $V = 2\Lambda_0 a^3 + 6a^2a$, where the second term can be generated by a sigma model \cite{20}. For negative cosmological constant $\Lambda_0 = -3/\ell^2$, the potential vanishes at $A = \alpha \ell$. In fact, for $\lambda = 0$, an exact solution can be found, namely $a = A \sin(t/\ell)$. The solution appears to be cyclic, but the corresponding universe is not owing to the curvature singularity at $a = 0$. When $\lambda$ is included, the $\dot{a}^6$ term has no effect at $a = A$, where $\dot{a} = 0$. However, there is a turning point $a_0$:

$$0 < a_0 = \frac{\sqrt{3} \alpha \ell \lambda^{1/4}}{\sqrt{2 \ell^2 + 3 \sqrt{\lambda}}} < A,$$

where the universe bounces. The solution is depicted in Fig. 3.

III. Cosmological time crystals from massive gravity: In \cite{12}, time crystals typically have two jumping points. In our gravity model, for $V(a)$ satisfying NEC, there can only be one, causing bounce of the universe. In order to reproduce the analogous behaviors of \cite{12}, we have to consider theory beyond Einstein. We find such a solution exists in massive gravity of \cite{21}, together with the Riemann cubics:

$$\mathcal{L} = \sqrt{-g} \left( R - 2\Lambda_0 + m^2(\mathcal{U}_2 + c_3 \mathcal{U}_3 + c_4 \mathcal{U}_4) \right) + \lambda \mathcal{L}_{(3)},$$

$$\mathcal{U}_2 = [\mathcal{K}]^2 - [\mathcal{K}^2], \quad \mathcal{U}_3 = [\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3],$$

$$\mathcal{U}_4 = [\mathcal{K}]^4 - 6[\mathcal{K}^2][\mathcal{K}]^2 + 8[\mathcal{K}^3][\mathcal{K}^2] + 6[\mathcal{K}^4],$$

$$\mathcal{K}^\mu_\nu = \delta^\mu_\nu - \sqrt{g} \delta_{\alpha\beta} \partial_\alpha \phi^\mu \partial_\nu \phi^\beta \eta_{ab}, \quad \eta_{ab} = \text{diag}(-1, 1, 1, 1).$$

The rectangular brackets denote traces, e.g. $[\mathcal{K}] = \text{Tr}(\mathcal{K}) = \mathcal{K}^\mu_\mu$. For the fiducial reference...
\( \phi^a = a_m(0, x_1, x_2, x_3) \), the effective Lagrangian for the FLRW metric is given by (2.7) with

\[
V = 2\Lambda_0 a^3 + 6m^2 (a_m - a) \left( (4c_3 + 4c_4 + 2)a^2 - a_m (5c_3 + 8c_4 + 1)a + a_m^2 (c_3 + 4c_4) \right).
\]

(3.4)

For a concrete demonstration, we choose

\[
c_3 = \frac{2}{57}, \quad c_4 = \frac{7}{348}, \quad \Lambda_0 = \frac{5989}{2900}, \quad \lambda = 1, \quad m = \frac{1}{10}, \quad a_m = 10.
\]

(3.5)

The system has two turning points \((a_-, a_+) = (1, 3)\). The cosmology cycles between \(a_\pm\), as shown in the right plot of Fig. 2. Since we have \(a_\pm < a_m\) in this solution, there is no ghost excitation from the massive gravity sector [22].

4 Conclusions

Time crystals can arise when the time translational symmetry in the vacuum is spontaneously broken. A simple classical mathematical model [1, 12] involves a ghost-like kinetic \(-\dot{y}^2/2\) augmented by the higher-order \(\dot{y}^4/12\), such that the true vacuum is shifted down with non-vanishing velocity in a specific direction, hence breaking the time reflection symmetry. Such a system is hard to realize in classical mechanics, but it arises naturally in the effective Lagrangian in Einstein gravity on the FLRW metric, extended with appropriate higher-order curvature invariants.

We focused on a class of Riemann cubics and found that for matter satisfying NEC, the time crystal mechanism could generate bounces. The sudden change of the sign of \(\dot{a}\) at the bounce is the effect of tunneling from one vacuum to the other while keeping \(a^2\) continuous. Our analysis shows that this is a robust mechanism for bounce universes; it always occurs for positive energy density. Cyclic universes can also be constructed since shrinking \(a(t)\) from its maximum is consistent with NEC. We also considered massive gravity and constructed time crystals with two sudden reversing points.

Although we have restricted the Riemann cubics such that they do not generate linear ghosts in maximally-symmetric spacetimes, they likely create ghosts in our cosmological crystals. The ghost-free Gauss-Bonnet combination that can also generate time crystals in \(D \geq 5\) is unfortunately a total derivative in four dimensions. This suggests that ghost-free time crystals as cosmological models have to involve non-minimally coupled matter. Our work provides a guidance for such constructions.
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