Proportional Fair MU-MIMO in 802.11 WLANs

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Abstract
We consider the proportional fair rate allocation in an 802.11 WLAN that supports multi-user MIMO (MU-MIMO) transmission by one or more stations. We characterise, for the first time, the proportional fair allocation of MU-MIMO spatial streams and station transmission opportunities. While a number of features carry over from the case without MU-MIMO, in general neither flows nor stations need to be allocated equal airtime when MU-MIMO is available.

1 Introduction
The next generation of 802.11 WLANs are expected to support multi-user MIMO (MU-MIMO) transmission, whereby parallel transmissions can be simultaneously made to multiple stations. This significantly extends the MIMO support introduced by the 802.11n standard and is, for example, included as part of the current draft 802.11ac standard that aims to support wireless data rates in excess of 1 Gbps. MU-MIMO offers much greater flexibility in scheduling MIMO transmissions, but immediately raises the question of how best to allocate MIMO spatial streams amongst network flows so as to balance fairness and performance. In this paper we consider the proportional fair allocation in an 802.11 WLAN that supports multi-user MIMO (MU-MIMO) transmission by one or more stations. While proportional fairness in WLANs has been the subject of considerable interest in the literature, it has only recently been put on a rigorous basis in [2] and consideration of MU-MIMO is new.

The main contribution of the paper is to characterise, for the first time, the proportional fair allocation of spatial streams and station transmission opportunities in WLANs where one or more stations support MU-MIMO. We demonstrate that this allocation can be found using a simple distributed algorithm. We show that a number of features carry over from the case without MU-MIMO, specifically that the rate region boundary is characterised by the station total airtimes summing to unity and that when the offered load is unconstrained and stations carry the same number of flows then stations are assigned equal total airtime. Importantly, however, we find that MU-MIMO can lead to a qualitatively different allocation of airtime compared to the situation without MU-MIMO. Namely, in general neither flows nor stations need to be allocated equal airtime.
2 Related Work

In [3] the authors propose a novel MAC design with opportunistic MU-MIMO scheduling based on channel sounding feedback, where packets are selected depending on their transmission duration and type of traffic. In [4] is also proposed a novel MAC design for MU-MIMO that focuses on issues such as MAC ACKing of MU-MIMO transmissions. Packets are scheduled for transmission using a weighted queueing mechanism that considers both packets acknowledgements and type of traffic. However, in both [3] [4] fairness and allocation of MU-MIMO transmission patterns amongst flows is not considered. The work in [5] focuses on packet aggregation in an IEEE 802.11ac AP, and considers a fixed MU-MIMO schedule where one flow is allocated per spatial stream. Regarding utility fairness in WLANs, in [2] is presented the first rigorous analysis of proportional fairness in 802.11 WLANs where transmissions are to a single destination.

3 Network Model

3.1 Preliminaries

We take as our starting point the network model in [2]. Consider an 802.11 WLAN with \( n \) stations, where each station \( i \) attempts to transmit at each MAC slot with probability \( \tau_i \). We assume that \( CW_{\text{min}} = CW_{\text{max}} \), so that the attempt probability is independent of the success or failure of the last transmission. Moreover, it is also assumed that there are no hidden terminals, so all nodes in the network can sense any ongoing transmission. Because of this, a collision can only happen if two or more stations transmit in the same slot.

The probability that a transmission by station \( i \) is successful is the probability that only station \( i \) transmits and is given by \( P_{\text{succ},i} = \tau_i \prod_{k=1,k\neq i}^{n} (1 - \tau_k) \). The probability that a MAC slot is idle is given by the probability that none of the stations transmit, \( P_{\text{idle}} = \prod_{k=1}^{n} (1 - \tau_k) \). Finally, the probability that a transmission by station \( i \) collides is \( P_{\text{coll},i} = 1 - P_{\text{succ},i} - P_{\text{idle}} \). The throughput of station \( i \) is then given by

\[
S_i(\tau) = \frac{P_{\text{succ},i} D_i}{\sigma P_{\text{idle}} + T_s (1 - P_{\text{idle}})},
\]

where \( \sigma \) is the duration of an idle slot, \( T_s \) the duration of a busy slot (either successful or collision) and \( D_i \) is the size in bits of the frame payload of station \( i \).

A change of variable that is useful for the analysis is \( x_i = \tau_i / (1 - \tau_i) \), where \( x_i \in [0, \infty) \) when \( \tau_i \in [0, 1] \). With this change of variable we have that \( P_{\text{idle}} = 1 / \prod_{k=1}^{n} (1 + x_k) \), \( P_{\text{succ},i} = x_i P_{\text{idle}} \), and

\[
S_i(x) = \frac{x_i D_i}{X(x) T_s},
\]

(1)
where $X(x) = a + \prod_{k=1}^{n}(1 + x_k) - 1$ with $a = \sigma/T_s$ and $x = [x_1, \ldots, x_n]^T$. Notice that $x_i/X(x)$ is the successful airtime for station $i$, and $D_i/T_s$ the rate. Hence, the total airtime ($T_i$) of station $i$ is given by the airtime spent on successful transmissions and collisions

$$T_i = \frac{x_i}{X(x)} \left(1 + \frac{P_{coll,i}}{1 - P_{coll,i}}\right).$$

(2)

### 3.2 Extension to MU-MIMO

The network model can be generalised to encompass MU-MIMO, where stations can transmit multiple spatial streams simultaneously. Let $F_i$ be the set of flows carried by station $i$, and $F = \cup_{i=1}^{n} F_i$ the set of flows in the WLAN. We let vector $v_{ik}$ describe the $k^{th}$ MU-MIMO transmission pattern on station $i$, where $v_{ik}$ has $|F_i|$ elements, and element $v_{ikf}$ defines the number of spatial streams allocated to flow $f$ in this pattern.

We collect the set of $K_i$ possible transmission patterns for station $i$ together to form matrix $V_i$, where the $k^{th}$ row of $V_i$ describes the $k^{th}$ pattern, $k = 1, \ldots, K_i$. See for example Figure 1. The set of allowable transmission patterns will be determined by the network characteristics, i.e. number of antennas of the stations, channel conditions and protocol constraints. For example, the draft 802.11ac restricts the use of MU-MIMO to the AP and allows at most 8 spatial streams with at most 4 streams for one client station. However, to keep our analysis as general as possible we will not make any assumptions about the structure of matrix $V$.

![Figure 1: Example of MU-MIMO transmission matrix $V_i$ where each row represents a possible MU-MIMO transmission pattern for station $i$.](image)

The number of bits transmitted by each spatial stream is affected by several factors such as noise, fading, external interference and inter-user interference. Because of this, a station may need to use different rates across spatial streams allocated to flows in order to achieve a low loss rate. Let $d_{ikf}$ be the average number of bits transmitted per spatial stream for flow $f$ in transmission pattern $k$ on station $i$. Similar to transmission patterns, we denote as $d_{ik}$ the vector of $K_i$ elements that contains the number of bits transmitted per spatial stream by the $k^{th}$ transmission pattern. Finally, we collect vectors $d_{ik}$ to form matrix $D_i$. 

3
Before transmitting a station selects a transmission pattern. Let \( \pi_{ik} \) denote the fraction of transmission opportunities that pattern \( k \) is selected by station \( i \), with \( \sum_{k=1}^{K_i} \pi_{ik} = 1 \). We collect the \( \pi_{ik} \) for station \( i \) together in vector \( \pi_i \). We can then express the throughput of flow \( f \) on station \( i \) as

\[
s(f) = \frac{x_i}{X(x)} \frac{1}{T_s} \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf},
\]

where \( \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf} \) is the average number of spatial streams used by flow \( f \) in station \( i \) and \( \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf} \) is the average number of bits sent for a flow \( f \) in a successful transmission.

Since spatial streams are transmitted in parallel, a MU-MIMO transmission occupies the same amount of airtime as a single spatial stream and so the total airtime \( T_i \) used by station \( i \) is still given by (2).

4 Proportional Fair Rate Allocation

4.1 Utility Fair Optimisation

The proportional fair rate allocation is the solution to the utility-fair optimisation problem (P):

\[
\begin{align*}
\text{maximise} & \quad \sum_{f \in F} \tilde{s}(f) \\
\text{subject to} & \quad \tilde{s}(f) \leq \log \left( \frac{e^{\tilde{x}_i}}{X(e^x)} \frac{1}{T_s} \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf} \right), \quad f \in F_i \\
& \quad \tilde{s}(f) \leq \tilde{s}(f), \quad f \in F \\
& \quad \sum_{k=1}^{K_i} \pi_{ik} = 1, \quad i = 1, \ldots, n \\
& \quad \pi_{ik} \geq 0, \quad i = 1, \ldots, n
\end{align*}
\]

where \( \tilde{s}(f) = \log s(f) \), \( \tilde{x}_i = \log x_i \). Observe that we have introduced an offered load constraint, where \( \tilde{s}(f) \) is the maximum (log) offered load for flow \( f \). This allows our analysis to encompass both high-rate applications such as data or video, and low-rate applications such as voice. Our reason for formulating the optimisation task in terms of log-transformed variables \( \tilde{s}, \tilde{x} \) is the following:
Lemma 1 (Convexity).

\[-\tilde{x}_i - \log \left( \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf} \right) - \log \frac{1}{T_s} + \log X(e^{\tilde{x}}) \quad (9)\]

is convex in \(\tilde{x}\) and \(\pi\).

**Proof.** Observe that the first term is linear in \(\tilde{x}\) (and so convex), the second term is convex in \(\pi\) due to the convexity of the negative log function when composed with a linear map [6]. The last term is convex in \(\tilde{x}\) by Lemma 1 of [2]. \(\square\)

It follows from Lemma 1 that constraint (5) is convex. Since the objective and remaining constraints are linear in the transformed variables, the optimisation problem is convex and so a proportional fair allocation exists. The optimisation problem satisfies the Slater condition and so strong duality holds. The Lagrangian is

\[
\mathcal{L}(\tilde{x}, \tilde{s}, \pi, \mu, \lambda, \nu, \Theta) = \sum_{f \in F} \tilde{s}(f) + \sum_{i=1}^{n} \sum_{f \in F_i} \mu_f \tilde{s}(f) - \tilde{s}(f) 
+ \sum_{i=1}^{n} \sum_{f \in F_i} \lambda_f \left( \log \frac{e^{\tilde{x}_i} \sum_{k=1}^{K_i} \pi_{ik} v_{ikf} d_{ikf}}{X(e^{\tilde{x}})} \frac{1}{T_s} - \tilde{s}(f) \right) 
+ \sum_{i=1}^{n} \nu_i \left( 1 - \sum_{k=1}^{K_i} \pi_{ik} \right) + \sum_{i=1}^{n} \sum_{k=1}^{K_i} \theta_{ik} \pi_{ik}
\]

where multipliers \(\mu = [\mu_1, \ldots, \mu_{|F|}]^T\), \(\lambda = [\lambda_1, \ldots, \lambda_{|F|}]^T\), \(\Theta = [\theta_1, \ldots, \theta_n]^T\) and \(\nu = [\nu_1, \ldots, \nu_n]^T\) with \(\theta_i = [\theta_{i1}, \ldots, \theta_{i|K_i|}]^T\).

**Theorem 1 (Proportional Fairness).** Either all offered loads can be fully serviced (in which case fairness is irrelevant) or the MU-MIMO proportional fair rate allocation is characterised by: (i) the airtime allocated to station \(i\) is \(T_i = \frac{|F_i| - \sum_{f \in F_i} \mu_f}{|F_i| - \sum_{f \in F} \mu_f}\) where \(|F_i|\) is the number of flows carried by station \(i\) and \(|F|\) the total number of flows in the WLAN, (ii) the station total airtimes sum to unity \(\sum_{i=1}^{n} T_i = 1\), (iii) the allocation of MU-MIMO transmission patterns on station \(j\) satisfies

\[
\sum_{f \in F_j} \frac{(1 - \mu_f) v_{jlf} d_{jlf}}{\sum_{k=1}^{K_j} \pi_{jk} v_{jfk} d_{jfk}} = \nu_j - \theta_{jl} \quad l = 1, \ldots, K_j \quad (10)
\]
Proof. The main KKT conditions are:

\[ 1 - \lambda_f - \mu_f = 0, \]  
\[ \sum_{f \in F_j} \lambda_f - \sum_{i=1}^n \left( \frac{x_j}{X(x)} \prod_{k=1, k \neq j}^K (1 + x_k) \lambda_f \right) = 0, \]  
\[ \sum_{f \in F_j} \lambda_f \frac{v_{jll}d_{jlf}}{\prod_{k=1}^{K_j} \pi_{jk}v_{jkl}d_{jkl}} = \nu_j - \theta_{jl}. \]  

From the first KKT condition (11) we obtain that \( \lambda_f = 1 - \mu_f \). Claim (i): From the second KKT condition (12), rearranging terms we obtain

\[ \frac{|F_j| - \sum_{f \in F_j} \mu_f}{|F| - \sum_{f \in F} \mu_f} = \frac{x_j}{X(x)} \left( 1 + \frac{P_{coll,j}}{1 - P_{coll,j}} \right) =: T_j \]  

provided \( |F| - \sum_{f \in F} \mu_f \neq 0 \). Further, since \( \lambda_f = 1 - \mu_f \geq 0 \) we have \( 0 \leq \mu_f \leq 1 \) and so \( |F| - \sum_{f \in F} \mu_f = 0 \) if and only if \( \mu_f = 1 \) for all \( f \in F \). But by complementary slackness, \( \mu_f = 1 > 0 \) for all \( f \in F \) implies that constraint (6) is tight for all \( f \in F \) i.e. we have a trivial situation where all offered loads can be served and fairness is irrelevant. Claim (ii) that \( \sum_{i=1}^n T_i = 1 \) follows immediately from (14). Claim (iii) follows from the third KKT condition (13).

When the offered load of all flows is unconstrained (\( \hat{s}(f) = \infty \) for all \( f \in F \)), we have

**Corollary 1** (Unconstrained offered load). The MU-MIMO proportional fair rate allocation is characterised by: (i) the airtime allocated to station \( i \) is \( T_i = \frac{|F_i|}{|F|} \) where \( |F_i| \) is the number of flows carried by station \( i \) and \( |F| \) the total number of flows in the WLAN, (ii) the station total airtimes sum to unity \( \sum_{i=1}^n T_i = 1 \), (iii) the allocation of MU-MIMO transmission patterns on station \( j \) satisfies

\[ \sum_{f \in F_j} \frac{v_{jll}d_{jlf}}{\prod_{k=1}^{K_j} \pi_{jk}v_{jkl}d_{jkl}} = \nu_j - \theta_{jl} \quad l = 1, \ldots, K_j \]

Proof. When offered load constraint (6) is loose for all flows, by complementary slackness the multipliers \( \mu_f = 0 \) for all flows.

When all stations carry the same number of flows and the offered load is unconstrained, then it follows from Corollary 1 that the proportional fair rate allocation assigns stations equal total airtime. However, when stations carry different numbers of flows the station airtime may be different. Further, depending on the available MU-MIMO transmission patterns it need not be the case that the flows are allocated equal total airtimes, even for flows sharing the same station – see Section 5 for an example. This is in marked contrast to the situation when MU-MIMO transmissions are not supported,
see [2]. Property (iii), that station airtimes sum to unity, in Theorem 1 and Corollary 1 extends to MU-MIMO WLANs the observation in [7] that this airtime constraint characterises the WLAN rate region boundary.

4.2 MU-MIMO Transmission Patterns

The proportional fair transmission pattern conditions (10) can be expressed in matrix form as

\[
A_j \text{diag}(\lambda_j) (A_j^T \pi_j)^{-*} = \nu_j \mathbf{1} - \theta_j, \quad j = 1, \ldots, n
\]  

(16)

where \(A_j := V_j \circ D_j\), \(x^{-*} := [\frac{1}{x_1}, \ldots, \frac{1}{x_n}]^T\) for vector \(x = [x_1, \ldots, x_n]^T\), \(\mathbf{1}\) denotes the all ones column vector and \(\lambda_j\) is the vector with elements \(\lambda_f = 1 - \mu_f, f \in F_j\).

When the offered load is unconstrained then \(\text{diag}(\lambda_j)\) equals the identity matrix. When \(A_j\) has full column rank \(|F_j|\) (this is commonly satisfied e.g. when the set of possible transmit patterns admits the option to transmit each flow separately in which case \(A_j\) contains the \(|F_j| \times |F_j|\) identity matrix), then we can write \(A_j := \begin{bmatrix} X \\ Y \end{bmatrix}\) where \(X\) is full rank and the rows of \(Y\) are linear combinations of the rows of \(X\). This partitioning can always be achieved simply by ordering the rows of \(A_j\) appropriately. Condition (16) becomes

\[
\begin{bmatrix} X \\ Y \end{bmatrix} (A_j^T \pi_j)^{-*} = \nu_j \mathbf{1} - \theta_j
\]  

(17)

Premultiplying both sides by \([ X^{-1} 0 ]\) and re-arranging,

\[
A_j^T \pi_j = (\begin{bmatrix} X^{-1} & 0 \end{bmatrix} (\nu_j \mathbf{1} - \theta_j))^{-*}
\]  

(18)

Given \(\nu_j\) and \(\theta_j\), vectors \(\pi_j\) satisfying (18) can be found using gaussian elimination. When \(A_j\) is non-singular, then the solution to (18) is unique. However, in general more than one such vector will exist and any such vector is proportional fair.

When the offered load is constrained, the above analysis extends immediately provided all elements of \(\lambda_j\) are non-zero. When one or more elements of \(\lambda_j\) is zero, the analysis becomes slightly more complex. In this case condition (18) imposes \(\text{rank}(A_j \text{diag}(\lambda_j))\) constraints which may be reformulated similarly to (18), but this may be less than the number of flows \(|F_j|\) on station \(j\) even when \(A_j\) is full column rank. However, for elements \(f\) of \(\lambda_j\) which are zero the multiplier \(\mu_f = 1\) since \(\lambda_f = 1 - \mu_f\) from the KKT conditions.

Hence, by complementary slackness offered load constraint (18) is tight for flow \(f\) i.e. we have the additional equality constraints that \(\frac{x^T}{X(\mathbf{x})} \sum_{k=1}^{L_f} \pi_{ik} v_{ikf} d_{ikf} = e^s\) thereby recovering \(|F_j|\) constraints for each station as required.
4.3 Distributed Algorithms

From the above analysis, given multipliers $\mu$, $\nu$ and $\Theta$ we can determine $\pi$ and $x$. The optimal multipliers $\mu$, $\nu$ and $\Theta$ can be found using standard subgradient descent algorithms which are amenable to decentralised implementation. Specifically, the sub-gradients for $\mu$, $\nu$ and $\Theta$ can all be evaluated using local information on a station and so require no message-passing. As discussed in Section 4.2, the MU-MIMO transmit pattern can also be determined using local information. Determining the station transmission attempt probabilities $x$ requires meeting the constraint that the sum of airtimes sums to unity, and so requires knowledge of all station airtimes in the WLAN. However, as discussed in [2], decentralised approximations can be found based on local observations of channel idle time.

5 Examples

5.0.1 Unequal airtime

We begin by presenting a simple example demonstrating that when MU-MIMO is available the proportional fair allocation need not assign equal airtime to flows. Consider a WLAN with an MU-MIMO equipped AP that carries 3 flows ($f_1$, $f_2$, $f_3$) transmitted to three client stations. The offered load is unconstrained. The matrix of available MU-MIMO transmission patterns at the AP is

\[ V = \begin{bmatrix} 1 & 4 & 3 \\ 5 & 1 & 0 \\ 3 & 0 & 3 \end{bmatrix} \]  \hspace{1cm} (19) \]

and $D$ is the all ones matrix, i.e. $A = V$.

Since the AP is the only transmitter, the optimal $P_{coll,AP} = 0$, $\tau_{AP} = 1$, and AP total airtime $T_{AP} = 1$. Solving optimisation problem (P), the proportional fair allocation of MU-MIMO transmission patterns is $\pi = [0.79, 0.21, 0]^T$. The appropriate definition to use for flow airtime is not clear when MU-MIMO is used. One option is the total airtime that would be needed by flow $f$ in order to obtain the same throughput when using a single spatial stream, which is given by $T_i \sum_{k=1}^{K_i} \pi_{ik} v_{ikf}$ and is proportional to the average number of spatial streams allocated to the flow. In the present example, this is 1.85 for flow 1, 3.36 for flow 2 and 2.36 for flow 3. Another option is the fraction of station $i$ airtime $\sum_{k=1}^{K_i} \pi_{ik} v_{ikf} \sum_{f \in F_i} \sum_{k=1}^{K_i} \pi_{ik} v_{ikf}$ used by flow $f$ spatial streams in this example is 0.24, 0.45 and 0.31 for, respectively, flows $f_1$, $f_2$ and $f_3$. A third option is the fraction of station transmission opportunities at which a flow transmits, and in this example we have that flows $f_1$ and $f_2$ are scheduled in every transmission, while flow $f_3$ is only scheduled in 79% of transmissions. Observe that none of these flow airtimes are equal at the proportional fair allocation. This is perhaps unsurprising since it is the station total airtime that corresponds to the shared network resource being consumed.
and so to the “cost” of transmissions. Indeed this is reflected in Theorem 1. Since flow transmissions occur in parallel, they share the same station airtime. For a given station airtime, the proportional fair allocation of spatial streams maximises the sum of log flow rates, and this need not correspond to allocating the same number of spatial streams or the same fraction of transmission opportunities to flows.

5.0.2 IEEE 802.11ac

Consider a WLAN consisting of an IEEE 802.11ac AP transmitting 2 flows to two client stations. Flow $f_1$ is a file transfer with unconstrained offered load and flows $f_2$ carries video traffic with an offered load of 1.5 Mbps. Client 1 and 2 transmit flows $f_3$ and $f_4$ respectively with unconstrained offered load. The channel bandwidth is 20 MHz, guard interval is 800 ns and the MCS at the AP are BPSK 1/2 (6.5 Mbps) and QPSK 1/2 (13 Mbps) depending on the transmission pattern used. Both clients transmit a single spatial stream with BPSK 1/2. The PHY and MAC header overheads are omitted for simplicity. The matrix of possible MU-MIMO transmission patterns at the AP is

$$\mathbf{V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = 10^6 \cdot \begin{bmatrix} 13 & 0 \\ 0 & 13 \\ 6.5 & 6.5 \\ 6.5 & 6.5 \end{bmatrix}. \tag{20}$$

Solving the optimisation problem (P) we obtain that the airtimes of the AP and the clients are 0.355 and 0.323 respectively, which are provided by $\tau_{AP} = 0.157$ and $\tau_{client} = 0.143$. The proportional fair allocation of MU-MIMO transmission patterns at the AP is $\mathbf{\pi} = [0, 0, 0.88, 0.12]^T$, and the average number of spatial streams allocated to the AP flows is $\mathbf{\pi}^T \mathbf{V} = [2.12, 0.88]$. The average throughput of flows $f_1$ and $f_2$ at the AP are given by $\mathbf{\pi}^T (\mathbf{V} \circ \mathbf{D})$ by the AP successful airtime, which are 3.59 and 1.5 Mbps respectively. The throughput of flows $f_3$ and $f_4$ is 1.52 Mbps.

6 Conclusions

We consider the proportional fair rate allocation in an 802.11 WLAN that supports multi-user MIMO (MU-MIMO) transmission by one or more stations. We characterise, for the first time, the proportional fair allocation of MU-MIMO spatial streams and station transmission opportunities. While a number of features carry over from the case without MU-MIMO, in general neither flows nor stations need to be allocated equal airtime when MU-MIMO is available.
References

[1] Kelly F.P., Maulloo A.K., and Tan D.K.H. Rate control for communication networks: shadow prices, proportional fairness and stability. *Journal of the Operational Research Society*, 49(3):237–252, 1998.

[2] A. Checco and D.J. Leith. Proportional fairness in 802.11 wireless lans. *Communications Letters, IEEE*, 15(8):807 –809, august 2011.

[3] L.X. Cai, H. Shan, W. Zhuang, X. Shen, J.W. Mark, and Z. Wang. A distributed multi-user mimo mac protocol for wireless local area networks. In *Global Telecommunications Conference, 2008. IEEE GLOBECOM 2008. IEEE*, pages 1–5. IEEE, 2008.

[4] M.X. Gong, E. Perahia, R. Stacey, R. Want, and Shiwen Mao. A csma/ca mac protocol for multi-user mimo wireless lans. In *Global Telecommunications Conference (GLOBECOM 2010), 2010 IEEE*, pages 1 –6, dec. 2010.

[5] B. Bellalta, J. Barcelo, D. Staehle, A. Vinel, and M. Oliver. On the performance of packet aggregation in ieee 802.11ac mu-mimo wlans. *Communications Letters, IEEE*, 16(10):1588 –1591, october 2012.

[6] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge university press, 2004.

[7] Vijay G. Subramanian and Douglas J. Leith. Convexity conditions for 802.11 wlans. *CoRR*, abs/1206.3120, 2012.