Lower and upper bounds on the energy of braided magnetic fields

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Abstract. Let $B$ be a class of topologically equivalent regular magnetic fields occupying a cylindrical domain of axis $Oz$ and produced by ideal MHD deformations of the uniform field $B_0\hat{z}$, and let $B_{\text{ff}} \subset B$ be constituted of the fields of $B$ which are force-free. We derive topological lower and upper bounds on the energy of the fields in $B$ and in $B_{\text{ff}}$, respectively. We also establish new formulae for the field line helicity and the relative magnetic helicity of the fields in $B$, and give a sufficient criterion of ideal MHD linear stability for the fields in $B_{\text{ff}}$.

1. Introduction
In the simple model of solar coronal loop proposed by Parker (Parker (1994), and references therein), the magnetic field $B$ is taken to occupy a cylindrical domain $D$ of axis $Oz$, filled up with a low beta perfectly conducting plasma, and to satisfy at any given time two conditions: it is obtained from the uniform vertical field $B_0\hat{z}$ by an ideal MHD deformation produced by continuous footpoints displacements on the lower and upper parts of the boundary of $D$, and it is in a force-free equilibrium state. One is thus immediately confronted with a nonstandard problem: is it really possible to find a force-free configuration having an arbitrarily prescribed topology? Parker argued that such a configuration does exist, but that it is singular when it is forced to be fully 3D by the braiding imposed to the lines: in that case, it needs to contain infinitely thin current sheets, i.e., surfaces across which $B$ is discontinuous. If a small resistivity is accounted for, current sheets are replaced by thin current layers in which intense magnetic energy dissipation eventually occurs. This results into plasma heating and Parker suggested that the corona maintains its high temperature by this mechanism. Actually, Parker’s conclusion about the necessary appearance of current sheets in the ideal situation has been criticized by many authors, and strong arguments have been advanced in favor of the existence of smooth topologically constrained 3D equilibria at least when the field deformation is not too large. An alternative scenario for generating current sheets has also been proposed by Ng & Bhattacharjee (1998). These authors argue that a progressively deformed smooth equilibrium reaches at some stage an unstable configuration from which it needs to relax to a necessarily singular lower energy state. The debate is still going on, and the reader may get a flavor of it by looking at some recent references (e.g., Aly & Amari (2010); Huang et al. (2010); Janse et al. (2010)).

The aim of this Communication is to present some new general properties of Parker’s model. We consider the set of all the regular magnetic fields which can be obtained by a continuous ideal deformation of $B_0$. We divide it into classes $B$ of topologically equivalent fields and take the point of view – opposite to Parker’s conclusion – that each of them contains at least one
2. Assumptions

Hereafter, \((x, y, z)\) are Cartesian coordinates and \(D\) is the cylinder \(S_0 \times ]0, h[ \subset \mathbb{R}^3\) of base \(S_0 \subset \{(x, y, 0)\}\), axis \(Oz\), and height \(h\). The boundary \(\partial D\) of \(D\) is constituted of \(S_0\), \(S_h = S_0 + h \hat{z}\), and a vertical surface \(S\). We denote as \(\hat{n}\) the exterior unit normal to \(\partial D\) and as \(|Y|\) the “natural” measure of the set \(Y\) (e.g., \(|D|\) is the volume of \(D\)). If \(X\) is a vector field, we set \(X = X_\perp + X_\parallel \hat{z}\) (\(r = r + z \hat{z}\) for the position vector), and \(\nabla X = \sum \partial_i X_i \hat{e}_i \otimes \hat{e}_j\) \((i, j = x, y, z)\).

We take \(D\) to contain a regular magnetic field \(B\) embedded in a low beta perfectly conducting plasma. \(B\) is assumed to have been produced by a continuous deformation, constrained by the frozen-in law, of the uniform field \(B_0 = B_0 \hat{z}\), with \(B_0 > 0\) a constant. This deformation transports an element of plasma from its initial position \(r_0 \in D\) (resp., \(S_0, S_h, S\)) to the new position \(r(0) = (r, z)(r_0) \in D\) (resp., \(S_0, S_h, S\)). The mapping \(r\) is one-to-one and orientation preserving, whence \(J = \det \mathbf{F} > 0\), where \(\mathbf{F} = \nabla_0 r\) and \(\nabla_0 r\) is the gradient with respect to \(r_0\).

The horizontal positions of the footpoints of \(B_0\) on \(S_0\) and \(S_h\) are thus transported from \(r_0\) to \(\mathbf{R}(r_0, 0)\) and \(\mathbf{R}(r_0, h)\), respectively, while the magnetic lines on \(S\) acquire their new positions by slipping tangentially to that surface. As it is well known, \(B\) is explicitly given by

\[
B[r(0)] = \frac{B_0 \cdot F(r_0)}{J(r_0)} = \frac{B_0}{J(r_0)} \frac{\partial r}{\partial z_0}(r_0). \tag{1}
\]

We say that two fields \(B\) and \(B'\) produced by the mappings \(r\) and \(r'\), respectively, are topologically equivalent if they can be continuously transformed into each other by an ideal MHD deformation keeping fixed the positions of the footpoints. This implies in particular that \(\mathbf{R}(r_0) = \mathbf{R}'(r_0)\) for \(r_0 \in S_0 \cup S_h\). We denote as \(B\) a class of equivalent fields and as \(B_{ff}\) the subset of \(B\) containing the fields that are force-free and then satisfy

\[
\nabla \times B = \alpha B \tag{2}
\]

for some function \(\alpha\). As \(\nabla \cdot B = 0\), \(B \cdot \nabla \alpha = 0\) and \(\alpha\) is constant along any magnetic field line. Our assumption here is that \(B_{ff}\) is nonempty – at least for a large class of \(B\). A field \(B\) belonging to \(B_{ff}\) can be possibly constructed as the final state of one of the two following processes: (i) the first one is an MHD viscous evolution with zero resistivity in which the footpoints are fixed on \(S_0\) and \(S_h\). The initial state is selected in \(B\) and then the field stays in that set at all subsequent times (Moffatt, 1985). (ii) The second one is a quasi-static evolution during which the field passes through a sequence of force-free configurations, with the initial field being the uniform \(B_0\). This evolution is driven by imposing the footpoints on \(S_0\) and \(S_h\) to move horizontally at the adequately prescribed velocity \(v(r, 0, t)\) and \(v(r, h, t)\), respectively.

One of our essential goal hereafter is to establish lower bounds on the energy \(W\) of the fields of \(B\), and upper bounds on that of the fields in \(B_{ff}\). \(W\) is given by the various expressions

\[
W = \int_D \frac{B^2}{8\pi} dv = W_0 + w = W_0 + w_\perp + w_z = \int_D \omega_0 dv_0, \tag{3}
\]

where we have set \(B = B_0 \hat{z} + b = (B_0 + b_z) \hat{z} + b_\perp\) and

\[
W_0 = \frac{B_0^2 |D|}{8\pi}, \quad w_k = \int_D \frac{b_k^2}{8\pi} dv \quad \text{for} \quad k = \perp, z, \quad \text{and} \quad \omega_0 = \frac{B_0^2}{8\pi J} |\frac{\partial F}{\partial z_0}|^2, \tag{4}
\]

and used that \(\int_{S_z} b_z ds = 0\) by flux conservation (\(S_z\) is the cross section of \(D\) at height \(z\)).
3. Some considerations on the topology of the fields in \( \mathcal{B} \)

3.1. Basic notions and characterization of the topology

We associate to the arbitrary field \( \mathbf{B} \in \mathcal{B} \) four functions which play an important role hereafter:

- **Boundary flux distribution.** It is just the vertical component \( B_z \) of \( \mathbf{B} \) on \( S_0 \) and \( S_h \).
- **Magnetic mapping.** It is the one-to-one mapping \( \mathbf{M} : S_0 \rightarrow S_h \) which associates to the lower endpoint \( \mathbf{r} = \mathbf{R} \) of the line \( \mathcal{L} = \mathcal{L}(\mathbf{R}) \) on \( S_0 \) its upper endpoint \( \mathbf{M}(\mathbf{r}) = \mathbf{R}^h + h \mathbf{h} \) on \( S_h \).
- **Winding number \( n(\mathbf{R}) \) of the line \( \mathcal{L}(\mathbf{R}) \) drawn on \( S \).** It is the algebraic number of turns,

\[
n(\mathbf{R}) = \frac{1}{|\partial S_0|} \int_{\mathcal{L}(\mathbf{R})} dq = \frac{1}{|\partial S_0|} \int_{\mathcal{L}(\mathbf{R})} \frac{B_q}{B} dl \in \mathbb{R},
\]

made by \( \mathcal{L}(\mathbf{R}) \) around \( S \) \((q, z)\) are orthonormal coordinates on \( S \), with \( \nabla_s q = \mathbf{z} \times \mathbf{n} \).
- **Relative twist \( \phi(\mathbf{R}, \mathbf{R}') \) of the pair of magnetic lines \( (\mathcal{L}(\mathbf{R}), \mathcal{L}(\mathbf{R}')) \).** When all the lines of \( \mathbf{B} \) are pointing upwards at all the points of \( D (B_z > 0 \text{ in } D) \), \( \phi(\mathbf{R}, \mathbf{R}') \) is defined to be the total angle by which the horizontal vector connecting \( \mathcal{L}(\mathbf{R}) \) to \( \mathcal{L}(\mathbf{R}') \) at height \( z \) rotates when moving vertically from \( z = 0 \) to \( z = h \). We refer to Berger & Prior (2006) for the general definition of \( \phi \), which accounts for the possibility of having \( B_z < 0 \) in some part of \( D \).

These functions can be checked to not depend on the particular field \( \mathbf{B} \) which is considered in \( \mathcal{B} \), and they can be taken to be properties of the class \( \mathcal{B} \) itself.

Finally, we claim that the topology of the fields in \( \mathcal{B} \) can be fully characterized by the magnetic mapping, \( \mathbf{M} \), and the winding number, \( n_1 = n(\mathbf{R}_1) \), of a particular line \( \mathcal{L}(\mathbf{R}_1) \) drawn on \( S \). Roughly speaking, we can chose for \( \mathcal{L}(\mathbf{R}_1) \) a simple curve on \( S \) starting at \( \mathbf{R}_1 \in \partial S_0 \) and ending on \( \partial S_h \) after making \( n_1 \) turns around, and then use continuity for connecting together in a topologically unique way the points \( \mathbf{R} \) and \( \mathbf{M}(\mathbf{R}) \) by pairwise disjoint simple curves.

3.2. Relative magnetic helicity

We now define the **relative magnetic helicity**, \( H \), of a field \( \mathbf{B} \in \mathcal{B} \) (Berger & Field, 1984). This important quantity is known to stay invariant when a field is submitted to an ideal MHD deformation keeping fixed the footpoints and it thus takes the same value for all the fields in \( \mathcal{B} \).

Let us introduce the unique potential field \( \mathbf{B}_\pi \) \((\nabla \times \mathbf{B}_\pi = 0)\) with the same boundary flux distribution as the fields in \( \mathcal{B} \) \((\mathbf{B}_\pi \notin \mathcal{B} \text{ in general})\), and its unique potential vector \( \mathbf{C}_\pi \) \((\mathbf{B}_\pi = \nabla \times \mathbf{C}_\pi)\) satisfying \( \nabla \cdot \mathbf{C}_\pi = 0 \text{ in } D \) and \( \mathbf{h} \cdot \mathbf{C}_\pi = 0 \text{ on } \partial D \). Next choose a vector potential \( \mathbf{C} \) for \( \mathbf{B} \) \((\mathbf{B} = \nabla \times \mathbf{C})\) such that \( \mathbf{h} \times (\mathbf{C} - \mathbf{C}_\pi) = 0 \text{ on } \partial D \) (this defines \( \mathbf{C} \) up to an additive \( \nabla f \), with \( f = \text{const. on } \partial D \)). Then \( H \) is given by (Berger, 1988)

\[
H = \int_D \mathbf{C} \cdot dB = \int_{S_0} \lambda B_z ds, \quad \text{where } \lambda(\mathbf{R}) = \int_{\mathcal{L}(\mathbf{R})} \mathbf{C} \cdot dl.
\]

The function \( \lambda \) (which is independent of the gauge chosen for \( \mathbf{C} \)) is called either field line helicity (Berger, 1988) or topological flux function (Yeates & Hornig, 2013). It can be given an alternative expression by proceeding as follows. We first note that \( \mathbf{B} \) and then also \( \mathbf{B}_\pi \) have equal fluxes through, respectively, the closed oriented curve \( \mathcal{C}_0 \subset S_0 \) and its image by the magnetic mapping, \( \mathcal{C}_h = \mathbf{M}(\mathcal{C}_0) \subset S_h \). This implies \( \int_{\mathcal{C}_0} \mathbf{C}_\pi \cdot dl = \int_{\mathcal{C}_h} \mathbf{C}_\pi \cdot dl \) by Stokes theorem, whence (by setting \( F^h(\mathbf{R}) = F(\mathbf{R}^h(\mathbf{R}), h) \))

\[
\forall \mathcal{C}_0 : \int_{\mathcal{C}_0} dl \cdot (\mathbf{C}_\pi \cdot \nabla \perp \mathbf{R}^h \cdot \mathbf{C}_\pi^h) = 0 \Rightarrow \exists \psi : \mathbf{C}_\pi^h - \nabla \perp \mathbf{R}^h \cdot \mathbf{C}_\pi^h = -\nabla \perp \psi.
\]

Next we choose an open curve \( \mathcal{C}_0 \subset S_0 \) connecting \( \mathbf{R}_1 \) to \( \mathbf{R} \), and note that the flux of \( \mathbf{B} \) through the contour \( \mathcal{C}_1 = \mathcal{C}_0 \cup \mathcal{L}(\mathbf{R}) \cup -\mathcal{C}_h \cup -\mathcal{L}(\mathbf{R}_1) \) \((\mathcal{C}_h = \mathbf{M}(\mathcal{C}_0) \subset S_h)\) vanishes. Hence
\( \lambda(R_1) - \psi(R_1) = \lambda(R) - \psi(R) \) by Stokes theorem and the previous result. As \( \psi \) is defined up to an additive constant, we can impose \( \psi(R_1) = \lambda(R_1) \) at some specific point \( R_1 \) and we get \( \lambda(R) = \psi(R) \) for all \( R \in S_0 \). Choosing \( R_1 \in \partial S_0 \), we obtain similarly

\[
\lambda(R_1) = \psi(R_1) = \int_0^h C_{\pi}(R_1, z) \, dz + \int_{C_h(R_1)} C_{\pi q} \, dq \quad (8)
\]

by considering the contour \( C_2 = C_\psi(R_1) \cup C^h(R_1) \cup -C(R_1) \), with \( C_\psi(R_1) \) the vertical segment on \( S \) above \( R_1 \) and \( C^h(R_1) \) the curve connecting \( (R_1, h) \) to \( (R^h(R_1), h) \) after running \( n(R_1) \) times along \( \partial S_h \). Whence

\[
\lambda(R) = \psi(R) = \int_0^h C_{\pi}(R_1, z) \, dz + \int_{C_h(R_1)} C_{\pi q} \, dq - \int_{C(R_1, R)} dl \cdot (C_{\pi \perp} - \nabla \perp R^h \cdot C^h \pi \perp), \quad (9)
\]

where \( C(R_1, R) \) connects \( R_1 \) to \( R \). This shows that all the fields \( B \in B \) have the same line helicity \( \lambda \). Some other properties of \( \lambda \) (with \( \lambda \) possibly computed from a potential vector \( A \not= C \)) have been discussed by Yeates & Hornig (2013) (see also Yeates & Hornig, these Proceedings).

Using Eq. (9) in Eq. (6) leads to a new formula in which \( H \) is directly expressed in terms of the field topology. We thus recover the fact that \( H \) has the same value for all \( B \in B \). The helicity formula can be transformed by integrating by parts, and one gets for instance

\[
H = \int_{S_0} [C_{\pi \perp} \times (\nabla \perp R^h \cdot C_{\pi \perp}^h)] \cdot \hat{z} \, ds + n \Phi_0^2 \quad (10)
\]

in the simple case where \( B_{z}(R, 0) = B_{z}(R, h) \) and all the lines on \( S \) have the same winding number \( n \in \mathbb{Z} \) (\( \Phi_0 = B_0|S_0| \) is the magnetic flux through \( S_0 \)).

### 4. Some properties of the fields in \( B_{ff} \)

#### 4.1. Some remarks on the force-free equations

Eq. (2) can be given the alternative form

\[
\nabla \cdot T = 0, \quad \text{where} \quad T = -(B^2/8\pi)\delta + B \otimes B / 4\pi \quad (11)
\]

(\( \delta \) denotes the unit tensor, i.e., \( (\delta)_{ij} = \delta_{ij} \) is Kronecker symbol). By using Eq. (1) and the standard relation \( \nabla_0 \cdot \text{Cof F} = 0 \) satisfied by the cofactor of \( F \), one may also write

\[
\nabla_0 \cdot T_0 = 0, \quad \text{where} \quad T_0 = \text{Cof F} \cdot T = -\frac{B_0^2}{8\pi J^2} \left| \frac{\partial \mathbf{r}}{\partial s_0} \right|^2 \text{Cof F} + \frac{B_0^2}{4\pi J^2} \hat{z} \otimes \frac{\partial \mathbf{r}}{\partial s_0}. \quad (12)
\]

Let us make three important points about these equations:

- Let us use the equation of a magnetic line \( \mathcal{L} \) of \( B \), \( dr / dl = B / B \), in Eq. (11) (\( l \) is an arclength along \( \mathcal{L} \)). We thus obtain

\[
\frac{d}{dl} (B \frac{dr}{dl}) = \nabla B, \quad (13)
\]

which is identical to the equation of propagation of a light ray in a medium with refractive index \( n = B \). Hence the magnetic lines of \( B \) also obey a Fermat principle: with \( B \) given, the line \( \mathcal{L}(R) \) makes stationary the functional \( \int_{\mathcal{L}(R)} B \, dl \) defined on the set of curves \( \mathcal{C}(R) \) connecting \( R \) and \( M(R) \). Alternatively, one can say that \( \mathcal{L}(R) \) is a geodesic line of the conformal metric \( B^2 \, dl^2 \). The ray equation above was previously written by Parker (1994) for the potential field \( B_\perp = \nabla \psi_\perp \). In that case, the lines are orthogonal to the equipotential surfaces \( \{ \psi_\perp = \tau \} \) as the light rays are orthogonal to the wave fronts. But no family of normal surfaces does exist for the lines of \( B \in B_{ff} \), as such a field has nonzero abnormality \( B \cdot \nabla \times B / B^2 = \alpha \).
The term \( B \) is formally similar to the equation of equilibrium determining the deformation of an elastic body with respect to a configuration of reference, with \( T_0 = T_0(F) \) corresponding to the first Piola-Kirchhoff stress tensor. The analogy is even with the equilibrium of an hyperelastic body, as \( T_0 \) is related to the stored energy density \( \omega \) (see Eq. (4)) by \( (T_0)_{ij} = \partial \omega / \partial (\partial x_j / \partial x_i) \) (see, e.g., Ciarlet (1988)). This remark opens the possibility of transferring some results from the well-developed theory of elasticity to the theory of topologically constrained MHD equilibria. Such equilibria have, however, some unusual properties from the point of view of elasticity. In particular, there are finite displacements that do not change the energy (those for which each plasma element is moved along a line of \( B \); this degeneracy disappears when plasma pressure is taken into account).

The structure of the force-free fields in \( B_{ff} \) is often studied in the reduced MHD approximation, in which one has \( B_z = B_0 \) and \( \nabla \times b_\perp = 0 \) in \( D \) while Eq. (11) reduces to \( B_0 \partial_2 b_\perp + b_\perp \cdot \nabla \times b_\perp + \nabla f = 0 \), for some function \( f \) (van Ballegooijen, 1985; Parker, 1994; Ng & Bhattacharjee, 1998). Under the substitutions \( z/B_0 \rightarrow t, b_\perp \rightarrow v, \) and \( f \rightarrow p/\rho, \) these equations become identical to the Euler equations for a 2D incompressible fluid of velocity \( v \), pressure \( p \), and uniform mass density \( \rho \). Those are usually solved in \( S_0 \), say, under two different types of boundary conditions. Either one seeks for a solution having a prescribed vorticity \( \omega = \hat{z} \cdot \nabla \times v \) (or velocity \( v \)) at \( t = 0 \) (initial value problem, most often considered). Or one works in the Lagrangian framework, in which the unknown is the mapping \( \mathbf{R}(\mathbf{R}_0) \) giving the position at time \( t \) of the fluid element located at \( \mathbf{R}_0 \) at \( t = 0 \), and one seeks for a solution with a prescribed \( \mathbf{R}(\mathbf{R}) \) at some \( T \). Such a solution \( \mathbf{R}(\mathbf{R}) \) can be interpreted as a geodesic line in the (infinite dimensional) set of area preserving diffeomorphisms of \( S_0 \) (Arnold & Khesin, 1998). When reformulated in our MHD context, these problems amount to determine a field \( \mathbf{B} \) having, respectively, a given \( \alpha \) on \( S_0 \) and a prescribed magnetic mapping. The second problem coincides with ours here (but for the imposition of the winding number \( n_1 \)), and we can benefit of the many results obtained so far in the hydrodynamical framework. For instance, we may state the existence of a unique regular field minimizing the energy as long as the imposed deformation is not too large.

### 4.2. A consequence of current conservation

For a force-free field \( \mathbf{B} \) in \( D \), the total vertical electric current \( I(z) \) through \( S_z \) is independent of \( z \) (there is no current flowing through \( S \)), and we obtain with help from Ampère’s law

\[
\int_{\partial S_z} b_q \, dq = \frac{4\pi}{c} I(0). \tag{14}
\]

On the other hand, the total currents flowing through, respectively, the arbitrary closed contour \( C_0 \subset S_0 \) and its magnetic image \( C_h = M(C_0) \subset S_h \) are equal, while no current flows through the contours \( C_1 \) and \( C_2 \) defined in Sect. 3.2. Proceeding as therein (with \( \mathbf{B} \) substituted for \( \mathbf{C} \)), we thus obtain

\[
\exists \chi(\mathbf{R}) : \mathbf{b}_\perp(\mathbf{R}_0, 0) - (\nabla \times \mathbf{R}_0^h \cdot \mathbf{b}_\perp)^h(\mathbf{R}) = -\nabla \times \chi(\mathbf{R}), \tag{15}
\]

\[
\int_{\mathcal{L}(\mathbf{R})} B \, dl = \chi(\mathbf{R}) + B_0 h, \tag{16}
\]

\[
\chi(\mathbf{R}) = \int_0^h b_z(\mathbf{R}, z) \, dz + \int_{\mathcal{C}^h(\mathbf{R})} b_q \, dq \quad \text{for} \quad \mathbf{R} \in \partial S_0. \tag{17}
\]

The term \( B_0 h \) in Eq. (16) just fixes the gauge of \( \chi \).

Using Eq. (16) in the expression \( W = \int_{S_0} (\int_{\mathcal{L}} B \, dl) B_z \, ds/8\pi \) of the energy leads to

\[
w = \frac{1}{8\pi} \int_{S_0} \chi B_z \, ds. \tag{18}
\]
4.3. Global equilibrium and virial theorem
Integrating the \( z \)-component of Eq. (11) between \( S_0 \) and \( S_z \) and using Gauss theorem leads to
\[
\int_{S_z} (B^2_\perp - B^2_z) \, ds = \int_{S_0} (B^2_\perp - B^2_z) \, ds,
\]
from which we get immediately
\[
w_\perp - w_z = \frac{\hbar}{8\pi} \int_{S_0} (b^2_\perp - b^2_z) \, ds = \frac{\hbar}{8\pi} \int_{S_h} (b^2_\perp - b^2_z) \, ds.
\]

Similarly, multiplying Eq. (11) scalarly by \( r \), integrating the result by parts in \( D \), and using once more Gauss theorem, we obtain the well-known scalar virial relation
\[
W = \int_{\partial D} \left( \frac{B^2}{8\pi} \hat{n} \cdot r - \left( \hat{n} \cdot B \right) \left( B \cdot r \right) \right) \, ds.
\]

Another interesting virial-type relation (analogous to the one derived by Green (1973) for homogeneous hyperelastic bodies) can be established by starting from Eq. (12), multiplying it scalarly by \( r_0 \cdot F - r \), and integrating the result with respect to \( r_0 \) over \( D \). By using in particular the relation between \( T_0 \) and \( \omega_0 \) quoted above, one gets after some algebra
\[
3W = \int_{\partial D} \hat{n}_0 \cdot \left[ T_0 \cdot (r - r_0 \cdot F) + \omega_0 r_0 \right] \, ds_0.
\]

4.4. A new energy formula
Using the decomposition \( B = B_0 + b \) and Eq. (20) in Eq. (21) and combining the resulting expression with Eqs. (17), (18), and (15), one gets after some algebra the new energy formula containing information of topological nature
\[
3w + w_\perp - w_z = - \int_{S_0} b^2_\perp \frac{R_h}{8\pi} \, ds + \frac{B_0}{4\pi} \int_{\partial S_h} R \cdot \hat{n} \left( \int_{\partial h(R)} b_0 \, dq' \right) \, dq
\]
\[
+ \int_{S_0} \left( R_h - R \cdot \nabla_\perp R_h \right) \cdot \frac{b^2_\perp B_z}{4\pi} \, ds + \int_{S_0} \left( 2\chi + R \cdot \nabla_\chi \right) b^2_\perp \, ds.
\]

It is also possible to arrive at this relation, a special case of which was previously derived in Aly (2005), by starting from Eq. (22).

5. Lower bounds on the energy of the fields in \( B \)
5.1. A nontopological lower bound
As it is well known, the energy \( W \) of an arbitrary field \( B \in B \) is always larger than the energy \( W_\pi \) of the potential field \( B_\pi \) introduced in Sect. 3.2, whence \( W_\pi - W_0 = w_\pi \leq w \). Of course, one has \( B_\pi = B_0 \) and \( w_\pi = 0 \) when the flux distribution is uniform on \( S_0 \) and \( S_h \), i.e., \( B_z \mid_{S_0,S_h} \equiv B_0 \).

When \( B_z \mid_{S_0,S_h} \neq B_0 \), the lower bounds given below are relevant only when they exceed \( w_\pi \).

5.2. A first topological lower bound
The magnetic energy \( w \) admits the lower bound
\[
\frac{w_\pi}{W_0} = \left[ \int_{S_0} \left( 1 + \frac{\xi^2}{\hbar} \right)^{1/2} \frac{d s_0}{|S_0|} \right]^2 - 1 \leq \frac{w}{W_0},
\]
where \( \xi(\mathbf{R}_0) = \mathbf{R}(\mathbf{R}_0, h) - \mathbf{R}(\mathbf{R}_0, 0) \) represents the horizontal separation between the two footpoints of the line of \( \mathbf{B} \) emerging at \( \mathbf{R}_0 \) in the initial uniform configuration. Eq. (24) is obtained by writing \( \xi(\mathbf{R}_0) + h\mathbf{\hat{z}} \) as the integral of \( \partial \mathbf{\Pi}(\mathbf{R}_0, z_0)/\partial z_0 \) with respect to \( z_0 \), applying Schwartz inequality to that integral, injecting the result into the last expression of \( W \) in Eq. (3), applying once more Schwartz inequality, and using the relation \( \int_D J \, d\mathbf{n}_0 = |D| \).

This lower bound is interesting when the deformation of the field is sufficiently small. But it certainly strongly underestimates \( w \) when \( \mathbf{B} \) is much deformed. For instance, \( w_1 \) keeps the same value if we add to the field an overall twist of \( 2\pi \mathbf{n} \), with \( |\mathbf{n}| \in \mathbb{N} \) arbitrarily large.

### 5.3. A second topological lower bound

Our second lower bound does not suffer the previous limitation but it is derived under the particular assumption that \( B_z = B_0 + b_z \geq B_1 > 0 \) in \( D \). This bound is a generalization of the one derived by Berger (1993) under the much more severe restriction \( B_2 \equiv B_0 \) in \( D \). Under our assumption the line \( \mathcal{L}(\mathbf{R}) \) can be represented by a function \( \mathbf{R}^z(\mathbf{R}) \), while the relative twist of the two lines \( \mathcal{L}(\mathbf{R}) \) and \( \mathcal{L}(\mathbf{R}') \) (see Sect. 3.1) can be expressed as

\[
\phi(\mathbf{R}, \mathbf{R}') = \int_0^h \frac{\partial \varphi^z}{\partial z} (\mathbf{R}, \mathbf{R}') \, dz. \tag{25}
\]

\( \varphi^z(\mathbf{R}, \mathbf{R}') \) is the angle of \( \mathbf{X}^z(\mathbf{R}, \mathbf{R}') = \mathbf{R}^z(\mathbf{R}') - \mathbf{R}^z(\mathbf{R}) \) and \( \mathbf{\hat{x}} \), and one has with \( \mathbf{\hat{u}} = \mathbf{\hat{z}} \times \mathbf{X}^z/|\mathbf{X}^z| \)

\[
\frac{\partial \varphi^z}{\partial z}(\mathbf{R}, \mathbf{R}') = \left[ \left( \frac{\mathbf{b}_\perp}{|\mathbf{b}_\perp|} \right) (\mathbf{R}^z(\mathbf{R}'), z) - \left( \frac{\mathbf{b}_\perp}{|\mathbf{b}_\perp|} \right) (\mathbf{R}^z(\mathbf{R}), z) \right] \cdot \left( \frac{\mathbf{\hat{u}}}{|\mathbf{\hat{u}}|} \right) (\mathbf{R}, \mathbf{R}') . \tag{26}
\]

To derive a lower bound on \( w \), we start from the quantity

\[
w^* = \frac{1}{16\pi R_0} \int_{S_0 \times S_0} |\phi(\mathbf{R}, \mathbf{R}')| B_z(\mathbf{R}, 0) B_z(\mathbf{R}', 0) \, ds \, ds' \tag{27}
\]

which takes the same value for all the \( \mathbf{B} \in \mathcal{B} \) \((2R_0 \) is the diameter of \( S_0 \)). Using Eqs. (25)-(26) in Eq. (27) and applying the triangle inequality leads after some simple algebra to

\[
8\pi R_0 w^* \leq \int_D \int_{S_z} \left( \int_{S_z} \frac{\mathbf{\hat{u}} \cdot \mathbf{\hat{b}}_\perp}{X} \, ds' \right) \, dv + \int_D \int_{S_z} \frac{\mathbf{\hat{b}}'_\perp}{X} \, ds' \, dv, \tag{28}
\]

where \( X = |\mathbf{R}' - \mathbf{R}| \) and \( \mathbf{\hat{b}}'_\perp = b_\perp(\mathbf{R}', z) \). The first integral in the right-hand side is similar to the one appearing in Berger (1993) and we proceed essentially as therein to bound it from above.

To bound the second integral, we use Schwartz inequality and the easily obtained inequality \( \int_{S_0} \int_{S_0} f(\mathbf{R}')/|\mathbf{R}' - \mathbf{R}| \, ds' \, ds \leq (\nu R_0)^2 \int_{S_0} (f(\mathbf{R})/|\mathbf{R}' - \mathbf{R}|) \, ds' \, ds' \) valid for any function \( f \) \((\nu = \nu(S_0) \leq 2\pi \) is a geometric constant). We thus get eventually

\[
\frac{w_2}{W_0} = \min_{s \geq 0} \left[ s + \frac{(w^*/W_0)^2}{(\mu + \sqrt{\nu^2})^2} \right] \leq \frac{w}{W_0}, \tag{29}
\]

where \( \mu = \mu(S_0) \) is another geometric constant. When \( S_0 \) is a disk, \( \nu = 2\pi \) and \( \pi(\mu R_0)^2 = \int_{S_0} m^2 \, ds \), with \( m = m(\mathbf{R}) = 2[\arcsin t/t + (1 - t^2)^{1/2}] \), \( t = r/R_0 \).

\( w_2/B_0 \) can be replaced by the slightly less accurate but often more convenient lower bound

\[
\frac{w_3}{W_0} = \max \left\{ \left( \frac{w^*}{\mu W_0} \right)^2 \left[ 1 + \left( \frac{2\nu w^*}{\mu^2 W_0} \right)^2 \right]^{1/2} + 1 \right\}^{-1} \sqrt{\frac{2w^*}{\nu W_0} - \frac{\mu^2}{\nu^2}} \leq \frac{w}{W_0}. \tag{30}
\]

These bounds are still valid if we substitute the crossing number \( \mathcal{C}(\mathbf{R}, \mathbf{R}') = \int_0^h |\partial z \varphi^z| \, dz \) of a pair of lines for \( |\phi(\mathbf{R}, \mathbf{R}')| \) in the definition of \( w^* \). The formulae obtained that way naturally reduce to Berger’s when \( B_z \equiv B_0 \) on \( S_0 \) and \( S_h \), and \( w^* \) is small.
6. Upper bound on the energy of the fields in \( B_{ff} \)

We next present an upper bound on the energy \( w \) of any force-free field \( B \in B_{ff} \). We make here the following additional assumptions on \( B_{ff} \) and \( S_0 \): (i) \( B_z \equiv B_0 \) and then \( b_z \equiv 0 \) on \( S_0 \) and \( S_h \); (ii) all the lines drawn on \( S \) have the same winding number \( n(R) = n \), with \( n \in \mathbb{Z} \) but possibly in the case where \( S_0 \) is a disk; and (iii) the domain \( S_0 \) is strictly star-shaped, i.e., \( \hat{n} \cdot R \geq R_1 > 0 \) on \( \partial S_0 \) for an adequately chosen origin of the frame. Eq. (23) thus reduces to

\[
4w - 2w_z = \left[ \frac{B_0}{4\pi} \int_{\partial S_h} (R \cdot \hat{n}) \left( \int_{c_h(R)} b'_q \, dq' \right) \, dq - \int_S \frac{b^2}{8\pi} R \cdot \hat{n} \, ds \right]
\]

\[
+ \frac{B_0}{4\pi} \int_{S_0} (R^h - R \cdot \nabla_\perp R^h) \cdot b^h \, ds - \int_S \frac{b^2}{8\pi} R \cdot \hat{n} \, ds.
\]

Taking into account the nonpositivity of the last term in the right-hand side of Eq. (31) and bounding the other terms with help from Eqs. (14) and (20), Schwartz inequality, and the elementary inequality \( 2xy \leq x^2 + y^2 \), we obtain eventually

\[
\frac{4w - 2w_z}{W_0} \leq 2N + 2L \sqrt{\frac{w - 2w_z}{W_0}},
\]

where \( w - 2w_z \geq 0 \) by Eq. (20) and

\[
L^2 = \frac{1}{|S_0| h^2} \int_{S_0} |R \cdot \nabla_\perp R^h - R^h|^2 \, ds \quad \text{and} \quad N = \frac{4\pi \delta n^2 |S_0|}{h^2}.
\]

The factor \( \delta \) is purely geometric \((\delta = \delta(S_0))\). When \( S_0 \) is a disk, one has \( \delta = 1 \) and \( L \) vanishes if the boundary motions are pure rigid rotations.

A simple discussion of inequality (32) thus allows to conclude that

\[
\frac{w}{W_0} \leq \frac{w}{W_0} = \begin{cases} 
\frac{(L^2 + 4N + \sqrt{L^2(L^2 + 8N)})}{8} & \text{when } N \leq L^2, \\
\frac{(L^2 + 2N)}{3} & \text{when } L^2 < N,
\end{cases}
\]

\[
\frac{w_z}{W_0} \leq \frac{1}{24} (L^2 + 8N).
\]

A similar upper bound on \( w \) may be established in the framework of reduced MHD (with then \( w_z = 0 \)), and for arbitrary flux distributions on the lower and upper boundaries, in which case \( \pi \) is given by an expression much more complex than the one above. We also mention that an alternative energy bound can be derived in a simple way when \( B \) is the final state of a quasi-static evolution during which the field passes through a sequence of regular force-free configurations \( B(t), \ 0 \leq t \leq T \), with \( B(0) = B_0 \). An equation of evolution can be obtained for the energy \( w(t) \) by starting from the ideal induction equation \( \partial_t B(t) = \nabla \times (\mathbf{v} \times B(t)) \) and the equilibrium equation (2) satisfied by \( B(t) \), and one can bound its right-hand side by applying Schwartz inequality and Eq. (20). One gets

\[
4\pi \frac{dw(t)}{dt} = \left( \int_{S_h} - \int_{S_0} \right) (\mathbf{v} \cdot b_\perp B_z)(t) \, ds \leq G_1(t) \sqrt{8\pi w(t) + G_2(t)},
\]

where the functions \( G_k \) can be computed from \( \mathbf{v} \), and an upper bound follows by time integration from 0 to \( T \). This bound does not suppose the assumptions (i)-(iii) to be verified, but it has the defect of depending on the specific velocity field that is applied on the boundary to produce \( B \) (of course, there are an infinity of such fields and it seems difficult to determine an optimal one among them).
7. A stability criterion for the fields in $B_{ff}$

We consider finally the important problem of the ideal stability properties of a force-free magnetic field $B \in B_{ff}$ (we recall that $\alpha = \alpha(L) = B \cdot \nabla \times B / B^2$). At the linear level, stability can be determined by using the energy principle (Bernstein et al., 1958): an arbitrary displacement $\xi(r)$, satisfying $\xi = 0$ on $S_0 \cup S_1$ (line-tying condition) and $\hat{n} \cdot \xi = 0$ on $S$, is applied to the plasma elements in $D$ and one looks at the sign of the resulting second variation of the energy

$$
\delta^2 W[\xi] = \frac{1}{8\pi} \int_D [||\nabla \times A||^2 - \alpha A \cdot \nabla \times A] dv,
$$

where $A = \xi \times B$. (37)

$B$ is linearly stable if $\delta^2 W[\xi] \geq 0$ for every $\xi$, while it is unstable if there exists a $\xi$ such that $\delta^2 W[\xi] < 0$. Following Aly (1990), one can reexpress $\delta^2 W$ as a functional of the field perturbation, $\delta b = \nabla \times A$, with the admissible $\delta b$ being constrained to satisfy $\nabla \cdot \delta b = 0$ in $D$, $\hat{n} \cdot \delta b = 0$ on $\partial D$, and $\int_S \hat{n} \cdot \delta b ds = 0$ for any surface $S \subset D$ bounded by a magnetic line $L$ and a curve on $\partial D$ connecting the footpoints of $L$. The method is as follows. One introduces the Euler representation $B = \nabla U \times \nabla V$ of the equilibrium field, writes $A = \lambda \nabla U + \mu \nabla V$, and expresses the coefficients $\lambda$ and $\mu$ as integrals of $\delta b \cdot \nabla V / B$ and $\delta b \cdot \nabla U / B$, respectively, along a magnetic line. Whence $A = A[\delta b]$, and $\delta^2 W = \delta^2 W[\delta b]$ indeed.

It can be shown that $\delta^2 W[\delta b] \geq 0$ for any $\delta b \neq 0$ if one has on every line $L$

$$
|\alpha(L)| \leq \left( \int_L \frac{|\nabla U|^2}{B} dl \int_L \frac{|\nabla V|^2}{B} dl \right)^{-1/2}.
$$

(38)

$B \in B_{ff}$ is then linearly stable if $|\alpha(L)|$ is not larger than some fraction of $|L|^{-1}$. This result is quite general, but it is not fully satisfying as the stability criterion is expressed in terms of $\alpha$ rather than in terms of topological quantities characterizing $B$. Another sufficient condition of stability, also expressed in terms of $\alpha$, can be derived by the method used in Aly (1990a).

8. Conclusion

In this Communication, we have reported some new general results on the model introduced long ago by Parker (1994) for studying a basic mechanism of plasma heating in confined magnetic structures. On the one hand, we have derived new formulae for the field line helicity and the relative helicity of the fields belonging to a class $B$ of topologically equivalent regular fields, and established lower bounds on their energy. In particular, we have generalized Berger’s bound (Berger, 1993) to the case of a field with a nonuniform $z$-component. On the other hand, we have considered the force-free fields contained in $B$, assuming the existence of at least one. We have made some new points about the analogies existing between the force-free equations and those of geometrical optics, elastostatics, and 2D time-dependent hydrodynamics, established upper bounds on their energy, and proposed a sufficient criterion of ideal MHD linear stability.

We note that studying the relations between the energy of a magnetic field and its topology is also quite important for constructing models of solar large scale eruptive processes. These phenomena exhibit huge expansions of the magnetic structures and one needs for studying them to introduce models in which the force-free coronal magnetic field is not confined as in Parker’s, but rather occupies an unbounded domain (e.g., the exterior of a sphere, or a half-space). By developing considerations similar to those in this paper, it turns out to be possible to establish basic inequalities which imply in particular that the radial scale of the field eventually increases very fast (at least as $\exp(t/T)^2$) when energy is continuously transferred from the subphotospheric layers into the corona (Aly & Amari (2007), and references therein).

Finally we recall that the problem of the constraints imposed by topology on magnetic energy has also been addressed in the framework of knot theory. In that case, the fields one considers are
not tied to a boundary as in Parker’s model, but are confined within a tube constructed about a closed curve of a given topological type (or a set of tubes associated to a link). One of the basic issues is then to find the (non force-free!) configurations accessible by ideal MHD incompressible motions which minimize energy, and much efforts have been done to establish topological energy lower bounds on the energy minimum (Arnold & Khesin (1998), and references therein).

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