Background of the $su(2)$-Algebraic Many-Fermion Models in the Boson Realization

Construction of Minimum Weight States by Means of an Auxiliary $su(2)$-Algebra and Its Related Problems

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The $su(2)$-algebraic many-fermion model is formulated so as to be able to get the unified understanding of the structures of three simple models: the single-level pairing, the isoscalar proton-neutron pairing and the two-level Lipkin model. The basic idea is to introduce an auxiliary $su(2)$-algebra, any generator of which commutes with any generator of the starting $su(2)$-algebra. With the aid of this algebra, the minimum weight states are completely determined in simple form. Further, concerning the two algebras, boson realization is presented. Through this formulation, the behavior of the total fermion in the Lipkin model is notably different from that in the other two models. As the supplementary problem, the boson-fermion realization and the Lipkin model in the isovector pairing model are investigated.

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§1. Introduction

It is well known that, with the aid of boson operators, we can describe various phenomena of nuclear and hadron physics, successfully. Especially, the studies of microscopic structures of the boson operators trace back to the year 1960. This year, Marumori, Arvieu & Veneroni, and Baranger\(^1\) proposed independently a theory, which is called the quasi-particle random phase approximation. With the aid of this theory, we could understand microscopic structure of the boson operators describing the collective vibrational motion observed in the spherical nuclei. In succession, the success of the theory has stimulated the studies of higher order corrections and one of the goals is the boson expansion theory: Belyaev & Zelevinsky, Marumori, Yamamura (one of the present authors) & Tokunaga, J. da Providência (one of the present authors) & Weneser, and Marshalek.\(^2\) We can find various further studies concerning the boson expansion theory in the review article by Klein & Marshalek.\(^3\) Especially, this review concentrated on the boson realization of Lie algebra governing many-fermion system under consideration. The above is a rough sketch of the boson expansion theory at the early stage. After these studies, too many papers have been published and it is impossible to follow them completely. Then, hereafter, we will
narrow down the discussion to the $su(2)$-algebraic many-fermion model and its boson realization.

We know three simple many-fermion models which obey the $su(2)$-algebra: (1) the single-level pairing model, (2) the isoscalar proton-neutron pairing model, and (3) the two-level Lipkin model. Hereafter, we abbreviate (1), (2) and (3) as (1) the pairing model, (2) the isoscalar pairing model and (3) the Lipkin model. The $su(2)$-algebra is composed of three generators $\tilde{S}_\pm, \tilde{S}_0$, which are of the bilinear forms for the fermion operators and $\tilde{S}_+$ plays a role of block for the orthogonal set built on the minimum weight state. Further, each model contains the total fermion number operator $\tilde{N}$. It may be interesting to see that these three models are in the triangle relation with one another. (i) The isoscalar pairing and the Lipkin model consist of two single-particle levels, but the pairing model is treated in one single-particle level. (ii) The operators $\tilde{S}_\pm$ as the building block in the Lipkin and the pairing model are the particle-hole pair and the fermion pair coupled to angular momentum $J = 0$, respectively. But, the proton-neutron pair in the isoscalar pairing model does not couple to $J = 0$. (iii) In the pairing and the isoscalar model, $\tilde{S}_0$ is a linear function of $\tilde{N}$. But, the Lipkin model does not contain $\tilde{N}$ explicitly, in other words, $\tilde{S}_0$ and $\tilde{N}$ are independent of each other. The above is the triangle relation of the three models.

For the above triangle, (i) and (ii) are not so serious as (iii), because (i) and (ii) merely determine the framework of the models. As was already mentioned, $\tilde{S}_0$ is a linear function of $\tilde{N}$ in the pairing and the isoscalar pairing model. Therefore, since $\tilde{N}$ commutes with the Hamiltonian which is widely adopted, the change of the eigenvalue of $\tilde{S}_0$ automatically leads to the change of the total fermion number $\tilde{N}$. In the Lipkin model, $\tilde{S}_0$ is a linear function of the difference between the fermion number operators of the two levels and $\tilde{N}$ is the sum of both fermion number operators. Therefore, the change of the eigenvalue of $\tilde{S}_0$ corresponds to the change of the difference between the fermion numbers of the two levels. The above suggests to us that, in the Lipkin model, the information on $\tilde{N}$ is contained fully in the minimum weight state, but, we do not know in which form $\tilde{N}$ is contained. On the other hand, we know that the minimum weight states in the pairing and the isoscalar pairing model depend on the fermion numbers partially through the seniority numbers. From the above reason, we are forced to reconsider the minimum weight states in the $su(2)$ models and it may also be interesting to investigate how $\tilde{S}_\pm, \tilde{S}_0$ in the boson realization are influenced by the minimum weight states. In this sense, we must recognize that the $su(2)$-algebraic many-fermion models contain still an open question in spite of long research history.

The main aim of this paper is to give a possible answer of the above-mentioned question. In order to arrive at the goal, we must reformulate the $su(2)$-algebraic many-fermion model in a rather general scheme. We start from preparing many-fermion system which is confined in 4$\Omega_0$ single-particle states ($\Omega_0$; integer or half-integer). It depends on the model under consideration. Following a certain idea which will be mentioned concretely in §2, we construct the $su(2)$-generators $\tilde{S}_\pm, \tilde{S}_0$. Further, we introduce another $su(2)$-algebra, the generators of which are denoted
as $\tilde{R}_{\pm,0}$. The most important condition in our scheme is that both algebras are connected to each other through the commutation relation [ any of $\tilde{R}_{\pm,0}$, any of $\tilde{S}_{\pm,0}$] = 0. If we follow the idea for constructing $\tilde{S}_{\pm,0}$ and $\tilde{R}_{\pm,0}$, it can be shown that there does not exist any other $su(2)$-algebra which is independent of $\tilde{R}_{\pm,0}$ and satisfies the above commutation relation. Conventionally, the minimum weight state $|m_0\rangle$ is determined through the relations $\tilde{S}_-|m_0\rangle = 0$ and $\tilde{S}_0|m_0\rangle = -s|m_0\rangle$. In addition to the above, we require the conditions $\tilde{R}_-|m_0\rangle = 0$ and $\tilde{R}_0|m_0\rangle = -r|m_0\rangle$. Then, $(\tilde{R}_+)^{r+r_0}|m_0\rangle (-r \leq r_0 \leq r)$ is also the minimum weight state for $\tilde{S}_{\pm,0}$. Further, in our scheme, we obtain the relation $s + r = \Omega_0$ under a certain scheme, which will be indicated concretely in §2 by the name of the aligned scheme. Usually, $s$ and $2r$ are called the magnitude of the quasi-spin and the seniority number. With the help of the condition we required newly, the minimum weight states can be completely derived without any device. In the Lipkin model, it can be shown that $\tilde{R}_0$ is a linear function of $\tilde{N}$ and then, $r_0$ is given as a function of $N$ and we can determine the minimum weight state as a function of $N$. Since [ any of $\tilde{R}_{\pm,0}$, any of $\tilde{S}_{\pm,0}$] = 0, we can apply the idea of the boson realization to each algebra and through the relation $s + r = \Omega_0$, both algebras are coupled with each other. As supplementary arguments, we take up two subjects. One is related to the boson-quasifermion realization for the $su(2)$-model. With the aid of this idea, we can describe many-fermion systems which do not obey the $su(2)$-algebra exactly. The other is concerned with the Lipkin model obeying the sub-algebra of the $so(5)$-algebra which describes the isovector pairing model. In this treatment, it is shown that $N$ is in the closed relation to the reduced isospin which characterizes the minimum weight state of the $so(5)$-algebra.

In §2, we present a general scheme of our idea concretely. Sections 3, 4 and 5 are devoted to applying the general scheme in §2 to the pairing, the isoscalar and the Lipkin model, respectively. The difference of the Lipkin model from the other two is clarified. In §6, the boson-quasifermion realization is formulated in rather general form. In §7, after the $so(5)$-algebra is recapitulated, the Lipkin model is treated in the form different from that given in §5. In §8, concluding remarks are mentioned.

§2. General scheme

Our description of the $su(2)$-algebraic models starts from giving a general scheme. We treat many-fermion system which is confined in $4\Omega_0$ single-particle states. Here, $\Omega_0$ denotes integer or half-integer which depends on the model under investigation. Since $4\Omega_0$ is an even number, all single-particle states are divided into equal parts, $P$ and $\overline{P}$. Therefore, as a partner, each single-particle state belonging to $P$ can find a single-particle state in $\overline{P}$. It is natural that we must make rules for finding the partners uniquely. We express the partner of the state $\alpha$ belonging to $P$ as $\overline{\alpha}$ and fermion operators in $\alpha$ and $\overline{\alpha}$ as $(\tilde{c}_\alpha, \tilde{c}_\alpha^*)$ and $(\tilde{c}_{\overline{\alpha}}, \tilde{c}_{\overline{\alpha}}^*)$, respectively.

For the above system, we define the following operators:

$$
\tilde{S}_+ = \sum_\alpha s_\alpha \tilde{c}_\alpha^* \tilde{c}_\alpha^* , \quad \tilde{S}_- = \sum_\alpha s_\alpha \tilde{c}_\alpha \tilde{c}_\alpha , \quad \tilde{S}_0 = \frac{1}{2} \sum_\alpha (\tilde{c}_\alpha^* \tilde{c}_\alpha + \tilde{c}_{\overline{\alpha}}^* \tilde{c}_{\overline{\alpha}}) - \Omega_0 . \quad (2.1)
$$
The symbol $s_\alpha$ denotes real number satisfying

$$s_\alpha^2 = 1.$$  \hspace{1cm} (2.2)

The sum $\sum_\alpha (\sum_\pi)$ is carried out in all single-particle states in $P(\overline{P})$ and, then, we have

$$\sum_\alpha 1 = 2\Omega_0.$$  \hspace{1cm} (2.3)

It is easily verified that the operators $\tilde{S}_\pm$ form the $su(2)$-algebra:

$$[\tilde{S}_+, \tilde{S}_-] = 2\tilde{S}_0,$$
$$[\tilde{S}_0, \tilde{S}_\pm] = \pm \tilde{S}_\pm.$$  \hspace{1cm} (2.4a)

The Casimir operator $\tilde{S}^2$ is defined as

$$\tilde{S}^2 = \tilde{S}_+\tilde{S}_- + \tilde{S}_0(\tilde{S}_0 - 1),$$
$$[\tilde{S}_{\pm,0}, \tilde{S}^2] = 0.$$  \hspace{1cm} (2.4b)

Usually, for the $su(2)$-algebraic model, the Hamiltonian is expressed in terms of $\tilde{S}_\pm$. Associating to the above $su(2)$-algebra, we introduce another $su(2)$-algebra, the generators of which are defined in the form

$$\tilde{R}_+ = \sum_\alpha \tilde{c}_\alpha^* \tilde{c}_\alpha, \quad \tilde{R}_- = \sum_\alpha \tilde{c}_\alpha^* \tilde{c}_\alpha, \quad \tilde{R}_0 = \frac{1}{2}\sum_\alpha (\tilde{c}_\alpha^* \tilde{c}_\alpha - \tilde{c}_\alpha \tilde{c}_\alpha^*).$$  \hspace{1cm} (2.5)

$$[\tilde{R}_+, \tilde{R}_-] = 2\tilde{R}_0,$$
$$[\tilde{R}_0, \tilde{R}_\pm] = \pm \tilde{R}_\pm.$$  \hspace{1cm} (2.6a)

$$\tilde{R}^2 = \tilde{R}_+\tilde{R}_- + \tilde{R}_0(\tilde{R}_0 - 1).$$  \hspace{1cm} (2.6b)

The following relation may be indispensable to understand our idea:

$$[\text{any of } \tilde{R}_{\pm,0}, \text{any of } \tilde{S}_{\pm,0}] = 0.$$  \hspace{1cm} (2.7)

The algebra $(\tilde{R}_{\pm,0})$ plays a role auxiliary to the algebra $(\tilde{S}_{\pm,0})$ which must plays a central role for describing the dynamics induced by the $su(2)$-Hamiltonian. The relation (2.7) tells us that the above two algebras seems to be independent of each other. But, as will be later shown, they are not completely independent. Hereafter, at some occasions, we will use the terminologies $S$-spin and $R$-spin for $(\tilde{S}_{\pm,0})$ and $(\tilde{R}_{\pm,0})$, respectively. As far as the authors know, we have never encountered any investigation based on the explicit use of the algebra $(\tilde{R}_{\pm,0})$ for the $su(2)$-algebraic many-fermion models. In this connection, we must mention that there does not exist any $su(2)$-algebra, the generators of which are expressed in terms of bilinear form such as $\sum_\alpha r_\alpha \tilde{c}_\alpha^* \tilde{c}_\alpha$ ($r_\alpha^2 = 1$) and commute with those of $(\tilde{R}_{\pm,0})$ defined in the relation (2.5).

Now, let us search a possible type of orthogonal set produced by the above algebras. For this aim, we introduce the following states in the scheme which may
be permitted to call a possible type of the aligned scheme for the arrangement of fermion creation operators:

\[
|m_0\rangle = \begin{cases} 
|0\rangle, & (r = 0) \\
\prod_{i=1}^{2r} c_{\alpha_i}^\dagger |0\rangle (= |r; (\overline{\alpha})\rangle). & (2r = 1, 2, \ldots, 2\Omega_0)
\end{cases}
\] (2.8a)

Here, \((\overline{\alpha})\) denotes the configuration

\[
(\overline{\alpha}) = \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_{2r}.
\] (2.8b)

The state \(|0\rangle\) is the vacuum of \((c_\alpha, c_\alpha^\dagger)\) and \((\tilde{c}_{\alpha}, \tilde{c}_{\alpha}^\dagger)\). Beforehand, it may be convenient to specify the rule of how to arrange the single-particle states in \(|m_0\rangle\) appropriately. For choosing \((\overline{\alpha})\) for a given value of \(r\), there exist \((2\Omega_0!) / (2r!) (2\Omega_0 - 2r)!\) possibilities. Then, the set \(|r; (\overline{\alpha})\rangle\) forms an orthogonal set:

\[
(r; (\overline{\alpha}) | r'; (\overline{\alpha}')) = \begin{cases} 
\prod_{i=1}^{2r} \delta_{\alpha_i, \alpha'_i}, & (r = r') \\
0, & (r \neq r')
\end{cases}
\] (2.9)

Further, the following relations are easily verified:

\[
\tilde{R}_- |r; (\overline{\alpha})\rangle = 0, \quad \tilde{S}_- |r; (\overline{\alpha})\rangle = 0,
\] (2.10)

\[
\tilde{R}_0 |r; (\overline{\alpha})\rangle = -r|r; (\overline{\alpha})\rangle, \quad \tilde{S}_0 |r; (\overline{\alpha})\rangle = -s|r; (\overline{\alpha})\rangle. \quad (s = \Omega_0 - r)
\] (2.11a)

It is important to see that \(|r; (\overline{\alpha})\rangle\) is a minimum weight state of \(R\) - and the \(S\)-spin and the eigenvalues of \(\tilde{R}_0\) and \(\tilde{S}_0\) are not independent of each other, but restricted to

\[
r + s = \Omega_0.
\] (2.12)

It should be noted that \(\Omega_0\) is the most basic parameter in the present many-fermion system and its value is automatically fixed for a given model. Therefore, we cannot choose the values of \(r\) and \(s\) independently of each other. This is the reason why the two algebras are not completely independent of each other. The eigenstate of \(\tilde{R}^2, \tilde{S}^2, \tilde{R}_0\) and \(\tilde{S}_0\) with the eigenvalues \(r(r+1), s(s+1)\) \((s = \Omega_0 - r)\), \(r_0\) and \(s_0\) is expressed as

\[
|s(= \Omega_0 - r) s_0, r r_0; (\overline{\alpha})\rangle = (\tilde{S}_+)^{s+s_0} |r r_0; (\overline{\alpha})\rangle, \quad (2.13a)
\]

\[
|rr_0; (\overline{\alpha})\rangle = (\tilde{R}_+)^{r+r_0} |r; (\overline{\alpha})\rangle, \quad (2.13b)
\]

\[
r = 0, 1/2, 1, \cdots, \Omega_0 - 1/2, \Omega_0, \quad r_0 = -r, -r + 1, \cdots, r - 1, r,
\]

\[
s = \Omega_0 - r, \quad s_0 = -s, -s + 1, \cdots, s - 1, s. \quad (2.13c)
\]
The state given in the relation (2.13a) can be rewritten as

$$|s = \Omega_0 - r\rangle s_0, rr_0; (\overline{\sigma}) = |\Omega_0 \sigma, rr_0; (\overline{\sigma})\rangle = \left(\tilde{S}_+\right)^{\sigma} |rr_0; (\overline{\sigma})\rangle ,$$

$$\sigma = 0, 1, \cdots , 2(\Omega_0 - r) .$$

We can see that the state $|\Omega_0 \sigma, rr_0; (\overline{\sigma})\rangle$ is expressed in terms of three quantum numbers $(\sigma, \tau r_0)$ except the quantum numbers $(\overline{\sigma})$ and the parameter $\Omega_0$. Here and hereafter, we omit the normalization constant for any state. It should be noted that $|rr_0; (\overline{\sigma})\rangle$ is the minimum weight state for the $S$-spin which is deeply connected to the dynamics. In this paper, we call possible type of the aligned scheme simply the aligned scheme. However, probably, we can find the minimum weight states which do not belong to the form introduced in the relation (2.8a). The simple example will be shown in §8.

As is clear from the above argument, our idea consists of two steps. The first is to determine the minimum weight states, in which $(\tilde{R}_{\pm,0})$ plays a central role. The second is to construct the orthogonal set connected with the minimum weight states obtained in the first, in which $(\tilde{S}_{\pm,0})$ plays a central role. The formalism developed in the above is constructed in the frame of one kind of the degree of freedom, $\tilde{c}_\alpha, \tilde{c}_\overline{\alpha}^\ast, \tilde{c}_\tau$ and $\tilde{c}_\overline{\tau}$. Then, it may be interesting to describe the present system in terms of two kinds of degrees of freedom, in which one is for the first and the other is for the second. For the above idea, the use of the boson realizations of the $su(2)$-algebras may be a possible candidate.

The relations (2.4), (2.6) and (2.7) suggest to us that the counterparts of $\tilde{R}_{\pm,0}$ and $\tilde{S}_{\pm,0}$ can be expressed in the following form:

$$\tilde{R}_+ = d_P^\ast \hat{d}_P , \quad \tilde{R}_- = d_P^\ast \hat{d}_P , \quad \tilde{R}_0 = \frac{1}{2}(d_P^\ast \hat{d}_P - \hat{d}_P^\ast d_P) ,$$

$$\tilde{S}_+ = \hat{a}^\ast \hat{b} , \quad \tilde{S}_- = \hat{b}^\ast \hat{a} , \quad \tilde{S}_0 = \frac{1}{2}(\hat{a}^\ast \hat{a} - \hat{b}^\ast \hat{b}) .$$

Here, $(\hat{d}_P, \hat{d}_P^\ast)$ etc. denote boson operators. The above is well known by the name of the Schwinger boson representation of the $su(2)$-algebra.\(^8\) Under the one-to-one correspondence to the original fermion space, we must construct the orthogonal set in the boson space. First, we set up the following correspondence:

$$|0\rangle \sim (\hat{b}^\ast)^{2\Omega_0} |0\rangle .$$

Here, $|0\rangle$ denotes the boson vacuum. Next, we notice that the state $|r; (\overline{\sigma})\rangle$ is obtained by operating $2r$ fermion creation operators in $\overline{P}$, i.e., $\prod_{i=1}^{2r} \tilde{c}_\overline{\tau}_i$, on the vacuum $|0\rangle$. This procedure may be transcribed in the boson space in the following manner: The counterpart of $|r; (\overline{\sigma})\rangle$ may be obtained by performing $2r$-time operation of $(\hat{d}_P^\ast \hat{b})$ on $(\hat{b}^\ast)^{2\Omega_0} |0\rangle$. This is formulated as

$$|r; \overline{P}\rangle = (\hat{d}_P^\ast \hat{b})^{2r} \cdot (\hat{b}^\ast)^{2\Omega_0} |0\rangle = (\hat{d}_P^\ast)^{2r} (\hat{b}^\ast)^{2(\Omega_0 - r)} |0\rangle .$$

Strictly speaking, $|r; \overline{P}\rangle$ is not counterpart of $|r; (\overline{\sigma})\rangle$, because $|r; \overline{P}\rangle$ does not contain $(\overline{\sigma})$. But, the dynamics induced by the $S$-spin depends on the minimum weight state

\[
|s(= \Omega_0 - r\rangle s_0, rr_0; (\overline{\sigma}) = |\Omega_0 \sigma, rr_0; (\overline{\sigma})\rangle = \left(\tilde{S}_+\right)^{\sigma} |rr_0; (\overline{\sigma})\rangle ,
\]

\[
\sigma = 0, 1, \cdots , 2(\Omega_0 - r) .
\]
only through $r$. Later, it will be shown. Therefore, at the present framework, it may be not always necessary to make $|r; \overline{P}\rangle$ depend on $(\overline{a})$. In §6, we will contact again with this point. With the aid of the relation (2.14), we can prove that $|r; \overline{P}\rangle$ satisfies the same relations as the relation (2.10) and (2.11). Further, we have

$$|rr_0; \overline{P}\rangle = \left(\hat{R}_+\right)^{r+\overline{r}_0} |r; \overline{P}\rangle = (\hat{d}_p)^{r+\overline{r}_0} (\hat{d}^*_p)^{r-\overline{r}_0} (\hat{b}^*)_2(\Omega_0-r)|0\rangle .$$

(2.17)

It may be self-evident that the counterpart of $|s(= \Omega_0 - r) s_0, rr_0; (\overline{a})\rangle$ is obtained in the following from:

$$|s(= \Omega_0 - r) s_0, rr_0; \overline{P}\rangle = \left(\hat{S}_+\right)^{s+s_0} |rr_0; \overline{P}\rangle = (\hat{a}^*_s)^{s+s_0} (\hat{b}^*)_s^{s-s_0} (\hat{d}_p^*)_s^{r+\overline{r}_0} (\hat{d}^*_p)^{r-\overline{r}_0}|0\rangle . \quad (s = \Omega_0 - r)$$

(2.18)

The state (2.18) satisfies the relation (2.13c). Formally, the eigenstate of the $R$- and the $S$-spin with the eigenvalues $(r, r_0)$ and $(s, s_0)$ can be expressed in the form

$$|ss_0, rr_0) = (\hat{a}^*)_s^{s+s_0} (\hat{b}^*)_s^{s-s_0} (\hat{d}_p^*)_s^{r+\overline{r}_0} (\hat{d}^*_p)^{r-\overline{r}_0}|0\rangle .$$

(2.19)

If we add the condition (2.12), the state (2.19) is reduced to the state (2.18). In this sense, the set $\{|ss_0, rr_0); r + s = \Omega_0\}$ forms the physical boson space and the condition (2.12) holds the key to the solution of our problem. The state (2.19) under the condition (2.12) is specified by three quantum numbers. As for the three, it may be interesting to consider which quantum numbers are possible. We pay attention to the total fermion number which is a constant of motion in the widely known $su(2)$-model. As can be suggested in the relation (2.1), $\tilde{S}_0$ is connected to $\tilde{N}$ in the form

$$\tilde{S}_0 = \frac{1}{2} \tilde{N} - \Omega_0 , \quad \text{i.e.,} \quad s_0 = \frac{1}{2} N - \Omega_0 .$$

(2.20a)

Here, of course, $\tilde{N}$ and $N$ denote the total fermion number operator and its eigenvalue, respectively. Another idea is to connect $\tilde{R}_0$ to $\tilde{N}$:

$$\tilde{R}_0 = \frac{1}{2} \tilde{N} - \Omega_0 , \quad \text{i.e.,} \quad r_0 = \frac{1}{2} N - \Omega_0 .$$

(2.20b)

If we combine the relation (2.20) with the condition (2.12), the expression (2.19) becomes the following:

$$|\Omega_0 N, rr_0) = (\hat{a}^*)^{\frac{1}{2} N - r} (\hat{b}^*)^{\frac{1}{2} (4\Omega_0 - N) - r} (\hat{d}_p^*)_r^{r+\overline{r}_0} (\hat{d}^*_p)^{r-\overline{r}_0}|0\rangle ,$$

(2.21a)

$$|\Omega_0 N, ss_0) = (\hat{a}^*)_s^{s+s_0} (\hat{b}^*)_s^{s-s_0} (\hat{d}_p^*)_s^{r+\overline{r}_0} (\hat{d}^*_p)^{r-\overline{r}_0}|0\rangle .$$

(2.21b)

The forms (2.21a) and (2.21b) are based on the relations (2.20a) and (2.20b), respectively. The state (2.21) will be discussed in §§3–5.

Another idea for escaping from the restriction (2.12) is to adopt following representation for $(\hat{S}_{\pm,0})$:

$$\hat{S}_+ = \hat{A}^* \cdot \sqrt{2\hat{S} - \hat{A}^* \hat{A}} ,$$

$$\hat{S}_- = \sqrt{2\hat{S} - \hat{A}^* \hat{A}} \cdot \hat{A} ,$$

(2.22)
Here, \((\hat{A}, \hat{A}^\dagger)\) denotes boson operator and \(\hat{S}\) is defined as
\[
\hat{S} = \Omega_0 - \hat{R}, \quad \hat{R} = \frac{1}{2}(\hat{d}_P^* \hat{d}_P + \hat{d}_P \hat{d}_P^*) . \tag{2.23}
\]

As an operator identity, we have
\[
\hat{R}^2 = \hat{R}(\hat{R} + 1), \quad \hat{S}^2 = \hat{S}(\hat{S} + 1), \quad \hat{S} = \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) . \tag{2.24}
\]

The relation (2.24) tells us that \(\hat{R}\) and \(\hat{S}\) can be regarded as the operators expressing the magnitudes of the \(R\)- and the \(S\)-spin, respectively, if they are independent of each other. In order to connect the \(S\)-spin to the \(R\)-spin, we adopt the form (2.23) as the operator for the magnitude of the \(S\)-spin, which comes from the relation (2.12). If \(\hat{S}\) is replaced with \(c\)-number \(s\), the form (2.20) becomes the conventional Holstein-Primakoff representation.\(^9\)

In this sense, we take the form (2.20) into the Holstein-Primakoff representation. We can see that the \(S\)-spin depends only on \(r\). The counterpart of \(|s(= \Omega_0 - r) s_0, r r_0; (\overline{\tau})\rangle\) for the Holstein-Primakoff representation is given as
\[
|\Omega_0 \sigma, r r_0 \rangle = (\hat{A}^\sigma)^{r+r_0}(\hat{d}_P^\sigma)^{r-r_0}|0\rangle, \quad (\hat{A}|0\rangle = \hat{d}_P|0\rangle = \hat{d}_P^*|0\rangle = 0) \tag{2.25}
\]

This form comes from the form (2.13).

The present boson representation seems to be apparently not so new. But, in reality, it contains new features. On the occasion of investigating the boson realization of the \(su(2)\)-algebraic many-fermion models, the present framework, which consists of the two steps, teaches us that it may be necessary to take into account not only the algebra \((\hat{S}_{\pm}, \hat{0})\) but also \((\hat{R}_{\pm}, \hat{0})\). Especially, it is interesting to investigate how the operator \(\hat{R}_0\) influences the results. In \$3-5, we will present the results given in three concrete models in relation to the effects of \(\hat{R}_0\).

\section{3. The pairing model}

The first example of the application of the general scheme given in \$2 is the single-level pairing model.\(^4\) This model may be one of most basic models in nuclear physics. It consists of identical fermions in one single-particle level with the degeneracy \(2j + 1\) \((= 2\Omega, \; j; \text{half-integer})\). We denote the fermion operators as \((\tilde{c}_m, \; \tilde{c}_m^\dagger)\), where \(m = \pm 1/2, \; \pm 3/2, \cdots, \; \pm (j - 1), \; \pm j\). For the above system, we define the following operators:
\[
\tilde{S}_+ = \frac{1}{2} \sum_m (-)^j m \tilde{c}_m^\dagger \tilde{c}_{-m}, \quad \tilde{S}_- = \frac{1}{2} \sum_m (-)^j m \tilde{c}_{-m} \tilde{c}_m, \quad \tilde{S}_0 = \frac{1}{2} \sum_m \tilde{c}_m^\dagger \tilde{c}_m - \frac{1}{2} \Omega . \tag{3.1}
\]
The operator $\tilde{S}_+$ plays a role of creation of the Cooper pair and the set $(\tilde{S}_{\pm,0})$ forms the $su(2)$-algebra (2.4). The expression (3.1) can be rewritten to

$$
\tilde{S}_+ = \sum_{m>0} (-)^j m \tilde{c}^*_m \tilde{c}_{-m}, \quad \tilde{S}_- = \sum_{m>0} (-)^j m \tilde{c}_{-m} \tilde{c}_m ,
$$

$$
\tilde{S}_0 = \frac{1}{2} \sum_{m>0} (\tilde{c}^*_m \tilde{c}_m + \tilde{c}^*_{-m} \tilde{c}_{-m}) - \frac{1}{2} \Omega .
$$

(3.2)

The expression (3.2) is reduced to the form (2.1), if $m, -m, (-)^j m$ and $\Omega$ read, for $m > 0$,

$$
m \rightarrow \alpha , \quad -m \rightarrow \bar{\alpha} , \quad (-)^j m \rightarrow s_\alpha , \quad \Omega \rightarrow 2\Omega_0 .
$$

(3.3)

Hereafter, at some occasions, we use $\bar{m}$ for $-m$. The form (3.2) suggests to us that $P$ and $\bar{P}$ consist of positive $m$ and negative $m$, respectively, and the states $m$ and $-m$ are in the relation of the partner. Practically, this choice of the partner may be unique.

Under the reading (3.3), we can define $\tilde{R}_{\pm,0}$ in the form

$$
\tilde{R}_+ = \sum_{m>0} \tilde{c}^*_m \tilde{c}_{-m} , \quad \tilde{R}_- = \sum_{m>0} \tilde{c}^*_{-m} \tilde{c}_m , \quad \tilde{R}_0 = \frac{1}{2} \sum_{m>0} (\tilde{c}^*_m \tilde{c}_m - \tilde{c}^*_{-m} \tilde{c}_{-m}) .
$$

(3.4)

The operators $\tilde{R}_{\pm,0}$ obey the $su(2)$-algebra and satisfy the relation (2.7). In this connection, we know another $su(2)$-algebra, which satisfies the relation (2.7):

$$
\tilde{j}_+ = \sum_m \mu_m(j) \tilde{c}^*_m \tilde{c}_{m-1} , \quad \tilde{j}_- = \sum_m \mu_m(j) \tilde{c}^*_{m-1} \tilde{c}_m , \quad \tilde{j}_0 = \sum_m m \tilde{c}^*_m \tilde{c}_m ,
$$

$$
\mu_m(j) = \sqrt{(j+m)(j-m+1)} .
$$

(3.5)

The set $(\tilde{j}_{\pm,0})$ is the angular momentum operator. The set $(\tilde{S}_{\pm,0})$ is scalar for $(\tilde{j}_{\pm,0})$ and both sets commute with each other. However, as can be seen in the expression (3.5), the set $(\tilde{j}_{\pm,0})$ is not suitable for our present discussion.

Now, let us discuss the seniority scheme which characterizes the present model. The state introduced in the relation (2.13b) can be expressed as $|rr_0;(\bar{m})\rangle$ in the present notation. Here, $(\bar{m})$ denotes the configuration $\bar{m}_1, \bar{m}_2, \cdots, \bar{m}_{2r}$. It satisfies

$$
\tilde{S}_-|rr_0;(\bar{m})\rangle = 0 ,
$$

$$
\tilde{R}_0|rr_0;(\bar{m})\rangle = r_0|rr_0;(\bar{m})\rangle ,
$$

(3.7a)

$$
\tilde{S}_0|rr_0;(\bar{m})\rangle = -s|rr_0;(\bar{m})\rangle . \quad (s = \frac{\Omega}{2} - r)
$$

(3.7b)

The relation (3.6) indicates that $|rr_0;(\bar{m})\rangle$ does not contain the Cooper pair. With the definition of $\tilde{R}_0$ and $\tilde{S}_0$, the relation (3.7) leads us to

$$
(\tilde{N}_+ + \tilde{N}_-)|rr_0;(\bar{m})\rangle = 2r|rr_0;(\bar{m})\rangle ,
$$

(3.8a)

$$
(\tilde{N}_+ - \tilde{N}_-)|rr_0;(\bar{m})\rangle = 2r_0|rr_0;(\bar{m})\rangle .
$$

(3.8b)
Here, $\tilde{N}_+$ and $\tilde{N}_-$ denote the fermion number operators of $P$ and $\overline{P}$, respectively:
\[
\tilde{N}_+ = \sum_{m>0} \tilde{c}_m^* \tilde{c}_m, \quad \tilde{N}_- = \sum_{m>0} \tilde{c}_{-m}^* \tilde{c}_{-m}.
\] (3.9)

As was already mentioned, $|rr_0; (m\rangle$ does not contain any Cooper pair and $(\tilde{N}_+ + \tilde{N}_-)$ denotes the total fermion number. Therefore, we can conclude that $2r$ denotes the number of fermions which do not couple to the Cooper pair, that is, the seniority number or the number of the unpaired fermion. The role of $r_0$ can be interpreted as follows: At $r_0 = -r$, all unpaired fermions belong to $\overline{P}(m<0)$. By successive operation of $\tilde{R}_+$, the number of the unpaired fermion in $P$ increases and passes through the point $r_0 = 0$ ($2r = \text{even}$) or $r_0 = \pm 1/2$ ($2r = \text{odd}$), where the unpaired fermions occupy $\overline{P}$ and $P$ in equal weight. Finally, at $r_0 = r$, all unpaired fermions belong to $\overline{P}(m>0)$. As was mentioned in the above, we can learn how the structure of the minimum weight state changes with respect to the increase of $r_0$ from $-r$ to $r$.

By operating $\tilde{S}_+$ on $|rr_0; (m\rangle$ successively, we obtain the state $|s (= \Omega/2 - r) s_0, rr_0; (m\rangle\rangle$. The relation (2.13) gives us
\[
0 \leq 2r \leq \Omega, \quad -\left(\frac{\Omega}{2} - r\right) \leq s_0 \leq \frac{\Omega}{2} - r.
\] (3.10)

The definition of $\tilde{S}_0$ shown in the relation (3.2) leads us to
\[
s_0 = \frac{N}{2} - \frac{\Omega}{2}.
\] (3.11)

Here, $N$ denotes the total fermion number in the state $|s (= \Omega/2 - r) s_0, rr_0 : (m\rangle\rangle$. With the use of the relations (3.10) and (3.11), we have
\[
\text{if} \quad 0 \leq N \leq \Omega, \quad 0 \leq 2r \leq N, \\
\text{if} \quad \Omega \leq N \leq 2\Omega, \quad 0 \leq 2r \leq 2\Omega - N.
\] (3.12)

Under the inequality (3.12), automatically, the first of the relation (3.10) can be derived. The parameters $\Omega$, $N$ and $2r$ are even or odd numbers and by taking into account this property, the relation (3.12) gives us the following:

1. $N$ : even,
\[
2r = \begin{cases} 
0, 2, \cdots, N & (0 \leq N \leq \Omega) \\
0, 2, \cdots, \Omega & (N = \Omega, \Omega : \text{even}) \\
0, 2, \cdots, (2\Omega - N) & (\Omega < N \leq 2\Omega)
\end{cases}
\] (3.13a)

2. $N$ : odd,
\[
2r = \begin{cases} 
1, 3, \cdots, N & (1 \leq N \leq \Omega) \\
1, 3, \cdots, \Omega & (N = \Omega, \Omega : \text{odd}) \\
1, 3, \cdots, (2\Omega - N) & (\Omega < N \leq 2\Omega - 1)
\end{cases}
\] (3.13b)

The above is the well-known rule in the pairing model. Thus, we could interpret the seniority scheme in terms of the $su(2)$-algebra $(\tilde{R}_{\pm,0})$. In this sense, the algebra $(\tilde{R}_{\pm,0})$ may be permitted to call the seniority algebra or the seniority spin.
Finally, we must contact with the boson realization for the pairing model. Main features have already been discussed in §2. For the comparison with the other model presented in §§4 and 5, only we show the counterpart of $|s(= \Omega/2 - r) s_0, r r_0; (\overline{m})\rangle$ in the Schwinger boson representation in terms of total fermion number $N$. The expression (2.21a) with $\Omega_0 = \Omega/2$ gives us

$$|\Omega/2, N, r r_0\rangle = (\hat{a}^*)^{(N-2r)}(\hat{b}^*)^{(2\Omega-N-2r)}(\hat{d}_{\overline{m}}^*)^{r+r_0}(\hat{d}_{\overline{m}}^*)^{r-r_0}|0\rangle .$$  (3.14)

Of course, we used the relation (3.11). Since $r$ is positive and the exponents of $\hat{a}^*$ and $\hat{b}^*$ should be positive or zero, we have $0 \leq 2r \leq N$ and $0 \leq 2r \leq 2\Omega - N$, which lead us to the relation (3.12). In the present case, the state (2.25) is expressed as

$$|\Omega/2, \sigma, r r_0\rangle = (\hat{A}^*)^\sigma(\hat{d}_{\overline{m}}^*)^{r+r_0}(\hat{d}_{\overline{m}}^*)^{r-r_0}|0\rangle , \quad \sigma = 0, 1, 2, \cdots, (\Omega - 2r) .$$  (3.15)

The total fermion number $N$ is related to $\sigma$ in the following:

$$N = 2(r + \sigma) \quad \text{for} \quad 0 \leq N \leq \Omega , \quad (3.16a)$$

$$N = 2(\Omega - (r + \sigma)) \quad \text{for} \quad \Omega \leq N \leq 2\Omega . \quad (3.16b)$$

The above is the outline of the pairing model based on the general framework in §2.

§4. The isoscalar pairing model

In addition to the pairing model, we know a many-fermion model obeying the $su(2)$-algebra, which is called the isoscalar proton-neutron pairing model$^5$ (in this paper, abbreviated as the isoscalar pairing model). In this model, two single-particle levels, which we call the $p$- and the $n$-level, are occupied by protons and neutrons, respectively. The degeneracies are the same as each other: $2\Omega = 2j + 1$ ($j$; half-integer). The proton-neutron pairs coupled in the isoscalar type play an central role in this model. Of course, the pairs obey the $su(2)$-algebra.

Let us start from giving the isospin operator:

$$\tilde{T}_+ = \sum_m \tilde{c}_{pm}^* \tilde{c}_{nm} , \quad \tilde{T}_- = \sum_m \tilde{c}_{nm}^* \tilde{c}_{pm} , \quad \tilde{T}_0 = \frac{1}{2} \sum_m (\tilde{c}_{pm}^* \tilde{c}_{pm} - \tilde{c}_{nm}^* \tilde{c}_{nm}) , \quad (4.1)$$

$$[\tilde{T}_+, \tilde{T}_-] = 2\tilde{T}_0 , \quad [\tilde{T}_0, \tilde{T}_{\pm}] = \pm \tilde{T}_\pm . \quad (4.2)$$

Here, $m = \pm \frac{1}{2}, \pm \frac{3}{2}, \cdots, \pm (j - 1), \pm j$. For the above isospin operator, we can give the fermion-pair in the isoscalar type $S_{\pm,0}$:

$$\tilde{S}_+ = \sum_m s_m \tilde{c}_{pm}^* \tilde{c}_{n-m} , \quad \tilde{S}_- = \sum_m s_m \tilde{c}_{n-m}^* \tilde{c}_{pm} ,$$

$$\tilde{S}_0 = \frac{1}{2} \sum_m (\tilde{c}_{pm}^* \tilde{c}_{pm} + \tilde{c}_{nm}^* \tilde{c}_{nm}) - \Omega . \quad (4.3)$$

The set $(\tilde{S}_{\pm,0})$ forms the $su(2)$-algebra and commutes with $(\tilde{T}_{\pm,0})$ under the condition

$$s_{\overline{m}} = s_m . \quad (s_m^2 = 1) \quad (4.4)$$
The case \( s_m = -s_m \), for example, such as \( s_m = (-)^{j-m} \), leads us to the isovector type. In the case \( s_m = (-)^{j-m} \), \( s_m = -1 \) and \( s_m = +1 \) appear alternatively as \( m \) increases. In the present case, \( s_m = -1 \) and \( s_m = +1 \) can be freely chosen and, then, without loss of generality, we can set \( s_m = +1 \) for all \( m \). Hereafter, we will adopt the expression (4.3) with \( s_m = +1 \).

Concerning the idea for defining the \( su(2) \)-algebra \( \tilde{R}_{\pm,0} \), we have two possibilities:

(1) For \( m > 0 \), \( p, m \to \alpha \), \( p, -m \to \overline{\alpha} \), and \( n, m \to \alpha \), \( n, -m \to \overline{\alpha} \),

\[
s_m(= 1) \to s_\alpha, \quad \Omega \to \Omega_0. \tag{4.5}
\]

(2) For all \( m \), \( p, m \to \alpha \), \( n, -m \to \overline{\alpha} \), \( s_m(= 1) \to s_\alpha, \quad \Omega \to \Omega_0. \tag{4.6}
\]

For the possibility (1), \( \tilde{R}_{\pm,0} \) can be expressed in the form

\[
\tilde{R}_+ = \sum_{m > 0} \left( \tilde{c}_{pm}^* \tilde{c}_{m\overline{m}} + \tilde{c}_{nm}^* \tilde{c}_{n\overline{m}} \right), \quad \tilde{R}_- = \sum_{m > 0} \left( \tilde{c}_{pm}^* \tilde{c}_{m\overline{m}} + \tilde{c}_{nm}^* \tilde{c}_{n\overline{m}} \right),
\]

\[
\tilde{R}_0 = \frac{1}{2} \sum_{m > 0} \left( (\tilde{c}_{pm}^* \tilde{c}_{m\overline{m}} + \tilde{c}_{nm}^* \tilde{c}_{n\overline{m}}) - (\tilde{c}_{pm}^* \tilde{c}_{m\overline{m}} + \tilde{c}_{nm}^* \tilde{c}_{n\overline{m}}) \right). \tag{4.7}
\]

However, the form (4.7) does not satisfy the relation (2.7), which is the most fundamental in our formalism. From the above reason, we have to renounce the possibility (1). In the possibility (2), \( \tilde{R}_{\pm,0} \) can be expressed as

\[
\tilde{R}_+ = \sum_{m} \tilde{c}_{pm}^* \tilde{c}_{n\overline{m}}, \quad \tilde{R}_- = \sum_{m} \tilde{c}_{n\overline{m}}^* \tilde{c}_{pm},
\]

\[
\tilde{R}_0 = \frac{1}{2} \sum_{m} (\tilde{c}_{pm}^* \tilde{c}_{pm} - \tilde{c}_{nm}^* \tilde{c}_{nm}). \tag{4.8}
\]

We can prove that the expression (4.8) satisfy the relation (2.7). On the basis of the relation (4.8), we will continue our discussion.

In a way similar to the case of the pairing model, we introduce the proton and the neutron number operator in the form

\[
\tilde{N}_p = \sum_{m} \tilde{c}_{pm}^* \tilde{c}_{pm}, \quad \tilde{N}_n = \sum_{m} \tilde{c}_{n\overline{m}}^* \tilde{c}_{n\overline{m}}. \tag{4.9}
\]

Operating \( \tilde{N}_p \) and \( \tilde{N}_n \) on the minimum weight state in the present case \( |rr_0; (n\overline{m}) \rangle \), we have

\[
(\tilde{N}_p + \tilde{N}_n)|rr_0; (n\overline{m}) \rangle = 2r|rr_0; (n\overline{m}) \rangle, \tag{4.10a}
\]

\[
(\tilde{N}_p - \tilde{N}_n)|rr_0; (n\overline{m}) \rangle = 2r_0|rr_0; (n\overline{m}) \rangle. \tag{4.10b}
\]

Here, \((n\overline{m})\) denotes the configuration \( n\overline{m}_1, n\overline{m}_2, \ldots, n\overline{m}_{2r} \). We can see that the relation (4.10) has the same structure as that in the pairing model shown in the relation (3.8). Therefore, the interpretation given in §3 is available without any alternation. The quantum number \( 2r \) indicates the seniority number, the number of
the fermions which do not couple to the proton-neutron pair in the isoscalar type and it is distributed to \((r+r_0)\) protons and \((r-r_0)\) neutrons. By operating \(\hat{S}^+_s\) successively on the state \(|rr_0; (n\overline{m})\rangle\), we have \(|s(= \Omega - r) \ s_0, rr_0; (n\overline{m})\rangle\) and operating \(\tilde{N}(= \tilde{N}_p + \tilde{N}_n)\) on this state, we obtain the relation

\[
N = 2s_0 + 2\Omega .
\]  

(4.11)

Noting \(-s \leq s_0 \leq s\) and \(s = \Omega - r(\geq 0)\), from the relation (4.11) we obtain the following inequality:

\[
\begin{align*}
&\text{if} \quad 0 \leq N \leq 2\Omega \quad , \quad 0 \leq 2r \leq N , \\
&\text{if} \quad 2\Omega \leq N \leq 4\Omega \quad , \quad 0 \leq 2r \leq 4\Omega - N .
\end{align*}
\]  

(4.12)

The above corresponds to the relation (3.12) in the pairing model. By discriminating the case \(N = \text{even}\) and the case \(N = \text{odd}\), the inequality (4.12) leads us to

\[
(1) \ N : \text{even}, \\
2r = \begin{cases} 
0, 2, \cdots, N , & (0 \leq N \leq 2\Omega - 2) \\
0, 2, \cdots, 2\Omega , & (N = 2\Omega) \\
0, 2, \cdots, 4\Omega - N . & (2\Omega + 2 \leq N \leq 4\Omega)
\end{cases}
\]  

(4.13a)

\[
(2) \ N : \text{odd}, \\
2r = \begin{cases} 
1, 3, \cdots, N , & (1 \leq N \leq 2\Omega - 1) \\
1, 3, \cdots, 4\Omega - N . & (2\Omega + 1 \leq N \leq 4\Omega - 1)
\end{cases}
\]  

(4.13b)

It may be interesting to compare the above result with that shown in the relation (3.13). The quantity \(\Omega\) in the pairing model corresponds to \(2\Omega\) in the present model and, then, even if \(\Omega\) is odd, \(2\Omega\) is always even. This difference appears in both relations. The proton and the neutron number contained in the state \(|s(= \Omega - r) \ s_0, rr_0; (n\overline{m})\rangle\), \(N_p\) and \(N_n\), are given as

\[
N_p = \Omega + s_0 + r_0 , \quad N_n = \Omega + s_0 - r_0 .
\]  

(4.14)

In the case \(\Omega_0 = \Omega\), the counterpart of \(|s(= \Omega - r) \ s_0, rr_0; (n\overline{m})\rangle\) in the Schwinger boson representation is obtained from the relation (2.21):

\[
|\Omega N; rr_0\rangle = (\hat{a}^*)^{\frac{1}{2}(N-2r)}(\hat{b}^*)^{\frac{1}{2}(4\Omega-N-2r)}(\hat{d}_p)^{r+r_0}(\hat{d}_p^\dagger)^{r-r_0}|0\rangle .
\]  

(4.15)

If \(4\Omega\) is replaced with \(2\Omega\), the state (4.15) becomes the same form as that given in the relation (3.14). The state (4.15) gives us the relations \(r \geq 0\), \(N - 2r \geq 0\) and \((4\Omega - N) - 2r \geq 0\), which are reduced to the relation (4.12). In the present case, the state (2.25) becomes

\[
|\Omega \sigma, rr_0\rangle = (\hat{A}^\dagger)^{\sigma}(\hat{d}_p^\dagger)^{r+r_0}(\hat{d}_p)^{r-r_0}|0\rangle , \quad \sigma = 0, 1, 2, \cdots, 2(\Omega - r) .
\]  

(4.16)

The total fermion number \(N\) is related to \(\sigma\) in the following:

\[
N = 2(r + \sigma) \quad \text{for} \quad 0 \leq N \leq 2\Omega ,
\]  

(4.17a)

\[
N = 2(2\Omega - (r + \sigma)) \quad \text{for} \quad 2\Omega \leq N \leq 4\Omega .
\]  

(4.17b)

The general framework in §2 gives us the above result in the case of the isoscalar pairing model.
§5. The Lipkin model

In this section, we will investigate the model proposed by Lipkin, Meshkov and Glick, which is, usually called the Lipkin model. It aims at schematic description of the particle-hole excitation. The Lipkin model consists of two single-particle levels with the same degeneracy \( 2j + 1 = 2\Omega \), \( j \) : half-integer. We discriminate the two levels as the \( p \)- and the \( h \)-level. The fermion operators in the \( p \)- and the \( h \)-level are denoted as \( \tilde{\gamma}_p^m, \tilde{\gamma}_p^* \) and \( \tilde{\gamma}_h^m, \tilde{\gamma}_h^* \), respectively, where \( m = -j, -j + 1, \cdots, j - 1, j \). For the above system, we define the following operators:

\[
\tilde{S}_+ = \sum_m \tilde{\gamma}_p^m \tilde{\gamma}_h^m, \quad \tilde{S}_- = \sum_m \tilde{\gamma}_h^m \tilde{\gamma}_p^m,
\]

\[
\tilde{S}_0 = \frac{1}{2} \sum_m (\tilde{\gamma}_p^m \tilde{\gamma}_h^m - \tilde{\gamma}_h^m \tilde{\gamma}_p^m) .
\]

They obey the \( su(2) \)-algebra (2.4). Further, we introduce the total fermion number operator \( \tilde{N} \), which is given as

\[
\tilde{N} = \sum_m (\tilde{\gamma}_p^m \tilde{\gamma}_p^m + \tilde{\gamma}_h^m \tilde{\gamma}_h^m) .
\]

In the pairing and the isoscalar pairing model, \( \tilde{N} \) is expressed as \( \tilde{N} = 2\tilde{S}_0 + 2\Omega \), in which \( \tilde{N} \) and \( \tilde{S}_0 \) depend on each other and we have \([ \tilde{N}, \tilde{S}_\pm ] = \pm 2\tilde{S}_\pm (\neq 0) \). But, in the Lipkin model, \( \tilde{N} \) is independent of \( \tilde{S}_0 \). It can be seen in the relations (5.1) and (5.2) and we have \([ \tilde{N}, \tilde{S}_\pm ] = 0 \). These points distinguish the Lipkin model from the other two models.

We introduce the particle and the hole operators \( \tilde{c}_p^m, \tilde{c}_p^* \) and \( \tilde{c}_h^m, \tilde{c}_h^* \) which are related to

\[
\tilde{\gamma}_p^m = \tilde{c}_p^m , \quad \tilde{\gamma}_h^m = (-)^{j-m} \tilde{c}_h^* .
\]

Then, \( \tilde{S}_{\pm,0} \) and \( \tilde{N} \) can be rewritten as

\[
\tilde{S}_+ = \sum_m (-)^{j-m} \tilde{c}_p^m \tilde{c}_h^* , \quad \tilde{S}_- = \sum_m (-)^{j-m} \tilde{c}_h^m \tilde{c}_p^* ,
\]

\[
\tilde{S}_0 = \frac{1}{2} \sum_m (\tilde{c}_p^m \tilde{c}_p^m + \tilde{c}_h^m \tilde{c}_h^m) - \Omega ,
\]

\[
\tilde{N} = \sum_m (\tilde{c}_p^m \tilde{c}_p^m - \tilde{c}_h^m \tilde{c}_h^m) + 2\Omega .
\]

The Hamiltonian in the Lipkin model is expressed in terms of the above \( \tilde{S}_{\pm,0} \), where \( \tilde{S}_\pm \) are expressed in the form of the particle-hole pairs. Comparison between the relations (4.3) and (5.4) is interesting. If the index \( n \) and the factor \( s_m(= +1) \) in the relation (4.3) read the index \( h \) and the factor \((-)^{j-m}\), respectively, the relation (4.3) becomes identical with the relation (5.4). Therefore, we can apply the two
possibilities (4.5) and (4.6) for defining \( \tilde{R}_{\pm,0} \) to the present case:

1. For \( m > 0, \quad p, m \rightarrow \alpha, \quad p, \overline{m} \rightarrow \overline{\alpha}, \quad \text{and} \quad h, m \rightarrow \alpha, \quad h, \overline{m} \rightarrow \overline{\alpha}, \) \((-)^{j-m} \rightarrow s_{\alpha}, \quad \Omega \rightarrow \Omega_0 \). \hspace{1cm} (5.6)

2. For all \( m, p, m \rightarrow \alpha, \quad h, m \rightarrow \alpha, \) \((-)^{j-m} \rightarrow s_{\alpha}, \quad \Omega \rightarrow \Omega_0 \). \hspace{1cm} (5.7)

For the possibility (1), \( \tilde{R}_{\pm,0} \) can be expressed as

\[
\tilde{R}_+ = \sum_{m>0} (\tilde{c}^*_p \tilde{c}_m + \tilde{c}^*_h \tilde{c}_{\overline{m}}), \quad \tilde{R}_- = \sum_{m>0} (\tilde{c}^*_p \tilde{c}_m + \tilde{c}^*_h \tilde{c}_{\overline{m}}),
\]
\[
\tilde{R}_0 = \frac{1}{2} \sum_{m>0} ((\tilde{c}^*_p \tilde{c}_m + \tilde{c}^*_h \tilde{c}_{\overline{m}}) - (\tilde{c}^*_p \tilde{c}_m + \tilde{c}^*_h \tilde{c}_{\overline{m}})). \hspace{1cm} (5.8)
\]

For the possibility (2), \( \tilde{R}_{\pm,0} \) can be expressed as

\[
\tilde{R}_+ = \sum_m \tilde{c}^*_p \tilde{c}_m, \quad \tilde{R}_- = \sum_m \tilde{c}^*_h \tilde{c}_{\overline{m}},
\]
\[
\tilde{R}_0 = \frac{1}{2} \sum_m (\tilde{c}^*_p \tilde{c}_m - \tilde{c}^*_h \tilde{c}_{\overline{m}}). \hspace{1cm} (5.9)
\]

Different from the case of the isoscalar pairing model, we can prove that the relation (5.8) and (5.9) satisfy the relation (2.7). As will be discussed fully in §7, the \( su(2) \)-algebra for the Lipkin model shown in the relation (5.4) is a sub-algebra of the \( so(5) \)-algebra, the typical example of which is the isovector pairing model. In the \( so(5) \)-algebra for the isovector pairing model, only the possibility (1) is available. Therefore, the possibility (1) in the Lipkin model may be better to discuss in relation to the \( so(5) \)-algebra and from the above-mentioned reason, in this section, we adopt the possibility (2) and the configuration \((h\overline{m}) = h\overline{m}_1, \ h\overline{m}_2, \ldots, \ h\overline{m}_{2r}\) is used. We have already shown that the Lipkin model is analogous to the isoscalar pairing model except the treatment of the total fermion number, which is expressed in the form

\[
\tilde{N} = 2\tilde{R}_0 + 2\Omega. \hspace{1cm} (5.10)
\]

In the isoscalar pairing model, \( \tilde{N} \) is given as

\[
\tilde{N} = 2\tilde{S}_0 + 2\Omega. \hspace{1cm} (5.11)
\]

In §2, we have already mentioned that there exist two forms for expressing \( \tilde{N} \) in the relations (2.20a) and (2.20b). Certainly, in the Lipkin model, we have the relation (5.10) which has been notified in the relation (2.20b). Increases of the total fermion numbers are carried out in terms of the successive operations of \( \tilde{R}_+ \) and \( \tilde{S}_+ \), respectively.

Conventionally, the case of the closed-shell system has been treated in the Lipkin model. It corresponds to the case where the \( h \)-level is fully occupied in the ground state, if the interaction is switched off. As the interest of physics, it may be acceptable because originally this model was proposed with the aim of the schematic
understanding of collective motion induced by the particle-hole pairs. In the framework developed in §2, we will discuss the problem of how to generalize the above-mentioned situation. Of course, it may be based on the theoretical interest. First, we note the relations

\[ 2\tilde{S}_0 = (\tilde{N}_p + \tilde{N}_h) - 2\Omega \]  
\[ 2\tilde{R}_0 = \tilde{N}_p - \tilde{N}_h \]  
i.e., \[ \tilde{N} = (\tilde{N}_p - \tilde{N}_h) + 2\Omega \]  

Here, \( \tilde{N}_p \) and \( \tilde{N}_h \) denote

\[ \tilde{N}_p = \sum_m \tilde{c}^*_p c_p \]  
\[ \tilde{N}_h = \sum_m \tilde{c}^*_h c_h \]  

The minimum weight state of the \( S \)-spin, \( |rr_0; (h\bar{m})\rangle \), is the eigenstate of \( \tilde{S}_0 \) and \( \tilde{R}_0 \) with the eigenvalues \(-s\) and \( r_0\), respectively. Therefore, \( |rr_0; (h\bar{m})\rangle \) is the eigenstate of \( \tilde{N}_p \) and \( \tilde{N}_h \), the eigenvalues of which are denoted as \( n_p \) and \( n_h \), respectively. The relation (5.12) gives as

\[ 2s = 2\Omega - (n_p + n_h) \]  
\[ N = 2\Omega + (n_p - n_h) \]  

The relation (5.14) leads us to

\[ 2s = 4\Omega - N - 2n_h = N - 2n_p \]  

Since \( \Omega, n_p \) and \( n_h \) are positive integers and \( s \) should be positive, the relation (5.15) gives us the following:

(1) \( N \) : even,

\[ 2s = \begin{cases} 
0, 2, \cdots, N, & (0 \leq N \leq 2\Omega - 2) \\
0, 2, \cdots, 2\Omega, & (N = 2\Omega) \\
0, 2, \cdots, 4\Omega - N, & (2\Omega + 2 \leq N \leq 4\Omega)
\end{cases} \]  

(2) \( N \) : odd,

\[ 2s = \begin{cases} 
1, 3, \cdots, N, & (1 \leq N \leq 2\Omega - 1) \\
1, 3, \cdots, 4\Omega - N, & (2\Omega + 1 \leq N \leq 4\Omega - 1)
\end{cases} \]  

The above corresponds to the relation (4.13). On the other hand, with the use of the relations (5.14b) and \( 2s \geq 0 \) in the relation (5.15), we derive the result

\[ 0 \leq n_h \leq 2\Omega - \frac{N}{2}, \quad 0 \leq n_p \leq \frac{N}{2} \]  
if \( 0 \leq N < 2\Omega \), \( n_p < n_h \),

if \( N = 2\Omega \), \( n_p = n_h \),

if \( 2\Omega < N \leq 4\Omega \), \( n_p > n_h \).  

It may be instructive to draw the relations (5.17) and (5.18) in figure. The closed areas depicted by marks of “hat (\( \hat{\} \))” (Figs. 1(a) and (c)) and the thick line (Fig. 1(b)) satisfy the relations (5.17) and (5.18).
On the basis of the above argument, we will discuss several points concretely. As was already mentioned, conventionally, the Lipkin model has been treated in the case $N = 2\Omega$ and if the interaction is switched off, the $h$-level is fully occupied in the ground state. This suggests to us the relation $n_p = n_h = 0$ and the relation (5.14) gives us $2s = 2\Omega$, that is, $2s = 2\Omega = N$. In relation to this case, the following cases may be interesting: Even if $n_h \neq 0$, the case $n_p = 0$ leads us to $2s = N \left( N = 2\Omega - n_h < 2\Omega \right)$. Further, the case $(n_p \neq 0, n_h = 0)$ gives us $2s = 4\Omega - N \left( N = 2\Omega + n_p > 2\Omega \right)$. These cases indicate that $2s$ is expressed only in terms of $\Omega$ and $N$. However, in other cases, $2s$ is not so simple as that in the above cases. For example, if $N = 2\Omega$, we have $n_p = n_h (= n_0)$ and $2s = N - 2n_0$. The above argument may be helpful for specifying $2s$ for the Holstein-Primakoff boson realization. For this task, the relations (5.14) and (5.15) are useful.

In a manner similar to the case of the isoscalar pairing model, we can express the counterpart of $|s(= \Omega - r) s_0, rr_0; (h\bar{m})\rangle$ in the Schwinger boson representation in the following form:

$$|\Omega N; ss_0\rangle = (\hat{a}^*)^{s+s_0} (\hat{b}^*)^{s-s_0} (\hat{d}_p^*)^{\frac{1}{2}(N-2s)} (\hat{d}_F^*)^{\frac{1}{2}(4\Omega-N-2s)}|0\rangle .$$  \hspace{1cm} (5.19)

The state (5.19) gives us the relations $s \geq 0$, $N - 2s \geq 0$ and $(4\Omega - N) - 2s \geq 0$, which are reduced to the relations (5.17) and (5.18). Here, we used the relation (5.14). It may be interesting to see that if $s$ is replaced with $r$, the relation (5.16) becomes the relation (4.13). The state (2.25) in the previous cases is rather different from the present case. It may be expressed in the form

$$|\Omega\sigma, rN\rangle = (\hat{A}^*)^\sigma (\hat{d}_p^*)^{r (N-2\Omega)} (\hat{d}_F^*)^{r - \frac{1}{2}(N-2\Omega)}|0\rangle ,$$

$$\sigma = 0, 1, 2, \cdots, 2(\Omega - r) .$$ \hspace{1cm} (5.20)

The total fermion number $N$ is contained in the part of the minimum weight state.

§6. Boson-quasifermion realization

As a supplementary argument, we consider the boson-fermion realization. In §2, we presented a possible boson realization of the $su(2)$-algebraic models for many-
fermion system. As can be seen in the relations (2.14a) and (2.25), at the first step, we express the minimum weight states in terms of the bosons $\tilde{d}_\rho, \tilde{d}_\rho^*$ and, at the second step, the orthogonal set constructed on each minimum weight state is described in terms of the boson $\tilde{A}^\ast$. This point may be interesting, because the two steps are carried out independently or separately from each other. The use of $(\tilde{d}_\rho, \tilde{d}_\rho^*, \tilde{d}_\rho^*, \tilde{d}_\rho)$ enables us to specify the minimum weight states in terms of $(r, r_0)$. In the conventional treatment for the $su(2)$-algebraic models, only the $S$-spin is the object of the investigation and its Holstein-Primakoff representation is obtained in the relation (2.22) by replacing $\tilde{S}$ with the magnitude of the $S$-spin, $s$. But, the task to connect $s$ to the seniority number is performed by each device for each model. In our case, without any device, the use of the $R$-spin gives us the relation $s = \Omega_0 - r$. Further, the $R$-spin orders us to use $r_0$ which may be regarded as one of the quantum numbers specifying the minimum weight states. The conventional treatment does not contain $r_0$. In this sense, compared with the conventional one, ours gives us somewhat detailed, but interesting information on the minimum weight states.

In order to get more detailed information, it may be desirable to specify the minimum weight states in terms of the quantum numbers $\alpha, \bar{\alpha}$, etc. under the two step scheme. For this aim, the following two treatments are instructive: 1) the boson-quasifermion mapping for the pairing model and 2) the quantization of the Dirac bracket appearing in the canonical form of the extended TDHF method including the Grassmann variables for the pairing and the Lipkin model. The former has been presented by Suzuki and Matsuyanagi\(^{10}\) and later by Hasegawa and Kanesaki\(^{11}\) and the latter by Kuriyama and one of the present authors (M.Y.).\(^{12}\) With the aim of completing the above-mentioned scheme, we introduce the operators $(\tilde{b}_\alpha, \tilde{b}_\alpha^\ast)$ and $(\tilde{b}_{\bar{\alpha}}, \tilde{b}_{\bar{\alpha}}^\ast)$ governed by the following conditions:

\begin{equation}
\{ \tilde{b}_\alpha, \tilde{b}_\beta^\ast \} = \delta_{\alpha\beta} - s_\alpha \tilde{b}_\alpha (2\tilde{S})^{-1} s_\beta \tilde{b}_\beta^\ast ,
\end{equation}

\begin{equation}
\{ s_\alpha \tilde{b}_{\bar{\alpha}}, s_\beta \tilde{b}_{\bar{\alpha}}^\ast \} = \delta_{\alpha\beta} - \tilde{b}_\alpha^\ast (2\tilde{S})^{-1} \tilde{b}_\beta^\ast ,
\end{equation}

\begin{equation}
\{ \tilde{b}_\alpha, s_\beta \tilde{b}_\beta^\ast \} = s_\alpha \tilde{b}_\alpha (2\tilde{S})^{-1} \tilde{b}_\beta^\ast ,
\end{equation}

\begin{equation}
\{ s_\alpha \tilde{b}_{\bar{\alpha}}, \tilde{b}_\beta^\ast \} = \tilde{b}_\alpha^\ast (2\tilde{S})^{-1} s_\beta \tilde{b}_{\bar{\alpha}}^\ast ,
\end{equation}

\begin{equation}
\{ \tilde{b}_\alpha, \tilde{b}_\beta \} = \{ \tilde{b}_{\bar{\alpha}}, \tilde{b}_{\bar{\alpha}}^\ast \} = 0 .
\end{equation}

Here, $\{ \tilde{A}^\ast, \tilde{B} \}$ denotes anti-commutator for $\tilde{A}$ and $\tilde{B}$. The operator $\tilde{S}$ is defined as

\begin{equation}
\tilde{S} = \Omega_0 - \frac{1}{2} \sum_\alpha (\tilde{b}_\alpha^\ast \tilde{b}_\alpha + \tilde{b}_{\bar{\alpha}}^\ast \tilde{b}_{\bar{\alpha}}) .
\end{equation}

The above anti-commutation relations are closely related to the constraints appearing in the canonical form of the constraint system presented by Dirac, that is, the Dirac brackets.\(^{13}\)

Next, we define $\tilde{\chi}_{\pm}$ in the following bi-linear form:

\begin{equation}
\tilde{\chi}_+ = \sum_\alpha s_\alpha \tilde{b}_\alpha^\ast \tilde{b}_{\bar{\alpha}}^\ast , \quad \tilde{\chi}_- = \sum_\alpha s_\alpha \tilde{b}_\alpha \tilde{b}_{\bar{\alpha}} .
\end{equation}
With the use of the relations (6.1)–(6.3), we can prove the relation
\[
\tilde{b}_\alpha \tilde{\chi}_+ |\bar{0}\rangle = s_\alpha \tilde{b}_\alpha \tilde{\chi}_+ |\bar{0}\rangle = 0 \quad \text{and} \quad (\tilde{b}_\alpha |\bar{0}\rangle = s_\alpha \tilde{b}_\alpha |\bar{0}\rangle = 0) \quad (6.6)
\]
The relation (6.6) leads us to
\[
(\tilde{0}|\tilde{\chi}_- \cdot \tilde{\chi}_+ |\bar{0}\rangle = 0 \quad \text{i.e.,} \quad \tilde{\chi}_+ |\bar{0}\rangle = 0 \quad (6.7)
\]
Therefore, in the space spanned by \((\tilde{b}_\alpha^*, \tilde{b}_\alpha^*\rangle\), any state which contains \(\tilde{\chi}_+\) vanishes. It indicates that \(\tilde{\chi}_+\) play a role of the constraints of the above-mentioned Dirac’s formalism. The relation (6.3) tells us that all states constructed by \((\tilde{b}_\alpha, \tilde{b}_\alpha^*)\) are anti-symmetric with respect to the quantum numbers specifying the single-particle states, i.e., fermion type. The relations (6.1) and (6.2) determine the normalization. In this sense, \(\tilde{b}_\alpha, \tilde{b}_\alpha^*, \tilde{b}_\sigma^*\) and \(\tilde{b}_\alpha^*\) can be called the quasi-fermion operators. However, the relation (6.7) suggests to us that, compared with the original fermion system presented by \((\tilde{c}_\alpha^*, \tilde{c}_\beta^*)\), one degree of freedom is reduced in the system given by \((\tilde{b}_\alpha^*, \tilde{b}_\alpha^*)\). In order to cancel this discrepancy, we introduce a new degree of freedom in terms of boson \((\hat{A}, \hat{A}^*)\) satisfying
\[
[ \hat{A}, \hat{A}^* ] = 1 , \quad [ \tilde{b}_\alpha, \hat{A}^* ] = [ \tilde{b}_\sigma, \hat{A}^* ] = [ \tilde{b}_\alpha, \tilde{b}_\sigma ] = [ \hat{A}, \tilde{b}_\alpha ] = [ \tilde{b}_\alpha, \tilde{b}_\sigma ] = 0 . \quad (6.8)
\]
With the use of \(\tilde{b}_\alpha, s_\alpha \tilde{b}_\alpha^*, \tilde{b}_\alpha^*, s_\alpha \tilde{b}_\alpha^*\), \(\hat{A}\) and \(\hat{A}^*\), we define the operators \(\tilde{c}_\alpha^*\) and \(s_\alpha \tilde{c}_\alpha^*\) in the form
\[
\tilde{c}_\alpha^* = \sqrt{1 - \frac{\hat{A}^* \hat{A}}{2s}} \tilde{b}_\alpha + s_\alpha \tilde{b}_\alpha^* \frac{\hat{A}}{\sqrt{2s}} ,
\]
\[
s_\alpha \tilde{c}_\alpha^* = -\tilde{b}_\alpha^* \frac{\hat{A}}{\sqrt{2s}} + \sqrt{1 - \frac{\hat{A}^* \hat{A}}{2s}} s_\alpha \tilde{b}_\alpha^* . \quad (6.9)
\]
We can prove the relation
\[
\{ \tilde{c}_\alpha^* , \tilde{c}_\beta^* \} = \{ s_\alpha \tilde{c}_\alpha^* , s_\beta \tilde{c}_\beta^* \} = \delta_{\alpha\beta} , \\
\{ \tilde{c}_\alpha^* , s_\beta \tilde{c}_\beta^* \} = \{ s_\alpha \tilde{c}_\alpha^* , \tilde{c}_\beta^* \} = 0 , \\
\{ \tilde{c}_\alpha^* , \tilde{c}_\beta \} = \{ \tilde{c}_\alpha^* , \tilde{c}_\beta^* \} = \{ \tilde{c}_\alpha^* , \tilde{c}_\beta^* \} = 0 . \quad (6.10)
\]
Here, (and hereafter), we omit the terms related to \(\tilde{\chi}_\pm\). The relation (6.10) gives us the following identity:
\[
\tilde{c}_\alpha = \tilde{c}_\alpha^* , \quad s_\alpha \tilde{c}_\alpha^* = s_\alpha \tilde{c}_\alpha^* . \quad (6.11)
\]
The above indicates that the original fermion operators \(\tilde{c}_\alpha, \tilde{c}_\alpha^*, s_\alpha \tilde{c}_\alpha^*\) and \(s_\alpha \tilde{c}_\alpha^*\) are connected with \(\tilde{b}_\alpha, \tilde{b}_\alpha^*, s_\alpha \tilde{b}_\alpha^*, s_\alpha \tilde{b}_\alpha^*\), \(\hat{A}\) and \(\hat{A}^*\) through the relations (6.9) and (6.11). If \(\hat{A}/\sqrt{2s}\) and \(\hat{A}^*/\sqrt{2s}\) can be regarded as c-numbers, the relation (6.9) ((6.11)) is reduced to the Bogoliubov transformation and \(\tilde{b}_\alpha\) etc. becomes the quasi-particle operators.
With the use of the relations (6.9) and (6.11), we have the following relation:

\[
\hat{S}_+ = \hat{A}^* \cdot \sqrt{2 \tilde{S} - \hat{A}^* \hat{A}} , \quad \hat{S}_- = \sqrt{2 \tilde{S} - \hat{A}^* \hat{A}} \cdot \hat{A} ,
\]

\[
\hat{S}_0 = \hat{A}^* \hat{A} - \tilde{S} .
\]  

(6.12)

Here, \( \tilde{S} \) is defined in the relation (6.4). With the use of the relation (6.12), \( \hat{S}^2 \) can be calculated as

\[
\hat{S}^2 = \tilde{S}(\tilde{S} + 1) .
\]  

(6.13)

Therefore, \( \tilde{S} \) indicates the operator for the magnitude of the \( S \)-spin. On the other hand, \( \tilde{R}_{\pm,0} \) can be expressed as

\[
\tilde{R}_+ = \sum_\alpha \tilde{b}_\alpha^* \tilde{b}_\alpha , \quad \tilde{R}_- = \sum_\alpha \tilde{b}_\alpha^* \tilde{b}_\alpha ,
\]

\[
\tilde{R}_0 = \frac{1}{2} \sum_\alpha (\tilde{b}_\alpha^* \tilde{b}_\alpha - \tilde{b}_\alpha \tilde{b}_{\alpha}^*) .
\]  

(6.14)

It may be interesting, but natural that the seniority algebra can be expressed in terms of the quasi-fermions. It does not depend on \( (\hat{A}, \hat{A}^*) \). Further, for the magnitude of the \( R \)-spin, we have

\[
\tilde{R} = \Omega_0 - \tilde{S} , \quad \tilde{R} = \frac{1}{2} \sum_\alpha (\tilde{b}_\alpha^* \tilde{b}_\alpha + \tilde{b}_\alpha \tilde{b}_{\alpha}^*) .
\]  

(6.15)

On the other hand, \( \hat{S}_{\pm,0} \) are expressed in terms of \( \hat{A} \) and \( \hat{A}^* \) and through \( \tilde{S} = \Omega_0 + \tilde{R} \), they depend on \( \tilde{b}_\alpha^* \) etc. It is in the same situation as that in the case of the boson realization. The comparison of the forms (6.14) and (6.15) with the relations (2.14a) and (2.23) leads us to the following:

\[
\hat{d}_p \hat{d}_p^* \sim \sum_\alpha \tilde{b}_\alpha^* \tilde{b}_\alpha , \quad \hat{d}_p^* \hat{d}_p \sim \sum_\alpha \tilde{b}_\alpha \tilde{b}_{\alpha}^* ,
\]

\[
\frac{1}{2} (\hat{d}_p^* \hat{d}_p - \hat{d}_p \hat{d}_p^*) \sim \frac{1}{2} \sum_\alpha (\tilde{b}_\alpha^* \tilde{b}_\alpha - \tilde{b}_\alpha \tilde{b}_{\alpha}^*) .
\]  

(6.16a)

Further, we have

\[
\frac{1}{2} (\hat{d}_p^* \hat{d}_p + \hat{d}_p \hat{d}_p^*) \sim \frac{1}{2} \sum_\alpha (\tilde{b}_\alpha^* \tilde{b}_\alpha + \tilde{b}_\alpha \tilde{b}_{\alpha}^*) .
\]  

(6.16b)

The relation (6.16) suggests to us that the use of the quasi-fermions permits us to complete our aim mentioned in the introductory part of this section. At the ending of this section, we add a small remark: In order to treat fermions in the frame of classical mechanics, Casalbuoni introduced the Grassmann variables,\(^{14}\) which are quantized in Ref. 12).
§7. The Lipkin model — Part II

In this section, we consider some features different from those in §5. Following the promise in §5, we will discuss the Lipkin model in relation to the isovector pairing model, which is formulated in terms of the so(5)-algebra. This model consists of ten generators, in which the fermion-pair type generators are as follows:

\[ \tilde{Q}^*_+ = \sum_m (-)^{m}\tilde{c}_{pm}^*\tilde{c}_{pm}^\dagger, \quad \tilde{Q}^*_0 = \sum_m (-)^{m}\tilde{c}_{nm}\tilde{c}_{nm}^\dagger, \quad \tilde{Q}^*_0 = \sum_m (-)^{m}\tilde{c}_{nm}\tilde{c}_{nm}^\dagger, \quad \tilde{Q}^*_0 = \sum_m (-)^{m}\tilde{c}_{nm}\tilde{c}_{nm}^\dagger. \] (7.1a)

\[ \tilde{Q}^- = \sum_m (-)^{m}\tilde{c}_{nm}\tilde{c}_{nm}^\dagger, \quad \tilde{Q}^- = \sum_m (-)^{m}\tilde{c}_{nm}\tilde{c}_{nm}^\dagger. \] (7.1b)

Other four generators have already appeared in the discussion on the isoscalar pairing model as the relations (4.1) and (4.3):

\[ \tilde{\tau}^+_0 = \sum_m \tilde{c}_{pm}\tilde{c}_{pm}^\dagger, \quad \tilde{\tau}^- = \sum_m \tilde{c}_{pm}\tilde{c}_{pm}^\dagger, \quad \tilde{\tau}^+_0 = \frac{1}{2} \sum_m (\tilde{c}_{pm}\tilde{c}_{pm} - \tilde{c}_{nm}\tilde{c}_{nm}), \quad \tilde{\tau}^- = \frac{1}{2} \sum_m (\tilde{c}_{pm}\tilde{c}_{pm} + \tilde{c}_{nm}\tilde{c}_{nm}) - \Omega. \] (7.2)

The sets \((\tilde{Q}^\pm, 0)\) and \((\tilde{Q}^\pm, 0)\) form the isovectors with respect to the isospin \((\tilde{\tau}^\pm, 0)\). The Casimir operator of the so(5)-algebra, \(\hat{\Gamma}_{so(5)}\), is expressed as

\[ \hat{\Gamma}_{so(5)} = \frac{1}{2} (\tilde{Q}^+_0\tilde{Q}^+_0 + \tilde{Q}^-_0\tilde{Q}^-_0) + \tilde{Q}^+_0\tilde{Q}^-_0 + \tilde{\tau}^+_0\tilde{\tau}^-_0 + \tilde{\sigma}(\tilde{\sigma} - 3) + \tilde{\tau}^+_0(\tilde{\tau}^-_0 - 1) \]

\[ = \frac{1}{2} (\tilde{Q}^+_0\tilde{Q}^+_0 + \tilde{Q}^-_0\tilde{Q}^-_0) + \tilde{Q}^+_0\tilde{Q}^-_0 + \tilde{\tau}^+_0\tilde{\tau}^-_0 + \tilde{\sigma}(\tilde{\sigma} - 3) + \tilde{\tau}^+_0(\tilde{\tau}^-_0 + 1). \] (7.4)

The su(2)-generators which commute with the above ten generators are as follows:

\[ \tilde{R}^\pm,0 \equiv \tilde{R}^\pm_0(p) + \tilde{R}^\pm_0(n), \] (7.5)

\[ \tilde{R}^p_0 = \sum_{m>0} \tilde{c}_{pm}\tilde{c}_{pm}^\dagger, \quad \tilde{R}^-_0 = \sum_{m>0} \tilde{c}_{pm}\tilde{c}_{pm}^\dagger, \] (7.6a)

\[ \tilde{R}^+_0 = \sum_{m>0} \tilde{c}_{nm}\tilde{c}_{nm}^\dagger, \quad \tilde{R}^-_0 = \sum_{m>0} \tilde{c}_{nm}\tilde{c}_{nm}^\dagger, \] (7.6b)

The above su(2)-generators are copied from the relation (4.7).
We can see that the set \((\tilde{Q}_0^+, \tilde{Q}_0^-, \tilde{\Sigma})\) forms the \(su(2)\)-algebra and under the following reading, this set is reduced to the Lipkin model shown in the relations (5.4) and (5.5):

\[
n \rightarrow h, \quad \tilde{Q}_0^+ \rightarrow \tilde{S}_+, \quad \tilde{Q}_0^- \rightarrow \tilde{S}_-, \quad \tilde{\Sigma} \rightarrow \tilde{S}_0, \quad \tilde{\tau}_0 \rightarrow \frac{\tilde{N}}{2} - \Omega.
\]  

Therefore, it is possible to formulate the Lipkin model as the sub-algebra of the \(so(5)\)-algebra which describes the isovector pairing model.

Let us search the minimum weight state for the \(so(5)\)-algebra. For this aim, we set up the relations

\[
\begin{align}
\tilde{Q}_\pm|0\rangle &= 0, \\
\tilde{R}_-|0\rangle &= 0, \\
\tilde{\Sigma}|0\rangle &= -s|0\rangle, \\
\tilde{R}_0|0\rangle &= -\tau|0\rangle.
\end{align}
\]

Concerning the isospin, we treat two cases separately:

\[
\begin{align}
\text{case (1)} &; \quad \tilde{\tau}_-|0\rangle = 0, & \quad \tilde{\tau}_0|0\rangle &= -\tau|0\rangle, \quad (7.10a) \\
\text{case (2)} &; \quad \tilde{\tau}_+|0\rangle = 0, & \quad \tilde{\tau}_0|0\rangle &= +\tau|0\rangle. \quad (7.10b)
\end{align}
\]

With the use of the expressions (7.1)–(7.3) and (7.5) with (7.6), we obtain the following form for \(|0\rangle\):

\[
|m_0\rangle = \left\{ \begin{array}{l}
|r, \tau; (pn\bar{\mu}, n\bar{m})\rangle = \left( \prod_{j=1, \mu_j > 0}^{r-\tau} \tilde{c}_{n\bar{m}_j}^* \tilde{c}_{n\bar{m}_j} \right) \left( \prod_{i=1, m_i > 0}^{2\tau} \tilde{c}_{m_i}^* \right) |0\rangle, \\
|r, \tau; (n\bar{p}\mu, p\bar{m})\rangle = \left( \prod_{j=1, \mu_j > 0}^{r-\tau} \tilde{c}_{p\bar{m}_j}^* \tilde{c}_{p\bar{m}_j} \right) \left( \prod_{i=1, m_i > 0}^{2\tau} \tilde{c}_{m_i}^* \right) |0\rangle.
\end{array} \right.
\]

(7.11)

Here, the upper and the lower form in the relation (7.11) are obtained for the cases (1) and (2), respectively. The symbol \((pn\bar{\mu}, n\bar{m})\) denotes the configuration \(p\bar{m}_1, p\bar{m}_2, \cdots, p\bar{m}_{r-\tau}, n\bar{m}_1, n\bar{m}_2, \cdots, n\bar{m}_{2\tau}\) and \((n\bar{p}\mu, p\bar{m})\) is given by exchanging \(p\) and \(n\) in \((pn\bar{\mu}, n\bar{m})\). Further, we have

\[
r + s = \Omega, \quad \tau \leq r, \quad \text{the eigenvalue of } \tilde{F}_{so(5)} = s(s + 3) + \tau(\tau + 1). \quad (7.12)
\]

We can learn that \(2r\) and \(\tau\) indicate the seniority number and the reduced isospin which characterize the \(so(5)\)-algebra. Therefore, by operating \((\tilde{R}_+)^{r+\tau_0}\) on the state (7.11), the minimum weight state of the \(so(5)\)-algebra is obtained:

\[
\begin{align}
|rr_0, \tau; (pn\bar{\mu}, n\bar{m})\rangle &= (\tilde{R}_+)^{r+\tau_0}|r, \tau; (pn\bar{\mu}, n\bar{m})\rangle \quad \text{for the case (1)}, \quad (7.14a) \\
|rr_0, \tau; (n\bar{p}\mu, p\bar{m})\rangle &= (\tilde{R}_+)^{r+\tau_0}|r, \tau; (n\bar{p}\mu, p\bar{m})\rangle \quad \text{for the case (2)}. \quad (7.14b)
\end{align}
\]

The orthogonal set for the \(so(5)\)-algebra is constructed by operating \(\tilde{Q}_\pm_{0, \pm}\) and \(\tilde{\tau}_\pm\) (for the state (7.14a)) and \(\tilde{\tau}_-\) (for the state (7.14b)) on the state (7.14). Totally, it is
specified by six quantum numbers except the extra quantum numbers specifying the minimum weight state, for example, such as \( r_0 \). In this paper, we omit the concrete procedure for this construction, because we are interested in the sub-algebra, i.e., the \( su(2) \)-algebra.

On the basis of the above results on the \( so(5) \)-algebra, we will discuss the Lipkin model under the reading (7.7). It may be self-evident that the states (7.14a) and (7.14b) are the minimum weight states for the algebra \( \tilde{S}_{\pm,0} \). The eigenvalues of \( \tilde{S}_0 \) and \( \tilde{R}_0 \) are given by \(-s\) and \(-r\), respectively, which are related to each other under the relation (7.12) and, then, the physical meanings are the same as those given in §5. The states (7.14a) and (7.14b) are also the eigenstates of \( \tilde{\tau}_0 \) with the eigenvalues \(-\tau\) and \(+\tau\), respectively. As is shown in the relation (7.7), total fermion number operator \( \tilde{N} \) is expressed as \( \tilde{N} = 2\Omega + 2\tilde{\tau}_0 \). Therefore, for the states (7.14a) and (7.14b), the fermion numbers \( N \) are given by \( N = 2\Omega - 2\tau \) and \( N = 2\Omega + 2\tau \), respectively. This implies that \( \tilde{\tau} \) plays the same role as that of \( \tilde{R}_0 \) in §5. Then, the role of \( (\tilde{R}_{\pm,0}) \) in the present case may be interesting.

The states (7.14a) and (7.14b) \( (r_0 = -r) \) are expressed in terms of the operators \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\), where \( \overline{m} < 0 \), i.e., \( m > 0 \). By operating \( \tilde{R}_+ \) successively, these states change their structures and they are expressed not only by \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\) but also \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\). Finally, at \( r_0 = r \), the minimum weight states are expressed only in terms of \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\). Therefore, for example, at \( r_0 = 0 \) which appears in the case \( r = \) even, the state (7.14) contains \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\) and \((\tilde{c}_{pm}^*, \tilde{c}_{nm}^*)\) in equal weight. In the case \( r = \) odd, the situation similar to the case \( r = \) even appears at \( r_0 = \pm 1/2 \). We observe these features in the pairing model given in §3. The minimum weight state in §5 does not contain such a distinction. In this sense, the minimum weight states in §§5 and 7 may be equivalent to each other, but, the state in §7 contains the information other than the state in §5.

The idea discussed in the above suggests to us that we formulate the pairing model in §3 in the present framework. We note that the operators \((\tilde{Q}_+, \tilde{Q}_-, (\tilde{\Sigma} - \tilde{\tau}_0 - \Omega)/2)\) form the \( su(2) \)-algebra. If these operators read \((\tilde{S}_+, \tilde{S}_-, \tilde{S}_0)\), respectively, and the \( p \)-level is vacant, it is reduced to the pairing model in §3. If \( \tau \) is equal to \( r \) in the state given in the upper of (7.11), the \( p \)-level becomes vacant and this case corresponds to the minimum weight state of the pairing model.

Now, with the aid of the reading (7.7), we consider the boson realization of the Lipkin model based on the isovector pairing model. First, we introduce the counterparts of \( \tilde{R}_{\pm,0} \) and \( \tilde{S}_{\pm,0} \) in the boson space. The counterpart of \( \tilde{R}_{\pm,0} \) shown in the relation (7.5) is given in the Schwinger boson representation:

\[
\hat{R}_{\pm,0} = \hat{R}_{\pm,0}(p) + \hat{R}_{\pm,0}(h) ,
\]

\[
\hat{R}_+(p) = \hat{d}_p^*(p)\hat{d}_p(p) , \quad \hat{R}_-(p) = \hat{d}_p^*(p)\hat{d}_p(p) ,
\]

\[
\hat{R}_0(p) = \frac{1}{2}(\hat{d}_p^*(p)\hat{d}_p(p) - \hat{d}_p^*(p)\hat{d}_p(p)) ,
\]

\[
\hat{R}_+(h) = \hat{d}_h^*(h)\hat{d}_h(h) , \quad \hat{R}_-(h) = \hat{d}_h^*(h)\hat{d}_h(h) ,
\]

\[
\hat{R}_0(h) = \frac{1}{2}(\hat{d}_h^*(h)\hat{d}_h(h) - \hat{d}_h^*(h)\hat{d}_h(h)) .
\]
The magnitudes of the $R_p$- and the $R_h$-spin are given by

$$
\hat{R}(p) = \frac{1}{2}(\hat{d}_p(p)\hat{d}_p(p) + \hat{d}_\tau(p)\hat{d}_\tau(p)) \quad \text{and} \quad \hat{R}(h) = \frac{1}{2}(\hat{d}_p^*(h)\hat{d}_p(h) + \hat{d}_\tau^*(h)\hat{d}_\tau(h)) .
$$

(7.16c)

The case of the $S$-spin is expressed as

$$
\hat{S}_+ = \hat{a}^* \hat{a}^* \quad \text{and} \quad \hat{S}_- = \hat{b}^* \hat{b}^* \quad \text{and} \quad \hat{S}_0 = \frac{1}{2}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) .
$$

(7.17a)

The magnitude of the $S$-spin is given by

$$
\hat{S} = \frac{1}{2}(\hat{a}^* \hat{a}^* + \hat{b}^* \hat{b}^*) .
$$

(7.17b)

Here, $(\hat{d}_p(p), \hat{d}_p^*(p))$ etc. denote boson operators. In order to get the counterpart of the state $|m_0\rangle$ shown in the relation (7.11), we must investigate the coupling scheme of the $R_p$- and the $R_h$-spin governing the state $|m_0\rangle$. The connection of the $S$-spin with the $R$-spin is given by the relation (7.12).

The state $|m_0\rangle$ given in the relation (7.11) satisfies the relation

$$
\begin{align*}
\langle \tilde{R}_-(p)|m_0\rangle &= \tilde{R}_-(h)|m_0\rangle = 0 , \quad \text{i.e.,} \quad \tilde{R}_-|m_0\rangle = 0 , \\
\langle \tilde{R}_0(p)|m_0\rangle &= -r_p|m_0\rangle , \quad \tilde{R}_0(h)|m_0\rangle = -r_h|m_0\rangle , \\
\text{i.e.,} \quad \tilde{R}_0|m_0\rangle &= -(r_p + r_h)|m_0\rangle .
\end{align*}
$$

(7.18b)

Here, $r_p$ and $r_h$ are given as

$$
\begin{align*}
r_p &= \frac{1}{2}(r - \tau) , \quad r_h = \frac{1}{2}(r + \tau) \quad \text{for the upper state of (7.11)} , \\
r_p &= \frac{1}{2}(r + \tau) , \quad r_h = \frac{1}{2}(r - \tau) \quad \text{for the lower state of (7.11)} .
\end{align*}
$$

(7.19b)

The relation (7.19) leads us to

$$
r_p + r_h = r , \quad \text{i.e.,} \quad (r_p + r_h)(r_p + r_h + 1) = r(r + 1) .
$$

(7.20)

The above discussion gives us the following picture for the coupling scheme: The directions of both spins are the same as each other.

Under the above preparation, we will investigate the other kind of the boson realization for the Lipkin model. First, we notice the following operator identity:

$$
\hat{R}^2 = \hat{T}^2 , \quad \hat{T}^2 = -\hat{T}_+\hat{T}_- + \hat{T}_0(\hat{T}_0 - 1) .
$$

(7.21)

Here, $\hat{R}^2$ denotes the Casimir operator of the $su(2)$-algebra, the generators of which are defined in the relations (7.15) and (7.16). The operator $\hat{T}^2$ denotes the Casimir operator of the $su(1,1)$-algebra, the generators of which are defined as follows:

$$
\begin{align*}
\hat{T}_+ &= \hat{d}_p(p)\hat{d}_p^*(h) - \hat{d}_p^*(h)\hat{d}_p(p) , \quad \hat{T}_- &= \hat{d}_\tau(h)\hat{d}_p(p) - \hat{d}_p(p)\hat{d}_\tau(h) , \\
\hat{T}_0 &= \frac{1}{2}(\hat{d}_p(p)\hat{d}_p(p) + \hat{d}_\tau(p)\hat{d}_\tau(p) + \hat{d}_p^*(h)\hat{d}_p(h) + \hat{d}_\tau^*(h)\hat{d}_\tau(h)) + 1 .
\end{align*}
$$

(7.22a)
The set \((\hat{T}_\pm, 0)\) satisfies
\[
[\hat{T}_+, \hat{T}_-] = -2\hat{T}_0, \quad [\hat{T}_0, \hat{T}_\pm] = \pm\hat{T}_\pm. \tag{7.22b}
\]

Detailed argument can be found in the paper by the present authors (J. da P. & M. Y.) together with Kuriyama.\(^{15}\) It may be interesting to see that \(\hat{T}_0\) can be expressed in terms of the operators for the magnitudes of the \(R_p\)- and the \(R_h\)-spins:
\[
\hat{T}_0 = \hat{R}(p) + \hat{R}(h) + 1, \quad \text{i.e.,} \quad \hat{T}_0(\hat{T}_0 - 1) = (\hat{R}(p) + \hat{R}(h))(\hat{R}(p) + \hat{R}(h) + 1). \tag{7.23}
\]

The term \((\hat{R}(p) + \hat{R}(h))\) indicates the simple sum of the magnitudes of the two \(su(2)\)-spins. Let the eigenstate of \(\hat{R}(p), \hat{R}(h), \hat{R}_2\) and \(\hat{R}_0, |\lambda; r prhr><r|\), satisfy the relation
\[
\hat{T}_-|\lambda; r prhr><r| = 0, \quad (\lambda; r prhr)|\hat{T}_+ = 0. \tag{7.24}
\]

Then, for the state \(|\lambda; r prhr><r|\), we have
\[
r(r + 1) = (r_p + r_h)(r_p + r_h + 1). \tag{7.25}
\]

The relation (7.25) is nothing but the result (7.20) and, then, the condition (7.23) presents us the picture that the directions of the \(R_p\)- and the \(R_h\)-spin are the same as each other.

The relation (7.24) suggests to us that we adopt the idea presented by Dirac for the constraint system. In §6, we have adopted this idea for the case of many-fermion system. We require the following constraints:
\[
\hat{T}_- \approx 0, \quad \hat{T}_+ \approx 0. \tag{7.26}
\]

Then, we define the Dirac bracket for \(A\) and \(B\), which is denoted as \([ [A, B] ]\), in the form
\[
[[A, B]] = [A, B] + [A, \hat{T}_+] \left( [\hat{T}_+, \hat{T}_-] \right)^{-1} [\hat{T}_-, B] + [A, \hat{T}_-] \left( [\hat{T}_-, \hat{T}_+] \right)^{-1} [\hat{T}_+, B]. \tag{7.27a}
\]

By using the ordinary boson commutation relations, for example, such as \([d_P(p), d_P^*(p)] = 1\), we calculate the right-hand side of the relation (7.27a). Consequently, we can express \([[A, B]]\) as a function of bosons \(d_P^*(p)\) etc., which is denoted as \(\hat{C}_{AB}\), namely, we have
\[
[[A, B]] = \hat{C}_{AB}. \tag{7.27b}
\]

Under the above result, we set up the commutation relation for \(A\) and \(B\) in the following form:
\[
[A, B] = \hat{C}_{AB}. \tag{7.28}
\]
For example, we have

\[ [ \hat{d}_P(p), \hat{d}_P^*(p) ] = 1 - \hat{d}_P^*(h) \left[ 2(\hat{R}(p) + \hat{R}(h) + 1) \right]^{-1} \hat{d}_P(h). \] (7.29)

For the \textit{R}-spin, we adopt the Schwinger boson representation shown in the forms (7.15) and (7.16) with the commutation relations (7.29), etc. For the \textit{S}-spin, we adopt the Holstein-Primakoff representation in the following form:

\[
\hat{S}_+ = \hat{A}^* \sqrt{2(\Omega - (\hat{R}(p) + \hat{R}(h)))} - \hat{A}^* \hat{A}, \\
\hat{S}_- = \sqrt{2(\Omega - (\hat{R}(p) + \hat{R}(h)))} - \hat{A}^* \hat{A} \cdot \hat{A}, \\
\hat{S}_0 = \hat{A}^* \hat{A} - (\Omega - (\hat{R}(p) + \hat{R}(h))).
\] (7.30)

Of course, we used the relation

\[
\hat{S} = \Omega - \hat{R}, \quad \hat{R} = \hat{R}(p) + \hat{R}(h).
\] (7.31)

Thus, we obtained the boson realization of the Lipkin model which is closely related to the isovector pairing model.

§8. Concluding remarks

In this paper, we formulated the \textit{su}(2)-algebraic many-fermion model in a rather general scheme. In our idea, the \textit{su}(2)-algebra (\(\tilde{R}_{\pm,0}\)), which we called the auxiliary algebra, plays a decisive role for determining the minimum weight state. Through the use of this algebra, we can learn various aspects of the models, some of which are newly derived. Further, we showed that the idea adopted in the \textit{su}(2)-algebra is also applicable to the \textit{so}(5)-algebra which treats the isovector pairing model and the Lipkin model is formulated as a sub-algebra under the reading (7.7).

Finally, we will give two comments. In the relations (2.1) and (2.5), we make the following replacement:

\[
\tilde{c}_\alpha^* = \tilde{\gamma}_{+\alpha}, \quad s_\alpha \tilde{c}_\alpha^* = \tilde{\gamma}_{-\alpha},
\] (8.1)

The operators (\(\tilde{\gamma}_{+\alpha}, \tilde{\gamma}_{+\alpha}^*\)) and (\(\tilde{\gamma}_{-\alpha}, \tilde{\gamma}_{-\alpha}^*\)) are also fermions. The sets (\(\tilde{S}_{\pm,0}\)) and (\(\tilde{R}_{\pm,0}\)) can be rewritten in the form

\[
\tilde{S}_+ = \sum_\alpha \tilde{\gamma}_{+\alpha}^* \tilde{\gamma}_{-\alpha}, \quad \tilde{S}_- = \sum_\alpha \tilde{\gamma}_{-\alpha}^* \tilde{\gamma}_{+\alpha}, \quad \tilde{S}_0 = \frac{1}{2} \sum_\alpha (\tilde{\gamma}_{+\alpha}^* \tilde{\gamma}_{+\alpha} - \tilde{\gamma}_{-\alpha}^* \tilde{\gamma}_{-\alpha}),
\] (8.2)

\[
\tilde{R}_+ = \sum_\alpha s_\alpha \tilde{\gamma}_{+\alpha}^* \tilde{\gamma}_{-\alpha}, \quad \tilde{R}_- = \sum_\alpha s_\alpha \tilde{\gamma}_{-\alpha} \tilde{\gamma}_{+\alpha}, \quad \tilde{R}_0 = \frac{1}{2} \sum_\alpha (\tilde{\gamma}_{+\alpha}^* \tilde{\gamma}_{+\alpha} + \tilde{\gamma}_{-\alpha}^* \tilde{\gamma}_{-\alpha}) - \Omega_0.
\] (8.3)

For the forms (8.2) and (8.3), let us adopt the reading

\[
\tilde{\gamma} \to \hat{c}, \quad \tilde{\gamma}^* \to \hat{c}^*, \quad +\alpha \to \alpha, \quad -\alpha \to \alpha^\dagger.
\] (8.4)
Then, it can be seen that the roles of \((\tilde{S}_{\pm,0})\) and \((\tilde{R}_{\pm,0})\) mentioned in §2 are reversed. Consequently, with the aid of the auxiliary algebra shown in the form (8.3), we can describe the \(su(2)\)-algebraic model expressed in terms of the form (8.2). For example, the Lipkin model can be described without introducing the particle and hole operators shown in the relation (5.3). The above is the first comment.

The basic idea presented in this paper is found out in the use of the \(R\)-spin and the states in the aligned scheme defined in the relations (2.5) and (2.8), respectively. As the second comment, we must mention that our present description of the \(su(2)\)-algebraic many-fermion model may be incomplete. As an example, let us take up two-fermion states. In the case of \(|m_0\rangle = \tilde{c}_\gamma^* \tilde{c}_\delta^* |0\rangle\), we have 

\[
\tilde{R}_+ |m_0\rangle = (\tilde{c}_\gamma^* \tilde{c}_\delta^* + \tilde{c}_\delta^* \tilde{c}_\gamma^*) |0\rangle
\]

and 

\[
(\tilde{R}_+)^2 |m_0\rangle = \tilde{c}_\gamma^* \tilde{c}_\delta^* |0\rangle.
\]

A two-fermion state orthogonal to \(\tilde{R}_+ |m_0\rangle\) is 

\[
|m'\rangle = (\tilde{c}_\gamma^* \tilde{c}_\delta^* - \tilde{c}_\delta^* \tilde{c}_\gamma^*) |0\rangle,
\]

which leads us to 

\[
\tilde{R}_+ |m'\rangle = 0,
\]

i.e., \(r = 0\). In the case \(\gamma \neq \delta\), 

\[
|m'\rangle \text{ is orthogonal to } \tilde{S}_+ |0\rangle.
\]

The seniority number of 

\[
|m'\rangle \text{ is } 2, \quad s = \Omega_0 - 1, \quad \text{but } r = 0, \quad \text{i.e., } s + r = \Omega_0 - 1 \neq \Omega_0.
\]

The above indicates that 

\[
|m'\rangle \text{ is not in the aligned scheme.}
\]

This is the second comment. In the future, we will report on the minimum weight states which will be obtained by extending the role of the \(R\)-spin discussed in this paper.

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