The size-Ramsey number of powers of paths

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Abstract
Given graphs $G$ and $H$ and a positive integer $q$, say that $G$ is $q$-Ramsey for $H$, denoted $G \rightarrow (H)_q$, if every $q$-coloring of the edges of $G$ contains a monochromatic copy of $H$. The size-Ramsey number $r(H)$ of a graph $H$ is defined to be $r(H) = \min\{|E(G)|: G \rightarrow (H)_2\}$. Answering a question of Conlon, we prove that, for every fixed $k$, we have $r(P_n^k) = O(n)$, where $P_n^k$ is the $k$th power of the $n$-vertex path $P_n$ (i.e., the graph with vertex set $V(P_n)$ and all edges $\{u, v\}$ such that the distance between $u$ and $v$ in $P_n$ is at most $k$). Our proof is probabilistic, but can also be made constructive.

KEYWORDS
powers of paths, Ramsey, size-Ramsey

1 INTRODUCTION

Given graphs $G$ and $H$ and a positive integer $q$, say that $G$ is $q$-Ramsey for $H$, denoted $G \rightarrow (H)_q$, if every $q$-coloring of the edges of $G$ contains a monochromatic copy of $H$. When $q = 2$, we simply write $G \rightarrow H$. In its simplest form, the classical theorem of Ramsey [24] states that for any $H$ there exists an integer $N$ such that $K_N \rightarrow H$. The Ramsey number $r(H)$ of a graph $H$ is defined to be the smallest such $N$. Ramsey problems have been well studied and many beautiful
techniques have been developed to estimate Ramsey numbers. For a detailed summary of developments in Ramsey theory, see the excellent survey of Conlon et al [7].

A number of variants of the classical Ramsey problem are also under active study. In particular, Erdős et al [12] proposed the problem of determining the smallest number of edges in a graph \( G \) such that \( G \rightarrow H \). Define the size-Ramsey number \( \hat{r}(H) \) of a graph \( H \) to be

\[
\hat{r}(H) := \min |E(G)|: G \rightarrow H.
\]

In this paper, we are concerned with finding bounds on \( \hat{r}(H) \) in some specific cases.

For any graph \( H \), it is not difficult to see that \( \hat{r}(H) \leq \left( \frac{r(H)}{2} \right) \). A result due to Chvátal (see, eg, [12]) shows that in fact this bound is tight for complete graphs. For the \( n \)-vertex path \( P_n \), Erdős [11] asked the following question.

**Question 1.1.** Is it true that

\[
\lim_{n \to \infty} \frac{\hat{r}(P_n)}{n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{\hat{r}(P_n)}{n^2} = 0?
\]

Answering Erdős’ question, Beck [3] proved that the size-Ramsey number of paths is linear, that is, \( \hat{r}(P_n) = O(n) \), by means of a probabilistic construction. Alon and Chung [2] provided an explicit construction of a graph \( G \) with \( O(n) \) edges such that \( G \rightarrow P_n \). Recently, Dudek and Prałat [10] gave a simple alternative proof for this result [21]. More generally, Friedman and Pippenger [14] proved that the size-Ramsey number of bounded-degree trees is linear [8,15,17] and it is shown in [16] that cycles also have linear size-Ramsey numbers.

A question posed by Beck [4] asked whether \( \hat{r}(G) \) is linear for all graphs \( G \) with bounded maximum degree. This was negatively answered by Rödl and Szemerédi, who showed that there exists an \( n \)-vertex graph \( H \) and maximum degree 3 such that \( \hat{r}(H) = \Omega(n \log^{1/60} n) \). The current best upper bound for bounded-degree graphs is proved in [19], where it is shown that for every \( \Delta \) there is a constant \( c \) such that for any graph \( H \) with \( n \) vertices and maximum degree \( \Delta \),

\[
\hat{r}(H) \leq cn^{2-1/\Delta} \log^{1/\Delta} n.
\]

For further results on size-Ramsey numbers, the reader is referred to [5,18,25].

Given an \( n \)-vertex graph \( H \) and an integer \( k \geq 2 \), the \( k \)th power \( H^k \) of \( H \) is the graph with vertex set \( V(H) \) and all edges \( \{u, v\} \) such that the distance between \( u \) and \( v \) in \( H \) is at most \( k \). Answering a question of Conlon [6], we prove that all powers of paths have linear size-Ramsey numbers. The following theorem is our main result.

**Theorem 1.2.** For any integer \( k \geq 2 \),

\[
\hat{r}(P_n^k) = O(n).
\]

Since \( C_n^k \subseteq P_n^{2k} \), the next corollary follows directly from Theorem 1.2.
Corollary 1.4. For any integer \( k \geq 2 \),

\[
\hat{r}(C_n^k) = O(n).
\] (1.2)

Throughout the paper, we use big \( O \) notation with respect to \( n \to \infty \), where the implicit constants may depend on other parameters. For a path \( P \), we write \( |P| \) for the number of vertices in \( P \). For simplicity, we omit floor and ceiling signs when they are not essential.

The paper is structured as follows: in Section 2, we introduce some preliminary definitions and give an outline of the proof; the proof of Theorem 1.2 is given in Section 3; in Section 4, we mention some related open problems.

# 2 | OUTLINE OF THE PROOF

To prove Theorem 1.2, we will show that there exists a graph \( G \) with \( O(n) \) edges such that \( G \to P_n^k \).

To construct \( G \) we begin by taking a pseudorandom graph \( H \) with bounded degree. The existence of such an \( H \) will be proved in Lemma 3.1. Given \( H^k \), we then take a complete blow-up, defined as follows.

**Definition 2.1.** Given a graph \( H \) and a positive integer \( t \), the complete \( t \) blow-up of \( H \), denoted \( H_t \), is the graph obtained by replacing each vertex \( v \) of \( H \) by a complete graph with \( r(K_t) \) vertices, the cluster \( C(v) \), and by adding, for every \( \{u, v\} \in E(H) \), every edge between \( C(u) \) and \( C(v) \).

Note that we replace each vertex with a clique on \( r(K_t) \) vertices rather than \( t \) vertices as might have been expected.

The following immediate fact states that the complete blow-ups of powers of bounded-degree graphs have a linear number of edges. This makes them valid candidates for showing \( \hat{r}(P_n^k) = O(n) \).

**Fact 2.2.** Let \( k, t, a, \) and \( b \) be positive constants. If \( H \) is a graph with \( |V(H)| = an \) and \( \Delta(H) \leq b \), then \( |E(H_t^k)| = O(n) \).

The heart of the proof is to show that, given any 2-coloring of the edges of \( H_t^k \), we can find a monochromatic copy of \( P_t \). To do this we will use the fact that \( H \) satisfies a particular property (Lemma 3.2). We shall also make use of the following result.

**Theorem 2.3** (Pokrovskiy [23, Theorem 1.7]). Let \( k \geq 1 \). Suppose that the edges of \( K_n \) are colored with red and blue. Then \( K_n \) can be covered by \( k \) vertex-disjoint blue paths and a vertex-disjoint red balanced complete \( (k + 1) \)-partite graph.

We remark that we do not need the full strength of this result, in the sense that we do not need the complete \((k + 1)\)-partite graph to be balanced; it suffices for us to know that the vertex classes are of comparable cardinality. Such a result can be derived easily by iterating Lemma 1.5 in [23], for which Pokrovskiy gives a short and elegant proof (see also [22, Lemma 1.10]).

We shall also use the classical Kővári–T. Sós–Turán theorem [20], in the following simple form.
Theorem 2.4. Let $G$ be a balanced bipartite graph with $t$ vertices in each vertex class. If $G$ contains no $K_{s,s}$, then $G$ has at most $4t^{2-s}$ edges.

Let us now give a brief outline of how we find our monochromatic copy of $P_n^k$ in a 2-edge colored $H_t^k$. Suppose the edges of $H_t^k$ have been colored red and blue by an arbitrary coloring $\chi$. Recall that $H_t^k$ is obtained by blowing up $H^k$; in particular, the vertices $v$ of $H^k$ become large complete graphs $C_v$ in $H_t^k$. By the choice of parameters, Ramsey’s theorem tells us that each such $C_v$ contains a monochromatic copy $B_v$ of $K_t$. We may assume without loss of generality that at least half of the $B_v$ are blue.

Let $F$ be the subgraph of $H$ induced by the vertices $v$ such that $B_v$ is blue. We shall define an auxiliary edge-coloring $\chi'$ of $F$. By using Theorem 2.3 we shall be able to find either (i) a blue $P_n$ in $F$ under $\chi'$ or (ii) a $P_n$ in $F$ (not in $F^k$) with certain additional properties. The path in (ii) will be found applying Lemma 3.2 with the sets $A_i$ being the vertex classes of a red complete $(k + 1)$-partite subgraph of $F^k$. This red complete $(k + 1)$-partite subgraph of $F^k$ will be found using Theorem 2.3, applied to a suitable red/blue colored complete graph (we complete $F^k$ with its auxiliary coloring $\chi'$ to a red/blue colored complete graph by considering nonedges of $F^k$ red).

In case (i), where we find a blue $P_n$ in $F$ under the coloring $\chi'$, we shall be able to find a blue $P_n^k$ in $H_t^k$. In case (ii), the properties of the path $P_n$ found in $F$ will ensure the existence of a red $P_n^k$ in $F^k \subseteq H_t^k$. The idea of defining an auxiliary graph on monochromatic cliques as above was used in [1].

## 3 | PROOF OF THEOREM 1.2

Our first lemma guarantees the existence of bounded-degree graphs with the pseudorandomness property we require.

**Lemma 3.1.** For every positive constants $\varepsilon$ and $a$, there is a constant $b$ such that, for any large enough $n$, there is a graph $H$ with $v(H) = an$ such that

1. For every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geq \varepsilon n$, we have $|E_H(S, T)| > 0$.
2. $\Delta(H) \leq b$.

**Proof:** Fix positive constants $\varepsilon$ and $a$. Let $c = 4a/\varepsilon^2$ and $b = 4ac$ and consider a sufficiently large $n$. Let $G = G(2an, p)$ be the binomial random graph with $p = c/n$. By Chernoff’s inequality, with high probability we have $|E(G)| < (4a^2c)n$. Moreover, with high probability $G$ satisfies (1) (with $H = G$) by the following reason: let $X_G$ be the number of pairs of disjoint subsets of $V(G)$ of size $\varepsilon n$ with no edges between them. Then, from the choice of $c$ and using Markov’s inequality, we have

$$\Pr[X_G \geq 1] \leq \mathbb{E}[X_G] \leq \left(\frac{2an}{\varepsilon n}\right)^{(cn)^2} < 2^{4an} \cdot e^{-\frac{c^2n}{n}} = o(1).$$

Thus, there is a graph $G$ with $|E(G)| < (4a^2c)n$ and $X_G = 0$.

Now let $H$ be a subgraph of $G$ obtained by iteratively removing a vertex of maximum degree until exactly $an$ vertices remain. Then $\Delta(H) \leq b$, as otherwise, from the choice of $b$ we would have deleted more than $b \cdot an > |E(G)|$ edges from $G$ during the iteration,
which contradicts property (1). Moreover, as $H$ is an induced subgraph of $G$, (1) is maintained. This completes the proof of the lemma. □

We now show that any graph satisfying the hypothesis of Lemma 3.1 and property (1) also satisfies an additional property.

Algorithm 1.

Input: a graph $H$ with $v(H) = n$ satisfying (1) and sets $A_i \subseteq V(H)$ ($1 \leq i \leq k + 1$) with $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $|A_i| \geq \varepsilon n$ for all $i$.

Output: a path $P_n = (x_1, \ldots, x_n)$ in $H$ with $x_i \in A_j$ for all $i$, where $j \equiv i \pmod{k + 1}$.

1. foreach $1 \leq i \leq k + 1$ do
2. \hspace{1em} $U_i \leftarrow A_i$; \hspace{1em} $D_i \leftarrow \emptyset$
3. while $|D_i| \leq |A_i|/2$ for all $i$ do
4. \hspace{1em} pick $x_1 \in U_1$ and let $P = (x_1)$; \hspace{1em} $r \leftarrow 1$; \hspace{1em} $U_1 \leftarrow U_1 \setminus \{x_1\}$
5. while $1 \leq |P| < n$ do
6. \hspace{2em} if $\exists u \in U_{r+1}$ with $(x_r, u) \in E(H)$ then
7. \hspace{3em} $x_{r+1} \leftarrow u$; \hspace{3em} $U_{r+1} \leftarrow U_{r+1} \setminus \{u\}$
8. \hspace{3em} $P \leftarrow (x_1, \ldots, x_{r+1})$; \hspace{3em} $r \leftarrow r + 1$
9. \hspace{2em} else
10. \hspace{3em} $D_{r} \leftarrow D_{r} \cup \{x_r\}$
11. \hspace{3em} $P \leftarrow (x_1, \ldots, x_{r-1})$; \hspace{3em} $r \leftarrow r - 1$
12. if $|P| = n$ then
13. \hspace{1em} return $P$ \hspace{1em} // path has been found
14. STOP with failure \hspace{1em} // this will not happen

Lemma 3.2. For every integer $k \geq 1$ and every $\varepsilon > 0$, there exists $a_0 > 0$ such that the following holds for any $a \geq a_0$. Let $H$ be a graph with $an$ vertices such that for every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geq \varepsilon n$ we have $|E_H(S, T)| > 0$. Then, for every family $A_1, \ldots, A_{k+1} \subseteq V(H)$ of pairwise disjoint sets each of size at least $\varepsilon an$, there is a path $P_n = (x_1, \ldots, x_n)$ in $H$ with $x_i \in A_j$ for all $1 \leq i \leq n$, where $j \equiv i \pmod{k + 1}$.

To prove Lemma 3.2, we analyze a depth-first search algorithm, adapting a proof idea in [5, Lemma 4.4]. More specifically, we run an algorithm (stated formally as Algorithm 1). Our algorithm receives as input a graph $H$ with $v(H) = n$ satisfying property (1), and a family of pairwise disjoint sets $A_1, \ldots, A_{k+1} \subseteq V(H)$ with $|A_i| \geq \varepsilon an$ for all $i$. The output of $\mathcal{A}$ is a path $P_n = (x_1, \ldots, x_n)$ in $H$ with $x_i \in A_j$ for all $1 \leq i \leq n$, where $j \equiv i \pmod{k + 1}$.

As it runs, the algorithm builds a path $P = (x_1, \ldots, x_r)$ with $x_i \in A_j$ for all $i$ and $j$ with $j \equiv i \pmod{k + 1}$. Furthermore, it maintains sets $U_j$ and $D_j \subseteq A_j$ for all $j$, with the property that $U_j, D_j$, and $V(P) \cap A_j$ form a partition of $A_j$ for every $j$. The cardinality of the sets $U_j$ decreases as the algorithm runs, while the $D_j$ increases. As the algorithm runs, we have $r = |P| < n$ and it searches for an edge $(x_r, u) \in E(H)$ where $u$ belongs to the set $U_{r+1}$ of unused vertices in $A_{r+1}$. If such a vertex $u \in U_{r+1}$ is found, then $P$ is made one vertex longer by adding $u$ to it. If there is no such vertex $u$, then $x_r$ is declared a dead end and it is put into $D_r$. Moreover,
the path $P$ is shortened by one vertex; it becomes $P = (x_1, \ldots, x_{r-1})$. Our algorithm iterates this procedure. If we find a path $P$ with $n$ vertices this way, then we are done.

We now analyze Algorithm 1.

Proof of Lemma 3.2. We will prove that Algorithm 1 returns a path $P$ on line 13 as desired, instead of terminating with failure on line 14.

Fix an integer $k \geq 1$ and $\varepsilon > 0$. Let

$$a_0 = 2 + \frac{4}{\varepsilon(k + 1)}, \tag{3.1}$$

fix $a \geq a_0$, and let $n$ be sufficiently large. Let $H$ be a graph with $an$ vertices satisfying property (1), that is, for every pair of disjoint sets $S, T \subseteq V(H)$ with $|S|, |T| \geq \varepsilon n$ we have $|E_H(S, T)| > 0$. Let $A_1, \ldots, A_{k+1} \subseteq V(H)$ be a family of pairwise disjoint sets each of size at least $\varepsilon n$.

First recall that $U_0, D_1, \ldots, D_{k+1}$ form a partition of $A_i$ for every $i$. Since the path $P$ is always empty on line 4, at this point we have $|U_0| \geq |A_1| - |D_1| \geq |A_1|/2 > 0$. Then, line 4 is always executed successfully.

Suppose now that $\mathcal{A}$ stops with failure on line 14. Then, for some $i$, say $i = r$, the set $D_r = D_r$ became larger than $|U_0|/2 \geq \varepsilon n/2 \geq \varepsilon n$. Furthermore, we have $|P| < n$ and $|D_{r+1}| \leq |A_{r+1}|/2$ (indices modulo $k + 1$) and hence,

$$|U_{r+1}| \geq |A_{r+1}| - |D_{r+1}| - |V(P) \cap A_{r+1}| \geq \frac{1}{2}|A_{r+1}| - \left[ \frac{n}{k + 1} \right] \geq \frac{1}{2}\varepsilon n - \frac{2n}{k + 1} > \varepsilon n.$$

Note that this is the only place where the exact value of $a_0$ is used. Applying property (1) to the pair $(D_r, U_{r+1})$, we see that there is an edge $\{x, u\} \in E(H)$ with $x \in D_r$ and $u \in U_{r+1}$. Consider the moment in which $x$ was put into $D_r$. This happened on line 10, when $P$ had $x$ as its foremost vertex and $\mathcal{A}$ was trying to extend $P$ further into $U_{r+1}$. At this point, because of the edge $\{x, u\} \in E(H)$, we must have had $u \notin U_{r+1}$ (see line 6). Since the set $U_{r+1}$ decreases as $\mathcal{A}$ runs, this is a contradiction and hence $\mathcal{A}$ does not terminate on line 14.

Since $\sum_{i \leq k+1} (|D_i| - |U_i|)$ increases as Algorithm 1 runs, we know the algorithm terminates. Therefore, we conclude that it returns a suitable path $P$ as claimed. □

We are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Fix $k \geq 1$ and let $\varepsilon = 1/3(k + 1)$. Let $a_0$ be the constant given by an application of Lemma 3.2 with parameters $k$ and $\varepsilon$. Set $a = \max\{6k, a_0\}$ and let $b$ be given by Lemma 3.1 for this choice of $a$. Moreover, let $H$ be a graph with $|V(H)| = an$ and $\Delta(H) \leq b$ be as in Lemma 3.1. Finally, put $t = (64k)^{2k}$ and $s = 2k$.

Let $H_t^k$ be a complete-$t$-blow-up of $H^k_t$, as in Definition 2.1, and let $\chi: E(H_t^k) \rightarrow \{\text{red}, \text{blue}\}$ be an edge-coloring of $H_t^k$. We shall show that $H_t^k$ contains a monochromatic copy of $P_n^k$ under $\chi$. By the definition of $H_t^k$, any cluster $C(v)$ contains a monochromatic copy $B(v)$ of $K_t$. Without loss of generality, the set $W := \{v \in V(H): B(v)\text{is blue}\}$ has cardinality at least $v(H)/2$. Let $F := H[W]$ be the
subgraph of $H$ induced by $W$, and let $F'$ be the subgraph of $F^k_i \subseteq H^k_t$ induced by $\bigcup_{w \in W} V(B(w))$.

Given the above coloring $\chi$, we define a coloring $\chi'$ of $F^k$ as follows. An edge $\{u, v\} \in E(F^k)$ is colored blue if the bipartite subgraph $F'[V(B(u)), V(B(v))]$ of $F'$ naturally induced by the sets $V(B(u))$ and $V(B(v))$ contains a blue $K_{s,s}$. Otherwise $\{u, v\}$ is colored red.

Claim 3.4. Any $2$-coloring of $E(F^k)$ has either a blue $P_n$ or a red $P^k_n$.

Proof. We apply Theorem 2.3 to $F^k$, where if an edge is not present in $F^k$, then we consider it to be in the red color class. If $F^k$ contains a blue copy of $P_n$, then we are done. Hence we may assume $F^k$ contains a balanced, complete $(k + 1)$-partite graph $K$ with parts $A_1, \ldots, A_{k+1}$ on at least $v(F^k) - kn \geq an/2 - kn$ vertices, with no blue edges between any two parts. As $a \geq 6k$, each one of these parts has size at least

$$\frac{1}{k + 1} \left(1 - \frac{2}{k}\right)n \geq \varepsilon n. \quad (3.2)$$

By Lemma 3.2 applied to the collection of sets of vertices $A_1, \ldots, A_{k+1}$ of $F \subseteq H$ (specifically $F$ and not $F^k$), we see that $F[V(K)]$ contains a path with $n$ vertices such that any consecutive $k + 1$ vertices are in distinct parts of $K$. Therefore $F^k[V(K)]$ contains a copy of $P^k_n$ in which every pair of adjacent vertices are in distinct parts of $K.$ By the definition of $K$, such a copy is red. \hfill \square

By Claim 3.4, $F^k$ contains a blue copy of $P_n$ or a red copy of $P^k_n$ under the edge-coloring $\chi'$. Thus, we can split our proof into these two cases.

Case 1. First suppose $F^k$ contains a blue copy $(x_1, \ldots, x_n)$ of $P_n$. Then, for every $1 \leq i \leq n - 1$, the bipartite graph $F'[V(B(x_i)), V(B(x_{i+1}))]$ contains a blue copy of $K_{s,s}$, with, say, vertex classes $X_i \subseteq V(B(x_i))$ and $Y_{i+1} \subseteq V(B(x_{i+1}))$. As $|X_i| = |Y_i| = s = 2k$ for all $2 \leq i \leq n - 1$, we can find sets $X'_i \subseteq X_i$ and $Y'_i \subseteq Y_i$ such that $|X'_i| = |Y'_i| = k$ and $X'_i \cap Y'_i = \emptyset$ for all $2 \leq j \leq n - 1$. Let $X'_1 = X_1$ and $Y'_n = Y_n$.

We now show that the set $U := \bigcup_{i=1}^{n-1} X'_i \cup \bigcup_{i=2}^{n} Y'_i$ provides us with a blue copy of $P^k_{2kn}$ in $F' \subseteq H^k_t$. Note first that $|U| = 2k + 2k(n - 2) + 2k = 2kn$. Let $u_1, \ldots, u_{2kn}$ be an ordering of $U$ such that, for each $i$, every vertex in $X'_i$ comes before any vertex in $Y'_{i+1}$ and after every vertex in $Y'_i$. By the definition of the sets $X'_i$ and $Y'_i$ and the construction of $F' \subseteq F^k_t \subseteq H^k_t$, each vertex $u_i$ is adjacent in blue to $\{u_j \in U : 1 \leq |i - j| \leq k\}$. Thus, $U$ contains a blue copy of $P^k_{2kn}$, as claimed.

Case 2. Now suppose $F^k$ contains a red copy $P$ of $P^k_n$. That is, $F^k$ contains a set of vertices $\{x_1, \ldots, x_n\}$ such that $x_i$ is adjacent in red to all $x_j$ with $1 \leq |i - j| \leq k$. We shall show that, for each $1 \leq i \leq n$, we can pick a vertex $y_i \in V(B(x_i))$ so that $y_1, \ldots, y_n$ define a red copy of $P^k_n$ in $F' \subseteq F^k_t \subseteq H^k_t$. We do this by applying the local lemma [13] (a greedy strategy also works).

We have to show that it is possible to pick the $y_i (1 \leq i \leq n)$ in such a way that $\{y_i, y_j\}$ is a red edge in $F'$ for every $i$ and $j$ with $1 \leq |i - j| \leq k$. Let us choose $y_i \in V(B(x_i))$ $(1 \leq i \leq n)$ uniformly and independently at random. Let $e = \{x_i, x_j\}$ be an edge in $P \subseteq F^k$. We know that $e$ is red. Let $A_e$ be the event that $\{y_i, y_j\}$ is a blue edge in $F'$. Since the edge $e$ is red, we know that the bipartite graph $F'[V(B(x_i)), V(B(x_j))]$ contains no blue $K_{s,s}$. Theorem 2.4 then tells us that $\Pr[A_e] \leq 4t^{-1/s}$. 

The events \( A_e \) are not independent, but we can define a dependency graph \( D \) for the collection of events \( A_e (e \in E(P)) \) by adding an edge between \( A_e \) and \( A_f \) if and only if \( e \cap f \neq \emptyset \). Then \( \Delta(D) \leq 4k \). Given that

\[
4\Delta \mathbb{P}[A_e] \leq 64kt^{-1/3} = 1, \tag{3.3}
\]

for all \( e \), the local lemma tells us that \( \mathbb{P} \left[ \bigcap_{e \in E(P)} \overline{A_e} \right] > 0 \), and hence a simultaneous choice of the \( y_i \) (1 \( \leq i \leq n \)) as required is possible. This completes the proof of Theorem 1.2.

Throughout our proof we have used probabilistic methods to show the existence of \( G \). We now briefly discuss how our proof could be made constructive. For instance, it suffices to take for \( H \) a suitable \((n, d, \lambda)\)-graph as in Alon and Chung [2], namely, it is enough to have \( \lambda = O(\sqrt{d}) \) and \( d \) large enough with respect to \( k \) and \( 1/\varepsilon \).

4 | OPEN QUESTIONS

We make no attempts to optimize the constant given by our argument, so the following question is of interest.

**Question 4.1.** For any integer \( k \geq 2 \), what is \( \limsup_{n \to \infty} \hat{r}(P_n^k)/n \)?

It is also interesting to consider what happens when more than two colors are at play. For \( q \in \mathbb{N} \), let \( \hat{r}_q(H) \) denote the \( q \)-color size-Ramsey number of \( H \), that is, the smallest number of edges in a graph that is \( q \)-Ramsey for \( H \).

**Conjecture 4.2.** For any \( q, k \in \mathbb{N} \) we have \( \hat{r}_q(P_n^k) = O(n) \).

It is conceivable that in hypergraphs the size-Ramsey number (defined analogously as for graphs) of tight paths may be linear. Let \( H_n^{(k)} \) denote the tight path of uniformity \( k \) on \( n \) vertices; that is, \( V(H_n^{(k)}) = [n] \) and \( E(H_n^{(k)}) = \{\{1,\ldots,k\},\{2,\ldots,k+1\},\ldots,\{n-k+1,\ldots,n\}\} \). The following question appears as Question 2.9 in [9].

**Question 4.3.** For any \( k \in \mathbb{N} \), do we have \( \hat{r}(H_n^{(k)}) = O(n) \)?

Finally, we note that for fixed \( k \), our main result implies the linearity of the size-Ramsey number for the grid graphs \( G_{k,n} \), the cartesian product of the paths \( P_k \) and \( P_n \). Indeed our main result implies the linearity of the size-Ramsey number for any sequence of graphs with bounded bandwidth. For the \( d \)-dimensional grid graph \( G_n^d \), obtained by taking the cartesian product of \( d \) copies of \( P_n \), we raise the following question.

**Question 4.4.** For any integer \( d \geq 2 \), is \( \hat{r}(G_n^d) = O(n^d) \)?

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