SURFACES AND FRONTS WITH HARMONIC-MEAN CURVATURE ONE IN HYPERBOLIC THREE-SPACE

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ABSTRACT. We investigate surfaces with constant harmonic-mean curvature one (HMC-1 surfaces) in hyperbolic three-space. We allow them to have certain kinds of singularities, and discuss some global properties. As well as flat surfaces and surfaces with constant mean curvature one (CMC-1 surfaces), HMC-1 surfaces belong to a certain class of Weingarten surfaces. From the viewpoint of parallel surfaces, CMC-1 surfaces and HMC-1 surfaces are representative among this class.

1. INTRODUCTION

In the differential geometry of surfaces in hyperbolic three-space \( \mathbb{H}^3 \), surfaces with constant mean curvature one (CMC-1 surfaces, for short) are one of the central subjects \([\text{Br}], \text{UY}, \text{CHR}\), etc. The theory of flat surfaces in \( \mathbb{H}^3 \) is also developing, thanks to the appearance of a representation formula due to Gálvez, Martínez and Milán \([\text{GMM1}]\). From the viewpoint of global theory for flat surfaces, one should generalize the category of surfaces to that of fronts. (Roughly speaking, a front is a surface with certain kinds of singularities.) Any complete flat surface in \( \mathbb{H}^3 \) must be a horosphere or a hyperbolic cylinder, however, many complete flat fronts exist in \( \mathbb{H}^3 \) (see \([\text{KUY2}]\)).

On the other hand, Gálvez, Martínez and Milán \([\text{GMM2}]\) also studied a wider class of surfaces in \( \mathbb{H}^3 \), including both CMC-1 surfaces and flat surfaces. It is a class of Weingarten surfaces satisfying \( \alpha(H - 1) = \beta K \) for some constants \( \alpha \) and \( \beta \). Here, \( H \) denotes the mean curvature, and \( K \) is the Gaussian curvature. Indeed, the following theorem is shown in \([\text{GMM2}]\):

Gálvez-Martínez-Milán’s formula \([\text{GMM2}]\). Let \( M \) be a non-compact, simply-connected surface and \( f : M \to \mathbb{H}^3 \) a Weingarten surface satisfying \( \alpha(H - 1) = \beta K \), where \( \alpha \) and \( \beta \) are real constants with \( \alpha \neq 2\beta \). Then, there exist a meromorphic curve \( G : M \to \text{SL}(2, \mathbb{C}) \) and a pair \((h, \theta)\) consisting of a meromorphic function \( h \) and a holomorphic one-form \( \theta \) on \( M \), such that the immersion \( f \) and its unit normal field \( \nu \) can be recovered as \( f = G \mathcal{H} G^* \) and \( \nu = \mathcal{H} \mathcal{H}^* \), where

\[
\mathcal{H} = \begin{bmatrix}
1 + \frac{\varepsilon |h|^2}{1 + \varepsilon |h|^2} & -\varepsilon \bar{h} \\
-\varepsilon h & 1 + \varepsilon |h|^2
\end{bmatrix}
\quad \text{and} \quad
\tilde{\mathcal{H}} = \begin{bmatrix}
\frac{1 - \varepsilon^2 |h|^2}{1 + \varepsilon |h|^2} & \varepsilon \bar{h} \\
\varepsilon h & -1 - \varepsilon |h|^2
\end{bmatrix}
\]

with \( \varepsilon = \alpha/(\alpha - 2\beta) \) and \( 1 + \varepsilon |h|^2 > 0 \). Moreover, the curve \( G \) satisfies

\[
G^{-1} dG = \begin{bmatrix}
0 & \theta \\
\frac{dh}{dh} & 0
\end{bmatrix}.
\]
The following formulas hold:

\[(1.3)\] \[I = (1 - \varepsilon)\theta dh + \left(\frac{(1 - \varepsilon)^2|h|^2}{(1 + \varepsilon|h|^2)^2} + (1 + \varepsilon|h|^2)|\theta|^2\right) + (1 - \varepsilon)\bar{\theta}d\bar{h},\]

\[(1.4)\] \[\alpha I - 2\beta I = (\alpha - 2\beta)\left(1 + \varepsilon|h|^2|\theta|^2 - \frac{(1 - \varepsilon)^2|h|^2}{(1 + \varepsilon|h|^2)^2} \right),\]

where \(I\) and \(\mathbb{II}\) denote the first and second fundamental forms.

Conversely, let \(M\) be a Riemann surface, \(\mathcal{G} : M \to \text{SL}(2, \mathbb{C})\) a meromorphic curve and \((h, \theta)\) a pair as above satisfying \((1.2)\) and such that \((1.4)\) is a positive definite metric. Then \(f := \mathcal{G}\mathcal{H}\mathcal{G}^* : M \to \mathbb{H}^3\) (\(\mathcal{H}\) as in \((1.1)\)), is a Weingarten surface satisfying \(\alpha(H - 1) = \beta K\) with \(I\) and \(\alpha I - 2\beta I\) given by \((1.3)\) and \((1.4)\).

In the statement above, \(\text{SL}(2, \mathbb{C})\) denotes the \(2 \times 2\) complex special linear group, i.e., the complex Lie group consisting of \(2 \times 2\) matrices with determinant 1, and we regard the hyperbolic 3-space \(\mathbb{H}^3\) as \(\text{SL}(2, \mathbb{C})/\text{SU}(2)\). (See Section 3.2 for details.)

For the ratio \([\alpha : \beta]\) in \(\mathbb{R}P^1\), let \(W_{[\alpha;\beta]}\) be the set of Weingarten surfaces satisfying \(\alpha(H - 1) = \beta K\), and set

\[\mathcal{W} := \bigcup_{[\alpha;\beta] \in \mathbb{R}P^1} W_{[\alpha;\beta]}\]

It is remarkable that \(\mathcal{W}\) is closed under parallel transforms, that is, any parallel surface of any surface in \(\mathcal{W}\) is always in \(\mathcal{W}\). More precisely, dividing \(\mathcal{W}\) into four subclasses

\[\mathcal{W}^0 := W_{[0:1]}, \quad \mathcal{W}^1 := \bigcup_{\lambda < 1/2} W_{[1:\lambda]}, \quad \mathcal{W}^2 := W_{[1:1/2]}, \quad \mathcal{W}^3 := \bigcup_{\lambda > 1/2} W_{[1:\lambda]},\]

we can prove that each \(\mathcal{W}^j\) is closed under parallel transforms. (See Theorem 2.3 and Theorem 2.5.) Hence, we can roughly say that CMC-1 surface theory represents the theory of surfaces in \(\mathcal{W}^1\). For instance, one can construct a Weingarten surface satisfying \(H - 1 = \lambda K\) (\(\lambda < 1/2\)), though it may have singularities, by constructing any CMC-1 surface first and by parallelly transforming it appropriately. By the same reasoning, the theory of surfaces in \(\mathcal{W}^3\) can be represented by one special type of surfaces. We will take \(W_{[1:1]}\) as that representative for \(\mathcal{W}^3\), because surfaces in \(W_{[1:1]}\), i.e., Weingarten surfaces satisfying \(H - 1 = K\), have another special geometric meaning: the sum of the reciprocals of the principal curvature is constantly 2. In other words, the harmonic mean of the principal curvature functions is constantly 1. We also call them \(\text{surfaces with constant harmonic-mean curvature one}\) (HMC-1 surfaces, for short).

For the reason mentioned above, we will study HMC-1 surfaces in this paper. Although many works have been done on CMC-1 surfaces, HMC-1 surfaces have received less attention. For example, there is Epstein’s work \([12]\), however, it seems lesser-known. (In classical Euclidean surface theory, the radii of principal curvature were considered as the fundamental entities. There seem to be some works about the mean radius of principal curvatures, or equivalently, about the harmonic mean of principal curvatures; e.g., Christoffel’s theorem about rigidity of surfaces (cf. \([8]\ pp.299–302)\).)

In Section 2, we discuss background material for the sake of precisely understanding the contents mentioned in this introduction. Section 3 is devoted to deriving
the formula due to Gálvez, Martínez and Milán, but for the case of HMC-1 surfaces. Like the case of flat surfaces, it is more natural to consider HMC-1 fronts rather than HMC-1 surfaces. HMC-1 fronts are defined in Section 4. Some global properties are discussed and some examples are provided there.

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2. Background

2.1. Basics. Let \( L^4 \) denote the Minkowski 4-space with the Lorentzian inner product \( \langle \cdot, \cdot \rangle_L \) of signature \((-+)++\). Let \( \mathcal{F} \) be the set of positively oriented and positively time-oriented frames \((e_0, e_1, e_2, e_3)\) in \( L^4 \) satisfying

\[
\langle e_\alpha, e_\beta \rangle_L = \begin{cases} 
-1 & \text{if } \alpha = \beta = 0, \\
0 & \text{if } \alpha \neq \beta, \\
1 & \text{if } \alpha = \beta > 0.
\end{cases}
\]

The indices \( \alpha \) and \( \beta \) run over \( 0, 1, 2, 3 \), while the indices \( i, j \) and \( k \) run over \( 1, 2, 3 \). We shall use Einstein’s convention, that is, the symbol \( \sum \) may be omitted for sums over indices.

Regarding \( e_\alpha: (e_0, e_1, e_2, e_3) \ni \mathcal{F} \mapsto e_\alpha \in L^4 \) \((\alpha = 0, 1, 2, 3)\) as \( L^4 \)-valued functions, \( de_\alpha \) \((\alpha = 0, 1, 2, 3)\) are \( L^4 \)-valued one-forms on \( \mathcal{F} \). The connection forms \( \omega^\beta_\alpha \) are defined by \( de_\alpha = e_\beta \otimes \omega^\beta_\alpha \). We write \( \omega^i \) for \( \omega^0_0 \). Differentiating \( (2.1) \), we have

\[
\omega^0_0 = 0, \quad -\omega^0_1 + \omega^0_3 = 0, \quad \omega^i_0 + \omega^j_0 = 0, \quad \omega^i_0 + \omega^j_0 = 0, \\
(2.3) \quad de_0 = e_i \otimes \omega^j, \quad de_i = e_0 \otimes \omega^j + e_j \otimes \omega^i.
\]

Again, differentiating \( (2.3) \), we have the structure equations:

\[
(2.4) \quad d\omega^i = -\omega^j_k \wedge \omega^k_i, \quad d\omega^i_j = -\omega^k_i \wedge \omega^k_j - \omega^i_k \wedge \omega^k_j.
\]

The hyperbolic 3-space \( \mathbb{H}^3 \) is the upper half component of the two-sheeted hyperboloid in \( L^4 \), i.e.,

\[
\mathbb{H}^3 = \{ x = (x_0, x_1, x_2, x_3) \in L^4 ; \langle x, x \rangle_L = -1, x_0 > 0 \}
\]

with the metric induced by \( \langle \cdot, \cdot \rangle_L \). \( \mathbb{H}^3 \) is a space form of constant negative curvature \(-1\). As usual, we regard \( e_0: \mathcal{F} \to \mathbb{H}^3 \subset L^4 \) as the oriented orthonormal frame bundle of \( \mathbb{H}^3 \).

Let \( M \) be a connected, oriented surface, and \( f: M \to \mathbb{H}^3 \) an immersion. Let \( \{e_1, e_2\} \) be a local orthonormal frame on \( U \subset M \), and let \( \nu \) denote a unit normal field. Regarding them as \( L^4 \)-valued functions, we consider a map

\[
(e_0, e_1, e_2, e_3) := (f, e_1, e_2, \nu): U \to \mathcal{F}.
\]

We shall use the same notation for differential forms on \( \mathcal{F} \) and forms on \( U \) pulled back by this map. Since \( \langle \nu, df \rangle_L = 0 \),

\[
(2.5) \quad 0 = \langle \nu, df \rangle_L = \langle e_3, de_0 \rangle_L = \omega^3.
\]

From now on, we shall use the following convention on the ranges of indices: \( 1 \leq i, j, k \leq 2 \). It follows from \( (2.2), (2.3) \) and \( (2.5) \) that

\[
\omega^0_0 = 0, \quad -\omega^0_1 + \omega^0_3 = 0, \quad -\omega^0_2 + \omega^0_1 = 0, \quad \omega^1_0 + \omega^1_0 = 0, \quad \omega^2_0 + \omega^2_0 = 0, \quad \omega^3_0 + \omega^3_0 = 0,
\]

\[
(2.6) \quad de_0 = e_i \otimes \omega^j, \quad de_i = e_0 \otimes \omega^j + e_j \otimes \omega^i + e_3 \otimes \omega^3, \quad de_3 = e_j \otimes \omega^3.
\]
And the structure equations (2.4) become

\begin{align}
  d\omega^i &= -\omega^j \wedge \omega^i, \quad 0 = d\omega^3 = -\omega^3 \wedge \omega^i, \\
  d\omega^i_2 &= -\omega^3 \wedge \omega^2 - \omega^i \wedge \omega^2, \quad d\omega^3 = -\omega^3 \wedge \omega^i_2. 
\end{align}

Following Bryant’s notation (Br1), we introduce two complex-valued one-forms \(\omega := \omega^1 + \sqrt{-1}\omega^2\), \(\pi := \omega^3 - \sqrt{-1}\omega^3\), and a complex vector \(e := (e_1 - \sqrt{-1}e_2)/2\). Then (2.7) and (2.8) are rewritten as

\begin{align}
  d\omega &= \sqrt{-1}\omega^2 \wedge \omega, \quad \omega \wedge \pi + \bar{\omega} \wedge \bar{\pi} = 0, \\
  d\omega_2 &= -\frac{1}{2}(\pi \wedge \bar{\pi} + \omega \wedge \bar{\omega}), \quad d\pi = -\sqrt{-1}\omega^3 \wedge \pi. 
\end{align}

The first fundamental form \(I = \langle de_0, de_0 \rangle_L\) is given by

\[I = \langle e_1 \otimes \omega^1, e_1 \otimes \omega^1 \rangle_L = (\omega^1)^2 + (\omega^2)^2 = \omega \bar{\omega} = |\omega|^2.\]

The Gaussian curvature \(K\) is determined by \(d\omega_2 = K(\sqrt{-1}/2) \omega \wedge \bar{\omega}\). Hence, it follows from (2.10) that

\[K + 1) \omega \wedge \bar{\omega} + \pi \wedge \bar{\pi} = 0.\]

The second fundamental form \(II = -(de_0, de_3)_L\) is given by

\[II = -(e_1 \otimes \omega^1, e_1 \otimes \omega^3) = \omega^1 \omega^3 + \omega^2 \omega^3 = \frac{1}{2}(\omega \pi + \bar{\omega} \bar{\pi}) = \text{Re}(\omega \pi).\]

If we set \(\omega^3 = h_{ij} \omega^i\), then \(h_{12} = h_{21}\) and \(II = h_{11}(\omega^1)^2 + 2h_{12} \omega^1 \omega^2 + h_{22}(\omega^2)^2\). Moreover,

\[\pi = \frac{1}{2}((h_{11} - h_{22}) - 2\sqrt{-1}h_{12}) \omega + \frac{1}{2}((h_{11} + h_{22}) \bar{\omega}).\]

Setting \(q = ((h_{11} - h_{22}) - 2ih_{12})/2\) and \(H = (h_{11} + h_{22})/2\), we can write

\[\pi = q\omega + H\bar{\omega}.\]

Here, \(H\) is the mean curvature. The second fundamental form \(II\) is written as

\[II = \frac{q}{2} \omega \omega + H \omega \bar{\omega} + \frac{q}{2} \bar{\omega} \bar{\omega}.\]

It follows from (2.11) and (2.12) that

\[K = -1 + H^2 - |q|^2 = -1 + \det(h_{ij})\]

As a corollary,

\[H^2 - K - 1 \geq 0\]

holds at every point \(p \in M\), with equality if and only if \(p\) is an umbilic point.

The third fundamental form \(III = (de_3, de_3)_L\) is given by

\[III = \langle e_1 \otimes \omega^3, e_1 \otimes \omega^3 \rangle_L = (\omega^3)^2 = \pi \bar{\pi} = |\pi|^2.\]

The ideal boundary \(\partial \mathbb{H}^3\) is considered as the quotient space \(N^3/\sim\), where

\[N^3 = \{ x = (x_0, x_1, x_2, x_3) \in L^4 ; \langle x, x \rangle_L = 0, \; x_0 > 0 \}\]

and \(x \sim y\) if \(x = \lambda y\) for some positive constant \(\lambda\). In other words, \(\partial \mathbb{H}^3\) consists of positive null half-lines in \(L^4\). \(N^3/\sim\) is diffeomorphic to the 2-sphere, and a natural conformal structure on \(N^3/\sim\) is given by the induced metric on \(N^3\). Hence, \(\partial \mathbb{H}^3(= N^3/\sim)\) is identified with the conformal 2-sphere. By definition, the hyperbolic Gauss
maps are \(G^\pm = [e_0 \pm e_3] : M \to \partial \mathbb{H}^3\), where \([v]\) denotes the line spanned by \(v \in \mathbb{L}^1\). Because we mainly treat \(G^+ = [e_0 + e_3]\), we simply write \(G\) for \(G^+\).

The conformal structure on \(M\) induced by \(G\) is the conformal class determined by \(\langle d(e_0 + e_3), d(e_0 + e_3) \rangle_L\). Indeed, it is computed as follows:

**Lemma 2.1.**

\[
\langle d(e_0 + e_3), d(e_0 + e_3) \rangle_L = |\omega - \bar{\pi}|^2 = 2(H - 1) I - K I.
\]

**Proof.** Since \(de_0 + de_3 = e \otimes (\omega - \bar{\pi}) + \bar{e} \otimes (\bar{\omega} - \pi)\) holds by (2.10), the first equality is obvious. The second equality follows from a straightforward computation using (2.12), (2.13) and (2.14). \(\square\)

Similarly, the third fundamental form \(\mathbb{I} = \langle de_3, de_3 \rangle_L\) is computed as (2.16)

\[
\mathbb{I} = \langle de_3, de_3 \rangle_L = 2H I - (K + 1) I.
\]

**Proposition 2.2.** The Gaussian curvature \(\tilde{K}\) of the pseudometric \(\langle d(e_0 + e_3), d(e_0 + e_3) \rangle_L\) is

\[
\tilde{K} = \frac{K}{K - 2(H - 1)}.
\]

**Proof.** Setting \(\alpha = \omega - \bar{\pi}\), we have

\[
da \alpha = d\omega - d\bar{\pi} = \sqrt{-1} \omega^2 \wedge \omega - (\sqrt{-1} \omega^2 \wedge \bar{\pi}) = \sqrt{-1} \omega^2 \wedge \alpha.
\]

Hence, we can consider \(\omega^2\) as the connection form of the metric \(|\alpha|^2\). On the other hand, by (2.12) and (2.14),

\[
\alpha \wedge \bar{\alpha} = (\omega - \bar{\pi}) \wedge (\bar{\omega} - \pi) = \{(1 - H)\omega - \bar{e}\bar{\omega}\} \wedge \{(1 - H)\bar{\omega} - q\omega\}
\]

\[
= \{(1 - H)^2 - |q|^2\} \omega \wedge \bar{\omega} = (K - 2H + 2) \omega \wedge \bar{\omega}.
\]

Therefore

\[
d\omega^2 = K \cdot \frac{-1}{2} \omega \wedge \bar{\omega} = \frac{K}{K - 2H + 2} \cdot \frac{-1}{2} \alpha \wedge \bar{\alpha},
\]

which proves the assertion. \(\square\)

### 2.2. Parallel surfaces.

A map \(f_t := \cosh t f + \sinh t \nu\) is called the parallel surface of \(f\) at distance \(t\). It is easily verified that \(f_t : M \to \mathbb{H}^3\) and that \(f_t(p)\) is joined to \(f(p)\) by a hyperbolic line segment of length \(t\). In general, \(f_t\) may fail to be an immersion. In fact, \(f_t\) is an immersion if and only if \(\cosh t \omega^i + \sinh t \omega_3^i \neq 0\) for every \(p \in M\), because

\[
df_t = \cosh t df + \sinh t d\nu = \cosh t de_0 + \sinh t de_3
\]

\[
= \cosh t e_i \otimes \omega^i + \sinh t e_i \otimes \omega_3^i = e_i \otimes (\cosh t \omega^i + \sinh t \omega_3^i).
\]

In this section, we assume that \(f_t\) is an immersion, unless otherwise stated.

The first fundamental form \(I_t = \langle df_t, df_t \rangle_L\) is

\[
I_t = (\cosh t \omega^1 + \sinh t \omega_3^1)^2 + (\cosh t \omega^2 + \sinh t \omega_3^2)^2,
\]

hence, the \(\theta^i := \cosh t \omega^i + \sinh t \omega_3^i\) \((i = 1, 2)\) form an orthonormal frame of \(f_t\). It follows from the structure equations (2.10) and (2.14) that

\[
d\theta^i = -\omega^i_j \wedge \theta^j.
\]
Thus $\omega_1^2$ is also a connection form of $I_t = (\theta^1)^2 + (\theta^2)^2$. Denoting the Gaussian curvature of $I_t$ by $K_t$, we have

$$d\omega_1^2 = K_t \theta^1 \wedge \theta^2 = (K \omega^1 \wedge \omega^2).$$

Using $K = -1 + \det(h_{ij})$ and $2H = h_{11} + h_{22}$, we have

$$\theta^1 \wedge \theta^2 = \{\cosh^2 t - 2H \cosh t \sinh t + (K + 1) \sinh^2 t\} \omega^1 \wedge \omega^2.$$ 

Therefore

$$d\omega_1^2 = \frac{K}{K \sinh^2 t - 2H \cosh t \sinh t + \cosh^2 t + \sinh^2 t} \theta^1 \wedge \theta^2.$$ 

This implies that

$$K_t = \frac{K}{K \sinh^2 t - 2H \cosh t \sinh t + \cosh^2 t + \sinh^2 t}.$$ 

Thus, since $(K_t) = K_t$, we have

$$K_t = \frac{K}{K \sinh^2 t + 2H \cosh t \sinh t + \cosh^2 t + \sinh^2 t} = K.$$ 

This formula together with (2.17) implies that

$$H_t = \frac{H(\cosh^2 t + \sinh^2 t) - (2 + K) \cosh t \sinh t}{K \sinh^2 t - 2H \cosh t \sinh t + \cosh^2 t + \sinh^2 t}.$$ 

The formulas (2.17) and (2.18) yield the following well-known theorem:

**Theorem 2.3.**

(1) All parallel surfaces of a flat surface are also flat.

(2) A family of parallel surfaces of a surface with constant mean curvature $(|H| > 1)$ contains a surface with constant Gaussian curvature $(K > 0)$, and vice versa.

We can rewrite (2.18) as

$$H_t - 1 = \frac{(\cosh t + \sinh t)(\cosh t + \sinh t)(H - 1) - K \sinh t}{K \sinh^2 t - 2(H - 1) \cosh t \sinh t + (\cosh t - \sinh t)^2}.$$ 

Multiplying $K$ on both sides, we have

$$K(H_t - 1) = K_t e^t \{e^t(H - 1) - K \sinh t\}.$$ 

For example, if we assume that the original surface $f$ has constant mean curvature one (CMC-1), then

$$H_t - 1 = (-e^t \sinh t)K_t,$$

thus, $f_t$ is a Weingarten surface. The family of Weingarten surfaces satisfying $H - 1 = \lambda K$ for some constant $\lambda$ includes the following interesting surfaces:

- If $\lambda = 0$, then $f$ is a CMC-1 surface.
- If $\lambda = 1/2$, then at least one of the principal curvatures equals 1.
- If $\lambda = 1$, then the sum of the reciprocals of the principal curvature is the constant value 2, that is, $f$ is a surface with constant harmonic-mean curvature one (HMC-1).
These are verified by $K = -1 + \kappa_1\kappa_2$, $2H = \kappa_1 + \kappa_2$ and that the harmonic-mean curvature is $2/(\kappa_1^{-1} + \kappa_2^{-1})$, where $\kappa_i$ ($i = 1, 2$) denote the principal curvatures.

Conversely, we assume that the original surface $f$ satisfies $H - 1 = \lambda K$ for some constant $\lambda$. Then the parallel surface $f_t$ satisfies

$$H_t - 1 = \frac{(2\lambda - 1)e^{2t} + 1}{2}K_t,$$

and hence is the same kind of Weingarten surface. Since $\lambda_t := \{(2\lambda - 1)e^{2t} + 1\}/2$ satisfies $(2\lambda_t - 1) = (2\lambda - 1)e^{2t}$, the following lemma is clear:

**Lemma 2.4.** (1) If $\lambda = 1/2$, then $\lambda_t = 1/2$ for all $t$.
(2) If $\lambda < 1/2$, then $\lambda_t < 1/2$ for all $t$, and $\lambda_t = 0$ for some unique $t$.
(3) If $\lambda > 1/2$, then $\lambda_t > 1/2$ for all $t$, and $\lambda_t = 1$ for some unique $t$.

Therefore, we have the following theorem:

**Theorem 2.5.** (1) Let $f$ be a surface satisfying that at least one of the principal curvatures equals 1, i.e., a Weingarten surface with $H - 1 = K/2$. Then all parallel surfaces of $f$ also satisfy that at least one of the principal curvatures equals 1.
(2) Let $f$ be a Weingarten surface with $H - 1 = \lambda K$ for some constant $\lambda(>1/2)$. Then the family of parallel surfaces of $f$ consists of Weingarten surfaces with $H - 1 = \lambda K$ ($\lambda > 1/2$). This family includes a single HMC-1 surface.
(3) Let $f$ be a Weingarten surface with $H - 1 = \lambda K$ for some constant $\lambda(<1/2)$. Then the family of parallel surfaces of $f$ consists of Weingarten surfaces with $H - 1 = \lambda K$ ($\lambda < 1/2$). This family includes a single CMC-1 surface.

Theorem 2.5 is well-known, whereas Theorem 2.6 seems to be lesser-known.

### 2.3. Weingarten surfaces satisfying $H - 1 = \lambda K$.

Throughout this section, $f: M \to \mathbb{H}^3$ denotes a Weingarten surface satisfying $H - 1 = \lambda K$ for some constant $\lambda$, unless otherwise stated.

It follows from Lemma 2.1 that

$$\langle d(e_0 + e_3), d(e_0 + e_3) \rangle_L = 2(H - 1)I - KI = -K(I - 2\lambda I).$$

Hence, if we endow $M$ with the “metric” $I - 2\lambda I$, then the hyperbolic Gauss map $G$ is conformal. However, we need to check that $I - 2\lambda I$ is indeed a metric:

**Lemma 2.6.** If $\lambda \neq 1/2$, then $I - 2\lambda I$ is either positive or negative definite.

**Proof.** $I - 2\lambda I$ is definite if and only if

$$\det \begin{bmatrix} 1 - 2\lambda h_{11} & -2\lambda h_{12} \\ -2\lambda h_{12} & 1 - 2\lambda h_{22} \end{bmatrix} > 0.$$

This condition (2.19) is equivalent to $4\lambda^2(K + 1) - 2\lambda(2H) + 1 > 0$. Moreover, from the assumption $H - 1 = \lambda K$, this is equivalent to $(2\lambda - 1)^2 > 0$.

As stated before, the special case $\lambda = 0$ concerns CMC-1 surfaces. The following proposition is shown in [157] when $\lambda = 0$, and can be proved by the same argument. Thus, the proof is omitted here.

**Proposition 2.7.** Let $f: M \to \mathbb{H}^3$ be a Weingarten surface satisfying $H - 1 = \lambda K$ for some constant $\lambda(\neq 1/2)$. Then $I - 2\lambda I$ determines a conformal structure on $M$, and the hyperbolic Gauss map $G: (M, I - 2\lambda I) \to \partial \mathbb{H}^3$ is conformal.

Conversely, if an immersed surface $f: M \to \mathbb{H}^3$ satisfies...
(i) $1 - 2\lambda_0 \mathbb{I}$ is definite, and
(ii) $G: (M, 1 - 2\lambda_0 \mathbb{I}) \to \partial \mathbb{H}^3$ is conformal
for some constant $\lambda_0$, then $f$ is a totally umbilic surface or a Weingarten surface satisfying $H - 1 = \lambda_0 K$.

The following propositions follow easily from (2.15) and Proposition 2.2, respectively.

**Proposition 2.8.** If $f: M \to \mathbb{H}^3$ is a Weingarten surface satisfying $H - 1 = \lambda K$ ($\lambda \neq 1/2$), then the Gaussian curvature $K$ satisfies the following inequalities:

(i) If $\lambda = 0$, then $K \leq 0$.
(ii) If $\lambda < 1/2(\neq 0)$, then $K \leq 0$ or $K \geq (1 - 2\lambda)/\lambda^2$.
(iii) If $\lambda > 1/2$, then $K \leq (1 - 2\lambda)/\lambda^2$ or $K \geq 0$.

And the mean curvature $H$ satisfies the following:

(i) If $0 < \lambda < 1/2$, then $H \leq 1$ or $H \geq (1 - \lambda)/\lambda$.
(ii) If $\lambda < 0$ or $\lambda > 1/2$, then $H \leq (1 - \lambda)/\lambda$ or $H \geq 1$.

**Proposition 2.9.** For a Weingarten surface satisfying $H - 1 = \lambda K$ ($\lambda \neq 1/2$), the pseudometric $(d(e_0 + e_3), d(e_0 + e_3))_L$ has constant curvature $1/(1 - 2\lambda)$.

### 3. Surfaces with constant harmonic-mean curvature one

#### 3.1. Basics
In this section, we study the case $\lambda = 1$ for Weingarten surfaces satisfying $H - 1 = \lambda K$, that is, the case $H - 1 = K$. As stated in the previous section, a Weingarten surface satisfying $H - 1 = K$ has constant harmonic-mean curvature one, and we call it an HMC-1 surface.

By Lemma 2.1 and (2.16), an HMC-1 surface satisfies

$$\langle de_0 + de_3, de_0 + de_3 \rangle_L = K I - 2 \mathbb{I},$$

(3.1)

$$\langle III \rangle = \langle de_3, de_3 \rangle_L = -H I - 2 \mathbb{I}.$$  

(3.2)

These two quadratic differentials are conformally equivalent. Following Bryant's notation (13), we set $\eta := (\omega_1^3 - \omega_3^3) - \sqrt{-1}(\omega_2^2 - \omega_2^3)$ ($= \hat{\omega} - \pi$).

The formulas (3.1) and (3.2) become

$$|\eta|^2 = (\omega_1^3 - \omega_3^3)^2 + (\omega_2^2 - \omega_2^3)^2 = -K(1 - 2 \mathbb{I}),$$

(3.3)

$$|\pi|^2 = (\omega_1^3)^2 + (\omega_2^3)^2 = -H(1 - 2 \mathbb{I}).$$

(3.4)

Consequently, we obtain the following lemma:

**Lemma 3.1.** $\eta(p) = 0$ if and only if $p$ is an umbilical point with $\mathbb{I}_p = I_p$, i.e., a point where $K = 0$ and $H = 1$. $\pi(p) = 0$ if and only if $p$ is a totally geodesic point, i.e., a point where $K = -1$ and $H = 0$.

**Lemma 3.2.** The Gaussian and mean curvatures $K, H$ are given by

$$K = \frac{|\eta|^2}{|\pi|^2 - |\eta|^2}, \quad H = \frac{|\pi|^2}{|\pi|^2 - |\eta|^2}.$$

**Proof.** It follows from (3.3) and (3.4) that $H|\eta|^2 = K|\pi|^2$. This formula implies the assertion, since $H - 1 = K$. $\square$
Note that
\[ K \leq -1 \ (H \leq 0) \text{ or } K \geq 0 \ (H \geq 1) \]
for HMC-1 surfaces, because of Proposition 2.8. It follows from Lemma 3.2 that \( K \leq -1 \) if and only if \(|\pi|^2 < |\eta|^2|\), and that \( K \geq 0 \) if and only if \(|\pi|^2 > |\eta|^2|\).

**Lemma 3.3.**

(3.5) \( \bar{q}\eta = -K \bar{\pi} \).

**Proof.** By (2.14), we have \(|q|^2 = H^2 - (K + 1) = H^2 - H(H - 1) = HK \). Moreover, using (2.12), we have \( \eta = \bar{\omega} - \pi = \bar{\omega} - (q\omega + H\bar{\omega}) = -q\omega + (1 - H)\bar{\omega} \), therefore, \( \bar{q}\eta = -|q|^2\omega - K\bar{q}\omega = -HK\omega - K\bar{q}\omega = (-K)(H\omega + \bar{q}\omega) = (-K)\bar{\pi} \). \( \square \)

Now we shall equip \( M \) with the complex structure that is compatible with the conformal structure \([I - 2I]\). Here, we give the orientation of \( M \) so that \( \eta \) is a \((1,0)\)-form. Note that \( \bar{\pi} \) is also a \((1,0)\)-form, because of (3.5).

**Lemma 3.4.** Let \( I^{2,0} \) and \( \underline{I}^{2,0} \) denote the \((2,0)\)-parts of the complexification of the fundamental forms \( I \) and \( \underline{I} \), respectively. Then

(3.6) \( I^{2,0} = 2 \underline{I}^{2,0} = \eta \bar{\pi} \),

and \( \eta \bar{\pi} \) is a holomorphic quadratic differential on \( M \).

**Proof.** (3.6) is obtained by rewriting \( I \) and \( \underline{I} \) with \( \eta = \bar{\omega} - \pi \). Indeed,

\[ I (= \omega \bar{\omega}) = \eta \bar{\pi} + |\pi|^2 + |\eta|^2 + \bar{\eta}\pi, \quad \underline{I} (= \text{Re}(\omega\pi)) = \frac{1}{2}(\eta \bar{\pi} + 2|\pi|^2 + \bar{\eta}\pi). \]

It follows from the formulas in (2.10), (2.11) that \( d\bar{\pi} = \sqrt{-1}\omega^1_2 \wedge \bar{\pi} \) and \( d\eta = -\sqrt{-1}\omega^1_2 \wedge \eta \). They imply that \( \eta \bar{\pi} \) is holomorphic. \( \square \)

**Proposition 3.5.** The pseudometric \(|\pi|^2 \) has the Gaussian curvature \( K/(K + 1) \) of \( M \).

**Proof.** Since \( d\bar{\pi} = \sqrt{-1}\omega^1_2 \wedge \bar{\pi} \), we may regard \( \omega^1_2 \) as the connection form for \( |\pi|^2 \). Moreover, its exterior differential is computed as

\[ d\omega^1_2 = -\frac{1}{\sqrt{-1}}(\pi \wedge \bar{\pi} + \omega \wedge \bar{\omega}) = -\frac{1}{\sqrt{-1}}(\pi \wedge \bar{\pi} + \frac{1}{K + 1}\bar{\pi} \wedge \pi) \]

\[ = -\frac{1}{\sqrt{-1}}\left(\frac{-K}{K + 1}\bar{\pi} \wedge \pi\right) = \frac{K}{K + 1}\frac{\sqrt{-1}}{2}\bar{\pi} \wedge \pi, \]

because of (2.10) and (2.11), which proves the assertion. \( \square \)

**Lemma 3.6.**

\[ \bar{\partial}\partial(e_0 + e_3) = \frac{1}{2}e_0 \otimes (\bar{\eta} \wedge \eta). \]
Proof. Because \( d(e_0 + e_3) = e \bar{\eta} + \bar{e} \eta \), we have \( \partial (e_0 + e_3) = \bar{\eta} \). Furthermore, taking \( \partial \), we can calculate as follows:

\[
\begin{align*}
\overline{\partial} \partial (e_0 + e_3) &= \overline{\partial} \bar{\eta} \wedge \eta + \bar{\eta} \wedge \eta \\
&= d \bar{\eta} \wedge \eta + \bar{\eta} \wedge \eta \quad \text{(since \( \eta \) is a (1,0)-form,)} \\
&= \left( e_0 \otimes \frac{1}{2} \omega + \sqrt{-1} \bar{\eta} \wedge \omega_2^1 + e_3 \otimes \frac{1}{2} \bar{\pi} \right) \wedge \eta + \bar{\eta} \otimes \left( -\sqrt{-1} \omega_2^1 \wedge \eta \right) \\
&= \frac{1}{2} e_0 \otimes (\omega \wedge \eta) \quad \text{(since \( \pi \) is a (1,0)-form,)} \\
&= \frac{1}{2} e_0 \otimes ((\omega - \bar{\pi}) \wedge \eta) \quad \text{(since \( \bar{\pi} \) is a (1,0)-form,)} \\
&= \frac{1}{2} e_0 \otimes (\bar{\eta} \wedge \eta).
\end{align*}
\]

\( \square \)

3.2. An overview on the work by Gálvez, Martínez and Milán. In this section we give an overview on deriving Gálvez-Martínez-Milán’s formula stated in the introduction, restricting ourselves to HMC-1 surfaces.

First of all, we review the matrix model for \( \mathbb{H}^3 \). We identify \( \mathbb{L}^4 \) with \( \text{Herm}(2) \), the set of \( 2 \times 2 \) Hermitian matrices, via

\[
\mathbb{L}^4 \ni x = (x_0, x_1, x_2, x_3) \leftrightarrow X = \begin{bmatrix} x_0 + x_3 & x_1 + \sqrt{-1} x_2 \\ x_1 - \sqrt{-1} x_2 & x_0 - x_3 \end{bmatrix} \in \text{Herm}(2).
\]

Since \( \langle x, y \rangle_L = -\det X \) and \( 2 x_0 = \text{tr} X \),

\[
\begin{align*}
\mathbb{H}^3 &= \{ X \in \text{Herm}(2) ; \det X = 1, \text{tr} X > 0 \} \\
&= \{ aa^* ; a \in \text{SL}(2, \mathbb{C}) \} = \text{SL}(2, \mathbb{C})/\text{SU}(2), \\
\partial \mathbb{H}^3 &= \{ X \in \text{Herm}(2) ; \det X = 0, \text{tr} X > 0 \}/\sim \\
&= \left\{ aa^* ; a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{C}^2 \setminus \{0\} \right\}/\sim \\
&= (\mathbb{C}^2 \setminus \{0\})/(\mathbb{C} \setminus \{0\}) = \mathbb{CP}^1,
\end{align*}
\]

where \( a^* \) is the conjugate transpose of \( a \), and \( \mathbb{CP}^1 \) is the complex projective line.

Hereafter, we will consider \( \mathbb{H}^3 \) to be \( \text{SL}(2, \mathbb{C})/\text{SU}(2) \), and \( \partial \mathbb{H}^3 \) to be \( \mathbb{CP}^1 \).

Recall that \( G = [e_0 + e_3] : M \to \mathbb{CP}^1 (= \partial \mathbb{H}^3) \) is a conformal map. Thus

\[
e_0 + e_3 = \Lambda \begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix}
\]

(3.7)

for some holomorphic functions \( A, B \) and a positive function \( \Lambda \). Note that \( A, B \) and \( \Lambda \) have an ambiguity, but \( A|A|^2, \Lambda|B|^2 \) and \( \Lambda AB \) are well-defined on \( M \).

Let \( z \) be a local holomorphic coordinate on \( M \), and let the lower suffix denote the partial derivative. Differentiating (3.7) twice, we have

\[
(e_0 + e_3)_{zz} = \begin{bmatrix} A & A_z \\ B & B_z \end{bmatrix} \begin{bmatrix} \bar{A}z & \bar{A}_z \\ \bar{B}_z & \bar{B} \end{bmatrix}.
\]

Hence, this and Lemma 3.6 imply that

\[
e_0 = \frac{2}{|\eta/dz|^2} \begin{bmatrix} A & A_z \\ B & B_z \end{bmatrix} \begin{bmatrix} \bar{A}_z \bar{A} & \bar{A}_z \Lambda \\ \bar{B}_z \bar{B} & \bar{B}_z \Lambda \end{bmatrix}.
\]
Here we have assumed (and will continue to assume) that $\eta$ is not identically zero. This assumption means that the surface is not a horosphere.

Setting 
\[ g := \begin{bmatrix} A & \Lambda_z \\ B & B_z \end{bmatrix}, \quad \delta := \frac{2}{|\eta/dz|^2}, \quad \Omega := \delta \begin{bmatrix} \Lambda_{z\bar{z}} & \Lambda_z \\ \Lambda_{\bar{z}} & \Lambda \end{bmatrix}, \]
we have the following formulas:

\begin{equation}
(3.8) \quad e_0 = g\Omega g^*,
\end{equation}

\begin{equation}
(3.9) \quad e_3 = (e_0 + e_3) - e_0 = g \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} g^* - g\Omega g^* = g\tilde{\Omega} g^*,
\end{equation}

where 
\[ \tilde{\Omega} = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} - \Omega = \begin{bmatrix} \Lambda - \delta \Lambda_{z\bar{z}} & -\delta \Lambda_z \\ -\delta \Lambda_{\bar{z}} & -\delta \Lambda \end{bmatrix}. \]

**Lemma 3.7.**

\[ |\eta/dz|^2 = |\det g|^2 \Lambda^2 = 4(\log \Lambda)_{z\bar{z}}. \]

**Proof.**

\begin{align*}
\langle e_0, e_0 \rangle_L &= -\det e_0 = -\det(g\Omega g^*) = -|g|^2 \delta^2 (\Lambda_{z\bar{z}} \Lambda - \Lambda_z \Lambda_{\bar{z}}), \\
\langle e_3, e_3 \rangle_L &= -\det e_3 = -\det(g\tilde{\Omega} g^*) = -|g|^2 \left\{ -\delta \Lambda^2 + 2\delta^2 (\Lambda_{z\bar{z}} \Lambda - \Lambda_z \Lambda_{\bar{z}}) \right\}.
\end{align*}

Subtracting and adding these, we have

\begin{align*}
-2 &= -|\det g|^2 \delta \Lambda^2, \quad \text{i.e., } |\eta/dz|^2 = |\det g|^2 \Lambda^2, \\
0 &= -|\det g|^2 \left\{ -\delta \Lambda^2 + 2\delta^2 (\Lambda_{z\bar{z}} \Lambda - \Lambda_z \Lambda_{\bar{z}}) \right\}, \quad \text{i.e., } \Lambda^2 = 2\delta (\Lambda_{z\bar{z}} \Lambda - \Lambda_z \Lambda_{\bar{z}}).
\end{align*}

By Proposition 2.2, the pseudometric $|\eta|^2 = \langle d(e_0 + e_3), d(e_0 + e_3) \rangle_L$ has constant curvature $-1$ for an HMC-1 surface. It follows from the Frobenius theorem that there exists a holomorphic map $h$ from the universal cover $\tilde{M}$ to the Poincaré disk $\mathbb{D}$ such that the pull-back of the Poincaré metric via $h$ coincides with $|\eta|^2$, that is,

\begin{equation}
(3.10) \quad 1 - |h|^2 > 0, \quad |\eta|^2 = \frac{4|h|^2}{(1 - |h|^2)^2}.
\end{equation}

Exchanging with $h$, we reexamine (3.8) and (3.9). Since the pair $A, B$ has the ambiguity of multiplication by non-zero holomorphic functions, we can start with the assumption

\begin{equation}
(3.11) \quad (\det g =) AB_z - A_z B = h_z.
\end{equation}

It follows from Lemma 3.7 and (3.10) that

\begin{equation}
(3.12) \quad \Lambda = \frac{2}{1 - |h|^2}.
\end{equation}

It is straightforward to calculate that

\begin{align*}
\Lambda_z &= \frac{2h_z \bar{h}}{(1 - |h|^2)^2}, \\
\Lambda_{z\bar{z}} &= \frac{2|h|^2(1 + |h|^2)}{(1 - |h|^2)^3}.
\end{align*}
Hence, we have
\[ \theta(3.16) \]
Substituting (3.10), (3.12), (3.13), (3.14) into (3.8), (3.9), respectively, we have
\[ \text{Lemma 3.8.} \]
As a by-product of this formula, we obtain the following lemma:

Introducing the three matrices
\[ G := \begin{bmatrix} A & A_z/h_z \\ B & B_z/h_z \end{bmatrix}, \quad \mathcal{H} := \begin{bmatrix} 1 + |h|^2 \\ -|h|^2 \end{bmatrix}, \quad \tilde{\mathcal{H}} := \begin{bmatrix} 1 \\ -1 + |h|^2 \end{bmatrix}, \]
we can write
\[ e_0 = G\mathcal{H}G^*, \quad e_3 = \tilde{G}\tilde{\mathcal{H}}G^*. \]
By straightforward calculation using (3.11), i.e., \( A_z B - AB_z = \dot{h}_z \), we have
\[ G^{-1}dG = \begin{bmatrix} 0 & \theta \\ dh & 0 \end{bmatrix}, \quad \text{where } \theta = \frac{B_z A_{zz} - A_z B_{zz}}{(h_z)^2}dz. \]
The one-form \( \theta \) is also written as
\[ (3.15) \quad \theta = \frac{1}{A} d \left( \frac{dA}{dh} \right) = \frac{1}{B} d \left( \frac{dB}{dh} \right). \]
Note that \( \theta \) is a one-form defined on \( \tilde{M} \).

In the following, we describe the fundamental forms in terms of \( h \) and \( \theta \). It is not difficult to calculate that
\[ de_0 = G \begin{bmatrix} ((|h|^2 - 1)\bar{\theta} & 2d\bar{h}/(1 - |h|^2) + (1 - |h|^2)\bar{\theta} \\ 0 & 2d\bar{h}/(1 - |h|^2) + (1 - |h|^2)\bar{\theta} \end{bmatrix} G^*. \]
Hence, we have
\[ (3.16) \quad I = -\det(de_0) = \left| \frac{2}{1 - |h|^2}dh + (1 - |h|^2)\bar{\theta} \right|^2 = \frac{4|\dot{h}|^2}{(1 - |h|^2)^2} + 2\theta dh + 2\theta d\bar{h} + (1 - |h|^2)^2|\theta|^2. \]
As a by-product of this formula, we obtain the following lemma:

**Lemma 3.8.** \( \theta dh \) and \( (1 - |h|^2)^2|\theta|^2 \) are well-defined on \( \tilde{M} \).

It is not difficult to calculate that
\[ de_3 = G \begin{bmatrix} ((|h|^2 - 1)\bar{\theta} & (|h|^2 - 1)\bar{\theta} \\ 0 & (|h|^2 - 1)\bar{\theta} \end{bmatrix} G^*. \]
Hence, we have
\[ \boxed{\quad \boxed{ \mathbf{I} = -\det(de_3) = (1 - |h|^2)^2 |\theta|^2. \quad (3.17) } \}
It follows that
\[ I - 2 \mathbf{I} = |de_0|^2 + \{ |de_0 + de_3|^2 - |de_0|^2 - |de_3|^2 \} = |de_0 + de_3|^2 - |de_3|^2 \]
\[ (3.17) \quad \boxed{\quad \boxed{ \mathbf{I} - 2 \mathbf{I} = \frac{4|\dot{h}|^2}{(1 - |h|^2)^2} - (1 - |h|^2)^2|\theta|^2. \quad (3.17) } \}

From the argument above, one can understand the Gálvez-Martínez-Milán formula stated in the introduction.

We finish this section by providing some other formulas. By \(3.16\) and \(3.17\), we have

\[
I = \theta dh + (1 - |h|^2)^2 + \bar{\theta} dh.
\]

In particular,

\[
|\pi|^2 = \|I\|^1,1 = (1 - |h|^2)^2.
\]

It follows from Lemma 3.2, \(3.10\) and \(3.18\) that

\[
K = \frac{4|dh|^2}{(1 - |h|^2)^4|\theta|^2 - 16|dh|^2}, \quad H = \frac{(1 - |h|^2)^4|\theta|^2 - 16|dh|^2}{(1 - |h|^2)^4|\theta|^2 - 4|dh|^2}.
\]

### 3.3. Improvement of the representation formula.

We shall give a slight improvement of the Gálvez-Martínez-Milán formula, limiting ourselves to HMC-1 surfaces, and make it clear what is a local invariant and what is a global invariant.

The hyperbolic Gauss map \(G = A/B\) is globally-defined on \(M\). We can represent \(G\) using \(G\) as follows:

**Lemma 3.9.**

\[
G = (-G_h)^{-3/2} \left[ \frac{-GG_h + GG_{hh}/2 - G_h^2}{-G_h} - \frac{G}{G_h/2} \right],
\]

where \(G_h = dG/dh\), \(G_{hh} = d^2G/dh^2\).

**Proof.** \(G\) is computed as

\[
G = \begin{bmatrix} A & dA/dh \\ B & dB/dh \end{bmatrix} = \begin{bmatrix} GB & d(GB)/dh \\ B & dB/dh \end{bmatrix} = \frac{1}{B} \begin{bmatrix} GB^2 & BD(GB)/dh \\ B^2 & BdB/dh \end{bmatrix}.
\]

On the other hand, substituting \(A = BG\) to \(AdB - BdA = dh\), we have

\[
B^2 = -\frac{dh}{dG} = -\frac{1}{G_h}.
\]

Eliminating \(B\) from \(3.20\) with this, we have the assertion. \(\Box\)

Thus, for an HMC-1 surface \(f\), we can make a representation formula \(f = G\hat{H}G^*\) with \(G\) as in \(3.16\) and

\[
\hat{H} = \begin{bmatrix} 1 + |h|^2 & \bar{h} \\ 1 - |h|^2 & h \end{bmatrix}.
\]

from a meromorphic function \(G\) on \(M\) and a holomorphic map \(h: \tilde{M} \to \mathbb{D}\). However, it is not defined on \(M\) yet (merely on \(\tilde{M}\), in general). We need to find the condition that \(f\) is single-valued on \(M\). Indeed, we prove:

**Proposition 3.10.** \(f = G\hat{H}G^*\) with \(G, \hat{H}\) as in \(3.16\) and \(3.22\) is single-valued on \(M\) if and only if the pseudometric \(|\eta|^2 = 4|dh|^2/(1 - |h|^2)^2\) is single-valued on \(M\).
Proof. Suppose that $|\eta|^2 = 4|dh|^2/(1 - |h|^2)^2$ is single-valued on $M$.

By (3.12) and Lemma 3.9, we have

\[
\Lambda|A|^2 = \frac{2}{1 - |h|^2} |G|^2 - G h|^{-1} = \frac{2}{1 - |h|^2} |G|^2 |dh| \frac{dG}{|dG|} = 2 |\Lambda| B|^2.
\]

Thus, all $\lambda|A|^2$, $\lambda|B|^2$, $\lambda A \bar{B}$ are single-valued on $M$. In other words, $e_0 + e_3$ is single-valued on $M$ (because of (3.7)). On the other hand, $\eta \wedge \bar{\eta}$ is also single-valued on $M$. Therefore, recalling the formula $\partial \bar{\partial} (e_0 + e_3) = \frac{1}{2} e_0 \otimes \bar{\eta} \wedge \eta$, we can conclude that $e_0(=f)$ is also single-valued on $M$. □

Therefore we have:

**Theorem 3.11.** Let $G$ be a meromorphic function on a Riemann surface $M$, and $|\eta|^2$ a pseudometric on $M$ of constant curvature $-1$. Suppose that the quadratic differential form (3.17) is definite. Then $f := G \bar{G}^*$, determined by (3.10), (3.19) and (3.22), is an HMC-1 immersion from $M$ to $\mathbb{H}^3$.

Conversely, any HMC-1 surface (except a horosphere) has this parametrization in terms of $(G, h)$.

**Remark.** (1) It has been already proved in [KUY1] that the solution to the differential equation (1.2) is described as (3.19).

(2) Under the condition $h(z) = z$, the formula $f := G \bar{G}^*$ with (3.19) and (3.22), was already seen in [GMM2, Theorem 4], where the condition $h(z) = z$ is caused by their assumption that $M$ is simply-connected and complete.

As compared with it, Theorem 3.11 is devoted to surfaces of non-trivial topology. The period condition is clarified, indeed, it is that $|\eta|^2$ is single-valued on $M$.

The one-form $\theta$ can be calculated from the Schwarzian derivative as follows:

**Lemma 3.12.**

\[
\theta = -\frac{1}{2} \{G; h\} dh = -\frac{1}{2} \left\{ \frac{G_{hh}}{G_h} \right\}_h - \frac{1}{2} \left( \frac{G_{hh}^2}{G_h^2} \right)_h \right\} dh,
\]

where $\{G; h\}$ denotes the Schwarzian derivative of $G$ with respect to $h$.

**Proof.** Differentiating (3.21) $B^2 = -1/G_h$ with respect to $h$, we have

\[
dB/dh = \frac{1}{2} B \left( \frac{G_{hh}}{G_h^2} \right)_h.
\]

Differentiating this again, we have

\[
\frac{d}{dh} \left( \frac{dB}{dh} \right) = \frac{1}{2} \left\{ \frac{1}{2} \frac{B^2}{G_h} \left( \frac{G_{hh}}{G_h^2} \right)_h + \frac{1}{B} \frac{G_{hh}}{G_h} \left( \frac{G_h}{G_h} \right)_h \right\}.
\]

Therefore, it follows from (3.15) that

\[
\theta = \frac{1}{2} \left\{ \frac{1}{2} \frac{B^2}{G_h} \left( \frac{G_{hh}}{G_h^2} \right)_h + \frac{1}{B} \frac{G_{hh}}{G_h} \left( \frac{G_h}{G_h} \right)_h \right\} dh.
\]
Again, using $B^2 = -1/G_h$, we obtain
\[
\theta = \frac{1}{2} \left\{ \frac{1}{2} \left( \frac{G_{hh}}{(G_h)^2} - \left( \frac{G_{hh}}{G_h} \right) h \right) \right\} dh = -\frac{1}{2} \{ G; h \} dh.
\]
\[\square\]

4. Fronts with constant harmonic-mean curvature one

4.1. Definition. Let $M$ be a Riemann surface. Given a meromorphic function $G: M \to \mathbb{C} \cup \{\infty\}$ and a pseudometric $|\eta|^2$ of constant curvature $-1$ on $M$, we can define a map $f = \mathcal{G}H^G: M \to \mathbb{H}^3$ using (3.10), (3.19) and (3.22). Since $\mathcal{G}$ has poles $\{p_i\}$ in general, $f$ should be considered a map on $M \setminus \{p_i\}$. However, in such a case, we retake $M$ to be $M \setminus \{p_i\}$.

We call $f$ an HMC-1 map associated with $(G, |\eta|^2)$. By definition, the regular image of an HMC-1 map forms an immersed surface with constant harmonic-mean curvature one whose unit normal vector field is $\nu = \mathcal{G} \left[ \frac{1}{-h - 1 + |h|^2} \right] \mathcal{G}^*$. Though $f$ may fail to be an immersion, the unit normal $\nu$ is defined across the singularities. Hence, the following definition does make sense.

An HMC-1 map $f$ is called an HMC-1 front if $(f, \nu): M \to T_1 \mathbb{H}^3(\cong T_1^* \mathbb{H}^3)$ is an immersion, where $T_1 \mathbb{H}^3$ ($T_1^* \mathbb{H}^3$) denotes the unit (co)tangent bundle over $\mathbb{H}^3$. (The term front comes from wave fronts in the theory of singularities.) It is obvious from the definition that the formulas for HMC-1 surfaces in the previous section can be applied for HMC-1 fronts.

**Proposition 4.1.** For an HMC-1 map $f: M \to \mathbb{H}^3$, the following three conditions are equivalent:

1. $f$ is an HMC-1 front.
2. The $(1,1)$-part $I^{1,1} = |\eta|^2 + |\pi|^2$ of the first fundamental form $I = |\eta + \pi|^2$ is a Riemannian metric on $M$.
3. $\mathcal{G}: M \to \text{SL}(2, \mathbb{C})$ is non-singular.

**Proof.** We can put (1)–(3) in different words as follows:

1. $|df|^2$ and $|dv|^2$ never vanish simultaneously, that is, $|2dh/(1-|h|^2) + (1-|h|^2)\theta|^2$ and $(1-|h|^2)^2 |\theta|^2$ never vanish simultaneously.
2. $I^{1,1} = 4|dh|^2/(1-|h|^2)^2 + (1-|h|^2)^2 |\theta|^2$ never vanishes.
3. Either $\theta \neq 0$, or both $\theta = 0$ and $dh \neq 0$.

Then, it is not difficult to see the equivalency. \[\square\]

**Remark.** $T_1 \mathbb{H}^3$ ($\cong T_1^* \mathbb{H}^3$) has a canonical Riemannian metric, which is called the Sasakian metric. We denote by $I^S$ the pull-back of the Sasakian metric via the map $(f, \nu)$. $I^S$ is a Riemannian metric on $M$ for a front $f: M \to \mathbb{H}^3$. Indeed, $I^S = |df|^2 + |dv|^2 = |\eta + \pi|^2 + |\pi|^2$. $I^S$ is not conformally equivalent to $I^{1,1}$ in general.

It is clear from Proposition 4.1 that a singularity of an HMC-1 front is a point where

\[(\eta + \pi) \wedge (\bar{\eta} + \bar{\pi}) = 0 \iff \eta \wedge \bar{\eta} + \pi \wedge \bar{\pi} = 0 \iff |\eta|^2 = |\pi|^2.
\]

**Proposition 4.2.** There are no compact HMC-1 fronts.
Proof. Suppose, by way of contradiction, that there exists a compact HMC-1 front \( f : M \to \mathbb{H}^3 \).

It follows from Lemma 3.6 that

\[
(e_0 + e_3)_{zz} = \frac{|e/dz|^2}{2} e_0.
\]

Taking the trace of both side, we have

\[
(\text{tr}(e_0 + e_3))_{zz} = \frac{|e/dz|^2}{2} \text{tr} e_0 = |e/dz|^2 x_0 \geq 0.
\]

Hence, \( \text{tr}(e_0 + e_3) \) is a subharmonic function on \( M \). It must be constant, since \( M \) is compact. Again by (4.1), we have \( \eta = 0 \), a contradiction. \( \square \)

4.2. **Weak completeness.** We say that an HMC-1 front \( f : M \to \mathbb{H}^3 \) is weakly complete if \( I^{1,1} \), the (1,1)-part the first fundamental form \( I \), is a complete Riemannian metric on \( M \) (cf. [KRSUY]).

**Proposition 4.3.** For an HMC-1 front, weak completeness is equivalent to the completeness of \( I^S \), the induced metric of the Sasakian metric.

**Proof.** Let \( \gamma : [0, \infty) \to M \) be an arbitrary divergent path. Recall that \( I^S = |\eta + \pi|^2 + |\pi|^2 \) and \( I^{1,1} = |\eta|^2 + |\pi|^2 \). We wish to prove that, if \( \gamma \) has infinite length with respect to one of the two metrics, then it also has infinite length with respect to the other metric. When \( \int_\gamma |\pi| = \infty \), it is trivial that \( \gamma \) has infinite length with respect to both metrics \( I^S \) and \( I^{1,1} \). Hence, we have only to give a proof under the assumption \( \int_\gamma |\pi| < \infty \).

(a) Suppose that \( I^{1,1} \) is complete. Clearly, \( \int_\gamma |\eta| = \infty \). If we divide the interval \([0, \infty)\) so that

\[
[0, \infty) = J_+ \cup J_-, \quad \text{where} \quad J_+ = \{ |\eta| \geq |\pi| \}, \quad J_- = \{ |\eta| < |\pi| \},
\]

then \( \int_{J_+} |\eta| = \infty \), because \( \int_{J_-} |\eta| < \int_{J_-} |\pi| < \infty \). Thus,

\[
\int_\gamma \sqrt{|\eta + \pi|^2 + |\pi|^2} \geq \int_\gamma |\eta + \pi| \geq \int_\gamma |\eta| - |\pi| \geq \int_{J_+} |\eta| - \int_{J_+} |\pi| = \infty - (\text{finite value}) = \infty.
\]

Therefore, \( I^S \) is complete.

(b) Conversely, we suppose \( I^S \) is complete. Clearly, \( \int_\gamma |\eta + \pi| = \infty \). If we divide the interval \([0, \infty)\) so that

\[
[0, \infty) = J'_+ \cup J'_-, \quad \text{where} \quad J'_+ = \{ |\eta + \pi| \geq |\pi| \}, \quad J'_- = \{ |\eta + \pi| < |\pi| \},
\]

then \( \int_{J'_+} |\eta + \pi| = \infty \), because \( \int_{J'_-} |\eta + \pi| < \int_{J'_-} |\pi| = \infty \). Thus

\[
\int_\gamma \sqrt{|\eta|^2 + |\pi|^2} \geq \int_\gamma |\eta| \geq \int_\gamma |\eta + \pi| - |\pi| \geq \int_{J'_+} |\eta + \pi| - \int_{J'_+} |\pi| = \infty.
\]

Therefore \( I^{1,1} \) is complete. \( \square \)
Note that an HMC-1 front is weakly complete if it is complete (in the usual sense), because \( I^2 = |df|^2 + |dv|^2 \) is complete if \( I = |df|^2 \) is complete.

4.3. HMC-1 fronts of finite topology. There are two kind of ends for (weakly) complete HMC-1 fronts of finite topology. One is conformally equivalent to the punctured disk \( \Delta^* = \{ z : 0 < |z| < 1 \} \), and the other is conformally equivalent to the annulus \( A_r = \{ z : r < |z| < 1 \} \). We shall call the former a puncture-type end, and the latter an annular end. For a puncture-type end \( \Delta^* \), we also call a point \( z = 0 \) an end. For an annular end \( A_r \), we also call the boundary \( |z| = r \) an end.

**Theorem 4.4.** Let \( f : M \to \mathbb{H}^3 \) be a weakly complete HMC-1 front of finite topology. Then

1. the set of singularities never accumulate to a puncture-type end, and
2. the Gaussian curvature \( K(z) \) converges to 0 as \( z \) tends to a puncture-type end.

Hence, the mean curvature \( H(z) \) converges to 1.

**Proof.** Let \( \Delta^* \subset M \) be a puncture-type end. If we assumed \( \lim_{z \to 0} h(z) \in \partial \mathbb{D} \), then some portion of a neighborhood of 0 is not contained in the image of \( h \). This is a contradiction. Therefore \( h(0) \in \mathbb{D} \). Hence, \( \lim_{z \to 0} |\eta|^2 = \lim_{z \to 0} 4|dh|^2/(1 - |h|^2)^2 < \infty \).

On the other hand, \( I^{1-} = |\eta|^2 + |\pi|^2 = 4|dh|^2/(1 - |h|^2)^2 + (1 - |h|^2)^2 |\theta|^2 \) is complete at 0. Thus \( \lim_{z \to 0} |\pi|^2 = \infty \).

Therefore, \( |\pi|^2 \neq |\eta|^2 \) near 0, and \( \lim_{z \to 0} K = \lim_{z \to 0} |\eta|^2/(|\pi|^2 - |\eta|^2) = 0 \). \( \square \)

Note that, compared with Proposition 4.4 (1), it can occur that the set of singularities accumulate toward an annular end (see Example 3 below).

**Proposition 4.5.** Let \( f : M \to \mathbb{H}^3 \) be a weakly complete HMC-1 front of finite topology, and \( V(\equiv A_r) \) an annular end. Then the Gaussian curvature \( K(z) \) converges to \( -1 \) as \( z \) tends to a point \( z_0 \in \partial A_r \), with \( |z_0| = r \), unless \( \lim_{z \to z_0} M = \infty \). Hence, the mean curvature \( H(z) \) converges to 0.

**Proof.** Let \( A_r \subset M \) be an annular end, and take an arbitrary point \( z_0 \in \partial A_r \) with \( |z_0| = r \). Then \( \lim_{z \to z_0} h \in \partial \mathbb{D} \). (If we assumed \( \lim_{z \to z_0} h \in \mathbb{D} \), then the image of some neighborhood of \( z_0 \) is also contained in \( \mathbb{D} \). At every point \( w \) in the neighborhood, \( \lim_{z \to w} |\theta| \) must be infinity, since all divergent paths have infinite length. However, this is impossible because \( \theta \) is a holomorphic one-form.) Hence, \( \lim_{z \to z_0} |\eta|^2 = \infty \). Therefore, \( \lim_{z \to z_0} K = \lim_{z \to z_0} |\eta|^2/(|\pi|^2 - |\eta|^2) = -1 \), unless \( \lim_{z \to z_0} M = \lim_{z \to z_0} |\pi|^2 = \infty \). \( \square \)

From Proposition 4.4 and Proposition 4.5, we may use the adjective horospherical for a puncture-type end, and hemispherical for an annular end.

4.4. Examples. We show some examples of (weakly) complete HMC-1 fronts of finite topology.

In this section, we denote by \( ds^2_{\mathbb{H}} \) the Poincaré metric on the unit disk, and in the figures, the hyperbolic three-space \( \mathbb{H}^3 \) is realized by the Poincaré ball model.

**Example 1.** For a positive number \( \alpha \), consider

\[
G(z) = z, \quad |\eta|^2 = \frac{4|\alpha|^2|z|^{2\alpha-2}}{(1 - |z|^{2\alpha})^2} |dz|^2 \quad (= h^* ds^2_{\mathbb{H}} \text{ where } h(z) = z^\alpha)
\]
on
\[ M = \begin{cases} \Delta = \{ z; |z| < 1 \} & \text{if } \alpha = 1, \\ \Delta \setminus \{0\} & \text{otherwise}. \end{cases} \]

Then the HMC-1 front \( f : M \to \mathbb{H}^3 \) associated with \( (G, |\eta|^2) \) satisfies
\[
\theta = \frac{1 - \alpha^2}{4\alpha} z^{-\alpha-1} dz, \quad dh = \alpha z^{\alpha-1} dz, \quad Q(= \eta \bar{\pi}) = \frac{1 - \alpha^2}{2z^2} dz^2, \\
|\pi|^2 = \frac{|\alpha^2 - 1|^2 (1 - |z|^{2\alpha})}{16|\alpha|^2} |dz|^2.
\]

If \( \alpha = 1 \), then \( f \) is complete and totally geodesic. If \( \alpha \neq 1 \), then \( f \) is weakly complete and has a horospherical end at \( z = 0 \) and a hemispherical end at \( |z| = 1 \). Its singular locus is the circle
\[
|z| = \left( -\sqrt{\frac{2\alpha^2}{\alpha^2 - 1}} + \sqrt{\frac{2\alpha^2}{\alpha^2 - 1} + 1} \right)^{1/\alpha}.
\]

\( K \geq 0 \) inside this circle, and \( K \leq -1 \) outside the circle.

\( (G, h) = (z, z) \) \hspace{1cm} \( (G, h) = (z, z^2) \)

\( (G, h) = (z, \sqrt{z}) \) \hspace{1cm} \( (G, h) = (z, \sqrt{z}) \) (half cut)

**Figure 1.**
**Example 2.** For a non-zero complex number $k$, consider $G(z) = \exp(kz)$, $|\eta|^2 = ds_H^2$ on $\Delta$, then the associated HMC-1 front satisfies

$$
\theta = \frac{k^2}{4} dz, \quad dh = dz, \quad Q = \frac{k^2}{2} dz^2, \quad |\pi|^2 = \frac{|k|^4}{16} (1 - |z|^2^2)|dz|^2.
$$

Its singular locus is $|z|^2 = 1 - 2\sqrt{2}/|k|$. In particular, it has no singularities if $|k| < 2\sqrt{2}$.

![Figure 2](image)

**Example 3.** Consider $G(z) = z + 1/z$, $|\eta|^2 = ds_H^2$ on $\Delta$. Then the associated HMC-1 front satisfies

$$
\theta = \frac{3}{(z^2 - 1)^2} dz, \quad dh = dz, \quad Q = \frac{6}{(z^2 - 1)^2} dz^2, \quad |\pi|^2 = \frac{9(1 - |z|^2^2)^2}{|z^2 - 1|^4}|dz|^2.
$$

Its singular locus is

$$
C : 2|z^2 - 1| = 3(1 - |z|^2)^2.
$$

$f$ has an annular end at $|z| = 1$, and the singular locus $C$ accumulates at $z = \pm 1$.

**References**

[Bry] R. Bryant, *Surfaces of mean curvature one in hyperbolic space*, in *Théorie des variétés minimales et applications*, Astérisque, **154–155** (1988), 321–347.

[CHR] P. Collin, L. Hauswirth, and H. Rosenberg, *The geometry of finite topology Bryant surfaces*, Ann. of Math. (2) **153** (2001), no. 3, 623–659.

[E] C. L. Epstein, *Envelopes of Horospheres and Weingarten Surfaces in Hyperbolic 3-Space*, unpublished.

[GMM1] J. A. Gálvez, A. Martínez and F. Milán, *Flat surfaces in hyperbolic 3-space*, Math. Ann., **316** (2000), 419–435.

[GMM2] J. A. Gálvez, A. Martínez and F. Milán, *Complete linear Weingarten surfaces of Bryant type. A Plateau problem at infinity*, Trans. Amer. Math. Soc., **356** (2004), no. 9, 3405–3428.

[KRSUY] M. Kokubu, W. Rossman, K. Saji, M. Umehara and K. Yamada, *Singularities of flat fronts in hyperbolic space*, to appear in Pacific J. Math.

[KUY1] M. Kokubu, M. Umehara and K. Yamada, *An elementary proof of Small’s formula for null curves in PSL(2, C) and an analogue for Legendrian curves in PSL(2, C)*, Osaka J. Math., **40**(3) (2003), 697–715.
\[(G, h) = (z + 1/z, z)\] singular locus in \(\Delta\)

Figure 3.

[KUY2] M. Kokubu, M. Umehara and K. Yamada, Flat fronts in hyperbolic 3-space, Pacific J. Math., 216 (2004), no.1, 149–175.

[S] M. Spivak, A comprehensive introduction to differential geometry, Vol. V, Publish or Perish, Inc., (1979).

[UY] M. Umehara and K. Yamada, Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space, Ann. of Math. (2) 137 (1993), no.3, 611–638.

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