Abstract. We give a relationship between the classical scissors congruence groups (for the spherical and hyperbolic geometries) and the algebraic $K$-theory of the complex numbers. These results can be seen as modified versions of conjectures of Goncharov \cite{Gon99}. We prove, in particular, that the homology of the “Dehn complex” of Goncharov splits as a summand of the twisted homology of the isometry group. We also show an agreement between certain regulator maps and the volume homomorphisms out of the scissors congruence groups.

Introduction

Scissors congruence asks the following question: given two polytopes $P$ and $Q$, when is it possible to decompose $P$ into finitely many polytopes and form $Q$ out of the pieces? More formally, is it possible to write $P = \bigcup_{i=1}^n P_i$ and $Q = \bigcup_{i=1}^n Q_i$ such that $P_i \cong Q_i$ for all $i$, and such that $\text{meas}(P_i \cap P_j) = \text{meas}(Q_i \cap Q_j) = 0$ for all $i \neq j$? This question can be asked in any dimension and any geometry, but the classical contexts of concern are Euclidean, spherical and hyperbolic geometries.

This question is extremely old, stemming from an ancient Greek definition of area and volume. If two polytopes are scissors congruent then their volumes are equal. The reverse implication is not true, however, above dimension 2; at that point, a second measure, called the Dehn invariant, appears. For three-dimensional polyhedra (in Euclidean, spherical or hyperbolic space) this invariant is defined as follows:

$$D(P) = \sum_{e \text{ edge of } P} \text{len}(e) \otimes \theta(e) \in \mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}.$$ 

Here, $\theta(e)$ is the dihedral angle at $e$; in other words, it is the arc length of the intersection with $P$ of a small circle around $e$. In dimension $n$ it is possible to define other Dehn invariants, by picking a dimension $\ell$ and writing a similar sum over all faces of $P$ of dimension $\ell$; the measure of the angle will then be a portion of the sphere in dimension $n - \ell - 1$. By the Dehn–Sydler theorem \cite{Syd65, Jes68} in Euclidean space in dimensions 3 and 4, two polytopes are scissors congruent if and only if their volumes and Dehn invariants are equal. By work of Dupont and Sah \cite{DS82} the same is true in 3-dimensional spherical and hyperbolic space. We thus have the following conjecture:

**Generalized Hilbert’s Third Problem** \cite[Question 1]{DS82}. In Euclidean, spherical, and hyperbolic geometries, do the volume and generalized Dehn invariant separate the scissors congruence classes of polytopes?

Remark. Spherical scissors congruence groups are often used to measure angles, and are thus normalized so that the class of the entire sphere, as well as classes which arise from suspensions of classes in a lower-dimensional sphere, are 0. The Dehn invariant, and thus the generalized version of Hilbert’s third problem, can also be defined for the reduced spherical scissors congruence classes.

One of the main results of this paper is the following:

**Theorem A.** Generalized Hilbert’s third problem in odd-dimensional reduced spherical and hyperbolic space is true modulo torsion if certain Cheeger–Chern–Simons invariants \cite[Sect. 5]{DK90} are injective. In the reduced spherical case the converse also holds.

Cheeger–Chern–Simons invariants are expected to be injective by Ramakrishnan’s conjecture \cite[7.1.2]{Ram89}. It is known that the classical scissors congruence groups for $S^0$ and $\mathcal{H}^n$ for $n = 1, 2, 3$ are torsion-free, and that in fact all Euclidean scissors congruence groups are real vector spaces. Thus it is not unreasonable to
expect that all scissors congruence groups in positive dimensions are torsion-free, and thus that rationalization does not change the effectiveness of the volume and Dehn invariants.

This theorem is proved in Theorem 4.5.

Motivated by the theory of mixed Tate motives, in [Gon99] Goncharov explored a different perspective on this problem. In the paper, Goncharov constructs a map from the kernel of the Dehn invariant (when restricted to simplices with algebraic vertices) into a slice of the algebraic $K$-theory of $\mathbb{Q}$. Goncharov also conjectures that the scissors congruence groups of $2n - 1$-dimensional hyperbolic (resp. spherical) polytopes should give information about the $n$-th weight part of $K(\mathbb{C})$. More concretely, Goncharov conjectures [Gon99] Conjecture 1.8 that there exists a map

$$H_{i}P_\ast(S^{2n-1}) \rightarrow (gr_n^r K_{n+i}(\mathbb{C})_Q \otimes \epsilon(n))^+,$$

and moreover [Gon99] Conjecture 1.9 that this map is an isomorphism. Here, $P_\ast(S^{2n-1})$ is a chain complex constructed out of iterations of the Dehn invariant starting at $P_\ast(S^{2n-1})$ (see Definition 4.2), $\epsilon(n)$ is twisting factor and the $+\ast$ denotes taking the 1-eigenspace of the action of complex conjugation. Part of the deep interest of this conjecture is its connection to the Beilinson–Soulé conjecture, which states that $gr_j^r K_{2n+1}(R)_Q = gr_j^r K_{2n}(R)_Q = 0$ when $j \leq i$. The complex on the left-hand side of this equation only has nonzero groups for $i = 1$ through $n$. Thus, if this conjecture is true for all $i$ and $n$, the Beilinson–Soulé conjecture for odd-graded $K$-theory groups automatically follows.

We have proved the following alternate form of [Gon99] Conjecture 1.8:

**Theorem B.** There exist zigzags

$$(gr_n^r K_{n+i}(\mathbb{C})_Q \otimes \epsilon(n))^+ \leftrightarrow (C_{n,i} \otimes \epsilon(n))^+ \rightarrow H_{i}P_\ast(S^{2n-1})$$

and

$$(gr_n^r K_{n+i}(\mathbb{C})_Q \otimes \epsilon(n))^− \leftrightarrow (C_{n,i} \otimes \epsilon(n))^− \rightarrow H_{i}P_\ast(\mathcal{H}^{2n-1}).$$

Here, $C_{n,i}$ are groups obtained from the homotopy groups of $BGL(n; \mathbb{C})^+.$

See Theorem 7.2 for a more precise statement.

The main technical result allowing us to conclude Theorem A is the following:

**Theorem C.** Let $X = S^{2n-1}$ or $\mathcal{H}^{2n-1}$, and let $I(X)$ be the isometry group of $X$. Then $H_{i}P_\ast(X)$ is a direct summand of $H_{i+n}(I(X); \mathbb{Q})$.

This is Theorem 5.5.

Our theorems follow from a homotopical analysis of a topological space constructed out of maps analogous to the Dehn invariant. The key observation is that in a topological context homotopy coinvariants and the construction of the “total complex” that Goncharov uses to define $P_\ast$ commute past one another; thus we can compute the homotopy type of a space modeling this complex explicitly. We produce a spectral sequence whose lowest nonzero row is the complex $P_\ast(S^{2n-1})$ (resp. $P_\ast(\mathcal{H}^{2n-1})$); the fact that we can identify the homotopy type of the “total complex” allows us to directly relate this to the homology of $O(2n)$. The map from algebraic $K$-theory induced by the hyperbolic map

$$M \mapsto \left[ \frac{M}{(MT)_{-1}} \right].$$

This approach suggests that scissors congruence is more closely related to the Hermitian $K$-theory of $\mathbb{C}$ than to the usual $K$-theory.

**Outline of the Proof.** Here we offer an outline of the proof of Theorem C, since it has a number of moving parts and makes extensive use of homotopy theory.

The observation that gets the proof off of the ground is that the Dehn complex is the total complex of an $(n - 1)$-cube of spaces, prior to taking homology. Each space in this cube arises as a homotopy coinvariant of a group action; the key observation
here is that the cube can be constructed in such a way that the same group is acting on each space, and all maps in the cube are equivariant. We can then commute the homotopy cofiber past the group action, and analyze them independently. We denote this space \((Y^X)_{hI(X)}\) (it is defined in Section 5). Sections 1, 2, and 3 are devoted to the construction of this cube.

In the homotopical analysis of \((Y^X)_{hI(X)}\) a minor miracle occurs: the space is weakly equivalent to a sphere with \(I(X)\) acting on it (almost) trivially. We offer two proofs of this fact in Section 6. This allows for significant simplification of the spectral sequences that compute them. The spectral sequences that we use are the homotopy orbit spectral sequence (Proposition A.5) and the spectral sequence for the total homotopy cofiber of a cube (Section A.3). An analysis of the total homotopy cofiber spectral sequence (Lemma 5.4) allows us to immediately conclude that the Dehn complex splits off of \(hI(X)\). We can also use the homotopy orbit spectral sequence to compute \(H_*(([Y^X]_{hI(X)})\) to obtain a shift of \(H_*(I(X); \mathbb{Q})\); this allows us to conclude Theorem [A] together with this fact produces Theorem [A].

Theorem [A] is discussed in Section 7. This theorem follows from a comparison of the homology of \(I(X)\) with the homology of the general linear group, which produces the (rational) \(K\)-theory.

Remark. In this paper we mostly focus on spherical and hyperbolic geometries, as well as work over \(\mathbb{R}\) and \(\mathbb{C}\), as these were our main examples of interest. However, most of our techniques do not rely on either these choices of geometry or the choice of field. In future work we hope to work out further implications of these approaches in other fields, geometries, and isometry groups.

Organization. In Section 1 we set up the basic topological objects of study \(F_X\) whose homology will be related to scissors congruence groups. In Section 2 we prove that the homology of these spaces is isomorphic to the classical scissors congruence groups. In Section 2 we define the classical and topological Dehn invariants. In Section 4 we introduce the complex \(P\) and the Dehn cube; Sections 5 and 6 analyze the structure of the spectral sequence and prove the main technical ingredient for Theorem [A]. Section 7 connects the main technical result to Goncharov’s conjectures and proves the main theorems.

Notation and conventions. All groups are considered discrete unless explicitly stated otherwise.

We work in the category of pointed topological spaces. Thus homology is reduced, and all constructions on spaces are pointed. In particular, homotopy coinvariants are taken in a correctly-pointed manner, so that \(*_{hG} \simeq *\) and \((S^0)_{hG} \simeq (BG)_+\).

We denote by \(X^n\) either \(H^n\) or \(S^n\). In each case, we think of \(X^n\) as sitting inside \(\mathbb{R}^{n+1}\), with subspaces being given by subspaces of \(\mathbb{R}^n\) through the origin (which intersect, respectively, the plane where \(x_{n+1} = 1\), the hyperboloid \(-x_0^2 + x_1^2 + \cdots + x_n^2\), and the sphere in a nonempty set). When the dimensions is clear from context we write \(X\) instead of \(X^n\).

It is important to note that all homology is taken with rational coefficients. In the interest of readability we omit the rationals from our notation.

For any abelian group \(A\), we write \(A' \overset{\text{def}}{=} A \otimes \mathbb{Q}\).

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1. Geometry, flags, and configurations

Our main goal in this section is to establish the basic definitions of the objects we will be using, as many of these definitions are not (quite) standard. We work over any infinite base field \(k\).

Definition 1.1 (Based on Cat04 Definition 1.0.3)). A geometry \(X\) is a quadratic space \((E, q)\) over \(k\), where \(q\) is totally nondegenerate, together with its isometry group \(I(X)\). The dimension of \(X\) is \(\dim E - 1\). The points of \(X\) are the points of \(\mathbb{P}(E)\), the projectivization of \(E\). We write \(I(X)\) for the isometry group of \(X\): the subgroup of \(GL(n + 1; k)\) which restricts to isometries between points in \(X\).

When we wish to emphasize that a geometry \(X\) has dimension \(n\), we write it as \(X^n\).

Definition 1.2. The usual geometries are the spherical geometry \(S^n\), given by the quadratic form \(x_0^2 + \cdots + x_n^2\), and the hyperbolic geometry \(H^n\), given by the quadratic form \(-x_0^2 + x_1^2 + \cdots + x_n^2\).
In later sections, we will often be considering maps of the form $X^n \rightarrow X^n \times S^k$. A map of this sort states that we fix a type of geometry (spherical or hyperbolic), and both $X$’s are of this same type, of dimensions $n$ and $a$, respectively.

**Definition 1.3.** For a geometry $(E, q)$, where $q$ has signature $(n_-, n_+)$, a subspace $U$ of $X$ is a linear subspace of $E$ such that the restriction of $q$ to $U$ is totally nondegenerate and such that the signature $(m_-, m_+)$ of $q|_U$ has $m_-=n_-$. The dimension of a subspace $U$ of $X$ is one less than the dimension of the representing linear subspace.

When $k$ contains $\sqrt{-1}$, the condition on the signature is vacuous.

**Remark 1.4.** The condition on the signature may appear artificial, but it is necessary in order to model the types of subspaces we are concerned with. We can think of a geometry of dimension $n$ as sitting inside $k^{n+1}$ as a subspace. In the case when $k$ is not algebraically closed, a plane of dimension $m$ may not intersect this subspace in a subspace of dimension $m-1$, as desired. The condition on the signature ensures that this will happen in the cases of concern in this paper.

**Remark 1.5.** Many of the definitions and results in this paper will also work for the Euclidean geometry, as well as for geometries with signatures other than $(0, n+1)$ and $(1, n)$. However, there are enough subtleties and differences between these cases that in this paper we focus exclusively on the spherical and hyperbolic cases.

The key structure that we require from the quadric is the presence of an orthogonal complement for any subspace and the notion of a projection onto the orthogonal complement.

**Definition 1.6.** Let $U$ be an $i$-dimensional subspace of $X$, represented by a linear subspace $V$ of $E$. We define the orthogonal complement $U^\perp$ of $U$ to be the subspace of $X$ represented by $V^\perp$.

If $U$ and $U'$ are subspaces of $X$, represented by $V$ and $V'$, then we write $U \perp U'$ if $V \perp V'$. We write $U \oplus U'$ for the subspace represented by $V \oplus V'$. If $U \perp U'$ we write $U \oplus U'$ to emphasize this fact.

For subspaces $U \subseteq U'$ of $X$, we write

$$ pr_{U'}U' \overset{\text{def}}{=} U' \cap U^\perp. $$

The isometry group of $pr_{U'}U'$ is taken to be the subgroup isometry group of $U'$ that fixes $U$.

We will be using the following three properties of subspaces:

**Lemma 1.7.** Let $X^n$ be a usual geometry and let $U^i$ be a subspace of $X$. Then $\dim U^\perp = n - i - 1$. For a subspace $V$ containing $U$, $V$ is uniquely determined by $U$ and $U^\perp \cap V$. In addition, the induced quadratic form on $pr_U V$ is spherical.

We now relate the simplicial set of configurations to a flag complex associated to a geometry.

**Definition 1.8.** Let $X$ be a geometry of dimension $n$ over $k$.

Let $T^m(X)$ be the simplicial set whose $i$-simplices are sequences $U_0 \subseteq \cdots \subseteq U_i$ of nonempty subspaces of $X$ of dimension at most $m$. The $j$-th face map deletes $U_j$; the $j$-th degeneracy repeats $U_j$. The isometry group $I(X)$ acts on $T^m(X)$.

We define the flag complex of $X$ by

$$ F_X \overset{\text{def}}{=} T^m(X)/T^{m-1}(X), $$

with the inherited $I(X)$-action. More explicitly, the $i$-simplices of $F_X$ are sequences $U_0 \subseteq \cdots \subseteq U_i$ (plus a basepoint), where each $U_j$ is a nonempty subspace of $X$ and $U_i = X$. The face maps and degeneracies work as before, with the caveat that if $U_{i-1} \neq X$ then $d_i$ sends the simplex $U_0 \subseteq \cdots \subseteq U_i$ to the basepoint.

**Definition 1.9.** $T^{m}(X)$ is the simplicial set whose $i$-simplices are given by the subset of $X^{i+1}$ of those tuples $(x_0, \ldots, x_i)$ such that any subset of the tuple has a nondegenerate span of dimension at most $m$. The face maps are given by dropping the appropriate coordinate; the degeneracies are given by repeating the appropriate coordinate.

We begin with a basic lemma about the homotopy type of $T^{m}(X)$ in the absence of dimension restrictions:

**Lemma 1.10.**

$$ T^{m}(X) \simeq \ast. $$
Proof. By [Cat04] Proposition 2.2.1, since $k$ is infinite $\tilde{H}_i(T^{\dim X}_p(X)) = 0$. (In fact, Cathelineau proves this only with rational coefficients, but his proof works equally well integrally.) To see that $T^{\dim X}_p(X)$ is contractible it suffices to check that it is simply-connected. By [Cat04] Proposition 2.2.2] for any pair of points $(x, y)$ spanning a subspace of $X$, the subset $W_{x, y}$ of those points in $X$ such that $(x, y, w)$ spans a subspace is a Zariski-open subspace of $X$. Suppose that we are given a loop represented by the sequence of 1-simplices $(x_0, x_1), (x_1, x_2), \ldots , (x_0, x_j)$. Then, since $k$ is infinite, there exists a point $w$ such that $(x_j, w)$ spans a subspace for all $i$, and the loop is homotopic to a loop of the form $(x_0, w), (w, x_0)$. This is contracted by the 2-simplex $(x_0, w, x_0)$, so $T^{\dim X}_p(X)$ is contractible.

We turn our attention to showing that $T^{\dim}_p(X)$ and $T^m(X)$ are homotopy equivalent, thus justifying our focus on $T^m(X)$. There is a standard map which proves this on homology; see for example [Dup01 (3.8)]. We lift map to simplicial sets. For any simplicial set $K$, let $Sd K$ be the barycentric subdivision of $K$. [GJ99] Section III.4]. Define the map $h:Sd T^{\dim}_p(X) \to T^m(X)$ to be the map induced by taking a set of points in $X$ to their span. An $i$-simplex in $Sd T^{\dim}_p(X)$ is a sequence $\vec{x}_0 \subseteq \vec{x}_1 \subseteq \cdots \subseteq \vec{x}_i$, where $\vec{x}_j$ is a tuple in $X$ and $\vec{x}_j$ is an (ordered) subset of $\vec{x}_j$ for all $j$. Taking the spans of each of these gives an $i$-simplex in $T^m(X)$; as taking convex hulls is $G$-equivariant, this action is $G$-equivariant.

Proposition 1.11. The map

$$h:Sd T^{\dim}_p(X) \longrightarrow T^m(X)$$

induced by taking tuples in $X$ to their spans is a $G$-equivariant weak equivalence.

Proof. We use Theorem A [GG87], p.578]. Given a $q$-simplex in $T^m(X)$, $y: \Delta^q \to T^m(X)$, represented by $(U_0 \subseteq \cdots \subseteq U_q)$, the (left) naive homotopy fiber $[HKM+04]$ Defn. 3.1] is the simplicial set

$$(h|y)_p = \{(x, h x \equiv y): x \in Sd T^{\dim}_p(X)\}$$

More concretely, $(h|y)_p$ is the simplicial set $Sd T^{\dim}_p(U_0)$; as this is isomorphic to $Sd T^m(U_0)$ it is contractible by Lemma [1.10].

Since $h$ is compatible with both the dimension filtration and the $G$-action, we can use it to construct a $G$-equivariant equivalence between quotients.

Theorem 1.12. For all $m \geq 0$, the map $h$ induces a $G$-equivariant equivalence

$$T^{\dim}_p(X)/T^{\dim,m-1}_p(X) \simeq T^m(X)/T^{m-1}(X).$$

Proof. We have the $G$-equivariant commutative diagram

$$\begin{array}{ccc}
Sd T^{\dim,m-1}_p(X) & \xrightarrow{h} & T^{m-1}(X) \\
\downarrow & & \downarrow \\
Sd T^{\dim}_p(X) & \xrightarrow{h} & T^m(X)
\end{array}$$

where the vertical maps are both cofibrations. Taking vertical homotopy cofibers gives the desired result.

2. FROM SCISSORS CONGRUENCE TO FLAG COMPLEXES

The goal of this section is to relate the homology of flag complexes to scissors congruence groups. Thus, in this section, we fix $k = \mathbb{R}$.

The first step in analyzing scissors congruence groups is constructing a model for the scissors congruence groups which is easy to work with both algebraically and topologically. To do this, we focus on flag complexes, which are simultaneously geometric (they form a simplicial set, and thus a topological space), algebraic (their top homology group is the Steinberg module, which keeps track of how flags of linear subspaces of a vector space interact), and categorical (they can usually be modeled as the nerve of a category). Thus our first goal is to relate flag complexes to scissors congruence groups; we follow the methods of [Dup01], Chapter 2.3]. For scissors congruence to be defined we need a notion of a geometry to work within, as well as a notion of “inside” and “outside” for polytopes; thus we will need to be working inside an ordered field.

The basic building block of a polytope (and thus of a scissors congruence group) is a simplex, which can be defined as a convex hull.
Definition 2.1. Suppose $X$ has dimension $n$. A convex hull of a tuple $(a_0, \ldots, a_m)$ of points in $X$ is the subset of $X$ represented by the cone
\[
\left\{ \sum_{i=0}^{m} c_i b_i \in \mathbb{R}^{n+1} \mid c_i \geq 0 \ \forall i, \ b_i \in a_i \subseteq \mathbb{R}^{n+1} \right\}.
\]

An $m$-simplex in $X$ is the convex hull of a tuple $(a_0, \ldots, a_m)$ which is not contained in an $m-1$-dimensional subspace of $X$. An $m$-polytope in $X$ is a finite union of $m$-simplices; note that we make no assumptions of convexity or connectedness. When $m = n$ we omit it from the terminology and refer simply to “simplices in $X$” or “polytopes in $X$.”

Remark 2.2. When $X$ is hyperbolic, a simplex is uniquely determined by its vertices in the following sense. The hyperboloid $-x_0^2 + x_1^2 + \cdots + x_n^2$ has two connected components, and we think of $X$ as one of these components and a point of $X$ as the intersection of the representing line with this component. A tuple of points $(a_0, \ldots, a_n)$ in $X$ thus defines a tuple of vectors in $\mathbb{R}^{n+1}$, and thus the positive cone above is well-defined.

When $X$ is spherical, there are $2^{n+1}$ possible choices of “sign” of the representatives $b_i$. Thus a simplex is no longer uniquely defined by its vertices.

We can now define the scissors congruence group of $X$:

Definition 2.3. Let $X$ be a usual geometry and let $G$ be a subgroup of $I(X)$. Then the scissors congruence group of $X$ relative to $G$, denoted $\mathcal{P}(X, G)$, is the free abelian group generated by polytopes in $X$ modulo the relations

- $[P \cup Q] = [P] + [Q]$ if $P \cap Q$ is contained in a finite union of $m-1$-dimensional subspaces.
- $[P] = [g \cdot P]$ for any $g \in G$. Here $g$ acts on $P$ pointwise; as it is in $I(X)$ it takes convex hulls to convex hulls.

When $G = I(X)$ we omit it from the notation.

Note that
\[
\mathcal{P}(X, G) \cong H_0(G, \mathcal{P}(X, 1)).
\]

The main result in this section is that the homology of $F_X$ is closely related to the scissors congruence groups of $X$: we follow the proof of [Dup01] Theorem 2.10. The following theorem shows that in the Euclidean and hyperbolic cases the scissors congruence groups $\mathcal{P}(X, G)$ can be computed as the $G$-coinvariants of the homology of a quotient of $\text{Tpl}^n(X)$.

Theorem 2.5. [Dup01] Theorem 2.10 Then the map taking a tuple of points to its convex hull defines a $I(\mathcal{H}^n)$-equivariant isomorphism
\[
H_n(\text{Tpl}^n(\mathcal{H}^n)/\text{Tpl}^{n-1}(\mathcal{H}^n))^t \longrightarrow \mathcal{P}(\mathcal{H}^n, 1).
\]

Here, $^t$ means that the action is twisted by the determinant: for any $g \in I(\mathcal{H}^n)$, $g$ acts on a homology on the left by $(-1)^{\text{det} g}$ as well as by the usual action of $\mathcal{H}^n$.

The case $X = S^n$ is more complicated, as convex hulls are now only well-defined up to a certain equivalence relation.

Definition 2.6. Let
\[
\Sigma: \bigoplus_{\substack{V \subseteq \mathbb{R}^{n+1} \cap S^n \atop \dim V = n}} \mathcal{P}(V \cap S^n, 1) \longrightarrow \mathcal{P}(S^n, 1)
\]
be the “suspension” map taking a simplex in $V \cap S^n$ to the union of the two simplices defined by the choice of representatives in $V^\perp$. We denote by $\mathcal{P}(S^n, G)$ the cokernel of the induced map
\[
\Sigma: H_0(G, \bigoplus_{\substack{V \subseteq \mathbb{R}^{n+1} \cap S^n \atop \dim V = n}} \mathcal{P}(V \cap S^n, 1)) \longrightarrow H_0(G, \mathcal{P}(S^n, 1)).
\]

The key observation is the following:
Theorem 2.7 ([Dup01 Corollary 5.18]). The map taking a simplex to its convex hull induces a $O(n+1)$-equivariant isomorphism

$$H_0(\text{Tpl}_n^\ast(S^n)/\text{Tpl}_{n-1}^\ast(S^n)) \cong (\text{coker } \Sigma).$$

In particular, since $\Sigma$ is $O(n+1)$-equivariant, $\Sigma$ induces an isomorphism on coinvariants

$$H_0(O(n+1), H_n(\text{Tpl}_n^\ast(S^n)/\text{Tpl}_{n-1}^\ast(S^n)) \cong \mathcal{P}(S^n, O(n+1))/(\text{image } \Sigma).$$

The quotient $\mathcal{P}(S^n)/(\text{image } \Sigma)$ turns out to be the more “correct” notion of scissors congruence of the sphere, as it is most often used to measure angles; the entire sphere is the “total” angle, and should therefore not be present in the final calculations. When we discuss the Dehn invariant in Section 3 this will become clearer, as Dehn invariants are only well-defined inside these reduced scissors congruence groups.

The above implies that scissors congruence information is contained inside $H_n(\text{Tpl}_n^\ast(X)/\text{Tpl}_{n-1}^\ast(X))$. In fact, this is the only nonzero homology group of this space. Although this is nontrivial directly from the definition, it follows from the fact that $\text{Tpl}_n^\ast(X)$ is homotopy equivalent to $T_n^\ast(X)$: since all nondegenerate simplices of $T_m^\ast(X)$ have dimension at most $m$, all homology above degree $m$ must vanish.

Theorems 2.5 and 2.7 together with 2.4 demonstrate that scissors congruence groups are group homology with coefficients in $H_n(\text{F}_X)_G$. In order to model the twist on the topological level, we introduce an extra “twisting” dimension.

Definition 2.8. Let $S^\sigma$ be the simplicial set $\Delta^1 \cup_{\partial \Delta^1} \Delta^1$. This is a model of a circle with two 0-simplices and two 1-simplices. There is an action of $\mathbb{Z}/2$ on $S^\sigma$ given by swapping the two 1-simplices.

Explicitly, this model of $S^\sigma$ has 0-simplices $\{\ast, \ast\}$ (with $\ast$ as the basepoint) and $n$-simplices

$$\{\ast, *, \pm 1, \ldots, \pm n\}.$$

For the nondegenerate 1-simplices $\pm 1$ we define $d_0(\pm 1) = \ast$ and $d_1(\pm 1) = \ast$. For a general $n$-simplex $e_i$ (with $i > 0$ and $\epsilon = \pm 1$), $d_j(e_i) = \epsilon(i - 1)$ if $j < i$ and $e_i$ otherwise, with $d_0(e_1) = \ast$. The $\mathbb{Z}/2$-action fixes $\ast$ and $\ast$ and takes $e_i$ to $(-\epsilon)i$.

The group $I(X)$ acts diagonally on $S^\sigma \wedge F_X$, and $H_0(\text{F}_X) \cong H_{n+1}(S^\sigma \wedge F_X)$ as groups. As $I(X)$-modules, these differ only by the action on $S^\sigma$, which adds a twist by the determinant. In particular, this means that

$$H_0(G, H_n(\text{F}_X)) \cong H_0(G, H_{n+1}(S^\sigma \wedge F_X)).$$

From the homotopy orbit spectral sequence (see Proposition 3.4), we have

$$H_0(G, H_{n+1}(S^\sigma \wedge F_X)) \cong H_{n+1}((S^\sigma \wedge F_X)_{hG}).$$

Summarizing the results in this section, we get the following:

Proposition 2.9. Let $X$ have dimension $n$ and let $G$ be a subgroup of the isometry group of $X$. When $X = H^n$

$$\mathcal{P}(X, G)_\mathbb{Q} \cong H_{n+1}((S^\sigma \wedge F_X)_{hG}).$$

When $X = S^n$ we get

$$\overline{\mathcal{P}}(S^n, G)_\mathbb{Q} \cong H_{n+1}((S^\sigma \wedge F_{S^n})_{hG}).$$

Remark 2.10. One may be inspired by Proposition 2.9 to define $\mathcal{P}(X, G)_\mathbb{Q}$ for an $n$-dimensional geometry $X$ over a general field $k$ and $G \leq I(X)$ to be $H_{n+1}((S^\sigma \wedge F_X)_{hG})$. Although we do not know of a geometric interpretation for this group for $k \neq \mathbb{R}$, these groups may still prove to be of interest.

In the case when $X = S^{2n}$ there is a much stronger result:

Proposition 2.11. When $n \geq 1$,

$$(S^\sigma \wedge F_{S^{2n}})_{hO(2n+1)} \cong \mathbb{Q} \ast.$$

In particular, $\overline{\mathcal{P}}(S^{2n})_\mathbb{Q} = 0$.

The final statement in the proposition is a direct consequence of [Sah79 Proposition 6.2.2].

Proof. The matrix $-I \in O(2n+1)$ acts on all homology groups of $S^\sigma \wedge F_{S^{2n}}$ by $-1$. Thus by center kills [Dup01 Lemma 5.4], $H_i(O(2n+1), H_{2n}(S^\sigma \wedge F_{S^{2n}})) = 0$ for all $i$. Thus by the homotopy orbit spectral sequence (Proposition 3.5), $\overline{H}_i((S^\sigma \wedge F_{S^{2n}})_{hO(2n+1)}) = 0$ for all $i$. Since $(S^\sigma \wedge F_{S^{2n}})_{hO(2n+1)}$ is simply-connected (as suspensions and homotopy coinvariants commute), it must be contractible. □
3. Dehn invariants

The statement (rephrased in modern terminology) of Hilbert’s third problem is extremely simple:

Do there exist two polyhedra with the same volume which are not scissors congruent?

The answer, given in 1901 by Dehn is “yes”: the cube and regular tetrahedron are not scissors congruent, even if they have the same volume. Dehn proved this statement by constructing a second invariant of polyhedra (these days called the “Dehn invariant”) which is zero on a cube and nonzero on any regular tetrahedron. This invariant takes values in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$—a difficult group to map into in any case, but even more startling given that tensor products were only originally defined in 1938. In this section we give a definition of the classical Dehn invariant (extended to arbitrary dimensions by Sah in [Sah79]) and construct a derived model.

**Definition 3.1.** Let $X^n$ be a usual geometry, and consider $\mathcal{P}(X)$. For any integer $0 < i < n$, we define the $i$-th classical Dehn invariant in the following manner. Since $\mathcal{P}(X)$ is generated by simplices, it suffices to define it on simplices. For a simplex $\sigma$ in $X$ with vertices $\{x_0, \ldots, x_n\}$, we define

$$\hat{D}_i(\sigma) = \sum_{\substack{I \subset \{0, \ldots, n\} \setminus \{i\} \neq \emptyset \atop \dim U = i + 1}} [x_I \otimes [pr_{U^\perp}(x_I)] \in \mathcal{P}(X^i, I(X^i)) \otimes \overline{\mathcal{P}}(S^{n-i-1}, I(S^{n-i-1})).$$

Here, $x_I$ is the set $\{x_i : i \in I\}$, $[x_I]$ is the class of the simplex with vertices $x_I$ in $X^i$, and $[pr_{U^\perp}(x_I)]$ is the class in $\overline{\mathcal{P}}(S^{n-i-1}, I(S^{n-i-1}))$ of the simplex spanned by the projections of the $x_I$. For a more detailed discussion of this, see [Sah79 Section 6.3].

In order to begin relating the classical Dehn invariant to a space of flags, we first construct the Dehn invariant as the $I(X)$-coinvariants of a map of groups; this, together with Theorem 2.1 will give us the tools to rephrase it in terms of flags. Fix a subspace $U$ of $X$ with $\dim U = i$; write $U^\perp$ for the $S^{n-i-1}$-sphere orthogonal to $U$. Here, we are considering $X$ as embedded inside $\mathbb{R}^{n+1}$, with $U$ defined by an $i+1$-dimensional subspace $V$; $U^\perp$ is the unit sphere in $V^\perp$. We define a map

$$\hat{D}_U: \mathcal{P}(X, 1) \longrightarrow \mathcal{P}(U, 1) \otimes \overline{\mathcal{P}}(U^\perp)$$

in the following manner. For a simplex $\sigma$ with vertices $\{x_0, \ldots, x_n\}$ in $X$, suppose that $U = \text{span}(x_0, \ldots, x_i)$. Let $\tau$ be the projection of $\{x_{i+1}, \ldots, x_n\}$ to $U^\perp$. We define

$$\hat{D}_U([x_0, \ldots, x_n]) = [x_0, \ldots, x_i] \otimes [\tau].$$

For any simplex $\{x_0, \ldots, x_n\}$ such that there do not exist $0 \leq j_0 < \cdots < j_i \leq n$ with $U = \text{span}(x_{j_0}, \ldots, x_{j_i})$, we define

$$\hat{D}_U([x_0, \ldots, x_n]) = 0.$$

**Lemma 3.2.** With this definition,

$$\mathcal{P}(X, 1) \bigoplus_{U \subseteq X \atop \dim U = i} \mathcal{P}(U, 1) \otimes \overline{\mathcal{P}}(U^\perp, 1)$$

is well-defined and $I(X)$-equivariant. After taking $I(X)$-coinvariants (and rationalizing) this map becomes $(\hat{D}_i)_Q$.

**Proof.** First, note that this map is well-defined. To check this we must check that for any simplex $\sigma$, all but finitely many of the $\hat{D}_U$ are 0. This is because there are only finitely many subspaces $U$ which are the span of a subset of $\{x_0, \ldots, x_n\}$.

The action of $I(X)$ on the left is simply an action on tuples. The action on the right is a bit more complicated: it acts on the indices of the sum, and acts within each group, as well. However, as we can think of simplices in $U$ as $i$-simplices in $X$ that happen to be contained in $U$, on each individual simplex the action is the same: simply acting on each vertex of the simplex. In addition, since the $I(X)$-action commutes with orthogonal projection the map is $I(X)$-equivariant.

Thus it remains to check that the rationalized $I(X)$-coinvariants of this map are $\hat{D}_i$. The left-hand side is $\mathcal{P}(X)_Q$. $I(X)$ identifies all of the summands on the right-hand side, and the stabilizer of any fixed $U$ is
that the scissors congruence groups are Q-vector spaces, this step would not be necessary.

The classical Dehn invariant can be iterated in the following sense. Suppose that $i < j$; then the following square commutes:

\[
\begin{array}{ccc}
\mathcal{P}(X^n) & \xrightarrow{\tilde{D}_i} & \mathcal{P}(X^i) \otimes \tilde{P}(S^{n-i-1}) \\
\mathcal{P}(X^j) \otimes \tilde{P}(S^{n-j-1}) & \xleftarrow{\tilde{D}_j} & \mathcal{P}(X^i) \otimes \tilde{P}(S^{j-i-1}) \otimes \tilde{P}(S^{n-j-1})
\end{array}
\]

For a simplex $(x, y) \in X_i \wedge Y_j$, the face maps $d_\ell$ are defined to be $d_m \times 1: X_i \wedge Y_j \to X_{i-1} \wedge Y_j$ when $\ell \leq i$, and $1 \times d_{\ell-i-1}: X_i \wedge Y_j \to X_i \wedge Y_{j-1}$ otherwise. If $i = \ell = 0$ or $j = m - 1 - \ell = 0$ then the face map takes the simplex to the basepoint. Degeneracies are defined analogously, with the first $i + 1$ acting on the $x$-coordinate, and the last $m - i - 1$ acting on the $y$-coordinate.

For those unfamiliar with reduced joins, an introduction and proofs of the most relevant properties of the reduced join are in Section A.1. The most important feature of reduced joins is their relationship to smash products; the proof is given in Section A.1.

**Lemma 3.6.** Let $X$ and $Y$ be pointed simplicial sets. The map $f: S^1 \wedge X \wedge Y \to X \wedge Y$ given by sending $(i, x, y) \in (S^1 \wedge X \wedge Y)_n$ to $(d^{i+1}_0 x, d^{i+1}_0 y)$ is a simplicial weak equivalence.

Dehn invariants act in relation to a particular dimension of subspace. As we wish to define them on the space $F_X$ equivariantly with respect to the $I(X)$-action, we begin by defining a Dehn invariant relative to a single subspace.

**Definition 3.7.** Let $U$ be a proper $i$-dimensional subspace of $X$. We define the derived Dehn invariant relative to $U$ $D_U: F_X \to F_X \bar{\wedge} F_{U^\perp}$ by

\[
D_U(U_0 \subseteq \cdots \subseteq U_n) = \left\{ \begin{array}{ll}
\ast & \text{if j s.t. } U_j = U \\
(U_0 \subseteq \cdots \subseteq U_j) \wedge (pr_{U^\perp} U_{j+1} \subseteq \cdots \subseteq pr_{U^\perp} U_n) & \text{if } j = \max\{i \mid U_i = U\}.
\end{array} \right.
\]

**Lemma 3.8.** Let $U$ be a proper subspace of $X$. $D_U$ is well-defined. If we let $G_U$ be the subgroup of $I(X)$ of those elements fixing $U$ then $G_U$ acts on $U^\perp$ and $D_U$ is $G_U$-equivariant.

**Proof.** That $D_U$ is compatible with the simplicial structure is direct from the definitions.

To check that $D_U$ is $G_U$-equivariant, note that if there is no $\ell$ with $U_\ell = U$ then this is also true after applying $g$, and since $g$ fixes $\ast$ this commutes with $D_U$. Now suppose that such an $\ell$ exists, and let $g \in G_U$. Then

\[
D_U(g \cdot (U_0 \subseteq \cdots \subseteq U_i)) = (U_0 \subseteq \cdots \subseteq U_i) \wedge (pr_{U^\perp} g \cdot U_{j+1} \subseteq \cdots \subseteq pr_{U^\perp} g \cdot U_i) = g \cdot D_U(U_0 \subseteq \cdots \subseteq U_i).
\]
Note that this derived Dehn invariant can also be iterated:

**Lemma 3.9.** Let $U \subseteq V$ be subspaces of $X$. Then the following diagram commutes:

\[
\begin{array}{c}
F_X \xrightarrow{D_V} F_U \xrightarrow{\sim} F_{U^\perp} \\
\downarrow D_V \quad \downarrow \quad \quad \downarrow 1 \circ D_{U \cap V} \\
F_V \xrightarrow{\sim} F_{V^\perp} \xrightarrow{D_V \circ 1} F_U \xrightarrow{\sim} F_{U^\perp \cap V} \xrightarrow{\sim} F_{V^\perp}
\end{array}
\]

**Proof.** Fix any $m$-simplex $U_0 \subseteq \cdots \subseteq U_m$. Suppose $U_i = U$ and $U_j = V$. For any subspace $W$ of $X$, if $pr_{U^\perp}(w) \subseteq V \cap U^\perp$ then we must have $W \subseteq V$. In particular,

\[
(1 \circ D_{U \cap V} \circ D_U)(U_0 \subseteq \cdots \subseteq U_m) \\
= 1 \circ D_{U \cap V}((U_0 \subseteq \cdots \subseteq U_i) \wedge (pr_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq pr_{U^\perp}(U_m))) \\
= (U_0 \subseteq \cdots \subseteq U_i) \wedge (pr_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq pr_{U^\perp}(U_j)) \wedge (pr_{V^\perp}pr_{U^\perp}(U_{j+1}) \subseteq \cdots \subseteq pr_{V^\perp}pr_{U^\perp}(U_m)) \\
= (U_0 \subseteq \cdots \subseteq U_i) \wedge (pr_{U^\perp}(U_{i+1}) \subseteq \cdots \subseteq pr_{U^\perp}(U_j)) \wedge (pr_{V^\perp}(U_{j+1}) \subseteq \cdots \subseteq pr_{V^\perp}(U_m))
\]

where the last step follows because $V^\perp \subseteq U^\perp$. This is clearly equal to the composition around the bottom, as desired. \qed

**Definition 3.10.** Let $0 \leq i \leq n$. We define the **dimension-$i$ derived Dehn invariant** to be

\[
D_i; F_X \xrightarrow{\vee D_U} \bigvee_{U \subseteq X \atop \dim U = i} F_U \xrightarrow{\sim} F_{U^\perp}.
\]

Note that this is well-defined: every simplex contains at most one space of dimension $i$, and thus only a single dimension-$i$ component will be nontrivial on it.

More formally, we can think of the wedge in the codomain as the subspace of the product of those tuples where at most one of the components is not equal to the basepoint. The map then $\prod_U D_U$ factors through the inclusion $\bigvee_U F_U \xrightarrow{\sim} \prod_U F_U \xrightarrow{\sim} F_{U^\perp}$, and $D_i$ is defined to be this factorization.

**Lemma 3.11.** $D_i$ is well-defined and $I(X)$-equivariant.

Thus we have a Dehn invariant for a fixed dimension. Moreover, this Dehn invariant can be put into a square similar to (3.4). When $i < j$ the diagram

\[
\begin{array}{c}
F_X \xrightarrow{D_i} \bigvee_{U \subseteq X \atop \dim U = i} F_U \xrightarrow{\sim} F_{U^\perp} \\
\downarrow D_i \quad \downarrow \quad \quad \downarrow 1 \circ D_{U \cap V} \\
\bigvee_{V \subseteq X \atop \dim V = j} F_V \xrightarrow{\sim} F_{V^\perp} \xrightarrow{D_{j-i} \circ 1} \bigvee_{U \subseteq V \atop \dim U = i} F_U \xrightarrow{\sim} F_{U^\perp \cap V} \xrightarrow{\sim} F_{V^\perp}
\end{array}
\]

commutes and is $I(X)$-equivariant.

We wrap up this section by checking that this definition of the Dehn invariant is compatible with the classical Dehn invariant.

**Lemma 3.13.** Suppose $\dim X = n$ and $k = \mathbb{R}$. Then $H_0(I(X); H_{n+1}(S^\sigma \wedge D_i))$ is the rationalized classical Dehn invariant.

**Proof.** Rewriting $D_i$, using Lemma 3.2, we see that it suffices to construct an $I(X)$-equivariant diagram relating $\bigoplus_U D_U$ to $H_{n+1}(S^\sigma \wedge D_i)$. 

For a geometry $W$ of dimension $i$, write
\[ R_n(W) \overset{\text{def}}{=} \text{Sd Tpl}^i(W)/\text{Sd Tpl}^{i-1}(W). \]

We define $D_i^{R}: R_n(X) \to \bigvee R_\ell(U) \oplus R_\ell(U^\perp)$ in the following manner. A $j$-simplex of $R_n(W)$ is a sequence $T_0 \subseteq \cdots \subseteq T_j$ of tuples of points in $W$ such that the span of $T_j$ is $W$. If there exists a maximal $\ell$ such that $\dim \text{span} T_\ell = i$, we map this $j$-simplex to the simplex $(T_0 \subseteq \cdots \subseteq T_\ell) \wedge (\text{span} T_{\ell+1} \subseteq \cdots \subseteq \text{span} T_j)$, indexed by $\text{span} T_\ell$. Otherwise, we map to the basepoint. This is a well-defined simplicial map for the same reason that $D_i$ is.

Consider the following diagram:

\[
\begin{array}{ccc}
H_{n+1}(S^\sigma \wedge F_X) & \overset{H_{n+1}(S^\sigma \wedge D_i)}{\longrightarrow} & H_{n+1}\left( S^\sigma \wedge \bigvee_{U \subseteq X} \bigwedge_{\dim U = i} R_\ell(U) \oplus R_\ell(U^\perp) \right) \\
\downarrow h & & \downarrow h \\
H_{n+1}(S^\sigma \wedge R_\ell(X)) & \overset{D_i^{R}}{\longrightarrow} & H_{n+1}\left( S^\sigma \wedge \bigvee_{U \subseteq X} \bigwedge_{\dim U = i} R_\ell(U) \oplus R_\ell(U^\perp) \right) \\
\downarrow p & & \downarrow p \\
\mathcal{P}(X,1) \overset{D_Q}{\longrightarrow} & \bigoplus_{U \subseteq X} \mathcal{P}(U,1) \otimes \overline{\mathcal{P}}(S^{n-i-1},1)
\end{array}
\]

Here, the vertical maps $h$ are induced by the map $h$ in Theorem [1.12] (and are thus isomorphisms). The vertical maps $p$ are isomorphisms by Proposition [2.9]. Since all maps in this diagram are $I(X)$-equivariant, the lemma follows.

\[ \square \]

Remark 3.14. Classically, the tensor product would be replaced by the smash product, and choosing instead the reduced join may appear to be a perverse choice: the reduced join of simplicial sets is not symmetric in the simplicial sets, and using it to model a symmetric structure like the tensor product feels unnatural. And, indeed, the authors spent considerable time on attempts to rework this material using a smash product. Unfortunately (or, possibly, incredibly interestingly), it does not seem possible to construct a topological model of the Dehn invariant using a smash product of spaces.

An interesting corollary of this is that the constructions in this section are fundamentally unstable. Smash products of spaces lift naturally to smash products of spectra, and therefore give some hope that analogous constructions could be lifted to stable models of scissors congruence (such as those arising from $\mathcal{CZ}$ [Zak17]). Unfortunately, this does not appear to be the case, and the natural question arises: how stable is the Dehn invariant? What parts of it can be seen stably? And which portions are irredeemably unstable?

4. A CURIOUS CHAIN COMPLEX

Let $X$ be a usual geometry and let $k = R$.

Consider the classical Dehn invariant $D_i$. Using the square [3.4], we see that by varying over all possible values of $0 < i < 2n-1$, the Dehn invariant produces a commutative cube. When $j$ is even, $\mathcal{P}(S^j) = 0$ (Proposition [2.11]); thus in fact this cube will be nonzero only along those dimensions that produce no odd-dimensional sphere, which gives an $n-1$-cube. Goncharov considers the total complex of this cube in [Gon99], we refer to this complex as the Dehn complex and denote it by $\mathcal{P}_*(X)$. The goal of this section is to develop a tool for analyzing this complex using total homotopy cofibers of cubes.

**Definition 4.1.** Denote by $\mathcal{T}_n$ the category whose objects are sequences $(b, a_1, \ldots, a_i)$ of nonnegative integers such that $b + a_1 + \cdots + a_i = 2n-1$ and in which all $a_j$ are even. There exists a morphism $(b, a_1, \ldots, a_i) \to (b', a'_1, \ldots, a'_i)$ if there exist indices $0 \leq i_0 < \cdots < i_\ell = i$ such that $b = b' + a'_1 + \cdots + a'_{i_0}$ and $a_j = a'_{j-i_0+1} + \cdots + a'_{i_j}$.
Note that $\mathcal{I}_n$ is an $n-1$-cube.

**Definition 4.2.** We define the Dehn complex to be the total complex of the cube $D: \mathcal{I}_n \to \text{AbGp}$ given by

$$D(b, a_1, \ldots, a_i) = \mathcal{P}(X^b)_Q \bigotimes_{j=1}^{i} \mathcal{P}(S^{a_j-1})_Q.$$  

The image of $D$ on the map $(b, a_1, \ldots, a_j + a_{j+1}, \ldots, a_i) \to (b, a_1, \ldots, a_i)$ is $1 \otimes \cdots \otimes D_{a_j} \otimes \cdots \otimes 1$.

The equivariant Dehn cube is the cube $D': \mathcal{I}_n \to \text{AbGp}$ given by

$$D'(b, a_0, \ldots, a_i) = \bigoplus_{W \oplus_i \mathcal{V}_j = X} \mathcal{P}(W, 1)_Q \bigotimes_{j=1}^{i} \mathcal{P}(V_j, 1)_Q.$$  

By the same reasoning as in the proof of Lemma 3.2 we see that $H_0(I(X), D') = D$.

To see the connection between the Dehn complex and the equivariant Dehn cube, we need the notion of total homotopy cofiber. For a more in-depth discussion of these ideas, see [MV15, Section 5.9].

**Definition 4.3.** Let $I$ be the category $0 \to 1$. An $n$-cube in $\mathcal{C}$ is a functor $I^n \to \mathcal{C}$. Suppose that $\mathcal{C}$ is a model category. Write $I^n$ for the full subcategory of $I^n$ which does not contain the object $(1, \ldots, 1)$.

Let $F: I^n \to \mathcal{C}$ be a functor. The total homotopy cofiber $\text{thocofib} F$ is the homotopy cofiber of the map $\text{hocofib} F|_I \to F(1, \ldots, 1)$.

The important example we need is the following:

**Example 4.4.** Let $F: I^n \to \text{Mod}_R$ be a functor. Then the total complex of $F$ is quasi-isomorphic to $\text{thocofib} F[0]$, where $F[0](A) = A[0] \in \text{Ch}_R$.

Thus the Dehn complex is obtained by constructing a cube of coinvariants of homology groups and taking its total complex. The goal of this section is to show how to commute taking coinvariants and total homotopy cofibers past one another: to construct, $I(X)$-equivariantly, a cube of spaces that produce this cube after taking homology, coinvariants, and then the total complex. Since homotopy coinvariants and the total complex (which is a total homotopy cofiber) commute past one another, in future sections we will do these operations in the opposite order to relate the homology of the Dehn complex to algebraic $K$-theory.

We proceed as in previous sections: by replacing $\mathcal{P}(X)$ with $F_X$. For convenience and clarity, we introduce the notation $F_{\bar{A}}$ and $G_{\bar{A}}^\star$.

**Definition 4.5.** Let $\bar{A} \in \mathcal{I}_n$ have $\bar{A} = (b, a_1, \ldots, a_i)$. We define

$$F_{\bar{A}} \overset{\text{def}}{=} \bigvee_{W \oplus_i V_j = X} F_W \star \bigotimes_{j=1}^{i} F_{V_j}$$

and

$$G_{\bar{A}} \overset{\text{def}}{=} \bigwedge_{W \oplus_i V_j = X} F_W \wedge \bigwedge_{j=1}^{i} (S^* \wedge F_{V_j}).$$

We can then duplicate the construction of the Dehn complex in spaces:

**Definition 4.6.** We define the functor $Y: \mathcal{I}_n \to \text{Top}$ by

$$\bar{A} \mapsto S^\star \wedge F_{\bar{A}},$$

with morphisms given by the appropriate $D_i$. We define the Dehn space $Y^X$ by

$$Y^X = \text{thocofib} Y.$$

\(^1\) Or any other category in which you can define homotopy cofibers; the particular examples we care about in this paper are simplicial sets and chain complexes.
Our goal is now to construct a natural isomorphism \( \mathbf{D} \rightarrow H_0(I(X), H_{2n}(Y)) \). To construct this, note that it suffices to construct an equivariant natural isomorphism \( \alpha : \mathbf{D}' \rightarrow H_{2n}(Y) \).

We have

\[
\mathbf{D}'(\vec{A}) = \bigoplus_{W \oplus^+ \oplus^* V_j = X} \mathcal{P}(W) \mathcal{Q} \otimes \bigoplus_{j=1}^i \mathcal{P}(V_j) \mathcal{Q} = \bigoplus_{W \oplus^+ \oplus^* V_j = X} H_{b+1}(S^\sigma \wedge F_W) \otimes \bigoplus_{j=1}^i H_{a_j}(S^\sigma \wedge F_{V_j})
\]

\[
\cong H_{2n} \left( \bigvee_{W \oplus^+ \oplus^* V_j = X} S^\sigma \wedge F_W \wedge \bigwedge_{j=1}^i (S^\sigma \wedge F_{V_j}) \right) = H_{2n}(S^\sigma \wedge G_{\vec{A}}).
\]

However, we do not have a model of the Dehn invariant on the spaces inside the \( H_{2n} \) above and must therefore work on the homological level.

For \( \vec{A} = (b, a_1, \ldots, a_i) \in \mathcal{I}_n \) we define

\[
f_{\vec{A}} : S^\sigma \wedge G_{\vec{A}} \rightarrow S^\sigma \wedge F_{\vec{A}}
\]

as follows. Let \( \gamma : S^\sigma \wedge S^\sigma \rightarrow S^\sigma \wedge S^1 \) be defined by \((a, b) \mapsto (|a|, |b|) \), and \( \gamma(\ast) = \ast \). For any simplicial sets \( K \) and \( L \), let \( f : S^1 \wedge K \wedge L \rightarrow K \wedge L \) take \((a, x, y)\) to \((d_{a+1}^b x, d_{a+1}^b y)\); by Lemma 4.9 it is a weak equivalence. We define \( f_{\vec{A}} \) inductively, as an i-fold composition of maps of the following form:

\[
S^\sigma \wedge K \wedge S^\sigma \wedge L \rightarrow S^\sigma \wedge S^\sigma \wedge K \wedge L \rightarrow S^\sigma \wedge S^1 \wedge K \wedge L \rightarrow S^\sigma \wedge K \wedge L.
\]

**Lemma 4.7.** \( f_{\vec{A}} \) is an \( I(X) \)-equivariant rational weak equivalence.

**Remark 4.8.** In fact, \( f_{\vec{A}} \) is a weak equivalence after inverting 2.

In an ideal world, we could define \( \alpha_{\vec{A}} = H_{2n}(f_{\vec{A}}) \). However, in our case it is not that simple, as this is in fact not a weak equivalence. In the case when \( \vec{A} = (2n - 1) \) we have \( F_{\vec{A}} = G_{\vec{A}} = F_X \). Fix \( b \), and consider the Dehn invariant \( D_b \) given by the morphism \((2n - 1) \rightarrow (b, a)\). We have the following noncommutative diagram:

\[
\begin{array}{ccc}
H_{2n}(S^\sigma \wedge G_{(b, a)}) & \xrightarrow{\gamma} & H_{2n}(f_{(b, a)}) \\
\hat{D}_b \downarrow & & \downarrow H_{2n}(f_{(b, a)}) \\
H_{2n}(S^\sigma \wedge F_X) & \rightarrow & H_{2n}(S^\sigma \wedge F_{(b, a)})
\end{array}
\]

The difference between the two compositions is the doubling of the circle in the map \( \gamma \). Thus the difference between the two maps is a multiplication by 2. This is true in general; for \( j > 0 \) each Dehn invariant touches only one of the coordinates, and thus a similar noncommutativity occurs. However, as we are working rationally, this is easily fixed:

**Lemma 4.9.** \( \alpha_{\vec{A}} \stackrel{\text{def}}{=} 2^{-1} H_{2n}(f_{\vec{A}}) \) is a natural isomorphism \( \mathbf{D}' \rightarrow H_{2n}(Y) \).

Using the homotopy orbit spectral sequence (Proposition 4.5) we have thus proved the following:

**Theorem 4.10.** Let \( X \) be a usual geometry of dimension \( 2n - 1 \). The Dehn complex is isomorphic to the total complex of the \( n - 1 \)-cube given by the functor \( \mathcal{I}_n \rightarrow \mathbf{AbGp} \)

\[
\vec{A} \rightarrow H_{2n}((S^\sigma \wedge F_{\vec{A}})_{hI(X)}).
\]

5. **Large cubes and the Dehn complex**

A priori, the homology of the Dehn complex is mysterious. However, Goncharov conjectures that there is a canonical homomorphism [Conj. 1.8]

\[
H_*(\mathcal{P}_*(S^{2n-1})) \rightarrow (\text{gr}_n^X K_{n+1}(C) \mathcal{Q} \otimes \epsilon(n))^+.
\]
that is in fact an isomorphism \cite[Conj. 1.9]{Gon99}. Here, $\text{gr}_n^\gamma$ is the $n$-th graded part of the $\gamma$-filtration, $\epsilon(n)$ is a copy of $Q$ with complex conjugation acting on it by $(-1)^n$, and $\cdot^+$ denotes taking the $+1$-eigenspace of the action by complex conjugation. Goncharov conjectures the existence of a similar map for $H^{2n-1}$; in this form the codomain changes from the $+1$-eigenspace to the $-1$-eigenspace of complex conjugation, but otherwise remains the same.

In this section we use Theorem \ref{thm:thcofib} to begin to analyze the structure of this complex. The key idea here is the following:

The total complex of a cube is the total homotopy cofiber of the cubical diagram inside the category of chain complexes.

With this observation, we note that the total homotopy cofiber commutes with homotopy coinvariants of a group action. The difficulty in analyzing the homotopy type of the complex lies in the fact that homology does not commute with either limits or colimits.

To help analyze the structure of the complex we will instead study the functor $Y$ and the Dehn space $Y^X$ (Definition \ref{defn:dehn}). If we remove the homology from the statement of Theorem \ref{thm:thcofib} we would like to study the topology of the cofib of $\mathbb{Y}_{hI(X)}$. Since both homotopy coinvariants and the total homotopy cofiber are homotopy colimits, they commute past one another; thus $\text{thcofib} \mathbb{Y}_{hI(X)} \simeq (Y^X)_{hI(X)}$.

**Theorem 5.1.** There is an equivalence

$$(Y^X)_{hI(X)} \simeq (S^\sigma \wedge S^{n-1})_{hI(X)}.$$ Here $I(X)$ acts by det on the $S^\sigma$-coordinate and trivially on the $S^{n-1}$-coordinate.

This result is so surprising, and is so close to the core of our understanding of Goncharov’s conjecture, that we provide two proofs of it, one direct from the combinatorics of the simplicial sets and one more conceptual and tied to the proof of the Solomon–Tits theorem. Both proofs are in Section \ref{section:proofs}.

From this we directly conclude the following:

**Theorem 5.2.**

$$H^i_{hI(X)}(Y^X) \cong H_{i-n}(I(X); Q^I).$$

**Proof.** By the above theorem, $H^i_{hI(X)}(Y^X) \cong (S^\sigma \wedge S^{n-1})_{hI(X)}$. Thus

$$H^i_{hI(X)} \cong H_i(S^\sigma \wedge S^{n-1})_{hI(X)}.$$ By the homotopy orbit spectral sequence (Proposition \ref{prop:orbit}), we have

$$H_i((S^\sigma \wedge S^{n-1})_{hI(X)}) \cong H_{i-n}(I(X); H_{n}(S^\sigma \wedge S^{n-1})) \cong H_{i-n}(I(X); Q^I),$$

as desired. \hfill $\square$

To connect the homotopy type of $Y^X_{hI(X)}$ to the Dehn complex, we use the spectral sequence for the total homotopy type of a cube proved in Proposition \ref{prop:cube}. Applying this to the case of $Y^X_{hI(X)}$ we get

$$E^1_{p,q} = \bigoplus_{\bar{A}=(b_0, a_1, \ldots, a_{n-1}, -p)} \tilde{H}_q(\mathcal{Y}(<\bar{A}>)) \longrightarrow \tilde{H}_{p+q}(Y^X_{hI(X)}).$$

Note that when $q < 2n$ each entry is 0. When $q = 2n$ the row of the spectral sequence is exactly a shift of the Dehn complex. In particular, this tells us that there is a surjection $H_*(Y^X_{hI(X)}) \longrightarrow H_*(\mathcal{P}_*(X))$. However, because of the particular structure of the Dehn complex, we can conclude more:

**Theorem 5.3.** Let $X$ be spherical or hyperbolic. Then $H_*\mathcal{P}_*(X^{2n-1})Q$ is naturally a direct summand of $H_{*+2n}((Y^{X^{2n-1}})_{hI(X^{2n-1})})$.

This theorem is a simple corollary of the following technical result:

**Lemma 5.4.** Let $G$ be a discrete group. Let $F: I^n \longrightarrow G\text{Top}$ be an $n$-cube such that $H_* F(i)$ is concentrated in degree $k$ for all $i \in I^n$. Let $C_G$ be the total complex of $H_0(G; H_k F(\cdot)): I^n \longrightarrow \text{AbGp}$. Then $H_* C_G$ is a direct summand of $H_{*+k}(\text{thcofib} F_{hG})$. 

Proof. Since $F(i)$ is $k$-connected, $H_0(G; H_k(F(i))) \cong H_k(F(i)_hG)$. We thus consider the spectral sequence in Proposition $[\text{A.3}]$; this has

$$E_{p,q}^1 = \bigoplus_{i=n-p} \tilde{H}_q(F(i)_hG),$$

and converges to $H_{p+q}(\text{thocofib } F)$. Note that the $q = k$-row of this spectral sequence is exactly $C_\ast$, with $d^1$ as the differential. We have another spectral sequence with

$$\tilde{E}_{p,q}^1 = \bigoplus_{i=n-p} H_q(F(i)),$$

converging to $H_{p+q}(\text{thocofib } F)$. The natural map $F(i) \to F(i)_hG$ gives a map of spectral sequences which is surjective at the $q = k$-row. Note, however, that $\tilde{E}$ is concentrated at the $q = k$-row, since $H_\ast(F(i))$ is concentrated in degree $k$ for all $i$. Thus all differentials out of this row are 0, and therefore the same must be true in $E$.

We can consider $C_\ast$ to be a spectral sequence concentrated in the $q = k$-row, with all differentials above $d^1$ equaling 0. Then there is a map of spectral sequences $E \to C$ given by the identity on the $q = k$-row and 0 otherwise. By the above analysis this map has a section; thus on the $E^\infty$-page it also has a section, and thus $H_\ast C_\ast$ splits off of $H_\ast(\text{thocofib } F)$; the indexing is off by $k$ because $C_\ast$ is concentrated in the $q = k$-row.

We are now ready to prove the theorem.

Proof of Theorem 5.3. By Theorem 4.10 $P_\ast(X)$ is rationally isomorphic to $\text{thocofib} \left( \mathcal{A} \xrightarrow{} H_{2n} \left( \left( S^\sigma \wedge \mathcal{A} \right)_I(X) \right) \right)$ in the category of chain complexes.

We can apply Lemma 5.4 to this situation, as each $X(i)$ in the $n-1$-cube has homology concentrated in degree $2n$. This gives that $H_\ast P_\ast(X)$ is a direct summand of $H_\ast + 2n(\text{thocofib } X(i)_I)$

The upshot of the above two theorems is the following:

**Theorem 5.5.** $H_\ast P_\ast(X)$ is a direct summand of $H_\ast + n(I(X); \mathbb{Q}^i)$.

Proof. We have shown that $H_\ast P_\ast(X)$ splits off of $H_{2n+i}(Y^X_I(X))$ in Theorem 5.3. We have also shown that $H_{2n+i}(Y^X_I(X)) \cong H_{n+i}(I(X); \mathbb{Q}^i)$. Thus, we have the statement of the theorem.

Below is a picture of the spectral sequence for $H_\ast(Y^X_I(X))$. The red indicates the non-zero entries. The Dehn complex is the thick blue line sitting at the line $q = 2n$. 

$q$

3n

2n

$p$

2n 3n
6. Two proofs of Theorem 5.1

In this section we prove Theorem 5.1. We begin with some notation. Let $F: \mathcal{I}_n \to \text{Top}$, be defined by

$$F(b, a_1, \ldots, a_k) \overset{\text{def}}{=} S^\sigma \wedge \bigvee_{W \oplus^+ \bigoplus V_i = \mathbb{R}^{n+1}} F_W \overset{k}{\star} \star F_{V_i}$$

and $F': \mathcal{I}_n \to \text{Top}_*$ be defined by

$$F'(\tilde{A}) = F(\tilde{A})_{hI(X)}.$$

We set

$$Z \overset{\text{def}}{=} \text{thocofib} F \quad \text{and} \quad Z' \overset{\text{def}}{=} \text{thocofib} F'.$$

Note that

$$Z_{hI(X)} \cong Z'.$$

We have the following surprising identification:

**Proposition 6.1.**

$$Z \cong S^\sigma \wedge S^{2n-1}.$$

*Here, the $I(X)$-action is trivial on the $S^{2n-1}$-coordinate and acting by the determinant on $S^\sigma$.***

In Subsections 6.1 and 6.2 we give two different proofs of this proposition; for now we assume it and complete the proof of Theorem 5.1.

Rationally, by Lemma 4.7

$$\left( S^\sigma \wedge \bigvee_{W \oplus^+ \bigoplus V_i = \mathbb{R}^{n+1}} F_W \overset{k}{\star} \star F_{V_i} \right)_{hI(X)} \cong (S^\sigma \wedge F_W)_{hI(X)} \wedge \bigwedge_{i=1}^k (S^\sigma \wedge F_{V_i})_{hO(a_i)},$$

If any of the $a_i$ are odd then by Proposition 2.11 $(S^\sigma \wedge F_{V_i})_{hO(a_i)} \cong \ast$; thus all entries in $F'$ at points with any odd $a_i$-coordinate must be contractible. If we thus compute $Z'$ by first taking homotopy cofibers in all of the odd dimensions we get a cube with a single entry $y_{hI(X)}^X$ (at the source) and all other entries contractible; thus

$$Z' \cong \Sigma^n (y_{hI(X)}^X).$$

By the homotopy orbit spectral sequence (see Proposition A.5) and Proposition 6.1

$$H_i(Z_{hI(X)}) \cong H_{i-2n}(I(X); \mathbb{Q}^t).$$

Thus

$$H_i(Y_{hI(X)}) \cong H_{i+n}(Z') \cong H_{i+n}(Z_{hI(X)}) \cong H_{i-n}(I(X); \mathbb{Q}^t),$$

completing the proof of the theorem.

We now present two different proofs of Proposition 6.1. The first is based on the combinatorics of the simplicial sets $S^\sigma \wedge F_{F'}$; it is direct and relies on very few technicalities. The second is more conceptual, and produces a proof of the Solomon–Tits theorem along the way. Although the basic underlying ideas of the proofs are quite similar, the approaches differ enough that we felt it was useful to present both approaches. None of the results of this section are necessary for understanding later sections, and thus the reader more interested in the connections to algebraic $K$-theory may wish to skip this section.

6.1. Combinatorics. We begin by computing the homotopy cofiber of a single Dehn invariant. In order to be able to do this for any general map in the cube, we first generalize the definition of the Dehn invariant.

Let $W, U_1, \ldots, U_j$ be any decomposition of $X$ into orthogonal subspaces. Define

$$d_j = \dim W + \sum_{\ell=1}^j \dim U_\ell \quad j \geq 0.$$
(Thus $d_0 = \dim W$ and $d_i = \dim X$.) Let $\ell$ be an integer distinct from $d_1, \ldots, d_i$. Let $j$ be the minimal index such that $d_j > \ell$. For convenience, we define

$$D_\ell: F_W \ast F_{U_1} \ast \cdots \ast F_{U_j} \longrightarrow \bigvee_{\substack{V_j \subseteq U_j \\ \dim V_j = \ell - d_j, \ldots, 1}} F_W \ast F_{U_1} \ast \cdots \ast F_{U_j} \ast F_{V_j \cap U_j} \ast \cdots \ast F_{U_i}$$

to be $1 \ast \cdots \ast D_{\ell - d_j - 1} \ast \cdots \ast 1$.

**Definition 6.2.** For any subset $I \subseteq \{1, \ldots, 2n - 1\}$ let $N_I F_X$ be the subspace of $F_X$ containing no subspace with dimension contained in $I$.

This definition gives us a convenient way to identify the total homotopy cofiber of a Dehn cube. To compute the total homotopy cofiber we need the following special case:

**Proposition 6.4.** Let $X$ be a pointed simplicial set, and let $Y_1, \ldots, Y_n$ be subspaces of $X$. Write $P(n)$ for the partial order of subsets of $\{1, \ldots, n\}$. We define a functor

$$F: P(n) \longrightarrow \mathbf{sSet} \quad \text{by} \quad I \longmapsto X / \bigcup_{i \in I} Y_i,$$

with the induced morphisms given by the quotient maps. Then

$$\text{thocofib} F \simeq \Sigma^n \bigcap_{i=1}^n Y_i.$$

**Proof.** We prove this by induction on $n$. When $n = 0$ the cube is trivial and the statement holds. When $n = 1$ the cube is $X \longrightarrow X / Y$, and the total homotopy cofiber is $\Sigma Y$, as desired.

Now consider the general case. We can compute the total homotopy cofiber iteratively [MV15 Proposition 5.9.3] by first taking cofibers in the direction of “adding $n$ to a set”: the direction in which each subset $J \in P(n)$ with $n \notin J$ is mapped to $J \cup \{n\}$. Taking the homotopy cofiber in each of these directions produces the cube $G: P(n - 1) \longrightarrow \mathbf{sSet}$ given by

$$I \longmapsto \Sigma Y_n / \bigcup_{i \in I} \Sigma(Y_i \cap Y_n).$$

This is an $n - 1$-cube of the same type as in the proposition; by the induction hypothesis,

$$\text{thocofib} G \simeq \Sigma^{n-1} \bigcap_{i=1}^{n-1} \Sigma(Y_i \cap Y_n).$$

We can think of $\Sigma(Y_i \cap Y_n)$ as sitting inside $\Sigma X$ as $\Sigma Y_i \cap \Sigma Y_n$; then

$$\Sigma^{n-1} \bigcap_{i=1}^{n-1} \Sigma(Y_i \cap Y_n) = \Sigma^{n-1} \bigcap_{i=1}^{n-1} (\Sigma Y_i) \cap (\Sigma Y_n) = \Sigma^{n-1} \bigcap_{i=1}^{n-1} \Sigma Y_i = \Sigma^n \bigcap_{i=1}^n Y_i,$$

as desired. 

**Proposition 6.4.** Let $I \subseteq \{1, \ldots, 2n - 1\}$. Then the cube formed by the $D_i$’s for $i \in I$ has total homotopy cofiber $\Sigma^{|I|} N_I F_X$.

**Proof.** For conciseness, write $D_I$ for the composition of the $D_i$ for $i \in I$. Since we know that the Dehn cube commutes, the order of composition is irrelevant. Let $I = \{i_0, \ldots, i_{j-1}\}$. We claim that

$$D_I: F_X \longrightarrow \bigvee_{\substack{W \oplus U_1 \oplus \cdots \oplus U_j = X \\ \dim W = i_0 \quad \dim U_i = i_i - i_{i-1} - \cdots - i_0, \quad i < j}} F_X \ast F_{U_1} \ast \cdots \ast F_{U_j}$$

is isomorphic to the map

$$F_X \longrightarrow F_X / N_I F_X$$
via the isomorphism

\[
F_X \cong F_{U_1} \ast \ldots \ast F_{U_j} \rightarrow F_X / N_1 F_X
\]

taking an \( \ell_j \)-simplex

\[
(U_0 \subseteq \cdots \subseteq U_{\ell_0}, U_{\ell_0+1} \subseteq \cdots \subseteq U_{\ell_1}, \ldots, U_{\ell_{j-1}+1} \subseteq \cdots \subseteq U_{\ell_j})
\]
to the flag

\[
U_0 \subseteq \cdots \subseteq U_{\ell_0} \subseteq U_{\ell_0} \oplus U_{\ell_0+1} \subseteq U_{\ell_0} \oplus U_{\ell_0+2} \subseteq \cdots \subseteq U_{\ell_0} \oplus \cdots \oplus U_{\ell_j}.
\]

Note that every flag in the image contains subspaces of all dimensions contained in \( I \), and any face map that removes one of them takes the simplex to the basepoint. In addition, from the construction of the map we see that it is bijective on simplices, showing that it is an isomorphism of simplicial sets.

An analogous analysis (with more annoying simplices) shows that in fact each Dehn invariant is quotienting out by a subspace. Note, also, that

\[
N_1 F_X \cong \bigcap_{i \in I} N_{(i)} F_X.
\]

Thus the Dehn cube is isomorphic to a cube of the form in Lemma 6.3. Applying the lemma we see that the total homotopy cofiber is

\[
\Sigma^{n-1} \bigcap_{i=1}^{n-1} N_{(i)} F_X = \Sigma^{n-1} N_1 F_X,
\]

as desired. \( \square \)

We can now use this to prove Proposition 6.1.

**Proof of Proposition 6.1.** By Proposition 6.4, \( Z \simeq S^\sigma \wedge \Sigma^{2n-1} N_{0,\ldots,2n-2} F_X \). However, \( N_{0,\ldots,2n-2} F_X \cong S^0 \), as it has exactly two simplices in each dimension: the basepoint and \( X = \cdots = X \). The \( S^\sigma \) commutes out, as none of its simplices are involved with the calculation. Thus

\[
Z \simeq S^\sigma \wedge S^{2n-1}.
\]

\( \square \)

### 6.2. The Solomon–Tits Theorem and the Dehn Invariant

In this section, we extend a variant of Quillen’s proof of the Solomon–Tits theorem and show how it relates directly to the total cofiber of Dehn invariants. Similar proofs appear in a work by Kahn [Kah11] and Rognes [Rog00] (the authors learned it from the latter). Both are cleaner rephrasings of Quillen’s proof [Qui73].

**Definition 6.5.** Define \( F_{U,V} \) to be the pointed simplicial set whose non-basepoint \( i \)-simplices are of the form \( U_0 \subseteq \cdots \subseteq U_i \), where \( U_i = V \) and \( U \subseteq U_0 \).

Now let \( W \) be any subspace of \( V \). We define \( F^W_V \) to be the subspace of \( F_V \) containing the basepoint and all simplices \( U_0 \subseteq \cdots \subseteq U_i \) such that \( U_0 \cap W \neq 0 \). More generally, for any \( U \subseteq W \subseteq V \) we define \( F^W_{U,V} \) to be the subspace of \( F_V \) containing the basepoint and all simplices \( U_0 \subseteq \cdots \subseteq U_i \) such that \( U_0 \cap W \supseteq U \).

**Lemma 6.6.** Suppose \( U \oplus W = V \). Then

\[
F_{U,V} \cong F_{0,W}.
\]

**Proof.** The isomorphism is given explicitly by the mutually inverse maps

\[
F_{U,V} \rightarrow F_{0,W} \quad U_0 \subseteq \cdots \subseteq U_i \quad (U_0 \cap W) \subseteq \cdots \subseteq (U_i \cap W) \text{ and }
F_{0,W} \rightarrow F_{U,V} \quad W_0 \subseteq \cdots \subseteq W_i \quad (W_0 \oplus U) \subseteq \cdots \subseteq (W_i \oplus U).
\]

\( \square \)

The following fact is critical. It is deceptively simple, but allows for homotopy theoretic control of the Dehn invariant.

**Lemma 6.7.** For any nontrivial subspace \( W \) of \( V \), \( F^W_V \) is contractible. More generally, for any \( U \subseteq W \subseteq V \), \( F^W_{U,V} \) is contractible.
Proof. Write $\text{Sub}^\wedge(U,V)$ for the full subcategory of subspaces $W'$ of $V$ such that $U \subsetneq W' \cap W$; we write $\text{Sub}^\wedge(U,V)$ when we wish to exclude $V$.

We have that $F^\wedge_{U,V} \cong N\text{Sub}^\wedge(U,V)/N\text{Sub}^\wedge(U,V)$. Since $\text{Sub}^\wedge(U,V)$ has a terminal object, it is contractible, and thus $F^\wedge_V \cong \Sigma N\text{Sub}^\wedge(U,V)$; it therefore suffices to show that $N\text{Sub}^\wedge(U,V)$ is contractible.

Consider the functor

$$F: \text{Sub}^\wedge(U,V) \longrightarrow \text{Sub}^\wedge(U,W) \quad W' \longrightarrow W' \cap W.$$ 

There is also an inclusion functor $\iota: \text{Sub}(U,W) \longrightarrow \text{Sub}^\wedge(U,V)$, which is right adjoint to $F$. Thus $\text{Sub}^\wedge(U,V)$ is a retract of $\text{Sub}^\wedge(U,W)$, and in particular $F$ is a weak equivalence. Since $\text{Sub}^\wedge(U,W)$ has a terminal object it is contractible, as desired.

We can now use the machinery defined above to prove a “generalized” Solomon–Tits theorem. The usual Solomon–Tits theorem follows from this generalization as a corollary; see Corollary 6.9. It also has the distinct advantage of exhibiting a direct connection to the construction of the Dehn invariant.

Theorem 6.8 (Generalized Solomon–Tits). Let $V'$ be any nontrivial proper subspace of $V$. Then there is a weak equivalence

$$F_V \xrightarrow{\sim} \bigvee_{W \subseteq V' = V} F_W \ast F_{W,V}$$

More generally, for any fixed $U$ and any $U \subsetneq V' \subsetneq V$ there exists a weak equivalence

$$F_{U,V} \xrightarrow{\sim} \bigvee_{W \subseteq V' = V} F_{U,W} \ast F_{W,V}$$

Proof. The first statement follows from the second by setting $U = 0$ so we focus on the second. The space $F^\wedge_{U,V}$ is contractible and we observe that

$$F^\wedge_{U,V}/F^\wedge_{U,V} \cong \bigvee_{W \subseteq V' = V} F_{U,W} \ast F_{W,V}.$$ 

Indeed, to see this isomorphism we note that every non-basepoint simplex in $F_{U,V}/F^\wedge_{U,V}$ must be of the form $U_0 \subseteq \cdots \subseteq U_j \subset U_{j+1} \subseteq \cdots \subseteq U_i$, where $U_j \cap V' = 0$ and $U_{j+1} \cap V' \neq 0$, with $0 \leq j < i$. We can then map this simplex to the simplex $(U_0 \subseteq \cdots \subseteq U_j, U_{j+1} \subseteq \cdots \subseteq U_i)$, which gives the isomorphism.

Applying this inductively we can conclude the Solomon–Tits theorem.

Corollary 6.9 (Solomon–Tits). Suppose that $\dim V = n$ and fix any maximal flag $V_1 \subseteq \cdots \subseteq V_{n-1} \subseteq V$. There is a weak equivalence

$$F_V \xrightarrow{\sim} \bigvee_{U_1 \subsetneq \cdots \subsetneq U_n} F_{U_1,U_2} \ast \cdots \ast F_{U_{n-1},U_n}$$

where the wedge is taken over strings of $n$ subspaces $U_1 \subset \cdots \subset U_n$ such that $U_i \cap V_{n-i} = V$; here we write $V_0 = 0$.

In particular, $F_V$ has the homotopy type of a wedge of spheres.

Proof. That the map exists and is a weak equivalence follows by induction.

To see the last part, we note that each space $F_{U_1,U_2}$ has the homotopy type of $S^0$. Since $S^0 \ast \cdots \ast S^0$ has the homotopy type of $S^{n-1}$, the homotopy type of $F_{V\cap}$ is a wedge of copies of $S^{n-1}$. This is as expected since it is a suspension of the usual Solomon–Tits building.

For the remainder of this section, we assume that we work inside of a vector space $V$ of dimension $d$ that is equipped with a quadratic form so that we have a notion of orthogonal projection. Thus, in Corollary 6.9
we may identify \( F_{U_i U_{i+1}} \) with \( F_{U_i} \cap U_{i+1} \). The Solomon–Tits theorem is then equivalent to the statement that the map

\[
(6.10) \quad F_V \xrightarrow{\sim} \bigvee_{L_1 \oplus \cdots \oplus L_{d} = V} F_{L_1} \star \cdots \star F_{L_d}.
\]

is a weak equivalence for any fixed maximal flag \( V_1 \subset \cdots \subset V_{d-1} \).

We now use the above work to get purchase on the Dehn invariant. The intuition is that the Solomon–Tits theorem breaks up the flag complex into sequences of joins of zero dimensional spaces. The Dehn map accomplishes a similar goal by using the extra structure of geometry.

**Example 6.11.** For this example, we work over \( \mathbb{R} \), set \( V = \mathbb{R}^2 \), and fix a line \( \ell \) in \( \mathbb{R}^2 \). Consider the simplest Dehn map

\[
F_{\mathbb{R}^2} \xrightarrow{D_1} \bigvee_{L \oplus L_1 = \mathbb{R}^2} F_L \star F_{L_1}.
\]

By generalized Solomon–Tits,

\[
F_{\mathbb{R}^2} \xrightarrow{\sim} \bigvee_{L \oplus \ell = \mathbb{R}^2} F_L \star F_{\ell} \xrightarrow{\sim} \bigvee_{L \oplus \ell = \mathbb{R}^2} S^1.
\]

This equivalence is modeled by modding out by the contractible subspace defined above. Examining the diagram below, where the vertical string of maps is a cofiber sequence, and the horizontal map is given by the Dehn invariant

\[
\begin{array}{c}
F_{\mathbb{R}^2} \xrightarrow{\sim} \\
\downarrow \\
F_{\mathbb{R}^2} \xrightarrow{D_1} \bigvee_{L \oplus L_1 = \mathbb{R}^2} F_L \star F_{L_1} \xrightarrow{\sim} \\
\downarrow \approx \\
\bigvee_{L \oplus \ell = \mathbb{R}^2} F_L \star F_{\ell}.
\end{array}
\]

we get an induced inclusion

\[
\bigvee_{L \oplus \ell = \mathbb{R}^2} F_L \star F_{\ell} \xrightarrow{\sim} \bigvee_{L \oplus L_1 = \mathbb{R}^2} F_L \star F_{L_1},
\]

where we identify factors by projecting \( \ell \) away from \( L \) and onto \( L_1 \). It follows that the only term present in the latter wedge that is not present in the former wedge is the one indexed by \( \ell \oplus L_1 \). Thus the cofiber of this is the single sphere \( F_\ell \star F_{L_1} \simeq S^1 \). Thus, at least in this dimension, the total homotopy cofiber of the Dehn cube (Dehn line) is a single circle.

Following this example, we compute the cofiber of a single Dehn invariant of the full flag. Note that this in theorem, we are not placing any restrictions on the dimension of \( U \) (i.e. it may be even or odd).

**Theorem 6.12.** For a single Dehn invariant \( D_k \), we have

\[
\text{hocofib} \left( F_V \xrightarrow{D_k} \bigvee_{\dim U = k} F_U \star F_{U_\perp} \right) \simeq \bigvee_{\dim U = k \atop U \oplus V_{d-k} \neq V} F_U \star F_{U_\perp}.
\]
Proof. The proof proceeds in complete analogy with Ex. 6.11. We consider the diagram

\[
\begin{array}{c}
F_V \xrightarrow{D_k} \bigvee \text{dim } U = k \ F_U \xrightarrow{\perp} F_U^\perp \\
\approx \\
\bigvee \text{dim } U = k
\end{array}
\]

The top horizontal map \( D_k \) is the Dehn map, the vertical map on the left is the quotient by the contractible subspace \( F_V^\perp \) and the bottom horizontal map is the inclusion induced by projecting \( F_U^\perp \) onto \( F_U \).

In the bottom two entries, both spaces are indexed spaces of dimension \( k \) in \( V \), but on the right we are indexing over all spaces of dimension \( k \) in \( V \). The terms left over after taking the cofiber are those \( U \) such that \( U \oplus V_{d-k} \neq V \), i.e. \( U \) of dimension \( k \) such that \( U \cap V_{d-k} \neq 0 \).

For any space \( U \) let \((U)_1 \subset \cdots \subset (U)_{\text{dim } U} = U\).

The proof of Theorem 6.12 suggests the following important consequence.

**Theorem 6.13.** Fix a maximal flag \((U)_j\) for \( U \). The Dehn invariant \( D_k \) can be modeled as an inclusion of 

\[
\bigvee_{E_1} F_{L_0} \xrightarrow{\perp} \cdots \xrightarrow{\perp} F_{L_{\text{dim } U}} \xrightarrow{\perp} \bigvee_{E_2} F_{L_0} \xrightarrow{\perp} \cdots \xrightarrow{\perp} F_{L_{\text{dim } U}}^\perp
\]

where \( E_2 \) is the following expression

\[
L_1^0 \oplus \cdots \oplus^\perp L_{\text{dim } U} = U
\]

\[
L_1^0 \oplus \cdots \oplus^\perp L_{\text{dim } U}^0 = U^\perp
\]

\[
L_1^0 \oplus \cdots \oplus^\perp L_j \oplus (U)_j = U \text{ for each } j
\]

\[
L_1^0 \oplus \cdots \oplus^\perp L_j \oplus (U^\perp)_j = U^\perp \text{ for each } j
\]

and \( E_1 \) is the same as \( E_2 \) with the following extra conditions

\[
L_1^0 \oplus \cdots \oplus^\perp L_b \oplus V_{d-b} = V \text{ for each } 1 \leq b \leq \text{dim } U
\]

Proof. The proof of Theorem 6.12 shows that the Dehn invariant \( D_k \) can be modeled homotopically by an inclusion

\[
\bigvee_{U \oplus V_{d-k} = V^*} F_U \xrightarrow{\perp} F_U^\perp \bigvee_{\text{dim } U = k} F_U \xrightarrow{\perp} F_U^\perp
\]

Applying Corollaries 6.9 and 6.10 to break up each of \( F_U, F_U^\perp \) gives the desired result.

In order to understand the spaces comprising \( Z \), it will be useful to use the Solomon-Tits theorem to express each of them as joins of copies of \( S^0 \).

**Lemma 6.14.** Let \( U_1 \oplus \cdots \oplus^\perp U_j = V \) be an orthogonal decomposition of \( V \). Let \((U_1)_k\) denote any maximal flag of \( U_1 \). Then

\[
\bigvee_{U_1 \oplus \cdots \oplus^\perp U_j = V} F_{U_1} \xrightarrow{\perp} \cdots \xrightarrow{\perp} F_{U_j} \xrightarrow{\perp} \bigvee_{E} F_{L_1} \xrightarrow{\perp} \cdots \xrightarrow{\perp} F_{L_j} \xrightarrow{\perp} \bigvee_{L_{\text{dim } U_1} \oplus \cdots \oplus^\perp L_{\text{dim } U_j}} F_{L_1} \xrightarrow{\perp} \cdots \xrightarrow{\perp} F_{L_j}
\]
where $E$ is the following expression

$$L_1^1 \oplus \ldots \oplus L_{\dim U_1}^1 \oplus \ldots L_j^1 \oplus \ldots \oplus L_{U_j}^j = V$$

$$L_1^1 \oplus \ldots \oplus L_{\dim U_1}^1 = U_1$$

$$L_1^1 \oplus \ldots \oplus L_{\dim U_1}^i = U_i$$

$$L_j^1 \oplus \ldots \oplus L_{U_j}^j = U_j$$

for each $1 \leq b \leq \dim U_1$

Proof. This follows from an application of (6.10) to each of the spaces $F_{U_i}$. \hfill \square

We are now ready to prove Proposition 6.1.

Proof of Proposition 6.1. All of our subspaces are defined by linear subspaces in a vector space of dimension $d = 2n$. Each of the maps in $Z$ can be modeled as an inclusion

$$\bigvee_{E_i} F_{L_1} \star \ldots \star F_{L_{\dim V}} \longrightarrow \bigvee_{E_2} F_{L_1} \star \ldots \star F_{L_{\dim U}} \star F_{L_1} \star \ldots \star F_{L_{\dim U^{-1}}}$$

where the only thing that changes are the restrictions in $E_1$ and $E_2$. There are strictly fewer restrictions on sums of lines in $E_2$ — the spaces $U_i$ need not satisfy any conditions with respect to the maximal flag on the ambient space. This is the key difference.

Once we have modelled every Dehn invariant by an inclusion, every diagram as in (3.12) is a latching diagram and thus Reedy cofibrant. Need more here to ensure cofibrancy.

To take the ambient space. This is the key difference.

We now focus on the Dehn invariant on these two-dimensional subspaces. For all two-dimensional subspaces $W$ containing a preferred line $\ell$,

$$F_W \sim \bigvee_{L \oplus \ell = W} F_L \star F_L$$

The only line that does not appear in this wedge is $\ell$. Thus, when we quotient by $\bigvee_{W \oplus L_2, \ldots \oplus L_{2n}} F_W \star F_{L_2} \ldots \star F_{L_{2n}}$ we obtain

$$\bigvee_{\ell \oplus L_2, \ldots \oplus L_{2n}} S^0 \star F_{L_2} \star \ldots \star F_{L_{2n}}$$

The rest of the Dehn invariants only affect the remaining lines. Continuing to quotient, we are left with a single sphere $S^0 \star \ldots \star S^0$ indexed by successive orthogonal complements in our distinguished flag. \hfill \square
7. Goncharov’s Conjectures

In [Gon99], Goncharov has a series of three conjectures about possible connections between the Dehn complex and the algebraic $K$-theory of $C$. We give a summary of these conjectures here. Our notation does not exactly agree with Goncharov’s; in particular, Goncharov’s Dehn complex is cohomologically graded and 1-indexed, while ours is homologically graded and 0-indexed. We number the parts of our summary by the number of the conjecture in [Gon99].

**Conjecture 7.1** ([Gon99] Conjectures 1.7-1.9]). Let $P_i(X^{2n-1})$ be the Dehn complex for the geometry $X^{2n-1}$.

1. There exist homomorphisms
   \[ H_i P_i(S^{2n-1}) \longrightarrow (gr_n^\gamma K_{n+i}(C) \otimes \epsilon(n))^+ \]
   and
   \[ H_i P_i(H^{2n-1}) \longrightarrow (gr_n^\gamma K_{n+i}(C) \otimes \epsilon(n))^-. \]

2. When $i = n-1$ these homomorphisms are injective, and the volume map is compatible with the Borel (resp. Beilinson) regulator.

3. These maps are isomorphisms for all $i$.

Here, $gr_n^\gamma$ is the $n$-th graded part of the $\gamma$-filtration, and $\epsilon(n)$ is the vector space $\mathbb{Q}$ with $\mathbb{Z}/2$ acting on it via multiplication by $(-1)^n$. The sign in the superscript indicates taking the $\pm 1$ eigenspace with respect to the action by complex conjugation.

For an exposition of the $\gamma$-filtration, see for example [Gra94]. Goncharov proves (1.8) in the case when $C$ is replaced with $\mathbb{Q}$, $i = n-1$, and simplices in the Dehn complex are restricted to those with algebraic vertices [Gon99, Theorem 1.6].

Inspired by the conjectures, we can prove the following alternative connection between the algebraic $K$-theory of $C$ and the scissors congruence groups. We must first change from the $\gamma$-filtration to the rank filtration. The rank filtration on $K_*(C)$ is defined by
\[ F_i K_*(C) \overset{\text{def}}{=} \text{im} \left( \pi_0 BGL(i; C)^+ \longrightarrow \pi_0 BGL(C)^+ \right). \]
Then
\[ gr^\gamma_n K_*(C) = \text{coker}(F_{n-1} K_*(C) \longrightarrow F_n K_*(C)). \]
We wish to compare this with the data found just in $BGL(n; C)^+$; we thus define
\[ C_{n,i} \overset{\text{def}}{=} \text{coker} \left( \pi_{n+1}(BGL(n-1; C)^+) \longrightarrow \pi_{n+1}(BGL(n; C)^+) \right). \]
Due to the difference between $gr^\gamma_n K_*(C) \otimes \epsilon(n)$ and $C_{n,i}$ we end up with a zigzag where the left-hand side is surjective.

**Theorem 7.2.** There exist zigzags
\[ (gr^\gamma_n K_{n+i}(C) \otimes \epsilon(n))^+ \longleftrightarrow (C_{n,i} \otimes \epsilon(n))^+ \longrightarrow H_i P_i(S^{2n-1}) \]
and
\[ (gr^\gamma_n K_{n+i}(C) \otimes \epsilon(n))^-. \]

**Proof.** We focus on proving the first zigzag exists. The proof for the second is almost identical; we discuss the necessary changes at the end of the proof.

The left-hand side of the zigzag exists by definition. By Theorem 5.3 it suffices to prove that the zigzag exists when the right-hand codomain is replaced with $H_{i+n}(O(2n); \mathbb{Q}^n)$. Since we are working rationally, we have
\[ H_{n+i}(O(2n; \mathbb{C}); \mathbb{Q}^n)^+ \cong H_{n+i}(O(2n; \mathbb{Q})^n), \]
where the $^+$ indicates taking the fixed points of complex conjugation. We are therefore going to construct a map
\[ C_{n,i} \otimes \epsilon(n) \longrightarrow H_{n+i}(O(2n; \mathbb{C}); \mathbb{Q}^n) \]
which is equivariant with respect to complex conjugation. We have the hyperbolic map
\[ h: GL(n; \mathbb{C}) \longrightarrow SO(n, n; \mathbb{C}) \quad \text{given by} \quad M \longmapsto \left( \begin{smallmatrix} M & 0 \\ 0 & (MT)^{-1} \end{smallmatrix} \right) \]
which is also equivariant with respect to complex conjugation. Here, $SO(n, n; C)$ is the group of linear maps preserving the quadratic form $x_1 x_n + \cdots + x_n x_2$.

We wish to relate the homology of $SO(n, n; C)$ to the homology of $O(2n; C)$ with twisted coefficients. As discussed in [Cat07, Above Theorem 1.4], $H_*(O(2n; C); Q^t) \cong H_*(SO(2n; C))^+$, where here the $+$ indicates the $+1$-eigenspace of the $\mathbb{Z}/2$-action by conjugation by any matrix of determinant $-1$. In particular, $H_*(O(2n; C); Q^t)$ is a direct summand of $H_*(SO(2n))$.

To go from $SO(n, n; C)$ to $SO(2n; C)$ we use the following explicit construction. Let $D_n = (d_{ij})_{i,j=1}^{2n}$ be the matrix defined by

$$d_{ij} = \begin{cases} 
1/\sqrt{2} & \text{if } i = j, 1 \leq i \leq n \\
i/\sqrt{2} & \text{if } i = j, n+1 \leq i \leq 2n \\
-1/\sqrt{2} & \text{if } i = j + n \\
0 & \text{otherwise.}
\end{cases}$$

When $n = 1$ this is the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & i \\ i \\ 1 \\ 1 \end{pmatrix}.$$

$D_n$ is defined so that conjugation by $D_n$ gives an isomorphism $SO(n, n; C) \rightarrow SO(2n; C)$. Let $h_{D_n}: GL(i, C) \rightarrow SO(2i; C)$ be the map given by $h$ composed with conjugation by $D_i$. We thus have the following diagram, in which everything is tensored with $Q$ and we assume $n \geq 2$:

$$\pi_*(BGL(n-1; C)) \xrightarrow{(h_{D_{n-1}})_*} \pi_*(BGL(n-2; C)) \xrightarrow{\text{prim}} H_*(SO(2n-2; C)) \xrightarrow{H_*(SO(2n-2; C); Q^t)} 0$$

The right-hand map is 0 by [Cat07] Theorem 1.6. Thus, taking vertical cokernels, we get a map

$$C_{n,i} \xrightarrow{H_*(O(2n; C); Q^t)}.$$  

Unfortunately, this map is not equivariant with respect to the action by complex conjugation. If we define $E_n = \text{diag}(I_n, -I_n)$, we get that

$$\overline{D}_n = E_n D_n.$$  

When $n$ is even, this map is conjugation by a matrix of determinant 1 (and is therefore trivial on homology), but when $n$ is odd it is not. To make the map equivariant, we tensor $C_{n,i}$ on the left with $\varepsilon(n)$, which makes it equivariant (as the right-hand side is the $-1$-eigenspace of conjugation by a matrix of determinant $-1$). This means that the map

$$C_{n,i} \otimes \varepsilon(n) \xrightarrow{H_*(O(2n; C); Q^t)}$$

is equivariant, as desired. Taking the fixed points under complex conjugation gives the first part of the theorem.

To show the hyperbolic part of the theorem, exactly one thing needs to change: $D_n$ should have $n-1$ blocks of the type illustrated, and one block which looks like

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$  

With this transformation, we must set $E_n = \text{diag}(-I_{n-1}, I_{n+1})$; thus complex conjugation acts by conjugation by a matrix of determinant $(-1)^{n-1}$, which analogously to the above gives us an equivariant map of the form

$$C_{n,i} \otimes \varepsilon(n-1) \xrightarrow{H_*(O(1, 2n-1; C); Q^t)}.$$  

However,

$$(C_{n,i} \otimes \varepsilon(n-1))^+ \cong (C_{n,i} \otimes \varepsilon(n))^-,\)$$

which gives us the form desired by the theorem. □
Remark 7.4. If we choose $E$ to map $SO(n, n)$ to $SO(n+k, n−k)$ for some $k \neq 0, 1$, an interesting phenomenon occurs: depending on the parity of $k$, the left-hand side of the zigzag is equal to either the $k = 0$ or $k = 1$ case, while the right-hand side instead contains the homology of other orthogonal groups. It would be interesting to determine what these maps look like, and what kinds of phenomena they detect.

Part of Goncharov’s conjectures concern the relationship of the Borel regulator out of $K_{2n−1}(C)$ with the volume in the scissors congruence group. We are able to resolve these conjectures in modified form by standard relatively standard constructions with Cheeger-Chern-Simons classes. By work of Dupont-Hain-Zucker [DHZ00], the Borel element is twice the Cheeger-Chern-Simons class.

We use an idea, due to Dupont which works for more general homogeneous spaces. This idea is explained briefly in [Dup01] Sect. 10, which is a digest of much more general theorems in [DK90] and [DHZ00] Section 5.

The Lie group $O(2n)$ acts on $O(2n)/O(2n − 1) \cong S^{2n−1}$ in the usual way. The idea is to use this action to obtain maps from $H_*(O(2n; R)^δ; Q^i) \to R$ which are invariants of homology, and so give elements of cohomology. The group $H_*(O(2n; R)^δ, Q^i)$ is computed via the bar construction and a cycle in degree $2n − 1$ is represented by a sequence of elements of $(g_1, \ldots, g_{2n−1})$. Then pick a point $x_0$ and define a geodesic simplex in $S^{2n−1}$ by $Δ^{2n−1} = (x_0, g_1x_0, g_2g_1x_0, \ldots, g_n \cdots g_1x_0)$. Given the volume form $v \in Ω^{2−i}(S^{2n−1})$ (which we normalize to so that $\int_{S^{2n−1}} v_{S^{2n−1}} = 1$ we can then define a map $I(ω): B_{2n−1}(O(2n)^δ, Q^i) \to R$ $I(v_{S^{2n−1}})(g_1, \ldots, g_{2n−1}) = \int_{Δ_{2n−1}} v_{S^{2n−1}}$.

Dupont and Kamber [DK90] Thm. 5.3 show that this is a cocycle mod $Z$, and determines an element in $H_{2n−1}(O(2n, R); Q^i)$. This technique is quite general and applies to highly connected homogeneous spaces. By considering $H^n = O^+(n, 1)/O(n)$ one obtains a similar hyperbolic statement.

**Theorem 7.5.** There are commutative diagrams

\[
\begin{array}{ccc}
\text{ker } D_{2n−1} & \longrightarrow & H_{2n−1}(O(2n; R)^δ, Q^i) \\
\downarrow \text{vol} & & \downarrow \text{CCS} \\
R/(2π)^nZ & \longrightarrow & \end{array}
\]

and

\[
\begin{array}{ccc}
\text{ker } D_{2n−1} & \longrightarrow & H_{2n−1}(O(2n, 1; R)^δ, Q^i) \\
\downarrow \text{vol} & & \downarrow \text{CCS} \\
R & \longrightarrow & \end{array}
\]

**Proof.** We give the proof for the $S^{2n−1}$ case — the proof for hyperbolic space is identical. We claim that the edge homomorphism onto the $E_1$-page in the spectral sequence of Theorem 5.3 is given by the map described above, and gives a surjection $H_{2n−1}(O(2n; R)^δ; Q^i) \to \ker D_{2n−1}$ (a similar claim is made in [Dup01] Sect. 10, Rmk. 3). Theorem 5.3 shows that this map is split by the inclusion. The rest of the diagram follows from the discussion above. The normalization in the target for the spherical case is chosen to agree with the convention that $v_{S^{2n−1}}$ has integral periods.

Consider the spectral sequence for $Z_{hO(2n)}$, where $Z$ is the space defined in the beginning of Section 10. This spectral sequence in a shift by $n$ to the right of the spectral sequence for $v_{S^{2n−1}}$; see the diagram on page 65. The spectral sequence induces the following diagram:

\[
\begin{array}{ccc}
H_{2n−1}(O(2n); Q^i) & \longrightarrow & H_0(O(2n), H_{2n}(S^\sigma \wedge F_{S^{2n−1}})) \\
\uparrow \cong & & \uparrow \cong \\
H_{4n}(S^\sigma \wedge Z_{hO(2n)}) & \longrightarrow & H_{2n}(S^\sigma \wedge F_{S^{2n−1}})_{hO(2n)})
\end{array}
\]

(7.6)

Here the map along the bottom is an edge homomorphism in the spectral sequence. The homology group $H_{4n}(Z_{hO(2n)})$ maps surjectively onto the first non-trivial filtration term in the $E^∞$-page, and so maps on to $E^∞_{2n, 2n}$. We also know that the $E^∞$-page in that row is the same as the $E^2$ page and so $H_{4n}(Z_{hO(2n)})$ surjects.
on to \( \ker D_{2n-1} \) which in turn injects into \( \mathcal{P}(S^{2n-1}) \), which is on the \( E^1 \) page. The map from \( H_{4n}(Z_hO(2n)) \) onto the \( E^1 \)-page is induced by the quotient map

\[
Z \longrightarrow \Sigma^{2n-1}F_{S^{2n-1}}
\]

(the latter is the first filtration quotient in the filtration for the total homotopy cofiber spectral sequence). Note that since \( Z \simeq S^{2n-1} \) the map above is determined by a map \( S^{2n-1} \longrightarrow \Sigma^{2n-1}F_{S^{2n-1}} \).

Upon applying homotopy coinvariants and \( H_{4n-1} \) the above becomes the top horizontal line in the following diagram:

\[
\begin{array}{cccccc}
H_{4n-1}(Z_{hO(2n)}) & \longrightarrow & H_{4n-1}(\Sigma^{2n-1}(S^\sigma \wedge F_{S^{2n-1}})_{hO(2n)}) & \cong & H_{2n}((S^\sigma \wedge F_{S^{2n-1}})_{hO(2n)}) & \cong \\
\cong & & \cong & & \cong & \\
H_{4n-1}(S^\sigma \wedge S^{2n-1})_{hO(2n)} & \longrightarrow & H_{2n}\left((S^\sigma \wedge H_{S^{2n-1}})_{hO(2n)}\right) & \cong & H_{0}(O(2n); H_{2n}(S^\sigma \wedge F_{S^{2n-1}})) & \\
\cong & & \cong & & \cong & \\
H_{2n}((S^\sigma)_{hO(2n)}) & \longrightarrow & H_{0}(O(2n); H_{n-1}(F_{S^{2n-1}}) \otimes \mathbf{Q}^t) & \\
\cong & & \cong & & \cong & \\
H_{2n-1}(O(2n); \mathbf{Q}^t) & \longrightarrow & H_{0}(O(2n); H_{2n-1}(F_{S^{2n-1}}) \otimes \mathbf{Q}^t)
\end{array}
\]

The vertical identifications are all given by the identifications of \( Z \) and suspension isomorphisms. The vertical identification on the bottom left is given by the fact that \( S^\sigma \) is the suspension of the \( \mathbb{Z}/2 \)-space given by two points that are swapped by the sign representation (or by the homotopy orbit spectral sequence).

We now examine the bottom map on the chain level. On the chain level, this is a map between \( C_*(O(2n)) \otimes Q_{(O(2n))} \mathbf{Q}^t \) and \( C_*(F_{S^{2n-1}}) \otimes Q_{(O(2n))} \mathbf{Q}^t \). To produce maps between these it suffices to produce maps \( C_*(O(2n)) \longrightarrow C_*(F_{S^{2n-1}}) \) up to degree \( 2n-1 \). The latter chain complex is \((2n-1)\)-acyclic. Thus, one can successively lift any map \( C_0(O(2n)) \longrightarrow C_0(F_{S^{2n}}) \) to a chain map \( C_i(O(2n)) \longrightarrow C_i(F_{S^{2n}}) \) for any \( i \leq 2n-1 \) and any two such maps are homotopic. For a choice of line \( L \) one such map is visibly given on generators by

\[
(g_0, \ldots, g_{2n-1}) \mapsto g_0L + g_0g_1L + \cdots + g_0 \cdots g_{2n-1}L.
\]

where the right hand side represents a flag.

Homotopically, this amounts to the following. In map \( S^{2n-1} \longrightarrow \Sigma^{2n-1}F_{S^{2n-1}} \), \( O(2n) \) acts trivially on the source \( S^{2n-1} \) and and in the target it acts trivially on \( \Sigma^{2n-1} \) and on \( F_{S^{2n-1}} \) by permuting the indexing set. In other words, \( O(2n) \) does not act on the constituent spheres in \( F_{S^{2n-1}} \), but just on the indexing set.

So, the map on homology is essentially a map on indexing sets.

We end this section by noting that the following result of Cathelineau can be easily recovered.

**Theorem 7.7.** [Cat03, Thm. 10.1.1, 10.2.1] The homology of the Dehn complex in low degrees is computed as

\[
\begin{align*}
H_0(P^{S}(n)) & \cong H_n(O(2n); \mathbf{Q}^t) \\
H_1(P^{S}(n)) & \cong H_{n+1}(O(2n); \mathbf{Q}^t)
\end{align*}
\]

**Proof.** There are no other terms on the relevant diagonals in the spectral sequence.

**APPENDIX A. TECHNICAL MISCELLANY**

**A.1. Reduced joins.** In this section we restate the definition of a reduced join and prove several important properties.
Definition A.1. We define $X \ast Y$ to be the simplicial set with
\[
(X \ast Y)_n = \bigwedge_{i=0}^{n-1} X_i \wedge Y_{n-i-1}.
\]
On a simplex $(x, y) \in X_i \wedge Y_{n-i-1}$, the map $d_j$ is defined to be $d_j \colon X_i \wedge Y_{n-i-1} \to X_{i-1} \wedge Y_{n-i-1}$ if $j \leq i$ and $1 \wedge d_{j-i-1} : X_i \wedge Y_{n-i-1} \to X_i \wedge Y_{n-i-2}$ if $j \geq i + 1$. The degeneracies are defined similarly.

Lemma A.2. Reduced joins distribute over wedge products.

Proof. We have
\[
\left( \bigvee_{\alpha \in A} (X_\alpha \ast Y) \right)_n = \bigvee_{\alpha \in A} \left( \bigvee_{i+j=n-1} (X_\alpha)_i \wedge Y_j = \bigwedge_{i+j=n-1} \left( \bigvee_{\alpha \in A} X_\alpha \right)_i \wedge Y_j = \left( \bigwedge_{\alpha \in A} X_\alpha \right) Y \right)_n.
\]

Since each step of this expression commutes with simplicial maps, the two are isomorphic as simplicial sets. \qed

Lemma A.3. Let $f \colon X \to Y$ be a quotient of simplicial sets. Then the map $f \ast 1 \colon X \ast Z \to Y \ast Z$ is also a quotient of simplicial sets. $(f \ast 1)^{-1}() = f^{-1}(* \ast Z).

Proof. It suffices to show that every nonbasepoint simplex in the codomain has a unique preimage in the domain. Consider a non-basepoint $n$-simplex $x \in Y \ast Z$; this is a pair of the form $(y_i, z_j)$ with $y_i \in Y_i$, $z_j \in Z_j$ and $i + j = n - 1$. As $y_i \in Y_i$ is non-basepoint, it has a unique preimage $x_i \in X_i$. As the given map takes $(x, z)$ to $(f(x), z)$ the preimage of $(y_i, z_j)$ is exactly $(f^{-1}(y_i), z_j)$, which is unique.

The simplices that map to the basepoint are exactly those that $f$ maps to the basepoint, with anything in the $Z$-coordinate. \qed

We end by giving a map relating the smash product and the reduced join.

Lemma A.4. Let $X$ and $Y$ be pointed simplicial sets. The map $f : S^1 \wedge X \wedge Y \to X \ast Y$ given by sending $(i, x, y) \in (S^1 \wedge X \wedge Y)_n$ to $(d_i^{n-i+1} x, d_0^{n+1} y)$ is a simplicial weak equivalence.

Proof. The fact that $f$ is well-defined is direct from the definition. We define $X \ast_w Y$ to be the double mapping cylinder of the diagram
\[
X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y.
\]

We can thus think of $X \ast_w Y$ as the quotient of $I \times X \wedge Y$ given by the mapping cylinder relations $(x, 0, y) \sim (x', 0, y)$ and $(x, 1, y) \sim (x, 1, y')$ for all $x, x' \in X$ and $y, y' \in Y$. Consider the following commutative square:

\[
\begin{array}{ccc}
X \ast_w Y & \xrightarrow{g} & S^1 \wedge X \wedge Y \\
| & f' & | \\
X \ast Y & \xrightarrow{g'} & X \ast Y
\end{array}
\]

The maps $g$ and $g'$ are both weak equivalences because they are quotients by contractible subspaces. The map $f'$ is a weak equivalence by [FG04, Corollary 3.4]. Thus, by 2-of-3, $f$ is a weak equivalence, as desired. \qed

A.2. Homotopy coinvariants. All of the results in this section are well-known to experts, although we could not find references for them for the specific cases we were interested in.

We begin by working out a notion of homotopy orbits and the homotopy orbit spectral sequence for pointed spaces. We are working in the category of pointed topological spaces. For a pointed space $X$ we can then define
\[
X_{hG} \overset{\text{def}}{=} (EG_+ \wedge X)/G.
\]

Note, in particular, that $*_hG \cong *$ and $S^0_{hG} \cong BG_+$. We would like to thank Cary Malkiewich for this proof.

\[2\text{Nondegenerately basepointed compactly generated Hausdorff spaces.}\]
Proposition A.5. Let $G$ be a discrete group acting on a nondegenerately pointed space $X$ with good $G$-action around the basepoint. There is a spectral sequence

$$H_p(G, \bar{H}_q(X)) \rightarrow \bar{H}_{p+q}(X_{hG}).$$

The proposition holds for all simplicial sets with $G$-action, which is the case of concern in this paper.

Proof. Consider $X$ as an unpointed space; write this space $\overline{X}$. Note that $(\overline{X}_{hG}, BG)$ is a good pair, with $X_{hG} \simeq \overline{X}_{hG}/BG$. We then have two spectral sequences

$$H_p(G, H_q(X)) \rightarrow H_{p+q}(X_{hG})$$

and

$$H_p(G, H_q(\overline{X}_{hG})) \rightarrow H_{p+q}(BG).$$

The second is a retract of the first; if we take the other summand, we get a spectral sequence

$$H_p(G, H_q(X)) \rightarrow \bar{H}_{p+q}(X_{hG}),$$

as desired. \hfill \Box

There is a particular model of homotopy coinvariants that is particularly useful for us. Suppose that $X$ is a pointed simplicial set with a $G$-action. Let $C^X_n$ be the category whose objects are the $n$-simplices of $X$, and whose morphisms are given by the action of $G$ (away from the basepoint). Then $C^X_n$ is a simplicial category.

Proposition A.6. $X_{hG} \simeq |C^X_\ast|$. 

Proof. This follows directly from the definition of $X_{hG}$ and the model of $C^X_n$ as the diagonal of a bisimplicial set which is $X_{hG}$ in one direction and $EG$ in the other. \hfill \Box

Proposition A.7. Let $G$ be a group acting on a pointed simplicial set $X$. Suppose that $Y_\ast$ is a subspace of $X$, such that the following two conditions hold:

1. If $g \in G$ is such that there exists a (non-basepoint) simplex $y \in Y$, such that $g \cdot y \in Y$, then for all $y' \in Y$, $g \cdot y' \in Y$.
2. For all $n$ and for all $x \in X_n$ there exists $g \in G$ such that $g \cdot x \in Y_n$.

Let $H$ be the subgroup of $G$ that takes $Y_\ast$ to $Y_\ast$. Then

$$(X_\ast)_{hG} \simeq (Y_\ast)_{hH}.$$ 

Proof. Let $Z_\ast$ be the bisimplicial set whose $(n, m)$-simplex consist of diagrams

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} x_m,$$

where the $x_i \in X_n$ for $i = 0, \ldots, m$ and $g_i \cdot x_{i-1} = x_i$. Then $|Z_\ast| \simeq (X_\ast)_{hG}$. In addition, if we let $W_\ast$ be the sub-bisimplicial set containing those diagrams where the $x_i \in Y$ and the $g_i \in H$ then $|W_\ast| \simeq Y_{hH}$. Thus it suffices to check that the inclusion $W_\ast \rightarrow Z_\ast$ induces an equivalence on geometric realization. To prove this, it suffices (by [GJ99, Proposition IV.1.9]) to show that for all $n$, $W_n \rightarrow Z_n$ is a weak equivalence of simplicial sets.

$Z_n$ (resp. $W_n$) is the nerve of the objects whose nerve are $X_n$ (resp. $Y_n$) and whose morphisms are induced by the action of $G$ (resp. $H$); call these categories $\mathcal{C}$ and $\mathcal{D}$. $\mathcal{C}$ is clearly a subcategory of $\mathcal{C}$; thus to show that the map induces an equivalence on nerves it suffices to check that the inclusion is full and essentially surjective. That it is full follows from condition (1), since since if we are given $y, y' \in Y_n$ then any $g$ such that $g \cdot y = y'$ is in $H$. That it is essentially surjective follows from condition (2), since every element of $X_n$ is isomorphic via the action of $G$ to an element of $Y_n$. \hfill \Box

A.3. The spectral sequence for the total homotopy cofiber of a cube.

The technical result that we need in order to understand the Dehn cube is the spectral sequence for the total homotopy cofiber of a cube. As the usual spectral sequence is stated only for ordinary, rather than reduced, homology, we state our analog here. We use the notation introduced in Section 4.

Proposition A.8. Let $F: \mathcal{I}_n \rightarrow \mathbf{Top}$ be a functor. There is a spectral sequence

$$\bigoplus_{\tilde{A} = (b, a_1, \ldots, a_{n-1})} \bar{H}_q(F(\tilde{A})) \rightarrow \bar{H}_{p+q}({\text{thocofib}} F).$$
Proof. By [MV15 Proposition 9.6.14], for a functor $G: \mathcal{I}_n \to \text{Top}$ there is a spectral sequence

$$\bigoplus_{\vec{A} = (b, a_1, \ldots, a_n - p - 1)} H_q(G(\vec{A})) \longrightarrow H_{p+q}(\text{thocofib} G).$$

Each of the spaces we have is pointed, thus the functor $C: \mathcal{I}_n \to \text{Top}$ defined by $C(\vec{A}) = \ast$ is a retract of $G$. In particular, this means that the spectral sequence given by the kernel of the induced map $G \to C$ is also a spectral sequence, which converges to $\ker(H_{p+q}(\text{thocofib} G) \to H_{p+q}(\text{thocofib} C))$. Since thocofib $C \simeq \ast$, this reduces to the desired spectral sequence. \hfill \Box

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