Abstract

The 'square root' of the interacting Dirac equation is constructed. The obtained equations lead to the Yang-Mills superfield with the appropriate equations of motion for the component fields.
1. Introduction.

The idea of taking the square root of the Dirac equation was originally proposed in Ref. [1]. It was shown that natural scene for such construction is superspace, and the new field equations were proposed. Such idea follows directly from the analogous procedure performed by Dirac on the Klein-Gordon equation [2].

The construction was re-considered in Ref. [3] and a set of new free field equations was solved. The solution turned out to be the Maxwell superfield with appropriate equations of motion for the component fields i.e. the Maxwell equations (the "photon") and the massless Dirac equation (the "photino").

In the present paper we demonstrate how to construct the square root of the interacting (with non-abelian vector gauge field) Dirac equation. We also construct the corresponding field equations and study their solutions.

2. Solution.

Let us define the covariant derivative:

$$D_\mu = \partial_\mu + ig\hat{A}_\mu,$$

and write the Dirac equation coupled to the vector gauge field $\hat{A}_\mu \equiv A^a_\mu(x)T^a$ (with the Hermitian matrices $T^a$ being the gauge group generators) as:

$$(i \gamma_\mu D_\mu - m) \Psi = 0.$$  \hspace{1cm} (2)

Using the two-component notation in the chiral representation one can rewrite this equation as:

$$-\left( i\bar{\sigma}_\mu \sigma_\alpha D_\mu \frac{m}{\bar{\sigma}_\alpha} \right) \left( \varphi_\alpha \bar{\chi}^{\dot\alpha} \right) \equiv D_{\text{int}} \left( \varphi_\alpha \bar{\chi}^{\dot\alpha} \right) = 0.$$  \hspace{1cm} (3)

The Lorenz indices are denoted here by $\mu, \nu, \lambda$ and $\rho$, the spinor indices by $\alpha$ and $\beta$.

We are now looking for the operator $S_{\text{int}}$ satisfying:

$$S_{\text{int}} S_{\text{int}}^\dagger = D_{\text{int}}.$$  \hspace{1cm} (4)

The solution is analogical to the non-interacting case when expressed in terms of the covariant spinorial derivatives:

$$D_\alpha = \left( \partial/\partial \theta^{\alpha} + i\sigma_\mu \alpha \bar{\theta}^{\dot{\alpha}} \partial_\mu \right) + ig\hat{A}_\alpha, \quad \hat{A}_\alpha \equiv A^a_\alpha T^a, \quad \bar{D}_{\dot{\alpha}} = (D_\alpha)^*,$$  \hspace{1cm} (5)
and reads:
\[
S_{\text{int}} = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\mathcal{D}^\alpha \\
\mathcal{\bar{D}}^\dot{\alpha}
\end{array} \right).
\]
Indeed, using the anticomutation relation:
\[
\{ \mathcal{D}^\alpha, \mathcal{\bar{D}}^\dot{\alpha} \} = -2i\sigma^{\mu\dot{\alpha}} \mathcal{D}_\mu
\]
we reproduce the Dirac operator \( \mathcal{D}_{\text{int}} \) with, in analogy to the free field case, an operator \( M_{\text{int}} \) appearing in place of the mass \( m \). In the present case it is defined as:
\[
M_{\text{int}} = -\frac{1}{4} (\mathcal{D}^2 + \mathcal{\bar{D}}^2).
\]
We conclude that the operator \( S_{\text{int}} \) is the square root of the interacting Dirac operator \( \mathcal{D}_{\text{int}} \) when acting within the space of fields \( \Lambda \) fulfilling the ‘mass condition’ [3]:
\[
M_{\text{int}}\Lambda = m\Lambda.
\]
On this space we can therefore consider the equation:
\[
S_{\text{int}}\Lambda = 0
\]
as a square root of the interacting Dirac equation.
Following [1] and [3] we are interested in the solutions of Eq. (10) having the form:
\[
\Lambda = \begin{pmatrix} W_\alpha \\ \bar{H}^\dot{\alpha} \end{pmatrix},
\]
Using Eq. (6) we obtain the following set of equations:
\[
S_{\text{int}} \begin{pmatrix} W_\alpha \\ \bar{H}^\dot{\alpha} \end{pmatrix} = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
\mathcal{D}^\alpha \\
\mathcal{\bar{D}}^\dot{\alpha}
\end{array} \right) \begin{pmatrix} W_\alpha \\ \bar{H}^\dot{\alpha} \end{pmatrix} = 0
\]
together with:
\[
M_{\text{int}} \begin{pmatrix} W_\alpha \\ \bar{H}^\dot{\alpha} \end{pmatrix} = m \begin{pmatrix} W_\alpha \\ \bar{H}^\dot{\alpha} \end{pmatrix}.
\]
Note that due to Eq. (4) the above superfields obey also the Dirac equation. Using the techniques analogous to those of Ref. [3] we obtain:
\[
\mathcal{D}^\beta (\mathcal{D}_\dot{\epsilon} W_\alpha) = \mathcal{\bar{D}}^\dot{\beta} (\mathcal{D}_\beta \bar{W}_\dot{\epsilon}) = \mathcal{D}_\mu (\mathcal{D}_\beta \bar{W}_\dot{\epsilon}) = 0,
\]
\[
\mathcal{D}^\beta (\mathcal{D}_\beta \bar{H}^\dot{\epsilon}) = \mathcal{\bar{D}}^\dot{\beta} (\mathcal{D}_\epsilon \bar{H}^\dot{\epsilon}) = \mathcal{D}_\mu (\mathcal{D}_\beta \bar{H}^\dot{\epsilon}) = 0.
\]
Let us look closer at the last equality in Eq. (15). Multiplying it by \( \partial_\nu \) and subtracting from \( \partial_\mu \mathcal{D}_\nu (\tilde{\mathcal{D}}_\alpha W_\alpha) \) we obtain:

\[ \hat{F}_{\mu\nu} (\tilde{\mathcal{D}}_\alpha W_\alpha) = 0, \]  

(16)

where \( \hat{F}_{\mu\nu} \) is the gauge field tensor:

\[ \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + ig \left[ \hat{A}_\mu, \hat{A}_\nu \right]. \]

(17)

The case \( \hat{F}_{\mu\nu} = 0 \) corresponds to the free field system and was solved in Ref. [3]. With the gauge interactions switched on we have thus:

\[ \tilde{\mathcal{D}}_\alpha W_\alpha = 0. \]

(18)

In analogous way we obtain:

\[ \mathcal{D}_\alpha \tilde{\mathcal{H}}^{\dot{\alpha}} = 0. \]

(19)

The mass condition can be now simplified. Multiplying the upper Eq. (13) by \( \mathcal{D}^\alpha \) we obtain:

\[ m \mathcal{D}^\alpha W_\alpha = 0. \]

(20)

Out of two possible choices let us first assume \( m \neq 0, \mathcal{D}^\alpha W_\alpha = 0 \). We use the Dirac equation to prove that:

\[ -2i\bar{\sigma}^{\mu\dot{\alpha}} \mathcal{D}_\mu W_\alpha = \left\{ \mathcal{D}^\alpha, \tilde{\mathcal{D}}^{\dot{\alpha}} \right\} W_\alpha = 2m \tilde{\mathcal{H}}^{\dot{\alpha}} = 0, \]

(21)

and thus \( \mathcal{H}^{\dot{\alpha}} = 0 \). Using the lower Eq. (13) and the second component of the Dirac equation we arrive analogically at \( W_\alpha = 0 \). Therefore the nontrivial solutions to our system require the mass to vanish:

\[ m = 0. \]

(22)

As an immediate consequence Eqs. (13) reduce to:

\[ \mathcal{D}^2 W_\alpha = \tilde{\mathcal{D}}^2 \tilde{\mathcal{H}}^{\dot{\alpha}} = 0. \]

(23)

Repeating analogous considerations which have led to Eqs. (18, 19) we obtain:

\[ \mathcal{D}^\alpha W_\alpha = \tilde{\mathcal{D}}^{\dot{\alpha}} \tilde{W}^{\dot{\alpha}} = 0, \]

\[ \tilde{\mathcal{D}}_\dot{\alpha} \mathcal{H}^{\dot{\alpha}} = \mathcal{D}^\alpha \mathcal{H}_\alpha = 0. \]

(24)
The above equations together with the equations (18), (19) define the supermultiplets $W_\alpha$ and $\bar{H}^{\dot{\alpha}}$ and their dynamic equations. The solution has the form of the Yang-Mills superfield [4][5]:

$$W_\alpha = -i\tilde{\lambda}_\alpha (y) + \left[ \delta_\alpha^\beta \hat{d} (y) - \frac{i}{2} (\sigma^\mu \tilde{\sigma}^\nu)_{\alpha}^{\beta} \hat{F}_{\mu\nu} (y) \right] \theta_\beta$$

$$+ \theta \theta^{\mu} \sigma_{\alpha\dot{\alpha}} \partial_\mu \tilde{\lambda}^{\dot{\alpha}} (y),$$

(25)

with:

$$y^\mu = x^\mu + i\theta \sigma^\mu \bar{\theta},$$

$$\tilde{\lambda}_\alpha = \lambda_a \sigma^a, \quad \hat{d} = d^a \sigma^a,$$

$$\partial_\mu \tilde{\lambda}^{\dot{\alpha}} = \partial_\mu \tilde{\lambda}^{\dot{\alpha}} - ig [\hat{A}_\mu, \tilde{\lambda}^{\dot{\alpha}}],$$

and analogous expression for the field $\bar{H}^{\dot{\alpha}}$. The fields $\hat{F}_{\mu\nu}, \tilde{\lambda}_\alpha$ and $\hat{d}$ satisfy corresponding dynamic equations:

$$\partial_\mu \hat{F}_{\mu\nu} = 0, \quad \epsilon^{\mu\nu\lambda\rho} \partial_\nu \hat{F}_{\lambda\rho} = 0,$$

$$\sigma^{\mu}_{\alpha\dot{\alpha}} \partial_\mu \tilde{\lambda}^{\dot{\alpha}} = 0, \quad \hat{d} = 0.$$

3. Conclusions.

In this paper we show how to define the square root of the Dirac equation coupled to the non-abelian gauge field. The obtained equations lead to the Yangs-Mills superfield with appropriate equations of motion for the component fields.

It may be interesting to extend the results of this paper in several directions. It is worth studying others form of solutions, proposed in Ref. [1]. This problem is currently under investigation.

Starting from the Dirac equation on the curved space-time, one may hope to obtain some form of supergravity.

It seems also interesting to extend our consideration to higher space-time dimensions.

References

[1] J. Szwed, Phys. Lett. B 181, 305, (1986).
[2] P. A. M. Dirac, Proc. R. Soc. (London) A 117, 610, (1928); A 118, 351, (1928).

[3] A. Bzdak, L. Hadasz, Phys. Lett. B 582, 113, (2004).

[4] J. Wess, J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, Princeton, 1983.

[5] S. Weinberg, *The Quantum Theory of Fields. Vol. III Supersymmetry*, Cambridge University Press, Cambridge, 2000.