In-medium effect on the thermodynamics and transport coefficients in van der Waals hadron resonance gas

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An extension of the van der Waals hadron resonance gas (VDWHRG) model which includes in-medium thermal modification of hadron masses, the TVDWHRG model, is considered in this paper. Based on the 2+1 flavor Polyakov Linear Sigma Model (PLSM) and the scaling mass rule for hadrons we obtain the temperature behavior of all hadron masses for different fixed baryon chemical potentials \( \mu_B \). We calculate various thermodynamic observables at \( \mu_B = 0 \) GeV in TVDWHRG model. An improved agreement with the lattice data by TVDWHRG model in the crossover region \( (T \sim 0.16 \sim 0.19 \text{ GeV}) \) is observed as compared to those by VDWHRG and Ideal HRG (IHRG) models. We further discuss the effects of in-medium modification of hadron masses and VDW interactions on the transport coefficients such as shear viscosity \( (\eta) \), scaled thermal \( (\lambda/T^2) \) and electrical \( (\sigma_{el}/T) \) conductivities in IHRG model at different \( \mu_B \), by utilizing quasi-particle kinetic theory with relaxation time approximation.

I. INTRODUCTION

Strongly interacting matter created in ultra-relativistic heavy-ion experiments at the Relativistic Heavy-Ion Collider (RHIC) of BNL and the Large Hadron Collider (LHC) of CERN has attracted intense theoretical and experimental investigations. The study of strongly interacting matter can give a deep understanding of Quantum Chromodynamics (QCD) phase diagram and equation of state (EOS) of hot and dense matter. Lattice QCD simulation as a reliable tool to study QCD thermodynamics have demonstrated that at finite temperature and vanishing baryon chemical potential \( \mu_B \) there exists a smooth crossover (phase transition from hadronic matter to a chirally symmetric Quark-Gluon Plasma (QGP)) ranging from 0.15 to 0.2 GeV [1,2]. Ideal Hadron Resonance Gas (IHRG) model is a widely used effective model of QCD which provides a remarkable good description of the lattice data [3,5] at low temperature \( (T < 0.15 \text{ GeV}) \) and zero \( \mu_B \). However, IHRG model fails to fit with the lattice QCD data in the crossover region \( (T = 0.16 \sim 0.19 \text{ GeV}) \). So an extended IHRG model called VDWHRG model which includes both the long distance attractive and the short distance repulsive van der Waals (VDW) type interactions between (anti)baryons is implemented [6,7]. The results of thermodynamic quantities within VDWHRG model are closer to the lattice data in crossover region than IHRG model.

In addition to the consideration that interactions between hadrons play a crucial role in crossover region, we also take into account the effect that hadrons may melt into quark and gluon constituents at high temperature and baryon chemical potential/density. As we known that spontaneous chiral symmetry breaking is an important aspect of QCD vacuum at low energy which is assumed to contribute to the masses of hadrons [8–10]. With the increase of temperature or baryon chemical potential, chiral symmetry will be restored which implies that constituent quark with nonzero mass reduces to be massless. Once the constituent quark mass is relevant to temperature and baryon chemical potential, the masses of hadrons are dependent on temperature and baryon chemical potential naturally. In the literature two main effective QCD-like models, the Polyakov Nambu-Jona-Lasinio (PNJL) [11–14] and Polyakov linear \( \sigma \) model (PLSM) [15–20] are widely used. These models are successful in explaining the dynamics of spontaneous breaking and restoration of chiral symmetry, and can also describe the thermal evolution of meson masses in hot and dense matter. So it is essential to replace vacuum hadron masses with temperature and chemical potential dependent masses to explore thermal hadron mass effect on the thermodynamic quantities and transport coefficients in hot and dense hadronic matter. The transport properties of strongly interacting matter play a significant role in describing the dynamical evolution of hot and dense matter. Among various transport coefficients, the dissipative coefficients like shear and bulk viscosities of QGP and hadronic matter which have been widely investigated by several research groups [21–24]. Recently shear and bulk viscosities of hot hadronic matter in VDWHRG model have been investigated [25,26]. Another important but less concerned transport coefficients are electrical \( (\sigma_{el}) \) and thermal \( (\lambda) \) conductivity which are also estimated by various methods in QGP [27–30], hot pion gas [31,32] and excluded volume HRG (EHRG) model [33]. However, so far most of these works did not take into account the influence of thermal hadron masses.

In this work, we use 2+1 flavor Polyakov Linear Sigma Model (PLSM) combined with the generalized mass scaling rule of hadrons to obtain the thermal behavior of hadron masses. We apply this effect to the calculation of the thermodynamic quantities in VDWHRG model and compare them with the lattice data. We also estimate the transport coefficients like shear viscosity, electrical...
and thermal conductivities of hadronic matter composed of quasi-particles with $T$ and $\mu_B$ dependent masses in VDWHRG model. Our calculations of transport coefficients are based on Boltzmann equation in relaxation time approximation.

The arrangement of this paper is as follows. In Sec. II we review the ideal and interacting HRG models. In Sec. III we discuss the analytical expressions for the medium modifications of hadron masses at finite temperature and baryon chemical potential. In Sec. IV we present the formulas of the transport coefficients in the quasi-particle kinetic theory under relaxation time approximation. In Sec. V we show our numerical results, and Sec. VI we summarize our studies.

II. HADRON RESONANCE GAS

A. Ideal hadron resonance gas model

All thermodynamic quantities in IHRG model can be obtained from the sum of the logarithm of grand canonical partition function over all hadrons and resonances [35]

$$\ln Z^{id} = \sum_i \ln Z_i^{id}(T, \mu_i, m_i).$$

(1)

For particle species $i$,

$$\ln Z_i^{id} = \pm \frac{V g_i}{(2\pi)^3} \int d^3p \ln \left[ 1 \pm \exp\left(-E_i - \mu_i/T\right) \right],$$

(2)

where $id$ refer to the ideal (non-interacting) gas and $V$ is the volume of system. $g_i$ stands for the degeneracy factor which satisfies the relation $g_i = (2J_i + 1)$. $J_i$ is angular momentum of hadron $i$. The sign $\pm$ is positive for fermions and negative for bosons. $E_i = \sqrt{p^2 + m_i^2}$ denotes energy of the single particle. $m_i$ presents mass of hadron $i$ which is usually taken as the vacuum mass. However in this paper we also consider the effect of finite temperature or chemical potential on masses of hadrons. $\mu_i = \bar{B}_i \mu_B + S_i \mu_S + Q_i \mu_Q$ is the chemical potential, $B_i, S_i, Q_i$ are the baryon charge, strangeness charge and electric charge of particle species $i$ respectively, $\mu_B/\mu_S/\mu_Q$ give the corresponding chemical potentials. We assume $\mu_S = \mu_Q = 0$ which is a reasonable approximation in heavy-ion collision experiments [39]. The thermodynamic quantities (pressure, energy density and number density) in IHRG model can be given by [26]

$$P^{id} = T \frac{\partial \ln Z^{id}}{\partial V} = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_i} f_i^{id}(T, \mu_i),$$

(3)

$$\epsilon^{id} = \frac{1}{V} \left( \frac{\partial \ln Z^{id}}{\partial T} \right) = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} E_i f_i^{id}(T, \mu_i),$$

(4)

$$n^{id} = \frac{T}{V} \left( \frac{\partial \ln Z^{id}}{\partial \mu_i} \right)_{V,T} = \sum_i g_i \int \frac{d^3p}{(2\pi)^3} f_i^{id}(T, \mu_i),$$

(5)

where $f_i^{id}$ is ideal Fermi or Bose distribution function $f_i^{id} = 1/\left(\exp\left[(E_i - \mu_i)/T\right] \pm 1\right)$.

B. Interacting hadron resonance gas

In this work, we consider more realistic system of hadrons which includes both the short-distance repulsive and the long-distance attractive interactions between hadrons. There are different phenomenological excluded-volume models to simulate the repulsive interaction of hadrons such as van der Waals [37] and Carnahan-Starling excluded-volume models [38] with quantum statistics. For the attractive interaction, four various forms have been discussed [6] [39–41]; van der Waals, Redlich-Kwong-Soave, Peng-Robinson and Clausius models. Therefore to take into account both the repulsive and attractive interactions different interacting hadron gas models could be employed: the VDW, RKS, PR, Clausius, VDW-CS, RKS-CS, PR-CS and Clausius-CS models. In interacting hadron resonance gas model, interactions only exist between baryon-baryon pairs and antibaryon-antibaryon pairs while the baryon-antibaryon, meson-baryon and meson-meson interactions are neglected [6]. So the total pressure in grand canonical ensemble can be written as following [7]

$$P(T, \mu) = P_M(T, \mu) + P_B(T, \mu) + P_{\bar{B}}(T, \mu),$$

(6)

with

$$P_M(T, \mu) = \sum_{z \in M} P_z^{id}(T, \mu_z),$$

(7)

$$P_B(T, \mu) = [F(h_B) - h_B P'(h_B)] \sum_{z \in B} P_z^{id}(T, \mu_z^{B*}) + n_B^2 u'(n_B),$$

(8)

$$P_{\bar{B}}(T, \mu) = [F(h_{\bar{B}}) - h_{\bar{B}} P'(h_{\bar{B}})] \sum_{z \in B} P_z^{id}(T, \mu_z^{B*}) + n_{\bar{B}}^2 u'(n_{\bar{B}}),$$

(9)

where $\mu$ can be baryon potential, strangeness potential or electric charge potential. The subscripts $M, B, \bar{B}$ stand for mesons, baryons and antibaryons respectively. The constructed functions $F(h_{\bar{B}}(\bar{B}))$ and $u(n_{\bar{B}}(\bar{B}))$ are related to the repulsive and attractive interactions of (anti)baryon pairs respectively. The analytical forms of $F(h_{\bar{B}}(\bar{B}))$ and $u(n_{\bar{B}}(\bar{B}))$ are different according to the choice of real gas models listed previously. $n_{\bar{B}}(\bar{B})$ and $h_{\bar{B}}(\bar{B})$ are respectively the total density of (anti)baryon and packing ratio of all (anti)baryonic volume occupied in total system volume which satisfies a relation $h_{\bar{B}}(\bar{B}) = \frac{\pi}{2} n_{\bar{B}}(\bar{B})$. For the total number density of baryons $n_B$ which can be obtained by using $n_B = \partial/P_B/\partial \mu_B$:

$$n_B(T, \mu) = F(h_B) \sum_{z \in B} n_z^{id}(T, \mu_z^{B*}).$$

(10)
And the shifted chemical potential of baryon $\mu_z^{B*}$ is given by

$$\mu_z^{B*} - \mu_z = \frac{b}{4} F'(h_B) \sum_{z \in B} P_{z}^{id}(T, \mu_z^{B*})$$
$$- u(n_B) - n_B u'(n_B).$$  \hspace{1cm} (11)

The key is to obtain $\mu_z^{B*}$. At given $T$ and $\mu$, $\mu_z^{B*}$ can be calculated by solving Eqs. (10-11) numerically. Accordingly, other thermodynamic quantities such as the entropy density $s_B = (\partial P_B / \partial T)_\mu$ and the energy density can be determined.

$$s_B(T, \mu) = F(h_B) \sum_{z \in B} s_{z}^{id}(T, \mu_z^{B*}),$$  \hspace{1cm} (12)

$$\epsilon_B(T, \mu) = F(h_B) \sum_{z \in B} \epsilon_{z}^{id}(T, \mu_z^{B*}) + n_B u(n_B).$$  \hspace{1cm} (13)

Eqs. (10-13) are also applicable to antibaryons. In this work, we use VDW model: $F(h_B(B)) = 1 - 4h_B(B)$ and $u(n_B(B)) = - a n_B(B)$. The parameters $a$ and $b$ are determined by reproducing the properties of nuclear matter in its ground state. \hspace{1cm} (12) The values of $a$ and $b$ are different according to the choice of real gas models. \hspace{1cm} (13)

III. MASS SENSITIVITY OF HADRONS AT FINITE TEMPERATURE AND BARYON CHEMICAL POTENTIAL

In present work, we obtain temperature and baryon chemical potential dependent masses of the pseudo scalar ($\pi, \eta, \eta'$, $K$) and scalar ($\sigma, a_0, f_0, \kappa$) mesons in the framework of $2+1$ flavor Polyakov Linear Sigma Model (PLSM). The scalar and pseudo scalar meson masses are defined by the curvature of the temperature and quark chemical potential dependent thermodynamic grand potential $\Omega(T, \mu_{fl}(fl = u, d, s))$ with respect to corresponding scalar field $\alpha_{x, x} = \sigma_x$ and pseudo scalar field $\alpha_{p, x} = \pi_x (x, y = 0, \ldots, 8)$ which can be expressed as

$$m_i^{2}(T, \mu_{fl}) = \frac{\partial^2 \Omega(T, \mu_{fl})}{\partial \alpha_{i,x} \partial \alpha_{i,y}}\bigg|_{\min} ; i = s, p,$$  \hspace{1cm} (14)

where $\min$ denotes to minimize the above expression and $i = s(p)$ corresponds to the scalar (pseudo scalar) mesons. The detailed description of PLSM can be find in Refs. \hspace{1cm} (16-20). The values of all involved parameters and the choice of forms of Polyakov loop potential in our calculation are taken from Ref. \hspace{1cm} (15). For masses of all baryons and other mesons, we use the generalized scaling rule of hadron masses in Refs. \hspace{1cm} (44-46) which can be expressed as

$$M_{B/M}(T, \mu_B) = M_{B/M}(0, 0) + (N_q - N_s) \delta M_q(T, \mu_B) + N_s \delta M_s(T, \mu_B),$$  \hspace{1cm} (15)

where the subscript $B/M$ stands for baryon/meson, $M_{q/s}$ is the light/strange constituent quark mass. $\delta M_{q/s}$ denotes the variation of the constituent quark mass due to temperature and baryon chemical potential. $N_q/s$ is the number of light/strange quark in a given hadron. Fig. 1 shows the normalized masses of light constituent quark $M_{u,d} / M_{u,d}^0 (a)$ and strange constituent quark $M_s / M_s^0 (b)$ as a function of temperature $T$ for different values of baryon chemical potential $\mu_B$ in PLSM. The temperature behavior of the normalized constituent quark mass shows a smooth crossover (chiral transition). The starting temperature at which the light/strange constituent quark masses begin to melt is about $T \sim 160/180$ MeV (not chiral pseudo-critical temperature) at $\mu_B = 0$ GeV. And as the increase of $\mu_B$, $M_{u,d} / M_{u,d}^0$ decreases earlier.

Fig. 2 (a-h) shows the thermal evolution of the pseudo scalar mesons ($\pi, K, \eta', \eta$) and scalar mesons ($a_0, \kappa, \sigma, f_0$) calculated in the PLSM. The masses of these states degenerate at about $T \sim 160$ MeV for $\mu_B = 0$ and 0.1 GeV cases. For $\mu_B = 0.2$ and 0.3 GeV, these states degenerate at $T \sim 130$ MeV and $T < 100$ MeV respectively. The melting behavior of hadrons can affect the thermodynamic quantities and transport coefficients of hadronic matter, which can been seen later in present work.
IV. TRANSPORT COEFFICIENTS

Transport coefficients in the medium composed of quasi-particles whose masses depend on temperature and chemical potential can be derived by utilizing the relativistic kinetic theory under relaxation time approximation \[21, 22, 47\]. The general expressions of shear viscosity \(\eta\), thermal conductivity \(\lambda\) and electrical conductivity \(\sigma_{el}\) can be written as \[21, 47\]

\[
\eta = \frac{1}{15T} \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^4}{E_i^2} \tau_i f_i(1 \pm f_i^d),
\]

\[
\lambda = \left(\frac{w}{n_B T}\right)^2 \sum_i g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_i^2} \tau_i \left( B_i - \frac{n_B E_i}{w}\right)^2 \times f_i^d(1 \pm f_i^d),
\]

\[
\sigma_{el} = \frac{1}{3T} \sum_i g_i e_i^2 \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{E_i^2} \tau_i f_i(1 \pm f_i^d).
\]

Here, \(e_i\) and \(B_i\) are the electric charge and baryon charge of hadron species \(i\) respectively, \(w\) is enthalpy density. The sign \(\pm\) corresponds to bosons and fermions respectively. \(\tau_i\) is the average relaxation time of hadron species \(i\). We assume only elastic scattering between hadrons, so \(\tau_i^{-1}\) for the process of \(i(p_1) + j(p_2) \rightarrow i(p_3) + j(p_4)\) can be given by \[43\]

\[
\tau_i^{-1} = \sum_j \frac{g_j}{1 + \delta_{ij}} \int \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1),
\]

where \(d^3p_k/(2\pi)^3/(2E_k)\), \(p_{tot} = p_1 + p_2 - p_3 - p_4.\) The factor \(1/(1 + \delta_{ij})\) is to avoid double counting for identical incoming particles. In Eq. (19), the average of the initial degeneracy factor and the sum of final degeneracy factor are implicitly included in the matrix element (\(\hat{M}\)). Using the formula of scattering cross section \[49\]

\[
\sigma_{ij} = \frac{\int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2}{4 \sqrt{(p_1 \cdot p_2)^2 - m_i^2 m_j^2}},
\]

then we can rewrite \(\tau_i^{-1}\) and take thermal averaging

\[
\tau_i^{-1} \equiv \sum_j \frac{n_j}{1 + \delta_{ij}} \langle \sigma_{ij} v_{ij} \rangle,
\]

where \(n_j = g_j \int d^3p_2/(2\pi)^3 f_j(p_2)\) is the number density of particle \(j\). It is important to note that if particle \(j\) is a baryon/antibaryon the detailed form of the number density can be modified in interacting hadron resonance gas

\[
n_j(T, \mu_j) = \begin{cases} n_j^d(T, \mu_j), & \text{in ideal HRG}; \\ n_j^d(T, \mu_j^{BB}), & \text{in interacting HRG}. \end{cases}
\]

The Lorentz scalar flow factor is defined as

\[
v_{ij} = \sqrt{(p_1 \cdot p_2)^2 - m_i^2 m_j^2} \frac{E_1 E_2}{E_0}. \tag{23}
\]

Therefore the thermal average cross section can be written in the following form under Maxwell-Boltzmann distribution approximation after some uncomplicated simplification

\[
\langle \sigma_{ab} v_{ij} \rangle = \frac{\int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1) f_i^d(p_2)}{\int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) f_i^d(p_2)}
\]

\[
= \frac{\int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1) f_i^d(p_2) \beta e^{-\beta E_i}}{\beta \int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1) f_i^d(p_2) \beta e^{-\beta E_i}}
\]

\[
= \frac{\beta \int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1) f_i^d(p_2) \beta e^{-\beta E_i}}{\beta \int \prod_{k=2}^4 \frac{d^3p_k}{2E_k} (2\pi)^4 \delta^4(p_{tot}) |\hat{M}|^2 f_j(p_1) f_i^d(p_2) \beta e^{-\beta E_i}}. \tag{24}
\]

where \(\sqrt{S}\) is center-of-mass energy, \(S_0 = (m_i + m_j)^2\), \(\gamma(S) = [S - (m_i + m_j)^2]/[S - (m_i - m_j)^2]\) and \(\beta = 1/T\). \(K_\nu\) is the modified Bessel function of order \(\nu\). We regard all hadrons as hard spheres which have the same radius \(r_n\) as nucleons, so \(\sigma\) is a constant with \(\sigma = 4\pi r_n^2\).

V. RESULTS AND DISCUSSIONS

In the following, the extension of VDWHRG model which includes thermal evolution of hadron masses is considered. We refer to this model as Thermal VDWHRG (TVDWHRG) model. In the treatment of HRG model we include all vacuum masses of hadrons and resonances (the zero width approximation is applied) up to 2.0 GeV which are listed in the Particle Data Group Book of 2014 \[50\]. The parameters of the VDW model yield \(a \approx 239\) MeV fm\(^3\) and \(b \approx 3.42\) fm\(^3\) \(= \frac{4\pi r_n^3}{3}\) \((r_n\) is radius of nucleons\) from the fit to the properties of nuclear matter at zero temperature \[43\]. The temperature dependences of the scaled pressure \(P/T^4\), the energy density \(\epsilon/T^4\), the entropy density \(s/T^3\) and speed of sound squared \(c_s^2 = dP/d\epsilon\) at \(\mu_B = 0\) GeV within IHRG, VDWHRG and TVDWHRG models are depicted in Fig. 5. It is noted that from Fig. 3(a-c), comparing with the results in IHRG model, the pressure, the energy density and the entropy density within VDWHRG and TVDWHRG models have a modest suppression due to the interactions of baryon-baryon pairs and antibaryon-antibaryon pairs are stronger at \(T > 0.16\) GeV. The pressure \(P/T^4\), energy density \(\epsilon/T^4\) and entropy density \(s/T^3\) within TVDWHRG model have a better agreement with the lattice data of the Wuppertal-Budapest \[8\] and the Hot QCD collaborations \[14\] up to \(T = 0.19\) GeV. The quantitative difference of results between VDWHRG model and TVDWHRG model at the case of \(\mu_B = 0\) GeV and \(T > 0.16\) GeV mainly comes from the decrease of mass of hadrons which leads to an increase of \(\exp[-m(T)/T]\). Fig. 5(d) shows the speed of sound squared, \(c_s^2\), which is consistent with the lattice
data in TVWDHRG model at $T = 0.165 \sim 0.18$ GeV while $c_T^2$ within all considered HRG models gives a bad fit to the lattice data of Wuppertal-Budapest collaboration at $T = 0.135 \sim 0.155$ GeV. In addition, there are no significant difference in the pressure, the energy density and the entropy density by all considered models at $T < 0.165$ GeV. It can be explained in two aspects. (i) The interactions of baryon pairs and antibaryon pairs are relatively weak at $\mu_B = 0$ GeV where the contribution of mesons is dominant compared with that of (anti)baryons \[24.\] (ii) At $T < 0.165$ GeV, the masses of hadrons are almost not affected by temperature which can be seen from Fig. [1] and Fig. [2].

![Graphs showing pressure as a function of temperature for different chemical potentials](image)

Fig. [2] shows the pressure as a function of temperature for $\mu_B = 0.1(a), 0.2(b)$ and $0.3(c)$ GeV. As can be noted from Fig. [2](a-c), the pressure is underestimated by all models at $T < 0.16$ GeV. For the cases of $\mu_B = 0.1$ and $0.2$ GeV, the pressure within TVDWHRG model fits well with the lattice data at $T = 0.16 \sim 0.19$ GeV than within VDWHRG model or IHRG model which can be seen in Fig. [2](a) and (b). However at $\mu_B = 0.3$ GeV, the pressure fails to simulate the result of lattice data within all considered HRG models. There are two possible reasons for the failure. (i) The parameters of van der Waals model may vary with the choice of baryon chemical potential \[20.\] (ii) It is a challenging task for lattice QCD simulation due to said sigh problem at nonzero $\mu_B$. The existing lattice data is only estimated up to $\mu^2$ \[4.\] Therefore, in the case of nonzero chemical potential, we do not pay more attention to comparing our results with the lattice data and we just qualitatively explore the effects of thermal hadron masses and VDW interactions on the thermodynamic quantities and transport coefficients of hot hadronic matter.

Here we refer to hadron resonance gas model which only considers the effect of thermal hadron masses as Thermal HRG (THRG) model. The temperature dependence of shear viscosity $\eta$ at $\mu_B = 0.1, 0.2, 0.3, 0.35$ GeV for all considered HRG models is depicted in Fig. [5]. As we can be seen from Fig. [5](a), $\eta$ increases monotonically as $T$ increases and as $\mu_B$ increases in IHRG and THRG models. The magnitude of $\eta$ in THRG model considering the effect of thermal hadron masses is weakly enhanced at $\mu_B = 0.1$ GeV compared to IHRG model case, which is similar to the result of Ref. \[26.\] and the enhancement in $\eta$ is more obvious with the increase of $\mu_B$. When the VDW interactions are taken into account in the calculation of $\eta$ (as in Fig. [5](b)), $\eta$ increases obviously at higher
FIG. 3. (Color online) The temperature dependences of different thermodynamics within IHRG model (black dashed lines), VDWHRG model (blue dashed lines) and TVDWHRG model (red solid lines) at $\mu_B = 0$. The lattice QCD results are taken from the Wuppertal-Budapest [3] (red solid circle symbol with error bar) and the HotQCD collaborations [4] (blue uptriangle symbol with error bar).

FIG. 4. (Color online) The temperature dependence of the scaled pressure ($P/T^4$) within IHRG model (dashed black lines), VDWHRG model (wide dashed purple lines) and TVDWHRG model (solid red lines) at $\mu_B = 0.1(a)$, 0.2(b) and 0.3 GeV (c). The lattice QCD results (red symbol with error bar) are taken from Ref. [4].
FIG. 5. (Color online) Left panel (a) shows the temperature dependence of shear viscosity within IHRG (lines) and THRG (symbols) model for \( \mu_B = 0.1 \) (black solid line and square symbol), 0.2 (red dotted line and circular symbol), 0.3 (blue dashed line and uptriangle symbol) and 0.35 (green dashed-dotted line and downtriangle symbol) GeV. Right panel (b) shows the temperature dependence of \( \eta \) within VDWHRG (lines) and TVDWHRG (symbols) models for \( \mu_B = 0.1, 0.2, 0.3 \) and 0.35 GeV.

FIG. 6. (Color online) Same as Fig. 5 for scaled thermal conductivity \( \lambda/T^2 \).

FIG. 7. (Color online) Same as Fig. 5 for scaled electrical conductivity \( \sigma_{el}/T \).
The temperature dependence of scaled thermal conductivity $\lambda/T^2$ for fixed baryon chemical potentials $\mu_B = 0.1$, 0.2, 0.3, 0.35 GeV with all considered HRG models is plotted in Fig. 6. As can be seen from Fig. 6(a) without the effect of VDW interactions of hadrons, $\lambda/T^2$ ratio decreases as $T$ increases and decreases as $\mu_B$ increases in IHRG and THRG models. Taking into account of medium effect can lead a certain amount of suppression in $\lambda/T^2$ compared to IHRG model case especially at $\mu_B > 0.2$ GeV.

In this work we investigate the thermodynamics and transport coefficients with TVDWHRG model, which is the extension of VDWHRG model by including the effect of temperature $T$ and baryon chemical potential $\mu_B$ dependent hadron masses. In TVDWHRG model thermal hadron masses can obtained by $2+1$ flavor Polyakov linear $\sigma$ model combined with the scaling rule of hadron masses. We estimate the thermodynamics like the pressure, energy density, entropy density and speed of sound square in TVDWHRG model and compare them with the lattice data. It has been shown that the thermodynamics in TVDWHRG model give a better agreement with the available lattice data than in VDWHRG model at $\mu_B = 0$ GeV. We also investigate the transport coefficients like shear viscosity ($\eta$), scaled electrical ($\sigma_{el}/T$) conductivity and scaled thermal conductivity ($\lambda/T^2$) of hadronic matter in all considered HRG models using the quasi-particle kinetic theory under relaxation time approximation up to $T = 0.185$ GeV. Taking into account the effects of VDW interactions and thermal hadron masses, the transport coefficients may be modified considerably. When we only consider the effect of $T$ and $\mu_B$ dependent hadron masses on transport coefficients, the magnitude of $\sigma_{el}/T$ for fixed $\mu_B$ is relatively suppressed in THRG model while $\eta$ is enhanced in THRG model compared to IHRG model case. $\lambda/T^2$ have a significant suppression in THRG model compared to IHRG model case for $\mu_B > 0.2$ GeV. The suppression or enhancement of transport coefficients due to thermal mass effect is more pronounced with the increase of $\mu_B$ because the variation of thermal hadron masses is more sensitive with the increase of $\mu_B$. The general behavior of transport coefficients in THRG and IHRG models is similar. However when we consider the VDW interactions into the calculation of $\eta$, $\lambda/T^2$ and $\sigma_{el}/T$ within TVDWHRG and VDWHRG models, which leads to a significant enhancement in transport coefficients at higher $T$ ($T > 0.16$ GeV) especially for $\mu_B = 0.3$ and 0.35 GeV. The VDW interactions are strengthened with the increase of $\mu_B$ which can lead to the curves of $\sigma_{el}/T$ for different $\mu_B$ cross with each other. The effect of VDW interactions is enhanced further if we consider both the effect of VDW interactions between hadrons (as in Fig. 7(b)) and thermal hadron masses which even leads to the curves for different $\mu_B$ cross with each other and $\sigma_{el}/T$ increases with the increase of $\mu_B$ at higher $T$ ($T > 0.16$ GeV). In addition, we observe that the effect of VDW interactions on $\sigma_{el}/T$ is improved by the inclusion of medium effect for $\mu_B = 0.3$ and 0.35 GeV at higher $T$ ($T > 0.16$ GeV). Without the consideration of thermal hadron masses, all transport coefficients in our calculation are not very sensitive to the effect of VDW interactions at lower temperature which is similar to the results of thermodynamic quantities.

VI. CONCLUSION

In this work we investigate the thermodynamics and transport coefficients with TVDWHRG model, which is the extension of VDWHRG model by including the effect of temperature $T$ and baryon chemical potential $\mu_B$ dependent hadron masses. In TVDWHRG model thermal hadron masses can obtained by $2+1$ flavor Polyakov linear $\sigma$ model combined with the scaling rule of hadron masses. We estimate the thermodynamics like the pressure, energy density, entropy density and speed of sound square in TVDWHRG model and compare them with the lattice data. It has been shown that the thermodynamics in TVDWHRG model give a better agreement with the available lattice data than in VDWHRG model at $\mu_B = 0$ GeV. We also investigate the transport coefficients like shear viscosity ($\eta$), scaled electrical ($\sigma_{el}/T$) conductivity and scaled thermal conductivity ($\lambda/T^2$) of hadronic matter in all considered HRG models using the quasi-particle kinetic theory under relaxation time approximation up to $T = 0.185$ GeV. Taking into account the effects of VDW interactions and thermal hadron masses, the transport coefficients may be modified considerably. When we only consider the effect of $T$ and $\mu_B$ dependent hadron masses on transport coefficients, the magnitude of $\sigma_{el}/T$ for fixed $\mu_B$ is relatively suppressed in THRG model while $\eta$ is enhanced in THRG model compared to IHRG model case. $\lambda/T^2$ have a significant suppression in THRG model compared to IHRG model case for $\mu_B > 0.2$ GeV. The suppression or enhancement of transport coefficients due to thermal mass effect is more pronounced with the increase of $\mu_B$ because the variation of thermal hadron masses is more sensitive with the increase of $\mu_B$. The general behavior of transport coefficients in THRG and IHRG models is similar. However when we consider the VDW interactions into the calculation of $\eta$, $\lambda/T^2$ and $\sigma_{el}/T$ within TVDWHRG and VDWHRG models, which leads to a significant enhancement in transport coefficients at higher $T$ ($T > 0.16$ GeV) especially for $\mu_B = 0.3$ and 0.35 GeV. The VDW interactions are strengthened with the increase of $\mu_B$ which can lead to the curves of $\sigma_{el}/T$ for different $\mu_B$ cross with each other. The effect of VDW interactions is enhanced further if we consider both the effect of VDW interactions between hadrons (as in Fig. 7(b)) and thermal hadron masses which even leads to the curves for different $\mu_B$ cross with each other and $\sigma_{el}/T$ increases with the increase of $\mu_B$ at higher $T$ ($T > 0.16$ GeV). In addition, we observe that the effect of VDW interactions on $\sigma_{el}/T$ is improved by the inclusion of medium effect for $\mu_B = 0.3$ and 0.35 GeV at higher $T$ ($T > 0.16$ GeV). Without the consideration of thermal hadron masses, all transport coefficients involved in our calculation are not very sensitive to the effect of VDW interactions at lower temperature which is similar to the results of thermodynamic quantities.

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