NON-SURJECTIVE GAUSSIAN MAPS FOR SINGULAR CURVES ON K3 SURFACES

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Abstract. Let \((S, L)\) be a polarized K3 surface with \(\text{Pic}(S) = \mathbb{Z}[L]\) and \(L \cdot L = 2g - 2\), let \(C\) be a nonsingular curve of genus \(g - 1\) and let \(f : C \to S\) be such that \(f(C) \in |L|\). We prove that the Gaussian map \(\Phi_{\omega_C(-T)}\) is non-surjective, where \(T\) is the degree two divisor over the singular point \(x\) of \(f(C)\). This generalizes a result of Kemeny with an entirely different proof. It uses the very ampleness of \(C\) on the blown-up surface \(\tilde{S}\) of \(S\) at \(x\) and a theorem of L’vovski.

1. Introduction

Let \(C\) be a complex projective nonsingular curve of genus \(g\). Let \(L, A\) be invertible sheaves on \(C\) and let

\[ R(L, A) := \ker[H^0(C, L) \otimes H^0(C, A) \to H^0(C, LA)] \]

Then we can define a Gaussian map

\[ \Phi_{L, A} : R(L, A) \to H^0(C, \omega_C LA) \]

in a well-known way that will be recalled in \([12]\). If \(L = A\) the map \(\Phi_{L, L}\) has the same image as its restriction to \(\bigwedge^2 H^0(C, L) \subset R(L, L)\), which is denoted by:

\[ \Phi_L : \bigwedge^2 H^0(C, L) \to H^0(C, \omega_C L^2) \]

If we take \(L = \omega_C\) then the map:

\[ \Phi_{\omega_C} : \bigwedge^2 H^0(C, \omega_C) \to H^0(C, \omega_C^3) \]

is called the Wahl map.

The following result, due to Wahl (see \([W]\) and also \([BM]\) for a different proof), gives a necessary condition for a nonsingular curve to be hyperplane section of a K3 surface:

**Theorem 1** (Wahl). Every nonsingular curve in a very ample linear system \(|L|\) on a K3 surface \(S\) has non-surjective Wahl map.

This result has been generalized by L’vovski (see \([L]\) and also \([BF]\) for an elementary detailed proof) in the following form:

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Theorem 2 (L’vovski). Let $C$ be a smooth curve of genus $g > 0$ and let $A$ be a very ample line bundle on $C$ embedding $C$ in $\mathbb{P}^n$, $n \geq 3$. If $C \subset \mathbb{P}^n$ is scheme-theoretically a hyperplane section of a smooth surface $X \subset \mathbb{P}^{n+1}$ then the Gaussian map $\Phi_{\omega_C,A}$ is non-surjective.

In this paper we will focus on singular curves on a K3 surface $S$. Our starting point is the recent work [Ke] by Kemeny. Let $V^i_g$ be the moduli space of triples $(S, L, f : C \to S)$, where $(S, L)$ is a polarized K3 surface with $L \cdot L = 2g - 2$, $C$ is a smooth curve of genus $g - 1$ and $f$ is an unramified stable map, birational onto its image, such that $f(C) \in |L|$. Then the following holds:

Theorem 3 ([Ke], Theorem 1.7). Fix an integer $g \geq 14$. Then there is an irreducible component $I^0 \subseteq V^1_g$ such that for a general triple $(S, L, f : C \to S) \in I^0$ the Gaussian map $\Phi_{\omega_C(-T)}$ is non-surjective, where $T = P + Q \subseteq C$ is the divisor over the node of $f(C)$.

The component $I^0$ appearing in the statement might a priori include all 1-nodal curves in $|L|$, but this is not proved in [Ke]. The proof is rather indirect and relies on the fact that $H^0(C, f^* T_S) = 0$ for the general triple $(S, L, f : C \to S) \in I^0$ (see [Ke], Lemma 3.17).

Our main result is the following more general statement, which is an exact analogue of Theorem 3 for singular curves:

Theorem 4. Fix an integer $g \geq 9$. Let $(S, L)$ be a polarized K3 surface such that $\text{Pic}(S) = \mathbb{Z}[L]$ and $L \cdot L = 2g - 2$. Let $C$ be a smooth curve of genus $g - 1$ endowed with a morphism $f : C \to S$ birational onto its image and such that $f(C) \in |L|$. If $T = P + Q \subseteq C$ is the divisor over the singular point of $f(C)$, then the Gaussian map $\Phi_{\omega_C(-T)}$ is non-surjective.

Note that we are not making any generality assumption on the triple $(S, L, f : C \to S)$. In particular we are not assuming that $f(C)$ is a nodal curve: the hypothesis that $C$ has genus $g - 1$ just implies that $f(C)$ is either 1-nodal or has an ordinary cusp. Note also, by contrast, that the normalization $C$ of a 1-nodal curve on a K3 surface tends to have surjective Wahl map $\Phi_{\omega_C}$, and therefore, by Theorem 3 not to be embeddable in any K3 surface. In fact, the following result holds:

Theorem 5 (Sernesi [S]). Let $(S, L)$ be a general primitively polarized K3 surface of genus $g + 1$. Assume that $g = 40, 42$ or $\geq 44$. Then the Wahl map of the normalization of any 1-nodal curve in $|L|$ is surjective.

In outline, the proof of Theorem 4 goes as follows.

We prove that on the blow-up $\sigma : \tilde{S} \to S$ at the singular point of $f(C)$ the line bundle $H := \sigma^* L(-2E)$ is very ample, where $E$ denotes the exceptional divisor of the blow-up $\sigma : \tilde{S} \to S$. This fact is a special case of Theorem 10 which gives a more general very-ampleness criterion of independent interest.

Hence we can apply Theorem 2 to

$$A := H \cdot C = C \cdot C = (C + E) \cdot C - E \cdot C = \omega_C(-T)$$

and obtain that the Gaussian map $\Phi_{\omega_C,-\omega_C(-T)}$ is non-surjective.

Finally, we prove that $\text{coker}(\Phi_{\omega_C(-T)})$ surjects onto $\text{coker}(\Phi_{\omega_C,-\omega_C(-T)})$ (Theorem 3). In particular, $\Phi_{\omega_C(-T)}$ is non-surjective.

We work over the field $\mathbb{C}$ of complex numbers.
2. Conormal sheaves and projections

Let \( C \) be as in the Introduction. It turns out to be natural, for our purposes, to introduce the so-called syzygy sheaves. We define the syzygy sheaf \( M_L \) of a globally generated invertible sheaf \( L \) by the exact sequence:

\[
0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \to L \to 0
\]

If \( L \) is very ample the above sequence is a twist of the dualized Euler sequence, and we get

\[
M_L = \Omega^1_{\mathbb{P}|C} \otimes L
\]

where \( \mathbb{P} \cong \mathbb{P}H^0(C, L)^\vee \). Therefore the conormal sequence of \( C \subset \mathbb{P} \) twisted by \( L \) takes the following form:

\[
0 \to N^\vee_{C/P} \otimes L \to M_L \to \omega_C L \to 0
\]

Now let \( A \) be another invertible sheaf on \( C \) and tensor (2) by \( A \):

\[
0 \to N^\vee_{C/P} \otimes LA \to M_L \otimes A \overset{\rho}{\to} \omega_C LA \to 0
\]

Then \( H^0(C, M_L \otimes A) = \mathcal{R}(L, A) \) and the map induced by \( \rho \) on global sections:

\[
\Phi_{L,A} : H^0(C, M_L \otimes A) \to H^0(C, \omega_C LA)
\]

is the Gaussian map of \( L, A \). When \( L = A \) we have:

\[
H^0(C, M_L \otimes L) = I_2 \bigoplus \bigwedge^2 H^0(C, L)
\]

where

\[
I_2 = \ker[S^2H^0(C, L) \to H^0(C, L^2)]
\]

Since \( I_2 \subset \ker(\Phi_{L,L}) \) the map \( \Phi_{L,L} \) has the same image as its restriction to \( \bigwedge^2 H^0(C, L) \subset \mathcal{R}(L, L) \), which is denoted by:

\[
\Phi_L : \bigwedge^2 H^0(C, L) \to H^0(C, \omega_C L^2)
\]

Now let \( L \) be very ample, \( P \in C \) and assume that \( L(-P) \) is also very ample. Then we have embeddings:

\[
\varphi_L : C \to \mathbb{P}^r, \quad \varphi_{L(-P)} : C \to \mathbb{P}^{r-1}
\]

where \( h^0(C, L) = r + 1 \). The following proposition relates the conormal sheaves \( N_{C/P}^\vee \) and \( N_{C/P^{r-1}}^\vee \).

**Proposition 6.** There is an exact sequence:

\[
0 \to N_{C/P^{r-1}}^\vee \otimes L(-P) \to N_{C/P}^\vee \otimes L \to \mathcal{O}_C(-2P) \to 0
\]

**Proof.** There is an exact sequence

\[
0 \to M_{L(-P)} \to M_L \to \mathcal{O}_C(-P) \to 0
\]
induced by the inclusion $H^0(C, L(-P)) \subset H^0(C, L)$. Recalling (1) we get a commutative and exact diagram whose first two rows are twisted conormal sequences:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N_C^\vee \otimes L(-P) & \rightarrow & M_{L(-P)} & \rightarrow & \omega_C L(-P) & \rightarrow & 0 \\
0 & \rightarrow & N_C^\vee \otimes L & \rightarrow & M_{L} & \rightarrow & \omega_C L & \rightarrow & 0 \\
0 & \rightarrow & \mathcal{O}_C(-2P) & \rightarrow & \mathcal{O}_C(-P) & \rightarrow & \omega_C L \otimes \mathcal{O}_P & \rightarrow & 0 \\
\end{array}
\]

The first column gives the sequence (4).

**Corollary 7.** Let $C, L$ be as before and suppose that $P, Q \in C$ are points such that $L(-P - Q)$ is very ample. Then there is an exact sequence:

\[(5) \quad 0 \rightarrow N_C^\vee \otimes L(-P - Q) \rightarrow N_C^\vee \otimes L \rightarrow \mathcal{O}_C(-2P) \bigoplus \mathcal{O}_C(-2Q) \rightarrow 0\]

**Proof.** Left to the reader.

**3. A comparison result between Gaussian maps**

After the preliminaries collected in the previous section, we are ready to prove the following result:

**Theorem 8.** Let $C$ be a projective nonsingular curve of genus $g$, $T = P + Q$ an effective divisor of degree 2 on $C$. Assume that $\text{Cliff}(C) \geq 3$. Then there is a surjection:

\[\text{coker}(\Phi_{\omega_C(-T)}) \rightarrow \text{coker}(\Phi_{\omega_C, \omega_C(-T)}) \rightarrow 0\]

In particular, if $\Phi_{\omega_C, \omega_C(-T)}$ is not surjective then $\Phi_{\omega_C(-T)}$ is not surjective.

**Proof.** The hypothesis $\text{Cliff}(C) \geq 3$ implies that $\omega(-T)$ is very ample and maps $C \subset \mathbb{P}^{g-3}$. We have an exact sequence ([189], Lemma 1.4.1):

\[0 \rightarrow M_{\omega_C(-T)} \rightarrow M_{\omega_C} \rightarrow \mathcal{O}_C(-P) \bigoplus \mathcal{O}_C(-Q) \rightarrow 0\]
which, twisted by \( \omega_C(-T) \), appears as the middle column in the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N_{C/Pg-3}^\vee \otimes \omega_C^2(-2T) & \rightarrow & M_{\omega_C(-T)} \otimes \omega_C(-T) & \rightarrow & \omega_C^2(-2T) & \rightarrow & 0 \\
0 & \rightarrow & N_{C/Pg-1}^\vee \otimes \omega_C^2(-T) & \rightarrow & M_{\omega_C} \otimes \omega_C(-T) & \rightarrow & \omega_C^2(-T) & \rightarrow & 0 \\
0 & \rightarrow & \omega_C(-T-2P) \oplus \omega_C(-T-2Q) & \rightarrow & \omega_C(-T-P) \oplus \omega_C(-T-Q) & \rightarrow & \omega_C^2(-T) \otimes O_T & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

where the first column is \[ \text{[5]} \] for \( L = \omega_C \), twisted by \( \omega_C(-T) \). The homomorphisms \( a \) and \( b \) induce \( \Phi_{\omega_C(-T)} \omega_C(-T) \) and \( \Phi_{\omega_C} \omega_C(-T) \) respectively on global sections. Therefore, taking cohomology, we obtain the following diagram:

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{coker}(\Phi_{\omega_C(-T)}) & \rightarrow & H^1(N_{C/Pg-3}^\vee \otimes \omega_C^2(-2T)) & \rightarrow & H^1(\Omega_{pg-3} \otimes \omega_C^2(-2T)) & \rightarrow & 0 \\
0 & \rightarrow & \text{coker}(\Phi_{\omega_C} \omega_C(-T)) & \rightarrow & H^1(N_{C/Pg-1}^\vee \otimes \omega_C^2(-T)) & \rightarrow & H^1(\Omega_{pg-1} \otimes \omega_C^2(-T)) & \rightarrow & 0 \\
0 & \rightarrow & \ker(H^1(g)) & \rightarrow & H^1(\omega_C(-T-2P)) \oplus H^1(\omega_C(-T-2Q)) & \rightarrow & H^1(\Omega_{pg} \otimes \omega_C^2(-T)) & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Since \( \text{Cliff}(C) \geq 3 \) it follows that \( H^1(g) \) is an isomorphism, thus \( \zeta \) is surjective. \( \square \)

Theorem [8] can be generalized in several ways, for example, using similar methods and induction on \( n \) one can also prove the following:

**Theorem 9.** Let \( C \) be a projective nonsingular curve of genus \( g \), \( T = P_1 + \cdots + P_n \) an effective divisor of degree \( n \geq 1 \). Let \( L \) be an invertible sheaf on \( C \) of degree \( d \geq 2g + 1 + n \). Then there is a surjection:

\[
\text{coker}(\Phi_{L(-T)}) \rightarrow \text{coker}(\Phi_{L,L(-T)}) \rightarrow 0
\]

We will not pursue this here.

**4. Very ampleness on blown-up surfaces**

For the proof of Theorem 9 we will need a special case of the following result of independent interest:

**Theorem 10.** Let \( S \) be a K3 surface such that \( \text{Pic}(S) = \mathbb{Z}[L] \) for some ample invertible sheaf \( L \). Assume that \( L \cdot L = 2g - 2 \geq (\ell + 1)^2 + 3 \) for some \( \ell \geq 1 \). Let
$x \in S$, $\sigma : \tilde{S} \to S$ the blow-up of $S$ at $x$ and $E \subset \tilde{S}$ the exceptional curve. Then the sheaf $H := \sigma^* L(-\ell E)$ is very ample on $\tilde{S}$.

Proof. We follow closely the application of Reider’s method in [V], proof of Theorem 16. We must prove that for each subscheme $Z \subset \tilde{S}$ of length two we have $H^1(\tilde{S}, I_Z \otimes H) = 0$.

By contradiction, assume that $H^1(\tilde{S}, I_Z \otimes H) \neq 0$ for some $Z$. By Serre duality we have:

$$H^1(\tilde{S}, I_Z \otimes H) = \text{Ext}^1(I_Z, E - H)$$

and therefore there is a non-split exact sequence:

$$(6) \quad 0 \to \sigma^* L^{-1}((\ell + 1)E) \to E \to I_Z \to 0$$

where $E$ is torsion free. We have:

$$c_1(E)^2 = 2g - 2 - (\ell + 1)^2, \quad c_2(E) = 2$$

$$\chi(\text{Hom}(E, E)) = 4\chi(\mathcal{O}_S) + c_1^2(E) - 4c_2(E) = 2g - 2 - (\ell + 1)^2 \geq 3$$

Therefore by Serre duality:

$$2h^0(\text{Hom}(E, E)) \geq h^0(\text{Hom}(E, E)) + h^0(\text{Hom}(E, E)) \geq 3$$

It follows that there is a homomorphism $\phi : E \to E$ which is not proportional to the identity. By the usual trick we can assume that $\phi$ is generically of rank one (see [V], proof of Proposition 15). $\ker(\phi)$ and $\text{im}(\phi)$ are torsion free rank one, therefore of the form $A \otimes I_W$ and $B \otimes I_W'$ respectively, for some invertible sheaves $A, B$ which are of the form:

$$A = \sigma^* L^\alpha(\beta E), \quad B = \sigma^* L^{-1-\alpha}((\ell + 1 - \beta)E)$$

From the exact sequence

$$0 \to A \otimes I_W \to E \to B \otimes I_W' \to 0$$

we compute:

$$(7) \quad 2 = c_2(E) = \deg(W) + \deg(W') - \beta(\ell + 1 - \beta) \geq -\beta(\ell + 1 - \beta)$$

Indeed, since $A \otimes I_W \subset E$, from (6) we see that we must have $\alpha \leq 0$. Similarly $-\alpha - 1 \leq 0$ because $B \otimes I_W' \subset E(E)$. Therefore:

$$-1 \leq \alpha \leq 0,$$

proving (7).

Suppose $\alpha = 0$. Then we have an inclusion $\mathcal{O}_{\tilde{S}}(\beta E) \otimes I_W \subset I_Z$, which implies $\beta \leq 0$. If $\beta < 0$ then (7) gives a contradiction. If $\beta = 0$ we get an inclusion $I_W \subset I_Z$. This implies that the pullback homomorphism:

$$\psi : \text{Ext}^1(I_Z, E - H) \to \text{Ext}^1(I_W, E - H)$$

maps (6) to zero, thus $\psi$ is not injective. But $\psi$ is dual to:

$$\psi^\vee : H^1(\tilde{S}, I_W \otimes H) \to H^1(\tilde{S}, I_Z \otimes H)$$

which is henceforth not surjective. On the other hand the diagram:

\begin{equation}
\begin{array}{ccc}
0 & \longrightarrow & I_W \otimes H \\
\downarrow & & \downarrow \\
I_Z \otimes H & \longrightarrow & H |_Z \\
\end{array}
\end{equation}

shows that we have an exact sequence

\begin{equation}
0 \longrightarrow I_W \otimes H \longrightarrow I_Z \otimes H \longrightarrow \mathcal{T} \longrightarrow 0
\end{equation}

with \( \mathcal{T} \) torsion: therefore \( \psi'^\vee \) is surjective, and we have a contradiction. So the case \( \alpha = 0 \) cannot occur.

Suppose \( \alpha = -1 \). In this case we use the inclusion \( B \otimes I_W \subseteq \mathcal{E}(E) \). We obtain an inclusion \( I_{W'}((\ell - \beta)E) \subseteq \mathcal{E} \) which implies

\[ I_{W'}((\ell - \beta)E) \subseteq I_Z \]

and therefore \( \ell - \beta \leq 0 \). The case \( \ell - \beta = 0 \) gives an inclusion \( I_{W'} \subseteq I_Z \) and is treated using a diagram analogous to (8), leading to a contradiction as before.

If \( \ell - \beta < 0 \) then \( \beta \geq \ell + 1 \). If \( \beta > \ell + 1 \) then (7) gives a contradiction. If \( \beta = \ell + 1 \) then (7) gives:

\[ \text{deg}(W) + \text{deg}(W') = 2 \]

If \( \text{deg}(W) > 0 \) then \( A \otimes I_W = \sigma^*L^{-1}(\ell + 1)E) \otimes I_W \subseteq \sigma^*L^{-1}(\ell + 1)E) \) and therefore \( \phi(\sigma^*L^{-1}(\ell + 1)E) \) is a torsion subsheaf of \( \mathcal{E}(E) \), a contradiction. Then \( W = 0 \) and \( W' = Z \). This gives \( I_Z \subseteq \mathcal{E}(E) \), viz. \( I_Z(-E) \subseteq \mathcal{E} \). This implies that the pullback

\[ \theta : \text{Ext}^1(I_Z, E - H) \longrightarrow \text{Ext}^1(I_Z(-E), E - H) \]

maps (6) to zero, thus the dual map:

\[ \theta'^\vee : H^1(\tilde{S}, I_Z \otimes (H - E)) \longrightarrow H^1(\tilde{S}, I_Z \otimes H) \]

is not surjective. But \( \text{coker}(\theta'^\vee) \subseteq H^1(E, I_Z \otimes \mathcal{O}_E(H)) = 0 \) because \( I_Z \otimes \mathcal{O}_E(H) \) is an invertible sheaf of degree \( \geq 0 \) on \( E \). We have a contradiction and the theorem is proved.

**Remarks 11.**

(i) For the first values of \( \ell \) the condition of the theorem gives:

\[
\begin{align*}
\ell = 1 & : \quad g \geq 5 \\
\ell = 2 & : \quad g \geq 7 \\
\ell = 3 & : \quad g \geq 11.
\end{align*}
\]

(ii) The case \( \ell = 1, g = 5 \) has already been considered in [B].

(iii) An interesting implicit consequence of Theorem 10 is the following existence result:
Under the assumptions of Theorem 10 for a given $\ell \geq 2$, through each point $x \in S$ there exist integral curves in $|L|$ having an ordinary multiple point of multiplicity exactly $\ell$ at $x$ and no other singularities.

(iv) One can combine the main result of [GL] with Theorem 1.4 of [FKP] to deduce, in certain ranges of $g, \ell$, that $\sigma^* L(-\ell E)$ is not only very ample but even embeds $\tilde{S}$ as an arithmetically Cohen-Macaulay surface. This is the case e.g. for $\ell = 1$, $g \geq 5$ and for $\ell = 2$ and $g \geq 9$.

In the next section we are going to apply Theorem 10 in the case $\ell = 2$.

5. Proof of Theorem 10

Let $x \in S$ be the singular point of $f(C)$, let $\sigma : \tilde{S} \to S$ be the blow-up at $x$ and $E \subset \tilde{S}$ the exceptional curve. Then $C \in |\sigma^* L(-2E)|$ and Theorem 10 implies that $\sigma^* L(-2E)$ is very ample, thus $C$ is a hyperplane section of $\tilde{S} \subset \mathbb{P}^{g-2}$ embedded by $\sigma^* L(-2E)$. Let $T := f^*(x) = P + Q = C \cdot E$. Then

$$\omega_C(-T) = (C + E) \cdot C - C \cdot E = C \cdot C$$

Therefore $\Phi_{\omega_C, \omega_C(-T)}$ is not surjective, by Theorem 2. Now we use Theorem 1.4 of [FKP]. Since $g \geq 9$ we have, in the notation of [FKP]:

$$\rho_{\text{sing}}(g, 1, 4, g-1) = \rho(g - 1, 1, 4) + 1 < 0$$

and

$$\rho_{\text{sing}}(g, 2, 6, g-1) = \rho(g - 1, 2, 6) + 1 < 0$$

Therefore by [FKP], Theorem 1.4, we have $W_1^1(C) = \emptyset = W_0^2(C)$, thus $\text{Cliff}(C) \geq 3$. We can then apply Theorem 8 to deduce that $\Phi_{\omega_C(-T)}$ is not surjective either. □

References

[BF] Ballico E., Fontanari C.: Gaussian maps, the Zak map and projective extensions of singular varieties. Result. Math. 44 (2003), 29-34.

[B] Bauer I.: Inner projections of algebraic surfaces: a finiteness result. J. Reine Angew. Math. 460 (1995), 1-13.

[BM] Beauville A., Merindol J.Y.: Sections hyperplanes des surfaces K3. Duke Math. J. 55 (1987), 873-878.

[FKP] Flamini F., Knutsen A.L., Pacienza G.: Singular curves on a K3 surface and linear series on their normalizations. International J. of Math. 18 (2007), 671-693.

[GL] Green M., Lazarsfeld R.: On the projective normality of complete linear series on an algebraic curve, Inventiones Math. 83 (1986), 73-90.

[Ke] Kemeny M.: The moduli of singular curves on K3 surfaces. J. de Math. Pures et appliquées 104 (2015), 882-920.

[L] L’vovsky S.: Extensions of projective varieties and deformations. I, II. Michigan Math. J., 39 (1992), 41–51, 65–70.

[L89] Lazarsfeld R.: A sampling of vector bundle techniques in the study of linear series. In Riemann Surfaces - Proceedings of the College on Riemann Surfaces, ICTP Trieste, 1987. World Scientific (1989).

[S] Sernesi E.: The Wahl map of one-nodal curves on K3 surfaces. To appear on Contemporary Mathematics. arXiv:1701.04801.

[V] Voisin C.: Segre classes of tautological bundles on Hilbert schemes of surfaces. arXiv:1708.06325.

[W] Wahl J.: The jacobian algebra of a graded Gorenstein singularity. Duke Math. J. 55 (1987), 849-872.
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