Statistical Inference for Linear Mediation Models with High-dimensional Mediators and Application to Studying Stock Reaction to COVID-19 Pandemic

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Abstract

Mediation analysis draws increasing attention in many scientific areas such as genomics, epidemiology and finance. In this paper, we propose new statistical inference procedures for high dimensional mediation models, in which both the outcome model and the mediator model are linear with high dimensional mediators. Traditional procedures for mediation analysis cannot be used to make statistical inference for high dimensional linear mediation models due to high-dimensionality of the mediators. We propose an estimation procedure for the indirect effects of the models via a partial penalized least squares method, and further establish its theoretical properties. We further develop a partial penalized Wald test on the indirect effects, and prove that the proposed test has a $\chi^2$ limiting null distribution. We also propose an $F$-type test for direct effects and show that the proposed test asymptotically follows a $\chi^2$-distribution under null hypothesis and a noncentral $\chi^2$-distribution under local alternatives. Monte Carlo simulations are conducted to examine the finite sample performance of the proposed tests and compare their performance with existing ones. We further apply the newly proposed statistical inference procedures to study stock reaction to COVID-19 pandemic via an empirical analysis of studying the mediation effects of financial metrics that bridge company’s sector and stock return.

\textbf{JEL classification:} C12; C13

\textbf{Keywords} Mediation Analysis; Penalized Least Squares; Sparsity; Wald test.
1 Introduction

Since the seminal work of Baron & Kenny (1986), mediation analysis has been used in various scientific research, such as economics, psychology, pedagogy, and behavioral science (Conti et al., 2016; Chernozhukov et al., 2021; Mackinnon, 2008; Hayes, 2013; Vanderweele, 2015). It is designed to investigate the mechanisms whereby exposure variables affect an outcome through intermediate variables, which are termed as mediators. For instance, in the field of policy evaluation, while there certainly is no shortage of techniques assessing effects of policies or other treatments on an outcome (Imbens, 2004; Donald & Hsu, 2014; Athey et al., 2018; Ai et al., 2021), mediation analyses move a step further to disentangle such effect into indirect effects through mediators, such as certain economic indices, and direct effects. Numerous statistical inference procedures for mediation models with low-dimensional mediators have been extensively studied (Preacher & Hayes, 2008; Vanderweele & Vansteelandt, 2014). See Ten Have & Joffe (2010) and Preacher (2015) for brief reviews about inferences under low-dimensional mediation models.

On account of modern data-collecting technology, mediation analysis extends its territory to quantitative finance, genomics, internet analysis, biomedical research, among other data-intensive fields. This brings in high-dimensional mediators and requires attention on high-dimensional mediation model (HDMM), where the number of potential mediators is much larger than the sample size. Our work is motivated by such a high-dimensional mediation structure when studying the effects of company’s belonging sector on stock return via influencing various financial metrics during the COVID-19 period. Direct effects of sectors, as well as financial statements, on stock performance have been extensively studied in literature. See for instance Fama & French (1993); Graham et al. (2002); Callen & Segal (2004); Edirisinghe & Zhang (2008); Dimitropoulos & Asteriou (2009); Fama & French (2015); Khan & Khokhar (2015); Enke & Thawornwong (2005); Huang et al. (2019). Yet as to be evidently shown by the empirical analysis in section 3.2, the companies’ belonging sectors also significantly affect stock returns indirectly through certain financial metrics in the statements. In our analysis, 550 financial indexes are involved, based on only 490 companies, resulting in high dimensional mediators.

The high-dimensionality, on every account, poses both computational and statistical challenges for carrying out efficient mediation analysis. For instance, the traditional structural equation modeling fails due to the rank-deficiency of the observed covariance matrix. However, notwithstanding the high dimensional mediation structure, the number of truly active mediators is typically assumed small and less than the sample size. This is referred to as the sparsity assumption in the literature, although the sparsity pattern is unknown and thus to be recovered. See, for example,
Fan et al. (2020) and references therein. Many existing methods in literature break through such obstacle by utilizing the dimension reduction techniques in regular linear models. For example, Huang & Pan (2016) and Chen et al. (2018) adopted principal components analysis to compress the dimensionality of mediators, and applied bootstrap for inference. These methods are intuitive and simple to implement, but lack theoretical justification about asymptotic distributions of the test statistics. As an extension of Huang & Pan (2016), Zhao et al. (2020) further introduced sparse principal component analysis to mediation models. Zhang et al. (2016) used a two-stage technique with (a) first screening out “unimportant” mediators, and then (b) applying existing procedures for the post-screened outcome model. Zhou et al. (2020) introduced debiased penalized estimators for the direct and indirect effects, with theoretical guarantees of the related tests. However, their method involves estimating high dimensional matrices, leading to potentially unstable estimates and expensive computation. Furthermore, imposing penalization on all parameters reduces the efficiency of estimators, and hence tests. There are many developments on this topic in the recent literature (Chakrabortty et al., 2018; Derkach et al., 2019; Song et al., 2020).

In this paper, we propose new statistical inference procedures for HDMM. Statistical inference for high-dimensional data has been an active research topic in the literature (Belloni et al., 2014; Zhang & Zhang, 2014; van de Geer et al., 2014; Javanmard & Montanari, 2014; Shi et al., 2019; Fan, et al., 2020a,b). However, there are much less work on statistical inference for HDMM. To our best knowledge, Zhou et al. (2020) is the only one on testing hypothesis on indirect effect with solid theoretical analysis. Our inference procedure on indirect effect is distinguished from Zhou et al. (2020) in that we observe the indirect effect in HDMM indeed is a low dimensional parameter and is the difference between the total effect and the direct effect in the HDMM. This motivates us to estimate the total effect via least squares method and the direct effect by partial penalized least squares method, and then estimate the indirect effect by the difference between the estimates of the total effect and the direct effect. We establish the asymptotical normality of the indirect effect estimate and further develop a Wald test for the indirect effect.

We estimate the direct effect in the HDMM by partial penalized least squares method, and propose an $F$-type test for it. The statistical inference on the direct effect essentially is the same as statistical inference on low dimensional coefficients in high-dimensional linear models. This topic has been studied under the setting in which the covariate vector in the high-dimensional linear models is fixed design (Zhang & Zhang, 2014; van de Geer et al., 2014; Shi et al., 2019). Due to the nature of HDMM, the design matrix in HDMM must be random rather than fixed since mediators are random. Thus, the statistical setting studied in this paper is different from the one in Shi et al. (2019), in which the covariate vector is assumed to be fixed design. We study the asymptotical
property of the proposed estimator in the random-design setting. The random design imposes challenges in deriving the rate of convergence and asymptotical normality of the partial penalized least squares estimates. Under mild regularity conditions, we prove the sparsity and establish the rate of convergence of the partial penalized least squares estimate. We further establish an asymptotical representation of the estimate. Based on the asymptotical representation, we can easily derive the asymptotical normality of the estimate and derive the asymptotical distributions of the proposed test for the direct effect under null hypothesis and under local alternative.

We show that the proposed estimate of indirect effect is asymptotically more efficient than the one proposed in Zhou et al. (2020), and indeed is asymptotically efficient under normality assumption. This is because the debias step of debiased Lasso inflates the asymptotical variance of the resulting estimate. We conduct Monte Carlo simulation studies to assess the finite sample performance of the proposed estimate in terms of bias and variance and to examine Type I error and power of the proposed test. We also conduct numerical comparisons among the proposed estimate, the oracle estimate and the estimate proposed in Zhou et al. (2020). Our numerical comparison indicates that the proposed estimate performs as well as the oracle one, and outperforms the estimate proposed by Zhou et al. (2020).

We utilize the proposed method to study the mediator role of financial metrics that bridge company’s sector and stock return. We select six financial metrics out of all the 550 that indeed mediate the pathways linking company sector and stock return, with interestingly and informatively financial interpretations. We also compare the metrics selected using our data during the COVID-19 period and those classical findings in existing works, including Fama & French (2015), Ediris:inghe & Zhang (2008), among others. We indeed discover some unique patterns and features due to the pandemic. Moreover, according to the proposed tests for effects of sector, both its direct effect and indirect effect via financial metrics are statistically significant. Therefore, evaluating the selected financial metrics, as well as the sector information, might help investors to make wiser investment decisions and choose stocks especially during the pandemic.

The rest of this paper is organized as follows. In section 2, we propose a new statistical inference procedure for the indirect effect and establish its theoretical properties. We also construct an $F$-type test for the direct effect. Section 3 presents numerical studies and a real data example. Conclusion and discussion are given in section 4. All proofs are presented in Appendix.
2 Tests of hypotheses on indirect and direct effects

Consider the mediation models

\[ y = \alpha^T_0 m + \alpha^T_1 x + \varepsilon_1, \quad (2.1) \]
\[ m = \Gamma^T x + \varepsilon, \quad (2.2) \]

where \( y \) is the outcome, \( m \) is the \( p \)-dimensional mediator, \( x \) is the \( q \)-dimensional exposure variable, and \( a^T \) denotes transpose of \( a \). We in this paper assume \( p \) is high dimensional, while \( q \) is fixed and finite. Correspondingly, \( \alpha_0 \) and \( \alpha_1 \) are \( p \)- and \( q \)-dimensional regression coefficient vectors, and \( \Gamma \) is a \( q \times p \) coefficient matrix. Following the literature on high-dimensional mediation model (Zhang et al. 2016; van Kesteren & Oberski 2019; Zhou et al. 2020), we impose a sparsity assumption that only a small proportion of entries in \( \alpha_0 \) are nonzero. This implies that the corresponding variables in \( m \) are actually relevant to \( y \). Notably, from equation (2.2), \( m \) must be random. We further assume that \( \varepsilon_1 \) and \( \varepsilon \) are independent random errors with \( \text{var}(\varepsilon_1) = \sigma^2_1 \) and \( \text{cov}(\varepsilon) = \Sigma^* \); \( \varepsilon_1 \) is independent of \( m, x \), and \( \varepsilon \) is independent of \( x \).

Plugging (2.2) into (2.1) yields

\[ y = (\beta + \alpha_1)^T x + \varepsilon_1 + \varepsilon_2 = \gamma^T x + \varepsilon_3, \quad (2.3) \]

where \( \beta = \Gamma \alpha_0, \varepsilon_2 = \alpha^T_0 \varepsilon \) with \( \text{var}(\varepsilon_2) = \sigma^2_2 = \alpha^T_0 \Sigma^* \alpha_0, \gamma = \beta + \alpha_1, \) and \( \varepsilon_3 = \varepsilon_1 + \varepsilon_2 \) is the total random error. Following the literature (Imai et al. 2010; Vanderweele & Vansteelandt 2014), we refer \( \beta \) to the indirect effect of \( x \) on \( y \) mediated by \( m \), \( \alpha_1 \) to the direct effect, and \( \gamma = \alpha_1 + \beta \) to the total effect. A causal interpretation of \( \beta \) and \( \alpha_1 \) is briefly discussed in the Appendix.

2.1 Estimating indirect and direct effects

In practice, of interest is to test whether there exists significant (joint) indirect effect or not. This can be formulated as the following hypothesis testing problem

\[ H_0 : \beta = 0 \text{ versus } H_1 : \beta \neq 0. \quad (2.4) \]

When both \( p \) and \( q \) are finite-dimensional, \( \beta \) can be estimated through \( \hat{\beta} = \hat{\Gamma} \hat{\alpha}_0 \), where \( \hat{\Gamma} \) and \( \hat{\alpha}_0 \) are \( \sqrt{n} \)-consistently estimated from models (2.1) and (2.2). That is, \( \hat{\Gamma} = \Gamma + E_\gamma \) and \( \hat{\alpha}_0 = \alpha_0 + e_\alpha \), where \( E_\gamma = O_P(1/\sqrt{n}) \) and \( e_\alpha = O_P(1/\sqrt{n}) \) are estimation errors. Then

\[ \|\hat{\beta} - \beta\| \leq \|\Gamma e_\alpha\| + \|E_\gamma \alpha_0\| + \|E_\gamma e_\alpha\| = O_P(1/\sqrt{n}), \quad (2.5) \]
where \( \| \cdot \| \) stands for the Euclidean norm.

When \( p \) is high-dimensional, however, the right-hand side of (2.5) is no longer \( O_p(1/\sqrt{n}) \). This results in potentially non-ignorable estimation error of \( \hat{\beta} \). Moreover, \( \beta \) is challenging to be estimated through \( \Gamma \alpha_0 \) as it involves estimation of a high-dimensional matrix and a high-dimensional vector, though, interestingly, \( \beta = \Gamma \alpha_0 \) is \( q \)-dimensional, fixed and finite.

As a key observation from (2.3), the indirect effect \( \beta = \gamma - \alpha_1 \), is the difference between the total effect and direct effect. This motivates us to estimate \( \beta \) by separately estimating \( \gamma \) via (2.3) and \( \alpha_1 \) via (2.1), respectively, rather than estimating the high-dimensional \( \Gamma \) and \( \alpha_0 \).

Suppose that \( \{m_i, x_i, y_i\}, i = 1, \cdots, n \) is a random sample from (2.1) and (2.2). Let \( y = (y_1, \cdots, y_n)^T \) and \( X = (x_1, \cdots, x_n)^T \). Then we estimate \( \gamma \) by its least squares estimate

\[
\hat{\gamma} = (X^T X)^{-1} X^T y. \tag{2.6}
\]

While for the estimator of \( \alpha_1 \), due to the high-dimensionality of \( \alpha_0 \), we propose the following partial penalized least squares method:

\[
(\hat{\alpha}_1, \hat{\alpha}_0) = \arg \min_{\alpha_1, \alpha_0} \frac{1}{2n} \| y - M \alpha_0 - X \alpha_1 \|^2 + \sum_{j=1}^p p_\lambda(|\alpha_{0j}|), \tag{2.7}
\]

where \( M = (m_1, \cdots, m_n)^T \) and \( p_\lambda(\cdot) \) is a penalty function with a tuning parameter \( \lambda \). The regularization is only applied to the high-dimensional yet sparse \( \alpha_0 \). We opt not penalize \( \alpha_1 \) to achieve local power on the direct effect \( \alpha_1 \) and the indirect effect \( \beta \) under local alternatives. See Theorem 2 and Corollary 1 below for more details. Thus, our proposal is different from Zhou et al. (2020), in which the central idea is to develop a debiased estimator not of \( \alpha_0 \) or \( \beta \), but of \( \hat{\Sigma}_{XM} \alpha_0 \) with \( \hat{\Sigma}_{XM} = E[\mathbf{x} \mathbf{m}^T] \). This may lead to less efficient estimators due to debiasing, as discussed in the next subsection.

### 2.2 Theoretical results

In this section, we investigate statistical properties of the estimators. We first present some notations and assumptions. For the penalty function, it is assumed that \( p_\lambda(t_0) \) is increasing and concave in \( t_0 \in [0, \infty) \), and has a continuous derivative \( p'_\lambda(t_0) \) with \( p'_\lambda(0+) > 0 \). Denote \( \rho(t_0, \lambda) = p_\lambda(t_0)/\lambda \) for \( \lambda > 0 \). Further, \( \rho'(t_0, \lambda) \) is increasing in \( \lambda \in (0, \infty) \) and \( \rho'(0+, \lambda) \) does not depend on \( \lambda \). Define \( \bar{\rho}(\mathbf{v}, \lambda) = \{\sgn(v_1)\rho'(|v_1|, \lambda), \cdots, \sgn(v_l)\rho'(|v_l|, \lambda)\}^T \) for any vector \( \mathbf{v} = (v_1, \cdots, v_l)^T \), where \( \sgn(\cdot) \) is the sign function. Define the local concavity of \( \rho(\cdot) \) at \( \mathbf{v} \) as

\[
\kappa(\rho, \mathbf{v}, \lambda) = \lim_{\epsilon \to 0^+} \max_{1 \leq j \leq l} \sup_{t_1 < t_2 \in [|v_j| - \epsilon, |v_j| + \epsilon]} \frac{\rho'(t_2, \lambda) - \rho'(t_1, \lambda)}{t_2 - t_1}.
\]
Let $\theta = (\alpha_1^T, \alpha_0^T)^T$ and $\theta_0 = (\alpha_1^T, \alpha_0^T)^T$, the true value of $\theta$. Further let $\hat{\theta} = (\hat{\alpha}_1^T, \hat{\alpha}_0^T)$ be the estimator of $\theta_0$. Denote $A = \{j : \alpha_{0j}^* \neq 0\}$, and $s = |A|$ is the number of elements in $A$. Moreover, $\hat{\theta} = (\alpha_1^T, \alpha_0^T)$. Then $\hat{\theta}_0, \hat{\theta}$ are similarly defined. Let $M_j^T$ denote the $j$th column of $M$. Let $M_A^T$ be the submatrix of $M$ formed by columns in $A$. $m_{i,A}$ is the $i$th column of the matrix $M_A^T$. Similarly, let $\alpha^*_{0,A}$ be the subvector of $\alpha_0^*$ formed by elements in $A$. Define $A^c = [1, \cdots, p] - A$ as the complement set of $A$. Define $N_0 = \{\delta \in \mathbb{R}^s : \|\delta - \alpha^*_{0,A}\|_2 \leq d_n\}$.

Let $\Sigma = \{\Sigma_{MM}, \Sigma_{MX}, \Sigma_{XX}\}$. Denote

$$
\Sigma = \begin{pmatrix}
\Sigma_{XX} & \Sigma_{XM} \\
\Sigma_{MX} & \Sigma_{MM}
\end{pmatrix}.
$$

In this paper, for any vector $\mathbf{v} = (v_1, \cdots, v_i)^T$, $\|\mathbf{v}\|_\infty = \max_i |v_i|$ and $\|\mathbf{v}\|_2 = (\mathbf{v}^T \mathbf{v})^{1/2}$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of the matrix $A$, respectively. $\|A\|_{2,\infty} = \sup_{\|\mathbf{v}\|_2 = 1} \|A\mathbf{v}\|_\infty$. Further $a \gg b$ means $\lim_{n \to \infty} a/b = \infty$. We impose the following conditions:

A1. $\lambda_{\min}(\Sigma) \geq c > 0, \lambda_{\max}(\Sigma) = O(1)$, and $\|M_{A^c}^T(X, M_A^T)\|_{2,\infty} = O_P(n)$.

A2. Let $d_n$ be the half minimum signal of $\alpha^*_{0,A}$, i.e. $d_n = \min_{j \in A} |\alpha^*_{0j}|/2$. Assume that $d_n \gg \lambda_n \gg \max\{\sqrt{s/n}, \sqrt{\log p/n}\}, p'_n(d_n) = o((ns)^{-1/2}), \lambda_n \kappa_0 = o(1)$ where $\kappa_0 = \max_{\delta \in N_0} \kappa(\rho, \delta, \lambda_n)$.

A3. For some $\varpi > 2$, there exists a positive sequence $K_n$ such that $E[\|m_{A^c}z_{\infty}\|_{\infty}] \leq K_n^\varpi$ and $K_n^2 \log p/n^{1-2/\varpi-\varsigma} \to 0$ for some arbitrary small $\varsigma > 0$. Further assume that $\max_{1 \leq j \leq p+q} E(z_{j}^4) < C < \infty$, here $z = (m \cdot x)$, $z_j$ is the $j$-th component of $z$.

To emphasize the dependence on the sample size, in the above conditions and the Appendix, we use $\lambda_n$ to denote the tuning parameter. The first two conditions are mild and commonly assumed. See for instance [Fan & Lv] (2011). Condition A2 imposes a minimal signal condition on nonzero elements in $\alpha_0$, but not on $\alpha_1$. Since our primary interest is to make statistical inference on direct effect $\alpha_1$ and indirect effect $\beta = \gamma - \alpha_1$, and $\alpha_0$ may be treated as a nuisance parameter in this model. Thus, Condition A2 is reasonable in practice. Condition A3 is imposed for establishing sparsity result. Compared with existing literature, A3 is very mild. In fact, to simplify the proof, some papers assume that all covariates are uniformly bounded - see for instance [Wang et al.] (2012). Under bounded covariates condition, A3 reduces to $E(|\varepsilon_{1,\infty}|) \leq C$ by taking $K_n$ as a constant. Furthermore, the dimension of $p$ is allowed to be an exponential order of the sample size $n$ according to conditions A2 and A3.
**Theorem 1** Suppose that Conditions (A1)-(A3) hold, and \( s = o(n^{1/2}) \), then with probability tending to 1, \( \hat{\alpha}_0 \) must satisfy (i) \( \hat{\alpha}_{0,A^c} = 0 \). (ii) \( ||\hat{\alpha}_{0,A} - \alpha_{0,A}||_2 = O_P(\sqrt{s/n}) \). Let \( \epsilon_1 = (\varepsilon_{11}, \cdots, \varepsilon_{n1})^{T} \). If further \( s = o(n^{1/3}) \), we obtain that

\[
\sqrt{n}(\hat{\vartheta} - \vartheta) = \frac{1}{\sqrt{n}} \Sigma^{-1} \begin{pmatrix} X^{T} \epsilon_1 \\ M_{A}^{T} \epsilon_1 \end{pmatrix} + o_P(1).
\]

The above results provide the sparsity of \( \alpha_0 \), the convergence rate of \( \hat{\alpha}_{0,A} \) and the asymptotic representation of \( \hat{\vartheta} \), respectively.

Based on the results in Theorem 1, we further obtain the following corollary:

**Corollary 1** Suppose that Conditions (A1)-(A3) hold, and \( s = o(n^{1/3}) \), we have

\[
\sqrt{n}(\hat{\alpha}_1 - \alpha_1^*) \rightarrow N(0, \sigma_1^2(\Sigma_{XX}^{-1} + B)), \quad \sqrt{n}(\hat{\beta} - \beta^*) \rightarrow N(0, \sigma_2^2 \Sigma_{XX}^{-1} + \sigma_1^2 B),
\]

where \( B = \Sigma_{XX}^{-1} \Sigma_{XM}(\Sigma_{MM} - \Sigma_{MX} \Sigma_{XX}^{-1} \Sigma_{XM})^{-1} \Sigma_{MX} \Sigma_{XX}^{-1} \), and \( \beta^* \) is the true value of \( \beta \).

This corollary presents the asymptotic normalities of the estimators \( \hat{\alpha}_1 \) and \( \hat{\beta} \). We next make theoretical comparison with the estimators in Zhou et al. (2020). Note that the asymptotic variance matrices of \( \hat{\alpha}_1^2 \) and \( \hat{\beta}_1^2 \) in Zhou et al. (2020) are \( \sigma_1^2(\Sigma_{XX}^{-1} + \tilde{B}) \) and \( \sigma_2^2 \Sigma_{XX}^{-1} + \sigma_2^2 \tilde{B} \), respectively, where \( \tilde{B} = \Sigma_{XX} \Sigma_{XM}(\tilde{\Sigma}_{MM} \Sigma_{XX}^{-1} \Sigma_{XM})^{-1} \Sigma_{MX} \Sigma_{XX}^{-1} \). To show our proposed estimators are more efficient than those proposed in Zhou et al. (2020), it suffices to show that \( \tilde{B} < B \). Note that \( \Sigma_{XX}^{-1} + B = (I_q, 0_{q\times s}) \Sigma_{XX}^{-1} (I_q, 0_{q\times s})^{T} \), and

\[
\Sigma_{XX}^{-1} + \tilde{B} = (I_q, 0_{q\times s})(\Sigma - E[xm_{A}^{T}]E[m_{A}m_{A}^{T}]^{-1}E[m_{A}^{T}x^{T}])^{-1}(I_q, 0_{q\times s})^{T}.
\]

Thus, \( \tilde{B} < B \) since \( (\Sigma - E[xm_{A}^{T}]E[m_{A}m_{A}^{T}]^{-1}E[m_{A}^{T}x^{T}])^{-1} > \Sigma_{XX}^{-1} \). Hence our proposed estimators are more efficient than those proposed in Zhou et al. (2020). This should not be surprised because the debias Lasso inflates its asymptotical variance in the debiased step for high-dimensional linear model (van de Geer et al., 2014). The proposed partial penalized least squares method does not penalize \( \alpha_1 \), and hence the debiased step becomes unnecessary.

Under normality assumption that \( \varepsilon_1 \sim N(0, \sigma_1^2) \) and \( \varepsilon \sim N(0, \Sigma^*) \), it can be shown that our proposed estimators are indeed asymptotically efficient. Under the normality assumption, the
maximum likelihood estimator (MLE) of \( \alpha_1, \alpha_{0,A} \) in the oracle model knowing \( \alpha_{0,A^c} = 0 \) satisfies
\[
\left( X^T(y - M_A\hat{\alpha}^M_{0,A} - X\hat{\alpha}^M_1) \right) = 0. \tag{2.8}
\]
This implies that \( \hat{\theta}^M = (\hat{\alpha}^M_1, \hat{\alpha}^M_{0,A}) \) has the same asymptotic distribution as \( \hat{\theta} \).

Since the MLE of \( \Gamma_A \) is \( \hat{\Gamma}_A^M = (X^TX)^{-1}X^TM_A \), the MLE of \( \beta \) can be written as
\[
\hat{\beta}^M = \hat{\Gamma}_A^M\hat{\alpha}^M_{0,A} = (X^TX)^{-1}X^TM_A\hat{\alpha}^M_{0,A}. \tag{2.9}
\]
By the definition of \( \hat{\gamma} \) and \( \hat{\alpha}_1 \), it follows that
\[
\hat{\beta} = \hat{\gamma} - \hat{\alpha}_1 = (X^TX)^{-1}X^Ty - (X^TX)^{-1}X^T(y - M_0\hat{\alpha}_0) = (X^TX)^{-1}X^TM\hat{\alpha}_0. \tag{2.10}
\]
Recall that Theorem 1 indicates that with probability tending to 1, \( \hat{\alpha}_{0,A^c} = 0 \), and hence
\[
\hat{\beta} = (X^TX)^{-1}X^TM\hat{\alpha}_{0,A^c}. \tag{2.11}
\]
Note that \( \hat{\alpha}_{0,A} \) and \( \hat{\alpha}^M_{0,A} \) have the same asymptotic distribution. Consequently, \( \hat{\beta} \) has the same asymptotic distribution as \( \hat{\beta}^M \). Thus it is asymptotically efficient.

### 2.3 Test for indirect effect

To form the test statistic for the indirect effect \( \beta \), we first study its asymptotic variance matrix. Let \( \hat{A} = \{ j : \hat{\alpha}_{0j} \neq 0 \} \). With probability tending to 1, we have \( \hat{A} = A \). Then the variance matrix \( \Sigma \) and \( \sigma^2 \) can be estimated by the estimated sample version and the mean squared errors, respectively.
\[
\hat{\Sigma} = \frac{1}{n} \left( \begin{array}{cc} X^TX & X^TM_A \end{array} \right), \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n-s-q} \| y - M\hat{\alpha}_0 - X\hat{\alpha}_1 \|^2,
\]
where \( s = |\hat{A}|. \) As is shown, \( \hat{\sigma}^2 = \sigma^2 + o_P(1) \). In fact, when \( s = o(n^{1/2}) \), we have \( \hat{\sigma}^2 = \sigma^2 + O_P(n^{-1/2}) \). Alternatively, we can estimate \( \sigma^2 \) using refitted cross-validation \((\text{Fan et al. } 2012)\) or the scaled lasso \((\text{Sun & Zhang } 2013)\).

As to \( \sigma^2_2 \), we first estimate \( \sigma^2 = \text{var}(\varepsilon) = \sigma^2_1 + \sigma^2_2 \) by the classic least squares residual variance estimator \( \hat{\sigma}^2 \) based on model (2.3). Thus \( \hat{\sigma}^2_2 = \hat{\sigma}^2 - \hat{\sigma}^2_1 \). In practice, \( \hat{\sigma}^2_1 \) may sometimes be larger than \( \hat{\sigma}^2_2 \), where we would simply set \( \hat{\sigma}^2_2 = 0 \). This is possible when no mediators are relevant. That is, \( \alpha_0 = 0 \), and hence \( \sigma^2_2 \) indeed equals zero.

According to Corollary 1, the asymptotic variance matrices of \( \hat{\alpha}_1 \) and \( \hat{\beta} \) can be consistently estimated by:
\[
\hat{\sigma}^2_1(I_q, 0_{q \times \hat{s}})^T; \hat{\sigma}^2_2 \hat{\Sigma}_X^{-1} + \hat{\sigma}^2_1[(I_q, 0_{q \times \hat{s}})^T \hat{\Sigma}_X^{-1}) ] - \hat{\Sigma}_X^{-1}, \tag{2.12}
\]
where $\hat{\Sigma}_{XX} = X^T X / n$. Then Wald test statistic for the hypotheses in (2.4) can be derived as

$$S_n = n \hat{\beta}^T \left\{ \hat{\sigma}^2 \hat{\Sigma}^{-1}_{XX} + \hat{\sigma}_1^2 [(I_q, 0_{q \times \delta}) \hat{\Sigma}^{-1}(I_q, 0_{q \times \delta})^T - \hat{\Sigma}^{-1}_{XX}] \right\}^{-1} \hat{\beta}.$$  

Clearly, under $H_0$, $S_n \to \chi^2_q$, a chi-square random variable with $q$ degrees of freedom.

To investigate the local power of $S_n$, we consider the local alternative hypotheses $H_{1n}: \beta = \Delta / \sqrt{n}$, where $\Delta$ is a constant vector. From Corollary 1, under such local alternative hypotheses, $S_n \to \chi^2_q(\Delta^T (\sigma^2 \Sigma^{-1}_{XX} + \sigma_1^2 B)^{-1} \Delta)$, a chi-square random variable with $q$ degrees of freedom and noncentrality parameter $\Delta^T (\sigma^2 \Sigma^{-1}_{XX} + \sigma_1^2 B)^{-1} \Delta$. Thus, $S_n$ can detect local effects that converge to 0 at root-$n$ rate.

### 2.4 $F$-type Test on direct effect

It is of interest to test the following hypothesis

$$H_{02}: \alpha_1 = 0 \text{ versus } H_{12}: \alpha_1 \neq 0. \quad (2.13)$$

(2.1) and (2.2) are called complete or full mediation models under $H_{02}$, while incomplete or partial mediation models under $H_{12}$.

Testing the hypothesis in (2.13) essentially is to test low dimensional regression coefficients in linear regression model (2.1). This has been studied when the covariates in (2.1) are fixed design (Zhang & Zhang, 2014; van de Geer et al., 2014; Shi et al., 2019). Due to the nature of mediation model, the covariates in (2.1) are random design. The fixed-design assumption on $m$ is inappropriate in mediation models.

We will propose an $F$-type test for (2.13), and further show that the proposed $F$-test asymptotically has a chi-square distribution with $q$ degrees of freedom under $H_{02}$, and a noncentral chi-square distribution with $q$ degrees of freedom under $H_{12}$. Similar to $F$-test, we need to calculate the residual sum of squares (RSS) under the null and alternative hypotheses. Under $H_{02}$, the penalized least squares function for model (2.1) becomes

$$\frac{1}{2n} \| y - M \hat{\alpha}_0 \|^2 + \sum_{j=1}^p p_X(|\alpha_{0j}|). \quad (2.14)$$

Denote by $\hat{\alpha}_0$ the resulting penalized least squares estimator. Then the RSS under $H_{02}$ is $RSS_0 = \| y - M \hat{\alpha}_0 \|^2$. Under $H_{12}$, we can estimate $\hat{\alpha}_0$ and $\alpha_1$ by the partial penalized least squares method in (2.7). Then we calculate $RSS_1 = \| y - M \hat{\alpha}_0 - X \hat{\alpha}_1 \|^2$, the RSS under $H_{12}$.

The $F$-type test for hypothesis (2.13) is defined to be

$$T_n = \frac{(RSS_0 - RSS_1)}{RSS_1/(n-q)}. \quad (2.15)$$
Theorem 2 below shows that the asymptotical null distribution of $T_n$ is a chi-square distribution with $q$ degrees of freedom. To evaluate the local power of $T_n$ under local alternative hypotheses, we impose the following assumption.

A4. Consider local alternative hypotheses $H_{1n} : \alpha_1 = h_n$. Assume that $\|h_n\|_2 = O(\sqrt{1/n})$.

**Theorem 2** Suppose that Conditions (A1)-(A4) hold, and $s = o(n^{1/3})$. It follows that

$$\sup_x |P(T_n \leq x) - P(\chi_q^2(nh_n^T\Phi^{-1}h_n/\sigma_1^2) \leq x)| \to 0. \quad (2.16)$$

Here $\Phi = (I_q, 0_{q \times s})\Sigma^{-1}(I_q, 0_{q \times s})^T$ and $\chi_q^2(nh_n^T\Phi^{-1}h_n/\sigma_1^2)$ is a chi square random variable with $q$ degrees of freedom and noncentrality parameter $nh_n^T\Phi^{-1}h_n/\sigma_1^2$.

Theorem 2 implies that under $H_{02}$, $T_n$ asymptotically follows $\chi_q^2$ distribution, which does not depend on any parameter in the model. This is similar to the Wilks phenomenon for likelihood ratio test in classical statistical setting. In other words, the Wilks phenomenon still holds in this high dimensional mediation model. Theorem 2 also implies that $T_n$ can detect local alternatives that are distinct from the null hypothesis at the rate of $1/\sqrt{n}$.

### 2.5 Algorithm and tuning parameter selection

To compute the partial penalized estimators $\hat{\alpha}_1$ and $\hat{\beta}$, we apply the local linear approximation algorithm (LLA) in Zou & Li (2008) with the SCAD penalty in Fan & Li (2001),

$$p_\lambda'(t) = \lambda I(t \leq \lambda) + \frac{(a\lambda - t)_+}{(a - 1)\lambda} I(t > \lambda),$$

and set $a = 3.7$. The tuning parameter $\lambda$ for our method is chosen based on the high-dimensional BIC (HBIC) method in Wang et al. (2013). For a fixed regularization parameter $\lambda$, define

$$(\hat{\alpha}_0^\lambda, \hat{\alpha}_1^\lambda) = \min_{\alpha_0, \alpha_1} \frac{1}{2n}\|y - M\alpha_0 - X\alpha_1\|_2^2 + \sum_{j=1}^p p_\lambda(|\alpha_{0,j}|).$$

The minimization of the partial penalized least squares method can be carried out as follows.

1. Get initial values for $\alpha_0^{(0)}, \alpha_1^{(0)}$ by minimizing a partial $L_1$-penalized least squares: $(\hat{\alpha}_0^{(0)}, \hat{\alpha}_1^{(0)}) = \min_{\alpha_0, \alpha_1} \frac{1}{2n}\|y - M\alpha_0 - X\alpha_1\|_2^2 + \lambda \sum_{j=1}^p |\alpha_{0,j}|.$

2. Solve $(\alpha_0^{(k+1)}, \alpha_1^{(k+1)}) = \min_{\alpha_0, \alpha_1} \frac{1}{2n}\|y - M\alpha_0 - X\alpha_1\|_2^2 + \sum_{j=1}^p p_\lambda(|\alpha_{0,j}^{(k)}|)|\alpha_{0,j}|$ for $k = 1, 2, \cdots$, until $\{(\alpha_0^{(k)}, \alpha_1^{(k)})\}$ converges.
In practice, we use a data-driven method to choose the tuning parameter $\lambda$. Following Wang et al. (2013), we use the HBIC criterion to choose $\lambda$. The HBIC score is defined as $\text{HBIC}(\lambda) = \log(\|y - M\alpha_0 - X\alpha_1\|_2^2) + \text{df} \log(\log(n)) \log(p+q)/n$, where df is the number of variables with nonzero coefficients in $(\alpha_0^T, \alpha_1^T)^T$. Minimizing $\text{HBIC}(\lambda)$ yields a selection of $\lambda$.

3 Numerical studies

In this section, we examine the finite sample performance of the proposed procedures via Monte Carlo simulation studies and illustrate the proposed procedure by a real data example.

3.1 Simulation studies

We first examine finite sample performances of the proposed partial-penalization based test statistics, along with comparisons with the oracle test statistics which know the true set $A = \{j : \alpha_{0j}^* \neq 0\}$, denoted as $S^O_n$ and $T^O_n$ as a benchmark, and the debiased test statistics $S^Z_n$ and $T^Z_n$ in Zhou et al. (2020), denoted by Zhou et al.’s method in the tables and figures in this section. Note that Zhou et al. (2020) focuses on the test of indirect effects. One can derive a valid Wald test for direct effects based on the asymptotical normality established in their paper.

Example 1. In this example, we set $n = 300$, $q = 1$, and $p = 500$. $x \sim N(0, 1)$ and $m = \Gamma^T x + \varepsilon$, where $\varepsilon \sim N(0, \Sigma^*)$ with $\Sigma^*$ being an AR correlation structure. That is, the $(i,j)$-element of $\Sigma^*$ equals $\rho|i-j|$ and $\rho$ is set to be 0.5. Take $\Gamma = c_1(\tau_1, \cdots, \tau_p)^T$, where $\tau_k = 0.2k$ for $k = 1, \cdots, 5$, and when $k > 5$, $\tau_k$’s are independently generated from $N(0, 0.1^2)$. Set $c_1 = 0$ to examine Type I error rate and $c_1 = \pm 0.1, \pm 0.2, \cdots, \pm 1$ for power when testing the indirect effects.

We generate the response $y$ from model $y = \alpha_0^T m + \alpha_1^T x + \varepsilon_1$, where $\varepsilon_1 \sim N(0, 0.5^2)$, $\alpha_0 = [1, 0.8, 0.6, 0.4, 0.2, 0, \cdots, 0]^T$ and $\alpha_1 = c_2$ is set in the same fashion as $c_1$. The simulation results are based on 500 replications. The significance level is set to be 0.05.

We first compare the performances of $S_n$, $S^O_n$ and $S^Z_n$ for testing the indirect effect $\beta$. We set $c_2 = 0.5$ and $\beta = \Gamma\alpha_0 = 1.4c_1$. The left panel of Figure 1 depicts power functions of the three tests versus the values of $c_1$ over $[-0.3, 0.3]$. All the three tests gain larger powers as $|c_1|$ increases. $S_n$ performs as well as the oracle $S^O_n$, and is generally more powerful than $S^Z_n$. For instance, when $c_1 = -0.2$, the empirical power of $S^Z_n$ is 0.516, while the empirical powers of $S_n$ and $S^O_n$ are 0.596. These observations are in consistent with the theoretical results in Section 2.
Figure 1: Left panel is the empirical sizes and powers of $S_n$, $S_Z^n$, and $S_O^n$ at level $\alpha = 0.05$ over 500 replications for testing indirect effect when $\alpha_1 = 0.5$. Solid line, dotted line and solid line marked by ‘*’ represent the sizes and powers of $S_n$, $S_O^n$, and $S_Z^n$, respectively. Right panel is empirical sizes and powers of $T_n$, $T_Z^n$, and $T_O^n$ at level $\alpha = 0.05$ over 500 replications for testing direct effect when $\beta = 0.7$. The solid line, dotted line, and solid line marked by ‘*’ represent the sizes and powers of $T_n$, $T_O^n$, and $T_Z^n$, respectively.

Next, we turn to test the direct effect. Set $c_1 = 0.5$. And $c_2$ is taken from $0, \pm 0.1, \pm 0.2, \cdots, \pm 1$, where $c_2 = 0$ corresponds to the null hypothesis. The right panel of Figure 1 depicts the power function of the three tests versus the values of $c_2$ over $[-0.3, 0.3]$. The proposed test $T_n$ performs almost the same as the oracle one, and is obviously more powerful than the test $T_Z^n$ proposed in Zhou et al. (2020), whose power curve is asymmetric. In fact, when $c_2 = -0.2$, the empirical powers of our test statistic $T_n$ and the oracle test $T_O^n$ are about 1, while that of $T_Z^n$ is only about 0.780.

Furthermore, $T_Z^n$ performs unstably according to our simulation studies. To gain insight of this, we explore more on $\hat{\alpha}_1^Z, \hat{\beta}^Z$. The estimates $\hat{\alpha}_1, \hat{\beta}$ and $\hat{\alpha}_O, \hat{\beta}_O$ are reported in Table 1 from which it can be seen that the biases of $\hat{\alpha}_1, \hat{\beta}$ and $\hat{\alpha}_O, \hat{\beta}_O$ are very small, while $\hat{\alpha}_1^Z$ has a large bias. This may be due to that the direct effect $\alpha_1$ is also penalized in Zhou et al. (2020)’s estimation procedure based on scaled lasso. This makes sense only if the direct effect is expected to be zero. As seen in Table 1 the bias of $\hat{\alpha}_1^Z$ is very small when $c_2 = 0$, yet inversely when $c_2 \neq 0$. Table 1 also reports standard errors of corresponding estimates. Both the proposed method and oracle outperform Zhou et al. (2020), especially when estimating $\alpha_1$.

To assess the accuracy of variance estimation of $\hat{\alpha}_1$ and $\hat{\beta}$, Table 2 reports their estimated standard errors in two ways. As to each method - new, oracle and Zhou et al.’s method, the first
Table 1: Estimated biases and standard deviations (in parentheses) of different methods with different $c_1$ and $c_2$. Except for $c_1$ and $c_2$, the values in this table equals 100 times of the actual ones.

| $c_1$ | $c_2$ | New method $\tilde{\alpha}_1$ | $\tilde{\beta}$ | Oracle $\alpha^o_1$ | $\beta^o$ | Zhou et al.’s method $\tilde{\alpha}_1^Z$ | $\tilde{\beta}^Z$ |
|-------|-------|-------------------------------|-----------------|----------------------|----------|----------------------------------------|-----------------|
| -0.8  | 0.5   | -0.23 (4.15)                 | -0.22 (3.73)    | -0.11 (4.11)         | -0.35 (13.70) | -11.7 (5.56)                          | 11.31 (14.05)   |
| -0.4  | 0.5   | 0.18 (3.13)                  | -0.33 (11.98)   | 0.25 (3.68)          | -0.40 (11.95) | -3.49 (5.10)                          | 3.37 (12.20)    |
| 0     | 0.5   | -0.02 (2.99)                 | 0.39 (12.61)    | -0.00 (2.99)         | 0.37 (12.63)  | -0.13 (8.65)                          | 0.47 (15.00)    |
| 0.4   | 0.5   | 0.02 (3.15)                  | 0.08 (11.83)    | -0.02 (3.11)         | 0.12 (11.81)  | -0.60 (5.31)                          | 0.77 (12.66)    |
| 0.8   | 0.5   | 0.31 (3.79)                  | 0.26 (12.69)    | 0.16 (3.72)          | 0.42 (12.63)  | -1.57 (8.57)                          | 2.19 (15.05)    |
| 0.5   | -0.8  | 0.16 (3.38)                  | 0.79 (11.62)    | 0.11 (3.37)          | 0.85 (11.64)  | 16.37 (5.61)                          | -7.63 (13.13)   |
| 0.5   | -0.4  | -0.01 (3.43)                 | 0.16 (12.58)    | -0.09 (3.36)         | 0.26 (12.57)  | 16.05 (4.00)                          | -8.08 (13.64)   |
| 0.5   | 0     | 0.10 (3.35)                  | -0.15 (12.52)   | 0.01 (3.33)          | -0.06 (12.52) | 0.66 (5.66)                           | -0.71 (13.82)   |
| 0.5   | 0.4   | 0.35 (3.39)                  | 0.01 (12.26)    | 0.32 (3.37)          | 0.04 (12.26)  | -0.96 (5.69)                          | 1.30 (13.10)    |
| 0.5   | 0.8   | 0.13 (3.29)                  | 0.24 (12.10)    | 0.05 (3.26)          | 0.32 (12.17)  | -0.53 (5.58)                          | 0.84 (12.86)    |

The column lists the empirical standard deviations of point estimates $\tilde{\alpha}_1$ or $\tilde{\beta}$ over 500 replications (they are also recorded in parentheses of Table 1); for the second column, we estimate standard errors of $\hat{\alpha}_1$ and $\hat{\beta}$ using formula (2.12) in each simulation run, and reports the average together with standard deviations (in parentheses) over the 500 runs. Note that the R package “freebird” [Zhou et al. (2020)] does not provide the estimated standard error of $\hat{\alpha}_1$. From Table 2, for the new method and oracle, the standard errors estimated by Monte Carlo simulations are close to those calculated from formulas; while the two versions of [Zhou et al. (2020)] depart more.

Table 2: Estimated standard deviations and average estimated standard errors with their standard deviations (in parentheses) over 500 replications with different $c_1$ and $c_2$. Except for $c_1$ and $c_2$, the values in this table equals 100 times of the actual ones.

| Direct effect ($\hat{\alpha}_1$) | Indirect Effect ($\hat{\beta}$) |
|----------------------------------|---------------------------------|
| $c_1$ | $c_2$ | New method | Oracle | New method | Oracle | Zhou et al.’s method |
|------|------|------------|--------|------------|--------|----------------------|
|      |      | std se(se) |        | std se(se) |        | std se(se)           |
| -0.8 | 0.5  | 4.15       | 3.85 (0.23) | 4.11 | 3.89 (0.23) | 13.73 | 12.56 (0.72) | 13.70 | 12.56 (0.72) | 14.05 | 13.43 (1.03) |
| -0.4 | 0.5  | 3.13       | 3.10 (0.18) | 3.08 | 3.17 (0.18) | 11.98 | 12.38 (0.73) | 11.95 | 12.38 (0.73) | 12.20 | 13.14 (0.85) |
| 0    | 0.5  | 2.99       | 2.90 (0.17) | 2.99 | 2.91 (0.17) | 12.61 | 12.26 (0.66) | 12.63 | 12.26 (0.66) | 15.00 | 13.12 (2.62) |
| 0.4  | 0.5  | 3.15       | 3.18 (0.18) | 3.11 | 3.19 (0.18) | 11.83 | 12.35 (0.71) | 11.81 | 12.35 (0.71) | 12.66 | 13.09 (0.82) |
| 0.8  | 0.5  | 3.79       | 3.88 (0.23) | 3.72 | 3.88 (0.23) | 12.69 | 12.47 (0.73) | 12.63 | 12.47 (0.73) | 15.05 | 13.37 (1.79) |
| 0.5  | -0.8 | 3.38       | 3.31 (0.19) | 3.37 | 3.32 (0.19) | 11.62 | 12.43 (0.71) | 11.64 | 12.42 (0.71) | 13.13 | 14.30 (0.76) |
| 0.5  | -0.4 | 3.43       | 3.30 (0.19) | 3.36 | 3.31 (0.20) | 12.58 | 12.30 (0.70) | 12.57 | 12.30 (0.70) | 13.64 | 13.19 (0.71) |
| 0.5  | 0    | 3.35       | 3.32 (0.18) | 3.33 | 3.33 (0.18) | 12.52 | 12.35 (0.75) | 12.52 | 12.34 (0.75) | 13.82 | 13.78 (3.73) |
| 0.5  | 0.4  | 3.39       | 3.32 (0.19) | 3.37 | 3.33 (0.19) | 12.26 | 12.39 (0.71) | 12.26 | 12.39 (0.71) | 13.10 | 13.14 (0.75) |
| 0.5  | 0.8  | 3.29       | 3.33 (0.20) | 3.26 | 3.34 (0.20) | 12.10 | 12.37 (0.74) | 12.17 | 12.37 (0.74) | 12.86 | 13.27 (1.31) |

Furthermore, Figure 2 visually compares the standard deviations of $\hat{\beta}$ over 500 point estimates using the new method (x-axis) with those using oracle or Zhou et al.’s method (y-axis), respectively. Each blue diamond or red dot in the figure corresponds to each of the 21 different simulation
Table 3: Comparison results of the average computing time (in seconds) over 500 replications.

| $c_1$ | $c_2$ | New method | Zhou et al.’s method |
|-------|-------|------------|----------------------|
| -0.8  | 0.5   | 1.38       | 1,207.88             |
| -0.4  | 0.5   | 1.47       | 1,327.82             |
| 0     | 0.5   | 1.31       | 1,197.66             |
| 0.4   | 0.5   | 1.52       | 1,614.84             |
| 0.8   | 0.5   | 1.22       | 1,332.24             |
| 0.5   | -0.8  | 1.35       | 1,192.32             |
| 0.5   | -0.4  | 1.33       | 1,329.48             |
| 0.5   | 0     | 1.48       | 1,544.23             |
| 0.5   | 0.4   | 1.50       | 1,790.34             |

settings - when holding $c_2 = 0.5$, vary $c_1 = 0, \pm 0.1, \cdots, \pm 1$ in (a) and holding $c_1 = 0.5$, vary $c_2 = 0, \pm 0.1, \cdots, \pm 1$ in (b). The figures imply that the estimated standard errors of the new method are close to oracle, and are generally smaller than those of Zhou et al.’s method. This in turn intuitively illustrates the precision of proposed estimators.

Lastly, Table 3 reports the computing times, where the new method is nearly 1000 times faster than Zhou et al.’s method. The proposed method is very fast and stable because initialized by LASSO estimator, LLA algorithm converges in one step.

Example 2. In this example, we examine the finite sample performances of proposed method when heavy-tail errors are encountered. Specifically, assume now $\varepsilon_1 \sim t_6/\sqrt{6}$. The multiplier $\sqrt{6}$ ensures the equality of variance of $\varepsilon_1$ to that when $\varepsilon_1 \sim N(0, 0.5^2)$. All other settings are identical.
to those in Example 1. We first investigate the performances of $S_n, S_n^O$ and $S_n^Z$ for testing indirect effect $\beta$ via the left panel of Figure 3. The proposed test $S_n$ performs as well as the oracle one $S_n^O$ in terms of controlling Type-I error rate ($c_1 = 0$) and possessing much larger power than $S_n^Z$ (when $c_1 \neq 0$), especially when $c_1 < 0$. Similar phenomena are observed in the right panel of Figure 3 when examining $T_n, T_n^O$ and $T_n^Z$. The proposed test $T_n$ performs as well as the oracle one, and is more powerful than the test $T_n^Z$. In fact, when $c_2 = -0.2$, the empirical powers of our test statistic $T_n$ and the oracle test $T_n^O$ are about 1, while that of $T_n^Z$ is only about 0.756. In addition, we also evaluate the accuracy and precision of $\hat{\alpha}_1$ and $\hat{\beta}$ through Tables 4 and 5. The overall pattern in these two tables with $\varepsilon_1 \sim t_6/\sqrt{6}$ is very similar to that for $\varepsilon_1 \sim N(0, 0.5^2)$. In sum, the proposed method retains its validity for heavy-tailed error distributions.

Table 4: Estimated biases and standard deviations (in parentheses) of different methods with different $c_1$ and $c_2$ when $\varepsilon_1 \sim t_6/\sqrt{6}$. Except for $c_1$ and $c_2$, the values in this table equals 100 times of the actual ones.

| $c_1$ | $c_2$ | $\hat{\alpha}_1$ | $\hat{\beta}$ | $\hat{\alpha}_1^O$ | $\hat{\beta}^O$ | $\hat{\alpha}_1^Z$ | $\hat{\beta}^Z$ |
|-------|-------|------------------|--------------|----------------|---------------|------------------|--------------|
| -0.8  | 0.5   | 0.14(4.06)       | -0.30(12.46) | 0.22(3.93)     | -0.38(12.43)  | -13.93(6.09)    | 13.50(12.84)  |
| -0.4  | 0.5   | 0.01(1.93)       | -0.14(6.24)  | 0.06(1.89)     | -0.19(6.23)   | -3.34(2.81)     | 3.23(6.43)    |
| 0     | 0.5   | 0.16(3.03)       | -0.36(12.21) | 0.14(3.01)     | -0.34(12.21)  | -1.13(4.68)     | 0.86(12.74)   |
| 0.4   | 0.5   | 0.16(3.29)       | -0.36(12.30) | 0.09(3.26)     | -0.28(12.29)  | -0.77(5.19)     | 0.52(13.01)   |
| 0.8   | 0.5   | 0.28(3.07)       | -0.26(6.67)  | 0.21(3.02)     | -0.18(6.63)   | 0.75(4.06)      | -0.70(7.15)   |
| 0.5   | -0.8  | 0.19(3.44)       | -0.37(12.34) | 0.10(3.40)     | -0.28(12.33)  | 6.50(5.61)      | -6.73(12.89)  |
| 0.5   | -0.4  | 0.16(3.45)       | -0.32(12.22) | 0.09(3.41)     | -0.25(12.20)  | 5.92(12.67)     | -6.16(16.26)  |
| 0.5   | 0     | 0.19(3.42)       | -0.34(12.34) | 0.09(3.41)     | -0.25(12.30)  | 0.70(4.56)      | -0.95(12.95)  |
| 0.5   | 0.4   | 0.20(3.44)       | -0.39(12.39) | 0.09(3.41)     | -0.28(12.33)  | -1.20(5.30)     | 0.93(12.98)   |
| 0.5   | 0.8   | 0.18(3.44)       | -0.34(12.32) | 0.09(3.41)     | -0.25(12.30)  | -1.17(5.29)     | 0.96(13.07)   |

### 3.2 Real data analysis

We apply the proposed method to an empirical analysis to examine whether financial statements items and metrics mediate the relationship between company sectors and stock price recovery after COVID-19 pandemic outbreak. While investors and researchers have reached a consensus ages ago that stock returns highly rely on companies’ belonging sectors, recent studies more focus on using financial statements or market conditions to predict stock returns. Fama & French (1993)’s pioneering proposal of the three-factor model started this era, which captures patterns of return using market return, firm size and book-to-market ratio factors. Callen & Segal (2004) showed that accruals, cash flow, growth in operating income significantly influence stocks return. Edirisinghe
Table 5: Estimated standard deviations and average estimated standard errors with their standard deviations (in parentheses) of different methods with different $c_1$ and $c_2$ when $\varepsilon_1 \sim t_6/\sqrt{6}$. Except for $c_1$ and $c_2$, the values in this table equals 100 times of the actual ones.

|                | Direct effect ($\hat{\alpha}$) | Indirect Effect ($\hat{\beta}$) |
|----------------|---------------------------------|---------------------------------|
|                | New method | Oracle | New method | Oracle | Zhou et al.’s method |
| $c_1$ | $c_2$ | std     | se(std)    | std     | se(std)    | std     | se(std)    | std     | se(std)    |
| -0.8          | 0.5        | 4.06    | 3.87(0.26) | 3.93    | 3.88(0.27) | 12.46   | 12.55(0.70) | 12.43   | 12.55(0.70) |
| -0.4          | 0.5        | 1.93    | 1.94(0.14) | 1.89    | 1.95(0.14) | 6.24    | 6.29(0.33)  | 6.23    | 6.29(0.33)  |
| 0             | 0.5        | 3.03    | 2.91(0.21) | 3.01    | 2.92(0.21) | 12.21   | 12.29(0.72) | 12.21   | 12.29(0.72) |
| 0.4           | 0.5        | 3.29    | 3.17(0.23) | 3.26    | 3.18(0.23) | 12.30   | 12.35(0.72) | 12.29   | 12.35(0.72) |
| 0.8           | 0.5        | 3.07    | 2.92(0.22) | 3.02    | 2.93(0.22) | 6.67    | 6.66(0.30)  | 6.63    | 6.66(0.30)  |
|                | 0.5        | -0.8    | 3.44    | 3.31(0.24) | 3.40    | 3.32(0.24) | 12.34   | 12.39(0.71) | 12.33   | 12.39(0.71) |
|                | 0.5        | -0.4    | 3.45    | 3.31(0.24) | 3.41    | 3.32(0.24) | 12.32   | 12.39(0.71) | 12.30   | 12.39(0.71) |
|                | 0.5        | 0       | 3.42    | 3.31(0.24) | 3.41    | 3.32(0.24) | 12.34   | 12.39(0.71) | 12.30   | 12.39(0.71) |
|                | 0.5        | 0.4     | 3.44    | 3.31(0.24) | 3.41    | 3.32(0.24) | 12.39   | 12.39(0.71) | 12.33   | 12.39(0.71) |
|                | 0.5        | 0.8     | 3.44    | 3.31(0.24) | 3.41    | 3.32(0.24) | 12.32   | 12.39(0.71) | 12.30   | 12.39(0.71) |

& Zhang (2007, 2008) developed a relative financial strength metric based on data envelopment analysis (Farrell, 1957; Charnes et al., 1978), and found that return on assets and solvency ratio has high correlation with stock price return. To enhance prediction accuracy, deep neural network and data mining techniques were developed, with model inputs as historical financial statements and output as stock price return (Enke & Thawornwong, 2005; Huang et al., 2019; Lee et al., 2019). Meanwhile, it is reasonable to hypothesize that companies’ sectors affect stock performances via influencing the associated financial metrics. Few existing works, however, study the mediating effects of such financial metrics. Hence our analysis aims to fill in this gap, and use the proposed mediation analysis to select important financial metrics, as well as to test the direct and indirect effects of companies’ sectors on returns.

In addition, we in this analysis are specifically interested in the stock performance of S&P 500 component companies during the COVID-19 pandemic period. As is known, the outbreak of the COVID-19 dealt a shock to the U.S. economy with unprecedented speed, and the government had to take a lockdown to stop spread of virus. The lockdown took a toll in the U.S. economy: business were closed, millions of people lost jobs and the price of an oil futures contract fell below zero. The crisis spread to the U.S. stock market, dragging down the major index S&P 500 by 33.92%. To help businesses, households and the economy, the Federal Reserve and the White House launched various rescue programs and take measures to stabilize energy prices from the end of March, 2020. Therefore, all these events and measures led the U.S. stock market to a V-shape pattern, thanks to which, the general financial rules from classical literature may not directly apply any more. Admittedly, a number of recent literature studied the economic reaction to COVID-19 pandemic from sector or company level data (Ramelli & Wagner, 2020; Zhang et al., 2020; Baker, 2021).
Empirical size and powers

Figure 3: Left panel is empirical sizes and powers of $S_n$, $S^Z_n$ and $S^O_n$ when $\varepsilon_1 \sim t_{\alpha}/\sqrt{6}$ at level $\alpha = 0.05$ over 500 replications for testing indirect effect when $\alpha_1 = 0.5$. Dotted line, solid line, and solid line marked by ‘*’ represent the sizes and powers of $S_n$, $S^O_n$ and $S^Z_n$, respectively. Right panel is empirical sizes and powers of $T_n$, $T^Z_n$ and $T^O_n$ for testing direct effect when $\beta = 0.7$. The dotted line, solid line, and solid line marked by ‘*’ represent the sizes and powers of $T_n$, $T^O_n$ and $T^Z_n$, respectively.

et al., 2020; Gormsen & Koijen, 2020; De Vito & Gómez, 2020). Thorbecke (2020) analyzed sector-specific and macroeconomic variables as contributing factors to stock return in COVID-19 downturn and found that idiosyncratic factors negatively affected energy and consumer cyclical sectors. Hassan et al. (2020) investigated companies’ transcripts of quarterly earnings call from January to September 2020 to investigate senior management’s and major market participants’ opinions about future prospects. They discovered several important factors related to accounting and business fundamentals, including supply chain, production and operations and financing, that are highly associated with stock market recovery from COVID-19. However, these methods mainly rely on prior financial knowledge to select low dimensional data for modeling, while ignore important company level factors. Besides, these methods only consider the relation of stock return to either sector level or company level while failing to recognize that the company’s financial plays a role in mediating stock sector effects to stock price return. Therefore, we use the proposed method to study the financial statement items or metrics that mediate the relationship between firm sectors and stock performance in this special period. This work may then shed light on how to select valuable stocks during a pandemic or any adverse event likewise.

In the mediation models, the response is taken to be the stock return from its highest price before the pandemic in February, 2020 to April 30th, 2020. The closed price is adjusted for both dividends and splits. The potential mediators in $m$ are 550 accounting metrics from financial
statements of associated companies, scratched from Yahoo Finance on April 30, 2020. We obtain firms’ annual reports from fiscal year 2015 to 2019 and the first three quarterly reports in 2019. We use the firms’ latest annual report to compute financial metrics and use previous annual reports to compute average growth rate of each financial metrics. The exposure variables in \( x \), are companies’ sectors according to Global Industry Classification Standard (GICS) that are coded as dummy variables. GICS classifies companies into eleven sectors: basic materials, communication services, consumer cyclical, consumer defensive, energy, financial services, healthcare, industrials, real estate, technology and utilities. We set energy sector as baseline level.

Table 6 presents the estimated direct and indirect effects of companies’ sectors, together with their standard errors. We also calculate Wald’s test for the indirect effect and generalized likelihood test for direct effect, with \( p \)-values smaller than \( 1 \times 10^{-9} \) and \( 10^{-15} \), respectively, indicating both the direct and indirect effect are significant. As for direct effect, stocks in sectors such as healthcare and technology are more likely to outperform benchmark than ones from utilities sector. Furthermore, sectors influence the stocks performance partly through business operation reflected by selected financial metrics, and the indirect effects are significantly positive.

The selected mediating metrics, their associated estimated coefficients in model (2.1), as well
as their brief descriptions, are presented in Table 7. These selected metrics are of their own significance. For instance, the first three chosen metrics in Table 7, namely return on assets, gross margin and annual growth rate of operating income, reflect firms’ revenue. Return on assets is an indicator of how well a firm utilizes its assets, by determining how profitable a firm is relative to its total assets. A firm with a higher return-on-assets value is preferred, as the firm squeezes more out of limited resources to make a profit. Gross margin is the portion of sales revenue a firm retains after subtracting costs of producing the goods it sells and the services it provides. It measures the gross profit of a firm. A firm that has higher gross margin is more likely to retain more profit for every dollar of good sold. Annual growth rate of operating income shows the firm’s growth of generating operating income compared with previous year. Operating income measures the amount of profit realized from a business’s operation, after deducting operating expenses such as wages, depreciation, and cost of goods sold. A firm with high growth of operating income can avoids unnecessary production costs, and improve core business efficiency. In a word, a firm with higher return on assets, gross margin and growing operating income is considered profitable, and hence, is likely to attract investors.

On the other hand, both the average growth rate of quick ratio and debt to assets are indicators of financial leverage of a firm. Quick ratio of a firm is defined as the dollar amount of liquid assets dividing that of current liabilities, where liquid assets are the portion of assets that can be quickly converted into cash with minimal impact on the price received in open market, while current liabilities are a firm’s debts or obligations to be paid to creditors within one year. Thus a large quick ratio indicates that the firm is fully equipped with enough assets to be instantly liquidated to pay off its current liabilities. Debt to assets is the total amount of debt relative to assets owned by a firm. It reflects a firm’s financial stability. Therefore, a firm with a higher quick ratio or a lower debt to assets might be more likely to survive when it is difficult to finance through borrowing and cover its debts, thus are more favorable to investors during the economy lockdown.

Lastly, receivables turnover quantifies a firm’s effectiveness in collecting its receivables or money owed by clients. It shows how well a firm uses and manages the credit it extends to customers and how quickly that short-term debt is paid. Receivables turnover can be negative when net credit sale is negative because the client pre-pay for the product or service. A negative receivables turnover means that the firm are less susceptible to counter-party credit risk because it already receives the cash from its client before delivering the service or shipping out the product. This is especially important during liquidity dry periods when the clients may default or delay payment due to lack of cash. Therefore, a firm that has a negative receivables turnover is preferred.

On all accounts, one might incorporate the analysis results as reference when seeking for a
Table 7: Selected importance mediators and their coefficients

| Selected mediator                  | Estimated coefficient (std) | Description                                                                 |
|-----------------------------------|----------------------------|-----------------------------------------------------------------------------|
| Return on assets                  | 0.4246 (0.0379)            | Net income divided by the total assets                                       |
| Gross margin                      | 0.0841 (0.0393)            | The difference between the revenue and cost of goods sold divided by revenue |
| AGR* Operating Income             | 0.1063 (0.0347)            | Revenues subtract the cost of goods sold and operating expenses              |
| AGR* Quick ratio                  | 0.1194 (0.0345)            | Total current assets minus inventory divided by total current liabilities     |
| Debt to assets                    | -0.1209 (0.0369)           | Total debts divided by total assets                                          |
| Receivables turnover (days)       | -0.0947 (0.0346)           | Average receivables divided by net credit sales times 360 days               |

AGR: average growth rate, calculated as the average of growth rates for the metrics from 2015 to 2019.

stock portfolio during the financial crisis caused by pandemic. First, the sectors in ‘Healthcare’, ‘Consumer defensive’, ‘Communication service’, ‘Utility’ and ‘Technology’ have the top five positive direct effects on stock return. In terms of the financial metrics, we may focus on those reported in Table 7 to filter stocks. For example, we shall select firms that have higher values in AGR operating income, gross margin, quick ratio, and return on assets but lower values in debt to assets and receivable turnover.

Moreover, we compare our findings with those selected in established models. For instance, our method picks profitability factors like return on assets, which is also selected in Fama & French (2015), as profitability is the core of a firm’s stock performance. But we do not include metrics representing size of firm, valuation of stock price or investment that were covered by Fama & French (2015). For firm size factor, there is no evidence that small-size firms recovered faster or slower than larger-size ones. For valuation of stock price factor, previous price valuation ratio changed significantly due to stock price change and is no longer reliable to predict future stock return. For investment factor, it is less important for a short-term stock price movement. Compared with Edirisinghe & Zhang (2008), our method also picks profitability (return on assets), liquidity (quick ratio) and solvency (debt to assets) metrics, as in Edirisinghe & Zhang (2008). During the crisis, a firm facing liquidity crunch could not access to credit. Therefore, a firm with sufficient cash and less debt is more easily to survive and less likely to be forced to liquidate valuable assets
at unfavorable prices. And its stock would be safer and more attractive to investors. But we did not select metrics of earnings per share or about capital intensity as in Edirisinghe & Zhang (2008). The lockdown dramatically changes a firm’s revenue structure and capital allocation, and hence reduces predictive capability of these metrics to short-term recovery.

4 Conclusion

In this paper, we propose statistical inference procedures for the indirect effects in high dimensional mediation model. We introduce a partial penalized least squares method and study its statistical properties under random design. We show that the proposed estimators are more efficient than existing ones. We further propose a partial penalized Wald test to detect the indirect effect, with a $\chi^2$ limiting null distribution. In this paper, we also propose an $F$-type test for the direct effect and reveal Wilks phenomenon in the high-dimensional mediation model. We further utilize the proposed inference procedures to analyze the mediation effects of various financial metrics on the relationship between company’s sector and the stock return.

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Appendix

A.1 Proofs of Theorems

Define

$$Q_n(\theta) = \frac{1}{2n} \|y - M\alpha_0 - X\alpha_1\|_2^2 + \sum_{j=1}^p \lambda(\|\alpha_0,j\|).$$

**Proof of Theorem 1:** To enhance the readability, we divide the proof of Theorem 1 into three steps. In the first step, we show that there exists a local minimizer $\hat{\theta}$ of $Q_n(\theta)$ with the constraints
\( \alpha_{0,A^c} = 0 \), such that \( \| \theta - \theta_0 \|_2 = O_P(\sqrt{s/n}) \). In the second step, we prove that \( \theta \) is indeed a local minimizer of \( Q_n(\theta) \). This implies \( \hat{\theta} = \theta \). In the final step, we derive the asymptotic expansion of \( \hat{\theta} \).

**Step 1: Consistency in the \((s + q)\)-dimensional subspace:** We first constrain \( Q_n(\theta) \) on the \((s + q)\)-dimensional subspace of \( \{ \theta \in R^{p+q} : \alpha_{0,A^c} = 0 \} \). This constrained partial penalized least squares function is given by

\[
\bar{Q}_n(\vartheta) = \frac{1}{2n} \| y - M_A\delta - X\alpha_1 \|_2^2 + \sum_{j=1}^s p_\lambda(|\delta_j|).
\]

Here \( \vartheta = (\alpha_1^T, \delta^T)^T \) and \( \delta = (\delta_1, \ldots, \delta_s)^T \). We now show that there exists a strict local minimizer \( \bar{\vartheta} \) of \( \bar{Q}_n(\vartheta) \) such that \( \| \bar{\vartheta} - \vartheta_0 \|_2 = O_P(\sqrt{s/n}) \). To this end, we consider an event

\[
H_n = \{ \min_{\vartheta \in \partial N_\tau} \bar{Q}_n(\vartheta) > \bar{Q}_n(\vartheta_0) \}.
\]

where \( N_\tau = \{ \vartheta \in R^{s+q} : \| \vartheta - \vartheta_0 \|_2 \leq \tau \sqrt{s/n} \} \) with \( \tau \in (0, \infty) \), and \( \partial N_\tau \) denotes the boundary of the closed set \( N_\tau \). Clearly, on the event \( H_n \), there exists a local minimizer of \( \bar{Q}_n(\vartheta) \) in \( N_\tau \). Thus, we only need to show that \( P(H_n) \to 1 \) as \( n \to \infty \) when \( \tau \) is large. To this aim, we next analyze the function \( \bar{Q}_n \) on the boundary \( \partial N_\tau \).

For any \( \vartheta \), it follows from a second order Taylor’s expansion that

\[
\bar{Q}_n(\vartheta) - \bar{Q}_n(\vartheta_0) = -(\vartheta - \vartheta_0)^T \nu + \frac{1}{2}(\vartheta - \vartheta_0)^T D(\vartheta - \vartheta_0).
\]

(A.1)

Here

\[
\nu = \left( \frac{1}{n} X^T (y - M_A\alpha_0^* - X\alpha_1^*) \right),
\]

and

\[
D = \frac{1}{n} \left( X^T X \quad X^T M_A \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \Lambda(\alpha_0^*) \end{array} \right)
\]

where \( \alpha_0^* \) lies in the line segment jointing \( \delta \) and \( \alpha_0^* \), and \( \Lambda(\alpha_0^*) \) is a diagonal matrix with nonnegative diagonal elements. Clearly \( \alpha_0^* \in N_0 \). By condition (A2), the maximum eigenvalue of \( \Lambda(\alpha_0^*) \) is upper bounded by \( \lambda_n \kappa_0 \). Recall that

\[
\Sigma = \left( \begin{array}{cc} \Sigma_{XX} & \Sigma_{XM} \\ \Sigma_{MX} & \Sigma_{MM} \end{array} \right).
\]
Further note that

\[
P(||D_1 - \Sigma||_2 \geq \eta) \leq \frac{1}{\eta^2} E[||D_1 - \Sigma||_2^2] \leq \frac{cn}{\eta^2 n^2} E[\sum_{i,j} m_{1i} m_{1j} - E(m_{1i} m_{1j})]^2
\]

\[
+ \sum_{i=1}^{s} \sum_{j=1}^{q} [m_{1i} x_{1j} - E(m_{1i} x_{1j})]^2 + \sum_{i,j} [x_{1i} x_{1j} - E(x_{1i} x_{1j})]^2 = \frac{cs^2}{\eta^2 n}.
\]

Thus \( ||D_1 - \Sigma||_2 = O_P(s / \sqrt{n}) = o_P(1) \), when \( s = o(n^{1/2}) \).

Since \( \lambda_{\min}(\Sigma) \geq \bar{c} \) and \( \lambda_n \kappa_0 = o(1) \), we have:

\[
\lambda_{\min}(D) \geq \bar{c} > 0.
\] (A.2)

Consequently, we obtain

\[
\min_{\vartheta \in \partial N_r} Q_n(\vartheta) - Q_n(\vartheta_0) \geq \min_{\vartheta \in \partial N_r} \left( -\|\vartheta - \vartheta_0\|_2 \|\nu\|_2 + \frac{1}{2} \|\vartheta - \vartheta_0\|_2^2 \right)
\]

\[
= - \sqrt{\frac{s}{n}} \tau \|\nu\|_2 + \frac{1}{2} \frac{s}{n} \tau^2 \bar{c}.
\]

By the Markov inequality, it entails that

\[
P(H_n) \geq P(||\nu||_2^2 \leq \frac{1}{2 \sqrt{\frac{s}{n}} \tau \bar{c}}) \geq 1 - \frac{4nE\|\nu\|_2^2}{s \tau^2 \bar{c}^2}.
\] (A.3)

In the following, we aim to show that \( E\|\nu\|_2^2 = O(s/n) \).

Note that

\[
\nu = \begin{pmatrix} \frac{1}{n} X^T \epsilon_1 \\ \frac{1}{n} M_T^T \alpha_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda_n \rho(\alpha_0, A) \end{pmatrix} = \nu_1 - \nu_2,
\]

Then by condition (A1),

\[
E\|\nu_1\|_2^2 = \frac{1}{n} \text{tr} \left[ E \left( \begin{pmatrix} X^T \epsilon_1 \\ M_T^T \alpha_1 \end{pmatrix} \begin{pmatrix} X^T \epsilon_1 \\ M_T^T \alpha_1 \end{pmatrix}^T \right) \right]
\]

\[
= \frac{\sigma^2}{n} \text{tr}(\Sigma) \leq \sigma^2 \frac{s + q}{n} \lambda_{\max}(\Sigma) = O(\frac{s}{n}).
\]

It follows from the concavity of \( \rho(\cdot) \), \( d_n < |\alpha_{0j,A}| \), and condition (A2) that:

\[
\|\nu_2\|_2^2 \leq (s^{1/2} p_A(d_n))^2 = o(\frac{1}{n}).
\]

Consequently, step 1 is completed.

Step 2: Sparsity: According to Theorem 1 in Fan and Lv (2011), it suffices to show that with probability tending to 1, we have:

\[
\frac{1}{n} \|M_A^T(y - M\bar{\alpha}_0 - X\bar{\alpha}_1)\|_\infty \ll \lambda_n.
\] (A.4)
Here $\bar{\theta} = (\bar{\alpha}_1, \bar{\alpha}_0^T)^T$ satisfies that $\bar{\alpha}_{0,c} = 0$ and $\|\bar{\theta} - \theta_0\|_2 = O_P(\sqrt{s/n})$. Note that

$$M_{\mathcal{A}^e}^T(y - M\alpha_0 - X\bar{\alpha}_1) = M_{\mathcal{A}^e}^T\epsilon_1 - M_{\mathcal{A}^e}^T(X, M_\mathcal{A})(\bar{\theta} - \theta_0).$$

(A.5)

For the second term,

$$\|M_{\mathcal{A}^e}^T(X, M_\mathcal{A})(\bar{\theta} - \theta_0)\|_\infty \leq \|M_{\mathcal{A}^e}^T(X, M_\mathcal{A})\|_{2,\infty}\|\bar{\theta} - \theta_0\|_2 = O_P(\sqrt{ns}).$$

Next we come to determine the rate of the first term $\|M_{\mathcal{A}^e}^T\epsilon_1\|_\infty$.

Let $a_n = n^{1/\omega+c}K_n$, $b = \sqrt{Cn\log p}$ with $C$ being large enough and note that

$$m_{ij}\epsilon_{i1} = m_{ij}\epsilon_{i1}I(|m_{ij}\epsilon_{i1}| \leq a_n) - E[m_{ij}\epsilon_{i1}I(|m_{ij}\epsilon_{i1}| \leq a_n)]
+ m_{ij}\epsilon_{i1}I(|m_{ij}\epsilon_{i1}| > a_n) - E[m_{ij}\epsilon_{i1}I(|m_{ij}| > a_n)]
=: \epsilon_{ij,1} + \epsilon_{ij,2}.$$

We have

$$P \left( \left| \sum_{i=1}^n m_{ij}\epsilon_{i1} \right| > b, \text{ for some } j \in \mathcal{A}^c \right)
\leq P \left( \left| \sum_{i=1}^n \epsilon_{ij,1} \right| + \left| \sum_{i=1}^n \epsilon_{ij,2} \right| > b, \text{ for some } j \in \mathcal{A}^c \right)
\leq P \left( \left| \sum_{i=1}^n \epsilon_{ij,1} \right| > b/2, \text{ for some } j \in \mathcal{A}^c \right) + P \left( \left| \sum_{i=1}^n \epsilon_{ij,2} \right| > b/2, \text{ for some } j \in \mathcal{A}^c \right)
=: P_1 + P_2.$$

Firstly consider the term $P_1$. Note that $\epsilon_{ij,1}, \ldots, \epsilon_{nj,1}$ are independent centered random variables a.s. bounded by $2a_n$ in absolute value. Then the Bernstein inequality yields that

$$P_1 \leq 2(p - s) \max_j \exp \left\{ - \frac{b^2/4}{2nE(\epsilon_{j,1}^2) + 2 \cdot 2a_n \cdot b/(2 \cdot 3)} \right\}
\leq 2p \max_j \exp \left\{ - \frac{C\log p/4}{2E(\epsilon_{j,1}^2) + 2a_n \sqrt{C\log p/n}/3} \right\} \to 0.$$

Next we turn to consider $P_2$. First note that

$$P_2 \leq P \left( \max_{i=1}^n \sum_{j=1}^n |m_{ij}\epsilon_{i1}I(|m_{ij}\epsilon_{i1}| > a_n) + \max_{j=1}^n nE[|m_{ij}\epsilon_{i1}|I(|m_{ij}\epsilon_{i1}| > a_n)] > b/2 \right)$$

Further note that

$$E^2[|m_{ij}\epsilon_{i1}|I(|m_{ij}\epsilon_{i1}| > a_n)] \leq E[m_{ij}^2\epsilon_{i1}^2]P(|m_{ij}\epsilon_{i1}| > a_n) \leq E[m_{ij}^2\epsilon_{i1}^2] \frac{E[|m_{ij}\epsilon_{i1}|^\omega]}{a_n^\omega}.$$
We then conclude that
\[
\max_j nE[|m_j\varepsilon_1| I(|m_j\varepsilon_1| > a_n)] \leq \max_j n\sqrt{E[m_j^2\varepsilon_1^2] E[|m_j\varepsilon_1|^\infty]} a_n^\infty = o(\sqrt{n}).
\]

From this, we then have
\[
P_2 \leq P\left(\sum_{i=1}^n \max_j |m_{ij}\varepsilon_{i1}| I(|m_{ij}\varepsilon_{i1}| > a_n) > b/4\right)
\leq P \left(\max_j |m_{ij}\varepsilon_{i1}| > a_n \text{ for some } i\right)
\leq n E[\max_j |m_j\varepsilon_1|^\infty] a_n^\infty = o(1).
\]

Thus \(\|M_A^T\varepsilon_1\|_\infty = O_P(\sqrt{n \log p})\).

Consequently, given condition A2, step 2 is finished.

**Step 3: Asymptotic expansions:** Steps 1 and 2 show that \(\hat{\alpha}_{0,A} = 0\) with probability tending to 1, and further \(\|\hat{\alpha}_{0,A} - \alpha_{0,A}^*\|_2 = O_P(\sqrt{s/n})\).

First denote
\[
\hat{L}(\vartheta_0) = \begin{pmatrix} X^T(y - M_A\alpha_{0,A}^* - X\alpha_1^*) \\ M_A^T(y - M_A\alpha_{0,A}^* - X\alpha_1^*) \end{pmatrix} = \begin{pmatrix} X^T\varepsilon_1 \\ M_A^T\varepsilon_1 \end{pmatrix}.
\] (A.6)

For \(\hat{\vartheta}\), denote
\[
\hat{L}(\hat{\vartheta}) = \begin{pmatrix} X^T(y - M_A\alpha_{0,A} - X\hat{\alpha}_1) \\ M_A^T(y - M_A\alpha_{0,A} - X\hat{\alpha}_1) \end{pmatrix} = \begin{pmatrix} 0 \\ n\lambda_n\tilde{p}(\alpha_{0,A}) \end{pmatrix}.
\] (A.7)

Notice that
\[
\hat{L}(\vartheta_0) = \hat{L}(\hat{\vartheta}) + nD_1(\hat{\vartheta} - \vartheta_0).
\]

Or equivalently we have
\[
\frac{1}{\sqrt{n}}(\hat{L}(\vartheta_0) - \hat{L}(\hat{\vartheta})) = \Sigma\sqrt{n}(\hat{\vartheta} - \vartheta_0) + (D_1 - \Sigma)\sqrt{n}(\hat{\vartheta} - \vartheta_0).
\]

Recall that \(\|D_1 - \Sigma\|_2 = O_P(s/\sqrt{n})\), and \(\|\hat{\vartheta} - \vartheta_0\| = O_P(\sqrt{s/n})\). Then we have
\[
(D_1 - \Sigma)\sqrt{n}(\hat{\vartheta} - \vartheta_0) = o_P(1),
\]
when \(s = o(n^{1/3})\). Thus, we have
\[
\sqrt{n}(\hat{\vartheta} - \vartheta_0) = \Sigma^{-1}\frac{1}{\sqrt{n}}(\hat{L}(\vartheta_0) - \hat{L}(\hat{\vartheta})) + o_P(1).
\]
Under condition (A2), we have $\|\hat{\alpha}_{0,A} - \alpha_{0,A}^*\|_\infty = O_P(\sqrt{s/n}) \ll d_n$. This implies that

$$\min_{j \in A} |\hat{\alpha}_{0j,A}| > \min_{j \in A} |\alpha_{0j,A}^*| - d_n = d_n.$$ 

By the concavity of $p(\cdot)$ and condition (A2), we obtain that

$$\|n\lambda_n\hat{p}(\hat{\alpha}_{0,A})\|_2 \leq ns^{1/2}p'_\lambda_n(d_n) = o(n^{1/2}).$$

Since $\lambda_{\max}(\Sigma^{-1}) = O(1)$, it follows that

$$\sqrt{n} (\hat{\theta} - \theta_0) = \sum^{-1} \frac{1}{\sqrt{n}} \hat{L}(\theta_0) + o_P(1). \quad (A.8)$$

**Proof of Corollary 1:** Recall that

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{XX}^{-1} + \Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} \Sigma_{MX} \Sigma_{XX}^{-1} - \Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} & 0 \\ -\Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} & -\Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} \end{pmatrix}.$$ 

Here $\Sigma_{MM,X} = \Sigma_{MM} - \Sigma_{MX} \Sigma_{XX}^{-1} \Sigma_{XM}$.

As a result, it follows that

$$\sqrt{n} (\hat{\alpha}_1 - \alpha_1^*) = (I_{q \times q}, 0_{q \times s}) \Sigma^{-1} \frac{1}{\sqrt{n}} \hat{L}(\theta_0) + o_P(1)$$

$$= \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} X^T \epsilon_1 + \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} (\Sigma_{MX} \Sigma_{XX}^{-1} X^T - M_A^T) \epsilon_1 + o_P(1). \quad (A.9)$$

The asymptotic variance matrix of $\hat{\alpha}_1$ is

$$\sigma_1^2 (I_{q \times q}, 0_{q \times s}) \Sigma^{-1} (I_{q \times q}, 0_{q \times s})^T = \sigma_1^2 \left( \Sigma_{XX}^{-1} + \Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} \Sigma_{MX} \Sigma_{XX}^{-1} \right).$$

Recall that

$$\sqrt{n} (\hat{\gamma} - \gamma^*) = \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} X^T (\epsilon_1 + \epsilon_2) + o_P(1). \quad (A.10)$$

Consequently we obtain that

$$\sqrt{n} (\hat{\beta} - \beta^*) = \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} X^T \epsilon_2 + \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} \Sigma_{MX} \Sigma_{MM,X}^{-1} (M_A^T - M_X \Sigma_{XX}^{-1} X^T) \epsilon_1 + o_P(1)$$

$$= \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} \sum_{i=1}^n W_{1i} + \frac{1}{\sqrt{n}} \Sigma_{XX}^{-1} \Sigma_{XM} \Sigma_{MM,X}^{-1} \sum_{i=1}^n W_{2i} + o_P(1). \quad (A.11)$$

Here $W_{1i} = x_i \varepsilon_{2i}$ and $W_{2i} = (m_{i,A} - \Sigma_{MX} \Sigma_{XX}^{-1} x_i) \varepsilon_{1i}$.

It is easy to show that $E[W_{1i}] = E[x_i E(\varepsilon_{2i} | x_i)] = 0$. Similarly, we have $E[W_{2i}] = E[(m_{i,A} - \Sigma_{MX} \Sigma_{XX}^{-1} x_i) E(\varepsilon_{1i} | x_i, m_{i,A})] = 0$. 

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Further we obtain that \( \text{var}(W_{1i}) = \sigma_2^2 \Sigma_{XX}, \text{var}(W_{2i}) = \sigma_1^2 \Sigma_{MM.X}, \) and
\[
\text{cov}(W_{1i}, W_{2i}) = E[x_i \varepsilon_{2i}(m_{i,A} - \Sigma_{MX} \Sigma_{XX}^{-1} x_i) \varepsilon_{1i}]
= E[x_i \varepsilon_{2i}(m_{i,A} - \Sigma_{MX} \Sigma_{XX}^{-1} x_i) E(\varepsilon_{1i} | x_i, m_{i,A}, \varepsilon_{2i})] = 0.
\]

As a result, it follows that
\[
\sqrt{n}(\hat{\beta} - \beta^*) \rightarrow N(0, \sigma_2^2 \Sigma_{XX}^{-1} + \sigma_1^2 \Sigma_{XX}^{-1} \Sigma_{MM.X} \Sigma_{XX}^{-1}). \tag{A.12}
\]

**Proof of Theorem 2:** Similar to the arguments in the proof of Theorem 1, we can also show that \( \hat{\alpha}_{0,A} = 0 \) with probability 1, and further \( \|\hat{\alpha}_{0,A} - \alpha^*_{0,A}\|_2 = O_P(\sqrt{s/n}). \)

Denote \( \Delta \hat{\theta} = \hat{\theta} - \bar{\theta} = (\Delta \hat{\theta}_1, \Delta \hat{\theta}_2) \) and
\[
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix}^{-1}.
\]

It is noted that
\[
\begin{pmatrix}
0 \\
n \lambda_n \bar{\rho}(\hat{\alpha}_{0,A})
\end{pmatrix} = \hat{L}(\hat{\theta}) - \bar{L}(\bar{\theta}) - nD_1 \Delta \hat{\theta} = \begin{pmatrix}
L_1(\hat{\theta}) \\
n \lambda_n \bar{\rho}(\hat{\alpha}_{0,A})
\end{pmatrix} - \Sigma \Delta \hat{\theta} - (D_1 - \Sigma)n \Delta \hat{\alpha}. \tag{A.13}
\]

Here \( D_1 = \begin{pmatrix}
X^T X & X^T M_A \\
M_A^T X & M_A^T M_A
\end{pmatrix} / n. \)

From the proof of Theorem 1, it is known that \( \|n \lambda_n \bar{\rho}(\hat{\alpha}_{0,A})\|_2 = o_p(n^{1/2}) \) and similarly \( \|n \lambda_n \bar{\rho}(\hat{\alpha}_{0,A})\|_2 = o_p(n^{1/2}). \)

Further recall that \( \|D_1 - \Sigma\|_2 = O_P(s/\sqrt{n}) \) and \( \|\Delta \hat{\theta}\|_2 = \|\hat{\theta} - \bar{\theta}_0\| = O_P(\sqrt{s/n}). \)

Thus under condition that \( s = o(n^{1/3}), \) we have
\[
o_P(1) = \begin{pmatrix}
\frac{1}{\sqrt{n}} L_1(\hat{\theta}) \\
0
\end{pmatrix} - \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22}
\end{pmatrix} \begin{pmatrix}
\sqrt{n} \Delta \hat{\theta}_1 \\
\sqrt{n} \Delta \hat{\theta}_2
\end{pmatrix}, \tag{A.14}
\]

from which we have
\[
\sqrt{n} \Delta \hat{\theta}_2 = -\Sigma_{22}^{-1} \Sigma_{21} \sqrt{n} \Delta \hat{\theta}_1 + o_P(1), \text{ and } \sqrt{n} \Delta \hat{\theta}_1 = \Sigma_{11} \frac{1}{\sqrt{n}} L_1(\hat{\theta}) + o_P(1). \tag{A.15}
\]

Note that \( \sqrt{n} \Delta \hat{\theta}_1 = \sqrt{n}(\hat{\alpha}_1 - \alpha^*_1) + \sqrt{n} h_n = O_P(1) \) from Corollary 1. Thus we get \( \sqrt{n} \Delta \hat{\theta}_2 = O_P(1), \) which further implies that \( \Delta \hat{\theta}_2^T n \lambda_n \bar{\rho}(\hat{\alpha}_{0,A}) = o_P(1). \)

Now we are ready to investigate the asymptotic distribution of \( T_n. \)

Under the event \( \hat{\alpha}_{0,A} = \alpha_{0,A} = 0 \) and recalling equation \( (A.15), \) we can show that
\[
\text{RSS}_1 - \text{RSS}_0 = -2 \Delta \hat{\theta}^T \hat{L}(\hat{\theta}) + \Delta \hat{\theta}^T n D_1 \Delta \hat{\theta}
= -n \Delta \hat{\theta}_1^T (\Sigma_{11})^{-1} \Delta \hat{\theta}_1 + o_P(1). \tag{A.16}
\]
Now denote $\Phi = (I_q, 0_{q \times s})\Sigma^{-1}(I_q, 0_{q \times s})^T$. It is easy to know that $\Phi = \Sigma^{11}$. From the proof of Corollary 1, it is known that $\sqrt{n}(\hat{\alpha}_1 - \alpha_1^\star) \to N(0, \sigma^2_1 \Phi)$. Thus we obtain that

$$\text{RSS}_0 - \text{RSS}_1 = \|\Phi^{-1/2}[\sqrt{n}(\hat{\alpha}_1 - \alpha_1^\star)] + \sqrt{n}\Phi^{-1/2}\hat{h}_n\|^2_2 + o_P(1).$$  \hspace{1cm} (A.17)

On the other hand, we have

$$\frac{\text{RSS}_1}{n - q} = \frac{1}{n - q}\|y - M\hat{\alpha}_0 - X\hat{\alpha}_1\|^2_2 = \frac{1}{n - q}\|y - M\alpha_0^* - X\alpha_1^*\|^2_2 - 2\frac{1}{n - q}(\hat{\vartheta} - \vartheta_0)^T\hat{L}(\vartheta_0) + \frac{1}{n - q}(\hat{\vartheta} - \vartheta_0)^TnD_1(\hat{\vartheta} - \vartheta_0) = I_1 - 2I_2 + I_3.$$

It is easy to know that $I_1 \to \sigma_1^2$, while $I_3 \leq \|\hat{\vartheta} - \vartheta_0\|_2\|D_1\|_2 = O_P(s/n) = o_P(1)$. Further note that

$$I_2 \leq \|\hat{\vartheta} - \vartheta_0\|_2\|\hat{L}(\vartheta_0)\|_2/(n - q) = O_P((1/n)\sqrt{s/n\sqrt{s}}) = O_P(s\sqrt{s}/n) = o_P(1).$$

In sum, it follows that

$$\frac{\text{RSS}_1}{n - q} = \sigma_1^2 + o_P(1).$$  \hspace{1cm} (A.18)

As a result, we have

$$T_n = \frac{\text{RSS}_0 - \text{RSS}_1}{\text{RSS}_1/(n - q)} \to \chi^2_q(n\hat{h}_n^T\Phi^{-1}\hat{h}_n/\sigma_1^2).$$

### A.2 Natural direct and indirect effects

Under the independence conditions of random errors in the models, the sequential ignorability assumption \cite{Imai2010} holds, and the natural direct and indirect effects can be identified. As argued by \cite{Imai2010}, only the sequential ignorability assumption is needed and neither the linearity nor the no-interaction assumption is required for the identification of mediation effects. However, in the situation with high dimensional mediators, it would be very challenging if not impossible to make inference about the mediation models without linearity nor the no-interaction assumption. The linearity and the no-interaction assumptions are widely adopted in recent studies about HDMM \cite{Zhang2016, Kesteren2019, Zhou2020}.

To define the natural direct and natural indirect effects, we give some notation first. Let $y(x^*, m^*)$ denote the potential outcome that would have been observed had $x$ and $m$ been set to $x^*$ and $m^*$, respectively, and $m(x^*)$ denotes the potential mediator that would have been observed
had \( x \) been set to \( x^* \). Following Imai et al. (2010), Vanderweele & Vansteelandt (2014), and others, for \( x = x_1 \) versus \( x_0 \), the natural direct effect is defined as \( E[y(x_1, m(x_0)) - y(x_0, m(x_0))] \). While the indirect effect is defined as \( E[y(x_1, m(x_1)) - y(x_1, m(x_0))] \). Then the total effect \( E[y(x_1, m(x_1)) - y(x_0, m(x_0))] \) is the sum of the natural direct and indirect effect. Vanderweele & Vansteelandt (2014) showed that

\[
E[y(x_1, m(x_0)) - y(x_0, m(x_0))] = \alpha_1^T (x_1 - x_0)
\]

\[
E[y(x_1, m(x_1)) - y(x_1, m(x_0))] = (\Gamma \alpha_0)^T (x_1 - x_0).
\]

Thus \( \alpha_1 \) can be interpreted as the average natural direct effect, and \( \beta = \Gamma \alpha_0 \) can be interpreted as the average natural indirect effect, of a one-unit change in the exposure \( x \).

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