Three combinatorial formulas for type $A$ quiver polynomials and $K$-polynomials

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Type A quiver loci

- A **quiver** $Q$ is a finite directed graph, and a **representation** of $Q$ is an assignment of vector space to each vertex and linear map to each arrow.
- $Q$ is of **type A** if its underlying graph is a type A Dynkin diagram.
- Once the vector spaces $K^{d_0}, \ldots, K^{d_n}$ at the vertices are fixed, the collection of representations is an algebraic variety, denoted by $\text{rep}_Q(d)$. This variety carries the action of a **base change group**:

$$GL(d) := GL(d_0) \times GL(d_1) \times \cdots \times GL(d_n).$$

- These orbit closures are called **quiver loci**.

**Example**

A representation of an equioriented type A quiver:

$$K^{d_0} \xrightarrow{V_1} K^{d_1} \xrightarrow{V_2} K^{d_2} \cdots \xrightarrow{V_n} K^{d_n}.$$  

Here, $V_i$ is a $d_{i-1} \times d_i$ matrix, and $\text{rep}_Q(d)$ is the affine space of all sequences $(V_1, \ldots, V_n)$. The base change group $GL(d)$ acts by:

$$(g_0, g_1, \ldots, g_{n-1}, g_n) \cdot (V_1, \ldots, V_n) = (g_0 V_1 g_1^{-1}, \ldots, g_{n-1} V_n g_n^{-1}).$$
The equioriented setting is well-understood. In particular:

- Orbits are determined by ranks of all products $V_i V_{i+1} \cdots V_j$, $i \leq j$.
- (Zelevinsky '85) The collection of these rank conditions is equivalent to certain Schubert-type rank conditions on an opposite Schubert cell in a partial flag variety. Eg. if $Q$ has three arrows,

$$
(V_1, V_2, V_3) \xrightarrow{\zeta} \begin{bmatrix}
0 & 0 & V_1 & I_{d_0} \\
0 & V_2 & I_{d_1} & 0 \\
V_3 & I_{d_2} & 0 & 0 \\
I_{d_3} & 0 & 0 & 0
\end{bmatrix} \subseteq \begin{bmatrix}
* & * & * & I_{d_0} \\
* & * & I_{d_1} & 0 \\
* & I_{d_2} & 0 & 0 \\
I_{d_3} & 0 & 0 & 0
\end{bmatrix} \cong P \setminus PwB_–.
$$

This map $\zeta$ is an equioriented Zelevinsky map.

- (Lakshmibai-Magyar '98) The Zelevinsky map is scheme-theoretic isomorphism which takes each orbit closure to a Schubert variety intersected with an opposite Schubert cell. Consequently, these quiver loci are normal and Cohen-Macaulay with rational singularities, F-split...

- The coordinate rings of equioriented type $A$ quiver loci are naturally multigraded, and there exist multiple combinatorial formulas for their multidegrees and $K$-polynomials.

**Goal:** Generalize to all orientations.
A type $A$ quiver is **bipartite** if every vertex is a source or sink:

$$
\begin{align*}
&\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ }$
\end{align*}
$$

$GL(d)$-orbits of bipartite type $A$ quivers are completely determined by ranks of particular matrices: given an interval $[i, j] \subseteq Q$, define the matrix

$$
Z_{[i,j]} = \begin{pmatrix}
V_{i+2} & V_{i+1} \\
\vdots & \vdots \\
V_{j-2} & V_{j-1} & V_j \\
V_i & V_{i+1} & \cdots & V_{i+2}
\end{pmatrix}.
$$

Let $r_{[i,j]} := \text{rank } Z_{[i,j]}$, and let $r$ be the array of all $r_{[i,j]}$. Then, two representations in $\text{rep}_Q(d)$ lie in the same $GL(d)$-orbit if and only if they have the same rank array $r$. 
The bipartite Zelevinsky map

Theorem (Kinser-R)

- There is a closed immersion from each representation space of a bipartite type A quiver to an opposite Schubert cell of a partial flag variety.
- This bipartite Zelevinsky map identifies each quiver locus with a Schubert variety intersected with the above opposite Schubert cell.
- Consequently, quiver loci are normal and C-M with rational singularities, F-split, orbit closure containment is determined by Bruhat order.

Example

The image of \((V_1, V_2, V_3, V_4, V_5, V_6)\) under the bipartite Zelevinsky map is:

\[
\begin{pmatrix}
0 & 0 & V_1 & l_{d_0} & 0 & 0 & 0 & 0 \\
0 & V_3 & V_2 & 0 & l_{d_2} & 0 & 0 \\
V_5 & V_4 & 0 & 0 & 0 & l_{d_4} & 0 \\
V_6 & 0 & 0 & 0 & 0 & 0 & l_{d_6} \\
l_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l_{d_3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & l_{d_5} & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\subseteq \begin{pmatrix} \ast & I \\ I & 0 \end{pmatrix} \cong P \backslash P v_0 B^-.
The maximal torus $T \subseteq \text{GL}(d)$ consisting of matrices which are diagonal in each factor induces a multigrading on $K[\text{rep}_Q(d)]$ which makes the ideals of orbit closures homogeneous:

$$
\begin{align*}
&y_3 &\beta_3 &\alpha_3 &y_2 &\beta_2 &\alpha_2 &y_1 &\beta_1 &\alpha_1 &y_0 \\
&x_3 & & &x_2 & & &x_1 & & &
\end{align*}
$$

Associate an alphabet $s^j$ to the vertex $x_j$, and an alphabet $t^i$ to the vertex $y_i$:

$$s^j = s^j_i, s^j_2, \ldots, s^j_{d(x_j)} \quad \text{and} \quad t^i = t^i_1, t^i_2, \ldots, t^i_{d(y_i)}.$$

The coordinate function $f^{\alpha_k}_{ij}$ (picking out $(i,j)$-entry of $M_{\alpha_k}$) has degree $t^k_i - s^k_j$, and $f^{\beta_k}_{ij}$ has degree $t^k_i - s^k_j$.

With respect to the natural torus action on the opposite cell $[\begin{bmatrix} * & l_{dy} \\ l_{dx} & 0 \end{bmatrix}]$, the bipartite Zelevinsky map is $T$-equivariant.
The \textbf{K-theoretic quiver polynomial} $K\mathcal{Q}_r(t/s)$ (resp., \textbf{quiver polynomial} $\mathcal{Q}_r(t-s)$) is the $K$-polynomial (resp., multidegree) of the quiver locus $\Omega_r$ with respect to its embedding in $\text{rep}_Q(d)$ and multigrading above.

Let $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$ be alphabets. Denote by $G_w(A; B)$ the \textbf{double Grothendieck polynomial} associated to $w$: if $w_0$ the longest element of the symmetric group $S_m$ then

$$G_{w_0}(A; B) = \prod_{i+j \leq m} \left(1 - \frac{a_i}{b_j}\right),$$

and $G_{s_iw}(A; B) = \partial_i G_w(A; B)$ whenever $\ell(s_iw) < \ell(w)$.

The \textbf{double Schubert polynomial} $G_v(A; B)$ of a permutation $v$ is obtained from $G_v(A; B)$ by substituting $1 - \star$ for each variable $\star$, and then taking lowest degree terms.
The bipartite ratio formulas

- Let $r$ be an array of ranks that determines a bipartite quiver orbit.
- Let $\nu(r)$ be the associated Zelevinsky permutation.
- Let $\nu_*$ be the Zelevinsky permutation of the big $GL(d)$-orbit (which has closure $\text{rep}_Q(d)$).

**Theorem (Kinser-Knutson-R)**

$$K_{Q_r}(t/s) = \frac{\mathcal{G}_{\nu(r)}(t, s; s, t)}{\mathcal{G}_{\nu_*}(t, s; s, t)} \quad \text{and} \quad Q_r(t - s) = \frac{\mathcal{G}_{\nu(r)}(t, s; s, t)}{\mathcal{G}_{\nu_*}(t, s; s, t)}.$$

**Main idea of proof.**

Use the bipartite Zelevinsky map along with [Woo-Yong '12] on K-polynomials and multidegrees of Kazhdan-Lusztig varieties.
Pipe dreams and lacing diagrams

Consider the dimension vector \( \mathbf{d} = (2, 2, 2, 3, 2, 2, 1) \), so that representations have the form:

\[
\begin{align*}
K^2 & \xrightarrow{V_{\beta_3}} V_{\alpha_3} \\
K^2 & \xrightarrow{V_{\beta_2}} V_{\alpha_2} \\
K^3 & \xrightarrow{V_{\beta_1}} V_{\alpha_1}
\end{align*}
\]

Work with the orbit through:

\[
P = \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right)
\]

This sequence of partial permutations can be visualized with a lacing diagram:
Pipe dreams and lacing diagrams

The Zelevinsky image of the associated quiver locus is a Kazhdan-Lusztig variety which has pipe dreams supported inside the diagram of \( v_0 \) (i.e. the northwest quadrant of \( \begin{bmatrix} * & l_{dy} \\ l_{dx} & 0 \end{bmatrix} \)). For example:

![Pipe Dreams Diagram](image)

Denote by \( \text{Pipes}(v_0, v(r)) \) all pipe dreams of \( v(r) \) supported inside the Rothe diagram for \( v_0 \). Let \( P_* \) be the pipe dream which has a + at position \((i, j)\) if and only if \((i, j)\) lies outside of the “zig-zag” region.

**Lemma**

*Every element of \( \text{Pipes}(v_0, v(r)) \) contains \( P_* \) as a subdiagram, and furthermore \( \text{Pipes}(v_0, v_*) = \{ P_* \} \).*
Theorem (Bipartite Pipe formula, Kinser-Knutson-R)

For any rank array $r$, we have

$$KQ_r(t/s) = \sum_{P \in \text{Pipes}(v_0, v(r))} (-1)^{|P|-l(v(r))} (1 - t/s)^{P \setminus P^*}$$

and

$$Q_r(t - s) = \sum_{P \in \text{RedPipes}(v_0, v(r))} (t - s)^{P \setminus P^*}.$$

Theorem (Bipartite component formula, Buch-Rimányi, Kinser-Knutson-R)

$$KQ_r(t/s) = \sum_{w \in KW(r)} (-1)^{|w|-l(v(r))} \mathcal{G}_w(t, s)$$

and

$$Q_r(t - s) = \sum_{w \in W(r)} \mathcal{G}_w(t, s).$$
From the bipartite orientation to arbitrary orientation

Associate a bipartite type $A$ quiver to an arbitrarily oriented quiver by inserting vertices and arrows. Let $Q$ be the quiver:

We construct an associated bipartite quiver $\tilde{Q}$ by adding two new vertices $w_1, w_3$, and two new arrows $\delta_1, \delta_3$. 
Theorem (Kinser-R)

Let $Q$ be a quiver of type $A$, and $\widetilde{Q}$ the associated bipartite quiver defined above. Let $U$ be the open set in $\text{rep}_{\widetilde{Q}}(d)$ where the maps over the added arrows are invertible. Then there is a morphism $\pi: U \rightarrow \text{rep}_{Q}(d)$ which is equivariant with respect to the natural projection of base change groups $GL(\widetilde{d}) \rightarrow GL(d)$. Each orbit closure $\overline{O} \subseteq \text{rep}_{Q}(d)$ for an arbitrary type $A$ quiver is isomorphic to an open subset of an orbit closure of $\text{rep}_{\widetilde{Q}}(\widetilde{d})$, up to a smooth factor. Namely, we have

$$\overline{\pi^{-1}(O)} \cong G^* \times \overline{O},$$

where the closure on the left hand side is taken in $U$. 
Substitution to obtain formulas for arbitrary orientation

We can show that the $K$-polynomial of an orbit closure for $Q$ is obtained from the $K$-polynomial of its corresponding orbit closure for $\tilde{Q}$ by substitution of variables.
Thank you.