ON A PROBLEM OF HOFFSTEIN AND KONTOROVICH

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Abstract. Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ and $d$ be a fundamental discriminant. Hoffstein and Kontorovich ask for a bound on the least $|d|$ (if it exists) such that the central value $L(1/2, \pi \otimes \chi_d) \neq 0$. The bound should be given in terms of the weight, Laplace eigenvalue and/or level of $\pi$.

Let $f$ be a holomorphic twist-minimal newform of even weight $\ell$, odd cubefree level $N$, and trivial nebentypus. When $\pi \cong \pi_f$ and the squarefree part of $N$ is of appropriate size, we conditionally improve upon level aspect results of Hoffstein and Kontorovich under subconvexity (with a sub-Weyl exponent) for automorphic $L$-functions. As a consequence we conditionally prove that given an elliptic curve $E/\mathbb{Q}$ of conductor $N$, there exists a small twist that has Mordell–Weil rank equal to zero.

1. Introduction

1.1. Statement of results. Hoffstein and Kontorovich [HK10] pose the following

Problem. Let $\pi$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ and $d$ be a fundamental discriminant. If they exist, establish a bound for the least value of $|d|$ (relative to the data attached to $\pi$) such that

\[ L(1/2, \pi \otimes \chi_d) \neq 0. \]  

(1.1)

In this paper we focus on holomorphic cuspidal newforms on $GL_2$ over $\mathbb{Q}$. Let $N$ will be a cubefree odd integer having factorisation $N = N_0N_1^2$, $\mu^2(N_0) = \mu^2(N_1) = 1$, (1.2)

and $\ell$ be an even integer. Let $S_\ell(N)$ denote the space of holomorphic cusp forms on $\Gamma_0(N)$ with weight $\ell$ and trivial nebentypus. Similarly, let $S_\ell^{\text{new}}(N)$ be the space of such arithmetically normalised newforms (first Fourier coefficient is normalised to be 1). For each $f \in S_\ell^{\text{new}}(N)$, let $\pi_f \cong \otimes_p \pi_{f,p}$ denote the corresponding cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$.

Let $c(\pi_f) := \prod c(\pi_{f,p}) = N$ denote the arithmetic conductor of $\pi_f$. It is said that $f \in S_\ell^{\text{new}}(N)$ is twist-minimal if $c(\pi_f \otimes \psi) \geq c(\pi_f)$ for all Dirichlet characters $\psi$.

In the case of holomorphic cusp forms, Waldspurger’s formula [Wal81] and the Riemann–Roch theorem imply that that there exists a quadratic twist of fundamental discriminant $d$ satisfying $|d| \ll_\varepsilon (\ell N)^{1+\varepsilon}$ and (1.1). This bound is on par with what a “convexity bound” for the relevant multiple Dirichlet series would imply (see section 1.2 for more discussion).

Hoffstein and Kontorovich [HK10] prove bounds of the above quality for general $\pi$ on $GL_r(\mathbb{A}_\mathbb{Q})$, $r = 1, 2, 3$. The investigations in [HK10] are restricted to $r \leq 3$ in order to

2010 Mathematics Subject Classification. Primary 11F03, 11F66, 11F68.

Key words and phrases. $L$-function, quadratic twist, ranks, modular forms, elliptic curves, multiple Dirichlet series.
guarantee the relevant multiple Dirichlet series $Z(s,w;\pi)$ have meromorphic continuation past the point $(1/2,1)$.

Conditional on a strong uniform quantitative subconvexity hypothesis for automorphic $L$-functions, we improve the bound $|d| \ll (\ell N)^{1+\varepsilon}$ in the level aspect, for some levels $N$.

**Theorem 1.1.** Let $N = N_0 N_1^2$ be an odd cube-free odd integer and $f \in S_{\text{new}}^e(N)$ be twist minimal. Suppose there is $0 < \delta_1 < 1/4$ such that for all positive fundamental discriminants $d$ we have

$$L(1/2 + it, \pi_f \otimes \chi_d) \ll_{\varepsilon} \varepsilon(\pi_f \otimes \chi_d)^{1/4 - \delta_1 + \varepsilon}(1 + |t|)^{A},$$

for some $A > 0$. Suppose there exists a $0 < \delta_2 < 1/4$ such that

$$L(1/2 + it, \text{Sym}^2 \pi_f) \ll_{\varepsilon} \varepsilon(\text{Sym}^2 \pi_f)^{1/4 - \delta_2 + \varepsilon}(1 + |t|)^{A},$$

for some $A > 0$.

Then there exists a fundamental discriminant $d_0$ satisfying

$$1 \leq d_0 \ll_{\varepsilon} N^{\varepsilon}\left(N_0^{7/4 - 5\delta_1} N_1^{7/4 - 5\delta_1} + \frac{N^{1 - 6\delta_2}}{N_0^{2\delta_2}}\right)$$

and $(d_0, 2N) = 1$, such that

$$L(1/2, \pi_f \otimes \chi_{d_0}) \neq 0.$$ 

The constant in (1.5) is ineffective in terms of $\varepsilon > 0$.

**Remark.**

- Theorem 1.1 gives a power saving improvement over [HK10] when $N$ has squarefree part

$$N_0 \ll N^{(7\delta_1 - 5/4)/(7/4 - 5\delta_1) + o(1)}.$$  

Thus Theorem 1.1 requires a strong input: $5/28 = 0.178 \ldots < \delta_1 < 1/4$ and $\delta_2 > 0$.

- The limiting case $\delta_1 = 1/4$ (i.e. the generalised Lindelöf hypothesis) gives a power saving improvement as soon as

$$1 < N_0 \ll N^{1 - o(1)}.$$  

- Michel and Venkatesh [MV10] solved the uniform subconvexity problem on $\text{GL}_2(\mathbb{A}_Q)$. However, we are far away from exhibiting the sub-Weyl exponent needed for Theorem 1.1. Subconvexity of the symmetric square is another major open problem.

- It is likely that an improved Heath–Brown quadratic large sieve [HB95] for modular $L$-functions would also yield similar results.

- Existence of certain $d$ satisfying (1.1) can be ruled out by root number considerations. Suppose $N$ were a perfect square in Theorem 1.1 and $\pi_f$ has root number $\varepsilon(\pi_f) = -1$. Then $\varepsilon(\pi_f \otimes \chi_d) = \varepsilon(\pi_f)\chi_d(-N) = -1$ for all fundamental discriminants $d > 0$ satisfying $(d, 2N) = 1$. Then all such central values $L(1/2, \pi_f \otimes \chi_d)$ vanish.

- Let $F$ be a number field. Problems concerning the existence of $d$ (and infinitude of such $d$) such that (1.1) holds for a fixed cuspidal automorphic representation $\pi$ of $\text{GL}_2(\mathbb{A}_F)$ was resolved by Friedberg and Hoffstein in [FH95]. This was built on a host of works [BFH90b, BFH90a, Iwa90, MM91]. Ono and Skinner [OS98] give lower bounds on the number fundamental discriminants $0 < |d| \leq X$ such that (1.1) holds for a fixed holomorphic $\pi$ on $\text{GL}_2(\mathbb{A}_Q)$. 
Let $E/\mathbb{Q}$ be an elliptic curve with Weierstrass equation $y^2 = x^3 + ax + b$, having conductor $N$. Let $E^{(d)}/\mathbb{Q}$ define the twisted curve given by the equation $dy^2 = x^3 + ax + b$. An immediate consequence of Theorem 1.1, the modularity theorem [BCDT01], and a result of Kolyvagin [Kol88] is

**Corollary 1.** Let $N = N_0 N_1^2$ be an odd cubefree integer and $E/\mathbb{Q}$ be a twist minimal elliptic curve of conductor $N$. Suppose $L(s, \pi_E \otimes \chi_d)$ satisfies (1.3) for all fundamental discriminants $d$, and that $L(s, \text{Sym}^2 \pi_E)$ satisfies (1.4). Then there exists a fundamental discriminant $d_0$ satisfying the conditions in (1.5) such that $E^{(d_0)}$ has Mordell–Weil rank equal to zero.

Note that primes $p \mid N_0$ (resp. $p \mid N_1$) correspond to primes of multiplicative reduction (resp. additive reduction) for $E/\mathbb{Q}$. Thus the savings obtained in Corollary 1 are governed by the reduction types of bad primes.

Impressive work of Petrow [Pet14] conditionally establishes (under GRH) the existence of a fundamental discriminant $d_0$ satisfying

$$0 < d_0 \ll A \frac{N \ell}{(\log N \ell)^A}, \quad \varepsilon(f \otimes \chi_{d_0}) = -1, \quad \text{and} \quad (d_0, 2N) = 1,$$

such that the derivative $L'(1/2, \pi_f \otimes \chi_{d_0}) \neq 0$. It is expected that the methods of this paper would carry over to that situation as well.

We close by mentioning that a power saving improvement on the “convexity” bound $|d| \ll_{\varepsilon} (\ell N)^{1+\varepsilon}$ in the weight aspect would have interesting applications to non-vanishing of certain $\text{GL}_3 \times \text{GL}_2$ $L$-functions. For this, one may consult work of Liu and Young [LY14].

**1.2. Heuristics and outline.** Here we outline the main ideas in this paper, ignoring most technicalities (i.e. the presence of smooth functions). We remind the reader that $\ell$ is an arbitrary but fixed positive even integer, and that $N$ is an odd cubefree integer allowed to move and has factorisation (1.2).

For now we ignore the requirement that $d$ be a fundamental discriminant (it will be addressed in momentarily).

In order to obtain Theorem 1.1, one could try to prove an asymptotic formula

$$\sum_{d \sim X} \frac{1}{d^{1/2}} L(1/2, \pi_f \otimes \chi_d) = T_{\pi_f}(X) + E_{\pi_f}(X),$$

where $T_{\pi_f}(X)$ and $E_{\pi_f}(X)$ are the main and error terms respectively. Typically $T_{\pi_f} = c_{\pi_f} \cdot X^{1/2+o(1)}$ where $|c_{\pi_f}| \gg \varepsilon N^{-\varepsilon}$. If $T_{\pi_f}(X)$ dominates the error term for some range, then the existence of a $d \sim X$ satisfying (1.1) would immediately follow.

Each $L$-function on the left side of (1.6) has conductor $\asymp_\ell X^2 N$. Let $1 \leq R \ll_\ell X^2 N$ be a parameter chosen later. We use the unbalanced approximate functional equation [IK04, Theorem 5.3] to open each summand in (1.6) (this is morally the same as applying Voronoi summation). Interchanging the order of summation gives

$$\sum_{d \sim X} \frac{1}{d^{1/2}} L(1/2, \pi_f \otimes \chi_d) \approx \sum_{1 \leq n \ll R} \frac{\lambda_f(n)}{n^{1/2}} \sum_{d \sim X} \frac{\chi_d(n)}{d^{1/2}}
+ \varepsilon(f) \sum_{1 \leq n \ll X^2 N/R} \frac{\lambda_f(n)}{n^{1/2}} \sum_{d \sim X} \frac{\chi_d(-Nn)}{d^{1/2}},$$

(1.7)
where \( \varepsilon(\pi_f \otimes \chi_d) = \varepsilon(f)\chi_d(-N) \) is the root number of \( L(1/2, \pi_f \otimes \chi_d) \).

Applying Poisson summation to the first (resp. second) \( d \) summation in gives a dual sum whose length is \( |d| \ll \varepsilon R/X \) (resp. \( |d| \ll \varepsilon XN^2/R \)). In order to gain from this move we would need \( X \gg \varepsilon N^2 \). This is a deadlock for our problem.

To circumvent this we use the conductor drop coming from the factorisation \( N = N_0N_1^2 \).

Observe that \( \chi_d(-N) = \chi_d(-N_0N_1^2) = \chi_d(-N_0) \), and repeating the above Poisson step, we only need \( X \gg (NN_0)^{1/2} \) to shorten both summations.

Post Poisson, one would expect the main terms to come from the zero frequencies from both sums, so we ignore these terms for now. Subconvexity of the symmetric square (1.4) ensures the error incurred from the contour shifting required for main term extraction is acceptable. We point out that [Li79, Example 1] gives \( \varepsilon(\text{Sym}^2\pi_f) = N_0^2N_1^3 \) (as opposed to the full \( N^2 \)), and so we also benefit from this conductor drop.

We choose \( R := X(NN_0)^{1/2} \) to balance the lengths of both \( d \) summations. Interchanging the orders of summation post Poisson and application of the hypothesis (1.3) would in theory yield a result. In reality, there is a conductor raising penalty incurred by the Möbius inversion of the condition \( (d,2N) = 1 \) (equivalent to \( (d,2\text{rad}(N)) = 1 \)), and this is handled in the endgame calculation.

Unfortunately not all integers are fundamental discriminants, and to make the above approach rigorous we use certain combinatorial weights \( \mathcal{P}_d(1/2; \pi_f) \) [BFH04, CG07, CG10, Dia19] \( (\mathcal{P}_d(s; \pi_f) \) is Dirichlet polynomial) coming from the theory of multiple Dirichlet series. We consider the perturbed moment

\[
\sum_{d = d_0d_2^2 \sim X \atop (d,2N) = 1} \mathcal{P}_d(1/2; \pi_f) L(1/2, \pi_f \otimes \chi_{d_0}).
\]

Mellin inversion and other standard moves bring into play a multiple Dirichlet series \( Z(s, w; \pi_f) \). For \((s, w) \in \mathbb{C}^2\) in an appropriate region of absolute convergence,

\[
Z(s, w; \pi_f) \approx \sum_{d \geq 1 \atop d = d_0d_2^2 \atop (d,2N) = 1} \frac{\mathcal{P}_d(s; \pi_f) L(s, \pi_f \otimes \chi_{d_0})}{d^w},
\]

and for another region,

\[
Z(s, w; \pi_f) \approx \sum_{n \geq 1 \atop n = n_0n_1^2 \atop (n,2N) = 1} \frac{\mathcal{Q}_n(w; \pi_f) L(w, \chi_{n_0}\chi_N)}{n^s},
\]

where the \( \mathcal{Q}_n(w; \pi_f) \) are certain combinatorial weights.

The series \( Z(s, w; \pi_f) \) has finite group of functional equations isomorphic to the dihedral group of order 8 (Weyl group of the root system \( C_2 \)) with generators

\[
\gamma_1(s, w) = (1 - s, w + 2s - 1) \quad \text{and} \quad \gamma_2(s, w) = (s + w - 1/2, 1 - w).
\]

Consequently \( Z(s, w; \pi_f) \) has full meromorphic continuation to \( \mathbb{C}^2 \) with well understood polar hyperplanes.

The notion of a “convexity” bound for \( Z(s, w; \pi_f) \) is a priori not well defined. It depends on what is assumed about each summand (both \( L(s, \pi_f \otimes \chi_{d_0}) \) and \( L(w, \chi_{n_0}\chi_N) \)) in various
regions of absolute convergence for $Z(s, w; \pi_f)$. If one assumes the full Lindelöf hypothesis for each $L$-function (in the $d$ and $N$-aspects), this would give

$$Z(1/2, 1/2 + it; \pi_f) \ll_{\ell, \varepsilon, A} N^{1/2 + \varepsilon} (1 + |t|)^A,$$

for some $A > 0$ (see [HK10, Proposition 3.20]). A Mellin inversion and contour shifting argument to the half line as in [HK10, Remark 4.2] would yield the bound $|d| \ll_{\ell, \varepsilon} N^{1 + \varepsilon}$ mentioned in the introduction. Thus it is natural to ask whether one can conditionally go beyond this bound, as we have done in this paper.

A differing school of thought ([Blo11] and [Dah18]) is that the “convexity” bound should correspond to a Lindelöf-on-average bound in the $d$-aspect, and the convexity bound for the other parameters, in the regions of absolute convergence. The advantage of this regime is that these assumptions can be established unconditionally using Heath-Brown’s quadratic large sieve [HB95]. In this weaker setting, Blomer [Blo11] established “subconvex” bounds for GL$_1$ multiple Dirichlet series in the $t$-aspect, and his student Dahl [Dah18] for the same series in the level aspect.

The approximate functional equation/Voronoi move in our heuristic is mimicked by application of the functional equation corresponding to $\gamma_1$. Similarly, the Poisson step is mimicked by application of $\gamma_2$. Blomer observed [Blo11] the equivalence between Poisson and $\gamma_2$, and we make crucial use of this insight in this paper.

An alternative approach could be to Möbius invert the squarefree condition as in [Pet14]. However such an involved computation is not necessarily given that we are only concerned with non-vanishing, and not the full moment sieved to fundamental discriminants.

The global descriptions of the combinatorial weights we use were first discovered by Bump, Friedberg and Hoffstein [BFH04, Theorem 1.2] using brute force methods. We choose to build our weights locally using work of Chinta and Gunnells [CG07, CG10] and Diacunu [Dia19]. One pleasing novelty of this approach is how the algebraic structure of these weights naturally give each term in the Euler product of $L(s, \text{Sym}^2 \pi_f)$ (see Lemma 3.6). The main term of the first moment involves the constant $L(1, \text{Sym}^2 \pi_f)$.

The methods in this paper could probably be extended to the GL$_r(\mathbb{A}_Q)$ cases for $r = 1, 3$.

Section 2 contains basic $L$-functions facts. Section 3 records the relevant multiple Dirichlet series we use and a careful derivation of the scattering matrices that appear in the functional equations for them. Section 4 makes the Voronoi and Poisson heuristic above rigorous and Section 5 contains the proof of Theorem 1.1.

This work is intended be a pleasant interaction between the multiple Dirichlet series and approximate functional equation perspectives, and the equivalences between them.

Acknowledgements

I warmly thank Jeffrey Hoffstein for introducing me to his open question and for many fruitful discussions. I am also grateful to Maksym Radziwill and Matthew Young for their helpful comments, and Henri Darmon for an insight regarding root numbers (communicated via Maksym Radziwill). I would also like to thank Alex Kontorovich for his helpful feedback on my manuscript.

Conventions

Unless otherwise stated, the implied constants are allowed to depend on $\ell$ and an arbitrarily small constant $\varepsilon > 0$. The quantity $\varepsilon > 0$ may differ in each instance it appears.
2. Preliminaries

2.1. \(L\)-functions. Let the Fourier expansion of \(f \in S^\text{new}_\ell(N)\) be given by

\[
f(\tau) := \sum_{n=1}^{\infty} \lambda_f(n)n^{(\ell-1)/2}e^{2\pi in\tau} \in S^\text{new}_\ell(N), \quad \lambda_f(1) = 1, \quad \tau \in \mathbb{H}. \tag{2.1}
\]

The \(L\)-function attached to \(\pi_f\) is

\[
L(s, \pi_f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_{p \nmid N} \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1} \prod_{p | N} \left(1 - \frac{\lambda_f(p)}{p^s}\right)^{-1}, \quad \text{Re } s > 1. \tag{2.2}
\]

Each Euler factor on the far right side of (2.2) is denoted \(L(s, \pi_f, \frac{q}{N})\). The \(L\)-function \(L(s, \pi_f)\) has analytic continuation to all of \(\mathbb{C}\), and satisfies the functional equation [IK04, Theorem 14.17]

\[
\Lambda(s, \pi_f) = \varepsilon(\pi_f)\Lambda(1 - s, \tilde{\pi}_f),
\]

where \(\varepsilon(\pi_f)\) is the root number (\(|\varepsilon(\pi_f)| = 1\)), \(\tilde{\pi}_f\) denotes the contragredient, and

\[
\Lambda(s, \pi_f) := c(\pi_f)^{s/2}\pi^{-s}\Gamma\left(s + \frac{\ell-1}{2}\right)\Gamma\left(s + \frac{\ell+1}{2}\right)L(s, \pi_f).
\]

Since \(f\) has trivial nebentypus, \(\pi_f \cong \tilde{\pi}_f\) (\(\pi_f\) is self-contragredient).

Let \(\chi\) be a primitive Dirichlet character with conductor \(Q\), and

\[
L(s, \pi_f \otimes \chi) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s} = \prod_{p \nmid N} \left(1 - \frac{\lambda_f(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s}}\right)^{-1} \prod_{p | N} \left(1 - \frac{\lambda_f(p)\chi(p)}{p^s}\right)^{-1}, \quad \text{Re } s > 1.
\]

If \((N, Q) = 1\), then by [IK04, Proposition 14.20] we have

\[
\Lambda(s, \pi_f \otimes \chi) = \varepsilon(\pi_f \otimes \chi)\Lambda(1 - s, \tilde{\pi}_f \otimes \chi), \tag{2.3}
\]

with \(c(\pi_f \otimes \chi) = NQ^2\) and root number

\[
\varepsilon(\pi_f \otimes \chi) = \varepsilon(\pi_f)\chi(N)g_\chi^2, \tag{2.4}
\]

where \(g_\chi\) is the normalised Gauss sum attached to \(\chi\). A more convenient formula is

\[
\varepsilon(\pi_f \otimes \chi_d) = \varepsilon(\pi_f)\chi_d(-N). \tag{2.5}
\]

If \(\varepsilon(\pi_f \otimes \chi_d) = -1\) then \(L(1/2, \pi_f \otimes \chi_d) = 0\).

2.2. Dirichlet \(L\)-functions. Let \(\chi\) be a character modulo \(Q\). The Dirichlet \(L\)-function

\[
L(w, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^w}, \quad \text{Re}(w) > 1,
\]

has meromorphic continuation to all of \(\mathbb{C}\). It has simple pole at \(w = 1\) when \(\chi = 1\) is the principal character modulo \(Q\) and is entire if \(\chi \neq 1\).

It \(\chi\) is primitive modulo \(Q\) then we have the functional equation

\[
L(w, \chi) = \frac{g_\chi}{i^s}\left(\frac{Q}{\pi}\right)^{1/2-w}\frac{\Gamma\left(\frac{1-w+a}{2}\right)}{\Gamma\left(\frac{w+a}{2}\right)}L(1 - w, \overline{\chi}), \quad w \in \mathbb{C}, \tag{2.6}
\]
2.3. Real characters. Following [DGH03] and [Blo11] we will use a slightly different notation for real characters in the remainder of the paper.

Let $d$ and $n$ be odd positive integers with factorisations
\[ d = d_0d_1^2 \quad \text{and} \quad n = n_0n_1^2, \quad \text{where} \quad \mu^2(d_0) = \mu^2(n_0) = 1. \] (2.7)

Write
\[ \chi_d(n) := \left( \frac{d}{n} \right) = \tilde{\chi}_n(d). \]

The character $\chi_d$ is the Jacobi–Kronecker symbol of conductor $d_0$ if $d \equiv 1 \pmod{4}$ and $4d_0$ if $d \equiv 3 \pmod{4}$. We have $\chi_d(-1) = 1$, so $\chi_d$ is even. By quadratic reciprocity we have
\[ \tilde{\chi}_n = \begin{cases} \chi_n, & n \equiv 1 \pmod{4} \\ \chi_{-n}, & n \equiv 3 \pmod{4}. \end{cases} \] (2.8)

For $a \in \{\pm 1, \pm 2\}$, let $\chi_a$ denote the four characters modulo 8. That is, $\chi_1$ is the trivial character, $\chi_{-1}$ is induced from the non-trivial character modulo 4, $\chi_2 = 1$ if and only if $n \equiv 1, 7 \pmod{8}$ and $\chi_{-2}(n) = 1$ if and only if $n \equiv 1, 3 \pmod{8}$.

All real primitive characters can be constructed from products $\chi_{d_0}\chi_a$ with $d_0$ odd and squarefree and $a \in \{\pm 1, \pm 2\}$.

In the body of the paper we will also require the $d$ from (2.7) to satisfy $(d, 2N) = 1$. Let
\[ \Div(N) := \{a \cdot c : a \in \{\pm 1, \pm 2\} \quad \text{and} \quad c \mid \rad(N)\}, \]
where $\rad(m) := \prod_{p \mid m} p$ denotes the usual radical of an integer $m$.

When working with multiple Dirichlet series, we write primitive real Dirichlet characters using the convention
\[ \chi_{d_0}\chi_{ac} \quad \text{where} \quad (d_0, 2N) = 1, \quad \text{and} \quad ac \in \Div(N). \] (2.9)

2.4. Subconvexity hypotheses. Let $N$ be as in (1.2), $f \in S^\text{new}_f(N)$ be twist minimal, $ac \in \text{Dic}(N)$, and $d_0$ squarefree with $(d_0, 2N) = 1$. Then we have $c(\pi_f \otimes \chi_{d_0}\chi_{ac}) \mid 8Ncd_0^2$ by [AL78, Proposition 3.1]. Thus we can relax the subconvexity hypothesis (1.3) becomes
\[ L(1/2 + it, \pi_f \otimes \chi_{d_0}\chi_{ac}) \ll_{\ell, A, \varepsilon} (Ncd_0^2)^{1/4 - \delta_1 + \varepsilon}(1 + |t|)^A. \] (2.10)

We have $c(\Sym^2\pi_f) = N_0^2N_1^3$ by [Li79, Example 1], and so the subconvexity hypothesis (1.4) becomes
\[ L(1/2 + it, \Sym^2\pi_f) \ll_{\ell, A, \varepsilon} (N_0^2N_1^3)^{1/4 - \delta_2 + \varepsilon}(1 + |t|)^A, \] (2.11)
for some $A > 0$.

A consequence of (2.10), dyadic partition of unity, Mellin inversion and a trivial estimation of Euler factors attached to primes $p \mid 2N$ is the following

**Lemma 2.1.** Suppose $f \in S^\text{new}_f(N)$ satisfies (2.10) and has Fourier expansion (2.1). Then for $ac \in \Div(N)$, $d_0$ squarefree with $(d_0, 2N) = 1$, $Q \geq 1$, $t \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\sum_{1 \leq n \leq Q \atop (n, 2N) = 1} \frac{\lambda_f(n)\chi_{ac}(n)\chi_{d_0}(n)}{n^{1/2 + it}} \ll_{\ell, A, \varepsilon} Q^\varepsilon (Ncd_0^2)^{1/4 - \delta_1 + \varepsilon}(1 + |t|)^A,
\]
for some $A > 0$.

3. Construction of the Multiple Dirichlet Series

The classical approach to multiple Dirichlet series and automorphic forms is well summarised by [BFH04, BBC + 06].

Global formulas for the combinatorial weights we need were originally discovered by Bump, Friedberg and Hoffstein [BFH04, Theorem 1.2] using brute force methods. We opt to describe the multiple Dirichlet series locally using the recipe briefly described on [CG07, pg. 331]. Such a description makes clear the relationship between the local representations $\pi_{f,p}$ and the $p$-part of the multiple Dirichlet series relevant to our situation.

3.1. Chinta–Gunnells action in the case $A_3$. The power of the Chinta–Gunnells construction [CG07, CG10] is that it works for arbitrary simply laced root systems. For simplicity we focus on the Dynkin diagram $A_3$, which is sufficient for our purposes. A more general summary can be found in [Dia19, Section 2.1].

Note that the multiple Dirichlet series we use in Section 3.2 is a two variable specialisation of a three variable series whose group of functional equations is isomorphic to the Weyl group of $A_3$, denoted $W(A_3)$. Hence the specialised series has group of functional equations isomorphic to $W(C_2)$ ($C_2$ is the Dynkin fold of $A_3$), as mentioned in section 1.2.

After fixing an ordering of the roots, let $A_3 = A_3^+ \cup A_3^-$ denote a decompositon into positive and negative roots. Let $\alpha_1, \alpha_2$ and $\alpha_3$ be simple roots (where $\alpha_3$ corresponds to the central node of the Dynkin diagram of $A_3$) and $\sigma_i \in W(A_3)$ be the simple reflection through a hyperplane perpendicular to $\alpha_i$. The simple reflections $\sigma_i$ generate the Weyl group and satisfy

$$(\sigma_i \sigma_j)^{r_{ij}} = 1,$$

with $r_{ii} = 1$, $r_{12} = r_{21} = 2$ and $r_{13} = r_{31} = r_{23} = r_{32} = 3$.

The action of simple reflections on roots is given by

$$\sigma_i \alpha_j = \begin{cases} 
\alpha_i + \alpha_j, & \text{if } \alpha_i \text{ and } \alpha_j \text{ are adjacent} \\
-\alpha_j, & \text{if } i = j \\
\alpha_j, & \text{otherwise.}
\end{cases}$$

Let $\Lambda(A_3)$ be the root lattice of $A_3$. Each element $\lambda \in \Lambda(A_3)$ has a unique representation as an integral combination of simple roots

$$\lambda = k_1 \alpha_1 + k_2 \alpha_2 + k_3 \alpha_3. \quad (3.1)$$

Set $z := (z_1, z_2, z_3)$ and for $\lambda \in \Lambda(A_3)$, set $z^\lambda := z_1^{k_1} z_2^{k_2} z_3^{k_3}$. Fix a parameter $q \geq 1$ (we will later take $q = p$ prime). Define $\epsilon_i z = z'$, where

$$z'_j = \begin{cases} 
-z_j, & \text{if } i \text{ and } j \text{ are adjacent} \\
z_j, & \text{otherwise},
\end{cases}$$

and $\alpha_i z = \tilde{z}$, where

$$\tilde{z}_j = \begin{cases} 
\sqrt{q} z_i z_j, & \text{if } i \text{ and } j \text{ are adjacent} \\
1/(q z_j), & \text{if } i = j \\
z_j, & \text{otherwise.}
\end{cases}$$
For $h \in \mathbb{C}(z)$, let
\[
  h_i^\pm(z) := \frac{1}{2}(h(z) \pm h(a_i z)).
\]

(3.2)

The action of a simple reflection $\sigma_i$ on $h \in \mathbb{C}(z)$ is given by
\[
  (h|\sigma_i)(z) := -\frac{1 - qz_i}{qz_i(1 - z_i)}h_i^+(\sigma_i z) + \frac{1}{\sqrt{qz_i}}h_i^-(\sigma_i z).
\]

(3.3)

This extends to a $W(A_3)$-action on $\mathbb{C}(z)$ [CG07, Lemma 3.2].

Using the Weyl group action in (3.3), Chinta and Gunnells [CG07, CG10] constructed a $W(A_3)$ invariant function $g_{A_3}(z) \in \mathbb{C}(z)$ such that
- $g_{A_3}(0, q) = 1$;
- for each $i = 1, 2, 3$, the function $(1 - z_i) \cdot g_{A_3}(z; q)|_{z_j = 0}$ for all $j$ adjacent to $i$ is independent of $z_i$.

The rational function satisfying the above conditions is unique [Whi14, Whi16].

A straightforward computation verifies that
\[
g_{A_3}(z; q) := \frac{1 - z_1 z_3 - z_2 z_3 + z_1 z_2 z_3 + qz_1 z_2 z_3^2 - qz_1^2 z_2 z_3^2 - qz_1 z_2^2 z_3 + qz_1^2 z_2^2 z_3^2}{(1 - z_1)(1 - z_2)(1 - z_3)(1 - qz_1^2 z_3^2)(1 - qz_2^2 z_3^2)(1 - qz_1^2 z_2^2 z_3^2)}
\]

(3.4)

is the desired function [CG07, Example 3.7] (our vertices are labelled differently). Expand $g_{A_3}(z; q)$ in a power series
\[
g_{A_3}(z; q) = \sum_{k_1, k_2, j \geq 0} a(k_1, k_2, j; q)z_1^{k_1}z_2^{k_2}z_3^j.
\]

(3.5)

Define the symmetric polynomials $P_j(z_1, z_2; q) \in \mathbb{C}[z_1, z_2]$ by the expression
\[
g_{A_3}(z; q) := g_1^+(z; q) + g_1^-(z; q)
= (1 - z_1)^{-1}(1 - z_2)^{-1}\sum_{j \text{ even}} P_j(z_1, z_2; q)z_3^j + \sum_{j \text{ odd}} P_j(z_1, z_2; q)z_3^j.
\]

(3.6)

Similarly, polynomials $Q_k(z_3; q) \in \mathbb{C}[z_3]$ can be defined by the relationship
\[
g_{A_3}(z; q) := g_3^+(z; q) + g_3^-(z; q)
= (1 - z_3)^{-1}\sum_{k=(k_1, k_2) \text{ even}} Q_k(z_3; q)z_1^{k_1}z_2^{k_2} + \sum_{k=(k_1, k_2) \text{ odd}} Q_k(z_3; q)z_1^{k_1}z_2^{k_2},
\]

(3.7)

where $|k| := k_1 + k_2$. Since $q$ is invariant under the action generated by (3.3), we have the formal functional equations
\[
P_j(z_1, z_2; q) = (\sqrt{q}z_1)^{j-a}P_j\left(\frac{1}{qz_1}, z_2; q\right) = (\sqrt{q}z_2)^{j-a}P_j\left(z_1, \frac{1}{qz_2}; q\right),
\]

(3.8)

and
\[
Q_k(z_3; q) = (\sqrt{q}z_3)^{|k|a}Q_k\left(\frac{1}{qz_3}; q\right),
\]

(3.9)

where $a_n = 0$ or $1$ according to whether $n$ is even or odd respectively.
3.2. Definition of series and some properties. Suppose \( a_1, c_1, a_2, c_2 \in \text{Div}(N) \). Our arguments will use the following multiple Dirichlet series (formally defined as)

\[
Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) := \sum_{m_1, m_2, d \geq 1 \atop \gcd(m_1m_2d, 2N) = 1} \frac{H^{\pi_f}(m_1, m_2, d)\chi_{a_2c_2}(d)\chi_{a_1c_1}(m_1m_2)}{(m_1m_2)^{s}d^w}, \quad (3.10)
\]

where the coefficients \( H^{\pi_f}(m_1, m_2, d) \) will now be defined using the recipe of [CG07, Section 4].

The function \( H^{\pi_f}(m_1, m_2, d) \) on quadruples of odd integers satisfies a twisted multiplicativity property. For \((m_1m_2d, m_1'm_2'd) = 1\) we have

\[
H^{\pi_f}(m_1m_1', m_2m_2', dd') = H^{\pi_f}(m_1, m_2, d)H^{\pi_f}(m_1', m_2', d') \left( \frac{d}{m_1m_2} \right) \left( \frac{d'}{m_1'm_2'} \right). \quad (3.11)
\]

Given property (3.11), it suffices to define \( H^{\pi_f}(p^{k_1}, p^{k_2}, p^j) \) for all primes \( p \nmid 2N \). These coefficients are recorded by the generating function (called the \( p \)-part):

\[
Z_p^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) := \sum_{k_1, k_2, j \geq 0} \frac{H^{\pi_f}(p^{k_1}, p^{k_2}, p^j)\chi_{a_2c_2}(p^j)\chi_{a_1c_1}(p^{k_1+k_2})}{p^{(k_1+k_2)s+jw}} = g_{A_3} \left( \chi_{a_1c_1}(p)\alpha_p p^{-s}, \chi_{a_1c_1}(p)\beta_p p^{-s}, \chi_{a_2c_2}(p)p^{-w}; p \right),
\]

where \( \alpha_p, \beta_p \) are the Satake parameters attached to \( \pi_{f,p} \). In other words,

\[
H^{\pi_f}(p^{k_1}, p^{k_2}, p^j) := a(k_1, k_2, j; p)\alpha_p^{k_1}\beta_p^{k_2} \quad \text{for all primes } p \nmid 2N \quad \text{and} \quad j, k_1, k_2 \geq 0, \quad (3.12)
\]

where \( a(k_1, k_2, j; p) \) are the coefficients in the power series expansion in (3.5).

Using

\[
Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) = \prod_p Z_p^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f),
\]

we see that \( Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) \) converges absolutely for \( \Re(s), \Re(w) \gg 1 \).

We also have

\[
H^{\pi_f}(p^{k_1}, p^{k_2}, p^j) = \alpha_p^{k_1}\beta_p^{k_2}, \quad \text{when} \quad \min(k_1 + k_2, j) = 0; \quad (3.13)
\]

\[
H^{\pi_f}(p^{k_1}, p^{k_2}, p^j) = 0, \quad \text{when} \quad \min(k_1 + k_2, j) = 1; \quad (3.14)
\]

\[
H^{\pi_f}(p^{k_1}, p^{k_2}, p^j) = 0, \quad \text{when} \quad k_1 + k_2 \equiv j \equiv 1 \pmod{2}. \quad (3.15)
\]
For \( d, m_1, m_2 \) positive integers with \((dm_1m_2, 2N) = 1\), consider the following Dirichlet polynomials built from the \( P_j \) and \( Q_k \) given in (3.6) and (3.7):

\[
\mathcal{P}_d(s, \chi_{a_1c_1}; \pi_f) := \prod_{p^j|d} P_j \left( \chi_{a_1c_1}(p) \chi_{dp^{-j}}(p) \alpha_pp^{-s}, \chi_{a_1c_1}(p) \chi_{dp^{-j}}(p) \beta_pp^{-s}; p \right)
\times \prod_{p^j|d} P_j(\alpha_pp^{-s}, \beta_pp^{-s}; p);
\tag{3.16}
\]

\[
\mathcal{Q}_{m_1,m_2}(w, \chi_{a_2c_2}; \pi_f) := \prod_{p^{k_1}|m_1} \alpha_p^{k_1} \beta_p^{k_2} Q_k(\chi_{a_2c_2}(p) \chi_{m_1m_2p^{-k_1-k_2}}(p)p^{-w}; p)
\times \prod_{p^{k_2}|m_2} \alpha_p^{k_1} \beta_p^{k_2} Q_k(p^{-w}; p);
\tag{3.17}
\]

\[
\bar{\mathcal{Q}}_n(w, \chi_{a_2c_2}; \pi_f) := \sum_{m_1m_2=n} \mathcal{Q}_{m_1,m_2}(w, \chi_{a_2c_2}; \pi_f).
\tag{3.18}
\]

Observe that

\[
\bar{\mathcal{Q}}_n(w, \chi_{a_2c_2}; \pi_f) := \prod_{p^k|n} \left( \sum_{k_1+k_2=k, k_1,k_2 \geq 0} \alpha_p^{k_1} \beta_p^{k_2} Q_k(\chi_{a_2c_2}(p) \chi_{np^{-k}}(p)p^{-w}; p) \right)
\times \prod_{p^k|n} \left( \sum_{k_1+k_2=k, k_1,k_2 \geq 0} \alpha_p^{k_1} \beta_p^{k_2} Q_k(p^{-w}; p) \right).
\tag{3.19}
\]

The Dirichlet polynomials \( \mathcal{P}_d \) (resp. \( \bar{\mathcal{Q}}_n \)) inherit functional equations from the local ones for \( P_j \) (resp. \( Q_k \)) given in (3.8) (resp. (3.9)).

**Lemma 3.1.** Let \( f \in \mathcal{S}_f^{\text{new}}(N) \) and \( a_1c_1, a_2c_2 \in \text{Div}(N) \). Suppose that \( d = d_0d_1^2 \) and \( n = n_0n_1^2 \) where \((dn, 2N) = 1\) and \( \mu^2(d_0) = \mu^2(n_0) = 1\). Then

\[
\mathcal{P}_d(s, \chi_{a_1c_1}; \pi_f) = d_1^{2-4s} \mathcal{P}_d(1-s, \chi_{a_1c_1}; \pi_f),
\tag{3.20}
\]

and

\[
\bar{\mathcal{Q}}_n(w, \chi_{a_2c_2}; \pi_f) = n_1^{-2w} \bar{\mathcal{Q}}_n(1-w, \chi_{a_2c_2}; \pi_f),
\tag{3.21}
\]

where \( \mathcal{P}_d(s, \chi_{a_1c_1}; \pi_f) \) and \( \bar{\mathcal{Q}}_n(w, \chi_{a_2c_2}; \pi_f) \) are defined by (3.16) and (3.18).

The Dirichlet polynomials \( \mathcal{P}_d \) and \( \bar{\mathcal{Q}}_n \) satisfy crude bounds.

**Lemma 3.2.** Let \( f \in \mathcal{S}_f^{\text{new}}(N) \) and \( a_1c_1, a_2c_2 \in \text{Div}(N) \). Suppose that \( d = d_0d_1^2 \) and \( n = n_0n_1^2 \) where \((dn, 2N) = 1\) and \( \mu^2(d_0) = \mu^2(n_0) = 1\). Then

\[
|\mathcal{P}_d(s, \chi_{a_1c_1}; \pi_f)| \ll \epsilon \left\{ \begin{array}{ll} d_1^\epsilon, & \text{Re}(s) \geq \frac{1}{2} \\ d_1^{2-4\text{Re}(s)+\epsilon}, & \text{Re}(s) < \frac{1}{2} \end{array} \right.,
\tag{3.22}
\]
\[ |\tilde{Q}_n(w, \chi_{a_2c_2}; \pi_f)| \ll_{\varepsilon} \begin{cases} n_1^{1/2 + \varepsilon}, & \text{Re}(w) \geq \frac{1}{2}, \\ n_1^{3/2 - 2\text{Re}(w) + \varepsilon}, & \text{Re}(w) < \frac{1}{2}, \end{cases} \] 

(3.23)

where \( P_d(s, \chi_{a_1c_1}; \pi_f) \) and \( \tilde{Q}_n(w, \chi_{a_2c_2}; \pi_f) \) are defined by (3.16) and (3.18). The implied constants depend only on \( \varepsilon > 0 \).

**Proof.** This follows from a straightforward modification of the argument in [Dia19, Appendix B] using the maximum principle and Cauchy’s inequality. \( \blacksquare \)

The Dirichlet polynomials \( P_d(s, \chi_{a_1c_1}; \pi_f) \) and \( \tilde{Q}_n(w, \chi_{a_2c_2}; \pi_f) \) are in known in the literature as **correction polynomials**. Their purpose is to give two different representations of (3.10), each absolutely convergent in different tube domains in \( \mathbb{C}^2 \) [BFH04, HK10].

**Lemma 3.3.** Let \( f \in S^\new\ell(N) \) and \( a_1c_1, a_2c_2 \in \text{Div}(N) \). We have

\[ Z(N)(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) = \sum_{d \geq 1} \frac{L^{(2N)}(s, \pi_f \otimes \chi_{a_1c_1d_0}) \chi_{a_2c_2}(d) P_d(s, \chi_{a_1c_1}; \pi_f)}{d^w}, \] 

(3.24)

on the domain

\[ \Omega_1 := \{(s, w) \in \mathbb{C}^2 : 2 \text{Re}(s) + \text{Re}(w) > 2\} \cap \{(s, w) : \text{Re}(w) > 1\}, \] 

(3.25)

and

\[ Z(N)(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) = \sum_{n \geq 1} \frac{L^{(2N)}(w, \chi_{a_2c_2} \chi_{n0}) \chi_{a_1c_1}(n) \tilde{Q}_n(w, \chi_{a_2c_2}; \pi_f)}{n^s}, \] 

(3.26)

on the domain

\[ \Omega_2 := \{(s, w) \in \mathbb{C}^2 : \text{Re}(s) + \text{Re}(w) > \frac{3}{2}\} \cap \{(s, w) : \text{Re}(s) > 1\}, \] 

(3.27)

with exception of a polar hyperplane \( w = 1 \) when \( a_2 = c_2 = 1 \). Thus the functions

\[ (w - 1)Z(N)(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f) \] 

(3.28)

are holomorphic on \( \Omega_1 \cup \Omega_2 \).

**Proof.** The formulas (3.24) and (3.26) hold for \( \text{Re}(s), \text{Re}(w) \gg 1 \) by a straightforward modification of the computations in [Dia19, Section 3].

Then (3.24) (resp. (3.26)) can be extended to hold on the domain (3.25) (resp. (3.27)) using Heath–Brown’s quadratic large sieve [HB95, Corollary 3] together with (2.3) and (3.20) (resp. (2.6) and (3.21)), and the bound (3.22) (resp. (3.23)). \( \blacksquare \)
3.3. Functional equations, meromorphic continuation and residues. Recall that recall $N$ is as in (1.2) and $f \in S^\text{new}_\ell(N)$ is twist minimal.

Our argument in Section 4 requires the exact scattering matrix for the functional equations involving $Z^\lambda(s, w; \chi_{a_1c_1}, \chi_{a_1c_1}; \pi_f)$. Their precise shape is determined by the factorisation of $N$ in (1.2) (in other words the ramified local representations $\pi_{f,p}$).

Following [DGH03, Section 4], we will store the multiple Dirichlet series defined above in vector form. Denote

$$\overline{Z}^\lambda(s, w; \chi_{\text{Div}(N)}, \chi_{a_1c_1}; \pi_f), \quad \text{(resp. } \overline{Z}^\lambda(s, w; \chi_{a_2c_2}, \chi_{\text{Div}(N)}; \pi_f))$$

the $4|\text{rad}(N)| \times 1$ column vector whose entries are

$$Z^\lambda(s, w; \chi^{(j)}, \chi_{a_1c_1}; \pi_f) \quad \text{(resp. } Z^\lambda(s, w; \chi_{a_2c_2}, \chi^{(j)}; \pi_f)),$$

where $\chi^{(j)}$ for $j = 1, \ldots, 4|\text{rad}(N)|$ range over the characters $\chi_{a_2c_2}$ (resp. $\chi_{a_1c_1}$).

Let $\gamma_1, \gamma_2 : C^2 \to C^2$ be two involutions defined by

$$\gamma_1(s, w) = (1 - s, w + 2s - 1) \quad \text{and} \quad \gamma_2(s, w) = (s + w - 1/2, 1 - w).$$

The symmetry group generated by these two involutions is isomorphic to the dihedral group of order 8.

A version of the following lemma is an implicit in [HK10].

**Lemma 3.4.** Let $f \in S^\text{new}_\ell(N)$ be twist minimal and $\Omega_1$ (resp. $\Omega_2$) be as in (3.25) (resp. (3.27)) respectively. For each $a_1c_1 \in \text{Div}(N)$, there exists a $4|\text{rad}(N)| \times 4|\text{rad}(N)|$ matrix $\Phi_{a_1c_1}(s; \pi_f)$ of meromorphic functions in $s$ such that for all $(s, w) \in \Omega_1$, we have

$$\overline{Z}^\lambda(s, w; \chi_{\text{Div}(N)}, \chi_{a_1c_1}; \pi_f) = \Phi_{a_1c_1}(s; \pi_f) \overline{Z}^\lambda(1 - s, w + 2s - 1; \chi_{\text{Div}(N)}, \chi_{a_1c_1}; \pi_f). \quad (3.29)$$

For each $a_2c_2 \in \text{Div}(N)$, there exists a $4|\text{rad}(N)| \times 4|\text{rad}(N)|$ matrix $\Psi_{a_2c_2}(w; \pi_f)$ of meromorphic functions in $w$ such that for all $(s, w) \in \Omega_2$ we have

$$\overline{Z}^\lambda(s, w; \chi_{a_2c_2}, \chi_{\text{Div}(N)}; \pi_f) = \Psi_{a_2c_2}(w; \pi_f) \overline{Z}^\lambda(s + w - 1/2, 1 - w; \chi_{a_2c_2}, \chi_{\text{Div}(N)}; \pi_f). \quad (3.30)$$

For twist minimal $f \in S^\text{new}_\ell(N)$ (trivial nebentypus), [LW12, Proposition 2.8] implies that

$$N_0 = \prod_{p \mid |N| \text{ special representation}} p, \quad (3.31)$$

and

$$N_1 = \prod_{p \mid |N| \text{ supercuspidal representation}} p. \quad (3.32)$$

For each $p \mid N_0$, let $\alpha_p$ be the Satake parameter attached to the special representation $\pi_{f,p}$. For these primes we have $\alpha_p^2 = p^{-1}$.

For $c_1, c_2, c_2' \mid \text{rad}(N)$, define

$$\mathcal{N}(\pi_f)_{c_1c_2c_2'} := \prod_{p \mid |N|} p.$$

For a Dirichlet character $\chi$, let $\overline{c}(\chi) := (c(\chi), 8)$. 


Lemma 3.5. Let $N$ be as in (1.2), $f \in \mathcal{S}_f^{\mathrm{new}}(N)$ be twist minimal, and $\mathcal{N}(\pi_f)_{c_1c_2c'_2}$ be as in (3.33). Then for $a_1c_1, a'_1c'_1, a_2c_2, a'_2c'_2 \in \text{Div}(N)$, we have the formulae

$$
\Phi_{a_1c_1}(s; \pi_f)_{a_2c_2a'_2c'_2}
= 2^{-2} \varepsilon(\pi_f \otimes \chi_{X-1(c_1)c_1}) \cdot \delta_{\mathcal{N}(\pi_f)_{c_1c_2c'_2}} \frac{\chi_{\chi-1(c_1)c_1}}{\chi_{\chi-1(c_1)c_1}} \cdot \chi_{\chi-1(c_1)c_1}( -c(\pi_f \otimes \chi_{X-1(c_1)c_1}) )
\times (c(\pi_f \otimes \chi_{X-1(c_1)c_1}))^{1/2-s/p-1+2s} \frac{\Gamma\left(1-\frac{s+c_1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \chi_{a_1c_1}(\mathcal{N}(\pi_f)_{c_1c_2c'_2})
\times \left\{ \tilde{c}(\chi_{X-1(c_1)c_1})^{1-2s} \left[ \frac{L(1-s, \pi_f \otimes \chi_{a_1c_1})}{L(s, \pi_f \otimes \chi_{a_1c_1})} + \frac{\chi_{a_2c_2}(7) L(1-s, \pi_f \otimes \chi_{7a_1c_1})}{L(s, \pi_f \otimes \chi_{7a_1c_1})} \right] \right\}
\times \prod_{p \mid \mathcal{N}(\pi_f)_{c_1c_2c'_2}} \frac{\alpha_p (p^{-1} - s - p^{-s})}{1 - p^{-3+2s}},
\right.
\end{equation}

and

$$
\Psi_{a_2c_2}(w; \pi_f)_{a_1c_1a'_1c'_1}
= 2^{-2} \frac{1}{c_2} \cdot \frac{1}{\pi} \cdot \frac{1}{\pi} \cdot \frac{1}{w} \chi_{a_2c_2} \left( \frac{c_1c'_1}{(c_1, c'_1)^2} \right)
\prod_{p \mid (c_1, c'_1)c_1} \frac{p^{-1} - w - p^{-w}}{1 - p^{-2(1-w)}}
\prod_{p \mid \text{rad}(N)/\{c_1, c'_1\}} \frac{1 - p^{-1}}{1 - p^{-2(1-w)}}
\times \left\{ \tilde{c}(\chi_{a_2c_2}) \frac{1}{\Gamma\left(\frac{1-w}{2}\right)} \left[ \frac{L(1-w, \chi_{a_2c_2})}{L(w, \chi_{a_2c_2})} + \chi_{a_1c_1}(5) \frac{L(1-w, \chi_{a_2c_2})}{L(w, \chi_{a_2c_2})} \right] \right\}
\times \left\{ \tilde{c}(\chi_{-a_2c_2}) \frac{1}{\Gamma\left(\frac{1+w}{2}\right)} \left[ \chi_{a_1c_1}(5) \frac{L(1-w, \chi_{a_2c_2})}{L(w, \chi_{a_2c_2})} \right] \right\},
\right.
\end{equation}

Proof. These computations are a generalisation of the ideas in proof of [DW18, Theorem 2.3]. Fix a full set of squarefree positive representatives for $\left( \frac{2}{\text{rad}(N)} \right)^{\times} / \left( \frac{2}{\text{rad}(N)\mathcal{N}} \right)^{\times}$, denote it by $C_{\text{rad}(N)}$.

In order to use (2.3), write

$$
\pi_f \otimes \chi_{\text{ad}0} = (\pi_f \otimes \chi_{X-1(c)c_1}) \otimes \chi_{X-1(c)\text{ad}0}.
$$
Denote
\[ \Lambda_N(s, \pi_f \otimes \chi_{acd_0}) := \left( \frac{\pi}{e(\chi_{-1}(c_{d_0}) \sqrt{c(\pi_f \otimes \chi_{-1}(c))}} \right)^{-s} \Gamma\left( \frac{s + \ell - 1}{2} \right) \Gamma\left( \frac{s + \ell + 1}{2} \right) \times \prod_{\substack{p \mid 2N \\ p \mid (\chi_{acd_0})}} L(s, \pi_{f,p} \otimes \chi_{acd_0}). \]

For \( p \mid N_1 \) we have \( L(s, \pi_{f,p}) = 1 \), and for \( p \mid N_0 \) we have \( L(s, \pi_{f,p}) = (1 - \alpha_p p^{-s})^{-1} \). Thus
\[ \prod_{\substack{p \mid 2N \\ p \mid (\chi_{acd_0})}} L(s, \pi_{f,p} \otimes \chi_{acd_0}) = L(s, \pi_{f,2} \otimes \chi_{acd_0}) \cdot \prod_{\substack{p \mid N_0 \\ p \mid (\chi_{acd_0})}} \frac{1}{1 - \alpha_p \chi_{acd_0}(p)p^{-s}}. \]

For \( D \in C_{\text{rad}(N)} \) and \( (s, w) \in \Omega_1 \), the linear combination
\[ 2^{-\omega(\text{rad}(N)) - 2} \chi_{a_2c_2}(D) \sum_{a'_2c'_2 \in \text{Div}(N)} \chi_{a'_2c'_2}(D) Z^{(N)}(s, w; a'_2c'_2, a_1c_1; \pi_f) \]
isolates summands the summands of \( Z^{(N)}(s, w; a_2c_2, a_1c_1; \pi_f) \) that have \( dD \) congruent to a square modulo \( \text{rad}(N) \). We now combine (2.3) and Lemma 3.1 to obtain the following observation. For \( (s, w) \in \Omega_1 \), the function
\[ \Lambda_N(s, \pi_f \otimes a_1c_1 D) \sum_{a'_2c'_2 \in \text{Div}(N)} \chi_{a'_2c'_2}(D) Z^{(N)}(s, w; a'_2c'_2, a_1c_1; \pi_f), \]
is invariant under the involution \( \gamma_1 \) (note that \( \gamma_1(\Omega_1) = \Omega_1 \)), exactly up to the root number (cf. (2.5))
\[ \varepsilon(\pi_f \otimes \chi_{-1(c_1)c_1}) \chi_{-1(c_1)c_1} a_1 D (-c(\pi_f \otimes \chi_{-1(c_1)c_1})). \]

For \( (s, w) \in \Omega_1 \), write
\[ Z^{(N)}(s, w; a_2c_2, a_1c_1; \pi_f) = 2^{-\omega(\text{rad}(N)) - 2} \sum_{D \in C_{\text{rad}(N)}} \chi_{a_2c_2}(D) \times \sum_{a'_2c'_2 \in \text{Div}(N)} \chi_{a'_2c'_2}(D) Z^{(N)}(s, w; a'_2c'_2, a_1c_1; \pi_f). \]

Using (3.37), (3.38) and (3.39), we see that the following holds for all \( (s, w) \in \Omega_1 \),
\[ Z^{(N)}(s, w; a_2c_2, a_1c_1; \pi_f) = 2^{-\omega(\text{rad}(N)) - 2} \varepsilon(\pi_f \otimes \chi_{-1(c_1)c_1}) \chi_{-1(c_1)c_1} (1 - c(\pi_f \otimes \chi_{-1(c_1)c_1})) \times \sum_{a'_2c'_2 \in \text{Div}(N)} Z^{(N)}(1 - s, w + 2s - 1; a'_2c'_2, a_1c_1; \pi_f) \times \sum_{D \in C_{\text{rad}(N)}} \chi_{a_2c_2}(D) \chi_{a'_2c'_2}(D) \chi_D (-c(\pi_f \otimes \chi_{-1(c_1)c_1})) \times \frac{\Lambda_N(1 - s, \pi_f \otimes a_1c_1 D)}{\Lambda_N(s, \pi_f \otimes a_1c_1 D)}. \]
Thus by (3.36),
\[
\frac{\Lambda_N(1 - s, \pi_f \otimes \chi_{a_1c_1D})}{\Lambda_N(s, \pi_f \otimes \chi_{a_1c_1D})} = \pi^{-1+2s} \cdot \tilde{c}(X_{\chi_{a_1c_1D}})^{1-2s} \cdot \left( c(\pi_f \otimes X_{\chi_{a_1c_1D}})^{1/2-s} \right.
\]
\[
\times \frac{\Gamma\left(\frac{1-s+f+c}{2}\right)}{\Gamma\left(\frac{s+f+c}{2}\right)} \cdot \frac{L(1-s, \pi_{f,2} \otimes \chi_{a_1c_1D})}{L(s, \pi_{f,2} \otimes \chi_{a_1c_1D})} \times \prod_{p \mid N_0 \atop p \nmid c_1} \frac{1-p^{-2} + \alpha_p \chi_{a_1c_1D}(p)(p^{-1-s} - p^{-s})}{1-p^{-3+2s}}.
\]

(3.41)

After substitution of (3.41) into (3.40), and expansion of the products, we see that each term will vanish when the summation over $D$ is performed unless the total character in $D$ is trivial. This is only possible for summands corresponding to $c_2' \mid \text{rad}(N)$ that satisfy

\[
\mathcal{N}(\pi_f)_{c_1} c_2' \mid \frac{N_0}{(c_1, N_0)}.
\]

Thus we obtain (3.34).

Note that (3.35) follows from an analogous, but simpler computation.

**Proposition 3.1.** Let $f \in S^\text{new}(N)$ be twist minimal. For each $a_1c_1, a_2c_2 \in \text{Div}(N)$, the function

\[
w(w-1)(2s+w-1)(2s+w-2)Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f)
\]

has a holomorphic extension to all of $\mathbb{C}^2$. For each $(z, w) \in \mathbb{C}^2$, there exists a constant $C := C(\text{Re}(z), \text{Re}(w), \ell)$ such that we have

\[
|w(w-1)(2s+w-1)(2s+w-2)Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f)|
\]

\[
\ll_{\pi_f, c_1, c_2, \text{Re}(z), \text{Re}(w)} \left| (1 + |\text{Im}(s)|)(1 + |\text{Im}(w)|) \right|^C.
\]

(3.43)

**Proof.** This a straightforward modification of the several complex variable arguments in [Blo11, Lemma 2] and [DGH03, Section 4.3].

Both (3.34) and (3.35) are both holomorphic functions for $\text{Re}(s) < 1$ and $\text{Re}(w) < 1$ respectively.

One iteratively applies (3.29) and (3.30) to holomorphically extend each of the functions in (3.42) to all of $\mathbb{C}^2 \setminus P^*$, where

\[
P^* := \{(s, w) : (\text{Re } s, \text{Re } w) \in P\} \subseteq \{(s, w) : |\text{Re}(s)|^2 + |\text{Re}(w)|^2 \leq 10\},
\]

and $P \subseteq \mathbb{R}^2$ is a certain closed polygon. Bochner’s Theorem [Boc38] then holomorphically extends the functions in (3.42) to all of $\mathbb{C}^2$. One can then use [DGH03, Propositions 4.6 and 4.7] and the argument on [DGH03, pg. 341] to establish (3.43).

**Remark 3.1.** We point out that the bound (3.43) is not used in obtaining the main results of this paper. Its purpose is to ensure absolute convergence for certain contour integrals in $s$ and $w$ after Mellin inversion of smooth functions.

**Lemma 3.6.** Let $f \in S^\text{new}(N), a_1c_1, a_2c_2 \in \text{Div}(N)$, and consider $Z^{(N)}(s, w; \chi_{a_2c_2}, \chi_{a_1c_1}; \pi_f)$ on the domain $(s, w) \in \Omega_2$ given in (3.27). If $\chi_{a_2c_2}$ is non-trivial, then the function
$Z^{(N)}(s, w; \chi_{a_2 c_2}, \chi_{a_1 c_1}; \pi_f)$ is holomorphic on $\Omega_2$. If $a_2 = c_2 = 1$, then there is a polar hyperplane at $w = 1$ with residue

$$\text{Res}_{w=1} Z^{(N)}(s, w; 1, \chi_{a_1 c_1}; \pi_f) = L^{(2N)}(2s, \text{Sym}^2 \pi_f) \prod_{p|2N} (1 - p^{-1}).$$

**Proof.** Recall the representation (3.26), valid for $(s, w) \in \Omega_2$. The only summands of (3.26) that have poles are those with $a_2 = c_2 = n_0 = 1$, and these are each simple and come from the Dirichlet $L$–function.

Then

$$\text{Res}_{w=1} Z^{(N)}(s, w; 1, \chi_{a_1 c_1}; \pi_f) \quad (3.44)$$

$$= \prod_{p|2N} (1 - p^{-1}) \sum_{m_1, m_2 \geq 1 \atop (m_1 m_2, 2N) = 1} \frac{Q_{m_1, m_2}(1; 1; \pi_f)}{(m_1 m_2)^s}$$

$$= \prod_{p|2N} (1 - p^{-1}) \prod_{p|2N} \left(1 + \sum_{k_1+k_2 \geq 2 \atop k_1+k_2 \text{ even}} \frac{\alpha_p^{k_1} \beta_p^{k_2} Q_{k_1, k_2}(p^{-1}; p)}{p^{s(k_1+k_2)}} \right)$$

$$= \prod_{p|2N} (1 - p^{-1}) \prod_{p|2N} \left(1 - p^{-1}\right) g_3^{+}(\alpha_p p^{-s}, \beta_p p^{-s}, p^{-1}; p)$$

$$= \prod_{p|2N} (1 - p^{-1}) \prod_{p|2N} \left(1 - p^{-2s}\right)^{-1}(1 - \alpha_p^2 p^{-2s})^{-1}(1 - \beta_p^2 p^{-2s})^{-1},$$

$$= L^{(2N)}(2s, \text{Sym}^2 \pi_f) \prod_{p|2N} (1 - p^{-1}). \quad (3.45)$$

Note that the third to last display follows from (3.7), and the penultimate display from direct computation using (3.2) and (3.4). \hfill \Box

### 4. Voronoi and Poisson summation with combinatorial weights

Let $W : (0, \infty) \to \mathbb{R}$ be a smooth function compactly supported on $[1, 2]$. For a given twist minimal $f \in S^\text{new}_\ell(N)$, we asymptotically evaluate

$$M_{\pi_f}(X) := \sum_{d=d_0 d_1 \atop \mu^2(d_0) = 1 \atop (d, 2N) = 1} W\left(\frac{d}{X}\right) \frac{1}{d^{1/2}} L^{(2N)}(1/2, \pi_f \otimes \chi_{d_0}) \mathcal{P}_d(1/2; 1; \pi_f), \quad X \to \infty, \quad (4.1)$$

with error term uniform in both $X$ and $N$.

The conductor of each $L(1/2, \pi_f \otimes \chi_{d_0})$ in (4.1) is $\asymp \ell X^2 N$. Let $R$ be a parameter satisfying $1 \leq R \ll \ell X^2 N$. For $\varepsilon > 0$ fixed and small, let $U := (X N)^\varepsilon$ and $\gamma$ denote the straight line
segment \([\varepsilon - iU, \varepsilon + iU]\). For \(a'_1 c'_1, a'_2 c'_2 \in \text{Div}(N)\), consider the expressions

\[
S_{\pi_f}(X, R, a'_1 c'_1) := \frac{1}{c'_1^{1/2}} \int_{\gamma} \int_{\gamma} \sum_{1 \leq d \leq U R c'_1 / X} \mathcal{P}_d(1/2 + s - w, \chi_{a'_1 c'_1}; \pi_f) \frac{d^{1/2+w}}{d^{1/2+w}} \times \sum_{1 \leq n \leq U R / (n, 2N) = 1} \frac{\lambda_f(n) \chi_{a'_1 c'_1}(n) \chi_{d_0}(n)}{n^{1/2+s-w}} \left| \frac{dw \, ds}{s} \right| ;
\]

and

\[
\tilde{S}_{\pi_f}(X, R, a'_1 c'_1, a'_2 c'_2) := \delta(c'_1, c'_2) = 1 \cdot \delta(c'_1 | N_0) \cdot \frac{1}{c'_1^{1/2} N_0} \int_{\gamma} \int_{\gamma} \sum_{1 \leq d \leq U X N_0 c'_1 / R} \mathcal{P}_d(1/2 + w - s; \chi_{a'_1 c'_1}; \pi_f) \chi_{a'_2 c'_2}(d) \frac{d^{1/2+2s-w}}{d^{1/2+2s-w}} \times \sum_{1 \leq n \leq U X N_0 c'_1 / R / (c'_2, 2N) = 1} \frac{\lambda_f(n) \chi_{a'_1 c'_1}(n) \chi_{d_0}(n)}{n^{1/2+w-s}} \left| \frac{ds \, dw}{s} \right|.
\]

**Proposition 4.1.** Let \(N\) be as in (1.2), \(f \in S^\text{new}_f(N)\) be twist minimal, and \(M_{\pi_f}\) be as in (4.1). Suppose that the statements in (2.10) and (2.11) hold with exponents \(0 < \delta_1, \delta_2 < 1/4\) respectively.

Then for \(\varepsilon > 0\), \(X \geq 1\), and \(1 \leq R \ll \sqrt{X^2 N}\),

\[
M_{\pi_f}(X) = X^{1/2} \cdot (1 + \delta_N = \varepsilon(\pi_f)) \cdot \hat{W}(1/2) \cdot L^{(2N)}(1, \text{Sym}^2 \pi_f) \cdot \prod_{p | 2N} (1 - p^{-1})
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \sum_{a'_1 c'_1 \in \text{Div}(N)} S_{\pi_f}(X, R, a'_1 c'_1) \right)
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \sum_{a'_1 c'_1, a'_2 c'_2 \in \text{Div}(N)} \tilde{S}_{\pi_f}(X, R, a'_1 c'_1, a'_2 c'_2) \right)
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \left( \frac{(N_0^2 N_1^3)^{1/4 - \delta_2} X^{1/2}}{R_1^{1/4}} + \frac{(N_0^2 N_1^3)^{1/4 - \delta_2} R^{1/4}}{N^{1/4} N_0^{3/4}} + 1 \right) \right),
\]

where \(U = (X N)^\varepsilon\).

**Remark 4.1.** Consider the extremal case when \(\varepsilon(\pi_f) = N\) is a perfect square. When one twists \(\pi_f\) by an a Dirichlet character \(\chi_{d_0}\) with \(d_0 > 0\) (i.e. it is even) and \((d_0, 2N) = 1\), then \(\varepsilon(\pi_f \otimes \chi_{d_0}) = \varepsilon(\pi_f)\).

In this case, we would expect the main term to double when \(\varepsilon(\pi_f) = 1\), and vanish identically when \(\varepsilon(\pi_f) = -1\). This explains the factor \(1 + \delta_N = \varepsilon(\pi_f)\) in Proposition 4.1.
Proof. Let $H$ an even holomorphic function that satisfies $H(0) = 1$ and the growth estimate

$$|H(z)| \ll_{\text{Re}(z), C} (1 + |z|)^{-C},$$

for any $C > 0$.

Consider the integral

$$J_{\pi_f}(X, R) := \frac{1}{(2\pi i)^2} \int_{(4.6)} \int_{(2.5)} \widehat{W}(w) H(s) Z^{(N)}(1/2 + s, 1/2 + w; 1, 1; \pi_f) X^w R^s \frac{ds}{s} dw.$$  \hspace{1cm} (4.5)

We move the $s$-contour to Re($s$) = $-2$ and encounter a pole at $s = 0$. The residue is

$$\frac{1}{2\pi i} \int_{(4.6)} \widehat{W}(w) Z^{(N)}(1/2, 1/2 + w; 1, 1; \pi_f) X^w dw = M_{\pi_f}(X),$$

where the equality follows from Lemma 3.3 and Mellin inversion.

We make the change of variable $s \to -s$ in the shifted integral, and then apply (3.29). The net result is

$$M_{\pi_f}(X) = J_{\pi_f}(X, R) + \sum_{a_2' c_2' \in \text{Div}(N)} K_{\pi_f}(X, R, a_2' c_2'),$$ \hspace{1cm} (4.6)

where

$$K_{\pi_f}(X, R, a_2' c_2') := \frac{1}{(2\pi i)^2} \int_{(4.6)} \int_{(2.5)} \widehat{W}(w) H(s) \Phi_{11}(1/2 - s; \pi_f) 1_{a_2' c_2'}
\times Z^{(N)}(1/2 + s, 1/2 + w - 2s; \chi_{a_2' c_2'}, 1; \pi_f) X^w R^s \frac{ds}{s} dw.$$  \hspace{1cm} (4.7)

4.1. Treatment of $J_{\pi_f}(X, R)$. We interchange the $s$ and $w$ integrations in (4.5) by Fubini’s Theorem, and then move the $w$-contour to Re($w$) = $-1$. We encounter a pole at $w = 1/2$, whose residue can be computed using Lemma 3.6. No other poles are encountered because the domain of integration remains in $\Omega_2$ when this perturbation of contour is made (see Lemmas 3.3 and 3.6).

We make the change of variable $w \to -w$ in the shifted integral, and then apply (3.30). The net result is

$$J_{\pi_f}(X, R) = \frac{1}{2\pi i} \widehat{W}(1/2) X^{1/2} \prod_{p \leq 2N} (1 - p^{-1}) \int_{(2.5)} H(s) L^{(2N)}(1 + 2s, \text{Sym}^2 \pi_f) R^s \frac{ds}{s}
+ \sum_{a_1' c_1' \in \text{Div}(N)} I_{\pi_f}(X, R, a_1' c_1'),$$ \hspace{1cm} (4.8)

where

$$I_{\pi_f}(X, R, a_1' c_1') := \frac{1}{(2\pi i)^2} \int_{(2.5)} \int_{(1)} \widehat{W}(-w) H(s) \Psi_{11}(1/2 - w; \pi_f) 1_{a_1' c_1'}
\times Z^{(N)}(1/2 + s - w, 1/2 + w; 1, \chi_{a_1' c_1'}; \pi_f) X^{-w} R^s \frac{ds}{s} dw.$$
4.1.1. **First main term.** We move the $s$-contour in the first term of (4.8) to $\text{Re}(s) = -1/4$, encountering a pole at $s = 0$. We obtain

$$
\frac{1}{2\pi i} \hat{W}(1/2) X^{1/2} \prod_{p|2N} (1 - p^{-1}) \int_{(2)} H(s)L^{(2N)}(1 + 2s, \text{Sym}^2 \pi_f) R^s \frac{ds}{s} = X^{1/2} \hat{W}(1/2) L^{(2N)}(1, \text{Sym}^2 \pi_f) \prod_{p|2N} (1 - p^{-1}) + O_{\varepsilon, \varepsilon}\left(U(N_0^2 N_1^3)^{1/4 - \delta_2} X^{1/2} R^{-1/4}\right),
$$

(4.9)

where the error term follows from hypothesis (2.11) and a trivial estimation of the missing Euler factors. Thus (4.9) gives one of the main terms and one of the error terms in Proposition 4.1.

4.1.2. **Treatment of $I_{\pi_f}(X, R, a_1 c'_1)$.** Using Lemma 3.3 we obtain

$$
I_{\pi_f}(X, R, a_1 c'_1) := \frac{1}{(2\pi i)^2} \sum_{d \geq 1}^{d \geq 1} \frac{1}{d^{1/2}} \sum_{(n, 2N) = 1}^{(n, 2N) = 1} \frac{\lambda_f(n) \chi_{a_1 c'_1}(n) \chi_{a_0}(n)}{n^{1/2}}
$$

$$
\times \int_{(2)} \int_{(1)} \hat{W}(-w) H(s) \Psi_{11}(1/2 - w; \pi_f)_{11a_1 c'_1}
$$

$$
\times \mathcal{P}_d(1/2 + s - w, \chi_{a_1 c'_1}; \pi_f) \left(\frac{dX}{n}\right)^{-w} \left(\frac{n}{R}\right)^{-s} dw ds. \quad (4.10)
$$

Observe from (3.35) that

$$
\Psi_{a_2 c'_2}(1/2 - w; \pi_f)_{11a_1 c'_1} = \delta_{(c'_1, c'_2)} = c'_2 c'_1^{-1/2 + w} A(w), \quad \text{Re}(w) \geq \varepsilon, \quad (4.11)
$$

where $A(w)$ is a holomorphic in the given half-plane. For each $w \in \mathbb{C}$ with $\text{Re}(w) \geq \varepsilon$, there exists a constant $C_1 := C_1(\text{Re}(w))$ such that

$$
|A(w)| \ll \text{Re}(w), \varepsilon \ N^\varepsilon (1 + |\text{Im}(w)|)^{C_1}. \quad (4.12)
$$

We move the $s$-contour in (4.10) to $\text{Re}(s) = B_1 + 2$ for sufficiently large $B_1 > 0$. Recall the divisor bound $|\lambda_f(n)| \leq d(n)$ [Del74] and Lemma 3.2 for the weights $\mathcal{P}_d$. The net contribution to $I_{\pi_f}(X, R, a_1 c'_1)$ from all $n \geq UR$ is $O_{\varepsilon, B_1}(\|X N\|^{-B_2})$ and some $B_2 > 0$. We truncate the $n$ sum (4.10) to the range $1 \leq n \leq UR$.

We next move the $w$-contour to $\text{Re}(w) = \text{Re}(s) = B_1 + 2$. We again use divisor bounds, as well as (4.11) and (4.12). We truncate the $d$-sum to the range $1 \leq d \leq UR c'_1 / X$, incurring a negligible error.

We now move the $s$ and $w$ contours to $\text{Re}(s) = \text{Re}(w) = \varepsilon$. By the rapid decay of $\hat{W}$ and $H$ we can truncate the $s, w$-integrations in (4.10) to $|\text{Im}(s)|, |\text{Im}(w)| \leq U$ with negligible error. We interchange the finite summations with the absolutely convergent integrals, and use (4.11) and (4.12) in (4.10). We obtain the error terms involving $S_{\pi}(X, R, a_1 c'_1)$ in Proposition of 4.1.

4.2. **Treatment of $K_{\pi_f}(X, R, a_2 c'_2)$**.
4.2.1. Second main term. We interchange the $s$ and $w$ integrations in (4.7) by Fubini’s Theorem, and then move the $w$-contour to $\text{Re}(w) = 4$ for each (4.7). Observe that the domain of integration remains in $\Omega_2$ (see Lemmas 3.3 and 3.6).

We encounter the polar hyperplane $1/2 + w - 2s = 1$ when $a_2c_2' = 11$, otherwise there are no poles encountered for this move when $a_2c_2' \neq 11$. We can compute its residue using Lemma 3.6. Observe that (3.34) gives

$$
\Phi_{11}(1/2; \pi_f)_{1111} = \delta_{N=\mathbb{C}} \cdot \varepsilon(\pi_f).
$$

The residue is

$$
\frac{1}{2\pi i} X^{1/2} \prod_{p|2N} (1-p^{-1}) \int_{(2)} \hat{W}(1/2 + 2s) H(s) \Phi_{11}(1/2 - s; \pi_f)_{1111}
$$

$$
\times L^{(2N)}(1 + 2s, \text{Sym}^2 \pi_f) X^{2s} R^{-s} \frac{ds}{s}
$$

$$
= \delta_{N=\mathbb{C}} \cdot \varepsilon(\pi_f) X^{1/2} L^{(2N)}(1, \text{Sym}^2 \pi_f) \frac{\hat{W}(1/2)}{2N} \prod_{p|2N} (1-p^{-1})
$$

$$
\quad + O_{\ell,\varepsilon}(U N_0^2 N_1^3)^{1/4-\delta_2} N^{-1/4} N_0^{-3/4} R^{1/4},
$$

(4.13)

where the error term follows from (3.34) and the hypothesis (2.11). Substituting (4.13) back into (4.6) gives the second main term and another error term in Proposition 4.1.

For each $a_2c_2' \in \text{Div}(N)$ we are left to estimate

$$
L_{\pi_f}(X, R, a_2c_2') := \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(4)} \hat{W}(w) H(s) Z^{(N)}(1/2 + s, 1/2 + w - 2s; \chi_{a_2c_2'}, 1; \pi_f)
$$

$$
\times \Phi_{11}(1/2 - s; \pi_f)_{11a_2c_2'} X^w R^{-s} dw \frac{ds}{s}.
$$

(4.14)

We handle this in the next argument.

4.2.2. Remaining terms. Applying (3.30) to (4.14) gives

$$
L_{\pi_f}(X, R, a_2c_2') := \frac{1}{(2\pi i)^2} \sum_{a_2'c_2' \in \text{Div}(N)} \int_{(2)} \int_{(4)} \hat{W}(w) H(s) \Phi_{11}(1/2 - s; \pi_f)_{11a_2'c_2'}
$$

$$
\times \Psi_{a_2'c_2'}(1/2 + w - 2s; \pi_f)_{11a_2'c_2'}
$$

$$
\times Z^{(N)}(1/2 + w - s, 1/2 + 2s - w; \chi_{a_2'c_2'}, \chi_{a_2'c_2'}; \pi_f)
$$

$$
\times X^w R^{-s} dw \frac{ds}{s}.
$$

(4.15)

We will again use the decay of $\Phi$ coming from the $\pi_{f,p}$ that are special representations. A particular case of the formula (3.34) can be written as

$$
\Phi_{11}(1/2 - s; \pi_f)_{11a_2'c_2'} = N^s \delta_{c_2'N_0} \left(\frac{N_0}{c_2'}\right)^{-1+s} B(s), \quad \text{Re}(s) \geq \varepsilon,
$$

(4.16)

where $B(s)$ is a holomorphic in the given half plane. For each $s \in \mathbb{C}$ with $\text{Re}(s) \geq \varepsilon$, there exists a constant $C_2 := C_2(\text{Re}(s), \ell)$ such that

$$
|B(s)| \ll_{\text{Re}(s), \ell, \varepsilon} N^\varepsilon (1 + |\text{Im}(s)|)^{C_2}.
$$

(4.17)
We use Lemma 3.3 to open the multiple Dirichlet series in (4.15). We then argue similarly to Section 4.1, except now we appeal to (4.11), (4.12), (4.16) and (4.17). We obtain the error terms $\tilde{S}_{\pi_f}(X, R, a_1 c_1', a_2 c_2')$ in Proposition 4.1. This completes the proof.

5. Endgame

Proof of Theorem 1.1. We use Proposition 4.1 with $X \geq 1$ and $1 \leq R \ll \varepsilon X^2 N$ to be chosen later.

We then apply Lemma 2.1 to estimate the $S_{\pi_f}$ and $\tilde{S}_{\pi_f}$ terms defined in (4.2) and (4.3) respectively. We also exploit that the fact that $c'_1 | \text{rad}(N) = N_0 N_1$ in both (4.2) and (4.3). The net result is

\[
M_{\pi_f}(X) = X^{1/2} \cdot (1 + \delta_{N=\square} \cdot \varepsilon(\pi_f)) \cdot \tilde{W}(1/2) \cdot L^{(2N)}(1, \text{Sym}^2 \pi_f) \cdot \prod_{p \mid 2N} (1 - p^{-1})
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \left( \frac{R}{X} \right)^{1-2\delta_1} (NN_0^3 N_1^{1/4-\delta_1} + \frac{XN N_0}{R} )^{1-2\delta_1} (NN_0^3 N_1^{1/4-\delta_1} ) \right)
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \left( \frac{(NN_0^3 N_1^{1/4-\delta_2})^{X^{1/2}}}{R^{1/4}} + \frac{(NN_0^3 N_1^{1/4-\delta_2})^{R^{1/4}}}{N_0^{3/4}} + 1 \right) \right).
\]

We choose $R := X(NN_0)^{1/2}$ to balance the error terms in the middle display above. This yields

\[
M_{\pi_f}(X) = X^{1/2} \cdot (1 + \delta_{N=\square} \cdot \varepsilon(\pi_f)) \cdot \tilde{W}(1/2) \cdot L^{(2N)}(1, \text{Sym}^2 \pi_f) \cdot \prod_{p \mid 2N} (1 - p^{-1})
\]

\[
+ O_{\varepsilon, \varepsilon} \left( U \left( N^{9/8-(7/2)\delta_1} N_0^{7/8-(5/2)\delta_1} + \frac{X^{1/4} N_0^{1/4-(3/2)\delta_2}}{N_0^{3/4}} \right) \right). \tag{5.1}
\]

A Theorem of Hoffstein and Lockhart [HL94, pg. 164] ensures that

\[
L(1, \text{Sym}^2 \pi_f) \gg_{\varepsilon, \varepsilon} N^{-\varepsilon},
\]

with ineffective constant depending on $\varepsilon > 0$. Taking

\[
X \gg_{\varepsilon} N^\varepsilon \left( N^{9/4-7\delta_1} N_0^{7/4-5\delta_1} + \frac{N^{1-6\delta_2}}{N_0^{2\delta_2}} \right)
\]

shows that the error terms in (5.1) are smaller than the main term for $M_{\pi_f}(X)$. Hence there is a fundamental discriminant $d$ in the desired range such that

\[
L^{(2N)}(1/2, \pi_f \otimes \chi_d) \neq 0.
\]

A trivial estimation of the missing Euler factors yields Theorem 1.1.

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