Isotropic Metric in the Theory of General Relativity

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Abstract

We explain why the isotropic metric is quite appropriate to put the physical meaning of spacial variables in the theory of general relativity. Using the isotropic metric, we conclude that i) $g_{00}$ does not become positive even inside the black hole, ii) there exists the center of the Universe if the curvature of the Universe $k \neq 0$, iii) the Universe is spacially finite but not closed for $k > 0$.

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§1. Introduction

The theory of general relativity has a long history after Einstein’s work in 1916 [1]. But even now, there remains some ambiguity to interpret the physical meaning of variables in the metric. The origin of this ambiguity comes from the general coordinate transformation in the theory of general relativity. The quite beautiful symmetry of the theory of general relativity, the general coordinate invariance, makes the physical meaning of variables ambiguous, because we can freely change variables. Then the problem of time [4], for example, becomes the issue in the theory of general relativity.

In this paper, we will explain why the isotropic metric [2, 3] is quite appropriate to put the physical meaning of spatial variables in the theory of general relativity.

Using the isotropic metric, we conclude that

i) $g_{00}$ does not become positive even inside the black hole,

ii) there exists the center of the Universe if the curvature of the Universe $k \neq 0$,

iii) the Universe is spatially finite but not closed for $k > 0$.

§2. Physical Interpretation of the Isotropic Metric

– Isotropic Metric in Cartesian Coordinate –

We decompose the metric in the theory of general relativity into the time and spatial metric $d\sigma^2$ in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx^0dx^0 + 2g_{0i}dx^0dx^i + d\sigma^2.$$  \hspace{1cm} (1)

In order to interpret the meaning of spatial variables physically, we use the Cartesian coordinate. We connect the Cartesian coordinate and the polar coordinate just in the same way as in the flat space,

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$  \hspace{1cm} (2)

The polar coordinate in the theory of general relativity is ambiguous to put the physical meaning, because it is just one set of variables and nothing else in the curved space-time. While, the Cartesian coordinate has the quite definite physical meaning. Therefore we will always put the physical meaning for spatial variables by transforming into the Cartesian coordinate. (It may be the issue whether the Cartesian coordinate only has the physical
meaning or not, but at least the metric becomes uniquely fixed by rewriting into the Cartesian coordinate.)

Isotropy of the space

If the overall factor of the spatial metric is angle ($\theta$ and $\phi$) independent in comparison with the spatially flat metric in the polar coordinate, we call that such space is isotropic. Isotropic spatial metric is therefore given by

$$d\sigma^2 = g_{ij}dx^i dx^j = f(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

$$= f\left(\sqrt{x^2 + y^2 + z^2}\right) (dx^2 + dy^2 + dz^2). \quad (3)$$

Uniqueness of the spatial metric

We will explain how to uniquely determine physical spatial variables if the spatial metric has "isotropic" symmetry of the form

$$d\sigma^2 = g(R)dR^2 + h(R)(R^2 d\Theta^2 + R^2 \sin^2 \Theta d\Phi^2), \quad (4)$$

where we assume that the asymptotic flatness, $g_{ij}(R, \Theta, \Phi) \to \delta_{ij}$ as $R \to \infty$, is satisfied in order to interpret the variables $R, \Theta, \Phi$ physically.

Then we can uniquely determine the change of variables $r = r(R), \Theta = \theta, \Phi = \phi$ in such a way as Eq. (4) becomes in the form of Eq. (3) by keeping the asymptotic flatness $g_{ij}(r, \theta, \phi) \to \delta_{ij}$ as $r \to \infty$.

We can put the physical meaning of spatial variables in this isotropic metric, because we can easily transform the isotropic metric in polar coordinate into the isotropic metric in the Cartesian coordinate. Therefore there is no ambiguity of the physical interpretation of spatial variables, such as the radius of the black hole.

Homogeneity of the space

We call that the metric is homogeneous if the metric in Cartesian coordinate is invariant under the global parallel translation.
\[ x \rightarrow x' = x + x_0, \]
\[ y \rightarrow y' = y + y_0, \]
\[ z \rightarrow z' = z + z_0, \]
\[(5)\]

where \( x_0, y_0, z_0 \) are constants.

Then the homogeneous spatial metric is given by

\[ d\sigma^2 = g_{ij}dx^i dx^j = (\text{const.})(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \ d\phi^2) \]
\[ = (\text{const.})(dx^2 + dy^2 + dz^2), \]
\[(6)\]

where (const.) is the \( r, \theta, \phi \) or \( x, y, z \) independent constant.

§3. Isotropic Metric in the Black Hole

The standard expression of Schwarzschild metric, which also expresses the metric of the black hole, is given by

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu \]
\[ = -(1 - 2M/R)dt^2 + \left( \frac{dR^2}{1 - 2M/R} + R^2 d\Theta^2 + R^2 \sin^2 \Theta \ d\Phi^2 \right). \]
\[(7)\]

We can connect this metric with the isotropic metric by the relation \( R = r(1+M/2r)^2, \Theta = \theta, \Phi = \phi \). Writing \( R = r \left( 1 + \frac{M}{r} + \frac{M^2}{4r^2} \right) \) and noticing that \( r = \sqrt{x^2 + y^2 + z^2} \) has a physical meaning so that \( r \) must satisfy \( r \geq 0 \), we have \( R = M + r + \frac{M^2}{4r} \geq 2M \), which means that \( g_{00} \leq 0 \) even inside the black hole. (If \( R \) is allowed to become \( R < 2M \), then we may interpret that the role of the space and time is exchanged for \( d\Theta = 0, d\Phi = 0 \), but this is not the case.) Under the above change of variables, the spatial metric is given by

\[ ds^2 = -\left( \frac{1 - M/2r}{1 + M/2r} \right)^2 dt^2 + (1 + M/2r)^4(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \ d\phi^2) \]
\[ = -\left( \frac{1 - M/2r}{1 + M/2r} \right)^2 dt^2 + (1 + M/2r)^4(dx^2 + dy^2 + dz^2), \]
\[(8)\]
where \( r = \sqrt{x^2 + y^2 + z^2} \). At \( r = M/2 \), which is equivalent \( R = 2r = M \), we have \( g_{00} = 0 \) but otherwise \( g_{00} > 0 \), so that even inside the black hole, the role of the space and time is not exchanged.

\[ g_{00} = 0 \]

### §4. Isotropic Metric in the Cosmology

We apply the theory of general relativity to the cosmology, and take the Friedmann Universe, and the metric[3] of that Universe is given by

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
\]

\[
= -dt^2 + a(t)^2 \left( \frac{dR^2}{1 - kR^2} + R^2 d\Theta^2 + R^2 \sin^2 \Theta d\Phi^2 \right)
\]

\[
= -dt^2 + a(t)^2 \left( \frac{dr^2 + r^2 d\Theta^2 + r^2 \sin^2 \theta d\phi^2}{(1 + kr^2/4)^2} \right)
\]

\[
= -dt^2 + a(t)^2 \left( \frac{dx^2 + dy^2 + dz^2}{1 + k(x^2 + y^2 + z^2)/4} \right),
\]

where

\[
R = \frac{r}{(1 + kr^2/4)}, \quad \Theta = \theta, \quad \Phi = \phi,
\]

and \( r = \sqrt{x^2 + y^2 + z^2} \).

From this expression, we can see that \( 0 \leq R = \frac{1}{(1/r + kr/4)} \leq 1/\sqrt{k} \) and \( R = 1/\sqrt{k} \) does not correspond to the infinity far point but only finite distant point \( r = \sqrt{x^2 + y^2 + z^2} = 2/\sqrt{k} \). As \( r \) changes as \( 0 \rightarrow 2/\sqrt{k} \rightarrow +\infty \), the corresponding \( R \) changes as \( 0 \rightarrow 1/\sqrt{k} \rightarrow 0 \). Therefore \( r = 0 \) and \( r = \infty \) becomes degenerate in \( R \) variable. We also notice that there exists the center of the Universe for \( k \neq 0 \), because the spacial metric has no invariance under the global parallel translation \( x \rightarrow x + x_0, y \rightarrow y + y_0, z \rightarrow z + z_0 \).
We consider \( k > 0 \) case, and first consider the total volume of the Universe,

\[
V = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \sqrt{\det g_{ij}}
\]

\[
= \int_0^{+\infty} dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{r^2 \sin \theta}{(1 + kr^2/4)^3}
\]

\[
= \frac{16\pi}{k^{3/2}} \int_0^{+\infty} d\xi \frac{\sqrt{\xi}}{(1 + \xi)^3} = \frac{2\pi^2}{k^{3/2}}.
\]

(12)

It may possible to calculate the total volume \( V' \) of the Universe with variables \( R, \Theta, \Phi \) in the form

\[
V' = \int_0^{\sqrt{k}} dR \int_0^\pi d\Theta \int_0^{2\pi} d\Phi \frac{R^2 \sin \Theta}{\sqrt{1 - kR^2}}
\]

\[
= \frac{4\pi}{k^{3/2}} \int_0^1 dx \frac{x^2}{\sqrt{1 - x^2}} = \frac{\pi^2}{k^{3/2}}.
\]

(13)

We notice that \( V \) and \( V' \) gives the different value and have the relation \( V = 2V' \). There is the reason of this factor 2 difference. For given \( R \), there exist two \( r_+, r_- \) with \( 0 \leq r_- \leq 2/\sqrt{k} \leq r_+ \), which satisfy \( R = \frac{1}{(1/r_- + kr_-/4)} = \frac{1}{(1/r_+ + kr_+/4)} \), that is, the mapping between \( r \) and \( R \) is 2 to 1 except at \( r = 2/\sqrt{k} \). For example, \( r = 2/\sqrt{k} \) correspond to single \( R = 1/\sqrt{k} \), so that the correspondence is 1 to 1 only at this point. While \( r = 0 \) and \( r = +\infty \) both corresponds to \( R = 0 \). Thus the correct total volume of the Universe becomes \( V \) instead of \( V' \).

Therefore we can see that the total volume of the Universe is finite. We notice that the Universe is not closed by looking at the expression of the spacial metric, which is written by the Cartesian coordinate

\[
d\sigma^2 = a(t)^2 \left( \frac{dx^2 + dy^2 + dz^2}{[1 + k(x^2 + y^2 + z^2)/4]^2} \right),
\]

(14)

which gives \( g_{ij}(x, y, z, t) = a(t)^2 \delta_{ij} / \left[ 1 + k(x^2 + y^2 + z^2)/4 \right]^2 \). From this expression, we can interprete that \( i) \) there is the center of the Universe for \( k \neq 0 \), because the spacial metric
$g_{ij}$ is not symmetric under $x \to x + x_0$, $y \to y + y_0$, $z \to z + z_0$. ii) this spatial metric $g_{ij}$ is not periodic under $x, y, z \to \pm \infty$.

Therefore we can conclude that the Universe is finite but not closed for $k > 0$.

### Finite Universe in polar coordinate

We will explain that the Universe is finite but not closed by using the polar coordinate also. We define $\sin \chi = \sqrt{kR}$, then we have $\cos \chi = \pm \sqrt{1 - kR^2}$ but how should we choose the sign of $\pm$? As $R$ changes from 0 to $1/\sqrt{k}$ and cover that range only once, we must choose the range of $\chi$ as $0 \leq \chi \leq \pi/2$ and we must choose $\cos \chi = \sqrt{1 - kR^2}$ in order to make that parametrization meaningful. In this range of $\chi$, the mapping $R \leftrightarrow \chi$ and $\chi \leftrightarrow R$ both become single valued. By using the relation $R = \frac{1}{(1/r + kr/4)}$, i) $r = 0$ gives $R = 0$ and $\chi = 0$, ii) $r = 2/\sqrt{k}$ gives $R = 1/\sqrt{k}$ and $\chi = \pi/2$, iii) $r = +\infty$ gives $R = 0$ and $\chi = 0$. We will compare this parametrization of spatial metric with the parametrization of the half $S^3$.

### Parametrization of the half $S^3$

We can parametrize the half $S^3$ in the form

\begin{align}
    w_1 &= \frac{\sin \chi \sin \Theta \cos \Phi}{\sqrt{k}}, \\
    w_2 &= \frac{\sin \chi \sin \Theta \sin \Phi}{\sqrt{k}}, \\
    w_3 &= \frac{\sin \chi \cos \Theta}{\sqrt{k}}, \\
    w_4 &= \frac{\cos \chi}{\sqrt{k}},
\end{align}

with

\begin{equation}
    0 \leq \chi \leq \pi/2, \quad 0 \leq \Theta \leq \pi \quad 0 \leq \Phi \leq 2\pi.
\end{equation}

We call the above parametrization as the half $S^3$, because the range of $\chi$ in closed $S^3$ is $0 \leq \chi \leq \pi$, while $\chi$ in Eq. (16) takes only the half compared to $S^3$. Then we can understand that the above half $S^3$ is not closed.

In these variables, we have $w_1^2 + w_2^2 + w_3^2 + w_4^2 = 1/k$ and the spatial metric is given by
\[ d\sigma^2 = a(t)^2 \left( dw_1^2 + dw_2^2 + dw_3^2 + dw_4^2 \right) \]
\[ = a(t)^2 \left( \frac{dR^2}{1 - kR^2} + R^2 d\theta^2 + R^2 \sin \theta^2 d\phi^2 \right). \]  

(17)

We will explain some interesting paths in the followings.

path(1)

We consider a path, going around the circle at finite distance \( x^2 + y^2 = 4/k \) with \( z = 0 \) in Cartesian coordinate, which is equivalent to the path changing \( \phi \) from 0 to \( 2\pi \) with fixed \( r = 2/\sqrt{k}, \theta = \pi/2 \) in polar coordinate. As before, we connect the Cartesian coordinate and the polar coordinate in the form \( x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta \).

This path is equivalent to the path changing \( \Phi \) from 0 to \( 2\pi \) with fixed \( \chi = \pi/2 \) \( (R = 1/\sqrt{k}), \Theta = \pi/2 \) in \( \chi, \Theta, \Phi \) variables. This path is the great circle even for the non-closed half \( S^3 \), that is, though \( \chi \) changes in the range \( 0 \leq \chi \leq \pi/2 \), the above path becomes the great circle. Of course this path is the great circle on the closed \( S^3 \) also.

path(2)

What we are interested in is the path

\[ x : 0 \to 2/\sqrt{k} \to +\infty \to 2/\sqrt{k} \to 0 \to -2/\sqrt{k} \to -\infty \to -2/\sqrt{k} \to 0, \]
\[ y = z = 0. \]

In polar coordinate, this path is expressed as the result of the following steps:

Step 1)

\[ r : 0 \to 2/\sqrt{k} \to +\infty \to 2/\sqrt{k} \to 0, \]
\[ \theta = \pi/2, \quad \phi = 0, \]

Step 2)

\[ \phi : 0 \to \pi, \]
\[ r = 0, \quad \theta = \pi/2, \]

Step 3)

\[ r : 0 \to 2/\sqrt{k} \to +\infty \to 2/\sqrt{k} \to 0, \]
\[ \theta = \pi/2, \quad \phi = \pi. \]
Using $\chi, \Theta, \Phi$ variables on the non-closed half $S^3$, this path is given by

Step 1)

$\chi: 0 \to \pi/2 \to 0 \to \pi/2 \to 0$

$\Theta = \pi/2, \Phi = 0$

Step 2)

$\Phi: 0 \to \pi$

$\chi = 0, \Theta = \pi/2$

Step 3)

$\chi: 0 \to \pi/2 \to 0 \to \pi/2 \to 0$

$\Theta = \pi/2, \Phi = \pi$

and this path is not the great circle, but the path going to the edge ($\chi = \pi/2$) and coming back to the origin. In this way, the above path is not the closed path on the closed manifold, but the path going outside and coming back to the origin on the non-closed half $S^3$.

While on the closed $S^3$, where $\chi$ takes $0 \leq \chi \leq \pi$, we can take the path

Step 1)

$\chi: 0 \to \pi/2 \to \pi \to \pi/2 \to 0$

$\Theta = \pi/2, \Phi = 0$

Step 2)

$\Phi: 0 \to \pi$

$\chi = 0, \Theta = \pi/2$

Step 3)

$\chi: 0 \to \pi/2 \to \pi \to \pi/2 \to 0$

$\Theta = \pi/2, \Phi = \pi$

and this is the great circle on the closed $S^3$.

§5. Summary

In this paper, we interprete the meaning of spacial variables physically by rewriting into the Cartesian coordinate. In this context, the isotropic metric is quite appropriate to put
the physical meaning of spacial variables in the theory of general relativity. The metric also becomes unique after rewriting into the Cartesian coordinate. Using the isotropic metric, we conclude that $i)$ $g_{00}$ does not become positive even inside the black hole, $ii)$ there exists the center of the Universe if the curvature of the Universe $k \neq 0$, $iii)$ the Universe is spacially finite but not closed for $k > 0$.

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