Duality-invariant topological quantum phase of a moving dyon

Ricardo Heras*

Department of Physics and Astronomy,
University College London, London WC1E 6BT, UK

We derive the duality-invariant quantum phase that accumulates a dyon upon encircling a solenoid enclosing electric and magnetic fluxes. The phase is topological because it depends on the number of windings the dyon carries out around the solenoid and is independent of the shape of the dyon trajectory. The phase is nonlocal because there is no generalised Lorentz force locally acting on the dyon and then the electric and magnetic fluxes have no local consequences at any point of the dyon trajectory. We also obtain the energy levels of the proposed dyon-solenoid configuration and envision a quantum two-slit interference experiment associated to the new phase. We show that the duality symmetry of the new phase unifies the Aharonov-Bohm phase with its corresponding dual phase, the Dowling-Williams-Franson phase. This duality symmetry also suggests two new alternative interpretations of the Aharonov-Bohm phase.

I. INTRODUCTION

In 1959 Aharonov and Bohm [1] introduced the quantum phase that bears their name today. After its experimental verification [2, 3], the Aharonov-Bohm (AB) phase: \( \delta_{\text{AB}} = q\Phi_e/(\hbar c) \), in which an electric charge \( q \) encircles an infinitely-long solenoid enclosing the magnetic flux \( \Phi_m \), initiated the subject of topological quantum phases in electromagnetic configurations. In 1984 Aharonov and Casher [4] introduced another quantum phase, the Aharonov-Casher (AC) phase: \( \delta_{\text{AC}} = 4\pi\mu\lambda_e/(\hbar c) \), in which a magnetic moment of magnitude \( \mu \) moves around, and parallel to, an infinitely-long rod endowed with the linear electric charge density \( \lambda_e \). He and McKellar [5] in 1993 and independently Wilkens [6] in 1994 introduced the dual of the AC phase, the He-McKellar-Wilkens (HMW) phase: \( \delta_{\text{HMW}} = -4\pi d\lambda_m/(\hbar c) \), in which an electric moment of magnitude \( d \) moves around, and parallel to, an infinitely-long rod having the linear magnetic charge density \( \lambda_m \). In 1999 Dowling et al. [7] suggested the dual of the AB phase, the Dowling-Williams-Franson (DWF) phase: \( \delta_{\text{DWF}} = -g\Phi_e/(\hbar c) \), in which a magnetic charge \( g \) encircles an infinitely-long solenoid enclosing the electric flux \( \Phi_e \). Topological electromagnetic quantum phases and geometric phases [8] have become increasingly influential in several areas of physics (for a recent review see [9]).

In this paper we take a step forward in the subject of electromagnetic quantum phases and derive the first duality-invariant quantum phase: \( \delta = n(q\Phi_e - g\Phi_m)/(\hbar c) \) that accumulates a dyon [10–12], a hypothetical particle possessing an electric charge \( q \) and a magnetic charge \( g \), that moves around an infinitely-long dual solenoid enclosing an electric flux \( \Phi_e \) due to a current of magnetic charges and a magnetic flux \( \Phi_m \) due to a current of electric charges. In getting this new phase we consider the well-known problem that there is no standard Lagrangian for the Lorentz force including both electric and magnetic charges [13, 14]. However, we construct here a consistent Lagrangian for the generalised Lorentz force acting on the dyon on account of the restrictions of the proposed configuration formed by the dyon and the dual solenoid. We show that the introduced phase is topological because it depends on a winding number \( n \) and is nonlocal because there are no forces locally acting on the dyon due to the electric and magnetic fluxes. We also show that this phase is gauge invariant. We then calculate the energy levels associated to the proposed configuration and show that they exhibit the nonlocal effect of the electric and magnetic fluxes. In the next step we outline a hypothetical quantum two-slit interference experiment associated to the proposed phase. The novel feature of the introduced phase is that it is invariant under a general \( U(1) \) electromagnetic duality transformation group having the angle \( \theta \) as a parameter. We demonstrate that this \( U(1) \) symmetry of the new phase unifies the AB and DWF phases: for a value of the angle \( \theta \) the proposed phase gives the AB phase and for another value of this angle it gives the DWF phase. Finally, we show that the duality symmetry of the proposed phase suggests two new alternative interpretations of the AB phase. In the first interpretation, the AB phase arises from the nonlocal interaction between a dyon, whose electric and magnetic charges are connected by a duality transformation, and the magnetic flux of the dual solenoid. In the second interpretation, the AB phase arises from the nonlocal interaction between the electric charge of the dyon and the electric and magnetic fluxes of the solenoid which are connected by another duality transformation.

II. DUAL ELECTRODYNAMICS

Consider the configuration formed by a dyon carrying an electric charge \( q \) and a magnetic charge \( g \), moving around an infinitely-long dual solenoid, i.e., a solenoid in which there are currents due to the motion of independent electric and magnetic charges, of radius \( R \), laying along the \( z \)-axis and having uniform internal electric and magnetic fluxes \( \Phi_e = \pi R^2 E \) and \( \Phi_m = \pi R^2 B \), where \( E \) and \( B \) are the associated electric and magnetic fields. For simplicity
we will call this configuration the dyon-solenoid configuration on the understanding that the solenoid is a dual-solenoid. In this paper we adopt Gaussian units and use cylindrical coordinates unless otherwise specified. The current densities of the solenoid are given by
\[ J_s = \frac{e\Phi_s \delta(\rho - R)\hat{\phi}}{4\pi^2 R^2}, \quad J_n = -\frac{e\Phi_s \delta(\rho - R)\hat{\phi}}{4\pi^2 R^2}, \] (1)
where \( \delta \) is the Dirac delta function. We observe that the electric current \( J_s \) is a magnetisation current [15]: \( J_s = e\nabla \times M \), where \( M = \Phi_s \Theta(R - \rho)\hat{z}/(4\pi^2 R^2) \) is the confined magnetisation vector with \( \Theta \) being the Heaviside step function with the values \( \Theta = 1 \) if \( R > \rho \) and \( \Theta = 0 \) if \( R < \rho \). Likewise, the magnetic current \( J_n \) is a polarisation current \( J_n = e\nabla \times P \), where \( P = -\Phi_s \Theta(R - \rho)\hat{z}/(4\pi^2 R^2) \) is the confined polarisation vector. The corresponding Maxwell’s equations read
\[ \nabla \cdot E = 0, \quad \nabla \times E = \frac{\Phi_s \delta(\rho - R)\hat{\phi}}{\pi R^2}, \] (2)
\[ \nabla \cdot B = 0, \quad \nabla \times B = \frac{\Phi_s \delta(\rho - R)\hat{\phi}}{\pi R^2}, \] (3)
whose solutions are given by
\[ E = \frac{\Phi_s \Theta(R - \rho)\hat{z}}{\pi R^2}, \quad B = \frac{\Phi_s \Theta(R - \rho)\hat{z}}{\pi R^2}. \] (4)
From (4) we observe that the fields \( E \) and \( B \) are confined in the solenoid and vanish outside of it. We also note that the fields \( E \) and \( B \) are connected with the polarisation and magnetisation vectors \( P \) and \( M \) through the relations \( E = -4\pi P \) and \( B = 4\pi M \). Using the homogeneous equations \( \nabla \cdot B = 0 \) and \( \nabla \cdot E = 0 \), we introduce the vector potentials \( A \) and \( C \) via the relations
\[ B = \nabla \times A, \quad E = -\nabla \times C. \] (5)
Inserting these relations in the inhomogeneous equations given in (2) and (3), using the double-curl identity \( \nabla^2 F = \nabla(\nabla \cdot F) - \nabla \times (\nabla \times F) \), and adopting the Coulomb gauge conditions
\[ \nabla \cdot A = 0, \quad \nabla \cdot C = 0, \] (6)
we obtain the Poisson equations for the vector potentials
\[ \nabla^2 A = -\frac{\Phi_s \delta(\rho - R)\hat{\phi}}{\pi R^2}, \quad \nabla^2 C = \frac{\Phi_s \delta(\rho - R)\hat{\phi}}{\pi R^2}. \] (7)
The solution of these Poisson equations are given by
\[ A = \left( \frac{\Phi_s \Theta(R - \rho)}{2\pi \rho} + \frac{\Phi_s \rho \Theta(R - \rho)}{2\pi R^2} \right) \hat{\phi}, \] (8)
\[ C = \left( -\frac{\Phi_s \Theta(R - \rho)}{2\pi \rho} - \frac{\Phi_s \rho \Theta(R - \rho)}{2\pi R^2} \right) \hat{\phi}. \] (9)
A formal derivation of (8) has recently been presented [15]. Following this derivation we can also show (9). Outside the dual solenoid \( (\rho > R) \) the electric and magnetic fields vanish as seen in (4) and in this region the potentials \( A \) and \( C \) given in (8) and (9) reduce to
\[ A = \frac{\Phi_s \hat{\phi}}{2\pi \rho}, \quad C = -\frac{\Phi_s \hat{\phi}}{2\pi \rho}. \] (10)
These potentials are pure gauge potentials:
\[ A = \nabla \chi, \quad C = \nabla \xi, \] (11)
where the functions \( \chi \) and \( \xi \) are given by
\[ \chi = \frac{\Phi_s \phi}{2\pi}, \quad \xi = -\frac{\Phi_s \phi}{2\pi}, \] (12)
with \( \phi \) being the azimuthal coordinate. From (6) and (11) it follows that the functions \( \chi \) and \( \xi \) satisfy the equations \( \nabla^2 \chi = 0 \) and \( \nabla^2 \xi = 0 \). We also notice that the functions \( \chi \) and \( \xi \) are multi-valued \([i.e., \chi(\phi) \neq \chi(\phi + 2\pi)]\) and their gradients \( \nabla \chi \) and \( \nabla \xi \) are singular functions in agreement with the fact that the dyon-solenoid configuration lies on a non-simply connected region.
We note that the fields, potentials, currents and fluxes associated to the dual solenoid are invariant under the electromagnetic duality transformations:
\[ E \rightarrow B, \quad B \rightarrow -E, \quad A \rightarrow C, \quad C \rightarrow -A, \quad J_s \rightarrow J_n, \quad J_n \rightarrow -J_s, \quad \Phi_s \rightarrow \Phi_n, \quad \Phi_n \rightarrow -\Phi_s. \] (13)
When a dyon is acted by electric and magnetic fields it undergoes a generalised Lorentz force
\[ F = q\left( E + \frac{v}{c} \times B \right) + g\left( B - \frac{v}{c} \times E \right), \] (14)
where \( v \) is the dyon’s non-relativistic velocity. However, this force is zero in the region outside the dual solenoid because the electric and magnetic fields are zero in this region. In other words, we have a force-free region outside the solenoid.
On the other hand, the electric and magnetic charges of the dyon satisfy the duality transformations:
\[ q \rightarrow g, \quad g \rightarrow -q. \] (15)
We should emphasize that in principle we have two independent duality symmetries: One associated to the solenoid and represented by (13) and the other one associated to the dyon and represented by (15). However, the relevant electromagnetic relations associated to the dyon-solenoid configuration will be duality invariant when both duality symmetries involve the same parameter of the transformation. We will see in Sec. VIII that the duality transformations (13) and (15) are a particular case of a more general duality transformation group, namely, a \( U(1) \) group characterised by an angle \( \theta \) as parameter. Equations (13) and (15) implicitly involve the parameter \( \theta = \pi/2 \). The electrodynamic quantities associated to the dyon-solenoid configuration are invariant under
this $U(1)$ duality transformation group and in particular are invariant under the combined action of the duality transformations given in (13) and (15). One example of these relevant relations is the generalised Lorentz force given by (14), which is invariant under the transformations (13) and (15). Another example is the phase \( \delta = n(q\Phi_a - g\Phi_b)/(\hbar c) \), which will be derived in Sec. IV, and which is also invariant under the transformations (13) and (15).

III. LAGRANGIAN FORMULATION

Rohrlich demonstrated that there is no standard Lagrangian from which the generalised Lorentz force (14) can be deduced [13]. This is a well-known problem of the classical electrodynamics with magnetic monopoles. Rosenbaum [14] went further and showed that there is no standard Lagrangian that leads to the generalised Lorentz force (14), unless an extra condition not derivable from the action principle is assumed. In four-dimensional notation the extra condition reads \( J^\mu e_\mu = J^\nu e_\nu \) which must be valid at each point, for all \( \mu \) and \( \nu \). In the case of time-independent currents, this condition is satisfied if the electric current \( J_e \) is proportional to the magnetic current \( J_m \). In our configuration the currents (1) of the dual solenoid satisfy the proportionality relation: \( J_m = -(\Phi_e/\Phi_n) J_m \) because \( \Phi_e/\Phi_n \) is a constant. Therefore we should be able to construct a suitable Lagrangian that yields a generalised Lorentz force associated to our dyon-solenoid configuration.

Let us first proceed with some degree of generality by considering the Lagrangian

\[
L = \frac{mv^2}{2} + \frac{v}{c} \left( Q_1 A_1 + Q_2 A_2 \right) + \mathbf{x} \cdot \left[ \nabla \times (Q_2 A_1 - Q_1 A_2) \right],
\]

where \( m \) is the mass of a non-relativistic particle characterised by the constant parameters \( Q_1 \) and \( Q_2 \), \( v = dx/dt \) is the particle’s velocity with \( \mathbf{x} \) being its position, and \( A_1 \) and \( A_2 \) are prescribed time-independent vector functions. These vector functions are restricted to satisfy the condition that their curls \( \nabla \times A_1 \) and \( \nabla \times A_2 \) are constant functions of space and time. We note that the Lagrangian (16) is invariant under the duality transformations: \( Q_1 \to Q_2, Q_2 \to -Q_1, A_1 \to A_2 \) and \( A_2 \to -A_1 \). The corresponding Euler-Lagrange equations are given by

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \mathbf{v}} \right) - \frac{\partial L}{\partial \mathbf{x}} = \mathbf{F} - Q_1 \left( -\nabla \times A_2 + \frac{v}{c} \times (\nabla \times A_1) \right) - Q_2 \left( \nabla \times A_1 - \frac{v}{c} \times (-\nabla \times A_2) \right) = 0,
\]

from (17) it follows that the particle undergoes the force

\[
\mathbf{F} = Q_1 \left[ -\nabla \times A_2 + \frac{v}{c} \times (\nabla \times A_1) \right] + Q_2 \left( \nabla \times A_1 - \frac{v}{c} \times (-\nabla \times A_2) \right),
\]

Accordingly, under the specified restrictions, the generalised duality-invariant Lorentz-like force (18) has been derived from the Lagrangian (16).

Let us now apply (16) and (18) to the case of a dyon interacting with constant electric and magnetic fields. We write \( Q_1 = g, Q_2 = g, A_1 = \mathbf{A}, \) and \( A_2 = \mathbf{C} \). The corresponding Lagrangian of the system reads

\[
L = \frac{mv^2}{2} + \frac{v}{c} \cdot (g\mathbf{A} + g\mathbf{C}) + \mathbf{x} \cdot \left[ \nabla \times (g\mathbf{A} - g\mathbf{C}) \right],
\]

which leads to a particular case of the generalised Lorentz force acting on the dyon:

\[
\mathbf{F} = q \left( \mathbf{E} + \frac{v}{c} \times \mathbf{B} \right) + g \left( \mathbf{B} - \frac{v}{c} \times \mathbf{E} \right).
\]

The forces (14) and (20) exhibit the same form but the latter is of a more general character. The latter applies only when the electric and magnetic fields are constant vectors and given by the relations \( \mathbf{B} = \nabla \times \mathbf{A} \) and \( \mathbf{E} = -\nabla \times \mathbf{C} \). In the dyon-solenoid configuration the fields \( \mathbf{E} \) and \( \mathbf{B} \) satisfy these relations and the fields are equal to zero outside the dual solenoid and are constant inside of it. However, since the dyon moves outside the dual solenoid, the generalised Lorentz force acting on the dyon is zero. We then conclude that (19) is a suitable Lagrangian to describe our dyon-solenoid configuration.

The Lagrangian (19) is not invariant under space translations because of the presence of the vector \( \mathbf{x} \). But this is not a serious inconvenience. What matters is that the corresponding equation of motion remains invariant under space translations. In the Appendix A we apply the transformation \( \mathbf{x} \to \mathbf{x} + \varepsilon \mathbf{n} \) to the Lagrangian (19) (here \( \varepsilon \mathbf{n} \) an infinitesimal constant vector in the \( \mathbf{n} \) direction) and obtain the relation \( L(\mathbf{x} + \varepsilon \mathbf{n}, v, t) = L(\mathbf{x}, \mathbf{v}, t) + dL(\mathbf{x}, \mathbf{v}, t)/dt \), i.e., the Lagrangian changes by a total derivative. As is well-known, when a total derivative is added to a Lagrangian the corresponding equation of motion is unchanged. Therefore the transformation \( \mathbf{x} \to \mathbf{x} + \varepsilon \mathbf{n} \) does not change the equation of motion.

The canonical momentum derived from (19) reads

\[
\mathbf{p} = mv + \frac{1}{c} (g\mathbf{A} + g\mathbf{C}).
\]

The Hamiltonian of the force-free configuration reads

\[
H = \frac{1}{2m} \left( \mathbf{p} - \frac{1}{c} (g\mathbf{A} + g\mathbf{C}) \right)^2.
\]

Equations (19)-(22) are invariant under the duality transformations given in (13) and (15). We then conclude that the classical canonical treatment for the electromagnetic dyon-solenoid configuration is duality-invariant.
IV. DUALITY-INVARIENT QUANTUM PHASE

The canonical substitution \( p \rightarrow -i\hbar \nabla \) and the Hamiltonian (22) allow us to construct the duality-invariant Schrödinger equation for the non-relativistic dyon

\[
i\hbar \frac{\partial \Psi}{\partial t} = \frac{1}{2m} \left( -i\hbar \nabla - \frac{1}{c}(qA + gC) \right)^2 \Psi. \tag{23}\]

Since outside the dual solenoid the magnetic and electric fields vanish \( B = \nabla \times A = 0 \) and \( E = -\nabla \times C = 0 \), it follows that the solution of (23) can be constructed by multiplying the free solution \( \Psi_0 \) by a suitable phase factor, i.e., this solution is given by

\[
\Psi = e^{(i/\hbar)c \int_C (qA + gC) \cdot dl'} \Psi_0, \tag{24}\]

where the line integral in the phase is taken along a path of the dyon from the point \( \mathcal{O} \) to the point \( \mathbf{x} \). There is no restriction on the path taken by the dyon from the points \( \mathcal{O} \) to \( \mathbf{x} \), except that such path must never cross the dual solenoid. As the dyon encircles the solenoid, its wave function picks up the phase:

\[
\delta = \frac{1}{\hbar c} \int_C (qA + gC) \cdot dl', \tag{25}\]

and then (24) takes the form \( \Psi = e^{i\delta} \Psi_0 \). Inserting the potentials \( A = \Phi_e \phi'/(2\pi \rho') \) and \( C = -\Phi_e \phi/(2\pi \rho') \), and using \( dl' = \rho' d\phi' \phi \) in (25) we obtain

\[
\delta = \frac{(q\Phi_e - g\Phi_e)}{2\pi \hbar c} \oint_C d\phi'. \tag{26}\]

As the dyon moves around the dual solenoid (see Fig. 1), the azimuthal coordinate \( \phi' \) must vary continuously in time; thus we allow it to take on all values and do not restrict it to lie in the interval \([0, 2\pi]\), i.e., \( \oint_C d\phi' = \int_{t_1}^{t_2} (d\phi'/dt) dt = \phi'(t_2) - \phi'(t_1) \). Since the curve \( C \) is closed, this change must be an integer multiple of \( 2\pi \), i.e., \( \phi'(t_1) - \phi'(t_2) = 2\pi n \), where \( n \) is the winding number of the dyon path \( C \). The integration over \( \phi' \) is \( \oint_C d\phi' = 2\pi n \), and thus (26) gives the accumulated quantum phase

\[
\delta = \frac{n}{\hbar c} (q\Phi_n - g\Phi_n). \tag{27}\]

This is the main result of this paper. The novelty of the phase \( \delta \) is that it is invariant under the duality transformations \( \Phi_e \rightarrow \Phi_n, \Phi_n \rightarrow -\Phi_e \) and \( q \rightarrow g, g \rightarrow -q \) given in (13) and (15), respectively. The phase \( \delta \) is the first example of a manifestly duality-invariant electromagnetic quantum phase. The theoretical importance of this new phase lies on the fact that its duality symmetry allows us to give a unified description of the AB and DWF phases as well as two new interpretations of the AB phase. These results will be presented in detail in sections VIII and IX. We should also note that although Dowling et al. [7] showed that the HMW and DWF phases are derived trivally from the AC and AB phases via duality transformations, none of these four electromagnetic quantum phases are duality-invariant like the phase \( \delta \).

To finish this section we note that the phase \( \delta \) is gauge-invariant. To show that this is the case we must take into account two aspects: (i) Since the potentials (10) are in the Coulomb gauges \( \nabla \cdot A = 0 \) and \( \nabla \cdot C = 0 \) it follows that a further gauge transformation applied to these potentials must be a restricted gauge transformation, i.e., one in which the associated gauge functions \( \Lambda_1 \) and \( \Lambda_2 \) satisfy the Laplace equations \( \nabla^2 \Lambda_1 = 0 \) and \( \nabla^2 \Lambda_2 = 0 \). (ii) The corresponding gauge functions \( \Lambda_1 \) and \( \Lambda_2 \) must be single-valued functions and their gradients \( \nabla \Lambda_1 \) and \( \nabla \Lambda_2 \) must be non-singular. Let us now consider the restricted gauge transformations of the Coulomb-gauge potentials in (10):

\[
\tilde{A} = A + \nabla \Lambda_1, \quad \tilde{C} = C + \nabla \Lambda_2, \tag{28}\]

where \( \Lambda_1 \) and \( \Lambda_2 \) are single-valued gauge functions satisfying \( \nabla^2 \Lambda_1 = 0 \) and \( \nabla^2 \Lambda_2 = 0 \) with \( \nabla \Lambda_1 \) and \( \nabla \Lambda_2 \) non-singular. From (25) and (28) it follows that the gauge invariance of the new phase

\[
\frac{1}{\hbar c} \oint (q\tilde{A} - g\tilde{C}) \cdot dl' = \frac{1}{\hbar c} \oint (qA - gC) \cdot dl', \tag{29}\]

where we have used the fact that \( \oint \nabla' \Lambda_1 \cdot dl' = 0 \) and \( \oint \nabla' \Lambda_2 \cdot dl' = 0 \) because \( \Lambda_1 \) and \( \Lambda_2 \) are single-valued.

V. TOPOLOGY AND NONLOCALITY

The phase \( \delta \) is topological because it depends on the number of windings the dyon carries out around the solenoid and is independent of the shape of the dyon trajectory. However, to what extent is this topological property a defining feature of the phase \( \delta \)? Let us discuss this question in more detail. Crucial for the existence of the phase \( \delta \) is the fact that the dyon-solenoid configuration is defined in a non-simply connected region which includes...
also see that the phase invariant Schrödinger equation given in (23). We can respond this theorem is used together with the potentials (10) to obtain the Hamiltonian of the system

\[ H = \frac{1}{2m} \left( p_{\phi}^2 - \frac{(q\Phi_n - g\Phi_e)}{2\pi c b} \right)^2, \]

where \( p_{\phi} = mb(d\phi/dt) + (q\Phi_n - g\Phi_e)/(2\pi cb) \) is the canonical momentum. Since the associated wave function depends only on the azimuthal coordinate \( \psi = \psi(\phi) \) it follows that \( \nabla \rightarrow (\phi/b)(d/d\phi) \). The canonical substitution is \( p_{\phi} \rightarrow -i\hbar(1/b)(d/d\phi) \) which is used in (32) to obtain the time-independent Schrödinger equation

\[ \frac{1}{2m} \left( -\frac{i\hbar}{b} d/d\phi - \frac{(q\Phi_n - g\Phi_e)}{2\pi c b} \right)^2 \psi(\phi) = E_\ell \psi(\phi), \]

where \( E_\ell \) are the corresponding energy levels. To solve (33) it is convenient to write this equation in the form

\[ \frac{d^2 \psi}{d\phi^2} - \frac{i(q\Phi_n - g\Phi_e)}{\pi \hbar c} \frac{d\psi}{d\phi} + \frac{2mb^2 E_\ell}{\hbar^2} \left( \frac{(q\Phi_n - g\Phi_e)}{2\pi \hbar c} \right)^2 \psi = 0. \]

Normalised solutions of (34) are of the form

\[ \psi(\phi) = \frac{e^{i\ell\phi}}{\sqrt{2\pi b}}, \]

with \( \ell \) given by

\[ \ell = \frac{(q\Phi_n - g\Phi_e)}{2\pi \hbar c} \pm \frac{b}{\hbar} \sqrt{2m E_\ell}. \]

Single-valuedness of the wave function: \( \psi(\phi) = \psi(\phi + 2\pi) \) requires the condition that \( \ell \) be an integer. This condition (36) yield the duality-invariant energy levels

\[ E_\ell = \frac{\hbar^2}{2mb^2} \left( \ell - \frac{(q\Phi_n - g\Phi_e)}{2\pi \hbar c} \right)^2. \]

The energy levels clearly depend on the fluxes \( \Phi_n \) and \( \Phi_e \) even though the dyon is moving in a region for which these fluxes are excluded. This exhibits the nonlocal effect of these fluxes on the energy levels.
VII. QUANTUM INTERFERENCE EFFECT

A quantum interference effect associated to the phase $\delta$ can be envisioned by considering a hypothetical two-slit experiment in which dyons are emitted from a source, pass through two slits on a first screen and are finally detected on a second screen (See Fig. 3). The total wave function reads $\Psi = \Psi_1 + \Psi_2$ with $\Psi_1$ and $\Psi_2$ being the wave functions of the dyons passing through the first and second slit. In this case the probability density reads $P = |\Psi_1 + \Psi_2|^2$. We now modify the set-up by inserting a dual solenoid between the two screens. From the solution (24) each of the wave functions picks up a phase. The total wave function of the dyons can now be expressed as

\[
\Psi = (\Psi_1 + e^{i(\bar{q}/\hbar)}f(qA+gC)\cdot d\Psi_2) \times e^{i(\bar{q}/\hbar)\int f(qA+gC)\cdot d\Psi_1},
\]

where we have used the property $\int f_2 - \int f_1 = \oint$. The probability density is now given by $P = |\Psi_1 + e^{i\delta}\Psi_2|^2$, where $\delta = (q\Phi_n - g\Phi_e)/(\hbar c)$. This means that the wave functions create an interference pattern attributable to the electric and magnetic fluxes even though the dyons are moving in a region for which these fluxes are excluded. In this case the difference of the two dyon paths $\oint_2 - \oint_1 = \oint$ forms a closed loop with the winding number $n = 1$.

The presence of the phase $\delta$ manifests in the shift of the whole interference pattern (see Fig. 3). Following the standard calculation given by the AB effect [20], this shift is given by

\[
\Delta x = \frac{l\lambda}{d} \delta = \frac{l(q\Phi_n - g\Phi_e)}{mv cd},
\]

where $l$ is the distance between the two screens, $d$ is the separation between the slits on the first screen, $\lambda = \hbar/(mv)$ is the reduced de Broglie wavelength of the dyons with charges $q$ and $g$ and mass $m$ and velocity $v$.

VIII. $U(1)$ DUALITY SYMMETRY

We observe that the phase $\delta$ in (27) is invariant under the general $U(1)$ electromagnetic duality transformations

\[
q + ig = e^{-i\theta}(q + ig'), \quad \Phi_e + ig\Phi_n = e^{-i\theta}(\Phi'_e + ig\Phi'_n),
\]

where $\theta$ is an arbitrary real angle. This means that the phase $\delta = (q\Phi_n - g\Phi_e)/(\hbar c)$ takes the same form for primed quantities, that is,

\[
\delta' = \frac{n(q\Phi'_n - g\Phi'_e)}{\hbar c}.
\]

The transformations in (40) can explicitly be written as

\[
q' = q\cos \theta + g' \sin \theta, \quad g' = -q' \sin \theta + g' \cos \theta,
\]

\[
\Phi'_e = \Phi_e \cos \theta + \Phi'_n \sin \theta, \quad \Phi'_n = -\Phi'_e \sin \theta + \Phi'_n \cos \theta.
\]

The corresponding inverse transformations are given by

\[
q' = q \cos \theta - g \sin \theta, \quad g' = q \sin \theta + g \cos \theta,
\]

\[
\Phi'_e = \Phi_e \cos \theta - \Phi'_n \sin \theta, \quad \Phi'_n = \Phi_e \sin \theta + \Phi'_n \cos \theta.
\]

By exploiting the arbitrariness of the angle $\theta$, we can show that the phase $\delta = (q\Phi_n - g\Phi_e)/(\hbar c)$ unifies the AB phase [1]: $\delta_{AB} = q\Phi_n/(\hbar c)$ and its corresponding dual, the DWF phase [7]: $\delta_{DWF} = -q\Phi_e/(\hbar c)$.

We have two procedures to obtain the phase $\delta_{AB}$ from the duality-invariant phase $\delta'$:

(Ia) If we assume that all dyons have the same ratio of magnetic charge to electric charge: $g'/q' = \text{constant}$ and considering that $\theta$ is an arbitrary angle, we can write

\[
\frac{g'}{q'} = \tan \theta,
\]

which implies $\theta = \arctan(g'/q')$. The relation (46) and the second transformation in (42) imply the vanishing of the magnetic charge $g$ of the dyon. In this case the transformations in (44) become:

\[
q' = q \cos \theta, \quad g' = q \sin \theta.
\]

These transformations and the second transformation in (43) yield

\[
q'\Phi'_n - g'\Phi'_e = q(-\Phi'_e \sin \theta + \Phi'_n \cos \theta) = q\Phi_n.
\]

Using this relation the phase $\delta'$ becomes the AB phase:

\[
\left[\frac{n(q\Phi'_n - g\Phi'_e)}{\hbar c}\right]_{\theta = \arctan(g'/q')} = \frac{nq\Phi_n}{\hbar c}.
\]

(Ib) We can choose the angle $\theta$ to satisfy

\[
\frac{\Phi'_e}{\Phi'_n} = -\tan \theta,
\]

which implies $\theta = \arctan(-\Phi'_e/\Phi'_n)$. The relation (50) and the first transformation in (43) imply the vanishing of the
electric flux \( \Phi_e \) of the dual solenoid. Hence the transformations in (45) become

\[
\Phi'_e = -\Phi_n \sin \theta, \quad \Phi'_n = \Phi_n \cos \theta. \tag{51}
\]

These transformations and the first transformation in (42) yield

\[
q' \Phi'_n - g' \Phi'_e = (q' \sin \theta + g' \cos \theta) \Phi_n = q \Phi_n, \tag{52}
\]

and therefore the phase \( \delta' \) becomes again the AB phase:

\[
\left[ \frac{n(q' \Phi'_n - g' \Phi'_e)}{\hbar c} \right]_{\theta = \tan^{-1}(-\Phi'_e/\Phi'_e)} = \frac{nq \Phi_n}{\hbar c}. \tag{53}
\]

We note that (46) and (50) imply the relation

\[
g' \Phi'_n + q' \Phi'_e = 0. \tag{54}
\]

Since the flux \( \Phi'_n \) defines a current of electric charges \( q' \) and the flux \( \Phi'_e \) defines a current of magnetic charges \( g' \) then (54) indicates that in the time-independent regime there is no force acting between electric and magnetic charges.

Analogously, we have two ways to obtain the phase \( \delta_{\text{DWF}} \) from the duality-invariant phase \( \delta' \):

(Iiia) We choose the angle \( \theta \) to satisfy

\[
g' = -\cot \theta, \tag{55}
\]

which implies \( \theta = \cot^{-1}(-q' / g') \). The relation (55) and the first transformation in (42) imply the vanishing of the electric charge \( q \) of the dyon and therefore the transformations in (44) reduce to

\[
q' = -g \sin \theta, \quad g' = g \cos \theta. \tag{56}
\]

These transformations and the first transformation in (43) imply

\[
q' \Phi'_n - g' \Phi'_e = -g(\Phi'_e \cos \theta + \Phi'_n \sin \theta) = -g \Phi_e. \tag{57}
\]

Using this relation, the phase \( \delta' \) becomes the DWF phase:

\[
\left[ \frac{n(q' \Phi'_n - g' \Phi'_e)}{\hbar c} \right]_{\theta = \cot^{-1}(-q' / g')} = -\frac{ng \Phi_e}{\hbar c}. \tag{58}
\]

(Iiib) We now choose the angle \( \theta \) to satisfy

\[
\Phi'_e = \Phi'_n \cos \theta, \tag{59}
\]

which implies \( \theta = \cot^{-1}(\Phi'_e / \Phi'_n) \). This relation and the second transformation in (43) imply the vanishing of the magnetic flux \( \Phi_e \) of the dual solenoid. The transformations in (45) become

\[
\Phi'_e = \Phi_e \cos \theta, \quad \Phi'_n = \Phi_n \sin \theta. \tag{60}
\]

These relations together with the second transformation in (42) imply

\[
q' \Phi'_n - g' \Phi'_e = -(-q' \sin \theta + g' \cos \theta) \Phi_n = -g \Phi_e, \tag{61}
\]

and then the phase \( \delta' \) becomes the DWF phase:

\[
\left[ \frac{n(q' \Phi'_n - g' \Phi'_e)}{\hbar c} \right]_{\theta = \cot^{-1}(\Phi'_e / \Phi'_n)} = -\frac{ng \Phi_e}{\hbar c}. \tag{62}
\]

As expected, (55) and (59) also imply the relation (54).

From (I) and (II) we can see how the \( U(1) \) electromagnetic duality of the phase \( \delta' \) displays all its power: for the angle \( \theta \) defined either by \( \theta = \tan^{-1}(g' / q') \) or \( \theta = \tan^{-1}(-\Phi'_e / \Phi'_n) \) we obtain the AB phase \( \delta_{\text{AB}} = q \Phi_n / (\hbar c) \) and for the angle \( \theta \) defined either by \( \theta = \cot^{-1}(-g' / q') \) or \( \theta = \cot^{-1}(\Phi'_e / \Phi'_n) \) we obtain the DWF phase \( \delta_{\text{DWF}} = -g \Phi_e / (\hbar c) \). It is in this sense that we say that the phase \( \delta' \) unifies the AB and DWF phases.

**IX. TWO INTERPRETATIONS OF THE AB PHASE BASED ON \( U(1) \) DUALITY SYMMETRY**

The transformations in (47) imply

\[
q = \sqrt{q'^2 + g'^2}, \tag{63}
\]

and therefore we can write

\[
\sqrt{q'^2 + g'^2} \Phi_n = q \Phi_n. \tag{64}
\]

This relation suggests an alternative interpretation of the AB phase: Considering the right-hand side of (64), we can say that the AB phase \( \delta_{\text{AB}} = q \Phi_n / (\hbar c) \) is originated by the nonlocal interaction between the electric charge \( q \) and the enclosed magnetic flux \( \Phi_n \) of a dual solenoid.

This is in essence the standard nonlocal interpretation of the AB phase [21]. However, considering the left-hand side of (64) we can express the AB phase as

\[
\delta_{\text{AB}} = \frac{\sqrt{q'^2 + g'^2} \Phi_n}{\hbar c}, \tag{65}
\]

and claim that this phase is originated by the nonlocal interaction between a dyon having the electric charge \( q' = q \cos \theta \) and the magnetic charge \( g' = q \sin \theta \) with the enclosed magnetic flux \( \Phi_n \) of a dual solenoid.

Analogously, from (51) it follows that

\[
\Phi_n = \sqrt{\Phi'_e^2 + \Phi'_n^2}, \tag{66}
\]

which allows to express the AB phase as

\[
\delta_{\text{AB}} = \frac{q \sqrt{\Phi'_e^2 + \Phi'_n^2}}{\hbar c}; \tag{67}
\]

We can interpret (67) by saying that the AB phase is originated by the nonlocal interaction between the electric charge \( q \) of a dyon and the electric flux \( \Phi'_e = -\Phi_n \sin \theta \) and the magnetic flux \( \Phi'_n = \Phi_n \cos \theta \) of a dual solenoid.
X. DISCUSSION AND CONCLUSION

In absence of dyons and magnetic monopoles we could not observe the phase \( \delta \), at least not directly. In this sense the phase \( \delta \) is of speculative nature. However, if we assume an elementary dyon with charges \( q = e \) and \( g = g_0 \) where \( e \) and \( g_0 \) are the elementary electric and magnetic charges (this assumption is feasible because our basic equations do not violate CP symmetry [22]) then the phase \( \delta \) becomes

\[
\delta = \frac{n \pi R^2 e}{\hbar c} \left( B - \frac{1}{2\alpha} E \right),
\]

where we have assumed that \( e \) and \( g_0 \) are connected by \( g_0 = e/(2\alpha) \) with \( \alpha \) being the fine structure constant, as dictated by the Dirac quantisation condition [23, 24]. The field \( B \) is the confined magnetic field of a magnetic solenoid and the field \( E \) is the confined electric field of an electric solenoid. If a dual solenoid could be constructed in such a way that it encloses both \( E \) and \( B \) fields satisfying \( B - E/2\alpha \neq 0 \) then we could have evidence of the phase \( \delta \) and therefore of the elusive magnetic monopoles.

Summarising, we have shown that a dyon moving around a solenoid enclosing electric and magnetic fluxes accumulates a duality-invariant quantum phase which is topological, nonlocal, and gauge-invariant. Although speculative in nature, the new phase is of theoretical importance because its \( U(1) \) electromagnetic duality symmetry allows us to unify the AB and DWF phases and have two new interpretations of the AB phase.

Appendix A: Translational symmetry

Consider the Lagrangian (19)

\[
L = \frac{mv^2}{2} + \frac{v}{c} \cdot (qA + gC) + \frac{1}{2} \cdot (\nabla \times (gA - qC)),
\]

where

\[
A(x) = \frac{\Phi_a}{2\pi R^2} \hat{\rho}, \quad C(x) = -\frac{\Phi_e}{2\pi R^2} \hat{\phi}.
\]

Let us apply an infinitesimal space translation transformation to the Lagrangian (A1):

\[
x \rightarrow x + \varepsilon \hat{n},
\]

where \( \varepsilon \) is an infinitesimal quantity and \( \varepsilon \hat{n} \) is a constant vector in the \( \hat{n} \) direction. The first term of (A1) is trivially invariant under the transformation (A3) because the velocity vector \( v(t) \) does not have an explicit dependency on the position. For the second term in (A1) we need to apply the transformation (A3) to the potentials in (A2).

This is simpler to do in Cartesian coordinates. Inserting \( \rho = \sqrt{x^2 + y^2} \) and \( \phi = (-y\hat{x} + x\hat{y})/\sqrt{x^2 + y^2} \) in (A2) the potentials now read

\[
A(x) = \frac{\Phi_a(-y\hat{x} + x\hat{y})}{2\pi R^2}, \quad C(x) = -\frac{\Phi_e(-y\hat{x} + x\hat{y})}{2\pi R^2}.
\]

Now, we calculate \( A(x + \varepsilon \hat{n}) \) and \( C(x + \varepsilon \hat{n}) \),

\[
A(x + \varepsilon \hat{n}) = \frac{\Phi_n}{2\pi R^2}(-(y + \varepsilon y)\hat{x} + (x + \varepsilon x)\hat{y})
= A(x) + \frac{\Phi_a(\varepsilon y \hat{y} - \varepsilon x \hat{x})}{2\pi R^2}
= A(x) + \nabla \left[ \frac{\Phi_e(\varepsilon x y - \varepsilon y x)}{2\pi R^2} \right].
\]

\[
C(x + \varepsilon \hat{n}) = -\frac{\Phi_e}{2\pi R^2}(-(y + \varepsilon y)\hat{x} + (x + \varepsilon x)\hat{y})
= C(x) + \frac{\Phi_e(\varepsilon x y - \varepsilon y x)}{2\pi R^2}.
\]

Using the second line of (A5) and (A6) in the second term of the Lagrangian (A1) it follows

\[
\frac{v}{c} \cdot (qA(x + \varepsilon \hat{n}) + gC(x + \varepsilon \hat{n})) = \frac{v}{c} \cdot ((q\Phi_a - g\Phi_e)(\varepsilon x \hat{y} - \varepsilon y \hat{x})
= \frac{v}{c} \cdot (qA(x) + gC(x)) + \frac{d}{dt} \left[ \frac{(q\Phi_a - g\Phi_e)(\varepsilon x y - \varepsilon y x)}{2\pi R^2} \right].
\]

From (A7) and (A8) it follows that the transformation (A3) applied to the Lagrangian (A1) gives

\[
L(x + \varepsilon \hat{n}, v, t) = L(x, v, t) + \frac{dK(x, t)}{dt},
\]

where

\[
K(x, t) = \frac{(q\Phi_a - g\Phi_e)(\varepsilon x y - \varepsilon y x)}{2\pi R^2 c} + t(\varepsilon \hat{n}) \cdot \left[ \nabla \times (gA(x) - qC(x)) \right].
\]

Since the transformed Lagrangian (A10) differs only by a total derivative with respect to the Lagrangian (A1), it follows that the equation of motion for both Lagrangians is identical, i.e., (A1) and (A10) are equivalent Lagrangians. Therefore, the transformation (A3) is a symmetry of the equation of motion.
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