Type I, II, III and IV $q$-negative binomial distributions of order $k$

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March 19, 2024

Abstract

The distributions of waiting times in variations of the negative binomial distribution of order $k$. In one variation, we apply different enumeration schemes on the runs of successes. In another variation, binary trials with a geometrically varying probability of ones were performed. The exact distribution of the waiting time for the $r$-th occurrence of a success run of a specified length (e.g., nonoverlapping, overlapping, at least, exact, and $\ell$-overlapping) in a $q$-sequence of binary trials. First, the waiting time for the $r$-th occurrence of a success run with the “nonoverlapping” counting scheme was examined. Theorem 3.1 gives a probability function of the Type I $q$-negative binomial distribution of order $k$. Next, we consider the waiting time for the $r$-th occurrence of a success run with the “at least” counting scheme. Theorem 4.1 gives a probability function of the Type II $q$-negative binomial distribution of order $k$. Next, we consider the waiting time for the $r$-th occurrence of a success run with the “overlapping” counting scheme. Theorem 5.1 gives a probability function of the Type III $q$-negative binomial distribution of order $k$. Next, we consider the waiting time for the $r$-th occurrence of a success run with the “exact” counting scheme. Theorem 6.1 gives a probability function of the Type IV $q$-negative binomial distribution of order $k$. Next, we consider the waiting time for the $r$-th occurrence of a success run with the “$\ell$-overlapping” counting scheme, which is a more generalized counting scheme. Theorem 7.1 gives a probability function of the $q$-negative binomial distribution of order $k$ in the $\ell$-overlapping case.

The main theorems are Type I, II, III, and IV $q$-negative binomial distributions of order $k$ and the $q$-negative binomial distribution of order $k$ in the $\ell$-overlapping case. This work, examined sequences of independent binary zero and one trials with not necessarily identical distribution with the probability of ones varying according to a geometric rule. The exact formulas for the distributions were obtained using enumerative combinatorics.

Keywords: Waiting time problems, Type I, II, III, and IV $q$-negative binomial distribution of order $k$, $q$-Negative binomial distribution of order $k$ in the $\ell$-overlapping case, Runs, Binary trials, $q$-Distributions

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1 Introduction

Charalambides (2010b) studied discrete $q$-distributions on Bernoulli trials with geometrically varying success probabilities. Let us consider a sequence $X_1,...,X_n$ of zero(failure)-one(success) Bernoulli trials, such that the trials of the subsequence after the $(i-1)$-th zero until the $i$-th zero are independent with equal failure probability. The $i$-th geometric sequences of trials is the subsequence after the $(i-1)$-th zero and until the $i$-th zero, for $i > 0$, and the subsequence after the $(j-1)$'st zero and until the $j$'th zero, for $j > 0$ are independent for all $i \neq j$ (i.e., the $i$-th and $j$-th geometric sequences are independent) with probability of zeroes at the $i$-th geometric sequence of trials.

$$q_i = 1 - \theta q^{i-1}, \quad i = 1, 2, ..., \quad 0 \leq \theta \leq 1, \quad 0 \leq q < 1.$$ (1.1)

The probability of failures in the independent geometric sequences of trials increases geometrically with rate $q$. Let $F_j = \sum_{m=1}^{j} (1 - X_m)$ denote the number of zeroes in the first $j$ trials. Because the probability of a zero at the $i$-th geometric sequence of trials is in fact the conditional probability of the occurrence of a zero at any trial $j$ given the occurrence of $(i-1)$ zeroes in the previous trials. We can express this situation as follows.

$$q_{j,i} = p(X_j = 0 \mid F_j = i - 1) = 1 - \theta q^{i-1}, \quad i = 1, 2, ..., j, \quad j = 1, 2, ....$$ (1.2)

Here, (1.1) is exactly equal to the conditional probability given in (1.2). For a clear understanding, we consider an example with $n = 18$, i.e., the binary sequence 111011110111100110. Each subsequence has its own success and failure probabilities according to a geometric rule.

The stochastic model (1.1) or (1.2) has interesting applications. It was studied as a reliability growth model by Dubman and Sherman (1969) and applied to a $q$-boson theory in physics by Jing and Fan (1994) and Jing (1994). More specifically, the $q$-binomial distribution was introduced as a $q$-deformed binomial distribution, to set up a $q$-binomial state. The stochastic model (1.1) was also applied as a sequential-intervention model to start-up demonstration tests, as proposed by Balakrishnan et al. (1995).

The stochastic model (1.1) is the $q$-analog of the classical binomial distribution with geometrically varying probability of zeroes, which is a stochastic model of independent and identically distributed (IID) trials with the following failure probability:

$$\pi_j = P(X_j = 0) = 1 - \theta, \quad j = 1, 2, ..., \quad 0 < \theta < 1.$$ (1.3)

As $q$ tends toward 1, the stochastic model (1.1) reduces to IID (Bernoulli) model (1.3), because $q_i \to \pi_i, \quad i = 1, 2, ...$ or $q_{j,i} \to 1 - \theta, \quad i = 1, 2, ..., j, \quad j = 1, 2, ....$
Discrete $q$-distributions based on the stochastic model of the sequence of independent Bernoulli trials have been explored by numerous researchers. For a lucid review and comprehensive list of publications in this area, the interested reader may consult the monographs by Charalambides (2010b,a, 2016).

From mathematical and statistical viewpoints, Charalambides (2016), in the preface of his book remarked upon the advantage of using a stochastic model of a sequence of independent Bernoulli trials, in which the probability of success in a trial is assumed to vary with the number of trials and/or the number of successes. Such a model is advantageous because it allows us to incorporate the experience gained from previous trials and/or successes. If the probability of success at a trial is a general function of the number of trials and/or the number of successes, little can be inferred about the distributions of the various random variables that can be defined using this model. The assumption that the probability of success (or failure) at a trial varies geometrically, with the rate (proportion) $q$, leads to the introduction of discrete $q$-distributions.

Let us consider a distribution that is related to the success run analog of the classical negative binomial distribution. Let $X_1, X_2, \ldots$ be a sequence of binary trials with two possible outcomes (i.e., success or failure) in each trial. In this work, we examined the waiting time until the $r$-th (where $r$ is a positive integer) appearance of a success run of length $k$ by considering the enumeration scheme (i.e., nonoverlapping, at least, overlapping, exact, or $\ell$-overlapping scheme). Notably, the special case $r = 1$ reduces to the geometric distribution of order $k$ (the distribution of the number of trials until the success run of length $k$, denoted as $T_k$). When considering the waiting distribution, different counting schemes are used, and each scheme generates a different type of waiting time distribution.

There are several ways to count a scheme. Each counting scheme depends on different conditions: whether overlapping counting is permitted and whether counting starts from scratch when a certain kind or size of run has been enumerated. Feller (1968) proposed a classical counting method: once $k$ consecutive successes are observed, the number of occurrences of $k$ consecutive successes is counted, and the counting procedure starts anew (i.e., from scratch). This is called a nonoverlapping counting scheme or Type I distributions of order $k$. In another scheme, success runs of length greater than or equal to $k$ preceded and followed by a failure or by the beginning or by the end of the sequence are counted (see, e.g., Mood (1940)), and this is usually called the at least counting scheme or Type II distribution of order $k$. Ling (1988) suggested the overlapping counting scheme, where in an uninterrupted sequence of $m \geq k$ successes preceded and followed by a failure or by the beginning or by the end of the sequence are counted. It accounts for $m - k + 1$ success runs of length of $k$ and is also referred to as Type III distributions of order $k$. Mood (1940) suggested an exact counting scheme, wherein success runs of length exactly $k$ preceded and succeeded by failure or by nothing are counted. This scheme is also referred to as Type IV distributions of order $k$.

It is well known that the negative binomial distribution arises as the distribution of the sum of $r$ independent random variables following the geometric distribution with parameter $p$. The ran-
dom variable $W_{r,k}^{(a)}$ denoted by the waiting time for the $r$-th occurrence of a success run with the counting scheme used $a = I$, which indicates the nonoverlapping counting scheme; $a = II$, which indicates the at least counting scheme; $a = III$, which indicates the overlapping one; and $a = IV$, which indicates the exactly one these schemes are denoted as $W_{r,k}^{(I)}, W_{r,k}^{(II)}, W_{r,k}^{(III)}$, and $W_{r,k}^{(IV)}$, respectively. In addition, if the sequence is an IID sequence of random variables $X_1, X_2, \ldots$, then the distributions of $W_{r,k}^{(I)}, W_{r,k}^{(II)}, W_{r,k}^{(III)}$ and $W_{r,k}^{(IV)}$ will be referred to as Type $I$, $II$, $III$ and $IV$ negative binomial distributions of order $k$ and will be denoted as $NB_k^{(I)}(r, \theta), NB_k^{(II)}(r, \theta), NB_k^{(III)}(r, \theta)$, and $NB_k^{(IV)}(r, \theta)$, respectively.

When the sequence is a $q$-geometric model, the distributions of $W_{r,k}^{(I)}, W_{r,k}^{(II)}, W_{r,k}^{(III)}$ and $W_{r,k}^{(IV)}$ have been called Type $I$, $II$, $III$ and $IV$ $q$-negative binomial distributions of order $k$, respectively. They can be denoted as $q - NB_k^{(I)}(r, \theta), q - NB_k^{(II)}(r, \theta), q - NB_k^{(III)}(r, \theta)$ and $q - NB_k^{(IV)}(r, \theta)$, respectively.

According to the four aforementioned counting schemes, the random variables of the number of $k$-success run of length $k$ have an overlapping (common) part of length at most $k$.

According to the counting scheme mentioned earlier, the random variables of the number of $\ell$-overlapping success runs of length $k$ counted in $n$ outcomes are denoted as $N_{n,k,\ell}$. Moreover, if the sequence is an IID sequence of random variables, $X_1, X_2, \ldots, X_n$, then distributions of $N_{n,k,\ell}$...
will be referred to as binomial distributions of order \( k \) in the \( \ell \)-overlapping case and denoted as \( B_{k,\ell}(n, \theta) \).

When the sequence is a \( q \)-geometric model, the distributions of \( W_{r,k,\ell} \) and \( N_{n,k,\ell} \) are called \( q \)-negative binomial distribution of order \( k \) in the \( \ell \)-overlapping case and \( q \)-binomial distribution of order \( k \) in the \( \ell \)-overlapping case, respectively. They are denoted as \( q - NB_{k,\ell}(r, \theta) \) and \( q - B_{n,k,\ell}(n, \theta) \), respectively.

To explain this better, let us assume that \( n = 15 \) binary trials, numbered from 1 to 15, are performed, and we obtain the following outcomes 11111011110111011. Then, the \( \ell \)-overlapping \( 1 \)-runs of length 4 are as follows: 1,2,3,4; 3,4,5,6; 8,9,10,11 for \( \ell = 2 \), and 1,2,3,4; 2,3,4,5; 3,4,5,6; 8,9,10,11 for \( \ell = 3 \). Hence, \( N_{15,4,2} = 3 \) and \( N_{15,4,3} = 4 \).

Let \( N_n^{(a)} \), \( a = I, II, III \) be a random variable denoting the number of occurrences of runs in the sequence of \( n \) trials; \( N_n^{(a)} \), \( a = I, II, III \), which is coincident with \( N_n, G_n, \) and \( M_n, \) respectively. The random variable \( N_n \) is closely related to the random variable \( W_{r,k} \) (see Feller (1968)). We have the following dual relationship \( N_n^{(a)} < r \) if and only if \( W_{r,k} > n \) for \( a = I, II, III \). Now, the probability function of the \( q \)-binomial distribution of order \( k \), can be derived easily using the dual relationship between the binomial and negative binomial distributions of order \( k \):

\[
P_{q,\theta} \left( N_n^{(I)} < r \right) = P_{q,\theta} \left( W_{r,k} > n \right), \quad P_{q,\theta} \left( N_n^{(II)} < r \right) = P_{q,\theta} \left( W_{r,k}^{(II)} > n \right)
\]

\[
P_{q,\theta} \left( N_n^{(III)} < r \right) = P_{q,\theta} \left( W_{r,k}^{(III)} > n \right).
\]

However, the above mentioned dual relationship cannot be considered by the Type IV enumeration scheme. Because \( W_{r,k}^{(IV)} > n \) implies \( N_n^{(IV)} < r \), instead of an iff relationship.

The \( q \)-negative binomial distribution with parameters \( q, \theta, k, \) and \( r \) is a distribution of the length of a sequence of Bernoulli trials with geometrically increasing failure probability until the \( r \)-th appearance of a success run of length \( k \). Here, \( 0 < q, \theta < 1 \), and \( r \) and \( k \) are positive integers. In this study, we examined the waiting time distribution for the \( r \)-th appearance of a success run of length \( k \) (nonoverlapping, at least, overlapping, and \( \ell \)-overlapping), and the probability of ones vary according to a geometric rule. The remainder of this paper is organized as follows: In Section 2, we introduce the basic definitions and necessary notations that will be useful throughout this article. In Section 3, we will study the Type I \( q \)-negative binomial distribution of order \( k \). We derive the exact probability function of the Type I \( q \)-negative binomial distribution of order \( k \) via combinatorial analysis. In Section 4, we will study the Type II \( q \)-negative binomial distribution of order \( k \). We derived the exact probability function of the Type II \( q \)-negative binomial distribution of order \( k \) via combinatorial analysis. In Section 5, we examine the Type III \( q \)-negative binomial distribution of order \( k \). We derive the exact probability function of the Type III \( q \)-negative binomial distribution of order \( k \) via combinatorial analysis. In Section 6, we will study the Type IV \( q \)-negative binomial distribution of order \( k \). We derive the exact probability function of the Type IV \( q \)-negative binomial distribution of order \( k \) using combinatorial analysis.

In Section 7, we examine the waiting time for the \( r \)-th \( \ell \)-overlapping occurrence of a success run of length \( k \) in a \( q \)-geometric sequence. The exact probability function of the \( q \)-negative binomial distribution of order \( k \) in the \( \ell \)-overlapping case was derived via combinatorial analysis.

5
2 Terminology and notation

We recall some definitions, notations, and known results that will be used in this paper. Throughout this work, we suppose that $0 < q < 1$. First, we introduce the following notation.

- $S_n$: the total number of successes in $X_1, X_2, \ldots, X_n$;
- $F_n$: the total number of failures in $X_1, X_2, \ldots, X_n$.

Next, we introduce some basic $q$-sequences and functions and their properties, which are useful in the rest of the paper. The $q$-shifted factorials are defined as

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).
\]  

(2.1)

Let $m$, $n$, and $i$ be positive integers and $z$ and $q$ be real numbers, with $q \neq 1$. The number $[z]_q = (1 - q^z) / (1 - q)$ is called the $q$-number, and in particular, $[z]_q$ is called the $q$-integer. The $m$th order factorial of the $q$-number $[z]_q$, which is defined by

\[
[z]_{m,q} = \prod_{i=1}^{m} [z - i + 1]_q = [z]_q[z - 1]_q \cdots [z - m + 1]_q
\]

\[
= \frac{(1 - q^z)(1 - q^{z-1}) \cdots (1 - q^{z-m+1})}{(1 - q)^m}, \quad z = 1, 2, \ldots, m = 0, 1, \ldots, z.
\]  

(2.2)

is called $q$-factorial of $z$ of order $m$. In particular, $[m]_{q,1} = [1]_{q,2} [2]_{q,3} \cdots [m]_{q,m}$ is called $q$-factorial of $m$. The $q$-binomial coefficient (or Gaussian polynomial) is defined as follows:

\[
\begin{aligned}
\left[ \begin{array}{c} n \\ m \end{array} \right]_q &= [n]_{m,q} [m]_{q}! [n-m]_{q}! \\
&= \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-m+1})}{(1 - q^m)(1 - q^{m-1}) \cdots (1 - q)} \\
&= \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, \quad m = 1, 2, \ldots,
\end{aligned}
\]  

(2.3)

The $q$-binomial ($q$-Newton’s binomial) formula is expressed as follows:

\[
\prod_{i=1}^{n}(1 + qz^{i-1}) = \sum_{k=0}^{n} q^{k(k-1)/2} \left[ \begin{array}{c} n \\ k \end{array} \right]_q z^k, \quad -\infty < z < \infty, \quad n = 1, 2, \ldots.
\]  

(2.4)

For $q \to 1$, the $q$-analogs correspond to their classical counterparts, that is,

\[
\lim_{q \to 1} \left[ \begin{array}{c} n \\ r \end{array} \right]_q = \binom{n}{r}
\]

Now, let us consider the sequence of independent geometric sequences of trials with the probability of failure at the $i$th geometric sequence of trials given by (1.1) or (1.2). We now focus on studying the number of successes in a given number of trials in this stochastic model.
Definition 2.1. Let $Z_n$ be the number of successes in a sequence of $n$ independent Bernoulli trials, with the probability of success at the $i$-th geometric sequence of trials given by (1.1) or (1.2). The distribution of the random variable $Z_n$ is called the $q$-binomial distribution, with parameters $n$, $\theta$, and $q$.

Let us introduce a $q$-analog of the binomial distribution. The probability function of the number $Z_n$ of successes in $n$ trials $X_1, \ldots, X_n$ is given by

$$P_{q,\theta}(Z_n = r) = \binom{n}{r}_q \theta^r (1 - \theta)^{n-r}, \quad r = 0, 1, \ldots, n,$$

with parameters $n$ and $\theta$. The $q$-binomial distribution was studied by Charalambides (2010b) and is related to the $q$-Bernstein polynomial. Jing (1994) introduced probability function (2.5) as a $q$-deformed binomial distribution, and derived the recurrence relation of its probability distribution. In the sequel, $P_{q,\theta}(\cdot)$ and $P_{\theta}(\cdot)$ denote the probabilities related to the stochastic models (1.1) and (1.3), respectively.

### 3 Type I $q$-negative binomial distribution of order $k$

In this section, we will study the Type I $q$-negative binomial distribution of order $k$. Let us consider the waiting time for the $r$-th occurrence of a success run of length $k$. For $r \in \mathbb{N}$ and $k \in \mathbb{N}$, let $W_{r,k}^{(I)}$ be the waiting time for the $r$-th appearance of a run of successes of length $k$. We use the nonoverlapping counting scheme (Type I enumeration scheme; Feller (1968)). In this scheme, once $k$ consecutive successes show up in a run of successes of length $k$ are counted, the counting procedure starts anew (from scratch). The support (range set) of $W_{r,k}^{(I)}$, $\mathcal{R}(W_{r,k}^{(I)})$ is given by

$$\mathcal{R}(W_{r,k}^{(I)}) = \{kr, kr + 1, \ldots\}.$$

We now state a useful definition and lemma for the proofs of the theorem in the sequel.
Definition 3.1. For $0 < q \leq 1$, define

$$A^k_q(r,s,t) = \sum_{y_1, y_2, \ldots, y_r} q^{y_2 + 2y_3 + \cdots + (r-1)y_r},$$

where the summation is performed over all integers $y_1, \ldots, y_r$ satisfying

$$y_1 + y_2 + \cdots + y_r = s,$$

$$\left\lfloor \frac{y_1}{k} \right\rfloor + \left\lfloor \frac{y_2}{k} \right\rfloor + \cdots + \left\lfloor \frac{y_r}{k} \right\rfloor = t,$$

and

$$y_j \geq 0, \quad j = 1, \ldots, r.$$

The following gives a recurrence relation useful for computing of $A^k_q(r,s,t)$.

Lemma 3.1. [Yalcin and Eryilmaz (2014)] For $0 < q \leq 1$, $A^k_q(r,s,t)$ obeys the following recurrence relation,

$$A^k_q(r,s,t) = \begin{cases} 
\sum_{j=0}^{k-1} q^{(r-1)j} A^k_q(r-1,s-j,t) \\
+ \sum_{j=k}^{s} q^{(r-1)j} A^k_q(r-1,s-j,t-\left\lfloor \frac{j}{k} \right\rfloor), & \text{if } r > 1, \ s \geq 0 \ and \ t \geq 0 \\
1, & \text{if } r = 1, \ s \geq 0 \ and \ t = \left\lfloor \frac{s}{k} \right\rfloor \\
0, & \text{otherwise.}
\end{cases}$$

Remark 1. We observe that $A^1_k(r,s,t)$ is the number of integer solutions $(y_1, \ldots, y_r)$ of

$$y_1 + y_2 + \cdots + y_r = s,$$

$$\left\lfloor \frac{y_1}{k} \right\rfloor + \left\lfloor \frac{y_2}{k} \right\rfloor + \cdots + \left\lfloor \frac{y_r}{k} \right\rfloor = t,$$

and

$$y_j \geq 0, \quad j = 1, \ldots, r,$$

indicating the total number of arrangements of $s$ balls in $r$ distinguishable cells. Hence, $t$ nonoverlapping runs of balls of length $k$ is given by

$$A^1_k(r,s,t) = \binom{r+t-1}{t} S(s-kt,r,k-1),$$

where $S(a, b, c)$ denotes the total number of integer solutions $x_1 + x_2 + \cdots + x_a = c$ such that $0 < x_i < b$ for $i = 1, 2, \ldots, a$. The number can be expressed as

$$S(a, b, c) = \sum_{j=0}^{a} (-1)^j \binom{a}{j} \binom{c-j(b-1)-1}{a-1}.$$

For example, see Charalambides (2002).
The probability function of the Type I $q$-negative binomial distribution of order $k$ is obtained from the following theorem. It is evident that

$$P_{q,\theta}(W_{r,k}^{(I)} = n) = 0 \text{ for } 0 \leq n < rk$$

and hence, we focus on determining the probability mass function (PMF) for $n \geq rk$.

**Theorem 3.1.** The PMF $w_q^{(I)}(n;r;k;\theta) = P_{q,\theta}(W_{r,k}^{(I)} = n)$ is given by

$$w_q^{(I)}(n;r;k;\theta) = \begin{cases} \sum_{i=1}^{n-rk} \theta^n q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) A_q^k(i, n-k-i, r-1) & \text{if } n \geq rk + 1, \\ \theta^{nk} & \text{if } n = rk, \\ 0, & \text{if } n < rk. \end{cases}$$

**Proof.** First, we examine $w_q^{(I)}(rk;r;k;\theta)$. Clearly, $w_q^{(I)}(rk;r;k;\theta) = (\theta q^0)^k = \theta^k$. Hereinafter, we assume $n > rk$. By the definition of $W_{r,k}^{(I)}$ every sequence of $n$ binary trials belonging to event $W_{r,k}^{(I)} = n$ must end with $k$ successes. The event $W_{r,k}^{(I)} = n$ can be expressed as

$$\left\{ W_{r,k}^{(I)} = n \right\} = \left\{ N_{n-k,k} = r-1 \land X_{n-k+1} = \cdots = X_n = 1 \right\}.$$

We partition the event $W_{r,k}^{(I)} = n$ into disjointed events given by $F_n = i$, for $i = 1, \ldots, n - rk$. Adding the probabilities, we have

$$P_{q,\theta}(W_{r,k}^{(I)} = n) = \sum_{i=1}^{n-rk} P_{q,\theta}(N_{n-k,k} = r-1 \land F_n = i \land X_{n-k+1} = \cdots = X_n = 1).$$

If the number of zeroes in the first $n - k$ trials is equal to $i$, that is, $F_{n-k} = i$, then for each of the $(n - k + 1)$ to $n$-th trials, the probability of success is

$$p_{n-k+1} = \cdots = p_n = \theta q^i.$$ 

This probability can be rewritten as follows.

$$P_{q,\theta}(W_{r,k}^{(I)} = n) = \sum_{i=1}^{n-rk} P_{q,\theta}(N_{n-k,k} = r-1 \land F_{n-k} = i) \times P_{q,\theta}(X_{n-k+1} = \cdots = X_n = 1 \mid F_{n-k} = i) = \sum_{i=1}^{n-rk} P_{q,\theta}(N_{n-k,k} = r-1 \land S_n = i) \left( \theta q^i \right)^k.$$
An element of the event $\left\{ W_{r,k}^{(I)} = n, F_n = i \right\}$ is an ordered sequence consisting of $n - i$ successes and $i$ failures such that the length of the success run is nonnegative integer, $r$ nonoverlapping runs of success of length $k$ and end with $k$ successes. The number of these sequences can be derived as follows: First, we distribute the $i$ failures. Note that $i$ failures form $i + 1$ cells. Next, we will distribute the $n - i - k$ successes into $i + 1$ distinguishable cells as follows.

\[
\begin{array}{cccc}
1 \cdots 101 \cdots 10 \cdots 01 \cdots 101 \cdots 1 & 1 \cdots 1 \\
\vdots & \vdots \\
y_1 & y_2 & y_i & y_{i+1} & k
\end{array}
\]

with $i$ 0s and $n - i$ 1s, where the length of the first 1-run is $y_1$, the length of the second 1-run is $y_2$, ..., the length of the $(i + 1)$-th 1-run is $y_{i+1}$. The probability of the event $\left\{ W_{r,k}^{(I)} = n, F_n = i \right\}$ is given by

\[
(\theta q^0)^y_1 (1 - \theta q^0)(\theta q^1)^y_2 (1 - \theta q^1) \cdots (\theta q^{i-1})^y_i (1 - \theta q^{i-1})(\theta q^i)^y_{i+1+k}.
\]

Using simple exponentiation algebra arguments to simplify,

\[
\theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) q^{y_2 + 2y_3 + \cdots + iy_{i+1}}.
\]

However, $y_j$s are nonnegative integers such that $y_1 + y_2 + \cdots + y_{i+1} = n - k - i$ and

\[
\left\lfloor \frac{y_1}{k} \right\rfloor + \left\lfloor \frac{y_2}{k} \right\rfloor + \cdots + \left\lfloor \frac{y_{i+1}}{k} \right\rfloor = r - 1
\]

so that

\[
P_{n,\theta} \left( W_{r,k}^{(I)} = n, F_n = i \right)
= \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1 + y_2 + \cdots + y_{i+1} = n-k-i} \sum_{y_1 \geq 0, y_j \geq 0} q^{y_2 + 2y_3 + \cdots + iy_{i+1}}.
\]

Summing with respect to $i = 1, \ldots, n-rk$, we get

\[
\sum_{i=1}^{n-rk} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1 + y_2 + \cdots + y_{i+1} = n-k-i} \sum_{y_1 \geq 0, y_j \geq 0} q^{y_2 + 2y_3 + \cdots + iy_{i+1}}
\]

From lemma 3.1 we can rewrite the above as follows:

\[
\sum_{i=1}^{n-rk} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) A_q^k(i, n - k - i, r - 1).
\]

Thus, the proof is completed.
For \( q = 1 \), from Theorem [3.1], the PMF of the Type I negative binomial distribution of order \( k \) in Bernoulli trials with success probability \( \theta \) is obtained as follows:

**Corollary 3.1.** The PMF \( w^{(I)}(n; r, k; \theta) = P_{\theta} \left( W^{(I)}_{r,k} = n \right) \) is given by

\[
P_{\theta} \left( W^{(I)}_{r,k} = n \right) = \begin{cases} 
\sum_{i=1}^{n-rk} \binom{n-i}{i} \left( 1 - \theta \right)^i \theta^{n-i} A^k_i(i, n-k-i, r-1) & \text{if } n \geq rk + 1, \\
\frac{\theta^{n}}{k^r} & \text{if } n = rk, \\
0 & \text{if } n < rk.
\end{cases}
\]  

(3.1)

Notably, Equation (3.1) in Corollary 4.1 is an alternative to the formula proposed by Philippou (1984) (See Definition 2.1) and computationally simpler.

**Remark 2.** A random variable related to \( W_{r,k} \) is \( N_{n,k} \) denotes the number of occurrences of a success run of length \( k \) in the sequence of \( n \) trials. Because the events \( (N_{n,k} \geq r) \) and \( (W^{(I)}_{r,k} \leq n) \) are equivalent, an alternative formula for the PMF of Type I \( q \)-negative binomial distribution of order \( k \), can be easily obtained, using the dual relationship between the binomial and negative binomial distribution of order \( k \). This alternative formula is given by

\[
P_{q,\theta} \left( N_{n,k} \geq r \right) = P_{q,\theta} \left( W^{(I)}_{r,k} \leq n \right).
\]

Consequently, the PMF \( w_q^{(I)}(n; k, r; \theta) = P_{q,\theta}(W^{(I)}_{r,k} = n) \) is implicitly determined by

\[
w_q^{(I)}(n; k, r; \theta) = \sum_{x=0}^{r-1} f_q^{(I)}(x; n-1, k; \theta) - f_q^{(I)}(x; n, k; \theta) \text{ for } n \geq rk \text{ and } r \geq 1,
\]

(3.2)

where the probabilities \( f_q^{(I)}(x; n-1, k; \theta) = P_{q,\theta}(N_{n-1,k} = x) \) and \( f_q^{(I)}(x; n, k; \theta) = P_{q,\theta}(N_{n,k} = x) \) were obtained by Yalçın and Eryilmaz (2014) Theorem 2:

\[
f_q^{(I)}(x; n, k; \theta) = \sum_{i=0}^{n-xk} \binom{n-i}{i} \prod_{j=1}^{i} (1 - \theta q^{j-1}) A^k_i(i + 1, n - i, x) \text{ for } x = 0, 1, \ldots, \left\lfloor \frac{n}{k} \right\rfloor.
\]

(3.3)

Usually, the obtained expression (3.3) for \( P_{q,\theta}(W^{(I)}_{r,k} = n) \) is computationally faster than that obtained using (3.1).

**4 Type II \( q \)-negative binomial distribution of order \( k \)**

In this section, we examine the Type II \( q \)-negative binomial distribution of order \( k \). Let us consider the waiting time for the \( r \)-th occurrence of a success run of length at least \( k \). For \( r \in N \) and \( k \in N \), let \( W^{(II)}_{r,k} \) be the waiting time for the \( r \)-th appearance of a run of successes of length at least \( k \). We use the at least counting scheme (Type II enumeration scheme, by Mood (1940),
i.e., a run of successes of length greater than or equal to $k$ preceded and succeeded by failure or by nothing. The support (range set) of $W_{r,k}^{(II)}$, $\mathfrak{R}(W_{r,k}^{(II)})$ is given by

$$\mathfrak{R}(W_{r,k}^{(II)}) = \{ r(k+1) - 1, r(k+1), \ldots \}.$$ 

We now make a useful definition and lemma for the proofs of the Theorem in the sequel.

**Definition 4.1.** For $0 < q \leq 1$, we define

$$B^k_q(r,s,t) = \sum_{y_1, y_2, \ldots, y_r} q^{y_2 + 2y_3 + \cdots + (r-1)y_r},$$

where the summation is over all integers $y_1, \ldots, y_r$ satisfying

$$y_1 + y_2 + \cdots + y_r = s,$$

$$\sum_{i=1}^{r} I(y_i - k) = t,$$

$$y_j \geq 0, \quad j = 1, \ldots, r,$$

where $I(j)$ stands for the indicator function such that $I(j) = 1$, if $j \geq 0$; $0$, otherwise.

The following gives a recurrence relation useful for the computation of $B^k_q(r,s,t)$.

**Lemma 4.1.** [Makri and Psillakis (2016)] For $0 < q \leq 1$, $B^k_q(r,s,t)$ obeys the following recurrence relation,

$$B^k_q(r,s,t) = \left\{ \begin{array}{ll} \sum_{j=0}^{s-(t-1)k} q^{(r-1)j} B^k_q(r-1,s-j,t-I(j-k)), & \text{if } r > 1, \ s \geq tk \text{ and } t \leq r \\ 1, & \text{if } r = 1, \ s \geq k \text{ and } t = 1 \\ 0, & \text{otherwise.} \end{array} \right.$$ 

**Remark 3.** We observe that $B^k_1(r,s,t)$ is the number of integer solutions $(y_1, \ldots, y_r)$ of

$$y_1 + y_2 + \cdots + y_r = s,$$

$$\sum_{i=1}^{r} I(y_i - k) = t,$$

$$y_j \geq 0, \quad j = 1, \ldots, r.$$ 

This indicates the total number of arrangements of $s$ balls in $r$ distinguishable cells such that each of exactly $t$ of them receives at least $k$ balls. This number is given by

$$B^k_1(r,s,t) = \binom{r}{t} H_{r-t}(s-tk, r, k-1).$$
where \( H_m(\alpha, r, k) \) denotes the number of allocations of \( \alpha \) indistinguishable balls into \( r \) distinguishable cells, where each of the \( m \) (\( 0 \leq m \leq r \)) specified cells is occupied by at most \( k \) balls. The number can be expressed as

\[
H_m(\alpha, r, k) = \sum_{j=0}^{\lfloor \frac{\alpha}{r+1} \rfloor} (-1)^j \binom{m}{j} \left( \frac{\alpha - (k+1)j + r - 1}{\alpha - (k+1)j} \right).
\]

(Makri et al. (2007b)).

The probability function of the Type II \( q \)-negative binomial distribution of order \( k \) is obtained from Theorem 4.1. Clearly,

\[
P_{q,\theta}(W^{(II)}_{r,k} = n) = 0 \quad \text{for} \quad 0 \leq n < r(k+1) - 1
\]
and so we focus on determining the PMF for \( n \geq r(k+1) - 1 \).

**Theorem 4.1.** The PMF \( w_q^{(II)}(n; r, k; \theta) = P_{q,\theta}(W^{(II)}_{r,k} = n) \) is given by

\[
P_{q,\theta}(W^{(II)}_{r,k} = n) =
\begin{cases}
\sum_{i=r-1}^{n-rk} \theta^{n-i}q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) B_q(i, n-k-i, r-1) & \text{if } n > r(k+1) - 1, \\
\sum_{i=r-1}^{n-rk} \theta^{n-i}q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) B_q(i, n-k-i, r-1) = \theta^{r(k+1)-1} & \text{if } n = r(k+1) - 1, \\
0 & \text{if } n < r(k+1) - 1.
\end{cases}
\]

**Proof.** We first consider \( w_q^{(II)}(r(k+1) - 1; r, k; \theta) \). Clearly, \( w_q^{(II)}(r(k+1) - 1; r, k; \theta) = (\theta q^0)^{r(k+1)-1} = \theta^{r(k+1)-1} \). Hereinafter, \( n > r(k+1) - 1 \). From the definition of \( W^{(II)}_{r,k} \) every sequence of \( n \) binary trials belonging to event \( W^{(II)}_{r,k} = n \) must end with \( k \) successes preceded by a failure. The event \( W^{(II)}_{r,k} = n \) can be expressed as

\[
\left\{ W^{(II)}_{r,k} = n \right\} = \left\{ G_{n-k-1,k} = r - 1 \land X_{n-k} = 0 \land X_{n-k+1} = \cdots \land X_n = 1 \right\}.
\]

We partitioned the event \( W^{(II)}_{r,k} = n \) into disjointed events given by \( F_n = i \), for \( i = r-1, \ldots, n-rk \). Adding the probabilities we have

\[
P_{q,\theta}(W^{(II)}_{r,k} = n) = \sum_{i=r-1}^{n-rk} P_{q,\theta}(G_{n-k-1,k} = r - 1 \land X_{n-k} = 0 \land F_n = i \land X_{n-k+1} = \cdots = X_n = 1).
\]

If the number of zeroes in the first \( n-k-1 \) trials is equal to \( i-1 \), that is, \( F_{n-k-1} = i-1 \), then in each of the \( (n-k+1) \) to \( n \)-th trials the probability of success is

\[
p_{n-k+1} = \cdots = p_n = \theta q^i.
\]
This probability can be written as follows.

\[ P_{q, \theta} \left( W_{r,k}^{(II)} = n \right) \]

\[ = \sum_{i=r-1}^{n-rk} P_{q, \theta} \left( G_{n-k-1,k} = r - 1 \land F_{n-k-1} = i - 1 \right) \times P_{q, \theta} \left( X_{n-k} = 0 \land X_{n-k+1} = \cdots = X_{n} = 1 \mid F_{n-k-1} = i - 1 \right) \]

\[ = \sum_{i=r-1}^{n-rk} P_{q, \theta} \left( G_{n-k-1,k} = r - 1 \land F_{n-k-1} = i - 1 \right) \left( 1 - \theta q^{i-1} \right) \left( \theta q^{i} \right)^{k}. \]

An element of the event \( \{ W_{r,k}^{(II)} = n, F_n = i \} \) is an ordered sequence that consists of \( n - i \) successes and \( i \) failures such that the length of the success run is a nonnegative integer, and \( r \) nonoverlapping runs of success of length at least \( k \) end with \( k \) successes preceded by a failure. The number of these sequences can be derived as follows: First, we distribute the \( i \) failures. Note that \( i \) failures form \( i + 1 \) cells. Next, we distribute the \( n - i - k \) successes into \( i \) distinguishable cells as follows.

```
1\ldots101\ldots10\ldots01\ldots101\ldots101\ldots01\ldots1
```

with \( i \) 0s and \( n - i \) 1s, where the length of the first 1-run is \( y_1 \), the length of the second 1-run is \( y_2 \); ...; the length of the \( (i) \)-th 1-run is \( y_i \). The probability of the event \( \{ W_{r,k}^{(II)} = n, F_n = i \} \) is given by

\[(\theta q^{0})^{y_1}(1 - \theta q^{0})(\theta q^{1})^{y_2}(1 - \theta q^{1})\cdots(\theta q^{i-1})^{y_i}(1 - \theta q^{i-1})(\theta q^{i})^{k}.\]

Using simple exponentiation algebra arguments to simplify, we get

\[\theta^{n-i} q^{ik} \prod_{j=1}^{i} \left( 1 - \theta q^{j-1} \right) q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.\]

However, the \( y_j \)'s are nonnegative integers such that \( y_1 + y_2 + \cdots + y_i = n - k - i \) and

\[I(y_1 - k) + \cdots + I(y_i - k) = r - 1\]

so that

\[P_{q, \theta} \left( W_{r,k}^{(II)} = n, F_n = i \right) \]

\[= \theta^{n-i} q^{ik} \prod_{j=1}^{i} \left( 1 - \theta q^{j-1} \right) \sum_{y_1 + y_2 + \cdots + y_i = n - k - i} \sum_{I(y_1 - k) + \cdots + I(y_i - k) = r - 1} q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.\]
Summing the above with respect to $i = r - 1, \ldots, n - rk$ yields

$$\sum_{i=r-1}^{n-rk} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1, \ldots, y_i = n-k-i} q^{y_1+y_2+\cdots+y_i} \prod_{y_1+y_2+\cdots+y_i=n-k-i} \prod_{y_1 = 0, \ldots, y_i = 0}$$

From lemma 4.1, we can rewrite the above relation as follows:

$$\sum_{i=r-1}^{n-rk} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) B_q^k(i, n-k-i, r-1).$$

Thus, the proof is completed.

For $q = 1$, from Theorem 4.1, the PMF of the Type II negative binomial distribution of order $k$ in Bernoulli trials with success probability $\theta$ is obtained as follows:

**Corollary 4.1.** The PMF $w^{(II)}(n; r; k; \theta) = P_\theta \left(W^{(II)}_{r,k} = n\right)$ is given by

$$P_\theta \left(W^{(II)}_{r,k} = n\right) = \begin{cases} \sum_{i=r-1}^{n-rk} \theta^{n-i} (1 - \theta)^i B_q^k(i, n-k-i, r-1) & \text{if } n > r(k+1) - 1, \\ \theta^{r(k+1) - 1} & \text{if } n = r(k+1) - 1, \\ 0, & \text{if } n < r(k+1) - 1. \end{cases}$$

**Remark 4.** $G_{n,k}$ is a random variable related to $W^{(II)}_{r,k}$. It denotes $G_{n,k}$ the number of occurrences of a success run of length at least $k$ in a sequence of $n$ trials. Because the events $(G_{n,k} \geq r)$ and $(W^{(II)}_{r,k} \leq n)$ are equivalent, an alternative formula for the PMF of Type II $q$-negative binomial distribution of order $k$, can be easily obtained using the dual relation between the binomial and negative binomial distribution of order $k$ is given by

$$P_q, \theta \left(G_{n,k} \geq r\right) = P_q, \theta \left(W^{(II)}_{r,k} \leq n\right).$$

Consequently, the PMF $w_q^{(II)}(n; k; r; \theta) = P_q, \theta \left(W^{(II)}_{r,k} = n\right)$ is implicitly determined by

$$w_q^{(II)}(n; k; r; \theta) = \sum_{x=0}^{n-rk} f_q^{(II)}(x; n-k, \theta) - f_q^{(II)}(x; n-k, \theta)$$

for $n \geq r(k+1) - 1$ and $r \geq 1$, (4.1)

where the probabilities $f_q^{(II)}(x; n-k, \theta) = P_q, \theta(G_{n-k} = x)$ and $f_q^{(II)}(x; n-k, \theta) = P_q, \theta(G_{n-k} = x)$ are obtained from Makri and Psillakis (2016).

$$f_q^{(II)}(x; n-k, \theta) = \sum_{i=0}^{n-k} \theta^{n-i} \prod_{j=1}^{i} (1 - \theta q^{j-1}) B_q^k(i+1, n-i, x)$$

for $x = 0, 1, \ldots, \left[n + 1\right]$. (4.2)

Usually, the obtained expression (4.2) for $P_q, \theta \left(W^{(II)}_{r,k} = n\right)$ is computationally faster than that obtained using (4.1).
5 Type III $q$-negative binomial distribution of order $k$

In this section, we study the Type III $q$-negative binomial distribution of order $k$. Let us consider the waiting time for the $r$-th occurrence of the overlapping success run of length $k$. For $r \in \mathbb{N}$ and $k \in \mathbb{N}$, let $W_{r,k}^{(III)}$ be the waiting time for the $r$-th appearance of the overlapping run of success of length $k$. We use the overlapping counting scheme (Type III enumeration scheme proposed by Ling (1988)); i.e., an uninterrupted sequence of $m \geq k$ successes preceded and followed by a failure or by the beginning or end of the sequence. The support (range set) of $W_{r,k}^{(III)}$, $\mathcal{R}(W_{r,k}^{(III)})$ is given by

$$\mathcal{R}(W_{r,k}^{(III)}) = \{k + r - 1, k + r, \ldots\}.$$ 

We list a useful definition and lemma for the proofs of Theorem in the sequel.

**Definition 5.1.** For $0 < q \leq 1$, we define

$$C_k^q(r,s,t) = \sum_{y_1, y_2, \ldots, y_r} q^{y_2 + 2y_3 + \cdots + (r-1)y_r}.$$ 

Here, the summation is taken over all integers $y_1, \ldots, y_r$ satisfying

$$y_1 + y_2 + \cdots + y_r = s,$$

$$C(y_1) + \cdots + C(y_r) = t, \text{ and}$$

$$y_j \geq 0, \quad j = 1, \ldots, r.$$ 

Further,

$$C(j) = \begin{cases} j - k + 1, & \text{if } j \geq k, \\ 0, & \text{otherwise} \end{cases}$$

Lemma 5.1 gives a recurrence relation useful for the computation of $C_k^q(r,s,t)$.

**Lemma 5.1.** [Yalcin (2013)] For $0 < q \leq 1$, $C_k^q(r,s,t)$ obeys the following recurrence relation,

$$C_q(r,s,t) =$$

$$\begin{cases} \sum_{j=0}^{k-1} q^{(r-1)/j} C_q(r-1, s-j, t) + \\
\sum_{j=k}^{s} q^{(r-1)/j} C_q(r-1, s-j, t-j+k-1), & \text{if } r > 1, s \geq 0 \text{ and } t \geq 0 \\
1, & \text{if } r = 1, s \geq k \text{ and } t = s-k+1 \\
0, & \text{or } r = 1, 0 \leq s < k \text{ and } t = 0, \\
& \text{otherwise}. \end{cases}$$
Remark 5. We observe that $C_1(r, s, t)$ is the number of integer solutions $(y_1, \ldots, y_r)$ of

$$y_1 + y_2 + \cdots + y_r = s,$$

$$C(y_1) + \cdots + C(y_r) = t,$$ and

$$y_j \geq 0, \quad j = 1, \ldots, r,$$

where

$$C(j) = \begin{cases} 
  j - k + 1, & \text{if } j \geq k, \\
  0, & \text{otherwise}
\end{cases}$$

This means that the total number of arrangements of $s$ balls in $r$ distinguishable cells, yielding $t$ overlapping runs of balls of length $k$ is given by

$$C_1(r, s, t) = \min(r, t) \sum_{a=1}^{\min(r, t)} \binom{r}{a} \binom{t-a}{a-1} S(r-a, k+1, s+r-t-ak),$$

where

$$S(a, b, c) = \sum_{j=0}^{a} (-1)^j \binom{a}{j} \binom{c-j(b-1)-1}{a-1}$$

(Charalambides (2002)).

The probability function of the Type III $q$-negative binomial distribution of order $k$ is obtained from Theorem 5.1, and clearly,

$$P_{q, \theta}(W_{r,k}^{(III)} = n) = 0 \text{ for } 0 \leq n < k + r - 1$$

Hence, we focus on determining the PMF for $n \geq k + r - 1$.

Theorem 5.1. The PMF $w_q^{(III)}(n; r, k; \theta) = P_{q, \theta}(W_{r,k}^{(III)} = n)$ is given by

$$P_{q, \theta}(W_{r,k}^{(III)} = n) = \begin{cases} 
  \sum_{t=k}^{k+r-1} \sum_{i=\left[\frac{n-k-r+1}{k+r-1}\right]}^{\frac{n-k-r+1}{k+r-1}} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) C_q(i, n-t-i, r-C(t)) & \text{if } n > k + r - 1, \\
  \theta^{k+r-1} & \text{if } n = k + r - 1, \\
  0, & \text{if } n < k + r - 1.
\end{cases}$$

Proof. We first consider $w_q^{(III)}(k+r-1; r, k; \theta)$. Clearly, $w_q^{(III)}(k+r-1; r, k; \theta) = (\theta q^0)^{k+r-1} = \theta^{k+r-1}$. Hereinafter, $n > k + r - 1$. By the definition of $W_{r,k}^{(III)}$ every sequence of $n$ binary trials belonging to the event $W_{r,k}^{(III)} = n$ must end with $k$ successes, and the $r$-th overlapping success
runs occur in the $n$-th trial. Let us consider $W_{r,k}^{(III)} = n$ end with $t$ successes. The event $W_{r,k}^{(III)} = n$ can be expressed as follows:

$$\left\{ W_{r,k}^{(III)} = n \right\} = \bigcup_{t=k}^{k+r-1} \left\{ M_{n-t,k} = r - C(t) \land X_{n-t} = 0 \land X_{n-t+1} = \cdots = X_n = 1 \right\}.$$ 

We partition the event $W_{r,k}^{(III)} = n$ into disjointed events given by $F_n = i$, for $i = \lfloor \frac{s-k-r}{k} \rfloor + 1, \ldots, n - (k + r - 1)$. By adding the probabilities, we get

$$P_{q,\theta}(W_{r,k}^{(III)} = n) = \sum_{t=k}^{k+r-1} \sum_{i=\lfloor \frac{s-k-r}{k} \rfloor + 1}^{n-(k+r-1)} P_{q,\theta}(M_{n-t,k} = r - C(t) \land X_{n-t} = 0 \land F_{n-t} = i) \times P_{q,\theta}(X_{n-t+1} = \cdots = X_n = 1 \mid F_{n-t} = i)$$

$$= \sum_{t=k}^{k+r-1} \sum_{i=\lfloor \frac{s-k-r}{k} \rfloor + 1}^{n-(k+r-1)} P_{q,\theta}(M_{n-t,k} = r - C(t) \land X_{n-t} = 0 \land F_{n-t} = i) \left( \theta q^i \right)^k.$$

An element of the event $\left\{ W_{r,k}^{(III)} = n, F_n = i \right\}$ is an ordered sequence consisting of $n - i$ successes and $i$ failures, such that the length of the success run is nonnegative integer, has $r$ overlapping runs of success of length $k$, and ends with $t$ ($t = k, \ldots, k + r - 1$) successes. The number of these sequences can be derived as follows: First, we distribute the $i$ failures, and $i$ failures form $i + 1$ cells. Next, we distribute the $n - i - t$ successes into $i$ distinguishable cells as follows.

$$\begin{array}{cccccc}
1 & \ldots & 101 & \ldots & 01 & \ldots & 10
\end{array}$$

with $i$ 0s and $n-i$ 1s, where the length of the first one-run is $y_1$; the length of the second one-run is $y_2$, ..., and the length of the $(i)$-th one-run is $y_i$. The probability of the event $\left\{ W_{r,k}^{(III)} = n, F_n = i \right\}$ is given by

$$(\theta q^0)^{y_1}(1 - \theta q^0)(\theta q^1)^{y_2}(1 - \theta q^1) \cdots (\theta q^{i-1})^{y_i}(1 - \theta q^{i-1})(\theta q^t)^i.$$
Simple exponentiation algebra arguments are used to simplify the above probability.

\[
\theta^{y_1 + \cdots + y_i + r} \prod_{j=1}^{i} (1 - \theta q^{j-1}) q^{y_2 + 2y_3 + \cdots + (i-1)y_i}
\]

\[
\theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
\]

However, \( y_j \) are nonnegative integers such that \( y_1 + y_2 + \cdots + y_i = n - t - i \) and

\[
C(y_1) + \cdots + C(y_i) + C(t) = r,
\]

so that

\[
P_{q,\theta} \left( W_{r,k}^{(III)} = n, F_n = i \right) = \sum_{t=k}^{k+r-1} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{C(y_1) + \cdots + C(y_i) + C(t) = r} q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
\]

Summing the above with respect to \( i = \left\lceil \frac{n-k-r}{k} \right\rceil + 1, \ldots, n - (k + r - 1) \), then

\[
\sum_{t=k}^{k+r-1} \sum_{i=\left\lceil \frac{n-k-r}{k} \right\rceil + 1}^{n-(k+r-1)} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{C(y_1) + \cdots + C(y_i) + C(t) = r} q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
\]

From Lemma 5.1, we can rewrite the above as follows:

\[
\sum_{t=k}^{k+r-1} \sum_{i=\left\lceil \frac{n-k-r}{k} \right\rceil + 1}^{n-(k+r-1)} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) C_q(i, n - t - i, r - C(t)).
\]

Thus, the proof is completed.

\[ \square \]

For \( q = 1 \), from Theorem 5.1, the PMF of the Type III negative binomial distribution of order \( k \) in Bernoulli trials with success probability \( \theta \) is obtained as follows:

**Corollary 5.1.** The PMF \( W_{r,k}^{(III)}(n; r, k; \theta) = P_{\theta} \left( W_{r,k}^{(III)} = n \right) \) is given by

\[
P_{\theta} \left( W_{r,k}^{(III)} = n \right) = \begin{cases} 
\sum_{i=k}^{k+r-1} \sum_{j=\left\lceil \frac{n-k-r}{k} \right\rceil + 1}^{n-(k+r-1)} \theta^{n-i} (1 - \theta)^i C(i, n - t - i, r - C(t)) & \text{if } n > k + r - 1, \\
\theta^{k+r-1} & \text{if } n = k + r - 1, \\
0 & \text{if } n < k + r - 1.
\end{cases}
\]

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Remark 6. $M_{n,k}$ is a random variable related to $W_{r,k}^{(III)}$. It denotes $M_{n,k}$ the number of occurrences of the overlapping success run of length $k$ in the sequence of $n$ trials. Because the events $(M_{n,k} ≥ r)$ and $(W_{r,k}^{(III)} ≤ n)$ are equivalent, an alternative formula for the PMF of Type III $q$-negative binomial distribution of order $k$, can be easily obtained, using the dual relation between the binomial and negative binomial distribution of order $k$ as follows:

$$P_{q,θ}(M_{n,k} ≥ r) = P_{q,θ}(W_{r,k}^{(III)} ≤ n).$$

Consequently, the PMF $w_q^{(III)}(n;k;r;θ) = P_{q,θ}(W_{r,k}^{(III)} = n)$ is implicitly determined by

$$w_q^{(III)}(n;k;r;θ) = \sum_{x=0}^{r-1} f_q^{(III)}(x;n-1,k;θ) - f_q^{(III)}(x;n,k;θ) \text{ for } n ≥ r(k+1) - 1 \text{ and } r ≥ 1, \quad (5.1)$$

where the probabilities $f_q^{(III)}(x;n-1,k;θ) = P_{q,θ}(M_{n-1,k} = x)$ and $f_q^{(III)}(x;n,k;θ) = P_{q,θ}(M_{n,k} = x)$ are as obtained by Yalcin (2013):

$$f_q^{(III)}(x;n,k;θ) = \sum_{i=0}^{n-k-x+1} θ^{n-i} \prod_{j=1}^{i} (1 - θq^{j-1})C_q^k(i+1,n-i,x) \text{ for } x = 0,1,\ldots,\left[\frac{n+1}{k+1}\right]. \quad (5.2)$$

Usually, the obtained expression (5.2) for $P_{q,θ}(W_{r,k}^{(III)} = n)$ is computationally faster than that obtained using (5.1).

6 Type IV $q$-negative binomial distribution of order $k$

We examined the Type IV $q$-negative binomial distribution of order $k$. First, we consider the waiting time for the $r$-th occurrence of the success run of length exactly $k$. For $r ∈ N$ and $k ∈ N$, let $W_{r,k}^{(IV)}$ be the waiting time for the $r$-th appearance of the run of successes of length exactly $k$. We use the length exactly $k$ counting scheme (Type IV enumeration scheme, as proposed by Mood (1940)). In this scheme, a success run of length exactly $k$ is preceded and succeeded by failure or by nothing. The support (range set) of $W_{r,k}^{(IV)}$, $\mathcal{R}(W_{r,k}^{(IV)})$ is given by

$$\mathcal{R}(W_{r,k}^{(IV)}) = \{r(k+1) - 1, r(k+1),\ldots\}.$$

We now present a useful definition and lemma for the proofs of Theorem 6.1 in the sequel.

Definition 6.1. For $0 < q ≤ 1$, we define the polynomial

$$D_q^k(r,s,t) = \sum_{y_1,y_2,\ldots,y_r} q^{y_1+2y_2+\cdots+(r-1)y_r},$$

where the summation is preformed over all integers $y_1,\ldots,y_r$ that satisfy

$$y_1 + y_2 + \cdots + y_r = s,$$
\[
\delta_{k,y_1} + \cdots + \delta_{k,y_r} = t, \quad \text{and} \quad y_j \geq 0, \quad j = 1, \ldots, r.
\]

\text{where,}
\[
\delta_{i,j} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j.
\end{cases}
\]

Now, we obtain a recurrence relation useful in the computation of \(D_q^k(r,s,t)\).

**Lemma 6.1.** [Oh and Jang (2022)] For \(0 < q \leq 1\), \(D_q^k(r,s,t)\) obeys the following recurrence relation:

\[
D_q^k(r,s,t) = \begin{cases} 
1, & \text{for } r = 1, s = k, t = 1 \\
0, & \text{otherwise.}
\end{cases}
\]

\[
= \begin{cases} 
\sum_{j=0}^{k-1} q^{j(r-1)}D_q^k(r-1,s-j,t) + q^{k(r-1)}D_q^k(r-1,s-k,t-1), & \text{for } r \geq 2, s \geq tk, t \leq r \\
\sum_{j=k+1}^{s} q^{j(r-1)}D_q^k(r-1,s-j,t), & \text{for } r = 1, 0 \leq s < k, t = 0 \\
& \text{or } r = 1, s > k, t = 0 \\
& \text{or } r = 1, s = k, t = 0
\end{cases}
\]

**Remark 7.** We observe that \(D_1^k(r,s,t)\) is the number of integer solutions \((y_1, \ldots, y_r)\) of

\[
y_1 + y_2 + \cdots + y_r = s,
\]

\[
\delta_{k,y_1} + \cdots + \delta_{k,y_r} = t, \quad \text{and} \quad y_j \geq 0, \quad j = 1, \ldots, r.
\]

where
\[
\delta_{i,j} = \begin{cases} 
1, & \text{if } i = j \\
0, & \text{if } i \neq j.
\end{cases}
\]

This means that the number of allocations of \(s\) balls into \(r\) cells so that each of exactly \(t\) of them receives exactly equal to \(k\) balls is given by

\[
D_1^k(r,s,t) = \binom{r}{t} A(s-tk, r-t, k),
\]

where \(A(\alpha, r, k) = \sum_{j=0}^{[\alpha/k]} (-1)^j \binom{r}{j} \left( \frac{\alpha-(k+1)j+r-1}{\alpha-jk} \right)\) (see Makri et al. (2007b)).

The probability function of the Type IV \(q\)-negative binomial distribution of order \(k\) is obtained from the following theorem [6.1] and clearly,

\[
P_{q,\theta} W_{r,k}^{(IV)} = n = 0 \text{ for } 0 \leq n < r(k+1) - 1
\]

Hence, we focus on determining the PMF for \(n \geq r(k+1) - 1\).
Theorem 6.1. The PMF $w_q^{(IV)}(n; r, k; \theta) = P_{q, \theta} \left( W_{r, k}^{(IV)} = n \right)$ is given by

$$P_{q, \theta} \left( W_{r, k}^{(IV)} = n \right) = \begin{cases} \sum_{i=r-1}^{n-rk} \theta^{n-i} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) D_q(i, n-k-i, r-1) & \text{if } n > r(k+1) - 1 \\ \theta^{r(k+1)-1} & \text{if } n = r(k+1) - 1, \\ 0, & \text{if } n < r(k+1) - 1. \end{cases}$$

Proof. We first examine $w_q^{(IV)}(r(k+1) - 1; r, k; \theta)$. Clearly, $w_q^{(IV)}(r(k+1) - 1; r, k; \theta) = (\theta q^0)^{r(k+1)-1} = \theta^{r(k+1)-1}$. Hereinafter, $n > r(k+1) - 1$. By the definition of $W_{r, k}^{(IV)}$, every sequence of $n$ binary trials belonging to the event $W_{r, k}^{(IV)} = n$ must end with $k$ successes. The event $W_{r, k}^{(IV)} = n$ can be expressed as

$$\{ W_{r, k}^{(IV)} = n \} = \{ E_{n-k, k} = r-1 \land X_{n-k} = 0 \land X_{n-k+1} = \cdots = X_n = 1 \}.$$

We partition the event $W_{r, k}^{(IV)} = n$ into disjointed events given by $F_n = i$, for $i = r-1, \ldots, n-rk$. Adding the probabilities, we get

$$P_{q, \theta} \left( W_{r, k}^{(IV)} = n \right) = \sum_{i=r-1}^{n-rk} P_{q, \theta} \left( E_{n-k, k} = r-1 \land X_{n-k} = 0 \land \land X_{n-k+1} = \cdots = X_n = 1 \right).$$

If the number of zeroes in the first $n-k$ trials is equal to $i$, that is, $F_{n-k} = i$, then in each of the $(n-k+1)$ to $n$-th trials, the probability of success is

$$p_{n-k+1} = \cdots = p_n = \theta q^i.$$ We can now rewrite this as follows:

$$P_{q, \theta} \left( W_{r, k}^{(IV)} = n \right) = \sum_{i=r-1}^{n-rk} P_{q, \theta} \left( E_{n-k, k} = r-1 \land F_{n-k} = i \right)$$

$$\times P_{q, \theta} \left( X_{n-k} = 0 \land X_{n-k+1} = \cdots = X_n = 1 \mid F_{n-k} = i \right)$$

$$= \sum_{i=r-1}^{n-rk} P_{q, \theta} \left( E_{n-k, k} = r-1 \land S_n = i \right) \left( 1 - \theta q^{i-1} \right) \left( \theta q^i \right)^k.$$

An element of the event $\{ W_{r, k}^{(IV)} = n, F_n = i \}$ is an ordered sequence that consists of $n - i$ successes and $i$ failures such that the length of the success run is a nonnegative integer; $r$ nonoverlapping runs of success of length exactly $k$ end with exactly length $k$ successes. The number of
these sequences can be derived as follows: First, we distribute the $i$ failures. Note that $i$ failures form $i + 1$ cells. Next, we distribute the $n - i - k$ successes into $i$ distinguishable cells as follows:

\[
\begin{array}{cccc}
\hline
1 & \ldots & 1 & 01 \ldots 10 \ldots 1 \ldots 101 \ldots 1 \\
y_1 & y_2 & y_{j-1} & y_i \\
\hline
\end{array}
\]

with $i$ 0s and $n - i$ 1s. Here, the length of the first one-run is $y_1$, and the length of the second one-run is $y_2$, ..., the length of the $(i)$-th one-run is $y_i$. The probability of the event $\{W_{r,k}^{(IV)} = n, \ F_n = i\}$ is given by

\[
(\theta q^0)^{y_1}(1 - \theta q^0)(\theta q^1)^{y_2}(1 - \theta q^1) \cdots (\theta q^{i-1})^{y_i}(1 - \theta q^{i-1})(\theta q^i)^k.
\]

Using simple exponentiation algebra arguments to simplify

\[
\theta^{n-i}q^k \prod_{j=1}^{i} (1 - \theta q^{j-1})q^{y_2+2y_3+\cdots+(i-1)y_i}.
\]

However, $y_i$s are nonnegative integers such that $y_1 + y_2 + \cdots + y_i = n - k - i$ and

\[
\delta_k, y_1 + \delta_k, y_2 + \cdots + \delta_k, y_i = r - 1
\]

so that

\[
P_q, \theta \left( W_{r,k}^{(IV)} = n, \ F_n = i \right) = \theta^{n-i}q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1+y_2+\cdots+y_i=n-k-i} \sum_{\delta_k, y_1+\delta_k, y_2+\cdots+\delta_k, y_i=r-1} q^{y_2+2y_3+\cdots+(i-1)y_i}.
\]

Summing the above with respect to $i = r - 1, \ldots, n - rk$, then

\[
\sum_{i=r-1}^{n-rk} \theta^{n-i}q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1+y_2+\cdots+y_i=n-k-i} \sum_{\delta_k, y_1+\delta_k, y_2+\cdots+\delta_k, y_i=r-1} q^{y_2+2y_3+\cdots+(i-1)y_i}.
\]

From lemma 6.1, we can rewrite the above as follows:

\[
\sum_{i=r-1}^{n-rk} \theta^{n-i}q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) D_q(i, n - k - i, r - 1).
\]

Thus, the proof is completed.
For \( q = 1 \), from Theorem [6.1] the PMF of the Type IV negative binomial distribution of order \( k \) in Bernoulli trials with success probability \( \theta \) is obtained as follows:

**Corollary 6.1.** The PMF \( w^{(IV)}(n; r, k; \theta) = P_\theta \left(W^{(IV)}_{r_k} = n\right) \) is given by

\[
P_\theta \left(W^{(IV)}_{r_k} = n\right) =
\begin{cases}
\sum_{i=r-1}^{n-r_k} \theta^{n-i}(1-\theta)^i D_1(i, n-k-i, r-1) & \text{if } n > r(k+1) - 1 \\
\frac{\theta^{r(k+1)-1}}{i} & \text{if } n = r(k+1) - 1 \\
0, & \text{if } n < r(k+1) - 1.
\end{cases}
\]

### 7 \( q \)-negative binomial distribution of order \( k \) in the \( \ell \)-overlapping case

We examined the \( q \)-negative binomial distribution of order \( k \) in the \( \ell \)-overlapping case. Let us consider the waiting time for the \( r \)-th occurrence of the \( \ell \)-overlapping success run of length exactly \( k \). For \( r \in N \) and \( k \in N \), let \( W_{r,k,\ell} \) be the waiting time for the \( r \)-th appearance of the \( \ell \)-overlapping run of successes of length \( k \). We will use the \( \ell \)-overlapping counting scheme, i.e., a success run of length \( k \) each of which may have an overlapping (common) part of length at most \( \ell \) \((\ell = 0, 1, \ldots, k-1)\) with the previous run of success of length \( k \) that has already been enumerated. The support (range set) of \( W_{r,k,\ell} \), \( \mathcal{R}(W_{r,k,\ell}) \) is given by

\[
\mathcal{R}(W_{r,k,\ell}) = \{\ell + r(k - \ell), \ell + r(k - \ell) + 1, \ldots\}.
\]

We now present a useful definition and lemma for the proofs of the theorem 7.1 in the sequel.

**Definition 7.1.** For \( 0 < q \leq 1 \), define

\[
E^k_q(r,s,t) = \sum_{y_1,y_2,\ldots,y_r} q^{y_2+2y_3+\cdots+(r-1)y_r},
\]

where the summation is performed over all integers \( y_1, \ldots, y_r \) satisfying

\[
y_1 + y_2 + \cdots + y_r = s, \quad D(y_1) + \cdots + D(y_r) = t, \quad \text{and} \quad y_j \geq 0, \quad j = 1, \ldots, r.
\]

Here,

\[
D(j) = \begin{cases}
\left[\frac{j-1}{k-1}\right], & \text{if } j \geq k, \\
0, & \text{otherwise}
\end{cases}
\]
Now, we present a recurrence relation useful for computing $E^{k,l}_q(r,s,t)$.

**Lemma 7.1.** ([Kinaci et al. 2016]) For $0 < q \leq 1$, $E^{k,l}_q(r,s,t)$ obeys the following recurrence relation:

$$E^{k,l}_q(r,s,t) = \begin{cases} 
\sum_{j=0}^{k-1} q^{r-1-j} E^{k,l}_q(r-1,s-j,t) + \\
\sum_{j=k}^{s} q^{r-1-j} E^{k,l}_q(r-1,s-j,t-D(j)), & \text{if } r > 1, s \geq 0 \text{ and } t \geq 0 \\
1, & \text{if } r = 1, s \geq k \text{ and } t = \frac{r-l}{k-l} \\
0, & \text{or } r = 1, 0 \leq s < k \text{ and } t = 0, \text{ otherwise.}
\end{cases}$$

**Remark 8.** Note that $E^{k,l}_1(r,s,t)$ is the number of integer solutions $(y_1, \ldots, y_r)$ of

$$y_1 + y_2 + \cdots + y_r = s,$$
$$D(y_1) + \cdots + D(y_r) = t,$$
$$y_j \geq 0, \quad j = 1, \ldots, r.$$

Here,

$$D(j) = \begin{cases} 
\left\lceil \frac{j-l}{k-l} \right\rceil, & \text{if } j \geq k, \\
0, & \text{otherwise}
\end{cases}$$

indicating that the total number of arrangements of $s$ balls in $r$ distinguishable cells, yielding $t$ $l$-overlapping runs of balls of length $k$ is given by

$$E^{k,l}_1(r,s,t) = \sum_{a=1}^{\min(r,t)} \binom{r}{a} \binom{t-1}{a-1} C(s-al-(k-l)s; a, r-a; k-l-1, k-1).$$

Here, $C(\alpha; i, r-i; m-a, n-1)$ denotes the total number of integer solutions arrangements of $\alpha$ indistinguishable balls into $r$ distinguishable cells, $i$ of which have capacity $m-1$, and each of the rest of $r-i$ has capacity $n-1$. This number can be expressed as follows:

$$C(\alpha; i, r-i; m-a, n-1) = \sum_{j_1=0}^{\lfloor (\alpha-mj_1)/n \rfloor} \sum_{j_2=0}^{j_1} (-1)^{j_1+j_2} \binom{i}{j_1} \binom{r-i}{j_2} \binom{\alpha-mj_1-nj_2+r-1}{r-1}$$

(See [Makri et al. 2007a]).

The probability function of the $q$-negative binomial distribution of order $k$ in the $l$-overlapping case is obtained from the following theorem as shown below:

$$P_{q,\theta}(W_{r,k,\ell} = n) = 0 \text{ for } 0 \leq n < l + r(k-l),$$

and hence, we focus on determining the PMF for $n \geq l + r(k-l)$. 

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Theorem 7.1. The PMF \( w_{q,\ell}(n; r, k; \theta) = P_{q,\theta}(W_{r,k,\ell} = n) \) is given by

\[
P_{q,\theta}(W_{r,k,\ell} = n) = \begin{cases} 
\sum_{i=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x-1-r(k-l)}{k} \right\rceil + 1}^{n-r(k-l)+l} \theta^{n-i}q^{i} \prod_{j=1}^{\ell} (1 - \theta q^{j-1}) \times \\
E_q^{\ell,i}(i, n-t-i, r-D(t)), & \text{if } n \geq l + r(k-l) \\
\theta^{l+r(k-l)}, & \text{if } n = l + r(k-l) \\
0, & \text{if } n < l + r(k-l).
\end{cases}
\]

Proof. We first examine \( w_q^{(\ell)}(l+r(k-l); r, k; \theta) \). Clearly, \( w_q^{(\ell)}(l+r(k-l); r, k; \theta) = (\theta q^\ell)^{l+r(k-l)} = \theta^{l+r(k-l)} \). Hereafter, we assume \( n > l + r(k-l) \). From the definition of \( W_{r,k,\ell} \), every sequence of \( n \) binary trials belonging to the event \( W_{r,k,\ell} = n \) must end with \( k \) successes, and the \( r-\ell \) overlapping success runs occur in the \( n \)th trial. Let us consider \( W_{r,k,\ell} = n \) end with \( t \) successes. The event \( W_{r,k,\ell} = n \) can be expressed as

\[
\{ W_{r,k,\ell} = n \} = \bigcup_{t=k}^{r(k-l)+l} \{ N_{n-t,k,\ell} = r-D(t) \land X_{n-t} = 0 \land X_{n-t+1} = \cdots = X_n = 1 \}.
\]

We partition the event \( W_{r,k,\ell} = n \) into disjointed events given by \( F_n = i \), for \( i = \left\lceil \frac{x-1-r(k-l)}{k} \right\rceil + 1, \ldots, n-r(k-l)-l \). By adding the probabilities, we get

\[
P_{q,\theta}(W_{r,k,\ell} = n) = \sum_{t=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x-1-r(k-l)}{k} \right\rceil + 1}^{n-l-r(k-l)} P_{q,\theta}(N_{n-t,k,\ell} = r-D(t) \land X_{n-t} = 0 \land F_n = i \land X_{n-t+1} = \cdots = X_n = 1).
\]

If the number of zeroes in the first \( n-t \) trials is equal to \( i \), that is, \( F_{n-t} = i \), then in each of the \((n-t+1)\) to \( n \)th trials, the probability of success is

\[
p_{n-t+1} = \cdots = p_n = \theta q^i.
\]

The above expression can be rewritten as follows.

\[
P_{q,\theta}(W_{r,k,\ell} = n) = \sum_{t=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x-1-r(k-l)}{k} \right\rceil + 1}^{n-l-r(k-l)} P_{q,\theta}(N_{n-t,k,\ell} = r-D(t) \land X_{n-t} = 0 \land F_{n-t} = i)
\]

\[
\times P_{q,\theta}(X_{n-t+1} = \cdots = X_n = 1 \mid F_{n-t} = i)
\]

\[
= \sum_{t=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x-1-r(k-l)}{k} \right\rceil + 1}^{n-l-r(k-l)} P_{q,\theta}(N_{n-t,k,\ell} = r-D(t) \land X_{n-t} = 0 \land F_{n-t} = i) \left( \theta q^i \right)^k.
\]
An element of the event $\{W_{r,k,\ell} = n, \ F_n = i\}$ is an ordered sequence that consists of $n - i$ successes and $i$ failures such that the length of the success run is nonnegative integer with $r$ overlapping runs of success of length $k$ and end with $t(t = k, \ldots, r(k - \ell) + \ell)$ successes. The number of these sequences can be derived as follows: First, we distribute the $i$ failures; $i$ failures form $i + 1$ cells. Next, we distribute the $n - i - t$ successes into $i$ distinguishable cells as follows.

$$
\begin{array}{cccc}
1 & \ldots & 101 & \ldots \ 0 \ldots 10 \ldots 01 \ldots 10 \ldots 1 \ldots 1 \ldots 1
\end{array}
$$

with $i$ 0s and $n - i$ 1s, where the length of the first one-run is $y_1$, the length of the second one-run is $y_2$, ..., the length of the $(i)$-th one-run is $y_i$. The probability of the event $\{W_{r,k,\ell} = n, \ F_n = i\}$ is given by

$$
(\theta q^0)^{y_1} (1 - \theta q^0) (\theta q^1)^{y_2} (1 - \theta q^1) \cdots (\theta q^{i-1})^{y_i} (1 - \theta q^{i-1}) (\theta q^t)^t.
$$

We use simple exponentiation algebra arguments to simplify

$$
\theta^{y_1 + \cdots + y_i + t} q^i \prod_{j=1}^{i} (1 - \theta q^{j-1}) q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
$$

However, $y_j$s are nonnegative integers such that $y_1 + y_2 + \cdots + y_i = n - t - i$ and

$$
D(y_1) + \cdots + D(y_i) + D(t) = r
$$

so that

$$
P_{q, \theta} (W_{r,k,\ell} = n, \ S_n = i) = \sum_{t=k}^{r(k-l)+l} \theta^{n-i} q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1 + y_2 + \cdots + y_i = n - t - i} \sum_{y_1 \geq 0, \ldots, y_i \geq 0} q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
$$

Summing the above with respect to $i = \left\lceil \frac{x - l - 1 - r(k-l)}{k} \right\rceil + 1, \ldots, n - l - r(k-l)$, we get

$$
\sum_{t=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x - l - 1 - r(k-l)}{k} \right\rceil + 1} \theta^{n-i} q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) \sum_{y_1 + y_2 + \cdots + y_i = n - t - i} \sum_{y_1 \geq 0, \ldots, y_i \geq 0} q^{y_2 + 2y_3 + \cdots + (i-1)y_i}.
$$

From lemma [7.1] we can rewrite as follows.

$$
\sum_{t=k}^{r(k-l)+l} \sum_{i=\left\lceil \frac{x - l - 1 - r(k-l)}{k} \right\rceil + 1} \theta^{n-i} q^k \prod_{j=1}^{i} (1 - \theta q^{j-1}) E_q^{k,l}(i, n - t - i, r - D(t)).
$$

Thus, the proof is completed.
For $q = 1$, from Theorem 7.1 the PMF of the negative binomial distribution of order $k$ for $\ell$-overlapping success runs of length $k$ in Bernoulli trials with success probability $\theta$ is obtained as follows:

**Corollary 7.1.** The PMF $w^{(\ell)}(n; r, k; \theta) = P_\theta(W_{r,k,\ell} = n)$ is given by

$$P_\theta(W_{r,k,\ell} = n) =
\begin{cases}
\sum_{i=0}^{n-r(k-\ell)-l} \theta^{n-i} (1 - \theta)^i \mathbb{E}_1^{k,\ell} (i, n - t - i, r - D(t)) & \text{if } n > l + r(k - l), \\
\theta^{l+r(k-l)} & \text{if } n = l + r(k - l), \\
0 & \text{if } n < l + r(k - l).
\end{cases}$$

**Remark 9.** $N_{n,k,\ell}$ is a random variable related to $W_{r,k,\ell}$, and it denotes $N_{n,k,\ell}$ the number of occurrences of success run of length $k$ in the sequence of $n$ trials. Because the events $(N_{n,k,\ell} \geq r)$ and $(W_{r,k,\ell} \leq n)$ are equivalent, an alternative formula for the PMF of $q$-negative binomial distribution of order $k$ in the $\ell$-overlapping case, can be easily obtained using the dual relation between the binomial and negative binomial distribution of order $k$ in the $\ell$-overlapping case as follows:

$$P_{q,\theta} (N_{n,k,\ell} \geq r) = P_{q,\theta} (W_{r,k,\ell} \leq n).$$

Consequently, the PMF $w_q^{(\ell)}(n; k, r, \ell; \theta) = P_{q,\theta}(W_{r,k,\ell} = n)$ is implicitly determined by

$$w_q^{(\ell)}(n; k, r, \ell; \theta) = \sum_{x=0}^{r-1} f_q^{(\ell)}(x; n - 1, k, \ell; \theta) - f_q^{(\ell)}(x; n, k, \ell; \theta) \text{ for } n \geq \ell + r(k - \ell) \text{ and } r \geq 1, \quad (7.1)$$

where the probabilities $f_q^{(\ell)}(x; n - 1, k, \ell; \theta) = P_{q,\theta}(N_{n-1,k,\ell} = x)$ and $f_q^{(\ell)}(x; n, k, \ell; \theta) = P_{q,\theta}(N_{n,k,\ell} = x)$ already obtained by Kinaci et al. (2016) as follows:

$$f_q^{(\ell)}(x; n, k, \ell; \theta) = \sum_{i=0}^{v(x)} \theta^{n-i} \prod_{j=1}^{i} (1 - \theta q^{j-1}) \mathbb{E}_q^{k,\ell} (i + 1, n - i, x) \text{ for } x = 0, 1, \ldots, \left\lfloor \frac{n - \ell}{k - \ell} \right\rfloor, \quad (7.2)$$

where $v(x) = \begin{cases}
  n & \text{if } x = 0 \\
  n - (x(k - \ell) + \ell) & \text{otherwise}.
\end{cases}$

Usually, the obtained expression (7.2) for $P_{q,\theta}(W_{r,k,\ell} = n)$ is computationally faster than that obtained using (7.1).

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