Spin-weighted Green’s functions
in a conical space

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Abstract

We give an analysis of the spin-weighted Green’s functions well-defined in a conical space. We apply these results in the case of a straight cosmic string and in the Rindler space in order to determine generally the Euclidean Green’s functions for the massless spin-$\frac{1}{2}$ field and for the electromagnetic field. We give also the corresponding Green’s functions at zero temperature. However, except for the scalar field, it seems that these Euclidean Green’s functions do not correspond to the thermal Feynman Green’s functions.

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A conical space is an Euclidean space with a conical-type line singularity which can be described by the metric

$$ds^2 = d\rho^2 + B\rho^2 d\varphi^2 + (dx^1)^2 + \cdots + (dx^{n-2})^2$$  \hspace{1cm} (1)

in a coordinate system \((\rho, \varphi, x^i), i = 1, \ldots, n-2\), such that \(\rho > 0\) and \(0 \leq \varphi < 2\pi\), the hypersurfaces \(\varphi = 0\) and \(\varphi = 2\pi\) being identified. Metric (1) is characterized by a constant \(B\) supposed strictly positive; it reduces to the Euclidean metric for \(B = 1\).

In the massless case, the spin-weighted Green’s functions \(G_{(s)}\), \(s\) being integer or half-integer, obeys the equation \([1]\)

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{B^2 \rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{2is}{B\rho^2} \frac{\partial}{\partial \varphi} - \frac{s^2}{\rho^2} + \Delta_{n-2}\right)G_{(s)} = -\frac{1}{B\rho} \delta(\rho - \rho_0) \delta(\varphi - \varphi_0) \delta^{(n-2)}(x^i - x^i_0) \hspace{1cm} (2)$$

where \(\Delta_{n-2}\) is the usual Laplacian operator. For \(s\) half-integer, we suppose that \(G_{(s)}\) is antiperiodic in \(\varphi\) with period \(2\pi\) and for \(s\) integer that \(G_{(s)}\) is periodic in \(\varphi\) with period \(2\pi\). We impose that \(G_{(s)}\) vanishes when the points \((\rho, \varphi, x^i)\) and \((\rho_0, \varphi_0, x^i_0)\) are infinitely separated. Furthermore, we require that \(G_{(s)}\) is bounded in the limit where \(\rho \to 0\). These conditions should be uniquely determined the solution. We remark that \(G_{(s)} = G_{(-s)}\).

The spin-weighted Green’s functions can be directly related to the Green’s function \(D_\Phi\) of the Laplacian operator for metric (1), i.e. equation (2) with \(s = 0\), having the following conditions of periodicity in the coordinate \(\varphi\)

$$D_\Phi(\rho, \varphi + 2\pi, x^i) = \exp(2i\pi \Phi) D_\Phi(\rho, \varphi, x^i)$$

$$\frac{\partial}{\partial \varphi} D_\Phi(\rho, \varphi + 2\pi, x^i) = \exp(2i\pi \Phi) \frac{\partial}{\partial \varphi} D_\Phi(\rho, \varphi, x^i) \hspace{1cm} (3)$$

and vanishing when the points \((\rho, \varphi, x^i)\) and \((\rho_0, \varphi_0, x^i_0)\) are infinitely separated. We remark that \(D_\Phi = \overline{D}_{-\Phi}\). From equation (2), it is easy to show that

$$G_{(s)} = \exp[-iBs(\varphi - \varphi_0)] D_{1/2 + Bs}(\rho, \varphi - \varphi_0, x^i) \hspace{0.5cm} s = \pm 1/2, \pm 3/2, \ldots$$

$$G_{(s)} = \exp[-iBs(\varphi - \varphi_0)] D_{Bs}(\rho, \varphi - \varphi_0, x^i) \hspace{0.5cm} s = 0, \pm 1, \ldots \hspace{1cm} (4)$$

We point out that \(D_\Phi\) depends only on the fractional part of \(\Phi\), denoted \(\gamma\) such that \(0 \leq \gamma < 1\). Henceforth, we write \(\gamma = [\Phi]\). A detailed analysis of the Green’s function \(D_\Phi\)
has been done by Guimarães and Linet\[2\]. In consequence, the study of the spin-weighted Green’s functions $G_{(s)}$ results immediately from relations (4).

Equation (2) appears in the study of the field theories on the spacetime of a straight cosmic string for different values of the spin. The Euclidean Green’s function $S_E$ for a massless spin-$\frac{1}{2}$ field is given by

$$S_E = \left(e^{\mu^a \gamma_a \partial_\mu} + \frac{\gamma^\perp}{2\rho}\right) \left(\mathcal{R}G_{(-1/2)} + \gamma^\perp \gamma^2 \mathcal{G}_{(-1/2)}\right)$$

(5)

in polar vierbein $e^\mu_a\[3, 4\]$. The Euclidean Green’s function $G_{E\mu\mu_0}$ for the electromagnetic field can be found by the following relation [5]

$$G_{E\rho\rho_0} - i\frac{\rho_0}{B\rho} G_{E\varphi\rho_0} = \mathcal{G}_{(-1)}$$

(6)

the other components being

$$G_{E\rho\varphi_0} = -\frac{\rho_0}{\rho} G_{E\varphi\rho_0} \quad \text{and} \quad G_{E\varphi\varphi_0} = B^2 \rho_0 G_{E\rho\rho_0}$$

(7)

Similar formulas with $s = 2$ exist for the linearized gravitational perturbations. Now, we have determined $D_\Phi$ in closed form at four dimensions [2]

$$D_\Phi = \frac{e^{i(\varphi - \varphi_0)\gamma} \sinh[(\eta(1 - \gamma)/B) + e^{-i(\varphi - \varphi_0)(1 - \gamma)} \sinh(\eta\gamma/B)]}{8\pi^2 B\rho\rho_0 \sinh \eta \left[\cosh(\eta/B) - \cos(\varphi - \varphi_0)\right]} \quad \text{with} \quad \gamma = [\Phi]$$

(8)

where

$$\cosh \eta = \frac{\rho^2 + \rho_0^2 + (x^1 - x_0^1)^2 + (x^2 - x_0^2)^2}{2\rho_0}$$

(9)

therefore we obtain the general expression of $S_E$ and $G_{E\mu\mu_0}$ in terms of $B$ by formulas (4) and (3). For a cosmic string the physical value of $B$ is such that $B \leq 1$. In this case $[1/2 - B/2] = 1/2 - B/2$ and $[B] = B$ and then $S_E$ reduces to the result of Frolov and Serebriany [3] and $G_{E\mu\mu_0}$ to the one of Allen et al [3].

Another physical case is the one of the field theories at finite temperature $T$ on the Rindler space which is described by the Euclidean metric

$$ds^2 = d\xi^2 + \xi^2 d\tau^2 + (dx^1)^2 + (dx^2)^2$$

(10)

in a coordinate system $(\xi, \tau, x^1, x^2)$ such that $\xi > 0$ and $0 \leq \tau < 2\pi$. To analyse these theories, it is necessary to know the spin-weighted Green’s functions antiperiodic in $\tau$ for
s half-integer and periodic in \( \tau \) for s integer, the period being \( \beta = 1/kT \). The equivalence with the previous problem is obtained by setting

\[
\rho = \xi \quad \text{and} \quad \varphi = \frac{2\pi}{\beta} \tau \quad \text{with} \quad 0 \leq \tau < \beta \quad \text{and} \quad B = \frac{\beta}{2\pi}
\]  

In notations (11), we rewrite the Green’s function \( D_\Phi \)

\[
D_\Phi = \frac{e^{2i\pi \gamma (\tau - \tau_0)/\beta} \sinh[2\pi \eta(1 - \gamma)/\beta] + e^{-2i\pi (1 - \gamma)(\tau - \tau_0)/\beta} \sinh(2\pi \eta \gamma/\beta)}{4\pi \beta \xi_0 \sinh \eta \left[ \cosh(2\pi \eta/\beta) - \cos(2\pi (\tau - \tau_0)/\beta) \right]} \quad \text{with} \quad \gamma = [\Phi]
\]

and also formulas (12)

\[
G_s = \exp[-is(\tau - \tau_0)]D_{1/2, \beta s/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \quad s = \pm 1/2, \pm 3/2, \ldots
\]

\[
G_s = \exp[-is(\tau - \tau_0)]D_{\beta s/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \quad s = 0, \pm 1, \ldots
\]

For the massless spinor Green’s function at finite temperature, we obtain

\[
S_\beta = \left( e^{2\gamma/2} \partial_\mu + \frac{\gamma \lambda}{2\xi} \right) \left[ \Re \left( \exp[i(\tau - \tau_0)/2] D_{1/2-\beta/4\pi}(\xi, \tau - \tau_0, x^1, x^2) \right) \right]
\]

\[
G_{\beta\xi_0} = \Re \left( \exp[i(\tau - \tau_0)/2] D_{\beta/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \right)
\]

\[
G_{\beta\tau_0} = -\xi \Im \left( \exp[i(\tau - \tau_0)/2] D_{\beta/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \right)
\]

\[
G_{\beta\tau\tau_0} = \xi_0 G_{\beta\xi_0} \quad \text{and} \quad G_{\beta\xi\tau_0} = -\frac{\xi_0}{\xi} G_{\beta\tau\tau_0}
\]

After some calculations, we see that (14) gives for \( \beta \leq 2\pi \) the known result [3]. For the electromagnetic field, we obtain the components of the vector Green’s function

\[
G_{\beta\xi_0} = \Re \left( \exp[i(\tau - \tau_0)/2] D_{\beta/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \right)
\]

\[
G_{\beta\tau_0} = -\xi \Im \left( \exp[i(\tau - \tau_0)/2] D_{\beta/2\pi}(\xi, \tau - \tau_0, x^1, x^2) \right)
\]

\[
G_{\beta\tau\tau_0} = \xi_0 G_{\beta\xi_0} \quad \text{and} \quad G_{\beta\xi\tau_0} = -\frac{\xi_0}{\xi} G_{\beta\tau\tau_0}
\]

giving for \( \beta \leq 2\pi \) the known result [7].

In the Rindler space, the question of the validity of the expression of the thermal Green’s functions for \( \beta > 2\pi \) is crucial because for finding the thermal average of the energy-momentum tensor we have needed of the limit of the spin-weighted Green’s functions at zero temperature, i.e. \( \beta \to \infty \). For the scalar field \( s = 0 \), we know the Euclidean Green’s function \( G_{(0)\infty} \) at zero temperature [8]

\[
G_{(0)\infty} = \frac{\eta}{4\pi^2 \xi_0 \sinh \eta \left[ \eta^2 + (\tau - \tau_0)^2 \right]}
\]
With the aid of (16), we can define a Green’s function satisfying conditions (3) by putting
\[
\sum_{p=-\infty}^{+\infty} \exp(-2i\pi \Phi p) G(0)_{\infty}(\xi, \tau - \tau_0 + \beta p, x^1, x^2)
\]
(17)

In virtue of the uniqueness of the solution, we may identify (17) with \( D\Phi \). Hence, we find the spin-weighted Green’s functions \( G_{(s)\infty} \) at zero temperature as
\[
G_{(s)\infty} = \exp[-is(\tau - \tau_0)] G(0)_{\infty}(\xi, \tau - \tau_0, x^1, x^2)
\]
(18)

Taking into account (17), we get the basic property for a Green’s function at zero temperature
\[
G_{(s)} = \sum_{p=-\infty}^{+\infty} (-1)^p G_{(s)\infty}(\xi, \tau - \tau_0 + \beta p, x^1, x^2) \quad s = \pm 1/2, \pm 3/2, \ldots
\]
\[
G_{(s)} = \sum_{p=-\infty}^{+\infty} G_{(s)\infty}(\xi, \tau - \tau_0 + \beta p, x^1, x^2) \quad s = 0, \pm 1, \ldots
\]
(19)

We can then deduce the Green’s functions at zero temperature for the massless spin-\( \frac{1}{2} \) field and for the electromagnetic field.

In the case of the scalar field, the canonical quantification at finite temperature yields the thermal Feynman Green’s function. A rotation of Wick in the Feynman Green’s function gives the Euclidean Green’s function \( G(0) \) which is bounded in the limit where \( \xi \to 0 \).

For the massless spinor field and the electromagnetic field, several authors, working in the framework of the canonical quantification, obtain as expression of the Euclidean Green’s functions for any \( \beta \) the same form as the one for \( \beta \leq 2\pi \). We emphasize that \( D\Phi \) given by (12) vanishes at \( \xi = 0 \). When we replace \( \gamma \) by \( \Phi \), this expression gives also an Euclidean Green’s function but it blows up at \( \xi = 0 \) for \( \beta > 2\pi \). So, it seems that the Euclidean Green’s functions derived from the thermal Feynman Green’s functions are ill-defined at \( \xi = 0 \) for \( \beta > 2\pi \).

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