Research Article

On Impulsive Boundary Value Problem with Riemann-Liouville Fractional Order Derivative

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Our manuscript is devoted to investigating a class of impulsive boundary value problems under the concept of the Riemann-Liouville fractional order derivative. The subject problem is of implicit type. We develop some adequate conditions for the existence and uniqueness of a solution to the proposed problem. For our required results, we utilize the classical fixed point theorems from Banach and Schaefer. It is to be noted that the impulsive boundary value problem under the fractional order derivative of the Riemann-Liouville type has been very rarely considered in literature. Finally, to demonstrate the obtained results, we provide some pertinent examples.

1. Introduction

The fractional order differential equations (abbreviated as FODEs) are the generalization of the ordinary differential equations of the integer order. In the 17th century (1665), the great mathematicians Newton, Leibnitz and L’Hospital introduced for the first time the idea of fractional order differential equations (FODEs). Later on, in 1823, another mathematician by the name of Lacroix, introduced the fractional derivative [1] of simple power function. Furthermore, this area has been studied by many researchers because it has significant applications in various fields of science and technology in mathematical modeling of different fields of Science and Technology. For instance, some phenomena including the diffusion process [2], some chemical processes of electrochemistry [3], infectious disease in biology [4], signal and image processing [5], dynamic processes [6], and systems control theory [7] can be excellently described by using FODEs instead of the ordinary derivative. For further applications of FODEs, we refer to [8–13] and the references therein.

On the other hand, an interesting and important branch recently got warm attention known as impulsive differential equations (IDEs). In recent times, the said area has been increasingly used to model many physical and social phenomena in social sciences in a very interesting way. Currently in the said area, significant contribution has been done by various researchers like Simeonov and Bainov [14], Benchohra et al. [15], Lakshmikantham et al. [16], and Samoilenko and Perestyuk [17]. Benchohra and Slimani [18] has initiated the study of FODEs under impulsive conditions by using fractional derivatives of the Caputo and Riemann-Liouville type with order \( \alpha \in (0, 1) \). In addition, some applications of IDEs have been studied in various scientific disciplines such as biology, geography, engineering, dynamics, physics, geology, and management sciences. In terms of the important applications of IDEs, due to important applications of IDEs this field has a lot of significance and concentration (see [19–22]). For general research and significance, we refer some more important publications [23–26]. Many researchers have recently studied nonlinear FODEs with different kinds of boundary and initial conditions. Boundary value problems have significant applications in various fields of dynamics and fluid mechanics as well as engineering disciplines. Here, it is
remarkable that problems under integral boundary conditions have some important applications in fluid mechanics, chemical engineering, thermelasticity, flow of groundwater, population dynamics, and more (see [27–29]). Furthermore, we also refer some significance of FODEs under integral boundary conditions as discussed in [27–31].

Recently, due to increasing applications of FODEs to model real-world problems more comprehensively, researchers are taking keen interest to investigate different areas of fractional calculus. In particular, the use of FODEs in mathematical modeling of infectious diseases and other biological phenomena has got more attention. Various researchers have studied the fractional predator-prey pathogen model, the n-predator-prey model with herd behavior, etc. (see [32–34]). Further, there is also modeling of the interaction between tumor growth and the immune system, edge-detecting techniques, an infectious disease on a predator-prey model, etc. (we refer to [35–40]).

As stated earlier, the area devoted to IDEs with a fractional order has many applications. These differential equations can be modeled to those evolutionary processes which are subjected to abrupt changes in their states. Recently, some authors have used IDEs for the mathematical modeling of certain biological events. It is remarkable that impulsive differential equations are used in mathematical models which give rise to some important dual-layered impulsive systems. The said systems will open new doors in the future to develop a general mathematical theory for the said systems. For instance, the author of [41] has obtained very interesting results in this regard for various kinds of biological models of infectious diseases. Here, we remark that a very basic and important qualitative problem in the investigation of IDEs with a fractional order concerns the existence theory of solutions. For these purposes, researchers have used the classical fixed point theory and some tools of nonlinear analysis. For instance, in [42], the authors have applied fixed point results to develop the corresponding existence theory of solutions by using the Caputo derivative of the fractional order. In the same line, in [43], the authors have used the Picard-type analysis to investigate the stochastic-type IDEs of a fractional order by using the Caputo operator. In all these papers, the Caputo operator has been increasingly used. It is to be noted that the fractional order derivative of the Riemann-Liouville type has been very rarely used in IDEs.

Authors [44] have established existence theory for fractional order IDEs with initial conditions by using the fixed point theory. The authors in [45] investigated the following problem of IDEs under the fractional order derivative of the Riemann-Liouville type as

\[
\begin{align*}
\mathbb{R}L D^\alpha_0 r(z) &= f(z, r(z)), \quad z \in [0, T], z \notin z_m, 1 < \alpha_m \leq 2, z \in J, z \notin z_m, \\
\Delta r(z_m) &= \psi_m(r(z_m)), \quad m = 1, 2, 3, \cdots, q, \\
\Delta^\alpha r(z_m) &= \psi_m^*(r(z_m)), \quad m = 1, 2, 3, \cdots, q, \\
r(0) &= 0, \quad D^{\alpha-1}_0 r(0) = \beta,
\end{align*}
\]

where \( \mathbb{R}L \) is denoted as the Riemann-Liouville fractional order derivative, \( J = [0, 1] \), \( f : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is a continuous function. Furthermore, \( \psi_m, \psi_m^* : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions for \( m = 1, 2, \cdots, q, \) and \( \Delta r(z_m) = I^{1-\alpha}_0 r(z_m) - I^{1-\alpha}_0 r(z_m), \Delta^\alpha r(z_m) = I^{2-\alpha}_0 r(z_m) - I^{2-\alpha}_0 r(z_m) \) with \( r(z_m) = \lim_{h \rightarrow 0} r(z_m + h), r(z_m^*) = \lim_{h \rightarrow 0} r(z_m + h), m = 1, 2, \cdots, q, \) for \( 0 = z_0 < z_1 < z_2 \cdots z_{q+1} = 1. \) And also, where \( I^{1-\alpha}_0, I^{2-\alpha}_0 \) are denoted as the Riemann-Liouville integral of fractional order \( 1 - \alpha < 0, 2 - \alpha > 0 \) on \( J, \) respectively. To establish the required results, we utilize the Scheaffer fixed point theorem to investigate sufficient conditions for the existence of at least one solution to the problem under consideration (2). Furthermore, the criterion of uniqueness is derived by using the Banach contraction theorem. For the demonstration of our results, we provide some concrete examples.

2. Preliminaries

In this section, we provide some important results, basic definitions, and lemmas from the literature of fractional calculus [1, 3, 10, 11], which are needed in this manuscript.

Let \( J = [0, 1], J_0 = (z_0, z_1], \) and \( J_m = (z_m, z_{m+1}] \) for \( m = 1, 2, 3 \cdots q. \) Suppose that \( PC(J, \mathbb{R}) = \{r : J \rightarrow \mathbb{R}; r \in \mathbb{C}((z_m, z_{m+1}), \mathbb{R}), m = 0, 1, 2, \cdots, q + 1 \} \) and \( r(z_m^*) \) and \( r(z_m) \) exist with \( r(z_m) = \{r(z_m), m = 1, 2, \cdots, q \} \). Note that \( PC(J, \mathbb{R}) \) is a Banach space of piece-wise continuous function with norm \( ||r|| = \sup_{z \in J} |r(z)|. \)

Definition 1. The integral of the Riemann-Liouville fractional order \( \beta > 0 \) of a continuous function \( f : (0, \infty) \rightarrow \mathbb{R} \) is defined by

\[
\mathbb{D}_0^\beta f(z) = \frac{1}{\Gamma(\beta)} \int_0^z (z-s)^{\beta-1} f(s) ds, \quad z \in (0, 1).
\]

Therefore, the right side is point-wise defined on \( (0, \infty) \), where \( \Gamma \) is the symbol of gamma function defined as \( \Gamma(\beta) = \int_0^\infty e^{-s} s^{\beta-1} ds. \)
Lemma 4. Then, FODE

\[ RD^\beta f(z) = \left( \frac{d}{dz} \right)^n \int_0^z \frac{(z-s)^{n-\beta-1}}{\Gamma(n-\beta)} f(s) ds, \quad n-1 < \beta < n, z \in (0, 1), \]

where \( n = [\beta] + 1 \).

Definition 3. If the function \( f : (c, d) \rightarrow \mathcal{R} \) is at least \( n \)-times differentiable, then the Caputo fractional derivative of order \( \beta > 0 \) is defined as

\[ CD^\beta f(z) = \int_c^z \frac{(z-s)^{n-\beta-1}}{\Gamma(n-\beta)} f^n(s) ds, \quad n-1 < \beta < n, z \in (c, d), \]

where \( n = [\beta] + 1 \).

Lemma 4 [11]. Suppose \( \beta > 0 \), and \( r \in C(b, d) \cap L(b, d) \). Then, FODE

\[ RD^\beta r(z) = 0, \]

has a unique solution given by

\[ r(z) = d_i(z-b)^{\beta_i-1} + d_{i-1}(z-b)^{\beta_{i-1}-1} + \cdots + d_{n}(z-b)^{\beta_n-1}, \]

where \( d_i \in \mathcal{R}, i = 1, 2, \ldots, n, \) and \( n-1 < \beta < n \).

Lemma 5 [11]. In particular, \( \beta > 0 \) and \( r \in C[0, 1] \cap L[0, 1] \). We have

\[ ID^\beta r(z) = r(z) + a_1z^{\beta-1} + a_2z^{\beta-2} + \cdots + a_nz^{\beta-n}, \]

where \( a_k \in \mathcal{R}, k = 1, 2, \ldots, n, \) and \( n-1 < \beta < n \).

3. Main Works

To convert our considered problem into an impulsive fractional integral equation, the given Lemma is provided.

Lemma 6. The solution of the given linear IDE of the fractional order with the Riemann-Liouville derivative

\[ \begin{cases}
RD^\alpha r(z) = \sigma(z), & 1 < \alpha \leq 2, z \in \mathcal{J}, z \neq z_m, \\
\Delta r(z_m) = \psi_m(r(z_m)), & m = 1, 2, 3, \ldots, q, \\
\Delta^\alpha r(z_m) = \psi_m^*(r(z_m)), & m = 1, 2, 3, \ldots, q, \\
T^{1-a} r(0) = 0, & T^{2-a} r(1) = 0,
\end{cases} \]

is given by

\[ r(z) = \left\{ \begin{array}{ll}
\int_{z_m}^{z} \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds + z_m \left\{ \int_{z_m}^{z} (1-s)^{\alpha-1} \sigma(s) ds + \left( 1-z_m \right) \left( \int_{z_m}^{z} (z-s)^{\alpha-1} \sigma(s) ds - \psi_m(r(z_m)) \right) \right\} \\
+ \left( \int_{z_m}^{z} (z-s)^{\alpha-1} \sigma(s) ds - \psi_m(r(z_m)) \right) z^\alpha \Gamma(1+\alpha), \\
\int_{z_m}^{z} \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - [ \{(z-z_m)^{\alpha-1} + (z-z_m)^{\alpha-2} + z_m(z-z_m)^{\alpha-2}\} (1-z_m) ] \\
\times \left( \int_{z_m}^{z} (z-s)^{\alpha-1} \sigma(s) ds - \psi_m(r(z_m)) \right) - [ \{(z-z_m)^{\alpha-2} + (z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\} ] \\
\times \left( \int_{z_m}^{z} (z-s)^{\alpha-1} \sigma(s) ds - \psi_m(r(z_m)) \right) - \{(z-z_m)^{\alpha-1} + z_m(z-z_m)^{\alpha-2}\} \left( \int_{z_m}^{z} (1-s)^{\alpha-1} \sigma(s) ds \right), & z \in (z_m, \infty].
\end{array} \right. \]

Proof. Suppose \( r(z) \) is a solution to Problem (9); then, taking the Riemann-Liouville integral on both sides to using Lemma 5, there exist some constants \( c_0, c_1 \in \mathcal{R} \) such that

\[ r(z) = \int_0^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - c_0 \zeta^{\alpha-1} - c_1 \zeta^{\alpha-2}, \quad z \in [0, z_m]. \]

Again taking the Riemann-Liouville integral to using Lemma 9, for some constant \( d_0, d_1 \in \mathcal{R} \), we have

\[ r(z) = \int_0^z \frac{(z-s)^{\alpha-1}}{\Gamma(\alpha)} \sigma(s) ds - d_0 (z-z_m)^{\alpha-1} - d_1 (z-z_m)^{\alpha-2}, \quad z \in (z_m, z_1]. \]

Now, by using the impulsive conditions, we have \( \Delta r(z_1) = I^{1-a} r(z_1) - I^{1-a} r(z_1) = \psi_1(r(z_1)) \) and \( \Delta^\alpha r(z_1) = I^{2-a} r(z_1') - I^{2-a} r(z_1') = \psi_1^*(r(z_1)) \), and we find that

\[ -d_0 = \int_0^{z_1} (z_1-s)^{\alpha-1} \sigma(s) ds - c_0 - c_1 z_1^{\alpha-1} + \psi_1(r(z_1)), \]

\[ -d_1 = \int_0^{z_1} (z_1-s)^{\alpha-1} \sigma(s) ds - c_0 z_1 - c_1 + \psi_1^*(r(z_1)). \]
Thus, putting the values in (12), we have

\[ r(z) = \int_{z_1}^{z} \frac{(z-s)^{α-1}}{Γ(α)} \sigma(s)ds - (z-z_1)^{α-1} \]

\[ \times \left( \int_{0}^{z} (z-s)^{-2} \sigma(s)ds - \psi_m(r(z)) \right) - (z-z_1)^{α-2} \]

\[ \times \left( \int_{0}^{z} (z-s)^{-2} \sigma(s)ds - \psi_m^*(r(z)) \right) - c_1 \left\{ z_1^{-1}(z-z_1)^{α-1} + (z-z_1)^{α-2} \right\}, \quad z \in (z_1, z_2]. \]

The above process can be repeated in this way until we obtain the solution \( r(z) \) for \( z \in (z_m, z_{m+1}) \) as

\[ r(z) = \int_{z_m}^{z} \frac{(z-s)^{α-1}}{Γ(α)} \sigma(s)ds - (z-z_m)^{α-1} \]

\[ \times \left( \int_{0}^{z_m} (z_m-s)^{-2} \sigma(s)ds - \psi_m(r(z_m)) \right) - (z-z_m)^{α-2} \]

\[ \times \left( \int_{0}^{z_m} (z_m-s)^{-2} \sigma(s)ds - \psi_m^*(r(z_m)) \right) - c_1 \left\{ z_m^{-1}(z-z_m)^{α-1} + (z-z_m)^{α-2} \right\}, \quad z \in (z_m, z_{m+1}). \]

Now, applying boundary condition \( I^{1-α} r(0) = 0 \) and \( I^{2-α} r(1) = 0 \) to get the values of constant \( c_0 \) and \( c_1 \), we have

\[ c_0 = 0, \]

\[ c_1 = z_m \int_{0}^{z_m} (1-s)^{2} \sigma(s)ds - z_m(1-z_m) \]

\[ \times \left( \int_{0}^{z_m} (z_m-s)^{-2} \sigma(s)ds - \psi_m(r(z_m)) \right) - z_m \left( \int_{0}^{z_m} (z_m-s)^{2} \sigma(s)ds - \psi_m^*(r(z_m)) \right). \]

The values of \( c_0, c_1 \) putting in (11) and (15), one can obtain (10). On the contrary, suppose \( r(z) \) is a solution of the impulsive fractional integral equation (10). Following the direct calculation, we see that (10) satisfies the problem (9).

For simplification, we use the following notations:

\[ \sigma_1 = \sup_{z \in [0,1]} \left\{ (z-z_m)^{α-1} + \left\{ (z-z_m)^{α-1} + z_m(z-z_m)^{α-2} \right\} (1-z_m) \right\}, \]

\[ \sigma_2 = \sup_{z \in [0,1]} \left\{ (z-z_m)^{α-2} + \left\{ (z-z_m)^{α-1} + z_m(z-z_m)^{α-2} \right\} \right\}, \]

\[ \sigma_3 = \sup_{z \in [0,1]} \left\{ (z-z_m)^{α-1} + z_m(z-z_m)^{α-2} \right\}. \]
Theorem 7. Under hypotheses (H$_1$)-(H$_4$) and if the following condition holds
\[
\left[ \frac{K^*}{1 - L^*} \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_1}{3} \right) \Gamma(\alpha + 1) \right] \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_1}{3} \right) + (\sigma_1 N_1^* + \sigma_2 N_2^*) < 1,
\]
(22)
then, there exists a unique solution for Problem (2) on $\mathcal{J}$.

Proof. Let $r, \bar{r} \in PC(\mathcal{J}, \mathcal{R})$ for some $z \in \mathcal{J}$, we have
\[
|M_r(z) - M_r(z)|
\leq \int_{z_n}^c \left( \frac{z-s}{\Gamma(\alpha)} \right)^{a-1} \left| f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s)) \right| ds + \left| (z - z_m)^{a-1} \left\{ (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right\} (1 - z_m) \right|
\times \left( \int_{z_{m-1}}^{z_m} (z_m - s) \left| f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s)) \right| ds + \left| \psi_m(r(z)) - \psi_m(\bar{r}(z)) \right| \right)
+ \left| (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right|
\times \left( \int_{z_m}^1 (1-s)^{a-1} \left\{ f(s, r(s), \tau(s)) - f(s, \bar{r}(s), \bar{\tau}(s)) \right\} ds \right),
\]
(23)
which further gives
\[
|M_r(z) - M_r(z)|
\leq \int_{z_n}^c \left( \frac{z-s}{\Gamma(\alpha)} \right)^{a-1} \left| \tau(s) - \bar{\tau}(s) \right| ds + \left| (z - z_m)^{a-1} \left\{ (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right\} (1 - z_m) \right|
\times \left( \int_{z_{m-1}}^{z_m} (z_m - s) \left| \tau(s) - \bar{\tau}(s) \right| + \left| \psi_m(r(z)) - \psi_m(\bar{r}(z)) \right| \right)
+ \left| (z - z_m)^{a-2} + \left\{ (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right\} \right|
\times \left( \int_{z_m}^1 (1-s)^{a-1} \left\{ \tau(s) - \bar{\tau}(s) \right\} ds + \left| \psi_m(r(z)) - \psi_m(\bar{r}(z)) \right| \right)
+ \left| (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right|
\times \left( \int_{z_m}^1 (1-s)^{a-1} \left\{ \tau(s) - \bar{\tau}(s) \right\} ds \right),
\]
(24)
where $\tau, \bar{\tau} \in \mathcal{C}(\mathcal{J}, \mathcal{R})$ are given by
\[
\tau(z) = f(z, r(z), \tau(z)),
\bar{\tau}(z) = f(z, \bar{r}(z), \bar{\tau}(z)).
\]
By using hypothesis (H$_2$), we have
\[
|\tau(z) - \bar{\tau}(z)| = \left| f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z)) \right| 
\leq K^* |r(z) - \bar{r}(z)| + L^* |\tau(z) - \bar{\tau}(z)|.
\]
(26)
Repeating this process, we get
\[
|\tau(z) - \bar{\tau}(z)| \leq \frac{K^*}{1 - L^*} |r(z) - \bar{r}(z)|.
\]
(27)
Therefore, for every $z \in \mathcal{J}$ and from (24), using hypothesis (H$_3$), (H$_4$), and (27), one has
\[
|M_r(z) - M_r(z)|
\leq \frac{K^*}{1 - L^*} \int_{z_n}^c \left( \frac{z-s}{\Gamma(\alpha)} \right)^{a-1} \left| r(s) - \bar{r}(s) \right| ds + \left| (z - z_m)^{a-1} \left\{ (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right\} (1 - z_m) \right|
\times \left( \int_{z_{m-1}}^{z_m} (z_m - s) \left| r(s) - \bar{r}(s) \right| ds + N_1^* |r(z) - \bar{r}(z)| \right)
+ \left| (z - z_m)^{a-2} + \left\{ (z - z_m)^{a-1} + z_m(z - z_m)^{a-2} \right\} \right|
\times \left( \int_{z_m}^1 \left(1-s\right)^{a-1} \left| r(s) - \bar{r}(s) \right| ds + N_2^* |r(z) - \bar{r}(z)| \right)
\times \left( \int_{z_m}^1 \left(1-s\right)^{a-1} \left| r(s) - \bar{r}(s) \right| ds + N_2^* |r(z) - \bar{r}(z)| \right)
\times \left( \int_{z_m}^1 \left(1-s\right)^{a-1} \left| r(s) - \bar{r}(s) \right| ds \right).
\]
(28)
Upon further simplification, (29) yields
\[
\|M_r - M_{\bar{r}}\|
\leq \frac{K^*}{1 - L^*} \left( \frac{z - z_m}{\Gamma(\alpha + 1)} \right) \|r - \bar{r}\| + \sigma_1 \left( \frac{K^*}{1 - L^*} \left( \frac{z - z_m}{\Gamma(\alpha + 1)} \right)^2 \|r - \bar{r}\| \right)
+ \sigma_2 \left( \frac{K^*}{1 - L^*} \left( \frac{z - z_m}{\Gamma(\alpha + 1)} \right)^3 \|r - \bar{r}\| + N_2^* \|r - \bar{r}\| \right)
+ \sigma_3 \left( \frac{K^*}{1 - L^*} \right) \left( \frac{z - z_m}{\Gamma(\alpha + 1)} \right)^3 \|r - \bar{r}\|.
\]
(29)
Hence, from (29), we have
\[
\|M_r - M_{\bar{r}}\|
\leq \left[ \frac{K^*}{1 - L^*} \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) \right] \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (\sigma_1 N_1^* + \sigma_2 N_2^* + \sigma_3 N_3^*) \|r - \bar{r}\|.
\]
(30)
By (22), operator $M$ is a contraction. Thus, according to Banach’s contraction principle, operator $M$ has a unique fixed point which is the unique solution to Problem (2).
Next, we will prove that Problem (2) has at least one solution for this, and we use Schaefer’s fixed point theorem. Let the given hypotheses hold true:

(H2) There exist \( x, y, h \in PC(\mathcal{F}, \mathcal{R}) \), with

\[
x^* = \sup_{z \in [0, 1]} x(z),
\]
\[
y^* = \sup_{z \in [0, 1]} y(z),
\]
\[
h^* = \sup_{z \in [0, 1]} |h(z)| < 1,
\]

such that

\[
|f(z, r(z), \tau(z))| \leq x(z) + y(z)|r(z)| + h(z)|\tau(z)|,
\]

for \( z \in \mathcal{F}, r \in PC(\mathcal{F}, \mathcal{R}), \) and \( \tau \in \mathcal{R} \).

(H2) The function \( \psi_m : PC(\mathcal{F}, \mathcal{R}) \rightarrow \mathcal{R} \) is continuous, and there exist constants \( A'_m, B'_m > 0 \) such that

\[
|\psi_m(r(z))| \leq A'_m|(r(z)| + B'_m,
\]

for every \( r \in PC(\mathcal{F}, \mathcal{R}), \) \( m = 1, \ldots, q \).

(H2) The function \( \psi_m^* : PC(\mathcal{F}, \mathcal{R}) \rightarrow \mathcal{R} \) is continuous, and there exist constants \( A''_m, B''_m > 0 \) such that

\[
|\psi_m^*(r(z))| \leq A''_m|(r(z)| + B''_m,
\]

for every \( r \in PC(\mathcal{F}, \mathcal{R}), \) \( m = 1, \ldots, q \).

Theorem 8. If the hypotheses (H1), (H2), (H2)-(H2) are satisfied, then Problem (2) has at least one solution.

Proof. The proof is performed in several steps.

(Step 1) The operator \( M \) is continuous

Assume \( \{r_n\} \) be a sequence such that \( r_n \longrightarrow r \) on \( PC(\mathcal{F}, \mathcal{R}) \).

For \( z \in \mathcal{F} \), we have

\[
|M_{r_n}(z) - M_r(z)|
\]

\[
= \int_{z_n}^z (z-s)^{n-1} |r_n(s) - r(s)| ds + \int_{z_n}^z |r_n(z) - r(z)| ds + |r_n(z)| + |r(z)| + \int_{z_n}^z |\psi_m(r_n(z)) - \psi_m(r(z))| ds,
\]

where \( \tau_n(z), \tau(z) \in PC(\mathcal{F}, \mathcal{R}) \) are given by

\[
\tau_n(z) = f(z, r_n(z), \tau_n(z)), \quad \tau(z) = f(z, r(z), \tau(z)).
\]

Now, from assumption (H2), we have

\[
|\tau_n(z) - \tau(z)| = |f(z, r_n(z), \tau_n(z)) - f(z, r(z), \tau(z))| \\
\leq K^* \|r_n - r\| + L^* |\tau_n(z) - \tau(z)|.
\]

Repeating this process, we get

\[
|\tau_n(z) - \tau(z)| \leq \frac{K^*}{1 - L^*} \|r_n - r\|.
\]

Since \( r_n \longrightarrow r, \tau_n(z) \longrightarrow \tau(z) \) as \( n \longrightarrow \infty \) for every \( z \in \mathcal{F} \). We know that every convergent sequence is bounded, so for this, let \( \zeta > 0 \) such that for every \( z \in \mathcal{F} \), we have \( |\tau_n(z)| \leq \zeta \) and \( |\tau(z)| \leq \zeta \). Then, we have

\[
(z-s)^{n-1}|\tau_n(z) - \tau(z)| \leq (z-s)^{n-1}||\tau_n(z)| + |\tau(z)|| \\
\leq 2\zeta(z-s)^{n-1},
\]

\[
(z_n - s)^{n-1}|\tau_n(z) - \tau(z)| \leq (z_n - s)^{n-1}||\tau_n(z)| + |\tau(z)|| \\
\leq 2\zeta(z_n - s)^{n-1},
\]

for every \( z \in \mathcal{F} \); the function \( s \longrightarrow 2\zeta(z-s)^{n-1} \) and \( s \longrightarrow 2\zeta(z_n - s)^{n-1} \) are integrable on \( [0, 1] \), upon the use of these facts and the Lebesque-dominated convergence theorem in (35). After using assumptions (H2)-(H2), we see that

\[
|M_{r_n}(z) - M_r(z)| \longrightarrow 0, \quad n \longrightarrow \infty,
\]

and hence, we have

\[
\|M_{r_n} - M_r\| \longrightarrow 0, \quad n \longrightarrow \infty.
\]

Therefore, operator \( M \) is continuous.

(Step 2) The operator \( M \) assigns bounded sets to bounded sets on \( PC(\mathcal{F}, \mathcal{R}) \). Just prove it for any \( \zeta^* > 0 \), there exists a positive constant \( E^* \), such that for every \( r \in D = \{r \in PC(\mathcal{F}, \mathcal{R}) : \|r\| \leq \zeta^* \} \), we have \( \|M_r\| \leq E^* \). To derive this result for each \( z \in \mathcal{F} \), one has
where \( \tau(z) \in \mathcal{C}(\mathcal{F}, \mathcal{R}) \) is given by
\[
\tau(z) = f(z, r(z), \tau(z)).
\] (43)

By hypothesis \((H_3)\) and for every \( z \in \mathcal{F} \), we have
\[
|\tau(z)| = |f(z, r(z), \tau(z))| \leq |x(z) + y(z)\tau(z) + \psi(z)| \\
\leq |x(z) + y(z)| |r(z)| + |h(z)| |\tau(z)| \\
\leq |x(z) + y(z)| + h(z)|\tau(z)| \leq x^* + y^* \xi^* + h^*|\tau(z)|.
\] (44)

Thus, we have
\[
|\tau(z)| \leq \frac{x^* + y^* \xi^*}{1 - h^*} := R^*.
\] (45)

Therefore, from (42) by using (45), one has
\[
|Mr(z)| \leq \frac{R^*}{1/(\alpha + 1)} + \frac{\sigma_1 R^*}{2} + \frac{A_1^* \xi^*}{3} + B_1^* + \frac{\sigma_2 R^*}{3} \\
+ A_2^* \xi^* + B_2^* + \frac{\sigma_3 R^*}{3}.
\] (46)

Hence, one has
\[
\|Mr\| \leq \left[R^* \left(\frac{1}{1/(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3}\right) + (A_1^* + A_2^* \xi^* + B_1^* + B_2^*)\right] = E^*.
\]

Thus, the operator \( M \) is bounded.

(Step 3) The operator \( M \) assigns bounded sets to equicontinuous sets of \( PC(\mathcal{F}, \mathcal{R}) \). Let \( z_1, z_2 \in \mathcal{F} \), and \( z_1 \leq z_2 \), \( D \) is a bounded set as in Step 2, and let \( r \in D \); then, we have

Clearly, in the inequality (49), the right hand side tends to zero as \( z_1 \rightarrow z_2 \). Hence, \( [Mr(z_2) - Mr(z_1)] \rightarrow 0 \) as \( z_1 \rightarrow z_2 \). As a consequence of the passage from Step 1 to Step 3 combined with the Arzella-Ascoli theorem, we
conclude that $M : PC(\mathcal{J}, \mathcal{R}) \rightarrow PC(\mathcal{J}, \mathcal{R})$ is completely continuous.

(Step 4) In the last step, we need to show that the set $F = \{ r(z) \in PC(\mathcal{J}, \mathcal{R}): r(z) = \mu M(r(z)) \}$, for some $0 < \mu < 1$ is bounded. Let $r \in F$; then, $r(z) = \mu M(r(z))$ for some $0 < i < 1$. Therefore, for every $z \in \mathcal{J}$, we have

\[
r(z) = \mu M(r(z)) = \mu \int_{\mathcal{Z}} \frac{(z-s)^{a-1}}{I(a)} \tau(s) ds - \mu \left( \int_{\mathcal{Z}} (z - m)^{a-1} + \left( (z - z_m)^{a-1} + z_m(z - z_m)^{-2} \right) (1 - z_m) \right) \times \left( \int_{\mathcal{Z}} (z_m - s) \tau(s) ds - \psi_m(r(z_m)) \right) - \mu \left( (z - z_m)^{a-1} + z_m(z - z_m)^{-2} \right) \left( \int_{\mathcal{Z}} (1 - s)^2 \tau(s) ds \right).
\]

(50)

Now, we have

\[
|r(z)| = |\mu M(r(z))| \leq \int_{\mathcal{Z}} \frac{(z-s)^{a-1}}{I(a)} |\tau(s)| ds + |(z - m)|^{-1} + \left( (z - z_m)^{a-1} + z_m(z - z_m)^{-2} \right) (1 - z_m) \times \left( \int_{\mathcal{Z}} (z_m - s) |\tau(s)| ds + |\psi_m(r(z_m))| \right) + |(z - z_m)^{a-1} + \left( (z - z_m)^{a-1} + z_m(z - z_m)^{-2} \right) \times \left( \int_{\mathcal{Z}} (z_m - s)^2 |\tau(s)| ds + |\psi_m(r(z_m))| \right) + |(z - z_m)^{a-1} + z_m(z - z_m)^{-2}| \left( \int_{\mathcal{Z}} (1 - s)^2 |\tau(s)| ds \right).
\]

(51)

Using (45) and hypotheses $(H_6)$ and $(H_7)$ in (51), we get

\[
|r(z)| \leq \frac{R^*}{I(a + 1)} + \frac{\sigma_1 R^*}{2} + A_1^* \xi^* + B_1^* + \frac{\sigma_2 R^*}{3} + A_2^* \xi^* + B_2^* + \frac{\sigma_3 R^*}{3}.
\]

(52)

Hence, one has from where

\[
||r|| \leq \left[ R^* \left( \frac{1}{I(a + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (A_1^* + A_2^*) \xi^* + B_1^* + B_2^* \right] = Z^*.
\]

(53)

Hence, the given set $F$ is bounded as a result of the Schaefer fixed point theorem, and we conclude that operator $M$ has at least one fixed point. Hence, the corresponding Problem (2) has at least one solution.

4. Examples

Here, we provide two pertinent examples to verify the previous results.

Example 1. Consider the following IDE under the Riemann-Liouville-type integral boundary condition and the Riemann-Liouville fractional order derivative

\[
\Delta r \left( \frac{1}{5} \right) = \psi_1 \left( \frac{1}{5} \right) = \frac{\sin \left( r(1/5) \right)}{30},
\]

\[
\Delta^* r \left( \frac{1}{5} \right) = \psi_1^* \left( \frac{1}{5} \right) = \frac{e^{-r(1/5)}}{15},
\]

\[
1^{-}\Delta r(0) = 0, 1^{-}\Delta^* r(1) = 0.
\]

(54)

where $\alpha = (1/2)$, we set

\[
f(z, r(z), \tau(z)) = \frac{z + \cos \left( r(\tau(z)) \right) + \cos \left( \tau(z) \right)}{90 + z^2}, \quad r(z) \in PC(\mathcal{J}, \mathcal{R}), \quad \tau(z) \in \mathcal{R}, \text{ and } z \in [0, 1].
\]

(55)

Clearly $f$ is a jointly continuous function. Now for every $r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R})$, $\tau(z), \bar{\tau}(z) \in \mathcal{R}$, and $z \in [0, 1]$, we have

\[
|f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \leq \frac{z + \cos \left( r(\tau(z)) \right) + \cos \left( \tau(z) \right)}{90 + z^2} - \frac{z + \cos \left( \bar{r}(\bar{\tau}(z)) \right) + \cos \left( \bar{\tau}(z) \right)}{90 + z^2} = \frac{\cos \left( r(\tau(z)) \right) - \cos \left( \tau(z) \right) + \cos \left( \bar{r}(\bar{\tau}(z)) \right) - \cos \left( \bar{\tau}(z) \right)}{90 + z^2} \leq \frac{\cos \left( r(\tau(z)) \right) - \cos \left( \tau(z) \right)}{90 + z^2} + \frac{\cos \left( \bar{r}(\bar{\tau}(z)) \right) - \cos \left( \bar{\tau}(z) \right)}{90 + z^2}.
\]

(56)

\[
|f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z))| \leq \frac{1}{90} (|r - \bar{r}| + |\tau - \bar{\tau}|).
\]

Which satisfies hypothesis $(H_2)$ with $K^* = L^* = (1/90)$. Now, we set

\[
\Delta r \left( \frac{1}{5} \right) = \psi_1 \left( \frac{1}{5} \right) = \frac{\sin \left( r(1/5) \right)}{30}, \quad r \in PC(\mathcal{J}, \mathcal{R}).
\]

(57)
Then, for \( r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R}) \), we have
\[
\left| \psi_1^a \left( r \left( \frac{1}{5} \right) \right) - \psi_1^a \left( \bar{r} \left( \frac{1}{5} \right) \right) \right| = \left| \sin \left( r(1/5) \right) - \sin \left( \bar{r}(1/5) \right) \right| \leq \frac{1}{30} |r - \bar{r}|. \tag{58}
\]

Therefore, with \( N^*_1 = (1/30) \), hypothesis \( (H_3) \) is satisfied. Next, we set
\[
\Delta^* r \left( \frac{1}{5} \right) = \psi_1^a \left( r \left( \frac{1}{5} \right) \right) = \frac{e^{-r(1/5)}}{15}, \quad r \in PC(\mathcal{J}, \mathcal{R}). \tag{59}
\]

Then, for \( r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R}) \), we have
\[
\left| \psi_1^a \left( r \left( \frac{1}{5} \right) \right) - \psi_1^a \left( \bar{r} \left( \frac{1}{5} \right) \right) \right| = \left| e^{-r(1/5)} - e^{-\bar{r}(1/5)} \right| \leq \frac{1}{15} |r - \bar{r}|. \tag{60}
\]

Thus, with \( N^*_2 = (1/15) \), hypothesis \( (H_4) \) is satisfied. Further, we need to satisfy the given condition of Theorem 7, by
\[
\left[ K^* \frac{1}{1 - L^*} \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{\sigma_1}{2} + \frac{\sigma_2}{3} + \frac{\sigma_3}{3} \right) + (\sigma_1 N^*_1 + \sigma_2 N^*_2) \right] = \frac{1}{90} \left( \frac{1}{\Gamma(\alpha + 1)} + \frac{35}{20} + \frac{55}{30} + \frac{28}{30} + \frac{35}{300} + \frac{55}{150} \right) < 1. \tag{61}
\]

Therefore, all the hypotheses and conditions of Theorem 7 are satisfied. Therefore, the considered problem (54) has a unique solution on \( \mathcal{J} \).

**Example 2.** Consider another example of IDE under the Riemann-Liouville-type integral boundary condition and the Riemann-Liouville fractional order derivative:

\[
\begin{align*}
\mathcal{D}_t^a r(z) &= e^{-z} + e^{2z} \sin \left( \sqrt{r(z)} \right) + \sin \left( \sqrt{\mathcal{D}_t^a r(z)} \right) \frac{35 + z^3}{35 + z^3}, \quad 1 < \alpha \leq 2, z \in [0, 1], z \neq \frac{1}{7}, \\
\Delta r \left( \theta \right) &= \psi_1^a \left( r \left( \theta \right) \right) = \frac{\tan \left( r(1/7) \right)}{25 + \tan \left( r(1/7) \right)}, \\
\Delta^* r \left( \theta \right) &= \psi_1^a \left( r \left( \theta \right) \right) = \frac{e^{r(\theta)}}{55 + 30e^{r(\theta)}}, \\
I^{1-a} r(0) &= 0, I^{2-a} r(1) = 0.
\end{align*}
\]  

where \( \alpha = (3/2) \) and \( \mathcal{D}_t^a \) denotes the Riemann-Liouville fractional order derivative. We set
\[
f(z, r(z), \tau(z)) = e^{-z} + e^{2z} \sin \left( \sqrt{r(z)} \right) + \sin \left( \sqrt{\tau(z)} \right) \frac{35 + z^3}{35 + z^3},
\]
\[
r(z) \in PC(\mathcal{J}, \mathcal{R}), \quad \tau(z) \in \mathcal{R}, \quad z \in [0, 1].
\]  

Clearly \( f \) is a jointly continuous function. Now for every \( r(z), \bar{r}(z) \in PC(\mathcal{J}, \mathcal{R}), \quad \tau(z), \bar{\tau}(z) \in \mathcal{R}, \) and \( z \in [0, 1] \), we have
\[
\left| f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z)) \right| = \left| e^{-z} + e^{2z} \sin \left( \sqrt{r(z)} \right) + \sin \left( \sqrt{\tau(z)} \right) \frac{35 + z^3}{35 + z^3} - e^{-z} + e^{2z} \sin \left( \sqrt{\bar{r}(z)} \right) + \sin \left( \sqrt{\bar{\tau}(z)} \right) \frac{35 + z^3}{35 + z^3} \right|
\]
\[
\leq \left| e^{-z} \left\{ \sin \left( \sqrt{r(z)} \right) - \sin \left( \sqrt{\bar{r}(z)} \right) \right\} \frac{35 + z^3}{35 + z^3} \right| + \left| \sin \left( \sqrt{\tau(z)} \right) - \sin \left( \sqrt{\bar{\tau}(z)} \right) \right| \frac{35 + z^3}{35 + z^3}
\]
\[
\leq e^{-2z} \left| \sqrt{r(z)} - \sqrt{\bar{r}(z)} \right| + \frac{1}{35 + z^3} \left| \sqrt{\tau(z)} - \sqrt{\bar{\tau}(z)} \right|
\]
\[
\left| f(z, r(z), \tau(z)) - f(z, \bar{r}(z), \bar{\tau}(z)) \right| \leq \frac{1}{35} \left( |r - \bar{r}| + |\tau - \bar{\tau}| \right).
\]  

This satisfies hypothesis \( (H_2) \), with \( K^* = L^* = (1/35) \).
Now, another hypotheses for every $r(z) \in PC(\mathcal{J}, \mathcal{R})$, $\tau(z) \in \mathcal{R}$, and $z \in [0, 1]$, we have
\[
|f(z, r(z), \tau(z))| = \frac{e^{-z} + e^{-2z} \sin \left( \sqrt{\tau(z)} \right) + \sin \left( \sqrt{\tau(z)} \right)}{35 + z^3} \leq \frac{e^{-z}}{35 + z^3} + \frac{e^{-2z}}{35 + z^3} \sqrt{\tau(z)} + \frac{1}{35 + z^3} \sqrt{\tau(z)} \leq \frac{e^{-z}}{35 + z^3} + \frac{e^{-2z}}{35 + z^3} |r(z)| + \frac{1}{35 + z^3} |\tau(z)|.
\]
(65)

Thus, hypothesis ($H_5$) is satisfied with $x(z) = (e^{-z}/35 + z^3)$, $y(z) = (e^{-2z}/35 + z^3)$, and $h(z) = (1/35 + z^3)$. Now, we set
\[
\Delta \frac{1}{7} = \psi_1 \left( \frac{1}{7} \right) = \frac{\tan \left( \frac{r(1/7)}{25} \right)}{r(1/7)}, \quad r \in PC(\mathcal{J}, \mathcal{R}).
\]
(66)

Then, for every $r \in PC(\mathcal{J}, \mathcal{R})$, we have
\[
\left| \psi_1 \left( \frac{1}{7} \right) \right| = \left| \frac{\tan \left( \frac{r(1/7)}{25} \right)}{r(1/7)} \right| \leq \frac{1}{25} |r| + 1.
\]
(67)

Therefore, hypothesis ($H_6$) is satisfied with $A_1^* = (1/25)$ and $B_1^* = 1$. Next, we set
\[
\Delta^* r \left( \frac{1}{7} \right) = \psi_1^* \left( \frac{1}{7} \right) = \frac{e^{r(1/7)}}{55 + 30e^{r(1/7)}}, \quad r \in PC(\mathcal{J}, \mathcal{R}).
\]
(68)

Then, for every $r \in PC(\mathcal{J}, \mathcal{R})$, we have
\[
\left| \psi_1^* \left( \frac{1}{7} \right) \right| = \left| \frac{e^{r(1/7)}}{55 + 30e^{r(1/7)}} \right| \leq \frac{1}{55} |r| + \frac{1}{30}.
\]
(69)

Thus, hypothesis ($H_7$) is satisfied with $A_1^* = (1/55)$ and $B_1^* = (1/30)$. Therefore, all of the hypotheses of Theorem 8 are satisfied, and therefore, the considered problem (62) has at least one solution on $\mathcal{J}$.

5. Conclusion

IDEs of the fractional order have received proper attention due to their important applications in various fields of applied sciences. In the past, most studies have been done using the Caputo-type fractional derivatives to handle IDEs. In very few papers, investigating IDEs of the fractional order was done using the Riemann-Liouville derivative. Therefore, we have established successfully some important results devoted to the existence theory of a solution to the considered nonlinear implicit IDE with the Riemann-Liouville-type integral boundary conditions under the Riemann-Liouville fractional order derivative. The corresponding results for the existence and uniqueness of the solution have been archived by utilizing the classical Schiefer and Banach contraction fixed point theorems. For the demonstration of our results, we have enriched the paper by providing two pertinent examples.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

No conflict of interest exist.

Authors’ Contributions

Equal contribution has been done by all the authors.

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References

[1] J. T. Machado, V. Kiryakova, and F. Mainardi, “Recent history of fractional calculus,” Communications in Nonlinear Science and Numerical Simulation, vol. 16, no. 3, pp. 1140–1153, 2011.
[2] R. Metzler and J. Klafter, “Boundary value problems for fractional diffusion equations,” Physica A: Statistical Mechanics and its Applications, vol. 278, no. 1-2, pp. 107–125, 2000.
[3] K. B. Oldham, “Fractional differential equations in electrochemistry,” Advances in Engineering Software, vol. 41, no. 1, pp. 9–12, 2010.
[4] F. A. Rihan, “Numerical modeling of fractional-order biological systems,” Abstract and Applied Analysis, vol. 2013, Article ID 816803, 11 pages, 2013.
[5] J. Sabatier, O. P. Agrawal, and J. A. T. Machado, Advances in Fractional Calculus, Springer, Dordrecht, 2007.
[6] V. E. Tarasov, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, Heidelberg, Higher Education Press, Beijing, 2010.
[7] B. M. Vintagre, I. Podlybni, A. Hernandez, and V. Felic, “Some approximations of fractional order operators used in control theory and applications,” Fractional Calculus Applied Analysis, vol. 3, no. 3, pp. 231–248, 2000.
[8] G. A. Anastassiou, “On right fractional calculus,” Chaos, Solitons and Fractals, vol. 42, no. 1, pp. 365–376, 2009.
[9] D. Baleanu, Z. B. Guvenc, and J. A. T. Machado, New Trends in Nanotechnology and Fractional Calculus Applications, Springer, New York, 2010.
[10] E. Hiff, Application of Fractional Calculus in Physics, Word Scientific, Singapore, 2000.
[11] A. A. Kilbas, H. M. Srivasta, and J. J. Trujillo, Theory and Application of Fractional Differential Equations, North-Holland Mathematics Studies, North-Holland, Amsterdam, 2006.
[12] M. D. Ortigueira, Fractional Calculus for Scientists and Engineers, Springer, Dordrecht, Lecture Notes in Electrical Engineering edition, 2011.
[13] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[14] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect. Stability, Theory and Applications*, Ellis Horwood Series in Mathematics and Its Applications, Ellis Horwood Limited; New York etc.: Halsted Press, Chichester, 1989.

[15] M. Benchohra, J. Henderson, and V. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications, Hindawi Publishing Corporation, New York, 2006.

[16] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, World Scientific, Singapore etc., 1989.

[17] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, World Scientific Publishing Co., Inc., River Edge, New Jersey, 1995, (Translated from the Russian).

[18] M. Benchohra and B. A. Slimani, "Impulsive fractional differential equations," submitted.

[19] A. Ali, F. Rabiei, and K. Shah, "On Ulam's type stability for a class of impulsive fractional differential equations with nonlinear integral boundary conditions," The Journal of Nonlinear Sciences and Applications, vol. 10, no. 9, pp. 4760–4775, 2017.

[20] Y. V. Rogovchenko, "Impulsive evolution systems: main results and new trends," *Dynamics of Continuous, Discrete and Impulsive Systems series*, vol. 3, no. 1, pp. 57–88, 1997.

[21] K. Shah, K. Ali, and S. Bushnaq, "Hyers-Ulam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions," *Mathematical Methods in the Applied Sciences*, vol. 41, pp. 1–15, 2018.

[22] K. Shah, H. Khalil, and R. A. Khan, "Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations," *Chaos, Solitons & Fractals*, vol. 77, pp. 240–246, 2015.

[23] J. X. Sun, *Nonlinear Functional Analysis and Its Application*, Science Press, Beijing, 2008.

[24] G. Wang, L. Zhang, and G. Song, "Extremal solutions for the first order impulsive functional differential equations with upper and lower solutions in reversed order," *Journal of Computational and Applied Mathematics*, vol. 235, no. 1, pp. 325–333, 2010.

[25] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems. Theory and Applications*, Kluwer Academic, Dordrecht, 1997.

[26] X. Zhang, P. Agarwal, Z. Liu, and H. Peng, "The general solution for impulsive differential equations with Riemann-Liouville fractional-order $q \in (1, 2)$," *Open Mathematics*, vol. 13, no. 1, pp. 908–923, 2015.

[27] R. P. Agarwal, M. Benchohra, and S. Hamani, "Boundary value problems for fractional differential equations," *Georgian Mathematical Journal*, vol. 16, no. 3, pp. 401–411, 2009.

[28] A. Cabada and G. Wang, "Positive solutions of nonlinear fractional differential equations with integral boundary value conditions," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 1, pp. 403–411, 2012.

[29] B. Ahmad and J. J. Nieto, "Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions," *Boundary Value Problems*, vol. 2009, Article ID 708576, 11 pages, 2009.

[30] B. Ahmad, J. J. Nieto, and A. Alsaedi, "Existence and uniqueness of solutions for nonlinear fractional differential equations with non-separated type integral boundary conditions," *Acta Mathematica Scientia*, vol. 31, no. 6, pp. 2122–2130, 2011.

[31] B. Ahmad and S. Sivasundaram, "Existence of solutions for impulsive integral boundary value problems of fractional order," *Nonlinear Analysis: Hybrid Systems*, vol. 4, no. 1, pp. 134–141, 2010.

[32] W. M. Ahmad and R. el-Khazali, "Fractional-order dynamical models of love," *Chaos, Solitons & Fractals*, vol. 33, no. 4, pp. 1367–1375, 2007.

[33] K. Adolffsson, M. Enelund, and P. Olsson, "On the fractional order model of viscoelasticity," *Mechanics of Time-dependent Materials*, vol. 9, no. 1, pp. 15–34, 2005.

[34] E. Ahmed, A. Hashish, and F. A. Rihan, "On fractional order cancer model," *Journal of Fractional Calculus and Applications*, vol. 3, no. 2, pp. 1–6, 2012.

[35] X. Yang, X. Jiang, and J. Kang, "Parameter identification for fractional fractal diffusion model based on experimental data," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 29, no. 8, p. 083134, 2019.

[36] C. Cattani, H. M. Srivastava, and X.-J. Yang, *Fractional Dynamics*, De Gruyter Open Poland, 2016.

[37] M. A. Dokuyucu and H. Dutta, "A fractional order model for Ebola virus with the new Caputo fractional derivative without singular kernel," *Chaos, Solitons & Fractals*, vol. 134, p. 109717, 2020.

[38] M. Arfan, H. Alrabaiah, M. U. Rahman et al., "Investigation of fractal-fractional order model of COVID-19 in Pakistan under Atangana-Baleanu Caputo (ABC) derivative," *Results in Physics*, vol. 24, p. 104046, 2021.

[39] A. Atangana, A. Akgül, and K. M. Owolabi, "Analysis of fractal fractional differential equations," *Alexandria Engineering Journal*, vol. 59, no. 3, pp. 1117–1134, 2020.

[40] A. Atangana, M. A. Khan, and Fatmawati, "Modeling and analysis of competition model of bank data with fractal-fractional Caputo-Fabrizio operator," *Alexandria Engineering Journal*, vol. 59, no. 4, pp. 1985–1998, 2020.

[41] R. Miron, "Impulsive differential equations with applications to infectious diseases, doctoral dissertation," Université d‘Ottawa/University of Ottawa, Canada, 2014.

[42] Asma, A. Ali, K. Shah, and F. Jarad, "Ulam-Hyers stability analysis to a class of nonlinear implicit impulsive fractional differential equations with three point boundary conditions," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 7, 2019.

[43] D. Yang and J. R. Wang, "Non-instantaneous impulsive fractional-order implicit differential equations with random effects," *Stochastic Analysis and Applications*, vol. 35, no. 4, pp. 719–741, 2017.

[44] Q. Chen, A. Debbouche, Z. Luo, and J. R. Wang, "Impulsive fractional differential equations with Riemann-Liouville derivative and iterative learning control," *Chaos, Solitons & Fractals*, vol. 102, pp. 111–118, 2017.

[45] W. Tukunthorn, S. K. Ntouyas, and J. Tariboon, "Impulsive multiorders Riemann-Liouville fractional differential equations," *Discrete Dynamics in Nature and Society*, vol. 2015, Article ID 603893, 9 pages, 2015.