Power law decay of entanglement quantifiers in a single agent to a many body system coupling

Ohad Shpielberg

1Haifa Research Center for Theoretical Physics and Astrophysics, University of Haifa, Mt. Carmel, Haifa 31905, Israel
(Dated: April 5, 2022)

Manipulating many body quantum systems is a challenge. A useful way to achieve it would be to entangle the system to a diluted system, with a small particle number. Preparation of such entangled states can be facilitated as ground state of a many body Hamiltonian or the steady state of a many body open quantum system. Here we study two-site lattice models with a conserved boson number, biased to display a large occupancy in one of the sites. The Von Neumann entanglement entropy as well as the Logarithmic negativity show a typical power law decay in $R$, the occupancy ratio between the two sites. These results imply that it is feasible to entangle a large many body system to a single atom, as recently reported experimentally.

I. INTRODUCTION

Entanglement is a key resource in quantum information [1–3], quantum computing [4, 5] and quantum metrology [6]. Recently, there has been significant advancement in generating, manipulating and measuring entangled many body states; both experimentally and theoretically [7–10]. Both preparation of the entangled state and its validation using, e.g. entanglement witnesses [11], are challenging aspects in many body systems and are the focus of ongoing research [12–15]. Preparation of a desired entangled state can be realized as the ground state of a carefully designed Hamiltonian. Therefore, understanding the entanglement properties of ground states is of practical importance. Significant effort has been directed for extended systems, especially in 1D. The ground state of a Hamiltonian with local interactions typically exhibits an area law in the bipartite Von Neumann entanglement entropy [16], contrary to the generic volume law of typical quantum states. However, the area law does not characterise a system composed of a few sites, where each site can occupy a large number of particles (see Fig. 1). For two sites, the average Von Neumann entanglement entropy is known [17, 18], but the characteristic properties of the ground state entanglement remain largely unexplored.

A particularly appealing case is for a single agent in system $B$ to be entangled to a large number of particles in system $A$. Such entanglement allows to manipulate the many body system via the single agent. This setup was experimentally demonstrated in [19], where a single photon was entangled with roughly 3,000 atoms. At this point, it is unknown whether there is a limit to the number of particles that could realistically be entangled with a single agent. It is further unknown whether our setup leads to typical ground state entanglement properties.

To answer the above questions, the entanglement needs to be quantified [20, 21]. Choosing an appropriate entanglement quantifier depends on the intended application, e.g. entanglement distillation to produce Bell states. However, the entanglement quantifier can alternatively be chosen to accommodate fast calculations of known density matrices, e.g. the Logarithmic negativity $E_{\text{ln}}$. For pure states, the Von Neumann entanglement entropy $E_{\text{vn}}$ serves both purposes.

Keeping in mind the motivation of entangling a single agent to a many body system, we turn to a simpler theoretical setup. In this work we consider $N$ bosons, occupying a two-site system. We study the ground state entanglement properties when the system is tuned to display an overwhelming majority of bosons occupying site $A$ (see Fig. 1). To make this statement precise, let $\hat{n}_{A,B}$ be the corresponding number operators. Then, $R = \langle \hat{n}_A \rangle / \langle \hat{n}_B \rangle$ is the ratio of the particle occupancies. We study the large $R$ behaviour of both the ground state Von Neumann entanglement entropy and Logarithmic negativity of the Bose-Hubbard Hamiltonian. For open quantum systems, the same setup can be considered, where the steady state takes the role of the ground state. We study the steady state Logarithmic negativity of a Lindblad super-operator model – the quantum
asymmetric inclusion process at large $R$ values [22] [23].

Both models are studied at different scaling regimes. Nevertheless, they all consistently lead to a power law decay in the entanglement quantifiers. Quantitatively, the Von Neumann entanglement entropy $E_{\text{vn}} \sim \frac{\log R}{R^\alpha}$ and the Logarithmic negativity $E_{\text{ln}} \sim \frac{1}{R^\alpha}$ for $R \gg 1$. See Tables I and II for a summary of the results.

We argue that the power law decay is typical, as we have considered two disparate models and different scaling regimes for each model. The slow power law decay, contrasting with an exponential decay, answers in a quantifiable way how realistic it is to entangle a single, or a few atoms to a highly occupied many body state. The exponent $\alpha$ is non-universal. Therefore, interacting systems that result in small $\alpha$ values are favorable to facilitate entanglement between the diluted system to the large occupancy system.

The structure of this paper is as follows. In Sec. III we present the Hamiltonian and Lindblad models and summarise the main results. Sec. IV presents in full the analytical and numerical treatment of the systems under study. Finally, Sec. V recaps the main findings and their physical relevance and suggests future directions.

II. MODELS AND RESULTS

The aim of this work is to quantify the bipartite entanglement of a composite $AB$ system at large $R$ values. Therefore, it is natural to study lattice models, where the distinction between the two subsystems is clear cut. In particular, we study lattice models with two sites, $A$ and $B$.

To demonstrate the power law behaviour is typical, two disparate lattice models are considered. First, the ground state entanglement of the two-site Bose-Hubbard model is extensively studied. Second, we consider a generalization of the asymmetric inclusion process [24] to the quantum realm via a Lindblad equation, dubbed here the quantum asymmetric inclusion process (QASIP). We then study the entanglement properties of the steady state at large $R$.

A. The two-site Bose-Hubbard model

The Bose-Hubbard model is a simple yet rich many body lattice model of spin-less bosons. It allows studying the superfluid-insulator transition [24] and can be experimentally implemented using optical lattices [26] [27]. The particular case of the two-site Bose-Hubbard model was extensively used in the literature to study tunneling effects between potential wells [28] as well as fragmentation [29]. Importantly for our purposes, the two site Bose-Hubbard model is expected to be both analytically tractable and to present typical physical behaviour in terms of the ground state entanglement. Hence, it serves as the starting point of our analysis.

The two-site Bose-Hubbard Hamiltonian is given by

$$H_{\text{BH}} = -J(\hat{b}_A^\dagger \hat{b}_B + \hat{b}_B^\dagger \hat{b}_A) - \mu \hat{n}_A + \frac{U}{2} \sum_{i=A,B} \hat{n}_i - \hat{n}_i^2, \tag{1}$$

where $J$ is the hopping matrix element between neighboring sites and $U$ determines the strength of the on-site interaction. The operators $\hat{b}_i, \hat{b}_i^\dagger$ are the site-dependent bosonic creation and annihilation operators and $\hat{n}_i = \hat{b}_i^\dagger \hat{b}_i$ is the number operator for $i = A, B$. In an optical lattice, the potential wells are represented by the two sites [28]. The potential offset between the two asymmetric potential wells is given by $\mu$. It furthermore allows to imbalance the system towards large $R$ values.

It is useful to note that the total particle number, $\hat{N} = \hat{n}_A + \hat{n}_B$ is conserved. Therefore, we analyze the ground state with $N$ bosons. In what follows, we consider two scaling schemes leading to large $R$ values.

First, taking large $\mu$ values and keeping $N, J$ and $U$ fixed, a perturbative treatment leads to $R = \mu^2 / J^2 + O(\mu)$. See Sec. III. In this limit, and as long as $\frac{\sqrt{N}}{\mu} \frac{U N^2}{\mu} < 1$, we find analytically the power law behaviour described in Fig. 2. These results are also numerically corroborated in Fig. 2. Note that the logarithmic correction in the Von Neumann entanglement hardly changes the behaviour from a clean power law.

Second, we consider the large $N$ limit, with fixed $\mu, J$ and $U$. A perturbative approach is harder in this case as the Hilbert space of the effective Hamiltonian depends on the particle number $N$ (see Sec. III). See Fig. 3, Table III and the appendix C for the numerical analysis of the large $N$ limit.

In conclusion, the ground state of the two-site Bose-Hubbard model leads to a power law behaviour of both the Von Neumann entanglement entropy and the Logarithmic negativity at large $R$ values in two different scaling schemes.

Next, we explore the large $R$ steady state entanglement in an open quantum system setup: the quantum asymmetric inclusion process.
\[ \Gamma = \mu \propto \alpha \]

Lindblad equation.

\[ R \approx E \ln R \]

- \[ \varepsilon, \eta, \gamma \]

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.

\[ \varepsilon, \eta, \gamma \]

could be restored.
The QASIP, like the Bose-Hubbard model can be shown to conserve the particle number $N = \hat{n}_A + \hat{n}_B$ (see the appendix B). However, a related but more general property exists for the QASIP. In the number operator basis, we can write the density matrix as

$$\hat{\rho} = \sum_S a_S \hat{\rho}_S, \quad \text{where}$$
$$\hat{\rho}_S = \sum_{x,y=0} \varrho_S(x,y) |x,S-x\rangle \langle y,S-y|.$$  

Here, $S$ takes non-negative integer values and $a_S$ are non-negative prefactors that sum to 1. Note that the hermitian of the density matrix $\hat{\rho}_S$ implies $\varrho_S^*(x,y) = \varrho_S(y,x)$ and unity trace implies $\sum_S \varrho_S(x,x) = 1$.

The dynamics of eq. (3) is restricted to the subspace of $\hat{\rho}_S$:

$$\partial_t \varrho_S (x, y) = -\eta(x-y)^2 \varrho_S(x, y)$$
$$+ \gamma x y \varrho_S(x-1, y-1)$$
$$- \frac{1}{2} \varrho_S (x^2 + y^2) \varrho_S(x, y)$$
$$- i\varepsilon \sum_{z=\pm 1} x_z \varrho_S(x+z, y)$$
$$- y_z \varrho_S(x, y+z)$$

$$X_+ = \sqrt{(X+1)(S-X)}$$
$$X_- = \sqrt{X(S-X+1)}, \quad \text{where} \quad X = x, y.$$  

Namely, we have replaced the treatment of the infinite dimensional density matrix $\hat{\rho}$, with a treatment of finite dimensional $(S+1)^2$ density matrices $\hat{\rho}_S$ at fixed $S$.

The conserved number $S$ of $\hat{\rho}_S$ equals the number of particles in the system as

$$\text{Tr} \hat{\rho}_S \hat{N} = \sqrt{\text{Tr} \hat{\rho}_S \hat{N}^2} = S.$$  

Eq. (5) implies therefore that the process conserves the particle number. From hereon out, we may replace $S$ by $N$.

In eq. (5), there are different scaling schemes leading to the large $R$ limit at the steady state. The Table I summarises the power law behaviour of the Logarithmic negativity in three different scaling regimes. The results in Table I were verified both analytically and numerically. See Fig. 4 for the large $\gamma$ limit and Fig. 5 for the large $\eta$ limit. For the large $N$ only numerical evidence is currently present, see Fig. 6.

In conclusion, the steady state QASIP in eq. (3) exhibits a power law decay in the Logarithmic negativity, similarly to the Bose-Hubbard dynamics.

In the next section, we provide a detailed derivation of the results for the Bose-Hubbard model and for the QASIP.
TABLE II. The power law behaviour of the Logarithmic negativity in the steady state QASIP. For the large $N$ limit, the exponents are evaluated numerically.

|   | $R$ | $E_{\ln}(R) = \Gamma R^{-\alpha}$ |
|---|-----|---------------------------------|
| $\gamma \gg 1$ | $\frac{\gamma^2 N^2}{4\epsilon_0^2}$ | $\Gamma = \frac{2\sqrt{N}}{\epsilon_0^{2/4}}, \alpha = 1/2$ |
| $\eta \gg 1$ | $\frac{\eta^2 N^2}{2\epsilon_0^2}$ | $\Gamma = \frac{2N^{3/4}}{\epsilon_0^{2/4}}, \alpha = 1$ |
| $N \gg 1$ | $\propto N^\beta, \beta \approx 1.988$ | $\alpha \approx 0.236$ |

III. ANALYTICAL AND NUMERICAL ANALYSIS

In Sec. II, we have introduced two lattice models: the Bose-Hubbard model and the quantum asymmetric inclusion process. The power law behaviour of the Von Neumann entanglement and the Logarithmic negativity was summarized in Tables I and II. In this section, we describe the analysis of these results in detail.

A. Bose-Hubbard model

To analyze the entanglement properties of the two-site Bose-Hubbard model in eq. \((1)\), we need to find the $N$-particle ground state of the Hamiltonian. Given a description of the ground state, finding the ratio $R$ and the entanglement quantities $E_{\text{ve}}$, $E_{\text{ln}}$ becomes straightforward, but sometimes technically cumbersome. See the appendix [A].

The Hilbert space of $N$ particles for the Bose-Hubbard Hamiltonian eq. \((1)\) is spanned by the $N+1$ Fock states $|n_A, n_B\rangle = \frac{1}{\sqrt{n_A!n_B!}} (\hat{b}_A)^{n_A} (\hat{b}_B)^{n_B} |0, 0\rangle$. Namely,

$$|\psi_N\rangle = \sum_{k=0}^{N} a_k |k, N-k\rangle, \quad \text{where} \sum_{k=0}^{N} |a_k|^2 = 1. \quad (7)$$

Then, the $N$ particle Bose-Hubbard Hamiltonian can be written as a $N+1 \times N+1$ matrix. Finding the ground state can be done analytically and for all $\mu, U, J$ values when we set $N = 1$. This simple case will reveal the intuitive scaling limits leading to the large $R$ behaviour, both for the Bose-Hubbard model as well as for the quantum asymmetric inclusion process in the next subsection.

Indeed, for $N = 1$, the Bose-Hubbard Hamiltonian can be represented by the $2 \times 2$ matrix

$$H_{\text{BH}}^{(N=1)} = \begin{pmatrix} -\mu & -J \\ -J & 0 \end{pmatrix} \quad (8)$$

where the wavefunction is in its most general form

$$|\psi_{N=1}\rangle = \cos \zeta |10\rangle + e^{i\phi_1} \sin \zeta |01\rangle = \begin{pmatrix} \cos \zeta \\ e^{i\phi_1} \sin \zeta \end{pmatrix} \quad (9)$$

for real $\zeta, \phi_1$ values and the right hand side of eq. \((9)\) is in a vector notation corresponding to the matrix in eq. \((8)\). Clearly, the ratio is $R = \cot^2 \zeta$. The lowest eigenvalue of eq. \((8)\) is $\epsilon = -\frac{1}{2}(\mu + \sqrt{\mu^2 + 4J^2})$ with the ground state $|\psi_{N=1}\rangle = \frac{1}{\sqrt{N}}(-\epsilon |10\rangle + J|01\rangle)$ and $\Lambda^2 = \epsilon^2 + J^2$ is a normalization constant. This implies that $R = e^2/J^2$. Therefore, in this particular case, $\epsilon > 1$ only if $\mu/J \gg 1$ and leads to $R = (\mu/J)^2 + O(\mu)$ asymptotically.

The Von Neumann entanglement for the ground state in eq. \((9)\) is

$$E_{\text{ve}} = -2 \cos^2 \zeta \ln(\cos \zeta) - 2 \sin^2 \zeta \ln(\sin \zeta). \quad (10)$$

Using trigonometric identities, we recover $\cos^2 \zeta = \frac{R}{1+R}$ and $\sin^2 \zeta = \frac{1}{1+R}$. Asymptotically for large $R$, $E_{\text{ve}} = \ln R$ to leading order as reported in Table II.

To find the Logarithmic negativity, we need to write the partially transposed density matrix

$$\rho^\text{PT} = \cos^2 \zeta |10\rangle \langle 10| + \sin^2 \zeta |01\rangle \langle 01| + \cos \zeta \sin \zeta (e^{i\phi_1} |00\rangle \langle 11| + e^{-i\phi_1} |11\rangle \langle 00|). \quad (11)$$

The eigenvalues of the partially transposed density matrix are $\cos^2 \zeta, \sin^2 \zeta$ and $\pm \frac{\sqrt{1-\cos^2 4\zeta}}{2\sqrt{2}}$. Therefore, the Logarithmic negativity is $E_{\text{ln}} = \log_2(1 + \frac{1-\cos^2 4\zeta}{\sqrt{2}})$. For large $R$ and to leading order, we find $E_{\text{ln}} = \frac{2}{\ln 2} - R^{-1/2} + O(R^{-1})$ as reported in Table II.

For $N > 1$, the ground state solution becomes cumbersome, but still requires dealing with a $N+1 \times N+1$ Bose-Hubbard matrix. The numerical code that produced Fig. 2 and 7 finds the ground state of the Bose-Hubbard matrix at some finite $N$. Then, it calculates the Von Neumann entanglement and the Logarithmic negativity.

From the $N = 1$ example, we have seen that the large $\mu$ limit leads to a large bias $R \gg 1$. This happens when the $\mu$ term dominates the energy of the ground state, i.e. for $\mu \gg \sqrt{N}J, UN^2$. Another large $R$ limit is recovered for finite $\mu, U, J > 0$ and large $N$. This limit, the particles condense due to the strong attractive energy $\sim UN^2$. The symmetry is broken by the potential offset $\mu$, leading to condensation of the particles in site $A$ and to large $R$ values.

A perturbative approach for the large $N$ limit is non-trivial due to as the dimension of the effective Hilbert space changes with $N$. However, a direct numerical analysis clearly reveals the power law behaviour in this limit. See Table I, Sec. III and the appendix [C] for more details. In what follows, we consider the large $\mu$ limit and evaluate the ground state and the entanglement quantifiers using a perturbative approach.

Let us develop a standard perturbation theory for the Hamiltonian $H = H_0 + \frac{1}{\mu} H_1 = \frac{1}{\mu} H_{\text{BH}}$

$$H_0 = -\hat{n}_A \quad H_1 = H_{\text{BH}} - \mu H_0, \quad (12)$$

at large $\mu$. 

The eigenstates of $H_0$ are $|\phi_n^{(0)}\rangle = |n, N-n\rangle$ with energies $\epsilon_n^{(0)} = -n$. The first order correction to the ground state is

$$|\phi_N\rangle = |N,0\rangle + \lambda |N - 1, 1\rangle + O(\lambda^2)$$

where $\lambda = J/\mu N$ assumed to be small as well as $UN^2/\mu \ll 1$. At this limit we find $R = \langle \hat{n}_A \rangle / \langle \hat{n}_B \rangle = N/\lambda^2$. So, we can approximate at small $\lambda$, $R = \mu^2/J^2 + O(\mu)$. The Von Neumann entanglement can be calculated for the ground state $|\phi_N\rangle$

$$E_{vn} = -\frac{1}{1 + \lambda^2} \ln \left( 1 + \lambda^2 \right) - \frac{\lambda^2}{1 + \lambda^2} \ln \left( 1 + \lambda^2 \right)$$

$$= -\lambda^2 \ln \lambda^2 + O(\lambda^3).$$

At this limit, we find the reported scaling

$$E_{vn} = \frac{N}{R} \ln \frac{N}{R}$$

for leading order. This is corroborated numerically in Fig. 2 and summarised in Table I. Recall that in this scaling, $N$ may be large, but $R \gg N$.

To find the Logarithmic negativity, we need to calculate the eigenvalues of the partially transposed density matrix of $|\phi_N\rangle$. For $N > 1$, the only non-zero eigenvalues are $\frac{1}{1+\lambda^2}$, $\frac{\lambda^2}{1+\lambda^2}$, $\frac{\lambda^2}{1+\lambda^2}$. This leads to

$$E_{ln} = \log_2 \left( 1 + \frac{2\lambda}{1 + \lambda^2} \right) = \frac{2}{\ln 2} \sqrt{\frac{N}{R}}$$

For large $R$ values, where the perturbation theory applies, the Logarithmic negativity dominates the Von Neumann entanglement as it should [21][37]. Again, we refer to Fig. 2 to see the excellent agreement with the numerical evaluation.

Other scaling schemes, leading to large $R$ values can exist. Nevertheless, the power law behaviour of the entanglement quantifiers is believed to persist, based on the $N = 1$ exactly solvable cases.

We turn to study the large $R$ entanglement properties of a completely different setup – the quantum asymmetric inclusion process.

### B. The QASIP

To analyze the steady state entanglement properties of the QASIP at large $R$, we need to find the steady state density matrix with a fixed $S$, i.e. $\mathcal{L}_{QASIP}(\hat{\rho}_S) = 0$. Namely, we wish to find $\hat{\rho}_S(x,y)$ such that the right hand side of eq.\hspace{1pt}[5] vanishes.

As in the Bose-Hubbard model, it is useful to study first the simple case of $S = 1$. Here, $x, y = \{0, 1\}$ and demanding a steady state in eq.\hspace{1pt}[5] leads to

$$\left( \begin{array}{ccc} -\gamma & i\varepsilon & -i\varepsilon \\ i\varepsilon & -\frac{1}{2}\gamma - \eta & 0 \\ -i\varepsilon & 0 & -\frac{1}{2}\gamma - \eta \\ \gamma & -i\varepsilon & i\varepsilon \end{array} \right) \left( \begin{array}{c} \rho_1(0,0) \\ \rho_1(0,1) \\ \rho_1(1,0) \\ \rho_1(1,1) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right).$$

Solving eq.\hspace{1pt}[17], we find the steady state solution for $S = 1$

$$\mathcal{N}_1 \hat{\rho}_1 = 4\varepsilon^2 |0,1\rangle \langle 0,1| + 2\gamma \varepsilon (|1,0\rangle \langle 0,1| - |0,1\rangle \langle 1,0|)$$

$$+ (4\varepsilon^2 + \gamma^2 + 2\gamma \eta) |1,0\rangle \langle 1,0|.$$
that to first order, there are yet no corrections to the occupancies. Hence, we solve to second order in $1/\gamma$ obtaining
\begin{equation}
\dot{\rho}^{(2)}_\gamma = \frac{4\varepsilon^2}{\gamma S} |S-1,1\rangle \langle S-1,1| \label{eq:23} \\
+ \frac{4i\varepsilon \eta}{S\sqrt{2}} |S-1,1\rangle \langle S,0|
- \frac{2\sqrt{2}\varepsilon^2}{\sqrt{S(S-1)}} |S,0\rangle \langle S-2,2| + \text{h.c.}
\end{equation}
To second order, we find $\mathcal{N}_\gamma = \frac{\varepsilon^2}{\gamma S} + 1$. From the perturbative solution of eq.\ref{eq:19}, we find that
\begin{equation}
R = S - 1 + \frac{S^2 \varepsilon^2}{4\varepsilon^2} + O(\gamma) \approx \frac{S^2 \varepsilon^2}{4\varepsilon^2}. \label{eq:24} 
\end{equation}
This approximation also implies the assumption $\varepsilon^2 \ll S\gamma^2$. Also, the Logarithmic negativity can be calculated as there are at most four non-zero eigenvalue for partially transposed density matrix for any $S$ value. We find to leading order
\begin{equation}
E_{\ln} = \frac{4\varepsilon}{\gamma \sqrt{S} \ln 2} + O\left(\frac{1}{\gamma^2}\right) \approx \frac{2\sqrt{S}}{\ln 2} \frac{1}{\sqrt{R}} + O\left(\frac{1}{R}\right). \label{eq:25} 
\end{equation}
As noted in Sec.\ref{sec:2}, the $E_{\ln}(R)$ power law behaviour was verified numerically in Fig.\ref{fig:4}.

### 2. Large dephasing limit

Here we consider the large $\eta$ limit with fixed $S, \gamma, \varepsilon$. We write the density matrix as a perturbative series in $1/\eta$
\begin{equation}
\dot{\rho}_\gamma = \frac{1}{\mathcal{N}_\eta} (\dot{\rho}^{(0)}_\gamma + \frac{1}{\eta} \dot{\rho}^{(1)}_\gamma + \frac{1}{\eta^2} \dot{\rho}^{(2)}_\gamma) + O(1/\eta^3), \label{eq:26} 
\end{equation}
where here $\mathcal{N}_\eta$ is a normalization constant ensuring the truncated density matrix has trace 1. Again, the perturbative series implies the order by order steady state solutions
\begin{equation}
0 = \mathcal{L}_D(\dot{\rho}^{(0)}_\gamma) \label{eq:27} \\
0 = \frac{1}{\eta} \mathcal{L}_D(\dot{\rho}^{(1)}_\gamma) + (\mathcal{L}_E + \mathcal{H}_{\text{int}}) \dot{\rho}^{(0)}_\gamma \label{eq:28} \\
0 = \frac{1}{\eta} \mathcal{L}_D(\dot{\rho}^{(2)}_\gamma) + (\mathcal{L}_E + \mathcal{H}_{\text{int}}) \dot{\rho}^{(1)}_\gamma \label{eq:29} 
\end{equation}
Eq.\ref{eq:27} admits a degenerate solution
\begin{equation}
\dot{\rho}^{(0)}_\gamma = \sum_{k=0}^S a_k |k, S-k\rangle \langle k, S-k| \label{eq:30} 
\end{equation}
with $a_k$ non-negative coefficients. This degeneracy is broken in the next order, i.e. eq.\ref{eq:28}. We find $\dot{\rho}^{(0)}_\gamma = |S,0\rangle \langle S,0|$, however the degeneracy moves to the next order
\begin{equation}
\dot{\rho}^{(1)}_\gamma = \sum_{k=0}^S b_k |k, S-k\rangle \langle k, S-k| + i\varepsilon \sqrt{S} |S,0\rangle \langle S-1,1| - |S-1,1\rangle \langle S,0|, \label{eq:31} 
\end{equation}
where $b_k$ are again non-negative coefficients. To evaluate to leading order $R$, we have to break the degeneracy in $b_k$. This breaking is obtained at the next order, i.e. eq.\ref{eq:29}, where we find $b_k = 6\sqrt{S-1} \frac{\varepsilon^2}{\gamma}$ and
\begin{equation}
\dot{\rho}^{(2)}_\gamma = -\frac{\varepsilon^2 \sqrt{S-1} \sqrt{S}}{2\gamma} |S-2,2\rangle \langle S,0| + i\gamma \varepsilon \frac{S^{3/2}}{\sqrt{S}} |S-1,1\rangle \langle S,0| \\
- i\frac{2\sqrt{2}\varepsilon^2 \gamma S - 1}{\gamma} |S-2,2\rangle \langle S-1,1| + \sum_{k=0}^S c_k |k, S-k\rangle \langle k, S-k| + \text{h.c.} \label{eq:32} 
\end{equation}
The degeneracy in the non-negative terms $c_k$ is broken at the third order of the expansion. To leading order in $\eta$, we find $\mathcal{N}_\eta = 1 + 2\varepsilon^2 \gamma S$. Therefore, to leading order
\begin{equation}
R = \frac{\gamma \eta S}{2\varepsilon^2} + S - 1 \approx \frac{\gamma \eta S}{2\varepsilon^2}. \label{eq:33} 
\end{equation}
Again, the spectrum of the partially transposed density matrix is composed of only four non-zero eigenvalues for any $S > 1$: $-\frac{\gamma \varepsilon \sqrt{S}}{\gamma \eta + 2\varepsilon^2}, \frac{\gamma \varepsilon \sqrt{S}}{\gamma \eta + 2\varepsilon^2}, \frac{\varepsilon^2 S}{\gamma \eta + 2\varepsilon^2}, \frac{\varepsilon^2 S}{\gamma \eta + 2\varepsilon^2}$. The Logarithmic negativity is thus given by
\begin{equation}
E_{\ln} = \log_2 \left(1 + \frac{2\gamma \varepsilon S}{\gamma \eta + 2\varepsilon^2}\right) \approx \frac{2\varepsilon S}{\gamma \eta + 2\varepsilon^2} = \frac{\gamma S^{3/2}}{\varepsilon \ln 2} \frac{1}{R}. \label{eq:34} 
\end{equation}
As noted in Sec.\ref{sec:2} the $E_{\ln}(R)$ power law behaviour was verified numerically in Fig.\ref{fig:5}.

### C. Large number of particles

We also studied the scaling limit $S \gg 1$ and finite $\eta, \gamma, \varepsilon$. Analytically, a perturbative solution in this case becomes hard due to the change in the state space, similarly to the Bose-Hubbard case. Nevertheless, it is possible to numerically find the steady state and calculate the Logarithmic negativity even for large $S$ values. This was carried out numerically (Fig.\ref{fig:6}) and reported on in Sec.\ref{sec:4}.

### IV. DISCUSSION

State of the art experimental techniques allows to entangle a single agent to thousands of atoms \cite{19}. How-
ever, it was unclear whether one could push the experimental techniques to significantly increase the number of atoms entangled to the agent.

Here, we have explored the theoretical bounds on entangling one or a few agents to a many body system. The ground state of a two-site Bose-Hubbard model, with an occupancy bias $R \gg 1$ leads to a power law decay in the Logarithmic negativity and the Von Neumann entanglement entropy in different scaling limits. Furthermore, the steady state of the QASIP biased to large $R$ values also exhibits a power law decay in the Logarithmic negativity. We stress that while the power law behaviour is typical, the exponent depends on the scaling limits, see Tables I and II.

From the slow decay of the entanglement, it is now clear it is typically possible to entangle thousands of atoms to a single agent. Furthermore, designing systems with slow entanglement decay (small $\alpha$) allows to entangle more particles in the many body system to the one agent (or a few). It would be particularly appealing to tangle more particles in the many body system to the steady state of the QASIP biased to large $R$ values which is an entangle quantifier.

In this section, we provide a brief introduction to the entanglement quantifiers used in this text: the Von Neumann entanglement entropy and the Logarithmic negativity. The purpose of quantifiers is to distinguish between entangled to non-entangled states (separable) and furthermore to suggest a hierarchy of values for entangled states. Here we do not aim to give an exhaustive account of quantum quantifiers, but to motivate the usage of the Von Neumann entanglement entropy and the Logarithmic negativity in the case at hand.

For pure states, all entanglement measures are defined to correspond to the Von Neumann entanglement entropy [21]. In bipartite system $AB$, 

$$E_{\text{vn}}(\rho_{AB}) = - \text{Tr} \rho_A \ln \rho_A = - \text{Tr} \rho_B \ln \rho_B, \quad (A1)$$

where $\rho_A = \text{Tr}_B \rho_{AB}$ is the reduced density matrix. $E_{\text{vn}} > 0$ only for non-separable pure states.

In terms of wave functions (which are pure states), the Schmidt decomposition using orthonormal states imply $|\psi\rangle = \sum_i \alpha_i |u_i\rangle_A \otimes |v_i\rangle_B$. Then, we find $E_{\text{vn}}(|\psi\rangle) = - \sum_i |\alpha_i|^2 \log |\alpha_i|^2$.

Entanglement is harder to quantify for mixed states. Many different measures for the entanglement exists. Typically, entanglement measures are given in the form of some minimization problem, making them hard to calculate. Instead, we will use the Logarithmic negativity which is an entanglement monotone and not a measure. Namely, for pure state the Logarithmic negativity does not correspond to the Von Neumann entanglement entropy (except for specific cases). However, it is straightforward to calculate the Logarithmic negativity, making it a favorable entanglement quantifier.

The Logarithmic Negativity is given by

$$E_{\text{ln}}(\rho) = \log_2 \|\rho^{PT}\|_1, \quad (A2)$$

where $\|\rho^{PT}\|_1 \equiv \text{Tr} \sqrt{\rho A\rho A}$. Intuitively speaking, the Logarithmic negativity counts the amount of negative eigenvalues in the partially transposed density matrix relating it to the Peres–Horodecki criterion [39 40]. We note that a positive Logarithmic negativity values insures non-separability, but a vanishing value does not guarantees separability.

The Logarithmic Negativity is an entanglement monotone [20 21 41], which implies that on average, under locally quantum operations and classical communication (LOCC), the Logarithmic Negativity does not increase. Furthermore, the Logarithmic negativity was shown to

$$E_{\text{ln}}(\rho) = \log_2 \|\rho^{PT}\|_1, \quad (A2)$$

where $\rho^{PT}$ is the partially transposed density matrix, and $\|A\|_1 \equiv \text{Tr} \sqrt{AA^T}$. Intuitively speaking, the Logarithmic negativity counts the amount of negative eigenvalues in the partially transposed density matrix relating it to the Peres–Horodecki criterion [39 40]. We note that a positive Logarithmic negativity values insures non-separability, but a vanishing value does not guarantees separability.
be an upper bound for the distillation entanglement, connecting it to useful quantum operations using maximally entangled states [12]. Since the distillation entanglement is an entanglement measure, it is evident that for pure states, the Von Neumann entanglement entropy is bounded by the Logarithmic negativity. This fact provides a consistency check in our numerical assessment.

Appendix B: The Lindblad adjoint dynamics

The purpose of this section is to introduce the Heisenberg operator evolution picture for the Lindblad dynamics.

For an observable $\hat{O}$ (explicitly time-independent), we have the expectation value $\langle \hat{O} \rangle = \text{Tr} \hat{O} \rho$. Therefore,

$$\partial_t \langle \hat{O} \rangle = \text{Tr} \hat{O} \partial_t \rho = \text{Tr} \hat{O} \mathcal{L}(\rho),$$

(B1)

where $\mathcal{L}(\rho)$ is a Lindblad super-operator of eq. (3). Then, the formal adjoint $\mathcal{L}^\dagger$ is defined such that

$$\partial_t \langle \hat{O} \rangle = \text{Tr} \mathcal{L}^\dagger (\hat{O}) \rho.$$  

(B2)

For the Lindblad super-operator in eq. (3), it implies the Heisenberg picture

$$\partial_t \hat{O} = \mathcal{L}^\dagger (\hat{O}) = -\mathcal{H}(\hat{O}) + \sum_k \hat{L}_k^\dagger \hat{O} \hat{L}_k - \frac{1}{2} \{\hat{L}_k^\dagger \hat{L}_k, \hat{O}\}.$$

(B3)

It is rather straight-forward to see that $\partial_t \hat{N} = 0$ for the QASIP.

Appendix C: Additional numerical data for the Bose-Hubbard model

Here we present further technical details on the numerical analysis of the Bose-Hubbard model.

For large $N$ values, the Von Neumann entanglement entropy is easier to obtain than the logarithmic negativity. In the Appendix A it was shown that the pure state Von Neumann entanglement entropy can be obtained from the ket state. However, the logarithmic negativity requires finding the spectrum of the partially transposed density matrix. The complexity of handling density matrices is certainly higher than that of handling ket states, hence the lower values that were reached for the logarithmic negativity.

In Fig. 7 the scaling $R \propto N^\beta$ is presented in the large $N$ limit for three values of the parameters $(J, U, \mu)$. The values are picked to produce large $R$ values and to span over a few length scales in $R$, providing a reliable prediction for the exponents.

![Graph showing scaling $R \propto N^\beta$ for $N \geq 1$](image)

**FIG. 7.** In the large $N$ limit of the two-site Bose-Hubbard model, numerical evaluation show that $R \propto N^\beta$ with $\beta \approx 2.15$. The three parameters in the range $N \in [1, 400]$, are picked to allow for a sufficiently large $R$ values.

In Fig. 8 the large $N$ limit of the QASIP model, it is numerically corroborated that $R \propto N^\beta$ where $\beta \approx 1.988$. Different $\eta, \gamma$ values were tested, leading to the large $R$ limit at $\varepsilon = 1$, $N \in [1, 25]$.

![Graph showing scaling $R \propto N^\beta$ for $N \geq 1$](image)

**FIG. 8.** In the large $N$ limit of the QASIP model, it is numerically corroborated that $R \propto N^\beta$, $\beta \approx 1.988$. Different $\eta, \gamma$ values were tested, leading to the large $N$ limit at $\varepsilon = 1$, $N \in [1, 25]$.

Appendix D: Additional numerical data for the QASIP model

Here we present further technical details on the numerical analysis of the QASIP model.

In Fig. 8 the large $N$ scaling of $R \propto N^\beta$ is plotted for different $\gamma$ values. Since testing the logarithmic negativity in large $N$ values is numerically challenging, the parameters were chosen to facilitate as large $R$ as possible which improves exponent fitting.

---

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. 81, 865 (2009).

[2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Reviews of modern physics 80, 517 (2008).

[3] V. Vedral, Decoding reality: the universe as quantum information (Oxford University Press, 2018).

[4] M. A. Nielsen and I. Chuang, “Quantum computation and quantum information,” (2002).

[5] J. Preskill, Quantum 2, 79 (2018).

[6] L. Pezze, A. Smerzi, M. K. Oberthaler, R. Schmied, and P. Treutlein, Reviews of Modern Physics 90, 035005 (2018).
