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ON SERRE’S UNIFORMITY CONJECTURE FOR SEMISTABLE ELLIPTIC CURVES OVER TOTALLY REAL FIELDS

SAMUELE ANNI AND SAMIR SIKSEK

Abstract. Let $K$ be a totally real field, and let $S$ be a finite set of non-archimedean places of $K$. It follows from the work of Merel, Momose and David that there is a constant $B_{K,S}$ so that if $E$ is an elliptic curve defined over $K$, semistable outside $S$, then for all $p > B_{K,S}$, the representation $\rho_{E,p}$ is irreducible. We combine this with modularity and level lowering to show the existence of an effectively computable constant $C_{K,S}$, and an effectively computable set of elliptic curves over $K$ with CM $E_1, \ldots, E_n$ such that the following holds. If $E$ is an elliptic curve over $K$ semistable outside $S$, and $p > C_{K,S}$ is prime, then either $\rho_{E,p}$ is surjective, or $\rho_{E,p} \sim \rho_{E_i,p}$ for some $i = 1, \ldots, n$.

1. Introduction

Let $K$ be a number field. We write $G_K = \text{Gal}(\overline{K}/K)$ for the absolute Galois group of $K$. For an elliptic curve $E/K$, we write $\rho_{E,p}$ for the associated representation of $G_K$ on the $p$-torsion of $E$:

$$\rho_{E,p} : G_K \to \text{Aut}(E[p]) \cong \text{GL}_2(\mathbb{F}_p).$$

We recall the following celebrated theorem of Serre.

Theorem 1 (Serre [20 Théorème 2]). Let $K$ be a number field and $E$ an elliptic curve over $K$ without CM. Then there is a constant $C_{E,K}$ such that for all $p > C_{E,K}$ the representation $\rho_{E,p}$ is surjective.

Serre’s Uniformity Conjecture (originally formulated by Serre as a question [20 § 4.3] and [27]) asserts the existence of a constant $C_K$, depending only on $K$, such that if $E$ is an elliptic curve over $K$ without complex multiplication, and $p > C_K$ is a prime, then the representation $\rho_{E,p}$ is surjective. Mazur [21] proved that $\rho_{E,p}$ is irreducible for any prime $p > 163$ and elliptic curve $E$ over $\mathbb{Q}$. Recently, Bilu, Parent and Rebolledo [3] proved, for $p \geq 11$, $p \neq 13$, and $E/\mathbb{Q}$ without complex multiplication, that the image of the representation is also not contained in the normalizer of a split Cartan subgroup of $\text{GL}_2(\mathbb{F}_p)$.

No analogues of the above-mentioned theorems of Mazur and of Bilu, Parent and Rebolledo are known for elliptic curves over general number fields. The strongest
known result is Merel’s Uniform Boundedness Theorem [22], which asserts the following: for \( d \geq 1 \), there is a constant \( B_d \) such that if \( E \) is an elliptic curve over a number field \( K \) of degree \( d \), and \( p > B_d \) is a prime, then \( E(K)[p] = 0 \). A number of irreducibility results are however known for semistable elliptic curves over number fields, whose proofs make essential use of Merel’s Theorem. For example, Kraus [20, Appendix B] shows that if \( K \) is a number field that does not contain the Hilbert class field of an imaginary quadratic field, then there is a constant \( B_K \) such that for a prime \( p > B_K \) and a semistable elliptic curve \( E/K \), the representation \( \rho_{E,p} \) is irreducible.

As noted by Serre [21, Theorem 4], Mazur’s Theorem cited above implies the following: if \( E/\mathbb{Q} \) is a semistable elliptic curve without complex multiplication, then the representation \( \rho_{E,p} \) is surjective for any prime \( p \geq 11 \). To motivate our present work, it is appropriate to give a sketch of the argument. By Mazur’s Theorem, we may suppose that \( \rho_{E,p} \) is irreducible. As \( \mathbb{Q} \) has a real embedding, \( \rho_{E,p} \) is therefore absolutely irreducible (e.g. [25, Lemma 5]). If \( \rho_{E,p} \) is not surjective, then its image is contained in the normalizer \( N_{ns} \) of non-split Cartan subgroup \( C_{ns} \) or the normalizer \( N_s \) of a split Cartan subgroup \( C_s \). In either case, the representation \( \rho_{E,p} \) induces a quadratic character \( \psi : G_\mathbb{Q} \to N_s/C_s \cong \{\pm 1\} \). This character is unramified away from the archimedean and additive places. As \( E \) is semistable, we see that \( \psi \) is unramified away from \( \infty \), and as the narrow class number of \( \mathbb{Q} \) is 1, we have \( \psi = 1 \). It follows that the image of \( \rho_{E,p} \) is contained in \( C_s \) or \( C_{ns} \). These groups are absolutely reducible, giving a contradiction. Over a number field \( K \), the argument breaks down. First the narrow class number of \( K \) maybe greater than 1. Moreover, let \( L \) be the narrow class field of \( K \). If the image of \( \rho_{E,p} \) is contained in the normalizer of a Cartan subgroup, then \( \rho_{E,p}(G_L) \) is contained in a Cartan subgroup: \( C_s \) or \( C_{ns} \). If the former, then we can conclude the argument using (say) Kraus’ result, provided \( L \) does not contain the Hilbert class field of an imaginary quadratic field. In the latter case, we do the same if \( L \) has some real embedding. It is clear, however, that the argument does not hold in general.

In this paper, we restrict ourselves to totally real fields \( K \). This allows us to apply modularity and level lowering theorems to semistable elliptic curves \( E/K \) whose mod \( p \) image is contained in the normalizer of a Cartan subgroup.

**Theorem 2.** Let \( K \) be a totally real field, and let \( S \) be a finite set of non-archimedean places of \( K \). There are an effectively computable constant \( C_{K,S} \), depending only on \( K \) and \( S \), and a finite computable set \( E_1, \ldots, E_n \) of elliptic curves over \( K \) with complex multiplication such that the following holds: if \( E \) is an elliptic curve over \( K \) semistable outside \( S \), and \( p > C_{K,S} \) is prime, then either \( \rho_{E,p} \) is surjective, or \( \rho_{E,p} \sim \rho_{E_i,p} \) for some \( i = 1, \ldots, n \).

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2. Irreducibility of mod \( p \) representations of elliptic curves

To deal with the Borel images we shall invoke the following theorem due to Freitas and Siksek [13], but is in fact a corollary of the ideas of David [8] and Momose [23] building on Merel’s Uniform Boundedness Theorem [22].
Theorem 3 ([13 Theorem 1]). Let $K$ be a totally real Galois number field of degree $d$, with ring of integers $O_K$ and Galois group $G = \text{Gal}(K/\mathbb{Q})$. Let $\mathfrak{S} = \{0, 12\}^G$, which we think of as the set of sequences of values 0, 12 indexed by $\tau \in G$. For $s = (s_\tau) \in \mathfrak{S}$ and $\alpha \in K$, define the twisted norm associated to $s$ by
\[
\mathcal{N}_s(\alpha) = \prod_{\tau \in G} \tau(\alpha)^{s_\tau}.
\]
Let $\epsilon_1, \ldots, \epsilon_{d-1}$ be a basis for the unit group of $K$ (modulo $\pm 1$), and define
\[
A_s := \text{Norm}(\gcd((\mathcal{N}_s(\epsilon_1) - 1)O_K, \ldots, (\mathcal{N}_s(\epsilon_{d-1}) - 1)O_K)).
\]
Let $B$ be the least common multiple of the $A_s$ taken over all $s \neq (0)_{\tau \in G}$, $(12)_{\tau \in G}$. Then $B \neq 0$. Moreover, let $p \nmid B$ be a rational prime, unramified in $K$, such that $p \geq 17$ or $p = 11$. If $E/K$ is an elliptic curve semistable at all $v \mid p$ and $\overline{\rho}_{E,p}$ is reducible then $p < (1 + 3^{6d})$, where $h$ is the class number of $K$.

3. Modularity

Let $K$ be a totally real number field, and let $E$ be an elliptic curve over $K$. Recall that $E$ is modular if there exists a Hilbert cuspidal eigenform $f$ over $K$ of parallel weight 2, with rational Hecke eigenvalues, such that the Hasse–Weil $L$-function of $E$ is equal to the Hecke $L$-function of $f$. It is conjectured that all elliptic curves over totally real fields are modular, and, recently, modularity has been proved for elliptic curves over real quadratic fields, see [15].

For what follows, we need a suitable modularity lifting theorem. The following such theorem is derived in [15] as a relatively straightforward consequence of a deep theorem of Breuil and Diamond [5, Théorème 3.2.2], which builds on the work of Kisin [19], Gee [17], and Barnet-Lamb, Gee and Geraghty [1, 2].

Theorem 4 ([13 Theorem 2]). Let $E$ be an elliptic curve over a totally real number field $K$, and let $p \neq 2$ be a rational prime. Suppose $\overline{\rho}_{E,p}$ is modular in the following sense: $\overline{\rho}_{E,p} \sim \overline{\rho}_{f,\omega}$ for some Hilbert cuspidal eigenform over $K$ of parallel weight 2, where $\omega \mid p$. Suppose moreover that $\overline{\rho}_{E,p}(G_{K(\zeta_p)})$ is absolutely irreducible. Then $E$ is modular.

Proposition 3.1. Let $K$ be a totally real field. Let $p \geq 7$ be a prime that is unramified in $K$. Suppose that $E$ is semistable at some prime $v$ of $K$ above $p$, and that moreover $\overline{\rho}_{E,p}$ is irreducible but not surjective. Then $E$ is modular.

Proof. Write $G := \overline{\rho}_{E,p}(G_K)$. As $v \mid p$ is unramified, we have $K \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$, and so det $\overline{\rho}_{E,p} = \chi : G_K \to \mathbb{F}_p^*$ is surjective, where $\chi$ is the mod $p$ cyclotomic character. By assumption $\overline{\rho}_{E,p}$ is irreducible but not surjective, and so $G$ does not contain $\text{SL}_2(\mathbb{F}_p)$. It follows [20, §2] that $G$ is contained in the normalizer of a Cartan subgroup, or its projectivization $\mathbb{P}G := G/(G \cap \mathbb{F}_p^*)$ is isomorphic to $A_4$, $S_4$ or $A_5$. In particular, $G$ does not contain elements of order $p$.

Write $I_v \subset G_K$ for the inertia subgroup at $v$. As $E$ is semistable at $v$ and $v$ is an unramified prime, we have (using [20, §1.11, §1.12] and the fact that $G$ does not contain elements of order $p$):
\[
\overline{\rho}_{E,p}|_{I_v} \sim \begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \overline{\rho}_{E,p}|_{I_v} \otimes \mathbb{F}_p \mathbb{P}F_{p^2} \sim \begin{pmatrix} \omega & 0 \\ 0 & \omega p \end{pmatrix};
\]
here $\omega$ is a level 2 fundamental character $I_v \to \mathbb{F}_p^*$. More precisely, if $E$ has good ordinary or multiplicative reduction at $v$ then we are in the first case of (1), and
if $E$ has good supersingular reduction at $v$ then we are in the second case. We observe from (1) that $F \rho G$ contains an element of order $p - 1$ or $p + 1$. Since $p \geq 7$, we see that $F \rho G$ is not isomorphic to $A_4$, $S_4$, and $A_5$. It follows that $G$ is contained in the normalizer $N_\nu$ of a Cartan subgroup $C_\nu$. The representation $\overline{\rho}_{E, p}$ is irreducible, and as $K$ is totally real, $\overline{\rho}_{E, p}$ must be absolutely irreducible (e.g. [23, Lemma 5]). Thus the image $G$ is contained in $N_\nu$ but not in $C_\nu$.

Now, as $\overline{\rho}_{E, p}$ has solvable image, we can view it as a totally odd irreducible Artin representation. By a standard argument (c.f. [10] Proof of Lemma 4.2), we have $\overline{\rho}_{E, p} \sim \overline{\rho}_{f, \varpi}$, for some Hilbert modular form $f$ over $K$, of parallel weight 2, and $\varpi \mid p$.

By Theorem 4.1 in order to show that $E$ is modular it is sufficient to show that $\overline{\rho}_{E, p}(G_{K(\sqrt{p})})$ is absolutely irreducible. Suppose otherwise. It follows [15] Lemma 4.2 that $G^+ := G\cap GL_2^+(\mathbb{F}_p)$ is absolutely reducible, where $GL_2^+(\mathbb{F}_p)$ is the subgroup of $GL_2(\mathbb{F}_p)$ consisting of matrices with square determinant. Suppose $E$ has good ordinary or multiplicative reduction at $v$ and so we are in the first case of (1). Let $g$ be a generator of $\mathbb{F}_p^*$. Then, with a suitable choice of basis for $E[p]$, the image $G$ contains all matrices of the form $A_r := \begin{pmatrix} g^r & 0 \\ 0 & 1 \end{pmatrix}$; these share the eigenvectors $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. As $G$ is absolutely irreducible, it must contain some matrix $B$ whose eigenvectors $\neq u, v$. It follows that $G^+$ contains the pair of matrices $A_2$, $BA_2$, where $s = 0$ or $1$ according to whether $\det(B)$ is a square or non-square. It is easy to check that these do not have common eigenvectors, contradicting the absolute reducibility of $G^+$. If $E$ has good supersingular reduction at $v$ then we are in the second case of (1). It is now easy to check, similarly to the above, that $G^+$ is absolutely irreducible, giving a contradiction. This completes the proof.  

4. Level lowering

In this section, $K$ is a totally real field, and $S$ a finite set of non-archimedean primes of $K$. Moreover, $p \geq 7$ is a rational prime that is unramified in $K$ such that $v \notin S$ for all $v \mid p$.

**Lemma 4.1.** Let $E$ be an elliptic curve defined over $K$ that is semistable outside $S$. Suppose that $\overline{\rho}_{E, p}$ is irreducible but not surjective. Then

(i) $\overline{\rho}_{E, p}$ is unramified at all $\mu \notin S$, $\mu \nmid p$;

(ii) $\overline{\rho}_{E, p}$ is finite at all $v \mid p$.

**Proof.** Let $v \mid p$. We would like to prove (ii), which is certainly true if $E$ has good reduction at $v$. By hypothesis, $E$ is semistable at $v$, and so we may assume that $E$ has multiplicative reduction at $v$. Write $G_v \subset G_K$ for the decomposition group at $v$. By the proof of Proposition 3.1 we know that $G = \overline{\rho}_{E, p}(G_K)$ does not contain any elements of order $p$. It immediately follows that $\overline{\rho}_{E, p}(I_\mu)$ is “peu ramifié”, proving (ii).

Let $\mu$ be a non-archimedean prime of $K$, not in $S$, and not above $p$. Then $E$ is semistable at $\mu$, and so the inertia subgroup $I_\mu \subset G_K$ acts unipotently on $E[p]$. As $G$ does not contain elements of order $p$, we have $\overline{\rho}_{E, p}(I_\mu) = 1$, proving (i). \qed

Now let

$$\mathcal{M} = \prod_{\nu \in S} (2^{+6} v_3(2) + 3 v_3(3),$$
Lemma 4.2. Assume the hypotheses of Lemma [4.4]. Then there exists a Hilbert eigenform \( f \) over \( K \) of parallel weight 2 and level dividing \( \mathcal{M} \) such that \( \mathfrak{f}_{E,p} \sim \mathfrak{f}_{E,S} \) where \( \mathfrak{p} \mid p \) is a prime of \( \mathbb{Q}_l \), the field generated by the eigenvalues of \( f \).

Proof. Let \( \mathcal{N} \) be the conductor of \( E \). The additive part of \( \mathcal{N} \) divides \( \mathcal{M} \) (e.g. [28, Theorem IV.10.4]). By Proposition 3.1 and Theorem 4, there is a Hilbert eigenform \( f_0 \) over \( K \), with rational eigenvalues, level \( \mathcal{N} \) and parallel weight 2 such that \( \mathfrak{f}_{E,p} \sim \mathfrak{f}_{E,S} \). By Lemma 4.1, we have \( \mathfrak{f}_{E,p} \) is finite at all \( v \mid p \), and unramified at all \( \mu \nmid \mathcal{M} \). Applying level lowering theorems due Fujiwara [16], Jarvis [18], and Rajaei [24], we may remove these primes from the level (without changing the weight); the argument is practically identical to that in [14, Theorem 7], and so we omit the details.

Remark. Chen [6] observes that if \( E \) is an elliptic curve over \( \mathbb{Q} \), and \( \mathfrak{f}_{E,p} \) has image contained in the normalizer of a Cartan subgroup, then \( p \) is a congruence prime for the newform attached to \( E \). Our Lemma 4.2 encompasses Chen’s observation.

5. Proof of Theorem 2

Assume the hypotheses of Theorem 2 in particular, let \( E \) be an elliptic curve semistable outside \( S \). With the help of Theorem 3 we know that there is an effectively computable constant \( C_{K,S} \) such that if \( p > C_{K,S} \) then \( p \) is unramified in \( K \), all the primes \( v \mid p \) satisfy \( v \nmid S \), and \( \mathfrak{f}_{E,p} \) is irreducible. Suppose \( \mathfrak{f}_{E,p} \) is not surjective. We now apply Lemma 4.2 to deduce that \( \mathfrak{f}_{E,p} \sim \mathfrak{f}_{E,S} \) for some cuspidal Hilbert eigenform of parallel weight 2 and level dividing \( \mathcal{M} \). There are certainly only finitely many such eigenforms. We would like to increase \( C_{K,S} \) by an effectively computable amount so that the conclusion of Theorem 2 holds. Crucial to the effectivity is the existence of an algorithm [9] for determining the eigenforms \( f \) of a given weight and level, as well as their Hecke eigenvalues at given primes, and the fields generated by these eigenvalues. We will eliminate all such eigenforms \( f \) with \( \mathbb{Q}_l \not\subseteq \mathbb{Q} \), where \( \mathbb{Q}_l \) is the field generated by the eigenvalues of \( f \). So suppose that \( \mathfrak{f}_{E,p} \sim \mathfrak{f}_{E,S} \) where \( \mathbb{Q}_l \not\subseteq \mathbb{Q} \). Let \( l \) be the prime ideal of smallest possible norm such that \( l \nmid S \) and \( a_l(f) \nmid 1 \). If \( l \nmid p \), then \( p \mid \text{Norm}_{K/Q}(l) \) and so we obtain a contradiction by supposing that \( C_{K,S} > \text{Norm}_{K/Q}(l) \). We may therefore suppose that \( l \mid p \). Comparing the traces of the images of Frobenius at \( l \) in the representations \( \mathfrak{f}_{E,p} \) and \( \mathfrak{f}_{E,S} \), we have either \( a_l(f) = a_l(E) \pmod{\mathfrak{p}} \) if \( E \) has good reduction at \( l \), or \( a_l(f) \equiv \pm \text{Norm}_{K/Q}(l) + 1 \pmod{\mathfrak{p}} \) if \( E \) has multiplicative reduction at \( l \). In the former case, by the Hasse–Weil bounds, \( p \) divides

\[
\prod_{l \mid p} \text{Norm}_{Q_l/Q}(a_l(f) - t), \quad B = 2(\text{Norm}_{K/Q}(l))^{1/2}.
\]

As \( a_l(f) \not\in \mathbb{Q} \), all the terms in the product are non-zero, and so this gives a bound on \( p \). By taking \( C_{K,S} \) larger than this product we obtain a contradiction. If \( E \) has multiplicative reduction at \( l \), then \( p \) divides

\[
\text{Norm}_{Q_l/Q}(a_l(f) - \text{Norm}_{K/Q}(l) - 1) \cdot \text{Norm}_{Q_l/Q}(a_l(f) + \text{Norm}_{K/Q}(l) + 1)
\]

and again we obtain a contradiction by taking \( C_{K,S} \) larger than this product. Thus we are reduced to finitely many forms \( f \) satisfying \( \mathbb{Q}_l = \mathbb{Q} \).

So far we proved that there are an effectively computable constant \( C_{K,S} \) and a finite computable set \( f_1, \ldots, f_n \) of Hilbert eigenforms over \( K \) of parallel weight 2 with \( \mathbb{Q} \)-rational eigenvalues such that the following holds: if \( E \) is an elliptic curve
over $K$ semistable outside $S$, and $p > C_{K,S}$ is prime, then either $\rho_{E,p}$ is surjective, or $\rho_{E,p} \sim \rho_{i,p}$ for some $i = 1, \ldots, n$.

Next we have to show that the surviving forms $\mathfrak{f}$ have CM, possibly after enlarging $C_{K,S}$ by an effective amount. In fact, by a theorem of Dimitrov ([12, Theorem 2.1], [11, § 3]), if $\mathfrak{f}$ does not have CM, there is a constant $B_1$ such that for $p > B_1$ and $\varpi | p$, the image of $\rho_{\mathfrak{f},p}$ contains a conjugate of $\text{SL}_2(F_p)$. It is however unclear to us as to whether Dimitrov’s proof can be made effective, and so we proceed in a more elementary manner.

By the proof of Proposition 3.1 the image of $\rho_{E,p}$ is dihedral, and so there is a quadratic character $\psi$ such that $\rho_{E,p} \sim \rho_{E,p} \otimes \psi$. It is immediate from Lemma 4.1 that $\psi$ is unramified away from $S$, the archimedean primes, and the primes $v | p$. Suppose $v | p$. Comparing the restriction of the representation to the inertia subgroup at $v$, displayed in (1), and the restriction of the twisted representation by $\psi$, it is easy to deduce that the quadratic character $\psi$ is unramified at $v$. Hence, its conductor divides $\prod_{i \in S} [1 + 2^v(2)]$ and so $\psi$ belongs to a finite effectively computable set of characters.

Suppose $\rho_{E,p} \sim \rho_{f,p}$ where now $\mathfrak{f}$ has rational eigenvalues. Let $g = \mathfrak{f} \otimes \psi$. If $g = \mathfrak{f}$ then $\mathfrak{f}$ has CM as desired. Thus we may suppose $g \neq \mathfrak{f} \otimes \psi$. Let $\mathfrak{l}$ be the prime ideal of $K$ of smallest possible norm so that $1 \notin S$ and $a_\mathfrak{l}(\mathfrak{f}) \neq a_\mathfrak{l}(g)$. As before, if $\mathfrak{l} | p$ or if $E$ has multiplicative reduction at $\mathfrak{l}$, then we obtain a bound on $p$. We therefore suppose that $\mathfrak{l} \nmid p$ and $E$ has good reduction at $\mathfrak{l}$. From the relations $\rho_{E,p} \sim \rho_{f,p} \otimes \psi$ and $g = \mathfrak{f} \otimes \psi$ we have $a_\mathfrak{l}(\mathfrak{f}) \equiv a_\mathfrak{l}(g) \pmod{p}$. As $a_\mathfrak{l}(\mathfrak{f}) \neq a_\mathfrak{l}(g)$ we obtain a bound on $p$.

Now as the surviving forms $\mathfrak{f}_i$ are CM Hilbert eigenforms over $K$ of parallel weight 2 with $\mathbb{Q}$-rational eigenvalues. As explained in [4, § 2.2], they correspond to CM elliptic curves $E_i$ over $K$. The conductors of $E_i$ are the levels of $\mathfrak{f}_i$. As there is an effective algorithm to determine elliptic curves of a given conductor (c.f. [7]), the proof is complete.

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