A simple model of additive process mechanics for controlling the evolution of stresses in filament-wound cylindrical composites during the manufacturing process

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Abstract. A relatively simple mathematical model is proposed and developed that allows using additive process mechanics approaches to trace the evolution of technological stress fields in cylindrical composites during their manufacture by slow multi-layer winding of fibre-reinforced material with an arbitrary variation in the pre-tension force and fibre packing density. The constructed closed analytical dependences make it possible to state problems on controlling the distributions of these stresses.

1. Introduction

Composites are widely used in creating modern materials and manufacturing products in various sectors of industry. This is largely due to the fact that their use allows to achieve the required strength and stiffness indicators of the resulting materials and products with a significant gain in weight and geometric parameters. At the same time, we can point out another remarkable feature of composites: many of the processes that are already used in practice or only potentially conceivable for their manufacture open up fundamental capabilities for controlled or even targeted formation of the required characteristics of the technological stress-strain state in the resulting materials and products. Such processes, of course, include all possible processes of additive — in the most general sense of the term — manufacturing of composites. It is obvious that in order to successfully use the mentioned capabilities, it is necessary to understand the mechanisms that cause the occurrence and development of mechanical stresses in specific composite manufacturing processes, and, moreover, to develop a mathematical description of these mechanisms.

It should be noted that traditional approaches to the study of mechanical problems associated with additive manufacturing processes operate with standard categories of mechanics of deformable solids, augmenting them with certain corrections designed to take into account the shape and properties of the solid that change during the manufacturing process. However, such approaches cannot be fully adequate due to the very special kinematics of the processes of deformation of solids subjected to unceasing supplementation with new material elements. Such solids, according to the already established terminology, are called built-up or growing solids. For their mechanical description, it is necessary to develop fundamentally new approaches, which is what the scientific works of the
corresponding research school established in Russia under the leadership of Professor A.V. Manzhirov are devoted to (some of the latest results in this area can be found, for example, in [1–14]). The research presented in this paper is based on these approaches.

One of the long-used methods for additive manufacturing of composite materials and products is the method of multi-layer winding of the being reinforced material on a certain form. Pre-tension of the wound layers causes the emergence and further evolution of technological stress fields in the entire solid formed by winding. The nature and result of evolution will obviously depend in a decisive way on many specific parameters of the technological process under consideration. However, qualitative and, especially, quantitative forecasting of this dependence is impossible without adequate mathematical modelling of this process.

This work is devoted to the construction and analysis of one of the simplest mathematical models of additive process mechanics for the technological process of composite manufacturing by multilayer power winding of a material being reinforced with parallel fibres onto a slowly rotating circular cylindrical form.

2. Calculation model
The simulated process is sequential applying material layers of relatively small thickness onto the outer surface of the manufactured solid. Each such layer contains one round of reinforcing fibres in the radial direction. The rounds of the fibres are oriented perpendicular to the generatrix of the current cylindrical application surface and are located along this surface without a fixed step and without any reference to the location of the rounds in the layer of material having been previously applied onto the solid. The fibre position in the binder stratum of each layer, in the radial direction, is also not clearly defined and can change even within a single layer. In the described case, we can assume that the equivalent mechanical properties of the solid obtained as a result of winding will be transversally isotropic with the local axis of isotropy coinciding with the circumferential direction [15]. Indeed, under the assumptions made, the neighborhood of an arbitrary point in any axial section of this solid is normally permeated by a large number of reinforcing fibres rather randomly distributed over this neighborhood.

We believe that it is acceptable to change the thickness of the elementary layers of material applied during the winding process. However, this change occurs independently of the cycles of wrapping these layers around the formed cylindrical solid. In this case, the total thickness of a large number of sufficiently thin elementary layers will, on average, remain the same in all axial sections throughout the entire process. This means that this process can be modelled as axisymmetric.

The problem on deforming the being formed solid will be considered in plane and quasi-static formulation. The first assumes a sufficient length of this solid along the axis of its rotation, the second assumes a smooth change in the rotation speed during the winding of the layers of additional material onto the solid, as well as the insignificance of dynamic effects caused by bringing these layers into contact with the being formed solid. We consider the rotation itself sufficiently slow in order to be able to not take into account the impact on the material of centrifugal forces of inertia (an example of solving the problem on layer-by-layer formation of a rotating isotropic viscoelastic cylinder taking into account centrifugal forces is presented in paper [16]). In addition, due to the relative smallness of the thickness of the applied individual layers, the following two simplifications will also be justified. On the one hand, it is correct to model the studied process of solid growth as a continuous one, in which an infinitely thin layer of new material is added to this solid for each infinitesimal time interval. On the other hand, we can refuse to take into account possible retarded deformation of the material, focusing only on the analysis of pure elastic deformation, since, as shown in [17], the influence of the deformation aftereffect on the process of stress development in a growing solid is leveled at a very low rate of growth of the latter.

We will investigate small strain in the being manufactured cylindrical solid. Then we can assume as given the function \( R(t) \) describing the law of change over time \( t \) of the radius \( R \) of the outer surface of the growing solid under consideration, and this function will be strictly increasing. Indeed,
this radius changes over time due to two factors: adding new material to the solid and the deformation of this solid. With a small strain, the latter factor can be ignored in comparison with the former.

In the proposed model, we will also not take into account the pliability of the form used, considering it to be essentially more rigid than the material wound on it. Thus, the internal radius \( R_0 \) of the being formed cylindrical composite will be considered constant. Note that this assumption is not of principle in the constructed model: it can be easily abandoned at the expense of complexity of all the dependencies obtained as a result of solving the corresponding mechanical problem.

Let \( \varphi \) hereinafter be the circumferential, \( z \) — the longitudinal, and \( \rho \) — the radial coordinate counted in a system that rotates along with the rigid cylindrical form on which the material is wound.

The time in relation to which the growth process under study is considered is denoted by \( t \).

As indicated above, the ort that defines the local isotropy axis indicates the direction of change of the curvilinear coordinate \( \varphi \) at each point. So orts that indicate the direction of change of coordinates \( z \) and \( \rho \) set a local orthogonal basis in the isotropy plane passing through this point.

We denote by \( v_\varphi, v_z, v_\rho \) the components of the velocity field associated with the deformation displacements of the points inside the growing solid under consideration, in the introduced moving frame of reference \((\varphi, z, \rho)\). The plane strain state and the axial symmetry of the process give

\[
v_\varphi = 0, \quad \frac{\partial v_\rho}{\partial z} = \frac{\partial v_z}{\partial \varphi} = 0; \quad \frac{\partial v_\rho}{\partial \varphi} = \frac{\partial v_z}{\partial \rho} = 0.
\]

Ignoring those deformation displacements of the points of the being formed solid, which twirl it in the considered coordinate system around the axis of symmetry, we will also have a mirror symmetry of the mentioned velocity field, that is

\[
v_\rho = 0.
\]

As a result, \( v_\rho \), depending on \( t \) and \( \rho \) will be the only non-zero component of this velocity field.

Thus, the strain rate tensor will have only two non-zero physical components in the coordinate system used:

\[
D_{\varphi\rho} = \frac{v_\rho}{\rho}, \quad D_{\rho\rho} = \frac{\partial v_\rho}{\partial \rho}.
\]

These components must be associated with the rates of change of the normal components of the stress tensor, based on the equations of state of a transversely isotropic elastic medium:

\[
\frac{\partial \sigma_{\varphi\rho}}{\partial t} = C_1 D_{\varphi\rho} + C_2 D_{\rho\rho}, \quad \frac{\partial \sigma_{z\rho}}{\partial t} = C_2 D_{\varphi\rho} + C_3 D_{\rho\rho}, \quad \frac{\partial \sigma_{\rho\rho}}{\partial t} = C_2 D_{\varphi\rho} + C_4 D_{\rho\rho},
\]

where \( C_k \) are the elastic constants. The tangent components of the stress tensor in the coordinate system used will be zero at all points of the solid.

Naturally, the stress equilibrium equation and its corollary for the stress rates remain valid, so

\[
\frac{\partial^2 \sigma_{\rho\rho}}{\partial \rho \partial t} + \frac{1}{\rho} \left( \frac{\partial \sigma_{\rho\rho}}{\partial t} - \frac{\partial \sigma_{\varphi\rho}}{\partial \varphi} \right) = 0, \quad R_0 < \rho < R(t).
\]

The rigid coupling of the first wound layer of material with the form surface is expressed by the boundary condition

\[
v = 0, \quad \rho = R_0.
\]
In the here employed model of continuous growth, the outer surface of the being formed solid, onto which the layers of additional material are applied during the winding process, moves in space in the normal direction due to constant replenishment with new points, in the vicinity of which some initial stress state is predetermined:

\[ \sigma_{\varphi\varphi} = \sigma_s(R(t)), \quad \sigma_{zz} = \sigma_{r\varphi} = 0, \quad \rho = R(t). \]  \hspace{1cm} (5)

This stress state is characterized by the function \( \sigma_s(R) \) that sets the pre-tension in the circumferential direction of the applied layer of material at the moment when the outer radius of the solid being formed by winding is equal to \( R \). Denoting this moment in time by \( \tau_*(\rho) \), we will have the following identity fair for the entire cylindrical growing solid under consideration, that is, for all layers of radius \( \rho \) included in its composition:

\[ R(\tau_*(\rho)) = \rho. \]

On the basis of this identity, setting \( t = \tau_*(\rho) \) in (5), we obtain

\[ \sigma_{\varphi\varphi} = \sigma_s(\rho), \quad \sigma_{zz} = \sigma_{r\varphi} = 0, \quad t = \tau_*(\rho). \]  \hspace{1cm} (6)

It is obvious that initial conditions (6) unique to the deformable solid are equivalent to boundary conditions (5) on the moving growth surface.

3. Results and discussion

We can show (but omit the corresponding derivations here because of the limited paper) that specific initial conditions (6) considered together with the stress equilibrium equation lead to the condition on the moving growth surface that sets the initial rate of change of radial stresses at points attached to it:

\[ \frac{d\sigma_{r\varphi}}{dt} = s(t), \quad \rho = R(t), \]  \hspace{1cm} (7)

where the known function \( s(t) \) in the right-hand side is determined by the particular programs of solid growth, \( R(t) \), and of wound material pre-tension, \( \sigma_s(R) \):

\[ s(t) = \frac{\sigma_s(R(t))}{R(t)} \frac{dR(t)}{dt}. \]

The formulation of condition (7) closes the statement of the boundary value problem for evaluation of the rates of change of the stress-strain state characteristics for the solid under consideration during its continuous growth. The problem is given by differential equation (3) supplemented with relations (2) and (1), and by boundary conditions (4) and (7) on the stationary and moving boundaries of the growing solid in the considered coordinate system, respectively.

It turns out, it is possible to construct the exact analytical solution of the above non-classical mechanical problem (the constructing procedure is also to omit here). The following rates of change of all stress tensor non-zero components during the growth process will correspond to this solution:

\[ \frac{d\sigma_{\varphi\varphi}}{dt} = \frac{[R(t)/R_0]^{1/2} s(t)}{(C_2 + \gamma C_4)[R(t)/R_0]^{1/2} - C_2 + \gamma C_4} \left[ \frac{C_1 - \gamma C_2}{(\rho/R_0)^{1/2}} - \left( \frac{C_1 + \gamma C_2}{(\rho/R_0)^{1/2}} \right)^{1/2} \right] \equiv S_\varphi(\rho, t), \]

\[ \frac{d\sigma_{zz}}{dt} = \frac{[R(t)/R_0]^{1/2} s(t)}{(C_2 + \gamma C_4)[R(t)/R_0]^{1/2} - C_2 + \gamma C_4} \left[ \frac{C_2 - \gamma C_4}{(\rho/R_0)^{1/2}} - \left( \frac{C_2 + \gamma C_4}{(\rho/R_0)^{1/2}} \right)^{1/2} \right] \equiv S_\rho(\rho, t), \]  \hspace{1cm} (8)

\[ \frac{d\sigma_{r\varphi}}{dt} = \frac{[R(t)/R_0]^{1/2} s(t)}{(C_2 + \gamma C_4)[R(t)/R_0]^{1/2} - C_2 + \gamma C_4} \left[ \frac{C_2 - \gamma C_4}{(\rho/R_0)^{1/2}} - \left( \frac{C_2 + \gamma C_4}{(\rho/R_0)^{1/2}} \right)^{1/2} \right] \equiv S_r(\rho, t), \]
where \( \gamma = \sqrt{C_1/C_4} \) is the dimensionless material constant which can be called, in its mechanical sense, the index of anisotropy of the composite material obtained as a result of winding (it would be \( \gamma = 1 \) if the material were fully isotropic).

After solving the problem given by (3), (2), (1), (4), and (7), it is possible to determine how the fields of technological stresses in the resulting composite will evolve during its winding. To do this, one should integrate expressions (8) in time taking into account initial conditions (6):

\[
\begin{align*}
\sigma_p &= \sigma_c(\rho) + \int_{\tau_c(\rho)}^t S_p(\rho, \tau) \, d\tau, \\
\sigma_z &= \int_{\tau_c(\rho)}^t S_z(\rho, \tau) \, d\tau, \\
\sigma_\rho &= \int_{\tau_c(\rho)}^t S_\rho(\rho, \tau) \, d\tau. 
\end{align*}
\]

The initial stresses in the applied elementary layers of additional material, caused by pre-tension of the reinforcing fibres, can be calculated in the proposed model of mechanics of continuous additive processes as follows. Let \( F(R) \) be the pre-tension force acting on the fibre wound onto the surface of the being formed solid when this solid has an external radius \( R \). Let \( \kappa(R) \) be the packing density of reinforcing fibres (the number of fibres per unit area of the axial section) in the vicinity of this fibre. Then the function \( \sigma_c(R) \) in boundary condition (5) can be set as

\[
\sigma_c(R) = \kappa(R) F(R).
\]

The analytical dependences obtained above make it possible to formulate various problems on technological control of the process stresses developing in filament-wound cylindrical composites. One of the goals of such controlling may be to minimize the risk of destruction of the composite both during its manufacture and in particular conditions of further work [18]. Constructed formulas (8), (9) also allow us to determine the evolution of the contact pressure from the wound layer of composite material onto the form used during its winding, which is extremely important for analyzing the stability and wear of the latter [19–23]. In addition, the developed model provides a simple apparatus for solving the problem on predicting the residual stresses in manufactured composites of the considered kind after their separation from the form having been used as technological tool. These stresses are unavoidable in every additive process of obtaining materials due to the very specifics of such processes and are explained by strain incompatibility of different material elements making up the solid being formed in a such process [24].

4. Conclusion

A non-classical mathematical model of solid mechanics has been developed to predict and control the evolution of technological stresses in cylindrical composites during their additive manufacturing by multi-layer winding of fibre-reinforced material. The model takes into account the geometric scheme of fibre packing and the possibility to arbitrarily variate the force of their pre-tension and the packing density. The corresponding boundary value problem in terms of the stress-strain state characteristics rates in the being manufactured composite solid is stated. This problem exact solution is constructed, on the basis of which the development of stress fields in the obtained composite is reconstituted throughout the entire manufacturing process for concrete programs of changing the control parameters of the simulated technology.

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