BOSON STARS: EARLY HISTORY AND RECENT PROSPECTS

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Boson stars are descendants of the so–called geons of Wheeler, except that they are built from scalar particles instead of electromagnetic fields. If scalar fields exist in nature, such localized configurations kept together by their self-generated gravitational field can form within Einstein’s general relativity. In the case of complex scalar fields, an absolutely stable branch of such non-topological solitons with conserved particle number exists. Our present surge stems from the speculative possibility that these compact objects could provide a considerable fraction of the non-baryonic part of dark matter. In any case, they may serve as a convenient “laboratory” for studying numerically rapidly rotating bodies in general relativity and the generation of gravitational waves.

1 Introduction

If scalar fields exist in nature, soliton-type configurations kept together by their self-generated gravitational field can form absolutely stable boson stars (BS), resembling neutron stars. They are descendants of the so–called geons of Wheeler.

We will review the history of these hypothetical stars, starting 1968 with the work of Kaup as well as that of Ruffini and Bonazzola. In building macroscopic boson stars, a nonlinear Higgs type potential was later considered as an additional repulsive interaction. Thereby the Kaup limit for boson stars can even exceed the limiting mass of 3.23 $M_{\odot}$ for neutron stars.

Moreover, in the spherically symmetric case, we have shown via catastrophe theory that these boson stars have a stable branch with a wide range of masses and radii.

Recently, we constructed for the first time the corresponding localized rotating configurations via numerical integration of the coupled Einstein–Klein–Gordon equations. Due to gravito–magnetic effect, the ratio of conserved angular momentum and particle number turns out to be an integer $a$, the azimuthal quantum number of our soliton–type stars. The resulting axisymmetric metric, the energy density and the Tolman mass are completely regular. Moreover, we analyze the differential rotation and stability of such fully relativistic configurations.

The present surge stems from the possibility that these localized objects could provide a considerable fraction of the non-baryonic part of dark matter.
1.1 Geons in general relativity

Transferring the ideas of Mach and Einstein to the microcosmos, the curving up of the background metric should be self-consistently produced by the stress-energy content $T_{\mu\nu}$ of matter via the Einstein equations with cosmological term. In some geometrodynamical models, extended particles owning internal symmetries were classically described by objects which closely resemble geons or wormholes. The geon, i.e. a gravitational electromagnetic entity, was originally devised by Wheeler to be a self-consistent, nonsingular solution of the otherwise source-free Einstein–Maxwell equations having persistent large-scale features. It realizes to some extent the proposal of Einstein and Rosen in their 1935 paper:

"Is an atomistic theory of matter and electricity conceivable which, while excluding singularities in the field, makes use of no other fields than those of the gravitational field ($g_{\mu\nu}$) and those of the electromagnetic field in the sense of Maxwell (vector potentials $\phi_\mu$)?"

The Lagrangian density of gravitational coupled Maxwell field $A := A_\mu dx^\mu$ reads

$$\mathcal{L}_{\text{geon}} = \frac{1}{2\kappa} \sqrt{|g|} R - \frac{1}{4} \sqrt{|g|} F_{\mu\nu} F^{\mu\nu},$$

where $\kappa = 8\pi G$ is the gravitational constant in natural units, $g$ the determinant of the metric $g_{\mu\nu}$, $R := g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} \left( \partial_\nu \Gamma^\sigma_{\mu\alpha} - \partial_\mu \Gamma^\sigma_{\nu\alpha} + \Gamma^\sigma_{\mu\beta} \Gamma^{\beta\alpha}_{\nu\sigma} - \Gamma^\sigma_{\nu\beta} \Gamma^{\beta\alpha}_{\mu\sigma} \right)$ the curvature scalar with Tolman’s sign convention, and $F = dA = (1/2) F_{\mu\nu} dx^\mu \wedge dx^\nu$. Greek indices $\mu, \nu, \cdots$ are running from 0 to 3.

Such a geon provides a well-defined model for a classical body in general relativity exhibiting “mass without mass”. If spherically symmetric geons would stay stable, the possibility would arise to derive the equations of motions for the center of gravity solely from Einstein’s field equations without the need to introduce field singularities. In a sense this approach also achieves some of the goals of the so-called unitary field theory.

Geons, as we are using the term, are gravitational solitons, which are held together by self-generated gravitational forces and are composed of localized fundamental classical fields. The coupling of gravity to neutrino fields has already been considered by Brill and Wheeler. It lays the appropriate groundwork for an extension to nonlinear spinor geons satisfying the combined Einstein–Dirac equations. In previous papers however, algebraic complications resulting from the spinor structure as well as from the internal symmetry are avoided by considering, instead, interacting scalar fields coupled to gravity. In order to maintain a similar dynamics, a scalar self-interaction $U(\Phi)$ is assumed which can be formally obtained by ”squaring” the fundamental nonlinear spinor equation. Klein–Gordon geons have been previously constructed by Kaup. However, the additional nonlinearity of the scalar fields turns out to be an important new ingredient.

1.2 Do scalar fields exist in nature?

The physical nature of the spin–0–particles out of which the boson star (BS) is presumed to consist, is still an open issue. Until now, no fundamental elementary scalar particle has been found in accelerator experiments, which could serve as the
main constituent of the boson star. In the theory of Glashow, Weinberg, and Salam, a Higgs boson–dublett ($\Phi^+, \Phi^0$) and its anti-dublett ($\Phi^-, \overline{\Phi}^0$) are necessary ingredients in order to generate masses for the $W^\pm$ and $Z^0$ gauge vector bosons. After symmetry breaking, only one scalar particle, the Higgs particle $H := (\Phi^0 + \overline{\Phi}^0)/\sqrt{2}$, remains free and occurs as a state in a constant scalar field background. Nowadays, it is indicated by the rather heavy top quark of 176 GeV/$c^2$, one expects the mass of the Higgs particle to be close to 1000 GeV/$c^2$. However, above 1.2 TeV/$c^2$ the self–interaction $U(\Phi)$ of the Higgs field is so large that the perturbative approach of the standard model becomes unreliable. Therefore a conformal extension of the standard model with gravity included may be necessary, see. High–energy experiments at the LHC at Cern will reveal if these Higgs particles really exist in nature.

As free particles, the Higgs boson is unstable with respect to the decays $H \rightarrow W^+ + W^-$ and $H \rightarrow Z^0 + Z^0$. In a compact object like the boson star, these decay channels are expected to be in equilibrium with the inverse process $Z^0 + Z^0 \rightarrow H$, for instance. This is presumably in full analogy with the neutron star or quark star, where one finds an equilibrium of $\beta$– and inverse $\beta$–decay of the neutrons or quarks and thus stability of the macroscopic star with respect to radioactive decay. Nishimura and Yamaguchi constructed a neutron star using an equation of state of an isotropic fluid built from Higgs bosons.

2 Boson stars

In a perspective paper Kaup has studied for the first time the full generally relativistic coupling of linear Klein–Gordon fields to gravity in a localized configuration. It is already realized that no Schwarzchild type event horizon occurs in such numerical solutions. Moreover, the problem of the stability of the resulting scalar geons with respect to radial perturbations is treated. It is shown that such objects are resistant to gravitational collapse (related works include Refs. 24, 34, 133). These considerations are on a semiclassical level, since the Klein–Gordon field is treated as a classical field. However, using a Hartree–Fock approximation for the second quantized two–body problem, Ruffini and Bonazzola showed that the same coupled Einstein–Klein–Gordon equations apply. Exact but singular solutions of the coupled Maxwell–Einstein–Klein–Gordon equation have been constructed before by Das.

The Lagrangian density of gravitationally coupled complex scalar field $\Phi$ reads

$$\mathcal{L}_{BS} = \frac{\sqrt{|g|}}{2\kappa} \left\{ R + \kappa \left[ g^{\mu\nu}(\partial_\mu \Phi^*)(\partial_\nu \Phi) - U(|\Phi|^2) \right] \right\}.$$  \hspace{1cm} (2)

Using the principle of variation, one finds the coupled Einstein–Klein–Gordon equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T^\mu_\nu(\Phi),$$  \hspace{1cm} (3)

$$\left( \square + \frac{dU}{d|\Phi|^2} \right) \Phi = 0,$$  \hspace{1cm} (4)
where
\[ T_{\mu\nu}(\Phi) = \frac{1}{2}(\partial_{\mu}\Phi^*)(\partial_{\nu}\Phi) + (\partial_{\mu}\Phi)(\partial_{\nu}\Phi^*) - g_{\mu\nu}L(\Phi)/\sqrt{|g|} \quad (5) \]

is the stress–energy tensor and \( \Box := \left(1/\sqrt{|g|}\right) \partial_{\mu} \left(\sqrt{|g|} g^{\mu\nu} \partial_{\nu}\right) \) the generally covariant d’Alembertian.

The stationarity ansatz
\[ \Phi(r, t) = P(r)e^{-i\omega t} \quad (6) \]
describes a spherically symmetric bound state of the scalar field with frequency \( \omega \).

In the case of spherical symmetry, the line-element reads
\[ ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} \left[ dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right], \quad (7) \]
in which the functions \( \nu = \nu(r) \) and \( \lambda = \lambda(r) \) depend on the Schwarzschild type radial coordinate \( r \).

In the years following the geons of Wheeler, some efforts were also made in order to find a (semi–) classical model describing elementary particles. In 1968, Kaup presented the notion of the ‘Klein–Gordon geon’, which nowadays has been christened mini–boson star. It can be regarded as a macroscopic quantum state.

As in the case of a prescribed Schwarzschild background, the spacetime curvature affects the resulting Schrödinger equation for the radical function \( P(r) \) essentially via an external gravitational potential. Indeed Feinblum and McKinley found eigensolutions with nodes corresponding to the principal quantum number \( n \) of the H–atom. They also realized that localized solutions fall off asymptotically as \( P(r) \sim (1/r) \exp(-\sqrt{m^2 - E^2} r) \) in a Schwarzschild-type asymptotic background.

The energy–momentum tensor becomes diagonal, i.e. \( T_{\mu\nu}(\Phi) = \text{diag} (\rho, -p_r, -p_\perp, p_\perp) \) with
\[ \rho = \frac{1}{2}(\omega^2 P^2 e^{-\nu} + P'^2 e^{-\lambda} + U), \]
\[ p_r = \rho - U, \]
\[ p_\perp = p_r - P'^2 e^{-\lambda}. \quad (8) \]
This form is familiar from fluids, except that the radial and tangential pressure generated by the scalar field are in general different, i.e. \( p_r \neq p_\perp \), due to the different sign of \( (P')^2 \) in these expressions.

In general, the resulting system of three coupled nonlinear equations for the radial parts of the scalar and the (strong) gravitational tensor field has to be solved numerically. In order to specify the starting values for the ensuing numerical analysis, asymptotic solutions at the origin and at spatial infinity are instrumental.

That the stress–energy tensor of a BS, unlike a classical fluid, is in general anisotropic has already been noticed by Kaup. In contrast to neutron stars, where the ideal fluid approximation demands an isotropic symmetry for the pressure, for spherically symmetric boson stars there are different stresses \( p_r \) and \( p_\perp \) in radial or tangential directions, respectively. Ruffini and Bonazzola introduced
the notion of fractional anisotropy $a_f := (p_r - p_\perp)/p_r = P^2e^{-\lambda}/(\rho - U)$ which depends essentially on the self-interaction; cf. Ref. \[44\].

So the perfect fluid approximation is totally inadequate for boson stars. Actually, Ruffini and Bonazzola \[110, 12\] used the formalism of second quantization for the complex Klein–Gordon field and observed an important feature: If all scalar particles are within the same ground state $|\Phi> = (N, n, l, a) = (N, 0, 0, 0)$, which is possible because of Bose–Einstein statistics, then the semi-classical Klein–Gordon equation of Kaup is recovered in the Hartree–Fock approximation. In contrast to the Newtonian approximation, the full relativistic treatment avoids an unlimited increase of the particle number and negative energies, but induces critical masses and particle numbers with a global maximum.

There exists a decisive difference between self–gravitating objects made of fermions or bosons: For a many fermion system the Pauli exclusion principle forces the typical fermion into a state with very high quantum number, whereas many bosons can coexist all in the same ground state (Bose–Einstein condensation). This also reflects itself in the critical number of stable configurations:

- $N_{\text{crit}} \simeq (M_{\text{Pl}}/m)_3$ for fermions
- $N_{\text{crit}} \simeq (M_{\text{Pl}}/m)_2$ for massive bosons without self–interaction.

Cold mixed boson–fermion stars have been studied by Henrique et al. \[55\] and Jetzer \[61\].

2.1 Gravitational atoms as boson stars

In a nutshell, a boson star is a stationary solution of a (non-linear) Klein–Gordon equation in its own gravitational field; cf. \[91, 93\]. We treat this problem in a semi-classical manner, because effects of the quantized gravitational field are neglected. Therefore, a (Newtonian) boson star is also called a gravitational atom \[35\]. Since a free Klein–Gordon equation for a complex scalar field is a relativistic generalization of the Schrödinger equation, we consider for the ground state a generalization of the wave function

$$|N, n, l, a>: \Phi = R_n^a(r) Y_l^{[a]}(\theta, \varphi)e^{-i(E_n/\hbar)t}$$

$$= \frac{1}{\sqrt{4\pi}} R_n^a(r) P_l^{[a]}(\cos \theta) e^{ia\varphi} e^{-i(E_n/\hbar)t} \quad (9)$$

of the hydrogen atom. Here $R_n^a(r)$ is the radial distribution, $Y_l^{[a]}(\theta, \varphi)$ the spherical harmonics, $P_l^{[a]}(\cos \theta)$ are the normalized Legendre polynomials, and $|a| \leq l$ are the quantum numbers of azimuthal and angular momentum.

Thus ‘gravitational atoms’ represent coherent quantum states, which nevertheless can have macroscopic size and large masses. The gravitational field is self-generated via the energy–momentum tensor, but remains completely classical, whereas the complex scalar fields are treated to some extent as Schrödinger wave functions, which in quantum field theory are referred to as semi-classical.

Motivated by Heisenberg’s non-linear spinor equation \[54, 94\], additional self-interacting terms describing the interaction between the bosonic particles in a “geon”
type configuration were first considered by Mielke and Scherzer \cite{95}, where also solutions with nodes, i.e. “principal quantum number” \( n > 1 \) and non-vanishing angular momentum \( l \neq 0 \) for a t’Hooft type monopole ansatz \( \Phi^I \sim R(r) P^I_l(\cos \theta) \) were found. An analytical solution of the coupled Einstein-scalar-field system also exists. Recently Rosen \cite{109} reviewed his old idea of an elementary particle built out of scalar fields within the framework of the Klein–Gordon geon or the mini–boson star.

Further analysis is needed in order to understand these highly interesting instances of a possible fine structure in the energy levels of gravitational atoms. In view of these rich and prospective structures, are quantum geons \cite{141} capable of internal excitations?

In building macroscopic boson stars, Colpi et al. \cite{20} used a Higgs–type self-interaction in order to accommodate a repulsive force besides gravity. This repulsion between the constituents is instrumental to blow up the boson star so that much more particles will have room in the confined region. Thus the maximal mass of a BS can reach or even extend the limiting mass of 3.23 \( M_\odot \) for neutron stars \cite{28} with realistic equations of state \( p = p(\rho) \) for which the (phase) velocity of sound is \( v_s = \sqrt{dp/d\rho} \leq c \). However, this fact depends on the strength of the self-interaction. The exciting possibility of having cold stars with very large \( n \) may add another thread to the question of black hole formation.

The work of Friedberg et al. \cite{39} renewed the interest in the study of boson stars. They investigated the Newtonian limit, analyzed in more detail the solutions with higher nodes of Feinblum et al. \cite{34} and Mielke and Scherzer \cite{95}, and in a preliminary form, stability questions. Several surveys \cite{62,80,129} summarize the present status of the non-rotating case.

2.2 Critical masses of boson stars

The Noether theorem associates with each symmetry a locally conserved current \( \partial_\mu j^\mu = 0 \) and “charge”. The first “constant of motion” of our coupled system of equations is given by the invariance of the Lagrangian density under a global phase transformation \( \Phi \rightarrow \Phi e^{-i\vartheta} \) of the complex scalar field. From the associated Noether current \( j^\mu \) arises the particle number:

\[
N := \int j^0 d^3x \, , \quad j^\mu = \frac{i}{2} \sqrt{|g|} g^{\mu\nu}[\Phi^* \partial_\nu \Phi - \Phi \partial_\nu \Phi^*] \, . \tag{10}
\]

For the total gravitational mass of localized solutions we use Tolman’s expression \cite{135,47}:

\[
M = \int (2T^0_0 - T^\mu_\mu) \sqrt{|g|} d^3x \, . \tag{11}
\]

Since boson stars are macroscopic quantum states, they are prevented from complete gravitational collapse by the Heisenberg uncertainty principle.

This provides us also with crude mass estimates: For a boson to be confined within the star of radius \( R_0 \), the Compton wavelength has to satisfy \( \lambda_\Phi = (2\pi\hbar/mc) \leq 2R_0 \). On the other hand, the star’s radius should be of the order of the last stable Kepler orbit \( 3R_\odot \) around a black hole of Schwarzschild radius
In order to avoid an instability against complete gravitational collapse.

For a mini–boson star of effective radius \( R_0 \approx (\pi/2)^2 R_S \) close to its Schwarzschild radius one obtains the estimate

\[
M_{\text{crit}} \approx (2/\pi) M_{\text{Pl}}^2/m \geq 0.633 M_{\text{Pl}}^2/m, \tag{12}
\]

cf. Ref. 62, which provides a rather good upper bound on the so-called Kaup limit. The correct value in the second expression was found only numerically as a limit of the maximal mass of a stable mini–boson star. Here \( M_{\text{Pl}} := \sqrt{\hbar c/G} \) is the Planck mass and \( m \) the mass of a bosonic particle. For a mass of \( m = 30 \text{ GeV}/c^2 \), one can estimate the total mass of this mini–boson star to be \( M \approx 10^{10} \text{ kg} \) and its radius \( R_0 \approx 10^{-17} \text{ m} \). This amounts to a density \( 10^{48} \) times that of a neutron star.

This result was later extended by Colpi et al. 20 for the bosonic potential

\[
U(|\Phi|) = m^2 |\Phi|^2 + (\lambda/2)|\Phi|^4. \tag{13}
\]

with an additional quartic self–interaction. Since \( |\Phi| \sim M_{\text{Pl}}/\sqrt{8\pi} \) inside the boson star, one finds the energy density

\[
\rho \approx m^2 M_{\text{Pl}}^2 (1 + \Lambda/8), \quad \text{where} \quad \Lambda := \frac{\lambda}{4\pi} \frac{M_{\text{Pl}}^2}{m^2}. \tag{14}
\]

This corresponds to a star formed from non–interacting bosons with rescaled mass \( m \to m/\sqrt{1 + \Lambda/8} \). Consequently, the maximal mass of a stable BS scales with the coupling constant \( \Lambda \) approximately as

\[
M_{\text{crit}} \approx \frac{2}{\pi} \sqrt{1 + \Lambda/8} \frac{M_{\text{Pl}}^2}{m} \to \frac{1}{\pi\sqrt{2}} \frac{\Lambda M_{\text{Pl}}^2}{m} \quad \text{for} \quad \Lambda \to \infty, \tag{15}
\]

cf. Fig. 2 from Colpi et al. 20.

For \( m \approx 1 \text{ GeV}/c^2 \) of the order of the proton mass and \( \Lambda \approx 1 \), this is in the range of the Chandrasekhar limiting mass \( M_{\text{Ch}} := M_{\text{Pl}}^3/m^2 \approx 1.5 M_\odot \), where \( M_\odot \) denotes the mass of the sun.

| Compact Object | Critical mass \( M_{\text{crit}} \) | Particle Number \( N_{\text{crit}} \) |
|----------------|---------------------------------|----------------------------------|
| Fermion Star:  | \( M_{\text{Ch}} := M_{\text{Pl}}^3/m^2 \) | \( \sim (M_{\text{Pl}}/m)^3 \) |
| Mini–BS:       | \( M_{\text{Kaup}} = 0.633 M_{\text{Pl}}^2/m \) | \( 0.653 (M_{\text{Pl}}/m)^2 \) |
| Boson Star:     | \( (1/\pi\sqrt{8\pi})\sqrt{\Lambda} M_{\text{Pl}}^3/m^2 \) | \( \sim (M_{\text{Pl}}/m)^3 \) |
| Soliton Star:   | \( 10^{-2}(M_{\text{Pl}}^3/m^3 \Phi_0^2) \) | \( 2 \times 10^{-3}(M_{\text{Pl}}^3/m^2 \Phi_0^2) \) |

In astrophysical terms, this maximal mass is \( M_{\text{crit}} \approx 0.06 \sqrt{\Lambda} M_{\text{Pl}}^3/m^2 \approx 0.1 \sqrt{\Lambda} (\text{GeV}/\text{mc}^2)^2 M_\odot \).

For light scalars, this value can even exceed the limiting mass of \( 3.23 M_\odot \) for a neutron star (NS). For cosmologically relevant (invisible) axions of \( m_a \approx 10^{-5} \) GeV/mc, one obtains

\[
M_{\text{crit}} \approx 0.06 \sqrt{\Lambda} M_{\text{Pl}}^3/m^2 \approx 0.1 \sqrt{\Lambda} (\text{GeV}/\text{mc}^2)^2 M_\odot.
\]
eV an axion star with the ridiculously large mass of $M_{\text{crit}} \sim 10^{27} \sqrt{\lambda} M_\odot$ would be possible and stable.\footnote{13}

For a \textit{dilaton star} built from a very light dilaton $\chi$ of mass $m_{\text{dil}} = 10^{-11} \text{ eV}/c^2$, Gradwohl and Kälbermann\footnote{48} found

$$M_{\text{crit}} = 7 \sqrt{\lambda} M_\odot, \quad R_{\text{crit}} = 40 \sqrt{\lambda} \text{ km}, \quad (16)$$

where $\bar{\lambda}$ is the rescaled coupling constant of the $\chi^4$ interaction.

Therefore, if scalar fields would exist in nature, such compact objects could even question for massive compact objects the observational black hole paradigm in astrophysics.

### 2.3 Stability and catastrophe theory

For such soliton-type configurations kept together by their self-generated gravitational field, the issue of stability is crucial. In the spherically symmetric case, it was shown by Gleiser\footnote{44, 45}, Jetzer\footnote{61}, and Lee & Pang\footnote{79} that boson stars having masses below the Kaup limit are stable against small radial perturbation. More recently, we have demonstrated via catastrophe theory\footnote{74, 119, 75} that this \textit{stable branch} is even absolutely stable. Moreover, our present surge stems from the possibility that these compact objects with a wide range of masses and radii could provide a considerable fraction of the non-baryonic part of dark matter\footnote{126, 117}; see below. Charged boson stars and their induced vacuum instabilities have been studied in Refs.\footnote{59, 60, 64}. The problem of non-radial pulsations of a boson star has been mathematically formulated\footnote{71}.

### 2.4 Boson star formation

The possible abundance of solitonic stars with astrophysical mass but microscopic size could have interesting implications for galaxy formation, the microwave background, and formation of protostars.

Therefore it is an important question if boson stars can actually form from a primordial bosonic “cloud”\footnote{134}. (The primordial formation of non-gravitating non-topological solutions was studied by Frieman et al.\footnote{41}.)

As Seidel and Suen\footnote{124, 126} have shown, cf. Fig.\footnote{5}, there exists a dissipation-less relaxation process they call \textit{gravitational cooling}. Collisionless star systems are known to settle to a centrally denser system by sending some of their members to larger radius. Likewise, a bosonic cloud will settle to a unique boson star by ejecting part of the scalar matter. Since there is no viscous term in the KG equation\footnote{4}, the radiation of the scalar field is the only mechanism. This was demonstrated numerically by starting with a spherically symmetric configuration with $M_{\text{initial}} \geq M_{\text{Kaup}}$, i.e. which is more massive then the Kaup limit. Actually such oscillating and pulsating branches have been predicted earlier in the stability analysis of Kusmartsev, Mielke, and Schunck\footnote{73, 119} by using catastrophe theory. Oscillating soliton stars were constructed by using real scalar fields which are periodic in time\footnote{22}. Without spherical symmetry, i.e. for $\Phi \sim R_a(r) Y_l^a(\theta, \varphi)$, the emission of gravitational waves would also be necessary.
For a real (pseudo–) scalar field like the axion, the outcome is quite different. The axion has the tendency to form compact objects (oscillatons) in a short time scale. Due to its intrinsic oscillations it would be, contrary to a BS, unstable. Since the field disperses to infinity, finite non-singular self-gravitating solitonic objects cannot be formed with a massless Klein–Gordon field \[ \phi \] , but, instead, solutions with an infinite range can be found where the mass increases linearly \[ \Delta = \frac{(n-1)(M_{\text{Pl}}/m)^2}{2} \]. These solutions can be used to fit the observed rotation curves for dwarf and spiral galaxies; see also the contribution to the Dark Matter session of Schunck in these proceedings. Similar investigations using excited BS states were used in \[ \text{[23, 24, 25]} \]. In Ref. \[ \text{[26]} \] a different mechanism for forming axion miniclusters and starlike configurations was proposed. For fermionic soliton stars, the temperature dependence in the forming of cold configurations has also been studied \[ \text{[27]} \].

2.5 Gravitational waves

A boson star is an extremely dense object, since non-interacting scalar matter is very “soft”. However, these properties are changed considerably by a repulsive self-interaction \( U(\Phi) \).

In the last stages of boson star formation, one expects that first a highly excited configuration forms in which the quantum numbers \( n, l \) and \( a \) of the gravitational atom, i.e. the number \( n - 1 \) of nodes, the angular momentum and the azimuthal angular dependence \( e^{ia\phi} \) are non-zero.

In a simplified picture of BS formation, all initially high modes have eventually to decay into the ground state \( n = l = a = 0 \) by a combined emission of scalar radiation and gravitational radiation.

In a Newtonian approximation \[ \text{[28]} \], the energy released by scalar radiation from states with zero quadrupole moment can be estimated by

\[
E_{\text{rad}} \sim (n-1)M_{\text{Pl}}^2/m, \quad \Delta N \sim (n-1)(M_{\text{Pl}}/m)^2.
\]
This is accompanied by a loss of boson particles with the rate \( \Delta N \) given above.

The lowest mode which has quadrupole moment and therefore can radiate gravitational waves is the \( 3d \) state with \( n = 3 \) and \( l = 2 \). For \( \Delta j = 2 \) transitions, it will decay into the \( 1s \) ground state with \( n = 1 \) and \( l = 0 \) while preserving the particle number \( N \). The radiated energy is quite large, i.e., \( E_{\text{rad}} = 2.9 \times 10^{22} \) (GeV/mc\(^2\)) Ws. Thus the final phase of the BS formation would terminate in an outburst of gravitational radiation despite the smallness of the object.

### 2.6 Rotating boson stars

In recent papers \([12], [16], [20], [23]\), we proved numerically that rapidly rotating boson stars with \( a \neq 0 \) exist in general relativity. Because of the finite velocity of light and the infinite range of the scalar matter within the boson star, our localized configuration can rotate only differentially, but not uniformly. Thus our new axisymmetric solution of the coupled Einstein–Klein–Gordon equations represent the field-theoretical pendant of rotating neutron stars which have been studied numerically for various equations of state and different approximation schemes \([40], [22], [32]\) as a model for (millisecond) pulsars; cf. the paper of Schunck and Mielke in these proceedings.

Kobayashi et al. \([67]\) tried to find slowly rotating states (near the spherically symmetric ones) of the boson star, but they failed. The reason for that is a quantization of the relation of angular momentum and particle number \([24]\). In Newtonian theory, boson stars with axisymmetry have been constructed by several groups. Static axisymmetric boson stars, in the Newtonian limit \([23]\) and in general relativity (GR) \([24]\), show that one can distinguish two classes of boson stars by their parity transformation at the equator. In both approaches only the negative parity solutions revealed axisymmetry, while those with positive parity merely converged to solutions with spherical symmetry. The metric potentials and the components of the energy-momentum tensor are equatorially symmetric despite of the antisymmetry of the scalar field. In the Newtonian description, Silveira and de Sousa \([22]\) followed the approach of Ferrell and Gleiser \([14]\) and constructed solutions which have no equatorial symmetry at all. Hence, in GR, we have to separate solutions with and without equatorial symmetry. In a more recent paper, Ryan \([111]\) investigated the gravitational radiation of macroscopic boson stars (with large self-interaction) by taking into account the reduction of the differential equations in this scenario.

### 2.7 Gravitational evolution and observation of boson stars

Recently, several papers appeared which investigated the evolution of boson stars if the external gravitational constant changes its value with time \([137], [138]\), for an earlier investigation cf. \([14]\). This can be outlined within the theory of Jordan–Brans–Dicke or a more general scalar tensor theory. The results show that the mass of the boson star decreases due to a space-depending gravitational constant, given through the Brans–Dicke scalar. The mass of a boson star with constant central density is influenced by a changing gravitational constant. Moreover, the possibility of a gravitational memory of boson stars or a formation effect upon their surrounding has been analyzed as well \([38]\).
The issue of observation has also been recently discussed. Direct observation of boson stars seems to be impossible in the near future. But two effects could possibly give indirect hints. In the outer regions, the rotation velocity of baryonic objects surrounding the boson star can reveal the star’s mass. Assuming that the scalar matter of the BS interacts only gravitationally, we would have a transparent BS detecting a gravitational redshift up to values of $z = 0.68$ observable by radiating matter moving in the strong gravitational potential. For further investigations of rotation curves, cf. Ref. 128, 76.

3 Dilaton stars and kinks

Real massless scalar fields coupled to Einstein gravity are known to admit exact solutions. Already in 1959, Buchdahl \cite{13} found a continuous two-parameter family of static solutions: Accordingly, any static vacuum solution $g_{\mu\nu} = (g_{00}, g_{AB})$ of Einstein’s theory can be mapped into the solution

$$\begin{cases} (g_{00}^{\beta}, g_{00}^{1-\beta} g_{AB}) \\ \Phi = \lambda \ln g_{00} \end{cases}$$

with $\beta = \pm \sqrt{1 - 2\lambda^2}$ (18)

of the Einstein–KG system. For an extension to conformally coupled scalar fields, see Bekenstein \cite{7}.

In the framework of the Jordan–Brans–Dicke–Thiry theory, these solutions appear already in Ref. 28 and correspond to those found by Majumdar \cite{85} for the Einstein–Maxwell system. Later they were rederived by Wyman \cite{145} and for a special case recovered independently by Baekler et al. \cite{4}. Further closed analytical expressions of the so-called Wyman solution \cite{145} are constructed via Computer Algebra in Ref. 113. Generalizations to spacetimes of arbitrary dimensions are reconsidered by Xanthopoulos and Zannias \cite{147} in the spherically symmetric isotropic case, see also Ref. 146 in the case of a conformally coupled scalar field. The global initial value problem for a self-gravitating massless real scalar field has been analyzed by Christodoulou \cite{16} in a spherically symmetric spacetime, see also Choptuik \cite{15} and Gürses \cite{49}. Gürses \cite{49} found conformally flat solutions. For all solutions the scalar field develops a logarithmic singularity at the origin, for some solutions the metric becomes there also singular leading to a naked singularity.

A generally relativistic Klein–Gordon field with an effective $\Phi^3$ self-interaction for an interior ball has also been analyzed \cite{17}. In order to avoid a singular configuration at the origin, a repulsive (or “ghostlike”) scalar field has been chosen as a source of Einstein’s equations. In a further step, Kodama et al. \cite{68,69,70} constructed spherically symmetric kink-type solutions for a repulsive scalar field with a $U(\Phi) \sim (\text{const.} - \Phi^2)^2$ self-coupling (compare also with Ellis \cite{30}). As is common for kinks, the radial function at spatial infinity is chosen to be $\pm \text{const.}$ characterizing this nonlinear model. The constant is necessary in order to eliminate the induced cosmological constant which otherwise would occur for the constant solution characterizing the kink solution asymptotically.

These type of solution, however, suffer from a dynamical instability, see Jetzer and Scialom \cite{63}. In flat spacetime, according to Derrick’s theorem \cite{26}, no stable
time-dependent solutions of finite energy exist for a non-linearly coupled real scalar field.

Other scalar fields arise from axion \( \chi \), inflaton or dilaton fields \( \Phi \) with their corresponding compact objects. Recently, stationary axisymmetric solution of the Einstein–dilaton–axion action are obtained by García et al. \[43\].

In the process of a Kaluza–Klein type dimensional reduction of supergravity or superstring models there arises the dilaton field \( \chi \) as part of the higher–dimensional metric. These real scalar fields couple to gravity in the non–minimal \( \chi^2 R \) fashion, resembling the Brans–Dicke field of scalar–tensor theories. The corresponding dilaton stars \[48\] are stable because of a conserved dilaton current and charge \( Q_{\text{dil}} \) in such models.

In an extended model \[132\] with Higgs field \( \Phi^I \) and dilaton coupling, there occur, however, unstable branches with diverging mass \( M \) for high central values \( |\Phi^I(\theta)| \) of the Higgs field.

4 Other gravitational solitons

To some extent Wheeler’s concept of geons \[140\] has anticipated the (nonintegrable) soliton solutions \[90\] of classical nonlinear field theories. As mentioned in the Introduction, a geon or gravitational soliton originally was meant to consist of a spherical shell of electromagnetic radiation held together by its own gravitational attraction. In the idealized case of a thin spherical geon, cf. Pfister \[103\], the corresponding metric functions have the values \( \exp(\nu_c) = 1/9 \) well inside and \( \exp(\nu) = \exp(\lambda) = 1 - 2m(r)/r \) well outside the active region. The trapping area for the electromagnetic wave trains has a radius of \( r_{\text{act}} = 9m/4 \). This result has been confirmed by applying Ritz variational principles \[33\]. Although this procedure is rather artificial, thereby one obtains a “bag–like” object \[14\] having inside a portion of an Einstein microcosmos and outside a Schwarzschild manifold as background spacetime.

Configurations with toroidal or linear electromagnetic waves have been constructed by Ernst \[33\], the cylindrical geons of Melvin \[87\] are stable against gravitational collapse under large radial perturbations. Neutrino geons have been analyzed by Brill and Wheeler \[9\]. Brill and Hartle \[10\] could even demonstrate the existence of gravitational solitons constructed purely from gravitational waves. By expanding the occurring gravitational waves in terms of tensor spherical harmonics, it can be shown \[106\] that the radial function experiences the same effective potential except that an additional factor appears in front of the contributions from the background metric. In a recent paper, the possibility of black holes formed by collapsed gravitational waves has been discussed \[57,143\]. Although these objects are weakly unstable, they could contribute to dark matter \[57\].

For the generally relativistic kink of Kodama \[69\], the radial solution becomes zero at a certain radius \( r_0 \) at which the background geometry develops a Schwarzschild type horizon. (Geon–type solutions exhibiting an event horizon may be termed “black solitons” \[112\].) The boundary condition at \( r_0 \), however, allows an extension of these solutions into a three-manifold consisting of two asymptotically Euclidean spaces connected by an Einstein–Rosen bridge \[29\]. Arguments are given that this
extended, nonsingular configuration is stable with respect to radial oscillations. It should be noted that such solutions cannot be constructed for the wormhole topology $R \times S^1 \times S^2$ which would be obtainable by identifying the asymptotically flat regions. The reason simply being that the radial functions of the kink has an opposite sign in the other sheet of the Universe.

New wormhole type solutions are discussed by Ellis and by Thorne et al., cf. also Ref. Their throats will, however, be kept open by “exotic matter” which violates the weak energy condition $T_{\mu\nu}u^\mu u^\nu \geq 0$ for timelike vectors $u^\mu$. Then such wormhole configurations would allow closed timelike curves and the paradoxical possibility of a “time machine”.

Although we have no intention to give a complete review, we would like to mention that other studies on geons involve massless scalar fields, coupled Einstein–Maxwell–Klein–Gordon systems, the generally relativistic Dirac equation in an external gravitational field, or even combined Dirac–Einstein–Maxwell field equations.

According to a result of Brill, a massless scalar field can even be geometrized in the sense of the already unified field theory or geometrodynamics of Rainich, Misner, and Wheeler. Loosely speaking, this means that the scalar field can be completely read off from the “footprints” it leaves on the geometry.

The non-topological solitons (NS) of Rosen as well as of Lee and Wick can be regarded as the non-gravitational precursors of boson stars. For a specific Higgs type self-interaction $U(\Phi)$, they are localized solutions of a non-linear Klein–Gordon equation in flat spacetime. Spherically symmetric solutions in a prescribed gravitational background such as that of Schwarzschild or constant curvature were presented in Refs.

Similar configurations are called Q–balls, which are stabilized by the conserved (baryon number) charge $Q$, fermion Q–balls, neutrino balls, and quark nuggets in the case of spinors. Bound further by their self-generated gravitational field, such Q–stars may model neutron stars with an equation of state usually not accessible in the laboratory. Therefore, their mass can also exceed the Chandrasekhar limit of $\sim 3 M_\odot$ for neutron stars.

In quantum chromodynamics (QCD), nowadays the most prominent model for strong interactions, the dynamics of the mediating vector gluons is determined by an action modelled after Maxwell’s theory of electromagnetism. The resulting model is a gauge theory of the Yang–Mills type. However, it is known that in such sourceless non–Abelian gauge theories there are no classical glueballs which otherwise would be an indication for the occurrence of confinement in the quantized theory. The reason simply is that nearby small portions of the Yang–Mills fields always point in the same direction in internal space and therefore must repel each other as like charges.

Monopole type solution of the Einstein–Yang–Mills system are found numerically by Bartnik and McKinnon in which gravity balances the repulsion of the internal gauge fields. An interesting attempt to determine the solution analytically in terms of a series expansion and nonlinear recursion relations is given by Schunck.
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