On the past asymptotic dynamics of non-minimally coupled dark energy

Genly Leon

Department of Mathematics, Universidad Central de Las Villas, Santa Clara CP 54830, Cuba

E-mail: genly@uclv.edu.cu

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Abstract

We apply dynamical system techniques to investigate cosmological models inspired in scalar–tensor theories written in the Einstein frame. We prove that if the potential and the coupling function are sufficiently smooth functions, the scalar field almost always diverges into the past. The dynamics of two important invariant sets are investigated in some detail. By assuming some regularity conditions for the potential and the coupling function, a dynamical system is constructed, well suited to investigating the dynamics where the scalar field diverges, i.e. near the initial singularity. The critical points therein are investigated and the cosmological solutions associated with them are characterized. We find that our system admits scaling solutions. Some examples are taken from the bibliography to illustrate the major results. Also we present asymptotic expansions for the cosmological solutions near the initial spacetime singularity, which extend, in a way, the previous results of other researchers.

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1. Introduction

It is believed that the field responsible for early-time inflation as well as the field driving the current accelerated expansion is in the form of a scalar field (see [1] for the observational status of the acceleration of expansion). Theories including scalar fields, such as scalar–tensor theories (STT) of gravity [2, 3], can be supported by fundamental physical theories such as superstring theory [4]. Otherwise, scalar fields can be viewed merely as convenient heuristic models to elucidate certain (qualitative) dynamical features of the early- and/or the late-time universe1. Quintessential dark energy (DE) models [6], for instance, are (heuristical proposals)

1 See [5] and references therein for the analysis of a scalar field responsible for both the early- and the late-time inflationary expansion in the context of brane cosmology.
described by an ordinary scalar field minimally coupled to gravity. A wise selection of the scalar field self-interacting potentials can drive the current accelerated expansion. Other scalar field models have been treated in the literature (see, for instance, the reviews [7]).

The natural generalizations to models of quintessence evolving independently from the background matter are models which exhibit non-minimal coupling between both components. Experimental tests in the solar system impose severe restrictions on the possibility of non-minimal coupling between DE and ordinary matter fluids [8]. However, we argue that, due to the unknown nature of dark matter (DM), it is possible to have additional (non-gravitational) interactions between the DE and the DM components. This argument does not enter into conflict with the experimental data, but, when the stability of dark energy potentials in quintessence models is considered, the dark matter–dark energy coupling might be troublesome [9].

The interaction between dark energy and dark matter appears when we apply conformal transformations in scalar–tensor theories (STT)\(^2\). About these theories, it is known that they survive several observational tests including solar system tests [10] and Big-Bang nucleosynthesis constraints [12]. Its simplest proposal is the Brans–Dicke theory (BDT) [2], in which a scalar field, \(\chi\), acts as the source for the gravitational coupling with a varying Newtonian ‘constant’ \(G \sim \chi^{-1}\). More general STT with a non-constant BD parameter \(\omega(\chi)\), and non-zero self-interaction potential \(V(\chi)\), have been formulated, and also survive astrophysical tests [8].

The action for a general class of STT, written in the so-called Einstein frame (EF), is given by [13]

\[
S_{EF} = \int_{M_4} d^4x \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2}(\nabla \phi)^2 - V(\phi) + \chi(\phi) - 2 L_{\text{matter}} \right\}.
\]  

(1)

In this equation \(R\) is the curvature scalar, \(\phi\) is the scalar field, related via conformal transformations to the dilaton field, \(\chi\). \(V(\phi)\) is the quintessence self-interaction potential, \(\chi(\phi)^{-2}\) is the coupling function, \(L_{\text{matter}}(\mu, \nabla \mu, \chi^{-1} g_{\alpha\beta})\) is the matter Lagrangian, \(\mu\) is a collective name for the matter degrees of freedom.

By considering the conformal transformation \(\bar{g}_{\alpha\beta} = \chi(\phi)^{-1} g_{\alpha\beta}\) and defining the Brans–Dicke coupling ‘constant’ \(\omega(\chi)\) in such a way that \(d\phi = \pm \sqrt{\omega(\chi)} + 3/2 \chi^{-1} d\chi\) and recalling \(\bar{V}(\chi) = \chi^2 V(\phi(\chi))\) the action (1) can be written in the Jordan frame (JF) as (see [14])

\[
S_{JF} = \int_{M_4} d^4x \sqrt{|\bar{g}|} \left\{ \frac{1}{2} \bar{R} - \frac{1}{2} \frac{\omega(\chi)}{\chi}(\nabla \chi)^2 - \bar{V}(\chi) + L_{\text{matter}}(\mu, \nabla \mu, \bar{g}_{\alpha\beta}) \right\}.
\]  

(2)

Both frames are both formally and physically equivalent [15]. This fact removes previous doubts and fully establishes equivalence at the classical level; however, this does not guarantee physical equivalence at the quantum level [16].

By making use of the conformal equivalence between the Einstein and Jordan frames we can find, for example, that the theory formulated in the EF with the coupling function \(\phi(\chi) = \chi_0 \exp((\phi - \phi_0)/\sigma)\), \(\sigma = \pm \sqrt{\omega_0} + 3/2\) and potential \(V(\phi) = \beta \exp((\alpha - 2)\sigma / (\phi - \phi_0))\) corresponds to the Brans–Dicke theory (BDT) with a power-law potential, i.e., \(\omega(\chi) = \omega_0, \bar{V}(\chi) = \beta \chi^\alpha\). Exact solutions with exponential couplings and exponential potentials (in the EF) were investigated in [17].

In the STT given by (2), the energy–momentum of the matter fields is separately conserved. However, when written in the EF (1), this is no longer the case, although the overall energy

\(^2\) See [11] for applications of conformal transformations in both relativity and cosmology.
density is conserved. In fact in the EF we find that

\[ Q_\beta = -\frac{1}{2} T \frac{\partial \phi \chi(\phi)}{\chi(\phi)} \nabla_\beta \phi, \quad T = T^\alpha_\alpha \]

where

\[ T^{\alpha \beta} = \frac{1}{|g|} \delta^{\alpha \beta} \left\{ \sqrt{|g|} \chi^{-2} \mathcal{L}(\mu, \nabla \mu, \chi^{-1} g_{\mu \nu}) \right\}. \]

In the present investigation we study FRW spacetimes with flat spatial slices, i.e., we consider the line element:

\[ ds^2 = -dt^2 + a(t)^2 \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right). \] (3)

We use a system of units in which \( 8\pi G = c = \hbar = 1 \).

The simplest way of modelling matter and energy in the universe is to consider that the energy–momentum tensor \( T^{\alpha \beta} \) is in the form of a perfect fluid

\[ T^{\alpha \beta} = \text{diag}(\rho, p, p, p), \]

where \( \rho \) and \( p \) are respectively the isotropic energy density and the isotropic pressure (consistently with the FRW metric, pressure is necessarily isotropic [18]). For simplicity we will assume a barotropic equation of state \( p = (\gamma - 1) \rho \).

In this case, the vector field \( Q_\mu \) has only one non-null component, the time component, so it can be written as \( Q_\beta = (Q_0, 0, 0, 0) \). \( Q_0 \) can be considered in some way as a rate of energy exchange between the scalar field and the background.

One of the first papers to take seriously the possibility of interaction in scalar field cosmologies, from the dynamical system perspective, was [19]. In that paper we can find a review on the subject. There the interaction terms (in the flat FRW geometry) \(-Q_0 = -\alpha \rho \) were investigated, where \( \alpha \) is a constant, \( \phi \) is the scalar field, \( \rho \) is the energy density of background matter and \( H \) stands for the Hubble parameter. The first choice corresponds to an exponential coupling function \( \chi(\phi) = \chi_0 \exp(2\alpha \phi/(4 - 3\gamma)) \). The second case corresponds to the choice \( \chi = \chi_0 a^{-2\alpha/(4-3\gamma)} \) (and then \( \rho \propto a^{\alpha-3\gamma} \)), where \( a \) denotes the scale factor of the universe (recall that the former derivations are only valid in a flat FRW model). Other phenomenological coupling functions were studied elsewhere. We would like to draw the attention of the reader to a physically well-motivated approach to the coupling function in [20]. In that paper a coupling term of the form \(-Q_0 = -\alpha \rho \) was investigated, where \( \alpha \) is a constant (\( \Gamma \) in their notation). As commented in that reference, if \( \alpha > 0 \), the model can describe either the decay of dark matter into radiation, the decay of the curvaton field into radiation or the decay of dark matter into dark energy (see section III of [20] for more information and for useful references). In [21], the authors construct a family of viable scalar–tensor models of dark energy (which includes pure \( f(R) \) theories and quintessence). They consider a coupling between the scalar field and the non-relativistic matter in the Einstein frame of the type—in our notation—\( \chi(\phi) = e^{-Q_0 \phi} \), with \( Q \) constant. By investigating a phase space the authors obtain that the model possesses a phase of late-time acceleration preceded by a standard matter era, while at the same time satisfying the local gravity constraints (LGC). In fact, by studying the evolution of matter density perturbations and employing them, the authors place bounds on the coupling of the order \( |Q| < 2.5 \times 10^{-3} \) (for the massless case). By a chameleon mechanism the authors show that these models can be made compatible with LGC even when \( |Q| \) is of the order of unity if the scalar-field potential is chosen to have a sufficiently large mass in the high-curvature regions.

About the dynamics of coupled dark energy, it was found (see [14] and references therein) that typically at early times \( (t \to 0) \) the BDT solutions are approximated by the vacuum
solutions and at late times \((t \to \infty)\) by matter-dominated solutions, in which the matter is dominated by the BD scalar field (denoted by \(\chi\) in the Jordan frame). Exact perfect fluid solutions in STT of gravity with a non-constant BD parameter \(\omega(\chi)\) have been obtained by various authors (see [22]). Coupled quintessence was investigated also in [23] by using dynamical system techniques.

In order to classify the global behavior of the solutions of (1) a detailed knowledge of the form of the scalar field potential (and of the coupling function \(\chi\)) is required. However, up to the present, there exists no consensus about the specific functional form of \(V(\phi)\) (and of \(\chi(\phi)\). As a consequence, it would be of interest to classify the dynamical behavior of solutions without specifying the functional form of the potential function (and of the coupling function). In the literature on general relativity (GR) several attempts have been made in this more general direction. The dynamical system techniques have proven very useful to do so [24–26]. In this respect we know the dynamical behavior of scalar field spacetimes for a wide class of non-negative potentials (within Einstein gravity (EG)).

In this investigation we would like to study, from the dynamical systems point of view, a phenomenological model inspired in a STT with action (1), where the matter and the (quintessence) scalar field are coupled in the action (1) through the scalar–tensor metric \(\chi(\phi)^{-1}g_{ab}\) [13]. We consider arbitrary functional forms for the self-interaction potential and the coupling function for the scalar field \(\phi\). When we take the conformal transformation allowing writing the action in the JF as in (2) the coupling function \(\chi\) should be interpreted as the dilation (BD) field and the corresponding \(\omega(\chi)\) as the varying BD parameter.

By employing Hubble-normalized dynamical variables in addition to the scalar field we find, by investigating the phase space, that the scalar field almost always diverges into the past, allowing us to identify this regime with the physical region in the vicinity of the initial spacetime singularity. For finite values of the scalar field we find that the late-time (early time) attractor is associated with the minimum of the logarithm of the potential (coupling) function.

By assuming some general regularity conditions for the potential and for the coupling function when \(\phi \to \infty\), and using the formalism developed in [24], we are able to explore the phase space corresponding to this limit. We are able to compute the critical points corresponding to that limit and to characterize the cosmological solutions associated with them. Scaling solutions do arise in this regime. Also we are able to characterize the initial singularity in STT. In appendix A an example taken from the literature is given to illustrate the formalism developed for analyzing the region where the scalar fields diverge.

2. The model

In the model described by the action (1), the background energy density does not necessarily correspond to a dark matter component and, analogously, the scalar field energy density does not necessarily correspond to a dark energy component. However, as done many times in the literature (see, for instance, [17, 28]), we assume that \(\phi\) is a quintessence scalar field, which is coupled metrically to a background of a perfect fluid. This model is viable phenomenologically. The possibility of a universal coupling of dark energy to all sorts of matter, including baryons (but excluding radiation) is studied in [29].

3 In [27], the non-negativity of the potential is relaxed and new results within the context of EG generalizing those in [25] are obtained.

4 In appendix A a summary of the main results of the theory of dynamical systems that are used in this paper is provided.
2.1. The field equations

By using the line element (3), the Einstein’s field equations (derived by varying the action (1)) are: (a) the Raychaudhuri equation
\[ \dot{H} = -\frac{1}{2}(\gamma \rho + \dot{\phi}^2), \tag{4} \]
where the dot denotes the derivative with respect to the cosmic time \( t \), \( \rho \) is the energy density of dark matter, (b) the Friedmann equation
\[ 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho, \tag{5} \]
and the continuity equation
\[ \dot{\rho} = -3\gamma H\rho - \frac{1}{2}(4 - 3\gamma) \rho \dot{\phi} \frac{\chi'(\phi)}{\chi(\phi)}. \tag{6} \]

This equation can be integrated in quadratures to give the useful relation
\[ \rho = \rho_0 a^{-3\gamma} \chi^{-2+3\gamma/2}. \]

The equation of motion of the scalar field (written as two differential equations, one for \( \phi \) and the other for \( \dot{\phi} \)) reads
\[ \frac{d\phi}{dt} = \dot{\phi}, \tag{7} \]
\[ \frac{d\dot{\phi}}{dt} = -3H\dot{\phi} - V'(\phi) + \frac{1}{2}(4 - 3\gamma) \rho \frac{\chi'(\phi)}{\chi(\phi)}, \tag{8} \]
where the coma denotes the derivative with respect to the scalar field.

Observe that, if \( \chi = \text{const} \), the equations for the minimally coupled theory are recovered.

From equations (4), (6)–(8) we see that \((H, \rho, \phi, \dot{\phi}) \in \mathbb{R}^4\) remains in the hypersurface defined by the restriction (5). Thus, defining an autonomous system in the phase space
\[ \Omega = \{ (H, \rho, \phi, \dot{\phi}) \in \mathbb{R}^4 : 3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho \}. \tag{9} \]

We shall consider the following general assumptions:
\( V(\phi) \in C^3 \) and \( V(\phi) \geq 0, \chi(\phi) \in C^3 \) with \( \chi(\phi) > 0, \rho \geq 0, 0 < \gamma < 2, \gamma \neq 4/3 \) instead of considering specific choices for both the potential and the coupling function.

3. Qualitative analysis on the Hubble normalized state space

In this section we rewrite equations (4), (6)–(8) as an autonomous system defined on a state space by introducing Hubble-normalized variables. These variables satisfy an inequality arising from the Friedmann equation (5). We analyze the cosmological model by investigating the flow of the autonomous system in a phase space by using dynamical system tools. In order to make the paper self-contained we offer in appendix A some terminology and results from the theory of dynamical systems that we use in the demonstration of our main results.

3.1. Normalized variables

In order to analyze the initial singularity and the late-time behavior it is convenient to normalize the variables, since in the vicinity of a hypothetical initial singularity physical variables would typically diverge, whereas at late times they commonly vanish [30].
Let us introduce the following normalized variables,
\[ x = \frac{1}{H}, \quad y = \frac{\dot{\phi}}{\sqrt{6}H}, \quad z = \frac{\sqrt{\rho}}{\sqrt{3}H} \] (10)
and the time coordinate
\[ d\tau = \frac{3}{H} dt. \] (11)

Besides, if no additional information is available on the functional forms of the coupling and the potential, the most natural variable to add to the former ones is the scalar field itself.

### 3.2. The autonomous system

The field equations (4), (6)–(8) can be used to obtain evolution equations for the variables (10) and the scalar field \( \phi \),

\[ x' = \frac{1}{2}x(2y^2 + z^2 \gamma), \] (12)
\[ y' = y^3 + \frac{1}{2}(z^2 \gamma - 2)y - \frac{x^2 \partial_\phi V(\phi)}{3\sqrt{6}} + \frac{(4 - 3\gamma)z^2}{2\sqrt{6}} \partial_\phi \ln \chi(\phi), \] (13)
\[ z' = \frac{1}{2}z(2y^2 + (z^2 - 1)\gamma) - \frac{(4 - 3\gamma)yz}{2\sqrt{6}} \partial_\phi \ln \chi(\phi), \] (14)
\[ \phi' = \sqrt{\frac{2}{3}} y. \] (15)

where the prime denotes the derivative with respect to \( \tau \). This is an autonomous system where the variables are subject to the constraint
\[ y^2 + z^2 + \frac{1}{3}x^2 V(\phi) = 1. \] (16)

Observe that \( y^2 + z^2 \leq 1 \), since \( V(\phi) \) is non-negative and the restriction (16) holds.

By the 2 in appendix A it is possible to prove that any combination of the sets
\[ x < 0, \quad x = 0, \quad x > 0, \quad z < 0, \quad z = 0, \quad z > 0 \]
is an invariant set of the flow of our dynamical system provided \( \chi \) is at least of class \( C^2 \).

Taking into account this result we can restrict our attention to the flow restricted to the phase space:
\[ \Sigma = \{(\phi, x, y, z) \in \mathbb{R}^4 : x \geq 0, z \geq 0, y^2 + z^2 + 1/3x^2 V(\phi) = 1\}. \] (17)

Observe that if \( x \neq 0 \) and \( V(\phi) > 0 \) we can use the constraint (16) as a definition for \( x \). In this way the evolution equation of \( x \) decouples from the evolution equation of the other variables. Equation (13) can be rewritten as
\[ y' = -\frac{(1 - y^2 - z^2)}{\sqrt{6}} \partial_\phi \ln V(\phi) + \frac{(4 - 3\gamma)z^2}{2\sqrt{6}} \partial_\phi \ln \chi(\phi) + y^3 + \frac{1}{2}(z^2 \gamma - 2)y. \] (18)

Hence, we may then study the dynamical system on \( \mathbb{R}^3 \) given by equations (12), (18) and (14) with the constraint \( y^2 + z^2 < 1 \).
3.3. Main lemma

First, we want to prove the lemma 3.1 by stating that the orbits passing through an arbitrary point \( p \in Z^+X^0 \) interpolate between a regime where the Hubble parameter diverges (containing an initial singularity into the past) and a regime where the background density is negligible (into the future). Those orbits represent cosmological solutions with non-vanishing dimensionless background energy density and finite and positive Hubble parameter. The result stated by the above lemma is obtained by constructing a monotonic function defined on an invariant set and by applying the LaSalle monotonicity principle (theorem 2 in appendix A). This lemma is very important as a tool for investigating the past attractor or \( \omega \)-limit set of the flow (see definitions 1 and 2 in appendix A): it is necessarily located at the invariant set \( \{ x = 0 \} \).

Lemma 3.1. Let be \( Z^+ = \{(x, y, z, \phi) \in \Sigma : z > 0 \} \) and let be \( X^0 = \{(x, y, z, \phi) \in \Sigma : x = 0 \} \). Then for all \( p \in Z^+ - X^0 \) the \( \alpha \)- and \( \omega \)-limit sets of \( p \) are such that \( \alpha(p) \subset X^0 \) and \( \omega(p) \subset \partial Z^+ \), where \( \partial Z^+ \) denotes the boundary of \( Z^+ \).

Proof. By proposition 2 in appendix A we have that \( S = Z^+ - X^0 = \{(x, y, z, \phi) \in \Sigma : z > 0, x > 0 \} \) is an invariant set of the flow (12)–(15). Let the function \( Z(x, y, z, \phi) = \left( \frac{z}{x} \right)^2 \chi(\phi)^{2+2\gamma/2} \) with \( \chi(\phi) > 0 \), be defined on \( S \). It is a monotonic decreasing function (in the direction of the flow) in \( S \), since its directional derivative through the flow is \( Z' = -\gamma Z \) (which is obviously negative). The rank of \( Z \) is \( (0, \infty) \). Let be \( s \in \bar{S} - S = \partial Z^+ \cup X^0 \). It is verified that \( Z(s) \to 0 \) as \( s \to \partial Z^+ \) and \( Z(s) \to \infty \) as \( s \to X^0 \). Hence, applying the LaSalle monotonicity principle (see theorem 2, in appendix A), we have for all \( p \in Z^+ - X^0 \) that \( \alpha(p) \subset X^0 \) and \( \omega(p) \subset \partial Z^+ \), as required. \( \square \)

3.4. The dynamics restricted to the invariant sets \( x = 0 \) and \( z = 0 \).

Now we will characterize the dynamics in both the invariant sets \( X^0 = \{(x, y, z, \phi) \in \Sigma : x = 0 \} \) and \( Z^0 = \{(x, y, z, \phi) \in \Sigma : z = 0 \} \).

The dynamics in the invariant set \( X^0 \) is governed by the differential equations

\[
y' = \frac{1}{2}(1 - y^2) \left( y(y - 2) + \frac{4 - 3\gamma}{\sqrt{6}} \frac{\chi'(\phi)}{\chi(\phi)} \right), \tag{19}
\]

\[
\phi' = \sqrt{\frac{2}{3}} y, \tag{20}
\]

plus the algebraic equation:

\[
y^2 + z^2 = 1. \tag{21}
\]

The only critical point of this system (with \( \phi \) bounded) is the critical point \( Q \) with coordinates \((x, y, z, \phi) = (0, 0, 0, \phi_1)\) with \( \chi'(\phi_1) = 0 \) and \( \chi(\phi_1) \neq 0 \).

The critical point \( Q \) represents matter-dominated cosmological solutions with the Hubble parameter diverging. Since \( \partial_{\phi} \chi(\phi_1) = 0 \), they are solutions with minimally coupled scalar field (with negligible kinetic energy). The potential function is also unimportant in the dynamics.

\( ^5 \) See definition 4 in appendix A.
The eigenvalues of the linearization around $Q$ are $\frac{1}{2}, -\gamma, \Delta_1 \pm \sqrt{\Delta_1^2 + 4\Delta_2 z_0^2 \frac{V''(\phi_1)}{V' \phi}}$, where $\Delta_1 = (-2 + \gamma)/4 < 0$, and $\Delta_2 = (4 - 3\gamma)/6$.

Then the local stability of $Q$ in the invariant set $X^0$ is as follows (we are assuming that the barotropic index $\gamma$ satisfies $0 < \gamma < 2$):

(i) $Q$ is a stable focus if $0 < \gamma < 4/3$ and $\chi''(\phi_1) < -\frac{\Delta_1^2 \chi(\phi_1)}{\Delta_2}$ or $4/3 < \gamma < 2$ and $\chi''(\phi_1) > -\frac{\Delta_1^2 \chi(\phi_1)}{\Delta_2}$.

(ii) $Q$ is a stable node if $0 < \gamma < 4/3$ and $0 > \chi''(\phi_1) \geq -\frac{\Delta_1^2 \chi(\phi_1)}{\Delta_2}$ or $4/3 < \gamma < 2$ and $0 < \chi''(\phi_1) \leq -\frac{\Delta_1^2 \chi(\phi_1)}{\Delta_2}$.

(iii) $Q$ is a saddle point if $0 < \gamma < 4/3$ and $\chi''(\phi_1) > 0$ or $4/3 < \gamma < 2$ and $\chi''(\phi_1) < 0$.

(iv) $Q$ is non-hyperbolic, if $\chi''(\phi_1) = 0$, in which case, there exists a one-dimensional stable manifold which is tangent to the axis $y$ at $Q$ (the associated eigenvector is $e_y = (1, 0)$). There exists also a one-dimensional center manifold tangent to the line $(1 - \gamma/2)y - \sqrt{2/3}\phi = 0$ at $Q$.

The dynamics in the invariant set $Z^0$ is governed by the differential equations:

$$y' = (y^2 - 1) \left(y + \frac{\sqrt{6} \partial_\phi V(\phi)}{6 V(\phi)}\right),$$

and (20), plus the equation

$$y^2 + 1/3x^3 V(\phi) = 1,$$

where $V(\phi)$ is given as input.

The only critical point (with $\phi$ bounded) in the invariant set $Z^0$ is the critical point $P$ with coordinates $x = \frac{1}{V(\phi_2)}$, $y = 0$, $z = 0$, $\phi = \phi_2$ with $\chi'(\phi_2) \neq 0$, $V'(\phi_2) = 0$. The eigenvalues of the linearization around $P$ are: $0$, $-\chi$, $-\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \frac{4}{3} V''(\phi_2) / V(\phi_2)}$. The zero eigenvalue has the associated eigendirection $e_y$.

The local behavior of the critical point $P$ in the invariant set $Z^0$ is as follows:

(i) $P$ is a saddle if $V''(\phi_2) < 0$.

(ii) $P$ is a stable node if $0 < V''(\phi_2) \leq \frac{1}{2} V(\phi_2)$, and

(iii) $P$ is a stable focus if $V''(\phi_2) > \frac{1}{2} V(\phi_2)$.

(iv) $P$ is non-hyperbolic (in the invariant set $Z^0$) if $V''(\phi_2) = 0$.

When the orbits located at $Z^0$ approach this critical point, the energy density of DM and the kinetic energy density of DE tend to zero. In this case the energy density of the universe will be dominated by the potential energy of DE. Hence, the universe would be expanding forever in a de Sitter phase.

3.5. Conditions for the divergence of the scalar field backward in time

Now we will prove theorem 3.2 which, essentially, states that if the potential and the coupling function are sufficiently smooth functions, then for almost all the points lying in a four-dimensional state space, the scalar field diverges when the orbit through $p$ is followed backward in time. This theorem is an extension of theorem 1 in [24] to STT.

**Theorem 3.2.** Assume that $\chi(\phi)$ and $V(\phi)$ are positive functions of class $C^3$. Let $p$ be a point in $\Sigma$, and let $O^{-}(p)$ be the past orbit of $p$ under the flow of (12)–(15) with constraint (16). Then $\phi$ is almost always unbounded on $O^{-}(p)$ for almost all $p$. 

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**Proof.** Let \( p \in \Sigma \) such that \( \phi \) is bounded on \( O^- (p) \). It follows that \( O^- (p) \) is contained on a compact subset of (the closure of) \( \Sigma \). Hence, that trajectory must asymptotically approach some limit set \( \alpha (p) \) (see the analogous of theorem 1 in appendix A for \( \alpha \) limit sets). From (12) it follows that \( x \) is a monotonic increasing function through the flow and then it must be constant in \( \alpha (p) \).

There are two possibilities (i) \( y = z = 0 \) at \( \alpha (p) \) or (ii) \( x = 0 \) at \( \alpha (p) \).

The only invariant set with \( y = z = 0 \) (and \( \phi \) bounded) is the critical point \( P \).

Now we will prove that \( P \) is not the past asymptote of an open set of orbits of \( \Sigma \).

Observe that at least one of its associated eigenvalues has always negative real part. Hence, by the center manifold theorem (theorem 1 in appendix A), we can conclude that there exists an invariant stable manifold, \( E^s \), of \( P \), intersecting \( P \). The existence of a stable manifold of dimension \( r > 0 \) implies that all solutions asymptotically approaching \( P \) in the past (i.e., those ones approaching \( P \) as \( \tau \to -\infty \)) are in an invariant unstable manifold or a center manifold of dimension \( 4 - r < 4 \).

Then the only possibility is that \( \alpha (p) \) is contained in \( x = 0 \) (i.e., it is contained in the circumference \( y^2 + z^2 = 1 \)).

The only invariant sets for the flow given by (19)–(21) are the hypersurfaces \( |y| = 1 \) and the critical point \( Q \). Then there are two possibilities: \( \alpha (p) = Q \) or the \( \alpha \)-limit set of \( p \) lies on the hypersurface \( |y| = 1 \). In the last case, from (20), we have that \( \phi \) is unbounded on \( \alpha (p) \), a contradiction.

In order to complete the proof we need only to demonstrate that \( Q \) is not the past asymptote to an open set of trajectories in \( \Sigma \).

In fact, at least one of its associated eigenvalues has negative real part. In view of the center manifold theorem, we can conclude that there exists a local stable manifold, \( E^s \), that intersects \( Q \) of dimension \( s > 0 \). All orbits close to \( Q \) in this set exponentially approach \( Q \) as \( \tau \to +\infty \). As before, the existence of a local stable manifold of dimension \( s > 0 \) implies that all solutions past asymptotic to \( Q \) must lie on an unstable manifold or center manifold of dimension \( 4 - s < 4 \). □

Theorem 3.2 allow us to conclude that in order to investigate the generic asymptotic behavior of the system (12)–(15) with restriction (16) it is necessary to study the region where \( \phi = \pm \infty \). However, as has been investigated in [32] (where results from [25] are extended), the region \( \phi = \pm \infty \) is not exclusively associated with the asymptotic behavior toward the past. In fact, the scalar field can diverge toward the future, provided additional requirements under the potential and the coupling function are fulfilled. But, this will be the purpose of a forthcoming paper.

### 3.6. The flow near \( \phi = +\infty \)

In this section we will investigate the flow near \( \phi = +\infty \) following the nomenclature and formalism introduced in [24]. Analogous results hold near \( \phi = -\infty \).

By assuming that \( V, \chi \in E^2_k \), with exponential orders \( N \) and \( M \) (the set of all class-\( k \) WBI functions; see definitions 5 and 6 in appendix B) respectively, we can define a dynamical system well suited to investigate the dynamics near the initial singularity. We will investigate the critical points therein, particularly, those representing scaling solutions and one associated with the initial singularity.

Let \( \Sigma _e \subset \Sigma \) be the set of points in \( \Sigma \) for which \( \phi > \epsilon ^{-1} \), where \( \epsilon \) is any positive constant which is chosen sufficiently small so as to avoid any points where \( V \) or \( \chi = 0 \), thereby ensuring
that $\overline{W}_V(\varphi)$ and $\overline{W}_X(\varphi)$ are well-defined (see definitions of $W$’s and of hatted functions in appendix B).

We now make the coordinate transformation

$$\begin{align*}
(x, y, z, \phi) & \mapsto (x, y, z, \psi) \\
\psi = f(\phi)
\end{align*}$$

(24)
on $\Sigma_e$, where $f(\phi)$ tends to zero as $\phi$ tends to $+\infty$ and has been chosen so that the conditions (i)–(iii) of definition 6 in appendix B are satisfied with $k = 2$.

Substituting these new coordinates into equations (12), (14) and (18) we obtain the three-dimensional dynamical system:

$$y' = y^3 + \frac{1}{2}(z^2y - 2)y - \frac{(1 - y^2 - z^2)}{\sqrt{6}}(\overline{W}_V + N) + \frac{z^2(4 - 3y)}{2\sqrt{6}}(\overline{W}_X + M),$$

(25)

$$z' = \frac{1}{2}(2z^2 + (z^2 - 1)y) + \frac{yz(-4 + 3y)}{2\sqrt{6}}(\overline{W}_X + M),$$

(26)

$$\psi' = \sqrt{\frac{2}{3}} f' y.$$  

(27)

We may identify $\Sigma_e$ with its projection into $\mathbb{R}^3$ so that we have $\Sigma_e = \{ 0 < \varphi < f(\varphi^{-1}), 0 \leq y^2 + z^2 < 1 \}$. The variable $x$ can be treated as a function on $\Sigma_e$ defined by the constraint equation which becomes

$$y^2 + z^2 + 1/3x^2 = 1.$$  

(28)

The directional derivative of $x$ along the flow generated by (25)–(27) may be obtained directly by equation (12).

Since $f'$, $\overline{W}_V$ and $\overline{W}_X$ are $C^2$ at $\varphi = 0$ we may extend (25)–(27) onto the boundary of $\Sigma_e$ to obtain a $C^2$ system on the closure of $\Sigma_e$, i.e., $\overline{\Sigma}_e$. From definition 6, $f'$, $\overline{W}_V$ and $\overline{W}_X$ vanish at the origin and are each of second order or higher in $\varphi$ and $f'$ is negative on $\Sigma_e$.

3.6.1. Critical points. The system (25)–(27) admits the critical points labeled by $P_i$, $i \in \{1, 2, 3, 4, 5, 6\}$. In the following we discuss the existence and the stability conditions for the critical points. In table 1 are displayed the values of some cosmological magnitudes of interest for the critical points (the deceleration parameter, the effective equation of state (EoS) parameter for the total matter, etc).

(i) The critical point $P_1$ with coordinates $y = -1, z = 0$ and $\varphi = 0$ exists for all the values of the free parameters. The eigenvalues of the linearized system around $P_1$ are

| Point | $y$ | $z$ | $\Omega_{de}$ | $w_{tot}$ | Acceleration? |
|-------|-----|-----|---------------|-----------|---------------|
| $P_1$ | $-1$ | $0$ | $1$           | $1$       | No            |
| $P_2$ | $1$  | $0$ | $1$           | $1$       | No            |
| $P_3$ | $\delta$ | $\sqrt{1 - \delta^2}$ | $\delta^2$ | $y + (y - 1)\delta$ | $0 < y < \frac{1}{2}$ and $|M| < \Gamma$ |
| $P_4$ | $-\frac{N}{\sqrt{2}}$ | $0$ | $1$           | $-1 + \frac{N^2}{2}$ | $N^2 < 2$ |
| $P_{5,6}$ | $-\frac{6\sqrt{2}}{\beta}y^2$ | $\pm \sqrt{\frac{2(2\alpha + \beta_1 - 3\beta_2)}{\beta}}$ | $-2\alpha + \beta_1 \pm \frac{3\alpha \beta_2}{\beta}$ | $1 + \frac{\beta_2 y - (\beta_2 + \beta_3) y^2 - 3\beta_2^2}{\beta^2}$ | $\frac{\beta}{\beta_2} - 1$ |

Table 1. The properties of the critical points for the system (25)–(27). We use the notations $\alpha = 3(N(y - 2) + M(3y - 4))$, $\beta = 2(2N - M(3y - 4))$, $\delta = \frac{N(3y - 4)}{\sqrt{3y - 2} - 4 - 3y}$, and $\Gamma = \frac{\sqrt{2}\sqrt{2(2\alpha + \beta_1 - 3\beta_2)}}{\beta}$.
\( \lambda_{1,1} = 2 - \sqrt{2/3}N, \lambda_{1,2} = \frac{2 - \gamma}{2} - \frac{M(4 + 3y)}{2x}, \) and \( \lambda_{1,3} = 0. \) Hence the critical point is non-hyperbolic, then the Hartman–Groban theorem does not apply. By the center manifold theorem there exist:

(a) a stable invariant subspace of dimension two (tangent to the \( y-z \) plane) if: (i) the potential is a WBI function (see definition 5 in appendix B) of exponential order \( N > \sqrt{6} \) and the coupling function is a WBI function of exponential order \( M < -\frac{\sqrt{6}(y-2)}{3y-4} \) (provided \( 0 < \gamma < \frac{4}{3} \)), or (ii) the barotropic index satisfies \( \frac{4}{3} < \gamma < 2 \), the potential is a WBI function of exponential order \( N > \sqrt{6} \) and the coupling function is a WBI function of exponential order \( M > -\frac{\sqrt{6}(y-2)}{3y-4} \) (respectively, \( M < -\frac{\sqrt{6}(y-2)}{3y-4} \)) provided \( 0 < \gamma < \frac{4}{3} \) (respectively, \( \frac{4}{3} < \gamma < 2 \));

(b) an unstable invariant subspace of dimension two (tangent to the \( y-z \) plane) provided the potential is a WBI function of exponential order \( N < \sqrt{6} \) and the coupling function is a WBI function of exponential order \( M < -\frac{\sqrt{6}(y-2)}{3y-4} \) (respectively, \( M > -\frac{\sqrt{6}(y-2)}{3y-4} \)) provided \( 0 < \gamma < \frac{4}{3} \) (respectively, \( \frac{4}{3} < \gamma < 2 \));

(c) a one-dimensional center manifold which is tangent to the critical point in the direction of the axis \( \varphi \). This center manifold can be two-dimensional or even three-dimensional (see the discussion on point 3).

(ii) The critical point \( P_3 \) with coordinates \( y = 1, z = 0 \) and \( \varphi = 0 \) exists for all the values of the free parameters. The eigenvalues of the linearized system around \( P_3 \) are \( \lambda_{2,1} = 2 + \sqrt{2/3}N, \) \( \lambda_{2,2} = \lambda_{1,2} \) and \( \lambda_{3,3} = 0 \) (see point 1). Hence the critical point is non-hyperbolic, then the Hartman–Groban theorem does not apply. By the center manifold theorem there exist:

(a) a stable invariant subspace of dimension two (tangent to the \( y-z \) plane) if: (i) \( N < -\sqrt{6}, M > \frac{\sqrt{6}(y-2)}{3y-4} \) for \( 0 < \gamma < \frac{4}{3} \), or (ii) \( \frac{4}{3} < \gamma < 2, N < -\sqrt{6} \) and \( M < \frac{\sqrt{6}(y-2)}{3y-4} \);  

(b) an unstable invariant subspace of dimension two (tangent to the \( y-z \) plane) provided \( N > -\sqrt{6} \), and \( M < \frac{\sqrt{6}(y-2)}{3y-4} \) (respectively \( M > \frac{\sqrt{6}(y-2)}{3y-4} \)) provided \( 0 < \gamma < \frac{4}{3} \) (respectively \( \frac{4}{3} < \gamma < 2 \));

(c) a one-dimensional center manifold which is tangent to the critical point in the direction of the axis \( \varphi \). This center manifold can be two-dimensional or even three-dimensional (see the discussion on point 3).

In the following section we shall study the initial spacetime (big bang) singularity. The critical points \( P_{1,2} \) can account for that singularity. They are in the same phase portrait for the values \( -\sqrt{6} < N < -\sqrt{6} \) and \( -\sqrt{6} < M < -\sqrt{6} \) and \( 0 < \gamma < \frac{4}{3} \) (in which case they have a two-dimensional unstable manifold and a one-dimensional center respectively). It is easy to show that the Hubble parameter (and the matter density) of the cosmological solutions associated with these points diverges into the past. The scalar field also diverges, it equals \( +\infty \) (respectively \( -\infty \)) for \( P_1 \) (respectively \( P_2 \)). However, even in this case, the past attractor corresponds to \( P_1 \) since \( \ddot{f} < 0 \) and for \( y > 0 \) the orbits enter the phase portrait and \( P_2 \) acts as a saddle. The last point can be a past attractor only on a set of measure zero (if \( \varphi = 0 \)).

(iii) The critical point \( P_4 \) with coordinates \( y = \frac{M(4+3y)}{\sqrt{6}(2-3y)}, z = \sqrt{1 - \frac{M(4+3y)}{6(2-3y)}} \) and \( \varphi = 0 \) exists if \( 0 < \gamma < \frac{4}{3} \) and \( -\frac{\sqrt{6}(2-3y)}{4-3y} \leq M \leq \frac{\sqrt{6}(2-3y)}{4-3y} \). The eigenvalues of the matrix of derivatives evaluated at the critical point are \( \lambda_{3,1} = \frac{6(y-2)^2 - M^2(4+3y)^2}{12(y-2)^2}, \lambda_{3,2} = -\frac{3M^2}{4} + (M+N)M + \frac{2(N-M)^2M}{3(y-2)^2} = y \) and \( \lambda_{3,3} = 0. \) Hence the critical point is non-hyperbolic, then
the Hartman–Grobman theorem does not apply. Under the above existence conditions we find, by the center manifold theorem, that there exists a stable manifold of dimension two for the values of the parameters: (i) \( M < 0 \) and \( N > \frac{M'(4-3y^2-6y-2y')}{2(My-y^2)} \), or (ii) \( M > 0 \) and \( N < \frac{M'(4-3y^2-6y-2y')}{2(My-y^2)} \). Otherwise there exists an unstable manifold of dimension one (in this case the stable subspace is one-dimensional). The center manifold is in both cases one-dimensional. If \( M = \pm \frac{\sqrt{6(2+y)}}{4+3y} \) this critical point reduces to \( P_{1,2} \). In this case the center subspace is two-dimensional and is spanned by the eigenvectors \( e_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_\varphi = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \). The center manifold is tangent to the center subspace at the critical point. If additionally \(|\gamma| = \sqrt{6} \), the center manifold is three-dimensional.

(iv) The critical point \( P_4 \) with coordinates \( y = -\frac{N}{\sqrt{6}}, z = 0 \) and \( \varphi = 0 \) exists if \(|\gamma| \leq \sqrt{6} \). Observe that this point reduces to \( P_{1,2} \) if \( N^2 = 6 \). The matrix of derivatives evaluated at the critical point has the eigenvalues \( \lambda_{4,1} = \frac{1}{2}(N^2 - 6) \leq 0 \), \( \lambda_{4,2} = \frac{1}{2}N(2M + N) - \frac{1}{2}(MN + 2) \gamma \) and \( \lambda_{4,3} = 0 \). Hence the critical point is non-hyperbolic and, as before, the Hartman–Grobman theorem does not apply. However, we can use the center manifold theorem to investigate the stability of this critical point. The structure of the center manifold is as follows:

(a) if \( \lambda_{4,1} < 0 \) and \( \lambda_{4,2} \neq 0 \) the center manifold is spanned by \( e_\gamma \). Then it is one-dimensional. Before analyzing this case in detail, we will provide additional information about the structure of the center manifold.

(b) if \( M = \frac{2(N^2-3y)}{N(3y-4)} \) and \( N^2 < 6 \), the center subspace is spanned by the eigenvectors \( e_\gamma \) and \( e_\varphi \).

(c) if \( N^2 = 6 \) and \( M \neq \pm \frac{\sqrt{6(2+y)}}{4+3y} \) it is spanned by the eigenvectors \( e_\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) and \( e_\varphi \).

(d) if \( N^2 = 6 \) and \( M = \pm \frac{\sqrt{6(2+y)}}{4+3y} \), the center manifold is three-dimensional.

The local behavior described in the above cases (excluding the first case) is in some way special. It requires fine tuning of the free parameter. However, the typical behavior (in the invariant set \( z = 0 \)) is the existence of a one-dimensional center manifold \( C_N \) through \( P_4 \), which is tangent to the \( z \)-axis (if \( \lambda_{4,1} < 0 \) and \( \lambda_{4,2} \neq 0 \)). \( C_N \) is an exponential attractor on a sufficiently small neighborhood of \( P_4 \). It is intuitively obvious from the geometry (for instance, observe figure B1) that any solutions past asymptotic to \( P_4 \) must lie on the center manifold.

Let us investigate the case in which \( \lambda_{4,1} < 0 \) and \( \lambda_{4,2} \neq 0 \). Of course, in this case the stable manifold is at least one-dimensional (and as we mentioned before the center manifold is one-dimensional).

The structure of the stable subspace is as follows:

(a) if the potential is of exponential order zero (\( N = 0 \)), then the critical point has coordinates \( (0,0,0) \). The eigenvalues of the linearization are \( (-1,0,-\frac{\gamma}{2}) \) and in this case, the stable subspace is generated by the eigenvectors \( e_\gamma, e_\varphi \);

(b) if \( 0 < \gamma < \frac{1}{4}, -\sqrt{6} < N < 0 \), and \( M > \frac{2(N^2-3y)}{N(3y-4)} \); or

(c) if \( \frac{1}{4} < \gamma < 2, -\sqrt{6} < N < 0 \), and \( M < \frac{2(N^2-3y)}{N(3y-4)} \); or

(d) if \( 0 < \gamma < \frac{4}{7}, 0 < N < \frac{4}{7} \), and \( M < \frac{2(N^2-3y)}{N(3y-4)} \); or

(e) if \( \frac{4}{7} < \gamma < 2, 0 < N < \sqrt{6} \), and \( M > \frac{2(N^2-3y)}{N(3y-4)} \) the stable subspace is generated by the eigenvectors \( e_\gamma, e_\varphi \).
(f) By interchanging $>$ and $<$ in the inequalities for $M$ in cases (b)–(e) we find that the stable manifold is one-dimensional and is tangent to the critical point in the direction of $e_\gamma$ (accordingly, the unstable subspace is spanned by $e_\gamma$).

(v) The critical points $P_{5,6}$ with coordinates $y = \frac{\sqrt{\nu}}{M(3\gamma - 4) - 2M}$, $z = \frac{\pm \sqrt{3M(2M + N) - 6(2M - 4)\gamma}}{2N + M(3\gamma - 4) - 2M}$ (respectively) exist if the following conditions are simultaneously satisfied: $4N(2M + N) - 6(2M + 4\gamma) > 0$, $\mp (2N + M(4 - 3\gamma)) > 0$ and $\frac{4N^2 + M(8 - 6\gamma)N + 6(\gamma - 2)\gamma}{(2N + M(3\gamma - 4))}$ $\leq 1$ (i.e., the critical points are real-valued, and they are inside the cylinder $\Sigma_e$).

The associated eigenvalues are

$$\lambda_{5,6}^\pm = \frac{a}{b} \pm \frac{\sqrt{8(\beta^2 + 2\gamma^2)\alpha^2 - 2\beta(\gamma - 4)(\beta^2 - 216\gamma^2)\alpha - (\gamma - 2)(\beta^2 - 216\gamma^2)\beta}}{6\sqrt{6}\beta}$$

and $\lambda_{5,6} = 0$, where $a = 3(N(\gamma - 2) + M(3\gamma - 4))$ and $b = 2(2N - M(3\gamma - 4))$. Assuming that the conditions for existence are satisfied, we can analyze the stability of the critical points by means of the center manifold theorem. We find that the non-null eigenvalues cannot be either complex conjugates with positive real parts or real-valued with different sign, then the unstable subspace of $P_{5,6}$ is the empty set. Then the stable subspace is two-dimensional (provided $\lambda_{5,6}$ is the only null eigenvalue). When the orbits are restricted to this invariant set, the point $P_{5,6}$ acts as a stable spiral (if the eigenvalues are complex conjugated) or as a node (if the eigenvalues are negative reals). The conditions on the parameters for those cases are very complicated to display here.

If $M = \frac{2(\gamma - 3)}{N(\gamma - 4)}$, the $P_{5,6}$ reduces to $P_4$ but in this case, the center manifold is two-dimensional and is spanned by $e_\gamma$, $e_\phi$.

3.7. The flow near $\phi = -\infty$

With the purpose of complementing the global analysis of the system it is necessary to investigate its behavior near $\phi = -\infty$. It is an easy task since the system (4), (6)–(8) is invariant under the transformation of coordinates

$$(\phi, \phi) \to -(\phi, \phi), \quad V \to U, \quad \chi \to \Xi,$$

where $U(\phi) = V(-\phi)$ and $\Xi(\phi) = \chi(-\phi)$. Hence, for a particular potential $V$, and a particular coupling function $\chi$, the behavior of the solutions of the equations (4), (6)–(8) around $\phi = -\infty$ is equivalent (except for the sign of $\phi$) to the behavior of the system near $\phi = \infty$ with potential and coupling functions $U$ and $\Xi$, respectively.

If $U$ and $\Xi$ are of class $C^1$, the preceding analysis in $\Sigma_e$ can be applied (with an adequate choice of $\epsilon$).

In the following we will denote by $E^k$ the set of class-$C^k$ functions well behaved in both $+\infty$ and $-\infty$. We will use Latin uppercase letters with subscripts $+\infty$ and $-\infty$, respectively, to indicate the exponential orders of $E^k$ functions in $+\infty$ and $-\infty$.

3.8. The global geometric structure of the phase space

Let $\Omega(x_0)$ be the region of the phase space given by (17) with $x < x_0$, then since $x$ is monotonic decreasing, this set equals the union of its past orbits.

The procedure for defining a coordinate system near $-\infty$, given in section 3.7, can be used to embed $\Omega(x_0)$ as a compact differentiable four-dimensional manifold $\Sigma(x_0)$ such that the vector field defined by (12)–(15) can be smoothly extended over $\Sigma(x_0)$.

With this purpose we define an atlas as follows.
First, the interior of $\Sigma(x_0)$ is defined as the set
\[
\{(\phi, x, y, z) \in \mathbb{R}^4 : 0 < x < x_0, y^2 + z^2 < 1\}
\]
where we use the local chart (coordinate system) given by (10). Obviously, this set is bounded in the variables $x, y, z$. We define a second local chart $(\phi, x, y, z)$ in the open subset of this set for which $x > \epsilon$ for $\epsilon$ small enough, by (24). An analogous local chart $(\phi, x, y, z)$ can be defined near $\phi = -\infty$ by the procedure given in section 3.7.

The construction is completed by attaching a boundary which is defined taking the union of $x = 0, x = x_0, \phi = 0$ and the circumference $y^2 + z^2 = 1$ to each local chart. By construction, $\Sigma(x_0)$ is compact and it is embedded in $\mathbb{R}^4$.

The vector field defined by (12)–(15) can be smoothly extended over the boundary of $\Sigma(x_0)$ such that $\Sigma(x_0)$ is the union of its past orbits. $\Omega(x_0)$ is a three-dimensional hypersurface embedded in $\Sigma(x_0)$. It is important to note that $\Omega(x_0)$ approaches the non-physical boundary along the intersection of the plane $x = 0$ with the plane $\phi = 0$ and the circumference $y^2 + z^2 = 1$. This set is called non-physical boundary of $\Omega$ and denoted by $\partial\Omega$.

4. The initial spacetime singularity

In this section we will study the initial spacetime (big-bang) singularity. The critical points $P_1, P_2$ can represent such a singularity. They are at the same phase space for the values of $M, N$ and $\gamma$ in the intervals $-\sqrt{6} < N < \sqrt{6}, -\sqrt{\frac{6}{\gamma - 4}} < M < \sqrt{\frac{6}{\gamma - 4}}$ and $0 < \gamma < \frac{4}{3}$ (in this case, they have an unstable two-dimensional manifold and a center one-dimensional manifold). It is easy to show that the Hubble parameter and the matter energy density of the associated cosmological solutions diverge toward the past. The scalar field diverges too, and it is equal to $+\infty$ and $-\infty$ for $P_1$ and $P_2$ respectively. However, even in this case, the possible past attractor corresponds to $P_1$ since $f' < 0$ whereas for $y > 0$ the orbits enter the phase space and $P_2$ acts as a saddle. The critical point $P_2$ can act as a past attractor only in a set of initial conditions of measure zero (when $\phi = 0$).

4.1. Analysis near $P_1$

From the analysis in section 3.6.1, it seems reasonable to think that the initial spacetime singularity can be associated with the critical point $P_1$. Its unstable manifold is two-dimensional provided $N < \sqrt{6}$. The asymptotic behavior of neighboring solutions to $P_1$ can be approximated, for $\tau$ negative large enough, as
\[
y(\tau) = -1 + O(e^{\lambda_1 \tau}), \quad z(\tau) = O(e^{\lambda_1 \tau}).
\]
By substitution of (29) into (15), and integrating the resulting equation, we obtain
\[
\phi(\tau) = \sqrt{\frac{2}{3}}(-\tau + \bar{\phi}) + O(e^{\lambda_1 \tau}).
\]
Then by expanding around $\tau = -\infty$ up to first order, we get
\[
\phi = f\left(\sqrt{\frac{2}{3}}(-\tau + \bar{\phi}) + O\left(\frac{1}{\tau}\right)^2\right) + O(e^{\lambda_1 \tau}) = f\left(\sqrt{\frac{2}{3}}(-\tau + \bar{\phi})\right) + O(e^{\lambda_1 \tau}) + h,
\]
where $h$ denotes higher order terms to be discarded.

Then we have a first-order solution to (25)–(27). Also, by substitution of (29) into (12) and solving the resulting differential equation with the initial condition $x(0) = x_0$ we get the
first-order solution
\[ x = x_0 e^\tau. \] (31)

Then we have
\[ t - t_i = \frac{1}{3} \int \frac{x(\tau)}{x} d\tau = \frac{1}{3} x_0 e^{\frac{\tau}{\sqrt{3}}} \]
for simplicity let us set \( t_i = 0 \).

Neglecting the error terms, we have the following expressions,
\[ H = x^{-1} = (x_0 e^{\tau})^{-1} = \frac{1}{3t}, \]
\[ \phi = \frac{2}{3} (-\tau + \phi) = -\sqrt{\frac{2}{3}} \ln \frac{t}{c}, \]
\[ \dot{\phi} = -\sqrt{\frac{2}{3}} t^{-1}, \rho = 0, \]
where \( c = 1/3x_0 e^{\phi} \).

This asymptotic solution corresponds to the exact solution of (4), (6)–(8) when \( V \) vanishes identically and \( \chi \) is a constant (the minimal coupling case). Hence, there exists a generic class of massless minimally coupled scalar field cosmologies in the vicinity of the initial spacetime singularity.

The above idea can be stated, more precisely, as the
Theorem 4.1. Let \( V \in \mathcal{E}_2^+ \) with the exponential order \( N \) satisfying \( N < \sqrt{6} \) and let \( \chi \in \mathcal{E}_2^+ \) with the exponential order \( M \) such that

(i) \( 0 < \gamma < \frac{4}{3} \) and \( M > -\frac{\sqrt{6}(\gamma - 2)}{\gamma^2 - 4} \) or
(ii) \( \frac{4}{3} < \gamma < 2 \) and \( M < -\frac{\sqrt{6}(\gamma - 2)}{\gamma^2 - 4} \).

Then there exists a neighborhood \( N(P_1) \) of \( P_1 \) such that for each \( p \in N(P_1) \) the orbit \( \psi_p \) past asymptotic to \( P_1 \) and the associated cosmological solution is:
\[ H = \frac{1}{3t} + O(\epsilon_V(t)), \] (33)
\[ \phi = -\sqrt{\frac{2}{3}} \ln \frac{t}{c} + O(t\epsilon_V(t)), \] (34)
\[ \dot{\phi} = -\sqrt{\frac{2}{3}} t^{-1} + O(\epsilon_V(t)), \] (35)
\[ \rho = \chi_0^2 t^{-\frac{\gamma}{2}} \chi \left( -\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{\gamma}{2}} (1 + O(t\epsilon_V(t))) \] (36)

where \( \epsilon_V(t) = tV(-\sqrt{\frac{2}{3}} \ln \frac{t}{c}) \).

Before proceeding to the proof of this theorem, let us make a few comments. Since \( V \in \mathcal{E}_2^+ \) has exponential order \( N \), then by applying theorem 2 in [24] we have
\[ \lim_{t \to 0} t^\alpha V \left(-\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right) = \lim_{\phi \to \infty} e^{-\sqrt{\frac{2}{3}} \phi} V(\phi) = 0, \quad \forall \alpha > \frac{2}{\sqrt{3}} N. \]

Then for \( N < \sqrt{6} \) the error terms \( O(\epsilon_V(t)) \) and \( O(t\epsilon_V(t)) \) are dominated by the first-order terms. If \( N < \sqrt{3} \) both error terms tend uniformly to zero.

On the other hand, since \( \chi \in \mathcal{E}_2^+ \) has exponential order \( M \), then \( t^{-\gamma} \chi \left(-\sqrt{\frac{2}{3}} \ln \frac{t}{c} \right)^{\frac{\gamma}{2}} \) tends uniformly to zero, as \( t \to 0 \) in cases (i) \( 0 < \gamma < \frac{4}{3} \) and \( M > -\frac{\sqrt{6}(\gamma - 2)}{\gamma^2 - 4} \) or (ii) \( \frac{4}{3} < \gamma < 2 \) and \( M < -\frac{\sqrt{6}(\gamma - 2)}{\gamma^2 - 4} \).

In the above cases, the matter tends uniformly to zero as \( P_1 \) is approached.
Proof of theorem 4.1. From equation (12) and using (28) as a definition for $y$ we get the equation

$$\frac{d\ln x}{d\tau} = \left(\frac{\gamma}{2} - 1 \right) z^2 + 1 - \frac{1}{3} x^2 V(\phi).$$

Using the first-order expressions $z = O(e^{\lambda_1 \tau})$ and $x = x_0 e^{\tau}$ we have the differential equation for $x$:

$$\frac{d\ln x}{d\tau} = 1 - \frac{1}{3} x_0^2 V(\phi) e^{2\tau} + h. \quad (37)$$

where we denote by $h$ any collection of higher order terms to be discarded.

By integrating both sides of (37) we find the solution

$$x = x_0 e^{\tau} \left(1 - \frac{x_0^2}{3\lambda_{1,1}} V(\phi) e^{2\tau}\right) + h. \quad (38)$$

In the above deduction we have used the auxiliary result proved in [24]:

$$\int V(\phi) e^{2\tau} d\tau = \frac{V(\phi) e^{2\tau}}{\lambda_{1,1}} + h, \quad (39)$$

which is valid if $V \in \mathcal{E}^2$ with exponential order $N < \sqrt{6}$. We have used, also, the approximation $e^u \approx 1 + u$.

Now we want to derive a second-order expression for $t$,

$$t = \frac{1}{3} \int x(\tau) d\tau = \frac{1}{3} \int \left( x_0 e^{\tau} \left(1 - \frac{x_0^2}{3\lambda_{1,1}} V(\phi) e^{2\tau}\right) \right) d\tau + h$$

$$= \frac{1}{3} x_0 e^{\tau} \left(1 - \frac{x_0^2}{9\lambda_{1,1}} V(\phi) e^{2\tau}\right) + h. \quad (40)$$

In the above deduction we have used the auxiliary result proved in [24]:

$$\int V(\phi) e^{3\tau} d\tau = \frac{1}{3} V(\phi) e^{3\tau} + h \quad (41)$$

Equation (40) may be inverted, to second order, to give

$$x_0 e^{\tau} = 3 \left(t + \frac{V(\phi)}{\lambda_{1,1}} \right)^3 + h.$$

Substituting this result into (38) we get

$$x(t) = 3t - \frac{6V(\phi)t^3}{\lambda_{1,1}} + h. \quad (42)$$

and then

$$H(t) = \frac{1}{x(t)} = \frac{1}{3t} + \frac{2V(\phi)t}{3\lambda_{1,1}} + h. \quad (43)$$

Equation (25) can be written as

$$\frac{d\ln y}{d\tau} = \left(-1 + \frac{\gamma}{2} \right) z^2 - \frac{1}{3} x^2 V(\phi) - \frac{(1 - y^2 - z^2)}{\sqrt{6}y} (W + N) + \frac{z^2(4 - 3\gamma)}{2\sqrt{6}y} (W - M).$$
By the same arguments as in the deductions of (38) and (42) we get

\[ y = -1 + \frac{x_0^2}{3\lambda_{1,1}} e^{2\tau} V(\phi) + h \]

\[ = -1 + \frac{3V(\phi)t^2}{\lambda_{1,1}} + h. \quad (44) \]

Combining expansions (44) and (42) in \( \dot{\phi}(t) = \sqrt{\frac{3}{2}} y \), we find

\[ \dot{\phi}(t) = -\sqrt{\frac{3}{2}} \left( \frac{1}{t} - \frac{tV(\phi)}{\lambda_{1,1}} \right) + h. \quad (45) \]

This equation can be integrated up to second order to get

\[ \phi(t) = -\sqrt{\frac{3}{2}} \left( \frac{\ln t}{e} - \frac{V(\phi)t^2}{2\lambda_{1,1}} \right) + h. \quad (46) \]

Equation (26) can be written as

\[ \frac{d\ln z}{d\tau} = -\frac{1}{3} x^2 V(\phi) + \left( 1 - \frac{y}{2} \right) (1 - z^2) + \frac{y(-4 + 3y)}{2\sqrt{6}} W(\phi), \]

where we have used the constraint equation (28) as a definition for \( y^2 \).

By the same arguments as in the deductions of (38) and (42) we get

\[ z = z_0 \exp \left( \lambda_{1,2} \tau + \frac{4 - 3y}{2\sqrt{6}} \int W(\phi) \, d\tau \right) \left( 1 - \frac{x_0^2}{3\lambda_{1,1}} V(\phi) e^{2\tau} \right) + h. \quad (47) \]

By definition

\[ W(\phi) = \frac{\chi^\prime(f^{-1}(\phi))}{\chi(f^{-1}(\phi))} = -M. \]

Then by using the first-order expression

\[ f^{-1}(\phi) = \phi = \sqrt{\frac{3}{2}} (-\tau + \phi_0) + O(\phi^{\lambda_{1,1}\tau}) + h \]

(as derived in former sections) and integrating out the resulting expression with respect to \( \tau \) we get the estimation

\[ \int W(\phi) \, d\tau = -\sqrt{\frac{3}{2}} \ln \chi(\phi) \, d\tau - M \tau + h. \quad (48) \]

By substitution of (48) into (47) we get

\[ z = z_0 e^{t^{1/2} \chi(\phi) e^{2\tau}} \left( 1 - \frac{x_0^2}{3\lambda_{1,1}} V(\phi) e^{2\tau} \right) + h \]

\[ = \chi_0 t^{1/2} \chi(\phi)^{1/2} + h \]

(49)

where \( \chi_0 = z_0 t^{1/2} \chi(\phi)^{1/2} \).

Combining the expansions for \( z \) and \( x \) in \( \rho = \frac{3\lambda_{1,1}}{x} \), we find

\[ \rho = \frac{1}{3} \left( 1 + \frac{4}{\lambda_{1,1}} V(\phi)t^2 \right) \chi(\phi)^{1/2} \chi_0 t^{-\gamma} + h. \quad (50) \]

Observe that the second term, \( h \), on the right-hand side of (46) tends to zero when \( t \to 0 \). This allows us to Taylor-expand \( V \) and \( \chi \) around \( \phi^* = -\sqrt{\frac{3}{2}} \ln \frac{1}{z} \) to get

\[ V(\phi(t)) = V(\phi^*) (1 + \alpha W(\phi^*) V(\phi)t^2) + h \]

(51)
and
\begin{equation}
\chi(\phi(t)) = \chi(\phi^*)(1 + \alpha W_j(\phi^*)V(\phi)^2) + h
\end{equation}
(52)
where \(\alpha\) is a constant. By substituting equations (51) and (52) into equations (43), (45), (46), (50) the theorem is proven. \(\square\)

4.2. A global singularity theorem

Finally, we will state (without a rigorous proof) a global singularity theorem which is in some way an extension of theorem 6 in [24] (page 3501). It is not totally an extension of this theorem, since in our framework it is very difficult to prove that the correspondence with the massless minimally coupled scalar field cosmologies is one-to-one.

The theorem states the following:

**Theorem 4.2.** Let \(V \in \mathcal{E}^2\) be such that \(N_{2,\infty}^2 < 6\) and \(\chi \in \mathcal{E}^2\) such that

(i) \(0 < \gamma < \frac{4}{3}\) and \(M_{2,\infty} > \frac{\sqrt{6(\gamma - 2)}}{\gamma - 4}\) or

(ii) \(\frac{4}{3} < \gamma < 2\) and \(M_{2,\infty} \leq \frac{\sqrt{6(\gamma - 2)}}{\gamma - 4}\).

Then it is verified asymptotically that:

\begin{align*}
H &= \frac{1}{3t} + O\left(\epsilon_\psi(t)^\pm\right), \\
\phi &= \pm\sqrt{\frac{3}{2}} \ln \frac{t}{\hat{c}} + O\left(t\epsilon_\psi(t)^\pm\right), \\
\dot{\phi} &= \pm\sqrt{\frac{3}{2}} t^{-1} + O\left(\epsilon_\psi(t)^\pm\right), \\
\rho &= \chi_0^2 t^{-\gamma} \left(\pm\sqrt{\frac{3}{2}} \ln \frac{t}{\hat{c}}\right)^{\frac{\gamma - 2}{2}} (1 + O\left(t\epsilon_\psi(t)^\pm\right)),
\end{align*}

where \(\epsilon_\psi(t)^\pm = t V\left(\pm\sqrt{\frac{3}{2}} \ln \frac{1}{t}\right)\).

**Sketch of the proof.** Following the reasoning [24], it is sufficient to demonstrate that almost all the solutions are past asymptotic to the critical point \(P_1\) (in \(\infty\) or in \(-\infty\)). Since \(x\) is monotonic, it is sufficient to consider solutions in \(\Omega(x_0) \subset \Sigma(x_0)\) where \(x_0\) is arbitrary. Since \(\Sigma(x_0)\) is compact and it contains its past orbits, then all the points \(p\) must have an \(\alpha\)-limit set, \(\alpha(p)\). Particularly, for the points in the physical space \(\Omega(x_0)\), theorem 3.2 implies that \(\alpha(p)\) must contain almost always a critical point with \(\phi = 0 (\phi = \pm \infty)\). By the discussion in section 3.8, each point with \(\phi = 0\) being a limit point of the physical trajectory must be part of the non-physical boundary \(\partial\Omega(x_0)\) and then must have \(x = 0\). Since \(x\) is monotonically increasing, the set \(\alpha(p)\) must be contained completely in the plane \(x = 0\), or namely in \(\partial\Omega(x_0)\). It can be proved that the only conceivable generic past attractor is the critical points \(P_1\) in \(\pm \infty\) (the other critical points cannot be generic sources by our previous linear analysis).

5. Conclusions

In this paper we have investigated models with additional (non-gravitational) interaction between DE and DM. This kind of interaction is justified if the interacting components are
of unknown nature, as is the case for the DM and the DE, the dominant components in the cosmic fluid. We have investigated these models from the dynamical systems viewpoint. The functional form of the potential and the coupling function is arbitrary from the beginning. Some general results are obtained and proved by considering general hypotheses on these input functions.

We have proved (by using normalized variables) that the scalar field typically diverges into the past. This is formulated in theorem 3.2. It is an extension of theorem 1 in [24] to the non-minimally coupled scalar field setting.

In lemma 3.1 it is proved that the orbit passing through an arbitrary point $p \in \Sigma$ (representing cosmological solutions with non-vanishing dimensionless background energy density and positive finite Hubble parameter) is past asymptotic to a regime where the Hubble parameter diverges containing an initial singularity into the past, and is future asymptotic to a regime where the background density is negligible into the future. This result is obtained by constructing a monotonic function defined on an invariant set and by applying the LaSalle monotonocity principle (theorem 4.12 [31]).

We have proved theorem 3.2 that makes it clear that in order to investigate the generic past asymptotic behavior of our system we must seek the limit where the scalar field diverges.

By assuming some regularity conditions on the potential and the coupling function in that regime we have constructed a dynamical system (well suited to investigate the dynamics where the scalar field diverges, i.e. near the initial singularity). The critical points therein are investigated and the cosmological solutions associated with them are characterized. We find the existence of three critical points $P_3, P_5$ and $P_6$. They are in the boundary of the phase space $\Sigma_c$. They represent cosmological scaling solutions (where the contribution of the dimensionless potential energy is negligible). By tuning the free parameters they can be accelerating. In contrast in [24] there exists only one (in our notation, $P_4$) representing an accelerating cosmology. The solutions associated with $P_{1,2}$ ($P_e$ in the notation in [24]) represent stiff and then decelerating solutions (actually solutions associated with a massless scalar field).

We have proved a theorem (theorem 4.1) which is an extension of theorem 4 in [24] to the STT framework. Also, we sketch the proof of the global singularity theorem 4.2. Theorem 4.2 indicates that the past asymptotic structure of non-minimally coupled scalar field theories with FRW metric, as in the FRW general relativistic case, is independent of the exact details of the potential and/or the details of the background matter and the coupling function.

This is a conjecture with solid theoretical and numerical foundations (see figures B1 and B2 in appendix B). To prove that the family of solutions which asymptotically approach $P_1$ are completely characterized by the solution space of the massless scalar field cosmological model (i.e., $V$ and $\chi$ and then \( \rho \), being dynamically insignificant in the neighborhood of the singularity $P_1$) it is required to prove that this correspondence is one-to-one and continuous, which is hard to do in our scenario.

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Appendix A. Some terminology and results from the dynamical systems theory

In the following $\phi(t, x)$ denotes the flow generated by the vector field (or differential equation)

$$
\dot{x}(t) = f(x(t)), \quad x(t) \in \mathbb{R}^n.
$$

(A.1)

### A.1. Limit sets

**Definition 1** (definition 8.1.1 [36], p 104). A point $x_0 \in \mathbb{R}^n$ is called an $\omega$-limit point of $x(t) \in \mathbb{R}^n$, denoted by $\omega(x)$, if there exists a sequence $\{t_i\}, t_i \to \infty$ such that $\phi(t_i, x) \to x_0$.

$\alpha$-limits are defined similarly by taking a sequence $\{t_i\}, t_i \to -\infty$.

**Definition 2** (definition 8.1.2 [36], p 105). The set of all $\omega$-limit points of a flow or map is called a $\omega$-limit set. The $\alpha$-limit is similarly defined.

**Proposition 1** (proposition 8.1.3 [36], p 105). Let $\phi_t(\cdot)$ be a flow generated by a vector field and let $M$ be a positively invariant compact set for this flow (see definition 3.0.3, p 28 [36]). Then for $p \in M$, we have

(i) $\omega(p) \neq \emptyset$

(ii) $\omega(p)$ is closed

(iii) $\omega(p)$ is invariant under the flow, i.e., $\omega(p)$ is a union of orbits.

(iv) $\omega(p)$ is connected.

A similar result follows for $\alpha$-limit sets provided the hypothesis of the proposition is satisfied for the time reversed flow (see proposition 1.1.14 in [37]).

### A.2. Center manifolds

A general vector field can be transformed locally in the neighborhood of a fixed point into a vector field of the form

$$
\dot{x} = Ax + f(x, y),
$$

(A.2)

$$
\dot{y} = By + g(x, y), \quad (x, y) \in \mathbb{R}^c \times \mathbb{R}^s,
$$

(A.3)

where

$$
f(0, 0) = g(0, 0) = Df(0, 0) = Dg(0, 0) = 0,
$$

(A.4)

$A$ is a $c \times c$ matrix having eigenvalues with zero real parts, $B$ is a $s \times s$ matrix with negative real parts, and $f$ and $g$ are $C^r$ functions ($r \geq 2$).

The center manifold for the vector field (A.3) is defined as follows:

**Definition 3** (definition 18.1.1 [36], p 246). An invariant manifold will be called a center manifold for (A.3) if it can locally be represented as follows,

$$
W_{\text{loc}}^c(0) \{ (x, y) \in \mathbb{R}^c \times \mathbb{R}^s | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0 \},
$$

for $\delta$ sufficiently small.

The conditions $h(0) = 0, Dh(0) = 0$ imply that $W_{\text{loc}}^c(0)$ is tangent to $E^c$ at $(x, y) = (0, 0)$, where $E^c$ is the center subspace, i.e., the invariant set spanned by the eigenvectors whose associated eigenvalues have zero real parts.
Theorem 1 (center manifold (theorem 2.7.1 [38])). Let \( \phi_t \) be the flow of a vector field, \( \mathbf{X} \), then there exists locally a center manifold, \( W_{loc}^c \), containing the origin and invariant under \( \phi_t \) such that \( W_{loc}^c \) has tangent space \( E^c \) at \( x = 0 \). This manifold is \( C^k \) for all \( k \in \mathbb{N} \), but its domain of definition can depend on \( k \). Furthermore, there are locally smooth stable and unstable manifolds, \( W_{loc}^s \) and \( W_{loc}^u \), which contain \( x = 0 \), are invariant under \( \phi_t \), have tangent spaces \( E^s \) and \( E^u \), respectively, and are such that \( \phi_t|_{W_{loc}^s} \) is a contraction while \( \phi_t|_{W_{loc}^u} \) is an expansion.

A.3. Monotone functions and monotonicity principle

Definition 4 (definition 4.8 [31], p 93). Let \( \phi_t \) be a flow on \( \mathbb{R}^n \), let \( S \) be an invariant set of \( \phi_t \) and let \( Z : S \rightarrow \mathbb{R} \) be a continuous function. \( Z \) is a monotonic decreasing (increasing) function for the flow \( \phi_t \) means that for all \( x \in S, Z(\phi_t(x)) \) is a monotonic decreasing (increasing) function of \( t \).

Proposition 2 (proposition 4.1 [31], p 92). Consider a differential equation \( x' = f(x), x \in \mathbb{R}^n \) with flow \( \phi_t \). Let \( Z : \mathbb{R}^n \rightarrow \mathbb{R} \) be a \( C^1(\mathbb{R}^n) \) function which satisfies \( Z' = \alpha Z \), where \( \alpha : \mathbb{R}^n \rightarrow \mathbb{R} \) is a continuous function. Then the subsets of \( \mathbb{R}^n \) defined by \( Z > 0, Z = 0, \) or \( Z < 0 \) are invariant sets for \( \phi_t \).

Theorem 2 (monotonicity principle [31], p 103). Let \( \phi_t \) be a flow on \( \mathbb{R}^n \) with \( S \) an invariant set. Let \( Z : S \rightarrow \mathbb{R} \) be a \( C^1(\mathbb{R}^n) \) function whose range is the interval \( (a, b) \) where \( a \in \mathbb{R} \cup \{-\infty\}, b \in \mathbb{R} \cup \{+\infty\}, \) and \( a < b \). If \( Z \) is decreasing on orbits in \( S \), then for all \( x \in S, \omega(x) \subset \{s \in \overline{S} - S | \lim_{y \rightarrow s} Z(y) \neq b \} \) and \( \alpha(x) \subset \{s \in \overline{S} - S | \lim_{y \rightarrow s} Z(y) \neq a \} \).

Appendix B. Regularity conditions of the potential and the coupling function at infinity

In this section we present rigorous statements of what we mean by regularity conditions. We present some worked examples that we shall use to construct a toy model. Our purpose is illustrated by the techniques for analysis in the region \( \phi = +\infty \). In principle the analysis is general enough to be applied to more physically interesting situations.

Definition 5 (see [24]). Let \( V : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^2 \) non-negative function. Let there exist some \( \phi_0 > 0 \) for which \( V(\phi) > 0 \) for all \( \phi > \phi_0 \) and some number \( N \) such that the function \( W_V : (\phi_0, \infty) \rightarrow \mathbb{R}, W_V(\phi) = \partial_\phi V(\phi) - N \) satisfies

\[
\lim_{\phi \rightarrow \infty} W_V(\phi) = 0.
\] (B.1)

Then we say that \( V \) is well behaved at infinity (WBI) of exponential order \( N \).

It is important to point out that \( N \) may be 0, or even negative. Indeed the class of WBI functions of order 0 is of particular interest, containing all non-negative polynomials as remarked in [24].

In that reference a procedure for classifying the smoothness of WBI functions at infinity was defined. If we have some coordinate transformation \( \phi = f(\phi) \) which maps a neighborhood of infinity to a neighborhood of the origin, then if \( g \) is a function of \( \phi, \overline{\phi} \) is the function of \( \phi \) whose domain is the range of \( f \) plus the origin, which takes the values;
Figure B1. Orbits in the invariant set \( \{ z = 0 \} \subset \Sigma \), for the model with coupling function (B.2) potential (B.3). We select the values of the parameters: \( \epsilon = 1.00, \mu = 2.00, A = 0.50, \alpha = 0.33, B = 0.5 \), and \( \phi_0 = 0 \). Observe that (i) almost all the orbits are past asymptotic to \( P_1 \); (ii) \( P_2 \) is a saddle, and (iii) the center manifold of \( P_4 \) attracts all the orbits in the \( \{ z = 0 \} \). However, it is no longer an attractor in the invariant set \( z > 0, \phi = 0 \) (see figure B2).

![Figure B1](image1.png)

Figure B2. Orbits in the invariant set \( \{ \phi = 0 \} \subset \Sigma \), for the model with coupling function (B.2) potential (B.3). We select the values of the parameters: \( \epsilon = 1.00, \mu = 2.00, A = 0.50, \alpha = 0.33, B = 0.5 \), and \( \phi_0 = 0 \). In the figure (i) \( P_{1,2} \) are local past attractors, but \( P_1 \) is the global past attractor; (ii) \( P_{3,4} \) are saddles and (iii) \( P_5 \) is a local future attractor.

![Figure B2](image2.png)

\[
\bar{g}(\phi) = \begin{cases} 
g(f^{-1}(\phi)), & \phi > 0 \\
\lim_{\phi \to \infty} g(\phi), & \phi = 0 
\end{cases}
\]

**Definition 6** [see (24)]. A \( C^k \) function \( V \) is class \( k \) WBI if it is WBI and if there exists \( \phi_0 > 0 \) and a coordinate transformation \( \phi = f(\phi) \) which maps the interval \([\phi_0, \infty)\) onto \((0, \epsilon)\), where \( \epsilon = f(\phi_0) \) and \( \lim_{\phi \to \infty} f = 0 \), with the following additional properties:

(i) \( f \) is \( C^{k+1} \) and strictly decreasing.

(ii) the functions \( \bar{W}_V(\phi) \) and \( \bar{T}(\phi) \) are \( C^k \) on the closed interval \([0, \epsilon]\).

(iii) \( \frac{\partial \bar{W}}{\partial \phi}(0) = \frac{\partial \bar{T}}{\partial \phi}(0) = 0 \).
We designate the set of all class-\(k\) WBI functions \(E^+_k\). In table 1 of [24] a diverse range of qualitative behavior within the framework of \(E^+_k\) potentials is incorporated.

### B.1. Worked examples

#### B.1.1. Power-law coupling function.

Let us consider the coupling function
\[
\chi(\phi) = \left(\frac{3\alpha}{8}\right)\phi_0(\phi - \phi_0)^{\frac{3}{\alpha}}, \quad \alpha > 0, \text{const., } \phi_0 > 0.
\]

(B.2)

Observe that
\[
\frac{d\ln \chi(\phi)}{d\chi} = \frac{2}{\alpha(\phi - \phi_0)} \neq 0
\]
for all finite values of \(\phi\). Then the critical point \(Q\) located at the invariant set \(X^0\) does not exist for this choice of coupling function. By our analysis in section 3, the early time dynamics is associated with the limit where the scalar field diverges.

This choice produces a coupling BD parameter given by
\[
2\omega(\chi) + 3 = \frac{4}{3}\alpha \left(\frac{\chi}{\chi_0}\right)^{\alpha}.
\]

These types of power-law couplings were investigated in [33] from the astrophysical viewpoint. For scalar–tensor theories without potential, the cosmological solutions for the matter domination era (in a Robertson–Walker metric) are
\[
a(t) \propto (\ln t)^{\alpha - 1}/3\alpha t^2, \quad \phi(t) \propto (\ln t)^{1/\alpha}.
\]
The values of the parameter \(\alpha\) in concordance with the predictions of \(4H\) are \(\alpha = 1, 0.33, 3\) (see table 4.2 in [33]).

#### B.1.2. The Albrecht–Skordis potential.

Albrecht and Skordis [34] have proposed a particularly attractive model of quintessence. It is driven by a potential which introduces a small minimum to the exponential potential:
\[
V(\phi) = e^{-\mu\phi}(A + (\phi - B)^2).
\]

(B.3)

Unlike previous quintessence models, late-time acceleration is achieved without fine tuning of the initial conditions. The authors argue that such potentials arise naturally in the low-energy limit of \(M\)-theory. The constant parameters, \(A\) and \(B\), in the potential take values of order 1 in Planck units, so there is also no fine tuning of the potential (we also suppose that \(\mu \neq 0\)).

They show that, regardless of the initial conditions, \(\rho\) scales, with \(\rho \propto \rho_0 \propto t^{-2}\) during the radiation and matter eras, but leads to permanent vacuum domination and accelerated expansion after a time which can be close to the present.

The extremes of the potential (B.3) are located at \(\phi^\pm = \frac{1 + \mu \pm \sqrt{1 - A\mu^2}}{\mu^2} \cdot \frac{1}{\mu^2}\). They are real if \(1 \geq \mu^2 A\). The local minimum (respectively, local maximum) is located at \(\phi^-\) (respectively \(\phi^+\)) since
\[
\pm V''(\phi^\pm) = -2V_0\sqrt{1 - A\mu^2} e^{-\mu \pm \sqrt{1 - A\mu^2}} < 0.
\]

As we investigated in section 3, the late-time dynamics in the invariant set \(Z^0\) is associated with the extremes of the potential. When we restrict ourselves to this invariant set, we find that the critical point associated with \(\phi^+\) is always a saddle point of the corresponding phase portrait. The critical point associated with \(\phi^+\) could be either a stable node or a stable spiral if
\[
\frac{8(3 + 2\mu^2)}{(3 + 4\mu^2)^2} < A \leq \frac{1}{\mu^2}.
\]
or
\[ A < \frac{8(3 + 2\mu^2)}{(3 + 4\mu^2)^2}. \]

The early time dynamics in this invariant set corresponds to the limit \( \phi \to +\infty \).

Let us concentrate now on how to apply the mathematics introduced in this section.

Observe first that
\[
W_\chi (\phi) = \frac{2}{\alpha (\phi - \phi_0)} \Rightarrow \lim_{\phi \to +\infty} W_\chi (\phi) = 0 \tag{B.4}
\]
and
\[
W_V (\phi) = \frac{2(\phi - B)}{A + (B - \phi)^2} \Rightarrow \lim_{\phi \to +\infty} W_V (\phi) = 0. \tag{B.5}
\]

In other words, the coupling function (B.2) and the potential (B.3) are WBI of exponential orders \( M = 0 \) and \( N = -\mu \), respectively.

It is easy to prove that power-law coupling and the Albrecht–Skordis potential are at least \( \mathcal{E}_2 \), under the admissible coordinate transformation
\[
\psi = \phi^{-1} = f (\phi). \tag{B.6}
\]

Using the above coordinate transformation we find
\[
\overline{W}_\chi (\psi) = \begin{cases} 
\frac{2\psi}{\alpha (1 - \psi\phi_0)}, & \psi > 0 \\
0, & \psi = 0
\end{cases} \tag{B.7}
\]
\[
\overline{W}_V (\psi) = \begin{cases} 
-\frac{2\psi (B\psi - 1)}{A\psi^2 + (B\psi - 1)^2}, & \psi > 0 \\
0, & \psi = 0
\end{cases} \tag{B.8}
\]
and
\[
\overline{f'} (\psi) = \begin{cases} 
-\psi^2, & \psi > 0 \\
0, & \psi = 0
\end{cases} \tag{B.9}
\]

**B.2. The dynamics in the limit \( \phi \to +\infty \): an example**

We will present here a toy model by choosing a power-law coupling and Albrecht–Skordis potential with the only purpose of illustrating our previous results. Due to the generality of our study it can be applied for more realistic scalar–tensor cosmologies.

Let us consider a toy model with the coupling function (B.2) and the potential (B.3).

Observe that this toy model corresponds to a theory written in the Jordan frame with the action
\[
S_{JF} = \int_{M_4} d^4x \sqrt{|g|} \left\{ \frac{1}{2} \bar{R} - \frac{1}{2} \frac{\omega (\chi)}{\chi} (\bar{\nabla} \chi)^2 - \bar{V} (\chi) + L_{\text{matter}} (\mu, \nabla \mu, \bar{g}_{\alpha \beta}) \right\} \tag{B.10}
\]
with BD coupling
\[
\omega (\chi) = -\frac{3}{2} + \frac{2}{3} \alpha \left( \frac{\chi}{\chi_0} \right)^{\alpha},
\]
and potential
\[
\bar{V} (\chi) = \chi^2 e^{-\mu \phi (\chi)} (A + (\phi(\chi) - B)^2)
\]
where
\[ \phi(\chi) = \phi_0 + \sqrt{\frac{8}{3\alpha}} \left( \frac{\chi}{\chi_0} \right)^{3/2}. \]

We have also considered the conformal metric
\[ \bar{g}_{\alpha\beta} = \chi (\phi)^{-1} g_{\alpha\beta}. \]

In this example, the evolution equations for \( y, z \) and \( \psi \) are given by equations (25)–(27) with \( M = 0, N = -\mu \) and \( \bar{W}_y, \bar{W}_z, \) and \( \bar{F} \), given respectively by (B.7), (B.8) and (B.9). The state space is defined by \( \Sigma_\varepsilon = \{(y, z, \psi) : 0 \leq y^2 + z^2 \leq 1, 0 \leq \psi \leq \varepsilon \} \).

The critical points of the system (25)–(27) in this example are \( P_{1,2} = (\pm 1, 0, 0), P_3 = (0, 1, 0), P_4 = \left( \frac{\sqrt{y}}{\sqrt{z}}, 0, 0 \right) \) and \( P_{5,6} = \left( \frac{\sqrt{y}}{\sqrt{2}}, \pm \frac{\sqrt{-12y + 4z}}{2z}, 0 \right) \). The points \( P_{1,2,3} \) exist for all the values of the free parameters. The critical point \( P_4 \) exists for \( \mu \leq 6 \). The critical point \( P_5 \) exists if \( \mu \leq -\sqrt{3y} \) whereas the critical point \( P_6 \) exists if \( \mu \geq \sqrt{3y} \). We will characterize the critical points \( P_{5,6} \) in more detail (for the analysis of the other critical points we refer the reader to table 1). The critical points \( P_{5,6} \) corresponds to those studied in the book [14] (see equation 4.23, p 49) with the identifications \( \Psi = y, \Phi^2 = \frac{\psi(\phi)}{3H^2} = \frac{2\gamma(\gamma - 1)}{2\mu^2} \) and \( k = -\mu \). As stated in that reference the scalar field ‘inherits’ the equation of state of the fluid, i.e., \( \psi_0 = \gamma \). Then these solutions represent cosmological kinetic-matter scaling solutions\(^6\) (the potential energy density is negligible). These critical points represent accelerating cosmologies for \( 0 < y < \frac{2}{3} \). The eigenvalues of the matrix of derivatives evaluated at \( P_{5,6} \) are \( \left\{ 0, -\frac{2y^2}{\mu^2} \pm \frac{1}{2\mu} \sqrt{(2 - \gamma)(24y^2 + \mu^2(2 - 9\gamma))} \right\} \). The orbits initially in the stable subspace of \( P_{5,6} \) spiral-in around \( P_{5,6} \) if \( \mu^2 > 24y^2/(2 - 9\gamma) \) provided \( \frac{2}{3} < y < 2, \gamma \neq \frac{2}{3} \). Otherwise \( P_{5,6} \) looks like a stable node for the orbits lying in the stable subspace. The center subspace is tangent to the critical points in the direction of the \( \psi \) axis.

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\(^6\) See [35] for a notion of ‘scaling’ solutions, particularly, kinetic-matter scaling solutions.
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