Lp-THEORY FOR A CAHN-HILLIARD-GURTIN SYSTEM

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ABSTRACT. In this paper we study a generalized Cahn-Hilliard equation which was proposed by Gurtin [8]. We prove the existence and uniqueness of a local-in-time solution for a quasilinear version, that is, if the coefficients depend on the solution and its gradient. Moreover we show that local solutions to the corresponding semilinear problem exist globally as long as the physical potential satisfies certain growth conditions. Finally we study the long-time behaviour of the solutions and show that each solution converges to a equilibrium as time tends to infinity.

1. INTRODUCTION

We start with the derivation of the classical Cahn-Hilliard equation. Consider the free energy functional of the form

\( F(\psi) = \int_{\Omega} \left( \frac{1}{2} |\nabla \psi|^2 + \Phi(\psi) \right) \, dx, \)

where \( \Omega \) is a bounded, open and connected subset of \( \mathbb{R}^n \) with boundary \( \Gamma := \partial \Omega \in C^3 \). We assume that the order parameter \( \psi \) is a conserved quantity. The according conservation law reads

\( \partial_t \psi + \text{div} \, j = 0, \)

where \( j \) is a vector field representing the phase flux of the order parameter. The next step is to combine the two quantities \( j \) and \( \mu \). Similar to Fourier’s law in the derivation of the heat equation one typically assumes that \( j \) is given by

\( j = -\nabla \mu, \)

a postulated relation. Finally we have to derive an equation for \( \mu \). The chemical potential \( \mu \) is given by the variational derivative of \( F \), i.e.

\( \mu = \frac{\delta F}{\delta \psi} = -\Delta \psi + \Phi'(\psi). \)

If \( F \) is of the form (1.1) this yields the classical Cahn-Hilliard equation.

In the early nineties GURTIN [8] proposed a generalized Cahn-Hilliard equation, which is based on the following objections:

- Fundamental physical laws should account for the work associated with each operative kinematical process;
- There is no clear separation of the balance law (1.2) and the constitutive equation (1.3);

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Date: May 5, 2014.

2000 Mathematics Subject Classification. 35K55, 35B38, 35B40, 35B65, 82C26.

Key words and phrases. Cahn-Hilliard-Gurtin equation, quasilinear elliptic-parabolic system, optimal regularity, global existence, convergence to steady states, Lojasiewicz-Simon inequality.
Forces that are associated with microscopic configurations of atoms are not considered in the derivation of the classical Cahn-Hilliard equation. According to Gurtin there should exist so called 'microforces' whose work accompanies changes in the order parameter $\psi$. The microforce system is characterized by the microstress $\xi \in \mathbb{R}^n$ and scalar quantities $\pi$ and $\gamma$ which represent internal and external microforces, respectively. The main assumption in [8] is that $\xi$, $\pi$ and $\gamma$ satisfy the (local) microforce balance
\begin{equation}
\text{div} \, \xi + \pi + \gamma = 0,
\end{equation}
which can be motivated from a static point of view, see [8] for more details. In a next step we want to derive constitutive equations, which relate the quantities $j$, the flux of the order parameter, $\xi$ and $\pi$ to the fields $\psi$ and $\mu$. The technique used in [8] for this derivation is based on the balance equation (1.4) and a (local) dissipation inequality, which is a direct consequence of the first and the second law of thermodynamics, that is, the energy balance
\begin{align}
\frac{d}{dt} \int_{\Omega} e \, dx &= -\int_{\partial \Omega} q \cdot \nu \, d\sigma + \int_{\Omega} r \, dx + \mathcal{W}(\Omega) + \mathcal{M}(\Omega),
\end{align}
and
\begin{align}
\frac{d}{dt} \int_{\Omega} S \, dx &\geq -\int_{\partial \Omega} q \cdot \nu \, d\sigma + \int_{\Omega} r \, dx,
\end{align}
cf. [8, Appendix A]. The second law of thermodynamics is also known as the Clausius-Duhem inequality. Here $e$ is the internal energy, $S$ is the entropy, $\theta$ is the absolute temperature, $q$ is the heat flux, $r$ is the heat supply, $\mathcal{W}(\Omega)$ is the rate of working on $\Omega$ of all forces exterior to $\Omega$ and $\mathcal{M}(\Omega)$ is the rate at which energy is added to $\Omega$ by mass transport. Let $F$ be the free energy density, depending on the vector $z = (\psi, \nabla \psi, \mu, \nabla \mu, \partial_t \psi)$. Then the second law of thermodynamics (in its mechanical version as considered by Gurtin [8]) reads
\begin{align}
\frac{d}{dt} \int_{\Omega} F(z) \, dx &\leq -\int_{\partial \Omega} \mu j(z) \cdot \nu \, d\sigma + \int_{\partial \Omega} \xi \cdot \nabla \psi \, d\sigma + \int_{\Omega} \mu m \, dx + \int_{\Omega} \gamma \partial_t \psi \, dx,
\end{align}
with $m$ being the external mass supply. Making use of Green's formula, we obtain
\begin{align}
\frac{d}{dt} \int_{\Omega} F(z) \, dx &\leq -\int_{\Omega} (\nabla \mu \cdot j(z) + \mu \text{div} \, j) \, dx
\end{align}
\begin{align}
&+ \int_{\Omega} (\text{div} \, \xi \partial_t \psi + \xi \cdot \nabla \partial_t \psi) \, dx + \int_{\Omega} \mu m \, dx + \int_{\Omega} \gamma \partial_t \psi \, dx.
\end{align}
in presence of external mass supply $m$, (1.2) will be modified to
\begin{equation}(1.5)\quad \partial_t \psi + \text{div} \, j = m.\end{equation}
In view of (1.4) and (1.5) we obtain the dissipation inequality
\begin{align}
\frac{d}{dt} \int_{\Omega} F(z) \, dx &\leq \int_{\Omega} (\mu \partial_t \psi - j \cdot \nabla \mu - \pi \partial_t \psi + \xi \cdot \nabla \partial_t \psi) \, dx.
\end{align}
This in turn yields the following local dissipation inequality
\begin{align}
\partial_t F(z) &\leq \mu \partial_t \psi - j \cdot \nabla \mu - \pi \partial_t \psi + \xi \cdot \nabla \partial_t \psi,
\end{align}
for all fields $\psi$ and $\mu$, this means, we have
\begin{align}(1.6)\quad (\partial_\psi F + \pi - \mu) \psi + (\partial_\psi F - \xi) \cdot \nabla \psi + \partial_\mu F \mu + \partial_\mu F \nabla \mu + \partial_\psi F \nabla \psi + \nabla \mu \cdot j &\leq 0,
\end{align}
where \( \dot{u} = \partial_t u \) and \( \ddot{u} = \partial_t^2 u \) for a smooth function \( u \). This local inequality needs to be satisfied for all smooth fields \( \psi \) and \( \mu \). Hence we have necessarily

\[
F(z) = F(\psi, \nabla \psi) \quad \text{and} \quad \xi(\psi, \nabla \psi) = \partial_v F(\psi, \nabla \psi)
\]

and there remains the inequality

\[
(\partial_v F + \pi - \mu) \psi + \nabla \mu \cdot j \leq 0
\]

whose general solution is given by (cf. [8, Appendix B])

\[
\partial_v F + \pi - \mu = -\beta \psi - c \cdot \nabla \mu \quad \text{and} \quad \mu = \psi - B \nabla \mu,
\]

with constitutive moduli \( \beta (z) \) (scalar), \( a(z), c(z) \) (vectors), \( B(z) \) (matrix) and the constraint that the matrix

\[
\begin{bmatrix}
\beta & c^T \\
a & B
\end{bmatrix}
\]

is positive semidefinite. For convenience we assume that \( \beta \) is constant and \( a, c \) and \( B \) do only depend on \( x \) instead of \( z \), whence we deal with an approximation of the constitutive moduli \( \beta (z), a(z), c(z), B(z) \). In particular, if the free energy density \( F \) is given by \( F(\psi, \nabla \psi) = \frac{1}{2} |\nabla \psi|^2 + \Phi(\psi) \) we obtain the following semilinear Cahn-Hilliard-Gurtin equations,

\[
\begin{aligned}
\partial_t \psi - \text{div}(B \nabla \mu) - \text{div}(a \partial_t \psi) &= f, & t > 0, \; x \in \Omega, \\
\mu - c \cdot \nabla \mu + \Delta \psi - \beta \partial_t \psi - \Phi'(\psi) &= g, & t > 0, \; x \in \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^n \) is open and bounded with boundary \( \Gamma = \partial \Omega \in C^3 \). We want to emphasize that for the special case \( B = I, a = c = 0 \) and \( \beta = 0 \), we obtain the classical Cahn-Hilliard equation or the viscous Cahn-Hilliard equation if \( \beta > 0 \).

Let us point out that we will also deal with a quasilinear version of \( \text{(1.3)} \) in Section 5. To be precise, we will consider the system

\[
\begin{aligned}
\partial_t \psi - \text{div}(b(x, \psi, \nabla \psi) \nabla \mu) - \text{div}(a(x, \psi, \nabla \psi) \partial_t \psi) &= f, & t > 0, \; x \in \Omega, \\
\mu - c(x, \psi, \nabla \psi) \cdot \nabla \mu + \Delta \psi - \beta \partial_t \psi - \Phi'(\psi) &= g, & t > 0, \; x \in \Omega.
\end{aligned}
\]

In this paper, we are interested in solutions of \( \text{(1.8)} \) and \( \text{(1.9)} \) subject to the Neumann boundary conditions for \((\psi, \mu)\), having optimal \( L_p \)-regularity in the sense

\[
\psi \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^2_p(\Omega)),
\]

and

\[
\mu \in L_p(J; H^2_p(\Omega)),
\]

for given functions \( f \in L_p(J; L_p(\Omega)) \) and \( g \in L_p(J; H^1_p(\Omega)) \), where \( J = [0, T] \). We will always use the following assumptions for the semilinear problem \( \text{(1.8)} \):

- \( a, c \in C^1(\overline{\Omega}), \)
- \( \text{div} \; a(x) = \text{div} \; c(x) = 0 \; \text{in} \; \Omega, \)
- \( a(x)|v(x)| = c(x)|v(x)| = 0 \; \text{on} \; \partial \Omega, \)
- \( \beta > 0, \; B = bI, \; \text{with} \; b \in C^1(\overline{\Omega}), \)
- \( \beta \geq 0, \; a \geq 0, \) and \( \beta = 0, \)
- \( \text{if} \; \beta > 0, \)
- \( \text{there is a constant} \; \varepsilon > 0, \) such that the estimate

\[
\beta \varepsilon^2 + (a + c|z|)z + (Bz|z|) \geq \varepsilon(z_0^2 + |z_1|^2)
\]

is valid for all \((z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n\) and all \( x \in \Omega. \)
In Section 2, where we consider $\Omega = \mathbb{R}^n$, we allow for general, positive definite matrices $B$. It is also possible to consider those matrices in all other sections but for the sake of convenience we restrict ourselves to the case $B = bI$. Actually this allows to draw back the problem in the half space $\mathbb{R}^n_+$ to the whole space $\mathbb{R}^n$ by means of reflection methods.

Results on existence and uniqueness can be found e.g. in the papers of Bonfoh & Miranville [3], Miranville [10], [11], Miranville, Piétrus & Rakotoson [12] and Miranville & Zelik [14]. In any of these papers the authors use a variational approach and energy estimates to obtain global well-posedness in an $L^2$-setting, with periodic boundary conditions for a cuboid in $\mathbb{R}^3$. The qualitative behavior of solutions of the Cahn-Hilliard-Gurtin equation has been investigated in [3], [12] and [13]. In [3] and [12] the authors proved the existence of finite dimensional attractors, whereas Miranville & Rougirel [13] showed that each solution converges to a steady state, again with the help of the Lojasiewicz-Simon inequality. One assumption of Miranville & Rougirel [13] is that the norms $|a|$, $|c|$ and $|B - I|$ are bounded by a possibly small constant. In the present paper we will give an alternative proof for the relative compactness of the orbit $\{\psi(t)\}_{t \geq 0}$ in $H_2^1(\Omega)$ with the help of semigroup theory and a priori estimates (see Proposition 7.1).

The present paper is structured as follows. In Section 2 we deal with a corresponding linearized system to (1.9) in the full space $\mathbb{R}^n$ with constant coefficients. Section 3 is devoted to the analysis of the linearized system with constant coefficients in the half space $\mathbb{R}^n_+$. Making use of the optimal regularity results of Sections 2 and 3 we apply the method of localization and some perturbation results in Section 4 to derive optimal $L^p$-regularity for the linearized Cahn-Hilliard-Gurtin equations (i.e. (1.8) with $\Phi' = 0$) in an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial \Omega \in C^3$. In Section 5 we prove the existence and uniqueness of a local-in-time solution of (1.9). For this purpose it is crucial to have the optimal $L^p$-regularity result from Section 4 at our disposal. To the knowledge of the author there are no results on the local well-posedness of (1.9) but only for the case where $a$, $c$ and $b$ depend solely on the order parameter $\psi$, cf. Miranville [15]. In Section 6 we investigate the global well-posedness of the semilinear system (1.8). The basic tools are a priori estimates and the Gagliardo-Nirenberg inequality. Finally, in Section 7, we show that each solution $\psi(t)$ of (1.8) converges to a steady state in $H_2^1(\Omega)$ as $t \to \infty$. To this end we will use relative compactness results and the Lojasiewicz-Simon inequality.

2. The Linear Cahn-Hilliard-Gurtin Problem in $\mathbb{R}^n$

In this section we will solve the full space problem

\begin{equation}
\begin{aligned}
\partial_t u - \text{div}(au\partial_t u) = \text{div}(B\nabla \mu) + f, & \quad t > 0, \ x \in \mathbb{R}^n, \\
\mu - c \cdot \nabla \mu = \beta \partial_t u - \Delta u + g, & \quad t > 0, \ x \in \mathbb{R}^n, \\
u(0) = u_0, & \quad t = 0, \ x \in \mathbb{R}^n,
\end{aligned}
\end{equation}

(2.1)

where $\beta > 0$, $a, c \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times n}$. Note that the matrix (1.7) is positive semidefinite if and only if

$$\beta z_0^2 + (a + c|z_1)z_0 + (Bz_1|z_1) \geq 0$$

holds for all $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$ and all $x \in \Omega$. Here $(\cdot|\cdot)$ denotes the usual scalar product in $\mathbb{C}^n$ and the vector fields $a, c$ as well as the matrix valued function $B$ are assumed to be smooth. In the sequel we will use a slightly stronger assumption.
(H) There is a constant $\varepsilon > 0$, such that
\[
\beta z_0^2 + (a + c|z_1)z_0 + (Bz_1|z_1) \geq \varepsilon(z_0^2 + |z_1|^2)
\]
is valid for all $(z_0, z_1) \in \mathbb{R} \times \mathbb{R}^n$ and all $x \in \Omega$.

The following result is useful for the analysis of (2.1) (see also [13, Lemma 5.1]).

**Proposition 2.1.** Let (H) hold. Then
\[
(\beta B\xi|\xi) - \frac{1}{4}((a \otimes c + c \otimes a)\xi|\xi) \geq \varepsilon \beta |\xi|^2,
\]
for all $\xi \in \mathbb{R}^n$.

**Proof.** Hypothesis (H) reads
\[
\beta z_0^2 + (d|z_1)z_0 + (Bz_1|z_1) \geq \varepsilon(z_0^2 + |z_1|^2),
\]
where $d := a + c$. Observe that the left side of this inequality can be rewritten as
\[
\left(\sqrt{\beta z_0} + \frac{1}{2\sqrt{\beta}}(d|z_1)\right)^2 + \left(B - \frac{1}{4\beta}(d \otimes d)\right)z_1|z_1).
\]
For a fixed $z_1 \in \mathbb{R}^n$ we choose $z_0 \in \mathbb{R}$ in such a way that the squared bracket is equal to 0. Thus we obtain the estimate
\[
(\beta Bz_1|z_1) - \frac{1}{4}((d \otimes d)z_1|z_1) \geq \varepsilon \beta |z_1|^2,
\]
valid for all $z_1 \in \mathbb{R}^n$. By the definition of $d$ it holds that
\[
d \otimes d = a \otimes c + c \otimes a + a \otimes a + c \otimes c,
\]
hence we obtain the identity
\[
\beta B - \frac{1}{2}(a \otimes c + c \otimes a) = \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a \otimes a + c \otimes c - a \otimes c - c \otimes a)
\]
\[
= \beta B - \frac{1}{4}(d \otimes d) + \frac{1}{4}(a - c) \otimes (a - c).
\]
Since the matrix $(a - c) \otimes (a - c)$ is positive semi-definite we finally obtain the assertion. \hfill \Box

Here is the main result on optimal $L_p$-regularity of (2.1).

**Theorem 2.2.** Let $1 < p < \infty$ and assume that (H) holds true. Then (2.1) admits a unique solution
\[
u \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)) =: Z^1,
\]
\[
\mu \in L_p(J; H^2_p(\mathbb{R}^n)) =: Z^2,
\]
if and only if the data is subject to the following conditions.

(i) $f \in L_p(J; L_p(\mathbb{R}^n)) =: X^1$,
(ii) $g \in L_p(J; H^1_p(\mathbb{R}^n)) =: X^2$,
(iii) $u_0 \in B^{3/2-p}_pp(\mathbb{R}^n) =: X_p$. 

Proof. Necessity is clear by substituting the solution \((u, \mu) \in Z^1 \times Z^2\) into the equations \((2.1)\) \ref{2.1}. This yields the desired regularity for the functions \(f, g\). The regularity for the initial value \(u_0\) follows from the trace theorem
\[
H^1_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)) \hookrightarrow C(J; B^{3-2/p}_pp(\mathbb{R}^n)),
\]
where \(B^{3-2/p}(\mathbb{R}^n) = (H^1_p(\mathbb{R}^n), H^2_p(\mathbb{R}^n))_{1-1/p,p}\) is the real interpolation space with exponent \(1-1/p\) and parameter \(p\).

To prove sufficiency of the conditions (i)-(iii), we first apply the operator \((I - \Delta)^{-1/2}\) to both equations in \((2.1)\) and define the new functions \(w = (I - \Delta)^{-1/2}u\), \(\eta = (I - \Delta)^{-1/2}\mu\), \(\tilde{f} = (I - \Delta)^{-1/2}f\), \(\tilde{g} = (I - \Delta)^{-1/2}g\) and \(w_0 = (I - \Delta)^{-1/2}u_0\). Then it holds that
\[
\tilde{f} \in L_p(J; H^1_p(\mathbb{R}^n)), \quad \tilde{g} \in L_p(J; H^2_p(\mathbb{R}^n)), \quad w_0 \in B^{3-2/p}_pp(\mathbb{R}^n)
\]
and we are looking for a solution \((w, \eta)\) of the system
\[
w_t - \text{div}(aw_t) = \text{div}(B\nabla \eta) + \tilde{f}, \quad t > 0, \quad x \in \mathbb{R}^n, \\
\eta - c \cdot \nabla \eta = \beta w_t - \Delta w + \tilde{g}, \quad t > 0, \quad x \in \mathbb{R}^n, \\
w(0) = w_0, \quad t = 0, \quad x \in \mathbb{R}^n,
\]
in the regularity class
\[
w \in H^1_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)), \\
\eta \in L_p(\mathbb{R}^+; H^2_p(\mathbb{R}^n)).
\]
In a next step we want to eliminate the functions \(\tilde{g}\) and \(w_0\). To achieve this, let \(w^*\) be the unique solution of the problem
\[
\beta w_t^* - \Delta w^* = -\tilde{g}, \quad t > 0, \quad x \in \mathbb{R}^n, \\
w^*(0) = w_0, \quad t = 0, \quad x \in \mathbb{R}^n,
\]
with regularity
\[
w^* \in H^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)),
\]
if and only if \(\tilde{g} \in L_p(J \times \mathbb{R}^n)\) and \(w_0 \in B^{3-2/p}_pp(\mathbb{R}^n)\). Here \(J\) denotes the interval \([0, T]\). If we even have \(\tilde{g} \in L_p(J; H^2_p(\mathbb{R}^n))\) and \(w_0 \in B^{3-2/p}_pp(\mathbb{R}^n)\) then by regularity theory we obtain
\[
w^* \in H^1_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)).
\]
The pair of functions \((v, \eta) = (w - w^*, \eta)\) should now solve the problem
\[
\partial_t v - \text{div}(av\partial_t v) = \text{div}(B\nabla \eta) + F, \quad t > 0, \quad x \in \mathbb{R}^n, \\
\eta - c \cdot \nabla \eta = \beta \partial_t v - \Delta v, \quad t > 0, \quad x \in \mathbb{R}^n, \\
v(0) = 0, \quad t = 0, \quad x \in \mathbb{R}^n,
\]
where \(F\) is defined by
\[
F = \tilde{f} + w^*_t - \text{div}(aw^*_t) \in L_p(J; H^1_p(\mathbb{R}^n)).
\]
In order to solve \((2.3)\) we take the Laplace transform in the time variable and the Fourier transform in the spatial variable to obtain
\[
\lambda(1 - i(a|\xi|))\hat{v} = -(B\xi|\xi|)\hat{\eta} + \hat{F}, \\
(1 - i(c|\xi|))\hat{\eta} = (\beta \lambda + |\xi|^2)\hat{v}.
\]
This system of algebraic equations can be written in matrix form
\[
\begin{bmatrix}
\lambda(1 - i(a|\xi|)) & (B_\xi|\xi|) \\
-(\beta \lambda + |\xi|^2) & (1 - i(c|\xi|))
\end{bmatrix}
\begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix}
= \begin{bmatrix}
\hat{F} \\
0
\end{bmatrix},
\]
where \(\lambda \in \Sigma_\phi\), \(\phi > \pi/2\) and \(\xi \in \mathbb{R}^n\) such that \(|\lambda| + |\xi| \neq 0\). Hence the unique solution to these equations is given by
\[
\begin{bmatrix}
\hat{v} \\
\hat{\eta}
\end{bmatrix}
= \frac{1}{m(\lambda, \xi)}
\begin{bmatrix}
(1 - i(c|\xi|)) & -(B_\xi|\xi|) \\
(\beta \lambda + |\xi|^2) & \lambda(1 - i(a|\xi|))
\end{bmatrix}
\begin{bmatrix}
\hat{F} \\
0
\end{bmatrix},
\]
provided
\[m(\lambda, \xi) := \det M(\lambda, \xi) \neq 0.\]
To see this we consider the function \(\tilde{m}(\lambda, \xi) := m(\lambda, \xi)/\lambda\) given by
\[\tilde{m}(\lambda, \xi) = 1 - (a|\xi|(c|\xi|) + \beta(B_\xi|\xi| - i(a + c)|\xi|) + \beta(B_\xi|\xi||\xi|^2/\lambda = z_1(\xi) + z_2(\lambda, \xi),\]
where \(z_2(\lambda, \xi) := \beta(B_\xi|\xi|)|\xi|^2/\lambda\). Let \(\phi_j = \arg z_j\) then a short computation shows that
\[|z_1 + z_2| \geq C(\phi_1, \phi_2)(|z_1| + |z_2|),\]
provided that \(|\phi_1 - \phi_2| < \pi\). Here
\[C(\phi_1, \phi_2) := \frac{1}{\sqrt{2}} \min\{1, (1 + \cos(\phi_1 - \phi_2))^{1/2}\}\]
From Proposition 2.1 and the Cauchy-Schwarz inequality we obtain
\[\left|1 - (a|\xi|(c|\xi|) + \beta(B_\xi|\xi|)\right| \leq C|a + c| \frac{|\xi|}{1 + |\xi|^2} \leq C|a + c| < \infty,\]
hence \(|\phi_1| \leq \sigma < \pi/2\) for all \(\xi \in \mathbb{R}^n\). Since \(|\phi_2| = |\arg \lambda| < \phi\) we have
\[|\phi_1 - \phi_2| \leq \sigma + \phi < \pi,\]
provided \(\phi > \pi/2\) is sufficiently close to \(\pi/2\) and this in turn yields together with Proposition 2.1
\[|\tilde{m}(\lambda, \xi)| = |z_1 + z_2| \geq C(|z_1| + |z_2|) \geq C(1 + |\xi|^2 + |\xi|^4/|\lambda|)
\]
or equivalently
\[(2.4) \quad |m(\lambda, \xi)| \geq C(|\lambda|(1 + |\xi|^2) + |\xi|^4).\]
Observe that the converse is also true, i.e. there is a constant \(C > 0\) such that
\[|m(\lambda, \xi)| \leq C(|\lambda|(1 + |\xi|^2) + |\xi|^4).
\]
In particular it holds that \(m(\lambda, \xi) = 0\) if and only if \(|\lambda| + |\xi| = 0\).
Next, let \(v_0, v_1 \in H^1(J; H^1_0(\mathbb{R}^n)) \cap L^2(J; H^2_0(\mathbb{R}^n))\) be the unique solutions of
\[\partial_t(I - \Delta)v_0 + \Delta^2v_0 = F - c \cdot \nabla F, \quad t > 0, \quad x \in \mathbb{R}^n,
\]
\[v_0(0) = 0,
\]
and
\[\partial_t(I - \Delta)v_1 + \Delta^2v_1 = (I - \Delta)^{1/2}F, \quad t > 0, \quad x \in \mathbb{R}^n,
\]
\[v_1(0) = 0.
\]
The existence of \(v_0\) and \(v_1\) may be seen by the Dore-Venni-Theorem. It follows that
\[\partial_t(I - \Delta)v + \Delta^2v = S(\partial_t(I - \Delta) + \Delta^2)v_0,
\]
and
\[(I - \Delta)^{3/2} \eta = S(I - \Delta)(\beta \partial_x - \Delta)v_1,\]
where the linear operator $S$ is defined by its Fourier-Laplace symbol
\[\hat{S}(\lambda, \xi) = \frac{\lambda(1 + |\xi|^2) + |\xi|^4}{m(\lambda, \xi)}.\]

Note that the assertion of Theorem 2.2 follows if we can show that $S$ is a bounded operator from $L_p(J; L_p(\mathbb{R}^n))$ to $L_p(J; L_p(\mathbb{R}^n))$. This will be a consequence of the classical Mikhlin multiplier theorem and the Kalton-Weis Theorem 4.5. It is not difficult to show that the symbol $\hat{S}(\lambda, \xi)$ satisfies the Mikhlin condition
\[(M) \quad \max_{|\alpha| \leq [n/2]+1} \sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |\partial_\xi^\alpha \hat{S}(\lambda, \xi)| < \infty,
\]
where $\alpha \in \mathbb{N}^n_0$ is a multiindex and $|s|$ denotes the largest integer not exceeding $s \in \mathbb{R}$.

The classical Mikhlin multiplier theorem then implies that $\hat{S}$ is a Fourier multiplier in $L_p(\mathbb{R}^n; \mathbb{C})$ w.r.t. the variable $\xi$ and this yields a holomorphic uniformly bounded family $\{\hat{S}(\lambda)\}_{\lambda \in \Sigma_\rho} \subset \mathcal{B}(L_p(\mathbb{R}^n; \mathbb{C})), \rho > \pi/2$. By Theorem 3.2 this family is also $\mathcal{R}$-bounded in $L_p(J; L_p(\mathbb{R}^n; \mathbb{C}))$ (for the notion of $\mathcal{R}$-boundedness we refer the reader to [5]). Finally, since the operator $\partial_t$ admits a bounded $\mathcal{H}\infty$-calculus with angle $\pi/2$ we obtain from Theorem 4.5 the desired property of the operator $S$. For the functions $u = (I - \Delta)^{1/2}w$ and $\mu = (I - \Delta)^{1/2}\eta$, this yields
\[u \in H^1_p(J; \mathcal{H}^1_p(\mathbb{R}^n)) \cap L_p(J; \mathcal{H}^2_p(\mathbb{R}^n)),\]
as well as
\[\mu \in L_p(J; \mathcal{H}^2_p(\mathbb{R}^n)),\]
and the proof is complete.

For later purpose we need a perturbation result. To be precise we consider coefficients $a, c$ and $B$ with a small deviation from constant ones, i.e.
\[a(x) = a_0 + a_1(x), \quad c(x) = c_0 + c_1(x), \quad B(x) = B_0 + B_1(x),\]
with $a_1, c_1 \in W^1_\infty(\mathbb{R}^n; \mathbb{R}^n)$, $B_1 \in W^1_\infty(\mathbb{R}^n; \mathbb{R}^{n \times n})$ and
\[|a_1|_\infty + |c_1|_\infty + |B_1|_\infty \leq \omega.\]
Furthermore we assume that $\text{div } a_1(x) = \text{div } c_1(x) = 0$ for a.e. $x \in \mathbb{R}^n$ and that the quadruple $(\beta, a_0, c_0, B_0)$ satisfies (H). Observe that if $\omega > 0$ is sufficiently small, then $(\beta, a(x), c(x), B(x))$ satisfy (H) as well for all $x \in \overline{\Omega}$, with a possibly smaller constant $\varepsilon > 0$.

We have the following result.

**Corollary 2.3.** Under the above assumptions on the coefficients the statement of Theorem 2.2 remains true, if $\omega > 0$ is sufficiently small.

**Proof.** By a shift of the function $u$ we may assume that $u_0 = g = 0$. For the time being we consider an interval $J_\delta = [0, \delta]$, with a suitable small $\delta > 0$, to be chosen later. The corresponding function spaces are denoted by $X^1_\delta$ and $Z^2_\delta$. Moreover
\[0Z^1_\delta : = \{ u \in Z^1_\delta : u|_{t=n} = 0 \}.
\]
Assume that we already know a solution $(u, \mu) \in 0Z^1_\delta \times Z^2_\delta$ of (2.1). Thanks to Theorem 2.2 we have a solution operator $S \in \mathcal{B}(X^1_\delta \times X^2_\delta \times X_p; 0Z^1_\delta \times Z^2_\delta)$ for the
constant coefficient case \((\beta, a_0, c_0, B_0)\). With the help of \(S\) we write the solution in the following way.

\[
\begin{bmatrix}
u \\
\mu
\end{bmatrix} = S \begin{bmatrix} f \\
0
\end{bmatrix} + ST \begin{bmatrix} u \\
\mu
\end{bmatrix},
\]

where

\[
T \begin{bmatrix} u \\
\mu
\end{bmatrix} := \begin{bmatrix} \text{div}(a_1(x)\partial_1 u) + \text{div}(B_1(x)\nabla \mu) \\
c_1(x) \cdot \nabla \mu
\end{bmatrix}.
\]

From the boundedness of \(S\) and since \(\text{div} a_1(x) = 0\), \(x \in \mathbb{R}^n\), we obtain the estimate

\[
|\{u, \mu\}|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} \leq C(|f|_{\mathbb{Z}_1^4} + |T(u, \mu)|_{\mathbb{Z}_2^4})
\]

\[
\leq C(|f|_{\mathbb{Z}_1^4} + \omega|\{u, \mu\}|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} + \|\nabla \mu\|_{L_p(J_\delta; L_2(\mathbb{R}^n))}),
\]

for some constant \(C > 0\). The problem is that the term \(\|\nabla \mu\|\) does not become small in \(L_p(J_\delta; L_2(\mathbb{R}^n))\), since the function \(\mu\) has no regularity w.r.t. the variable \(t\). However, we have the following result.

**Proposition 2.4.** Let \((u, \mu) \in \mathbb{Z}_1^4 \times \mathbb{Z}_2^4\) be a solution of (2.1) with \(g = u_0 = 0\). Assume furthermore that the (variable) coefficients satisfy the above assumptions. Then there exists a constant \(C > 0\), independent of \(J_\delta\), such that the estimate

\[
|\mu|_{L_p(J_\delta; H^1_2(\mathbb{R}^n))} \leq C(|f|_{\mathbb{Z}_1^4} + |\omega|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} + |u|_{L_p(J_\delta; H^2_2(\mathbb{R}^n))})
\]

is valid.

**Proof.** The proof follows the lines of the proof of Proposition 3.3. \qed

Owing to (2.5) and Proposition 2.4 we obtain the estimate

\[
|\{u, \mu\}|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} \leq C(|f|_{\mathbb{Z}_1^4} + |\omega|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} + |u|_{L_p(J_\delta; H^2_2(\mathbb{R}^n))}).
\]

The mixed derivative theorem and Sobolev embedding yield \(0H^1_0(J_\delta; H^1_2(\mathbb{R}^n)) \cap L_p(J_\delta; H^{\delta/2}_2(\mathbb{R}^n)) \hookrightarrow 0H^{1/2}_p(J_\delta; H^{\delta}_2(\mathbb{R}^n)) \hookrightarrow L_{2\delta}(J_\delta; H^{\delta}_2(\mathbb{R}^n))\), hence by Hölder’s inequality we obtain

\[
|u|_{L_p(J_\delta; H^{\delta}_2(\mathbb{R}^n))} \leq C|u|_{\mathbb{Z}_2^4} + \|\omega\|_{\mathbb{Z}_1^4 \times \mathbb{Z}_2^4} + |u_0|_{X_\delta},
\]

where \(C > 0\) is some constant. The latter estimates show that the operator \(L \in B(Z_\delta^1 \times Z_\delta^2; X_\delta^1 \times X_\delta^2 \times X_\delta^p)\), defined by

\[
L \begin{bmatrix} u \\
\mu
\end{bmatrix} = \begin{bmatrix} \partial_t u - \text{div}(a \partial_1 u) - \text{div}(B \nabla \mu) \\
\mu - c \cdot \nabla \mu - \beta \partial_1 u + \Delta u
\end{bmatrix},
\]

is injective and has closed range, hence \(L\) is a semi-Fredholm operator. Replacing the coefficients \((\beta, a, c, B)\) by

\[
(\beta, a_\tau, c_\tau, B_\tau) := (1 - \tau)(\beta, a_0, c_0, B_0) + \tau(\beta, a, c, B), \tau \in [0, 1]
\]

we may conclude from the considerations above that for each \(\tau \in [0, 1]\) the corresponding operator \(L_\tau\) is semi-Fredholm as well. The continuity of the Fredholm index yields that the index of \(L_1 = L\) is 0, since \(L_0\) is an isomorphism, by Theorem
A successive application of the above procedure yields the claim for the time interval $J = [0, T]$. The proof is complete.

3. THE LINEAR CAHN-HILLIARD-GURTIN PROBLEM IN $\mathbb{R}^n_+$

In order to treat the case of a half space, we consider first constant coefficients which are subject to the following assumptions: $B = bI$ and $(a|e_n) = (c|e_n) = 0$, where $e_n := [0, \ldots, 0, -1]^T$ is the outer unit normal at $\partial \mathbb{R}^n_+$. Furthermore we assume that $(\beta, a, c, B)$ satisfy (H), whence it holds that $b \geq \varepsilon > 0$. Moreover the boundary conditions on $a$ and $c$ yield that the last components of $a$ and $c$ are identically zero. We are interested to solve the following system in $\mathbb{R}^n_+$.

$$
\begin{align*}
\partial_t u - \text{div}(a\partial_t u) &= b\Delta u + f, \quad t > 0, \quad (x', y) \in \mathbb{R}^n_+, \\
\mu - c \cdot \nabla u &= \beta \partial_t u - \Delta u + g, \quad t > 0, \quad (x', y) \in \mathbb{R}^n_+, \\
\partial_y \mu &= h_1, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0, \\
\partial_y u &= h_2, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0, \\
u(0) &= u_0, \quad t = 0, \quad (x', y) \in \mathbb{R}^n_+.
\end{align*}
$$

Note that the conormal boundary condition $(b\nabla e_n) = h_1$ is equivalent to $-b\partial_y \mu = h_1$, where $b > 0$ is constant. Hence it suffices to consider the boundary condition $\partial_y \mu = h_1$ with some scaled function $h_1$. Concerning optimal $L_p$-regularity of (3.1) we have the following result.

**Theorem 3.1.** Let $1 < p < \infty$, $p \neq 3/2$ and assume that (H) holds true. Then (3.1) admits a unique solution

$$
\begin{align*}
u \in H^1_p(J; H^1_p(\mathbb{R}^n_+)) \cap L_p(J; H^3_p(\mathbb{R}^n_+)) =: Z^1, \\
\mu \in L_p(J; H^2_p(\mathbb{R}^n_+)) =: Z^2,
\end{align*}
$$

if and only if the data is subject to the following conditions.

(i) $f \in L_p(J; L_p(\mathbb{R}^n_+)) =: X^1$,  
(ii) $g \in L_p(J; H^1_p(\mathbb{R}^n_+)) =: X^2$, 
(iii) $h_1 \in L_p(J; W^{1-1/p}(\mathbb{R}^{n-1})) =: Y^1$, 
(iv) $h_2 \in W^{1-1/p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W^{2-1/p}(\mathbb{R}^{n-1})) =: Y^2$, 
(v) $u_0 \in B^{3-2/p}(\mathbb{R}^n_+) =: X_p$. 
(vi) $\partial_y u_0 = h_2|_{t=0}$ if $p > 3/2$.

**Proof.** The necessity part follows from the equations and trace theory, cf. [6]. Concerning sufficiency, we first reduce (3.1) to the case $h_1 = h_2 = u_0 = 0$. For this purpose we solve the elliptic problem

$$
\begin{align*}
(I - \Delta_{x'})\eta - \partial_y^2 \eta &= 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0, \\
\partial_y \eta &= h_1, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0.
\end{align*}
$$

Define $\tilde{L} := (I - \Delta_{x'})^{1/2}$ in $L_p(\mathbb{R}^{n-1})$, with $D(\tilde{L}) = H^1_p(\mathbb{R}^{n-1})$ and let $L$ denote the natural extension of $\tilde{L}$ to $L_{p,\text{loc}}(\mathbb{R}^+; L_p(\mathbb{R}^{n-1}))$, that is $D(L) = L_{p,\text{loc}}(\mathbb{R}^+; H^1_p(\mathbb{R}^{n-1}))$ and $Lu = \tilde{L}u$ for each $u \in D(\tilde{L})$. Then the unique solution $\eta$ of (3.2) is given by

$$
\eta(y) = -L^{-1}e^{-Ly}h_1.
$$
Since $h_1 \in L_p(J; W^{1-1/p}_p(\mathbb{R}^n)) = D_L(1 - 1/p, p)$, we have $e^{-L_y h_1} \in D(L)$ and therefore $\eta \in L_p(J; H^2_p(\mathbb{R}^n))$, with $\partial_y \eta_{|y=0} = h_1$. In order to remove $h_2$ and $u_0$, we solve the initial boundary value problem

$$
\begin{align*}
\beta \partial_t v - \Delta_y v - \partial_y^2 v &= 0, \quad t > 0, \quad y > 0, \\
\partial_y v &= h_2, \quad t > 0, \quad y > 0, \\
v(0) &= u_0, \quad t = 0, \quad y > 0, \\
v_0 &= u, \quad t = 0, \quad y > 0.
\end{align*}
$$

(3.3)

To this end we extend $u_0 \in B_{pp}^{3-2/p}(\mathbb{R}^n_+)$ to a function $\tilde{u}_0 \in B_{pp}^{3-2/p}(\mathbb{R}^n)$ and solve the heat equation

$$
\begin{align*}
\beta \partial_t v - \Delta v &= 0, \quad t > 0, \quad x \in \mathbb{R}^n, \\
\tilde{v}(0) &= \tilde{u}_0, \quad t = 0, \quad x \in \mathbb{R}^n,
\end{align*}
$$

in $L_p(J; H^1_p(\mathbb{R}^n))$. This yields a solution

$$
\tilde{v} \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)).
$$

If $v_1 := P \tilde{u}$ denotes the restriction of $\tilde{v}$ to the half space $\mathbb{R}^n_+$, the function $v_2 := v - v_1$ should solve the initial boundary value problem

$$
\begin{align*}
\beta \partial_t v_2 - \Delta_y v_2 - \partial_y^2 v_2 &= 0, \quad t > 0, \quad y > 0, \\
\partial_y v_2 &= \tilde{h}_2, \quad t > 0, \quad y > 0, \\
v_2(0) &= 0, \quad t = 0, \quad y > 0,
\end{align*}
$$

(3.4)

where $\tilde{h}_2 := h_2 - \partial_y v_1_{|y=0}$. Set $v_3 = (I - \Delta_y)^{1/2} v_2$. Then $v_3$ is a solution of

$$
\begin{align*}
\beta \partial_t v_3 - \Delta_y v_3 - \partial_y^2 v_3 &= 0, \quad t > 0, \quad y > 0, \\
\partial_y v_3 &= h_3, \quad t > 0, \quad y > 0, \\
v_3(0) &= 0, \quad t = 0, \quad y > 0,
\end{align*}
$$

(3.5)

with $h_3 = (I - \Delta_y)^{1/2} \tilde{h}_2 \in W^{1/2-1/p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n))$. We define $L = (\beta \partial_t - \Delta_y)^{1/2}$ with natural domain

$$
D(L) = W^{1/2}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^1_p(\mathbb{R}^n)).
$$

Then, the unique solution $v_3$ of (3.5) is given by

$$
\begin{align*}
v_3(y) &= -L^{-1} e^{-L_y_h_3},
\end{align*}
$$

and $h_3 \in D_L(1 - 1/p, p)$. This yields

$$
\begin{align*}
v_3 \in W^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)).
\end{align*}
$$

On the other hand, if we consider the function $v_4 := \partial_y v_2$ as the solution of

$$
\begin{align*}
\beta \partial_t v_4 - \Delta_y v_4 - \partial_y^2 v_4 &= 0, \quad t > 0, \quad y > 0, \\
v_4 &= \tilde{h}_2, \quad t > 0, \quad y > 0, \\
v_4(0) &= 0, \quad t = 0, \quad y > 0,
\end{align*}
$$

(3.6)

we obtain $v_4(y) = e^{-L_y \tilde{h}_2}$ and $\tilde{h}_2 \in D_L(2 - 1/p, p)$. This yields

$$
\begin{align*}
v_4 \in W^1_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; H^2_p(\mathbb{R}^n)).
\end{align*}
$$

From the regularity of $v_3$ and $v_4$ we may conclude that

$$
\begin{align*}
v_2 \in W^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)).
\end{align*}
$$
Now the functions $u_1 := u - v$ and $\mu_1 := \mu - \eta$, with $v = v_1 + v_2$, should solve the system

$$
\partial_t u_1 - \text{div}(a\partial_t u_1) = b\Delta u_1 + f_1, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
$$

$$
\mu_1 - c \cdot \nabla u_1 = \beta \partial_t u_1 - \Delta u_1 + g_1, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
$$

(3.7)

$$
\partial_y \mu_1 = 0, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0,
$$

$$
\partial_y u_1 = 0, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0,
$$

$$
u_1(0) = 0, \quad t = 0, \quad x' \in \mathbb{R}^{n-1}, \quad y > 0,
$$

with some modified data $f_1 \in X^1$ and $g_1 \in X^2$. In a next step we extend the functions $f_1$ and $g_1$ w.r.t. the spatial variable to $\mathbb{R}^n$ by even reflection, i.e. we set

$$
f_2(t, x', y) = \begin{cases} f_1(t, x', y), & \text{if } y \geq 0 \\ f_1(t, x', -y), & \text{if } y < 0 \end{cases} \quad \text{and} \quad g_2(t, x', y) = \begin{cases} g_1(t, x', y), & \text{if } y \geq 0 \\ g_1(t, x', -y), & \text{if } y < 0 \end{cases}.
$$

Thanks to Theorem 2.2 we can solve the full space problem

$$
\partial_t u_2 - \text{div}(a\partial_t u_2) = b\Delta u_2 + f_2, \quad t > 0, \quad x \in \mathbb{R}^n,
$$

(3.8)

$$
\mu_2 - c \cdot \nabla u_2 = \beta \partial_t u_2 - \Delta u_2 + g_2, \quad t > 0, \quad x \in \mathbb{R}^n,
$$

$$
u_2(0) = 0, \quad t = 0, \quad x \in \mathbb{R}^n,
$$

since $f_2 \in L_p(J; L_p(\mathbb{R}^n))$ and $g_2 \in L_p(J; H^1_p(\mathbb{R}^n))$. This yields a unique solution $u_2 \in H^1_p(J; H^1_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n))$ and $\mu_2 \in L_p(J; H^2_p(\mathbb{R}^n))$.

By Theorem 2.2 at this point we emphasize that the equations (3.7)$_{1,2}$ are invariant w.r.t. even reflection on the hyper surface $\mathbb{R}^{n-1} \times \{0\}$ in the normal variable $y$, due to the structure of the coefficients. This in turn implies that the solution $(u_2, \mu_2)$ is symmetric, w.r.t the variable $y$ and this yields necessarily, $\partial_y u_2|_{y=0} = \partial_y \mu_2|_{y=0} = 0$.

Denoting by $P$ the restriction of the solution $(u_2, \mu_2)$ to the half space $\mathbb{R}^n_+$, it follows that $(u_1, \mu_1) = P(u_2, \mu_2)$ is the unique solution of (3.7) and therefore $u = u_1 + u_2$ and $\mu = \eta + \mu_1$ is the unique solution of (3.1). The proof is complete.

For later purposes we will need the following perturbation result. Let $B_0 = b_0 I$,

$$
a(x) = a_0 + a_1(x), \quad c(x) = c_0 + c_1(x), \quad B(x) = B_0 + B_1(x), \quad D(x) = I + D_1(x)
$$

with $a_1, c_1 \in W^1_{\infty}(\mathbb{R}^n_+, \mathbb{R}^n)$, $B_1 \in W^1_{\infty}(\mathbb{R}^n_+, \mathbb{R}^{n \times n})$, $D_1 \in W^2_{\infty}(\mathbb{R}^n_+, \mathbb{R}^{n \times n})$ and

$$
|a_1|_{\infty} + |c_1|_{\infty} + |B_1|_{\infty} + |D_1|_{\infty} \leq \omega,
$$

for some $\omega > 0$. Let furthermore $\text{div} a_1(x) = \text{div} c_1(x) = 0$ for a.e. $x \in \mathbb{R}^n_+$ and

$$
(a_0 |\nu(x)) = (a_1(x)|\nu(x)) = (c_0 |\nu(x)) = (c_1(x)|\nu(x)) = 0.
$$

If the constant coefficients $(\beta, a_0, c_0, B_0)$ satisfy Hypothesis (H) we have the following result.

**Corollary 3.2.** Let $1 < p < \infty$, $p \neq 3/2$, $\beta > 0$ and suppose that the data satisfies the conditions (i)-(v) of Theorem 2.1 and $(D \nabla u_0|_{\eta=0}) = h_2|_{\eta=0}$ if $p > 3/2$. Under the above assumptions on the coefficients $(a, c, B, D)$, there exists a unique solution

$$
u \in H^3_p(J; H^3_p(\mathbb{R}^n_+)) \cap L_p(J; H^3_p(\mathbb{R}^n_+)),
$$

$$
\mu \in L_p(J; H^2_p(\mathbb{R}^n_+)),
$$
of the system
\[ \partial_t u - \text{div}(a \partial_t u) = \text{div}(B \nabla \mu) + f, \quad t > 0, \quad (x', y) \in \mathbb{R}_+^n, \]
\[ \mu - (c \nabla \mu) = \beta \partial_t u - \text{div}(D \nabla u) + g, \quad t > 0, \quad (x', y) \in \mathbb{R}_+^n, \]
(3.9)
\[ (B \nabla u) = h_1, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0, \]
\[ (D \nabla u) = h_2, \quad t > 0, \quad x' \in \mathbb{R}^{n-1}, \quad y = 0, \]
\[ u(0) = u_0, \quad t = 0, \quad (x', y) \in \mathbb{R}_+^n, \]
provided \( \omega > 0 \) is sufficiently small.

Proof. First of all, we reduce (3.9) to the case \( u_0 = 0 \) as follows. Extend the initial data \( u_0 \in B_{\delta}^{3-2/p}(\mathbb{R}_+^n) \) to some \( \tilde{u}_0 \in B_{\delta}^{3-2/p}(\mathbb{R}^n) \) and solve the heat equation
\[ \partial_t v - \Delta v = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \]
\[ v(0) = \tilde{u}_0, \quad x \in \mathbb{R}^n, \]
to obtain a unique solution
\[ v \in H^1_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; H^3_p(\mathbb{R}^n)) = Z^1, \]
for some interval \( J = [0, T] \). If \( (u_0, \mu) \in Z^1 \times Z^2 \) is a solution of (3.9), then the shifted function \( (u - v, \mu) \in [0]Z^1 \times Z^2 \) solves (3.9) with \( u_0 = 0 \) and some modified functions \( \tilde{f}, \tilde{g}, \tilde{h}_2 \) depend only on \( f, g, h_2 \) and the fixed function \( \tilde{v} \in Z^1 \) from above. In the sequel we will not rename the functions \( u, f, g \) and \( h_2 \).

By the structure of the coefficients and by trace theory we obtain the estimate
\[ \|(u, \mu)|_{Z^1 \times Z^2} \]
\[ \leq C(|f|_{X^1_2} + |g|_{X^2_2} + |h_1|_{Y^1_2} + |h_2|_{Y^2_2} + \omega)(u, \mu)|_{Z^1_2 \times Z^2_2} + |u|_{L_p(J; H^2_p(\mathbb{R}^n))} + |\nabla \mu|_{L_p(J; L_p(\mathbb{R}^n))}, \]
with a constant \( C > 0 \) which does not depend on \( \delta > 0 \) since \( u_0 = 0 \). The derivation of this estimate follows the lines of the proof of Corollary 2.3. The term \( |u|_{L_p(J; H^2_p(\mathbb{R}^n))} \) is of lower order and may be estimated by
\[ |u|_{L_p(J; H^2_p(\mathbb{R}^n))} \leq \delta^{1/2} \mu |_{Z^1_2 \times Z^2_2}, \]
hence this term may be compensated by the left side of the latter estimate if \( \delta > 0 \) is small enough. If in addition \( \omega > 0 \) is sufficiently small, the same is true for \( \omega(u, \mu)|_{Z^1_2 \times Z^2_2}. \) To estimate the term \( |\nabla \mu| \) in \( L_p(J; L_p(\mathbb{R}^n)) \), we use the following proposition whose proof is given in the Appendix.

**Proposition 3.3.** Let \( (u, \mu) \in Z^1_2 \times Z^2_2 \) be a solution of (3.9) with \( u_0 = 0 \). Then there exists a constant \( C > 0 \), independent of \( J_\delta \), such that the estimate
(3.10)
\[ |\mu|_{L_p(J; H^1_2(\mathbb{R}^n))} \leq C(|f|_{X^1_2} + |g|_{X^2_2} + |h_1|_{Y^1_2} + |u|_{L_p(J; H^2_2(\mathbb{R}^n))}) \]
is valid.

Now the claim follows by applying a similar homotopy argument as in the proof of Corollary 2.3. □
4. Bounded domains, Localization

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain with boundary \( \partial \Omega \in C^3 \). In this section we solve the system

\[
\begin{align*}
\partial_t u - \text{div}(a\partial_t u) &= \text{div}(b\nabla \mu) + f, \quad t > 0, \ x \in \Omega, \\
\mu - c \cdot \nabla \mu &= \beta \partial_t u - \Delta u + g, \quad t > 0, \ x \in \Omega, \\
b\nabla \mu \cdot \nu &= h_1, \quad t > 0, \ x \in \partial \Omega, \\
\partial_n u &= h_2, \quad t > 0, \ x \in \partial \Omega, \\
u(0) &= u_0, \quad t = 0, \ x \in \Omega,
\end{align*}
\]

(4.1)

with coefficients \( a, c \in [C^1(\overline{\Omega})]^n \) and \( b \in C^1(\overline{\Omega}) \). We furthermore assume that \( \text{div} a(x) = \text{div} c(x) = 0, \ x \in \Omega, \ \{a(x)|\nu(x)\} = \{c(x)|\nu(x)\} = 0, \ x \in \partial \Omega \) and (\( \beta, a, c, b \)) satisfy (H). Before we start with the localization procedure we prove two lemmata, which are interesting for their own.

**Lemma 4.1.** Let \( 1 < p < \infty, \ p \neq 3/2, \ J = [0, T] \) and \( \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \in C^3 \). Then for each \( \beta > 0 \) the initial-boundary value problem

\[
\begin{align*}
\beta \partial_t u - \Delta u &= f, \quad t \in J, \ x \in \Omega, \\
\partial_n u &= g, \quad t \in J, \ x \in \partial \Omega, \\
u(0) &= u_0, \quad t = 0, \ x \in \Omega,
\end{align*}
\]

(4.2)

admits a unique solution

\[ u \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)), \]

if and only if the data are subject to the following conditions.

(i) \( f \in L_p(J; H^1_p(\Omega)), \)

(ii) \( g \in W^{1-1/2p}_p(J; L_p(\partial \Omega)) \cap L_p(J; W^{2-1/p}_p(\partial \Omega)), \)

(iii) \( u_0 \in B^{3-2/p}_p(\Omega), \)

(iv) \( \partial_n u_0 = g|_{t=0}, \) provided \( p > 3/2. \)

**Proof.** The 'only if' part follows from the equations and well known result in trace theory. Indeed, given a solution

\[ u \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)), \]

of (4.2) it follows directly that \( f \in L_p(J; H^1_p(\Omega)), \) Furthermore it holds that

\[ H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)) \hookrightarrow C(J; (H^1_p(\Omega); H^3_p(\Omega))_{1-1/p,p}) = C(J; B^{3-2/p}_p(\Omega)), \]

by trace- and interpolation theory. Hence \( u(0) \in B^{3-2/p}_p(\Omega). \) Finally observe that

\[ \nabla u \in H^1_p(J; L_p(\Omega)) \cap L_p(J; H^2_p(\Omega)). \]

Taking the trace of \( \nabla u \) on \( \partial \Omega \) yields

\[ \nabla u|_{\partial \Omega} \in W^{1-1/2p}_p(J; L_p(\partial \Omega)) \cap L_p(J; W^{2-1/p}_p(\partial \Omega)), \]

the required regularity for \( g. \) Finally, since

\[ W^{1-1/2p}_p(J; L_p(\partial \Omega)) \cap L_p(J; W^{2-1/p}_p(\partial \Omega)) \rightarrow C(J; B^{2-3/p}_p(\partial \Omega)), \]

it follows that \( \partial_n u(0) = g|_{t=0} \) in case \( p > 3/2. \) To prove sufficiency of the conditions (i)-(iv), note that by the results of Sections 2 & 3 the unique solution of the corresponding full space and half space problem to (4.2) possess the desired regularity.
Then the claim for a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial \Omega \in C^3$ follows from localization, change of coordinates and perturbation theory, cf. [5].

The second lemma provides maximal regularity of (4.1) in case $a = c = 0$ and $b = 1$, the so-called viscous Cahn-Hilliard equation in its linear form.

**Lemma 4.2.** Let $1 < p < \infty$, $p \neq 3/2$, $J = [0, T]$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial \Omega \in C^3$. Then for each $\beta > 0$ the system

\[
\begin{align*}
\partial_t u - \Delta \mu &= f, \quad t \in J, \ x \in \Omega, \\
\mu - \beta \partial_t u + \Delta u &= g, \quad t \in J, \ x \in \Omega, \\
\partial_\nu \mu &= h_1, \quad t \in J, \ x \in \partial \Omega, \\
\partial_\nu u &= h_2, \quad t \in J, \ x \in \partial \Omega, \\
u(0) &= u_0, \quad t = 0, \ x \in \Omega,
\end{align*}
\]

admits a unique solution

\[
\begin{align*}
u &\in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)), \\
\mu &\in L_p(J; H^2_p(\Omega)),
\end{align*}
\]

if and only if the data are subject to the following conditions.

(i) $f \in L_p(J; L_p(\Omega))$,
(ii) $g \in L_p(J; H^1_p(\Omega))$,
(iii) $h_1 \in L_p(J; W^{1-1/p}_p(\partial \Omega))$,
(iv) $h_2 \in W^{1-2p}_p(J; L_p(\partial \Omega)) \cap L_p(J; W^{2-1/p}_p(\partial \Omega))$,
(v) $u_0 \in B^{p-2/p}_p(\Omega)$,
(vi) $\partial_\nu u_0 = h_2|_{t=0}$, provided $p > 3/2$.

**Proof.** By Lemma 4.1 there exists a unique solution

\[
v \in H^1_p(J; H^1_p(\Omega)) \cap L_p(J; H^3_p(\Omega)).
\]

of the problem

\[
\begin{align*}
\beta \partial_t v - \Delta v &= -g, \quad t \in J, \ x \in \Omega, \\
\partial_\nu v &= h_2, \quad t \in J, \ x \in \partial \Omega, \\
v(0) &= u_0, \quad t = 0, \ x \in \Omega.
\end{align*}
\]

Hence, w.l.o.g. we may assume $g = h_2 = u_0 = 0$ in (4.2), with $f$ being replaced by some modified function $f \in L_p(J; L_p(\Omega))$, which depends at most on the fixed functions $f$ and $v$.

Now we want to reduce (4.3) to a single equation for $u$. Suppose that we already know a solution of (4.3). Inserting (4.3) into (4.3) yields the elliptic problem

\[
\begin{align*}
\mu - \beta \Delta \mu &= \beta f - \Delta u, \quad t \in J, \ x \in \Omega, \\
\partial_\nu \mu &= h_1, \quad t \in J, \ x \in \partial \Omega,
\end{align*}
\]

for the function $\mu$. It is well-known that for each $\beta > 0$ the latter problem admits a unique solution $\mu \in L_p(J; H^2_p(\Omega))$, provided $(\beta f - \Delta u) \in L_p(J; L_p(\Omega))$ and $h_1 \in L_p(J; W^{1-1/p}_p(\partial \Omega))$. Denoting by $S$ the corresponding solution operator, we may write

\[
\mu = S \begin{bmatrix} \beta f \\ h_1 \end{bmatrix} - S \begin{bmatrix} \Delta u \\ 0 \end{bmatrix}.
\]
Inserting this expression into \( (4.3) \), we obtain the problem

\[
\begin{align*}
\beta \partial_t u - \Delta u &= h - \hat{S} u, \quad t \in J, \ x \in \Omega, \\
\partial_x u &= 0, \quad t \in J, \ x \in \partial \Omega,
\end{align*}
\]

(4.4)

where \( h := S(\beta f, h_1) \) and \( \hat{S} u := S(\Delta u, 0) \). Since \( S \) is a bounded linear operator from \( L_p(J; L_p(\Omega)) \times L_p(J; W_p^{1-1/p}(\partial \Omega)) \) to \( L_p(J; H_p^2(\Omega)) \) it follows that \( \hat{S} \) is bounded and linear from \( L_p(J; H_p^2(\Omega)) \) to \( L_p(J; H_p^2(\Omega)) \). Thanks to Lemma 4.3 there exists a solution operator \( \mathcal{T} \) of (4.2) which is a linear and bounded mapping from

\[
L_p(J; H_p^1(\Omega)) \times \omega W_p^{1-1/2p}(J; L_p(\partial \Omega)) \cap L_p(J; W_p^{2-1/p}(\partial \Omega)) \times B_p^{3-2/p}(\Omega)
\]
to \( \omega H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) \). With the help of \( \mathcal{T} \) we may write

\[
u = \mathcal{T} \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} - \mathcal{T} \begin{bmatrix} \hat{S} u \\ 0 \\ 0 \end{bmatrix}.
\]

We estimate

\[
|\mathcal{T}(\hat{S} u, 0, 0)|_Z \leq C|\hat{S} u|_{L_p(J; H_p^1(\Omega))} \leq C|u|_{L_p(J; H_p^2(\Omega))} \leq CT^{1/2p}|u|_{L_p(J; H_p^2(\Omega))} \leq CT^{1/2p}|u|_Z,
\]

by Hölder’s inequality. Here the constant \( C > 0 \) does not depend on \( T > 0 \), since the time traces at \( t = 0 \) are zero. A Neumann series argument yields a unique solution \( u \in Z \) of (4.4) on a (possibly) small time interval \( J = [0, T] \). Since (4.4) is linear and invariant with respect to time shifts, the solution exists global in time.

□

The main result of this section reads as follows.

**Theorem 4.3.** Let \( 1 < p < \infty, \ p \neq 3/2, \ J = [0, T] \). Suppose furthermore that \( a, c \in [C^1(\Omega)]^n, \ b \in C^1(\Omega) \). Then (4.1) admits a unique solution

\[
u \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) = \mathbb{Z}^1, \quad \mu \in L_p(J; H_p^3(\Omega)) = \mathbb{Z}^2,
\]

if and only if the data are subject to the following conditions.

(i) \( f \in L_p(J; L_p(\Omega)) = X^1 \),
(ii) \( g \in L_p(J; H_p^1(\Omega)) = X^2 \),
(iii) \( h_1 \in L_p(J; W_p^{1-1/p}(\Gamma)) = Y^1 \),
(iv) \( h_2 \in W_p^{1-2/2p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{2-1/p}(\Gamma)) = Y^2 \),
(v) \( u_0 \in B_p^{3-2/p}(\Omega) = X_p \),
(vi) \( \partial_x u_0 = h_2 |_{t=0} \) if \( p > 3/2 \).

**Proof.** By Lemma 4.2 we may first reduce (4.1) to the case \( h_1 = h_2 = u_0 = 0 \) and some modified functions \( f, g \) in the right regularity classes. We cover \( \Omega \) by finitely many open sets \( U_k, \ k = 1, \ldots, N \), which are subject to the following conditions.

(i) \( U_k \cap \Gamma = \emptyset \) and \( U_k = B_{r_k}(x_k) \) for all \( k = 1, \ldots, N \);
(ii) \( U_k \cap \Gamma \neq \emptyset \) for \( k = N+1, \ldots, N \).
We choose next a partition of unity \( \{ \varphi_k \}_{k=1}^N \) such that \( \sum_{k=1}^N \varphi_k(x) = 1 \) on \( \overline{\Omega} \), \( 0 \leq \varphi_k(x) \leq 1 \) and \( \text{supp} \varphi_k \subset U_j \). Note that \((u, \mu)\) is a solution of \((4.1)\) if and only if
\[
\partial_t u_k - \text{div}(a \partial_t u_k) = \text{div}(b \nabla \mu_k) + f_k + F_k(u, \mu), \quad t \in [0, \delta], \ x \in \Omega \cap U_k, \ 1 \leq k \leq N,
\]
\[
\mu_k - c \cdot \nabla \mu_k = \beta \partial_t u_k - \Delta u_k + g_k + G_k(u, \mu), \quad t \in [0, \delta], \ x \in \Omega \cap U_k, \ 1 \leq k \leq N
\]
(4.5)
\[
b \nabla \mu_k \cdot \nu = (b \nabla \varphi_k \cdot \nu) \mu, \quad t \in [0, \delta], \ x \in \Gamma \cap U_k, \ N_1 + 1 \leq k \leq N
\]
\[
\partial_x u_k = u \partial_x \varphi_k, \quad t \in [0, \delta], \ x \in \Gamma \cap U_k, \ N_1 + 1 \leq k \leq N
\]
\[
u_k(0) = 0, \quad t = 0, \ x \in \Omega \cap U_k.
\]
Here we have set \( u_k = u \varphi_k \), \( \mu_k = \mu \varphi_k \), \( f_k = f \varphi_k \), \( g_k = g \varphi_k \). The terms \( F_k(u, \mu) \) and \( G_k(u, \mu) \) are defined by
\[
F_k(u, \mu) = -(a \cdot \nabla \varphi_k) \partial_t u - (\nabla b \cdot \nabla \varphi_k) \mu - 2b \nabla \varphi_k \cdot \nabla \mu - b \mu \Delta \varphi_k,
\]
and
\[
G_k(u, \mu) = -(c \cdot \nabla \varphi_k) \mu + 2b u \nabla \varphi_k + u \Delta \varphi_k.
\]
In case \( k = 1, ..., N_1 \) we have no boundary conditions, i.e. we only have to consider the first two equations in \((4.5)\). In order to treat these local problems with the help of Corollary 2.3 we extend the coefficients from \( B_{r_k}(x_k) \) to \( \mathbb{R}^n \) in such a way that \( \text{div} \tilde{a}(x) = \text{div} \tilde{c}(x) = 0, \ x \in \mathbb{R}^n \), holds for the extended coefficients \( \tilde{a} \) and \( \tilde{c} \). Note that w.l.o.g. we may assume \( x_k = 0 \). This follows by a translation in \( \mathbb{R}^n \).

We use the following extension \( \tilde{a} \) of \( a \) (or \( \tilde{c} \) of \( c \)).

\[
\tilde{a}^k(x) = \begin{cases} a(x), & x \in \overline{B_{r_k}(0)}; \\ a \left( \frac{r^2}{r^2} x \right) - 2 \left( \xi \left( a \left( \frac{r^2}{r^2} x \right) \right) \right) \xi + R(r, \xi) \xi, & x \in \mathbb{R}^n \setminus \overline{B_{r_k}(0)}; \end{cases}
\]

where \( r = |x| \), \( \xi = x/|x| \) and \( \xi_j, a_j \) denote the components of \( \xi \) and \( a \), respectively.

The task is to compute the scalar valued function \( R(r, \xi) \). Since \( \text{div} a(x) = 0, \ x \in \Omega \), the divergence of \( a \left( \frac{r^2}{r^2} x \right) \) and \( \xi \left( a \left( \frac{r^2}{r^2} x \right) \right) \) may be computed to the result
\[
\text{div} \left[ a \left( \frac{r^2}{r^2} x \right) \right] = -2 \frac{r^2}{r^2} \sum_{i,j} \xi_i \xi_j \partial_j a_i \left( \frac{r^2}{r^2} x \right)
\]
and
\[
\text{div} \left[ \xi \left( a \left( \frac{r^2}{r^2} x \right) \right) \right] = \frac{(n-1)}{r} \left( \xi \left( a \left( \frac{r^2}{r^2} x \right) \right) \right) - r^2 \frac{r^2}{r^2} \sum_{i,j=1}^n \xi_i \xi_j \partial_j a_i \left( \frac{r^2}{r^2} x \right).
\]

The divergence of the last term \( R(r, \xi) \xi \) is given by
\[
\text{div} \left[ R(r, \xi) \xi \right] = \partial_r R(r, \xi) + \frac{n-1}{r} R(r, \xi).
\]
Finally, this yields that \( \text{div} \tilde{a}^k(x) = 0 \) if and only if the function \( R = R(r, \xi) \) solves the ordinary differential equation
\[
\partial_r R(r, \xi) + \frac{(n-1)}{r} R(r, \xi) = 2 \left( a \left( \frac{r^2}{r^2} \xi \right) \right), \quad r \geq r_k.
\]
In order to achieve \( \tilde{a}^k, \tilde{c} \in W_\infty^1(\mathbb{R}^n; \mathbb{R}^n) \), we require \( \tilde{a}^k(r_k \xi) = a(r_k \xi) \). This yields the initial condition \( R(r_k, \xi) = 2(a(r_k \xi) | \xi) \), hence the function \( R = R(r, \xi) \) is explicitly given by
\[
R(r, \xi) = \frac{r^{n-1}}{r^{n-1}} R(r_k, \xi) + \frac{2(n-1)}{r^{n-1}} \int_{r_k}^r s^{n-2} \left( a \left( \frac{r^2}{s} \xi \right) \right) \xi \ ds, \quad r \geq r_k.
\]
Then the differential operators \( u \) will be done with the help of a suitable transformation. Let
\[
\frac{r_k^2}{s^2} \left( a \left( \frac{r_k^2}{s^2} \right) \right) - 2 \left( a(r_k \xi) \right) \xi
\]
we may write
\[
\tilde{a}_k(x) = a \left( \frac{r_k^2 x}{r^2} \right) - 2 \left( a \left( \frac{r_k^2}{r^2} \right) - a(r_k \xi) \right) \xi
\]
\[
+ \frac{2(n - 1)}{r^{n-1}} \int_{r_k}^r s^{n-2} \left( a \left( \frac{r_k^2}{s} \xi \right) - a(r_k \xi) \xi \right) \, ds,
\]
in case \(|x| > r_k\). Owing to this identity and the assumption \( a, c \in C^1(\overline{\Omega}) \), it is evident that there holds
\[
|\tilde{a}_k(x) - a(0)| + |\tilde{c}_k(x) - c(0)| \leq \omega,
\]
for all \( x \in \mathbb{R}^n \), where \( \omega > 0 \) can be made as small as we wish, by decreasing the radius \( r_k \) of the charts \( U_k \), \( k \in \{1, \ldots, N_1\} \).

For the coefficient function \( b \) we use the reflection method from [5], i.e. we set
\[
\tilde{b}_k(x) = \begin{cases} b(x), & x \in B_{r_k}(0), \\ b \left( \frac{r_k^2 x}{r^2} \right), & x \in \mathbb{R}^n \setminus B_{r_k}(0). \end{cases}
\]
(4.7)

It may be readily checked that \( \tilde{b}_k \in W_{1}^{1}(\mathbb{R}^n) \) and that
\[
|b(0) - \tilde{b}_k(x)| \leq \omega, \quad x \in \mathbb{R}^n,
\]
with the same \( \omega > 0 \) as above. Hence for each chart \( U_k, \ k \in \{1, \ldots, N_1\} \) we have coefficients which fit into the setting of Corollary 2.3. Therefore we obtain corresponding solution operators \( S_k^F \) of (4.5) such that
\[
\begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = S_k^F \begin{bmatrix} f_k + F_k(u, \mu) \\ g_k + G_k(u, \mu) \end{bmatrix},
\]
for each \( k \in \{1, \ldots, N_1\} \).

For the remaining charts \( U_k, \ k \in \{N_1+1, \ldots, N\} \) we obtain problems in perturbed half spaces with inhomogeneous Neumann boundary conditions. For the further analysis we have to understand how to treat (4.1) in such a setting. To this end we fix a point \( x_0 \in \partial \Omega \) and a chart \( U(x_0) \cap \partial \Omega \neq \emptyset \). After a composition of a translation and a rotation in \( \mathbb{R}^n \), we may assume that \( x_0 = 0 \) and \( \nu(x_0) = [0, \ldots, 0, -1] = \epsilon_n \). Consider a graph \( \rho \in C^0(\mathbb{R}^{n-1}) \), having compact support, such that
\[
\{(x', x_n) \in \overline{U(x_0)} : x_n = \rho(x') \} = \partial \Omega \cap \overline{U(x_0)}.
\]
Note that by decreasing the size of the charts we may assume that \(|\nabla x' \rho|_{\infty} \) is as small as we like, since \( \nabla_{x'} \rho(0) = 0 \).

For the time being, we only know that \( \text{div} \, a(x) = \text{div} \, c(x) = 0 \) for all \( x \in U(x_0) \cap \Omega \).

So we have to extend the coefficients \( a \) and \( c \) in a suitable way. To this end we first transform the crooked boundary \( U(x_0) \cap \partial \Omega \) to a straight line in \( \mathbb{R}^{n-1} \times \{0\} \). This will be done with the help of a suitable transformation. Let \( u(x', x_n) = v(g(x)) = v(x', x_n - \rho(x')) \) and \( \mu(x) = \eta(g(x)) = \eta(x', x_n - \rho(x')) \) and \( B_{\rho_0}(x_0) = g(U(x_0)) \).

Then the differential operators \( a \cdot \nabla u \) and \( c \cdot \nabla \mu \) transform as follows.
\[
a(x) \cdot \nabla u(x) = a(x) \cdot (Dg(x)^T \nabla v(g(x))) = (Dg(x) a(x)) \cdot \nabla v(g(x)) = \tilde{a}(g(x)) \cdot \nabla v(g(x)),
\]
and
\[
c(x) \cdot \nabla \mu(x) = c(x) \cdot (Dg^T(x) \nabla \eta(g(x))) = (Dg(x) c(x)) \cdot \nabla \eta(g(x)) = \tilde{c}(g(x)) \cdot \nabla \eta(g(x)),
\]
with \( \tilde{a}(x) := Dg(x)a(g^{-1}(x)) \) and \( \tilde{c}(x) := Dg(x)c(g^{-1}(x)) \). The transformed Laplace operator reads

\[
\Delta u = \text{div}(DgDg^T \nabla v).
\]

Similarly we obtain

\[
\text{div}(b \nabla \mu) = \text{div}(\tilde{B} \nabla \eta),
\]

where \( \tilde{B}(x) := b(g^{-1}(x))Dg(x)Dg^T(x), x \in g(U(x_0)) \cap \mathbb{R}_+^n \). Here the matrix \( Dg \) is given by

\[
Dg(x) = \begin{bmatrix}
I_{n-1} & 0 \\
-\nabla_x \rho(x')^T & 1
\end{bmatrix}, x' \in \mathbb{R}^{n-1},
\]

where \( I_{n-1} \) is the identity matrix in \( \mathbb{R}^{(n-1) \times (n-1)} \). Observe that the normal \( \nu \) at \( U(x_0) \cap \partial \Omega \) is given by

\[
\nu(x', \rho(x')) = \frac{1}{\sqrt{1 + |\nabla_x \rho|^2}} \begin{bmatrix}
\nabla_x \rho \\
-1
\end{bmatrix}.
\]

Therefore it holds that

\[
\sqrt{1 + |\nabla_x \rho(x')|^2}(Dg)^{-1} \nu = [0, \ldots, 0, -1]^T = e_n,
\]

hence the transformed boundary conditions are \( \tilde{B} \nabla \eta \cdot e_n = \sqrt{1 + |\nabla \rho(x')|^2} \Theta^{-1} h_1 \) and

\[
DgDg^T \nabla \cdot e_n = \sqrt{1 + |\nabla \rho(x')|^2} \Theta^{-1} h_2.
\]

Here \( \Theta^{-1} \) is defined by \( (\Theta^{-1} u)(x) := u(g^{-1}(x)), x \in \mathbb{R}^n_+ \).

By construction, the transformed coefficients satisfy \( \text{div} \tilde{a}(x) = \text{div} \tilde{c}(x) = 0 \) for all \( x \in B_{r_0}(x_0) \cap \mathbb{R}^n_+ \) and \( \tilde{a}(x)|e_n) = \tilde{c}(x)|e_n) = 0 \) for all \( x \in B_{r_0}(x_0) \cap \partial \mathbb{R}^n_+ \). Now we are in a position to use the extension \( (4.9) \) in order to extend \( \tilde{a} \) and \( \tilde{c} \) to the whole of \( \mathbb{R}^n_+ \), such that the divergence condition \( \text{div} \tilde{a}(x) = \text{div} \tilde{c}(x) = 0 \) is preserved for \( x \in \mathbb{R}^n_+ \). It is furthermore clear by the structure of \( (4.9) \) that \( \tilde{a}(x)|e_n) = \tilde{c}(x)|e_n) = 0 \) holds for all \( x \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\} \). The coefficient matrix \( \tilde{B} \) can be extended to a matrix \( \tilde{B} \) on \( \mathbb{R}^n_+ \) by the reflection method \( (4.7) \). In particular it holds that \( \tilde{B}(x_0) = B(x_0) = B(x_0) = b(x_0)I \). By construction.

Therefore we have to solve the following perturbed problem in the half space \( \mathbb{R}^n_+ \):

\[
\partial_t v - \text{div}(\tilde{a} \partial_t v) = \text{div}(\tilde{B} \nabla \eta) + \Theta^{-1} f, \quad t \in [0, \delta], x \in \mathbb{R}^n_+,
\]

\[
\eta - \tilde{c} \cdot \nabla \eta = \beta \partial_t v - \text{div}(Dv \nabla v) + \Theta^{-1} g, \quad t \in [0, \delta], x \in \mathbb{R}^n_+,
\]

(4.9) \quad \begin{align*}
(\tilde{B} \nabla \eta)|e_n) &= \sqrt{1 + |\nabla \rho(x')|^2} \Theta^{-1} h_1, \quad t \in [0, \delta], x' \in \mathbb{R}^{n-1}, \quad y = 0, \\
(Dv)|e_n) &= \sqrt{1 + |\nabla \rho(x')|^2} \Theta^{-1} h_2, \quad t \in [0, \delta], x' \in \mathbb{R}^{n-1}, \quad y = 0,
\end{align*}

\[
v(0) = 0, \quad t = 0, x \in \mathbb{R}^n_+,
\]

with \( D := DgDg^T \in W^{1,2}_p(\mathbb{R}^{n-1}) \) and some functions \( (f, g, h_1, h_2) \in X^1 \times X^2 \times Y^1 \times Y^2 \) such that \( h_2|_{t=0} = 0 \). From the extension method above it follows that

\[
|\tilde{a}(x) - a(x_0)| + |\tilde{c}(x) - c(x_0)| + |\tilde{B}(x) - B(x_0)| \leq \omega,
\]

for all \( x \in \mathbb{R}^n_+ \), where we can choose \( \omega > 0 \) arbitrarily small, by decreasing the radius \( r_0 > 0 \) of the ball \( B_{r_0}(x_0) = g(U(x_0)) \). Furthermore it holds that \( |D(x) - I| \leq \omega, x \in \mathbb{R}^n_+ \), since we may choose \( |\nabla \rho|\infty \) as small as we wish. An application of Corollary \( 3.2 \) yields a unique solution operator \( S^H \) of (4.9), hence \( \Theta S^H \) is the corresponding solution operator for the chart \( U(x_0) \). At this point we want to remark that the function \( \sqrt{1 + |\nabla_x \rho|^2} \) is a multiplier for the spaces \( W^{1-1/p}_p(\mathbb{R}^{n-1}) \) and \( W^{2-1/p}_p(\mathbb{R}^{n-1}) \), since \( \rho \in C^3(\mathbb{R}^{n-1}) \) has compact support.
This above computation yields solution operators \( \Theta_k S_k^H \) for the charts \( U_k, k \in \{N_1 + 1, \ldots, N\} \), hence we may write

\[
(4.10) \quad \begin{bmatrix} u_k \\ \mu_k \end{bmatrix} = \Theta_k S_k^H \begin{bmatrix} \Theta_k^{-1} (f_k + F_k(u, \mu)) \\ \Theta_k^{-1} (g_k + G_k(u, \mu)) \\ \Theta_k^{-1} (B \nabla \varphi \cdot \nu) \\ \Theta_k^{-1} (u \partial \varphi) \end{bmatrix},
\]

for each \( k \in \{N_1 + 1, \ldots, N\} \). Summing (4.8) and (4.10) over all charts \( U_k, k \in \{1, \ldots, N\} \), we obtain

\[
(4.11) \quad \begin{bmatrix} u \\ \mu \end{bmatrix} = \sum_{k=1}^{N_1} S_k^F \begin{bmatrix} f_k + F_k(u, \mu) \\ g_k + G_k(u, \mu) \end{bmatrix} + \sum_{k=N_1+1}^{N} \Theta S_k^H \begin{bmatrix} \Theta^{-1} (f_k + F_k(u, \mu)) \\ \Theta^{-1} (g_k + G_k(u, \mu)) \\ \Theta^{-1} (B \nabla \varphi \cdot \nu) \\ \Theta^{-1} (u \partial \varphi) \end{bmatrix},
\]

since \( \{\varphi_k\}_{k=1}^{N} \) is a partition of unity. By the boundedness of the solution operators we obtain the estimate

\[
(4.12) \quad |(u, \mu)|_{Z^1_x \times Z^2_\delta} \leq M(|f|_{X^1_0} + |g|_{X^2_0} + |u|_{L_p(J_0; H^2_\delta(\Omega))} + |\partial \mu|_{L_p(J_0; L^p_\mu(\Omega))}) + |\mu|_{L_p(J_0; H^2_\delta(\Omega))},
\]

for some constant \( M > 0 \) which is independent of the interval \( J_0 = [0, \delta] \) under consideration. The term \(|u|_{L_p(J_0; H^2_\delta(\Omega))}\) may be estimated by \( \delta^{1/p} C |u|_{Z^2_\delta} \) with some constant \( C > 0 \) being independent of \( J_0 \). To estimate the remaining terms we need the following result.

**Proposition 4.4.** There exists a constant \( M > 0 \), independent of \( J_0 \), such that

\[
|\mu|_{L_p(J_0; H^2_\delta(\Omega))} + |\partial \mu|_{L_p(J_0; L^p_\mu(\Omega))} \leq M(|f|_{X^1_0} + |g|_{X^2_0} + |h_1|_{Y^1_\delta} + |h_2|_{Y^2_\delta} + |u_0|_{X^1_0}).
\]

**Proof.** The proof follows the lines of the proof of Proposition 3.3. \( \square \)

Choosing \( \delta > 0 \) sufficiently small, we obtain from (4.12) and Proposition 4.4 the estimate

\[
|(u, \mu)|_{Z^1_x \times Z^2_\delta} \leq M(|f|_{X^1_0} + |g|_{X^2_0} + |h_1|_{Y^1_\delta} + |h_2|_{Y^2_\delta} + |u_0|_{X^1_0}),
\]

for a solution of (4.11). This shows that the bounded operator \( L : Z^1_x \times Z^2_\delta \rightarrow X^1_\delta \times X^2_\delta \times Y_\delta \) defined by

\[
L(u, \mu) = \begin{bmatrix} \partial u - \text{div}(a \partial u) - \text{div}(B \nabla \mu) \\ \mu - (c \cdot \nabla u) - \beta \partial \mu + \Delta u \\ B \nabla \mu \cdot \nu \\ \partial \nu u \\ u|_{t=0} \end{bmatrix},
\]

is injective and has closed range, i.e. it is semi Fredholm. Here \( Y_\delta \) is defined by

\[ Y_\delta := \{(h_1, h_2, u_0) \in Y^1_\delta \times Y^2_\delta \times X_p : \partial \nu u_0 = h_2|_{t=0}, \ p > 3/2\}, \]

which is a closed linear subspace of the Banach space \( Y^1_\delta \times Y^2_\delta \times X_p \). To show surjectivity, we apply again the Fredholm argument to the set of data

\[
(\beta, a, c, B, \tau) = (1 - \tau)(\beta, 0, 0, I_n) + \tau(\beta, a, c, B), \quad \tau \in [0, 1].
\]

The corresponding operators \( L_\tau \) are semi Fredholm by the above procedure and by Lemma 1.2 the operator \( L_0 \) is bijective. The continuity of the Fredholm index thus yields that the index of \( L_1 = L \) is 0 and therefore the operator \( L \) is bijective as well.
A successive application of the above arguments yields existence of a unique solution $(u, \mu)$ of (4.1) on an arbitrary bounded interval $[0, T]$. This completes the proof of Theorem 4.3. \hfill \Box

5. LOCAL WELL-POSEDNESS

Let $p > n + 2$, $f \in X^1$, $g \in X^2$, $h_j \in Y^j$, $j = 1, 2$ and $\psi_0 \in X_p$ be given such that the compatibility condition $\partial_t \psi_0 = h_2|_{t=0}$ is satisfied. In this section we consider the quasilinear system

$$\begin{align*}
\partial_t \psi - \text{div}(a(x, \psi, \nabla \psi) \partial_t \psi) &= \text{div}(b(x, \psi, \nabla \psi) \nabla \mu) + f, \quad t > 0, \ x \in \Omega, \\
\mu - c(x, \psi, \nabla \psi) \cdot \nabla \mu &= \beta \partial_t \psi - \Delta \psi + \Phi'(\psi) + g, \quad t > 0, \ x \in \Omega, \\
b(x, \psi, \nabla \psi) \partial_t \mu &= h_1, \quad t > 0, \ x \in \Gamma, \\
\partial_t \psi &= h_2, \quad t > 0, \ x \in \Gamma, \\
\psi(0) &= \psi_0, \quad t = 0, \ x \in \Omega,
\end{align*}$$

(5.1)

where $\Phi \in C^3(\mathbb{R})$. Assume that we have given vector fields $a, c \in C^1(\overline{\Omega}; C^2(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n))$ and a scalar valued function $b \in C^1(\overline{\Omega}; C^2(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}))$ such that

$$\begin{align*}
\tilde{a}(x) := a(x, \psi_0(x), \nabla \psi_0(x)), \quad \tilde{b}(x) := b(x, \psi_0(x), \nabla \psi_0(x)), \\
\tilde{c}(x) := c(x, \psi_0(x), \nabla \psi_0(x))
\end{align*}$$

(5.2)

satisfy the conditions

$$\begin{align*}
\text{div} \tilde{a}(x) &= \text{div} \tilde{c}(x) = 0, \ x \in \Omega, \\
\text{(2.1)}(\tilde{a}(x)|\nu(x)) &= (\tilde{c}(x)|\nu(x)) = 0, \ x \in \partial \Omega.
\end{align*}$$

(5.3)

Suppose furthermore that $(\beta, \tilde{a}, \tilde{c}, \tilde{b})$ are subject to Hypothesis (H) for each $x \in \overline{\Omega}$. Observe that for $p > n + 2$ we have $\psi_0 \in X_p = B_{pp}^{3/2}(\Omega) \hookrightarrow C^2(\overline{\Omega})$, hence $\tilde{a}, \tilde{c} \in [C^1(\overline{\Omega})]^n$ and $\tilde{b} \in C^1(\overline{\Omega})$ and therefore the coefficients, frozen at $\psi_0$, satisfy the assumptions in Theorem 4.3. Thanks to Theorem 4.3 we may define a pair of functions $(u^*, v^*) \in Z^1 \times Z^2$ as the unique solution of the linearized system

$$\begin{align*}
u^* - \text{div}(\tilde{a}u^*) &= \text{div}(\tilde{b}v^*) + f, \quad t > 0, \ x \in \Omega, \\
\nu^* - \tilde{c} \cdot \nabla v^* &= \beta u^* - \Delta u^* + g, \quad t > 0, \ x \in \Omega, \\
\tilde{b} \nabla v^\cdot \nu &= h_1, \quad t > 0, \ x \in \Gamma, \\
\partial_v u^* &= h_2, \quad t > 0, \ x \in \Gamma, \\
u^*(0) &= \psi_0, \quad t = 0, \ x \in \Omega.
\end{align*}$$

(5.5)

We set

$$\begin{align*}
E_1 &= Z^1(T) \times Z^2(T), \quad \mathcal{E}_1 = \{(u, v) \in E_1 : u|_{t=0} = 0\}, \\
E_0 &= X^1(T) \times X^2(T) \times Y^1(T) \times Y^2(T), \quad \mathcal{E}_0 = \{(f, g, h_1, h_2) \in E_0 : h_2|_{t=0} = 0\}
\end{align*}$$

and denote by $\| \cdot \|$ and $\| \cdot \|_0$ the canonical norms in $E_1$ and $E_0$, respectively. We define a linear operator $L : \mathcal{E}_1 \rightarrow \mathcal{E}_0$ by

$$\begin{align*}
L(u, v) &= \begin{bmatrix}
\partial_t u - \text{div}(\tilde{a} \partial_t u) - \text{div}(\tilde{b} \nabla v) \\
v - \tilde{c} \cdot \nabla v - \beta \partial_t u + \Delta u \\
\tilde{b} \nabla v \cdot \nu \\
\partial_v u
\end{bmatrix}
\end{align*}$$
and a nonlinear function $G : 0E_1 \times E_1 \to 0E_0$ by

$$G((u, v), (u^*, v^*)) = \begin{bmatrix} G_1((u, v), (u^*, v^*)) + G_2((u, v), (u^*, v^*)) \\ G_3((u, v), (u^*, v^*)) + G_4((u, v), (u^*, v^*)) \\ G_5((u, v), (u^*, v^*)) \\ 0, \end{bmatrix}$$

where

$$G_1(u, u^*) = \text{div}[(a(x, u + u^*, \nabla(u + u^*)) - \tilde{a})\partial_t(u + u^*)],$$

$$G_2((u, v), (u^*, v^*)) = \text{div}[(b(x, u + u^*, \nabla(u + u^*)) - \tilde{b})\nabla(v + v^*)],$$

$$G_3((u, v), (u^*, v^*)) = (c(x, u + u^*, \nabla(u + u^*)) - \tilde{c}) \cdot \nabla(v + v^*),$$

and

$$G_5((u, v), (u^*, v^*)) = [\tilde{b} - b(x, u + u^*, \nabla(u + u^*))] \nabla(v + v^*) \cdot v.$$

Considering $L$ as an operator from $0E_1$ to $0E_0$, we obtain from Theorem 4.3 that $L$ is a bounded isomorphism and by the open mapping theorem $L$ is invertible with bounded inverse $L^{-1}$. It is easily seen that $(\psi, \mu) := (u + u^*, v + v^*)$ is a solution of (5.6) if and only if

$$L(u, v) = G((u, v), (u^*, v^*)) \text{ or equivalently } (u, v) = L^{-1}G((u, v), (u^*, v^*)).$$

Consider a ball $B_r \subset 0E_1$ where $r \in (0, 1]$ will be fixed later. Define a nonlinear operator by $T(u, v) := L^{-1}G((u, v), (u^*, v^*))$. To apply the contraction mapping principle we have to show that $T B_r \subset B_r$ and that there exists a constant $\kappa < 1$ such that the contractive inequality

$$(5.6) \quad |T(u, v) - T(\tilde{u}, \tilde{v})|_1 \leq \kappa |(u, v) - (\tilde{u}, \tilde{v})|_1$$

holds for all $(u, v), (\tilde{u}, \tilde{v}) \in B_r$. The following proposition is crucial to prove the desired properties of the operator $T$.

**Proposition 5.1.** Let $p > n + 2, J = [0, T]$ and assume $\Phi \in C^3(-\mathbb{R})$. Then there exists a constant $C > 0$, independent of $T$ and $r$, and functions $\mu_j = \mu_j(T) \in C[0, T \to 0]$ such that for all $(\tilde{u}_1, \tilde{v}_1), (\tilde{u}_2, \tilde{v}_2) \in B_r$ the following statements hold.

(i) $|G_1(\tilde{u}_1, u^*) - G_1(\tilde{u}_2, u^*)|_{X^1} \leq C(r + \mu_1(T))|((\tilde{u}_1, \tilde{v}_1) - (\tilde{u}_2, \tilde{v}_2)|_1$;

(ii) $|G_2((\tilde{u}_1, \tilde{v}_1), (u^*, v^*)) - G_2((\tilde{u}_2, \tilde{v}_2), (u^*, v^*))|_{X^1} \leq C(r + \mu_2(T))|((\tilde{u}_1, \tilde{v}_1) - (\tilde{u}_2, \tilde{v}_2)|_1$;

(iii) $|G_3((\tilde{u}_1, \tilde{v}_1), (u^*, v^*)) - G_3((\tilde{u}_2, \tilde{v}_2), (u^*, v^*))|_{X^1} \leq C(r + \mu_3(T))|((\tilde{u}_1, \tilde{v}_1) - (\tilde{u}_2, \tilde{v}_2)|_1$;

(iv) $|G_4(\tilde{u}_1, u^*) - G_4(\tilde{u}_2, u^*)|_{X^2} \leq C\mu_4(T)|((\tilde{u}_1, \tilde{v}_1) - (\tilde{u}_2, \tilde{v}_2)|_1$;

(v) $|G_5((\tilde{u}_1, \tilde{v}_1), (u^*, v^*)) - G_5((\tilde{u}_2, \tilde{v}_2), (u^*, v^*))|_{X^1} \leq C(r + \mu_5(T))|((\tilde{u}_1, \tilde{v}_1) - (\tilde{u}_2, \tilde{v}_2)|_1$.

**Proof.** Define the ball $B_r(u^*, v^*) \subset E_1$ by means of

$$B_r(u^*, v^*) := \{(u, v) \in E_1 : (u, v) = (\tilde{u}, \tilde{v}) + (u^*, v^*), \ (\tilde{u}, \tilde{v}) \in B_r\}.$$

Let $(u_j, v_j) \in B_r(u^*, v^*)$, $j \in \{1, 2\}$. Observe that

$$|u_j - u^*|_{X^1, p} \leq C_0 |u_j - u^*|_{Z^1} \leq r,$$
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with some $C_0 > 0$, which is independent of $T > 0$. This yields

$$|u_j|_{\infty, x, p} \leq C T + |u^*|_{\infty, x, p} \leq C_0 + |u^*|_{\infty, x, p} =: R,$$

since $r \in (0, 1]$. To prove the first part, note that

$$\text{div}[(a(x, u_1, \nabla u_1) - \tilde{a})\partial_t u_1] - \text{div}[(a(x, u_2, \nabla u_2) - \tilde{a})\partial_t u_2]$$

$$= (a(x, u_1, \nabla u_1) - \tilde{a}) \cdot \nabla \partial_t u_1 - (a(x, u_2, \nabla u_2) - \tilde{a}) \cdot \nabla \partial_t u_2$$

$$+ \text{div}(a(x, u_1, \nabla u_1) - \tilde{a})\partial_t u_1 - \text{div}(a(x, u_2, \nabla u_2) - \tilde{a})\partial_t u_2.$$

Next we have

$$(a(x, u_1, \nabla u_1) - \tilde{a}) \cdot \nabla \partial_t u_1 - (a(x, u_2, \nabla u_2) - \tilde{a}) \cdot \nabla \partial_t u_2$$

$$= (a(x, u_1, \nabla u_1) - a(x, u_2, \nabla u_2)) \cdot \nabla \partial_t u_1 + (a(x, u_2, \nabla u_2) - \tilde{a}) \cdot (\nabla \partial_t u_1 - \nabla \partial_t u_2).$$

Therefore we may estimate

$$|(a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)) \cdot \nabla \partial_t u_1|_{X^1}$$

$$\leq |a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} (|\nabla \partial_t u_1 - \nabla \partial_t u^*|_{X^1} + |\nabla \partial_t u^*|_{X^1})$$

$$\leq L(R)C(r + \mu(T))(|u_1 - u_2|_{\infty, \infty} + |\nabla u_1 - \nabla u_2|_{\infty, \infty})$$

$$\leq L(R)C(r + \mu(T))|u_1 - u_2|_{Z^1},$$

as well as

$$|(a(\cdot, u_2, \nabla u_2) - \tilde{a}) \cdot (\nabla \partial_t u_1 - \nabla \partial_t u_2)|_{X^1}$$

$$\leq (|a(\cdot, u_1, \nabla u_1) - a(\cdot, u^*, \nabla u^*)|_{\infty, \infty} + |a(\cdot, u^*, \nabla u^*) - \tilde{a}|_{\infty, \infty})(|\nabla \partial_t u_1 - \nabla \partial_t u_2|_{X^1})$$

$$\leq L(R)C(r + \mu(T))|u_1 - u_2|_{Z^1},$$

where $\mu(T) := \max\{|\nabla \partial_t u^*|_{X^1}, |u^* - \psi_0|_{\infty, x} \} \rightarrow 0$ as $T \rightarrow 0$ since $u^* \in Z^1$ is fixed and $u^*|_{t=0} = \psi_0$. For the remaining terms we use the identity

$$\text{div}(a(x, u, \nabla u)) = \text{div}_x a(x, u, \nabla u) + \partial_q a(u, \nabla u) : \nabla^2 u,$$

where $a = a(x, z, q), q = [q_1, \ldots, q_n]^{\top}$

$$\partial_q a(u, \nabla u) : \nabla^2 u := \sum_{i,j=1}^n \partial_q a_j(u, \nabla u) \partial_i \partial_j u.$$

Furthermore we make use of

$$\text{div}(a(x, u_1, \nabla u_1) - \tilde{a})\partial_t u_1 - \text{div}(a(x, u_2, \nabla u_2) - \tilde{a})\partial_t u_2$$

$$= (\text{div}(a(x, u_1, \nabla u_1)) - \text{div}(a(x, u_2, \nabla u_2)))\partial_t u_1 + \text{div}(a(x, u_2, \nabla u_2) - \tilde{a})(\partial_t u_1 - \partial_t u_2).$$

Let us first estimate $\text{div}(a(x, u_1, \nabla u_1)) - \text{div}(a(x, u_2, \nabla u_2))$ in $L_\infty(0, T; L_\infty(\Omega))$. By [5.7] we obtain

$$|\text{div}(a(\cdot, u_1, \nabla u_1)) - \text{div}(a(\cdot, u_2, \nabla u_2))|_{\infty, \infty}$$

$$\leq |\partial_x a(\cdot, u_1, \nabla u_1) - \partial_x a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla u_1|_{\infty, \infty} + |\partial_x a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla u_1 - \nabla u_2|_{\infty, \infty}$$

$$+ |\partial_q a(\cdot, u_1, \nabla u_1) - \partial_q a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla^2 u_1|_{\infty, \infty} + |\partial_q a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla^2 u_1 - \nabla^2 u_2|_{\infty, \infty}$$

$$\leq C(R, u^*)|u_1 - u_2|_1.$$
with \( C > 0 \) being independent of \( T \) and \( |\partial_t u^*|_{X^1} \to 0 \) as \( T \to 0 \). In a similar way we obtain
\[
|\text{div}(a(\cdot, u_2, \nabla u_2)) - \text{div}(a(\cdot, \psi_0, \nabla \psi_0))|_{\infty, \infty}
\leq |\partial_t a(\cdot, u_2, \nabla u_2) - \partial_t a(\cdot, \psi_0, \nabla \psi_0)|_{\infty, \infty} |\nabla \psi_0|_{\infty, \infty} + |\partial_2 a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla u_2 - \nabla \psi_0|_{\infty, \infty}
\]
\[
+ |\partial_q a(\cdot, u_2, \nabla u_2) - \partial_q a(\cdot, \psi_0, \nabla \psi_0)|_{\infty, \infty} |\nabla^2 \psi_0|_{\infty, \infty} + |\partial_q a(\cdot, u_2, \nabla u_2)|_{\infty, \infty} |\nabla^2 u_2 - \nabla^2 \psi_0|_{\infty, \infty}
\]
\[
\leq C(R, u^*) |u_2 - \psi_0|_{\infty, X^p}
\]
\[
\leq C(R, u^*)(r + |u^* - \psi_0|_{\infty, X^p}).
\]
Note that \( |u^* - \psi_0|_{\infty, X^p} \to 0 \) as \( T \to 0 \) since \( u^*_i |_{t=0} = \psi_0 \). Finally it holds that \( |\partial_t u_1 - \partial_t u_2|_{X^1} \leq |u_1 - u_2|_{X^1} \). This proves (i). Statements (ii) and (iii) follow in a very similar way, while (v) follows from trace theory and (ii). To prove (iv), we use the condition \( \Phi \in C^3(\mathbb{R}) \) to conclude
\[
|\Phi'(u_1) - \Phi'(u_2)|_{X^2} \leq T^{1/p}(|\Phi'(u_1) - \Phi'(u_2)|_{\infty, \infty} + |\nabla \Phi'(u_1) - \nabla \Phi'(u_2)|_{\infty, \infty})
\]
\[
\leq T^{1/p} C(R)(|u_1 - u_2|_{\infty, \infty} + |\nabla u_1 - \nabla u_2|_{\infty, \infty} + |u_2|_{\infty, \infty} |\nabla u_1 - \nabla u_2|_{\infty, \infty})
\]
\[
\leq T^{1/p} C(R, u^*) |u_1 - u_2|_{X^2}.
\]
where \( C(R, u^*) > 0 \) does not depend on \( T > 0 \) and \( r \in (0, 1] \). The proof is complete.

With the help of Proposition \ref{L2} we are able to prove the desired properties of the operator \( T \) defined above. We first care about the contractive mapping property.

(5.8)
\[
|T(u_1, v_1) - T(u_2, v_2)|_{1} \leq |L^{-1}|G((u_1, v_1), (u^*, v^*)) - G((u_2, v_2), (u^*, v^*))|_{0}
\]
\[
\leq |L^{-1}||G_1(u_1, u^*) - G_1(u_2, u^*)|_{X^1}
\]
\[
+ |G_2((u_1, v_1), (u^*, v^*)) - G_2((u_2, v_2), (u^*, v^*))|_{X^1}
\]
\[
+ |G_3((u_1, v_1), (u^*, v^*)) - G_3((u_2, v_2), (u^*, v^*))|_{X^2}
\]
\[
+ |G_4(u_1, u^*) - G_4(u_2, u^*)|_{X^2}
\]
\[
+ |G_5((u_1, v_1), (u^*, v^*)) - G_5((u_2, v_2), (u^*, v^*))|_{Y^1}
\]
\[
\leq C(r + \mu(T))(|u_1, v_1|_{1} - |u_2, v_2|_{1}).
\]
where \( \mu = \mu(T) \) is a function with the property that \( \mu(T) \to 0 \) as \( T \to 0 \) and \( C > 0 \) is a constant which does not depend on \( T > 0 \). Thus, if \( T > 0 \) and \( r \in (0, 1] \) are sufficiently small we obtain (5.9). The self mapping property can be shown in a similar way. The above computation yields

(5.9)
\[
|T(u, v)|_{1} \leq |T(u, v) - T(0, 0)|_{1} + |T(0, 0)|_{1}
\]
\[
\leq C \left( (r + \mu(T))(|u, v|_{1} + |G(0, 0), (u^*, v^*))|_{0}ight)
\]
\[
\leq C \left( (r + \mu(T))r + |G(0, 0), (u^*, v^*))|_{0}. \right)
\]
Since \( G((0, 0), (u^*, v^*)) \) is a fixed function in \( E_0 \) it follows that \( |G((0, 0), (u^*, v^*))|_{0} \to 0 \) as \( T \to 0 \), whence \( T_{B_T} \subset B_1 \), provided that \( T > 0 \) and \( r \in (0, 1] \) are small enough. The contraction mapping principle yields a unique fixed point \( (\hat{u}, \hat{v}) \in E_1 \) or equivalently \( (\psi, \mu) := (\hat{u} + u^*, \hat{v} + v^*) \in E_1 \) is the unique local solution of \( (5.10) \). Therefore we have the following result.
Theorem 5.2. Let $p > n + 2$, $J_0 = [0, T_0]$ and suppose that $\Phi \in C^3(\mathbb{R})$, $a, c \in C^1(\Omega; C^2(\mathbb{R}^n; \mathbb{R}^n))$ and $b \in C^1(\Omega; C^2(\mathbb{R}^n; \mathbb{R}))$. Then there exists an interval $J = [0, T] \subset J_0$, such that (5.1) admits a unique solution
\[
\psi \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)) = Z^1, \quad \mu \in L_p(J; H_p^2(\Omega)) = Z^2,
\]
if the data are subject to the following conditions.

(i) $f \in L_p(J; L_p(\Omega)) = X^1$,
(ii) $g \in L_p(J; H_p^1(\Omega)) = X^2$,
(iii) $h_1 \in L_p(J; W_p^{1-1/p}(\Gamma)) = Y^1$,
(iv) $h_2 \in W_p^{1-1/p}(J; L_p(\Gamma)) \cap L_p(J; W_p^{2-1/p}(\Gamma)) = Y^2$,
(v) $\psi_0 \in B_p^{3-2/p}(\Omega) = X_p$,
(vi) $\partial_r \psi_0 = h_2 |_{t = 0}$,
(vii) $(\beta, \tilde{a}, \tilde{b}, \tilde{c})$ satisfy (H) for all $x \in \bar{\Omega}$ as well as (5.3) and (5.4).

Remark 5.3. An inspection of the proof of Theorem 5.2 shows that the assumption $p > n + 2$ can be relaxed to $p > (n + 2)/3$ in the semilinear case, i.e. if $(a, b, c)$ are independent of $\psi$ and $\nabla \psi$. Indeed, it remains to estimate the nonlinearity $\Phi(\psi)$ in $L_p(0, T; H_p^1(\Omega))$. However, in the sequel we will always assume the stronger condition $p > n + 2$.

6. Global Well-Posedness

Let $n \leq 3$ and $p > n + 2$ according to Theorem 5.2. In this section we consider the semilinear version of (5.1), i.e. we assume that $a = a(x)$, $c = c(x)$ and $B = b(x)I$. Then, a successive application of Theorem 5.2 yields a maximal interval of existence $J_{\text{max}} = [0, T_{\text{max}})$ for the solution $(\psi, \mu)$ of (5.1), i.e. (5.1) admits a unique solution
\[
\psi \in H_p^1(J; H_p^1(\Omega)) \cap L_p(J; H_p^3(\Omega)), \quad \mu \in L_p(J; H_p^2(\Omega)),
\]
for each interval $J = [0, T] \subset J_{\text{max}}$.

Suppose $T_{\text{max}} < \infty$ and let $J = [0, T] \subset [0, T_{\text{max}})$. We start with an a priori estimate for the solution $\psi \in Z^1$ on the maximal interval of existence $J_{\text{max}}$. To do so we multiply (6.1) by $\mu$, (6.2) by $-\partial_t \psi$ and integrate by parts to obtain

\[
\frac{d}{dt} \left( \int_\Omega \partial_t \psi \mu + (B \nabla \mu | \nabla \psi) + (a | \nabla \mu) \partial_t \psi \right) dx = \int_\Omega \mu f \ dx + \int_\Gamma \mu h_1 \ d\Gamma \tag{6.1}
\]
and

\[
\int_\Omega \left( -\partial_t \psi \mu + (c | \nabla \mu) \partial_t \psi + \beta | \nabla \psi |^2 + \frac{1}{2} \frac{\partial}{\partial t} | \nabla \psi |^2 + \partial_t \Phi(\psi) \right) dx = \int_\Gamma \partial_t \psi h_2 \ d\Gamma - \int_\Omega \partial_t \psi g \ dx, \tag{6.2}
\]

since $(a | \nu) = 0$ on $\partial \Omega$. Adding (6.1) and (6.2) yields the equation

\[
\frac{d}{dt} \left( \frac{1}{2} | \nabla \psi |^2 + \int_\Omega \Phi(\psi) \ dx \right) + \beta | \nabla \psi |^2 + (a + c | \partial_t \psi \nabla \mu |)^2 + (B \nabla \mu | \nabla \mu)^2
\]
\[
= \int_\Omega \mu f \ dx + \int_\Gamma \mu h_1 \ d\Gamma + \int_\Gamma \partial_t \psi h_2 \ d\Gamma - \int_\Omega \partial_t \psi g \ dx.
\]

From Assumption (H) with $z_0 = \partial_t \psi$ and $z_1 = \nabla \mu$ it follows that
\[
\beta | \partial_t \psi |^2 + (a + c | \partial_t \psi \nabla \mu |)^2 + (B \nabla \mu | \nabla \mu)^2 \geq \varepsilon (| \partial_t \psi |^2 + | \nabla \mu |^2).
\]
For the first and the second integral in (6.3) we apply Hölder’s inequality as well as the Poincaré-Wirtinger inequality to obtain
\[
\int_\Omega \mu f \, dx \leq C|f|^2_2 \left( |\nabla \mu|^2 + |\int_\Omega \mu \, dx| \right) \quad \text{and} \quad \int_\Gamma \mu h_1 \, d\Gamma \leq C|h_1|^2_2, \Gamma \left( |\nabla \mu|^2 + |\int_\Omega \mu \, dx| \right).
\]

The integral \( \int_\Omega \mu \, dx \) can be computed in the following way. Since \( \text{div} \, c = 0 \) in \( \Omega \) and \( (c|\nu) = 0 \) on \( \Gamma \) we have
\[
\int_\Omega (c|\nabla \mu) \, dx = \int_\Gamma (c|\nu) \mu \, d\Gamma - \int_\Omega \mu \text{div} \, c \, dx = 0,
\]
hence it follows from (5.1) and (6.4) 1
\[
\Phi(s) \geq -\frac{\eta}{2} s^2 - c_0, \quad s \in \mathbb{R},
\]
where \( c_0 > 0 \) and \( 0 < \eta < \lambda_1 \), with \( \lambda_1 > 0 \) being the first nontrivial eigenvalue of the negative Neumann Laplacian and
\[
|\Phi'(s)| \leq (c_1 \Phi(s) + c_2 s^2 + c_3)^\theta, \quad \text{for all} \ s \in \mathbb{R},
\]
and some constants \( c_1 > 0, \ \theta \in (0,1) \). This yields
\[
|\int_\Omega \mu \, dx| \leq \left( c_1 \Phi(\psi) + c_2 |\psi|^2 + c_3 \right)^\theta \, dx \, + \, C(|g|_1 + |h_1|_1, \, \Gamma + |f|_1).
\]

By the last estimate, Young’s inequality and the Poincaré inequality it holds that
\[
\int_\Omega \mu f \, dx + \int_\Gamma \mu h_1 \, d\Gamma \leq C(\delta) \left( |\nabla \psi|^2_2 + \int_\Omega \Phi(\psi) \, dx + |f|^2_2 + |h_1|^2_2, \Gamma + |g|^2_2 + 1 \right) + \delta|\nabla \mu|^2_2,
\]
where \( \delta := \max\{2, \frac{1}{\theta-\gamma}\} \) and \( \delta > 0 \) may be arbitrarily small. For the term \( \int_\Omega \partial_t \psi g \, dx \) in (6.3) we apply Young’s inequality one more time to obtain
\[
\int_\Omega \partial_t \psi \, dx \leq \delta|\partial_t \psi|^2_2 + C(\delta)|g|^2_2.
\]
Integrating (6.3) with respect to \( t \) and choosing \( \delta > 0 \) small enough, we obtain together with (6.6) and (6.7) the estimate
\[
\int_0^t \frac{1}{2} |\nabla \psi(t)|^2_2 + \int_\Omega \Phi(\psi(t)) \, dx + C_1 (|\partial_t \psi|^2_2 + |\nabla \mu|^2_2) \leq C_2 \left( \int_0^t \left( \frac{1}{2} |\nabla \psi(\tau)|^2_2 + \Phi(\psi(\tau)) \right) \, d\tau + |f|^2_2 + |h_1|^2_2, \Gamma + |g|^2_2 + 1 \right) + \int_0^t \int_\Gamma \partial_t \psi h_2 \, d\Gamma \, d\tau.
\]
In order to treat the last double integral, we have to assume more regularity for the function \( h_2 \). To be precise, we assume that
\[
h_2 \in H^1_p(J; L_p(\Gamma)) \cap L^p(J; W^{2,1/p}_p(\Gamma)) \hookrightarrow C(J; L_p(\Gamma)).
\]
Due to this fact, we may integrate the last term in (6.5) by parts to the result

\[(6.9) \int_0^t \int_\Gamma \partial_t \psi h_2 \ d\Gamma \ d\tau = \int_\Gamma \psi(t)h_2(t) \ d\Gamma - \int_\Gamma \psi_0 h_2|_{t=0} \ d\Gamma - \int_0^t \int_\Gamma \psi \partial_t h_2 \ d\Gamma \ d\tau,\]

where we also made use of Fubini’s theorem. For the first term we use Young’s inequality, the embedding \(H^1_0(\Omega) \hookrightarrow L^2(\Gamma)\) and the fact that

\[(6.10) \int_\Omega \psi(t) \ dx = \int_\Omega \psi_0 \ dx + \int_0^t \int_\Omega f \ dx \ d\tau + \int_0^t \int_\Gamma h_1 \ d\Gamma \ d\tau.\]

This yields

\[
\int_\Gamma \psi(t)h_2(t) \ d\Gamma \leq \delta \|\psi(t)\|^2_{H^1_0(\Omega)} + C(\delta)\|\psi_0\|^2_{H^1_0(\Gamma)} \\
\leq \delta C|\nabla \psi(t)|^2 + C(\delta) (|h_2|^2_{\infty,2,\Gamma} + |f|_{1,1} + |h_1|_{1,1,\Gamma} + |\psi_0|_1).
\]

Observe that we have \(h_2|_{t=0} = \partial_\nu \psi_0 \in B_{pp}^{3-3/p}(\Gamma) \hookrightarrow L^2(\Gamma)\) and, by trace theory,

\(B_{pp}^{3-3/p}(\Omega) \hookrightarrow B_{pp}^{3-3/p}(\Gamma) \hookrightarrow L^2(\Gamma)\).

It follows that the integral \(\int_\Gamma \psi_0 h_2|_{t=0} \ d\Gamma\) converges. Finally, concerning the last term in (6.9) we apply Young’s inequality one more time to the result

\[
\int_0^t \int_\Gamma \partial_t h_2 \ d\Gamma \ d\tau \leq \frac{1}{2} \int_0^t |\psi(\tau)|^2_{H^1_0(\Omega)} \ d\tau + \frac{1}{2} \|\partial_t h_2\|^2_{2,2,\Gamma} \\
\leq C \int_0^t |\nabla \psi(\tau)|^2 \ d\tau + C(T, f, h_1, \partial_t h_2, \psi_0),
\]

where we used again (6.10). Set

\[
E(u) = \frac{1}{2} |\nabla u|^2 + \int_\Omega \Phi(u) \ dx, \quad u \in H^1_0(\Omega).
\]

Then by the above estimates there exist some constants \(C_j > 0\) such that

\[
E(\psi(t)) + C_1(|\partial_\nu \psi|^2_{2,2,\Gamma} + |\nabla \mu|^2_{2,2,\Gamma}) \leq C_2 \int_0^t E(\psi(\tau)) \ d\tau + C_3(T, f, g, h_1, h_2, \partial_t h_2, \psi_0),
\]

for all \(t \in [0, T]\), provided that \(\delta > 0\) is sufficiently small. With the help of (6.4) it follows that \(E(u)\) is bounded from below for all \(u \in H^1_0(\Omega)\), hence we may apply Gronwall’s lemma to the result that \(E(\psi(\cdot))\) is bounded on \(J_{\max} = [0, T_{\max})\). Applying (6.4) one more time and using the fact that \(|\int_\Omega \psi(t, x) \ dx| \leq C\) it holds that

\[
\psi \in L^\infty(J_{\max}; H^1_0(\Omega)).
\]

Note that in the semilinear case the following estimate for the maximal solution \((\psi, \mu)\) of (6.1) holds

\[(6.11) \|\psi\|_{L^2(T)} + |\mu|_{L^2(T)} \leq C \left( \|\Phi(\psi)\|_{X^1(T)} + |f|_{X^1(T)} + |g|_{X^2(T)} + |h_1|_{Y^1(T)} + |h_2|_{Y^2(T)} + |\psi_0|_{X_p} \right).
\]

Here the constant \(C > 0\) does not depend on \(T \in (0, T_{\max})\). Suppose that \(\Phi'(\psi)\) satisfies the estimate

\[(6.12) \|\Phi'(\psi)\|_{X^2(T)} \leq C(T)\|\psi\|^n_{L^\infty(0, T_{\max}; H^1_0(\Omega))},\]

\(n \geq 1\). Then for all \(\psi \in C([0, T_{\max}) \cap \bar{J}_{\max}) \cap L^\infty(J_{\max}; H^1_0(\Omega))\) we have

\[
\Phi'(\psi)(\cdot) \in L^\infty(J_{\max}; H^1_0(\Omega)).
\]
for some $\kappa \in (0, 1)$ and $m > 0$, where $C(T) > 0$ and $\sup_{T \in [0, T_{\max})} C(T) < \infty$. Substituting (6.12) into (6.11) yields

$$|\psi|_{Z^1(T)} \leq M \left(1 + |\psi|_{Z^1(T)}^2\right),$$

where $M > 0$ does not depend on $T \in (0, T_{\max})$. This in turn yields that $|\psi|_{Z^1(T_{\max})}$ is bounded, since $\kappa \in (0, 1)$. Therefore $\psi(T_{\max}) \in B_{3p/2}^{n-2/p}(\Omega)$ is well-defined and we may continue the maximal solution $(\psi, \mu)$ beyond the point $T_{\max}$, which is a contradiction to the maximality of $T_{\max}$.

It remains to show the validity of (6.12). We start with the term $\nabla \Phi'(\psi) = \Phi''(\psi) \nabla \psi$ in $L_p(\Omega)^n$. It holds that

$$|\Phi''(\psi) \nabla \psi|_{p} \leq |\Phi''(\psi)|_{3p/2} |\nabla \psi|_{3p},$$

by Hölder’s inequality. Assume that there exists a constant $C > 0$ such that

$$|\Phi''(s)| \leq C(1 + |s|^\alpha), \quad \text{for all } s \in \mathbb{R} \text{ and some } \alpha \geq 1,$$

where $\alpha < 4$ in case $n = 3$. Then we have

$$|\Phi''(\psi) \nabla \psi|_{p} \leq C(1 + |\psi|_{3p/2}^\alpha) |\nabla \psi|_{3p}.$$

Applying the Gagliardo-Nirenberg interpolation inequality we obtain

$$|\psi|_{3p/2} \leq C|\psi|_{H^{3p/2}_0(\Omega)}^{\alpha} \|\psi\|_{q}^{1-a},$$

provided

$$\frac{n}{q} - \frac{2n}{3ap} = a \left(3 - \frac{n}{p} + \frac{n}{q}\right), \quad a \in [0, 1].$$

On the other side we obtain

$$|\nabla \psi|_{3p} \leq C|\psi|_{H^{3p/2}_0(\Omega)}^b \|\psi\|_{q}^{1-b},$$

provided

$$1 - \frac{n}{3p} + \frac{n}{q} = b \left(3 - \frac{n}{p} + \frac{n}{q}\right), \quad b \in [1/3, 1].$$

Chose $q$ in such a way, that $H_{3p/2}^{3p/2}(\Omega) \hookrightarrow L_q(\Omega)$, i.e. $n/q \geq n/2 - 1$. Thus $q$ may be arbitrarily large if $n \in \{1, 2\}$ and $q \leq 6$ in case $n = 3$. If $n = 3$, let

$$\frac{an}{2} < q < \min \left\{6, \frac{3ap}{2}\right\}, \quad \text{while in case } n = 1, 2 \text{ we require }$$

$$\frac{an}{2} < q < \frac{3ap}{2}. \quad \text{This is possible, since } n < 3p \text{ for } n \leq 3 \text{ and } an/2 < 6 \text{ if } n = 3, \text{ since in this case we assume } \alpha < 4. \text{ Now it follows that }$$

$$|\psi|_{3p/2}^{\alpha} |\nabla \psi|_{3p} \leq C|\psi|_{H^{3p/2}_0(\Omega)}^{\alpha + b} \|\psi\|_{q}^{1+\alpha(1-a) - b}.$$

To gain something from this inequality we require $\alpha + b < 1$ which is equivalent to

$$\left(3 - \frac{n}{p} + \frac{n}{q}\right) > \alpha \left(n - \frac{2n}{3ap} + 1 - \frac{n}{3p} + \frac{n}{q} - 1 - \frac{n}{p} + (1 + a) \frac{n}{q}\right).$$

This in turn yields $\alpha < 2q/n$ which is certainly true by (6.14) and (6.15). With $\kappa := \alpha + b \in (0, 1)$ we obtain the estimate

$$|\nabla \Phi'(\psi(t))|_{L_p(\Omega)^n} \leq C|\psi(t)|_{H^{3p/2}_0(\Omega)}^{\alpha} \|\psi(t)\|_{H^{3p/2}_0(\Omega)}^b,$$
valid for a.e. $t \in [0, T] \subset [0, T_{\max})$ and some $m > 0$. Similarly one obtains

$$|\Phi'(\psi(t))|_{L^p(\Omega)} \leq C|\psi(t)|^{n_c}_{H^1_0(\Omega)}|\psi(t)|^m_{H^1_0(\Omega)},$$

for a.e. $t \in [0, T] \subset [0, T_{\max})$. Finally this yields

$$(6.17) \quad |\Phi'(\psi(t))|_{H^1_0(\Omega)} \leq C|\psi(t)|^{n_c}_{H^1_0(\Omega)}|\psi(t)|^m_{H^1_0(\Omega)},$$

for a.e. $t \in [0, T] \subset [0, T_{\max})$. Integration of the $p$-th power of (6.17) and Hölder’s inequality imply

Remark 6.2.

In conclusion we have the following result.

**Theorem 6.1.** Let $p > n + 2$, $n \leq 3$, $q = \max\{2, \frac{1}{p-1}\}$, with $\theta$ from (6.5). Suppose that $a, c \in C^1(\overline{\Omega})$ and $b \in C^1(\overline{\Omega})$ satisfy condition (H) as well as $\text{div} \ a(x) = \text{div} \ c(x) = 0$, $x \in \Omega$ and $(a(x)|\nu(x)) = (c(x)|\nu(x)) = 0$, $x \in \partial \Omega$. Assume furthermore that $\Phi \in C^3(\mathbb{R})$ satisfies (6.4), (6.5) and (6.13). Then there exists a unique global solution $(\psi, \mu)$ of (6.1) on $J_0 = [0, T_0]$, with

$$\psi \in H^1_p(J_0; H^1_p(\Omega)) \cap L_p(J_0; H^3_p(\Omega))$$

and

$$\mu \in L_p(J_0; H^2_p(\Omega)),$$

provided that the data are subject to the following conditions.

1. $f \in L_p(J_0; L_p(\Omega)) \cap L_q(J_0; L_2(\Omega))$,
2. $g \in L_p(J_0; H^1_p(\Omega))$,
3. $h_1 \in L_p(J_0; W^{1-1/p}_p(\Gamma)) \cap L_q(J_0; L_2(\Gamma))$,
4. $h_2 \in H^1_p(J_0; L_p(\Gamma)) \cap L_p(J_0; W^{2-1/p}_p(\Gamma))$, 
5. $\psi_0 \in B^{-2/p}_{pp}$,
6. $\partial_\nu \psi_0 = h_2|_{t=0}$.

The solution depends continuously on the given data and if $f = g = h_1 = h_2 = 0$, the map $\psi_0 \mapsto \psi(t)$, $t \in \mathbb{R}_+$, defines a global semiflow on the natural phase manifold defined by (v) $\in$ (vi).

**Remark 6.2.** The assertion of Theorem 6.1 remains true if we assume that $p > (n+2)/3$, which is sufficient for the well-posedness of the semilinear model by Remark 5.3.

7. Asymptotic Behavior

In this last section we will give a qualitative analysis of global solutions of the Cahn-Hilliard-Gurtin system

$$\partial_t \psi - \text{div}(a \partial_t \psi) = \text{div}(b \nabla \mu), \quad t > 0, \quad x \in \Omega,$$

$$\mu - c \cdot \nabla \mu = \beta \partial_t \psi - \Delta \psi + \Phi'(\psi), \quad t > 0, \quad x \in \Omega,$$

$$(7.1) \quad \begin{aligned}
B \nabla \mu \cdot \nu &= 0, \quad t > 0, \quad x \in \Gamma, \\
\partial_\nu \psi &= 0, \quad t > 0, \quad x \in \Gamma,
\psi(0) &= \psi_0, \quad t = 0, \quad x \in \Omega.
\end{aligned}$$

To be more precise we will show that each trajectory converges to a stationary point, i.e. to a solution of the corresponding stationary system. The so called Lojasiewicz-Simon inequality will play an important role in the proof of this assertion. Assume that $a, c \in C^1(\overline{\Omega})$ and $b \in C^1(\overline{\Omega})$ with $\text{div} \ a(x) = \text{div} \ c(x) = 0$, $x \in \Omega$ and $(a(x)|\nu(x)) = (c(x)|\nu(x)) = 0$, $x \in \partial \Omega$. Suppose that the data $(\beta, a, c, B)$ satisfy
condition (H) for all \( x \in \overline{\Omega} \). Moreover we assume that \( \Phi \in C^3(\mathbb{R}) \) and that it satisfies the estimate
\[
|\Phi''(s)| \leq C(1 + |s|^\gamma), \quad \text{for all } s \in \mathbb{R},
\]
and some constant \( C > 0 \). Here \( \gamma \geq 1 \) is arbitrary if \( n \in \{1, 2\} \) and \( \gamma < 3 \) if \( n = 3 \). At this point we want to remark that (7.2) already implies (6.13).

Let \( \psi_0 \in H^2_0(\Omega) \) such that \( \partial_x \psi_0 = 0 \) and let \( (\psi, \mu) \) be the unique global solution of (7.1). We recall from Section 6 the energy functional
\[
E(u) = \frac{1}{2} |\nabla u|^2 + \int_{\Omega} \Phi(u) \, dx,
\]
defined on the energy space
\[
V := \{ u \in H^1_2(\Omega) : \int_{\Omega} u \, dx = 0 \}.
\]
Note that due to (7.1), and the boundary condition (6.13), we obtain \( \int_{\Omega} \psi \, dx \equiv \int_{\Omega} \psi_0 \, dx \), since \((a(x)|\nu(x)) = 0 \) on \( \Gamma \). If we perform a shift of \( \psi \) by means of \( \psi = \psi - c \), where \( c := \int_{\Omega} \psi_0 \, dx \), it follows that \( \tilde{\psi} \) is again a solution of (7.1), provided that the physical potential \( \Phi \) is replaced by \( \tilde{\Phi}(s) = \Phi(s + c) \). Additionally it holds that \( \int_{\Omega} \tilde{\psi} \, dx = 0 \). It follows from (6.3) that \( E(\psi(\cdot)) \) satisfies the equation
\[
\frac{d}{dt} E(\psi(t)) + \beta |\partial_t \psi(t)|^2 + (a + c|\partial_t \psi(t)| |\nabla \mu(t)|^2) + (B|\nabla \mu(t)||\nabla \mu(t)|)^2 = 0,
\]
for all \( t \in \mathbb{R}_+ \). Making again use of Hypothesis (H) we obtain the inequality
\[
(7.3) \quad \frac{d}{dt} E(\psi(t)) + \varepsilon (|\partial_t \psi(t)|^2 + |\nabla \mu(t)|^2) \leq 0,
\]
which holds for all \( t \in \mathbb{R}_+ \). Integrating with respect to \( t \) and making use of (6.3) as well as of the Poincaré inequality we obtain the a priori estimates
\[
\psi \in L_\infty(\mathbb{R}_+; H^1_2(\Omega)) \quad \text{and} \quad \partial_t \psi, |\nabla \mu| \in L_2(\mathbb{R}_+ \times \Omega).
\]

**Proposition 7.1.** The orbit \( \{ \psi(t) \}_{t \in \mathbb{R}_+} \) is relatively compact in \( V \).

**Proof.** We rewrite equation (7.1) as follows
\[
\beta \partial_t \psi - \Delta \psi + \psi = \mu - \overline{\mu} - (c(x)|\nabla \mu|) + \psi - \Phi'(\psi),
\]
where \( \overline{\mu} = \frac{1}{|\Omega|} \int_{\Omega} \Phi'(\psi) \, dx \). By the energy estimates above and the Poincaré-Wirtinger inequality it holds that
\[
f := \mu - \overline{\mu} + (c|\nabla \mu|) \in L_2(\mathbb{R}_+; L_2(\Omega)).
\]
Furthermore we have
\[
g := \overline{\mu} + \Phi'(\psi) \in L_\infty(\mathbb{R}_+; L_q(\Omega)),
\]
where \( q = 6/(\gamma + 2) \) is determined by the growth condition (7.2) on \( \Phi \). The operator \( A := \Delta - I \) with domain
\[
D(A) = \{ u \in H^2_0(\Omega) : \partial_x u = 0 \text{ on } \Gamma \}
\]
generates an exponentially stable, analytic \( C_0 \)-semigroup \( \{ T(t) \}_{t \in \mathbb{R}_+} \) in \( L_\mu(\Omega) \). Therefore
\[
T(\cdot) * f \in H^1_2(\mathbb{R}_+; L_2(\Omega)) \cap L_2(\mathbb{R}_+; H^2_0(\Omega)) \Rightarrow C_0(\mathbb{R}_+; H^1_2(\Omega)).
\]
For the function $g$ we apply elementary semigroup theory to obtain

$$T(\cdot) * g \in C_0(\mathbb{R}_+; H^s_2(\Omega)),$$

for each $s \in (0, 2)$. The space $H^s_2(\Omega)$ embeds compactly into $H^1_2(\Omega)$, if $s$ is chosen close enough to 2. This completes the proof of relative compactness, since $T(\cdot)\psi_0 \in C_0(\mathbb{R}_+; H^1_2(\Omega))$.

The following proposition provides some properties of the $\omega$-limit set

$$\omega(\psi) = \{ \varphi \in V : \exists (t_n) \nearrow \infty, \text{ s.t. } \psi(t_n) \to \varphi \text{ in } V \}.$$

**Proposition 7.2.** Suppose that $(\psi, \mu)$ is a global solution of (7.1) and let $\Phi$ satisfy Hypotheses (6.4) and (7.2). Then the following statements hold:

(i) The mapping $t \mapsto E(\psi(t))$ is nonincreasing and the limit $\lim_{t \to \infty} E(\psi(t)) =: E_\infty \in \mathbb{R}$ exists.

(ii) The $\omega$-limit set $\omega(\psi) \subset V$ is nonempty, connected, compact and $E$ is constant on $\omega(\psi)$.

(iii) Every $\psi_\infty \in \omega(\psi)$ is a strong solution (in the sense of $L_2$) of the stationary problem

$$-\Delta \psi_\infty + \Phi'(\psi_\infty) = \mu_\infty, \quad x \in \Omega,$$

$$\partial_\nu \psi_\infty = 0, \quad x \in \Gamma,$$

where $\mu_\infty = \frac{1}{|\Omega|} \int_\Omega \Phi'(\psi_\infty) \, dx = \text{const.}$

(iv) Each $\psi_\infty \in \omega(\psi)$ is a critical point of $E$, i.e. $E'(\psi_\infty) = 0$ in $V^*$, where $V^*$ is the topological dual space of $V$.

**Proof.** Inequality (7.3) implies that $E(\psi(t))$ is nonincreasing with respect to $t$. Furthermore by (6.4) it follows that $E(u)$ is bounded from below for all $u \in V$. This proves (i). Assertion (ii) follows easily from well-known facts in the theory of dynamical systems.

Let $\psi_\infty \in \omega(\psi)$. Then there exists a sequence $(t_n) \nearrow \infty$ such that $\psi(t_n) \to \psi_\infty$ in $V$ as $n \to \infty$. Since $\partial_\nu \psi \in L_2(\mathbb{R}_+ \times \Omega)$ it follows that $\psi(t_n + s) \to \psi_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$ and by relative compactness also in $V$. Integrating (7.3) from $t_n$ to $t_n + 1$ we obtain

$$E(\psi(t_n + 1)) - E(\psi(t_n)) + \epsilon \int_0^1 \int_\Omega \left( |\nabla \mu(t_n + s, x)|^2 + |\partial_\nu \psi(t_n + s, x)|^2 \right) \, dx \, ds \leq 0.$$

Letting $t_n \to +\infty$ yields

$$|\nabla \mu(t_n + \cdot, \cdot)|, \partial_\nu \psi(t_n + \cdot, \cdot) \to 0 \quad \text{in } L_2([0, 1] \times \Omega).$$

This in turn yields a subsequence $(t_{n_k})$ such that $|\nabla \mu(t_{n_k} + s)|, \partial_\nu \psi(t_{n_k} + s) \to 0$ in $L_2(\Omega)$ for a.e. $s \in [0, 1]$. We fix such an $s$, say $s^* \in [0, 1]$. The Poincaré-Wirtinger inequality implies that

$$|\mu(t_{n_k} + s^*) - \mu(t_{n_k} + s^*)|_2 \leq C_p \left( |\nabla \mu(t_{n_k} + s^*) - \nabla \mu(t_{n_k} + s^*)|_2 + \int_\Omega |\Phi'(\psi(t_{n_k} + s^*)) - \Phi'(\psi(t_{n_k} + s^*))|_2 \, dx \right),$$

since $\int_\Omega \mu \, dx = \int_\Omega \Phi'(\psi) \, dx$. Letting $k, l \to \infty$ and making use of (7.2) it follows that $\mu(t_{n_k} + s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by $\mu_\infty$. Since the gradient is a closed operator in $L_2(\Omega; \mathbb{R}^n)$ it holds that $\mu_\infty \in H^1_2(\Omega)$.
and $\nabla \mu_\infty = 0$. Thus $\mu_\infty = \text{const.}$ and we have the identity $\mu_\infty = \frac{1}{|\Omega|} \int_\Omega \Phi'(\psi_\infty) \, dx$. Finally we multiply (7.4) by a function $\varphi \in V$ in $L_2(\Omega)$ to the result

$$
(7.5) \quad \langle (\mu(t_{n_k} + s^*), \varphi)_2 + (c \cdot \nabla \mu(t_{n_k} + s^*), \varphi)_2
= \beta(\partial_t \psi(t_{n_k} + s^*), \varphi)_2 - (\Delta \psi(t_{n_k} + s^*), \varphi)_2 + (\Phi'(\psi(t_{n_k} + s^*)), \varphi)_2.
$$

Taking the limit $t_{n_k} \to \infty$ we obtain

$$
a(\psi(t_{n_k} + s^*), \varphi) \to (\mu_\infty - \Phi'(\psi_\infty), \varphi)_2,
$$

where $a : V \times V \to \mathbb{R}$ is defined by $a(u, v) = \langle \nabla u, \nabla v \rangle_2$ and $(\cdot, \cdot)_2$ denotes the scalar product in $L_2(\Omega)$. Since $\Phi'(\psi_\infty) \in L_q(\Omega)$ with $q = 6/(\gamma + 2)$ it follows that $\psi_\infty \in D(A_q) = \{u \in H^2_q(\Omega) : \partial_\nu u = 0\}$, where $A_q$ is the part of the operator $A$ in $L_q(\Omega)$ which is induced by the form $a(u, v)$. Observe that $q > 6/5$ by assumption, whence we may apply a bootstrap argument to conclude $\psi_\infty \in H^2(\Omega)$ and $\partial_\nu \psi_\infty = 0$ on $\Gamma$ (recall that $q > 1$ may be arbitrarily large in case $n \in \{1, 2\}$). Going back to (7.5) we obtain for $(t_{n_k}) \not\to \infty$ the identity

$$
(\nabla \psi_\infty, \nabla \varphi)_2 + (\Phi'(\psi_\infty), \varphi)_2 = (\mu_\infty, \varphi)_2,
$$

for all functions $\varphi \in V$. This yields (iii) after integration by parts. To prove (iv) observe that by Proposition 5.2 the functional $E$ is twice continuously Fréchet differentiable and its first derivative is given by

$$
\langle E'(u), h \rangle_{V^*, V} = \int_\Omega \nabla u \nabla h \, dx + \int_\Omega \Phi'(u) h \, dx, \quad u, h \in V.
$$

Integration by parts finally yields assertion (iv).

At this point we could simply refer to the paper of Miranville & Rougirel [13] to prove the main Theorem 7.4 below. However, for the sake of completeness we provide a proof of this result. The next proposition is the key for the proof of the convergence of the orbit $\{\psi(t)\}_{t \geq 0}$ towards a stationary state as $t \to \infty$.

**Proposition 7.3** (Lojasiewicz-Simon inequality). Let $\varphi \in \omega(\psi)$ and assume in addition to (6.4) and (7.2) that $\Phi$ is real analytic. Then there exist constants $s \in (0, \frac{1}{2}]$, $C, \delta > 0$ such that

$$
|E(u) - E(\varphi)|^{1-s} \leq C|E'(u)|_{V^*},
$$

whenever $|u - \varphi|_V \leq \delta$.

**Proof.** This is Proposition 6.6 in [4].

Now we are in a position to state the main result of this section.

**Theorem 7.4.** Let $\Phi$ satisfy the conditions (6.4) and (7.2). Assume in addition that $\Phi$ is real analytic. Then the limit

$$
\lim_{t \to \infty} \psi(t) =: \psi_\infty
$$

exists in $V$ and $\psi_\infty$ is a strong solution of the stationary problem (7.4).
Proof. Since each element $\varphi \in \omega(\psi)$ is a critical point of $E$, Proposition 7.3 implies that the Lojasiewicz-Simon inequality is valid in some neighborhood of $\varphi \in \omega(\psi)$. By Proposition 7.2 (ii) the $\omega$-limit set is compact, hence there exists $N \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{N} B_{\delta_j}(\varphi_j) \supset \omega(\psi),$$

where $B_{\delta_j}(\varphi_j) \subset V$ are open balls with center $\varphi_j \in \omega(\psi)$ and radius $\delta_j$. Additionally in each ball the Lojasiewicz-Simon inequality is valid. It follows from Proposition 7.2 (i) and (ii) that the energy functional $E$ is constant on $\omega(\psi)$, i.e. $E(\varphi) = E_\infty$, for all $\varphi \in \omega(\psi)$. Thus there exists an open set $U \supset \omega(\psi)$ and uniform constants $s \in (0, \frac{1}{2}]$ $C, \delta > 0$ with

$$|E(u) - E_\infty|^{1-s} \leq C|E'(u)|_{V^*},$$

for all $u \in U$. A well-known result in the theory of dynamical systems states that the $\omega$-limit set is an attractor for the orbit $\{\psi(t)\}_{t \in \mathbb{R}_+}$. To be precise this means

$$\lim_{t \to \infty} \text{dist}(\psi(t), \omega(\psi)) = 0 \quad \text{in} \ V.$$

This implies that there exists some time $t^* \geq 0$ such that $\psi(t) \in U$ for all $t \geq t^*$ and thus the Lojasiewicz-Simon inequality holds for the solution $\psi(t)$, i.e.

$$|E(\psi(t)) - E_\infty|^{1-s} \leq C|E'(\psi(t))|_{V^*}, \quad t \geq t^*.$$  

(7.6)

Define a function $H : \mathbb{R}_+ \to \mathbb{R}_+$ by $H(t) = (E(\psi(t)) - E_\infty)^s$. Then with (7.3) and (7.6) it holds that

$$-\frac{d}{dt}H(t) = (E(\psi(t)) - E_\infty)^{s-1}\left(-\frac{d}{dt}E(\psi(t))\right)$$

$$\geq \frac{\beta(\partial_t \psi(t))^2 + |\nabla \mu(t)|^2}{(E(\psi(t)) - E_\infty)^{1-s}}$$

$$\geq C_\varepsilon \frac{\beta(\partial_t \psi(t))^2 + |\nabla \mu(t)|^2}{|E'(\psi(t))|_{V^*}}.$$  

(7.7)

The first Fréchet derivative of $E$ in $V$ reads

$$\langle E'(u), h \rangle_{V^*, V} = \int_{\Omega} \nabla u \cdot \nabla h \, dx + \int_{\Omega} \Phi'(u) h \, dx,$$

for all $(u, h) \in V \times V$. Setting $u = \psi(t)$ and making use of (7.4) we obtain with the help of Hölder’s inequality, Poincaré’s inequality and integration by parts

$$\langle E'(\psi(t)), h \rangle_{V^*, V} = \int_{\Omega} (\mu(t) - \bar{\mu}(t)) h \, dx - \int_{\Omega} \nabla \mu(t) h \, dx - \beta \int_{\Omega} \partial_t \psi(t) h \, dx$$

$$\leq C(\|\nabla \mu(t)\|_2 + |\partial_t \psi(t)|_2)\|h\|_2,$$

since $\text{div} \ c(x) = 0$, $x \in \Omega$ and $(c(x)|\nabla \psi(x)|_2 = 0$, $x \in \partial \Omega$. Taking the supremum in (7.8) over all functions $h \in V$ with norm less than 1 it follows that

$$\|E'(\psi(t))\|_{V^*} \leq C(\|\nabla \mu(t)\|_2 + |\partial_t \psi(t)|_2).$$

We insert this estimate into (7.7) to obtain

$$-\frac{d}{dt}H(t) \geq C_\varepsilon (\|\nabla \mu(t)\|_2 + |\partial_t \psi(t)|_2).$$

Integrating this inequality from $t^*$ to $\infty$ it follows that $|\partial_t \psi(t)|_2, \|\nabla \mu(t)\|_2 \in L_1(\mathbb{R}_+)$, since $H(t) > 0$. This implies that the limit $\lim_{t \to \infty} \psi(t) =: \psi_\infty$ exists firstly in $L_2(\Omega)$
but by relative compactness also in \( V \). Finally, by Proposition 7.2 (iii) the limit \( \psi_\infty \) is a solution of the stationary problem (7.4). The proof is complete. \( \square \)

8. Appendix

Proof of Proposition 3.3

We substitute (3.9) into (3.9) to obtain the elliptic problem (8.1)

\[
\mu + A(x, \partial)\mu = \text{div}(a \text{div}(D\nabla u)) - \text{div}(D\nabla u) + \tilde{f}, \quad x \in \mathbb{R}^n_+; \quad B(x, \partial)\mu = h_1, \quad x \in \partial \mathbb{R}^n_+,
\]

with

\[
\tilde{f} = \beta f + a \cdot \nabla g - g \in L_p(J_3 \times \mathbb{R}^n_+).
\]

Here the differential operators \( A(x, \partial) \) and \( B(x, \partial) \) are defined by

\[
A(x, \partial)\mu := -(a + c) \cdot \nabla \mu + \text{div}(a(c \cdot \nabla \mu)) - \text{div}(\beta B \nabla \mu), \quad x \in \mathbb{R}^n_+,
\]

and

\[
B(x, \partial)\mu := B \nabla \mu \cdot \nu, \quad x \in \partial \mathbb{R}^n_+,
\]

valid for all \( \mu \in H^2(\mathbb{R}^n_+) \). It will be convenient to rewrite the operator \( A(x, \partial) \) as follows. \( A(x, \partial) = A_0(x, \partial) + A_1(x, \partial) \), where

\[
A_0(x, \partial)\mu := -\text{div}(\beta B \nabla \mu) - \frac{1}{2}(a \otimes c + c \otimes a) \nabla \mu, \quad x \in \mathbb{R}^n_+,
\]

and

\[
A_1(x, \partial)\mu := a \cdot (\nabla c \nabla \mu) - \frac{1}{2} \text{Div}(a \otimes c + c \otimes a) \cdot \nabla \mu - (a + c) \cdot \nabla \mu, \quad x \in \mathbb{R}^n_+.
\]

Actually this splitting shows that problem (8.1) is indeed elliptic by Assumption (H) and Proposition 2.1 provided \( \omega > 0 \) is sufficiently small. Will will now proceed in several steps.

Step 1. In this first step we want to reduce (8.1) to the case of homogeneous boundary conditions \( B(x, D)\mu = 0 \). Consider the elliptic problem with constant coefficients

\[
\lambda \mu - \text{div}(\tilde{B}_0 \nabla \mu) = f, \quad x \in \mathbb{R}^n_+, \quad (\tilde{B}_0 \nabla \mu | \nu) = g, \quad x \in \partial \mathbb{R}^n_+,
\]

where \( \tilde{B}_0 := \beta B_0 - \frac{1}{2}(a_0 \otimes c_0 + c_0 \otimes a_0) \) and \( \lambda \in \mathbb{R} \) is a parameter. Note that (8.2) is an elliptic problem with a conormal boundary condition and constant coefficients. Thanks to Proposition 2.1 the matrix \( \tilde{B}_0 \) is positive definite. By well known results it follows that for each \( f \in L_p(\mathbb{R}^n_+) \) and \( g \in W^{1-1/p}_p(\partial \mathbb{R}^n_+) \) problem (8.2) has a unique solution \( \mu \in H^2(\mathbb{R}^n_+) \), provided \( \lambda > 0 \). We remind that the variable coefficients

\[
a(x) = a_0 + a_1(x), \quad c(x) = c_0 + c_1(x), \quad B(x) = B_0 + B_1(x),
\]

have a small deviation from the constant ones \( a_0, c_0, B_0 \), i.e.

\[
|a_1|_\infty + |c_1|_\infty + |B_1|_\infty \leq \omega,
\]

with \( \omega > 0 \) being sufficiently small. Furthermore we have \( a, c \in W^{1}_\infty(\mathbb{R}^n_+; \mathbb{R}^n), B \in W^{1}_\infty(\mathbb{R}^n_+; \mathbb{R}^{n \times n}) \). Therefore we may apply perturbation theory to conclude that there exists \( \lambda_0 > 0 \) such that for each \( f \in L_p(\mathbb{R}^n_+) \) and \( g \in W^{1-1/p}_p(\partial \mathbb{R}^n_+) \) the elliptic problem

\[
\lambda_0 \mu - \text{div}(\tilde{B} \nabla \mu) = f, \quad x \in \mathbb{R}^n_+, \quad \tilde{B} \nabla \mu \cdot \nu = g, \quad x \in \partial \mathbb{R}^n_+,
\]
has a unique solution \( \mu \in H^2_p(\mathbb{R}^n_+) \), provided \( \lambda \geq \lambda_0 \) and \( \omega > 0 \) is sufficiently small. Here \( \tilde{B} := \beta B - \frac{1}{2}(a \otimes c + c \otimes a) \). Note that \( A_0(x, \partial)\mu = -\text{div}(\tilde{B}\nabla \mu) \) and

\[
(\tilde{B}\nabla \mu|\nu) = (B\nabla \mu|\nu) = B(x, \partial)\mu,
\]

since \((a|\nu) = (c|\nu) = 0 \) by assumption. Moreover, the operator \( A_1(x, \partial) \) defined above contains only terms of lower order with \( L_\infty \)-coefficients. Thus, applying perturbation theory one more time, there exists \( \lambda_1 > 0 \) such that for each \( f \in L_p(\mathbb{R}^n_+) \) and \( g \in W_p^{1-1/p}(\partial \mathbb{R}^n_+) \) the problem

\[
(\lambda \mu + A(x, \partial)\mu = f, \quad x \in \mathbb{R}^n_+; \quad B(x, \partial)\mu = g, \quad x \in \partial \mathbb{R}^n_+),
\]

has a unique solution \( \mu \in H^2_p(\mathbb{R}^n_+) \), provided \( \lambda \geq \lambda_1 \) and \( \omega > 0 \) is sufficiently small.

**Step 2.** The results of this first step enable us to reduce (S.1) to the case of homogeneous boundary conditions. In this step we show that the \( L_\mu \)-realization of the boundary value problem \((A, B)\) with domain

\[
D(A) = \{ u \in H^2_p(\mathbb{R}^n_+) : B(x, \partial)u = 0 \},
\]

is dissipative. First, let \( p \geq 2 \). Integration by parts yields

\[
\text{Re} \int_{\mathbb{R}^n_+} \text{div}\tilde{w} |w|^{p-2} \, dx
= -\int_{\mathbb{R}^n_+} |w|^{p-4} \text{Re} \left( \frac{p}{2}(\tilde{B}\nabla w \cdot \nabla \tilde{w}) |w|^2 + \left(\frac{p}{2} - 1\right) (\tilde{B}\nabla w \cdot \nabla \tilde{w}) \tilde{w}^2 \right) \, dx
\]

for all \( w \in D(A) \), since \( \text{div} a(x) = \text{div} c(x) = 0 \) in \( \Omega \) and \( (a(x)|\nu(x)) = (c(x)|\nu(x)) = 0 \) on \( \partial \Omega \). Setting \( \nabla \tilde{w} = u + iv \) and \( w = b_1 + ib_2 \) with \( u, v \in \mathbb{R}^n, b_j \in \mathbb{R} \), we obtain the estimate

\[
\text{Re} \left( \frac{p}{2}(\tilde{B}\nabla w \cdot \nabla \tilde{w}) |u|^2 + \left(\frac{p}{2} - 1\right) (\tilde{B}\nabla w \cdot \nabla \tilde{w}) \tilde{w}^2 \right)
\geq \varepsilon \beta \left( |u|^2 + |v|^2 \right) (b_1^2 + b_2^2)
\geq \varepsilon \beta \left( |u|^2 + |v|^2 \right) (b_1^2 + b_2^2) = \varepsilon \beta |\nabla \tilde{w}|^2 |w|^2.
\]

Here we made use of Proposition 2.1. This shows that \( A \) is dissipative in \( L_\mu(\Omega) \) for \( p \geq 2 \). If \( p \in (1, 2) \) we replace \(|w|\) by \( w_x := \sqrt{|w|^2 + \varepsilon} \) for \( \varepsilon > 0 \) in the calculations involving \( A \) and then pass to the limit as \( \varepsilon \searrow 0 \).

The dissipativity of \( A \) allows us to set \( \lambda = 1 \) in (S.3). By the same arguments one can show that the \( L_\mu \)-realization \( A_0 \) of the elliptic boundary value problem \((A_0, B)\) with domain \( D(A_0) = D(A) \) is dissipative, too. Indeed, \( A_0 \mu = \text{div}(\tilde{B}\nabla \mu) \) and \( \tilde{B} \) defined above is a positive definite and symmetric matrix by Proposition 2.1. Therefore we may also set \( \lambda = 1 \) in (S.3).

**Step 3.** By the results of Steps 1 & 2 we may decompose the unique solution \( \mu \in H^2_p(\mathbb{R}^n_+) \) of (S.1) into \( \mu = \mu_1 + \mu_2 \), where \( \mu_1, \mu_2 \in H^2_p(\mathbb{R}^n_+) \) are the unique solutions of the elliptic problems

\[
(8.5) \quad \mu_1 + A_0(x, \partial)\mu_1 = \text{div} \left( (a \text{div}(D\nabla u)) \right), \quad x \in \mathbb{R}^n_+; \quad B(x, \partial)\mu_1 = 0, \quad x \in \partial \mathbb{R}^n_+,
\]

and

\[
(8.6) \quad \mu_2 + A(x, \partial)\mu_2 = f - \text{div}(D\nabla u) - A_1(x, D)\mu_1, \quad x \in \mathbb{R}^n_+; \quad B(x, \partial)\mu_2 = g, \quad x \in \partial \mathbb{R}^n_+.
\]
For $\mu_2$ we have the estimate
\[ |\mu_2| H^2_\mu(\mathbb{R}^n_+) \leq C \left( |u| H^2_\mu(\mathbb{R}^n_+) + |f| L^p(\mathbb{R}^n_+) + |g| W^{1-1/p}(\partial \mathbb{R}^n_+) + |\mu_1| H^2_\mu(\mathbb{R}^n_+) \right), \]
with some constant $C > 0$, since $A_3(x, \partial)$ consists solely of lower order terms with $L_\infty$-coefficients. To obtain the desired estimate for $\mu$, we therefore have to prove the estimate
\[ |\mu_1| H^2_\mu(\mathbb{R}^n_+) \leq C |u| H^2_\mu(\mathbb{R}^n_+), \]
for the solution $\mu_1$ of (5.5), where $C > 0$. For this purpose note that the $L_p$-realization $A_0$ of $(A_0, \mathcal{B})$ generates a $C_0$-semigroup in $E_0 := L^p(\mathbb{R}^n_+)$ and $\lambda + A_0$ is a linear isomorphism from $E_1 := D(A_0)$ to $E_0$ for each $\lambda \in \rho(-A_0)$, the resolvent set of $A_0$. Let $E_{1/2} := [E_0, E_1]_{1/2}$, $E_{-1/2} := (E^*_{1/2})'$ where $E^* = E'$ and denote by $A_{-1/2}$ the $E_{-1/2}$-realization of $A_0$. Here the symbol $[,]_{1/2}$ denotes the complex interpolation functor of exponent $1/2$. It follows from [2, Theorem V.2.1.3 & Corollary V.2.1.4] that the operator $A_{-1/2}$ is the generator of a $C_0$-semigroup with $\rho(A_{-1/2}) = \rho(A_0)$ and $\lambda + A_{-1/2} : E_{1/2} \rightarrow E_{-1/2}$ is a linear isomorphism for each $\lambda \in \rho(-A_0)$. It remains to determine the spaces $E_{1/2}$ and $E_{-1/2}$. To compute $E_{1/2}$, we have to interpolate Sobolev spaces involving boundary conditions. This has been done e.g. in [15] and [1]. Following these results it holds that
\[ E_{1/2} = [E_0, E_1]_{1/2} = H^1_p(\mathbb{R}^n_+). \]
Actually in [1] this result was proven for $C^1$-coefficients but the result remains true for $W^{1}_\infty$-coefficients. From this characterization we obtain
\[ E_{-1/2} = \left( H^1_p(\mathbb{R}^n_+) \right)', \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty. \]
Set $F = a \operatorname{div}(\nabla u) \in H^1_p(\mathbb{R}^n_+)$ and $f = \operatorname{div} F \in L^p(\mathbb{R}^n_+)$. Then $\mu_1 \in H^2_p(\mathbb{R}^n_+)$ is a solution of the abstract equation $A_1 + A_0 \mu_1 = f$. We claim that this $f$ can be identified with a linear functional in $E_{-1/2}$ (for which we will write $f$ again). Indeed the mapping
\[ \varphi \mapsto \int_{\mathbb{R}^n_+} f \varphi \, dx =: \langle f, \varphi \rangle_{E_{-1/2}, H^1_p}, \]
defines a linear functional on $H^1_p(\mathbb{R}^n_+)$, since by Hölder’s inequality it holds that
\[ |f|_{E_{-1/2}} = \sup_{|\varphi| \leq 1} \|\langle f, \varphi \rangle_{E_{-1/2}, H^1_p} \|_{H^1_p} = \sup_{|\varphi| \leq 1} \left| \int_{\mathbb{R}^n_+} f \varphi \, dx \right| \leq |f|_{L^p(\mathbb{R}^n_+)}. \]
Integrating by parts we obtain furthermore
\[ |f|_{E_{-1/2}} = \sup_{|\varphi| \leq 1} \left| \int_{\mathbb{R}^n_+} \nabla F \varphi \, dx \right| = \sup_{|\varphi| \leq 1} \left| \int_{\mathbb{R}^n_+} (F \nabla \varphi) \, dx \right| \leq \sup_{|\varphi| \leq 1} \left| F \right|_{L^p(\mathbb{R}^n_+)} \left| \varphi \right|_{H^1_p(\mathbb{R}^n_+)} = \left| F \right|_{L^p(\mathbb{R}^n_+)} \left| \varphi \right|_{H^1_p(\mathbb{R}^n_+)}. \]
where we also made use of \((F[ν]) = \langle a[ν] \rangle \text{div}(D∇u) = 0\). Since \(A_{-1/2}\) is the \(E_{-1/2}\)-realization of \(A_0\) (hence an extension of \(A_0\)) with \(1 ∈ \rho(-A_{-1/2}) = \rho(-A_0)\) and since \(f = \text{div}(μΔu) ∈ E_{-1/2}\), we obtain a constant \(C > 0\) such that the estimate

\[
|μ1|_{H^1_p(ℝ^n)} ≤ C|μ1|_{E_{-1/2}} ≤ C|f|_{L_p(ℝ^n)} ≤ C|μ|_{H^1_p(ℝ^n)},
\]

for the solution \(μ1 ∈ H^1_p(ℝ^n)\) of \((8.5)\) is valid. From the estimates for \(μ1\) and \(μ2\) and the embedding \(H^1_p(ℝ^n) → H^1_p(ℝ^n)\), we obtain a constant \(C > 0\) such that

\[
|μ|_{H^1_p(ℝ^n)} ≤ C \left( |f|_{L_p(ℝ^n)} + |g|_{H^1_p(ℝ^n)} + |h1|_{W^{1-1/p}_{p}(ℝ^n)} + |u|_{H^2_p(ℝ^n)} \right).
\]

In the case that the functions depend on the parameter \(t\) it follows that the estimate \((8.7)\)

\[
|μ(t)|_{H^1_p(ℝ^n)} ≤ C \left( |f(t)|_{L_p(ℝ^n)} + |g(t)|_{H^1_p(ℝ^n)} + |h1(t)|_{W^{1-1/p}_{p}(ℝ^n)} + |u(t)|_{H^2_p(ℝ^n)} \right),
\]

holds for a.e. \(t ∈ J = [0,T]\) where the constant \(C > 0\) is uniform in \(t\) since the coefficients of the differential operators considered above are independent of \(t\) as well. Taking the \(p\)-th power and integrating \((8.7)\) with respect to \(t\), we obtain \((8.8)\)

\[
|μ|_{L_p(J;H^1_p(ℝ^n))} ≤ C \left( |f|_{L_p(J;ℝ^n)} + |g|_{L_p(J;H^1_p(ℝ^n))} + |h1|_{L_p(J;W^{1-1/p}_{p}(ℝ^n))} + |u|_{L_p(J;H^2_p(ℝ^n))} \right).
\]

Finally the estimate for \(∂_tu \in L_p(J;L_p(ℝ^n))\) follows from \((8.1)\) and \((8.3)\). The proof is complete.

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