Scaling Exponents in Quantum Gravity near Two Dimensions

Hikaru Kawai*, Yoshihisa Kitazawa†, and Masao Ninomiya#

Abstract

We formulate quantum gravity in $2 + \epsilon$ dimensions in such a way that the conformal mode is explicitly separated. The dynamics of the conformal mode is understood in terms of the oversubtraction due to the one loop counter term. The renormalization of the gravitational dressed operators is studied and their anomalous dimensions are computed. The exact scaling exponents of the 2 dimensional quantum gravity are reproduced in the strong coupling regime when we take $\epsilon \to 0$ limit. The theory possesses the ultraviolet fixed point as long as the central charge $c < 25$, which separates weak and strong coupling phases. The weak coupling phase may represent the same universality class with our Universe in the sense that it contains massless gravitons if we extrapolate $\epsilon$ up to 2.

* Department of Physics, University of Tokyo, Hongo, Tokyo 113, Japan
  E-mail address : TKYVAX$hepnet::KAWAI

† Department of Physics, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152, Japan
  E-mail address : TITVSO::KITAZAWA.decnet

# Uji Research Center, Yukawa Institute for Theoretical Physics, Kyoto University, Uji 611, Japan
  E-mail address : NINOMIYA@JPNYITP.bitnet
1. Introduction

Our attempt to understand quantum gravity is plagued by multitudes of difficulties such as non-renormalizability and the instability of the conformal mode. In order to circumvent these problems, various approaches to make sense of quantum gravity have been proposed with considerable success. We may cite string theory and topological gravity as examples. Of all these attempts, notable progress in our understanding of quantum gravity is brought about by the exact solution of two dimensional quantum gravity. Although it has been a toy model, this has provided us precious insight into more general universal classes of quantum gravity.

In view of the existence of two dimensional quantum theory of gravitation, it is reasonable to expect that consistent quantum gravity theory exists in 2+\(\epsilon\) dimensions as well. In this context one may draw an analogy with nonlinear sigma models. Nonlinear sigma models are renormalizable in two dimensions and they are asymptotically free. Furthermore they possess well defined 2+\(\epsilon\) dimensional expansion. Such an expansion provides us with information about realistic nonlinear sigma models which are relevant to critical phenomena in Nature.

In the constructive (lattice) approach to quantum gravity, dynamical triangulation method has turned out to be effective to integrate over the metric. This point has been demonstrated by the success of the matrix model approach to the 2 dimensional quantum gravity. The dynamical triangulation method has been extended to higher dimensions such as 3 and 4 dimensional quantum gravity [1, 2]. They have indicated the existence of two distinct phases in 3 and 4 dimensional quantum gravity. This feature appears to be in accord with the 2 + \(\epsilon\) dimensional expansion approach [3 - 6]. We believe it is important to develop this analytical approach further.

The organization of this paper is as follows. In section 2, we formulate quantum gravity in 2+\(\epsilon\) dimensions in such a way that the conformal mode is explicitly separated. In section 3, the one loop counter term is computed in our formulation. The dynamics of the conformal mode is understood in terms of the oversubtraction due to the one loop counter term. In section 4, the renormalization of the gravitational
dressed operators is studied and their anomalous dimensions are computed. The two dimensional quantum gravity turns out to be $\epsilon \to 0$ limit of the strong coupling regime. The section 5 is devoted to the conclusions and discussions. We explain the basic physical picture of the $2 + \epsilon$ dimensional quantum gravity. The appendix contains the short summary of the background field method in our formulation.

2. Separation of conformal mode in $2 + \epsilon$ dimensional gravity

Firstly we would like to seek a proper formalism of $D = 2 + \epsilon$ dimensional quantum gravity. From the study of two dimensional quantum gravity, we have learned that the conformal mode of the metric is the important dynamical degree of freedom in two dimensions. Therefore our strategy is to adopt a parametrization and a gauge which singles out the conformal mode. Let us write the metric as follows

$$g_{\mu\nu} = \hat{g}_{\mu\rho} (e^{h})^\rho_{\nu} e^{-\phi} \quad (2.1)$$

where $\hat{g}_{\mu\nu}$ is the background metric and the $h^\mu_{\ ,\nu}$ field is taken to be traceless $h^\mu_{\ ,\mu} = 0$. Hence

$$\sqrt{g} = \sqrt{\hat{g}} e^{-\frac{\epsilon}{2} \phi} \quad . \quad (2.2)$$

We define further

$$\tilde{g}_{\mu\nu} = \hat{g}_{\mu\rho} (e^{h})^\rho_{\nu}$$

$$= \hat{g}_{\mu\nu} + h_{\mu\nu} + \frac{1}{2} h_{\mu\rho} \hat{g}^{\rho\sigma} h_{\sigma\nu} + \cdots \quad (2.3)$$

where tensor indices are raised and lowered by the background metric $\hat{g}_{\mu\nu}$ and $h_{\mu\nu}$ is symmetric in $\mu$ and $\nu$.

The Einstein action becomes in terms of these variables

$$\int d^Dx \sqrt{g} R = \int d^Dx \sqrt{\hat{g}} e^{-\frac{\epsilon}{2} \phi} \tilde{R}$$

$$- \int d^Dx \sqrt{g} e^{-\frac{\epsilon}{2} \phi} \frac{1}{4} \epsilon (D - 1) \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad . \quad (2.4)$$

We expand the action in terms of $h^\mu_{\ ,\nu}$ and $\phi$ fields and drop the linear term in these
In this paper we use the notation of t’Hooft and Veltman [7]. In particular, $\mu$ denotes the covariant derivative with respect to the background metric.

When we compute the one loop counter term, we use the background field method. This is an efficient and manifestly gauge invariant method to compute the effective action. We drop the term linear in quantum fields in the action, because it is implicitly assumed that such fields are coupled to sources which drive them to assume their background form. Therefore if we use the conventional coupling of the source to the metric $g_{\mu\nu}J^{\mu\nu}$, we should have dropped the linear term in $h'_{\mu\nu}$, where $g_{\mu\nu} = \hat{g}_{\mu\nu} + h'_{\mu\nu}$, rather than $h^\mu_{\nu}$ and $\phi$ fields separately. However we have checked that the difference does not lead to any change in the one loop counter term. See the appendix for more detailed discussions.

In order to cancel the last two terms in the quadratic action, we choose the following gauge fixing term

$$\int d^Dx \sqrt{\hat{g}} \left\{ \frac{1}{4} h^\mu_{\nu,\rho} h^\nu_{\mu,\rho} + \frac{1}{2} \hat{R}^\sigma_{\mu\nu\rho\sigma} h^\rho_{\mu} h^\nu_{\nu} - \frac{\epsilon}{4(D-1)} \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon}{8} \phi^2 \hat{R} + \frac{\epsilon}{2} \phi h^\mu_{\nu} \hat{R}^\nu_{\mu} + \epsilon \frac{\phi}{2} h^\mu_{\mu} h^\nu_{\nu} \right\} + \cdots. \quad (2.5)$$

Including this term, the total quadratic action is

$$\int d^Dx \sqrt{\hat{g}} \left\{ \frac{1}{4} h^\mu_{\nu,\rho} h^\nu_{\mu,\rho} + \frac{1}{2} \hat{R}^\sigma_{\mu\nu\rho\sigma} h^\rho_{\mu} h^\nu_{\nu} - \frac{\epsilon}{4} D \hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{\epsilon}{2} \phi h^\mu_{\nu} \hat{R}^\nu_{\mu} + \epsilon \frac{\phi}{8} \phi^2 \hat{R} \right\}. \quad (2.7)$$

For later convenience we write down some of the interaction vertices which are
readable from eq. (2.4)

\[
\int d^Dx \sqrt{g} \left\{ \frac{1}{8} \epsilon^2 (D-1) \phi \dot{g}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{32} \epsilon^3 (D-1) \phi^2 \dot{\phi}^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \\
+ \frac{1}{4} \epsilon (D-1) h^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{8} \epsilon (D-1) h^{\mu}_\rho h^{\rho \nu} \partial_\mu \phi \partial_\nu \phi \\
+ \cdots \right\} .
\] (2.8)

Under the general coordinate transformation,

\[
g_{\mu \nu} \rightarrow g_{\mu \nu} + \partial_\mu \epsilon^\rho g_{\rho \nu} + g_{\mu \rho} \partial_\nu \epsilon^\rho + \epsilon^\mu \partial_\mu g_{\rho \nu}
\] (2.9)

\(h^{\mu \nu}\) and \(\phi\) fields transform as follows:

\[
\delta h^{\mu \nu} = (e^{-h})^\mu_\rho (e^h)^\rho_\sigma \epsilon^\tau_\sigma (\dot{g} e^h)_{\tau \nu} + \epsilon^\mu_{\nu} \\
+ (e^{-h})^\mu_\rho (e^h)^\rho_\nu \epsilon^\sigma_\sigma - \frac{2}{D} \epsilon^\rho_\rho \delta^\mu_{\nu} \\
- \frac{1}{D} (e^{-h})^\rho_\sigma (e^h)^\sigma_\mu \epsilon^\tau_\nu \delta^\mu_{\tau} \\
\delta \phi = \epsilon^\mu \partial_\mu \phi - \frac{2}{D} \epsilon^\mu_{\mu} \epsilon^\nu_\nu - \frac{1}{D} (e^{-h})^\mu_\nu (e^h)^\nu_\mu \epsilon^\rho .
\] (2.10)

Under these transformations, the gauge fixing term changes as

\[
\delta (h^{\nu}_{\mu, \nu} + \frac{\epsilon}{2} \partial_\mu \phi) \\
= \epsilon_{\mu, \nu} - \hat{R}^\nu_{\mu, \nu} \epsilon^\nu + \frac{\epsilon}{2} \left( \partial_\nu \phi \epsilon^\nu \right)_{, \mu} + \cdots .
\] (2.11)

Therefore the ghost action is

\[
\sqrt{g} \{ \psi^* \mu, \nu \psi^\mu_\mu_\nu - \psi^* \mu \hat{R}^\nu_\mu \psi^\nu_\nu - \frac{\epsilon}{2} \left( \partial_\nu \phi \psi^* \mu_\mu \psi^\nu + \cdots \right}\}
\] (2.12)

where only the coupling of the \(\phi\) field to the ghost field is shown and it is proportional to \(\epsilon\).
$h_{\mu\nu}$ is a symmetric matrix and $\hat{g}^{\mu\nu}h_{\mu\nu} = 0$. We introduce a traceless symmetric matrix $H_{\mu\nu} (\delta^{\mu\nu}H_{\mu\nu} = 0)$ and express $h_{\mu\nu}$ as

$$h_{\mu\nu} = \hat{g}^{\rho\sigma}H_{\rho\sigma} - \frac{1}{D}\delta^{\mu\nu}\hat{g}^{\rho\sigma}H_{\rho\sigma}.$$  \hfill (2.13)

Then

$$\sqrt{\hat{g}}h_{\mu\nu,\rho}h^{\mu\nu,\rho} = \sqrt{\hat{g}}(\delta^{\mu\nu}\delta^{\rho\sigma} - \frac{1}{D}\hat{g}_{\mu\nu}\hat{g}_{\rho\sigma})H_{\mu\nu,\rho}H^{\mu\nu,\rho}$$  \hfill (2.14)

where indices are raised by $\hat{g}^{\mu\nu}$ as usual.

In order to proceed further, we expand the background metric around the flat metric

$$\hat{g}_{\mu\nu} = \delta_{\mu\nu} + \hat{h}_{\mu\nu}.$$  \hfill (2.15)

The kinetic terms of $h^{\mu\nu}$, $\psi_\mu$ and $\phi$ fields can be expanded in $\hat{h}_{\mu\nu}$ as well,

$$\frac{1}{4}\sqrt{\hat{g}}h_{\mu\nu,\rho}h^{\mu\nu,\rho} = \frac{1}{4}(\delta^{\mu\nu}\delta^{\rho\sigma} - \frac{1}{D}\delta^{\mu\nu}\delta_{\rho\sigma})(\partial_{\alpha}H^{\rho\sigma} + \hat{\eta}_{\alpha,\rho}^{\sigma}H^{\rho'\sigma'})$$

$$- (\partial_{\alpha}H_{\mu\nu} - \hat{h}_{\mu\nu,\rho}H^{\mu\nu,\rho}) - \frac{1}{4}\hat{S}_{\alpha\beta}\partial_{\mu}H_{\rho\sigma}\partial_{\nu}H^{\rho\sigma} + \cdots$$  \hfill (2.16)

where

$$\hat{\eta}_{\alpha,\rho}^{\sigma'}_{\rho} = \hat{\Gamma}_{\alpha}^{\rho'}_{\rho}\delta^{\sigma'}_{\sigma} + \hat{\Gamma}_{\alpha}^{\rho'}_{\sigma}\delta^{\sigma'}_{\rho}$$

$$\hat{S}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{1}{D}\delta_{\alpha\beta}\hat{h}_{\mu\nu}$$  \hfill (2.17)

and for $\psi_\mu$

$$\sqrt{\hat{g}}\hat{g}^{\alpha\beta}\psi_{\mu,\alpha}^*\psi_{\beta}$$

$$= (\partial_{\alpha}\psi_{\mu}^* - \hat{\eta}_{\alpha,\rho}^{\mu}\psi_{\rho}^*)(\partial_{\alpha}\psi_{\mu} + \hat{\eta}_{\alpha,\sigma}^{\mu}\psi_{\sigma})$$

$$ - \hat{S}_{\alpha\beta}\partial_{\alpha}\psi_{\mu}\partial_{\beta}\psi_{\mu} + \cdots$$  \hfill (2.18)

with

$$\hat{\eta}_{\alpha,\rho}^{\mu} = \hat{\Gamma}_{\alpha}^{\rho}_{\mu}.$$  \hfill (2.19)
For $\phi$

$$\frac{D}{8} \sqrt{\hat{g}} \hat{g}^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi = \frac{D}{8} \epsilon \partial_\alpha \phi \partial_\alpha \phi - \frac{D}{8} \epsilon \hat{S}^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi + \cdots . \quad (2.20)$$

The propagators are

$$< H_{\mu \nu}(P) H_{\rho \sigma}(-P) > = \frac{1}{P^2} \left( \delta_{\mu \rho} \delta_{\nu \sigma} + \delta_{\mu \sigma} \delta_{\nu \rho} - \frac{2}{D} \delta_{\mu \nu} \delta_{\rho \sigma} \right)$$

$$< \psi^\mu(P) \psi^*_\nu(-P) > = \frac{1}{P^2} \delta^\mu_\nu \quad (2.21)$$

$$< \phi(P) \phi(-P) > = -\frac{1}{P^2} \frac{4}{\epsilon D} .$$

As is seen, the propagator of the $\phi$ field has a $\frac{1}{\epsilon}$ singularity. However this singularity is likely to be controllable since the coupling of the $\phi$ field are suppressed by powers of $\epsilon$. We demonstrate this point later by concrete calculations.

3. Dynamics of conformal mode at one loop level

We are now in the position to evaluate the one loop divergences. It is found that there is no $\eta_\mu$ dependence in the counter term. To verify this, it is convenient to adopt the doubling trick and complexify the $H_{\mu \nu}$ field. Then the same argument goes through with [4]. $h^{\mu \nu}$ field gives rise to the following tadpole divergence

$$\sqrt{\hat{g}} \frac{1}{2} \hat{R}^{\sigma \mu \nu \rho} < h^{\rho \sigma} h^{\mu \nu} > = \sqrt{\hat{g}} \frac{1}{2 \pi} \frac{\hat{R}}{\epsilon} . \quad (3.1)$$

A tadpole divergence also arises due to the ghost field

$$\sqrt{\hat{g}} \hat{R}^{\nu} \alpha < \psi^* \alpha \psi \nu > = \sqrt{\hat{g}} \frac{1}{2 \pi} \frac{\hat{R}}{\epsilon} . \quad (3.2)$$

The remaining divergences come from the well known conformal anomaly due to free fields. One loop divergence due to a free scalar field is

$$- \frac{1}{24 \pi} \sqrt{\hat{g}} \frac{\hat{R}}{\epsilon} . \quad (3.3)$$

The remaining labor in determining the one loop counter term is to count the number of degrees of freedom. The $\phi$ field contributes one and the $h^{\mu \nu}$ field has
two dynamical degrees of freedom in two dimensions. The complex two component ghost field should be counted as $-4$. In addition we include matter fields with the central change $c$.

In this way we have found that the one loop counter term is

$$\frac{25 - c}{24\pi} \frac{1}{\epsilon} \sqrt{g} \hat{R}.$$  \hspace{1cm} (3.4)

The bare coupling is

$$\frac{1}{G_0} = \mu^\epsilon \left( \frac{1}{G} - \frac{25 - c}{24\pi} \frac{1}{\epsilon} \right)$$ \hspace{1cm} (3.5)

where $G$ is the renormalized coupling. The $\beta$ function is determined by

$$\mu \frac{\partial}{\partial \mu} \left( \frac{1}{G_0} \right) = 0$$ \hspace{1cm} (3.6)

yielding

$$\beta = \epsilon G - \frac{25 - c}{24\pi} G^2.$$ \hspace{1cm} (3.7)

The $\beta$ function shows that quantum gravity in $2 + \epsilon$ dimensions possesses an ultraviolet fixed point $G^* = \frac{24\pi}{25-c} \epsilon$ as long as $c < 25$.

We would like to recall [6] that the $\beta$ function depends on which interaction is compared with the gravitational interaction. It is shown later that the coefficient of the Thirring interaction, which becomes dimensionless in two dimensions, is automatically fixed to be unity. Therefore our computation scheme has avoided the double expansion in $\epsilon$ and the central charge $c$.

Let us examine more closely what we have done in the one loop renormalization of the theory. We have considered the tree level action

$$\frac{\mu^\epsilon}{G} \int d^D x \left\{ \sqrt{g} R + \sqrt{g} \frac{1}{2\alpha} (h^\nu_{\mu,\nu} + \frac{\epsilon}{2} \partial_\mu \phi)(h^\rho_{\rho,\rho} + \frac{\epsilon}{2} \partial^\rho \phi) \right\}$$  \hspace{1cm} (3.8)

where $\alpha$ is put to 1 in our Feynman type gauge. We have found that the one loop bare action is the same form with the tree level action with the substitution $\mu^\epsilon/G \rightarrow$
$1/G_0$. With this coupling constant renormalization, we obtain the following finite effective action at the one loop level

$$
\frac{1}{G} \mu^4 \int d^D x \sqrt{\hat{g}} \hat{R} + \text{finite terms.}
$$

(3.9)

The counter term $-\int d^D x \frac{25 - c}{24\pi} \frac{\mu^4}{\epsilon} \sqrt{\hat{g}} \hat{R}$ is appropriate to the $h_{\mu\nu}$ field but it is in fact an oversubtraction for the conformal mode $\phi$. As it can be seen from eq.(2.4), the conformal mode $\phi$ is suppressed at least by single power of $\epsilon$. Therefore there is no divergence which involves the conformal mode $\phi$ at the one loop level.

Nevertheless we are subtracting a finite term for $\phi$, namely $\int d^D x \frac{25 - c}{24\pi} \frac{1}{4} \partial_\mu \phi \partial_\mu \phi$ which is contained in the counter term. It is an oversubtraction since at the tree level, the kinetic term of the conformal mode is $O(\epsilon)$.

It is an important point and we would like to explain it in detail. Consider the perturbative evaluation of the effective action around the flat metric. The background metric can be decomposed into the conformal mode $\bar{\phi}$ and the traceless symmetric matrix $\bar{h}_{\mu\nu}$, $\bar{g}_{\mu\nu} = e^{-\bar{\phi}}(e^{\bar{h}})_{\mu\nu}$. The singularity of the two point function of $\bar{h}_{\mu\nu}$ field in the one loop effective action is $O(\frac{1}{\epsilon})$. However the two point function of $\bar{\phi}$ in the effective action at the one loop level is $O(\epsilon)$ since there is a factor $\epsilon$ suppression for each $\bar{\phi}$ field. This is very strange since the kinetic term of $\bar{\phi}$ which is contained in the one loop counter term is $O(1)$.

The resolution of this puzzle must be that in the full effective action, a finite nonlocal term is present which precisely cancels the $O(1)$ term which involves the conformal mode. In fact the Liouville action is such a term,

$$
- \frac{25 - c}{96\pi} \mu^4 \int d^D x \sqrt{\hat{g}} \hat{R} \frac{1}{\Delta} \hat{R}
$$

$$
= - \frac{25 - c}{96\pi} \int d^D x e^{-\frac{2\bar{\phi}}{\Delta}} \left( \frac{1}{\Delta} \hat{R} - 2\bar{\phi} \hat{R} + \bar{\phi} \Delta \bar{\phi} + O(\epsilon) \right)
$$

(3.10)

After subtracting the one loop local counter term from the effective action, the nonlocal Liouville term remains in the effective action.

The remarkable point is that the one loop counter term and the nonlocal Liouville term are the same as long as the conformal mode is involved at $O(1)$. In
the conventional field theory, the counter term is present to cancel the divergent part of the effective action. Therefore the counter term is not a large quantity in spite of the $\frac{1}{\epsilon}$ pole. The situation here is different. As long as the conformal mode is concerned, we are subtracting the $O(1)$ quantity from the $O(\epsilon)$ quantity. In this sense, the counter term for the conformal mode is a large quantity and we are performing an oversubtraction.

We are forced to adopt such an oversubtraction in order to respect the general covariance of the theory. If we consider the multiple insertions of this counter term, it could cause extra singularities in $\frac{1}{\epsilon}$.

We argue that this problem can be taken care of by using the following bare coupling in the calculation,

$$\mu^\epsilon G_0 = \frac{G}{1 - \frac{25 - c}{24\pi} G} = G^\sum_{n=0}^{\infty} \left( \frac{25 - c}{24\pi} \frac{G}{\epsilon} \right)^n$$  \hspace{1cm} (3.11)

For the propagator of the conformal mode, the use of the above mentioned bare coupling is nothing but the insertion of the counter term $\int d^D x \frac{25 - c}{24\pi} \frac{1}{4} \partial_\mu \phi \partial_\mu \phi$ infinite times. Since the counter term dominates over the tree term, such resummation is necessary for the conformal mode.

The gauge parameter $\alpha$ will be renormalized also. In order to perform such renormalization, the quantum propagator (versus background) for $h_{\mu\nu}$ should be calculated. Although we do not need such a knowledge in this paper, it will be necessary in the two loop renormalization of the theory[8].

In the following discussion concerning the operator renormalization, we perform calculations in terms of $G_0 \mu^\epsilon$. As it will be demonstrated shortly such a calculation is free from the oversubtraction problem. In particular we can consider $\epsilon \to 0$ limit (2 dimensional quantum gravity).
4. Gravitational anomalous dimensions and two dimensional limit

We next introduce the cosmological constant term $\int d^D x \sqrt{g}$ into the action (2.4) and evaluate the anomalous dimension $\gamma_{\Delta_0}$, where $\Delta_0 = 0$ denotes the canonical dimension. In the present article it is assumed that this operator is multiplicatively renormalizable. In our parametrization (2.1), the cosmological constant operator takes the form (2.2). In the following we compute the one and two loop corrections to $\sqrt{g}$ by taking into account only gravitational fluctuations.

To evaluate the divergent part of each graph, we have made assumption that $G_0 \mu^\epsilon$ is of order $\epsilon (= D - 2)$. Due to the relation eq.(3.5), it is the case as long as $G \gtrsim \epsilon$. At the one loop level we obtain

$$< \frac{1}{2!} \left( \frac{D}{2} \phi \right)^2 > = \frac{G_0 \mu^\epsilon}{2\pi \epsilon} \left( \frac{1}{\epsilon} + \text{const.} \right). \quad (4.1)$$

At the two loop order there are 9 graphs as depicted in Fig.1. We notice from (2.8) that $\phi$ propagator (2.21) is order $\frac{1}{\epsilon}$ and $3\phi$ and $4\phi$ vertices are proportional to $\epsilon^2$ and $\epsilon^3$ respectively, while the $\phi - 2h$ and $2\phi - 2h$ vertices are proportional to $\epsilon$. Using these facts we may determine the strength of the $\frac{1}{\epsilon}$ singularity for each graph. Under the assumption $G_0 \mu^\epsilon \sim 0(\epsilon)$, graphs (d),(e) and (g) turn out to be finite. The divergent part of graphs (c) cancels with that of (f). The ghost contribution graph (h) is also finite due to the suppression factor $\epsilon$ in eq.(2.12). The same is true for the matter contribution depicted by graph (i). The remaining divergences of the two loop graphs consist of the graphs (a) and (b). The divergence of graph (a) is given by

$$\frac{1}{2} \left( \frac{G_0 \mu^\epsilon}{2\pi \epsilon} \right)^2 \left( \frac{1}{\epsilon} + \text{const.} \right)^2. \quad (4.2)$$

For graph (b) we obtain

$$- \frac{1}{2 \epsilon} \left( \frac{G_0 \mu^\epsilon}{2\pi \epsilon} \right)^2. \quad (4.3)$$

Summarizing the calculation up to the two loop order, we obtain the renormal-
ized cosmological term \( Z \sqrt{g} e^{-\frac{D}{2} \phi} \), where \( Z \) is given by

\[
Z = 1 - \left( \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} \right) \left( \frac{1}{\epsilon} + \text{const.} \right) + \frac{1}{2} \left( \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} \right)^2 + \frac{1}{2} \left( \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} \right)^2 \left( \frac{1}{\epsilon} + \text{const.} \right)^2 .
\] (4.4)

In order to evaluate the anomalous dimension \( \gamma_{\Delta_0=0} \) defined by

\[
\gamma_{\Delta_0} = \mu \frac{\partial}{\partial \mu} \log Z
\] (4.5)

we make an expansion of \( \log Z \) as

\[
\log Z = -\frac{1}{\epsilon} \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} + \frac{1}{2} \left( \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} \right)^2 + O(G_0^3) .
\] (4.6)

Thus \( \gamma_{\Delta_0=0} \) is obtained as

\[
\gamma_{\Delta_0=0} = -\frac{G_0 \mu^\epsilon}{2 \pi \epsilon} + \left( \frac{G_0 \mu^\epsilon}{2 \pi \epsilon} \right)^2 + O(G_0^3) .
\] (4.7)

The one loop coupling \( G_0 \mu^\epsilon \) has already been computed in eq.(3.5). However when evaluating \( \gamma_{\Delta_0=0} \) in eq.(4.7) we assume that in the bare Lagrangian the counter term is dominant so that \( G_0 \) is given by

\[
\frac{1}{G_0} = -\frac{25 - c \mu^\epsilon}{24 \pi \epsilon}
\] (4.8)

This assumption is equivalent to say that the renormalized coupling constant \( G \) is not small \( G \gg \epsilon \). In this case we obtain

\[
\gamma_{\Delta_0=0} = \frac{4}{Q^2} + \left( \frac{4}{Q^2} \right)^2 + O((\frac{4}{Q^2})^3)
\] (4.9)

where \( Q = \sqrt{\frac{25 - c}{3}} \) in the notation [9].
For a general operator $\int d^{D}x \sqrt{g}^{1-\Delta_{0}}\Phi$ of a spinless field $\Phi$ with a canonical dimension $2\Delta_{0}$, we have performed a similar calculation up to two loops for the operator $\sqrt{g}^{1-\Delta_{0}} = \sqrt{g}^{1-\Delta_{0}} e^{-\frac{D}{2}(1-\Delta_{0})\phi}$. The anomalous dimension is obtained as

$$\gamma_{\Delta_{0}} = \frac{4}{Q^{2}}(1 - \Delta_{0})^{2} + \left(\frac{4}{Q^{2}}\right)^{2}(1 - \Delta_{0})^{3} + O\left(\frac{4}{Q^{2}}\right)^{3}.$$  \hspace{1cm} (4.10)

Let us recall the fermion field coupled to gravity[6]. We can redefine the fermion field in such a way that the kinetic term decouples from the conformal mode at $D = 2$. The fermion mass term with the canonical dimension $2\Delta_{0} = 1$ receives the gravitational dressing of the following form in terms of the rescaled field

$$\int d^{D}x \sqrt{\hat{g}}^{\frac{1}{2}}\bar{\Psi}\Psi$$  \hspace{1cm} (4.11)

In our gauge the coefficient of the kinetic term of the rescaled fermion field is automatically fixed to be unity since the conformal mode decouples at $D = 2$. Therefore we only need to consider the renormalization of the operator $\sqrt{\hat{g}}^{(1-\Delta_{0})}$. We now compare our result (4.10) with the exact solution of two dimensional gravity [9 - 11]. In their conformal gauge approach, the cosmological term $\int d^{2}x \sqrt{\hat{g}} e^{-\phi}$ receives a dressing from gravitational fluctuations and becomes $\int d^{2}x \sqrt{\hat{g}} e^{\alpha\phi}$. Similarly the operator $\int d^{2}x (\sqrt{\hat{g}} e^{-\phi})^{1-\Delta_{0}}\Phi$, where $\Phi$ is a general spinless primary field with scaling dimension $2\Delta_{0}$, becomes $\int d^{2}x \sqrt{\hat{g}}^{1-\Delta_{0}} e^{\beta\phi}\Phi$. The parameters $\alpha$ and $\beta$ have been exactly computed:

$$\alpha = -\frac{Q}{2} \left\{ 1 - \sqrt{1 - \frac{8}{Q^{2}}} \right\}$$

$$\beta = -\frac{Q}{2} \left\{ 1 - \sqrt{1 - \frac{8(1 - \Delta_{0})}{Q^{2}}} \right\}.$$  \hspace{1cm} (4.12)

In order to compare our result (4.10), we make use of the following relation

$$\frac{\beta}{\alpha} = \frac{2(1 - \Delta_{0}) + \gamma_{\Delta_{0}}}{2 + \gamma_{\Delta_{0}}=0}$$  \hspace{1cm} (4.13)

which denotes the scaling dimension of the operator $\int d^{2}x (\sqrt{\hat{g}} e^{-\phi})^{1-\Delta_{0}}\Phi$ when we choose as the standard scale that of the cosmological term.
This relation can be derived with recourse to the scaling argument together with the renormalization group analysis. Let us consider the following correlation functions

\[ \langle \Pi_i \int d^D x (\sqrt{\hat{g}} e^{-\frac{D}{2} \phi})^{1-\Delta_i} \Phi_i \rangle |_{\mu} . \]  

We rescale the background metric as \( \hat{g}_{\mu\nu} \rightarrow \lambda \hat{g}_{\mu\nu} \). The correlation function changes

\[ \Pi_i \lambda^{(1-\Delta_i)} \langle \Pi_j \int d^D x (\sqrt{\hat{g}} e^{-\frac{D}{2} \phi})^{1-\Delta_j} \Phi_j \rangle |_{\mu \lambda^{\frac{1}{2}}} \]  

where the renormalization scale also changes as \( \mu \rightarrow \mu \lambda^{\frac{1}{2}} \). The renormalization group predicts

\[ (4.15) = \Pi_i \lambda^{(1-\Delta_i)} \langle \Pi_j \int d^D x (\sqrt{\hat{g}} e^{-\frac{D}{2} \phi})^{1-\Delta_j} \Phi_j \rangle |_{\mu} . \]

The running of the coupling constant \( G \) can be neglected as long as \( G \gg \epsilon \). If we choose the cosmological term as the standard scale, we obtain (4.13).

Let us first evaluate (4.13) for the exact solution (4.12). By expanding in powers of \( \frac{1}{Q^2} \), the result is

\[ \frac{\beta}{\alpha} = \frac{1 - \Delta_0 + \frac{4}{2} \frac{1}{Q^2} (1 - \Delta_0)^2 + \frac{1}{2} \left( \frac{4}{2} \right)^2 (1 - \Delta_0)^3 + O\left( \frac{4}{Q^2} \right)^3}{1 + \frac{4}{2} \frac{1}{Q^2} + \frac{1}{2} \left( \frac{4}{2} \right)^2 + O\left( \frac{4}{Q^2} \right)^3} . \]

On the other hand, if we use the perturbative values (4.9) and (4.10) in (4.13), the result agrees with (4.17). Therefore our computational scheme is shown to give a \( \frac{1}{Q^2} = \frac{3}{25-\epsilon} \) expansion.

A comment is in order. We have assumed counter term dominance in the one loop bare Lagrangian so \( G_0 \) is given by (4.8). Therefore \( G \) is assumed to be much larger than \( \epsilon \). However we may use the proposal of the present authors (H.K. and M.N.[6]) to replace \( G \) by the fixed point of the \( \beta \) function, \( G^* = \frac{24\pi}{25-\epsilon} \epsilon \). We then
obtain
\[ \gamma_{\Delta=0}(G^*) = -\frac{4}{Q^2} - \frac{4}{Q^2} + \left(\frac{4}{Q^2}\right)^2 + O\left(\frac{4}{Q^2}\right)^3. \] (4.18)

Similarly
\[ \gamma_{\Delta=0}(G^*) = -\frac{4}{Q^2}(1 - \Delta_0) - \frac{4}{Q^2}(1 - \Delta_0) + \left(\frac{4}{Q^2}\right)(1 - \Delta_0)^3. \] (4.19)

This result disagrees with that of the exact solution (4.17). Therefore, the conjecture that two dimensional gravity corresponds to the \( D = 2 \) limit of \( D = 2 + \epsilon \) dimensional gravity at its fixed points is regrettably incorrect. It rather appears to correspond to the strong coupling phase of the \( D = 2 + \epsilon \) dimensional quantum gravity.

Turning back to our two loop calculation, the following important observation should be pointed out: The graphs containing the \( h_{\mu\nu} \), such as (c),(e) and (f), do not play any role in the renormalization and thus we are allowed to consider only the conformal mode \( \phi \). This is what occurs in two dimensional gravity in conformal gauge. Our observation would be further confirmed by deriving the exact solution (4.12) in our scheme. In fact in \( 2 + \epsilon \) dimensions, we can derive (4.12) by considering only the \( \phi \) mode with the assumption that the one loop counter term dominates in the bare Lagrangian. By dropping the \( h_{\mu\nu} \) field in Einstein action (2.4), we keep the following action for the \( \phi \) field
\[ \sqrt{g}R \simeq \sqrt{\hat{g}}\hat{R}e^{-\frac{\epsilon}{4}\phi} - \frac{\epsilon(D-1)}{4}\sqrt{g} e^{-\frac{\epsilon}{4}\phi} \hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \] (4.20)

For later convenience a new variable \( \psi \) is introduced through
\[ e^{-\frac{\epsilon}{4}\phi} = 1 + \frac{\epsilon}{4}\psi \] (4.21)
such that
\[ \sqrt{g}R \simeq \sqrt{\hat{g}}\hat{R}(1 + \frac{\epsilon}{4}\psi)^2 - \frac{\epsilon(D-1)}{4}\sqrt{g} \hat{g}^{\mu\nu}\partial_\mu\psi\partial_\nu\psi. \] (4.22)

Here, as in the perturbative calculation, we assume that the following counter term
(3.4) is dominant:

\[ \mathcal{L}_{c.t.} = - \frac{25 - c}{24\pi} \frac{\mu^\epsilon}{\epsilon} \sqrt{\hat{g}} \hat{R} (1 + \frac{\epsilon}{4} \psi)^2 + \frac{25 - c D - 1}{24\pi} \frac{\mu^\epsilon}{4} \sqrt{\hat{g}} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi. \]  

(4.23)

This is free field theory for which the \( \psi \) propagator is

\[ <\psi(P)\psi(-P)> = \frac{24\pi}{25 - c} \frac{D - 1}{P^2}. \]  

(4.24)

Thus, in computing the expectation value of the composite operator

\[ <\sqrt{g}^{1-\Delta_0}>= \sqrt{g}^{1-\Delta_0} <\exp \{ \frac{4}{\epsilon} (1 - \Delta_0) \log(1 + \frac{\epsilon}{4} \psi) \}> \]  

(4.25)

for a general operator \( \int d^D x (\sqrt{g})^{1-\Delta_0} \Phi \), Wick’s contraction theorem can be applied. With each contraction, we associate a factor coming from the divergent part of the \( \psi \) loop

\[ \frac{24\pi}{25 - c} \frac{2}{D - 1} \int \frac{d^D P}{(2\pi)^D} \frac{1}{P^2} = - \frac{24}{25 - c} \frac{1}{\epsilon} + 0(\epsilon^0). \]  

(4.26)

In order to evaluate exactly the divergent part of (4.25), we may consider a zero dimensional model for which the action is given by

\[ S = \frac{1}{2} \frac{c - 25}{24} \epsilon \psi^2. \]  

(4.27)

So eq.(4.25) reduces to the ordinary integration

\[ <\sqrt{g}^{1-\Delta_0}> = \frac{1}{Z} \int_{-\infty}^{\infty} d\psi \exp \left\{ \frac{4}{\epsilon} (1 - \Delta_0) \log(1 + \frac{\epsilon}{4} \psi) - \frac{1}{2} \frac{c - 25}{24} \epsilon \psi^2 \right\} \]  

(4.28)

where

\[ Z = \int_{-\infty}^{\infty} d\psi \exp(- \frac{1}{2} \frac{c - 25}{24} \epsilon \psi^2). \]  

(4.29)
By introducing another new variable $\rho = \frac{\epsilon}{4}\psi$, (4.28) reads

$$< \sqrt{g^{1-\Delta_0}} >= \text{const.} \int_{-\infty}^{\infty} d\rho \exp\left[\frac{1}{\epsilon}\{4(1 - \Delta_0)\log(1 + \rho) - \frac{c - 25}{3}\rho^2}\right]. \quad (4.30)$$

Let us evaluate (4.30) by means of the saddle point method. The saddle point $\rho_0$ is given by

$$\rho_0 = \frac{1}{2}\{-1 \pm \sqrt{1 - \frac{8(1 - \Delta_0)}{Q^2}}\} \quad (4.31)$$

which satisfies the equation

$$\frac{\partial}{\partial \rho_0}\{4(1 - \Delta_0)\log(1 + \rho_0) - \frac{c - 25}{3}\rho_0^2\} = 0. \quad (4.32)$$

Evaluating (4.30) at $\rho = \rho_0$, we obtain the renormalization of the $\sqrt{g^{1-\Delta_0}}$ operator as $Z_{\Delta_0}\sqrt{g^{1-\Delta_0}}$ where

$$Z_{\Delta_0} = \exp\left\{-\frac{4}{\epsilon}(1 - \Delta_0) \log(1 + \rho_0) + \frac{8\pi}{G_0}\mu^{-\epsilon}\rho_0^2\right\}. \quad (4.33)$$

Thus the anomalous dimension is given by

$$\gamma_{\Delta_0} = \mu \frac{\partial}{\partial \mu} \log Z_{\Delta_0} = \rho_0^2 Q^2 = -2(1 - \Delta_0) - Q^2 \rho_0. \quad (4.34)$$

Inserting (4.34) into (4.13) we obtain the exact solution provided that in (4.31) $\rho_0 = \frac{1}{2}\{-1 + \sqrt{1 - \frac{8(1 - \Delta_0)}{Q^2}}\}$ is chosen.
5. Conclusions and Discussions

In this concluding section, we would like to clarify the basic theoretical structure and physical picture of the quantum gravity in $2 + \epsilon$ dimensions.

Since the quantum gravity is not renormalizable in 4 dimensions, we have pursued the $2 + \epsilon$ dimensional expansion of quantum gravity which is power counting renormalizable.

However the presence of the $\frac{1}{\epsilon}$ pole in the propagator of the conformal mode $\phi$ makes the renormalization program difficult. The problem is not the existence of the $\frac{1}{\epsilon}$ pole itself, since it is possible to choose a gauge as we have done in which the interactions of $\phi$ are suppressed by powers of $\epsilon$.

The real problem is how to handle the oversubtractions of $\phi$. As we have explained before, the one loop counter term is an oversubtraction for $\phi$. If we consider insertions of the one loop counter term for $\phi$ $n$ times, it causes the extra singularities of $O((\frac{\mu}{\epsilon})^n)$. This singularity cannot be subtracted by local counter terms in general since the integrand itself is divergent before the loop momenta integrations. Hence it cannot be made finite by differentiating the external momenta. However these singularities do not originate from the high momentum loop integrations and should not be regarded as the ultraviolet singularities. Therefore we have said that this problem is resolved by the resummation of the one loop counter term insertions for $\phi$. Putting it in another way we claim that the real expansion parameter of the theory is not only $G$ but also $\kappa$ where $\frac{1}{\kappa}$ is the effective inverse propagator of the conformal mode. We recall our bare Lagrangian

$$\frac{1}{G_0} \sqrt{g} R + \text{gauge fixing term.} \quad (5.1)$$

The one loop quantum correction is

$$\frac{\mu^\epsilon}{\epsilon} \frac{25 - c}{24\pi} \sqrt{g} \bar{R} + \text{finite terms.} \quad (5.2)$$

By choosing $\frac{1}{G_0} = \mu^\epsilon (\frac{1}{G} - \frac{25 - c}{24\pi} \frac{1}{\epsilon})$, the theory becomes finite up to the one loop level. We stress that the only $h_{\mu\nu}$ field possesses the one loop divergence. For $\phi$,
the “tree level” Lagrangian is
\[
\mu^\epsilon \left( \frac{25 - c}{24\pi} - \frac{\epsilon}{G} \right) \left( \frac{1}{4} \sqrt{\bar{g}} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \phi \tilde{R} \right) + \cdots \tag{5.3}
\]
and the one loop correction is \( O(\epsilon) \). In other word at the one loop level \( \kappa \) is nothing but \( G_0 \mu^\epsilon \)
\[
\frac{1}{\kappa} = -\frac{1}{G} + \frac{25 - c}{24\pi \epsilon}. \tag{5.4}
\]
In this sense, the \( h_{\mu\nu} \) field and \( \phi \) field are renormalized in very different ways.

Based on these understandings of the theoretical structure, we present our basic physical picture of the quantum gravity in \( 2 + \epsilon \) dimensions. The dynamics of \( h_{\mu\nu} \) field resembles the spin system in \( 2+\epsilon \) dimensions. When the gravitational coupling \( G \) is weak, the quantum gravity is in the ordered phase. When \( G \) is strong it is in the disordered phase. The ultraviolet fixed point is located at \( G_c = O(\epsilon) \). In the weak coupling regime, \( G \) vanishes in the infrared limit. The opposite situation takes place in the strong coupling regime. When \( G < G_c \), the theory contains massless gravitons if we can extrapolate up to \( \epsilon = 2 \). It is physically the most interesting phase. On the other hand the dynamics of the conformal mode is governed by \( \kappa \). \( \kappa \) is of \( O(\epsilon) \) away from the ultraviolet fixed point, but it becomes larger near that point. Therefore the conformal mode has nontrivial dynamics near the ultraviolet fixed point.

Let us make the conformal mode \( \phi \) massive by adding the cosmological constant term \( \Lambda \int d^D x \sqrt{g} \) to the action. If we do so, \( \Lambda \) acts as the infrared cutoff like the electron mass of QED and it sets the scale of the physics (and the Universe). By rescaling the metric \( g_{\mu\nu} \), we fix \( \Lambda = 1 \) further. Then the bare coupling becomes \( G_0 \Lambda \tilde{\phi} \). This is a free parameter of our theory which controls the size of the Universe.

In the infrared limit \( \Lambda \to 0 \), the effective bare coupling \( G_0 \Lambda \tilde{\phi} \) vanishes. In the previous section, the anomalous dimensions are computed in power series of \( G_0 \mu^\epsilon \sim G_0 \Lambda \tilde{\phi} \). In the weak coupling limit, \( h_{\mu\nu} \) field also become classical in the infrared limit. Therefore all anomalous dimensions due to gravitational dressing vanish when \( \Lambda \to 0 \) in the weak coupling regime. The physics there is described
in terms of classical Einstein theory with small cosmological constant $G_0 \Lambda^\epsilon$. This regime indeed resembles our Universe. Since $G_0 \Lambda^\epsilon$ is a free parameter in $2 + \epsilon$ dimensional quantum gravity, we can tune it to have a large Universe like our own. On the other hand in the ultraviolet limit formally $G_0 \Lambda^\epsilon \rightarrow \infty$ which means $\kappa$ becomes larger in that limit. Let us introduce the effective central charge $c_{\text{eff}}$ in such a way that

$$\frac{\epsilon}{\kappa} = \frac{25 - c_{\text{eff}}}{24\pi}.$$ (5.5)

In terms of $c_{\text{eff}}$, the ultraviolet limit corresponds to $c_{\text{eff}} = 25$ limit. In this sense, when $\epsilon$ is small enough the dynamics of the conformal mode at the ultraviolet fixed point is very close to that of the critical string and hence calculable. The $O(\epsilon)$ corrections are calculable in the sense of $\frac{1}{c}$ expansion[12].

The 2d gravity is scale invariant since all correlation functions scale if we change the cosmological constant which controls the size of the Universe. In $2 + \epsilon$ dimensions, the scale invariance is broken since the dynamics of $h_{\mu\nu}$ and $\phi$ fields change in a nontrivial way if we change the cosmological constant.

If we would like to have a constructive definition of the quantum gravity in 3 or 4 dimensions, the theory should have ultraviolet fixed points. This condition puts the constraints in the matter content of the theory such that $c < 25$. The number 25 is comfortably large. It makes the applicability of $2 + \epsilon$ dimensional expansion of quantum gravity up to 4 dimensions plausible. For such an enterprise, we make the following predictions:

The theory possesses two distinct phases, namely weak and strong coupling phases. We can control the theory by tuning the inverse of the bare gravitational coupling constant $\frac{1}{G_0} \Lambda^{-\epsilon}$. The continuum limit which resembles our Universe can be taken by approaching the ultraviolet fixed point from the weak coupling side.

In the strong coupling phase near the phase transition point, $c_{\text{eff}}$ is in the range $1 < c_{\text{eff}} < 25$. Therefore the system may be in the branched polymer phase.

In fact these predictions are in accord with recent numerical simulations[13].
Finally $c < 25$ constraint may put the upper bound to the quark-lepton species if we extrapolate $\epsilon$ up to 2. Therefore the quantum gravity may explain why there are only three generations of quarks and leptons in the Universe.

Acknowledgements

We are grateful to D. Lancaster for his reading the manuscript. This work is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture.

Appendix  Background field method in terms of $\phi$ and $h_{\mu\nu}$

Let us consider the generating function of the connected Green’s functions in a background gauge,

$$
e^{-W_{b.g.}} = \int \exp\left[-S(g) - G.F.(\hat{g}, h') + J^{\mu\nu} \cdot h'_{\mu\nu}\right],$$  \hspace{1cm} (A.1)

where $S$ is the Einstein action, G.F. denotes the gauge fixing term which depends on the background $\hat{g}$ and $h'_{\mu\nu}$ field is defined as $g_{\mu\nu} = \hat{g}_{\mu\nu} + h'_{\mu\nu}$. $\int$ implies the functional integration over the fields in the theory including the ghost determinant. $J^{\mu\nu} \cdot h'_{\mu\nu}$ implies $\int d^Dx J^{\mu\nu}(x)h'_{\mu\nu}(x)$. Since our gauge fixing term is a simple function of $h_{\mu\nu}$ and $\phi$ fields in addition to $\hat{g}_{\mu\nu}$, it is certainly a function of $h'_{\mu\nu}$ and $\hat{g}_{\mu\nu}$. The effective action $\Gamma$ is obtained after the Legendre transform

$$\Gamma = W_{b.g.} + J^{\mu\nu} < h'_{\mu\nu} >$$

$$< h'_{\mu\nu} > = -\frac{\delta W_{b.g.}}{\delta J^{\mu\nu}}$$  \hspace{1cm} (A.2)

We may shift the quantum field such that $h'_{\mu\nu} \rightarrow g_{\mu\nu} - \hat{g}_{\mu\nu}$. Then

$$e^{-W_{b.g.}} = \int \exp\left[-S(g) - G.F.(\hat{g}, g - \hat{g}) + J^{\mu\nu} \cdot (g - \hat{g})_{\mu\nu}\right]$$

$$\Gamma = W_{b.g.} + J^{\mu\nu} \cdot (< g_{\mu\nu} > - \hat{g}_{\mu\nu})$$

$$= W + J^{\mu\nu} \cdot < g_{\mu\nu} >$$  \hspace{1cm} (A.3)

where $W$ is the conventional generating function with an unconventional gauge fixing term $G.F.(\hat{g}, g - \hat{g})$. In this way one can see that the background field method
is an efficient way to compute the conventional effective action in an unconventional gauge. By coupling the source term, we can expand the action around the nontrivial background

\[ S(g) + \text{G.F.} - J^{\mu\nu} \cdot g_{\mu\nu} \]
\[ = S(\hat{g}) + (\frac{\delta S}{\delta g})^{\mu\nu} \cdot h'_{\mu\nu} + \frac{\delta^2 S}{\delta^2 g} \cdot h'^2 + \cdots \]  
\[ + \text{G.F.} - J^{\mu\nu} \cdot \hat{g}_{\mu\nu} \]  

(A.4)

When we compute the effective action in the background field method, the linear term in \( h'_{\mu\nu} \) field in the action is dropped. The logic behind it is the following relation

\[ J^{\mu\nu} = \frac{\delta \Gamma}{\delta g^{\mu\nu}} = \frac{\delta S}{\delta g^{\mu\nu}} + \text{higher order terms in } h \]  

(A.5)

Therefore if we use the conventional coupling of the source to the metric \( g_{\mu\nu} \cdot J^{\mu\nu} \), we should drop the linear term in \( h'_{\mu\nu} \). In this paper we have adopted the parametrization of the metric \( g_{\mu\nu} = (\hat{g}e^h)_{\mu\nu}e^{-\phi} \), hence

\[
h'_{\mu\nu} = h_{\mu\nu} - \phi \hat{g}_{\mu\nu} + \frac{1}{2} (h^2)_{\mu\nu} - h_{\mu\nu} \phi + \frac{1}{2} \phi^2 \hat{g}_{\mu\nu} + \cdots \]  

(A.6)

If we drop only the linear term in \( h_{\mu\nu} \) and \( \phi \) in the action, we have to add the following term in the action

\[ - \frac{\delta S}{\delta g_{\mu\nu}} \cdot \left( \frac{1}{2} (h^2)_{\mu\nu} - h_{\mu\nu} \phi + \frac{1}{2} \phi^2 \hat{g}_{\mu\nu} + \cdots \right) \]
\[ = -\sqrt{\hat{g}} \left( \frac{1}{2} \hat{R} \hat{g}^{\mu\nu} - \hat{R}'^{\mu\nu} \right) \cdot \left( \frac{1}{2} (h^2)_{\mu\nu} - h_{\mu\nu} \phi + \frac{1}{2} \phi^2 \hat{g}_{\mu\nu} + \cdots \right) \]  

(A.7)

This additional term leads to the one loop divergence due to the conformal mode

\[ -\sqrt{\hat{g}} \frac{\epsilon}{2} \hat{R} \cdot <\phi^2> \]  

(A.8)

However we also have the contribution to \( \Gamma \) from the source term

\[ J^{\mu\nu} \cdot <h'_{\mu\nu}> \]
\[ = \frac{\delta S}{\delta g_{\mu\nu}} \cdot <\frac{1}{2} (h^2)_{\mu\nu} - h_{\mu\nu} \phi + \frac{1}{2} \phi^2 \hat{g}_{\mu\nu} + \cdots> \]  
\[ = \sqrt{\hat{g}} \frac{\epsilon}{2} \hat{R} \cdot <\phi^2> \]  

(A.9)

These two divergences cancel each other. In conclusion, we can simply drop the
linear term in $h_{\mu\nu}$ and $\phi$ in the action to compute the one loop divergence in the effective action.
Figure caption

Fig. 1 The two loop correction graphs to $\sqrt{g}$ which is shown by a cross. The solid and wavy lines denote $\phi$ and $h_{\mu\nu}$ propagators, while the dashed and dash-and-dotted lines are $\psi^\mu$ and matter propagators.

References

[1] M.E. Agishtein and A.A. Migdal, PUPT-1287 (1991).
[2] J. Ambjørn and S. Versted, NBI-HE-91-45 (1991).
[3] S. Weinberg, in General Relativity, an Einstein Centenary Survey, eds. S.W. Hawking and W. Israel (Cambridge University Press, 1979) p.790.
[4] R. Gastmans, R. Kallosh and C. Truffin, Nucl. Phys. B133 (1978) 417.
[5] S.M. Christensen and M.J. Duff, Phys. Lett. B79 (1978) 213.
[6] H. Kawai and M. Ninomiya, Nucl. Phys. B336 (1990) 115.
[7] G. ’t Hooft and M. Veltman, Ann. Isnt. Henri Poincare 20 (1974) 69.
[8] L.F. Abbott, Nucl. Phys. B185 (1980) 189.
[9] J. Distler and H. Kawai, Nucl. Phys. B321 (1989) 509.
[10] F. David, Mod. Phys. Lett. A3 (1988) 1651.
[11] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 819.
[12] H. Kawai, Y. Kitazawa and M. Ninomiya, paper in preparation.
[13] M.E. Agishtein and A.A. Migdal, PUPT-1311 (1992).