Invertible minor assignment: sparse deformations of determinant expansions and their hyperdeterminants

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Abstract

We introduce an algebraic model based on the expansion of the determinant of two matrices, one of which is generic, to check the additivity of $\mathbb{Z}^d$-valued set functions. Each individual term of the expansion is deformed through a monomial factor in $d$ indeterminates with exponents defined by the set function. A family of sparse polynomials is derived from Grassmann-Plücker relations, and their compatibility is linked to the factorisation of hyperdeterminants in the ring of Laurent polynomials.

It is proved that, in broad generality, this deformation returns a determinantal expansion if and only if it is induced by a diagonal matrix of monomials acting as a kernel into the initial determinant expansion, which guarantees the additivity of the set function. The hypotheses underlying this result are tested through the construction of counterexamples, and their implications are explored in terms of complexity reduction, with special attention to permutations of families of subsets.

Keywords: Sparse polynomials, Plücker relations, hyperdeterminant, integrability

1 Introduction

1.1 Motivation

The notion of integrability is often connected to different, but consistent representations of a system, which result in complexity reduction or computational efficiency. In this paper, we focus on a combinatorial notion of integrability related to set functions: for a given set $\mathcal{X}$, a family $\mathcal{X}$ of subsets of $\mathcal{X}$, and a function $\Psi : \mathcal{X} \rightarrow \mathbb{Z}^d$, $d \geq 1$, we ask whether $\Psi$ is induced by a function $\psi : \mathcal{X} \rightarrow \mathbb{Z}^d$, i.e. whether we can reconstruct $\Psi(\mathcal{I})$ from the data $\psi(\mathcal{I}) = \{\psi(\alpha) : \alpha \in \mathcal{I}\}$.

For this purpose, we introduce an algebraic model based on determinant expressions to encode the information regarding set functions. The symmetries of the determinant entail computation advantages, especially when compared to other immanants such as the permanent [17]. A more general connection between determinantal varieties and universal integrable systems is given by the bilinear form of the Kadomtsev-Petviashvili II (KP) hierarchy, which expresses Grassmann-Plücker relations for an infinite-dimensional space [6]. An instance is provided by the first term of the hierarchy, namely the KP II equation:

$$D_{\text{KP}}(\tau, \tau) := (D_x^4 - 4 \cdot D_x D_t + 3 \cdot D_y^2) \tau \cdot \tau = 0$$  \hspace{1cm} (1.1)

where

$$D_x(f \cdot g) := (\partial_x f) \cdot g - f \cdot (\partial_x g) = (\partial_{x_1} - \partial_{x_2}) f(x_1) g(x_2)|_{x_1 = x_2}.$$  \hspace{1cm} (1.2)
is the Hirota derivative. A special class of solutions (τ-functions) of the bilinear KP II hierarchy includes
Wronskian solutions of the type
\[ \tau(x) = \det(A \cdot \Theta(x) \cdot K) \]  \hspace{1cm} (1.3)
where \( A \) is a matrix of constant coefficients,
\[ \Theta := \text{diag} \left( \exp \left( \sum_{r=1}^{d} \kappa_{r}^{I} x_{r} \right), \ldots, \exp \left( \sum_{r=1}^{d} \kappa_{n}^{I} x_{r} \right) \right) \]  \hspace{1cm} (1.4)
and \( K \) is the Vandermonde matrix associated with the \( n \)-tuple \( (\kappa_{1}, \ldots, \kappa_{n}) \). For these solutions of the KP II hierarchy, the determinant (1.3) is converted into an exponential sum through the well-known Cauchy-Binet expansion for two matrices \( A, K \in \mathbb{R}^{k \times n} \) \( (k \leq n) \)
\[ \det(A \cdot K) = \sum_{I \in \wp_{k}[n]} \Delta_{A}(I) \cdot \Delta_{K}(I). \]  \hspace{1cm} (1.5)
where \( \wp_{k}[n] := \{ I \subseteq \{1, \ldots, n\} : \#I = k \} \) and \( \Delta_{A}(I) \) (respectively, \( \Delta_{K}(I) \)) is the maximal minor of \( A \) extracted from columns (respectively, rows) indexed by \( I \subseteq \{1, \ldots, n\} \). From (1.5), the solution (1.3) is expressed as
\[ \det(A \cdot \Theta(x) \cdot K) = \sum_{I \in \wp_{k}[n]} \Delta_{A}(I) \cdot \Delta_{K}(I) \cdot e^{\sum_{\alpha \in I} x \cdot K(\alpha)}. \]  \hspace{1cm} (1.6)
In addition to its use in algebraic computations, the expansion (1.5) also has a relevant geometric interpretation, since it arises in the study of principal angles between subspaces of a vector space [11] and consequent applications [5].

These special characteristics of this determinantal expansion and their relations with integrability have led us to consider the information-theoretic properties of exponential sums of the type (1.6): in particular, we can explore the information content in the \( \Delta_{A}(I) \cdot \Delta_{K}(I) \) \( \tau \)-functions of the bilinear KP II hierarchy includes deformations of each product of minors \( \Delta_{A}(I) \cdot \Delta_{K}(I) \) in (1.5) described by a monomial \( t_{a_{1}(I)}^{1} \ldots t_{a_{d}(I)}^{d} \) in \( d \) indeterminates. This deformation can be described by a mapping that associates each subset \( I \) indexing the minor \( \Delta_{A}(I) \) of \( A \) with an element \( \Psi(I) := (a_{1}(I), \ldots, a_{d}(I)) \in \mathbb{Z}^{d} \). From this, we can explore a combinatorial notion of integrability related to subsets \( I \), namely, the existence of a “potential” \( \psi : \{1, \ldots, n\} \rightarrow \mathbb{Z}^{d} \) such that \( \Psi(I) \) can be uniquely recovered by the values \( \psi(\alpha) \) with \( \alpha \in I \).

In this way, another concept related to integrability emerges, that is \( \text{curvature} \). In the present model, curvature has a set-theoretic meaning: for all \( \mathcal{H} \subseteq \{1, \ldots, n\} \) with \( \# \mathcal{H} = k - 2 \) and \( \alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in \{1, \ldots, n\} \setminus \mathcal{H} \), the object of interest is the quantity
\[ \Psi(\mathcal{H} \cup \{\alpha_{1}, \alpha_{2}\}) + \Psi(\mathcal{H} \cup \{\beta_{1}, \beta_{2}\}) - \Psi(\mathcal{H} \cup \{\alpha_{1}, \beta_{2}\}) - \Psi(\mathcal{H} \cup \{\beta_{1}, \alpha_{2}\}) \]  \hspace{1cm} (1.8)
whenever the arguments lie in the domain of \( \Psi \), i.e. the associated minors do not vanish. In this context, integrability relates to the additivity of \( \Psi \), and we will use the two terms interchangeably.
The algebraic structure resulting from this model combines two quadratic contributions, that are the Grassmann-Plücker relations for two matrices, through the Cauchy-Binet expansion of the determinant of their product. This results in a set of quartic equations that constrain the form of the minors and, hence, the deformations preserving the determinant form.

1.2 Overview and main result

The main contribution of this work is to prove the following result:

**Theorem 1.** Let \( L(t), R(t) \) be two matrices depending on \( d \) indeterminates \( t = (t_1, \ldots, t_d) \) such that \( L(1), R(1)^T \in \mathbb{C}^{k \times n} \) have full rank and \( R(t) \) is generic. If the terms in the Cauchy-Binet expansion of \( L(t) \cdot R(t) \) satisfy the monomial condition

\[
\Delta_{L(t)}(\mathcal{I}) \cdot \Delta_{R(t)}(\mathcal{I}) = g_{\mathcal{I}} \cdot \prod_{u \in \{1, \ldots, d\}} \mu^{\mathcal{I};u}(\mathcal{I}), \quad \mathcal{I} \in \wp_k[n],
\]  

(1.9)

where \( d(\mathcal{I}, \alpha) \in \mathbb{Z} \) and \( g_{\mathcal{I}} \in \mathbb{C} \), then the existence of a generic \((2 \times 2)\)-submatrix of \( L(t) \) or \( L(t)^T \) guarantees the existence of \( m_0 \in \mathbb{Z}^d \) and a map \( \psi : \{1, \ldots, n\} \rightarrow \mathbb{Z}^d \) such that the two pairs \((L(t), R(t))\) and \((L(1), \text{diag}(t^{\psi(1)}, \ldots, t^{\psi(n)}) \cdot R(1))\) give the same Cauchy-Binet expansion up to a common factor \( t^{m_0} \), i.e.

\[
\Delta_{L(t)}(\mathcal{I}) \cdot \Delta_{R(t)}(\mathcal{I}) = t^{m_0} \cdot \Delta_{L(1)}(\mathcal{I}) \cdot \Delta_{R(1)}(\mathcal{I}) \cdot \prod_{\alpha \in \mathcal{I}} t^{\psi(\alpha)}, \quad \mathcal{I} \in \wp_k[n].
\]  

(1.10)

The exact definition of the generic \((2 \times 2)\)-submatrix mentioned in Theorem 1, which will be referred to as an explainable local key, will be provided in the following sections.

Theorem 1 is non-trivial especially when there are not enough non-vanishing minors of \( L(t) \), and we will provide counterexamples where the number of non-vanishing minors does not suffice to guarantee the integrability of set functions. In more detail, the possibility to recover (1.10) comes from the information provided by (1.9), which is hidden when \( \Delta_{L(t)}(\mathcal{I}) = 0 \), and this opens up the way for deviations from the form (1.10). This observation highlights the information-theoretic interpretation of the model: the information source is a fixed matrix \( L(t) \), and a generic matrix \( R(t) \) is selected to explore the structure of \( L(t) \) by means of available information (1.9).

The assumptions of Theorem 1 specify neither the form of \( g_{\mathcal{I}} \) nor the exponents \( d(\mathcal{I}, u) \), which makes the Cauchy-Binet expansion a sparse polynomial \([8]\) with a given upper bound for its sparsity. This argument does not rely on the association \( \mathcal{I} \mapsto (g_{\mathcal{I}}, d(\mathcal{I}; \cdot)) \) nor on potential reductions arising from the cancellation of terms in the expansion (1.5): if we are able to find an assignment of non-vanishing terms \( g_{\mathcal{I}} \) that is consistent with (1.5) and the assumptions of Theorem 1, then the reduction (1.10) follows.

As a consequence, we will discuss applications of Theorem 1 with special attention to information-theoretic properties of determinantal constraints when the set function \( \Psi \) corresponds to a permutation of subsets \( \hat{\Psi} : \wp_k[n] \rightarrow \wp_k[n] \). In this case, the complexity reduction involves those permutations that are induced by a permutation \( \hat{\psi} : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \). Furthermore, we will show that the present model relates to the factorisation of hyperdeterminants \([4]\) in the Laurent ring \( \mathbb{C}(t) \). Hyperdeterminants affect the integrability of the model especially when an extremal number of vanishing minors of \( L(t) \) is attained, i.e. when \( L(t) = (1_k | 1_k) \in \mathbb{C}^{k \times (2k)} \): in this case, the model is equivalent to a principal minor assignment with variable (monomial) minors of \( r(t) \), where \( R(t) = (1_k | r(t))^T \in \mathbb{C}(t)^{(2k) \times k} \). The assumption regarding monomial deformation of terms in the Cauchy-Binet expansions is equivalent to the invertibility in the ring \( \mathbb{C}(t) \) of the non-vanishing principal minors of the composed matrix

\[
\begin{pmatrix}
0_k & L(t) \\
R(t) & 0_n
\end{pmatrix}
\]

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indexed by subsets \( \{1, \ldots, k\} \cup I \) with \( I \subseteq \{k+1, \ldots, k+n\} \) and \( \#I = k \). This connection between the present model and non-constant, invertible minor assignment in a given ring will be the basis for further studies.

1.3 Organization of the paper

The paper is divided into two main parts: following Section 2, where we fix the notation that will be used thereafter, in the first part we derive the polynomial equations that are the basis for our study (Section 3) and the constraints on algebraic extensions (Section 4) resulting from these equations. In Section 5, we explore the constraints for non-vanishing minors in the three-term Grassmann-Plücker relations, both describing allowed configurations (Subsections 5.1-5.2) and systems where multiple distinct configurations are possible (Subsection 5.3). In the second part, we include vanishing minors: in Section 6, we explore basic conditions guaranteeing the integrability of the monomial deformation in the above-mentioned sense. We conclude with comments regarding applications and future work in Section 7.

2 Preliminaries

2.1 Notation

Let \( \wp([n]) \) be the power set of \( [n] := \{1, \ldots, n\} \) and \( \wp_k([n]) := \{I \subseteq [n] : \#I = k\} \subseteq \wp([n]) \). We adopt the shortening expressions

\[
I_\alpha^\beta := I \setminus \{\alpha\} \cup \{\beta\}, \quad I \in \wp([n]), \alpha \in I, \beta \notin I
\]

and, similarly, \( I_\beta := I \cup \{\beta\}\), \( I_\alpha := I \setminus \{\alpha\}\), \( I_{\alpha_1\alpha_2} := I \setminus \{\alpha_1, \alpha_2\}\), etc. The notation \( I_\alpha^\beta \) implicitly assumes that \( \alpha \in I \) and \( \beta \notin I \), unless \( \alpha = \beta \) where \( I_\alpha^\alpha := I \). We denote as \( I^c := [n] \setminus I \) the complement of \( I \) in \( [n] \). When both a set \( V \) and a labelling of its elements \( V = \{\nu_1, \ldots, \nu_k\} \) are fixed, we also write \( (2.1) \) as

\[
V_i^\alpha := V \setminus \{\nu_i\} \cup \{\alpha\}, \quad i \in V, \alpha \in V^c.
\]

For a given matrix \( A \in \mathbb{C}^{k \times n} \), we consider the matroid

\[
\mathcal{G}(A) := \{I \in \wp_k([n]) : \Delta_A(I) \neq 0\}.
\]

In particular, the following exchange relation holds [13]:

\[
\forall A, B \in \mathcal{G}(A), \alpha \in A \setminus B : \exists \beta \in B \setminus A. A^\beta_\alpha \in \mathcal{G}(A).
\]

We adopt the notation \( A_{\mathcal{A},\mathcal{B}} \) to denote the submatrix of \( A \) with rows extracted from \( \mathcal{A} \subseteq [k] \) and columns extracted from \( \mathcal{B} \subseteq [n] \).

In order to connect previous works on exponential sums, starting from the indeterminates \( x \) in (1.6) we move to polynomial rings through the parametrisation \( t_u := e^{\varphi_u(x)}, u \in [d] \), where \( \varphi_u \) are functions of these indeterminates. So, we introduce the \( d \)-tuple

\[
t := (t_u : u \in [d]).
\]

For each index \( w \in [d] \), the symbol \( t_{w}^- \) denotes the \((d-1)\)-tuple obtained from \( t \) through the omission of the \( w \)-th component. Given the ring \( \mathbb{C}(t) := \mathbb{C}[t, t^{-1}] \) of Laurent polynomials in \( t_1, \ldots, t_d \), let \( \mathbb{F} \) be the field of fraction of \( \mathbb{C}(t) \). Given two polynomials \( P, Q \), the symbol \( P|Q \) means that \( P \) is a factor of \( Q \).

Let \( \Psi \) refer to a set function returning the exponents \( d(I; \alpha) \) in (1.9) for a general non-vanishing
polynomial in the indeterminates $t$: using the standard inner product $\langle \cdot, \cdot \rangle$ between polynomials, this map can be formulated as

$$P \in \mathbb{C}(t) \mapsto \Theta(P) := \{ m \in \mathbb{C}(t) : \exists m^{-1}, \langle P, m \rangle = \langle m, m \rangle \}$$  (2.5)

$$P \in \mathbb{C}(t) \mapsto \Psi(P) := \{ e \in \mathbb{Z}^d : t^e \in \Theta(P) \}$$  (2.6)

with $t^e := \prod_{i=1}^d t_i^{e_i}$. Since a finite number of exponential functions with pairwise distinct exponents are linearly independent over $\mathbb{C}$, the previous definition is well-posed. When $\Psi(P) \subseteq \{0,1\}^d$, it reduces to the characteristic function of a finite subset of $\varphi[d]$. When $P$ is invertible in $\mathbb{C}(t)$, the evaluation $\psi(P)$ is defined by $\Psi(P) := \{\psi(P)\}$.

We will make use of monomial orders: let us recall that a monomial order $\preceq$ is a total order on the class of monic monomials that is compatible with multiplication:

$$\forall x, y, z \text{ monic monomials : } x \preceq y \Rightarrow x \cdot z \preceq y \cdot z.$$  (2.7)

With such notation, the two sides of (1.9) are expressed as

$$g(I) := C(t) \cdot g_\Psi(I) \cdot \prod_{\alpha \in \Psi(I)} t_\alpha,$n

$$h(I) := \Delta_{L(t)}(I) \cdot \Delta_{R(t)}(I), \quad I \in \varphi_k[n].$$  (2.8)

The set

$$\chi(I_{\alpha\beta}^{ij}) := \{ h(I) \cdot h(I_{\alpha\beta}^{ij}), h(I_{\alpha}^{ij}), h(I_{\beta}^{ij}), h(I_{\alpha}^{ij}) \}$$  (2.9)

is called observable when $\chi(I_{\alpha\beta}^{ij}) \neq \{0\}$ and integrable when

$$\#\Psi(\chi(I_{\alpha\beta}^{ij}) \setminus \{0\}) = 1,$$  (2.10)

which implies that $\chi(I_{\alpha\beta}^{ij})$ is observable too. We say that $I \in \Phi(L(t))$ is an integrable basis when (2.10) holds for all the observable sets $\chi(I_{\alpha\beta}^{ij})$ with basis $I$. We call $h(I) \cdot h(I_{\alpha\beta}^{ij})$ the central term of $\chi(I_{\alpha\beta}^{ij})$; furthermore, we refer to a term in $\chi(I_{\alpha\beta}^{ij})$ that is algebraically independent of the other two as a unique term.

The subsets $I$ are considered unordered: signed minors are introduced as

$$\Delta_A(\alpha, \beta|H) := \Delta_A(H_{\alpha\beta}) \cdot (-1)^{1+S(\alpha, H)+S(\beta, H)} \cdot \text{sign} (\alpha - \beta), \quad H \in \varphi_{k-2}[n]$$  (2.11)

where

$$S(\alpha, H) := \# \{ \beta \in H : \beta < \alpha \}, \quad \alpha \in [n] \setminus H$$  (2.12)

takes into account the role of permutations of elements of $[n]$ and leads to the following expression for the three-terms Plücker relations [4]

$$\Delta_A(\delta_1, \delta_2|H) \cdot \Delta_A(\delta_3, \delta_4|H) = \Delta_A(\delta_1, \delta_3|H) \cdot \Delta_A(\delta_2, \delta_4|H) - \Delta_A(\delta_1, \delta_4|H) \cdot \Delta_A(\delta_2, \delta_3|H)$$  (2.13)

for any $H \in \varphi_{k-2}[n]$ and pairwise distinct elements $\delta_a \in [n] \setminus H$, $a \in [4]$.

### 2.2 Assumptions

We assume that there are no null columns in $L(t)$, since they do not affect the terms occurring in the Cauchy-Binet expansion. This assumption implies that, for each $I \in \Phi(L(t))$, we can choose a function
$r : [n] \rightarrow I$ such that $\Delta_{L(t)}(T^{(\alpha)}_{\alpha}) \neq 0$ for all $\alpha \in [n]$. Similarly, we assume that for each $i \in I$ there is $\alpha \in I^c$ such that $\Delta_{L(t)}(T^{(\alpha)}_{\alpha}) \neq 0$, otherwise, we can equivalently express the family of monomials (1.9) into a sparse deformation of (1.5) with a reduced number of rows.

Regarding the matrix $R(t)$, we assume that it is generic, namely, all its maximal minors are non-vanishing for a generic choice of $t$. Besides generalising the properties of the Vandermonde matrix $K$ for solitonic solutions mentioned in Subsection 1.1, together with the monomial condition (1.9) this request guarantees the invertibility in $C(t)$ of all the deformed terms $h(I)$ in (2.8) associated with invertible minors $\Delta_{L(t)}(I)$, independently of the knowledge of the matroid $\Theta(L(t))$.

### 3 Coupled Plücker relations

Let us consider the Grassmann-Plücker relation in the form (2.13) for $L(t)$, namely,

$$
\Delta_{L(t)}(\mathcal{H}_{ij}) \cdot \Delta_{L(t)}(\mathcal{H}_{\alpha\beta}) = c_1 \Delta_{L(t)}(\mathcal{H}_{ij}) \cdot \Delta_{L(t)}(\mathcal{H}_{\alpha\beta}) + c_2 \Delta_{L(t)}(\mathcal{H}_{ij}) \cdot \Delta_{L(t)}(\mathcal{H}_{\alpha\beta})
$$

(3.1)

where $\mathcal{H} \in \mathcal{V}_{k-2}[n]$, $i, j, \alpha, \beta \in \mathcal{H}^c$ and the signs

$$
c_1 := \text{sign}[(i - j) \cdot (\alpha - \beta) \cdot (i - \beta) \cdot (\alpha - j)],
$$

$$
c_2 := \text{sign}[(i - j) \cdot (\alpha - \beta) \cdot (i - \alpha) \cdot (j - \beta)]
$$

(3.2)

take into account the permutation of indices $i, j, \alpha, \beta$ with respect to a fixed order. The same relation holds for $R(t)$. Multiplying term-by-term the resulting equations (3.1) for $L(t)$ and $R(t)$, we get

$$
h(I) \cdot h(T^{(\alpha)}_{\alpha\beta}) = h(T^{(\alpha)}_{\alpha\beta}) \cdot h(T^{(\beta)}_{\beta\alpha}) + h(T^{(\beta)}_{\beta\alpha}) \cdot h(T^{(\alpha)}_{\alpha\beta})
$$

$$
+ c_1 c_2 \Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta}) \cdot \Delta_{R(t)}(T^{(\beta)}_{\beta\alpha}) \cdot \Delta_{L(t)}(T^{(\alpha)}_{\alpha\beta}) \cdot \Delta_{L(t)}(T^{(\beta)}_{\beta\alpha})
$$

$$
+ c_1 c_2 \Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta}) \cdot \Delta_{R(t)}(T^{(\beta)}_{\beta\alpha}) \cdot \Delta_{L(t)}(T^{(\alpha)}_{\alpha\beta}) \cdot \Delta_{L(t)}(T^{(\beta)}_{\beta\alpha})
$$

(3.3)

since $c_1, c_2$ only depend on the indices $i, j, \alpha, \beta$. This expression can be written as

$$
h(I) \cdot h(T^{(\alpha)}_{\alpha\beta}) = h(T^{(\alpha)}_{\alpha\beta}) \cdot h(T^{(\beta)}_{\beta\alpha}) + h(T^{(\beta)}_{\beta\alpha}) \cdot h(T^{(\alpha)}_{\alpha\beta})
$$

$$
+ c_1 c_2 \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})} \cdot h(T^{(\beta)}_{\beta\alpha}) \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})} \cdot h(T^{(\alpha)}_{\alpha\beta})
$$

$$
+ c_1 c_2 \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})} \cdot h(T^{(\beta)}_{\beta\alpha}) \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})} \cdot h(T^{(\alpha)}_{\alpha\beta}).
$$

(3.4)

since we are assuming that all the minors $\Delta_{R(t)}(I)$, $I \in \mathcal{V}_k[n]$, are non-vanishing for a generic choice of $t$. We can define the $Y$-terms, or cross-ratios, as

$$
Y(T)^{(\alpha\beta)}_{\alpha\beta} := c_1 c_2 \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})} \cdot \frac{\Delta_{R(t)}(T^{(\alpha)}_{\alpha\beta})}{\Delta_{R(t)}(T^{(\beta)}_{\beta\alpha})}
$$

(3.5)

Furthermore, we introduce

$$
Y(I) \ := \ \{ Y(T)^{(\alpha\beta)}_{\alpha\beta} : \chi(T)^{(\alpha\beta)}_{\alpha\beta} \text{ is observable} \}.
$$

(3.6)

The dependence of $Y(T)^{(\alpha\beta)}_{\alpha\beta} := Y(I)^{(\alpha\beta)}_{\alpha\beta}$ on the basis $I$ will be implicit when no ambiguity arises. Being $Y(T)^{(\alpha\beta)}_{\alpha\beta} = (Y(T)^{(\alpha\beta)}_{\alpha\beta})^{-1}$, the assumption in Subsection 2.2 regarding $R(t)$ is equivalent to the existence of each $Y(I)$.
For each generalised permutation matrix $D(t)$ dependent on $t$, the expressions (3.5) are invariant with respect to the action

$$L(t) \mapsto L(t) \cdot D(t)^{-1}, \quad R(t) \mapsto D(t) \cdot R(t).$$

(3.7)

Using this invariance, we can fix a form for matrices $R(t)$ and $L(t)$ while preserving the variables $Y$. In particular, choosing

$$D(t) := \text{diag} \left( R_{k+1,1}(t) \ldots R_{k+1,k}(t) \ 1 \ R_{k+1,1}(t) \ldots R_{k+1,k}(t) \ R_{k+1,1}(t) \ldots R_{k+1,k}(t) \ R_{k+1,1}(t) \ldots R_{k+1,k}(t) \right)$$

(3.8)

together with the right-multiplication of $R(t)$ by

$$d(t) := \text{diag} \left( R_{k+1,1}(t) \ R_{k+1,2}(t) \ldots R_{k+1,k}(t) \right)$$

(3.9)

the matrix $R(t)$ takes the form

$$R(t) \mapsto D(t) \cdot R(t) \cdot d(t) =$$

$$\begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
1 & R_{k+1,2}R_{k+2,3} \ldots R_{k+1,k} & \ldots & R_{k+1,2}R_{k+2,3} \ldots R_{k+1,k} \\
\vdots & \vdots & \ddots & \vdots \\
1 & R_{k+1,2}R_{k+2,3} \ldots R_{k+1,k} & \ldots & R_{k+1,2}R_{k+2,3} \ldots R_{k+1,k}
\end{pmatrix}.$$ 

(3.10)

In this way, we can re-define the matrices $R(t)$ and $L(t)$ so that, for each $j \in \{2, \ldots, k\}$ and $\alpha \in \{2, \ldots, n - k\}$ we get

$$R_{\alpha j}(t) \mapsto \frac{\Delta R(t)(Y_{k+1}^{\alpha j})}{\Delta R(t)(Y_{k+1}^{\alpha j})} \cdot \frac{\Delta R(t)(Y_{k+1}^{\alpha j})}{\Delta R(t)(Y_{k+1}^{\alpha j})} = c_1 c_2 \cdot Y_{10}^{1j}.$$ 

(3.11)

Remark 2. The previous representation allows us discussing the duality defined by a pair of matrices $(L^\perp(t), R^\perp(t))$ representing the orthogonal complements of the subspaces associated with $(L(t), R(t))$, respectively. Given any $I \in \mathcal{G}(L(t))$, an appropriate choice of a permutation matrix $P$ lets us map $I$ to $[k]$ through $(L(t), R(t)) \mapsto (L(t) \cdot P, P^{-1} \cdot R)$ while preserving the expansion (1.5) up to a permutation of its terms. Then, we consider the reduced row echelon form $L(t) \mapsto: (\hat{1}_{k, l})$ of $L(t) \cdot P$ and the reduced column echelon form $\hat{R}(t) := (\hat{1}_{k, r})^T$ of $P^{-1} \cdot R(t)$. A well-known result (see e.g. [13, 7] and reference therein for more details) asserts that $L^\perp(t)$ (respectively, $R^\perp(t)$) spans the same subspaces as $\hat{L}(t)$ (respectively, $\hat{R}(t)$) and, moreover, there exist two non-vanishing coefficients $C_L, C_R$ such that

$$\Delta_{\hat{L}(t)}(J) = C_L \cdot \Delta_{L(t)}(J), \quad \Delta_{\hat{R}(t)}(J) = C_R \cdot \Delta_{R(t)}(J)$$

for all $J \in \mathcal{G}[n]$, with $\hat{X} = \text{diag}((-1)^{i+1})_{i \in [n]}, \text{alt}(\hat{L}(t)) := (\hat{1}^T[-1_{n-k}])^T \Xi, \text{alt}(\hat{R}(t)) := (\hat{r}^T[-1_{n-k}])^T$. In particular, if $n < 2 \cdot k$ we can work with matrices $\text{alt}(L(t))$ and $\text{alt}(R(t))$ that preserve the minors in (1.5), up to a common multiplicative factor. The minors extracted from these matrices have dimension $(n - k, n)$ instead of $(k, n)$. Therefore, we can assume $n \geq 2 \cdot k$ without loss of generality.

The present notation for (3.2) gives

$$c_1 c_2 = -\text{sign}[\alpha \cdot (j - \beta) \cdot (j - \beta)],$$

(3.12)
which is equal to $-1$ when upper indices $i, j$ are always smaller than lower indices $\alpha, \beta$: unless otherwise stated, we will assume that this condition holds without loss of generality due to the invariance (3.7).

It is also easy to check that

$$
Y^{ij}_{\alpha\beta} Y^{ij}_{\beta\gamma} = -Y^{ij}_{\alpha\gamma},
$$

(3.13)

$$
Y^{im}_{\alpha\beta} Y^{mj}_{\alpha\beta} = -Y^{ij}_{\alpha\beta}.
$$

(3.14)

Iterating (3.13), we obtain an identity that will be used several times in the rest of this work, which we will refer to as a quadrilateral decomposition:

$$
Y^{ij}_{\alpha\beta} = -Y^{ij}_{\alpha\beta} \cdot Y^{ij}_{\beta\gamma}
$$

$$
= -Y^{im}_{\alpha\beta} \cdot Y^{mj}_{\alpha\beta} \cdot Y^{im}_{\beta\delta} \cdot Y^{jm}_{\delta\beta}
$$

(3.15)

for all $i, j, m \in I$ and $\alpha, \beta, \delta \in \mathcal{I}^C$.

Then, the identity (3.3) is equivalent to

$$
h(I) \cdot h(T^i_{\alpha\beta}) = h(I^i_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) + Y^{ij}_{\alpha\beta} \cdot h(I^j_{\alpha\beta}) + Y^{ij}_{\alpha\beta} \cdot h(I^j_{\alpha\beta}) + h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta})
$$

(3.16)

where the dependence on $t$ is implicit.

Remark 3. Note that $Y^{ij}_{\alpha\beta} \neq -1$ for all $I \in \mathcal{I}_{k}[n], i, j \in I$ and $\alpha, \beta \in \mathcal{I}^C$, unless $i = j$ or $\alpha = \beta$. Indeed, $Y^{ij}_{\alpha\beta} = -1$ is equivalent to

$$
\Delta_{R(t)}(I^i_{\alpha\beta}) \cdot \Delta_{R(t)}(I^j_{\alpha\beta}) = -c_1 c_2 \cdot \Delta_{R(t)}(T^i_{\alpha\beta}) \cdot \Delta_{R(t)}(T^j_{\alpha\beta})
$$

which implies $\Delta_{R(t)}(I)\cdot \Delta_{R(t)}(T^i_{\alpha\beta}) = 0$ by the Plücker relations (3.1), and this contradicts the assumption $\Delta_{R(t)}(T^i_{\alpha\beta}) \neq 0$. Indeed, (3.13) and (3.14) are special instances of (3.15): these correspond to the degenerate cases where the lower, respectively the upper indices in some $Y$-terms coincide, leading to the only allowed cases of $Y$-terms equal to $-1$, according to the definition (3.5) and Remark 3.

In the case $\{0\} \subset \chi(I^i_{\alpha\beta})$, (3.16) is equivalent to $P(Y^{ij}_{\alpha\beta}) = 0$ where $P$ is a monic polynomial over $\mathbb{C}(t)$, since $h(I^i_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) \neq 0$ is invertible in $\mathbb{C}(t)$ by assumption; so $Y^{ij}_{\alpha\beta}$ is integral over $\mathbb{C}(t)$, and we introduce the notation

$$
Y^{ij}_{\alpha\beta} = e^{ij}_{\alpha\beta} \cdot \sqrt{B^{ij}_{\alpha\beta}} + A^{ij}_{\alpha\beta}
$$

(3.17)

where $e^{ij}_{\alpha\beta} \in \{+1, -1\}$ and

$$
A^{ij}_{\alpha\beta} := h(I) \cdot h(I^i_{\alpha\beta}) - h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) - h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}),
$$

(3.18)

$$
B^{ij}_{\alpha\beta} := \left(h(I) \cdot h(I^i_{\alpha\beta}) - h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) - h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta})\right)^2
$$

$$
-4 \cdot h(I^i_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}) \cdot h(I^j_{\alpha\beta}).
$$

(3.19)

For future convenience, we also introduce the following function based on (3.19)

$$
B(x, y, z) := (x - y - z)^2 - 4yz.
$$

(3.20)

In (3.17) we have fixed $\sqrt{B^{ij}_{\alpha\beta}}$ as one of the two roots of $B^{ij}_{\alpha\beta}$. In particular, we refer to terms $Y^{ij}_{\alpha\beta} \notin \mathbb{F}$ as radical $Y$-terms. Also for the $A$-terms (3.18) and the $B$-terms (3.19) the dependence on $I$ will be
where no ambiguity arises. From (3.16) it follows that any radical Y-term \( Y^{ij}_{\alpha\beta} \) satisfies

\[
h(\mathcal{I}) \cdot h(\mathcal{I}^{ij}_{\alpha\beta}) \cdot h(\mathcal{I}^{j}_{\alpha}) \cdot h(\mathcal{I}^{i}_{\beta}) \cdot h(\mathcal{I}^{j}_{\beta}) \neq 0.
\]

**Definition 4.** We say that \( \mathcal{I} := \{i, j\} \times \{\alpha, \beta\} \in \mathcal{I} \times \mathcal{T}^c \) is a local key (for the set \( \mathcal{I} \in \mathfrak{G}(\mathbf{L}(t)) \)) if (3.21) holds. A weak local key is a set \( \mathfrak{w} := \{a, b\} \times \{\gamma, \delta\} \in \mathcal{I} \times \mathcal{T}^c \) such that at least two non-vanishing products can be extracted from \( \chi(\mathcal{I}^{ij}_{\alpha\beta}) \).

Note that the choice of the sign \( \pm \) in (3.17) is related to the exchange \( \alpha \Leftrightarrow \beta \). Indeed, the signs (3.2) transform under \( \alpha \Leftrightarrow \beta \) as \( c_1 \leftrightarrow c_2 \), hence the \( c_1c_2 \) is preserved. Furthermore, one has

\[
1 = \frac{-\sqrt{B^{ij}_{\alpha\beta} + A^{ij}_{\alpha\beta}}}{2 \cdot h(\mathcal{I}^{j}_{\beta}) \cdot h(\mathcal{I}^{i}_{\alpha})} \cdot \frac{\sqrt{B^{ij}_{\alpha\beta} + A^{ij}_{\alpha\beta}}}{2 \cdot h(\mathcal{I}^{j}_{\beta}) \cdot h(\mathcal{I}^{i}_{\alpha})}.
\]

The exchange of the roles of \( \alpha \) and \( \beta \) affects (3.16) through \( Y^{ij}_{\alpha\beta} \mapsto Y^{ij}_{\beta\alpha} = (Y^{ij}_{\alpha\beta})^{-1} \) and the change of sign \( \pm \mapsto \mp \) in (3.17).

From the Plücker relations (3.1), we find:

\[
Y(\mathcal{I}^{ij}_{\alpha\beta}) = -c_2 \frac{\Delta_R(\mathcal{I}) \cdot \Delta_R(\mathcal{I}^{ij}_{\alpha\beta})}{\Delta_R(\mathcal{I}^{j}_{\beta}) \cdot \Delta_R(\mathcal{I}^{i}_{\alpha})} = -Y(\mathcal{I}^{ij}_{\alpha\beta}) - 1,
\]

\[
Y(\mathcal{I}^{ij}_{\beta\alpha}) = c_1 \cdot \frac{\Delta_R(\mathcal{I}) \cdot \Delta_R(\mathcal{I}^{ij}_{\beta\alpha})}{\Delta_R(\mathcal{I}^{j}_{\beta}) \cdot \Delta_R(\mathcal{I}^{i}_{\alpha})} = \frac{1}{1 + \left(Y(\mathcal{I}^{ij}_{\alpha\beta})\right)^2}.
\]

Along with (3.22), the transformations (3.23) and (3.24) define a set of local transformations (with respect to the basis \( \mathcal{I} \) and indices \( i, j, \alpha, \beta \)) that will be useful in the following. This also holds for three-term sets (2.9): using the notation

\[
(a(\mathcal{I})^{ij}_{\alpha\beta}, u(\mathcal{I})^{ij}_{\alpha\beta}, d(\mathcal{I})^{ij}_{\alpha\beta}) := \left(h(\mathcal{I})h(\mathcal{I}^{ij}_{\alpha\beta}), h(\mathcal{I}^{i}_{\alpha}) \cdot h(\mathcal{I}^{j}_{\beta}), h(\mathcal{I}^{i}_{\beta}) \cdot h(\mathcal{I}^{j}_{\alpha})\right)
\]

the exchange \( i \Leftrightarrow \alpha \) defines the change of basis from \( \mathcal{I} \) to \( \mathcal{J} := \mathcal{I}^{i}_{\alpha} \), which affects the terms in \( \chi(\mathcal{I}^{ij}_{\alpha\beta}) \) as follows:

\[
\mathcal{I} = \mathcal{J}^{o}_{i}, \quad \mathcal{I}^{ij}_{\alpha\beta} = \mathcal{J}^{ij}_{o}, \quad a(\mathcal{I})^{ij}_{\alpha\beta} = u(\mathcal{J})^{ij}_{i},
\]

\[
\mathcal{I}^{o}_{\alpha} = \mathcal{J}, \quad \mathcal{I}^{j}_{\beta} = \mathcal{J}^{o}_{j}, \quad u(\mathcal{I})^{ij}_{\alpha\beta} = o(\mathcal{J})^{ij}_{j},
\]

\[
\mathcal{I}^{o}_{\beta} = \mathcal{J}^{o}_{j}, \quad \mathcal{I}^{j}_{\alpha} = \mathcal{J}^{i}, \quad d(\mathcal{I})^{ij}_{\alpha\beta} = d(\mathcal{J})^{ij}_{i}.
\]

**Remark 5.** There are three basic non-equivalent actions that can be carried out on the indices of \( \chi(\mathcal{I}^{ij}_{\alpha\beta}) \):

- the identity, the exchange of upper indices \( i \Leftrightarrow j \) (equivalent to the parallel exchange \( \alpha \Leftrightarrow \beta \)), and
- the exchange of basis \( \mathcal{I} \mapsto \mathcal{J}^{o}_{i} \) (equivalent to the parallel exchange \( \mathcal{I} \mapsto \mathcal{J}^{i}_{j} \)). Other operations can be obtained from the composition of the previous ones: for instance, \( \mathcal{I} \mapsto \mathcal{J}^{o}_{i} \) is obtained from the composition of \( i \Leftrightarrow j \), the above-mentioned exchange of basis, and a second exchange \( i \Leftrightarrow \alpha \).

Looking at the action of these transformations on Y-terms, they are obtained from the composition of the inversion \( f_h(Y) := Y \mapsto Y^{-1} \) associated with \( i \Leftrightarrow j \), and the transformation \( f_v(Y) := -1 - Y \) associated with \( i \Leftrightarrow \alpha \) as in (3.23). These two functions are involutions, and they can be placed in correspondence with transpositions in the permutation group \( S_3 \): the identity is mapped to the trivial permutation, \( f_h \) is mapped to the transposition \( (1 2) \), and \( f_v \) is mapped to \( (2 3) \). In this way, a combination of these two functions corresponds to the product of the associated transpositions, and the distinct combinations are recovered decomposing the elements in \( S_3 \) in terms of \( (1 2) \) and \( (2 3) \). In
4 Constraints on algebraic extensions

This section formalises some simple results, which will nevertheless be relevant in the rest of the discussion. For a given invertible \( G_{ij}^{\alpha\beta} \in \mathbb{C}(t) \), we introduce

\[
\hat{\chi}(I|_{\alpha\beta}^{ij}) := \left\{ (G_{ij}^{\alpha\beta})^{-1} \cdot X, \, X \in \chi(I|_{\alpha\beta}^{ij}) \right\}. \tag{4.1}
\]

We anticipate that the scaling (4.1) will be used later in this work with different definitions of the monomial \( G_{ij}^{\alpha\beta} \). In this subsection, we consider the “ground” monic monomial in \( \mathbb{C}(t) \)

\[
G_{ij}^{\alpha\beta} := t^{\min \{ \Psi(h(I) \cdot h(I_{0,\alpha}^{ij})), \Psi(h(I_{0,\beta}^{ij}) \cdot h(I_{0,\alpha}^{ij})), \Psi(h(I_{0,\beta}^{ij}) \cdot h(I_{0,\beta}^{ij})) \}}
= \prod_{c=1}^{d} t^{\min \{ \Psi(h(I) \cdot h(I_{0,\alpha}^{ij})_{c}), \Psi(h(I_{0,\beta}^{ij}) \cdot h(I_{0,\alpha}^{ij})_{c}), \Psi(h(I_{0,\beta}^{ij}) \cdot h(I_{0,\beta}^{ij})_{c}) \}}. \tag{4.2}
\]

Note that \( B_{ij}^{\alpha\beta} \) and \( G_{ij}^{\alpha\beta} \) are symmetric under the permutation of the elements of \( \chi(I|_{\alpha\beta}^{ij}) \), so these quantities are preserved by the simultaneous application of a bijection \( \pi : \{i, j, \alpha, \beta\} \rightarrow \{i, j, \alpha, \beta\} \) and the change of basis induced by \( \pi \).

We start from the following three lemmata, which contain very basic results that will repeatedly recalled in the rest of the paper.

**Lemma 6.** For all \( I \in \mathfrak{G}(\mathbb{L}(t)) \), if \( 0 \in \chi(I|_{\alpha\beta}^{ij}) \) and \( \#\chi(I|_{\alpha\beta}^{ij}) > 1 \), then \( \alpha_{ij}^{\alpha\beta} \) or \( \alpha_{ij}^{\beta\alpha} \) is a constant different from \( \{0, -1\} \), a Laurent monomial, or a binomial.

**Proof.** The proof is a straightforward calculation: from (3.16), at most one term in \( \chi(I|_{\alpha\beta}^{ij}) \) vanishes at \( \chi(I|_{\alpha\beta}^{ij}) \neq \{0\} \). In particular, (3.16) can be expressed as

\[
0 = (Y_{ij}^{\alpha\beta} + 1) \cdot \left( h(I_{\beta}^{ij}) \cdot h(I_{0,\alpha}^{ij}) + \frac{1}{Y_{ij}^{\alpha\beta}} \cdot h(I_{0,\beta}^{ij}) \cdot h(I_{0,\beta}^{ij}) \right) \tag{4.3}
\]
when \( h(I) \cdot h(I_{0,\alpha}^{ij}) = 0 \), or as

\[
h(I) \cdot h(I_{0,\alpha}^{ij}) = (Y_{ij}^{\alpha\beta} + 1) \cdot h(I_{\beta}^{ij}) \cdot h(I_{0,\beta}^{ij}) \tag{4.4}
\]
at \( h(I_{0,\beta}^{ij}) \cdot h(I_{0,\beta}^{ij}) = 0 \). We find that \( Y_{ij}^{\alpha\beta} \) can be a monomial

\[
Y_{ij}^{\alpha\beta} = \frac{h(I_{\beta}^{ij}) \cdot h(I_{0,\beta}^{ij})}{h(I_{\beta}^{ij}) \cdot h(I_{\beta}^{ij})} \tag{4.5}
\]
in the case (4.3), or a binomial

\[
Y_{ij}^{\alpha\beta} = \frac{h(I) \cdot h(I_{0,\beta}^{ij})}{h(I_{\beta}^{ij}) \cdot h(I_{\beta}^{ij})} - 1 \tag{4.6}
\]
from (4.4), unless the ratios in (4.5) or (4.6) are constant, which means that \( Y_{ij}^{\alpha\beta} \in \mathbb{C} \setminus \{0, -1\} \) according to Remark 3. The case \( h(I_{\beta}^{ij}) \cdot h(I_{0,\beta}^{ij}) = 0 \) is analogous to \( h(I_{\beta}^{ij}) \cdot h(I_{0,\beta}^{ij}) = 0 \) under exchange of the indices \( \alpha \leftrightarrow \beta \), which returns the reciprocal of the expression (4.6). \( \square \)
Lemma 7. If (3.21) holds and $B^{ij}_{\alpha\beta}$ is a perfect square in $\mathbb{C}(t)$, then $Y^{ij}_{\alpha\beta} \in \mathbb{C}$. Also, each observable set $\chi(I|_{\alpha\beta})$ satisfying $Y^{m}_{\gamma\delta} \in \mathbb{C}$ is integrable.

Proof. Let $B^{ij}_{\alpha\beta} = P^2$ for some $P \in \mathbb{C}(t)$. From the definitions (3.18)-(3.19), we get

$$\left(A^{ij}_{\alpha\beta} - P\right) \cdot \left(A^{ij}_{\alpha\beta} + P\right) = 4 \cdot h(I_j) \cdot h(J_i) \cdot h(I_j) \cdot h(I_j). \tag{4.7}$$

So both $A^{ij}_{\alpha\beta} - P$ and $A^{ij}_{\alpha\beta} + P$ are invertible in $\mathbb{C}(t)$ since $\mathbb{C}(t)$ is a unique factorization domain, hence their sum $2A^{ij}_{\alpha\beta}$ (and their difference $2P$) has sparsity at most 2. From (3.18), this means that at least two of the elements in $\chi(I|_{\alpha\beta})$ are linearly proportional. According to (3.26), we can change the basis $I \mapsto J$ with $J \in \{I, J_i, J_j\}$ in order to get $h(J) \cdot h(J_j) = c \cdot h(J_i) \cdot h(J_j)$, $c \in \mathbb{C} \setminus \{0\}$, while preserving the factors of $B^{ij}_{\alpha\beta}$ in $\mathbb{C}(t)$. We label the invertible element $\gamma := h(J_i) \cdot h(J_j) \cdot h(J_j)^{-1} \cdot h(J_i)^{-1}$ to express

$$\frac{B^{ij}_{\alpha\beta}}{h(J_i)^2 \cdot h(J_j)^2} = (1 - c^2) \cdot \tau^2 - 2(c + 1) \cdot \tau + 1. \tag{4.8}$$

Being $c \neq 0$, $B^{ij}_{\alpha\beta}$ is a squarefree polynomial in $\mathbb{C}(t)$ only if $\tau \in \mathbb{C}$: by definition, this means that $\chi(I|_{\alpha\beta})$ is integrable, i.e. (5.1) holds and, hence, $Y^{ij}_{\alpha\beta} \in \mathbb{C}$.

We can also consider other observable sets $\chi(I|_{\alpha\beta})$ satisfying $Y^{m}_{\gamma\delta} \in \mathbb{C}$ and note that they are integrable, as follows from the constrained forms (4.5) or (4.6) when $0 \in \chi(I|_{\alpha\beta})$. \hfill $\square$

Lemma 8. If (3.21) holds for $\chi(I|_{\alpha\beta})$, then the existence of $\gamma \in \mathbb{T}$ such that $0 \in \chi(I|_{\alpha\beta})$ implies $Y^{ij}_{\alpha\beta} \in \mathbb{C}$.

Proof. If $h(I_j) \cdot h(I_j) = 0$, then we take $l \in I$ such that $h(I_j) \neq 0$, which exists due to the argument in Subsection 2.2, and it may coincide with one of the two indices $i, j$. For instance, if $l = j$, then $h(I_j) = 0$ and both $Y^{ij}_{\alpha\gamma}$ and $Y^{ij}_{\beta\gamma}$ are algebraic due to Lemma 6. Thus, $Y^{ij}_{\alpha\beta} = -Y^{ij}_{\beta\gamma} \cdot Y^{ij}_{\alpha\gamma}$ is algebraic too. This also holds when $h(I_j) = h(I_j) = 0$: in that event, $Y^{ij}_{\beta\gamma} \cdot Y^{ij}_{\alpha\gamma} \cdot Y^{ij}_{\alpha\gamma}$ are all algebraic by Lemma 6, thus

$$Y^{ij}_{\alpha\beta} = -\frac{Y^{ij}_{\alpha\gamma}}{Y^{ij}_{\beta\gamma}} = -\frac{Y^{ij}_{\alpha\gamma} \cdot Y^{ij}_{\beta\gamma}}{Y^{ij}_{\alpha\gamma} \cdot Y^{ij}_{\beta\gamma}} \in \mathbb{F}. \text{ This is compatible with (3.17) if and only if } B^{ij}_{\alpha\beta} \text{ is a perfect square in } \mathbb{C}(t), \text{ which implies } Y^{ij}_{\alpha\beta} \in \mathbb{C} \text{ due to Lemma 7.}$$

When $h(I_j) \cdot h(I_j) = 0$, we can work with $J := J_i$: indeed, $Y(J)^{ij}_{\alpha\beta}$ still satisfies (3.21) due to (3.26) and, similarly, $0 \in \chi(J|_{\alpha\beta})$ being $h(J_i) = h(I_j)^2 = 0$. So we can repeat the previous argument to get $Y^{ij}_{\alpha\beta} \in \mathbb{C}$ and, from (3.23), $Y^{ij}_{\alpha\beta} \in \mathbb{C}$. \hfill $\square$

Definition 9. We call $(Y^{ij}_{\alpha\beta}, Y^{ij}_{\beta\gamma}, Y^{ij}_{\alpha\gamma})$ a radical triple if all its components are radical Y-terms derived from (3.16).

Lemma 10. All the observable Y-terms with basis I lie in the same quadratic extension of $\mathbb{F}$.

Proof. We can assume the existence of a term $B^{ij}_{\alpha\beta}$ that is not a perfect square in $\mathbb{C}(t)$, otherwise all the Y-terms lie in $\mathbb{F}$ and the thesis follows. Since $\mathbb{C}(t)$ is a unique factorization domain, we can express

$$B^{ij}_{\alpha\beta} := \left(Q^{ij}_{\alpha\beta}\right)^2 \cdot D \tag{4.9}$$

where $Q^{ij}_{\alpha\beta}, D \in \mathbb{C}(t)$ and $D$ is a squarefree polynomial, namely, there exists no $P \in \mathbb{C}(t)$ such that $P^2 | D$. So the thesis is equivalent to

$$Y^{ij}_{\alpha\beta} \in \mathbb{F}(\sqrt{Y^{ij}_{\alpha\beta}}) = \mathbb{F}(\sqrt{D}) \tag{4.10}$$

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for all $\gamma, \delta \in \mathcal{I}^c$, $\gamma \neq \delta$, where $\mathbb{F}(\sqrt{y_{ij}^{\alpha\beta}})$ is the algebraic extension of $\mathbb{F}$ by a square root of $y_{ij}^{\alpha\beta}$.

All the sets $\chi(\mathcal{I}_{ij}^{\alpha\beta})$, $\delta \in \{\alpha, \beta\}$, are local keys by contraposition of Lemma 8. Then, we start from the instance $\delta \in \{\alpha, \beta\}$ and note that we only have to consider radical triples $(y_{ij}^{\alpha\beta}, y_{ij}^{\alpha\gamma}, y_{ij}^{\alpha\delta})$; indeed, if $y_{ij}^{\alpha\delta} \in \mathbb{F}$ for some $\delta \in \{\alpha, \beta\}$, then (3.13) trivially returns the thesis (4.10) for this instance. Adapting (3.17) for the three terms $y_{ij}^{\alpha\beta}, y_{ij}^{\alpha\gamma}$, and $y_{ij}^{\alpha\delta}$ in the expansion of $y_{ij}^{\alpha\beta} = -y_{ij}^{\alpha\gamma}y_{ij}^{\alpha\delta}$, we get

$$y_{ij}^{\alpha\gamma} \in \mathbb{F}(\sqrt{B_{ij}^{\alpha\beta}}) \Leftrightarrow y_{ij}^{\alpha\delta} \in \mathbb{F}(\sqrt{B_{ij}^{\alpha\beta}}).$$

(4.11)

Using again the assumption $y_{ij}^{\alpha\beta}, y_{ij}^{\alpha\gamma}, y_{ij}^{\alpha\delta} \notin \mathbb{F}$, we find (4.10) at $\delta \in \{\alpha, \beta\}$. This argument can be adapted to triples $(y_{ij}^{\alpha\beta}, y_{ij}^{m1}, y_{ij}^{m2})$ by transposition of upper and lower indices. Consequently, for all $\gamma_1, \gamma_2 \in \mathcal{I}^c$ and $m_1, m_2 \in \mathcal{I}$, this leads to

$$y_{ij}^{\gamma_1\gamma_2} = -y_{ij}^{\gamma_1\alpha} \cdot y_{ij}^{\gamma_2\alpha} \in \mathbb{F}(\sqrt{D}), \quad y_{ij}^{m_1m_2} = -y_{ij}^{m_1}\cdot y_{ij}^{m_2} \in \mathbb{F}(\sqrt{D}).$$

(4.12)

Then, we move to terms $y_{ij}^{m_1m_2}$ with $\{\alpha, \beta\} \neq \{\gamma_1, \gamma_2\}$ and $\{i, j\} \neq \{m_1, m_2\}$: from the previous argument, we infer $y_{ij}^{\gamma_1\gamma_2} \notin \mathbb{F}(\sqrt{D})$ for all $\sigma \in \{\alpha, \beta\}$ and $\gamma \in \{\gamma_1, \gamma_2\}$; moreover, $y_{ij}^{\gamma_1\gamma_2} \notin \mathbb{F}$ for at least one choice of $\gamma_1, \gamma_2 \in \{\alpha, \beta\}$. Starting from these two $Y$-terms, keeping fixed the lower indices, and taking into account (4.12), we find

$$y_{ij}^{\gamma_1\gamma_2} = y_{ij}^{\gamma_1\sigma_1} \cdot y_{ij}^{\gamma_2\sigma_2} \cdot y_{ij}^{\gamma_1\gamma_2} \in \mathbb{F}(\sqrt{D}).$$

(4.13)

The previous lemma states that all the $B$-terms are Laurent polynomials in $t$ that are not perfect squares in $\mathbb{C}(t)$ have the same squarefree part $D$.

**Proposition 11.** The quantity $D$ in (4.9) is the same for each choice of the basis $\mathcal{J} \in \mathfrak{G}(\mathcal{L}(t))$.

*Proof.* Here we make explicit the dependence of $Y(\mathcal{I})^{ij}_{\alpha\beta}$ on the basis $\mathcal{I}$. For any $\mathcal{J} \in \mathfrak{G}(\mathcal{L}(t))$, we denote as $D(\mathcal{J})$ the squarefree part of any $B$-term associated with a radical $Y$-term with the convention that the constant term of $D(\mathcal{J})$ is 1. In particular, $D(\mathcal{J}) = 1$ will denote that there is no radical $Y$-term arising from a local key with basis $\mathcal{J}$. Let us label the difference set $\mathcal{I} \setminus \mathcal{J} := \{m_1, \ldots, m_r\}$, $r \leq k$. The set $\mathfrak{G}(\mathcal{L}(t))$ is a matroid, hence the exchange property (2.4) implies that there exists a labelling $\mathcal{J} \setminus \mathcal{I} := \{\delta_1, \ldots, \delta_r\}$ such that

$$\mathcal{L}_u := \mathcal{I} \setminus \{m_1, \ldots, m_u\} \cup \{\delta_1, \ldots, \delta_u\} \in \mathfrak{G}(\mathcal{L}(t)), \quad u \in [r].$$

(4.14)

Note that $\delta_u \neq \delta_t$ for all $t < u$, since $\delta_t \in \mathcal{L}_{u-1}$ and $\delta_u \notin \mathcal{L}_{u-1}$. We set $\mathcal{L}_0 := \mathcal{I}$ to unify the notation.

Now consider any $o \in [r]$; the thesis holds for $\mathcal{L}_0$ and $\mathcal{L}_o$ when $D(\mathcal{L}_{o-1}) = D(\mathcal{L}_o) = 1$, so let us assume $D(\mathcal{L}_{o-1}) \neq 1$ or $D(\mathcal{L}_o) \neq 1$, say $D(\mathcal{L}_{o-1}) \neq 1$ through an appropriate choice of the labelling of $\{\mathcal{I}, \mathcal{J}\}$. This means that there exists a radical term $Y(\mathcal{L}_{o-1})^{ij}_{\alpha\beta}$ with $i, j \in \mathcal{L}_{o-1}$ and $\alpha, \beta \in \mathcal{L}_{o-1}^c$. We now invoke the decompositions (3.15)

$$Y(L_{o-1})^{ij}_{\alpha\beta} = -Y(L_{o-1})^{m_1}_{\alpha\delta_1} \cdot Y(L_{o-1})^{m_2}_{\alpha\delta_2} \cdot Y(L_{o-1})^{m_3}_{\delta_1\beta} \cdot Y(L_{o-1})^{m_4}_{\delta_2\beta}. \quad (4.15)$$

Since $\chi(\mathcal{L}_{o-1})^{ij}_{\alpha\beta}$ satisfies (3.21) by assumption and $h((L_{o-1})^{m_4}_{\delta_2\beta}) \neq 0$ by construction (4.13), all the factors in the right-hand side of (4.14) derive from observable sets. By Lemma 10, they lie in the same quadratic extension of $\mathbb{F}$; furthermore, from $Y(L_{o-1})^{ij}_{\alpha\beta} \notin \mathbb{F}$, at least one of these $Y$-terms lies not in $\mathbb{F}$ too. These two properties are preserved under the transformation rules (3.23)-(3.24), which let us...
move from \(\mathcal{L}_{o-1}\) to \(\mathcal{L}_o\), finding \(D(\mathcal{L}_{o-1}) = D(\mathcal{L}_o)\). Concatenating these equalities for all \(\alpha \in [r]\), we get \(D(\mathcal{I}) = D(\mathcal{J})\).

A consequence of the previous proposition is that the set \(\Psi(D)\) of monomials appearing in \(D\) is uniquely characterized by any radical \(Y\)-term, modulo multiplication by invertible elements in \(\mathbb{C}(t)\), independently on the choice of \(\mathcal{I} \in \mathfrak{S}(\mathcal{L}(t))\).

5 Reduction of set functions: quadratic case

In this section, we consider a general, but fixed, element \(\mathcal{I} \in \mathfrak{S}(\mathcal{L}(t))\) and cases where \((3.21)\) condition holds, so the integrability condition \((2.10)\) reads

\[
\Psi(\mathcal{I}) + \Psi(\mathcal{I}_{\alpha_3}^{ij}) = \Psi(\mathcal{I}_d) + \Psi(\mathcal{I}_{\beta}^j) + \Psi(\mathcal{I}_{\alpha_3}^i).
\]

(5.1)

5.1 Allowed configurations for radical terms

Proposition 12. If \(Q_{\alpha_3}^{ij}\) in \((4.9)\) is not invertible in \(\mathbb{C}(t)\), then it is a binomial.

Proof. Considering the definition \((4.2)\), we introduce

\[
\hat{A}_{\alpha_3}^{ij} := \frac{A_{\alpha_3}^{ij}}{G_{\alpha_3}^{ij}}, \quad \hat{B}_{\alpha_3}^{ij} := \frac{B_{\alpha_3}^{ij}}{(G_{\alpha_3}^{ij})^2}, \quad \hat{Q}_{\alpha_3}^{ij} := \frac{Q_{\alpha_3}^{ij}}{G_{\alpha_3}^{ij}}.
\]

(5.2)

Whenever \(D = 1\), we find \((\hat{Q}_{\alpha_3}^{ij})^2 = \hat{B}_{\alpha_3}^{ij}\), which means \(Q_{\alpha_3}^{ij}, B_{\alpha_3}^{ij} \in \mathbb{C}\) by Lemma 7, while \(Q_{\alpha_3}^{ij}\) is invertible in \(\mathbb{C}(t)\) at \(D = \hat{B}_{\alpha_3}^{ij}\). Focusing on the remaining cases, we take \(p \in [d]\) satisfying \(\partial_p \hat{Q}_{\alpha_3}^{ij} \neq 0\), where \(\partial_p\) denotes the partial derivative with respect to \(t_p\). We move to the ring \(\mathbb{C}[t_p][t_p]\) of monovariate polynomials in \(t_p\) with coefficients in \(\mathbb{C}[t] = \mathbb{C}\). In particular, we look at \(\hat{B}_{\alpha_3}^{ij}(t_p), \hat{Q}_{\alpha_3}^{ij}(t_p), D(t_p)\) as polynomials in \(\mathbb{C}[t_p][t_p]\), so the normalisation \((5.2)\) and the factorisation \((4.9)\) entail that \(\hat{Q}_{\alpha_3}^{ij}(t_p)\) is a common factor of \(\hat{B}_{\alpha_3}^{ij}(t_p)\) and \(\partial_p \hat{B}_{\alpha_3}^{ij}(t_p)\). By \((5.2)\) there is a term in \(\chi(\mathcal{I}_{\alpha_3}^{ij})\) that is constant with respect to \(t_p\), therefore we can write

\[
\hat{B}_{\alpha_3}^{ij}(t_p) = c_1 t_p^{d_1} + c_2 t_p^{d_2} + c_3 - 2c_1 c_2 t_p^{d_1+d_2} - 2c_1 c_3 t_p^{d_1} - 2c_2 c_3 t_p^{d_2} \quad \text{and} \quad \hat{Q}_{\alpha_3}^{ij}(t_p) = \hat{B}_{\alpha_3}^{ij}(t_p) - \partial_p \hat{B}_{\alpha_3}^{ij}(t_p).
\]

(5.3)

(5.4)

with \(c_u \in \mathbb{C}[t_p], E_u = c_u t_p^{d_u}, u \in [3]\) and \(d_1 \geq d_2 \geq d_3 = 0\). Each common factor of \(\hat{B}_{\alpha_3}^{ij}\) and \(\partial_p \hat{B}_{\alpha_3}^{ij}\) also divides the combination

\[
\frac{t_p}{d_1} \partial_p \hat{B}_{\alpha_3}^{ij}(t_p) - \hat{B}_{\alpha_3}^{ij}(t_p) = (c_1 t_p^{d_1} - c_2 t_p^{d_2} + c_3) (c_1 t_p^{d_1} + c_2 (1 - 2d_1^{-1}d_2) t_p^{d_2} - c_3).
\]

(5.5)

However, we also find

\[
\hat{B}_{\alpha_3}^{ij}(t_p) - (c_1 t_p^{d_1} - c_2 t_p^{d_2} + c_3)^2 = -4c_1 c_3 t_p^{d_1},
\]

(5.6)

\[
\hat{B}_{\alpha_3}^{ij}(t_p) - (c_1 t_p^{d_1} - c_2 t_p^{d_2} - c_3)^2 = -4c_2 c_3 t_p^{d_2}.
\]

(5.7)

From \((5.6)\) and \(c_3 \neq 0\) we see that \(\hat{B}_{\alpha_3}^{ij}(t_p)\) and \(c_1 t_p^{d_1} - c_2 t_p^{d_2} + c_3\) are coprime in \(\mathbb{C}[t_p][t_p]\), thus from \((5.5)\) any common factor between \(\hat{B}_{\alpha_3}^{ij}(t_p)\) and \(\partial_p \hat{B}_{\alpha_3}^{ij}(t_p)\) also divides

\[
w_1(t_p) := c_1 t_p^{d_1} + c_2 (1 - 2d_1^{-1}d_2) t_p^{d_2} - c_3.
\]

13
If $d_1 = d_2$ we find the reductions

\[
\begin{align*}
    w_{1,d_1=d_2}(t_p) & := (c_1 - c_2) t_p^{d_1} - c_3, \\
    B(t_p)_{d_1=d_2} & := (c_1 - c_2)^2 t_p^{2d_1} + c_3^2 - 2(c_1 + c_2)c_3 t_p^{d_1}.
\end{align*}
\]

(5.8)

(5.9)

Considering their combination

\[
- B(t_p)_{d_1=d_2} + w_{1,d_1=d_2}(t_p)^2 = 4c_2 c_3 \cdot t_p^{d_1}
\]

we find that $w_{1,d_1=d_2}(t_p)$ and $B(t_p)_{d_1=d_2}$ are coprime. Thus, $d_1 \neq d_2$ and we look at the combination

\[
- t_p \cdot \partial_t \hat{B}_{ij}^{1\beta} - \frac{2c_3 d_2 - 2(d_1 - d_2) \cdot (c_1 t_p^{d_1} - c_2 t_p^{d_2} - c_3)}{d_1 - d_2} \cdot d_1 \cdot w_1(t_p)
\]

\[
\begin{align*}
    & = 2(d_1 - d_2) (c_1 t_p^{d_1} - c_2 t_p^{d_2} - c_3) (c_2 \cdot t_p^{d_2} - c_3 d_1^2 (d_1 - d_2)^{-2}).
\end{align*}
\]

(5.10)

(5.11)

Noting that $c_1 t_p^{d_1} - c_2 t_p^{d_2} - c_3$ and $\hat{B}_{ij}^{1\beta}(t_p)$ are coprime by (5.7), any common factor has to divide

\[
w_2(t_p) := c_2 \cdot t_p^{d_2} - c_3 d_1^2 (d_1 - d_2)^{-2}.
\]

(5.12)

Finally, each common divisor of $\hat{B}_{ij}^{1\beta}(t_p)$ and $\partial_t \hat{B}_{ij}^{1\beta}(t_p)$ also divides

\[
w_3(t_p) := w_1(t_p) - (1 - 2d_1^{-1} d_2) \cdot w_2(t_p)
\]

\[
\begin{align*}
    & = c_1 t_p^{d_1} - (d_1 - d_2)^{-2} c_3 d_1^2.
\end{align*}
\]

(5.13)

The two factors (5.12) and (5.13) have a non-trivial common factor only if the condition

\[
c_1^{-d_2} (d_1 - d_2)^{-2d_2} c_3^{-1} d_1^{2d_2} = c_2^{-d_2} c_3^{-1} d_1^{2d_1} (d_1 - d_2)^{-2d_1}
\]

is satisfied, equivalently only if

\[
\frac{E_2^{d_1}}{E_1^{d_1}} = \frac{E_2^{d_2}}{(d_1 - d_2)^2 (d_1 - d_2)} \cdot \frac{E_1^{d_2}}{d_2^{d_2}}.
\]

(5.14)

(5.15)

If this relation holds, then we can introduce

\[
\varrho := d_2 / d_1, \quad \varrho_0 := \gcd(d_1, d_2) / d_1
\]

and move back to the ring $\mathbb{C}[t]$, since the ratios $\varrho_0, \varrho \in (0, 1) \cap \mathbb{Q}$ and the set $\{E_1, E_3\} \subseteq \hat{\chi}(I_{ij}^{1\beta})$ are uniquely defined by (5.15), which does not depend on the choice of $t_p$. In particular, under the condition (5.15) the non-trivial square factors of $\hat{B}_{ij}^{1\beta}$ have to divide

\[
\left(E_1^{\varrho_0} - (\varrho^{-1} - 1)^{-2} \varrho_0 \cdot E_3^{\varrho_0} E_3^{\rho_0} \right)^2.
\]

(5.16)

(5.17)

For a given choice of the complex phase of $E_3^{\varrho_0}$, the compatibility of (5.12) and (5.13) determines the complex phase of $E_2^{\varrho_0}$, in line with the uniqueness of the factorisation in $\mathbb{C}(t)$. Therefore, we can express $\hat{B}_{ij}^{1\beta}$, up to invertible elements in $\mathbb{C}(t)$, in terms of the monomial

\[
r := (\varrho^{-1} - 1)^{2} \varrho_0 \cdot E_3^{-\varrho_0} E_1^{\rho_0}
\]

(5.18)

whose complex phase is well-defined too. It is easily checked that $\hat{B}_{ij}^{1\beta}$ has a double root at $r = 1$ and, hence, the factor $\hat{Q}_{ij}^{1\beta}$ is given by (5.17).
As a consequence, situations where \( \hat{Q}^{ij}_{\alpha \beta} \) is not invertible in \( \mathbb{C}(t) \) are associated with two monomials

\[
E_u = c_u t_u^u \in \chi(I^{ij}_{\alpha \beta}), \quad u \in \{1, 3\}, \quad c_u \in \mathbb{C} \setminus \{0\} \tag{5.19}
\]

where \( t_1 \) and \( t_3 \) are two disjoint tuples of variables extracted from \( t \), and \( e_u \in \mathbb{N}^{d_u} \). This leads us to introduce the following notation.

**Definition 13.** Given linearly independent \( f_1, f_3 \in \mathbb{Z}^d \), we say that a monomial \( x \) is \((f_1, f_3)\)-coherent if \( \Psi(x) \) lies in the \( \mathbb{Z} \)-submodule of \( \mathbb{Z}^d \) spanned by \( f_1 \) and \( f_3 \). In other words, \( x \) is \((f_1, f_3)\)-coherent if \( x = c \cdot t_1^{f_1} t_3^{f_3} f_5 \) for some \( c \in \mathbb{C} \setminus \{0\} \) and weights \( q_1, q_3 \in \mathbb{Z} \). A set \( M \) of monomials will be said \((f_1, f_3)\)-coherent if all its elements are \((f_1, f_3)\)-coherent, and \( M \) is \((f_1, f_3)\)-homogeneous if it is \((f_1, f_3)\)-coherent and the sum of weights \( q_1 + q_3 \) is independent of the choice of \( x = c \cdot t_1^{f_1} t_3^{f_3} f_5 \in M \).

**Corollary 14.** When \( \hat{Q}^{ij}_{\alpha \beta} \) is not invertible in \( \mathbb{C}(t) \), \( \Psi(D) \) is \((f_1, f_3)\)-homogeneous where \( \{f_1, f_3\} = \Psi(\hat{Q}^{ij}_{\alpha \beta}) \).

**Proof.** Let us consider the two sub-tuples \( t_1, t_3 \) of \( t \) introduced in (5.19). Both \( \Theta(\hat{Q}^{ij}_{\alpha \beta}) \) and \( \Theta(\hat{Q}^{ij}_{\alpha \beta}) \) are \((\varrho_0 e_1, \varrho_0 e_3)\)-homogeneous by (5.4) and (5.15). For any monomial order \( \preceq \) on \( \Theta(D) \cup \Theta(\hat{Q}^{ij}_{\alpha \beta}) \), if there exists a set \( I \subseteq \Theta(D) \) of monomials that make \( \Theta(D) \) \((\varrho_0 e_1, \varrho_0 e_3)\)-inhomogeneous, then we can look at the minimum of such monomials with respect to \( \preceq \). Specifically, the term

\[
\left( \min \Theta(\hat{Q}^{ij}_{\alpha \beta}) \right) \cdot (\min I) \in \Theta(\hat{Q}^{ij}_{\alpha \beta})
\]

makes \( \Theta(\hat{Q}^{ij}_{\alpha \beta}) \) \((\varrho_0 e_1, \varrho_0 e_3)\)-inhomogeneous too, i.e. a contradiction. Hence, \( \Theta(D) \) is \((\varrho_0 e_1, \varrho_0 e_3)\)-homogeneous, which implies that \( \Psi(D) \) lies in the \( \mathbb{Z} \)-submodule generated by \( \{\varrho_0 e_1, \varrho_0 e_3\} = \Psi(\hat{Q}^{ij}_{\alpha \beta}) \). \( \square \)

**Remark 15.** In addition to the final remarks in the proof of Proposition 12 regarding \( \hat{Q}^{ij}_{\alpha \beta} \), Corollary 14 guarantees that also \( D \) can be expressed as a polynomial in the variable in (5.18) when \( \hat{Q}^{ij}_{\alpha \beta} \) is not invertible.

In these situations, the form (5.17) of \( \hat{Q}^{ij}_{\alpha \beta} \) depends on the choice of labelling of the elements of \( \{t_1, t_3\} \); different choices are represented by the exchange \( t_1 \leftrightarrow t_3 \), which induces \( \varrho \leftrightarrow 1 - \varrho \) consistently with (5.15) and preserves the factors of \( \hat{Q}^{ij}_{\alpha \beta} \), since it only contributes as a constant factor \( (\varrho^{-1} - 1)^2 e_0 \) multiplying \( \hat{Q}^{ij}_{\alpha \beta} \) in (5.17). In terms of the monovariate notation (5.18), this exchange leads to the change of variable \( r \mapsto r^{-1} \).

It is worth noting that (5.15) can be seen as a normalisation condition: specifically, it implies

\[
\frac{E_2^{d_1}}{E_3^{d_1 - d_2} E_1^{d_2}} = \left( \frac{d_1 - d_2}{d_1} \right)^{-2(d_1 - d_2)} \left( \frac{d_2}{d_1} \right)^{-2d_2} \in \mathbb{Q}_+ \tag{5.20}
\]

which leads to

\[
\frac{1}{2d_1} \log \left( \frac{E_2^{d_1}}{E_3^{d_1 - d_2} E_1^{d_2}} \right) = -H \left( \frac{d_2}{d_1} \right) \tag{5.21}
\]

where \( H(p) \) is the entropy associated with a Bernoulli random variable with parameter \( p \).

**Example 16.** The first non-trivial case where \( \hat{Q}^{ij}_{\alpha \beta} \neq 1 \) is found at \( d_1 = 2d_2 \) in (5.15), which leads to the ratios \( \gcd(d_1, d_2)/d_1 = \frac{1}{2} \) and \( d_2/(d_1 - d_2) = 1 \). Thus, we get

\[
D = \varepsilon_1^2 - 6\varepsilon_1\varepsilon_3 + \varepsilon_3^2 \tag{5.22}
\]
where $\varepsilon_u^2 = E_u$, $u \in \{1, 3\}$, and the factorization (4.9) is given by

$$B_{\alpha\beta}^{ij} = (\varepsilon_1 - \varepsilon_3)^2 \cdot (\varepsilon_1^2 - 6\varepsilon_1\varepsilon_3 + \varepsilon_3^2).$$  \hspace{1cm} (5.23)$$

This example also shows that ambiguity can arise when one reconstructs $B_{\alpha\beta}^{ij}$ from $D$: indeed, the case $D = \varepsilon_1^2 - 6\varepsilon_1\varepsilon_3 + \varepsilon_3^2$ can be associated with both the configurations $\chi_{1\alpha}(I_{\alpha\beta}^{ij}) = \{\varepsilon_1, 2\varepsilon_1, \varepsilon_3\}$ and $\chi_{1\beta}(I_{\alpha\beta}^{ij}) = \{\varepsilon_1, 2\varepsilon_3, \varepsilon_3\}$ with $\hat{Q}_{\alpha\beta}^{ij} = 1$, as well as $\chi_{11}(I_{\alpha\beta}^{ij}) = \{\varepsilon_1^2, 4\varepsilon_1\varepsilon_3, \varepsilon_3^2\}$ with $\hat{Q}_{\alpha\beta}^{ij} = \varepsilon_1 - \varepsilon_3$. We highlight that, in such cases of ambiguity with $\hat{Q}_{\alpha\beta}^{ij} := \varepsilon_1 - \varepsilon_3$, the set $\chi(I_{\alpha\beta}^{ij})$ has to be $(\Psi(\varepsilon_1), \Psi(\varepsilon_3))$-homogeneous as well, in line with the proof of Corollary 14.

### 5.2 Recovering local from global data

Now we address the reverse problem of recovering local data that generate $Y$-terms, i.e. the sets $\chi(I_{\alpha\beta}^{ij})$, starting from the global information provided by $D$. In the present context, the term global means independency from the choice of the basis and the indices of the $Y$-term.

**Lemma 17.** If $\#\Theta(D) = 2$, then $B_{\alpha\beta}^{ij}$ is squarefree in $\mathbb{C}(t)$ and there exist two distinct terms $E_u, E_w \in \chi(I_{\alpha\beta}^{ij})$ and $c_S \in \{1, -1\}$ such that $E_u = c_S \cdot E_w$.

**Proof.** From (5.17), the only possibility for $D$ when $\#\Theta(D) = 2$ and $B_{\alpha\beta}^{ij}$ has a non-invertible square factor $\hat{Q}_{\alpha\beta}^{ij} \in \mathbb{C}(t)$ is

$$\Psi(D) = \{2 \cdot (1 - \varrho_0) \cdot e_1, 2 \cdot (1 - \varrho_0) \cdot e_3\}.$$  \hspace{1cm} (5.24)$$

We can order the terms in $\hat{B}_{\alpha\beta}^{ij}, \hat{Q}_{\alpha\beta}^{ij}$, and $D$ based on their weight $\nu_1(\cdot)$ with respect to $t_1$: specifically, from Definition 13 we can normalise $\nu_1(E_1) = 1$, while $\nu_1(E_3) = 0$ and, from (5.15), $\nu_1(E_2) = \varrho$. The induced order is total since $\hat{B}_{\alpha\beta}^{ij}, \hat{Q}_{\alpha\beta}^{ij}$, and $D$ are $(\varrho_0e_1, \varrho_0e_3)$-homogeneous.

This allows comparing the exponents in $\nu_1(\Theta(\hat{B}_{\alpha\beta}^{ij}))$ with the corresponding ones in the expansion $\nu_1(\Theta(D \cdot (\hat{Q}_{\alpha\beta}^{ij})^2))$: the minimum is 0 for both sets, while the least non-vanishing weights are

$$\min_1 \nu_1(\Theta(D)) \setminus \{0\} = 2 \cdot (1 - \varrho_0),$$  \hspace{1cm} \min_1 \nu_1(\Theta((\hat{Q}_{\alpha\beta}^{ij})^2)) \setminus \{0\} = \varrho_0,$$

$$\min_1 \nu_1(\Theta(\hat{B}_{\alpha\beta}^{ij})) \setminus \{0\} = \varrho.$$  \hspace{1cm} (5.25)$$

In order to satisfy (4.9), two elements in $\{2 \cdot (1 - \varrho_0), \varrho_0, \varrho\}$ have to coincide; however, from (5.16) we infer

$$0 \leq \varrho_0 \leq \frac{1}{2}, \hspace{1cm} \varrho_0 \leq \varrho < 1.$$  \hspace{1cm} (5.25)$$

This implies $\varrho < 1 \leq 2 \cdot (1 - \varrho_0)$, so we get $\varrho_0 = \varrho$, i.e. $d_J|d_I$. We can repeat this argument considering the order induced by the valuation $\nu_3(\cdot)$ with respect to $t_3$, which corresponds to the transformations $e_1 \leftrightarrow e_3$ and $\varrho \leftrightarrow 1 - \varrho$, according to (5.15). Thus, we also get $\varrho_0 = 1 - \varrho$, which means $\varrho = \varrho_0 = \frac{1}{2}$. This leads to (5.22) and does not satisfy $\#\Theta(D) = 2$. Therefore $D = \hat{B}_{\alpha\beta}^{ij}$ and, in order to get $\#\Theta(D) = 2$, we find two linearly dependent elements of $\hat{\chi}(I_{\alpha\beta}^{ij})$, say $E_2 = c_S \cdot E_3$, where $c_S \in \mathbb{C}$ satisfies $D = (c_S - 1)^2 E_3^2 - 2(c_S + 1)E_3E_1 + E_1^2$. This is a binomial if and only if $c_S \in \{1, -1\}$. \hfill $\square$

A possible issue arising from this process has been highlighted in Example 16. Before analysing it in more detail, the results in Proposition 12 and Remark 17 suggest the following definition.

**Definition 18.** [Type and class of radical terms] We say that a configuration $\chi(I_{\alpha\beta}^{ij})$ (or the associated polynomials $B_{\alpha\beta}^{ij}$ and $Y_{\alpha\beta}^{ij}$) is of $G$-type (generic type) if $\#\Psi(D) > 2$ and is of $S$-type (singular type) if $\#\Psi(D) = 2$. A $G$-type configuration $\chi(I_{\alpha\beta}^{ij})$ is class-I if $Q_{\alpha\beta}^{ij}$ is a monomial, and it is class-II if $\hat{Q}_{\alpha\beta}^{ij}$ is a binomial (5.17).
We stress that the type of a configuration is independent of the set $\chi(I_{\alpha,\beta}^{ij})$, since it only depends on $D$, while this may not hold for the class, as is shown in Example 16.

**Proposition 19.** For $G$-type configurations, we can reconstruct $\hat{B}_{\alpha,\beta}^{ij}$ from $D$ and the knowledge of the class of $\hat{B}_{\alpha,\beta}^{ij}$.

**Proof.** We focus on class-II configurations, otherwise $\hat{B}_{\alpha,\beta}^{ij} = D$. Let $\Omega := \#\Theta(D) > 2$ and proceed as follows:

1. choose any variable $t_p$ such that $\partial_{t_p} D \neq 0$ and order the elements in $\Theta(D)$ based on their degree with respect to $t_p$. This order is total since all the elements in $\Psi(D)$ belong to a $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ generated by two vectors $\mathbf{f}_1, \mathbf{f}_3$, and $\Theta(D)$ is $(\mathbf{f}_1, \mathbf{f}_3)$-homogeneous by Corollary 14. Therefore, we can identify the maximum $\mathbf{d}_1$ and the minimum $\mathbf{d}_\Omega$ in $\Theta(D)$ with respect to this order. These two monomials also represent the maximum and the minimum with respect to one of the orders induced by the functions $\nu_1$ and $\nu_3$ introduced in the proof of Lemma 17, say $\nu_1$ with a choice of labelling. Starting from $D$, in this case the function $\nu_1 := \nu_{d_1}$ is normalised in order to have $\nu_{d_1}(\mathbf{d}_1) = 1$.

2. Denoting $q_u := \nu_{d_1}(\mathbf{d}_u)$, $u \in [\Omega]$, we now recover $\varrho_0 := \gcd(d_1, d_2)/d_1$ looking at the minimal gap

$$
\mu := \min_u \{q_u - q_{u+1}\}.
$$

(5.26)

Indeed, we first note that $\mu \leq \frac{1}{2}$, and from (5.15) the equality $\mu = \frac{1}{2}$ corresponds to

$$
\nu_{d_1}(E_1) \cdot (1 - \varrho) = \nu_{d_1}(E_1^2) - \nu_{d_1}(E_1 E_2) = \nu_{d_1}(E_2 E_3) - \nu_{d_1}(E_2^2) = \nu_{d_1}(E_1) \cdot \varrho,
$$

(5.27)

so we get $\varrho = \frac{1}{2}$ and recover $\varrho_0 = \mu = \frac{1}{2}$ too. Vice versa, if (5.27) does not hold, then we infer

$$
\mu \leq \min\{q_1 - q_2, q_\Omega - 1 - q_\Omega\} = \nu_{d_1}(E_1) \cdot \varrho_0.
$$

(5.28)

On the other hand, from Corollary 14 we know that $\Psi(D)$ lies in the $\mathbb{Z}$-submodule generated by $\Psi(\hat{G}_{\alpha,\beta}^{ij})$, thus $\mu$ is bounded from below by $\nu_{d_1}(\Theta(\hat{G}_{\alpha,\beta}^{ij})) = \nu_{d_1}(E_1) \cdot \varrho_0$. So

$$
\mu = \nu_{d_1}(E_1) \cdot \varrho_0.
$$

(5.28)

Being $\nu_{d_1}(E_1)^{-1} = \nu_{E_1}(\mathbf{d}_1) = 2 \cdot (1 - \varrho_0)$, we can recover $\varrho_0$ from $\mu$ using (5.28).

3. From $\mathbf{d}_1$ and $\varrho_0$, we can also get $E_1 = \mathbf{d}_1^{1/(1 - \varrho_0)}$. As already discussed in the proof of Proposition 12, at this step we introduce an arbitrary choice for the complex phase of $E_1$, which however also acts on $E_2$ and $E_3$, so it does not affect the parametrisation (5.18). Furthermore, from the knowledge of $\varrho_0$ we can consider an appropriate scaling for the elements $q_u$, i.e. $s_u := 2 \cdot (1 - \varrho_0) \cdot q_u$, $u \in [\Omega]$.

4. We now focus on $\varrho$ looking at $s_2$ and $s_{\Omega-1}$, which coincide at $\Omega = 3$. Clearly, $\varrho = \varrho_0$ when $\varrho_0 = \frac{1}{2}$, as shown in Example 16, so we assume $\varrho_0 < \frac{1}{2}$. When $s_2 < 2 - 3\varrho_0$, we can look at (5.27) and conclude $2 - \varrho_0 = 1 + \varrho$, hence, $\varrho = 1 - \varrho_0$. Similarly, we find $\varrho = \varrho_0$ when $s_{\Omega-1} > \varrho_0$. On the other hand, at $\varrho_0 < \frac{1}{2}$ we have $1 - \varrho \neq \varrho$, then the relations $s_2 = 2 - 3\varrho_0$ and $s_{\Omega-1} = \varrho_0$ imply that a cancellation occurs in the expansion of (4.9). We look at the quantity $\Xi := 4d_\Omega d_1 d_2^{-1} d_{\Omega-1}^{-1}$ and check the relations

$$
0 = E_1^{2\varrho_0} d_2 - 2 (\varrho^{-1} - 1)^{-2\varrho_0} E_1^{\varrho_0} E_3^{\varrho_0} d_1,
$$

(5.29)
in order to discriminate between different cases: \( \Xi \neq 1 \) means that exactly one between (5.29) and (5.30) vanishes, so \( \varrho \in \{ 1 - \varrho_0, \varrho_0 \} \). As noted in Remark 15, both these choices return the same factors for \( \tilde{Q}^i_{\alpha \beta} \). When \( \Xi = 1 \), (5.29) and (5.30) are consistent and define a unique value

\[
(q^{-1} - 1)^{-2} \varrho_0 E^0_3 = \frac{1}{2} E^0_1 d_3 d_1^{-1}.
\]

(5.31)

In this way, we have obtained all the information that uniquely defines \((\tilde{Q}^i_{\alpha \beta})^2\) and, from (4.9), \( B^i_{\alpha \beta} \) up to units.

**Theorem 20.** From any radical \( Y^i_{\alpha \beta} \) we can recover a finite number \( N_D \) of configurations of monomials \((E_1, E_2, E_3) \in \mathbb{C}(t)^3\) defining radical \( Y \)-term through

\[
Y(E_1, E_2, E_3) := \frac{E_1 - E_2 - E_3 + \varepsilon \sqrt{(E_1 - E_2 - E_3)^2 - 4 E_2 E_3}}{2 E_3}.
\]

(5.32)

In particular, there exist no radical terms if \( n - k > 2 N_D + 2 \).

**Proof.** All the \( B \)-terms are homogeneous quadratic expressions of the elements of \( (T^i_{\alpha \beta}) \), so they may be recovered from \( Y^i_{\alpha \beta} \) up to \( \hat{Q}^i_{\alpha \beta} \), which also includes the choice of an overall complex phase for \( E_1, E_3 \) mentioned before. We focus on \( \hat{\chi}(T^i_{\alpha \beta}) \), consider a given monomial order on \( \Theta(D) \), and label its elements accordingly, i.e. \( u < v \) implies \( d_u < d_v \), \( u, v \in [\Omega] \) with \( \Omega := \# \Theta(D) \).

Starting from class-II configurations, Proposition 19 entails that \( \varrho_0, \tilde{Q}^i_{\alpha \beta} \), and the set \( \{ \varrho, 1 - \varrho \} \) are uniquely defined by \( D \). For each individual choice of the value \( \varrho \) or \( 1 - \varrho \), we can reconstruct \( \hat{\chi}(T^i_{\alpha \beta}) \) from \( \Theta(\tilde{Q}^i_{\alpha \beta}) \) using (5.15). According to Remark 15, the two choices are related by \( \varrho \Leftrightarrow 1 - \varrho \) and induce the exchange of roots of \( B^i_{\alpha \beta} \) in (5.32).

Now we move to class-I configurations and look at \( d_1 = \max \Theta(D) \) and \( d_3 = \min \Theta(D) \), which coincide if and only if \( Y^i_{\alpha \beta} \) is \( \mathbb{C} \). We fully recover \( \hat{\chi}(T^i_{\alpha \beta}) = \{ E_1, E_2, E_3 \} \) up to common factors when \( \Omega \geq 4 \), since we can fix a phase for \( E_1 := d_1^\varepsilon \) and get \( E_2 = \frac{1}{2} E_1 \cdot d_2 d_1^{-1} \) and \( E_3 := 2 E_2 \cdot d_3 d_1^{-1} \). The occurrence \( \Omega = 3 \) means that two terms in \( \hat{\chi}(T^i_{\alpha \beta}) \) are linearly proportional: in order to unify the notation, we still denote as \( \varrho = 0 \), respectively \( \varrho = 1 \), the choice \( \Psi(E_2) = \Psi(E_3) \), respectively \( \Psi(E_2) = \Psi(E_1) \). Each of the two choices \( u \in \{1, 3\} \) allows us to uniquely recover the constant \( c \) from \( B(E_1, c \cdot E_u, E_3) = D \).

Possible ambiguity between the two classes may arise, as shown in Example 16. Thus, when \( \# \Theta(D) > 2 \), the possible configurations follows from: i. the knowledge of the class; ii. the choice in \( \{ \varrho, 1 - \varrho \} \) defining \( E_2 \ (\varrho \in [0, 1]) \); iii. the choice of the square root of \( B^i_{\alpha \beta} \); iv. the ordering of \( \hat{\chi}(T^i_{\alpha \beta}) \) defining \( (E_1, E_2, E_3) \).

When \( \# \Theta(D) = 2 \), the possible configurations for \( Y^i_{\alpha \beta} \) generated by \( D \) come from the following choices: i. the sign \( \varepsilon \) in (5.32); ii. the constant \( c \in \{1, -1\} \) mentioned in Lemma 17; iii. with the notation of Lemma 17, the choice of \( e \in \Theta(D) \) such that \( e = E_u = c \cdot E_u \), which will be referred to as \( \varrho \) in line with the previous notation; iv. the ordering \( (E_1, E_2, E_3) \). There are two choices for both the sign \( \varepsilon \) and identification of the independent term in \( \Theta(D) \); at \( c = +1 \) there are only three distinct configurations for \( (E_1, E_2, E_3) \) associated with cyclic subgroup of \( S_3 \), while there are six configurations at \( c = -1 \), namely, the full \( S_3 \) group.

Finally, let us denote as \( N_D \) the number of distinct forms (5.32) for radical terms. We suppose that there exists a radical term \( Y^i_{\alpha \beta} \). By contraposition of Lemma 8, all the sets \( \chi(T^i_{\alpha \beta}) \) and \( \chi(T^i_{\beta \gamma}) \) satisfy (3.21). Therefore, for each \( \gamma \in \mathcal{T} \), at least one between \( Y(T^i_{\alpha \gamma}) \notin F \) and \( Y(T^i_{\beta \gamma}) \notin F \) holds.
If \( n - k > 2N_D + 2 \), Dirichlet's box principle would imply that one between \( \alpha \) and \( \beta \), say \( \alpha \) with an appropriate labelling, satisfies \( Y_{\alpha}^{ij} \notin \mathcal{F} \) for \( N_D + 1 \) distinct indices \( \gamma_u \in \mathcal{F} \). From the previous argument, a second application of Dirichlet's box principle lets us infer that there exists two indices, say \( \gamma_p \) and \( \gamma_q \), such that \( Y_{\alpha}^{ij} = Y_{\beta}^{ij} \). The identity (3.13) implies \( Y_{\beta}^{ij} = -1 \), which contradicts Remark 3, thus \( n - k \leq 2N_D + 2 \). □

The proof of Theorem 20 makes manifest that multiple sets \( \bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) \) are compatible with a given \( D \) only through different choices of the class and the correspondence of monomials in \( \Theta(D) \) with the triple

\[
\bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) := (G_{\alpha \beta}^{ij})^{-1} \cdot \left( h(\mathcal{I}) \cdot h(\mathcal{I}_{\alpha \beta}^{ij}), h(\mathcal{I}_{\alpha}^{ij}) \cdot h(\mathcal{I}_{\beta}^{ij}), h(\mathcal{I}_{\alpha}^{ij}) \cdot h(\mathcal{I}_{\beta}^{ij}) \right). \tag{5.33}
\]

The set \( \bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) \) is completely determined when \( \Theta(D) \) spans a 3-dimensional sublattice of \( \mathbb{Z}^d \), while ambiguities come when a reduction happens in \( D \) and the span of \( \Theta(D) \) is 2-dimensional. Under the latter hypothesis, \( \bar{\Theta}(\mathcal{I}_{\alpha \beta}^{ij}) \) spans a 2-dimensional sublattice too, and it can be expressed using the monovariate notation (5.18). In particular, this argument applies when two terms in \( \bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) \) are proportional over \( \mathbb{C} \), so we briefly discuss this situation.

Remark 21. For a \( G \)-type configuration \( (E_1, E_2, E_3) = (r, c_1 \cdot r, q_1) \) with \( c_1 \in \mathcal{C} \), this constant can only assume values from a set \( \{k, k^{-1}\} \) that is uniquely defined by the knowledge of \( \bar{B}_{\alpha \beta}^{ij} \). The two choices correspond to different labelling \( \{E_1, E_2\} \) for the two proportional terms. Different configurations for \( \bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) \) are described by permutations of two triples \( (r, c_1 \cdot r, q_1) \) and \( (r, c_2, q_2) \) such that \( B(r, c_1 \cdot r, q_1) \) is equal to \( B(r, c_2, q_2) \), up to units. In turn, both \( (r, c_2, q_2) \) and \( q_1r^{-1} \cdot (r, c_2, q_2) = (q_1, c_2q_1r^{-1}, q_2q_1r^{-1}) \) return the same \( B \)-term, up to units. Therefore, the ratio \( c_2/q_2 = (c_2q_1r^{-1})/(q_2q_1r^{-1}) \) in the triple \( (r, c_2, q_2) \) belongs to \( \{k, k^{-1}\} \) as well. The explicit sets returning the same \( B \)-terms are

\[
C_1 := \{r, k \cdot g, g\}, \quad C_2 := \{(1 - k) \cdot g, \frac{r}{1 - k}, \frac{kr}{1 - k}\}. \tag{5.34}
\]

We also remark that the set \( C_2 \) can be obtained from \( C_1 \) starting from (3.13): two \( Y \)-terms associated with the same set \( C_1 \) but different roots of \( B \) generate a term derived from \( C_2 \), i.e.

\[
\sqrt{kg - g - r + \sqrt{B}} \cdot \frac{g - kg - r - \sqrt{B}}{2r} = \frac{k - 1}{2r} \cdot \left(1 - \frac{(1 - k)g - \frac{r}{1 - k}}{\frac{kr}{1 - k} - \sqrt{B}}\right) \tag{5.35}
\]

and, vice versa, starting from the right-hand side of the previous equation, we uniquely recover the product on the left-hand side in terms of configurations generated from \( C_1 \). We will further explore this aspect in the next section of this subsection.

For \( S \)-type systems, Proposition 12 and Corollary 14 imply that \( B_{\alpha \beta}^{ij} \) is squarefree in \( \mathbb{C}(t) \). We can write \( D = e_1^2 + e_2^2 \) with linearly independent monomials \( e_1, e_2 \). As in the proof of Theorem 20, the components of \( \bar{\chi}(\mathcal{I}_{\alpha \beta}^{ij}) := (E_1, E_2, E_3) \) are defined by the constant \( c_S \in \{1, -1\} \) in Lemma 17 and by the choice of \( p \in [2] \) in the relation \( \Theta(e_p) = \Theta(E_u) = \Theta(E_w) \) for two distinct \( u, w \in [3] \). This corresponds to the choices shown in Table 5.1. In analogy with (5.18), for \( S \)-type systems, we can consider the variables \( r := e_2^{-1}c_1 \) or \( r^{-1} \).
5.3 Bounds on the matrix dimensions

We now explore the constraints (5.32) for radical terms together with the condition (3.13). The details of this subsection do not affect the qualitative results in Theorem 20 regarding the restriction on dimensions in presence of radical terms. However, a deeper analysis of the implications of (3.13) can provide quantitative bounds and lead to a better comprehension of the possible combinations of monomials.

**Remark 22.** For future reference, we examine constraints concerning triples \((Y^{ij}_{a_1 a_2}, Y^{ij}_{a_2 a_3}, Y^{ij}_{a_1 a_3})\) that contain both radical and constant terms, say \(\kappa := -Y^{ij}_{a_2 a_3} \in \mathbb{C} \setminus \{0\}\) and \(Y^{ij}_{a_1 a_2}, Y^{ij}_{a_1 a_3} \notin \mathbb{F}^\dagger\).

i. From Remark 5, we get six transformations acting on \(\hat{\chi}(I|_{a_2 a_3})\), which correspond to the elements of \(S_3\) consistently with the compositions of permutations. In our case, the action of these transformations on \(Y^{ij}_{a_2 a_3}\) affects the constant \(\kappa\), for instance (3.23) and (3.24) act as \(-\kappa \mapsto \kappa - 1\) and \(-\kappa \mapsto (1 - \kappa^{-1})^{-1}\), respectively.

ii. The proportionality of the radical terms \(Y^{ij}_{a_1 a_2}\) and \(Y^{ij}_{a_1 a_3}\) means that the corresponding \(B\)-terms have the same factors in \(C(t)\), and an appropriate choice of the scaling (4.1) for the elements in \(\chi(I|_{a_2 a_3})\) lets us fix \(\hat{B}^{ij}_{a_1 a_3} = \hat{B}^{ij}_{a_1 a_3} := B\) without affecting the value of the associated \(Y\)-terms. When \(B\) is generated by three pairwise independent elements of \(\chi(I|_{a_1 a_2})\), they can be uniquely recovered from \(B\), e.g. adapting the steps in the proof of Proposition 19. This forces \(Y^{ij}_{a_1 a_2} = Y^{ij}_{a_1 a_3}\), in contradiction with Remark 3. Therefore, each of the sets \(\chi(I|_{a_1 a_2})\) and \(\chi(I|_{a_1 a_3})\) contains two proportional terms.

A similar argument holds for \(S\)-type configurations with \(c_S = 1\): from Table 5.1, we note that \(A^{ij}_{a_1 a_2}\) and \(A^{ij}_{a_1 a_3}\) have the same factors also under the exchanges of basis \(I \mapsto I_h\) and \(I \mapsto I_v\) only if \(Y^{ij}_{a_1 a_2} = Y^{ij}_{a_1 a_3}\), at odds with Remark 3.

iii. Finally, we recall Remark 21: when \(D\) is of \(G\)-type, the possible configurations are shown in (5.34). When \(D\) is of \(S\)-type, starting from the possible configurations in Table 5.1, we get a finite set of allowed values for \(Y^{ij}_{a_1 a_2}\). This set is also closed under the transformation rules (3.23) and (3.24), i.e. under the action of functions \(f_h\) and \(f_v\) introduced in Remark 5. This reduces the possible values for \(Y^{ij}_{a_2 a_3}\) to the set \([-\frac{1}{2}, 1, -2]\).

Starting from Remark 5, we can now analyse the additional constraints on \(Y^{ij}_{a_1 a_2} \in \mathbb{C} \setminus \{0\}\) generated by the existence of radical terms \(Y^{ij}_{a_1 a_2}, \gamma \in \mathcal{T}\). Prior to that, we introduce the notation that will be used in the rest of this subsection.

**Definition 23.** When two components of \(\vec{\chi}(I|_{a_1 a_2})\) are proportional, we denote as \(A^{ij}_{a_1 a_2} \in [3]\) the position of the unique independent component in \(\vec{\chi}(I|_{a_1 a_2})\) when \(Y^{ij}_{a_1 a_2} \notin \mathbb{F}\); we extend this definition setting \(A^{ij}_{a_1 a_2} = 0\) to indicate the condition \(Y^{ij}_{a_1 a_2} \in \mathbb{C}\).

Furthermore, we introduce the symbols

\[
\begin{align*}
h(T_{a_0})h(T'_{a_1}) &= c_{a_0 a_1} \cdot h(T_{a_1}h(T'_{a_1})) = c_{a_0 a_2} \cdot h(T_{a_2}h(T'_{a_2})) = c_{a_0 a_3} \cdot h(T_{a_3}h(T'_{a_3})) \\
\Rightarrow h(T_{a_1})h(T'_{a_2}) &= c_{a_1 a_2} c_{a_0 a_2} \cdot h(T_{a_2}h(T'_{a_2})),
\end{align*}
\]

(5.36)

So the symbols \(c_{a_0 a_2}\) satisfy \(c_{a_1 a_1} = 1\) and \(c_{a_1 a_2} = c_{a_1 a_3} = c_{a_2 a_3}\) for each choice of indices \(a_0, a_1, a_2\).

**Lemma 24.** For each radical term \(Y^{ij}_{a_1 a_2}\), there is at most one index \(\gamma\) such that \(Y^{ij}_{a_1 a_2} \notin \mathbb{C}\).

**Proof.** Let \((Y^{ij}_{a_1 a_2}, Y^{ij}_{a_2 a_3}, Y^{ij}_{a_1 a_3})\) be a triple containing both radical and constant \(Y\)-terms, all of which satisfy (3.21) by contraposition of Lemma 8, and label the indices so that \(Y^{ij}_{a_1 a_2} \notin \mathbb{F}\) and \(\kappa := -Y^{ij}_{a_2 a_3} \in \mathbb{C}\). Remark 22 (ii.) implies \(\#\Psi(B^{ij}_{a_1 a_2}) \leq 3\), as a consequence of the existence of two proportional terms
in \( \hat{\chi}(I^{ij}_{a_i1a_2}) \). Recalling the definitions of central and unique terms provided in Subsection 2.1, we can assume, moving to a basis \( J \in \{ I, J_{a_2}, I_{a_3} \} \) if necessary, that the central term in \( \hat{\chi}(J^{ij}_{a_i1a_2}) \) is unique while preserving the relations \( Y^{ij}_{a_i1a_2} \notin F \) and \( Y^{ij}_{a_3a_3} \in C \). Instantiating (5.36) at \((\delta_0, \delta_1, \delta_2) = (\alpha_2, \alpha_1, \alpha_3)\) and \( c_{a_1a_2}, c_{a_3a_3} \in C \), we get \( c_{a_1a_3} \in C \); thus, being \( Y^{ij}_{a_1a_3} \notin F \), the central term in \( \hat{\chi}(I^{ij}_{a_1a_2}) \) is unique too. Using the notation introduced in Definition 23, this argument can be summarised as

\[
Y^{ij}_{a_1a_2} \notin \mathbb{F}, \quad Y^{ij}_{a_2a_3} \in \mathbb{C} \Rightarrow \Lambda^{ij}_{a_1a_2} = \Lambda^{ij}_{a_1a_3}.
\] (5.37)

This generates two possible G-type configurations, namely the two sets \( C_1 \) and \( C_2 \) in (5.34). Under the previous assumption on the positions of the central terms in \( \hat{\chi}(I^{ij}_{a_1a_2}) \) and \( \hat{\chi}(I^{ij}_{a_1a_3}) \), the A-terms (3.18) corresponding to these two sets are \( r - (1 + \eta)g + (1 + \eta) \cdot r - (1 - \eta)^2 \cdot g \); being \( \kappa \neq -1 \) for G-type configurations, these polynomials have not the same factors. So, there cannot be another index \( \alpha_4 \neq \alpha_3 \) such that \( Y^{ij}_{a_2a_4} \in C \), since it would entail \( Y^{ij}_{a_2a_3} \in C = Y^{ij}_{a_2a_4} \in C \) contradicting Remark 3. For S-type configurations, the possible values for \( Y^{ij}_{a_2a_3} \) lie in \( \{-\frac{1}{2}, 1, -2\} \) by Remark 22. The existence of two distinct indices \( \alpha_3, \alpha_4 \) such that \( Y^{ij}_{a_2a_3}, Y^{ij}_{a_2a_4} \in C \) implies that \( Y^{ij}_{a_1a_3}, Y^{ij}_{a_1a_4} \) are proportional radical terms, then \( Y^{ij}_{a_2a_3}, Y^{ij}_{a_2a_4}, Y^{ij}_{a_1a_4} \in \{-\frac{1}{2}, 1, -2\} \). But \( Y^{ij}_{a_2a_4} = -Y^{ij}_{a_2a_3} Y^{ij}_{a_2a_4} \) and there exist no elements \( a, b, c \in \{-\frac{1}{2}, 1, -2\} \) such that \( ab = -c \).

\( \square \)

**Lemma 25.** The only form for \( D \) that is compatible with both classes I and II is (5.22). In particular, all the radial terms have the same class when \( D \) has not this form.

**Proof.** Let \( D \) be a squarefree polynomial in \( \mathbb{C}[t] \) such that there exist three invertible elements \( c_u \in \mathbb{C}(t), u \in \{3\} \), satisfying \( B(c_1, c_2, c_3) = D \). This corresponds to the class-I configuration, and we look for a non-invertible polynomial \( Q^{ij}_{a_3} \in \mathbb{C}(t) \) such that \( D \cdot (Q^{ij}_{a_3})^2 \) is an allowed class-II B-term. By Lemma 17, the existence of such \( Q^{ij}_{a_3} \) entails \( \#(\Theta(D)) \geq 3 \), while by Proposition 12 there exist \( f_1 \in \mathbb{C}_0^d, f_2 \in \mathbb{C}^d_0 \) with \( d_1 + d_3 \leq d \) such that \( Q^{ij}_{a_3} = c_1 t_1^{d_1} - c_3 t_3^{d_3} \) for some \( c_1, c_3 \in \mathbb{C} \setminus \{0\} \). From the proof of Corollary 14, we know that \( \Theta(D) \) has to be \( (f_1, f_3) \)-homogeneous. The set \( \{c_1, c_2, c_3\} \) has to be \( (f_1, f_3) \)-homogeneous as well, otherwise we can find a minimal term violating the coherence or the homogeneity condition also in \( \Theta(B(c_1, c_2, c_3)) = \Theta(D) \). This means that we can choose an appropriate labelling so that \( \Psi(c_u) = p \cdot f_u \) with \( u \in \{1, 3\} \) and \( p \in \mathbb{N} \), while \( \Psi(c_2) = q \cdot f_1 + (p - q) \cdot f_3 \), \( 0 \leq q \leq p \).

We focus on cases where (5.27) does not hold, otherwise we recover Example 16 as in the proof of Proposition 19. So, \( q \in \{1, p - 1\} \), say \( q = 1 \) without loss of generality, then we express \( D \) as a polynomial in (5.18): we now prove that the resulting B-term \( (r - 1)^2 \cdot D \), where \( D = B(r^2, c_1 r, c_2) \) for some \( \omega \in \mathbb{N} \), has sparsity strictly greater than 6 at \( \omega \geq 3 \), so it cannot be represented as a B-term. This claim is easily checked at \( \omega \in \{3, 4\} \), where (5.27) does not hold only if \( c_1 = -c_2 \) and, hence, we get

\[
(r - 1)^2 \cdot B(r^2, c_1 r, -c_1) = c_1^2 - 2c_1^2 \cdot r^2 + c_1 \cdot r^3 + c_1(c_1 - 6) \cdot r^4 + 6c_1 \cdot r^5 + (1 - 2c_1) \cdot r^6 - 2r^7 + r^8
\]

and

\[
(r - 1)^2 \cdot B(r^4, c_1 r, -c_1) = c_1^2 - 2c_1^2 \cdot r^2 + c_1(c_1 + 2) \cdot r^4 - 6c_1 \cdot r^5 + 6c_1 \cdot r^6 - 2c_1 r^7 + r^8 - 2r^9 + r^{10}.
\]

At most one coefficient can vanish in both these expressions, so the sparsity is at least 7. At \( \omega > 4 \), we get

\[
(r - 1)^2 \cdot B(r^\omega, c_1 r, c_2)
\]
$$= c_2^2 - 2c_2(c_1 + c_2)r + (c_1^2 + 4c_1c_2 + c_2^2)r^2 - 2c_1(c_1 + c_2)r^3 + c_1^2r^4 - 2c_2r^\omega$$
$$+ 2(2c_2 - c_1) \cdot r^{\omega + 1} + 2(2c_2 - c_1) \cdot r^{\omega + 2} - 2c_1 r^{\omega + 3} + r^{2\omega} - 2r^{2\omega + 1} + r^{2\omega + 2}$$

which has sparsity at least 7. The remaining exponent $\omega = 2$ returns $(r - 1)^2 \cdot B(r^2, c_1r, c_2)$, which is easily checked not to reproduce a $B$-term. Therefore, when $D$ is not in the form (5.22) we can recover the class of any radical term and all the radical terms have the same class.

**Proposition 26.** When $n - k \geq 5$, for all $\mathbf{I} \in \Psi(L(t))$, $i, j \in \mathbf{I}$, and $\alpha, \beta \in \mathcal{I}^c$, the triple $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta}$ returning a radical term $Y_{\alpha \beta}^{ij}$ contains two proportional, but not equal components.

**Proof.** Let $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = (E_1, E_2, E_3)$ violate the thesis, i.e. it contains pairwise independent monomials, or two components coincide. Being $Y_{\alpha \beta}^{ij} \notin \mathcal{F}$, this is summarised in the following condition:

$$\#\Psi(\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta}) = \#\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} > 1.$$  \hspace{1cm} (5.38)

As noted in Remark 22 (ii.), the conditions in (5.38) imply $Y_{\alpha \beta}^{ij} \notin \mathcal{I}^c$. Together with the assumption $Y_{\alpha \beta}^{ij} \notin \mathcal{I}^c$, we infer $Y_{\alpha \beta}^{ij} \notin \mathcal{F}$ too, by contraposition. We have

$$\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = (E_1, E_2, E_3),$$

$$\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} \neq (E_1, E_2, E_3),$$

$$\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} \neq (E_1, E_2, E_3).$$

First look at configurations where $Y_{\alpha \beta}^{ij} \notin \mathcal{F}$, $1 \leq u < w \leq 3$, generate the same multi-set, i.e. $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta}$ is a permutation of $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta}$, which also implies $B_{u \alpha \beta}^{ij} = B_{w \alpha \beta}^{ij} = B$. Under the latter condition, the configuration

$$\frac{E_1 - E_2 - E_3 + \sqrt{B}}{2E_x}, \frac{E_2 - E_1 - E_3 + \sqrt{B}}{2E_y}, \frac{E_3 - E_1 - E_2 + \sqrt{B}}{2E_z} = E_1E_2E_3$$

is the only one returning $Y_{\alpha \beta}^{ij} \cdot Y_{\alpha \beta}^{ij} \cdot Y_{\alpha \beta}^{ij} \in \mathcal{F}$, which is a necessary condition to get (3.13). The constraints $E_x \in \{E_2, E_3\}$, $E_y \in \{E_1, E_3\}$, and $E_z \in \{E_1, E_2\}$ imply $E_xE_yE_z = E_xE_1^{-1}$ for some $s, t \in [3]$, so this ratio never equals $-1$ under the hypothesis (5.38).

From the previous argument, we infer that multiple configurations for $\mathcal{N}_\gamma$ sets are involved, and this allows us to move to the polynomial ring $C(t)$ according to Remarks 15 and 21. We instantiate (5.36) at $i = \gamma_1, \gamma_2$ and change the basis and the indices using the transformations mentioned in Remark 5, if necessary, to fix the form of $Y_{\alpha \beta}^{ij}$. Specifically, we can deal with the two types of configurations individually due to Lemma 17. For $G$-type configurations satisfying (5.38), multiple $\mathcal{N}$ sets can arise when both classes I and II appear: from Lemma 25, this can be achieved only at (5.22), so we choose the form $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = (r^2, 4r, 1)$ and get $c_{\alpha \beta \gamma} \cdot c_{\alpha \beta \gamma} \cdot c_{\alpha \beta \gamma} = 4 \cdot r$. Hence, we infer $\{c_{\alpha \beta \gamma}, c_{\alpha \beta \gamma}\} = \{2 \cdot r, 2\}$, since the alternative $\{c_{\alpha \beta \gamma}, c_{\alpha \beta \gamma}\} = \{r^2, 4 \cdot r^{-1}\}$ is equivalent to (5.39); these values can be generated only by

$$\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = \{(r, 2 \cdot r, 1), (r, 2, 1)\}.$$  \hspace{1cm} (5.40)

For $S$-type systems, we choose the form $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = (-\frac{1}{2}, r^2, -\frac{1}{2})$, so we have $c_{\alpha \beta \gamma} = -4 \cdot r^2$: being the configuration $\{c_{\alpha \beta \gamma}, c_{\alpha \beta \gamma}\} = \{-4r^2, 1\}$ equivalent to (5.39), we infer $\{c_{\alpha \beta \gamma}, c_{\alpha \beta \gamma}\} = \{2 \cdot r, -2 \cdot r\}$. These values are generated by

$$\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = \left\{\left(\frac{1}{2}, r, -\frac{1}{2}\right), \left(-\frac{1}{2}, r, \frac{1}{2}\right)\right\}.$$  \hspace{1cm} (5.41)

In both cases, at $n - k \geq 5$ we can invoke Dirichlet’s box principle to identify two indices $\gamma_1, \gamma_2 \in (\mathcal{I}_{\alpha \beta})^c$ such that $\mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta} = \mathcal{N}(\mathbf{I})^{ij}_{\alpha \beta}$; the only possibility to make them different, according to Remark 3, is that $h(\mathbf{I}^{ij}_{\alpha \beta}) Y_{\gamma_1 \gamma_1}^{ij}$ and $h(\mathbf{I}^{ij}_{\alpha \beta}) Y_{\gamma_1 \gamma_2}^{ij}$ are conjugate over $\mathcal{F}$, hence

$$Y_{\gamma_1 \gamma_2}^{ij} = -Y_{\gamma_1 \gamma_1}^{ij} \cdot Y_{\gamma_1 \gamma_2}^{ij} = \frac{h(\mathbf{I}^{ij}_{\alpha \beta})}{h(\mathbf{I}^{ij}_{\alpha \beta})} \cdot (Y_{\alpha \beta}^{ij})^2.$$
This expression for $Y_{ij}^{ij}$ is not compatible with $Y_{ij}^{ij}$ as derived from one of the triples in (5.40) or (5.41), since $B_{ij}^{ij}$ has neither the same factors as $D$, nor it agrees with (5.23). Thus, $n-k \leq 4$. □

Configurations containing both class-I and class-II terms are allowed at $n-k \leq 4$, e.g.

$$\frac{-r-1 + \sqrt{D}}{2}, \frac{r-3 - \sqrt{D}}{2} = \frac{-r^2 - 4r - 1 + (r-1) \cdot \sqrt{D}}{2}.$$  

Proposition 27. If $n-k \geq 5$, no observable set generates a radical Y-term.

Proof. By Proposition 26, all the sets (2.9) contain two proportional monomials, so there are no class-II terms; by Proposition 19, this means that all the $B$-terms that are not a perfect square coincide, up to units, with a unique polynomial $B$. Any radical term $Y_{ij}^{ij}$ generates sets $\chi(I|_{\alpha ij})$ and $\chi(I_{ij}^{ij})$ satisfying (3.21) for all $\gamma$, by contraposition of Lemma 8.

These premises entail that there exist no radical triples with two coinciding values $\Lambda_{\alpha \alpha \alpha}$; indeed, supposing that $\Lambda_{\alpha \alpha \alpha} = \Lambda_{\alpha \alpha \alpha}$, through an appropriate re-labelling we can choose the basis $\mathcal{J} \in \{I, I_{ij}, I_{\alpha ij}\}$ so that the value of $\Lambda_{\alpha \alpha \alpha}$ is $\Lambda_{\alpha \alpha \alpha}$ (if $\mathcal{J} = I$) or $\Lambda_{\alpha \alpha \alpha}$ (if $\mathcal{J} = I_{ij}$) is 1 in the new basis. From (5.36) evaluated at $(\delta_{ij}, \delta_{ij}, \delta_{ij}) = (\alpha_1, \alpha_2, \alpha_3)$ with $c_{\alpha \alpha \alpha} \in \mathbb{C}$, $w \in \{2, 3\}$, we get $\Lambda_{\alpha \alpha \alpha} = 1$ too. On the other hand, Remark 21 implies that $c_{\alpha \alpha \alpha}, c_{\alpha \alpha \alpha} \in \{\kappa, \kappa^{-1}\}$, being $\kappa \neq 1$ by Proposition 26; here we unify the notation setting $\kappa := c_S = -1$ for $S$-type systems. Then, we get a contradiction, as $c_{\alpha \alpha \alpha} = c_{\alpha \alpha \alpha}, c_{\alpha \alpha \alpha}$ does not belong to $\{\kappa, \kappa^{-1}\}$.

Starting from $G$-type systems, there are at most two allowed configurations for sets (5.34). Therefore, at least two $Y$-terms in any radical triple, say $Y_{ij}^{ij}$ and $Y_{ij}^{ij}$, are associated with the same $\chi$-set, which we can assume equal to $C_1$ through an appropriate parametrisation of the variable $r$. The remaining term $Y_{ij}^{ij}$ has to belong to $C_2$, since we have already noted in the proof of Proposition 26 that (5.39) is the only configuration returning a monomial from $Y_{ij}^{ij}, Y_{ij}^{ij}, Y_{ij}^{ij}$, but it does not equal $-1$ at $\kappa \neq -1$.

On the other hand, fixing $Y_{ij}^{ij}$, we can recover the allowed configuration for $Y_{ij}^{ij}, Y_{ij}^{ij}$ as in (5.35). In principle, starting from $Y_{ij}^{ij}, Y_{ij}^{ij}$, this argument gives a bound of two indices $\gamma_1, \gamma_2$ such that the product $Y_{ij}^{ij} Y_{ij}^{ij}$, $u \in \{2, 3\}$, coincides with the left-hand side of (5.35), and at most two indices $\lambda_1, \lambda_2$ such that $Y_{ij}^{ij}, Y_{ij}^{ij} \in \mathbb{C}$, according to Lemma 24. We also note that

$$Y_{ij}^{ij} \in \mathbb{C}, \quad Y_{ij}^{ij} = 1 \Rightarrow Y_{ij}^{ij} = 1$$

due to (5.37). The previous argument implies $\Lambda_{ij}^{ij} \in \{2, 3\}$ and, also in this case, from $Y_{ij}^{ij} \in \mathbb{C} we get $\Lambda_{ij}^{ij} = \Lambda_{ij}^{ij}$. So, $Y_{ij}^{ij}$ and $Y_{ij}^{ij}$ are proportional over $\mathbb{C}$, and the unique terms in $\chi(I_{ij}^{ij})$ and $\chi(I_{ij}^{ij})$ have the same position: from the proof of Lemma 24, we infer that $\Lambda_{ij}^{ij}$ and $\Lambda_{ij}^{ij}$ have the same factors in $\mathbb{C}(r)$ only if $\chi(I_{ij}^{ij}) = \chi(I_{ij}^{ij})$. However, being $\kappa \neq 1$, we have

$$\Lambda_{ij}^{ij} = \Lambda_{ij}^{ij} \in \{2, 3\}, \quad \chi(I_{ij}^{ij}) = \chi(I_{ij}^{ij}) \Rightarrow \chi(I_{ij}^{ij}) = \chi(I_{ij}^{ij})$$

which means $Y_{ij}^{ij} = Y_{ij}^{ij}$ since $Y_{ij}^{ij} \in \mathbb{C}$, finding a contradiction with Remark 3. Thus, the alternatives $Y_{ij}^{ij} \in \mathbb{C}$ and $Y_{ij}^{ij} \notin \mathbb{C}$ are mutually exclusive. The same holds for $Y_{ij}^{ij} \in \mathbb{C}$ and $Y_{ij}^{ij} \notin \mathbb{C}$. Thus, at most four indices $\{\alpha \alpha \alpha, \alpha \alpha \alpha, \alpha \alpha \alpha, \alpha \alpha \alpha\}$, with $\xi_1 \in \{\gamma_1, \lambda_1\}$ and $\xi_2 \in \{\gamma_2, \lambda_2\}$, are allowed.

A similar argument can be carried out for $S$-type systems, where the two allowed configurations correspond to the choice $\kappa := c_S = -1$ in Table 5.1. First, we look at triples $(Y_{ij}^{ij}, Y_{ij}^{ij}), Y_{ij}^{ij}, Y_{ij}^{ij})$ containing both radical and constant components, say $Y_{ij}^{ij} \in \mathbb{C}$: then $Y_{ij}^{ij} \in \{-\frac{1}{2}, -1, -2\}$ as shown in Lemma 24, and $\chi(I_{ij}^{ij}) = \chi(I_{ij}^{ij})$ if and only if $Y_{ij}^{ij} = 1$. We also have

$$Y_{ij}^{ij} = 1 \Leftrightarrow \Lambda_{ij}^{ij} = \Lambda_{ij}^{ij} = 1$$

(5.43)
and, arguing as before, for each $Y_{ij}^u \notin \mathcal{F}$, assuming $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$ through an appropriate choice of indices and basis, we can state

$$\Lambda_{ij}^{\alpha_1\gamma_u} = \Lambda_{ij}^{\alpha_1\gamma_w} \in \{2, 3\} \Rightarrow Y_{ij}^{\alpha_1\gamma_u} \in \left\{-\frac{1}{2}, -2\right\} \Rightarrow \chi(I\alpha_1\gamma_u) = \chi(I\alpha_1\gamma_w)$$ (5.44)

where the first implication follows from (5.43), and the second implication also takes into account the equality $\hat{B}^{ij}_{\alpha_1\gamma_u} = \hat{B}^{ij}_{\alpha_1\gamma_w}$ for the configurations with $c_S = -1$ in Table 5.1. Then, from the premise in (5.44), we can infer (5.42), which leads to a contradiction as before. Therefore, for each $Y_{ij}^{\alpha_1\gamma_w}$ with $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$, consistency with Remark 3 implies that we can find at most two indices $\lambda_1, \lambda_2$ such that one between $Y_{ij}^{\alpha_1\lambda_1}, Y_{ij}^{\alpha_1\lambda_2} \in \mathbb{C}$ (by Lemma 24) and at most two indices $\gamma_1, \gamma_2$ such that $Y_{ij}^{\alpha_1\gamma_u} \notin \mathcal{F}$ for all $u, w \in [2]$. These two alternatives are mutually exclusive, since $Y_{ij}^{\alpha_1\gamma_u} \in \mathbb{C}$ and $\Lambda_{ij}^{\alpha_1\gamma_u} \in \{2, 3\}$ entails $\Lambda_{ij}^{\alpha_1\gamma_u} = \Lambda_{ij}^{\alpha_1\gamma_u}$, leading to a contradiction as in (5.44). The same argument can be carried out with $\alpha_1$ instead of $\alpha_2$. In conclusion, also in this case, we get at most 4 indices in $\mathcal{I}^C$ under the assumption of existence of radical $Y$-terms.

This bound represents the minimal required information to recover consistency conditions that force integrability, under Assumption (3.21), as it is shown in the following example.

**Example 28.** When $n - k = 4$ we can get radical terms through the following two configurations: $L_{ex,u} = E_{-1,u}$ and $R_{ex,u} = E_{1,u}^T$ with $u \in [2]$ and

$$E_{e,1} := \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & r + \varepsilon \cdot \sqrt{r^2 + 1} & -r - \varepsilon \cdot \sqrt{r^2 + 1} & -1 \end{pmatrix},$$

$$E_{e,2} := \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & r + \varepsilon \cdot \sqrt{r^2 + 1} & 1 + r + \varepsilon \cdot \sqrt{r^2 + 1} & 1 - r - \varepsilon \cdot \sqrt{r^2 + 1} \end{pmatrix}.$$  

It is easily checked by direct computation that all the minors of $R_{ex,u}$ are non-vanishing, the products $\Delta_{L_{ex,u}(\mathcal{I})} \cdot \Delta_{R_{ex,u}(\mathcal{I})}$, $\mathcal{I} \in \varphi_2[6]$, are monomials in $\mathbb{C}(r)$, but (5.1) does not hold for all the indices.

### 6 Reduction of set functions and hyperdeterminants: general case

We now look at those cases where reductions occur in (3.16), which yield algebraic solutions for the $Y$-terms. In this section, we focus on systems with $n - k \geq 5$. From Remark 2, this means that $n - k \geq \max\{k, 10\}$.

We provide a first instance of reduction, which comes from proportional columns in $L(t)$. We recall the invariance of the expansion of $\det(L(t) \cdot R(t))$ under generalised permutations (3.7); in particular, for each set $\mathcal{J} \subseteq [n]$ such that $\text{rank}(L(t)_{(k),\mathcal{J}}) = 1$, we can act via a generalised permutation in order to make all these columns equal. No bound exists on the number of non-constant $Y$-terms if we include matrices with several proportional columns, as is shown in the next example.

**Example 29.** Let us consider

$$L_0(t) := (1_k | \tilde{a}) \in \mathbb{C}(t)^{k \times (k+1)},$$

$$R_0(t) := (1_k | \tilde{r}_0(t))^T \in \mathbb{C}(t)^{(k+1)\times k}$$

where $\tilde{a} \in \mathbb{C}^k \setminus \{0_k\}$ and $\tilde{r}_0(t)$ is a vector of units in $\mathbb{C}(t)$, so $h([k])_{k+1} = a_i \cdot e_i(t)$. For each $s \geq 1$, we can choose generic vectors $\tilde{e}_{u_i}(t)$, $u_i \in [s]$ of invertible elements in $\mathbb{C}(t)$ such that each $\{\tilde{e}_{u_i}, u_i \in \mathcal{I}\}$ is linearly
independent over $\mathbb{C}(t)$ for all $\mathcal{I} \in \mathcal{E}_h[s]$. We construct matrices $L_u(t)$ and $R_u(t)$ as follows: append $s$ replicas of the column $(L_0(t))_u$ to $L_0(t)$ to generate a new matrix $L_u(t)$; then, iteratively append $c_{u+1}(t)$ to $R_u(t)$ to generate $R_{u+1}(t)$, for all $u \in [s-1]$. In this way, we obtain $L_u(t) \in \mathbb{C}(t)^{k \times (k+s+1)}$ such that the non-vanishing maximal minors $[k]_u$ are indexed by $(i, \alpha) \in [k] \times [k+1:k+s+1]$. The associated term $\Delta_{L_u(t)}([k]_u) \cdot \Delta_{R_u(t)}([k]_u)$ is invertible in $\mathbb{C}(t)$. However, the corresponding $Y$-terms are, in general, non-constant.

Therefore, in the rest of this section, we assume the following hypothesis.

**Assumption 30.** There exists $\mathcal{I} \in \mathcal{G}(L(t))$ such that the columns indexed by elements of $\mathcal{I}$ are not all pairwise proportional, i.e. there are not $n - k$ pairwise proportional columns.

**Definition 31.** Given $\mathcal{I}$, a row-choice is a mapping $r_{\mathcal{I}} : \mathcal{I}^c \rightarrow \mathcal{I}$ such that $h(I_{\mathcal{I}}^{\omega}) \neq 0$ for each $\omega \in \mathcal{I}^c$. Recalling the assumptions in Subsection 2.2, in particular the lack of null columns in $L(t)$, such a mapping always exists. When no ambiguity arises, we will keep the subscript $\mathcal{I}$ in $r_{\mathcal{I}}$ implicit. We will say that two rows $i$ and $j$ are parallel when $h(I_{\mathcal{I}}^{\alpha}) = 0$ for all $\alpha, \beta \in \mathcal{I}^c$.

We will investigate the propagation of accessible information on minors obtained through single exchange of indices. For this purpose, for every set $\mathcal{A} \subseteq \mathcal{I}$, we introduce

$$N_{\mathcal{I}, \mathcal{A}} := \{ \gamma \in \mathcal{I}^c : \forall i \in \mathcal{A} : h(I_{\gamma_i}^{\alpha}) = 0 \}, \quad F_{\mathcal{I}, \mathcal{A}} := \{ \gamma \in \mathcal{I}^c : \forall i \in \mathcal{A} : h(I_{\gamma_i}^{\alpha}) \neq 0 \}. \quad (6.1)$$

Dually, for all $\mathcal{H} \subseteq \mathcal{I}^c$ we denote

$$N^{\mathcal{I}, \mathcal{H}} := \{ m \in \mathcal{I} : \forall \alpha \in \mathcal{H} : h(I_m^{\alpha}) = 0 \}, \quad F^{\mathcal{I}, \mathcal{H}} := \{ m \in \mathcal{I} : \forall \alpha \in \mathcal{H} : h(I_m^{\alpha}) \neq 0 \}. \quad (6.2)$$

For sets $\mathcal{A} = \{ i, j \}$ and $\mathcal{H} = \{ \alpha, \beta \}$ we will explicitly write the indices as $N_{\mathcal{I}, \mathcal{A}} = N_{\mathcal{I}, ij}, F_{\mathcal{I}, \mathcal{A}} = F_{\mathcal{I}, ij}, N^{\mathcal{I}, \mathcal{H}} = N^{\mathcal{I}, \alpha \beta}$, and $F^{\mathcal{I}, \mathcal{H}} = F^{\mathcal{I}, \alpha \beta}$. In order to simplify the notation, also in this case, we will omit the subscript $\mathcal{I}$ when no ambiguity arises. Finally, we define

$$N(\mathcal{A}; \mathcal{H}) := \{ (m, \omega) : \omega \in N_{\mathcal{I}, \mathcal{A}} \text{ or } m \in N^{\mathcal{I}, \mathcal{H}} \}. \quad (6.3)$$

From the adjunction

$$\mathcal{H} \subseteq F_{\mathcal{A}} \Leftrightarrow \mathcal{A} \subseteq F^{\mathcal{H}} \quad (6.4)$$

we get consistently say that the set $\mathcal{A} \subseteq \mathcal{I}$ and $\mathcal{H} \subseteq \mathcal{I}^c$ are adjoint if $\mathcal{H} \subseteq F_{\mathcal{A}}$.

**Remark 32.** For every $\mathcal{A} \subseteq \mathcal{I}$ and $i \in \mathcal{A}$, the set $N_{\mathcal{I}, \mathcal{A}}$ is invariant under the shift of basis $\mathcal{I} \mapsto \mathcal{J} := I_{\alpha}$ whenever $h(I_{\alpha}^{\alpha}) \neq 0$: clearly $\alpha \notin N_{\mathcal{I}, \mathcal{A}}$ and, for all $\beta \in N_{\mathcal{I}, \mathcal{A}}$, from (3.26) we find $h(J_{\beta}^{\alpha}) = h(I_{\beta}^{\alpha}) = 0$. Then, for all $j \in \mathcal{A}'$, we get $h(J_{\beta}^{\alpha}) = h(I_{\beta}^{\alpha}) = 0$, being $h(I_{\beta}^{\alpha}) = h(I_{\beta}^{\alpha}) = 0$. So, $N_{\mathcal{I}, \mathcal{A}} \subseteq N_{\mathcal{J}, \mathcal{A}'}$; by symmetry under the exchanges $i \mapsto \alpha$ and $\mathcal{I} \mapsto \mathcal{J}$, we get

$$N_{\mathcal{I}, \mathcal{A}} = N_{\mathcal{J}, \mathcal{A}'} \quad (6.5)$$

Dually, for every $\mathcal{H} \subseteq \mathcal{I}^c$ and $\alpha \in \mathcal{H}$, the set $N^{\mathcal{H}}$ is invariant under the change of basis $\mathcal{I} \mapsto I_{\alpha}^{\alpha}$ whenever $h(I_{\alpha}^{\alpha}) \neq 0$.

For each local key $\mathcal{C}$, we will express the standard projections on the row-set $\mathcal{I}$ and column-set $\mathcal{I}^c$ as $\mathcal{C}_r$ and $\mathcal{C}_c$, respectively. Given a local key $\mathcal{C} = \mathcal{C}_r \times \mathcal{C}_c \subseteq \mathcal{I} \times \mathcal{I}^c$, we denote the associated set of null columns $N(\mathcal{C}_r; \mathcal{C}_c)$ as $N(\mathcal{C})$.

The existence of a local key represents a local property, since it refers to a submatrix of $L(t)$ and is independent of the full structure $\mathcal{G}(L(t))$. Furthermore, this property is independent on the set function
$\Psi$ related to minors in $\mathfrak{S}(L(t))$. In this sense, the existence of a local key is a structural property of the two original matrices that forces global integrability, as we will show. Before that, we state the following basic lemma for future convenience.

**Lemma 33.** For any local key $\epsilon, \gamma \notin \mathcal{N}_c$, $l \notin \mathcal{N}_c$, and $(i_u, \alpha_w) \in \epsilon (u, w \in [2])$, we have $Y^{i_1i_2}_{\gamma\alpha_w}$, $Y^{i_1u}_{\alpha_1\alpha_2} \in \mathbb{C}$.

**Proof.** For any local key $\{i, j\} \times \{\alpha, \beta\}$, we get $Y^{ij}_{\alpha\beta} \in \mathbb{C}$, as follows from Lemma 8 when there exists $\gamma$ such that $0 \in \chi(\mathcal{T}^{ij}_{\alpha\beta}) \cup \chi(\mathcal{T}^{ij}_{\gamma\alpha})$, and from Theorem 20 and Proposition 27 otherwise (being $\#\mathcal{F}_{ij} = \#\mathcal{C} > 5$ in that event).

Then, for each $\gamma \notin \mathcal{N}_c$, we can choose $u \in [2]$ such that $h(\mathcal{T}^{ij}_\gamma) \neq 0$, say $u = 2$ through an appropriate labelling of $\epsilon$. At $h(\mathcal{T}^{ij}_\gamma) = 0$, both $Y^{i_1i_2}_{\alpha_1\gamma}$ and $Y^{i_1i_2}_{\alpha_2\gamma}$ have the form (4.6), which is compatible with (3.13) and $Y^{i_1i_2}_{\alpha_1\alpha_2} \in \mathbb{C}$ only if $Y^{i_1i_2}_{\alpha_1\gamma}, Y^{i_1i_2}_{\alpha_2\gamma} \in \mathbb{C}$ too. At $h(\mathcal{T}^{ij}_\gamma) \neq 0$, the local key $\epsilon$ guarantees the existence of $w \in [2]$ so that $\chi(\mathcal{T}^{i_1i_2}_{\alpha_\omega})$ is a local key, say $w = 2$; then, the previous argument entails $Y^{i_1i_2}_{\alpha_2\gamma} \in \mathbb{C}$ and, from (3.13), $Y^{i_1i_2}_{\alpha_1\gamma} \in \mathbb{C}$ too. The argument leading to $Y^{i_1i_2}_{\alpha_1\alpha_2} \in \mathbb{C}, l \notin \mathcal{N}_c$ and $u \in [2]$, is analogous. \(\square\)

### 6.1 Principal Minors and Hyperdeterminants

We look at an algebraic condition involving “principal” $Y$-terms, i.e. quantities defined by pairs of corresponding indices.

**Lemma 34.** For $Y^{a_1a_3}_{\delta_1\delta_1}, Y^{a_1a_2}_{\delta_1\delta_2}, Y^{a_2a_3}_{\delta_2\delta_2} \in \mathbb{F}$, the term $Y^{a_1a_2}_{\delta_1\delta_1}$ lies in an algebraic extension of $\mathbb{F}$ of degree at most 2.

**Proof.** Let us introduce

$$m^{a_1a_2a_3}_{\delta_1\delta_2\delta_3} := \varepsilon^{a_1a_2a_3}_{\delta_1\delta_2\delta_3} \cdot \frac{\Delta_{R(t)(t^2)} \cdot \Delta_{R(t)(t^{a_1a_2a_3})}}{\Delta_{R(t)(t^{a_1a_3})} \cdot \Delta_{R(t)(t^{a_2a_3})} \cdot \Delta_{R(t)(t^{a_1a_2})}} \quad (6.6)$$

where

$$\varepsilon^{a_1a_2a_3}_{\delta_1\delta_2\delta_3} := -\text{sign} \left[ \prod_{u < w} (a_u - a_w) \cdot \prod_{z < x} (\delta_x - \delta_z) \cdot \prod_{r \neq s} (a_r - \delta_s) \right]. \quad (6.7)$$

Recalling the definitions (3.2) and (3.5), it can be easily verified that the following identity holds:

$$1 + Y^{a_1a_2}_{\delta_2\delta_1} + Y^{a_1a_3}_{\delta_1\delta_1} + Y^{a_2a_3}_{\delta_2\delta_2} \cdot Y^{a_1a_2}_{\delta_1\delta_3} + Y^{a_1a_3}_{\delta_1\delta_2} \cdot Y^{a_2a_3}_{\delta_2\delta_1} - \left(Y^{a_1a_2}_{\delta_1\delta_1}\right)^{-1} \cdot Y^{a_1a_3}_{\delta_1\delta_1} \cdot Y^{a_1a_2}_{\delta_2\delta_1} = m^{a_1a_2a_3}_{\delta_1\delta_2\delta_3}. \quad (6.8)$$

Therefore, $Y^{a_1a_2}_{\delta_1\delta_1} \cdot Y^{a_1a_2}_{\delta_1\delta_1}$ is a root of the quadratic polynomial

$$P^{a_1a_2a_3}_{\delta_1\delta_2\delta_3}(X) := X^2 + (1 + Y^{a_1a_2}_{\delta_2\delta_1} + Y^{a_1a_3}_{\delta_1\delta_1} + Y^{a_2a_3}_{\delta_2\delta_2} - m^{a_1a_2a_3}_{\delta_1\delta_2\delta_3}) \cdot X - Y^{a_1a_3}_{\delta_1\delta_1} \cdot Y^{a_1a_2}_{\delta_2\delta_1} \cdot Y^{a_2a_3}_{\delta_2\delta_2} \quad (6.9)$$

and the thesis follows. \(\square\)

**Remark 35.** Note that the discriminant of (6.9)

$$\Delta^{a_1a_2a_3}_{\delta_1\delta_2\delta_3} := \left(1 + Y^{a_1a_2}_{\delta_2\delta_1} + Y^{a_1a_3}_{\delta_1\delta_1} + Y^{a_2a_3}_{\delta_2\delta_2} - m^{a_1a_2a_3}_{\delta_1\delta_2\delta_3}\right)^2 + 4 \cdot Y^{a_1a_3}_{\delta_1\delta_1} \cdot Y^{a_1a_2}_{\delta_2\delta_1} \cdot Y^{a_2a_3}_{\delta_2\delta_2} \quad (6.10)$$

for binomials $Y^{a_1a_2}_{\delta_1\delta_1}, Y^{a_1a_2}_{\delta_1\delta_1}, Y^{a_2a_3}_{\delta_2\delta_2}$ is equal to the hyperdeterminant [4].

Now, let us use the notation $c_{1}^{(i, j, \alpha, \beta)}$ to refer to the sign $c_1$ in (3.2) with the specification $(i, j, \alpha, \beta) := (a_s, a_t, \delta_s, \delta_t)$; then, looking at (6.7), we note that

$$\varepsilon^{a_1a_2a_3}_{\delta_1\delta_2\delta_3} = c_1^{(1, 2)} \cdot c_1^{(1, 3)} \cdot c_1^{(2, 3)} \quad (6.11)$$

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while from (3.1) we derive
\[ Y_{\omega}^{a_i,\alpha_1} + 1 = c_{ij}^{(a,t)} \frac{\Delta_{R(t)}(I) \cdot \Delta_{R(t)}(I_{\delta_i,\delta_j})}{\Delta_{R(t)}(I_{\delta_i}) \cdot \Delta_{R(t)}(I_{\delta_j})}. \]  
(6.12)

Therefore, from (6.6), (6.11), and (6.12), we verify the identity
\[ m_{\lambda_1,\lambda_2,\lambda_3} = (Y(I_{\delta_1})^{a_1,\alpha_2} + 1) \cdot (Y(I_{\delta_2})^{a_2,\alpha_3} + 1) \cdot (Y(I_{\delta_3})^{a_3,\alpha_1} + 1) \]  
(6.13)
which entails that the factor
\[ g_{\lambda_1,\lambda_2,\lambda_3}^{a_1,\alpha_2,\alpha_3} := \frac{Y(I_{\delta_1})^{a_1,\alpha_2} + 1}{Y(I_{\delta_2})^{a_2,\alpha_3} + 1}. \]  
(6.14)
is invariant under permutations of the three pairs \((\alpha_1, \delta_1), \alpha_2, \delta_2, \alpha_3, \delta_3\).

**Lemma 36.** For each \(\alpha, \beta \in F_{ij}\) and \((m, \omega) \in N\{\{i, j\}; \{\alpha, \beta\}\}\) with \(h(I_{m,\omega}) \neq 0\), a non-constant term \(Y_{\omega}^{m,\omega}\) determines at most one allowed set \(\{Y_{\omega}^{m,\omega} : s \in c_r, \delta \in c_c\}\).

**Proof.** Given any \((m, \omega) \in N\{\{i, j\}; \{\alpha, \beta\}\}\) satisfying \(h(I_{m,\omega}) \neq 0\), we define
\[ Y_+ := \{Y_{\omega}^{m,\omega}, Y_{\beta,\omega}^{m,\omega}\}, \quad Y_- := \{Y_{\omega}^{m,\omega}, Y_{\beta,\omega}^{m,\omega}\}, \quad Y := Y_+ \cup Y_- \]  
and adopt the notation \(Y_{\omega}^{m,\omega} := \{Y_{\sigma,1}, Y_{\sigma,2}\}\) for each \(\sigma \in \{+, -\}\). From (3.13)-(3.14), we find
\[ Y_{\alpha,\beta}^{m,\omega} \cdot Y_{\omega}^{m,\omega} = -(Y_{\alpha,\omega}^{m,\omega} Y_{\beta,\omega}^{m,\omega}). \]  
(6.15)
Each term in \(Y_{\omega}\) is a constant or a binomial by Lemma 6, so \(Y_{\alpha,\beta}^{m,\omega} \in F\). From \(\alpha, \beta \in F_{ij}\), the same lemma and Lemma 33 imply that \(Y_{\alpha,\beta}^{m,\omega} \in \sigma^{-1}\) is invertible in \(C(t)\).

Under the condition \(\#(Y_{\omega} \cap C) = 0\), the factors of the product \(Y_{\sigma,1} \cdot Y_{\sigma,2}\) uniquely determine the singletons \(\Psi(Y_{\sigma,1} + 1)\) and \(\Psi(Y_{\sigma,2} + 1)\), so we consider the corresponding evaluations \(\psi(Y_{\sigma,1} + 1)\) and \(\psi(Y_{\sigma,2} + 1)\) as defined in Subsection 2.1. Therefore, starting from (6.15) and taking into account Remark 3, at \(Y_{\omega} \cap C = \emptyset\) we can infer
\[ \{\psi(Y_{+,1} + 1), \psi(Y_{+,2} + 1)\} = \{-\psi(Y_{-,1} + 1), -\psi(Y_{-,2} + 1)\}. \]  
(6.16)
We can extend the condition (6.16) whenever \(2 \#(Y_{\omega} \cap C)\), since it trivially holds at \(Y_{\omega} \subset C\), and it follows from Remark 3 when \(\#(Y_{\omega} \cap C) = 2\). Hence, at \(2 \#(Y_{\omega} \cap C)\), each \(Y\)-term in (6.15) is constant, or there exists a unit \(\tau \in C(t)\) such that
\[ (Y_{\omega}^+, Y_{\omega}^-) = \{(\tau^{-1} - 1 - \tau \theta - 1), \{\tau - 1, -\theta^{-1} - 1\}\}. \]  
(6.17)

At \(2 \#(Y_{\omega} \cap C)\), which implies \(\#(Y_{\omega} \cap C) = 3\), we can label \(\{+, -\} = \{a, b\}\) to find \(Y \in Y_{\omega}\) such that \(Y_b Y\) for all \(Y_b \in Y_{\omega}\). This means that there exist \(\varepsilon_1, \varepsilon_2 \in \{1, -1\}\) and \(c \in C\) such that
\[ \theta^{-1} = \varepsilon_2 \cdot c \cdot (1 + \varepsilon_1 - 2\varepsilon_2)/2, \]  
(6.18)
\[ (Y_{\omega}^+, Y_{\omega}^-) = \{(\tau^2 - 1 - \varepsilon_2), \{\tau - 1, -\varepsilon_1 - 1\}\}. \]  
(6.19)

With an appropriate labelling of \(c\), we can fix \(Y_{\omega}^{m,\omega} = \tau - 1\). With this notation, we use Lemma 34, in particular (6.9), to get another expression for \(Y_{\omega}^{m,\omega}\). Denoting
\[ Q_{\delta_1,\delta_2,\delta_3}^{a_1,\alpha_2,\alpha_3} := 1 + Y_{\delta_2,\delta_1}^{a_1,\alpha_2} + Y_{\delta_3,\delta_1}^{a_1,\alpha_3} + Y_{\delta_3,\delta_2}^{a_2,\alpha_3} - m_{\delta_1,\delta_2,\delta_3}^{a_1,\alpha_2,\alpha_3} \]  
(6.20)
from (6.9) instantiated at \((a_1, a_2, a_3) = (m, j, i)\) and \((\delta_1, \delta_2, \delta_3) = (\omega, \beta, \alpha)\) we infer the condition

\[
\left(2 \cdot \vartheta \cdot (\tau - 1) + Q_{\omega, \beta, \alpha}^{mji}\right)^2 = \Delta_{\omega, \beta, \alpha}^{mji}
\]

\[
\Rightarrow (\tau - 1) \cdot \left(\vartheta \cdot (\tau - 1) + Q_{\omega, \beta, \alpha}^{mji}\right) = Y_{\omega, \beta, \alpha}^{mji} \cdot Y_{\beta, \alpha}^{mji}.
\]

(6.21)

In order to find an expression for \(m_{\omega, \beta, \alpha}^{mji}\), we look at (6.13) obtaining

\[
m_{\omega, \beta, \alpha}^{mji} = g \cdot \left(Y(I)^{\gamma}_{ij}_{\beta, \alpha} + 1\right) \cdot (Y(I)^{m_{\omega}}_{\omega, \beta, \alpha} + 1) \cdot (Y(I)^{m_{\omega}}_{\beta, \alpha} + 1)
\]

(6.22)

where \(g = g_{\omega, \beta, \alpha}^{mji}\). At this point, we note that the condition \(\alpha, \beta \in \mathcal{F}_{ij}\) is preserved under the shift \(I \mapsto I^\omega\), being \((m, \omega) \in \mathcal{N}()\). Then, based on the configuration (6.19), we can use (6.15) and the transformation rule (3.24) to recover \(Y(I)^{m_{\omega}}_{\omega, \beta, \alpha} + 1\); consequently, from (6.13), we find

\[
m_{\omega, \beta, \alpha}^{mji} = \frac{c_\gamma + 1}{c_\gamma} \cdot \tau^{2\varepsilon_2} \left(1 - (c_\varepsilon + 1)\varepsilon_2\tau^{-\frac{1}{2}(c_\varepsilon - 2\varepsilon_2 + 1)}\right).
\]

(6.23)

Combining (6.19) and (6.23), we write (6.21) as

\[
\tau^{\frac{1}{2}(1 - \varepsilon_1)} \cdot \left(c_\varepsilon^2 \varepsilon_1 - 2\varepsilon_2 + 2 + (c_\varepsilon + 1)^2 \varepsilon_2 + \varepsilon_2^2 \tau^{\frac{1}{2}(c_\varepsilon + 1) - \varepsilon_2} - \varepsilon_2^{2\tau^{\frac{1}{2}(c_\varepsilon + 1) + \varepsilon_2}}\right) = \tau + 1.
\]

(6.24)

Evaluating both the sides of this equation at \(\tau = -1\), the definition \(\varepsilon_1, \varepsilon_2 \in \{-1, 1\}\) requires \(c_\varepsilon = -\frac{1}{2}\).

On the other hand, this value for \(c_\varepsilon\) is not compatible with the evaluation of the two sides of (6.24) at \(\tau = 1\), which excludes the configuration (6.19) and identify (6.17) as the sole allowed configuration. □

For each change of basis \(I \mapsto J := I^\gamma \in \mathfrak{G}(L(t))\), where \((\gamma, \gamma) \in \mathfrak{C}\) and \(c\) is a (weak) local key, we get a new (weak) local key

\[
c_\gamma := (c_\varepsilon)_{\gamma}^I \times (c_\gamma)_{\gamma}^J
\]

(6.25)

for the basis \(I^\gamma\). All these transformations preserve the set \(\mathcal{N}(c)\) by Remark 32, and at least one of them satisfies

\[
\prod_{(i, \sigma) \in \mathcal{E}} h_i^J \neq 0.
\]

(6.26)

**Definition 37.** A local key \(c\) is said explainable if

\[
\mathcal{N}(c) \neq (I \setminus c_r) \times (I^c \setminus c).
\]

(6.27)

Otherwise, it is referred to as unexplainable, or equivoc.

**Proposition 38.** Let \(c = \{i_1, i_2\} \times \{\alpha_1, \alpha_2\}\) be an explainable local key. Then, for all \(u, w \in [2]\), \(l_1, l_2 \in I\), \(\gamma_1, \gamma_2 \in I^c\), and for each \((m, \omega)\) such that \(\chi(I)^{m_{\omega}}\) is observable for some \((i, \alpha) \in c\), we have

\[
Y_{\gamma_1 \gamma_2}^{i_1 i_2} \cdot Y_{\alpha_1 \alpha_2}^{i_1 i_2} \cdot Y_{\omega, \beta, \alpha}^{m_{\omega}} \in \mathbb{C}.
\]

(6.28)

**Proof.** First, we note that the thesis holds for \(I\) if and only if it holds for \(J := I^\gamma\) with \((i, \alpha) \in c\): indeed, the transformation rule (3.24) gives

\[
Y(J)^{m_{\omega}}_{\omega, \beta, \alpha}, Y(J)^{m_{\omega}}_{\omega, \beta, \alpha}, Y(J)^{m_{\omega}}_{\omega, \beta, \alpha} \in \mathbb{C} \Leftrightarrow Y(I)^{i_1 i_2}_{\gamma_1 \gamma_2}, Y(I)^{i_1 i_2}_{\alpha_1 \alpha_2}, Y(I)^{i_1 i_2}_{\omega, \beta, \alpha} \in \mathbb{C}
\]

(6.29)

for all \(i, j \in c_r, \alpha, \beta \in c, \) so we infer

\[
Y(I)^{m_{\omega}}_{\beta, \alpha} = -Y(I)^{m_{\omega}}_{\beta, \alpha} \cdot Y(I)^{i_1 i_2}_{\gamma_1 \gamma_2} \cdot Y(I)^{i_1 i_2}_{\alpha_1 \alpha_2} \cdot Y(I)^{i_1 i_2}_{\omega, \beta, \alpha} \in \mathbb{C}
\]
where the coincidence of indices $\alpha = \beta$ or $i = j$ returns $Y$-terms equal to $-1$, in line with (3.5).

Lemma 33 guarantees $Y(I)_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$ for all $\gamma \notin \mathcal{N}_\varepsilon$ and, similarly, $Y(I)_{\alpha_1 \alpha_2}^{i_1 i_2} \in \mathbb{C}$ for all $I \notin \mathcal{N}_\varepsilon$. Thus, we can focus on $\mathcal{N}_\varepsilon$ and $\mathcal{N}_\varepsilon$ to prove $Y(I)_{\gamma \alpha}^{i_1 i_2}, Y(I)_{\alpha_1 \alpha_2}^{i_1 i_2} \in \mathbb{C}$ for all $I \in \mathcal{I}$, $\gamma \in \mathcal{I}^c$, which gives $Y(I)_{\gamma \alpha}^{i_1 i_2} = -Y(I)_{\gamma \alpha}^{i_1 i_2} Y(I)_{\alpha_1 \alpha_2}^{i_1 i_2} \in \mathbb{C}$ and, similarly $Y(I)_{\alpha_1 \alpha_2}^{i_1 i_2} \in \mathbb{C}$. From the assumptions in Subsection 2.2, for each $\omega \in \mathcal{N}_\varepsilon$, we can find $m \in I$ such that $h(I_m^\omega) \neq 0$, and vice versa. So we lose no generality by looking at pairs in $N(\varepsilon)$ with $h(I_m^\omega) \neq 0$: in fact, each $(m, \omega) \notin N(\varepsilon)$ is associated with an observable set $\chi(I_m^\omega) \in \mathcal{C}$ defined by an element of $\varepsilon$, say $(i_\varepsilon, \alpha_\varepsilon)$ through an appropriate labelling, satisfying $h(I_m^\omega) \cdot I \neq 0$. We have $h(I_m^\omega) \neq 0$ for at least one choice $H \in \{I, I_\alpha^\varepsilon\}$, then we recall the initial observation and update the definition of $I$, if necessary, to match the condition $h(I_m^\omega) \neq 0$. Whether $h(I_m^\omega) = 0$, $Y_{\gamma \alpha}^{i_1 i_2}$ takes the form (4.6) for both $u \in \{2\}$, while $Y_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$ by Lemma 33: this forces $Y_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$. An analogous argument holds if $h(I_m^\omega) = 0$. So we are led to $h(I_\alpha^\varepsilon) \cdot h(I_m^\omega) \neq 0$: being $\varepsilon$ a local key, we can choose $\omega \in \{2\}$ so that $h(I_m^\omega) \neq 0$: similarly, we can pick $\varepsilon \in \{2\}$ so that $h(I_\alpha^\varepsilon) \neq 0$. Thus, we find another local key $(i, i_\varepsilon, m) \times \{\omega, \omega\}$ and, from Lemma 33, we infer $Y_{\gamma \alpha}^{i_1} = Y_{\gamma \alpha}^{i_1 i_2} \cdot Y_{\gamma \alpha}^{i_1 i_2} \cdot Y_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$. Summarising, it is enough to show $Y(I_m^\omega) \in \mathbb{C}$ for a given $(i, \alpha)$ and for $(m, \omega) \notin N(\varepsilon)$ such that $h(I_m^\omega) \neq 0$, entailing $Y_{\gamma \alpha}^{i_1 i_2}, Y_{\alpha_1 \alpha_2}^{i_1 i_2}, Y_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$ from (6.17) with $Y_{\gamma \alpha}^{i_1 i_2} \in \mathbb{C}$.

Now, fixing $(i_\varepsilon, \gamma) \notin N(\varepsilon)$, we can choose the labelling in $\varepsilon$ so that $h(I_m^\omega) \cdot h(I_\alpha^\varepsilon) \neq 0$ holds, then we prove $Y(I_m^\omega) \in \mathbb{C}$. The elements $h(I_\alpha^\varepsilon), h(I_m^\omega)$, and $h(I_\beta^\varepsilon)$, $(j, \beta) \in \varepsilon$, are non-vanishing by hypothesis, and they remain so under the shift $I \mapsto I_{\alpha_1}^\varepsilon$: then, the change of basis $I \mapsto I_{\alpha_1}^\varepsilon$ generates a nonvanishing element $h(I_{\alpha_1}^\varepsilon)$ for each $x \in \{i_\varepsilon, l\}$ and $\xi \in \{\alpha_\varepsilon, \omega, \gamma\}$ with $h(I_{\alpha_1}^\varepsilon) = 0$ and, vice versa, it may lead to $h(I_{\alpha_1}^\varepsilon) = 0$ whenever $h(I_{\alpha_1}^\varepsilon) = 0$. Therefore, at least one of the choices $H \in \{I, I\}$ returns at least two non-vanishing terms among the three $h(I_{\alpha_1}^\varepsilon), h(I_{\omega}^\varepsilon), h(I_{\gamma}^\varepsilon)$. In this way, we can choose the basis $H$ to get at most one vanishing element $h(I_{\alpha_1}^\varepsilon)$ with $x \in \{i_\varepsilon, l\}$ and $\xi \in \{\alpha_\varepsilon, \omega, \gamma\}$, which returns five local keys satisfying (6.26), i.e. $c_1 := (c_\varepsilon)^{i_1} \times c_i$, and $c_2 := c_r \times (c_r)^{i_1}$ with $(i, \alpha) \in \varepsilon$. Being the property $Y(I_m^\omega) \in \mathbb{C}$ preserved under the shift $I \mapsto I_{\alpha_1}^\varepsilon$, we can assume $H = I$ without loss of generality.

Finally, we take $(m, \omega) \notin N(\varepsilon)$, so we have $h(I_m^\omega) = h(I_m^\omega) = 0$, or $h(I_m^\omega) = h(I_m^\omega) = 0$: we consider the instance $h(I_m^\omega) = h(I_m^\omega) = 0$, the alternative being analogous by transposition of upper and lower indices. In the present configuration, Lemma 36 returns (6.17) as the only allowed configuration for the factors in (6.15); in particular, we get $\psi(Y_{\alpha_1}^{m_i} + 1) = -\psi(Y_{\alpha_1}^{m_i} + 1)$ for both $s \in \{i_\varepsilon, l\}$. This gives $\psi(Y_{\alpha_2}^{m_1} + 1) = \psi(Y_{\alpha_2}^{m_1} + 1)$, so we look at (6.17) for the (weak) local key $c_{i_1}^{\gamma_1}$ to infer

$$\psi(Y_{\alpha_2}^{m_1} + 1) = \psi(Y_{\alpha_2}^{m_1} + 1) = \psi(Y_{\alpha_2}^{m_1} + 1), \quad Y_{\gamma_1}^{m_1} + 1 = (Y_{\gamma_1}^{m_1} + 1)^{-1}. \quad (6.30)$$

These relations entail $(m, \omega) \notin N(\varepsilon^:)$: otherwise, we could adapt the previous argument, starting from (6.17) to get $\psi(Y_{\alpha_1}^{m_i} + 1) = -\psi(Y_{\alpha_1}^{m_i} + 1)$ with $\sigma \in \{\alpha_\varepsilon, \omega, \gamma\}$, and finding

$$Y_{\alpha_2}^{m_1} = (Y_{\alpha_2}^{m_1} + 1)^{-1} - 1 = Y_{\alpha_2}^{m_1}$$

at odds with Remark 3. From $h(I_m^\omega) = h(I_m^\omega) = 0$, this means $h(I_m^\omega) \cdot h(I_m^\omega) \neq 0$ for some $j \in \varepsilon$, hence (6.26) holds for $\chi(I_m^\omega)$, and $Y_{\alpha_2}^{m_1}$ is a unit in $\mathbb{C}(t)$ by (4.5) or Lemma 33. Furthermore, (6.30) still holds considering $(m, \gamma) \notin N(\varepsilon)$ instead of $(m, \omega)$. Thus, we can combine (6.17) and (6.30), denoting $Y_{\alpha_2}^{m_1} := \tau - 1$ and $\vartheta := Y_{\alpha_2}^{m_1}$, to get

$$Y_{\gamma_1}^{m_1} = -Y_{\alpha_2}^{m_1} Y_{\alpha_2}^{m_1} = \frac{\tau - 1}{\vartheta + 1} \quad (6.31)$$

where $\tau$ is a unit and $\vartheta \in \mathbb{C} \setminus \{-1\}$ by Lemma 8 and Remark 3. This relation entails $Y_{\alpha_2}^{m_1} \notin \mathbb{C}(t)$ if $\tau \notin \mathbb{C}$, since $\tau - 1$ and $\vartheta + 1$ are coprime, non-invertible Laurent polynomials at $\vartheta \in \mathbb{C} \setminus \{-1\}$. On the other hand, the (weak) local key $c_{i_1}^{\gamma_1}$ satisfies (6.26), so it lies in $\mathbb{C}(t)$ by Lemma 8 and (4.5), which forces
\[ \tau \in \mathbb{C}, \text{ as well as } Y_{i\alpha}^{im} \in \mathbb{C} \text{ for all } (i, \alpha) \in \mathfrak{c}. \] By the same token, both \( Y_{i\gamma}^{jm} \) and \( Y_{\omega_\alpha \chi}^{im} = -Y_{i\gamma}^{jm} \cdot Y_{\omega_\alpha \chi}^{im} \) are units in \( \mathbb{C}(\mathfrak{t}) \), which is compatible with (4.6) only if \( Y_{\omega_\alpha \chi}^{im} \) and, hence, \( Y(I)_{i\alpha \omega}^{m \chi} \in \mathbb{C} \) are constant. \( \square \)

6.2 Identification of an integrable basis

Explainable local keys represent the local information that allows recovering the global integrability. For this reason, we make explicit the following hypothesis that we shall assume hereafter.

Assumption 39. There exists a set \( \mathcal{I} \in \mathfrak{I}(\mathfrak{L}(\mathfrak{t})) \) with an explainable local key.

Proposition 40. Let \( \mathfrak{c} \) denote an explainable local key. Then, \( Y_{\omega_\alpha \omega_\chi}^{m_1 m_2} \in \mathbb{C} \) for all \( (m_1, \omega_1), (m_2, \omega_2) \in \mathcal{N}(\mathfrak{c}) \) with \( h(I_{\omega_\alpha \omega_\chi}^{m}) \neq 0, s \in [2] \).

Proof. Let \( \mathfrak{c} = \{ i_1, i_2 \} \times \{ \alpha_1, \alpha_2 \} \) and \( m_1 \neq m_2 \). For any \((i, \alpha) \in \mathfrak{c} \) and \( s \in [2] \), the set \( \chi(I_{\omega_\alpha \omega_\chi}^{m}) \) is observable, then \( Y_{\omega_\alpha \omega_\chi}^{im} \in \mathbb{C} \) by Proposition 38; from Lemma 7, we find

\[
\forall s \in [2]: \quad \Psi(h(I_{\omega_\alpha \omega_\chi}^{m})) + \Psi(h(I)) = \Psi(h(I_{\alpha}^{m})) + \Psi(h(I_{\omega_\alpha \omega_\chi}^{m})). \tag{6.32}
\]

Now we note that the thesis holds when \( h(I_{\omega_\alpha \omega_\chi}^{m_1}) \cdot h(I_{\omega_\alpha \omega_\chi}^{m_2}) \neq 0 \): this condition entails that \( \chi(I_{\omega_\alpha \omega_\chi}^{m}) \) is observable for all \( s, u \in [2] \); then, Proposition 38 gives \( Y_{\omega_\alpha \omega_\chi}^{im} \in \mathbb{C} \) and, adapting (6.15), we find

\[
Y_{(m_1 m_2)}^{\omega_\alpha \omega_\chi} = -Y_{\omega_1 \omega_2}^{i_1 \alpha \omega_1} Y_{\omega_1 \omega_2}^{i_1 \alpha \omega_2} Y_{\omega_1 \omega_2}^{i_2 \omega_2 \omega_1} \in \mathbb{C}. \tag{6.33}
\]

So, we assume \( h(I_{\omega_\alpha \omega_\chi}^{m_1}) \cdot h(I_{\omega_\alpha \omega_\chi}^{m_2}) = 0 \) in the rest of the proof, which implies \( h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}) \neq 0 \).

We now consider the change of basis \( \mathcal{I} \mapsto I_{\omega_\alpha}^{m_1} \), observing that \( \mathfrak{c} \) remains an explainable local key being \((m_1, \omega_1) \in \mathcal{N}(\mathfrak{c})\). The set \( \chi(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}) \) is observable, since we are taking \( h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}) \neq 0 \). Thus, we can repeat the previous argument to get \( Y(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}) \in \mathbb{C} \) and, recalling Lemma 7 as before, we find

\[
\Psi(Y(I_{\omega_\alpha \omega_\chi}^{m_1 m_2})) = 0
\]

\[
\Leftrightarrow \quad \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2})) + \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_1})) = \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_2})) + \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}))
\]

\[
(\text{from 6.32}) \quad \Leftrightarrow \quad \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2})) = \Psi(h(I_{\alpha}^{m_1})) - \Psi(h(I)) + \Psi(h(I_{\omega_\alpha \omega_\chi}^{m_1 m_2}))
\]

which is equivalent to

\[
\Psi(m_{\omega_\alpha \omega_\chi}^{m_1 m_2}) = \Psi(Y_{(m_1 m_2)}^{\omega_\alpha \omega_\chi}). \tag{6.34}
\]

Being \( Y_{\omega_\alpha \omega_\chi}^{i_1 i_2} \in \mathbb{C} \) by Proposition 38, (6.34) and the assumption \( h(I_{\omega_\alpha \omega_\chi}^{m_1}) \cdot h(I_{\omega_\alpha \omega_\chi}^{m_2}) = 0 \) allow us to introduce the following notation: \( Y_{\omega_\alpha \omega_\chi}^{i_1 i_2} + 1 =: \tau \) with \( \tau \in \mathbb{C}(\mathfrak{t}) \) invertible, \( (Y_{\omega_\alpha \omega_\chi}^{i_1 m_1}, Y_{\omega_\alpha \omega_\chi}^{i_2 m_2}) = (c_{u,1} - 1, c_{u,2} - 1) \in \mathbb{C}^2 \), and \( m_{\omega_\alpha \omega_\chi}^{i_1 i_2} = c_{u,3} \cdot \tau \) with \( c_{u,3} \in \mathbb{C}, u \in [2] \).

Lemma 34 asserts that \( Y_{\omega_\alpha \omega_\chi}^{i_1 m_1} \) and \( Y_{\omega_\alpha \omega_\chi}^{i_2 m_2} \) are algebraic or belong to a quadratic extension of \( \mathbb{F}, u \in [2] \), depending on the existence of a factor with odd multiplicity of the discriminant (6.10)

\[
\Delta_u := (1 + Y_{\omega_\alpha \omega_\chi}^{i_1 m_1} + Y_{\omega_\alpha \omega_\chi}^{i_2 m_2} + Y_{\omega_\alpha \omega_\chi}^{i_1 m_2} - m_{\omega_\alpha \omega_\chi}^{i_1 i_2})^2 + 4 \cdot Y_{\omega_\alpha \omega_\chi}^{i_1 m_1} \cdot Y_{\omega_\alpha \omega_\chi}^{i_2 m_2} \cdot Y_{\omega_\alpha \omega_\chi}^{i_1 m_2} \tag{6.35}
\]

for both \( u \in \{1, 2\} \). With the notation introduced above, we find

\[
\Delta_u = \left(-c_{u,1} + c_{u,2} + \tau - c_{u,3} \cdot \tau\right)^2 - 4c_{u,2}(c_{u,1} - 1) \cdot (c_{u,3} \cdot c_{u,2}^{-1} - 1) \cdot \tau. \tag{6.36}
\]

In order to get \( Y_{\omega_\alpha \omega_\chi}^{i_1 m_1} \cdot Y_{\omega_\alpha \omega_\chi}^{i_2 m_2} = -Y_{\omega_\alpha \omega_\chi}^{i_1 i_2} \in \mathbb{C} \) we also have

\[
\sqrt{\Delta_1} \in \mathbb{C} \Leftrightarrow \sqrt{\Delta_2} \in \mathbb{C}.
\]

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When these discriminants are not perfect squares, we can exploit the symmetry of different expressions under the re-labelling \( m_1 \Leftrightarrow m_2 \) and \( \omega_1 \Leftrightarrow \omega_2 \): for each \( u \in [2] \), the terms

\[
\zeta^{(1)}_u := Y_{\omega_1 \omega_1}^{m_1 m_2} \cdot Y_{\omega_2 \omega_2}^{i_u m_1} , \quad \zeta^{(2)}_u := Y_{\omega_2 \omega_2}^{m_1 m_2} \cdot Y_{\omega_1 \omega_1}^{i_u m_2}
\]

have degree 2 over \( \mathbb{F} \); in particular, both of them are roots of the quadratic equation \( P_{\omega_1 \omega_2}^{m_1 m_2}(X) = 0 \), where \( P_{\omega_1 \omega_2}^{m_1 m_2} \) is a specification of (6.9). Furthermore, the coefficient

\[
Q_u := Q_{\omega_1 \omega_2}^{m_1 m_2} = 1 + Y_{\omega_1 \omega_1}^{i_u m_1} + Y_{\omega_2 \omega_2}^{i_u m_2} + Y_{\omega_2 \omega_2}^{m_1 m_2} - m_{\omega_1 \omega_2}^{i_u m_1 m_2}
= -2 + c_{u,1} + c_{u,2} + \tau - c_{u,3} \cdot \tau . \tag{6.37}
\]

is not vanishing, since this would mean \( c_{u,3} = 1 \) and hence, the right-hand side of (6.33) would have no roots in \((C_0)^d\), so it could not coincide with the left-hand side \( Y_{\omega_1 \omega_2}^{m_1 m_2} = \tau - 1 \).

We note that \( \zeta^{(1)}_u \) and \( \zeta^{(2)}_u \) are proportional over \( \mathbb{C} \), since \( Y_{\omega_1 \omega_1}^{i_1 m_1} = -Y_{\omega_2 \omega_2}^{i_2 m_1} \) and \( Y_{\omega_1 \omega_1}^{i_2 m_2} \in \mathbb{C} \). Being \( \sqrt{\Delta_1} \notin \mathbb{C} \), the constant of proportionality \( Y_{\omega_2 \omega_2}^{i_1 m_2} \) is uniquely defined by the coefficients \( 1 - c_{u,3} \) of \( \tau \) in (6.37), namely

\[
Y_{\omega_2 \omega_2}^{i_1 m_2} = \frac{Y_{\omega_1 \omega_1}^{i_1 m_1}}{Y_{\omega_2 \omega_2}^{i_2 m_1}} = \frac{1 - c_{u,3}}{1 - c_{2,3}} . \tag{6.38}
\]

From the previous observation regarding the roots of the polynomial \( P_{\omega_1 \omega_2}^{m_1 m_2} \) the same argument holds through the exchange \( (m_1, \omega_2) \Leftrightarrow (m_2, \omega_1) \) and, hence, the same constant of proportionality (6.38) is associated with \( Y_{\omega_1 \omega_1}^{i_1 m_2} \) too. This entails \( Y_{\omega_1 \omega_1}^{i_1 m_2} = -1 \), i.e. a contradiction.

Now we have to consider the case where the discriminants (6.36) are perfect squares in \( \mathbb{C}(t) \): from (6.13) we obtain

\[
c_{u,3}c_{u,2}^{-1} - 1 = Y(I_{\omega_2}^{m_2})_u m_1 \neq 0 , \quad c_{u,3}c_{u,1}^{-1} - 1 = Y(I_{\omega_1}^{m_1})_u m_2 \neq 0 . \tag{6.39}
\]

Together with the analogous condition \( c_{u,s} - 1 = Y(I_{\omega_2}^{m_2})_u \neq 0 , \quad u, s \in [2] \), (6.39) forces \( \Delta_u \) to be invertible in order to be a square: in particular, this means \( c_{u,1} = c_{u,2} \), which gives rise to the following chain of implications

\[
\forall u \in [2] : Y_{\omega_1 \omega_1}^{i_u m_1} = Y_{\omega_2 \omega_2}^{m_1 m_2}
\Rightarrow -Y_{\omega_1 \omega_1}^{i_1 m_1} \cdot Y_{\omega_1 \omega_1}^{m_1 m_2} = -Y_{\omega_2 \omega_2}^{i_1 m_2} \cdot Y_{\omega_2 \omega_2}^{m_1 m_2}
\Leftrightarrow Y_{\omega_1 \omega_1}^{i_1 m_2} = Y_{\omega_2 \omega_2}^{i_1 m_2}
\Rightarrow Y_{\omega_1 \omega_1}^{i_1 m_2} = -1
\]

contradicting Remark 3. \( \square \)

**Theorem 41.** Under Assumption 39, we can find an integrable basis \( \mathcal{I} \).

**Proof.** From Lemmata 6 and 7, the thesis is equivalent to the condition \( Y(I) \subset \mathbb{C} \), where \( Y(I) \) is defined in (3.6). Taking into account Proposition 38, this condition holds for all observable sets of the form (6.28). Therefore, we only have to look at observable sets where at most one index comes from \( \epsilon \): for each observable set \( \chi(I_{\omega_1 \omega_2}^{m_1 m_2}) \), we fix an appropriate labelling for \( \omega_1, \omega_2 \) and \( m_1, m_2 \) so that \( h(I_{\omega_1}^{m_1}) \neq 0 \), \( u \in [2] \).

We start from observable sets with one index coming from \( \epsilon \), say \( m_1 = i_1 \) and \( h(I_{\omega_1}^{i_1}) \cdot h(I_{\omega_2}^{m_2}) \neq 0 \) with a suitable labelling, since an analogous argument holds whether one between \( \omega_1 \) and \( \omega_2 \) lies in \( \epsilon \). We easily get \( Y_{\omega_1 \omega_1}^{i_1 m_2} \in \mathbb{C} \) if there exists \( \sigma \in \{ \alpha_1, \alpha_2, \omega_1 \} \) such that \( h(I_{\omega_1}^{\sigma}) \neq 0 \): in that event, we can pick \( w \in [2] \) such that \( \chi(I_{\alpha_0}^{i_1 m_2}) \) is observable; then, noting that \( \chi(I_{\alpha_0}^{i_1 m_2}) \) is observable as well, Proposition
38 gives $Y^{11}_{\omega_1\omega_2} = -Y^{11}_{\omega_1\alpha_2}Y^{11}_{\alpha_1\omega_2} \in \mathbb{C}$. Otherwise, we look the specification $g^{11}_{\alpha_1\omega_1\omega_2}$ of (6.14): as recalled in Remark 35, we have

$$g^{11}_{\alpha_1\omega_1\omega_2} = \frac{Y(I\alpha_1)}{Y(I\omega_2)} = \frac{Y(I\omega_2)}{Y(I\alpha_1)} \in \mathbb{C} \quad (6.40)$$

where the last expression is constant due to Proposition 38, since $c$ is still an explainable local key under the change of basis $I \mapsto I_{m_2}$ at $m_2 \in \mathcal{N}(\alpha_1, \omega_1)$. Consequently, focusing on the first equality in (6.40), $Y(I\alpha_1)^{11}_{\omega_1\omega_2} \in \mathbb{C}$ is and only if $Y(I\omega_2)^{11}_{\omega_1\omega_2} \in \mathbb{C}$. We find $h(\mathcal{H}m_2) = 0$ for any basis $h \in \{I, I\alpha_1\}$, and $Y(I\alpha_1)^{11}_{\omega_1\omega_2}, Y(I\omega_2)^{11}_{\omega_1\omega_2} \in \mathbb{C}$ by Proposition 38, while at least one choice of $h$ satisfies $h(\mathcal{H}m_2) \neq 0$: through an appropriate labelling, we let $I$ be such a set. Therefore, both $Y^{m_1}_{\omega_1\omega_2}$ and $Y^{m_2}_{\omega_1\omega_2}$ assume the form (6.6), which is compatible with $Y(I\alpha_1)^{11}_{\omega_1\omega_2} \in \mathbb{C}$ only if $Y^{m_2}_{\omega_1\omega_2} \in \mathbb{C}$.

Then, we move to the case where all the indices of the observable set are not taken from $c$: in particular, situations where $(m_1, \omega_1), (m_2, \omega_2) \in \mathcal{N}(c)$ entail $Y^{m_1m_2}_{\omega_1\omega_2} \in \mathbb{C}$ by Proposition 40. So, we can suppose the existence of $(m_s, \omega_s) \notin \mathcal{N}(c)$ for some $s \in [2]$, say $s = 1$ without loss of generality. Equivalently, this means that there exists $(i_u, \alpha_u) \in \mathbf{c}$ such that $h(I\alpha_u^m)h(I\omega_u^m) \neq 0$. So we can reduce the configuration to the previous one: the sets $\chi(I\alpha_u^{m_1m_2})$ and $\chi(I\omega_u^{m_1m_2})$ are observable with an index taken from $c$ ($\alpha_u$ and $i_u$, respectively), then the previous argument entails $Y^{m_1m_2}_{\omega_1\omega_2} \in \mathbb{C}$. On the other hand, we have $Y^{m_1}_{\omega_1\alpha_u}, Y^{m_2}_{\omega_1\alpha_u} \in \mathbb{C}$ by Proposition 38, being $\chi(I\alpha_u^{m_1})$ and $\chi(I\alpha_u^{m_2})$ observable. In conclusion, we get

$$Y^{m_1m_2}_{\omega_1\omega_2} = \frac{Y^{m_1m_2}_{\omega_1\alpha_u}Y^{m_1m_2}_{\omega_1\alpha_u}Y^{m_1m_2}_{\omega_1\omega_2}Y^{m_2m_2}_{\omega_1\omega_2} \in \mathbb{C}}.$$ 

\[
6.3 \text{ Propagation of Integrability between Different Bases}
\]

Finally, we extend the previous result and prove Theorem 1 under Assumption 39. For this purpose, we will prove a more general result on the propagation of the integrability property moving from a basis to another one.

**Lemma 42.** Let $\mathcal{H}, K \in \mathfrak{S}(\mathbf{L}(t))$ with $r := \#(\mathcal{H} \setminus K)$. Then, there exists a finite sequence $\mathcal{L}_0 := \mathcal{H}, \mathcal{L}_1, \ldots, \mathcal{L}_r := K$ of elements of $\mathfrak{S}(\mathbf{L}(t))$ such that $\#(\mathcal{L}_{u-1} \setminus \mathcal{L}_u) = 2$, $v \in [r]$. 

**Proof.** This is Lemma 6 in [3] and easily follows from the exchange property of matroids (2.4).

The existence of a set $\mathcal{I}$ satisfying Assumption 39 provides a basis for the integrability of observable sets with basis $\mathcal{I}$. In order to transfer the integrability property under changes of bases, we focus on the existence of a set $\{G \times \{\kappa_1, \kappa_2\}, G \in \mathcal{I} \}$ and $\kappa_1, \kappa_2 \in \mathcal{I}^c$, such that

$$h(\mathcal{I}_{\kappa_1}^G) \cdot h(\mathcal{I}_{\kappa_2}^G) \neq 0. \quad (6.41)$$

This set generates a symmetry that allow the transfer of integrability under the condition in Remark 3.

**Remark 43.** We note that the existence of two indices as in (6.41) holds for a set $\mathcal{I}$ if and only if it holds for any set $\mathcal{J} \in \mathfrak{S}(\mathbf{L}(t))$. We can easily prove this claim by contraposition: for a fixed $\mathcal{I} \in \mathfrak{S}(\mathbf{L}(t))$, we cannot find $G \in \mathcal{I}$ and $\kappa_1, \kappa_2 \in \mathcal{I}^G$ as in (6.41) if and only if for each $r \in \mathcal{I}$ there is at most one $\kappa \in \mathcal{I}$ such that $h(\mathcal{I}_r^\kappa) \neq 0$. From the assumption $n - k \geq k$ stated in Remark 2 and the absence of null columns as in Subsection 2.2, we infer that there exists a unique map $g : \mathcal{I}^r \to \mathcal{I}$ that returns $h(\mathcal{I}_{\kappa}^{g(\kappa)}) \neq 0$. The possible changes of bases correspond to the substitution of a set $\mathcal{A} \subseteq \mathcal{I}^r$ with $g(\mathcal{A})$, and vice versa: this structure is preserved under such exchanges.

**Theorem 44.** If there exists an integrable basis $\mathcal{I}$ and indices $G \in \mathcal{I}, \kappa_1, \kappa_2 \in \mathcal{I}^c$ such that (6.41) holds, then each basis in $\mathfrak{G}(\mathbf{L}(t))$ is integrable. In particular, under Assumption 39, each basis in $\mathfrak{G}(\mathbf{L}(t))$ is integrable.
Proof. Let $\mathcal{I}$ be integrable, and suppose, for the sake of contradiction, that the thesis does not hold for a given basis $\mathcal{J} \in \mathfrak{S}(\mathbf{L}(t))$. We can construct a finite sequence $\mathcal{L}_0 := \mathcal{I}$, $\mathcal{L}_1, \ldots, \mathcal{L}_r := \mathcal{J}$ of elements of $\mathfrak{S}(\mathbf{L}(t))$ such that $r = #(\mathcal{I}\Delta\mathcal{J})$ and $#(\mathcal{L}_{u-1}\Delta\mathcal{L}_u) = 2$, $u \in [r]$ as stated in Lemma 42. Say

$$q := \min \{u \in [r] : \mathcal{L}_{u-1} \text{ is integrable and } \mathcal{L}_u \text{ is non-integrable} \}. \quad (6.42)$$

In order to simplify the notation, we denote $\mathcal{A} := \mathcal{L}_{q-1}$ and $\mathcal{B} := \mathcal{L}_q$, with $\mathcal{B} = \mathcal{A}^\alpha$. The previous definition is equivalent to the existence of an observable set $\chi(\mathcal{B}_{ij})$ that is not integrable, i.e. $Y(\mathcal{B})_{ij} \notin \mathbb{C}$ by Lemma 7.

First, we express $Y(\mathcal{B})_{ij}^\alpha$ as a product of $Y$-terms of the form $Y_{\omega}^{\alpha\omega}$ with $\omega \in \{\alpha, \beta\}$ and $m \in \{i, j\}$ considering the decomposition

$$Y(\mathcal{B})_{ij}^\alpha = -Y(\mathcal{B})_{\alpha}^{\alpha\omega}Y(\mathcal{B})_{\beta}^{\alpha\omega}Y(\mathcal{B})_{\alpha\beta}Y(\mathcal{B})_{\beta\beta}^\omega. \quad (6.43)$$

At $v \in \{\alpha, \beta\}$, respectively $\omega \in \{i, j\}$, we can use (3.13), respectively (3.14), as special instances of (6.43). For instance, when $v, \omega \in \{\alpha, \beta\} \times \{i, j\}$ we recover the tautology $Y(\mathcal{B})_{ij}^\alpha = Y(\mathcal{B})_{ij}^\alpha$, while at $v \in \{\alpha, \beta\}$, say $v = \alpha$ through an appropriate labelling, we have the decomposition

$$Y(\mathcal{B})_{ij}^\alpha = -Y(\mathcal{B})_{\alpha}^{\alpha\omega}Y(\mathcal{B})_{\beta}^{\alpha\omega}Y(\mathcal{B})_{\alpha\beta}^\omega. \quad (6.44)$$

These decompositions allow moving from terms of the form $Y(\mathcal{B})_{ij}^{\alpha\omega}$ to the associated terms $Y(\mathcal{A})_{ij}^{\alpha\omega}$, $s \in \{i, j\}$ and $\tau \in \{\alpha, \beta\}$, through the transformation rules (3.23)-(3.24). Consequently, from $Y(\mathcal{B})_{ij}^\alpha \notin \mathbb{C}$, the decomposition (6.43), and the transformation rules, we derive

$$\exists s \in \{i, j\}, \tau \in \{\alpha, \beta\} : Y(\mathcal{A})_{ij}^{\alpha\omega} \notin \mathbb{C}. \quad (6.45)$$

Choosing a proper labelling of $\{i, j\}$ and $\{\alpha, \beta\}$, we can specify (6.45) assuming $Y(\mathcal{A})_{ij}^{\alpha\omega} \notin \mathbb{C}$ without loss of generality. In particular, from $Y(\mathcal{A})_{ij}^{\alpha\omega} \neq -1$ and the definition of $Y$-terms, we infer $\omega \neq i$ and $\beta \neq v$. By the hypothesis of integrability of $\mathcal{A}$, (6.45) is satisfied only if $\chi(\mathcal{A})_{ij}^{\alpha\omega}$ is not observable, which entails

$$h(\mathcal{A})_{ij}^\alpha = 0, \quad h(\mathcal{A})_{ij}^{\alpha\omega} = h(\mathcal{B})_{ij}^\omega = 0 \quad (6.46)$$

where the first condition follows from $h(\mathcal{A})_{ij} \neq 0$ by definition. On the other hand, the set $\chi(\mathcal{B})_{ij}^{\alpha\omega}$ is assumed observable: from (6.46), this means

$$h(\mathcal{B})_{ij}^\alpha \cdot h(\mathcal{B})_{ij}^\omega = h(\mathcal{A})_{ij}^{\alpha\omega} \cdot h(\mathcal{B})_{ij}^\omega \neq 0 \quad (6.47)$$

which makes the sets $\chi(\mathcal{A})_{ij}^{\alpha\omega}$ and $\chi(\mathcal{B})_{ij}^{\alpha\omega}$ observable, thus, $Y(\mathcal{A})_{ij}^{\alpha\omega}, Y(\mathcal{B})_{ij}^{\alpha\omega} \in \mathbb{C}$ by the integrability of $\mathcal{A}$.

We can easily see that the only configuration compatible with the condition of observability of $\chi(\mathcal{A})_{ij}^{\alpha\omega}$, the assumption $Y(\mathcal{A})_{ij}^{\alpha\omega} \notin \mathbb{C}$, and the definition $h(\mathcal{A})_{ij}^\alpha = h(\mathcal{B})_{ij}^\alpha \neq 0$ is described by a function $\varphi$ defined by $\varphi(\alpha) := i$, $\varphi(\beta) := j$, and $\varphi(\omega) := v$, such that the following holds:

$$\forall \delta \in \{\alpha, \beta, \omega\}, s \in \{i, j, v\} : h(\mathcal{A})_{ij}^\delta \neq 0 \Leftrightarrow s = \varphi(\delta). \quad (6.48)$$

In particular, the previous property uniquely defines the only configuration where $Y_{\omega}^{\alpha\omega}$ is not expressed as a product, obtained from iterations of (3.13) and (3.14), that solely involves $Y$-terms associated with
observable sets. As a consequence, we get \( v \notin \{ \alpha, \beta \} \) and \( \omega \notin \{ i, j \} \). Summarising, from (6.48) we get
\[
Y(A)^{ij}_{\alpha \beta}, Y(A)^{iv}_{\omega \alpha}, Y(A)^{iv}_{\omega \beta} \in \mathbb{C} \tag{6.49}
\]
since these \( Y \)-terms come from observable sets.

Now, we instantiate (6.9), at \((a_1, a_2, a_3) := (v, i, j)\) and \((\delta_1, \delta_2, \delta_3) := (\omega, \alpha, \beta)\); then, \( Y^{iv}_{\omega \beta} \) satisfies the quadratic equation
\[
P^{iv}_{\omega \beta}(X) := P^{iv}_{\omega \beta}(Y^{ij}_{\beta \alpha} \cdot X) = 0
\]
where \( P^{iv}_{\omega \beta} \) comes from (6.9). On the other hand, \( Y^{iv}_{\omega \beta} \notin \mathbb{F} \) implies that the minimal polynomial of \( Y^{iv}_{\omega \beta} \) over \( \mathbb{F} \) has degree at least 2. Therefore, \( P^{iv}_{\omega \beta} \) is the minimal polynomial of \( Y^{iv}_{\omega \beta} \). The coefficient of \( Y^{iv}_{\omega \beta} \) in \( P^{iv}_{\omega \beta} \) is not vanishing, since this would imply \( Y(A)^{iv}_{\omega \beta} \in \mathbb{C} \); hence, from (6.49), the discriminant (6.10) can never be a perfect square in \( \mathbb{C}(t) \). The inclusion of these conditions in (6.43) leads to \( Y^{iv}_{\omega \alpha} \neq Y^{iv}_{\omega \beta} \), so \( Y^{iv}_{\omega} \) is the unique conjugate root of \( Y^{iv}_{\omega \beta} \). From the symmetry of (6.48) and \( P^{iv}_{\omega \beta} \) under permutations of the indices \((i, j, v)\), this argument can be extended to infer
\[
Y^{g(\xi_2)g(\xi_3)}_{\xi_1 \xi_2 \xi_3} \notin \mathbb{F}
\]
for each permutation \((\xi_1, \xi_2, \xi_3)\) of \((\alpha, \beta, \gamma)\), where \( g \) is the row-choice in (6.48).

Here, we invoke the existence of elements \( g \in A \) and \( \kappa_1, \kappa_2 \in A^G \) such that (6.41) holds: elements of this type exist in \( I \), as follows from the existence of a local key; then, they exist in all the basis \( \mathcal{G}(L(t)) \) and, in particular, in \( A \) by Remark 43. Then, we extend the row-choice in (6.48), using the same symbol \( g \) with a slight abuse of notation, setting \( g(\kappa_1) = g(\kappa_2) := g \). For each \( \gamma_1, \gamma_2 \in \{ \alpha, \beta, \omega \}, \gamma_1 \neq \gamma_2 \), we can adapt the previous argument: given \( u \in [2] \), when \( Y(A)^{g(\gamma_1)}_{\gamma_1 \gamma_2} \notin \mathbb{C} \), the condition (6.48) also holds under the substitution of labels \{ \omega, \alpha, \beta \} with \{ \kappa_u, \gamma_1, \gamma_2 \}; in that event, we recover
\[
Y^{g(\gamma_1)g(\gamma_2)}_{\kappa_u \gamma_1 \gamma_2} = Y^{g(\gamma_1)g(\gamma_2)}_{\kappa_2 \gamma_1 \gamma_2},
\]
so \( \Delta^{g(\gamma_1)g(\gamma_2)}_{\kappa_2 \gamma_1 \gamma_2} \in \mathbb{C} \) since it is the only situation where it is a perfect square in \( \mathbb{C}(t) \) and the condition
\[
Y(A)^{g(\gamma_1)g(\gamma_1)}_{\kappa_u \gamma_1 \gamma_1}, Y(A)^{g(\gamma_1)g(\gamma_1)}_{\kappa_u \gamma_1 \gamma_2}, Y(A)^{g(\gamma_1)g(\gamma_2)}_{\kappa_u \gamma_2 \gamma_2} \in \mathbb{C}
\]
holds. This means
\[
\forall u, s \in [2] : Y^{g(\gamma_1)g}_{\kappa_u \gamma_2} \in \mathbb{C}. \tag{6.50}
\]
Note that this last argument is analogous to the one discussed in the proof of Proposition 40. Finally, the term
\[
Y(A)^{iv}_{\omega \beta} = \left( Y(A)^{iv}_{\alpha \kappa_2} \cdot Y(A)^{iv}_{\kappa_2 \beta} \right) \cdot \left( Y(A)^{iv}_{\beta \kappa_2} \cdot Y(A)^{iv}_{\kappa_2 \alpha} \right) \cdot Y(A)^{iv}_{\omega \alpha}
\]
is constant, since each factor in the right-hand side derives from an observable set or is of the form (6.50). This also includes the degenerate cases where \( g \in \{ v, i \} \), since this means that some factors are equal to \(-1\) according to the definition (3.5). Hence, we have obtained a contradiction, then the thesis holds. In particular, it holds when Assumption 39 is satisfied, since it guarantees the integrability of a basis \( I \) by Theorem 41.

We introduce the following binary relation on \([n]\):
\[
\alpha \lor \beta \triangleleft \exists I \in \mathcal{G}(L(t)) : T^{\beta}_{\beta} \in \mathcal{G}(L(t)) \text{ or } T^{\alpha}_{\alpha} \in \mathcal{G}(L(t)). \tag{6.52}
\]
The relation $\mathcal{I}_\beta^\alpha \in \mathfrak{S}(\mathbf{L}(t))$ implicitly assumes that $\alpha \in \mathcal{I}, \beta \notin \mathcal{I}$.

**Lemma 45.** Under the hypothesis of Theorem 44, for all $\alpha \forall \beta$, the quantity

$$\psi_2(\alpha; \beta) := \Psi(h(J)^{-1} \cdot h(J_\beta^\alpha))$$

(6.53)

does not depend on the choice of the set $\mathcal{J} \in \mathfrak{S}(\mathbf{L}(t))$ such that $J_\beta^\alpha \in \mathfrak{S}(\mathbf{L}(t))$ too. In particular, this holds if Assumption 39 is verified.

**Proof.** The argument generalises the proof of Proposition 9 in [3]: in particular, we will prove the equality of the quantity (6.53) evaluated at two distinct sets $\mathcal{I}_a$ and $\mathcal{J}_a$ satisfying (6.52), where $\mathcal{I}_a = \mathcal{I}$ is a distinguished, fixed set satisfying Assumption 39. Let us introduce $\mathcal{I}_b := \mathcal{I}_\beta^\alpha$, $\mathcal{J}_b := \mathcal{I}_\beta^\alpha$, and $\kappa := \#(\mathcal{I} \setminus \mathcal{J})$. For $x \in \{a, b\}$, we consider a finite sequence $\mathcal{I}_{x,0} := \mathcal{I}_x, \mathcal{I}_{x,1}, \ldots, \mathcal{I}_{x,\kappa} := \mathcal{J}_x$ satisfying the requirements in Lemma 42, thus $\alpha \in \mathcal{I}_{x,u}$ and $\beta \notin \mathcal{I}_{x,u}$ for all $u \in [\kappa]_0$. These sequences determine as many bijections $\pi_x, x \in \{a, b\}$, from $\mathcal{I}_a \setminus \mathcal{J}_a = \{\delta_1, \ldots, \delta_r\} = \mathcal{I}_b \setminus \mathcal{J}_b$ into $\mathcal{J} \setminus \mathcal{I} = \{\pi_x(\delta_1), \ldots, \pi_x(\delta_r)\}$ via $\{\pi_x(\delta_u)\} = \mathcal{I}_{x,u} \setminus \mathcal{I}_{x,u-1}$.

The proof proceeds by induction on $\kappa$: the base step $\kappa = 1$ follows from Theorem 41, since $\mathcal{I}_a, \mathcal{J}_a, \mathcal{J}_b \in \mathfrak{S}(\mathbf{L}(t))$ means that $\chi(\mathcal{I}_a|_{\mathcal{I}_a(\pi_\alpha)})$ is observable and Assumption 39 holds for $\mathcal{I}_a$. Now, let us suppose that the thesis holds for all $u < \kappa$ and look at any $4$-tuple $(\mathcal{I}_a, \mathcal{I}_b, \mathcal{J}_a, \mathcal{J}_b) \in \mathfrak{S}(\mathbf{L}(t))$ with $\frac{1}{2} \cdot \#(\mathcal{I} \setminus \mathcal{J}) = \kappa$. We pay attention to the $4$-tuple $(\mathcal{I}_a, \mathcal{I}_b, \mathcal{J}_a, \mathcal{J}_b) \in \mathfrak{S}(\mathbf{L}(t))$ and $\frac{1}{2} \cdot \#(\mathcal{I}_a \Delta \mathcal{I}_a, \mathcal{J}_b) = \kappa$. Using the inductive hypothesis, we get

$$\Psi(h(I_a,1)^{-1} h(I_b,1)) = \Psi(h(I_a,\kappa)^{-1} h(I_b,\kappa)).$$

Combining the previous equation with the relation

$$\Psi(h(I_a,0)^{-1} h(I_b,0)) = \Psi(h(I_a,1)^{-1} h(I_b,1))$$

which holds by Theorem 44, we get

$$\Psi(h(I_a)^{-1} h(I_b)) = \Psi(h(I_a)^{-1} h(I_b)).$$

(6.54)

**Remark 46.** The relation (6.52) is reflexive and symmetric (we can look at change of basis $\mathcal{I} \equiv \mathcal{I}_\beta^\alpha$ under the exchange $\alpha \equiv \beta$), so its transitive closure $\bar{\nu}$ defines an equivalence. Every pair $\alpha \bar{\nu} \omega$ is obtained from a finite sequence $(\delta_0, \delta_1, \ldots, \delta_n)$ where $\delta_0 = \alpha$, $\delta_n = \omega$, and $\delta_{i-1} \bar{\nu} \delta_i$ for all $i \in [n]$. It is easy to see that $\alpha \bar{\nu} \omega$ whenever there exists $\mathcal{I}, \mathcal{J} \in \mathfrak{S}(\mathbf{L}(t))$ that separate $\alpha$ and $\omega$, namely $\alpha \in \mathcal{I} \cap \mathcal{J}^c$ and $\omega \in \mathcal{J} \cap \mathcal{I}^c$; in fact, in this case, we can construct a sequence $\mathcal{I}_{a,0} := \mathcal{I}, \mathcal{I}_{a,1}, \ldots, \mathcal{I}_{a,\kappa} := \mathcal{J}$ and the bijection $\pi_a : \mathcal{I} \setminus \mathcal{J} \rightarrow \mathcal{J} \setminus \mathcal{I}$ as in the proof of Lemma 45; then, we can find a subsequence that starts and ends (not necessarily in this order) with $\mathcal{I}_{\pi_\alpha(\alpha)}^\alpha$ and $\mathcal{I}_{\pi_\alpha(\omega)}^{\pi_\alpha(\omega)}$, where all the adjacent elements of the subsequence are related by $\bar{\nu}$.

**Theorem 47.** If there exists a map $\Psi : \mathfrak{S}(\mathbf{L}(t)) \rightarrow \mathbb{Z}^d$ such that the terms in the Cauchy-Binet expansion of $\mathbf{L}(t) \cdot \mathbf{R}(t)$ satisfy the monomial deformation condition

$$\Delta_{\mathbf{L}(t)}(\mathcal{I}) \cdot \Delta_{\mathbf{R}(t)}(\mathcal{I}) = g_\mathcal{I} \cdot t^{\Psi(\mathcal{I})}, \quad \mathcal{I} \in \mathfrak{S}(\mathbf{L}(t)), \quad g_\mathcal{I} \in \mathbb{C}$$

(6.55)

then, under the hypothesis of Theorem 44, there exist full-rank constant matrices $\mathbf{A}, \mathbf{Q}^I \in \mathbb{C}^{k \times n}$, an
element $m_0 \in \mathbb{Z}^d$, and a map $\psi : [n] \to \mathbb{Z}^d$ such that

\[
\Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) = t^{m_0} \cdot \Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) \prod_{\alpha \in I} t^{\psi(\alpha)}, \quad \forall I \in \mathfrak{L}(n]. \quad (6.66)
\]

In particular, this holds if Assumption 39 is verified.

**Proof.** We introduce the sub-relations $\forall_H \subseteq \forall$ for each $H \in \mathfrak{J}(L(t))$:

\[
\alpha \forall_H \beta \quad \text{def} \quad H^\alpha_\beta \in \mathfrak{J}(L(t)) \text{ or } H^\alpha_\beta \in \mathfrak{J}(L(t)). \quad (6.57)
\]

For each $H \in \mathfrak{J}(L(t))$, we will denote as $\forall_H$ the transitive closure of $\forall_H$, which is an equivalence by definition. So we consider $I$ that satisfies Assumption 39, which implies that all the bases in $\mathfrak{J}(L(t))$ are integrable by Theorem 44, and focus on $\forall_I$. The quantity (6.53) satisfies

\[
\psi_2(\alpha; \alpha) = 0, \quad \text{(consistency with reflexivity)} \quad (6.58)
\]

\[
\psi_2(\alpha; \beta) = -\psi_2(\beta; \alpha) \quad \text{(consistency with symmetry and reflexivity)} \quad (6.59)
\]

where the second equation follows from (6.53) with $I^2$ instead of $I$. The function $\psi_2$ can be extended to make it compatible with $\forall$: indeed, for each $\alpha \forall_H \omega$, by definition there exists a finite sequence $(\delta_1, \ldots, \delta_n)$ with $\delta_1 = \alpha$, $\delta_n = \omega$ and $\delta_{i-1} \forall_I \delta_i$ for all $i \in [k]$, so we extend (6.53) by additivity:

\[
\bar{\psi}_2(\alpha; \omega) := \sum_{i=2}^{k} \psi_2(\delta_{i-1}; \delta_i). \quad (6.60)
\]

This definition is consistent, since each pair $(\delta_{i-1}; \delta_i)$ lies in the domain of $\psi_2$: (6.59) is preserved, as can be seen by considering the sequence in reversed order and using (6.60) to get $\bar{\psi}_2(\omega; \alpha) = -\bar{\psi}_2(\alpha; \omega)$. Furthermore, we can show that the definition (6.60) does not depend on the choice of the sequence $(\delta_1, \ldots, \delta_n)$, but only on its endpoints: due to (6.59), it is enough to show that $\bar{\psi}_2$ is consistent with (6.58) for each choice of the basis $I$, i.e. $\bar{\psi}_2$ vanishes for closed paths on indices. From

\[
\delta_i \in I \iff \delta_{i+1} \in I^C
\]

each closed path contains an odd number of indices, say $2 \cdot p + 1$ with $\delta_1 = \delta_{2p+1}$; then, we prove that $\bar{\psi}_2(\alpha; \alpha) = 0$ for all closed paths by induction on $p$. Since the case $p = 1$ is equivalent to (6.59), the base step is $p = 2$, where the thesis holds thanks to Theorem 44. For the induction step, we assume that the thesis holds for all $u \leq p$, and consider any closed sequence $(\delta_1, \ldots, \delta_{2p+3})$ with $\delta_1 = \delta_{2p+3}$ and $\delta_i \forall_I \delta_{i+1}$ for all $i \in [2p+2]$. In order to simplify the notation, we consider indices modulo $2p + 2$, i.e. $\delta_{2p+2+l} = \delta_l$ for all $l \in [2p+2]$, and act via a cyclic shift of labels, if necessary, in order to have $\delta_1 \in I$. If we can find $\delta_u, \delta_w$ such that $1 < w - u < 2p + 1$ and $\delta_u \forall_I \delta_w$, then we can write

\[
\bar{\psi}_2(\delta_1; \delta_1) = \left( \sum_{i=u+1}^{w} \psi_2(\delta_{i-1}; \delta_i) + \psi_2(\delta_w; \delta_u) \right) + \left( \sum_{j=w+1}^{2p+2+u} \psi_2(\delta_{j-1}; \delta_j) + \psi_2(\delta_u; \delta_w) \right).
\]

Each summand between brackets is a cycle with length not greater than $2p$, then the inductive hypothesis applies, and we get $\bar{\psi}_2(\delta_1; \delta_1) = 0$. Otherwise, $\delta_u \forall_I \delta_w$ never holds at $|w - u| \neq 1$: in this case, for all
Then, we fix an index $M \neq 0$ we get

$$h(I_{\bar{\delta}^2 u}^\delta) \cdot h(I_{\bar{\delta}^2 u+2 + M}^{\delta 1}) \neq 0 = h(I_{\bar{\delta}^2 u+2 + M}^\delta) \cdot h(I_{\bar{\delta}^2 u}^{\delta 1})$$

and the Plücker relations impose

$$h(I_{\bar{\delta}^2 u+1}^{\delta 2 u+2M + 1}) \neq 0. \quad (6.61)$$

Then, we fix an index $u$, say $u = 1$, and get

$$\psi_2(\delta_1; \delta_2) + \psi_2(\delta_2; \delta_3) + \psi_2(\delta_3; \delta_4) = \Psi \left( \frac{h(I_{\bar{\delta}^2 u}^{\delta 1})}{h(I)} \right) + \Psi \left( \frac{h(I_{\bar{\delta}^2 u}^{\delta 2})}{h(I)} \right) + \Psi \left( \frac{h(I_{\bar{\delta}^2 u}^{\delta 3})}{h(I)} \right) \quad (6.62)$$

Hence, the sequence $(\delta_1, \delta_4, \ldots, \delta_{2p+2}, \delta_1)$ is valid for the basis $I_{\bar{\delta}^2 u}^{\delta 1} \in \mathfrak{S}(\mathbf{L}(t))$ due to $(6.61)$ and gives rise to the same value for $\psi_2(\delta_1; \delta_1)$ by $(6.62)$ and Lemma 45. The length of the sequence $(\delta_1, \delta_4, \ldots, \delta_{2p+2}, \delta_1)$ is $2p$, therefore the inductive hypothesis applies, and the thesis is proved.

Finally, we can explicite the function $\psi$ mentioned in the thesis: let us fix an arbitrary set $\mathcal{I} \in \mathfrak{S}(\mathbf{L}(t))$, choose a representative $\overline{\mathcal{I}}_c$ for each equivalence class of $\overline{\mathcal{I}}$, and assign a $d$-tuple $\psi(\overline{\mathcal{I}}_c) \in \mathbb{Z}^d$ to every chosen representative. Then, for each $\alpha \in [n]$ belonging to the same class of $\overline{\mathcal{I}}_c$, we define

$$\psi(\alpha) := \psi(\overline{\mathcal{I}}_c) + \psi_2(\overline{\mathcal{I}}_c; \alpha). \quad (6.63)$$

The components of each pair $(i_u, \alpha_u)$ such that $I_{\alpha_u} \in \mathfrak{S}(\mathbf{L}(t))$ lie in the same equivalence class of $\overline{\mathcal{I}}$, then, we have

$$\tilde{\psi}_2(i_u; \alpha_u) = \psi_2(i_u; \overline{\mathcal{I}}_c) + \psi_2(\overline{\mathcal{I}}_c; \alpha_u) = \psi(\overline{\mathcal{I}}_c) - \psi(i_u) + \psi(\alpha_u) - \psi(\overline{\mathcal{I}}_c) = \psi(\alpha_u) - \psi(i_u). \quad (6.64)$$

In this way, we can evaluate $\Psi(\mathcal{J})$ from $\Psi(\mathcal{I})$ as follows: the Grassmann-Plücker relations imply that, for any $\alpha \in \mathcal{J} \setminus \mathcal{I}$, there exists $i \in \mathcal{I} \setminus \mathcal{J}$ with $I_i, J^\alpha_u \in \mathfrak{S}(\mathbf{L}(t))$. Then, we can choose an ordering $(\alpha_1, \ldots, \alpha_r)$ for the set $\mathcal{J} \setminus \mathcal{I}$ and construct a sequence of sets in $\mathfrak{S}(\mathbf{L}(t))$ via

$$\mathcal{T}_0 := \mathcal{J}, \quad \mathcal{T}_u := (\mathcal{T}_{u-1})_i^{\alpha_u} \quad (6.65)$$

where each $i_u$ satisfies $(\mathcal{T}_{u-1})_i^{\alpha_u}, \mathcal{T}_u^{i_u} \in \mathfrak{S}(\mathbf{L}(t))$. Clearly, $i_u = i_w$ implies $u = w$, since $u < w$ implies $i_u \in \mathcal{T}_{w-1}$ and $i_w \notin \mathcal{T}_{w-1}$. In conclusion, we can write

$$\Psi(\mathcal{J}) = \Psi(\mathcal{I}) + \sum_{u=0}^{r-1} \Psi(\mathcal{T}_u) - \Psi(\mathcal{T}_{u+1})$$

$$= \Psi(\mathcal{I}) + \sum_{u=0}^{r-1} \Psi(\mathcal{T}_u) - \Psi((\mathcal{T}_{u})_i^{\alpha_{u+1}})$$

$$= \Psi(\mathcal{I}) + \sum_{u=0}^{r-1} \Psi(\mathcal{T}_u) - \Psi((\mathcal{T}_{u})_i^{\alpha_{u+1}})$$

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minors of \((6.69)\) are invertible in the principal minors of \((6.69)\). Then, the monomial deformation hypothesis states that the principal this can be easily checked for values of \(k\).

For generic entries of \(A\), we note that (6.68) makes the terms (6.55) equivalent to the principal minors of (6.69). Then, the monomial deformation hypothesis states that the principal minors of (6.69) are invertible in \(C(t)\).

We can also find a family of counterexamples that does not satisfy the integrability condition \((1.10)\) when there exists an explainable local key.

\[
\Psi(I) + \sum_{u=0}^{r-1} \Psi(T^{u+1}_{\alpha_{u+1}}) - \Psi(I)
\]

\[
\Psi(I) + \sum_{u=1}^{r} \bar{\psi}(i_u; \alpha_u)
\]

\[
\Psi(I) + \sum_{u=0}^{r} \psi(\alpha_u) - \psi(i_u)
\]

getting (6.56) with \(m_0 := \Psi(I) - \sum_{u \in I} \psi(i_u)\).

\[\forall i \in I_u : h(I_u) \neq 0 \Rightarrow \forall w \neq u, j \in I_u : h(I_u) = 0. \quad (6.67)\]

For each surjective choice function \(\kappa : I \rightarrow [k]\), which defines a reduced row echelon form for \(L(t)_{[k] \times I}\), (6.67) means that the subspace spanned by the rows of \(L(t)\) in \(C^n\) is the direct sum of subspaces spanned by rows of \(L(t)_{\kappa(I_u) \times [n]}\), \(u \in [c]\). Vice versa, if the row span of \(L(t)\) is the direct sum of row spans of \(L(t)_{S_1 \times [n]}\) and \(L(t)_{S_2 \times [n]}\), with \(S_1 \cup S_2 = [k]\) and \(S_1 \cap S_2 = \emptyset\), then for each \(\mathcal{H} \in \mathcal{S}(L(t))\), \(\kappa : \mathcal{H} \rightarrow [k]\) surjective, and \(\alpha_u \in \kappa^{-1}(S_u)\) \((u \in [2])\) we get \(\alpha_1 \mathcal{H} \alpha_2\). Since the decomposition into direct sum of subspaces is independent on the choice of a basis from \(\mathcal{S}(L(t))\), for each \(\mathcal{H} \in \mathcal{S}(L(t))\) we have \(i \mathcal{H} \alpha = i \mathcal{H} \alpha\). Exchanging the roles of \(I\) and \(\mathcal{H}\), we get the reverse inclusion, thus \(\nabla_I = \nabla_{\mathcal{H}}\).

### 6.4 Counterexamples

The previous results highlight the information content that is required to recover additivity. In Section 5, where we have assumed (3.21), the amount of information is provided by the dimensionality of the matrices. When vanishing minors are considered, as in this section, the amount of information is provided by the existence of a local key (Assumption 39).

As in Example 28, we can find an example where the matroid \(\mathcal{S}(L(t))\) has not enough elements to guarantee additivity, i.e. Assumption 39 does not hold.

**Example 49.** Let us consider

\[
L_0 := (1_k | 1_k)
\]

so that \(\mathcal{S}(L_0)\) does not satisfy Assumption 39. Let \(S \in C^{k \times k}\) be a skew-symmetric constant matrix, \(1_k = (1, \ldots, 1)^T\) have all equal components in \(C^k\), and define

\[
r_0 := t \cdot 1_k \cdot 1_k^T + S
\]

\[
R_0 := (1_k | r_0)^T.
\]

For generic entries of \(A\), these matrices satisfy (6.55), but not (6.56), since the \(Y\)-terms are not constant: this can be easily checked for values of \(k \leq 6\). We note that (6.68) makes the terms (6.55) equivalent to the principal minors of (6.69). Then, the monomial deformation hypothesis states that the principal minors of (6.69) are invertible in \(C(t)\).

We can also find a family of counterexamples that does not satisfy the integrability condition (1.10) when there exists an explainable local key.
Proposition 50. We can find $R(t)$ that gives rise to a non-integrable configuration whether there exist a basis $\mathcal{I}$ and at most two indices $\alpha_1, \alpha_2 \in \mathcal{I}^C$ such that
\[ \exists i \in \mathcal{I}, \gamma_1, \gamma_2 \in \mathcal{I}^C, \gamma_1 \neq \gamma_2 : h(T_{\gamma_1}) \cdot h(T_{\gamma_2}) \neq 0 \quad \Rightarrow \quad \{\gamma_1, \gamma_2\} = \{\alpha_1, \alpha_2\}. \quad (6.71) \]

Proof. By Theorem 47, the lack of integrability entails that Assumption 39 does not hold; in particular, we will show that we can satisfy $(6.71)$ starting from a local key $\epsilon := \{i_1, i_2\} \times \{\alpha_1, \alpha_2\}$ and considering elements $(m_s, \omega_s) \in \mathcal{N}(\epsilon)$, so we can adopt the parametrisation $(6.17)$ with $\vartheta^2 := Y_{\alpha_1}^{i_1} \in \mathbb{C}$, namely
\[ Y_{\omega_1}^{i_1} := \tau_1^2 - 1, \quad Y_{\omega_2}^{i_1} := \frac{-\vartheta^2 \tau_1^2 - 1}{\vartheta^2}, \quad Y_{\omega_1}^{i_2} := \frac{-\vartheta^2 \tau_1^2 - 1}{\vartheta^2}, \quad Y_{\omega_2}^{i_2} := \frac{-\vartheta^2 \tau_1^2 - 1}{\vartheta^2}. \quad (6.72) \]

since the following argument can be adapted to the case where $Y_{\omega_1}^{i_1} = -\vartheta^2 \tau_1^2 - 1$ by transposition of operations on upper and lower indices. In line with The previous pattern returns
\[ Y_{\omega_1}^{i_1} := \tau_1^2 - 1, \quad Y_{\omega_2}^{i_1} := -Y_{\omega_1}^{i_1} = -Y_{\omega_2}^{i_1} = \frac{-\tau_1^2 - 1}{\tau_1} = \frac{-\tau_1^2}{\tau_1}. \quad (6.73) \]

As an initial remark, we note that, for each pair $(m_2, \omega_2) \in T\{i_1, i_2\} \times (\mathcal{I}^C)^{\alpha_1 \alpha_2}$, we get $m_1 \neq m_2$ and only if $\omega_1 \neq \omega_2$. Indeed, $m_1 = m_2 = m$ such that $h(T^m_{\omega_1}) \cdot h(T^m_{\omega_2}) \neq 0$ leads to the relation $Y_{\alpha_1}^{i_1} \cdot Y_{\alpha_2}^{i_2} = -Y_{\alpha_1}^{i_1} \cdot Y_{\alpha_2}^{i_2}$, which uniquely identifies $Y_{\alpha_1}^{i_1}$ independently of the choice of $s \in [2]$, due to the assumption $Y_{\alpha_1}^{i_1} \notin \mathbb{C}$ in the configuration $(6.17)$. So $Y_{\alpha_1}^{i_1} \in (\mathcal{I}^C)^{\alpha_1 \omega_1}$ is compatible with Remark 3 only if $\omega_1 = \omega_2$. The same argument holds for upper and lower indices by transposition.

Then, from $(6.13)$ and for all $(u, w) \in [2]^2$, we get the expressions
\[ m_{\alpha_1 \alpha_2}^{i_1 i_2} := \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 i} \right) \cdot \left( Y \left( T^m_{\omega_2} \right)^{\alpha_2 i} \right) \cdot \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 w} \right) + 1 \cdot \left( Y \left( T^m_{\omega_2} \right)^{\alpha_2 w} \right) \cdot \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 w} \right) + 1 \right) \right) \end{align*} \quad (6.74) \]
\[ m_{\alpha_1 \alpha_2}^{i_1 i_2} := \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 i} \right) \cdot \left( Y \left( T^m_{\omega_2} \right)^{\alpha_2 i} \right) \cdot \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 w} \right) + 1 \cdot \left( Y \left( T^m_{\omega_2} \right)^{\alpha_2 w} \right) \cdot \left( Y \left( T^m_{\omega_1} \right)^{\alpha_1 w} \right) + 1 \right) \right) \end{align*} \quad (6.75) \]

All the terms $Y_{\alpha_1 \alpha_2}^{i_1 i_2}$, $(i, \alpha) \in \epsilon$, lie in $F$: this has already been derived for $(m, \omega) \in \mathcal{N}(\epsilon)$, while for other cases we look at the relations $Y_{\alpha_1 \alpha_2}^{i_1 i_2} = -Y_{\alpha_1 \alpha_2}^{i_1 i_2} \cdot Y_{\alpha_1 \alpha_2}^{i_2 i_1} = -Y_{\alpha_1 \alpha_2}^{i_1 i_2} \cdot Y_{\alpha_1 \alpha_2}^{i_2 i_1}$. Being $Y_{\alpha_1 \alpha_2}^{i_1 i_2}, Y_{\alpha_1 \alpha_2}^{i_1 i_2} \in \mathbb{F}$, if $Y_{\alpha_1 \alpha_2}^{i_1 i_2} \notin \mathbb{F}$ we can use $(6.9)$ and find that $Y_{\alpha_1 \alpha_2}^{i_1 i_2} \cdot Y_{\alpha_1 \alpha_2}^{i_1 i_2}$ and $Y_{\alpha_1 \alpha_2}^{i_1 i_2} \cdot Y_{\alpha_1 \alpha_2}^{i_1 i_2}$ are proportional over $\mathbb{F}$, which implies
\[ \frac{\Delta_{\alpha_1 \alpha_2}^{i_1 i_2}}{(Q_{\alpha_1 \alpha_2}^{i_1 i_2})^2} = \frac{\Delta_{\alpha_1 \alpha_2}^{i_1 i_2}}{(Q_{\alpha_1 \alpha_2}^{i_1 i_2})^2}. \]

By the definition $(6.10)$ of $\Delta_{\alpha_1 \alpha_2}^{i_1 i_2}$, this relation is equivalent to
\[ \tau_1^{-4} = (Y_{\alpha_1 \alpha_2}^{i_1 i_2})^2 = \frac{(Q_{\alpha_1 \alpha_2}^{i_1 i_2})^2}{(Q_{\alpha_1 \alpha_2}^{i_1 i_2})^2} = \frac{Y_{\alpha_1 \alpha_2}^{i_1 i_2} Y_{\alpha_1 \alpha_2}^{i_2 i_1}}{Y_{\alpha_1 \alpha_2}^{i_1 i_2} Y_{\alpha_1 \alpha_2}^{i_2 i_1}} = Y_{\alpha_1 \alpha_2}^{i_1 i_2} Y_{\alpha_1 \alpha_2}^{i_2 i_1}. \]

This results in $Y_{\alpha_1 \alpha_2}^{i_1 i_2} = Y_{\alpha_1 \alpha_2}^{i_1 i_2}$ and contradicts Remark 3. Thus, there exists $\delta_{(u, w)} \in \mathbb{C}(t)$ satisfying $\Delta_{\alpha_1 \alpha_2}^{i_1 i_2} = \delta_{(u, w)}^2$.

Then, for any fixed $(m_2, \omega_2) \in \mathcal{N}(\epsilon)$, we have
\[ g_{(u, w)} := \frac{Y \left( T^m_{\omega_1} \right)^{\alpha_1 i} + 1}{Y \left( T^m_{\omega_2} \right)^{\alpha_1 i} + 1} = \frac{Y \left( T^m_{\omega_1} \right)^{\alpha_1 i} + 1}{Y \left( T^m_{\omega_2} \right)^{\alpha_1 i} + 1}. \]
Whenever $Y(I)_{\omega_2 \omega_1}^{m_1 m_2}$, $Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2}$ $\in \mathbb{C}$, we get $g_{(u, w)} \in \mathbb{C}$, which also entails
\[
g(1, 1) = Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1 \quad Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1 = \vartheta^4.
\] (6.76)

Setting $Y_{\omega_2 \omega_1}^{m_1 m_2} = : \xi^2$, we express $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ as
\[
\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2} = (\xi^2 + \tau_2^2 - \tau_2^2 - g(1, 1) \cdot \tau_2^2 \xi^2)^2 - 4 \tau_2^2 \xi^2 \cdot (\tau_2^2 - 1) \cdot (g(1, 1) \tau_2^2 - 1).
\] (6.77)

From (6.77), the quantity $q_2 := \xi^2 + \tau_2^2 - \tau_2^2 - g(1, 1) \cdot \tau_2^2 \xi^2$ verifies
\[
(q_2 - \delta(1, 1)) \cdot (q_2 + \delta(1, 1)) = 4 \tau_2^2 \xi^2 \cdot (\tau_2^2 - 1) \cdot (g(1, 1) \tau_2^2 - 1).
\]

The left-hand side of this expression is invertible in $\mathbb{C}(t)$, so both $q_2 - \delta(1, 1)$ and $q_2 + \delta(1, 1)$ are invertible too: therefore, $(q_2 - \delta(1, 1)) + (q_2 + \delta(1, 1)) = 2 \cdot q_2$ has sparsity at most 2. Being $\tau_2 \in \mathbb{C}$ and $g(1, 1) \in \mathbb{C}$, whenever $\xi \notin \mathbb{C}$, this constraint on the sparsity of $q_2$ is satisfied only if $\xi^2 = -\tau_2^2$ or $g(1, 1) \tau_2^2 \xi^2 = -1$; these two conditions are equivalent through the change of basis $I \mapsto T_{\alpha_1}^{m_1}$, which acts as $\tau_2 \mapsto \tau_1^{-2}$ and $\xi^2 \mapsto g(1, 1) \xi^2$. Focusing on the pair $(i_2, \alpha_1)$, it induces the transformations $\tau_2 \mapsto -\vartheta^4 \tau_2^2$ and $g(1, 1) \mapsto g(2, 1) = \vartheta^4 g(1, 1)$, while $\xi^2$ does not change; then, looking again at (6.77) to evaluate $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$, the resulting constraints are $\xi^2 = \vartheta^2 \tau_2^2$ or $g(1, 1) \vartheta^2 \tau_2^2 \xi^2 = 1$, which are not compatible with the ones obtained from $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ at $\omega^2 \in \mathbb{C} \setminus \{-1\}$. Therefore, we infer $\xi \in \mathbb{C}$: this condition returns a polynomial $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ with degree at most 2 in $\tau_1^2$. When $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ is quadratic in $\tau_2^2$, its discriminant is
\[
16 \cdot Y(I)_{\omega_2 \omega_1}^{m_1 m_2} Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} Y(I)_{\alpha_1 \omega_1}^{m_1 m_2} Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} Y(I)_{\omega_2 \omega_1}^{m_1 m_2} - 1 \cdot Y(I)_{\omega_2 \omega_1}^{m_1 m_2} - 1 \neq 0
\]
due to Remark 3. So $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ can be a perfect square in $\mathbb{C}(t)$ only if it is invertible; from (6.77), this can happen only if $q_2 = 0$, i.e. $Y_{\omega_2 \omega_1}^{m_1 m_2} + 1 = \tau_2^2 = \xi^2$ and $g(1, 1) = \xi^{-4}$. Also in this situation, we can consider $\Delta_{\alpha_1 \omega_1 \omega_2}^{m_1 m_2}$ to get analogous conditions: in particular, $Y_{\omega_2 \omega_1}^{m_1 m_2} + 1 = \xi^2 = Y_{\omega_2 \omega_1}^{m_1 m_2} + 1$, which means $Y_{\omega_2 \omega_1}^{m_1 m_2} = -1$, i.e. a contradiction.

Now, we set $\chi_w := \delta_{1, w} - \delta_{2, w} \cdot \vartheta^2$, where $w \in [2]$ and $\delta_{i, w}$ is the Kronecker Delta, and express (6.8) in the equivalent form
\[
1 + Y_{\omega_1 \alpha_2}^{m_2} + Y_{\omega_1 \alpha_2}^{m_2} + Y_{\omega_2 \alpha_1}^{m_2} \cdot (1 + Y_{\omega_1 \alpha_2}^{m_2} + Y_{\omega_2 \alpha_1}^{m_2}) = m_{\omega_1 \alpha_2}^{m_1 m_2}.
\] (6.78)

This relation lets us exclude the configuration where $\psi(Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1) = \psi(Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1)$ and $\psi(Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1) = \psi(Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1)$ both hold; indeed, from (6.74)-(6.75), these constraints would return
\[
\frac{m_{\omega_2 \omega_1}^{m_1 m_2}}{m_{\omega_2 \omega_1}^{m_1 m_2}} = \frac{Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1}{Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1} \cdot \frac{Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1}{Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1} = -\vartheta^2 (\vartheta^{-2} + 3) \cdot (\chi_w \tau_2^2)^2
\]
\[
= \frac{Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1}{Y(T_{\omega_1}^{m_1})_{\omega_2 \alpha_2}^{m_2} + 1} \cdot \frac{Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1}{Y(I)_{\omega_2 \omega_1}^{m_1 m_2} + 1} = -\vartheta^2 (\vartheta^{-2} + 3) \cdot (\chi_w \tau_2^2)^2
\] (6.79)

for both $u \in [2]$; therefore, we infer $\vartheta^4 (\vartheta^{-2} + 3) = \vartheta^2 (\vartheta^{-2} + 3)$. Furthermore, we get
\[
\frac{g(1, 2)}{g(1, 2)} = \frac{Y(T_{\omega_1}^{m_1})_{\omega_1 \omega_2}^{m_1 m_2}}{Y(T_{\omega_1}^{m_1})_{\omega_1 \omega_2}^{m_1 m_2}} = (g(1, 1) \tau_2^2)^{-1} \tau_2^2 = g(1, 1).
\] (6.80)

From (6.78) instantiated at $(a_1, a_2, a_3) = (m_2, m_1, i_1)$ and $(\delta_1, \delta_2, \delta_3) = (\omega_2, \omega_1, \alpha_w)$, we get
\[
-1 + \xi^2 + \chi_w \cdot \vartheta^2 + (\tau_2^2)^{-1} - 1 \cdot (1 + Y_{\omega_1 \omega_2}^{m_2} + Y_{\omega_1 \omega_2}^{m_1 m_2}) = m_{\omega_1 \omega_1}^{m_1 m_2}
\] (6.81)
Let \( t_1 \) be any root of \( \tau _1^2 - 1 \), and denote as \( t_2^2, z^2 \), and \( \hat{g}_{(u,w)} \) the evaluation of \( \tau _2^2, \xi^2 \), and \( g_{(u,w)} \) at this point, respectively. Evaluating (6.81) at \( t_1 \) for both \( w \in [2] \), from (6.79)-(6.80) we get

\[
-1 + z^2 + t_2^2 = \varepsilon \cdot \vartheta^2 \cdot z^2 t_2^3, \quad -1 + z^2 - \vartheta^2 t_2^2 = \varepsilon \cdot \vartheta^2 \cdot z^2 (-\vartheta^2 t_2^2)
\]

for a given \( \varepsilon \in \{1,-1\} \). Solving for \( t_2^2 \) and \( \vartheta^2 \), we find \( \vartheta^2 = -1 \), which contradicts \( Y_{i_1i_2}^{1} \neq -1 \). Having proved that at least one between \( Y(I)_{i_2m_2} \) and \( Y(I)_{i_1m_1} \) is not constant, this argument lets us consider a shift of the type

\[
\mathcal{I} \leftarrow \mathcal{J} \in \{ \mathcal{I}, \mathcal{I}^{m_1} \mathcal{I}^{m_2}, \mathcal{I}^{m_1 m_2} \}
\]

(6.82)

such that, in the updated basis, we get \( Y(I)_{i_2m_2}^{1}, Y(I)_{i_1m_1}^{1} \notin \mathcal{C} \) and \( Y(I)_{i_1m_2}^{1} + 1 = (Y(I)_{i_2m_2}^{1} + 1)^{-1} \).

Focusing on \( \kappa := (m_{i_2m_2}^{1})^{-1} m_{i_1m_1}^{1} \), the condition \( \kappa \neq 1 \) means that \( Y(I)_{i_2m_2}^{1} \in \mathcal{C} \) if and only if \( Y(I)_{i_2m_2}^{1} \in \mathcal{C} \) and, from (6.74)-(6.75) we infer \( \tau _1^2, \tau _2^2 \in \mathcal{C} \). Also here, we can consider (6.78) instantiated at \( (a_1, a_2, a_3) = (m_2, m_1, i_1) \) and \( (\delta_1, \delta_2, \delta_3) = (\omega_2, \omega_1, \alpha_0) \) for both \( w \in [2] \), then we evaluate these expressions at \( t_1 \) and denote as \( C \) the evaluation of the ratio \( (m_{i_2m_2}^{1})^{-1} \cdot m_{i_1m_1}^{1} \), obtaining

\[
-1 + t_2^2 + z^2 = C \cdot (-1 + t_2^2 + z^2) = m_{i_1m_1}^{1} \mid t_1 \neq 1.
\]

Remark 3 entails \( \mu \neq 1 \). and, together with the condition \( -1 + t_2^2 + x^2 \neq 0 \), this excludes the configuration derived from \( Y(I)_{i_2m_2}^{1} \notin \mathcal{C} \). However, this would imply that \( Y(I)_{i_2m_2}^{1} \in \mathcal{C} \) and, hence, \( Y_{i_1m_1}^{1} \), contradicting the original hypothesis \( Y_{i_1m_1}^{1} \notin \mathcal{C} \). Therefore, \( \mu = 1 \).

Finally, we solve the system

\[
\frac{m_{i_1m_1}^{1}}{m_{i_2m_2}^{1}} = K_w \cdot m_{i_1m_1}^{1}, \quad w \in [2]
\]

in the indeterminates \( Y(I)_{i_2m_2}^{1}, Y(I)_{i_1m_1}^{1} \) with coefficients \( K_1, K_2 \in \mathcal{C} \), these two \( Y \)-terms lie in \( \mathcal{C} \) too, as well as \( Y(I)_{i_1m_1}^{1} \) and, hence, \( Y_{i_1m_1}^{1} \), contradicting the original hypothesis \( Y_{i_1m_1}^{1} \notin \mathcal{C} \). Therefore, \( \mu = 1 \).

This configuration gives

\[
m_{i_1m_1}^{1} = \frac{(\mu \cdot t_2 \tau _1 + \tau _1 \tau _2)}{t_1^2 \tau _2^2}.
\]

Being \( Y_{i_1m_1}^{1}, Y_{i_1m_2}^{1} \neq 0 \) by hypothesis, we find \( \mu \tau _2 \tau _1 \neq 0 \neq \mu \tau _1 - \tau _2 \); therefore, \( m_{i_2m_2}^{1} \) is invertible in \( \mathcal{C}(t) \) only if \( \mu \tau _2 \tau _1, \tau _1^2, \) and \( \tau _2^2 \) are proportional over \( \mathcal{C} \). So, we get

\[
m_{i_1m_1}^{1} \cdot m_{i_2m_2}^{1} \in \mathcal{C}
\]

and, furthermore, we can fix a monomial \( \zeta \in \mathcal{C}(t) \) such that \( \tau _u = c_u \cdot \zeta \) with \( c_u \in \mathcal{C} \) and \( u \in (i_1,i_2)^{2} \).

We extend this definition setting \( c_u := 0 \) at \( u \in (i_1,i_2)^{2} \) and \( c_u := 1 \). In this way, we can extend the initial observation to \( m_s \neq m_t \) if and only if \( \omega_s \neq \omega_t \) for all the pairs \( (m_s, \omega_s) \) such that \( h(I)_{j_2}^{1} \neq 0 \).

This argument can be repeated to recover the same form for each pair of indices \( i_s, i_w \) such that

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\{i_s, i_w\} \times \{\alpha_1, \alpha_2\} is a local key. In particular, we can have a given set \( \mathcal{C} \subseteq I \) with \( \#\mathcal{C} = D \) such that \( \{i_1, i_s\} \times \{\alpha_1, \alpha_2\} \) is a local key with associated term \( Y_{i_1 i_s}^{\alpha_1 \alpha_2} = \vartheta_s^2 \), \( s \in \mathcal{C} \). In this way, we get a non-integrable configuration, where the obstruction to integrability can be “localised” on a fixed set of entries in \( R(t) \): indeed, we change the gauge (3.11) considering a diagonal matrix \( Z \) acting via (3.10), so \( R(t) \) takes the block form

\[
R(t) = \begin{pmatrix}
- (\vartheta_s^2 + 1)^{-1} \tau^{-2} & e_1 \times D & 0_{1 \times (k-2)} & 0 & -1 & Z^{(2)} \\
1_{D \times 1} & s_{D \times D} & 0_{D \times (k-2)} & 1_{D \times 1} & 0_{D \times 1} & 1_{D \times (n-k-2)}
\end{pmatrix}
\]

where, setting \( \mu_{1,u,w} := \Delta_{1u}^m w_\omega \), we specify

\[
(Z^{(2,2)})_{m_u \omega_u} := 1 + \text{sign}(w - u) \cdot \frac{c_{wH(u,w)}}{c_u}, \quad m_u \in T_{i_1} \setminus \mathcal{C}, \quad w_\omega \in (i_{\alpha \beta})^\mathcal{C},
\]

\[
(Z^{(2)})_{i_1 u} := -(Z^{(2)})_{m_u \omega_u} \cdot e_u^2,
\]

\[
s_{s_1 s_2} := \delta_{s_1 s_2} \cdot (\vartheta_s^2 + 1)^{-1} = 1 - \delta_{s_1 s_2} \cdot 1_{s_1},
\]

\[
e_s := \delta_{s_1} \cdot (\vartheta_s^2 + 1)^{-1} \tau^{-2}.
\]

7 Applications and future work

This work focused on the exploration of an algebraic condition related to the sparsity of deformations of determinantal expansions for the complexity reduction of integer-valued set functions. As mentioned in the Introduction, this model is equivalent to the study of configurations compatible with the invertibility of principal minors of a variable matrix in a given ring.

During the investigation, some relevant issues and connections have arisen, which drive future studies. We conclude this paper briefly mentioning some of these applications.

7.1 Complexity reduction from permutations of sets to permutations of elements

The first implication of the previous results regards the communication of information of a given combinatorial structure: specifically, we can embed the information on permutations of elements of a given set in permutations of subsets. This lets us take advantage of the apparent complexity arising from this embedding, still allowing to recover the original permutation. The retrieval of the original permutation can be done through the control matrix \( R(t) \), as we now discuss.

First, we notice that Assumption 30, which entails Theorem 47, holds independently of the exponents \( \Psi(I) \) and the matrix \( R(t) \), since the existence of an explainable local key only regards the independence structure (i.e. the matroid) of \( L(1) \). This also reflects in the conclusion (6.56), where the matrix \( A \) is independent of the exponents \( \psi(1) \) and the matrix \( Q \). Proposition 50 exploits the possibility to choose the matrix \( R(t) \) in order to construct a counterexample for Theorem 47 when Assumption 39 does not hold. Here, Thus, we use the knowledge that the structural Assumption 39 is satisfied, considering multiple models derived from different choices of \( \Psi \) and \( R(t) \).

We now adapt each permutation \( \hat{\Psi} : \mathfrak{S}(L(t)) \rightarrow \mathfrak{S}(L(t)) \) to our formalism through the composition

\[
\Psi := 1 \circ \hat{\Psi} : \varphi_k[n] \rightarrow \{0, 1\}^n
\]
where 1 is the map associating each set with its characteristic function. This defines a set function with values in $\mathbb{Z}^n$. With these premises, we have the following:

**Corollary 51.** Let $\hat{\Psi} : \mathfrak{S}(A(t)) \rightarrow \mathfrak{S}(A(t))$ be a permutation. Using the correspondence (7.1), for any mapping $g : \varphi_k[n] \rightarrow \mathbb{C}$ there exist two $(k \times n)$-dimensional matrices $A(t), Q^2(t)$ satisfying

$$
\Delta_{A(t)}(I) \cdot \Delta_{Q^2(t)}(I) = g(I) \cdot t^{\Psi(I)}, \ I \in \varphi_k[n],
$$

(7.2)

if and only if there exist matrices $a, q^2 \in \mathbb{C}^{k \times n}$ and a permutation $\psi \in S_n$ such that the pairs $(A(t), Q(t))$ and $(a, \text{diag}(t_{\psi(1)}, \ldots, t_{\psi(n)}) \cdot q)$ have the same Cauchy-Binet expansion, i.e.

$$
\Delta_{A(t)}(I) \cdot \Delta_{Q(t)}(I) = \Delta_a(I) \cdot \Delta_q(I) \cdot \prod_{\alpha \in I} t_{\psi(\alpha)}, \ I \in \varphi_k[n].
$$

(7.3)

From an information-theoretic perspective, we are assuming the existence of $a$, even if this matrix is unknown, so $g(I) = \Delta_a(I) \cdot \Delta_q(I)$, where $q = Q(I)$. In the following discussion, we will use the same symbol for $\Psi$ and $\hat{\Psi}$ in (7.1) with a slight abuse of notation.

Corollary 51 suggests an approach to check if a given permutation $\Psi$ of $\mathfrak{S}(a)$ is induced by a permutation $\psi \in S_n$: the information on the permutation $\Psi$ is encoded in the products $g(I) \cdot t^{\Psi(I)}$ in (7.2) and, based on the previous observation, we have the freedom to choose different matrices $Q(t)$ and evaluation points $t_0$ in order to infer information on $\Psi$ from these products. So, we will refer to a given choice of a matrix $q$ and evaluation points $t_0$ as a *query*, which returns the unordered multi-set of values $g(I) \cdot t_0^{\Psi(I)}, I \in \varphi_k[n]$. For a given query $(q, t_0)$ and a permutation $\Psi \in \mathcal{S}_{\mathfrak{S}(A(t))}$, we set

$$
\mathcal{G}(\Psi; q, t_0) := \left\{ \left| g(I) \cdot t_0^{\Psi(I)} \right| \right\} \setminus \{0\}, \ \Delta(\Psi; q, t_0) := \sum_{I \in \varphi_k[n]} g(I) \cdot t_0^{\Psi(I)}.
$$

(7.4)

A possible approach to check if $\Psi$ is induced by a permutation $\psi$ of the elements of $[n]$ proceeds as follows:

1. Using a query $(q_*, t_*)$, we gain information on $a$, specifically, the bounds $\lambda := \min\{|\Delta_a(I)| \setminus \{0\}$ and $\mu := \max\{|\Delta_a(I)|\}$. As will be manifest in the next steps, this information allows choosing evaluation points $t_0$ “generically”, i.e. in such a way that different permutations $\Psi_1 \neq \Psi_2$ produce different sums $\Delta(\Psi_1; q, t_0) \neq \Delta(\Psi_2; q, t_0)$.

2. We choose $q$ so that

$$
\frac{\mu}{\lambda} < \min \left\{ |\Delta_q(J)^{-1} \cdot \Delta_q(I)| : J, I \in \varphi_k[n], I \neq J, |\Delta_q(J)| \leq |\Delta_q(I)| \right\}.
$$

(7.5)

A possible choice is given by a Vandermonde matrix with a suitable choice of parameters. This entails $|\Delta_a(J)\Delta_a(I)^{-1}| < |\Delta_q(I)\Delta_q(J)^{-1}|$, whenever $I \neq J$ and $|\Delta_q(J)| \leq |\Delta_q(I)|$; as a consequence, all the non-vanishing coefficients $g(I)$ are different.

3. Subsequently, we denote $\lambda_q := \min\{|\Delta_q(I)|\}$ and $\mu_q := \max\{|\Delta_q(I)|\}$, and choose $t_0$ so that $|t_{0, n+1}|/|t_{0, n}| \gg \delta := 2\mu\mu_q/(\lambda\lambda_q)$; this provides an order for $\mathcal{G}(\Psi; q, t_0)$ that is compatible with the lexicographic order induced by $|t_{0, 1}| < \cdots < |t_{0, n}|$ and makes the sums $\sum_{g \in Z} g$ for $Z \subseteq \mathcal{G}(\Psi; q, t_0)$ pairwise different. In particular, the order obtained from this query produces the mapping $\Gamma : \varphi_k[n] \rightarrow \mathbb{C}$ such that $\Gamma(I) := g_{\Psi^{-1}(I)}$: being the terms in $\mathcal{G}(\Psi; q, t_0)$ distinct, $\Gamma_0 := \Gamma|_{\Gamma^{-1}(\mathbb{C} \setminus \{0\})}$ is injective.
4. Now, we define $I^{(0)} := \Gamma_0^{-1}(\arg\min\mathcal{G}(q; t_0))$ and, for all $u \in [k + 1; n]$, 
$$T_u := \delta \cdot \prod_{\beta = u - k}^{u - 1} |t_0, \beta|, \quad I^{(u)} := \Gamma_0^{-1}(\arg\min\mathcal{G}(q; t_0) \cap [T_u, \infty]).$$

If $\Psi$ is induced by a permutation $\psi_{hp} \in S_n$, then
$$\forall u \in [k + 1; n] : \quad I^{(u)} \setminus I^{(u - 1)} = \{\psi_{hp}^{-1}(u)\}.$$  \hspace{1cm} (7.6)

The condition (7.6) is necessary for the existence of $\psi_{hp} \in S_n$ inducing $\Psi$. Similarly, we can iteratively construct $\psi_{hp}^{-1}(u)$ for all $u \in [k]$ from the sets $\mathcal{G}(q; t_0) \cap [\Gamma(I^{(u - 1)}), \Gamma(I^{(u)})]$. 

5. If we pass the checking in the previous step, then we also find $\psi_{hp}^{-1}$ and, hence, we recover $\psi_{hp}$. Then, we can verify with a single check if $\Psi$ coincides with the function $\Psi_{hp} := \sum_{u \in I} \psi(u)$ constructed from $\psi_{hp}$ by additivity: indeed, only $\Psi_{hp}$ can produce a determinant expansion by Corollary 51, whose hypotheses can be checked with the knowledge we have recovered in the previous steps, and $\Delta(\Psi; t_0) = \Delta(\psi_{hp}; t_0)$ only if $\Psi = \psi_{hp}$. Thus, we can control this last equality to tell if $\Psi$ is induced by a permutation $\psi \in S_n$ and, if so, returns such a permutation $\psi_{hp}$.

Additional a priori knowledge on the domain of the entries of $a$, e.g. $Z$ or an algebraic number field, could be used to extend the previous argument in the context of secure communication, enabling specific methods to select the queries based on the available information.

7.2 Chu Spaces

The previous subsection deals with standard set membership. Theorem 47 may be considered as a generalisation encompassing non-standard set membership, which includes multisets and hybrid sets [10]. More generally, the present model can be viewed in terms of non-classical set membership. In particular, we this framework lends itself to connections with Chu spaces [15]: they are defined as triples $(\mathcal{E}, r, S)$ where $\mathcal{E}$ and $S$ are sets of points and states, respectively, and $r : \mathcal{E} \times S \rightarrow K$ is a $K$-valued relation between points and states, where $K$ is the referred to as the valuation set.

We remark that, using the dictionary of Chu spaces, a matrix $L(t)$ without repeated columns is said extensional. In the present context, we may say that a matrix $L(t) \in C(t)^{k \times n}$, $k \leq n$, is “projectively extensional” if it has no proportional columns: such a property is relevant in our framework, since it provides a condition (Assumption 30) for the existence of the reduction (6.56), as it is shown in Example 29. Non-classical behaviours generalising the membership relation, as well as (non-)extensional representations, were also discussed in the context of tropical algebra in [2].

This digression on Chu spaces is also motivated by the connection between the definition of integrability discussed in this paper and adjointness conditions: given two Chu spaces $(\mathcal{E}_u, r_u, S_u)$, $u \in [2]$, a Chu morphism between them is defined as a pair of functions $(f^-, f^+)$ satisfying the adjointness condition
$$r_1(\alpha, f^+=IJ) = r_2(f^-(\alpha), J)$$  \hspace{1cm} (7.7)
(f^-, f^-) is a Chu morphism between ([n], Ψ_*, ϕ_k[n]) and the usual membership space ([n], ∈, ϕ_k[n]).

7.3 Factors of sparse hyperdeterminants

The emergence of the hyperdeterminant (35) and its factorisation properties in $\mathbb{C}(t)$ to infer algebraic properties of $Y$-term deserve more attention. Indeed, these properties open up the way to deviations from integrability when Assumption 39 is not satisfied, as it is shown in Example 49.

In this regard, we have already noticed the relevance of the squarefree parts and common factors of hyperdeterminants, here expressed as sparse polynomials in $\mathbb{C}(t)$, obtained from invertible principal minors. This object may connect to other topics of interest in different, but related domains where hyperdeterminants are efficiently used, i.e. the study of polynomial relations among principal minors of a generic matrix [9], and the entropy region of Gaussian random variables [16].

Moreover, hyperdeterminants play a major role in quantum theories [12], as well as Chu spaces [1, 14]. This suggests us to further investigate algebraic models and (deviations from) the above-mentioned combinatorial notion of integrability to explore some relevant features of quantum models such as entanglement and contextuality. We will discuss these issues in a future work.

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