Optimal preconditioners for systems defined by functions of Toeplitz matrices

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Abstract

We propose several circulant preconditioners for systems defined by some functions $g$ of Toeplitz matrices $A_n$. In this paper we are interested in solving $g(A_n)x = b$ by the preconditioned conjugate method or the preconditioned minimal residual method, namely in the cases when $g(z)$ are the functions $e^z$, $\sin z$ and $\cos z$. Numerical results are given to show the effectiveness of the proposed preconditioners.

Keywords: Toeplitz matrices, Functions of matrices, Circulant preconditioners, PCG, PMINRES

MSC: 65F08, 65F10, 65F15, 15A16, 15B05

1. Introduction

Motivated by [16] in which the authors proposed some optimal preconditioners for certain functions of general matrices, we show that $g(c(A_n))$, where $c(A_n)$ is the optimal circulant preconditioner for $A_n$ first proposed in [9], is an effective preconditioner for $g(A_n)$. Specifically we are interested in the cases when $g(z)$ are the trigonometric functions $e^z$, $\sin z$ and $\cos z$.

A crucial application of $e^{A_n}$, for example, arises from the discretisation of integro-differential equations with a shift-invariant kernel [17]. Solving those equations is often required in areas such as the option pricing [12, 28]. Related work on computing the exponential of a block Toeplitz matrix arising in approximations of Markovian fluid queues can also be found in [2].
Over the past few decades, preconditioning for Toeplitz matrices with circulant matrices has been extensively studied. Strang [30] and Olkin [22] were the first to propose using circulant matrices as preconditioners in this context. Theoretical results that guarantee fast convergence with circulant preconditioners were later given by [7]. Other circulant preconditioners such as optimal preconditioners [9], Huckle’s preconditioners [14] and superoptimal preconditioners [31] were then developed for certain classes of Hermitian and positive definite Toeplitz matrices generated by positive functions $f$. The restriction on $f$ was later relaxed for example in [5, 29, 11]. Some work had also been done on preconditioning for Hermitian indefinite Toeplitz systems [6], non-Hermitian Toeplitz systems [15] and nonsymmetric Toeplitz systems [25]. For references on the development of preconditioning of Toeplitz matrices we refer to [20, 4].

Throughout this work we consider the case when $f$ is a $2\pi$-periodic continuous complex-valued function as analysed in [8], thus the corresponding Toeplitz matrix $A_n[f]$ is in general complex and non-Hermitian. Consequently $g(A_n[f])$ is also a non-Hermitian complex matrix. We let $c_n[f]$ be the optimal circulant preconditioner [9] derived from $A_n[f]$. Using $g(c_n[f])$ as the preconditioner, we can then apply CG to the normal equations system

$$[g(c_n[f])^{-1}g(A_n[f])^*g(c_n[f])^{-1}g(A_n[f])]x = [g(c_n[f])^{-1}g(A_n[f])^*g(c_n[f])^{-1}b].$$

We also consider the special case in which we can use MINRES for the Hermitian indefinite $g(A_n[f])$ with the preconditioner $g(c_n[f])$.

Given a circulant matrix $C_n$, we remark that $g(C_n)$ is also a circulant matrix. By the diagonalisation $C_n = F_n^*A_nF_n$, where $F_n$ is a Fourier matrix [10] in which the entries are given by $[F_n]_{jk} = \frac{1}{\sqrt{n}}e^{-2\pi ijk/n}$ with $j, k = 0, 1, \ldots, n - 1$, we have

$$g(C_n) = F_n^*g(A)F_n.$$  

Therefore, for any vector $d$ the product $g(C_n)^{-1}d$ can be efficiently computed by several Fast Fourier Transforms (FFTs) in $O(n \log n)$ operations [3].

It must be noted however that fast matrix vector multiplication with the matrix $g(A_n[f])$ is not readily archived by circulant embedding. For $e^{A_n[f]}$, the matrix vector multiplication can be computed efficiently for example by a fast algorithm in $O(n \log n)$ operations [18].

This paper is outlined as follows. In section 2 we first provide some lemmas on bounds for the spectra of $A_n[f]$ and $c_n[f]$. In section 3 we also give several lemmas on functions of matrices. In section 4 we provide the
main results on the preconditioned matrix \( g(c_n[f])^{-1}g(A_n[f]) \). In section 5 we present numerical results to demonstrate the effectiveness of the proposed preconditioners.

2. Spectra of \( c_n[f] \) and \( A_n[f] \)

Denote by \( C_{2\pi} \) the Banach space of all \( 2\pi \)-periodic continuous complex-valued functions equipped with the supremum norm \( \| \cdot \|_\infty \). For all \( f \in C_{2\pi} \), we let

\[
a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)e^{-ik\theta} d\theta, \quad k = 0, \pm 1, \pm 2, \ldots,
\]

be the Fourier coefficients of \( f \). Let \( A_n[f] \) be the \( n \)-by-\( n \) complex Toeplitz matrix with the \((j,k)\)-th entry given by \( a_{j-k} \). The function \( f \) is called the generating function of the matrix \( A_n[f] \). We then have

\[
A_n[f] = \begin{bmatrix}
a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \\
a_{-1} & a_0 & a_{-1} & \cdots & a_{-n+2} \\
\vdots & a_1 & a_0 & \cdots & \vdots \\
a_{n-2} & \cdots & \cdots & a_1 \\
a_{n-1} & a_{n-2} & \cdots & a_1 & a_0
\end{bmatrix}.
\]

We also let \( c_n[f] \) be the \( n \)-by-\( n \) optimal circulant preconditioner \([9]\) for \( A_n[f] \), namely

\[
c_n[f] = \begin{bmatrix}
c_0 & c_{n-1} & \cdots & c_2 & c_1 \\
c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
\vdots & c_1 & c_0 & \cdots & \vdots \\
c_{n-2} & \cdots & \cdots & c_1 \\
c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{bmatrix}
\]

defined by

\[
c_k = \begin{cases} 
(n-k)a_k + ka_{k-n}, & 0 \leq k < n, \\
c_{n+k}, & -n < k < 0.
\end{cases}
\]

Lemma 2.1. \([8, Lemma 1 and 3]\) If \( f \in C_{2\pi} \) we have

\[
\|A_n[f]\|_2 \leq 2\|f\|_\infty \quad \text{and} \quad \|c_n[f]\|_2 \leq 2\|f\|_\infty \quad n = 1, 2, \ldots.
\]
Lemma 2.1 states that the 2-norm of the circulant matrix and that of the Toeplitz matrix generated by \( f \) are bounded by a constant which is independent of \( n \).

**Lemma 2.2.** [[8, Theorem 1]] Let \( f \in C_{2\pi} \). Then for all \( \epsilon > 0 \) there exists a positive integer \( M > 0 \) such that for \( n > 2M \), we have

\[
c_n[p_M] - A_n[p_M] = U_n - W_n,
\]

where \( p_M \) is a trigonometric polynomial such that \( \|f - p_M\|_\infty < \epsilon \), \( A_n[p_M] \in \mathbb{C}^{n \times n} \) is the Toeplitz matrix generated by \( p_M \), \( c_n[p_M] \in \mathbb{C}^{n \times n} \) is the optimal circulant preconditioner for \( A_n[p_M] \),

\[
\text{rank} \ U_n \leq 2M
\]

and

\[
\|W_n\|_2 \leq \frac{1}{n} M(M + 1)(\epsilon + \|f\|_\infty).
\]

Lemma 2.2 indicates that the difference between the circulant matrix and the Toeplitz matrix generated by a trigonometric approximation to \( f \) can be decomposed into the sum of a matrix of low rank and a matrix of small norm. In the next section, this lemma is used to prove that the difference between the matrix exponential of a circulant matrix and that of a Toeplitz matrix can also be decomposed in a similar fashion.

### 3. Preliminaries on matrix functions

In this section we introduce the preliminaries that will be used in the following section.

**Definition** [[13]] For any \( A_n \in \mathbb{C}^{n \times n} \),

\[
e^{A_n} = I_n + A_n + \frac{1}{2!} A_n^2 + \frac{1}{3!} A_n^3 + \cdots,
\]

\[
\cos A_n = I_n - \frac{1}{2!} A_n^2 + \frac{1}{4!} A_n^4 - \frac{1}{6!} A_n^6 + \cdots
\]

and

\[
\sin A_n = A_n - \frac{1}{3!} A_n^3 + \frac{1}{5!} A_n^5 - \frac{1}{7!} A_n^7 + \cdots.
\]
Lemma 3.1. [13, Theorem 10.2] For any $A_n, B_n \in \mathbb{C}^{n \times n}$
\[ e^{(A_n+B_n)t} = e^{A_n t} e^{B_n t} \]
for all $t$ if and only if $A_n B_n = B_n A_n$.

Definition [13] Given a vector norm on $\mathbb{C}^n$, the corresponding \textit{subordinate matrix norm} is defined by
\[ \| A_n \| = \max_{x \neq 0} \frac{\| A_n x \|}{\| x \|}. \]

Definition [13] The norm $\| \cdot \|$ is called \textit{consistent} if
\[ \| AB \| \leq \| A \| \| B \| \]
for all $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$.

Lemma 3.2. [13, Problem 10.3] For any subordinate matrix norm and any $A_n, B_n \in \mathbb{C}^{n \times n}$, we have
\[ \| e^{A_n} - e^{B_n} \| \leq \| A_n - B_n \| e^{\max(\| A_n \|, \| B_n \|)}. \]

Lemma 3.3. [13, Theorem 10.1] For $A_n \in \mathbb{C}^{n \times n}$, let
\[ P_{r,s} = \sum_{i=0}^{r} \frac{1}{i!} \left( \frac{A_n}{s} \right)^i s. \]

Then for any consistent matrix norm
\[ \| e^{A_n} - P_{r,s} \| \leq \| A_n \|^{r+1} \frac{e^{\| A_n \|}}{s^r (r+1)!} \]
and
\[ \lim_{r \to \infty} P_{r,s} = \lim_{s \to \infty} P_{r,s} = e^{A_n}. \]

Lemma 3.4. [13, Theorem 10.10] For $A_n \in \mathbb{C}^{n \times n}$ and any subordinate matrix norm,
\[ e^{-\| A_n \|} \leq \| e^{A_n} \| \leq e^{\| A_n \|} \quad n = 1, 2, \ldots. \]

Lemma 3.5. [24] For any circulant matrix $C_n \in \mathbb{C}^{n \times n}$, the \textit{absolute value circulant matrix} $|C_n|$ is defined to be
\[ |C_n| = (C_n^* C_n)^{\frac{1}{2}} = (C_n C_n^*)^{\frac{1}{2}} = F_n^* |\Lambda_n| F_n, \]
where $|\Lambda_n|$ is the diagonal matrix in the eigenvalue decomposition of $C_n$ with all entries replaced by their magnitudes.
4. Spectra of the preconditioned matrices

In this section, for Hermitian $g(A_n[f])$ we show the preconditioned matrix

$$g(c_n[f])^{-1}g(A_n[f])$$

can be decomposed into the sum of a unitary matrix, a matrix of low rank and a matrix of small norm for sufficiently large $n$ under some assumptions. For non-Hermitian $g(A_n[f])$, we consider its normal equations system and also show that the preconditioned matrix can also be decomposed in a similar way.

4.1. Spectra of $(e^{c_n[f]} - e^{A_n[f]})^{-1}$

We first provide some lemmas concerning to the matrix exponential and then give the main results concerning to the preconditioned matrix $(e^{c_n[f]} - e^{A_n[f]})^{-1}$.

**Corollary 4.1.** Let $f \in C_{2\pi}$. Let $A_n[f] \in \mathbb{C}^{n \times n}$ be the Toeplitz matrix generated by $f$ and $c_n[f] \in \mathbb{C}^{n \times n}$ be the optimal circulant preconditioner for $A_n[f]$. We have

$$\| (e^{c_n[f]} - 1 )^{-1} \|_2 \leq e^{2\|f\|_{\infty}} \quad n = 1, 2, \ldots.$$  

**Proof** By Lemma 3.1, we have

$$e^{c_n[f]}e^{-c_n[f]} = e^{c_n[f] - c_n[f]} = I_n.$$  

Thus $e^{-c_n[f]}$ is the inverse of $e^{c_n[f]}$. We then have

$$\| (e^{c_n[f]} - 1 )^{-1} \|_2 = \| e^{-c_n[f]} \|_2.$$  

Using lemmas 2.1 and 3.4, we have

$$\| e^{-c_n[f]} \|_2 \leq e^{\|c_n(f)\|_2} \leq e^{2\|f\|_{\infty}}.$$  

□

We are now ready to give our main results on the spectrum of $(e^{c_n[f]} - 1)e^{A_n[f]}$.

**Theorem 4.2.** Let $f \in C_{2\pi}$. Let $A_n[f] \in \mathbb{C}^{n \times n}$ be the Toeplitz matrix generated by $f$ and $c_n[f] \in \mathbb{C}^{n \times n}$ be the optimal circulant preconditioner for $A_n[f]$. For all $\epsilon > 0$, there exist positive integers $N$ and $M$ such that for all $n > N$

$$e^{c_n[f]} - e^{A_n[f]} = R_n[f] + E_n[f],$$  

where

$$\text{rank } R_n[f] \leq 2M,$$

$$\| E_n[f] \|_2 \leq \epsilon.$$  

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Proof Since \( f \in C_{2\pi} \), by Weierstrass theorem [26, Theorem 6.1], for any \( \epsilon > 0 \) there exists \( M \in \mathbb{N} \) and a trigonometric polynomial

\[
p_M(\theta) = \sum_{k=-M}^{k=M} \rho_k e^{ik\theta}
\]

such that

\[
\| f - p_M \|_{\infty} \leq \epsilon. \tag{1}
\]

For all \( n > 2M \) we decompose

\[
e^{cn[f]} - e^{A_n[f]} = \underbrace{e^{cn[f]} - e^{cn[p_M]}}_{G_1} + \underbrace{e^{cn[p_M]} - e^{A_n[p_M]}}_{B} + \underbrace{e^{A_n[p_M]} - e^{A_n[f]}}_{G_2},
\]

where \( A_n[p_M] \in \mathbb{C}^{n \times n} \) is the Toeplitz matrix generated by \( p_M \) and \( c_n[p_M] \in \mathbb{C}^{n \times n} \) is the optimal circulant preconditioner for \( A_n[p_M] \).

We first want to show that \( G_1 + G_2 \) is of small norm. Using Lemmas 2.1, 3.2 and (1) we have

\[
\| G_1 + G_2 \|_2 \leq \| e^{cn[f]} - e^{cn[p_M]} \|_2 + \| e^{A_n[f]} - e^{A_n[p_M]} \|_2
\]

\[
\leq \| c_n[f] - c_n[p_M] \|_2 e^{\max(\| c_n[f] \|_2, \| c_n[p_M] \|_2)} + \| A_n[f] - A_n[p_M] \|_2 e^{\max(\| A_n[f] \|_2, \| A_n[p_M] \|_2)}
\]

\[
\leq (\| c_n[f] - c_n[p_M] \|_2 + \| A_n[f] - A_n[p_M] \|_2) e^{2\max(\| f \|_{\infty}, \| p_M \|_{\infty})}
\]

\[
\leq (2 \| f - p_M \|_{\infty} + 2 \| f - p_M \|_{\infty}) e^{2\max(\| f \|_{\infty}, \| p_M \|_{\infty})}
\]

\[
\leq (4 e^{2\max(\| f \|_{\infty}, \| p_M \|_{\infty})}) \epsilon. \tag{2}
\]

We further rewrite

\[
B = e^{cn[p_M]} - \sum_{i=0}^{K} \frac{1}{i!} c_n[p_M]^i + \sum_{i=0}^{K} \frac{1}{i!} c_n[p_M]^i - \sum_{i=0}^{K} \frac{1}{i!} A_n[p_M]^i + \sum_{i=0}^{K} \frac{1}{i!} A_n[p_M]^i - e^{A_n[p_M]},
\]

where \( K \) is a positive integer.

We are now to measure the norm of \( B_1 + B_2 \). Using Lemmas 2.1, 3.3 and
we have

\[ \| B_1 + B_2 \|_2 \leq \| e^{c_n[pM]} - \sum_{i=0}^{K} \frac{1}{i!} c_n[pM]^i \|_2 + \| \sum_{i=0}^{K} \frac{1}{i!} A_n[pM]^i - e^{A_n[pM]} \|_2 \]

\[ \leq \frac{\| c_n[pM] \|_2^{K+1}}{(K+1)!} \epsilon^{\| c_n[pM] \|_2} + \frac{\| A_n[pM] \|_2^{K+1}}{(K+1)!} \epsilon^{\| A_n[pM] \|_2} \]

\[ \leq \frac{\| c_n[pM] \|_2^{K+1}}{(K+1)!} + \frac{\| A_n[pM] \|_2^{K+1}}{(K+1)!} \epsilon^{2\| pM \|_\infty} \]

\[ \leq \frac{2\| pM \|_\infty^{K+1}}{(K+1)!} 2\epsilon^{2\| pM \|_\infty} =: \epsilon_K \]

which converges to zero as \( K \) goes to infinity. Therefore for a given \( \epsilon_K > 0 \), there exists an integer \( K \) such that

\[ \| B_1 + B_2 \|_2 \leq \epsilon_K \leq \epsilon. \]  \hspace{1cm} \text{(3)}

We next show that \( D \) can be decomposed into a sum of a matrix of low rank and a matrix of small norm. Firstly, we observe that

\[ c_n[pM] - A_n[pM] = U_n - W_n, \]

where

\[
U_n = \begin{bmatrix}
\frac{n-M}{n} \rho_M & \cdots & \frac{n-1}{n} \rho_1 \\
\vdots & \ddots & \vdots \\
\frac{n-M}{n} \rho_{-M} & \cdots & \frac{n-M}{n} \rho_M
\end{bmatrix}
\]
and

\[ W_n = \begin{bmatrix}
0 & \frac{1}{n} \rho_{-1} & \cdots & \frac{M}{n} \rho_{-M} \\
\frac{1}{n} \rho_1 & \ddots & \ddots & \cdots \\
\vdots & & \ddots & \ddots \\
\frac{M}{n} \rho_M & \cdots & \cdots & \frac{M}{n} \rho_{-M}
\end{bmatrix}. \]

Rewrite \( D \) as

\[
D = \sum_{i=0}^{K} \frac{1}{i!} c_n[p_M]^i - \sum_{i=0}^{K} \frac{1}{i!} A_n[p_M]^i \\
= \sum_{i=1}^{K} \frac{1}{i!} (c_n[p_M]^i - A_n[p_M]^i) \\
= \sum_{i=1}^{K} \frac{1}{i!} \left( \sum_{j=0}^{i-1} c_n[p_M]^j (U_n - W_n) A_n[p_M]^{i-1-j} \right) \\
= \sum_{i=1}^{K} \frac{1}{i!} \left( \sum_{j=0}^{i-1} c_n[p_M]^j U_n A_n[p_M]^{i-1-j} \right) + \sum_{i=1}^{K} \frac{1}{i!} \left( \sum_{j=0}^{i-1} c_n[p_M]^j W_n A_n[p_M]^{i-1-j} \right).
\]

\[ R_n[f] \]
Using Lemmas 2.1 and 2.2, we can estimate the norm of $J$:

$$
\|J\|_2 = \| \sum_{i=1}^{K} \frac{1}{i!} \sum_{j=0}^{i-1} c_n[p_M]^j W_n A_n[p_M]^{i-1-j} \|_2 \\
\leq \sum_{i=1}^{K} \frac{1}{i!} \| \sum_{j=0}^{i-1} c_n[p_M]^j W_n A_n[p_M]^{i-1-j} \|_2 \\
\leq \| W_n \|_2 \sum_{i=1}^{K} \frac{1}{i!} \sum_{j=0}^{i-1} \| c_n[p_M] \|_2 \| A_n[p_M] \|_2 \|^{i-1-j} \\
\leq \| W_n \|_2 \sum_{i=1}^{K} \frac{1}{i!} \sum_{j=0}^{i-1} (2\|p_M\|_\infty)^j (2\|p_M\|_\infty)^{i-1-j} \\
= \| W_n \|_2 \sum_{i=1}^{K} \frac{1}{(i-1)!} (2\|p_M\|_\infty)^{i-1} \\
\leq \| W_n \|_2 e^{2\|p_M\|_\infty} (\epsilon + \|f\|_\infty) e^{2\|p_M\|_\infty} \\
\leq \frac{1}{n} M (M + 1) (\epsilon + \|f\|_\infty) e^{2\|p_M\|_\infty}. \quad (4)
$$

We now show that rank $R_n[f] \leq 2KM$ by first investigating the structure of $R_n[f]$. Similar to the approach used in the proof of Lemma 3.11 in [21], simple computations show that

$$
c_n[p_M]^{\alpha} U_n A_n[p_M]^{\beta} =
\begin{bmatrix}
\Diamond & \cdots & \Diamond & \cdots & \Diamond \\
\vdots & \Diamond & \cdots & \Diamond & \cdots \\
\Diamond & \cdots & \Diamond & \cdots & \Diamond \\
\end{bmatrix},
$$
where the diamonds represent the non-zero entries which appear only in the four \((\alpha + 1)M\) by \((\beta + 1)M\) blocks in the corners, provided that \(n\) is larger than \(2 \max(\alpha + 1, \beta + 1)M\). Since the rank of 
\[
R_n[f] = \sum_{i=1}^{K} \frac{1}{i!} \left( \sum_{j=0}^{i-1} c_n[p_M]^{j} U_n A_n[p_M]^{i-1-j} \right)
\]
is determined by that of \(\sum_{j=0}^{K-1} c_n[p_M]^{j} U_n A_n[p_M]^{K-1-j}\) which is a block matrix with only four non-zero \(KM\) by \(KM\) blocks in its corners, it follows that the rank of \(R_n[f]\) is less than or equal to \(2KM\) if we assume \(n > 2KM\).

Considering (4), we pick
\[
N := \max \{ M(M + 1)(1 + \frac{\|f\|_{\infty}}{\epsilon})e^{2\|p_M\|_{\infty}}, 2KM \},
\]
and it follows that for all \(n > N\) we have \(\|J\|_2 \leq \epsilon\). Further combining this result with (2) and (3), we conclude that for all \(n > N\)
\[
\|E_n[f]\|_2 = \|G_1 + B_1 + J + B_2 + G_2\|_2 \leq (4e^{2\max(\|f\|_{\infty},\|p_M\|_{\infty})} + 2)\epsilon.
\]

\[\square\]

**Corollary 4.3.** Let \(f \in C_{2\pi}\). Let \(A_n[f] \in \mathbb{C}^{n \times n}\) be the Toeplitz matrix generated by \(f\) and \(c_n[f] \in \mathbb{C}^{n \times n}\) be the optimal circulant preconditioner for \(A_n[f]\). For all \(\epsilon > 0\), there exist positive integers \(N\) and \(M\) such that for all \(n > N\)
\[
(e^{c_n[f]} - 1) e^{A_n[f]} = I_n + \hat{R_n}[f] + \hat{E_n}[f],
\]
where
\[
\text{rank } \hat{R_n}[f] \leq 2M, \\
\|\hat{E_n}[f]\|_2 \leq \epsilon.
\]

**Proof** By Theorem 4.2, we know that for all \(\epsilon > 0\), there exist positive integers \(N\) and \(M\) such that for all \(n > N\)
\[
e^{c_n[f]} - e^{A_n[f]} = R_n[f] + E_n[f],
\]
where
\[
\text{rank } R_n[f] \leq 2M,
\]

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\[ \|E_n[f]\|_2 \leq \epsilon. \]

By Corollary 4.1 we know that \(\|(e^{cn[f]})^{-1}\|_2\) is uniformly bounded with respect to \(n\), so that we have

\[
(e^{cn[f]})^{-1}e^{An[f]} = I_n + (e^{cn[f])^{-1}(e^{An[f]} - e^{cn[f])} = I_n + (e^{cn[f])^{-1}(-R_n[f]) + (e^{cn[f])^{-1}(-E_n[f]).
\]

The result follows. \(\square\)

**Remark** Since \(A_n[f]\) is Hermitian when \(f\) is real-valued, we can write \(A_n[f] = Z_n^TD_nZ_n\) where \(D_n\) is a diagonal matrix with real eigenvalues \(d_i\) being the eigenvalues of \(A_n[f]\). We immediately see that \(e^{A_n[f]} = Z_n^Te^{D_n}Z_n\) is positive definite as its eigenvalues are all of the form \(e^{d_i} > 0\). Thus CG can be used in this case.

Next, for the more general case when \(e^{A_n[f]}\) is non-Hermitian, we can use CG for the normal equations system with the preconditioner \(e^{cn[f]}\).

**Corollary 4.4.** Let \(f \in C_{2\pi}\). Let \(A_n[f] \in \mathbb{C}^{n \times n}\) be the Toeplitz matrix generated by \(f\) and \(c_n[f] \in \mathbb{C}^{n \times n}\) be the optimal circulant preconditioner for \(A_n[f]\). For all \(\epsilon > 0\), there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[
[(e^{cn[f])^{-1}e^{A_n[f]}][[(e^{cn[f])^{-1}e^{A_n[f]} = I_n + \tilde{R}_n[f] + \tilde{E}_n[f],
\]

where

\[
\text{rank } \tilde{R}_n[f] \leq 4M,
\]

\[
\|\tilde{E}_n[f]\|_2 \leq \epsilon.
\]

**Proof** By Corollary 4.3 we know that for all \(\epsilon > 0\) there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[
(e^{cn[f]})^{-1}e^{A_n[f]} = I_n + \tilde{R}_n[f] + \tilde{E}_n[f],
\]

where

\[
\text{rank } \tilde{R}_n[f] \leq 2M,
\]

\[
\|\tilde{E}_n[f]\|_2 \leq \epsilon.
\]
We then have
\[
((e^{c_n[f]} - 1)e^{A_n[f]})(e^{c_n[f]} - 1)e^{A_n[f]}
= (I_n + \tilde{R}_n[f] + \tilde{E}_n[f])(I_n + \tilde{R}_n[f] + \tilde{E}_n[f])
\]
\[
= I_n + \tilde{R}_n[f]^*(I_n + \tilde{R}_n[f] + \tilde{E}_n[f]) + (I_n + \tilde{E}_n[f]^*)\tilde{R}_n[f]
+ \tilde{E}_n[f] + \tilde{E}_n[f]^* \tilde{E}_n[f].
\]

It immediately follows that \(\text{rank } \tilde{R}_n[f] \leq 4M\) and \(\|E_n[f]\|_2 \leq \epsilon^2 + 2\epsilon.\)

Since \(((e^{c_n[f]} - 1)e^{A_n[f]})(e^{c_n[f]} - 1)e^{A_n[f]}\) in Corollary 4.4 is Hermitian, by Weyl's theorem we know that its eigenvalues are mostly close to 1 when \(n\) is sufficiently large.

4.2. Spectra of \((\sin c_n[f])^{-1}\sin A_n[f] \text{ and } (\cos c_n[f])^{-1} \cos A_n[f]\)

In this subsection we directly show that similar results hold for \((\sin c_n[f])^{-1}\sin A_n[f] \text{ and } (\cos c_n[f])^{-1} \cos A_n[f]\) using the theorems on \((e^{c_n[f]} - 1)e^{A_n[f]}\).

**Theorem 4.5.** Let \(f \in C_{2\pi}\). Let \(A_n[f] \in \mathbb{C}^{n \times n}\) be the Toeplitz matrix generated by \(f\) and \(c_n[f] \in \mathbb{C}^{n \times n}\) be the optimal circulant preconditioner for \(A_n[f]\). For all \(\epsilon > 0\), there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[\sin c_n[f] - \sin A_n[f] = R_n[f] + E_n[f],\]

where

\[\text{rank } R_n[f] \leq 2M,\]
\[\|E_n[f]\|_2 \leq \epsilon.\]

**Proof** By Theorem 4.2, we know that for all \(\epsilon > 0\) there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[e^{c_n[f]} - e^{A_n[f]} = R_n[f] + E_n[f],\]

where

\[\text{rank } R_n[f] \leq 2M,\]
\[\|E_n[f]\|_2 \leq \epsilon.\]
Using the fact that $\sin A = \frac{e^{iA} - e^{-iA}}{2i}$ for any $A_n \in \mathbb{C}^{n \times n}$, we write

$$
\sin c_n[f] - \sin A_n[f] = \left(\frac{e^{ic_n[f]} - e^{-ic_n[f]}}{2i} - \frac{e^{1A_n[f]} - e^{-1A_n[f]}}{2i}\right)
$$

$$
= \left(\frac{e^{ic_n[f]} - e^{1A_n[f]}}{2i} - \frac{e^{-ic_n[f]} - e^{-1A_n[f]}}{2i}\right)
$$

$$
= \left(\frac{e^{cn[i]} - e^{A_n[i]}}{2i} - \frac{e^{cn[-i]} - e^{A_n[-i]}}{2i}\right)
$$

$$
= \left(\frac{R_n[i] + E_n[i]}{2i}\right) - \left(\frac{R_n[-i] + E_n[-i]}{2i}\right)
$$

$$
= \left(\frac{R_n[i] - R_n[-i]}{2i}\right) + \left(\frac{E_n[i] - E_n[-i]}{2i}\right).
$$

Since $R_n[i]$ and $R_n[-i]$ are both block matrices with only four non-zero $M$ by $M$ blocks in their corners, we see that the rank of $R_n[i]$ is less than or equal to $2M$. Also

$$
\|E_n[f]\|_2 = \left\|\frac{E_n[i] - E_n[-i]}{2i}\right\|_2
$$

$$
\leq \frac{\|E_n[i]\|_2 + \|E_n[-i]\|_2}{2}
$$

$$
\leq \epsilon.
$$

□

**Corollary 4.6.** Let $f \in C_{2\pi}$. Let $A_n[f] \in \mathbb{C}^{n \times n}$ be the Toeplitz matrix generated by $f$ and $c_n[f] \in \mathbb{C}^{n \times n}$ be the optimal circulant preconditioner for $A_n[f]$. If $\|\sin c_n[f]\|_2^{-1}$ is uniformly bounded with respect to $n$, then for all $\epsilon > 0$ there exist positive integers $N$ and $M$ such that for all $n > N$

$$
(sinc_n[f])^{-1}(\sin A_n[f]) = I_n + \tilde{R}_n[f] + \tilde{E}_n[f],
$$

where

$$\text{rank} \tilde{R}_n[f] \leq 2M,$$

$$\|\tilde{E}_n[f]\|_2 \leq \epsilon.$$
Proof By Theorem 4.5, we know that for all $\epsilon > 0$, there exist positive integers $N$ and $M$ such that for all $n > N$

$$\sin c_n[f] - \sin A_n[f] = R_n[f] + \mathcal{E}_n[f],$$

where

$$\text{rank } R_n[f] \leq 2M,$$

$$\|\mathcal{E}_n[f]\|_2 \leq \epsilon.$$

By the assumption that $\|(\sin c_n[f])^{-1}\|_2$ is uniformly bounded with respect to $n$, we have

$$(\sin c_n[f])^{-1} \sin A_n[f] = I_n + (\sin c_n[f])^{-1} (\sin A_n[f] - \sin c_n[f])$$

$$= I_n + (\sin c_n[f])^{-1} (-R_n[f]) + (\sin c_n[f])^{-1} (-\mathcal{E}_n[f]).$$

The result follows. □

Remark From

$$\|(\sin c_n[f])^{-1}\|_2 = \max_i \left| \frac{1}{\sin \lambda_i} \right|$$

we know that $\|(\sin c_n[f])^{-1}\|_2$ could be arbitrarily large since $\sin \lambda_i$ could be close to zero, where $\lambda_i$ is the $i$-th eigenvalue of $c_n[f]$. Therefore, we have needed to assume that $\|(\sin c_n[f])^{-1}\|_2$ is uniformly bounded with respect to $n$.

Consider now the special case when $\sin A_n[f]$ is Hermitian. Unlike the case with the matrix exponential, we cannot use CG for $\sin A_n[f]$ as it is not positive definite in general. By the diagonalisation of $\sin A_n[f] = Z_n^T (\sin D_n) Z_n$ where $D_n$ is a diagonal matrix with real eigenvalues $d_i$ being the eigenvalues of $A_n[f]$, as before, we see that its eigenvalues are all of the form $-1 \leq \sin d_i \leq 1$. Krylov subspace methods like MINRES [23] together with a Hermitian positive definite preconditioner $|\sin c_n[f]|$ should be used [32] Section 5], where $|\sin c_n[f]|$ is the absolute value circulant preconditioner [24] [19] of $\sin c_n[f]$.

Corollary 4.7. Let $f \in C_{2\pi}$ be a real-valued function. Let $A_n[f] \in \mathbb{C}^{n \times n}$ be the Toeplitz matrix generated by $f$ and $c_n[f] \in \mathbb{C}^{n \times n}$ be the optimal circulant preconditioner for $A_n[f]$. If $\|(\sin c_n[f])^{-1}\|_2$ is uniformly bounded with respect
to \( n \), then for all \( \epsilon > 0 \) there exist positive integers \( N \) and \( M \) such that for all \( n > N \)

\[
| \sin c_n[f] |^{-1} \sin A_n[f] = Q_n + \tilde{\mathcal{R}}_n[f] + \tilde{\mathcal{E}}_n[f],
\]

where \( Q_n \) is unitary and Hermitian,

\[
\text{rank } \tilde{\mathcal{R}}_n[f] \leq 2M,
\]

\[
\| \tilde{\mathcal{E}}_n[f] \|_2 \leq \epsilon.
\]

**Proof** Using a similar approach proposed in [24], we want to show that \( | \sin c_n[f] | \) is an effective Hermitian positive definite preconditioner for \( \sin A_n[f] \) under the assumptions.

As \( \sin c_n[f] \) is a circulant matrix we write \( \sin c_n[f] = F_n^* (\sin \Lambda_n) F_n \) where \( \sin \Lambda_n \) is the diagonal matrix with the eigenvalues of \( \sin c_n[f] \). We then immediately have

\[
| \sin c_n[f] | = F_n^* | \sin \Lambda_n | F_n
\]

\[
= F_n^* (\sin \Lambda_n) F_n F_n^* (\text{sign}(\sin \Lambda_n))^{-1} F_n
\]

\[
= \sin c_n[f] Q_n,
\]

where \( \text{sign}(\sin \Lambda_n) = \text{diag}(\frac{\sin \lambda_i}{|\sin \lambda_i|}) = \text{diag}(\pm 1) \) and \( Q_n \) is both unitary and involutory (i.e. \( Q_n^2 = I_n \)). It is noted that \( | \sin \lambda_i | \neq 0 \) for \( i = 1, 2, \ldots, n \) by the assumption, so that \( \text{sign}(\sin \Lambda_n) \) is well defined.

By Corollary 4.6 we know that for all \( \epsilon > 0 \) there exist positive integers \( N \) and \( M \) such that for all \( n > N \)

\[
(\sin c_n[f])^{-1} (\sin A_n[f]) = I_n + \tilde{\mathcal{R}}_n[f] + \tilde{\mathcal{E}}_n[f],
\]

where

\[
\text{rank } \tilde{\mathcal{R}}_n[f] \leq 2M,
\]

\[
\| \tilde{\mathcal{E}}_n[f] \|_2 \leq \epsilon.
\]

Using (5), we have

\[
| \sin c_n[f] |^{-1} \sin A_n[f] = Q_n \sin c_n[f]^{-1} \sin A_n[f] = Q_n + \underbrace{Q_n \tilde{\mathcal{R}}_n[f]}_{\tilde{\mathcal{R}}_n[f]} + \underbrace{Q_n \tilde{\mathcal{E}}_n[f]}_{\tilde{\mathcal{E}}_n[f]}.
\]
Since $Q_n$ is unitary, we know
\[
\text{rank } \tilde{R}_n[f] = \text{rank}(Q_n \tilde{R}_n[f]) = \text{rank } \tilde{R}_n[f] \leq 2M
\]
and
\[
\|\tilde{E}_n[f]\|_2 = \|Q_n \tilde{E}_n[f]\|_2 = \|\tilde{E}_n[f]\|_2 \leq \epsilon.
\]
The result follows. \hfill \Box

Remark Since $|\sin c_n[f]|$ is also a circulant matrix, $|\sin c_n[f]|^{-1}d$ for any vector $d$ can be efficiently computed by several FFTs in $O(n \log n)$ operations.

For the more general case when $\sin A_n[f]$ is non-Hermitian, we can use CG for the normal equations system with the preconditioner $\sin c_n[f]$.

Corollary 4.8. Let $f \in C_{2\pi}$. Let $A_n[f] \in \mathbb{C}^{n \times n}$ be the Toeplitz matrix generated by $f$ and $c_n[f] \in \mathbb{C}^{n \times n}$ be the optimal circulant preconditioner for $A_n[f]$. If $\|(|\sin c_n[f]|^{-1})\|_2$ is uniformly bounded with respect to $n$, then for all $\epsilon > 0$ there exist positive integers $N$ and $M$ such that for all $n > N$
\[
(|\sin c_n[f]|^{-1} \sin A_n[f])^* (|\sin c_n[f]|^{-1} \sin A_n[f]) = I_n + \bar{R}_n[f] + \bar{E}_n[f],
\]
where
\[
\text{rank } \bar{R}_n[f] \leq 4M,
\]
\[
\|\bar{E}_n[f]\|_2 \leq \epsilon.
\]

Proof By Corollary 4.6 we know that for all $\epsilon > 0$ there exist positive integers $N$ and $M$ such that for all $n > N$
\[
(|\sin c_n[f]|^{-1} \sin A_n[f])^* (|\sin c_n[f]|^{-1} \sin A_n[f]) = I_n + \tilde{R}_n[f] + \tilde{E}_n[f],
\]
where
\[
\text{rank } \tilde{R}_n[f] \leq 2M,
\]
\[
\|\tilde{E}_n[f]\|_2 \leq \epsilon.
\]
We then have

\[
[(\sin c_n[f])^{-1} \sin A_n[f]]^*[(\sin c_n[f])^{-1} \sin A_n[f]]
= (I_n + \hat{R}_n[f] + \hat{E}_n[f])^*(I_n + \hat{R}_n[f] + \hat{E}_n[f])
\]

\[
= I_n + \hat{R}_n[f] + \hat{E}_n[f] + (I_n + \hat{E}_n[f])^* \hat{R}_n[f] \hat{E}_n[f]
\]

It immediately follows that \(\text{rank } R_n[f] \leq 4M\) and \(\|E_n[f]\|_2 \leq \epsilon^2 + 2\epsilon\). □

Since \([(\sin c_n[f])^{-1} \sin A_n[f]]^*[(\sin c_n[f])^{-1} \sin A_n[f]]\) in Corollary 4.8 is Hermitian, by Weyl's theorem, we know that its eigenvalues are mostly close to 1 when \(n\) is sufficiently large.

Because \(\cos A_n = e^{-iA_n} + e^{iA_n}\) for any \(A_n \in \mathbb{C}^{n \times n}\), we have the following similar theorem and corollaries for \(\cos A_n[f]\).

**Theorem 4.9.** Let \(f \in C_{2\pi}\). Let \(A_n[f] \in \mathbb{C}^{n \times n}\) be the Toeplitz matrix generated by \(f\) and \(c_n[f] \in \mathbb{C}^{n \times n}\) be the optimal circulant preconditioner for \(A_n[f]\). For all \(\epsilon > 0\), there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[
\cos c_n[f] - \cos A_n[f] = R_n[f] + E_n[f],
\]

where

\[
\text{rank } R_n[f] \leq 2M, \quad \|E_n[f]\|_2 \leq \epsilon.
\]

**Corollary 4.10.** Let \(f \in C_{2\pi}\). Let \(A_n[f] \in \mathbb{C}^{n \times n}\) be the Toeplitz matrix generated by \(f\) and \(c_n[f] \in \mathbb{C}^{n \times n}\) be the optimal circulant preconditioner for \(A_n[f]\). If \(\|\cos c_n[f]\|_2^{-1}\) is uniformly bounded with respect to \(n\), then for all \(\epsilon > 0\) there exist positive integers \(N\) and \(M\) such that for all \(n > N\)

\[
|\cos c_n[f]|^{-1} \cos A_n[f] = Q_n + \tilde{R}_n[f] + \tilde{E}_n[f],
\]

where \(Q_n\) is unitary and Hermitian,

\[
\text{rank } \tilde{R}_n[f] \leq 2M, \quad \|\tilde{E}_n[f]\|_2 \leq \epsilon.
\]
For the more general case when \( \cos A_n[f] \) is non-Hermitian, we can use CG for the normal equations system with the preconditioner \( \cos c_n[f] \).

**Corollary 4.11.** Let \( f \in C_{2\pi} \). Let \( A_n[f] \in \mathbb{C}^{n \times n} \) be the Toeplitz matrix generated by \( f \) and \( c_n[f] \in \mathbb{C}^{n \times n} \) be the optimal circulant preconditioner for \( A_n[f] \). If \( \| (\cos c_n[f])^{-1} \|_2 \) is uniformly bounded with respect to \( n \), then for all \( \epsilon > 0 \) there exist positive integers \( N \) and \( M \) such that for all \( n > N \)

\[
[(\cos c_n[f])^{-1} \cos A_n[f]]^* [(\cos c_n[f])^{-1} \cos A_n[f]] = I_n + \overline{\mathcal{R}}_n[f] + \mathcal{E}_n[f],
\]

where

\[
\text{rank } \overline{\mathcal{R}}_n[f] \leq 4M,
\]

\[
\| \mathcal{E}_n[f] \|_2 \leq \epsilon.
\]

Since \( [(\cos c_n[f])^{-1} \cos A_n[f]]^* [(\cos c_n[f])^{-1} \cos A_n[f]] \) in Corollary 4.11 is Hermitian, by Weyl’s theorem, we know that its eigenvalues are mostly close to 1 when \( n \) is sufficiently large.

5. Extension to analytic functions of Toeplitz matrices

**Lemma 5.1.** [13, Theorem 1.18] Let \( h \) be analytic on an open subset \( \Omega \subseteq \mathbb{C} \) such that each connected component of \( \Omega \) is closed under conjugation. Consider the corresponding matrix function \( h \) on its natural domain in \( \mathbb{C}^{n \times n} \), the set \( D = \{ A_n \in \mathbb{C}^{n \times n} : \Lambda(A_n) \subseteq \Omega \} \). Then the following are equivalent:

(a) \( h(A_n^*) = h(A_n)^* \) for all \( A_n \in D \).

(b) \( h(A_n) = \overline{h(A_n)} \) for all \( A_n \in D \).

(c) \( h(\mathbb{R}^{n \times n} \cap D) \subseteq \mathbb{R}^{n \times n} \).

(d) \( h(\mathbb{R} \cap \Omega) \subseteq \mathbb{R} \).

**Lemma 5.2.** [13, Theorem 4.7] Suppose \( h \) has a Taylor series expansion

\[
h(z) = \sum_{k=0}^{\infty} a_k (z - \alpha)^k,
\]

where \( a_k = h^{(k)}(\alpha) k! \), with radius of convergence \( r \). If \( A_n \in \mathbb{C}^{n \times n} \) then \( f(A_n) \) is defined and is given by

\[
h(A_n) = \sum_{k=0}^{\infty} a_k (A_n - \alpha I_n)^k
\]
if and only if the distinct eigenvalues $\lambda_1, \cdots, \lambda_s$ of $A_n$ satisfy one of the conditions

(a) $|\lambda_i - \alpha| < r,$

(b) $|\lambda_i - \alpha| = r$ and the series for $h^{(n_i-1)}(\lambda)$, where $n_i$ is the index of $\lambda_i$, is convergent at the point $\lambda = \lambda_i$, $i = 1, \ldots, s$.

Lemma 5.3. \[13, Theorem 4.8\] Suppose $h$ has a Taylor series expansion

$$h(z) = \sum_{k=0}^{\infty} a_k (z - \alpha)^k,$$

where $a_k = \frac{h^{(k)}(\alpha)}{k!}$, with radius of convergence $r$. If $A_n \in \mathbb{C}^{n \times n}$ with $\rho(A_n - \alpha I_n) < r$ then for any matrix norm $\| \cdot \|$,

$$\| h(A_n) - \sum_{k=0}^{K-1} a_k (A_n - \alpha I_n)^k \| \leq \frac{1}{K!} \max_{0 \leq t \leq 1} \| (A_n - \alpha I_n)^K h^{(K)}(\alpha I_n + t(A_n - \alpha I_n)) \|.$$

We first assume that the condition in Lemma 5.2 is satisfied so $h(A_n)$ can be represented by a Taylor series of $A_n$. Replacing Lemma 3.3 with Lemma 5.3, we can show that $h(c_n[f]) - h(A_n[f])$ can be decomposed into a sum of a matrix of certain rank and a small norm matrix in a similar manner to theorem 4.2. Further assuming the boundedness of $h(c_n[f])$, we can prove a similar decomposition for $h(c_n[f])^{-1}h(A_n[f])$ or its normal equations matrix.

6. Numerical results

In this section, we demonstrate the effectiveness of our proposed preconditioners $g(c_n[f])$ for the systems $g(A_n[f])x = b$ using CG, MINRES and GMRES [27]. Throughout all numerical tests, $e^{A_n[f]}$ is computed by the MATLAB built-in function expm whilst $\sin A_n[f]$ and $\cos A_n[f]$ are computed by funm. Also, we use the function pcg to solve the Hermitian positive definite systems

$$g(A_n[f])x = b,$$

and

$$g(A_n[f])^*g(A_n[f])x = g(A_n[f])^*b,$$

where $b$ is generated by the function randn(n,1), with the zero vector as the initial guess. For Hermitian indefinite systems, we use the function minres.
As a comparison, GMRES is also used for some systems and it is executed by gmres. The stopping criterion used is

\[ \frac{\|r_j\|_2}{\|b\|_2} < 10^{-7}, \]

where \( r_j \) is the residual vector after \( j \) iterations.

Example 1: We first consider \( e^{A_n[f]} \), where \( A_n[f] \) is generated by several functions \( f \) with moderate \( \|f\|_\infty \). Table 1 shows the numbers of iterations needed for \( e^{A_n[f]} \) with \( A_n[f] \) generated by \( f(\theta) = \frac{4}{3} \theta \cos(\theta) \) with or without preconditioners. It is clear that the proposed preconditioner is efficient for speeding up the rate of convergence of CG. In Figure 1 (a) and (b), the contrast between the spectra of the matrices is shown. In Figure 1 (c), we see that the eigenvalues of the preconditioned matrix are highly clustered near 1. By the analysis on the rate of convergence of preconditioned CG for highly clustered spectrum given in [1], the preconditioned matrix can be regarded as having an "efficient" condition number \( b/a \), where \([a, b]\) is the closed interval in which most of the eigenvalues are clustered. Therefore, a fast convergence rate for preconditioned CG is expected due to the cluster of eigenvalues at 1 and the small number of outliers.

Table 1: Numbers of iterations with CG for \( e^{A_n[f]} \) with the generating \( f(\theta) = \frac{4}{3} \theta \cos(\theta) \).

| \( n \) | \( I_n \) | \( e^{c_n[f]} \) |
|---|---|---|
| 128 | 224 | 20 |
| 256 | 325 | 21 |
| 512 | 414 | 25 |
| 1024 | 491 | 26 |

In Figure 2, we further show the spectrum of \( e^{A_n[f]} \) before and after applying the preconditioner \( e^{c_n[f]} \) with different \( n \). We observe that the highly clustered spectra seem independent of \( n \).
Figure 1: The spectrum of (a) $e^{A_n[f]}$ and that of (b) $(e^{c_n[f]} - 1)^{-1} e^{A_n[f]}$. (c) The zoom-in spectrum of (b). $A_n[f]$ is generated by $f(\theta) = \frac{4}{3} \theta \cos(\theta)$ and $n = 512$.

Figure 2: The spectrum of $e^{A_n[f]}$ and that of $(e^{c_n[f]} - 1)^{-1} e^{A_n[f]}$ (a) $n = 256$, (b) $n = 1024$ or (c) $n = 4096$. $A_n[f]$ is generated by $f(\theta) = \frac{4}{3} \theta \cos(\theta)$. 
Example 2: Table 2 (a) and (b) show the numerical results using CG and GMRES for the normal equations matrices of $e^{A_n[f]}$ with $A_n[f]$ generated by $f(\theta) = 2\theta \cos(\theta) + \theta i$, respectively. Again, we observe that the preconditioners are efficient for speeding up the rate of convergence.

Table 2: Numbers of iterations with (a) CG for $(e^{A_n[f]}e^{A_n[f]})^*$ and (b) GMRES for $e^{A_n[f]}$ with $A_n[f]$ generated by $f(\theta) = 2\theta \cos(\theta) + \theta i$.

| $n$  | $I_n$  | Preconditioner |
|------|--------|----------------|
| 128  | 14219  | 75             |
| 256  | 78645  | 96             |
| 512  | >100000| 145            |
| 1024 | >100000| 110            |

| $n$  | $I_n$  | Preconditioner |
|------|--------|----------------|
| 128  | 128    | 21             |
| 256  | 248    | 23             |
| 512  | 477    | 25             |
| 1024 | 891    | 27             |
Example 3: We next consider the matrix sine functions. Table 3 (a) shows the numerical results using MINRES for $\sin A_n[f]$ with $A_n[f]$ generated by $f(\theta) = -(\frac{\theta^2}{2\pi} + \frac{1}{10\pi})$. As the matrix in this case is symmetric negative definite, we also show numerical results using CG as a comparison in Table 3 (b).

Table 3: Numbers of iterations with (a) MINRES and (b) CG for $\sin A_n[f]$ with $A_n[f]$ generated by $f(\theta) = -(\frac{\theta^2}{2\pi} + \frac{1}{10\pi})$.

| $n$  | $I_n$ | $|\sin c_n[f]|$ |
|------|-------|-----------------|
| 128  | 138   | 15              |
| 256  | 227   | 14              |
| 512  | 238   | 11              |
| 1024 | 243   | 10              |

| $n$  | $I_n$ | $|\sin c_n[f]|$ |
|------|-------|-----------------|
| 128  | 138   | 16              |
| 256  | 235   | 14              |
| 512  | 253   | 11              |
| 1024 | 255   | 10              |
Example 4: Table 4 shows the numerical results for $(\sin A_n[f])^* \sin A_n[f]$ with $A_n[f]$ generated by $f(\theta) = -\left(\frac{\theta^2}{2\pi} + \frac{1}{10}\right)$. Since the normalised matrices are highly ill-conditioned, CG without preconditioner requires large numbers of iteration to get the solutions to the desired tolerance. However, the numbers of iterations are reduced significantly with our proposed preconditioner.

Table 4: Numbers of iterations with (a) CG for $(\sin A_n[f])^* \sin A_n[f]$ and (b) GMRES for $\sin A_n[f]$ with $A_n[f]$ generated by $f(\theta) = -\left(\frac{\theta^2}{2\pi} + \frac{1}{10}\right)$.

| $n$ | $I_n$ | Preconditioner |
|-----|-------|----------------|
| (a) |       |                |
| 128 | 1094  | 31             |
| 256 | 3238  | 27             |
| 512 | 4844  | 22             |
| 1024| 10152 | 16             |

| $n$ | $I_n$ | Preconditioner |
|-----|-------|----------------|
| (b) |       |                |
| 128 | 128   | 16             |
| 256 | 255   | 17             |
| 512 | 384   | 13             |
| 1024| 463   | 10             |
Example 5: Lastly, we consider the matrix cosine functions. Table 5 and 6 show the numerical results for symmetric matrix $\cos A_n[f]$ with $A_n[f]$ generated by $f(\theta) = (\frac{\pi}{2} - \frac{1}{10\pi}) \cos (\theta^2) - \frac{\pi}{4}$ and for $(\cos A_n[f])^* \cos A_n[f]$ with $A_n[f]$ generated by $f(\theta) = (\frac{\pi}{2} - \frac{1}{10\pi}) \cos (\theta^2) + \frac{2}{\pi} i$, respectively. In Figure 3 (a) and (b), we also show the spectrum of the matrices before and after applying the preconditioner $|\cos A_n[f]|$. In the zoom-in spectrum shown in Figure 3 (c), we observe that the eigenvalues of the preconditioned matrix are mostly $\pm 1$. We conclude that our proposed preconditioners appear effective for these systems defined by matrix cosine functions of Toeplitz matrices.

Table 5: Numbers of iterations with MINRES for $\cos A_n[f]$ with $A_n[f]$ generated by $f(\theta) = (\frac{\pi}{2} - \frac{1}{10\pi}) \cos (\theta^2) - \frac{\pi}{4}$.

| $n$  | $I_n$ | $|\cos c_n[f]|$ |
|------|-------|----------------|
| 128  | 77    | 30             |
| 256  | 139   | 36             |
| 512  | 261   | 38             |
| 1024 | 506   | 42             |

Table 6: Numbers of iterations with (a) CG for $(\cos A_n[f])^* \cos A_n[f]$ and (b) GMRES for $\cos A_n[f]$ with $A_n[f]$ generated by $f(\theta) = (\frac{\pi}{2} - \frac{1}{10\pi}) \cos (\theta^2) + \frac{2}{\pi} i$.

| $n$  | $I_n$ | Preconditioner |
|------|-------|----------------|
| (a)  |       |                |
| 128  | 191   | 21             |
| 256  | 435   | 22             |
| 512  | 973   | 22             |
| 1024 | 2092  | 23             |

| $n$  | $I_n$ | Preconditioner |
|------|-------|----------------|
| (b)  |       |                |
| 128  | 126   | 16             |
| 256  | 249   | 16             |
| 512  | 492   | 17             |
| 1024 | 972   | 17             |
Figure 3: The spectrum of (a) \( \cos A_n[f] \) and that of (b) \( |\cos c_n[f]|^{-1} \cos A_n[f] \). (c) The zoom-in spectrum of (b). \( A_n[f] \) is generated by \( f(\theta) = (\pi/2 - 1/10^4) \cos (\theta^2) - \pi/4 \) and \( n = 512 \).

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