THE DUALIZING COMPLEX OF F-INJECTIVE AND DU BOIS SINGULARITIES

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Abstract. Let \((R, \mathfrak{m}, k)\) be an excellent local ring of equal characteristic. Let \(j\) be a positive integer such that \(H^i_\mathfrak{m}(R)\) has finite length for every \(0 \leq i < j\). We prove that if \(R\) is F-injective in characteristic \(p > 0\) or Du Bois in characteristic 0, then the truncated dualizing complex \(\tau_{> -j} \omega^R_Q\) is quasi-isomorphic to a complex of \(k\)-vector spaces. As a consequence, F-injective or Du Bois singularities with isolated non-Cohen-Macaulay locus are Buchsbaum. Moreover, when \(R\) has \(F\)-rational or rational singularities on the punctured spectrum, we obtain stronger results generalizing [Ma15] and [Ish84].

1. Introduction

In this paper, we study the dualizing complexes of F-injective and Du Bois singularities. We prove the following result.

Main Theorem A (Theorem 3.4, Corollary 4.3). Suppose that \((R, \mathfrak{m}, k)\) is a Noetherian local ring such that \(H^i_\mathfrak{m}(R)\) has finite length\(^1\) for every \(0 \leq i < j\). Suppose one of the following conditions hold:

(a) \(R\) is of characteristic \(p\) and is F-injective.
(b) \(R\) is essentially of finite type over a field of characteristic zero and is Du Bois.

Then the truncated dualizing complex \(\tau_{> -j} \omega^R_Q\) is quasi-isomorphic to a complex of \(k\)-vector spaces. Equivalently, \(\tau^\left< -j \right> R\Gamma_\mathfrak{m}(R)\) is quasi-isomorphic to a complex of \(k\)-vector spaces. In particular, these truncated complexes split into a direct sum of their cohomologies.

In the case that \(j = \dim R\), the condition proven in the theorem is also known as Buchsbaum [SV86], [Sch82]. Buchsbaum should be thought of as a minimal generalization of the Cohen-Macaulay condition. In the special case of \(j = \dim R\), our result says that:

(a) F-injective singularities with isolated non-Cohen-Macaulay locus are Buchsbaum, re-proving a result of [Ma15].

\(^1\)Or dually, that \(H^{-i} \omega^R_Q\) has finite length for \(0 \leq i < j\) (provided a dualizing complex exists). Under mild conditions this is equivalent to saying that the non-Cohen-Macaulay locus on \(\text{Spec}(R)\) has codimension \(j\).
Du Bois singularities with isolated non-Cohen-Macaulay locus are Buchsbaum, generalizing the case of normal isolated singularities [Ish84] and answering a question of Shunsuke Takagi. Moreover, in characteristic $p > 0$, we define the $*$-truncation or tight closure truncation $\tau^{<d,*}R\Gamma_mR$, to be the object $D$ in the derived category of $R$-modules such that we have an exact triangle:

$$D \rightarrow R\Gamma_mR \rightarrow (H^d_m(R)/0^*_R)[−d] \rightarrow .$$

This complex $\tau^{<d,*}R\Gamma_mR$ has the property that its lower cohomologies are $H^*_R$, while its top cohomology is $0^*_R$. Indeed, this complex is just Matlis dual to $C$ wherein

$$\tau(\omega_R)[d] \rightarrow \omega_R \rightarrow C \rightarrow .$$

is a triangle and $\tau(\omega_R)$ is the parameter test module [Smi94], [BST15]. Our second main result is the following:

**Main Theorem B** (Theorem 3.6). Let $(R, m, k)$ be an excellent equidimensional Noetherian local ring that is $F$-rational on the punctured spectrum. If $R$ is $F$-injective then $\tau^{<d,*}R\Gamma_mR$ is quasi-isomorphic to a complex of $k$-vector spaces. In particular, $\tau^{<d,*}R\Gamma_mR$ splits into a direct sum of its cohomologies.

We note that if $\tau^{<d,*}R\Gamma_mR$ splits into its cohomologies, then so does $\tau^{<d}(\tau^{<d,*}R\Gamma_mR) = \tau^{<d}R\Gamma_mR$. Hence Theorem B generalizes Theorem A in the case $j = d = \dim R$ (and $R$ is $F$-rational on the punctured spectrum), because it encodes information about the top local cohomology. We also mention that the full complex $R\Gamma_mR$ never splits into its cohomologies (in the derived category) unless $R$ is already Cohen-Macaulay, see [Corollary 5.2].

In characteristic zero, we also obtain an analogous result that in some ways is even stronger since it holds even without a Du Bois / $F$-injective hypothesis. Indeed if $\pi : W \rightarrow X$ is a resolution of singularities of some reduced scheme $X$, then we have the exact triangle

$$\pi_!\omega_W[1] \rightarrow R\mathcal{H}om_\mathcal{O}_X(\Omega^0_X, \omega_X^*) \rightarrow C \rightarrow .$$

When $X$ has Du Bois singularities $\Omega^0_X \simeq_{qis} \Omega_X$ and the complex $C$ is analogous to the $C$ described above in characteristic $p > 0$. However, even without this assumption we have the following theorem.

**Main Theorem C** (Theorem 4.1, Corollary 4.4). With notation as above, suppose that $X = \text{Spec } R$ is the spectrum of a local ring $(R, m, k)$ of essentially finite type over a field of characteristic zero and $H^{-i}(C)$ has finite length for all $i$ (this last condition happens automatically if $R$ has rational singularities on the punctured spectrum). Then $C$ is quasi-isomorphic to a complex of $k$-vector spaces and in particular, it splits into a direct sum of its cohomologies.

1.1. This paper’s history. The first version of this paper that appeared on the arXiv was written only by the second and third authors. Shortly after it appeared the first author contacted the second and third authors with an alternate proof of Main Theorem A (at that time, we required the residue field to be perfect). Together, all the authors generalized the strategy of the new proof to the case where the residue field is not necessarily perfect. The
original proof can now be found in Section 3.1. The first author also suggested a more conceptual proof of Corollary 5.2 which lead to the more generalized statement Proposition 5.1.

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2. Preliminaries

Throughout this paper, all rings are commutative with multiplicative identity 1 and all schemes are separated. In most cases, we work with Noetherian rings and we also assume all Noetherian rings and schemes have dualizing complexes. When working with dualizing complexes on Noetherian local rings, we always assume that they are normalized, which means the first nonzero cohomology is in degree $-\dim R$. In characteristic $p > 0$, we will typically assume that our Noetherian rings are $F$-finite, which means that the Frobenius morphism is a finite morphism. We recall that $F$-finite Noetherian rings are always excellent [Kun76] and always have dualizing complexes [Gab04].

For a ring $R$, we use $D(R)$ to denote the derived category of $R$-modules. For an ideal $I \subseteq R$ we write $D_I(R)$ for the full subcategory of $D(R)$ spanned by objects $K \in D(R)$ such that $K$ restricts to 0 in $D(U)$, where $U = \text{Spec}(A) - \text{Spec}(R/I)$

We remind the reader of the definitions of $F$-injective and Du Bois singularities.

Definition 2.1 ($F$-injective singularities, [Fed83]). Suppose $(R, m, k)$ is an $F$-finite Noetherian local ring of characteristic $p > 0$. Then $R$ is $F$-injective if $H^i_m(R) \xrightarrow{F} H^i_m(F_*R)$ is injective for all $i \geq 0$. Or dually that $H^{-i}F_\ast \omega^q_R \xrightarrow{\partial} H^{-i}\omega^q_R$ is surjective for all $i$.

Definition 2.2 (Du Bois singularities, [DB81, KS11]). Suppose that $X$ is a scheme essentially of finite type over a field of characteristic zero. We say that $X$ is Du Bois if the canonical map $O_X \xrightarrow{\pi} \Omega^0_X$ is a quasi-isomorphism. A quick way to define $\Omega^0_X$ is as follows. If $X \subseteq Y$ can be embedded in a smooth scheme and $\pi : \tilde{Y} \to Y$ is a log resolution of $X \subseteq Y$, or an embedded resolution of $X \subseteq Y$ such that the reduced exceptional divisor of $\pi$ is SNC and intersects the strict transform $\tilde{X}$ in a SNC divisor. In either case, let $\tilde{X} = (\pi^{-1}(X))_{\text{red}}$. Then $\Omega^0_X = R\pi_* O_{\tilde{X}}$, see [KS11, Theorem 6.4]

We assume that the readers are already familiar with some of the basic properties of $F$-injective and Du Bois singularities, see for instance [Fed83, KS11].

The following result was previously used to give another proof of Schenzel’s homological criterion of Buchsbaum singularities [Sch82].

Lemma 2.3 (Proposition 2.4.3 in [SV86]). Let $(R, m, k)$ be a Noetherian local ring and let $f^* : K^* \to L^*$ be a homomorphism of complexes of $R$-modules such that

(a) $K^*$ is a complex of $k$-vector spaces.
(b) $H^i(f^*) : H^i(K^*) \to H^i(L^*)$ is a surjective homomorphism for every $i$.

Then $L^*$ is quasi-isomorphic to a complex of $k$-vector spaces.

In this article we will need the following slight generalization of the dual version of this lemma in the derived category and for potentially non-Noetherian rings.
Lemma 2.4. Let \((R, m, k)\) be a local ring. Fix \(K \in D(R), L \in D(k)\). Assume there exists a map \(f: K \rightarrow L\) in \(D(R)\) such that the induced maps \(H^i(K) \rightarrow H^i(L)\) are injective. Then \(K\) comes from an object of \(D(k)\) under the forgetful functor \(D(k) \rightarrow D(R)\), i.e., \(K\) is quasi-isomorphic to a complex of \(k\)-vector spaces.

Proof. We can choose an isomorphism \(L \cong \oplus_i H^i(L)[-i]\) in \(D(k)\). As the map \(f: H^i(K) \rightarrow H^i(L)\) is injective, we can choose a decomposition \(H^i(L) \cong f(H^i(K)) \oplus Q_i\). These choices give a map \(L \cong \oplus_i H^i(L)[-i] \rightarrow \oplus_i f(H^i(K))[-i] =: K'\) in \(D(k)\). The composite \(K \rightarrow L \rightarrow K'\) in \(D(R)\) is a quasi-isomorphism by construction, so \(K \cong K'\) comes from \(D(k)\). \(\Box\)

3. \(F\)-injective singularities

In this section, we prove our main results in characteristic \(p\). The proofs in this section rely crucially on the following observation concerning perfect rings:

Proposition 3.1. Let \(A \rightarrow B\) be a surjection of perfect \(\mathbb{F}_p\)-algebras. Let \(K \in D^b(A)\) be a complex such that each \(H^i(K)\) is a \(B\)-module. Then \(K \cong K \otimes_A^L B\) via the canonical map, and thus \(K\) comes from \(D^b(B)\) via the forgetful functor \(D^b(B) \rightarrow D^b(A)\).

Proof. We must check that the canonical map \(K \rightarrow K \otimes_A^L B\) is an isomorphism for \(K\) as above. This assertion is stable under exact triangles, so we reduce to \(K = M[0]\) being a \(B\)-module \(M\) placed in degree 0. But then \(K \otimes_A^L B \cong M[0] \otimes_B (B \otimes_A^L B)\), so the claim follows from [BS16, Lemma 5.10], which implies that \(B \otimes_A^L B \cong B\) via the multiplication map. \(\Box\)

Remark 3.2. Let \(A \rightarrow B\) be a surjection of perfect \(\mathbb{F}_p\)-algebras. Using the same method used to prove [Proposition 3.1] one can show the following: the forgetful functor \(D(B) \rightarrow D(A)\) is fully faithful, and the essential image comprises those \(K \in D(A)\) such that each \(H^i(K)\) is a \(B\)-module. We do not prove this here as we do not need it.

Using [Proposition 3.1] we obtain a criterion for when a “finite length \(\phi\)-complex” is pushed forward from the closed point:

Lemma 3.3. Let \((R, m, k)\) be a local ring of characteristic \(p > 0\) with absolute Frobenius \(F: R \rightarrow R\). Fix \(K \in D(R)\) such that each \(H^i(K)\) has finite length. Assume we are given a map \(\phi_K: K \rightarrow F_*K\) which is injective on each \(H^i\). Then \(K\) comes from \(D(k)\).

Proof. Let \(R_\infty\) be the perfection of \(R\), so \(R_\infty \cong \lim F_*^e R\) is the direct limit of copies of \(R\) along (twists of) the Frobenius map \(F: R \rightarrow F_*^e R\). In particular, \(R_\infty\) is a local ring (as Frobenius is a homeomorphism), is perfect (by construction), has maximal ideal \(m_\infty := \lim F_*^e m\) (with the same transition maps as before), and has residue field \(k_\infty\), the perfection of \(k\). Define \(K_\infty\) to be the homotopy-colimit (i.e. derived colimit)

\[
\text{hocolim}_e F_*^e K := \text{hocolim} \left( K \rightarrow F_*^e K \rightarrow F_*^{e2} K \rightarrow \ldots \right)
\]

as in [spa15 Tag 0A5K]. Then \(K_\infty \in D(R_\infty)\). Note that we have

\[
H^i(K_\infty) := \lim F_*^e H^i(K) := \lim \left( H^i(K) \rightarrow F_* H^i(K) \rightarrow F_*^2 H^i(K) \rightarrow \ldots \right),
\]

as cohomology commutes with direct limits. Also, the canonical map \(K \rightarrow K_\infty\) is injective on each \(H^i(-)\) by the assumption on \(\phi_K\) (and because \(F_*^e\) is exact). We claim that \(K_\infty\)
lies in the essential image of the forgetful functor \( D(k_{\infty}) \to D(R_{\infty}) \); this suffices to prove the result by [Lemma 2.4](#). To see this, using [Proposition 3.1](#) for the map \( R_{\infty} \to k_{\infty} \), it is enough to show that \( H^i(K_{\infty}) \) is killed by \( m_{\infty} \subset R_{\infty} \). By the formula for \( H^i(K_{\infty}) \) above and the formula \( m_{\infty} = \lim_m F^i_m m \) (with the transition maps being Frobenius), it suffices to show that \( m \) kills \( H^i(K) \). As \( K \) has finite length homology, there exists some \( c \) such that \( m^c \) kills \( H^i(K) \). But then \( m \) kills \( F^e H^i(K) \) if \( e \geq \log_p(c) \) since \( F^e(m) \subset m^c \) for such \( e \). As the map \( H^i(K) \to F^e H^i(K) \) induced by an \( e \)-fold iterate of \( \phi_K \) is injective, it then follows that \( m \) kills \( H^i(K) \), as wanted. \( \square \)

This lemma specializes to prove the following result:

**Theorem 3.4.** Let \((R, m, k)\) be a Noetherian local ring of characteristic \( p > 0 \). Assume there exists some \( j \geq 0 \) such that \( H^i_m(R) \) has finite length for every \( i < j \), and that the Frobenius on \( R \) induces an injective map on \( H^i_m(R) \) for \( i < j \) (e.g., \( R \) is \( F \)-injective). Then \( \tau^{<j} R \Gamma_m(R) \subset D(R) \) comes from an object of \( D(k) \). In other words, \( \tau^{<j} R \Gamma_m(R) \) and \( \tau_{>-\infty} \omega_R^d \) are quasi-isomorphic to complexes of \( k \)-vector spaces.

**Proof.** Set \( K = \tau^{<j} R \Gamma_m(R) \). Then the Frobenius on \( R \) induces a map \( \phi_K : K \to F^e K \) that satisfies the hypotheses of [Lemma 3.3](#) by assumption. Thus by [Lemma 3.3](#) \( \tau^{<j} R \Gamma_m(R) \) comes from \( D(k) \). \( \square \)

As an immediate consequence of the above theorem, we reproive and in fact generalize the main result of [Ma15](#). We note that this result is also obtained using different methods in [QS16](#).

**Corollary 3.5.** Let \((R, m, k)\) be a Noetherian local ring of characteristic \( p > 0 \) and dimension \( d \) such that \( H^i_m(R) \) has finite length for every \( i < d \). Suppose Frobenius acts injectively on \( H^i_m(R) \) for every \( i < d \) (e.g., \( R \) is \( F \)-injective). Then \( R \) is Buchsbaum.

**Proof.** Applying [Theorem 3.4](#) to \( j = d \), \( \tau^{<d} R \Gamma_m(R) \) is quasi-isomorphic to a complex of \( k \)-vector spaces. This implies \( R \) is Buchsbaum by Schenzel’s criterion [Sch82](#). \( \square \)

Next we prove a stronger version of [Theorem 3.4](#) when \( R \) is \( F \)-rational on the punctured spectrum. Basically we will show that, in this case, if we truncate \( R \Gamma_m R \) at the \( d \)-th spot (resp., \( \omega_R^d \) at the \(-d\)-th spot) “up to tight closure”, it is still quasi-isomorphic to a complex of \( k \)-vector spaces.

We introduce some notations. We let \((R, m, k)\) be a reduced and equidimensional local ring of characteristic \( p > 0 \) and dimension \( d \). Let \( R \Gamma_m R = 0 \to G^0 \to G^1 \to \cdots \to G^{d-1} \to G^d \to 0 \). We define

\[
\tau^{<d,*} R \Gamma_m R = 0 \to G^0 \to G^1 \to \cdots \to G^{d-1} \to (\text{im} \phi)^*_{G^d} \to 0
\]

to be the \(*\)-truncation of \( R \Gamma_m R \) at the \( d \)-th spot (where \((-)^*\) denotes tight closure). This is a well-defined object in the derived category of \( R \)-modules: it is the natural object such that we have an exact triangle\(^2\)

\[
\tau^{<d,*} R \Gamma_m R \to R \Gamma_m R \to H^d_m(R) / \mathcal{O}^*_{H^d_m(R)}[-d] \xrightarrow{+1}.
\]

\(^2\text{Equivalently, } \tau^{<d,*} R \Gamma_m(R)[1] \in D(R) \text{ is the cone of the canonical composite map } R \Gamma_m(R) \to H^d_m(R)[-d] \to (H^d_m(R) / \mathcal{O}^*_{H^d_m(R)})[-d]. \]

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Dually, let \( \omega_R^* = 0 \to I^{-d} \to I^{-d-1} \to \cdots \to I^{-1} \to I^0 \to 0 \) be the normalized dualizing complex and let \( \tau(\omega_R) \subseteq \omega_R \) be the parameter test submodule of \( R \) (see [Smi94] or Section 2 of [BST15] for definitions and details). We define
\[
\tau_{\geq -d}^* \omega_R^* = 0 \to I^{-d}/\tau(\omega_R) \to I^{-d-1} \to \cdots \to I^{-1} \to I^0 \to 0
\]
to be the \( * \)-truncation of \( \omega_R^* \) at \( -d \)-th spot. Again this is the object in the derived category that completes the triangle:
\[
\tau(\omega_R)[d] \to \omega_R^* \to \tau_{\geq -d}^* \omega_R^* \to 0.
\]

It is easy to check that \( \tau_{\leq -d}^* \text{R} \Gamma^m R \) is the Matlis dual of \( \tau_{\geq -d}^* \omega_R^* \), as \( \tau(\omega_R)^{\vee} \cong H^d_m(R)/H_{hd}^d(R) \).

We assume the readers are familiar with basic tight closure theory [HH90], in particular the Lemma 3.7.

Alternative proofs of the main theorems when \( k \) is perfect. In this subsection we give a more elementary, but in some ways more involved, proof of Theorem 3.4. We start with some general lemmas.

**Lemma 3.7.** Let \( (A, m, k) \to (S, n) \) be a module-finite ring homomorphism with \( A \) Noetherian and regular. Then the map \( \text{Ext}^i_A(k, S) \to H^i_m(S) \) is the Matlis dual of the map \( H^{-i}(\omega_S^*) \to H^{-i}(\omega_S^* \otimes^L_A k) \).

**Proof.** First notice that
\[
\text{R} \text{Hom}_A(k, S) = \text{R} \text{Hom}_A(k, \text{R} \text{Hom}_A(\omega_S^*, \omega_A^*)) = \text{R} \text{Hom}_A(k \otimes^L_A \omega_S^*, \omega_A^*).
\]
Since \( A \) is regular, \( k \otimes^L_A \omega_S^* \) is a bounded complex. By local duality, \( H^i(\text{R} \text{Hom}_A(k \otimes^L_A \omega_S^*, \omega_A^*)) \) is the Matlis dual of \( H^{-i}(\text{R} \text{Hom}_A(\omega_S^* \otimes^L_A k)) = H^{-i}(\omega_S^* \otimes^L_A k) \).

The map \( \text{Ext}^i_A(k, S) \to H^i_m(S) \) can be identified with the natural map
\[
H^i(\text{R} \text{Hom}_A(k, S)) \to H^i(\lim_{n \to} \text{R} \text{Hom}_A(A/m^n, S)) \cong H^i(\text{R} \Gamma^m S).
\]
Hence by local duality this is the Matlis dual of \( H^{-i}(\omega_S^* \otimes^L_A k) \leftarrow H^{-i}(\omega_S^*) \). \( \square \)

**Lemma 3.8.** Let \( (A, m, k) \to (R, m, k) \) be a surjective homomorphism between \( F \)-finite Noetherian local rings of characteristic \( p > 0 \). Let \( G^* \) be a bounded complex of \( R \)-modules such that \( H^{-i}G^* \) has finite length. If \( A \) is regular, then \( H^{-i}(F^e G^*) \to H^{-i}(F^e G^* \otimes^L_A k) \) is injective for \( e \gg 0 \).
Proof. By the projection formula, it is enough to show $H^{-i}(G^*) \to H^{-i}(G^* \otimes_A^L A/m^{[p^e]})$ is injective for $e \gg 0$. Set $\text{dim } A = n$. We take a minimal free resolution of $A/m^{[p^e]}$ over $A$:

$$P^* = 0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^{-1} \xrightarrow{\partial} P^0(=A) \to 0,$$

where we use $P^{-i}$ to emphasize that these modules are in cohomology degree $-i$. The map $g$ is represented by a minimal generating set of $m^{[p^e]}$, say $[y_1^e, \ldots, y_n^e]$. We consider the double complex $G^* \otimes_A P^*$:

$$\cdots \to G^{-i-1} \otimes P^{-1} \to G^{-i} \otimes P^{-1} \to G^{-i+1} \otimes P^{-1} \to \cdots$$

We know that the map $H^{-i}(G^*) \to H^{-i}(G^* \otimes_A^L A/m^{[p^e]})$ is induced by the natural map of the $-i$-th cohomology of the bottom row to the $-i$-th cohomology of the total complex of $G^* \otimes_A P^*$. Pick $x \in \ker \alpha_i$, if $(x, 0, \ldots, 0) \in \oplus_s(G^{-i+s} \otimes P^{-s})$ is a boundary in the total complex, then we have:

$$x \in (\text{im } \alpha_{i+1} + \text{im}(\text{id} \otimes g)) \cap \ker \alpha_i$$

$$= (\text{im } \alpha_{i+1} + m^{[p^e]}G^{-i}) \cap \ker \alpha_i$$

$$\subseteq (\text{im } \alpha_{i+1} + m^{[p^e]}G^{-i}) \cap \ker \alpha_i$$

Hence for $e \gg 0$, by Artin-Rees lemma, we have $x \in \text{im } \alpha_{i+1} + m^{[p^e]-l} \ker \alpha_i$ for some fixed $l > 0$. Since $H^{-i}(G^*) = \ker \alpha_i / \text{im } \alpha_{i+1}$ has finite length by assumption, for $e \gg 0$ we have $m^{[p^e]-l} \ker \alpha_i \subseteq \text{im } \alpha_{i+1}$. Thus for $e \gg 0$, $x \in \text{im } \alpha_{i+1}$, that is, $(x, 0, \ldots, 0) \in \oplus_s(G^{-i+s} \otimes P^{-s})$ is a boundary in the total complex if and only if $x$ is already a boundary in $G^*$. This proves $H^{-i}(G^*) \to H^{-i}(G^* \otimes_A^L A/m^{[p^e]})$ is injective for $e \gg 0$. \hfill $\Box$

The following lemma is crucial. The idea comes from [Han99, Proposition 6.3.5]

Lemma 3.9. Let $(R, m, k)$ be an $F$-finite Noetherian local ring of characteristic $p > 0$ such that $H^i_m(R)$ has finite length for every $0 \leq i < j$. If $R$ is $F$-injective and $k$ is perfect, then depth$(F^e_R/R) \geq j$ for every $e \geq 1$.

Proof. First of all, $R$ is $F$-finite and $F$-injective, which implies $R$ is reduced, for example see [SZ13]. Now the short exact sequence:

$$0 \to R \to F^e_*R \to (F^e_*R)/R \to 0$$

induces a long exact sequence on local cohomology:

$$\cdots \to H^i_m(R) \xrightarrow{\phi_i} H^i_m(F^e_*R) \to H^i_m((F^e_*R)/R) \to \cdots$$

Since $k$ is perfect, we have $l_R(H^i_m(R)) = l_{F^e_*R}(H^i_m(F^e_*R)) = l_R(H^i_m(F^e_*R))$. Since $R$ is $F$-injective, for every $0 \leq i < j$, the map $\phi_i$ is an injective map between $R$-modules of the same length, and hence bijective. Chasing the long exact sequence we then know that $H^i_m((F^e_*R)/R) = 0$ for every $0 \leq i < j$, which proves that depth$(F^e_*R/R) \geq j$. \hfill $\Box$
Now we give an alternative proof of Theorem 3.4 when $R$ is $F$-injective and $k$ is perfect.

**Theorem 3.10.** Let $(R, m, k)$ be a Noetherian local ring of characteristic $p > 0$ such that $H^i_m(R)$ has finite length for every $0 \leq i < j$. If $R$ is $F$-injective with $k$ perfect, then $\tau_{> j} \omega^*_R$ is quasi-isomorphic to a complex of $k$-vector spaces.

**Proof.** Since whether $\tau_{> j} \omega^*_R$ or $\tau_{< j} \Omega_G R$ is quasi-isomorphic to a complex of $k$-vector spaces is unaffected under completion, we may complete $R$ and assume $(R, m, k)$ is a complete local ring with $k$ perfect. In particular, $R$ is finite and by Cohen’s structure theorem we can fix $(A, m, k) \to (R, m, k)$ with $A$ a regular local ring. By Lemma 3.8 applied to $G = \omega^*_R$, we can pick $e \gg 0$ such that $H^{-i}(F^*_e \omega^*_R) \to H^{-i}(F^*_e \omega^*_R \otimes^L_A k)$ is injective for every $0 \leq i < j$ (note that $H^{-i} \omega^*_R$ has finite length because its dual $H^i_m(R)$ has finite length for every $0 \leq i < j$). Now by Lemma 3.7 applied to $S = F^*_e R$, we have $\text{Ext}^i_A(k, F^*_e R) \to H^i_m(F^*_e R)$ is surjective for every $0 \leq i < j$.

By Lemma 3.9 we know that $\text{depth}_A(F^*_e R)/R \geq j$. Now the short exact sequence

$$0 \to R \to F^*_e R \to (F^*_e R)/R \to 0$$

induces the following commutative diagram:

$$
\begin{array}{cccccc}
0 = \text{Ext}^{-i-1}_A(k, (F^*_e R)/R) & \longrightarrow & \text{Ext}^i_A(k, R) & \longrightarrow & \text{Ext}^i_A(k, F^*_e R) & \longrightarrow & \text{Ext}^i_A(k, (F^*_e R)/R) = 0 \\
0 = H^{-i-1}_m((F^*_e R)/R) & \longrightarrow & H^i_m(R) & \longrightarrow & H^i_m(F^*_e R) & \longrightarrow & H^i_m((F^*_e R)/R) = 0 \\
\end{array}
$$

A diagram chase shows that the surjectivity of $\text{Ext}^i_A(k, F^*_e R) \to H^i_m(F^*_e R)$ implies the surjectivity of $\text{Ext}^i_A(k, R) \to H^i_m(R)$. Thus Lemma 3.7 implies that $H^{-i}(\omega^*_R) \to H^{-i}(\omega^*_R \otimes^L_A k)$ is injective and thus so is $H^{-i}(\tau_{> j} \omega^*_R) \to H^{-i}(\tau_{> j} \omega^*_R \otimes^L_A k)$. Now Lemma 2.4 shows that $\tau_{> j} \omega^*_R$ is quasi-isomorphic to a complex of $k$-vector spaces.

**Remark 3.11.** In the previous version of this paper (available on the arXiv) we gave a proof of Theorem 3.6 again in the case where the residue field is perfect. However, the proof was rather involved and so we omit it here.

### 4. Du Bois singularities

We now prove our main results for Du Bois singularities. Suppose $X$ is a reduced scheme essentially of finite type over a field $k$ of characteristic zero and $\pi : W \to X$ is a resolution of singularities. Then by for instance [KST11, Theorem 4.2.2], there exists a natural map $\Omega^0_W \to R\pi_* \Omega^0_X$. Since $W$ is smooth, and hence has Du Bois singularities, $\Omega^0_W \simeq_{qis} \Omega_W$ and hence there is a natural map

$$(4.0.1) \quad \Omega^0_{\underline{X}} \to R\pi_* \Omega_W.$$

**Theorem 4.1.** Suppose that $(R, m, k)$ is a reduced local ring essentially of finite type over a field of characteristic zero. Let $k : W \to X = \text{Spec} R$ be a resolution of singularities. Suppose that

$$
R \kappa_* \omega^*_W \to R \mathcal{H}om_X(\Omega^0_{\underline{X}}, \omega^*_X) \to C \overset{+1}{\to}
$$

is a triangle where the first map is the Grothendieck dual of $(4.0.1)$. Suppose that $H^{-i}C$ has finite length for all $0 \leq i < j$ for some $j$ (equivalently, it is supported within $V(m)$ for those
same $i$). Then $\tau_{> - j} C$ is quasi-isomorphic to a complex of $k$-vector spaces and in particular, $\tau_{> - j} C \simeq_{\text{qis}} \oplus_{0 \leq i < j} H^{-i}(C)[i]$ is a direct sum of its cohomologies.

**Proof.** Let $(A, m) \to (R, m)$ be a surjection with $A$ regular. Let $X = \text{Spec } R$ and $Y = \text{Spec } A$. We take a log resolution $\pi : \tilde{Y} \to Y$ of $(Y, X)$ obtained in the following way. First form an embedded resolution of $X$ in $Y$, $\kappa : Y' \to Y$, in such a way that the exceptional divisor is SNC and intersects the strict transform of $X$ with normal crossings in a SNC divisor. Then blow up the strict transform $W$ of $X$ in $Y'$, see for instance [BEV05], obtaining a log resolution

$$\pi : \tilde{Y} \to Y' \to Y$$

of $(Y, X)$ and also of $(Y', W)$. Note $\kappa = \kappa|_{W} : W \to X$ is a resolution of singularities. We observe also that $\tilde{X} = \rho^{-1}W$ is the union of components of $\overline{X} = \pi^{-1}(X)_{\text{red}}$ whose $\pi$-image dominates a component of $X$. Write $\overline{X} = \tilde{X} + E + F$ where $E = \pi^{-1}(V(m))_{\text{red}}$.

**Claim 4.2.** With notation as above $R_{\pi} \omega^*_{\tilde{X}} \simeq_{\text{qis}} R_{\kappa} \omega^*_W$.

**Proof of claim.** There is a map $\rho : \tilde{X} \to W$ by construction. It suffices to show that $R_{\rho} \omega^*_{\tilde{X}} \simeq_{\text{qis}} \omega^*_W$. But $W$ is smooth and hence Du Bois and recall that $(\tilde{Y}, \tilde{X}) = (\rho^{-1}(W)_{\text{red}})$ is a log resolution of $(Y', W)$. Therefore by applying the criterion from [Definition 2.2] we see that $R_{\rho} \omega^*_{\tilde{X}} \simeq_{\text{qis}} \omega^*_{W}$. Applying Grothendieck duality proves the claim. □

We next note that $R_{\pi} \omega^*_{\tilde{X}} \simeq_{\text{qis}} R_{\kappa} \omega^*_W$.

Furthermore observe that the map

$$R_{\pi} \omega^*_W \to R_{\kappa} \omega^*_W$$

is induced by the inclusion $\tilde{X} \to \overline{X}$. By blowing up further (even at the $\kappa$-stage) if needed, we may assume that no stratum of $\tilde{X} + F$ lies over $m$. Indeed, if there are any such strata, simply blow them up so that their inverse images become parts of $E$.

With this notation, we observe that we have the short exact sequence:

$$0 \to \mathcal{O}_{E \cup F}(\tilde{X}) \to \mathcal{O}_X \to \mathcal{O}_{\tilde{X}} \to 0$$

and so by dualizing and pushing forward, we see that

$$C \simeq_{\text{qis}} R_{\pi} \omega^*_{E \cup F}(\tilde{X}).$$

From the short exact sequence

$$0 \to \mathcal{O}_E(\tilde{X} - F) \to \mathcal{O}_{E \cup F}(\tilde{X}) \to \mathcal{O}_F(\tilde{X}) \to 0$$

we obtain

$$R_{\pi} \omega^*_E(\tilde{X}) \to C \to R_{\pi} \omega^*_E(\tilde{X} + F) \to 1.$$
image by our construction. Hence we see that $H^{-i}R\pi_*\omega^*_F(\widetilde{X}) = 0$ for all $i < j$. Thus
\[ \tau_{> j} R\pi_*\omega^*_F(\widetilde{X}) \simeq_{\text{qis}} 0. \] It follows that
\[ \tau_{> j} C \simeq_{\text{qis}} \tau_{> j} R\pi_*\omega^*_F(\widetilde{X} + F). \]

But since $\pi(E) = V(\mathfrak{m})$, we see that $\mathfrak{m}$ annihilates the terms of $R\pi_*\omega^*_E(\widetilde{X} + F)$. In other words, $R\pi_*\omega^*_E(\widetilde{X} + F)$ is a complex of $k$-vector spaces. Thus $\tau_{> j} C$ is quasi-isomorphic to a complex of $k$-vector spaces as claimed.

While technical, this theorem immediately implies results analogous to those we proved previously for $F$-injective singularities.

**Corollary 4.3.** Suppose that $(R, \mathfrak{m}, k)$ is a reduced local ring essentially of finite type over a field of characteristic zero. Suppose that $X = \text{Spec } R$ has Du Bois singularities and that $H^i_\mathfrak{m}(R)$ has finite length for all $i < j \leq \dim R$. Then $\tau^{-j} R\Gamma_{\mathfrak{m}}(R)$ is quasi-isomorphic to a complex of $k$-vector spaces or dually, $\tau_{> j} \omega^*_X$ is quasi-isomorphic to a complex of $k$-vector spaces.

In particular, Du Bois singularities with isolated non-Cohen-Macaulay locus are Buchsbaum.

**Proof.** Because $X$ has Du Bois singularities, $\omega^*_X \simeq_{\text{qis}} R\mathcal{H}\text{om}_X(\mathcal{O}_X^0, \omega^*_X)$. We then notice that $\tau_{> j} C \simeq_{\text{qis}} \tau_{> j} \omega^*_X$ where $C$ is defined as in [Theorem 4.1] since $j \leq \dim R$. The result immediately follows.

Since the parameter test submodule $\tau(\omega_R)$ is well-known to be the characteristic $p > 0$ analogue of the Grauert-Riemenschneider canonical sheaf $\pi_*\omega_W = \omega^*_R$. The following result is an analogue of [Theorem 3.6]

**Corollary 4.4.** Suppose that $(R, \mathfrak{m}, k)$ is a reduced and equidimensional local ring essentially of finite type over a field of characteristic zero. Suppose that $X = \text{Spec } R$ has Du Bois singularities and that $X \setminus V(\mathfrak{m})$ has rational singularities. Let $\pi : W \to X$ is a resolution of singularities and consider the triangle:
\[ \pi_*\omega_W[d] \to \omega^*_X \to C \to \mathcal{O}_X^0. \]
Then $C$ is quasi-isomorphic to a complex of $k$-vector spaces and so is a direct sum of its cohomologies.

**Proof.** We first observe that $\pi_*\omega_W[d] \simeq_{\text{qis}} R\pi_*\omega^*_W$ by the Grauert-Riemenschneider vanishing theorem because $W$ is smooth. Since $X \setminus V(\mathfrak{m})$ has rational singularities, we see that $H^{-i}C$ is supported within $V(\mathfrak{m})$ for all $i < j = \dim X + 1$. Finally since $X$ has Du Bois singularities, $\mathcal{O}_X \simeq_{\text{qis}} \mathcal{O}_X^0$, and we see that $\omega^*_X \simeq_{\text{qis}} R\mathcal{H}\text{om}_X(\mathcal{O}_X^0, \omega^*_X)$. Putting this all together, we see that $C$ is the $C$ as in [Theorem 4.1] and that all the conditions of [Theorem 4.1] are satisfied with $j = \dim X + 1$ (and hence $C = \tau_{> j} C$). The conclusion follows.

5. Further questions

Buchsbaum singularities have the property that their truncated ($> \dim$) dualizing complexes are the direct sum of their cohomologies. One might ask if the full dualizing complex splits into a direct sum of all its cohomologies, especially in view of results such as [Kol86] or
Proposition 5.1. Let $R$ be a Noetherian ring, and $I \subset R$ an ideal such that $\text{Spec}(R/I)$ is connected. Then $\mathbf{R} \Gamma_I(R)$ is indecomposable in $D(R)$, i.e., it admits no non-trivial direct sum decomposition.

In the proof below, we write $D_I(R)$ for the full subcategory of $D(R)$ spanned by complexes $K$ such that $K$ restricts to 0 in $D(U)$, where $U = \text{Spec}(A) - \text{Spec}(R/I)$; this is equivalent to asking that each $H^i(K)$ is $I^\infty$-torsion.

Proof. Write $\widehat{R}$ for the $I$-adic completion of $R$, and let $K \mapsto \widehat{K}$ denote the derived $I$-adic completion functor $D(R) \to D(\widehat{R})$; as $R$ is Noetherian, we can calculate $\widehat{K}$ by applying the naive $I$-adic completion functor $M \mapsto \lim M/I^nM$ to terms of a flat representative $K^*$ of $K$ in $D(R)$ (see [spa15, Tag 091N]). In particular, the derived $I$-adic completion of $R$ is $\widehat{R}$, so the notation is consistent. We need the following fact: if $K \in D_I(R)$, then $\mathbf{R} \Gamma_I(\widehat{K}) \simeq K$; see [spa15, Tag 06AV] for a proof. In particular, the derived $I$-adic completion of any non-zero object in $D_I(R)$ is non-zero, so this functor takes decomposable objects of $D_I(R)$ to decomposable objects of $D(R)$.

Now consider $K = \mathbf{R} \Gamma_I(\widehat{R}) \in D_I(R)$. Using the Cech complex to calculate $\mathbf{R} \Gamma_I(\widehat{R})$, we see that $\widehat{K} \simeq \widehat{R}$. Now $\widehat{R} \in D(\widehat{R})$ is indecomposable by the hypothesis that $\text{Spec}(R/I)$ is connected: if $\widehat{R} \simeq M \oplus N$ in $D(\widehat{R})$, then each of $M$ and $N$ are finite projective $\widehat{R}$-modules (viewed as complexes placed in degree 0), and their ranks give locally constant $\mathbb{N}$-valued functions $r_M, r_N : \text{Spec}(\widehat{R}) \to \mathbb{N}$ such that $r_M + r_N = 1$. As $\text{Spec}(R/I)$ is connected, so is $\text{Spec}(\widehat{R})$, so all such locally constant functions are constant, so either $r_M = 1$ and $r_N = 0$ or vice versa. Thus, either $M \simeq \widehat{R}$ and $N = 0$, or vice versa. Now the fact quoted in the previous paragraph implies that $\mathbf{R} \Gamma_I(\widehat{R})$ is also indecomposable, proving the claim. \qed

As a special case of Proposition 5.1 (the case $I = m$), we have the following result which we expect is well known to experts but we do not know a reference.

Corollary 5.2. Let $(R, m)$ be a Noetherian local ring of dimension $d$ with a dualizing complex $\omega_R^\bullet$. Then $\omega_R^\bullet \simeq_{\text{qis}} \bigoplus_{i=0}^d (H^{-i} \omega_R^\bullet)[i]$ if and only if $R$ is Cohen-Macaulay. Equivalently, $\mathbf{R} \Gamma_m R \simeq_{\text{qis}} \bigoplus_{i=0}^d (H^i \omega_m^\bullet)[i]$ if and only if $R$ is Cohen-Macaulay.

This shows that one at least must truncate at the bottom degree of the dualizing complex. Of course, there are numerous rings $R$ whose truncated dualizing complexes split into direct sums of their cohomologies even if the non-Cohen-Macaulay locus is not isolated. For instance, suppose that $R$ is an $F$-injective or Du Bois $d$-dimensional ring whose non-Cohen-Macaulay locus is isolated. Let $S = R[x]$. Then the map $f : \text{Spec } S \to \text{Spec } R$ is smooth and hence $\omega_S^\bullet = f^! \omega_R^\bullet = f^* \omega_R^\bullet = S \otimes_R \omega_R^\bullet$, see for instance [Har66, Chapter VII, Section 4]. It follows that the non-Cohen-Macaulay locus of $S$ is not isolated, but that $\tau_{> d-1} \omega_S^\bullet$ is a direct sum of its cohomologies. This suggests the following question.

Question 5.3. Do any related conditions on singularities (for instance, $F$-injective, $F$-pure, Du Bois, log canonical, having only a unique $F$-pure or log canonical center) imply that the
truncated dualizing complex
\[ \tau_{> -d} \! \omega^*_R \]
is a direct sum of its cohomologies? In particular, does this behavior still occur even when the non-Cohen-Macaulay locus is not isolated?

Our characteristic zero result on Theorem 4.1 actually suggests the following question as well.

**Question 5.4.** Suppose that \((R, \mathfrak{m}, k)\) is an \(F\)-finite Noetherian local ring of characteristic \(p > 0\). Is there a complex \(B \in D^b(R)\) analogous to the characteristic zero object \(R \Gamma_m(\Omega_{\mathfrak{m}}^0, \omega_X^*)\), the Matlis dual of \(R \hom_X(\Omega_X^0, \omega_X^*)\). In particular, we would hope for the following properties (although some might be weakened).

(a) We have a functorial map \(R \Gamma_m(R) \to B\) in the derived category.

(b) The induced maps on cohomology \(H^i_m(R) \to H^i(B)\) are surjective with kernel equal to \(0_{H^d_m(R)}\), the Frobenius closure of zero.

(c) If \(D \to B \to (H^d_m(R)/0_{H^d_m(R)})[-d] \to H^d(B)\) is an exact triangle and \(H^i(D)\) has finite length for all \(i\), then \(D\) is quasi-isomorphic to a complex of \(k\)-vector spaces.

The statement (c) is the analog of Theorem 4.1. If the residue field \(k\) is perfect and each \(H^i_m(R)\) has finite length for \(i < \dim(R)\), then such a \(B\) exists: we may set \(B\) to be the “homotopy fibre product” \(R \Gamma_m(R_{\infty}) \times_{H^d_m(R_{\infty})} (H^d_m(R)/0_{H^d_m(R)})[-d]\) (defined formally as a suitable shifted cone). In general, however, we do not know if such a \(B\) exists.

**References**

[Amb03] F. Ambro: *Quasi-log varieties*, Tr. Mat. Inst. Steklova 240 (2003), no. Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 220–239. 1993751 (2004f:14027)

[BS16] B. Bhatt and P. Scholze: *Projectivity of the Witt vector affine grassmannian*, arXiv:1507.06490, to appear in Inventiones mathematicae.

[BST15] M. Blickle, K. Schwede, and K. Tucker: *F-singularities via alterations*, Amer. J. Math 137 (2015), no. 1, 61–109.

[BEV05] A. M. Bravo, S. Encinas, and O. Villamayor: *A simplified proof of desingularization and applications*, Rev. Mat. Iberoamericana 21 (2005), no. 2, 349–458. MR2174912

[DB81] P. Du Bois: *Complexe de de Rham filtré d’une variété singulière*, Bull. Soc. Math. France 109 (1981), no. 1, 41–81. MR613848 (82j:14006)

[EH08] F. Enescu and M. Hochster: *The Frobenius structure of local cohomology*, Algebra Number Theory 2 (2008), no. 7, 721–754.

[Fed83] R. Fedder: *F-purity and rational singularity*, Trans. Amer. Math. Soc. 278 (1983), no. 2, 461–480.

[Fuj14] O. Fujino: *Fundamental theorems for semi log canonical pairs*, Algebr. Geom. 1 (2014), no. 2, 194–228.

[Gab04] O. Gabber: *Notes on some t-structures*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734.

[Han99] D. Hanes: *Special conditions on maximal Cohen-Macaulay modules, and applications to the theory of multiplicities*, Thesis, University of Michigan (1999).

[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[HH90] M. Hochster and C. Huneke: *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. 3 (1990), no. 1, 31–116.
M.-N. Ishida: The dualizing complexes of normal isolated Du Bois singularities, Algebraic and Topological theories (1984), 387–390.

J. Kollár: Higher direct images of dualizing sheaves I, Ann. of Math. (2) 123 (1986), no. 1, 11–42.

J. Kollár and S. J. Kovács: Log canonical singularities are Du Bois, J. Amer. Math. Soc. 23 (2010), no. 3, 791–813. 2629988

S. J. Kovács and K. E. Schwede: Hodge theory meets the minimal model program: a survey of log canonical and Du Bois singularities, Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 51–94. 2796408 (2012k:14003)

E. Kunz: On Noetherian rings of characteristic p, Amer. J. Math. 98 (1976), no. 4, 999–1013.

L. Ma: F-injectivity and Buchsbaum singularities, Math. Ann 362 (2015), no. 1-2, 25–42.

P. H. Quy and K. Shimomoto: F-injectivity and Frobenius closure of ideals in Noetherian rings of characteristic $p > 0$, preprint.

P. Schenzel: Applications of dualizing complexes to Buchsbaum rings, Adv. Math. 44 (1982), no. 1, 61–77.

K. Schewde and W. Zhang: Bertini theorems for F-singularities, Proc. Lond. Math. Soc. (3) 107 (2013), no. 4, 851–874.

K. E. Smith: Tight closure of parameter ideals, Invent. Math. 115 (1994), no. 1, 41–60.

K. E. Smith: F-rational rings have rational singularities, Amer. J. Math. 119 (1997), no. 1, 159–180.

T. stacks project authors: The Stacks Project, Available at http://stacks.math.columbia.edu

J. Stückrad and W. Vogel: Buchsbaum rings and applications, Springer-Verlag, Berlin, 1986.