New Numerical Methods for Iterative or Perturbative Solution of Quantum Field Theory

Stephen C. Hahn and G. S. Guralnik
Department of Physics, Brown University, Providence RI 02912–1843

Abstract

A new computational idea for continuum quantum Field theories is outlined. This approach is based on the lattice source Galerkin methods developed by García, Guralnik and Lawson. The method has many promising features including treating fermions on a relatively symmetric footing with bosons. As a spinoff of the technology developed for “exact” solutions, the numerical methods used have a special case application to perturbation theory. We are in the process of developing an entirely numerical approach to evaluating graphs to high perturbative order.

1 General Approach

Conventional numerical evaluation of quantum field theory involves evaluating the path integral on a spacetime lattice using Monté Carlo integration methods. Monté Carlo methods have been very successful for many problems and have served to at least roughly confirm the standard model of particle interactions. However, many serious issues are either not yet solved by this approach or not likely to be calculable at all. These include dealing with actions that are not manifestly positive definite or which have important effects from the details of fermionic interactions beyond the quenched approximation. Consequently, it is worthwhile to search for supplementary methods of numerical calculation.

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1To appear in Proceedings of Fourth Workshop on Quantum Chromodynamics, American University of Paris, 1–6 June 1998.
2E-mail: hahn@het.brown.edu.
3E-mail: gerry@het.brown.edu.
The source Galerkin method, although at a relatively primitive stage of development, appears to work on both the lattice and the continuum while dealing with fermions as easily as bosons. Of course, the continuum version of source Galerkin completely avoids the usual lattice problem of fermionic multiplicity and the tricks used to minimize errors from this problem. Furthermore, our “source Galerkin” method is less restrictive as to the class of allowed actions. Source Galerkin tends to use significantly less compute time than Monté Carlo methods but can consume significantly larger amounts of memory. The formulation of a source Galerkin approximation to a given field theory can also require much more initial work and input to the calculation than the usual Monté Carlo approaches.

We confine this discussion to the continuum analysis of $\phi^4$ interactions; examples of lattice calculations have been given elsewhere.\[1, 2, 3\] Our continuum results to date include non-linear sigma models and four-fermion interactions. We have gauge theory calculations in progress.

We start with the differential equations satisfied by the vacuum function $Z$ for a scalar field $\phi$ with interaction $g\phi^4/4$ coupled to a scalar source $J(x)$ which satisfies the equation:

$$\left(-\partial^2_D + M^2\right) \frac{\delta Z}{\delta J(x)} + g \frac{\delta^3 Z}{\delta J^3(x)} - J(x)Z = 0 \quad (1)$$

The source Galerkin technique is designed to directly solve functional differential equations of this type. Before we proceed to outline a solution technique, it is essential to point out that this equation by itself does not uniquely specify a theory.\[4\] Since it is a set of third order differential equations at every point in space time, it possesses, a priori, an infinite solution set. The solution studied in this talk will be the usual one which corresponds to the symmetry preserving solution obtained from evaluating a path integral with real axis definitions for the regions of integration. This is the solution that is regular in the coupling, $g$, as it approaches zero.
If we write (1) in the form:
\[ \hat{E}_J Z(J) = 0 \]  
(2)

The source Galerkin method is defined by picking an approximation \( Z^*(J) \) to the solution \( Z(J) \) such that
\[ \hat{E}_J Z^*(J) = R \]  
(3)

where \( R \) is a residual dependent on \( J \). We pick the parameters of our approximation to make this residual as small as possible on the average. To give this statement a meaning, we define an inner product over the domain of \( J \): i.e. \( (A, B) \equiv \int d\mu(J) A(J) B(J) \).

In addition, we assume we have a collection of test functions which are members of a complete set: \( \{\varphi_i(J)\} \). The source Galerkin minimization of the residual \( R \) is implemented by setting the parameters of our test function \( Z^*(J) \) so that projections of test functions against the residual vanish so that \( ||Z^* - Z||_2 \to 0 \) as the number of test functions \( \to \infty \).

The equations defining the quantum field theory are differential equations in the field sources and spacetime. We can deal with the continuum by taking advantage of our knowledge of functional integration to evaluate integrals of the form:
\[ I = \int [dJ] \exp \left[ -J^2(x)/\epsilon^2 \right] P(J) \]  
(4)

and use this to define an inner product of sources on the continuum as follows:
\[ (J(x_1) \cdots J(x_n), J(y_1) \cdots J(y_m))_J = \begin{cases} \epsilon^{n+m} \delta_+ \{x_1 \cdots x_n y_1 \cdots y_m\} & n + m \text{ even} \\ 0 & \text{otherwise} \end{cases} \]  
(5)

where we have absorbed a factor of 2 by redefining \( \epsilon \). \( \delta_+ \) is defined by
\[ \delta_+ \{x\alpha\beta \cdots\} = \delta(x - \alpha)\delta_+ \{\beta \cdots\} + \delta(x - \beta)\delta_+ \{\alpha \cdots\} + \cdots, \]  
(6)
\[ \delta_+ \{x\alpha\} = \delta(x - \alpha). \]  
(7)
In addition to this inner product definition, we need good guesses for approximate form for $Z^*$ and numerical tools to calculate, symbolically or numerically, various functions and their integrals, derivatives, and so on. For most of our calculations, we have found it very useful to choose a lesser known class of functions, with very suitable properties for numerical calculation, known as Sinc functions. We take our notation for the Sinc functions from Stenger:

$$S(k, h)(x) = \frac{\sin(\pi(x - kh)/h)}{\pi(x - kh)/h}$$ \hspace{1cm} (8)

Sinc approximations satisfy many identities, which make these functions very easy to use for Galerkin methods, collocation, integration by parts, and integral equations.

With the definition of a norm and set of expansion functions, we can postulate an ansatz for $Z$:

$$Z^* = \exp \left[ \sum \int_{xy} J(x)G_2(x - y)J(y) + \cdots \right]$$ \hspace{1cm} (9)

This ansatz with only free field structure is very simple and one would not expect to describe the entirety of a given theory with it. However, the strength of the Galerkin method to force convergence to correct answers is often strong enough to yield remarkably good results for the lowest state of a theory. Extremely good approximations to theories are obtained as additional source structure combined with the appropriate spacetime structure is added to the ansatz.

In principle, it is possible to allow the functions multiplying the sources to be totally arbitrary and then to fix them by expanding in Sinc functions and apply the Galerkin technique. The associated computations soon demand more memory than is available from current computing systems. Instead we can use our knowledge of the spectral representations of field theory and graphical approaches (developed from perturbation theory but we emphasize that our approach is not perturbative) to introduce a beautiful
and intuitive approach of regulated Lehmann representations to produce candidates for \( Z^* \). These build the appropriate spacetime Lorentz structure into our approximations and make the operational cost of our numerical approach independent of spacetime dimension. Since any exact two-point function can be represented as a sum over free two-point functions an appropriate representation of the two-point function greatly facilitates this approach. We choose as the basis of our numerical solutions a regulated Euclidean propagator structure:

\[
\Delta(m; x) \equiv \int (dp) \frac{e^{ip \cdot x - p^2/\Lambda^2}}{p^2 + m^2} = \int (dp) \int_0^{\infty} ds \frac{\pi}{s + 1/\Lambda^2} \frac{e^{-sm^2}}{e^{s(p^2 + m^2)}} \]  

(10)

This cutoff in the Galerkin test function as well as the definition of inner product assures that we never have to address the issue of a divergence during the course of a calculation, just as in the case of lattice calculation where the lattice spacing serves as the cutoff.

This integral can be approximated using Sinc methods

\[
\Delta(m; x) \approx \frac{h}{(2\pi)^d} \sum_{k=-N}^{N} \frac{1}{e^{kh}} \left[ \frac{\pi}{z_k + 1/\Lambda^2} \right]^{d/2} \exp \left[ -z_k m^2 - \frac{x^2}{4(z_k + 1/\Lambda^2)} \right] , \]  

(12)

Here \( z_k = e^{kh} \). For practical purposes, a sum over eighty terms with \( \Lambda^2 = 10 \) gives accuracy out to twelve digits—more than adequate for most hardware floating-point representations. Thus we have a form for a two-point scalar Green function, regulated by the scale \( \Lambda^2 \) with constant computational cost regardless of spacetime dimension.

We can take derivatives explicitly or by construction:

\[
\partial^2 \Delta(m; x) = m^2 \Delta(m; x) - \delta(x) \]  

where \( \delta(x) = e^{-x^2\Lambda^2/4} \)
From this representation, we can directly construct a fermion two-point function:

\[ S(m; x) = (\gamma \cdot \partial - m)\Delta(m; x) \]  \hspace{1cm} (14)

These representations mean that free scalar and free fermion results are exact and immediate in any Galerkin evaluation of these trivial cases.

2 Results: \( \phi^4 \)

We itemize some results obtained using a regulated single propagator with parameters set by the Source Galerkin method. At lowest order, our ansatz for the generating functional is

\[ Z^* = \exp \int \frac{1}{2} j_x G_{xy} j_y. \]  \hspace{1cm} (15)

Results for this ansatz are given in Figure 1. These results are strikingly accurate and can matched up essentially exactly with results of Monté Carlo calculations in two and higher dimensions.
Figure 2: Additional connector-based ansätze for the four-point function, $H$, in $\lambda \phi^4$. In the bottom row, we have two contact ansätze on the left, followed by two mediated ansätze.

We can enhance these results by including additional 4 source terms in $Z^*$. Some simple additional terms that we include with weights and masses to be calculated using the Source Galerkin technique are the terms of the forms given in Figure 2. The effect of adding a fourth order term is shown in Figure 3.

3 Perturbation Theory

The representation that we have given for the free propagator (10) provides the basis for an entirely numerical expansion of any Feynman graph. We have examined the possibility of generating all graphs of a theory through automatic functional differentiation of a vacuum amplitude and then numerically analyzing the resultant expressions using Sinc expansions. This method generates and evaluates graphs at a speed which to our knowledge far exceeds any method used in the past. We believe that it will be possible to calculate to quite high orders in perturbation theory.
Figure 3: Comparison of four-$H$ approximation with lowest order and exact answer, in one-dimension ($\Lambda^2 = 70$).

4 Conclusions

We have discussed a method for numerical calculations for field theories on the continuum; this method being based on the original lattice source Galerkin technique.\[1\] We presented a Lorentz-invariant regulated representation derived from Lehmann representations. This approach has the computational advantages of minimal memory utilization and parallelizable algorithms and also allows direct representation of fermionic Green functions. Finally, a number of useful peripheral calculations can be made using this approximate representation: one can calculate diagrams in a regulated perturbation theory, as well as calculating dimensionally regularized loops numerically. In general, this technique of evaluating field theories takes advantage of the symmetries of the Lorentz group; future work includes the extension of the method to more general internal groups, such as gauge groups or supersymmetry.
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References

[1] S. García, G. S. Guralnik, and J. W. Lawson. A new approach to numerical quantum field theory. *Physics Letters*, B333:119, 1994.

[2] J. W. Lawson and G. S. Guralnik. Source Galerkin calculations in scalar field theory. *Nuclear Physics*, B459:589, 1996. [hep-th/9507130](http://arxiv.org/abs/hep-th/9507130).

[3] J. W. Lawson and G. S. Guralnik. New numerical method for fermion field theory. *Nuclear Physics*, B459:612, 1996. [hep-th/9507131](http://arxiv.org/abs/hep-th/9507131).

[4] S. García, Z. Guralnik, and G. S. Guralnik. Theta vacua and boundary conditions of the Schwinger–Dyson equations. [hep-th/9612079](http://arxiv.org/abs/hep-th/9612079), 1996.

[5] F. Stenger. *Numerical Methods Based on Sinc and Analytic Functions*. Springer Series in Computational Mathematics. Springer–Verlag, 1993.

[6] W.-M. Wang. Perturbative calculations in quantum field theory using Sinc functions. ScB Thesis, Brown University, 1998.

[7] R. Easther, G. S. Guralnik, and S. C. Hahn. Fast evaluation of Feynman diagrams. HET–1158, in preparation.
[8] S. C. Hahn. *Functional method of weighted residuals and quantum field theory*. PhD thesis, Brown University, 1998.