NON-SOLVABILITY IN THE FLAT CATEGORY OF ELLIPTIC OPERATORS WITH REAL ANALYTIC COEFFICIENTS

MARTINO FASSINA AND YIFEI PAN

Abstract. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open set. For an elliptic differential operator $L$ on $\Omega$ with real analytic coefficients and a point $p \in \Omega$, we construct a smooth function $g$ with the following properties: $g$ is flat at $p$ and the equation $Lu = g$ has no smooth local solution $u$ that is flat at $p$.

1. Introduction and Main Result

Let $f$ be a smooth complex-valued function defined on an open set $\Omega \subseteq \mathbb{R}^n$. We say that $f$ is flat at a point $p \in \Omega$ if its $k$-jet vanishes at $p$ for all $k \in \mathbb{N}$. Functions with this property are ubiquitous in mathematics. For instance, any smooth function with compact support $K$ is flat at every point of the boundary of $K$ (Lemma 3.5). Flat functions also play a role in the theory of PDEs, particularly in the study of the unique continuation property [2, 3, 11, 13]. In that context, the following question arises naturally: given a differential operator $L$ and the germ of a flat function $f$ at a point $p$, is there always a local solution $u$ to $Lu = f$ that is also flat at $p$? In this paper we show that the answer is negative for every elliptic operator $L$ with real analytic coefficients defined in an open set of $\mathbb{R}^n$, where $n \geq 2$. Here is our main result.

Main Theorem. (Theorem 2.8) Let $p \in \mathbb{R}^n$, $n \geq 2$, and let $L$ be an elliptic differential operator with real analytic coefficients defined on an open neighborhood $\Omega$ of $p$ in $\mathbb{R}^n$. Then there exists a germ of a flat function $g$ at $p$ with the property that there is no function $u$ flat at $p$ solving $Lu = g$.

Note that solvability here is meant in the sense of germs at $p$. Since $L$ is elliptic, then smooth local solutions always exist. Our theorem can be therefore restated by saying that there exists a smooth germ $g$ vanishing to infinite order at $p$ such that all smooth local solutions to $Lu = g$ vanish to finite order at $p$. We stress that this “flat non-solvability” phenomenon only occurs in dimension 2 or higher: in dimension 1 local flat solutions for flat data always exist (Remark 2.12).

2010 Mathematics Subject Classification. Primary 35J99.
Key words and phrases. Elliptic operators, flat functions.
The first author acknowledges support of NSF grant 13-61001.
Our proof of the main theorem is elementary, relying only on classical results in the theory of linear differential operators, for which we refer to [4]. These results, together with the necessary background definitions, are recalled in Section 2. There the proof of the main theorem is also carried out. The most technical parts of the proof can be found in Section 4 and Section 5, while in Section 3 we present some consequences of our result.

2. Non-solvability in the flat category

Let \( C^\infty(\Omega) \) be the ring of smooth complex-valued functions on an open set \( \Omega \subset \mathbb{R}^n \) and \( C^\infty_0 \) the corresponding ring of germs at 0. Here \( n \geq 2 \) is a positive integer that will be fixed throughout the paper. Unless otherwise stated, all the functions considered will be complex-valued. Recall that elements of \( C^\infty_0 \) are smooth functions defined on some open neighborhood of 0, and two smooth functions \( f \) and \( g \) coincide in \( C^\infty_0 \) if they agree on a neighborhood of 0. The ring \( C^\infty_0 \) is local with unique maximal ideal \( \mathfrak{m} = \{ f \in C^\infty_0 \mid f(0) = 0 \} \). Note, for \( f \in C^\infty_0 \), that
\[
\text{flat at } 0 \iff f \in \bigcap_{k=0}^{\infty} \mathfrak{m}^k.
\] (2.1)

Notation 2.1. We denote by \( c_f \) the ideal of germs of smooth functions that are flat at 0. By (2.1) we have
\[
c_f = \bigcap_{k=0}^{\infty} \mathfrak{m}^k.
\]

Taking the Taylor series expansion at the origin defines a ring homomorphism
\[
\sim : C^\infty_0 \longrightarrow \mathbb{C}[[x_1, \ldots, x_n]]
\]
\[
f \mapsto \tilde{f}
\] (2.2)
to the ring of formal power series \( \mathbb{C}[[x_1, \ldots, x_n]] \). The map in (2.2) is clearly not injective: its kernel is precisely the ideal of flat germs \( c_f \). In other words,
\[
f \in c_f \iff \tilde{f} = 0.
\]

It is a classical result of Borel [1] that the map in (2.2) is surjective, that is, every formal power series in \( \mathbb{C}[[x_1, \ldots, x_n]] \) is the Taylor series at the origin of a smooth function \( f \). We refer the reader to [9] for a modern proof relying on the Whitney’s Extension Theorem. At the risk of some redundancy, we state Borel’s result in a separate lemma for future use.

**Lemma 2.2.** (Borel) Let \( h \in \mathbb{C}[[x_1, \ldots, x_n]] \) be a formal power series. Then there exists a germ \( f \in C^\infty_0 \) whose Taylor expansion \( \tilde{f} \) at 0 is such that \( \tilde{f} = h \).

Let \( \mathcal{C}^\omega(\Omega) \) be the ring of complex-valued real analytic functions on an open set \( \Omega \subset \mathbb{R}^n \) and \( \mathcal{C}^\omega_0 \) the corresponding ring of germs at the origin. By definition,
\[
f \in \mathcal{C}^\omega_0 \iff \tilde{f}(x) \text{ converges to } f(x) \text{ for } x \text{ in a neighborhood of } 0.
\]
We now introduce differential operators on an open subset $\Omega \subset \mathbb{R}^n$.

**Notation 2.3.** We use the subscript notation for derivatives, writing $\partial_{x_j}$ in place of $\partial/\partial x_j$. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we let $D^\alpha := \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$. We write $|\alpha|$ for the length of $\alpha$, that is, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Recall that $L$ is a differential operator on $\Omega$ of order $m$ with real analytic coefficients if

$$L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in C^\omega(\Omega). \quad (2.3)$$

**Definition 2.4.** Let $L$ be a differential operators with real analytic coefficients on $\Omega \subset \mathbb{R}^n$ as in (2.3). We say that $L$ is **elliptic** if

$$\sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha \neq 0 \text{ for } x \in \Omega, \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

**Example 2.5.** The Laplacian $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ and its powers are elliptic operators in $\mathbb{R}^n$. The Cauchy-Riemann operator $\partial_x + i\partial_y$ is elliptic in $\mathbb{R}^2 \cong \mathbb{C}$.

The next lemma is the main technical point in the paper. The proof is presented in Section 4 and requires Borel’s Lemma 2.2.

**Lemma 2.6.** Let $L$ be an elliptic differential operator with real analytic coefficients defined on an open neighborhood of the origin in $\mathbb{R}^n$, where $n \geq 2$. Then there exist $G \in C^\infty_0$ and $g \in c_f$ such that $L(G) = g$ and the Taylor series expansion $\tilde{G}$ of $G$ at 0 is not convergent in any neighborhood of 0.

It is a deep classical result that elliptic differential operators with real analytic coefficients have analytic solutions for analytic data. This statement was first proved by Petrowsky [12] for homogeneous operators with constant coefficients. After a number of successive generalisations [6, 7, 8] it is nowadays a textbook result.

**Theorem 2.7.** [4, Theorem 7.5.1] Let $L$ be an elliptic differential operator with real analytic coefficients defined on an open set $\Omega \subset \mathbb{R}^n$. If $g \in C^\infty(\Omega)$ and $f$ is a distribution on $\Omega$ such that $Lf = g$ in $\Omega$ in the sense of distributions, then $f \in C^\infty(\Omega)$.

We are now ready to state and prove our main theorem. The proof relies on Lemma 2.6 and Theorem 2.7.

**Theorem 2.8.** Let $p \in \mathbb{R}^n$, where $n \geq 2$, and $L$ an elliptic differential operator with real analytic coefficients defined on an open neighborhood $\Omega$ of $p$ in $\mathbb{R}^n$. Then there exists a germ of a flat function $g$ at $p$ such that there is no smooth germ $u$ flat at $p$ solving $Lu = g$.

**Proof.** We can assume without loss of generality that $p$ is the origin. Let $g \in c_f$ and $G \in C^\infty_0$ be as in Lemma 2.6. Assume by contradiction that there exists a flat solution $u \in c_f$ to
$L u = g$. Then $\tilde{u} - \tilde{G} = \tilde{u} - \tilde{G} = \tilde{G}$. Recall that, by construction, the formal power series $\tilde{G}$ does not converge in any neighborhood of 0. Hence $u - G$ is not analytic at 0. The function $u - G$, however, satisfies $L(u - G) = 0$, and therefore Theorem 2.7 implies $u - G \in C^\omega_0$, which is a contradiction. □

Here is a special instance of Theorem 2.8 in the case of the Laplace operator $\Delta = \sum_{j=1}^{n} \partial_x^2$ in $\mathbb{R}^n$, for $n \geq 2$, and its powers.

**Corollary 2.9.** For every positive integer $m$, there exists a germ of a smooth function $g$ that is flat at 0 such that every local solution $u$ to $\Delta^m u = g$ vanishes to finite order at 0.

Another interesting case is represented by the Cauchy-Riemann operator $\partial_x + i\partial_y$ in $\mathbb{R}^2$.

**Corollary 2.10.** There exists a germ of a smooth function $g$ on $\mathbb{R}^2 \simeq \mathbb{C}$ that is flat at 0 and such that every local solution $u$ to $\bar{\partial} u = g \, dz$ vanishes to finite order at 0.

The previous statement can be easily promoted to several complex variables as follows.

**Corollary 2.11.** For every $n \in \mathbb{N}$, there exists a closed smooth $(0,1)$ form $\alpha = \sum_j \alpha_j \, d\bar{z}_j$ in $\mathbb{C}^n$ defined in a neighborhood of 0 such that:

- The functions $\alpha_j$ are all flat at 0.
- Every local solution to $\bar{\partial} u = \alpha$ vanishes to finite order at 0.

**Proof.** For every $j$, let $\alpha_j = g(z_j)$, where $g$ is as in Corollary 2.10. □

**Remark 2.12.** The hypothesis $n \geq 2$ in Theorem 2.8 cannot be dropped. Indeed, in dimension 1, every elliptic differential operator $L$ with smooth coefficients admits local flat solutions for flat data. To see this fact, let

$$L = \partial_x^n + a_{n-1}\partial_x^{n-1} + \cdots + a_1\partial_x + a_0, \quad a_j \in C^\infty(\mathbb{R})$$

and let $g \in c_f$ be the germ of a flat function at the origin. Let now $f$ be the (unique) smooth local solution at 0 of the Cauchy problem

$$\begin{cases}
L f = g \\
\partial_x^j f(0) = 0 & j = 0,1,\ldots,n-1.
\end{cases} \quad (2.4)$$

We claim that $f \in c_f$. Note that

$$\partial_x^j f = g - a_{n-1}\partial_x^{n-1}f - \cdots - a_0f. \quad (2.5)$$

Evaluating (2.5) at 0 we obtain $\partial_x^j f(0) = 0$. Taking derivatives of (2.5) and exploiting the fact that $g$ is flat at 0, one proves inductively that $\partial_x^k f(0) = 0$ for all $k$. Hence $f \in c_f$. 
3. Applications

3.a. Corollaries to Theorem 2.8. Let $L$ be an elliptic differential operator with real analytic coefficients defined on some open neighborhood $\Omega$ of 0 in $\mathbb{R}^n$, where $n \geq 2$. The next statements follow easily from Theorem 2.8.

**Corollary 3.1.** The following hold:

1. The linear map $L: c_f \rightarrow c_f$ is injective but not surjective.
2. The induced map on the quotient spaces $L: C_0^\infty/c_f \rightarrow C_0^\infty/c_f$ is surjective but not injective.

**Proof.** (1) Assume that $u \in c_f$ is such that $Lu = 0$. By Theorem 2.7, $u$ is real analytic at 0, and hence $u \equiv 0$ near 0. This proves that $L: c_f \rightarrow c_f$ is injective. The failure of surjectivity was proved in Theorem 2.8.

(2) Surjectivity follows from the classical existence theory [10]. To show that injectivity fails, it is enough to consider $g \in c_f$ as in Theorem 2.8. In fact, for every $u \in C_0^\infty$ such that $Lu = g$, we have $u \notin c_f$. □

**Corollary 3.2.** For every integer $N$ there exists a smooth germ $u \notin c_f$ such that $L^k u \notin c_f$ for $k = 0, \ldots, N$, but $L^{N+1} u \in c_f$.

**Proof.** Let $u \in C_0^\infty$ be such that $L^{N+1} u = g$, where $g \in c_f$ is as in Theorem 2.8. □

**Corollary 3.3.** For each $k \in \mathbb{N}$ there is a strict inclusion $L^{k+1} c_f \subset L^k c_f$.

**Proof.** For $k \in \mathbb{N}$, let $h = L^k g$, where $g \in c_f$ is as in Theorem 2.8. Assume by contradiction that $h \in L^{k+1} c_f$. Then there exists $u \in c_f$ such that $L^{k+1} u = h$. Since $L$ is injective on $c_f$, this implies that $Lu = g$, which is absurd by the choice of $g$. Hence $h \notin L^{k+1} c_f$, and the inclusion $L^{k+1} c_f \subset L^k c_f$ is strict. □

By Corollary 3.3 there is a strictly decreasing chain of $\mathbb{C}$-vector spaces

$$c_f \supset L c_f \supset L^2 c_f \supset L^3 c_f \supset \ldots$$

We now consider the $L$-invariant subspace $K$ of $c_f$ defined by $K := \bigcap_{k \in \mathbb{N}} L^k c_f$.

**Proposition 3.4.** Either $K = \{0\}$ or $K$ is an infinite dimensional $\mathbb{C}$-vector space.

**Proof.** Assume there exists a non-zero element $g \in K$. Then for every positive integer $k$ there exists an element $f_k \in c_f$ such that $L^k f_k = g$. It is easy to see that $f_k \in K$ for all $k$. Moreover, since $L: c_f \rightarrow c_f$ is injective (Corollary 3.1), then $f_k \neq 0$ for all $k$. We now claim that for each $m > 0$ the set $\{f_1, \ldots, f_m\}$ is linearly independent. By contradiction, assume that

$$f_1 = c_2 f_2 + \cdots + c_m f_m, \quad c_j \in \mathbb{C}.$$  

Then, recalling that $L f_{k+1} = f_k$ for each $k$, we have
Letting $A$ be the matrix of constants that appears on the right side of (3.1) and $Id$ the identity matrix of size $m - 1$, we have that the vector $f := (f_2, f_3, \ldots, f_m)$ satisfies the elliptic system $(LId - A)f = 0$. Recall that elliptic systems with real analytic coefficients have analytic solutions for analytic data [6]. Since $f_k \in c_f$ for all $k$, this implies that $f_2 = f_3 = \cdots = f_m = 0$, which is a contradiction. □

3.b. Solutions with compact support. We now present an application of the ideas presented in Section 2 to the study of compactly supported solutions to elliptic operators with real analytic coefficients. Recall that the support of a function $f$, which we denote as $\text{supp}(f)$, is the closure of the set where $f$ is non-zero. We write $\partial \text{supp}(f)$ for the boundary of the support. The connection with flatness is made clear in the next simple lemma.

Lemma 3.5. If $f \in C^\infty(\mathbb{R}^n)$ has compact support and $x_0$ is a point of the boundary of the support of $f$, then $f$ is flat at $x_0$.

Proof. Consider a sequence of points $x_j \in \mathbb{R}^n \setminus \text{supp}(u)$ such that $x_j \to x_0$ and observe, for every multi-index $\alpha$, that $D^\alpha f(x_0) = \lim_{x_j \to x_0} D^\alpha f(x_j) = 0$. □

Proposition 3.6. Let $L$ be an elliptic differential operator with real analytic coefficients defined on an open set $\Omega \subset \mathbb{R}^n$. Let $f$ be a smooth compactly supported function on $\Omega$ and $u$ a smooth compactly supported solution to $Lu = f$. Then

$$\partial \text{supp}(u) \subseteq \text{supp}(f) \subseteq \text{supp}(u).$$

Proof. The inclusion $\text{supp}(f) \subseteq \text{supp}(u)$ is clear. For the inclusion $\partial \text{supp}(u) \subseteq \text{supp}(f)$, consider a point $x_0 \in \partial \text{supp}(u)$. Since $u$ is flat at $x_0$ (Lemma 3.5) but not identically zero in any neighborhood of $x_0$, then $u$ is not analytic at $x_0$. Assume by contradiction that $x_0 \notin \text{supp}(f)$. Then $f \equiv 0$ in a neighborhood of $x_0$. Hence $u$ satisfies $Lu = 0$ in the sense of germs at $x_0$. By Theorem 2.7, $u$ is analytic at $x_0$, which is a contradiction. □

Remark 3.7. The topological relations described in Proposition 3.6 between the support of the initial datum and the support of the solution are well known in the case of the operator $\bar{\partial}$ (see for example [5, Proposition 1.1]).

4. Proof of Lemma 2.6

The proof of Lemma 2.6 relies on the following Proposition, whose proof is in turn given in Section 5.
Proposition 4.1. Let $L$ be an elliptic differential operator with real analytic coefficients defined on an open neighborhood $\Omega$ of the origin in $\mathbb{R}^n$, where $n \geq 2$. Then there exist a sequence of polynomials $(p_k)_{k \in \mathbb{N}} \subset \mathbb{C}[x_1, \ldots, x_{n-1}]$, a neighborhood $U$ of the origin in $\mathbb{R}^n$ and a sequence $(u_k)_{k \in \mathbb{N}}$ of real analytic functions in $U$ such that the following hold.

- For each $k$ we have $Lu_k = 0$.
- Each $u_k$ vanishes to order $k$ at the origin.
- For each $k$, the power series expansion at $0$ of $u_k$ converges to $u_k$ on $U$.
- For each $k$ the polynomial $p_k$ is homogeneous of degree $k$ and moreover $u_k = p_k$ on $x_n = 0$.

Proof of Lemma 2.6. Let $U \subset \mathbb{R}^n$, $(p_k)_{k \in \mathbb{N}} \subset \mathbb{C}[x_1, \ldots, x_{n-1}]$ and $(u_k)_{k \in \mathbb{N}} \subset C^\omega(U)$ be as in Proposition 4.1. Let $Z(p_k)$ be the zero set of $p_k$ in $\mathbb{R}^{n-1}$. Since each $Z(p_k)$ has empty interior in $\mathbb{R}^{n-1}$, the Baire category theorem implies that the union $\bigcup_{k \in \mathbb{N}} Z(p_k)$ is nowhere dense in $\mathbb{R}^{n-1}$. In particular, there exists a point $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \in \mathbb{R}^{n-1}$ arbitrarily close to the origin of $\mathbb{R}^{n-1}$ such that $p_k(\bar{x}_1, \ldots, \bar{x}_{n-1}) \neq 0$ for all $k$.

For each $k \in \mathbb{N}$ define

$$b_k := \frac{k!}{|p_k(\bar{x}_1, \ldots, \bar{x}_{n-1})|}$$

and consider the formal power series $\sum_k b_k u_k \in \mathbb{C}[[x_1, \ldots, x_n]]$. We first note that it is well defined. Indeed, for each integer $j$, only the functions $u_k$ with $k \leq j$ contain terms of order $j$. We claim that the formal power series $\sum_k b_k u_k$ does not converge in any neighborhood of $0$. To prove this fact, consider the point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n-1}, 0)$ and evaluate along the real line $L := \{x = t\bar{x} \mid t \in \mathbb{R}\} \subset \mathbb{R}^n$. Note that $L$ is contained in the hyperplane $x_n = 0$. We thus have

$$\sum_{k=0}^\infty |b_k u_k(t\bar{x})| = \sum_{k=0}^\infty |b_k p_k(t\bar{x}_1, \ldots, t\bar{x}_{n-1})| = \sum_{k=0}^\infty |b_k t^k p_k(\bar{x}_1, \ldots, \bar{x}_{n-1})| = \sum_{k=0}^\infty |t|^k k!,$$

which is clearly divergent for $t \in \mathbb{R} \setminus \{0\}$. By Borel’s Lemma 2.2, there exists $G \in C_0^\infty$ such that $\tilde{G} = \sum_k b_k p_k(x)$. Let now $g := L(G)$. It remains to prove that $g$ is flat at $0$. We thus need to show, for every multi-index $\alpha \in \mathbb{N}^n$, that $D^\alpha L(G)(0) = 0$. By Taylor’s theorem, there exists a neighborhood $U$ of the origin in $\mathbb{R}^n$ and a smooth function $\eta$ vanishing at $0$ to order at least $m + |\alpha| + 1$ such that

$$G = \sum_{k=1}^{m+|\alpha|} b_k u_k + \eta \text{ in } U.$$

We then have

$$D^\alpha L G = D^\alpha L \left( \sum_{k=1}^{m+|\alpha|} b_k u_k + \eta \right) = D^\alpha \left( \sum_{k=1}^{m+|\alpha|} b_k Lu_k \right) + D^\alpha L \eta \text{ in } U. \quad (4.1)$$
Evaluating (4.1) at 0, we obtain
\[ D^\alpha LG(0) = D^\alpha \left( \sum_{k=1}^{m+|\alpha|} b_k L u_k \right)(0) + D^\alpha L\eta(0) = 0, \]
where we have exploited that \( L u_k = 0 \) near 0 for each \( k \) and that \( \eta \) vanishes at 0 to order at least \( m + |\alpha| + 1 \). This concludes the proof. \( \square \)

5. Proof of Proposition 4.1

We follow Hörmander’s reasoning in the proof of [4, Theorem 5.11].

**Remark 5.1.** For the special case of \( L \) homogeneous with constant coefficients, Proposition 4.1 (and hence Lemma 2.6) also hold without the hypothesis of \( L \) being elliptic. In order to produce a sequence \( (u_k)_{k\in\mathbb{N}} \) of solutions to \( Lu = 0 \) with the property that each \( u_k \) vanishes to order \( k \) at the origin, one exploits the following observation. If \( L = \sum_{|\alpha|=m} a_{\alpha} D^{\alpha} \) with \( a_{\alpha} \in \mathbb{C} \) and \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \setminus \{0\} \) is such that \( \sum_{|\alpha|=m} a_{\alpha} \zeta^{\alpha} = 0 \), then \( u_k := (\zeta_1 x_1 + \cdots + \zeta_n x_n)^k \) satisfies \( Lu_k = 0 \). Hence the equation \( Lu = 0 \) has homogeneous polynomial solutions of arbitrary degree. This was a commonplace 19th century observation that we learned from Bruce Reznick [14, pag. 180].

**Notation 5.2.** Throughout this section we denote by ord\(_0(f)\) the order of vanishing at the origin of a function \( f \), by which we mean the smallest degree of a non-zero term in the Taylor series of \( f \) at 0.

We start by choosing for each \( k \in \mathbb{N} \) a homogeneous polynomial of degree \( k \) with complex coefficients \( p_k \in \mathbb{C}[x_1, \ldots, x_{n-1}] \). We will build a solution to
\[
\begin{aligned}
Lu &= 0 \\
u &= p_k \text{ on } x_n = 0,
\end{aligned}
\]
and call this solution \( u_k \).

- It will follow from our construction that ord\(_0(u_k) = k\).
- We will keep track of the domain of definition of \( u_k \) and prove that (if the polynomials \( p_k \) are chosen appropriately) it is independent of \( k \).

**Remark 5.3.** We will later see that not all sequences of polynomials \( p_k \) would serve our purpose, and we will need to require more conditions on them.

**Lemma 5.4.** Let \( L \) be an elliptic operator of order \( m \) defined on an open set \( \Omega \subset \mathbb{R}^n \). Then, letting \( \beta = (0, \ldots, 0, m) \), we can write \( L \) as
\[
L = D^\beta - \sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}.
\]
Proof. Let $L = \sum_{|\gamma|=m} a_\gamma D^\gamma + \ldots$, where dots stand for lower order terms. Let

$$p(x, \xi) := \sum_{|\gamma|=m} a_\gamma(x)\xi^\gamma, \quad x \in \Omega, \xi \in \mathbb{R}^n.$$ 

We claim that $a_\beta(x) \neq 0$ for all $x \in \Omega$. Indeed, if there exists $\bar{x} \in \Omega$ with $a_\beta(\bar{x}) = 0$, then $p(\bar{x}, (0, \ldots, 0, 1)) = 0$, thus contradicting the ellipticity of $L$. Since $a_\beta$ is non-vanishing in $\Omega$, we can divide by $a_\beta$ and rearrange the terms to obtain (5.2). □

Back to solving (5.1). We first make the substitution $v = u - p_k$. We now have to solve

$$\begin{cases}
L(v + p_k) = 0 \\
v = 0 \text{ on } x_n = 0.
\end{cases} \quad (5.3)$$

If we write $L$ as in Lemma 5.4, the equation $L(v + p_k) = 0$ becomes

$$D^\beta v = \sum_{|\alpha|\leq m, \alpha_n<m} a_\alpha D^\alpha v - Lp_k. \quad (5.4)$$

Note that since $p_k$ is homogeneous of order $k$ and $L$ is an operator of order $m$, then

$$\text{ord}_0(Lp_k) \geq \max\{0, k - m\}.$$ 

Notation 5.5. We denote by $\mathbb{D}_R$ a polycylinder of multi-radius $R = (R_1, \ldots, R_n)$ centered at the origin, that is, $\mathbb{D}_R = \{(x_1, \ldots, x_n) \in \mathbb{R}^n, |x_j| < R_j\}$.

Let $\mathbb{D}_R$ be a polycylinder such that for each $\alpha$ the Taylor series of $a_\alpha$ converges in $\mathbb{D}_R$. Note that it follows immediately that the function $-Lp_k$ also admits a representation as a convergent power series in $\mathbb{D}_R$, for each $k$ and regardless of the choice of $p_k$.

The next lemma shows how we can solve (5.3) by recursion.

Lemma 5.6. Fix $k \in \mathbb{N}$. Then there is a sequence of functions $\{v_\nu\}_{\nu \in \mathbb{N}}$ such that

$$\begin{cases}
D^\beta v_{\nu+1} = \sum_\alpha a_\alpha D^\alpha v_\nu - Lp_k \\
v_\nu = 0 \text{ on } x_n = 0.
\end{cases} \quad (5.5)$$

Moreover, the following properties hold.

- For each $\nu \in \mathbb{N}$, the function $v_\nu$ is a convergent power series in $\mathbb{D}_R$.
- For $\nu \geq 1$, we have $\text{ord}_0(v_\nu) \geq k$.

Proof. Let $v_0 = 0$ on $\mathbb{D}_R$. We look for solutions to

$$\begin{cases}
D^\beta v_1 = -Lp_k \\
v_1 = 0 \text{ on } x_n = 0.
\end{cases} \quad (5.6)$$
If \(-Lp_k(x) = \sum c_\gamma x^\gamma\) in \(\mathbb{D}_R\), then we choose the solution to (5.6) given by

\[
 v_1(x) := \sum c_\gamma x^{\gamma + \beta} \frac{\gamma_n!}{(\gamma_n + m)!}. \tag{5.7}
\]

Note that the series on the right side of (5.7) is still convergent in \(\mathbb{D}_R\). Moreover, recall that \(\text{ord}_0(Lp_k) \geq \max\{0, k - m\}\). Hence \(\text{ord}_0(v_1) \geq k\).

We now proceed by induction on \(\nu\). Assume that the lemma is proved for \(\nu\). We want \(v_{\nu+1}\) to solve (5.5). Let \(\sum c_\gamma x^\gamma\) be an expression for \(\sum_\alpha a_\alpha D^\alpha v_\nu - Lp_k\) as a convergent power series in \(\mathbb{D}_R\). Then let

\[
 v_{\nu+1}(x) = \sum c_\gamma x^{\gamma + \beta} \frac{\gamma_n!}{(\gamma_n + m)!}. 
\]

We see that \(\text{ord}_0(v_{\nu+1}) \geq m + \max\{\text{ord}_0(v_\nu) - m, \text{ord}_0(Lp_k)\}\). Applying the inductive hypothesis, we conclude that \(\text{ord}_0(v_{\nu+1}) \geq k\). \(\Box\)

**Remark 5.7.** The sequence \(v_\nu\) should really be called \(v^k_\nu\), since there is one sequence for each \(k\), or better for every polynomial \(p_k\). Note that we have not yet put any restriction on the choice of polynomials. We will use the notation \(v^k_\nu\) whenever we want to emphasize the dependence on \(p_k\).

With the sequence \(v_\nu\) from Lemma 5.6 available, the proof of Proposition 4.1 follows easily from the next lemma.

**Lemma 5.8.** There exists an open neighborhood \(U\) of 0 in \(\mathbb{R}^n\) and a choice of polynomials \(p_k\) such that for each \(k\) the sequence \(v^k_\nu\) converges uniformly on \(U\).

**Proof of Proposition 4.1.** For each \(k \in \mathbb{N}\), let \(v_k := \lim_{\nu \to \infty} v^k_\nu\). By Lemma 5.8 and Lemma 5.6, each \(v_k\) is a uniform limit of analytic functions in \(U\). Hence \(v_k\) is itself analytic in \(U\) and moreover, for every multi-index \(\alpha \in \mathbb{N}^n\), we have

\[
 D^\alpha v_k = \lim_{\nu \to \infty} D^\alpha v^k_\nu. 
\]

If we therefore take \(\lim_{\nu \to \infty}\) on both sides of the following equation

\[
 D^\beta v^k_{\nu+1} = \sum_\alpha a_\alpha D^\alpha v^k_\nu - Lp_k, 
\]

we see that \(v_k\) solves (5.3). Tracing back our substitution, we let \(u_k := p_k + v_k\). Note that each \(u_k\) is analytic in \(U\) and solves (5.1). In particular, \(Lu_k = 0\), as required. Now note that, for each multi-index \(\alpha\) of length \(|\alpha| < k\), we have

\[
 D^\alpha v_k(0) = \lim_{\nu \to \infty} D^\alpha v^k_\nu(0) = 0. \tag{5.8}
\]

The last equality in (5.8) follows from \(\text{ord}_0(v^k_\nu) \geq k\) (Lemma 5.6). We conclude that each \(v_k\) vanishes to order at least \(k\) at 0. Now consider \(u_k = p_k + v_k\). Recall that \(p_k\) is a homogeneous polynomial of degree \(k\) in the variables \(x_1, \ldots, x_{n-1}\). By construction, \(v_k = 0\) for \(x_n = 0\). In
particular, the Taylor series of $v_k$ at 0 does not have any term not involving the variable $x_n$. Therefore in the sum $v_k + p_k$ no cancellation happens and $u_k$ vanishes to order exactly $k$ at 0, as wanted. \hfill\Box

The only thing left to prove is Lemma 5.8. We first state and prove two simple auxiliary lemmas which appear in [4].

**Lemma 5.9.** Let $g$ be a $C^1$ function in one real variable $x$, for $|x| < R$, and assume

$$|g'(x)| \leq |x|^a, \quad |x| < R, \quad \text{and} \quad g(0) = 0,$$

where $a \geq 0$. It follows that

$$|g(x)| \leq \frac{|x|^{a+1}}{a+1}, \quad |x| < R.$$  

**Proof.** Follows immediately from $g(x) = \int_0^x g'(t)dt$. \hfill\Box

**Lemma 5.10.** Assume that $g(x) = \sum_j a_j x^j, a_j \in \mathbb{C}$ is a convergent power series in $|x| < R$ and

$$|g(x)| \leq (R - |x|)^{-a}, \quad |x| < R,$$

where $a \geq 0$. It follows that

$$|g'(x)| \leq e(1 + a)(R - |x|)^{-a-1}, \quad |x| < R.$$  

**Proof.** Complexify $g$ to a holomorphic function in $|\zeta| < R$. Let $0 < \epsilon < \rho = R - |\zeta|$ and let $|\zeta - \zeta_1| < \epsilon$. Since $-\epsilon \leq |\zeta| - |\zeta_1|$, adding $\rho$ on both sides we obtain

$$\rho - \epsilon \leq \rho + |\zeta| - |\zeta_1| = R - |\zeta_1|.$$  

It follows that

$$(R - |\zeta_1|)^{-a} \leq (\rho - \epsilon)^{-a}. \quad (5.9)$$

Combining (5.9) and the hypothesis, we have

$$|g(\zeta_1)| \leq (R - |\zeta_1|)^{-a} \leq (\rho - \epsilon)^{-a}.$$  

By Cauchy’s formula, we obtain

$$|g'(|\zeta|) \leq (\rho - \epsilon)^{-a} \epsilon^{-1}.$$  

Letting $\epsilon = \rho/(1 + a)$, we get

$$|g'(|\zeta|) \leq (1 + a)(1 + a^{-1}) \rho^{-a-1} \leq e(1 + a)\rho^{-a-1}.$$  

\hfill\Box

For each $k \in \mathbb{N}$, define the sequence of differences $w^k_{\nu} := v^k_{\nu+1} - v^k_{\nu}$. Since the $v^k_{\nu}$ satisfy (5.5), then for every $k$ we have

$$D^\beta w^k_{\nu+1} = \sum_\alpha D^\alpha w^k_{\nu}, \quad \nu = 0, 1, \ldots$$  

(5.10)
Remark 5.11. Note that Lemma 5.8 is proved if we can show that there exists a neighborhood $U$ of 0 in $\mathbb{R}^n$ and a choice of polynomials $p_k$ such that for all $k$ the series $\sum_\nu |w_\nu^k(x)|$ converges uniformly in $U$.

Recall that $\mathbb{D}_R$ is a polycylinder centered at 0 of multi-radius $R = (R_1, \ldots, R_n)$ such that all the functions $a_\alpha$ admit a representation as convergent power series in $\mathbb{D}_R$. From now on we assume, without loss of generality, that $R_j < 1$ for all $j$.

Notation 5.12. We denote by $D_\gamma^j$ the derivative $D^\gamma$, where $\gamma = (0, \ldots, 0, j)$.

Lemma 5.13. There exists a constant $C$ and a choice of the polynomials $p_k$ such that the following estimate holds for every $k$:

$$|D^\beta w_\nu^k(x)| \leq C^{\nu+1}|x_n|\nu d(x)^{-m\nu-1}, \quad x \in \mathbb{D}_R, \quad \nu = 0, 1, \ldots$$

(5.11)

Here $d(x) = \prod_{j=1}^{n-1}(R_j - |x_j|)$.

Proof. We will prove the lemma by induction on $\nu$.

We choose the polynomials $p_k$ so that $|Lp_k| \leq 1$ in $\mathbb{D}_R$ for all $k$. This is easily achieved by replacing each $p_k$ by $p_k/M$, where $M = \max\{|Lp_k(x)|, x \in \mathbb{D}_R\}$.

Recall that $w_0^k = v_1^k - v_0^k$, $v_0^k \equiv 0$ and $D^\beta v_1^k = -Lp_k$. Hence

$$|D^\beta w_0^k(x)| = |Lp_k(x)| \leq 1, \quad x \in \mathbb{D}_R.$$  

(5.12)

Note that, for each $i$,

$$\frac{R_i}{R_i - |x_i|} \geq 1, \quad x \in \mathbb{D}_R.$$  

Letting $C := \prod_{i=1}^{n-1} R_i$, we then have $Cd(x)^{-1} \geq 1$. Combining with (5.12), we get

$$|D^\beta w_0^k(x)| \leq Cd(x)^{-1}, \quad x \in \mathbb{D}_R, \quad k \in \mathbb{N},$$  

that is, (5.11) holds for $\nu = 0$.

We now want to prove that $C$ can be chosen sufficiently large so that (5.11) can be proved by recursion. Assume that (5.11) holds for $\nu$ and for every $k \in \mathbb{N}$. Repeated application of Lemma 5.9 gives, for $0 \leq j < m$ and $k \in \mathbb{N}$,

$$|D^j w_\nu^k(x)| \leq C^{\nu+1}|x_n|^{\nu+(m-j)}d(x)^{-m\nu-1}\left(\prod_{i=0}^{m-j-1} \frac{1}{\nu + i}\right), \quad x \in \mathbb{D}_R.$$  

(5.13)

Note that $|x_n| < 1$ and

$$\frac{1}{\nu + i} \leq \frac{1}{\nu}, \quad \text{for } \nu \in \mathbb{N}, \quad i = 0, \ldots, m - j - 1.$$  

Hence (5.13) becomes

$$|D^j w_\nu^k(x)| \leq C^{\nu+1}|x_n|^{\nu+1}d(x)^{-m\nu-1}\nu^{-j-m}, \quad x \in \mathbb{D}_R.$$  

(5.14)
To conclude the inductive step, we need to prove an estimate on $|D^{\beta}w_{\nu+1}^k(x)|$ for $x \in \mathbb{D}_R$. In virtue of (5.10), it is enough to have an estimate on each $|D^\alpha w_{\nu}^k(x)|$, where $|\alpha| \leq m$, $\alpha_n < m$. Assume that $\alpha_n = j < m$. A repeated application of Lemma 5.10 to (5.14) yields, for every $k \in \mathbb{N}$,

$$|D^\alpha w_{\nu}^k(x)| \leq C^{\nu+1}|x_n|^{\nu+1} \nu^{j-m} e^{\nu-1} |\alpha|-j d(x)^{-m\nu-1-|\alpha|} \left( \prod_{i=1}^{\alpha_n} (m\nu + 1 + i) \right), \quad x \in \mathbb{D}_R. \quad (5.15)$$

Here we have used that $(R_i - |x_i|) < 1$ for all $i$, and therefore

$$(R_i - |x_i|)^{-a} < (R_i - |x_i|)^{-a-1} \quad \text{for} \quad a \geq 0.$$ 

Equation (5.15) implies, for $x \in \mathbb{D}_R$ and $k \in \mathbb{N}$,

$$|D^\alpha w_{\nu}^k(x)| \leq C^{\nu+1}|x_n|^{\nu+1} \nu^{j-m} e^{m-j} d(x)^{-m\nu-1-(m-j)} (m\nu + m + 1)^{m-j} \leq C^{\nu+1}|x_n|^{\nu+1} e^{m-j} d(x)^{-m(\nu+1)-1} \left( m + \frac{m}{\nu} + \frac{1}{\nu} \right)^{m-j} \leq C^{\nu+1}|x_n|^{\nu+1} e^{m} d(x)^{-m(\nu+1)-1} (2m + 1)^{m}. \quad (5.16)$$

Let $A$ be a constant such that $\sum_{\alpha} |a_\alpha| \leq A$ in $\mathbb{D}_R$. Recalling that

$$D^\beta w_{\nu+1}^k = \sum_{\alpha} D^\alpha w_{\nu}^k$$

and combining with (5.16), we see that (5.11) holds for $\nu + 1$ provided that $C$ is chosen large enough so that $C \geq Ae^{m}(2m + 1)^{m}$. \hfill \Box

**Remark 5.14.** In Lemma 5.13 we could have employed any sequence of homogeneous polynomials $p_k$ for which there exists a constant $M$ such that $|Lp_k| \leq M$ in $\mathbb{D}_R$ for all $k$.

**Proof of Lemma 5.8.** As noted in Remark 5.11, it is enough to prove that there exists a neighborhood $U$ of 0 in $\mathbb{R}^n$ and a choice of polynomials $p_k$ such that for all $k$ the series $\sum_{\nu} |w_{\nu}^k(x)|$ converges uniformly in $U$. We exploit the estimate proved in Lemma 5.13. Consider the equation (5.14) for $j = 0$. Let $U$ be the neighborhood of 0 where $C|x_n|/d(x)^m < 1$. Then the series $\sum_{\nu} |w_{\nu}^k(x)|$ converges uniformly in $U$ for every $k \in \mathbb{N}$. \hfill \Box

6. Acknowledgements

The first author acknowledges useful conversations with Luca Baracco, Christine Laurent-Thiébaut and Bruce Reznick.
References

[1] Émile Borel, Sur quelques points de la théorie des fonctions. *Ann. Sci. École Norm. Sup.* (3) 12 (1895), 9–55.

[2] Torsten Carleman, Sur un problème d’unicité pur les systèmes d’équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.* 26 (1939), no. 17.

[3] Adam Coffman and Yifei Pan, Smooth counterexamples to strong unique continuation for a Beltrami system in $C^2$. *Comm. Partial Differential Equations* 37 (2012), no. 12, 2228–2244.

[4] Lars Hörmander, Linear partial differential operators, Die Grundlehren der mathematischen Wissenschaften, Bd. 116 *Academic Press, Inc. Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg* (1963).

[5] Christine Laurent-Thièbaut and Mei-Chi Shaw, Solving $\bar{\partial}$ with prescribed support on Hartogs triangles in $C^2$ and $CP^2$. Preprint, arXiv:1609.04194[math.DS].

[6] Charles B. Morrey Jr. and Louis Nirenberg, On the analyticity of the solutions of linear elliptic systems of partial differential equations. *Comm. Pure Appl. Math.* 10 (1957), 271–290.

[7] Charles B. Morrey Jr., On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. I. Analyticity in the interior. *Amer. J. Math.* 80 (1958), 198–218.

[8] Charles B. Morrey Jr., On the analyticity of the solutions of analytic non-linear elliptic systems of partial differential equations. II. Analyticity at the boundary. *Amer. J. Math.* 80 (1958), 219–237.

[9] Raghavan Narasimhan, Analysis on real and complex manifolds. North-Holland Mathematical Library, 35. *North-Holland Publishing Co., Amsterdam,* (1985).

[10] Louis Nirenberg, On Elliptic Partial Differential Equations. In: Faedo S. (eds) Il principio di minimo e sue applicazioni alle equazioni funzionali. C.I.M.E. Summer Schools, vol 17. *Springer, Berlin, Heidelberg,* (2011).

[11] Yifei Pan, Unique continuation for Schrödinger operators with singular potentials. *Comm. Partial Differential Equations* 17 (1992), no. 5-6, 953–965.

[12] Ivan G. Petrovsky, Sur l’analyticité des solutions des systèmes d’équations différentielles, *Rec. Math. N. S. [Mat. Sbornik]* (5) 47 (1939), 3–70.

[13] Murray H. Protter, Unique continuation for elliptic equations. *Trans. Amer. Math. Soc.* 95 (1960), 81–91.

[14] Bruce Reznick, Homogeneous polynomial solutions to constant coefficient PDE’s. *Adv. Math.* 117 (1996), no. 2, 179–192.

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

E-mail address: fassina2@illinois.edu

Department of Mathematical Sciences, Purdue University Fort Wayne, 2101 East Coliseum Boulevard, Fort Wayne, IN 46805, USA

E-mail address: pan@pfw.edu