On complex-time heat kernels of fractional Schrödinger operators via Phragmén–Lindelöf principle

KONSTANTIN MERZ

Abstract. We consider fractional Schrödinger operators with possibly singular potentials and derive certain spatially averaged estimates for its complex-time heat kernel. The main tool is a Phragmén–Lindelöf theorem for polynomially bounded functions on the right complex half-plane. The interpolation leads to possibly nonoptimal off-diagonal bounds.

1. Introduction

Phragmén–Lindelöf theorems (cf. [2,40,56,60]) are powerful complex analysis tools that extend the maximum modulus principle to certain unbounded domains. They are often used in the presence of exponential bounds, which are, for example, available in the analysis of heat kernels of elliptic second-order differential operators. Suppose, for example, that $e^{-tH}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^d)$ whose integral kernel satisfies

$$0 \leq e^{-tH}(x, y) \leq ct^{-\frac{d}{4}} \exp\left[-\frac{b|x-y|^2}{t}\right], \quad t > 0, \ x, y \in \mathbb{R}^d,$$

(1.1)

where $b, c$ are positive constants. In [24, Theorem 3.4.8] and [25, Lemma 9, Theorem 10] Davies used the Phragmén–Lindelöf principle to show that for all $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that

$$|e^{-zH}(x, y)| \leq c_\varepsilon(\text{Re} \ z)^{-\frac{d}{4}} \exp\left[-\text{Re} \left(\frac{b|x-y|^2}{(1+\varepsilon)z}\right)\right], \quad x, y \in \mathbb{R}^d$$

(1.2)

holds for all complex times $z \in \mathbb{C}_+ := \{z \in \mathbb{C} : \text{Re}(z) > 0\}$, whenever (1.1) holds.

Coulhon and Sikora [22, Section 4] reversed Davies’ ideas and used the Phragmén–Lindelöf principle to derive Gaussian heat kernel estimates like (1.2) from suitably weighted off-diagonal estimates—so-called Davies–Gaffney estimates (cf. [22, (3.2)] or (3.15) for $\theta = 0$ and $\alpha = 2$)—and on-diagonal estimates like $e^{-tH}(x, x) \leq ct^{-\frac{d}{2}}$. The

Mathematics Subject Classification: Primary 35K08; Secondary 35J10

Keywords: Complex-time heat kernel, Phragmén–Lindelöf principle, fractional Laplace, fractional Schrödinger operator, Davies–Gaffney estimate.
latter can often be derived from Sobolev inequalities corresponding to $H$ by Nash’s method, cf. Davies [24, Section 2.4] and Milman and Semenov [51].

Complex-time heat kernel estimates like (1.2) are of paramount importance in many problems in harmonic analysis and partial differential equations, e.g., in proving $L^p$ boundedness of spectral multipliers and convergence of Riesz means, and investigating maximal regularity properties of the Schrödinger evolution $e^{itH}$ for operators $H$ whose heat kernels satisfy sub-Gaussian estimates, see, e.g., [5–10,14,15,17–21,27,28,37,47,59].

On the other hand, there has been recent interest in the fractional Laplace operator $(-\Delta)^{\alpha/2}$ in $L^2(\mathbb{R}^d)$ for $\alpha > 0$ and its generated holomorphic semigroup $e^{-z(-\Delta)^{\alpha/2}}$ for $z \in \mathbb{C}_+$. For $\alpha > 0$, Blumenthal and Getoor [4] proved

$$|e^{-t(-\Delta)^{\alpha/2}}(x)| \leq c_d,\alpha \frac{t}{(t^{1/\alpha} + |x|)}^{d+\alpha}, \quad t > 0, \ x \in \mathbb{R}^d,$$

whereas for $\alpha = 1$ one has the explicit formula

$$e^{-t(-\Delta)^{1/2}}(x) = c_d \frac{t}{(t^2 + |x|^2)^{(d+1)/2}}$$

for an explicit constant $c_d > 0$. The slow decay of these kernels and their extension to $\mathbb{C}_+$—that we discuss momentarily—complicates many arguments in problems where heat kernel estimates are the central element of the analysis like [13,32,50]. In particular, they impede the generalization of the above-mentioned works on spectral multipliers for operators whose heat kernels only decay algebraically. As (1.2) indicates, complex-time heat kernels satisfy worse bounds as $|\arg(z)|$ increases. Thus, the derivation of sharp estimates becomes even more crucial in this scenario.

Zhao and Zheng [65, Theorem 1.3] recently proved uniform complex-time heat kernel estimates for $(-\Delta)^{\alpha/2}$ and all $\alpha > 0$ using the stationary phase method. While for $\alpha = 1$ one has the explicit formula

$$e^{-z(-\Delta)^{1/2}}(x) = c_d \frac{z}{(z^2 + |x|^2)^{(d+1)/2}}, \quad z \in \mathbb{C}_+, \ x \in \mathbb{R}^d,$$

the derivation of such estimates for $\alpha \neq 1$ is rather intricate. Pointwise estimates in the case $|\arg(z)| = \pi/2$ are, however, well known, see, e.g., Miyachi [52, Proposition 5.1], Wainger [63, pp. 41–52], Huang et al. [43] and [65, pp. 2–3].

Using perturbation theory, Zhao and Zheng extended their results and derived uniform complex-time heat kernel estimates for fractional Schrödinger operators

$$H_\alpha := (-\Delta)^{\alpha/2} + V \quad \text{in} \ L^2(\mathbb{R}^d)$$

when $V \in L^1_{\text{loc}}(\mathbb{R}^d : \mathbb{R})$ belongs to the higher-order Kato class $K_{\alpha}(\mathbb{R}^d)$ (cf. [26,35,66] for a precise definition). In this case, $V$ is infinitesimally form bounded with respect to $(-\Delta)^{\alpha/2}$ and $H_\alpha$ can be defined as the self-adjoint Friedrichs extension of the corresponding quadratic form with form core $C^\infty_c(\mathbb{R}^d)$. Their estimates for the kernel of the holomorphic extension of $e^{-tH_\alpha}$ to $\mathbb{C}_+$ read as follows.
Theorem 1.1 [65, Theorem 1.5]. Let $\alpha > 0$, $z = |z|e^{i\theta} \in \mathbb{C}_+$, and $V \in K_\alpha(\mathbb{R}^d)$. Then for any $\varepsilon \in (0, 1)$ there are $c_{d, \alpha} > 0$ and $\mu_{\varepsilon, V, d, \alpha} > 0$ such that
\[
|e^{-zH_\alpha}(x, y)| \leq c_{d, \alpha}e^{\mu_{\varepsilon, V, d, \alpha}|z|}(|\cos \theta|)^{-d} \left(\frac{\frac{1}{\alpha} - \frac{1}{2}}{\frac{d}{2} + \alpha - 1}\right) |z| |z|^{\frac{1}{\alpha}} + |x - y|^{d + \alpha}
\]
holds for all $x, y \in \mathbb{R}^d$.

Estimate (1.6) reflects the best possible off-diagonal pointwise decay for $|x - y| \gg |z|^{1/\alpha}$ and all $|\theta| < \pi/2$ (compare with (1.3)) and is particularly useful for $|z| \lesssim 1$. The parameter $\mu_{\varepsilon, V, d, \alpha}$ is defined in [65, p. 22]. Due to its complicated dependence on $\varepsilon$, an optimization with respect to $\varepsilon \in (0, 1)$ does not seem to be straightforward. Moreover, their proof gives $\mu_{\varepsilon, V, d, \alpha} > 0$ even if $V \geq 0$.

In this note, we derive a Phragmén–Lindelöf principle for polynomially bounded functions (Theorem 2.1) and apply it to obtain suitably weighted and averaged estimates for $e^{-zH_\alpha}$ that are not uniform in $\theta$ but do not deteriorate for $|z| \gg 1$ (Sects. 3.1–3.2). Moreover, we can allow for critically singular potentials $V$, like the Hardy potential $|x|^{-\alpha}$, which do not belong to $K_\alpha(\mathbb{R}^d)$. In this case, $e^{-tH_\alpha}$ is generally not $L^1 \to L^\infty$ bounded. On the downside, it is not clear whether the off-diagonal decay of our estimates is optimal, especially when $|\pi/2 - |\theta|| \ll 1$.

Organization and notation

In Sect. 2, we state and prove a Phragmén–Lindelöf theorem for polynomially bounded functions. We apply this theorem in Sect. 3 to derive estimates for the complex-time heat kernel of fractional Schrödinger operators with nonnegative potentials (Corollary 3.1 and Corollary 3.2 for a pointwise estimate), and potentials with a nonvanishing, possibly singular, negative part (Theorem 3.7 and Corollaries 3.11–3.12). The applicability of these bounds is discussed in Sect. 3.3. In Sect. 4, we collect some properties of the estimates—so-called dyadic Davies–Gaffney estimates (Definitions 3.3 and 3.9)—that we discuss when $V$ has a negative part.

We write $A \lesssim B$ for two nonnegative quantities $A, B \geq 0$ to indicate that there is a constant $C > 0$ such that $A \leq CB$. If $C = C_\tau$ depends on a parameter $\tau$, we write $A \lesssim_\tau B$. The dependence on fixed parameters like $d$ and $\alpha$ is sometimes omitted. The notation $A \sim B$ means $A \lesssim B \lesssim A$. All constants are denoted by $c$ or $C$ and are allowed to change from line to line. We abbreviate $A \wedge B := \min\{A, B\}$ and $A \vee B := \max\{A, B\}$. The Heaviside function is denoted by $\theta(x)$. We use the convention $\theta(0) = 1$. The indicator function and the Lebesgue measure of a set $\Omega \subseteq \mathbb{R}^d$ are denoted by $1_{\Omega}$ and $|\Omega|$, respectively. The Euclidean distance between two sets $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ is denoted by $d(\Omega_1, \Omega_2) := \inf_{x \in \Omega_1, y \in \Omega_2} |x - y|$. If $T : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded linear operator, we write $T \in B(L^p \to L^q)$ and denote its operator norm by $\|T\|_{p \to q}$. For $1 \leq p \leq \infty$ we write $p' = (1 - 1/p)^{-1}$. 


2. Phragmén–Lindelöf principle with polynomial bounds

**Theorem 2.1.** Let $X$ be a Banach space equipped with a norm $\| \cdot \|$ and $F : \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \to X$ be a holomorphic function satisfying

$$
\| F(|z|e^{i\theta}) \| \leq a_1 (|z| \cos \theta)^{-\beta_1} \quad \text{and} \quad \| F(|z|) \| \leq a_1 |z|^{-\beta_1} \left( \frac{a_2}{|z|} \right)^{-\beta_2} \left( \frac{a_3}{|z|} \right)^{\beta_3} 
$$

(2.1)

(2.2)

for some $a_1, a_2, a_3 > 0$, $\beta_1, \beta_2, \beta_3 \geq 0$, all $|z| > 0$, and all $|\theta| < \pi/2$. Then, for all $\varepsilon \in (0, 1)$ one has

$$
\| F(|z|e^{i\theta}) \| \leq a_1 (|z| \cos \theta)^{-\beta_1} \left[ 1 \wedge \varepsilon^{-\beta_1} \left( \left( \frac{a_2}{|z|} \right)^{-\beta_2} \left( \frac{a_3}{|z|} \right)^{\beta_3} \right)^{1-|\theta|/\gamma(\varepsilon,\theta)} \right] 
$$

(2.3)

for all $|\theta| < \pi/2$ and $|z| > 0$, where $\gamma(\varepsilon, \theta) := \varepsilon|\theta| + (1 - \varepsilon)\pi/2$.

**Proof.** For $\gamma \in (0, \pi/2)$ and $z = |z|e^{i\theta}$, let

$$
G(z) := z^{-\beta_1} F(z^{-1}) \cdot H_2(z) \cdot H_3(z) 
$$

(2.4)

with

$$
H_2(z) := \exp \left( \beta_2 \log(a_2z) \left( 1 + \frac{i}{2\gamma} \log(a_2z) \right) \right) = (a_2z)^{\beta_2 \left( 1 + \frac{i}{2\gamma} \log(a_2z) \right)} \quad (2.5)
$$

and

$$
H_3(z) := \exp \left( -\beta_3 \log(a_3z) \left( 1 + \frac{i}{2\gamma} \log(a_3z) \right) \right) = (a_3z)^{-\beta_3 \left( 1 + \frac{i}{2\gamma} \log(a_3z) \right)} \quad (2.6)
$$

Here $\log(|z|e^{i\theta}) := \log(|z|) + i\theta$ with $|\theta| < \pi$ is the principal branch of the logarithm. By the assumptions on $\| F(z) \|$ for $\theta = 0$ and $\theta = \gamma$, and the bounds

$$
|H_2(|z|)| \leq (a_2|z|)^{\beta_2}
$$

and

$$
|H_2(|z|e^{i\gamma})| = \left| \exp \left( \frac{i\beta_2}{2} \left( (\log |a_2z|)^2 / \gamma + \gamma \right) \right) \right| = 1
$$

and

$$
|H_3(|z|)| \leq (a_3|z|)^{-\beta_3} \quad \text{and} \quad |H_3(|z|e^{i\gamma})| = 1,
$$

respectively, we have

$$
\| G(|z|) \| \leq |z|^{-\beta_1} \cdot a_1 |z|^{\beta_1} (a_2|z|)^{-\beta_2} (a_2|z|)^{\beta_2} \cdot (a_3|z|)^{\beta_3} (a_3|z|)^{-\beta_3} \leq a_1
$$
and

\[ \| G(|z|e^{i\gamma}) \| \leq |z|^{-\beta_1} \cdot a_1 |z|^{\beta_1} (\cos \gamma)^{-\beta_1} = a_1 (\cos \gamma)^{-\beta_1}. \]

Combining the above two formulas using the three lines lemma shows

\[ \| G(|z|e^{i\theta}) \| \leq a_1 (\cos \gamma)^{-\beta_1 \theta / \gamma} \leq a_1 (\cos \gamma)^{-\beta_1} \]

for all \( 0 \leq \theta \leq \gamma \) and \( |z| > 0 \).

Plugging this estimate and the identities

\[
\begin{align*}
H_2(z^{-1})^{-1} & = \left| (a_2 z^{-1})^{-\beta_2 (1 + i \log(a_2 z^{-1}))} \right| = \left( a_2 |z|^{-1} \right)^{-\beta_2 (1 + \theta / \gamma)} \\
H_3(z^{-1})^{-1} & = \left( a_3 |z|^{-1} \right)^{\beta_3 (1 + \theta / \gamma)}
\end{align*}
\]

into the expression for \( F(z) \) in (2.4), i.e.,

\[ F(|z|e^{i\theta}) = z^{-\beta_1} G(z^{-1}) H_2(z^{-1})^{-1} \cdot H_3(z^{-1})^{-1}, \]

implies for \( -\gamma \leq \theta < 0 \) and \( |z| > 0 \),

\[ \| F(|z|e^{i\theta}) \| \leq a_1 |z|^{-\beta_1 (\cos \gamma)^{-\beta_1}} \left( a_2 |z|^{-1} \right)^{-\beta_2 (1 + \theta / \gamma)} \cdot \left( a_3 |z|^{-1} \right)^{\beta_3 (1 + \theta / \gamma)} . \]

(2.7)

By a reflection along the real axis, we conclude the corresponding bound with \( \theta \) replaced by \( -\theta \in [-\gamma, 0) \). Choosing \( \gamma = \gamma(\epsilon, \theta) = \epsilon \theta + (1 - \epsilon)\pi/2 \) for any \( 0 < \epsilon < 1 \) (which ensures \( |\theta| < \gamma < \pi/2 \)), let us estimate \( (\cos \gamma)^{-\beta_1} \leq \epsilon^{-\beta_1} (\cos \theta)^{-\beta_1} \) and conclude the proof of (2.3), upon taking the minimum between (2.7) and (2.1).

**Remarks 2.2.** (1) For \( \beta_3 = 0 \), one observes that the decay of \( |F(z)| \) in the regime \( a_2 \gg |z| \) becomes weaker as \( |\theta| \) increases.

(2) The choice \( \gamma = \epsilon |\theta| + (1 - \epsilon)\pi/2 \) suppresses the effect of large \( |\theta| \) for \( \epsilon \ll 1 \), but does not prevent the vanishing of decay as \( |\theta| \to \pi/2 \).

### 3. Application: semigroups of fractional Schrödinger operators

Let \( d \in \mathbb{N}, \alpha > 0, V \in L^1_{\text{loc}}(\mathbb{R}^d : \mathbb{R}) \) and recall the notation (1.5) for the fractional Schrödinger operator \( H_\alpha = (-\Delta)^{\alpha/2} + V \) in \( L^2(\mathbb{R}^d) \). We assume that \( V \) is such that the quadratic form of \( H_\alpha \) is nonnegative on \( C_c^\infty(\mathbb{R}^d) \) so that it gives rise to a self-adjoint operator by Friedrichs’ theorem. In particular, \( e^{-z H_\alpha} \) is a bounded, holomorphic semigroup on \( L^2(\mathbb{R}^d) \) whenever \( \text{Re}(z) > 0 \) [45, p. 493]. In the following, we derive certain weighted and averaged estimates for the extension of the semigroup \( e^{-t H_\alpha} \) to complex times \( z \in \mathbb{C}_+ \) and distinguish between the cases where \( V \geq 0 \) and where \( V \) has a nonvanishing negative part, respectively.
3.1. Nonnegative potentials

If $\alpha \in (0, 2)$, then $e^{-t(-\Delta)^{\alpha/2}}(x) \sim_{d, \alpha} t^{-d/\alpha} (1 + |x|/t^{1/\alpha})^{-d/\alpha}$, i.e., the kernel is in particular positive [4]. Therefore, Trotter’s formula implies for $V \geq 0$ the upper bound

$$e^{-tH_{\alpha}}(x, y) \lesssim_{d, \alpha} t^{-\frac{d}{\alpha}} \left(1 + \frac{|x|}{t^{1/\alpha}}\right)^{-d-\alpha}. \quad (3.1)$$

In turn, $(3.1)$ implies the following weighted $L^2 \to L^\infty$ estimate for any $\alpha \in (0, 2)$, $r > 0$,

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_y(r)} |\exp(-tH_{\alpha})(x, y)|^2 \, dx \lesssim_{d, \alpha} t^{-\frac{d}{\alpha}} \left(1 + \frac{r}{|z|^{1/\alpha}}\right)^{-d-2\alpha}. \quad (3.2)$$

The uniform complex-time heat kernel estimates (1.6) by Zhao and Zheng yield the following extension to $z \in \mathbb{C}_+$, i.e.,

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_y(r)} |\exp(-zH_{\alpha})(x, y)|^2 \, dx \lesssim_{d, \alpha, \epsilon} \epsilon^2 \alpha \cdot (\cos \theta)^{-d} \left(1 + \frac{r}{|z|^{1/\alpha}}\right)^{-d-2\alpha}. \quad (3.3)$$

In some applications, one is merely in possession of averaged and possibly weighted analogs of $(3.1)$, such as $(3.2)$. This is typically the case when $V$ has a singular negative part, which will be discussed in more detail in Sect. 3.2. The following corollary illustrates how the Phragmén–Lindelöf principle can be used to extend weighted $L^2 \to L^\infty$ estimates like $(3.2)$ to complex times.

**Corollary 3.1.** Let $\alpha \in (0, 2)$ and $V \geq 0$. Let further $z = |z| e^{i\theta}$ with $|\theta| \in [0, \pi/2)$, $r > 0$, $\epsilon \in (0, 1)$, and

$$\beta_{d, \alpha, \epsilon}(\theta) := (d + 2\alpha) \left(1 - \frac{|\theta|}{\epsilon |\theta| + (1 - \epsilon) \pi/2}\right) \geq 0. \quad (3.4)$$

Then,

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_y(r)} |\exp(-zH_{\alpha})(x, y)|^2 \, dx \lesssim_{d, \alpha, \epsilon} (|z| \cos \theta)^{-\frac{d}{\alpha}} \left(1 + \frac{r}{|z|^{1/\alpha}}\right)^{-\beta_{d, \alpha, \epsilon}(\theta)}. \quad (3.5)$$

Although the singularity of $(\cos \theta)^{-d/\alpha}$ in $(3.5)$ is less severe than in $(3.3)$, the decay in the region $r \gg |z|^{1/\alpha}$ becomes weaker as $|\theta|$ increases.

**Proof.** For $y \in \mathbb{R}^d$, define the analytic function $F_y : \mathbb{C}_+ \to \mathbb{C}$ by

$$F_y(z) = \left(\int_{\mathbb{R}^d} e^{-zH_{\alpha}}(x, y) f(x) \, dx\right)^2$$
for any normalized $f \in L^2(\mathbb{R}^d)$ with supp$f \subseteq \mathbb{R}^d \setminus B_y(r)$. For $|\theta| < \frac{\pi}{2}$ and $z = |z|e^{i\theta}$, we have, using the unitarity of $e^{-itH_a}$ and (3.1) with $t$ replaced by $|z| \cos \theta$,

$$\sup_{y \in \mathbb{R}^d} |F_y(z)| \leq \|e^{-zH_a}\|_{2 \to \infty}^2 \leq \|e^{-|z| \cos \theta H_a}\|_{2 \to \infty}^2 \|e^{-i|z| \sin \theta H_a}\|_{2 \to 2}^2 \leq C_{1,d,\alpha} (|z| \cos \theta)^{-\frac{d}{\alpha}}.$$

On the other hand, we have for $\theta = 0$,

$$\sup_{y \in \mathbb{R}^d} |F_y(|z|)| \leq \int_{\mathbb{R}^d \setminus B_y(r)} |e^{-|z|H_a}(x, y)|^2 \, dx \leq C_{2,d,\alpha} |z|^{-d/\alpha} \left(1 + \frac{r}{|z|^{1/\alpha}}\right)^{-d-2\alpha} \leq C_{2,d,\alpha} |z|^{-d/\alpha} \left(\frac{r_\alpha}{|z|}\right)^{-\frac{d}{\alpha} - 2\alpha}$$

by (3.2). Thus, we may apply Theorem 2.1 with $X = \mathbb{C}$, $a_1 = \max\{C_{1,d,\alpha}, C_{2,d,\alpha}\}$, $a_2 = r_\alpha$, $\beta_1 = d/\alpha$, $\beta_2 = (d + 2\alpha)/\alpha$, and $\beta_3 = 0$, and obtain

$$|F_y(z)| \lesssim_{d,\alpha,\epsilon} (|z| \cos \theta)^{-\frac{d}{\alpha}} \left[1 + \left(\frac{r_\alpha}{|z|}\right)^{-\frac{d+2\alpha}{\alpha}} \left(1 - \frac{|\theta|}{\epsilon|\theta| + (1-\epsilon)\pi/2}\right)^2\right],$$

which shows (3.5). \hfill \Box

Similarly, one can use Theorem 2.1 to derive pointwise estimates.

**Corollary 3.2.** Let $\alpha \in (0, 2)$ and $V \geq 0$. Let further $z = |z|e^{i\theta}$ with $|\theta| \in [0, \pi/2)$, $x, y \in \mathbb{R}^d$, $\epsilon \in (0, 1)$, and

$$\beta_{d,\alpha,\epsilon}(\theta) := (d + \alpha) \left(1 - \frac{|\theta|}{\epsilon|\theta| + (1-\epsilon)\pi/2}\right) \geq 0. \quad (3.6)$$

Then,

$$\left|e^{-zH_a}(x, y)\right| \lesssim_{d,\alpha,\epsilon} (|z| \cos \theta)^{-\frac{d}{\alpha}} \left(1 + \frac{|x - y|}{|z|^{1/\alpha}}\right)^{-\beta_{d,\alpha,\epsilon}(\theta)} \quad (3.7)$$

**Proof.** For $z = |z|e^{i\theta}$ with $|\theta| \in [0, \pi/2)$, we use $\|e^{-2\text{Im}(z)H_a}\|_{2 \to 2} = 1$ and estimate

$$\|e^{-zH_a}\|_{1 \to \infty} \leq \|e^{-\text{Re}(z)H_a/2}\|_{1 \to 2} \|e^{-\text{Re}(z)H_a/2}\|_{2 \to \infty} \leq C_{1,d,\alpha} (|z| \cos \theta)^{-\frac{d}{\alpha}},$$

which shows

$$\left|e^{-zH_a}(x, y)\right| \leq C_{1,d,\alpha} (|z| \cos \theta)^{-\frac{d}{\alpha}}, \quad x, y \in \mathbb{R}^d.$$ 

On the other hand, (3.1) also implies

$$\left|e^{-|z|H_a}(x, y)\right| \leq C_{2,d,\alpha} |z|^{-\frac{d}{\alpha}} \left(\frac{|x - y|^{\alpha}}{|z|}\right)^{-\frac{d+2\alpha}{\alpha}}, \quad x, y \in \mathbb{R}^d.$$
Thus, by Theorem 2.1 with \( X = \mathbb{C} \), \( a_1 = \max\{C_{1,d,\alpha}, C_{2,d,\alpha}\} \), \( a_2 = |x - y|^\alpha \), \( \beta_1 = d/\alpha \), \( \beta_2 = (d + \alpha)/\alpha \), and \( \beta_3 = 0 \), we obtain

\[
|e^{-zH_\alpha}(x, y)| \lesssim_{d,\alpha,\varepsilon} (|z| \cos \theta)^{-d/\alpha} \left[ 1 + \left( \frac{|x - y|^\alpha}{|z|} \right)^{1-\beta/2} \right] \]

for all \( x, y \in \mathbb{R}^d \). This shows (3.7). \( \square \)

3.2. Potentials with a negative part

We now consider the situation where the semigroup \( e^{-tH_\alpha} \) is not necessarily \( L^2 \to L^\infty \) bounded anymore. This typically occurs when \( V \) has a nonvanishing, singular negative part [49,51]. In these cases, \( e^{-tH_\alpha} \) may nevertheless satisfy certain weighted \( L^p \to L^q \) estimates with \( 1 < p \leq q < \infty \). To introduce the estimates that we discuss here, we denote by

\[
A_2(x, r, k) := B_\varepsilon(2^kr) \setminus B_\varepsilon(2^{k-1}r) \quad \text{with} \quad A_2(x, r, 0) := B_\varepsilon(r) \quad \text{and} \quad k \in \mathbb{N}_0
\]
dyadic annuli around \( x \in \mathbb{R}^d \).

**Definition 3.3.** Let \( r > 0 \), \( (T_r)_{r>0} \) be a family of linear bounded operators on \( L^2(\mathbb{R}^d) \), \( 1 \leq p \leq q \leq \infty \), \( \beta, \sigma > 0 \), and \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfy \( g(\lambda) \sim_\beta (1 + \lambda)^{-\beta} \). Then \( T_r \) is said to satisfy the *dyadic* \((p, q, \sigma)\) *Davies–Gaffney estimate* if there is a finite constant \( C_{DG} = C_{DG}(d, p, q, \beta, \sigma) > 0 \) such that

\[
\|1_{B_\varepsilon(r)}T_r1_{A_2(x, r, k)}\|_{p \to q} \leq C_{DG} r^{-d(1/p - 1/q)} g(2^k) 2^{kd/\sigma}, \quad x \in \mathbb{R}^d, k \in \mathbb{N}_0,
\]

and if \( \beta > d(1/\sigma + 1/q) \).

This definition is inspired by Davies [25] and the notion of generalized Gaussian estimates introduced by Blunck and Kunstmann, cf. [10, p. 920]. Heuristically, the projection onto \( B_\varepsilon(r) \) captures the singularity of \( T_r \) at \( x \), whereas the projection onto \( A_2(x, r, k) \) controls its decay at distance \( 2^kr \). We use dyadic annuli instead of annuli with constant thickness, since they allow to exploit decay more effectively. Similar estimates were, for example, used by Schreieck and Voigt [58] on \( L^p \) independence of the spectrum of Schrödinger operators with form small potentials and later systematically studied and exploited in a series of works by Blunck and Kunstmann [5–10] on spectral multiplier theorems for operators whose semigroups need not have a bounded kernel but obey the mentioned generalized Gaussian estimates.

Estimate (3.8) with \( p = 1, q = \infty \), and the relaxed assumption \( \beta \geq d/\sigma \) is equivalent to a pointwise estimate for the kernel of \( T_r \).

**Proposition 3.4.** Suppose \( r, \beta, \sigma > 0 \), \( \beta \geq d/\sigma \), and \( (T_r)_{r>0} \) is a family of linear operators in \( \mathcal{B}(L^1 \to L^\infty) \) with integral kernel \( T_r(x, y) \). Then, the following statements are equivalent.
(1) $T_r$ satisfies (3.8) with $p = 1$, $q = \infty$, and $C_{DG}$ replaced by $c_{\beta,d,\sigma} C_{DG}$ for some $c_{\beta,d,\sigma} > 0$.

(2) One has $|T_r(x, y)| \lesssim_{\beta,d,\sigma} C_{DG} r^{-d} (1 + |x - y|/r)^{-\beta-d/\sigma}$ for all $x, y \in \mathbb{R}^d$.

Proof. $(1) \Rightarrow (2)$: In this case, (3.8) asserts
\[
\sup_{y,z \in \mathbb{R}^d} |1_{B_r(x)}(y)| T_r(y, z) |1_{A_2(x,r,k)}(z)| \leq c_{\beta,d,\sigma} C_{DG} r^{-d} g(2k)^{2kd/\sigma}
\]
\[
\lesssim_{\beta,d,\sigma} C_{DG} r^{-d} 2^{-k(\beta-d/\sigma)}
\]
for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$. Choosing $y = x$ and $k \in \mathbb{N}_0$ such that $2k = \max\{1, \lfloor x - z\rfloor/r\}$ yields $|T_r(x, z)| \lesssim_{\beta,d,\sigma} C_{DG} r^{-d} (1 + \lfloor x - z\rfloor/r)^{-\beta-d/\sigma}$ for all $x, z \in \mathbb{R}^d$.

$(2) \Rightarrow (1)$: We estimate
\[
\|1_{B_r(x)} T_r 1_{A_2(x,r,k)}\|_{1 \to \infty} = \sup_{y,z \in \mathbb{R}^d} |1_{B_r(x)}(y)| T_r(y, z) |1_{A_2(x,r,k)}(z)|
\]
\[
\leq \sup_{y \in B_r(x)} \sup_{z \in \mathbb{R}^d \setminus B_r(2^{k-1}r)} |T_r(y, z)|
\]
\[
\lesssim_{\beta,d,\sigma} C_{DG} r^{-d} (1 + 2^k \theta(k-2))^{-\beta-d/\sigma},
\]
which concludes the proof. \qed

Further consequences of Definition 3.3 will be discussed in Sect. 4. They play an important role in the subsequent analysis.

The example of the semigroup of $H_a$ with the Hardy potential $V = a|x|^{-\alpha}$ illustrates that (3.8) is a reasonable assumption. Note that $|x|^{-\alpha} \notin K_{\alpha}(\mathbb{R}^d)$. The resulting operator
\[
\mathcal{L}_{a,\alpha} := (-\Delta)^{\alpha/2} + \frac{a}{|x|^\alpha} \text{ in } L^2(\mathbb{R}^d)
\]
for $\alpha \in (0, 2 \wedge d)$ and $a \geq a_\ast \equiv a_\ast(\alpha, d) > -\infty$ is sometimes called generalized or fractional Hardy operator. By the sharp Hardy–Kato–Herbst inequality, $\mathcal{L}_{a,\alpha}$ is bounded from below and nonnegative in the sense of quadratic forms if and only if $a \geq a_\ast$. We refer to Kato [45, Chapter 5, Equation (5.33)] (for $\alpha = 1$ and $d = 3$) and Herbst [39, Equation (2.6)] and also [31, 33, 46, 64] for proofs of this fact and the explicit expression of $a_\ast$.

For $a \in [a_\ast, 0]$, Bogdan et al. [11] proved pointwise heat kernel bounds, namely
\[
e^{-t \mathcal{L}_{a,\alpha}}(x, y) \sim_{d,a,\alpha} \left(1 + \frac{t^{1/\alpha}}{|x|}\right)^{\delta} \left(1 + \frac{t^{1/\alpha}}{|y|}\right)^{\delta} \frac{t^{d/\alpha}}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}, \quad t > 0, x, y \in \mathbb{R}^d \setminus \{0\},
\]
(3.10)
where $\delta = \delta(a, d, \alpha) \in [0, (d - \alpha)/2]$ satisfies $\delta(0, d, \alpha) = 0$, $\delta(a_\ast, d, \alpha) = (d - \alpha)/2$, and increases monotonously as $a$ decreases. An explicit formula for $\delta(a, d, \alpha)$ is, for example, contained in [11] or Frank et al. [31]. In particular, $e^{-t \mathcal{L}_{a,\alpha}}$ is $L^p \to L^p$ bounded for $a < 0$ if and only if $p \in (d/(d - \delta), d/\delta)$. This follows from (3.10), duality, and $(e^{-\mathcal{L}_{a,\alpha}} 1_{B_0(1)})(x) \gtrsim |x|^{-\delta}$. Moreover, $e^{-t \mathcal{L}_{a,\alpha}}$ satisfies the $L^2 \to L^\infty$ estimates in (3.2) if and only if $a \geq 0$. 
Example 3.5. Let $\alpha \in (0, 2 \land d)$, $t > 0$, and $r_t := t^{1/\alpha}$.

(1) If $a \in [a_*, 0)$ and $p \in (d/(d - \delta), 2]$, then $T_{r_t} := e^{-t L_{a, \alpha}}$ satisfies the dyadic $(p, p', p')$, $(p, 2, p')$, and $(2, p', 2)$ Davies–Gaffney estimate (3.8) with $g(\lambda) \sim_d a, (1 + \lambda)^{-d-a}$.

(2) If $a \geq 0$ and $1 \leq p \leq 2 \leq q < \infty$, then $T_{r_t} := e^{-t L_{a, \alpha}}$ satisfies the dyadic $(p, q, p')$ Davies–Gaffney estimate (3.8) with $g(\lambda) \sim_d a, (1 + \lambda)^{-d-a}$.

Proof. (1) By Hölder’s inequality and (3.10), one obtains for $x \in \mathbb{R}^d$ and $k \in \mathbb{N} \setminus \{1\}$

$$
\sup_{f \in L^p : \|f\|_p = 1} \left\| 1_{B_t(r_t)} e^{-t L_{a, \alpha}} 1_{A_2(x, r_t, k)} f \right\|_{p'}^{p'} \\
\leq \int_{|z-y| \leq 1} dz \left( 1 \vee \frac{r_t}{|z|} \right)^{\delta p'} \int_{|z-y| \in [2^{k-1}, 2^k]} dy \left( 1 \vee \frac{r_t}{|y|} \right)^{\delta p'} \frac{r_t^{2\alpha p'}}{(r_t + |z - y|)^{(d+a)}} \\
r_t^{-d(p' - 2)} (1 + 2^k)^{(d+a)p'} 2^{kd}
$$

where we used $|z - y| \geq |y - x| - r_t \geq (2^k - 1)r_t$. For $k \in \{0, 1\}$, it suffices to integrate $y$ over $|y - x|$, in which case the left side is bounded by a constant times $r_t^{-d(p' - 2)} \lesssim r_t^{-d(p' - 2)} (1 + 2^k)^{(d+a)p'} 2^{kd}$ as well. Taking the $p'$-th root yields the dyadic $(p, p', p')$ Davies–Gaffney estimate (3.8).

The $(p, 2, p')$ and $(2, p', 2)$ Davies–Gaffney estimates follow similarly. For $x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$, one has

$$
\sup_{f \in L^2 : \|f\|_2 = 1} \left\| 1_{B_t(r_t)} e^{-t L_{a, \alpha}} 1_{A_2(x, r_t, k)} f \right\|_{p'}^{p'} \\
\leq \int_{|z-y| \leq 1} dz \left( 1 \vee \frac{r_t}{|z|} \right)^{\delta p'} \left[ \int_{|z-y| \in [2^{k-1}, 2^k]} dy \left( 1 \vee \frac{r_t}{|y|} \right)^{2\delta} \frac{r_t^{2\alpha p'}}{(r_t + |z - y|)^{2(d+a)}} \right]^{p'/2} \\
\lesssim r_t^{-d(p' - 2)/2} (1 + 2^k)^{(d+a)p'} 2^{kd} r_t/p'
$$

which shows the $(2, p', 2)$ estimate. The $(p, 2, p')$ estimate is shown analogously.

(2) By (3.1) (Trotter’s formula) and a similar computation, one obtains

$$
\left\| 1_{B_t(r_t)} e^{-t L_{a, \alpha}} 1_{A_2(x, r_t, k)} \right\|_{p \to q} \lesssim r_t^{-d(1/p - 1/q)} (1 + 2^k)^{-(d+a)} \cdot 2^{kd/p'}
$$

for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$. □

Remarks 3.6. In fact, (3.10) allows to derive pointwise estimates for $e^{-z L_{a, \alpha}}$ using Davies’ method, cf. [51, Theorem A] and Bui and D’Ancona [13, Proposition 3.4] when $|\arg(z)| < \pi/4$.

We now use Theorem 2.1 to extend dyadic Davies–Gaffney estimates (3.8) for $e^{-t H_a}$ and $t > 0$ to complex times $z \in \mathbb{C}_+$. 

**Theorem 3.7.** Let $\alpha > 0$, $\beta > 0$, $1 \leq p \leq 2 \leq q \leq \infty$, and $\sigma, t > 0$. Suppose $e^{-tH_\alpha}$ satisfies the dyadic $(p, q, \sigma)$ Davies–Gaffney estimate (3.8) (Definition 3.3) with $r_t := t^{1/\alpha}$, i.e., there is a constant $C_{DG} > 0$ such that

$$\|1_{B_t}(r_t) e^{-tH_\alpha} 1_{A_2(x,r_t,k)}\|_{p \to q} \lesssim_{\text{DG}} C_{DG} r_t^{-d \left( \frac{\beta}{p} - \frac{\beta - \frac{d}{\sigma}}{2} \right)} 2^{-k(\beta - \frac{d}{\sigma})}, \quad x \in \mathbb{R}^d, k \in \mathbb{N}_0. \tag{3.11}$$

In case $p \in [1, 2)$ and $q \in (2, \infty]$, and $q \neq p'$, assume additionally the bounds

$$\max \{\|e^{-tH_\alpha}\|_{p \to p'}, \|e^{-tH_\alpha}\|_{q \to q}\} \lesssim_{d, p, q, \alpha, \beta, \sigma} 1. \quad \text{Then, for $z = |z|e^{i\theta} \in \mathbb{C}_+$, $\xi \geq 0$, $r_z := |z|^{1/\alpha}(\cos \theta)^{-\xi}$, $\varepsilon \in (0, 1)$, and}$

$$\tilde{\beta} = \tilde{\beta}_{d, \theta, \varepsilon}(\theta) := \left( \beta - \frac{d}{\sigma} \right) \left( 1 - \frac{|\theta|}{\varepsilon|\theta| + (1 - \varepsilon)\pi/2} \right) \geq 0, \tag{3.12}$$

one has

$$\|1_{B_z}(r_z) e^{-zH_\alpha} 1_{A_2(x,r_z,k)}\|_{p \to q} \lesssim_{d, \alpha, \beta, \sigma, \varepsilon} C_{DG} \left( |z| \cos \theta \right)^{-d \left( \frac{\beta}{p} - \frac{\beta - \frac{d}{\sigma}}{2} \right)} (\cos \theta)^{-d\frac{d}{\sigma} \cdot 2^{-k\tilde{\beta}}} \tag{3.13}$$

for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$. Moreover,

$$\|e^{-zH_\alpha}\|_{p \to q} \lesssim_{d, \alpha, p, q, \beta, \sigma} C_{DG} \left( |z| \cos \theta \right)^{-d \left( \frac{\beta}{p} - \frac{\beta - \frac{d}{\sigma}}{2} \right)} \tag{3.14}$$

**Remarks 3.8.**

(1) Examples 3.5 and (1.6) indicate that one will have $\beta = d + \alpha$ in many scenarios. Nevertheless, we prefer to keep $\beta$ as a free parameter here and in the following to illustrate that certain estimates for $e^{-tH_\alpha}$ can be extended to complex times under less severe decay conditions.

(2) We make some remarks on the choices $r_t = t^{1/\alpha}$ and $r_z = |z|^{1/\alpha}(\cos \theta)^{-\xi}$. The power $\alpha^{-1}$ reflects the scaling relation between time and space in $e^{-zH_\alpha}$ and is dictated by the order of the principal symbol of $(-\Delta)^{\alpha/2} + \min \{V, 0\}$. For $t > 0$, this is seen in (1.3) for $V = 0$, in (3.1) for $\alpha \in (0, 2)$ and $V \geq 0$, in (3.10) for $V = a|x|^{-\alpha}$ with $a \geq a_*$, and in Huang et al. [42, Theorem 1.3] when $V \in K_\alpha(\mathbb{R}^d)$ is a perturbation. For complex times, the power is expected to be $\alpha^{-1}$, too. This is confirmed by Corollaries 3.1 and 3.2 for $V \geq 0$, (1.6) for $V \in K_\alpha(\mathbb{R}^d)$, and estimates for $e^{-zH_\alpha}$ when $\alpha \in 2\mathbb{N}$, see (3.15).

On the other hand, a natural choice for $\xi$ is not obvious due to the complicated relation between $\theta$ and $|z|$ in (3.13). For $\alpha \in 2\mathbb{N}$, $V \geq 0$, $x, y \in \mathbb{R}^d$, $r, s > 0$, and $|x - y| > r + s$, Davies [25, Theorem 10] showed

$$\|1_{B_x}(r)e^{-|z|e^{i\theta}H_\alpha} 1_{B_y}(s)\|_{2 \to 2} \leq \exp \left( -c \left( \frac{d(B_x(r), B_y(s))}{|z|^{\frac{d}{\alpha}}(\cos \theta)^{-\frac{d}{\alpha} - 1}} \right) \right) \tag{3.15}$$
with similar estimates being available for more general (even singular) $V$, cf. [9, pp. 154-156]. This shows that $r_z = |z|^{-1/\alpha}(\cos \theta)^{-\zeta}$ with $\zeta \equiv (\alpha - 1)/\alpha > 0$ is a natural choice when $\alpha \in 2\mathbb{N}$. In fact, $\zeta > 0$ is necessary for (3.15) to hold on all of $\mathbb{C}_+$, which can be seen by fixing $x \neq y$ and letting $|\theta| \to \pi/2$.

(3) Estimate (3.15) and many related variants, such as [14, Proposition 4.1] by Carron et al. and [9, Theorem 2.1] by Blunck, were proved using Phragmén–Lindelöf principles, see Davies [25, Lemma 9] for the original version. The presence of exponential bounds for $\alpha$ with similar estimates being available for more general (even singular)

The proof of Theorem 3.7 is inspired by Blunck [9, Theorem 2.1] and uses two consequences (Propositions 4.3 and 4.5) of the dyadic Davies–Gaffney estimates that are contained in Sect. 4.

**Proof of Theorem 3.7.** For $\mu = 1/2 - 1/q$ and $\nu = 1/p - 1/2$, we have

$$
r_t^{d\nu} \|e^{-tH_a}\|_{2 \to q} \lesssim C_{DG}^{\mu/(\mu + \nu)} \quad \text{and} \quad r_t^{d\nu} \|e^{-tH_a}\|_{p \to 2} \lesssim C_{DG}^{\nu/(\mu + \nu)}, \quad t > 0
$$

(3.16)

by Proposition 4.3, whenever $p = 2$ and $q \in [2, \infty)$, or $p \in [1, 2]$ and $q = 2$, or $q = p'$. In all other cases, (3.16) follows from Riesz–Thorin interpolation between (4.3) in Proposition 4.3 and the $L^p \to L^p$ and $L^q \to L^q$ boundedness of $e^{-tH_a}$. Thus, we obtain for $z = |z|e^{i\theta} \in \mathbb{C}_+$ and $t \in (0, \text{Re}(z)/2),$

$$
\|e^{-zH_a}\|_{p \to q} \leq \|e^{-tH_a}\|_{2 \to q} \|e^{-(z-2t)H_a}\|_{2 \to 2} \|e^{-tH_a}\|_{p \to 2} \lesssim C_{DG} r_t^{-d\left(\frac{1}{p} - \frac{1}{q}\right)}.
$$

(3.17)

Combining (3.17) with $\lim_{t \to 0} r_t^{-1} = 2^{1/\alpha}(|z| \cos \theta)^{-\frac{1}{\alpha}}$ yields

$$
\|e^{-zH_a}\|_{p \to q} \lesssim C_{DG} (|z| \cos \theta)^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)},
$$

which proves (3.14). Moreover, for any ball $B = B_{x_0}(r_0)$ and its $k$-th dyadic annulus $A^{(k)}_2 = A_2(x_0, r_0, k)$ centered around $B$ (with $k \in \mathbb{N}_0$ and the convention $A^{(0)}_2 = B$), we obtain

$$
\|1_B e^{-zH_a} 1_{A^{(k)}_2}\|_{p \to q} \leq c_{d,\alpha,\beta,p,q,\sigma} C_{DG} (|z| \cos \theta)^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)}.
$$

(3.18)

On the other hand, (4.5) in Proposition 4.5 implies for $\theta = 0$,

$$
\|1_B e^{-tH_a} 1_{A^{(k)}_2}\|_{p \to q} \leq c_{d,\alpha,\beta,p,q,\sigma} C_{DG} t^{-\frac{d}{\alpha}\left(\frac{1}{p} - \frac{1}{q}\right)} \cdot \left(1 + \frac{|B|^{\frac{\beta}{2}}}{r_t^{\theta(k - 2) + \theta(1 - k)}} \right).
$$

(3.19)
Define the analytic function $F : \mathbb{C}_+ \to \mathcal{B}(L^p \to L^q)$ by

$$F(z) := (c_{d,\alpha,\beta,p,q,\sigma} \mathcal{D}G)^{-1} \left( 1 + \frac{|B|^\frac{q}{d}}{z} \right)^{-\frac{d}{\alpha}} 1_B e^{-\varepsilon H_a} 1_{A_2(k)}. \quad (3.20)$$

By (3.18) and (3.19), we have (using $|1 + |B|^{\alpha/d}|/z| \geq 1$ for $z \in \mathbb{C}_+$)

$$\|F(|z| e^{i\theta})\|_{p \to q} \leq (|z| \cos \theta)^{-\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)},$$

$$\|F(|z|)\|_{p \to q} \leq |z|^{-\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)} \left( \frac{d(B, A_2(k))}{|z|^{1/\alpha}} \right)^{-\beta} \left( \frac{|A_2(k)|^q}{|z|} \right)^{\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)}.$$

We now apply Theorem 2.1 with $a_1 = 1, \beta_1 = d(1/p - 1/q)/\alpha$, and $a_2 = d(B, A_2(k))\alpha$, $\beta_2 = \beta/\alpha, \alpha_3 = |A_2(k)|^q, \beta_3 = \frac{d}{\alpha}$ if $k \in \mathbb{N} \setminus \{1\}$, and $a_2 = a_3 = 1$ and $\beta_2 = \beta_3 = 0$ if $k \in \{0, 1\}$.

Abbreviating $\gamma_\varepsilon = \varepsilon |\theta| + (1 - \varepsilon)\pi/2$, we obtain

$$\|F(|z| e^{i\theta})\|_{p \to q} \lesssim \varepsilon (|z| \cos \theta)^{-\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)}$$

$$\times \left\{ \theta(1 - k) + \theta(k - 2) \left[ 1 \wedge \left( \frac{d(B, A_2(k))}{|z|} \right)^{-\beta} \left( \frac{|A_2(k)|^q}{|z|} \right)^{\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)} \right] \right\}$$

for all $|z| > 0$ and $|\theta| < \pi/2$.

By the definition (3.20) and

$$\left| 1 + \frac{|B|^{\alpha/d}}{z} \right| \lesssim \left( 1 + \frac{|B|}{|z|^{d/\alpha}} \right)^{1/q},$$

this implies

$$\|1_B e^{-\varepsilon H_a} 1_{A_2(k)}\|_{p \to q} \lesssim \mathcal{C}_D G(|z| \cos \theta)^{-\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)} \left( 1 + \frac{|B|}{|z|^{d/\alpha}} \right)^{\frac{1}{q}}$$

$$\times \left\{ \theta(1 - k) + \theta(k - 2) \left[ 1 \wedge \left( \frac{d(B, A_2(k))}{|z|} \right)^{-\beta} \left( \frac{|A_2(k)|^q}{|z|} \right)^{\frac{d}{\alpha} \left( \frac{1}{p} - \frac{1}{q} \right)} \right] \right\}.$$
Choosing $B \equiv B_x(r_z)$ and $A^{(k)}_2 \equiv A_2(x, r_z, k)$, and recalling $r_z = |z|^{1/2} (\cos \theta)^{-\zeta}$ and $r_{|z|} = |z|^{1/\alpha}$ yields

$$
\|1_{B_x(r_z)} e^{-zH_\sigma} 1_{A_2(x, r_z, k)}\|_{p \to q} 
\lesssim C_{DG} (|z| |\cos \theta|^{\frac{d}{\alpha}} (\frac{1}{\beta} - \frac{d}{\alpha}) \cdot \left(1 + \frac{r_z}{r_{|z|}}\right)^{\frac{d}{\alpha}} 
\times \left\{ \theta (1 - k) + \theta (k - 2) \left(\frac{2^k r_z}{r_{|z|}}\right)^{-(\beta - \frac{d}{\alpha}) \cdot (1 - \frac{3|q|}{p \rho})} \right\}
\lesssim C_{DG} (|z| |\cos \theta|^{\frac{d}{\alpha}} (\frac{1}{\beta} - \frac{d}{\alpha}) \cdot (\cos \theta)^{-\frac{d\zeta}{\alpha}} \cdot 2^{-k (\beta - \frac{d}{\alpha}) \left(1 - \frac{3|q|}{(1 + \sigma) \rho}\right)}
$$

for all $z \in \mathbb{C}_+$. Here we used $\beta \geq d/\sigma$ and $r_z \geq r_{|z|}$ since $\zeta \geq 0$. This shows (3.13) and concludes the proof of Theorem 3.7. 

For $p \in [1, 2]$, Theorem 3.7 can be used to extend dyadic $(2, p', \sigma)$ and $(p, 2, \tilde{\sigma})$ Davies–Gaffney estimates for $e^{-tH_\sigma}$ to complex times. In Proposition 4.6, we show that for $t > 0$, $\sigma = 2$, and $\tilde{\sigma} = p'$ they can be inferred from $(p, p', p')$ estimates under a slightly stronger decay condition.

**Definition 3.9.** Let $r > 0$, $(T_r)_{r > 0}$ be a family of linear bounded operators on $L^2(\mathbb{R}^d)$, $1 \leq p \leq q \leq \infty$, $\beta, \sigma > 0$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy $g(\lambda) \sim_\beta (1 + \lambda)^{-\beta}$. Then $T_r$ is said to satisfy the

1. **restricted dyadic $(p, q, \sigma)$ Davies–Gaffney estimate** if there is a finite constant $C_{DG} = C_{DG}(d, p, q, \beta, \sigma) > 0$ such that (3.8) holds, and if $\beta > d(1/p + 1/\sigma)$.
2. **dual dyadic $(p, p', p')$ Davies–Gaffney estimate** if $p \in [1, 2]$, if there is a finite constant $C_{DG} = C_{DG}(d, p, \beta, \sigma) > 0$ such that (3.8) holds, and if $\beta > d(1/2 + 1/p')$.

**Remarks 3.10.**

1. If $p \in [1, 2]$, then the restricted dyadic $(p, p', p')$ estimate implies the dual dyadic $(p, p', p')$ and the dyadic $(p, p', p')$ Davies–Gaffney estimate.
2. In Proposition 4.6, we show that dual dyadic $(p, p', p')$ estimates imply dyadic $(2, p', 2)$ and $(p, 2, p')$ estimates, whereas restricted dyadic $(p, p', p')$ estimates imply restricted dyadic $(2, p', 2)$ and $(p, 2, p')$ Davies–Gaffney estimates.
3. The notions of dyadic $(p, p, \sigma)$ and restricted dyadic $(p, p, \sigma)$ Davies–Gaffney estimates coincide.
4. The semigroup $e^{-tL_{a, a}}$ in Example 3.5 satisfies the restricted dyadic $(p, p', p')$ estimates whenever $a \in [-a_*, 0)$ and $p \in (d/(d - \delta), 2)$. Moreover, if $V \geq 0$, then $e^{-tH_\sigma}$ satisfies the restricted dyadic $(p, p', p')$ Davies–Gaffney estimates for all $p \in [1, 2]$.

The following corollary shows that dual dyadic $(p, p', p')$ Davies–Gaffney estimates can be used to derive complex-time $(2, p', 2)$ and $(p, 2, p')$ estimates.
Corollary 3.11. Let $0 < \alpha, \beta > 0$, $1 \leq p \leq 2$, and $t > 0$. Suppose $e^{-tH_\alpha}$ satisfies the dual dyadic $(p, p', p')$ Davies–Gaffney estimate (Definition 3.9) with $r \equiv r_t := t^{1/\alpha}$ and $g(\lambda) \sim_\beta (1 + \lambda)^{-\beta}$. Then, for $z = |z|e^{i\theta} \in \mathbb{C}^+$, $\zeta \geq 0$, $r_z := |z|^{1/\alpha} (\cos \theta)^{-\zeta}$, $\varepsilon \in (0, 1)$, and

$$
\tilde{\beta}^{(1)} := \tilde{\beta}^{(1)}_{d, \beta, \varepsilon}(\theta) := \left( \beta - \frac{d}{2} \right) \left( 1 - \frac{|\theta|}{\varepsilon |\theta| + (1 - \varepsilon)\pi/2} \right) \geq 0,
$$

$$
\tilde{\beta}^{(2)} := \tilde{\beta}^{(2)}_{d, \beta, p, \varepsilon}(\theta) := \left( \beta - \frac{d}{p'} \right) \left( 1 - \frac{|\theta|}{\varepsilon |\theta| + (1 - \varepsilon)\pi/2} \right) \geq 0,
$$

one has

$$
\|1_{B_z(r_z)} e^{-zH_\alpha} 1_{A_2(x, r_z, k)} \|_{2 \to p'} \lesssim_{d, \alpha, p, \beta, \varepsilon} C_{\text{DG}} \left( |z| \cos \theta \right)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{p'} \right)} \left( \cos \theta \right)^{-\frac{d\varepsilon}{2} \cdot 2^{-k\tilde{\beta}^{(1)}}} (3.22)
$$

and

$$
\|1_{B_z(r_z)} e^{-zH_\alpha} 1_{A_2(x, r_z, k)} \|_{p \to 2} \lesssim_{d, \alpha, p, \beta, \varepsilon} C_{\text{DG}} \left( |z| \cos \theta \right)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{p'} \right)} \left( \cos \theta \right)^{-\frac{d\varepsilon}{2} \cdot 2^{-k\tilde{\beta}^{(2)}}} (3.23)
$$

for all $x \in \mathbb{R}^d$ and $k \in \mathbb{N}_0$. Moreover,

$$
\|e^{-zH_\alpha} \|_{2 \to p'} = \|e^{-zH_\alpha} \|_{p \to 2} \lesssim_{d, \alpha, p, \beta} C_{\text{DG}} \left( |z| \cos \theta \right)^{-\frac{d}{2} \left( \frac{1}{p} - \frac{1}{p'} \right)} (3.24)
$$

Proof. By Proposition 4.6, the dual dyadic $(p, p', p')$ Davies–Gaffney estimate (3.8) implies the dyadic $(2, p', 2)$ and $(p, 2, p')$ Davies–Gaffney estimates. Thus, assumption (3.11) in Theorem 3.7 with $(p, q, \sigma) = (2, p', 2)$ or $(p, q, \sigma) = (p, 2, p')$ there is satisfied. The proof is concluded by an application of Theorem 3.7. □

While the $L^p \to L^q$ estimates (3.14) and $L^2 \to L^{p'}$ and $L^p \to L^2$ estimates (3.24) could be proved rather directly, one could have also obtained them by combining Proposition 4.3 with (3.13), (3.22), and (3.23). However, this argument requires a smallness assumption on $|\theta|$ and produces another nonpositive power of $\cos \theta$. On the other hand, it seems difficult to extend estimates for $\|e^{-tH_\alpha} \|_{p \to p}$ (cf. Corollary 4.8) to complex times without any restrictions on $|\theta|$ or $V$.

Corollary 3.12. Let $0 < \alpha, \beta > 0$, $1 \leq p \leq 2$, and $t > 0$.

1. Suppose $e^{-tH_\alpha}$ satisfies the dyadic $(p, 2, p')$ Davies–Gaffney estimate (Definition 3.3) or the dual dyadic $(p, p', p')$ Davies–Gaffney estimate (Definition 3.9)
with \( r \equiv r_1 := t^{1/\alpha} \) and \( g(\lambda) \sim (1 + \lambda)^{-\beta} \). Then, for \( z = |z| e^{i\theta} \in \mathbb{C}_+ \), \( \zeta \geq 0 \), and
\[
r_z := |z|^{1/\alpha} (\cos \theta)^{-\zeta}, \quad \epsilon \in (0, 1), \quad \text{and}
\]
\[
\tilde{\beta}(2) \equiv \tilde{\beta}^{(2)}_{d, \alpha, p, \beta, \epsilon}(\theta) := \left( \beta - \frac{d}{p'} \right) \left( 1 - \frac{|\theta|}{\epsilon|\theta| + (1 - \epsilon)\pi/2} \right) \geq 0,
\]
one has
\[
\| \mathbf{1}_{B_{\theta}(r)} e^{-z H_{a}} \mathbf{1}_{A_{2}(x, r_z, k)} f \|_{p \to p'} \lesssim_{d, \alpha, p, \beta, \epsilon} C_{DG} (\cos \theta)^{-d \left( \frac{1}{2} + \frac{1}{\alpha} \right) \left( \frac{1}{2} - \frac{1}{p'} \right)} \frac{d \zeta}{\pi}
\]
for all \( x \in \mathbb{R}^d \) and \( k \in \mathbb{N}_0 \).

(2) If \( e^{-t H_a} \) satisfies the restricted dyadic \((p, 2, p')\) or \((p, p', p')\) Davies–Gaffney estimate (Definition 3.9) and \(|\theta| < \pi/2\) satisfies \( \tilde{\beta}(2) > d/p \), i.e.,
\[
\frac{|\theta|}{\epsilon|\theta| + (1 - \epsilon)\pi/2} < 1 - \frac{d}{p} \left( \beta - \frac{d}{p'} \right)^{-1},
\]
then
\[
\| e^{-z H_a} \|_{p \to p} = \| e^{-z H_a} \|_{p' \to p} \lesssim_{d, \alpha, p, \beta, \epsilon} C_{DG} (\cos \theta)^{-d \left( \frac{1}{2} + \frac{1}{\alpha} \right) \left( \frac{1}{2} - \frac{1}{p'} \right)} \frac{d \zeta}{\pi}.
\]

(3) Suppose \( V \in K_{\alpha}(\mathbb{R}^d) \) and \( \mu_{\epsilon, V, d, \alpha} > 0 \) is the constant appearing in (1.6). Then for all \( z \in \mathbb{C}_+ \) and \( 0 < \epsilon \ll 1 \),
\[
\| e^{-z H_a} \|_{1 \to 1} \lesssim_{d, \alpha, p} e^{H_{\mu, V, d, \alpha} |z|} (\cos \theta)^{-d \left( \frac{1}{2} - \frac{1}{\alpha} \right) \left[ 1_{[\alpha < 1]} - \left( \frac{d}{2} + \alpha - 1 \right) 1_{[\alpha \geq 1]} \right]}.
\]

Remarks 3.13. (1) The power of \( \cos \theta \) in (3.27) could not be correct if the estimate held for all \( z \in \mathbb{C}_+ \). This can be seen by considering \( H_a = -\Delta \), since
\[
\| e^{z \Delta} \|_{1 \to 1} \lesssim (\cos \theta)^{-\frac{d}{2}} \quad \text{and} \quad \| e^{z \Delta} \|_{2 \to 2} \leq 1 \implies \| e^{z \Delta} \|_{p \to p} \lesssim (\cos \theta)^{-d \left( \frac{1}{2} - \frac{1}{p} \right)}
\]
for all \( z \in \mathbb{C}_+ \). Moreover, this upper bound is sharp as there is a matching lower bound, cf. Arendt et al. [1, Lemma 2.2].

(2) Since \( \beta > d \) in the case of restricted dyadic \((p, 2, p')\) or \((p, p', p')\) Davies–Gaffney estimates, there exist \(|\theta| \in [0, \pi/2)\) for which \( \tilde{\beta}(2) > d/p \) is satisfied.

Proof. To prove (3.25), it suffices to assume that \( e^{-t H_a} \) satisfies the dyadic \((p, 2, p')\) Davies–Gaffney estimate by Proposition 4.6. By Hölder’s inequality and (3.13) in Theorem 3.7 (which reduces to (3.23) in this case), we have
\[
\| \mathbf{1}_{B_{\theta}(r)} e^{-z H_a} \mathbf{1}_{A_{2}(x, r_z, k)} f \|_{p \to p} \leq |B_{\theta}(r_z)| \frac{2 - p}{2p} \| \mathbf{1}_{B_{\theta}(r)} e^{-z H_a} \mathbf{1}_{A_{2}(x, r_z, k)} f \|_2
\]
\[
\lesssim_{d, \alpha, p, \beta, \epsilon} C_{DG} \frac{d}{p} \left( \frac{1}{2} - \frac{1}{\alpha} \right)^{-d \left( \frac{1}{2} - \frac{1}{\alpha} \right) \left( \frac{1}{2} - \frac{1}{p'} \right)} \frac{d \zeta}{\pi} 2^{-k \tilde{\beta}(2)} \| f \|_p
\]
for any \( f \in L^p, x \in \mathbb{R}^d \), and \( k \in \mathbb{N}_0 \). This proves (3.25).

To prove (3.27), it suffices to assume that \( e^{-t H_a} \) satisfies the restricted dyadic \((p, 2, p')\) Davies–Gaffney estimate by Proposition 4.6. If \( \tilde{\beta}(2) > d/p \), then (3.25)
shows that \( e^{-zH} \) satisfies the (restricted) dyadic \((p, p', p' \cdot (1 - |\theta|/\gamma\varepsilon)^{-1})\) Davies–Gaffney estimate with \( c_{d, \alpha, p, \beta, \varepsilon} \) \( C_{DG} \) \( (\cos \theta)^{-d \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{d}{2} \right) \) instead of \( C_{DG} \), and \( \beta \) replaced by \( \beta (1 - |\theta|/\gamma\varepsilon) \), where \( \gamma\varepsilon = \varepsilon |\theta| + (1 - \varepsilon)\pi/2 \). Thus, (3.27) follows from (4.3) in Proposition 4.3. Estimate (3.28) follows from (1.6). □

3.3. Applicability of the obtained heat kernel bounds

In this subsection, we discuss the applicability of the complex-time bounds obtained in the previous subsections.

3.3.1. Regularization of Schrödinger groups

A selection of applications of \( L^p \to L^p \) bounds for complex-time heat kernels is contained in [54, Chapter 7]. Here we focus on one of them, namely \( L^p \to L^p \)-bounds of regularizations of Schrödinger groups. The following abstract result is due to Boyadzhiev–de Laubenfels [12]. See also Elmennaoui [29] for a similar result concerning Riesz means. In view of the ensuing sections, it is stated in a slightly more general form compared to the previous results.

**Theorem 3.14** [12, Theorem 2.1]. Let \((\Omega, \mu)\) be a measure space, \( p \in [1, \infty) \), \( A \geq 0 \) a nonnegative (and thereby self-adjoint) operator in \( L^2(\Omega) \), and \( \gamma > \delta \geq 0 \) such that \( \|e^{-zA}\|_{p \to p} \lesssim \gamma, \delta \) for all \( z = |z|e^{i\theta} \in \mathbb{C}_+ \). If \( \{e^{-zA}\}_{z \in \mathbb{C}_+} \) is bounded, strongly continuous, holomorphic of angle \( \pi/2 \) on \( L^p(\Omega) \), then

\[
\| (1 + A)^{-\gamma} e^{itA} \|_{p \to p} \lesssim_{\gamma, \delta, p} (1 + |t|)^{\gamma}, \quad t \in \mathbb{R}. \tag{3.29}
\]

Moreover, the map \( \mathbb{R} \ni t \mapsto (1 + A)^{-\gamma} e^{itA} \) is strongly continuous on \( L^p(\Omega) \).

**Remarks 3.15.** As noted in [9, p. 153], the fact that \( \{e^{-zA}\}_{z \in \mathbb{C}_+} \) is strongly continuous on \( L^p \) on all strict subsectors of \( \mathbb{C}_+ \) can be inferred from the assumptions in the theorem by arguing as in Ouhabaz [53], see also [54, Corollary 7.5]. The strong continuity of \( \mathbb{R} \ni t \mapsto (1 + A)^{-\gamma} e^{itA} \) follows from (3.29) and the strong continuity and holomorphy of \( \{e^{-zA}\}_{z \in \mathbb{C}_+} \) as in [54, p. 211].

Theorem 3.14 together with the pointwise bounds for \( e^{it(-\Delta)^{\alpha/2}} \) in [65] has the following consequence.

**Corollary 3.16.** Let \( \alpha > 0 \), \( p \in [1, \infty) \), and

\[
\gamma > \begin{cases} 
2d \left( \frac{1}{\alpha} - \frac{1}{2} \right) |1/p - 1/2| & \text{for } \alpha \in (0, 1), \\
(d - 1) |1/p - 1/2| & \text{for } \alpha = 1, \\
2 \left( \frac{d}{2} + \alpha - 1 \right) |1/p - 1/2| & \text{for } \alpha > 1.
\end{cases}
\]

Then, one has \( \|1 + (-\Delta)^{\alpha/2} e^{it(-\Delta)^{\alpha/2}}\|_{p \to p} \lesssim_{\alpha, \gamma, d, p} (1 + |t|)^{\gamma} \) for all \( t \in \mathbb{R} \).
Proof. For \( \alpha = 1 \), this is the content of [54, Theorem 7.20]. The claim for the other cases follows from Theorem 3.14, interpolation with \( \|e^{-z(-\Delta)^{\alpha/2}}\|_{2 \to 2} \leq 1 \), duality, and the bounds for \( e^{-z(-\Delta)^{\alpha/2}} \) in [65, Theorem 1.3], which yield
\[
\| \exp(-z(-\Delta)^{\alpha/2}) \|_{1 \to 1} \lesssim (\cos \theta)^{-d(\frac{1}{2} - \frac{1}{\alpha})}1_{(\alpha < 1)} - (d + \alpha - 1)1_{(\alpha \geq 1)}.
\]
This concludes the proof. \( \square \)

Unfortunately, the bounds in Corollary 3.12 are still too weak to apply Theorem 3.14 to \( A = H_\alpha \) with \( V \neq 0 \) due to the exponential growth in (3.28) and the fact that the bound (3.27) only holds for \( z \) inside a sector strictly contained in \( \mathbb{C}_+ \).

### 3.3.2. Multiplier theorems

The spectral theorem asserts that bounded and measurable functions of self-adjoint operators in Hilbert spaces are bounded operators. Proving a corresponding statement in Banach spaces is known to be much more delicate. For instance, Hörmander’s classical multiplier theorem [41] asserts the \( L^p(\mathbb{R}^d) \) boundedness of Fourier multipliers \( F(-\Delta) \) provided the multiplier \( F \) is sufficiently smooth. (In fact, as Fefferman [30] demonstrated, some smoothness of \( F \) is necessary for \( L^p \)-boundedness.) If \( F : [0, \infty) \to \mathbb{C} \) is a bounded, measurable function, then Hörmander’s theorem asserts that the operator \( F(-\Delta) \), which is initially defined via Plancherel’s theorem on \( L^2(\mathbb{R}^d) \), extends to an \( L^p(\mathbb{R}^d) \) bounded operator for all \( p \in (1, \infty) \) with
\[
\| F(-\Delta) \|_{p \to p} \lesssim \sup_{t > 0} \| \omega(\cdot) F(t\cdot) \|_{H^s(\mathbb{R})}
\]
for any fixed non-trivial “partition of unity” function \( \omega \in C_c^\infty(\mathbb{R}_+) \) which satisfies \( \sum_{k \in \mathbb{Z}} \omega(2^k t) = 1 \) for all \( t > 0 \), whenever \( s > d/2 \). For a selection of Hörmander multiplier theorems for Schrödinger operators \( H_{\alpha=2} \) in \( L^2(\mathbb{R}^d) \), we refer to [8,28,37].

For Schrödinger operators \( H_\alpha \) in \( L^2(\mathbb{R}^d) \) with \( \alpha \neq 2 \), there seem—to the best of the author’s knowledge—only two results available. Chen et al. [16, Section 5.3] proved a multiplier theorem for \( H_\alpha \) in \( L^2(\mathbb{R}^1) \) with \( \alpha > 1 \) and \( V \geq 0 \). On the other hand, [50, Theorem 2] contains a Hörmander multiplier theorem for \( H_\alpha \) in \( L^2(\mathbb{R}^d) \) with \( \alpha < \min\{2, d\} \) and potentials \( V(x) \) obeying \( \frac{a}{|x|^\alpha} \leq V(x) \leq \frac{\tilde{a}}{|x|^\alpha} \) for any \( \tilde{a} \geq a > 0 \). The latter result strongly relies on an abstract multiplier theorem by Hebisch [38] for operators whose heat kernels satisfy weighted ultracontractive estimates and a certain Hölder condition, which are tailored to (smoothing) Poisson-type heat kernel bounds.

Kriegler [47, Corollary 3.6] proved a Hörmader multiplier theorem for operators whose complex-time heat kernels satisfy Poisson-type bounds. Translated to the language of the present work, Kriegler’s theorem might apply to nonnegative operators \( H_{\alpha=1} = \sqrt{-\Delta} + V \) in \( L^2(\mathbb{R}^d) \) whose complex-time heat kernel satisfies the following bound: there is \( \beta \geq 0 \) such that for all \( x, y \in \mathbb{R}^d, z = |z|e^{i\theta} \in \mathbb{C}_+ = \{ z \in \mathbb{C}_+ : \text{Re}(z) > 0 \} \), one has
\[
|e^{-zH}(x, y)| \lesssim (\cos \theta)^{-\beta} \frac{|z|^\beta}{|z|^2 + |x - y|^2}^{(d+1)/2}.
\]
This assumption is satisfied when $V \equiv 0$ and seems natural when one works with $V \geq 0$ or potentials $V$ whose negative part is not too singular, in particular without Hardy singularities. However, neither the bounds in Theorem 1.1, nor those in Corollary 3.2 suffice to conclude a multiplier theorem for $H_\alpha$ with $V \neq 0$ using Kriegler’s result because of the exponential growth in (1.6) and the deteriorating decay in (3.7), respectively. We hope that this work stimulates further research in these directions.

3.3.3. Operators on manifolds

Li and Yau [48], Davies [23,24], and Sturm [61] proved pointwise sub-Gaussian heat kernel bounds for the negative of the Laplace–Beltrami operator $-\Delta_g \geq 0$ on Riemannian manifolds $(M,g)$ with lower bounded Ricci curvature. Sturm [62] obtained sub-Gaussian heat kernel bounds also for $-\Delta_g + V$ when $V$ belongs to the Kato class. Moreover, pointwise sub-Gaussian heat kernel estimates are available for uniformly elliptic second-order differential operators on domains in $\mathbb{R}^d$ with Dirichlet boundary conditions, cf. [54, Theorem 6.10]. These estimates were extended to complex times, e.g., by Carron–Coulhon–Ouhabaz [14, Proposition 4.1] (see also [54, Theorems 7.2–7.3]) and used to prove $L^p \rightarrow L^p$-estimates for complex-time heat kernels, cf. [14, Theorem 4.3] or [54, Theorem 7.4]. As discussed in Sect. 3.3.1, these estimates lead, among others, to $L^p \rightarrow L^p$-bounds for regularizations of the corresponding Schrödinger group, cf. [14, Theorem 5.2] or [54, Theorem 7.12]. It is natural to consider analogous questions for $(-\Delta_g)^{\alpha/2} + V$, where $(-\Delta_g)^{\alpha/2}$ is defined by the spectral theorem, or for fractional Schrödinger operators on domains $\Omega \subseteq \mathbb{R}^d$. We first discuss the latter. For sufficiently regular $V \in L^1_{\text{loc}}(\Omega : \mathbb{R})$ (e.g., $V \in L^{d/\alpha}_{\text{loc}}$ suffices), and $\psi$ belonging to the Sobolev space $H^\alpha(\mathbb{R}^d)$ and vanishing almost everywhere in $\mathbb{R}^d \setminus \Omega$, the quadratic form $\langle \psi, (-\Delta)^{\alpha/2} + V \rangle_{L^2(\mathbb{R}^d)}$ is bounded from below and closed in the Hilbert space $L^2(\Omega)$. Thus, this form generates a self-adjoint operator $H_\alpha^{(\Omega)}$ in $L^2(\Omega)$, which is also bounded from below. For $0 \leq V \in L^1_{\text{loc}}(\Omega)$, the Friedrichs extension automatically provides us a self-adjoint operator, whose heat kernel can be bounded using the maximum principle by

$$\exp(-t H_\alpha^{(\Omega)}(x,y)) \leq \exp(-t (-\Delta)^{\alpha/2}(x,y)), \quad x, y \in \Omega,$$

whenever $\alpha \in (0, 2)$. Since such bounds are the only input in the proof of Corollary 3.2, one obtains analogous complex-times heat kernel estimates.

**Corollary 3.17.** Let $\alpha \in (0, 2)$, $\Omega \subseteq \mathbb{R}^d$ be an open subset and $0 \leq V \in L^1_{\text{loc}}(\Omega)$. Let further $z = |z|e^{i\theta}$ with $|\theta| \in [0, \pi/2]$, $x, y \in \Omega$, $\epsilon \in (0, 1)$, and $\beta_{d,\alpha,\epsilon}(\theta)$ be defined as in Corollary 3.2. Then,

$$|\exp(-z H_\alpha^{(\Omega)}(x,y))| \lesssim (|z| \cos \theta)^{-d/\alpha} \left(1 + \frac{|x - y|}{|z|^{1/\alpha}}\right)^{-\beta_{d,\alpha,\epsilon}(\theta)}. \quad (3.30)$$

Regarding fractional powers of $-\Delta_g$ on compact $d$-dimensional Riemannian manifolds $(M,g)$, Gimperlein and Grubb [34, Theorem 4.2] exploited the subordination
principle (cf. [57, Chapter 5], see also [36,44,55] for sharp bounds and [3,55] for explicit expressions of the subordinator) to prove estimates for $\exp(-t(\Delta_g)^{\alpha/2})$ with $\alpha \in (0, 2)$ using those for $e^{tA_x}$. For $x, y \in M$ and $t > 0$, they obtained

$$e^{-t(\Delta_g)^{\alpha/2}}(x, y) \sim_{d,\alpha} \frac{t}{(\rho(x, y) + t^{1/\alpha})^\alpha} \cdot \left(1 + (\rho(x, y) + t^{1/\alpha})^{-d}\right),$$  \hspace{1cm} (3.31)

where $\rho(x, y)$ denotes the geodesic distance between $x$ and $y$. Moreover, they extended these estimates to complex times [34, Theorem 1] and showed for $z \in \mathbb{C}_+$,

$$|e^{-z(\Delta_g)^{\alpha/2}}(x, y)| \lesssim_{d,\alpha} (\cos \theta)^{-N} \frac{|z|}{(\rho(x, y) + |z|^{1/\alpha})^\alpha} \cdot \left(1 + (\rho(x, y) + |z|^{1/\alpha})^{-d}\right)$$  \hspace{1cm} (3.32)

with $N = \max\{d/\alpha, 7d/2+4\alpha+7\}$. In [34, Theorem 4.3], they also obtained real-time heat kernel estimates for $(-\Delta_g)^{\alpha/2} + V$ with $V$ being any, not necessarily self-adjoint, classical pseudodifferential operator of order $\alpha - 1$. Assuming additionally from now on that $V$ is such that $(-\Delta_g)^{\alpha/2} + V \geq 0$, then their result reads

$$|e^{-t((-\Delta_g)^{\alpha/2}+V)(x, y)}| \lesssim_{d,\alpha} \frac{t}{(\rho(x, y) + t^{1/\alpha})^\alpha} \cdot \left(1 + (\rho(x, y) + t^{1/\alpha})^{-d}\right) + \frac{t}{(\rho(x, y) + t^{1/\alpha})^{d+\alpha-1}}, \quad t > 0.$$  \hspace{1cm} (3.33)

Since $(-\Delta_g)^{\alpha/2} + V$ is self-adjoint in this situation, we can estimate

$$|e^{-2z((-\Delta_g)^{\alpha/2}+V)(x, y)}| \leq \|e^{-\text{Re}(z)((-\Delta_g)^{\alpha/2}+V)}\|_{L^1\to L^2}^2$$

$$\lesssim_{d,\alpha} 1 + (|z| \cos \theta)^{-d/\alpha} + (|z| \cos \theta)^{-d+1/\alpha}$$

$$\lesssim 1 + (|z| \cos \theta)^{-d/\alpha}$$  \hspace{1cm} (3.34)

for $z = |z|e^{i\theta} \in \mathbb{C}_+$. On the other hand, since $M$ is compact, (3.33) implies

$$|e^{-z((-\Delta_g)^{\alpha/2}+V)(x, y)}| \leq cd_{d,\alpha,M} |z|^{-d/\alpha} \left(\frac{\rho(x, y)^\alpha}{|z|}\right)^{-d+\alpha/\alpha},$$  \hspace{1cm} (3.35)

where $cd_{d,\alpha,M} > 0$ only depends on $d, \alpha$, and $\sup_{x,y\in M} \rho(x, y) < \infty$. We now prove a variant of Theorem 2.1 to obtain further estimates for $|e^{-z((-\Delta_g)^{\alpha/2}+V)(x, y)}|$.

**Lemma 3.18.** Let $X$ be a Banach space equipped with a norm $\| \cdot \|$ and $F : \mathbb{C}_+ = \{z \in \mathbb{C} : \text{Re}(z) > 0\} \rightarrow X$ be a holomorphic function satisfying

$$\|F(|z|^{1/\alpha})\| \leq 1 + a_1 (|z| \cos \theta)^{-\beta_1} \quad \text{and}$$

$$\|F(|z|)\| \leq a_1 |z|^{-\beta_1} \left(\frac{a_2}{|z|}\right)^{-\beta_2} \cdot \left(\frac{a_3}{|z|}\right)^{\beta_3}$$  \hspace{1cm} (3.36)

$$\|F(|z|)\| \leq a_1 |z|^{-\beta_1} \left(\frac{a_2}{|z|}\right)^{-\beta_2} \cdot \left(\frac{a_3}{|z|}\right)^{\beta_3}$$  \hspace{1cm} (3.37)
for some $a_1, a_2, a_3 > 0$, $\beta_1, \beta_2, \beta_3 \geq 0$, all $|z| > 0$, and all $|\theta| < \pi/2$. Then, for all $\varepsilon \in (0, 1)$ one has

$$
\|F(|z|e^{i\theta})\| \leq 2 \left| 1 + \frac{a_1^{1/\beta_1}}{z} \right|^{\beta_1} (e \cos \theta)^{-2\beta_1} \cdot \left( \frac{a_2}{|z|} \right)^{-\beta_2} \cdot \left( \frac{a_3}{|z|} \right)^{\beta_3} \frac{1-|\theta|/\gamma(\varepsilon, \theta)}{1}\cdot (3.38)
$$

for all $|\theta| < \pi/2$ and $|z| > 0$, where $\gamma(\varepsilon, \theta) := \varepsilon|\theta| + (1 - \varepsilon)\pi/2$.

**Proof.** The proof is similar to that of Theorem 2.1, so we merely indicate the needed modifications. We define the functions $H_2(z)$ and $H_3(z)$ as in (2.5)–(2.6), while we choose the function $G(z)$ as

$$
G(z) := (1 + a_1^{1/\beta_1}z)^{-\beta_1} \cdot F(z^{-1}) \cdot H_2(z) \cdot H_3(z).
$$

Then, we have $\|G(|z|e^{i\gamma})\| \leq 1$ similarly as before, but for $\gamma \in (0, \pi/2)$, we obtain

$$
\|G(|z|e^{i\gamma})\| \leq 1 + a_1 (|z|^{-1} \cos \gamma)^{-\beta_1} \leq (\cos \gamma)^{-\beta_1} (1 + a_1 |z|^{\beta_1}) \leq 2 (\cos \gamma)^{-2\beta_1}.
$$

Proceeding as in the proof of Theorem 2.1, we obtain for $-\gamma \leq \theta < 0$,

$$
\|F(|z|e^{i\theta})\| \leq 2 \left| 1 + \frac{a_1^{1/\beta_1}}{z} \right|^{\beta_1} \cdot e^{-2\beta_1 (\cos \theta)^{-2\beta_1}} \cdot \left( \frac{a_2}{|z|} \right)^{-\beta_2} \cdot \left( \frac{a_3}{|z|} \right)^{\beta_3} \frac{1+\theta/\gamma(\varepsilon, \theta)}{1}\cdot (3.38)
$$

Reflecting the estimate along the real axis yields (3.38). □

This lemma and the bounds (3.34)–(3.35) yield, as in the proof of Corollary 3.2, the following estimates.

**Corollary 3.19.** Let $\alpha \in (0, 2)$, $\varepsilon \in (0, 1)$, and $(M, g)$ be a $d$-dimensional compact Riemannian manifold with geodesic distance $\rho : M \times M \to \mathbb{R}_+$. Let $V$ be a classical pseudodifferential operator of order $\alpha - 1$ (in the sense of [34]) such that $(-\Delta_g)^{\alpha/2} + V \geq 0$ in $L^2(M)$. Then for all $x, y \in M$ estimate (3.33) holds and, for all $z = |z|e^{i\theta} \in \mathbb{C}_+$, and $\beta_{d, \alpha, \varepsilon}(\theta)$ as in (3.6) in Corollary 3.2, we have

$$
|e^{-z((-\Delta_g)^{\alpha/2} + V)}(x, y)|
$$

$$
\leq c_{d, \alpha, M} \min \left\{ 1 + (|z| \cos \theta)^{-\frac{d}{\alpha}}, (e \cos \theta)^{-\frac{2d}{\alpha}} (1 + |z|^{-1})^{\frac{d}{\alpha}} \cdot \left( \frac{\rho(x, y)}{|z|^{1/\alpha}} \right)^{-\beta_{d, \alpha, \varepsilon}(\theta)} \right\},
$$

(3.39)

where the constant $c_{d, \alpha, M} > 0$ only depends on $d$, $\alpha$, and $\sup_{x, y \in M} \rho(x, y) < \infty$.

When $\rho(x, y) < |z|^{1/\alpha}$, then the bound (3.34) for $e^{-z((-\Delta_g)^{\alpha/2} + V)}(x, y)$ is already suitable. On the other hand, the estimate

$$
\left( 1 + |z|^{-1} \right)^{\frac{d}{\alpha}} \cdot \left( \frac{\rho(x, y)}{|z|^{1/\alpha}} \right)^{-(d+\alpha)} \lesssim \frac{|z|}{\rho(x, y)^{d+\alpha}} + \frac{|z|^{1+d/\alpha}}{\rho(x, y)^{d+\alpha}} \leq \frac{|z|}{\rho(x, y)^{d+\alpha}} + \frac{|z|}{\rho(x, y)^{\alpha}}
$$

shows that—disregarding different negative powers of the prefactor $\cos(\theta)$ and pretending that $\beta_{d, \alpha, \varepsilon}(\theta)$ could be replaced by $d + \alpha$ in (3.39)—the right sides of (3.39) and (3.32) “qualitatively agree” when $\rho(x, y) > |z|^{1/\alpha}$. 


4. Consequences of dyadic Davies–Gaffney estimates

We collect some consequences of the dyadic Davies–Gaffney estimate (3.8) (Definitions 3.3 and 3.9). The dyadic partition \( 1 = \sum_{k \geq 0} 1_{A_{2k}(x,r,k)} \) for any \( x \in \mathbb{R}^d \) and \( r > 0 \) and the triangle inequality yield the following preliminary estimate.

**Corollary 4.1.** Let \( r > 0, (T_r)_{r > 0} \) be a family of linear operators that satisfy the dyadic \((p, q, \sigma)\) Davies–Gaffney estimate (3.8) (Definition 3.3). Then,

\[
\|1_{B_{x}(r)}T_r\|_{p \to q} \lesssim_{d, \beta, \sigma} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)}, \quad x \in \mathbb{R}^d.
\]  

(4.1)

In fact, for (4.1) to hold, one only needs \( \sum_{k \geq 0} g(k) 2^{kd/\sigma} < \infty \). We will upgrade this estimate in Proposition 4.3 with the help of the ball averages

\[
(N_{p,r}f)(x) := r^{-\frac{d}{p}}\|1_{B_{x}(r)}f\|_{L^p(\mathbb{R}^d)} \quad \text{and} \quad (N_{p,q,r}f)(x) := r^{-\frac{d}{q}}\|1_{B_{x}(r)}f\|_{L^p(\mathbb{R}^d)}
\]

(4.2)

for \( p, q \in [1, \infty) \). The following lemma by Blunck and Kunstmann summarizes useful estimates involving these averaging operators.

**Lemma 4.2.** [7, Lemma 3.3] Let \( 1 \leq p \leq q \leq \infty \) and \( r > 0 \). Then

1. \( (N_{p,r}f)(x) \lesssim_{p,q} (N_{q,r}f)(x) \) for all \( x \in \mathbb{R}^d \),
2. \( \|f\|_{L^p(\mathbb{R}^d)} \sim_p \|N_{p,r}f\|_{L^p(\mathbb{R}^d)} \) and
3. \( \|N_{p,q,r}f\|_{L^q(\mathbb{R}^d)} \lesssim_{p,q} \|f\|_{L^p(\mathbb{R}^d)} \).

**Proposition 4.3.** Let \( r > 0 \) and \((T_r)_{r > 0}\) be a family of linear operators that satisfy the dyadic \((p, q, \sigma)\) Davies–Gaffney estimate (3.8) (Definition 3.3). Then,

\[
\|T_r\|_{p \to q} \lesssim_{d, \beta, \sigma, p, q} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)}.
\]

(4.3)

If \( q = p' \) (so in particular \( 1 \leq p \leq 2 \)) and \( T_r \) is in addition self-adjoint in \( L^2(\mathbb{R}^d) \) and satisfies \( T_{r+s} = T_r T_s \) for all \( r, s > 0 \), then

\[
\|T_r\|_{p \to 2} = \|T_r\|_{2 \to p'} \lesssim_{d, \beta, \sigma, p} C_{DG}^{1/2} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)}.
\]

(4.4)

**Proof.** By \( \|f\|_q \sim \|N_{q,r}f\|_q \), a dyadic partition of unity, assumption (3.8), and \( \|N_{p,q,r}f\|_q \lesssim \|f\|_p \) for \( 1 \leq p \leq q \leq \infty \), we have

\[
\|T_r f\|_q \lesssim_q \|N_{q,r}T_r f\|_q \lesssim_q \left( \int_{\mathbb{R}^d} \left( r^{-d/q} \|1_{B_{x}(r)}\|_{L^p(\mathbb{R}^d)} \sum_{k=0}^{\infty} 1_{A_{2k}(x,r,k)} f\|_q \right)^q \, dx \right)^{1/q}
\]

\[
\leq C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \left( \int_{\mathbb{R}^d} \left( \lim_{k \to \infty} \sum_{k=0}^{\infty} g(2^k) 2^{kd\left(\frac{1}{d} + \frac{1}{q}\right)} (2^k r)^{-d\left(\frac{1}{q}\right)} \|1_{B_{x}(2^k r)} f\|_p \right)^q \, dx \right)^{1/q}
\]

\[
\leq C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \sum_{k=0}^{\infty} g(2^k) 2^{kd\left(\frac{1}{d} + \frac{1}{q}\right)} \|N_{p,q,2^k r} f\|_q
\]

\[
\lesssim_{p,q} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \sum_{k=0}^{\infty} g(2^k) 2^{kd\left(\frac{1}{d} + \frac{1}{q}\right)} \|f\|_p \lesssim_{d, \beta, q} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_p.
\]
This shows (4.3). Formula (4.4) follows from \( \|T_{2r}\|_{p\to p'} = \|T_r\|_{p\to 2}^2 = \|T_r\|_{2\to p'}^2 \).

We need one other consequence of (3.8) in the proof of Theorem 3.7. To that end, we record the following geometric observations.

**Lemma 4.4.** Let \( x, z \in \mathbb{R}^d \), \( r, r_0 > 0 \), and \( |x - z| \leq r + r_0 \).

1. If \( j, k \in \mathbb{N}\setminus\{1\} \) and \( A_2(z, r, j) \cap A_2(x, r_0, k) \neq \emptyset \), then \( 2^{k-3}r_0 \leq (2^j - 1)r \leq 2^{k+3}r_0 \).
2. If \( j \in \mathbb{N} \) and \( A_2(z, r, j) \cap B_x(r_0) \neq \emptyset \), then \( (2^j - 1)r \leq 2r_0 \), i.e., \( j \leq 1 + \log_2(1 + 2r_0/r) \).

**Proof.** By a translation, we can assume \( z = 0 \) without the loss of generality. In that case \( |x| \leq r + r_0 \). Let \( y \in \mathbb{R}^d \) denote those points belonging to the intersection of the geometric objects in claims (1) and (2). We use that, if \( a \leq b \) and \( c \leq d \) and, if \( [a, b] \cap [c, d] \neq \emptyset \), then either \( b \leq c \leq b \leq d \), or \( c \leq a \leq d \leq b \), or \( a \leq c \leq d \leq b \). If in addition \( a = 0 \) and \( b, c, d \geq 0 \), then \( c \leq b \).

1. If the annuli intersect, then \( |y| \in [2^{j-1}r, 2^jr] \) and \( |x - y| \in [2^{k-1}r_0, 2^kr_0] \). By the triangle inequality and the bounds on \( |x| \) and \( |y| \), we also have

\[
|y| \leq |x| + |y| \leq r_0 + (2^j + 1)r \quad \text{and} \quad |x - y| \geq \max(|y| - |x|, 0) \geq \max((2^j - 1)r - r_0, 0).
\]

Thus, \( |x - y| \in [2^{k-1}r_0, 2^kr_0] \cap [\max((2^j - 1)r - r_0, 0), (2^j + 1)r + r_0] \).

This shows that, if \( (2^j - 1)r > r_0 \), then either

a. \( 2^{k-1}r_0 \leq (2^j - 1)r - r_0 \leq 2^kr_0 \leq (2^j + 1)r + r_0 \), or
b. \( (2^j - 1)r - r_0 \leq 2^{k-1}r_0 \leq (2^j + 1)r + r_0 \leq 2^kr_0 \), or

c. \( 2^{k-1}r_0 \leq (2^j - 1)r - r_0 \leq (2^j + 1)r + r_0 \leq 2^kr_0 \), or

d. \( (2^j - 1)r - r_0 \leq 2^{k-1}r_0 \leq 2^kr_0 \leq (2^j + 1)r + r_0 \).

In particular, \( 2^{k-3}r_0 \leq 2^jr \leq 2^{k+3}r_0 \). If instead \( (2^j - 1)r \leq r_0 \), then \( (2^k - 1)r \leq (2^j + 1)r \), so in particular \( 2^{k-3}r_0 \leq 2^jr \leq 2^{k+3}r_0 \).

2. If \( y \in A_2(0, r, j) \cap B_x(r_0) \neq \emptyset \), then \( |y| \in [2^{j-1}r, 2^jr] \) and \( |x - y| \leq r_0 \). The triangle inequality and the bounds on \( |x - y| \) and \( |x| \) imply \( |y| = |y - x + x| \leq 2r_0 + r \) and therefore \( 2^{j-1}r \leq 2r_0 + r \).

**Proposition 4.5.** Let \( r > 0 \) and \( (T_r)_{r>0} \) be a family of linear operators that satisfy the dyadic \((p, q, \sigma)\) Davies–Gaffney estimate (3.8) (Definition 3.3). Then

\[
\|1_{B_x(r_0)}T_r1_{A_2(x, r_0, k)}\|_{p\to q} \leq_d d, \alpha, \beta, p, q \quad C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{q}\right)} \left(1 + \frac{|B_x(r_0)|}{r^d}\right)^{1/q} \times \left[\left(\frac{d(B_x(r_0), A_2(x, r_0, k))}{r}\right)^{-\beta} \cdot \left(\frac{|A_2(x, r_0, k)|}{r^d}\right)^{\frac{1}{q}} \theta(k - 2) + \theta(1 - k) \right] \quad (4.5)
\]
holds for all \( x \in \mathbb{R}^d, r_0 > 0, \) and \( k \in \mathbb{N}_0. \) In fact, the factor \( (1 + |B_x(r_0)|/r^d)^{1/q} \) can be removed for \( k \in \{0, 1\}, \) i.e.,

\[
\|1_{B_x(r_0)} T_r 1_{A_2(x,r_0,k)}\|_{p \to q} \leq d, \sigma, \beta, p, q \ C_{DG} r^{-d \left( \frac{1}{p} - \frac{1}{q} \right)} \left[ \left( \frac{d(B_x(r_0), A_2(x,r_0,k))}{r} \right)^{-\beta} \left( \frac{|A_2(x,r_0,k)|}{r^d} \right)^{\frac{1}{q}} \right] \times \left( 1 + \frac{|B_x(r_0)|}{r^d} \right)^{\frac{1}{q}} \theta(k - 2) + \theta(1 - k) \right] \tag{4.6}
\]

holds for all \( x \in \mathbb{R}^d, r_0 > 0, \) and \( k \in \mathbb{N}_0. \)

Recall that (3.8) only involved one radius. Here the radii \( r \) appearing in \( T_r, \) and \( r_0 \) appearing in \( B_x(r_0) \) and \( A_2(x,r_0,k) \) in formula (4.5) are independent of each other. Note also that the proof of (4.5) only needs \( \beta > d/\sigma. \)

**Proof.** To prove (4.5), we use \( \|f\|_q \sim \|N_{q,r} f\|_q, \) decompose dyadically, apply (3.8), and obtain

\[
\|1_{B_x(r_0)} T_r 1_{A_2(x,r_0,k)} f\|_q \lesssim \|N_{q,r} (1_{B_x(r_0)} T_r 1_{A_2(x,r_0,k)} f)\|_q \leq \left( \int_{\mathbb{R}^d} \left( r^{-d \frac{\sigma}{\sigma}} \|1_{B_z(r)} 1_{B_x(r_0)} T_r 1_{A_2(z,r,j)} 1_{A_2(x,r_0,k)} f\|_q \right)^q \, dz \right)^{1/q} \leq C_{DG} r^{-d \frac{\sigma}{\sigma}} \sum_{j=0}^{\infty} g(2^j) 2^j \left( \int_{|z-x| \leq r+r_0} \|1_{A_2(z,r,j)} 1_{A_2(x,r_0,k)} f\|_p^q \, dz \right)^{1/q} \tag{4.7}
\]

To estimate \((\ldots)^{1/q}\) on the right side, we use Lemma 4.4 with \(|x - z| \leq r + r_0\) and distinguish between the following cases.

1. If \( j, k \in \mathbb{N}\setminus\{1\}, \) and \( A_2(z, r, j) \cap A_2(x, r_0, k) \neq \emptyset, \) then \( 2^{k-3} r_0 \leq 2^j r \leq 2^{k+3} r_0 \) by (1) in Lemma 4.4.
2. If \( k \in \mathbb{N}\setminus\{1\}, j \in \{0, 1\}, \) and \( A_2(z, r, j) \cap A_2(x, r_0, k) \neq \emptyset, \) then \( B_z(2^j r) \cap A_2(x, r_0, k) \neq \emptyset \) and so \( 2^j r \geq 2^{k-3} r_0 \) by (2) in Lemma 4.4.
3. If \( k \in \{0, 1\}, \) we put no restriction on \( j \in \mathbb{N}_0. \)

Thus,

\[
\|1_{A_2(z,r,j)} 1_{A_2(x,r_0,k)} f\|_p \lesssim \|f\|_p \left[ 1_{\{2^j r \geq 2^{k-3} r_0\}} \theta(k - 2) + 1_{\{2^j r \leq 2^{k+3} r_0\}} \theta(j - 2) + \theta(1 - j)) + \theta(1 - k) \right].
\]

If \( k \in \{0, 1\}, \) we simply sum over all \( j \in \mathbb{N}_0. \) If \( k \in \mathbb{N}\setminus\{1\} \) and \( j \in \mathbb{N}\setminus\{1\}, \) we use \( g(2^j) 2^{j \frac{\sigma}{\sigma}} \lesssim \beta 2^{-j (\beta - \frac{d}{\sigma})} \) and that we are summing over dyadic numbers.
If $k \in \mathbb{N} \setminus \{1\}$ and $j \in \{0, 1\}$, we use $\beta \geq d/\sigma$ and $2^j r \geq 2^k r_0$ to estimate $g(2^j) \cdot 2^{\frac{kd}{p'}} \lesssim_{d, \beta, \sigma} (2^k r_0/r)^{-\beta + d/\sigma}$. Thus, (4.7) can be estimated by

$$
\|1_B_{r_0}T_r1_{A_2(x, r_0, k)}f\|_q \\
\lesssim_q C_{DG}r^{-\frac{d}{\sigma}} \left( \int_{|z-x| \leq r + r_0} dz \right)^{1/q} \|f\|_p \\
\times \sum_{j=0}^{\infty} \left[ 2^j \left( \frac{d}{\sigma} - \beta \right) \left( 1_{[2^j r_0 \leq 2^k r_0]} \theta(k - 2) (1_{[2^j r \leq 2^k r_0]} \theta(j - 2) + \theta(1 - j)) + \theta(1 - k) \right) \right]
$$

$$
\lesssim_{d, \beta, q} C_{DG} r^{-d \left( \frac{1}{p'} - \frac{1}{q} \right)} \cdot \left( 1 + \frac{r_0}{r} \right)^{\frac{d}{q}} \left[ \left( \frac{2^k r_0}{r} \right)^{-\beta + d/\sigma} \theta(k - 2) + \theta(1 - k) \right] \|f\|_p .
$$

Since $2^k r_0 \sim d(B_x(r_0), A_2(x, r_0, k)) \sim |A_2(x, r_0, k)|^{\frac{1}{d}}$ for $k \geq 2$, this proves (4.5).

The improved estimate (4.6) for $k \in \{0, 1\}$ follows from $\|1_B_{r_0}T_r1_{A_2(x, r_0, k)}\|_{p \rightarrow q} \leq \|T_r\|_{p \rightarrow q}$ and Proposition 4.3. This concludes the proof of Proposition 4.5. □

The proof of Corollary 3.11 relies on the following proposition which says that $(p, p', p')$ estimates imply $(2, p', 2)$ and $(p, 2, p')$ estimates, if one assumes additionally $\sum_{k \geq 0} g(2^k) 2^{kd \left( \frac{1}{p'} + \frac{1}{2} \right)} < \infty$. This assumption is contained in the notion of dual dyadic Davies–Gaffney estimates.

**Proposition 4.6.** Let $p \in [1, 2]$ and $r > 0$.

(1) If $(T_r)_{r>0}$ is a family of linear operators that satisfy the dual dyadic $(p, p', p')$ Davies–Gaffney estimate (3.8) (Definition 3.9), then $T_r$ satisfies the dyadic $(2, p', 2)$ Davies–Gaffney estimate (Definition 3.3), i.e.,

$$
\|1_{B_x(r)}T_r1_{A_2(x, r, k)}\|_{2 \rightarrow p'} \lesssim_{d, \beta, p} C_{DG} r^{-d \left( \frac{1}{p'} - \frac{1}{q} \right)} g(2^k) 2^{kd}, \quad x \in \mathbb{R}^d , k \in \mathbb{N}_0
$$

(4.8a)

and the dyadic $(p, 2, p')$ Davies–Gaffney estimate, i.e.,

$$
\|1_{B_x(r)}T_r1_{A_2(x, r, k)}\|_{p \rightarrow 2} \lesssim_{d, \beta, p} C_{DG} r^{-d \left( \frac{1}{p'} - \frac{1}{2} \right)} g(2^k) 2^{kd}, \quad x \in \mathbb{R}^d , k \in \mathbb{N}_0 .
$$

(4.8b)

(2) If $(T_r)_{r>0}$ satisfies the restricted dyadic $(p, p', p')$ Davies–Gaffney estimate (3.8) (Definition 3.9), then it satisfies the restricted dyadic $(2, p', 2)$ and $(p, 2, p')$ Davies–Gaffney estimates (4.8a) and (4.8b).

**Proof.** Since $\beta > d(1/2 + 1/p')$ for the dual dyadic $(p, p', p')$ estimates and $\beta > d$ for the restricted dyadic $(p, p', p')$ estimates, it suffices to show (4.8a) and (4.8b).
To prove (4.8a), we use \( \| f \|_{p'} \sim_p \| N_{p',r,f} \|_{p'} \) and obtain for \( f \in L^2 \),
\[
\| 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_{p'} \lesssim \| N_{p',r} 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_{p'}
= \left( \int_{\R^d} dz \left( r^{-d/p} \| 1_{B,(r)} T_r 1_{A(x,r,k)} \sum_{j \geq 0} 1_{A(z,r,j)} f \|_{p'} \right) \right)^{1/p'}
\leq C_{DG} r^{-d/p} \sum_{j \geq 0} g(2j)^{jd/d} \left( \int_{|x-z| \leq 2r} dz \| 1_{A(x,r,k)} 1_{A(z,r,j)} f \|_{p'} \right)^{1/p'}.
\]
We use Lemma 4.4 (with \(|x-z| \leq 2r\)) and distinguish between the following cases.

(1) If \( k, j \in \mathbb{N} \setminus \{1\} \) and \( A_2(x,r,k) \cap A_2(z,r,j) \neq \emptyset \), then \( 2^{k-3} \leq 2^j \leq 2^{k+3} \) by (1) in Lemma 4.4.

(2) If \( k \in \mathbb{N} \setminus \{1\}, j \in \{0,1\} \), and \( A_2(x,r,k) \cap A_2(z,r,j) \neq \emptyset \), then \( A_2(x,r,k) \cap B_2(2^j r) \neq \emptyset \) and so \( k \leq 4 \) by (2) in Lemma 4.4.

(3) If \( k \in \{0,1\}, j \in \mathbb{N}_0 \), and \( A_2(x,r,k) \cap A_2(z,r,j) \neq \emptyset \), then \( B_2(x,r,k) \cap A_2(z,r,j) \neq \emptyset \) and so \( j \leq 4 \) by (2) in Lemma 4.4.

Using this observation and Hölder’s inequality, we can estimate
\[
\| 1_{A_2(x,r,k)} 1_{A_2(z,r,j)} f \|_p \lesssim d,p,\beta 1_{|j-k| \leq 4} (2^j)^{(1/2 - 1/2)} \| f \|_2, \quad |x-z| \leq 2r
\]
and deduce (using \( g(2^j) \sim \beta 2^{-j}\beta \))
\[
\| 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_{p'} \lesssim C_{DG} r^{-d/p} \sum_{j \geq 0} g(2^j)^{jd/d} 2^{kd/2} 1_{|j-k| \leq 4} \left( \int_{|x-z| \leq 2r} dz \right)^{1/p'} \| f \|_2 \lesssim_p d,p,\beta C_{DG} r^{-d(1/2 - 1/2)p} g(2^j)^{kd/2} \| f \|_2.
\]
This concludes the proof of (4.8a).

The proof of (4.8b) is similar but uses also (1) of Lemma 4.2, i.e., \( (N_{2,r,f})(x) \lesssim_p (N_{p',r,f})(x) \). By the above reasoning and \( \| 1_{A_2(x,r,k)} 1_{A_2(z,r,j)} f \|_p \leq 1_{|j-k| \leq 4} \| f \|_p \) for \(|x-z| < 2r\), we obtain
\[
\| 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_2 \lesssim \| N_{2,r} 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_2 \lesssim \| N_{p',r} 1_{B,(r)} T_r 1_{A(x,r,k)} f \|_2
= \left( \int_{\R^d} dz \left( r^{-d/p} \| 1_{B,(r)} T_r 1_{A(x,r,k)} \sum_{j \geq 0} 1_{A_2(z,r,j)} f \|_{p'} \right) \right)^{1/2}
\leq C_{DG} r^{-d/p} \sum_{j \geq 0} g(2^j)^{jd/d} \left( \int_{|x-z| \leq 2r} dz \| 1_{A_2(x,r,k)} 1_{A_2(z,r,j)} f \|_{p'}^2 \right)^{1/2}
\leq C_{DG} r^{-d(1/2 - 1/2)p} (2^j)^{kd/2} \| f \|_p \lesssim C_{DG} r^{-d(1/2 - 1/2)p} g(2^j)^{kd/2} (2^j)^{kd/2} \| f \|_p.
\]
This concludes the proof of Proposition 4.6.
Combining Proposition 4.6 with Proposition 4.3 yields the following variant of (4.4), which does not need that $T_r$ has the semigroup property.

**Corollary 4.7.** Let $p \in [1, 2]$, $r > 0$, and $(T_r)_{r>0}$ be a family of linear operators that satisfy the dual dyadic $(p, p', p')$ Davies–Gaffney estimate (3.8) (Definition 3.9). Then,

$$\|T_r\|_{2 \to p'} \lesssim_{d, \beta, p} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{p'}\right)}$$  \hspace{1cm} (4.9a)

and

$$\|T_r\|_{p\to 2} \lesssim_{d, \beta, p} C_{DG} r^{-d\left(\frac{1}{p} - \frac{1}{2}\right)}$$ \hspace{1cm} (4.9b)

for all $x \in \mathbb{R}^d$.

We now show that restricted dyadic $(p, 2, p')$ and $(p, p', p')$ estimates imply (restricted) dyadic $(p, p, p')$ estimates and $L^p$ boundedness of $T_r$.

**Corollary 4.8.** Let $p \in [1, 2]$, $r > 0$, and $(T_r)_{r>0}$ be a family of linear operators that satisfy the restricted dyadic $(p, 2, p')$ or $(p, p', p')$ Davies–Gaffney estimate (3.8) (Definition 3.9). Then, $T_r$ satisfies the restricted dyadic $(p, p, p')$ Davies–Gaffney estimate

$$\|1_{B_x(r)} T_r 1_{A_{2}(x,r,k)} f\|_{p\to p} \lesssim_{d, \beta, p} C_{DG} 2^{\frac{k d}{2}} g(2^k), \quad x \in \mathbb{R}^d, k \in \mathbb{N}_0.$$  \hspace{1cm} (4.10)

Moreover,

$$\|T_r\|_{p \to p} = \|(T_r)^*\|_{p' \to p'} \lesssim_{d, \beta, p} C_{DG}.$$  \hspace{1cm} (4.11)

**Proof.** By Proposition 4.6, it suffices to assume that $(T_r)_{r>0}$ satisfies the restricted $(p, 2, p')$ Davies–Gaffney estimates.

By Hölder’s inequality and (4.8b), we have

$$\|1_{B_x(r)} T_r 1_{A_{2}(x,r,k)} f\|_p \leq \|B_x(r)\|_{\frac{2-p}{p}} \cdot \|1_{B_x(r)} T_r 1_{A_{2}(x,r,k)} f\|_2 \lesssim_{d, \beta, p} C_{DG} r^{\frac{2-p}{2p}} d \left(\frac{1}{p} - \frac{1}{2}\right) 2^k g(2^k) \|f\|_p$$

for any $f \in L^p$, $x \in \mathbb{R}^d$, and $k \in \mathbb{N}_0$. This proves (4.10). Formula (4.11) follows from (4.10) and Proposition 4.3. This concludes the proof. \hfill \Box

**Acknowledgements**

I wish to thank Volker Bach and Jean–Claude Cuenin for valuable discussions and an anonymous referee for many helpful remarks and suggestions.

**Funding Information** Open Access funding enabled and organized by Projekt DEAL.

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.
**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**REFERENCES**

[1] Wolfgang Arendt, Omar El-Mennaoui, and Matthias Hieber. Boundary values of holomorphic semigroups. *Proc. Amer. Math. Soc.*, 125(3):635–647, 1997.

[2] Joseph Bak and Donald J. Newman. *Complex Analysis*. Springer, New York, third edition, 2010.

[3] Harald Bergström. On some expansions of stable distribution functions. *Ark. Mat.*, 2:375–378, 1952.

[4] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Trans. Amer. Math. Soc.*, 95:263–273, 1960.

[5] S. Blunck and P. C. Kunstmann. Weighted norm estimates and maximal regularity. *Adv. Differential Equations*, 7(12):1513–1532, 2002.

[6] S. Blunck and P. C. Kunstmann. Weak type \((p, p)\) estimates for Riesz transforms. *Math. Z.*, 247(1):137–148, 2004.

[7] S. Blunck and P. C. Kunstmann. Generalized Gaussian estimates and the Legendre transform. *J. Operator Theory*, 53(2):351–365, 2005.

[8] Sönke Blunck. A Hörmander-type spectral multiplier theorem for operators without heat kernel. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2(3):449–459, 2003.

[9] Sönke Blunck. Generalized Gaussian estimates and Riesz means of Schrödinger groups. *J. Aust. Math. Soc.*, 82(2):149–162, 2007.

[10] Sönke Blunck and Peer Christian Kunstmann. Calderón-Zygmund theory for non-integral operators and the \(H^\infty\) functional calculus. *Rev. Mat. Iberoamericana*, 19(3):919–942, 2003.

[11] Krzysztof Bogdan, Tomasz Grzywny, Tomasz Jakubowski, and Dominika Pilarczyk. Fractional Laplacian with Hardy potential. *Communications in Partial Differential Equations*, 44(1):20–50, 2019.

[12] Khristo Boyadzhiev and Ralph de Laubenfels. Boundary values of holomorphic semigroups. *Proc. Amer. Math. Soc.*, 118(1):113–118, 1993.

[13] The Anh Bui and Piero D’Ancona. Generalized Hardy operators. *Preprint*, pages 1–24, April 2021.

[14] Gilles Carron, Thierry Coulhon, and El-Maati Ouhabaz. Gaussian estimates and \(L^p\)-boundedness of Riesz means. *J. Evol. Equ.*, 2(3):299–317, 2002.

[15] Peng Chen, El Maati Ouhabaz, Adam Sikora, and Lixin Yan. Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. *J. Anal. Math.*, 129:219–283, 2016.

[16] Peng Chen, El Maati Ouhabaz, Adam Sikora, and Lixin Yan. Spectral multipliers without semigroup framework and application to random walks. *J. Math. Pures Appl.*, 9(9), 143:162–191, 2020.

[17] Peng Chen, Xuan Thinh Duong, Zhijie Fan, Ji Li, and Lixin Yan. The Schrödinger equation in \(L^p\) spaces for operators with heat kernel satisfying Poisson type bounds. *J. Math. Soc. Japan*, 74(1):285–331, 2022.

[18] Peng Chen, Xuan Thinh Duong, Ji Li, and Lixin Yan. Sharp endpoint \(L^p\) estimates for Schrödinger groups. *Math. Ann.*, 378(1–2):667–702, 2020.

[19] Thierry Coulhon and Xuan Thinh Duong. Riesz transforms for \(1 \leq p \leq 2\). *Trans. Amer. Math. Soc.*, 351(3):1151–1169, 1999.
[20] Thierry Coulhon and Xuan Thinh Duong. Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss. *Adv. Differential Equations*, 5(1-3):343–368, 2000.

[21] Thierry Coulhon and Xuan Thinh Duong. Riesz transforms for $p>2$. *C. R. Acad. Sci. Paris Sér. I Math.*, 332(11):975–980, 2001.

[22] Thierry Coulhon and Adam Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc. (3)*, 96(2):507–544, 2008.

[23] E. B. Davies. Gaussian upper bounds for the heat kernels of some second-order operators on Riemannian manifolds. *J. Funct. Anal.*, 80(1):16–32, 1988.

[24] E. B. Davies. *Heat Kernels and Spectral Theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1990.

[25] E. B. Davies. Uniformly elliptic operators with measurable coefficients. *J. Funct. Anal.*, 132(1):141–169, 1995.

[26] E. B. Davies and A. M. Hinz. Kato class potentials for higher order elliptic operators. *J. London Math. Soc. (2)*, 58(3):669–678, 1998.

[27] Xuan T. Duong and Derek W. Robinson. Semigroup kernels, Poisson bounds, and holomorphic functional calculus. *J. Funct. Anal.*, 142(1):89–128, 1996.

[28] Xuan Thinh Duong, El Maati Ouhabaz, and Adam Sikora. Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.*, 196(2):443–485, 2002.

[29] Omar Elmennaoui. *Traces des semi-groupes holomorphes singuliers a l’origine et comportement asymptotique*. PhD thesis, Besançon, 1992.

[30] Charles Fefferman. The multiplier problem for the ball. *Ann. of Math. (2)*, 94:330–336, 1971.

[31] Rupert L. Frank, Elliott H. Lieb, and Robert Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.*, 21(4):925–950, 2008.

[32] Rupert L. Frank, Konstantin Merz, and Heinz Siedentop. Equivalence of Sobolev norms involving generalized Hardy operators. *International Mathematics Research Notices*, 2021(3):2284–2303, February 2021.

[33] Rupert L. Frank and Robert Seiringer. Non-linear ground state representations and sharp Hardy inequalities. *J. Funct. Anal.*, 255(12):3407–3430, 2008.

[34] Heiko Gimperlein and Gerd Grubb. Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators. *J. Evol. Equ.*, 14(1):49–83, 2014.

[35] A. Gulisashvili. On the Kato classes of distributions and the BMO-classes. In *Differential equations and control theory (Athens, OH, 2000)*, volume 225 of *Lecture Notes in Pure and Appl. Math.*, pages 159–176. Dekker, New York, 2002.

[36] John Hawkes. A lower Lipschitz condition for the stable subordinator. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 17:23–32, 1971.

[37] Waldemar Hebisch. A multiplier theorem for Schrödinger operators. *y Colloq. Math.*, 60/61(2):659–664, 1990.

[38] Waldemar Hebisch. Functional calculus for slowly decaying kernels. *Preprint*, pages 1–26, 1995.

[39] Ira W. Herbst. Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys.*, 53:285–294, 1977.

[40] Einar Hille. *Analytic Function Theory. Vol. II*. Introductions to Higher Mathematics. Ginn and Co., Boston, Mass.-New York-Toronto, Ont., 1962.

[41] Lars Hörmander. Estimates for translation invariant operators in $L^p$ spaces. *Acta Math.*, 104:93–140, 1960.

[42] Shanlin Huang, Ming Wang, Quan Zheng, and Zhiwen Duan. $L^p$ estimates for fractional Schrödinger operators with Kato class potentials. *J. Differential Equations*, 265(9):4181–4212, 2018.

[43] Tianxiao Huang, Shanlin Huang, and Quan Zheng. Inhomogeneous oscillatory integrals and global smoothing effects for dispersive equations. *J. Differential Equations*, 263(12):8606–8629, 2017.

[44] I. A. Ibragimov and Yu. V. Linnik. *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff Publishing, Groningen, 1971. With a supplementary chapter by I. A. Ibragimov and V. V. Petrov, Translation from the Russian edited by J. F. C. Kingman.

[45] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin-New York, second edition, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132.
[46] V. F. Kovalenko, M. A. Perelmuter, and Ya. A. Semenov. Schrödinger operators with $L^{1/2}(\mathbb{R}^d)$-potentials. J. Math. Phys., 22:1033–1044, 1981.

[47] Christoph Kriegler. Hörmander functional calculus for Poisson estimates. Integral Equations Operator Theory, 80(3):379–413, 2014.

[48] Peter Li and Shing-Tung Yau. On the parabolic kernel of the Schrödinger operator. Acta Math., 156(3-4):153–201, 1986.

[49] Vitali Liskevich, Zeev Sobol, and Hendrik Vogt. On the $L_p$-theory of $C_0$-semigroups associated with second-order elliptic operators. II. J. Funct. Anal., 193(1):55–76, 2002.

[50] Konstantin Merz. On scales of Sobolev spaces associated to generalized Hardy operators. Math. Z., 299(1):101–121, 2021.

[51] Pierre D. Milman and Yu. A. Semenov. Global heat kernel bounds via desingularizing weights. J. Funct. Anal., 212(2):373–398, 2004.

[52] Akihiko Miyachi. On some singular Fourier multipliers. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 28(2):267–315, 1981.

[53] El-Maati Ouhabaz. Gaussian estimates and holomorphy of semigroups. Proc. Amer. Math. Soc., 123(5):1465–1474, 1995.

[54] El Maati Ouhabaz. Analysis of Heat Equations on Domains, volume 31 of London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, 2005.

[55] Harry Pollard. The representation of $e^{-x^2}$ as a Laplace integral. Bull. Amer. Math. Soc., 52:908–910, 1946.

[56] Walter Rudin. Real and Complex Analysis. McGraw-Hill Book Co., New York, third edition, 1987.

[57] René L. Schilling, Renming Song, and Zoran Vondraček. y Bernstein Functions, volume 37 of De Gruyter Studies in Mathematics. Walter de Gruyter & Co., Berlin, second edition, 2012. Theory and Applications.

[58] Gabriele Schreieck and Jürgen Voigt. Stability of the $L_p$-spectrum of Schrödinger operators with form-small negative part of the potential. In Functional Analysis (Essen, 1991), volume 150 of Lecture Notes in Pure and Appl. Math., pages 95–105. Dekker, New York, 1994.

[59] Adam Sikora, Lixin Yan, and Xiaohua Yao. Sharp spectral multipliers for operators satisfying generalized Gaussian estimates. J. Funct. Anal., 266(1):368–409, 2014.

[60] Elias M. Stein and Rami Shakarchi. Complex Analysis, volume 2 of Princeton Lectures in Analysis. Princeton University Press, Princeton, NJ, 2003.

[61] Karl-Theodor Sturm. Heat kernel bounds on manifolds. Math. Ann., 292(1):149–162, 1992.

[62] Karl-Theodor Sturm. Schrödinger semigroups on manifolds. J. Funct. Anal., 118(2):309–350, 1993.

[63] Stephen Wainger. Special trigonometric series in $k$-dimensions. Mem. Amer. Math. Soc., 59:102, 1965.

[64] D. Yafaev. Sharp constants in the Hardy-Rellich inequalities. Journ. Functional Analysis, 168(1):212–144, 1999.

[65] Shiliang Zhao and Quan Zheng. Uniform complex time heat kernel estimates without Gaussian bounds. arXiv e-prints, arXiv:2012.08763, December 2020.

[66] Quan Zheng and Xiaohua Yao. Higher-order Kato class potentials for Schrödinger operators. Bull. Lond. Math. Soc., 41(2):293–301, 2009.