GEOMETRIC INTERPRETATION OF THE POISSON STRUCTURE IN AFFINE TODA FIELD THEORIES

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Abstract. We express the Poisson brackets of local fields of the affine Toda field theories in terms of the Drinfeld-Sokolov dressing operator. For this, we introduce a larger space of fields, containing “half screening charges” and “half integrals of motions”. In addition to local terms, the Poisson brackets contain nonlocal terms related to trigonometric $r$-matrices.

Introduction

Since the work of Zakharov and Shabat [23], the dressing techniques have played an important role in the theory of classical integrable systems. These techniques have been developed by Drinfeld and Sokolov in [7] in the framework of affine Toda field theories. Later, Feigin and one of us proposed ([13], [14]) another approach to these theories; this approach was shown ([8], [10]) to be equivalent to that of [7]. In those works, the space of local fields of the Toda theory (equivalently, the mKdV hierarchy) associated to an affine Lie algebra $\mathfrak{g}$ is described as the ring of functions on the coset space $N_+/A_+$ of a unipotent subgroup of the Kac-Moody group $G$ corresponding to $\mathfrak{g}$. The mKdV flows are then identified with the right action of the principal commutative Lie algebra $\mathfrak{a}$ normalizing $A_+$, $N_+$ being viewed as an open subset of the flag manifold of $G$. This leads to a system of variables, in which the flows become linear and hence can be integrated.

In the works on quantization of the Toda theories, an important role is played by the vertex operator algebra structure on the space of local fields. At the classical level, this gives rise to what we call here a vertex Poisson algebra (VPA) structure on the space of local fields of a Toda theory. The notion of the VPA structure coincides with the notion of “coisson algebra” (on the disc) introduced by Beilinson and Drinfeld in [5]. The goal of this work is to define this and related structures on the space of fields of a Toda theory in the Lie group terms using the identification described above.

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To this end, we make use of an idea introduced earlier by Feigin and one of us in [9], where a similar problem was solved in the setting of the classical lattice Toda theory associated to $\hat{sl}_2$. In that work, the space of local fields was extended by the “half screening charges” and “half integrals of motions”. The screening charges and integrals of motions (IM’s) are sums of the lattice translates of certain expressions, and their “halves” are just the sums of positive translates of the same expressions. The full space was then identified with the quotient $G/H$, where $H$ is the Cartan subgroup of $G$. The smaller spaces of fields, without half screening charges (resp. without half IM’s), can be obtained by taking the quotient of $G/H$ from the left by a Borel subgroup $B_-$ (resp. from the right by the positive part of the loop group of the Cartan subgroup). In this interpretation, the Poisson structure on the full space is given by the difference $\ell(R) - r(R^\infty)$, where $\ell(R)$ stands for the left action of the trigonometric $r$-matrix $R \in g^\otimes 2$ and $r(R^\infty)$ stands for the right action of its “infinitely twisted” version $R^\infty \in g^\otimes 2$. This leads to a description of the Poisson structures of the quotients.

In this work, we enlarge the space of local fields of the continuous affine Toda theories in a similar way, by adding continuous analogues of the half screening charges and IM’s. Using results of [9, 10], we identify the full space $\tilde{\pi}_0$ obtained this way with the space of functions on $B_- \times N_+$. We then study the structure of nonlocal VPA on $\tilde{\pi}_0$. The axioms for this structure are given in Section 1. The main feature is that the the Poisson bracket $\{u(x), v(y)\}$, where $u, v$ are elements of the algebra, can be expressed as a linear combination of local terms of the form $u_k(x)\partial^k_x\delta(x - y)$ with $k \geq 0$, and of nonlocal terms of the form $a(x)\partial^{-1}_x\delta(x - y)b(y)$.

A similar formalism was introduced by Radul [22] in the framework of formal variational calculus (see also [21]). Natural examples of nonlocal VPA’s are given by the higher Adler-Gelfand-Dickey (AGD) structures (i.e. structures obtained from the pair of the first two AGD structures by application of the Magri recursion procedure) introduced in [11, 1].

In Sections 2 and 3 we give a geometric description of the nonlocal VPA structure of $\tilde{\pi}_0 = \mathbb{C}[B_- \times N_+]$. A generic element $g = (b_-, n_+)$ of $B_- \times N_+$ can be considered as the product of expansions of the scattering matrix of the Lax operator from $-\infty$ to $x$ for a small spectral parameter $\lambda$, and from $x$ to $+\infty$ for a large $\lambda$.

The Poisson bracket on $N_+$ is obtained by a straightforward extension of the local VPA structure on $N_+/A_+$. On the other hand, according to [4] and in the spirit of [12], [3], we define the Poisson bracket on $B_-$ via the trigonometric $r$-matrix. We show that the trigonometric Poisson brackets on $B_-$ are compatible with the Poisson brackets of local fields (see Lemma 2.2). The Poisson brackets of $N_+$ and between $N_+$ and $B_-$ also have nonlocal parts which we determine in Lemmas 3.3 and 3.4. To derive the complete expression for the Poisson brackets (Thm. 3.1), we use the
evolution equation

\[ \partial_x g(x) = g(x)p_{-1}, \]

where \( p_{-1} \) is a degree \(-1\) element of \( \mathfrak{a} \).

After that we obtain another realization of the VPA structures on \( N_+ \) and \( N_+/A_+ \) purely in terms of the unipotent group elements (i.e. the Drinfeld-Sokolov dressing operators) – see Cor. 3.1 and formula (47). The latter formula could in principle be obtained directly in the framework of \( N_+ \), but the simple form (1) of the action of \( \partial_x \) on a generic element \( g \) of the whole Kac-Moody group \( G \) makes the derivation easier on \( G \).

Let us now say a few words about possible applications and extensions of this work. First, one may think of the following program of quantization of our results: to formulate quantum axioms corresponding to nonlocal VPA in the spirit of [5]; to quantize the geometric formulas (41), (47), (50) for nonlocal VPA structures; to realize these formulas in terms of the algebra of local fields of quantum Toda theories. The combination of small and large limits for the spectral parameter which we use is reminiscent of the work [4]. Second, it would be interesting to carry out the present work in the case of higher AGD structures; this would lead to a family of compatible nonlocal VPA structures on \( B_- \times N_+ \) (we construct such a family in sect. 4, but its connection with the AGD structures is not clear to us). Next, it would be interesting to obtain similar results for other soliton equations such as the nonlinear Schrödinger (NLS) equation; in that case \( \mathfrak{a} \) should be replaced by the loop algebra with values in the Cartan subalgebra \( \mathfrak{h} \). The geometric interpretation of the NLS variables analogous to the one used here, was obtained by Feigin and one of us [15] (the case of \( \hat{\mathfrak{sl}}_2 \) was also treated in [2]). Finally, the fact that the VPA structure given in Thm. 3.1 is left \( G \)-invariant, leads us to conjecture the existence of affine Weyl group symmetries of the mKdV hierarchies, mixing local and nonlocal terms (see remark 3.5). These symmetries are probably connected with the Darboux transformations.

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The first author would like to dedicate this paper to A. Guichardet, his senior colleague who is about to retire from Ecole Polytechnique.

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1In the course of writing this paper, we became aware of several works dealing with vertex operator structures containing logarithmic terms, see [19, 16] and references therein
1. Nonlocal vertex Poisson algebras.

Let $(R, \partial)$ be a differential ring, and let $A$ be the associated ring of formal pseudodifferential operators. The ring $A$ has generators $\tilde{\partial}, \tilde{\partial}^{-1}$, and $i(r)$ for $r \in R$, and relations $\tilde{\partial}\tilde{\partial}^{-1} = \tilde{\partial}^{-1}\tilde{\partial} = 1$, $[\partial, i(r)] = i(\partial r)$ for $r \in R$, and $i : R \to A$ is an algebra morphism. In what follows, we will denote by $\partial, \partial^{-1}$ and $r, \tilde{\partial}, \tilde{\partial}^{-1}$ and $i(r)$, respectively.

1.1. The modules $R_n$. Let $n \geq 1$ be an integer. We will use the following notation in the algebra $A^{\otimes n}$: $\partial_{x_i}, \partial_{x_i}^{-1}$ will stand for $\otimes_{j=1}^{n-1} r \otimes_{j=i+1}^{n} 1$, $\otimes_{j=1}^{i-1} 1 \otimes \partial \otimes_{j=i+1}^{n} 1$, $\otimes_{j=1}^{i-1} \partial \otimes_{j=i+1}^{n} 1$, respectively. For $T \in A$, we also denote $\otimes_{j=1}^{i-1} 1 \otimes T \otimes_{j=i+1}^{n} 1$ by $T(x_i)$.

We define $R_n$ to be the quotient of $A^{\otimes n}$ by the left ideal $I_n$, generated by $\sum_{i=1}^{n} \partial_i$ and $r(z_i) - r(z_j)$ for $r \in R$ and $i, j = 1, \ldots, n$. We will sometimes denote $R_n$ by $R_n(R)$. Let us consider $R_n$ as a left $A^{\otimes n}$-module and denote by $\delta_{x_1, \ldots, x_n}$ its generator $1 + I_n$. We then have the relations

\[(r(x_i) - r(x_j))\delta_{x_1, \ldots, x_n} = 0, \quad (\sum_{i=1}^{n} \partial_{x_i})\delta_{x_1, \ldots, x_n} = 0,
\]

so that $\delta_{x_1, \ldots, x_n}$ plays the role of the distribution $\prod_{i=2}^{n} \delta(x_1 - x_i)$.

The module $R_2$ admits the following simple description. Let $T \mapsto T^*$ be the anti-automorphism of $A$, defined by $r^* = r$ and $\partial^* = -\partial$. Let us endow $A$ with the $A^{\otimes 2}$-module structure, defined by $(a \otimes b)c = acb^*$, for $a, b, c \in A$. Then the linear map $op : R_2 \to A$, defined by $op((a \otimes b)\delta_{x_1x_2}) = ab^*$, is an isomorphism of $A^{\otimes 2}$-modules. The inverse map to $op$ is given by $op^{-1}(a) = (a \otimes 1)\delta_{x_1x_2} = (1 \otimes a^*)\delta_{x_1x_2}$.

1.2. Definition of the nonlocal VPA structure. A nonlocal VPA structure on $(R, \partial)$ is a linear map $P : R \otimes R \to R_2$, satisfying the following conditions:

\[P(ab \otimes c) = a(x_1)P(b \otimes c) + b(x_1)P(a \otimes c),\]

\[P(\partial a \otimes b) = \partial_{x_1}P(a \otimes b),\]

\[P(b \otimes a) = -\sigma(P(a \otimes b)),\]

for $a, b, c \in R$, where $\sigma$ is the involutive automorphism of $R_2$ defined by $\sigma((a \otimes b)\delta_{x_1x_2}) = (b \otimes a)\delta_{x_1x_2}$, for $a, b \in A$, and the Jacobi identity that we formulate below.

Let us define a map $P_{x,y,z} : R \otimes R \to R_3$ by the following rules (we attach indices $y, z$ to $R_2$ and $x, y, z$ to $R_3$):

\[P_{x,yz}(a \otimes \delta_{yz}) = 0, \quad P_{x,yz}(a \otimes \partial_y m) = \partial_y P_{x,yz}(a \otimes m),\]

\[P_{x,yz}(a \otimes b(y)T(z)\delta_{yz}) = (op \circ P)(a \otimes b)(x)T(z)\delta_{xyz} + b(y)P_{x,yz}(a \otimes T(z)\delta_{yz})\]

for $a, b \in R$, $T \in A$, $m \in R_2$. 

\[P_{x,yz}(a \otimes \delta_{yz}) = 0, \quad P_{x,yz}(a \otimes \partial_y m) = \partial_y P_{x,yz}(a \otimes m),\]

\[P_{x,yz}(a \otimes b(y)T(z)\delta_{yz}) = (op \circ P)(a \otimes b)(x)T(z)\delta_{xyz} + b(y)P_{x,yz}(a \otimes T(z)\delta_{yz})\]

for $a, b \in R$, $T \in A$, $m \in R_2$. 

One can check easily that $P_{x,yz}$ is well-defined by these conditions. This follows from the identity $P_{x,yz}(a \otimes \partial_y b(y)m) - P_{x,yz}(a \otimes b(y)\partial_y m) = P_{x,yz}(a \otimes (\partial b)(y)m)$, which can be checked by putting $m$ in the form $(1 \otimes T)\delta_{yz}, T \in \mathcal{A}$.

The Jacobi identity is then expressed as

$$P_{x,yz}(a \otimes P(b \otimes c)) = \sigma_{xy}[P_{x,yz}(b \otimes P(a \otimes c))] + \sigma_{xz}[P_{x,yz}(c \otimes P(b \otimes a))],$$

for any $a, b, c \in R$, where $\sigma_{xy}, \sigma_{xz}$ are the automorphisms of $\mathcal{R}_3$, defined by

$$\sigma_{xy}(T(x)U(y)V(z)) = U(x)T(y)V(z)\delta_{xyz},$$

and

$$\sigma_{xz}(T(x)U(y)V(z)) = V(x)U(y)T(z)\delta_{xyz},$$

for $T, U, V \in \mathcal{A}$.

**Remark 1.1** We may consider elements of $\mathcal{R}_2$ as kernels in $x_1$ and $x_2$, expressed as linear combinations of $r(x_1)\partial_{x_1}^k \delta(x_1 - x_2)$, and think of $P(a \otimes b)$ as $\{a(x_1), b(x_2)\}$. The terms with $k \geq 0$ are called local and the terms with $k < 0$ are called nonlocal.

The expression $P_{x,yz}(a \otimes m)$ should then be thought of as expressing the Poisson bracket of the form $\{a(x), m(y, z)\}$, where $m$ is some kernel.

Note that $P_{x,yz}$ also has the properties

$$P_{x,yz}(a \otimes \partial_z m) = \partial_z P_{x,yz}(a \otimes m),$$

$$P_{x,yz}(a \otimes T(y)b(z)\delta_{yz}) = (op \circ P)(a \otimes b)(x)T(z)\delta_{xyz} + b(z)P_{x,yz}(a \otimes T(y)\delta_{yz}).$$

For $a, b, c \in R$, $\{a(x), \{b(y), c(z)\}\}$ is expressed as $P_{x,yz}(a \otimes P(b \otimes c))$. On the other hand, $\{b(y), \{a(x), c(z)\}\}$ is expressed as $\sigma_{xy}[P_{x,yz}(b \otimes P(a \otimes c))]$, and $\{c(z), \{b(y), a(x)\}\}$ as $\sigma_{xz}[P_{x,yz}(c \otimes P(b \otimes a))]$. This explains the connection between the standard Jacobi identity for the Poisson brackets and formula (6).

1.3. **Connection with the Beilinson-Drinfeld formalism.** Let $\mathcal{A}_+$ be the subalgebra of $\mathcal{A}$, generated by $R$ and $\partial$ (the algebra of differential operators). Let us set $\mathcal{R}_2^+(R) = op^{-1}(\mathcal{A}_+)$. We will say that $P$ defines a local VPA, if $P$ takes values in $\mathcal{R}_2^+(R)$.

In this case, the notion described here coincides with that of “coisson algebra” (on the disc) of Beilinson and Drinfeld ([5]). Indeed, let $X = \text{Spec} \mathbb{C}[t]$, and $D_X$ be the ring of differential operators on $X$. Let $A$ be the algebra $R[t]$, considered as a $D_X$-module by the rule that $\frac{d}{dt}$ acts on $R$ as $\partial$. We extend $P$ to an operation

$$\{,\} \in \text{Hom}_{D_X \times X}(A \otimes A, \Delta_s A)$$

($\Delta : X \to X \times X$ denotes the diagonal embedding) as in [5], (0.1.7), in the following way. Let us call $t_1 = t \otimes 1$ and $t_2 = 1 \otimes t$ the coordinates of $X \times X$; $A \otimes A$ is identified with $R \otimes R[t_1, t_2]$, $\frac{\partial}{\partial t_1}$ and $\frac{\partial}{\partial t_2}$ acting on $R \otimes R$ as $\partial \otimes 1$ and $1 \otimes \partial$; on the other hand, $\Delta_s A$ is identified with $\mathcal{A}_+[t]$, with $t_{1,2}$ acting as $t$ and $\frac{\partial}{\partial t_{1,2}}$ as $\frac{\partial}{\partial t}$. We then set $\{at_1^n, bt_2^m\} = t^{n+m}P(a \otimes b)$, for $a, b \in R$.}


Remark 1.2 As explained in [5], some local VPAs can be obtained as a classical limit of a family of chiral algebras (or vertex operator algebras [6, 17]), in the same way as one obtains Poisson algebras as a classical limit of a family of associative algebras. In particular, the local VPA $\pi_0$ described below is the classical limit of the vertex operator algebra of a Heisenberg algebra (see Remark 2 in [14]).

It would be interesting to generalize the notion of chiral algebra to allow for non-locality.

1.4. A class of nonlocal VPA’s. In this section we give a construction of a class of nonlocal VPA’s. The results of this section will only be used in the proof of Prp. 3.3. However, the construction presented here might be of general interest.

Proposition 1.1. Let $E_k, k \geq 0$, be the subspace of $\text{Der}(R)^{\otimes 2}$, consisting of all tensors $\sum \alpha u_\alpha \otimes v_\alpha$, such that

$$\sum_\alpha (u_\alpha a)(\partial^i v_\alpha b) = 0, \quad \forall a, b \in R, \quad i = 0, \ldots, k.$$ 

Suppose we are given elements $\sum \alpha x_i^{(\alpha)} \otimes y_i^{(\alpha)}$ of $\text{Der}(R)^{\otimes 2}$ for all $i \geq -1$. Assume that

$$\sum_\alpha x_i^{(\alpha)} \otimes y_i^{(\alpha)} = (-1)^{i+1} \sum_\alpha y_i^{(\alpha)} \otimes x_i^{(\alpha)},$$

and that

$$\sum_\alpha [x_i^{(\alpha)}, \partial] \otimes y_i^{(\alpha)} \in \sum_\alpha x_{i-1}^{(\alpha)} \otimes y_{i-1}^{(\alpha)} + E_i$$

(here we set for $i \leq -2$, $\sum \alpha x_i^{(\alpha)} \otimes y_i^{(\alpha)} = 0$; so that $\sum_\alpha x_{-1}^{(\alpha)} \otimes y_{-1}^{(\alpha)} \in (\text{Der}(R)^{\partial})^{\otimes 2}$).

Then the formula

$$P(a \otimes b) = \sum_{i \geq -1} \sum_\alpha (x_i^{(\alpha)} a)(x_i^{(\alpha)} b)(y_i^{(\alpha)} b)(y_i^{(\alpha)} c)(z)(-1)^{i+j} \partial^i_x \partial^j_z \delta_{xyz},$$

for $a, b \in R$ defines a nonlocal VPA structure on $R$.

Proof. The first condition ensures the antisymmetry of $P$, the second condition is equivalent to the $\partial$-linearity condition (4). The first condition being satisfied, the Jacobi identity for $P$ is automatically satisfied: for example, the term $P_{xy,z}(P(a \otimes b) \otimes c)$ is equal to

$$\sum_{i,j,\alpha,\beta} (x_j^{(\alpha)} x_i^{(\alpha)} a)(x_j^{(\alpha)} b)(y_j^{(\beta)} c)(z)(-1)^{i+j} \partial^i_y \partial^j_z \delta_{xyz}$$

$$+ \sum_{i,j,\alpha,\beta} (x_i^{(\alpha)} a)(x_j^{(\beta)} y_i^{(\alpha)} b)(y_j^{(\beta)} c)(z)(-1)^{i+j} \partial^i_x \partial^j_z \delta_{xyz}$$

whose second term is cancelled by

$$\sum_{i,j,\alpha,\beta} (x_j^{(\beta)} x_i^{(\alpha)} b)(y_i^{(\alpha)} c)(z)(y_j^{(\beta)} a)(x)(-1)^{i+j} \partial^i_y \partial^j_z \delta_{xyz}$$
which is a cyclic permutation of the first one.

\[ \square \]

**Remark 1.3** Let us denote by \( \text{Der}(R) \) the Lie algebra of derivations of \( R \) and assume that we have \( \varpi \in (\text{Der}(R)^{\partial}) \otimes^2 \), and \( P_+ : R \otimes R \to A_+ \), such that writing \( \varpi = \sum_i \varpi_i \otimes \varpi'_i \) (\( \varpi_i, \varpi'_i \) derivations of \( R \) commuting with \( \partial \)),

\begin{equation}
\text{op} \circ P(a \otimes b) = P_+(a \otimes b) + \varpi_i(a) \partial^{-1} \varpi'_i(b).
\end{equation}

In view of the form of \( \varpi \), the nonlocal part of axioms (5), (6) is satisfied. The l.h.s. of the Jacobi identity contains terms of the form

\[ a(x)b(y)c(z)\partial^{-1}_{x_i} \partial^{-1}_{x_j}, \]

\[ a(x)b(y)c(z)\partial^{-1}_{x_i} \partial^k_{x_j}, \]

\[ a(x)b(y)c(z)\partial^k_{x_i} \partial^\ell_{x_j}, \]

\[ a, b, c \in R, \ k, \ell \geq 0, \ x_i \neq x_j \] run through \( x, y, z \). The sum of the terms of the first type then cancels automatically (see Lemma 2.3).

All nonlocal \( \text{VPA} \) structures that we study in this work will be of the type described here.

Note that in particular, if \( P_+ \) is 0, then \( P \) given by (8) always defines a \( \text{VPA} \) structure.

We will denote by \( A_{-1} \) the span of \( A_+ \) and the \( a\partial^{-1}b, a,b \in R \), and by \( \mathcal{R}^{-1}_2(R) \) the space \( \text{op}^{-1}(A_{-1}) \).

**1.5. Hamiltonian vector fields.** We wish to show briefly here how the notions introduced above can be related to the Gelfand-Dickey-Dorfman theory of formal variational calculus (see [18], and the introduction of [5]). Let us assume that \( P \) takes values in \( A_+ \). Define \( V_f \in \text{End}(R) \) by

\begin{equation}
V_f(a) = -(\text{op} \circ P(a \otimes f)) \cdot 1,
\end{equation}

the result of the action of the differential operator \( P(a \otimes f) \) on \( 1 \in R \). Then \( V_f \) is a derivation of \( R \), commuting with \( \partial \). \( V_f \) actually depends only on the class of \( f \) on \( R/\partial R \), and is called the Hamiltonian vector field corresponding to the density \( f \). We will also use the notation

\[ V_f(a) = \{ \int_{-\infty}^{\infty} f, a \} = -\{ a, \int_{-\infty}^{\infty} f \}, \]

where \( \int_{-\infty}^{\infty} f \) denotes the class of \( f \) in \( R/\partial R \).

It can be used to define a Lie algebra structure on \( R/\partial R \), by the rule,

\[ \{ \int_{-\infty}^{\infty} f, \int_{-\infty}^{\infty} g \} = \int_{-\infty}^{\infty} V_f(g), \quad \forall f, g \in R. \]

A derivation \( D \) of \( R \) which commutes with \( \partial \) defines an operation \( D_2 \) on \( \mathcal{R}_2 \), in the following way. We set \( D_2 \delta_{xy} = 0 \), and extend \( D_2 \) to the whole \( \mathcal{R}_2 \) by the condition that it commutes with \( \partial_x \) and \( \partial_y \), and the formulae

\[ D_2(a(x)m(x,y)) = a(x)D_2m(x,y) + (Da)(x)m(x,y), \]

where \( a, b, c \in R \), and \( k, \ell \geq 0 \). The sum of the terms of the first type then cancels automatically (see Lemma 2.3).
Proposition 1.2. Under the conditions above, $\mathcal{V}_f$ is an infinitesimal automorphism of $R$. Moreover, $f \mapsto \mathcal{V}_f$ is a Lie algebra homomorphism from $R/\partial R$ to $\text{Der}(R)^\partial$.

Proof. The first part is straightforward and the second is contained in [18].

\[D_2(b(y)m(x, y)) = b(y)D_2m(x, y) + (Db)(y)m(x, y)\]
for $m \in R_2$, $a, b \in \mathcal{A}$; this definition makes sense because $D$ commutes with $\partial$.

We then say that $D$ is an infinitesimal automorphism of $R$, if $P(Da \otimes b) + P(a \otimes Db) = D_2P(a \otimes b)$ for $a, b \in R$.

2. Nonlocal extensions of the VPA of local fields $\pi_0$.

2.1. Notation and definition of $\pi_0$. Let $\bar{g}$ be an affine Lie algebra, with generators $e_i, f_i, \alpha_i^\vee, i = 0, \ldots, l, d$, subject to the relations of [20]. Let $a_i, a_i^\vee$ be the labels of $\bar{g}$, $h$ be its Coxeter number; then $K = \sum a_i a_i^\vee$ is a generator of the center of $\bar{g}$. Let $h = \bar{h}$ be the Cartan subalgebra of $\bar{g}$ with the same generators except $d$, $g$ be the quotient of $\bar{g}$ by $CK$, and let us denote the same way elements of $\bar{g}$ and their images in $g$. Let $h$ be the Cartan subalgebra of $g$, generated by the $a_i^\vee$'s, $n_+$ and $n_-$ be the pronilpotent subalgebras generated by the $e_i$'s and the $f_i$'s respectively, and $\bar{b}_- = h \oplus n_-$. Let $\alpha_i$, $i = 0, \ldots, l$ be the simple roots of $\bar{g}$, positive with respect to this decomposition. For each $x \in g$ we can write $x = x_+ + x_-$, where $x_+ \in n_+, x_- \in b_-$. There is an invariant inner product $\langle \cdot, \cdot \rangle$ on $\bar{g}$; let us denote in the same way its restriction to $\bar{g}$. Let $\sigma$ be any section of $g$ to $\bar{g}$: the restriction of $\langle \cdot, \cdot \rangle$ to $\sigma(g)$ defines an inner product on $g$, independent of $\sigma$ and again denoted by $\langle \cdot, \cdot \rangle$.

Let $\bar{h}$ be the Cartan subalgebra of $\bar{g}$ spanned by $\alpha_i^\vee, i = 0, \ldots, l$ and $d$. The restriction of the inner product $\langle \cdot, \cdot \rangle$ to $\bar{h}$ is non-degenerate and hence defines an isomorphism $\bar{h} \simeq \bar{h}^*$. Let $\omega_i^\vee \in \bar{h}^*$ be the $i$th fundamental coweight, i.e. it satisfies $\langle \omega_i^\vee, \alpha_j \rangle = \delta_{i,j}, \omega_i^\vee, d = 0$. Denote by $h_i, h_i^\vee$ the elements of $\bar{h}$, which are the images of $\alpha_i, \omega_i^\vee \in \bar{h}^*$ under the isomorphism $\bar{h} \simeq \bar{h}^*$ (note that this is not a standard notation).

Let

$$p_{-1} = \sum_{i=0}^{l} \frac{(\alpha_i, \alpha_i)}{2} f_i,$$

and let $a$ be the centralizer of $p_{-1} \in g$. It is a commutative subalgebra of $g$, called the principal abelian subalgebra. We have $a = a_+ \oplus a_-$, where $a_+ \oplus a_- = a \cap \bar{n}_-$. Let $I$ be the set (with multiplicities) of integers, congruent to the exponents of $\bar{g}$ modulo $h$. Then $a_\pm$ is generated by elements $p_n, n \in \pm I$.

We normalize the $p_n, n \in \pm I$, such a way that

$$\langle p_n, p_{-n} \rangle = \frac{1}{h}, \quad n \in I,$$
where \( h \) is the Coxeter number of \( \tilde{g} \). Let \( \pi_0 \) be the free differential ring generated by \( u_i, i = 1, \ldots, l \); we have \( \pi_0 = \mathbb{C}[u_i, \partial u_i, \ldots] \).

**Proposition 2.1** ([5]). The space \( \pi_0 \) has a VPA structure, defined by the formula

\[
P(A \otimes B) = \sum_{1 \leq i, j \leq l} (\alpha_i, \alpha_j) \sum_{n \geq 0} m \geq 0 \left( \frac{\partial A}{\partial u_i^{(n)}} \partial^{n+1} \otimes \frac{\partial B}{\partial u_j^{(m)}} \partial^m \right) \delta_{xy}.
\]

**Remark 2.1** This structure is uniquely determined by the following formulas:

\[
P(A \otimes 1) = 0, \quad \forall A \in \pi_0,
\]

and

\[
P(u_i \otimes u_j) = (\alpha_i, \alpha_j) \partial \delta_{xy}.
\]

The space \( \pi_0 \) also has the structures of \( n^+ \)– and \( a^- \)–modules, defined in [13]. Below we will extend these structures in three ways.

**2.2. Extension by half IM’s.** Let \( B_- \) be the ind-algebraic group corresponding to \( b_- \), and \( N_+ \) be the pro-algebraic Lie group corresponding to \( n_+ \), \( A_+ \) its subgroup corresponding to \( a_+ \). Let \( G \) be the group corresponding to \( g \), containing \( B_- \) and \( N_+ \) as subgroups.

Recall from [13], [14] the identification of \( \pi_0 \) and \( \mathbb{C}[N_+/A_+] \) as rings and \( n_+ \)–modules. Moreover, the Lie algebra \( a_- \) acts on \( \mathbb{C}[N_+/A_+] \) from the right, since we can identify \( N_+ \) with an open subspace of \( B_+ \setminus G \). Let \( \partial_n \) be the derivation of \( \pi_0 \) corresponding to the right action of \( p_{-n} \) on \( N_+/A_+ \). In particular, \( \partial_1 = \partial \).

Consider for \( n \in I \), the hamiltonians \( H_n \in \pi_0 \) from [10]. They satisfy

\[
\epsilon_{-\alpha_i} e_i \cdot H_n = \partial A_n^{(i)},
\]

for certain \( A_n^{(i)} \in \pi_{-\alpha_i} \) (in the notation of [10]). We have an isomorphism \( \pi_0 \simeq \mathbb{C}[N_+/A_+] \) (see [13], [14]).

\[
\partial H_n = \partial H_n = \partial H_{n,m}
\]

for certain \( H_{n,m} \in \pi_0 \). Following [10], Sect. 4, define \( \pi_0^+ = \pi_0 \otimes \mathbb{C}[F_n]_{n \in I} \), and extend the action of \( \partial_n \) to it by the formula

\[
\partial_n F_m = H_{n,m}.
\]

(12) (so that \( F_n \) can be viewed as a “half integrals of motion” \( \int_{-\infty}^x H_n \)). In particular, \( \partial F_m = H_m \).

According to Thm. 5 and Prop. 9 of [10], the ring \( \pi_0^+ \) is isomorphic to \( \mathbb{C}[N_+] \), and the action of \( \partial_n \) on \( \pi_0^+ \) defined this way corresponds to the right action of \( p_{-n} \) on \( N_+ \).
Finally, the action of the generators of \( \mathfrak{n}_+ \) are defined by
\[
e_i \cdot F_n = \varepsilon^{-1} \alpha_i A_n^{(i)} = \phi_n(e_i)
\]
(in the notation of [10]).

Now we extend the VPA structure on \( \pi_0 \) to a nonlocal VPA structure on \( \pi_0^+ \). For \( a \in \pi_0, n \in I \), we have
\[
\{ \int_{-\infty}^{\infty} H_n, a \} = n \partial_n a,
\]
([7], prop. 4.5) so that
\[
P(H_n \otimes a) \in n \partial_n a(y) \delta_{xy} + \partial_x \mathcal{R}_x(a_\pi).
\]
Let \( i_n(a) \) be the element of \( \mathcal{R}_x(\pi_0) \), such that \( P(H_n \otimes a) = n \partial_n a(y) \delta_{xy} + \partial_x(i_n(a)) \) (this defines \( i_n(a) \) uniquely, since \( \xi \in \mathcal{A}, \partial \cdot \xi = 0 \) implies \( \xi = 0 \)). We then define
\[
P(F_n \otimes a) = n \partial_n a(y) \partial^{-1} \delta_{xy} + i_n(a).
\]
Next, we have:
\[
P(H_n \otimes H_m) = n \partial_n H_m(x) \delta_{xy} + \partial_x(i_n(H_m)).
\]
Integrating this expression w.r.t. the second variable, we obtain
\[
n \partial_n H_m(x) + \partial_x \left( \int_{-\infty}^{\infty} i_n(H_m)(x, y) dy \right).
\]
where
\[
\int_{-\infty}^{\infty} i_n(H_m)(x, y) dy
\]
is defined as \( a_0(x) \), where \( i_n(H_m) = \sum_{i \geq 0} a_i(x) \partial_x^i \delta_{xy} \).

On the other hand, we obtain from formula (14) that
\[
\{ H_n(x), \int_{-\infty}^{\infty} H_m \} = -m \partial_m H_n(x).
\]

Comparing the last two formulas we find: \( \int_{-\infty}^{\infty} i_n(H_m)(x, y) dy = -(n + m) H_{n,m}(x) \) (for degree reasons it cannot contain constant terms) and so \( i_n(H_m) = -(n + m) H_{n,m}(x) \partial_x \delta_{xy} + \partial_y Q_{n,m} \), with \( Q_{n,m} \in \mathcal{A}_+ \). Thus, we obtain:
\[
P(H_n \otimes H_m) = n(\partial_n H_m)(x) \delta_{xy} - (n + m) H_{n,m}(y) \partial_x \delta_{xy} + (\partial Q_{n,m})(x) \partial_y \delta_{xy}.
\]
Using formula \( H_i = \partial F_i \) we finally obtain:
\[
P(F_n \otimes F_m) = (m H_{n,m}(x) + n H_{n,m}(y)) \partial_x^{-1} \delta_{xy} + Q_{n,m}(x) \delta_{xy}.
\]

**Proposition 2.2.** The formulae (11), (16), (17) define a nonlocal VPA structure on \( \pi_0^+ \).
Proof. To establish the antisymmetry condition, we should check that
\[ P(F_n \otimes F_m) = -\sigma(P(F_m \otimes F_n)) \]
this identity is \( \partial_x^{-1}\partial_y^{-1} \) applied to the same identity, with \( H_n \) and \( H_m \) in place of \( F_n \) and \( F_m \), which is true. The same argument works with the bracket \( P(F_n \otimes a) \), \( a \in \pi_0 \), with only one application of \( \partial^{-1} \).

Let us pass to the Jacobi identity. We should check it for the tensors \( F_n \otimes F_m \otimes F_k \), \( F_n \otimes F_m \otimes a \), \( F_n \otimes a \otimes b \), \( a, b \in \pi_0 \). These identities are \( \partial_x^{-1}\partial_y^{-1}\partial_z^{-1} \), resp. \( \partial_x^{-1}\partial_y^{-1} \), \( \partial_x^{-1} \) applied to the same identities with \( H \)'s replacing the \( F \)'s, which are true. \( \Box \)

Proposition 2.3. The action of \( \partial_n \) on \( \pi_0^+ \) is an infinitesimal automorphism of the nonlocal VPA structure on \( \pi_0^+ \).

Moreover, we have for all \( a \in \pi_0^+ \),
\[ (18) \quad P(H_n \otimes a) = n\partial_n a(x)\delta_{xy} + \partial_x \rho_n(a), \]
with \( \rho_n(a) \in R_{2}^{-1}(\pi_0^+) \), such that
\[ \rho_n(a) = \sum_{m \in I} H_{n,m}(x)\rho_{n,m}(a)(y)\partial_x^{-1}\delta_{xy} + R_{2}^+(\pi_0^+), \]
where \( \rho_{n,m}'s \) are linear endomorphisms of \( \pi_0^+ \).

Proof. Since \( n\partial_n \) coincides with \( V_{H_n} \), it satisfies the infinitesimal automorphism identity on \( \pi_0^+ \). To see that this identity is also satisfied for the tensors \( F_n \otimes a \) \( (a \in \pi_0) \) and \( F_n \otimes F_m \), we remark that they are \( \partial_x^{-1} \), resp. \( \partial_x^{-1}\partial_y^{-1} \) applied to the similar identities for \( H_n \otimes a \), resp. \( H_n \otimes H_m \).

According to formula (15), equation (18) is satisfied for \( a \in \pi_0 \), with \( \rho_n(a) \) actually lying in \( R_{2}^+(\pi_0) \). Therefore (18) holds for \( a = F_m \), as can be seen by applying \( \partial_x \) to (17). By Leibnitz rule, if (18) is true for \( a, b \in \pi_0^+ \), then it is true for \( ab \). Moreover, we find that
\[ \rho_n(ab) = a(y)\rho_n(b) + b(y)\rho_n(a). \]
This proves (18) and the properties of \( \rho_n \) in general. \( \Box \)

Remark 2.2 Formula (18) can be viewed as a nonlocal substitute of the hamiltonian property of \( H_n \).

2.3. Extension by \( \mathbb{C}[B_-] \). Let \( \mathbb{C}[B_-] \) be the ring of algebraic functions on \( B_- \). Let \( \pi_0 = \pi_0 \otimes \mathbb{C}[B_-] \). We extend the actions of \( n_+ \) and \( a_- \) on \( \pi_0 \), to actions of \( \mathfrak{g} \) and \( a_- \) on \( \pi_0 \), in the following way.

Recall the identification as rings and \( n_+\)-modules, of \( \pi_0 \) with \( \mathbb{C}[N_+/A_+] \). Moreover, the \( a_- \)-action on \( \pi_0 \) is identified with its action by vector fields on \( B_- \setminus G/A_+ \) from the right, where \( N_+/A_+ \) is an open subset (recall that \( G \) is the group corresponding to \( \mathfrak{g} \)). The ring \( \pi_0 \) is then identified with the ring of functions on \( B_- \times N_+/A_+ \).
We define on it the actions of $\mathfrak{g}$ and $\mathfrak{a}_-$ as follows: the value of the vector field generated by $x \in \mathfrak{g}$ at $(b_-, n_+ A_+)$ is

$$ (r(\text{Ad}(b_-^{-1})(x))_-, \ell(\text{Ad}(b_-^{-1})(x))_+) $$

and the value of the vector field $\partial_n$ (denoted by $\partial$ for $n = 1$) generated by $p_n$, $n \in I$ is

$$ (r(\text{Ad}(n_+)(p_n)_-), \ell(\text{Ad}(n_+)(p_n)_+)); $$

here $r(y)$ and $\ell(y)$ denote the right and left vector fields generated by a Lie algebra element $y$. The proof of these formulas is analogous to the proof of Lemma 1 in [10].

In particular, for $n = 1$ we have according to Lemma 2 of [10]

$$ \partial n_+ = (n_+ p_{-1} n_+^1)_+ n_+ = -(n_+ p_{-1} n_+^1)_- n_+ + (n_+ p_{-1} n_+^1)n_+ = $$

$$ = -(p_{-1} + \sum_i u_i h^{\gamma}_i)n_+ + n_+ p_{-1}. $$

We also have:

$$ \partial b_- = b_-(n_+ p_{-1} n_+^1)_- = b_-(p_{-1} + \sum_i u_i h^{\gamma}_i). $$

The last two formulas should be considered in an arbitrary representation of $G$ of the form $V((\lambda))$, where $V$ is finite-dimensional (see [10]).

Set now

$$ P(b_- \otimes u_i) = -\partial_y \left[ \text{Ad}(b_- (y))(h_i) b_-(x) \partial_x^{-1} \delta_{xy} \right]. $$

By this formula we mean the following. In any representation of $B_-$ of the form $V((\lambda))$, an element $b_-$ of $B_-$ can be viewed as a matrix $(b_{-, kl})$ whose entries $b_{-, kl}$ are Taylor series in $\lambda^{-1}$ with coefficients in the ring $\mathbb{C}[B_-]$. Such functions in fact generate $\mathbb{C}[B_-]$. The left hand side of formula (23) is the matrix whose entries are $P(b_- \otimes u_i)$. The right hand side of the formula is also a matrix of the same size, whose entries are elements of $\mathcal{R}_2(\mathbb{C}[B_-])$. Via Leibnitz rule, formula (23) defines $P(f \otimes u_i) \in \mathcal{R}_2(\mathbb{C}[B_-])$ for any $f \in \mathbb{C}[B_-]$. We interpret similarly formulas below for $P(b_- \otimes b_-), P(g \otimes g)$, etc.

**Remark 2.3** In view of (22), we consider $b_-(x)$ as the ordered exponential

$$ P \exp \int_{-\infty}^{x} (p_{-1} + \sum_{i=1}^{l} u_i(z) h^{\gamma}_i) dz. $$

Note that in any representation $b_-(x)$ is represented by the matrix

$$ (\text{Id} - \sum_{i=0}^{l} \frac{(\alpha_i, \alpha_i)}{2} f_i S_i + \ldots) \exp \left( \sum_{i=1}^{l} h^{\gamma}_i \varphi_i(x) \right), $$

(24)
where $\varphi_i(x) = \int_{-\infty}^{x} u_i(y)dy$, and $S_i$ is the $i$th “half screening”

$$S_i = \int_{-\infty}^{x} e^{-\varphi_i(y)}dy.$$  

The reason for this terminology is that if in the last formula we integrate over a closed contour, we obtain the classical limit of the screening operator of conformal field theory; the sum of the screening operators coincides with the hamiltonian of the affine Toda field theory. The other terms in the first factor of formula (24) can be expressed as consecutive Poisson brackets of the $S_i$’s.

Poisson bracket (23) can be informally obtained as follows. Since

$$P((p-1 + \sum_{i=1}^{l} u_i h_j^\vee) \otimes \varphi_j) = -h_j \delta_{xy},$$

we can write

$$\{ b_-(x), \varphi_j(y) \} = \int_{-\infty}^{x} dz P \exp \int_{-\infty}^{z} (p-1 + \sum_{i=1}^{l} u_i h_i^\vee)(-h_j \delta_{xx}) P \exp \int_{z}^{x} (p-1 + \sum_{i=1}^{l} u_i h_i^\vee)$$

$$= -\left( P \exp \int_{-\infty}^{y} (p-1 + \sum_{i=1}^{l} u_i h_i^\vee) \right) h_j \left( P \exp \int_{y}^{x} (p-1 + \sum_{i=1}^{l} u_i h_i^\vee) \right) 1_{y<x}$$

$$= -b_-(y)h_j b_-(y)^{-1}b_-(x)1_{y<x},$$

where $1_{y<x}$ is the function of $x, y$ equal to 1 when $y < x$ and to 0 else. For $y$ fixed, this is a shifted Heaviside function in $x$; applying $\partial_x$ to it gives $\delta(x-y)$, so that we identify $1_{y<x}$ with $\partial_x^{-1} \delta_{xy}$. Formula (23) is obtained by applying $\partial_y$ to this identity. 

□

**Lemma 2.1.** Definition (23) is compatible with the $\partial$-linearity condition (4) on $P$.

**Proof.** Both left and right hand sides of (23) satisfy the same identity

$$\partial_x (lhs) = (lhs)(p-1 + \sum_{j} u_j(x)h_j^\vee) - b_-(x)h_i \partial_y \delta_{xy},$$

and

$$\partial_x (rhs) = (rhs)(p-1 + \sum_{j} u_j(x)h_j^\vee) - b_-(x)h_i \partial_y \delta_{xy}.$$  

□

Let us define for $b_- \in B_-$, and $x \in g$, $r(x)(b_-) = (Ad b_- (x))_\cdot b_-$ in any representation of $G$ (recall that $x_-$ stands for the projection of $x \in g$ on the second factor of the decomposition $g = n_+ \oplus b_-$). This formula defines the right action of $g$ on $B_-$ viewed as an open subset of $N_+ \backslash G$.

Let $\Delta_+$ be the set of positive roots of $\tilde{g}$, and let $e^a, e_\alpha, \alpha \in \Delta_+$, be dual bases of $n_+, n_-$ for the inner product $\langle \cdot, \cdot \rangle$. 


Let

\[ R^+ = \frac{1}{2} \left( \sum_{\alpha} e^\alpha \otimes e_\alpha + \sum_i h_i \otimes h_i^\vee - \sum_{\alpha} e_\alpha \otimes e^\alpha \right), \]

\[ R^- = \frac{1}{2} \left( \sum_{\alpha} e_\alpha \otimes e^\alpha - \sum_i h_i \otimes h_i^\vee - \sum_{\alpha} e^\alpha \otimes e_\alpha \right), \]

Let us define, in the tensor product of any pair of representations of \( G \),

\[ P(b_- \otimes b_-) = \left\{ ((r \otimes r)(R^-)(b_- (x) \otimes b_- (x)))(1 \otimes b_-(x)^{-1}b_-(y)) \right. \\
\left. - ((r \otimes r)(R^+)(b_-(y) \otimes b_-(y)))(b_-(y)^{-1}b_-(x) \otimes 1) \right\} \partial_x^{-1} \delta_{xy}, \]

where \( pr_- \) is the projection \( \mathfrak{g} \rightarrow \mathfrak{b}_- \) along \( \mathfrak{n}_- \). This formula can be derived using the argument of Remark 2.3. A similar formula has been obtained by Faddeev and Takhtajan [12] (in the rational case) and used by Bazhanov, Lukyanov and Zamolodchikov [4] (see also [3]).

Formula (27) can also be written as

\[ P(b_- \otimes b_-) = (pr_- \otimes pr_-)[\text{Ad}^{\otimes 2}(b_-(x))(R^-) - \text{Ad}^{\otimes 2}(b_-(y))(R^+)] \\
(b_-(x) \otimes b_-(y)) \partial_x^{-1} \delta_{xy}. \]

It satisfies the antisymmetry condition (21), because

\[ R^- = -R^{+(21)}. \]

Lemma 2.2. Formula (27) is compatible with the \( \partial \)-linearity condition (4) on \( P \).

Proof. The l.h.s. of (27) satisfies

\[ \partial_y (\text{lhs}) = (\text{lhs})(1 \otimes (p_{-1} + \sum_i u_i(y)h^\vee_i)) \]

\[ + \sum_i (b_-(x)h_i \delta_{xy} - \text{Ad}(b_-(y))[p_{-1}, h_i]b_-(x)) \otimes b_-(y)h_i \delta_x^{-1} \delta_{xy}, \]

and the r.h.s. satisfies

\[ \partial_y (\text{rhs}) = \left\{ ((r \otimes r)(R^-)(b_-(x) \otimes b_- (x))(1 \otimes b_-(x)^{-1}b_-(y)(p_{-1} \\
\sum_i h^\vee_i u_i(y)) \right) \\
- \left[ (r \otimes r)(R^+)[(b_-(y) \otimes b_-(y))(p_{-1} \otimes 1 + 1 \otimes p_{-1} - \sum_i u_i(y)h^\vee_i \otimes 1 \\
+ 1 \otimes h^\vee_i)](b_-(y)^{-1}b_-(x) \otimes 1) \\
+ [(r \otimes r)(R^+)(b_-(y) \otimes b_-(y))][p_{-1} + \sum_i u_i(y)h^\vee_i]b_-(y)^{-1}b_-(x) \otimes 1 \right] \}
\partial_x^{-1} \delta_{xy} \]

\[ - ([r \otimes r](R^- - R^+)(b_-(x) \otimes b_-(x)) \big) \delta_{xy} \]
The \( \delta_{xy} \)-terms coincide because \( R^+ - R^- = \sum_i h_i \otimes h_i^\dagger \). The \( \partial_{xy}^{-1} \delta_{xy} \)-terms containing \( u_i(y) \) are coincide, since \( R^+ \) is invariant by the conjugation by \( \mathfrak{h} \).

The identification of the remaining terms follows from the formula

\[
(29) \quad [p_{-1} \otimes 1 + 1 \otimes p_{-1}, R^+] = \sum_i [p_{-1}, h_i] \otimes h_i^\dagger.
\]

Let \( \rho : \mathfrak{g} \to \mathfrak{g} \) be the linear map defined by \( \rho(x) = (R^+, 1 \otimes x) \). Formula (29) means that \( [\text{ad}(p_{-1}), \rho] \) is the linear endomorphism of \( \mathfrak{g} \), equal to 0 on \( \mathfrak{n}_+ \) and \( \mathfrak{n}_- \) and to \( \text{ad}(p_{-1}) \) on \( \mathfrak{h} \), which can be easily verified.

**Remark 2.4** It follows from the proof that the brackets (27) can also be expressed replacing \( R^+ \) by \( R^+ + \kappa \sum_i e^i \otimes \epsilon_i \) and \( R^- + \kappa \sum_i e^i \otimes \epsilon_i, (e^i, (\epsilon_i)) \) dual bases of \( \mathfrak{g} \). This follows also from the fact that \( [\sum_i e^i \otimes \epsilon_i, b_\cdot \otimes b_\cdot] = 0 \) for any \( b_\cdot \in B_\cdot \).

We prefer the form given here because then (5) is manifestly satisfied.

Let us extend \( P \) to \( \mathbb{C}[B_\cdot] \otimes \pi_0 \) by linearity, to \( \mathbb{C}[B_\cdot] \otimes \pi_0 \) by formulas (23), (4) and Leibnitz rule, to \( \pi_0 \otimes \mathbb{C}[B_\cdot] \) by antisymmetry, and finally to \( (\mathbb{C}[B_\cdot] \otimes \pi_0) \otimes \pi_0 \) by the Leibnitz rule.

**Proposition 2.4.** The operation \( P \) on \( \mathbb{C}[B_\cdot] \otimes \pi_0 \) defined by formulas (11), (23), (27) and the rules above, satisfies the nonlocal VPA axioms.

**Proof.** By construction, \( P \) satisfies the \( \partial \)-linearity axiom. The Leibnitz rule is satisfied for the brackets involving \( b_\cdot \)'s since the right hand sides of (23) and (27) respect the tensor structure. For the other brackets, the Leibnitz rule is satisfied by construction. As was already mentioned, the antisymmetry of \( P \) follows from \( R^+ = R^- \). Let us check now the Jacobi identity. For the \( \mathbb{C}[B_\cdot] \) part it follows from the following lemma.

**Lemma 2.3.** Let \( (R, \partial) \) be a differential ring. Let \( R_0 \) be a subring of \( R \), such that \( R \) is spanned by elements of the form \( \partial a_i \), where \( a_i \in R_0 \).

Let \( \sum_i \omega_i \otimes \omega'_i \in S^2(\text{Der}(R_0)) \), and let us assume that there exists a linear map \( P : R \otimes R \to R_2 \), satisfying the Leibnitz rule and the \( \partial \)-linearity, such that

\[
P(a \otimes b) = \sum_i (\omega_i a)(x)(\omega'_i b)(y)\delta_{x_1}^{-1} \delta_{xy},
\]

for \( a, b \in R_0 \). Then \( P \) satisfies the Jacobi identity.

**Proof.** It is enough to check it for elements of \( R_0 \). We have

\[
P_{xy}P(a \otimes b) \otimes c = \sum_{i,j} (\omega_j \omega_i a)(x)(\omega'_j b)(y)(\omega'_i c)(z)\delta_{x_1}^{-1} \delta_{xyz} + (\omega_j a)(x)(\omega_j \omega'_i b)(y)(\omega'_i c)(z)\delta_{x_1}^{-1} \delta_{xy}^{-1} \delta_{xyz};
\]

the first term is cancelled by

\[
\sum_i (\omega_i c)(z)(\omega_j \omega'_i a)(x)(\omega'_j b)(y)\delta_{x_1}^{-1} \delta_{xyz}
\]
(which is obtained from the second by a cyclic permutation). \qed

The Jacobi identity in the case of the proposition is then proved as follows: introduce the variables $\varphi_i$, $i = 1, \ldots, l$; let $\tilde{R} = \mathbb{C}[B_-] \otimes \mathbb{C}[\varphi_i^{(k)}]$, and identify $R$ with a subalgebra of $\tilde{R}$ by $\partial \varphi_i = u_i$. Extend the bracket $P$ to $\tilde{R}$, by the rules

$$P(\varphi_i \otimes \varphi_j) = -(\alpha_i, \alpha_j)\partial_x^{-1}\delta_{xy},$$

and

$$P(b_- \otimes \varphi_i) = -\text{Ad}(b_-(y))(h_i)b_-(x)\partial_x^{-1}\delta_{xy}.$$ 

Let now $R_0 = \mathbb{C}[B_-] \otimes \mathbb{C}[\varphi_i]$; the restriction of $P$ to $R_0$ is of the form given in the lemma, so $P$ satisfies the Jacobi identity. \qed

### 2.4. Extension by $\mathbb{C}[B_-]$ and half IM's.

Consider now the algebra $\tilde{\pi}_0 = \pi_0 \otimes \mathbb{C}[F_n] \otimes \mathbb{C}[B_-]$. We extend the actions of $\partial$ and of the flows $\partial_n$ on it in the way compatible with the embeddings $\pi_0^+ \subset \tilde{\pi}_0$ and $\tilde{\pi}_0 \subset \tilde{\pi}_0$. We also define a nonlocal VPA structure on $\tilde{\pi}_0$, in such a way that the above embeddings are nonlocal VPA morphisms, and we define the brackets $P(F_n \otimes b_-)$ by

$$P(F_n \otimes b_-) = \partial_x^{-1}P(H_n \otimes b_-);$$

we then extend $P$ to $\tilde{\pi}_0^{\otimes 2}$ by antisymmetry and the Leibnitz rule (it is easy to check that (30) is compatible with the Leibnitz rule for the products of matrix elements of $b_-$).

**Proposition 2.5.** The operation $P$ defined by formulas (11), (16), (17), (23), (27), (30), antisymmetry and the Leibnitz rule, defines a nonlocal VPA structure on $\tilde{\pi}_0$.

**Proof.** The same as in Prop. 2.1. \qed

The rest of this section is devoted to proving that the image of $P$ actually lies in $\mathcal{R}_2^{-1}(\pi_0)$ (the definition of this space is in Rem. 1.3). It is enough to prove it for $P(F_n \otimes b_-)$.

**Lemma 2.4.** For certain $A_{ni} \in \mathcal{R}_2^+(\pi_0)$, we have

$$P(F_n \otimes (p_{-1} + \sum_{i=1}^{l} u_i h^\vee_i)) = \sum_{i=1}^{l} n(\partial_n u_i(y))h^\vee_i \partial_x^{-1}\delta_{xy} + A_{ni}h^\vee_i.$$ 

**Proof.** We have $\{\int_{-\infty}^{\infty} H_n, u_i\} = n\partial_n(u_i)$, so that $P(H_n \otimes u_i) = n\partial_n u_i(y)\delta_{xy} + \partial_x A_{ni}$, with $A_{ni} \in \mathcal{R}_2^+(\pi_0)$. Now

$$P(F_n \otimes u_i) = \partial_x^{-1}P(H_n \otimes u_i) = n(\partial_n u_i(y))\partial_x^{-1}\delta_{xy} + A_{ni},$$

and the statement follows. \qed

**Lemma 2.5.** $\partial_n(\sum_{i=1}^{l} u_i(x) h^\vee_i)$ is equal to the Cartan component of $- [p_{-1}, \text{Ad}(n_+(x))(p_{-n})]$. 


for any $n_+ \in N_+$ (here log stands for the inverse of the exponential map $\exp : n_+ \to N_+$, which is an isomorphism, and the index 1 means the component of principal degree one of an element of $\tilde{g}$). Hence
\[
\partial_n(\sum u_i(x)h_i) = -\partial_n([p_1, (\log n_+(x))_1]) = -[p_1, \partial_n(\log n_+(x))]_1 = -[p_1, (\Ad(n_+)\Ad(p_-))_1].
\]

Alternatively, this statement is a formulation of the Cartan part of the equation
\[
[p + p_- + \sum u_i h_i^\vee, \partial_n - (\Ad(n_+)\Ad(p_-))_+] = 0,
\]
which follows from the zero-curvature equation (see e.g. formula (10) of [10]) and the fact that $\Ad(n_+)\Ad(p_-)$ commutes with $[\partial + p_1 + \sum u_i h_i^\vee$ (see formula (13) of [10]).

Let us set $A_{ni} = \sum_{k \geq 0} A^{(k)}_{ni}(y)\delta_x \delta_{xy}$, with $A^{(k)}_{ni} \in \pi_0$.

**Lemma 2.6.** Let $Z \in \mathcal{R}_2(\pi_0)$ be such that
\[
(32) \quad \partial_y Z = Z(p_1 + \sum_{i=1}^l u_i(y)h_i^\vee) + b_-(y)\left(\sum_{i=1}^l n(\partial_n u_i)(y)h_i^\vee \partial_x^{-1}\delta_{xy} + A_{ni}h_i^\vee\right).
\]

Then
\[
(33) \quad Z = n[b_-(y)\Ad(n_+(y))(p_-)] - \Ad(b_-(x))(\Ad(n_+(x))(p_-))_+ b_-(y)\partial_x^{-1}\delta_{xy} + \Ad(b_-(x))(\sum_{i=1}^l A^{(0)}_{ni}(x)h_i^\vee) b_-(y)\delta_x \delta_{xy} - \sum_{k \geq 1} \delta_x \delta_{xy} \partial_x^{-1} A^{(k)}_{ni}(x) h_i^\vee b_-(y)\delta_x \delta_{xy}.
\]

**Proof.** Let
\[
Z_0 = n[b_-(y)\Ad(n_+(y))(p_-)] - \Ad(b_-(x))(\Ad(n_+(x))(p_-))_+ b_-(y)\partial_x^{-1}\delta_{xy}.
\]

Then
\[
\partial_y Z_0 - Z_0(p_1 + \sum_{i=1}^l u_i(y)h_i^\vee) = n\left\{b_-(y)\left[p_1 + \sum_{i=1}^l u_i(y)h_i^\vee, \Ad(n_+(y))(p_-)\right] \right\} \partial_x^{-1}\delta_{xy} = -nb_-(y)p_1, \left(\Ad(n_+(y))(p_-)\right) \right\} \partial_x^{-1}\delta_{xy} = nb_-(y)\partial_n(\sum_{i=1}^l u_i(y)h_i^\vee)\partial_x^{-1}\delta_{xy},
\]
where the last equality follows from Lemma 2.5. We then have

\[ \partial_y(Z - Z_0) - (Z - Z_0)(p_{-1} + \sum_{i=1}^{l} u_i(y))h_i^{\vee} = b_-(y) \sum_{i=1}^{l} A_n h_i^{\vee}, \]

so that

\[ \partial_y[(Z - Z_0)b_-(y)^{-1}] = \text{Ad}(b_-(y))\left[\sum_{i=1}^{l} A_n h_i^{\vee}\right], \]

and the result. \( \square \)

**Proposition 2.6.**

\[ P(F_n \otimes b_{-}) = n[b_-(y)(\text{Ad}(n_+(y))(p_{-n}))_-- \text{Ad}(b_-(x))(\text{Ad}(n_+(x))(p_{-n}))_-- b_-(y)]\partial_x^{-1}\delta_{xy} + \text{Ad}(b_-(x))\left(\sum_{i=1}^{l} A_{ni}^{(0)}(x)h_i^{\vee}\right)b_-(y)\partial_y^{-1}\delta_{xy} - \sum_{k \geq 0} \partial_x^{k}[\text{Ad}(b_-(x))\left(\sum_{i=1}^{l} A_{ni}^{(k+1)}(x)h_i^{\vee}\right)b_-(y)\delta_{xy}] \]

and so belongs to \( \mathcal{R}_{-1}(\tilde{\pi}_0) \).

*Proof.* Indeed, \( P(F_n \otimes b_{-}) \) defined by formula (30) satisfies (32), by the Leibnitz rule and Lemma 2.4, and we apply to it Lemma 2.6. \( \square \)

**Remark 2.5** It follows from Thm. 3.1 that the extension of \( \partial_n \) to \( \tilde{\pi}_0 \), defined again as \( r(p_{-n}) \), is an infinitesimal automorphism of the nonlocal VPA structure of \( \tilde{\pi}_0 \). \( \square \)

### 3. Geometric interpretation of the Poisson structures.

#### 3.1. Determination of the nonlocal terms.

In 2.3, we have defined a nonlocal VPA \( \tilde{\pi}_0 \). It is isomorphic as a differential algebra to \( \mathbb{C}[B_{-} \times N_+] \) with the derivation \( \partial \) defined by the right action of \( p_{-1} \) on \( B_{-} \times N_+ \). We will now describe \( P \) in these geometric terms.

Let us set \( t = \sum_{\alpha} e_{\alpha} \otimes e_{\alpha} + e_{\alpha} \otimes e_{\alpha} + \sum_i h_i \otimes h_i^{\vee} \). Recall that \( p_{-n} \) is a basis of \( a_{-} \) dual to \( p_n \) with respect to the inner product \( \langle \cdot, \cdot \rangle \).

**Lemma 3.1.**

\[ P(b_{-} \otimes b_{-}) = -\ell^{\otimes 2}(t)(b_{-}(x) \otimes b_{-}(y))\partial_x^{-1}\delta_{xy} \]
(pr_ \otimes pr_-)(\text{Ad}^2 b_-(x)R^- - \text{Ad}^2(b_-(y))(R^+)) = (pr_ - \otimes pr_-)(\text{Ad}^2(b_-(x))
\sum_\alpha e^\alpha \otimes e^\alpha) + (pr_ - \otimes pr_-)(\text{Ad}^2(b_-(y)) \sum_\alpha e^\alpha \otimes e^\alpha - \sum_i h_i \otimes h_i^\vee 
= \sum_\alpha (\text{Ad}(b_-(x))(e^\alpha)_-) \otimes \text{Ad}(b_-(x))(e_\alpha) + \sum_\alpha \text{Ad}(b_-(y))(e^\alpha) \otimes (\text{Ad}(b_-(y))(e^\alpha)_-) 
- \sum_i h_i \otimes h_i^\vee 
= - \text{Ad}(b_-(x))(\text{Ad}(b_-(x)^{-1})(e^\alpha)_-) \otimes e_\alpha - e_\alpha \otimes \text{Ad}(b_-(y))(\text{Ad}(b_-(y)^{-1})(e^\alpha)_-) 
- \sum_i h_i \otimes h_i^\vee.

The first equality is obtained using the following arguments: \( R^- = \sum_\alpha e^\alpha \otimes e_\alpha - \frac{1}{2} t, \)
\( R^+ = - \sum_\alpha e_\alpha \otimes e^\alpha + \frac{1}{2} t \); \( t \) is \( B_- \)-invariant, and \( (pr_ - \otimes pr_-)t = \sum_i h_i \otimes h_i^\vee \); the second equality is straightforward, and the last one is because for \( b_- \in B_- \), we have

\[
\sum_\alpha (\text{Ad}(b_-)(e^\alpha)_-) \otimes \text{Ad}(b_-)(e_\alpha) = - \sum_\alpha \text{Ad}(b_-)(\text{Ad}(b_-^{-1})(e^\alpha)_-) \otimes e_\alpha.
\]

Formula (35) can be shown as follows: let \( \xi \in n_+ \), then

\[
\langle \text{lhs of } (35), 1 \otimes \xi \rangle = (\text{Ad} b_- (\text{Ad} b_-^{-1}(\xi))_+)_-,
\]

and

\[
\langle \text{rhs of } (35), 1 \otimes \xi \rangle = - \text{Ad} b_- (\text{Ad} b_-^{-1}(\xi))_-,
\]

with \( x_+ = x - x_- \).

Formula (34) then follows from (28).

\[\square\]

**Lemma 3.2 ([10], Prop. 6).** In any representation of \( G \), \( n_+(x) \) has the form

\[
\bar{n}_+(x) \exp \left( \sum_{n \in \mathbb{C}[u^n]} \frac{1}{n} p_n F_n(x) \right),
\]

where \( \bar{n}_+(x) \) is a matrix of polynomials with entries in \( \mathbb{C}[u^{(n)}] \).

Recall from [10], Lemma 1, that in any representation of \( N_+ \) we have the following formula for the right action of \( x \in g \) on \( N_+ \subset B_- \backslash G \):

\[r(x)n_+ = (\text{Ad}(n_+)(x))_+n_+.
\]

**Lemma 3.3.**

\[
P(n_+ \otimes n_+) = r(a)[n_+(x) \otimes n_+(y)] \partial_x^{-1} \delta_{xy} + \text{local terms},
\]

where \( a = \sum_{n \in \mathbb{C}[u^{(-n)}]} p_n \otimes p_{-n} \).
Proof. Let us write \( n_+ \) in the form (36) and compute the nonlocal part of \( P(n_+ \otimes n_+) \). It comes from three different terms: \( P(F_n \otimes n_+) \), \( P(n_+ \otimes F_n) \), \( P(F_n \otimes F_m) \).

We have

\[
P(F_n \otimes n_+) = n \partial_n n_+(y) \partial_x^{-1} \delta_{xy} + \rho_n(n_+),
\]

by (18). Set \( F = \sum_{n \in I} \frac{1}{n} \rho_n F_n \). We then have \( \bar{n}_+ = n_+ e^{-F} \). The Leibnitz rule gives

\[
P(F_n \otimes \bar{n}_+) = n \partial_n n_+(y) e^{-F(y)} \partial_x^{-1} \delta_{xy} + \rho_n(n_+) e^{-F(y)} + (1 \otimes \bar{n}_+(y)) \sum_{m \in I} - \frac{1}{m} (1 \otimes \rho_m) P(F_n \otimes F_m). \]

Using (17), we obtain

\[
P(F_n \otimes \bar{n}_+) \in n \partial_n n_+(y) e^{-F(y)} \partial_x^{-1} \delta_{xy} + \rho_n(n_+) e^{-F(y)} + (1 \otimes \bar{n}_+(y)) \sum_{m \in I} - \frac{1}{m} (1 \otimes \rho_m) (m H_{n,m}(x) + n H_{n,m}(y)) \partial_x^{-1} \delta_{xy} + R_2^+(\pi_0),
\]

so that

\[
(38)
\]

\[
P(F_n \otimes \bar{n}_+) \in n((\text{Ad} \bar{n}_+(p_{-n})) \circ \bar{n}_+) - \bar{n}_+ \sum_{m \in I} \frac{1}{m} \rho_m H_{n,m}(y) \partial_x^{-1} \delta_{xy} + \rho_n(n_+) e^{-F(y)} - \sum_{m \in I} H_{n,m}(x)(\bar{n}_+(y) \rho_m) \partial_x^{-1} \delta_{xy} + R_2^+(\pi_0).
\]

(39)

Let us show that

\[
\mathcal{W} = \rho_n(n_+) e^{-F(y)} - \sum_{m \in I} H_{n,m}(x)(\bar{n}_+(y) \rho_m) \partial_x^{-1} \delta_{xy}
\]

belongs to \( R_2^+(\pi_0) \). Apply \( \partial_x \) to (38). The l.h.s. of the resulting identity belongs to \( R_2^+(\pi_0) \), as well as \( \partial_x (n((\text{Ad} \bar{n}_+(p_{-n})) \circ \bar{n}_+) - \bar{n}_+ \sum_{m \in I} \frac{1}{m} \rho_m H_{n,m}(y) \partial_x^{-1} \delta_{xy} \). It follows that

\[
\partial_x \mathcal{W} \in R_2^+(\pi_0).
\]

On the other hand, in view of Prop. 2.3, we can write the nonlocal part of \( \mathcal{W} \) as \( \sum_{m \in I} H_{n,m}(x) w_m(y) \partial_x^{-1} \delta_{xy} \). The result of the action of \( \partial_x \) on this nonlocal part should be local; since all \( \partial H_{n,m}, m \in I \), are independent, it follows that all \( w_m \)'s vanish, so that

\[
\mathcal{W} \in R_2^+(\pi_0).
\]

Formula (38) becomes

\[
P(F_n \otimes \bar{n}_+) \in n ((\text{Ad} \bar{n}_+(p_{-n})) \circ \bar{n}_+) - \bar{n}_+ \sum_{m \in I} \frac{1}{m} \rho_m H_{n,m}) \partial_x^{-1} \delta_{xy} + R_2^+(\pi_0).
\]
We derive from this a similar statement on $P(\bar{n}_+ \otimes F_n)$. Combining these with (17), and the fact that $P(\bar{n}_+ \otimes \bar{n}_+)$ is contained in $R_2(\pi_0)$, we obtain (37). □

Introduce the following notation. For any tensor $\gamma = \sum_i \gamma_i \otimes \gamma'_i$, and two operators $a$ and $b$, we write $a(\gamma^{(1)}) \otimes b(\gamma^{(2)})$ for $\sum_i a(\gamma_i) \otimes b(\gamma'_i)$.

**Lemma 3.4.**

\begin{equation}
P(b_- \otimes n_+) = -b_-(x)[\text{Ad}(b_-^{-1}(x))(t^{(1)})]_+ \otimes [\text{Ad}(b_-^{-1}(y))(t^{(2)})]_+ n_+(y) \partial_x^{-1}\delta_{xy} + \sum_{n \in I} b_-(x)\text{Ad}(n_+(x))(p_{-n}) \otimes n_+(y) p_n \partial_x^{-1}\delta_{xy} + \text{local terms};
\end{equation}

this equation should be understood in the tensor product of two representations of $g$.

The proof is given in Sect. 5.

**3.2. Determination of VPA structures.**

**Theorem 3.1.** The nonlocal VPA structure of $\bar{\pi}_0$ is expressed, via the identification of $\bar{\pi}_0$ with $\mathbb{C}[B_- \times N_+]$, by the formula

\begin{equation}
P(g \otimes g) = [-\ell^\otimes_2(t)(g(x) \otimes g(y)) + r^\otimes_2(a)(g(x) \otimes g(y))] \partial_x^{-1}\delta_{xy} + \sum_{n \geq 0} r^\otimes_2((\text{ad } p_{-1})^{-n-1} \otimes 1)(t-a)(g(x) \otimes g(y)) \partial_x^n \delta_{xy}.
\end{equation}

**Proof.** Let us denote by $g(x)$ the pair $(b_-(x), n_+(x))$; it lies in the variety $B_- \times N_+$. This variety is endowed with left and right actions of $g$, that we denote $\ell$ and $r$. They are defined as follows. The mapping $B_- \times N_+ \to G$, associating to $b_- n_+$ the product $b_- n_+$ embeds $B_- \times N_+$ in $G$ as a Schubert cell. The left and right actions of $g$ on $G$ can be restricted to $B_- \times N_+$; so that $\ell(x)$ is the vector field equal at $(b_-, n_+)$ to the sum of $r((\text{Ad } b_-(x))_-)$ on the first component of the product, and $\ell((\text{Ad } b_-(x))_+)$ on the second one, according to

\[x \cdot b_- n_+ = b_- [(\text{Ad } (b_-)(x))_- + (\text{Ad } (b_-)(x))_+] n_+.
\]

Likewise, $r(x)$ is the vector field equal at $(b_-, n_+)$ to the sum of $r((\text{Ad } n_+(x))_-)$ on the first component of the product, and $\ell((\text{Ad } n_+(x))_+)$ on the second one, since

\[b_- n_+ \cdot x = b_- [(\text{Ad } (n_+)(x))_- + (\text{Ad } (n_+)(x))_+] n_+.
\]

Formulas (27) and (53) then imply

\begin{equation}
P(g \otimes g) = [-\ell^\otimes_2(t)(g(x) \otimes g(y)) + r^\otimes_2(a)(g(x) \otimes g(y))] \partial_x^{-1}\delta_{xy} + \text{local terms}.
\end{equation}

The action of $\partial$ on $g$ coincides with $r(p_{-1})$, due to formulas (22) and (21), and we obtain

\begin{equation}
\partial g = r(p_{-1}) g.
\end{equation}
Therefore $P(g \otimes g)$ satisfies the differential equations

$$\partial_x P(g \otimes g) = (r(p_{-1}) \otimes 1)P(g \otimes g), \quad \partial_y P(g \otimes g) = (1 \otimes r(p_{-1}))P(g \otimes g).$$

We will use these equations to determine $P(g \otimes g)$ completely. Let us first determine the local part of the r.h.s. of (42). Denote it by $\mathcal{Y}$. We have

$$\partial_x \mathcal{Y} - (r(p_{-1}) \otimes 1)\mathcal{Y} = r^{\otimes 2}(t - a)(g(x) \otimes g(x))\delta_{xy},$$

(44)

$$\partial_y \mathcal{Y} - (1 \otimes r(p_{-1}))\mathcal{Y} = -r^{\otimes 2}(t - a)(g(x) \otimes g(x))\delta_{xy},$$

(45)

(we have used the $N_{+}$ and $B_{-}$-invariances of $t$ to replace $\ell^{\otimes 2}(t)$ by $r^{\otimes 2}(t)$, and that $(g(x) \otimes g(y))\delta_{xy} = (g(x) \otimes g(x))\delta_{xy}$).

Recall that $g = a \oplus \text{Im}(\text{ad} p_{-1})$; this is an orthogonal decomposition in $g$; $\text{ad} p_{-1}$ is an automorphism of $\text{Im}(\text{ad} p_{-1})$, and $t - a$ belongs to $(\text{Im}(\text{ad} p_{-1}))^{\otimes 2}$.

Then

$$\mathcal{Y}_0 = \sum_{n \geq 0} r^{\otimes 2}\left( (\text{ad} p_{-1})^{-n-1} \otimes 1 \right)(t - a)(g(x) \otimes g(y))\partial_x^n \delta_{xy}$$

is a solution to (44); to see it, one should apply the Leibnitz rule, formula (43) and note the cancellation of all terms except the one in $\delta_{xy}$. It is also a solution of (45) because $((\text{ad} p_{-1})^n \otimes 1)(t - a) = (-1)^n(1 \otimes (\text{ad} p_{-1})^n)(t - a)$. Indeed, this identity can be proved by writing $t - a = \sum_{\alpha} \tilde{e}^\alpha \otimes \tilde{e}_\alpha$, where $\tilde{e}^\alpha$, $\tilde{e}_\alpha$ are dual bases of $\text{Im}(\text{ad} p_{-1})$, and using the anti-selfadjointness of $\text{ad} p_{-1}$.

Let $P_0$ be the operation defined by (42) and (46). It defines a nonlocal VPA structure on $(\mathbb{C}[B_{-} \times N_{+}], r(p_{-1}))$ by virtue of Prop. 1.1 (the elements of $E_i$ in the second condition being $\ell^{\otimes 2}(t) - r^{\otimes 2}(t)$ for $i = 0$, and $0$ for $i > 0$).

The theorem follows from the following lemma which is proved in Sect. 5.

**Lemma 3.5.** $P_0$ is equal to $P$.

\[ \square \]

**Remark 3.1** The variety $B_{-} \times N_{+}$ can be considered as an open subset of $G$. One can show that formula (52) defines a nonlocal VPA structure on the whole group $G$.

\[ \square \]

It is easy to derive now a formula for the nonlocal VPA structure of $\mathbb{C}[N_{+}]$.

**Corollary 3.1.** The nonlocal VPA structure of $\pi_0^{+}$ is expressed, via the identification of $\pi_0^{+}$ with $\mathbb{C}[N_{+}]$, by the formula

$$P(n_+ \otimes n_+) = r^{\otimes 2}(a)(n_+(x) \otimes n_+(y))\partial_x^{-1}\delta_{xy}$$

(47)

$$+ \sum_{n \geq 0} r^{\otimes 2}\left( (\text{ad} p_{-1})^{-n-1} \otimes 1 \right)(t - a)(n_+(x) \otimes n_+(y))\partial_x^n \delta_{xy},$$
We now reformulate (47) so as to make it clear that the \( A_+ \)-invariant functions of \( N_+ \) only have local Poisson brackets expressed in terms of \( A_+ \)-invariant functions of \( N_+ \). Let us denote
\[
(t - a)_k = \left( (\text{ad} p_{-1})^{-k-1} \otimes 1 \right) (t - a).
\]

**Corollary 3.2.**

\[
P(n_+ \otimes n_+) = r^{\otimes 2}(a)(n_+)(x) \otimes n_+(y))\partial^{-1}_{xy}
\]
\[
+ \sum_{k \geq 0} \left\{ \text{Ad}(\tilde{n}_+)(x)\left[ (\partial_x - \sum_{n \in I} \frac{1}{n} H_n(x) \text{ad} p_n)^k ((t - a)_k^{(1)}) \right] \right\} n_+(x)
\]
\[
\otimes \left( \text{Ad}(\tilde{n}_+)(y)(t - a)_k^{(2)} \right) n_+(y)\delta_{xy}.
\]

**Proof.** Let \( F(x) = \sum_{n \in I} \frac{1}{n!} p_n F_n(x) \), and \( \alpha \in \tilde{g} \). Then
\[
\text{Ad}(e^{F(x)-F(y)})(\alpha \partial^n_{xy}) = [\partial_x - \text{ad}(F'(x))]^n(\alpha \delta_{xy}).
\]
Indeed, this is obvious for \( n = 0 \). Assume that it is true for \( n \), and apply \( \partial_x \) to the corresponding identity. We find
\[
\text{Ad}(e^{F(x)-F(y)})(\alpha \partial^n_{xy}) + [F'(x), \text{Ad}(e^{F(x)-F(y)})(\alpha \partial^n_{xy})] = \partial_x \{ [\partial_x - \text{ad}(F'(x))]^n(\alpha \delta_{xy}) \};
\]
but the l.h.s. of this identity is expressed as
\[
\text{Ad}(e^{F(x)-F(y)})(\alpha \partial^n_{xy}) + [F'(x), (\partial_x - \text{ad}(F'(x)))^n(\alpha \delta_{xy})],
\]
and we obtain (49) at step \( n + 1 \).

Now formula (48) follows directly from (47) and (49). In fact, the local part of formula (47) can be rewritten as
\[
\sum_{k \geq 0} (\text{Ad}(n_+(x) \otimes n_+(y))(t - a)_k)n_+(x) \otimes n_+(y)\partial_x^k \delta_{xy}.
\]
After substituting \( n_+(x) = \tilde{n}_+(x)e^{F(x)} \) in this formula we obtain
\[
\sum_{k \geq 0} \left( \text{Ad}(\tilde{n}_+(x) \otimes \tilde{n}_+(y)) \text{Ad}(e^{F(x)} \otimes e^{F(y)})(t - a)_k \right) n_+(x) \otimes n_+(y)\partial_x^k \delta_{xy},
\]
which is equal to
\[
\sum_{k \geq 0} \left( \text{Ad}(\tilde{n}_+(x) \otimes \tilde{n}_+(y)) (\text{Ad}(e^{F(x)-F}(y) \otimes 1)(t - a)_k \right) n_+(x) \otimes n_+(y)\partial_x^k \delta_{xy}
\]
by \( a_+ \)-invariance of \( (t - a)_k \). Applying formula (49) we obtain (48).
\[ \square \]

Now let \( V((\lambda)), W((\mu)) \) be \( \tilde{g} \)-modules, and \( v \in V, w \in W \) be such that \( a_+ v = 0, a_+ w = 0 \). The matrix coefficients of \( n_+(x)v, n_+(y)w \) are composed of functions on
$N_+/A_+$, or, equivalently, of elements of $\pi_0$. Moreover, all elements of $C[N_+/A_+]$ can be obtained this way. Formula (48) implies that

\begin{equation}
P(n_+ v \otimes n_+ w) = \sum_{k \geq 0} \left\{ \text{Ad}(\bar{n}_+(x)) \left[ (\partial_x - \sum_{n \in I} \frac{1}{n} H_n(x) \text{ad} p_n) \right]^k \right\}_+ n_+(x) v \otimes \left( \text{Ad}(\bar{n}_+(y))(t - a)^{(2)}_k \right)_+ n_+(y) w \delta_{xy};
\end{equation}

since $\bar{n}_+(x), \bar{n}_+(y)$ are matrices whose entries are elements of $\pi_0$, (50) shows at the same time that the Poisson brackets of the entries of $n_+(x)v$ and $n_+(y)w$ are local and expressed in terms of functions of $N_+/A_+$.

In the theory of the mKdV equations, an important role is played by the embedding of $N_+/A_+$ in $\tilde{g}$ as an $N_+$–coadjoint orbit of an element of $a$. This motivates us to compute the Poisson brackets of the matrices $\text{Ad}(n_+)(p_n)$. The result is a direct consequence of Cor. 3.1.

**Corollary 3.3.** For $n, m \in \pm I$,

\begin{equation}
P(\text{Ad}(n_+)(p_n) \otimes \text{Ad}(n_+)(p_m)) = \sum_{k \geq 0} \left[ \left\{ \text{Ad}(\bar{n}_+(x)) \left[ (\partial_x - \sum_{n \in I} \frac{1}{n} H_n(x) \text{ad} p_n) \right]^k \right\}_+ \text{Ad}(n_+(x))(p_n) \right] \otimes \left[ \left( \text{Ad}(\bar{n}_+(y))(t - a)^{(2)}_k \right)_+ \text{Ad}(n_+(y))(p_m) \right] \delta_{xy}.
\end{equation}

Here again, the locality of these brackets and the fact they are expressed in terms of $A_+$–invariant functions on $N_+$ is manifest.

**Remark 3.2** Formulas (47) and (50) give a geometric interpretation of the nonlocal VPA structures of $\pi_0^+$ and $\pi_0$ respectively. □

We now give a formula for $P(u_i \otimes n_+)$. Due to the presence of nonlocal quantities $F_n$ in $n_+$, this bracket contains nonlocal as well as local terms.

**Proposition 3.1.**

\begin{equation}
P(u_i \otimes n_+) = n_+(y) \partial_x \left( - (n_+(x)^{-1} h_i n_+(x)) a \partial_y^{-1} \delta_{xy} \right) + \sum_{n \geq 0} (- \text{ad } p_{-1})^{-n-1} \left( (n_+(x)^{-1} h_i n_+(x))_{\text{Im(ad } p_{-1})} \partial_x^n \delta_{xy} \right),
\end{equation}

where the indices $a$ and $\text{Im(ad } p_{-1})$ stand for the projections on the components of $g = a \oplus \text{Im(ad } p_{-1})$. 
Proof. Consider again the variable \( \varphi_i = \partial^{-1} u_i \). It is easier to compute \( \mathcal{G} = P(\varphi_i \otimes n_+) \) first. We have

\[
\partial_y \mathcal{G} = -(p_{-1} + \sum_i u_i(y) h_i^\gamma) \mathcal{G} + \mathcal{G} p_{-1} - h_i n_+(x) \partial_{xy}.
\]

Let \( \mathcal{H} = n_+(y)^{-1} \mathcal{G} \), then

\[
\partial_y \mathcal{H} = \mathcal{H}, p_{-1} - n_+(x)^{-1} h_i n_+(x) \partial_{xy}.
\]

A solution to this equation is

\[
\mathcal{H}_0 = -(n_+(x)^{-1} h_i n_+(x))_a \partial_{xy}^{-1} \delta_{xy} + \sum_{n \geq 0} (-\text{ad } p_{-1})^{n-1}((n_+(x)^{-1} h_i n_+(x))_\text{Im(ad } p_{-1})) \partial_{xy}^n \delta_{xy}.
\]

On the other hand, the nonlocal part of \( \mathcal{H} \) coincides with that of \( \mathcal{H}_0 \) because it is equal to the nonlocal part of \( \sum_{n \in I} \frac{1}{n} \text{p}_n P(\varphi_i \otimes F_n) \); but \( P(\varphi_i \otimes F_n) = -n \partial_n \varphi_i(x) + \text{local terms} \), since \( P(\varphi_i \otimes H_n) \in n \partial_n \varphi_i(x) + \partial_y (\mathcal{R}_2(\pi_0)) \); so this nonlocal part is expressed as

\[
-\sum_n \partial_n \varphi_i(x) p_n \partial_{xy}^{-1} \delta_{xy}.
\]

To establish the coincidence of the nonlocal parts of \( \mathcal{H} \) and \( \mathcal{H}_0 \), it remains to check the identity

\[
(51) \quad \partial_n \varphi_i(x) = \langle n_+^{-1}(x) h_i n_+(x), p_{-n} \rangle
\]

which amounts to

\[
\partial_n u_i(x) = -\langle h_i, \{ p_{-1} + \sum_i u_i(x) h_i^\gamma, n_+ p_{-n} n_+^{-1} \} \rangle;
\]

since by Lemma 2.5, \( \partial_n (\sum_i u_i(x) h_i^\gamma) \) coincides with the Cartan part of

\[
-\{ p_{-1}, n_+ p_{-n} n_+^{-1} \},
\]

this is verified. So (51) holds, and the nonlocal parts of \( \mathcal{H} \) and \( \mathcal{H}_0 \) coincide.

So \( \mathcal{H}_1 = \mathcal{H} - \mathcal{H}_0 \) has no nonlocal terms, and satisfies \( \partial_y \mathcal{H}_1 = [\mathcal{H}_1, p_{-1}] \); the arguments used to establish (62) then show that \( \mathcal{H}_1 = 0 \). \( \square \)

Remark 3.3 As we noted in the introduction, it is possible to derive (37) using Prop. 3.1 and a differential equation in the first variable, satisfied by \( P(n_+ \otimes n_+) \). But we find the present use of \( B_- \) more natural. \( \square \)

3.3. Gelfand-Dickey-Dorfman structure. According to Sect. 1.5, the VPA \( \pi_0 = \mathbb{C}[N_+/A_+] \) is endowed with a Gelfand-Dickey-Dorfman structure. We give here a geometric interpretation of it.

Proposition 3.2. Let \( V(\langle \lambda \rangle), W(\langle \mu \rangle) \) be \( \tilde{g} \)-modules, and \( v \in V, w \in W \) be such that \( a_+ v = 0, a_+ w = 0 \). We have:

\[
\mathcal{V}_{n_+ v \otimes (1 \otimes n_+ w)} = \sum_{k \geq 0} (-\partial)^k (\text{Ad}(n_+)(t - a)_k^{1+} n_+ v) \otimes (\text{Ad}(n_+)(t - a)_k^{2+} n_+ w)
\]

Proof. We apply the definition (9) of \( \mathcal{V}_j \) to formula (47). \( \square \)
3.4. Compatible nonlocal VPA structures. Let us show how the nonlocal VPA structure on $\mathbb{C}[B_- \times N_+]$, defined in Thm. 3.1, can be embedded into an infinite family of compatible nonlocal VPA structures (we call such a family compatible, if any linear combination of these structures is again a nonlocal VPA structure).

Let us identify $\mathfrak{g}$ with the subalgebra of the loop algebra $\tilde{\mathfrak{g}} \otimes \mathbb{C}((\lambda))$ of a finite dimensional semisimple Lie algebra $\tilde{\mathfrak{g}}$, consisting of the elements $x(\lambda)$ satisfying

$$x(\zeta \lambda) = x(\lambda)^\sigma,$$

$\sigma$ an automorphism of $\tilde{\mathfrak{g}}$ and $\zeta$ a root of unity of the same order $r$. There is an action of $\mathbb{C}((\lambda^r))$ on $\mathfrak{g}$. Let us denote by the same letter elements of $\mathbb{C}((\lambda^r))$ and the corresponding operators on $\mathfrak{g}$.

For $n \in \mathbb{Z}$, set

$$t_n = (\lambda^{rn} \otimes 1)t, \quad a_n = (\lambda^{rn} \otimes 1)a.$$

Note that $\text{ad} p_{-1}$ commutes with the action of $\lambda^{rn}$, so that

$$t_n - a_n \in \text{Im}(\text{ad} p_{-1})^{\otimes 2}.$$

Let us define an operation $P_n$ on $\mathbb{C}[B_- \times N_+]$, by the formula

$$P_n(g \otimes g) = [-\ell^{\otimes 2}(t_n)(g(x) \otimes g(y)) + r^{\otimes 2}(a_n)(g(x) \otimes g(y))]\partial_{x}^{-1}\delta_{xy}$$

$$+ \sum_{n \geq 0} r^{\otimes 2}((\text{ad} p_{-1})^{-n-1} \otimes 1)(t_n - a_n)(g(x) \otimes g(y))\partial_{x}^{n}\delta_{xy}.$$ (52)

**Proposition 3.3.** The formulae (52) define compatible nonlocal VPA structures on $\mathbb{C}[B_- \times N_+]$ (endowed with the derivation $r(p_{-1})$).

**Proof.** A combination of the brackets (52) corresponds to the same formula, with $t_n$ and $a_n$ replaced by $t_f$ and $a_f$ respectively, with $t_f = (f \otimes 1)t$ and $a_f = (f \otimes 1)a$, $f$ a certain element of $\mathbb{C}[\lambda^r, \lambda^{-r}]$. The resulting bracket satisfies the conditions of Prop. 1.1; the elements of $E_i$ in the second condition are $\ell^{\otimes 2}(t_f) - r^{\otimes 2}(t_f)$ for $i = 0$, and 0 for $i > 0$. \qed

**Remark 3.4** The variety $B_- \times N_+$ can be considered as an open subset of $G$. It has a compatible family of nonlocal VPA structures defined by (52). In the same way as in the proof of Prop. 3.3, one can show that formula (52) defines a compatible family of nonlocal VPA structures on the whole group $G$. \qed

**Remark 3.5** The extension to $G$ of the nonlocal VPA structure defined in Thm. 3.1 is clearly left $G$-invariant. It follows that left $G$-translations provide symmetries of the mKdV hierarchy, respecting the Poisson structure. Infinitesimal left translations by elements of $\mathfrak{n}_+$ correspond to the Toda flows; left translations by elements of $\mathfrak{b}_-$ do not change the variables $u_i$. A class of translations that would be interesting to study further are left translations by elements of the affine Weyl group; they should mix local and nonlocal variables while respecting the Poisson structure. A. Orlov pointed out to us that they probably coincide with the Darboux transformations. \qed
4. The proof of Lemma 3.4 and Lemma 3.5

4.1. Proof of Lemma 3.4. Equation (40) is rewritten as

\begin{equation}
\frac{\partial}{\partial y}(\text{lhs of (53)}) = (\text{lhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\frac{\partial}{\partial y}(\text{rhs of (53)}) = (\text{rhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\& - (p_{-1} + \sum_{i} u_{i}(y) \text{Ad}(b_{-}^{-1}(y))(\bar{e}^{\beta}))_{+}
\end{equation}

\begin{equation}
\& + \sum_{i,k} b_{-}(x) \partial_{k} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms},
\end{equation}

using (21) and (23), and

\begin{equation}
\frac{\partial}{\partial y}(\text{lhs of (53)}) = (\text{lhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\frac{\partial}{\partial y}(\text{rhs of (53)}) = (\text{rhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\& - (p_{-1} + \sum_{i} u_{i}(y) \text{Ad}(b_{-}^{-1}(y))(\bar{e}^{\beta}))_{+}
\end{equation}

\begin{equation}
\& + \sum_{i,k} b_{-}(x) \partial_{k} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms},
\end{equation}

using (21) and (22). Since

\begin{equation}
\frac{\partial}{\partial y}(\text{lhs of (53)}) = (\text{lhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\frac{\partial}{\partial y}(\text{rhs of (53)}) = (\text{rhs of (53)}) (1 \otimes p_{-1}) - 1 \otimes (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})
\end{equation}

\begin{equation}
\& - (p_{-1} + \sum_{i} u_{i}(y) \text{Ad}(b_{-}^{-1}(y))(\bar{e}^{\beta}))_{+}
\end{equation}

\begin{equation}
\& + \sum_{i,k} b_{-}(x) \partial_{k} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms},
\end{equation}

these two equations coincide. In (54), we denote by $x_{1}$ the part of $x \in \mathfrak{g}$, of principal degree 1. (54) is proved by pairing its right and left hand sides with $1 \otimes h_{i}$, $i = 1, \ldots, l$. It follows that the difference of the two sides of (53) satisfies

\begin{equation}
\frac{\partial}{\partial y}(\text{lhs of (53) - rhs of (53)}) = (\text{lhs of (53) - rhs of (53)}) (1 \otimes p_{-1})
\end{equation}

\begin{equation}
\& - (p_{-1} + \sum_{i} u_{i}(y)h_{i}^{\vee})(\text{lhs of (53) - rhs of (53)}) + \text{local terms}.
\end{equation}

On the other hand, we have

\begin{equation}
\frac{\partial}{\partial x}(\text{lhs of (53)}) = (\text{lhs of (53)}) (\sum_{i,k} b_{-}(x) h_{i}^{\vee} \partial_{k} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms},
\end{equation}

\begin{equation}
\frac{\partial}{\partial x}(\text{rhs of (53)}) = (\text{rhs of (53)}) (\sum_{i,k} b_{-}(x) h_{i}^{\vee} \partial_{k} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms},
\end{equation}

\begin{equation}
\& + \sum_{n \in I} b_{-}(x) \{ p_{-1} + \sum_{i} u_{i}(x) \text{Ad}(n_{+}(x))(p_{-n})_{-} \}
\end{equation}

\begin{equation}
\& + \sum_{i} \sum_{n \in I} b_{-}(x) \partial_{n} u_{i}(x) \otimes n_{+}(y) p_{n} \partial_{x}^{-1} \delta_{xy} + \text{local terms}.
\end{equation}

We have the equality

\begin{equation}
[p_{-1}, (\text{Ad}(n_{+}(x))(p_{-n}))_{1}] = \sum_{i} h_{i}^{\vee} \partial_{n} u_{i}(x),
\end{equation}

because of Lemma 2.5.
Therefore the right hand sides of the last two formulas coincide up to local terms, and
\[
\partial_x (\text{lhs of (53)} - \text{rhs of (53)}) = (\text{lhs of (53)} - \text{rhs of (53)})
\]
\[
(1 \otimes (p_{-1} + \sum u_i(x)h_i^r)) + \text{local terms}.
\]

Let \( \mathcal{X} = (1 \otimes n_+(y)^{-1})(\text{lhs of (53)} - \text{rhs of (53))}(1 \otimes b_-(x)^{-1}) \), then
\[
\partial_x \mathcal{X} = \text{local terms}
\]
by (56), and
\[
\partial_y \mathcal{X} = [\mathcal{X}, 1 \otimes p_{-1}] + \text{local terms}
\]
by (21) and (55). The first equation gives
\[
\partial_x \mathcal{X} = \sum_{n \geq 0} \mathcal{X}_n(y) \partial_x^n \delta_{xy},
\]
so
\[
\mathcal{X} = \mathcal{X}_0(y) \partial_x^{-1} \delta_{xy} + \text{local terms},
\]
and the second equation gives us
\[
\partial_y \mathcal{X}_0(y) = [\mathcal{X}_0(y), 1 \otimes p_{-1}].
\]
Let \( \xi \) be any element of the dual to \( \mathfrak{b}_- \), and \( \mathcal{X}_\xi(y) = (\xi \otimes 1)(\mathcal{X}_0(y)) \). Then \( \mathcal{X}_\xi(y) \) has values in \( \mathfrak{n}_+ \) and satisfies
\[
\partial_y \mathcal{X}_\xi(y) = [\mathcal{X}_\xi(y), p_{-1}];
\]
we then follow the proof of [10], lemma 3, to conclude that \( \mathcal{X}_\xi(y) \) is constant and lies in \( \mathfrak{a}_+ \). Recall how this can be done: decompose \( \mathcal{X}_\xi(y) \) in its homogeneous principal components \( \sum_i \mathcal{X}_{\xi,i}(y) \), and each component along the decomposition \( \text{Im(ad } p_{-1}) \oplus \mathfrak{a} \), as \( \mathcal{X}_{\xi,i}^1(y) + \mathcal{X}_{\xi,i}^2(y) \); let \( i \) be the smallest index, such that \( \mathcal{X}_{\xi,i}^1(y) \) is not zero; the equation implies that \( \partial_y \mathcal{X}_\xi(y) \) has a nonzero component of degree \( i - 1 \) in \( \text{Im ad p}_{-1} \), hence a contradiction. So \( \mathcal{X}_\xi(y) \) lies in \( \mathfrak{a} \); (57) then implies that it is constant. We finally obtain:
\[
\text{lhs of (53)} - \text{rhs of (53)} = \sum_{n \in I} x_n b_-(x) \otimes n_+(y) p_n \partial_x^{-1} \delta_{xy} + \text{local terms},
\]
with \( x_n \in \mathfrak{b}_- \). But there is only one possibility, \( x_n = 0 \), which is compatible with the following invariance property of \( P \).

Recall that for \( \xi \in \mathfrak{b}_- \), \( \ell(\xi) \) is the derivation of the algebra \( \tilde{\mathfrak{g}}_0 \) defined by the action of the left translation by \( \xi \), on \( \mathfrak{b}_- \times \mathfrak{N}_+ \). Since \( \ell(\xi) \) commutes with \( \partial \), it induces an endomorphism (also denoted by \( \ell(\xi) \)) of \( \mathcal{R}_2(\tilde{\mathfrak{g}}_0) \), according to the rules used in 2.1 in the case of \( \partial_n \). We then have:

**Lemma 4.1.** For \( a, b \in \tilde{\mathfrak{g}}_0 \), \( \xi \in \mathfrak{b}_- \),
\[
P(\ell(\xi)a \otimes b) + P(a \otimes \ell(\xi)b) = \ell(\xi)P(a \otimes b).
\]
Proof. For the brackets \( P(b_- \otimes b_-) \), this follows from (27) and the invariance of \( t \). We also have

\[
P(\ell(\xi)b_- \otimes u_i) = \partial_y ([\xi, \text{Ad}(b_-(y))(h_i)]b_-(x)\partial^{-1}_x \delta_{xy} + \text{Ad}(b_-(y))(h_i)\xi b_-(x)\partial^{-1}_x \delta_{xy}) = \ell(\xi)P(b_- \otimes u_i)
\]

so that

\[
P(H_n \otimes \ell(\xi)b_-) = \ell(\xi)P(H_n \otimes b_-),
\]

(in this equality, the second \( \ell(\xi) \) is \( \ell(\xi) \otimes 1 \) acting on \( \text{End}(V)((\xi) \otimes R_2(\pi_0)) \) and \( (\ell(\xi) \otimes 1)r_n = (1 \otimes \ell(\xi))r_n \) (equality in \( \text{End}(V)((\xi) \otimes R_2(\pi_0)) \); so that

\[
P(F_n \otimes \ell(\xi)b_-) = \ell(\xi)P(F_n \otimes b_-).
\]

Finally, the elements of \( \pi^+_0 \) are invariant under \( \ell(\xi) \), so the identity is trivially satisfied for their Poisson brackets. \( \square \)

Now (53) follows. \( \square \)

4.2. Proof of Lemma 3.5. The operation \( P_0 \) is defined by the identities

\[
P_0(b_- \otimes b_-) = -\ell^{(2)}(t)(b_-(x) \otimes b_-(y))\partial^{-1}_x \delta_{xy},
\]

\[
P_0(b_- \otimes n_+) = \sum_{n \geq 0}[b_-(n_+(-a)^{1}n_{-1}^{1})(x) \otimes (n_+(t-a)g(-1)n_{-1}^{1})(y)]\partial^{-1}_x \delta_{xy},
\]

where we denote \( (\text{ad } p_{-1})^{n-1} \otimes 1(t-a) \) by \( (t-a)_n \), any element \( \alpha \in g \otimes g \) is decomposed as \( \sum \alpha^{(1)} \otimes \alpha^{(2)} \), and

\[
P_0(n_+ \otimes n_+ = \sum_{n \geq 0}r^{(2)}(\text{ad } p_{-1})^{n-1} \otimes 1(t-a)(n_+(x) \otimes n_+(y))\partial^{-1}_x \delta_{xy}.
\]

By construction, \( P_0(g \otimes g) \) satisfies the identities

\[
\partial_x P_0(g \otimes g) = (r(p_{-1}) \otimes 1)P_0(g \otimes g), \quad \partial_y P_0(g \otimes g) = (1 \otimes r(p_{-1}))P_0(g \otimes g).
\]

Clearly, \( P_0(b_- \otimes b_-) \) coincides with \( P(b_- \otimes b_-) \). \( \mathcal{B} = P(b_- \otimes n_+) \) satisfies the equation

\[
\partial_y \mathcal{B} + (1 \otimes (p_{-1} + \sum_i u_i h_i^{\gamma}))\mathcal{B} - \mathcal{B}(1 \otimes p_{-1}) = \sum_i (\partial_y \text{Ad}(b_-(y))(h_i)b_-(x)\partial^{-1}_x \delta_{xy} + h_i^{\gamma}n_+(y))\delta_{xy},
\]

by virtue of (23) and (21). Let us determine an equation satisfied by \( \mathcal{B}_0 = P_0(b_- \otimes n_+) \). Let \( \mathcal{E} = P_0(b_- \otimes b_-) \). \( \mathcal{E} \) satisfies the equation

\[
\partial_y \mathcal{E} = \mathcal{E}(1 \otimes (p_{-1} + \sum_i u_i h_i^{\gamma})) - \partial_y [\text{Ad}(b_-(y))(h_i)b_-(x)\partial^{-1}_x \otimes b_-(y)]\delta_{xy}.
\]
Note that due to (61), $P_0(b_- \otimes g)$ satisfies
\[ \partial_y P_0(b_- \otimes g) = (1 \otimes r(p_{-1}))P_0(b_- \otimes g). \]
This implies, writing $B_0$ as $P_0(b_- \otimes b_-^1 g)$ [$B_- \times N_+$ has a left $B_-$-action, defined as the product of the left action of $B_-$ on itself and of the trivial one, that we use here], that $B_0$ satisfies
\[ \partial_y B_0 + (1 \otimes (p_{-1} + \sum_i u_i h_i^\gamma))B_0 - B_0(1 \otimes p_{-1}) = \sum_i (\partial_y \text{Ad}(b_-)(y))(h_i)b_-^i(x)\partial_x^{-1} \otimes h_i^\gamma n_+(y)) \delta_{xy}. \]

Let us set $B_1 = (1 \otimes n_+^i(y))(B - B_0)$, we obtain
\[ \partial_y B_1 + [1 \otimes p_{-1}, B_1] = 0. \]

The nonlocal parts of $B$ and $B_0$ coincide, so that $B_1$ contains only local terms; write $B_1 = \sum_{n \geq 0} B_1^{(n)}(x)\partial_x^n \delta_{xy}$ (each $B_1^{(n)}$ belongs to the tensor product of the tangent space $T_{b_-}(x)B_-$ to $B_-$ at $b_-(x)$ with $n_+$), we then get
\[ [1 \otimes p_{-1}, B_1^{(0)}(x)] = 0, \quad B_1^{(n)} = [1 \otimes p_{-1}, B_1^{(n+1)}(x)] \]
for $n \geq 0$, so $B_1^{(0)}(x)$ belongs to $T_{b_-}(x)B_- \otimes a$ by the first equation and to $T_{b_-}(x)B_- \otimes \text{Im}(\text{Ad} \ p_{-1})$ by the second one (specialized to $n = 0$), so that it is zero; repeating this argument for $B_1^{(1)}$, we find it to vanish as well, etc. So $B_1 = 0$ and
\[ (62) \quad P(b_- \otimes n_+) = P_0(b_- \otimes n_+). \]

Now, $B = P(b_- \otimes n_+)$ satisfies the equation
\[ (63) \quad \partial_x B = B((p_{-1} + \sum_i u_i(x)h_i^\gamma) \otimes 1) + \sum_i b_-^i(x) \otimes (h_i^\gamma P(u_i \otimes n_+)). \]

On the other hand, let $C = P(n_+ \otimes n_+)$ and $C_0 = P_0(n_+ \otimes n_+)$. $C$ satisfies the equation
\[ \partial_x C = -((p_{-1} + \sum_i u_i(x)h_i^\gamma) \otimes 1)C + C(p_{-1} \otimes 1) - \sum_i h_i^\gamma n_+(x) \otimes P(u_i \otimes n_+). \]

Let us determine an equation satisfied by $C_0$. Due to (61), $P_0(g \otimes n_+)$ satisfies
\[ \partial_x P_0(g \otimes n_+) = P_0(g \otimes n_+)(p_{-1} \otimes 1), \]
and writing $P_0(n_+ \otimes n_+)$ as $P_0(b_-^1 g \otimes n_+)$ (using the same left $B_-$-action as above) and using (63), we get
\[ \partial_x C_0 = -((p_{-1} + \sum_i u_i(x)h_i^\gamma) \otimes 1)C_0 + C_0(p_{-1} \otimes 1) - \sum_i h_i^\gamma n_+(x) \otimes P(u_i \otimes n_+). \]

$C$ and $C_0$ satisfy the same equation, so that $C_1 = (n_+(x)^{-1} \otimes 1)(C - C_0)$ (which belongs to the tensor product of $n_+$ with the tangent space to $N_+$ at $n_+(y)$) satisfies
\[ \partial_x C_1 = [C_1, p_{-1} \otimes 1]. \]
Since the nonlocal parts of $P_0$ and $P$ coincide, $C_1$ contains no nonlocal terms. We can use the same arguments as in the case of $B_1$, to conclude that $C_1 = 0$ and

$$P(n_+ \otimes n_+) = P_0(n_+ \otimes n_+). \quad (64)$$

Lemma 3.5 now follows from (62), (64).

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