EXPLICIT GALOIS OBSTRUCTION AND DESCENT FOR HYPERELLIPITIC CURVES WITH TAMELY CYCLIC REDUCED AUTOMORPHISM GROUP

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Abstract. This paper is devoted to the explicit description of the Galois descent obstruction for hyperelliptic curves of arbitrary genus whose reduced automorphism group is cyclic of order coprime to the characteristic of their ground field. Along the way, we obtain an arithmetic criterion for the existence of a hyperelliptic descent.

The obstruction is described by the so-called arithmetic dihedral invariants of the curves in question. If it vanishes, then the use of these invariants also allows the explicit determination of a model over the field of moduli; if not, then one obtains a hyperelliptic model over a degree 2 extension of this field.

Introduction

The classical problem of Galois descent, as first considered by Weil in [16], is the following: Let $X$ be a variety over an algebraically closed extension $K$ of a base field $k$. Suppose that $X$ is isomorphic with all its Galois conjugates under the action of $\text{Gal}(K|k)$, or in other words that $k$ is the field of moduli of $X$ for the extension $K|k$. Does there exist a model of $X$ over $k$?

Such a model of $X$ is then called a (Galois) descent. Generically, or more precisely, when the geometric automorphism group of $X$ is trivial, the answer to this question is affirmative [16], but this partial answer is unsatisfactory, as there are many interesting classes of varieties that allow non-trivial automorphisms. This paper considers one such class, namely that of hyperelliptic curves. The explicit form of their defining equations makes hyperelliptic curves the simplest class of curves after conics and elliptic curves (for which the answer to the above question is well-known to be affirmative). Due to the presence of the hyperelliptic involution, hyperelliptic curves never have a trivial automorphism group. This makes them a fundamental example for studying the descent problem.

This problem allows a further refinement for hyperelliptic curves; instead of merely asking for a model over $k$, one can ask for a model that is defined by a hyperelliptic equation, or alternatively, a homogeneous polynomial in two variables, over $k$. Let us call such a model a hyperelliptic descent. Asking for such a descent reduces one to the study of homogeneous binary forms. This is a great benefit.
since not only was the invariant theory of these forms extensively studied in the nineteenth century, but one can often also apply the method of covariants, as in [11].

The answer to the descent question depends heavily on the reduced automorphism group $\overline{G}$ of $X$, which is the geometric automorphism group $G = \text{Aut}_K(X)$ of $X$ quotiented by the hyperelliptic involution. Assume, as we will throughout this paper, that the characteristic of $k$ does not equal 2. Moreover, let us say that $\overline{G}$ is tamely cyclic if it is cyclic of order coprime to the characteristic of $K$. Then Huggins’ seminal work [9] shows that if $\overline{G}$ is tamely cyclic, the curve $X$ allows a hyperelliptic descent. For tamely cyclic $\overline{G}$, explicit counterexamples for descent were first constructed by Earle [5] and Shimura [13]. More recently, the full classification of the hyperelliptic curves that do not allow a hyperelliptic descent for the extension $\mathbb{C}|\mathbb{R}$ was initiated by Bujalance-Turbek [2] and completed by Huggins [8].

In Section 3, we give a complete answer to the descent problem in the case where $X$ is a hyperelliptic curve with tamely cyclic reduced automorphism group, for any extension $K|k$. The problem is naturally stratified by the genus $g$ of $X$ and the isomorphism class of the full automorphism group $G$ of $X$ (which under our hypotheses is either cyclic or a product of a cyclic group of order 2 and another cyclic group by [1, Satz.5.2]). We refer to Theorems 3.6 and 3.8 for precise statements, but depending on the pair $(g, G)$, either all the curves in the stratum descend or the obstruction is classified by the resolvability of a certain norm equation determined by what we call arithmetic dihedral invariants of $X$. If this obstruction vanishes, we show how a descent can be effectively constructed if $\overline{G}$ is non-trivial. In Subsection 3.3, we consider the slightly more complicated case of trivial $\overline{G}$, where arithmetic dihedral invariants are not available. Finally, in Subsection 3.4, we show how to construct counterexamples to descent beyond the case of the extension $\mathbb{C}|\mathbb{R}$: given any quadratic extension of fields $L|k$, Theorem 3.6 gives explicit and universal families of curves which are defined over $L$ and whose moduli are in $k$ but which do not descend to $k$.

The arithmetic dihedral invariants, which will be discussed in Section 2, are closely related with and indeed partially named after the dihedral invariants defined by Gutierrez and Shaska [6]. We note a few differences. First of all, the models from which we derive our arithmetic dihedral invariants are normalized in a weaker way than in loc. cit. This is our reason for calling our invariants ‘arithmetic’. Secondly, the arithmetic dihedral invariants allow us to treat effectively the reconstruction and parametrization of forms with given invariants, also in the non-generic cases where many of the coefficients in these normal forms are zero. Contrary to what is suggested in Lemma 3.2 and Theorem 4.5 in [6], such non-generic reconstruction is in fact more involved than that in the generic case.

Finally, the purported reconstruction in [6, Theorem 4.5] actually takes place over a quadratic extension of $k$, as was already pointed out in [10]. In particular, [6, Corollary 4.6] is incorrect, as can also be seen from the results in [9, Section 6] and our complete classification of the counterexamples in Theorem 3.6.

In general, the arithmetic dihedral invariants are homogeneous invariants for the action on binary forms of the subgroup of $\text{PGL}_2$ consisting of diagonal and anti-diagonal matrices. Therefore any invariant for $\text{PGL}_2$ can be written as a rational function in the arithmetic dihedral invariants. For those curves whose reduced automorphism group is tamely cyclic, the arithmetic dihedral invariants...
can conversely be written as expression in the homogeneous invariants for PGL_2. We perform these calculations explicitly for curves of genus 3.

Before defining the arithmetic dihedral invariants and proving the main theorem, we need a result relating the existence of a general descent with that of a hyperelliptic descent, which is given in Section 1. Building on results by Mestre [12] and Huggins [9], we shall show in Theorem 1.7 that these questions are equivalent except when the genus of X is odd and its reduced automorphism group is tamely cyclic of odd order, in which case it turns out that a descent always exists by Theorem 3.8. Furthermore, we completely classify the counterexamples to this equivalence in this remaining case in Theorem 3.12. Even more surprisingly, regardless of genus or automorphism group, the existence of a hyperelliptic descent of X can be characterized arithmetically. In order to formulate this precisely, consider the quotient $B = X/G$. It has a canonical descent $B_0$ to $k$, and it is well-known (see for example the discussion in [4, Section 2.9]) that $X$ descends if $B_0$ has a rational point over the field of moduli. In Theorem 1.9, we show that in fact the existence of such a rational point is equivalent with the existence of a hyperelliptic descent of $X$. In particular, we see that $X$ always admits a hyperelliptic equation over a degree 2 extension of $k$.

After the proof of the main Theorems 3.6 and 3.8, we briefly turn to algorithmic considerations and the implementation of our results in Section 4. Our Magma functionality is available online\(^1\). We also discuss how this implementation can be combined with the results of [10], which concludes our exploration of the arithmetic aspects of the moduli space of genus 3 curves by showing how to reconstruct a genus 3 curve from its invariants over an extension of the field of moduli of minimal degree. This additional functionality has been added to the package g3twists\(^2\).

Section 5 concludes the paper and briefly discusses the remaining open questions on the descent of hyperelliptic curves.

**Notation.** We let $K$ be an algebraically closed field whose characteristic does not equal 2, and we let $k$ be a subfield of $K$, which we will later assume to be infinite. We denote $\Gamma = \text{Gal}(K/k)$. The curves over $K$ and its subfields that are considered in this paper will be smooth, projective and geometrically irreducible throughout.

$X$ will always denote a hyperelliptic curve over $K$ of genus $g$ whose field of moduli with respect to the extension $K/k$ equals $k$. We denote the geometric automorphism group $\text{Aut}_K(X)$ of $X$ by $G$. The reduced automorphism group $\overline{G} = \text{Aut}_K(X)/\iota$ is the quotient of $G$ by the central element $\iota$ given by the hyperelliptic involution of $X$. In the second half of this paper, we will additionally suppose that $\overline{G}$ is tamely cyclic, i.e. cyclic of order coprime to the characteristic of $K$. Finally, given a curve $X$ and a divisor $D$, we denote the group of automorphisms $\alpha$ of $X$ over $K$ such that $\alpha(D) = D$ by $\text{Aut}_K(X, D)$.

We will occasionally construct a model of $X$ over an intermediate field $k \subset L \subset K$. When considering such curves over intermediate fields, we restrict our consideration of morphisms to those defined over $L$, unless explicitly specified otherwise; we denote the corresponding automorphism groups by $\text{Aut}_L(X)$ et cetera.

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\(^1\)http://iml.univ-mrs.fr/~ritzenth/programme/hyp-desc.tgz
\(^2\)http://iml.univ-mrs.fr/~ritzenth/programme/g3twists_v1.1.tgz
If \( \varphi : X \to Y \) is a morphism between algebraic curves, then the \textit{ramification divisor} of \( \varphi \) is the divisor of the points on \( X \) that ramify under \( \varphi \). The \textit{branch divisor} of \( \varphi \) is the image of this divisor under \( \varphi \).

We adopt the usual notation of denoting the Galois action by a superscript, \( e.g. \ f^\sigma \) for the conjugation on a binary form. We consider this as a left action, which leads to the somewhat counterintuitive equality \( f^\sigma \sigma^* = (f^\sigma)^* \). Finally, our notation \( C_n, D_n \) for groups is as in \([11]\), as are the notations \( A.f, f \sim g \).

1. Descent and Hyperelliptic Descent

Let the notation \( k \subset L \subset K \) and \( \Gamma \) be as in the introduction, and the let \( C \) be a curve over \( K \).

**Definition 1.1.** The \textit{field of moduli} of \( C \) with respect to the extension \( K|k \) is the fixed field of the group \( \{ \sigma \in \Gamma : C \cong_k C^\sigma \} \).

A \textit{model} of \( C \) over \( L \) is a curve \( C_0 \) over \( L \) such that \( C \) is isomorphic to \( C_0 \) over \( K \). The field \( L \) is then called a \textit{field of definition} for \( C \). If \( C \) has field of moduli \( k \) for the extension \( K|k \), then a \textit{descent} of \( C \) is a model of \( C \) over \( k \).

In this section, we consider a hyperelliptic curve \( X \) of genus \( g \) whose field of moduli for the extension \( K|k \) equals \( k \). For these curves, one can ask for a more specific form of descent:

**Definition 1.2.** Under the hypotheses above, a \textit{hyperelliptic descent} of \( X \) is a model \( X_0 \) of \( X \) over \( k \) that is defined by a homogeneous polynomial \( f_0(x, z) \) of degree \( 2g + 2 \) over \( k \) without repeated roots. More precisely, this is to say that \( X_0 \) is the desingularization of the curve \( y^2 = f_0(x, z) \) in \((1, 1, g + 1)\)-weighted projective \((x, z, y)\)-space over \( k \).

1.1. Equivalence between Descent and Hyperelliptic Descent

By the proof of \([9, \text{Corollary 1.6.6}] \), a hyperelliptic descent to \( k \) for \( X \) exists if \( k \) is finite; we therefore assume it to be infinite throughout this paper. A fundamental result of Mestre \([12]\) tells us that if \( g \) is even, then if \( X \) descends, it descends hyperelliptically. However, when \( g \geq 3 \) is odd, this need not be the case. A counterexample is given in the discussion after Proposition 4.14 in \([10]\). Due to the simpler nature of hyperelliptic descents, it would be useful to know in what other cases this equivalence remains true. The reasoning in \([10]\) shows that if the reduced automorphism group \( \overline{\Gamma} = \text{Aut}_K(X)/\iota \) is trivial, then counterexamples exist for all odd \( g \geq 3 \) as long as pointless conics over \( k \) exist, as for example in the typical case where \( k \) is a number field. We now turn to the case \( \overline{\Gamma} \cong C_3 \); this case will turn out to be fundamental.

**Lemma 1.3.** Let \( Q \) be a genus 0 curve over \( k \), and let \( \alpha \in \text{Aut}_k(Q) \) be an automorphism of order 2 of \( Q \). Then the quotient \( Q/\alpha \) is isomorphic to \( \mathbb{P}_k^1 \).

**Proof.** Consider an anticanonical embedding \( Q \to \mathbb{P}^2 \) realizing \( Q \) as a conic. By our choice of embedding, the automorphisms of \( Q \) extend to \( \mathbb{P}^2 \), and are therefore represented by elements of \( \text{GL}_3(k) \). Let \( M \in \text{GL}_3(k) \) be a matrix representing \( \alpha \). As \( \alpha \) is of order 2, \( M^2 \) is a scalar matrix \( \mu \). But then \( \det(M^2) = \det(M)^2 = \mu^2 \), so \( \mu \) is a square and we can choose a representative of \( \alpha \) such that \( M^2 = \text{id} \). Using a suitable choice of basis, we can therefore choose our embedding such that \( \alpha \) is given by \( (x : y : z) \to (-x : y : z) \). As such, the equation for \( Q \) is of the form \( x^2 = q(y, z) \) with \( q \) of degree 2. It is obvious from this that \( Q/\alpha \) is isomorphic to \( \mathbb{P}_k^1 \). \( \square \)
Lemma 1.4. Let $Q$ be a genus 0 curve over $k$, and let $\alpha \in \Aut_k(Q)$ be an automorphism of order 2 of $Q$. Then there exists a $k$-rational divisor $\mathcal{R}$ of degree 6 on $Q$ such that $\Aut_K(Q, \mathcal{R})$ is generated by $\alpha$.

Proof. Choose 3 rational points $p_1, p_2, p_3$ on $Q/\alpha$ that are not in the branch locus of the canonical map $\pi : Q \to Q/\alpha$. We show that for a sufficiently generic choice of $p_1, p_2, p_3$ the divisor $\mathcal{R} = \pi^{-1}(p_1 + p_2 + p_3)$ satisfies $\Aut_K(Q, \mathcal{R}) = \langle \alpha \rangle$.

Geometrically, our situation transforms as follows. Choose an isomorphism $\varphi : Q \to \mathbb{P}^1_k$ over $K$. Using the three-transitivity of $\Aut_K(\mathbb{P}^1_K)$, we may suppose that the projection $Q \to Q/\alpha$ is described by the map $\mathbb{P}^1_K \to \mathbb{P}^1_K$ $(x : z) \mapsto (x^2 : z^2)$ which is branched at $(0 : 1)$ and $(1 : 0)$. Choose three distinct points $q_1 = (\lambda : 1)$, $q_2 = (\mu : 1)$ and $q_3 = (\nu : 1)$ with $\lambda, \mu, \nu \neq 0, \infty$. Consider the morphism

$$(\lambda, \mu, \nu) \mapsto X_{\lambda, \mu, \nu} : y^2 = (x^2 - \lambda)(x^2 - \mu)(x^2 - \nu).$$

Outside a divisor $D$ of $\mathbb{A}^3_k = \{ (\lambda, \mu, \nu) \}$ the reduced automorphism group of $X_{\lambda, \mu, \nu}$ is $C_2$ [3]. We can pull back the divisor $D$ by the induced isomorphism $\varphi : Q/\alpha \to \mathbb{P}^1_k$ to the arithmetic situation $Q \to Q/\alpha$. Now by Lemma 1.3, $Q/\alpha \simeq \mathbb{P}^1_k \to \mathbb{A}^3_k$ hence, because $K$ is infinite and $\mathbb{A}^3_k$ is affine, we can find a point $(\lambda_0, \mu_0, \nu_0) \in \mathbb{A}^3_k \subset (Q/\alpha)^3$ outside the pullback of $D$ giving rise to a divisor $\mathcal{R}$ on $Q$ such that $\Aut_K(Q, \mathcal{R}) \simeq \Aut_K(X_{\lambda_0, \mu_0, \nu_0}/\alpha) \simeq C_2$.

Lemma 1.5. Let $Q$ be a genus 0 curve over $k$, and let $\alpha$ be an automorphism of order 2 of $Q$. If $Q \not\cong \mathbb{P}^1_k$, there exists a quadratic extension $L$ of $k$ and an isomorphism $\varphi : Q \to \mathbb{P}^1_k$ over $L$ such that $\varphi^\sigma = \varphi \alpha$ for the generator $\sigma$ of $\Gal(L|k)$.

Proof. Suppose that $Q$ is not isomorphic to $\mathbb{P}^1_k$. Choose $\mathcal{R}$ as in Lemma 1.4 and consider the pair $(Q, \mathcal{R})$, which is defined over $k$. Geometrically, there exists a degree 2 cover $X$ of $Q$ branched in $\mathcal{R}$, which has reduced geometric automorphism group $C_2$. Note that a priori $X$ and the cover are defined over an extension of $k$.

Regardless, the field of moduli of $X$ with respect to the extension $K|k$ equals $k$. Indeed, if we choose some $K$-isomorphism $\gamma : Q \to \mathbb{P}^1_K$, then we have

$$(\gamma(\mathcal{R}))^\sigma = \gamma^\sigma(\mathcal{R}) = \gamma(\mathcal{R})$$

for $\sigma \in \Gamma$. This shows that $\gamma(\mathcal{R})$, which is the branch locus of $X$, and $(\gamma(\mathcal{R}))^\sigma$, which is the branch locus of $X^\sigma$, differ by the automorphism $\gamma^\sigma \gamma^{-1}$ of $\mathbb{P}^1_K$. Hence $X^\sigma$ is $K$-isomorphic to $X$ since two hyperelliptic curves with isomorphic branch loci are isomorphic.

By [3, Theorem 6], this implies that the genus 2 curve $X$ is hyperelliptically defined over $k$. This in turn implies the existence of an isomorphism

$$\varphi : (Q, \mathcal{R}) \longrightarrow (\mathbb{P}^1_K, \mathcal{R}_0)$$

over some Galois extension $M$ of $k$. Then the map $\Gal(M|k) \to \Aut_K(Q, \mathcal{R}) = \langle \alpha \rangle$ that sends $\tau$ to $\varphi^{-1}\varphi^\tau$ is a homomorphism because $\Aut_K(Q, \mathcal{R}) = \Aut_k(Q, \mathcal{R})$. Indeed, we have $\varphi^{-1}\varphi^\tau \varphi^{-1} = \varphi^{-1}\varphi^\tau (\varphi^{-1})^\tau \varphi^\tau = \varphi^{-1}\varphi^\tau \varphi^{-1} \varphi^\tau$.

The kernel of this homomorphism is not all of $\Gal(M|k)$ because that would imply that $Q$ is isomorphic to $\mathbb{P}^1_k$. So it cuts out a quadratic extension $L$ over which $\varphi$ is defined and such that $\varphi^\sigma = \varphi \alpha$. 

Proposition 1.6. Suppose that the reduced automorphism group $\overline{\Aut}_K^0(X/k)$ of $X/k$ contains an element $\alpha$ of order 2 defined over $k$. Then $X$ descends hyperelliptically.
Proof. Let $Q = X/\iota$ along with its branch locus $R$. If $Q \cong \mathbb{P}_k^1$ we are done. We therefore assume that this is not the case. Let $\alpha$ be its non-trivial geometric automorphism. Then $\alpha$ is defined over $k$ by uniqueness, as are $Q$ and $R$ by hypothesis. Choose $L$ and $\varphi$ as in Lemma 1.5. The divisor $\varphi(R) = R_0$ is $L$-rational, but it is even $k$-rational since $R_{\sigma} = (\varphi(R))^\sigma = \varphi^\sigma(R^\sigma) = \varphi(\alpha(R)) = \varphi(R) = R_0$.

Now the $2:1$ cover of $\mathbb{P}_k^1$ with branch locus $R_0$ is $K$-isomorphic to $X$ and admits a hyperelliptic equation over $k$. □

From this proposition, it is easy to prove the following main result.

**Theorem 1.7.** Let $X$ be a hyperelliptic curve over $K$ of genus $g$ and with reduced automorphism group $\overline{G}$ whose field of moduli for the extension $K|k$ equals $k$. Then the existence of a descent of $X$ is equivalent to the existence of a hyperelliptic descent, except possibly when $g$ is odd and $\overline{G}$ is tamely cyclic of odd cardinality.

Proof. The case of even $g$ is due to Mestre in [12], and Huggins proved the result in the case where $\overline{G}$ is not tamely cyclic in [9, Theorem 5.4]. As for the case where $\overline{G}$ is tamely cyclic of even order, we get a pair $(Q, R)$ as above whose geometric automorphism group has even cardinality. It has a unique subgroup of order 2, which is necessarily defined over $k$. Hence so is its unique generator $\alpha$, and we can use Proposition 1.6. □

**Remark 1.8.** In the remaining case where $g$ and $\#G$ are both odd, we refer the reader to Theorem 3.8 for a criterion and to Example 4.6 for an explicit counterexample.

1.2. **An arithmetic criterion for hyperelliptic descent.** Using some notions from [4], we can now characterize arithmetically when $X$ allows a hyperelliptic descent. Denote the quotient $X/G$ by $B$. By construction, $B$ has a canonical Weil descent datum. Let $B_0$ be the corresponding model over $k$; it depends only on the $K$-isomorphism class of $X$. It is well-known (cf. the discussion in [4, Section 2.9]) that the existence of a $k$-rational point on $B_0$ implies that $X$ descends.

**Theorem 1.9.** Let $X$ be a hyperelliptic curve over $K$ of genus $g$ and with geometric automorphism group $G$ whose field of moduli for the extension $K|k$ equals $k$. Then $X$ descends hyperelliptically if and only if the canonical model $B_0$ of the quotient $B = X/G$ has a $k$-rational point.

Proof. If there exists a hyperelliptic descent, then $B_0$ has a rational point. Indeed, the curve $B_0$ can then be obtained as the quotient of $\mathbb{P}_k^1$ by the reduced automorphism group of the hyperelliptic descent. Note that this group is defined over $k$, though its elements might not be.

Conversely, if $B_0$ has a $k$-rational point, a descent $X_0$ exists. By Theorem 1.7, it then only remains to consider the case where the reduced automorphism group of $X_0$ is tamely cyclic of odd order. Let $Q_0$ be the quotient of $X_0$ by its canonical involution. We get a morphism of odd degree $Q_0 \to B_0$, whence a $k$-rational divisor $D$ of odd degree, $2n + 1$ say, on the conic $Q_0$. This implies that $Q_0$ is isomorphic with $\mathbb{P}_k^1$: if we let $\mathcal{K}$ be the canonical divisor of $Q_0$, then the $k$-rational divisor $D - n\mathcal{K}$ is linearly equivalent to a $k$-rational point on $Q_0$. □
The next proposition gives a concrete criterion for the presence of a rational point on \( B_0 \), which we will use in Section 3.

**Proposition 1.10.** Let \( L \) be a quadratic extension of \( k \), and let \( \sigma \) be the non-trivial element of \( \text{Gal}(L|k) \). Let \( \alpha \) be an element of \( \text{Aut}_k(\mathbb{P}^1_k) \) of order two, represented by an element \( M \) of \( \text{GL}_2(k) \). Then the descent of \( \mathbb{P}^1_k \) determined by the Weil cocycle obtained by inflating \( \sigma \mapsto \alpha \) is isomorphic to \( \mathbb{P}^1_k \) if and only if \(-\det(M)\) is a norm for the extension \( L|k \).

**Proof.** Since the characteristic polynomial of \( M \) is \( x^2 - \nu \) for a \( \nu \in k \), its Frobenius matrix equals \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The Weil cocycle translates as follows: we have to see whether there exists an invertible matrix \( N \) over \( L \) such that

\[
N^\sigma = \lambda N \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}.
\]

Writing out (1) and eliminating, we obtain the condition \( \lambda^\sigma \lambda = \nu \), which shows that our condition is necessary. Conversely, if such a \( \lambda \) exists, then we can take

\[
N = \begin{bmatrix} 1 & \lambda^\sigma \\ \beta & \lambda \end{bmatrix},
\]

where \( \beta \) is any generator of \( L \) over \( k \). \( \square \)

2. Invariants

Let \( X \) be a hyperelliptic curve of genus \( g \) over \( K \), defined by a homogeneous binary form \( f \) over \( K \) of degree \( 2g + 2 \), whose reduced automorphism group \( \mathcal{G} \) is tamely cyclic of order \( n \geq 2 \). In this section we will construct invariants of \( f \) that can be used to determine the descent obstruction for \( X \). To this end, we first construct geometric normal forms for \( f \). Modulo a normalization that we do not make, the discussion at the beginning of this section is analogous to that in \([6, \text{Section 2}]\).

Simultaneously diagonalizing the elements of \( \mathcal{G} \) by using the tameness hypothesis, we may suppose that \( \mathcal{G} = \langle \alpha \rangle \), where \( \alpha : \mathbb{P}^1_K \to \mathbb{P}^1_K \) is given by \( t \mapsto \zeta_n t \) for a primitive \( n \)-th root of unity \( \zeta_n \). Since the elements of \( \mathcal{G} \) then act freely outside of their fixed points \( 0 \) and \( \infty \) and the form \( f \) is of even degree without repeated roots, \( f \) has one of the following normal forms: either

\[
f = a_m x^{mn} + a_{m-1} x^{(m-1)n} z^n + \ldots + a_1 x^n z^{(m-1)n} + a_0 z^{mn},
\]

or

\[
f = x (a_m x^{mn} + a_{m-1} x^{(m-1)n} z^n + \ldots + a_1 x^n z^{(m-1)n} + a_0 z^{mn}),
\]

or finally

\[
f = x z (a_m x^{mn} + a_{m-1} x^{(m-1)n} z^n + \ldots + a_1 x^n z^{(m-1)n} + a_0 z^{mn}).
\]

A calculation shows that the automorphism group of a member of the family of hyperelliptic curves determined by (2) contains the group \( C_2 \times C_n \), generated by \( (x : z : y) \mapsto (\zeta_n x : z : y) \) and \( (x : z : y) \mapsto (x : z : -y) \), while the automorphism groups of the other families contain the group \( C_{2n} \), generated by the transformation \( (x : z : y) \mapsto (\zeta_n x : z : -y) \), respectively \( (x : z : y) \mapsto (\zeta_n x : z : \zeta_{2n} y) \).

Our methods now diverge from those of \([6]\): we do not further normalize to suppose \( a_m = a_0 = 1 \) so as to avoid breaking symmetry. This is not essential to our results, but it will make it easy to construct a normal form of \( f \) over an at worst quadratic extension of \( k \), as we shall see Proposition 3.2.
For now, we first focus on the case (2). Let \( T \subset \text{GL}_2(K) \) be the subset of diagonal matrices. Define the group of matrices \( D \) by
\[
D = \langle \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rangle.
\]
Then \( D \) is an extension of \( T \) by \( \mathbb{Z}/2\mathbb{Z} \). We now have

**Proposition 2.1.** Consider two normal forms \( f, f' \) as in (2) with reduced automorphism group \( C_n \). Suppose that \( A \in \text{GL}_2(K) \) is such that \( f' \sim A.f \). Then \( A \in D \).

**Proof.** This follows immediately from the fact, proved in [9, Lemma 3.3], that the normalizer in \( \text{PGL}_2(K) \) of the group \( G \) above equals \( D \). \( \square \)

In the next two subsections, we will assume at first that \( m = 2\ell \) is even. We postpone the case where \( m \) is odd until the end of this section.

### 2.1. Arithmetic diagonal invariants.

In line with the decomposition of \( D \) as extension, we first consider the action of \( T \). On the coefficients of the normal form (2), it is given by
\[
(a_m, a_{m-1}, \ldots, a_1, a_0) \mapsto \begin{cases} 
(\lambda^{m_1} a_m, \lambda^{(m-1)_1} a_{m-1}, \ldots, \lambda^{n_1} a_1, \mu^{m_2} a_0) & \\
(\lambda^{m_1} a_m, \lambda^{(m-1)_2} a_{m-1}, \ldots, \lambda^{n_2} a_1, \mu^{m_2} a_0) &
\end{cases}
\]
Consider the following subset of a generating system of homogeneous invariants for the action of \( T \):

- **Deg. 1:** \( J_1 = a_\ell, \)
- **Deg. 2:** \( J_{2,0} = a_2 a_0, \quad J_{2,1} = a_{2\ell-1} a_1, \ldots, J_{2,\ell-1} = a_{\ell+1} a_{\ell-1}, \)
- **Deg. 3:** \( J_3 = a_{\ell+2} a_{\ell-1}^2, \)
- **Deg. 4:** \( J_4 = a_{\ell+3} a_{\ell-1}^3, \)
- **Deg. \( \ell + 1:** \( J_{\ell+1} = a_{2\ell} a_{\ell-1}^\ell. \)

These expressions, the first index of whose subscripts indicates their degree and also their weight in the corresponding projective space, are clearly homogeneous invariants under the action of \( T \). We call them the *generic arithmetic diagonal invariants* (for the given integer \( m \) and automorphism group \( C_2 \times C_n \)). We now justify their name.

**Proposition 2.2.** Suppose that \( f \) and \( f' \) in (2) are such that
\[
a_{2\ell}, a_{2\ell-1}, \ldots, a_{\ell+2}, a_{\ell-1} \neq 0
\]
and
\[
a_{2\ell}', a_{2\ell-1}', \ldots, a_{\ell+2}', a_{\ell-1}' \neq 0.
\]
If the generic arithmetic diagonal invariants \( J \) and \( J' \) of \( f \) and \( f' \) define the same point in the corresponding weighted projective space, then there exists an \( A \in T \) such that \( f' \sim A.f \).

**Proof.** Since we are in a weighted projective space, we may suppose that the homogeneous invariants are equal. Then a suitable modification by a matrix of the form \( \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} \) can be used to ensure that \( a_{\ell-1} = a_{\ell-1}' \) without affecting the equality of the tuples of invariants \( J \) and \( J' \). But from the expressions defining the homogeneous invariants, it is obvious that under the hypotheses of the proposition, a form \( f \) is uniquely determined by its invariants and \( a_{\ell-1} \). \( \square \)
Remark 2.3. The proposition does not exclude that other transformations may transform \( f' \) and \( f \) into one another. For example, consider the case where \( f = a_0 x^4 + a_1 x^2 z^2 + a_0 z^4 \) and \( f' = a_0 x^4 + a_1 x^2 z^2 + a_2 z^4 \), which can be obtained from one another by the transformation \((x : z) \mapsto (z : x)\) as well as by \((x : z) \mapsto (rx : r^{-1}z)\) for \( r = \sqrt{a_0/a_2} \).

In the non-generic case \( \text{(i.e. when one of the conditions of Proposition 2.2 is not satisfied)} \), the construction of the appropriate arithmetic diagonal invariants is slightly more complicated. To proceed in these cases, note that the set of indices of the coefficients \( a_i \) of \( f \) that are non-zero do not change under the action of \( T \), and also that \( a_0 \) and \( a_2 \) are never zero.

Given \( f \), we let \( S = \{ s_1, \ldots, s_{\#S} \} \) be a set of the coefficient indices of \( f \) such that

1. \( a_s \neq 0 \) for all \( s \in S \) and
2. if one of \( a_i, a_{2\ell-i} \) is non-zero, then either \( i \) of \( 2\ell - i \) is in \( S \).

Conversely, if given a sequence \( S \), the form \( f \) satisfies \( a_s \neq 0 \) for all \( s \in S \), then we say that \( f \) is \( S \)-admissible.

We now construct the homogeneous invariants of \( T \) that are monomials in the \( \{ a_s : s \in S \} \).

**Proposition 2.4.** Under the hypotheses above, associate to \( S \) the single-row matrix \( M_S = (s_1 - \ell, \ldots, s_{\#S} - \ell) \) over \( \mathbb{Q} \). Then the elements of \( \ker(M_S) \cap \mathbb{Q}^{\#S} \) are in one-to-one correspondence with the homogeneous invariants of \( T \) for the family of forms corresponding to \( S \) that are monomials in \( \{ a_s : s \in S \} \), by the association

\[ v \leftrightarrow \prod_{i=1}^{\#S} a_{s_i}^{v_i} \]

**Proof.** This follows from the transformation behavior of the coefficients \( a_i \), which is given in (5). \( \square \)

That together with the invariants \( J_1, J_{2,0}, \ldots, J_{2,\ell-1} \) these new diagonal invariants allow one to reconstruct the binary form \( f \) still has a pleasantly direct proof, which also shows that a finite subset suffices for this purpose and indicates how to construct such a subset.

**Proposition 2.5.** Suppose that \( f \) and \( f' \) in (2) are both \( S \)-admissible. If the invariants of \( f \) and \( f' \) from Proposition 2.4 define the same point in the corresponding weighted projective space, then there exists an \( A \in T \) such that \( f' \sim A f \).

**Proof.** We will prove the proposition in the case \( \#S > 1 \), the remaining case being easier. If necessary, we will modify \( S \), by changing the entry \( 2\ell \) to 0 or inversely (which is always possible since \( a_0 a_{2\ell} \neq 0 \)), so as to suppose that \( M_S \) does not completely consist of either only strictly positive or only strictly negative elements.

Before starting the actual proof of the proposition, we construct a \( \mathbb{Z} \)-basis of the \( \mathbb{Z} \)-module \( K_S = \ker(M_S) \cap \mathbb{Z}^S \) used in Proposition 2.4. The module \( K_S \) is torsion-free, since it is a submodule of a torsion-free \( \mathbb{Z} \)-module. Furthermore, the quotient \( \mathbb{Z}^S/K_S \) is torsion-free as well. Indeed, suppose that \( nx \in K_S \) for some \( n \in \mathbb{Z} \) and \( x \in \mathbb{Z}^S \). Then \( M_S(nx) = 0 \), so \( M_S(x) = 0 \) and \( x \in \ker(M_S) \cap \mathbb{Z}^S = K_S \). We thus have an exact sequence of finitely generated free \( \mathbb{Z} \)-modules

\[ 0 \rightarrow K_S \rightarrow \mathbb{Z}^S \rightarrow \mathbb{Z}^S/K_S \rightarrow 0. \]

Choose a basis \( \{ v_i \}_{i=1}^{\#S-1} \) of \( K_S \). Then since the sequence above is split, there exists a vector \( w \in \mathbb{Z}^S \) such that \( \mathbb{Z}^S \) has basis \( \{ v_i \}_{i=1}^{\#S-1} \cup \{ w \} \).
We will now construct an element \( v \in K_S \) such that all the entries of \( v \) are strictly positive. To accomplish this, note that not all the entries of \( M_S \) have the same sign, since this would make the corresponding form \( f \) highly singular. Therefore, given an index \( i \) of \( M_S \), we can find another index \( j \) such that \( (M_S)_i \) and \( (M_S)_j \) have the opposite sign. Canceling appropriately gives an element of \( K_S \) having non-zero strictly positive coefficients at the indices \( i \) and \( j \) only, with values \((M_S)_j\) and \((- (M_S))_j\) of the same sign at these places, which we can send to its additive inverse if necessary to get an element of \( \mathbb{N}^S \cap K_S \) that is non-trivial at the index \( i \). Summing over the index of \( M_S \) gives the requested \( v \).

This allows us to further refine the basis \( \{ v_i \}_{i=1}^{#S-1} \cup \{ w \} \) of \( \mathbb{Z}^S \) such that all its elements are in \( \mathbb{N}^S \). Indeed, consider the element \( v \) constructed in the previous paragraph and choose \( v \) such that \( \mathbb{N}v_1 = \mathbb{Q}v \cap \mathbb{N}^S \). Then \( v_1 \) also has all of its entries strictly positive. Moreover, \( v_1 \) can again be completed to a basis \( \{ v_i \}_{i=1}^{#S-1} \) of \( K_S \) because an argument as in the previous paragraph yields a split exact sequence of free \( \mathbb{Z} \)-modules

\[
0 \rightarrow \mathbb{Z}v_1 \rightarrow K_S \rightarrow K_S/\mathbb{Z}v_1 \rightarrow 0.
\]

It then only remain to add sufficiently large multiples of \( v_1 \) to the other elements of the resulting basis. This yields the requested basis \( \{ v_i \}_{i=1}^{#S-1} \) of \( K_S \), which we can augment to a basis \( \{ v_i \}_{i=1}^{#S-1} \cup \{ w \} \) of \( \mathbb{Z}^S \) as before; moreover, by adding multiples of the \( v_i \) to \( w \), we can insure that \( w \) is in \( \mathbb{N}^S \) as well.

We now start the proof of the proposition. The monomials corresponding to the basis elements \( v_i \) by 2.4 now play the role of the generic arithmetic dihedral invariants. We let \( t = \prod_{i=1}^{#S} a_{S_i}^{-w_i} \) be the monomial corresponding to \( w \). Since \( w \) is not in \( K_S \), we can use matrices of the form \( \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \) to suppose that the value of \( t \) is the same for \( f \) and \( f' \) without affecting the value of the invariants corresponding to the \( v_i \). But this time, knowing \( t \) and the value of these invariants along with the invariants \( J_1, J_2, \ldots, J_{2,\ell-1} \) determines the coefficients of the forms involved. Indeed, because \( \mathbb{Z}^S \) has basis \( \{ v_i \}_{i=1}^{#S-1} \cup \{ w \} \), we can reconstruct the non-zero coefficients \( \{ a_s : s \in S \} \). The invariants \( J_1, J_2, \ldots, J_{2,\ell-1} \) then determine the other coefficients by property (2) of \( S \).

**Corollary 2.6.** The set of binary forms with given arithmetic diagonal invariants is a rational space of dimension 1.

**Proof.** Clear from the proof of Proposition 2.5: this set is be parametrized by the monomial corresponding to the complementary vector \( w \). \( \square \)

**Example 2.7.** The generic invariants of Proposition 2.2 correspond to \( S = (2\ell, 2\ell - 1, \ldots, \ell + 2, \ell - 1) \), so \( M_S = (\ell, \ell - 1, \ldots, 2, -1) \). The resulting kernel \( K_S \) has an ordered basis consisting of the positive elements

\[
(1, 0, 0, \ldots, 0, 0, \ell), \\
(0, 1, 0, \ldots, 0, 0, \ell - 1), \\
\ldots \\
(0, 0, 0, \ldots, 0, 2, 1).
\]

corresponding to the generic invariants \( J_{\ell+1}, J_\ell, \ldots, J_3 \), respectively. The complementary element \((0, 0, \ldots, 0, 1)\) corresponds to \( a_{-1} \), which can indeed be used to parametrize the corresponding rational spaces, as we have seen in the proof of Proposition 2.2.
Let $\ell = 6$ and take $S = (12, 8, 3, 1)$. A basis for $K_S$ in $\mathbb{N}^4$ is given by 
\begin{align*}
\{(3, 0, 1, 3), (3, 1, 0, 4), (5, 0, 0, 6)\},
\end{align*}
and a complementary element $w$ is furnished by $(1, 0, 0, 1)$. This shows that for forms $f$ in (2) with $\ell = 6$ and such that
\begin{align*}
 a_{10} = a_7 = a_5 = a_2 = 0
\end{align*}
and $a_8, a_3, a_1 \neq 0$, the invariants
\begin{align*}
 J_1 &= a_6, J_{2,0} = a_{12}a_0, J_{2,1} = a_{11}a_1, J_{2,3} = a_9a_3, J_{2,4} = a_{8}a_4, \\
 J_7 &= a_{12}^3a_3a_1^3, J_8 = a_{12}^3a_4a_1^4, J_{11} = a_{12}^5a_1^6
\end{align*}
suffice, and we can use $a_{12}a_1$ to parametrize the corresponding rational spaces.

It may seem unnatural to modify invariants depending on the form of $f$, but in practice this is very useful, since the parametrization from Corollary 2.6 is crucial for our reconstruction purposes. Note that once an initial form $f$ is given, one can determine which invariants are appropriate from the vanishing behavior of its coefficients.

We note that a uniform approach to the problem is also available, by constructing the invariant algebra of the action of $D$. This can be done by writing down the invariant monomials of given weight, adding the result to the set of generator if it is not an expression in the monomials already found. This process runs out of steam at some point since the algebra of invariants is finitely generated. A script to generate this invariant algebra is available online\(^1\). For the case $m = 8$, it is generated by the expressions,
\begin{align*}
 \text{Deg.1 :} & \quad a_4, \\
 \text{Deg.2 :} & \quad a_7a_1, a_6a_2, \quad a_{5}a_{3}, a_{8}a_{0}, \\
 \text{Deg.3 :} & \quad a_{8}a_{3}a_{1}, a_{7}a_{5}a_{0}, \quad a_{7}a_{3}a_{2}, a_{6}a_{5}a_{1}, \\
 & \quad a_{8}a_{2}, a_{6}a_{0}, \quad a_{6}a_{3}, a_{2}a_{2}, \\
 \text{Deg.4 :} & \quad a_{8}a_{6}a_{1}^2, a_{7}a_{2}a_{0}, \quad a_{8}a_{2}a_{2}a_{1}, a_{7}a_{a_{3}a_{0}}, \quad a_{7}a_{3}a_{2}, a_{6}a_{3}a_{1}, \\
 & \quad a_{8}a_{2}, a_{6}a_{2}a_{0}, \quad a_{7}a_{3}, a_{2}a_{1}, \\
 \text{Deg.5 :} & \quad a_{2}a_{2}a_{1}^2, a_{2}a_{3}a_{0}, \quad a_{8}a_{3}a_{1}^3, a_{2}a_{3}a_{0}, \\
 & \quad a_{2}^2a_{2}, a_{6}a_{1}, \quad a_{2}^2a_{2}, a_{6}a_{1}, \\
 \text{Deg.6 :} & \quad a_{2}^2a_{3}a_{1}, a_{2}a_{3}a_{0}, \quad a_{8}a_{3}a_{1}, a_{7}a_{3}a_{0}, \\
 \text{Deg.7 :} & \quad a_{2}^2a_{3}a_{1}, a_{7}a_{3}a_{0}.
\end{align*}

By a result of Wehlau \cite{wehlau2007}, this process always terminates at degree $m - 1$. The non-generic invariants obtained above are expressions in these monomials. Theoretically this approach is much more satisfying, but the results get unwieldy for bigger $m$, with the number of invariants running into the hundreds for $m \geq 12$.

2.2. Arithmetic dihedral invariants. Resuming the main thread of our argument, now that we have determined the invariants for the action of the normal subgroup $T \subset D$, we can construct the invariants for $D$ itself by a symmetrization. Before starting, we need an elementary result.
Lemma 2.8. Let $n$ be a positive integers, and let $X$ be the affine space with coordinates $(s_1, \ldots, s_n, t_1, \ldots, t_n)$. Define an action of $C_2$ on $X$ by $s_i \leftrightarrow t_i$. Consider the invariants $\{s_i + t_i\}_{i=1}^n \cup \{s_i t_j + s_j t_i\}_{i,j=1}^n$ of this action. Then the orbit of a point $x \in X$ is determined by these invariants.

Proof. An elementary argument using quadratic equations shows that the subset $\{s_i + t_i\}_{i=1}^n \cup \{s_i t_j + s_j t_i\}_{i,j=1}^n$ determines $x = (s_1, \ldots, s_n, t_1, \ldots, t_n)$ up to some sequence of exchanges $s_i \leftrightarrow t_i$.

We have to show that the additional invariants suffice to tell apart a sequence of such exchanges, except when either none or all of the $s_i$ and $t_i$ are exchanged. So suppose that we have two indices $i$ and $j$ where $s_i \neq t_i$ and $s_j \neq t_j$, and we exchange $s_i$ and $t_i$ while leaving the coordinates with index $j$ fixed. Then equality of the invariants yields $s_i t_j + s_j t_i = t_i t_j + s_j s_i$, hence $(s_i - t_i)(s_j - t_j) = 0$, contradicsion.

This suffices to determine the invariants we need from our previous results. In the generic case, $J'_i$ denote the transformation of the invariant $J_i$ under the involution $a_i \mapsto a_{m-i}$ on the coefficients. Let

\begin{align*}
I_1 &= J_1, \\
I_{2,0} &= J_{2,0}, I_{2,1} = J_{2,1}, \ldots, & I_{2,\ell-1} &= J_{2,\ell-1}, \\
I_{3,1} &= J_3 + J'_3, & I_{3,2} &= J_3 J'_3, \\
&\vdots & &\vdots \\
I_{\ell+1,\ell+1,1} &= J_{\ell+1} + J'_{\ell+1}, & I_{\ell+1,\ell+1,2} &= J_{\ell+1} J'_{\ell+1}, \\
I_{3,4} &= J_3 J'_4 + J'_3 J_4, & I_{3,5} &= J_3 J'_5 + J'_3 J_5, \ldots, & I_{3,\ell+1} &= J_3 J'_{\ell+1} + J'_3 J_{\ell+1}, \\
I_{4,5} &= J_4 J'_5 + J'_4 J_5, & I_{4,6} &= J_4 J'_6 + J'_4 J_6, \ldots, & I_{4,\ell+1} &= J_4 J'_{\ell+1} + J'_4 J_{\ell+1}, \\
&\vdots & &\vdots \\
I_{\ell,\ell+1} &= J_{\ell} J'_{\ell+1} + J'_{\ell} J_{\ell+1}.
\end{align*}

These expressions are clearly homogeneous invariants under the action of $D$. This is not a dihedral group, but after [6], we call them the generic arithmetic dihedral invariants (for the given integer $m$ and automorphism group $C_2 \times C_n$). As is illustrated in Example 4.1, we can usually get by with a further subset of these generic arithmetic dihedral invariants in our calculations.

Proposition 2.9. Suppose that $f$ and $f'$ in (2) are such that

1. either $a_{2\ell}, a_{2\ell-1}, \ldots, a_{\ell+1}, a_{\ell-1} \neq 0$ or $a_{\ell+1}, a_{\ell-1}, \ldots, a_0 \neq 0$ and
2. either $a'_{2\ell}, a'_{2\ell-1}, \ldots, a'_{\ell+1}, a'_{\ell-1} \neq 0$ or $a'_{\ell+1}, a'_{\ell-1}, \ldots, a'_0 \neq 0$.

If the generic arithmetic dihedral invariants $I$ and $I'$ of $f$ and $f'$ define the same point in the corresponding weighted projective space, then there exists an $A \in D$ such that $f' \sim A f$.

Proof. Note that the conditions of the Proposition 2.9 are indeed invariant under the action of $D$. Using Proposition 2.2 and Lemma 2.8. The conclusion is now immediate from the fact that the action of the matrix $[1 0 \ 0 1]$ sends $J_i$ to $J'_i$. \qed

For the non-generic cases, the symmetrization process is similar. We content ourselves with examples for octavics and even forms of degree 12.
Example 2.10. The arithmetic dihedral invariants of a general octavic form
\[ f = a_8 x^8 + a_7 x^7 z + \ldots + a_1 x z^7 + a_0 z^8 \]
are equal to,
\[
\begin{align*}
\text{Deg.1 : } & i_1 = a_4, \\
\text{Deg.2 : } & i_2 = a_0 a_8, \quad j_2 = a_1 a_7, \quad k_2 = a_2 a_6, \quad l_2 = a_3 a_5, \\
\text{Deg.3 : } & i_3 = a_0 a_5 a_7 + a_1 a_3 a_8, \quad j_3 = a_0 a_6^2 + a_2^2 a_8, \\
& k_3 = a_1 a_5 a_6 + a_2 a_3 a_7, \quad l_3 = a_2 a_5^2 + a_3^2 a_6, \\
\text{Deg.4 : } & i_4 = a_0 a_5^2 a_6 + a_2 a_3^2 a_8, \quad j_4 = a_0 a_3 a_6 a_7 + a_1 a_2 a_5 a_8, \\
& k_4 = a_0 a_2 a_7^2 + a_1^2 a_6 a_8, \quad l_4 = a_1 a_5^3 + a_3^3 a_7, \\
& m_4 = a_1 a_3 a_6^2 + a_2^2 a_5 a_7, \\
\text{Deg.5 : } & i_5 = a_0^2 a_6 a_7^2 + a_1^2 a_2 a_8^2, \quad j_5 = a_0 a_5^4 + a_3^4 a_8, \\
& k_5 = a_0 a_3 a_7^2 + a_1^2 a_5^2 a_8, \quad l_5 = a_1^2 a_6^3 + a_2^3 a_7^2, \\
\text{Deg.6 : } & i_6 = a_0^2 a_3 a_7^3 + a_1^3 a_5 a_8^2, \\
\text{Deg.7 : } & i_7 = a_0^3 a_7^4 + a_1^4 a_8^3.
\end{align*}
\]
And since \( k[a_0, a_1, \ldots, a_8]^{\text{PGL}_2(K)} \subset k[a_0, a_1, \ldots, a_8]^{D} \), the Shioda invariants can be expressed in these invariants. For instance the degree 2 Shioda invariant is equal to
\[
\frac{1}{70} i_1^2 + 2 i_2 - \frac{1}{4} j_2 + \frac{1}{14} k_2 -\frac{1}{28} l_2,
\]
the degree 3 invariant equals
\[
\frac{9}{34300} i_1^3 + \frac{3}{35} i_1 i_2 + \frac{9}{560} i_1 j_2 - \frac{33}{13720} i_1 k_2 - \frac{27}{27400} i_1 l_2 + \frac{3}{56} i_2^3 + \frac{9}{392} j_3 - \frac{3}{784} k_3 + \frac{9}{5488} l_3,
\]
et cetera.

When we specialize these invariants to even octavics, i.e. \( a_1 = a_3 = a_5 = a_7 = 0 \), only four of these dihedral invariants are non-zero: \( i_1, i_2, k_2 \) and \( j_3 \). Note that up to some geometric isomorphism, it is always possible to find a model of this form for a hyperelliptic curve of genus 3 with automorphism group which contains \( D_4 \). This yields a rational parameterization of the stratum \( D_4 \). Furthermore, we see proves that there exist \( q^3 - 2 q^2 + 3 \mathbb{F}_q \)-isomorphism classes of hyperelliptic curves of genus 3 with automorphism group \( D_4 \) over finite fields of characteristic greater than 7, and consequently that there exist \( q^5 - q^3 + q - 2 \) other classes in the \( C_2 \) stratum. These problems had been left open in [10].

For these curves, we can conversely write \( i_1 \) as the quotient of a degree 13 polynomial in the Shioda invariants over a degree 12 similar polynomial, \( i_2 \) as the quotient of a degree 5 polynomial in the Shioda invariants and \( i_1 \) over a degree 3 similar polynomial, \( k_2 \) as a degree 2 polynomial in the degree 2 Shioda invariant and \( i_1, i_2 \) and \( j_3 \) as a degree 3 polynomial in the degree 3 Shioda invariant and \( i_1, i_2, k_2 \). These formulas are too large to be written here but they are available online.

Example 2.11. Let \( m = 12 \), and consider the forms \( f \) in (2) such that both \( a_2 = a_5 = a_7 = a_{10} = 0 \) and either \( a_{11}, a_9, a_4 \neq 0 \) or \( a_8, a_3, a_1 \neq 0 \). The symmetrization of the invariants in Example 2.11 gives the following arithmetic dihedral invariants for this family:
\[
\begin{align*}
I_1 &= a_6, \\
I_{2,0} &= a_{12} a_0, \quad I_{2,1} = a_{11} a_1, \\
I_{2,3} &= a_9 a_3, \quad I_{2,4} = a_8 a_4.
\end{align*}
\]
\[ I_{7,7,1} = a_{11}^6 a_0 + a_{12}^6 a_1, \quad I_{7,7,2} = a_{11}^6 a_1 a_{12}, \]
\[ I_{8,8,1} = a_{11}^3 a_9 a_0 + a_{12}^3 a_9 a_1, \quad I_{8,8,2} = a_{12}^3 a_9 a_3 a_0, \]
\[ I_{11,11,1} = a_4^4 a_9 a_0 + a_{12}^3 a_9 a_1, \quad I_{11,11,2} = a_{12}^4 a_9 a_9 a_1 a_0, \]
\[ I_{7,8} = a_{12}^3 a_{11} a_{34} a_0^4 + a_5^5 a_{11} a_9 a_6 a_0, \quad I_{7,11} = a_{12}^3 a_{11} a_9 a_9 a_1 a_0, \]
\[ I_{8,11} = a_{12}^3 a_{11} a_9 a_9 a_1 a_0 + a_{12}^3 a_{12} a_9 a_1 a_1 a_0. \]

It remains to briefly discuss the case where \( m = 2\ell - 1 \) is odd, which gives slightly different arithmetic dihedral invariants. In the generic case \( a_{2\ell-1}, a_{2\ell-2}, \ldots, a_{\ell+1}, a_{\ell-1} \neq 0 \), they are given by symmetrizing the following homogeneous arithmetic diagonal invariants

\[
J_{2,0} = a_{2\ell-1} a_0, \quad J_{2,1} = a_{2\ell-2} a_1, \ldots, J_{2,\ell-1} = a_{\ell} a_{\ell-1},
\]
\[ J_{4} = a_{\ell+1} a_{\ell-1}^2, \]
\[ \vdots \]
\[ J_{2\ell} = a_{2\ell-1} a_{\ell-1}^{2\ell-1}, \]

again with respect to the involution \( a_i \mapsto a_{m-i} \). As in the case of even \( m \), one can show that these invariants generically determine \( f \) up to a transformation by diagonal matrices. The non-generic cases for odd \( m \) can be dealt with as for even \( m \), with the matrix \( M_S \) now being given by \((2(s_1 - \ell) + 1, \ldots, 2(s \neq S - \ell) + 1)\).

The story for the third normal form (4) is completely analogous to that of the first. Finally, the second normal form (3) turns out to be simpler than the other. In this case, the breaking of symmetry caused by the sole factor \( z \) causes the normalizer of the family to be \( T \) instead of \( D \), so we need not symmetrize and the arithmetic diagonal invariants suffice.

3. Explicit obstruction and descent

We will now use the arithmetic dihedral invariants from Section 2 to obtain descent criteria for hyperelliptic curves, depending on their reduced automorphism group \( G \) and their genus \( g \). If the obstruction vanishes, then we also indicate how an explicit descent can be obtained. Throughout, we let \( X \) be a hyperelliptic curve over \( K \) of genus \( g \) and with reduced automorphism group \( G \) whose field of moduli for the extension \( K|k \) equals \( k \).

We again denote the canonical model over \( k \) of \( B = X/G \) by \( B_0 \). We let \( R \) be the support of the branch divisor of the quotient map \( X \to B \). As \( R \) is mapped to its conjugates under the chosen isomorphisms \( X \to X^\sigma \), it transforms to a \( k \)-rational divisor \( R_0 \) on \( B_0 \). We have a decomposition \( R = S + T \), where \( T \) is the order 2 branch divisor of the tame cyclic cover \( X/\ell \to B \), and where \( S \) is contained in the image of the branch divisor of \( X \to X/\ell \). We get a corresponding decomposition \( R_0 = S_0 + T_0 \), which is a decomposition into \( k \)-rational divisors, as follows by applying the argument for \( R \) to \( T \).

In Theorem 1.7, we proved that the existence of a descent implies the existence of a hyperelliptic descent except possibly if \( g \) is odd and \( G \cong C_{2n} \) with \( n \) odd. We divide the issue of explicit descent in four categories:
(1) If $G$ is tamely cyclic and non trivial, we can then use arithmetic dihedral invariants to give criteria for the existence of a hyperelliptic descent and calculate the descent if this obstruction vanishes. This is Section 3.1.

(2) If $g$ is odd and $G \cong \mathbb{C}_2 \times \mathbb{C}_2$ with $n$ odd, we show in Section 3.2 that the curve always descend explicitly and we give couterelexamples to hyperelliptic descent in Section 3.4.

(3) If $G$ is trivial, we give in Section 3.3 a heuristic ‘generic’ method based on the covariant method developed in [11].

(4) If $G$ is not tamely cyclic, then [9, Theorem 5.4] shows that a descent always exists; we leave the explicit descent in this case unsolved for $g > 3$.

3.1. Explicit hyperelliptic descent. In this and the following subsection, we suppose that the reduced automorphism group $\overline{G}$ is non-trivial and tamely cyclic.

**Proposition 3.1.** Let $X$ be a hyperelliptic curve over $K$ whose field of moduli for the extension $K/k$ equals $k$, represented by a binary form $f$ in one of the normal forms $(2), (3), (4)$ over $K$. Then the arithmetic dihedral invariants $I(f)$ of $f$ can be represented by a $k$-rational tuple.

**Proof.** The arithmetic dihedral invariants of $f$ and its $\Gamma$-conjugates all define the same point in a weighted projective space. We can therefore use the uniqueness of the canonical representative from [10, Section 1.4].

**Proposition 3.2.** The arithmetic diagonal invariants $J(f)$ of $f$ can be represented by a tuple defined over a quadratic extension $L = k(\sqrt{d})$ of $k$, and $f$ is equivalent to a normal form $f_L$ over $L$.

**Proof.** The first part follows from the fact that the map between the corresponding moduli space is of degree 2 by Lemma 2.8. The second then follows by using the parametrization in Proposition 2.5.

**Example 3.3.** Suppose $G \cong \mathbb{C}_2 \times \mathbb{C}_2$ and $g = 3$, so that the normal form of $f$ is given by $f = a_1x^8 + a_3x^6z^2 + a_2x^4z^4 + a_1x^2z^6 + a_0z^8$. Then the generic arithmetic dihedral invariants are symmetrizations of $J_1 = a_2$, $J_{2,0} = a_4a_0$, $J_{2,1} = a_5a_1$ and $J_3 = a_6a_2^2$. In this case, $d$ is simple to determine; it is the quadratic extension incurred by passing from $I_{3,3,1}^0 = J_3 + J_3'$ and $I_{3,3,2}^0 = J_3J_3'$ to $J_3$ itself. This extension is determined by its discriminant $d = I_{3,3,1}^2 - 4I_{3,3,2}$.

**Example 3.4.** Suppose $G \cong \mathbb{C}_2 \times \mathbb{C}_2$ and $g = 4$, so that the normal form of $f$ is given by $f = a_5x^{10} + a_4x^8z^2 + a_3x^6z^4 + a_2x^4z^6 + a_1x^2z^8 + a_0z^{10}$. Then the generic arithmetic dihedral invariants are symmetrizations of $J_{2,0} = a_6a_0$, $J_{2,1} = a_5a_1$, $J_{2,2}a_4a_2$, $J_4 = a_5a_2^2$ and $J_6 = a_6a_2^3$. Generically, $d$ is given by $I_{4,1}^2 - 4I_{4,2}$. Though in particular cases this may not define a proper extension of the ground field $k$ and we may have to use $d = I_{6,6,1}^2 - 4I_{6,6,2}$ instead.

Let $f_L$ be the partial descent from Proposition 3.2. We may identify $X$ with the curve corresponding to this binary form. As usual, the isomorphisms between $X$ and its conjugates induce a canonical descent datum on the quotient $B = X/\text{Aut}(X)$, which yields a model $B_0$ of $B$ over $k$. Assuming as we did that the existence of a descent is equivalent to the existence of a hyperelliptic descent, Theorem 1.9 shows that $X$ descends if and only if $B_0$ has a $k$-rational point. To decide whether or not this is the case, we will apply Proposition 1.10 after first studying the resulting cocycle for $B$. 
So let $\sigma$ be the generator of the Galois group of the corresponding extension. We know that $f_L^r$ has the same arithmetic dihedral invariants as $f_L$. Let $S$ be the matrix $[1 \, 0 \, 1]$. Then either $f_L^r$ is in the orbit of $f_L$ or in the orbit of $S f_L$ under the action of \( T \).

Suppose that the normal form of $f_L$ is described by (2). We then either have
\[
(a_m^\sigma, a_{m-1}^\sigma, \ldots, a_1^\sigma, a_0^\sigma) \mapsto (\lambda^m a_m, \lambda^{(m-1)} a_{m-1}, \ldots, \lambda^n a_1, \mu^m a_m)
\]
(6)
or
\[
(a_m^\sigma, a_{m-1}^\sigma, \ldots, a_1^\sigma, a_0^\sigma) \mapsto (\lambda^m a_0, \lambda^{(m-1)} a_1, \ldots, \lambda^n a_{m-1}, \mu^m a_m).
\]
(7)
In both cases the induced cocycle on $B$ is chosen for our partial descent $f_L^r$ to $L$, the same isomorphism $X/\text{Aut}(X) \cong B \rightarrow B^\sigma \cong (X/\text{Aut}(X))^\sigma$ will be induced. For those $\sigma \in \Gamma$ that leave the normal form invariant unchanged, which is certainly the case for those $\sigma$ fixing $L$, we may therefore just as well take the isomorphism sending $f_L$ to $f_L^r$ to be the identity, which implies our claim.

By the previous paragraph, we can apply the inflation-restriction exact sequence. This shows that the cocycle in $H^1(\text{Gal}(K/k), \text{PGL}_2(K))$ under consideration is the inflation of a cocycle in $H^1(\text{Gal}(L/k), \text{PGL}_2(L))$. In the case (6), this cocycle is given by

$$
\sigma \mapsto \begin{bmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{bmatrix}$$

and in the case (7) by

$$
\sigma \mapsto \begin{bmatrix} 0 & \lambda^n \\ \lambda^n & 0 \end{bmatrix}.
$$

Suppose that we are in the first case. Then by dividing by $\lambda^n$, we obtain a cocycle of the form

$$
\sigma \mapsto \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix}.
$$

The Weil cocycle condition translates into the equality $r^\sigma r = 1$, so by Hilbert 90 the cocycle (9) is a coboundary. More precisely, the descent morphism is then given by a diagonal matrix, so in fact there will then exist a descent defined by a normal form over $k$, a representative of which over the field of moduli can be reconstructed by normalizing the arithmetic diagonal invariants and using the parametrization from Corollary 2.6.

In the second case, let $r = \mu^n/\lambda^n$. We get the normalized cocycle

$$
\sigma \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

The fact that (8) indeed defines a cocycle shows that $r^\sigma = r$, so $r \in k$. In the generic case, this $r$ allows a simple expression in terms of the arithmetic dihedral invariants.

Indeed, suppose first that $m = 2 \ell$ is even. Then if $a_{\ell-1}$, $a_\ell$ and $a_{\ell+1}$ are all nonzero, the transformation formula (7) shows that

$$
r = (a_\ell^2 a_{\ell-1})/(a_{\ell+1}^2 a_\ell) = a_{\ell-1}/a_{\ell+1}^2.
$$

The demand from Proposition 1.10 that this be a norm is verified if and only if $a_{\ell+1} a_{\ell-1} = I_{2,\ell-1}$ is. An explicit calculation of a descent obstruction in this case is given in Example 4.1.
Now suppose that \( m = 2\ell - 1 \) is odd. Then \( r = (a_{\ell+1}^r a_{\ell+1})/(a_{\ell+1}^r a_{\ell}) \) is a norm, so in the generic case that \( a_{\ell} \) and \( a_{\ell+1} \) are non-zero, a hyperelliptic descent always exists. An illustration of this is given in 4.2 in Section 4. In the non-generic case, the same argument shows that \( r^{2n-1} \) is a norm for the \( i \) such that \( a_{\ell+i} \) (and hence \( a_{\ell+1-i} \)) is non-zero, and the set of exponents of \( r \) thus obtained has greatest common divisor equal to one, since otherwise the binary form we started with would have more automorphisms. Therefore we arrive at the same conclusion in complete generality.

We can reason similarly for the other normal forms as well. A slightly amusing subcase is that where \( G \cong C_{2n} \) and \( n \mid 2g + 1 \). This is the case corresponding to the normal form (3) treated at the end of Section 2, whose normalizer equals \( T \) instead of \( D \). In this case, the proof of Lemma 3.1 shows that not merely \( I(f) \) but also \( J(f) \) is represented by a \( k \)-rational tuple \( J_0 \). Again using the parametrization from Corollary 2.6, we can construct a normal form \( f_0 \) over \( k \) with homogeneous invariants \( J_0 \). The hyperelliptic curve \( X_0 \) over \( k \) corresponding to \( f_0 \) is then a descent of \( X \). One such descent is calculated in Example 4.3.

Remark 3.5. The existence of a descent can sometimes also be proved by using [1] and a signature argument as in [10, Prop.4.3] to show that \( B_0 \) has a \( k \)-rational point. That there exist a hyperelliptic descent then follows from Theorem 1.9. However, the explicit construction of \( f_0 \) uses the arithmetic diagonal invariants and the parametrization from Corollary 2.6 in an essential way.

The theorem below gives the solution to the descent problem depending on \( g \) and \( G \). That the list of cases is complete follows from the discussion of the normal forms at the beginning of Section 2. We give the generic descent obstruction to keep its statement simple.

**Theorem 3.6.** Let \( X \) be a hyperelliptic curve over \( K \) of genus \( g \) and with automorphism group \( G \) whose field of moduli for the extension \( K \mid k \) equals \( k \), represented by a binary form \( f \) over \( K \). Let \( I \) be the tuple of arithmetic dihedral invariants over \( f \) and let \( I_0 \) be the normalization of \( I \). Let \( J \) be a tuple of arithmetic triangular invariants lying over \( I_0 \), and let \( J_0 \) be the normalization of \( I_0 \), so that \( X \) has a normal form over the at most quadratic extension \( L \) of \( k \) defined by \( J_0 \). Then the generic descent obstructions are as follows.

- If \( G \cong C_{2n} \), \( n \mid 2g + 1 \), then \( X \) admits a normal model of the form (3) over \( L = k \). The curve \( X \) descends hyperelliptically.
- If \( G \cong C_{2n} \) with \( n \) odd, \( n \mid g + 1 \), and \( m/2 = (g + 1)/n \) odd, then \( X \) admits a normal model of the form (2) over the extension \( L = k(\sqrt{I_{3,3,1}^2 - 4I_{3,3,2}}) \) of \( k \). The curve \( X \) admits a descent if and only if \( I_{2, m-2} \) is a norm from \( L \), in which case there exists a hyperelliptic descent as well.
- If \( G \cong C_{2n} \) with \( n \) odd, \( n \mid g + 1 \), and \( m/2 = (g + 1)/n \) even, then \( X \) admits a normal model of the form (2) over the extension \( L = k(\sqrt{I_{3,3,1}^2 - 4I_{3,3,2}}) \) of \( k \). The curve \( X \) always admits a descent, and it descends hyperelliptically if and only if \( I_{2, m-1} \) is a norm from \( L \).
- If \( G \cong C_{2n} \), \( n \mid g \), and \( m/2 = g/n \) odd, then \( X \) admits a normal model of the form (4) over the extension \( L = k(\sqrt{I_{3,3,1}^2 - 4I_{3,3,2}}) \) of \( k \). The curve \( X \) always admits a descent, and it descends hyperelliptically if and only if \( I_{2, m-1} \) is a norm from \( L \).
\begin{itemize}
\item If \( G \cong \mathbb{C}_{2n} \) with \( n \) odd, \( n \mid g \), and \( m/2 = g/n \) even, then \( X \) admits a normal model of the form (4) over the extension \( L = k(\sqrt{I_{3,3,1}^2 - 4I_{3,3,2}}) \) of \( k \). The curve \( X \) admits a descent if and only if \( 2 \beta - 1 \) is a norm from \( L \), in which case there exists a hyperelliptic descent as well.
\item If \( G \cong \mathbb{C}_{2n} \) with \( n \) even, \( n \mid 2g \), and \( m = 2g/n \) odd, then \( X \) admits a normal model of the form (4) over the extension \( L = k(\sqrt{I_{3,3,1}^2 - 4I_{3,3,2}}) \) of \( k \). The curve \( X \) admits a descent if and only if \( 2 \beta - 1 \) is a norm from \( L \), in which case it descends hyperelliptically as well.
\item If \( G \cong \mathbb{C}_2 \times \mathbb{C}_n \) with \( n \) even, \( n \mid 2(g + 1) \), and \( m = 2(g + 1)/n \) odd, then \( X \) admits a normal model of the form (2) over the extension \( L = k(\sqrt{I_{4,4,1}^2 - 4I_{4,4,2}}) \) of \( k \). The curve \( X \) admits a descent if and only if \( 2(q - 1) \) is a norm from \( L \), in which case it descends hyperelliptically as well.
\end{itemize}

It remains to show how to construct a descent if the criterion is verified. For this, we need the following proposition.

**Proposition 3.7.** Let \( D \) be a \( k \)-rational effective divisor of degree 2 on \( \mathbb{P}^1_k \). Then there exists a tamely cyclic cover \( \mathbb{P}^1_k \to \mathbb{P}^1_k \) of degree \( n \) that is defined over \( k \) and branches in \( D \).

**Proof.** The case where \( D = [p_1] + [p_2] \) with \( p_1, p_2 \in k \) being trivial, we are left with the case \( D = [\sqrt{d}] + [-\sqrt{d}] \) where \( d \) is a nonsquare in \( k \). In that case, consider the expansion of the expression \( (x + \sqrt{d}z)^n \) as \( p + q \sqrt{d} \), with \( p, q \in k[x, z] \). Then we can take \( (x : z) \mapsto (p : q) \) as our cover. Note that \( p \) and \( q \) do not contain a common factor. Indeed, this would be a factor of \( (x + \sqrt{d}z)^n \) as well, hence it would equal \( (x + \sqrt{d}z) \). But because \( p \) and \( q \) are defined over \( k \), they would then both be divisible by \( (x^2 - dz^2) \). Hence \( (x + \sqrt{d}z)^n \) as well, which is absurd. So \( (p : q) \) does indeed define a degree \( n \) cover of \( \mathbb{P}^1_k \) over \( k \).

To see that \( (p : q) \) is (tamely) cyclic, note that by construction the equation \( p(t, 1)/q(t, 1) = -\sqrt{d} \) has \( t = -\sqrt{d} \) as an \( n \)-fold solution. Therefore \( -\sqrt{d} : 1 \) is in the branch locus of \( (p : q) \), and hence \( \sqrt{d} \) as well since \( (p : q) \) is defined over \( k \). The Riemann-Hurwitz formula excludes the possibility of other points occurring in the branch locus of \( (p : q) \), which is therefore indeed given by \( D \). \( \square \)

Now first suppose that we are in the case of normal form (2). The divisor \( S \) on \( B \) is the divisor of zeroes of \( a_m x^m + a_{m-1} x^{m-1} z + \ldots + a_1 x z + a_0 \) on \( \mathbb{P}^1_k \). We descend this divisor to \( B_0 \) by applying the matrix \( N \) from the proof of Proposition 1.10. This gives a \( k \)-rational divisor \( S_0 \) on \( B_0 \cong \mathbb{P}^1_k \). The divisor \( T \), which corresponds to \( (1 : 0) + (0 : 1) \) in our normalization, is transformed to the \( k \)-rational divisor \( T_0 = (1 : \beta) + (1 : \beta^\sigma) \).
We now construct a model over \( k \) of the quotient map \( X/\iota \to B \). For this, we apply Proposition \( 3.7 \) with \( D = T_0 \). This gives a \( k \)-rational model of the branch divisor of \( X \to X/\iota \) by pulling back \( S_0 \) through the corresponding cover. Explicitly, using the notation of the proof of Proposition \( 3.7 \), if \( S_0 \) is the zero locus of the polynomial \( s_0 \), then this model is given by \( s_0(p, q) \). A hyperelliptic descent of \( X \) is then described by \( y^2 = s_0(p, q) \).

In the case of normal form (4), the pullback of \( S \) by the quotient map \( X/\iota \to B \) is properly contained in the branch locus of \( X \to X/\iota \); we have to add the two points in the ramification locus of \( X/\iota \to B \), which are \((1 : 0)\) and \((0 : 1)\) for our normal forms. If we construct the \( k \)-rational model of the cover \( X/\iota \to B \) as for the case (2) above, then this ramification locus transforms into the ramification locus of the \( k \)-rational cover from Proposition \( 3.7 \). Using the notation of that proposition, this ramification is given by \((1 : \beta) + (1 : \beta^\sigma)\), which is \( k \)-rational. So we again get a hyperelliptic descent.

As we have mentioned above, in the case of normal form (3) a direct descent can be performed using arithmetic diagonal invariants.

### 3.2. Explicit non-hyperelliptic descent

In the case where asking for a hyperelliptic descent and a general descent is not equivalent, it turns out that one can always descend.

**Theorem 3.8.** Let \( X \) be a hyperelliptic curve over \( K \) of genus \( g \) and with automorphism group \( G \) whose field of moduli for the extension \( K \mid k \) equals \( k \). Let \( g \) be the genus of \( X \), and let \( \overline{G} \) be its (tamely cyclic) reduced automorphism group. If \( g \) and \( \# \overline{G} \) are odd, then \( X \) descends.

**Proof.** As in Subsection 3.1, we again apply 3.2 to identify \( X \) with a normal form over a quadratic extension \( L \) of \( k \) defined by passing from the arithmetic dihedral invariants to the arithmetic diagonal invariants. Once more this reduces us to the consideration of a cocycle in \( H^1(\text{Gal}(L/k), \text{PGL}_2(L)) \) on the canonical quotient \( B = X/\text{Aut}(X) \cong \mathbb{P}_L^1 \) of \( X \), which is either of the form (9) or (10). If we are in the case (9), then we are done as before. So suppose that as in (10), the cocycle is given by

\[
\sigma \mapsto \begin{bmatrix} 0 & r \\ 1 & 0 \end{bmatrix}. \tag{11}
\]

Let us first explicitly construct a conic \( Q \) over \( k \) in the isomorphism class of \( B_0 \). We take \( Q \) to be given by the equation \( x^2 + \lambda y^2 + \mu z^2 = 0 \), where \( \lambda = 1/\ell \) and where \( \mu \) is such that \( L = k(\sqrt{-\mu}) \). Consider the morphism \( \varphi \) is given by the parametrization from the point \((\sqrt{-\mu} : 0 : 1) \in Q(L) \). Then one verifies that \( \varphi^\sigma = \varphi \alpha \) for the automorphism \( \alpha : x \mapsto r/x \) of \( \mathbb{P}_L^1 \) corresponding to (11).

So we can indeed take \( Q \) as our canonical model \( B_0 \) of \( B = X/\text{Aut}(X) \) over \( k \), and \( \varphi \) as a descent morphism \( B \to B_0 \). This morphism transforms the branch divisor \( T = (1 : 0) + (0 : 1) \) of the cyclic automorphism group acting on our normal forms into a \( k \)-rational divisor \( T_0 \). Indeed, we have

\[
T_0^\sigma = (\varphi([0] + [\infty]))^\sigma = \varphi^\sigma([0] + [\infty]) = \varphi(\alpha([0] + [\infty])) = \varphi([\infty] + [0]) = T_0.
\]

This allows us to once more construct on \( B_0 = Q \) a cyclic cover ramifying over \( T_0 \) that is a \( k \)-rational model of the cyclic cover \( X/\iota \to B \) of \( \mathbb{P}^1 \) ramifying over \( T \). Indeed, let \( f : \mathbb{P}^1_X \to \mathbb{P}^1_L \) be given by \( x \mapsto x^n/\ell^{(n-1)/2} \), and let \( f_0 = \varphi f \varphi^{-1} \). One verifies that \( f = \alpha f \alpha^{-1} \), which implies that \( f_0 = \varphi f \varphi^{-1} = (\varphi f \varphi^{-1})^{-1} = f_0^{-1} \). Therefore \( f_0 \) is the requested \( k \)-rational cyclic cover ramifying over \( T_0 \).
This shows that for the normal form (2), the ramification locus of \( X \to X/\iota \) has a \( k \)-rational model on \( B_0 \), where this time \( B_0 \) is not necessarily isomorphic with \( \mathbb{P}^1_k \). In the case of normal form (4), we again have to add the ramification divisor of \( X/\iota \to B \). In this case, the assertion remains true, since the ramification divisor of the cover constructed above is \( k \)-rational. More precisely, it is given by the two points on the line \( y = 0 \). We can therefore conclude as in [10, Proposition 4.14] because \( g \) is odd.

\[ \square \]

**Remark 3.9.** The proof of Theorem 3.8 also shows how to descend explicitly. We refer to Example 4.6 for more details.

### 3.3. The trivial case.

We conclude our discussion of descent criteria by considering the case where the reduced automorphism group \( \overline{G} \) of \( X \) is trivial. Depending on whether the genus \( g \) of \( X \) is even or odd, we obtain criteria for the existence of a descent or a hyperelliptic descent as special cases of Subsection 3.1 or 3.2, but such results were known before. Indeed, in [10, Proposition 4.14] it was proved that a descent always exists if \( g \) is odd. And if \( g \) is even, we recover that the curve \( X \) descends if and only if it descends hyperelliptically [12].

Still, to get an explicit descent is actually less easy, due to the absence of arithmetic dihedral invariants. There are two ways two get around this problem.

Let us assume that the binary form \( f \) defining \( X \) admits a non-trivial covariant \( c \) that is a binary sextic or octavic whose automorphism group is again trivial. Note that given an arbitrary sextic or octavic covariant \( c \), the results from [12] and [11] allow us to test effectively whether it has these properties or not. Given a \( c \) that passes muster, these same methods then also allow us

1. to compute the field of moduli \( k \) of \( C \), which coincides with the field of moduli of \( X \);
2. to compute the obstruction to descending the curve \( C \) hyperelliptically; and finally,
3. to explicitly descend \( C \) hyperelliptically if this obstruction vanishes.

We can then determine the descent obstruction for \( X \), and an explicit descent if this obstruction vanishes, as follows. First of all, if \( C \) admits a hyperelliptic descent, then it was proved in [11, Th.2.8] that this is also the case for \( X \), and moreover, *loc. cit.* shows how to obtain a descent of \( X \) explicitly from that of \( C \).

If \( C \) does not admit a hyperelliptic descent, then this is also the case for \( X \), as otherwise computing the covariant of the descent of \( X \) would give a hyperelliptic descent of \( C \). Hence if the genus of \( X \) is even and \( C \) has no hyperelliptic descent, then \( X \) does not descend at all.

It remains to treat the case where the genus of \( X \) is odd and that \( C \) does not admit a hyperelliptic descent. Regardless of whether \( C \) descends or not (and in fact, this may not be the case if \( c \) is a binary sextic), we can still use the canonical descent datum on the quotient \( C/\text{Aut}(C) \) to obtain a conic \( Q \) which is a \( k \)-rational model for both \( C/\text{Aut}(C) \) and \( X/\text{Aut}(X) \), along with a corresponding descent morphism \( \varphi : \mathbb{P}^1 \to Q \) over \( K \). By covariance, a coboundary for the canonical isomorphisms between \( C/\text{Aut}(C) \) and its Galois conjugates is also a coboundary for the canonical isomorphisms between the conjugates of \( X/\text{Aut}(X) \). Therefore the image of the
branch locus of the morphism $X \to X/\iota = X/\text{Aut}(X)$ transforms under $\varphi$ to $k$-rational divisor on $Q$. Since the genus of $X$ is odd, this suffices to descend $X$, as in the proof of [10, Prop.4.13].

At least in characteristic 0 and genus $g \leq 2^7$, a covariant $c$ with the properties requested above exists. Indeed, if we let $f$ be a binary form defining $X$, then the covariant form $c = (f, f)_{2g-2}$ is generically a non-singular binary octavic with trivial automorphism group. This can easily be proved by using a computer algebra package to randomly produce a single example of such an $f$ and resume the proof of [11, Prop.1.8]. To prove the generic existence of such a covariant for general genus, let alone for all hyperelliptic curves with trivial reduced automorphism group, seems to be more involved.

In the unlikely event that no suitable covariant is available whatsoever, one can proceed as follows. If $X$ is defined by a binary form $f$ over a finite Galois extension $M$ of $k$, then one can construct a suitable cocycle for $B = X/\text{Aut}(X) = X/\iota$ over $M$ by using the fast methods from [11]. Whether the canonical descent $B_0$ of $B$ is pointless or not, one can calculate the corresponding descent morphism $B \to B_0$, for example by the explicit methods in [7]. If the descent obstruction is trivial, then one again obtains a descent of $X$ by ramifying over the image of the branch locus of $X \to B$ under this morphism $B \to B_0$.

3.4. **Counterexamples.** To finish this section, we will show how to obtain explicit counterexamples to descent. We first treat some classical counterexamples to hyperelliptic descent, where $K = \mathbb{C}$ and $k = \mathbb{R}$. In this case, the classification of the curves that do not descend is known. These curves were first constructed by [2], the final correct statement being given by Huggins in [8]. We can slightly improve their results.

**Proposition 3.10.** Let $Q$ be the pointless conic over $\mathbb{R}$ defined by the homogeneous equation $x^2 + y^2 + z^2 = 0$. Let $(\mathbb{P}^1_k, \mathbb{R})$ be one of the divisors over $\mathbb{C}$ defined in [8, Proposition 5.0.5]. Consider the $\mathbb{C}$-morphism $\varphi : \mathbb{P}^1_k \to Q_\mathbb{C}$ given by $(s : t) \to (i(s^2 + t^2) : s^2 - t^2 : 2st)$.

Then $R_0 = \varphi(R)$ is an $\mathbb{R}$-rational divisor on $Q$ that defines a hyperelliptic curve over $\mathbb{C}$ whose moduli are for the extension $\mathbb{C}[R]$ are in $\mathbb{R}$ but which does not descend hyperelliptically. Up to isomorphism, all examples of such curves are obtained in this way. Such a curve still descends to $\mathbb{R}$ if and only if its genus and the cardinality of its reduced automorphism group $\overline{G}$ of $X$ are both odd.

**Proof.** Let $\sigma$ be the generator of $\text{Gal}(\mathbb{C}/\mathbb{R})$. Then $R^\sigma = \alpha(R)$, where $\alpha : \mathbb{P}^1_\mathbb{R} \to \mathbb{P}^1_\mathbb{R}$ is given by $(s : t) \to (-t : s)$. But we also have $\varphi^\sigma = \varphi\alpha$. Therefore

$$R^\sigma = \varphi^\sigma(R^\sigma) = \varphi(\alpha(\alpha(R))) = \varphi(R) = R$$

and hence $R$ is indeed $\mathbb{R}$-rational. This gives our realization of the curve as the cover of a conic. All is then clear from [8, Proposition 5.0.5], with our sharpening of the result following from Theorem 3.8. □

**Remark 3.11.** A signature argument can also be used to prove Proposition 3.10, except if $G \cong C_2^m$ with $n$ odd and either $g/n$ is odd or $(g+1)/n$ is even. Our results shows that in these cases, a descent is always possible, and how it can be obtained explicitly.
Having analyzed the obstruction to descent, it is now straightforward to construct curves whose field of moduli is not a field of definition.

**Theorem 3.12.** Let $L = k(\sqrt{d_1})$ be a quadratic extension of $k$, and choose $d_2 \in k$ such that $d_2$ is not a norm from $L$. Suppose that $u \in L$ is such that $N_{L/k}(u) = 1$. Let $m = 2\ell$ be an even number, and choose $a_m, \ldots, a_0$ in $L$ such that

$$a_\sigma^\ell = ua_\ell, a_{\ell-1} = ud_2a_{\ell+1}, \ldots, a_0 = ud_2a_m^\sigma$$

for the non-trivial element $\sigma$ of $\text{Gal}(L|k)$. Consider the binary normal forms

$$f = a_mx^{mn} + a_{m-1}x^{(m-1)n}z^n + \ldots + a_1xz^{(m-1)n} + a_0z^{mn},$$

$$g = xzf.$$  

Suppose that $\text{Aut}_K(f)$ is generated by $(x, z) \mapsto (\zeta nx, z)$. Then the curves corresponding to $f$ and $g$ have field of moduli for the extension $K|k$ and do not descend hyperelliptically to $k$. This exhausts the counterexamples to hyperelliptic descent that allow a normal form over a quadratic extension of $k$. The corresponding curves $X$ descend to $k$ without descending hyperelliptically if and only if their genera and $n$ are both odd.

**Proof.** We can use the norm criterion from Proposition 1.10 over $L$. The descent matrix is given by $\begin{bmatrix} 0 & d_2 \\ 1 & 0 \end{bmatrix}$, which by construction is not a norm from $L$. The universality statement is obvious from the constructions in this section. \hfill $\square$

**Remark 3.13.**

- The argumentation in the previous sections shows that this result yields all counterexamples of descent up to isomorphism, in the sense that every curve over $K$ whose field of moduli for the extension $K|k$ equals $k$ and yet does not descend is $K$-isomorphic to one of the curves in Theorem 3.12 for some quadratic extension $L$ of $k$.

- Conversely, if $\#G$ is even, then every curve over a given quadratic extension $L$ of $k$ can be transformed to a normal form over $L$. In other words, given a quadratic extension $L$ of $k$, any curve defined over $L$ whose field of moduli for the extension $K|k$ equals $k$ and yet does not descend is $L$-isomorphic, and not merely $K$-isomorphic, to one of the curves in Theorem 3.12.

- For genus 3, explicit equations for the open condition on the automorphism group given in Theorem 3.12 were determined in [10]. For general genus, it is usually easy to verify once the coefficients are given, by using the methods of [11].

This construction yields counterexamples for many more quadratic extensions than the usual $C|\mathbb{R}$. Moreover, the cases where $d_2$ is a norm from $L$ yield a host of examples for which it is anything but obvious that the resulting curves descend. Explicit descent calculations for the curves constructed in Theorem 3.12 are given in Examples 4.4, 4.5 and 4.6.

4. Implementation and Examples

For generic hyperelliptic curves $X$ with tamely cyclic reduced automorphism group $\overline{G}$, we have implemented the previous section in Magma. The resulting algorithms are available online.

The implementation is straightforward considering the constructive methods that we used; as soon as a generator $\alpha$ for $\overline{G}$ is known, which can be determined effectively by using the methods in [11], then one can diagonalize $\alpha$ over an at most...
quadratic extension and apply our results. The remaining steps (determining and normalizing the arithmetic invariants, parametrizing to determine a partial descent, solving a norm equation and if necessary constructing the necessary cover to define $X$ over $k$) are effective and efficient for ‘natural’ fields such as number fields.

We now give some examples of these computations. To begin with, we mention that usually, we do not need the full set of generic arithmetic dihedral invariants in our computations, and our algorithms take this into account. The following example illustrates this.

**Example 4.1.** Inspired by Theorem 3.6, let $d_1 = 2, d_2 = 3$, let

\[ a_6 = 7 + \sqrt{d_1}, a_5 = 3 - 2\sqrt{d_1}, a_4 = (1 + \sqrt{d_1}), a_3 = 12\sqrt{d_1}, a_2 - d_2 \overline{a_2} = -d_2(1 - \sqrt{d_1}), \]

\[ a_1 = -d_2^2a_2 = -d_2^2(3 + 2\sqrt{d_1}), a_0 = -d_2^2a_0 = -d_2^3(7 - \sqrt{d_1}). \]

Let

\[ f = a_6x^{12} + a_5x^{10}z^2 + a_4x^8z^4 + a_3x^6z^6 + a_2x^4z^8 + a_1x^2z^{10} + a_0z^{12}. \]

The corresponding hyperelliptic curve is determined by the following subset of the arithmetic dihedral invariants:

\[ (J_1, J_{2,0}, J_{2,1}, J_{2,2}, J_{3,3,1}, J_{3,3,2}, J_{3,4}). \]

Indeed, one shows by direct calculation that $J_3 \neq J_3'$ for $f$, hence also for all its transformations. Having chosen the order of the roots $J_3$ and $J_3'$ of the corresponding quadratic equation, the linear system

\[ J_i + J'_i = I_{i,i,1} \]

\[ J'_3J_i + J_3J'_i = I_{3,i} \]

is always invertible for $i > 3$, determining $J_i$ and $J'_i$ in terms of the choice of the order of $J_3$ and $J_3'$ and the invariants above. In particular, we need only normalize the invariants $(J_1, J_{2,0}, J_{2,1}, J_{2,2}, J_{3,3,1}, J_{3,3,2}, J_{3,4})$ to determine the field of moduli of our curve. This normalization is

\[ \left( 1, -3 \cdot 47, -1, 2, -1, 1, 2^{14}, 2, 2^{13}, 2^{13}, 2^{13}, 2^{13} \right), \]

so the field of moduli is the rational field. The norm criterion now shows that the curve does not descend, as expected.

Next is an example for the normal form (4).

**Example 4.2.** Consider the genus 3 hyperelliptic curve $X$ corresponding to the binary form

\[
(20456\sqrt{5} + 43640)x^8 + (-17772\sqrt{5} - 56716)x^7z + (28984\sqrt{5} + 3584)x^6z^2 + (25862\sqrt{5} - 95522)x^5z^3 + (67320\sqrt{5} - 136740)x^4z^4 + (84995\sqrt{5} - 193217)x^3z^5 + (75097\sqrt{5} - 167611)x^2z^6 + (38764\sqrt{5} - 86676)xz^7 + (7942\sqrt{5} - 17762)z^8
\]

over $\mathbb{Q}(\sqrt{5})$. This curve has an automorphism of order 2, and it allows a normal form (4) over the ground field given by

\[
xz((11270829\sqrt{5} - 25242007)x^6 + (1408299\sqrt{5} - 5284449)x^4z^2 + (-5642070\sqrt{5} - 12929374)x^2z^4 + (-204992252\sqrt{5} - 458411532)z^6).
\]
The normalized arithmetic dihedral invariants of this form now generate the field of moduli. They are given by

\[(I_{2,0}, I_{2,1}, I_{4,4,1}, I_{4,4,2}) = \left(\frac{2}{3}, 1, \frac{29}{32}, \frac{2}{3}\right).\]

As mentioned in Subsection 3.1, the descent obstruction for this normal form is trivial because \(m = 3\) is odd. In this particular case, this reflects itself in the fact that the arithmetic diagonal invariants are rational. They are given by

\[(J_{2,0}, J_{2,1}, J_4) = \left(\frac{2}{3}, 1, 3\right),\]

and this time we get the descent

\[xz \left(3x^6 + x^4z^2 + x^2z^4 + \frac{2}{9}z^6\right).\]

Finally, we perform the descent of a binary form in normal form (3).

**Example 4.3.** Consider the genus 4 hyperelliptic curve \(X\) corresponding to the binary form

\[
\begin{align*}
& (138076\sqrt{5} + 291100)x^{10} + (-120728\sqrt{5} - 370816)x^9z \\
+&(243042\sqrt{5} + 208878)x^8z^2 + (48987\sqrt{5} - 760529)x^7z^3 \\
+&(515947\sqrt{5} - 751581)x^6z^4 + (754227\sqrt{5} - 1880505)x^5z^5 \\
+&(1243617\sqrt{5} - 2713183)x^4z^6 + (1462433\sqrt{5} - 3287139)x^3z^7 \\
+&(1243263\sqrt{5} - 2777109)x^2z^8 + (625402\sqrt{5} - 1398734)xz^9 \\
+&(124654\sqrt{5} - 278722)z^{10}.
\end{align*}
\]

over \(\mathbb{Q}(\sqrt{5})\). This curve has an automorphism of order 3, and it allows a normal form (3) over the ground field given by

\[
z((91955817\sqrt{5} - 213442907)x^9 + (268416746\sqrt{5} + 589172042)x^6z^2 \\
+(-30323641593\sqrt{5} - 67805941509)x^3z^6 + (3073332514916\sqrt{5} + 6872180416996)xz^9).
\]

As mentioned in Subsection 3.1, after normalizing, the arithmetic diagonal invariants for this case will generate the field of moduli. In this case, these invariants are up to scalar given by

\[(J_{2,0}, J_{2,1}, J_4) = \left(\frac{2}{3}, 1, \frac{8}{3^2}\right).\]

Using the parametrization of Corollary 2.6 for the generic case, we obtain the following hyperelliptic descent of \(X\):

\[z \left(8 \frac{3^2}{3^2}x^9 + x^6z^3 + x^3z^6 + \frac{3}{4}z^9\right).\]

We now discuss the case of curves of genus 3, which was our initial motivation for this paper, the cases of genus 2 having been completely resolved already in [12] and [3].

The invariant theory of binary octavics was completely determined by Shioda in [14], and can be applied to solve the descent problem for genus 3 hyperelliptic curves with great efficiency. The steps for this are as follows:
• using the stratum equations from [10], determine the geometric automorphism group \( G \) of \( X \) from its Shioda invariants;
• if \( G \cong D_4 \), then use either the parametrizations or reconstruction methods from [10] or (in the case \( G \cong C_2^3 \)) the covariant descent method from [11];
• if \( G \cong D_4 \), then determine the arithmetic dihedral invariants of \( f \), directly or from its Shioda invariants (see Example 2.10), and apply the methods of this paper.

Note that the determination of the arithmetic dihedral invariants of \( f \) from its Shioda invariants in the final step of this algorithm can be done in polynomial time, since there are explicit formulas relying the two. Such formulas can again be found online.

**Example 4.4.** Again let \( d_1 = 2, d_2 = 3 \), let
\[
\begin{align*}
  a_4 &= 7 + \sqrt{d_1}, a_3 = 3 - 2\sqrt{d_1}, a_2 = 12\sqrt{d_1}, \\
  a_1 &= -d_2a_3 = -d_2(3 + 2\sqrt{d_1}), a_0 = -d_2a_4 = -d_2^3(7 - \sqrt{d_1}).
\end{align*}
\]
Construct the binary octavic
\[
f = a_4x^8 + a_3x^6z^2 + a_2x^4z^4 + a_1x^2z^6 + a_0z^8,
\]
or one of its transformations. The normalized Shioda invariants of this octic are given by
\[
-5 \cdot 7 \cdot 401^3/(3^5 \cdot 13^2 \cdot 23^2 \cdot 1667^2),
-5 \cdot 7 \cdot 401^3/(3^5 \cdot 13^2 \cdot 23^2 \cdot 1667^2),
2^4 \cdot 5^4 \cdot 7^{13} \cdot 401^4 \cdot 343951/(3^7 \cdot 13^4 \cdot 23^4 \cdot 1667^4),
2^4 \cdot 5^4 \cdot 7^{16} \cdot 401^5 \cdot 1663 \cdot 29947/(3^7 \cdot 13^5 \cdot 23^5 \cdot 1667^5),
2^3 \cdot 5^7 \cdot 7^{18} \cdot 47 \cdot 59 \cdot 401^6 \cdot 3271 \cdot 14653/(3^{11} \cdot 13^6 \cdot 23^6 \cdot 1667^6),
2^3 \cdot 5^7 \cdot 7^{22} \cdot 401^7 \cdot 166150639393/(3^{11} \cdot 13^7 \cdot 23^7 \cdot 1667^7),
-2^2 \cdot 5^7 \cdot 7^{25} \cdot 401^8 \cdot 25309 \cdot 148913 \cdot 395201/(3^{13} \cdot 13^8 \cdot 23^8 \cdot 1667^8),
2^6 \cdot 5^8 \cdot 7^{27} \cdot 17 \cdot 401^9 \cdot 4278649 \cdot 127546933/(3^{15} \cdot 13^9 \cdot 23^9 \cdot 1667^9),
-2^2 \cdot 5^8 \cdot 7^{30} \cdot 11 \cdot 61 \cdot 401^{10} \cdot 537787278082528849/(3^{17} \cdot 13^{10} \cdot 23^{10} \cdot 1667^{10}).
\]
This gives the normalized arithmetic dihedral invariants
\[
(I_1, I_{2,0}, I_{2,1}, I_{3,3,1}, I_{3,3,2}) = \left(1, \frac{-47}{25}, \frac{-1}{25}, \frac{101}{26}, \frac{-47}{215}, \frac{-768}{215}, \frac{-768}{215}\right),
\]
which are somewhat simpler. We have \( I_{3,3,1}^2 - 4I_{3,3,2} = 11^213^2/2^{13}3^2 \). This defines the quadratic extension \( \mathbb{Q}(\sqrt{2}) \) of the rational field. The invariant \( I_{2,1} \) is not a norm from this extension, so we see that no hyperelliptic descent exists, whence the non-existence of any descent at all.

We can still use the normalized arithmetic diagonal invariants to get a descent over the quadratic extension defined by these invariants. Up to switching \( J_3 \) and \( J'_3 \) we have
\[
(J_1, J_{2,0}, J_{2,1}, J_3) = \left(1, \frac{-143\sqrt{2} + 202}{768}, \frac{-47}{32}, \frac{-1}{96}\right),
\]
which, using the Corollary 2.6 in the generic case where the parameter is \( a_1 \), yields the partial hyperelliptic descent
\[
(-143/768\sqrt{2} + 101/384)x^8 - 1/96z^8 + z^2 + x^2z^4 + x^2z^4 + x^2z^4 + (1716\sqrt{2} + 2424)z^6
\]
4.5. Modifying $d_1 = 3, d_2 = 13$ in Example 4.4 so that the obstruction vanishes, we do get a descent to the rationals. Explicitly, this is constructed as follows. The descent matrix on the canonical quotient $B = X/\text{Aut}(X)$ is given by
\[
\begin{pmatrix}
0 & 144/13 \\
1 & 0
\end{pmatrix}
\]
We apply Proposition 1.10, taking $\lambda = (-60 - 24\sqrt{3})/13$ and $\beta = 1/\sqrt{3}$ in the proof. Transforming the quotient $B = X/G \cong \mathbb{P}^1_K$ by the corresponding $N$ from Proposition 1.10, the ramification divisor of $X/\iota \to B$ is mapped from the zero locus of
\[
1/5184(10309\sqrt{3} + 17745)x^4 + 13/144x^3z + x^2z^2 + xz^3 + (-244\sqrt{3} + 420)z^4
\]
into that of
\[
38x^4 + 320x^3z + 657x^2z^2 + 924xz^3 + 387z^4
\]
on the canonical model $B_0 = \mathbb{P}^1_{\mathbb{Q}}$ of $B$. The ramification locus of $X/\iota \to B$ transforms to $\{(\sqrt{3}:1),(-\sqrt{3}:1)\}$. Pulling the equation back by using the corresponding cover
\[(x:z) \mapsto (x^2 + 3z^2 : 2xz)
\]
from Proposition 3.7 and multiplying by a scalar gives the hyperelliptic descent
\[
y^2 = 19x^8 + 320x^7z + 1542x^6z^2 + 6576x^5z^3 + 12006x^4z^4 + 19728x^3z^5 + 13878x^2z^6 + 8640xz^7 + 1539z^8.
\]
4.6. Finally, by modifying Example 4.4 to
\[
f = a_4x^{12} + a_3x^9z^3 + a_2x^6z^6 + a_1x^3z^9 + a_0z^{12}
\]
we get a curve that does not descend hyperelliptically but which does descend as the cover of a conic. Our algorithms return a divisor on the conic $X^2 - 2Y^2 + 96Z^2 = 0$ over which we have to branch. The result, whose expression is slightly unwieldy, can be found online too.

5. Conclusions and remaining questions

In [10] and [11], effective parametrizations of the automorphism strata in genus 3 were determined, which return a model over the field of moduli as long as the reduced automorphism group is not $C_2$. These methods also return an equation for the curves with $C_2$, the problem being that this equation is determined over an extension of the field of moduli. The present work shows if such an equation can be hyperelliptically descended to a quadratic extension of the field of moduli, and if possible, how to determine a descent over the field of moduli itself.

This concludes our explicit arithmetic exploration of the moduli space of hyperelliptic genus 3 curves, at least when the characteristic of the ground field is 0 or bigger than 7. Given any tuple of Shioda invariants of a genus 3 curve, one can determine
\begin{itemize}
\item the automorphism group of the curve,
\item whether or not the curve descends to the field of moduli, and
\item a model of the curve over its field of moduli, if it exists.
\end{itemize}
When the characteristic of the ground field is positive and less than or equal to 7, a non-trivial effort is already needed to find the appropriate analogues of the Shioda invariants.

Turning to genus bigger than 3, our algorithms do not completely solve the descent problem for hyperelliptic curves. It gives methods when the field of moduli is not necessarily a field of definition, but this does not exhaust all aspects of the problem. For example, it remains to descend effectively if the reduced automorphism group is not tamely cyclic or is trivial, and to consider the case where the characteristic of $k$ equals 2. Table 1 gathers our state of knowledge (we emphasize what is proved in the present paper).

| $G$                     | Condition  | Descent $\Leftrightarrow$ Hyper. Descent | Obstruction to Descent | Effective Method |
|-------------------------|------------|------------------------------------------|------------------------|------------------|
| Not tamely cyclic       | -          | Yes ($\text{see [8]}$)                   | No ($\text{see [8]}$)  | ?                |
| Tamely cyclic and non-trivial | $g$ odd and $\#G > 1$ odd | No                                             | No                     | Yes              |
|                         | $g$ even or $\#G$ even | Yes                                             | Yes ($\text{computable}$) | Yes              |
| Trivial                 | $g$ odd    | No ($\text{see [10]}$)                   | No ($\text{see [10]}$) | Generically? ($\text{Yes if } g \leq 2^7$) |
|                         | $g$ even   | Yes ($\text{see [10]}$)                   | Yes ($\text{see [10]}$) | Generically? ($\text{Yes if } g \leq 2^7$) |

Table 1. Issues addressed in the present paper.

References

[1] R. Brandt and H. Stichtenoth. Die Automorphismengruppen hyperelliptischer Kurven. *Manuscripta Math.*, 55(1):83–92, 1986.
[2] E. Bujalance and P. Turbek. Asymmetric and pseudo-symmetric hyperelliptic surfaces. *Manuscripta Math.*, 108(1):1–11, 2002.
[3] G. Cardona and J. Quer. Field of moduli and field of definition for curves of genus 2. *Lecture Notes Ser. Comput.*, 13:71–83, 2005.
[4] P. Dèbes and J.-C. Douai. Algebraic covers: field of moduli versus field of definition. *Annales Sci. E.N.S.*, 30:303–338, 1997.
[5] C. Earle. On the moduli of closed Riemann surfaces with symmetry. *Ann. of Math. Studies*, 66:119–130, 1971.
[6] J. Gutierrez and T. Shaska. Hyperelliptic curves with extra involutions. *LMS J. Comput. Math.*, 8:102–115, 2005.
[7] R. A. Hidalgo and S. Reyes. A constructive proof of Weil’s Galois descent theorem. Preprint at [http://arxiv.org/abs/1203.6294](http://arxiv.org/abs/1203.6294).
[8] B. Huggins. *Fields of moduli and fields of definition of curves*. PhD thesis, University of California, Berkeley, California, 2005. [http://arxiv.org/abs/math.NT/0610247](http://arxiv.org/abs/math.NT/0610247).
[9] B. Huggins. Fields of moduli of hyperelliptic curves. *Math. Res. Lett.*, 14(2):249–262, 2007.
[10] R. Lercier and C. Ritzenthaler. Hyperelliptic curves and their invariants: geometric, arithmetic and algorithmic aspects. *Journal of Algebra*, 372:595–536, Dec. 2012.
[11] R. Lercier, C. Ritzenthaler, and J. Sijsling. Fast computation of isomorphisms of hyperelliptic curves and explicit descent. To appear in the proceedings of ANTS X.
[12] J.-F. Mestre. Construction de courbes de genre 2 à partir de leurs modules. In *Effective methods in algebraic geometry*, volume 94 of *Prog. Math.*, pages 313–334, Boston, 1991. Birkhäuser.
[13] G. Shimura. On the field of rationality for an abelian variety. *Nagoya Math. J.*, 45:167–178, 1971.
[14] T. Shioda. On the graded ring of invariants of binary octavics. *American J. of Math.*, 89(4):1022–1046, 1967.
[15] D. L. Wehlau. Constructive invariant theory for tori. *Ann. Inst. Fourier (Grenoble)*, 43(4):1055–1066, 1993.
[16] A. Weil. The field of definition of a variety. *American Journal of Mathematics*, 78:509–524, 1956.

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