QUANTUM FLUCTUATIONS OF SCALAR FIELD IN CONICAL SPACE

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Abstract

We consider vacuum polarization effect of a conformally coupled massless scalar field in the background produced by an idealized straight cosmic string. Using previous criterion we show the calculation of back reaction of the field to the metric in the context of semiclassical gravity theory is not valid in some regions due to large quantum fluctuations in the conical space.

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I. INTRODUCTION

What we are going to consider is a scalar field in the neighborhood of a straight, infinitely long static cosmic string, taken as an example to see how large the deviation from the results expected from the semiclassical gravity theory will be.

A cosmic string is one of several possible forms of topological defects formed during the phase transition in the early universe. There have been considerable interests in those objects especially due to the possibility that strings can serve as the seeds for galaxy formation (see [23, 24] for a full review).

However, a cosmic string is interesting in itself in that the spacetime outside a static infinite straight string is locally flat but globally conical with a deficit angle related to the linear mean density of the string [25]. Due to the boundary (Casimir) effect [4], even the stress tensor of a free scalar field will not vanish in this spacetime. Helliwell and Konkowski [14] (see also [19, 6, 7, 22]) first calculated the effect of vacuum fluctuations of a conformal scalar field outside a straight string. The calculated stress tensor is traceless, falls off as the fourth power of the distance from the string, and is proportional to the linear mean density of the string in the limit of small linear mean density. Most important of all, the energy density of the conformal scalar field is negative, which is another example of the existence of negative energy density in quantum field theory [8, 18]. Hiscock [15] then calculated the stress tensor due to Casimir effect in the conical space in the vicinity of an infinitely long straight cosmic string, then used the semiclassical gravity theory to determine the back-reaction to the background metric itself. The calculations are very similar to the calculations of quantum fields in the wedge formed by two perfectly conducting plates [4]. He found out there may be a repulsive gravitational force after the inclusion of the back-reaction. However, this approach is questionable since the validity of the semiclassical theory is not well founded for negative energy density cases. Here we are going to examine the validity of the correction to the background metric calculated from semiclassical theory for some range of the physical parameters. In order to get a quantitative measure of the deviation from the semiclassical gravity theory, we here study the gravitational radiation by quantum systems in [9]. We compare the predictions from the full quantum theory and the semiclassical theory in a linearized theory of gravity. We will examine the coherent states, which can be considered “the most classical” quantum states. This will give us some guidance in defining a numerical measure of the deviation since we know in the case of coherent states, this numerical measure should predict no violation of the semiclassical theory.

We are stimulated by the discussion in [9] to propose a numerical measure for the applicability (or non-applicability) of semiclassical gravity theory in various circumstances. We take the absolute value of the difference $\langle T_{\alpha\beta}(x_1) T_{\mu\nu}(x_2) \rangle - \langle T_{\alpha\beta}(x_1) \rangle \langle T_{\mu\nu}(x_2) \rangle$ first and then divide it by $\langle T_{\alpha\beta}(x_1) T_{\mu\nu}(x_2) \rangle$ to form a dimensionless quantity. The reason we choose the denominator to be $\langle T_{\alpha\beta}(x_1) T_{\mu\nu}(x_2) \rangle$ and not $\langle T_{\alpha\beta}(x_1) \rangle \langle T_{\mu\nu}(x_2) \rangle$ is to avoid artificial blowup when $\langle T_{\alpha\beta}(x) \rangle$ vanishes for some physical range of parameters.

We propose that the extent to which the semiclassical approximation is violated can be measured by the dimensionless quantity [16, 17]

$$\Delta_{\alpha\beta\mu\nu}(x, y) \equiv \left| \frac{\langle T_{\alpha\beta}(x) T_{\mu\nu}(y) \rangle - \langle T_{\alpha\beta}(x) \rangle \langle T_{\mu\nu}(y) \rangle}{\langle T_{\alpha\beta}(x) T_{\mu\nu}(y) \rangle} \right|. \quad (1.1)$$
This quantity is a dimensionless measure of the stress tensor fluctuations. (Note that it is not a tensor, but rather the ratio of tensor components.) If its components are always small compared to unity, then these fluctuations are small and we expect the semiclassical theory to hold. However, the numerous components and the dependence upon two spacetime points make this a rather cumbersome object to study. For simplicity, we will concentrate upon the coincidence limit, \( x \to y \), of the purely temporal component of the above quantity, that is

\[
\Delta(x) \equiv \left| \frac{\langle T_{00}^2(x) \rangle - \langle T_{00}(x) \rangle^2}{\langle T_{00}^2(x) \rangle} \right|.
\]  

(1.2)

The local energy density fluctuations are small when \( \Delta \ll 1 \), which we take to be a measure of the validity of the semiclassical theory. Note that we have used normal ordering with respect to the Minkowski vacuum state to define the various operators.

To derive the form of the metric of the spacetime around an idealized straight string, we start with the Nambu action for an infinitely thin relativistic line [21]

\[
S = -\mu \int dA = -\mu \int d^2\sigma \sqrt{-\gamma},
\]

(1.3)

where \( \mu \) is the linear density of the string and

\[
\gamma_{ab} = \partial_a x^\mu(\sigma) \partial_b x^\nu(\sigma) g_{\mu\nu}(x(\sigma))
\]

(1.4)

is the metric on the world sheet of the string embedded in the background spacetime with metric \( g_{\mu\nu}(x(\sigma)) \). Normally the magnitude of the linear density or equally the tension depends on the energy scale of the symmetry breaking which is responsible for the creation of the string. For a symmetry breaking at the grand unification scale, we have \( \mu \approx (10^{16} \text{GeV})^2 \approx 10^{22} \text{g cm}^{-1} \). The stress tensor of the string can be readily obtained by variation of the Nambu action with respect to the metric,

\[
T_{\mu\nu}(x) = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{g^{\mu\nu} = \eta^{\mu\nu}}.
\]

(1.5)

We can choose the coordinates \((\tau, \sigma)\) on the world sheet of the cosmic string such that

\[
\dot{x} \cdot x' = 0
\]

(1.6)

\[
\dot{x}^2 + x'^2 = 0,
\]

(1.7)

where \( x^i \) are the trajectories of the string. In the Lorentz-Hilbert gauge, the linearized Einstein equations can be solved. The 00 component of \( h_{\mu\nu} \) \( (g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}) \) is just twice the classical Newtonian potential. For a straight string on the \( z \)-axis, the only non-zero components of the stress tensor are \( T_{00} = -T_{33} = \mu \delta^2(x) \). From this, it is readily obtained that the classical Newtonian gravitational potential vanishes in this spacetime [25]. Assuming some cutoff distance (presumably the symmetry breaking scale when the string is created) of the Nambu action and taking terms up to second order in \( G_N \mu \), the metric of the spacetime around an idealized infinite straight string is solved to be

\[
ds^2 = -dt^2 + dr^2 + (1 - 4 G_N \mu)^2 r^2 d\varphi^2 + dz^2.
\]

(1.8)

If we make the substitution \( \theta = |1 - 4 G_N \mu| \varphi \), the metric can be recast into Minkowskian form. But the periodicity in the angular coordinate is changed into \( \alpha \equiv 2\pi|1 - 4 G_N \mu| \). In short, it is a conical space with deficit angle \( 8\pi G_N \mu \).
II. CONFORMAL SCALAR FIELD IN A CONICAL SPACE

In this section we are going to talk about the quantum field theory of a massless scalar field in the conical space around a long straight cosmic string, and the back-reaction of the scalar field to the conical space due to vacuum polarization. The stress tensor we use here will be the so-called “new improved energy-momentum tensor” in order to compare with other works. We will see that the change of the form of the stress tensor will not change our former conclusion concerning the relation between the negativness of the energy density and the violation of the semiclassical theory drastically. In the calculation, we use Feynman’s propagator instead of the Hadamard elementary function to illustrate the independence of the choice of the Green’s function in obtaining a local form of the stress tensor and the fluctuations. Usually the form of the stress tensor can be reduced to a form containing only a single undetermined function of the linear mean density by using axial symmetry, Lorentz symmetry along the z-axis, conformal scale invariance, tracelessness and conservation of energy-momentum [5,10,11,15]. However, we do not have this luxury here since the quantum fluctuations will not respect the original symmetry.

The dynamical equation of a massless scalar field in this metric is

\[ \Box \phi \equiv \left[ -\frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] \phi = 0 \tag{2.1} \]

The Feynman Green’s function obeys the equation

\[ \left[ -\frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right] G_F(x, x') = -g^{-1/2} \delta(x, x') \tag{2.2} \]

In the following text we will omit the subscript F in \( G_F \) for simplicity. The renormalized “new improved stress tensor” \[1,3] of the scalar field may be found from \( G(x, x') \) through

\[ \langle T_{\mu\nu} \rangle_{\text{Ren}} = -i \lim_{x' \to x} \left( \frac{2}{3} \nabla_\mu \nabla_\nu - \frac{1}{3} \nabla_\mu \nabla_\nu - \frac{1}{6} g_{\mu\nu} \nabla_\rho \nabla_\rho' \right) G_{\text{Ren}}(x, x'), \tag{2.3} \]

where \( G_{\text{Ren}}(x, x') \) is the renormalized Green’s function, which will be determined later. The Green’s function may be obtained by Schwinger’s formalism,

\[ G(x, x') = i \int_0^\infty ds \, e^{is\Box} \delta(x, x'). \tag{2.4} \]

We then expand the \( \delta \) function as a sum and integral of the mode functions, which is the completeness condition,

\[ \delta(x, x') = \frac{i}{\alpha} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp \sum_{n=-\infty}^{\infty} u(x, \omega, k, p, n) u^*(x', \omega, k, p, n). \tag{2.5} \]

Here

\[ u(x, \omega, k, p, n) = \left( \frac{p}{\alpha} \right)^{1/2} J_{|2\pi n/\alpha|}(pr) e^{i(kz-\omega t)} e^{i(2n\pi\theta/\alpha)} \tag{2.6} \]

are the eigenfunctions of the differential operator
\[ \Box \equiv -\frac{\partial^2}{\partial t^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{2.7} \]

and \( J \) is the Bessel function of the first kind, \( n \) is an integer, \( \omega \) and \( k \) are arbitrary real numbers, and \( p \) is real and positive. Replace this into the Schwinger’s proper time representation of Green’s function,

\[
G(x, x') = i \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} ds \ e^{i(\omega^2 - k^2 - p^2)} e^{-i\omega(t-t')} e^{ik(z-z')} \times 
\sum_{n=-\infty}^{\infty} J_{2n\pi/\alpha}(pr) J_{2n\pi/\alpha}(pr') e^{i(2n\pi/\alpha)(\theta-\theta')} . \tag{2.8} \]

In contrast, the Minkowski Green’s function is

\[
G_0(x, x') = \frac{i}{4\pi^2 (x - x')^2} = \frac{i}{4\pi^2 \left[ -(t-t')^2 + r^2 + r'^2 + (z - z')^2 + 2rr' \cos(t-t') \right]} . \tag{2.9} \]

Since the processes of taking coincidence limits in the \( t, r \) and \( z \) directions commute with the operation of the stress tensor operator, we have the essential quantity

\[
G(\theta, \theta') \equiv \lim_{(t',r',z') \to (t,r,z)} G(x, x') = \frac{i}{4\alpha^2 r^2} \csc^2 \left( \frac{\pi(\theta - \theta')}{\alpha} \right). \tag{2.10} \]

Taking \( \alpha \to 2\pi \) we have the Minkowskian correspondence (beware of a mistake in the coefficient in [14])

\[
G_0(\theta, \theta') \equiv \lim_{(t',r',z') \to (t,r,z)} G_0(x, x') = \frac{i}{16\pi^2 r^2} \csc^2 \left( \frac{\theta - \theta'}{2} \right). \tag{2.11} \]

Thus the renormalized Green’s function should be

\[
G_{\text{Ren}}(\theta, \theta') = G(\theta, \theta') - G_0(\theta, \theta'). \tag{2.12} \]

Due to the form of (2.8) and the dependence of the Green’s function on the geodesic distance, all the derivatives needed for calculating the stress tensor can be obtained from the derivative with respect to \( \theta \). They are

\[
\lim_{x' \to x} \partial^2_{x} G(x, x') = - \lim_{x' \to x} \partial_t \partial_t G(x, x') = \lim_{x' \to x} \partial_r \partial_r G(x, x') = - \lim_{x' \to x} \partial_{z} \partial_{z} G(x, x') = \frac{1}{3r^2} \lim_{\theta' \to \theta} (1 + \partial^2_{\theta}) G(\theta, \theta') , \tag{2.13} \]

and

\[
\lim_{x' \to x} \partial_r \partial_r G(x, x') = \frac{1}{3r^2} \lim_{\theta' \to \theta} (4 + \partial^2_{\theta}) G(\theta, \theta') . \tag{2.14} \]

Using
The Nambu action should be valid up to the symmetry breaking scale. Assuming the order of the grand unification scale for physical regions (i.e., where the Nambu action holds) of the parameter $\mu$, we can get the renormalized energy density

$$\lim_{\theta' \to \theta} \frac{\partial^2}{\partial \theta^2} G_{\text{Ren}}(\theta, \theta') = \frac{i}{480 \pi^2 r^4} \left(1 - 4 G_N \mu\right)^{-4} - 1,$$

(2.15)

we can get the renormalized energy density

$$\langle T_{00} \rangle = -i \lim_{x' \to x} \left( \frac{2}{3} \partial_t \partial_{t'} - \frac{1}{3} \partial_{t'} \partial_{t'} - \frac{1}{6} g_{00} \nabla \rho \nabla \theta' \right) G_{\text{Ren}}(x, x')$$

$$= -i \lim_{\theta' \to \theta} \left[ -\frac{1}{3r^2} \left(1 + \frac{\partial^2}{\partial \theta^2}\right) \right] G_{\text{Ren}}(\theta, \theta')$$

$$= - \frac{1}{1440 \pi^2 r^4} \left(1 - 4 G_N \mu\right)^{-4} - 1.$$

(2.16)

The stress tensor can be shown to be

$$\langle T_{\mu\nu} \rangle = \frac{1}{1440 \pi^2 r^4} \left(1 - 4 G_N \mu\right)^{-4} - 1 \text{ diag}[1, 1, -3r^2, 1].$$

(2.17)

It is worthwhile to notice that due to the boundary effect, the energy density is negative for physical regions (i.e., where the Nambu action holds) of the parameter $\mu$. Reinserting $h$ and $c$, we have

$$\rho = \langle T_{00} \rangle = - \frac{\hbar}{1440 \pi^2 r^4 c} \left(1 - 4 G_N \mu \right)^{-4} - 1 \approx -10^{-4} G_N \mu \hbar \frac{c}{r^4}.$$ 

(2.18)

We know that the linear mean density of the cosmic string produced at the grand unified scale gives $G_N \mu \approx 10^{-6}$. The numerical value is $\rho \approx -\frac{10^{-47} \text{cm} g}{r^4}$. And the assumption of the Nambu action should be valid up to the symmetry breaking scale. Assuming $r$ can be of the order of the grand unification scale $L \approx 10^{-30} \text{ cm}$, the upper limit of the value of the energy density is about $-10^{73} \text{g cm}^{-3}$, a pretty high density. Previously we have shown that for quantum states with negative energy density, the semiclassical gravity theory may not be trusted. The amount of deviation from the semiclassical theory for a scalar field in the vicinity of a cosmic string should be examined carefully to see if the semiclassical theory can be applied.

The expectation value of the square of energy density becomes [16][17]

$$\langle T_{00}^2(x) \rangle_{\text{Ren}} = \lim_{x_1, x_2, x_3, x_4 \to x} \left( \frac{2}{3} \partial_{t_1} \partial_{t_2} - \frac{1}{3} \partial_{t_3} \partial_{t_4} - \frac{1}{6} g_{\mu\nu} \nabla_{\rho} \nabla_{\rho'} \right)$$

$$\left( \frac{2}{3} \partial_{t_3} \partial_{t_4} - \frac{1}{3} \partial_{t_3} \partial_{t_4} - \frac{1}{6} g_{\mu\nu} \nabla_{\sigma} \nabla_{\sigma'} \right)$$

$$\left[ G_{\text{Ren}}(x_1, x_2) G_{\text{Ren}}(x_3, x_4) + G_{\text{Ren}}(x_1, x_3) G_{\text{Ren}}(x_2, x_4) + G_{\text{Ren}}(x_1, x_4) G_{\text{Ren}}(x_2, x_3) \right].$$

(2.19)

The result is

$$\langle T_{00}^2(x) \rangle_{\text{Ren}} = \langle T_{00}(x) \rangle^2_{\text{Ren}} + \frac{G_N^2 \mu^2 (1 - 2 G_N \mu)^2}{97200 (1 - 4 G_N \mu)^8 \pi^4 r^8}$$

$$\times \left( -2541 + 35748 G_N \mu - 199432 G_N^2 \mu^2 
+ 511744 G_N^3 \mu^3 - 511744 G_N^4 \mu^4 \right).$$

(2.20)
The second term in the equation is the fluctuating part. Due to the smallness of the value of the factor $G_N\mu$, the second term can be approximated by terms proportional to $G_N^2\mu^2$, of the same order of $\langle T_{00}(x)\rangle_{\text{Ren}}$. So even though it is inversely proportional to the eighth power of the distance from the string, it is not negligible in the semiclassical description. From the formula for $\Delta$, we have

$$\Delta = \frac{2541 - 35748G_N\mu + 199432(G_N\mu)^2 - 511744(G_N\mu)^3 + 511744(G_N\mu)^4}{2529 - 35652G_N\mu + 199048(G_N\mu)^2 - 510976(G_N\mu)^3 + 510976(G_N\mu)^4}$$

(2.21)

which is independent of $r$. The degree of the negativeness of the energy density depends both on the parameter $G_N\mu$ in the theory and the distance from the string. However, the measure $\Delta$ does not depend on the distance or the symmetry breaking scale $\mu$ crucially far from Planck scale, which is similar to the calculation for the two conducting plates, where $\Delta$ is independent of the separation of the plates. Besides, since $G_N\mu$ is pretty small, $\Delta \approx 1$. That is, no matter how far away from the string, semiclassical theory should not be trusted. Even though the quantum fluctuations get smaller with increasing distance, they are still comparable to the energy density.

From the dependence of $\Delta$ to $G_N\mu$ and $r$ in Planck units ($G_N\mu \to \mu$), we can easily see the value of $\Delta$ is small and insensitive to both $r$ and $\mu$ when away from Planck scale. The measure of deviation $\Delta$ is large when the value of the energy density is negative, independent of $r$. That means the deviation from semiclassical gravity theory is significant at all distances from the string.

**III. CONCLUSION AND DISCUSSION**

From this investigation we know that the back-reaction in conical spacetime can not be calculated from the naive semiclassical gravity theory.

Using a criterion obtained before, we are able to tell the range of applicability quantitatively. It is interesting to see that the degree of violation of the semiclassical theory does not depend on the linear density of the cosmic string (or equivalently, the symmetry breaking scale) or the distance from the string critically. This result indicate the back-reaction calculation in [15] based on the semiclassical theory of gravity should not to be trusted.

This is a good example in which semiclassical gravity theory can be violated even far from Planck scale. Negative energy density may be a sign for such violation. Once we encounter negative energy density, the validity of the semiclassical theory should be checked and some other extensions describing the fluctuating nature of the stress tensor and hence the spacetime should be used.

The interesting possible indications of the present result in the structure formation mechanism is to be investigated.

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