APPROXIMABLE TRIANGULATED CATEGORIES

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Abstract. In this survey we present the relatively new concept of approximable triangulated categories. We will show that the definition is natural, that it leads to powerful new results, and that it throws new light on old, familiar objects.

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1. Introduction

In this survey there is one major, recent technical tool—we present the concept of approximable triangulated categories. And then we will sketch some results, from the last few years, showing that this tool is useful.

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The definition of approximable triangulated categories relies on the following building blocks: compact generators in triangulated categories and $t$–structures. We have an extensive background section introducing these—we recommend that beginners skip the remainder of the introduction and move on to Section 2. The introduction of an article is normally the author's attempt to persuade the expert to read on—hence it tends to assume some familiarity with the existing theory, the expert will not want to be bored with stuff she already knows, she will want to see if this article contains anything new and interesting. The introduction to an article is often more intimidating than the body of the manuscript.

In addition to Section 2 the beginners might wish to look at the three appendices, which were written to answer questions from students to whom the material was new. Especially relevant is Appendix C where we draw up a dictionary between our approach and the more standard one in the literature—in the interest of efficiency we depart from the usual way to introduce derived categories. An interested beginner, who wants to explore this further by looking elsewhere, is encouraged to consult this dictionary.

Back to the experts: we plan to discuss approximability in triangulated categories and its applications, and we begin with a heuristic explanation of what approximability is all about.

Any $t$–structure on the triangulated category can be used to define a “metric”: two objects are close to each other if they agree up to a small “difference”. More precisely: the objects $x, y \in \mathcal{T}$ are close to each other if there exists in $\mathcal{T}$ a triangle $x \rightarrow y \rightarrow z \rightarrow$ with $z \in \mathcal{T} \leq -n$ for some large $n$. We declare that, the larger the integer $n$, the closer the points $x$ and $y$. In the world of metric spaces we are accustomed to the notion of equivalent metrics, and this naturally leads to the concept of equivalent $t$–structures.

We are also accustomed to expressing points in a metric space as limits of Cauchy sequences of simpler, more accessible points. For example the Taylor series approximates a function by polynomials, and the Fourier series approximates a function by finite sums of exponentials. There is a triangulated category version, we will explain it more fully in the body of the article. For the introduction the discussion below will give the gist of the construction, albeit a little vaguely and with details missing.

Discussion 1.1. If we plan to approximate objects of the triangulated category $\mathcal{T}$, by Cauchy sequences of simpler objects, then we first need to measure what we mean by “simplicity”—returning to the analogy of the previous paragraphs, we need to declare what will be the triangulated category replacement for the polynomials forming the partial sums in a Taylor series. In doing this we will slightly modify an idea due to Bondal and Van den Bergh [8]. We will start with a compact generator $G$ for the

\[1\text{The category } \mathcal{T} \text{ isn't quite a metric space, the obvious “metric” isn't symmetric. There is a rich literature on asymmetric metrics following Lawvere [26, 27], but the emphasis there goes in entirely different directions—it's the triangle inequality in Lawvere's asymmetric metrics that is emphasized, see for example Leinster [28]. Our metrics might be asymmetric, but they are decidedly non-archimedean—hence the triangle inequality is dumb, every triangle is isosceles.}\]
triangulated category and, for each integer $n > 0$, we will define two classes of objects $\langle G \rangle_{\leq n} \subset \langle G \rangle_{\geq n}$. These will be the objects obtainable from $G$ in $n$ allowable steps—the difference between $\langle G \rangle_{\leq n}$ and $\langle G \rangle_{\geq n}$ is that for the smaller $\langle G \rangle_{\leq n}$ there are fewer operations allowed.

Returning to the analogy with Taylor series: so far we have explained what will be our replacement for the polynomials of degree $\leq n$. We have already indicated the “metric” we plan to work with, it is the one determined by whatever $t$–structure we end up choosing. So it becomes interesting to know which objects in the triangulated category have Taylor series “converging” to them. And now we come to the (somewhat imprecise) definition: the triangulated category $\mathcal{T}$ is declared to be approximable if it has coproducts, and there exist in $\mathcal{T}$

(i) a compact generator $G$

(ii) a $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$

and these $t$–structure and generator can be chosen to satisfy

(iii) For some $n > 0$ we have $G[n] \in \mathcal{T}^{\leq 0}$ and $\text{Hom}(G[-n], \mathcal{T}^{\leq 0}) = 0$.

(iv) In the metric induced by the $t$–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of (ii), every object in $\mathcal{T}^{\leq 0}$ can be expressed as the limit of a sequence whose terms belong to $\bigcup_n \langle G \rangle_{\leq n}$.

**Remark 1.2.** It is a formal consequence of the definition that an approximable triangulated category is complete with respect to the metric of Discussion 1.1(iv)—any Cauchy sequence converges. Also: by definition, if $\mathcal{T}$ is approximable then $\mathcal{T}^{\leq 0}$ is contained in the closure of $\bigcup_n \langle G \rangle_{\leq n}$ with respect to the metric. It turns out that the closure is not much larger: it is nothing other than $\mathcal{T}^\sim = \bigcup_n \mathcal{T}^{\leq n}$.

One could also wonder what the closure of $\bigcup_n \langle G \rangle_{\leq n}$ might be—we will return to this later, it turns out to be a subcategory which, for many $\mathcal{T}$, has been studied extensively in the classical literature.

Now that we have a rough idea what approximability means, the first question we might ask ourselves is “Is the theory nonempty, are there any examples?”

**Example 1.3.** It turns out there are plenty of examples. If $R$ is any ring then $D(R)$, the unbounded derived category of complexes of left $R$–modules, is an example. So is the homotopy category of spectra, and so is the category $D(R\text{-Mod})$, provided $R$ is a dga such that $H^i(R) = 0$ for all $i > 0$. Here $D(R\text{-Mod})$ stands for the derived category whose objects are all left dg $R$–modules.

All of these are easy examples. It is a nontrivial theorem that, when $X$ is a quasicompact, separated scheme, the category $D_{\text{qc}}(X)$ is approximable. Here $D_{\text{qc}}(X)$ means the derived category, whose objects are cochain complexes of $\mathcal{O}_X$–modules with quasicoherent cohomology. It is also a nontrivial theorem that, under reasonable hypotheses, the recollement of two approximable categories is approximable.
Application 1.4. We have introduced a new gadget—namely approximable triangulated categories—and mentioned that there are plenty of interesting examples out there. But the skeptical reader will want to know what use this new toy might have: are there applications? Do we learn anything new, about the familiar old categories of Example 1.3, because we now know them to be approximable?

The answer is Yes. We list below five results we were recently able to prove, using the fact that $\mathcal{D}_{qc}(X)$ is approximable.

(i) Suppose $X$ is a quasicompact, separated scheme. Then the category $\mathcal{D}_{\text{perf}}(X)$ is strongly generated, in the sense of Bondal and Van den Bergh [8], if and only if $X$ can be covered by open affine subsets $U_i = \text{Spec}(R_i)$ with each $R_i$ of finite global dimension.

(ii) Suppose $X$ is a finite-dimensional, noetherian, separated scheme, and assume further that every closed, reduced, irreducible subscheme of $X$ has a regular alteration. Then the category $\mathcal{D}_{\text{coh}}^b(X)$ is strongly generated.

(iii) Suppose $X$ is a scheme proper over a noetherian ring $R$, and let $\mathfrak{y} : \mathcal{D}_{\text{coh}}^b(X) \to \text{Hom}_R[\mathcal{D}_{\text{perf}}^\text{op}(X), R-\text{Mod}]$ be the Yoneda map. That is: $\mathfrak{y}$ is the map taking an object $B \in \mathcal{D}_{\text{coh}}^b(X)$ to the functor $\mathfrak{y}(B) = \text{Hom}(-, B)$, viewed as an $R$–linear homological functor $\mathcal{D}_{\text{perf}}^\text{op}(X) \to R-\text{Mod}$.

Then $\mathfrak{y}$ is fully faithful, and the essential image of $\mathfrak{y}$ are the finite homological functors. A functor $H : \mathcal{D}_{\text{perf}}^\text{op}(X) \to R-\text{Mod}$ is finite if, for any object $A \in \mathcal{D}_{\text{perf}}^\text{op}(X)$, the $R-$modules $H(A[i])$ are all finite and all but finitely many of them vanish.

(iv) Suppose $X$ is a finite-dimensional scheme proper over a noetherian ring $R$, and assume further that every closed, reduced, irreducible subscheme of $X$ has a regular alteration. Let $\tilde{\mathfrak{y}} : \mathcal{D}_{\text{perf}}^\text{op}(X) \to \text{Hom}_R[\mathcal{D}_{\text{coh}}^b(X), R-\text{Mod}]$ be the Yoneda map. That is: $\tilde{\mathfrak{y}}$ is the map taking an object $A \in \mathcal{D}_{\text{perf}}^\text{op}(X)$ to the functor $\tilde{\mathfrak{y}}(A) = \text{Hom}(A, -)$, viewed as an $R$–linear homological functor $\mathcal{D}_{\text{coh}}^b(X) \to R-\text{Mod}$.

Then $\tilde{\mathfrak{y}}$ is fully faithful, and the essential image of $\tilde{\mathfrak{y}}$ are the finite homological functors.

(v) Suppose $X$ is a noetherian, separated scheme. Then the categories $\mathcal{D}_{\text{perf}}^\text{op}(X)$ and $\mathcal{D}_{\text{coh}}^b(X)$ determine each other. More explicitly: there is a recipe which takes a triangulated category $\mathcal{S}$ as input, and out of it cooks up another triangulated category denoted $\mathcal{S}(\mathcal{S})$. If we apply this recipe to $\mathcal{D}_{\text{perf}}^\text{op}(X)$ what comes out is $\mathcal{D}_{\text{coh}}^b(X)$, and if we apply it to $[\mathcal{D}_{\text{coh}}^b(X)]^\text{op}$ the output is $[\mathcal{D}_{\text{perf}}^\text{op}(X)]^\text{op}$.

In the body of the article we will say more about the theorems—for example we will remind the reader what it means for a triangulated category $\mathcal{T}$ to be “strongly generated”. For now we note only that (i), (ii), (iii), (iv) and (v) above represent sharp improvements over the existing literature. More precisely

(vi) There were versions of (i) and (ii) known before approximability, but they all assumed equal characteristic—the reader can find a sample of the known results in...
Bondal and Van den Bergh [8, Theorem 3.1.4], Iyengar and Takahashi [21, Corollary 7.2], Orlov [37, Theorem 3.27] and Rouquier [39, Theorem 7.38].

(vii) The only known versions of (iii) and (iv), prior to approximability, assumed that $R$ is a field. See Bondal and Van den Bergh [8, Theorem A.1] for (iii), and Rouquier [39, Corollary 7.51(ii)] for (iv).

(viii) The only known versions of (v) prior to approximability assumed either that $X$ is affine, see Rickard [38, Theorem 6.4], or that $X$ is projective over a field, see Rouquier [39, Remark 7.50].

The definition of approximability is the assumption that there exist a $t$–structure $(T^\leq, T^\geq)$ and a compact generator $G \in \mathcal{T}$ with some properties. It becomes natural to wonder how free we are in the choice of $t$–structure and compact generator. This leads to a string of surprising results.

**Facts 1.5.** Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator $G$. Then the following can be proved.

(i) There exists a preferred equivalence class of $t$–structures in $\mathcal{T}$. Here two $t$–structures are declared equivalent if they induce equivalent metrics.

(ii) Let us choose in $\mathcal{T}$ a compact generator $G$ and a $t$–structure $(T^\leq, T^\geq)$, and assume they satisfy the conditions in Discussion 1.1 (iii) and (iv)—that is: the pair $G$ and $(T^\leq, T^\geq)$ pass the test for checking the approximability of $\mathcal{T}$.

Then it’s automatic that $(T^\leq, T^\geq)$ belongs to the preferred equivalence class of $t$–structures. Hence the metric defined by any $t$–structure $(T^\leq, T^\geq)$, which can be used to test for approximability, is unique up to equivalence.

(iii) Suppose $\mathcal{T}$ is approximable. Then any $t$–structure $(T^\leq, T^\geq)$ in the preferred equivalence class and any compact generator $G$ satisfy the conditions in Discussion 1.1 (iii) and (iv).

Thus approximability is robust; it doesn’t really matter which $t$–structure and compact generator one chooses, as long as the $t$–structure belongs to the preferred equivalence class. Furthermore the categories

$$\mathcal{T}^- = \bigcup_n T_\leq^n, \quad \mathcal{T}^+ = \bigcup_n T_\geq^{-n}, \quad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

turn out to be intrinsic. They depend only on $\mathcal{T}$, not on the particular representative $(T^\leq, T^\geq)$ in the preferred equivalence class.

Now that we know the metric is intrinsic (up to equivalence), it makes sense to return to the question raised in Remark 1.2. What is the closure of $\bigcup_n \langle G \rangle \langle -n,n \rangle$? In view of the above we should not be surprised to learn

(iv) Define the category $\mathcal{T}^-_c$ to be the closure in $\mathcal{T}$ of $\bigcup_n \langle G \rangle \langle -n,n \rangle$. This category is intrinsic, it does not depend on the choice of compact generator $G$. And it follows that the category $\mathcal{T}^b_c = \mathcal{T}^-_c \cap \mathcal{T}^b$ must also be intrinsic.
Remark 1.6. It becomes interesting to figure out what these intrinsic subcategories are in the special cases of Example 1.3. Let us confine ourselves to just one case: assume $\mathcal{T} = \mathcal{D}_{qc}(X)$ with $X$ a separated, noetherian scheme. In this special case one can prove:

(i) The standard $t$–structure is in the preferred equivalence class. Hence the categories $\mathcal{T}^-, \mathcal{T}^+$ and $\mathcal{T}^b$ have their usual meanings: that is $\mathcal{T}^- = \mathcal{D}_{qc}^-(X)$, $\mathcal{T}^+ = \mathcal{D}_{qc}^+(X)$ and $\mathcal{T}^b = \mathcal{D}_{qc}^b(X)$.

(ii) The category $\mathcal{T}_c^-$ turns out to be $\mathcal{D}_{coh}^-(X)$, hence the category $\mathcal{T}_c^b = \mathcal{T}_c^- \cap \mathcal{T}_c^b$ is nothing other than $\mathcal{D}_{coh}^b(X)$.

It turns out that Applications 1.4 (iii), (iv) and (v) generalize greatly. The glorious, abstract versions for (iii) and (iv) go as follows. Let $R$ be a noetherian ring, and let $\mathcal{T}$ be an $R$–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator $G$, such that $\text{Hom}(G, G[n])$ is a finite $R$–module for all $n \in \mathbb{Z}$. Consider the two functors

$$Y : \mathcal{T}_c^\text{op} \to \text{Hom}_R(\mathcal{T}_c^\text{op}, R\text{-Mod}), \quad \tilde{Y} : \mathcal{T}_c^\text{op} \to \text{Hom}_R(\mathcal{T}_c^b, R\text{-Mod})$$

defined by the formulas $Y(B) = \text{Hom}(\mathcal{T}_c, B)$ and $\tilde{Y}(A) = \text{Hom}(A, \mathcal{T}_c)$. Note that, in these formulas, we permit all $A, B \in \mathcal{T}_c$. But the $(-)$ in the formula $Y(B) = \text{Hom}(\mathcal{T}_c, B)$ is assumed to belong to $\mathcal{T}_c$, whereas the $(-)$ in the formula $\tilde{Y}(A) = \text{Hom}(A, \mathcal{T}_c)$ must lie in $\mathcal{T}_c^b$. Now consider the following composites

$$\begin{align*}
\mathcal{T}_c^b \xrightarrow{i} \mathcal{T}_c^- \xrightarrow{Y} \text{Hom}_R(\mathcal{T}_c^\text{op}, R\text{-Mod}) \\
[\mathcal{T}_c^\text{op}] \xrightarrow{\tilde{i}} [\mathcal{T}_c^-]^{\text{op}} \xrightarrow{\tilde{Y}} \text{Hom}_R(\mathcal{T}_c^b, R\text{-Mod})
\end{align*}$$

where $i$ and $\tilde{i}$ are the obvious inclusions. We assert:

(iii) The functor $Y$ is full, and the essential image consists of the locally finite homological functors. A functor $H : [\mathcal{T}_c]^{\text{op}} \to R\text{-Mod}$ is locally finite if $H(A[n])$ is a finite $R$–module for every $n \in \mathbb{Z}$, and vanishes if $n \ll 0$.

The composite $Y \circ i$ is fully faithful, and the essential image consists of the locally finite homological functors.

(iv) Assume there exists an integer $N > 0$ and an object $G' \in \mathcal{T}_c^b$ with $\mathcal{T} = \langle G' \rangle_N^{(-\infty, \infty)}$ — this condition will be explained in the body of the paper, under the hypotheses placed on $X$ in Application 1.4(iv) the condition is satisfied by $\mathcal{T} = \mathcal{D}_{qc}(X)$, this may be found in [32, Theorem 2.3].

Then the functor $\tilde{Y}$ is full, and the essential image consists of the locally finite homological functors. The composite $\tilde{Y} \circ \tilde{i}$ is fully faithful, and the essential image consists of the finite homological functors.

As we have said, Application 1.4(v) also has a vast generalization, which goes as follows:

(v) With the notation as in Application 1.4(v) one has, for any approximable $\mathcal{T}$, a triangulated equivalence $\mathcal{G}(\mathcal{T}_c) \cong \mathcal{T}_c^b$. If the triangulated category $\mathcal{T}$ is not only
approximable but also noetherian, then one also has a triangulated equivalence
\[ G([\mathbb{T}_C^b])^{op} \cong [\mathbb{T}_C]^{op}. \]

The notion of noetherian triangulated categories in Remark 1.6(v) is new, and was
inspired by the result. Noetherianness is a condition that seems natural, and guarantees
that there will be plenty of objects in \( \mathbb{T}_C^b \). Without some noetherian hypothesis, the only
obvious object in \( \mathbb{T}_C^b \) is zero.

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presenting parts or all of the material.

2. Background

It’s time to speak to the non-experts—the readers familiar with triangulated cate-
gories, compact generators and \( t \)-structures are advised to skip ahead to Section 3. In
this section we will present a quick reminder of the three concepts in the sentence above.
We plan to usually proceed from the concrete to the abstract: for most of the section
we study first an example, actually four examples—all four examples will be derived
categories \( D^C(A) \), we list the four in Example 2.3—and only then do we move on to
the general definitions. We should therefore begin by recalling what are the categories
\( D^C(A) \), first in generality that covers the four examples and more, and then narrowing
down to the specific ones that will interest us.

Example 2.1. Let \( A \) be an abelian category. The derived category \( D^C(A) \) is as follows:

(i) The objects are the cochain complexes in \( A \), that is diagrams in \( A \) of the form

\[
\cdots \rightarrow A^{-2} \rightarrow A^{-1} \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots
\]

where the composites \( A^i \rightarrow A^{i+1} \rightarrow A^{i+2} \) all vanish. The subscript \( C \) and super-
script \( C' \) stand for conditions. We may choose not to allow all cochain complexes,
when the mood strikes us we can capriciously impose any conditions on the cochain
complexes that our heart desires—subject to the mild hypotheses that guarantee
that the few operations we’re about to perform take complexes satisfying the re-
strictions to complexes satisfying the restrictions.

(ii) Cochain maps are morphisms in \( D^C(A) \), that is any commutative diagram

\[
\cdots \rightarrow A^{-2} \rightarrow A^{-1} \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots
\]

\[
\cdots \rightarrow B^{-2} \rightarrow B^{-1} \rightarrow B^0 \rightarrow B^1 \rightarrow B^2 \rightarrow \cdots
\]
is a morphism from the top to the bottom row—as long as the rows are cochain
complexes satisfying the restrictions, that is objects in \( D^C(A) \).
But then we formally invert the cohomology isomorphisms. In the literature the cohomology isomorphisms often go by the name “quasi-isomorphisms”.

**Explanation 2.2.** Given a category $\mathcal{C}$ and a collection $S$ of morphisms in $\mathcal{C}$, an old theorem of Gabriel and Zisman [15] tells us that there exists a functor $F : \mathcal{C} \to S^{-1}\mathcal{C}$ so that

(i) If $s \in S \subset \text{Mor}(\mathcal{C})$ then $F(s)$ is invertible.
(ii) Any functor $F' : \mathcal{C} \to B$, with $F'(S)$ contained in the isomorphisms of $B$, factors uniquely as $\mathcal{C} \xrightarrow{F} S^{-1}\mathcal{C} \xrightarrow{F'} B$.

What we mean when we say that in $D(\mathcal{C})$ we “formally invert” the cohomology isomorphisms is: let $\mathcal{C}$ be the category with the same objects as $D(\mathcal{C})$ but where the morphisms are the cochain maps, and let $S$ be the collection of cochain maps inducing cohomology isomorphisms. Then $D(\mathcal{C})$ is defined to be $S^{-1}\mathcal{C}$.

In principle categories of the form $S^{-1}\mathcal{C}$ can be dreadful—the morphisms are equivalence classes of composable strings, where each string is a sequence whose pieces are either morphisms in $\mathcal{C}$ or inverses of elements of $S$. The Hom-sets needn’t be small, and in general it can be a nightmare to decide when two such strings are equivalent, meaning define the same morphism in $S^{-1}\mathcal{C}$. For categories like $D(\mathcal{C})$ the calculus of fractions happens not to be too bad, there is a literature dealing with it. The interested reader is referred to Hartshorne [18] or Verdier [42] for the original presentations, or Gelfand and Manin [16], Kashiwara and Schapira [22] or Weibel [43] for more modern treatments.

In this survey we skip the discussion of the calculus of fractions. This means that the reader will be asked to believe several computations along the way—when these occur there will be a footnote to the effect.

**Example 2.3.** In this survey, the key examples to keep in mind are:

(i) If $R$ is a ring, $D(R)$ will be our shorthand for $D(R\text{-Mod})$; the abelian category $\mathcal{A}$ is the category of all left $R$–modules, and since there are no superscripts or subscripts decorating the $D$ we impose no conditions. All cochain complexes of left $R$-modules are objects of $D(R)$.

Now let $X$ be a scheme. The abelian category, in all three examples below, is the category of sheaves of $\mathcal{O}_X$–modules. It’s customary to abbreviate what should be written $D(\mathcal{O}_X\text{-Mod})$ to just $D(X)$, and we will follow this custom.

But all three categories we will look at are decorated, there are restrictions. We list them below.

(ii) The objects in $D_{\text{qc}}(X)$ are cochain complexes of $\mathcal{O}_X$–modules, and the only condition we impose is that the cohomology sheaves must be quasicohherent.

(iii) The objects of $D_{\text{perf}}(X)$ are the perfect complexes. A cochain complex of $\mathcal{O}_X$–modules is perfect if it is locally isomorphic to a bounded complex of vector bundles. More precisely: an object $P \in D_{\text{qc}}(X)$ belongs to $D_{\text{perf}}(X)$ if there exists an open cover of $X$ of the form $X = \bigcup_i U_i$ such that, if $u_i : U_i \to X$ is the inclusion,
then the obvious functor \( u_i^* : D_{qc}(X) \rightarrow D_{qc}(U_i) \) takes \( P \in D_{qc}(X) \) to an object \( u_i^*(P) \in D_{qc}(U_i) \) which is isomorphic in \( D_{qc}(U_i) \) to a bounded complex of vector bundles.

(iv) Assume \( X \) is noetherian. The objects of \( D^b_{coh}(X) \) are the complexes of \( \mathcal{O}_X \)-modules with coherent cohomology—as indicated by the subscript—and this cohomology vanishes in all but finitely many degrees, the superscript \( b \) stands for “bounded”.

Remark 2.4. If we’re going to be working with categories like \( D^E_\mathcal{C}(A) \), it is natural to wonder what useful structure they might have. The next definition spells out the answer.

The idea is simple enough: we started with the category \( \mathcal{C} \) whose objects are the same as those of \( D^E_\mathcal{C}(A) \), but the morphisms were honest cochain maps. We then performed the construction of Explanation 2.2 formally inverting the class \( S \) of cohomology isomorphisms, to form \( D^E_\mathcal{C}(A) = S^{-1}\mathcal{C} \). The information retained isn’t much more than the cohomology of the complex. Ordinary homological algebra teaches us that there are really only two things you can do with cohomology:

(i) Shift the degrees.
(ii) Form the the long exact sequence in cohomology that comes from a short exact sequence of cochain complexes.

The structure of a triangulated category, formalized in Definition 2.5 (i) and (ii) below, encapsulates this: Definition 2.5 (i) gives the shifting of degrees, while Definition 2.5 (ii) is the abstract version of the long exact sequence in cohomology coming from a short exact sequence of cochain complexes. See Example 2.7 for more detail: we spell out the recipe that endows \( D^E_\mathcal{C}(A) \) with the structure of a triangulated category, and do so by steering as close as possible to the simple, motivating idea.

Definition 2.5. To give the additive category \( \mathcal{T} \) the structure of a triangulated category we must:

(i) Specify an invertible additive endofunctor \( \mathcal{T} \rightarrow \mathcal{T} \). In this article we will denote it \( [1] \) and have it act on the right: thus it takes the object \( X \) and the morphism \( f \) in \( \mathcal{T} \) to \( X[1] \) and \( f[1] \), respectively.

Before we continue the definition we set up

Notation 2.6. For the purpose of the current definition (Definition 2.5) we adopt the following convention. With \( [1] : \mathcal{T} \rightarrow \mathcal{T} \) the endofunctor of Definition 2.5 (i), a candidate triangle is any three composable morphisms \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \) in the category \( \mathcal{T} \). The candidate triangles form a category, a morphism of candidate triangles is defined to be a commutative diagram in \( \mathcal{T} \):

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
X' & \xrightarrow{f'} & Y'
\end{array}
\quad\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{w} & & \downarrow{w[1]} \\
Y' & \xrightarrow{g'} & Z'
\end{array}
\quad\begin{array}{ccc}
Z & \xrightarrow{h} & X[1] \\
\downarrow{u[1]} & & \\
X' & \xrightarrow{h'} & X'[1]
\end{array}
\]
which we view as a morphism from the top to the bottom row. The composition of
morphisms of candidate triangles is the obvious.

**Continuation of Definition 2.5**, now that the notation has been explained. In
addition to the invertible endomorphism \([1 : T \to T]\) of (i) we must specify
(ii) A full subcategory of the category of candidate triangles, whose objects will be called
exact triangles. [In some parts of the literature they go by the name distinguished
triangles.]

For \(T\) to qualify as a triangulated category the data of (i) and (ii) above must satisfy the
following axioms:

**[TR1]**: Any candidate triangle isomorphic to an exact triangle is an exact triangle.
For any object \(X \in T\) the diagram \(0 \to X \overset{\text{id}}{\to} X \to 0\) is an exact triangle.
Any morphism \(f : X \to Y\) may be completed to an exact triangle \(X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1]\).

**[TR2]**: Any rotation of an exact triangle is exact. That is: \(X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} X[1]\)
is an exact triangle if and only if \(Y \overset{-g}{\to} Z \overset{-h}{\to} X[1] \overset{-f[1]}{\to} Y[1]\) is.

**[TR3+4]**: Given a commutative diagram, where the rows are exact triangles,

\[
\begin{array}{ccc}
X & \overset{f}{\to} & Y \\
\downarrow u & & \downarrow v \\
X' & \overset{f'}{\to} & Y'
\end{array}
\quad
\begin{array}{ccc}
Y & \overset{g}{\to} & Z \\
\downarrow v & & \downarrow w \\
Y' & \overset{g'}{\to} & Z'
\end{array}
\quad
\begin{array}{ccc}
Z & \overset{h}{\to} & X[1] \\
\downarrow h & & \downarrow w \\
Z' & \overset{h'}{\to} & X'[1]
\end{array}
\]

we may complete it to a morphism of exact triangles, that is a commutative
diagram

\[
\begin{array}{ccc}
X & \overset{f}{\to} & Y \\
\downarrow u & & \downarrow v \\
X' & \overset{f'}{\to} & Y'
\end{array}
\quad
\begin{array}{ccc}
Y & \overset{g}{\to} & Z \\
\downarrow v & & \downarrow w \\
Y' & \overset{g'}{\to} & Z'
\end{array}
\quad
\begin{array}{ccc}
Z & \overset{h}{\to} & X[1] \\
\downarrow h & & \downarrow w \\
Z' & \overset{h'}{\to} & X'[1]
\end{array}
\]

Moreover: we can do it in such a way that

\[
\begin{pmatrix}
-g & 0 \\
v & f'
\end{pmatrix}
\begin{pmatrix}
-h & 0 \\
w & g'
\end{pmatrix}
\begin{pmatrix}
-f[1] & 0 \\
u[1] & h'
\end{pmatrix}
\]

is an exact triangle.

**Example 2.7.** We have asserted that the category \(D^e_c(A)\) is triangulated. It is only
fair to tell the reader what is the endofunctor \([1 : D^e_c(A) \to D^e_c(A)]\) and what are the
exact triangles. The endofunctor $[1]$, called the shift or suspension, is easy: it takes the cochain complex $A^*$, that is the diagram

$\cdots \rightarrow A^{-2} \xrightarrow{\partial^{-2}} A^{-1} \xrightarrow{\partial^{-1}} A^0 \xrightarrow{\partial^0} A^1 \xrightarrow{\partial^1} A^2 \xrightarrow{\partial^2} \cdots$

to the cochain complex $(A[1])^*$ below:

$\cdots \rightarrow A^{-1} \xrightarrow{-\partial^{-1}} A^0 \xrightarrow{-\partial^0} A^1 \xrightarrow{-\partial^1} A^2 \xrightarrow{-\partial^2} A^3 \xrightarrow{-\partial^3} \cdots$

In words: we shift the complex to the left by one, that is $(A[1])^n = A^{n+1}$, and the maps all change signs. This deals with objects.

If $f^* : A^* \rightarrow B^*$ is a cochain map

$\cdots \rightarrow A^{-2} \xrightarrow{\partial^{-2}} A^{-1} \xrightarrow{\partial^{-1}} A^0 \xrightarrow{\partial^0} A^1 \xrightarrow{\partial^1} A^2 \xrightarrow{\partial^2} \cdots$

$\downarrow f^{-2} \quad \downarrow f^{-1} \quad \downarrow f^0 \quad \downarrow f^1 \quad \downarrow f^2 \quad \downarrow f^3 \quad \cdots$

$\cdots \rightarrow B^{-2} \xrightarrow{\partial^{-2}} B^{-1} \xrightarrow{\partial^{-1}} B^0 \xrightarrow{\partial^0} B^1 \xrightarrow{\partial^1} B^2 \xrightarrow{\partial^2} \cdots$

then $(f[1])^*$ is the cochain map

$\cdots \rightarrow A^{-1} \xrightarrow{-\partial^{-1}} A^0 \xrightarrow{-\partial^0} A^1 \xrightarrow{-\partial^1} A^2 \xrightarrow{-\partial^2} A^3 \xrightarrow{-\partial^3} \cdots$

$\downarrow f^{-1} \quad \downarrow f^0 \quad \downarrow f^1 \quad \downarrow f^2 \quad \downarrow f^3 \quad \downarrow f^4 \quad \cdots$

$\cdots \rightarrow B^{-1} \xrightarrow{-\partial^{-1}} B^0 \xrightarrow{-\partial^0} B^1 \xrightarrow{-\partial^1} B^2 \xrightarrow{-\partial^2} B^3 \xrightarrow{-\partial^3} \cdots$

This defines what the functor $[1]$ does to cochain maps, and we extend to arbitrary morphisms in $\mathbf{D}_{\mathbb{E}}^\mathcal{E}(\mathcal{A})$ by the universal property of the localization process.

To spell this out a bit, as in Explanation 2.2 let $\mathcal{C}$ be the category with the same objects as $\mathbf{D}_{\mathbb{E}}^\mathcal{E}(\mathcal{A})$ but where the morphisms are the cochain maps. We have defined a functor $[1] : \mathcal{C} \rightarrow \mathcal{C}$, which takes the class $S \subset \text{Mor}(\mathcal{C})$ of cohomology isomorphisms to itself. The composite $\mathcal{C} \xrightarrow{[1]} \mathcal{C} \xrightarrow{F} S^{-1}\mathcal{C}$ is a functor from $\mathcal{C}$ to the category $S^{-1}\mathcal{C} = \mathbf{D}_{\mathbb{E}}^\mathcal{E}(\mathcal{A})$, which takes the morphisms in $S$ to isomorphisms. By the universal property it factors uniquely through $F$, that is there exists a commutative square

$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{[1]} & \mathcal{C} \\
F \downarrow & & \downarrow F \\
S^{-1}\mathcal{C} & \xrightarrow{\exists!} & S^{-1}\mathcal{C}
\end{array}$

We declare $[1] : \mathbf{D}_{\mathbb{E}}^\mathcal{E}(\mathcal{A}) \rightarrow \mathbf{D}_{\mathbb{E}}^\mathcal{E}(\mathcal{A})$ to be the unique map $S^{-1}\mathcal{C} \rightarrow S^{-1}\mathcal{C}$ making the square commute.

It remains to describe the exact triangles—Remark 2.4 provided the intuition, it told us that the exact triangles should be the formalization of the long exact sequence in
cohomology coming from a short exact sequence of cochain complexes. We propose to give the skeleton of the construction below, and the reader interested in more detail is referred to the appendices.

Suppose therefore that we are given a commutative diagram in $\mathcal{A}$

\[
\cdots \rightarrow X^{-2} \rightarrow X^{-1} \rightarrow X^0 \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\cdots \rightarrow Y^{-2} \rightarrow Y^{-1} \rightarrow Y^0 \rightarrow Y^1 \rightarrow Y^2 \rightarrow \cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\cdots \rightarrow Z^{-2} \rightarrow Z^{-1} \rightarrow Z^0 \rightarrow Z^1 \rightarrow Z^2 \rightarrow \cdots 
\]

where the rows are objects of $D^C_\xi(\mathcal{A})$, that is cochain complexes satisfying the hypotheses.

So far we may view the above as morphisms $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^*$ in the category $D^C_\xi(\mathcal{A})$.

Assume further that, for each $i \in \mathbb{Z}$, the sequence $X^i \xrightarrow{f^i} Y^i \xrightarrow{g^i} Z^i$ is split exact—it’s easier to deal with degreewise split short exact sequences, in Appendix B the reader will see that, up to isomorphism in $D(R)$, this suffices. We next want to mimic the process that produces the differential of the long exact sequence in cohomology. Choose, for each $i \in \mathbb{Z}$, a splitting $\theta^i : Z^i \rightarrow Y^i$ of the map $g^i : Y^i \rightarrow Z^i$. Now for each $i$ we have the diagram

\[
\begin{array}{ccc}
Z^i & \xrightarrow{\theta^i} & Y^i & \xrightarrow{g^i} & Z^i \\
\downarrow \partial^i_2 & & \downarrow \partial^i_1 & & \downarrow \partial^i_2 \\
Z^{i+1} & \xrightarrow{g^{i+1}} & Y^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1}
\end{array}
\]

If we delete the middle column the resulting square commutes—the composites of the horizontal maps are identities. If we delete the left column the square is commutative because it is part of the diagram defining the cochain map $g^*$. It follows that, in the diagram below,

\[
\begin{array}{ccc}
Z^i & \xrightarrow{\theta^i} & Y^i \\
\downarrow \partial^i_2 & & \downarrow \partial^i_1 \\
Z^{i+1} & \xrightarrow{g^{i+1}} & Z^{i+1}
\end{array}
\]

the two composites from top left to bottom right are equal. Thus the difference $\theta^{i+1}\partial^i_Z - \partial^i_Y \theta^i$ is annihilated by the map $g^{i+1} : Y^{i+1} \rightarrow Z^{i+1}$, hence $\theta^{i+1}\partial^i_Z - \partial^i_Y \theta^i$ must factor uniquely through the kernel of $g^{i+1}$, it can be written uniquely as the composite $Z^i \xrightarrow{b^i}$
Thus we have constructed in the category \( D_C'(A) \) a diagram

\[
\cdots \longrightarrow Z^{-2} \xrightarrow{\partial^{-2}_Z} Z^{-1} \xrightarrow{\partial^{-1}_Z} Z^0 \xrightarrow{\partial^0_Z} Z^1 \xrightarrow{\partial^1_Z} Z^2 \xrightarrow{\partial^2_Z} \cdots
\]

\[
\cdots \longrightarrow X^{-1} \xrightarrow{-\partial^{-1}_X} X^0 \xrightarrow{-\partial^0_X} X^1 \xrightarrow{-\partial^1_X} X^2 \xrightarrow{-\partial^2_X} X^3 \xrightarrow{-\partial^3_X} \cdots
\]

Thus we have constructed in the category \( D_C'(A) \) a diagram \( X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{h^*} X^*[1] \). We declare

(i) The exact triangles in \( D_C'(A) \) are all the isomorphs, in \( D_C'(A) \), of diagrams that come from our construction.

It needs to be checked that [TR1], [TR2] and [TR3+4] of Definition 2.5 are satisfied, the reader can amuse herself with this.

For future reference we recall:

**Notation 2.8.** If \( T \) is a triangulated category and \( n \in \mathbb{Z} \) is an integer, then \([n] \) will be our shorthand for the endofunctor \([1]^n : T \rightarrow T \). Also: we will often lazily abbreviate “exact triangle” to just “triangle”.

**Definition 2.9.** A full subcategory \( S \subset T \) is called triangulated if \( 0 \in S \), if \( S[1] = S \), and if, whenever \( X, Y \in S \) and there exists in \( T \) a triangle \( X \rightarrow Y \rightarrow Z \rightarrow X[1] \), we must also have \( Z \in S \). The subcategory \( S \) is thick if it is triangulated, as well as closed in \( T \) under direct summands.

Now that we have recalled the notion of triangulated categories, as well as thick and triangulated subcategories, it is time to remember the other two building blocks of the theory we plan to introduce: compact generators and \( t \)-structures. We begin with

**Definition 2.10.** Let \( T \) be a triangulated category with coproducts. An object \( C \in T \) is compact if the functor \( \text{Hom}(C,-) \) respects coproducts. A set of compact objects \( \{G_i, i \in I\} \) is said to generate the category \( T \) if the following equivalent conditions hold

(i) If \( X \in T \) is an object, and if \( \text{Hom}(G_i, X[n]) \cong 0 \) for all \( i \in I \) and all \( n \in \mathbb{Z} \), then \( X \cong 0 \).

(ii) If a triangulated subcategory \( S \subset T \) is closed under coproducts and contains the objects \( \{G_i, i \in I\} \), then \( S = T \).

If the category \( T \) contains a set of compact generators it is called compactly generated.

**Remark 2.11.** The equivalence of (i) and (ii) in Definition 2.10 is not meant to be obvious, but it is a standard result. We will mostly be interested in the situation where the category \( T \) is compactly generated and, moreover, the set of compact generators may be chosen to consist of a single element. That is: for some compact object \( G \in T \) the set \( \{G\} \) generates, as in Definition 2.10 (i) or (ii).
Example 2.12. The categories $\mathbf{D}(R)$ and $\mathbf{D}_{\text{qc}}(X)$ of Example 2.3 (i) and (ii) both have coproducts: the coproduct of a family of cochain complexes

$$\cdots \to A_{\lambda}^{-2} \to A_{\lambda}^{-1} \to A_{\lambda}^0 \to A_{\lambda}^1 \to A_{\lambda}^2 \to \cdots$$

turns out to be nothing other than

$$\cdots \to \bigoplus_{\lambda \in \Lambda} A_{\lambda}^{-2} \to \bigoplus_{\lambda \in \Lambda} A_{\lambda}^{-1} \to \bigoplus_{\lambda \in \Lambda} A_{\lambda}^0 \to \bigoplus_{\lambda \in \Lambda} A_{\lambda}^1 \to \bigoplus_{\lambda \in \Lambda} A_{\lambda}^2 \to \cdots$$

It’s clear that the formula above does not work for the categories $\mathbf{D}_{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}(X)$ of Example 2.3 (iii) and (iv), if we take a giant direct sum of complexes satisfying the restrictions the resulting complex will fail to satisfy the restrictions. And it’s not just that the formulas don’t work, the categories $\mathbf{D}_{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}(X)$ don’t have coproducts.

Now for compact generators. If $R \in \mathbf{D}(R)$ stands for the cochain complex

$$\cdots \to 0 \to 0 \to R \to 0 \to 0 \to \cdots$$

that is the complex whose only nonzero entry is the module $R$ in degree 0, then it can be shown that there is an isomorphism of functors $\text{Hom}_{\mathbf{D}(R)}(R, -) \cong H^0(-)$. The functor $H^0(-)$ obviously respects coproducts, hence so does $\text{Hom}_{\mathbf{D}(R)}(R, -)$; that is the object $R \in \mathbf{D}(R)$ is compact.

Next observe that, if $X \in \mathbf{D}(R)$ is an object such that $H^n(X) \cong \text{Hom}(R, X[n]) \cong 0$ for all $n \in \mathbb{Z}$, then $X$ is acyclic; its cohomology all vanishes. The cochain map $0 \to X$ is an isomorphism in cohomology, hence an isomorphism in $\mathbf{D}(R)$. That is: $X \cong 0$. Thus the compact object $R \in \mathbf{D}(R)$ satisfies Definition 2.10(i), it is a compact generator.

The category $\mathbf{D}(R)$ is compactly generated, and more precisely we have learned that the object $R \in \mathbf{D}(R)$ is a single compact generator.

Not so easy is the fact that, if $X$ is a quasicompact, quasiseparated scheme, then the category $\mathbf{D}_{\text{qc}}(X)$ also has a single compact generator. This is a theorem, proved in Bondal and Van den Bergh [8, Theorem 3.1.1(ii)].

Notation 2.13. Let $\mathcal{T}$ be a triangulated category with coproducts. It is standard to denote by $\mathcal{T}^c$ the full subcategory, whose objects are the compact objects in $\mathcal{T}$. It isn’t difficult to show that $\mathcal{T}^c$ is always a thick subcategory of $\mathcal{T}$, as in Definition 2.9. In the case where $\mathcal{T} = \mathbf{D}_{\text{qc}}(X)$ the category $\mathcal{T}^c$ turns out to be the $\mathbf{D}_{\text{perf}}(X)$ of Example 2.3 (iii), the reader can find this fact in Bondal and Van den Bergh [8, Theorem 3.1.1(i)].

We also need to recall $t$-structures, and we plan to begin with the concrete. But first a reminder.

---

2 In this example it helps to know the calculus of fractions of $\mathbf{D}_{\mathcal{E}}(A) = S^{-1}\mathcal{E}$ mentioned in Example 2.2. After all we are making assertions about morphisms in $\mathbf{D}_{\mathcal{E}}(A)$: to say that an object is a coproduct is a universal property for certain morphisms. Moreover we also make an assertion about $\text{Hom}_{\mathbf{D}(R)}(R, -)$. 

Reminder 2.14. Let $\mathcal{T}$ be an triangulated category, and $\mathcal{A}$ an abelian category. A functor $H : \mathcal{T} \to \mathcal{A}$ is called homological if, for every triangle $A \to B \to C \to A[1]$ in $\mathcal{T}$, the sequence $H(A) \to H(B) \to H(C)$ is exact in $\mathcal{A}$.

From the axiom [TR2] of Definition 2.5—the axiom telling us that any rotation of a triangle is a triangle—it follows that the functor $H$ must take a triangle in $\mathcal{T}$ to a long exact sequence.

Example 2.15. It follows from the axioms of triangulated categories that all representable functors are homological. That is: if $\mathcal{T}$ is a triangulated category and $A \in \mathcal{T}$ is an object, then Hom$(A, -)$ and Hom$(-, A)$ are, respectively, homological functors $\mathcal{T} \to \mathcal{A}$ and $\mathcal{T}^{\text{op}} \to \mathcal{A}$, where $\mathcal{A}$ is the category of abelian groups.

The functor $H : \mathbf{D}(R) \to R-\text{Mod}$, taking a complex to its zeroth cohomology, is homological. In Example 2.12 we were told that $H(-) \cong \text{Hom}(R, -)$, that is the functor $H$ is a special case of the previous paragraph, it is a representable functor.

On the categories $\mathbf{D}_{\text{qc}}(X)$, $\mathbf{D}_{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}^{b}(X)$ the homological functor we will usually consider is traditionally denoted $\mathcal{H}$, and takes its values in the abelian category $\mathcal{O}_X-\text{Mod}$ of sheaves of $\mathcal{O}_X$–modules. Again: the functor $\mathcal{H}$ just takes a complex of sheaves to the zeroth cohomology sheaf.

The fact that $H$ and $\mathcal{H}$ are homological is by the construction of triangles, see Example 2.7—it comes down to the statement that the long exact sequence coming from a short exact sequence of cochain complexes is exact.

And finally we turn to $t$–structures, introducing them by example.

Example 2.16. In the category $\mathcal{T} = \mathbf{D}(R)$ we define two full subcategories by the formula

(i) $\mathcal{T}^{\leq 0} = \{ A \in \mathbf{D}(R) \mid H(A[i]) = 0 \text{ for all } i > 0 \}$

(ii) $\mathcal{T}^{\geq 0} = \{ A \in \mathbf{D}(R) \mid H(A[i]) = 0 \text{ for all } i < 0 \}$

While in the case where $\mathcal{T}$ is either of the categories $\mathbf{D}_{\text{qc}}(X)$ or $\mathbf{D}_{\text{coh}}^{b}(X)$, the formula is

(iii) $\mathcal{T}_{\leq 0} = \{ A \in \mathcal{T} \mid \mathcal{H}(A[i]) = 0 \text{ for all } i > 0 \}$

(iv) $\mathcal{T}_{\geq 0} = \{ A \in \mathcal{T} \mid \mathcal{H}(A[i]) = 0 \text{ for all } i < 0 \}$

These pairs of subcategories, in each of $\mathbf{D}(R), \mathbf{D}_{\text{qc}}(X)$ and $\mathbf{D}_{\text{coh}}^{b}(X)$, define a $t$–structure. For each of the three categories the particular $t$–structure above is traditionally called the standard $t$–structure. The category $\mathbf{D}_{\text{perf}}^{b}(X)$ does not usually have a nontrivial $t$–structure.

Let us next give the formal definition:

Definition 2.17. A $t$–structure on a triangulated category $\mathcal{T}$ is a pair of full subcategories $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ satisfying

(i) $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$

3For the non-algebraic-geometers: the letter $H$ is taken, it usually means another homological functor.
(ii) $\text{Hom}\left(\mathcal{T}^{\leq 0}[1], \mathcal{T}^{\geq 0}\right) = 0$

(iii) Every object $B \in \mathcal{T}$ admits a triangle $A \rightarrow B \rightarrow C \rightarrow$ with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

Remark 2.18. It can be checked that the pairs of subcategories of Example 2.16 satisfy parts (i), (ii) and (iii) of Definition 2.17, they do provide $t$-structures on each of $\mathcal{D}(R)$, $\mathcal{D}_{\text{qc}}(X)$ and $\mathcal{D}^b_{\text{coh}}(X)$.

We have now introduced all the players: triangulated categories, compact generators and $t$-structures. We end the section recalling certain standard shorthand conventions.

Notation 2.19. Let $\mathcal{T}$ be a triangulated category with a $t$-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$. Then

(i) For any integer $n \in \mathbb{Z}$ we set $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$

(ii) Furthermore, we adopt the conventions $\mathcal{T}^- = \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\leq n}$, $\mathcal{T}^+ = \bigcup_{n \in \mathbb{N}} \mathcal{T}^{\geq -n}$, $\mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$.

3. Approximability—the intuition, which comes from $\mathcal{D}(R)$

In the last section we recalled, for the benefit of the non-expert, some standard facts about triangulated categories, compact generators and $t$-structures—as well as the special cases that play a big role in this article, namely $\mathcal{D}(R)$, $\mathcal{D}_{\text{qc}}(X)$, $\mathcal{D}^\text{perf}(X)$ and $\mathcal{D}^b_{\text{coh}}(X)$. It’s time to move on to the subject matter of this article: approximability. As we’ve tried to do throughout, we will proceed from the concrete to the abstract. Let us therefore first study what it all means for the category $\mathcal{D}(R)$, when $R$ is a ring.

The category $\mathcal{D}(R)$ has a standard $t$-structure, see Example 2.16, Definition 2.17 and Remark 2.18. Suppose we are given an object $F^* \in \mathcal{D}(R)^{\leq 0}$, meaning a cochain complex

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$

such that $H^i(F^*) = 0$ for all $i > 0$. Then $F^*$ has a projective resolution. We can produce a cochain map

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots$$
inducing an isomorphism in cohomology, and so that each $P^i$ is a projective $R$–module. This gives us, in the category $D(R)$, an isomorphism $P^* \to F^*$. Now consider

```
\cdots \to 0 \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0 \to \cdots \\
\cdots \to P^{-n-1} \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0 \to \cdots \\
\cdots \to P^{-n-1} \to 0 \to \cdots \to 0 \to 0 \to 0 \to \cdots
```

This yields a pair of cochain maps $E^* n \xrightarrow{f_n} P^* \xrightarrow{g_n} D^*_n$ so that, in each degree $i$, the maps $E^i n \xrightarrow{f_i} P^i \xrightarrow{g_i} D^i_n$ deliver a split exact sequence of $R$–modules. Example 2.7 constructs for us a cochain map $h^* n : D^* n \to E^*[1]$ so that the diagram $E^* n \xrightarrow{f_n} P^* \xrightarrow{g_n} D^*_n \xrightarrow{h_n} E^*[1]$ is an exact triangle. The isomorphism $P^* \to F^*$ in the category $D(R)$, coupled with the fact that any isomorph of a triangle is a triangle, produces in $D(R)$ a triangle which we will write $E^* n \xrightarrow{f_n} F^* \xrightarrow{g_n} D^*_n \xrightarrow{h_n} E^*[1]$.

**Summary 3.1.** Given an object $F^* \in D(R)^{\leq 0}$ and an integer $n \geq 0$ we have constructed, in $D(R)$, a triangle $E^* n \xrightarrow{f_n} F^* \xrightarrow{g_n} D^*_n \xrightarrow{h_n} E^*[1]$. This triangle is such that $D^*_n \in D(R)^{\leq -n-1}$, while $E^* n$ is not too complicated.

In the Introduction we mentioned that we will view the objects $D^*_n$ as “small” with respect to the metric induced by the $t$–structure. Up to an arbitrarily small “correction term” $D^*_n$, we have a way of approximating the object $F^*$ by the object $E^* n$ which we view as simpler. In order to formalize the idea we need to make precise what we mean by saying that $E^* n$ is “not too complicated”. We will do this in the next section.

### 4. Measuring the Complexity of an Object

As we have said in the Introduction, measuring how complicated an object is will involve a small tweak of an idea from Bondal and Van den Bergh \[8\]. We remind the reader.

**Reminder 4.1.** Let $\mathcal{T}$ be a triangulated category, possibly with coproducts, and let $\mathcal{A}, \mathcal{B} \subset \mathcal{T}$ be full subcategories. We define the full subcategories

\[
\begin{aligned}
(i) \quad \mathcal{A} \ast \mathcal{B} &= \left\{ x \in \mathcal{T} \mid \text{there exists in } \mathcal{T} \text{ a triangle } a \to x \to b \right\} \\
&\quad \text{with } a \in \mathcal{A} \text{ and } b \in \mathcal{B} \\
(ii) \quad \text{add}(\mathcal{A})&: \text{this consists of all finite coproducts of objects of } \mathcal{A}. \\
(iii) \quad \text{Assume } \mathcal{T} \text{ has coproducts. Define } \text{Add}(\mathcal{A}) &\text{ to consist of all coproducts of objects of } \mathcal{A}. \\
(iv) \quad \text{smd}(\mathcal{A}) &\text{: the category of all direct summands of objects of } \mathcal{A}.
\end{aligned}
\]
Remark 4.2. Reminder 4.1(i) is as in [6, 1.3.9], while Reminder 4.1(iv) is identical with [S] beginning of 2.2. Reminder 4.1(ii) and (iii) follow the usual conventions in representation theory; in [S] beginning of 2.2 the authors adopt the (unconventional) notation that add(\(\mathcal{A}\)) and Add(\(\mathcal{A}\)) are also closed under the suspension—thus add(\(\bigcup_{n=-\infty}^{\infty} \mathcal{A}[n]\)). The definitions that follow are therefore slightly different from [S], and it is this small tweak that makes all the difference—with the tweaked definitions, approximations turn out to exist in great generality.

And now we come to the key definition: we’re about to measure how much effort goes into constructing an object \(X\) out of some given full subcategory \(\mathcal{A} \subset \mathcal{I}\). In practice our usual choice for \(\mathcal{A}\) will be a \(\mathcal{A} = \{G\}\), the subcategory with just a single object \(G\), which we will often assume to be a compact generator.

Definition 4.3. Let \(\mathcal{I}\) be a triangulated category, possibly with coproducts, let \(\mathcal{A} \subset \mathcal{I}\) be a full subcategory and let \(m \leq n\) be integers, possibly infinite. We define the full subcategories

(i) \(\mathcal{A}[m,n] = \bigcup_{i=m}^{n} \mathcal{A}[-i]\).

(ii) \(\langle \mathcal{A} \rangle_{1}^{[m,n]} = \operatorname{smd}[\operatorname{add}(\mathcal{A}[m,n])].\)

(iii) \(\langle \mathcal{A} \rangle_{1}^{[m,n]} = \operatorname{smd}[\operatorname{Add}(\mathcal{A}[m,n])].\) [This definition assumes \(\mathcal{I}\) has coproducts].

Now let \(\ell > 0\) be an integer, and assume the categories \(\langle \mathcal{A} \rangle_{k}^{[m,n]}\) and \(\overline{\langle \mathcal{A} \rangle}_{k}^{[m,n]}\) have been defined for \(k\) in the range \(1 \leq k \leq \ell\). We proceed inductively to set

(iv) \(\langle \mathcal{A} \rangle_{\ell+1}^{[m,n]} = \operatorname{smd}[\langle \mathcal{A} \rangle_{1}^{[m,n]} \ast \langle \mathcal{A} \rangle_{\ell}^{[m,n]}].\)

(v) \(\overline{\langle \mathcal{A} \rangle}_{\ell+1}^{[m,n]} = \operatorname{smd}[\langle \mathcal{A} \rangle_{1}^{[m,n]} \ast \overline{\langle \mathcal{A} \rangle}_{\ell}^{[m,n]}].\) [This definition assumes \(\mathcal{I}\) has coproducts].

Example 4.4. Let us go back to our favorite example \(\mathbf{D}(R)\). Suppose \(\mathcal{A} = \{R\}\) is the category with a single object \(R\), and we will now proceed to say something about the subcategories \(\langle R \rangle_{\ell}^{[-n,0]} \subset \mathbf{D}(R)\). Let us start with

(i) \(\langle R \rangle_{1}^{[-n,0]}\): this turns out to be the category of all isomorphs in \(\mathbf{D}(R)\) of the cochain complexes

\[
\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow 0 \cdots \longrightarrow 0 \longrightarrow P^{-1} \longrightarrow 0 \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots
\]

with \(P^i\) finitely generated and projective.

This much is basically true by construction. We start with the object \(R\), and in Definition 4.3(i) we form the category \(R[-n,0] = \{R[i], 0 \leq i \leq n\}\) with finitely many objects. And then Definition 4.3(ii) allows us to first form finite coproducts of objects in \(R[-n,0]\), meaning cochain complexes

\[
\cdots \longrightarrow 0 \longrightarrow P^{-n} \longrightarrow 0 \longrightarrow P^{-1} \longrightarrow 0 \longrightarrow P^0 \longrightarrow 0 \longrightarrow \cdots
\]
with each $P^i$ a finitely generated free module, and then we are permitted direct summands in $D(R)$ of the above. It may be shown that these are all isomorphic to complexes as above, but where we allow the $P^i$ to be finitely generated and projective.

This was the easy part. Now the categories $\langle R \rangle_{\ell}^{[-n,0]}$ grow as $\ell$ grows, but it’s a little unclear how fast. They all contain $\langle R \rangle_1^{[-n,0]}$, and are all contained in the subcategory $S \subset D(R)$ of objects isomorphic in $D(R)$ to cochain complexes

$$\cdots \to 0 \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0 \to \cdots$$

with $P^i$ finitely generated and projective. Unlike the category $\langle R \rangle_{\ell}^{[-n,0]}$ of (i) above, for objects in $S$ the maps $P^i \to P^{i+1}$ are unconstrained—beyond (of course) the standing assumption that all composites $P^i \to P^{i+1} \to P^{i+2}$ must vanish, the objects of $S \subset D(R)$ must be cochain complexes.

What turns out to be true is (ii) $\langle R \rangle_{n+1}^{[-n,0]} = S$; hence $\langle R \rangle_{\ell}^{[-n,0]} = S$ for all $\ell \geq n + 1$.

We leave to the reader the proofs of the assertions made in this Example.

**Example 4.5.** Let us stay with our favorite example $D(R)$, and let us continue to put $A = \{R\}$, that is $A$ is the full subcategory of $D(R)$ with the single object $R$. We now want to work out what are the categories $\langle R \rangle_{\ell}^{[−n,0]}$. The discussion turns out to be much the same as in Example 4.4, and the summary of the results is

(i) The category $\langle R \rangle_1^{[−n,0]}$ consists of all isomorphs in $D(R)$ of cochain complexes

$$\cdots \to 0 \to P^{-n} \to 0 \to \cdots$$

with $P^i$ projective.

(ii) The category $\langle R \rangle_{n+1}^{[−n,0]}$ consists of all isomorphs in $D(R)$ of cochain complexes

$$\cdots \to 0 \to P^{-n} \to \cdots$$

with $P^i$ projective. Moreover: if $\ell \geq n + 1$ then $\langle R \rangle_{\ell}^{[−n,0]} = \langle R \rangle_{n+1}^{[−n,0]}$.

Thus the objects in both $\langle R \rangle_{n+1}^{[−n,0]}$ and $\langle R \rangle_{n+1}^{[−n,0]}$ are isomorphic in $D(R)$ to complexes of projectives vanishing outside the interval $[−n,0]$, and the difference is that in $\langle R \rangle_{n+1}^{[−n,0]}$ the projective modules are not constrained to be finitely generated.

**Conclusion 4.6.** In the new notation we have introduced, Summary 3.1 and Example 4.5 combine to say: for any object $F \in D(R)_{\leq 0}$ and any integer $n \geq 0$ there exists a triangle

$$E_n \xrightarrow{f} F \xrightarrow{g} D_n \xrightarrow{h} E_n[1]$$

with $D_n \in D(R)_{\leq −n−1}$ and $E_n \in \langle R \rangle_{n+1}^{[−n,0]}$.

---

*The proofs are easy for the reader familiar with the calculus of fractions of Explanation 2.2. Other readers are asked to accept the assertions on faith.*
Remark 4.7. Let $D(R\text{-proj})_{\leq 0} \subset D(R)$ be the full subcategory, whose objects are the isomorphs in $D(R)$ of cochain complexes

$$\cdots \longrightarrow P^{-n-1} \longrightarrow P^{-n} \longrightarrow \cdots \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow 0 \longrightarrow \cdots$$

with $P^i$ finitely generated and projective. Summary 3.1 and Example 4.4 combine to say: for any object $F \in D(R\text{-proj})_{\leq 0}$ and any integer $n \geq 0$ there exists a triangle

$$E_n \xrightarrow{f} F \xrightarrow{g} D_n \xrightarrow{h} E_n[1]$$

with $D_n \in D(R)_{\leq -n-1}$ and $E_n \in \langle R \rangle_{n+1}^{-n,0}$.

We will return to the category $D(R\text{-proj})_{\leq 0}$ and to its relative $D(R\text{-proj})_{\leq 0}^\sim = \bigcup_{n \in \mathbb{N}} D(R)_{\leq -n} \subset \langle R \rangle_{n+1}^{-n,0}$ much later in the article.

5. The formal definition of approximability

Now that we are thoroughly prepared, approximability becomes easy to formulate precisely:

Definition 5.1. Let $\mathcal{I}$ be a triangulated category with coproducts. It is approximable if there exists a compact generator $G \in \mathcal{I}$, a $t$–structure $(\mathcal{I}^{\leq 0}, \mathcal{I}^{\geq 0})$, and an integer $A > 0$ so that

(i) $G[A] \in \mathcal{I}^{\leq 0}$ and $\text{Hom}(G[-A], \mathcal{I}^{\leq 0}) = 0$.

(ii) For every object $F \in \mathcal{I}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D \longrightarrow E[1]$, with $D \in \mathcal{I}^{\leq -1}$ and $E \in \langle G \rangle_{A}^{[-A,A]}$.

Example 5.2. Let $R$ be a ring. In Example 2.12 we learned that the object $R \in D(R)$ is a compact generator, we will take this to be our $G$ of Definition 5.1. For the $t$-structure we choose the standard one, see Example 2.16. And for our integer we set $A = 1$.

It’s clear that $R[1] \in D(R)_{\leq 0}$ and that $\text{Hom}(R[-1], D(R)_{\leq 0}) = 0$. This establishes Definition 5.1(i). Finally suppose we are given an object $F \in D(R)_{\leq 0}$. By Conclusion 4.6 with $n = 0$, there must exist a triangle $E \longrightarrow F \longrightarrow D \longrightarrow E[1]$ with $D \in D(R)_{\leq -1}$ and $E \in \langle R \rangle_{1}^{[0,0]} \subset \langle R \rangle_{1}^{-1,1}$. This proves that Definition 5.1(ii) holds.

Thus the category $D(R)$ is approximable.

Remark 5.3. The reader might be disappointed: until now we have been stressing that approximability will allow us to obtain arbitrarily good estimates of the objects in any approximable category $\mathcal{I}$, and in Example 5.2 we see that the definition only involves a zero-order approximation.

Don’t let this disturb you, it’s easy to iterate and estimate the given object $F$ to arbitrarily high order. This will manifest itself in our theorems.
Remark 5.4. In Example 2.3 we told the reader that, in this survey, the key examples of triangulated categories will be $\mathbf{D}(R)$, $\mathbf{D}_{\text{qc}}(X)$, $\mathbf{D}_{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}^b(X)$. The definition of approximability is tailored so that the category $\mathbf{D}(R)$ is obviously approximable, as we have seen in Example 5.2. What about the other three?

The categories $\mathbf{D}_{\text{perf}}(X)$ and $\mathbf{D}_{\text{coh}}^b(X)$ cannot possibly be approximable, in Example 2.12 we learned that they don’t even have coproducts.

It is a non-obvious theorem that, as long as the scheme $X$ is quasicompact and separated, the category $\mathbf{D}_{\text{qc}}(X)$ is approximable. And it should come as a surprise—after all approximability was modeled on the idea of taking a projective resolution of a bounded-above cochain complex and then truncating, and this is a construction that can only work in the presence of enough projectives. There aren’t enough projectives in either the category of sheaves of $\mathcal{O}_X$–modules, or in its subcategory of quasicoherent sheaves.

In Example 2.12 we mentioned that even the existence of a single compact generator in $\mathbf{D}_{\text{qc}}(X)$ isn’t obvious, it’s a theorem of Bondal and Van den Bergh. The existence proof isn’t particularly constructive—it doesn’t give us much of a handle on this compact generator. And the definition of approximability is the assertion that the compact generator may be chosen to satisfy several useful properties; it decidedly isn’t clear how to prove any of them.

Given that it is going to entail real effort to prove that $\mathbf{D}_{\text{qc}}(X)$ is approximable, it shouldn’t come as a surprise that there are far-reaching consequences.

And now it’s time for

6. The main theorems

Remark 6.1. The theorems break up into three groups, namely

(i) Theorems that produce more examples of approximable categories. So far we have discussed in some detail the example $\mathbf{D}(R)$, and then made some passing comments about $\mathbf{D}_{\text{qc}}(X)$. See Remark 5.4

(ii) Formal consequences of approximability—that is structure that comes for free, which every approximable category has.

(iii) Applications to concrete examples, which teach us new and interesting facts about old and familiar categories.

In this section we will list the results of type (i) and (iii) by group, doing little more than giving formal statements. In the remainder of the article we will first expand on the results in group (iii), saying something about what was known before and about the proofs, both of the new and the old versions—presumably the reader is most likely to be persuaded by the theory if she can see applications that matter.

And then, towards the end of the article, we will give results in group (ii). We hope that by then, with the reader’s interest piqued by the group (iii) applications, she will have the patience to also read the structural theorems.
Facts 6.2. (The main theorems—sources of more examples). The following statements are true:

(i) If $\mathcal{T}$ has a compact generator $G$, so that $\text{Hom}(G,G[n]) = 0$ for all $n \geq 1$, then $\mathcal{T}$ is approximable.

Special cases of (i) include:

The category $\mathcal{T} = \mathbf{D}(R)$ and the compact generator $G = R$, in other words we recover Example 5.2 as a special case of (i). More generally: if $R$ is a dga, and $H^n(R) = 0$ for all $n > 0$, then the category $\mathcal{T} = \mathbf{D}(R–\text{Mod})$ with $G = R$ is an example. Further examples come from topology, for instance we can let $\mathcal{T}$ be the homotopy category of spectra and let $G = \mathbb{S}^0$ be the zero-sphere.

The proof of (i) is basically trivial, there is a brief discussion in [35, Remark 3.3].

(ii) Let $X$ be a quasicompact, separated scheme. Then the category $\mathbf{D}_{\text{qc}}(X)$ is approximable.

In Remark 5.3 we noted that this isn’t an easy fact, it is after all counterintuitive—it says that, in the category $\mathbf{D}_{\text{qc}}(X)$, one can pretend to have enough projectives—at least for some purposes. The proof isn’t trivial.

If $X$ is a separated scheme, of finite type over a noetherian ring, then the reader can find a proof [32, Theorem 5.8]. It constitutes the main technical lemma of the paper, the rest amounts to applications. The generalization to quasicompact, separated schemes is by a trick which may be found in [35, Example 3.6].

(iii) Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \leftarrow \mathcal{S} \rightarrow \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and any compact object $H \in \mathcal{S}$ has the property that $\text{Hom}(H,H[i]) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

Once again this isn’t obvious, it requires proof. The reader can find it in [3] Theorem 4.1—it is the main theorem of the article.

So far the majority of the interesting applications has been to algebraic geometry—it’s the example in Fact 6.2(ii) that has proved useful. But the subject is in its infancy, it is to be hoped that there will be applications to come, in other contexts.

Facts 6.3. (The main theorems—applications). Assertions (i) and (ii) below are [32, Theorem 0.5 and Theorem 0.15], respectively. Assertion (iii) follows from [35, Corollary 0.5], while assertion (iv) follows from [34, Theorem 0.2]. Assertion (v) is a consequence of [33, Proposition 0.15], together with the elaboration and discussion in the couple of paragraphs immediately following the statement of the Proposition. Anyway: all of (v) may be found in [33].

In Explanation 6.4 the reader is reminded what the various technical terms in the statements below mean.
(i) Let $X$ be a quasicompact, separated scheme. The category $D_{\text{perf}}(X)$ is strongly generated if and only if $X$ has an open cover by affine schemes $\text{Spec}(R_i)$, with each $R_i$ of finite global dimension.

(ii) Let $X$ be a separated scheme, and assume it is noetherian, finite-dimensional, and that every closed, reduced, irreducible subscheme of $X$ has a regular alteration. Then the category $D^b_{\text{coh}}(X)$ is strongly generated.

(iii) Let $X$ be a scheme proper over a noetherian ring $R$. Let $\tilde{Y}$ be the Yoneda map

$$D^b_{\text{coh}}(X) \xrightarrow{\tilde{Y}} \text{Hom}_R\left([D_{\text{perf}}(X)]^{\text{op}}, R-\text{Mod}\right)$$

That is: the map $\tilde{Y}$ sends the object $B \in D^b_{\text{coh}}(X)$ to the functor $\tilde{Y}(B) = \text{Hom}(-, B)$, viewed as an $R$–linear homological functor $[D_{\text{perf}}(X)]^{\text{op}} \rightarrow R$–Mod.

Then $\tilde{Y}$ is fully faithful, and the essential image is the set of finite $R$–linear homological functors $H: D_{\text{perf}}(X)^{\text{op}} \rightarrow R$–mod. An $R$–linear homological functor is finite if, for all objects $C \in D_{\text{perf}}(X)$, the $R$–module $\oplus_n H(C[n])$ is finite.

(iv) Suppose $X$ is finite dimensional scheme proper over a noetherian ring $R$, and assume further that every closed, reduced, irreducible subscheme of $X$ has a regular alteration. Let $\overline{Y}$ be the Yoneda map

$$[D_{\text{perf}}(X)]^{\text{op}} \xrightarrow{\overline{Y}} \text{Hom}_R\left(D^b_{\text{coh}}(X), R-\text{Mod}\right)$$

That is: the map $\overline{Y}$ takes an object $A \in D_{\text{perf}}(X)$ to the functor $\overline{Y}(A) = \text{Hom}(A, -)$, viewed as an $R$–linear homological functor $D^b_{\text{coh}}(X) \rightarrow R$–Mod.

Then $\overline{Y}$ is fully faithful, and the essential image of $\overline{Y}$ are the finite homological functors.

(v) Let $X$ be a noetherian, separated scheme. There is a recipe which takes the triangulated category $D_{\text{perf}}(X)$ as input, and out of it constructs the triangulated category $D^b_{\text{coh}}(X)$. And there is a recipe going back: from the triangulated category $[D^b_{\text{coh}}(X)]^{\text{op}}$ as input the machine spews out $[D_{\text{perf}}(X)]^{\text{op}}$.

**Explanation 6.4.** We remind the reader what the terms used in the theorems mean.

Let $S$ be a triangulated category, and let $G \in S$ be an object. Then

(i) $G$ is a classical generator if $S = \bigcup_n \langle G \rangle^{[-n,n]}$.

(ii) $G$ is a strong generator if there exists an integer $\ell > 0$ with $S = \bigcup_n \langle G \rangle^{[-n,n]}$. The category $S$ is called strongly generated if it has a strong generator.

(iii) Suppose $X$ is a noetherian scheme, finite-dimensional, reduced and irreducible. A regular alteration of $X$ is a generically finite, proper, surjective morphism $\tilde{X} \rightarrow X$ with $\tilde{X}$ regular.

The non-expert deserves some explanation of (iii): we all know what a resolution of singularities is, but the known existence theorems are too restrictive (for our purposes). Of course resolutions of singularities are conjectured to exist quite generally, unfortunately
what has been proved so far is limited to equal characteristic zero, or to schemes of very low dimension. Regular alterations are less restrictive, and the known existence theorems are much more general—see de Jong [13, 14].

As it turns out, in the proofs of Facts 6.3 (ii) and (iv) regular alterations suffice. The non-expert should therefore view the condition imposed on the noetherian scheme $X$, in Facts 6.3 (ii) and (iv), as a mild technical hypothesis.

**Remark 6.5.** The reader should note that Fact 6.2 asserts that the category $D_{qc}(X)$ is approximable, and now we’re telling the reader that the consequences—Facts 6.3 (i), (ii), (iii), (iv) and (v)—are all assertions about the categories $D^{perf}(X)$ and $D^{b}_{coh}(X)$. A technical, formal statement, about the huge category $D_{qc}(X)$, turns out to have a string of powerful consequences about the much smaller categories $D^{perf}(X)$ and $D^{b}_{coh}(X)$, that many people have been studying for decades.

**7. More about the strong generation of $D^{perf}(X)$ and $D^{b}_{coh}(X)$**

As promised, we will now say a little more about Facts 6.3 (i) and (ii). In this section we will survey what was known before, the basic idea of the old proofs, and how the proof based on approximability departs from the older methods.

The non-algebraic-geometers are advised to skip the discussions of the proofs. The brief summary is that the proofs based on approximability are short, simple, sweet and work in great generality—the hard work goes into proving that the category $D_{qc}(X)$ is approximable. After that it’s all downhill.

Let us begin with Facts 6.3(i), we recall the statement for the reader’s convenience:

**Theorem 7.1.** Assume $X$ is quasicompact, separated scheme. Then $D^{perf}(X)$ is strongly generated if and only if $X$ may be covered by open affine subsets $\text{Spec}(R_i)$, with each $R_i$ of finite global dimension.

**Remark 7.2.** If $X$ is noetherian and separated, this simplifies to saying that $D^{perf}(X)$ is strongly generated if and only if $X$ is regular and finite dimensional.

**Historical Survey 7.3.** When $X = \text{Spec}(R)$ is affine Theorem 7.1 is old: it was first proved by Kelly [25], see also Street [40]. The result was rediscovered by Christensen [10, Corollary 8.4] and later Rouquier [39, Proposition 7.25].

Bondal and Van den Bergh [8, Theorem 3.1.4] proved the first global version: if $X$ is a separated scheme, smooth over a field $k$, then the category $D^{perf}(X)$ is strongly generated. The case where $X$ is assumed of finite type over a field and regular [regularity is weaker than smoothness] follows from either Rouquier [39, Theorem 7.38] or Orlov [37, Theorem 3.27].

This summarizes the results known before approximability. Note that, with the exception of Kelly’s, the old results all assumed equal characteristic and that $X$ is noetherian. By contrast Theorem 7.1 works fine in the mixed characteristic, non-noetherian situation.
Discussion of the Proofs, Old and New 7.4. By combining Kelly’s old theorem \cite{25} with the main theorem of Thomason and Trobaugh \cite{41}, one easily deduces one of the implications in Theorem 7.1 if $X$ is quasicompact and separated, and $\text{Spec}(R)$ embeds in $X$ as an open, affine subset, then $R$ must be of finite global dimension. The reader can find the argument spelled out in more detail in (for example) \cite{32, Remark 0.11}.

Now for the tricky direction of Theorem 7.1, the direction saying that, if $X$ is quasicompact and separated, and admits a cover by open affines $\text{Spec}(R_i)$ with each $R_i$ of finite global dimension, then it follows that $\mathsf{D}_{\text{perf}}(X)$ is strongly generated. As we have already said: the case where $X$ is affine is contained in Kelly’s old theorem.

We remind the reader: Bondal and Van den Bergh \cite[Theorem 3.1.4]{8} proved the first global version. They proved that, if $X$ is a separated scheme, smooth over a field $k$, then the category $\mathsf{D}_{\text{perf}}(X)$ is strongly generated. Their proof relies on the fact that, if $\delta : X \rightarrow X \times_k X$ is the diagonal embedding, then the functor $R\delta_\ast$ respects perfect complexes. It is a characterization of smoothness for $R\delta_\ast$ to respect perfect complexes—hence the argument isn’t one that readily lends itself to generalizations.

Nevertheless there were improvements. The case where $X$ is assumed of finite type over a field and regular follows from either Rouquier \cite[Theorem 7.38]{39} or Orlov \cite[Theorem 3.27]{37}. Both proofs still use a diagonal argument—Rouquier’s approach refines Bondal and Van den Bergh’s by stratifying $X$, while the refinement in Orlov’s article is not quite so easy to sum up briefly. It was Orlov’s clever approach to the problem that inspired the idea of approximability.

It remains to give the reader some idea how approximability helps in the proof of Theorem 7.1. And the main point is that approximability allows us to reduce the general case to the case of an affine scheme, where we can use the old theorem of Kelly’s. For the reader’s convenience and because the proof is so easy, we prove below the variant of Kelly’s theorem we will actually use.

**Theorem 7.5.** Suppose $R$ is an associative ring, and $\mathsf{D}(R)$ its derived category. Let $n \geq 0$ be an integer, and let $F \in \mathsf{D}(R)$ be an object such that the projective dimension of $H^i(F)$ is $\leq n$ for all $i \in \mathbb{Z}$. Then $F \in \langle R \rangle^{-\infty}_{n+1}$.

Before proving the theorem we remind the reader [who is familiar with the calculus of fractions in derived categories]: any morphism $P \rightarrow H^i(E)$ in $\mathsf{D}(R)$, for any projective $R$-module $P$ and any $E \in \mathsf{D}(R)$, lifts uniquely to a cochain map

\[
\begin{array}{cccccccc}
\cdots & & 0 & & 0 & & P & & 0 & & 0 & & \cdots \\
& \downarrow & & & & & & \downarrow & & & & \downarrow & & \\
& & & E^i & & E^i & & E^{i+1} & & E^{i+2} & & \cdots
\end{array}
\]
Proof. We prove the theorem by induction on $n$. Suppose first that $n = 0$; hence $H^i(F)$ is projective for every $i \in \mathbb{Z}$. The identity map $H^i(F) \to H^j(F)$ lifts to a cochain map

$$
\cdots \to 0 \to 0 \to H^i(F) \to 0 \to 0 \to \cdots
$$

and when we combine, for every $i \in \mathbb{Z}$, we obtain a cochain map

$$
\cdots \to H^{-2}(F) \to H^{-1}(F) \to H^0(F) \to H^1(F) \to H^2(F) \to \cdots
$$

This is an isomorphism in cohomology, hence an isomorphism in $D(R)$.

Now suppose $n \geq 0$, and we know the result for every $\ell$ with $0 \leq \ell \leq n$. We wish to show it for $n + 1$. Suppose therefore that we are given an object $F \in D(R)$ with $H^i(F)$ of projective dimension $\leq n + 1$ for every $i$. Choose for every $i$ a projective module $P^i$ and a surjection $P^i \to H^i(F)$. Now form the corresponding cochain map

$$
\cdots \to 0 \to 0 \to P^i \to 0 \to 0 \to \cdots
$$

and combine over $i$ to form

$$
\cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to P^2 \to \cdots
$$

giving a map $P \to F$, which we complete to a triangle $P \to F \to Q$. Clearly $P \in (R)^{-\infty,\infty}_1$, and the long exact sequence in cohomology gives that $H^i(Q)$ is of projective dimension $\leq n$. Hence $F$ belongs to $(R)^{-\infty,\infty}_1 \ast (R)^{(-\infty,\infty)}_{n+1} \subset (R)^{(-\infty,\infty)}_{n+2}. \ \Box$

At this point the non-algebraic-geometer (who hasn’t yet done so) is advised to skip ahead to Theorem 7.12. What will come between now and then is largely aimed to show that the approximability of $D_{qc}(X)$ makes the reduction to Kelly’s old theorem straightforward and easy—hopefully the sketch we give will make this transparent to the experts, but for non-algebraic-geometers it might be mystifying. Anyway: the reduction depends on the following little lemma—the reader should note the way approximability enters the proof of the lemma, this is the only point where approximability will be used.
Lemma 7.6. Let $X$ be a quasicompact, separated scheme, let $G \in \text{D}_{\text{qc}}(X)$ be a compact generator, and let $u : U \rightarrow X$ be an open immersion with $U$ quasicompact. Then the object $Ru_*\mathcal{O}_U \in \text{D}_{\text{qc}}(X)$ belongs to $\langle G \rangle^{[−n,n]}_n$ for some integer $n > 0$.

Proof. It is relatively easy to show that, for some sufficiently large integer $\ell > 0$, we have $\text{Hom}(Ru_*\mathcal{O}_U, \text{D}_{\text{qc}}(X)^{≤−\ell}) = 0$. By the approximability of $\text{D}_{\text{qc}}(X)$ we may choose an integer $n$ and a triangle $E \rightarrow Ru_*\mathcal{O}_U \rightarrow D$ with $D \in \text{D}_{\text{qc}}(X)^{≤−\ell}$ and $E \in \langle G \rangle^{[−n,n]}_n$.

But the map $Ru_*\mathcal{O}_U \rightarrow D$ must vanish by the choice of $\ell$, making $Ru_*\mathcal{O}_U$ a direct summand of the object $E \in \langle G \rangle^{[−n,n]}_n$. \hfill $\Box$

Sketch 7.7. We should indicate how Theorem 7.1 follows from the combination of Lemma 7.6 and Kelly’s old theorem. Let $X$ and the open affine cover by $U_i = \text{Spec}(R_i)$ be as in the hypotheses of Theorem 7.1. Because $X$ is quasicompact we may, possibly after passing to a subcover, assume that our cover is finite; write the cover as $\{U_i, 1 \leq i \leq r\}$.

Now choose a compact generator $G \in \text{D}_{\text{qc}}(X)$. The Lemma allows us to choose, for each open subset $U_i$, an integer $n_i$ so that $Ru_i*\mathcal{O}_{U_i} \in \langle G \rangle^{[−n_i,n_i]}_{n_i}$. Let $n$ be the maximum of the finitely many $n_i$; then $Ru_i*\mathcal{O}_{U_i} \in \langle G \rangle^{[−n,n]}_n \subset \langle G \rangle^{(−\infty,\infty)}_{−\infty}$ for every $i$ in the finite set.

Next, for each $i$ we know that $U_i = \text{Spec}(R_i)$ with $R_i$ is of finite global dimension, and Theorem 7.5 tells us that we may choose an integer $\ell > 0$ so that, for every one of the finitely many $i$ with $1 \leq i \leq r$, we have $\text{D}_{\text{qc}}(U_i) = \langle \mathcal{O}_{U_i} \rangle^{(−\infty,\infty)}_{\ell}$. It follows that

$$Ru_i*\text{D}_{\text{qc}}(U_i) = Ru_i*[\langle \mathcal{O}_{U_i} \rangle^{(−\infty,\infty)}_{\ell}] \subset \langle Ru_i*\mathcal{O}_{U_i} \rangle^{(−\infty,\infty)}_{\ell} \subset \langle G \rangle^{(−\infty,\infty)}_{\ell n}$$

Let $\mathcal{V} = \text{add}[\bigcup_{i=1}^{r} Ru_i*\text{D}_{\text{qc}}(U_i)]$, with the notation as in Reminder 4.1(ii). By the displayed inclusion above $[\bigcup_{i=1}^{r} Ru_i*\text{D}_{\text{qc}}(U_i)] \subset \langle G \rangle^{(−\infty,\infty)}_{\ell n}$, and as $\langle G \rangle^{(−\infty,\infty)}_{\ell n}$ is closed under (finite) coproducts it follows that $\mathcal{V} \subset \langle G \rangle^{(−\infty,\infty)}_{\ell n}$.

It’s an exercise to show that

$$\text{D}_{\text{qc}}(X) = \bigvee \mathcal{V} \ast \bigvee \cdots \ast \mathcal{V}$$

with the notation as in Reminder 4.1(i). Hence $\text{D}_{\text{qc}}(X) = \langle G \rangle^{(−\infty,\infty)}_{\ell n}$. We have proved a statement about $\text{D}_{\text{qc}}(X)$, and in Notation 2.13 we learned that $\text{D}^{\text{perf}}(X)$ is equal to the subcategory of compact objects in $\text{D}_{\text{qc}}(X)$. Standard compactness arguments tell us that from the equality $\text{D}_{\text{qc}}(X) = \langle G \rangle^{(−\infty,\infty)}_{\ell n}$ we can formally deduce the equality $\text{D}^{\text{perf}}(X) = \bigcup_{m \geq 0} \langle G \rangle^{[−m,m]}_{\ell n}$. \hfill $\Box$

---

5This isn’t immediate from the definition of approximability, but follows from the structural theorems. We are using the fact that $Ru_*\mathcal{O}_U \in \text{D}_{\text{qc}}(X)^{≤m}$ for some $m > 0$, coupled with the fact that one can approximate objects in $\text{D}_{\text{qc}}(X)^{≤0}$ to arbitrary order, not just to order zero as given in the definition. See Sketch 8.10(i) for more detail.
We want to highlight the power of approximability. Sketch 7.7 was meant to show the expert that Theorem 7.1 is easy to deduce by combining Kelly’s old theorem with Lemma 7.6, and the proof of Lemma 7.6 displays how the lemma follows immediately from the fact that $D_{qc}(X)$ is approximable.

While we’re into exhibiting the power of approximability, let us mention another corollary of Lemma 7.6—and therefore another easy consequence of approximability.

**Theorem 7.8.** Suppose $f : X \rightarrow Y$ is a separated morphism of quasicompact, quasiseparated schemes. If $Rf_* : D_{qc}(X) \rightarrow D_{qc}(Y)$ takes perfect complexes to complexes of bounded–below Tor-amplitude then $f$ must be of finite Tor-dimension.

**Reminder 7.9.** We owe the reader a glossary of the technical terms in the statement of Theorem 7.8.

(i) Given a morphism of schemes $f : X \rightarrow Y$, for any $x \in X$ there is an induced ring homomorphism $\mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ of the stalks. The map $f$ is of finite Tor-dimension at $x$ if $\mathcal{O}_{X,x}$ has a finite flat resolution over $\mathcal{O}_{Y,f(x)}$.

(ii) The map $f$ is of finite Tor-dimension if it is of finite Tor-dimension at every $x \in X$.

(iii) The complex $C \in D_{qc}(Y)$ is of bounded-below Tor-amplitude if, for every open immersion $u : U \rightarrow Y$ with $U = \text{Spec}(R)$ affine, the complex $u^*C \in D_{qc}(U) \cong D(R)$ is isomorphic to a bounded-below $K$–flat complex.

**Historical Survey 7.10.** We should tell the reader what was known in the direction of Theorem 7.8. If the schemes $X$ and $Y$ are noetherian and $f : X \rightarrow Y$ is of finite type, then the converse of Theorem 7.8 is known and old—the reader may find it in Illusie [19, Corollaire 4.8.1]. The direction proved in Theorem 7.8 was open for a long time, the first progress was in [29]. But the statement in [29] is much narrower than Theorem 7.8, it is confined to the situation where $f$ is proper.

**Sketch 7.11.** We should give the reader some idea why Theorem 7.8 follows easily from Lemma 7.6—this discussion is for algebraic geometers, the non-specialists are advised to skip ahead to Theorem 7.12.

It’s obviously local in $Y$ to determine if $f$ is of finite Tor-dimension. Using the main theorem of Thomason and Trobaugh [41], it’s also local in $Y$ to determine whether $Rf_*$ takes perfect complexes to complexes of bounded-below Tor-amplitude. Hence we may assume $Y$ is affine, therefore separated. As $f$ is separated we deduce that $X$ must be separated.

We are given that $Rf_*$ takes perfect complexes to complexes of bounded-below Tor-amplitude, and wish to show that $f$ is of finite Tor-dimension. Being of finite Tor-dimension is local in $X$; it suffices to show that, for each of open immersion $u : U \rightarrow X$ with $U$ affine, the composite $U \xrightarrow{u} X \xrightarrow{f} Y$ is of finite Tor-dimension. By Lemma 7.6 there exists a perfect complex $G \in D_{qc}(X)$ and an integer $n > 0$ with $Ru_*\mathcal{O}_U \in \langle G \rangle_{-n,n}$. 

Therefore

$$(fu)_*\mathcal{O}_U \cong R(fu)_*\mathcal{O}_U \cong Rfu_*\mathcal{O}_U \in Rf_*[\langle \mathcal{G}/n, -n, n \rangle \mathcal{G}] \subset \langle Rf_*\mathcal{G} \rangle_{[-n,n]}$$

But $Rf_*\mathcal{G}$ is of bounded below Tor-amplitude by hypothesis, and in forming $\langle Rf_*\mathcal{G} \rangle_{[-n,n]}$ we only allow $Rf_*\mathcal{G}[i]$ with $-n \leq i \leq n$, coproducts, extensions and direct summands. Hence the objects of $\langle Rf_*\mathcal{G} \rangle_{[-n,n]}$ have Tor-amplitude uniformly bounded below.

It’s time to turn our attention to Fact 6.3(ii), we remind the reader of the statement:

**Theorem 7.12.** Let $X$ be a separated, noetherian, finite-dimensional scheme, and assume that every closed, reduced, irreducible subscheme of $X$ has a regular alteration. Then the category $D^b_{coh}(X)$ is strongly generated.

**Historical Survey 7.13.** We should tell the reader what was known in the direction of Theorem 7.12. We have already alluded to the fact that, when $X$ is regular and finite-dimensional, the inclusion $D^{perf}(X) \rightarrow D^b_{coh}(X)$ is an equivalence and Theorem 7.1 tells us that the equivalent categories $D^{perf}(X) \cong D^b_{coh}(X)$ are strongly generated. Using a stratification, of a possibly singular $X$, Rouquier [39, Theorem 7.38] built and substantially extended on the argument in Bondal and Van den Bergh [8, Theorem 3.1.4] to show that $D^b_{coh}(X)$ is strongly generated whenever $X$ is a separated scheme of finite type over a perfect field $k$. The preprint by Keller and Van den Bergh [24, Proposition 5.1.2] generalized to separated schemes of finite type over arbitrary fields, but this Proposition disappeared in the passage to the published version [23]. The reader might also wish to look at Lunts [30, Theorem 6.3] for a different approach to the proof, but still using stratifications. If we specialize the result of Rouquier, extended by Keller and Van den Bergh, to the case where $X = \text{Spec}(R)$ is an affine scheme, we learn that $D^b(R\text{-mod})$ is strongly generated whenever $R$ is of finite type over a field $k$.

Note that, while Theorem 7.1 is easy and classical in the case where $X$ is affine, Theorem 7.12 is *neither easy nor classical for affine $X$*. In recent years there has been interest among commutative algebraists in understanding this better: the reader is referred to Aihara and Takahashi [2], Bahlekeh, Hakimian, Salarian and Takahashi [5] and Iyengar and Takahashi [21] for a sample of the literature. There is also a connection with the concept of the radius of the (abelian) category of modules over $R$; see Dao and Takahashi [11, 12] and Iyengar and Takahashi [21]. The union of the known results seems to be that $D^b(R\text{-mod})$ is strongly generated if $R$ is an equicharacteristic excellent local ring, or essentially of finite type over a field—see [21, Corollary 7.2]. In [21, Remark 7.3] it is observed that there are examples of commutative, noetherian rings for which $D^b(R\text{-mod})$ is *not* strongly generated.

The structure of the proof of Theorem 7.12 (see Sketch 7.14) is that one passes to regular alterations of $X$ and its closed subschemes. Assuming $X$ affine is no help with the approximability proof of Theorem 7.12—when $X$ is affine and singular we end up
proving a result in commutative algebra, but the technique of the proof passes through non-affine schemes.

Unlike all the pre-approximability results, except Kelly’s, Theorems 7.11 and 7.12 do not assume equal characteristic.

**Sketch 7.14.** We should tell the reader a little about the proof. But first we should make it clear that Theorem 7.12 will not be proved using approximability directly, instead we will prove it as a corollary of Theorem 7.11 which followed from approximability. Precisely: Theorem 7.1 and Theorem 7.12 are identical when $X$ is a finite-dimensional, regular, noetherian, separated scheme. And the idea is to reduce to this case.

Resolutions of singularities might look tempting, but in mixed characteristic they are known to exist only in low dimension. So the key is that we can get by with regular alterations—the hypotheses of the theorem say that they exist for every closed subvariety of $X$, and it turns out that Theorem 7.12 can be deduced from this using induction on the dimension of $X$ and two old theorems of Thomason’s.

This survey has been stressing that the hard work goes into proving approximability, the consequences are all easy corollaries. Theorem 7.12 must count as an exception, the argument is tricky. It might be relevant to note that in this field—noncommutative algebraic geometry—there are quite a number of theorems that are known in characteristic zero with proofs that rely on resolutions of singularities, and conjectured in positive characteristic. I wasn’t the first to come up with the idea of trying to use de Jong’s theorem, in other words trying to prove these conjectures using regular alterations. So far Theorem 7.12 is the only success story. It isn’t regular alterations alone that do the trick, it’s the combination of regular alterations and support theory—in this case support theory manifests itself as the two old theorems of Thomason’s.

**Problem 7.15.** There is a non-commutative version—Kelly’s old theorem doesn’t assume commutativity. This raises the obvious question: to what extent do the more recent theorems extend beyond commutative algebraic geometry?

Perhaps we should explain, and for simplicity let us stick to the case where $X = \text{Spec}(R)$ is affine. As we have presented the theory, up to now, we have implicitly been assuming that the ring $R$ is commutative. But what Kelly proved doesn’t depend on commutativity—the reader can see this for herself, just look at the proof of Theorem 7.5.

Let $R$ be any associative ring and let $D^b(R\text{-proj})$ be the derived category of bounded complexes of finitely generated, projective $R$–modules. Kelly’s 1965 theorem says that $D^b(R\text{-proj})$ has a strong generator if and only if $R$ is of finite global dimension.

All the later theorems listed above, including the recent ones whose proof relies on approximability, assume commutativity. In particular: assume $R$ is a commutative, noetherian ring, of finite type over an excellent ring of dimension $\leq 2$. Theorem 7.12, in the special case where $X = \text{Spec}(R)$, tells us that the category $D^b_{\text{coh}}(X) \cong D^b(R\text{-mod})$ is strongly generated. The category $D^b(R\text{-mod})$ has for objects the bounded complexes of finite $R$–modules.
Is the commutativity hypothesis necessary in the above? Is there some large class of noncommutative, noetherian rings for which $D^b(R\text{-mod})$ is strongly generated? The proof in the commutative case, which goes by way of the regular alterations of de Jong, doesn’t seem capable of a noncommutative extension.

**Remark 7.16.** Recall: a strong generator in $\mathcal{S}$ is an object $G \in \mathcal{S}$ such that, for some integer $\ell > 0$, we have $\mathcal{S} = \bigcup_n \langle G \rangle_{\ell}^{[-n,n]}$. One can ask for estimates on $\ell$. This leads to the definitions

(i) Given objects $G, F \in \mathcal{S}$, the $G$-level of $F$ is the smallest integer $\ell$ such that $F \in \bigcup_n \langle G \rangle_{\ell+1}^{[-n,n]}$. This notion is due to Avramov, Buchweitz and Iyengar [4].

(ii) Let $G$ be an object of $\mathcal{S}$. The generation time of $G$ is the smallest $\ell$ for which $\mathcal{S} = \bigcup_n \langle G \rangle_{\ell+1}^{[-n,n]}$. The set of all possible generation times, taken over all strong generators $G \in \mathcal{S}$, is known as the Orlov spectrum of $\mathcal{S}$. These notions first appeared in Orlov [36].

(iii) The Rouquier dimension of $\mathcal{S}$ is the smallest integer $\ell$ such that there exists a $G$ with $\mathcal{S} = \bigcup_n \langle G \rangle_{\ell+1}^{[-n,n]}$. This integer first appeared in Rouquier [39]—Rouquier’s name for this number was just plain “dimension”.

There are several conjectures, and many papers estimating these numbers—almost all in the equal characteristic case, after all until recently there was no existence theorem of strong generators in mixed characteristic. One can ask if the theorems surveyed in this article give good bounds in mixed characteristic—and the short answer is No. In more detail:

(iv) If we assume that $X$ is regular and quasiprojective, then the proof of Application 1(i) is effective. It gives an explicit upper bound on the Rouquier dimension of $D^\text{perf}(X) = D^b_{\text{coh}}(X)$. But the bound is dreadful.

(v) If we drop the quasiprojectivity hypothesis, and/or if we allow singularities, then the proof becomes ineffective. It proves the existence of an integer $\ell > 0$ and a generator $G$ with $D^b_{\text{coh}}(X) = \bigcup_n \langle G \rangle_{\ell+1}^{[-n,n]}$, but there is no estimate on $\ell$.

**Elaboration 7.17.** The following is for the benefit of the readers who would like Remark 7.16 spelt out a little more.

In [32, Section 4] the reader can find the argument that proves the approximability of $D_{\text{qf}}(X)$ when $X$ is quasiprojective—and in [32, Proposition 4.4] it’s made clear that the estimates are explicit. And, assuming $X$ is not only quasiprojective but also regular, Sketch 7.7 shows us how to pass from the estimates given by approximability to explicit estimates on the $\ell$ for which $D^\text{perf}(X) = \bigcup_n \langle G \rangle_{\ell+1}^{[-n,n]}$. Still assuming that $X$ is quasiprojective and regular, a careful reading of the proof of [32, Proposition 4.4] will show us that these crude estimates can easily be improved. But our point for now is that the proof is effective, it gives bounds.

The general proof of approximability, for quasicompact, separated $X$, is by reduction to the quasiprojective case. This reduction goes by
(i) We first use noetherian approximation to reduce to the case where $X$ is of finite type over $\mathbb{Z}$.

(ii) Next we use induction on the dimension coupled with Chow’s Lemma to reduce to the quasiprojective case.

The way to use noetherian approximation is straightforward enough—we don’t go into detail, leaving this to the reader’s imagination. In principle this part could be made effective, but not in a way that yields good bounds. Still: so far there are bounds of some sort.

But the true subtlety arises in (ii), with the way we use Chow’s Lemma. Chow’s Lemma produces for us a birational, projective morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ quasiprojective. And two remarks are in order

(iii) Chow’s Lemma is where we have to assume $X$ separated. This is the point of the proof where it doesn’t suffice for $X$ to be quasiseparated.

(iv) Since $\tilde{X}$ is quasiprojective the category $\mathcal{D}_{\text{qc}}(\tilde{X})$ is approximable. To deduce from this useful information about $\mathcal{D}_{\text{qc}}(X)$ we need to take a compact generator $\tilde{G} \in \mathcal{D}_{\text{qc}}(\tilde{X})$, and approximate $R\pi_*\tilde{G}$ using a compact generator $G \in \mathcal{D}_{\text{qc}}(X)$.

There is a way to achieve (iv), it relies on [29, Theorem 4.1]. But the proof of [29, Theorem 4.1] is homotopy-theoretic, it hinges on the fact that, for a compact object $G$, the functor $\text{Hom}(G, -)$ commutes with homotopy colimits. Ignoring the technicality that we work with homotopy colimits and not ordinary colimits, the point is the following. Any map from a compact object $G$ to an object $\text{colim} T_i$ factors through some $T_i$, but we have no control over how large $i \in \mathbb{N}$ will have to be. Thus any argument which has, embedded in it, the appeal to such homotopy colimit arguments, is inherently and hopelessly ineffective. This explains why we lose control when $X$ isn’t assumed quasiprojective.

If we allow $X$ to become singular, then Sketch 7.14 hints how to reduce to the regular case using regular alterations. As was said in Sketch 7.14, the proof appeals to two old theorems of Thomason’s, both of which depend on homotopy-colimit arguments. Hence this passage is also ineffective beyond salvation.

8. More about finite $R$–linear functors $H : [\mathcal{D}^{\text{perf}}(X)]^{\text{op}} \to R\text{-Mod}$ and $\tilde{H} : \mathcal{D}^{b}_{\text{coh}}(X) \to R\text{-Mod}$

It’s time to expand on Facts 6.3 (iii) and (iv). We begin by recalling the statements.

Theorem 8.1. Let $X$ be a finite-dimensional scheme proper over a noetherian ring $R$. Let $\mathcal{Y}$ the Yoneda map

$$\begin{align*}
\mathcal{D}^{b}_{\text{coh}}(X) & \xrightarrow{\mathcal{Y}} \text{Hom}_R\left(\mathcal{D}^{\text{perf}}(X)^{\text{op}}, R\text{-Mod}\right) \\
\end{align*}$$
taking $B \in D^b_{\text{coh}}(X)$ to the functor $\text{Hom}(-, B)$, and let $\tilde{\mathcal{Y}}$ be the Yoneda map

$$[D^\text{perf}(X)]^{\text{op}} \xrightarrow{\tilde{\mathcal{Y}}} \text{Hom}_R\left(D^b_{\text{coh}}X, R\text{-Mod}\right)$$

taking $A \in D^\text{perf}(X)$ to the functor $\text{Hom}(A, -)$. Assuming every closed subvariety of $X$ admits a regular alteration both functors are fully faithful, and in each case the essential image is the set of finite $R$–linear homological functors. Recall: an $R$–linear homological functor $H : S \rightarrow R\text{-Mod}$ is finite if, for all objects $C \in S$, the $R$–module $\oplus_n H(C[n])$ is finite.

For the functor $\mathcal{Y}$ the assertion is true even without the hypotheses of finite-dimensionality and the existence of regular alterations.

**Historical Survey 8.2.** We remind the reader what was known before.

(i) If $X$ is proper over $R$, if $A \in D^\text{perf}(X)$ and if $B \in D^b_{\text{coh}}(X)$, then

$$\text{Hom}(A[i], B) \cong H^{-i}(A^i \otimes B)$$

is a finite $R$–module for every $i$ and vanishes outside a bounded range. This much was proved by Grothendieck [17, Théorème 3.2.1].

Translating to the language of Theorem 8.1 given objects $A \in D^\text{perf}(X)$ and $B \in D^b_{\text{coh}}(X)$, then $\mathcal{Y}(B) = \text{Hom}(-, B)$ is a finite homological functor on $[D^\text{perf}(X)]^{\text{op}}$, while $\mathcal{Y}(A) = \text{Hom}(A, -)$ is a finite homological functor on $D^b_{\text{coh}}(X)$. This much has been known since 1961.

(ii) As long as $R$ is a field, Bondal and Van den Bergh [8, Theorem A.1] proved that every finite homological functor on $[D^\text{perf}(X)]^{\text{op}}$ is $\text{Hom}(-, B)$ for some $B \in D^b_{\text{coh}}(X)$. In the language of Theorem 8.1 they proved that the essential image of $\mathcal{Y}$ consists of the finite homological functors.

(iii) Still assuming $R$ is a field, the assertion of Theorem 8.1 about the functor $\tilde{\mathcal{Y}}$ can be found in Rouquier [39, Corollary 7.51(ii)]—although the author of the present article doesn’t follow the argument in [39] that briefly outlines how a proof might go, it’s too skimpy.

If $R$ is a field Theorem 8.1 improves on what was known about the functor $\mathcal{Y}$ by showing that it’s fully faithful. And for $R$ more general Theorem 8.1 is new, for both the functor $\mathcal{Y}$ and the functor $\tilde{\mathcal{Y}}$.

And now the time has come to tell the reader something about the proof of Theorem 8.1. It turns out that the theorem is an immediate corollary of a far more general fact, and the discussion of this result brings us naturally to the structure that all approximable categories share. Let us begin in even greater generality, not assuming all the hypotheses of approximability.

**Definition 8.3.** Let $\mathcal{T}$ be a triangulated category, and let $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ and $(\mathcal{T}^{'\leq 0}, \mathcal{T}^{'\geq 0})$ be two $t$–structures on $\mathcal{T}$. We declare them equivalent if there exists an integer $A > 0$ with $\mathcal{T}^{-A} \subset \mathcal{T}^{'\leq 0} \subset \mathcal{T}_1^{\leq A}$.
The definition agrees with the intuition of the Introduction: each \( t \)-structure defines a kind of (directed) metric, and we’d like declare \( t \)-structures equivalent whenever they induce equivalent metrics. And now we recall

**Remark 8.4.** Let \( \mathcal{T} \) be a triangulated category with coproducts, and let \( G \in \mathcal{T} \) be a compact generator. From Alonso, Jeremías and Souto [3, Theorem A.1] we learn that \( \mathcal{T} \) has a unique \( t \)-structure \( (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) \) generated by \( G \).

It is not difficult to show that, if \( G \) and \( H \) are two compact generators for \( \mathcal{T} \), then the \( t \)-structures \( (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) \) and \( (\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}) \) are equivalent. Thus up to equivalence there is a preferred \( t \)-structure on \( \mathcal{T} \), namely \( (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) \) where \( G \) is a compact generator. We say that a \( t \)-structure \( (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) \) is in the preferred equivalence class if it is equivalent to \( (\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}) \) for some compact generator \( G \), hence for every compact generator.

**Discussion 8.5.** Given a \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) it is customary to define the categories

\[
\mathcal{T}^{-} = \bigcup_n \mathcal{T}^{\leq n}, \quad \mathcal{T}^{+} = \bigcup_n \mathcal{T}^{\geq -n}, \quad \mathcal{T}^{b} = \mathcal{T}^{-} \cap \mathcal{T}^{+}
\]

as in Notation 2.19. It’s obvious from Definition 8.3 that equivalent \( t \)-structures yield identical \( \mathcal{T}^{-} \), \( \mathcal{T}^{+} \) and \( \mathcal{T}^{b} \).

Now assume we are in the situation of Remark 8.4, that is \( \mathcal{T} \) has coproducts and there exists a single compact generator \( G \). Then there is a preferred equivalence class of \( t \)-structures and, correspondingly, preferred \( \mathcal{T}^{-} \), \( \mathcal{T}^{+} \) and \( \mathcal{T}^{b} \). These are intrinsic, they’re independent of any choice. In the remainder of the article we only consider the “preferred” \( \mathcal{T}^{-} \), \( \mathcal{T}^{+} \) and \( \mathcal{T}^{b} \).

Slightly more sophisticated is the category \( \mathcal{T}^{-c} \) below.

**Definition 8.6.** Let \( \mathcal{T} \) be a triangulated category with coproducts, and assume it has a compact generator \( G \). Choose a \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) in the preferred equivalence class. The full subcategory \( \mathcal{T}^{-c} \) is defined by

\[
\mathcal{T}^{-c} = \left\{ F \in \mathcal{T} \left| \begin{array}{l}
\text{For all integers } n > 0 \text{ there exists a triangle } \\
E \to F \to D \to E[1]
\end{array} \right. \right. \\
\text{with } E \text{ compact and } D \in \mathcal{T}^{\leq -n-1}
\right\}
\]

We furthermore define \( \mathcal{T}^{-c} = \mathcal{T}^{b} \cap \mathcal{T}^{-c} \).

**Remark 8.7.** Intuitively the category \( \mathcal{T}^{-c} \) is the closure, with respect to the metric induced by the \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \), of the subcategory \( \mathcal{T}^{c} \) of all compact objects in \( \mathcal{T} \). It’s obvious that the category \( \mathcal{T}^{-c} \) is intrinsic, after all equivalent metrics will lead to the same closure. And as \( \mathcal{T}^{-c} \) and \( \mathcal{T}^{b} \) are both intrinsic, so is their intersection \( \mathcal{T}^{b} \).

We have defined all this intrinsic structure, assuming only that \( \mathcal{T} \) is a triangulated category with coproducts and with a single compact generator. In this generality we know that the subcategories \( \mathcal{T}^{-} \), \( \mathcal{T}^{+} \) and \( \mathcal{T}^{b} \) are thick. For the subcategories \( \mathcal{T}^{-c} \) and \( \mathcal{T}^{b} \) one proves
Proposition 8.8. If $\mathcal{I}$ has a compact generator $G$, such that $\text{Hom}(G, G[n]) = 0$ for $n \gg 0$, then the subcategories $\mathcal{I}_c^-$ and $\mathcal{I}_c^b$ are thick.

Remark 8.9. If $\mathcal{I}$ is approximable then, by Definition 5.1(i), there is an integer $A > 0$, a compact generator $G \in \mathcal{I}$ and a $t$–structure $(\mathcal{I}^{-\leq 0}, \mathcal{I}^{\geq 0})$, so that $\text{Hom}(G[-A], \mathcal{I}^{-\leq 0}) = 0$ and $G[A] \in \mathcal{I}^{-\leq 0}$. Hence $G[n] \in \mathcal{I}^{-\leq 0}$ for all $n \geq A$ and $\text{Hom}(G, G[n]) = 0$ for all $n \geq 2A$; Proposition 8.8 therefore tells us that the subcategories $\mathcal{I}_c^-$ and $\mathcal{I}_c^b$ are thick whenever $\mathcal{I}$ is approximable.

Of course it would be nice to be able to work out examples: what does all of this intrinsic structure come down to in special cases? This is where approximability helps. We first note

Proposition 8.10. Assume the category $\mathcal{I}$ is approximable; see Definition 5.1. We recall part of the definition: the category $\mathcal{I}$ is approximable if it has a compact generator $G$, a $t$–structure $(\mathcal{I}^{-\leq 0}, \mathcal{I}^{\geq 0})$ and an integer $A > 0$ satisfying some properties, see Definition 5.1 (i) and (ii) for the properties.

Then any $t$–structure, which comes as part of a triad satisfying the properties of Definition 5.1 (i) and (ii), must be in the preferred equivalence class. Furthermore: for any compact generator $G'$ and any $t$–structure $(\mathcal{I}^{-\leq 0}, \mathcal{I}^{\geq 0})$ in the preferred equivalence class, there is an integer $A' > 0$ so that the properties of Definition 5.1 (i) and (ii) hold.

In practice this means that, in proving that $\mathcal{I}$ is approximable, we must produce at least one useful $t$–structure that we know belongs to the preferred equivalence class. After all: this $t$–structure must be manageable enough to lend itself to a proof that the conditions in Definition 5.1(i) and (ii) hold. Note that the proof of Alonso, Jeremías and Souto [3, Theorem A.1] yields a $t$–structure $(\mathcal{I}_{G}^{-\leq 0}, \mathcal{I}_{G}^{\geq 0})$ in the preferred equivalence class, but the construction is a little opaque—it shows existence and uniqueness, but usually doesn’t give us much of a handle on $(\mathcal{I}_{G}^{-\leq 0}, \mathcal{I}_{G}^{\geq 0})$. So while we know that $t$–structures in the preferred equivalence class exist, this needn’t be especially useful in working with them.

Let $X$ be a quasicompact, separated scheme. We have told the reader that [32, Theorem 5.8] combined with [35, Example 3.6] prove that $\mathcal{I} = D_{qc}(X)$ is approximable; the $t$–structure used in the proof happens to be the standard $t$–structure of Example 2.16(iii) and (iv). We remind the reader: the standard $t$–structure on $D_{qc}(X)$ has

$$D_{qc}(X)^{\leq 0} = \{ F \in D_{qc}(X) \mid \mathcal{H}^i(F) = 0 \text{ for all } i > 0 \}$$
$$D_{qc}(X)^{\geq 0} = \{ F \in D_{qc}(X) \mid \mathcal{H}^i(F) = 0 \text{ for all } i < 0 \}$$

where $\mathcal{H}^i$ is the functor taking a cochain complex to its $i^{th}$ cohomology sheaf. Proposition 8.10 now informs us that the standard $t$–structure must belong to the preferred equivalence class. Hence the categories $\mathcal{I}^-$, $\mathcal{I}^+$ and $\mathcal{I}^b$ are the usual: we have $\mathcal{I}^- = D_{qc}^-(X)$, $\mathcal{I}^+ = D_{qc}^+(X)$ and $\mathcal{I}^b = D_{qc}^b(X)$. The subcategories $D_{qc}(X)$, $D_{qc}^+(X)$ and $D_{qc}^b(X)$ of
$D_{qc}(X)$ are traditionally defined to be

\[ D_{qc}^{-}(X) = \{ F \in D_{qc}(X) \mid H^i(F) = 0 \text{ for all } i \gg 0 \} \]
\[ D_{qc}^{+}(X) = \{ F \in D_{qc}(X) \mid H^i(F) = 0 \text{ for all } i \ll 0 \} \]
\[ D_{qc}^{b}(X) = D_{qc}^{-}(X) \cap D_{qc}^{+}(X) \]

Next we ask ourselves: what about $\mathcal{T}^{-}$ and $\mathcal{T}^{b}$? We begin with the affine case.

**Exercise 8.11.** Let $R$ be a ring. Prove that, in the category $\mathcal{J} = D(R)$, the subcategory $\mathcal{T}^{-}$ agrees with the $D^{-}(R$–proj) of Remark [7]

**Observation 8.12.** Now assume $R$ is a commutative ring, and let $X = \text{Spec}(R)$. Then [7 Theorem 5.1] tells us that the natural functor $D(R) \rightarrow D_{qc}(X)$ is an equivalence of categories. Putting $\mathcal{J} = D_{qc}(X) \cong D(R)$, we learn from Exercise 8.11 what the category $\mathcal{T}^{-}$ is.

Now let $X$ be any quasicompact, separated scheme. If $u : U \rightarrow X$ is an open immersion, then the functor $u^* : D_{qc}(X) \rightarrow D_{qc}(U)$ respects the standard $t$-structure and sends compact objects in $D_{qc}(X)$ to compact objects in $D_{qc}(U)$. Hence $u^*D_{qc}(X)^{-}_{c} \subset D_{qc}(U)^{-}_{c}$. Thus every object in $D_{qc}(X)^{-}_{c}$ must be "locally in $D^{-}(R$–proj)" meaning that for every open immersion $u : \text{Spec}(R) \rightarrow X$ we must have that $u^*D_{qc}(X)^{-}_{c} \subset D^{-}(R$–proj). The objects "locally in $D^{-}(R$–proj)" were first studied by Illusie [19 20] in SGA6. They have a name, they are the pseudocoherent complexes.

The next result is not so obvious. In [20 Theorem 4.1] the reader can find a proof that

**Proposition 8.13.** Let $X$ be a quasicompact, separated scheme. Then every pseudocoherent complex belongs to $D_{qc}(X)^{-}_{c}$. Coupled with Observation 8.12 this teaches us that the objects of $D_{qc}(X)^{-}_{c}$ are precisely the pseudocoherent complexes.

**Remark 8.14.** From now on we will assume the scheme $X$ noetherian and separated. In this case pseudocoherence simplifies: we have $D_{qc}(X)^{-}_{c} = D_{coh}^{-}(X)$. The objects $F \in D_{qc}(X)^{-}_{c}$ are the complexes whose cohomology sheaves $H^n(F)$ are coherent for all $n$, and vanish if $n \gg 0$. And $D_{qc}(X)^{b}_{c}$ is also explicit: it is our old friend, the category traditionally denoted $D_{coh}^{b}(X)$—we first met $D_{coh}^{b}(X)$ in Example [2.3(iv), and it figures prominently in the statement of Theorem 8.1

**Remark 8.15.** In Remark 8.14 we observed that the category $D_{coh}^{b}(X)$ has an intrinsic description as a subcategory of $\mathcal{J} = D_{qc}(X)$, it is $\mathcal{T}^{b}$. The category $D_{perf}(X)$ also has an intrinsic description, it’s the subcategory $\mathcal{T}^{c}$ of all compact objects in $\mathcal{J} = D_{qc}(X)$, see Notation [2.13]. With $\mathcal{J} = D_{qc}(X)$, Theorem 8.1 is a statement about the categories $\mathcal{T}^{c} = D_{perf}(X)$ and $\mathcal{T}^{b} = D_{coh}(X)$—and it turns out to be a special case of the following, infinitely more general assertion.

**Theorem 8.16.** Let $R$ be a noetherian ring, and let $\mathcal{J}$ be an $R$–linear, approximable triangulated category. Suppose there exists in $\mathcal{J}$ a compact generator $G$ so that $\text{Hom}(G,G[n])$
is a finite $R$–module for all $n \in \mathbb{Z}$. Consider the two functors

$$Y : \mathcal{T}_c^b \to \text{Hom}_R([\mathcal{T}_c^c]^{\text{op}}, \text{R–Mod}), \quad \tilde{Y} : \mathcal{T}_c^c \to \text{Hom}_R([\mathcal{T}_c^b]^{\text{op}}, \text{R–Mod})$$

defined by the formulas $Y(B) = \text{Hom}(-, B)$ and $\tilde{Y}(A) = \text{Hom}(A, -)$. Note that, in these formulas, we permit all $A, B \in \mathcal{T}_c^c$. But the $(-)$ in the formula $Y(B) = \text{Hom}(-, B)$ is assumed to belong to $\mathcal{T}_c^c$, whereas the $(-)$ in the formula $\tilde{Y}(A) = \text{Hom}(A, -)$ must lie in $\mathcal{T}_c^b$. Now consider the following composites

$$\begin{array}{ccc}
\mathcal{T}_c^b & \xrightarrow{i} & \mathcal{T}_c^c \\
\mathcal{T}_c^c & \xrightarrow{\tilde{Y}} & \text{Hom}_R([\mathcal{T}_c^c]^{\text{op}}, \text{R–Mod}) \\
\mathcal{T}_c^c & \xrightarrow{Y} & \text{Hom}_R([\mathcal{T}_c^b]^{\text{op}}, \text{R–Mod})
\end{array}$$

We assert:

(i) The functor $Y$ is full, and the essential image consists of the locally finite homological functors [see Explanation 8.17 for the definition of locally finite functors]. The composite $\tilde{Y} \circ i$ is fully faithful, and the essential image consists of the finite homological functors.

(ii) Assume there exists an integer $N > 0$ and an object $G' \in \mathcal{T}_c^b$ with $\mathcal{T} = (G')_N^{-\infty, \infty}$. Then the functor $\tilde{Y}$ is full, and the essential image consists of the locally finite homological functors. The composite $Y \circ \tilde{i}$ is fully faithful, and the essential image consists of the finite homological functors.

**Explanation 8.17.** In the statement of Theorem 8.16, the locally finite functors $[\mathcal{T}_c^c]^{\text{op}} \to \text{R–Mod}$ are those functors $H$ such that

(i) $H(\text{A}[i])$ is a finite $R$–module for every $i \in \mathbb{Z}$ and every $A \in \mathcal{T}_c^c$.

(ii) For fixed $A \in \mathcal{T}_c^c$ we have $H(\text{A}[i]) = 0$ if $i \ll 0$.

while the locally finite functors $H : \mathcal{T}_c^b \to \text{R–Mod}$ are those satisfying

(iii) $H(\text{A}[i])$ is a finite $R$–module for every $i \in \mathbb{Z}$ and every $A \in \mathcal{T}_c^b$.

(iv) For fixed $A \in \mathcal{T}_c^b$ we have $H(\text{A}[i]) = 0$ if $i \ll 0$.

The careful reader will observe that these definitions aren’t dual. The finiteness of $H(\text{A}[i])$ for every $A$ and every $i$ is, of course, obviously self-dual. But the vanishing isn’t. We might be tempted to unify the definitions into

(v) Let $\mathcal{S}$ be a triangulated category. A homological functor $H : \mathcal{S} \to \text{R–Mod}$ is locally finite if $H(\text{A}[i])$ is a finite $R$–module for every $i \in \mathbb{Z}$ and every $A \in \mathcal{S}$, and for fixed $A$ it vanishes when $i \ll 0$.

But this definition is wrong for $[\mathcal{T}_c^c]^{\text{op}}$, because its suspension functor is the negative of that of $\mathcal{T}_c^c$.

The way to unify the two definitions is to think of them as continuity with respect to a metric. This might become a little clearer in the next section.

**Remark 8.18.** From what we have said already it’s clear that the statement about $Y$ in Theorem 8.1 is a special case of Theorem 8.16(i). To deduce the assertion of Theorem 8.1...
about the functor \( \overline{Y} \), from Theorem \ref{t:8.16}(ii), we need to know that the category \( \mathcal{D}_\text{col}_b(X) \) contains an object \( G' \) with \( \mathcal{D}_\text{qc}(X) = \langle G' \rangle_{N=\infty}^{(-\infty,\infty)} \). This is proved in \[32\] Theorem 2.3.

**Sketch 8.19.** We should say something about the proof of Theorem \ref{t:8.16}—to keep the discussion focused let us restrict our attention to the proof of Theorem \ref{t:8.16}(i). For the purpose of this discussion let us fix a compact generator \( G \in \mathcal{T} \) and a \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) in the preferred equivalence class. Proposition \ref{p:8.10} tells us that we may choose an integer \( A > 0 \) so that the properties of Definition \ref{d:5.1} (i) and (ii) hold. An easy induction on the integer \( m \) leads to the following consequence of Definition \ref{d:5.1}(ii):

(i) For every integer \( m > 0 \) and every object \( F \in \mathcal{T}^{\leq 0} \) there exists a triangle \( E_m \rightarrow F \rightarrow D_m \rightarrow E[1] \) with \( D_m \in \mathcal{T}^{\leq -m} \) and \( E \in \langle G \rangle_{mA}^{[1-m-A,A]} \).

This much is easy. Not quite so straightforward is the following:

(ii) There exists an integer \( B \), depending only on \( A \), with the following property. For any integer \( m > 0 \) and any object \( F \in \mathcal{T}^{\leq 0} \cap \mathcal{T}_c \) there exists a triangle \( E_m \rightarrow F \rightarrow D_m \rightarrow E[1] \) with \( D \in \mathcal{T}^{\leq -m} \) and \( E \in \langle G \rangle_{mB}^{[1-m-B,B]} \).

(iii) In fact more is true: the objects \( E_m \), in either (i) or (ii) above, can be chosen to form a sequence \( E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots \) mapping to \( F \), and such that \( F \) is the homotopy colimit of the sequence. It is in this sense that the Introduction should be understood: we have expressed \( F \) as the homotopy colimit of the (directed) Cauchy sequence \( \{ E_m \} \).

The reader might wish to go back to Examples \ref{e:4.4} and \ref{e:4.5}, in which we explicitly worked out what the abstract theory comes down to in the special case where \( \mathcal{T} = \mathcal{D}(R) \), the \( t \)-structure is the standard one, and the compact generator \( G \) is the object \( R \in \mathcal{D}(R) \). In the terminology of (i) and (ii) above, Examples \ref{e:4.4} and \ref{e:4.5} amount to showing that \( A = B = 1 \) works for the special case.

Now back to the proof of Theorem \ref{t:8.16}(i). The fact that \( \mathcal{Y} \) is full on the category \( \mathcal{T}_c \) and fully faithful on the category \( \mathcal{T}_c^b \) turns out to be a straightforward consequence of (iii) above. It remains to show that the essential image of \( \mathcal{Y} \) is as claimed. One containment is easy: the fact that the essential image is contained in the locally finite (respectively finite) functors follows directly from the the hypothesis that \( \mathcal{T} \) has a compact generator \( G \) so that \( \text{Hom}(G,G[n]) \) is a finite \( \text{R}-\text{module} \) for all \( n \in \mathbb{Z} \), coupled with the fact that approximability implies the vanishing of \( \text{Hom}(G,G[n]) \) for \( n \geq 0 \).

It remains to prove the opposite containment. Fix a locally finite homological functor \( \mathcal{H} : (\mathcal{T}_c)^{\text{op}} \rightarrow \text{R-Mod} \). We need to exhibit an object \( F \in \mathcal{T}_c \) and an isomorphism \( \mathcal{H} \cong \mathcal{Y}(F) \). This actually suffices: it’s easy to show that if \( \mathcal{Y}(F) \) is finite then \( F \in \mathcal{T}_c^b \).

Modifying an old idea of Adams \[1\] one can produce, for each integer \( m > 0 \), an object \( F_m \in \mathcal{T}_c^b \), a morphism \( \mathcal{Y}(F_m) \rightarrow \mathcal{H} \), and show that this morphism is surjective when restricted to \( \cup_n \langle G \rangle_{m}^{[-n,n]} \subset \mathcal{T}_c^b \). Shifting \( \mathcal{H} \) if necessary, we may furthermore assume that \( F_m \in \mathcal{T}_c^b \cap \mathcal{T}_c^{\leq 0} \) for all \( m > 0 \). By (ii) above we may, for each integer \( m \), construct a triangle \( E_m \rightarrow F_m \rightarrow D_m \rightarrow E_m[1] \) with \( E_m \in \langle G \rangle_{mB}^{[1-m-B,B]} \) and \( D_m \in \mathcal{T}_c^{\leq -m} \). One
then needs to show the existence of an increasing sequence of integers \( \{m_1 < m_2 < m_3 < \cdots \} \) such that there is a Cauchy sequence \( E_{m_1} \rightarrow E_{m_2} \rightarrow E_{m_3} \rightarrow \cdots \) converging in \( T_c \) to an object \( E \) with a surjection \( \mathcal{Y}(E) \rightarrow H \).

This is the hard part. Once we have produced a surjection \( \mathcal{Y}(E) \rightarrow H \), one performs some tricks to deduce an isomorphism \( \mathcal{Y}(F) \cong H \).

9. **The categories \( \mathsf{D}^{\text{perf}}(X) \) and \( \mathsf{D}^{\text{b}}_{\text{coh}}(X) \) determine each other**

It remains to discuss Fact 6.3(v). We remind the reader: this is the assertion that \( \mathsf{D}^{\text{perf}}(X) \) and \( \mathsf{D}^{\text{b}}_{\text{coh}}(X) \) determine each other, as triangulated categories.

**Historical Survey 9.1.** Probably the first result in this direction deals with the case of affine schemes and may be found in Rickard \[38, Theorem 6.4\]. Rickard’s result tells us that, if \( R \) and \( S \) are noetherian rings, then

\[
\mathsf{D}^{\text{b}}(R^{\text{proj}}) \cong \mathsf{D}^{\text{b}}(S^{\text{proj}}) \iff \mathsf{D}^{\text{b}}(R^{\text{mod}}) \cong \mathsf{D}^{\text{b}}(S^{\text{mod}})
\]

The way Rickard’s theorem was proved was by analyzing the triangulated equivalences—in other words Rickard developed a Morita theory, and showed that the data that produces an equivalence on the left is the same as the data producing an equivalence on the right. Thus the proof does guarantee that a triangulated equivalence on the left will produce a triangulated equivalence on the right, and vice versa. But the new result is better in giving explicit recipes for passing back and forth between \( \mathsf{D}^{\text{b}}(R^{\text{proj}}) \) and \( \mathsf{D}^{\text{b}}(R^{\text{mod}}) \); there is only one scheme and two derived categories, not two schemes and four derived categories. Moreover: the new result doesn’t assume the schemes to be affine.

A different result by Rouquier \[39, Remark 7.50\] asserts that, if \( X \) and \( Y \) are projective over a field \( k \), then

\[
\mathsf{D}^{\text{coh}}_{\text{b}}(X) \cong \mathsf{D}^{\text{b}}_{\text{coh}}(Y) \iff \mathsf{D}^{\text{perf}}(X) \cong \mathsf{D}^{\text{perf}}(Y)
\]

And we know both the result and the proof from the previous section: the category \( \mathsf{D}^{\text{coh}}_{\text{b}}(X) \) can be described as the category of finite homological functors on \( [\mathsf{D}^{\text{perf}}(X)]^{\text{op}} \), and the category \( [\mathsf{D}^{\text{perf}}(X)]^{\text{op}} \) can be described as the finite homological functors on \( \mathsf{D}^{\text{coh}}_{\text{b}}(X) \). This time there is no Morita theory as background, we explicitly know how to construct the categories out of each other. The drawbacks of the result, compared to the new one, are

(i) In Rouquier’s statement the schemes were assumed proper over a field, and even the improved representability theorems of the last section only work under the assumption of properness over some commutative, noetherian ring \( R \).

(ii) The triangulated structure isn’t obvious. The category of (finite) homological functors on an \( R \)–linear triangulated category doesn’t usually carry a triangulated structure.
Thus the old results only work if the schemes are either affine or proper—whereas the new result has no such restrictions—and even in the special circumstances where one of the old results holds, the conclusion is less powerful than what’s in Fact 6.3(v).

Discussion 9.2. In the Introduction we already mentioned that a heuristic way to think of approximability is to consider the “metric” defined by a $t$-structure $(\mathcal{I}^{\leq 0}, \mathcal{I}^{\geq 0})$, and maybe study the limits of Cauchy sequences with respect to this metric. The reader is referred back to Discussion 1.1. Phrased in this language, the category $T^{-c}$ can be defined as the closure in $T$ of the subcategory $T^{c}$ with respect to the metric—see Fact 1.5(iv). This suggests the recipe for constructing $T^{-b}$ out of $T^{c}$, it remains to flesh it out a little.

Definition 9.3. Let $S$ be a triangulated category. A metric on $S$ is a sequence of additive subcategories $\{M_{i}, i \in \mathbb{N}\}$, satisfying

(i) $M_{i+1} \subset M_{i}$ for every $i \in \mathbb{N}$.
(ii) $M_{i} + M_{i} = M_{i}$, with the notation as in Reminder 4.1.

A metric $\{M_{i}\}$ is declared to be finer than the metric $\{N_{i}\}$ if, for every integer $i > 0$, there exists an integer $j > 0$ with $M_{j} \subset N_{i}$; we denote this partial order by $\{M_{i}\} \preceq \{N_{i}\}$. The metrics $\{M_{i}\}$, $\{N_{i}\}$ are equivalent if $\{M_{i}\} \preceq \{N_{i}\} \preceq \{M_{i}\}$.

Example 9.4. Suppose $\mathcal{I}$ is an approximable triangulated category, and let $(\mathcal{I}^{\leq 0}, \mathcal{I}^{\geq 0})$ be a $t$-structure in the preferred equivalence class. Out of this data we can construct two examples of $S$’s with metrics:

(i) Let $S$ be the subcategory $T^{c} \subset \mathcal{I}$, and put $M_{i} = T^{c} \cap \mathcal{I}^{\leq -i}$.
(ii) Let $S$ be the subcategory $[T^{b}]^{\text{op}}$, and put $M_{i}^{\text{op}} = T^{b} \cap \mathcal{I}^{\leq -i}$.

It’s obvious that equivalent $t$-structures define equivalent metrics. Thus up to equivalence we have a canonical metric on $T^{c}$ and a canonical metric on $[T^{b}]^{\text{op}}$. But the definition depends on the embedding into $\mathcal{I}$, which is the category with the $t$-structure.

Definition 9.5. Let $S$ be a triangulated category with a metric $\{M_{i}\}$. A Cauchy sequence in $S$ is a sequence $E_{1} \rightarrow E_{2} \rightarrow E_{3} \rightarrow \cdots$ such that, for every pair of integers $i > 0$, $j \in \mathbb{Z}$, there exists an integer $M > 0$ such that, in any triangle $E_{m} \rightarrow E_{m'} \rightarrow D_{m,m'}$ with $M \leq m < m'$, the object $\Sigma^{j}D_{m,m'}$ lies in $M_{i}$.

It is clear that the Cauchy sequences depend only on the equivalence class of the metric.

Construction 9.6. Now suppose $S$ is an essentially small triangulated category with a metric. If we write $\text{Mod-}S$ for the category of additive functors $S^{\text{op}} \rightarrow \text{Ab}$, then the Yoneda functor is a fully faithful embedding $Y : S \rightarrow \text{Mod-}S$. We remind the reader: the formula is $Y(A) = \text{Hom}(-, A)$.

In the abelian category $\text{Mod-}S$ we can form colimits. We define $\mathcal{L}(S) \subset \text{Mod-}S$ to be the full subcategory of all colimits in $\text{Mod-}S$ of Cauchy sequences in $S$, and define
\( \mathcal{S}(\mathcal{S}) \subset \mathcal{L}(\mathcal{S}) \) to be the full subcategory

\[
\mathcal{S}(\mathcal{S}) = \mathcal{L}(\mathcal{S}) \cap \left( \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} [Y(\Sigma^jM_i)]^{\perp} \right)
\]

Recall: an object \( M \in \text{Mod-}\mathcal{S} \) belongs to \( [Y(\Sigma^jM_i)]^{\perp} \) if, for every object \( m \in \Sigma^jM_i \), we have \( \text{Hom}(Y(m), M) = 0 \).

It’s clear that the subcategories \( \mathcal{S}(\mathcal{S}) \subset \mathcal{L}(\mathcal{S}) \subset \text{Mod-}\mathcal{S} \) depend only on the equivalence class of the metric.

The first result, which may be found in [33, Theorem 2.11], asserts:

**Theorem 9.7.** The category \( \mathcal{S}(\mathcal{S}) \) is a triangulated category, with the triangles being the colimits in \( \text{Mod-}\mathcal{S} \) of Cauchy sequences of triangles in \( \mathcal{S} \).

Thus the triangulated structure on \( \mathcal{S}(\mathcal{S}) \) also depends only on the equivalence class of the metric.

And the second result, which is in [33, Example 4.2 and Proposition 5.6], gives:

**Theorem 9.8.** With the metrics as in Example 9.4 (i) and (ii), we have triangulated equivalences

(i) \( \mathcal{S}(\mathcal{T}^c) = \mathcal{T}_c^b \).

(ii) If \( \mathcal{T} \) is noetherian then \( \mathcal{S}(\mathcal{T}_c^b)^{\text{op}} = (\mathcal{T}_c^c)^{\text{op}} \).

**Notation 9.9.** We owe the reader an explanation of the hypothesis in Theorem 9.8(ii).

Let \( \mathcal{T} \) be an approximable triangulated category, and let \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) be a \( t \)-structure in the preferred equivalence class. The category \( \mathcal{T} \) is noetherian if there exists an integer \( N > 0 \) so that, for every object \( F \in \mathcal{T}_c^c \), there exists a triangle \( F' \to F \to F'' \) in \( \mathcal{T}_c^c \) with \( F' \in \mathcal{T}_c^c \cap \mathcal{T}^{\leq N} \) and \( F'' \in \mathcal{T}_c^c \cap \mathcal{T}^{\geq 0} \).

It’s clear that replacing the \( t \)-structure \( (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}) \) by an equivalent one will only have the effect of changing the integer \( N \); the definition doesn’t depend on the choice of \( t \)-structure in the preferred equivalence class.

**Remark 9.10.** The way the metrics were presented, in Example 9.4, depends on the embedding of \( \mathcal{T}^c \) and \( \mathcal{T}_c^b \) in \( \mathcal{T} \). After all they are defined in terms of the preferred equivalence class of \( t \)-structures, which makes sense only in \( \mathcal{T} \).

But there are recipes that cook up equivalent metrics directly from \( \mathcal{T}^c \) and \( \mathcal{T}_c^b \). The reader is referred to [33, Remark 4.7 as well as Propositions 4.8 and 6.5].

**Remark 9.11.** In this survey we have often said that, so far, the main applications of the theory have been to algebraic geometry—it’s the fact that \( \mathcal{D}_{qc}(X) \) is approximable which has had the far-reaching consequences to date. In this context Theorem 9.8 is the first exception, it has striking consequences for categories not coming from algebraic geometry. From the theorem we learn the following (among other things).
(i) Let $R$ be any ring, possibly noncommutative. The recipe takes the triangulated category $\mathbf{D}^b(R\text{-proj})$ and out of it constructs the triangulated category $\mathbf{D}^-(R\text{-proj}) \cap \mathbf{D}^b(R\text{-Mod})$. The objects of this intersection are the bounded-above cochain complexes of finitely generated, projective $R$–modules, with bounded cohomology.

If $R$ is a coherent ring this category is equivalent to $\mathbf{D}^b(R\text{-mod})$.

(ii) If $R$ is a coherent (possibly noncommutative) ring, then the recipe takes $[\mathbf{D}^b(R\text{-mod})]^{\text{op}}$ and out of it constructs $[\mathbf{D}^b(R\text{-proj})]^{\text{op}}$.

(iii) Out of the homotopy category of finite spectra we construct the homotopy category of spectra with finitely many nonzero stable homotopy groups, all of them finitely generated.

(iv) Out of the homotopy category of spectra with finitely many nonzero stable homotopy groups, all of them finitely generated, we construct the homotopy category of finite spectra.

Remark 9.12. When I posted the current article on the archive Steve Lack wrote to inform me that metrics in categories aren’t new. From Richard Garner I later learned that completing with respect to such metrics has also been done before. Hence I wrote the survey [31], which offers a much-expanded treatment of the material discussed in Section 9 and includes some remarks about the category-theory literature that preceded.

Appendix A. Some dumb maps in $\mathbf{D}_c^e(\mathcal{A})$, and the proof that the third map of the triangle is a cochain map

For any object $A$ in the abelian category $\mathcal{A}$, we will write $\tilde{A}$ for the cochain complex
\[ \cdots \rightarrow 0 \rightarrow A \rightarrow A \rightarrow 0 \rightarrow 0 \rightarrow \cdots \]
with $A$ in degrees $-1$ and $0$. Since the cohomology of the complex $\tilde{A}$ vanishes, in the derived category $\mathbf{D}_c^e(\mathcal{A})$ the morphism $0 \rightarrow \tilde{A}$ is an isomorphism. We reiterate: $\tilde{A}$ is nothing more than a complicated representative of the isomorphism class of the zero object in $\mathbf{D}_c^e(\mathcal{A})$. With the conventions of Example 2.7 and Notation 2.8, for any integer $i \in \mathbb{Z}$ we may form the object $\tilde{A}[i]$, which is the cochain complex
\[ \cdots \rightarrow 0 \rightarrow A \rightarrow A \rightarrow 0 \rightarrow 0 \rightarrow \cdots \]
with $A$ in degrees $-i - 1$ and $-i$, and where $\varepsilon = (-1)^i$.

Let $X^* \in \mathbf{D}_c^e(\mathcal{A})$ be any object, meaning $X^*$ is a cochain complex
\[ \cdots \rightarrow X^{-2} \rightarrow X^{-1} \rightarrow X^{-1} \rightarrow X^0 \rightarrow X^0 \rightarrow X^1 \rightarrow X^1 \rightarrow X^2 \rightarrow \cdots \]
The cochain maps $X^* \to \tilde{A}[-i]$ are in bijection with morphisms $\theta^i : X^i \to A$ in $\mathcal{A}$, the bijection takes $\theta^i$ to the cochain map

$$
\begin{array}{cccccccc}
\cdots & \to & X_{i-2} & \overset{\partial_{X}^{i-2}}{\to} & X_{i-1} & \overset{\partial_{X}^{i-1}}{\to} & X_{i} & \overset{\partial_{X}^{i}}{\to} & X_{i+1} & \overset{\partial_{X}^{i+1}}{\to} & \cdots \\
\downarrow & & \varepsilon \theta \partial_{X}^{i-1} & & \downarrow & & \theta^i & & \downarrow & & \cdots \\
\cdots & \to & 0 & \to & A & \to & A & \to & 0 & \to & \cdots 
\end{array}
$$

Given two cochain complexes $X^*$ and $Y^*$, as well as a morphism $\theta^i : X^i \to Y^{i-1}$ in $\mathcal{A}$, we may form the composite

$$
\begin{array}{cccccccc}
\cdots & \to & X_{i-2} & \overset{\partial_{X}^{i-2}}{\to} & X_{i-1} & \overset{\partial_{X}^{i-1}}{\to} & X_{i} & \overset{\partial_{X}^{i}}{\to} & X_{i+1} & \overset{\partial_{X}^{i+1}}{\to} & \cdots \\
\downarrow & & \varepsilon \theta_{X}^{-1} & & \downarrow & & \id & & \downarrow & & \cdots \\
\cdots & \to & 0 & \to & X_{i} & \to & X_{i} & \to & 0 & \to & \cdots \\
\downarrow & & \varepsilon \theta & & \downarrow & & \delta_{Y}^{i-1} \theta^i & & \downarrow & & \cdots \\
\cdots & \to & \tilde{Y}_{i-2} & \overset{\partial_{Y}^{i-2}}{\to} & \tilde{Y}_{i-1} & \overset{\partial_{Y}^{i-1}}{\to} & \tilde{Y}_{i} & \overset{\partial_{Y}^{i}}{\to} & \tilde{Y}_{i+1} & \overset{\partial_{Y}^{i+1}}{\to} & \cdots 
\end{array}
$$

which is manifestly a cochain map; we will denote it by $\tilde{\theta}^i$. Of course in the category $\mathcal{D}_{\xi}'(\mathcal{A})$ the cochain map $\tilde{\theta}^i$ must be equal to the zero map, all we have done is produced many representatives of the zero map. We may also sum over $i \in \mathbb{Z}$; given any collection of morphisms $\theta^i : X^i \to Y^{i-1}$ in $\mathcal{A}$ we may form $\oplus_{i=-\infty}^{\infty} \tilde{\theta}^i$, which is a cochain map $\tilde{\theta}^* : X^* \to Y^*$. Needless to say the map $\tilde{\theta}^*$ also vanishes in $\mathcal{D}_{\xi}'(\mathcal{A})$.

Now go back to Example 2.7: in the example we construct a diagram

$$
\begin{array}{cccccccc}
\cdots & \to & \tilde{Z}^{-2} & \overset{\partial_{Z}^{2}}{\to} & \tilde{Z}^{-1} & \overset{\partial_{Z}^{1}}{\to} & \tilde{Z}^{0} & \overset{\partial_{Z}^{0}}{\to} & \tilde{Z}^{1} & \overset{\partial_{Z}^{1}}{\to} & \tilde{Z}^{2} & \to & \cdots \\
\downarrow & & h^{-2} & & h^{-1} & & h^{0} & & h^{1} & & h^{2} & & \cdots \\
\cdots & \to & \tilde{X}^{-1} & \overset{-\partial_{X}^{-1}}{\to} & \tilde{X}^{0} & \overset{-\partial_{X}^{0}}{\to} & \tilde{X}^{1} & \overset{-\partial_{X}^{1}}{\to} & \tilde{X}^{2} & \overset{-\partial_{X}^{2}}{\to} & \tilde{X}^{3} & \to & \cdots \\
\downarrow & & f^{-1} & & f^{0} & & f^{1} & & f^{2} & & f^{3} & & \cdots \\
\cdots & \to & \tilde{Y}^{-1} & \overset{-\partial_{Y}^{-1}}{\to} & \tilde{Y}^{0} & \overset{-\partial_{Y}^{0}}{\to} & \tilde{Y}^{1} & \overset{-\partial_{Y}^{1}}{\to} & \tilde{Y}^{2} & \overset{-\partial_{Y}^{2}}{\to} & \tilde{Y}^{3} & \to & \cdots 
\end{array}
$$

The construction is such that, if we delete the middle row, then the composite is $\tilde{\theta}^*$ for the explicit $\theta^*$ chosen in the Example. In particular: deleting the middle row gives a commutative diagram. Deleting the top row yields a commutative diagram, we are left with nothing other than the given cochain map $f^*[1]$. We conclude that, for each $i$, the
composites from top left to bottom right in the diagram

\[
\begin{array}{c}
Z^i \xrightarrow{\partial_Z} Z^{i+1} \\
\downarrow h^i \\
X^{i+1} \xrightarrow{-\partial_X^{i+1}} X^{i+2} \\
\downarrow f^{i+2} \\
Y^{i+2}
\end{array}
\]

must agree. Since the map \( f^{i+2} \) is a (split) monomorphism the square commutes, and we conclude that the diagram

\[
\begin{array}{c}
\cdots \xrightarrow{} Z^{-2} \xrightarrow{\partial_Z^{-2}} Z^{-1} \xrightarrow{\partial_Z^{-1}} Z^0 \xrightarrow{\partial_Z^0} Z^1 \xrightarrow{\partial_Z^1} Z^2 \xrightarrow{} \cdots \\
\downarrow h^{-2} \\
\cdots \xrightarrow{} X^{-1} \xrightarrow{-\partial_X^{-1}} X^0 \xrightarrow{-\partial_X^0} X^1 \xrightarrow{-\partial_X^1} X^2 \xrightarrow{-\partial_X^2} X^3 \xrightarrow{} \cdots
\end{array}
\]

commutes. Thus \( h^* \) is indeed a cochain map, as promised in Example 2.7.

**Appendix B. The assumption that the short exact sequences of cochain complexes are degreewise split is harmless**

Given a cochain complex \( X^* \), an object \( A \in \mathcal{A} \) and a morphism \( \theta^i : X^i \to A \), in Appendix [A] we constructed a corresponding cochain map \( X^* \to \tilde{A}[-i] \). In the special case where \( A = X^i \) and \( \theta^i \) is the identity \( \text{id} : X^i \to X^i \) the general recipe specializes to the cochain map \( \rho^i \) below

\[
\begin{array}{c}
\cdots \xrightarrow{} X^{-2} \xrightarrow{\partial_X^{-2}} X^{-1} \xrightarrow{\partial_X^{-1}} X^0 \xrightarrow{\partial_X^0} X^i \xrightarrow{\partial_X^i} X^{i+1} \xrightarrow{\partial_X^{i+1}} X^{i+2} \xrightarrow{} \cdots \\
\downarrow \varepsilon \partial_X^{-1} \\
\cdots \xrightarrow{} 0 \xrightarrow{} X^i \xrightarrow{} X^i \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} \cdots
\end{array}
\]

For this section the key point is that, in degree \( i \), the cochain map \( \rho^i : X^* \to \tilde{X}^i[-i] \) is a split monomorphism. Taking the direct sum over \( i \) produces

\[
X^* \longrightarrow \bigoplus_{i \in \mathbb{Z}} \tilde{X}^i[-i]
\]

We denote this map \( \rho^* : X^* \to \tilde{X}^* \) and observe that

(i) The object \( \tilde{X}^* = \bigoplus_{i \in \mathbb{Z}} \tilde{X}^i[-i] \) vanishes in \( \mathcal{D}_c^r(A) \).
(ii) The cochain map \( \rho^* \) is a split monomorphism in each degree.
Now suppose we’re given a short exact sequence of cochain complexes $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^*$. We may form the commutative diagram of cochain complexes, where the rows are exact

\[
\begin{array}{cccccccccc}
0 & \rightarrow & X^* & \xrightarrow{(p^* \, f^*)} & \tilde{X}^* \oplus Y^* & \xrightarrow{\tilde{g}^*} & \tilde{Z}^* & \rightarrow & 0 \\
0 & \rightarrow & X^* & \xrightarrow{f^*} & Y^* & \xrightarrow{g^*} & Z^* & \rightarrow & 0
\end{array}
\]

We know that the vertical maps $\text{id} : X \rightarrow X$ and $\pi : \tilde{X}^* \oplus Y^* \rightarrow Y^*$ both induce isomorphisms in cohomology. The 5-lemma, applied to the long exact sequences in cohomology that come from the short exact sequences of cochain complexes in the rows, tells us that the vertical morphism $\varphi^*$ also induces an isomorphism in cohomology. Hence the top row is isomorphic in $\mathcal{D}_C^C(A)$ to the bottom row, and the top row is degreewise split. This is the sense in which we said, back in Example 2.7, that up to isomorphism in $\mathcal{D}_C^C(A)$ we may assume our short exact sequence of cochain complexes is degreewise split.

**Appendix C. Translating the approach to derived categories we presented here to the more standard one in the literature**

Our presentation of derived categories has been minimalist—we have tried to be accurate without providing any information that wasn’t absolutely indispensable. This means that the student who is seeing this for the first time, and would like to look for more detail elsewhere in the literature, might have a hard time reconciling what’s here with other, more expansive accounts. This appendix was written to help.

**Definition C.1.** Suppose $\mathcal{A}$ is an abelian category. Following tradition we define the two categories $\mathcal{C}_C^C(A)$ and $\mathcal{D}_C^C(A)$ to have the same objects: cochain complexes in $\mathcal{A}$ subject to identical restrictions. The morphisms are

(i) In the category $\mathcal{C}_C^C(A)$ a morphism is a cochain map.

(ii) We have already met $\mathcal{D}_C^C(A)$: it is obtained from $\mathcal{C}_C^C(A)$ by formally inverting the morphisms in $\mathcal{C}_C^C(A)$ inducing isomorphisms in cohomology.

**Remark C.2.** We give the following list, translating the constructions of the previous two appendices to more standard language.

(i) Given a pair of objects $X^*$ and $Y^*$ in $\mathcal{C}_C^C(A)$, that is a pair of cochain complexes $X^*$ and $Y^*$, as well as a sequence of morphisms $\{\theta^i : X^i \rightarrow Y^{i-1}, \, i \in \mathbb{Z}\}$ in the category $\mathcal{A}$, Appendix A constructed for us (among other things) a morphism $\tilde{\theta}^* : X^* \rightarrow Y^*$ in the category $\mathcal{C}_C^C(A)$. It is traditional to say that the cochain map $\tilde{\theta}^*$ is *null homotopic*, and the sequence of maps $\{\theta^i : X^i \rightarrow Y^{i-1}, \, i \in \mathbb{Z}\}$ is called a *homotopy of $\theta^*$ with the zero map*. More generally: two cochain maps
\(f^*, g^* : X^* \to Y^*\) are declared to be homotopic to each other if there exists a homotopy of \(f^* - g^*\) with the zero map. That is: if \(f^* - g^* = \tilde{\theta}^*\) with \(\tilde{\theta}^*\) as in Appendix A.

(ii) In Appendix B we constructed, for every cochain complex \(X^*\), another cochain complex \(\tilde{X}^*\) and a map \(\rho : X^* \to \tilde{X}^*\) in the category \(\mathbf{C}^e_\epsilon(A)\). Note that, by the construction of the map \(\tilde{\theta}^* : X^* \to Y^*\) of (i) above (see Appendix A), the map \(\tilde{\theta}^*\) comes with a factorization \(X^* \xrightarrow{\rho^*} \tilde{X}^* \xrightarrow{\tilde{\theta}^*} Y^*\) in the category \(\mathbf{C}^e_\epsilon(A)\).

(iii) Given a cochain map \(f^* : X^* \to Y^*\), the object \(\tilde{X}^* \oplus Y^*\) is isomorphic in \(\mathbf{C}^e_\epsilon(A)\) to a complex traditionally called the mapping cylinder of \(f^*\). This is obviously dumb terminology since \(\tilde{X}^* \oplus Y^*\) manifestly doesn’t depend on \(f^*\). But the isomorph in \(\mathbf{C}^e_\epsilon(A)\), that is traditionally presented as the mapping cylinder, seems to depend on \(f^*\)—needless to say, the isomorphism with \(\tilde{X}^* \oplus Y^*\) involves \(f^*\).

(iv) In Appendix B we considered the degreewise split short exact sequence of cochain complexes

\[
0 \to X^* \xrightarrow{(\rho^*, f^*)} \tilde{X}^* \oplus Y^* \xrightarrow{\tilde{\theta}^*} \tilde{Z}^* \to 0
\]

The cochain complex \(\tilde{Z}^*\) is isomorphic in \(\mathbf{C}^e_\epsilon(A)\) to a complex traditionally known as the mapping cone on \(f^*\).

(v) When \(f^* : X^* \to Y^*\) is the identity \(\text{id} : X^* \to X^*\), the mapping cone is isomorphic in \(\mathbf{C}^e_\epsilon(A)\) to \(\tilde{X}^*\). Thus the object we call \(\tilde{X}^*\) is traditionally called the mapping cone on the identity. Once again it is traditional to give an isomorphic complex, which is more complicated-looking than the \(\tilde{X}^*\) of Appendix B.

**Definition C.3.** Suppose \(A\) is an abelian category. Following tradition we define, in addition to the two categories of Definition C.2, yet another category \(\mathbf{K}^e_\epsilon(A)\). It has the same objects as \(\mathbf{C}^e_\epsilon(A)\) or \(\mathbf{D}^e_\epsilon(A)\). However:

(i) In the category \(\mathbf{K}^e_\epsilon(A)\) a morphism is a homotopy equivalence class of cochain maps.

Thus two cochain maps are declared to be equal in \(\mathbf{K}^e_\epsilon(A)\) if they’re homotopic.

**Lemma C.4.** The natural functor \(F : \mathbf{C}^e_\epsilon(A) \to \mathbf{D}^e_\epsilon(A)\), that is the universal functor out of \(\mathbf{C}^e_\epsilon(A)\) which sends every cohomology isomorphism to an isomorphism, factors uniquely through \(\mathbf{K}^e_\epsilon(A)\).

**Proof.** Let \(X^*\) be any object in \(\mathbf{C}^e_\epsilon(A)\), and consider the following morphisms in \(\mathbf{C}^e_\epsilon(A)\)

\[
\begin{array}{cccc}
X^* & \xrightarrow{(\rho^*, \text{id})} & \tilde{X}^* \oplus X^* & \xrightarrow{(0, \text{id})} & X^*
\end{array}
\]
The two composites are equal, in each case the composite is the identity map \( \text{id} : X^* \to X^* \). But the map \((0, \text{id}) : \tilde{X}^* \oplus X^* \to X^* \) is an isomorphism in cohomology [because \( \tilde{X}^* \) is acyclic], therefore \( F : \mathcal{C}^\xi(A) \to \mathcal{D}^\xi(A) \) takes \((0, \text{id})\) to an isomorphism in \( \mathcal{D}^\xi(A) \). Hence the functor \( F : \mathcal{C}^\xi(A) \to \mathcal{D}^\xi(A) \) must take the two cochain maps
\[
\begin{pmatrix}
\rho^*\\
\text{id}
\end{pmatrix}
\xrightarrow{X^* \xrightarrow{\ constitute morphisms in } \tilde{X}^* \oplus X^*}
\begin{pmatrix}
0\\
\text{id}
\end{pmatrix}
\]
to equal morphisms in \( \mathcal{D}^\xi(A) \). But then, for any homotopy \( \{ \theta^i : X^i \to Y^{i-1}, i \in \mathbb{Z} \} \) as in Remark C.2(i), the two composites
\[
\begin{pmatrix}
\rho^*\\
\text{id}
\end{pmatrix}
\xrightarrow{X^* \xrightarrow{\ constitute morphisms in } \tilde{X}^* \oplus X^*}
\begin{pmatrix}
0\\
\text{id}
\end{pmatrix}
\]

must map under \( F : \mathcal{C}^\xi(A) \to \mathcal{D}^\xi(A) \) to equal morphisms in \( \mathcal{D}^\xi(A) \). If \( \{ \theta^i : X^i \to Y^{i-1}, i \in \mathbb{Z} \} \) is a homotopy with \( f^* - g^* = \theta^* = \theta \rho^* \) we discover that, in the category \( \mathcal{D}^\xi(A) \), we must have \( F(f^*) = F(g^*) \). \( \square \)

Remark C.5. It now follows easily that \( \mathcal{D}^\xi(A) \) can also be obtained from \( \mathcal{K}^\xi(A) \) by formally inverting the maps inducing isomorphisms in cohomology. This brings us to the traditional approach: one proves first that the category \( \mathcal{K}^\xi(A) \) is triangulated, then formally inverts. And there is a general theorem of Verdier, giving conditions under which the process of formally inverting morphisms takes one triangulated category to another.

Remark C.6. In Example 2.7 we started in \( \mathcal{C}^\xi(A) \) with a degree-wise split short exact sequence of cochain complexes \( X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \) and, after choosing for every \( i \in \mathbb{Z} \) a degree-wise splitting \( \theta^i : Z^i \to Y^i \), we extended in \( \mathcal{C}^\xi(A) \) to form the sequence of morphisms \( X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{h^*} \Sigma X^* \). The triangles in \( \mathcal{K}^\xi(A) \) can be defined to

\( \text{Recall Remark C.2(iii): for any cochain map } f^* : X^* \to X^* \text{ there is a cochain complex traditionally called the } \text{mapping cylinder of } f^*, \text{ and all it is, as an object of the category } \mathcal{C}^\xi(A), \text{ is a complicated isomorph of } \tilde{X}^* \oplus X^*. \text{ If } f^* : X^* \to X^* \text{ is the identity map then the two morphisms } X^* \xrightarrow{\ constitute morphisms in } \tilde{X}^* \oplus X^*, \text{ studied in the proof of Lemma C.4, traditionally go by the name “the inclusions of the front and back faces” of the mapping cylinder.}

Note that, although the mapping cylinder of \( f^* \) is independent of \( f^* \) up to isomorphism in \( \mathcal{C}^\xi(A) \), the inclusion of the back face depends on \( f^* \).
be the isomorphs in $K^C'_C(A)$ of the $X^* \xrightarrow{f^*} Y^* \xrightarrow{g^*} Z^* \xrightarrow{h^*} \Sigma X^*$ that come from the construction of Example 2.7.

We leave it to the reader to check (if she wishes) that the triangles we construct are isomorphic in $K^C'_C(A)$ to the standard ones in the literature. The interested reader can amuse herself by furthermore checking that the axioms of triangulated categories are satisfied, using either the description of the triangles in the paragraph above or the more standard one in the literature.

Even if the reader is feeling lazy today, the honest truth is that the existing literature won’t help—it tends to leave this verification to the reader, there would be no benefit in working with the standard description of triangles. Doing this will not reduce the amount of labor the indolent reader is asked to perform, the checking will still be her task, only the details that need to be proved will shift a little.

The good news is that, once the reader has verified that the category $K^C'_C(A)$ is triangulated, the passage from $K^C'_C(A)$ to $D^C'_C(A)$ becomes well-documented. The theorem of Verdier, alluded to in Remark C.5, is explicit enough to provide helpful information about the calculus of fractions involved—and this may be found in any of the standard treatments in the literature.

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