Gradient Flows of Modified Wasserstein Distances and Porous Medium Equations with Nonlocal Pressure

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Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday

Abstract
We study families of porous medium equations with nonlocal pressure. We construct their weak solutions via JKO schemes for modified Wasserstein distances. We also establish the regularization effect and decay estimates for the $L^p$ norms.

Keywords JKO scheme · Modified Wasserstein distances · Fractional Laplacian

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1 Introduction and Main Results
Discretizing a time-dependent PDE as a series of variational problem has been a common practice for a while. For example, consider the diffusion equation

$$\partial_t u(t, x) - \Delta u(t, x) = 0, \quad x \in \mathbb{R}^d.$$ 

The classical discretization scheme reads as follows

$$u^k = \arg\min_u F(u), \text{ with } F(u) := \left\{ \frac{1}{\tau} \|u - u^{k-1}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \right\}.$$

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This comes from the idea that diffusion equation is the gradient flux of the Dirichlet integral \[ \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \]. It also can be seen by considering the additive perturbation \( u^k + t\phi \) of the solution \( u^k \), where \( \phi \in C^\infty_c(\mathbb{R}^d) \). Then, the equation \( \frac{d}{dt} F(u^k + t\phi) \big|_{t=0} = 0 \) implies
\[
\int_{\mathbb{R}^d} [(u^k - u^{k-1})\phi + \tau \nabla u^k \nabla \phi] dx = 0.
\]
That is, \( u^k \) satisfies the backward finite difference scheme for the diffusion equation \( u^k = u^{k-1} + \tau \Delta u^k \).

A novel discretization scheme, called JKO scheme, was proposed by Jordan–Kinderlehrer–Otto in the seminal paper \([13]\), for \( \tau > 0 \),
\[
u^k = \arg\min_{u \in \mathcal{P}_2(\mathbb{R}^d)} F_w(u), \quad \text{where } F_w(u) := \left\{ \frac{1}{2\tau} W_2^2(u, u^{k-1}) + \int_{\mathbb{R}^d} u \log u dx \right\},
\]
where \( W_2 \) is the Wasserstein distance and \( \mathcal{P}_2(\mathbb{R}^d) \) is the space of probability Borel measures on \( \mathbb{R}^d \) with finite second-order moment (see Section 3).

The main idea in proving (1.1) is the perturbation by domain deformations \([13]\). Namely, given \( \eta \in C^\infty_c(\mathbb{R}^d) \), let \( \Phi_t(x) = x + t\eta(x) \) and \( u_t \) be the push forward of \( u^k \) under \( \Phi_t \). That is, for all \( \phi \in C^\infty_c(\mathbb{R}^d) \),
\[
\int_{\mathbb{R}^d} \phi(x) u_t(x) dx = \int_{\mathbb{R}^d} \phi(\Phi_t(x)) u^k(x) dx.
\]
Then, one can obtain a few key formulas
\[
\frac{du_t}{dt} = -\text{div}(\eta u_t),
\]
and, for \( \eta = \nabla \phi \)
\[
\frac{d}{dt} \left[ \frac{1}{2\tau} W_2^2(u_t, u^k) \right] \big|_{t=0} = \int_{\mathbb{R}^d} \frac{u^k - u^{k-1}}{\tau} \phi dx + \Lambda(\tau),
\]
where \( \Lambda(\tau) \to 0 \) as \( \tau \to 0 \), in some proper sense, and
\[
\frac{d}{dt} \int_{\mathbb{R}^d} u_t \log u_t dx \big|_{t=0} = -\int_{\mathbb{R}^d} \Delta u^k \phi dx,
\]
see \([13]\) for more details. The equation \( \frac{dF_w(u_t)}{dt} = 0 \) (coming from the variational formulation (1.1)), then heuristically gives the backward finite difference approximation
\[
u^k \approx u^{k-1} + \tau \Delta u^k.
\]
Moreover, let us define \( u_\tau(t) = u^k \) for \( t \in [k\tau, (k+1)\tau) \). Then, \( u_\tau(t) \to u(t) \) weakly in \( L^1(\mathbb{R}^d) \) for almost all \( t \in \mathbb{R} \) (see \([13]\)), where \( u \) satisfies the variational form of the diffusion equation
\[
\int_0^\infty \int_{\mathbb{R}^d} u(\phi_t - \Delta \phi) dx dt = \int_{\mathbb{R}^d} u(0)\phi(0) dx.
\]
Therefore, the JKO scheme is a proper discretization for the diffusion equation.

Let us review a few extensions of the JKO scheme. First, we consider the equation
\[
\partial_t u - \text{div}(u \nabla (\nabla (\nabla u))^{-s}) = 0,
\]
where \((\nabla)^{-s}u\) with \( 0 < s < \min\{1, \frac{d}{2}\}\) denotes the inverse of the fractional Laplacian operator.
Using the following JKO scheme
\[
  u^k := \arg\min_{u \in \mathcal{P}(\mathbb{R}^d)} \left\{ \frac{1}{2\tau} \mathcal{W}_2^2(u, u^{k-1}) + F(u) \right\},
\]
where \( F(u) = \frac{1}{2} \|u\|_{\dot{H}^{-s}(\mathbb{R}^d)}^2 \) is the square norm of the homogeneous Sobolev space \( \dot{H}^{-s}(\mathbb{R}^d) \), Lisini, Mainini, and Segatti solved (1.2) for \( 0 < s < \min\{1, d/2\} \) [14].

Next, we consider the equation
\[
  \partial_t u + \text{div} \left[ h(u)|\xi|^{p-2}\xi \right] = 0,  \quad (1.3)
\]
where \( p > 1, \xi = -\nabla \left[ \frac{\delta F}{\delta u} \right] \) and \( h : [0, \infty) \to [0, \infty) \) is a nonlinear increasing function. Here \( \frac{\delta F}{\delta u} \) is the first variation of a function \( F : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\} \).

Then, the corresponding JKO scheme for (1.3) reads as follows
\[
  u^k := \arg\min_{u \in \mathcal{P}(\mathbb{R}^d)} \left\{ \frac{1}{2\tau} \mathcal{W}_m^2(u, u^{k-1}) + F(u) \right\},
\]
Choosing \( h(u) = u^\alpha \), \( p = 2 \), and \( F(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \), (1.3) becomes the thin-film like equation
\[
  \partial_t u - \text{div}[u^\alpha \nabla(-\Delta)u] = 0.
\]

**Statement of the main results** In this paper, we would like to develop JKO schemes for modified Wasserstein distances for the following fractional equations
\[
  \partial_t u - \text{div}(u^\alpha \nabla(-\Delta)^{-s}u) = 0 \quad \text{in} \quad \mathbb{R}^d, \quad (1.4)
\]
and
\[
  \partial_t u + (-\Delta)^{1-s}u = 0 \quad \text{in} \quad \mathbb{R}^d, \quad (1.5)
\]
for \( 0 < \alpha \leq 1 \) and \( 0 < s < \min\{1, \frac{d}{2}\} \).

The first problem (1.4) has been studied by Caffarelli and Vázquez in [4] with \( \alpha = 1 \), where the existence of solutions was proved for non-negative bounded initial data. Regularity and asymptotic behavior of these solutions are established in [5–7]. Later, the existence of solutions for (1.4) has been solved for the case \( 0 < \alpha < 2 \) [20, 21], and for all \( \alpha > 0 \) [16, 22]. The second problem (1.5) was studied in [11]. The proof there consists in introducing a non-local Wasserstein distance and the entropy functionals.

Our approaches to solving (1.4) and (1.5) are different from [4, 16, 20–22] and [11] as we use JKO schemes from modified Wasserstein distances and the homogeneous Sobolev norm.

Our JKO scheme is defined inductively as follows. Given a step size \( \tau > 0 \) and a weight function \( m : (0, +\infty) \to \mathbb{R}_+ \), we start from some \( u_0 \in \mathcal{P}(\mathbb{R}^d) \) with \( \frac{1}{2} \|u_0\|_{\dot{H}^{-s}(\mathbb{R}^d)}^2 < +\infty \) and define
\[
  u^0_\tau := u_0, \quad u^k_\tau := \arg\min_{u \in \mathcal{P}(\mathbb{R}^d)} \left\{ \frac{1}{2\tau} \mathcal{W}_m^2(u, u^{k-1}) + \frac{1}{2} \|u\|_{\dot{H}^{-s}(\mathbb{R}^d)}^2 \right\}, \quad (1.6)
\]
where \( W_m \) is the modified Wasserstein distance (see Section 2) with respect to our chosen weight functions \( m \).

The approximation \( \tilde{u}_\tau : [0, \infty) \rightarrow \mathcal{P}(\mathbb{R}^d) \) is defined by
\[
\tilde{u}_\tau(t) := u^k_t \quad \text{for} \quad (k - 1)\tau < t \leq k\tau. \tag{1.7}
\]

In our JKO scheme (1.6), the weight function \( m \) (thus the modified Wasserstein distance) actually depends on the time-step \( \tau \). It allows us to overcome the degeneracy and to deal with both (1.4) and (1.5) in a unified way. Note that the idea of incorporating the time-step into a transport cost functional is not new. It was used in [15] to show existence of solutions of degenerate parabolic equations of the form
\[
\partial_t u = -\text{div}(h(u)\nabla(\Delta u)) \quad \text{for certain mobility functions} \ h.
\]

This idea was also used to solve the Kramer equations [10, 12], and certain dissipative variational evolution equations with gradient structure arising from large-deviation principles for sequences of Markov jump processes [19]. To our best knowledge, no ones have used this method to solve (1.4) and (1.5) before.

For \( 0 < s < \frac{d}{2} \), our energy function \( F(u) = \frac{1}{2} \| u \|_{H^{-s}(\mathbb{R}^d)}^2 \) admits an alternative representation (see (2.1))
\[
F(u) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_{d,s}(x - y) u(x)u(y)dxdy,
\]
with \( K_{d,s}(x) = \frac{\pi^{d/2}}{\pi^{d/4}\Gamma(s)} \Gamma\left(\frac{d-2s}{d}\right) \), where \( \Gamma \) is the Gamma function. Hence our function \( F(u) \) enlightens the structure of an interaction energy.

Now we state our main results. The first one is Theorem 1.1 containing all the properties of our gradient flow solutions of (1.4).

**Theorem 1.1** Let \( u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap (L^2 \cap H^{-s})(\mathbb{R}^d), \ d \geq 2 \) and \( 0 < \alpha \leq 1, \ 0 < s < \min\{1, \frac{d}{2}\} \). Then for every \( \tau > 0 \), the scheme (1.6) always has a solution \( u_{\tau} \), with \( m(r) = (r + \tau \frac{1}{r})^\alpha \). In addition, for every sequence \( \tau_n \downarrow 0 \), there exist a subsequence, still denoted by \( \tau_n \), and a function \( u : [0, +\infty) \rightarrow \mathcal{P}(\mathbb{R}^d) \) such that

i) \( \tilde{u}_{\tau_n} \rightarrow u \) strongly in \( L^2(0, T; H^0(\mathbb{R}^d)) \), \( \tilde{u}_{\tau_n}^\alpha \nabla(\Delta)^{-s} \tilde{u}_{\tau_n} \rightarrow u^\alpha \nabla(\Delta)^{-s} u \) strongly in \( L^1(\mathbb{R}^d \times (0, T)) \), \( \bar{u}_{\tau_n} \rightarrow u \) weakly in \( L^2(0, T; H^{1-s}(\mathbb{R}^d)) \) for every \( T > 0, \ 0 \leq \beta < 1 - s \),

ii) The function \( u \) is a weak solution to the equation
\[
\partial_t u - \text{div}(u^\alpha \nabla(\Delta)^{-s} u) = 0
\]
in the following weak sense
\[
\int_0^{+\infty} \int_{\mathbb{R}^d} \partial_t \psi u dxdt - \int_0^{+\infty} \int_{\mathbb{R}^d} \nabla_x \psi \cdot u^\alpha \nabla(\Delta)^{-s} udxdt = 0,
\]
for every \( \psi \in C_c(\mathbb{R}^d) \),

iii) For any \( k_0, L \in \mathbb{N}, \ 1 \leq p \leq 2^L \), there exists \( c > 0 \) such that
\[
\| \tilde{u}_\tau(t) \|_{L^p} \leq (\lambda(k_0, L)(t + k_0\tau))^{\frac{(1-\frac{1}{p})d}{\alpha + 2(1-s)}},
\]
where
\[
\lambda(k_0, L) = \frac{c}{\sup_{1 \leq n \leq L} \sup_{k \geq k_0} k \left( 1 + \frac{1}{d-1} \right)}.
\]
and
\[ (\lambda(k_0, L) - \frac{d}{\alpha + 2d(1 - s)}) \geq \|u_0\|_{L^\infty}, \]
iv) For any \( t_1 > 0 \) and for any \( \Lambda > 0 \), it holds that
\[ \sup_{t \geq t_1} \| (\bar{u}_\tau(t) - \Lambda)_+ \|_{L^2}^2 + \int_{t_1+\tau}^{\infty} \| (\bar{u}_\tau(t) - \Lambda)_+ \|^{\alpha + 2} \frac{dt}{L^{\frac{\alpha}{\alpha + 2d}(1 - s)}} \leq C \| (\bar{u}_\tau(t_1) - \Lambda)_+ \|_{L^2}^2. \]

Moreover, if \( \tau < 2^{-4} \) and \( u(0) \in L^2 \), there exists \( \Lambda_0 = \Lambda_0(\|u_0\|_{L^2}, s, \alpha, d) > 0 \) such that
\[ \sup_{t \geq 1} \| (\bar{u}_\tau(t) - \Lambda_0)_+ \|_{L^2}^2 + \int_{1}^{\infty} \| (\bar{u}_\tau(t) - \Lambda_0)_+ \|^{\alpha + 2} \frac{dt}{L^{\frac{\alpha}{\alpha + 2d}(1 - s)}} \leq C \tau^\frac{(d+1)}{2(1-s)} + 2. \]

Let us describe our strategy to prove Theorem 1.1. First, we establish the regularity of our minimizers \( u^k \). With suitable initial data \( u_0 \), our minimizers \( u^k \) not only belong to \( \check{H}^{-1}(\mathbb{R}^d) \) but also \( \check{H}^{1-s}(\mathbb{R}^d) \). Moreover, we can control \( \| u^k \|_{\check{H}^{1-s}(\mathbb{R}^d)} \) in terms of our potential function \( U \), defined in (3.1), in Lemma 4.1. Furthermore, as \( s \leq 1 \), for every sequence \( \tau_n \searrow 0 \), there exist a subsequence, still denoted by \( \tau_n \), and \( u : [0, +\infty) \rightarrow \mathcal{P}(\mathbb{R}^d) \) such that
\[ \tilde{u}_{\tau_n} \rightarrow u \text{ strongly in } L^2(0, T; \check{H}^\beta(\mathbb{R}^d)) \text{ for every } T > 0, 0 \leq \beta < 1 - s. \]

This task can be achieved as we have a flow interchange as in Lemma 3.4 for our scheme. In order to get our flow interchange, we need to verify that the associated semigroup \( S_\delta \) of our auxiliary function \( V_\delta \) is a \( \lambda \)-flow under our weighted Wasserstein distance. The definition of \( \lambda \)-flow will be presented at Definition 3.2. To do this, we establish the Eulerian calculus for our weighted Wasserstein metric in Lemma 3.3. Note that the flow interchange technique was introduced by McCann, Matthes, and Savaré [17], and the Eulerian calculus for the usual Wasserstein metric was introduced by Otto and Westdickenberg [18] and was developed later by Daneri and Savaré [9]. These ideas have been applied to study PDEs in [14] and [15] as applications of JKO schemes related to the standard Wasserstein metric and weighted Wasserstein metric, respectively. Our achieved regularity estimates of \( \{\tilde{u}_{\tau_n}\} \) are good enough to obtain a weak solution \( u \) of (1.4) via a compactness argument as \( \tau_n \rightarrow 0 \).

The other important features in our Theorem 1.1 are the decay rate at infinity of \( L^p \) norms and the almost boundedness of solutions \( u \) and \( u_\tau \) that we stated in iii) and iv), respectively. The decay rate of \( L^p \) norms was already proved in [7] and later in [14] for the usual homogeneous porous medium equation with nonlocal pressure with \( \alpha = 1 \). To get iii) and iv) of Theorem 1.1, first we establish \( L^{\frac{2d}{\alpha - 2d(1 - s)}} \) estimates of \( G(u^k) \) as in Lemma 4.3 for our minimizing movements scheme as follows
\[ C(d, s) \tau \| G(u^k) \|^2 \leq \int_{\mathbb{R}^d} g(u^{-1}_\tau(x)) dx - \int_{\mathbb{R}^d} g(u^k_\tau(x)) dx, \tag{1.9} \]
where \( G(r) = \int_0^r \sqrt{m(z)} g''(z) dz \) with \( g \in C^2([0, \infty), \mathbb{R}_+) \) is convex such that \( g(0) = g'(0) = g''(0) = 0 \). This estimate is obtained by applying the flow interchange technique. Then for the case \( p < +\infty \), combining (1.9) for the function \( g(z) = z^p \) with interpolation inequalities, we get our estimates of the \( L^p \)-norms. On the other hand, applying (1.9) for the function \( g(z) = (z - \Lambda)_+^p \), we will get iv) of Theorem 1.1. Then the almost boundedness of solutions \( u \) and \( u_\tau \) is a consequence of iv) by the following argument.

Letting \( \tau \rightarrow 0 \), from (1.8), we obtain
\[ \lim_{\tau \rightarrow 0} \sup_{t \geq 1} \| (\bar{u}_\tau(t) - \Lambda_0)_+ \|_{L^2}^2 + \int_1^\infty \| (\bar{u}_\tau(t) - \Lambda_0)_+ \|^{\alpha + 2} \frac{dt}{L^{\frac{\alpha}{\alpha + 2d}(1 - s)}} = 0, \]
which implies
\[ \sup_{t \geq 1} ||(u(t) - \Lambda_0)_+||_{L^2}^2 = 0. \]
That is, \( u(t) \leq \Lambda_0 \) for all \( t \geq 1 \).

Let us denote
\[ A_\tau = \{ x : (\bar{u}_\tau(t) - \Lambda_0)_+ \geq 1, \forall t \geq 1 \}. \]
Then, also from (1.8),
\[ |A_\tau| \leq C \tau^{d(\alpha+1)/(\alpha d - 1) + 2}. \]
So,
\[ (\bar{u}_\tau(t) - \Lambda_0)_+ \leq 1, \text{ i.e., } u_\tau(t) \leq \Lambda_0 + 1 \]
for any \( t \geq 1 \), and \( x \in \mathbb{R}^d \setminus A_\tau \).

Our second main result is the existence of weak solutions of (1.5)

**Theorem 1.2** Let \( u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap (L^2 \cap \dot{H}^{-s})(\mathbb{R}^d), 0 < s < \min\{1, d/2\} \). Let \( \{\tau_n\} \) be a positive sequence converging to 0, \( \bar{u}_{\tau_n} \) be the solution of the scheme (1.6) and (1.7) with \( m_n(r) = (r + 1)^{\frac{1}{10}} \) and \( u : [0, +\infty) \to \mathcal{P}(\mathbb{R}^d) \) as in Lemma 4.1. Then, there exists a subsequence of \( \bar{u}_{\tau_n} \) still denoted by \( \bar{u}_{\tau_n} \) converging to a solution \( u \) of the equation
\[ \partial_t u + (-\Delta)^{1-s} u = 0 \] (1.10)
in the following weak sense
\[ \int_0^{+\infty} \int_{\mathbb{R}^d} \partial_t \psi u dx dt - \int_0^{+\infty} \int_{\mathbb{R}^d} \psi (-\Delta)^{1-s} u dx dt = 0, \forall \psi \in C^\infty_c((0, +\infty) \times \mathbb{R}^d). \]

To show Theorem 1.2, we use the same strategy as proving Theorem 1.1 when we apply the weight function \( m(r) = (r + 1)^{\frac{1}{10}} \) instead of \( m(r) = (r + \tau_{\frac{10}{1}})^{\alpha} \).

Our paper is organized as follows. In Section 2, we review the definition of the modified Wasserstein distance and provide its approximation. In Section 3, we prove the existence and uniqueness of minimizers for solutions of our scheme and then we establish our flow interchange for the scheme via the Eulerian calculus of the modified Wasserstein distances. Finally, in Section 4, we will prove our main results.

**2 Preliminaries**

First, we review fractional Sobolev spaces. The Fourier transform of \( f \in L^1(\mathbb{R}^d) \) is defined by \( \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx \). We denote by \( \mathcal{S}(\mathbb{R}^d) \) the Schwartz space of smooth functions on \( \mathbb{R}^d \) with rapid decay at infinity, and by \( \mathcal{S}'(\mathbb{R}^d) \) its dual space. For every \( r \in \mathbb{R} \), the fractional Sobolev space \( H^r(\mathbb{R}^d) \) and the homogeneous fractional Sobolev space \( \dot{H}^r(\mathbb{R}^d) \)
are defined respectively by

\[ H^r(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : \hat{f} \in L^1_{\text{loc}}(\mathbb{R}^d), \quad \| f \|_{H^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\xi|^2)^r |\hat{f}(\xi)|^2 d\xi < +\infty \right\}, \]

\[ H^r(\mathbb{R}^d) := \left\{ f \in S'(\mathbb{R}^d) : \hat{f} \in L^1_{\text{loc}}(\mathbb{R}^d), \quad \| f \|_{\dot{H}^r(\mathbb{R}^d)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2r} |\hat{f}(\xi)|^2 d\xi < +\infty \right\}. \]

If \( r < \frac{d}{2} \) then the space \( \dot{H}^r(\mathbb{R}^d) \) is a Hilbert space with the scalar product

\[ \langle v, w \rangle_r := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2r} \hat{v}(\xi) \hat{w}(\xi) d\xi. \]

From [1, Proposition 1.37], we can imply that if \( r \in (0, 1) \), then there exists a constant \( C_{d,r} > 0 \) such that

\[ \langle v, w \rangle_r = C_{d,r} \| u \|_{H^r(\mathbb{R}^d)} \]

for every \( u \in \dot{H}^r(\mathbb{R}^d) \), where \( q := \frac{2d}{d-2r} \).

Given \( 0 < s < \min\{1, \frac{d}{2}\} \), the \( s \)-fractional Laplacian \((-\Delta)^s \) on \( \mathbb{R}^d \) is defined by means of Fourier transform as follows

\[ ((-\Delta)^s)u(x) = |x|^{2s} \hat{u}(x) \text{ for every } x \in \mathbb{R}^d. \]

As \( 0 < s < \frac{d}{2} \), we have that

\[ (-\Delta)^{-s}u(x) = \frac{1}{\gamma(s,d)} \int_{\mathbb{R}^d} \frac{u(y)}{|x - y|^{d-2s}} dy, \]

where \( \gamma(s,d) := \pi^{d/2} 2^s \Gamma(s) \Gamma\left(\frac{d-2s}{2}\right) \) [4, Appendix]. Therefore, our energy function

\[ F(u) = \frac{1}{2} \| u \|_{H^{-s}(\mathbb{R}^d)}^2 = \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \]

\[ = \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} ((-\Delta)^{-s})u(x) \hat{u}(\xi) d\xi \]

\[ = \frac{1}{2} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (-\Delta)^{-s}u(x)u(x) dx \]

\[ = \frac{1}{2} \frac{1}{\gamma(d,s)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(y)}{|x - y|^{d-2s}} dy u(x) dx \]

\[ = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_{d,s}(x - y) u(x) u(y) dy dx, \]

where \( K_{d,s}(x) := \frac{1}{\gamma(d,s)} x^{2s-d}. \)

Now let us review the definition of the modified Wasserstein distances introduced in [8] and studied in [3]. We denote by \( \mathcal{P}(\mathbb{R}^d) \) the space of probability Borel measures on \( \mathbb{R}^d \) and
\( \mathcal{P}_2(\mathbb{R}^d) \) the space of \( \mu \in \mathcal{P}(\mathbb{R}^d) \) with finite second-order moment. Let \( \mathcal{L}^d \) be the Lebesgue measure on \( \mathbb{R}^d \).

The Wasserstein distance on \( \mathcal{P}_2(\mathbb{R}^d) \) is defined by
\[
W_2(\mu^0, \mu^1) = \min_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\pi(x, y) : \pi \in \Gamma(\mu^0, \mu^1) \right\}^{1/2},
\]
where
\[
\Gamma(\mu^0, \mu^1) = \left\{ \pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : \pi(A \times \mathbb{R}^d) = \mu^0(A), \pi(\mathbb{R}^d \times A) = \mu^1(A) \forall \text{Borel } A \subset \mathbb{R}^d \right\}.
\]

The Wasserstein distance also admits a dynamical system description as follows [2]:
\[
W_2^2(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} \rho_s(x) |v_s(x)|^2 dx dt : \right. \\
\left. \partial_t \rho_s + \text{div}(\rho_s v_s) = 0, \mu_0 = \rho_0 \mathcal{L}^d, \mu_1 = \rho_1 \mathcal{L}^d \right\}.
\]

Let \( X \) be a metric space, we denote by \( \mathcal{M}^+_\text{loc}(X) \) (resp. \( \mathcal{M}^+(X) \)) the space of all nonnegative (resp. all finite nonnegative) Radon measures on \( X \). We denote by \( \mathcal{M}^\text{loc}(X, \mathbb{R}^d) \) (resp. \( \mathcal{M}(X, \mathbb{R}^d) \)) the space of \( \mathbb{R}^d \)-valued Radon measures on \( X \) (resp. the space of \( \mathbb{R}^d \)-valued Radon measures on \( X \) with finite total variation).

For two nonnegative functions \( f, g \), our notation \( f \sim g \) means that there exist \( C_1, C_2 > 0 \) such that \( C_1 f \leq g \leq C_2 f \).

We denote by CE the set of \( (\mu_t)_{t \in [0,1]} \subset \mathcal{M}^\text{loc}_\text{loc}(\mathbb{R}^d) \) and \( (v_s)_{s \in [0,1]} \subset \mathcal{M}^\text{loc}(\mathbb{R}^d, \mathbb{R}^d) \) such that
1. \( t \mapsto \mu_t \) is weakly* continuous in \( \mathcal{M}^\text{loc}_\text{loc}(\mathbb{R}^d) \);
2. \( t \mapsto v_t \) is Borel and \( \int_0^1 |v_t| (\mathbb{R}^d) dt < \infty \);
3. \( (\mu_t, v_t)_{t \in [0,1]} \) is a distributional solution of the following continuity equation (CE)
\[
\partial_t \mu_t + \nabla \cdot v_t = 0 \text{ in } \mathbb{R}^d \times (0, 1),
\]
which means
\[
\int_0^1 \int_{\mathbb{R}^d} \partial_t \varphi(x, t) d\mu_t(x) dt + \int_0^1 \int_{\mathbb{R}^d} \nabla \varphi(x, t) \cdot dv_t(x) dt = 0
\]
for every \( \varphi \in \mathcal{C}_c^1(\mathbb{R}^d \times (0, 1)) \).

Given \( \nu^0, \nu^1 \in \mathcal{M}^\text{loc}_\text{loc}(\mathbb{R}^d) \), we denote by CE(\( \nu^0 \to \nu^1 \)) the subset of CE such that \( \mu_0 = \nu^0, \mu_1 = \nu^1 \).

Let \( m : [0, +\infty) \to (0, +\infty) \) be a concave and nondecreasing function. The action density function \( G : [0, \infty) \times \mathbb{R}^d \to [0, +\infty) \) is defined by
\[
G(s_1, s_2) := \frac{|s_2|^2}{m(s_1)} \text{ for every } (s_1, s_2) \in [0, \infty) \times \mathbb{R}^d.
\]

Given measures \( \mu \in \mathcal{M}^\text{loc}_\text{loc}(\mathbb{R}^d) \) and \( \nu \in \mathcal{M}^\text{loc}_\text{loc}(\mathbb{R}^d, \mathbb{R}^d) \), we consider the Lebesgue decompositions \( \mu = \rho \mathcal{L}^d + \mu^\perp \) and \( \nu = \omega \mathcal{L}^d + \nu^\perp \). Here, \( \rho \in L^1_\text{loc}(\mathbb{R}^d, \mathcal{L}^d; \mathbb{R}), \) \( \omega \in L^1(\mathbb{R}^d, \mathcal{L}^d; \mathbb{R}) \), and \( \nu^\perp \) is the singular part of \( \nu \) with respect to the Lebesgue measure.
\( \mathcal{L}^d \) of \( \mathbb{R}^d \). We define \( \mathcal{G}(\mu, v) \) as follows:

\[
\mathcal{G}(\mu, v) := \begin{cases} 
\int_{\mathbb{R}^d} G(\rho, \omega)dx & \text{if } v^\perp = 0, \\
\infty & \text{otherwise}.
\end{cases}
\]

Now, by our choice of function \( m \), applying [8, Theorem 3.1 and Example 3.3]], we get that the action density function satisfies conditions (3.1a), (3.1b), and (3.1c) in [8]. Hence, we are ready to define the modified Wasserstein (pseudo) distance on \( \mathcal{M}^+_\text{loc}(\mathbb{R}^d) \) ([8, Definition 5.1]). Given \( \mu^0, \mu^1 \in \mathcal{M}^+_\text{loc}(\mathbb{R}^d) \), we define

\[
W_m(\mu^0, \mu^1) := \inf \left\{ \left( \int_0^1 \mathcal{G}(\mu_t, v_t)dt \right)^{1/2} : (\mu, v) \in \text{CE}(\mu^0 \rightarrow \mu^1) \right\}
\]

\[
= \inf \left\{ \int_0^1 (\mathcal{G}(\mu_t, v_t))^{1/2}dt : (\mu, v) \in \text{CE}(\mu^0 \rightarrow \mu^1) \right\}.
\]

The equality in the above definition follows from [8, Theorem 5.4]. If the set \( \text{CE}(\mu^0 \rightarrow \mu^1) \) is empty, we put \( W_m(\mu^0, \mu^1) = +\infty \).

In our JKO scheme (1.6), given \( \tau > 0 \), we choose weight functions \( m(r) = (r + \tau^{1/\alpha})^\alpha \) for (1.4) or \( m(r) = (r + 1)^{1/\alpha} \) for (1.5). For \( 0 < \alpha \leq 1 \) and \( 0 < \tau < 1 \), our weight functions \( m(r) = (r + \tau^{1/\alpha})^\alpha \) and \( m(r) = (r + 1)^{1/\alpha} \) are concave and nondecreasing.

The following lemma provides an approximation of the modified Wasserstein distance.

**Lemma 2.1** Let \( m \in C^\infty(\mathbb{R}^d_+, \mathbb{R}^+ \} be such that \( \inf_{x \in \mathbb{R}^d_+} m(x) > 0. \) Let \( \mu^0, \mu^1 \in \mathcal{P}(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \) be such that \( W_m(\mu^0, \mu^1) < \infty. \) Then there exist \( \rho_n \in C^\infty([0, 1] \times \mathbb{R}^d) \) and \( \phi_n \in C^\infty([0, 1] \times \mathbb{R}^d) \cap L^\infty([0, 1] \times \mathbb{R}^d) \) such that

- \( \rho_n(t) \in \mathcal{P}(\mathbb{R}^d) \) for every \( t \in [0, 1] \), \( ||\rho_n(0) - \mu^0||_{L^1(\mathbb{R}^d)} + ||\rho_n(1) - \mu^1||_{L^1(\mathbb{R}^d)} \rightarrow 0 \) as \( n \rightarrow \infty \),
- \( (\rho_n, \phi_n) \) satisfies \( \partial_t \rho_n(t, x) = -\text{div}(m(\rho_n(t, x))\nabla \phi_n(t, x)) \) and

\[
W_m^2(\mu^0, \mu^1) = \lim_{n \rightarrow +\infty} \int_0^1 \int_{\mathbb{R}^d} m(\rho_n(t, x))|\nabla \phi_n(t, x)|^2dxdt.
\]

**Proof** We can choose \( \rho_n(0), \rho_n(1) \in \mathcal{P}(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) such that \( ||\rho_n(0) - \mu^0||_{L^1(\mathbb{R}^d)} + ||\rho_n(1) - \mu^1||_{L^1(\mathbb{R}^d)} \rightarrow 0 \) as \( n \rightarrow \infty \), and \( W_m^2(\mu^0, \mu^1) = \lim_{n \rightarrow \infty} W_m^2(\rho_n(0), \rho_n(1)) \).

For every \( n \in \mathbb{N} \), there exist \( \rho_n \in C^\infty([0, 1] \times \mathbb{R}^d) \) and \( \phi_n \in C^\infty([0, 1] \times \mathbb{R}^d, \mathbb{R}^d) \) such that \( \partial_t \rho_n(t, x) = -\text{div}(\phi_n(t, x)) \) and

\[
\int_0^1 \int_{\mathbb{R}^d} \frac{|\phi_n(t, x)|^2}{m(\rho_n(t, x))}dxdt \leq W_m^2(\rho_n(0), \rho_n(1)) + \frac{1}{n}.
\]

(2.2) Let \( \phi_n(t, x) \) be the smooth solution to \( \text{div}(m(\rho_n(t, x))\nabla \phi_n(t, x)) = \text{div}(\phi_n(t, x)) \). Using \( \phi_n \) as a test function of this equation to get

\[
\int_{\mathbb{R}^d} m(\rho_n(t, x))|\nabla \phi_n(t, x)|^2dx = \int_{\mathbb{R}^d} \phi_n(t, x)\nabla \phi_n(t, x)dx.
\]

By Hölder inequality,

\[
\int_{\mathbb{R}^d} m(\rho_n(t, x))|\nabla \phi_n(t, x)|^2dx 
\leq \left( \int_{\mathbb{R}^d} \frac{|\phi_n(t, x)|^2}{m(\rho_n(t, x))}dx \right)^{1/2} \left( \int_{\mathbb{R}^d} m(\rho_n(t, x))|\nabla \phi_n(t, x)|^2dx \right)^{1/2}.
\]
Combining this with (2.2) and the definition of $W^2_m(\rho_n(0), \rho_n(1))$, one obtains

$$W^2_m(\rho_n(0), \rho_n(1)) \leq \int_0^1 \int_{\mathbb{R}^d} m(\rho_n(t, x))|\nabla \phi_n(t, x)|^2 dx dt$$

$$\leq W^2_m(\rho_n(0), \rho_n(1)) + \frac{1}{n}.$$ 

Letting $n \to \infty$, we obtain the result. The proof is complete.

3 JKO Schemes, Flow Interchange, and Eulerian Calculus

We begin this section with the following lemma proving the uniqueness of solutions of the scheme (1.6).

Lemma 3.1 For every $\tau > 0$, every $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap \dot{H}^{-s}(\mathbb{R}^d)$, and every $k \in \mathbb{N}$, the scheme (1.6) has a unique solution $u^k_\tau$.

Proof By [8, Theorems 5.5 and 5.6] and [14, Proposition 3.1], the function $u \mapsto \frac{1}{2\tau} W^2_m(u, u^{k-1}_\tau) + \frac{1}{2} \|u\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$ is nonnegative, lower semicontinuous with respect to the weak* topology and has weak* compact sublevels. Therefore, the scheme (1.6) has minimizers. As $W^2_m(\cdot, \cdot)$ is convex [8, Theorem 5.11] and the map $u \mapsto \frac{1}{2} \|u\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$ is strictly convex, we get that the map $\frac{1}{2\tau} W^2_m(u, u^{k-1}_\tau) + \frac{1}{2} \|u\|^2_{\dot{H}^{-s}(\mathbb{R}^d)}$ is strictly convex and hence we obtain the uniqueness of minimizers.

Let $U : [0, +\infty) \to [0, +\infty)$ be the function satisfying $U''(s) := \frac{1}{m(s)}$ with $U'(0) = U(0) = 0$. We define

$$U(u) := \int_{\mathbb{R}^d} U(u(x)) dx$$

for every $u \in \mathcal{P}(\mathbb{R}^d)$. Since $U$ is convex, the function $U$ is lower semicontinuous with respect to the weak convergence on $\mathcal{P}(\mathbb{R}^d)$.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a test function and let $\delta > 0$. Let $S_\delta$ be the semigroup defined by $S_{\delta,t} v_0 = v_t$ for every $t > 0$ with $v_t$ being the unique distribution solution, i.e., $v_t \in \mathcal{P}(\mathbb{R}^d)$, of the following equation with initial data $v_0 \in \mathcal{P}(\mathbb{R}^d)$

$$\partial_t v_t - \text{div}(m(v_t) \nabla \phi) - \delta \Delta v_t = 0 \text{ in } (0, +\infty) \times \mathbb{R}^d.$$ 

Let $\rho \in C_c^\infty([0, 1] \times \mathbb{R}^d)$ be such that $\rho(t) \in \mathcal{P}(\mathbb{R}^d)$ for every $t \in [0, 1]$. For every $h, t > 0$, we put

$$\rho^h(t) = S_{\delta,h t} \rho(t) \in \mathcal{P}(\mathbb{R}^d).$$

Let $\phi^h$ be the unique smooth solution to

$$\partial_t \rho^h(t, x) = -\text{div}(m(\rho^h(t, x)) \nabla \phi^h(t, x)) \text{ in } [0, 1] \times \mathbb{R}^d.$$ 

We define

$$A^h(t) = \int_{\mathbb{R}^d} m(\rho^h(t, x)) |\nabla \phi^h(t, x)|^2 dx.$$ 

Let us recall the definition of $\lambda$-flows on metric spaces [15, Section 3.2].
Definition 3.2 Let $E : (\mathcal{P}(\mathbb{R}^d), W_m) \to (-\infty, +\infty]$ be a proper lower semicontinuous function with proper domain $\text{Dom}(E) := \{ u \in \mathcal{P}(\mathbb{R}^d) : E(u) < +\infty \}$, and $\lambda \in \mathbb{R}$. A continuous semigroup $S_t : \text{Dom}(E) \to \text{Dom}(E)$, $t \geq 0$, is a $\lambda$-flow for $E$ if it satisfies
\[
\frac{1}{2} \limsup_{h \to 0} \frac{W_m^2(S_h(u), v) - W_m^2(u, v)}{h} + \frac{\lambda}{2} W_m^2(u, v) + E(u) \leq E(v)
\]
for every $u, v \in \text{Dom}(E)$ with $W_m(u, v) < +\infty$.

Let $V_\delta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ be the function defined by
\[
V_\delta(v) := \langle \varphi, v \rangle + \delta U(v) \text{ for every } v \in \mathcal{P}(\mathbb{R}^d).
\]
(3.3)
To prove that $S_\delta$ is a $\lambda$-flow for $V_\delta$, we need the following lemma which is the Eulerian calculus in our metric. Its proof is an adaptation of the proof of [15, formula 76].

Lemma 3.3 Let $m(r) = (r + \tau^{1/10})^\alpha$ or $m(r) = (r + 1)^{\tau^{-10}}$. Then for every $t \in [0, 1]$ and $h \geq 0$, we have
\[
\frac{1}{2} \partial_t A^h(t) + \partial_t V_\delta(\rho^h(t)) \leq -\lambda_\delta t A^h(t),
\]
(3.4)
where
\[
\lambda_\delta = -\frac{1}{2\delta} \|\nabla \varphi\|_L^2 \sup_{a > 0} (m(a)|m''(a)|) - \|D^2 \varphi\|_L \sup_{a > 0} |m'(a)|.
\]

Proof First, we have
\[
\partial_t \rho^h(t) = t(\partial_z S_\delta(z)z = h)(\rho(t)) = t \text{div}(m(\rho^h(t))\nabla \varphi) + \delta t \Delta \rho^h(t).
\]
(3.5)
Hence, thanks to (3.2)
\[
\partial_t \rho^h(t) = \delta \Delta \rho^h(t) + \text{div}(m(\rho^h(t))\nabla \varphi)
\]
\[
- \delta t \text{div} \left[ \text{div}(m(\rho^h(t))\nabla \phi^h(t))m'(\rho^h(t))\nabla \varphi \right]
\]
\[
- \delta t \Delta \left[ \text{div}(m(\rho^h(t))\nabla \phi^h(t)) \right].
\]
(3.6)
For every $t \in [0, 1]$ and $h \geq 0$, we have
\[
\partial_t V_\delta(\rho^h(t)) = \langle \varphi, \partial_t \rho^h(t) \rangle + \delta \int_{\mathbb{R}^d} U'(\rho^h(t))\partial_t \rho^h(t)
\]
\[
= -\langle \varphi, \text{div}(m(\rho^h(t))\nabla \phi^h(t)) \rangle - \delta \int_{\mathbb{R}^d} U'(\rho^h(t))\text{div}(m(\rho^h(t))\nabla \phi^h(t))
\]
\[
= \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \varphi \nabla \phi^h(t) + \delta \int_{\mathbb{R}^d} U''(\rho^h(t))m(\rho^h(t))\nabla \phi^h(t) \nabla \rho^h(t).
\]
Since $U''(\rho^h_n(t))m(\rho^h_n(t)) = 1$, we get
\[
\partial_t V_\delta(\rho^h(t)) = \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \varphi \nabla \phi^h(t) + \int_{\mathbb{R}^d} \rho^h(t) \Delta \phi^h(t).
\]
(3.7)
On the other hand,
\[
\frac{1}{2} \partial_t A^h(t) = -\frac{1}{2} \int_{\mathbb{R}^d} \partial_h m(\rho^h)\nabla \phi^h |^2 + \int_{\mathbb{R}^d} \partial_h \left( m(\rho^h)\nabla \phi^h \right) \nabla \phi^h
\]
\[
= -\frac{1}{2} \int_{\mathbb{R}^d} \partial_h \rho^h \rho^h m'(\rho^h) |\nabla \phi^h |^2 + \int_{\mathbb{R}^d} (\partial_t \rho^h \rho^h)\phi^h.
\]
(3.8)
By (3.5), we obtain

\[- \frac{1}{2} \int_{\mathbb{R}^d} \partial_t \rho^h m'(\rho^h)|\nabla \phi^h|^2 \]
\[- \frac{t}{2} \int_{\mathbb{R}^d} \left[ \text{div}(m(\rho^h(t))\nabla \varphi) + \delta \Delta \rho^h(t) \right] m'(\rho^h)|\nabla \phi^h|^2 \]
\[- \frac{t}{2} \int_{\mathbb{R}^d} \left[ m(\rho^h(t))\nabla \varphi + \delta \nabla \rho^h(t) \right] \nabla \left( m'(\rho^h)|\nabla \phi^h|^2 \right) \]
\[- \frac{t\delta}{2} \int_{\mathbb{R}^d} m''(\rho^h)|\nabla \rho^h(t)|^2|\nabla \phi^h|^2 + \frac{t\delta}{2} \int_{\mathbb{R}^d} m'(\rho^h)\nabla \rho^h(t)\nabla(|\nabla \phi^h|^2) \]
\[- \delta t \int_{\mathbb{R}^d} \text{div}(m(\rho^h(t))\nabla \phi^h(t))\Delta \phi^h \]
\[-\partial_t V(\rho^h(t)) - t \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \phi^h(t) \nabla \left[ m'(\rho^h(t))\nabla \phi^h \right] \]
\[- \partial_t \nabla \delta(\rho^h(t)) - t \int_{\mathbb{R}^d} m(\rho^h(t))m'(\rho^h(t))\nabla \phi^h(t) \nabla \left( \nabla \phi^h \right) \]
\[- t \int_{\mathbb{R}^d} m' \nabla \phi^h(t) \nabla \left( \nabla \phi^h \right) \nabla \phi^h \]
\[- \delta t \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \phi^h(t) \nabla (\Delta \phi^h) \]

(3.9)

It follows from (3.6) that

\[ \int_{\mathbb{R}^d} (\partial_t \partial_h \rho^h) \phi^h \]
\[ = \delta \int_{\mathbb{R}^d} \rho^h(t) \Delta \phi^h - \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \varphi \nabla \phi^h(t) \]
\[ + t \int_{\mathbb{R}^d} \text{div}(m(\rho^h(t))\nabla \phi^h(t))m'(\rho^h(t))\nabla \varphi \nabla \phi^h \]
\[ - \delta t \int_{\mathbb{R}^d} \text{div}(m(\rho^h(t))\nabla \phi^h(t))\Delta \phi^h \]
\[ \equiv - \partial_t \nabla \delta(\rho^h(t)) - t \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \phi^h(t) \nabla \left[ m'(\rho^h(t))\nabla \varphi \nabla \phi^h \right] \]
\[ + \delta t \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \phi^h(t) \nabla (\Delta \phi^h) \]
\[ = - \partial_t \nabla \delta(\rho^h(t)) - t \int_{\mathbb{R}^d} m(\rho^h(t))m'(\rho^h(t))\nabla \phi^h(t) \nabla \left( \nabla \phi^h \right) \]
\[- \int_{\mathbb{R}^d} m'(\rho^h(t))m''(\rho^h(t))(\nabla \phi^h(t)\nabla \rho^h(t)\nabla \varphi \nabla \phi^h) \]
\[ + \delta t \int_{\mathbb{R}^d} m(\rho^h(t))\nabla \phi^h(t) \nabla (\Delta \phi^h). \]
Combining this with (3.9), (3.8), and (3.7), one finds
\[
\frac{1}{2} \partial_t h^A_t(t) + \partial_t \mathbf{V}_\delta (\rho^h(t)) + \frac{t \delta}{2} \int_{\mathbb{R}^d} m''(\rho^h(t)) |\nabla \rho^h(t)|^2 |\nabla \phi^h|^2
\]
\[
+ t \delta \int_{\mathbb{R}^d} m(\rho^h(t)) \left[ -\Delta \left( \frac{1}{2} |\nabla \phi^h|^2 \right) + \nabla \phi^h(t) \nabla (\Delta \phi^h) \right]
\]
\[
+ \frac{t}{2} \int_{\mathbb{R}^d} m(\rho^h(t)) m''(\rho^h(t)) \left( \nabla \phi \nabla \rho^h |\nabla \phi^h|^2 - 2 |\nabla \phi^h(t) \nabla \rho^h| (\nabla \phi \nabla \phi^h) \right)
\]
\[
+ t \int_{\mathbb{R}^d} m(\rho^h(t)) m'(\rho^h(t)) \left( \nabla \phi \nabla \left( \frac{1}{2} |\nabla \phi^h|^2 \right) - \nabla \phi^h(t) \nabla (\nabla \phi \nabla \phi^h) \right).
\]
Since \( m'' \leq 0 \) and
\[
-\Delta \left( \frac{1}{2} |\nabla \phi^h|^2 \right) + \nabla \phi^h(t) \nabla (\Delta \phi^h) = -|\nabla^2 \phi^h|^2 \quad \text{(the Bochner formula)},
\]
\[
|\nabla \phi \nabla \rho^h |\nabla \phi^h|^2 - 2 |\nabla \phi^h(t) \nabla \rho^h| (\nabla \phi \nabla \phi^h) \bigg| \leq 2 |\nabla \phi^h| |\nabla \phi^h|^2 |\nabla \rho^h|,
\]
\[
|\nabla \phi \nabla \left( \frac{1}{2} |\nabla \phi^h|^2 \right) - \nabla \phi^h(t) \nabla (\nabla \phi \nabla \phi^h) \bigg| \leq |\nabla^2 \phi^h| |\nabla \phi^h|^2.
\]
one has
\[
\frac{1}{2} \partial_t h^A_t(t) + \partial_t \mathbf{V}_\delta (\rho^h(t)) \leq -\frac{t \delta}{2} \int_{\mathbb{R}^d} m''(\rho^h(t)) |\nabla \rho^h(t)|^2 |\nabla \phi^h|^2
\]
\[
+ t \delta \int_{\mathbb{R}^d} m(\rho^h(t)) \left( m''(\rho^h(t)) \right)^{1/2} |\nabla \phi^h| |\nabla \phi^h| \left( m''(\rho^h(t)) \right)^{1/2} |\nabla \phi^h| |\nabla \rho^h| \bigg| |\nabla \phi^h|^2 |\nabla \phi^h|^2
\]
\[
+ t \int_{\mathbb{R}^d} m(\rho^h(t)) m'(\rho^h(t)) |D^2 \phi^h| |\nabla \phi^h|^2
\]
\[
\leq \frac{t}{2 \delta} \int_{\mathbb{R}^d} m(\rho^h(t))^2 m''(\rho^h(t)) |\nabla \phi^h|^2 |\nabla \phi^h|^2
\]
\[
+ t \int_{\mathbb{R}^d} m(\rho^h(t)) m'(\rho^h(t)) |D^2 \phi^h| |\nabla \phi^h|^2.
\]
Thus, we deduce that
\[
\frac{1}{2} \partial_t h^A_t(t) + \partial_t \mathbf{V}_\delta (\rho^h(t)) \leq -\lambda_\delta t A^h_t(t)
\]
with
\[
\lambda_\delta = -\frac{1}{2 \delta} \| \nabla \phi^h \|_{L^\infty}^2 \sup_{a \geq 0} (m(a) |m''(a)|) - \| D^2 \phi^h \|_{L^\infty} \sup_{a \geq 0} |m'(a)|.
\]
The proof is complete. \( \square \)

Now in the following lemma, we are ready to establish our crucial flow interchange of our scheme.

**Lemma 3.4** Let \( \varphi \in C^\infty_c(\mathbb{R}^d) \) be a test function. Let \( m(r) = (r + r^{1/10})^\alpha \) or \( m(r) = (r + 1)^\alpha \). Let \( \delta > 0 \), we define the function \( \mathbf{V}_\delta : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \) by
\[
\mathbf{V}_\delta(v) := \langle \varphi, v \rangle + \delta U(v) \text{ for every } v \in \mathcal{P}(\mathbb{R}^d).
\]
Then there holds
\begin{equation}
V_\delta \left( u_\tau^k \right) - V_\delta \left( u_\tau^{k-1} \right) \leq -\tau \delta \| u_\tau^k \|_{H^{1-}\tau}^2 + \tau \left\{ \text{div} \left( m(u_\tau^k) \nabla (\Delta)^{-s} u_\tau^k \right), \varphi \right\} - \frac{\lambda_\delta}{2} W_m^2 \left( u_\tau^k, u_\tau^{k-1} \right),
\end{equation}
where
\begin{equation}
\lambda_\delta = -\frac{1}{2\delta} \| \nabla \varphi \|_L^2 \sup_{a>0} \text{sup} (m(a)|m''(a)|) - \| D^2 \varphi \|_L^2 \sup_{a>0} |m'(a)|.
\end{equation}

In particular,
\begin{equation}
V_\delta \left( u_\tau^k \right) - V_\delta \left( u_\tau^{k-1} \right) \leq \tau \left\{ \text{div} \left( m(u_\tau^k) \nabla (\Delta)^{-s} u_\tau^k \right), \varphi \right\} - \frac{\tau \lambda_\delta}{2} \left( \| u_\tau^{k-1} \|_{H^{1-}\tau}^2 - \| u_\tau^k \|_{H^{1-}\tau}^2 \right).
\end{equation}

**Proof** First, (3.12) follows from (3.11) and the fact that
\begin{equation}
\frac{1}{2\tau} W_m^2 \left( u_\tau^k, u_\tau^{k-1} \right) + \frac{1}{2} \| u_\tau^k \|_{H^{1-}\tau}^2 \leq \frac{1}{2} \| u_\tau^{k-1} \|_{H^{1-}\tau}^2.
\end{equation}
We now prove (3.11). We consider the equation
\begin{equation}
\partial_t v_t - \text{div}(m(v_t) \nabla \varphi) - \delta \Delta v_t = 0 \text{ in } (0, +\infty) \times \mathbb{R}^d.
\end{equation}
Clearly, (3.13) has a unique solution with initial data $v_0 \in \mathcal{P} (\mathbb{R}^d)$.

We define the semigroup $S_h$ by $S_{\delta,t} v_0 = v_t$ for every $t > 0$. By the definition of $u_\tau^k$ as a minimizer of (1.6)
\begin{equation}
\frac{1}{2\tau} W_m^2 \left( u_\tau^k, u_\tau^{k-1} \right) = \frac{1}{2} \| u_\tau^k \|_{H^{1-}\tau}^2 \leq \frac{1}{2\tau} W_m^2 \left( S_{\delta,h}(u_\tau^k), u_\tau^{k-1} \right) + \frac{1}{2} \| S_{\delta,h}(u_\tau^k) \|_{H^{1-}\tau}^2,
\end{equation}
and
\begin{equation}
\lim_{h \to 0} \frac{\| S_{\delta,h}(u_\tau^k) \|_{H^{1-}\tau}^2 - \| u_\tau^k \|_{H^{1-}\tau}^2}{2h} = \left\{ \left( -\Delta \right)^{-s} u_\tau^k, \left( \partial_h S_{\delta,h}(u_\tau^k) \right)_{h=0} \right\} = \left( -\Delta \right)^{-s} u_\tau^k, \text{div} \left( m(u_\tau^k) \nabla \varphi \right) + \delta \Delta u_\tau^k = -\delta \| u_\tau^k \|_{H^{1-}\tau}^2 + \left\{ \text{div} \left( m(u_\tau^k) \nabla (\Delta)^{-s} u_\tau^k \right), \varphi \right\}.
\end{equation}
So, it is enough to show that
\begin{equation}
\frac{1}{2} \limsup_{h \to 0} \frac{W_m^2 \left( S_{\delta,h}(\xi), \mu \right)^2 - W_m^2 (\xi, \mu)^2}{h} + \frac{\lambda_\delta}{2} W_m^2 (\mu, \xi)^2 + V_\delta (\xi) \leq V_\delta (\mu)
\end{equation}
for any $\mu, \xi \in \mathcal{P} (\mathbb{R}^d)$ with $V_\delta (\mu), V_\delta (\xi), W_m^2 (\mu, \xi) < \infty$.

Let $\rho_n$ be in Lemma 2.1 with $\mu^0 = \mu, \mu^1 = \xi$. Set for $h > 0, t > 0,
\rho_n^h (t) = S_{\delta,ht} v_0 \in \mathcal{P} (\mathbb{R}^d),
with $v_0 = \rho_n (t)$.

Let $\phi_n^h$ be a unique solution to
\begin{equation}
\partial_t \rho_n^h (t, x) = -\text{div} \left( m \left( \rho_n^h (t, x) \right) \nabla \phi_n^h (t, x) \right) \text{ in } [0, 1] \times \mathbb{R}^d.
\end{equation}
Set

\[ A_n^h(t) = \int_{\mathbb{R}^d} m \left( \rho_n^h(t, x) \right) |\nabla \phi_n^h(t, x)|^2 \, dx. \]

We have

\[ W_m^2(\mu, \xi)^2 = \lim_{n \to \infty} \int_0^1 A_n^0(t) \, dt. \tag{3.15} \]

From Lemma 3.3, we get that

\[ \frac{1}{2} \partial_t A_n^h(t) + \partial_t \left( \rho_n^h(t) \right) \leq -\lambda \delta t A_n^h(t) \tag{3.16} \]

for every \( n \in \mathbb{N} \), \( t \in [0, 1] \), and \( h \geq 0 \).

Now, we show that (3.16) implies (3.14). Indeed, (3.16) gives for \( c = ||\phi||_{L^\infty} \)

\[ \frac{1}{2} \partial_t \int_0^1 e^{2 \lambda \delta t h} A_n^h(t) \, dt \leq -\int_0^1 e^{2 \lambda \delta t h} \partial_t \left( V_\delta \left( \rho_n^h(t) + c \right) \right) \, dt \]

\[ = V_\delta \left( \rho_n(0) + c e^{2 \lambda \delta} \left( V_\delta \left( \rho_n^h(1) + c \right) \right) \right) + 2 \lambda \delta h \int_0^1 e^{2 \lambda \delta t h} \left( V_\delta \left( \rho_n^h(t) + c \right) \right) \, dt \]

\[ \leq V_\delta \left( \rho_n(0) + c e^{2 \lambda \delta} \left( V_\delta(S_{\delta, h} \rho_n(1)) + c \right) \right), \]

where we have used the fact that \( V_\delta(S_{\delta, h} \rho_n(1)) + c \) is decreasing.

For every positive function \( \theta \in C^1([0, 1]) \), we have

\[ \partial_t \rho_n^h \left( \tilde{\theta}^{-1}(t), x \right) = -\text{div} \left( m \left( \rho_n^h(\tilde{\theta}^{-1}(t), x) \right) \nabla \left( (\tilde{\theta}^{-1})'(t) \phi_n^h(\tilde{\theta}^{-1}(t), x) \right) \right), \]

and

\[ \rho_n^h(\tilde{\theta}^{-1}(0), x) = \rho_n^h(0, x), \quad \rho_n^h(\tilde{\theta}^{-1}(1), x) = \rho_n^h(1, x), \]

with

\[ \tilde{\theta}(t) = \left[ \int_0^1 \frac{1}{\theta(z)} \, dz \right]^{-1} \int_0^t \frac{1}{\theta(z)} \, dz, \text{ and } \tilde{\theta}^{-1} \text{ is the inverse of } \tilde{\theta}. \]

So,

\[ W_m^2(\rho_n(0), S_{\delta, h} \rho_n(1)) \]

\[ \leq \int_0^1 \int_{\mathbb{R}^d} (\tilde{\theta}^{-1})'(z) \, m \left( \rho_n^h(\tilde{\theta}^{-1}(z), x) \right) |\nabla \phi_n^h(\tilde{\theta}^{-1}(z), x)|^2 \, dx \, dz \]

\[ = \int_0^1 \int_{\mathbb{R}^d} (\tilde{\theta}^{-1})'(\tilde{\theta}(t)) m \left( \rho_n^h(t, x) \right) |\nabla \phi_n^h(t, x)|^2 \, dx \, dt \]

\[ = \int_0^1 \frac{1}{\theta(r)} \, dr \int_0^1 \theta(t) A_n^h(t) \, dt. \]
Applying this to $\theta(t) = e^{2x\delta t}$ to get
\[
W^2_m(\rho_n(0), S_{\delta,h}\rho_n(1)) \leq \frac{e^{-2\lambda\delta h} - 1}{-2\lambda\delta h} \int_0^1 e^{2\lambda\delta th} A_n^h(t) dt. \quad (3.18)
\]
Combining (3.17) with (3.18) yield
\[
\frac{-\lambda\delta h}{-2\lambda\delta h - 1} W^2_m(\rho_n(0), S_{\delta,h}\rho_n(1)) \\
\leq \frac{1}{2} \int_0^1 A_n^h(t) dt + h(V_\delta(\rho_n(0)) + c) - \frac{1}{-2\lambda h} (V_\delta(S_{\delta,h}\rho_n(1)) + c).
\]
From (3.15), lower semicontinuity of $W^2_m$, continuity of $S_{\delta,h}$ and $V_\delta$ under $\|\cdot\|_{L^1(\mathbb{R}^d)}$, letting $n \to \infty$, we get
\[
\frac{-\lambda\delta h}{-2\lambda h - 1} W^2_m(\mu, S_{\delta,h}\xi) \\
\leq \frac{1}{2} W^2_m(\mu, \xi)^2 + h(V_\delta(\mu) + c) - \frac{1}{-2\lambda h} (V_\delta(S_{\delta,h}\xi) + c).
\]
So,
\[
\frac{-\lambda\delta h}{e^{-2\lambda h} - 1} W^2_m(\mu, S_{\delta,h}\xi) - W^2_m(\mu, \xi) \\
+ \frac{1}{h} \left[ \frac{-\lambda\delta}{e^{-2\lambda h} - 1} - \frac{1}{2} \right] W^2_m(\mu, \xi) \\
\leq V_\delta(\mu) + c - \frac{1}{-2\lambda h} (V_\delta(S_{\delta,h}\xi) + c).
\]
Letting $h \to 0$, we obtain (3.14). The proof is complete.

\section{Proofs of the Main Results}

Now we are ready to prove our main results.

\textbf{Lemma 4.1} Let $u_0 \in P_2(\mathbb{R}^d) \cap (L^2 \cap \dot{H}^{-s})(\mathbb{R}^d)$. Let $m(r) = (r + \tau^{1/10})^a$ or $m(r) = (r + 1)^{\frac{1}{m}}$. Let $\tau > 0$ and $\{u^{k}_\tau\}$ be the sequence and $\bar{u}_\tau$ be the solution of the scheme (1.6) and (1.7). Then $u^{k}_\tau \in \dot{H}^{1-s}(\mathbb{R}^d)$ for every $k \geq 1$ and there holds,
\[
\tau \|u^{k}_\tau\|^2_{\dot{H}^{1-s}} \leq U(u^{k-1}_\tau) - U(u^{k}_\tau). \quad (4.1)
\]
In particular, for any $k \in \mathbb{N}$,
\[
U(u^{k}_\tau) + \int_0^{k\tau} \|\bar{u}_\tau(t)\|^2_{\dot{H}^{1-s}} dt \leq U(u^{0}_\tau) = U(u_0). \quad (4.2)
\]
Furthermore, for every sequence $\tau_n \searrow 0$, there exist a subsequence, still denoted by $\tau_n$, and $u : [0, +\infty) \rightarrow P(\mathbb{R}^d)$ such that
\[
\bar{u}_{\tau_n} \rightarrow u \text{ strongly in } L^2(0, T; \dot{H}^\beta(\mathbb{R}^d)) \text{ for every } T > 0, 0 \leq \beta < 1 - s, \quad (4.3)
\]
\[
\bar{u}_{\tau_n} \rightarrow u \text{ weakly in } L^2(0, T; \dot{H}^{1-s}(\mathbb{R}^d)) \text{ for every } T > 0.
\]
\textbf{Proof} Let $P_t$ be the semigroup associated to the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^d$. By the definition of $u^{k}_\tau$ as a minimizer of (1.6)
\[
\frac{1}{2\tau} W^2_m(u^{k}_\tau, u^{k-1}_\tau) + \frac{1}{2}\|u^{k}_\tau\|^2_{\dot{H}^{-s}} \leq \frac{1}{2\tau} W^2_m(P^h(u^{k}_\tau), u^{k-1}_\tau) + \frac{1}{2}\|P^h(u^{k}_\tau)\|^2_{\dot{H}^{-s}},
\]

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and
\[
\lim_{h \to 0} \frac{||P_h(u^k_t)||_{H^{-s}}^2 - ||u^k_t||_{H^{-s}}^2}{2h} = \langle (-\Delta)^{-s} u^k_t, (\partial_t P_h(u^k_t)) \rangle_{h=0} = -||u^k_t||_{H^{-1-s}}^2.
\]

So, to obtain (4.1), we need to show that
\[
\frac{1}{2} \limsup_{h \to 0} \frac{W_m^2(P_h(\xi), \mu)^2 - W_m^2(\xi, \mu)^2}{h} + U(\xi) \leq U(\mu)
\]
for any \( \mu, \xi \in P(\mathbb{R}^d) \) with \( V_\delta(\mu), V_\delta(\xi) < \infty, W_m(\mu, \xi) < \infty \).

Let \( \rho_n \) be in Lemma 2.1 with \( \mu^0 = \mu, \mu^1 = \xi \). For \( h > 0 \), let \( \rho_h^r(t) = P_{ht}(\rho^r(t)) \in P(\mathbb{R}^d) \) and \( \phi^h_r \) be the unique solution to \( \partial_t \rho_h^r = -\text{div}(m(\rho_h^r) \nabla \phi^h_r) \) in \([0, 1] \times \mathbb{R}^d \).

Applying Lemma 3.3 for \( \varphi = 0 \), we get that
\[
\frac{1}{2} \partial_t \int_{\mathbb{R}^d} m(\rho_h^r(t, x)) \left| \nabla \phi^h_r(t, x) \right|^2 dx + \partial_t U(\rho_h^r(t)) \leq 0.
\]

Similar to the proof of Lemma 3.4, (4.4) follows from (4.5).

By definition (1.7),
\[
\int_0^{\tau} ||\bar{u}_r(t)||_{H^{-1-s}}^2 dt = \sum_{j=1}^{k} \tau ||u^j_t||_{H^{-1-s}}^2.
\]

So, (4.2) follows from this and (4.1).

From (4.2), \( \{\bar{u}_{r_n}\} \) is bounded and hence converges weakly to \( u \) in \( L^2((0, T), H^{1-s}(\mathbb{R}^d)) \) for \( T < \infty \). By compact embeddings of Sobolev spaces and the dominated convergence theorem, we deduce that \( \bar{u}_{r_n} \) converges weakly to \( u \) in \( L^2((0, T), H^\beta(\mathbb{R}^d)) \) for \( T < \infty \), and \( 0 \leq \beta < 1 - s \). The proof is complete.

\( \square \)

**Lemma 4.2** Let \( u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap (L^2 \cap H^{-s})(\mathbb{R}^d) \). Let \( \{\tau_n\} \) be a positive sequence converging to \( 0, \bar{u}_\tau \) be the solution of the scheme (1.6) and (1.7) with \( m_\tau(r) = (r + \tau^{\frac{1}{10}})^\alpha \). Then \( u \) is a weak solution to
\[
\partial_t u - \text{div}(u^\alpha \nabla (-\Delta)^{-s} u) = 0,
\]
(4.6)

**Proof** As the set of all finite sums of functions of the type \( \psi(t, x) = \eta(t) \varphi(x) \) with \( \eta \in C^\infty_c(0, +\infty) \) and \( \varphi \in C^\infty(\mathbb{R}^d) \) is dense in \( C^\infty_c((0, +\infty) \times \mathbb{R}^d) \), to prove (4.6), we only need to prove that
\[
- \int_0^{+\infty} \eta'(t) \langle u(t), \varphi \rangle dt = \int_0^{+\infty} \eta(t) \langle \text{div}(u^\alpha(t) \nabla (-\Delta)^{-s} u(t)), \varphi \rangle dt
\]
for every \( \eta \in C^\infty_c(0, +\infty) \) and \( \varphi \in C^\infty(\mathbb{R}^d) \).

First, we recall \( ||u_0||_{H^{-s}}^2, U(u_0) < \infty \). Given a test function \( \varphi \in C^\infty_c(\mathbb{R}^d) \), and \( \delta, \tau > 0, m(r) = (r + \tau^{\frac{1}{10}})^\alpha \). Applying Lemma 3.4, we have
\[
\lambda_\delta \sim -\alpha \left( \frac{1}{\delta \tau^{\frac{1}{10}}} + \alpha \tau^{\frac{\alpha-1}{10}} \right).
\]

From Lemma 4.1, we know that \( \bar{u}_{r_n} \to u \) strongly in \( L^2((0, T), H^\beta(\mathbb{R}^d)) \) for every \( T > 0 \) and \( 0 \leq \beta < 1 - s \). Let \( \delta_n = \tau_n^{\frac{1}{10}} \) and \( \eta \in C^\infty_c(\mathbb{R}^+, \mathbb{R}^+) \). Let \( \tilde{\eta}_n : (0, +\infty) \to (0, +\infty) \) be the simple function defined by
\[
\tilde{\eta}_n(t) := \eta((k - 1)\tau_n) \text{ for every } (k - 1)\tau < t \leq k\tau, k \in \mathbb{N}.
\]
By (1.7),
\[-\int_0^{+\infty} \eta'(t) \mathbf{V}_{\delta_n}(\bar{u}_{\tau_n}(t)) dt = -\sum_{k=1}^{\infty} \int_{(k-1)\tau}^{k\tau} \eta'(t) \mathbf{V}_{\delta_n}(u_{\tau_n}^k) dt = \sum_{k=1}^{\infty} (\eta((k-1)\tau_n) - \eta(k\tau_n)) \mathbf{V}_{\delta_n}(u_{\tau_n}^k) = \sum_{k=1}^{\infty} \eta((k-1)\tau_n) \left( \mathbf{V}_{\delta_n}(u_{\tau_n}^k) - \mathbf{V}_{\delta_n}(u_{\tau_n}^{k-1}) \right).\]

Thanks to (3.12) in Lemma 3.4 and \( ||u_{\tau_n}^{k-1}||_{\mathcal{H}^{-s}}^2 - ||u_{\tau_n}^{k}||_{\mathcal{H}^{-s}}^2 \geq 0 \), there exists \( C(\varphi) > 0 \) such that
\[-\int_0^{+\infty} \eta'(t) \mathbf{V}_{\delta_n}(\bar{u}_{\tau_n}(t)) dt \leq \sum_{k=1}^{\infty} \eta((k-1)\tau_n) \left( \text{div} \left( \mathbf{u}_{\tau_n} + \tau_n^{1/10} \right)^\alpha \nabla (-\Delta)^{-s} u_{\tau_n}^k, \varphi \right) + C(\varphi) \tau_n \alpha \left( \frac{1}{\delta_n \tau_n} + \tau_n^{-\alpha} \right) \sum_{k=1}^{\infty} \eta((k-1)\tau_n) \left( ||u_{\tau_n}^{k-1}||_{\mathcal{H}^{-s}}^2 - ||u_{\tau_n}^{k}||_{\mathcal{H}^{-s}}^2 \right) \leq \int_0^{\infty} \eta_n(t) \left( \text{div} \left( \mathbf{u}_{\tau_n}(t) + \tau_n^{1/10} \right)^\alpha \nabla (-\Delta)^{-s} \mathbf{u}_{\tau_n}(t), \varphi \right) dt + C(\varphi) ||\eta||_{L^\infty} \left( \frac{2\alpha+3}{\tau_n^{0.10}} + \tau_n^{0.7} \right) ||u_0||_{\mathcal{H}^{-s}}^2.\]

This implies that there exist \( C, C(\eta, u_0) > 0 \) such that
\[-\int_0^{+\infty} \eta'(t) \langle \bar{u}_{\tau_n}(t), \varphi \rangle dt \leq \int_0^{\infty} \eta_n(t) \left( \text{div} \left( \mathbf{u}_{\tau_n}(t)^\alpha \nabla (-\Delta)^{-s} \mathbf{u}_{\tau_n}(t), \varphi \right) dt + C(\varphi) ||\eta||_{L^\infty} \left( \frac{2\alpha+3}{\tau_n^{0.10}} + \tau_n^{0.7} \right) ||u_0||_{\mathcal{H}^{-s}}^2 + \delta_n \int_0^{+\infty} |\eta'(t)||U(u_0)| dt + \int_0^{\infty} \int_{\mathbb{R}^d} \eta_n(t) \left( \mathbf{u}_{\tau_n}(t) + \tau_n^{1/10} \right)^\alpha - \mathbf{u}_{\tau_n}(t) \left| \nabla (-\Delta)^{-s} \mathbf{u}_{\tau_n}(t) \right| \left| \nabla \varphi \right| dx dt \leq \int_0^{\infty} \eta_n(t) \left( \text{div} \left( \mathbf{u}_{\tau_n}(t)^\alpha \nabla (-\Delta)^{-s} \mathbf{u}_{\tau_n}(t), \varphi \right) dt + C(\varphi) ||\eta||_{L^\infty} \left( \frac{2\alpha+3}{\tau_n^{0.10}} + \tau_n^{0.7} \right) ||u_0||_{\mathcal{H}^{-s}}^2 + C(\eta, u_0) \tau_n^{1/2} + C(1) \tau_n^{1/10} \int_0^{\infty} \int_{\mathbb{R}^d} \eta_n(t)\left| \nabla (-\Delta)^{-s} \mathbf{u}_{\tau_n}(t) \right| \left| \nabla \varphi \right| dx dt.\]

By (4.3) and letting \( n \to \infty \),
\[-\int_0^{+\infty} \eta'(t) \langle u(t), \varphi \rangle dt \leq \int_0^{\infty} \eta(t) \langle \text{div} \left( u^\alpha(t) \nabla (-\Delta)^{-s} u(t), \varphi \right) dt.\]
for any \( \varphi \in C_c^\infty(\mathbb{R}^d) \) and \( \eta \in C_c^\infty(\mathbb{R}^d_+,\mathbb{R}^d_+) \), which gives

\[
- \int_0^{+\infty} \eta'(t) \langle u(t), \varphi \rangle dt = \int_0^\infty \eta(t) \left( \text{div} \left( u^\alpha(t) \nabla (-\Delta)^{-\frac{1}{2}} u(t) \right), \varphi \right) dt
\]

for every \( \varphi \in C_c^\infty(\mathbb{R}^d) \) and \( \eta \in C_c^\infty(\mathbb{R}^d_+) \). This implies \( \eta = \varphi \) where we have used the fact that

\[
\phi = \int \frac{1}{1+|\nabla u|} u \quad \text{and} \quad \int \frac{1}{1+|\nabla u|} u \sqrt{1+|\nabla u|^2} \leq C(\eta, \varphi)
\]

Proof of Theorem 1.2 First, by interpolation inequality, \( ||u_0||_{H^{1+r}}^2 \leq \lambda^2 \). Given a test function \( \varphi \in C_c^\infty(\mathbb{R}^d) \), and \( \delta, \tau > 0 \), \( m(r) = (r+1)^{\frac{d}{2}} \). Applying Lemma 3.4, we have

\[
\lambda \sim -\tau \left( \frac{1}{\delta} + 1 \right).
\]

Let \( \delta_n = \tau_n \). As the proof of Lemma 4.2, there exists \( C(\varphi) > 0 \) such that

\[
- \int_0^{+\infty} \eta'(t) \nabla_{\delta_n} (\bar{u}_n(t)) dt \\
\leq \sum_{k=1}^{\infty} \eta((k-1)\tau_n) \tau_n \left( \text{div} \left( u^k_n + 1 \right) \nabla (-\Delta)^{-\frac{1}{2}} u^k_n, \varphi \right) \\
+ C(\varphi) \tau_n \sum_{k=1}^{\infty} \eta((k-1)\tau_n) \left( ||u^k_n||^2_{H^{1+r}} - ||u^k_n||^2_{H^{1+r}} \right)
\]

This implies

\[
- \int_0^{+\infty} \eta'(t) \langle \bar{u}_n(t), \varphi \rangle dt \leq \int_0^\infty \eta_n(t) \langle (-\Delta)^{-\frac{1}{2}} \bar{u}_n(t), \varphi \rangle dt \\
+ C(\varphi) ||\eta||_{L^\infty} \tau_n^{\frac{1}{10}} ||\varphi||_{H^{1+r}}^2 \\
+ \eta_n(t) \int \int_{\mathbb{R}^d} \eta_n(t) (\bar{u}_n(t) + 1)^{\frac{1}{2}} \log(\bar{u}_n(t) + 1) |\nabla (-\Delta)^{-\frac{1}{2}} \bar{u}_n(t)| |\nabla \varphi| dx dt,
\]

where we have used the fact that

\[
0 \leq (a + 1)^{\frac{1}{1+10}} - 1 \leq \tau_n (a + 1)^{\frac{1}{1+10}} \log(a + 1).
\]

Moreover, as \( \eta \in C_c^\infty(\mathbb{R}^d_+,\mathbb{R}^d_+) \), there exists \( T > 0 \), \( C(\eta) > 0 \) such that for \( n \gg 1 \),

\[
\int_0^{+\infty} \int_{\mathbb{R}^d} \eta_n(t) \langle \bar{u}_n(t) + 1 \rangle \log(\bar{u}_n(t) + 1) |\nabla (-\Delta)^{-\frac{1}{2}} \bar{u}_n(t)| |\nabla \varphi| dx dt
\]

Hölder

\[
\leq C(\eta, \varphi) + C(\eta, \varphi) \int_0^T ||\bar{u}_n(t)||_{H^{1+r}}^2 dt \leq C(\eta, \varphi).
\]
Thus,
\[- \int_0^{+\infty} \eta'(t) \langle \tilde{u}_n(t), \varphi \rangle dt \leq - \int_0^{+\infty} \eta(t) \langle (-\Delta)^{1-s} \tilde{u}_n(t), \varphi \rangle dt + C(\varphi, \eta, u_0) \tau_n^{1/10}.
\]
By (4.3) and letting \( n \to \infty \),
\[- \int_0^{+\infty} \eta'(t) \langle u(t), \varphi \rangle dt \leq - \int_0^{+\infty} \eta(t) \langle (-\Delta)^{1-s} u(t), \varphi \rangle dt
\]
for any \( \varphi \in C^\infty_c(\mathbb{R}^d) \) and \( \eta \in C^\infty_c(\mathbb{R}_+, \mathbb{R}_+) \), which gives
\[\int_0^{+\infty} \eta'(t) \langle u(t), \varphi \rangle dt = \int_0^{+\infty} \eta(t) \langle (-\Delta)^{1-s} u(t), \varphi \rangle dt
\]
for every \( \varphi \in C^\infty_c(\mathbb{R}^d) \) and \( \eta \in C^\infty_c(\mathbb{R}_+) \). Thus, we obtain (1.10). The proof is complete.

\[\square\]

Let \( p \in (1, +\infty) \) and let \( f_p : [0, +\infty) \to \mathbb{R} \) be the function defined by \( f_p(r) := r^p \). Let \( \tau > 0 \) and \( m(r) = (r + \tau \frac{1}{p})^a \), we consider the function \( P_p : [0, +\infty) \to \mathbb{R} \) defined by
\[P_p(r) := \int_0^r m(z) f''_p(z) dz.
\]
Let \( U : P_2(\mathbb{R}^d) \to (-\infty, +\infty] \) be the map defined by
\[U(u) := \int_{\mathbb{R}^d} f_p(u(x)) dx.
\]

**Lemma 4.3** Let \( u_0 \in \mathcal{P}(\mathbb{R}^d) \cap (L^2 \cap \dot{H}^{-s})(\mathbb{R}^d) \) such that \( \|u_0\|_{\dot{H}^{-s}(\mathbb{R}^d)}^2 < +\infty \). Let \( \tau > 0 \) and \( \{ u^k_\tau \} \) be the sequence and \( \bar{u}_\tau \) be the solution of the scheme (1.6) and (1.7) with \( m(r) = (r + \tau \frac{1}{p})^a \). Let \( g \in C^2([0, \infty), \mathbb{R}_+) \) be convex such that \( g(0) = g'(0) = g''(0) = 0 \). Then,
\[0 \leq \tau \| G(u^k_\tau) \|_{\dot{H}^{-1}(\mathbb{R}^d)}^2 \leq \int_{\mathbb{R}^d} g(u^k_{\tau-1}(x)) dx - \int_{\mathbb{R}^d} g(u^k_\tau(x)) dx \text{ for every } k \geq 1, \quad (4.7)
\]
with \( G(r) = \int_0^r \sqrt{m(z)} g''(z) dz \). In particular,
\[C(d, s) \tau \| G(u^k_\tau) \|_{L^2(\mathbb{R}^d)}^2 \leq \|u^k_{\tau-1}\|^p_{L^p(\mathbb{R}^d)} - \|u^k_\tau\|^p_{L^p(\mathbb{R}^d)}. \quad (4.8)
\]

**Proof** For \( \delta > 0 \), we consider the map \( U\delta : P_2(\mathbb{R}^d) \to (-\infty, +\infty] \) defined by
\[U\delta(u) := \int_{\mathbb{R}^d} g(u(x)) dx + \delta U(u).
\]
We consider
\[\partial_t v_t - \Delta G(v_t) - \delta \Delta v_t = 0 \text{ in } (0, +\infty) \times \mathbb{R}^d, \quad (4.9)
\]
where
\[G(r) = \int_0^r m(z) g''(z) dz.
\]
Clearly, (4.9) has a unique solution with initial data \( v_0 \in \mathcal{P}(\mathbb{R}^d) \).

We define the semigroup \( K_\delta \) by \( K_\delta v_0 = v_t \) for every \( t > 0 \). By the definition of \( u^k_\tau \) as a minimizer of (1.6)
\[\frac{1}{2\tau} W_m^2(u^k_\tau, u^{k-1}_\tau) + \frac{1}{2} \| u^k_\tau \|_{\dot{H}^{-s}}^2 \leq \frac{1}{2\tau} W_m^2(K_{\delta, h}(u^k_\tau), u^{k-1}_\tau) + \frac{1}{2} \| K_{\delta, h}(u^k_\tau) \|_{\dot{H}^{-s}}^2,
\]
and
\[ \lim_{h \to 0} \frac{||K_{\delta,h}(u^k_t)||^2_{H^{-s}} - ||u^k||^2_{H^{-s}}}{2h} = \left\langle (-\Delta)^{-s} u^k_t, \left( \partial_h K_{\delta,h} \left( u^k_t \right) \right)_{h=0} \right\rangle = \left\langle (-\Delta)^{-s} u^k_t, \Delta G(u^k_t) + \delta \Delta u^k_t \right\rangle = -\delta \| u^k_t \|^2_{H^{1-s}} - \left\langle u^k_t, (-\Delta)^{-s} G \left( u^k_t \right) \right\rangle. \]

So, we get that
\[ 0 \leq \tau \left( u^k_t, (-\Delta)^{-s} G \left( u^k_t \right) \right) \leq \limsup_{h \to 0} \frac{W^2_m(K_{\delta,h}(t), u^k)}{2h} - W^2_m(t, u^k). \tag{4.10} \]

We prove that
\[ \limsup_{h \to 0} \frac{W^2_m(K_{\delta,h}(\xi), \mu) - W^2_m(\xi, \mu)}{2h} + U_{\delta}(\mu) \leq U_{\delta}(\mu) \tag{4.11} \]
for any \( \mu, \xi \in \mathcal{P}(\mathbb{R}^d) \) with \( U_{\delta}(\mu), U_{\delta}(\xi) < \infty \).

Indeed, let \( \rho_n \) be in Lemma 2.1 with \( \mu^0 = \mu, \mu^1 = \xi \). Set for \( h > 0, t > 0, \rho^h_n(t) = K_{\delta,h}(t) \phi^h_n(t) \in \mathcal{P}(\mathbb{R}^d) \). Let \( \phi^h_n \) be a unique solution to \( \partial_t \rho^h_n(t) = -\text{div}(m(\rho^h_n(t)) \nabla \phi^h_n(t)) \) in \([0, 1] \times \mathbb{R}^d\).

We prove that
\[ \frac{1}{2} \partial_h \int_{\mathbb{R}^d} m \left( \rho^h_n(t, x) \right) \left\| \nabla \phi^h_n(t, x) \right\|^2 \, dx + \partial_h U_{\delta} \left( \rho^h_n(t) \right) \leq 0 \tag{4.12} \]
for any \( t \in [0, 1] \) and \( h \geq 0 \). Indeed,

**Step 1:** One has
\[ \partial_t U_{\delta} \left( \rho^h_n(t) \right) = \left\langle g' \left( \rho^h_n(t) \right), \partial_t \rho^h_n(t) \right\rangle + \delta \left\langle U' \left( \rho^h_n(t) \right), \partial_t \rho^h_n(t) \right\rangle = -\left\langle g' \left( \rho^h_n(t) \right), \text{div} \left( m \left( \rho^h_n(t) \right) \nabla \phi^h_n(t) \right) \right\rangle \]
\[ -\delta \left\langle U' \left( \rho^h_n(t) \right), \text{div} \left( m \left( \rho^h_n(t) \right) \nabla \phi^h_n(t) \right) \right\rangle = \int_{\mathbb{R}^d} m \left( \rho^h_n(t) \right) g'' \left( \rho^h_n(t) \right) \nabla \rho^h_n(t) \nabla \phi^h_n(t) \]
\[ + \delta \int_{\mathbb{R}^d} U'' \left( \rho^h_n(t) \right) m \left( \rho^h_n(t) \right) \nabla \phi^h_n(t) \nabla \rho^h_n(t). \]

Since \( G'(\rho^h_n(t)) = m(\rho^h_n(t)) g''(\rho^h_n(t)), U''(\rho^h_n(t)) m(\rho^h_n(t)) = 1 \),
\[ \partial_t U_{\delta} \left( \rho^h_n(t) \right) = - \int_{\mathbb{R}^d} G \left( \rho^h_n(t) \right) \Delta \phi^h_n(t) \right\rangle - \delta \int_{\mathbb{R}^d} \rho^h_n(t) \Delta \phi^h_n(t). \tag{4.13} \]

**Step 2:** By (3.8),
\[ \frac{1}{2} \partial_h \int_{\mathbb{R}^d} m \left( \rho^h_n(t, x) \right) \left\| \nabla \phi^h_n(t, x) \right\|^2 \, dx = -\frac{1}{2} \int_{\mathbb{R}^d} \partial_h \rho^h_n \left( \rho^h_n \right) \left\| \nabla \phi^h_n \right\|^2 + \int_{\mathbb{R}^d} \left( \partial_t \rho^h_n \right) \phi^h_n. \]

Note that
\[ \partial_t \rho^h_n(t) = t \Delta G(\rho^h_n(t)) + \delta t \Delta \rho^h_n(t). \tag{4.14} \]
Hence

$$\partial_t \partial_h \rho_n^h(t) = \delta \Delta \rho_n^h(t) + \Delta G \left( \rho_n^h(t) \right)$$

$$- t \Delta \left[ \text{div} \left( m \left( \rho_n^h(t) \right) \nabla \phi_n^h(t) \right) \left( G' \left( \rho_n^h(t) \right) + \delta \right) \right]. \quad (4.15)$$

By (4.14) and $m'' \leq 0, \ G' \geq 0$, we get

$$- \frac{1}{2} \int_{\mathbb{R}^d} \partial_h \rho_n^h m' \left( \rho_n^h \right) \left| \nabla \phi_n^h \right|^2$$

$$= - \frac{1}{2} \int_{\mathbb{R}^d} \Delta \left( G \left( \rho_n^h(t) \right) + \delta \rho_n^h(t) \right) m' \left( \rho_n^h \right) \left| \nabla \phi_n^h \right|^2$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \nabla \left( G \left( \rho_n^h(t) \right) + \delta \rho_n^h(t) \right) \nabla \left( m' \left( \rho_n^h \right) \left| \nabla \phi_n^h \right|^2 \right)$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} \left( G' \left( \rho_n^h(t) \right) + \delta \right) m'' \left( \rho_n^h \right) \left| \nabla \rho_n^h(t) \right|^2 \left| \nabla \phi_n^h \right|^2$$

$$+ \frac{1}{2} \int_{\mathbb{R}^d} \left( G' \left( \rho_n^h(t) \right) + \delta \right) m' \left( \rho_n^h \right) \nabla \rho_n^h(t) \nabla \left( \left| \nabla \phi_n^h \right|^2 \right)$$

$$\leq - \frac{1}{2} \int_{\mathbb{R}^d} \tilde{G} \left( \rho_n^h \right) + \delta m \left( \rho_n^h \right) \Delta \left( \frac{1}{2} \left| \nabla \phi_n^h \right|^2 \right),$$

where $\tilde{G}(r) = \int_0^r G'(a)m'(a)da$.

By (4.15),

$$\int_{\mathbb{R}^d} \left( \partial_t \partial_h \rho_n^h \right) \phi_n^h$$

$$= \int_{\mathbb{R}^d} \left( G \left( \rho_n^h(t) \right) + \delta \rho_n^h(t) \right) \Delta \phi_n^h$$

$$- t \int_{\mathbb{R}^d} \text{div} \left( m \left( \rho_n^h(t) \right) \nabla \phi_n^h(t) \right) \left( G' \left( \rho_n^h(t) \right) + \delta \right) \Delta \phi_n^h$$

$$\overset{(4.13)}{=} - \partial_t U_\delta \left( \rho_n^h(t) \right) + t \int_{\mathbb{R}^d} \left( G' \left( \rho_n^h(t) \right) + \delta \right) m \left( \rho_n^h(t) \right) \nabla \phi_n^h(t) \nabla \left( \Delta \phi_n^h \right)$$

$$+ t \int_{\mathbb{R}^d} G'' \left( \rho_n^h(t) \right) m \left( \rho_n^h(t) \right) \left( \nabla \phi_n^h(t) \nabla \rho_n^h(t) \right) \Delta \phi_n^h$$

$$= - \partial_t U_\delta \left( \rho_n^h(t) \right) + t \int_{\mathbb{R}^d} \left( G' \left( \rho_n^h(t) \right) + \delta \right) m \left( \rho_n^h(t) \right) \nabla \phi_n^h(t) \nabla \left( \Delta \phi_n^h \right)$$

$$+ t \int_{\mathbb{R}^d} \left( \nabla \phi_n^h(t) \nabla \tilde{G} \left( \rho_n^h(t) \right) \right) \Delta \phi_n^h,$$

where $\tilde{G}(r) = \int_0^r G''(a)m(a)da \geq 0.$
Since $\tilde{G}(r) = G'(r)m(r) - \tilde{G}(r)$,
\[
\int_{\mathbb{R}^d} \left( \partial_t \rho_n^h \right) \phi_n^h \\
= -\partial_t U_\delta \left( \rho_n^h(t) \right) + t \int_{\mathbb{R}^d} \left( \nabla \phi_n^h(t, x) \right)^2 dx + \partial_t U_\delta \left( \rho_n^h(t) \right)
\]
\[
\leq -\partial_t U_\delta \left( \rho_n^h(t) \right) + t \int_{\mathbb{R}^d} \left( \rho_n^h(t) + \delta m \left( \rho_n^h(t) \right) \right) \nabla \phi_n^h(t, x) \nabla \Delta \phi_n^h.
\]
Therefore,
\[
\frac{1}{2} \partial_h \int_{\mathbb{R}^d} m \left( \rho_n^h(t, x) \right) \nabla \phi_n^h(t, x) \left( \nabla \phi_n^h(t, x) \right)^2 dx + \partial_t U_\delta \left( \rho_n^h(t) \right)
\]
\[
\leq t \int_{\mathbb{R}^d} \left( \tilde{G} \left( \rho_n^h(t) \right) + \delta m \left( \rho_n^h(t) \right) \right) \nabla \phi_n^h(t) \nabla \Delta \phi_n^h - \Delta \left( \frac{1}{2} \left| \nabla \phi_n^h \right|^2 \right).
\]
Since
\[
-\Delta \left( \frac{1}{2} \left| \nabla \phi_n^h \right|^2 \right) + \nabla \phi_n^h(t) \nabla \Delta \phi_n^h = -D^2 \phi_n^h \leq 0,
\]
we get (4.12).

As the proof of step 2 of Lemma 3.4, (4.11) follows from (4.12).

Combining (4.10) and (4.11), we obtain
\[
0 \leq \tau \left\langle u_t^{k}, (\Delta)^{-1-s} \left( G(u_t^k) \right) \right\rangle \leq U_\delta \left( u_t^{k-1} \right) - U_\delta \left( u_t^{k} \right).
\]
Letting $\delta \to 0$, we get
\[
0 \leq \tau \left\langle u_t^{k}, (\Delta)^{-1-s} \left( G(u_t^k) \right) \right\rangle \leq \int_{\mathbb{R}^d} g \left( u_t^{k-1}(x) \right) dx - \int_{\mathbb{R}^d} g \left( u_t^{k}(x) \right) dx. \tag{4.16}
\]
Since $G'(t) = \sqrt{G(t)}$, one has
\[
\left( G(a) - G(b) \right)^2 = \left( \int_a^b G'(t) dt \right)^2 \leq \left( \int_a^b 1 dt \right) \left( \int_a^b \left( G'(t) \right)^2 dt \right) = (b-a)(G(b) - G(a))
\]
for any $b \geq a$. So,
\[
\left\| G(u_t^k) \right\|_{H^{1-t}(\mathbb{R}^d)}^2 = \left\langle G(u_t^k), (\Delta)^{-1-s} \left( G(u_t^k) \right) \right\rangle \leq \left\langle u_t^{k}, (\Delta)^{-1-s} \left( G(u_t^k) \right) \right\rangle. \tag{4.17}
\]
Thus, (4.7) follows from (4.16) and (4.17). The proof is complete.

Lemma 4.4 Let $u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap \dot{H}^{-s}(\mathbb{R}^d)$ and let $\tau > 0$. $\{u_t^k\}$ be the sequence and $\overline{u}_\tau$ be the solution of the scheme (1.6)–(1.7) with $m(r) = (r + \tau \mathbb{I})^a$. Let $d \geq 2$, $0 < s < 1$ and $C(d, s)$ as in Lemma 3.4. Then for any $L \in \mathbb{N}$, $1 \leq p \leq 2^L$, we have
\[
\left\| \overline{u}_\tau(t) \right\|_{L^p(\mathbb{R}^d)} \leq (\lambda(t + k_0 \tau))^{- \frac{(1-1/p)d}{d+2^L(t-s)}}, \tag{4.18}
\]

where
\[ \lambda = \frac{C(d, s)}{\sup_{1 \leq n \leq L} \sup_{k \geq k_0} k \left( 1 + \frac{1}{k-1} \right)^{\frac{(2^n-1)d}{d\alpha + 2(1-\tau)}} - 1} \]
and
\[ (\lambda \tau)^{-\frac{d}{d\alpha + 2(1-\tau)}} \geq ||u_0||_\infty. \]

Proof Applying (4.8) to \( g(z) = z^p \) for \( p > 2 \) to get
\[ C(d, s) \tau \left\| G(u^k_\tau) \right\|_{L^p(\mathbb{R}^d)}^2 \leq \left\| u^{k-1}_\tau \right\|_{L^p(\mathbb{R}^d)}^p - \left\| u^k_\tau \right\|_{L^p(\mathbb{R}^d)}^p. \]
From now on, for simplicity, we write \( ||f||_q \) for \( ||f||_{L^q(\mathbb{R}^d)} \). Since \( G(r) = \int_0^1 \sqrt{m(z)} g''(z)dz \geq \frac{2\sqrt{p(p-1)}}{\alpha + p} \frac{\alpha + p}{\frac{2(1-\tau)}{\alpha} - \frac{1}{\alpha}} \), there exists \( c > 0 \) such that
\[ c \tau \left\| u^k_\tau \right\|_{\frac{\alpha + p}{\frac{2(1-\tau)}{\alpha} - \frac{1}{\alpha}}} \leq \left\| u^{k-1}_\tau \right\|_p - \left\| u^k_\tau \right\|_p. \]
By interpolation inequality, one has for \( 1 \leq q < p \)
\[ c \tau \left\| u^k_\tau \right\|_{\frac{\alpha + p}{\frac{2(1-\tau)}{\alpha} - \frac{1}{\alpha}}} \leq \left\| u^{k-1}_\tau \right\|_p - \left\| u^k_\tau \right\|_p. \]
Now we apply this to \( p = 2^n \) and \( q = 2^{n-1} \) for \( n = 1, \ldots, \)
\[ c \tau \left\| u^k_\tau \right\|_{2^n} \leq \left\| u^{k-1}_\tau \right\|_{2^{n-1}} - \left\| u^k_\tau \right\|_{2^n}. \]
For every \( \lambda > 0, n, k \in \mathbb{N} \), we consider
\[ g_{2^n,k} = (\lambda(k + k_0)\tau)^{\frac{(1-2^{-n})d}{d\alpha + 2(1-\tau)}} \text{ for } k_0 \geq 1. \]
One has for \( k \geq 1 \) and \( n \geq 1, \)
\[ \frac{g_{2^n,k-1} - g_{2^n,k}}{2^{n+2^n+2^{n+1}(1-\tau)} - \frac{\alpha + 2^{n+1}(1-\tau)}{d}} g_{2^n-1,k} = \lambda \tau \left( k - 1 + k_0 \right)^{\frac{(2^n-1)d}{d\alpha + 2(1-\tau)}} - \left( k + k_0 \right)^{\frac{(2^n-1)d}{d\alpha + 2(1-\tau)}} \]
\[ = \lambda \tau (k + k_0) \left( 1 + \frac{1}{k - 1 + k_0} \right)^{\frac{(2^n-1)d}{d\alpha + 2(1-\tau)}} - 1 \].
If \( \lambda = \lambda(k_0, L) = \frac{c}{\sup_{1 \leq n \leq L} \sup_{k \geq k_0} k \left( 1 + \frac{1}{k-1} \right)^{\frac{(2^n-1)d}{d\alpha + 2(1-\tau)}} - 1} \) and
\[ (\lambda k_0 \tau)^{-\frac{d}{d\alpha + 2(1-\tau)}} \geq ||u_0||_\infty, \]
then, for any \( n = 1, \ldots, L \)
\[ c \tau \left\| g_{2^n,k} \right\|_{2^n} \geq g_{2^n,k-1} - g_{2^n,k}. \]
and \( g_{2^n,0} \geq ||\vec{u}_\tau(0)||_{2^n}, g_{1,k} = 1 \). Therefore, using induction, it is easy to check that for any \( n = 1, \ldots, L \)

\[
||u^k_{t}||_{2^n} \leq g_{2^n,k} = (\lambda(k + k_0)\tau)^{-\frac{(1-2^{-m})d}{d\alpha+2d}}.
\]

This gives

\[
||\vec{u}_\tau(t)||_{2^n} \leq (\lambda(t + k_0\tau))^{-\frac{(1-2^{-m})d}{d\alpha+2d}}
\]

for any \( n = 1, \ldots, L \). By interpolation inequality, we obtain

\[
||\vec{u}_\tau(t)||_{L_p(\mathbb{R}^d)} \leq (\lambda(t + k_0\tau))^{-\frac{(1-1/p)\alpha}{d\alpha+2d}}
\]

for any \( 1 \leq p \leq 2^L \). So, we finish to prove (4.18). The proof is complete. \( \square \)

**Lemma 4.5** Let \( d \geq 2, 0 < s < 1 \) and \( C(d,s) \) be as in Lemma 3.4. Let \( u_0 \in \mathcal{P}_2(\mathbb{R}^d) \cap \mathcal{H}^{-s}(\mathbb{R}^d) \) and let \( \tau > 0 \), \( \{u^k_t\} \) be the sequence and \( \vec{u}_\tau \) be the solution of the scheme (1.6)–(1.7) with \( m(r) = (r + \tau^{-1})^a \). Then there exists \( C > 0 \) such that for any \( t_1 > 0 \) and \( \Lambda > 0 \), we have

\[
\sup_{t \geq t_1} ((\vec{u}_\tau(t) - \Lambda) + ||\vec{u}_\tau(t) - \Lambda||_{2^n}^2 + \int_{t_1 + \tau}^\infty ||(\vec{u}_\tau(t) - \Lambda) + ||^{(a^2+2d)}_{d\alpha+2d} dt \leq C||((\vec{u}_\tau(t_1) - \Lambda) + ||_{2^n}^2. \tag{4.19}
\]

Moreover, if \( \tau < 2^{-d} \) and \( u_0 \in L^2 \), there exists \( \Lambda_0 = \Lambda(||u_0||_{L^2}, s, \alpha, d) > 0 \) such that

\[
\sup_{t \geq 1} ((\vec{u}_\tau(t) - \Lambda_0) + ||\vec{u}_\tau(t) - \Lambda_0||_{2^n}^2 + \int_1^\infty ||(\vec{u}_\tau(t) - \Lambda_0) + ||^{(a^2+2d)}_{d\alpha+2d} dt \leq C\tau^{d/(2\alpha+2d)+2}. \tag{4.20}
\]

**Proof** Let \( \Lambda > 0 \). Applying (4.8) to \( g(z) = (z - \Lambda) \) for \( p > 2 \) to get

\[
C(d,s)\tau \left\| G \left( u^k_t \right) \right\|^2_{\frac{2d}{d-2(1-s)}} \leq \left\| u^{k-1}_t - \Lambda \right\|_p - \left\| u^k_t - \Lambda \right\|_p.
\]

Letting \( p \to 2 \), there exists \( c > 0 \) such that

\[
c\tau \left\| u^k_t - \Lambda \right\|_2^2 + \int_{t_1 + \tau}^\infty \left\| u^{k-1}_t - \Lambda \right\|_{2^n}^2 - \left\| u^k_t - \Lambda \right\|_{2^n}^2,
\]

for any \( \Lambda > 0 \). So,

\[
\sup_{t \geq t_1} ((\vec{u}_\tau(t) - \Lambda) + ||\vec{u}_\tau(t) - \Lambda||_{2^n}^2 + \int_{t_1 + \tau}^\infty ||(\vec{u}_\tau(t) - \Lambda) + ||^{(a^2+2d)}_{d\alpha+2d} dt \leq 3||((\vec{u}_\tau(t_1) - \Lambda) + ||_{2^n}^2.
\]

for any \( t_1 \geq 0 \) and \( \Lambda > 0 \). This implies (4.19).

Next, let us set \( T_n = 1 - 2^{-n}, \Lambda_n = \Lambda(1 - 2^{-n}) \), and \( u_{\tau,n}(t) = (\vec{u}_\tau(t) - \Lambda_n) + \) for every \( 1 \leq n \leq \left[ \frac{1}{\log(2)} \right] - 3 \). And, we denote a level set of energy by

\[
U_n = \sup_{t \geq T_n} ||u_{\tau,n}(t)||_{2^n}^2 + \int_{T_n}^\infty ||u_{\tau,n}(t)||^{(a^2+2d)}_{d\alpha+2d} dt.
\]

We have

\[
U_1 \leq C||u(0)||_{2^n}^2 \tag{4.21}
\]

and

\[
U_n \leq C\inf_{t \in [T_{n-1}, T_n]} ||u_{\tau,n}(t)||_{2^n}^2 \leq C2^n||u_{\tau,n}(t)||_{L^2((T_{n-1}, \infty) \times \mathbb{R}^d)}^2,
\]
Since
\[
\begin{aligned}
&\left\{ \begin{array}{l}
u_{\tau,n} \leq \nu_{\tau,n-1}, \\
\nu_{\tau,n} > 0 \leq \nu_{\tau,n-1} > \Lambda^{2^{-n}}, \\
\|\nu_{\tau,n}\|_{L^{d(\alpha+2)+4(1-s)}}^{d(\alpha+2)+4(1-s)} ([T_{n-1},\infty) \times \mathbb{R}^d) \leq C U_n \frac{d+2(1-s)}{d},
\end{array} \right.
\end{aligned}
\]

one has for any \( n \geq 1 \)
\[
U_n \leq C 2^n \|\nu_{\tau,n-1}(t)\|_{L^2([T_{n-1},\infty) \times \mathbb{R}^d)}^2 \leq C 2^n \left( \frac{2^n}{\Lambda} \right) \frac{d(\alpha+1)}{d} U_{n-1} \frac{d+2(1-s)}{d}.
\]

Combining this with (4.21), we can find \( \Lambda_0 = \Lambda(\|u_0\|_{L^2}, d, s, \alpha) > 0 \) such that
\[
U_n \leq 2^{-c_0 n}, \quad c_0 = \frac{d(\alpha+1)}{2(1-s)} + 2
\]

for any \( n = 2, \ldots, \left\lfloor \frac{\log(\tau)}{\log(2)} \right\rfloor - 3 \). Therefore,
\[
\sup_{t \geq 1} \left\| \left( \nu_{\tau} - \Lambda_0 \right)_+ \right\|_2^2 + \int_1^{\infty} \left\| \left( \nu_{\tau} - \Lambda_0 \right)_+ \right\|_{d(\alpha+2)/d}^{d(\alpha+2)+4(1-s)} \frac{d}{d+2(1-s)} dt 
\leq 2^{-c_0 \left\lfloor \frac{\log(\tau)}{\log(2)} \right\rfloor - 3} \leq C\tau^{c_0},
\]

which implies (4.20). The proof is complete.

Proof of Theorem 1.1 i), ii), iii), and iv) follow from Lemmas 4.1, 4.2 and 4.4, 4.5.

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