Developing algebraic activity through conjecturing about relationships

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Abstract
This manuscript contributes to research on how algebraic thinking about operations and properties can develop, and relevant forms of curricular activity. The key question asked is: how can students’ early algebraic activity be fostered through focusing on relations involving operations and properties? We adopt Radford’s three aspects of algebraic thinking (indeterminacy, denotation and analyticity), which focus on considering, symbolising and operating on unknown objects. We investigate the work of a class of 11–12 year old students in re-analysing data collected over 20 years ago as part of a project exploring algebraic activity. Our findings point to the proposal that a focus on conjecturing about relationships can be a powerful route into early algebra. On the basis of our analysis we propose an extension to Radford’s third aspect of algebraic thinking (analyticity) to include structuring a mathematical space, e.g., through a set of conjectures.

Keywords Early algebra · Relations · Mathematical space · Deduction · Abduction · Conjecturing

1 Introduction

One conclusion from Kieran et al., (2016, p.15) was that little theorization had taken place regarding operations and properties as a route towards early algebraic activity. This article aims to offer both conceptual development and empirical evidence contributing to our understanding of how symbolizing relationships is linked to algebraic thinking. The question we address is: how can students’ early algebraic activity be fostered through focusing on relations involving operations and properties?

In the next section, we review past work on early algebra. We then consider briefly some definitions of algebra in order to situate our stance and set up a framework we use for analysis. Having established our definition of algebra, we then turn to some empirical data. We briefly describe our methodology and then analyse the written work of a class of 11–12 year old students as they engage in doing mathematics. The data was collected 20 years ago as part of a previous research project on developing algebraic activity.

This empirical evidence leads us to conclusions about how algebraic thinking can develop.

2 Past work on early algebra

In relation to developing early algebra, Carraher & Schliemann (2018) call for “widening the focus of mathematical problem solving beyond operations” (p. 112) to include the study of quantitative relationships, that still pose substantial challenges to young learners. Cai & Knuth (2011, p.x) propose the need, within the school curriculum, for better integrating opportunities for students to engage in algebraic activity. We concur with both these calls.

In her conclusion to the influential book “Early Algebraization”, Kieran (2011, p.581) identified seven themes of research. These themes were: thinking about the general in the particular; thinking rule-wise about patterns; thinking relationally about quantity, number, and numerical operations; thinking representationally about the relations in problem situations; thinking conceptually about the procedural; anticipating, conjecturing, and justifying; gesturing, visualizing, and languaging. Our focus is on “thinking relationally about quantity, number and numerical operations”, since it is in that strand we believe we have something novel to offer around operations and properties, as a route to early
algebraic activity, although we end up linking this strand to “conjecturing” also.

Empson et al., (2011) define relational thinking as the use of “fundamental properties of operations and equality to analyze a problem in the context of a goal structure” (p.409). We mean something slightly different when we write about thinking relationally; for us, we use this phrase more simply to indicate thinking about relations between objects, in contrast to thinking about the objects themselves. With Coles (2017), we assume that there is some ambiguity here, or at least that it is possible to consider almost any phenomenon as an object, or alternatively, in relation to other objects. For instance, a number can be considered as an object (perhaps denoting a quantity of discrete things) or as a relation (for instance, if the focus is on that number’s place in a sequence or pattern). When dealing with an operation or procedure, such as a function taking a starting number to an ending number, it is possible to focus on the start or end numbers as objects with properties, or to focus on a relation involving that operation, i.e., a relationship between the numbers (we take relation and relationalism as synonymous in a mathematical context). Such a focus on relations might be in the form of conjecturing a relationship. And, for instance, if two different functions are considered, it is possible to consider the functions as objects and focus on relations between them. So, being an object or a relation is not an ontological claim or property but rather a feature of how a mathematical concept is used.

As Cai & Knuth (2011) note, early algebra is not a new idea, and in China and Russia, from the 1950 and 1960s, algebra was introduced in a relational manner in elementary schools. A key influence on Russian early algebra practices was the work of Davydov (1990), who developed a curriculum (with Elkonin) where number was introduced as measure. Before number symbols were introduced to students, they worked on representing the relations between quantities. Students are initially invited to develop their own symbolism and later introduced to algebraic symbols, to codify relations such as a length “A” being greater than a length “B” (written, A > B). Algebraic relations are worked on by students before arithmetic (see Schmittau 2011). So, whereas a lot of important work on early algebra has been concerned with the shift from arithmetic to algebra, there are other forms of curriculum organisation where such a shift is not needed.

Freiman & Fellus (2021) take the view that we have distinct algebraic and arithmetic modes of thinking and that these modes are non-sequential. In other words, there is nothing inevitably about the move from one mode to the other. We also take it as an implication that there is no a priori reason for one mode to appear first in the curriculum. In our view, the mode of thinking that students develop in Davydov’s curriculum is close to what Squalli & Bronner (2017) term “analytical”, where one acts upon unknowns, as if they were known. When children look at two lengths and write A > B and, later, A = B + C (when the gap between A and B is labelled C), there is an intriguing interplay between the known and the unknown, which we see as one of the sources of the power of the Davydov curriculum. The lengths are visible to students and they can literally see (if they are attending to length) how the combination of B and C makes A. However, there is also an unknown element, in that they do not actually measure the lengths to find their values. The diagrams they use become schematics or psychological tools, which allow them to reason about unknowns, using something visible.

A special issue of Educational Studies in Mathematics (volume 106, issue 3) on Davydov’s approach indicates the enduring appeal of this thinking (Venenciano et al., 2021). A characteristic of the whole curriculum is that children are exposed to general structures first (such as relations between unequal and equal of lengths) and only then move on to particular instances (e.g., considering measured lengths). Working on algebra before arithmetic is one instance of this overall structuring of learning mathematics. Results from the use of Davydov’s curriculum can be startling in terms of the sophistication of thinking children can show from a young age (Schmittau, 2011).

Another influential perspective on early algebra comes from Kaput (2000), who makes suggestions for its teaching, including building on students’ natural linguistic and cognitive powers and informal knowledge. The idea of building on informal knowledge relates to Davydov’s work on measure and making use of children’s awareness of “greater than” and “less than”. Developing these ideas in a later publication, Kaput (2008) proposed two core aspects of using algebra in the early grades to support reasoning: (i) generalization and the expression of generalization in increasingly systematic, conventional symbol systems; and, (ii) syntactically guided action on symbols within organized systems of symbols. What is not specified, in Kaput’s suggestion, is what is being symbolised and we have come to view this as a crucial question, in relation to opportunities for algebraic activity. Indeed, one thing we aim to explore in this article, in considering how students’ early algebraic activity can be fostered through focusing on relations involving operations and properties, is the role of what is being symbolised.

3 Definitions of algebraic activity

There have been many characterisations of algebraic activity and we are not attempting to be comprehensive in this short section, rather we consider and compare three views that
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have influenced our work. The first is the definition adopted for the research project on which we report later on, from Kieran (1996). Kieran separated algebraic activity into three areas: generational, transformational and global meta-level. Generational activity might be seen in the generation of an algebraic rule for a number pattern. Transformational activity would be involved in, e.g., expanding or factorising an algebraic expression. Global meta-level activity includes: “analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, justifying, proving, and predicting” (Kieran, 2011, p.580); and we would want to add ‘conjecturing’ to this list. The definition prompts us to consider algebraic activity beyond the manipulation of symbols. Noticing structures may be done visually, for example, and expressed verbally; such thinking could count as algebraic activity, even before its codification into more succinct symbols.

Also influential on our thinking has been the work of Mason et al. (2005) and his linking of algebraic activity to processes of generalisation (an element of global meta-level activity). Mason was influenced by Gattegno (1987) in viewing generalising, and algebraic activity more broadly, as something available to all learners, one of the “powers of children” (p.5). Gattegno (1987) invited us to consider how children learn their home language(s), and to notice their algebraic activity. A child learning English as the language spoken at home will often make a grammatical mistake such as saying “I runned”. They are unlikely to have heard this said, suggesting such a phrase was not learnt by imitation. Rather, it appears we must concede that the child making this mistake has noticed a pattern in word endings, in particular contexts. They have generalised this pattern (adding “ed”) and applied their generalisation in a novel context (“I run”). In the case of “I runned” the application does not fit standard grammar, but Gattegno’s analysis points to algebraic features of what children have done to make such mistakes. And it is on the basis of such analyses that Gattegno proposes algebraic activity is a functioning, which anyone who speaks a language is capable of using. It is again clear that for Gattegno, as for Mason, and in Kieran’s global meta-level, that algebraic activity need not involve the use of the standard symbols of algebra.

The third influence on our perspective about algebraic activity comes from the longitudinal study of Radford (e.g., Radford, 2006; 2014) in which he proposes three elements to algebraic thinking (we take algebraic activity and algebraic thinking to be synonymous): indeterminacy, denotation and analyticity, which we will now unpack. Indeterminacy means that algebraic thinking involves considering unknowns, this could be, for example, unknown numbers or variables. Denotation is the condition that the indeterminate quantities need to somehow be symbolised (but not necessarily using the standard letter symbols of algebra). Analyticity means that indeterminate quantities are treated as though they are known in order to operate on them (for instance adding or dividing). Radford (2008) distinguishes forms of reasoning, with abduction being linked to the noticing of pattern or commonality and only deduction (p.85) being associated with analyticity (and algebraic thinking). The three aspects of: considering unknowns; symbolising the unknown; and, operating on the unknown, are key ones we bring to our analysis of data, augmented by ideas from Kieran and Mason. Radford writes about algebra in terms of unknown objects and relations between them. As alluded to above, we will be arguing that there are subtle distinctions between “objects” and “relations between objects”, in terms of what gets symbolized.

4 Methodology

The data analysed for this article comes from the written work of a class of 30 students aged 11–12, during their first year of secondary schooling in England, from 1998. Eighteen books remained in the archive to which we had access. This data was collected as part of a project (involving the first author) aiming to address the question of how to develop a need for algebra and we have drawn on the first author’s contextual knowledge of the project and project school. Sutherland’s (1991) prompt captures well the question the original study aimed to answer: “Can we develop a school algebra culture in which pupils find a need for algebraic symbolism to express and explore their mathematical ideas?” (p. 46). Data from the project has been re-analysed, using the conceptual developments outlined above. The reasons for returning to this data will hopefully become apparent as we describe the data itself. The overall framing of the original project (from 1998/9) was enactivist (Maturana & Varela, 1987). Enactivism is a theory of cognition which takes as a starting point the equivalence of knowing and doing “all doing is knowing, all knowing is doing” (Maturana & Varela, 1987, p.27).

Our current work is also influenced by ideas of enactivism, in terms of our approach to analysis. However, we recognise that when working with data that has already been collected, and so focusing just on the analysis, a theory as broad as enactivism under-specifies what needs to be done. However, Coles (2015) articulated five mechanisms for analysing language and communication, consistent with an enactivist view of learning. These mechanisms were developed with the analysis of language in mind. There is not space to consider all five, but two of them were relevant to working with written scripts and these are ideas that guided our work for this article. The two relevant mechanisms are:
the systematic search for pattern; and, meta-communication (Coles, 2015, p.239). We explain them briefly, in turn.

The systematic search for pattern implies a principled splitting or segmenting of the data to allow a focus on observable similarities and differences. For instance, there could be a focus on the use of a particular word or phrase. Multiple perspectives on the data (Reid, 1996, p.206) are needed to support the noticing of pattern, since different people will inevitably notice different things in any data. However, the key to making the search for pattern systematic is that once a pattern has been noticed, it is described in a manner that is then observable by others.

Secondly, an enactivist approach to communication, drawing on one of its influences in systems theory (e.g., Bateson 1972), will pay attention to when communications (written or spoken) are about the communications taking place. A meta-communication is just this idea – a communication about communication. Meta-communication is typically non-verbal; for instance, it is meta-communicative messages that indicate whether a stare is one of affection or aggression. In a written context, however, we only have access to verbalised messages and so this mechanism for analysis is to look out for any writing which refers to itself in some way. For instance, a student writing “I am going to try it out …” is a written communication about the written communications which follow and so would count as meta-communicative. We will draw out, later in the article, some speculations about potential links between meta-communication and developing algebraic activity.

There is a coherence between our enactivist commitment and Radford’s definitions of algebraic activity, in that the focus is on activity. In other words, there is no sense of attempting to interpret what is going on inside the heads of students. We analyse what students do and interpret such actions as synonymous with knowing. We recognise that Radford developed his ideas about algebra within a cultural-historical perspective (Roth & Radford, 2011), influenced by dialectical materialism, and that the roots of enactivism are in traditions of phenomenology and cybernetics. In terms of networking these theories, we do not attempt a synthesis or integration (Bikner-Ahsbahs & Prediger, 2010) of these quite different traditions but, rather, are combining and coordinating the use of different analytical tools “for the sake of a practical problem or the analysis of a concrete empirical phenomenon” (p.493). In our case, the concrete phenomenon are the books of students.

4.1 The dataset

In our re-analysis of the project data, we have focused on the work done by the class between September and December 1998, which was the first one third of the academic year. The reason for this choice is that students did their work in exercise books containing 24 pages, which typically was enough for classwork over a 4 to 6-month period, and while we have a majority of one class’s books from the start of the year (which was September), we do not have so many of their follow on books (some students may have kept these at the end of the academic year in July).

From students’ books it is possible to distinguish eight extended tasks worked on over this period; an extended task being one that lasts over a sequence of lessons (the longest one lasting 7 lessons). From these eight tasks, we focus on the first one. This task was the first one of the year (a task called “1089”). We focus on this one because students at that time in England were unlikely to have met any formal algebra in primary school and, as the data is from the start of secondary school, we therefore have data about their first entry into early algebra in a school setting.

A word about the writing in students’ books is needed at this point, because in our experience, the extent and kind of writing present is unusual. A typical example of a page from a student’s book is in Fig. 1.

We offer Fig. 1, not in the expectation the text can be read, but to give a sense of the balance of number work (for this task) and writing about that work. Throughout all the books such a balance is typical. It is the fact this balance of writing is unusual, in our experience in a mathematics exercise book, that we have found it worthwhile to consider data from so long ago. We believed there may have been a link between students’ writing and their use of algebra and that there may be practices here that are perhaps relatively under-recognised in research on algebra.

The teacher placed emphasis with the class on them writing down their ideas and engaged in a written dialogue, through comments made regularly on their work, supporting and encouraging such writing. The emphasis on writing was consistent with a focus in this class on processes of “thinking mathematically”. Tasks were designed and offered which allowed students elements of choice, within a constrained structure. As described in the original research project, there were also regular class discussions in which students would share their ideas and what they had found out, and when the teacher might offer a new skill or create a common focus on a particular question or result (Brown & Coles, 2008). The classroom environment was one, therefore, where ideas would spread from student to student, but where students were supported in staying with their own questions and writing down what they were doing and thinking. The writing in the books therefore does not necessarily indicate an idea original to that student, but rather an idea that made sense to that student.
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and how those relations were being treated. At this stage there was a recursive process of finding categories that made sense of the data and altering those categories in light of the data.

Before getting to the students’ work itself, we first detail the task we have analysed. The “1089” task involves an arithmetical “trick” where you take a 3-digit number (whose first digit is bigger than the last), reverse the digits and subtract the second number from the first. This gives you a 2 or 3-digit answer. You reverse the digits of the answer and add it to that answer. The result will be 198 or 1089, if you have done the subtraction and addition correctly.

The task was usually set up in a manner that attempted at some drama – where the class would be surprised at the extent of similarity in getting 1089 (to which most starting numbers will end) from all their different starting points (see Brown & Coles 2008, p.88). The initial challenge given to students (as described in Brown & Coles (2008), which has a write up from the time) was to “find one that doesn’t work” (i.e., a starting number that did not end with 1089, or 198). After working for a couple of lessons on this 3-digit challenge, students would move on to consider the situation when starting with 4 or 5-digit numbers (in which cases, different final numbers obtain, but still a constrained set), where a challenge was to predict, from the starting number, which final number will result.

4.2 Methods

As authors, we based our analysis on the students’ books. We also used the school mathematics department’s write ups of how the tasks were to be offered to classes, to explain the task itself. To make our search for patterns systematic we highlighted, on PDF copies of students’ books, instances of a consideration of relations between objects. As described in Sect. 2, we take a relation to mean any instance where what was in focus was a connection between two objects (such as numbers) as opposed to a focus on the objects themselves. A student writing “I’m doing 4-digit numbers” is attending to objects (the numbers he is “doing”) and a property of those objects (the number of digits). A student writing “If … then …” is describing a relation between the objects in the statement. We initially looked at students’ books together. There were some ambiguous cases and our choice was to include statements in the analysis rather than exclude if we were unsure. We then highlighted, independently, instances of considering relations, on several students’ books, and checked our results to confirm what we were taking to be a consideration of relations. We found good agreement and so proceeded to carry out the highlighting of relations separately before confirming together all of our highlighting and judgments. We then made a further pass through the books to highlight, in a different colour, any instances of written meta-communication.

With the highlighting done, of all of the aspects of students’ books that we wanted to analyse, this was a systematic element of the process complete and we then needed to look for further patterns in what relations students noticed...
5 Findings

Looking at the students’ books, and analysing any evidence of considering relations between objects, we found three patterns in their work (i.e., commonalities across books). The categories were: (1) noticing relations and patterns; (2) making, testing and structuring of conjectures; (3) proving conjectures. Within these categories we then considered Radford’s three aspects of algebraic activity. We exemplify each of these three categories below. In keeping with our overall enactivist approach to language, we also include a section where we have focused on evidence of students’ meta-communications in their writing (5.4). In Sect. 5.5, we then offer summary data across all student books. We use two initials to refer to different students. We present data which constitutes typical examples of each category.

5.1 Noticing patterns and relations

Students reasoned from relations about the specific numbers that they tried in the task algorithm. For example, student DE started with the number 201 and found that the algorithm went to 1089. DE tried to find some numbers that did not go to 1089 (in response to the class challenge) and chose the numbers 999 and 101 which led to 000. DE found a relation between the first digit and the last digit of the number 101 and 999 that did not end in 1089.

This number (101) is a bit similar to the one in the top left hand corner (999). It doesn’t end up 1089 but the first number is not bigger than the last.

The relation between digits is derived from determinate numbers 101 and 999, and although we might interpret a more general statement, this is not actually stated, so we view DE’s statement as an example of identifying a relation between determinate objects and one that is pre-generalization, i.e., there is no statement of a property that holds beyond the numbers tried out already. This kind of reasoning is not only evident about one or two numbers. A different student, DC, tried out 28 three-digit numbers and found that all the numbers went to 1089. The student wrote:

I did 28 numbers but it kept going 1089 and I can’t seem to change the number in the middle.

In a three-digit number case, the “number in the middle” likely refers to what happens in the answer to the initial subtraction (and hence in its reverse, which will not change the middle digit), which is that if a 3-digit number results from the initial subtraction, the middle digit will be a 9. DC found this relation between numbers in the 28 examples tried.

Therefore, in terms of our definition of algebra, we observed a pattern of students considering relations between determinate objects (numbers, in the cases above). We do not consider such statements to be examples of algebraic thinking, since there is no indeterminacy. These statements are about numbers and results obtained and do not reach forward into the unknown or make a statement that could be tested so there is no generalization or conjecturing; we do not consider any of Radford’s three conditions for algebraic thinking to be met.

5.2 Making, testing and structuring conjectures

Students showed evidence of testing relations and conjectures they found. For instance, after working on the 5-digit problem, CC noticed a relation between the second and third and fourth digits of numbers which result in 109,989. CC wrote about a relation between indeterminant digits, and then tested the relation by trying out different and new five-digit numbers, to check whether the relation indeed applied to other unknown numbers:

I think that the theory for the 109,989 column is that the 2nd digit and the 3rd digit and 4th digit is the same … I am going to try out this sum [10,200] because it doesn’t fit into my theory in the 109,989 … I think my theory is wrong.

CC here thinks her theory is wrong because she seems to want only numbers where the three middle digits are the same, to result in 109,989. The number 10,200 also ends up at that result, hence she sees her theory as wrong (for 5-digit numbers all that is needed, in fact, is that the 2nd and 4th digits are the same, to ensure the result of 109,989). The personal voice is strong in this excerpt.

There were a variety of ways in which relations were tested. Several students tested similar conjectures to CC’s by seeing if predictions were accurate for a new starting number. MW attempted to find counterexamples to test a relation. MW found a relation that operating on 3-digit numbers always results in 1089 and tested the relation with other numbers until finding 3-digit numbers where the sum is not 1089, as counterexamples. The word “counterexample” was introduced by the teacher, in the context of the 1089 problem.

Students also looked across relations and made predictions about relations. In other words, looking across relations was not only to generate a relation between objects but also to conjecture a relation between relations. For example, student DE wrote:
I think that there will be about 4 answers for the six-digit ones because there were three answers for the 4 and 5 digit one. And I don’t think there will be other answers for 6. I think that there will be an answer that is 990,099.

For CC, MW and DE a relation involving indeterminate objects is treated as an object, in the sense of being something that can be examined and tested. Hence, we view them as fulfilling the criteria of indeterminacy (they involve indeterminate objects) and denotation (the objects are symbolised). There is some analyticity involved in working out what is an appropriate number to test a conjecture, but perhaps this is a proto form of deductive reasoning and analyticity. In other words, we consider the examples so far take us to the threshold of algebraic thinking, arising from a consideration of relations involving operations on indeterminate objects.

There were also cases in this category which we consider show greater signs of analyticity. After performing the algorithm on many numbers, some students wrote about relations involving operations on unknowns, by categorising and generalising their work. For instance, SM considered a relation between digits:

If you have 4 digits and the first number is bigger … you would either get 10,989, 9999, or 10,890… if it goes in 10,890 you would get 2 numbers in the middle that are the same, and also if it goes in 9999 then the second number is always smaller than the third digit … and 10,890 the second number is bigger than the 3rd number.

SM worked with many four-digits numbers and (correctly) categorised the operation results into three groups which end up at 9999, 10,890, and 10,989. Based on the similarities of numbers in each category and the differences of numbers across categories, SM found a pattern in the 2nd and 3rd digits and conjectured a relation between those digits in starting numbers and the final result. The conjecture is a proposed generalisation for all numbers, those numbers being expressed in a general non-specific manner. Students were invited by the teacher to re-create it (a successful attempt is shown in Fig. 2). SM attempted to generalise the pattern they are claims about operations performed on any starting number. Another student example of a similar structuring is MW who wrote: “I now know how to predict the outcome” and then proceeded to outline the same three conjectures for a 4-digit number.

The question of the extent to which statements such as the ones above from SM and MW show analyticity is a complex one. There does not appear to be any operating with unknowns, which is one way Radford explains analyticity. However, what we do observe is what we describe as an activity of structuring the mathematical space. Any task entails a mathematical structure, i.e., a set of relations between elements of the task. That structure can become apparent to a student through acts of structuring (e.g., making predictions). We take a mathematical space (Brown, 1995) to denote what gets structured by such acts. A mathematical space is therefore what students produce patterns and conjectures about as they hit up against constraints and generalise underlying relations between unknowns. For example, the 1089 task generates a space in which patterns can be discerned, where there is some predictability and the possibility of deduction and certainty. The structuring of a mathematical space implies that a set of conjectures more or less completely covers a set of local possibilities within that space. So, working on 4-digit starting numbers, it turns out there are three possible results, which map on to the three possible relations between the 2nd and 3rd digits. When students offer a complete structuring, e.g., covering the possibilities that $b > c$, $b < c$ and $b = c$, we infer some elements of deduction. The phrase of MW’s “I now know” indicates to us that MW has understood they has covered all possibilities. We therefore argue that structuring a mathematical space, through a set of conjectures, offers opportunities to engage in the beginnings of deductive reasoning (and hence analyticity and hence algebra), arising out of the abductive reasoning used to develop the conjectures.

5.3 Proving conjectures

Radford’s algebraic element ‘analyticity’ is about operating on indeterminate objects in a deductive manner. The two earlier categories involved noticing, conjecturing, testing and structuring. In this final category we exemplify students’ work on proving.

Students themselves wondered why relations occurred (for instance, why do the 3-digit numbers come to 1089) and so in one of the last lessons on the task, the teacher showed an algebraic proof (Brown & Coles, 2008). This proof was worked on with the whole class, alongside a numerical example, the proof was then rubbed off the board and students were invited to re-create it (a successful attempt is shown in Fig. 2). SM attempted to generalise the pattern...
be the same and in the two sums you always need to
borrow from the column up the sum that you have to
add in. The middle number will always be nine and
nine and nine is 18 and the answer to the whole sum
will always be 1089.

The precise meaning is not clear to us, but we note the text
here has been desubjectified (Radford, 2001), which points
towards a form of generalization.

5.4 Meta-communication in algebraic writing

Across students’ books evidence of meta-communication,
or communication about other communications, generally
came in their commentary on what they were doing. For
instance, some students would write down something about
what they were going to do, before doing it (and this was
encouraged by the teacher): “I’m now going to try to …”
(Student CB). Other students commented when an idea or
prediction turned out correct: “I’m going to … yes my the-
ory works” (Student DE), or did not work, “I’m going to
try to … wrong” (Student MW). And finally, some students
made comments looking back at their work: “I tried picking
out” “We’ve been looking at …” (Student CB).

While meta-communication is not included as one of our
three aspects of algebraic activity, we do see links between
meta-communication about work in mathematics and meta-
cognition, which is in turn linked to skills of problem-
solving (an aspect of Kieran’s global meta-level algebraic
activity). In terms of analysis, we struggled to distinguish
when a written comment expressed a meta-communication
or not. What we have come to believe is that, in the context
of the work in these mathematics books, everything writ-
ten was a meta-communication (including all the examples
just above). We infer that there must have been something
about the way in which writing was set up by the teacher
and expected to be done, which meant that the comments
were always reflections on what had been done or what
was about to be done, and hence they were metacommu-
icative. We reflect on the potential role of this writing in
the next section, but we do not further analyse the forms of
meta-communication.

5.5 Summary of data and discussion

Having exemplified what we took as evidence for the differ-
ent aspects of algebraic activity, we now offer a summary of
our analysis of the writing in the students’ books (Table 1).
The numbers refer to the number of instances of writing
about a relation. When writing was extended (e.g., including
several sentences), we might interpret mention of two dif-
ferent relations, but in general, students wrote their ideas in

Fig. 2 Student SM’s recreation of the teacher’s proof of the 3-digit
result

Fig. 3 Student SM’s own proof of a 2-digit result

between starting numbers and outcome results by using let-
ter symbols to prove the relations. Place value notation is
used within this proof (i.e., the number A B C could also be
thought of as, 100 A + 10B + C).

Some students went on to adapt the 3-digit proof to other
cases, for example, that all 2-digit numbers result in 99 (see
Fig. 3).

The use of letters (Figs. 2 and 3) is no longer about let-
ters simply as labels for one digit or another. The letters here
serve functional roles, with place value properties, allowing
the application of algorithms for addition and subtraction.
This seems to us clear evidence of students operating with
unknowns and meeting all three criteria for algebraic activ-
ity (indeterminacy, denotation, analyticity), i.e., considering
unknowns; symbolising the unknown; and, operating on the
unknown.

Other students extended the proof to 4-digits and one or
more of the different results that can occur. However, evi-
dence of analyticity through proof did not always involve
letter symbols. DE tried to prove why the three-digit opera-
tion resulted in 1089 as follows:

It will always turn out as 1089 if the sum is written
correct because the number in the middle column will
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The evidence of structuring and proving conjectures was less frequent and also tended to be more involved, hence it made more sense to simply report if there was evidence of these with a yes/no across all the work on the 1089 task.

There are several immediate things we note from Table 1.

The first is there were many more instances of students writing about relations involving indeterminate objects (and operations on them) than determinate. Each comment about relations involving indeterminate objects shows at least some of the aspects of our definition of algebraic thinking and in some cases also global meta-level activity and all students did this. The second immediate observation is that the 16 out of 18 students showed some further forms of analyticity (testing or structuring a conjecture) and 10 showed evidence of deductive thinking and hence all three of Radford’s forms of algebraic activity.

Given our focus on early algebra, we offer some more detail about what these 16 students did, in terms of an analytical treatment of relations and conjectures (offering one example from each student, that showed their most algebra-like activity).

### Table 2: How students treated relations analytically

**Testing a conjecture by trying to find counterexamples**
- CB: “Today I’m going to carry on trying to find a sum that doesn’t end up as 1089”
- TP: “I could not find one that did not add up to 1089”

**Reasoning about conjectures**
- CC: “I think there will be 7 answers for 6 digit numbers because there is 1 answer for 3 digit numbers and 3 answers for 4 digit numbers and 5 answers for 5 digit numbers”
- BP: “I think if you use five-digit numbers, that there will be five different answers. I think that because when you use three digit numbers there is only one answer and when you use four digit numbers there is three answers so I think it goes up in two’s”

**Structuring a mathematical space**
- TK: All three possible conjectures about 4-digits
- SE: All three possible conjectures about 4-digits

**Deductive proof**
- AM: 3-digit symbolic proof
- CW: 4-digit symbolic proofs
- NP: 2, 3, 4, 5 digits symbolic proofs
- CZ: 3-digit symbolic proof
- DC: Attempting a symbolic proof
- DE: Attempting a symbolic proof, testing a relation, and an inductive generalization
- LT: 2-digits symbolic proof
- MW: Predict and test a relation, and a symbolic proof
- SM: Make a prediction and attempt a symbolic proof (2,3,4 digit)
- CP: 2-digit symbolic proof, and an inductive generalization

short paragraphs and we were able to analyse whether that paragraph or sentence showed evidence of a relation being discussed or not. The evidence of structuring and proving conjectures was less frequent and also tended to be more involved, hence it made more sense to simply report if there was evidence of these with a yes/no across all the work on the 1089 task.

There are several immediate things we note from Table 1. The first is there were many more instances of students writing about relations involving indeterminate objects (and operations on them) than determinate. Each comment about relations involving indeterminate objects shows at least some of the aspects of our definition of algebraic thinking and in some cases also global meta-level activity and all students did this. The second immediate observation is that the 16 out of 18 students showed some further forms of analyticity (testing or structuring a conjecture) and 10 showed evidence of deductive thinking and hence all three of Radford’s forms of algebraic activity.

Given our focus on early algebra, we offer some more detail about what these 16 students did, in terms of an analytical treatment of relations and conjectures (offering one example from each student, that showed their most algebra-like activity). We have split the making, testing and structuring of relations into three patterns (see Table 2). The first pattern of testing a conjecture we view as a global meta-level activity, but we are less clear if this can be considered as showing analyticity in relation to an unknown. The second pattern of reasoning about a conjecture might include testing it, or an inductive generalisation – again evidence of global-meta level (analysing relations). The third pattern is structuring a mathematical space, which we argue shows important forms of at least the beginnings of analyticity.
There would be consider-
ations about relations who showed least evidence of treating
relations analytically. This finding points to a potential role
for writing and meta-communication, in developing alge-
bra. There is perhaps something unsurprising if, as a student,
you are not writing down much about what you are doing,
or noticing, that it then becomes harder to consider, analyti-
cally, those patterns or relations that you have noticed.

We want to flag up student LT. This student arrived at
secondary school with low levels of prior attainment. From
their book we observe a large proportion of the arithmeti-
cal calculations carried out contained inaccuracies and LT
had difficulties in spelling. And yet, despite this, the student
wrote about relations between indeterminate objects and
engaged in conjecturing. For instance, in Fig. 4, LT is mak-
ing a prediction that with 4-digit numbers if the middle two
digits are the same, then they will end up at the same final
answer.

The way of working in the classroom, with common col-
lections of data (Brown & Coles, 2008), appears to have
supported this student in being able to think algebraically,
or approach thinking algebraically, based on the results of
others, when the inaccuracies of their own work would have
made this almost impossible to do alone (i.e., no patterns
would have been apparent).

Another student we want to highlight is NP, who extended
work on proof to cover 2, 3, 4 and 5-digit cases. Not all of
these were carried through accurately, but an example of
one which was is in Fig. 5. In this example the student has
shown that 4-digit numbers with the two middle digits the
same will all end up at 10,989.

The student in Fig. 5 has been able to work with letter
notation, reasoning for instance that “c + 10 – a” add “a – 1
– c” will give the result 9 (in the bottom row on the right).
The student has been able to compute an algorithmic pro-
cess for subtraction and addition and apply this to the case
of an arbitrary number (a b b c). There would be consider-
able detail we could go into, in relation to the uses of stan-
ard algebra in this case; what we see here is an example of
treating a relation analytically. The range of mathematical
sophistication across students’ books is perhaps apparent in
Figs. 4 and 5.

6 Conclusion

In this paper, we have re-analysed students’ mathematical
exercise books from a project that took place more than
twenty years ago. The reason for doing this is that, despite
the gap of time, we feel that the books show evidence of
ways of working on mathematics that are quite unusual.
Our focus has been on the question: how can students’ early
algebraic activity be fostered through focusing on relations
involving operations and properties? We have shown that
the students’ books contain evidence of forms of early alge-
bra that relate to the development of algebra from conjectur-
ing about relationships.

All students who participated in this study found relations
involving indeterminate objects. We have evidence that all
students noticed some structures, generated some patterns
and found some relations (see Table 1). We proposed the
notion of structuring a mathematical space, e.g., via a set of
conjectures, as a form of proto-analytical thinking, showing
elements of deduction. When students wrote about relations
involving indeterminate objects, they showed they were able
to generalise from what they had done, often in the form of a
conjecture about untried numbers (e.g., all 4-digit numbers
with the 2nd and 3rd digits the same will come to 10,989).
The expression of generality is closely related to algebraic
thinking (Mason, 2018) and points towards some analytic
handling of relationships. Given this task would have been
students’ first introduction to algebra in school we believe
the evidence from the books is striking, in terms of how
successful the approach appears to have been at provoking
aspects of algebraic thinking, through a focus on relations
involving operations and properties.

We have found it helpful to contrast the approach exem-
plified in the books to more typical routes into algebra
through the generalisation of number patterns from figures
(Radford, 2001). For instance, students may be presented
with a sequence of figures of squares joined together, made
of matchsticks, and asked to find the relation between the figure number and the number of matchsticks. There are some similarities in that there is scope in such tasks for commenting about determinate objects (a particular figure) and about indeterminate objects (e.g., to notice a property of all figures). What gets symbolised is usually a property of the indeterminate object (such as the number of matchsticks on the top row). However, in the 1089 task, students’ attention is on relations between an indeterminate object (their starting number) and a determinate object (a single or small set of finishing numbers). We feel it is this subtle difference (a focus on an unknown object, or on a relationship involving an unknown object) which is part of the explanation for what we observe as the power of the students’ route into early algebra. Relationships involving operations on unknown objects, although abstract, can be quite visible. A table of values showing that the numbers 4221, 7445, 8331, 6662 all end up at 10,989 invites a generalisation, e.g., conjecturing a relationship involving unknown objects (all 4-digit numbers where \( b = c \) end 10,989), it also might occasion consideration of what other relations between \( b \) and \( c \) could obtain.

In this study, one low attaining student (LT) performed many arithmetic operations incorrectly but showed evidence of important elements of algebra; the student found relations between indeterminate objects, which we argue indicates at least 2 of the 3 conditions for algebraic thinking. At the other end of the range of prior attainment, we also have evidence of several students engaging in novel symbolic moves. Students who were able to take and adapt the 3-digit algebraic proof and apply this in a novel situation showed levels of sophistication which are unusual in classes of year (or grade) 7, in our experience (Fig. 5). The 1089 task contained myriad relationships which could be noticed and, in this sense, afforded or occasioned the expression of generalisations. We are also struck by the evidence that algebraic activity in the books we have analysed does not appear to occur in some kind of transition away from arithmetic. The activities we have analysed as being algebraic, or fulfilling some of the conditions for algebra, is on relations between numbers and relations between arithmetical processes. However, once noticed, these generalisations are immediately tested with arithmetic. So, while arithmetic may provide a grounding to algebraic activity, in this task, arithmetic is never left behind.

Although the 1089 task can be applied to numbers with any amount of digits, there appears to have been something significant about an initial constraint for the whole class to consider 3-digit numbers only (which is evident from the students’ books). With this constraint, it was possible for students to experience the making and testing of conjectures and perhaps to develop some expectations about regularities, when they came to consider other numbers of digits. We imagine these regularities provided their own feedback to students, i.e., getting an answer that did not fit any pattern might suggest a calculation error. The potential for novelty is apparent in the diversity and range of questions and issues which students considered. And yet, this range of activity is within tight bounds set up by the teacher. In other words, despite the freedom for students to choose particular areas of interest and focus, all the activity entailed a lot of practice at addition, subtraction, and the making and testing of conjectures – key processes that, according to departmental documentation, the school wanted to establish at the start of the academic year.

Written meta-communications were found in all students’ writing. To report on their work, students needed to look back on what they had done and what they had noticed (e.g., from doing the algorithm). To test their conjectures, students again needed to consider what they were doing and the results obtained. Some students wrote their own feedback and, in general, the action of writing in students’ books was communicative and recursive. There are detailed comments from the teacher in each book, which we have chosen not to analyse, given our focus for this article. We imagine that the written dialogue through the books was a strong encouragement to students to engage in writing in lessons, in the knowledge that it would be read and receive comment.

The 1089 task itself does not sit easily in terms of typical categories used in the literature. We do not consider the task a typical problem-solving one, given it lasted over several weeks of work. There were clearly problem-solving elements to what students were doing, but the task does not fit into categorisations of whether skills are introduced first, or whether the problem solving comes first. While there was an initial lesson on the task that involved a direct teaching of the algorithm, the class’ work on the task moved frequently between moments of focus on a skill and times when students followed their own lines of inquiry (Brown & Coles, 2008). The teaching of skills of algebraic proof would be one example evident in books from late in the project. In other words, there appears to be no clear or distinct separation between a skills focus and a problem-solving focus. A key element is that the task entails the exploration of a mathematical structure with assumptions and boundaries (Coles & Sinclair, 2019) and with patterns and conjectures to uncover. We also imagine a key point is that, from the books, it is clear students were encouraged to ask their own questions and supported to work on those questions.

We view one contribution of this article as showing how a focus on working with relations involving operations and properties can be a prompt to early algebra, via conjecturing about those relationships. We propose that analyticity
can mean operating with unknowns (as Radford has it) and also (perhaps in a proto manner) activities of structuring a mathematical space, for instance by conjecturing a (locally) complete set of relations and testing them. We believe our results indicate deductive and abductive reasoning may be more entangled than previously considered. We have shown how a form of deductive structuring of a mathematical space can arise directly out of more abductive conjecturing.

We recognise the limitation in our work that we have looked at the classroom of one teacher. We are not wanting to suggest that it is an easy or straight-forward task to establish the classroom culture the student books speak to – a classroom in which students’ thinking is valued and encouraged, within tight constraints. Rather, what we hope to have offered is more of the form of an existence proof of what a mixed group of 11 and 12-year old students can do, within a few lessons of their first encounter with algebra in school, when introduced to algebra in the context of focusing on relations involving operations and properties, and where there is an invitation to conjecture about those relationships. The potential for the further development of algebraic thinking, through a focus on relations, is work that calls for new research.

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