On the r-mode spectrum of relativistic stars

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Abstract

We present a mathematically rigorous proof that the r-mode spectrum of relativistic stars to the rotational lowest order has a continuous part. A rigorous definition of this spectrum is given in terms of the spectrum of a continuous linear operator. This study verifies earlier results by Kojima concerning the nature of the r-mode spectrum.

Key words: stars: neutron – stars: oscillations – stars: rotation.

1 INTRODUCTION

The recently discovered r-mode instability (Andersson 1998; Friedman & Morsink 1998) in rotating neutron stars has significant implications for the rotational evolution of a newly born neutron star. The r modes are unstable due to the Chandrasekhar–Friedman–Schutz (CFS) mechanism (Chandrasekhar 1970; Friedman & Schutz 1978). Two independent computations by Andersson, Kokkotas & Schutz (1999a) and Lindblom, Owen & Morsink (1998) have found that the r-mode instability is responsible for slowing down a rapidly rotating, newly born neutron star to rotation rates comparable with that of the initial period of the Crab pulsar (~19 ms). This is achieved by the emission of current-quadrupole gravitational waves, which reduce the angular momentum of the star. The instability is active for as long as its growth time is shorter than the damping time due to the viscosity of neutron–star matter. The r-mode instability explains why only slowly rotating pulsars are associated with supernova remnants. The r-mode instability does not allow millisecond pulsars to be formed after an accretion-induced collapse of a white dwarf (Andersson et al. 1999a). It seems that millisecond pulsars can only be formed by the accretion-induced spin-up of old, cold, neutron stars. Additionally, this instability should be active in accreting neutron stars, and can limit their rotation provided that the stars are hotter than about $2 \times 10^7$ K and also in the rapidly spinning neutron stars in low-mass X-ray binaries (LMXB) (Andersson et al. 1999b). Finally, while the initially rapidly rotating star spins down, an energy equivalent to roughly 1 per cent of a solar mass is radiated in gravitational waves, making the process an interesting source of detectable gravitational waves (Owen et al. 1998).

Oscillations of stars are commonly described by the Lagrangian displacement vector $\xi$, which describes the displacement of a given fluid element due to the oscillation. Since $\xi$ is a vector on the $(\theta, \phi)$ 2-sphere, it can be written as a sum of spheroidal and toroidal components (or polar and axial components, in a different terminology). In a non-rotating star, the usual f, p and g modes of oscillation are purely spheroidal, characterized by the indices $(l, m)$ of the spherical harmonic function $Y_l^m$. In a rotating star, modes that reduce to purely spheroidal modes in the non-rotating limit also acquire toroidal components. Conversely, r modes in a non-rotating star are purely toroidal modes with vanishing frequency.\footnote{In the relativistic case the picture is similar (Thorne & Campolattaro 1967), but in this case there exists an additional family of quasi-normal modes, called space–time or w modes (Chandrasekhar & Ferrari 1991b; Kokkotas 1994).}

In a rotating star, the displacement vector acquires spheroidal components, and the frequency in the rotating frame, to first order in the rotational frequency $\Omega$ of the star, becomes

$$\sigma_l = \frac{2m\Omega}{l(l+1)} ,$$

for a given $(l, m)$ mode. An inertial observer measures a frequency of

$$\sigma_l = m\Omega - \sigma_l,$$

As mentioned earlier, when the star is set in slow rotation the axial (toroidal) modes are no longer degenerate, but instead the new family of r modes emerges, which are horizontal displacements on equipotential surfaces. In this case the axial (toroidal) perturbations with spherical harmonic indices $(l, m)$ induce polar (spheroidal) perturbations with harmonic indices $(l \pm 1, m)$ and vice versa.

The picture described above is purely Newtonian; the calculation was performed using Newtonian slowly rotating stellar models and the power radiated in gravitational waves was estimated using the quadrupole formula. The two approximations (Newtonian theory and slow rotation) that were used give only a qualitative picture, while the quantitative results would change if general relativity were used. The assumption of slow rotation is a
robust approximation because the expansion parameter \( \epsilon = \Omega \sqrt{R^3/M} \) is usually very small and the fastest spinning known pulsar has \( \epsilon \sim 0.3 \).

The perturbation equations for slowly rotating relativistic stars were derived by Kojima (1992) (see also Chandrasekhar & Ferrari 1991a). Kojima (1993) also calculated the effect of slow rotation on \( f \) modes. Andersson (1998) found the \( r \)-mode instability using the same set of equations (Kojima 1992), although his calculations overestimate the growth rate of the instability.

An important difference between Newtonian and general relativistic calculations is the dragging of the inertial frames, which might produce significant changes in the frequency spectra. It is exactly this that we are studying here. To be more specific, Kojima (1998) suggested that if one calculates the \( r \)-mode frequencies using general relativity to lowest order in \( \Omega \) the spectrum becomes continuous; this is in contrast to the calculations from Newtonian theory, where the spectrum is discrete and the frequencies are given by formula (1). In this article we prove in a mathematically rigorous way that Kojima’s suggestion for the existence of a continuous part within the spectrum is true.

Continuous spectra have been found in many cases in the study of differentially rotating fluids (Schutz & Verdaguer 1983; Verdaguer 1983; Balbinksi 1984a,b). The continuous spectra in these cases were again seen for the \( r \) modes, together with a wealth of interesting features such as the passage of low-order \( r \) modes from the discrete spectrum into the continuous one as the differential rotation increases; and the presence of low-order discrete \( p \) modes in the middle of the continuous spectrum in the more rapidly rotating discs (Schutz & Verdaguer 1983). The stars under consideration here have no differential rotation, and the existence of a continuous part of the spectrum is attributed to the dragging of the inertial frames due to general relativity.

## 2 Perturbation Equations

Since our calculations will be based on the equations of Kojima (Kojima 1997, 1998) and are presented in detail there, here we only briefly describe the perturbation equations.

We assume that the star is uniformly rotating with angular velocity \( \Omega \sim \mathcal{O}(\epsilon) \), where \( \epsilon \), as stated earlier, is small compared with unity. The metric is given by

\[
\mathrm{d}s^2 = -e^\sigma \mathrm{d}t^2 + e^\sigma e^\Upsilon \mathrm{d}r^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) - 2\omega e^\sigma \sin^2 \theta \mathrm{d}t \mathrm{d}\phi,
\]

where \( \omega \sim \mathcal{O}(\epsilon) \) describes the dragging of the inertial frame. If we include the effects of rotation only to order \( \epsilon \) the configuration is still spherical, because the deformation is of order \( \epsilon^2 \) (Harte 1967). The star is then described by the standard Tolman–Oppenheimer–Volkov (TOV) equations (see Chapter 23.5 of Misner, Thorne & Wheeler 1973) plus an equation for \( \omega \),

\[
(jr^4 \sigma')' - 16\pi (\rho + p) e^\sigma j r^4 \sigma = 0,
\]

where we have defined

\[
\sigma = \Omega - \omega,
\]

a prime denotes the derivative with respect to \( r \), and

\[
j = e^{-(\lambda + \eta)/2}.
\]

In the vacuum outside the star, \( \sigma \) can be written

\[
\sigma = \Omega - \frac{2j}{r},
\]

where \( J \) is the angular momentum of the star. The function \( \sigma \), both inside and outside the star, is a function of \( r \) only, and continuity of \( \sigma \) at the boundary (surface of the star, \( r = R \)) requires that \( \sigma_R = 6JR^{-4} \). Additionally, \( \sigma \) is a monotonically increasing function of \( r \) limited to

\[
\sigma_0 \leq \sigma \leq \Omega,
\]

where \( \sigma_0 \) is the value at the centre.

For the study of the perturbations of slowly rotating relativistic stars we should expand all perturbation functions in spherical harmonics and additionally assume a harmonic dependence of time, \( \exp[-i(\sigma - m\phi)] \). Here, \( \sigma \) is the oscillation frequency, normalized in units of \( (M/R^3)^{1/2} \). To lowest order for rotating stars, in accordance with Newtonian theory, we expect the toroidal perturbations of the fluid to have finite frequencies of order \( \Omega \) (or \( \epsilon \)). Then, from the six functions \( h_0, h_1, H_0, H_1, H_2 \), describing the metric perturbations in the Regge–Wheeler gauge (Thorne & Campolattaro 1967), only one, \( h_0 \), is of the same order as the function describing the toroidal fluid motions. All other perturbation functions plus the variations of the pressure \( \delta p \) and density \( \delta \rho \) are of higher order and thus can be omitted. We should point out that this approximation is valid only for the study of the \( r \) modes, and is not appropriate for the study of other fluid or \( m \) modes.

In a recent article, Lockitch & Friedman (1999) suggest that, in general, rotation mixes all axial and polar perturbations, and introduce the idea of hybrid modes. This mixing eliminates all purely polar modes, but there is still a set of purely axial modes. They suggest that this set of purely axial modes should not exist for relativistic slowly rotating stars, at least in the barotropic case. As we show here in the specific approximation described above, it is possible that the \( r \)-mode spectrum has a continuous part.

Using the above assumptions the master equation governing quasi-toroidal oscillations is given by (Kojima 1997, 1998)

\[
q\Phi + (\sigma - \mu) \left[ e^\sigma \Phi - \frac{1}{r^f} (r^f \sigma')' \right] = 0,
\]

where

\[
\Phi = h_0 r^{-f},
\]

and

\[
v = \frac{e^\Upsilon}{r^f} (l-1)(l+2),
\]

\[
q = \frac{1}{r^f} (r^f \sigma')' = 16\pi (\rho + p) e^\sigma \sigma ,
\]

\[
\mu = -\frac{(l+1)}{2m} (\sigma - m\Omega).
\]

Equation (9) in the Newtonian limit \( j \rightarrow 1 \), \( q \rightarrow 0 \) and \( \sigma \rightarrow \Omega \) reduces to a simple condition for the \( r \)-mode frequency, \( \sigma - \mu = 0 \), which is identical with (2). To this order of \( \Omega \) the eigenfrequency of the \( r \) modes is independent of the radial dependence of the eigenfunction \( \Phi(r) \) [or \( h_0(r) \)] and in this sense the eigenfrequency is infinitely degenerate.

The master equation (9) is not a regular eigenvalue problem since the coefficient \( (\sigma - \mu) \) becomes singular inside the star for

\[2\] The toroidal displacement vector is defined as \( \xi = (0, U_{\ell m} \sin^{-1} \theta \phi, -U_{\ell m} \sin \phi) Y^\ell_m \).
The corresponding representation of the natural numbers (including zero), all real numbers greater than zero, and the complex numbers, respectively. With respect to the real numbers, the definition used will be clear from the context. For each \( k \in \mathbb{N} \), the symbol \( C^k(I, \mathbb{C}) \) denotes the linear space of \( k \)-times continuously differentiable complex-valued functions on \( I \).

Throughout the paper, Lebesgue integration theory in the formulation of Riesz & Sz-Nagy (1955) is used. Compare with respect to this also Chapter III in Hirzebruch & Scharlau (1971) and Appendix A in Weidmann (1976). Following common usage there is no difference made between an almost everywhere (with respect to the chosen measure) defined function \( f \) and its associated equivalence class (consisting of all almost everywhere defined functions that differ from \( f \) only on a set of measure zero). In this sense \( L^2_c(I, r^j) \) denotes the Hilbert space of complex-valued, square-integrable functions (with respect to the measure \( r^j \) ) on the real line. The scalar product \( \langle \cdot \rangle \) on \( L^2_c(I, r^j) \) is defined by

\[
\langle f, g \rangle = \int_0^\infty r^j g(r) f(r) \, dr
\]

for all \( f, g \in L^2_c(I, r^j) \). Finally, \( L^2_c(I^2) \) denotes the Hilbert space of complex-valued, with respect to the Lebesgue measure, square-integrable functions on the two-dimensional interval \( I^2 \).

3 THE SPECTRUM OF KOJIMA’S MASTER EQUATION

In the following and in the remainder of the paper we consider the cases \( l \geq 1 \) only. As a first step, we rewrite (9) by introducing a new dependent variable \( \varphi \) defined by

\[
\varphi \equiv \varphi \Phi - \frac{1}{r^j} (r^j \Phi')'.
\]

The corresponding representation of \( \varphi \) in terms of \( \Phi \) can be performed with the help of the special linear independent solutions \( \Phi_1, \Phi_2 \) of the differential equation

\[
\frac{1}{r^j} (r^j \Phi')' - \varphi \Phi = \Phi'' + \left( \frac{4}{r^j} + \frac{j}{j} \right) \Phi' - \frac{(l - 1)(l + 2)}{r^2} \Phi = 0,
\]

given in the next section. This representation is given by

\[
\Phi(r) := -\frac{1}{W} \left[ \Phi_2(r) \int_0^r U_{ri} j \varphi dr' + \Phi_1(r) \int_r^\infty U_{ri} j \varphi dr' \right],
\]

where \( W \) is the Wronskian of \( \Phi_1 \) and \( \Phi_2 \), defined by

\[
W := r^j (\Phi_1 \Phi_2' - \Phi_2 \Phi_1').
\]

Roughly, \( \Phi_1 \) is an element of \( L^2_c(I, r^j) \) for small \( r \), and \( \Phi_2 \) is an element of \( L^2_c(I, r^j) \) for large \( r \). Apart from an irrelevant factor, such functions are uniquely defined by these demands. The use of these functions in the inversion (17) of (15) is necessary because of the demand (boundary condition) that \( \Phi \) is an element of \( L^2_c(I, r^j) \).

Introducing the new variable \( \varphi \) into (9) leads to the following equation for \( \varphi \):

\[
-\frac{q}{W} \left[ \Phi_2(r) \int_0^r U_{ri} j \varphi dr' + \Phi_1(r) \int_r^\infty U_{ri} j \varphi dr' \right] + (\sigma(r) - m) \varphi(r) = 0,
\]

for all \( r \in I \).

In the second step, (19) is turned into a regular spectral problem for the continuous linear operator \( A \) on \( L^2_c(I, r^j) \) defined as follows.

For every \( f \in L^2_c(I, r^j) \) we define \( Af \in L^2_c(I, r^j) \) by

\[
(Af)(r) := -\frac{q}{W} \left[ \Phi_2(r) \int_0^r U_{ri} j f dr' + \Phi_1(r) \int_r^\infty U_{ri} j f dr' \right] + \sigma(r)f(r),
\]

for all \( r \in I \). That this indeed defines a continuous linear operator on the whole of \( L^2_c(I, r^j) \) can be seen as follows. First, obviously, since \( \sigma \) is bounded continuous, by

\[
T_{ef} := \sigma f, \quad f \in L^2_c(I, r^j),
\]

there is a continuous linear operator defined on \( L^2_c(I, r^j) \). Second, in Appendix A it will be shown that from

\[
(Bf)(r) := -\frac{q}{W} \left[ \Phi_2(r) \int_0^r U_{ri} j f dr' + \Phi_1(r) \int_r^\infty U_{ri} j f dr' \right],
\]

for all \( r \in I \) and every \( f \in L^2_c(I, r^j) \) there is even defined a Hilbert–Schmidt operator \( B \) on \( L^2_c(I, r^j) \). Hence \( B \) is not only continuous but in addition compact and Hilbert–Schmidt. As a consequence, (20) defines a continuous linear operator on \( L^2_c(I, r^j) \), being equal to the sum of \( T_{ef} \) and \( B \).

The determination of the spectrum of (9) is now reduced to finding the spectrum \( \sigma(A) \) of the continuous linear (non-self-adjoint) operator \( A \). Since \( A \) is continuous it follows that \( \sigma(A) \) is a bounded subset of the complex plane contained in a circle around the origin with the radius given by the operator norm \( ||A|| \) of \( A \) (Reed & Simon 1978). The so-called essential spectrum of the operator \( T_{ef} \) is given by the values

\[
\sigma_0 \leq \mu = -\frac{l(l + 1)}{2m} (\sigma - m\Omega) \leq \Omega.
\]

(see Reed & Simon 1978, 1980) Moreover it is known (see, for example, Reed & Simon 1978, Corollary 2 of Theorem XIII.14 in Vol. IV) that the essential spectrum is invariant under perturbations by a compact linear operator such as \( B \). Hence the essential spectrum \( \sigma_{ess}(A) \) of \( A \), which is a part of \( \sigma(A) \), is also given by the range of values (23). The complement \( \sigma_{res}(A) = \sigma(A) \setminus \sigma_{ess}(A) \) (which is possibly empty) consists of isolated eigenvalues of finite multiplicity (Reed & Simon 1978). Using an argument of Kojima (1998) it follows that these eigenvalues are real [hence \( \sigma(A) \) is also real] and are contained in the interval (\( \Omega, ||A|| \)).

4 DETERMINATION OF \( \Phi_1 \) AND \( \Phi_2 \)

Here we distinguish two cases. The first uses the following

\[\text{[Reference: Reed & Simon (1978) on p. 106 of Vol. IV.]}\]
assumptions on $\rho$ and $p$. Both $\rho$ and $p$ are continuous real-valued functions on $I$ satisfying the Tolman–Oppenheimer–Volkov equations, and in addition are such that the limits 

$$\lim_{r \to 0} \rho(r) \quad \text{and} \quad \lim_{r \to 0} \rho(r)$$

both exist, and 

$$\rho(r) = 0 \quad \text{and} \quad p(r) = 0 \quad \text{for} \quad r \geq R,$$

(25)

are both satisfied, where $R$ denotes the radius of the star. Under these conditions the existence of the special solutions $\Phi_1, \Phi_2$ of the differential equation (16) is given by a theorem of Dunkel (1912) (see also Levinson 1948; Bellman 1949; Hille 1969) on linear first-order systems of differential equations having asymptotically constant coefficients at $+\infty$. By transformation one obtains from this a theorem where the singular point is finite. This theorem – which for the reader’s convenience is given in Appendix A – generalizes well-known results on weakly singular linear first-order systems with analytic coefficients. The determination of $\Phi_1$ and $\Phi_2$ proceeds as follows. First, from the TOV equations it follows that $\lambda, j$ are continuously differentiable functions on $I$ such that both 

$$\frac{j'}{j} \quad \text{and} \quad \frac{e^h - 1}{r}$$

are continuous as well as Lebesgue-integrable near 0, and that both 

$$r^2 \frac{j'}{j} \quad \text{and} \quad r(e^h - 1)$$

are continuous as well as Lebesgue-integrable near $\infty$. A consequence of the theorem given in Appendix A is the existence of linearly independent solutions $\Phi_1, \Phi_3$ of (16) satisfying 

$$\lim_{r \to 0} r^{l+1} \Phi_1(r) = 1,$$

$$\lim_{r \to 0} r^{l-1} \Phi'_1(r) = 1,$$

$$\lim_{r \to 0} r^{l+2} \Phi_3(r) = 1,$$

$$\lim_{r \to 0} r^{l+3} \Phi'_3(r) = -(l+2),$$

(28)

and of linearly independent solutions $\Phi_2, \Phi_4$ of (16) satisfying 

$$\lim_{r \to \infty} r^{l+1} \Phi_2(r) = 1,$$

$$\lim_{r \to \infty} r^{l+3} \Phi'_2(r) = -(l+2),$$

$$\lim_{r \to \infty} r^{l-1} \Phi_4(r) = 1,$$

$$\lim_{r \to \infty} r^{l-1} \Phi'_4(r) = l - 1.$$ 

(29)

In the next step we conclude that $\Phi_1, \Phi_2$ are linearly independent. The remaining solutions $\Phi_3, \Phi_4$ will be important in the proof of the compactness of $B$ (see Appendix).

Up to now everything said in this section is also valid for the case $l = 0$. For the following proof of linear independence we have to assume that $l \geq 1$. This is assumed in the remainder of the paper.

The proof of the linear independence of $\Phi_1, \Phi_2$ proceeds indirectly. We assume that there is a non-vanishing real $\alpha$ such that $\Phi_1 = \alpha \Phi_2$. Using this along with (28), (29), Lebesgue’s dominated convergence theorem and the monotonous convergence theorem we conclude that 

$$0 = \int_0^\infty \Phi_1 \left[ -(r^2 j \Phi_1) + (l - 1)(l + 2) r^2 j e^h \Phi_1 \right] dr$$

$$= \lim_{n \to \infty} \left[ -r^2 j \Phi_1 \Phi_1^{(n)} + \int_{r^n}^\infty r^2 j \Phi_1^2 \Phi_1^{(n)} dr \right]$$

$$+ (l - 1)(l + 2) \int_0^\infty r^2 j e^h \Phi_1^2 dr$$

$$= \int_0^\infty r^2 j \Phi_1^2 dr + (l - 1)(l + 2) \int_0^\infty r^2 j e^h \Phi_1^2 dr,$$

where we have made use of the fact that the function 

$$r^2 j e^h \Phi_1^2$$

(30)

is Lebesgue-integrable on $I$. This can be concluded from the facts that the function $j$ has a continuous extension onto the closed interval $[0, \infty)$, that the functions $\lambda$ and $j$ are constant for $r \geq R$, and that 

$$\lim_{r \to 0} e^{\lambda(r)} = 1, \lim_{r \to \infty} e^{\lambda(r)} = 1.$$ 

(31)

All these facts are consequences of the TOV equations and the assumptions made on $p$ and $\rho$. Hence from (30) follows the contradiction that the function $\Phi_1$ is trivial. As a consequence the functions $\Phi_1, \Phi_2$ are linearly independent.

The second case considers slowly rotating homogeneous stellar models. In this case the functions $e^h$ and $j$ can be given explicitly in terms of elementary functions (see for example Chandrasekhar & Miller 1974). It turns out that $\lambda, j$ are continuous functions and that $j$ is continuously differentiable on $I \setminus R$ such that the limits at $R$ of the derivatives of $j$ from the right and from the left differ from each other. The functions (26) are also Lebesgue-integrable near 0 and the functions (27) are also Lebesgue-integrable near $\infty$. In addition, (31) is also valid for this case. A consequence of the theorem in Appendix A is the existence of continuously differentiable functions $\Phi_1, \Phi_2$ on $I$ which are two times continuously differentiable on $I \setminus R$, satisfy (16) on $I \setminus R$, and in addition satisfy (28), (29). The linear independence of these functions can be proved completely analogously to the previously considered case.

5 DISCUSSION

In the previous section we showed for a wide class of background models for slowly rotating stars that for each $r_0 \in I$ the corresponding $\mu = \pi r_0$ [or rather the corresponding $\sigma$ according to (13)] belongs to the spectrum of (9). Furthermore we achieved a precise definition for the spectrum of (9) as the spectrum of the linear operator $A$.

This is an interesting new result that needs further study. There are still questions to be answered in order to understand the effect of the frame dragging induced by general relativity on the spectrum of the rotating stars. For example, one cannot exclude the possibility that isolated eigenvalues might also exist. Furthermore, the specific form of equation (9) is found under the assumption that the $r$-mode frequencies are of order $\Omega$. This...
implies that a few terms involving toroidal motions and all the terms related to spheroidal motions have been omitted either because they were of higher order or because they contribute via higher harmonics \((l \pm 1, m)\). This is not the case with the hybrid modes found by Lockitch & Friedman (1999), where the spheroidal motions have been taken into account. If these extra terms are included the form of the equation will change and the effect on the spectrum will be the emergence of the hybrid modes; however, the underlying nature of the spectrum remains the same and the techniques applied here can still be used.

Moreover, in the more general case, there is an imaginary part for each frequency which corresponds to the damping or growth of the fluid motions due to the emission of gravitational radiation or due to the viscosity, and it will be interesting to examine whether the spectrum will still be continuous.

In other words, the discovery of the existence of a continuous part of the r-mode spectrum is just the first step towards understanding the real nature of the spectrum. The idea of hybrid modes (Lockitch & Friedman 1999; Lindblom & Ipser 1999) already adds new features to the spectrum that have been overlooked in all previous studies. More work towards identifying possible isolated eigenvalues of the spectrum should be done, and additionally there is a need to examine if this specific nature of the spectrum is preserved when terms of order \(\Omega^2\) are included in equation (9); the coexistence of continuous and discrete parts of the spectrum is not at all impossible.

In conclusion, we would like to point out that the existence of a continuous part of the spectrum might affect some of the astrophysical estimations being made for the growth time of the r-mode instability, and consequently all the Newtonian estimations made earlier by Andersson et al. (1999a,b) and Kokkotas & Stergioulas (1999).

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REFERENCES

Andersson N., 1998, ApJ, 502, 708
Andersson N., Kokkotas K. D., Schutz B. F., 1999a, ApJ, 510, 2
Andersson N., Kokkotas K. D., Stergioulas N., 1999b, ApJ, 516, 307
Balbínski E., 1984a, MNRAS, 209, 145
Balbínski E., 1984b, MNRAS, 209, 721
Bellman R., 1949, A Survey of the Theory of the Boundedness, Stability, and Asymptotic Behaviour of Solutions of Linear and Nonlinear Differential and Difference Equations. Office of Naval Research, Washington DC
Chandrasekhar S., 1970, Phys. Rev. Lett., 24, 61
Chandrasekhar S., Ferrari V., 1991a, Proc. R. Soc. London, A, 433, 423
Chandrasekhar S., Ferrari V., 1991b, Proc. R. Soc. London, A, 434, 449
Chandrasekhar S., Miller J. C., 1974, MNRAS, 167, 63
Dunkel O., 1912, Am. Acad. Arts Sci. Proc., 38, 341
Friedman J. L., Schutz B. F., 1978, ApJ, 22, 281
Friedman J. L., Morris K., 1998, ApJ, 502, 7145
Hartle J. B., 1967, ApJ, 150, 1005
Hille E., 1969, Lectures on Ordinary Differential Equations. Addison-Wesley, Reading

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Hirzefrucht F., Scharf W., 1971, Einführung in die Funktionalanalyse. BI, Mannheim
Kojima Y., 1992, Phys. Rev. D, 46, 4289
Kojima Y., 1993, ApJ, 414, 247
Kojima Y., 1997, Prog. Theor. Phys. Suppl., 128, 251
Kojima Y., 1998, MNRAS, 293, 49
Kokkotas K.D., 1994, MNRAS, 268, 1015; Erratum: 1995, 277, 1599
Kokkotas K. D., Stergioulas N., 1999, A&A, 341, 110
Levinson N., 1948, Duke Math. J., 15, 111
Lindblom L., Ipser J. R., 1999, Phys. Rev. D, 59, 044009
Lindblom L., Owen B. J., Morris K., 1998, Phys. Rev. Lett., 80, 4843
Lockitch K.H., Friedman J. L., 1999, ApJ, in press (astro-ph/9812019)
Misner C. W., Thorne K. S., Wheeler J. A., 1973, Gravitation, W.H. Freeman, New York
Owen B., Lindblom L., Cutler C., Schutz B. F., Vecchio A., Andersson N., 1998, Phys. Rev. D, 58, 084020
Schutz B. F., Verdager E., 1983, MNRAS, 202, 881
Reed M., Simon B., 1978, Methods of Modern Mathematical Physics, Vol. IV. Academic, New York
Reed M., Simon B., 1980, Methods of Modern Mathematical Physics, Vol. I. Academic, New York
Riesz F., Sz-Nagy B., 1955, Functional Analysis. Unger, New York
Thorne K. S., Campolattaro A., 1967, ApJ, 149–591
Verdager E., 1983, MNRAS, 202, 903
Weidmann J., 1976, Lineare Operatoren in Hilberträumen. Teubner, Stuttgart

APPENDIX A: PROOF OF THE THEOREM

The variant of the theorem of Dunkel (1912) (see also Levinson 1948; Bellman 1949; Hille 1969) used in Section 4 is the following.

Theorem: Let \(n \in \mathbb{N}\); \(a, t_0 \in \mathbb{R}\) with \(a < t_0\); \(\mu \in \mathbb{N}\); \(\alpha_\mu := 1\) for \(\mu = 0\), and \(\alpha_\mu := \mu\) for \(\mu \neq 0\). In addition let \(A_0\) be a diagonalizable complex \(n \times n\) matrix, and \(e_1', \ldots, e_n'\) be a basis of \(C^n\) consisting of the eigenvectors of \(A_0\). Furthermore, for each \(j \in \{1, \ldots, n\}\) let \(\lambda_j\) be the eigenvalue corresponding to \(e_j'\), and \(P_j\) be the matrix representing the projection of \(C^n\) onto \(C_j e_j'\) with respect to the canonical basis of \(C^n\). Finally, let \(A_j\) be a continuous map from \((a, t_0)\) into the complex \(n \times n\) matrices \(M(n \times n, C)\) for which there is a number \(c \in (a, t_0)\) such that the restriction of \(A_{1k}\) to \([c, t_0)\) is Lebesgue-integrable for each \(j, k \in 1, \ldots, n\).

Then there is a \(C^1\) map \(R: (a, t_0) \to M(n \times n, C)\) with \(\lim_{t_0 \to a} R(t) = 0\) for each \(j, k \in 1, \ldots, n\) and such that \(u: (a, t_0) \to M(n \times n, C)\) defined by

\[
\begin{align*}
\sum_{j=1}^n \lambda_j (t_0 - t)^{-\mu} (E + R(t)) P_j & \quad \text{for } \mu = 0, \\
\sum_{j=1}^n \exp(\lambda_j (t_0 - t)^{-\mu}) (E + R(t)) P_j & \quad \text{for } \mu \neq 0
\end{align*}
\]

for all \(t \in (a, t_0)\) (where \(E\) is the \(n \times n\) unit matrix) maps into the invertible \(n \times n\) matrices and satisfies

\[
u(t) = \left(\frac{\alpha_\mu}{(t_0 - t)^{\mu + 1}} A_0 + A_T(t)\right) u(t)
\]

for each \(t \in (a, t_0)\).

Now we show that by (22) for all \(t \in I\) and every \(f \in L^2(I, r^2)\) there is defined a Hilbert–Schmidt operator \(B\) on \(L^2(I, r^2)\). Using the unitary transformation \(U\) from \(L^2(I, r^2)\) to \(L^2(I, r^2)\) given by

\[
Uf := r^2 \sqrt{f}, \quad f \in L^2(I, r^2)
\]
it is easily seen that this is equivalent to showing that the integral operator $\text{Int}(K')$ with the kernel function $K'$, where
\[
K'(r, r') := -\frac{q(r)(r')^2(j(r)j(r'))^{1/2}}{W} \begin{cases} 
\Phi_2(r)\Phi_1(r') & \text{for } r' \leq r \\
\Phi_1(r)\Phi_2(r') & \text{for } r' > r,
\end{cases}
\]  
for all $r \in I$ and $r' \in I$, defines a Hilbert–Schmidt operator on $L^2_C(I)$. It is well known (see for example Reed & Simon 1978, 1980), that this is equivalent to showing that $K'$ is an element of $L^2_C(I')$. Since $K'$ is continuous this follows if we can show that $|K'|^2$ is integrable over $I'$. For this we notice that as a consequence of (28) and (29) there are positive real $c_1, c_2$ such that
\[
|\Phi_1| \leq c_1 r^{j-1}, \quad |\Phi_2| \leq c_2 r^{-(j+2)}.
\]
From this and since $j$ is (by the TOV equations) bounded, the integrability of $|K'|^2$ follows as an application of Lebesgue’s dominated convergence theorem if the integrability of the following auxiliary function $H$ can be shown:
\[
H(r, r') := |q(r)|^2 \begin{cases} 
 r^{-2}(r')^{2(i+1)} & \text{for } r' \leq r \\
 r^{2(i+1)}(r')^{-2j} & \text{for } r' > r.
\end{cases}
\]  
(\ref{eq:A6})

Now for each $r \in I$,
\[
\int_0^\infty H(r, r') \, dr' = \frac{2(2j + 1)}{(2j - 1)(2j + 3)} r^2 |q(r)|^2,
\]
and this expression is integrable over $I$ since $q$ is continuous and has a compact support. Hence the integrability of $H$ follows from Tonelli’s Theorem. Taking all this into account, we conclude that $B$ defines a Hilbert–Schmidt operator on $L^2_C(I, r^j)$.

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