Optimal Rates of Sketched-regularized Algorithms for Least-Squares Regression over Hilbert Spaces

Junhong Lin 1  Volkan Cevher 1

Abstract

We investigate regularized algorithms combining with projection for least-squares regression problem over a Hilbert space, covering nonparametric regression over a reproducing kernel Hilbert space. We prove convergence results with respect to variants of norms, under a capacity assumption on the hypothesis space and a regularity condition on the target function. As a result, we obtain optimal rates for regularized algorithms with randomized sketches, provided that the sketch dimension is proportional to the effective dimension up to a logarithmic factor. As a byproduct, we obtain similar results for Nyström regularized algorithms. Our results provide optimal, distribution-dependent rates that do not have any saturation effect for sketched/Nyström regularized algorithms, considering both the attainable and non-attainable cases.

1. Introduction

Let the input space $H$ be a separable Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle_H$, and the output space $\mathbb{R}$. Let $\rho$ be an unknown probability measure on $H \times \mathbb{R}$. In this paper, we study the following expected risk minimization,

$$\inf_{\omega \in H} \mathcal{E}(\omega), \quad \mathcal{E}(\omega) = \int_{H \times \mathbb{R}} (\omega(x) - y)^2 d\rho(x,y),$$

(1)

where the measure $\rho$ is known only through a sample $z = \{z_i = (x_i, y_i)\}_{i=1}^n$ of size $n \in \mathbb{N}$, independently and identically distributed (i.i.d.) according to $\rho$.

The above regression setting covers nonparametric regression over a reproducing kernel Hilbert space (Cucker & Zhou, 2007; Steinwart & Christmann, 2008), and it is close to functional regression (Ramsay, 2006) and linear inverse problems (Engl et al., 1996). A basic algorithm for the problem is ridge regression, and its generalization, spectral algorithm. Such algorithms can be viewed as solving an empirical, linear equation with the empirical covariance operator replaced by a regularized one, see (Caponnetto & Yao, 2006; Bauer et al., 2007; Gerfo et al., 2008; Lin et al., 2018) and references therein. Here, the regularization is used to control the complexity of the solution to against over-fitting and to achieve best generalization ability.

The function/estimator generated by classic regularized algorithm is in the subspace $\text{span}\{\mathbf{x}\}$ of $H$, where $\mathbf{x} = \{x_1, \cdots, x_n\}$. More often, the search of an estimator for some specific algorithms is restricted to a different (and possibly smaller) subspace $S$, which leads to regularized algorithms with projection. Such approaches have computational advantages in nonparametric regression with kernel methods (Williams & Seeger, 2000; Smola & Schölkopf, 2000). Typically, with a subsample/sketch dimension $m < n, S = \text{span}\{\tilde{x}_j : 1 \leq j \leq m\}$ where $\tilde{x}_j$ is chosen randomly from the input set $\mathbf{x}$, or $S = \text{span}\{\sum_{j=1}^m G_{ij} x_j : 1 \leq i \leq m\}$ where $G = [G_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ is a general randomized matrix whose rows are drawn according to a distribution. The resulted algorithms are called Nyström regularized algorithm and sketched-regularized algorithm, respectively.

Our starting points of this paper are recent papers (Bach, 2013; Alaoui & Mahoney, 2015; Yang et al., 2015; Rudi et al., 2015; Myleiko et al., 2017) where convergence results on Nyström/sketched regularized algorithms for learning with kernel methods are given. Particularly, within the fixed design setting, i.e., the input set $\mathbf{x}$ are deterministic while the output set $\mathbf{y} = \{y_1, \cdots, y_n\}$ treated randomly, convergence results have been derived, in (Bach, 2013; Alaoui & Mahoney, 2015) for Nyström ridge regression and in (Yang et al., 2015) for sketched ridge regression. Within the random design setting (which is more meaningful (Hsu et al., 2014) in statistical learning theory) and involving a regularity/smoothness condition on the target function (Smale & Zhou, 2007), optimal statistical results on generalization error bounds (excess risks) have been obtained in (Rudi et al., 2015) for Nyström ridge re-
Optimal Rates of Sketched-regularized Algorithms for Least-Squares Regression over Hilbert Spaces

2. Learning with Projected-regularized Algorithms

In this section, we introduce some notations as well as auxiliary operators, and present projected-regularized algorithms.
It is easy to see that Problem (1) is equivalent to

$$\inf_{f \in H_\rho} \mathcal{E}(f), \quad \mathcal{E}(f) = \int_{H \times \mathbb{R}} (f(x) - y)^2 d\rho(x, y),$$  

(7)

The function that minimizes the expected risk over all measurable functions is the regression function (Cucker & Zhou, 2007; Steinwart & Christmann, 2008), defined as,

$$f_\rho(x) = \int_{\mathbb{R}} y d\rho(y|x), \quad x \in H, \rho_X\text{-almost every.}$$  

(8)

A simple calculation shows that the following well-known fact holds (Cucker & Zhou, 2007; Steinwart & Christmann, 2008), for all \(f \in L^2_{\rho_X}\), \(\mathcal{E}(f) - \mathcal{E}(f_\rho) = \|f - f_\rho\|_\rho^2\). Then it is easy to see that (7) is equivalent to \(\inf_{f \in H_\rho} \|f - f_\rho\|_\rho^2\). Under Assumption (2), \(H_\rho\) is a subspace of \(L^2_{\rho_X}\). Using the projection theorem, one can prove that a solution \(f_H\) for the problem (7) is the projection of the regression function \(f_\rho\) onto the closure of \(H_\rho\) in \(L^2_{\rho_X}\), and moreover, for all \(f \in H_\rho\) (Lin & Rosasco, 2017),

$$S^*_\rho f_\rho = S^*_\rho f_H,$$  

(9)

and

$$\mathcal{E}(f) - \mathcal{E}(f_H) = \|f - f_H\|_\rho^2.$$  

(10)

Note that \(f_H\) does not necessarily be in \(H_\rho\).

Throughout this paper, \(S\) is a closed, finite-dimensional subspace of \(H\), and \(P\) is the projection operator onto \(S\) or \(P = I\).

### 2.2. Projected-regularized Algorithms

In this subsection, we demonstrate and introduce projected-regularized algorithms.

The expected risk \(\overline{\mathcal{E}}(\omega)\) in (1) can not be computed exactly. It can be only approximated through the empirical risk \(\overline{\mathcal{E}}_n(\omega) = \frac{1}{n} \sum_{i=1}^n (\omega(x_i) - y_i)^2\). A first idea to deal with the problem is to replace the objective function in (1) with the empirical risk. Moreover, we restrict the solution to the subspace \(S\). This leads to the projected empirical risk minimization, \(\inf_{\omega \in S} \overline{\mathcal{E}}_n(\omega)\). Using \(P^2 = P\), a simple calculation shows that a solution for the above is given by \(\hat{\omega} = P\hat{\omega}\), with \(\hat{\omega}\) satisfying \(P \mathcal{T}_\rho \hat{\omega} = \mathcal{P} S^*_\rho \hat{\mathcal{Y}}\). Motivated by the classic (iterated) ridge regression, we replace \(P \mathcal{T}_\rho P\) with a regularized one, and thus leads to the following projected (iterated) ridge regression.

#### Algorithm 1. The projected (iterated) ridge regression algorithm of order \(\tau\) over the samples \(\mathbf{z}\) and the subspace \(S\)

is given by \(f^\tau_\rho = S^\tau \omega^\tau, \) where

$$\omega^\tau = P \mathcal{G}_\lambda(PT_{\tau}P) \mathcal{P} S^*_\rho \mathcal{Y}, \quad \mathcal{G}_\lambda(u) = \sum_{i=1}^\tau \lambda^{i-1}(\lambda + u)^{-i}. \quad \tag{11}$$

#### Remark 1.

1) Our results not only hold for projected ridge regression, but also hold for a general projected-regularized algorithm, in which \(\mathcal{G}_\lambda\) is a general filter function. Given \(\Lambda \subset \mathbb{R}_+, \) a class of functions \(\{\mathcal{G}_\lambda : [0, \kappa^2] \rightarrow [0, \infty], \lambda \in \Lambda\}\) are called filter functions with qualification \(\tau (\tau \geq 1)\) if there exist some positive constants \(E, F < \infty\) such that

$$\sup_{\lambda \in \Lambda} \sup_{u \in [0, \kappa^2]} |\mathcal{G}_\lambda(u)(u + \lambda)| \leq E. \quad \tag{12}$$

and

$$\sup_{\alpha \in [0, \tau]} \sup_{\lambda \in \Lambda} \sup_{u \in [0, \kappa^2]} |1 - \mathcal{G}_\lambda(u)(u + \lambda)^\alpha \lambda^{-\alpha} | \leq F. \quad \tag{13}$$

2) A simple calculation shows that

$$\mathcal{G}_\lambda(u) = \frac{1 - q^\tau}{u} = \frac{\sum_{i=0}^{\tau-1} q^i}{u + \lambda^\tau}, \quad q = \frac{\lambda}{\lambda + u}. \quad \tag{14}$$

Thus, \(\mathcal{G}_\lambda(u)\) is a filter function with qualification \(\tau, E = \tau, F = 1\). When \(\tau = 1,\) it is a filter function for classic ridge regression and the algorithm is projected ridge regression.

3) Another typical filter function studied in the literature is \(\mathcal{G}_\lambda(u) = u^{-1} 1_{(u \geq \lambda)}\), which corresponds to principal component (spectral cut-off) regularization. Here, \(1\) denotes the indication function. In this case, \(E = 2, F = 2^\tau\) and \(\tau\) could be any positive number.

In the above, \(\lambda\) is a regularization parameter which needs to be well chosen in order to achieve best performance. Throughout this paper, we assume that \(1/n \leq \lambda \leq 1\).

The performance of an estimator \(f^\tau_\rho\) can be measured in terms of excess risk (generalization error), \(\mathcal{E}(f^\tau_\rho) - \inf_{f \in H_\rho} \mathcal{E} = \mathcal{E}(\omega^\tau) - \inf_{f \in H_\rho} \mathcal{E},\) which is exactly \(\|f^\tau_\rho - f_H\|_\rho^2\) according to (10). Assuming that \(f_H \in H_\rho, \) i.e., \(f_H = S_\rho \omega_s\) for some \(\omega_s \in H\) (in this case, the solution with minimal \(H\)-norm for \(f_H = S_\rho \omega_s\) is denoted by \(\omega_H\)), it can be measured in terms of \(H\)-norm, \(\|\omega^\tau - \omega_H\|_H\), which is closely related to \(\|\mathcal{L}^{-\tau/2} S_\rho (\omega^\tau - \omega_H)\|_H = \|\mathcal{L}^{-\tau/2} (f^\tau_\rho - f_H)\|_\rho\) according to (5). In what follows, we will measure the performance of an estimator \(f^\tau_\rho\) in terms of a broader class of norms, \(\|\mathcal{L}^{-\alpha} (f^\tau_\rho - f_H)\|_\rho\), where \(\alpha \in [0, \frac{1}{2}]\) is such

---

1) Let \(L\) be a self-adjoint, compact operator over a separable Hilbert space \(H, \mathcal{G}_\lambda(L)\) is an operator on \(L\) defined by spectral calculus: suppose that \(\{\sigma_i, \psi_i\}\) is a set of normalized eigen-pairs of \(L\) with the eigenfunctions \(\{\psi_i\}\), forming an orthonormal basis of \(H,\) then \(\mathcal{G}_\lambda(L) = \sum \mathcal{G}_\lambda(\sigma_i) \psi_i \otimes \psi_i.\)
that $L^{-a}f_H$ is well defined. But one should keep in mind that all the derived results also hold if we replace $\|L^{-a}(x_H = f_H)\|_\rho$ with $\|T^{1/2}L^{-a}H^{-1}\|_H$ in the attainable case, i.e., $f_H \in H_\rho$. We will report these results in a longer version of this paper. Convergence with respect to different norms has its strong backgrounds in convex optimization, inverse problems, and statistical learning theory. Particularly, convergence with respect to target function values and $H$-norm has been studied in convex optimization. Interestingly, convergence in $H$-norm can imply convergence in target function values (although the derived rate is not optimal), while the opposite is not true.

3. Convergence Results

In this section, we first introduce some basic assumptions and then present convergence results for projected-regularized algorithms. Finally, we give results for sketched/Nyström regularized algorithms.

3.1. Assumptions

In this subsection, we introduce three standard assumptions made in statistical learning theory (Steinwart & Christmann, 2008; Cucker & Zhou, 2007; Lin et al., 2018). The first assumption relates to a moment condition on the output value $y$.

Assumption 1. There exist positive constants $Q$ and $M$ such that for all $l \geq 2$ with $l \in \mathbb{N}$,

$$\int_{\mathbb{R}} |y|^l d\rho(y|x) \leq \frac{1}{2} l!M^{l-2}Q^2,$$  \hspace{1cm} (15)

$\rho_X$-almost surely.

Typically, the above assumption is satisfied if $y$ is bounded almost surely, or if $y = \langle \omega_s, x \rangle_H + \epsilon$, where $\epsilon$ is a Gaussian random variable with zero mean and it is independent from $x$. Condition (15) implies that the regression function is bounded almost surely, using the Cauchy-Schwarz inequality.

The next assumption relates to the regularity/smoothness of the target function $f_H$.

Assumption 2. $f_H$ satisfies

$$\int_H (f_H(x) - f_\rho(x))^2 x \otimes x d\rho_X(x) \leq B^2 T,$$  \hspace{1cm} (16)

and the following Hölder source condition

$$f_H = L^\zeta g_0, \quad \|g_0\|_\rho \leq R.$$  \hspace{1cm} (17)

Here, $B$, $R$, $\zeta$ are non-negative numbers.

Condition (16) is trivially satisfied if $f_H - f_\rho$ is bounded almost surely. Moreover, when making a consistency assumption, i.e., $\inf_{H_\rho} \mathcal{E} = \mathcal{E}(f_\rho)$, as that in (Smale & Zhou, 2007; Caponnetto, 2006; Caponnetto & De Vito, 2007; Steinwart et al., 2009), for kernel-based non-parametric regression, it is satisfied with $B = 0$. Condition (17) characterizes the regularity of the target function $f_H$ (Smale & Zhou, 2007). A bigger $\zeta$ corresponds to a higher regularity and a stronger assumption, and it can lead to a faster convergence rate. Particularly, when $\zeta \geq 1/2$, $f_H \in H_\rho$ (Steinwart & Christmann, 2008). This means that the expected risk minimization (1) has at least one solution in $H$, which is referred to as the attainable case. Finally, the last assumption relates to the capacity of the space $H$ ($H_\rho$).

Assumption 3. For some $\gamma \in [0, 1]$ and $c_\gamma > 0$, $T$ satisfies

$$\text{tr}(T(T + \lambda I)^{-1}) \leq c_\gamma \lambda^{-\gamma}, \quad \text{for all } \lambda > 0.$$  \hspace{1cm} (18)

The left hand-side of (18) is called degrees of freedom (Zhang, 2005), or effective dimension (Caponnetto & De Vito, 2007). Assumption 3 is always true for $\gamma = 1$ and $c_\gamma = \kappa^2$, since $T$ is a trace operator. This is referred to as the capacity independent setting. Assumption 3 with $\gamma \in [0, 1]$ allows to derive better rates. It is satisfied, e.g., if the eigenvalues of $T$ satisfy a polynomial decaying condition $\sigma_i \sim i^{-1/\gamma}$, or with $\gamma = 0$ if $T$ is finite rank.

3.2. Results for Projected-regularized Algorithms

We are now ready to state our first result as follows. Throughout this paper, $C$ denotes a positive constant that depends only on $\kappa^2, c_\gamma, \gamma, B, M, Q, R, \zeta$ and $\|T\|$, and it could be different at its each appearance. Moreover, we write $a_1 \lesssim a_2$ to mean $a_1 \leq C a_2$.

Theorem 1. Under Assumptions 1, 2 and 3, let $\lambda = n^{\theta-1}$ for some $\theta \in [0, 1]$, $\gamma \geq \zeta$, and $a \in [0, \frac{1}{2} \land \zeta]$. Then the following holds with probability at least $1 - \delta (0 < \delta < 1)$.

1) If $\zeta \in [0, 1]$,

$$\|L^{-a}(x_H = f_H)\|_\rho \lesssim \lambda^{-a} \log^2 \frac{3}{\delta} t_{\theta/n},$$

$$\times \left( \lambda^\zeta + \frac{1}{\sqrt{n \lambda^\gamma}} + \lambda^{\zeta-1} \left( \Delta_5 + \Delta_5^{\zeta-1} \lambda^a \right) \right).$$  \hspace{1cm} (19)

2) If $\zeta \geq 1$ and $\lambda \geq n^{-1/2}$,

$$\|L^{-a}(x_H = f_H)\|_\rho \lesssim \lambda^{-a} \log^2 \frac{3}{\delta}$$

$$\times \left( \lambda^\zeta + \frac{1}{\sqrt{n \lambda^\gamma}} + \left( \Delta_5 + \lambda \Delta_5^{\zeta-1} \lambda^1 + \Delta_5^{\zeta-1} \lambda^\gamma \right) \right).$$  \hspace{1cm} (20)

Here, $\Delta_5$ is the projection error $\|(I - P)T^{1/2}\|^2$ and

$$t_{\theta/n} = [1 \lor (\theta^{-1} \land \log n^\gamma)].$$  \hspace{1cm} (21)
The above result provides high-probability error bounds with respect to variants of norms for projected-regularized algorithms. The upper bound consists of three terms. The first term depends on the regularity parameter $\zeta$, and it arises from estimating bias. The second term depends on the sample size, and it arises from estimating variance. The third term depends on the projection error. Note that there is a trade-off between the bias and variance terms. Ignoring the projection error, solving this trade-off leads to the best choice on $\lambda$ and the following results.

**Corollary 2.** Under the assumptions and notations of Theorem 1, let $\lambda = n^{-\frac{1}{2(1+\Delta_\delta)}}$. Then the following holds with probability at least $1 - \delta$.

1) If $2\zeta + \gamma \leq 1$,

$$\|L^{-1}(f_\lambda^* - f_H)\|_\rho \lesssim n^{-(\zeta - 1)} (1 + (\gamma \log n)^{1-a})(1 + \lambda^{-1} \Delta_\delta) \log^2 \frac{3}{\delta}.$$ (22)

2) If $\zeta \in [0, 1]$ and $2\zeta + \gamma > 1$,

$$\|L^{-1}(f_\lambda^* - f_H)\|_\rho \lesssim n^{-(\zeta - a)} (1 + \lambda^{-1} \Delta_\delta) \log^2 \frac{3}{\delta}.$$ (23)

3) If $\zeta \geq 1$,

$$\|L^{-1}(f_\lambda^* - f_H)\|_\rho \lesssim \lambda^{-a} \log^2 \frac{3}{\delta} \times \left( n^{-(\zeta - a)} + \Delta_\delta \left( 1 + \left( \frac{\lambda}{\Delta_\delta} \right)^{\lambda^{-1}} + \left( \frac{\lambda}{\Delta_\delta} \right)^a \right) \right).$$ (24)

Comparing the derived upper bound for projected-regularized algorithms with that for classic regularized algorithms in (Lin et al., 2018), we see that the former has an extra term, which is caused by projection. The above result asserts that projected-regularized algorithms perform similarly as classic regularized algorithms if the projection operator is well chosen such that the projection error is small enough.

In the special case that $P = I$, we get the follow result.

**Corollary 3.** Under the assumptions and notations of Theorem 1, let $\lambda = n^{-\frac{1}{2(1+\Delta_\delta)}}$ and $P = I$. Then with probability at least $1 - \delta$,

$$\|L^{-1}(f_\lambda^* - f_H)\|_\rho \lesssim \log^2 \frac{3}{\delta} \left( n^{-(\zeta - a)} (1 + (\gamma \log n)^{1-a}), \text{ if } 2\zeta + \gamma \leq 1, \text{ if } 2\zeta + \gamma > 1 \right).$$ (25)

The above result recovers the result derived in (Lin et al., 2018). The convergence rates are optimal as they match the mini-max rates with $\zeta \geq 1/2$ derived in (Caponnetto & De Vito, 2007; Blanchard & Mucke, 2016).

### 3.3. Results for Sketched-regularized Algorithms

In this subsection, we state results for sketched-regularized algorithms.

In sketched-regularized algorithms, the range of the projection operator $P$ is the subspace $\text{range}\{S_n^*G\}$, where $G \in \mathbb{R}^{m \times n}$ is a sketch matrix satisfying the following concentration inequality: For any finite subset $E$ in $\mathbb{R}^n$ and for any $t > 0$,

$$\Pr(||Ga||_2^2 - ||a||_2^2 \geq t||a||_2^2) \leq 2|E|e^{-c_0^2 \log^{3/2} n}.$$ (26)

Here, $c_0$ and $\beta$ are universal non-negative constants. Many matrices satisfy the concentration property.

- **Subgaussian sketches.** Matrices with i.i.d. subgaussian (such as Gaussian or Bernoulli) entries satisfy (26) with some universal constant $c_0$ and $\beta = 0$. More general, if the rows of $G$ are independent (scaled) copies of an isotropic $\psi_d$ vector, then $G$ also satisfies (26) (Mendelson et al., 2008).

- **Randomized orthogonal system (ROS) sketches.** As noted in (Krahmer & Ward, 2011), matrix that satisfies restricted isometry property from compressed sensing with randomized column signs satisfies (26). Particularly, random partial Fourier matrix, or random partial Hadamard matrix with randomized column signs satisfies (26) with $\beta = 4$ for some universal constant $c_0$. Using OS sketches has an advantage in computation, as that for suitably chosen orthonormal matrices such as the DFT and Hadamard matrices, a matrix-vector product can be executed in $O(n \log m)$ time, in contrast to $O(nm)$ time required for the same operation with generic dense sketches.

The following corollary shows that sketched-regularized algorithms have optimal rates provided the sketch dimension $m$ is not too small.

**Corollary 4.** Under the assumptions of Theorem 1, let $S = \text{range}\{S_n^*G\}$, where $G \in \mathbb{R}^{m \times n}$ is a randomized matrix satisfying (26). Let $\lambda = n^{-\frac{1}{2(1+\Delta_\delta)}}$ and

$$m \gtrsim \log^\beta n \begin{cases} 
\frac{\gamma (\zeta - a)}{\lambda} \log^2 \frac{3}{\delta} & \text{if } 2\zeta + \gamma \leq 1, \\
\frac{\gamma (\zeta - a)}{\lambda} \log^2 \frac{3}{\delta} & \text{if } 2\zeta + \gamma \leq 1,
\end{cases}$$ (27)

Then with confidence at least $1 - \delta$, the following holds

$$\|L^{-1}(f_\lambda^* - f_H)\|_\rho \lesssim \log^2 \frac{3}{\delta} \left( n^{-(\zeta - a)} (1 + (\gamma \log n)^{2-a}), \text{ if } 2\zeta + \gamma \leq 1, \text{ if } 2\zeta + \gamma > 1 \right).$$ (28)
The above results assert that sketched-regularized algorithms converge optimally, provided the sketch dimension is not too small, or in another words the error caused by projection is negligible when the sketch dimension is large enough. Note that the minimal sketch dimension from the above is proportional to the effective dimension $\lambda^{-\gamma}$ up to a logarithmic factor for the case $\zeta \leq 1$.

**Remark 2.** Considering only the case $\zeta = 1/2$ and $a = 0$, (Yang et al., 2015) provides optimal error bounds for sketched ridge regression within the fixed design setting.

### 3.4. Results for Nyström Regularized Algorithms

As a byproduct of the paper, using Corollary 2, we derive the following results for Nyström regularized algorithms.

**Corollary 5.** Under the assumptions of Theorem 1, let $S = \text{span}\{x_1, \ldots, x_m\}$, $2\zeta + \gamma > 1$, and $\lambda = n^{-\frac{1}{\bar{\zeta}+\gamma}}$. Then with probability at least $1 - \delta$,

$$\|L^{-a}(f_x^* - f_H)\|_\rho \lesssim n^{-\frac{\bar{\zeta}}{\bar{\zeta}+\gamma}} \log^3 \frac{3}{\delta},$$

provided that

$$m \gtrsim (1 + \log n)^{\frac{\bar{\zeta}}{\bar{\zeta}+\gamma}}\left\{\begin{array}{ll}
\frac{\log n}{n^{1/\bar{\zeta}+\gamma}} & \text{if } \zeta \geq 1, \\
\frac{1}{n^{1/\bar{\zeta}+\gamma}} & \text{if } \zeta \leq 1.
\end{array}\right.$$

**Remark 3.**
1) Considering only the case $1/2 \leq \zeta \leq 1$ and $a = 0$, (Rudi et al., 2015) provides optimal generalization error bounds for Nyström ridge regression. This result was further extended in (Myleiko et al., 2017) to a general Nyström regularized algorithm with a general source assumption indexed with an operator monotone function (but only in the attainable cases). Note that as in classic ridge regression, Nyström ridge regression saturates over $\zeta \geq 1$, i.e., it does not have a better rate even for a bigger $\zeta \geq 1$.
2) For the case $\zeta \geq 1$ and $a = 0$, (Myleiko et al., 2017) provides certain generalization error bounds for plain Nyström regularized algorithms, but the rates are capacity-independent, and the minimal projection dimension $O(n^{\frac{2}{\bar{\zeta}+\gamma}})$ is larger than ours (considering the case $\gamma = 1$ for the sake of fairness).

In the above lemma, we consider the plain Nyström subsampling. Using the ALS Nyström subsampling (Drineas et al., 2012; Gittens & Mahoney, 2013; Alalu & Mahoney, 2015), we can improve the projection dimension condition to (27).

**ALS Nyström Subsampling** Let $K = S_x S_x^*$. For $\lambda > 0$, the leveraging scores of $K(K + \lambda I)$ is the set $\{I_i(\lambda)\}_{i=1}^n$ with

$$I_i(\lambda) = (K(K + \lambda I)^{-1})_{ii}, \quad \forall i \in [n].$$

The $L$-approximated leveraging scores (ALS) of $K(K + \lambda I)$ is a set $\{\hat{I}_i(\lambda)\}_{i=1}^n$ satisfying $L^{-1}I_i(\lambda) \leq \hat{I}_i(\lambda) \leq LI_i(\lambda)$, for some $L \geq 1$. In ALS Nyström subsampling regime, $S = \text{range}\{\tilde{x}_1, \ldots, \tilde{x}_m\}$, where each $\tilde{x}_j$ is i.i.d. drawn according to

$$\mathbb{P}(\tilde{x} = x_i) \sim \hat{I}_i(\lambda).$$

**Corollary 6.** Under the assumptions of Theorem 1, let $\lambda = n^{-\frac{1}{(2\zeta + \gamma)^{\gamma}}}$. Then with probability at least $1 - \delta$, (28) holds provided that

$$m \gtrsim L^2 \log^2 \frac{3}{\delta} \log \left[\frac{n^{\frac{1}{\bar{\zeta}+\gamma}}}{n^{-\frac{1}{\bar{\zeta}+\gamma}}}\right] \left\{\begin{array}{ll}
\frac{n^\gamma}{n^{1/\bar{\zeta}+\gamma}} & \text{if } \zeta + \gamma \leq 1, \\
\frac{1}{n^{-\frac{1}{\bar{\zeta}+\gamma}}} & \text{if } \zeta \geq 1.
\end{array}\right.$$

All the results stated in this section will be proved in the next section.

### 4. Proof

In this section, we prove the results stated in Section 3. We first give some deterministic estimates and an analytics result. We then give some probabilistic estimates. Applying the probabilistic estimates into the analytics result, we prove the results for projected-regularized algorithms. We finally estimate the projection errors and present the proof for sketched-regularized algorithms.

#### 4.1. Deterministic Estimates

In this subsection, we introduce some deterministic estimates. For notational simplicity, throughout this paper, we denote

$$T_\lambda = T + \lambda I, \quad T_{\kappa\lambda} = T_\kappa + \lambda I.$$

We define a deterministic vector $\omega_\lambda$ as follows,

$$\omega_\lambda = G_\lambda(\mathcal{T})S_{\rho}f_H. \quad (30)$$

The vector $\omega_\lambda$ is often called population function. We introduce the following lemma. The proof is essentially the same as that for Lemma 26 from (Lin & Cevher, 2018). We thus omit it.

**Lemma 7.** Under Assumption 2, the following holds.
1) For any $\zeta - \tau \leq a \leq \zeta$,

$$\|L^{-a}(S_\rho \omega_\lambda - f_H)\|_\rho \leq R\lambda^{-\zeta - a}. \quad (31)$$

2)

$$\|T^{a-1/2} \omega_\lambda\|_H \leq \tau R \left\{\begin{array}{ll}
\lambda^{\zeta+a-1} & \text{if } -\zeta \leq a \leq 1 - \zeta, \\
\lambda^{\zeta} & \text{if } a \geq 1 - \zeta.
\end{array}\right.$$
The above lemma provides some basic properties for the population function. It will be useful for the proof of our main results. The left hand-side of (31) is often called true bias.

Using the above lemma and some basic operator inequalities, we can prove the following analytic, deterministic result.

**Proposition 8.** Under Assumption 2, let

\[
1 + \|T^\frac{1}{2} - T^\frac{1}{2}\|_2^2 + \|T^\frac{1}{2} - T^\frac{1}{2}\|_2^2 \leq \Delta_1,
\]

\[
\|T^\frac{1}{2}((T^\omega_\lambda - S^\omega_\lambda y) - (T^\omega_\lambda - S^\omega_\lambda f_H))\| \leq \Delta_2,
\]

\[
\|T - T^\omega_\lambda\|_2 \leq \Delta_3,
\]

\[
\|T^\frac{1}{2} (T - T^\omega_\lambda) - \Delta_4,
\]

\[
\|(I - P)T^\frac{1}{2}\|^2 \leq \Delta_5.
\]

Then, for any \(0 \leq \eta \leq \zeta \wedge \frac{1}{2}\), the following holds.

1) If \(\zeta \in [0, 1]\),

\[
\|L^{-a}(S_p \omega_\lambda - f_H)\|_2 \leq \zeta \|T^\omega_\lambda - \Delta_1 \|^a
\]

\[
\times \left( \Delta_2 + 2(\tau + 1)R \lambda^\zeta + \rho R \lambda^\zeta + (\Delta_5 + \Delta_5^{1-a} \lambda^a) \right).
\]

(33)

2) If \(\zeta \geq 1\),

\[
\|L^{-a}(S_p \omega_\lambda - f_H)\|_2 \leq \zeta \|T^\omega_\lambda - \Delta_1 \|^a
\]

\[
\times \left( \Delta_2 + 3R \lambda^\zeta + \kappa^2(\zeta - 1) \lambda R \gamma \Delta_4 + \tau \Delta_5
\]

\[
+ \lambda \Delta_3 + \Delta_5 \right) \Delta_5^{1-a} \lambda^a + \Delta_3^{1-a} \lambda^a + \Delta_3^{1-a} \lambda^a\right).
\]

(34)

We have with probability at least \(1 - \delta\),

\[
\|(T + \lambda)^{1/2}(T^\omega_\lambda + \lambda)^{-1/2}\|_2 \leq 3a_n, \lambda, \gamma(\theta)(1 \vee n^{\theta - 1}),
\]

and

\[
\|(T + \lambda)^{-1/2}(T^\omega_\lambda + \lambda)^{1/2}\|_2 \leq 4/3a_n, \lambda, \gamma(\theta)(1 \vee n^{\theta - 1}).
\]

**Lemma 10.** Let \(0 < \delta < 1/2\). It holds with probability at least \(1 - \delta\):

\[
\|T - T^\omega_\lambda\|_2 \leq \frac{6\delta^2}{\sqrt{n}} \log \frac{2}{\delta}.
\]

Here, \(\|\cdot\|_2\) denotes the Hilbert-Schmidt norm.

**Lemma 11.** Under Assumption 3, let \(0 < \delta < 1/2\). It holds with probability at least \(1 - \delta\):

\[
\|T^\frac{1}{2} (T - T^\omega_\lambda)\|_2 \leq 2\kappa \left( \frac{2\kappa}{n\sqrt{\lambda}} + \sqrt{\frac{c_2}{n\lambda^\gamma}} \right) \log \frac{2}{\delta}.
\]

The proof of the above lemmas can be done simply applying concentration inequalities for sums of Hilbert-space-valued random variables. We refer to (Lin & Rosasco, 2017) for the proofs.

**Lemma 12.** (Lin et al., 2018) Under Assumptions 1, 2 and 3, let \(\omega_\lambda\) be given by (30). For all \(\delta \in [0, 1/2]\), the following holds with probability at least \(1 - \delta\):

\[
\|T^\frac{1}{2}((T^\omega_\lambda - S^\omega_\lambda y) - (T^\omega_\lambda - S^\omega_\lambda f_H))\|_2 \leq \left( \frac{C_1}{n\lambda^\gamma(1-\zeta)} + \sqrt{\frac{C_2\lambda^{2\zeta}}{n\lambda^\gamma}} + \frac{C_3}{n\lambda^\gamma} \right) \log \frac{2}{\delta}.
\]

Here, \(C_1 = 4(M + R\gamma 2\zeta - 1)\), \(C_2 = 96R^2\kappa^2\) and \(C_3 = 32(3B^2 + 4Q^2)c_\gamma\).

With the above probabilistic estimates and the analytics result, Proposition 8, we are now ready prove results for projected-regularized algorithms.

**Proof of Theorem 1.** We use Proposition 8 to prove the result. We thus need to estimate \(\Delta_1, \Delta_2, \Delta_3, \text{ and } \Delta_4\). Following from Lemmas 9, 10, 11 and 12, with \(n^{-1} \leq \lambda \leq 1\), we know that with probability at least \(1 - \delta\),

\[
\Delta_1 \lesssim t_{\theta, n} \log \frac{3}{\delta},
\]

\[
\Delta_2 \lesssim \left( \frac{1}{n\lambda^\gamma(1-\zeta)} + \lambda^\zeta + \frac{1}{\sqrt{n\lambda^\gamma}} \right) \log \frac{3}{\delta},
\]

\[
\Delta_3 \lesssim \frac{1}{\sqrt{n}} \log \frac{3}{\delta},
\]

(38)

\[
\Delta_4 \lesssim \frac{1}{\sqrt{n\lambda^\gamma}} \log \frac{3}{\delta}.
\]

The results thus follow by introducing the above estimates into (33) or (34), combining with a direct calculation and \(1/n \leq \lambda \leq 1\).
4.3. Proof for Sketched-regularized Algorithms

In order to use Corollary 2 for sketched-regularized algorithms, we need to estimate the projection error. The basic idea is to approximate the projection error in terms of its ‘empirical’ version, \( \| (I - P)T^\perp_x \|_2^2 \). The estimate for \( \| (I - P)T^\perp_x \|_2^2 \) is quite lengthy and it is divided into several steps.

**Lemma 13.** Let \( 0 < \delta < \theta < 1 \) and \( \theta \in [0, 1] \). Given a fixed \( x \in H^n \), assume that for \( \lambda > 0 \),

\[
N_x(\lambda) := \text{tr}((T_x + \lambda)^{-1}T_x) \leq b_\gamma \lambda^{-\gamma}
\]

holds for some \( b_\gamma > 0 \), \( \gamma \in [0, 1] \). Then there exists a subset \( U_x \) of \( \mathbb{R}^{m \times n} \) with measure at least 1 \(-\delta \), such that for all \( G \in U_x \),

\[
\| (I - P)T^\perp_x \|_2^2 \leq 6 \lambda,
\]

provided that

\[
m \geq 144\log^3 n \lambda^{-\gamma} \log \frac{3}{\delta} (1 + 12b_\gamma) \log 2.
\]

Under the condition (39), Lemma 13 provides an upper bound for \( \| (I - P)T^\perp_x \|_2^2 \), which will be used to control the projection error using the following lemma.

**Lemma 14.** Let \( P \) be a projection operator in a Hilbert space \( H \), and \( A, B \) be two semidefinite positive operators on \( H \). For any \( 0 \leq s, t \leq \frac{1}{2} \), we have

\[
\| A^s (I - P) A^t \| \leq \| A - B \|^{s+t} + \| B^s (I - P) B^t \|^{s+t}.
\]

The left-hand side of (39) is called empirical effective dimension. It can be estimated as follows.

**Lemma 15.** Under Assumption 3, let \( \lambda = n^{-\theta} \) for some \( \theta \in [0, 1] \), and \( 0 < \delta < \theta \) with confidence \( 1 - \delta \),

\[
\text{tr}((T_x + \lambda)^{-1}T_x) \leq 3(4\kappa^2 + 2\kappa \sqrt{c_\gamma} + c_\gamma) \log \frac{4}{\delta} a_{n,\delta/2,\gamma}(\theta) \lambda^{-\gamma},
\]

where \( a_{n,\delta/2,\gamma}(\theta) \) is given as in Lemma 9.

The above lemma improves Proposition 1 of (Rudi et al., 2015). It does not require the extra assumption that the sample size is large enough, and our proof is simpler.

Now we are ready to estimate the projection error and give the proof for sketched-regularized algorithms.

**Proof of Corollary 4.** Let \( \lambda' = n^{-\theta'} \), with

\[
\theta' = \begin{cases} 
1, & \text{if } 2\zeta + \gamma \leq 1, \\
\frac{\zeta - 1}{\zeta} \left( \frac{1}{(1 - \zeta)(2\zeta + \gamma)} \right), & \text{if } \zeta \geq 1,
\end{cases}
\]

Following from Corollary 2, Lemmas 9 and 15, we know that there exists a subset \( \Omega \) of \( Z^n \) with measure at least \( 1 - 3\delta \), such that for all \( z \in \Omega \), (22) (or (23), or (24)), (41) (with \( \theta \) and \( \lambda \) replaced by \( \theta' \) and \( \lambda' \) in (41), respectively), and

\[
\| T_{\eta - \delta/2,\gamma}(x)^{1/2} T_{\eta - \delta/2,\gamma}^\perp \|_2^2 \lesssim \log \frac{3}{\delta} \left( 1 + \log \frac{3}{\delta} \right) \log \frac{3}{\delta},
\]

provided \( m \geq n^{\theta'_\gamma} \log^3 n \log \frac{3}{\delta} \), which is guaranteed by Condition (27). Note that,

\[
\| (I - P)T^\perp_x \|_2^2 \leq \| (I - P)T^\perp_x \|_2^2 \| T^\perp_x \|_2^2 \| T^\perp_x \|_2^2
\]

Introducing with (42) and (43), combining with (22) (or (23), or (24)), and by a simple calculation, one can prove the desired results.

The proof of Corollaries 5 and 6 will be given in the appendix due to space limitation.

5. Conclusion

In this paper, we prove optimal statistical results with respect to variants of norms for sketched or Nyström regularized algorithms. Our contributions are mainly on theoretical aspects. First, our results for sketched-regularized algorithms generalize previous results (Yang et al., 2015) from the fixed design setting to the random design setting. Moreover, our results involve the regularity/smoothness of the target function and thus can have a faster convergence rate. Second, our results cover the non-attainable cases, which have not been studied before for both Nyström and sketched regularized algorithms. Third, our results provide the first optimal, capacity-dependent rates even when \( \zeta \geq 1 \). This may suggest that sketched/Nyström regularized algorithms have certain advantages in comparison with distributed learning algorithms (Zhang et al., 2015), as the latter suffer a saturation effect over \( \zeta = 1 \). A future direction is to extend our analysis to learning with random features, see (Sriperumbudur & Sterge, 2017; Lin & Rosasco, 2018) and references therein.
Acknowledgements
This work was sponsored by the Department of the Navy, Office of Naval Research (ONR) under a grant number N62909-17-1-2111. It has also received funding from Hasler Foundation Program: Cyber Human Systems (project number 16066), and from the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation program (grant agreement n 725594-time-data).

References
Alaoui, Ahmed and Mahoney, Michael W. Fast randomized kernel ridge regression with statistical guarantees. pp. 775–783, 2015.

Bach, Francis. Sharp analysis of low-rank kernel matrix approximations. In Conference on Learning Theory, pp. 185–209, 2013.

Bach, Francis. On the equivalence between kernel quadrature rules and random feature expansions. Arxiv, 2015.

Baraniuk, Richard, Davenport, Mark, DeVore, Ronald, and Wakin, Michael. A simple proof of the restricted isometry property for random matrices. Constructive Approximation, 28(3):253–263, 2008.

Bauer, Frank, Pereverzev, Sergei, and Rosasco, Lorenzo. On regularization algorithms in learning theory. Journal of Complexity, 23(1):52–72, 2007.

Blanchard, Gilles and Mucke, Nicole. Optimal rates for regularization of statistical inverse learning problems. arXiv preprint arXiv:1604.04054, 2016.

Caponnetto, Andrea. Optimal learning rates for regularization operators in learning theory. Technical report, 2006.

Caponnetto, Andrea and De Vito, Ernesto. Optimal rates for the regularized least-squares algorithm. Foundations of Computational Mathematics, 7(3):331–368, 2007.

Caponnetto, Andrea and Yao, Yuan. Adaptation for regularization operators in learning theory. 2006.

Cucker, Felipe and Zhou, Ding Xuan. Learning theory: an approximation theory viewpoint, volume 24. Cambridge University Press, 2007.

Dicker, Lee H, Foster, Dean P, and Hsu, Daniel. Kernel ridge vs. principal component regression: Minimax bounds and the qualification of regularization operators. 2016.

Dicker, Lee H, Foster, Dean P, and Hsu, Daniel. Kernel ridge vs. principal component regression: Minimax bounds and the qualification of regularization operators. Electronic Journal of Statistics, 11(1):1022–1047, 2017.

Drineas, Petros, Magdon-Ismail, Malik, Mahoney, Michael W, and Woodruff, David P. Fast approximation of matrix coherence and statistical leverage. Journal of Machine Learning Research, 13(Dec):3475–3506, 2012.

Engl, Heinz Werner, Hanke, Martin, and Neubauer, Andreas. Regularization of inverse problems, volume 375. Springer Science & Business Media, 1996.

Fujii, Junichi, Fujii, Masatoshi, Furuta, Takayuki, and Nakamoto, Ritsu.o. Norm inequalities equivalent to Heinz inequality. Proceedings of the American Mathematical Society, 118(3):827–830, 1993.

Gerfo, L Lo, Rosasco, Lorenzo, Odone, Francesca, De Vito, Ernesto, and Verri, Alessandro. Spectral algorithms for supervised learning. Neural Computation, 20(7):1873–1897, 2008.

Gittens, Alex and Mahoney, Michael W. Revisiting the nystrom method for improved large-scale machine learning. arXiv preprint arXiv:1303.1849, 2013.

Hansen, Frank. An operator inequality. Mathematische Annalen, 246(3):249–250, 1980.

Hsu, Daniel, Kakade, Sham M, and Zhang, Tong. Random design analysis of ridge regression. Foundations of Computational Mathematics, 14(3):569–600, 2014.

Krahmer, Felix and Ward, Rachel. New and improved johnson–lindenstrauss embeddings via the restricted isometry property. SIAM Journal on Mathematical Analysis, 43(3):1269–1281, 2011.

Lin, Junhong and Cevher, Volkan. Optimal convergence for distributed learning with stochastic gradient methods and spectral-regularization algorithms. arXiv preprint arXiv:1801.07226, 2018.

Lin, Junhong and Rosasco, Lorenzo. Optimal rates for multi-pass stochastic gradient methods. Journal of Machine Learning Research, 18(97):1–47, 2017.

Lin, Junhong and Rosasco, Lorenzo. Generalization properties of doubly stochastic learning algorithms. Journal of Complexity, 47:42–61, 2018.

Lin, Junhong, Rudi, Alessandro, Rosasco, Lorenzo, and Cevher, Volkan. Optimal rates for spectral-regularized algorithms with least-squares regression over hilbert spaces. arXiv preprint arXiv:1801.06720, 2018.

Mendelson, Shahar, Pajor, Alain, and Tomczak-Jaegermann, Nicole. Uniform uncertainty principle for bernoulli and subgaussian ensembles. Constructive Approximation, 28(3):277–289, 2008.
Minsker, Stanislav. On some extensions of Bernstein’s inequality for self-adjoint operators. *arXiv preprint arXiv:1112.5448*, 2011.

Myleiko, GL, Pereverzyev Jr, S, and Solodky, SG. Regularized Nyström subsampling in regression and ranking problems under general smoothness assumptions. 2017.

Pinelis, IF and Sakhanenko, AI. Remarks on inequalities for large deviation probabilities. *Theory of Probability & Its Applications*, 30(1):143–148, 1986.

Ramsay, James O. *Functional data analysis*. Wiley Online Library, 2006.

Rudi, Alessandro, Camoriano, Raffaello, and Rosasco, Lorenzo. Less is more: Nyström computational regularization. *Advances in Neural Information Processing Systems*, pp. 1657–1665, 2015.

Smale, Steve and Zhou, Ding-Xuan. Learning theory estimates via integral operators and their approximations. *Constructive approximation*, 26(2):153–172, 2007.

Smola, Alex J and Schölkopf, Bernhard. Sparse greedy matrix approximation for machine learning. 2000.

Sriperumbudur, Bharath and Sterge, Nicholas. Approximate kernel pca using random features: Computational vs. statistical trade-off. *arXiv preprint arXiv:1706.06296*, 2017.

Steinwart, Ingo and Christmann, Andreas. *Support Vector Machines*. Springer Science & Business Media, 2008.

Steinwart, Ingo, Hush, Don R, and Scovel, Clint. Optimal rates for regularized least squares regression. In *Conference On Learning Theory*, 2009.

Tropp, Joel A. User-friendly tools for random matrices: An introduction. Technical report, DTIC Document, 2012.

Williams, Christopher KI and Seeger, Matthias. Using the Nyström method to speed up kernel machines. In *Advances in Neural Information Processing Systems*, pp. 661–667. MIT press, 2000.

Yang, Yun, Pilanci, Mert, and Wainwright, Martin J. Randomized sketches for kernels: Fast and optimal non-parametric regression. *arXiv preprint arXiv:1501.06195*, 2015.

Zhang, Tong. Learning bounds for kernel regression using effective data dimensionality. *Neural Computation*, 17 (9):2077–2098, 2005.

Zhang, Yuchen, Duchi, John C, and Wainwright, Martin J. Divide and conquer kernel ridge regression: a distributed algorithm with minimax optimal rates. *Journal of Machine Learning Research*, 16:3299–3340, 2015.
Supplementary: Optimal Rates of Sketched-regularized Algorithms for Least-squares Regression over Hilbert Spaces

In this appendix, we first prove the lemmas stated in Section 4 and Corollaries 5 and 6. We then review how the regression setting considered in this paper covers non-parametric regression with kernel methods.

A. Proofs for Section 4

For notational simplicity, we denote

\[ R_\lambda(u) = 1 - G_\lambda(u)u, \]

and

\[ N(\lambda) = \text{tr}(T(T + \lambda)^{-1}). \]

To proceed the proof, we need some basic operator inequalities.

**Lemma 16.** (*Fujii et al., 1993*) Let \( A \) and \( B \) be two positive bounded linear operators on a separable Hilbert space. Then

\[ \|A^sB^s\| \leq \|AB\|^s, \quad \text{when } 0 \leq s \leq 1. \]

**Lemma 17.** Let \( H_1, H_2 \) be two separable Hilbert spaces and \( S : H_1 \to H_2 \) a compact operator. Then for any function \( f : [0, \|S\|] \to [0, \infty[ \),

\[ f(SS^*)S = Sf(S^*S). \]

**Proof.** The result can be proved using singular value decomposition of a compact operator. \( \square \)

**Lemma 18.** Let \( A \) and \( B \) be two non-negative bounded linear operators on a separable Hilbert space with \( \max(\|A\|, \|B\|) \leq \kappa^2 \) for some non-negative \( \kappa^2 \). Then for any \( \zeta > 0 \),

\[ \|A^\zeta - B^\zeta\| \leq C_{\zeta, \kappa}\|A - B\|^\zeta, \]

where

\[ C_{\zeta, \kappa} = \begin{cases} 1 & \text{when } \zeta \leq 1, \\ 2^{z\kappa} & \text{when } \zeta > 1. \end{cases} \]

**Proof.** The proof is based on the fact that \( u^\zeta \) is operator monotone if \( 0 < \zeta \leq 1 \). While for \( \zeta \geq 1 \), the proof can be found in, e.g., (*Dicker et al., 2016*). \( \square \)

**Lemma 19.** Let \( X \) and \( A \) be bounded linear operators on a separable Hilbert space. Suppose that \( X \geq 0 \) and \( \|A\| \leq 1 \). Then for any \( s \in [0, 1] \),

\[ X^sA^sX \leq (X^sAX)^s. \]

**Proof.** Following from (*Hansen, 1980*) and the fact that the function \( u^s \) with \( s \in [0, 1] \) is operator monotone. \( \square \)

A.1. Proof of Proposition 8

Adding and subtracting with the same term, and using the triangle inequality, we have

\[ \|L^{-a}(S_p\omega_S^\kappa - f_H)\| \leq \|L^{-a}S_p(\omega_S^\kappa - \omega_\lambda)\| + \|L^{-a}(S_p\omega_\lambda - f_H)\|. \]

Applying Part 1) of Lemma 7 to bound the last term, with \( 0 \leq a \leq \zeta \),

\[ \|L^{-a}(S_p\omega_\lambda - f_H)\| \leq \|L^{-a}S_p(\omega_S^\kappa - \omega_\lambda)\| + R\lambda^{\zeta-a} \leq \|L^{-a}S_pT^{a-\frac{1}{2}}\|\|T^{\frac{1}{2}-a}(\omega_S^\kappa - \omega_\lambda)\| + R\lambda^{\zeta-a}. \]

Using the spectral theorem for compact operators, \( L = S_pS_p^* \), and \( T = S_p^*S_p \), we have

\[ \|L^{-a}S_pT^{a-\frac{1}{2}}\| \leq 1, \]
and thus
\[ \| L^{-a}(\mathcal{S}_\rho \omega^\xi - f_H)\|_\rho \leq \| T^{\frac{1}{2} - a}(\omega^\xi - \omega_\lambda)\|_H + R\lambda^{\xi - a}. \]

Adding and subtracting with the same term, and using the triangle inequality,
\[ \| L^{-a}(\mathcal{S}_\rho \omega^\xi - f_H)\|_\rho \leq \| T^{\frac{1}{2} - a}(\omega^\xi - P_\omega \lambda)\|_H + \| T^{\frac{1}{2} - a}(I - P)\omega_\lambda\|_H + R\lambda^{\xi - a}. \]

Since \( P \) is an orthogonal projected operator and \( a \in [0, \frac{1}{2}] \), we have
\[
\begin{align*}
\| T^{\frac{1}{2} - a}(I - P)\omega_\lambda\|_H &= \| T^{\frac{1}{2}-(1-2a)}(I - P)^{(1-2a)}(I - P)\omega_\lambda\|_H \\
&\leq \| T^{\frac{1}{2}-(1-2a)}(I - P)^{(1-2a)}(I - P)T^{\frac{1}{2}} \| \| T^{-\frac{1}{2}}\omega_\lambda\|_H \\
&\leq \| T^{\frac{1}{2}}(I - P)^{(1-2a)}\| \| (I - P)T^{\frac{1}{2}} \| \| T^{-\frac{1}{2}}\omega_\lambda\|_H \\
&= \Delta^{\frac{1}{2} - a}\tau R\kappa^{\frac{1}{2}(\xi - 1)+\lambda^{\xi - 1}}. 
\end{align*}
\]

(where for the last second inequality, we used Lemma 16 and Part 2 of Lemma 7), and we subsequently get that
\[
\| L^{-a}(\mathcal{S}_\rho \omega^\xi - f_H)\|_\rho \leq \| T^{\frac{1}{2} - a}(\omega^\xi - P_\omega \lambda)\|_H + \tau R\kappa^{\frac{1}{2}(\xi - 1)+\lambda^{\xi - 1}} - \Delta^{\frac{1}{2} - a} + R\lambda^{\xi - a}. 
\]

Since for all \( \omega \in H \), and \( a \in [0, \frac{1}{2}] \),
\[
\begin{align*}
\| T^{\frac{1}{2} - a}\omega\|_H &\leq \| T^{\frac{1}{2} - a}_{\xi \lambda}T^{\frac{1}{2} - a}_{\xi \lambda} \| \| T^{\frac{1}{2} - a}_{\xi \lambda}\omega\|_H \\
&\leq \lambda^{\frac{1}{2} - a}\| T^{\frac{1}{2} - a}_{\xi \lambda}T^{\frac{1}{2} - a}_{\xi \lambda} \| \| T^{\frac{1}{2}}\omega\|_H \\
&\leq \lambda^{\frac{1}{2} - a}\| T^{\frac{1}{2}}_{\xi \lambda}T^{\frac{1}{2}}_{\xi \lambda} \| \| 1^{1-2a}\| \| T^{\frac{1}{2}}_{\xi \lambda}\omega\|_H \\
&\leq \lambda^{\frac{1}{2} - a}\| T^{\frac{1}{2}}_{\xi \lambda}\omega\|_H 
\end{align*}
\]

(where we used Lemma 16 for the last second inequality), we get
\[
\| L^{-a}(\mathcal{S}_\rho \omega^\xi - f_H)\|_\rho \leq \lambda^{\frac{1}{2} - a}\| T^{\frac{1}{2}}_{\xi \lambda}(\omega^\xi - P_\omega \lambda)\|_H + \tau R\kappa^{\frac{1}{2}(\xi - 1)+\lambda^{\xi - 1}} - \Delta^{\frac{1}{2} - a} + R\lambda^{\xi - a}. \quad (47)
\]

In what follows, we estimate \( \| T^{\frac{1}{2}}_{\xi \lambda}(\omega^\xi - P_\omega \lambda)\|_H \).

Introducing with (11), with \( P^2 = P \),
\[
\| T^{\frac{1}{2}}_{\xi \lambda}(\omega^\xi - P_\omega \lambda)\|_H = \| T^{\frac{1}{2}}_{\xi \lambda}P(\mathcal{G}_\lambda(P_T x)P_{\xi}^\ast \bar{y} - P_\omega \lambda)\|_H .
\]

Since for any \( \omega \in H \),
\[
\| T^{\frac{1}{2}}_{\xi \lambda}P_\omega\|_H^2 = \langle PT_{\xi \lambda}P_\omega, \omega \rangle_H \leq \langle (P_T x + \lambda)\omega, \omega \rangle_H = \| (PT_{\xi \lambda}P + \lambda)\frac{1}{2}\omega\|_H^2 ,
\]
and we thus get
\[
\| T^{\frac{1}{2}}_{\xi \lambda}(\omega^\xi - P_\omega \lambda)\|_H \leq \| U^{\frac{1}{2}}_{\lambda}(\mathcal{G}_\lambda(\mathcal{U})P_{\xi}^\ast \bar{y} - P_\omega \lambda)\|_H ,
\]
where we denote
\[
\mathcal{U} = PT_{\xi \lambda}P, \quad \mathcal{U}_\lambda = \mathcal{U} + \lambda . \quad (48)
\]

Subtracting and adding with the same term, and applying the triangle inequality, with the notation \( R_\lambda \) given by (44) and \( P^2 = P \), we have
\[
\| T^{\frac{1}{2}}_{\xi \lambda}(\omega^\xi - P_\omega \lambda)\|_H \leq \| U^{\frac{1}{2}}_{\lambda}(\mathcal{G}_\lambda(\mathcal{U})P_{\xi}^\ast \bar{y} - T_{\xi \lambda}P_\omega \lambda)\|_H + \| U^{\frac{1}{2}}_{\lambda}R_\lambda(\mathcal{U})P_\omega \lambda\|_H . \quad (49)
\]
We will estimate the above two terms of the right-hand side.

**Estimating** $\|\text{Term A}\|_H$:

Note that

\[
(U^* \lambda) G_{\lambda}(U) PT_{x,\lambda} \leq ||U^* \lambda G_{\lambda}(U) PT_{x,\lambda}|| \leq \|[U^* \lambda G_{\lambda}(U)]\|^2,
\]

where we used $P^2 = P \leq I$ for the last inequality. Thus, combing with $\|A\| = \|A^* A\|^{\frac{1}{2}}$, 

\[
\|U^* \lambda G_{\lambda}(U) PT_{x,\lambda}\| \leq ||U^* \lambda G_{\lambda}(U)||.
\]

Using the spectral theorem, with $\|U\| \leq ||T_x\| \leq \kappa^2$ (implied by (6)), and then applying (12), 

\[
||U^* \lambda G_{\lambda}(U) PT_{x,\lambda}|| \leq \sup_{\omega \in [0, \kappa^2]} |(u + \lambda)G_{\lambda}(u)| \leq \tau.
\]

Using the above inequality, and by a simple calculation,

\[
||\text{Term A}||_H \leq ||U^* \lambda G_{\lambda}(U) PT_{x,\lambda}|| \leq \tau \|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

Adding and subtracting with the same terms, and using the triangle inequality,

\[
||\text{Term A}||_H \leq \tau \|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

where we used $T = S^*_\rho S_\rho$ for the last inequality. Applying Part 1) of Lemma 7 and $\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq 1$, 

\[
||\text{Term A}||_H \leq \tau \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

In what follows, we estimate $\|\|S_{\lambda} \lambda - T_x P \omega\|\|$, considering two different cases.

**Case** $\zeta \leq 1$.

We have

\[
\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

Since $P$ is a projection operator, $(I - P)^2 = I - P$, and we thus have

\[
\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

where for the last inequality, we used Part 2) of Lemma 7. Note that for any $\omega \in H$ with $\|\omega\|_H = 1$,

\[
\|\|T_{\lambda} \lambda (I - P) \omega\|\|^2 = (T_{\lambda} \lambda (I - P) \omega, (I - P) \omega)_H = \|T_{\lambda} \lambda (I - P) \omega\|_H^2 + \lambda \|\omega\|_H^2 \leq \|T_{\lambda} \lambda (I - P) \omega\|_H^2 + \lambda \|\omega\|_H^2 \leq \|T_{\lambda} \lambda (I - P) \omega\|_H^2 + \lambda \|\omega\|_H^2 \leq \|T_{\lambda} \lambda (I - P) \omega\|_H^2 + \lambda \|\omega\|_H^2.
\]

It thus follows that

\[
\|T_{\lambda} \lambda (I - P) \omega\|_H \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

and thus

\[
\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

\[
\|\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]

\[
\|\|\|S_{\lambda} \lambda - T_x P \omega\|\| \leq \Delta_1 (\Delta_2 + R\lambda^\zeta) + \tau \|\|\|S_{\lambda} \lambda - T_x P \omega\|\|.
\]
Introducing the above into (50), we know that Term. A can be estimated as (ζ ≤ 1)
\[ \|\text{Term. A}\|_H \leq \tau \Delta_1^\frac{1}{2} \left( \Delta_2 + (\tau + 1)R\lambda^\zeta + \tau R\lambda^{\zeta-1}\Delta_5 \right). \] (52)

Case ζ ≥ 1.

We first have
\[ \|T_{x\lambda}^{-\frac{1}{2}}T_x(I - P)\omega_\lambda\|_H \leq \Delta_1^\frac{1}{2} \|T_{x\lambda}^{-\frac{1}{2}}T_x(I - P)\omega_\lambda\|_H \]
\[ \leq \Delta_1^\frac{1}{2} \left( \|T_{x\lambda}^{-\frac{1}{2}}(I - T)(I - P)\omega_\lambda\|_H + \|T_{x\lambda}^{-\frac{1}{2}}T(I - P)\omega_\lambda\|_H \right) \]
\[ \leq \Delta_1^\frac{1}{2} \left( \Delta_4 \|I - P\|\omega_\lambda\|_H + \|T^{\frac{1}{2}}(I - P)\omega_\lambda\|_H \right). \]

Since P is a projection operator, \((I - P)^2 = I - P\), we thus have
\[ \|T_{x\lambda}^{-\frac{1}{2}}T_x(I - P)\omega_\lambda\|_H \leq \Delta_1^\frac{1}{2} \left( \Delta_4 \|I - P\|\omega_\lambda\|_H + \|T^{\frac{1}{2}}(I - P)\omega_\lambda\|_H \right) \]
\[ \leq \Delta_1^\frac{1}{2} (\kappa \Delta_4 + \Delta_5) \|T^{\frac{1}{2}}\omega_\lambda\|_H, \]
where we used (3) for the last inequality. Applying Part 2) of Lemma 7, we get
\[ \|T_{x\lambda}^{-\frac{1}{2}}T_x(I - P)\omega_\lambda\|_H \leq \Delta_1^\frac{1}{2} (\kappa \Delta_4 + \Delta_5) \tau\kappa^{2(\zeta-1)} R. \]

Introducing the above into (50), we get for ζ ≥ 1,
\[ \|\text{Term. A}\|_H \leq \tau \Delta_1^\frac{1}{2} \left( \Delta_2 + R\lambda^\zeta + (\kappa \Delta_4 + \Delta_5) \tau\kappa^{2(\zeta-1)} R \right). \] (53)

Estimating \( \|\text{Term. B}\|_H \):

We estimate \( \|\text{Term. B}\|_H \), considering two different cases.

Case I: ζ ≤ 1.

We first have
\[ \mathcal{U}_x^\frac{1}{2} R_\lambda(U)PT_{x\lambda}^\frac{1}{2}(U^\frac{1}{2} R_\lambda(U)PT_{x\lambda}^\frac{1}{2})* = \mathcal{U}_x^\frac{1}{2} R_\lambda(U)(U + \lambda P^2)R_\lambda(U)U_x^\frac{1}{2} \]
\[ \leq (R_\lambda(U)U_x)^2, \]
where we used \( P^2 = P \leq I \) for the last inequality. Thus, according to \( \|A\| = \|AA^*\|^{\frac{1}{2}} \),
\[ \|\mathcal{U}_x^\frac{1}{2} R_\lambda(U)PT_{x\lambda}^\frac{1}{2}\| \leq \|R_\lambda(U)U_x\|. \]

Using the spectral theorem and (13), and noting that \( \|U\| \leq \|P\|^2 \|T_x\| \leq \kappa^2 \) by (6), we get
\[ \|\mathcal{U}_x^\frac{1}{2} R_\lambda(U)PT_{x\lambda}^\frac{1}{2}\| \leq \sup_{u \in [0, \lambda^2]} |R_\lambda(u)(u + \lambda)| \leq \lambda. \]

Using the above inequality and by a direct calculation,
\[ \|\text{Term. B}\|_H \leq \|\mathcal{U}_x^\frac{1}{2} R_\lambda(U)PT_{x\lambda}^\frac{1}{2}\| \|T_{x\lambda}^{-\frac{1}{2}}T_{x\lambda}^\frac{1}{2}\| \|T^{\frac{1}{2}}\omega_\lambda\|_H \leq \lambda \Delta_1^\frac{1}{2} \|T^{\frac{1}{2}}\omega_\lambda\|_H. \]

Applying Part 2) of Lemma 7, we get
\[ \|\text{Term. B}\|_H \leq \tau R\lambda^\zeta \Delta_5^\frac{1}{2}. \] (54)

Applying the above and (52) into (49), we know that for any \( \zeta \in [0, 1], \)
\[ \|T_{x\lambda}^{-\frac{1}{2}}(\omega_x^\lambda - P\omega_\lambda)\|_H \leq \tau \Delta_1^\frac{1}{2} \left( \Delta_2 + (2\tau + 1)R\lambda^\zeta + \tau R\Delta_5\lambda^{\zeta-1} \right). \]

Using the above into (47), we can prove the first desired result.

Case II: ζ ≥ 1

We denote
\[ \mathcal{V} = T_x^\frac{1}{2} PT_x^\frac{1}{2}, \quad \mathcal{V}_\lambda = \mathcal{V} + \lambda. \] (55)
Noting that $U = PT_x P = PT_x^2 (PT_x^2)^*$, thus following from Lemma 17 (with $f(u) = (u + \lambda)^2 R_\lambda(u)$ and $P^2 = P$, \[\|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| = \|U_\lambda^2 R_\lambda(U)(PT_x^2) T_x^{\frac{\zeta - 1}{2}}\| = \|(PT_x^2) V_\lambda R_\lambda(V) T_x^{\frac{\zeta - 1}{2}}\|.\]

Adding and subtracting with the same term, using the triangle inequality,
\[\|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \leq \|PT_x^2 V_\lambda R_\lambda(V) V^{\zeta - 1}\| + \|PT_x^2 V_\lambda^2 R_\lambda(V)(T_x^{\frac{\zeta - 1}{2}} - V^{\zeta - 1})\| \leq \|PT_x^2 V_\lambda R_\lambda(V) V^{\zeta - 1}\| + \|PT_x^2 V_\lambda^2 R_\lambda(V)||T_x^{\frac{\zeta - 1}{2}} - V^{\zeta - 1}\|.
\]

Using Lemma 18, with (6) and $\|V\| \leq \|T_x\| \leq \kappa^2$,
\[\|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \leq \|PT_x^2 V_\lambda R_\lambda(V) V^{\zeta - 1}\| + \|PT_x^2 V_\lambda^2 R_\lambda(V)||\kappa^{2(\zeta - 2)} + \|T_x - V\|^{(\zeta - 1)^{\lambda}}.\]

Using $\|A\| = \|A^* A\|^{\frac{1}{2}}, P^2 = P$, the spectral theorem, and (13), for any $s \in [1, \tau]$,
\[\|PT_x^2 V_\lambda R_\lambda(V) V^{s - 1}\| = \|V^{s - 1} R_\lambda(V) V R_\lambda(V) V^{s - 1}\|^{\frac{1}{2}} \leq \sup_{u \in [0, \kappa^2]} |R_\lambda(u) u^{s - \frac{1}{2}} (u + \lambda)^{\frac{1}{2}}| \leq \lambda^s,
\]

and thus we get
\[\|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \leq \lambda^s + \lambda \kappa^{2(\zeta - 2)} \|T_x - V\|^{(\zeta - 1)^{\lambda}}.\]

Using Lemma 14, $(I - P)^2 = I - P$ and $\|A^* A\| = \|A\|^2$, we have
\[\|T_x - V\| = \|T_x^2 (I - P) T_x^{\frac{\zeta - 1}{2}}\| \leq \|T_x - T\| + \|T_x^2 (I - P) T_x^{\frac{\zeta - 1}{2}}\| \leq \Delta_3 + \Delta_5,
\]

and we thus get
\[\|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \leq \lambda^s + \lambda \kappa^{2(\zeta - 2)} (\Delta_3 + \Delta_5)^{\lambda}.\]

Now we are ready to estimate $\|\text{Term.B}\|_H$. By some direct calculations and Part 2) of Lemma 7,
\[\|\text{Term.B}\|_H \leq \|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \|T_x^{\frac{\zeta - 1}{2}} - \omega_\lambda\|_H \leq \|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| \|\tau R\|.
\]

Adding and subtracting with the same term, and using the triangle inequality,
\[\|\text{Term.B}\|_H \leq \tau R \left( \|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| + \|U_\lambda^2 R_\lambda(U)\| ||T_x^{\frac{\zeta - 1}{2}} - \omega_\lambda\|_H \right).\]

Using the spectral theorem, with $\|U\| \leq \|T_x\| \leq \kappa^2$ by (6) and (13),
\[\|U_\lambda^2 R_\lambda(U)\| = \sup_{u \in [0, \kappa^2]} |R_\lambda(u) (u + \lambda)^{\frac{1}{2}}| \leq \lambda^s,
\]

and we thus get
\[\|\text{Term.B}\|_H \leq \tau R \left( \|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| + \lambda^s ||T_x^{\frac{\zeta - 1}{2}} - \omega_\lambda\|_H \right).\]

Applying Lemma 18, with (3) and (6),
\[\|\text{Term.B}\|_H \leq \tau R \left( \|U_\lambda^2 R_\lambda(U) PT_x^{\frac{\zeta - 1}{2}}\| + \lambda^s \kappa^2 (\zeta - 3) + \Delta_3 (\zeta - 1)^{\lambda}\right).\]

Introducing with (56),
\[\|\text{Term.B}\|_H \leq \tau R \left( \lambda^s + \kappa^2 (\zeta - 2) + \lambda (\Delta_3 + \Delta_5) (\zeta - 1)^{\lambda} + \kappa^2 (\zeta - 3) + \lambda^s \Delta_3 (\zeta - 1)^{\lambda}\right).
\]

Introducing the above inequality and (53) into (49), noting that $\Delta_1 \geq 1$ and $\kappa^2 \geq 1$, we know that for any $\zeta \geq 1$,
\[\|T_x^{\frac{1}{2}} (\omega_\lambda^s - P \omega_\lambda)\|_H \leq \tau \Delta_3^2 \left( \Delta_2 + 2R \lambda^\zeta + \kappa^2 (\zeta - 1) R(\kappa R \Delta_4 + \tau \Delta_5 + \lambda (\Delta_3 + \Delta_5) (\zeta - 1)^{\lambda} + \lambda^s \Delta_3 (\zeta - 1)^{\lambda}\right).
\]

Using the above into (47), and by a simple calculation, we can prove the second desired result.
A.2. Proof of Lemma 13

Let $S_x = UΣV^*$ be the singular value decomposition of $S_x$, where $V : \mathbb{R}^r \to H, U \in \mathbb{R}^{n \times r}$ and $Σ = \text{diag}(σ_1, σ_2, \ldots, σ_r)$ with $V^*V = I_r, U^*U = I_r$ and $σ_1 ≥ σ_2, \ldots, σ_r > 0$. In fact, we can write $V = [v_1, \ldots, v_r]$ with

$$V a = \sum_{i=1}^{r} a(i)v_i, \quad \forall a \in \mathbb{R}^r,$$

with $v_i \in H$ such that $⟨v_i, v_j⟩_H = 0$ if $i \neq j$ and $⟨v_i, v_i⟩_H = 1$. Similarly, we write $U = [u_1, \ldots, u_r]$, and

$$S_x = \sum_{i=1}^{r} σ_i ⟨v_i, ·⟩_H u_i = \sum_{i=1}^{r} σ_i u_i \otimes v_i.$$

For any $μ ≥ 0$, we decompose $S_x$ as $S_{1,μ} + S_{2,μ}$ with

$$S_{1,μ} = \sum_{σ_i > μ} σ_i u_i \otimes v_i, \quad S_{2,μ} = \sum_{σ_i ≤ μ} σ_i u_i \otimes v_i,$$

and we will drop $μ$ to write $S_{j,μ}$ as $S_j$ when it is clear in the text. Denote $d$ the cardinality of $\{σ_i : σ_i > μ\}$. Correspondingly,

$$S_1 = U_1Σ_1V_1^*, \quad S_2 = U_2Σ_2V_2^*, \quad (57)$$

where $V_1 = [v_1, \ldots, v_d]$, $V_2 = [v_{d+1}, \ldots, v_r]$, $U_1 = [u_1, \ldots, u_d]$, $U_2 = [u_{d+1}, \ldots, u_r]$, $Σ_1 = \text{diag}(σ_1, \ldots, σ_d)$, and $Σ_2 = \text{diag}(σ_{d+1}, \ldots, σ_r)$. As the range of $P$ is $\text{range}(S_{2}^*G^*)$, we can let

$$P = P_1 + P_2,$$

where $P_1$ and $P_2$ are projection operators on $\text{range}(S_{1}^*G^*)$ and $\text{range}(S_{2}^*G^*)$, respectively.

As

$$T_x = S_x^*S_x = (UΣV^*)^*UΣV^* = VΣ^2V^*,$$

we have

$$\| (I - P)T_x^\frac{1}{2} \| = \| (I - P)VΣV^* \| = \| (I - P_1 - P_2) \sum_{i=1}^{2} V_iΣ_iV_i^* \|.$$

As $P_1$ is a projection operator on $\text{range}(S_{1}^*G^*)(\subseteq \text{range}(V_1))$ and $\text{range}(S_{2}^*G^*)(\subseteq \text{range}(V_2))$, and $V_1^*V_2 = 0$, we know that $P_1V_2 = 0$ when $i \neq j$. Thus, it follows that

$$\| (I - P)T_x^\frac{1}{2} \| = \sum_{i=1}^{2} \| (I - P_i)(V_iΣ_iV_i^*) \|$$

$$≤ \sum_{i=1}^{2} \| (I - P_i)(V_iΣ_iV_i^*) \|$$

As $Σ_2 = \text{diag}(σ_{d+1}, \ldots, σ_r)$ with $σ_r ≤ · · · ≤ σ_{d+1} ≤ μ$, we get

$$\| (I - P)T_x^\frac{1}{2} \| \leq \| (I - P_1)(V_1Σ_1V_1^*) \| + μ. \quad (58)$$

As $P_1$ is the projection operator on $\text{range}(S_{1}^*G^*)$, letting $W = GS_1$ and for any $λ > 0$,

$$P_1 = W^*(WW^*)^1W ≥ W^*(WW^* + λI)^{-1}W = W^*W(W^*W + λI)^{-1},$$

and thus

$$I - P_1 ≥ I - W^*W(W^*W + λI)^{-1} = λ(W^*W + λI)^{-1}.$$
As for notational simplicity, we write

\[ T_1^\perp (I - P_1) T_1^\perp \leq \lambda T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp, \]

where for notational simplicity, we write

\[ T_1 = (V_1 \Sigma_1 V_1^*)^2. \]  

(59)

Combing with

\[ \|(I - P)T_1^\perp\|^2 = \|T_1^\perp (I - P)2T_1^\perp\| = \|T_1^\perp (I - P)T_1^\perp\|, \]

we know that

\[ \|(I - P)T_1^\perp\|^2 \leq \lambda\|T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp\| \leq \lambda\|T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp\|. \]

As

\[ T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp = \left(T_1^\perp (W^*W + \lambda I)T_1^\perp\right)^{-1} = \left(I - T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp\right)^{-1}, \]

and if

\[ \|T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp\| \leq c < 1, \]  

(60)

then according to Neumann series,

\[ \|(I - P)T_1^\perp\|^2 \leq \lambda\|T_1^\perp (W^*W + \lambda I)^{-1} T_1^\perp\| \leq (1 - c)^{-1} \lambda, \]

(61)

If we choose \( \mu = \sqrt{\lambda} \), and introduce the above with \( c = \frac{1}{\lambda} \) into (58), one can get

\[ \|(I - P)T_2^\perp\|^2 \leq (\sqrt{2} + 1)^2 \lambda \leq 6\lambda, \]

(62)

which leads to the desired bound.

In what follows, we show that (60) with \( c = \frac{1}{\lambda} \) holds with high probability under the constraint (40). Recall (59) and that

\[ W = GS_1 + S_1 \]

Thus, \( T_1 = V_1 \Sigma_1 V_1^* V_1 \Sigma_1 V_1^* = V_1 \Sigma_1 V_1^* \), and

\[ W^*W = S_1^* G^* G S_1 = V_1 \Sigma_1 U_1^* G^* G U_1 \Sigma_1 V_1^*. \]

Therefore, with \( V_1^* V_1 = I \),

\[ T_1^\perp (W^*W) T_1^\perp = V_1 (\Sigma_1^2 + \lambda I)^{-1/2} \Sigma_1^2 V_1 (I - U_1^* G^* G U_1) \Sigma_1^2 V_1 (\Sigma_1^2 + \lambda I)^{-1/2} \Sigma_1^2 V_1 = V_1 (\Sigma_1^2 + \lambda I)^{-1/2} \Sigma_1^2 (I - U_1^* G^* G U_1) \Sigma_1^2 (\Sigma_1^2 + \lambda I)^{-1/2} \Sigma_1^2 V_1. \]

(63)

It follows that

\[ \|T_1^\perp (W^*W) T_1^\perp\| \leq \|V_1\|(\Sigma_1^2 + \lambda I)^{-1/2} \Sigma_1^2 \|\|I - U_1^* G^* G U_1\|\|\Sigma_1^2\| \leq \|I - U_1^* G^* G U_1\|. \]

Using \( U_1^* U_1 = I \),

\[ \|I - U_1^* G^* G U_1\| = \|U_1^* (I - G^* G) U_1\| = \max_{a \in R^d, ||a||_2 = 1} |(U_1^* (I - G^* G) U_1 a, a)| = \max_{a \in R^d, ||a||_2 = 1} ||U_1 a||_2^2 - ||G U_1 a||_2^2. \]

Based on a standard covering argument (Baraniuk et al., 2008), we know that

\[ \max_{a \in R^d, ||a||_2 = 1} ||U_1 a||_2^2 - ||G U_1 a||_2^2 \leq \frac{1}{2} \]

with probability at least

\[ 1 - 2(72)^d \exp \left( - \frac{m}{12^2 \sigma_0^2 \log^2 n} \right) \geq 1 - \delta, \]
provided that
\[ m \geq 144c_1^2 \log^d n \left( \log \frac{2}{\delta} + 6d \log 2 \right). \]  
(64)

Note that by (39)
\[ b_2 \lambda^{-\gamma} \geq \text{tr}(T \tau T^{-1}) = \sum_i \frac{\sigma_i^2}{\sigma_i^2 + \lambda} \geq \sum_{i>\lambda} \frac{\sigma_i^2}{\sigma_i^2 + \lambda} \geq \frac{d}{2}. \]

Thus, a stronger condition for (64) is (40). The proof is complete.

A.3. Proof of Lemma 14

Since \( P \) is a projection operator, \((I - P)^2 = I - P\). Then
\[ \|A^*(I - P)A^t\| = \|A^*(I - P)(I - P)A^t\| \leq \|A^*(I - P)\|\|(I - P)A^t\|. \]

Moreover, by Lemma 16,
\[ \|A^*(I - P)\| = \|A^{\frac{1}{2}}(I - P)^{\frac{1}{2}}\| \leq \|A^{\frac{1}{2}}(I - P)\|^2. \]

Similarly, \(\|(I - P)A^t\| \leq \|(I - P)A^t\|^{2t}\). Thus,
\[ \|A^*(I - P)A^t\| \leq \|A^{\frac{1}{2}}(I - P)\|^2\|(I - P)A^t\|^2 = \|(I - P)A^t\|^{2(t + s)}. \]

Using \(\|D\|^2 = \|D^*D\|\),
\[ \|A^*(I - P)A^t\| \leq \|(I - P)A(I - P)\|^t. \]

Adding and subtracting with the same term, using the triangle inequality, and noting that \(\|I - P\| \leq 1\) and \(s + t \leq 1\),
\[ \|A^*(I - P)A^t\| \leq \|(I - P)A(I - P)\|^t + \|(I - P)B(I - P)\|^{t + s} \]
\[ \leq \|A - B\|^{s + t} + \|(I - P)B(I - P)\|^{s + t}, \]
which leads to the desired result using \(\|D^*D\| = \|DD^*\|\).

A.4. Proof of Lemma 15

To prove the result, we need the following concentration inequality.

**Lemma 20.** Let \(w_1, \ldots, w_m\) be i.i.d random variables in a separable Hilbert space with norm \(\| \cdot \|\). Suppose that there are two positive constants \(B\) and \(\sigma^2\) such that
\[ \mathbb{E}[\|w_1 - \mathbb{E}[w_1]\|^l] \leq \frac{1}{2} l! B^{l-2} \sigma^2, \quad \forall l \geq 2. \]  
(65)

Then for any \(0 < \delta < 1/2\), the following holds with probability at least \(1 - \delta\),
\[ \left\| \frac{1}{m} \sum_{k=1}^m w_m - \mathbb{E}[w_1] \right\| \leq 2 \left( \frac{B}{m} + \frac{\sigma}{\sqrt{m}} \right) \log \frac{2}{\delta}. \]

In particular, (65) holds if
\[ \|w_1\| \leq B/2 \text{ a.s., and } \mathbb{E}[\|w_1\|^2] \leq \sigma^2. \]  
(66)

The above lemma is a reformulation of the concentration inequality for sums of Hilbert-space-valued random variables from (Pinelis & Sakhanenko, 1986). We refer to (Smale & Zhou, 2007; Caponnetto & De Vito, 2007) for the detailed proof.
Proof of Lemma 15. We first use Lemma 20 to estimate \( \text{tr}(T^{-\frac{1}{2}}_x (T_x - T) T^{-\frac{1}{2}}_x) \). Note that

\[
\text{tr}(T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x) = \frac{1}{n} \sum_{j=1}^{n} \|T^{-\frac{1}{2}}_x x_j\|_H^2 = \frac{1}{n} \sum_{j=1}^{n} \xi_j,
\]

where we let \( \xi_j = \|T^{-\frac{1}{2}}_x x_j\|_H^2 \) for all \( j \in [n] \). Besides, it is easy to see that

\[
\text{tr}(T^{-\frac{1}{2}}_x (T_x - T) T^{-\frac{1}{2}}_x) = \frac{1}{n} \sum_{j=1}^{n} (\xi_j - \mathbb{E}[\xi_j]).
\]

Using Assumption (2),

\[
\xi_1 \leq \frac{1}{\lambda} \|x_1\|_H^2 \leq \frac{\kappa^2}{\lambda},
\]

and

\[
\mathbb{E}[\|\xi_1\|^2] \leq \frac{\kappa^2}{\lambda}\mathbb{E}[\|T^{-\frac{1}{2}}_x x_1\|_H^2] \leq \frac{\kappa^2 N(\lambda)}{\lambda}.
\]

Applying Lemma 20, we get that there exists a subset \( V_1 \) of \( Z^n \) with measure at least \( 1 - \delta \), such that for all \( z \in V_1 \),

\[
\text{tr}(T^{-\frac{1}{2}}_x (T_x - T) T^{-\frac{1}{2}}_x) \leq 2 \left( \frac{2\kappa^2}{n\lambda} + \sqrt{\frac{\kappa^2 N(\lambda)}{n\lambda}} \right) \log \frac{\delta}{\delta}.
\]

Combining with Lemma 9, taking the union bounds, rescaling \( \delta \), and noting that

\[
\text{tr}(T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x) = \|T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x\|^2 \leq \frac{\|T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x\|^2}{\|T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x\|^2} \text{tr}(T^{-\frac{1}{2}}_x T_x T^{-\frac{1}{2}}_x) = \left( \text{tr}(T^{-\frac{1}{2}}_x (T_x - T) T^{-\frac{1}{2}}_x) + N(\lambda) \right)
\]

we get that there exists a subset \( V \) of \( Z^n \) with measure at least \( 1 - \delta \), such that for all \( z \in V \),

\[
\text{tr}((T_x + \lambda)^{-1} T_x) \leq 3a_{n,\delta/2,\gamma}(\theta) \left( 2 \left( \frac{2\kappa^2}{n\lambda} + \sqrt{\frac{\kappa^2 N(\lambda)}{n\lambda}} \right) \log \frac{\delta}{\delta} + N(\lambda) \right),
\]

which leads to the desired result using \( \lambda \leq 1 \), \( n\lambda \geq 1 \) and Assumption 3. \( \square \)

A.5. Proof for Corollary 5

Proof. Using a similar argument as that for (61), with \( W = S_{\bar{x}} \), where \( \bar{x} = \{x_1, \cdots, x_m\} \), we get for any \( \eta > 0 \),

\[
\|(I - P)T^{\frac{1}{2}}\|^2 \leq \eta \|(T_{\bar{x}} + \eta)^{-1/2}(T + \eta)^{1/2}\|^2.
\]

Letting \( \eta = \frac{1}{m} \), and using Lemma 9, we get that with probability at least \( 1 - \delta \),

\[
\|(I - P)T^{\frac{1}{2}}\|^2 \leq \frac{1}{m} \log \frac{3m^\gamma}{\delta}.
\]

Combining with Corollary 3, one can prove the desired result. \( \square \)

A.6. Proof of Corollary 6

We first note that in an L-ALS Nyström subsampling regime, \( S \) can be rewritten as \( S = \mathbb{range}(S_{\bar{x}} G^\top) \), where each row \( \frac{1}{\sqrt{m}} a_i^\top \) of \( G \) is i.i.d. drawn according to

\[
\mathbb{P}(a = \frac{1}{\sqrt{q_i}} e_i) = q_i,
\]
We first introduce the following basic probabilistic estimate.

**Lemma 21.** Let \(X_1, \ldots, X_m\) be a sequence of independently and identically distributed self-adjoint Hilbert-Schmidt operators on a separable Hilbert space. Assume that \(E[X_1] = 0\), and \(\|X_i\| \leq B\) almost surely for some \(B > 0\). Let \(V\) be a positive trace-class operator such that \(E[X_i^2] \leq \mathcal{V}\). Then with probability at least \(1 - \delta\), \((\delta \in [0, 1])\), there holds

\[
\left\| \frac{1}{m} \sum_{i=1}^{m} X_i \right\|^2 \leq \frac{2B^2}{3m} + \sqrt{\frac{2\mathcal{V}B}{m}}, \quad \beta = \log \frac{4\text{tr}\mathcal{V}}{\|\mathcal{V}\|\delta}.
\]

The above lemma was first proved in (Hsu et al., 2014; Tropp, 2012) for the matrix case, and it was later extended to the general operator case in (Minsker, 2011), see also (Rudi et al., 2015; Bach, 2015; Dicker et al., 2017). We refer to (Rudi et al., 2015; Dicker et al., 2017) for the proof.

Using the above lemma, and with a similar argument as that for Lemma 13, we can estimate the empirical version of the projection error as follows.

**Lemma 22.** Let \(0 < \delta < 1\) and \(\theta \in [0, 1]\). Given a fix input subset \(x \subseteq H^n\), assume that for \(\lambda \in [0, 1]\), (39) holds for some \(b, \gamma > 0\). Then there exists a subset \(U_\mathcal{X}\) of \(\mathbb{R}^{n \times n}\) with measure at least \(1 - \delta\), such that for all \(G \in U_\mathcal{X}\),

\[
\|(I - P)X \|^2 \leq 3\lambda,
\]

provided that

\[
m \geq 8b_\gamma\lambda^{-}\gamma L^2 \log \frac{8b_\gamma\lambda^{-}\gamma}{\delta}.
\]

**Proof.** If we choose \(u = 0\) in the proof of Lemma 13, then \(\mathcal{S}_\mathcal{X} = \mathcal{S}_1\) and \(\mathcal{S}_2 = 0\). Similarly, \(\mathcal{T}_\mathcal{X} = T_1\). In this case, (63) reads as

\[
T_{\mathcal{X}^\lambda}^{-\frac{1}{2}}(T_\mathcal{X} - W^*W)T_{\mathcal{X}^\lambda}^{-\frac{1}{2}} = V(\Sigma^2 + \lambda I)^{-1/2}\Sigma(I - U^*G^*GU)\Sigma(\Sigma^2 + \lambda I)^{-1/2}V^*.
\]

Thus, using \(V^*V = I, U^*U = I\) and \(U\) is of full column rank,

\[
\left\| T_{\mathcal{X}^\lambda}^{-\frac{1}{2}}(T_\mathcal{X} - W^*W)T_{\mathcal{X}^\lambda}^{-\frac{1}{2}} \right\| \leq \left\| U^*U(\Sigma^2 + \lambda I)^{-1/2}\Sigma U^*(I - G^*G)U\Sigma(\Sigma^2 + \lambda I)^{-1/2}U^*U \right\|
\]

\[
\leq \left\| U(\Sigma^2 + \lambda I)^{-1/2}\Sigma U^*(I - G^*G)U\Sigma(\Sigma^2 + \lambda I)^{-1/2}U^* \right\|
\]

Using \(K := K_{xx} = \mathcal{S}_\mathcal{X}^*\mathcal{S}_\mathcal{X} = U\Sigma^2U^*\), we get

\[
\left\| T_{\mathcal{X}^\lambda}^{-\frac{1}{2}}(T_\mathcal{X} - W^*W)T_{\mathcal{X}^\lambda}^{-\frac{1}{2}} \right\| \leq \left\| (K(K + \lambda I)^{-1})^{1/2}(I - G^*G)\left(K(K + \lambda I)^{-1}\right)^{1/2} \right\|
\]

Letting \(X_i = (K(K + \lambda I)^{-1})^{1/2}a_ia_i^*(K(K + \lambda I)^{-1})^{1/2}\), it is easy to prove that \(E[a_ia_i^*] = I\), according to the definition of ALS Nyström subsampling. Then the above inequality can be written as

\[
\left\| T_{\mathcal{X}^\lambda}^{-\frac{1}{2}}(T_\mathcal{X} - W^*W)T_{\mathcal{X}^\lambda}^{-\frac{1}{2}} \right\| \leq \frac{1}{m} \sum_{i=1}^{m} \left\| (E[X_i] - X_i) \right\|
\]

A simple calculation shows that

\[
\|X_i\| = \max_{j \in [n]} \left( K(K + \lambda I)^{-1} \right)_{jj} \leq \max_{j \in [n]} \left( \frac{1}{m} \sum_{k} \lambda_k l_k(\lambda) \right)_{jj} = \max_{j \in [n]} \frac{l_j(\lambda)}{q_j} \leq L^2 \sum_{j} l_j(\lambda) = L^2 \text{tr}(K K^{-1}_\lambda),
\]

where \(K_{xx} = \mathcal{S}_\mathcal{X}^*\mathcal{S}_\mathcal{X} = U\Sigma^2U^*\).
and
\[ \mathbb{E}[\lambda_i^2] = \mathbb{E}[a_i^* \{ (K(K + \lambda I)^{-1}) a_i \}] \leq L^2 \text{tr}(KK^{-1}) \mathbb{E}[\lambda_i] = L^2 \text{tr}(KK^{-1})KK^{-1}. \]

Thus,
\[ \| \mathbb{E}[\lambda_i] - \lambda_i \| \leq \mathbb{E}[\| \lambda_i \|] \leq 2L^2 \text{tr}(KK^{-1}), \]
and
\[ \mathbb{E}[(\lambda_i - \mathbb{E}[\lambda_i])^2] \leq \mathbb{E}[\lambda_i^2] \leq L^2 \text{tr}(KK^{-1})KK^{-1}. \]

Letting \( \mathcal{V} = L^2 \text{tr}(KK^{-1})KK^{-1} \), we have
\[ \| \mathcal{V} \| \leq L^2 \text{tr}(KK^{-1}), \]
and
\[ \text{tr}(\mathcal{V}) = \text{tr}(KK^{-1}) \left( 1 + \frac{\lambda}{\|K\|} \right). \]

Applying Lemma 21, noting that \( \text{tr}(KK^{-1}) = \text{tr}(T_K T_K^{-1}) \) and \( \|K\| = \|T_K\| \) as \( T_K = S_x^* S_x \), we get that there exists a subset \( U_x \subseteq R^{m \times n} \) with measure at least \( 1 - \delta \) such that for all \( G \in U_x \),
\[ \|T_{\lambda \epsilon}^{-\frac{1}{2}(T_K - W^* W) T_{\lambda \epsilon}^{-\frac{1}{2}}r_{\lambda \epsilon}^+} \| \leq \frac{4L^2 \text{tr}(T_K T_K^{-1})\beta}{3m} + \sqrt{\frac{2L^2 \text{tr}(T_K T_K^{-1})\beta}{m}}, \quad \beta = \log \frac{4\text{tr}(T_K T_K^{-1})(1 + \lambda/\|T_K\|)}{\delta}. \]

If \( \lambda \leq \|T_K\| \), using Condition (39), we have
\[ \beta \leq \log \frac{4b_\epsilon \lambda^{-\gamma} (1 + \lambda/\|T_K\|)}{\delta} \leq \log \frac{8b_\epsilon \lambda^{-\gamma}}{\delta}, \]
and, combining with (68),
\[ \frac{4L^2 \text{tr}(T_K T_K^{-1})\beta}{3m} + \sqrt{\frac{2L^2 \text{tr}(T_K T_K^{-1})\beta}{m}} \leq \frac{2}{3}. \]

Thus,
\[ \left\| T_{\lambda \epsilon}^{-1/2}(T - M) T_{\lambda \epsilon}^{-1/2} \right\| \leq \frac{2}{3}, \quad \forall G \in U_x. \]

Following from (60) and (61), one can prove (67) for the case \( \lambda \leq \|T_K\| \). The proof for the case \( \lambda \geq \|T_K\| \) is trivial:
\[ \| (I - P) T_{\lambda \epsilon}^{1/2} \|^2 \leq \| I - P \|^2 \| T_{\lambda \epsilon}^{1/2} \|^2 \leq \| T_K \| \leq \lambda. \]

The proof is complete. \( \Box \)

With the above lemma, and using a similar argument as that for Corollary 4, we can prove Corollary 6. We thus skip it.

### 2. Learning with Kernel Methods

Let the input space \( \Xi \) be a closed subset of Euclidean space \( \mathbb{R}^d \), the output space \( Y \subseteq \mathbb{R} \). Let \( \mu \) be an unknown but fixed Borel probability measure on \( \Xi \times Y \). Assume that \( \{(\xi_i, y_i)\}_{i=1}^n \) are i.i.d. from the distribution \( \mu \). A reproducing kernel \( K \) is a symmetric function \( K : \Xi \times \Xi \to \mathbb{R} \) such that \( (K(u_i, u_j))_{i,j=1}^n \) is positive semidefinite for any finite set of points \( \{u_i\}_{i=1}^n \) in \( \Xi \). The kernel \( K \) defines a reproducing kernel Hilbert space (RKHS) \( (H_K, \| \cdot \|_K) \) as the completion of the linear span of the set \( \{K(\xi, \cdot) : \xi \in \Xi\} \) with respect to the inner product \( \langle K(\xi), K(u) \rangle_K := K(\xi, u) \). For any \( f \in H_K \), the reproducing property holds: \( f(\xi) = \langle K(\xi), f \rangle_K \).

**Example B.1 (Sobolev Spaces).** Let \( X = [0, 1] \) and the kernel
\[ K(x, x') = \begin{cases} (1 - y)x, & x \leq y; \\ (1 - x)y, & x \geq y. \end{cases} \]

Then the kernel induces a Sobolev Space \( H = \{ f : X \to \mathbb{R} \mid f \text{ is absolutely continuous}, f(0) = f(1) = 0, f \in L^2(X) \} \).
In learning with kernel methods, one considers the following minimization problem

$$\inf_{f \in \mathcal{H}_K} \int_{\mathcal{X} \times \mathcal{Y}} (f(\xi) - y)^2 d\mu(\xi, y).$$

Since $f(\xi) = \langle K_\xi, f \rangle_K$ by the reproducing property, the above can be rewritten as

$$\inf_{f \in \mathcal{H}_K} \int_{\mathcal{X} \times \mathcal{Y}} (\langle f, K_\xi \rangle_K - y)^2 d\mu(\xi, y).$$

Letting $X = \{K_\xi : \xi \in \mathcal{X}\}$ and defining another probability measure $\rho(K_\xi, y) = \mu(\xi, y)$, the above reduces to the learning setting in Section 2.