K3 SURFACES AND CUBIC FOURFOLDS WITH A MOTIVE OF ABELIAN TYPE

MICHELE BOLOGNESI AND CLAUDIO PEDRINI

Abstract. Starting from known 4-dimensional families of K3 surfaces, we construct two families of cubic fourfolds whose motive is of Abelian type. Cubics from the first family are smooth, and their Chow motive is finite dimensional and Abelian. Those from the second family are singular, and their motives are Schur-finite and Abelian in Voevodsky’s triangulated category of motives.

INTRODUCTION

According to Kimura’s Conjecture on finite dimensional motives and by the results in [An] the Chow motive $h(S)$ of a complex K3 surface $S$ should be of abelian type. Also, by the work of Kuga and Satake, if one assumes the Hodge conjecture, every K3 surface over $\mathbb{C}$ is of abelian Hodge type, i.e. there exists an algebraic correspondence between any K3 surface $S$ and an associated abelian variety, the Kuga-Satake variety $K(S)$. This would imply that the motive of any such K3 surface is abelian, i.e. it lies in the subcategory $M^{Ab}_{rat}(\mathbb{C})$ of the (covariant) category $M_{rat}(\mathbb{C})$ of Chow motives generated by the motives of curves.

On the other hand, cubic fourfolds have been a very active field of research in the last few years, for several reasons. The rationality of the generic cubic is a very classical, and still unanswered, question in algebraic geometry. A lot of energy has been spent in order to find a good invariant that would detect the required birational properties, and tentatives have been made via Hodge theory, derived categories, Chow motives, algebraic cycles, etc. In most of these papers K3 surfaces appear as an important presence (in the cohomology, derived category, related Hyper-Kähler varieties, ...) whenever the cubic fourfolds are rational, or suspected to be rational.

Also in this paper K3 surfaces are the main tool to study the motive of certain cubic fourfolds, but under a different perspective. The goal of this paper is to showcase two new families of cubic fourfolds $X \subset \mathbb{P}^5$ whose motives are finite dimensional and of abelian type. Special families of cubic fourfolds with the same property have been described by R. Laterveer in [Lat 1] and [Lat 2], using particular shapes of their equations.

The cubics from the first family are smooth, and their Chow motive is finite dimensional and abelian. Here the relation with K3 surfaces is given by the well-known Chow-Künneth decomposition of the motive of a smooth cubic fourfold, in terms of the transcendental motive of the associated K3 surface, and some results of Garbagnati-Sarti and Laterveer. The associated K3 surfaces are in fact a 4-dimensional family $\mathcal{F}$ of smooth octic surfaces in $\mathbb{P}^5$, that are complete intersections of 3 quadrics in $\mathbb{P}^5$, see Sect. 1.1.
Theorem 0.1. Let $S$ be any K3 surface in $\mathcal{F}$, let $Q_1, Q_2$ and $Q_3$ be the three quadrics in $\mathbb{P}^5$ cutting out $S$, and let $X$ be a smooth cubic fourfold. If the equation of $X$ can be written as

$$L_1Q_1 + L_2Q_2 + L_3Q_3 = 0,$$

where the $L_i$ are three independent linear forms, then the Chow motive $h(X)$ is finite dimensional and of abelian type.

The cubics from the second family are singular, and rational. For any cubic $X$ of this family, there exists in fact a birational map $\psi_X : \mathbb{P}^4 \dashrightarrow X$ given by the full linear system of cubics through one 15-nodal, sextic K3 surface in $\mathbb{P}^4$ (see Sect. 2.3). Let us denote by $\mathcal{G}$ the family of these sextic surfaces; it is not hard to see that it is 4-dimensional. The motive of a desingularization of such a surface has been recently proven to be finite dimensional and of abelian type [ILP]. This allows us to show that the motive of the corresponding cubic fourfold is Schur-finite inside Voevodsky’s triangulated category of motives $\mathcal{DM}_{\mathbb{Q}}(\mathbb{C})$.

Theorem 0.2. Let $S$ be any surface from the family $\mathcal{G}$, and $X$ the cubic fourfold obtained via $\psi_X$, then $X$ has Schur finite motive in $\mathcal{DM}_{\mathbb{Q}}(\mathbb{C})$, belonging to the subcategory $\mathcal{DM}_{\mathbb{Q}}^\text{Ab}$, generated by the motives of curves.

Plan of the paper. In Sect. 1 we show that every K3 surface $S$ in the first family $\mathcal{F}$ has a motive of abelian type and there is a Kuga-Satake correspondence between $h(S)$ and the motive of a Prym variety $P$ of genus 4. Then we describe the second family $\mathcal{G}$ of K3 surfaces $S$ in $\mathbb{P}^4$ with 15 nodes and show that the motive of a desingularization of $S$ is of abelian type.

In Sect. 2 we show how to construct cubic fourfolds from K3 surfaces from the families $\mathcal{F}$ and $\mathcal{G}$, and show that their motives are of Abelian type as well.

We would like to thank Igor Dolgachev, Alice Garbagnati and Sandro Verra for very appreciated conversations and suggestions about the topics of this paper.

1. K3 surfaces

Known examples of K3 surfaces with a motive of abelian type are the Kummer surfaces, K3 surfaces with Picard rank $\geq 19$ (see [Ped]). In this paper we consider two families $\mathcal{F}$ and $\mathcal{G}$ of K3 surfaces with Picard rank 16, constructed in [GS].

In [GS] the authors construct families of K3 surfaces $S$, with $\rho(S) = 16$, admitting a symplectic action of the group $G = (\mathbb{Z}_2)^4$. The following lemma shows that, the transcendental motive $t_2(S)$, as defined in [KMP], only depends on the motive of the quotient surface $S/G$. Note that the motive $h(S/G)$ can be represented in $M_{\text{rat}}(\mathbb{C})$ see [Fu, 16.1.13].

Lemma 1.1. Let $S$ be a K3 surface over $\mathbb{C}$ with a finite group $G$ of symplectic automorphisms. Let $Y$ be a minimal desingularization of the quotient surface $S/G$. Then

$$t_2(S) \simeq t_2(Y)$$

in $M_{\text{rat}}(\mathbb{C})$.

Proof. From the results in [Huy 2], every symplectic automorphism $g \in G$ acts trivially on $A_0(S)$, so that $A_0(S)^G = A_0(S)$. Since $G$ is symplectic there are a finite number $\{P_1, \ldots, P_k\}$ of isolated fixed points for $G$ on $S$. Let $Y$ be a
minimal desingularization of the quotient surface $S/G$. The maps $f : S \to S/G$ and $g : Y \to S/G$ yield a rational map $S \dashrightarrow Y$ which is defined outside $\{P_1, \ldots, P_k\}$. Since $t_2(-)$ is a birational invariant for smooth projective surfaces we get a map $\theta : t_2(S) \to t_2(Y)$ such that $\theta$ is the map onto a direct summand, see [Ped, Prop.1]. Hence we can write

$$t_2(Y) = t_2(S) \oplus N.$$ 

Since $A_i(t_2(S)) = 0$, for $i \neq 0$, and $A_0(S)^G = A_0(S)$, we get $A_i(N) = 0$, for all $i$. From [GG, Lemma 1] we get $N = 0$, hence $t_2(S) = t_2(S)^G \simeq t_2(Y)$.

\[1.1. \textbf{Intersection of three quadrics.}\] The moduli space of K3 surfaces with Picard number 16, admitting a symplectic action of the group $G = (\mathbb{Z}_2)^4$, has a countable number of connected components of dimension 4. One connected component $\mathcal{F}$ maybe realized by considering a complete intersection $S$ of three quadrics $Q_1, Q_2, Q_3$ in $\mathbb{P}^5$, see [GS, 10.2]. Let us fix the three quadric equations

\begin{equation}
\sum_{0 \leq i \leq 5} a_i x_i^2 = 0; \quad \sum_{0 \leq i \leq 5} b_i x_i^2 = 0; \quad \sum_{0 \leq i \leq 5} c_i x_i^2 = 0
\end{equation}

with complex parameters $a_i, b_i, c_i$ and $i = 0, \ldots, 5$. The group $G$ is realized as the transformations of $\mathbb{P}^5$ changing an even number of signs in the coordinates $\{x_0 : x_1 : x_2 : x_3 : x_4 : x_5\}$. The dimension of the family $\mathcal{F}$ of these K3 surfaces is 4, see [GS, 10.2].

By Lemma 1.1 we get $t_2(S) = t_2(Y)$, with $Y$ a desingularization of $S/G$. An important feature of these surfaces is described in [Lat 3, Thm.3.1].

\textbf{Theorem 1.3.} A K3 surface belonging to the family $\mathcal{F}$ has a motive of abelian type.

More precisely one can show, using the results in [Para], that the motive $h(S)$ belongs to the subcategory of $\mathcal{M}_{\text{ram}}^{ab}$ generated by the motive of a Prym variety associated to the surface $Y$. By [GS, 10.2] the quotient $S/G$ is a double cover of $\mathbb{P}^2$ branched at six lines $l_i$ meeting in 15 points. The vector space $NS(Y) \otimes \mathbb{Q}$ of the desingularization $Y$ of $S/G$ is generated by the 15 classes $E_{i,j}$ of the 15 exceptional curves over the intersection points $l_i \cap l_j$ and by the class $h$ of the inverse image of a general line in $\mathbb{P}^2$. The motives of the surfaces $S$ and $Y$ have a Chow-K"unneth decomposition as in [KMP, 7.2.2]

\begin{align}
(1.4) \quad & h(S) = 1 \oplus L^2 \otimes \rho(S) \oplus t_2(S) \oplus L^2; \\
(1.5) \quad & h(Y) = 1 \oplus L^2 \otimes \rho(Y) \oplus t_2(Y) \oplus L^2;
\end{align}

where $\rho(S) = \rho(Y) = 16$ and $t_2(Y) = t_2(S)$, by 1.1. Therefore $h(Y) = h(S)$. By the results in [Para] there exists a surface $W$ which is a desingularization of the quotient $(C \times C)/F$ with $C$ a curve and $F$ a finite group such that $Y = W/i$, where $i$ is a symplectic involution. The curve $C$ has genus 5 and has an automorphism $h$ of order 4 such that the quotient $C/h$ is an elliptic curve $E$. The finite group $F$ acting on $C \times C$ is generated by the automorphism $(h, h^{-1})$ and the involution $(c_1, c_2) \to (c_2, c_1)$. Let $P$ be the connected component of the identity in the kernel
of the natural homomorphism \( \text{Jac} \ C \rightarrow \text{Pic} \ E \). The Prym variety \( P \) is an abelian variety of genus 4 and the Kuga-Satake variety \( K(Y) \) is a sum of copies of \( P \).

**Proposition 1.6.** The motive \( h(S) \) belongs to the subcategory of \( \mathcal{M}_{\text{rat}}^{\text{Ab}} \) generated by the motive \( h(P) \).

**Proof.** By the Main Theorem in [Para] there exists an algebraic cycle \( \Gamma \in A^2(P \times Y) \) such that the associated map \( \Gamma^* : H^2(Y, \mathbb{Q}) \rightarrow H^2(P, \mathbb{Q}) \) induces an inclusion between the lattices of transcendental cycles. Therefore the vector space \( H^2_{tr}(Y, \mathbb{Q}) \) is a direct summand of \( H^2_{tr}(P, \mathbb{Q}) \). Since \( P \) is an abelian variety its Chow motive \( h(P) \) has a C-K decomposition

\[
    h(P) = \sum_{0 \leq i \leq 8} h_i(P).
\]

The correspondence \( \Gamma \in A^4(P \times Y) \) gives a map \( h(P) \rightarrow h(Y) \), in the covariant category \( \mathcal{M}_{\text{rat}}(\mathbb{C}) \). By composing with the inclusion \( h_2(P) \subset h(P) \) and the projection \( h(Y) \rightarrow t_2(Y) \) we get a map of motives

\[
    \gamma : h_2(P) \rightarrow t_2(Y)
\]

such that the corresponding map at the level cohomology is split surjective. The motives \( h(P) \) and \( t_2(Y) = t_2(S) \) lie in the subcategory \( \mathcal{C} \subset \mathcal{M}_{\text{rat}}(\mathbb{C}) \) generated by finite dimensional motives. Let \( \mathcal{N} \) be the largest tensor ideal in \( \mathcal{C} \) such that the quotient category \( \mathcal{C}/\mathcal{N} \) is semi simple. The ideal \( \mathcal{N} \) corresponds to numerical equivalence of cycles, the functor \( \mathcal{C} \rightarrow \mathcal{C}/\mathcal{N} \) is conservative and reflects split epimorphisms, see [AK, 1.4.4 and 8.2.4]. Therefore the map \( \gamma \) is split surjective in \( \mathcal{M}_{\text{rat}}(\mathbb{C}) \). This proves that \( t_2(Y) = t_2(S) \) is a direct summand of \( h_2(P) \), hence the motive \( h(S) = h(Y) \) lies in the subcategory of \( \mathcal{M}_{\text{rat}}^{\text{Ab}} \) generated by the motive of the Prym variety \( P \). \( \square \)

1.2. **Degree six surfaces in \( \mathbb{P}^4 \).** A further example of a family \( \mathcal{G} \) of K3 surfaces with a motive of abelian type is given by degree 6 surfaces \( S \) in \( \mathbb{P}^4 \) with 15 ordinary nodes. The desingularization of these surfaces has Picard rank 16 as well. By a result in [GS, 8.2] these surfaces are the quotient of a K3 surface \( X \) by the action of a symplectic group \( G \simeq (\mathbb{Z}_2)^4 \) of automorphisms. We have a diagram

\[
    \begin{array}{ccc}
    \tilde{X} & \longrightarrow & X \\
    \downarrow \pi & & \downarrow f \\
    Y & \longrightarrow & S \\
    \end{array}
\]

where \( \tilde{X} \) is the blow-up of the fixed points under the action of \( G \) and \( Y \) contains 15 rational curves coming from the resolution of the singularities of \( S = X/G \). The map \( \pi \) is 16 : 1 outside the branch locus. By Lemma 1.1, the maps in (1.8) induce an isomorphism of motives \( h(X) \simeq h(Y) \).

The results of Paranjape have been recently extended in [ILP, Cor. 6.16].

**Theorem 1.9.** Let \( S \) be general K3 surface of degree 6 with 15 ordinary nodes, i.e singularities of type \( A_1 \). Then the motive of a desingularization of \( S \) is finite dimensional and of abelian type.

In particular every surface in the family \( \mathcal{G} \) has a motive of abelian type.
2. Cubic fourfolds with associated K3 surfaces in \( \mathcal{F} \) or \( \mathcal{G} \)

Not many families of cubic fourfolds with finite dimensional motive or of abelian type are known in the literature (see for example [Lat 1, Lat 2, ABP]).

In this section we use finite dimensionality of K3 surfaces belonging to the families \( \mathcal{F} \) and \( \mathcal{G} \) to construct cubic fourfolds with a finite dimensional motive. In Thm. 2.3 we consider a family of special cubic fourfolds containing a plane, hence belonging to \( \mathcal{C}_8 \). On the other hand, in Prop. 2.7 we study cubic fourfolds associated to K3 surfaces belonging to \( \mathcal{G} \). In the second case the motive of the singular cubic 4-fold is Schur-finite in Voevodsky’s triangulated category of motives \( \mathcal{DM}_{\mathbb{Q}}(\mathbb{C}) \).

2.1. Cubics with associated octic K3 surface. As it is well known, the Chow motive of a smooth cubic fourfold \( X \subset \mathbb{P}^5 \) has a Chow-K"unneth decomposition as follows:

\[
(2.1) \quad h(X) = 1 \oplus L \oplus (L^2)^{\oplus \rho_2} \oplus t(X) \oplus L^3 \oplus L^4,
\]

where \( \rho_2 \) is the rank of \( A^2(X) \) and all the summands of \( h(X) \), but possibly \( t(X) \), are finite dimensional and of abelian type. The motive \( t(X) \) is the transcendental motive of \( X \) (see [BP] for more details).

Let us now consider the following family of smooth cubic fourfolds. Let \( Q_1, Q_2 \) and \( Q_3 \) the three quadrics defined in Eq. 1.2. We are interested in smooth cubic fourfolds that are defined by the homogeneous equations

\[
(2.2) \quad F(X) := L_1Q_1 + L_2Q_2 + L_3Q_3 = 0,
\]

where the \( L_i \) are generic linear forms such that the cubic polynomial \( F(X) \) is non-singular.

**Theorem 2.3.** Smooth cubic fourfolds defined by equations as in (2.2) have a finite dimensional Chow motive of abelian type.

**Proof.** Let \( X \) be a smooth cubic fourfold defined by equation (2.2). The three linear forms \( L_i \) cut out a projective plane \( P := \{L_1 = L_2 = L_3 = 0\} \), that is contained in \( X \), hence \( X \) belongs to the Hassett divisor \( \mathcal{C}_8 \), parametrizing cubic fourfolds containing a plane. Let \( S \) be the K3 surface defined by the intersection of the 3 quadrics \( Q_1, Q_2, Q_3 \). The quadrics \( Q_i \) define a linear system of dimension two \( |Q| = \mathbb{P}^2 \). This projective plane naturally contains a discriminant curve \( D \) of degree 6, that parametrizes singular quadrics. The curve \( D \) defines another K3 surface \( T \) which is the minimal resolution of singularities of the double cover of \( \mathbb{P}^2 \) ramified in \( D \). The K3 surface \( T \) is the moduli of sheaves \( \mathcal{E} \) on \( S \) with rank \( r = 2 \), Euler characteristic \( \chi = 4 \) and first Chern class \( c_1(\mathcal{E}) = H \), where \( H \) is the primitive polarization of \( S \) of degree 8, see [MN]. It is a well known result due to Mukai and Orlov saying that two K3 surfaces are derived equivalent if and only if one is a moduli space of stable sheaves on the other, see [Huy 1]. Therefore the category \( \mathcal{D}^b(S) \) is equivalent to \( \mathcal{D}^b(T) \). By [Huy 3] the two K3 surfaces \( S \) and \( T \) have isomorphic motives. Therefore the transcendental motives \( t_2(S) \) and \( t_2(T) \) are isomorphic in \( \mathcal{M}_{rat}(\mathbb{C}) \).

If we project off \( P \), the cubic fourfold \( X \) is displayed as (birational to) a quadric surface fibration \( \pi : \tilde{X} \to |Q| \simeq \mathbb{P}^2 \). Here \( \tilde{X} \) is the blow-up of \( X \) along \( P \). The
fibre of $\pi$ over a point $x \in |Q| = \mathbb{P}^2$ is the residual surface of the intersection $\mathbb{P}^3_{<r,p>} \cap X$. It is straightforward to see that the discriminant divisor of $\pi$ is once again the sextic curve $D \subset \mathbb{P}^2$. The double cover $S' \rightarrow \mathbb{P}^2$ branched along $D$ is a (possibly singular) K3 surface, whose minimal desingularization is the K3 surface $T$. The motive $h(X)$ has a Chow-Künneth decomposition as (2.2). By [Büll, Sect.3] there is an isomorphism $t(X) \simeq t_2(T)(1)$. Therefore the motive $h(X)$ is of abelian type if and only if $t_2(T)$ is of abelian type. From the isomorphism

$$t_2(T) \simeq t_2(S)$$

and from [Lat 3] we get that $t_2(T)$ is of abelian type. Therefore the motive of $X$ is of abelian type in $\mathcal{M}_{rat}(\mathbb{C})$. \begin{proof}

\textbf{Proposition 2.4.} Cubic fourfolds defined as in (2.2) form a 4-dimensional family.

\textit{Proof.} The octic K3 surfaces from the family $\mathcal{F}$ are associated to these cubic fourfolds, in the sense of Hodge theory [Has, Sect. 5]. The moduli map that associates the K3 to a cubic fourfolds is injective, hence we conclude. \end{proof}

\textbf{2.2. Singular cubics and 15-nodal K3 surfaces.} The relation between smooth cubic fourfolds and polarized K3 surfaces, proved in [Has], has been recently extended to the case of fourfolds with isolated ADE singularities. A.K. Stegmann in [Steg] constructs the moduli space of cubic fourfolds with a certain combination of isolated ADE singularities as a GIT quotient and compares it to the moduli space of certain quasi-polarizes K3 surface of degree 6, proving that the moduli spaces are isomorphic. Here we consider the motive of a cubic fourfold $X$ with isolated singularities associated to a K3 surfaces of degree 6 with 15 nodes belonging to $\mathcal{G}$. Since we are now working with singular varieties, we need to change the category of motives from $\mathcal{M}_{rat}(\mathbb{C})$ to Voevodsky’s triangulated category of motives $\text{DM}_Q(\mathbb{C})$.

Let $X \subset \mathbb{P}^5$ be a cubic fourfold with isolated singularities. Projection from a double point $p \in X$ gives a birational map $\pi_p : X \rightarrow \mathbb{P}^4$ which can be factored as

$$\tilde{X}_p \xrightarrow{q_1} X : \tilde{X}_p \xrightarrow{q_2} \mathbb{P}^4$$

where $q_1$ is the blow-up of the singular point $p$ and $q_2$ is the blow-down of the lines contained in $X$ passing through $p$. These lines are parametrized by a normal surface $S_p$ of degree 6 in $\mathbb{P}^4$. The surface $S_p$ is a complete intersection of a quadric $Q_p$ and a cubic $C_p$. The quadric $Q_p$ is completely determined by $S_p$ while the cubic $C_p$ in $\mathbb{P}^4$ containing $S_p$ is uniquely determined modulo those cubics containing the quadric $Q_p$. Conversely, starting from a (2, 3)-complete intersection in $\mathbb{P}^4$, the blow up of the sextic surface followed by the contraction of the strict transform of the quadric yields a singular cubic fourfold with a double point, which is the image of the quadric. The type of this singularity depends on the rank of the quadric.

The cubic fourfold $X$ and the complete intersection $S_p$ can both have singularities, and still the birational transformation here above holds true. One of the main results in [Steg] is that there is a natural correspondence between singularities on the cubic fourfold and on the sextic surface, including the type of singularities.

Suppose that the surface $S_p$ has only isolated singularities. Since the singularities of $S_p$ are simple the minimal resolution of $S_p$ is a K3 surface. If $S_p$ is singular at a point $y$ then either $y = \pi_p(p')$, with $p' \neq p$ a singular point of $X$, or $y$ is singular for the quadric $Q_p$. In this second case the cubic $C_p$ cannot be singular at $y$,
because otherwise $X$ would be singular along the line $\overline{pj}$ while $X$ has only isolated singularities. Therefore the singularities of $X - \{p\}$ are in 1-1 correspondence with the singularities of $S_p$ not contained in $\text{Sing} Q_p$. Let $E_p$ be the exceptional divisor of the the blow-up $\tilde{X}_p \to X$ at $p$. Then $E_p$ is isomorphic to $Q_p$ and the singularities of $\tilde{X}_p$ on $E_p$ correspond to the singularities of $S_p$ which are contained in $\text{Sing} Q_p$. If $X$ has only a single $A_1$ singularity $p$ then the surface $S_p$ is a smooth K3 surface, see \cite[5.1 and 5.2]{Steg}. Note that, due to a result of Varchenko (see \cite[Theorem on the Upper Bound, p. 2781]{Varch}) the maximal number of isolated singularities which can occur on a cubic fourfold is 15.

Let $\text{DM}_Q(C)$ be Voevodsky’s triangulated category of motives (with $Q$-coefficients). There is a fully faithful embedding

\[
F : \mathcal{M}_{rat}(C) \to \text{DM}_Q(C).
\]

For every complex variety $V$ one can define a motive $M(V) \in \text{DM}_Q(C)$ and every blow-up diagram

\[
\begin{array}{ccc}
E & \xrightarrow{j} & Y = \text{Bl}_Z \\
\downarrow & & \downarrow \\
Z & \xrightarrow{j} & X
\end{array}
\]

induces a distinguished triangle in $\text{DM}_Q(C)$, see [Voev, (4.1.3)]

\[
(2.6) \quad M(E) \xrightarrow{\sigma} M(Z) \oplus M(Y) \xrightarrow{j \times \sigma} M(X) \longrightarrow M(E)[1]
\]

where $E$ is the exceptional divisor.

**Proposition 2.7.** Let $X \subset \mathbb{P}^5$ be a cubic fourfold with isolated singularities. Assume that for some singular point $p$ of $X$ the associated surface $S_p$ belongs to the family $\mathcal{G}$. Then the motive $M(X) \in \text{DM}_Q(C)$ is Schur-finite and lies in the triangulated tensor category $\text{DM}^\text{Ab}_Q(C)$ generated by the motives of curves.

**Proof.** Since $\tilde{X}_p$ is the blow-up of $\mathbb{P}^4$ along $S_p$ there is a diagram

\[
\begin{array}{ccc}
E_{S_p} & \longrightarrow & \tilde{X}_p = \text{Bl}_{S_p} \\
\downarrow & & \downarrow \\
S_p & \longrightarrow & \mathbb{P}^4
\end{array}
\]

where $E_{S_p}$ is the exceptional divisor of the blow-up $\tilde{X}_p \to \mathbb{P}^4$. Therefore from 2.6 we get a distinguished triangle in $\text{DM}_Q(C)$

\[
(2.8) \quad M(E_{S_p}) \longrightarrow M(S_p) \oplus M(\tilde{X}_p) \longrightarrow M(\mathbb{P}^4) \longrightarrow M(E_{S_p})[1].
\]

The exceptional divisor $E_{S_p}$ is a $\mathbb{P}^4$-bundle over $S_p$. By 1.1 and 1.9 the motive $h(S_p)$ equals the motive of a smooth K3 surface and is finite-dimensional. Therefore its image in $\text{DM}_Q(C)$ under the functor 2.5 is finite dimensional. By the projective bundle theorem, that is valid also in $\text{DM}_Q(C)$, see [Voev, (4.1.11)] , we get

\[
M(E_{S_p}) \simeq M(S_p)(1)[2]
\]
Therefore the motives $M(E_{S_i})$, $M(S_p)$ and $M(P^4)$ are finite dimensional in $\text{DM}_Q(C)$. Finite dimensional objects in the triangulated category $\text{DM}_Q(C)$ are also Schur-finite and Schur-finiteness has the two out of three property in $\text{DM}_Q(C)$, see [Maz, Prop. 5.3]. Therefore from 2.8 we get that the motive $M(\tilde{X}_p)$ is Schur-finite.

The blow-up $q_1 : \tilde{X}_p \rightarrow X$ induces a distinguished triangle in $\text{DM}_Q(C)$ as in 2.6

$$M(E_p) \rightarrow M(\{p\}) \oplus M(\tilde{X}_p) \rightarrow M(X) \rightarrow M(E_p)[1]$$

The exceptional divisor $E_p \subset \tilde{X}_p$ is isomorphic to the quadric $Q_p \subset P^4$. Therefore $M(E_p)$ is Schur-finite in $\text{DM}_Q(C)$. The motives $M(E_p)$, $M(\{p\})$ and $M(\tilde{X}_p)$ are Schur-finite. By the the two out of three property we get that the motive $M(X)$ is Schur-finite. Moreover, since the motives of $E_p$, $\{p\}$ and $\tilde{X}_p$ are of abelian type in $M_{\text{rat}}(C)$, the motive $M(X)$ belongs to the triangulated tensor subcategory $\text{DM}^\text{ab}_Q(C)$ of $\text{DM}_Q(C)$ generated by the motives of curves, see [Ay, Prop. 1.5.6].

2.3. A family of cubic fourfolds with associated 15-nodal, degree six K3 surface. The following construction gives an example of a dimension 4 family of singular cubic fourfolds, obtained from K3 surfaces belonging to the family $G$ of 15-nodal K3 surfaces. Therefore, by Prop. 2.7 the motive of all these fourfolds in $\text{DM}_Q(C)$ is Schur-finite.

The mere existence of a family of dimension 4 of 15-nodal K3 surfaces of degree 6 is assured by [GS, Thm. 8.3], but for the lack of a precise reference of our knowledge, we give here a geometric construction of such surfaces.

Proposition 2.9. A sextic, 15-nodal complete intersection surface in $P^4$ is a double cover of $P^2$ ramified along a degenerate sextic $F$ given by two conics and two lines. Such surfaces form a 4-dimensional family. The sextic $F$ has two everywhere tangent conics.

Proof. Let us consider such a K3 surface $S \subset P^4$ and project it onto $P^2$ from a line joining two nodes. Call them $q_1$ and $q_2$. Since both the nodes have multiplicity 2, this exhibits $S$ as a double cover of the plane, ramified along a sextic curve $F$.

For the generic desingularized K3 surface $\tilde{S}$ in our family, we have that $NS(S) \otimes Q$ is generated by the classes $H$, $E_1$, ..., $E_{15}$, where $H^2 = 6$ is the natural polarization and the $E_i$ are the exceptional divisors with $E_i^2 = -2$. Hence the polarization on $\tilde{S}$ that gives the map to $P^2$ is $|H - E_1 - E_2|$, and the family has dimension 4.

Since the projective model in $P^4$ has 15 nodes, the ramification sextic cannot be smooth. In fact, $F$ has 13 nodes, that are the images of the nodes $q_3$, ..., $q_{15}$, not contained inside the line center of projection. Hence it is reducible, and it splits as two conics and two lines. The images of the two exceptional divisors $E_1$ and $E_2$, exactly as in the well-known case of the double cover of $P^2$ defined by a Kummer surface, are sent to two conics $C_1$ and $C_2$ in $P^2$, which are everywhere tangent to $F$. That is, $C_i \cap F = 2\Delta_i$, where the $\Delta_i$ are degree 6 divisors. The strict transform in $\tilde{S}$ of each $C_i$ is the union of two $(-2)$-curves that intersect in 6 points. One of these two curves corresponds to (one of the) node(s) from which we project.

Remark 2.10. We also observe that a naive parameter count gives $5 + 5 + 2 + 2 - 8 = 6$ for the two lines and two conics up to projectivity. The remaining two conditions to descend to the dimension of the family are given by the tangency conditions.
Now, the construction of the corresponding family of cubic fourfolds is the same as in the one nodal case (see for example [Kuz, Lemma 5.1]). Take any K3 surface $S \subset \mathbb{P}^4$ in the family $\mathcal{G}$ and consider the map

$$\psi: \mathbb{P}^4 \dashrightarrow \mathbb{P}^5,$$

given by the full linear system of cubics through $S$. This induces a birational transformation that consists first in blowing-up $S$, and then contract to one point all the trisecant lines to $S$. The union of these trisecant lines is exactly the strict transform of the only quadric through $S$, which is then contracted on a (singular) point of $X$, giving the birational transformation already described in Sect. 2.2. By this construction and Prop. 2.7, we obtain the following.

**Theorem 2.12.** Let $S \subset \mathbb{P}^4$ be any K3 surface from the family $\mathcal{G}$, and let $X$ be a singular cubic fourfold obtained from $S$ via the map $\psi$ of Eq. 2.11. Then $X$ has Schür finite motive in $\text{DM}_Q(C)$ and belong to the subclass $\text{DM}^{Ab}_Q$.

**References**

[An] Y. André, *Pour une théorie inconditionnelle des motifs*, Publ. Math. IHÉS (1996), 1-48.

[ABP] H. Awada, M. Bolognesi, and C. Pedrini, *Families of special cubic fourfolds with Chow motive of abelian type*, arXiv:2007.07193v1 [math.AG], 14 Jul 2020.

[AK] Y. André and B. Kahn, *Nilpotence, radicaux et structures monoidales*, Rend. Sem. Mat. Univ. Padova 108 (2002), 107-291, With an appendix by Peter O’Sullivan.

[Ay] *The Motivic vanishing cycles and the conservative conjecture*, in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS 344, Cambridge University Press, 2008.

[BP] M. Bolognesi and C. Pedrini, *The transcendental motive of a cubic fourfold*, J. of Pure and App. Algebra, 224, nr. 8, (2020).

[Büll] T. H. Büll, *Motives of moduli spaces on K3 surfaces and of cubic fourfolds*, Manuscripta Math. 161 (2020), no. 1-2, 109-124.

[Fu] W. Fulton *Intersection Theory*, Springer-Verlag, Heidelberg-New-York (1984).

[GS] A. Garbagnati and A. Sarti, *Kummer surfaces and K3 surfaces with a $\mathbb{Z}/2\mathbb{Z}$ symplectic action*, Rocky Mountain J. Math. Volume 46, Number 4 (2016), 1141-1205.

[GG] S. Gorchinskiy and V. Gulestikii, *Motives and representability of algebraic cycles on threefolds over a field*, J. Alg. Geometry, 21 (2012) 343-373.

[Has] B. Hassett, *Special cubic 4-folds*, Compositio Math. (2000) no.1, 1-23.

[Huy 1] D. Huybrechts, *Derived and Abelian equivalence of K3 surfaces*, J. of Alg. Geometry, 17, (2008) 375-400.

[Huy 2] D. Huybrechts, *Symplectic automorphisms of K3 surfaces of arbitrary order*, Math. Res. Lett., 19 (2012), 947-951.

[Huy 3] D. Huybrechts, *Motives of derived equivalent K3 surfaces*, Abhandlungen aus dem math. Sem. Univ. Hamburg, 88, (2018), 201-207.

[ILP] C. Ingalls, A. Logan, and O. Patashnick, *Explicit coverings of families of elliptic surfaces by square of curves*, arXiv:2009.07807v1 [math.AG] 16 Sep 2020.

[KMP] B. Kahn, J. Murre, and C. Pedrini, *On the transcendental part of the motive of a surface*, pp. 143-202 in "Algebraic cycles and Motives Vol II", London Math. Soc. LNS 344, Cambridge University Press, 2008.

[Kuz] A. Kuznetsov, *Derived categories of cubic fourfolds*, in: Bogomolov F., Tschinkel Y. (eds) "Cohomological and Geometric Approaches to Rationality Problems". Progress in Mathematics, vol 282. Birkhäuser Boston.

[Lat 1] R. Laterveer, *A family of cubic fourfolds with finite-dimensional motive*, J. Math. Soc. Japan, Volume 70, no.4 (2018) 1453-1473.

[Lat 2] R. Laterveer, *A remark on the motive of the Fano variety of a cubic fourfold*, Ann. Math. Qué. 41 (2017), no. 1, 141–154.

[Lat 3] R. Laterveer, *A family of K3 surfaces having finite-dimensional motive*, Arch.Math., 108 (2016) 515-524.
[MN] C. Madonna and V. Nikulin, *On a classical correspondence between K3 surfaces*, Proc. Steklov Math. Inst. vol. 241 (2003) 120-153

[Maz] C. Mazza, *Schur Functors and Motives*, K-Theory 33 (2004) 89-106.

[Para] K. Paranjape, *Abelian varieties associated to certain K3 surfaces*, Compositio Mathematica 68 (1988) 11-22.

[Ped] C. Pedrini, *On the finite dimensionality of a K3 surface*, Manuscripta Math. 138 (2012), 59–72.

[Steg] A-K Stegmann, *Cubic fourfolds with ADE singularities and K3 surfaces*, Doktorin dissertation, Universität Hannover, (2020).

[Varch] A. N. Varchenko, *Asymptotic integrals and Hodge structures*, J. Math. Sci. 27, (1984).

[Voev] V. Voevodsky, *Triangulated category of motives over a field*, in ”Cycles, Transfers and Motivic Homologies Theory”, Annals of Math. Studies 143, Princeton University Press (2000).