Conformal Transformations of the Wigner Function and Solutions of the Quantum Corrected Vlasov Equation*

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Abstract

We study conformal properties of the quantum kinetic equations in curved spacetime. A transformation law for the covariant Wigner function under conformal transformations of a spacetime is derived by using the formalism of tangent bundles. The conformal invariance of the quantum corrected Vlasov equation is proven. This provides a basis for generating new solutions of the quantum kinetic equations in the presence of gravitational and other external fields. We use our method to find explicit quantum corrections to the class of locally isotropic distributions, to which equilibrium distributions belong. We show that the quantum corrected stress–energy tensor for such distributions has, in general, a non–equilibrium structure. Local thermal equilibrium is possible in quantum systems only if an underlying spacetime is conformally static (not stationary). Possible applications of our results are discussed.

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I Introduction

In a regime where the curvature of a spacetime is less than the Planck scale, a semi-
classical approximation to the quantum gravity is considered to be sufficient [1]. This
approximation is valid for "nearly" classical systems, for which quantum fluctuations in the
stress-energy tensor are small enough [3]. In such cases the method of Wigner functions
[4, 5, 6] is particularly convenient. It allows one to derive, from quantum field equations,
quantum kinetic equations for Wigner functions. Iterative solutions of the kinetic equations
give quantum corrections to classical distribution functions, which are used to compute ex-
pectation values of local observables such as the stress-energy tensor, the number–flux vector
etc. The standard definition of the flat–space Wigner function exploits the Wigner–Weyl
transformation [3] which fails in curved spacetime. Different approaches have been proposed
to extend the notion of the Wigner–Weyl transformation to curved spacetime. In Ref. [8], a
Wigner–Weyl–type transformation is defined by considering the bundle of geodesic lines on
a manifold. Ref. [9] utilizes Riemann normal coordinate systems and adiabatic expansions
similar to one used by [12] for studying ultraviolet properties of Green functions in curved
spacetime. In Ref. [10], a covariant Wigner function is defined by using the formalism of
tangent bundles. All three approaches mentioned, being intrinsically different, are locally
equivalent to each other [11], in the sense of adiabatic expansions, and yield the same quan-
tum corrected Vlasov–type kinetic equations. The mathematical structure of the equations
is rather complicated, even at lowest adiabatic order. This makes difficult the study of
properties of generic solutions of the quantum kinetic equations in an arbitrary background
spacetime. No doubt that any explicit computation of the Wigner function may be useful
for better understanding of quantum kinetic processes in gravitating systems.

If a spacetime manifold is endowed with some symmetries then both classical and quantum kinetic equations can often be solved explicitly. By now, only in a very few cases explicit solutions of the quantum corrected Vlasov equation in curved spacetime have been obtained. They include the Friedmann–Robertson–Walker cosmology [13, 14], stationary spacetimes [15] and some others [16]. A common property of those solutions is that the quantum corrections are local in phase space. Once a solution of the Vlasov equation (a classical distribution function) is known, one can compute, to a given adiabatic order, the Wigner function which corresponds to this solution, by applying a certain finite order differential operator to the classical distribution function. This property implies that one can express the expectation value of the stress–energy tensor or of any other local observable in terms of moments of the classical distribution function multiplied by some local geometrical quantities. This facilitates to a great extent the analysis of the back–reaction problem in those cases.

The natural question arises how symmetries of a spacetime manifest themselves in Wigner functions. In the present paper, we consider a particular, but important, type of geometrical symmetries, the conformal symmetry. For simplicity, attention will be restricted to the case of a scalar field, spin–$\frac{1}{2}$ and spin–1 fields can be treated similarly [11, 16]. The conformal symmetry is usually understood as the invariance of the generalized massless Klein–Gordon equation (with the term $\frac{1}{6} R \varphi$ involved [1]) under the conformal rescaling of the metric and of the scalar field: $g_{\alpha \beta} \rightarrow a^2 g_{\alpha \beta}, \varphi \rightarrow \varphi/a$. If the scalar field is coupled to an external potential, the latter must also be transformed in such a way that the scalar field equation remains invariant under this extended group of conformal transformations.

Two different views at the conformal symmetry are possible. When the external po-
tential has a direct physical meaning, the conformal symmetry is a dynamical symmetry of the physical system under consideration. This is the case, for example, in the $\varphi^4$ theory where the expectation value $\langle \varphi^3 \rangle$ plays a role of an external potential for quantum perturbations.

The second point of view is formal. One can treat, for example, the mass of the scalar field as an external potential and use the conformal symmetry as a convenient tool for generating new solutions of the field equation [17].

In this paper, we study transformation properties of the covariant Wigner function under the conformal transformation. As we have mentioned, different curved-space extensions of the flat-space Wigner function are possible, all of them giving similar adiabatic expansions. The use of the tangent bundle technique is particularly convenient for exploring mathematical properties of the Wigner function. By using the formalism of tangent bundles we derive, to second adiabatic order, the transformation law for the Wigner function and prove the conformal invariance of the quantum-corrected Vlasov equation. We then use our result to find explicit quantum corrections to the class of distribution functions which are locally isotropic in momentum space, i.e. have the form $F_{cl}(x, p) = F(x, u^\alpha(x)p_\alpha)$. As it was shown by [18, 19], in the case where external fields other than gravitation are absent, such form of the distribution function imposes the following restriction on the underlying space-time: it must be or stationary or conformally static (with a special form of the conformal factor). In the presence of an external potential this condition has to be modified. We do this by using the conformal properties of the Vlasov equation. We also analyse the structure of the quantum corrected stress-energy tensor and number-flux vector, and show that the quantum corrections lead, in general, to non-vanishing heat flux and viscosity, even if the classical distribution implies local thermal equilibrium.
The paper is organized as follows. In Sec. II, we outline the formalism of the Wigner function in curved spacetime. Section III derives the transformation law for the Wigner function. The lowest order quantum corrections to the isotropic distributions are evaluated in Sec. IV, and the quantum corrected stress–energy tensor and number–flux vector are calculated in Sec. V. Section VI contains some remarks concerning possible applications of our results. In Appendix A, technical details used for the evaluation of the transformation law are presented, and in Appendix B, the conformal invariance of the quantum corrected Vlasov equation is proven. Appendix C recalls some properties of the stress–energy tensor.

Conventions used throughout are the following. Greek indices run from 0 to 3, Latin indices from 1 to 3. The signature for the metric tensor is (+ − − −). The Riemann tensor \( R^\mu_{\nu\alpha\beta} \) is defined by

\[
[\nabla_\alpha, \nabla_\beta] X^\mu = R^\mu_{\nu\alpha\beta} X^\nu .
\] (1.1)

The Ricci tensor is \( R_{\alpha\beta} = R^\nu_{\alpha\nu\beta} \), the scalar curvature \( R = R^\alpha_{\alpha} \).

II The covariant Wigner function

To keep the paper self contained, we shall outline our definition and some properties of the covariant Wigner function in curved spacetime [10, 11]. Let \( \varphi(x) \) be a scalar field on a spacetime manifold \( \mathcal{M} \) with the metric \( g_{\alpha\beta}(x) \), and \( T_x(\mathcal{M}) \) be the tangent space at a point \( x \). The horizontal lift of the covariant derivative operator \( \nabla_\alpha \) to the tangent bundle \( T(\mathcal{M}) \) is defined by [20]

\[
\hat{\nabla}_\alpha = \nabla_\alpha - \Gamma^\beta_{\alpha\nu} y^\nu \frac{\partial}{\partial y^\beta} ,
\] (2.1)
where $g^\alpha$ are components of a tangent vector pertaining to $T_x(M)$, and $\Gamma^\beta_{\alpha\nu}(x)$ are the Christoffel symbols. Let us introduce the horizontal lift of the scalar field:

$$\Phi(x, y) = \exp(g^\alpha \hat{\nabla}_\alpha) \varphi(x) .$$  \hspace{1cm} (2.2)

Then, the covariant Wigner function is defined as follows:

$$f(x, p) = (\pi \hbar)^{-4} \sqrt{-g(x)} \int_{T_x(M)} d^4 y \ e^{-2iy^\alpha p_\alpha/\hbar} \langle \Phi(x, -y) \Phi^\dagger(x, y) \rangle ,$$  \hspace{1cm} (2.3)

where the bracket stand for an ensemble averaging with some density matrix defined on a Cauchy hypersurface \[7\], the dagger indicates the hermitian conjugation.

If the field $\varphi(x)$ is a solution of the generalized Klein–Gordon equation:

$$\left( \hbar^2 \nabla_\alpha \nabla^\alpha - \frac{1}{6} \hbar^2 R(x) + V(x) \right) \varphi(x) = 0 ,$$  \hspace{1cm} (2.4)

then the Wigner function obeys the quantum corrected Liouville–Vlasov equation supplemented by the quantum corrected mass–shell constraint \[8, 9, 11\]. The explicit form of the equations to second adiabatic order is given in \[Appendix\]. We note here that a Wigner function satisfying these equations (to second adiabatic order) is represented as the series in derivatives of the Dirac $\delta$–function \[9, 11\]:

$$f(x, p) = \left( F_{cl}(x, p) + \hbar^2 F_{qu}(x, p) \right) \delta(\Omega)+\hbar^2 F_1(x, p) \delta'(\Omega)+\hbar^2 F_2(x, p) \delta''(\Omega)+\hbar^2 F_3(x, p) \delta'''(\Omega) ,$$  \hspace{1cm} (2.5)

with the argument of the $\delta$–function being

$$\Omega = g^{\alpha\beta}(x)p_\alpha p_\beta - V(x) .$$  \hspace{1cm} (2.6)

The functions $F_n(x, p)$, $n = 1, 2, 3$, are expressed in terms of the classical distribution function $F_{cl}(x, p)$, in such a way that the quantum corrected mass–shell constraint is satisfied
(see Appendix B), assuming that $F_{cl}(x, p)$ obeys the Vlasov equation:

$$\delta(\Omega) \left( \hat{\mathcal{L}} F_{cl}(x, p) \right) = 0 . \quad (2.7)$$

Here $\hat{\mathcal{L}}$ is the Liouville–Vlasov operator [21, 22, 24]:

$$\hat{\mathcal{L}} = p^\alpha D_\alpha + \frac{1}{2} V_{\alpha\beta} \frac{\partial}{\partial p_\alpha} , \quad (2.8)$$

$D_\alpha$ being the horizontal lift of the covariant derivative operator to the cotangent bundle [20]:

$$D_\alpha = \nabla_\alpha + \Gamma^\mu_{\alpha\beta} p_\mu \frac{\partial}{\partial p_\beta} . \quad (2.9)$$

The function $F_{qu}(x, p)$ is found by integrating the quantum transport equation (see Appendix B) along classical trajectories [9, 11].

Equation (2.7) defines the classical distribution function only on the mass shell $\Omega = 0$. This is sufficient so far as one is interested in classical observables defined on the mass shell [22]. In quantum kinetic theory the situation is different [11]. Though the function $F_{cl}(x, p)$ is tied up with the $\delta$–function in the expansion (2.5), its properties out of the mass shell influence upon both the off–shell quantum corrections, the $F_n$’s, and the on–shell quantum corrections, $F_{qu}(x, p)$. The latter becomes evident if we get rid of the $\delta$–function in Eq. (2.7).

The function $F_{cl}(x, p)$ must satisfy the generalized Vlasov equation:

$$\hat{\mathcal{L}} F_{cl}(x, p) = \Omega \Delta_F(x, p) , \quad (2.10)$$

$\Delta_F$ being an arbitrary function which is assumed to be non–singular on the mass shell.

Then the equation governing the evolution of $F_{qu}(x, p)$ explicitly involves the function $\Delta_F$ (see Appendix B). In the next section we shall see that, in general, one can not get rid of $\Delta_F$. Even if the right–hand side of Eq. (2.10) happens to vanish for some system, for systems
conformally related to the one under consideration, the Liouville–Vlasov operator annihilates $F_{cl}$ on the mass shell only.

### III The Wigner function and conformal transformations of a spacetime

Let us now consider the conformal transformation on a spacetime manifold (we shall use throughout an overline for quantities in the conformally related spacetime):

$$
\begin{align*}
ga_{\alpha \beta}(x) &= a(x)^2 \overline{g}_{\alpha \beta}(x), \\
\varphi(x) &= \overline{\varphi}(x)/a(x), \\
V(x) &= \overline{V}(x)/a(x)^2,
\end{align*}
$$

(3.1)
a(x) being an arbitrary smooth function. Equation (2.4) is known to be invariant under such a transformation [1]. If $\varphi(x)$ is a solution of Eq. (2.4) on the manifold $M$ with the metric $g_{\alpha \beta}(x)$ and in the presence of the external potential $V(x)$, then $\overline{\varphi}(x)$ is a solution of the conformally transformed equation on the manifold $\overline{M}$ with the metric $\overline{g}_{\alpha \beta}(x)$ and in the presence of the external potential $\overline{V}(x)$, and vice versa.

One can expect that the Wigner function $f(x, p)$ associated with the field $\varphi$ must be somehow connected to the Wigner function $\overline{f}(x, p)$ associated with the field $\overline{\varphi}$. In the classical limit ($\hbar \to 0$) the connection is very simple. The mass shell (2.6) is only scaled under the conformal transformation (we treat covariant components of the momentum vector, $p_\alpha$, and contravariant components of the vector $y^\alpha$ as parameters which are not changed under conformal transformations, therefore, making difference between the tangent and cotangent
spaces). Thus,

$$\Omega = \Omega \, a^{-2}.$$  \hspace{1cm} (3.2)

Next, under the conformal transformation (3.1), the Liouville–Vlasov operator is transformed as follows:

$$\hat{\mathcal{L}} = \hat{\mathcal{L}} \, a^{-2} + a_\alpha \frac{\partial}{\partial p_\alpha} \Omega ,$$  \hspace{1cm} (3.3)

where

$$a_\alpha = \frac{\partial}{\partial x^\alpha} \ln a .$$  \hspace{1cm} (3.4)

By comparing Eqs. (3.2), (3.3) with Eqs. (B2) and (B1), one can conclude that the transformation law for the Wigner function should be

$$f(x, p) = a(x)^2 \overline{f}(x, p) + \text{quantum corrections} .$$  \hspace{1cm} (3.5)

We could also arrive at Eq. (3.5) by writing the transformation law for the Liouville–Vlasov operator in the form

$$\hat{\mathcal{L}} = a^{-2} \hat{\mathcal{L}} + \Omega a_\alpha \frac{\partial}{\partial p_\alpha} .$$  \hspace{1cm} (3.6)

From Eqs. (2.7) and (3.6) it follows that the classical distribution function remains invariant under the conformal transformation. Then Eqs. (2.5) and (3.2) imply (3.5).

One can also see from Eq. (3.6) that, unlike Eq. (2.7), the off–shell Vlasov equation (2.10) is not invariant under the conformal transformation. Namely, the function $\Delta_F$ is changed as follows:

$$\Delta_F(x, p) = \overline{\Delta_F}(x, p) + a_\alpha \frac{\partial}{\partial p_\alpha} F_{cl}(x, p) .$$  \hspace{1cm} (3.7)

In order to find the quantum corrections in Eq. (3.5), we must turn to the definition of the covariant Wigner function (2.3). First, we write the transformation law for the operator
\( \hat{\nabla}_\alpha \), Eq. (2.1), when acting to a scalar function in the tangent bundle. Under the conformal transformation of the metric, Eq. (3.1), it is transformed as follows (cf. Eq. (B14)):

\[
\hat{\nabla}_\alpha = \hat{\nabla}_\alpha - a^\nu y^\alpha \frac{\partial}{\partial y^\nu} - a_\alpha y^\nu \frac{\partial}{\partial y^\nu} + y_\alpha a^\nu \frac{\partial}{\partial y^\nu}.
\]  

(3.8)

Let us now write the field (2.2) in the following way (compare Eq. (3.1)):

\[
\Phi(x, y) = \left( e^{y^\alpha \hat{\nabla}_\alpha} a^{-1} e^{-y^\alpha \hat{\nabla}_\alpha} \right) \left( e^{y^\alpha \hat{\nabla}_\alpha} e^{-y^\alpha \hat{\nabla}_\alpha} \right) \left( e^{y^\alpha \hat{\nabla}_\alpha} \varphi(x) \right).
\]

(3.9)

The last factor on the right-hand-side of Eq. (3.9) is, by the definition, \( \Phi(x, y) \). The second factor, which is an operator acting in the tangent bundle, can be expanded as follows (see Appendix A):

\[
\hat{Z}(x, y) := e^{y^\alpha \hat{\nabla}_\alpha} e^{-y^\alpha \hat{\nabla}_\alpha}
\]

\[
= 1 + A^\alpha \left( \frac{1}{2} \hat{\nabla}_\alpha - \frac{\partial}{\partial y^\alpha} \right) + B^\alpha \left( \frac{1}{3} \hat{\nabla}_\alpha - \frac{1}{2} \frac{\partial}{\partial y^\alpha} \right)
\]

\[
+ \frac{1}{2} A^\alpha A^\beta \left( \frac{1}{4} \hat{\nabla}_\alpha \hat{\nabla}_\beta - \frac{\partial}{\partial y^\beta} \hat{\nabla}_\alpha + \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) + \text{terms of higher orders},
\]

(3.10)

where \( A^\alpha \) and \( B^\alpha \) are given in Appendix A.

Lastly, the first factor on the right-hand side of Eq. (3.9) is a function which involves the conformal factor \( a(x) \) and its derivatives. The known formula gives the expansion:

\[
A(x, y) := e^{y^\alpha \hat{\nabla}_\alpha} a^{-1} e^{-y^\alpha \hat{\nabla}_\alpha}
\]

\[
= a^{-1} \left( 1 - y^\alpha a_\alpha + \frac{1}{2} (y^\alpha a_\alpha)^2 - \frac{1}{2} y^\alpha y^\beta a_{\alpha;\beta} \right) + \text{terms of higher orders},
\]

(3.11)

where \( a_{\alpha;\beta} := \nabla_\beta a_\alpha \). On the right-hand side of Eqs. (3.10), (3.11), we have kept only those terms which yield lowest order quantum corrections to Eq. 3.5.
We are now prepared to write down the transformation law for the Wigner function, correct to second adiabatic order. Before doing that we note the following identity [11]:

$$\frac{\partial}{\partial y^\alpha} \Phi(x, y) = \tilde{\nabla}_\alpha \Phi(x, y) + \text{terms of second or higher orders}.$$ (3.12)

Then, with the aid of the results of [Appendix A], we finally get:

$$f(x, p) = a^2 \left( 1 + \frac{\hbar^2}{4} a_{\alpha;\beta} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} + \frac{\hbar^2}{8} A^\alpha(\partial p) D_\alpha - \frac{\hbar^2}{24} B^\alpha(\partial p) p_\alpha \right) \tilde{f}(x, p) + \text{terms of higher adiabatic order},$$ (3.13)

where the operator $D_\alpha$ is conformally related to (2.9), and $A^\alpha(\partial p)$ and $B^\alpha(\partial p)$ are the following operators (cf. Eqs. (A2),(A3)):

$$A^\alpha(\partial p) = (2 a_{\mu;\nu} \delta^\alpha_\gamma - \bar{\sigma}^\alpha_{\mu} g_{\nu\gamma} - 8 a_{\mu} a_{\nu} \delta^\alpha_\gamma + 2 \bar{\sigma}^\alpha_\nu a_{\beta} g_{\mu\gamma} \delta^\alpha_\gamma + 4 \bar{\sigma}^\alpha_\mu a_{\beta} g_{\nu\gamma} \delta^\alpha_\gamma + 4 \bar{\sigma}^\alpha_{\mu} a_{\nu} \bar{\sigma}^\alpha_{\gamma}) \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\gamma}.$$ (3.14)

$$B^\alpha(\partial p) = (2 a_{\gamma;\mu} \delta^\alpha_\nu - \bar{\sigma}^\alpha_{\mu} \bar{g}_{\nu\gamma} - 8 a_{\mu} a_{\nu} \delta^\alpha_\gamma + 2 \bar{\sigma}^\alpha_\nu \bar{a}_{\beta} \bar{g}_{\mu\gamma} \delta^\alpha_\gamma + 4 \bar{\sigma}^\alpha_{\mu} \bar{a}_{\beta} \bar{g}_{\nu\gamma} \delta^\alpha_\gamma + 4 \bar{\sigma}^\alpha_{\mu} \bar{a}_{\nu} \bar{g}^\alpha_{\gamma}) \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\gamma}.$$ (3.15)

We have denoted $\sigma_{\mu;\nu} := \nabla_{\nu} a_{\mu}$ and $\bar{\sigma}^\alpha := \bar{g}^\alpha_{\mu;\nu} a_{\beta}$. Equations (3.13)–(3.15) give the transformation law for the Wigner function which we have been seeking, correct to second adiabatic order. It is worth noting that the transformation law has been derived irrespective of any specific form of quantum kinetic equations, and it holds independently of whether or not the Wigner functions satisfy these equations.

One could follow a different way. Given equations which govern the evolution of a Wigner function, one may seek a transformation which leaves the equations invariant under the conformal transformation of the metric and the external potential. This is what we did in order to obtain the transformation law at zeroth order, Eq. (3.5). However, this program meets serious problems at higher orders because the structure of the quantum kinetic equations in curved spacetime is very complicated. Instead in [Appendix B] we prove, to second order, the
conformal invariance of the collisionless quantum kinetic equations by using the transformation law for the Wigner function.

One may make sure that Eqs. (3.13)–(3.15) yield the right transformation law for physical observables. For example, the number–flux vector defined by

\[
\langle J_\alpha(x) \rangle = \int \frac{d^4 p}{\sqrt{-g(x)}} p_\alpha f(x, p),
\]

is transformed as follows:

\[
(-g)^{1/4} \langle J_\alpha \rangle = (-\bar{g})^{1/4} \langle \bar{J}_\alpha \rangle.
\]

The transformation law (3.17) implies that the number of "particles" (defined as in kinetic theory) in a comoving three–volume,

\[
dN := \langle J^\alpha \rangle d\Sigma_\alpha,
\]

is an invariant of the group of conformal transformations. Here

\[
d\Sigma_\alpha := \sqrt{-g} \varepsilon_{\alpha\sigma\mu\nu} dx^\sigma dx^\mu dx^\nu.
\]

The transformation law for the stress–energy tensor depends upon an explicit form of the external potential. For a scalar field coupled to curvature one gets (see Appendix E)

\[
(g/\bar{g})^{1/4} \langle T_{\alpha\beta} \rangle = \langle \bar{T}_{\alpha\beta} \rangle + \frac{\hbar^2}{2} (6\xi - 1) \left( 2 a a_{\alpha(\gamma} \nabla_{\beta)} - a \bar{g}_{\alpha\beta} a_{\nu} \nabla^\nu - \bar{g}_{\alpha\beta} a_{\nu} \nabla^\nu \right) \left( \bar{N}/a \right),
\]

where \( \xi \) is the nonminimal gravitational coupling constant and

\[
\bar{N}(x) = \int \frac{d^4 p}{\sqrt{-\bar{g}(x)}} \bar{f}(x, p).
\]

IV Quantum corrections to isotropic distributions

Let us consider a system in external fields \( g_{\alpha\beta} \) and \( V \) whose classical distribution function (which is assumed obeying the Vlasov equation) is locally isotropic in momentum space, that is \( F_{cl}(x, p) = F(x, u^\alpha(x)p_\alpha) \), \( u^\alpha(x) \) being some world velocity field. Of course, the external fields must be consistent with the symmetry of the distribution function. From the
results of Refs. [18, 19] and from the conformal invariance of the Vlasov equation it follows
that only two possibilities exist:

A. The metric is conformally stationary, i.e. the line element takes the form (in preferred
coordinates)

$$ds^2 = a^2(dt^2 + 2\bar{g}_{0i}dx^0dx^i + \bar{g}_{ij}dx^idx^j),$$

(4.1)

with $\bar{g}_{0i}$ and $\bar{g}_{ij}$ being functions of the spatial coordinates $x^i$ alone, and either the potential
$V$ equals zero or the function $V := V a^2$ does not depend on time;

B. The metric is conformally static, that is $\bar{g}_{0i} = 0$ in some coordinate system, and the
function $\bar{V}$ depends on time alone (since the Vlasov equation is linear in $V$, one can conclude
that in Case B the function $\bar{V}$ may be of the form [19] $\bar{V} = \bar{V}_1(t) + \bar{V}_2(x^i)$; we shall not
deal with this more general situation here).

In accordance with our strategy, we can solve the quantum kinetic equation in an
auxiliary spacetime with the line element

$$ds^2 = dt^2 + 2\bar{g}_{0i}dx^0dx^i + \bar{g}_{ij}dx^idx^j,$$

(4.2)

and then use Eq. (3.13) to calculate the Wigner function in the physical spacetime.

In the spacetime with the metric $\bar{g}_{\alpha\beta}$ there exist a unit time–like Killing vector $\bar{\xi}^\alpha$,
$\bar{\xi}^\alpha\bar{\xi}_\alpha = 1$ ($\bar{\xi}^\alpha = \delta_0^\alpha$ in the preferred coordinates), which obeys the equation [23]:

$$\nabla_\alpha \bar{\xi}_\beta + \nabla_\beta \bar{\xi}_\alpha = 0 \quad \text{in Case A},$$

(4.3)

$$\text{or } \nabla_\alpha \bar{\xi}_\beta = 0 \quad \text{in Case B}.$$  (4.4)

The potential must satisfy the conditions:

$$\bar{\xi}^\alpha \nabla_\alpha = 0 = \bar{\xi}^\alpha \bar{\xi}^\beta \nabla_{\alpha\beta} = \ldots \quad \text{in Case A},$$

(4.5)

$$\Delta_{\alpha\beta} \nabla_\alpha = 0 = \Delta_{\alpha\beta} \nabla_{\alpha\beta} = \ldots \quad \text{in Case B}.$$  (4.6)
Here

\[ \Delta^{\alpha\beta} := \xi^\alpha \xi^\beta - g^{\alpha\beta} \] (4.7)

is the space–like projection operator and \( \nabla_\alpha := \nabla_\alpha \nabla \) etc.

In the two cases considered, the locally isotropic distribution functions \[19\] take the form (one can see that in Case B the distribution function below has the desired form \( F(x, u^\alpha(x)p_\alpha) \) on the mass shell):

\[
F_{cl} = F(\xi^\alpha p_\alpha) \quad \text{in Case A,}
\]

\[
F_{cl} = F(\sqrt{\Delta^{\alpha\beta}} p_\alpha p_\beta) \quad \text{in Case B}.
\]

We proceed now to the derivation of the quantum corrections to the classical distribution functions. Both the function (4.8) and the function (4.9) satisfy the off–shell Vlasov equation in the external fields \( g_{\alpha\beta} \) and \( \nabla \), with \( \Delta_F \) being zero (see Eq. (2.10)). Note, however, that for systems conformally related to the ones under consideration, \( \Delta_F \neq 0 \), as it follows from Eq. (3.7).

The semiclassical Wigner functions which correspond to the distribution functions (4.8) and (4.9) in the spacetime \( \mathcal{M} \) are represented by the same expansions as in Eq. (2.5), with the argument of the \( \delta \)–function being

\[
\overline{\Omega} = \sqrt{\Delta^{\alpha\beta}} p_\alpha p_\beta - \nabla.
\] (4.10)

The off–shell quantum corrections are obtained by substituting the classical distribution functions (4.8) and (4.9) into Eq. (B6). Both in Case A and in Case B, they can be written in the form:

\[
\mathcal{F}_1 = -\frac{1}{6} \left( \mathcal{F} + 2\mathcal{F}_{\alpha,\beta} p_\mu p_\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} \right) F_{cl},
\]

13
\[ F_2 = \left( -\frac{1}{3} R_{\mu\nu} p_{\mu} p_{\nu} + \frac{1}{4} \nabla_\alpha^2 + \frac{1}{2} \nabla_\nu^\mu p_\mu \frac{\partial}{\partial p_\nu} \right) F_{cl}, \]
\[ F_3 = \frac{1}{12} \left( 2 \nabla_\mu^\nu p_{\mu} p_{\nu} - \nabla^\alpha_\nu \nabla^{\alpha}_\nu \right) F_{cl}. \] (4.11)

Next, the on–shell quantum corrections are found from Eq. (B7). It follows from the conditions listed in Eqs. (4.5) and (4.6) that the equation for \( F_{qu} \) can be integrated as if the potential \( V \) were constant. A solution of this equation which corresponds to the classical distribution (4.8) was found in Ref. [15]. It reads (with \( \xi_\alpha \xi_\alpha = 1 \))
\[ F_{qu} = -\frac{1}{6} R_{\alpha\beta} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} F_{cl} \quad \text{in Case A.} \] (4.12)

Let us now consider Case B. It follows from Eq. (4.10) that the following identities hold [23]:
\[ \xi_\alpha R_{\alpha\beta\mu\nu} = 0 = \xi_\sigma R_{\alpha\beta\mu\nu;\sigma}. \] (4.13)

By using Eqs. (4.4), (4.6) and (4.13) one readily gets that the function
\[ F_{qu} = -\frac{1}{12} R_{\alpha\beta} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} F_{cl} \quad \text{in Case B} \] (4.14)
satisfies the quantum kinetic equation (B7), with \( F_{cl} \) being of the form (4.9).

Equations (4.11) and (4.12) give the second order quantum corrections to the locally isotropic distribution (4.8) in a stationary spacetime in the presence of a static potential (Case A), while Eqs. (4.11) and (4.14) give the quantum corrections to the distribution (4.9) in a static spacetime in the presence of a homogeneous potential (Case B). As we have said, the Wigner functions for systems conformally related to the ones which have been considered in this section are obtained by using Eqs. (3.1) and (3.13). In the next section we shall analyse the structure of the quantum corrected number–flux vector and stress–energy tensor in these cases.
V Quantum corrections to a perfect fluid

The distributions (4.8) and (4.9) both imply that the classical stress–energy tensor has a perfect fluid structure [19],

\[ \langle T_{\alpha\beta}^{cl} \rangle = (\varpi_{cl} + \varpi^\alpha_{cl} \varpi^\beta_{cl} - \varpi^\alpha_{cl} \varpi^\beta_{cl} ), \quad (5.1) \]

the classical energy density \( \varpi_{cl} \) and pressure \( \varpi_{cl} \) being given by

\[ \varpi_{cl} = 2\pi \int_0^\infty dp \, p^2 \left( \sqrt{V + p^2} \right)^{1/2} F_{cl}(V, p) , \]
\[ \varpi^\alpha_{cl} = \frac{1}{3} 2\pi \int_0^\infty dp \, p^4 \left( \sqrt{V + p^2} \right)^{-1/2} F_{cl}(V, p) , \quad (5.2) \]

with (compare Eqs. (4.8), (4.9))

\[ F_{cl}(V, p) = \begin{cases} 
F(\sqrt{V + p^2}) & \text{in Case A} \\
F(p) & \text{in Case B} . 
\end{cases} \quad (5.3) \]

The classical number–flux vector associated with the locally isotropic distributions reads:

\[ \langle J^\alpha_{cl} \rangle = n_{cl} \varpi^\alpha_{cl} , \quad (5.4) \]

the classical number density being

\[ n_{cl} = 2\pi \int_0^\infty dp \, p^2 F_{cl}(V, p) . \quad (5.5) \]

It should be mentioned that we have taken into account positive "energies" only, \( \bar{E} = \sqrt{V + p^2} \), when integrating over the mass shell. This is tantamount to multiplying the distribution function by the step function [24] \( \Theta(\varpi^\alpha_{cl} p_\alpha) \). Though \( \varpi^\alpha_{cl} p_\alpha \) is not a constant of motion in Case B, the step function is "ignored" by the Liouville–Vlasov operator, neither it contributes to the quantum kinetic equation, since differentiation of the step function
gives \( \delta(\xi^{\alpha} p_{\alpha}) \) which vanishes on the mass shell (we assume that, in the classical limit, \( \nabla \) is non–negative everywhere in the spacetime). Note that negative ”energies”, \( E = -\sqrt{\nabla + p^2} \), correspond to antiparticles [6].

The quantum corrections to Eqs. (5.1),(5.4) are easily obtained by substituting the Wigner functions found in the preceding section into Eqs. (C4) and (3.16). We shall first consider Case B. The second order quantum corrections to the number flux (5.4) can be shown to vanish. The quantum corrected stress–energy tensor can be written, after some simple algebra, in the following form:

\[
\langle T^{\alpha\beta} \rangle = \langle T^{\alpha\beta}_{cl} \rangle + \hbar^2 \left( \xi - \frac{1}{6} \right) R^{\alpha\beta} M_{1,1} + \hbar^2 \hat{\nabla} \Delta^{\alpha\beta} \left( \frac{1}{2} \left( \xi - \frac{1}{6} \right) M_{1,2} - \frac{1}{24} \nabla M_{1,3} \right) \\
- \hbar^2 \hat{\nabla}^2 \Delta^{\alpha\beta} \left( \frac{3}{4} \left( \xi - \frac{1}{6} \right) M_{1,3} - \frac{1}{96} M_{2,4} - \frac{1}{16} \nabla^2 M_{1,4} \right) + 1 \frac{32}{32} \hat{\nabla} \xi^{\alpha} \xi^{\beta} M_{1,3} .
\] (5.6)

Here \( \hat{\nabla} := \xi^{\alpha} \nabla_{\alpha} \nabla \) etc., \( \Delta^{\alpha\beta} \) is defined by Eq. (4.7), and the M’s denote the integrals:

\[
M_{n,k} = 2\pi \int_{0}^{\infty} dp \ p^{2n} (\nabla + p^2)^{1/2-k} F(p) .
\] (5.7)

The stress–energy tensor (5.6) has an ”almost” perfect fluid form, except the second term which involves \( R^{\alpha\beta} \). The latter has in general off–diagonal components. Note, however, that the Einstein equations impose severe restrictions on the underlying spacetime. Suppose, for example, that the conformal factor in Eq. (4.1) depends on time only: \( a = a(t) \) (recall that we are considering Case B, that is \( g_{0i} = 0 \) in the preferred coordinates). If the total stress–energy tensor of classical matter is assumed to have a perfect fluid form then the physical spacetime is necessary a Robertson–Walker one [19]. Then \( R^{\alpha\beta} = 2K \Delta^{\alpha\beta} \), with \( K = -1, 0, 1 \), that is the quantum–corrected stress–energy tensor preserves the perfect fluid structure in this case.

Let us write down the stress–energy tensor with the second order quantum corrections
in a Robertson–Walker spacetime with the line element

\[ ds^2 = a^2 \left( dt^2 - \frac{dr^2}{1 - Kr^2} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \right). \] (5.8)

Since the Bianchi identities hold automatically for the stress–energy tensor derived in kinetic theory, it suffices to know the energy density \( \varepsilon := \langle T_0^0 \rangle \). Making use of Eqs. (3.18), (5.2) and (5.6) gives

\[ \varepsilon = \frac{1}{a^4} M_{1,0} + \frac{\hbar^2}{2a^4} (1 - 6\xi) \left( \frac{a^2}{a^2} - K \right) M_{1,1} \\
+ \frac{\hbar^2}{4} (1 - 6\xi) \frac{a}{a^3} (\dot{U} + 2 \dot{a} \frac{a}{U}) M_{1,2} + \frac{\hbar^2}{32} (\dot{U} + 2 \dot{a} \frac{a}{U})^2 M_{1,3}. \] (5.9)

Here dots stand for the time derivative, and the M’s are given by Eq. (5.7), with \( V = Ua^2 \), \( U \) being the classical part of the external potential in the Robertson–Walker spacetime (see Appendix C). Equation (5.9) extends the result of Ref. [13] to the case of a variable potential and the nonminimal coupling to curvature. Physical manifestations of the quantum kinetic corrections in the Friedmann–Robertson–Walker cosmology have been examined in Ref. [14]. It was shown there that the quantum corrected energy density causes an apparent time variation of the value of the gravitational constant in the early universe.

We turn now to Case A of the preceding section. The structure of the quantum corrections is more complicated in this case. To avoid lengthy expressions, we shall confine our attention to systems for which \( V = 0 \). Substituting the distribution function (4.8) and the quantum corrections to it given by Eqs. (4.11), (4.12) into Eqs. (3.16) and (C4) gives the quantum corrected number–flux vector and stress–energy tensor in a stationary spacetime (recall that \( \bar{\xi}^\alpha \bar{\xi}_\alpha = 1 \)):

\[ \langle \mathbf{J}^\alpha \rangle = \bar{\xi}^\alpha M_2 - \frac{\hbar^2}{6} \Sigma^\mu_\alpha \bar{\xi}^\nu \overline{T}_{\mu
u} M_0, \] (5.10)
\[
\langle T^{\alpha\beta} \rangle = \frac{1}{3} (4 \xi^\alpha \xi^\beta - \mathcal{T}^{\alpha\beta}) (M_3 - \frac{\hbar^2}{2} \xi^\mu \xi^\nu \mathcal{R}_{\mu\nu} M_1)
\]
\[
+ \frac{2\hbar^2}{3} (\xi^{(\alpha} \mathcal{R}^{\beta)}_{\mu} \xi_\mu - \xi_\mu \xi_\nu \mathcal{R}^{\alpha\mu\nu}) M_1 .
\]  
(5.11)

Here \(\mathcal{R}^{\alpha\mu\beta\nu}\) is the Riemann tensor of the stationary spacetime, and

\[
M_n = 2\pi \int_0^\infty d p p^n F(p) .
\]  
(5.12)

The remarkable feature of the quantum corrections in Eqs. (5.10), (5.11) is that they have a non-equilibrium structure. To show this explicitly, let us compute the invariants of the number–flux vector and stress–energy tensor. The hydrodynamical velocity associated with the number flux (5.10) is

\[
v^\alpha = \xi^\alpha - \frac{\hbar^2}{6} \xi^{\mu} \xi^\nu \mathcal{R}_{\mu\nu} M_0/M_2 .
\]  
(5.13)

If we decompose the stress–energy tensor (5.11) like in Eq. (C8) we find next, with the aid of Eq. (C10), that the eigenvector of \(\langle T^{\alpha\beta} \rangle\) is

\[
u^\alpha = \xi^\alpha - \frac{\hbar^2}{4} \xi^{\mu} \xi^\nu \mathcal{R}_{\mu\nu} M_1/M_3 .
\]  
(5.14)

In general, \(\nu^\alpha\) does not coincide with \(v^\alpha\). This implies that the heat flux defined by

\[
q^\alpha = (\xi + \nu^\alpha) (\nu^\alpha - v^\alpha)
\]  
(5.15)

does not vanish. Indeed, substituting Eqs. (5.13), (5.14) into (5.15) gives

\[
q^\alpha = \frac{\hbar^2}{9} \left(2M_0M_3/M_2 - M_1\right) \xi^{\mu} \xi^\nu \mathcal{R}_{\mu\nu} .
\]  
(5.16)

The combination of the moments on the right–hand side of Eq. (5.16) can be zero for a very special choice of the function \(F(p)\). It is surprising that the heat flux as well as the viscosity (see Appendix C) do not vanish even when the classical distribution function describes local
thermal equilibrium \[23\]. Though the effect is purely quantum, it could play an important role in the early universe and in the evolution of massive stars. Some physical consequences of this will be considered elsewhere.

If (and only if) the manifold $\mathcal{M}$ is static, that is the classical fluid motion is irrotational, the quantum corrections in Eqs. (5.10),(5.11) vanish, which is consistent with Case B in the limit $\mathbf{\nabla} \to 0$, $\xi = 1/6$ (compare Eq. (5.6)).

Finally, the quantum corrected number–flux vector and stress–energy tensor in a conformally stationary spacetime can be easily found with the aid of Eqs. (3.17),(3.18). For the conformal coupling ($\xi = 1/6$) they are obtained by simply multiplying Eqs. (5.10) and (5.11), respectively, by $a^{-4}$ and $a^{-6}$.

VI Concluding remarks

In summary, we have considered conformal properties of the covariant Wigner function. The transformation law (3.13) relates Wigner functions in manifolds conformally related to each other, correct to second adiabatic order. Given a solution of quantum kinetic equations which govern the evolution of a quantum distribution function (a Wigner function) of a specific system, one is able to evaluate Wigner functions for a wide class of systems conformally related to the one under consideration.

There exist two possible applications of the result. First, our method allows one to simplify an original system by reducing it to a simpler one for which the evolution can be easily solved. As an example, we have found explicit solutions of the collisionless quantum kinetic equations in conformally stationary/static spacetimes and analyzed the structure of the quantum corrected stress–energy tensor and number–flux vector.
Second, in some theories of gravity, like Brans–Dicke–Jordan–type ones which have been attracting a great deal of attention in recent years in connection with the low energy limit of string theory (see, for instance, Ref. [26]), two metrics are involved: a physical metric which enters matter equations, and the so called Einstein metric which satisfies Einstein–like equations. The two metrics are usually conformally related to each other (though more general relations have been suggested, see Ref. [27]), and one might want to solve matter equations on the physical manifold and then express the solutions in terms of the metric associated with gravity. Our method described in the present paper is especially suitable for such theories.

We would like to finish our discussion by the following remark. It is well known [18, 19, 22, 24] that local thermal equilibrium in a relativistic gas is possible in a very restrictive class of physical spacetimes. As we have shown, in a quantum system thermal equilibrium is even more exceptional than in a classical one. Spacetimes in which quantum fields can be in local thermal equilibrium are restricted to conformally static ones. Although we have analyzed collisionless equations only, this is also correct for collision–dominated systems [24], because quantum corrections to the collision integral are supposed to be small compare with ones to the Liouville–Vlasov operator when a system is nearly equilibrium. Thus our results provide further motivation for studying non–equilibrium processes in quantum systems in curved spacetime.

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Appendix A : Evaluation of the transformation law for the Wigner function

One can derive Eq. (3.13) by applying the Campbell–Baker–Hausdorff formula \[29\]:

\[
\exp(\hat{D}) \exp(-\hat{D}) = \exp \left( \hat{D} - \frac{1}{2} [\hat{D}, \hat{D}] + \frac{1}{12} [\hat{D} + \hat{D}, [\hat{D}, \hat{D}]] + \ldots \right),
\]

(A1)

for the operators \( y^\alpha \nabla_\alpha \) and \( y^\alpha \nabla_\alpha \). Making use of Eq. (3.8), (A1) and taking into account the fact \([11]\) that the operator \( \nabla_\alpha \) commutes with \( \partial / \partial y^\alpha \) and annihilates \( y^\alpha \) immediately give Eq. (3.10) with

\[
A^\alpha = 2y^\alpha y^\nu a_\nu - y^\nu y_\nu a^\alpha ,
\]

(A2)

\[
B^\alpha = y^\mu \nabla_\mu A^\alpha - A^\mu \frac{\partial}{\partial y^\mu} A^\alpha
\]

\[
= 2y^\alpha y^\mu y^\nu (\nabla_\mu a_\nu) - y^\mu y^\nu y_\mu (\nabla_\mu a^\alpha) - 8y^\alpha (y^\nu a_\nu)^2 + 2y^\alpha y^\nu y_\nu a_\mu + 4y^\nu y_\nu y^\mu a_\mu a^\alpha.
\]

(A3)

Next, with the aid of Eqs. (3.10) and (3.12) we obtain:

\[
\hat{Z}(x, y) \Phi(x, y) = \left( 1 - \frac{1}{2} A^\alpha \hat{\nabla}_\alpha + \frac{1}{6} B^\alpha \frac{\partial}{\partial y^\alpha} + \frac{1}{8} A^\alpha A^\mu \hat{\nabla}_\alpha \hat{\nabla}_\mu \right) \Phi(x, y)
\]

\[+ \text{ terms of third or higher orders}. \] (A4)

Using Eqs. (3.9), (3.11), (A4) and analogous identities for the conjugate field, we arrive at

\[
\Phi(x, -y) \Phi^\dagger(x, y) = a^{-2} \left( 1 - y^\alpha y^\beta a_\alpha a_\beta - \frac{1}{2} A^\alpha \hat{\nabla}_\alpha - \frac{1}{6} B^\alpha \frac{\partial}{\partial y^\alpha} + \frac{1}{8} A^\alpha A^\mu \hat{\nabla}_\alpha \hat{\nabla}_\mu \right) \Phi(x, -y) \Phi^\dagger(x, y)
\]

\[+ \text{ terms of higher orders}. \] (A5)

Finally, substituting Eq. (A5) into the definition of the Wigner function (2.3) and integrating by parts give Eq. (3.13) (the last term in the bracket on the right–hand side of Eq. (A5) leads to terms of fourth adiabatic order which we do not keep).
Appendix B : Conformal invariance of the collisionless quantum kinetic equations

The equations for a Wigner function of a scalar field obeying the generalized Klein–Gordon equation (2.4) read [8, 9, 11], to second adiabatic order:

\[ \hat{\mathcal{L}} f(x, p) = \hbar^2 \hat{\Lambda} f(x, p), \] (B1)

\[ \Omega f(x, p) = -\hbar^2 \hat{\Pi} f(x, p), \] (B2)

where \( \Omega \) and \( \hat{\mathcal{L}} \) are respectively given by Eqs. (2.6) and (2.8), and the operators on the right–hand side of Eqs. (B1) and (B2) are

\[
\hat{\Lambda} = \frac{1}{6} R_{\nu \mu} p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} D^\nu - \frac{1}{24} R_{\alpha \mu \nu \sigma} p^\nu p^\mu \frac{\partial^3}{\partial p_\alpha \partial p_\beta \partial p_\sigma}, 
\]

\[
\hat{\Pi} = -\frac{1}{4} D^\alpha D_\alpha - \frac{1}{6} R - \frac{1}{12} R_{\alpha \mu \nu \sigma} p^\nu p^\mu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} - \frac{1}{48} V_{\alpha \beta \sigma} \frac{\partial^3}{\partial p_\alpha \partial p_\beta \partial p_\sigma} - \frac{1}{4} R_{\mu \nu} p^\mu \frac{\partial}{\partial p_\nu} + \frac{1}{8} V_{\alpha \beta} \frac{\partial^2}{\partial p_\alpha \partial p_\beta}. \] (B3)

Here \( D_\alpha \) is defined by Eq. (2.9), semicolons signify the covariant differentiation associated with the metric \( g_{\mu \nu} \).

The operators \( \hat{\Lambda} \) and \( \hat{\Pi} \) satisfy the following identity [11]:

\[ [\hat{\Lambda}, \Omega] = [\hat{\mathcal{L}}, \hat{\Pi}], \] (B5)

which allows one to look for solutions to Eqs. (B1), (B2) of the form (2.5) [9, 11]: The F’s in Eq. (2.5) are given by

\[ F_1(x, p) = \hat{\Pi} F_\alpha(x, p), \]
\[ F_2(x,p) = \frac{1}{2} [\hat{\Pi}, \Omega] F_{cl}(x,p), \]
\[ F_3(x,p) = \frac{1}{6} [[\hat{\Pi}, \Omega], \Omega] F_{cl}(x,p), \quad \text{(B6)} \]

while the function \( F_{qu}(x,p) \) satisfies the equation [\[\text{[\Pi]}\]]:

\[ \hat{\mathcal{L}} F_{qu}(x,p) = \hat{\Lambda} F_{cl}(x,p) + \hat{\Pi} \Delta_{F}(x,p), \quad \text{(B7)} \]

\( \Delta_{F}(x,p) \) being defined by Eq. \((2.10)\).

Consider now a conformally related spacetime with the metric \( g_{\alpha\beta} = g_{\alpha\beta}/a^2 \) and suppose that \( \mathcal{F}(x,p) \) satisfies the quantum corrected Vlasov equation and mass-shell constraint in the presence of the external fields \( \mathcal{G}_{\alpha\beta} \) and \( \nabla = Va^2 \). As we have shown in Sec. [\[\text{[\Pi]}\]], the conformal transformation of the spacetime induces the following transformation of the Wigner function defined in the cotangent bundle:

\[ f(x,p) = a^2 (1 + \hbar^2 \hat{C}) \mathcal{F}(x,p), \quad \text{(B8)} \]

where the operator \( \hat{C} \) is, to second adiabatic order (cf. Eq. \((3.13)\)),

\[ \hat{C} = \frac{1}{24} \left( -4 \bar{a}_{\mu;\nu} + \bar{a}_{\alpha\mu} g_{\mu\nu} + 28 a_\mu a_\nu - 10 \bar{a}^\alpha a_\alpha g_{\mu\nu} + 6 a_\mu \bar{D}_\nu - 3 \bar{g}_{\mu\nu} \bar{a}^\alpha \bar{D}_\alpha \right) \frac{\partial^2}{\partial p_\mu \partial p_\nu} \\
+ \frac{1}{24} \left( -2 \bar{a}_{\mu;\nu} p_\sigma + \bar{a}_{\alpha\mu} g_{\mu\nu} p_\alpha + 8 a_\mu a_\nu p_\sigma - 2 \bar{a}^\alpha a_\alpha g_{\mu\nu} p_\sigma - 4 \bar{g}_{\mu\nu} a_\alpha p_\alpha \right) \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\sigma}. \quad \text{(B9)} \]

If one takes into account the transformation laws for the Wigner function, Eq. \((\text{B8})\), and for the Liouville–Vlasov operator, Eq. \((\text{3.3})\), and uses the equations for the function \( \mathcal{F}(x,p) \), one arrives at the following equations:

\[ \hat{\mathcal{L}} f(x,p) = \hbar^2 \left( \hat{\lambda} + [\hat{\mathcal{L}}, \hat{C}] - a_\alpha \frac{\partial}{\partial p_\alpha} (\hat{\Pi} + [\hat{\Pi}, \hat{C}]) \right) f(x,p) \\
+ \text{terms of higher orders}, \quad \text{(B10)} \]
\[ \Omega f(x,p) = -\hbar^2 \left( \hat{P} + [\hat{C},\Omega] \right) f(x,p) + \text{terms of higher orders}. \]  

Now, our task is to prove that these equations are tantamount, to second adiabatic order, to Eqs. (B1), (B2). Tedious but straightforward calculations give the following results:

\[ \hat{P} + [\hat{C},\Omega] = \hat{P} a^2 + a_\alpha \frac{\partial}{\partial p_\alpha} \hat{L} + \frac{1}{4} (4a_\mu a_\nu - \bar{\alpha} a_\alpha g_{\mu\nu} - \bar{\alpha}_{\mu\nu}) \frac{\partial^2}{\partial p_\mu \partial p_\nu} \Omega, \]  

\[ \hat{L} + [\hat{L},\hat{C}] = (\hat{L} + a_\alpha \frac{\partial}{\partial p_\alpha} \hat{P}) a^2 + \frac{1}{4} (a_\mu, a_\alpha g_{\mu\nu}) \frac{\partial^2}{\partial p_\mu \partial p_\nu} \hat{L} - \frac{1}{24} (a_{\mu;\nu} - 8a_{\mu\nu} a_\sigma + 4a^{\nu} a_{\mu;\sigma} - 6a^{\nu} a_\alpha a_\mu g_{\nu\sigma}) \frac{\partial^3}{\partial p_\mu \partial p_\nu \partial p_\sigma} \Omega. \]  

When deriving Eqs. (B12), (B13) we have used the transformation laws for the Christoffel symbols and Riemann tensor [23]:

\[ \Gamma^\alpha_{\mu\nu} - \Gamma^\nu_{\mu\nu} = \delta^\alpha_\mu a_\nu + \delta^\nu_\mu a_\alpha - \bar{g}_{\mu\nu} \bar{\alpha}^\alpha, \]  

\[ a^{-2} R_{\alpha\mu\beta\nu} - \bar{R}_{\alpha\mu\beta\nu} = 2a_\nu [a, \bar{g}_{\mu\beta}] - 2a_{\beta[a} g_{\mu]\beta]. \]  

where \( a_{\alpha\beta} := a_{\alpha}, a_{\beta} + \frac{1}{2} \bar{g}_{\alpha\beta} \bar{\alpha} a_\alpha \). Recall also that \( a_\alpha := \partial_\alpha \ln a, \bar{\alpha} := \bar{\alpha} a_\beta a_\beta \) and semicolons in Eqs. (B9), (B12)–(B15) signify covariant differentiation associated with the metric \( g_{\alpha\beta} \).

Substituting Eqs. (B12), (B13) into Eqs. (B10), (B11) completes the proof.

Appendix C : The stress–energy tensor

Consider, for simplicity, the theory of a real scalar field described by the action:

\[ S = \frac{1}{2} \int d^4 x \sqrt{-g} \left( \hbar^2 \nabla^\alpha \varphi \nabla_\alpha \varphi + \hbar^2 \xi R \varphi^2 - U \varphi^2 \right). \]
$R$ being the Ricci scalar, $U$ an external potential not involving a metric dependence.

The variation of the action with respect to the metric tensor yields the stress–energy tensor \([1]\):

$$T_{\mu\nu}(\xi, V) = \bar{h}^2 \nabla_\mu \varphi \nabla_\nu \varphi + h^2 \xi R_{\mu\nu} \varphi^2 - \frac{1}{2} g_{\mu\nu} (h^2 \nabla^a \varphi \nabla_a \varphi + \frac{1}{6} h^2 R \varphi^2 - V \varphi^2) - h^2 \xi (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^a \nabla_a) \varphi^2. \quad (C2)$$

We have introduced, for convenience, the generalized external potential \([28]\)

$$V = U - \bar{h}^2 (\xi - \frac{1}{6}) R. \quad (C3)$$

The expectation value of the stress–energy tensor can be expressed in terms of the covariant Wigner function \([2.3]\) as follows \([1]\):

$$\langle T_{\mu\nu}(\xi, V) \rangle = \int \frac{d^4 p}{\sqrt{-g(x)}} (p_\mu p_\nu + \bar{h}^2 \xi R_{\mu\nu}) f(x, p)$$

$$+ \bar{h}^2 (\frac{1}{4} - \xi) (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^a \nabla_a) \int \frac{d^4 p}{\sqrt{-g(x)}} f(x, p). \quad (C4)$$

The action \((C1)\) is invariant under the conformal transformation \([3.1]\). Notice that the potential $U$ is transformed as the following:

$$U = \bar{U} / a^2 + \bar{h}^2 (1 - 6\xi) a^{-3} \nabla^a \nabla_a a. \quad (C5)$$

Since the transformation law for the potential explicitly involves the dependence on the metric, the stress–energy tensor is not conformally invariant. With the aid of Eq. \([3.13]\) we obtain the following relation:

$$a^2 \langle T_{\mu\nu} \rangle - \langle T_{\mu\nu} \rangle =$$

$$h^2 (6\xi - 1) \left( a(\mu \nabla_\nu) - \frac{1}{2} \mathcal{F}_{\mu\nu} a_\alpha \nabla^\alpha - a_{\mu a_\nu} - \frac{1}{2} \mathcal{F}_{\mu\nu} (\nabla^\alpha a_\alpha) \right) \int \frac{d^4 p}{\sqrt{-g(x)}} f(x, p). \quad (C6)$$
Simple algebra then leads to Eq. (3.18). When deriving Eq. (C6) we have used the identity (B15) and the following property of the operator (2.9):

\[
\int \frac{d^4p}{\sqrt{-g}} D_\alpha f(x,p) = \nabla_\alpha \int \frac{d^4p}{\sqrt{-g}} f(x,p) , \tag{C7}
\]

which was proven in Ref. [11].

From Eq. (C6) it follows that the stress–energy tensor (multiplied by the density \((-g)^{1/4}\)) is conformally invariant, but not tracefree, if \(\xi = 1/6\). \(T^\alpha_\alpha\) vanishes only if \(V = 0\).

Any physical stress–energy tensor can be decomposed as follows [6]:

\[
T_{\mu\nu} = (\varepsilon + P)u_\mu u_\nu - P g_{\mu\nu} + \Pi_{\mu\nu} , \tag{C8}
\]

where \(\varepsilon\) and \(P\) are, respectively, the energy density and the isotropic pressure, \(\Pi_{\mu\nu}\) is the tracefree viscosity tensor \([\Pi^\alpha_\alpha = 0 = u^\alpha \Pi_{\alpha\beta}]\), and \(u_\nu\) is a unit time–like vector \([u^\alpha u_\alpha = 1]\).

The perfect fluid stress–energy tensor takes the form:

\[
0 T_{\mu\nu} = (\varepsilon + P) u_\mu u_\nu - P g_{\mu\nu} . \tag{C9}
\]

For small deviations from the perfect fluid structure, \(T_{\mu\nu} = 0 T_{\mu\nu} + \frac{1}{3} T_{\mu\nu}\), one easily finds, to first order in the deviations:

\[
\begin{align*}
\varepsilon & = \varepsilon + u^\alpha u^\beta T_{\alpha\beta} \\
P & = P + \frac{1}{3} \Delta^\alpha T_{\alpha\beta} \\
u_\mu & = u_\mu - \Delta^\beta T_{\mu\beta} / (\varepsilon + P) \\
\Pi_{\mu\nu} & = (\Delta^\alpha \Delta^\beta - \frac{1}{3} \Delta_{\mu\nu} \Delta^\alpha) T_{\alpha\beta} ,
\end{align*} \tag{C10}
\]

where the projection operator has been introduced:

\[
\Delta_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu} . \tag{C11}
\]
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