Fusion categories via string diagrams

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Abstract

We use the string diagram calculus to give graphical proofs of the basic results of Etingof, Nikshych and Ostrik [7] on fusion categories. These results include: the quadruple dual is the identity, the ratio of the global dimension to the Frobenius-Perron dimension is an algebraic integer, and Ocneanu rigidity. We introduce the pairing convention as a convenient graphical framework for working with fusion categories, and use this framework to obtain explicit equations for pivotal structures and the pivotal operators.

1 Introduction

In this paper we fix our ground field to be \( \mathbb{C} \). A fusion category is a rigid semisimple linear monoidal category with finitely many isomorphism classes of simple objects and whose unit object is simple. A basic example of a fusion category is the category of representations of a finite group; in this way the study of fusion categories can be regarded as a common generalization of group theory and representation theory.

The foundational results about fusion categories were obtained by Etingof, Nikshych and Ostrik [7]. These results include the fact that the quadruple dual is the identity, positivity of the paired dimensions, the fact that the ratio of the global dimension to the Frobenius-Perron dimension is an algebraic integer, and Ocneanu rigidity.

These results were proved in [7] using the theory of weak Hopf algebras. The idea is that since every fusion category can be expressed as the category of finite-dimensional representations of some semisimple weak Hopf algebra [7], one may prove results about fusion categories by translating them into the language of weak Hopf algebras. One motivation for this paper, building on [8] [15] [17] [4], was to give a unified account of how these results can be proved directly in the fusion category itself, using the graphical calculus of string diagrams [12] [11] [23].

Another motivation has been the string-net description for the vector spaces in the Turaev-Viro model of 3-dimensional topological quantum field theory [13], where the graphical calculus plays a prominent role.

The pairing convention

In the string diagram calculus, the objects \( V \) of the fusion category \( C \) are used as labels for strands in the plane. The main feature is rigidity — the notion that the strands are oriented, and that there are left and right dual structure
maps, drawn as
\[
\left( \begin{array}{c}
V \\
V^*
\end{array} \right) \quad \text{and} \quad \left( \begin{array}{c}
V \\
V^*
\end{array} \right)
\]
respectively, satisfying ‘snake equations’ such as
\[
\begin{array}{c}
V \\
\downarrow
\end{array} = \begin{array}{c}
V^* \\
\downarrow
\end{array} .
\]
One can also form closed loops, which involves pairing left and right dual structure maps together:
\[
\begin{array}{c}
V^* \\
\downarrow
\end{array} \quad \begin{array}{c}
V \\
\downarrow
\end{array} \in \text{Hom}(1, 1).
\]
In order to make this well-defined, it is usually imagined that one needs a pivotal structure on the fusion category. The main idea we use in this paper is that, in fact, there is a well-defined way to make sense of diagrams which employ both left and right dual structure maps together, such as (1), even without a pivotal structure. The rule (at least for simple objects \(V\)) is that if left and right dual structure maps appear in a diagram, then they cannot be chosen independently but must be correlated in the following way: when paired together, they must give the fusion dimension of \(V\). (The fusion dimension is a certain positive number canonically associated to \(V\) in a fusion category; it does not make use of a pivotal structure). We call this the pairing convention, and it allows us to uniquely evaluate diagrams such as
\[
\begin{array}{c}
I \\
\downarrow
\end{array} \quad \begin{array}{c}
V^* \\
\downarrow
\end{array} \quad \begin{array}{c}
V \\
\downarrow
\end{array}
\]
in a ‘bare’ fusion category, which would otherwise not be well-defined.

**Pivotal operators**

Using the pairing convention, we can give a simple diagrammatic definition of the pivotal operators of a fusion category, which are certain canonically defined linear maps
\[
T_{BC}^A : \text{Hom}(A, B \otimes C) \to \text{Hom}(A, B \otimes C)
\]
associated to every triple of objects \(A, B\) and \(C\) in a fusion category. One advantage of our framework is that these maps are canonical, not depending on any choices of duals for their definition. The pivotal operators play the role of the double dual functor, and the string diagram calculus then gives a simple proof that \(T_{BC}^A\) squares to the identity, see Theorem 16. The idea of this graphical proof is originally due to Hagge and Hong [8].
Main results

Our main results are threefold. Firstly, we give graphical proofs of the following results from \[7\]:

- the quadruple dual is the identity (Theorem 16 and Proposition 53),
- positivity of the paired dimensions (Theorem 26),
- a pseudo-unitary fusion category is spherical (Corollary 58),
- the sphericalization of a fusion category is spherical (Corollary 68),
- the ratio $\frac{D}{\Delta}$ of the global dimension $D$ to the Frobenius-Perron dimension $\Delta$ is an algebraic integer (Proposition 74),
- Ocneanu rigidity, i.e. the vanishing of the Davydov-Yetter cohomology (Theorem 81).

Secondly, we formulate the graphical calculus in an geometric way, allowing us to show that string diagrams taking labels in a fusion category are invariant under the group of \textit{rigid spin diffeomorphisms} (Theorem 5). Finally, we explicitly compute the pivotal operators as a product of the pivotal indicators with the monodromy of the apex associator operators (Theorem 43), and write down explicit equations for a pivotal structure (Theorem 57).

Outline of paper

In Section 2 we give our conventions on string diagrams, and express their invariance under different groups of diffeomorphisms. In Section 3.2 we define the pairing convention and the pivotal operators on a fusion category, and prove results about them. In Section 4 we write down explicit equations for pivotal structures, discuss the sphericalization of a fusion category, and prove that $\frac{D}{\Delta}$ is an algebraic integer. In Section 5 we prove Ocneanu rigidity.

2 String diagrams

In this section we explain our conventions regarding the string diagram calculus, and express its invariance under the appropriate diffeomorphism groups.

2.1 Conventions

The string diagram calculus for monoidal categories is well-known (see \[23\] for an overview). Each diagram refers to a certain morphism in $C$. Our diagrams go from top to bottom, so that a morphism $f : A \to B \otimes C$ is drawn as

In these pictures there is no explicit parenthesis scheme on the input and output tensor products. Thus it is often supposed that one needs a a strict monoidal
category in order to make this calculus well-defined. In fact, this is not so
— the calculus makes perfect sense when reasoning about equations between morphisms, which is all we will ever need. For instance, suppose $f : A \to A' \otimes E$, $g : B \otimes C \to D \otimes C'$, $h : E \otimes D \to B'$, $k : A \otimes (B \otimes C) \to F \otimes C'$, $l : F \to A' \otimes B'$, and consider the following equation:

How are we to interpret this? This equation should be regarded as an infinite number of equations, one for each fixed parenthesis scheme of the input and output morphisms, which are taken to be the same on both sides of the equation. For each such parenthesis scheme, one should insert an appropriate sequence of associators and unit isomorphisms in order to make each side well-typed. Coherence for monoidal categories [16] guarantees that the evaluation of the resultant morphism is independent of this choice. Each equation implies all the others (see [2, Chap. 4]).

In a fusion category, the associators feature explicitly in the graphical calculus as follows. Suppose we fix a representative set of simple objects $X_i, i \in I$, and choose trivalent bases $e_\alpha : X_i \to X_j \otimes X_k$ drawn as

for each hom-set $\text{Hom}(X_i, X_j \otimes X_k)$. (We will always choose the canonical basis elements for the 1-dimensional hom-spaces $\text{Hom}(1, 1 \otimes 1)$, $\text{Hom}(1, 1 \otimes X_i)$ and $\text{Hom}(1, X_i \otimes 1)$). Then we can form two different bases for $\text{Hom}(X_i, X_j \otimes (X_k \otimes X_l))$ and so we have a change of basis transformation

Note that according to our diagram conventions, in the parenthesis scheme $X_j \otimes (X_k \otimes X_l)$ for the output objects, the left-hand side diagram of (3) evaluates as

while the right-hand side diagram evaluates as

where $a$ is the associator of the fusion category. We call the scalars $(F_{ijkl})_{\gamma n \delta}^{\alpha m \beta}$ the associator matrix elements.
Given a trivalent basis choice, we define the dual basis \( e^\beta : X_j \otimes X_k \to X_i \) as the one satisfying \( e^\beta \circ e^\alpha = \delta_{\alpha\beta} \text{id}_{X_i} \). Graphically, this is drawn as:

\[
\begin{array}{c}
X_i \\
X_j \\
X_k
\end{array}
\quad \text{and} \quad
\begin{array}{c}
X_i \\
X_j \\
X_k
\end{array}
\quad \text{satisfying} \quad
\begin{array}{c}
X_i \\
X_j \\
X_k = \delta_{\alpha\beta} X_i.
\end{array}
\]

We will often write both \( e^\alpha \) and \( e^\alpha \) simply as \( \alpha \), with the source and target distinguishing them; also we often write \( i \) instead of \( X_i \). Basis vectors satisfying provide a resolution of the identity,

\[
\sum_{i, \alpha} \alpha = j \quad \text{and} \quad k.
\]

Rigidity in a fusion category means that for each object \( V \) there exists an object \( V^* \) which can be equipped with unit and counit maps \( \eta : 1 \to V^* \otimes V \), \( \epsilon : V \otimes V^* \to 1 \) such that the rigidity equations hold:

\[
\begin{array}{c}
V \\
V^*
\end{array}
\quad = \quad
\begin{array}{c}
V \\
V^*
\end{array}
\quad \text{and} \quad
\begin{array}{c}
V \\
V^*
\end{array}
\quad = \quad
\begin{array}{c}
V^*
\end{array}
\]\n
For a general monoidal category, we would also demand that every object \( V \) also admits a left dual \( *V \) with unit and counit maps \( n : 1 \to V \otimes *V \), \( \epsilon : *V \otimes V \to 1 \) satisfying the reflected versions of (6). However, in the semisimple case, if \( V^* \) can be equipped as a right dual of \( V \), then it can also be equipped as a left dual.

Our approach is to try and avoid making fixed initial choices of duals for each object, to keep constructions as canonical as possible (thus we do not refer to a right dual functor \( * : C \to C^{\text{op}} \)). Note that we have added orientations to the strands in (6), this is simply a visual aid.

**Example 1.** The Yang-Lee category (see eg. [26]) has two simple objects 1 (drawn as a dotted line) and \( \tau \) (drawn as a solid line) with \( \tau^2 = 1 + \tau \). So, the only nontrivial trivalent basis elements are

\[
\begin{array}{c}
\tau
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\tau
\end{array}.
\]
The associators are

\begin{align}
\begin{array}{ll}
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} & = a \\
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} & + \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} \\
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} & = a - a \\
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}
\end{array}
\end{align}

and

\begin{align}
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} = \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}
\end{align}

where \( a = -\frac{1}{2}(1 + \sqrt{3}) \).

**Example 2.** The category \( E \) associated with the even part of the \( E_6 \) subfactor \([8, 22]\) has 3 simple objects: 1 (drawn as a dotted line), \( x \) (drawn as a solid line), \( y \) (drawn as a wiggly line) and fusion rules \( xy = yx = x \), \( xx = 1 + 2x + y \), \( yy = 1 \). So, the nontrivial trivalent basis elements are:

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}, \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}, \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}, \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture}
\end{center}

Some relevant associators are

\begin{align}
\begin{array}{ll}
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} = \frac{1}{d} \left( \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} + \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} \right) \\
& + \frac{1}{\sqrt{2}v} \left( \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} + \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} - \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} \right)
\end{array}
\end{align}

and:

\begin{align}
\begin{array}{ll}
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} = \frac{1}{\sqrt{2}} e^{\frac{-7\pi i}{12}} \left( \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} + \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} \right) \\
\begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} = \frac{1}{\sqrt{2}} e^{\frac{-7\pi i}{12}} \left( -i \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} + i \begin{tikzpicture}
\draw (0,0) -- (0,0.5);
\draw (0.5,0) -- (0.5,0.5);
\end{tikzpicture} \right)
\end{array}
\end{align}

Here \( d = 1 + \sqrt{3} \) and \( v = \sqrt{d} \). See [9] for a full list. Note that we are using the inverses of the matrices in [9] due to our conventions.

### 2.2 Diffeomorphism invariance of string diagrams

More formally, the invariance of the evaluation of string diagrams under diffeomorphisms of the plane can be expressed as follows. (Our motivation here is [13].)

We imagine each diagram as living in an outer disk \( D \), with strands labelled by objects of the category, and coupons \( C_i \) (drawn as discs or square regions) labelled by morphisms, which go from the tensor product of the objects labelling the strands in the northern hemisphere of the coupon to the tensor product of the objects labelling the strands in the southern hemisphere of the coupon. We
define a rigid isotopy of such a diagram to be a smooth path of diffeomorphisms \( \phi_t : D \to D \), \( t \in [0, 1] \) such that \( \phi_0 \) is the identity, \( \phi_1 \) is the identity on the boundary of \( D \) for all \( t \) and such that \( \phi_t \) restricted to each coupon \( C_i \) is a translation for all \( t \).

The following theorem is well-known and combines coherence for monoidal categories, rigidity, and the interchange law.

**Theorem 3** (see eg. [23]). The evaluation of a string diagram taking labels in a rigid monoidal category is invariant under rigid isotopy.

The string diagram calculus for a fusion category, and for a pivotal fusion category, are invariant under larger groups of diffeomorphisms. We will discuss pivotal structures in Section 4 but it is best to collect the results together here, for easier comparison with Theorem 3.

Recall that a spin structure on an oriented surface \( \Sigma \) can be defined as a homotopy class of trivializations of the stabilized tangent bundle \( T\Sigma \oplus \mathbb{R} \). Let \( S \) be a string diagram in the plane, with outer boundary disc \( D \) and coupons \( C_i \subset D \). Write \( X = D \setminus \cup_i C_i \). The constant frame field \( e \) on \( X \) is the constant trivialization of the trivial vector bundle \( X \times \mathbb{R} \) over \( X \), with \( e_x = (e_1, e_2, e_3) \) for all \( x \in X \), where \( e_1, e_2 \) and \( e_3 \) are the standard basis for \( \mathbb{R}^3 \). Given a diffeomorphism \( \phi \) of the disc \( D \), we define the push-forward \( \phi_*(e) \) of the constant frame field to be the orthonormal frame field on \( D \) given by

\[
\phi_*(e)(x) = \left( \frac{(\phi_*)_{x}(e_1)}{|(\phi_*)_{x}(e_1)|}, v, e_3 \right)
\]

where \( v \) is the unique vector completing the orthonormal oriented frame.

**Definition 4.** A rigid diffeomorphism of a string diagram \( S \) in the plane is a diffeomorphism \( \phi : D \to D \) satisfying:

- \( \phi|_{\partial D} = \text{id} \),
- \( \phi|_{C_i} \) is a translation for each coupon \( C_i \).

If in addition

- \( \phi_*(e) \) is homotopic to the constant frame field on \( D \setminus \cup_i \phi(C_i) \),

then we say that \( \phi \) is a rigid spin diffeomorphism.

In Section 3.2 we will introduce the pairing convention, which will allow us to make sense of pushing forward string diagrams taking labels a fusion category under arbitrary rigid diffeomorphisms. We have the following result.

**Theorem 5.** The evaluation of a string diagram taking labels in a fusion category is invariant under rigid spin diffeomorphisms.

**Proof.** Follows from Theorem 16 and \( \pi_1 SO(3) = \mathbb{Z}/2 \).

If we put a pivotal structure on the fusion category, then there is a different way of pushing forward string diagrams along rigid diffeomorphisms, this time using the pivotal structure.

**Theorem 6.** The evaluation of a string diagram taking labels in a pivotal fusion category is invariant under rigid diffeomorphisms.
Proof. If we use the pivotal structure \( \gamma \) (and not the pairing convention, which in general gives a different answer) to evaluate the right-hand side of (15), the resulting map \( \gamma^T_{jk} \) will be the identity.

3 The pivotal operators on a fusion category

In this section we define the pairing convention and the pivotal operators on a fusion category. We prove that the paired dimensions are positive, and we compute the pivotal operators in terms of the apex associator monodromy and the pivotal indicators. We show that the pivotal operators are monoidal and identify them with the double dual functor.

3.1 Fusion and Frobenius-Peron dimensions

We recall the following from [7]. In a fusion category over \( \mathbb{C} \), every simple object \( X_i \) has two canonical dimensions, which are positive real numbers: its Frobenius-Perron dimension \( d^+_i \) and its fusion dimension \( d^i \).

**Definition 7** (see [7]). The Frobenius-Perron dimensions \( d^+_j \) of the simple objects \( X_j \) are the unique positive real numbers satisfying

\[
 d^+_j d^+_k = \sum_i N^i_{jk} d^+_i
\]

where \( N^i_{jk} = \dim \text{Hom}(X_i, X_j \otimes X_k) \). That is, they furnish the unique homomorphism from the Grothendieck ring of \( \mathbb{C} \) to \( \mathbb{C} \) taking positive real values on the simple objects \( X_i \).

To define the fusion dimensions, we first need to define paired dimensions.

**Definition 8.** Let \( X_i \) and \( X^*_i \) be dual simple objects in \( \mathbb{C} \). Their paired dimension is

\[
 d_{\{i,i^*\}} = \eta \epsilon \circ\nabla X_i \bigcirc X^*_i
\]

where \( (\eta,\epsilon) \) is some choice of unit and counit maps exhibiting \( X^*_i \) as a right dual of \( X_i \), and \( (\n,\epsilon) \) is some choice of unit and counit maps exhibiting \( X^*_i \) as a left dual of \( X_i \).

Observe that the product (14) is independent of the choices of unit and counit maps made in the definition because it is invariant under rescaling

\[
 \eta \mapsto \lambda \eta, \epsilon \mapsto \frac{1}{\lambda} \epsilon, \n \mapsto \mu \n, \epsilon \mapsto \frac{1}{\mu} \epsilon.
\]

Note that the paired dimensions are certainly nonzero complex numbers. Indeed, \( \epsilon \circ \eta \) is zero if and only if one of \( \epsilon \) or \( \eta \) is zero (by semisimplicity), which would contradict the duality equations (6). Similarly for \( \epsilon \circ \n \). In Section 3.5 we will show how to compute the paired dimensions explicitly in terms of the associator matrix elements.

In fact, we will show in Theorem 26 that the paired dimensions are always positive real numbers. Anticipating this, we make the following definition.
**Definition 9.** The fusion dimension $d_i$ of a simple object $X_i$ in a fusion category over $\mathbb{C}$ is the positive square root of its paired dimension $d_{(i, i^*)}$.

In Corollary 34, we show how to read off the fusion dimensions directly from the associators.

**Example 10.** In the Yang-Lee category, $d_\tau^+ = \frac{1}{2}(1 + \sqrt{5})$ and we read off from (7) that $d_\tau = \frac{1}{2}(-1 + \sqrt{5})$. In category $\mathcal{E}$, $d_j^+ = 1 + \sqrt{3}$ and $d_j^- = 1$, while we read off from (10) that $d_x = d_x^+; \text{ similarly } d_y = d_y^+ = 1$.

A fusion category is called pseudo-unitary if its paired dimensions are the squares of its Frobenius-Perron dimensions, i.e. $d_{(i, i^*)} = (d_i^+)^2$ for all simple objects $X_i$.

### 3.2 The pivotal operators

**Definition 11.** A root choice on a fusion category is a symmetric choice $\{d_i\}$ of square roots of the paired dimensions; that is, one which satisfies $d_i^2 = d_{(i, i^*)}$ and $d_i = d_{i^*}$ for all $i \in I$.

**Remark 12.** Once we show (Theorem 26) that the paired dimensions are positive real numbers, we will always work with the canonical root choice on a fusion category given by the positive square roots.

The utility of a root choice is that it allows the following diagrammatic convention.

**Pairing convention.** Whenever a unit and counit $(\xrightarrow{\cup} X_i, X_i \xleftarrow{\cap})$ expressing $X_i^*$ as a right dual of $X_i$ appears together in some equation or statement with a unit and counit $(\xrightarrow{\cup} X_i, X_i^* \xleftarrow{\cap})$ expressing $X_i^*$ as a left dual of $X_i$, it will always be understood that the former is arbitrary while the latter is determined uniquely by the requirement that

$$\xrightarrow{\cup} X_i = d_i \quad \text{(or equivalently } \xrightarrow{\cup} X_i^* = d_i).$$

The pairing convention is the key diagrammatic idea in this paper. Using this convention, we only need fix a root choice (and not a pivotal structure, which is not known to exist in general) on the fusion category in order to unambiguously perform string diagram calculations where both left and right duals appear. Note that the pairing convention extends uniquely to the whole category and not just the simple objects, see Section 3.7.

Using this convention, we can diagrammatically define involutions on the fusion vector spaces.

**Definition 13.** Let $\{d_i\}$ be a root choice on $\mathcal{C}$. The pivotal operators

$$T_{jk}^i : \text{Hom}(X_i, X_j \otimes X_k) \to \text{Hom}(X_i, X_j \otimes X_k)$$

...
are defined by

\[ X_i \xrightarrow{\Phi} X_j \xleftarrow{\wedge} X_k \]  \hspace{1cm} (15)

where the pairing convention has been used.

That is, to evaluate the right hand side of (15), make an arbitrary choice of right duals \((X^*_i, \eta_i, \epsilon_i), (X^*_j, \eta_j, \eta_j)\) and \((X^*_k, \eta_k, \epsilon_k)\) for \(X_i, X_j\) and \(X_k\), and then choose the left dual structure maps using the pairing convention.

Note that changing the root choice by setting \(d_i \mapsto x_i d_i\) for some signs \(x_i = \pm 1\) with \(x_i = x_j^*\) will change the sign of the pivotal operators according to \(T^i_{jk} \mapsto x_i x_j x_k T^i_{jk}\).

**Lemma 14.** In an arbitrary trivalent basis \(\{e_\alpha: X_i \to X_j \otimes X_k\}\) with corresponding dual basis \(\{e^\beta: X_j \otimes X_k \to X_i\}\), the matrix elements of \(T^i_{jk}\) compute as

\[ e^\beta T^i_{jk} e_\alpha = \frac{1}{d_i} \]

**Proof.**

**Lemma 15.** In the pairing convention, the following graphical rules hold:
Proof. Because

\[ \lambda = \lambda \quad \text{for some } \lambda \in \mathbb{C}, \text{ and } i = i^* \.
\]

\begin{proof}

Theorem 16 (cf. [8, Thm 3]). The operator \( T_{jk}^i \) is an involution — that is, \( (T_{jk}^i)^2 = \text{id} \).

Proof. The operator \( T^2 \) sends

\[
\begin{array}{c}
\xymatrix{
  i 
  \\
  j 
  \\
  k 
  \\
}
\end{array}
\]

Its matrix elements are thus:

\[
\begin{array}{c}
\xymatrix{
  i 
  \\
  j 
  \\
  k 
  \\
}
\end{array}
\]
Remark 17. This calculation is essentially the Dirac belt trick proving that \( \pi_1 SO(3) = \mathbb{Z}/2\mathbb{Z} \), and provides a link between fusion categories and spin structures, as in Theorem 5. See also [8] and Remark 50.

Remark 18. If \( C = \text{Rep} H \) for a semisimple Hopf algebra \( H \), then the identity \( (T_{jk})^2 = \text{id} \) corresponds to the Larson-Radford formula \( S^2 = \text{id} \) (see [7] and references therein). The string diagram argument above can be regarded as giving a graphical proof of Radford’s formula.

Remark 19. The proof in [8] proceeds by first passing to a strictified skeletal category equivalent to \( C \). In our approach, the pairing convention, together with our conventions on string diagrams from Section 2.1, allows us to work directly in the category \( C \).

Since \( T_{jk}^i \) is an involution, the vector space \( \text{Hom}(X_i, X_j \otimes X_k) \) has a basis of eigenvectors \( e_{\alpha} \) whose eigenvalues are \( \pm 1 \). We call such a basis a pivotal basis for the fusion category \( C \).

Definition 20. The pivotal symbols \( \epsilon_{jk,\alpha}^i = \pm 1 \) are the eigenvalues of the pivotal operators, \( T_{jk}^i e_{\alpha} = \epsilon_{jk,\alpha}^i e_{\alpha} \).

Thus, in a fusion category, the vector spaces \( \text{Hom}(X_i, X_j \otimes X_k) \) decompose into the positive and negative eigenspaces of \( T_{jk}^i \):

\[
\text{Hom}(X_i, X_j \otimes X_k) = \text{Hom}(X_i, X_j \otimes X_k)_+ \oplus \text{Hom}(X_i, X_j \otimes X_k)_- \quad (16)
\]

Note that this decomposition is canonical since our convention will be to make the unique root choice where the \( d_i \) are positive real numbers, as guaranteed by Theorem 26 below.

Lemma 21. In a pivotal basis, the pivotal symbols can be computed as follows:

\[
\epsilon_{jk,\alpha}^i = \frac{1}{d_i}
\]

Proof. Follows immediately from Lemma 14.

Lemma 22. The pivotal operators \( T_{11}^1 \) and \( T_{ii}^1 \) are the identity maps for all \( i \in I \).

Proof. Follows from an elementary string diagram argument.
3.3 Positivity of paired dimensions

In this subsection, we show that the paired dimensions $d_{\{i,i^*\}}$ are positive real numbers.

The first step is to show that a root choice gives rise to a ‘twisted’ homomorphism from the Grothendieck ring to $\mathbb{C}$, which is to be compared with \cite{13}.

**Proposition 23** (cf. \cite{7}, pg. 593). In any root choice $\{d_i\}$, we have

$$d_j d_k = \sum_i \Tr(T^i_{jk}) d_i.$$

**Proof.**

$$d_j d_k = \begin{array}{c}
\begin{tikzpicture}
  \node (i) at (0,0) {$j$};
  \node (j) at (2,0) {$k$};
  \draw (i) to (j);
\end{tikzpicture}
\end{array}
$$

$$= \begin{array}{c}
\begin{tikzpicture}
  \node (i) at (0,0) {$j^*$};
  \node (j) at (2,0) {$j^*$};
  \draw (i) to (j);
\end{tikzpicture}
\end{array}
$$

$$= \sum_{i,\alpha} k^* i^{\alpha} j^* i^\alpha d_i \quad \text{(by Lemma 21)}
$$

$$= \sum_i \Tr(T^i_{jk}) d_i.$$

**Corollary 24.** Let $C$ be a pseudo-unitary fusion category. Then in the root choice $d_i = d_i^+$, the pivotal operators $T^i_{jk}$ are the identity maps.

**Proof.** Combining \cite{13} with Proposition 23, we have

$$d_j d_k = \sum_i \Tr(T^i_{jk}) d_i \leq \sum_i N^i_{jk} d_i = d_j d_k$$

and hence we must have $T^i_{jk} = \text{id}$. \hfill $\square$

**Lemma 25** (cf. \cite{7}, pg 594]). The numbers $\Tr(T^i_{jk})$ have the following symmetry properties:

(i) $\Tr(T^i_{jk}) = \Tr(T^{k^*}_{i^*j})$ (Conjugate cyclic)

(ii) $\Tr(T^i_{jk}) = \Tr(T^{j^*}_{k^*i})$ (Conjugate symmetric)
Proof. To establish (i), suppose that \( \{ e_\alpha : X_i \to X_j \otimes X_k \} \) are eigenvectors of \( T_{jk}^i \), so that \( T_{jk}^i e_\alpha = \epsilon_{j,k,\alpha} e_\alpha \). Choose arbitrary right duals \( (X_i^*, \eta_i, \epsilon_i) \) and \( (X_k^*, \eta_k, \epsilon_k) \) for \( X_i \) and \( X_k \). Then the basis for \( \text{Hom}(X_k^*, X_i^* \otimes X_j^*) \) given by

\[
 f_\alpha := \frac{1}{\epsilon_{j,k,\alpha}} \epsilon_{j,k,\alpha} f_\alpha.
\]

are also eigenvectors of \( T_{k,j}^{i*} \) with the same eigenvalues as the \( e_\alpha \), since

\[
 T_{k,j}^{i*} (f_\alpha) = \epsilon_{j,k,\alpha} \epsilon_{j,k,\alpha} f_\alpha.
\]

The proof of (ii) is similar, except one uses the dual basis \( \{ e^\alpha : X_j \otimes X_k \to X_i \} \) to define a basis \( \{ g^\alpha \} \) for \( \text{Hom}(X_i^*, X_k^* \otimes X_j^*) \) by setting

\[
 g^\alpha := \frac{1}{\epsilon_{j,k,\alpha}} \epsilon_{j,k,\alpha} g^\alpha.
\]

A similar string diagram argument then establishes that \( T_{k,j}^{i*} g^\alpha = \epsilon_{j,k,\alpha} g^\alpha \).

\[ \square \]

**Theorem 26** (cf. [7, Thm 2.3]). The paired dimensions \( d_{\{i,i^*\}} \) are real and positive.

**Proof.** We organize the various roots \( d_i \equiv \bigotimes_i \) into a column vector:

\[
 d = \left( \begin{array}{c}
 \bigotimes X_i \\
 \bigotimes X_{i_2} \\
 \vdots \\
 \bigotimes X_n \\
 \end{array} \right)^T.
\]

Define the matrices \( A_j \) via \( [A_j]_{ik} = \text{Tr}(T_{jk}^i) \). Then Proposition 23 says that \( d \) is a simultaneous eigenvector of each \( A_j \) with eigenvalue \( d_j \), i.e.

\[
 A_j d = d_j d.
\]

Thus we have \( A_j A_j d = d_j d_j d \) and \( d_{\{j,j^*\}} \) is an eigenvalue of the matrix \( A_j A_{j^*} \). But \( A_j^* = A_j^T \), by the symmetry properties established in
Lemma 25

\[ [A_j^*]^k = \text{Tr}(T_{j^\ast,k}) \]
\[ = \text{Tr}(T_{i^\ast,j^\ast}^k) \]
\[ = \text{Tr}(T_j^k) \]
\[ = [A_j^*]^k. \]

Thus \( d_{(j,j^\ast)} \) is an eigenvalue of the positive definite real matrix \( A_jA_j^T \) and is therefore real and positive.

Remark 27. From now on, we will always work in the canonical root choice on a fusion category given by the positive square roots of the paired dimensions, \( d_i = \sqrt{d_{(i,i^\ast)}}. \)

3.4 Monodromy interpretation of the pivotal operators

We can write the pivotal operators as a product of cyclic operators as follows. Write \( V_{ijk} = \text{Hom}(1, X_i \otimes (X_j \otimes X_k)). \)

Definition 28. In a fusion category, the cyclic operators \( C_{ijk}: V_{ijk} \to V_{kij} \) are defined as follows, using the pairing convention:

The monodromy of the cyclic operators is the operator \( T_{ijk} \) on \( V_{ijk} \) given as the composite

\[ T_{ijk} = V_{ijk} \xrightarrow{C_{ijk}} V_{kij} \xrightarrow{C_{kij}} V_{jki} \xrightarrow{C_{jki}} V_{ijk}. \]

Write \( Y \) and \( Y^{-1} \) for the ‘yanking’ maps

\[ Y : \text{Hom}(X_i, X_j \otimes X_k) \rightleftharpoons \text{Hom}(1, X_i^* \otimes X_j \otimes X_k) \]

defined by making some choice of structure maps equipping \( X_i^* \) as the right dual of \( X_i \). The following lemma is immediate.

Lemma 29. The pivotal operator \( T_{ijk}^i \) computes as the conjugate of the monodromy of the cyclic operators:

\[ T_{ijk}^i = Y^{-1}T_{i^\ast,jk}Y \]

Note that the statement of the lemma is well-defined since the right-hand side is independent of the right dual structure maps needed to define \( Y \) (the dependency cancels due to the \( Y^{-1} \)).

Remark 30. In the special case \( i = j = k \), the trace of the cyclic operator \( C_{iii} \) is closely related, but not equal, to the 3rd Frobenius-Schur indicator \( \nu_3(X_i) \) of \( X_i \), as defined in [20]. The indicator \( \nu_3(X_i) \) needs a pivotal structure \( \gamma \) for its definition, whence we will write it as \( \nu_3^\gamma(X_i) \). However, the cyclic operator \( C_{iii} \) is canonically defined on the fusion category without a pivotal structure.
general, given a pivotal structure $\gamma$ on a fusion category, every self-dual simple object $X_i$ has an associated sign $s_i^\gamma = \pm 1$, the sign of the quantum dimension of $X_i$ in the pivotal structure $\gamma$. The relationship is then

$$\text{Tr}(C_{iii}) = s_i^\gamma \nu_3(X_i).$$

3.5 The pivotal indicators

In the next two subsections we make some explicit choices in order to write down explicit formulas for the paired dimensions and the pivotal operators in terms of the associator matrices.

Let $C$ be a fusion category, with representative simple objects $X_i, i \in I$. Thus we have a map $*: I \to I$ with $** = \text{id}$. Choose for each $i \in I$ a nonzero map $\eta_i : 1 \to X_i^* \otimes X_i$:

\[ X_i^* \longrightarrow \downarrow X_i \]

Define $\hat{\eta}_i : X_i^* \otimes X_i \to 1$ as the linear dual of $\eta_i$, so that

\[ X_i^* \xrightarrow{\eta_i} X_i \xrightarrow{\hat{\eta}_i} 1 \]

for all $i \in I$. Define the complex numbers $a_i$ as the coefficients appearing in the associator expansion

\[ X_i \xrightarrow{a_i} X_i^* \otimes X_i \xrightarrow{\eta_i} 1 \]  \hspace{1cm} + \text{other terms.} \hspace{1cm} (17) \]

Similarly define $b_i$ as the coefficients appearing in the inverse associator expansion

\[ X_i \xrightarrow{b_i} X_i^* \otimes X_i \xrightarrow{\eta_i} 1 \]  \hspace{1cm} + \text{other terms.} \hspace{1cm} (18) \]

Clearly, these coefficients are precisely the numbers appearing in

\[ X_i \xrightarrow{a_i} X_i^* \]  \hspace{1cm} = X_i \]

\[ X_i \xrightarrow{b_i} X_i^* \]  \hspace{1cm} = X_i^* \]
In particular, they are nonzero as $C$ is rigid (if they were zero, it would be impossible to choose counit maps satisfying the rigidity equations).

**Example 31.** In the Yang-Lee category, we read off from (7) that $a_x = -\frac{1}{2}(1 + \sqrt{5})$. In category $E$, we read off from (10) that $a_x = 1 + \sqrt{3}$; similarly $a_y = 1$.

**Lemma 32.** $b_i = a_i^*$ for all $i \in I$.

**Proof.** Adapted from [1, Prop 5.3.13]. Consider evaluating the morphism

![Diagram]

in two different ways. On the one hand, by dragging the right-most $\eta$ downwards, and then using (19a), it equals $a_i \eta_i$. On the other hand, by dragging the left-most $\eta$ downwards, and then using (19b), it equals $b_i^* \eta_i$. Hence $a_i = b_i^*$.

For each $i \in I$, define $\epsilon_i : X_i \otimes X_i^* \to 1$ by $X_i \downarrow \otimes \ X_i^* = \frac{1}{a_i} X_i \downarrow \otimes X_i^*$. We have just proved the following:

**Lemma 33.** For each $i \in I$, $(\eta_i, \epsilon_i)$ satisfy the rigidity equations furnishing $X_i^*$ as a right dual of $X_i$.

**Corollary 34.** The paired dimensions compute as $d_{(i,i^*)} = \frac{1}{a_i a_i^*}$.

**Lemma 35.** There exists a choice of basis of the $\eta_i$ such that $a_i = a_i^*$ for all $i \in I$.

**Proof.** If $X_i$ is self-dual, then $i = i^*$ so we are done. Partition the non self-dual objects into ordered pairs $(X_i, X_i^*)$. Scale the $\eta_i$ by setting $\eta'_i = \eta_i$ and $\eta'^*_i = \lambda_i \eta_i^*$, where $\lambda_i$ is a root of $\lambda_i^2 = \frac{a_i}{a_i^*}$. In this new basis we have $a_i' = a_i^*$. In such a basis we have $a_{i}^{-2} = d_{(i,i^*)}$ for all $i \in I$. By Theorem 26 the paired dimension $d_{(i,i^*)}$ is a positive real number, with the fusion dimension $d_i$ defined as its positive square root. Hence $a_i = \pm \frac{1}{d_i}$. If $X_i$ is not self-dual and $a_i = -d_i$, then we can remove this sign by setting $\eta'_i = -\eta_i$, $\eta'^*_i = \eta_i^*$, after which we have $a_i' = d_i$. We call such a basis choice $\eta_i \in \text{Hom}(1, X_i \otimes X_i)$ satisfying $a_i > 0$ for all non self-dual $X_i$ a fair basis.

Note that if $X_i$ is self-dual then this sign cannot be removed. We record this as a definition.
Definition 36. Let $C$ be a fusion category over $\mathbb{C}$. The pivotal indicator $p(X_i)$ of a simple object $X_i$ is defined as follows. If $X_i$ is not self-dual, $p(X_i) = 1$. If $X_i$ is self-dual, then $p(X_i)$ is the sign of the coefficient $a_i$ appearing in the associator expansion

$$X_i = a_i + \text{other terms.}$$

where $\eta_i : 1 \to X_i \otimes X_i$ is some nonzero vector (the number $a_i$ is independent of this choice).

Example 37. In the Yang-Lee category, we read off from (7) that $p(\tau) = -1$. In category $\mathcal{E}$, we read off from (10) that $p(x) = 1$, similarly $p(y) = 1$.

Remark 38. As in Remark 30, the pivotal indicator $p(X_i)$ of a simple object is closely related, but not equal to, the 2nd Frobenius-Schur indicator $\nu_2(X_i)$, as defined in [20]. The former is defined using only the underlying fusion category, while the latter depends on a pivotal structure $\gamma$, whence we can write it as $\nu_{2\gamma}(X_i)$. The relationship is $p(X_i) = s_i^\gamma \nu_2(X_i)$, where $s_i^\gamma = \pm 1$ is the sign of the quantum dimension of $X_i$ in the pivotal structure $\gamma$.

3.6 Formula for the pivotal operators

We now show how to compute the pivotal operators directly in terms of the associator matrix elements.

Definition 39. The apex associators are the associator matrix elements where the top strand is the identity

$$= \sum_{\beta} (S_{ijk})_{\beta\alpha} \eta_k^\alpha \eta_i^\beta.$$ 

That is, $(S_{ijk})_{\beta\alpha} = (F_{ijk})_{\eta_k^\alpha \eta_i^\beta}$. The apex associator monodromy $A_{ijk}$ is the product of matrices $S_{jki} S_{kij} S_{ijk}$.

Example 40. In the Yang-Lee category, we read off from (9) that $S_{\tau\tau\tau} = 1$. For category $\mathcal{E}$, we read off from (11) and (12) that $S_{xxx} = 1$ and the apex associator monodromy is $A_{xxx} = S_{xxx}^3 = \text{id.}$

Thus they form a sharp angle diagrammatically, hence the name.
Remark 41. In all examples that the author knows of, the apex associator monodromy in a fair basis is the identity.

**Proposition 42.** In a fair basis, the cyclic operators $C_{ijk}$ compute as

$$C_{ijk} = p_k S_{ijk}$$

where $p_k$ is the pivotal indicator of $X_k$ and $S_{ijk}$ are the apex associators.

**Proof.** By definition, and using the pairing convention, the bending matrix elements $(C_{ijk})_{\beta\alpha}$ are the coefficients appearing in the expansion

$$d_k = \sum_\beta (C_{ijk})_{\beta\alpha}.$$ 

We can rewrite the left-hand side using the apex associators (20):

$$\text{LHS} = \sum_\beta (S_{ijk})_{\beta\alpha} = d_k a_k^* \sum_\beta (S_{ijk})_{\beta\alpha}.$$ 

The second equality uses (19b) and Lemma 32. In a fair basis, $d_k a_k^* = p_k$ and we are done.

We have thus proved the following result.

**Theorem 43.** In a fair basis, the pivotal operators $T_{ijk}$ can be written as

$$T_{ijk} = p_i p_j p_k A_{ijk}$$

(21)

**Remark 44.** The above formula refines a formula of Wang [26, Prop 4.17], which treats the multiplicity-free case and does not include the pivotal indicators.

**Example 45.** In the Yang-Lee category, $T_{\tau\tau\tau} = (-1)^3 = -1$. In category $\mathcal{E}$, all the pivotal operators are the identity.

### 3.7 The pivotal operators are monoidal

We want to precisely relate the pivotal operators with the ‘double dual functor’. We first need to show that the pivotal operators can be viewed as equipping the identity functor with coherence isomorphisms making it into a monoidal functor.
Lemma 46. Suppose \( F,G : D \to D' \) are linear functors between semisimple categories. Let \( \theta_i : F(X_i) \to G(X_i) \) be a collection of maps, where \( X_i \) ranges over the representatives of the simple objects of \( D \). Then there exists a unique natural transformation \( \theta : F \Rightarrow G \) such that \( \theta_{X_i} = \theta_i \).

Proof. Let \( A \in C \), and for each \( i \), choose a basis \( \{ e_\alpha : X_i \to A \} \) for \( \text{Hom}(X_i,A) \) with corresponding dual basis \( \{ e^\beta : A \to X_i \} \). Set \( \theta_A = \sum_\alpha \theta(e_\alpha) \circ \theta_i \circ F(e^\alpha) \).

This allows us to extend the pivotal operators \( T_{jk} \), initially defined only on the simple objects, to a natural transformation \( T : \otimes \Rightarrow \otimes \) from the tensor functor \( \otimes : C \boxtimes C \to C \) to itself.

Corollary 47. The pivotal operators \( T_{jk} \) extend uniquely to a natural transformation \( T : \otimes \Rightarrow \otimes \) from the tensor functor \( \otimes : C \boxtimes C \to C \) to itself.

Proof. The pivotal operators \( T_{jk} : \text{Hom}(X_i,X_j \otimes X_k) \to \text{Hom}(X_i,X_j \otimes X_k) \) induce maps
\[
T_{i,j} : X_i \otimes X_j \to X_i \otimes X_j
\]
for each pair of simple objects \( X_i, X_j \), defined uniquely by the requirement that for each simple object \( X_k \), post-composition with \( T_{i,j} \) is equal to \( T_{i,k} \). Now apply Lemma 46 to the case \( D = C \boxtimes C, C' = C \), and \( F = G \) given by the tensor functor \( \otimes : C \boxtimes C \to C \).

Recall that a monoidal functor \( (F,T,\phi) : C \to D \) between monoidal categories \( C \) and \( D \) consists of a functor \( F : C \to D \) together with natural isomorphisms \( T_{A,B} : F(A) \otimes F(B) \to F(A \otimes B) \) and \( \phi : 1_D \to F(1_C) \) satisfying certain coherence equations.

Proposition 48. The pivotal operators \( T_{jk} \), when extended to a natural transformation \( \{T_{A,B} : A \otimes B \to A \otimes B\} \), obey the coherence isomorphisms making \( T := (\text{id},T,\text{id}) : C \to C \) a monoidal functor.

Proof. By Lemma 46, we only need to check the coherence equations on the simple objects. The unit coherence equations are \( T_{1,1} = \text{id} \), which is automatically satisfied due to the rigidity equations (6). The coherence conditions on \( T \) are:

\[
\begin{align*}
F(X_j \otimes X_k) \otimes F(X_l) & \xrightarrow{T_{X_j \otimes X_k, X_l}} F((X_j \otimes X_k) \otimes X_l) \\
& \xrightarrow{T_{X_j, X_k \otimes X_l}} F(X_j \otimes (X_k \otimes X_l)) \\
& \xrightarrow{a_{F(X_j), F(X_k), F(X_l)}} F(X_j \otimes (X_k \otimes X_l)) \\
& \xrightarrow{\text{id} \otimes T_{X_k, X_l}} F(X_j) \otimes F(X_k \otimes X_l)
\end{align*}
\]

(22)
Apply this diagram to the basis vectors (2), working in a pivotal basis to simplify the calculation. It becomes the requirement that

$$
\sum_{n, \gamma, \delta} (F_{ijkl}^{i})_{\alpha m, \beta n, \delta} = 0
$$

To prove (23), insert the graphical definition (15) of the pivotal operators into (23), and expand out the left hand side using the associator expansion (3). As in the proof of Theorem 16, we obtain two diagrams which are rigidly isotopic, hence equal.

Corollary 49. In a pivotal basis, the pivotal symbols satisfy

$$
\epsilon_{i j, \alpha}^{m} \epsilon_{i k, \beta}^{n} = \epsilon_{i l, \gamma}^{n} \epsilon_{i j, \delta}^{m} \text{ whenever } (F_{ijkl}^{i})_{\alpha m, \beta n, \delta} \neq 0.
$$

Proof. This is precisely what (23) says, in a pivotal basis. □

Remark 50. The equation (24) satisfied by the $\epsilon_{i j, \alpha}$ is formally similar to the equation satisfied by the 2nd Stiefel-Whitney 2-cocycle $w_2(M) \in H^2(M, \mathbb{Z}/2\mathbb{Z})$ of an oriented manifold $M$ in Čech cohomology [19]. It can be thought of as a ‘$\mathbb{Z}/2$ version’ of the 2-cocycle condition in Davydov-Yetter cohomology of the fusion category, see Example 78.

Let us summarize the results of this section explicitly.

Definition 51. Let $C$ be a fusion category over $\mathbb{C}$. The pivotal endofunctor $T : C \to C$ of $C$ is the monoidal functor whose underlying functor is the identity functor, and whose monoidal coherence isomorphisms are the pivotal operators $T_{ijk}^i$.

We have proved the following.

Theorem 52 (cf. [7, Thm 2.6]). Every fusion category $C$ over $\mathbb{C}$ comes equipped with a canonical monoidal endofunctor $T : C \to C$, the pivotal endofunctor, whose underlying functor is the identity and which satisfies $T^2 = id$.

3.8 The pivotal operators are the double dual functor

We now relate our approach to the traditional one, by showing that the pivotal operators are simply the coherence isomorphisms of the double dual functor, with respect to a judicious choice of duals.

To define the dual functor $*: C \to C^{\text{op}}$, one needs to make a choice, for every object $V \in C$, of a triple

$$
(V^*, \eta_V : 1 \to V^* \otimes V, \epsilon_V : V \otimes V^* \to 1)
$$

which equips $V^*$ as a right dual of $V$. It is then a routine but tedious calculation to check that this makes $*: C \to C^{\text{op}}$ into a well-defined functor, which is equipped with canonical isomorphisms $V^* \otimes W^* \to (W \otimes V)^*$ making it into a monoidal functor. In particular, given such a choice of right duals, we have the associated double dual monoidal endofunctor $**$ of $C$, and coherence isomorphisms $\omega_{V,W} : V^{**} \otimes W^{**} \to (V \otimes W)^{**}$.
Proposition 53. Let $C$ be a fusion category. There exists a choice of right duals on $C$ such that $** = T$, the canonical pivotal endofunctor of $C$.

Proof. We can partition the class of objects of $C$ into singletons and pairs in such a way that if $V$ is self-dual, then it is in a singleton (where we set $V^* := V$), else it forms part of a pair $(V, \bar{V})$ of dual objects (where we set $V^* := \bar{V}$ and $\bar{V}^* := V$). It remains to choose the unit and counit maps.

Start with the simple objects. If $X$ is a self-dual simple object, make an arbitrary choice of maps $(\eta_X : 1 \to X \otimes X, \epsilon_X : 1 \to X)$ satisfying the rigidity equations, and set $(\eta_{X^*}, \epsilon_{X^*}) = (p_X \eta_X, p_X \epsilon_X)$, where $p_X$ is the pivotal indicator of $X$. This ensures that a closed loop labelled by $X$ evaluates to $d_X$, the fusion dimension of $X$ (and not its negative), as the partner convention demands.

If $X$ is a non-self-dual simple object, again make an arbitrary choice of unit and counit maps $(\eta_X : 1 \to X \otimes X, \epsilon_X : 1 \to X \otimes X)$ satisfying the rigidity equations, and set $\eta_{X^*}, \epsilon_{X^*}$ to be the unique maps satisfying the partner convention, i.e. $\epsilon_{X^*} \eta_X = d_X = \epsilon_X \eta_{X^*}$.

Now, choose representative simple objects $X_i, i \in I$, and for each pair of non-self-dual objects $(V, \bar{V})$, make an arbitrary choice of basis $e^\alpha : X_i \to V$ and $f^\alpha : X_i \to \bar{V}$, with associated dual bases $e^\beta, f^\beta$ as usual. Define $\eta_V : 1 \to V \otimes V$ and $\epsilon_V : V \otimes \bar{V} \to 1$ by

and make similar definitions for $\eta_{V^*}, \epsilon_{V^*}$. Do the same for the self-dual non-simple objects, adding the pivotal indicators as before. It is now a routine calculation to check that $**$ is the identity functor, and that $\omega_{V,W}$ computes as the pivotal operator $T_{V,W}$.

4 Pivotal structures

In this section we write down explicit equations for pivotal structures, define fusion homomorphisms, and discuss the sphericalization of a fusion category. Finally we show that the ratio $\frac{D}{\Delta}$ in a fusion category is an algebraic integer.

4.1 Explicit equations for pivotal structures

In Section 3.7 we showed that, by using the pairing convention, every fusion category over $C$ comes equipped with a canonical monoidal endofunctor $T : C \to C$, the pivotal endofunctor, whose underlying functor is simply the identity, with the pivotal operators $T_{jk}$ supplying the coherence isomorphisms.

On the other hand, a pivotal structure on a fusion category is usually defined by making choices of right duals $(V^*, \eta_V, \epsilon_V)$ for every object $V$ as in Section 3.8, giving rise to a monoidal double dual functor $** : C \to C$, and then declaring that a pivotal structure is a monoidal natural isomorphism $\gamma : \text{id} \Rightarrow **$. In Section 3.8 we showed that under a judicious choice of right duals, the double
dual functor \( \ast \ast \) computes as \( T \). In other words, although the first dual functor \( \ast : \mathcal{C} \to \mathcal{C}^{\text{op}} \) requires choices, the second dual functor \( \ast \ast : \mathcal{C} \to \mathcal{C} \) is canonical.

Thus in this paper we adopt the following definition.

**Definition 54.** A pivotal structure on a fusion category over \( \mathcal{C} \) is a monoidal natural isomorphism \( \gamma : \text{id} \Rightarrow T \), where \( T \) is the canonical pivotal endofunctor of the category.

For an alternative formulation in terms of *even-handed structures*, see [2]. Under this definition, quantum dimensions are defined as follows.

**Definition 55.** Let \( \gamma \) be a pivotal structure on a fusion category. The quantum dimension of an object \( V \) with respect to \( \gamma \) is defined as

\[
\dim_{\gamma}(V) = V^{\ast} \quad \text{where the pairing convention has been used. The pivotal structure \( \gamma \) is spherical when } \dim_{\gamma}(V) = \dim_{\gamma}(V^{\ast}) \text{ for all objects } V.
\]

The following is clear.

**Lemma 56.** The quantum dimension of a simple object \( X_i \) with respect to a pivotal structure \( \gamma \) computes as \( \dim_{\gamma}(X_i) = \gamma_i d_i \) where \( d_i \) is the fusion dimension of \( X_i \).

Under these conventions, we have the following.

**Theorem 57.** A pivotal structure on fusion category corresponds to a collection of numbers \( \gamma_i \in U(1) \), \( i \in I \) satisfying

\[
\gamma_j \gamma_k \text{id} = \gamma_i T_{jk}^i \quad \text{whenever } X_i \text{ is a summand of } X_j \otimes X_k \quad (25)
\]

where the \( T_{jk}^i \) are the pivotal operators of the fusion category. Moreover, the pivotal structure is spherical precisely when \( \gamma_i = \pm 1 \) for all \( i \in I \).

**Proof.** By Lemma 46, \( \gamma \) is uniquely determined by its components \( \gamma_i = \gamma_{X_i} \) on the representative simple objects \( X_i \). The coherence equation for being a monoidal natural isomorphism reduces to the scalars \( \gamma_i \in \mathbb{C}^{\times} \) obeying (25). Given a solution to (25), the collection \( \{\pm \gamma_i\} \) forms a finite subgroup of \( \mathbb{C}^{\times} \) and hence \( \gamma_i \in U(1) \). Since \( T_{11}^i = \text{id} \) from Lemma 22 we have \( \gamma_1 = 1 \), and similarly since \( T_{ii}^i = \text{id} \), we have \( \gamma_i = \gamma_i \) for all \( i \in I \). From Lemma 56 if \( \gamma \) is spherical, then \( \gamma_i d_i = \gamma_i d_i \) whence \( \gamma_i = \pm 1 \) since \( d_i = d_i \). Hence \( \gamma_i = \pm 1 \).

**Corollary 58** (cf. [7, Prop 8.23]). A pseudo-unitary fusion category admits a canonical spherical structure.

**Proof.** Follows from Corollary 24.

**Definition 59.** We say that a fusion category \( \mathcal{C} \) over \( \mathbb{C} \) is orientable if the pivotal symbols \( \epsilon_{j;k,\alpha} \) do not depend on \( \alpha \), that is, \( T_{jk}^i = \epsilon_{j;k} \text{id} \) for some signs \( \epsilon_{j;k} = \pm 1 \).
Clearly we have the following.

**Lemma 60.** If a fusion category is not orientable, then it does not admit a pivotal structure.

Recall from Theorem 43 that the pivotal operators can be expressed in terms of the apex associator monodromy operators.

**Proposition 61.** If the apex associator monodromy $A_{ijk} = \text{id}$ for all $i,j,k \in I$, then the fusion category admits a canonical spherical structure.

**Proof.** If $A_{ijk} = \text{id}$, then from Theorem 43,

$$T^i_{jk} = p_i p_j p_k \text{id}$$

where the $p_i = \pm 1$ are the pivotal indicators. Hence setting $\gamma_i = p_i$ will solve (25).

### 4.2 Fusion homomorphisms

The following property of quantum dimensions is well-known.

**Lemma 62.** Given a pivotal structure $\gamma$ on a fusion category $C$, the quantum dimension map $\text{dim}_\gamma : K(C) \rightarrow \mathbb{C}$ which sends $[V] \mapsto \text{dim}_\gamma(V)$ satisfies the following:

- It is a ring homomorphism,
- $\text{dim}_\gamma[X_i] \text{dim}_\gamma[X^*_i] = d_{\{i,i^*\}}$ for all simple objects $X_i$.

This motivates the following definition.

**Definition 63.** A function $f : K(C) \rightarrow \mathbb{C}$ from the Grothendieck ring of a fusion category to the ground field is called a fusion homomorphism if it is a ring homomorphism and if $f[X_i]f[X^*_i] = d_{\{i,i^*\}}$ for all simple objects $X_i$.

For a fusion category $C$, it is interesting to consider whether the injective map

$$\text{Pivotal structures}(C) \rightarrow \{\text{Fusion homomorphisms } f : K(C) \rightarrow \mathbb{C}\}.$$ 

is surjective. The following case is instructive. Fix a finite group $G$.

**Proposition 64.** The pivotal structures on $\text{Rep}(G)$ are in 1-1 correspondence with the fusion homomorphisms from $K(\text{Rep}(G))$ to $\mathbb{C}$.

**Proof.** We have

$$\text{Pivotal structures}(\text{Rep}(G)) \cong \text{Aut}_{\otimes}(\text{id})$$

$$\cong Z(G)$$

$$= \{g \in G : |\text{Tr}_{V_i}(g)| = \text{dim} V_i \text{ for all irreducibles } V_i\}$$

$$\cong \{\text{Fusion homomorphisms } f : [\text{Rep}(G)] \rightarrow \mathbb{C}\}.$$ 

The first isomorphism uses the fact that $\text{Rep}(G)$ comes with a canonical pivotal structure, and that in general the set of pivotal structures is a torsor for
The second isomorphism is a result of Müger [18]. The equality in the third line is a basic result of representation theory [10, Cor 2.28]. The final isomorphism uses two facts. Firstly, every ring homomorphism 

\[ [\text{Rep}(G)] \to \mathbb{C} \]

must take the form \( V \mapsto \text{Tr}_V(g) \) for some fixed \( g \in G \) — because a character like this certainly is a ring homomorphism, and there are as many such distinct characters as there are conjugacy classes in the group, which must exhaust all the ring homomorphisms since \( [\text{Rep}(G)] \mathbb{C} \) is isomorphic to the space of functions on the conjugacy classes. Secondly, in \( \text{Rep}(G) \) the paired dimensions \( d_{i,i'} \) are just \( \dim(V_i)^2 \), so that a fusion homomorphism must satisfy \( |f(V_i)| = \dim V_i \).

### 4.3 Sphericalization of a fusion category

We have seen in Corollary 52 that every fusion category \( C \) over \( \mathbb{C} \) comes equipped with a canonical monoidal action of \( \mathbb{Z}/2\mathbb{Z} \). The ‘equivariantization’ with respect to this action defines a new fusion category admitting a canonical spherical structure.

Recall (see eg. [21]) that a monoidal action of a group \( G \) on a monoidal category \( C \) is a monoidal functor \( F : BG \to \text{Aut}_\otimes(C) \), where \( BG \) is the group \( G \) thought of as a one-object category, and \( \text{Aut}_\otimes(C) \) is the monoidal category of monoidal endofunctors of \( C \).

**Definition 65.** The equivariantization \( C^G \) is the monoidal category defined as follows:

- An object of \( C^G \) consists of an object \( X \in C \) together with isomorphisms \( s_g : F_g(X) \to X \) for each \( g \in G \) such that the diagram

\[
\begin{array}{ccc}
F_g F_h(X) & \xrightarrow{F_g(s_h)} & F_g(X) \\
\gamma(g,h)_X & \downarrow & \downarrow s_g \\
F_{gh}(X) & \xrightarrow{s_{gh}} & X
\end{array}
\]

(26)

commutes for each \( g, h \in G \).

- A morphism in \( C^G \) is a morphism \( f : X \to Y \) in \( C \) satisfying

\[
f \circ s_g^X = s_y^g \circ F_g(f)
\]

(27)

for each \( g \in G \).

- The tensor product is defined by \( (X, s^X) \otimes (Y, s^Y) = (X \otimes Y, s^{X \otimes Y}) \) where \( s^{X \otimes Y}_g \) is the composite

\[
F_g(X \otimes Y) \xrightarrow{(T_g)^{-1}_{X,Y}} F_g(X) \otimes F_g(Y) \xrightarrow{s^X_g \otimes s^Y_g} X \otimes Y
\]

(28)

where \( T_{X,Y}^g : F_g(X) \otimes F_g(Y) \Rightarrow F_g(X \otimes Y) \) are the coherence isomorphisms equipping \( F_g \) as a monoidal functor.
Definition 66. The sphericalization $\tilde{C}$ of a fusion category $C$ is its equivariantization with respect to the canonical $\mathbb{Z}/2\mathbb{Z}$ action on it.

The advantage of our approach is that the action of $\mathbb{Z}/2\mathbb{Z}$ is especially simple, as the group acts by identity functors, with the monoidal coherence isomorphisms encoded in the pivotal operators $T_{jk}^i$.

Lemma 67. The sphericalization $\tilde{C}$ of $C$ has simple objects $X_i^s$, where $X_i$ is a simple object in $C$, and $s_i = \pm 1$. The fusion hom-vector spaces in $\tilde{C}$ compute in terms of the decompositions of the fusion hom-sets in $C$ (see (16)) as

$$\text{Hom}_{\tilde{C}}(X_i^s, X_j^r \otimes X_k^t) = \text{Hom}_C(X_i, X_j \otimes X_k)_{s_i s_j s_k}.$$ (29)

The associator in $\tilde{C}$ is the pullback of the associator in $C$. The forgetful tensor functor $F : \tilde{C} \to C$ sends $X_i^r \mapsto X_i$. The pivotal symbols $\epsilon$ of $\tilde{C}$ compute in terms of the pivotal symbols $\epsilon$ of $C$ as

$$\epsilon_{(i,s)}(j,s)(k,s_k) = s_i s_j s_k.$$ (30)

Proof. Write $\mathbb{Z}/2\mathbb{Z} = \{1, -1\}$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $C$ is strict as a group action, so $\gamma = \text{id}$ in (26). For a simple object $X_i$, write $s_i = X_i^{1}$. Then (26) becomes $s_i^2 = 1$ so $s_i = \pm 1$. Let $e_\alpha : X_i \to X_i \otimes X_k$ be a pivotal basis. From (27), $e_\alpha \in \text{Hom}(X_i^s, X_j^r \otimes X_k^t)$ if and only if

$$X_i \xrightarrow{e_\alpha} X_j \otimes X_k \xrightarrow{T_{X_j \otimes X_k}} X_j \otimes X_k \xrightarrow{s_j s_k}$$ (31)

commutes. By definition (see Corollary 47), $T_{X_j \otimes X_k} \circ e_\alpha = \epsilon_{jk, \alpha} e_\alpha$, so that (31) gives (29), as well as (30). \qed

Corollary 68 (cf. [7, Prop 5.14]). The sphericalization $\tilde{C}$ of a fusion category $C$ carries a canonical spherical structure, in which the quantum dimension of a simple object $X_i^r$ in $\tilde{C}$ computes in terms of the fusion dimension $d_i$ of $X_i$ in $C$ as

$$\dim(X_i^r) = s_i d_i.$$ (32)

Proof. By (29), $\tilde{C}$ is orientable. So from Theorem 57, a spherical structure amounts to a choice of signs $t_{(i,s_i)} = \pm 1$ satisfying

$$t_{(j,s_j)} t_{(k,s_k)} = s_i s_j s_k t_{(i,s_i)}$$

whenever $\text{Hom}(X_i, X_j \otimes X_k)_{s_i s_j s_k}$ is nonzero. Clearly the choice $t_{(i,s_i)} = s_i$ provides a solution, and (32) follows from the definition of the quantum dimension. \qed

Example 69. The sphericalization of the Yang-Lee category has simple objects $1^+, 1^-, \tau^+, \tau^-$. The fusion rules are $\tau^+ \tau^+ = 1^-, 1^- 1^- = 1^+, \tau^+ 1^- = 1^- \tau^+ = \tau^+$, and $\dim(\tau^\pm) = \pm d_\tau = \pm (\phi - 1)$. 

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Lemma 70. The Frobenius-Perron and fusion dimensions in $\tilde{C}$ are the same as in $C$, that is, $\bar{d}_{(X_i,s_i)}^{r} = d_{X_i}^{r}$ and $d_{(X_i,s_i)} = d_i$.

Proof. We claim that setting $d_{(X_i,s_i)}^{r} = d_{X_i}^{r}$ solves (13) and hence is the unique solution, by the Frobenius-Perron theorem. Indeed,

$$d_{(X_j,s_j)}^{r}d_{(X_k,s_k)}^{r} = \sum_i N_{jk}^i d_i^r$$

$$= \sum_i \left[ (N_{jk}^i)_{s_j s_k+} + (N_{jk}^i)_{s_j s_k-} \right] d_i^r \quad \text{for all } s_j, s_k = \pm 1$$

$$= \sum_{(i,s_i)} (N_{jk}^i)_{s_j s_k s_i} d_{(X_i,s_i)}^{r}.$$

For the fusion dimensions, the associator in $\tilde{C}$ is the pullback of that in $\tilde{C}$, so the paired dimensions are same as in $C$, i.e. $d_{(X_i,s_i),(X_i^*,s_i)} = d_{(i,i^*)}$. □

4.4 Algebraic integers

Recall that a modular category is a ribbon fusion category whose $s$-matrix

$$s_{ij} = \begin{array}{c}
\circlearrowleft\circlearrowright\\
1 \\

\end{array}$$

is invertible. (For consistency, we have used the pairing convention in the above diagram, so we must explicitly include the pivotal structure maps $\gamma$). The following results are well-known, with string diagram proofs given in [1].

Theorem 71. In a modular category, the following identities hold.

- (Verlinde’s formula) $\sum_k N_{ij}^k s_{kr} = \frac{s_{ir} s_{jr}}{s_{1r}}$.

- (Circumcision) $(s^2)_{ij} = D \delta_{ij}$.

In a modular category, the global dimension and the Frobenius-Perron dimension are closely related.

Lemma 72. In a modular category, with Frobenius-Perron dimension $\Delta$ and global dimension $D$, there exists a simple object $X_r$ such that $D/\Delta = \dim(X_r)^2$.

Proof. Follows from the Frobenius-Perron theorem, Verlinde’s formula, and Circumcision, as in [7, pg 622]. □

If $C$ is a $\mathbb{C}$-linear spherical fusion category, then it is straightforward to check that $Z(C)$ is a $\mathbb{C}$-linear, braided, rigid, spherical abelian category whose monoidal unit is simple. It is not immediately clear that $Z(C)$ is a fusion category, that is, semisimple with finitely many simple objects, or that it is modular, that is, that the $s$-matrix is invertible.
Theorem 73. $Z(C)$ is a modular category. Moreover, $\Delta_{Z(C)} = \Delta_C^2$ and $D_{Z(C)} = D_C^2$.

Theorem 73 was first proved by Müger [17] and more recently in a different approach by Bruguières and Virelizier [3, 4]. Both proofs are expressed entirely graphically in string diagrams.

Proposition 74 (cf. [7, Prop 8.22]). The ratio $D_C / \Delta_C$ of the global dimension of a fusion category $C$ to its Frobenius-Perron dimension is an algebraic integer.

Proof. We can assume $C$ is spherical. If not, then pass to its sphericalization $\tilde{C}$. By Lemma 70, $D_{\tilde{C}} = 2D_C$ and $\Delta_{\tilde{C}} = 2\Delta_C$, so that their ratio is the same.

Now assume $C$ is spherical. From Theorem 73,

$$\frac{D_{Z(C)}}{\Delta_{Z(C)}} = \left(\frac{D_C}{\Delta_C}\right)^2$$

so it suffices to show that the left hand side is an algebraic integer, which follows immediately from Theorem 73 and Lemma 72, since the quantum dimension of a simple object is an algebraic integer (since quantum dimensions furnish a homomorphism out of the Grothendieck ring, and hence are the eigenvalues of left-multiplication matrices).

Remark 75. Proposition 74 can be a powerful method to show that a given fusion category is pivotal. The idea is to use it to show that some other ratio is an actual integer, which places tight restrictions on the pivotal symbols. For instance, this method was used in [24, 25] to show that all generalized near-group categories admit spherical structures.

5 Ocneanu rigidity

In [15] Appendix E6] Kitaev gave a diagrammatic proof that the Davydov-Yetter cohomology [27, 28] of a unitary fusion category vanishes in positive degrees. In this section we show that the pairing convention enables us to extend Kitaev’s graphical proof to all fusion categories.

5.1 The tangent complex

Define the functor

$$T_n : \underbrace{C \boxtimes \cdots \boxtimes C}_{n \text{ times}} \to C$$

by $T(A_1, \ldots, A_n) = A_1 \otimes \cdots \otimes A_n$, parenthesized from left-to-right for definiteness, and define $C^n = \text{End}(F_n)$, the vector space of natural endomorphisms of $F_n$. For $n = 0$, set $C^0 = \mathbb{C}$. In terms of representative simple objects $X_i, i \in I$, we have

$$C^n = \bigoplus_{i_1, \ldots, i_n} \text{Hom}(X_1 \otimes \cdots \otimes X_n, X_1 \otimes \cdots \otimes X_n).$$
In string diagrams, we can write the components of \( c \in C^n \) as
\[
\begin{array}{c}
\vdots \\
\hat{c} \\
\vdots
\end{array}
\begin{array}{c}
i_1 \ldots i_n \\
\vdots
\end{array}
\]

For instance, a 1-cochain \( c \in C^1 \) is simply a collection of scalars \( c_i \in \mathbb{C}, i \in I \). A 2-cochain \( A \in C^2 \) is a collection of morphisms \( A_{ij} : X_i \otimes X_j \to X_i \otimes X_j \); this amounts to a collection of linear operators
\[
A_{ij} : \text{Hom}(X_k, X_i \otimes X_j) \to \text{Hom}(X_k, X_i \otimes X_j)
\]
(33)

**Definition 76.** The tangent complex of a fusion category is the sequence of \( \mathbb{C} \)-vector spaces and linear maps
\[
C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \to \ldots, \quad d^n = \sum_{k=0}^{n+1} (-1)^k f^k_n
\]
where the maps \( f^k_n : C^n \to C^{n+1} \) are defined by
\[
\begin{array}{c}
\vdots \\
\hat{c} \\
\vdots
\end{array}
\begin{array}{c}
i_1 \ldots i_{n+1} \\
\vdots
\end{array}
\begin{array}{c}
\vdots \\
\hat{c} \\
\vdots
\end{array}
\begin{array}{c}
i_1 \ldots i_{n+1} \\
\vdots
\end{array}
\]
(34)

(Note that for \( c \in C^0 = \mathbb{C}, (f^0_0(c))_i = \text{id}_{X_i} = (f^0_0(c))_i \) so that \( d^0 = 0 \).)

**Lemma 77.** The tangent complex is a complex, i.e. \( d^{n+1}d^n = 0 \).

**Proof.** Follows from the identity
\[
f^{n+1}_m f^n_k = f^{n+1}_{m+1} f^n_k \quad 0 \leq k \leq m \leq n + 1.
\]
(34)

For \( k = m = 0 \) or \( k = m = n + 1 \), (34) is just the resolution of the identity (5). For \( k \neq m \), (34) is just the interchange law in a monoidal category. In the other cases, (34) takes the following form (we do the case \( k = m = 1, n = 2 \)):
Pre-composing both sides of (35) with

and inserting the associator expansion (3), the left and right sides become the same expression, namely

\[
\sum_{m,\alpha,\beta} \left( F_{\alpha\beta} \right)_{\gamma\theta\omega} \theta_{\omega}.
\]

**Example 78.** A 1-cocycle \( c \in Z^1(C) \) is a collection of numbers \( c_i \) such that
\[ c_k = c_i + c_j \]
whenever \( X_k \) is a summand of \( X_i \otimes X_j \). A 2-cocycle \( A \in Z^2(C) \) is a collection of linear operators \( A_{\gamma\delta}^{ij} \) as in (33) such that

\[
\sum_{\gamma, m, \delta} (F_{\gamma m}^{ij})_{\alpha t \beta} = 0.
\]

Expanded out fully, this is the equation

\[
\sum_{\alpha} (A_{\gamma i}^{\theta \delta})_{\omega \alpha} F_{\theta \alpha}^{\gamma \theta \omega} - \sum_{\beta} (A_{\alpha \beta}^{\delta \theta})_{\gamma t \omega} F_{\delta \theta \omega}^{\gamma \theta \omega} + \sum_{\gamma} (A_{m k}^{\alpha \delta})_{\gamma \theta \omega} F_{\delta \theta \omega}^{\gamma \theta \omega} - \sum_{\delta} (A_{\iota l}^{n \delta})_{\delta \theta \omega} F_{\delta \theta \omega}^{\gamma \theta \omega} = 0.
\]

for all values of the free indices, where \( F \equiv F_{ij}^{s} \).

**Definition 79.** The Yetter-Davydov cohomology \( H^1(C) \) of a fusion category \( C \) is the cohomology of its tangent complex.

The low-dimensional cohomology groups have the following interpretations \[27, 28, 5, 15\]. The group \( H^1(C) \) classifies first-order deformations of \( \text{Aut}_\otimes(\text{id}_C) \), \( H^2(C) \) classifies first-order deformations of the tensor product functor, and \( H^3(C) \) classifies first-order deformations of the associator.

### 5.2 Vanishing of cohomology

We will need the following ‘handleslide’ identity.
Lemma 80. In the pairing convention for the graphical calculus, we have

\[
\sum_{\alpha, p} \frac{\alpha}{p} \cdot p^* = \sum_k d_k \cdot k^*
\]

where \(d_p\) are the fusion dimensions, \(X\) is an arbitrary object, and \(f : X_k \otimes X \rightarrow X_k \otimes X\) is an arbitrary morphism.

Proof.

\[
\text{LHS} = \sum_{p, \alpha} d_p \cdot p^* = d_k \sum_{p, \alpha} (37)
\]

where in the second equality the pairing convention is being used, and in the third equality we have defined new basis vectors \(f^\alpha \in \text{Hom}(X_{p^*}, X_i \otimes X_{k^*})\) and \(f^\beta \in \text{Hom}(X_i \otimes X_{k^*}, X_{p^*})\) by the formulas

\[
\begin{align*}
\sum_{p, \alpha} d_p &:= \sum_k d_k \cdot d_{p^*} \\
\sum_{p, \alpha} &:= \sum_k d_k \cdot d_{p^*} (38)
\end{align*}
\]
where the cup and cap maps are the same ones used in (37). Now we claim that $f_\alpha$ and $f_\beta$ form a dual basis. Indeed,

\[ f_\alpha \cdot f_\alpha^* = \delta_{\alpha \beta} \]

as can be seen by closing the loop on the left and applying the definition of the pairing convention. Hence the $f_\alpha f_\alpha^*$ term in the right hand side of (37) provides a resolution of the identity, proving the lemma.

**Theorem 81** (cf. [7, Thm 2.27]). $H^n(C) = 0$ for all $n > 0$.

**Proof.** We define an operator $\chi^n : C^n \to C^{n-1}$ by

\[
\chi^n(c) = \frac{1}{D} \sum_p d_p p^* \]

where on the right hand side we are using the pairing convention (else this diagram would not be well-defined), $d_p$ is the fusion dimension of $X_p$, and $D$ is the global dimension of $C$. We will show that

\[
\chi^{n+1} f_k = \begin{cases} 
\text{id}_{C^n} & \text{if } k = 0 \\
\frac{1}{D} f_{k-1} \chi^n & \text{if } k, n > 0
\end{cases}
\]

which will imply that $d\chi + \chi d = \text{id}$, and hence the cohomology vanishes. The case $k = 0$ of (39) is simply the relation $D = \sum_p d_p^2$, which is the definition of $D$. The cases $k > 1$ are tautologies. The nontrivial case is $k = 1$, which is precisely the handleslide identity from Lemma 80.

**Example 82.** For a 2-cocycle $A \in \mathcal{Z}^2(C)$, that is, a collection of linear operators $A_{ijk}$ satisfying (36), the vanishing of $H^2(C)$ means that $A_{ijk}$ must take the form

\[ A_{ijk} = (c_j - c_i + c_k) \text{id} \]

for some scalars $c_i, c_j, c_k$. This conclusion is perhaps not immediately evident from (36). Even if we assume that the $A_{ijk}$ in (36) are diagonal (which is also not immediately clear from $dA = 0$) i.e. $(A_{ijk})_{\alpha \beta} = \delta_{\alpha \beta} a_{ijk, \alpha}^1$, then (36) becomes the equation

\[ a_{ijk, \omega}^1 - a_{ij, \beta}^s + a_{mk, \gamma}^s - a_{ij, \delta}^m = 0 \quad \text{whenever } (F_{ijk})^{s \omega}_{\delta \gamma} \neq 0. \]  

The vanishing of $H^2(C)$ means that the equations (41) must force the $a_{ijk, \alpha}$ to all be equal for different $\alpha$ and furthermore take the form (40). Thus Ocneanu rigidity implies in particular that sufficiently many associators must be nonzero.
References

[1] Bojko Bakalov and Alexander Kirillov. Lectures on Tensor Categories and Modular Functors. American Mathematical Society, 2001.

[2] Bruce Bartlett. On unitary 2-representations of finite groups and topological quantum field theory. PhD thesis, Department of Pure Mathematics, University of Sheffield, 2009. arXiv:math/0512103

[3] Alain Bruguières and Alexis Virelizier. Quantum double of Hopf monads and categorical centers. Transactions of the American Mathematical Society, 364(3):1225–1279, 2012.

[4] Alain Bruguières and Alexis Virelizier. On the center of fusion categories. Pacific Journal of Mathematics, 264:1–30, 2013. arXiv:1203.4180

[5] Alexei Davydov. Twisting of monoidal structures. 1997. arXiv:q-alg/9703001

[6] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. Dualizable tensor categories. 2013. arXiv:1312.7188

[7] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. Annals of Mathematics, 162:581–642, 2005. arXiv:math/0203060 doi:10.4007/annals.2005.162.581

[8] Tobias Hagge and Seung-Moon Hong. Some non-braided fusion categories of rank 3. Communications in Contemporary Mathematics, 11(4):615–637, 2009. arXiv:0704.0208

[9] Seung-Moon Hong. On symmetrization of 6j-symbols and Levin-Wen hamiltonian. 2013. arXiv:0907.2204

[10] I. Martin Isaacs. Character Theory of Finite Groups. Dover Publications, 1976.

[11] André Joyal and Ross Street. The geometry of tensor calculus II. Available at web.science.mq.edu.au/street/GTCII.pdf

[12] André Joyal and Ross Street. The geometry of tensor calculus I. Advances in Mathematics, 88:55–112, 1991. doi:10.1016/0001-8708(91)90003-p

[13] Alexander Kirillov Jr. String-net model of Turaev-Viro invariants. 2011. arXiv:1106.6033

[14] Robion C. Kirby. The Topology of 4-Manifolds. Springer-Verlag, 1989.

[15] Alexei Kitaev. Anyons in an exactly solved model and beyond. Annals of Physics, 321:2–111, 2006. arXiv:cond-mat/0506438

[16] Saunders Mac Lane. Categories for the Working Mathematician. Springer, 1997. 2nd edition.

[17] Michael Müger. From subfactors to categories and topology II: The quantum double of tensor categories and subfactors. Journal of Pure and Applied Algebra, 180(1):159–219, 2003. arXiv:0111205
[18] Michael Müger. On the center of a compact group. *International Mathematics Research Notices*, 2004(51):2751–2756, 2004.

[19] Gregory L. Naber. *Topology, Geometry and Gauge fields: Interactions*. Springer, 2011.

[20] Siu-Hung Ng and Peter Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. 2005. [arXiv:math/0503167]

[21] Dmitri Nikshych. Non-group-theoretical semisimple Hopf algebras from group actions on fusion categories. *Selecta Mathematica*, 14(1):145–161, 2008. [arXiv:0712.0585]

[22] V. Ostrik. Pivotal fusion categories of rank 3 (with an Appendix written jointly with Dmitri Nikshych). 2013. [arXiv:1309.4822]

[23] Peter Selinger. *New Structures for Physics*, volume 813, chapter “A survey of graphical languages for monoidal categories”, pages 289–355. Springer, 2011. [arXiv:arXiv:0908.3347]

[24] Josiah Thornton. On braided near-group categories. 2011. [arXiv:1102.4640]

[25] Josiah Thornton. *Generalized Near-Group Categories*. PhD thesis, University of Oregon, 2012.

[26] Zhenghan Wang. *Topological Quantum Computation*. Number 112 in Conference Board of the Mathematical Sciences. Cambridge University Press, 2010.

[27] David N Yetter. Braided deformations of monoidal categories and Vassiliev invariants. In *Higher Category Theory*, volume 230 of *AMS Contemporary Mathematics*, pages 117–134, 1998.

[28] David N Yetter. Abelian categories of modules over a (lax) monoidal functor. *Advances in Mathematics*, 174(2):266–309, 2003.

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