Maximum of an Airy process plus Brownian motion and memory in KPZ growth

Pierre Le Doussal

1 CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France

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We obtain several exact results for universal distributions involving the maximum of the Airy\(_2\) process minus a parabola and plus a Brownian motion, with applications to the 1D Kardar-Parisi-Zhang (KPZ) stochastic growth universality class. This allows to obtain (i) the universal limit, for large time separation, of the two-time height correlation for droplet initial conditions, e.g. \(C_\infty = \lim_{t_1, t_2 \to +\infty} h(t_1)h(t_2) / h(t_1)^\frac{3}{2}\), with \(C_\infty \approx 0.623\), as well as conditional moments, which quantify ergodicity breaking in the time evolution; (ii) in the same limit, the distribution of the midpoint position \(x(t_1)\) of a directed polymer of length \(t_2\), and (iii) the height distribution in stationary KPZ with a step. These results are derived from the replica Bethe ansatz for the KPZ continuum equation, with a "decoupling assumption" in the large time limit. They agree and confirm, whenever they can be compared, with (i) our recent tail results for two-time KPZ with de Nardis [4], checked in experiments with Takeuchi [2], (ii) a recent result of Maes and Thiery [3] on midpoint position.

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Stochastic processes, such as the Brownian motion, are useful unifying mathematical tools to describe universal behavior of complex systems. In recent years, the Airy\(_2\) process, introduced in [4, 23], appeared in several contexts in physics and mathematics. Its simplest definition (see e.g. [5]) involves the Dyson Brownian motion (DBM) [5]: consider a large Hermitian random matrix \(H\) whose independent entries (both real and imaginary parts) perform independent stationary Orstein-Ulhenbeck processes (i.e. Brownian motions equilibrated in a harmonic well).

The Airy\(_2\) process describes the evolution of the largest eigenvalue of \(H\) (centered and scaled). The Airy\(_2\) process appears as a limit process in directed last passage percolation [7], non-intersecting Brownian bridges [4, 7-13], random tilings [14], interacting particle transport in 1D [15, 16], quantum dynamics of fermions [17-19] and stochastic growth models, either discrete [20] or the continuum 1D Kardar-Parisi-Zhang (KPZ) equation [21, 22] (for a review see [23, 24]). In fact the Airy\(_2\) process is a hallmark of the very broad 1D-KPZ universality class, which arises in all these models.

Models in the 1D-KPZ class usually allow for the definition of a height field \(h(x, t)\), which undergoes stochastic growth. The prominent example is the continuum KPZ equation [21], where \(h(x, t)\) is an interface height at point \(x \in \mathbb{R}\), evolving as a function of time \(t\) as

\[
\partial_t h(x, t) = \nu \partial_x^2 h(x, t) + \frac{\lambda_0}{2} (\partial_x h(x, t))^2 + \sqrt{D} \xi(x, t) \tag{1}
\]

driven by unit white noise \(\xi(x, t)\). For the curved (i.e. droplet) initial condition (IC) it is known (in some cases proved) that it converges at large time \(t \to +\infty\) (rescaled and centered) to [4, 7, 25, 26]

\[
(\Gamma(t))^{-\frac{1}{2}} \left( h_{\text{drop}}(x, t) - v_\infty t \right) \simeq \mathcal{A}_2(\hat{x}) - \hat{x}^2, \quad \hat{x} = \frac{Ax}{2t^{\frac{3}{2}}} \tag{2}
\]

where \(\mathcal{A}_2(\hat{x})\) is the Airy\(_2\) process, as an identity between processes (i.e. as \(\hat{x}\) is varied). Since \(\mathcal{A}_2\) is stationary (statistical translational and reflection invariant in \(\hat{x}\)) the \(-\hat{x}^2\) term embodies the mean parabolic profile. We use units such that the non-universal constants \(\Gamma = A = 1\), i.e. \(\lambda_0 = D = 2\) and \(\nu = 1\) for the KPZ equation [4], and set \(v_\infty = 0\) (upon a shift of \(h\)). The equilibrium measure of the DBM being the Gaussian unitary ensemble (GUE) measure for \(H\), the fluctuations of the Airy\(_2\) process, hence of the KPZ height from (2), at any given point, e.g. \(x = 0\), is the Tracy-Widom (TW) distribution for the largest eigenvalue of a GUE random matrix [27]. Its cumulative distribution function (CDF) is explicitly known as a Fredholm determinant

\[
\text{Prob}(\mathcal{A}_2(0) < \sigma) = F_2(\sigma) = \det(I - P_x K_{\lambda_1}) \tag{3}
\]

where \(K_{\lambda_1}(u, v) = \int_{-\infty}^{+\infty} dy Ai(y+u)Ai(y+v)\) is the Airy kernel, \(P_x\) being here the projector on \([\sigma, +\infty[\). Furthermore, from properties of the DBM, the Airy\(_2\) process is determinantal, i.e. any \(p\)-point joint CDF (JDVF) of \(\mathcal{A}_2(\hat{x})\) can be written as \(p \times p\) matrix Fredholm determinants, in term of an extended Airy kernel [4, 22].

Although much studied, and fully characterized by its determinantal structure, important open questions remain about the Airy process, with applications to the 1D KPZ class. First, for more general initial conditions \(h(x, t = 0)\), the value at a given point, e.g. \(x = 0\), is obtained from the variational problem [28]

\[
t^{-\frac{1}{3}} h(0, t) \simeq \max_{\hat{y}} \left( \mathcal{A}_2(\hat{y}) - \hat{y}^2 + h_0(\hat{y}) \right) \tag{4}
\]

when a limit exists for the rescaled IC \(h_0(\hat{y}) \simeq t^{-\frac{1}{3}} h(2^{2/3} \hat{y}, 0)\). Droplet subclass IC’s correspond to \(h_0(0) = 0\) and \(h_0(\hat{y}) = -\infty\) for \(\hat{y} \neq 0\), recovering [2], while flat subclass IC’s correspond to \(h_0(\hat{y}) = 0\). The CDF of \(h(0, t)\) and of the argmax in (4) (i.e. the endpoint distribution a directed polymer - see below) for flat IC, and other results such as intermediate classes of IC, have been obtained from exact solutions of models in the
KPZ class at large $t$, or from powerful methods directly on the Airy process which allow to treat a large class of $h_0$ [23,26,28]. The latter, however, do not readily extend to random initial conditions, such as the Brownian IC, related to the important stationary KPZ subclass.

The aim of this Letter is to study some properties of the optimization problem

$$\max_{\hat{y}} \left( A_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y}) \right)$$

where $B(\hat{y})$ is the two-sided unit Brownian motion. Eq. [3] defines $A_{\text{stat}}(\theta)$, the Airy process associated to stationary KPZ equation, at $t = 0$. We first describe our three main results and applications, then give explicit formula, and finally sketch the replica Bethe ansatz method.

Our first result is the probability distribution function (PDF) of the position $\hat{y}_m$ of the maximum in [3], i.e.

$$\hat{y}_m = \arg\max_{\hat{y} \in \mathbb{R}} \left( A_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y}) \right)$$

This distribution arises in the midpoint probability of a directed polymer (DP) in the white noise $d = 1 + 1$ random potential $\xi$. Recall that the partition sum $Z(x, t|y, 0)$ of continuum directed paths from $(y, 0)$ to $(x, t)$ defined as

$$Z(x, t|y, 0) := \int_{x(0) = y}^{x(t) = x} Dxe^{-\int_0^t d\tau[\frac{1}{4}(\frac{\partial}{\partial x})^2 - \sqrt{2} \xi(x(\tau), \tau)]}$$

equals $e^{h(x,t)}$ where $h(x, t)$ is the solution of [1] with droplet initial condition (centered at $y$). Consider a DP from $(0, 0)$ to $(0, t_2)$ and ask about the PDF, $P_{t_1,t_2}(y)$, of the position $x(t_1) = y$ at intermediate time $t_1$, see Fig. 1. In the limit of large times $t_1, t_2$, with $\hat{y} = y/(2t_1^{1/3})$,

$$P_{t_1,t_2}(y)dy = \frac{Z(0,t_2|y,t_1)Z(t_1,0|0)}{Z(0,t_2|0,0)} dy \to P_\Delta(\hat{y})d\hat{y} \quad (8)$$

One finds (see below) that as $\Delta = t_2 - t_1 \to +\infty$, $P_{t_1,t_2}(y)$ concentrates on $\hat{y} = \hat{y}_m$ defined in (6), hence

$$\mathcal{P}(\hat{y})d\hat{y} := P_{+\infty}(\hat{y})d\hat{y} = \text{Prob}(\hat{y}_m \in [\hat{y}, \hat{y} + d\hat{y}]) \quad (9)$$

Here we calculate this distribution, which also arises in the study of the coalescence of optimal paths [29].

Our second result is the following joint CDF

$$G(\sigma_1, \sigma_2) := \sigma_1, \max_{\hat{y} \in \mathbb{R}} (A_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})) < \sigma_2$$

It is important in the study of the so-called persistence of correlations in the two-time KPZ problem for droplet initial conditions, which exhibits an interesting memory effect, also called ergodicity breaking [1,2,30,32]. Indeed, consider the rescaled heights at $t_1$ and at $t_2 > t_1$, in the limit where both times are large, with $\Delta = (t_2 - t_1)/t_1$ fixed

$$t_1^{-1/3}h(0,t_1) \simeq A_2(0) \quad (11)$$

$$t_1^{-1/3}h(0,t_2) \simeq \max_{\hat{y} \in \mathbb{R}} (A_2(\hat{y}) - \hat{y}^2 + \Delta \frac{\hat{y}^2}{\Delta^2}) \quad (12)$$

where $A_2$ and $\tilde{A}_2$ denote two independent Airy processes. The second line expresses that the height at $t_2$ is the sum of a first contribution from the time interval $[0,t_1]$ and the second from $[t_1,t_2]$ which, for a fixed intermediate point $y$, are independent, see Fig 1. Optimization over $\hat{y}$ correlates them. Obtaining the resulting joint PDF (JPDF) of the two rescaled heights for arbitrary $\Delta$ is a difficult problem [1,2,32,37]. In the limit of well separated times, i.e. large $\Delta$, using that the Airy process $\tilde{A}_2$ is locally a Brownian, one obtains the third line in (11), where $B$ and $A_2$ are independent processes [38]. As is clear from (11), $h(0,t_1) \sim t_1^{1/3}$, $h(0,t_2) \sim t_2^{1/3}$ are quite different in magnitude (for large $t_2/t_1$), but remain correlated by the $O(t_1^{1/3})$ term. To measure this memory effect one usually define the dimensionless ratio of the two covariances

$$C(t_1,t_2) = \frac{\frac{\partial^2}{\partial t_1^2} h(0,t_2)}{\frac{\partial}{\partial t_1} h(0,t_1)^2} \simeq C_\Delta \quad (12)$$

which, at large times $t_1, t_2 \to +\infty$, becomes a universal function $C_\Delta$ of $\Delta$. From (11) the latter has a finite limit

$$C_\infty = \int d\sigma_1 d\sigma_2 G(\sigma_1,\sigma_2) = \frac{\sigma_1 \sigma_2}{\kappa_{GUE}^p}$$

where $p(\sigma_1,\sigma_2) = \partial_{\sigma_1} \partial_{\sigma_2} G(\sigma_1,\sigma_2)$ is the JPDF associated to (10), obtained here exactly (here and below $\kappa_{GUE}^p$ is the $p$-th cumulant of the GUE-TW distribution).
Finally, our third main result is the CDF for the height $h(x,t)$ for an IC equal to a Brownian plus a step, corresponding to a rescaled IC $h_0(y) = \sqrt{2} B(y) - \tilde{H} \text{sgn}(\hat{y})$. It will be relevant for any KPZ system where two half spaces, each stationary, are put in contact at $t = 0$, with a mismatch in height $2\tilde{H} t^{1/3}$.

Before displaying our explicit formula, it is important to recall some known results about the stationary KPZ IC subclass, which corresponds to $h_0(y) = \sqrt{2} B(y)$ where $B(x)$ is a two sided unit Brownian $\langle dB(x)^2 \rangle = dz$ with $B(0) = 0$. It is realized, e.g. by the solution $h_{\text{stat}}(x,t)$ of the KPZ equation (1) with a unit two sided Brownian initial condition $h(x,t = 0) = B_0(x)$ \[39\] as

$$ t^{-\frac{1}{2}} h_{\text{stat}}(x,t) \simeq A_{\text{stat}}(\hat{x}) = \max_y \left( A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2} B(\hat{y}) \right) \tag{14} $$

where here the last equality is only at fixed $\hat{x}$ (not as a process). Let us now define the two functions

$$ B_w(v) = e^{\frac{1}{2} v^3 - u v} - \int_0^{+\infty} dy \text{Ai}(v + y) e^{wy} \tag{16} $$

$$ L_\hat{x}(\sigma) = \sigma - 1 - \hat{x}^2 + \int_{\sigma}^{+\infty} dv (1 - B_\hat{x}(v) B_{-\hat{x}}(v)) \tag{17} $$

It is known that the one-point CDF of the $A_{\text{stat}}$ process is given by the extended Baik-Rains (EBR) distribution \[23, 10, 46\], which has the following exact expression

$$ \text{Prob}(A_{\text{stat}}(\hat{x}) < \sigma) = F_0(\sigma - \hat{x}^2, \hat{x}) = \partial_\sigma (F_2(\sigma) Y_\hat{x}(\sigma)) $$

in terms of the auxiliary function

$$ Y_\hat{x}(\sigma) := 1 + L_\hat{x}(\sigma) - \text{Tr}[P_\sigma K_{A_1} (I - P_\sigma K_{A_1})^{-1} P_\sigma B_{-\hat{x}} B_\hat{x}] $$

where we denote $AB^T$ the projector $AB^T(u,v) = A(u) B(v)$. For $\hat{x} = 0$, the function $F_0(\sigma, 0) = F_0(\sigma) = \partial_\sigma (F_2(\sigma) Y_0(\sigma))$ is the CDF of the standard Baik-Rains (BR) distribution $F_0$. The PDF of argmax $\hat{y}_m$. We now display our first result. Let $H(\hat{x}) = \text{Prob}(\hat{y}_m > \hat{x})$ the CDF of the position $\hat{y}_m$ of the maximum defined by $T$. Our method, detailed below, gives

$$ H(-\hat{x}) = \int d\sigma F_2(\sigma) \left[ Y_{\hat{x}}(\sigma) \times \text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma (\hat{A}' + \hat{x} A) A^T ] + (\text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}^T ] - 1)^2 \right] $$

where $A'$ is the derivative of the Airy function. Interestingly, in a recent work, Maes and Thiery \[3\] noted that the distribution of argmax $\hat{y}_m$ can be related, using the Fluctuation-Dissipation theorems (FDT), to the variance of the height in stationary KPZ, defined as

$$ g(\hat{x}) = \langle \sigma^2 \rangle_{F_0, \hat{x}} - \langle \sigma \rangle_{F_0, \hat{x}}^2 \tag{20} $$

where $\langle O(\sigma) \rangle_{F_0, \hat{x}} = \int d\sigma O(\sigma) \partial_\sigma F_0(\sigma - \hat{x}^2, \hat{x})$ denotes an average over the extended Baik Rains distribution, which is an even function of $\hat{x}$. Note that the second term is simply $-\langle \sigma \rangle_{F_0, \hat{x}}^2 = -\hat{x}^4$. As a consequence of \[3\], the scaled PDF of the midpoint probability is predicted as

$$ \mathcal{P}(\hat{y}) = -H'(\hat{y}) = f_{\text{KPZ}}(\hat{y}) \quad , f_{\text{KPZ}}(\hat{y}) := \frac{1}{4} g''(\hat{y}) \tag{21} $$

where the notation $f_{\text{KPZ}}(\hat{y})$ for the second derivative of $g$ in \[20\] was introduced in the context of the PNG and TASEP models \[17\].

It is thus important to check whether our result \[19\], obtained through an independent and completely different route, agrees with this prediction. Using the identities \[38\]

$$ \partial_\sigma Y_{\hat{x}}(\sigma) = (\text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}^T ] - 1) \times (\text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}^T ] - 1) $$

$$ \partial_\sigma F_2(\sigma) = \text{Tr}[P_\sigma (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}] $$

a few algebraic manipulations \[38\] show that \[19\] can indeed be rewritten as

$$ H(\hat{x}) = \frac{1}{2} - \frac{1}{4} g'(\hat{x}) \tag{22} $$

where, we recall, $g'(\hat{x})$ is odd, and $g'(\pm\infty) = \pm 2$. Hence our result \[19\] provides an equivalent, though different form for the midpoint probability $\mathcal{P}(\hat{y}) = H(\hat{y})$. This provides a test of our method (the decoupling assumption, see below) and of the FDT for the KPZ problem. Note that the result of \[3\] extends to finite time, while our method deals with large times.

Joint PDF of Airy and Airy minus parabola plus Brownian and persistent KPZ two time correlations. We now give our result for the JCDF \[10\]. We find, for $\sigma_1 \leq \sigma_2$

$$ G(\sigma_1, \sigma_2) = F_2(\sigma_1) Y_0(\sigma_1) \text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}^T - 1] \tag{23} $$

$$ + F_2(\sigma_1) \text{Tr}[ (I - P_\sigma K_{A_1})^{-1} P_\sigma A B_{\hat{x}}^T - 1]^2 $$

where $A_{\sigma}(u) = A(u + \sigma)$ and $G(\sigma_1, \sigma_2) = F_0(\sigma_2)$ for $\sigma_1 \geq \sigma_2$. An extended result for $\hat{x} \neq 0$ is displayed in \[38\]. It is easy to check \[38\] the continuity, $G(\sigma, \sigma) = F_0(\sigma)$ using the identities \[22\]. It is also easy to see that the marginal CDF of $\sigma_1$, $G(\sigma_1, +\infty) = F_2(\sigma_1)$ is the GUE-TW and the marginal CDF of $\sigma_2$, $G(+\infty, \sigma_2) = F_0(\sigma_2)$ is the BR distribution.

We now apply this result to the large time separation limit of the two-time correlation in the 1D KPZ class.
Using integration by parts one obtains \[ \langle (\sigma_2 - \sigma_1)^2 \rangle = 2 \int_{-\infty}^{+\infty} d\sigma_2 \int_{-\infty}^{\sigma_2} d\sigma_1 (F_2(\sigma_1) - G(\sigma_1, \sigma_2)) \] where here and below \( \langle . . \rangle \) denotes averages w.r.t. \( p(\sigma_1, \sigma_2) = \partial_{\sigma_1} \partial_{\sigma_2} G(\sigma_1, \sigma_2) \), the associated JPDF. This allows us to compute the two-time persistent dimensionless covariance ratio \[ C_\infty = \langle (\sigma_2 \sigma_1)^c \rangle / \langle \sigma_1^2 \rangle = \langle (\sigma_2 \sigma_1) \rangle - \langle (\sigma_2 - \sigma_1)^2 \rangle / 2K_{2,\text{GUE}} \] up to \( O(\Delta^{-1}) \) terms \[ = 1 + \frac{(\kappa_{1,\text{GUE}}^2 + \kappa_{1,\text{BR}}^2)}{2K_{2,\text{GUE}}} - \frac{\langle (\sigma_2 - \sigma_1)^2 \rangle}{2K_{2,\text{GUE}}} \] \[ = 3.13598 \mp \frac{\langle (\sigma_2 - \sigma_1)^2 \rangle}{2K_{2,\text{GUE}}} \approx 0.6225 \pm 0.0015 \] using the known GUE-TW and BR cumulants, i.e. \( \langle \sigma_2 \rangle = \kappa_{1,\text{BR}} = 0, \kappa_{1,\text{GUE}}^2 = -1.7710868, \kappa_{2,\text{GUE}} = 0.81319 \) and \( \kappa_{2,\text{BR}} = 1.15039 \), and evaluated numerically (see Sec IV. 6 in \[ \text{[13]} \). Eq. (27) compares quite well with recent numerical simulations and experiments \[ \text{[49, 50].} \]

Let us recall our recent study \[ \text{[1, 2]} \) and reexamine the observables defined there, using our new exact results. There we defined the variables \( h_1 = \langle h(0, t_1) / t_1 \rangle \) and the scaled height difference \( h = \langle h(0, t_2) - h(0, t_1) / (t_2 - t_1) \rangle^{1/3} \) \[ \text{(28)} \] Defining the (unknown) exact JPDF \( P_\Delta(\sigma_1, \sigma) : = \lim_{t_1, t_2 = t_1(1 + \Delta) \to +\infty} \delta(h_1 - \sigma_1) \delta(h - \sigma) \), we derived an approximation of it, denoted \( P_\Delta^{(1)}(\sigma_1, \sigma) \), conjectured to be exact to leading order in large positive \( \sigma_1 \), for any fixed \( \sigma \) and \( \Delta \). It was shown in \[ \text{[2]} \) to be good enough an approximation to fit experiments and numerics in a broad range of values \( \sigma_1 > \sigma_1(1) = \kappa_{1,\text{GUE}} \). It is thus of great importance to check whether our present exact result, valid only for large \( \Delta \), but for any \( \sigma_1, \sigma \), confirms this predictions.

At large times, the height difference, from \[ \text{[11]} \), takes the form, up to \( O(\Delta^{-2/3}) \) terms \( h \approx A_2(0) + \Delta^{-1/3} (\sigma_2 - \sigma_1) \) \[ \text{(29)} \] where we denote (with a slight abuse of notations) the two random variables \[ \sigma_1 = A_2(0) \quad \sigma_2 = \max_{\bar{y} \in \mathbb{R}} (A_2(\bar{y}) - \bar{y}^2 + \sqrt{2B(\bar{y})}) \] \[ \text{(30)} \] and we recall that \( A_2(0) \) is a GUE-TW random variable independent of the \( O(\Delta^{-1/3}) \) term. The first consequence, averaging \[ \text{[29]} \), is that \( \bar{h} = \kappa_{1,\text{GUE}} (1 - \frac{1}{\Delta^{1/3}}) + O(\Delta^{-1}) \) \[ \text{(31)} \] since \( \langle \sigma_2 \rangle = \kappa_{1,\text{BR}} = 0 \) and \( \langle \sigma_1 \rangle = \kappa_{1,\text{GUE}} \), in agreement with the general formula for \( \bar{h} \) for any \( \Delta \) (see \[ \text{[48]} \) in \[ \text{[1]} \). Important quantities, introduced in \[ \text{[12]} \), are the conditional averages of \( h \), either for a fixed value of \( h_1 = \sigma_1 \), \( \bar{h}_{h_1 = \sigma_1} \), or, for a value larger than some threshold \( h_1 = \sigma_1 > \sigma_{1c} \), \( \bar{h}_{h_1 = \sigma_1 > \sigma_{1c}} \). From the above one predicts \[ \bar{h}_{h_1 = \sigma_1} \simeq \kappa_{1,\text{GUE}} + \frac{1}{\Delta^{1/3}} \langle (\sigma_2 - \sigma_1) \rangle + O(\frac{1}{\Delta^{1/3}}) \] \[ \text{(32)} \] where the conditional average w.r.t. \( p \), denoted as \[ \langle (\sigma_2 - \sigma_1) \rangle = \frac{1}{F_2(\sigma_1)} \int_{-\infty}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) p(\sigma_1, \sigma_2) \] \[ \text{(33)} \] can be calculated from \[ \text{(24)} \). For large positive \( \sigma_1 \) one shows from \[ \text{(24)} \) that (see \[ \text{[48]} \) where next order is also displayed) \[ p(\sigma_1, \sigma_2) - 2\sigma_2 K_{\text{Ai}}(\sigma_1, \sigma_2) - \text{Ai}(\sigma_2)^2 \] \[ \text{(34)} \] which leads to, for large positive \( \sigma_1 \) \[ \langle (\sigma_2 - \sigma_1) \rangle \simeq R_{1/3}(\sigma_1) \] \[ \text{(35)} \] \[ R_{1/3}(\sigma_1) : = \frac{\int_{-\infty}^{+\infty} dy \text{Ai}(y)^2 - \int_{-\infty}^{+\infty} dy K_{\text{Ai}}(y, y)}{K_{\text{Ai}}(\sigma_1, \sigma_1)} \] \[ \text{(36)} \] which is precisely the prediction obtained in \[ \text{[1]} \). This is encouraging evidence that the method of \[ \text{[1]} \) is good enough to capture, as claimed there, the tail of the two-time JPDF. We can thus calculate the conditional averages beyond the large positive \( \sigma_1 \) regime. We show in Fig. 2 the leading order of \[ \bar{h}_{h_1 > \sigma_{1c}} = \kappa_{1,\text{GUE}} + \Delta^{-1/3} (\sigma_2 - \sigma_1)_{\sigma_1 > \sigma_{1c}} + O(\Delta^{-2/3}) \] \[ \text{(37)} \] evaluated numerically \[ \text{[48]} \) from \[ \text{(24)} \), a quantity which can be measured accurately in experiments and numerics.

![Fig. 2](image-url)
We obtain here its large $\Delta$ limit,
\[
C_\infty(\sigma_{1c}) = \frac{\sigma_2 \sigma_1 > \sigma_1 > \sigma_{1c}}{\sigma_1^2 > \sigma_1 > \sigma_{1c}} \quad (39)
\]
a function of $\sigma_{1c}$ which interpolates between $C_\infty(\sigma_{1c} = -\infty) = C_\infty$ the unconditioned two-time covariance ratio obtained in [27] and $C_\infty(\sigma_{1c} = +\infty) = 1$. It is evaluated numerically [48] from [24] and plotted in Fig. 3.

We also explored the case where the longer polymer in Fig 1 is constrained to pass to the right of $0$, $y > 0$: we find $C_\infty \approx 0.6925$, i.e. the correlations are increased.

Stationary KPZ in presence of a step. Finally, the height $h(x, t)$ in the KPZ class with a step at $x = 0$ in the initial condition, and independent Brownian initial condition on each side, takes the scaling form at large $t$
\[
t^{-\frac{1}{3}} h(x = 2t^{\frac{2}{3}} \hat{x}, t) \approx \hat{h}(\hat{x}) \quad (40)
\]
\[
:= \max_y \left( A_2(\hat{x} - \hat{y}) - (\hat{x} - \hat{y})^2 + \sqrt{2B(\hat{y})} - \hat{H} \mathrm{sgn}(\hat{y}) \right)
\]

Defining $G_{\hat{H}}(\sigma_L) = \mathrm{Prob}(\hat{h}(\hat{x}) - \hat{H} + \hat{x}^2 < \sigma_L)$, we obtain
\[
G_{\hat{H}}(\sigma_L) = F_2(\sigma_L) \times \left( Y_2(\sigma_L) e^{2\hat{H}} \mathrm{Tr}[(I - P_{\sigma_L} K_{Ai})^{-1} P_{\sigma_L} A_{24} H A_i^T] \right.
\]
\[
+ (\mathrm{Tr}[(I - P_{\sigma_L} K_{Ai})^{-1} P_{\sigma_L} A_{24} H A_i^T] - 1)
\]
\[
\left. \times (e^{2\hat{H}} \mathrm{Tr}[(I - P_{\sigma_L} K_{Ai})^{-1} A_{24} H A_i^T] - 1) \right) \quad (41)
\]

Method. The method is based on the replica Bethe ansatz, which led to exact solutions for one-time observables for various initial conditions [51.44.58.68]. Since it is an extension of the calculation in [61] we only sketch the idea, with details in Sec. II-III of [48]. We define, jointly in the same noise realization, $h_1(x, t) = \ln Z(x, t|0, 0)$ and $h_{L,R}(x, t)$ the solutions of the KPZ equation with IC’s respectively droplet (1), and Brownian on $y < 0$ (L) and on $y > 0$ (R). We define the joint Laplace transform
\[
\hat{g}_t(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = e^{-\int_{b=1, L,R} u_b Z_b(x, t)} \quad (42)
\]
with $u_b = e^{-t^1/3(\sigma_{bL} - \hat{x}^2)}$. Its large time limit gives
\[
\hat{g}_\infty(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = G_{\hat{H}}(\sigma_1, \sigma_L, \sigma_R; \hat{x}) := \mathrm{Prob}(A_2(\hat{x}) < \sigma_1, \hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R)
\]
with
\[
\hat{h}_{L,R}(\hat{x}) = \max_{y < \hat{x}, y > 0} (A_2(y - \hat{x}) - (y - \hat{x})^2 + \sqrt{2B(y)}) \quad (44)
\]
containing all desired information (and more). To compute $\hat{g}_t$ we expand the exponential in [48], and write the joint moments as quantum mechanical expectations
\[
\prod_{b=1, L,R} Z_b(x, t)^{n_b} = \langle x, x | e^{-H_{nL} | n_L, n_1, n_R \rangle} \quad (45)
\]
where
\[
H_n = -\sum_{\alpha=1}^n \frac{\partial^2}{\partial x^2} - 2 \sum_{1 \leq \alpha < \beta \leq n} \delta(x_\alpha - x_\beta) \quad (46)
\]
is the Hamiltonian of the attractive Lieb Liniger $\delta$-Bose gas model [60], and $|n_L, n_1, n_R \rangle$ is a state with $n_1$ bosons at $y = 0$, and $n_{L,R}$ in $y < 0$ and $y > 0$ respectively, with $n = n_1 + n_L + n_R$. One inserts in [48] the known Bethe Ansatz eigenstates, each consisting of $1 \leq n_s \leq n$ strings (bound states) of $m_j \geq 1$ bosons with rapidities $\lambda_{j,a} = k_j - \frac{1}{2}(m_j + 1 - 2a)$, $a = 1, ..., m_j$, and $\sum_{j=1}^{n_s} m_j = n$. For the Brownian (and flat) IC the overlap of $|n_L, n_1, n_R \rangle$ with any Bethe state can be expressed explicitly, although as a complicated sum over products of Gamma functions, extending as in [61] the combinatoric method introduced in [54]. It can be simplified in the large $t$ limit through the decoupling assumption (which sets all inter-string double products to unity) as done in [3.54.68]. Summing over the eigenstates becomes possible and leads to a Fredholm determinant formula for $\hat{g}_\infty$ [136.137], and [140.141] in [48]. For regularization the calculation includes finite drifts $u_{L,R} > 0$ in the Brownian, and the (delicate) limit $u_{L,R} = 0^+$ converts the Fredholm determinant into a final expression [140] in [48] for $G_{\hat{H}}(\sigma_1, \sigma_L, \sigma_R; \hat{x})$. Specializing to $\sigma_L = \sigma_R = \sigma_2$ gives the result (160) in [48] for the JCFD $G_{\hat{H}}(\sigma_1, \sigma_2)$ for general $\hat{x}$, which reduces to (24) for $G = G_0$ for $\hat{x} = 0$. Specializing instead to $\sigma_1 = +\infty$, one obtains (i) the result (19) for the CDF of the argmax of Airy minus parabola plus Brownian and (21) for the limiting midpoint DP probability, from
\[
H(\hat{x}) = \int_{-\infty}^{+\infty} d\sigma_R \delta_{\sigma_R \hat{g}_\infty(+\infty, \sigma_L, \sigma_R; \hat{x})} \rangle |_{\sigma_L = \sigma_R} \]
(ii) the CDF of the KPZ height in presence of a step IC: setting $\sigma_R = \sigma_L + H$ one obtains $G_R(\sigma_L) = \hat{g}_R(\infty, \sigma_L, H, \hat{x})$ leading to the result \[1\].

In conclusion, from a replica Bethe ansatz calculation, using a decoupling assumption, we obtained several distributions involving the maximum of the Airy process minus parabola plus Brownian. This leads to exact universal results for two-time KPZ in the large time separation limit $\Delta = t_2 - t_1 >> 1$, which correctly match, and nicely complement, our recent tail results \[2\] for any $\Delta$, putting both methods on firmer ground. Taken together they should also lead to further accurate comparisons with experiments and numerics in the universal large time limit, and allow to test other observables, e.g. the effect of the endpoint position $\hat{x} \neq 0$ as predicted here and in \[1\].

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Supplemental material.

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SUPPLEMENTARY MATERIAL

We give here the details of the calculations and their applications, as described in the main text of the Letter. Calculations being performed in a slightly more general framework, further explicit results are being displayed (e.g. for $\hat{x} \neq 0$, triple JCDF’s, with and without Brownian), which are then specialized to the cases considered in the main text.

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I. CONTINUUM KPZ EQUATION, DIRECTED POLYMER AND AIRY PROCESS

We start from the 1D KPZ equation \[1\] and work in the units \(x^* = \frac{(2\nu)^2}{D\lambda_0}, t^* = \frac{2(2\nu)^3}{D^2\lambda_0}\) and \(h^* = \frac{2\nu}{\lambda_0}\), such that it reads

\[
\partial_t h(x, t) = \partial_{x^*}^2 h(x, t) + (\partial_x h(x, t))^2 + \sqrt{2} \eta(x, t)
\]

where \(\eta\) is a unit white noise \(\eta(x, t)\eta(x', t') = \delta(x - x')\delta(t - t')\). The KPZ equation is mapped, via the Cole-Hopf transformation, to the continuous directed polymer (DP) in a quenched random potential, such that \(h(x, t) = \ln Z(x, t)\) is (minus) the free energy of the DP of length \(t\) with one fixed endpoint at \(x\). For an arbitrary initial condition, the solution at time \(t\) can be written as:

\[
e^{h(x, t)} = Z(x, t) := \int dy Z_\eta(x, t|y, 0) Z_0(y) , \quad Z_0(y) = e^{h(y, t=0)}.
\]

where here and below we denote \(\int dy = \int_{-\infty}^{\infty} dy\). Here \(Z_\eta(x, t|y, 0)\) is the partition function of the continuum directed polymer in the random potential \(-\sqrt{2} \eta(x, t)\) with fixed endpoints at \((x, t)\) and \((y, 0)\):

\[
Z_\eta(x, t|y, 0) = \int_{x(0)=y}^{x(t)=x} Dxe^{-\int_0^t d\tau \left[ \frac{1}{2} \eta(x(t), \tau)^2 - \sqrt{2} \eta(x(t), \tau) \right]}
\]

which is the solution of the (multiplicative) stochastic heat equation (SHE):

\[
\partial_t Z = \nabla^2 Z + \sqrt{2} \eta Z
\]

with Ito convention and initial condition \(Z_\eta(x, t = 0|y, 0) = \delta(x-y)\). Equivalently, \(Z(x, t)\) is the solution of (50) with initial conditions \(Z(x, t = 0) = e^{h(x, t=0)} = Z_0(x)\). We will adopt the notation (for the solution of the droplet initial condition started in \(y\)):

\[
h_\eta(x, t|y, 0) = \ln Z_\eta(x, t|y, 0)
\]

although it is somewhat improper (it requires a short time regularization, irrelevant here). We will most often omit the ”environment” index \(\eta\). Here and below overbars denote averages over the white noise \(\eta\).

Note that the time reversed path sees the time reversed random potential \(\tilde{\eta}\), which has the same distribution as \(\eta\), hence in law

\[
Z_\eta(x, t|y, 0) = Z_{\tilde{\eta}}(y, t|x, 0) \equiv_{\text{inlaw}} Z_\eta(y, t|x, 0)
\]

a property used extensively below.

In order to translate our results below in terms of Airy processes, it is useful to recall that the following convergence to the Airy\(_2\) process, \(A_2(\tilde{x})\), is expected at large time \[41\] [15] [22] [26]

\[
h(x, t|y, 0) \simeq t^{1/3}(A_2(\tilde{x} - \tilde{y}) - (\tilde{x} - \tilde{y})^2 + o(t^{1/3})) , \quad \tilde{x} = \frac{x}{2t^{2/3}}, \quad \tilde{y} = \frac{y}{2t^{2/3}}
\]

where \(h(x, t|y, 0)\) is the droplet solution with arbitrary endpoints \[51\]. In terms of processes, this equivalence is only valid at either fixed \(y\) or fixed \(x\). The process as \((x, y)\) are both varied is called the Airy sheet (see e.g. \[23\] [29] [60]) and is not yet fully characterized.

The formula \[48\] for a general initial condition then leads, in the large time limit, to a variational formula in terms of an Airy process. Indeed the variations of \(h(x, t|y, 0)\) being \(O(t^{1/3})\) the integral in \[48\] becomes dominated by the maximum value of the integrand, hence we can write, in the sense of the one-point PDF at fixed \(x\)

\[
h(x, t) \simeq t^{1/3} \max_{\tilde{y}} A_2(\tilde{x} - \tilde{y}) - (\tilde{x} - \tilde{y})^2 + h_0(\tilde{y}) + o(t^{1/3})
\]
where we have defined the rescaled initial condition
\[ h_0(y) = t^{-1/3}h(2t^{2/3}y, 0) \] (55)
This allows to classify the initial conditions, depending on whether \( h_0(y) \) has a time independent limit. When \( h_0(y) \rightarrow 0 \) one is in the flat IC class, with the GOE TW distribution \( F_1 \) for \( h(x, t) \). The case of the Brownian initial condition is discussed below. Other intermediate classes have been identified and studied \([26, 28]\).

II. DEFINITION OF THE GENERATING FUNCTION AND ITS PHYSICAL CONTENT

We define, in the same environment \( \eta \), the three partition sums of all DP paths with one endpoint at \((x, t)\) and the second endpoint at \(t = 0\) either on the negative \( y < 0 \), or positive \( y > 0 \) axis, or at \( y = 0 \), together with their logarithms, as
\[ Z_L(x, t) = \int_{y < 0} Z(x, t|y, 0)e^{a_L B_0(y)+w_L y} \), \quad h_L(x, t) = \ln Z_L(x, t) \] (56)
\[ Z_R(x, t) = \int_{y > 0} Z(x, t|y, 0)e^{a_R B_0(y)+w_R y} \), \quad h_R(x, t) = \ln Z_R(x, t) \] (57)
\[ Z_c(x, t) = Z(x, t|0, 0) \), \quad h_c(x, t) = \ln Z_c(x, t) \] (58)
We are multiplying by an additional weight on the \( y \) axis, corresponding to the cases which we can solve. Each parameter \( a_R \) and \( a_L \) is chosen zero or unity. Here \( B_0(y) \) is a two-sided unit centered Brownian, with \( B_0(0) = 0 \), i.e. \( \langle B_0(y)B_0(y') \rangle = \min(y, y')\theta(y)\theta(y') + \min(-y, -y')\theta(-y)\theta(-y') \) (it can also be written as the sum of two independent one-sided Brownians on respectively the negative and positive half line). The parameters \( w_L, w_R \) (usually chosen positive) represent the drifts of the Brownians. The partition sums \( Z_{L,R}(x, t) \) give the relative weights that a DP path with one endpoint at \((x, t)\) has second endpoint at \(t = 0\) either on \((y < 0, 0)\) or \((y > 0, 0)\), in presence of the additional weights \( e^{a_L B_0(y)+w_L y}, e^{a_R B_0(y)+w_R y} \). More precisely,
\[ p_{y}(x, t) = Z_R(x, t)/(Z_L(x, t) + Z_R(x, t)) \] (59)
is the probability, in a given sample, that the DP with one fixed endpoint at \((x, t)\) ends up at \( y > 0 \) at \( t = 0 \). Equivalently, ”reversing time”, it is clear from \([48]\) that \( h_{L,R}(x, t) \) are also solutions of the KPZ equation with initial conditions (Brownian, flat or wedge) on the corresponding half-line, i.e. \( h_L(y, 0) = a_L B_0(y) + w_L y \) for \( y < 0 \) and \( h_L(y, 0) = -\infty \) for \( y > 0 \), and \( h_R(y, 0) = a_R B_0(y) - w_R y \) for \( y > 0 \) and \( h_R(y, 0) = -\infty \) for \( y < 0 \), respectively. In addition, \( h_c(x, t) \) is the solution with the droplet IC centered at 0. Both interpretations will be used below.

We will be interested in the joint distribution of these three partition sums. To this aim we define the following generating function in terms of scaled parameters
\[ \hat{g}_\sigma(\sigma_1, \sigma_0; \sigma_R; \tilde{x}) := \exp(-e^{-t^{1/3}(\sigma_1-\tilde{x}^2)}Z_c(x, t) - e^{-t^{1/3}(\sigma_L-\tilde{x}^2)}Z_L(x, t) - e^{-t^{1/3}(\sigma_R-\tilde{x}^2)}Z_R(x, t)) \] (60)
where the average is implicitly over the noise \( \eta \) and the Brownian IC. We obtain below a formula, Eqs. (136)-(137), for this generating function in the limit of large \( t \). In that limit it becomes equal to the following joint CDF
\[ \hat{g}_\infty(\sigma_0; \sigma_0; \sigma_R; \tilde{x}) = \lim_{t \to +\infty} \text{Prob} \left( t^{-1/3}(h_c(x, t) + \frac{x^2}{4t}) < \sigma_1, t^{-1/3}(h_L(x, t) + \frac{x^2}{4t}) < \sigma_L, t^{-1/3}(h_R(x, t) + \frac{x^2}{4t}) < \sigma_R \right) \] (61)
In the limit of large time, using (53), (54), (55), we can translate this equality in terms of Airy processes. One finds
\[ h_{L,R}(x, t) \simeq t^{1/3}\hat{h}_{L,R}(\tilde{x}), \quad h_c(x, t) \simeq t^{1/3}(\mathcal{A}_2(-\tilde{x}) - \tilde{x}^2) \] (62)
where the random variables \( \hat{h}_{L,R}(\tilde{x}) \) are defined as maxima over the following sum of processes in \( y \)
\[ \hat{h}_L(\tilde{x}) = \max_{\tilde{y}<0}(\mathcal{A}_2(\tilde{y} - \tilde{x}) - (\tilde{y} - \tilde{x})^2 + 2\tilde{w}_L\tilde{y} + a_L\sqrt{2B(\tilde{y})}) \] (63)
\[ \hat{h}_R(\tilde{x}) = \max_{\tilde{y}>0}(\mathcal{A}_2(\tilde{y} - \tilde{x}) - (\tilde{y} - \tilde{x})^2 - 2\tilde{w}_R\tilde{y} + a_R\sqrt{2B(\tilde{y})}) \]
at fixed $\hat{x}$. We used that $t^{-1/3}B_0(2t^{2/3}\hat{y}) = \sqrt{2}B(\hat{y})$ where $B(\hat{y})$ is another unit two-sided Brownian in the variable $\hat{y}$, with $B(0) = 0$. We also reversed for convenience the sign of the argument of the Airy process, which is statistically symmetric. Hence we have (the result for this quantity being given in Eqs. 136-137)

$$\hat{g}_\infty(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = \text{Prob} \left( A_2(-\hat{x}) < \sigma_1, \hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R \right)$$

(64)

We now specialize to the cases of most interest.

- **Joint CDF of $\text{Airy}_2$ at a point and value of a maximum involving $\text{Airy}_2$.**

  Let us set $\hat{x} = 0$ and $\sigma_L = \sigma_R = \sigma_2$.

$$\hat{g}_\infty(\sigma_1, \sigma_2, \sigma_2; 0) = \text{Prob} \left( A_2(0) < \sigma_1, \max(h_L(0), h_R(0)) < \sigma_2 \right)$$

(65)

$$= \text{Prob} \left( A_2(0) < \sigma_1, \max(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + V(\hat{y})), \hat{h}_R(0) < \sigma_2 \right)$$

(66)

$$V(\hat{y}) = 2\hat{w}_L\hat{y} + a_L\sqrt{2}B(\hat{y})\theta(-\hat{y}) + (-2\hat{w}_R\hat{y} + a_R\sqrt{2}B(\hat{y}))\theta(\hat{y})$$

(67)

Hence we obtain the JCDF of $A_2(0)$ and $\max_x(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + V(\hat{y}))$ in the same environment (same realization of the Airy$_2$ process) for a family of potentials $V(\hat{y})$.

An example of particular interest for the two-time KPZ problem in the large time separation limit, is the case $a_L = a_R = 1$ in the limit $\hat{w}_L = \hat{w}_R = 0^+$ where one recovers the JPDF defined in the main text (the result for it being displayed in [24])

$$G(\sigma_1, \sigma_2) := \text{Prob} \left( A_2(0) < \sigma_1, \max(\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})) < \sigma_2 \right)$$

(68)

$$= \lim_{\hat{w}_L \to 0^+, \hat{w}_R \to 0^+} \hat{g}_{+\infty,a_L,a_R = 1}(\sigma_1, \sigma_2, \sigma_2; 0)$$

where $B(\hat{y})$ is a doubled sided unit Brownian (with $B(0) = 0$). It is generalized to arbitrary $\hat{x}$ as

$$G_{\hat{x}}(\sigma_1, \sigma_2) := \text{Prob} \left( A_2(\hat{x}) < \sigma_1, \max(\mathcal{A}_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})) < \sigma_2 - \hat{x}^2 \right)$$

(69)

$$= \lim_{\hat{w}_L \to 0^+, \hat{w}_R \to 0^+} \hat{g}_{+\infty,a_L,a_R = 1}(\sigma_1, \sigma_2, \sigma_2; \hat{x})$$

which applies to the two-time KPZ problem (equivalently to the two-DP problem) with a shift $\hat{x}$ in the droplet initial condition (equivalently in the DP endpoint). Its exact expression for arbitrary $\hat{x}$ is given in [160].

- **PDF or argmax of $\text{Airy}_2$ minus a parabola plus a Brownian, and extensions.**

  Let us set $w_L = w_R = 0^+$, we can rewrite, defining $\hat{z} = \hat{y} - \hat{x}$

$$\hat{h}_L(\hat{x}) = \max_{\hat{z} < -\hat{x}}(\mathcal{A}_2(\hat{z}) - \hat{z}^2 + a_L\sqrt{2}B(\hat{x} + \hat{z}))$$

(70)

$$\hat{h}_R(\hat{x}) = \max_{\hat{z} > -\hat{x}}(\mathcal{A}_2(\hat{z}) - \hat{z}^2 + a_R\sqrt{2}B(\hat{x} + \hat{z}))$$

(71)

which can also be written as

$$\hat{h}_L(\hat{x}) = \max_{\hat{z} < -\hat{x}}(\mathcal{A}_2(\hat{z}) - \hat{z}^2 + a_L\sqrt{2}\hat{B}(\hat{z})) + a_L\sqrt{2}B(\hat{x})$$

(72)

$$\hat{h}_R(\hat{x}) = \max_{\hat{z} > -\hat{x}}(\mathcal{A}_2(\hat{z}) - \hat{z}^2 + a_R\sqrt{2}\hat{B}(\hat{z})) + a_R\sqrt{2}B(\hat{x})$$

(73)

where we have defined

$$\hat{B}(\hat{z}) := B(\hat{z} + \hat{x}) - B(\hat{x})$$

(74)

which is also a unit two-sided Brownian with $\hat{B}(0) = 0$, which is, however, correlated to $B$. Because of this correlation, there are only two main applications.
The first is for the PDF of argmax of Airy\(_2\) minus a parabola plus a Brownian, which is equivalent to the PDF of the midpoint of a DP with Brownian initial conditions (see discussion in the main text and Fig. 1). Consider now \(a_L = a_R = 1\). The term \(\sqrt{2}B(\dot{x})\) cancels in the difference \(\dot{h}_R(\dot{x}) - \dot{h}_L(\dot{x})\), i.e.

\[
\dot{h}_R(\dot{x}) - \dot{h}_L(\dot{x}) = \max_{\dot{z} > -\dot{x}} (A_2(\dot{z}) - \dot{z}^2 + \sqrt{2}B(\dot{z})) - \max_{\dot{z} < -\dot{x}} (A_2(\dot{z}) - \dot{z}^2 + \sqrt{2}B(\dot{z})) \tag{75}
\]

Hence we have

\[
\text{Prob}\left(A_2(-\dot{x}) < \sigma_1, \dot{h}_R(\dot{x}) - \dot{h}_L(\dot{x}) \in [\sigma, \sigma + d\sigma]\right) = \lim_{\dot{w}_L \to 0^+, \dot{w}_R \to 0^+} \int_{-\infty}^{+\infty} d\sigma_R d\sigma_L (\partial_{\sigma_L} \partial_{\sigma_R} g_{+\infty, a_L, \dot{R}} = 1(\sigma_1, \sigma_L, \sigma_R; \dot{x}))(\sigma_L - \sigma_R - \sigma)d\sigma \tag{76}
\]

An application is to the PDF of argmax of the following variational problem. Let us define (we have now suppressed the tilde subscript on \(B\))

\[
\dot{z}_m = \text{argmax}_{\dot{z} \in \mathbb{R}} \left(A_2(\dot{z}) - \dot{z}^2 + \sqrt{2}B(\dot{z})\right) \tag{77}
\]

Clearly one has

\[
H(-\dot{x}) := \text{Prob}(\dot{z}_m > -\dot{x}) = \text{Prob}(\dot{h}_R(\dot{x}) - \dot{h}_L(\dot{x}) > 0) = \lim_{\dot{w}_L \to 0^+, \dot{w}_R \to 0^+} \int_{-\infty}^{+\infty} d\sigma_R (\partial_{\sigma_R} g_{+\infty, a_L, \dot{R}} = 1(\sigma_1, \sigma_L, \sigma_R; \dot{x}))(\sigma_L - \sigma_R - \sigma) \tag{78}
\]

The calculation of \(H(-\dot{x})\) is performed in Section VII, where its properties are studied. The final formula for it is (79), equivalently (79) in the text.

In principle we can extract a bit more information, keeping \(\sigma_1\) arbitrary we obtain the JPDF

\[
\text{Prob}(A_2(-\dot{x}) < \sigma_1, \dot{z}_m > -\dot{x}) = \lim_{\dot{w}_L \to 0^+, \dot{w}_R \to 0^+} \int_{-\infty}^{+\infty} d\sigma_R (\partial_{\sigma_R} g_{+\infty, a_L, \dot{R}} = 1(\sigma_1, \sigma_L, \sigma_R; \dot{x}))(\sigma_L - \sigma_R - \sigma) \tag{79}
\]

Taking a derivative w.r.t. \(\sigma_1\) and dividing by \(\partial_{\sigma_1} F_2(\sigma_1)\) it gives the probability that the longer polymer in Fig. 1 passes to the right of the origin, given the value of the Airy process at this point (equivalently, by translational invariance, one can shift all endpoints by \(-\dot{x}\) in Fig. 1 and ask the DP to pass right of \(-\dot{x}\). We do not display the resulting formula here, as it is a simple exercise to get it from (75) following similar steps as in Section VII. 1.

• Joint PDF of argmax and max, equivalently of endpoint position and free energy of a DP.

Consider \(a_L = a_R = 0\) and focus on \(w_{L,R} \to 0^+\). Then we have

\[
\dot{h}_L(\dot{x}) = \max_{\dot{z} < -\dot{x}} (A_2(\dot{z}) - \dot{z}^2) \quad , \quad \dot{h}_R(\dot{x}) = \max_{\dot{z} > -\dot{x}} (A_2(\dot{z}) - \dot{z}^2) \tag{80}
\]

Let us define the value and position of the following variational problem

\[
\dot{h}_m = \max_{\dot{z} \in \mathbb{R}}(A_2(\dot{z}) - \dot{z}^2) \quad , \quad \dot{z}_m = \text{argmax}_{\dot{z} \in \mathbb{R}}(A_2(\dot{z}) - \dot{z}^2) \tag{81}
\]

It can also be seen as the endpoint position \(z_m\) and associated (minus) free energy \(h_m\) of a directed polymer from the point \((0, 0)\) to the line \((z, t)\) (with a free endpoint \(z \in \mathbb{R}\), i.e. the point-to-line problem), in the limit \(t \to +\infty\), expressed in rescaled variables \(\dot{h}_m = t^{-1/3}\dot{h}_m\) and \(\dot{z}_m = z_m / 2 t^{2/3}\).

Then we will obtain here the joint C-PDF of the position and the value of the maximum as

\[
\text{Prob}(\dot{z}_m > -\dot{x}, \dot{h}_m) = \text{Prob}(\dot{h}_R(\dot{x}) - \dot{h}_L(\dot{x}) > 0, \dot{h}_R(\dot{x}) = \dot{h}_m) \tag{82}
\]

\[
\int_{-\infty}^{+\infty} d\sigma_R \delta(\dot{h}_m - \dot{z}_m^2) \int_{-\infty}^{+\infty} d\sigma_L \partial_{\sigma_L} \partial_{\sigma_R} g_{+\infty, a_L, \dot{R}} = 0(\sigma_1, \sigma_L, \sigma_R; \dot{x}) = [\partial_{\sigma_R} g_{+\infty, a_L, \dot{R}} = 0(\sigma_1, \sigma_L, \sigma_R; \dot{x})] |_{\sigma_L = \sigma_R = \dot{h}_m + \dot{z}_m^2}
\]
using, in the integration by part, that $\hat{g}_\infty$ vanishes when any of its arguments is sent to $-\infty$. The formula for (82) obtained from the present method is given in Section XII. 2., formula (341) and (343).

It is interesting to note that formulae for the joint PDF of $(\hat{h}_m, \hat{z}_m)$ were obtained previously by very different methods: (i) within a rigorous approach in [62] as a formula involving the Airy function and the resolvent of an associated operator, and (ii) from studying non-crossing Brownian paths, by Schehr in [63] as a formula involving a solution to the Lax pair for the Painlevé II equation. It was later shown in [64] that these formulas are equivalent.

The present derivation is much closer in spirit to the one of Dostenko [58] for the position of the endpoint $\hat{z}_m$. Dotsenko formula for the PDF of $\hat{z}_m$ was shown in Appendix C of [65] to coincide with the predictions of [62] and [63] for the marginal distribution of $z_m$. Here however we obtain directly the joint PDF of $(\hat{h}_m, \hat{z}_m)$. Also our integrals have a slightly different form. We will not attempt here to show that this formula is equivalent to the ones of [62] and [63] for the JPDF, although it is likely to be correct.

Note that here we obtain the more general result, i.e the triple JPDF

$$\text{Prob}(A_2(-\hat{x}) < \sigma_1, \hat{x}_m > -\hat{x}, \hat{h}_m) = [\partial_{\sigma_R} \hat{g}_{+\infty, a_{L,R}=0}(\sigma_1, \sigma_L, \sigma_R; \hat{x})]_{\sigma_L=\sigma_R=\hat{h}_m+\hat{x}^2}$$

which contains both the above JPDF and also the JPDF of the KPZ heights for the droplet and flat IC in the same noise. Indeed in Section XII. 1. we will display $\text{Prob}(A_2(-\hat{x}) < \sigma_1, \max_{\hat{z} \in \mathbb{R}}(A_2(\hat{z} - \hat{z}^2)))$, which gives the large time limit of the JPDF of the scaled KPZ heights of the flat and droplet solutions in the same noise, equivalently of the point-to-line and point-to-point DP scaled free energy in the same random potential.
III. CALCULATION OF THE GENERATING FUNCTION

III.1. Moment expansion

For notational convenience we introduce a second set of rescaled parameters

\[ \lambda := 2^{-2/3}t^{1/3}, \quad s = 2^{2/3}(\sigma - x^2) , \quad \tilde{w}_{L,R} = \lambda w_{L,R} = 2^{-2/3}\hat{w}_{L,R} , \quad \tilde{x} = x/\lambda^2 = 2^{7/3}\hat{x} \]

(84)

where the parameter \( \lambda \) was introduced in Ref. \[51\],\[52\]. With these new parameters the generating function \[90\] can be written, and expanded in series, in terms of the joint moments, as follows

\[ \hat{g}(\sigma_1, \sigma_L, \sigma_R; \tilde{x}) = g_\lambda(s_1, s_L, s_R; \tilde{x}) := e^{-e^{\lambda s_1}Z_L(x,t) - e^{-\lambda s_L}Z_L(x,t) - e^{-\lambda s_R}Z_R(x,t)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Z_n \]

(85)

\[ Z_n := (e^{-\lambda s_1}Z_L(x,t) + e^{-\lambda s_L}Z_L(x,t) + e^{-\lambda s_R}Z_R(x,t))^n \]

(86)

From the definitions \[56\], upon averaging over \( \eta \) and \( B_0 \) (which we have chosen independent), the moments can be written as

\[ Z_n = \int dy_1...dy_n \prod_{n=1}^{\eta} Z_\eta(x,t|y_n,0) \prod_{\alpha=1}^{n} (e^{-\lambda s_L + a_L B_0(y_n)} + w_L y_\alpha \theta(-y_\alpha) + e^{-\lambda s_1} \delta(y_\alpha) + e^{-\lambda s_R + a_R B_0(y_n)} - w_R y_\alpha \theta(y_\alpha)) B_0 \]

(87)

III.2. Quantum mechanics and overlap

As is now well known \[67\],\[68\] the \( \eta \) average in the middle of \[87\] can be rewritten as the expectation value between initial and final states of the quantum-mechanical evolution operator associated to the attractive Lieb-Liniger (LL) Hamiltonian for \( n \) identical particles \[69\]:

\[ H_n = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_\alpha^2} - 2\tilde{c} \sum_{1 \leq \alpha < \beta \leq n} \delta(x_\alpha - x_\beta) , \quad \tilde{c} = 1 \]

(88)

The eigenfunctions are known from the Bethe ansatz \[66\]. They are parameterized by a set of rapidities \( \mu \equiv \{ \lambda_1, ... \lambda_n \} \) which are solution of a set of coupled equations, the Bethe equations (see below). The eigenfunctions are totally symmetric in the \( x_\alpha \), and in the sector \( x_1 \leq x_2 \leq \cdots \leq x_n \), take the (un-normalized) form

\[ \Psi_\mu(x_1, .. x_n) = \sum_{P \in S_n} A_P \prod_{j=1}^{n} c^j \sum_{\alpha=1}^{n} \lambda_{P_\alpha} x_\alpha , \quad A_P = \prod_{1 \leq \alpha < \beta \leq n} a_{\lambda_{P_\alpha}, \lambda_{P_\beta}} , \quad a_{\lambda, \lambda'} = \left(1 + \frac{i}{\lambda' - \lambda} \right) \]

(89)

They can be deduced in the other sectors from their full symmetry with respect to particle exchanges. The sum runs over all \( n! \) permutations \( P \) of the rapidities \( \lambda_\alpha \). The corresponding eigenenergies are \( E_\mu = \sum_{\alpha=1}^{n} \lambda_\alpha^2 \). One can then rewrite the moment as a sum over eigenstates

\[ Z_n = \langle x ... x | e^{-tH_n} | \Phi_0 \rangle = \sum_\mu \Psi_\mu(x ... x) e^{-tE_\mu} \langle \Psi_\mu | \Phi_0 \rangle = \sum_\mu \Psi_\mu^*(x ... x) e^{-tE_\mu} \langle \Phi_0 | \Psi_\mu \rangle \]

(90)

where we have used that \( Z_n \) is real, and for convenience we will work with the second (i.e. complex conjugate) expression. Here \( | \Phi_0 \rangle \) is the initial state (see below) and \( | x ... x \rangle \) is the final state, with all particles at the same point \( x \). Since this state is fully symmetric in exchanges of particles, only symmetric eigenfunctions will contribute and we can consider particles as bosons. In the formula \[90\] we first need:

\[ \Psi_\mu^*(x ... x) = n! e^{-ix \sum_\alpha \lambda_\alpha} \]

(91)

The wavefunction of the initial replica state is also symmetric and equal to:

\[ \Phi_0(Y) = \langle y_1 ... y_n | \Phi_0 \rangle = \prod_{\alpha=1}^{n} (e^{-\lambda s_L + a_L B_0 L(-y_\alpha)} + w_L y_\alpha \theta(-y_\alpha) + e^{-\lambda s_1} + e^{-\lambda s_R + a_R B_0 R(y_\alpha)} - w_R y_\alpha \theta(y_\alpha)) B_{0 L}, B_{0 R} \]

(92)
where here and below coordinate multiplets are denoted by capital letters, e.g. \( Y \equiv y_1, \ldots, y_n \). For convenience, we are using that \( B_0(y) = \theta(-y)B_{0L}(-y) + \theta(y)B_{0R}(y) \) where \( B_{0L,0R} \) are two independent one-sided unit Brownians (both on the positive axis).

Taking advantage of the symmetry of the wavefunctions, we rewrite the overlap of the initial state and any eigenstate

\[
\langle \Phi_0 | \Psi_\mu \rangle = \int dY \Psi_\mu(Y) \Phi_0(Y) = n! \sum_{n_1, n_L, n_R \geq 0} \frac{1}{n_1!} \sum_{n_1 + n_L + n_R = n} \int_{y_1 < \cdots < y_n < 0} \langle e^{B_{0L}(-y_1) + \cdots + B_{0L}(-y_n)} \rangle_{B_{0L}}
\]

\[
\times \int_{0 < y_{n-n_R+1} < \cdots < y_n} \langle e^{B_{0R}(y_{n-n_R+1}) + \cdots + B_{0R}(y_n)} \rangle_{B_{0R}} \Psi_\mu(y_1, \ldots, y_n, 0, 0, y_{n-n_R+1}, \ldots, y_n)
\]

Inserting the form (89) of the wavefunction, we obtain the overlap as

\[
\langle \Phi_0 | \Psi_\mu \rangle = n! \sum_{P \in S_n} A_P \sum_{n_1, n_L, n_R \geq 0} \frac{1}{n_1!} e^{-\lambda_1 s_1 - \lambda n_L s_L - \lambda n_R s_R} G^L_{n_L, w_L, a_L} [\lambda p_1, \ldots, \lambda p_n] G^R_{n_R, w_R, a_R} [\lambda p_{n-n_R+1}, \ldots, \lambda p_n]
\]

where, as in (61) we define

\[
G^L_{p, w, a} [\lambda_1, \ldots, \lambda_p] := \int_{y_1 < y_2 < \cdots < y_p < 0} e^{\sum_{j=1}^p (w + i \alpha_j) y_j} < e^{\sum_{j=1}^p a B_{0L}(-y_j)} >_{B_{0L}}
\]

\[
G^R_{p, w, a} [\lambda_1, \ldots, \lambda_p] := \int_{0 < y_1 < y_2 < \cdots < y_p} e^{\sum_{j=1}^p (w - i \alpha_j) y_j} < e^{\sum_{j=1}^p a B_{0R}(y_j)} >_{B_{0R}}
\]

with \( G^L_{0, w, a} = G^R_{0, w, a} = 1 \). Note that for each permutation \( P \) there are three groups of rapidities, of sizes \( n_L, n_1, n_R \) respectively, associated to \( y < 0, y = 0 \) and \( y > 0 \) respectively. Permutations \( P \in S_n \) exchanges rapidities within these groups, but not between them. The above integrals can be explicitly evaluated and furthermore, as discussed in (61), see Eqs. (57-62) there, for the two "solvable" cases, \( a = 0, 1 \) a "miracle" occurs allowing to perform exactly the summation over the permutations, leading to a factorized form (53-69)

\[
H_{p, w, a}^R [\lambda_1, \ldots, \lambda_p] := \sum_{P \in S_p} A_P G^R_{p, w, a} [\lambda p_1, \ldots, \lambda p_p] = \frac{2^p}{\prod_{j=1}^p (2w - 1 - 2i \alpha_j)}
\]

\[
H_{p, w, a}^L [\lambda_1, \ldots, \lambda_p] := \sum_{P \in S_p} A_P G^L_{p, w, a} [\lambda p_1, \ldots, \lambda p_p] = \frac{1}{\prod_{a=1}^p (w + i \alpha_a)} \prod_{1 \leq a < b \leq p} \frac{2w + i \alpha_a + i \alpha_b - 1}{2w + i \alpha_a + i \alpha_b}
\]

where we have introduced two new functions which depend only on the set of rapidities, not on their order. These miracle identities (with \( p = n \)) allow to obtain simple expressions for the terms where two of the three variables \( n_L, n_R, n_1 \) are zero in (94) but (a priori) not for the general term, since there are then permutations which exchange rapidities between the three groups of rapidities.

Evaluation becomes possible however when the eigenstates are strings. We now follow the strategy of (61).

### III.3. Strings and combinatorial identities

In the limit of infinite system size, the rapidities solution to the Bethe equations are the so-called strings (70), and the spectrum of \( H_n \) is as follows. A general eigenstate is built by partitioning the \( n \) particles into a set of \( 1 \leq n_s \leq n \) bound states called strings each formed by \( m_j \geq 1 \) particles with \( n = \sum_{j=1}^{n_s} m_j \). The rapidities associated to these states are written as

\[
\lambda_{j,a} = k_j - \frac{i}{2} (m_j + 1 - 2a)
\]

where \( k_j \) is a real momentum. Here, \( a = 1, \ldots, m_j \) labels the rapidities within the string \( j = 1, \ldots, n_s \). We will denote \( | \mu \rangle \equiv |k, m\rangle \) these strings states, labelled by the set of \( k_j, m_j, j = 1, \ldots, n_s \). Here and below the boldface represents vectors with \( n_s \) components. Inserting these rapidities in (80) leads to the Bethe eigenfunctions of the infinite system, and their corresponding eigenenergies:

\[
E_\mu = \frac{1}{12}nt + \tilde{E}(k, m) \quad \text{and} \quad \tilde{E}(k, m) := \sum_{j=1}^{n_s} m_j k_j^2 - \frac{1}{12}m_j^3
\]
We have separated a trivial part of the energy, which can be eliminated by defining

\[ Z_n = e^{-\frac{1}{12}t} \tilde{Z}_n, \quad Z(x,t) = e^{-\frac{1}{12}t} \tilde{Z}(x,t) = -\frac{1}{12} t + \tilde{h}(x,t) \]  

(100)
i.e. leading to a simple shift in the KPZ field. We will implicitly study in the remainder of the paper \( \tilde{Z}_n, \tilde{Z}(x,t) \) and \( \tilde{h}(x,t) \) but will remove the tilde in these quantities for notational simplicity (as mentioned in the text). The formula for the norm of the string states reads [21]:

\[ \frac{1}{||\mu||^2} = \frac{1}{n!L^n} \Phi(\mathbf{k}, \mathbf{m}) \prod_{j=1}^{n} \frac{1}{m_j^2}, \quad \Phi(\mathbf{k}, \mathbf{m}) = \prod_{1 \leq i < j \leq n_s} \Phi_{k_i,m_i,k_j,m_j}, \quad \Phi_{k_i,m_i,k_j,m_j} = \frac{4(k_i - k_j)^2 + (m_i - m_j)^2}{4(k_i - k_j)^2 + (m_i + m_j)^2} \]  

(101)

so that the formula [90] for the moments becomes for \( L \to +\infty \) (provided all limits exist)

\[ Z_n = \sum_{n_s=1}^{n} \frac{1}{n_s!} \sum_{(m_1,..m_{n_s})_{n_s}} \frac{1}{n_s!} \int \frac{dk_j}{2\pi m_j} \Phi(\mathbf{k}, \mathbf{m}) e^{-t\hat{E}(\mathbf{k}, \mathbf{m})} e^{-i \sum_{i<j} m_i k_j x} (\Phi_0|\mathbf{k}, \mathbf{m}) \]  

(102)

where the second sum is over the set of partitions, denoted \((m_1,..m_{n_s})_n\), of the integer \( n = \sum_{j=1}^{n_s} m_j \) into \( n_s \) parts, with each \( m_j \geq 1 \).

It remains to calculate the overlap, formula [94]. If the states are strings, the sum over permutations can be performed. Let us sketch the main idea, introduced by Dotsenko [24], and checked in more details in [31] (to which we refer for details). Consider a string state \( \Psi_{\mathbf{m}} = |\mathbf{k}, \mathbf{m}\rangle \) with rapidities given by \( k_i \). From the definition [89], the only permutations \( P \) which have a non vanishing amplitude \( A_P \), are those such that for each string the intra-string order of increasing imaginary part is maintained. Hence if one is given a set of \( 3n_s \) integers \((m_j^L, m_j^L, m_j^R), j = 1, \ldots, n_s\):

\[ 0 \leq m_j^L \leq m_j, \quad 0 \leq m_j^L \leq m_j, \quad 0 \leq m_j^R \leq m_j, \quad m_j^L + m_j^L + m_j^R = m_j \]  

(103)

which specifies how many particles in each string belong to each of the three groups, then one knows \textit{bijectively} the three sets of rapidities which belong of each group. For instance one knows that the first set of rapidities is:

\[ \Lambda_L = \{\lambda_{j,r_j}, j = 1, \ldots, n_s, r_j = 1, \ldots, m_j^L \} \]  

(104)

and that the second set is \( \Lambda_R = \{\lambda_{j,r_j}, j = 1, \ldots, n_s, r_j = m_j^L + m_j^L + 1, \ldots, m_j \} \). To treat these two sets on equal footing, it is convenient to introduce the notation:

\[ \lambda_{j,r_j} = k_j - \frac{i}{2}(m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m_j^L \]  

(105)

\[ \bar{\lambda}_{j,r_j} = k_j + \frac{i}{2}(m_j + 1 - 2r_j) \quad 1 \leq r_j \leq m_j^R \]  

(106)

Finally, the third set, \( \Lambda_1 \), is simply the complement of the first two sets.

Consider now the overlap [94] for a given string state \( |\mathbf{k}, \mathbf{m}\rangle \). The sum over \( P \in S_n \) can be made in two stages. In a first stage one fixes a set of \( 3n_s \) integers \( \mathbf{m}^{R,1,L} = \{m_j^{R,1,L}\}_{j=1,..n_s} \). That fixes which of the \( n \) rapidities of the string belong to each of the three groups \( \Lambda_L, \Lambda_1, \Lambda_R \). Hence there is no more permutations (i.e. with a non vanishing contribution) exchanging rapidities between the three groups, and the only remaining permutations are permutations inside each group. One thus performs the sum over permutations inside each group. It then remains to sum over the variables \( \mathbf{m}^{R,1,L} \), a sum which we perform in a second stage. One takes advantage that one can factor \( A_P \) as (with \( n = n_L + n_1 + n_R \))

\[ \prod_{1 \leq \alpha < \beta \leq n} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} = \prod_{1 \leq \alpha < \beta \leq n_L} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \prod_{n-n_R+1 \leq \alpha < \beta \leq n} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \prod_{n_L+1 \leq \alpha < \beta \leq n_L + n_1} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \times \hat{G} \]  

(107)

with

\[ \hat{G} = \prod_{n=L}^{n_L} \prod_{\alpha=1}^{n_1} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \times \prod_{n_L+1 \leq \alpha < \beta \leq n_L+1+n_1} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \times \prod_{n_1+1 \leq \alpha < \beta \leq n_1+1+n_1} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \]  

(108)

and one defines the two fully symmetric functions of their arguments

\[ H^L_{n_L} = \sum_{P \in S_{n_L}} \left( \prod_{1 \leq \alpha < \beta \leq n_L} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \right) G^L_{n_L,w_L,a_L} \]  

(109)

\[ H^R_{n_R} = \sum_{P \in S_{n_R}} \left( \prod_{n-n_R+1 \leq \alpha < \beta \leq n} a_{\lambda_{P_\beta}, \lambda_{P_\alpha}} \right) G^R_{n_R,w_R,a_R} \]
One can then evaluate \( H_{nL}^L \) on the set \( \Lambda_L \) and \( H_{nR}^R \) on the set \( \Lambda_R \). One thus defines:

\[
\tilde{H}^L[k, m, m^L] = H_{nL}^L[\lambda_1, \ldots, \lambda_{n_L}, \ldots, \lambda_{n_L}, m^L] \\
\tilde{H}^R[k, m, m^R] = H_{nR}^R[\lambda_1, \ldots, \lambda_{n_R}, \lambda_{n_L}, m^R] 
\] (110)

Note that the functions on the left do not explicitly depend any more on the choice \( n_R, n_1, n_L \), they depend on this choice only via \( G \)

\[
\sum_{\text{permutations in the middle group, corresponding to particle with time}} \text{evaluating the functions } \tilde{G} \text{ (calculations).}
\]

However we will not need its precise form in what follows. One can then evaluate \( \langle \Phi_0 | k, m \rangle = n! \sum_{n_L, n_1, n_R \geq 0} \sum_{n_L + n_1 + n_R = n} \tilde{H}^L_{nL, wL, aL}[k, m, m^L] \tilde{H}^R_{nR, wR, aR}[k, m, m^R] G[k, m^L, m^1, m^R] \) (113)

Here \( G[k, m^L, m^1, m^R] \) is equal to the product \( \hat{G} \) in \ref{105}, which, for a fixed set \( (m^L, m^1, m^R) \) is independent of the permutations (since these only exchange rapidities inside each group), which is why formula (113) holds. The explicit calculation of \( G \) yields a complicated product of Gamma functions (see \ref{64} and \ref{61} for examples of such calculations).

However we will not need its precise form in what follows.

Inserting \ref{113} into \ref{102} and \ref{60} gives an a priori exact expression for the generating function at arbitrary time. It requires evaluating the functions \( \tilde{H}^{L,R} \) by injecting the string rapidities \( (98) \) into the formula \ref{96} and \ref{97}, according to the rule \ref{109}. We find (with the help of formula \ref{60-63} in \ref{61})

\[
\tilde{H}^L_{nL, wL, aL}[k, m, m^L] = \prod_{j=1}^{n_L} S^w_{m^L_j, m_j, k} \prod_{1 \leq i < j \leq n_L} P^w_{m^L_i, m_j, k} \\
\tilde{H}^R_{nR, wR, aR}[k, m, m^R] = \prod_{j=1}^{n_R} S^w_{m^R_j, m_j, -k} \prod_{1 \leq i < j \leq n_R} D^w_{m^R_i, m_j, -k}
\]

where the single string factors are:

\[
S^w_{m^L, m, k} = S^w_{m^L, m, k} = \frac{2m^L}{\Gamma(2w + 2ik + m - m^L)} \\
S^w_{m^L, m, k} = S^w_{m^L, m, k} = \frac{2m^L}{\Gamma(2w + 2ik + m)}
\] (115)

Note that the calculation is very similar to the one leading to Eq. \ref{74-76} in \ref{61} with the important difference that the formula there apply only to the case \( m^1_j = 0 \). Hence the notations for the \( S \) and \( D \) factors have slightly different arguments. We will not give the analogous formula to \ref{77} for \( D \) as we will not need it.

### III.4. Large time limit and decoupling assumption: first form of the kernel

In the large time limit, as in \ref{61}, we assume that one can set the product of factors \( D \) and \( G \) to unity. This is of course a highly non-trivial and radical assumption, however it is justified a posteriori by the results. It will be checked in all cases where the solution is known by other means. This procedure follows what has been done in other works, where it was also checked against other methods \ref{54-61}. It is often called the "decoupling assumption" following the procedure introduced in \ref{55}. Here we use it in an a priori different form, similar to the one introduced in \ref{54}. Although these two implementations have been shown to coincide in some cases \ref{56-57}, to our knowledge there is no general check of that. Nevertheless we use the name "decoupling assumption".

Let us first obtain a closed expression once these factors are set to unity, and take the large \( \lambda \) limit in a second stage. Putting together formula \ref{113}, \ref{114}, \ref{115}, with \( D \to 1 \) and \( G \to 1 \), into \ref{55} and \ref{102}, we obtain

\[
g_\lambda(s_1, s_L, s_R; \bar{x}) = g_\lambda^{\text{dec}}(s_1, s_L, s_R; \bar{x}) = 1 + \sum_{n_s = 1}^{\infty} \frac{1}{n_s!} Z^{\text{dec}}_{\lambda}(n_s, s_1, s_L, s_R; \bar{x})
\] (116)
where the partition sum with fixed number of strings reads

$$Z^{\text{dec}}_\lambda(n_s, s_1, s_L, s_R; \bar{x}) = \prod_{j=1}^{n_s} \sum_{m_j=1}^{\infty} \frac{dk_j}{2\pi m_j} (-1)^{m_j} e^{-i\lambda m_j k_j} \Phi(k, m) e^{-tE(k, m)}$$  \hspace{1cm} (117)

$$\times \sum_{m^L + m^R = m} \prod_{j=1}^{n_s} S_{m^L, m^R, k_j}^{wL, aL} e^{-\lambda m^L_j s_L - \lambda m^R_j s_R}$$

Here the subscript "dec" reminds it that $y$ is not the exact expression, but that the decoupling approximation has been applied. We have used that summing over $n$ in the generating function allows in turn to sum freely on $n_L, n_1, n_R$ and freely on $m$. Following the same steps as in [51] Section III. E.1, i.e. using the standard determinant double-Cauchy identity:

$$\Phi(k, m) = \prod_{j=1}^{n_s} (2m_j) \frac{\det_{1 \leq i, j \leq n_s} \left[ \frac{1}{2i(k_i - k_j) + m_i + m_j} \right]}$$  \hspace{1cm} (118)

performing the rescaling $k_j \rightarrow k_j/\lambda$, denoting $\bar{x} = x/\lambda^2$ and performing the Airy trick, i.e. the identity $\frac{1}{\pi} \Im m = \int dy \text{Ai}(y) e^{\lambda m y}$ for $\lambda m > 0$, we obtain

$$Z^{\text{dec}}_\lambda(n_s, s_1, s_L, s_R; \bar{x}) = 2^{n^2} \prod_{j=1}^{n_s} \sum_{m_j=1}^{\infty} \frac{dk_j}{2\pi m_j} dy_j \text{Ai}(y_j) (-1)^{m_j} e^{-i\lambda m_j k_j} e^{4\lambda m_j k_j^2 + \lambda m_j y_j}$$

$$\times \det_{1 \leq i, j \leq n_s} \left[ \frac{1}{2i(k_i - k_j) + m_i + m_j} \right] \times \sum_{m^L + m^R = m} \prod_{j=1}^{n_s} S_{m^L, m^R, k_j}^{wL, aL} e^{-\lambda m^L_j s_L - \lambda m^R_j s_R}$$

Using standard manipulations [51] [52] the partition sum at fixed number of string $n_s$ can thus be expressed itself as a determinant:

$$Z^{\text{dec}}_\lambda(n_s, s_1, s_L, s_R; \bar{x}) = \prod_{j=1}^{n_s} \sum_{v_i > 0} \det_{1 \leq i, j \leq n_s} M^{\lambda}_{s_1, s_L, s_R; \bar{x}}(v_i, v_j)$$  \hspace{1cm} (120)

with the Kernel:

$$M^{\lambda}_{s_1, s_L, s_R; \bar{x}}(v_i, v_j) = \int \frac{dk}{2\pi} dy \text{Ai}(y + 4k^2 + ik\bar{x} + v_i + v_j) e^{-i(kv_i - k v_j)} \phi_{\lambda}(k, y - s_L, y - s_R, y - s_1)$$  \hspace{1cm} (122)

$$\phi_{\lambda}(k, y_L, y_R, y) = \sum_{m^L, m^R, m \geq 0, m^L + m^R + m \geq 1} (-1)^{m^L + m^R + m} S_{m^L, m^R, m}^{wL, aL} S_{m^L, m^R, m}^{wR, aR} e^{\lambda m^L y_L + \lambda m^R y_R + \lambda m y}$$  \hspace{1cm} (123)

where the $S$ factors are given explicitly in [115]. The generating function thus becomes a Fredholm determinant:

$$g^{\text{dec}}_{\lambda}(s_1, s_L, s_R; \bar{x}) = \det[I + P_0 M^{\lambda}_{s_1, s_L, s_R; \bar{x}} P_0]$$  \hspace{1cm} (124)

where $P_0(v) = \theta(v)$ is the projector on $[0, +\infty]$. Here, again, this expression is valid as soon as the factors $D$ and $G$ are set (arbitrarily) to unity.

To study the large time limit, we first rewrite:

$$\phi_{\lambda}(k, y_L, y_R, y) = -2 + 2 \sum_{m^L \geq 0, m^R \geq 0, m \geq 0} (-1)^{m^L + m^R + m} S_{m^L, m^R, m}^{wL, aL} S_{m^L, m^R, m}^{wR, aR} e^{\lambda m^L y_L + \lambda m^R y_R + \lambda m y}$$

and use the Mellin-Barnes identity:

$$\sum_{m=0}^{\infty} (-1)^m f(m) = \frac{1}{2i} \int_C \frac{dz}{\sin \pi z} f(z)$$  \hspace{1cm} (125)

where $C = \kappa + i\mathbb{R}$, $-1 < \kappa < 0$, valid provided $f(z)$ is meromorphic, with no pole for $z > \Re(\kappa)$, and sufficient decay at infinity. It allows to rewrite (for $2w_{L,R} + \kappa > 0$)

$$\phi_{\lambda}(k, y_L, y_R, y) = -2 + 2 \left(\frac{1}{2i}\right)^2 \int_C \frac{dz}{\sin \pi z} \int_C \frac{dz}{\sin \pi z} \left(\frac{1}{2i}\right) \int_C \frac{dz}{\sin \pi z} \int_C \frac{dz}{\sin \pi z} \int_C \frac{dz}{\sin \pi z} S_{m^L, m^R, m}^{wL, aL} S_{m^L, m^R, m}^{wR, aR} e^{\lambda z_L y_L + \lambda z_R y_R + \lambda z y} \hspace{1cm} (126)$$
Here the analytic continuation \( f(m) \to f(z) \) has been performed using the second expression in \[115\], and we now define

\[
S_{zL,zR,z,k}^{w,0} = 2^{2zL} \frac{(2w + 2ik + zR + z)}{(2w + 2ik + zR + zL + z)}
\]

\[
S_{zL,zR,z,k}^{w,1} = 2^{(w + ik + \frac{zR + zL + z}{2})} \frac{\Gamma(w + 2ik + z + zL + zR)}{\Gamma(w + 2ik + zR + zL + z)}
\]

We now rescale \( zL,R \to zL,R/\lambda \), and we study the large time limit \( \lambda \to +\infty \). We first recall the definition of the rescaled drifts:

\[
\hat{w}_L = w_L\lambda \quad , \quad \hat{w}_R = w_R\lambda
\]

and we use that for \( a = 0, 1 \):

\[
S_{zL,zR}^{w,\lambda,a} = 1 + \frac{(1 + a)zL}{2\hat{w} + 2ik + zR + z - a_zL}
\]

as can be seen from \[127\]. Thus we obtain the infinite \( \lambda \) limit in the form of a multiple contour integral:

\[
\phi_{\infty}(k, yL, yR, y) = -2 \int_{C'} \frac{dL}{2\pi i} \int_{C'} \frac{dR}{2\pi i} \int_{C'} \frac{dz}{2\pi i} \left( \frac{(1 + aL)zL}{2\hat{w} + 2ik + zR + z - aLzL} \right) e^{(1 + aR)zR} e^{(1 + a)zL} e^{(1 + a)zR}
\]

where \( C' = 0^- + i\mathbb{R} \). The calculation of this integral is performed in Section IX below.

Let us now describe the resulting expression for the infinite time limit of the generating function. One finds

\[
g_{\infty}(s_1, s_L, s_R; \hat{x}) = \text{Det}[I + P_0 M(s_1, s_L, s_R, \hat{x}) P_0]
\]

\[
M(s_1, s_L, s_R, \hat{x})(v, j) = \int_{2\pi} \frac{dk}{2\pi} \text{Ai}(y + 4k^2 + ik\hat{x} + v + j) e^{-2ik(v_j)} \phi_{\infty}(k, y - sL, y - s_L, s_L, y, y - s_1)
\]

where from now on we are omitting from now on the "dec" subscript, since we conjecture that it is the exact result.

From \[319\] we find

\[
\frac{1}{2} \phi_{\infty}(k, yL, yR, y) = -1 + \theta(-yL)\theta(-yR)\theta(-y) + (1 + aL + aR - 3aRaL)
\]

\[
\times \theta(yL + aL/yR)\theta(yR + aR/yL)\theta((1 + aR)yL + (1 + aL)yR - y)e^{-2(\hat{w}_L + ik)(yL + aL/yR) - 2(\hat{w}_R - ik)(yL + aL/yR)}
\]

\[-2aL\theta(-yL)e^{2yL(\hat{w}_L + ik)} \theta(-yR)(\hat{w}_R + ik) \theta(-yL - y) \theta(-yR - y) + (1 + aL)\theta(yL)e^{-2\max(yL,yR)}(\hat{w}_L + ik)
\]

\[-2aR\theta(-yL)e^{2yR(\hat{w}_R - ik)} \theta(-yR)(\hat{w}_R - ik) \theta(-yL - y) \theta(-yR - y) - (1 + aR)\theta(yR)e^{-2\max(yL,yR)}(\hat{w}_R - ik)
\]

\[+ \frac{2aRaL}{\hat{w}_L + \hat{w}_R} \theta(yL + yR) e^{(\hat{w}_L + ik)yL + (\hat{w}_R - ik)yR - (\hat{w}_L + \hat{w}_R)\max(yL,yR,yR)}
\]

This is a new result, and we can check that it reduces in some limits to the results of our previous work. Note that the limit \( \lim_{y\to-\infty} \phi_{\infty}(k, yL, yR, y) \to \phi_{\infty}(k, yL, yR) \) reproduces correctly the function given in Eq. (B6) in \[61\]. From \[313\] we then see that the limit \( \lim_{s_1 \to +\infty} g_{\infty}(s_1, s_L, s_R; \hat{x}) = g_{\infty}(s_L, s_R; \hat{x}) \) reproduces the result given in Appendix D.1 in \[61\]. On the other hand, if we set \( s_L = s_R = s \) and specialize to \( s_1 > s \), one can check that the present result reduces to the result, independent of \( s_1 \), obtained in \[61\] Section III. E.2. Eqs. (102)-(104). Although this was expected for \( s_1 \to +\infty \), as discussed just above, the fact that it holds for all \( s_1 > s \) is a non-trivial and needed property (see below) and thus an important test for the "decoupling assumption".

### III.5. Second form of the kernel: general triple joint CDF

We now rewrite the kernel \[131,133\] using Airy function identities recalled in Appendix C of \[61\]. These manipulations are performed in Section X below and are a generalization of similar steps as in Appendices C and D.2 of \[61\]. Here we only give the final result. It is expressed as

\[
g_{\infty}(s_1, s_L, s_R; \hat{x}) = \hat{g}_{\infty}(s_1, s_L, s_R; \hat{x})
\]
in terms of the variables

\[ \sigma_{L,R,1} = 2^{-2/3}(s_{L,R,1} + \frac{\hat{x}^2}{16}) \quad , \quad \hat{w} = 2^{2/3}\hat{w} \quad , \quad \hat{x} = 2^{2/3}\frac{\hat{x}}{8} \]  

(135)

The generating function \( \hat{g}_\infty \) is obtained as a Fredholm determinant

\[
\text{Prob} \left( A_2(\hat{x}) < \sigma_1, \hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R \right) = \hat{g}_\infty(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = \text{Det}[I - P_0 K_{\sigma_1, \sigma_L, \sigma_R} P_0]
\]

(136)

where we have recalled that it is also the triple joint CDF associated to the Airy process where \( h_{L,R}(\hat{x}) \) are defined in \([63]\). The associated kernel, written here in full generality, using the shorthand notation \( \sigma_m = \min(\sigma_1, \sigma_L, \sigma_R) \) reads

\[
K_{\sigma_1, \sigma_L, \sigma_R}(v_i, v_j) = \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + y)\text{Ai}(v_j + y) - \frac{a_R a_L e^{(\hat{w}_L + \hat{w}_R)\sigma_m - (\hat{w}_L + \hat{w}_R)\sigma_R}}{\hat{w}_L + \hat{w}_R} \text{Ai}(v_i + \sigma_L)\text{Ai}(v_j + \sigma_R)
\]

\[
- (1 + a_L + a_R - 3a_L a_R)e^{2(\hat{w}_L + \hat{w}_R)(\sigma + a_R \sigma_L) + 2(\hat{w}_R - \hat{x})(\sigma_L + a_L \sigma_R)} \int_{\max(\sigma, \sigma_L, \sigma_R)}^{+\infty} \frac{e^{-(\hat{w}_L + \hat{w}_R)\sigma + a_R \sigma_L}}{\hat{w}_L + \hat{w}_R} d\sigma
\]

\[
\times \text{Ai}(v_i) (1 - a_L + a_R) y + (1 - a_R) \sigma_L - (1 - a_L) \sigma_R \times \text{Ai}(v_j) (1 - a_L + a_R) y - (1 - a_R) \sigma_L + (1 - a_L) \sigma_R
\]

\[
\times e^{-2y(\hat{w}_L + \hat{w}_R + a_R (\hat{w}_L + \hat{x}) + a_L (\hat{w}_R - \hat{x}))}
\]

\[
+ a_L \text{Ai}(v_i + \sigma_L) \int_{-\infty}^{\sigma_m} dy \text{Ai}(v_j + y)e^{(\hat{w}_L + \hat{w}_R)(y - \sigma_L) + a_R \text{Ai}(v_j + \sigma_R) \int_{-\infty}^{\sigma_m} dy \text{Ai}(v_i + y)e^{(\hat{w}_R - \hat{x})(y - \sigma_R)}
\]

\[
+ (1 - a_L) \int_{\max(\sigma, \sigma_L, \sigma_R)}^{+\infty} dy \text{Ai}(v_i + y + \sigma_L)\text{Ai}(v_j - y + \sigma_L)e^{-2y(\hat{w}_L + \hat{x})}
\]

\[
+ (1 - a_R) \int_{\max(\sigma, \sigma_L, \sigma_R)}^{+\infty} dy \text{Ai}(v_i - y + \sigma_R)\text{Ai}(v_j + y + \sigma_R)e^{-2y(\hat{w}_R - \hat{x})}
\]

(137)

We note that the kernel depends only on the combinations \( \hat{w}_L + \hat{x} \) and \( \hat{w}_R - \hat{x} \), as required by the so-called STS symmetry (see Section XI and e.g. \([53]\) or \([61]\)).

We can note some desirable properties of this result \([136], \[137]\) for the joint CDF, obtained via the replica calculation. First it depends on \( \sigma_1 \) only when \( \sigma_1 \leq \min(\sigma_L, \sigma_R) \). For \( \sigma_1 > \min(\sigma_L, \sigma_R) \) all dependence in \( \sigma_1 \) disappears and the associated joint PDF is zero. It can be checked considering all four cases \( a_{L,R} = 0, 1 \). This is a required property from the definition of this CDF: indeed, considering the point \( \hat{y} = 0 \) in the quantity to be maximized in the definition \([63]\), one sees that one must have \( A_2(\hat{x}) \leq \min(\hat{h}_L(\hat{x}) + \hat{x}^2, \hat{h}_R(\hat{x}) + \hat{x}^2) \). Clearly this is a non-trivial check of the replica method and the decoupling assumption.

Another check of \([136], \[137]\) is that for \( \sigma_L, \sigma_R \to +\infty \) it should become equal to the CDF of the GUE TW distribution for the variable \( \sigma_1 \), i.e. one must have \( g_\infty(\sigma_1, +\infty, +\infty; 0) = F_2(\sigma_1) = \text{Det}[I - P_0, K_{\sigma_1, P_0}] \) for any value of \( a_{L,R} \in (0, 1)^2 \) and \( w_{L,R} \). It is easy to check that in this limit all terms in \([137]\), apart from the first one, vanish. Upon the shift \( v_{i,j} \to v_{i,j} - \sigma_1 \) the first term recovers exactly the Airy kernel \( K_{\sigma_1}(v_i, v_j) = \int_{0}^{+\infty} dy \text{Ai}(v_i + y)\text{Ai}(v_j + y) \) and the desired property is thus correct.

Finally, for \( w_L, w_R \to +\infty \) the maximization in \([63]\) leads to the position of the maximum at \( \hat{y} = 0 \), hence \( \hat{h}_L(\hat{x}) + \hat{x}^2 = \hat{h}_R(\hat{x}) + \hat{x}^2 = A_2(\hat{x}) \). Hence in that limit one should have \( \lim_{w_L, w_R \to +\infty} g_\infty(\sigma_1, \sigma_L, \sigma_R; 0) = F_2(\min(\sigma_1, \sigma_L, \sigma_R) \), where \( F_2 \) is again the CDF of the GUE-TW distribution. Clearly it works since again all terms apart the first one in \( K_{\sigma_1, \sigma_L, \sigma_R} \) in \([137]\) vanish in that limit.

We will specialize below to two main cases: the Brownian case \( a_L = a_R = 1 \) and the flat case \( a_R = a_L = 0 \). The mixed or crossover cases can be studied along similar lines, but we will not detail it here.

### III.6. Joint CDF for Airy minus parabola plus double-sided Brownian with drifts

We now specialize to \( a_L = a_R = 1 \), i.e. the double-sided Brownian case. Eq. \([137]\) simplifies into

\[
K_{\sigma_1, \sigma_L, \sigma_R}(v_i, v_j) = K_{\sigma_1}(v_i + \sigma_m, v_j + \sigma_m) - \frac{e^{(\hat{w}_L + \hat{w}_R)\sigma_m - (\hat{w}_L + \hat{w}_R)\sigma_R}}{\hat{w}_L + \hat{w}_R} \text{Ai}(v_i + \sigma_L)\text{Ai}(v_j + \sigma_R)
\]

\[
+ \text{Ai}(v_i + \sigma_L) \int_{-\infty}^{0} dy \text{Ai}(v_j + y + \sigma_m)e^{(\hat{w}_L + \hat{w}_R)(y + \sigma_m - \sigma_L)} + \text{Ai}(v_j + \sigma_R) \int_{-\infty}^{0} dy \text{Ai}(v_i + y + \sigma_m)e^{(\hat{w}_R - \hat{x})(y + \sigma_m - \sigma_R)}
\]
This can be written in a more compact form by defining (as in (16) in the text)

$$B_w(v) := e^{w^3/3 - v w} - \int_{0}^{+\infty} dy \text{Ai}(v+y)e^{wy} = \int_{-\infty}^{0} \text{Ai}(v+y)e^{wy}$$

(138)

where the last equality holds only for $v < 0$ (when the integral is convergent), and

$$\text{Ai}_{\sigma,w}(v) := \text{Ai}(v + \sigma)e^{-w\sigma}$$

(139)

One can then check that one can rewrite the generating function, hence the following triple joint CDF as

$$\text{Prob}(A_2(-\hat{x}) < \sigma_1, \quad \max_{\hat{y} < 0}(A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + 2\hat{w}_L\hat{y} + \sqrt{2B(\hat{y})}) < \sigma_L - \hat{x}^2, \quad \max_{\hat{y} > 0}(A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + 2\hat{w}_R\hat{y} + \sqrt{2B(\hat{y})}) < \sigma_R - \hat{x}^2)$$

$$= \hat{g}_\infty(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = \text{Det}[I - P_{\sigma_m}K_{\sigma_L - \sigma_m, \sigma_R - \sigma_m}P_{\sigma_m}]$$

(140)

with the kernel

$$K_{\sigma_L - \sigma_m, \sigma_R - \sigma_m}(v_i, v_j) = K_{\text{Ai}}(v_i, v_j) + A_{\sigma_L - \sigma_m, \sigma_R - \sigma_m}(v_i)B_{\hat{w}_L + \hat{x}}(v_j) + B_{\hat{w}_R - \hat{x}}(v_i)A_{\sigma_R - \sigma_m, \hat{w}_R - \hat{x}}(v_j)$$

$$- \frac{1}{\hat{w}_L + \hat{w}_R}A_{\sigma_L - \sigma_m, \hat{w}_L + \hat{x}}(v_i)A_{\sigma_R - \sigma_m, \hat{w}_R - \hat{x}}(v_j)$$

(141)

which has the form of the Airy kernel plus two projectors.

We will now study the limit where the drift go to zero $\hat{w}_{L,R} = 0^+$ which relates to stationary KPZ.

### III.7. Double-sided Brownian in limit $\hat{w}_{L,R} = 0^+$: Joint CDF of Airy and Airy minus parabola plus Brownian

We continue with the Brownian case and now perform the stationary limit $\hat{w}_{L,R} \to 0$. This limit cannot be performed naively and require some manipulations. Since the kernel (141) is singular as $\hat{w}_{L,R} \to 0$, to perform the limit one first rewrite it as

$$K_{\sigma_L - \sigma_m, \sigma_R - \sigma_m}(v_i, v_j) = K_{\text{Ai}}(v_i, v_j) + (\hat{w}_L + \hat{w}_R)B_{\hat{w}_L + \hat{x}}(v_i)B_{\hat{w}_R - \hat{x}}(v_j)$$

$$- \frac{A_{\sigma_L - \sigma_m, \hat{w}_L + \hat{x}}(v_i)}{\sqrt{\hat{w}_L + \hat{w}_R}} - \sqrt{\hat{w}_L + \hat{w}_R}B_{\hat{w}_R - \hat{x}}(v_j)$$

(142)

which has the form of the Airy kernel plus two projectors.

We will now use the determinant identity, in quantum mechanical notations

$$\text{Det}[A - |U\rangle\langle V| + |R\rangle\langle S|] = \text{Det}A \times \left((1 - \langle V|A^{-1}|U\rangle)(1 + \langle S|A^{-1}|R\rangle) + \langle S|A^{-1}|U\rangle\langle V|A^{-1}|R\rangle\right)$$

(143)

where $|U\rangle\langle V|$ stands for the projector operator $UV^T$, explicitly in coordinate representation $(UV^T)(v_i, v_j) = U(v_i)V(v_j)$. We will use indifferently either notations, e.g. $\langle V|A^{-1}|U\rangle \equiv \text{Tr}[A^{-1}UV^T]$. Here we apply the operator identity (143) to

$$A = I - C \quad C = P_{\sigma_m}K_{\text{Ai}}P_{\sigma_m}, \quad A^{-1} = I + C(I - C)^{-1}$$

(144)

$$|U\rangle = \sqrt{\hat{w}_L + \hat{w}_R}P_{\sigma_m}B_{\hat{w}_L + \hat{x}} \quad , \quad |R\rangle = P_{\sigma_m}A_{\sigma_L - \sigma_m, \hat{w}_L + \hat{x}} - \sqrt{\hat{w}_L + \hat{w}_R}P_{\sigma_m}B_{\hat{w}_R - \hat{x}}$$

(145)

$$\langle V| = \sqrt{\hat{w}_L + \hat{w}_R}B_{\hat{w}_L + \hat{x}}P_{\sigma_m} \quad , \quad \langle S| = A_{\sigma_R - \sigma_m, \hat{w}_R - \hat{x}}P_{\sigma_m} - \sqrt{\hat{w}_L + \hat{w}_R}B_{\hat{w}_L + \hat{x}}P_{\sigma_m}$$

(146)

and we use below the shorthand notation

$$A_{\sigma}(v) = A_{\sigma,0}(v) = \text{Ai}(v + \sigma)$$

(147)

We will now set $\hat{w}_L = \hat{w}_R = \hat{w}$ for simplicity. We start by estimating the small $\hat{w}$ expansion of

$$\langle V|U\rangle = 2\hat{w}\text{Tr}[P_{\sigma_m}B_{\hat{w}_L + \hat{x}}B_{\hat{w}_R - \hat{x}}] = 1 - 2\hat{w}(1 + \mathcal{L}_z(\sigma_m)) + O(\hat{w}^2)$$

(148)
see e.g. formula (285) in Appendix E of [1], where one shows that  
\[
\mathcal{L}_z(\sigma) = \sigma - 1 - i^2 + \int_{\sigma}^{+\infty} dv \, (1 - B_z(v)B_{-z}(v)) = \sigma - 1 - \hat{x}^2
\]  
\[
+ 2 \int_{\sigma}^{+\infty} du \int_{0}^{\infty} dy \cosh\left(\frac{y^2}{2} - (u + y)\hat{x}\right)Ai(u + y) + \int_{\sigma}^{+\infty} du \int_{0}^{\infty} dy_1 dy_2 e^{\hat{x}(y_1 - y_2)} Ai(u + y_1)Ai(u + y_2)
\]

This implies that the following term in (143) is $O(\hat{w})$ as $\hat{w} \to 0$  
\[
1 - \langle V | A^{-1} | U \rangle = 1 - \langle V | U \rangle - \langle V | C(I - C)^{-1} | U \rangle = 2\hat{w}(1 + \mathcal{L}_z(\sigma_1)) - 2\hat{w}\langle B_{\hat{w} + \hat{x}} | C(I - C)^{-1} | B_{\hat{w} - \hat{x}} \rangle + O(\hat{w}^2)
\]

since the operator $C$ in the numerator of the last trace makes it convergent, due to fast decay of Airy functions. Returning to (143) we can write  
\[
\langle S | A^{-1} | R \rangle = \frac{1}{2\hat{w}} \text{Tr}[P_{\sigma_m}(I - C)^{-1} A_{\sigma_L - \sigma_m, \hat{x}} + \hat{A}^T_{\sigma_R - \sigma_m, -\hat{x}}] - \text{Tr}[P_{\sigma_m}(I - C)^{-1} B_{\hat{w} - \hat{x}} A_{\sigma_R - \sigma_m, \hat{w} - \hat{x}}] + \langle V | U \rangle
\]

We see from (143) and (150) that in that expression we only need the leading term $O(1/\hat{w})$, which is the first term, since the traces here are all convergent due to the Airy functions, and since $\langle V | U \rangle \simeq 1$ from (148). We obtain  
\[
\langle S | A^{-1} | R \rangle = \frac{1}{2\hat{w}} \text{Tr}[P_{\sigma_m}(I - C)^{-1} A_{\sigma_L - \sigma_m, \hat{x}} + \hat{A}^T_{\sigma_R - \sigma_m, -\hat{x}}] + O(1)
\]

Now we need to evaluate  
\[
\langle V | A^{-1} | R \rangle = \text{Tr}[P_{\sigma_m}(I - C)^{-1} A_{\sigma_L - \sigma_m, \hat{x}} + \hat{A}^T_{\sigma_R - \sigma_m, -\hat{x}}] - \langle V | A^{-1} | U \rangle \simeq \text{Tr}[P_{\sigma_m}(I - C)^{-1} B_{\hat{x}} A_{\sigma_L - \sigma_m, \hat{x}}] - 1
\]

Similarly one has  
\[
\langle S | A^{-1} | U \rangle \simeq \text{Tr}[P_{\sigma_m}(I - C)^{-1} B_{\hat{w} - \hat{x}} A_{\sigma_R - \sigma_m, -\hat{x}}] - 1
\]

Putting all together in the limit $\hat{w} \to 0$, we obtain our main result for the triple joint CDF  
\[
G_{\hat{z}}(\sigma_1, \sigma_L, \sigma_R) := \text{Prob} \left( A_2(-\hat{x}) < \sigma_1, \max_{\hat{y} > 0}(A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})) < \sigma_L - \hat{x}^2, \max_{\hat{y} > 0}(A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})) < \sigma_R - \hat{x}^2 \right)
\]

in the form, where we recall that we denote $\sigma_m := \min(\sigma_1, \sigma_L, \sigma_R)$  
\[
G_{\hat{z}}(\sigma_1, \sigma_L, \sigma_R) = \lim_{\hat{w} \to 0} \tilde{g}_{\infty}(\sigma_1, \sigma_L, \sigma_R; \hat{x}) = \lim_{\hat{w} \to 0} \text{Det}\left[ I - \sigma_m \hat{K}_{\sigma_L - \sigma_m, \sigma_L - \sigma_m} P_{\sigma_m} \right]
\]

\[
= F_2(\sigma_m) Y_{\hat{z}}(\sigma_m) \text{Tr}\left[ (I - P_{\sigma_m} K_{\hat{x}})^{-1} P_{\sigma_m} A_{\sigma_L - \sigma_m} A_{\sigma_R - \sigma_m} \right] e^{\hat{x}(\sigma_R - \sigma_L)}
\]

\[
+ F_2(\sigma_m) e^{-\hat{x}(\sigma_R - \sigma_m)} \text{Tr}\left[ (I - P_{\sigma_m} K_{\hat{x}})^{-1} P_{\sigma_m} A_{\sigma_L - \sigma_m} B_{\hat{x}}^T e^{\hat{x}(\sigma_R - \sigma_m)} - 1 \right] e^{\hat{x}(\sigma_R - \sigma_m)} \text{Tr}\left[ (I - P_{\sigma_m} K_{\hat{x}})^{-1} P_{\sigma_m} A_{\sigma_R - \sigma_m} B_{\hat{x}}^T - 1 \right] - 1
\]

We have defined, as in the main text in Eq. (270), the function  
\[
Y_{\hat{z}}(\sigma) = 1 + L_{\hat{z}}(\sigma) - \text{Tr}[P_{\sigma} K_{\hat{x}}(I - P_{\sigma} K_{\hat{x}})^{-1} P_{\sigma} B_{\hat{w}} B_{\hat{w}}^T]
\]

We recall that the vector $B_{\hat{z}}(v)$ is given by (138) and $L_{\hat{z}}(\sigma)$ is given by (149) (both also defined in Eq. (16) in the main text) and that $A_{\sigma}(u) := Ai(u + \sigma)$. We also recall that $F_2(\sigma) = \text{Det}[I - P_{\sigma} K_{\hat{x}} P_{\sigma}]$ is the CDF of the GUE-TW distribution, and everywhere $K_{\hat{x}}$ is the Airy kernel.

Equations (155), (156) is the master formula from which we will now obtain the three main results announced in the text as particular cases, each being analyzed in details below.

Before we do so let us indicate a useful alternative formula for $Y_{\hat{z}}$. Let us introduce the following notation for the two vectors  
\[
D_{\hat{z}}(v) = e^{\frac{i}{3} - \hat{z}^2} \quad B_{\hat{z}}(v) = D_{\hat{z}}(v) - B_{\hat{z}}(v) = \int_{0}^{+\infty} dy Ai(v + y)e^{\hat{z}y}
\]

\[
+ 2 \int_{\sigma}^{+\infty} du \int_{0}^{\infty} dy \cosh\left(\frac{y^2}{2} - (u + y)\hat{x}\right)Ai(u + y) + \int_{\sigma}^{+\infty} du \int_{0}^{\infty} dy_1 dy_2 e^{\hat{x}(y_1 - y_2)} Ai(u + y_1)Ai(u + y_2)
\]
where \( \hat{B}_x(v) \) has a fast decay at \( v \to +\infty \). We see from (149) that we can rewrite

\[
Y_x(\sigma) = \sigma - \hat{x}^2 + \int_{-\infty}^{+\infty} dv \left( \hat{B}_x(v)D_{-\hat{x}}(v) + \hat{B}_x(v)D_{-\hat{x}}(v) - \hat{B}_x(v)\hat{B}_{-\hat{x}}(v) - \text{Tr}[P_{\sigma}K_{\text{Ai}}(I - P_{\sigma}K_{\text{Ai}})^{-1}P_{\sigma}D_{-\hat{x}}D_{\hat{x}}^T] \right)
- \text{Tr}[P_{\sigma}K_{\text{Ai}}(I - P_{\sigma}K_{\text{Ai}})^{-1}P_{\sigma}\hat{B}_x\hat{B}_{-\hat{x}}^T] + \text{Tr}[P_{\sigma}K_{\text{Ai}}(I - P_{\sigma}K_{\text{Ai}})^{-1}P_{\sigma}(D_{-\hat{x}}\hat{B}_{\hat{x}}^T + D_{-\hat{x}}\hat{B}_{\hat{x}}^T)]
\]

Using that \( K(I - K)^{-1} = (I - K)^{-1} - I \) we see that the traces involving the substraction \(-I\) cancel with the first line and it remains

\[
Y_x(\sigma) = \sigma - \hat{x}^2 - \text{Tr}[(I - P_{\sigma}K_{\text{Ai}})^{-1}P_{\sigma}(K_{\text{Ai}}P_{\sigma}D_{-\hat{x}}D_{\hat{x}}^T + \hat{B}_x\hat{B}_{-\hat{x}}^T - D_{-\hat{x}}\hat{B}_{\hat{x}}^T - D_{-\hat{x}}\hat{B}_{\hat{x}}^T)]
\]
a form which is useful for numerical evaluations.

**IV. MOMENTS OF THE JOINT CDF AND CONDITIONAL AVERAGES FOR 2-TIME KPZ AT LARGE TIME**

**IV. 1. Joint CDF of the Airy process and the max of Airy process minus parabola plus Brownian on the line**

We now study

\[
G_x(\sigma_1, \sigma_2) := \text{Prob}\left( A_x(\hat{x}) < \sigma_1, \max_y(A_y(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})) < \sigma_2 - \hat{x}^2 \right)
\]

and its application to 2-time KPZ. Setting \( \sigma_L = \sigma_R = \sigma_2 \) in (155) and (156) we obtain the prediction

\[
G_x(\sigma_1, \sigma_2) = \lim_{\hat{x} \to \sigma_1} \hat{g}_x(\sigma_1, \sigma_2, \sigma_2; \hat{x}) = F_2(\sigma_1)Y_x(\sigma_1)\text{Tr}[(I - P_{\sigma_1}K_{\text{Ai}})^{-1}P_{\sigma_1}A_{\sigma_2 - \sigma_1}A_{\sigma_2 - \sigma_1}^T]
+ F_2(\sigma_1)\left( e^{-\hat{g}_x(\sigma_2 - \sigma_1)}\text{Tr}[(I - P_{\sigma_1}K_{\text{Ai}})^{-1}P_{\sigma_1}A_{\sigma_2 - \sigma_1}^T - 1]\left( e^{\hat{g}_x(\sigma_2 - \sigma_1)}\text{Tr}[(I - P_{\sigma_1}K_{\text{Ai}})^{-1}P_{\sigma_1}A_{\sigma_2 - \sigma_1}^T - 1] \right) \right)
\]

where the last equality is valid for \( \sigma_1 \leq \sigma_2 \). For \( \sigma_1 > \sigma_2 \) we have \( G_x(\sigma_1, \sigma_2) = G_x(\sigma_2, \sigma_2) \). We recall that the vector \( \hat{B}_x(v) \) is given by (138) and \( Y_x(\sigma) \) is given by (157). In the case \( \hat{x} = 0 \), the formula (160) reduces to the result (21) given in the text, where we denote \( G(\sigma_1, \sigma_2) = G_0(\sigma_1, \sigma_2) \). It is easy to see that the marginal CDF of \( \sigma_1 \), \( G_x(\sigma_1, \sigma_2 = +\infty) = F_2(\sigma_1) \) is the GUE-TW since only the last term in (160) survives in that limit. We now show that the marginal CDF of \( \sigma_2 \) correctly coincides with the EBR distribution.

**IV. 2. Extended Baik-Rains limit**

Let us start by showing that our expression (160) for \( G_x(\sigma_2, \sigma_2) \) correctly recovers the (extended) Baik-Rains distribution, \( F_0(\sigma_2 - \hat{x}^2; \hat{x}) \), as required from its definition. Indeed the stationary Airy process at a given point \( \hat{x} \), \( A_{\text{stat}}(\hat{x}) \), is given by

\[
A_{\text{stat}}(\hat{x}) = \max_{\hat{y}}(A_{\hat{x}}(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y}))
\]

and it is known that its one point CDF is given by

\[
\text{Prob}(A_{\text{stat}}(\hat{x}) < \zeta = \sigma - \hat{x}^2) = F_0(\zeta; \hat{x}) = \partial_\sigma(F_2(\sigma)Y_x(\sigma))
\]

We thus need to show that Eq. (160) for \( \sigma_1 = \sigma_2 = \sigma \) reduces to

\[
G_x(\sigma, \sigma) = \partial_\sigma(F_2(\sigma)Y_x(\sigma))
\]

We first use the identity

\[
\partial_\sigma F_2(\sigma) = \partial_\sigma \text{Det}[I - P_0K_{\text{Ai}}^2P_0] = -\text{Tr}[P_0(I - P_0K_{\text{Ai}}^2)^{-1}P_0\partial_\sigma K_{\text{Ai}}^2] = \text{Tr}[P_0(I - P_0K_{\text{Ai}}^2)^{-1}P_0A_{\text{Ai}}^2A_{\text{Ai}}^T]
\]

using that \( \partial_\sigma K_{\text{Ai}}^2(v_1, v_j) = -A_{\text{Ai}}(v_i)A_{\text{Ai}}(v_j) \) and here and below we often use the shorthand notation

\[
K_{\text{Ai}}^2(v) = K_{\text{Ai}}(v + \sigma)
\]
Hence the first term in (160) reduces to $Y_2(\sigma_1)\partial_{\sigma_1}F_2(\sigma_1)$.

To show the desired result (162), we will now show that

$$\partial_\sigma Y_2(\sigma) = \langle \text{Tr}[(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B^T_{z}] - 1 \rangle \langle \text{Tr}[(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B^T_{-z}] - 1 \rangle$$

(166)

so that the second term in (160) for $\sigma_2 = \sigma_1$ reduces to $F_2(\sigma_1)\partial_{\sigma_1}Y_2(\sigma_1)$. Hence if (166) is true, so is (162).

To show (166) we note that from (270) we have

$$\partial_\sigma Y_2(\sigma) = \partial_\sigma L_\tilde{z}(\sigma) - \partial_\sigma \text{Tr}[P_\sigma K_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-z}B^T_{\tilde{z}}]$$

(167)

From the definition of $L_\tilde{z}(\sigma)$ in (149) and of $B_{\tilde{z}}(\sigma)$ in (138) we have

$$\partial_\sigma L_\tilde{z}(\sigma) = B_{\tilde{z}}(\sigma)B_{-\tilde{z}}(\sigma) = 1 - \text{Tr}[\tilde{A}_\sigma(B_{\tilde{z}} + B_{-\tilde{z}})] \quad , \quad \lim_{\sigma \to +\infty} B_{\tilde{z}}(\sigma)B_{-\tilde{z}}(\sigma) = 1$$

(168)

in the second equality we used the identity

$$\int_0^{+\infty} \int_0^{+\infty} dy_1 dy_2 \text{Ai}(\sigma + y_1)\text{Ai}(\sigma + y_2)e^{x(y_1 - y_2)} = \int_0^{+\infty} \int_0^{+\infty} du \text{Ai}(\sigma + u)\text{Ai}(\sigma + u + y)(e^{xy} + e^{-xy})$$

(169)

Using that

$$\partial_\sigma B_{\tilde{z}}(v) = \text{Ai}(v) - \hat{x} B_{\tilde{z}}(v) \quad , \quad \partial_\sigma B_{\tilde{z}}^2(v) = \text{Ai}_\sigma(v) - \hat{x} B_{\tilde{z}}^2(v)$$

(170)

we have

$$-\partial_\sigma \text{Tr}[P_\sigma K_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-\tilde{z}}B^T_{\tilde{z}}] = -\partial_\sigma \text{Tr}[P_\sigma K_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-\tilde{z}}B^T_{\tilde{z}}]$$

(171)

$$= -\text{Tr}[P_\sigma K_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-\tilde{z}}((\text{Ai}_\sigma - \hat{x} B^2_{\tilde{z}}) - \text{Tr}[P_\sigma K_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma (\text{Ai}_\sigma + \hat{x} B_{-\tilde{z}}))(B^2_{\tilde{z}})]$$

$$+ \text{Tr}[P_\sigma (I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma (\text{Ai}_\sigma B^2_{\tilde{z}})] \text{Tr}[P_\sigma (I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-\tilde{z}}\text{Ai}^T_{\tilde{z}}]$$

$$= \text{Tr}[\text{Ai}_\sigma P_0(B^2_{\tilde{z}} + B^T_{\tilde{z}})]$$

$$= (\text{Tr}[P_\sigma (I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma (\text{Ai}_\sigma B^2_{\tilde{z}})] - 1)\text{Tr}[P_\sigma (I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma B_{-\tilde{z}}\text{Ai}^T_{\tilde{z}}] - 1 - 1$$

we use $C(1 - C)^{-1} = (1 - C)^{-1} - 1$ and

$$\partial_\sigma P_\sigma K^2_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1} = (I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma \partial_\sigma K^2_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1}$$

(172)

$$= -(I - P_\sigma K_{\tilde{A}_1})^{-1} P_\sigma \partial_\sigma K^2_{\tilde{A}_1}(I - P_\sigma K_{\tilde{A}_1})^{-1}$$

(173)

using that $\partial_\sigma K^2_{\tilde{A}_1} = -\text{Ai}_\sigma \text{Ai}^T_{\tilde{z}}$ (as above). Adding the last two lines of (171) to (168) leads to cancellations and, upon rearranging, to (166). This completes the proof that $G_{\tilde{z}}(\sigma_1, \sigma_1) = F_0(\hat{\sigma}_1 - \hat{x}^2, \hat{x})$, i.e., it equals the CDF of the extended Balk-Rains distribution, which is the one point CDF of the stationary Airy process.

IV. 3. Application to two-time KPZ in the large time separation limit: general framework

Here we recall the connection between two time KPZ in the large time separation limit and the Airy process, with little details since it was explained in Sections 7.3-7.5 in [1] (see also [22]). The only difference with [1] is the way a non zero spatial shift $\hat{x}$ is introduced (see discussion below). Defining $h(x, t|y, 0)$ the solution of the KPZ equation (47) with the droplet initial condition centered at $y$ (see (51)), we have at large times $t_1, t_2 \gg 1$ with $\Delta = (t_2 - t_1)/t_1$ fixed

$$h_1 := \lim_{t_1 \to +\infty} t_1^{-1/3}h(0, t_1|x, 0) = A_2(-\hat{x}) - \hat{x}^2$$

(174)

$$h_2 := \lim_{t_1 \to +\infty} t_1^{-1/3}h(0, t_2|x, 0)_{t_2=(1+\Delta)t_1} = \max_{y \in \mathbb{R}} (A_2(y - \hat{x}) - (y - \hat{x})^2 + \Delta^4(A_2((\Delta^2 y)^2 - \Delta^2))$$

where $A_2$ and $\hat{A}_2$ are two independent Airy processes, and we recall that $\hat{x} = x/(2t_1^{2/3})$, $\hat{y} = y/(2t_1^{2/3})$. Note our unnatural choice to define $h_2$ using the scale $t_1^{1/3}$, which however is convenient for the analysis below. In Ref. [1] the variable $h$ was also defined as

$$h := \lim_{t_1 \to +\infty} \frac{h(0, t_2|x, 0) - h(0, t_1|x, 0)}{(t_2 - t_1)^{1/3}}|_{t_2=(1+\Delta)t_1}$$

(175)
We also defined there the (unknown) exact JPDF $P_{\Delta}(\sigma_1, \sigma) := \delta(h_1 - \sigma_1)\delta(h - \sigma)$ and derived an approximation of it, denoted $P_{\Delta}^{(1)}(\sigma_1, \sigma)$, conjectured to be exact to leading order in large positive $\sigma_1$ at fixed $\sigma$. It was shown in [2] to be good enough an approximation to fit experiments and numerics in a broad range of values $\sigma_1 > \langle \sigma_1 \rangle = \kappa_1^{\text{GUE}}$. Note that the approximation which leads to $P_{\Delta}^{(1)}$ there is quite different from our approach here (even if in both case the RBA was used), in particular no decoupling assumption was necessary there.

Let us also point out an exact result for the mean heights. From simple scaling, and convergence of one-point distributions to GUE-TW respectively at $t_1$ and $t_2$, we know that

$$h_1 = \kappa_1^{\text{GUE}} - \hat{x}^2,$$

$$h_2 = (1 + \Delta)^{1/3} \kappa_1^{\text{GUE}} - \frac{\hat{x}^2}{1 + \Delta}$$

which generalizes to non-zero $\hat{x}$ the results of Section 3.3.2 of [1] (the last term in [177] arises from the $x$-dependent shift $-x^2/(4t_2)$ in $h(0, t_2)$ and our chosen scaling with $t_1^{1/3}$). This implies that

$$\bar{h} = \frac{h_2 - h_1}{1 + \Delta^{1/3}} = (1 + \Delta)^{1/3} - \frac{1}{\Delta^{1/3}} \kappa_1^{\text{GUE}} + \frac{\Delta^{2/3}}{1 + \Delta} \hat{x}^2$$

which is an exact result valid for any $\Delta$.

We focus now on the limit of large time separation $\Delta \to +\infty$. Then, due to the property that the Airy process is locally Brownian, i.e. for fixed $\hat{z}$ and $a \ll 1$ [38]

$$\hat{A}_2(\hat{z} + a\hat{v}) = \hat{A}_2(\hat{z}) + \sqrt{2} a B(\hat{v}) + O(a)$$

where $B(\hat{v})$ (a unit two sided Brownian) and $\hat{A}_2(\hat{z})$ are mutually uncorrelated, one obtains

$$h_2 = \Delta^{1/3} \hat{A}_2(0) + \max_{\hat{y}} [A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2} B(\hat{y})] + O(\Delta^{-1/3})$$

We can also write, with some abuse of notations

$$h = \hat{A}_2(0) + \Delta^{-1/3}(\sigma_2 - \sigma_1) + O(\Delta^{-2/3})$$

where here $\sigma_1$ and $\sigma_2$ denote the random variables

$$\sigma_1 = \hat{A}_2(-\hat{x})$$

$$\sigma_2 - \hat{x}^2 = \max_{\hat{y}} [A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2} B(\hat{y})]$$

Their JCDF is $G_2(\sigma_1, \sigma_2)$ defined in (159), which we obtained explicitly here in Eq. (160) for arbitrary $\hat{x}$, and in (24) in the text for $\hat{x} = 0$. Note that it exhibits a non-trivial dependence in $\hat{x}$. In Ref. [1] instead, the dependence in the position of the final point at $t = t_2$ was studied, rather than in the position of the initial point at $t = 0$ as we do here. Both dependences however can be related using the STS symmetry (see below and Section XI).

Hence the knowledge of the JCDF $G_2(\sigma_1, \sigma_2)$, and its associated JPDF $p_2(\sigma_1, \sigma_2) = \partial_\sigma_1 \partial_\sigma_2 G_2(\sigma_1, \sigma_2)$, gives some information about the two time KPZ height in the large time separation regime. One can ask precisely what is the extent of this information? Since the subdominant $O(\Delta^{-2/3})$ term is actually correlated with the (large) leading term $\Delta^{1/3} \hat{A}_2(0)$ (in an unknown way), moments of $h$ higher than one cannot be simply obtained (see detailed discussion in Section IV. 7. 2.). Denoting here and below averages w.r.t. $p_2(\sigma_1, \sigma_2)$ as $\langle O(\sigma_1, \sigma_2) \rangle := \int d\sigma_1 d\sigma_2 O(\sigma_1, \sigma_2)p_2(\sigma_1, \sigma_2)$, the simplest observable which can be obtained is, by averaging (181)

$$\overline{h} = \kappa_1^{\text{GUE}} + \Delta^{-1/3} \langle \sigma_2 - \sigma_1 \rangle + O(\Delta^{-2/3})$$

Since the marginals of $p_2$ are GUE-TW for $\sigma_1$, and EBR for $\sigma_2$ respectively, with $\langle \sigma_2 \rangle = \hat{x}^2$ (see the two previous subsections), one obtains

$$\overline{h} = \kappa_1^{\text{GUE}} (1 - \Delta^{-1/3}) + \hat{x}^2 + O(\Delta^{-2/3})$$

which coincides with the large $\Delta$ expansion of (178). Hence from the simple average we do not learn anything new, it is simply a test of the method. A more interesting two-time KPZ observable that can be calculated at large $\Delta$ from $p_2(\sigma_1, \sigma_2)$, is the two time correlation $\overline{h h_1}$ obtained from $\langle \sigma_2 \sigma_1 \rangle$. An even more detailed information is contained
in the conditional average \((\sigma_1 - \sigma_2)\sigma_1\), see definition below. As discussed in [2], conditioning these observables to \(h_1 \geq \sigma_1\) leads to observables which can be (and in some cases, have been) efficiently compared to experiments and numerics. We study these observables below, keeping close spirit to the notations defined in [1] [2]. We first define them and show how they can be calculated in Sections IV. 4. and IV. 5. and in IV. 6. we perform a numerical evaluation of the result.

IV. 4. Two-time KPZ universal height covariance ratio

The two-time height correlation, normalized to its single time value, is defined for the droplet initial condition (centered at \(x\)) and in the large time limit as

\[
C_{\Delta} = \lim_{t_1 \to +\infty} \frac{\langle h(0, t_1 | x, 0)h(0, t_2 | x, 0) \rangle}{\langle h(0, t_1 | x, 0) \rangle^2} \bigg|_{t_2 = (1 + \Delta) t_1} = \frac{h_1 h_2}{h_1^2} \tag{185}
\]

Here we are interested in its large \(\Delta\) limit which, using (174) and (182), can be obtained as

\[
C_{\infty} = \frac{\langle \sigma_1, \sigma_2 \rangle - \langle \sigma_1 \rangle \langle \sigma_2 \rangle}{\langle \sigma_1^2 \rangle} = \int d\sigma_1 d\sigma_2 \sigma_1 (\sigma_2 - x^2) p_2(\sigma_1, \sigma_2) \tag{186}
\]

using that \(\langle \sigma_2 \rangle = \frac{x^2}{2}\). This can be rewritten as

\[
C_{\infty} = \frac{\langle \sigma_1^2 \rangle + \langle \sigma_2^2 \rangle - \langle \sigma_2 - \sigma_1 \rangle^2 - 2x^2 \kappa_{GUE}^{\text{GUE}}}{2 \kappa_{GUE}^{\text{GUE}}} = \frac{1}{2} + \frac{\left(\kappa_{GUE}^{\text{GUE}} - \frac{x^2}{2}\right)^2 + \kappa_{GUE}^{\text{GUE}}}{2 \kappa_{GUE}^{\text{GUE}}} - \frac{\langle \sigma_2 - \sigma_1 \rangle^2}{2 \kappa_{GUE}^{\text{GUE}}} \tag{187}
\]

For \(x = 0\) we recover the formula (19) of the text, and using

\[
\kappa_{GUE}^{\text{GUE}} = -1.771086807411 \tag{188}
\]

\[
\kappa_{GUE}^{\text{GUE}} = 0.81319479 , \quad \kappa_{GUE}^{\text{GUE}} = 1.15039 \tag{189}
\]

we obtain

\[
C_{\infty} = 3.13598 - \frac{\langle \sigma_2 - \sigma_1 \rangle^2}{2 \kappa_{GUE}^{\text{GUE}}} \tag{190}
\]

One needs to calculate \(\langle \sigma_2 - \sigma_1 \rangle^2\) numerically using the formula (190). This is done in Section IV. 6. below, leading to the result (209). To perform this calculation more conveniently we first show the identity (190) below, valid for any \(x\) (which reduces to (25) in the text for \(x = 0\)). To this purpose, since \(\lim_{x \to +\infty} G_{\tilde{F}}(\sigma_1, \sigma_2) = F_2(\sigma_1)\) with fast convergence (see previous sections), in order to deal with convergent integrals we define

\[
\hat{G}_\tilde{F}(\sigma_1, \sigma_2) = G_{\tilde{F}}(\sigma_1, \sigma_2) - F_2(\sigma_1) \tag{191}
\]

Then we have \(\lim_{x \to +\infty} \hat{G}_\tilde{F}(\sigma_1, \sigma_2) = 0\) with fast convergence, and \(\hat{G}_\tilde{F}(\sigma_1 \geq \sigma_2, \sigma_2) = F_0(\sigma_2 - x^2; \tilde{x}) - F_2(\sigma_1)\). We recall that in the text we denote \(G(\sigma_1, \sigma_2) = G_{\tilde{F} = 0}(\sigma_1, \sigma_2)\).

Let us start with the identity

\[
\beta_\sigma \partial_\sigma_1 [(\sigma_2 - \sigma_1)^2 \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] = 2 \beta_\sigma_2 [(\sigma_1 - \sigma_2) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] + \partial_\sigma_1 [(\sigma_2 - \sigma_1)^2 \partial_\sigma_1 \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] = 2 \beta_\sigma_2 [(\sigma_1 - \sigma_2) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] + 2 (\sigma_2 - \sigma_1)^2 \partial_\sigma_2 \partial_\sigma_1 \hat{G}_\tilde{F}(\sigma_1, \sigma_2) \tag{192}
\]

We want to integrate the last term over the sector \(\sigma_1 < \sigma_2\) since it vanishes for \(\sigma_1 > \sigma_2\) (see previous section). We note that the integral of the three other terms are either zero, or simpler, namely

\[
\int_{-\infty}^{+\infty} ds_2 \int_{s_2}^{+\infty} ds_1 \partial_\sigma_1 [(\sigma_2 - \sigma_1)^2 \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] = \int_{-\infty}^{+\infty} ds_2 \partial_\sigma_2 \int_{s_2}^{+\infty} ds_1 \partial_\sigma_1 [(\sigma_2 - \sigma_1) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] = 0 \tag{193}
\]

\[
2 \int_{-\infty}^{+\infty} ds_1 \int_{s_1}^{+\infty} ds_2 \partial_\sigma_1 [(\sigma_1 - \sigma_2) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] = 2 \int_{-\infty}^{+\infty} ds_1 [(\sigma_1 - \sigma_2) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] \bigg|_{s_1 = -\infty}^{+\infty} = 0 \tag{194}
\]

\[
2 \int_{-\infty}^{+\infty} ds_2 \int_{s_2}^{+\infty} ds_1 (\sigma_2 - \sigma_1) \partial_\sigma_1 \hat{G}_\tilde{F}(\sigma_1, \sigma_2) = 2 \int_{-\infty}^{+\infty} ds_2 \int_{s_2}^{+\infty} ds_1 \hat{G}_\tilde{F}(\sigma_1, \sigma_2) + 2 \int_{-\infty}^{+\infty} ds_2 [(\sigma_2 - \sigma_1) \hat{G}_\tilde{F}(\sigma_1, \sigma_2)] \bigg|_{s_1 = -\infty}^{s_1 = +\infty} = 0 \tag{195}
\]
Hence we have shown that

\[ \langle (\sigma_2 - \sigma_1)^2 \rangle := \int_{-\infty}^{+\infty} d\sigma_2 \int_{-\infty}^{\sigma_2} d\sigma_1 (\sigma_2 - \sigma_1)^2 \partial_\sigma_2 \partial_\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2) = -2 \int_{-\infty}^{+\infty} d\sigma_2 \int_{-\infty}^{\sigma_2} d\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2) \]  

(196)

which, for \( \hat{x} = 0 \), is precisely Eq. (25) of the text.

It is possible to perform exactly one of the integration in that double integral, leading to an explicit expression as a single integral. Starting from (160) we can integrate over \( \sigma_2 \) which leads to

\[
\int_{-\infty}^{+\infty} d\sigma_1 \int_{-\infty}^{+\infty} d\sigma_2 [\tilde{G}_\sigma(\sigma_1, \sigma_2) - F_2(\sigma_1)] = \int_{-\infty}^{+\infty} d\sigma_1 W_\sigma(\sigma_1)
\]  

(197)

\[
W_\sigma(\sigma_1) = F_2(\sigma_1) \times \left( Y_\sigma(\sigma_1) \text{Tr}(I - P_\sigma K_{\text{Al}})^{-1} P_\sigma K_{\text{Al}} + \text{Tr}[(I - P_\sigma K_{\text{Al}})^{-1} P_\sigma K_{\text{Al}}(I - P_\tau K_{\text{Al}})^{-1} P_\tau B_\infty B_{-\tau}^T \right] - \text{Tr}[(I - P_\sigma K_{\text{Al}})^{-1} P_\sigma (\int_0^{+\infty} d\sigma e^{-\hat{x}^2} A_{\sigma} B_{\hat{x}}^T)] - \text{Tr}[(I - P_\sigma K_{\text{Al}})^{-1} P_\sigma (\int_0^{+\infty} d\sigma e^{\hat{x}^2} A_{\sigma} B_{-\hat{x}}^T)]
\]

IV. 5. Conditional first moment and conditional correlation

Another observable defined in [112] is the average of \( h \) conditioned to an observed value of \( h_1 \), denoted \( \overline{h}_{h_1} \equiv E(h|h_1) \), where here \( E \) denotes expectation w.r.t. the KPZ noise. Since \( h_1 = \sigma_1 - \hat{x}^2 \) in loose notations, from (181) and (174) it obeys

\[
\overline{h}_{h_1=\sigma_1-\hat{x}^2} = \kappa_1^{\text{GUE}} + \Delta^{-1/3}\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1} + O(\Delta^{-2/3})
\]

(198)

To make contact with the notations of [112] we will define a function, \( R_{1/3}^{\text{exact}} \), equal to the conditional moment, defined as (see [32] in the text)

\[
R_{1/3}^{\text{exact}}(\sigma_1) := \langle \sigma_2 - \sigma_1 \rangle_{\sigma_1} := \frac{1}{\partial_\sigma_1 F_2(\sigma_1)} \int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) \partial_\sigma_2 \partial_\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2)
\]  

(199)

where the index ”exact” distinguish it from the approximation to this function obtained there, see below. We also define a second function

\[
\tilde{R}_{1/3}^{\text{exact}}(\sigma_1) := \langle \sigma_2 - \sigma_1 \rangle_{\sigma_1} \partial_\sigma_1 F_2(\sigma_1)
\]

(200)

which, upon integration by parts of (199) can be also written as

\[
\tilde{R}_{1/3}^{\text{exact}}(\sigma_1) = - \int_{\sigma_1}^{+\infty} d\sigma_2 \partial_\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2) = F_2(\sigma_1) - F_0(\sigma_1 - \hat{x}^2; \hat{x}) - \partial_\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 \tilde{G}_\sigma(\sigma_1, \sigma_2)
\]

(201)

a form more convenient for numerical evaluation. We have used that \( \tilde{G}_\sigma(\sigma_1, \sigma_2) = F_2(\sigma_1 - \hat{x}^2; \hat{x}) \) and we recall that \( \tilde{G}_\sigma(\sigma_1, \sigma_2) = \tilde{G}_\sigma(\sigma_1, \sigma_2) = F_2(\sigma_1) \) and that the JPDF is \( p_\sigma(\sigma_1, \sigma_2) = \partial_\sigma_2 \partial_\sigma_1 G_\sigma(\sigma_1, \sigma_2) = \partial_\sigma_2 \partial_\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2) \).

Another useful conditional expectation was defined in [112], conditioned to \( h_1 \geq \sigma_{1c} - \hat{x}^2 \) i.e. \( \sigma_1 > \sigma_{1c} \) (there for \( \hat{x} = 0 \)). One has again \( \overline{h}_{h_1\geq\sigma_1-\hat{x}^2} = \kappa_1^{\text{GUE}} + \Delta^{-1/3}\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 \geq \sigma_{1c}} + O(\Delta^{-2/3}) \), where such conditional averages are denoted as \( \langle \ldots \rangle_{\sigma_1 \geq \sigma_{1c}} \) and for the first moment it reads

\[
\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 \geq \sigma_{1c}} = \frac{1}{1 - F_2(\sigma_{1c})} \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) \partial_\sigma_2 \partial_\sigma_1 \tilde{G}_\sigma(\sigma_1, \sigma_2) = \frac{1}{1 - F_2(\sigma_{1c})} \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \tilde{R}_{1/3}^{\text{exact}}(\sigma_1)
\]

(202)

Using the form (201) obtained by integration by parts, it can be rewritten as a single integral

\[
\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 \geq \sigma_{1c}} = \frac{1}{1 - F_2(\sigma_{1c})} \int_{\sigma_{1c}}^{+\infty} d\sigma (F_2(\sigma) - F_0(\sigma - \hat{x}^2; \hat{x}) + \tilde{G}_\sigma(\sigma_{1c}, \sigma))
\]

(203)
Another important observable was considered in \cite{1,2}, namely a conditional variant of the two-time covariance ratio \cite{185}, defined as

\[
C_\Delta(\sigma_{1c}) := \frac{h_1 h_2 h_{1c} > \sigma_{1c} - \hat{\sigma}^2}{h_1^2 h_{1c} > \sigma_{1c} - \hat{\sigma}^2}
\] (204)

From the above considerations, the large \(\Delta\) limit, \(C_\infty(\sigma_{1c}) = \lim_{\Delta \to +\infty} C_\Delta(\sigma_{1c})\), can be obtained as

\[
C_\infty(\sigma_{1c}) = 1 + \frac{\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 > \sigma_{1c}} - \langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 > \sigma_{1c}}}{\langle \sigma_2^2 \rangle_{\sigma_1 > \sigma_{1c}} - \langle \sigma_1 \rangle^2_{\sigma_1 > \sigma_{1c}}}
\] (205)

a function which interpolates between \(C_\infty(\sigma_{1c} = -\infty) = C_\infty\) the unconditioned two-time covariance ratio studied above, and \(C_\infty(\sigma_{1c} = +\infty) = 1\) (see below). It can be written as (see (20) in \cite{2})

\[
C_\infty(\sigma_{1c}) = 1 + \frac{\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1) - N(\sigma_{1c})^{-1} \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1) \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \partial_\sigma F_2(\sigma_1)}{\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1^2 \partial_\sigma F_2(\sigma_1) - N(\sigma_{1c})^{-1} \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \partial_\sigma F_2(\sigma_1)\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \partial_\sigma F_2(\sigma_1)}
\] (206)

where we have defined \(N(\sigma_{1c}) = 1 - F_2(\sigma_{1c})\). In this expression, one can use (201) and further integrations by parts to evaluate

\[
\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1) = \int_{\sigma_{1c}}^{+\infty} d\sigma_2 [\sigma_1 \hat{G}_2(\sigma_{1c}, \sigma_2) + \sigma_2 F_2(\sigma_2) - F_0(\sigma_2 - \hat{\sigma}^2, \hat{\sigma})] + \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \hat{G}_2(\sigma_1, \sigma_2)
\] (207)

remembering that \(p_2(\sigma_1, \sigma_2) = 0\) for \(\sigma_1 > \sigma_2\). The integral \(\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1)\) was given in a simpler form above in \cite{201,202,203}.

IV. 6. Numerical evaluation

Here we use the general, and by now standard, method of numerical calculation of traces and determinants given in \cite{2}. It was also summarized near Eqs (21)-(22) in \cite{2} to which we refer for details.

To calculate \(C_\infty\) from \cite{187}, we have evaluated numerically, for \(\hat{\sigma} = 0\), the following integral

\[
-\frac{1}{2} \langle \sigma_2 - \sigma_1 \rangle^2 = \int_{-\infty}^{+\infty} d\sigma_2 \int_{-\infty}^{+\infty} d\sigma_1 (G(\sigma_1, \sigma_2) - F_2(\sigma_1))
\] (208)

Using formula (21) for \(G(\sigma_1, \sigma_2)\) we find \(-2.044 \pm 0.001\) for this quantity. As a check, we also performed an independent calculation using the formula \cite{197}, which led to \(-2.0445 \pm 0.0005\). Using formula \cite{190} we arrive at the estimate

\[
C_\infty = 0.6225 \pm 0.0015
\] (209)

for the infinite time-separation universal covariance ratio.

To evaluate, for \(\hat{\sigma} = 0\), the conditional average we define and calculate numerically the following integral (from 201)

\[
S(\sigma_{1c}) := \int_{\sigma_{1c}}^{+\infty} d\sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1) = \int_{\sigma_{1c}}^{+\infty} d\sigma (F_2(\sigma) - F_0(\sigma) + G(\sigma_{1c}, \sigma) - F_2(\sigma_{1c}))
\] (210)

from which we obtain the conditional average

\[
\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1 > \sigma_{1c}} = \frac{S(\sigma_{1c})}{1 - F_2(\sigma_{1c})}
\] (211)

The result is plotted in Fig. 2 of the main text, and we refer to the caption for comments. Here we give some numerical values

\[
\begin{pmatrix}
-4.5 & -4 & -3.5 & -3 & -2.5 & -2 & -1.5 & -1 & 0 & 1 & 2 & 3 & 4 \\
1.771 & 1.767 & 1.752 & 1.712 & 1.639 & 1.538 & 1.423 & 1.305 & 1.094 & 0.932 & 0.814 & 0.727 & 0.661
\end{pmatrix}
\] (212)

To evaluate the conditional covariance ratio \cite{205} we evaluate (from (201) and (210))

\[
\int_{\sigma_{1c}}^{+\infty} d\sigma_1 \sigma_1 \hat{R}^{\text{exact}}_{1/3}(\sigma_1) = \int_{\sigma_{1c}}^{+\infty} d\sigma_1 S(\sigma_1) + \sigma_{1c} S(\sigma_{1c})
\] (213)
We then rewrite (206) in a form more suitable for the numerical evaluation

\[ C_{\infty}(\sigma_{1c}) = 1 + \frac{\int_{\sigma_{1c}}^{+\infty} d\sigma_{1} S(\sigma_{1}) - S(\sigma_{1c})}{2 \int_{\sigma_{1c}}^{+\infty} d\sigma_{1} (\sigma_{1} - \sigma_{1c}) (1 - F_{2}(\sigma_{1})) - \frac{\int_{\sigma_{1c}}^{+\infty} d\sigma_{1} (1 - F_{2}(\sigma_{1}))^{2}}{1 - F_{2}(\sigma_{1c})} } \]  

(214)

The result is plotted in Fig. 3 of the main text in the interval \( \sigma_{1c} \in [-2.5, 4] \), the evaluation outside this interval would require enhanced numerical precision and was not attempted here.

Let us now, for sake of comparison, display the asymptotic formula for these observables obtained from the analysis of Section IV.7. below, which are in general agreement with the predictions of [1]. We display here the series expansions not displayed explicitly there. For the conditional mean one finds

\[ (\sigma_{2} - \sigma_{1})_{\sigma_{1} > \sigma_{1c}} = \frac{S_{\text{asympt}}(\sigma_{1c})}{\int_{\sigma_{1c}}^{+\infty} d\sigma K_{A_{1}}(\sigma, \sigma)} \]  

(215)

with

\[ S_{\text{asympt}}(\sigma_{1c}) = \int_{\sigma_{1c}}^{+\infty} d\sigma_{1} \int_{\sigma_{1c}}^{+\infty} d\sigma_{2} (2K_{A_{1}}(\sigma_{1}, \sigma_{2}) - K_{A_{1}}(\sigma_{2}, \sigma_{2})) \]

(216)

We obtain the series expansion at large argument

\[ S_{\text{asympt}}(y) = e^{-4y^{2}/3} \left( \frac{3}{32\pi y^{2}} - \frac{67}{256\pi y^{7/2}} + \frac{3887}{4096\pi y^{3}} + O\left(y^{-13/2}\right) \right) \]

\[ \int_{\gamma_{y}}^{+\infty} d\sigma K_{A_{1}}(\sigma, \sigma) = e^{-4y^{2}/3} \left( \frac{1}{16\pi y^{3/2}} - \frac{35}{384\pi y^{5/2}} + \frac{3745}{18432\pi y^{9/2}} + O\left(y^{-13/2}\right) \right) \]

(217)

(218)

Taking the ratio we find

\[ (\sigma_{2} - \sigma_{1})_{\sigma_{1} > \sigma_{1c}} = \frac{3}{2\sigma_{1c}^{7/2}} - \frac{2}{\sigma_{1c}^{3/2}} + \frac{473}{64\sigma_{1c}^{7/2}} - \frac{4953}{128\sigma_{1c}^{9/2}} + O(\sigma_{1c}^{-13/2}) \]

(219)

which is found to be a good approximation for \( \sigma_{1c} \geq 4 \). We also have (as also displayed in [1])

\[ R_{1/3}^{\text{exact}}(\sigma_{1}) = (\sigma_{2} - \sigma_{1})_{\sigma_{1}} = \frac{3}{2\sigma_{1}^{1/2}} - \frac{13}{8\sigma_{1}^{3/2}} + \frac{327}{64\sigma_{1c}^{7/2}} - \frac{1513}{64\sigma_{1c}^{9/2}} + O(\sigma_{1c}^{-13/2}) \]

(220)

For the conditional covariance ratio, the asymptotics gives

\[ C_{\infty}(\sigma_{1c}) = 1 - \frac{3}{4} \frac{1}{\sigma_{1c}^{3/2}} + \frac{35}{8\sigma_{1c}^{3}} - \frac{4023}{128\sigma_{1c}^{9/2}} + O\left(\frac{1}{\sigma_{1c}^{6}}\right) \]

(221)

which, however, appears to be a good approximation only for relatively large \( \sigma_{1c} \geq 4 \). Note from Fig. 2 and Fig. 3 that the full asymptotic formula, i.e. using the complete form (216), is a good approximation to the exact result in a much broader region, \( \sigma_{1c} \geq -0.5 \) and \( \sigma_{1c} \geq 0.5 \) respectively, than their power series asymptotics.

IV. 7. Large \( \sigma_{1} \) expansion of the joint CDF and of the conditional moments

IV. 7. 1. Leading order

Here we evaluate the JPDF \( p_{z}(\sigma_{1}, \sigma_{2}) = \partial_{\sigma_{1}} \partial_{\sigma_{2}} G_{z}(\sigma_{1}, \sigma_{2}) \), in the limit of large positive \( \sigma_{1} \gg 1 \) at fixed \( \sigma_{21} = \sigma_{2} - \sigma_{1} \). Note that we can restrict to \( \sigma_{21} \geq 0 \) since the JPDF vanishes for \( \sigma_{2} < \sigma_{1} \). We use the fact that \( A_{1}(\sigma_{1}) = O(\exp(-\frac{2}{3}\sigma_{1}^{3/2})) \), where \( O(.) \) means up to a prefactor which is an algebraic series in fractional powers of \( 1/\sigma_{1} \) which we do not write explicitly. One can thus organize the large \( \sigma_{1} \) expansion as a sum of terms of the type \( O(\exp(-\frac{2k}{3}\sigma_{1}^{3/2})) \), \( k = 1, 2, \ldots k_{\text{max}} \), and discard products of more than \( k_{\text{max}} \) integrals of Airy functions.
We start by focusing on $k_{\max} = 2$, i.e. up to product of two Airy functions only. Instead of writing $\sigma_2 = \sigma_1 + \sigma_{21}$ explicitly, it is convenient to use the $\sigma_2$ variable, keeping in mind that it is large and of the same order as $\sigma_1$. Let us perform the counting of the degree $k$ for each of the building blocks of (160) (and, for $x = 0$, of (241) in the text). We note that

$$F_2(\sigma_1) = \text{Det}[I - P_{\sigma_1}K_{Ai}P_{\sigma_1}] \simeq_{\sigma_1 \gg 1} 1 - \text{Tr}[P_{\sigma_1}K_{Ai}] = 1 - \int_{\sigma_1}^{+\infty} duK_{Ai}(u,u)$$

(222)

hence $F_2$ equals unity plus an infinite sum of even degrees $k \geq 2$. Similarly from (138), which we rewrite as

$$B_2(u) = e^{-\hat{x}u}(e^{\hat{x}3} - \int_u^{+\infty} dyAi(y) e^{\hat{\hat{y}}y})$$

(223)

i.e. we see that $B_2$ is the sum of a degree $k = 0$ and degree $k = 1$ term. From (149) we see that $L_2$ is a finite sum of degrees $k = 0, 1, 2$ and from (157), that $Y_2$ equals $1 + L_2$ plus an infinite sum of degrees $k \geq 2$.

The first line in (160) contains already two explicit Airy functions ($k = 2$), hence we only need $F_2(\sigma_1)Y_2(\sigma_1)$ to order $k = 0$. This means that in the first line we can replace $F_2(\sigma_1) \rightarrow 1$ and, from (157) and (149), $Y_2(\sigma_1) \rightarrow 1 + L_2(\sigma_1) \rightarrow \sigma_1 - \hat{x}^2$. Hence

First line in (160) $\simeq (\sigma_1 - \hat{x}^2)\text{Tr}[P_{\sigma_1}Ai_2(\sigma_2 - \sigma_1)T_{Ai_2(\sigma_2 - \sigma_1)}] = (\sigma_1 - \hat{x}^2)K_{Ai}(\sigma_2, \sigma_2) \rightarrow \sigma_1 K_{Ai}(\sigma_2, \sigma_2)$

(224)

in the last step we have discarded terms which do not depend on both $\sigma_1$ and $\sigma_2$, since they do not contribute to $p_2(\sigma_1, \sigma_2)$.

The second line of (160), upon expanding, contain four terms. The term $F_2(\sigma_1)$ alone can be discarded (it depends on only one of the two variables). The term product of two traces contains already two Airy functions ($k = 2$) hence we can replace in that term $F_2(\sigma_1) \rightarrow 1$ and $(I - P_{\sigma_1}K_{Ai})^{-1} \rightarrow I$ in the traces leading to

$$\text{Tr}[P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T] \text{Tr}[P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T] \simeq (\int_0^{+\infty}Ai(v + \sigma_2)e^{-\hat{x}v}(\int_0^{+\infty}Ai(v + \sigma_2)e^{\hat{x}v}) \rightarrow 0$$

(225)

In the first equivalence in (225) we have replaced $B_{2(\sigma_1)}(v)$ by its $k = 0$ piece, $e^{\hat{x}3} + e^{\hat{x}v}$, and shifted the integrals by $\sigma_1$ and in the last step we observed that the result does not depend on $\sigma_1$, hence can be discarded. In the cross terms we see that we can again set $F_2(\sigma_1) \rightarrow 1$ and $(I - P_{\sigma_1}K_{Ai})^{-1} \rightarrow I$ since their leading degree is already $k = 1$ and the Airy kernel increases the degree by $k \rightarrow k + 2$. The cross term becomes

$$-F_2(\sigma_1)(e^{-\hat{x}(\sigma_2 - \sigma_1)}\text{Tr}[(I - P_{\sigma_1}K_{Ai})^{-1}P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T] - e^{\hat{x}(\sigma_2 - \sigma_1)}\text{Tr}[(I - P_{\sigma_1}K_{Ai})^{-1}P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T])$$

(226)

$$\simeq -e^{-\hat{x}(\sigma_2 - \sigma_1)}\text{Tr}[P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T] - e^{\hat{x}(\sigma_2 - \sigma_1)}\text{Tr}[P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T]$$

Now we can calculate

$$-e^{-\hat{x}(\sigma_2 - \sigma_1)}\text{Tr}[P_{\sigma_1}Ai_{2(\sigma_2 - \sigma_1)}B_2^T] = -e^{-\hat{x}(\sigma_2 - \sigma_1)} \int_{\sigma_1}^{+\infty} duAi(u + \sigma_2 - \sigma_1)e^{-\hat{x}u}(e^{\hat{x}3} - \int_u^{+\infty} dyAi(y)e^{\hat{x}y})$$

(227)

$$= -\int_{\sigma_1}^{+\infty} duAi(u)e^{-\hat{x}u}(e^{\hat{x}3} - \int_{u + \sigma_1 - \sigma_2}^{+\infty} dyAi(y)e^{\hat{x}y})$$

where we have shifted $u \rightarrow u + \sigma_1 - \sigma_2$ in the second line: this shows that the first term does not depend on $\sigma_1$, hence we can discard it.

Putting together and taking the derivatives, we finally obtain the leading behavior of the JPDF for $\sigma_{21} \geq 0$ fixed and large positive $\sigma_1 \gg 1$, which we call $p_2^{(1)}$, as

$$p_2^{(1)}(\sigma_1, \sigma_2) = \partial_{\sigma_2}\partial_{\sigma_1}G_2(\sigma_1, \sigma_2) \simeq p_2^{(1)}(\sigma_1, \sigma_2) + O(\exp(-2\sigma_1^{3/2}))$$

(228)

$$p_2^{(1)}(\sigma_1, \sigma_2) = \partial_{\sigma_2}[K_{Ai}(\sigma_2, \sigma_2) - 2\cosh(\hat{x}(\sigma_1 - \sigma_2))K_{Ai}(\sigma_1, \sigma_2)]$$

(229)

$$= -Ai(\sigma_2)^2 - 2\sigma_2[\cosh(\hat{x}(\sigma_1 - \sigma_2))K_{Ai}(\sigma_1, \sigma_2)]$$

which is $O(\exp(-\frac{3}{4}\sigma_1^{3/2}))$ and we have neglected all terms with $k > k_{\max} = 2$ i.e. of order $O(\exp(-2\sigma_1^{3/2}))$. One can check that to this order

$$\int_{\sigma_1}^{+\infty} d\sigma_2 p_2^{(1)}(\sigma_1, \sigma_2) = K_{Ai}(\sigma_1, \sigma_1) = \partial_{\sigma_1}F_2^{(1)}(\sigma_1)$$

(230)
as required, since the exact sum rule is
\[
\int_{\sigma_1}^{+\infty} d\sigma_2 p_2(\sigma_1, \sigma_2) = \partial_{\sigma_1} F_2(\sigma_1) \tag{231}
\]
and $F_2^{(3)}$ is the leading tail approximation of the CDF of the GUE-TW distribution of the same order accuracy $O(\exp(-4\sigma^{3/2}))$ ($k = 2$).

Let us now examine the behaviour of the PDF approximant $p_{21}^{(1)}$ for $\sigma_{21} \to 0$. We recall that $p_x(\sigma_1, \sigma_2) = 0$ for $\sigma_{21} = \sigma_2 - \sigma_1 < 0$ and one finds
\[
p_{21}^{(1)}(\sigma_1, \sigma_2) = \frac{2}{3} \left( \sigma_1 (\sigma_1 + 3\hat{x}^2)A_i(\sigma_1)^2 - (\sigma_1 + 3\hat{x}^2)A'_i(\sigma_1)^2 - 2A_i(\sigma_1)A'_i(\sigma_1) \right) \sigma_{21} + O(\sigma_{21}^2) \tag{232}
\]
hence the PDF exhibits a linear cusp singularity at $\sigma_{21} = 0$, at least to this order of approximation. We note that the function $p_{21}^{(1)}(\sigma_1, \sigma_2)$ is positive only for $\hat{x}^2 < \hat{x}^*(\sigma_1)^2$ where
\[
[\hat{x}^*(\sigma_1)]^2 := -\frac{\sigma_1}{3} + \frac{2A_i(\sigma_1)A'_i(\sigma_1)}{3\sigma_1 A_i(\sigma_1)^2 - 3A'_i(\sigma_1)^2} \approx_{\sigma_1 \gg 1} \sigma_1 + \frac{1}{\sigma_1^{1/2}} - \frac{3}{4\sigma_1^2} + O(\frac{1}{\sigma_1^3}) \tag{233}
\]
is the value of $\hat{x}^2$ at which the $O(\sigma_{21})$ term in (232) changes sign. The function $\hat{x}^*(\sigma_1)^2$ has a minimum at $\sigma_1 = -1.41801$ where $(\hat{x}^*)^2 = 0.267978$. This change of sign, which does not occur for $\hat{x} = 0$, is not a contradiction, but simply means that $p_{21}^{(1)}(\sigma_1, \sigma_2)$ is not a uniformly good approximation in $\hat{x}$ to $p_x^{(1)}(\sigma_1, \sigma_2)$, i.e. the values of $\sigma_1$ needed for it to be accurate increase with $\hat{x}^2$. The large $\sigma_1$ asymptotics, at fixed $\sigma_{21} > 0$ reads
\[
p_{21}^{(1)}(\sigma_1, \sigma_2) \approx e^{-\frac{2}{3} \sigma_2^{3/2} - \frac{2}{3} \sigma_1^{3/2}} \left( \frac{\cosh(\sigma_2 \hat{x})}{4\pi \sigma_1^{1/2}} - \frac{\hat{x} \sinh(\sigma_2 \hat{x})}{4\pi \sigma_1} + O(\frac{f(\sigma_2 \hat{x})}{\sigma_1^{1/2}}) \right) + e^{-\frac{2}{3} \sigma_2^{3/2}} \left( -\frac{1}{4\pi \sigma_1^{1/2}} + O(\frac{\sigma_{21}}{\sigma_1^{3/2}}, \frac{1}{\sigma_1^2}) \right) \tag{234}
\]
which is, as required, positive in all cases.

We can now calculate the conditional first moment, $\langle \sigma_2 - \sigma_1 \rangle_{\sigma_1}$, defined in (199), to this (leading) order of the large $\sigma_1$ expansion. Specifically we compute the approximation to $\tilde{R}_{1/3}^{\text{exact}}(\sigma_1)$ defined in (200). That defines a function, which we call $\tilde{R}_{1/3}(\sigma_1)$ to stick with the notations of (1) (2). We find
\[
\tilde{R}_{1/3}(\sigma_1) := \int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) p_{21}^{(1)}(\sigma_1, \sigma_2) = \int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) \partial_{\sigma_2} [K_{\lambda i}(\sigma_2, \sigma_2) - 2 \cosh(\hat{x}(\sigma_1 - \sigma_2))K_{\lambda i}(\sigma_1, \sigma_2)]
\]
\[
= \int_{\sigma_1}^{+\infty} d\sigma_2 (2 \cosh(\hat{x}(\sigma_1 - \sigma_2))K_{\lambda i}(\sigma_1, \sigma_2) - K_{\lambda i}(\sigma_2, \sigma_2)) \tag{235}
\]
since the boundary term in the integration by part vanishes. For $\hat{x} = 0$, we can use the identity
\[
2 \int_{\sigma_1}^{+\infty} d\sigma K_{\lambda i}(\sigma_1, \sigma) = \left( \int_{\sigma_1}^{+\infty} dy A_i(y) \right)^2 \tag{236}
\]
to obtain the result (35) in the text. As noted there, remarkably, the function $\tilde{R}_{1/3}(\sigma_1)$ exactly coincide with the one obtained in (1) (and similarly for $R_{1/3}(\sigma_1) = \tilde{R}_{1/3}(\sigma_1)/K_{\lambda i}(\sigma_1, \sigma_1)$) formulae (173) there.

This agreement extends to finite $\hat{x}$. Indeed the result of (1), see formula (169) there, can be rewritten as
\[
\tilde{R}_{1/3}(\sigma_1) = -\partial_{\sigma_1} \int_{\sigma_1}^{+\infty} dy_1 \int_{\sigma_1}^{+\infty} dy_2 K_{\lambda i}(\sigma_1, \sigma_2) \cosh(\tilde{X}(y_1 - y_2)) - \int_{\sigma_1}^{+\infty} dy K_{\lambda i}(\sigma_1, \sigma_2) \tag{237}
\]
which is exactly the same formula as (235) is we identify $\tilde{X}$ there with $\hat{x}$ here. There, the variable $\tilde{X}$ was defined as $\tilde{X} \simeq x_2/(2t_{21}^{1/3})$ for large $\Delta$ (see formula (164) there) where $x_2$ is the position at time $t_2$. The STS allows to relate this to shifting the position at time $t = 0$, see Eq. (327) in Section XI below and the discussion around it (there $\hat{x}$ is denoted $X_0$).
One can now replace $\tilde{R}^{\text{exact}}_{1/3}(\sigma_1)$ by its approximation to any desired order (here the leading one for large $\sigma_1$) in the formula for the two-time conditional covariance ratio \[206\], and obtain the corresponding approximation for this observable, see Section IV. 6. where it is displayed. We thus fully confirm the correctness of the leading order prediction for $C_\infty(\sigma_{1c})$ obtained in \[1\], which was successfully tested in the experiments \[2\] and found accurate even beyond its naive range of validity (i.e. $\sigma_1$ large positive). Below, we obtain the next order corrections, which go beyond the method of \[1\]. Before doing so let us discuss higher moments.

**IV. 7. 2. Higher moments**

The higher moments of $p^{(1)}_k$ can also be calculated. Let us restrict to $\hat{x} = 0$ here. One has

$$\int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1)^2 p^{(1)}(\sigma_1, \sigma_2) = -2 \int_{\sigma_1}^{+\infty} d\sigma_2 (\sigma_2 - \sigma_1) [K_{\lambda_1}(\sigma_2, \sigma_2) - 2K_{\lambda_1}(\sigma_1, \sigma_2)]$$

(238)

which is found \textit{not} equal to $2\tilde{R}_{2/3}(\sigma_1)$ defined in \[1\]. This is in agreement with the discussion at the end of Section 7.4 in \[1\]. Indeed in the infinite times limit at fixed $\Delta = (t_2 - t_1)/t_1$, we have, for large $\Delta$

$$h \simeq \hat{A}_2(0) + \Delta^{-\frac{2}{3}}(\max_{\hat{y} \in \mathbb{R}} (A_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})) - A_2(0)) + C\Delta^{-\frac{2}{3}} + \ldots$$

(239)

denoting $\sigma$ the random variable $h$ as in the text, we rewrite in shorthand notations

$$\sigma = \chi_2 + \Delta^{-\frac{2}{3}}(\sigma_2 - \sigma_1) + C\Delta^{-\frac{2}{3}} + \ldots$$

(240)

where the JPDF of the variables $\sigma_1, \sigma_2$ is $p(\sigma_1, \sigma_2)$ obtained here exactly, the first term is the standard GUE-TW random variable, \textit{uncorrelated} from $\sigma_1, \sigma_2$. Very little is known however about the random variable $C$. In particular it is likely to be correlated with the first two terms. What information about the PDF of $h$ (or the JPDF of $h$ and $h_1$ can we then obtain from our exact knowledge of $p(\sigma_1, \sigma_2)$? Writing the cumulant generating function

$$\frac{\phi}{e^{\lambda}} = e^{i\lambda\chi_2 + i\lambda\Delta^{-\frac{2}{3}}(\sigma_2 - \sigma_1) + \lambda\Delta^{-\frac{2}{3}}c}$$

(241)

shows, by expansion in $\lambda$, that mixed correlations of $\chi_2$ and $C$ prevent to determine the cumulants higher than the second one solely from the knowledge of $p(\sigma_1, \sigma_2)$. One can also write the JPDF defined in \[1\] and recalled in the main text as

$$P_{\Delta}(\sigma_1, \sigma) = \delta(h_1 - \sigma_1)\delta(h - \sigma) = \delta(h_1 - \sigma_1)\delta(\chi_2 + \Delta^{-\frac{2}{3}}(\sigma_2 - \sigma_1) + C\Delta^{-\frac{2}{3}} + \ldots - \sigma)$$

$$= \partial_\sigma F_2(\sigma)\partial_\sigma F_2(\sigma_1) + \Delta^{-\frac{2}{3}}\partial_\sigma^2 F_2(\sigma)(\sigma_2 - \sigma_1)\partial_\sigma F_2(\sigma_1) + O(\Delta^{-\frac{2}{3}})$$

(242)

by expanding the delta function in powers of $\Delta^{-\frac{2}{3}}$. This result, for large $\sigma_1$, agrees with the first two term of the expansion of $P_{\Delta}^{(1)}(\sigma_1, \sigma)$ in Eq. (168) Section 6.6 of \[1\], using the result (35). The present result (242) however is exact for all $\sigma_1$. Conversely, the result of \[1\] is conjectured to be exact only for large $\sigma_1$ but to all orders in $\Delta^{-\frac{2}{3}}$, a conjecture which we are unable to check here beyond the order $O(\Delta^{-\frac{2}{3}})$, since it contains information about the term $C$ and subleading ones.

**IV. 7. 3. Next order**

To show that the expansion can be carried further we obtain now the JPDF to the next order $k = 3$ (three Airy functions) which we denote as $p^{(2)}_k$. Let us collect the terms. The first line in \[160\] gives

$$p^{(2)}_k(\sigma_1, \sigma_2)\big|_{\text{first line in } 160} \simeq 2\partial_\sigma_1 \partial_\sigma_2 \int_0^{+\infty} du \int_0^{+\infty} dy \cosh\left(\frac{\hat{y}^3}{3} - (u + y)\hat{x}\right)\text{Ai}(u + y)K_{\lambda_1}(\sigma_2, \sigma_2)$$

(243)

$$= 2\text{Ai}(\sigma_2)^2 \int_0^{+\infty} dy \cosh\left(\frac{\hat{y}^3}{3} - (\sigma_1 + y)\hat{x}\right)\text{Ai}(\sigma_1 + y)$$
The term in the second line of (160) which is a product of two traces gives (in loose notations)

\[ \text{Tr}[P_{\sigma_1}A_{\sigma_2-\sigma_1}B^T_{\sigma_2-\sigma_1}B^T_{\sigma_2-\sigma_1}] = -\text{sym}_x \left[ \text{Tr}[P_{\sigma_1}A_{\sigma_2-\sigma_1} \int_0^{+\infty} dy A_i(y + u)v \text{Tr}[P_{\sigma_1}A_{\sigma_2-\sigma_1}, e^{xv}] \right] \]

where we denote \( \text{sym}_x f(x) = f(x) + f(-x) \). Hence

\[ p_x^{(2)}(\sigma_1, \sigma_2)|_{\text{Second line in (160)}} = \text{sym}_x e^{\frac{i}{\pi} \partial_{\sigma_2} \left[ e^{x\sigma_1-\sigma_2} K_{\text{Ai}}(\sigma_1, \sigma_2) \times \int_{\sigma_2}^{+\infty} dv A_i(v)e^{xv} \right]} \]  

(244)

The sum of the two cross terms becomes

\[ -\text{sym}_x e^{\frac{i}{\pi} F_2(\sigma_1)e^{-x(\sigma_2-\sigma_1)} \text{Tr}(I + P_{\sigma_1}K_{\text{Ai}})P_{\sigma_1}A_{\sigma_2-\sigma_1}e^{-xu}] \]  

(245)

which gives two terms. The first one

\[ \text{sym}_x e^{\frac{i}{\pi} \sigma_2} K_{\text{Ai}}(\sigma_1, \sigma_1)A_i(\sigma_2)e^{-x\sigma_2} \]  

(246)

The second one is

\[ -\text{sym}_x e^{\frac{i}{\pi} e^{-x(\sigma_2-\sigma_1)} \text{Tr}[P_{\sigma_1}K_{\text{Ai}}P_{\sigma_1}A_{\sigma_2-\sigma_1}e^{-xu}] = -\text{sym}_x e^{\frac{i}{\pi} e^{-x(\sigma_2-\sigma_1)} \int_{\sigma_1}^{+\infty} du \int_{\sigma_1}^{+\infty} dv K_{\text{Ai}}(u,v)A_i(v) + \sigma_2 - \sigma_1 e^{-xu} \]  

(248)

Taking a derivative w.r.t. \( \sigma_1 \) gives

\[ p_x^{(2)}(\sigma_1, \sigma_2)|_{\text{last}} = \text{sym}_x e^{\frac{i}{\pi} \partial_{\sigma_2} \left[ e^{x\sigma_1-\sigma_2} K_{\text{Ai}}(\sigma_1, \sigma_1)A_i(\sigma_2)e^{-x\sigma_2} \right]} \]  

(249)

Putting together all terms, we find that the total can be rewritten as a derivative

\[ p_x^{(2)}(\sigma_1, \sigma_2) = \partial_{\sigma_2} Q(\sigma_1, \sigma_2) \]  

(250)

\[ Q(\sigma_1, \sigma_2) = \text{sym}_x e^{\frac{i}{\pi} \left[ (K_{\text{Ai}}(\sigma_1, \sigma_2)e^{-x(\sigma_2-\sigma_1)} - K_{\text{Ai}}(\sigma_2, \sigma_2) \right] \int_{\sigma_1}^{+\infty} du A_i(u)e^{-u\sigma_1} + (K_{\text{Ai}}(\sigma_1, \sigma_2)e^{-x(\sigma_2-\sigma_1)} - K_{\text{Ai}}(\sigma_1, \sigma_1) \int_{\sigma_2}^{+\infty} du A_i(u)e^{-u\sigma_2} \]  

(251)

which immediately implies that

\[ \int_{\sigma_1}^{+\infty} d\sigma_2 p_x^{(2)}(\sigma_1, \sigma_2) = 0 \]  

(252)

as required by (231), since the GUE-TW CDF has no term with three Airy function \( (k = 3) \). It is easy to see, using the symmetry of \( Q(\sigma_1, \sigma_2) \) in its arguments, that \( Q \) vanishes quadratically and that the probability vanishes linearly, at coinciding points, i.e. \( p_x^{(2)}(\sigma_1, \sigma_1) = 0 \), again with a cusp singularity. Finally note that the correction to the conditional first moment can be expressed using \( Q \) by integration by part, as

\[ \tilde{R}_{1/3}^{ex}(\sigma_1) = \tilde{R}_{1/3}(\sigma_1) - \int_{\sigma_1}^{+\infty} d\sigma_2 Q(\sigma_1, \sigma_2) + O(e^{-\frac{2\sigma_1}{3}}) \]  

(253)
V. HALF AXIS RESULTS

V.1. joint PDF of Airy and maximum of Airy minus parabola plus Brownian on the half-axis

Here we concentrate on the simplified case $\sigma_L = +\infty$. In that case, from \[140\]-\[141\] it is easy to see that the dependence in $\hat{x}, \hat{w}_R$ is only on the variable $\hat{x} - \hat{w}_R$ (from the STS symmetry). Hence we can set directly $\hat{w}_R = 0$ with no loss of information and we obtain the JPDF of Airy and the maximum of Airy minus parabola plus Brownian on the half-axis defined as

$$G^R_x(\sigma_1, \sigma_R) := \text{Prob}\left(A_2(-\hat{x}) < \sigma_1, \max_{\hat{y} > 0} (A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})) < \sigma_R - \hat{x}^2\right)$$

(254)

where

$$\hat{x} = \hat{g}_\infty(\sigma_1, +\infty, \sigma_R; \hat{x}) = \text{Det}[I - P_{\sigma_m} \hat{K}_{\sigma_R - \sigma_m} P_{\sigma_m}]$$

with $\sigma_m = \min(\sigma_1, \sigma_R)$ and the kernel

$$\hat{K}_{\sigma_R - \sigma_m}(v_i, v_j) = K_{\text{Ai}}(v_i, v_j) + B(\hat{v}) \text{Ai}(v_j + \sigma_R - \sigma_m) e^{\hat{x}(\sigma_R - \sigma_m)}$$

(255)

Using that the second term in the kernel is a projector, one can rewrite

$$G^R_x(\sigma_1, \sigma_R) = F_2(\sigma_m)(1 - e^{-\hat{x}(\sigma_R - \sigma_m)})\text{Tr}[(I - P_{\sigma_m} K_{\text{Ai}})^{-1} P_{\sigma_m} \text{Ai}_{\sigma_R - \sigma_m} B^T_{\hat{x}}]$$

(256)

where we recall that $\text{Ai}_{\gamma}(v) = \text{Ai}(\sigma + v)$, which agrees with the limit of formula \[156\] for $\sigma_L \to +\infty$. The marginal of $\sigma_1$ is the GUE-TW distribution $F_2(\sigma_1)$, and the marginal of $\sigma_R$ is the CDF of the one point distribution for the transition process $A_{\hat{x} - \text{stat}}$

$$\text{Prob}(A_{\hat{x} - \text{stat}} < \sigma_R - \hat{x}^2) = G^R_x(+\infty, \sigma_R) = G^R_x(\sigma_1 \geq \sigma_R, \sigma_R)$$

(257)

$$= F_2(\sigma_R)(1 - \text{Tr}[(I - P_{\sigma_R} K_{\text{Ai}})^{-1} P_{\sigma_R} \text{Ai} B^T_{\hat{x}}]) = \text{Det}[I - P_{\sigma_R} K_{\text{Ai}} - P_{\sigma_R} \text{Ai} B^T_{\hat{x}}]$$

At point $\hat{x} = 0$ this kernel is also the one of the BPP transition (also called GUE1) which can also be written as $F^R_1$, i.e. GOE$^2$, see e.g. \[73\].

V.2. Two-time persistent correlations with half axis constraint

Consider now the two-time problem, or the two directed polymer (DP) problem, where the first (shorter) DP goes from $(-\hat{x}, 0)$ to $(0, t_1)$ and the second DP goes from $(\hat{x}, 0)$ to $(0, t_2)$ with the constraint on its path, denoted $x(t)$, that $y = x(t) > 0$ (we denote $y$ its position at $t_1$). Following similar arguments as in the previous section, with the same definitions as in \[112\]-\[116\] of the rescaled heights one has now

$$h_1 = A_2(-\hat{x}) - \hat{x}^2$$

(258)

$$h_2 = \Delta^{1/3} A_2(0) + \max_{\hat{y} > 0} [A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})] + O(\Delta^{-1/3})$$

(259)

$$h = \tilde{A}_2(0) + \Delta^{-1/3}(\sigma_R - \sigma_1) + O(\Delta^{-2/3})$$

(260)

where $\sigma_1$ and $\sigma_R$ are the random variables

$$\sigma_1 = A_2(-\hat{x})$$

(261)

$$\sigma_R - \hat{x}^2 = \max_{\hat{y} > 0} [A_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})]$$

(262)

We can write again the persistent correlation ratio defined as the $\Delta \to +\infty$ limit of \[185\] as

$$C_\infty = \frac{\langle \sigma_1 \sigma_R \rangle - \langle \sigma_1 \rangle \langle \sigma_R \rangle}{\langle \sigma_1^2 \rangle \langle \sigma_R^2 \rangle} = \frac{\langle \sigma_1^2 \rangle + \langle \sigma_R^2 \rangle - \langle (\sigma_R - \sigma_1)^2 \rangle - 2\langle \sigma_R \rangle \kappa_{1\text{GUE}}} {2\kappa_{1\text{GUE}}^2}$$

(263)

$$= \frac{1}{2} + \frac{(\kappa_{1\text{GUE}})^2 + \langle \sigma_R^2 \rangle - 2\langle \sigma_R \rangle \kappa_{1\text{GUE}} - \langle (\sigma_R - \sigma_1)^2 \rangle} {2\kappa_{2\text{GUE}}}$$

We calculate,

$$\langle (\sigma_R - \sigma_1)^2 \rangle := \int_{-\infty}^{+\infty} d\sigma_R \int_{-\infty}^{\sigma_R} d\sigma_1 (\sigma_R - \sigma_1)^2 \partial_{\sigma_R} \partial_{\sigma_1} G^R_x(\sigma_1, \sigma_R) = -2 \int_{-\infty}^{+\infty} d\sigma_R \int_{-\infty}^{\sigma_R} d\sigma_1 (G^R_x(\sigma_1, \sigma_R) - F_2(\sigma_1))$$
One integration can be performed exactly. Indeed, we can define for \( \sigma_1 < \sigma_R \)

\[
W^R_x(\sigma_1) := \int_{\sigma_1}^{+\infty} d\sigma_R [G^R_1(\sigma_1, \sigma_R) - F_2(\sigma_1)] = -\int_{\sigma_1}^{+\infty} d\sigma_R F_2(\sigma_1) e^{\hat{\sigma}(\sigma_R-\sigma_1)} \text{Tr}[[I - P_\sigma, K_{Ai}]^{-1} P_\sigma, A_{\sigma=\sigma_R-\sigma_1}B^T_{\hat{\sigma}}] 
\]

where \( B_\hat{\sigma} \) and \( \hat{B}_\sigma \) are two vectors defined in (158). Then we have

\[
\langle (\sigma_R-\sigma_1)^2 \rangle = -2 \int_{-\infty}^{+\infty} d\sigma W^R_x(\sigma_1) 
\]

which is thus the simpler version of (197). Let us denote \( F^{\text{half}}_x(\sigma) = \text{Det}[I - P_\sigma, K_{Ai} - P_\sigma, A\mathbf{B}^T_{\hat{\sigma},\sigma}] \) then for \( p = 1, 2 \)

\[
\langle \sigma^p_R \rangle = p \int_0^{+\infty} d\sigma \sigma^{p-1} (1 - F^{\text{half}}_x(\sigma)) - p \int_{-\infty}^0 d\sigma \sigma^{p-1} F^{\text{half}}_x(\sigma) 
\]

we find for \( \hat{\sigma} = 0 \) the numerical estimate

\[
\langle \sigma_R \rangle \approx -0.49368, \quad \langle \sigma^2_R \rangle \approx 1.47525 
\]

and we find

\[
C_\infty \approx 0.6925 
\]

which is slightly larger than the result in the full space. It means that the persistent correlation, i.e. the memory effect is increased by the constraint. Most likely the finite overlap between the two corresponding optimal paths, which is responsible for this correlation, increases (the longer path midpoint \( y \) tending to get closer to the origin).

Finally, note that one could further restrict the longer DP to pass through \( y \), in which case the result for \( C_\infty \) would be simply equal to two point correlation of the Airy process at separation \( \hat{y} \), normalized to its value at \( \hat{y} = 0 \). Note also that the probability that the unconstrained DP (starting at \( \hat{x} \)) passes right or left of \( \hat{y} = 0 \) is related to the function \( H(\hat{x}) \) (see text).

**VI. EXTENDED BAIK-RAINS DISTRIBUTION AND ITS MOMENTS**

Here we recall the definition and the exact expression for the CDF of the extended Baik-Rains distribution (EBR), and provide simpler expressions for its low moments using integrations by parts.

By definition the EBR distribution is the one-point distribution of the stationary Airy process [23, 41, 42] (defined at the one point level in (15)) with associated CDF

\[
\text{Prob}(A_{\text{stat}}(\hat{x}) < \sigma) =: F_0(\sigma - \hat{x}^2, \hat{x}) = \partial_\sigma (F_2(\sigma)Y_{\hat{x}}(\sigma)) 
\]

where \( F_0(\sigma - \hat{x}^2, \hat{x}) = H(\sigma, \hat{x}, -\hat{x}) \) where the function \( H \) was defined in definition 3 in [40] (see also Sec. 2.4. and formula (63) in [13]). The explicit form (269) was obtained by the RBA method in [44], and we recall the definition of the function

\[
Y_{\hat{x}}(\sigma) := \sigma - \hat{x}^2 + \int_{-\infty}^{+\infty} dv (1 - B_{\hat{x}}(v)B_{-\hat{x}}(v)) - \text{Tr}[[P_\sigma, K_{Ai}(I - P_\sigma, K_{Ai}]^{-1} P_\sigma, A\mathbf{B}^T_{\hat{x},\sigma}] 
\]

and of the auxiliary function

\[
B_w(v) = e^{\frac{1}{2}w^3 - w} - \int_0^{+\infty} dy A_i(v + y)e^{wy} 
\]

For \( \hat{x} = 0 \), the function \( F_0(\sigma, 0) = F_0(\sigma) = \partial_\sigma (F_2(\sigma)Y_0(\sigma)) \) is the CDF of the standard Baik-Rains (BR) distribution \( F_0 \).

Integration by parts allow to simplify the expressions for the moments. Let us first recall the calculation of the first moment of the EBR distribution, which is simple

\[
\kappa_1^{\text{EBR}} = \langle \sigma \rangle_{F_0, \hat{x}} = \int d\sigma \sigma \partial_\sigma^2 [F_2(\sigma)Y_{\hat{x}}(\sigma)] = -\int_0^{+\infty} d\sigma \sigma \partial_\sigma [F_2(\sigma)Y_{\hat{x}}(\sigma)] - \int_{-\infty}^{+\infty} \partial_\sigma [F_2(\sigma)Y_{\hat{x}}(\sigma) - (\sigma - \hat{x}^2)] 
\]

\[
= -[F_2(\sigma)Y_{\hat{x}}(\sigma)]_{-\infty}^{+\infty} - [F_2(\sigma)Y_{\hat{x}}(\sigma) - (\sigma - \hat{x}^2)]_{0}^{+\infty} = \hat{x}^2 
\]
and \( \kappa^2_{BR} = 0 \). We have used that for large positive \( \sigma \), \( Y_{\hat{x}}(\sigma) \approx F_2(\sigma)Y_{\hat{x}}(\sigma) \approx \sigma - \hat{x}^2 \) up to fast decaying terms.

The second moment of EBR distribution can be written as

\[
\mu^2_{EBR} = \langle \sigma^2 \rangle_{F_0,\hat{x}} = \int d\sigma \sigma^2 \partial^2_{\sigma}[F_2(\sigma)Y_{\hat{x}}(\sigma)] = \int_0^0 d\sigma \sigma^2 \partial^2_{\sigma}[F_2(\sigma)Y_{\hat{x}}(\sigma)] + \int_{+\infty}^{+\infty} d\sigma \sigma^2 \partial^2_{\sigma}[F_2(\sigma)Y_{\hat{x}}(\sigma) - (\sigma - \hat{x}^2)]
\]

\[
= -2 \int_{-\infty}^{0} d\sigma \partial \partial_{\sigma}[F_2(\sigma)Y_{\hat{x}}(\sigma)] - 2 \int_{0}^{+\infty} d\sigma \partial \partial_{\sigma}[F_2(\sigma)Y_{\hat{x}}(\sigma) - (\sigma - \hat{x}^2)]
\]

\[
= 2 \int_{-\infty}^{0} d\sigma F_2(\sigma)Y_{\hat{x}}(\sigma) + 2 \int_{0}^{+\infty} d\sigma [F_2(\sigma)Y_{\hat{x}}(\sigma) - (\sigma - \hat{x}^2)]
\]

(273)

and one can check that the boundary terms vanish in each equality. This formula is useful for numerical evaluations.

**VII. PDF OF ARGMAX OF AIRY MINUS PARABOLA PLUS BROWNIAN**

**VII. 1. CDF of Argmax \( H(\hat{x}) \)**

Here we calculate the CDF

\[
H(-\hat{x}) := \text{Prob}(\hat{z}_m > -\hat{x}) \, , \, \hat{z}_m = \text{argmax}_{\hat{z} \in \mathbb{R}} \left( A_2(\hat{z}) - \hat{z}^2 + \sqrt{2}B(\hat{z}) \right)
\]

(274)

From Eq. (279) we first need to calculate \( \hat{\sigma}_{+\infty,\sigma_L,\hat{x}}^\pm(\sigma = +\infty, \sigma_L, \sigma_R; \hat{x}) \) in the limit \( \hat{w}_L \to 0^+, \hat{w}_R \to 0^+ \). We can thus use the more general result (156) in the particular case \( \sigma_1 = +\infty \). We only need the case \( \sigma_L \leq \sigma_R \) (see below), in which case \( \sigma_m = \min(\sigma_1, \sigma_L, \sigma_R) = \sigma_L \) and we obtain

\[
\lim_{\hat{w}_L=\hat{w}_R=\hat{w} \to 0^+} \hat{\sigma}_{+\infty,\sigma_L,\hat{x}}^\pm(\sigma_1 = +\infty, \sigma_L, \sigma_R; \hat{x}) = \text{Det}[I - P_0K_{\sigma}^L P_0]
\]

(275)

\[
\times \left( Y_{\hat{x}}(\sigma_L)e^{\hat{x}(\sigma_R - \sigma_L)} \text{Tr}[(I - P_0K_{\sigma}^L P_0) - 1]P_0 \text{Ai}_{\sigma_R} \text{Ai}_T^L \right)
\]

where we use the shorthand notations

\[
K_{\sigma}^L(v_1, v_j) = K_{\sigma}(v_1 + \sigma, v_j + \sigma) \, , \, B_{\sigma}^L(v) = B_{\sigma}(v + \sigma) \, , \, \text{Ai}(v) = \text{Ai}(v + \sigma)
\]

(276)

where \( B_{\sigma}(v) \) is given in 138 (Eq. 16 in the main text), and \( Y_{\hat{x}}(\sigma) \) was defined in 157 (Eq. 270 in the main text). In 275 we have shifted the arguments of the kernel by \( \sigma_m = \sigma_L \) to use projectors \( P_0 \), which simplifies the evaluation of derivatives w.r.t. \( \hat{x} \) and \( \sigma_L \) (see below). One can check (as above) using the identities (164, 166) that for \( \sigma_R = \sigma_L \) this becomes the CDF of the extended BR distribution

\[
\lim_{\hat{w}_L=\hat{w}_R=\hat{w} \to 0^+} \hat{\sigma}_{+\infty,\sigma_L,\hat{x}}^\pm(\sigma_1 = +\infty, \sigma_L, \sigma_R; \hat{x}) = F_0(\sigma_L - \hat{x}^2, \hat{x}) = \partial \partial_{\sigma}[F_2(\sigma_L)Y_{\hat{x}}(\sigma_L)]
\]

(277)

Let us now calculate

\[
H(-\hat{x}) = \lim_{\hat{w}_L \to 0^+, \hat{w}_R \to 0^+} \int_{-\infty}^{+\infty} d\sigma \partial \partial_{\sigma}[\hat{\sigma}_{+\infty,\sigma_L,\hat{x}}(\sigma = +\infty, \sigma_L, \sigma_R; \hat{x})]|_{\sigma_L = \sigma_R^-}
\]

(278)

Taking the derivative w.r.t. \( \sigma_R \) in 275 using 170 and rearranging we obtain the main result given in the text in 19 namely

\[
H(-\hat{x}) = \int d\sigma F_2(\sigma) \times \left( Y_{\hat{x}}(\sigma) \text{Tr}[(I - P_0K_{\sigma})^{-1}P_0(\text{Ai} + \hat{x}^2)\text{Ai}_T] \right)
\]

\[
+(\text{Tr}[(I - P_0K_{\sigma})^{-1}P_0\text{Ai}\text{Bi}_T^L] - 1) \text{Tr}[(I - P_0K_{\sigma})^{-1}P_0(\text{Ai} + \hat{x}^2)\text{Bi}_T^L]
\]

(279)
We will now show that

\[ H(\hat{x}) = \frac{1}{2} - \frac{1}{4} \partial_\hat{x} g(\hat{x}) \]  

(280)

where

\[ g(\hat{x}) = \langle \sigma^2 \rangle_{\hat{F}_{0,\hat{x}}} - \langle \sigma \rangle_{\hat{F}_{0,\hat{x}}}^2 = \int d\sigma \sigma^2 \partial_\sigma |F_2(\sigma)Y_\hat{x}(\sigma)| - \left( \int d\sigma \sigma \partial_\sigma^2 |F_2(\sigma)Y_\hat{x}(\sigma)| \right)^2 \]  

(281)

is the second cumulant of the EBR distribution (note that although we denote it by \( g(\hat{x}) \) it has nothing to do with the generating function \( g(\sigma, \sigma_L, \sigma_R; \hat{x}) \)). Since \( Y_{-\hat{x}}(\sigma) = Y_\hat{x}(\sigma) \) the function \( g(\hat{x}) \) is even in \( \hat{x} \) hence \( \partial_\hat{x} g(\hat{x}) \) is an odd function of \( \hat{x} \).

To show (280) we proceed in two steps. First we show that the even part of \( H(\hat{x}) \) is constant and equal to 1/2, i.e. \( H(-\hat{x}) + H(\hat{x}) = 1 \). We see that in the first line of (279), since \( Y_\hat{x}(\sigma) \) is even the term proportional to \( \hat{x} \) cancels in the sum and what remains in the integrand can be simply rewritten as \( Y_{-\hat{x}}(\sigma) \partial_\sigma^2 F_2(\sigma) \) using the following identity obtained by derivation of (164), see Eq. (327) in Appendix G2 of \[\text{I} \] for details

\[ F_2(\sigma) \text{Tr}[(1 - P_\sigma K_{AI})^{-1} P_\sigma A_i A_i^T] = \frac{1}{2} \partial_\sigma^2 F_2(\sigma) \]  

(282)

To deal with the symmetrized version of the second line in (279) we need some further identities. We first recall the identity (166)

\[ \partial_\sigma Y_\hat{x}(\sigma) = (\text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^T] - 1)(\text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^{T,-}] - 1) \]  

(283)

We will need to take a derivative of this expression. An intermediate formula is obtained, using (170), as

\[ \partial_\sigma \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^T] = \partial_\sigma \text{Tr}[(I - P_0 K_{AI})^{-1} P_0 A_i (B_x^T)^T] \]  

(284)

\[ = \text{Tr}[(I - P_0 K_{AI})^{-1} P_0 A_i (B_x^T)^T] - \text{Tr}[(I - P_0 K_{AI})^{-1} P_0 A_i (B_x^T)^T] \text{Tr}[(I - P_0 K_{AI})^{-1} P_0 A_i A_i^T] \]  

\[ + \text{Tr}[(I - P_0 K_{AI})^{-1} P_0 A_i (A_i - \hat{x} B_x^T)^T] \]  

\[ = \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i (B_x^T)^T] - (\text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i (B_x^T)^T] - 1) \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i A_i^T] \]  

(285)

\[ - \hat{x} \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^T] \]

Hence taking the derivative of (283) we find that

\[ F_2(\sigma) \partial_\sigma^2 Y_\hat{x}(\sigma) = -2 \partial_\sigma F_2(\sigma) \partial_\sigma Y_\hat{x}(\sigma) \]  

(286)

\[ + \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i (B_x^T)^T] - 1)(\text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i (B_x^{T,-})^T] + \hat{x} \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^{T,-}] - \hat{x} \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i (B_x^{T,-})^T] - \hat{x} \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^{T,-}]) \]

where to obtain the first term we used (164) and (283). Now we can check that the sum of the last two lines in (286) is precisely the integrand which appears from the symmetrized version of the second line in (279). This finally leads to

\[ H(-\hat{x}) + H(\hat{x}) = \int d\sigma [Y_{-\hat{x}} \partial_\hat{x}^2 F_2(\sigma) + F_2(\sigma) \partial_\hat{x}^2 Y_\hat{x}(\sigma) + 2 \partial_\sigma F_2(\sigma) \partial_\sigma Y_\hat{x}(\sigma)] = \int d\sigma \partial_\sigma F_0(\sigma - \hat{x}^2, \hat{x}) = 1 \]  

(287)

i.e. a total derivative, where we have used the definition (17) of the EBR distribution. This is the announced result for the symmetrized part.

We will now complete the proof of (280) by showing that the antisymmetric part can be written as

\[ \frac{1}{2}(H(-\hat{x}) - H(\hat{x})) = \frac{1}{2} \int d\sigma \frac{\sigma^2}{2} |F_2(\sigma)Y_\hat{x}(\sigma)| - \frac{1}{4} \partial_\hat{x} \left( \int d\sigma \sigma \partial_\sigma^2 |F_2(\sigma)Y_\hat{x}(\sigma)| \right)^2 = \frac{1}{4} \partial_\hat{x} g_\infty(\hat{x}) \]  

(288)

We will first obtain the following expression for the following derivative w.r.t. \( \hat{x} \)

\[ \partial_\hat{x} Y_\hat{x}(\sigma) = -2 \hat{x} + \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma (B_\hat{x} - B_{-\hat{x}})(A_i^T)^T] + \hat{x} \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma (B_{-\hat{x}} + B_\hat{x}) A_i^T] \]  

(289)

\[ - \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^T] \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^{T,-}] + \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^{T,-}] \text{Tr}[(I - P_\sigma K_{AI})^{-1} P_\sigma A_i B_x^T] \]
Let us recall the definition
\[ Y_\hat{z}(\sigma) := 1 + \mathcal{L}_\hat{z}(\sigma) - \Tr[P_\sigma K_{\hat{A}i}(I - P_\sigma K_{\hat{A}i})^{-1} P_\sigma B_{\hat{z}}B_{\hat{z}}^T] \]  
(290)

From the definition of \( B_\hat{z}(v) \) one first obtain the derivative formul\( a \)
\[ \partial_\hat{z}B_\hat{z}(v) = (\hat{z}^2 - v)B_\hat{z}(v) + A_\hat{i}(v) - \hat{z}A_\hat{i}(v) \]  
(291)
\[ \partial_\hat{z} - (v_2 - v_1)B_\hat{z}(v_1)B_{\hat{z}}(v_2) = (A_\hat{i}(v_1) - \hat{z}A_\hat{i}(v_1))B_{\hat{z}}(v_2) = B_{\hat{z}}(v_1)(A_\hat{i}(v_2) + \hat{z}A_\hat{i}(v_2)) \]  
(292)

From the definition \( \mathcal{L}_\hat{z}(\sigma) \) as well as \( \mathcal{L}_\hat{z}(\sigma) \) we thus obtain
\[ \partial_\hat{z}\mathcal{L}_\hat{z}(\sigma) = -2\hat{z} - \int_\sigma^{+\infty} du \partial_\hat{z}(B_\hat{z}(u))B_{\hat{z}}(u) \]  
(293)
\[ = -2\hat{z} - \int_\sigma^{+\infty} du[A_\hat{i}(u)(B_{\hat{z}}(u) - B_{\hat{z}}(u)) - \hat{z}A_\hat{i}(u)(B_{\hat{z}}(u) + B_{\hat{z}}(u))] \]  
(294)
\[ = -2\hat{z} + \hat{z}\Tr[P_\sigma(B_{\hat{z}} + B_{\hat{z}})A_\hat{i}^T] - \Tr[P_\sigma(B_{\hat{z}} - B_{\hat{z}})(A_\hat{i})^T] \]  
(295)

Now from \( \mathcal{L}_\hat{z}(\sigma) \) we have denoted \( D(v_1, v_2) = (v_2 - v_1)B_{\hat{z}}(v_1)B_{\hat{z}}(v_2) \). Let us now calculate \( \Tr[K_{\hat{A}i}(I - K_{\hat{A}i})^{-1}D] \). To this aim let us first calculate for any integer \( p \geq 2 \)
\[ \Tr[(P_\sigma K_{\hat{A}i})^{p-1}D] = \int_{v_1, v_2} \ldots \int_{v_1, v_2} K_{\hat{A}i}(v_1, v_2) \ldots K_{\hat{A}i}(v_{p-1}, v_p) B_{\hat{z}}(v_p) B_{\hat{z}}(v_1) \]  
(297)
\[ \sum_{k=0}^{p-2} \Tr[(P_\sigma K_{\hat{A}i})^{k}P_\sigma(A_\hat{i})^T - A_\hat{i}(A_\hat{i})^T](P_\sigma K_{\hat{A}i})^{p-2-k}P_\sigma B_{\hat{z}}B_{\hat{z}}^T \]  
(298)
we have rewritten the factor in the first line \( v_1 - v_p = v_1 - v_2 - v_3 - \ldots - v_{p-1} - v_p \) and used that
\[ K_{\hat{A}i}(v_1, v_2) = \frac{A_\hat{i}(v_1)A_\hat{i}(v_2) - A_\hat{i}(v_1)A_\hat{i}(v_2)}{v_1 - v_2} \]  
(299)

In summary \( D \) acts as an operator derivative, replacing one of the \( K_{\hat{A}i} \) by the antisymmetric combination \( A_\hat{i}(A_\hat{i})^T - A_\hat{i}(A_\hat{i})^T \). Hence we have
\[ \Tr[(I - P_\sigma K_{\hat{A}i})^{-1}P_\sigma K_{\hat{A}i}P_\sigma D] = \Tr((I - P_\sigma K_{\hat{A}i})^{-1}P_\sigma(A_\hat{i})^T - A_\hat{i}(A_\hat{i})^T)(I - P_\sigma K_{\hat{A}i})^{-1}P_\sigma B_{\hat{z}}B_{\hat{z}}^T] \]  
(300)
\[ = \Tr((I - P_\sigma K_{\hat{A}i})^{-1}P_\sigma A_\hat{i}B_{\hat{z}}^T \) \Tr((I - P_\sigma K_{\hat{A}i})^{-1}P_\sigma A_\hat{i}B_{\hat{z}}^T) \]  
using that \( (I - K)^{-1}K = (I - K)^{-1} - 1 \). Putting together \( \mathcal{L}_\hat{z}(\sigma) \), \( \mathcal{L}_\hat{z}(\sigma) \), \( \mathcal{L}_\hat{z}(\sigma) \) and \( \mathcal{L}_\hat{z}(\sigma) \) we finally obtain the wanted formula \( \mathcal{L}_\hat{z}(\sigma) \).

Let us now write explicitly from \( \mathcal{L}_\hat{z}(\sigma) \) the antisymmetric combination
\[ \frac{1}{2}(H(\hat{z}) - H(\hat{x})) = \int d\sigma \left( \hat{z}(\partial_\sigma F_2(\sigma)Y_\hat{z}(\sigma) + F_2(\sigma)\partial_\sigma Y_\hat{z}(\sigma) - F_2(\sigma) \right) \]  
(301)
\[ + F_2(\sigma)\hat{x} \Tr[(1 - P_\sigma K_{\hat{A}i})^{-1}P_\sigma B_{\hat{z}}B_{\hat{z}}^T] + \frac{1}{2}F_2(\sigma) \Tr[(1 - P_\sigma K_{\hat{A}i})^{-1}P_\sigma B_{\hat{z}}B_{\hat{z}}^T\) \Tr[(1 - P_\sigma K_{\hat{A}i})^{-1}P_\sigma B_{\hat{z}}B_{\hat{z}}^T] \]  
\[ + \frac{1}{2}F_2(\sigma) \Tr[(1 - P_\sigma K_{\hat{A}i})^{-1}P_\sigma B_{\hat{z}}B_{\hat{z}}^T] \]  
where the term \( \hat{z}\partial_\sigma F_2(\sigma)Y_\hat{z}(\sigma) \) is the contribution of the first line in \( \mathcal{L}_\hat{z}(\sigma) \), using \( \mathcal{L}_\hat{z}(\sigma) \). We have rearranged terms so as to make appear \( \partial_\sigma Y_\hat{z}(\sigma) \) which is given by \( \mathcal{L}_\hat{z}(\sigma) \).
Comparison of \([301]\) and of \([289]\) immediately shows that

\[
\frac{1}{2} (H(-\hat{x}) - H(\hat{x})) = \int d\sigma \left( \hat{x} \partial_\sigma (F_2(\sigma)Y_2(\sigma)) + \frac{1}{2} F_2(\sigma) \partial_\sigma Y_2(\sigma) \right)
\]

\[
= \int d\sigma \hat{x} (\partial_\sigma (F_2(\sigma)Y_2(\sigma)) - \theta(\sigma)) + \int d\sigma (\frac{1}{2} F_2(\sigma) \partial_\sigma Y_2(\sigma) + \theta(\sigma) \hat{x})
\]

(302)

where in the second line we have added and subtracted a Heaviside function so as to make each integral convergent. We have used that for large positive \(\sigma\), \(Y_2(\sigma) \simeq F_2(\sigma)Y_2(\sigma) \simeq \sigma - \hat{x}^2\) up to fast decaying terms. Now we can check that the second term is \(\frac{1}{4} \partial_\hat{x}\) applied to the last line of Eq. \([272]\), which defines the second moment of the EBR distribution. Hence we have

\[
\int d\sigma \frac{1}{2} F_2(\sigma) \partial_\sigma Y_2(\sigma) + \theta(\sigma) \hat{x} = \frac{1}{2} \partial_\hat{x} \int d\sigma \frac{\sigma^2}{2} \partial^2_\sigma [F_2(\sigma)Y_2(\sigma)] = \frac{1}{4} \partial_\hat{x} \langle \sigma^2 \rangle_{F_0,\hat{x}}
\]

(303)

Let us now recall the calculation of the first moment of the EBR distribution in \([272]\). The first term in \([303]\) is calculated in exactly the same way

\[
\hat{x} \int d\sigma (\partial_\sigma (F_2(\sigma)Y_2(\sigma)) - \theta(\sigma)) = \hat{x} \int d\sigma (\partial_\sigma (F_2(\sigma)Y_2(\sigma)) - \theta(\sigma) \partial_\sigma (\sigma - \hat{x}^2)) = -\hat{x}^3 = -\frac{1}{4} \partial_\hat{x} \langle \sigma^2 \rangle_{F_0,\hat{x}}
\]

(305)

since the second integral in \([305]\) is exactly minus the sum of the two last terms in the first line of \([272]\). Putting together \([303], [304]\) and \([305]\), we have shown

\[
\frac{1}{2} (H(-\hat{x}) - H(\hat{x})) = \frac{1}{4} \partial_\hat{x} \langle \sigma^2 \rangle_{F_0,\hat{x}} - \langle \sigma \rangle_{F_0,\hat{x}}^2 = \frac{1}{4} \partial_\hat{x} g(\hat{x})
\]

(306)

Since \(\partial_\hat{x} g(\hat{x})\) is odd this finally implies our desired result \([280]\).

Note that the function

\[
f_{\text{KPZ}}(y) = \frac{1}{4} \partial_y^2 g(y)
\]

(307)

was introduced by Prahofer and Spohn \([47]\) in the context of the PNG and TASEP models, and is known to be a probability distribution, i.e. positive with \(\int dy f_{\text{KPZ}}(y) = 1\). Some values for \(f_{\text{KPZ}}(y)\) (see e.g. \([3]\)) are \(f_{\text{KPZ}}(0) = 0.54\), and the second and fourth moments are 0.714 and 0.733 respectively, with a kurtosis of 2.812 – 3. It is interesting to have recovered this function here by a completely different calculation, on a different observable (the argmax of Airy plus Brownian), as the connection between the two observables was pointed out only very recently by Maes and Thiery \([3]\). They obtained this connection from fluctuation dissipation relations exploiting the stationarity of the Brownian initial condition, and using the Burgers equation.

VIII. STATIONARY KPZ IN PRESENCE OF A STEP

Here we provide the derivation of the result \([11]\) in the text for the CDF

\[
G_{\hat{H}}(\sigma_L) = \text{Prob}(\hat{h}_{\text{step}}(\hat{x}) - \hat{H} + \hat{x}^2 < \sigma_L)
\]

(308)

where \((\text{with } \hat{H} > 0 \text{ with no loss of generality})\)

\[
\hat{h}_{\text{step}}(\hat{x}) := \max_y \left( A_2(\hat{x} - \hat{y}) - (\hat{x} - \hat{y})^2 + \sqrt{2} B(y) - \hat{H} \text{sgn}(\hat{y}) \right)
\]

This is relevant for the KPZ class when the initial condition is stationary far on each side of \(x = 0\) with a mismatch of height around \(x = 0\). More precisely, whenever the rescaled initial condition has the form \(h_0(\hat{y}) = \sqrt{2} B(\hat{y}) - \hat{H} \text{sgn}(\hat{y})\). One example is the solution \(h(x,t)\) of the continuum KPZ equation \([11]\) with initial conditions (in our units \(\nu = 1, \lambda_0 = D = 2\) \(h(x,t = 0) = B_0(x) - \hat{H} \text{sgn}(x)\), i.e. equal to a two sided unit Brownian (with \(B_0(\hat{0}) = 0\)) with a downward step of size \(2\hat{H}\) at \(x = 0\), which is scaled as \(H = H^{1/3}\), with fixed \(\hat{H}\), so as to remain relevant in the large time limit which we study here. Other examples are discussed in \([61]\) (see Eq. (2) and discussion there) where the step is smooth but scales appropriately. In all these cases, at large time \(t\) one has \(\lim_{t \to +\infty} t^{-\frac{2}{3}} h(x = 2t^{\frac{2}{3}} \hat{x},t) = \hat{h}_{\text{step}}(\hat{x})\), where \(\hat{h}_{\text{step}}(\hat{x})\) is defined in \([209]\), a definition which holds at a given point \(\hat{x}\), not as a process in \(\hat{x}\) (the latter would require replacing \(A_2(\hat{x} - \hat{y})\) by the so-called Airy sheet).
Using Eq. (64) and the definitions (63) we see that the desired CDF is given by our generating function as follows

$$G_R(\sigma_L) = \text{Prob}(\hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R = \sigma_L + 2\hat{H})$$

$$= \lim_{\hat{w}_L,\hat{w}_R \to 0^+} \hat{g}_{\infty} a_L a_R = a_L a_R = \sigma_L + 2\hat{H}; \hat{x}$$

(309)

As we stressed in Section, we already calculated this generating function for $\hat{w}_{L,R} > 0$ in (61) (which is also a particular case of the more general calculation performed here). It lead to the more general result for Brownian IC plus a wedge plus a step (see Section II C.2 item 4 there). However the limit $\hat{w}_{L,R} = 0^+$ is quite non-trivial, and not obtained there. It is performed here in Section III. 7. Let us translate the result in the present setting. From (275) we obtain, after some rearrangement

$$G_R(\sigma_L) = \text{Det}[I - P_{\sigma_L} K_A P_{\sigma_L}]$$

$$\times \left( Y_3(\sigma_L) e^{2\hat{H}} \text{Tr}[(I - P_{\sigma_L} K_A)^{-1} P_{\sigma_L} A_2 \hat{H} A_1^T]$$

$$+ (\text{Tr}[(I - P_{\sigma_L} K_A)^{-1} P_{\sigma_L} A_1 (B_2)^T] - 1) e^{2\hat{H}} \text{Tr}[(I - P_{\sigma_L} K_A)^{-1} P_{\sigma_L} A_2 \hat{H} (B - \hat{x})^T] - 1) \right)$$

(310)

which leads to the result (41) in the text.

IX. CALCULATION OF THE AUXILIARY FUNCTION $\phi_{\infty}(k, y_L, y_R, y)$

Here we compute the integral defined in (130). Let us denote $A_{L,R} = 2\hat{w}_{L,R} \pm 2ik$ and assume Re($A_{L,R}$) > 0. We recall that $\{a_L, a_R\} \in \{0, 1\}^2$. We expand the product in (130), leading to four terms. We use the elementary integrals

$$\left( -\frac{1}{2\pi i} \right) \int_{C_1} \frac{dz}{z} e^{-z} = \theta(-y)$$

$$\left( -\frac{1}{2\pi i} \right) \int_{C_1} \frac{dz}{z} \frac{1}{z} e^{-z} = z e^{-z} - e^{-z} = \int_0^\infty e^{z} \frac{d\nu}{A} \theta(-y+v) = \frac{1}{A} (\theta(-y) + \theta(y) e^{-Ay}) \quad \text{Re}(A) > 0$$

(311)

(312)

Let us write

$$\frac{e^{-L y_L + z R y_R + z y}}{A_R + z L + z - a_R z} = \int_0^\infty e^{z} \frac{d\nu}{A} \theta(v-y) \theta(-a_R v - y_R) \theta(v-y)$$

(313)

Hence

$$\left( -\frac{1}{2\pi i} \right)^3 \int_{C_1} \frac{dz_L}{z_L} \int_{C_1} \frac{dz_R}{z_R} \int_{C_1} \frac{dz}{z} e^{-L y_L + z R y_R + z y} = \int_0^\infty e^{z} \frac{d\nu}{A} \theta(v-y) \theta(-a_R v - y_R) \theta(v-y)$$

(314)

Taking a derivative w.r.t. $y_R$ one obtains:

$$\left( -\frac{1}{2\pi i} \right)^3 \int_{C_1} \frac{dz_L}{z_L} \int_{C_1} \frac{dz_R}{z_R} \int_{C_1} \frac{dz}{z} e^{-L y_L + z R y_R + z y} = -\int_0^\infty e^{z} \frac{d\nu}{A} \theta(v-y) \theta(-a_R v - y_R) \theta(v-y)$$

(315)

$$= -\delta_{a_R, 0} \theta(y_R) e^{y_R A_R} \theta(-y_R - y) \theta(-y_R - y) - \delta_{a_R, 0} \delta(y_R) \frac{1}{A} e^{-\max(y_L, y, 0) A_R}$$

(316)

which allows to evaluate the two cross-terms in (130). We also need:

$$\frac{e^{-L y_L + z R y_R + z y}}{(A_L + z R + z - a_L z L)(A_R + z L + z - a_R z R)} = \int_{v_1 > 0, v_2 > 0} e^{-A_L v_1 - A_R v_2} e^{z_L (y_L - v_2 + a_L v_1) + z_R (y_R - v_2 + a_L v_1) - z (y_1 - v_2)}$$

(317)

which similarly leads to

$$\left( -\frac{1}{2\pi i} \right)^3 \int_{C_1} \frac{dz_L}{z_L} \int_{C_1} \frac{dz_R}{z_R} \int_{C_1} \frac{dz}{z} e^{-L y_L + z R y_R + z y}$$

$$= \int_{v_1 > 0, v_2 > 0} e^{-A_L v_1 - A_R v_2} \theta(v_1 - y_R - a_R v_2) \theta(v_2 - y_L - a_L v_1) \theta(v_1 + v_2 - y)$$

(317)
and taking two derivatives we obtain:
\[
\left(\frac{-1}{2\pi}\right)^3 \int_{c'}^{c} \frac{dz_L}{z_L} \int_{c'}^{c} \frac{dz_R}{z_R} \int_{c'}^{c} \frac{dz}{z} (A_L + z_R + z - a_L z_L) (A_R + z_L + z - a_R z_R) \\
= \int dv_1dv_2e^{-A_Lv_1-A_Rv_2} \delta(v_1 - y_R - a_Rv_2) \delta(v_2 - y_L - a_Lv_1) \theta(v_1 + v_2 - y) \theta(v_1) \theta(v_2) \\
+ \frac{a_R a_L}{A_L + A_R} \delta(y_L + y_R) e^{A_L y_L - A_R y_R} e^{-\theta(y_L + y_R)} e^{-\theta(y_L, y_R)} (318)
\]
upon enumeration of all four cases \(a_{L,R} = 0, 1\). Putting all together we finally obtain the general result
\[
\frac{1}{2} \phi_\infty(k, y_L, y_R, y) = -1 + \theta(-y_L) \theta(-y_R) \theta(-y) \\
-2a_L \theta(-y_L) e^{y_L A_L} \theta(-y_L - y_R) \theta(-y_R - y) - (1 - a_L) \delta(y_L) \frac{1}{A_L} e^{-\max(y_R, 0) A_L} \\
-2a_R \theta(-y_R) e^{y_R A_R} \theta(-y_R - y_L) \theta(-y_L - y) - (1 - a_R) \delta(y_R) \frac{1}{A_R} e^{-\max(y_L, 0) A_R} \\
+(1 + a_L + a_R - 3a_R a_L) \theta(y_L + a_R y_L) \theta(y_R + a_R y_R) \theta((1 + a_R) y_L + (1 + a_L) y_R - y) e^{-A_L(y_R + a_R y_R)} e^{-A_R(y_L + a_L y_L)} \\
+ \frac{4a_R a_L}{A_L + A_R} \delta(y_L + y_R) e^{\frac{x}{2}(A_L y_L + A_R y_R)} e^{-\theta(y_L, y_R)} (319)
\]
which leads to (133) in Section III. 4.

X. DERIVATION OF THE SECOND FORM OF THE KERNEL (137)

We now rewrite the kernel (131)-(133) by integrating over \(k\), using Airy function identities recalled in Appendix C of [61] and following similar steps as in Appendix D there. This gives \(M_{s_1, s_L, s_R, \tilde{z}}(v_i, v_j) = e^{\frac{x}{2}(v_j - v_i)} M_{s_1, s_L, s_R, \tilde{z}}(v_i, v_j)\) where
\[
M_{s_1, s_L, s_R, \tilde{z}}(v_i, v_j) = -\int dy \left(1 - \theta(\min(s_L, s_R, s_1) - y)\right) 2^{1/3} \text{Ai}(2^{1/3} y + \frac{x^2}{32}) \text{Ai}(2^{1/3} y + \frac{x^2}{32}) \\
-(1 + a_L + a_R - 3a_R a_L) 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_L(L - a_R) - s_R(1 - a_L) + (1 - a_L) y + \frac{x^2}{32}}{2}) \theta(y - \frac{s_L + a_R s_R}{1 + a_L}) \\
\times \text{Ai}(2^{1/3} y + \frac{R(1 - a_L) - s_L(1 - a_R) + (1 - a_R) y + \frac{x^2}{32}}{2}) \theta(y - \frac{s_L + a_R s_R}{1 + a_L}) \\
\times \theta(y - \frac{(1 + a_R) s_L + (1 + a_L) s_R - s_1}{1 + a_R + a_L}) e^{-2(\tilde{w}_L + a_R \tilde{w}_R + a_L \tilde{w}_R)(s_1) + 2\tilde{w}_L(s_L + a_R s_L) + 2\tilde{w}_R(s_R + a_L s_R)} \\
+ 2a_L \theta(s_L - y) \theta(s_L + s_R - 2y) \theta(s_L + s_R + 2y - 2) 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_L}{2}) \text{Ai}(2^{1/3} y + \frac{s_R}{2}) e^{2(\tilde{w}_L + \tilde{w}_R)(y - s_L)} \\
+ 2a_R \theta(s_R - y) \theta(s_L + s_R - 2y) \theta(s_L + s_R + 2y - 2) 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_R}{2}) \text{Ai}(2^{1/3} y + \frac{s_L}{2}) e^{2(\tilde{w}_L + \tilde{w}_R)(y - s_R)} \\
+ \frac{1}{2} (1 - a_R) \delta(y - s_L) e^{-2(\tilde{w}_L + \tilde{w}_R)(y - s_L)} \int_0^{+\infty} dr e^{-r(\tilde{w}_L + \tilde{w}_R)} \\
\times 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_L}{2} + \frac{s_R - s_L - y}{2}) \text{Ai}(2^{1/3} y + \frac{s_R}{2} + \frac{s_L - y}{2}) \\
+ \frac{1}{2} (1 - a_R) \delta(y - s_R) e^{-2(\tilde{w}_L + \tilde{w}_R)(y - s_R)} \int_0^{+\infty} dr e^{-r(\tilde{w}_L + \tilde{w}_R)} \\
\times 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_R}{2} + \frac{s_L - s_R - y}{2}) \text{Ai}(2^{1/3} y + \frac{s_L}{2} + \frac{s_R - y}{2}) \\
\times \delta(y - \frac{s_L + s_R}{2}) \frac{a_R a_L}{\tilde{w}_L + \tilde{w}_R} e^{-(\tilde{w}_L + \tilde{w}_R)(\min(s_L, s_R, s_1) - 2) 2^{1/3} \text{Ai}(2^{1/3} y + \frac{s_L}{2}) \text{Ai}(2^{1/3} y + \frac{s_R}{2}) e^{2(\tilde{w}_L + \tilde{w}_R)}} (320)
\]
We have written the terms almost in the same order as they appear in (133). We used (C1) of [61] for the first term in (320) with \(a = v_i + y/2, b = v_j + y/2\), for the second term with \(a = v_i + y + a_R a_L, b = s_L a_R - a_R a_L, a_R a_L\) and
Let us recall the "statistical tilt symmetry" (STS) symmetry. If \( h(x,t) \) is the solution of the KPZ equation (47) with initial condition \( h(x,0) = h_0(x) \) and white noise \( \eta \), then, for any \( u \), \( \tilde{h}(x,t) = h(x - ut, t) - xu/2 + ut^2/4 \) is also solution with a tilted white noise \( \tilde{\eta} \), which has the same correlation as \( \eta \), and initial condition \( \tilde{h}(x,0) = h_0(x) - xu/2 \). If the initial condition of \( h \) is droplet at \( x = x_0 = 0 \), then the initial condition of \( \tilde{h} \) is also droplet at \( x = x_0 = 0 \). Hence one has the joint equivalence in law

\[
\{h(x_1,t_1|0,0), h(x_2,t_2|0,0)\} \equiv \{h(x_1-ut_1,t_1|0,0) - \frac{ut_1}{2} + \frac{u^2t_1^2}{4}, h(x_2-ut_2,t_2|0,0) - \frac{ut_2}{2} + \frac{u^2t_2^2}{4}\} \tag{322}
\]

valid for any \( u \). Let us now define \( H_j = h(x_j, t_j|x_0,0), j = 1,2 \), and denote \( P_{x_0,x_1,x_2}(H_1, H_2) \) the associated JPDF. First one has the translational invariance property

\[
P_{x_0,x_1,x_2}(H_1, H_2) = P_{x_0+y,x_1+y,x_2+y}(H_1, H_2) \tag{323}
\]

for any \( y \). This can be combined with the STS symmetry (322), and we obtain, for any \( u, y \):

\[
P_{x_0,x_1,x_2}(H_1, H_2) = P_{x_0+y,x_1+y-ut_1,x_2+y-ut_2}(H_1 + \frac{u(x_1-x_0)}{2} - \frac{u^2t_1}{4}, H_2 + \frac{u(x_2-x_0)}{2} - \frac{u^2t_2}{4}) \tag{324}
\]

Choosing \( u = (x_1-x_0)/t_1 \) and \( y = -x_0 \) we can express the JPDF with general endpoints in terms of the one where only the last point varies, as

\[
P_{x_0,x_1,x_2}(H_1, H_2) = P_{0,0,x_2-x_1-\frac{t_2}{t_1}x_1-\frac{t_2}{t_1}x_0}(H_1 + \frac{(x_1-x_0)^2}{4t_1}, H_2 + \frac{(x_2-x_0)^2-X_2^2}{4t_2}) \tag{325}
\]

Choosing \( u = \frac{x_2-x_1}{t_2-t_1} \) and \( y = \frac{x_1-x_0}{t_2-t_1} \) we can express the JPDF with general endpoints in terms of the one where only the first point varies, as

\[
P_{x_0,x_1,x_2}(H_1, H_2) = P_{0,0,x_2-\frac{t_2}{t_1}x_1-\frac{t_2}{t_1}x_0}(H_1 + \frac{(x_1-x_0)(x_2-x_1)}{2(t_2-t_1)} - \frac{t_1(x_2-x_1)^2}{4(t_2-t_1)^2}, H_2 + \frac{(x_2-x_0)(x_2-x_1)}{2(t_2-t_1)} - \frac{t_2(x_2-x_1)^2}{4(t_2-t_1)^2}) \tag{326}
\]

It is clear from this expression that, in the limit of large times \( t_1, t_2 \) and large time separation \( t_2/t_1 \gg 1 \), to be in the regime where \( \tilde{X}_0 = X_0/(2t_1^{2/3}) \) is fixed, one needs to take \( (x_0, x_1, x_2) = (0,0, x_2 \approx 2\tilde{X}_0t_1^{-1/3}t_2) \), namely

\[
P_{0,0,x_2-2\tilde{X}_0t_1^{-1/3}t_2}(H_1, H_2) \simeq P_{X_0=2t_1^{2/3}\tilde{X}_0,0,0}(H_1 - t_1^{1/3}\tilde{X}_0^2, H_2 - \frac{x_2^2}{4t_2}) \tag{327}
\]

This is the regime where the variable defined in (1) \( \tilde{X} = x_2/(2(t_2-t_1)t_1^{-1/3}) \approx \tilde{X}_0 \) is fixed. It can thus be compared with our results in the present paper where we vary instead the first endpoint position, see discussion in Section IV. 7. 1. below Eq. (237).
XII. FLAT LIMIT: MAX AND ARGMAX OF ARY \_2 MINUS PARABOLA, AND ARY \_2 AT A POINT, JOINTLY

We now specify the result, to the case \( a_L = a_R = 0 \) (wedge) in the limit \( \hat{w}_R = \hat{w}_L = 0^+ \) (flat). In that case we denote
\[
\hat{h}_L(\hat{x}) = \max_{\hat{z} < \hat{x}} (A_2(\hat{z}) - \hat{z}^2), \quad \hat{h}_R(\hat{x}) = \max_{\hat{z} > \hat{x}} (A_2(\hat{z}) - \hat{z}^2)
\]
and we have an exact expression for the following triple joint CDF as a Fredholm determinant
in terms of the kernel

\[
K_{\sigma_1, \sigma_L, \sigma_R}(v_i, v_j) = \int_{\sigma_m}^{+\infty} dy \text{Ai}(v_i + y)\text{Ai}(v_j + y)
\]

\[
- e^{2\hat{z}(\sigma - \sigma_L)} \int_{\max(\sigma_L, \sigma_R, \sigma_L + \sigma_R - \sigma_1)}^{+\infty} dy \text{Ai}(v_i + y + \sigma_L - \sigma_R)\text{Ai}(v_j + y - \sigma_L + \sigma_R)
\]

\[
+ \int_{\max(\sigma_L - \sigma_R, \sigma_L - \sigma_1, 0)}^{+\infty} dy \text{Ai}(v_i + y + \sigma_L)\text{Ai}(v_j - y + \sigma_L)e^{-2y\hat{z}}
\]

\[
+ \int_{\max(\sigma_L - \sigma_R, \sigma_L - \sigma_1, 0)}^{+\infty} dy \text{Ai}(v_i - y + \sigma_R)\text{Ai}(v_j + y + \sigma_R)e^{+2y\hat{z}}
\]
where \( \sigma_m = \min(\sigma_1, \sigma_L, \sigma_R) \). We now consider two applications of this formula.

XII.1. JPDF of the max of Airy2 minus parabola and of Airy2 at a point, i.e. JPDF of flat and droplet KPZ heights, i.e. point-to-point and point-to-line DP free energies

Let us further specialize to the case \( \sigma_L = \sigma_R = \sigma \). We obtain
\[
\text{Prob}(A_2(-\hat{x}) < \sigma_1, \max_y (A_2(y) - y^2) < \sigma - \hat{x}^2) = \text{Det}[I - P_\sigma K_{\text{max}(\sigma - \sigma_1, 0)}] P_\sigma
\]
\[
K_{\sigma - \sigma_1}(v_i, v_j) = \int_{\sigma - \sigma_1}^{\sigma} dy \text{Ai}(v_i + y)\text{Ai}(v_j + y)
\]
\[
- \int_{\sigma - \sigma_1}^{\sigma} dy \text{Ai}(v_i + y + \sigma)\text{Ai}(v_j - y)e^{-2y\hat{z}}
\]
Note that for \( \sigma \leq \sigma_1 \), using the identity
\[
K_0(v_i, v_j) = \int_{-\infty}^{+\infty} dy \text{Ai}(v_i + y)\text{Ai}(v_j - y)e^{-2y\hat{z}} = 2^{-1/3}\text{Ai}(2^{-1/3}(v_i + v_j - 2\hat{x}^2))e^{\hat{z}(v_i - v_j)}
\]
and defining (with \( \sigma' = \sigma - \hat{x}^2 = 2^{-2/3}s \))
\[
K'(v_i, v_j) = K_0(v_i + \sigma, v_j + \sigma) = 2^{-1/3}\text{Ai}(2^{-1/3}(v_i + v_j + s))e^{\hat{z}(v_i - v_j)}
\]
one recovers, using a similarity transformation of the kernel, for \( \sigma_1 \geq \sigma \)
\[
\text{Prob}(A_2(-\hat{x}) < \sigma_1, \max_y (A_2(y) - y^2) < \sigma - \hat{x}^2) = \text{Prob}(\max_y (A_2(y) - y^2) < \sigma')
\]
\[
= \text{Det}[I - P_\sigma K_0 P_\sigma] = \text{Det}[I - P_\sigma K^\text{GOE} P_\sigma] = D_1(s = 2^{2/3}\sigma')
\]
where \( K^\text{GOE}(v_i, v_j) = \text{Ai}(v_i + v_j + s) \) is the GOE kernel. Hence we correctly recover the well known result that the maximum of Airy2 minus a parabola is distributed by the (scaled) GOE TW distribution. Here in addition we have obtained in \( \text{[332]} \) the JCDF of this maximum and the value of the Airy2 process at some arbitrary point \(-\hat{x} \). Hence, Eq. \( \text{[332]} \) also gives the large time limit of the joint CDF of the (properly centered and scaled) KPZ height fields with respectively flat and droplet initial conditions in the same realization of the noise, i.e. of the couple \((h_{\text{flat}}(0, t), h(0, t|x, 0))\), equivalently of the point-to-point and point-to-line DP free energies in the same random potential.
XI.2. JPDF of max and argmax, i.e. of endpoint position and free energy of a point-to-line DP

Let us now specialize to $\sigma_1 = +\infty$ and $\sigma_L < \sigma_R$. Then we obtain from (329)-(330)

$\dot{g}_\infty(+\infty, \sigma_L, \sigma_R; \dot{x}) = \text{Det}[I - P_{\sigma_L} K_{\sigma_R - \sigma_L} P_{\sigma_L}]$ (337)

$K_{\sigma_R - \sigma_L}(v_i, v_j) = \int_{-\infty}^{+\infty} \text{d}y \text{Ai}(v_i + y)\text{Ai}(v_j + y) - e^{2\dot{x}(\sigma_R - \sigma_L)} \int_{0}^{+\infty} \text{d}y \text{Ai}(v_i + y)\text{Ai}(v_j + y + 2(\sigma_R - \sigma_L))$ (338)

$+ \int_{0}^{+\infty} \text{d}y \text{Ai}(v_i + y)\text{Ai}(v_j - y)e^{-2\dot{x}} + e^{2\dot{x}(\sigma_R - \sigma_L)} \int_{0}^{+\infty} \text{d}y \text{Ai}(v_i - y)\text{Ai}(v_j + y + 2(\sigma_R - \sigma_L))e^{2\dot{x}}$ (339)

For $\sigma_L = \sigma_R$ we obtain the same kernel as in (333)

$K_0(v_i, v_j) = \int_{-\infty}^{+\infty} \text{d}y \text{Ai}(v_i + y)\text{Ai}(v_j - y)e^{-2\dot{x}} = 2^{-1/3}\text{Ai}(2^{-1/3}(v_i + v_j - 2\dot{x}^2))e^{2\dot{x}(v_i - v_j)}$ (340)

which, as noted above, is identical to the GOE kernel by a similarity transformation.

We can now obtain an explicit formula for the JPDF of max and argmax, defined as

$\hat{h}_m = \max_{\hat{z} \in \mathbb{R}}(A_2(\hat{z}) - \hat{z}^2) , \quad \hat{z}_m = \text{argmax}_{\hat{z} \in \mathbb{R}}(A_2(\hat{z}) - \hat{z}^2)$ (341)

Recalling Eq. (82) and, taking a derivative, we obtain (we will denote $\sigma_R = \sigma$ for notational simplicity below)

$\text{Prob}(\hat{z}_m > -\dot{x}, \hat{h}_m) = [\delta_{\sigma_R} \dot{g}_\infty(+\infty, \sigma_L, 0; \dot{x})]_{\sigma_L = \sigma_R - \sigma_R = \sigma = \hat{h}_m + \dot{x}^2} = \text{Det}[I - P_{\sigma_L} K_{\sigma_R - \sigma_L}(v_i, v_j)]_{\sigma_L = \sigma_R - \sigma_R = \sigma = \hat{h}_m + \dot{x}^2}$ (342)

where we have defined the derivative kernel $K^{(1)}(v_i, v_j) := -\partial_{\sigma_R} K_{\sigma_R - \sigma_L}(v_i, v_j)|_{\sigma_L = \sigma_R, \sigma_R = \sigma}$, which reads, explicitly

$K^{(1)}(v_i, v_j) = 2 \int_{0}^{+\infty} \text{d}y \text{Ai}(v_i + y)\text{Ai}'(v_j + y) + \dot{x}\text{Ai}(v_j + y) - 2 \int_{0}^{+\infty} \text{d}y \text{Ai}(v_i - y)e^{2\dot{x}}(\text{Ai}'(v_j + y) + \dot{x}\text{Ai}(v_j + y))$ (343)

We can rewrite this result in a slightly different form

$\text{Prob}(\hat{z}_m > -\dot{x}, \hat{h}_m) = \text{Det}[I - P_{\sigma} \hat{K}_{\sigma}]\text{Tr}[(I - P_{\sigma} \hat{K}_{\sigma})^{-1} P_{\sigma} K^{(1)}]|_{\sigma = \hat{h}_m + \dot{x}^2}$ (344)

with

$\hat{K}_{\sigma}(v_i, v_j) = K_0(v_i + \sigma, v_j + \sigma) , \quad K^{(1)}_{\sigma}(v_i, v_j) = K^{(1)}_{\sigma}(v_i + \sigma, v_j + \sigma)$ (345)

Let us recall the result of [62] (which was proved equivalent in [64] to the one of [63]) in the present notations. First the marginal distribution of the maximum is

$\text{Prob}(\hat{h}_m \leq \sigma') = F_1(2^{2/3}\sigma') , \quad F_1(s) = \text{Det}[I - P_{\sigma} K^\text{GOE}_s P_{\sigma}] , \quad K^\text{GOE}_s(v_1, v_2) = \text{Ai}(v_1 + v_2 + s)$ (346)

i.e. it is the GOE TW distribution (as also found here, see previous subsection). Note that the GOE kernel can also be rewritten in terms of $K_0$ as

$\text{Ai}(v_i + v_j + 2^{2/3}(\sigma - \dot{x}^2)) = 2^{1/3}K_0(2^{1/3}v_i + \sigma, 2^{1/3}v_j + \sigma)e^{-2^{1/3}(v_i - v_j)\dot{x}}$ (347)

Using this relation we can rewrite the formula in Theorem 2 of [62] as follows (after a similarity transformation). The formula for the JPDF of $\hat{h}_m$ and $\hat{z}_m$ (more precisely its density) is then

$P(\hat{h}_m, \hat{z}_m) = \text{Det}[I - P_{\sigma_L} \tilde{K}_{\sigma} P_{\sigma_L} + P_{\sigma} \tilde{\psi}_{\hat{z}_m, \hat{h}_m}^T \tilde{\psi}_{\hat{z}_m, \hat{h}_m} P_{\sigma_L} - F_1(2^{2/3}\hat{h}_m))$ (348)

$\tilde{\psi}_{\hat{z}_m, \hat{h}_m}(v) = 2\hat{z}_m \text{Ai}(v + \hat{h}_m + \hat{z}_m^2) + 2\text{Ai}'(v + \hat{h}_m + \hat{z}_m^2) = 2\hat{z}_m \text{Ai}(v) + 2\text{Ai}'(v)|_{\sigma = \hat{h}_m + \hat{z}_m^2}$ (349)

It remains to check that our result agrees with the one of [62], that is

$\partial_{\dot{x}} \text{Prob}(\hat{z}_m > -\dot{x}, \hat{h}_m) = P(\hat{h}_m, -\dot{x})$ (350)
To show the agreement is thus equivalent to show that

\[ \partial_\hat{x} \text{Det}[I - P_0 \hat{K}_\sigma] \text{Tr}[(I - P_0 \hat{K}_\sigma)^{-1} P_0 \hat{K}^{(1)}_\sigma]|_{\sigma = \hat{h}_m + \hat{x}^2} \]

\[ = \text{Det}[I - P_0 \hat{K}_\sigma P_0 + P_0(-2\hat{x} A_\sigma + 2A'_\sigma)(2\hat{x} A_\sigma + 2A'_\sigma)P_0] - \text{Det}[I - P_0 \hat{K}_\sigma P_0] \]

where we recall that \( \hat{K}_\sigma \) and \( \hat{K}^{(1)}_\sigma \) also depend explicitly on \( \hat{x} \) (not indicated for notational simplicity).

We will not attempt here to show that (350) is correct (preliminary investigations show that it may not be trivial), hence it is left for the future.