On the existence-uniqueness and exponential estimate for solutions to stochastic functional differential equations driven by G-Lévy process

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Abstract

The existence-uniqueness theory for solutions to stochastic dynamic systems is always a significant theme and has received a huge attention. The objective of this article is to study the mentioned theory for stochastic functional differential equations (SFDEs) driven by G-Lévy process. The existence-uniqueness theorem for solutions to SFDEs driven by G-Lévy process has been determined. The error estimation between the exact solution and Picard approximate solutions has been shown. In addition, the exponential estimate has been derived.

1 Introduction

Stochastic dynamic equations based on G-Brownian motion have been studied by several authors [1, 7, 10, 13, 26]. Among them, the existence-uniqueness, stability, moment estimates, continuity and differentiability properties of solution with respect to the initial data were studied in detail [4, 8, 14, 16, 17, 21, 25]. Stochastic differential equations based on Lévy process perform a leading role in a broad range of applications, containing financial mathematics for describing the observed reality of financial markets [2], physics for various phenomena [22], genetics for the movement designs of many animals [9] and biology for modeling the spread of diseases [12]. In [11] Hu and Peng initiated the G-Lévy process. In [19] Ren represented a sublinear expectation related to the framework of G-Lévy process as an upper-expectation. Paczka then inaugurated the integrals and the Itô formula based on the G-Lévy process [15]. The existence and exponential estimates for solutions to stochastic differential equations (SDEs) driven by G-Lévy process were established by Wang and Gao [23]. They also constructed the BDG-type inequality in the stated framework [23]. The existence theory for solutions to SDEs based on G-Lévy process having discontinuous coefficients was given by Wang and Yuan [24]. The quasi-sure exponential stability of SDEs in the framework of G-Lévy process was initiated by Shen et. al. [20]. To the best of our knowledge
no text is available on the study of existence-uniqueness and exponential estimates for solutions to stochastic functional differential equations (SFDEs) driven by G-Lévy process. Consequently, the current research is concentrated on this theme. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$. Consider $BC((\infty,0];\mathbb{R}^d)$, the family of bounded continuous $\mathbb{R}^d$-valued mappings $\psi$ defined on $(\infty,0]$ with norm $\|\psi\| = \sup_{\infty < t < 0} |\psi(t)| \mathbb{I}$. Let $F_t = \sigma \{ B(v) : 0 \leq v \leq t \}$ be the natural filtration defined on a complete probability space $(\mathcal{S}, \mathcal{F}, \mathcal{P})$. Assume that $\{F_t : t \geq 0\}$ assures the usual characteristics. Let $f : [0,T] \times BC((\infty,0];\mathbb{R}^d) \rightarrow \mathbb{R}^d$, $g : [0,T] \times BC((\infty,0];\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times m}$, $h : [0,T] \times BC((\infty,0];\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times m}$ and $K : [0,T] \times BC((\infty,0];\mathbb{R}^d) \rightarrow \mathbb{R}^{d \times m}$ be Borel measurable. We consider the following SFDE driven by G-Lévy process:

$$dx(t) = f(t,x_t)dt + g(t, x_t)dB(t) + \int_{\mathbb{R}_0^d} K(t, x_{t-}, z)L(dt, dz), \quad (1.1)$$

on $t \in [0,T]$ with initial condition $\zeta(0) \in \mathbb{R}^d$, $x_t = \{x(t + \theta), \infty < \theta \leq 0\}$ and $x_{t-}$ indicates the left limits of $x_t$. $B(t)$ is a d-dimensional G-Brownian motion. For all $x \in \mathbb{R}^d$, $f(,x), g(,x), h(,x) \in \mathcal{M}_2^G((\infty,T];\mathbb{R}^d)$ and $K(,x,.) \in \mathcal{H}_G^2((\infty,T] \times \mathbb{R}_0^d;\mathbb{R}^d)$. Equation (1.1) has the following initial condition.

$$x_0 = \zeta = \{\zeta(\theta) : \infty < \theta \leq 0\}, \quad (1.2)$$

is $\mathcal{F}_0$-measurable, $BC((\infty,0];\mathbb{R}^d)$-value random variable such that $\zeta \in \mathcal{M}_G^2((\infty,T];\mathbb{R}^d)$.

The rest of the article is arranged as follows. Fundamental results and definitions of the G-framework are given in section 2. The existence-and-uniqueness of solutions to SFDEs driven by G-Lévy process is studied in section 3. Here the boundedness of solutions is determined. The error estimation between the exact and approximate solutions is shown. The exponential estimate for solutions to SFDEs driven by G-Lévy process is constructed in section 4.

## 2 Fundamental settings

In this section, we include preliminary results and notions of the G-framework required for the subsequent sections of this article [3, 5, 6, 16]. Consider $\mathcal{S}_T = C_0([0,T],\mathbb{R}^d)$, the space of real valued continuous mappings on $[0,T]$ such that $w(0) = 0$ endowed with the distance

$$\rho(w^1, w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \max_{t \in [0,i]} |w^1(t) - w^2(t)| \wedge 1 \right).$$

Let for any $w \in \mathcal{S}_T$ and $t \geq 0$, $B(t, w) = w(t)$ be the canonical process. Let $\mathcal{F}_t = \sigma \{ B(v) : 0 \leq v \leq t \}$ be the filtration generated by canonical process $\{B(t), t \geq 0\}$ and $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$. For any $T > 0$, define $\mathcal{L}_{ip}(\mathcal{S}_T) = \{\phi(B(t_1), B(t_2), \ldots, B(t_d)) : d \geq 1, t_1,t_2,\ldots,t_d \in [0,T], \phi \in C_{b,Lip}(\mathbb{R}^{d \times m})\}$, where $C_{b,Lip}(\mathbb{R}^{d \times m})$ is a space of bounded Lipschitz functions. A functional $\mathcal{E}$ defined on $\mathcal{L}_{ip}(\mathcal{S}_T)$ is known as a sublinear expectation if it ensures the characteristics given as follows. For every $x, y \in \mathcal{L}_{ip}(\mathcal{S}_T)$

(1) **Monotonicity:** $\mathcal{E}[x] \geq \mathcal{E}[y]$ if $x \geq y$.

(2) **Constant Preserving:** For all $c \in \mathbb{R}$, $\mathcal{E}[c] = c$. 

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For $t \leq T$, $L_{ip}(S_t) \subseteq L_{ip}(S_T)$ and $L_{ip}(S) = \bigcup_{n=1}^{\infty} L_{ip}(S_n)$. For $p \geq 1$, $L_{ip}^p(S)$ indicates the completion of $L_{ip}(S)$ endowed with the Banach norm $\hat{\mathbb{E}}[|\eta|^p]^\frac{1}{p}$ and $L_{ip}^p(S_t) \subseteq L_{ip}^p(S_T) \subseteq L_{ip}^p(S)$ for $0 \leq t \leq T < \infty$. The triple $(S, L_{ip}(S_T), \mathbb{E})$ is recognized as a sublinear expectation space. For $p \geq 1$, a partition of $[0, T]$ is a finite order subset $\{A_n^n : n \geq 1\}$ so that $A_n^n : 0 = t_0 < t_1 < ... < t_n = T$. The space $\mathbb{M}_{G}^{p,0}([0, T])$, $p \geq 1$ of simple processes is defined by

$$\mathbb{M}_{G}^{p,0}([0, T]) = \left\{ \eta_t(z) = \sum_{i=0}^{N-1} \xi_i(t)(z)I_{[t_i, t_{i+1})}(t); \xi_i(t)(z) \in L_{ip}^p(\Omega_{t_i}) \right\}. \tag{2.1}$$

The completion of space (2.1) equipped with the norm $\|\eta\| = \left\{ \int_0^T \hat{\mathbb{E}}[|\eta(s)|^p]ds \right\}^{1/p}$ is indicated by $\mathbb{M}_{G}^{p,0}([0, T]), p \geq 1$.

**Definition 2.1.** Let $\eta_t \in \mathbb{M}_{G}^{p}([0, T], p \geq 1$. Then the G-Itô’s integral is defined by

$$\int_0^T \eta(s)dB(s) = \sum_{i=0}^{N-1} \xi_i \left( B(t_{i+1}) - B(t_i) \right).$$

**Definition 2.2.** For a partition $0 = t_0 < t_1 < ... < t_{N-1} = T$, the quadratic variation process $\{\langle B \rangle(t)\}_{t \geq 0}$ is defined by

$$\langle B \rangle(t) = \lim_{N \to \infty} \sum_{i=0}^{N-1} \left( B(t_i^{N}) - B(t_i^{N+1}) \right)^2 = B(t)^2 - 2 \int_0^t B(s)dB(s).$$

A mapping $\Pi_{0,T} : \mathbb{M}_{G}^{0,1}(0, T) \to L_{ip}^2(\mathcal{F}_T)$ is given by

$$\Pi_{0,T}(\eta) = \int_0^T \eta(s)d\langle B \rangle(s) = \sum_{i=0}^{N-1} \xi_i \left( \langle B \rangle(t_{i+1}) - \langle B \rangle(t_i) \right).$$

It can be extended to $\mathbb{M}_{G}^{1}(0, T)$ and for $\eta \in \mathbb{M}_{G}^{1}(0, T)$ this is still given by

$$\int_0^T \eta(s)d\langle B \rangle(s) = \Pi_{0,T}(\eta).$$

Let $Q$ be a weakly compact set that represent $\mathbb{E}$. The capacity $\hat{\nu}$ is given as the following

$$\hat{\nu}(A) = \sup_{p \in Q} \mathbb{P}(A), \quad A \in \mathcal{F}_T.$$

The set $A$ is polar if $\hat{\nu}(A) = 0$. A characteristic holds quasi-surely $(q,s)$ if it sustains outside a polar set.
Lemma 2.3. Let $x \in \mathcal{L}_G^p$ and $E|x|^p < \infty$. Then
\[
\hat{\nu}(|x| > c) \leq \frac{E[|x|^p]}{c},
\]
for any $c > 0$.

The proof of the lemmas 2.4 and 2.5 can be seen in [10].

Lemma 2.4. Let $\lambda \in \mathcal{M}_G^p(0,T)$, $p \geq 2$. Then
\[
E\left[ \sup_{0 \leq t \leq T} \left| \int_0^t \lambda(s)dB(s) \right|^p \right] \leq \alpha E\left[ \int_0^T |\lambda(s)|^2 ds \right]^\frac{p}{2},
\]
where $0 < \alpha = k_2 T^\frac{p}{2} < \infty$, $k_2$ is a positive constant depending on $p$.

Lemma 2.5. Let $\lambda \in \mathcal{M}_G^p(0,T)$, $p \geq 1$. Then
\[
E\left[ \sup_{0 \leq t \leq T} \left| \int_0^t \lambda(s)d(B,B)(s) \right|^p \right] \leq \beta E\left[ \int_0^T |\lambda(s)|^2 ds \right]^\frac{p}{2},
\]
where $0 < \beta = k_1 T^{p-1} < \infty$ and $k_1$ is a positive constant depending on $p$.

Definition 2.6. A stochastic process $\{x(t), t \geq 0\}$ defined on a sublinear expectation space $(\mathcal{S}, \mathcal{L}_p(\mathcal{S}_T), E)$ is known as a G-Lévy process if it ensures the upcoming five characteristics:

1. $x(t) = 0$.
2. For any $t, s \geq 0$, the increment $x(t+s) - x(s)$ is independent of $x(t_1), x(t_2), ..., x(t_n)$, $\forall n \in \mathbb{N}$ and $0 \leq t_1 \leq t_2, ..., \leq t_n \leq t$.
3. For every $s, t \geq 0$, the distribution $x(t+s) - x(s)$ does not depend on $t$.
4. For each $t \geq 0$ there exists a decomposition $x(t) = x^c(t) + x^d(t)$.
5. $(x^c(t), x^d(t))_{t \geq 0}$ is a $2d$-dimensional Lévy process satisfying
\[
\lim_{t \downarrow 0} \frac{E[|x^c(t)|^3]}{t} = 0, \quad E[|x^d(t)|] \leq \alpha t, \quad t \geq 0,
\]
where $\alpha$ is a constant depends on $x$.

If $\{x(t), t \geq 0\}$ satisfies only the first three properties i.e. 1-3, then it is the classical Lévy process. It is known that $x^c(t)$ is generalized G-Brownian motion and $x^d(t)$ is of finite variation, where $x^c(t)$ and $x^d(t)$ are continuous part and jump part respectively. Let $\mathcal{H}_G^b([0,T] \times \mathbb{R}^d)$ be a collection of all basic fields defined on $[0,T] \times \mathbb{R}^d \times \mathcal{S}$ of the form
\[
K(u,z)(w) = \sum_{i=1}^{n-1} \sum_{j=1}^{m} \Lambda_{i,j} 1_{(t_i,t_{i+1})}(u)\psi_j(z),
\]
where $u, m \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_n \leq T$, $\{\psi_j\}_{j=1}^m \subset C_{b,lip}(\mathbb{R}^d)$ are mappings with disjoint supports such that $\psi_j(0) = 0$ and $\Lambda_{i,j} = \phi_{i,j}(x_{t_1}, ..., x_{t_i} - x_{t_{i-1}})$, $\phi_{i,j} \in C_{b,lip}(\mathbb{R}^{d+1})$. The norm on this space is given by
\[
\|K\|_{\mathcal{H}_G^b([0,T] \times \mathbb{R}^d)} = E\left[ \int_0^T \sup_{v \in \nu} \int_{\mathbb{R}^d} |K(s,z)|^p v(dz)ds \right]^{\frac{1}{p}}, \quad p = 1, 2.
\]
Definition 2.7. The Itô integral of \( K \in \mathcal{H}^\delta_G([0, T] \times \mathbb{R}^d_0) \) w. r. t. jump measure \( L \) is given as follows
\[
\int_0^t \int_{\mathbb{R}^d_0} K(s, z)L(ds, dz) = \sum_{0 < s \leq t} K\left(s, \triangle x(s)\right), \text{ q.s.}
\]
where \( 0 \leq v < t \leq T \).

Let \( \mathcal{H}^p_G([0, T] \times \mathbb{R}^d_0) \) be the topological completion of \( \mathcal{H}^\delta_G([0, T] \times \mathbb{R}^d_0) \) under the norm \( \| K \|_{\mathcal{H}^\delta_G([0, T] \times \mathbb{R}^d_0)} \), \( p = 1, 2 \). We can sill extend the Itô integral to the space \( \| K \|_{\mathcal{H}^p_G([0, T] \times \mathbb{R}^d_0)}, p = 1, 2 \), where the extended integral has valves in \( L^p_G(\mathbb{S}_T) \), \( p = 1, 2 \). For the above integrals, we have he following BDG-type inequality. For the proof see [23].

Lemma 2.8. Let \( K(s, z) \in \mathcal{H}^2_G([0, T] \times \mathbb{R}^d_0) \). Then a càdlàg modification \( \hat{x}(t) \) of \( x(t) = \int_0^t \int_{\mathbb{R}^d_0} K(s, z)L(ds, dz) \) exists such that for all \( t \in [0, T] \) and \( p \geq 2 \)
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |\hat{x}(t)|^2 \right] \leq k_3 \mathbb{E}\left[ \int_0^t \int_{\mathbb{R}^d_0} K^2(s, z)\nu(dz)ds \right],
\]
where \( k_3 \) is a positive constant depending on \( T \).

3 Bounded-ness and existence-uniqueness results for SFDEs driven by G-Lévy process

In this section, we shall determine the boundedness and existence-uniqueness results for solutions to problem (1.1). Let us first see the definition of solutions to equation (1.1).

Definition 3.1. An \( \mathcal{F}_t \)-adapted càdlàg process \( x(t) \in \mathbb{M}^2_G((\mathbb{R}^d_0); T) \) is called a solution to (1.1) with the initial data (1.2) if it satisfies
\[
x(t) = \zeta(0) + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\langle B, B \rangle(s) + \int_0^t h(s, x_s)dB(s) + \int_0^t \int_{\mathbb{R}^d_0} K(s, x_s-, z)L(ds, dz).
\]

A solution \( x(t) \) of (1.1) is said to be unique if it is identical to any other solution \( y(t) \) of the stated equation i.e.
\[
\mathbb{E}[|x(t) - y(t)|^2] = 0,
\]
holds q.s.

All through this article we propose the upcoming linear growth and Lipschitz conditions respectively.

(A_1) For every \( x \in BC((\mathbb{R}^d_0); \mathbb{R}^d) \), a positive number \( c_1 \) exists so that
\[
| f(t, x) |^2 \vee | g(t, x) |^2 \vee | h(t, x) |^2 \vee \int_{\mathbb{R}^d_0} | K(t, x, z) |^2 \nu(dz) \leq c_1(1 + | x |^2)
\]
(A2) For all \( x, y \in BC((\infty, 0]; \mathbb{R}^d) \), a positive number \( c_2 \) exists so that
\[
| f(t, y) - f(t, x) |^2 \vee | g(t, y) - g(t, x) |^2 \vee | h(t, y) - h(t, x) |^2 \\
\vee \int_{\mathbb{R}^d} | K(t, y, z) - K(t, x, z) |^2 v(dz) \leq c_2 | y - x |^2.
\]

In the forthcoming lemma we prove that any solution \( x(t) \) of equation \((1.1)\) is bounded, in particular \( x(t) \in \mathbb{M}_2^2((\infty, T]; \mathbb{R}^d) \).

**Lemma 3.2.** Let \( x(t) \) be a solution of equation \((1.1)\) with initial data \((1.2)\) such that \( \mathbb{E}\|x\|^2 \leq \infty \). Assume that the growth condition \( A_1 \) holds. Then
\[
\mathbb{E}\left[ \sup_{-\infty \leq s \leq t} |x(s)|^2 \right] \leq \mathbb{E}\|\zeta\|^2 + 5(1 + c_kT)e^{5c_1kT},
\]
where \( k = (1 + k_1)T + k_2 + k_3 \) and \( k_1, k_2, k_3 \) are positive constants.

**Proof.** Consider equation \((1.1)\) and use the basic inequality \( |\sum_{i=1}^5 a_i|^2 \leq 5 \sum_{i=1}^5 |a_i|^2 \) to derive
\[
| x(t) |^2 \leq 5 | \zeta(0) |^2 + 5 \int_0^t | f(s, x_s) |^2 ds + 5 \int_0^t | g(s, x_s) |^2 ds + 5 \int_0^t \int_{\mathbb{R}^d} | K(s, x_s, z) |^2 v(dz) ds.
\]

From the G- expectation, lemmas 2.4, 2.5, 2.8 and the cauchy inequality we get
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |x(s)|^2 \right] \leq 5 \mathbb{E} | \zeta(0) |^2 + 5t \mathbb{E} \int_0^t | f(s, x_s) |^2 ds + 5k_1 \mathbb{E} \int_0^t | g(s, x_s) |^2 ds + 5k_2 \mathbb{E} \int_0^t | h(s, x_s) |^2 ds + 5k_3 \mathbb{E} \int_0^t \int_{\mathbb{R}^d} | K(s, x_s, z) |^2 v(dz) ds.
\]

In view of assumption \( A_1 \) we deduce that
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |x(s)|^2 \right] \leq 5 \mathbb{E} | \zeta(0) |^2 + 5c_1(T + k_1T + k_2 + k_3)T + 5c_1(T + k_1T + k_2 + k_3)T \int_0^t \mathbb{E}|x_s|^2 ds
\]
\[
\leq 5 \mathbb{E} | \zeta(0) |^2 + 5c_1(T + k_1T + k_2 + k_3)T
\]
\[
+ 5c_1(T + k_1T + k_2 + k_3) \int_0^t \left[ \mathbb{E} | \zeta(0) |^2 + \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^2 \right) \right] ds
\]
\[
\leq 5 \mathbb{E} | \zeta(0) |^2 + 5c_1kT + 5c_1kT\mathbb{E} | \zeta(0) |^2 + 5c_1 \int_0^t \mathbb{E} \left( \sup_{0 \leq u \leq s} |x(u)|^2 \right) ds
\]
where \( \hat{k} = T + k_1T + k_2 + k_3 \). From the Grownwall inequality it follows
\[
\mathbb{E}\left[ \sup_{0 \leq s \leq t} |x(s)|^2 \right] \leq 5(1 + c_k\hat{k}T)\mathbb{E} | \zeta(0) |^2 + c_k\hat{k}T e^{5c_1\hat{k}T}
\]
\[ (3.2) \]
Noticing that
\[ E\left[ \sup_{-\infty<s\leq t} |x(s)|^2 \right] \leq E\|\zeta\|^2 + E\left[ \sup_{0\leq s\leq t} |x(s)|^2 \right], \]
it yields
\[ E\left[ \sup_{-\infty<s\leq t} |x(s)|^2 \right] \leq E\|\zeta\|^2 + 5\{(1 + c_1\hat{k}T)E\|\zeta\|^2 + c_1\hat{k}T\}e^{5c_1\hat{k}T}. \]

Letting \( t = T \), we deduce the desired expression.

For \( t \in [0, T] \), define \( x^0(t) = \zeta(0) \) and \( x_0^1 = \zeta \). For each \( n = 1, 2, \ldots, \) we set \( x_0^n = \zeta \) and define the Picard iteration,
\[ x^n(t) = \zeta(0) + \int_0^t f(s, x_s^{n-1})ds + \int_0^t g(s, x_s^{n-1})d\langle B, B \rangle(s) + \int_0^t h(s, x_s^{n-1})dB(s) \]
\[ + \int_0^t \int_{\mathbb{R}^d} K(s, x_s^{n-1}, z)L(ds, dz) \quad t \in [0, T]. \] (3.3)

Next we prove the existence-uniqueness result and error estimation between the exact solution \( x(t) \) and Picard approximate solutions \( x^n(t), n \geq 1 \).

**Theorem 3.3.** Let assumptions \( A_1 \) and \( A_2 \) hold and \( E\|\zeta\|^2 < \infty \). Then equation (1.1) admits a unique càdlàg solution \( x(t) \in \mathcal{M}_2^G((-\infty, T]; \mathbb{R}^d) \). Moreover, for all \( n \geq 1 \), the Picard approximate solutions \( x^n(t) \) and unique exact solution \( x(t) \) of (1.1) satisfy that
\[ E\left[ \sup_{0\leq s\leq t} |x^n(s) - x(s)|^2 \right] \leq C[Mt]^n e^{Mt}, \]
where \( C = 4c_2[(T + Tk_1 + k_2 + k_3)(1 + E\|\zeta\|^2)T, M = 4c_2[(T + Tk_1 + k_2 + k_3) \) and \( k_1, k_2, k_3 \) are positive constants.]

**Proof.** Consider the Picard iteration sequence \( \{x^n, n \geq 1\} \) given by (3.3). Obviously \( x^0(t) \in \mathcal{M}_2^G((-\infty, T]; \mathbb{R}^d) \). From the fundamental inequality \( |\sum_{i=1}^5 a_i|^2 \leq 5 \sum_{i=1}^5 |a_i|^2 \), lemmas 2.4, 2.5, 2.8, the cauchy inequality and assumption \( A_1 \), we deduce
\[ E\left[ \sup_{0\leq s\leq t} |x^n(s)|^2 \right] \leq 5E\|\zeta\|^2 + 5c_1\hat{k}T + 5c_1\hat{k}TE\|\zeta\|^2 + 5c_1\hat{k} \int_0^t E\left[ \sup_{0\leq u\leq s} |x^{n-1}(u)|^2 \right]ds, \]
where \( \hat{k} = (1 + k_1)T + k_2 + k_3 \). Noticing that
\[ \max_{1 \leq n \leq j} E\left[ \sup_{0\leq s\leq t} |x^{n-1}(s)|^2 \right] \leq \max \left\{ E\|\zeta\|^2, \max_{1 \leq n \leq j} E\left[ \sup_{0\leq s\leq t} |x^n(s)|^2 \right] \right\} \]
\[ \leq E\|\zeta\|^2 + \max_{1 \leq n \leq j} E\left[ \sup_{0\leq s\leq t} |x^n(s)|^2 \right], \]
it follows
\[ \max_{1 \leq n \leq j} E\left[ \sup_{0\leq s\leq t} |x^n(s)|^2 \right] \leq 5E\|\zeta\|^2 + 5c_1\hat{k}T + 10c_1\hat{k}TE\|\zeta\|^2 + 5c_1\hat{k} \int_0^t \max_{1 \leq n \leq j} E\left[ \sup_{0\leq u\leq s} |x^n(u)|^2 \right]ds. \]
Similarly, we derive
\[ \max_{1 \leq n \leq j} \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^n(s)|^2 \right] \leq 5[(1 + 2c\bar{k}T)\mathbb{E}||\zeta||^2 + c\bar{k}Te^{c\bar{k}t}], \]
but \( j \) is arbitrary and letting \( t = T \) it follows
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq T} |x^n(s)|^2 \right] \leq 5[(1 + 2c\bar{k}T)\mathbb{E}||\zeta||^2 + c\bar{k}Te^{c\bar{k}T}]. \] (3.4)

From the sequence \( \{x^n(t); t \geq 0\} \) defined by (3.3), we have
\[ x^1(t) - x^0(t) = \int_0^t f(s, x^0_s)ds + \int_0^t g(s, x^0_s)d(B, B)(s) + \int_0^t h(s, x^0_s)dB(s) \]
\[ + \int_0^t \int_{\mathbb{R}^d} K(s, x^0_{s-}, z)L(ds, dz). \]

In view of G-expectation, lemmas 2.4, 2.5, 2.8, the Cauchy inequality and assumption A1, we derive
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^1(s) - x^0(s)|^2 \right] \leq 4c_1[(T + Tk_1 + k_2 + k_3)\int_0^t (1 + \mathbb{E}||\zeta||^2)ds \leq C, \]
where \( C = 4c_1(T + Tk_1 + k_2 + k_3)(1 + E\|\zeta\|^2)T \). Next by similar arguments and assumption A2, it follows
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^2(s) - x^1(s)|^2 \right] \leq 4c_2(T + Tk_1 + k_2 + k_3)\int_0^t \mathbb{E}|x^1_s - x^0_s|^2 ds \]
\[ \leq 4c_2(T + Tk_1 + k_2 + k_3)\int_0^t \mathbb{E}\left[ \sup_{0 \leq s \leq s} |x^1(v) - x^0(v)|^2 \right] ds \]
\[ \leq 4c_2(T + Tk_1 + k_2 + k_3)Ct. \]

Similarly, we derive
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^3(s) - x^2(s)|^2 \right] \leq C[4c_2((T + Tk_1 + k_2 + k_3))^{2}\frac{t^2}{2!}. \]

Thus for all \( n \geq 0 \), we claim that
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^{n+1}(s) - x^n(s)|^2 \right] ds \leq C\left[\frac{M^n}{n!}\right], \] (3.5)
where \( C = 4c_2((T + Tk_1 + k_2 + k_3)(1 + E\|\zeta\|^2)T \) and \( M = 4c_2(T + Tk_1 + k_2 + k_3) \). With the mathematical induction we verify that for all \( n \geq 0 \), (3.5) holds. For \( n = 0 \), it has been proved. Let (3.5) holds for some \( n \geq 0 \). By using similar arguments as above, we derive
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |x^{n+2}(s) - x^{n+1}(s)|^2 \right] ds \leq 4c_2(T + Tk_1 + k_2 + k_3)\int_0^t \mathbb{E}|x^{n+1}_s - x^n_s|^2 ds \]
\[ \leq M\int_0^t \mathbb{E}\left[ \sup_{0 \leq s \leq s} |x^{n+1}(v) - x^n(v)|^2 \right] ds \]
\[ \leq M\int_0^t C\left[\frac{M^n}{n!}\right] ds \leq C\left[\frac{M^{n+1}}{(n+1)!}\right] = C\left[\frac{M^{n+1}}{(n+1)!}\right]. \]
This shows that (3.5) holds for \( n + 1 \). Thus by induction (3.5) holds for all \( n \geq 0 \). By virtue of lemma 2.3 we acquire

\[
\hat{\nu} \left\{ \sup_{0 \leq s \leq T} |x^{n+1}(s) - x^n(s)|^2 > \frac{1}{2^n} \right\} \leq 2^n \mathbb{E} \left[ \sup_{0 \leq s \leq T} |x^{n+1}(s) - x^n(s)|^2 \right] \leq \frac{K[2 Mt]^n}{n!}.
\]

Since \( \sum_{n=0}^{\infty} \frac{K[2 Mt]^n}{n!} < \infty \), from the Borel-Cantelli lemma we get that for almost all \( w \) a positive integer \( n_0 = n_0(w) \) exists so that

\[
\sup_{0 \leq t \leq T} |x^{n+1}(t) - x^n(t)|^2 \leq \frac{1}{2^n}, \quad \text{as} \ n \geq n_0.
\]  \hspace{1cm} (3.6)

It implies that q.s. the partial sums

\[
x^0(t) + \sum_{i=0}^{n-1} [x^{i+1}(t) - x^i(t)] = x^n(t),
\]

are uniformly convergent on \( t \in (-\infty, T] \). Denote the limit by \( x(t) \). Then the sequence \( x^n(t) \) converges uniformly to \( x(t) \) on \( t \in (-\infty, T] \). It follows that \( x(t) \) is \( \mathcal{F}_t \)-adapted and càdlàg. Also, from (3.5), we can see that \( \{x^n(t) : n \geq 1 \} \) is a cauchy sequence in \( \mathcal{L}^2_G \). Hence \( x^n(t) \) converges to \( x(t) \) in \( \mathcal{L}^2_G \), that is,

\[
\mathbb{E}|x^n(t) - x(t)|^2 \to 0, \quad \text{as} \ n \to \infty.
\]

Taking limits \( n \to \infty \) from (3.4) we deduce

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} |x(s)|^2 \right] \leq 5((1 + 2c_1kT)\mathbb{E}\|\zeta\|^2 + c_1kT)e^{c_1kT}. \]  \hspace{1cm} (3.7)

Next we need to verify that \( x(t) \) satisfies equation (1.1). In view of assumption \( A_2 \) and using similar arguments as above, we derive

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^t |f(s, x^n(s)) - f(s, x^n(s))|ds \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^t |g(s, x^n(s)) - g(s, x^n(s))|d(B, B)(s) |^2 \right] \\
+ \mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^t |h(s, x^n(s)) - h(s, x^n(s))|dB(s) |^2 \right] + \mathbb{E} \left[ \sup_{0 \leq s \leq T} \int_0^t \int_{\mathbb{R}^d} |K(s, x^n_{s-}, z) - K(s, x_{s-}, z)|L(ds, dz) |^2 \right] \\
\leq T \int_0^t \mathbb{E}|f(s, x^n(s)) - f(s, x^n(s))|^2ds + Tk_1 \int_0^t \mathbb{E}|g(s, x^n(s)) - g(s, x^n(s))|^2ds \\
+ k_2 \int_0^t \mathbb{E}|h(s, x^n(s)) - f(s, x^n(s))|^2ds + k_3 \int_0^t \int_{\mathbb{R}^d} \mathbb{E}|K(s, x^n_{s-}, z) - K(s, x_{s-}, z)|^2v(dz)ds \\
\leq c_2(T + Tk_1 + k_2 + k_3) \int_0^t \mathbb{E}\left| \sup_{0 \leq v \leq s} |x^n(v) - x(v)|^2 \right| ds \to 0 \text{ as, } n \to \infty,
\]
which means

\[ x \]

From the Grownwall inequality and same initial data one can derive

\[ \text{By virtue of assumption} \]

\[ x \]

in other words

\[ \int_0^t f(s, x^n_s) \to \int_0^t f(s, x_s), \text{ in } \mathcal{L}_G^2; \]

\[ \int_0^t h(s, x^n_s) d\langle B, B \rangle(s) \to \int_0^t h(s, x_s) d\langle B, B \rangle(s), \text{ in } \mathcal{L}_G^2; \]

\[ \int_0^t g(s, x^n_s) \to \int_0^t g(s, x_s), \text{ in } \mathcal{L}_G^2; \]

\[ \int_0^t h(s, x^n_s) dB(s) \to \int_0^t h(s, x_s) dB(s), \text{ in } \mathcal{L}_G^2; \]

\[ \int_0^t \int_{\mathbb{R}_G^d} K(s, x^n_{s-1}, z) L(ds, dz) ds \to \int_0^t \int_{\mathbb{R}_G^d} K(s, x_{s-}, z) L(ds, dz), \text{ in } \mathcal{L}_G^2. \]

For \( t \in [0, T] \) taking limits \( n \to \infty \) in \([3.3]\) we derive

\[ \lim_{n \to \infty} x^n(t) = \zeta(0) + \int_0^t \lim_{n \to \infty} f(s, x^n_s) ds + \int_0^t \lim_{n \to \infty} g(s, x^n_s) d\langle B, B \rangle(s) + \int_0^t \lim_{n \to \infty} h(s, x^n_s) dB(s) \]

\[ + \int_0^t \lim_{n \to \infty} K(s, x^n_{s-1}, z) L(ds, dz), \]

which yields

\[ x(t) = \zeta(0) + \int_0^t f(s, x_s) ds + \int_0^t g(s, x_s) d\langle B, B \rangle(s) + \int_0^t h(s, x_s) dB(s) + \int_0^t \int_{\mathbb{R}_G^d} K(s, x_{s-}, z) L(ds, dz), \]

\( t \in [0, T]. \) This show that \( x(t) \) is the solution of \([1.1]. \) To prove the uniqueness let us assume that equation \([1.1] \) admits two solutions \( x(t) \) and \( y(t). \) Following similar arguments we derive

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |y(s) - x(s)|^2 \right] \leq 4t \int_0^t \mathbb{E} |f(s, y_s) - f(s, x_s)|^2 ds + 4k_1 t \int_0^t \mathbb{E} |g(s, y_s) - g(s, x_s)|^2 ds \]

\[ + 4k_2 t \int_0^t \mathbb{E} |f(s, y_s) - f(s, x_s)|^2 ds + 4k_3 \int_0^t \int_{\mathbb{R}_G^d} \mathbb{E} |K(s, y_{s-}, z) - K(s, x_{s-}, z)|^2 \mu(dz) ds \]

By virtue of assumption \( A_2, \) we deduce

\[ \mathbb{E} \left[ \sup_{0 \leq s \leq t} |y(s) - x(s)|^2 \right] \leq 4c_2 (T + k_1 T + k_2 + k_3) \int_0^t \mathbb{E} \left[ \sup_{0 \leq u \leq s} |y(u) - x(u)|^2 \right] ds. \]

From the Gronwall inequality and same initial data one can derive

\[ \mathbb{E} \left[ \sup_{-\infty < s \leq t} |y(s) - x(s)|^2 \right] = 0 \quad (3.8) \]

which means \( x(t) = y(t) \) quasi-surely, for all \( t \in (-\infty, T]. \) Finally, we have to prove the error
estimation. From equations (1.1) and (3.3), using similar arguments as earlier it follows
\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} |x^n(s) - x(s)|^2\right] \leq 4T \int_0^t \mathbb{E}\left[|f(s, x^n_s) - f(s, x_s)|^2\right] ds + 4k_1T \int_0^t \mathbb{E}\left[g(s, x^n_s) - g(s, x_s)\right] ds
\]
\[+ 4k_2 \int_0^t \mathbb{E}\left|h(s, x^n_s) - h(s, x_s)\right|^2 ds + 4k_3 \int_0^t \mathbb{E}\left|K(s, x^n_{s-}, z) - K(s, x_{s-}, z)\right|^2 v(dz) ds \]
\[\leq 4c_2 [(T + T k_1 + k_2 + k_3) \int_0^t \mathbb{E}\left[\sup_{0 \leq v \leq s} |x^n(v) - x(v)|^2\right] ds \]
\[\leq M \int_0^t \mathbb{E}\left[\sup_{0 \leq v \leq s} |x^n(v) - x^{n-1}(v)|^2\right] ds + M \int_0^t \mathbb{E}\left[\sup_{0 \leq v \leq s} |x^{n-1}(v) - x(v)|^2\right] ds.
\]
In view of (3.5), we obtain
\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} |x^n(s) - x(s)|^2\right] \leq \frac{C[Mt^n}{n!} + M \int_0^t \mathbb{E}\left[\sup_{0 \leq v \leq s} |x^n(v) - x(v)|^2\right] ds
\]
Consequently,
\[
\mathbb{E}\left[\sup_{0 \leq s \leq t} |x^n(s) - x(s)|^2\right] \leq \frac{C[Mt^n}{n!} e^{Mt},
\]
which yields the error estimation between the Picard approximate solutions $x^n(t)$, $n \geq 0$ and exact solution $x(t)$ of problem (1.1). \qed

4 Exponential estimates for SFDEs driven by G-Levy process

To show the exponential estimates let us assume that problem (1.1) has a unique solution $x(t)$ on $t \in [0, \infty)$. Now we derive the exponential estimate for (1.1) as follows.

Theorem 4.1. Let assumptions $A_1$ and $A_2$ hold. Then,
\[
\lim_{n \to \infty} \sup_{t} \frac{1}{t} \log |y(t)| \leq \frac{5}{2} c_1 k,
\]
where $k = (1 + k_1)T + k_2 + k_3$ and $k_1, k_2, k_3$ are positive constants.

Proof. From assertion (3.7), we know that
\[
\mathbb{E}\left[\sup_{0 \leq s \leq T} |x(s)|^2\right] \leq 5\left[(1 + 2c_1 \hat{K}T)\mathbb{E}\|\xi\|^2 + c_1 \hat{K}T\right] e^{c_1 kT}.
\] (4.1)
In view of (1.1), for each $m = 1, 2, 3, \ldots$, we have
\[
\mathbb{E}\left[\sup_{m-1 \leq t \leq m} |x(t)|^2\right] \leq 5\left[(1 + 2c_1 \hat{K}T)\mathbb{E}\|\xi\|^2 + c_1 \hat{K}T\right] e^{c_1 k m}.
\]
By using lemma 2.3 for any $\epsilon > 0$, we derive
\[
\dot{\nu}\left\{w : \sup_{m-1 \leq t \leq m} |x(t)|^2 > e^{(5c_1 k + \epsilon)m}\right\} \leq \frac{\mathbb{E}\left[\sup_{m-1 \leq t \leq m} |x(t)|^2\right]}{e^{(5c_1 k + \epsilon)m}} \leq 5\left[(1 + c_1 \hat{K}T)\mathbb{E}\|\xi\|^2 + c_1 \hat{K}T\right] e^{-\epsilon m}.
\]
Since the series $\sum_{m=1}^{\infty} 5[(1 + c_1 k T)E\|z\|^2 + c_1 k T]e^{-cm}$ is convergent, from the Borel-Cantelli lemma we derive that for almost all $w \in \Omega$, a random integer $m_0 = m_0(w)$ exists so that

$$\sup_{m-1 \leq t \leq m} |x(t)|^2 \leq e^{(5c_1 k + \epsilon)m}, \text{ as } m \geq m_0.$$  

This implies that for $m - 1 \leq t \leq m$ and $m \geq m_0$ we have

$$|x(t)| \leq e^{\frac{1}{2}(5c_1 k + \epsilon)m}.$$  

Hence,

$$\lim_{t \to \infty} \sup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq \frac{1}{2}(5c_1 k + \epsilon),$$

the desired expression follows because $\epsilon$ is arbitrary.

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