Complex sine-Gordon Theory for Coherent Optical Pulse Propagation

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ABSTRACT

It is shown that the McCall-Hahn theory of self-induced transparency in coherent optical pulse propagation can be identified with the complex sine-Gordon theory in the sharp line limit. We reformulate the theory in terms of the deformed gauged Wess-Zumino-Witten sigma model and address various new aspects of self-induced transparency.
Self-induced transparency (SIT), a phenomenon of anomalously low energy loss in coherent optical pulse propagation, was first discovered by McCall and Hahn[1] and the integrability of the SIT equation was demonstrated by employing the inverse scattering method[2]. When phase variation is ignored in the case for a symmetric frequency distribution $g(\Delta w)$ of inhomogeneous broadening, McCall and Hahn have proved an area theorem for pulse propagation. In the sharp line limit where the frequency distribution is sharply peaked at the carrier frequency $w_0$ such that $g(\Delta w) = \delta(w - w_0)$, the SIT equation reduces to the well-known sine-Gordon equation and the $2\pi$ area pulse becomes a 1-soliton of the sine-Gordon theory. However, when phase variation is included, the area theorem no longer holds and the structure of SIT in general has not been well understood except for the construction of explicit solutions by the inverse scattering method[2][3]. In particular, despite its integrability, the SIT theory in the sharp line limit has not been identified with a known 1+1 dimensional integrable field theory, which made a systematic understanding of SIT in terms of a lagrangian field theory impossible.

The purpose of this Letter is to show that the SIT theory with phase variation can be identified with the complex sine-Gordon theory in the sharp line limit. The complex sine-Gordon theory, a generalization of the sine-Gordon theory with a phase degree of freedom, can be reformulated in terms of a nonlinear sigma model which is known as the integrably deformed gauged Wess-Zumino-Witten (WZW) model associated with the coset SU(2)/U(1)[4][5]. This allows us to address various new aspects of SIT in terms of characteristics of the complex sine-Gordon theory; e.g. topological v.s. non-topological solitons, local gauge symmetry, the U(1)-charge conservation, the chiral symmetry and the Kramers-Wannier duality for dark v.s. bright solitons. We also explain the off-resonance effect and inhomogeneous broadening of SIT in the context of the local gauge symmetry of the present formulation.

The SIT equation is given by

$$\tilde{\partial}E + 2\beta < P > = 0$$

$$\partial D - E^*P - EP^* = 0$$

$$\partial P + 2i\Delta wP + 2ED = 0$$   \hspace{1cm} (1)$$

where $\Delta w = w - w_0$, $\partial \equiv \partial/\partial z$, $\tilde{\partial} \equiv \partial/\partial \bar{z}$, $z = t - x/c$, $\bar{z} = x/c$. $E$, $P$ and $D$ represent the electric field, the polarization and the population inversion respectively. The bracket denotes an averaging over the distribution function of inhomogeneous broadening. Since Eq.(1) is invariant under the interchange $(\beta, E, P, D) \leftrightarrow (\beta, E, -P, -D)$, we assume the coupling constant $\beta$ to be positive which we set to one by rescaling $E$, $P$ and $D$. In a simpler case where phase variation is ignored to make $E$ real and the frequency distribution is sharply peaked at the carrier frequency($\Delta w = 0$), we may parametrize $E$, $P$ and $D$ by

$$E = E^* = \partial\varphi \hspace{1cm} < P > = P = -\sin 2\varphi \hspace{1cm} D = \cos 2\varphi.$$   \hspace{1cm} (2)$$

Then the SIT equation reduces to the well-known sine-Gordon equation

$$\tilde{\partial}\varphi - 2\beta \sin 2\varphi = 0.$$   \hspace{1cm} (3)$$
In order to include phase variation as well as the off-resonance effect ($\Delta w \neq 0$), we assume $E$ to be complex and require that the distribution is sharply peaked not necessarily at the carrier frequency, i.e., $g(\Delta w) = \delta(\Delta w - \xi)$ for some constant $\xi$. Introduce a more general parameterization of $E, P$ and $D$ in terms of three scalar fields $\varphi, \theta$ and $\eta$,

$$E = e^{i(\theta - 2\eta)}(2\partial\eta \frac{\cos \varphi}{\sin \varphi} - i\partial \varphi), \quad P = ie^{i(\theta - 2\eta)} \sin 2\varphi, \quad D = \cos 2\varphi \ .$$

(4)

The main result is that the SIT equation given in Eq.(1) then changes into a couple of second order nonlinear differential equations known as the complex sine-Gordon equation,

$$\bar{\partial} \partial \varphi + 4 \frac{\cos \varphi}{\sin^2 \varphi} \partial \eta \bar{\partial} \eta - 2 \sin 2\varphi = 0 \quad (5)$$

$$\bar{\partial} \partial \eta - \frac{2}{\sin 2\varphi}(\partial \eta \partial \varphi + \partial \eta \bar{\partial} \varphi) = 0 \quad (6)$$

together with a couple of first order constraint equations,

$$2 \cos^2 \varphi \partial \eta - \sin^2 \varphi \partial \theta - 2 \xi \sin^2 \varphi = 0 \quad (7)$$

$$2 \cos^2 \varphi \bar{\partial} \eta + \sin^2 \varphi \bar{\partial} \theta = 0 \ . \quad (8)$$

The complex sine-Gordon equation first appeared in 1976 in a description of relativistic vortices in a superfluid [6], and also independently in a treatment of O(4) nonlinear sigma model[7]. Note that Eqs.(5) and (6) consistently reduce to the sine-Gordon equation when phase variation is ignored so that $\eta = 0, \theta = \pi/2$ and the system is on resonance($\xi = 0$). Earlier works on the complex sine-Gordon theory have focused only on Eqs.(5) and (6). However, the constraints in Eqs.(7) and (8) which are new expressions constitute an essential part of the SIT theory, particularly in connection with a local U(1)-gauge symmetry of SIT as explained later. Thus we will call Eqs.(5)-(8) as the complex sine-Gordon theory. The gauge symmetry structure as well as the integrability of Eqs.(5)-(8) may be best understood if we reformulate the complex sine-Gordon theory in terms of the action principle. The proper action is given by a group theoretical nonlinear sigma model action known as the deformed gauged WZW action defined as follows;

$$S(g, A, \bar{A}, \beta) = S_{WZW}(g) + \frac{1}{2\pi} \int \text{Tr}\{ - A \bar{\partial} g g^{-1} + \bar{A} g^{-1} \partial g + A g A g^{-1} - AA \} - S_{\text{potential}}$$

$$S_{\text{potential}} = \frac{\beta}{2\pi} \int \text{Tr} g T g^{-1} \bar{T}$$

(9)

where $S_{WZW}(g)$ is the conventional SU(2)-WZW action and $g$ is an SU(2) matrix function. Tr denotes the trace and $T = -\bar{T} = i\sigma_3 = \text{diag}(i, -i)$ for Pauli matrices $\sigma_i$. The local gauge fields $A = a(z, \bar{z})\sigma_3, \bar{A} = \bar{a}(z, \bar{z})\sigma_3$ are introduced to gauge the U(1) subgroup of SU(2).

Owing to the absence of the kinetic terms, the gauge fields $A, \bar{A}$ act as Lagrange multipliers which result in the constraint equations. One of the nice properties of our formulation is that the equation of motion arising from the action (9) takes a zero curvature form,

$$\delta_g S = -\frac{1}{2\pi} \int \text{Tr}[ \partial + g^{-1} \partial g + g^{-1} A g + \beta \lambda T, \bar{\partial} + \bar{A} + \frac{1}{\lambda} g^{-1} \bar{T} g ] g^{-1} \delta g = 0 \quad (10)$$

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where the constant $\lambda$ is a spectral parameter and the square bracket denotes the commutation. The constraint equations coming from the $A, \bar{A}$-variations are

$$\delta A S = \frac{1}{2\pi} \int \text{Tr} ( -\partial g g^{-1} + g \bar{A} g^{-1} - \bar{A} ) \delta A = 0$$

(11)

$$\delta \bar{A} S = \frac{1}{2\pi} \int \text{Tr} ( g^{-1} \partial g + g^{-1} A g - A ) \delta \bar{A} = 0.$$  

(12)

The action (9) is known to possess the local U(1)-vector gauge symmetry under the transform;

$$g \rightarrow h^{-1} g h, \quad A \rightarrow A + h^{-1} \partial h, \quad \bar{A} \rightarrow \bar{A} + h^{-1} \bar{\partial} h$$

where $h = \exp(\phi(z, \bar{z}) \sigma_3)$, as well as the global U(1)-axial vector gauge symmetry under $g \rightarrow h g h$ for a constant $h$. In order to identify Eqs.(10)-(12) with the SIT equation, we fix the vector gauge by choosing

$$A = \xi T, \quad \bar{A} = 0$$

(14)

for a constant $\xi$. Such a gauge fixing is possible due to the flatness of $A, \bar{A}$. Also, we parameterize the $2 \times 2$ matrix $g$ by

$$g = e^{i \eta \sigma_3} e^{i \varphi (\cos \theta \sigma_1 - \sin \theta \sigma_2)} e^{i \eta \sigma_3} = \begin{pmatrix} e^{2i \eta} \cos \varphi & i \sin \varphi e^{i \theta} \\ i \sin \varphi e^{-i \theta} & e^{-2i \eta} \cos \varphi \end{pmatrix}.$$  

(15)

Then the parametrization in Eq.(4) arises from an identification of $E, P$ and $D$ with $g$ through the relation

$$g^{-1} \partial g + \xi g^{-1} T g - \xi T = \begin{pmatrix} 0 & -E \\ E^* & 0 \end{pmatrix}, \quad g^{-1} T g = -i \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix}$$

(16)

where we have used the constraint equation (12). Also, the zero curvature equation (10) with the identification in Eq.(16) becomes

$$\left[ \partial + \left( \begin{array}{c} i \beta \lambda + i \xi \\ E^* \\ -i \beta \lambda - i \xi \end{array} \right), \bar{\partial} - \frac{i}{\lambda} \begin{pmatrix} D & P \\ P^* & -D \end{pmatrix} \right] = 0$$

(17)

whose components agree precisely with the SIT equation in the sharp line limit. The constraint equations (11) and (12), combined with Eq.(14), also reduce to Eqs.(7) and (8). Thus we have shown that the SIT equation consistently arises from the action (9) with the gauge fixing in Eq.(14). Moreover, the zero curvature equation (17) demonstrates the integrability of the SIT equation. The potential term in Eq.(9) changes into the population inversion $D$

$$S_{\text{potential}} = \int \frac{\beta}{\pi} \cos 2\varphi = \int \frac{\beta}{\pi} D,$$

(18)

which for $\beta = 1$ possesses degenerate vacua at

$$\varphi = \varphi_n = (n + \frac{1}{2}) \pi, \quad n \in \mathbb{Z} \quad \text{and} \quad \theta = \theta_0 \quad \text{for} \quad \theta_0 \text{ constant}.$$  

(19)
The soliton solutions interpolating different vacua can be obtained either by applying the dressing method or by using the Bäcklund transformation\cite{5}. In particular, the 1-soliton solution is given by

\[
\cos \varphi = \frac{b}{\sqrt{(a - \xi)^2 + b^2}} \text{sech}(2bz - 2bC\bar{z})
\]

\[
\eta = (a - \xi)z + (a - \xi)C\bar{z}
\]

\[
\theta = -\tan^{-1}\left[\frac{a - \xi}{b} \coth(2bz - 2bC\bar{z})\right] - 2\xi z + 2D\bar{z}
\]  

where \( a, b \) are arbitrary constants and

\[
C = \frac{1}{(a - \xi)^2 + b^2}, \quad D = 0.
\]  

(20)

In terms of \( E \),

\[
E = -2ib \text{ sech}(2bz - 2bC\bar{z})e^{-2i(a\bar{z} - D\bar{z} + (a - \xi)c\bar{z})}.
\]  

(22)

If \( a - \xi = 0 \), this solution interpolates between two different vacua \( \varphi_n \) and \( \varphi_{n+1} \), i.e. it becomes a topological 1-soliton (\( \Delta n = 1 \)). On resonance where \( a = \xi = 0 \), Eq.(20) reduces to the 1-soliton of the sine-Gordon equation, or a 2\( \pi \) pulse of SIT. If \( a - \xi \neq 0 \), the solution in Eq.(20) reaches to the same vacuum asymptotically as \( x \to \pm\infty \) so that the topological number is zero (\( \Delta n = 0 \)). Nevertheless, except for the topological number, this solution possesses all the properties of a soliton so that we call it a nontopological 1-soliton. It represents a localized pulse with a steadily varying phase. The time area of the pulse, which is defined by

\[
\Delta S = 2 \int |E| dt,
\]  

(23)

is still 2\( \pi \). It is important to mention that this 2\( \pi \) area is a mere coincidence and should not be confused with the 2\( \pi \) area of the topological one. Because of the interference between phases of each nontopological solitons, multi-nontopological solitons in general do not possess the area which is an integer multiple of 2\( \pi \) and the area theorem of McCall and Hahn in the case of inhomogenous broadening does not hold. The stability of nontopological solitons, unlike the topological case whose stability is due to the topological protection, arises from the U(1)-charge conservation law. Recall that the action (9), consequently Eqs.(5)-(8), are invariant under the axial vector transform \( g \to hgh \) or, equivalently,

\[
\eta \to \eta + \epsilon \quad \text{for } \epsilon \text{ constant}.
\]  

(24)

The corresponding Noether currents and the charge are given by

\[
J = \frac{\cos^2 \varphi}{\sin^2 \varphi} \partial \eta, \quad \bar{J} = \frac{\cos^2 \varphi}{\sin^2 \varphi} \bar{\partial} \eta,
\]

\[
Q = \int_{-\infty}^{\infty} dx (J + \bar{J})
\]  

(25)

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where $J, \bar{J}$ satisfy the conservation law, $\partial \bar{J} + \bar{\partial} J = 0$. In particular, the charge of the 1-soliton in Eq.(20) is

$$Q_{1\text{-sol}} = -c(\text{sign}[b \cdot (a - \xi)] \frac{\pi}{2} - \tan^{-1} \frac{a - \xi}{b}).$$

The stability of nontopological solitons can be shown either by using conservation laws in terms of charge and energy as given in [5], or by studying the behavior against small fluctuations[9].

The action (9) also possesses two different types of discrete symmetries. This can be seen most easily in a different gauge where $A = \bar{A} = 0$ which is connected to the gauge in Eq.(14) by an appropriate vector gauge transformation. The first case is the chiral symmetry under the interchange,

$$z \leftrightarrow \bar{z} \text{ and } g \leftrightarrow g^{-1}( \text{ or } \eta \leftrightarrow -\eta, \varphi \leftrightarrow -\varphi)$$

which is a characteristic of the WZW action. Unlike the case of a sigma model without the Wess-Zumino term, parity alone ($z \leftrightarrow \bar{z}$) is not a symmetry. In the SIT context, this is due to the slowly varying envelop approximation which breaks the parity invariance of the Maxwell-Bloch equation. The chiral symmetry generates a new solution from a known one. For example, the chiral transform of the 1-soliton in Eq.(20) in the resonant case ($\xi = 0$) is again a 1-soliton but with the replacement of constants $a, b$

$$a \rightarrow -\frac{a}{a^2 + b^2}, \quad b \rightarrow \frac{b}{a^2 + b^2},$$

which changes the shape of the pulse as well as the velocity by $v \rightarrow c - v$. The currents and the charge also change into

$$J \rightarrow -\bar{J}, \quad \bar{J} \rightarrow -J, \quad Q \rightarrow -Q.$$
Finally, we show that inhomogeneous broadening can be incorporated into our formulation naturally with minor modifications. We maintain the constraint equation (12) only and modify the zero curvature equation by
\[
\partial + g^{-1} g + \xi g^{-1} T g - \xi T + \tilde{\lambda} T , \quad \bar{\partial} + \frac{g^{-1} T g}{\lambda - \xi} = 0 \tag{32}
\]
where the constant \( \tilde{\lambda} \) is a spectral parameter which becomes \( \lambda + \xi \) in the sharp line limit. We make the same identification as in Eq.(16) and require that \( g^{-1} g + \xi g^{-1} T g - \xi T \) is \( \xi \)-independent since \( E \) is a \( \xi \)-independent macroscopic quantity. This results in the SIT equation with inhomogenous broadening as given in Eq.(1). Once again, by using the dressing method, the 1-soliton can be obtained which is the same as in Eq.(20) but with the replacement
\[
C = \left\langle \frac{1}{(a - \xi)^2 + b^2} \right\rangle , \quad D = (a - \xi) \left\langle \frac{1}{(a - \xi)^2 + b^2} \right\rangle - \left\langle \frac{a - \xi}{(a - \xi)^2 + b^2} \right\rangle . \tag{33}
\]
Eq.(14) shows that each frequency \( \xi \) corresponds to a specific gauge choice in our formulation therefore inhomogenous broadening is equivalent to averaging over different gauge fixings. This implies that the inhomogenously broadened case can not be treated by a single field theory. Nevertheless it is remarkable that the group theoretical parametrization of \( E, P \) and \( D \) is still valid. Another important feature of inhomogenous broadening is that it introduces an anomaly term \( M \) in the U(1)-current conservation such that \( \partial \bar{J} + \bar{\partial} J = M \) and
\[
M = 2 \cot \varphi \left[ \cos(\theta - 2\eta) < \sin(\theta - 2\eta) \sin 2\varphi > - \sin(\theta - 2\eta) < \cos(\theta - 2\eta) \sin 2\varphi > \right.
\;
\left. - (\cot^2 \varphi \partial \eta + \frac{1}{2} \bar{\partial} \theta \partial \varphi \right] . \tag{34}
\]
This anomaly vanishes in the sharp line limit due to the constraint Eq.(8). It also vanishes in the case of 1-soliton and the charge remains conserved. This may be compared with the conserved area of topological solitons in the presence of inhomogenous broadening. It is an open question whether there exists a similar theorem to the area theorem concerning about the stability of pulses with phase variation in terms of charge and anomaly.

We may choose different groups and coset structures for the deformed gauged WZW action\[8\] which in our group theoretical formulation of SIT leads to other cases than the two level SIT. These cases, together with other aspects of SIT which were not considered in this Letter, will appear in a longer version of this Letter\[9\].

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References

[1] S. L. McCall and E. L. Hahn, Phys. Rev. Lett. 18 908 (1967); Phys. Rev. 183 457 (1969).

[2] M. J. Ablowitz, D. J. Kaup and A. C. Newell, J. Math. Phys. 15 1852 (1974).

[3] G. L. Lamb, Jr., Phys. Rev. Lett. 31, 196 (1973).

[4] I. Bakas, Int. J. Mod. Phys. A 9 3443 (1994); Q-H. Park, Phys. Lett. B328 329 (1994).

[5] Q-H. Park and H. J. Shin, “Duality in complex sine-Gordon theory”, to be published in Phys. Lett. B. hep-th/9506087.

[6] F. Lund and T. Regge, Phys. Rev. D14 1524 (1976).

[7] K. Pohlmeyer, Commun. Math. Phys. 46 207 (1976).

[8] Q-H. Park and H. J. Shin, Phys. Lett. B347 73 (1995); “Classical matrix sine-Gordon theory”, hep-th/9505017; T. J. Hollowood, J. L. Miramontes and Q-H. Park, Nucl. Phys. B445 451 (1995).

[9] Q-H. Park and H. J. Shin, to appear.