Non-perturbative Renormalization of Quark Bilinear Operators and $B_K$ using Domain Wall Fermions

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(Dated: December 6, 2007)

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Abstract

We present a calculation of the renormalization coefficients of the quark bilinear operators and the $K - \bar{K}$ mixing parameter $B_K$. The coefficients relating the bare lattice operators to those in the RI/MOM scheme are computed non-perturbatively and then matched perturbatively to the $\overline{\text{MS}}$ scheme. The coefficients are calculated on the RBC/UKQCD 2+1 flavor dynamical lattice configurations. Specifically we use a $16^3 \times 32$ lattice volume, the Iwasaki gauge action at $\beta = 2.13$ and domain wall fermions with $L_s = 16$. 
I. INTRODUCTION

The RBC and UKQCD collaborations have recently performed the first simulations with 2+1 flavor domain wall fermions [1, 2, 3]. Much interesting phenomenology requires the conversion of bare lattice quantities to a less arbitrary and more perturbatively amenable continuum scheme. In particular, this is true for the determination of weak matrix elements such as $B_K$ and for the Standard Model parameters such as quark masses. Of course, physical quantities are independent of the choice of renormalization procedure, nevertheless theoretical predictions are often given in terms of the parameters of the theory ($\alpha_s$ and quark masses) which require renormalization. In addition, for many processes (e.g. $K - \bar{K}$ mixing) the amplitudes are factorized into products of perturbative Wilson coefficient functions and operator matrix elements which contain the long-distance effects. The Wilson coefficients and operator matrix elements need to be combined with both evaluated in the same renormalization scheme. The purpose of this paper is to determine the factors by which matrix elements computed in our numerical simulations should be multiplied in order to obtain those in the $\overline{\text{MS}}$ scheme which is conventionally used for the evaluation of the coefficient functions.

In principle, for a sufficiently small lattice spacing $a$ and a sufficiently large renormalization scale $\mu$, it is possible to perform the renormalization of the bare lattice operators using perturbation theory. However, in practice the coefficients of lattice perturbation theory are frequently large leading to a poor convergence of the series and even with attempts such as tadpole improvement to resum some of the large contributions, it appears that the typical $n$-loop correction is numerically of $O(\alpha_s^n)$, in contrast to continuum perturbation theory where the corresponding contributions are of $O((\alpha_s/4\pi)^n)$. A related difficulty is the choice of the best expansion parameter ($\alpha_s$), for example between some tadpole improved lattice coupling or the $\overline{\text{MS}}$ coupling. In practice, at one-loop order, different reasonable choices can lead to significantly different results. For the quark bilinear operators and $B_K$ considered in this paper, we present the perturbative results and illustrate these points in Section II.

The main purpose of this paper is to avoid the uncertainties present when using lattice perturbation theory by implementing the Rome-Southampton RI/MOM non-perturbative renormalization technique [4]. The key idea of this technique is to define a sufficiently simple renormalization condition such that it can be easily imposed on correlation functions.
in any lattice formulation of QCD, or indeed in any regularization - that is, the condition is regularization invariant (RI). We therefore introduce counter-terms for any regularization such that a Landau gauge renormalized n-point correlation function with standard MOM kinematics at some scale $\mu^2$ has its tree level value. This condition is simple to impose whenever the renormalized correlation function is known in any regularization. It applies equally well to both perturbative expansions to any order and to non-perturbative schemes such as the lattice, and thus RI/MOM is a very useful interface for changing schemes. In particular, only continuum perturbation theory and the lattice regularization are required to obtain physical results from a lattice calculation.

Our choice of lattice action is important for the efficacy of the RI/MOM technique. With domain-wall fermions, $O(a)$ errors and chiral symmetry violation can be made arbitrarily small at fixed lattice spacing by increasing the size of the fifth dimension. This allows us to avoid the mixing between operators which transform under different representations of the chiral-symmetry group; this is a very significant simplification compared to some other formulations of lattice QCD. The action and operators are also automatically $O(a)$ improved.

Another important property of DWF is the existence of (non-local) conserved vector and axial currents. This will be discussed in detail below.

In this paper we study the renormalization of the quark bilinear operators $\bar{\psi}\Gamma\psi$, where $\Gamma$ is one of the 16 Dirac matrices, and of the $\Delta S = 2$ four-quark operator $O_{LL}$. Table I contains a summary of our results, relating bare operators in the lattice theory with Domain Wall Fermions and the Iwasaki gauge action at $\beta = 2.13$ ($a^{-1} = 1.729(28)$ GeV, see Section III for further details) to those in two continuum renormalization schemes. Columns three through five give the three independent $Z$ factors which, when multiplying the appropriate bilinear lattice operator, convert that operator into one normalized according to either the RI/MOM or $\overline{\text{MS}}$(NDR) schemes. The final column contains the combination of factors needed to convert a lattice result for the parameter $B_K$ into the corresponding RI/MOM or $\overline{\text{MS}}$(NDR) value.

The plan of the remainder of the paper is as follows. In the following section (Section II) we start by reviewing the perturbative evaluation of the renormalization constants; the results can later be compared with those obtained using the non-perturbative procedures. In Section III we begin the description of the non-perturbative computations with a brief introduction to the details of our simulation and to the computation of the quark propagators.
which are the basic building blocks for all our subsequent calculations. In Section [IV] we give a short introduction to the regularization independent (RI/MOM) scheme. In this section we also discuss the renormalization of flavor non-singlet bilinear operators, including the check of the Ward-Takahashi identities. The discussion of the renormalization of the four-quark operators and the results for the renormalization constant for $B_K$ are presented in Section [V]. Section [VI] contains a brief summary and our conclusions.

II. PERTURBATION THEORY

Before proceeding to describe our non-perturbative evaluation of the renormalization constants we briefly review the corresponding (mean field improved) perturbative calculations. Specifically, we present perturbative estimates for the renormalization constants of the quark bilinears and $B_K$. These can then be compared to those obtained non-perturbatively below. The ingredients for the perturbative calculations and a detailed description of the procedure can be found in refs. [5, 6].

Writing the domain wall height as $M = 1 - \omega_0$, the bare value of $\omega_0$ in our simulation is $\omega_0 = -0.8$. The mean field improved value of $\omega_0$ is then given by

$$\omega_0^{MF} = \omega_0 + 4(1 - u) \simeq -0.303,$$

where the link variable is defined by $u = \mathcal{P}^{1/4}$ and $\mathcal{P} = 0.588130692$ is the value of the plaquette in the chiral limit.

We define the renormalization constant, $Z_{O_i}$, which relates the bare lattice operator, $O_i^{\text{Latt}}(a^{-1})$, to the corresponding renormalized one in the \text{MS} scheme at a renormalization scale of $\mu = a^{-1}$ by:

$$O_i^{\text{MS}}(a^{-1}) = Z_i O_i^{\text{Latt}}(a^{-1}).$$

Here $i = S, P, V, A, T$ for the scalar and pseudoscalar densities, vector and axial-vector currents and tensor bilinear and $i = B_K$ for the $\Delta S = 2$ operator which enters into the $K^0 - \bar{K}^0$ mixing amplitude (or more precisely for the ratio of the $\Delta S = 2$ operator and the square of the local axial current, which is the relevant combination for the determination of
The one-loop, mean field improved estimates for the $Z_i$ are:

$$Z_{S,P} = \frac{u}{1 - (\omega_0^{MF})^2} \frac{1}{Z_{w}^{MF}} \left( 1 - \frac{\alpha_s C_F}{4\pi} 5.455 \right)$$  \hspace{0.5cm} (3)$$

$$Z_{V,A} = \frac{u}{1 - (\omega_0^{MF})^2} \frac{1}{Z_{w}^{MF}} \left( 1 - \frac{\alpha_s C_F}{4\pi} 4.660 \right)$$  \hspace{0.5cm} (4)$$

$$Z_T = \frac{u}{1 - (\omega_0^{MF})^2} \frac{1}{Z_{w}^{MF}} \left( 1 - \frac{\alpha_s C_F}{4\pi} 3.062 \right)$$  \hspace{0.5cm} (5)$$

$$Z_{B_K} = 1 - \frac{\alpha_s}{4\pi} 1.470,$$  \hspace{0.5cm} (6)$$

where $C_F$ is the second Casimir invariant $C_F = (N^2 - 1)/2N$ for the gauge group $SU(N)$. Here $\sqrt{Z_w}$ is the quantum correction to the normalization factor $\sqrt{1 - \omega_0^2}$ of the physical quark fields (the factors depending on this normalization cancel in the evaluation of $Z_{B_K}$). At one-loop order in perturbation theory

$$Z_w = 1 + \frac{\alpha_s C_F}{4\pi} 5.251.$$  \hspace{0.5cm} (7)$$

In obtaining the coefficients in Eqs. (3) - (7) we have interpolated linearly between the entries for $M = 1.30$ and $M = 1.40$ in tables III and IV of ref. 5 to the mean-field value of $M = 1.303$. Since the mean-field value of $M$ is so close to the quoted values at $M = 1.30$, we prefer this procedure to using the general interpolation formula quoted in 5. The difference between the two procedures is negligible compared to the remaining systematic uncertainties.

In order to estimate the numerical values of the renormalization constants we have to make a choice for the expansion parameter, i.e. the coupling constant $\alpha_s$. Here we consider two of the possible choices, the mean-field value as defined in eq.(62) of ref. 6 and the $\overline{\text{MS}}$ coupling, both defined at $\mu = a^{-1}$. The mean field improved coupling constant is given by

$$\frac{1}{g_0^{2\text{MF}}(a^{-1})} = \frac{\mathcal{P}}{g_0^2} + d_g + c_p + N_f d_f,$$  \hspace{0.5cm} (8)$$

where $g_0$ is the bare lattice coupling constant ($g_0^2 = 6/\beta$), and the remaining parameters are defined in ref. 6 and take the numerical values $d_g = 0.1053$, $c_p = 0.1401$ and for $\omega_0^{MF} = -0.303$, $d_f = -0.00148$. We therefore obtain

$$\alpha_{\text{MF}}(1.729 \text{ GeV}) = 0.1769.$$  \hspace{0.5cm} (9)$$

Such a value of the coupling is significantly lower than that in the $\overline{\text{MS}}$ scheme at the same scale, for which we take, $\alpha_{\overline{\text{MS}}}(1.729 \text{ GeV}) = 0.3138$. 

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The difference in the two values of the coupling constant leads to a significant uncertainty in the estimates of the renormalization constants at this order, as can be seen in Table II. The need to eliminate this large uncertainty is the principle motivation for the use of non-perturbative renormalization. The entries in Table II are the factors by which the matrix elements of the bare lattice operators should be multiplied in order to obtain those in the $\overline{\text{MS}}(\text{NDR})$ scheme at the renormalization scale $\mu = 1.729 \text{GeV}$.

Finally we perform the renormalization group running from $\mu = 1.729 \text{GeV}$ to obtain the normalization constants at other scales, and in particular at the conventional reference scale of $\mu = 2 \text{GeV}$ (see Table III). In each case we use the highest order available for the anomalous dimension; two loops for $B_K$, three loops for the tensor operator and four loop for the scalar/pseudoscalar densities. This is the same procedure which we use for the non-perturbatively renormalized normalization constants below and the details and references to the anomalous dimensions are presented in sections IV F, IV G and V D below. The numbers in Table III are the factors by which the matrix elements of the bare lattice operators should be multiplied in order to obtain those in the $\overline{\text{MS}}(\text{NDR})$ scheme at $\mu = 2 \text{GeV}$. The entries in the first column indicate which coupling was used in matching between the bare lattice operators and the $\overline{\text{MS}}(\text{NDR})$ scheme at $\mu = 1.729 \text{GeV}$, i.e. before the running to other scales.

III. SIMULATION DETAILS

The calculations described below were performed on the 2+1 flavor dynamical lattice configurations generated by the RBC and UKQCD collaborations \cite{2}. The lattices were generated with the Iwasaki gauge action at $\beta = 2.13$ and the domain-wall fermion action with $L_s = 16$. The size of the lattices used in this work is $16^3 \times 32$. The lattice spacing is $a^{-1} = 1.729(28) \text{GeV}$ and the residual mass $m_{\text{res}} = 0.00315(2)$ in lattice units \cite{7}. We have 3 independent ensembles with light sea quark mass 0.01, 0.02 and 0.03 respectively. The strange sea quark mass is fixed at 0.04. For each ensemble, we have used 75 configurations, starting from trajectory number 1000 and with trajectory separation 40.

Following the Rome-Southampton RI/MOM non-perturbative renormalization procedure \cite{4, 8}, the lattices are first fixed in Landau gauge. Then, on each gauge-fixed configuration, we measure the point-point quark propagators $S(x, x_0)$ with periodic
boundary conditions in space and time, where \( x_0 \) is the source position and \( x \) is the sink. We have chosen four different sources to generate the propagators,

\[
x_0 \in \{(0,0,0,0), (4,4,4,8), (7,7,7,15), (12,12,12,24)\}.
\]

Next, a discrete Fourier transform is performed on the propagators,

\[
S(p, x_0) = \sum_x S(x, x_0) \exp\left[-ip \cdot (x - x_0)\right],
\]

where

\[
p_\mu = \frac{2\pi}{L_\mu} n_\mu,
\]

\( n_\mu \) is a four-vector of integers and

\[
L_x = L_y = L_z = 16 \quad L_t = 32.
\]

For the \( n_\mu \) we take values in the ranges

\[
n_x, n_y, n_z \in \{-2, -1, 0, 1, 2\} \quad \text{and} \quad n_t \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}
\]

and require that the squared amplitude of the lattice momenta is in the range \( 0 \leq p^2 \lesssim 2.5 \).

In this paper, for simplicity of notation we frequently use lattice units for dimensionful quantities such as \( p \) and \( m \). When we particularly wish to emphasize the nature of the discretization errors we explicitly reinstate the lattice spacing, writing for example, \((ap)^2\) or \((am)^2\).

**IV. RENORMALIZATION OF QUARK BILINEARS**

We now discuss how Green functions computed on the lattice can be used to obtain the non-perturbative renormalization constants relating bilinear operators defined on the lattice to those normalized first according to the RI/MOM and then the \( \overline{\text{MS}} \) scheme. In the first two subsections below, Sections 4A and 4B we briefly introduce the definitions and notation that we use in the rest of this paper. Some of these are extracted from the earlier RBC paper on quenched lattices \cite{huber06} and are included here for completeness.

The definition of renormalization factors \( Z_q \) and \( Z_m \) for the quark wave function and mass in the RI/MOM scheme are given in Section 4A. The basic amputated quark-bilinear vertex functions are defined in Section 4B and the conditions defining the RI/MOM
scheme are written down. Since our calculations are necessarily performed at finite momenta, the effects of chiral symmetry breaking coming from both the non-zero quark masses and spontaneous chiral symmetry breaking are visible. We discuss these effects in detail in Section IV C for the important case of the vector and axial vector vertex functions.

Next, as a consistency check for our methods, we discuss the accuracy with which our off-shell vertex amplitudes satisfy the axial and vector Ward-Takahashi identities in Sections IV D and IV E respectively. In the later section, the determination of $Z_S/Z_q$ is also discussed. In Section IV G we compare the observed scale dependence of $Z_m$ with that predicted by perturbation theory and interpret the differences as coming from $(a\mu)^2$ errors. These are removed to determine first $Z_{m_{\text{RI/MOM}}}$ and then $Z_{m_{\text{MS}}}$. A similar determination of $Z_{q_{\text{RI/MOM}}}$ and then $Z_{q_{\text{MS}}}$ is presented in Section IV G. Finally, in Section IV H results for the tensor vertex renormalization factor $Z_T$ are obtained.

### A. Quark mass and wavefunction renormalization

First, we define the renormalization coefficients for the quark field and the quark mass as the ratio between the renormalized quantities and their bare counterparts,

$$q_{\text{ren}}(x) = Z_q^{\frac{1}{2}} q_0(x)$$

$$m_{\text{ren}} = Z_m m_0.$$  \hspace{1cm} (15) (16)

where $q_{\text{ren}}$ and $q_0$ are the renormalized and the bare quark wavefunction, and $m_{\text{ren}}$ and $m_0$ are the renormalized and the bare quark mass. With domain-wall fermions,

$$m_0 = m_f + m_{\text{res}}$$  \hspace{1cm} (17)

where $m_f$ is the input quark mass and $m_{\text{res}}$ is the residual mass. The renormalized propagator (in momentum space) is

$$S_{\text{ren}}(p, m_{\text{ren}}) = Z_q S_0(p, m_0)|_{m_0=m_{\text{ren}}/Z_m}$$  \hspace{1cm} (18)

where $p$ is the momentum of the quark propagator.

Since domain wall quarks enter the calculations described here in three different ways we must be careful to clearly distinguish their three distinct masses. As described above, our calculations are performed with 2+1 flavors of dynamical quarks. We will use the variable $m_l$
to label the input mass $m_f$ for the light dynamical quarks and $m_s$ for that of the dynamical strange quark. Since we often evaluate products of propagators which depend on a third quark mass, that mass is labeled $m_{\text{val}}$. In some cases the limit $m_{\text{val}} \to 0$ may be an adequate definition of the chiral limit. However, in order to deal with simple results from which a weak quark mass dependence has been removed we will often consider the “unitary” case $m_{\text{val}} = m_t$ and take the limit $m_{\text{val}} = m_t \to 0$. Of course, in this case $m_s$ remains non-zero but since its value is never changed this causes no immediate confusion. Underlying the validity of the Rome-Southampton renormalization scheme is the use of infrared-regular renormalization kinematics. Therefore, as our renormalization scale $\mu$ becomes larger and future calculations more accurate, even this weak quark mass dependence will completely disappear.

As discussed in detail in [8], the renormalization technique requires the existence of a window of momenta such that

$$\Lambda_{\text{QCD}} \ll |p| \ll a^{-1}. \quad (19)$$

In practice, however, violation of these restrictions, especially at the higher boundary, has to be considered. Due to the spontaneous breaking of the chiral symmetry as illustrated by the non-trivial difference between $Z_q/Z_A$ and $Z_q/Z_V$ at low momenta (see section IV C) we have to rely on the calculation in the relatively high momentum region, where $(ap)^2 \gtrsim 1$. Fortunately, the effects from breaking the restriction imposed by the finite lattice spacing $a$ are small and predictable. They introduce an error of $\mathcal{O}((ap)^2)$ to the renormalization coefficients which can be removed by quadratic fitting to the momentum dependence. A more detailed investigation of this issue is presented in [8].

It is possible in principle to relax the constraint $|p| \gg \Lambda_{\text{QCD}}$ in eq.(19) by performing step scaling, i.e. by matching the renormalization conditions successively to finer (and also smaller in physical units) lattices. This is beyond the scope of this paper.

The regularization independent (RI/MOM) scheme is defined such that by adjusting the renormalization coefficients $Z_q$ and $Z_m$ at the renormalization scale $\mu$, and restricting $p$ in a suitable window, we have:

$$\lim_{m_{\text{ren}} \to 0} - \frac{i}{12} \text{Tr} \left( \frac{\partial S_{\text{ren}}^{-1}(p)}{\partial \phi} \right)_{p^2=\mu^2} = 1 \quad (20)$$

$$\lim_{m_{\text{ren}} \to 0} \frac{1}{12m_{\text{ren}}} \text{Tr} \left( S_{\text{ren}}^{-1}(p) \right)_{p^2=\mu^2} = 1. \quad (21)$$
Imposing these conditions on the lattice and taking into account the dynamical breaking of chiral symmetry at low energies, the additive mass renormalization $m_{\text{res}}$ and $O(a^2)$ lattice artifacts, we have the following asymptotic behavior for relatively large $p^2$:

$$\frac{1}{12} \text{Tr} \left( S_{\text{latt}}^{-1}(p) \right) = \frac{a^3 \langle \bar{q}q \rangle}{(ap)^2} C_1 Z_{q} + Z_m Z_{q} \{am_{\text{val}} + am_{\text{res}} \} + O \left( (ap)^2 \right).$$

(22)

The left-hand-side of Eq. (22) at each of the unitary points ($m_l = m_{\text{val}}$) is calculated as the inverse of the average over all propagators, where the average is performed over all sources and configurations:

$$S(p)^{-1} = \left\{ \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{n_{\text{source}}} \sum_{x_0} S_i(p, x_0) \right] \right\}^{-1}$$

(23)

where $n_{\text{source}} = 4$ and $i \in \{1, 2, \cdots, N\}$ labels each configuration. (For brevity, in this equation and in the following we will suppress the subscript “latt” on all the lattice propagators. In the following text, unless otherwise specified, all the propagators $S(p)$ without a subscript denote the lattice propagators.) Because of possible correlations between propagators with different sources calculated on the same configuration, each group of 4 propagators from the same configuration is considered as one jackknife bin in the single-elimination procedure.

In Figure 1 we plot the results for $\frac{1}{12} \text{Tr}[S^{-1}(p)]$ as a function of the momentum and tabulate the corresponding numerical values in Table IV. In addition to our results for non-zero quark mass, we also plot and tabulate the results extrapolated to the chiral limit where $m_l = m_{\text{val}} = -m_{\text{res}}$ for each momentum. Two quantities of interest can be deduced from the mass dependence shown in Figure 1. First, the chiral limit gives a measure of the spontaneous and explicit chiral symmetry breaking and is given in the left-most column of Table IV. Second determining the slope with respect to $m_{\text{val}}$ provides one method to calculate $Z_m Z_{q}$. This is used later in Section IV E to test the vector Ward Identity which relates this to a second method of computing $Z_m Z_{q}$.

**B. Renormalization of flavor non-singlet fermion bilinears**

We now consider the renormalization of quark bilinear operators of the form $\bar{u} \Gamma d$, where $\Gamma$ is one of the 16 Dirac matrices. The corresponding renormalization constant $Z_{\Gamma}$ is the factor relating the renormalized and bare bilinear operators:

$$[\bar{u} \Gamma d]_{\text{ren}}(\mu) = Z_{\Gamma}(\mu a)[\bar{u} \Gamma d]_0,$$

(24)
where $\mu$ is the renormalization scale and we will treat only local operators where the lattice fields $\bar{u}$ and $d$ in the bilinear operator $[\bar{u}\Gamma d]_0$ are evaluated at the same space-time point.

Following the Rome-Southampton prescription [4] for renormalization in the RI/MOM scheme, we define the bare Green functions between off-shell quark lines, and evaluate their momentum-space counterparts $G_{\Gamma,0}(p)$ on the lattice, averaged over all sources and gauge configurations,

$$G_{\Gamma,0}(p) = \frac{1}{N} \sum_{i=1}^{N} \left\{ \frac{1}{n_{\text{source}}} \sum_{x_0} \left[ S_i(p, x_0) \Gamma \left( \gamma_5 S_i(p, x_0)^\dagger \gamma_5 \right) \right] \right\} . \quad (25)$$

We then amputate this Green function using the averaged propagators,

$$\Pi_{\Gamma,0}(p) = S^{-1}(p) G_{\Gamma,0}(p) \left( \gamma_5 \left[ S^{-1}(p) \right]^\dagger \gamma_5 \right) , \quad (26)$$

where $S^{-1}(p)$ is calculated according to Eq. (23). The bare vertex amplitudes are obtained from the amputated Green functions as follows [4, 8]:

$$\Lambda_S(p) = \frac{1}{12} \text{Tr} \left[ \Pi_1(p) 1 \right] \quad (27)$$

$$\Lambda_P(p) = \frac{1}{12} \text{Tr} \left[ \Pi_{\gamma_5}(p) \gamma_5 \right] \quad (28)$$

$$\Lambda_V(p) = \frac{1}{48} \text{Tr} \left[ \sum_{\mu} \Pi_{\gamma_\mu}(p) \gamma_\mu \right] \quad (29)$$

$$\Lambda_A(p) = \frac{1}{48} \text{Tr} \left[ \sum_{\mu} \Pi_{\gamma_\mu \gamma_5}(p) \gamma_5 \gamma_\mu \right] \quad (30)$$

$$\Lambda_T(p) = \frac{1}{72} \text{Tr} \left[ \sum_{\mu, \nu} \Pi_{\sigma_{\mu \nu}}(p) \sigma_{\nu \mu} \right] . \quad (31)$$

The values of all the five bare vertex amplitudes at the unitary mass points $m_l = m_{\text{val}}$ are presented in Table V through Table VII. Finally, by requiring that the renormalized vertex amplitudes satisfy

$$\Lambda_{i,\text{ren}} = \frac{Z_i}{Z_q} \Lambda_i = 1 , \quad i \in \{ S, P, V, A, T \} , \quad (32)$$

we can calculate the relevant renormalization constants.

Equations (24) through Eq. (32) describe the schematic procedure used to calculate the renormalization coefficients of quark bilinears. In practice however, with finite quark masses and a limited range of momenta, we have to consider lattice artefacts and other systematic uncertainties. We explain the details in the following sections.
C. Chiral symmetry breaking and $Z_A - Z_V$

In this section we examine the effects of both the low-energy spontaneous chiral symmetry breaking present in QCD and our non-zero quark masses on the large-momentum, off-shell propagators which we are using to impose non-perturbative renormalization conditions. A good quantity to study in order to understand these effects is the difference of the off-shell vector and axial vector vertex functions.

1. Numerical results for $Z_A - Z_V$

In the limit of a small mass and a large momentum, we expect

$$Z_A = Z_V,$$

or equivalently,

$$\Lambda_A(p^2) = \Lambda_V(p^2)$$

for $p^2 \gg \Lambda_{QCD}^2, m^2$.

However, with finite quark masses and at relatively low momenta $\Lambda_V$ and $\Lambda_A$ may receive different contributions of the form

$$\frac{m_{\text{val}}^2}{p^2},$$

and

$$\frac{m_{\text{val}} \langle \overline{q}q \rangle}{p^4}.$$

Here we are exploiting the $SU_L(3) \times SU_R(3)$ chiral symmetry of large $L_s$ domain wall fermions which implies that a difference between $\Lambda_V$ and $\Lambda_A$ requires the mixing of $(8, 1)$ and $(1, 8)$ representations and hence involves a product of two quantities which transform as $(3, \overline{3})$ and $(\overline{3}, 3)$. This requires the two powers of $m_{\text{val}}$ in Eq. (35) and the product $m_{\text{val}} \langle \overline{q}q \rangle$ in Eq. (36). The extra factors of $1/p^2$ and $1/p^4$ come from naive dimensional analysis.

To determine how much chiral symmetry breaking is present in our calculation, we examine the relative difference between $\Lambda_A$ and $\Lambda_V$. In Figure 2 we plot the quantity $\frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2}$ as a function of momentum. At relative low momenta, $0.5 \leq (pa)^2 \leq 1$, we observe that this quantity is quite large ($\sim 5\%$). Furthermore, even when we extrapolate $\frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2}$ to the chiral limit, where the terms in Eqs. (35) and (36) both vanish, the difference between $\Lambda_A$ and $\Lambda_V$ does not vanish. Here to obtain the chiral limit shown in Figure 2 we perform a
linear extrapolation $m_{val} + m_{res} \to 0$. While a quadratic extrapolation gives a similar result, this linear choice is motivated by the analysis presented in Section IV C 3.

Since the explicit chiral symmetry breaking effects needed to split $\Lambda_A$ and $\Lambda_V$ can be argued to be $O(m_{res}^2)$, we would not expect this difference to reflect explicit, finite-$L_s$, domain wall chiral symmetry breaking. In fact, similar deviations between $\Lambda_V$ and $\Lambda_A$ are seen on lattice ensembles without fermion loops where explicit domain wall chiral symmetry breaking is expected to be smaller. This is shown in Figure 3 where we plot the same quantities from a quenched simulation using the DBW2 gauge action. Thus, it appears that this difference represents the high energy tail of QCD dynamical chiral symmetry breaking rather than the explicit chiral symmetry breaking coming from the finite value of $L_s$.

While the effects of spontaneous chiral symmetry breaking will not vanish in the limit $m_{val} + m_{res} \to 0$, it is unlikely that the substantial difference found for $\Lambda_A - \Lambda_V$ in the chiral limit can be explained by a dimension-6 condensate such as

$$\frac{\langle \bar{q}q \rangle^2}{p^6}$$

since it is suppressed by six powers of momentum and appears to be too small for the size of the breaking we have observed. We have also fit the quantity $\frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2}$ to different powers of $p$, as is shown in Figure 4, and it is clear that the momentum dependence of the chiral symmetry breaking term is dominated by $p^{-2}$ or $p^{-3}$, very different from $p^{-6}$ that naive dimensional analysis suggests should appear in the $\langle \bar{q}q \rangle^2$ term above.

2. Effects of exceptional momenta

In fact, we believe that the origin of the difference between $\Lambda_A$ and $\Lambda_V$ is different. Our choice of kinematics corresponds to so called “exceptional momenta”, i.e. a momentum transfer is zero. This invalidates the naive power counting estimates used above and permits the low-energy, spontaneous chiral symmetry breaking to split $\Lambda_V$ and $\Lambda_A$ with only a $1/p^2$ suppression for large $p$, as we now explain. Begin by considering a general, amputated Feynman graph $\Gamma$ with $F$ external fermion lines and $B$ external boson lines. Recall that for connected graphs the degree of divergence $d$ of $\Gamma$ is defined as $d = 4 - 3F/2 - B$. If the graph $\Gamma$ is disconnected then its degree of divergence is the sum of those of its connected components. Now imagine that each external line of $\Gamma$ carries an incoming momentum $\lambda p_i$.
for $1 \leq i \leq F + B$, where $\lambda$ is an over-all scale factor. The asymptotic behavior for large $\lambda$ of the amplitude corresponding to such a graph will be $\lambda^{d'}$ where $d'$ is the degree of divergence of a subgraph $\Gamma' \subseteq \Gamma$. This subgraph $\Gamma'$ must be chosen so that i) there exists a routing of the internal momenta within $\Gamma$ such that all lines carrying momenta proportional to $\lambda$ lie within $\Gamma'$ and ii) $\Gamma'$ possesses the least negative degree of divergence $d'$ of all those subgraphs satisfying i) \[10, 11\]. Note, that $\Gamma'$ may equal the original graph $\Gamma$ and may itself be disconnected.

The most familiar situation is the case of non-exceptional momenta, defined as a momentum configuration in which no proper partial sum of the external momenta $p_i$ vanishes. Under these circumstances all subgraphs $\Gamma'$ obeying i) must be connected. (Otherwise there would be zero momentum transfer between the groups of momenta entering each of the disconnected components.) This implies that the subgraph $\Gamma'$ with the least negative degree of divergence is one with no additional external lines beyond those already appearing in $\Gamma$ which in turn implies that this subgraph $\Gamma'$ is the entire graph $\Gamma$. For the case of the vertex graph of interest, we deduce a constant behavior (up to logarithms) since $d = 4 - 1 - 2 \cdot 3/2 = 0$. (Here it is convenient to view this vertex graph as resulting from a normal Feynman graph in which an external vector boson is coupled to the vertex so the rules discussed above directly apply.)

This analysis not only gives the leading asymptotic behavior but also insures that extracting a few extra factors of the mass $m$ or the chiral condensate $\langle \bar{q}q \rangle$ will make the degree of divergence of that graph more negative and hence make its asymptotic fall-off more rapid, in the fashion suggested by naive power counting. For the case of interest, we would like to restrict a subset of the internal fermion lines of our graph $\Gamma$ to carry only low momenta so that they will reflect the low-energy, spontaneous chiral symmetry breaking of QCD. By definition, these low momentum lines cannot enter the subgraph $\Gamma'$ discussed above whose degree of divergence determines the asymptotic behavior of amplitude being studied. In order to split $\Lambda_V$ and $\Lambda_A$, chiral symmetry breaking transforming as an $(8,8)$ under $SU(3) \times SU(3)$ is required. This in turn requires that this low energy, excluded subgraph must be joined to the remainder of the graph by at least four fermion lines.

Such a circumstance is illustrated by the general vertex graph $\Gamma$ in Figure 5 contained in the outer dashed box. Here we have identified a subgraph $\Gamma_2$ which carries only low momenta and can therefore transform as $(8,8)$ even in the limit of vanishing quark mass,
$m_{\text{val}} + m_{\text{res}} = 0$. For the case of non-exceptional momenta, we must apply Weinberg’s theorem to the subgraph $\Gamma'$, enclosed in the inner dashed box, through which, by assumption, all of the large momenta entering the vertex and the two external fermion lines must be routed. Because of its connections to the subgraph $\Gamma_2$, the subgraph $\Gamma'$ has six external fermion lines and one external boson line (connected to the vertex). The resulting degree of divergence is $d' = 4 - 1 - 6 \cdot 3/2 = -6$, justifying the naive $1/p^6$ behavior in Eq. (37).

However, in our case $\Lambda_V$ and $\Lambda_A$ are being evaluated with zero momentum entering the current vertex and with a vanishing sum of the two incoming fermion momenta—a configuration of exceptional momentum. For such a choice of external momenta we can divide the subgraph $\Gamma'$ identified above into two pieces $\Gamma_1$ and $\Gamma_3$. Because the momenta are exceptional with no large momenta entering the vertex, we can route all of the large momenta through $\Gamma_3$. Since $\Gamma_3$ has only four external fermion lines, its degree of divergence is $d_3 = 4 - 4 \cdot 3/2 = -2$ and the $1/p^6$ behavior above has been replaced by the much less suppressed $1/p^2$. If we think of the subgraph $\Gamma_2$ as a generalized chiral condensate $\langle 0|\overline{q}q|0\rangle$ we are seeing the asymptotic behavior

$$\frac{\langle 0|\overline{q}q|0\rangle}{p^2},$$

very consistent with our numerical results. Note the discrepancy in dimensions between Eqs. (37) and (38) will be made up by four powers of $\Lambda_{\text{QCD}}$, the momentum scale to which the subgraph $\Gamma_2$ is restricted.

A simple class of graph allowing this behavior can be seen in Figure 6. Here the large momentum carried by the two external fermion lines can be routed through the gluon propagator that is shown explicitly so that the upper part of the diagram carries only low momenta. The large momentum behavior of the gluon propagator gives the expected $1/p^2$ behavior. The two general fermion propagators shown with the shaded “blobs” carry small momenta and, as suggested by Eq. (22), can show $(3, \overline{3})$ or $(\overline{3}, 3)$ chiral symmetry violation even when $m_{\text{val}} + m_{\text{res}} = 0$.

To confirm this analysis, we have also calculated the difference between $\Lambda_A$ and $\Lambda_V$ with non-exceptional momenta. We have chosen 5 different momentum scales, each corresponding to a set of momenta which satisfy the condition $p_1^2 = p_2^2 = (p_1 - p_2)^2 = p^2$ for five values of $p^2$, as listed in Table VIII and Table IX. We then calculated $\Lambda_A$ and $\Lambda_V$ with the two external fermions carrying respectively $p_1$ and $p_2$. The result is plotted in Figure 7 which
shows that the chiral symmetry breaking vanishes almost completely with non-exceptional kinematics at medium to large momenta.

While it would be more satisfactory to perform the calculations presented in this paper using non-exceptional momenta, the resulting RI/MOM normalization conditions would not correspond to those for which perturbative matching calculations have been carried out. Thus, we would not be able to relate the quantities which we calculated to those defined in the $\overline{MS}$ scheme. (Of course, this difficulty will be removed when the necessary perturbative calculations have been extended to non-exceptional kinematics.) A second, less significant advantage of the exceptional momenta which we use is that the exceptional momentum conditions are satisfied by a much larger set of discrete lattice momenta permitting the RI/MOM condition to be satisfied for more fine-grained sequence of energy scales.

We now return to the calculation with exceptional momenta ($p_1 = p_2$), at the scale which we are most interested in, that is $\mu \simeq 2$ GeV or $(ap)^2 \simeq 1.3$, where $\Lambda_A$ and $\Lambda_V$ have a difference of about 1%. Since we have no means to determine which of these two quantities has less contamination from low energy chiral symmetry breaking we have decided to take the average $\frac{1}{2}(\Lambda_A + \Lambda_V)$ as the central value for both $Z_q/Z_A$ and $Z_q/Z_V$. The difference between $\Lambda_A$ or $\Lambda_V$ and $\frac{1}{2}(\Lambda_A + \Lambda_V)$ then provides an estimate for one systematic error in our final results. The value of $\frac{1}{2}(\Lambda_A + \Lambda_V)$ is plotted in Figure 8.

3. Chiral extrapolation to vanishing quark mass

As discussed above, our use of exceptional momenta implies a $1/p^2$ suppression for both terms behaving as $m_{\text{val}}$ and $m_{\text{val}}^2$. The added dimension of a $m_{\text{val}}^2$ term can be provided by a factor of $1/\Lambda_{\text{QCD}}$ without the need to introduce additional inverse powers of $p$. For our largest value of $m_{\text{val}}a = 0.03$, we might estimate $m_{\text{val}}/\Lambda_{\text{QCD}} \approx 0.2$. This suggests that we should expect a linear rather than quadratic behavior in $m_{\text{val}}$ to dominate the small quark mass limit.

The difference $\Lambda_A - \Lambda_V$ discussed above provides a good place to study this effect. This difference reflects the chiral symmetry breaking of interest and may make these effects stand out with possibly reduced errors because of the statistical correlations between the two quantities being subtracted. Figure 9 compares linear and quadratic fits to the dependence on the quark mass evaluated at unitary points with $m_{\text{val}} = m_t + m_{\text{res}}$ for $p = 2.04$ GeV. In
Table X we present the results of these two fits:

\[
\frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2} = c_0 + c_1 \frac{m\Lambda_{QCD}}{p^2},
\]

(39)

\[
\frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2} = c_0 + c_2 \frac{m^2}{p^2},
\]

(40)

for \(\Lambda_{QCD} = 319.5\) MeV. As can be seen in the Table the linear fits are favored. The linear fits have the smaller \(\chi^2\) and the coefficient \(c_1\) is significantly closer to an expected value of 1 than is the coefficient \(c_2\). Thus, based on both the theoretical expectation and this empirical evidence, we will adopt this linear description in the remainder of this paper and extrapolate our exceptional momentum amplitudes to the chiral limit using a linear ansatz. For the case at hand, Figure 10 shows this linear extrapolation for the average \(\frac{1}{2} (\Lambda_A + \Lambda_V)\) to the chiral limit for the momentum \(p = 2\) GeV. Figure 8 shows the results in the chiral limit as a function of momentum. The results in the chiral limit are also presented in Table XI.

D. Axial Ward-Takahashi identity

Performing an axial rotation on the propagator leads to a relation between \(\Lambda_P\) and \(\text{Tr} (S^{-1})\), the axial Ward-Takahashi identity [8]:

\[
\Lambda_P (p) = \frac{1}{12} \frac{\text{Tr} [S^{-1} (p)]}{(m_{val} + m_{res})}.
\]

(41)

In a truly chiral theory or the present DWF calculation in the limit \(L_s \to \infty\) (when chiral symmetry becomes exact and \(m_{res} = 0\)), this identity will be obeyed configuration by configuration. However, for finite \(L_s\) and \(m_{res} \neq 0\), this relation will hold only after a gauge field average, (e.g. \(m_{res}\) is only defined after such an average). Thus, we should check Eq. (41) on gauge-averaged amplitudes.

Figure 11 shows the difference between the l.h.s and r.h.s of Eq. (41), divided by their average. The case of \(m_l = 0.01\) shows relatively larger breaking of \(\leq 8\%\), while the other two masses result in a smaller breaking. Since for \(m = 0.01\), the \(m_{res}\) term, with a value of 0.00308, represents a 30\% effect, this suggests that the use of \(m_{res}\) in the context of Eq. (41) may be accurate at the 25\% level for this lattice spacing. Note, we expect violations coming from a dimension-five, anomalous chromo-magnetic term to be suppressed by a factor of \((pa)^2\) relative to those from \(m_{res}\), making this \(\leq 8\%\) estimate comfortably smaller than the
naive estimate of $(m_{\text{res}}/0.01) \cdot (pa)^2 \approx 30\%$. However, the suggested growth in the size of these violations with increasing $(pa)^2$ may be visible in Figure 11.

E. Vector Ward-Takahashi identity and the chiral limit of $\Lambda_S$

Similar to the case with $\Lambda_P$, from the continuum vector Ward-Takahashi identity, we have the relation between $\Lambda_S$ and $\text{Tr} \left(S^{-1}(p)\right)$ [8]:

$$\Lambda_S = \frac{1}{12} \partial \text{Tr} \left[S^{-1}(p)\right] \partial m_{\text{val}}. \quad (42)$$

We are able to check our data against this identity using the three sources, (0,0,0,0), (4,4,4,8) and (12,12,12,24) since it is only for these three sources that multiple valence mass data are available for each sea quark mass.

Figure 12 shows the difference between the two sides of Eq. (42) divided by their average. For all three sea quark masses the data agrees well with the vector Ward-Takahashi Identity (Eq. (42)) for medium to large momenta. Equation (42) implies the relation

$$Z_m = \frac{1}{Z_S}, \quad (43)$$

and will use this equation and a calculation of $\Lambda_S$ to determine the mass renormalization factor in the following sections.

To extrapolate $\Lambda_S$ to the chiral limit, we will improve upon the discussion in [8] in two regards. First, as explained above, we will exploit the asymptotic properties of Feynman amplitudes evaluated at exceptional momenta and assume that the leading mass dependence in the chiral limit will be linear in $m$. This is different from the $m^2$ dependence assumed in Ref. [8] where dimensional arguments, appropriate to the non-exceptional case and leading to the $m^2$ behavior in Eq. 35 were adopted.

Second, in contrast to that earlier quenched calculation we can examine the behavior of $Z_S$ as a function of both the valence and light dynamical quark masses, $m_{\text{val}}$ and $m_l$ respectively. In Figure 13 we plot $\Lambda_S$ as a function of both $m_{\text{val}}$ and $m_l$. The three curves are each a linear plus double pole fit to the valence quark mass dependence of the form:

$$\Lambda_S(m_{\text{val}}, m_l) = c_0(m_l) + c_1(m_l)m_{\text{val}} + c_{dp} \frac{m_l^2}{m_{\text{val}}}, \quad (44)$$

where we have allowed the coefficients $c_0$ and $c_1$ of the constant and linear terms to vary with the dynamical light quark mass. However, we have used a common double-pole term with
the $m_l^2$ behavior expected for a theory with two light flavors. Recall that this double pole term arises from topological near-zero modes [8] which for two light flavors will be suppressed by two powers of the light quark mass. The data in Figure 13 shows just this behavior with the sharp turn-over at the smallest value of $m_{val}$ increasing as the light dynamical mass $m_l$ increases.

This double-pole can be deduced from Eq. 42. As discussed in Ref. [8], the NLO, $1/p^2$ term derived from an operator product expansion of the quark propagator on the right-hand side of this equation is proportional to the chiral condensate $\langle \bar{q}q \rangle$ [12, 13]. Isolated, topological, near-zero modes of the sort that arise from a gauge field background with non-zero Pontryagin index will contribute a term to the chiral condensate which behaves as $1/m_{val}$. This implies that the derivative in Eq. 42 will yield the double pole, $1/m_{val}^2$ hypothesized in Eq. 44. Such a near-zero mode will also introduce a factor of $m_l$ into the fermion determinant of the QCD measure for each light flavor in the theory, hence the expected factor of $m_l^2$ in the numerator of Eq. 44. In Figure 14 we show the variation of the double pole coefficient with the momentum at which the coefficient of the double pole was extracted. Also shown in this figure is a fit to the expected $1/p^2$ behavior which describes the results very well.

This understanding of the double pole terms suggests that a good strategy for extracting the chiral limit of $\Lambda_S$ first takes the limit of vanishing $m_l$ to remove this NLO double pole term and then extrapolates to $m_{val} = 0$. In the present case, we perform the simpler linear fit using the unitary points to obtain $\Lambda_S = Z_q/Z_S$ since we do not have the complete partially quenched results for each of our four sources.

**F. Mass renormalization and renormalization group running**

To calculate the mass renormalization constant $Z_m$, as defined in Eq. (16), we can either directly take the derivative of $\text{Tr} \left[ S_{\text{latt}}^{-1}(p) \right]$, by following Eq. (22),

$$Z_m Z_q = \frac{1}{12} \frac{\partial \text{Tr} \left[ S_{\text{latt}}^{-1} \right]}{\partial m_{val}} \quad (45)$$

or we can use the Ward-Takahashi identity,

$$Z_m = \frac{1}{Z_S} \quad (46)$$
With the analysis described in the above sections, we find that the method with the smallest statistical uncertainty is to use $1/Z_S$ as the value of $Z_m$. To remove the factor $Z_q$ from $Z_q/Z_S$ (which is equal to $\Lambda_S$ and can be calculated as described in Section IV B), we use the ratio $Z_q/Z_A$ calculated in Section IV C, as well as the value $Z_A = 0.7161(1)$ obtained in Ref. [7] using hadronic matrix elements. We therefore determine $Z_m$ in the RI/MOM scheme by computing separately the three factors on the right-hand side of

$$Z_m^{\text{RI/MOM}}(p) = \left[\frac{Z_q}{Z_S}(p)\right] \left[\frac{Z_A}{Z_q}(p)\right] \left(\frac{1}{Z_A}\right).$$

(47)

Table XII contains the values of $Z_m^{\text{RI/MOM}}(p)$ for a variety of momentum scales.

After obtaining the lattice value of $Z_m^{\text{RI/MOM}}$ at different momenta, we divide it by the predicted renormalization group running factor to calculate the scale invariant quantity $Z_m^{\text{SI}}$. The four-loop running formula we use is [14]:

$$Z_m^{\text{SI}} = \frac{c(\alpha_s(\mu_0)/\pi)}{c(\alpha_s(\mu)/\pi)} Z_m^{\text{RI/MOM}}(\mu)$$

(48)

where $\mu_0$ is chosen such that $(a\mu_0)^2 = 2$, a value that lies within the fitting range used below. For completeness we present in the appendices the detailed procedure for running $\alpha_s$ at four-loops (Appendix A) and the form of running factors (Appendix B).

As Figure 15 shows, the quantity $Z_m^{\text{SI}}$ is remarkably independent of the scale $\mu$. However, in spite of the name, for other cases, the scale-invariant $Z$ factors do show noticeable scale dependence and an additional correction is warranted. (See, for example, Figure 17.) We believe that the primary reason for this lack of scale invariance is the presence of lattice artifacts, namely the finite lattice spacing which introduces a small error of $O((a\mu)^2)$. Such an error can be reduced by removing the $\mu^2$ dependence in $Z_m^{\text{SI}}$. To do this we fit this momentum dependent $Z_m^{\text{SI}}$ to the form $A + B (a\mu)^2$ over the momentum range $1.3 < (a\mu)^2 < 2.5$ and then take the $(a\mu)^2 \to 0$ limit of that fit to remove the $\mu^2$ momentum dependence. We interpret the outcome as the true $Z_m^{\text{SI}}$. Note, we are ignoring possible $\mu$ dependence arising from the absence of higher order terms in the matching factor. Such scale dependence can only be removed by even higher order computation of the perturbative matching factor and such a correction is expected to be very small. While this procedure represents a negligible correction for this case of $Z_m^{\text{SI}}$, it will have a more significant effect in the cases considered below.
Our ultimate goal is to determine $Z_{m}^{\overline{MS}}$ which connects the bare lattice quark mass to its continuum counterpart defined according to the $\overline{MS}$ scheme, at the renormalization scale $\mu = 2$ GeV, because the corresponding continuum renormalization is conventionally done in this scheme. So we again use Eq. (48) to calculate $Z_{m}^{RI/MOM}(2 \text{ GeV})$ from the scale-independent value of $Z_{m}^{SI}$. Then we multiply it with the three-loop matching factor, which will also be explained in Appendix B to match the $Z_{m}^{RI/MOM}(2 \text{ GeV})$ to the $\overline{MS}$ scheme. The final step is shown in Figure 16 and the results are given in Table XII. The renormalization constant at the desired scale is

$$Z_{m}^{\overline{MS}}(2 \text{ GeV}) = 1.656 \pm 0.048 \text{ (stat)} \pm 0.150 \text{ (sys).} \quad (49)$$

The systematic error is determined by adding in quadrature our estimates of three different types of systematic error which we will now discuss.

The first is the effect on $Z_{m}$ of the difference between determining $Z_{q}/Z_{A}$ from $\frac{1}{2}(A_{A} + A_{V})$ or from $A_{A}$. This contributes an error of $\pm 0.011$ to $Z_{m}^{\overline{MS}}(2 \text{ GeV})$. Next we must assign an error to our use of three-loop matching factor, given in Eq. (B6). Here we assign an error equal to the magnitude of the final, order $\alpha_{s}^{3}$ error in this perturbative expression, which is $\pm 0.103$. While this may be a conservative estimate of the omitted terms of order $\alpha_{s}^{4}$ and higher, it also is intended to include the errors introduced by the order $\alpha_{s}^{3}$ estimate of the perturbative running determine the intermediate SI step used to remove the $(a\mu)^{2}$ errors.

Finally we address the errors arising from our failure to extrapolate to the limit of vanishing strange quark mass. Recall, we have evaluated the chiral limit in which both the valence quark mass which enters our off-shell propagators and the dynamical light quark mass are extrapolated to zero. However, all of the gauge ensembles used in this calculation were computed with a non-zero strange quark mass $m_{s} = 0.04$. Since we are matching our Green functions to those computed in perturbation theory in the mass-independent, $m \rightarrow 0$, limit our non-zero value for $m_{s}$ implies an additional systematic error. Because the dynamical quarks enter only through loops, their effect is different from that of the valence quarks discussed above. They do not contribute chiral symmetry breaking effects in our matrix elements. However, because of low energy chiral symmetry breaking, we do expect the dynamical quark masses to appear linearly in a quantity such as $Z_{m}$ in the limit $m \rightarrow 0$. To estimate the size of this $O(m_{s})$ effect, we begin with the size of the observed
linear dependence, $\partial Z_m / \partial m \approx 5.4$ which comes from both the calculated valence and light dynamical mass dependence of $\Lambda_S$. This is then multiplied by $1/2$ because there is only one flavor of strange quark and by $m_s = 0.04$ giving an error in $Z_m$ of $\pm 0.108$. The total systematic error given in Eq. 49, $\pm 0.150$, is then the sum of these three errors in quadrature.

G. Quark wavefunction renormalization and renormalization group running

In Section IV C we calculated the ratio of renormalization constants

$$\frac{Z_q}{Z_A} = \frac{1}{2} (\Lambda_A + \Lambda_V). \tag{50}$$

To calculate $Z_q$, we multiply this quantity with $Z_A = 0.7161(1)$ obtained in Ref. 4. Thus, we have evaluated the quantity $Z_q$ in the RI/MOM scheme, which is shown in Table XIII. To calculate $Z_q$ in the $\overline{\text{MS}}$ scheme, we follow a similar procedure as in the previous section, and start by dividing $Z_q^{\text{RI/MOM}}$ by the perturbative running factor. As shown in Appendix C, the functional form of the running factor is quite similar to that of $Z_m$. The energy scale $\mu_0$ where $Z_q^{\text{SI}}$ is fixed to the $Z_q^{\text{RI/MOM}}$ value is again chosen such that $(a\mu_0)^2 = 2$.

The calculated values of $Z_q^{\text{SI}}$ vary slightly with momentum due to the presence of lattice artifacts. To remove these, we again fit the dependence to the form $A + B (a\mu)^2$ and extrapolate to $a = 0$. The procedure is shown in Figure 17. Finally, we take the scale-invariant $Z_q^{\text{SI}}$, run up to different scales in the RI/MOM scheme, and then apply the perturbative matching factor (Appendix C) to translate it to the $\overline{\text{MS}}$ scheme. The $\overline{\text{MS}}$ values are shown in Figure 18 and Table XIII. Of particular interest, the value at $\mu = 2$ GeV is

$$Z_q^{\overline{\text{MS}}} (2 \text{ GeV}) = 0.7726 \pm 0.0030 \text{ (stat)} \pm 0.0083 \text{ (sys)} \tag{51}$$

The systematic errors are estimated using the same procedure explained in Section IV F. They are the sum in quadrature of the estimated errors arising from the difference $\Lambda_A - \Lambda_V$ (0.0061), the use of a perturbative matching factor accurate to order $\alpha^3$ (0.0045) and our use of a non-zero sea quark mass (0.0035).

H. Tensor Current Renormalization and Renormalization Group Running

To calculate the tensor current renormalization constant $Z_T$, we follow a procedure similar to those of the previous two sections. For each dynamical quark mass, we combine the ratios
\[ \frac{Z_{\text{RI/MOM}}^T}{Z_A}(p) = \left[ \frac{Z_T}{Z_q}(p) \right] \left[ \frac{Z_q}{Z_A}(p) \right] . \]

Ultimately, we use the independent hadronic matrix element calculation of \( Z_A \), which gives \( Z_A = 0.7161(1) \), to obtain \( Z_T \). Table XIV shows the values obtained for the \( Z_{\text{RI/MOM}}^T \) in the chiral limit for a range of lattice momenta. As discussed above we have performed the chiral extrapolation using a linear functional form, and Figure 19 shows this linear extrapolation at the lattice momentum \( (a\mu)^2 = 1.388 \).

We obtain SI values for \( Z_T \) in the chiral limit by dividing out the tensor current perturbative running factor, the evaluation of which is described in Appendix E. Again, the SI values obtained in this way exhibit a dependence on the lattice momenta, and again we fit the momentum-dependent \( Z_T^{\text{SI}} \) to the form \( A + B(a\mu)^2 \) and extrapolate to \( (a\mu)^2 \rightarrow 0 \) to remove the lattice artifacts, as shown in Figure 20. Finally, we run the scale-invariant \( Z_T/Z_A \) back to different scales in the RI/MOM scheme and use the perturbative matching factor (Appendix E) to match to the \( \overline{\text{MS}} \) scheme. The \( \overline{\text{MS}} \) values are shown in Figure 21 and Table XIV. At \( \mu = 2\text{GeV} \), we obtain:

\[ Z_{\text{MS}}^T(2\text{GeV}) = 0.7950 \pm 0.0034(\text{stat}) \pm 0.0150(\text{sys}) . \]

The systematic errors are determined in the same fashion as in the previous two sections. Specifically the errors arising from the difference \( \Lambda_A - \Lambda_V \) (0.0054), the use of a perturbative matching factor accurate to order \( \alpha \) (0.014) and our use of a non-zero sea quark mass (0.0003) are added in quadrature.

V. RENORMALIZATION COEFFICIENTS FOR \( B_K \)

A. General procedure for computing the mixing coefficients

By the renormalization of \( B_K \) we mean the calculation of the renormalization coefficient for the operator

\[ \mathcal{O}_{V_V + A_A} = (\bar{s}\gamma^\mu (1 - \gamma_5) d) (\bar{s}\gamma_\mu (1 - \gamma_5) d) \]

which is the operator responsible for the mixing between \( K^0 \) and \( \bar{K}^0 \). Since for finite \( L_s \) our theory does not posses exact chiral symmetry we must consider the possibility that this
operator can mix with the four other $\Delta S = 2$ operators with a different chiral structure:

$$
\mathcal{O}_{VV-AA} = (\bar{s}\gamma^\mu (1 - \gamma_5) d) \left( \bar{s}\gamma_\mu (1 + \gamma_5) d \right) \tag{53}
$$

$$
\mathcal{O}_{SS+PP} = (\bar{s}(1 - \gamma_5) d) \left( \bar{s}(1 + \gamma_5) d \right) \tag{54}
$$

$$
\mathcal{O}_{SS-PP} = (\bar{s}(1 - \gamma_5) d) \left( \bar{s}(1 + \gamma_5) d \right) \tag{55}
$$

$$
\mathcal{O}_{TT} = (\bar{s}\sigma^{\mu\nu} d) \left( \bar{s}\sigma_{\mu\nu} d \right) \tag{56}
$$

where they are labeled by the chirality structure of the even-parity components. The odd-parity components of these operators are not important here since they don’t contribute to $K^0 \leftrightarrow \overline{K^0}$ mixing.

For domain-wall fermions, the mixing of $\mathcal{O}_{VV+AA}$ with these four operators with wrong chirality should be strongly suppressed by $\mathcal{O} \left( m_{\text{res}}^2 \right)$. However, chiral perturbation theory predicts that the $B$ parameters of the operators with the wrong chirality diverge in the chiral limit $\sim \frac{1}{m_s}$. To address this issue, we will describe a theoretical argument to estimate the size of the mixing terms and an actual calculation of these chirality-violating mixing coefficients on the 2+1 flavor dynamical lattices.

Following the Rome-Southampton prescription $\{4, 15\}$, we first calculate the $5 \times 5$ matrix,

$$
M_{ij} = \hat{P}_j \left[ \Gamma^{\text{latt}}_i \right] = (\Gamma^{\text{latt}}_i)_{ABCD}^{\alpha\beta\gamma\delta} \left( \hat{P}_j \right)_{\beta\alpha\delta\gamma} \tag{57}
$$

where $\Gamma^{\text{latt}}_i$ is the amputated, four-point Green function. The Green functions are first averaged over all sources and configurations, and then amputated using the averaged propagator, in a procedure similar to the calculation of two-point amputated Green functions $\Pi_{\Gamma,0} (p)$ in Section $\{4, 15\}$. $\hat{P}_j$ is a suitable projector, which projects out the component with the expected chirality (for example, the projector corresponding to $\mathcal{O}_{VV+AA}$ is $\gamma^\mu \otimes \gamma_\mu + \gamma^\mu \gamma_5 \otimes \gamma_\mu \gamma_5$). The subscripts $i, j \in \{VV + AA, VV - AA, SS - PP, SS + PP, TT\}$.

It is straightforward to calculate the mixing matrix at tree level which we denote as:

$$
F_{ij} = \hat{P}_j \left[ \Gamma^{\text{tree}}_i \right]. \tag{58}
$$

The RI/MOM renormalization condition which we adopt is then:

$$
\frac{1}{Z^2_q} Z_{ij} M_{jk} = F_{ik} \tag{59}
$$

or

$$
\frac{1}{Z^2_q} Z = FM^{-1}. \tag{60}
$$
B. Theoretical argument for the suppression of mixing coefficients

As can be seen from the structure of the four operators in Eqs. (53), (54), (55) and (56), if they are to mix with $\mathcal{O}_{VV+AA}$ defined in Eq. (52) then two quark fields must change chirality from left- to right-handed. For domain wall fermions such a mixing can arise from the explicit breaking of chiral symmetry coming from the finite separation between the left and right walls. The asymptotic behavior for large $L_s$ of the resulting mixing coefficients can be estimated using the transfer matrix $T$ for propagation in the $s$-direction introduced by Furman and Shamir [17]. The large-$L_s$ limit is then controlled by matrix elements of the operator $T^{L_s}$ and is dominated by those four-dimensional fermion modes corresponding to eigenvalues of the transfer matrix which lie near unity.

As described in detail in Ref. [18] and in the original references cited therein, these fermion modes are believed to fall into two classes: modes localized in space-time with corresponding $T$ eigenvalues falling arbitrarily close to unity and de-localized modes characterized by a mobility edge $\lambda_c > 0$ and with eigenvalues of $T$ lying below $e^{-\lambda_c}$ [19, 20, 21, 22]. Since two quarks must change chirality to produce the required operator mixing, for the case of de-localized modes such mixing will be suppressed by the two factors of $e^{-\lambda_c L_s}$ needed for the propagation between the left and right walls of these two fermions, consistent with our estimate above that such effects should be of order $m_{\text{res}}^2$.

However, the effects of the localized modes are more subtle. We must address the possibility raised by Golterman and Shamir [23] that the contribution of such a mode to $m_{\text{res}}$ is suppressed because such modes are relatively rare and the necessary coincidence with the location of the operators being mixed is unlikely. However, if present, such a mode can mix right- and left-handed fermions with little further suppression since the corresponding $T$ eigenvalue may be very close to unity. This raises the possibility that a single such mode, suppressed by a single factor of $m_{\text{res}}$ might be occupied by the two different quark flavors to provide the double chirality flip needed to mix the operators. Fortunately, as argued in Refs. [9] and [15], this is not possible because the mixing in question requires both a quark and an anti-quark or two quarks of the same flavor to propagate across the fifth dimension. This requires two distinct modes and hence incurs the double suppression which is well represented by the $m_{\text{res}}^2$ estimate above. Note, $m_{\text{res}}^2 \approx 10^{-5}$ which will introduce $O(0.1\%)$ errors in current calculations of $B_K$ [24] and will be too small to be seen in non-perturbative
C. Lattice calculation of mixing coefficients

With the procedure described in Section V A, we can directly calculate the mixing coefficients. In particular, we have calculated the off-diagonal terms in the matrix $FM^{-1}$. Figure 22 shows the mixing coefficient $FM_{V + AA, V - AA}^{-1}$ at different unitary masses. As in the earlier discussion of the $\Lambda_A - \Lambda_V$ difference, our use of exceptional external momenta permits both a linear and quadratic mass dependence. As was found in Ref. [15] and suggested by the mass dependence seen in Figure 23, a linear dependence appears reasonable and it is a linear form that we have used in determining the chiral limit shown in Figure 22.

As can be seen in Figure 22, at the chosen reference scale, $\mu \simeq 2$ GeV or $(\alpha p)^2 \simeq 1.4$, the mixing coefficient is about 0.7% and decreases when the scale is made larger. Similar to the discussion in Section IV C, we again propose that this non-zero mixing coefficient in the chiral limit has its source in our use of exceptional momenta. Again we can determine the asymptotic behavior of the amplitude in question by determining the least negative degree of divergence of a subgraph $\Gamma'$ through which all of the large external momenta can be arranged to flow. Here it is convenient to treat the operator $O_{LL}$, which is evaluated at zero momentum, as an internal vertex of dimension 6 rather than an unusual sort of external line. This alters the rules for computing the degree of divergence of a subgraph: now any connected subgraph with $F$ external fermion lines and $B$ external boson lines in which this new $O_{LL}$ vertex appears, must have degree of divergence $d = 6 - 3F/2 - B$ since $O_{LL}$ has a dimension two higher than the usual renormalizable coupling. (As before, the degree of divergence of a disconnected graph is the sum of the degrees of divergence of its connected components.)

As in the case of the vertex amplitude discussed in Section IV C, the appearance of exceptional momenta does not change the asymptotic behavior in the large $\lambda$ limit with external momenta $\lambda p_i$ for $1 \leq i \leq 4$. Even for our exceptional case $p_1 = p_3 = -p_2 = -p_4$, $\lambda^d$ scaling with $d = 6 - 4 \cdot 3/2 = 0$ is expected. However, derivatives with respect to the quark mass or the occurrence of factors of $\langle \overline{q}q \rangle$ will be strongly affected by this choice of external momenta. As is shown in Figure 24, we can identify a disconnected subgraph $\Gamma'$ through which all the large external momenta can be routed which has $d = (6 - 4 \cdot 3/2) +$
\[(4 - 4 \cdot 3/2) = -2.\] (Note, momentum conservation implies that if all of the large momenta can be routed within a disconnected diagram then the choice of external momenta must be exceptional.) This \(d = -2\) value implies a \(1/p^2\) behavior with only low momenta flowing through the omitted subgraph \(\Gamma_1\). Since \(\Gamma_1\) has four external lines it can translate standard QCD vacuum symmetry breaking into the chiral symmetry breaking that is required to produce the operator mixing shown in Figure 22.

Again, we confirm this conclusion, by recomputing the coefficient \(FM_{VV+AA,VV-AA}^{-1}\) at non-exceptional momenta, as shown in Figure 25. With that choice of momenta the mixing coefficient vanishes completely within our statistical accuracy.

The other chiral symmetry breaking mixing coefficients, \(FM_{VV+AA,SS\pm PP}^{-1}\) and \(FM_{VV+AA,TT}^{-1}\) are very similar to the case of \(FM_{VV+AA,VV-AA}^{-1}\) just discussed. These coefficients are plotted in Figure 26 to Figure 29. Since our theoretical argument implies that the mixing coefficients are very small, \(i.e.\) \(O(m_{\text{res}}^2)\) and our numerical results are consistent with this implication, it is safe to neglect them and calculate the renormalization coefficient for \(B_K\):

\[
Z_{B_K}^{\text{RI/MOM}} = \frac{Z_{VV+AA,VV+AA}}{Z_A^2}.
\]

(D. Calculation of \(Z_{B_K}\) and renormalization group running

Using Eq. (61), the value of \(Z_q/Z_A\) from Section IV C and the value of \(Z_{VV+AA,VV+AA}^{-1} = FM_{VV+AA,VV+AA}^{-1}\), we can calculate the lattice values of \(Z_{B_K}\) at different masses and momenta, as shown in Table XVI. To extrapolate to the chiral limit, we again use a linear function, for the same reasons as described in Section IV C. The linear mass fit at the scale \(\mu = 2\) GeV is illustrated in Figure 30 and the value of \(Z_{B_K}^{\text{RI/MOM}}\) in the chiral limit is shown in Figure 31 and Table XVII.

Similar to the procedure described in Section IV F in order to determine \(Z_{B_K}^{\overline{\text{MS}}}\) from \(Z_{B_K}^{\text{RI/MOM}}\), we first divide the \(Z_{B_K}^{\text{RI/MOM}}(\mu)\) by the predicted running factor at one-loop order and obtain the quantity \(Z_{B_K}^{\text{SI}}(\mu)\). Then we fit a quadratic function \(A + B (a\mu)^2\) over the region \(1.3 < (a\mu)^2 < 2.5\) to remove the \(O((a\mu)^2)\) dependence from \(Z_{B_K}^{\text{SI}}\) induced by the lattice artifacts. Finally, we restore its perturbative running in the \(\overline{\text{MS}}\) scheme to the scale \(\mu = 2\) GeV. The perturbative running and matching factors are presented in Appendix D.

The procedure of dividing by the running and removing the \((a\mu)^2\) dependence is shown
in Figure 32 and the result of restoring the running in the $\overline{\text{MS}}$ scheme is shown in Figure 33. Table XVI lists $Z_{B_K}^{\overline{\text{MS}}}$ at different momentum scales. The final $Z_{B_K}$ we need in the $\overline{\text{MS}}$ scheme and $\mu = 2 \text{ GeV}$ is

$$Z_{B_K}^{\overline{\text{MS}}} (2 \text{ GeV}) = 0.9276 \pm 0.0052(\text{stat}) \pm 0.0220(\text{sys}).$$

(62)

The systematic error is calculated, following the same procedure as has been used for the previous quantities, as a sum in quadrature of the amount the result changes when $\frac{1}{2} (\Lambda_A + \Lambda_V)$ is replaced by $\Lambda_A (0.0131)$, the size of the highest order perturbative correction being made (here of $O(\alpha_s)$) (0.0177) and the effect of our non-zero value for $m_s$ in the calculation of $Z_{B_K}$ (0.0007).

VI. CONCLUSIONS

We have presented a study of the renormalization coefficients $Z_q$, $Z_m$, $Z_T$ and $B_K$ on the $16^3 \times 32$, 2+1 flavor dynamical domain-wall fermion lattices with Iwasaki gauge action of $\beta = 2.13$ and $a^{-1} = 1.729(28) \text{ GeV}$ generated by the RBC and UKQCD collaborations. These coefficients are important components in calculations of a number of important physical quantities reported elsewhere [3, 24]. The procedure closely follows that used in an earlier study with quenched lattice configurations [8, 15]. In addition to providing the $Z$-factors necessary to support a variety of calculations on these lattice configurations, this paper also presents a number of new results which go beyond earlier work.

First, the troublesome double pole which appears in a quenched calculation of the quantity $\Lambda_S$ because of topological near zero modes is now highly suppressed by the 2+1 flavor determinant. This allows us to use $\Lambda_S$ for an accurate calculation of $Z_m$. Second we have identified the $O(5\%)$ large chiral symmetry breaking effects seen in the off-shell Green functions $\Lambda_V$ and $\Lambda_A$ as caused by our use of exceptional momenta. We have advanced both a theoretical discussion explaining the pattern of symmetry breaking which we have observed and a calculation with non-exceptional momenta in which these effects are dramatically reduced.

Third, for $Z_{B_K}$ we have presented both a theoretical argument and numerical calculations showing the mixing coefficients with the operators with the wrong chirality are very small so that the calculation of $Z_{B_K}$ can be simplified by neglecting these mixing coefficients. Finally
we have exploited the earlier perturbative work of others and evaluated the factors relating the normalization of operators defined in the $\overline{\text{MS}}$ and RI/MOM schemes determining $Z^{\overline{\text{MS}}}_m$, $Z^{\overline{\text{MS}}}_q$, $Z^{\overline{\text{MS}}}_{BK}$ and $Z^{\overline{\text{MS}}}_T$ from their non-perturbative RI/MOM counterparts to three, three, one and two loops respectively.

Acknowledgments

We thank our collaborators in the RBC and UKQCD collaborations for assistance and useful discussions. This work was supported by DOE grant DE-FG02-92ER40699 and PPARC grants PPA/G/O/2002/00465 and PP/D000238/1. We thank the University of Edinburgh, PPARC, RIKEN, BNL and the U.S. DOE for providing the facilities on which this work was performed. A.S. was supported by the U.S. Dept. of Energy under contracts DE-AC02-98CH10886, and DE-FG02-92ER40716.

Appendix A: THE QCD $\beta$ FUNCTIONS AND THE RUNNING OF $\alpha_s$

The four-loop QCD beta functions is calculated in [25] and the conventions we use are the same as in [14]:

$$
\begin{align*}
\beta_0 &= \frac{1}{4} \left( 11 - \frac{2}{3} n_f \right), \\
\beta_1 &= \frac{1}{16} \left( 102 - \frac{38}{3} n_f \right), \\
\beta_2 &= \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right), \\
\beta_3 &= \frac{1}{256} \left[ \frac{149753}{6} + 3564 \zeta_3 - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f \\ &+ \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3 \right].
\end{align*}
$$

\[\text{(A1)}\]

To calculate the coupling constant $\alpha_s (\mu)$ at any scales, we have used the four-loop (NNNLO) running formula for $\alpha_s$ [25]:

$$
\frac{\partial \alpha_s}{\partial \ln \mu^2} = \beta (\alpha_s)
\quad = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 - \beta_3 a_s^5 + \mathcal{O} (a_s^6)
$$

\[\text{(A2)}\]
where $a_s = \frac{\alpha_s}{\pi}$. (We have changed the normalization of $a_s$ to match the definition of the $\beta$-functions coefficients.) For a numerical implementation, we start from the world-average value at $\mu = M_Z \ [26],$

$$\alpha_s^{(5)}(M_Z) = 0.1176 \pm 0.002, \quad (A3)$$

where the superscript indicates that it is in the 5-flavor region, and run $\alpha_s$ across the $m_b$ and $m_c$ threshold with the matching conditions:

$$\alpha_s^{(5)}(m_b) = \alpha_s^{(4)}(m_b) \quad \text{and} \quad \alpha_s^{(4)}(m_c) = \alpha_s^{(3)}(m_c). \quad (A4)$$

Having computed $\alpha_s^{(3)}(m_c)$, we can calculate the coupling constant at any scale in the 3-flavor theory. For example,

$$\alpha_s^{(3)}(\mu = 2 \text{ GeV}) = 0.2904. \quad (A5)$$

**Appendix B: PERTURBATIVE RUNNING AND MATCHING FOR $Z_m$**

In [14], the renormalization group equation for $m_{\text{ren}}(\mu)$ is solved to four-loop order (NNNLO). Using the solution with our definition of the renormalization coefficients $Z_m$, we obtain:

$$Z_m^{\text{SI}} = \frac{c(\alpha_s(\mu_0)/\pi)}{c(\alpha_s(\mu)/\pi)} Z_m^{\text{RI/MOM}}(\mu) \quad (B1)$$

where the function $c(x)$ is given by:

$$c(x) = (x)^{\tilde{\gamma}_0} \left\{ 1 + \left( \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0 \right) x 
+ \frac{1}{2} \left[ \left( \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0 \right)^2 + \tilde{\gamma}_2 + \tilde{\beta}_1^2 \tilde{\gamma}_0 - \tilde{\beta}_1 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_0 \right] x^2 
+ \left[ \frac{1}{6} \left( \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0 \right)^3 + \frac{1}{2} \left( \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_0 \right) \left( \tilde{\gamma}_2 + \tilde{\beta}_1^2 \tilde{\gamma}_0 - \tilde{\beta}_1 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_0 \right) 
+ \frac{1}{3} \left( \tilde{\gamma}_3 - \tilde{\beta}_1^3 \tilde{\gamma}_0 + 2 \tilde{\beta}_1 \tilde{\beta}_2 \tilde{\gamma}_0 - \tilde{\beta}_3 \tilde{\gamma}_0 + \tilde{\beta}_1^2 \tilde{\gamma}_1 - \tilde{\beta}_2 \tilde{\gamma}_1 - \tilde{\beta}_1 \tilde{\gamma}_2 \right) \right] x^3 
+ \mathcal{O}(x^4) \right\}, \quad (B2)$$

with $\tilde{\beta}_i = \frac{\beta_i}{\beta_0}$ and

$$\tilde{\gamma}_i = \frac{\gamma_m^{\text{RI/MOM}(i)}}{\beta_0} \quad (B3)$$
The evaluation of the coefficients of the QCD $\beta$ function and the running of $\alpha_s$ are explained in Appendix A and the anomalous dimensions are

\[ \gamma_{RI/MOM(0)}^{m} = 1 \]
\[ \gamma_{RI/MOM(1)}^{m} = \frac{1}{16} \left( 126 - \frac{52}{9} n_f \right) \]
\[ \gamma_{RI/MOM(2)}^{m} = \frac{1}{64} \left[ \left( \frac{20911}{3} - \frac{3344}{3} \zeta_3 \right) + \left( -\frac{18386}{27} + \frac{128}{9} \zeta_3 \right) n_f + \frac{928}{81} n_f^2 \right] \]
\[ \gamma_{RI/MOM(3)}^{m} = \frac{1}{256} \left[ \left( \frac{300665987}{648} - \frac{15000871}{108} \zeta_3 + \frac{6160}{3} \zeta_5 \right) + \left( -\frac{7535473}{108} + \frac{627127}{54} \zeta_3 + \frac{4160}{3} \zeta_5 \right) n_f \right.
\[ \left. + \left( \frac{670948}{243} - \frac{6416}{27} \zeta_3 \right) n_f^2 - \frac{18832}{729} n_f^3 \right] \],

where $n_f = 3$.

When applying Eq. (B1), we need to choose a value of $\mu_0$, where the SI value is calculated. The exact value of $\mu_0$ is immaterial and for convenience we choose its value such that

\[ (a\mu_0)^2 = 2. \]  

(B5)

To match the renormalization coefficients $Z_m$ from RI/MOM scheme to $\overline{\text{MS}}$ scheme, we have applied the three-loop matching factor \cite{14} obtaining:

\[ \frac{Z_{m}^{\overline{\text{MS}}}}{Z_{m}^{RI/MOM}} = 1 + \frac{\alpha_s}{4\pi} \left[ -\frac{16}{3} \right] + \left( \frac{\alpha_s}{4\pi} \right)^2 \left[ \left( -\frac{1990}{9} + \frac{152}{3} \zeta_3 + \frac{89}{9} n_f \right) \zeta_3 + \frac{4160}{3} \zeta_5 \right] \]
\[ + \left( \frac{\alpha_s}{4\pi} \right)^3 \left[ -\frac{6663911}{648} + \frac{408007}{108} \zeta_3 - \frac{2960}{9} \zeta_5 + \frac{236650}{243} n_f \right. \]
\[ \left. - \frac{4936}{27} \zeta_3 n_f + \frac{80}{3} \zeta_4 n_f - \frac{8918}{729} n_f^2 - \frac{32}{27} \zeta_3 n_f^2 \right]. \]

(B6)

Appendix C: PERTURBATIVE RUNNING AND SCHEME MATCHING FOR $Z_q$

The renormalization group equation for $Z_q$ is very similar to that for $Z_m$ \cite{14} and we can reuse the solution of the equation from Appendix \cite{B} to write:

\[ Z_{q}^{\text{SI}} = \frac{c^{[\gamma_2]}(\alpha_s(\mu)/\pi)}{c^{[\gamma_2]}(\alpha_s(\mu)/\pi)} Z_{q}^{RI/MOM}(\mu) \]

(C1)
where the function $c[n] (x)$ has exactly the same functional form as $c (x)$ defined in Eq. (B2), but with the coefficients $\gamma_i$ of the anomalous dimension $\gamma_m$ replaced by those of $\gamma_2$:

$$\gamma_i = \frac{\gamma_{\text{RI/MOM}(i)}}{\beta_0}$$  \hspace{1cm} (C2)

The coefficients of the anomalous dimension $\gamma_2$ are [14]:

$$\gamma_{\text{RI/MOM}(0)}^{\text{RI/MOM}} = 0$$

$$\gamma_{\text{RI/MOM}(1)}^{\text{RI/MOM}} = \frac{N^2 - 1}{16 N^2} \left\{ \left[ \frac{3}{8} + \frac{11}{4} N^2 \right] + n_f \left[ -\frac{1}{2} N \right] \right\}$$

$$\gamma_{\text{RI/MOM}(2)}^{\text{RI/MOM}} = \frac{N^2 - 1}{64 N^3} \left\{ \left[ \frac{3}{16} + \frac{25}{3} N^2 + \frac{14225}{288} N^4 - 3 N^2 \zeta_3 - \frac{197}{16} N^4 \zeta_3 \right] + n_f \left[ -\frac{1}{3} N - \frac{611}{36} N^3 + 2 N^3 \zeta_3 \right] 
+ n_f^2 \left[ \frac{10}{9} N^2 \right] \right\}$$

$$\gamma_{\text{RI/MOM}(3)}^{\text{RI/MOM}} = \frac{N^2 - 1}{256 N^4} \left\{ \left[ \frac{1027}{128} + \frac{7673}{384} N^2 + \frac{17456}{1152} N^4 + \frac{3993865}{3456} N^6 
+ 25 \zeta_3 + 31 N^2 \zeta_3 - \frac{10975}{64} N^4 \zeta_3 - \frac{111719}{192} N^6 \zeta_3 
- 40 \zeta_5 - 60 N^2 \zeta_5 + \frac{5465}{64} N^4 \zeta_5 + \frac{20625}{128} N^6 \zeta_5 \right] + n_f \left[ \frac{1307}{48} N + \frac{557}{144} N^3 - \frac{172793}{288} N^5 
- 4 N \zeta_3 + 2 N^3 \zeta_3 + \frac{7861}{48} N^5 \zeta_3 
- 30 N^3 \zeta_5 - \frac{125}{4} N^5 \zeta_5 \right] 
+ n_f^2 \left[ \frac{521}{72} N^2 + \frac{259}{3} N^4 + 6 N^2 \zeta_3 - \frac{26}{3} N^4 \zeta_3 \right] 
+ n_f^3 \left[ \frac{86}{27} N^3 \right] \right\}$$  \hspace{1cm} (C3)

where $N = 3$, which represents the number of colors, and $n_f = 3$.

When we match $Z_q$ from RI/MOM scheme to $\overline{\text{MS}}$ scheme, the three-loop matching factor
\[
\frac{Z^{\text{MS}}_q}{Z_q^{\text{RI/MOM}}} = 1 + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[ -\frac{517}{18} + 12\zeta_3 + \frac{5}{3}n_f \right] \\
+ \left(\frac{\alpha_s}{4\pi}\right)^3 \left[ -\frac{1287283 - 2070n_f + 104n_f^2}{648 - 12\zeta_3 + \frac{79}{4}\zeta_4 - \frac{1165}{3}\zeta_5} \right. \\
\left. + \frac{18014}{81}n_f - \frac{368}{9}\zeta_3n_f - \frac{1102}{243}n_f^2 \right] . \tag{C4}
\]

**Appendix D: PERTURBATIVE RUNNING AND SCHEME MATCHING FOR Z_{BK}**

To remove (restore) the perturbative renormalization group running of \( Z_{BK} \), we use the one-loop renormalization group running formula \(^{15}\):

\[
Z_{BK}^{\text{SI}}(n_f) = w_{\text{scheme}}^{-1}(\mu, n_f) \cdot Z_{BK}^{\text{scheme}}(\mu, n_f) \tag{D1}
\]

where

\[
w_{\text{scheme}}^{-1}(\mu, n_f) = \alpha_s(\mu)^{-\gamma_0/2\beta_0} \left[ 1 + \frac{\alpha_s(\mu)}{4\pi} J_{\text{scheme}}^{(n_f)} \right] \tag{D2}
\]

and

\[
J_{\text{RI/MOM}}^{(n_f)} = -\frac{17397 - 2070n_f + 104n_f^2}{6 (33 - 2n_f)^2} + 8 \ln 2 \tag{D3}
\]

\[
J_{\text{MS}}^{(n_f)} = \frac{13095 - 1626n_f + 8n_f^2}{6 (33 - 2n_f)^2} \tag{D4}
\]

with \( n_f = 3 \) in our analysis.

**Appendix E: PERTURBATIVE RUNNING AND SCHEME MATCHING FOR Z_T**

The anomalous dimension of the tensor current in the \( \overline{\text{MS}} \) scheme is given at three-loops in \(^{27}\),

\[
\gamma_T^{\overline{\text{MS}}(0)} = \frac{1}{3} ,
\]

\[
\gamma_T^{\overline{\text{MS}}(1)} = \frac{2}{1627} (543 - 26n_f) ,
\]

\[
\gamma_T^{\overline{\text{MS}}(2)} = \frac{1}{64243} \left( \frac{1}{2} [157665 - 4176\zeta_3 - (2160\zeta_3 + 7860)n_f - 54n_f^2] \right) . \tag{E1}
\]
For consistency we have adjusted the normalization from that used in [27] such that $\gamma_{\text{MS}}$ satisfies the generic RG-equation for the renormalization constant $Z_{\Gamma}$ of the quark bilinear $\bar{\psi} \Gamma \psi$,

$$
\frac{\partial \ln Z_{\Gamma}}{\partial \ln \mu^2} = \gamma_{\Gamma}(a_s)
= -\gamma_{\Gamma}^{(0)} a_s - \gamma_{\Gamma}^{(1)} a_s^2 - \gamma_{\Gamma}^{(2)} a_s^3 + \mathcal{O}(a_s^4),
$$

(E2)

with $a_s = \alpha_s / \pi$.

The perturbative running for the tensor current has also been computed at three loops in the RI/MOM' scheme [27], and we use it to obtain the RI/MOM scheme anomalous dimension as follows. We consider the conversion function $C_{\Gamma}^{\text{RI/MOM}(\prime)}$ used to match the RI/MOM or RI/MOM' scheme to the $\overline{\text{MS}}$ scheme:

$$
Z_{\Gamma}^{\overline{\text{MS}}} = C_{\Gamma}^{\text{RI/MOM}(\prime)} Z_{\Gamma}^{\text{RI/MOM}(\prime)}.
$$

(E3)

Applying the above renormalization group equation (E2) we obtain

$$
\gamma_{\Gamma}^{\text{RI/MOM}(\prime)} = \gamma_{\Gamma}^{\overline{\text{MS}}} - \frac{\partial \ln C_{\Gamma}^{\text{RI/MOM}(\prime)}}{\partial \ln \mu^2}.
$$

(E4)

Since the only difference between the RI/MOM and RI/MOM' schemes lies in the definition of the quark field renormalization constants $Z_{2}^{\text{RI/MOM}(\prime)}$ and $Z_{2}^{\text{RI/MOM}}$, we write $C_{\Gamma}^{\text{RI/MOM}(\prime)} = C_{\Gamma} C_{2}^{\text{RI/MOM}(\prime)}$. The vertex part $C_{\Gamma}$ of the conversion function is common to both the RI/MOM and RI/MOM' schemes. It follows that

$$
\gamma_{\Gamma}^{\text{RI/MOM}(\prime)} - \gamma_{\Gamma}^{\text{RI/MOM}} = \frac{\partial \ln C_{2}^{\text{RI/MOM}}}{\partial \ln \mu^2} - \frac{\partial \ln C_{2}^{\text{RI/MOM}(\prime)}}{\partial \ln \mu^2}
= \gamma_{2}^{\text{RI/MOM}} - \gamma_{2}^{\text{RI/MOM}(\prime)}.
$$

(E5)

Since both functions $\gamma_{2}^{\text{RI/MOM}}$ and $\gamma_{2}^{\text{RI/MOM}(\prime)}$ are known [14], we can now compute the anomalous dimension of the tensor current in the RI/MOM scheme from the known one in the RI/MOM' scheme. We note that since the r.h.s. of (E5) is valid for any choice of $\Gamma$ on the l.h.s., one may use the identity

$$
\gamma_{\Gamma}^{\text{RI/MOM}} = \gamma_{\Gamma}^{\text{RI/MOM}(\prime)} - \left( \gamma_{\Gamma'}^{\text{RI/MOM}(\prime)} - \gamma_{\Gamma'}^{\text{RI/MOM}} \right).
$$

(E6)

In order to compute $\gamma_{\Gamma}^{\text{RI/MOM}}$ here we have used $\gamma_{2}^{\text{RI/MOM}}$ as in [C2] and $\gamma_{2}^{\text{RI/MOM}(\prime)}$ from
\[ \gamma_{RI/MOM}'(0) = 0, \]
\[ \gamma_{RI/MOM}'(1) = \frac{N^2 - 1}{16N^2} \left\{ \left[ \frac{3}{8} + \frac{11}{4} N^2 \right] + n_f \left[ -\frac{1}{2} N \right] \right\}, \]
\[ \gamma_{RI/MOM}'(2) = \frac{N^2 - 1}{64N^3} \left\{ \left[ \frac{3}{16} + \frac{233}{24} N^2 + \frac{17129}{288} N^4 - 3N^2 \zeta_3 - \frac{197}{16} N^4 \zeta_3 \right] \right. \]
\[ + \ n_f \left[ -\frac{7}{12} N - \frac{743}{36} N^3 + 2N^3 \zeta_3 \right] \]
\[ + \ n^2_f \left[ \frac{13}{9} N^2 \right] \left\}, \right. \]
\[ \gamma_{RI/MOM}'(3) = \frac{N^2 - 1}{256N^4} \left\{ \left[ \frac{1027}{128} + \frac{8069}{384} N^2 + \frac{240973}{1152} N^4 + \frac{5232091}{3456} N^6 \right] \right. \]
\[ + \ 25 \zeta_3 + 31N^2 \zeta_3 - \frac{12031}{64} N^4 \zeta_3 - \frac{124721}{192} N^6 \zeta_3 \]
\[ - \ 40 \zeta_5 - 60N^2 \zeta_5 + \frac{5465}{64} N^4 \zeta_5 + \frac{20625}{128} N^6 \zeta_5 \right] \]
\[ + \ n_f \left[ \frac{329}{12} N - \frac{1141}{144} N^3 - \frac{113839}{144} N^5 \right] \]
\[ - \ 4N \zeta_3 + 5N^3 \zeta_3 + \frac{2245}{12} N^5 \zeta_3 \]
\[ - \ 30N^3 \zeta_5 - \frac{125}{4} N^5 \zeta_5 \right] \]
\[ + \ n^2_f \left[ -\frac{515}{72} N^2 + \frac{1405}{12} N^4 + 6N^2 \zeta_3 - \frac{32}{3} N^4 \zeta_3 \right] \]
\[ + \ n^3_f \left[ -\frac{125}{27} N^3 \right] \right\}. \quad (E7) \]

In this way we obtain the anomalous dimension:
\[ \gamma_{RI/MOM}(0) = \frac{1}{3}, \]
\[ \gamma_{RI/MOM}(1) = \frac{1}{16} \frac{2}{27} (543 - 26n_f), \]
\[ \gamma_{RI/MOM}(2) = \frac{1}{64} \frac{1}{243} (478821 - 117648\zeta(3) + 6(384\zeta(3) - 8713)n_f + 928n^2_f), \quad (E8) \]

from which we compute the running of \( Z_T \) using \( (B2) \).

Combining \( (A2) \) and \( (E2) \) we compute the expression for the matching factor \( C_{RI/MOM} \).

After expanding in \( \alpha_s \) we obtain:
\[ \frac{Z_{RI/MOM}^{MS}}{Z_{RI/MOM}} = 1 + \frac{1}{81}(-4866 + 1656\zeta(3) + 259n_f) \left( \frac{\alpha_s}{4\pi} \right)^2. \quad (E9) \]
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Table I: The four factors $Z_{S,P}$, $Z_{V,A}$, $Z_T$ and $Z_{B_K}$ by which the matrix elements of the bare lattice bilinear operators and the ratio of matrix elements $B_K$ should be multiplied in order to obtain the corresponding quantities renormalized in the RI/MOM or $\overline{\text{MS}}(\text{NDR})$ schemes. The RI/MOM quantities are defined at a scale $\mu = 2.037$ GeV, an available lattice momentum. The $\overline{\text{MS}}(\text{NDR})$ quantities are provided at the scale $\mu = 2$ GeV. The first error given is statistical and the second systematic. This table summarizes the main results of this paper.

| Scheme     | Scale       | $Z_q$   | $Z_{S,P}$ | $Z_{V,A}$ | $Z_T$   | $Z_{B_K}$ |
|------------|-------------|---------|-----------|-----------|---------|-----------|
| RI/MOM     | 2.037 GeV   | 0.8086(28)(74) | 0.466(14)(31) | 0.7161(1) | 0.8037(22)(55) | 0.9121(38)(129) |
| $\overline{\text{MS}}$(NDR) | 2.00 GeV    | 0.7726(30)(83) | 0.604(18)(55) | 0.7161(1) | 0.7950(34)(150) | 0.9276(52)(220) |

Table II: The factors, computed in perturbation theory, by which the matrix elements of the bare lattice operators should be multiplied in order to obtain those in the $\overline{\text{MS}}$(NDR) scheme at the renormalization scale $\mu = 1.729$ GeV. This table shows that the difference in the choice of the strong coupling constant leads to large uncertainty in the renormalization constants.

| Coupling | $Z_{S,P}(1.729\text{ GeV})$ | $Z_{V,A}(1.729\text{ GeV})$ | $Z_T(1.729\text{ GeV})$ | $Z_{B_K}(1.729\text{ GeV})$ |
|----------|----------------------------|----------------------------|-------------------------|----------------------------|
| $\alpha_{\text{MF}}(1.729\text{ GeV})$ | 0.788                      | 0.801                      | 0.827                   | 0.979                      |
| $\alpha_{\overline{\text{MS}}}(1.729\text{ GeV})$ | 0.672                      | 0.693                      | 0.737                   | 0.963                      |

Table III: The perturbative renormalization constants at the conventional scale of $\mu = 2$ GeV by renormalization group running from $\mu = 1.729$ GeV. The entries in the first column indicate which coupling was used in matching between the bare lattice operators and the $\overline{\text{MS}}$(NDR) scheme at $\mu = 1.729$ GeV.

| Coupling | $Z_{S,P}(2\text{ GeV})$ | $Z_{V,A}(2\text{ GeV})$ | $Z_T(2\text{ GeV})$ | $Z_{B_K}(2\text{ GeV})$ |
|----------|-------------------------|-------------------------|---------------------|-------------------------|
| $\alpha_{\text{MF}}(1.729\text{ GeV})$ | 0.822                   | 0.801                   | 0.813               | 0.993                   |
| $\alpha_{\overline{\text{MS}}}(1.729\text{ GeV})$ | 0.701                   | 0.693                   | 0.725               | 0.977                   |
Table IV: The quantity $\frac{1}{12} \text{Tr}(S^{-1}_{\text{latt}})$ evaluated at the unitary mass points, $m_{\text{val}} = m_l$ and linearly extrapolated to the chiral limit $m_l = -m_{\text{res}}$.

|         | $(ap)^2$ | $m_l = 0.01$ | $m_l = 0.02$ | $m_l = 0.03$ | chiral limit |
|---------|--------|-------------|-------------|-------------|-------------|
| $0.347$ | 0.0839(16) | 0.1141(16) | 0.1327(23) | 0.0524(34) |
| $0.617$ | 0.0558(13) | 0.0810(15) | 0.0980(20) | 0.0283(28) |
| $0.810$ | 0.0450(12) | 0.0692(14) | 0.0849(18) | 0.0187(28) |
| $1.079$ | 0.03744(82) | 0.0583(11) | 0.0741(16) | 0.0130(20) |
| $1.234$ | 0.0342(12) | 0.0543(13) | 0.0704(16) | 0.0105(26) |
| $1.388$ | 0.03203(84) | 0.0512(11) | 0.0665(15) | 0.0092(20) |
| $1.542$ | 0.03051(75) | 0.04873(97) | 0.0634(14) | 0.0087(18) |
| $1.851$ | 0.02640(92) | 0.04472(99) | 0.0597(13) | 0.0047(20) |
| $2.005$ | 0.02615(73) | 0.04354(92) | 0.0575(13) | 0.0054(18) |
| $2.467$ | 0.0236(10) | 0.0404(10) | 0.0540(13) | 0.0040(20) |

Table V: The five bare vertex amplitudes $\Lambda_i, i \in \{S,P,V,A,T\}$ averaged over four sources, with $m_l = m_{\text{val}} = 0.01$.

|         | $(ap)^2$ | $\Lambda_S$ | $\Lambda_P$ | $\Lambda_V$ | $\Lambda_A$ | $\Lambda_T$ |
|---------|--------|-------------|-------------|-------------|-------------|-------------|
| $0.347$ | 2.125(86) | 6.72(19) | 1.1702(58) | 1.0675(43) | 0.8904(43) |
| $0.617$ | 1.945(51) | 4.45(11) | 1.1419(37) | 1.0938(30) | 0.9404(26) |
| $0.810$ | 1.856(37) | 3.677(81) | 1.1348(31) | 1.1025(27) | 0.9618(19) |
| $1.079$ | 1.758(27) | 3.022(57) | 1.1335(29) | 1.1135(27) | 0.9882(16) |
| $1.234$ | 1.715(24) | 2.792(50) | 1.1291(29) | 1.1137(27) | 0.9935(17) |
| $1.388$ | 1.677(21) | 2.600(43) | 1.1328(26) | 1.1191(24) | 1.0065(13) |
| $1.542$ | 1.642(19) | 2.448(38) | 1.1355(27) | 1.1240(25) | 1.0167(14) |
| $1.851$ | 1.599(16) | 2.239(32) | 1.1387(29) | 1.1301(27) | 1.0310(16) |
| $2.005$ | 1.578(15) | 2.154(28) | 1.1420(27) | 1.1342(26) | 1.0392(16) |
| $2.467$ | 1.532(13) | 1.979(23) | 1.1495(29) | 1.1434(29) | 1.0577(19) |
Table VI: The five bare vertex amplitudes $\Lambda_i$, $i \in \{S, P, V, A, T\}$ averaged over four sources, with $m_l = m_{val} = 0.02$.

| $(ap)^2$ | $\Lambda_S$ | $\Lambda_P$ | $\Lambda_V$ | $\Lambda_A$ | $\Lambda_T$ |
|---------|-------------|-------------|-------------|-------------|-------------|
| 0.347   | 1.828(45)   | 5.09(14)    | 1.1745(46)  | 1.0412(28)  | 0.8930(31)  |
| 0.617   | 1.774(30)   | 3.600(82)   | 1.1465(30)  | 1.0838(21)  | 0.9414(18)  |
| 0.810   | 1.721(24)   | 3.052(61)   | 1.1360(24)  | 1.0943(19)  | 0.9614(15)  |
| 1.079   | 1.655(19)   | 2.590(45)   | 1.1331(22)  | 1.1069(20)  | 0.9870(12)  |
| 1.234   | 1.637(16)   | 2.428(40)   | 1.1307(21)  | 1.1083(20)  | 0.9930(12)  |
| 1.388   | 1.608(15)   | 2.283(33)   | 1.1323(21)  | 1.1141(19)  | 1.0049(11)  |
| 1.542   | 1.581(14)   | 2.175(30)   | 1.1351(21)  | 1.1199(20)  | 1.0159(12)  |
| 1.851   | 1.552(11)   | 2.019(24)   | 1.1389(22)  | 1.1275(21)  | 1.0309(12)  |
| 2.005   | 1.532(11)   | 1.955(23)   | 1.1416(23)  | 1.1315(22)  | 1.0390(14)  |
| 2.467   | 1.4984(91)  | 1.819(18)   | 1.1498(26)  | 1.1422(25)  | 1.0580(17)  |

Table VII: The five bare vertex amplitudes $\Lambda_i$, $i \in \{S, P, V, A, T\}$ averaged over four sources, with $m_l = m_{val} = 0.03$.

| $(ap)^2$ | $\Lambda_S$ | $\Lambda_P$ | $\Lambda_V$ | $\Lambda_A$ | $\Lambda_T$ |
|---------|-------------|-------------|-------------|-------------|-------------|
| 0.347   | 1.723(56)   | 4.10(14)    | 1.1809(56)  | 1.0357(23)  | 0.9020(24)  |
| 0.617   | 1.702(37)   | 3.049(87)   | 1.1457(37)  | 1.0769(19)  | 0.9451(17)  |
| 0.810   | 1.663(28)   | 2.658(66)   | 1.1356(31)  | 1.0886(17)  | 0.9642(14)  |
| 1.079   | 1.610(21)   | 2.307(49)   | 1.1325(27)  | 1.1015(18)  | 0.9883(12)  |
| 1.234   | 1.591(18)   | 2.182(42)   | 1.1294(25)  | 1.1050(20)  | 0.9951(12)  |
| 1.388   | 1.569(15)   | 2.076(37)   | 1.1312(25)  | 1.1105(20)  | 1.0061(11)  |
| 1.542   | 1.548(13)   | 1.991(33)   | 1.1337(26)  | 1.1157(21)  | 1.0161(12)  |
| 1.851   | 1.520(10)   | 1.869(27)   | 1.1366(26)  | 1.1228(22)  | 1.0300(13)  |
| 2.005   | 1.5065(96)  | 1.820(25)   | 1.1395(27)  | 1.1271(24)  | 1.0382(15)  |
| 2.467   | 1.4764(78)  | 1.717(20)   | 1.1464(27)  | 1.1371(26)  | 1.0561(17)  |
Table VIII: Groups of non-exceptional momenta satisfying \( p_1^2 = p_2^2 = (p_1 - p_2)^2 \). The individual integers \( (n_x, n_y, n_z, n_t) \) should be multiplied by \( 2\pi/L_d \), with \( L_x = L_y = L_z = 16 \) and \( L_t = 32 \).

| \((ap)^2\) | \(p_1\) | \(p_2\) |
|------------|---------|---------|
| 0.617      | (1,1,1,2) | (1,-1,1,2) |
|            | (1,1,1,2) | (1,1,-1,2) |
|            | (1,1,1,2) | (-1,1,1,2) |
|            | (1,1,1,2) | (1,1,1,-2) |
|            | (1,1,1,2) | (0,0,0,4) |
|            | (1,1,1,2) | (0,0,2,0) |
|            | (1,1,1,2) | (0,2,0,0) |
|            | (1,1,1,2) | (2,0,0,0) |
| 0.925      | (-1,-1,-2,0) | (-2,-1,0,-2) |
|            | (-1,-1,-2,0) | (-2,-1,0,2) |
|            | (-1,-1,-2,0) | (-2,1,-1,0) |
|            | (-1,-1,-2,0) | (-1,-2,0,-2) |
|            | (-1,-1,-2,0) | (-1,-2,0,2) |
|            | (-1,-1,-2,0) | (-1,0,-1,-4) |
|            | (-1,-1,-2,0) | (-1,0,-1,4) |
|            | (-1,-1,-2,0) | (0,-1,-1,-4) |
|            | (-1,-1,-2,0) | (0,-1,-1,4) |
|            | (-1,-1,-2,0) | (0,1,2,-2) |
|            | (-1,-1,-2,0) | (0,1,2,2) |
|            | (-1,-1,-2,0) | (1,2,-1,0) |
|            | (-1,-1,-2,0) | (1,0,-2,-2) |
|            | (-1,-1,-2,0) | (1,0,-2,2) |
Table IX: Groups of non-exceptional momenta satisfying $p_1^2 = p_2^2 = (p_1 - p_2)^2$, continuing Table VIII

| $(ap)^2$ | $p_1$     | $p_2$     |
|----------|-----------|-----------|
| 1.234    | (0,2,2,0) | (2,2,0,0) |
|          | (0,2,2,0) | (0,2,0,4) |
|          | (0,2,2,0) | (0,0,2,4) |
|          | (0,2,2,0) | (-2,2,0,0)|
|          | (0,2,2,0) | (0,2,0,-4)|
|          | (0,2,2,0) | (2,0,2,0) |
|          | (0,2,2,0) | (0,0,2,-4)|
|          | (0,2,2,0) | (-2,0,2,0)|
| 1.542    | (1,1,2,4) | (2,1,2,-2)|
|          | (1,1,2,4) | (1,-2,2,2)|
|          | (1,1,2,4) | (-2,1,2,2)|
|          | (1,1,2,4) | (-2,1,1,4)|
|          | (1,1,2,4) | (1,2,2,-2)|
|          | (1,1,2,4) | (1,-2,1,4)|
|          | (1,1,2,4) | (2,1,-1,4)|
|          | (1,1,2,4) | (1,2,-1,4)|
| 2.467    | (2,2,2,4) | (2,2,-2,4)|
|          | (2,2,2,4) | (2,-2,2,4)|
|          | (2,2,2,4) | (-2,2,2,4)|
|          | (2,2,2,4) | (2,2,2,-4)|
Table X: Results from fitting the coefficient for mass term in \((\Lambda_A - \Lambda_V)/[(\Lambda_A + \Lambda_V)/2]\). The linear dependence is assumed to be \(c_1 \frac{m_{\Lambda_{QCD}}}{p^2}\) and the quadratic dependence is assumed to be \(c_2 \frac{m^2}{p^2}\). The respective \(\chi^2/d.o.f\) is also listed. Both the coefficient \(c_1\) more nearly agreeing with its expected value of 1 and the smaller \(\chi^2\) suggest that the linear description is to be preferred. We use the value \(\Lambda_{QCD} = 319.5\ MeV\).

| \((ap)^2\) | \(c_1\) | \(\chi^2/dof\)_1 | \(c_2\) | \(\chi^2/dof\)_2 |
|-----------|--------|-----------------|--------|-----------------|
| 0.347     | -3.84(75) | 3.0(3.4)       | -14.7(3.1) | 6.6(5.1) |
| 0.617     | -3.33(67) | 2.2(2.8)       | -12.9(2.8) | 5.4(4.4) |
| 0.810     | -3.06(56) | 1.2(2.1)       | -12.1(2.5) | 4.1(3.7) |
| 1.079     | -3.01(42) | 0.4(1.3)       | -12.4(2.0) | 3.3(3.6) |
| 1.234     | -2.96(47) | 6.2(5.0)       | -11.4(2.1) | 12.8(7.2) |
| 1.388     | -2.58(36) | 1.7(2.7)       | -10.5(1.7) | 6.2(4.8) |
| 1.542     | -2.49(34) | 0.4(1.4)       | -10.2(1.6) | 3.3(3.6) |
| 1.851     | -2.33(35) | 0.06(41)        | -9.5(1.6) | 1.8(2.4) |
| 2.005     | -2.21(28) | 0.02(23)        | -9.2(1.3) | 1.3(2.2) |
| 2.467     | -1.89(32) | 0.01(22)        | -7.4(1.4) | 0.7(1.5) |
Table XI: Values for $\frac{1}{2}(\Lambda_A + \Lambda_V)$, $\Lambda_S$, and $\Lambda_T$ extrapolated to the chiral limit using a linear mass fit.

| $(ap)^2$ | $\frac{1}{2}(\Lambda_A + \Lambda_V)$ | $\Lambda_S$   | $\Lambda_T$   |
|----------|--------------------------------------|----------------|----------------|
| 0.347    | 1.1211(56)                           | 2.28(14)       | 0.8800(66)     |
| 0.617    | 1.1226(49)                           | 2.060(88)      | 0.9363(36)     |
| 0.810    | 1.1228(41)                           | 1.952(66)      | 0.9593(25)     |
| 1.079    | 1.1275(43)                           | 1.836(47)      | 0.9873(22)     |
| 1.234    | 1.1242(44)                           | 1.784(43)      | 0.9915(22)     |
| 1.388    | 1.1292(40)                           | 1.736(37)      | 1.0061(18)     |
| 1.542    | 1.1333(43)                           | 1.694(32)      | 1.0169(20)     |
| 1.851    | 1.1381(47)                           | 1.644(27)      | 1.0319(24)     |
| 2.005    | 1.1417(46)                           | 1.617(25)      | 1.0400(27)     |
| 2.467    | 1.1504(51)                           | 1.564(22)      | 1.0593(33)     |
Table XII: The non-perturbative factor $Z_{m}^{\text{RI/MOM}}$ as a function of the scale $\mu$ calculated from $\Lambda_S$ and the corresponding values for $Z_{m}^{\text{MS}}$. Note that the values for $Z_{m}^{\text{MS}}$ given in column three are obtained from those in column two by applying the RI/MOM – MS perturbative matching factors after the $O(a\mu)^2$ lattice artifacts have been removed using an intermediate conversion to a scale-invariant scheme as described in the text.

| $\mu$(GeV) | $Z_{m}^{\text{RI/MOM}}$ | $Z_{m}^{\text{MS}}$ |
|------------|--------------------------|-------------------|
| 1.018      | 2.85(18)                 | 1.625(47)         |
| 1.358      | 2.56(11)                 | 1.758(51)         |
| 1.556      | 2.428(80)                | 1.731(51)         |
| 1.796      | 2.273(56)                | 1.690(49)         |
| 1.920      | 2.216(49)                | 1.669(49)         |
| 2.037      | 2.146(42)                | 1.651(48)         |
| 2.147      | 2.087(37)                | 1.634(48)         |
| 2.352      | 2.018(29)                | 1.605(47)         |
| 2.448      | 1.978(27)                | 1.593(46)         |
| 2.716      | 1.899(22)                | 1.562(46)         |
Table XIII: The non-perturbative factor $Z_{q}^{\text{RI/MOM}}$ as a function of the scale $\mu$ calculated from $\frac{1}{2} (\Lambda_A + \Lambda_V)$ and the corresponding values for $Z_{q}^{\overline{\text{MS}}}$. Note that the values for $Z_{q}^{\overline{\text{MS}}}$ given in column three are obtained from those in column two by applying the RI/MOM $\rightarrow \overline{\text{MS}}$ perturbative matching factors after the $O(a\mu)^2$ lattice artifacts have been removed using an intermediate conversion to a scale-invariant scheme as described in the text.

| $\mu(\text{GeV})$ | $Z_{q}^{\text{RI/MOM}}$ | $Z_{q}^{\overline{\text{MS}}}$ |
|-------------------|--------------------------|---------------------------------|
| 1.018             | 0.8028(40)               | 0.8010(31)                      |
| 1.358             | 0.8039(35)               | 0.7849(30)                      |
| 1.556             | 0.8041(30)               | 0.7798(30)                      |
| 1.796             | 0.8074(31)               | 0.7754(30)                      |
| 1.920             | 0.8050(32)               | 0.7736(30)                      |
| 2.037             | 0.8086(28)               | 0.7722(30)                      |
| 2.147             | 0.8115(31)               | 0.7710(30)                      |
| 2.352             | 0.8150(34)               | 0.7691(29)                      |
| 2.448             | 0.8176(33)               | 0.7684(29)                      |
| 2.716             | 0.8238(37)               | 0.7665(29)                      |
Table XIV: The non-perturbative factor $Z_T^{RI/MOM}$ as a function of the scale $\mu$ calculated from $\Lambda_T$ and the corresponding values for $Z_T^{MS}$. Note that the values for $Z_T^{MS}$ given in column three are obtained from those in column two by applying the RI/MOM – MS perturbative matching factors after the $O(a\mu)^2$ lattice artifacts have been removed using an intermediate conversion to a scale-invariant scheme as described in the text.

| $\mu$(GeV) | $Z_T^{RI/MOM}$ | $Z_T^{MS}$ |
|------------|----------------|------------|
| 1.018      | 0.9121(74)     | 0.8812(38) |
| 1.358      | 0.8583(46)     | 0.8355(36) |
| 1.556      | 0.8380(32)     | 0.8194(35) |
| 1.796      | 0.8177(27)     | 0.8048(34) |
| 1.920      | 0.8118(27)     | 0.7986(34) |
| 2.037      | 0.8037(22)     | 0.7935(34) |
| 2.147      | 0.7981(21)     | 0.7892(34) |
| 2.352      | 0.7899(18)     | 0.7821(33) |
| 2.448      | 0.7862(17)     | 0.7791(33) |
| 2.716      | 0.7779(16)     | 0.7719(33) |

Table XV: The quantity $Z_{B_K}^{RI/MOM}$ evaluated at the unitary points where $m_{val} = m_l = m$.

| $\mu$(GeV) | $m = 0.01$  | $m = 0.02$  | $m = 0.03$  |
|------------|-------------|-------------|-------------|
| 0.954      | 0.9663(69)  | 0.9737(52)  | 0.9538(44)  |
| 1.272      | 0.9347(39)  | 0.9387(35)  | 0.9315(30)  |
| 1.458      | 0.9266(31)  | 0.9289(30)  | 0.9245(26)  |
| 1.683      | 0.9189(25)  | 0.9189(25)  | 0.9167(24)  |
| 1.799      | 0.9151(22)  | 0.9137(23)  | 0.9126(23)  |
| 1.909      | 0.9114(23)  | 0.9106(22)  | 0.9102(22)  |
| 2.012      | 0.9085(22)  | 0.9077(20)  | 0.9078(21)  |
| 2.204      | 0.9045(23)  | 0.9026(20)  | 0.9035(21)  |
| 2.294      | 0.9018(20)  | 0.9004(19)  | 0.9020(20)  |
| 2.545      | 0.8974(21)  | 0.8953(19)  | 0.8978(19)  |
Table XVI: The non-perturbative factor $Z_{B_K}^{RI/MOM}$ as a function of the scale $\mu$ and the corresponding values for $Z_{B_K}^{\overline{MS}}$. Note that the values for $Z_{B_K}^{\overline{MS}}$ given in column three are obtained from those in column two by applying the RI/MOM – MS perturbative matching factors after the $O(a\mu)^2$ lattice artifacts have been removed using an intermediate conversion to a scale-invariant scheme as described in the text.

| $\mu$(GeV) | $Z_{B_K}^{RI/MOM}$ | $Z_{B_K}^{\overline{MS}}$ |
|------------|-------------------|-----------------|
| 1.018      | 0.985(11)         | 1.0016(56)      |
| 1.358      | 0.9397(61)        | 0.9651(54)      |
| 1.556      | 0.9295(48)        | 0.9507(54)      |
| 1.796      | 0.9208(42)        | 0.9370(53)      |
| 1.920      | 0.9168(38)        | 0.9311(52)      |
| 2.037      | 0.9121(38)        | 0.9261(52)      |
| 2.147      | 0.9088(37)        | 0.9217(52)      |
| 2.352      | 0.9045(39)        | 0.9145(52)      |
| 2.448      | 0.9011(35)        | 0.9114(51)      |
| 2.716      | 0.8961(38)        | 0.9038(51)      |
Figure 1: The quantity $\frac{1}{12} \text{Tr}(S^{-1}_{\text{latt}})$ plotted versus $(ap)^2$ for the unitary mass points $m_t = 0.01$, 0.02 and 0.03 and at the linearly extrapolated, chiral limit $m_t = -m_{\text{res}}$. 
Figure 2: The ratio \( \frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2} \) plotted as a function of momentum at the unitary mass points \( m_{val} = m_l \) and in the chiral limit evaluated by linear extrapolation in \( m_l \). The 5-10\% difference at low momentum decreases rapidly as the momentum increases. At the scale \( \mu \simeq 2 \text{ GeV} \), or \( (ap)^2 \simeq 1.4 \), the difference is about 1\%, which contributes to the systematic error in \( Z_{BK} \).
Figure 3: The difference $\Lambda_A - \Lambda_V$ computed using four different quenched DBW2 lattice ensembles. These ensembles have quite different lattice scales. In addition the values of $L_s$, the extent in the 5th dimension used in computing the DWF propagators, also varies significantly. This provides compelling evidence that the observed chiral symmetry breaking is not an explicit breaking from finite $L_s$, but rather represents the high energy tail of QCD dynamical chiral symmetry breaking which would vanish if we were able to perform the NPR calculation at high enough energy. The data shown come from Refs. [15, 28, 29].
Figure 4: The $\chi^2/d.o.f$ which results from fitting the momentum dependence of the quantity \[ \frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)^2} \] (extrapolated to the chiral limit) to the form $p^{-n}$. We conclude that the best choice for $n$ lies between 2 and 3 and that it is unlikely that the term $\langle \bar{q}q \rangle^2 / p^6$ gives the dominant contribution to this chiral symmetry breaking.
Figure 5: The division of a general vertex graph into subgraphs. If the four-legged, internal subgraph $\Gamma_2$ carries momenta $p \sim \Lambda_{\text{QCD}}$ it can introduce low energy, $(8,8)$ chiral symmetry breaking into such an amplitude even in the limit that the momenta external to the entire diagram $\Gamma$, included in the outer dashed box, grow large. As discussed in the text, such a limit will be suppressed by $1/p^6$ if the external momenta are non-exceptional but by only $1/p^2$ for the exceptional case.

Figure 6: Sample diagram in which two low-momentum ($k \simeq \Lambda_{\text{QCD}}$) fermion propagators appear in a graph which is suppressed at high momentum only by a single factor of $1/p^2$. 


Figure 7: The value of \( \frac{\Lambda_A - \Lambda_V}{(\Lambda_A + \Lambda_V)/2} \) calculated with non-exceptional kinematics, which requires the sum of any subset of external momenta be non-zero. With this condition the chiral symmetry breaking is highly suppressed (as compared to Fig. 2) and vanishes almost completely over the available momentum region.
Figure 8: The average $\frac{1}{2} (\Lambda_A + \Lambda_V)$ plotted as a function of momentum and evaluated for a unitary choice of masses and in the chiral limit. The chiral limit is taken using a linear fit.
Figure 9: Comparison of linear (eq. (39) – top panel) and quadratic (eq. (40) – bottom panel) fits to the dependence of the chiral symmetry breaking difference $\frac{(\Lambda_A - \Lambda_V)}{(\Lambda_A + \Lambda_V)}$ on the quark mass $m_{val} = m_t$ at the scale $\mu = 2.04$ GeV. These plots suggest that a linear description is more accurate. This conclusion is borne out by the properties of the actual fits shown in Table X.
Figure 10: A plot showing the linear extrapolation of $\frac{1}{2}(\Lambda_A + \Lambda_V)$ (evaluated at the scale $\mu = 2.04$ GeV) to the chiral limit. The three data points are evaluated at the unitary points $m_{\text{val}} = m_l$. 
Figure 11: The difference between the quantities $\Lambda_P$ and $\frac{1}{12} \frac{\text{Tr}(S^{-1}_{\text{lat}})}{m_1 + m_{\text{res}}}$, divided by their average, is plotted versus momentum for unitary quark masses. This provides a test of the axial Ward-Takahashi identity.
Figure 12: The difference between the quantities $\Lambda_S$ and $\frac{1}{12} \frac{\partial \text{Tr} [S^{-1}_{\text{lat}}(p)]}{\partial m_{\text{val}}}$, divided by their average for each sea quark mass. The difference appears to zero within errors. This is a test of the vector Ward-Takahashi identity. The plot uses propagators from three sources.
Figure 13: The double-pole fit for $\Lambda_S$ at $\mu = 2.04$ GeV. The expected decrease in the pronounced $m_{\text{val}}$ dependence as the dynamical light quark mass $m_l$ decreases is easily seen.
Figure 14: Momentum dependence of the double pole coefficient, $c_{dp}$, fit to the expected $p^{-2}$ behavior. Good agreement is seen.
Figure 15: The quantities $Z_{m}^{\text{RI/MOM}}(\mu)$ and $Z_{m}^{\text{SI}}(\mu)$ plotted versus the square of the scale $a\mu$. Here $Z_{m}^{\text{SI}}(\mu)$ is obtained by dividing $Z_{m}^{\text{RI/MOM}}(\mu)$ by the predicted perturbative running factor. Shown also is the linear extrapolation of $Z_{m}^{\text{SI}}(\mu) = Z_{m}^{\text{SI}} + c(a\mu)^{2}$ using the momentum region $1.3 < (a\mu)^{2} < 2.5$ to remove lattice artifacts.
Figure 16: The mass renormalization factor $Z_m$ expressed in the $\overline{\text{MS}}$ scheme. These results are obtained by applying the perturbative running factor to $Z_{m}^{\text{SI}}$. The value we are interested in is $Z_{m}^{\overline{\text{MS}}} (\mu = 2 \text{ GeV})$. The upper and lower curves show the statistical errors.
Figure 17: The quantities $Z_{q}^{\text{RI/MOM}}(\mu)$ and $Z_{q}^{\text{SI}}(\mu)$ plotted versus the square of the scale $a\mu$. Here $Z_{q}^{\text{SI}}(\mu)$ is obtained by dividing $Z_{q}^{\text{RI/MOM}}(\mu)$ by the predicted perturbative running factor. Shown also is the linear extrapolation of $Z_{q}^{\text{SI}}(\mu) = Z_{q}^{\text{SI}} + c(a\mu)^{2}$ using the momentum region $1.3 < (a\mu)^{2} < 2.5$ to remove lattice artifacts.
Figure 18: The wave function renormalization factor $Z_q$ expressed in the $\overline{\text{MS}}$ scheme. These results are obtained by applying the perturbative running factor to $Z_q^{\text{SI}}$. The value we are interested in is $Z_q^{\overline{\text{MS}}} (\mu = 2 \text{ GeV})$. The upper and lower curves show the statistical errors.
Figure 19: A plot of $\frac{1}{2}(\Lambda_A + \Lambda_V)/\Lambda_T$ as a function of quark mass as well as the linear extrapolation to the chiral limit, at $(ap)^2 = 1.388$, or $\mu = 2.04$ GeV
Figure 20: The quantities $Z_{RI/MOM}^T(\mu)$ and $Z_{SI}^T(\mu)$ plotted versus the square of the scale $a\mu$. Here $Z_{SI}^T(\mu)$ is obtained by dividing $Z_{RI/MOM}^T(\mu)$ by the predicted perturbative running factor. Shown also is the linear extrapolation of $Z_{SI}^T(\mu) = Z_{SI}^T + c(a\mu)^2$ using the momentum region $1.3 < (a\mu)^2 < 2.5$ to remove lattice artifacts.
Figure 21: The wave function renormalization factor $Z_T$ expressed in the $\overline{\text{MS}}$ scheme. These results are obtained by applying the perturbative running factor to $Z^{\text{SI}}_T$. The value we are interested in is $Z^{\text{MS}}_T (\mu = 2 \text{ GeV})$. The upper and lower curves show the statistical errors.
Figure 22: The mixing coefficient $F M_{V^+AA, V^-AA}^{-1}$ for our three unitary mass values and linearly extrapolated to the chiral limit.
Figure 23: Linear extrapolation of the mixing coefficient $F M_{VV+AA,VV−AA}^{-1}$ to the chiral limit using the three unitary mass values, at the momentum scale $\mu = 2.04$ GeV.
Figure 24: A possible identification of subgraphs appearing in the chirality violating mixing between $O_{LL}$ and other four-quark operators. The disconnected subdiagram $\Gamma'$ has degree of divergence $d = -2$ for the case of exceptional momenta shown here. This permits a complex pattern of low-energy, vacuum chiral symmetry breaking coming from the low-energy, four-quark subgraph $\Gamma_1$ to enter such an amplitude with only a mild $1/p^2$ suppression.
Figure 25: The mixing coefficient $F_{M^{-1}_{VV+AA,VV-AA}}$ calculated at non-exceptional momenta. When extrapolated to the chiral limit the mixing coefficient vanishes, which shows that chiral symmetry breaking as shown in Fig. 22 comes from the existence of a low-energy sub-diagram that enters because of the special choice of external momenta.
Figure 26: The mixing coefficient $F_{VV+AA,SS-PP}^{-1}$ for unitary choices of the mass.
Figure 27: The mixing coefficient $F_{VV+AA,SS-PP}^{-1}$ calculated at non-exceptional momenta. When extrapolated to the chiral limit the mixing coefficient vanishes, which shows that chiral symmetry breaking as shown in Fig. 26 comes from the existence of a low-energy sub-diagram that enters because of the special choice of external momenta.
Figure 28: The mixing coefficient $F M^{-1}_{VV+AA,SS+PP}$ for unitary choices of the mass. The coefficients are very tiny over the region of medium to large momenta.
Figure 29: The mixing coefficient $F_{VV+AA,TT}^{-1}$ for unitary choices of the mass. These coefficients agree well with zero.
Figure 30: Linear extrapolation of $Z_{B_K}$ to the chiral limit using unitary mass values and the scale $\mu = 2.04$ GeV.
Figure 31: The renormalization factor $Z_{B\kappa}^{RI/MOM}$ evaluated for unitary mass values and extrapolated to the chiral limit.
Figure 32: The quantities $Z_{B_K}^{RI/MOM}$ and $Z_{B_K}^{SI}$ plotted versus the square of the scale $a\mu$. Here $Z_{B_K}^{SI}$ is obtained by dividing $Z_{B_K}^{RI/MOM}$ by the predicted perturbative running factor. Shown also is the linear extrapolation of $Z_{B_K}^{SI}(\mu) = Z_{B_K}^{SI} + c(a\mu)^2$ using the momentum region $1.3 < (a\mu)^2 < 2.5$ to remove lattice artifacts.
Figure 33: The renormalization factor $Z_{B_K}$ expressed in the $\overline{\text{MS}}$ scheme. These results are obtained by applying the perturbative running factor to $Z_{B_K}^{\text{SI}}$. The value we are interested in is $Z_{B_K}^{\overline{\text{MS}}} (\mu = 2 \text{ GeV})$. The upper and lower curves show the statistical errors.