Continuum approach to real time dynamics of 1+1D gauge field theory: out of horizon correlations of the Schwinger model

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We develop a truncated Hamiltonian method to study nonequilibrium real time dynamics in the Schwinger model - the quantum electrodynamics in D=1+1. This is a purely continuum method that captures reliably the invariance under local and global gauge transformations and does not require a discretisation of space-time. We use it to study a phenomenon that is expected not to be tractable using lattice methods: we show that the 1+1D quantum electrodynamics admits the dynamical horizon violation effect which was recently discovered in the case of the sine-Gordon model. Following a quench of the model, oscillatory long-range correlations develop, manifestly violating the horizon bound. We find that the oscillation frequencies of the out-of-horizon correlations correspond to twice the masses of the mesons of the model suggesting that the effect is mediated through correlated meson pairs. We also report on the cluster violation in the massive version of the model, previously known in the massless Schwinger model. The results presented here reveal a novel nonequilibrium phenomenon in 1+1D quantum electrodynamics and make a first step towards establishing that the horizon violation effect is present in gauge field theory.

Introduction. - Computing real time dynamics of an interacting many-body quantum system is a notoriously difficult problem. It has been currently getting an overwhelming amount of attention due to the fast developing field of nonequilibrium physics both in high energy \cite{1-9} and condensed matter physics \cite{10-12} on one side and renewed interest in chaos and information scrambling on the other side \cite{13-17}. It is also becoming a matter of increased experimental importance \cite{18-21}. The set of tools to deal with the problem has been greatly enriched by developments and new insights in integrability theory \cite{22-24}, holography \cite{25-29} and numerical algorithms such as density matrix renormalisation group (DMRG) \cite{30,31}, tensor networks (TNS) \cite{32-34} and lattice gauge theory \cite{35,36}. Although in the present time, there is an abundance of excellent numerical methods available for discrete systems, the methods for the real time evolution directly in the continuum remain scarce and less developed.

A powerful class of algorithms are the truncated Hamiltonian methods (THM) \cite{37-45}. They are numerical methods for quantum field theories (QFT) that work in the continuum and do not require a discretisation of space-time. They can be applied to a wide set of tasks like computing spectra \cite{37,39-47} and level spacing statistics \cite{48,49}, studying symmetry breaking \cite{45}, correlation functions \cite{50,51}, real time dynamics \cite{50-54} and also gauge field theories \cite{55,56}. The class of methods originates from the truncated conformal space approach (TCSA) introduced by Yurov and Zamolodchikov \cite{37}. A QFT model on a compact domain is regarded as point along the renormalisation group (RG) flow from the ultra violet (UV) fixed point generated by a relevant perturbation. The conformal field theory (CFT) algebraic machinery is used to represent the Hamiltonian as a matrix in the basis of the UV fixed point CFT Hilbert space. Finally, an energy cutoff is introduced to obtain a finite matrix which enables numerical computation that indeed efficiently captures nonperturbative effects. More broadly, instead of CFT, any solvable QFT can be used as the starting point for the expansion.

One of the central properties of quantum physics out of equilibrium is the horizon effect introduced by Cardy and Calabrese \cite{57-59}. A quantum system is initially prepared in a short range correlated nonequilibrium state, \( \langle O(x) O(y) \rangle \propto e^{-|x-y|/\xi} \) with a local observable \( O \), the correlation length \( \xi \), and let to evolve dynamically for \( t > 0 \) - a protocol commonly termed a quantum quench. The horizon bound states that the connected correlations following the quench spread within the horizon: \( \langle [O(t,x) O(t,y)] \rangle < \kappa \xi c_{\text{max}}^2 e^{-(|x-y| - 2ct)/\xi} \) for some constant \( \kappa \), where \( \xi_h \) is called the horizon thickness and \( c \) is the maximal velocity of the theory - speed of light in QFT and the Lieb-Robinson (LR) velocity in discrete systems \cite{60}. The intuition is that correlations spread by pairs of entangled particles created in initially correlated region \( |x-y| \lesssim \xi \) and traveling to opposite directions. This bound has been rigorously proven in CFT \cite{57,58,61} and demonstrated, analytically and numerically in a large set of interacting systems \cite{61-92,92-97} as well as observed in experiments \cite{98-100}. It has therefore been believed to be a universal property of quantum physics.

In a recent publication together with Sotiriadis and Takács \cite{51}, we have demonstrated that the horizon bound can be violated in QFT with nontrivial topological properties. We have proved this in the case of the sine-Gordon (SG) field theory, a prototypical example of
The Schwinger model. - We focus here on the simplest example of a gauge field theory, the 1+1D quantum electrodynamics (QED), i.e. the (massive) Schwinger model:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} \left( i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m \right) \Psi,$$

with $\Psi = (\Psi_-, \Psi_+)^T$ the Dirac fermion, $m$ the electron mass and $e$ the electric charge. As a consequence of invariance under large gauge transformations, the model has infinitely degenerate vacuum states, the $\emptyset$ vacua for a parameter $\theta \in (0, 2\pi)$ that enters the bosonised form of the Hamiltonian and plays the physical role of the constant background electric field [113, 114]. The Schwinger model thus has two physical parameters, the ratio $m/e$ and $\theta$.

The massless $m = 0$ version of the model was solved exactly by Schwinger [115] and has a gap of $e/\sqrt{\pi}$ corresponding to a meson, a bound state of a fermion and an antifermion. The full massive $m > 0$ version of the model is not integrable and has a rich phase diagram where the number of mesons depends on the values of the parameters $m/e$ and $\theta$ [105, 106, 113, 114, 116–128]. The Schwinger model displays confinement and has been extensively studied for pair creation and string breaking [108–113, 129–137].

Finally, it is known that due to the vacuum degeneracy, the massless version of the Schwinger model exhibits cluster violation of correlators of chiral fermion densities $\rho_\pm(x) = N \left[ \bar{\psi}(x) \frac{1 \mp \beta}{2} \psi(x) \right]$, [138–140],

$$\langle \rho_-(x_1) \cdots \rho_-(x_n) \rho_+(y_1) \cdots \rho_+(y_n) \rangle,$$

closely related to the correlators from eq. (2). This makes the model a good candidate for the horizon violation. The cluster violation is also intimately related to confinement of gauge theories [141–144].

Here we study general quenches of the massive Schwinger model and focus on the spreading of the current-current correlators:

$$C_\mu(t, x, y) = \langle J^\mu(t, x) J^\mu(t, y) \rangle.$$

We prepare the system in the ground state of the model with the prequench values of the parameters $m_0/e_0$, $\theta_0$ and at time $t = 0$ switch the parameters to their postquench values $m/e$, $\theta$. As a consequence of the Lieb-Robinson bound [60, 101, 102] and the Araki theorem [103, 104], the horizon violation is expected not to be present in short-range interacting discrete systems with finite local Hilbert space dimension and is likely a genuinely field theoretical phenomenon. Therefore discretising a model and simulating using DMRG or TNS [35, 36, 105–112] is not an option so methods working directly in the continuum are needed and THM seem to be the best class of methods for the task.

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The method. - We implement a THM for the Schwinger model in finite volume $L$ with anti-periodic boundary conditions (Neveu-Schwarz sector). We eliminate the gauge redundancy of degrees of freedom along-side with the bosonisation of the model [145].

Choosing the Weyl (time) gauge, $A_t = 0$, and defining $A \equiv A_x$, the Hamiltonian of the model is

$$H = \int_0^L dx \left( \frac{1}{2} A^2 - \Psi^\dagger (i\partial_x - eA) \Psi \right).$$

Expanding the fermion currents $J_\sigma(x) = \Psi^\dagger_\sigma(x)\Psi_\sigma(x) = \frac{1}{2} \{Q_\sigma - \sigma \sum_{n>0} \sqrt{n} \left( \sigma_{\sigma,n} e^{-\sigma \sqrt{n} \phi} + b^\dagger_{\sigma,n} e^{\sigma \sqrt{n} \phi} \right) \}$ with the chirality $\sigma = \pm$, its modes obey bosonic canonical commutation relations. Further defining the $N_\sigma$ vacua as $|0; N_-\rangle \equiv \prod_{n=N_-}^\infty c^\dagger_{-,n} |0\rangle$, $|0; N_+\rangle \equiv \prod_{n=N_+}^\infty c^\dagger_{+,n} |0\rangle$, with $c_{\sigma,n}$ the fermion mode operators, the Hilbert space spanned by bosonic modes $b^\dagger_{\sigma,n}$ on top of $|0; N_-\rangle \otimes |0; N_+\rangle$ is equivalent to the Hilbert space spanned by $c^\dagger_{\sigma,n}$ acting on top of $|0\rangle$. This is the foundation for the bosonisation of the model. Because of the invariance under large gauge transformations, the true vacua of the system are the infinitely degenerate $\theta$ vacua $|\theta\rangle = \sum_{N \in \mathbb{Z}} e^{-iN\theta} |0; N\rangle$ for $\theta \in [0,2\pi)$. Gauge invariance further implies that the only mode of the EM potential $A$ that is not fixed by the Gauss law is the zero mode $\alpha = \frac{1}{L} \int_0^L dx A(x)$ along with its dual $i\partial_t \alpha = \int_0^L dx A(x)$.

By setting $B_0 = \sqrt{\frac{1}{2ML}} (-\sqrt{\pi} (Q_+ - Q_-) + \frac{\partial}{\partial \alpha})$, the part of the Hamiltonian involving the zero modes transforms into a harmonic oscillator with the mass $M = \frac{m}{\sqrt{\pi}}$. Complemented with a Bogoliubov transform of the nonzero momentum modes into massive bosonic modes:

$$B_{\sigma,n} = \frac{1}{\sqrt{Z_n}} \left( \sqrt{\frac{2}{\sqrt{\phi}}} + \sqrt{\frac{Z_n}{\sqrt{\phi}}} \right) b_{\sigma,n} - \frac{1}{2} \left( \sqrt{\frac{2}{\sqrt{\phi}}} - \sqrt{\frac{Z_n}{\sqrt{\phi}}} \right) b^\dagger_{\sigma,n},$$

with $k_n = \frac{2\pi n}{L}$ and $E_n = \sqrt{M^2 + k_n^2}$, the massless part of the Hamiltonian is transformed into the Hamiltonian of a massive free boson with the mass $M$. The mass term of the Hamiltonian is written in the bosonic form using the bosonisation relation $\Psi_\sigma(x) = F_\sigma \frac{1}{\sqrt{L}} e^{-\sigma (\sqrt{4\pi} \Phi(x) - \frac{x}{2})}$; with $\partial_x \Phi_\sigma(x) = \sqrt{\pi} J_\sigma(x) + \frac{\sigma}{\sqrt{4\pi}} A(x)$ the chiral boson field and $F_\sigma$ the Klein factor. Then using $F_\sigma F_{-\sigma}|\theta\rangle = e^{\sigma i\theta}|\theta\rangle$, the Schwinger model Hamiltonian takes the bosonised form

$$H = H_M + U,$$

$$H_M = M \left( B_{0}^\dagger B_0 + \sum_{n>0} E_n \left( B^\dagger_{+,n} B_{+,n} + B^\dagger_{-,n} B_{-,n} \right) \right) ,$$

$$U = -\frac{mM}{2\pi} e^{\gamma} \int_0^L dx : \cos \left( \sqrt{4\pi} \Phi(x) + \theta \right) :_M ,$$

with $\Phi(x) = \Phi_- + \Phi_+$ with Bogoliubov transformed modes, $\cdot :_M$ denotes normal ordering w.r.t. the mass $M$ and $\gamma$ is the Euler-Mascheroni constant.

The form of the Hamiltonian (5) offers a natural THM splitting into the massive free part and the cosine potential. To implement the numerical method, the cosine potential and the observables, are represented as matrices in the Hilbert space of the free part - the Fock space generated by applying the $B^\dagger_{+,n}$ modes on the $\theta$ vacuum. Finally, an energy cutoff $\langle \Psi | H_M | \Psi \rangle \leq E_{cut}$ is imposed on the states $|\Psi\rangle$ of the THM Hilbert space. Momentum conservation implied by translation invariance and the decoupling of the $B_0$ mode from the rest of the modes are used to further reduce the dimension of the Hilbert space by diagonalising each sector separately. We use the Hilbert spaces with up to 20 000 states per sector. The full details of the method can be found in the Supplemental Material [146].

Figure 2: The THM spectrum the Schwinger model at $m/e = 0.125$ in dependence of the system size $L$ in the 0, 1 and 2 sectors of the total momentum. The spectral lines are compared with the $L \to \infty$ results of the MPS computations [127] for the vector and the scalar particles and the TNS [128] for the heavy vector particle. On top of the spectrum, the dominant frequency of the oscillations of the out-of-horizon correlations are plotted.

Results. - Our THM implementation of the Schwinger model recovers the results from the literature for the meson masses and gives a region of highly dense states above them, referred to as the continuum in the $L \to \infty$ limit (fig. 2). This serves as a sanity check of the method. We are able to get the masses of the vector meson precisely, while our THM method seems to be slightly less precise for the scalar meson mass. We have been able to simulate large system sizes $L \gg \frac{1}{M}$ where the finite size effects are exponentially suppressed.

The results shown in fig. 3 indeed confirm that the Schwinger model exhibits the horizon violation effect - the correlation functions $C_x(t,x,y)$ are nonzero and oscillating for $|x - y| > 2t$. The effect is found in quenches in both $e/m$ and $\theta$ as well as in quenches to and from the massless Schwinger model. The sign of the out-of-horizon correlations changes depending whether the quenched parameter is increased or decreased. As is expected for periodic boundary conditions, the effect is present in the $C_x$ and not present in the $C_t$ channel.
To shed light on the origin of the effect, we study the clustering properties of correlators of chiral densities \((3)\) (fig. 3, lower right), more specifically, its component \(\langle \psi_\sigma^\dagger(x) \psi_{-\sigma}(x) \psi_{-\sigma}(y) \psi_\sigma(y) \rangle\). We find that the correlator violates clustering - when \(x\) and \(y\) are far apart, the correlator does not cluster into \(\langle \psi_\sigma^\dagger(x) \psi_{-\sigma}(x) \rangle \langle \psi_{-\sigma}(y) \psi_\sigma(y) \rangle\). In case of the massless Schwinger model, this clustering violation is well known and can be computed analytically [138–140], in case of the massive version of the model, this is to our knowledge a new result. Interestingly, in the massless case, the normal ordered version of the correlator does not exhibit the clustering violation while in the massive model, even the normal ordered correlator violates clustering. We expect that similarly as in the SG model [51], the nonlinear postquench dynamics rotates the initial clustering violation from such chiral correlators into the local nonchiral observables. We note that in case of the ground states of the massive model, we observe numerically a tiny clustering violation also in the \(C_\sigma\) correlators which is two orders of magnitude smaller than the cluster violation of \(\langle \rho_\sigma \rho_{-\sigma} \rangle\). We expect, however, that this is not a physical fact but an artifact of the THM truncation. Such tiny artifacts are common in derivative fields but do not falsely produce the horizon violation effect, as was for example verified in case of Klein-Gordon dynamics in the first version of the model [51, 147]. As well as that, our THM simulation of the Schwinger model displays the horizon violation in the quenches starting from the massless model, where there are no such artifacts in the initial state. So we expect that the effect originates fully from the cluster violation of the chiral terms.

Fig. 2 shows how the dominant frequencies of the oscillations compare to the spectrum of the model. Due to the simulation times limited to \(t \leq L/4\), we are only able to see a few oscillations. Therefore, the frequencies have considerable error bars \((\Delta \omega \approx 2\pi/L)\) half a frequency bin) and the values of the possible discrete frequencies move with \(L\) resulting in a chainsaw pattern. The error bars compare to both the scalar meson mass and twice the vector meson mass. Based on the mechanism of the effect in the SG model [51], it is expected that the frequencies correspond to twice the mass of the lightest meson. This is supported by computations at higher values of \(m/e\), where those masses can be better discriminated (Fourier spectrum in the upper right of fig. 3). This suggest that the horizon violation is mediated through correlated vector meson pairs entangled by the quench. In some cases even subdominant peaks appear close to twice the masses of heavier mesons in the frequency spectra, suggesting that they could also be contributing to the effect.

**Discussion.** - We stress again that the observed phenomenon is in no contradiction with relativistic causality as guaranteed by the Lorentz invariance of the model the micro causality of the fields. Rather, the violation of horizon can be likely traced back to the cluster violation of chiral fermion fields as in the SG model [51].

Using the simplest representative, we have hereby demonstrated that the horizon violation occurs in gauge field theory. In the future, it would be interesting to explore higher gauge theories like SU(2) or SU(3) or study the Wess–Zumino–Witten models. It would be of crucial
importance to answer whether the effect is present also in $D > 1 + 1$. There, gauge fields are dynamical, so the physics could be drastically different. Further analytical approaches should be found to get a better understanding of the effect in the Schwinger model.

![Diagram](image)

Figure 4: Decay of the anisotropic initial condition or a $\theta$ term in a toy universe as a quench that generates long range correlations through the horizon violation effect. Long range correlations are the price that the toy universe has to pay for the initial anisotropy.

The horizon violation presented in this work is a novel phenomenon in 1+1D quantum-electrodynamics. It is reasonable to expect that it could have interesting physical implications, in particular if it turns out that the effect is present also in higher dimensions. In condensed matter physics, phase transitions are an ubiquitous phenomenon and could serve as a trigger for horizon violation generating quenches. Here, already the $D = 1 + 1$ case could be an interesting candidate since at the present day there are numerous experiments available for probing 1+1D physics [10]. An especially important class are ultra cold atoms in atom chips, where one dimensional QFTs are directly realised and correlation functions can be measured both in equilibrium states and nonequilibrium dynamics [148]. In cosmology, there several candidates for quenches like the end of inflation, the QCD and the electroweak transitions and topological symmetry breaking in grand unified theories [149–152]. Consider also the following example illustrated in fig. 4: a toy universe is created with an anisotropic initial condition - a nonzero background electric field. This is a possibility since the zero background field case is a special, fine-tuned, value. In $D = 1 + 1$ the background electric field is stable while in $D = 1 + 3$, it decays through the electric breakdown of the vacuum [114]. The rapid decay of the background electric field would serve as a quench that causes a horizon violation effect in the QED degrees of freedom as we have seen here in the $\theta_0 \neq 0 \rightarrow \theta = 0$ quenches. This transforms the initial anisotropy of the toy universe into long range correlations. Similarly, in a higher gauge theory the effect could be triggered by a decay of the theta term which is linked in some models with the cosmological constant [153, 154]. It would be interesting to explore the possible predictions for traces of this effect in the cosmic microwave background.

Finally, it would be interesting to use THM to explore the confinement and string breaking phenomena in the Schwinger model and to use THM implementations [56] to study dynamics of higher gauge theories.

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Supplemental Material

Details of the THM for the Schwinger model

Here we discuss the details of the truncated Hamiltonian method (THM) implementation of the Schwinger model defined on an interval of length $L$ with anti-periodic boundary conditions.

Bosonisation

The Schwinger model, the quantum electrodynamics in $D = 1 + 1$, is defined by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi} (i \gamma^\mu \partial_\mu - e \gamma^\mu A_\mu - m) \Psi$$

where $\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}$ is the Dirac fermion field, $A_\mu$ the electromagnetic (EM) potential, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the EM tensor, $m$ is the electron mass and $e$ the electric charge. Choosing the Weyl (time) gauge, $A_0 = 0$, defining $A \equiv A_x$, the Hamiltonian density takes the form:

$$H = H_{\text{EM}} + H_F + H_m,$$

with

$$H_{\text{EM}} = \frac{1}{2} \dot{A}^2,$$

$$H_F = -\bar{\Psi} \gamma^1 (i \partial_x - eA) \Psi,$$

$$H_m = m \bar{\Psi} \Psi.$$

We use here the metric $g = \text{diag}(1, -1)$ and the gamma matrices $\gamma^0 = -\sigma_1$, $\gamma^1 = i \sigma_2$.

In order to treat a gauge field theory in Hamiltonian formalism, one has to remove the redundancy in the degrees of freedom coming from gauge invariance. For the Schwinger model this goes hand in hand with bosonisation. Bosonisation is an exact duality between fermionic and bosonic theories in 1+1D relativistic QFT, developed by Tomonaga, Mattis, Lieb, Mandelstam, Coleman, Haldane and others [113, 155–161]. We bosonise the Schwinger model here using the operatorial (constructive) approach of Iso and Murayama [145].

The eigenfunctions and eigenvalues of the massless fermion part of the Hamiltonian $H_F(x) = \sum_{\sigma = \pm} \sigma \Psi^\dagger(x) (i \partial_x - eA) \Psi(x)$ are $(i \partial_x - eA) \psi_n = \epsilon_n \psi_n$:

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{-i(\epsilon_n x + e \int_0^x dx A)},$$

$$\epsilon_n = \frac{2\pi}{L} \left( n + \frac{1}{2} \delta_b - \frac{e\alpha L}{2\pi} \right).$$

The eigenfunctions satisfy periodic boundary conditions (Ramond sector) for $\delta_b = 0$ and anti-periodic (Neveu-Schwarz sector) for $\delta_b = 1$. The fermion bosonises to a boson with periodic boundary conditions for both values of $\delta_b$ reflecting the fact that the bosonisation is an equivalence between bosons and fermions up to $\mathbb{Z}_2$. We will derive the equations for general $\delta_b$ and in the end take $\delta_b = 1$, the Neveu-Schwarz sector of the fermion, for the THM study. We quantise the fermion field by expansion

$$\Psi(x) = \sum_{n \in \mathbb{Z}} c_{-,n} \psi_n(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_{+,n} \psi_n(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with the canonical anticommutation relation $\{ c_{\sigma,n}, c^\dagger_{\rho,m} \} = \delta_{\sigma,\rho} \delta_{n,m}$. Then $H_F = H_+ + H_-$ with $H_\sigma = \sigma \sum_{n \in \mathbb{Z}} \epsilon_n c_{\sigma,n}^\dagger c_{\sigma,n}$.

We expand the EM potential and its conjugate dual as:

$$A(x) = \alpha + \sum_{n \neq 0} A_n e^{i \frac{2\pi}{L} nx},$$

$$\dot{A}(x) = i \frac{\delta}{\delta A(x)} = i \frac{\delta}{\delta \alpha} + \sum_{n \neq 0} \frac{\delta}{\delta A_n} e^{-i \frac{2\pi}{L} nx}.$$
and we shall see that as a consequence of the gauge invariance only the zero modes of the EM field $\alpha$ and $i \frac{\partial}{\partial \alpha}$ are dynamical \[145, 162].

Expanding the fermion currents

$$J_\sigma(x) = \Psi_\sigma^+(x)\Psi_\sigma(x) = \frac{1}{L} \left[ Q_\sigma - \sigma \sum_{n>0} \sqrt{n} \left( b_{\sigma,n} e^{-\sigma \text{in} \frac{2\pi}{L} x} + b^\dagger_{\sigma,n} e^{\sigma \text{in} \frac{2\pi}{L} x} \right) \right],$$ (11)

its modes $b_{\sigma,n} = \frac{-\sigma}{\sqrt{n}} \sum_{k \in \mathbb{Z}} c^\dagger_{\sigma,k} e^{\text{in} \frac{2\pi}{L} x}$ obey canonical commutation relations $[b_{\sigma,n}, b^\dagger_{\mu,m}] = \delta_{\sigma,\mu} \delta_{n,m}$. Further defining the $N_\sigma$ vacua as

$$|0; N_-\rangle \equiv \prod_{n=0}^\infty c^\dagger_{-,n} |0\rangle, \quad |0; N_+\rangle \equiv \prod_{n=-\infty}^{N_+-1} c^\dagger_{+,n} |0\rangle,$$

it can be shown that the Hilbert space spanned by excitations with all possible combinations of $b^\dagger_{\sigma,n}$ on top of $|0; N_-\rangle \otimes |0; N_+\rangle$ is equivalent to the Hilbert space spanned by all the possible combinations of $c^\dagger_{\sigma,n}$ on top of $|0\rangle$. This is the core of bosonisation.

The fermion number operators take the following expectation values on the $N_\sigma$ vacua,

$$\langle Q_\sigma \rangle_{N_\sigma} = \sigma \left( N_\sigma - \frac{e\alpha L}{2\pi} + \frac{1}{2} (\delta_b - 1) \right),$$

as can be shown by regularisation by Hurwitz zeta resummation. Similarly,

$$\langle H_\sigma \rangle_{N_\sigma} = \frac{2\pi}{L} \left[ \frac{1}{2} \langle Q_\sigma \rangle_{N_\sigma}^2 - \frac{1}{24} \right],$$

for $H = \int_0^L dx \mathcal{H}$.

**Gauge invariance.** - The fermionic Hilbert space combined with the Hilbert space generated by the modes of the EM modes display a redundancy of degrees of freedom as is characteristic for gauge invariant theories and we have to eliminate this redundancy. QED is invariant under the transformations

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \lambda(x), \quad \psi(x) \rightarrow e^{i e \lambda(x)} \psi(x),$$

for systems defined on the circle (and other topologies with nontrivial homotopic groups), the gauge transformations can be divided into small gauge transformations where both $\lambda(x)$ and $e^{i e \lambda(x)}$ are single valued and large gauge transformations where $e^{i e \lambda(x)}$ is single valued but $\lambda(x)$ is not. Mathematically speaking, small gauge transformations are homotopic to the identity of the Lie group and large gauge transformations are not.

Let’s begin with small gauge transformations. As a consequence of the Dirac conjecture \[163, 164\] these are represented in the Hilbert space by the operator $U(\lambda) = \exp(-i \int_0^L dx G(x) \lambda(x))$ with the Gauss law generator

$$G(x) = \partial_x \left( -i \frac{\delta}{\delta A(x)} \right) - e J_0(x)$$

with $J_0 = J_+ + J_-$. Requiring that the physical states are invariant under $G$ using the expansions (11), (10) gives

$$Q \mid \text{physical state} \rangle = 0$$

$$\left\{ \frac{\delta}{\delta A_n} + \frac{eL}{\sqrt{n} 2\pi} \left( b^\dagger_{-,n} - b_{-,n} \right) \right\} \mid \text{physical state} \rangle = 0$$

$$\left\{ \frac{\delta}{\delta A_{-n}} + \frac{eL}{\sqrt{n} 2\pi} \left( b^\dagger_{+,|n|} - b_{-,|n|} \right) \right\} \mid \text{physical state} \rangle = 0$$

(17)

For $Q = Q_+ + Q_-$. Taking into account (13), the first constraint means that $N_- = N_+ = N$ in physical states and we can define the $N$ vacua as

$$|0; N\rangle \equiv |0; N_-\rangle \otimes |0; N_+\rangle.$$

The second and the third constrain mean that all the nonzero momentum modes of the EM field are fixed by gauge invariance and the only dynamical modes of the EM field are the zero modes $\alpha$ and $i \frac{\partial}{\partial \alpha}$ \[145, 162\]. It is also easy to
see that under small gauge transformations the wave functions (8) transform as \( \psi_n(x) \rightarrow e^{ie\lambda(x) - i\epsilon\lambda(0)} \psi_n(x) \) and thus \( c_{\sigma,n} \rightarrow e^{i\epsilon(\lambda)} c_{\sigma,n} \). It is clear that the currents and its momentum modes, including the charges are invariant under all gauge transformations.

The homotopy group of \( U(1) \) symmetry is \( \pi_1(U(1)) = \mathbb{Z} \) and large gauge transformations are generated by

\[
\lambda(x) = \frac{2\pi}{cL} w x, \quad w \in \mathbb{Z}
\]

(19)

The wave functions \( \psi_n(x) \) are invariant under those thus the fermion operators transform as \( c_n \rightarrow c_{\sigma,n+w} \). Consequently, the \( N \) vacua transform as \( |0;N\rangle \rightarrow |0;N+w\rangle \). The large gauge transformations commute with the Hamiltonian so they can be diagonalised in the same basis. The eigenstates of large gauge transformations are the \( \theta \) vacua

\[
|\theta\rangle = \sum_{N \in \mathbb{Z}} e^{-iN\theta} |0;N\rangle, \quad \theta \in [0, 2\pi)
\]

(20)

which form a continuous degenerate family of ground states of the fermionic parts of the Schwinger model Hamiltonian. A ground state of the full Hamiltonian is obtained as a tensor product of the ground state of the EM part with a \( \theta \) vacuum.

**Hamiltonian.** Following from the gauge invariance constraints (17) we have

\[
H_{\text{EM}} = -\frac{1}{2L} \left[ \left( \frac{\partial}{\partial \alpha} \right)^2 + 2 \left( \frac{eL}{2\pi} \right)^2 \sum_{n>0} \frac{1}{n} \left( \sigma_{+,n}^\dagger - \sigma_{-,n}^\dagger \right) \left( \sigma_{+,n} - \sigma_{-,n} \right) \right].
\]

(21)

Taking into account that \( [H_F, b_{\sigma,n}^\dagger] = \frac{2\pi}{L} nb_{\sigma,n}^\dagger \) and deducing its zero mode content from (14), the Hamiltonian \( H_F \) can only take the form

\[
H_F = \frac{2\pi}{L} \sum_{\sigma = \pm} \left[ \frac{1}{2} Q_\sigma^2 - \frac{1}{24} + \sum_{n>0} nb_{\sigma,n}^\dagger b_{\sigma,n} \right].
\]

(22)

We can then split the massless part of the Schwinger model Hamiltonian into a part with zero modes and a part with nonzero momentum modes:

\[
H_{\text{EM}} + H_F = H_0 + \sum_{n>0} H_n - \frac{2\pi}{12L}
\]

\[
H_0 = \frac{2\pi}{L} \left( \frac{Q^2 + Q_5^2}{4} \right) - \frac{1}{2L} \left( \frac{\partial}{\partial \alpha} \right)^2
\]

\[
H_n = \frac{2\pi}{L} \left( b_{\sigma,n}^\dagger b_{\sigma,n} + b_{\sigma,n}^\dagger b_{\sigma.n} - \frac{e^2L}{4\pi n} \left( b_{+,n}^\dagger - b_{-,n}^\dagger \right) \left( b_{+,n} - b_{-,n} \right) \right),
\]

(23)

with \( Q_5 = Q_+ - Q_- \) which takes the value \( Q_5 = 2N - \frac{eL}{\pi} + \delta_b - 1 \) on physical states and we keep in mind that \( Q = 0 \) on physical states.

The zero mode Hamiltonian \( H_0 \) is the Hamiltonian of a massive harmonic oscillator with mass

\[
M = \frac{e}{\sqrt{\pi}}
\]

(24)

and can be written in the canonical form as

\[
H_0 = M \left( B_0^\dagger B_0 + \frac{1}{2} \right)
\]

(25)

with \( B_0 = \sqrt{\frac{1}{2\pi L}} \left( -\sqrt{\pi}Q_5 + \frac{\phi}{\sqrt{\pi}Q_5} \right), B_0^\dagger = \sqrt{\frac{1}{2\pi L}} \left( -\sqrt{\pi}Q_5 - \frac{\phi}{\sqrt{\pi}Q_5} \right) \).

The nonzero momentum Hamiltonians \( H_n \) can be diagonalised with a Bogoliubov transformation

\[
B_{\sigma,n} = \cosh(t_n)b_{\sigma,n} + \sinh(t_n)b_{\sigma,n}^\dagger
\]

\[
\cosh(t_n) = \frac{1}{2} \left( \frac{\sqrt{E_n}}{\sqrt{k_n}} + \frac{\sqrt{k_n}}{\sqrt{E_n}} \right)
\]

\[
\sinh(t_n) = \frac{1}{2} \left( \frac{\sqrt{E_n}}{\sqrt{k_n}} - \frac{\sqrt{k_n}}{\sqrt{E_n}} \right)
\]

(26)
with \( k_n = \frac{2\pi n}{L} \) and \( E_n = \sqrt{M^2 + k_n^2} \). Then

\[
H_n = E_n \left( B_{+,n}^1 B_{+,n} + B_{-,n}^1 B_{-,n} + 1 \right)
\]

(27)

and \( H_{EM} + H_F \) becomes the Hamiltonian of the free massive boson with the mass \( M \). This reproduces the Schwinger’s result that the QED in \( D = 1 + 1 \) is gaped even if the bare mass of the fermion is zero. The Bogoliubov operator of this transformation, \( U_n b_{\sigma,n} U_n^\dagger = B_{\sigma,n} \), is the squeezing operator \( U_n = \exp \left[ -t_n \left( B_{+,n}^1 B_{-,n}^1 - B_{+,n} B_{-,n} \right) \right] \) meaning that the vacua annihilated by the massive modes \( B_{\sigma,n} \) are the squeezed coherent \( \theta \) vacua

\[
|\theta\rangle_M = \left( \prod_{n>0} U_n \right) |\theta\rangle.
\]

(28)

It remains to treat the mass term in the Hamiltonian, \( H_m \). We can express it in terms of the bosonic momentum modes of the currents, \( b_{\sigma,n} \) using the relation

\[
\Psi_\sigma(x) = F_\sigma \frac{1}{\sqrt{L}} e^{-\sigma i (\sqrt{\pi} \Phi_\sigma(x) - \frac{x}{2} \delta_0(x))}.
\]

(29)

with

\[
\Phi_\sigma(x) = \frac{1}{\sqrt{4\pi}} \left\{ \frac{2\pi}{L} Q_\sigma x - i \sum_{n>0} \frac{1}{\sqrt{n}} \left( b_{\sigma,n} e^{-\sigma in \frac{2\pi}{L} x} - b_{\sigma,n}^\dagger e^{\sigma in \frac{2\pi}{L} x} \right) + \sigma e \int_0^x dx' A(x') \right\}
\]

(30)

and with the normal ordering with respect to the modes \( b_{\sigma,n} \), which is the bosonisation relation for a fermion coupled to the EM field. Here, the term with \( \delta_0 \) is to assure that the fermion field satisfies the correct boundary conditions and \( F_\sigma \) are the Klein factors satisfying

\[
[F_\sigma, A_m] = \left[ F_\sigma, \delta A_m \right] = 0,
\]

\[
[F_\sigma, b_{\rho,m}] = \left[ F_\sigma, b_{\rho,m}^\dagger \right] = 0,
\]

\[
\left[ Q_\sigma, F_\sigma^\dagger \right] = \delta_{\sigma,\rho} F_\rho^\dagger,
\]

\[
\left[ Q_\sigma, F_\rho \right] = -\delta_{\sigma,\rho} F_\rho,
\]

\[
\left\{ F_\sigma^\dagger, F_\rho \right\} = 2\delta_{\sigma,\rho},
\]

\[
F_\sigma^\dagger F_\rho = 1.
\]

(31)

Since a function of \( b_{\sigma,n} \) and \( b_{\sigma,n}^\dagger \) can never alter the fermion number, the Klein factors make sure that \( \Psi_\sigma(x) \) as defined above has the true fermionic character. Some authors prefer to use exponentials of the zero modes of the compactified massless boson field in place of the Klein factors and the two conventions are fully equivalent. In particular, it can be shown that \( \{ \Psi_\sigma(x), \Psi_\rho(y) \} = \delta_{\sigma,\rho} \delta(x-y) \). Using the relations \( \left[ \frac{\delta}{\delta A_m}, b_{\sigma,n}^\dagger \right] = \left[ \frac{\delta}{\delta A_m}, b_{+,-n} \right] = \left[ \frac{\delta}{\delta A_m}, b_{+,-n}^\dagger \right] = \frac{eL}{2\pi\sqrt{n}} \delta_{\sigma,\rho} \delta(x-y) \) which follow from (17) it’s easy to see that \( \left[ \frac{\delta}{\delta A(x)}, \Psi_\sigma(y) \right] = 0 \). Finally, considering that \( F_\sigma \to e^{i\lambda(0)} F_\sigma \) under gauge transformations, if follows that \( \Psi_\sigma(x) \) transforms as the fermion field. We also have, as follows from the second line of (31) that \( F_\sigma^\dagger F_{-\sigma} |0; N\rangle = |0; N + \sigma\rangle \) and thus

\[
F_\sigma^\dagger F_{-\sigma} |\theta\rangle_M = e^{\sigma i \theta} |\theta\rangle_M.
\]

(32)

Using the definition (30) we can also read out the fermionisation relation for the fermion field coupled to the EM field, the inverse of the bosonisation relation:

\[
\partial_x \Phi_\sigma(x) = \sqrt{\pi} J_\sigma(x) + \frac{\sigma e}{2\sqrt{\pi}} A(x).
\]

(33)

We can use the bosonisation relation (29) to express the mass term in the Hamiltonian as

\[
H_m = -m \frac{1}{L} e^{\sum_{n>0} \frac{i}{2} \left( 1 - \frac{2\pi n}{L} \right)} \int_0^L dx \sum_{\sigma = \pm} e^{\sigma i \frac{2\pi}{L} (1-\delta_0(x))}; e^{\sigma i \sqrt{4\pi} \Phi(x)}; M F_\sigma^\dagger F_{-\sigma},
\]

(34)

where we have defined \( \Phi(x) \equiv \Phi_+(x) + \Phi_-(x) \) and \( \bullet : M \) denotes normal ordering with respect to the massive modes \( B_{\sigma,n} \). The prefactor \( e^{\sigma i \frac{2\pi}{L} x} \) comes from commuting \( F_{-\sigma} \) past \( e^{i\sigma \sqrt{4\pi} \frac{Q}{2 \pi \sqrt{n}}} \) using (31). The prefactor \( e^{\sum_{n>0} \frac{i}{2} \left( 1 - \frac{2\pi n}{L} \right)} \) comes from substituting the Bogoliubov transform (26) into \( \Phi_\sigma(x) \) and then rearranging the expression for \( H_m \) into
the normal ordered form w.r.t. $M$. In the $L \to \infty$ limit these prefactors take the value $\frac{ML}{4\pi} e^\gamma$ where $\gamma = 0.5772 \ldots$ is the Euler-Mascheroni constant.

Finally, putting all the terms together, with the $L \to \infty$ expression for the prefactor in the mass term, the Schwinger model Hamiltonian takes the form

$$H = M \left( B_0^\dagger B_0 \right) + \sum_{n \geq 0} E_n \left( B_{+,n}^\dagger B_{+,n} + B_{-,n}^\dagger B_{-,n} \right) + \text{const}$$

$$- \frac{mM}{4\pi} e^\gamma \int_0^L dx \sum_{\sigma = \pm} e^{\sigma i \frac{2\pi}{L} (1 - \delta_b)x} : e^{\sigma i \sqrt{4\pi}\Phi(x)} :_M F_{\sigma}^+ F_{\sigma}^\dagger$$

where $\text{const} = \sum_{n \geq 0} E_n + \frac{1}{2} M$ only affects the ground state energy and will be irrelevant to us. The last "equality" is to be understood only up to the details of the modes captured through the above bosonisation procedure and has taken into account (32) and the fact that all the physical states are created on top of the $\theta$ vacuum. The parameter $\theta$ thus appears in the Hamiltonian and plays the role of the constant background electric field as first pointed out by Coleman [113, 114]. As is manifest in the first line, the zero mode $B_0$ does not enter in the cosine term and is a harmonic oscillator decoupled from the other degrees of freedom. For our THM implementation, we choose $\delta_b = 1$, the anti-periodic boundary conditions for the fermion, the Neveu-Schwarz sector.

**Hilbert space.** - As has been made explicit in the above discussion, the Hilbert space of the Schwinger model after eliminating the gauge redundancy takes the form of the tensor product of the Hilbert space of the zero modes with the Hilbert space generated by all the possible bosonic excitations on top of the theta vacuum. All together we can write any state in the Hilbert space in the form

$$|\vec{r}\rangle \equiv \frac{1}{N_{\vec{r}}} \left( B_0^\dagger \right)^{r_0} \prod_{n=1}^{\infty} \left( B_{+,n}^\dagger \right)^{r_{+,n}} \left( B_{-,n}^\dagger \right)^{r_{-,n}} |0\rangle_0 \otimes |\theta\rangle_M$$

where $\vec{r} = (r_0, r_{-1}, r_{-2}, \ldots, r_{+1}, r_{+2}, \ldots)$ is a vector of occupation numbers and $|0\rangle_0$ is the vacuum of the $B_0$ mode. The normalisation is $N_{\vec{r}}^2 = (r_0!) \prod_{k=1}^{\infty} (r_{k,-}!)(r_{k,+})!$.

**Truncated Hamiltonian method.**

The truncated Hamiltonian method (THM) consists of splitting the Hamiltonian into an analytically solvable and an unsolvable part, the perturbing operator and the observables as matrices in the eigenbasis of the solvable part. Finally, an energy cutoff is introduced which renders the matrices finite and enables numerical diagonalisation which is the key to nonperturbative treatment of a strong interaction with the perturbing potential. Then, expressing the perturbing operator and the observables as matrices in the eigenbasis of the solvable part, the THM implementation, we choose $\delta_b = 1$, the anti-periodic boundary conditions for the fermion, the Neveu-Schwarz sector.

**Matrix elements.** - The matrix elements are computed between general states of the Hilbert space $|\vec{r}\rangle$ and $|\vec{r}'\rangle$, defined in eq. (36). The required matrix elements are:

**Boson mode operators:**

$$\langle \vec{r}' | B_{\sigma,n}^\dagger | \vec{r} \rangle = \left( \prod_{\rho,k \neq \sigma,n} \delta_{\sigma,n} \right) \sqrt{(r_{\sigma,n} + 1) \delta_{r_{\sigma,n+1}, r_{\sigma,n}}}$$

$$\langle \vec{r}' | B_{\sigma,n} | \vec{r} \rangle = \left( \prod_{\rho,k \neq \sigma,n} \delta_{\sigma,n} \right) \sqrt{r_{\sigma,n} \delta_{r_{\sigma,n+1}, r_{\sigma,n}}}$$

**Boson number operator:**

$$\langle \vec{r}' | B_{\sigma,n}^\dagger B_{\sigma,n} | \vec{r} \rangle = r_{\sigma,n} \delta_{\vec{r}', \vec{r}}$$
Vertex operator: To implement the cosine potential we need the matrix elements

$$\langle \hat{r} \rangle : e^{\rho \sqrt{\Phi}(x)} : M F^*_\rho F - \rho \hat{r} \rangle =$$

$$= e^{\rho \theta} \langle \hat{r} \rangle \prod_{\sigma = \pm} e^{-i \sigma k_n x} B^\dagger_{\sigma, n} e^{-i \sigma k_n x} B_{\sigma, n} e^{i \sigma k_n x} | \hat{r} \rangle$$

$$= e^{\rho \theta} \delta_{\rho, \sigma, n} \prod_{\sigma = \pm} \prod_{n = 1}^{\infty} \frac{1}{\sqrt{r_{\sigma, n}}} e^{i \sigma k_n x (r_{\sigma, n} - r_{\sigma, n})}.$$
Recall that \( \lim_{L \to \infty} e^{\sum_{n>0} \frac{1}{2} \left(1 - \frac{k_n^2}{L^2}\right) e^\gamma} = \frac{ML}{4\pi} e^\gamma \) where \( \gamma \) is the Euler-Mascheroni constant so that the explicit \( L \) dependence cancels out. In fact, we use this limiting expression to get the results closer to the thermodynamic limit. The required matrix element is given by

\[
\langle \vec{p}' | e^{i \sqrt{\gamma} \Phi(x) - \Phi(y)} | \vec{p} \rangle = \\
= \delta_{\vec{p}', \vec{p}} \prod_{\sigma = \pm} \prod_{n=1}^{\infty} \frac{1}{\sqrt{\sigma_n}} \sum_{j_{\sigma,n}^{\partial} = 0}^{\infty} \sum_{j_{\sigma,n} = 0}^{\infty} (-1)^{j_{\sigma,n}} \left( \sqrt{\frac{2\pi}{L}} \rho \frac{E_k}{E_{\sigma,n} + j_{\sigma,n}} \right)^{j_{\sigma,n}^{\partial} + j_{\sigma,n}} \cdot \left( e^{i \sigma_k x} - e^{i \sigma_k y} \right)^{j_{\sigma,n}} \left( e^{-i \sigma_k x} - e^{-i \sigma_k y} \right)^{j_{\sigma,n}} \left( B_{\sigma,n} \right)^{j_{\sigma,n}^{\partial}} \left( B_{\sigma,n}^\dagger \right)^{j_{\sigma,n}} \left( B_{\sigma,n}^\dagger \right)^{j_{\sigma,n}^{\partial}} \] (45)

and the expectation value of the boson operators is given by eq. (41).

**Truncation.** We perform the THM truncation by choosing a value for the cutoff energy \( E_{\text{cut}} \) and keeping only those states of the Hilbert space \( |\vec{p}\rangle \) for which \( \langle \vec{p}' | H_{\text{EM}} + H_F | \vec{p} \rangle \leq E_{\text{cut}} \). This results in a better converging code than for example if truncating by keeping a fixed number of momentum modes. The truncation criterium depends on the charge \( e \) and the system size \( L \) as \( E_n = \sqrt{k_n^2 + \frac{e^2}{\sigma}} = \frac{2\pi}{L} \sqrt{n^2 + \frac{L^2}{(2\pi)^2} \frac{e^2}{\sigma}} \) and for fixed \( E_{\text{cut}} \) the number of states in the THM Hilbert space decreases with increasing \( e \) and \( L \). Therefore, in practice the truncation is done by choosing a desired number of states in the THM Hilbert space and then for a given \( e \) and \( L \) finding \( E_{\text{cut}} \) that gives us a Hilbert space size closest to the desired one. In that way we can assure that results obtained at different \( e \) and \( L \) are achieved with comparable Hilbert space sizes.

The size of the Hilbert space that has to be kept in the computer’s memory can be reduced by taking into account the symmetries of the model. Since the zero mode \( B_0 \) is decoupled from the rest of the modes, we can diagonalise the Hamiltonian in each of it’s sectors separately. In particular, for real time dynamics following quenches it is enough to keep the \( \langle B_0^\dagger B_0 \rangle = 0 \) sector where the initial states, the ground states, reside. Furthermore, because of the translation invariance of the model, the ground states are in the the zero total momentum sector \( \rho_{\text{tot}} = \sum_{\sigma = \pm} \sum_{n=1}^{\infty} k_n \langle B_{\sigma,n}^\dagger B_{\sigma,n} \rangle = 0 \) of the Hilbert space, which drastically reduces the number of states that have to be kept in the computer’s memory in order to compute the quench dynamics. We do, however, have to diagonalise the Hamiltonian also in the sectors with other values of the total momentum in order to compute the full spectrum of the model (excited states). For the results presented in this Letter, we use up to 20 000 states per sector.

In case of truncated conformal space approach (TCSA) methods, where the expansion is around a CFT, the renormalisation group theory guarantees that for relevant perturbing operators, the cut-away high energy part of the \( \mathcal{O} \) of the model (excited states). For the results presented in this Letter, we use up to 20 000 states per sector.

**Quench protocol.** In order to study the quench dynamics, one takes for the initial state the ground state \( |\Psi\rangle \) of the prequench Hamiltonian \( H(m_0/e_0, \theta_0, L) \) which can be found by numerical diagonalisation of the Hamiltonian. At \( t = 0 \), the parameters are quenched to the postquench values \( H(m/e, \theta, L) \). The dynamics is computed using the numerical exponentiation of the postquench Hamiltonian:

\[
|\Psi(t)\rangle = e^{-itH} |\Psi\rangle .
\] (46)

Finally, correlators are computed as expectation values on these states

\[
C_{\mu}(t, x, y) = \langle J^\mu(t, x) J^\mu(t, y) \rangle .
\] (47)

**Further results**

Here we list some further results adding more detail to those presented in fig. 3 in the main text.

The effect is found in quenches of either of the parameters of the system, \( e/m \) and \( \theta \) as well as in quenches to and from the massless Schwinger model. The sign of the out-of-horizon correlations changes depending whether the quenched parameter is increased or decreased. Fig 5 gives an overview of these observations.
Figure 5: Time dependent $\langle J^x(t,x)J^y(t,y) \rangle$ and $\langle J^t(t,x)J^t(t,y) \rangle$ correlations for different type of quenches in the Schwinger model (initial correlations subtracted): 1.) Quench in m/e with $m_0 = 0.25$, $m = 0.125$, $\theta_0 = \theta = 0$; 2.) Quench in $\theta$ with $\theta_0 = \pi/4$, $\theta = 0$, $m_0 = m = 0.125$; 3.) Quench from the massless Schwinger model with $m_0 = 0$, $m = 0.125$, $\theta_0 = \theta = 0$; 4.) Quench to the massless model $m_0 = 0$, $m = 0.125$, $\theta_0 = \theta = 0$. All with $e_0 = e = 1$, $L = 40$.

Fig. 6: The correlator $\langle \psi_\sigma^\dagger(x)\psi_{-\sigma}(x)\psi_\sigma^\dagger(y)\psi_{-\sigma}(y) \rangle$ exhibits clustering. While the clustering is restored by normal ordering for the massless model, it is violated in the massive model even for the normal ordered correlator. The magnitude of the correlators depends on both $e/m$ and $\theta$.

Figure 6: Cluster violation of the $\langle \psi_\sigma^\dagger(x)\psi_{-\sigma}(x)\psi_\sigma^\dagger(y)\psi_{-\sigma}(y) \rangle$ correlator at different values of the parameters: 1.) $m = 0$, $\theta = 0$; 2.) $m = 0.125$, $\theta = 0$; 3.) $m = 0.125$, $\theta = \pi/4$. All with $e_0 = e = 1$, $L = 40$.

Fig 7: In quenches to the special value of the parameter $\theta = \pi$, to the mass above the Ising phase transition point, the horizon dynamics is strongly suppressed, resembling the confined dynamics observed in [97]. Note that here we are plotting the $C_t$ correlator for which there is no horizon violation effect.

Figure 7: Suppression of the horizon spreading in quenches to the $\theta = \pi$ line above the Ising phase transition point. Here, $\theta_0 = 0$, $\theta = \pi$, $m_0 = m = 0.5615$, $e_0 = e = 1$, $L = 40$. 