Gentle Perturbations of the Free Bose Gas I.

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Abstract

It is demonstrated that the thermal structure of the noncritical free Bose Gas is completely described by certain periodic generalized Gaussian stochastic process or equivalently by certain periodic generalized Gaussian random field. Elementary properties of this Gaussian stochastic thermal structure have been established. Gentle perturbations of several types of the free thermal stochastic structure are studied. In particular new models of non-Gaussian thermal structures have been constructed and a new functional integral representation of the corresponding euclidean-time Green functions have been obtained rigorously.

Key words: free Bose Gas, $W^*$-KMS structure, periodic generalized stochastic process, gentle perturbations, multitime Green functions.

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1 Introduction

A variety of existence and analycity results — also constructive ones — have been rigorously obtained for some realistic models of nonrelativistic quantum matter in thermal equilibrium (see [1, 2, 3, 4, 5, 6, 7, 8]). Nevertheless, a number of basic questions on the origin of the fundamental macroscopic quantum phenomena such as superconductivity, superfluidity, etc. [9, 10] are still out of rigorous demonstrations in the above mentioned realistic treatment. Only for mean-field-like and exactly solvable models a mathematically well defined analysis of these phenomena has been performed [11, 12, 13]. It is worthwhile to mention here the recent activity on the superconductivity problem in Fermi matter models of physical interest [14, 15], which is based on the rigorous renormalization group approach invented by Gallavotti and his coworkers [16].

The main objective of the present series of papers is to construct a class of models of selfinteracting nonrelativistic Bose matter in a thermal equilibrium for which a rigorous discussion of the Bose-Einstein condensation, as well as other phase transitions, would be feasible. In order to approach this goal, we intend to use extensively methods inferred from the Constructive Euclidean QFT. In the first paper of the planned series the stochastic content of the fundamental $W^*$ — $KMS$ structure of a free, noncritical Bose gas [17] is described. We prove that the abelian sector of the Weyl algebra may be described by a certain generalized periodic stochastic process with values in $D'(\mathbb{R}^d)$ (the space of the Schwartz distributions) and, what is more, that a reconstruction of the whole thermal structure can be derived from it (see prop 2.5 below). Similar situation do also occur in the case of the critical Bose gas when the underlying process is nonergodic [18]. Having described a free Bose gas in terms of stochastic processes, one may perturb them with multiplicative (-like) functionals, thereby creating some new non-Gaussian thermal processes. Furthermore, given such a process, one is able to reproduce its $W^*$ KMS counterpart by methods of [17, 19, 20]. In this article we shall confine ourselves to the simplest case of perturbations, which we have called (after [21]) gentle perturbations of a free thermal process. Using standard tools of statistical mechanics [22] such as, for example, the Kirkwood-Salsburg analysis, the correlation inequalities of [3], homogeneous limits, we provide a class of Euclidean invariant models of selfinteracting Bose matter that can be controlled rigorously, as we shall demonstrate in
section 3.

The unbounded (of polynomial type) perturbations of a free thermal structure will be studied in another paper of this series [18]. In the critical region nonergodicity is preserved under gentle perturbations (cf. the second part of [18]), but whether this is related to arising of the Bose condensate in an interacting system remains to be resolved.

The pioneering paper of [21] and the following ones [23, 24, 25] have provided, among others, the major inspirations for our own Euclidean attitude to many bosons physics. The methods of classical statistical mechanics have been already applied to the studies of certain quantum systems in [24, 26, 27], and, to some extent, our approach to an interacting Bose gas resembles that of the authors just quoted.

2 Free Bose Gas(es). Euclidean Aspects

The main aim of this section is to point out certain stochastic aspects that arise in the Euclidean time of the thermal structure describing systems of noninteracting Bose particles being in the thermal equilibrium at (inverse) temperature $\beta > 0$ and the chemical activity $z$. Most of the results obtained below apply well to the case when the kinetic energy function $E(p)$ of the individual particle is such that:

(i) $\forall t \in \mathbb{R}_+ e^{iE(p)}$ is positive definite continuous function of $p \in \mathbb{R}^d$ or equivalently that

(ii) $\{e^{iE(-i\nabla)}, t \geq 0\}$ generates a semigroup of positivity preserving operators on $L_2(\mathbb{R}^d)$.

The most general form of such functions is given by the Levi-Khintchine formula (see e.g. [28, 29])

$$E(p) = a + i\mathbf{b} \cdot p + p \cdot C \cdot p - \int \left[e^{i\mathbf{p}x} - 1 - i\mathbf{p}h(x)\right]r(dx)$$  \hspace{1cm} (2.1)

where: $a$ is some real constant, $\mathbf{b}$ is some vector in $\mathbb{R}^d$, $C$ is some nonnegative definite matrix and $r$ is some nonnegative measure on $\mathbb{R}^d$, called the Levy measure, such that $\int_{\mathbb{R}^d} 1 \wedge |x|^2 r(dx) < \infty$, where $x \wedge y \equiv \min\{x, y\}$; $h$ is so called cut-off function with the compact support and satisfying $h(x) = x$ in
some neighbourhood of the origin (see e.g. [28] for explaining the role played by the cut-off function \( h \) in this scheme). In particular the functions \( \mathcal{E}(p) = |p|^\alpha; 0 < \alpha \leq 2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2} \) belong to this class. The common feature of all such functions is that the corresponding semigroups \( \{e^{-t\mathcal{E}(\cdot - \iota \nabla)}, t \geq 0\} \) are generated by stochastic Markov processes with stationary independent increments known as Levy processes [28, 29].

The kernels of the semigroups \( \{e^{-t\mathcal{E}(\cdot - \iota \nabla)}, t \geq 0\} \) denoted as \( \mathcal{K}_{t}^{(\mathcal{E})}(x, y) \) have explicit expressions throughout the corresponding path space integrals [29]. This enables us to apply the methods of [1] to reproduce (up to some extent) the basic results of [1, 2, 3, 4] for interacting gases with nonstandard kinetic energy. The corresponding results shall be reported elsewhere [30].

In the present paper we confine ourselves to the choice:
\( \mathcal{E}(p) = p^2 \) called the standard Bose gas, and
\( \mathcal{E}(p) = \sqrt{p^2 + m^2}, m \geq 0 \) called the semirelativistic Bose gas.

In the case of standard Bose gas the corresponding path space integral is well known as Wiener (conditioned) integral and in this case the corresponding transition function has a kernel

\[
\mathcal{K}_{t}^{s}(x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/(4t)^{1/2}}
\]

In the case of semirelativistic Bose gas the corresponding transition function has a kernel

\[
\mathcal{K}_{t}^{m}(x, y) = \frac{4}{\pi^{1/2}} \int_{0}^{\infty} d\tau \mathcal{K}_{\tau}^{s}(x-y) e^{-\tau^2/4\tau} e^{-m^2\tau} \tau^{3/2}
\]

with fast exponential decay as \( |x-y| \rightarrow \infty \) for \( m > 0 \) and in the case \( m = 0 \) equal to the well known symmetric Cauchy density:

\[
\mathcal{K}_{t}^{0}(x, y) = \frac{c \cdot t}{(t^2 + |x-y|^2)^{(d+1)/2}}
\]

### 2.1 Global Aspects

Let \( \mathcal{W}(\mathfrak{h}) \) be the abstract Weyl algebra built over the one-particle space \( \mathfrak{h} \equiv L_2(\mathbb{R}^d) \) equipped with the standard symplectic form \( \sigma(f, g) \equiv \text{Im} \langle f | g \rangle \). For a chosen kinetic energy function \( \mathcal{E}(p) \) as above we define free thermal state
\( \omega_0^{(\beta,\mu)} \) on the algebra \( \mathcal{W}(h) \):

\[
\omega_0^{(\beta,\mu)}(W(f)) \equiv \exp -\frac{1}{2} \int dp |\hat{f}(p)|^2 \hat{C}_0^{(\beta)}(p) \tag{2.5}
\]

where:

\[
\hat{C}_0^{(\beta)}(p) \equiv \frac{1 + ze^{-\beta \mathcal{E}(p)}}{1 - ze^{-\beta \mathcal{E}(p)}}, \tag{2.6}
\]

\( 0 < \beta \) is the (inverse) temperature, \( z \equiv e^{-\beta \mu} \) is the chemical activity and \( \mu \) is the chemical potential. The values of \( z \) (corresponding to the noncritical regime of the free Bose gas exclusively considered here) are restricted to:

\[
0 < \sup_p ze^{-\beta \mathcal{E}(p)} < 1
\]

which in the case \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = |p| \) corresponds to \( 0 < z < 1 \) (resp. \( \mu > 0 \)) and \( 0 < ze^{-\beta m} < 1 \) (resp. \( \mu > -m \)) if \( m > 0 \) and \( \mathcal{E}(p) = \sqrt{p^2 + m^2} \).

Some elementary properties of the free thermal kernel \( C_0^{(\beta)}(x) \) are collected in the following proposition.

**Proposition 2.1** For any noncritical value of \( z \) the corresponding free thermal kernels \( C_0^{(\beta)}(x) \) have the following properties:

(i) \( C_0^{(\beta)}(x) = \delta(x) + R_0^{(\beta)}(x) \), where \( R_0^{(\beta)}(x) > 0 \) for any \( x \in \mathbb{R}^d \) and \( R_0^{(\beta)}(x) \in S(\mathbb{R}^d) \) if \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2} \) with \( m > 0 \).

(ii) if \( \mathcal{E}(p) = |p| \) then \( C_0^{(\beta)}(x) = \delta(x) + R_0^{(\beta)}(x) \) where \( R_0^{(\beta)}(x) > 0 \) and \( R_0^{(\beta)} \in C_0(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \).

**Proof:**

From the assumption \( \sup_p ze^{-\beta \mathcal{E}(p)} < 1 \) we obtain an equality:

\[
\hat{C}_0^{(\beta)}(p) = 1 + \hat{R}_0^{(\beta)}(p) \tag{2.7}
\]

where \( \hat{R}_0^{(\beta)}(p) = 2 \sum_{n=1}^\infty z^n e^{-\beta n \mathcal{E}(p)} \).

From the positive-definiteness of the function \( p \in \mathbb{R}^d \longrightarrow e^{-t \mathcal{E}(p)} \) for each \( t > 0 \), it follows that for each \( n \), \( \exp(-\beta n \mathcal{E}(p)) \) is the Fourier transform of some positive measure \( d\mu_n^{(\beta)} \) on \( \mathbb{R}^d \). Moreover from the fact that \( \exp(-\beta n \mathcal{E}(p)) \in S(\mathbb{R}^d) \) in the case (i) it follows that \( d\mu_n^{(\beta)}(x) = \rho_n^{(\beta)}(x) d^d x \), with \( \rho_n^{(\beta)}(x) \in S(\mathbb{R}^d) \).
By elementary arguments it follows that also \( \sum_{n=1}^{\infty} z^n \exp -\beta n E(p) \in S(\mathbb{R}^d) \) in the case (i), therefore we conclude that all assertions of (i) are valid. The conclusions of (ii) follow from the explicit form (2.4) of the corresponding kernels and elementary arguments.

\[ \square \]

Let \((H_0, \Omega_0, \pi_0)\) be the corresponding GNS triplet obtained from \((\mathcal{W}(h), \omega_{0}^{(\beta,\mu)})\). Then defining \(\alpha_t^0(\pi_0(W(f))) = \pi_0(W(z^{-it/\beta} e^{itE(p)} f))\) we obtain an \(\sigma\)-weakly continuous group of automorphisms of \(\pi_0(\mathcal{W}(h))^\circ\). It is well known that the system \(\mathcal{C}_0 \equiv (H_0, \Omega_0, \alpha_t^0; \pi_0(\mathcal{W}(h))^\circ)\) forms a \(W^*\)-KMS system in the (inverse) temperature \(\beta\) (see i.e. [17]). The corresponding multitime Green functions of the system \(\mathcal{C}_0\) are given by:

\[
G_0((t_1, f_1), \ldots, (t_n, f_n)) = \prod_{1 \leq i \leq j \leq n} \left[ \exp i \sigma((t_i, f_i), (t_j, f_j)) \right] \times \exp -\frac{1}{2} \int f_i(p) \hat{f}_j(p) \hat{G}_0^\beta(t_i - t_j; p) dp
\]

where:

\[
\sigma((t_i, f_i), (t_j, f_j)) = \text{Im} < z^{-it_i/\beta} e^{itE(p)} \hat{f}_i | z^{-it_j/\beta} e^{itE(p)} \hat{f}_j >
\]

\[
\hat{G}_0^\beta(t; p) = \frac{z^{-it/\beta} e^{itE(p)} + z^{1+it/\beta} e^{-(\beta+it)E(p)}}{1 - ze^{-\beta E(p)}}
\]

By elementary arguments they can be extended analytically to the holomorphic functions \(G_0((\zeta_1, f_1), \ldots, (\zeta_n, f_n))\) of \(\zeta = (\zeta_1, \ldots, \zeta_n) \in T_n^\beta \equiv \{ \zeta^n = (\zeta_1, \ldots, \zeta_n) \in \mathcal{C}^n | \ldots \text{Im} \zeta_i < \text{Im} \zeta_{i+1} < \ldots, \sum_{i=1}^{n-1} (\text{Im} \zeta_{i+1} - \text{Im} \zeta_i) < \beta \} \) and continuous on \(T_n^\beta\). The restrictions of the analytically continued Green functions to the so called Euclidean region \(E_n^\beta \equiv \{ \mathbf{z} \in \mathbb{C}^n | \text{Re} z_i = 0; -\beta/2 \leq \text{Im} z_1 \leq \ldots \leq \text{Im} z_i \leq \text{Im} z_{i+1} \leq \ldots \leq \beta/2 \}\) will be called Euclidean Green functions of the free Bose gas and their full collection extended to \(\bigcup_{n \geq 0} \mathcal{W}(h)^{\times n}\) by linearity will be denoted by \(E^0\). The following abbreviations will be used:

\[
E_n^{\beta,+} = \{(S_1, \ldots, S_n) \in E_n^\beta | 0 \leq S_1\};
\]

5
Proposition 2.2 Let $^{E}G^{0} = \{^{E}G^{0}_{W_{1},\ldots,W_{k}}(S_{1},\ldots,S_{k}) \mid W_{i} \in W(h), (S_{1},\ldots,S_{k}) \in E_{k}^{\beta}\}$ be the collection of the Euclidean Green functions of the free Bose gas in the noncritical regime. Then the collection $^{E}G^{0}$ has the following properties:

EG(1) (i) for each fixed $W^{k} \in W(h)^{\times k}$ the map

$^{E}_{k} \ni S^{k} \rightarrow ^{E}G^{0}_{W^{k}}(S^{k})$

is continuous

(ii) for each fixed $S^{k} \in E_{k}^{\beta}$ the map

$W(h)^{\times k} \ni W^{k} \rightarrow ^{E}G^{0}_{W^{k}}(S^{k})$

is: multilinear and for any $f^{k} \in L_{2}(\mathbb{R}^{d})^{\times k}$ the map:

$L_{2}(\mathbb{R}^{d})^{\times k} \ni f^{k} \rightarrow ^{E}G^{0}_{f^{k}}(S^{k})$

is continuous and obeys the estimate $|^{E}G_{f^{k}}(S^{k})| \leq 1$. 
(iii) for any $S^k \in E^\beta_k$ and any $S \in [-\beta/2, \beta/2]$ such that $S_k + S \leq \beta/2$ the Euclidean Green functions are locally shift invariant i.e. for any $W^k \in \mathcal{W}(h)^{\times k}$:

$$E G^0_{W_k}(S^k + S) = E G^0_{W_k}(S^k)$$

where $S^k + S \equiv (S_1 + S, \ldots, S_k + S)$.

(iv) for any $W^k \in \mathcal{W}(h)^{\times k}$, any $S^k: \exists_{1 \leq i \leq k-1} S_i = S_{i+1}$ we have the equality:

$$E G^0_{W_k}(S^k) \equiv E G^0_{W_1}((i)S^{k-1})$$

where $W^{(i)} = (W_1, \ldots, W_{i-1}, W_i \cdot W_{i+1}, \ldots, W_k)$

$S^{(i)} = (S_1, \ldots, S_i, S_{i+2}, \ldots, S_k)$

(v) for any $W^k \in \mathcal{W}(h)^{\times k}: \exists_{1 \leq i \leq k}: W_i = 1$ the following equality holds:

$$E G^0_{W_k}(S^k) \equiv E G^0_{W_{i-1}}((i)S^{k-1})$$

where

$$W^{(k-1)} = (W_1, \ldots, W_{i-1}, W_{i+1}, \ldots, W_k)$$

$$S^{(k-1)} \equiv (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots S_k)$$

(vi) $E G^0_1(0) = 1$

**EG(2) (OS-positivity)**

For any terminating sequences

$$\sim W = (W^0, W^1, \ldots, W^k, \ldots), \quad \sim S = (S^0, \ldots, S^k, \ldots)$$

with

$$S^k \in E_{k}^{\beta, +} \text{ for all } k = 1, 2, \ldots$$

$$\sum_{k,l} E G^0_{W_k^*, W_l^*}(S^k, S^l) \geq 0 , \quad (2.20)$$

**EG(3)** For any terminating sequences

$$W = (W^0, W^1, \ldots, W^k, \ldots), \quad \sim S = (S^0, \ldots, S^k, \ldots)$$

with

$$S^k \in E_{k}^{\beta, +} \text{ for all } k = 1, 2, \ldots \text{ and for any } f \in L_2(\mathbb{R}^d):$$
\[
\sum_{k,l} E G^0_{\sim W k,i,\sim f, \sim W i}(S^{k,i}, 0, 0, S^l) \leq \sum_{k,l} E G^0_{\sim W k, \sim W l}(S^{k,i}, S^l) \quad (2.21)
\]

**EG(4) (weak form of the KMS condition)**

Let \( E \hat{G}^0_{W_0, \ldots, W_n}(S_1, \ldots, S_n) \equiv E G^0_{W_0, W_1, \ldots, W_n}(-\beta, S_1 - \frac{\beta}{2}, \ldots, S_n - \frac{\beta}{2}) \) for \( 0 \leq S_1 \leq \ldots \leq S_n \leq \beta \). Then for each \( n \), \( W_{n+1} \in \mathcal{W}(h)^{\times n} \)

\[
E \hat{G}^0_{W_{n+1}}(S^n) =
\]

\[
E \hat{G}^0_{W_{n+1}}(S^n) = \]

\[
\sum_{k,l} E G^0_{\sim W k,i,\sim f, \sim W i}(S^{k,i}, 0, 0, S^l) \quad (2.22)
\]

**EG(5) (Euclidean invariance and uniqueness of the vacuum)**

Under the natural action \( \tau_{\{a,A\}} \) of the Euclidean Group of Motions \( E(d) \) on the Weyl algebra \( \mathcal{W}(h) \) the Euclidean Green functions are

(i) invariant

(ii) have the cluster decomposition property, i.e.

\[
\lim_{|a| \rightarrow \infty} E G^0_{\tau_{\{a,0\}} W^k, W^l}(S^k, S^l) = E G^0_{W^k}(S^k) \cdot E G^0_{W^l}(S^l) \quad (2.23)
\]

**Proof:**

Let us consider the free gas GNS \( W^*-KMS \) structure \( \mathcal{A}_0 = (\mathcal{H}_0, \Omega_0, \alpha^0_0, \pi_0(\mathcal{W}(h))'' \). By the Araki theorem [31] the Euclidean Green functions are represented as:

\[
E G^0_{W^n}(S^n) = \Omega_0 | \alpha^0_{1s_1}(\pi_0((W_1)) \ldots \alpha^0_{n}(\pi_0(W_n)))\Omega_0 \rangle \quad (2.24)
\]

and by the very definition of \( \mathcal{A}_0 \):

\[
\omega_0^{(\beta,\mu)}(W(f)) = \langle \Omega_0, \pi_0(W(f))\Omega_0 \rangle \quad (2.25)
\]

Now everything follows easily from (2.24) and the Araki theorem. In particular the OS positivity \( EG(2) \) follows from the fact that the \textit{l.h.s.} of (2.20) can be written as:

\[
(l.h.s. \ of \ (2.20)) \equiv \left\langle \Omega_0 | (\sum_{k} \prod_{i_k=1}^{k} \alpha^0_{i_k s_k}(\pi_0(W(f^k_{i_k}))))^+ \cdot (\sum_{k} \prod_{i_k=1}^{k} \alpha^0_{i_k s_k}(\pi_0(W(f^k_{i_k}))))\Omega_0 \right\rangle \quad (2.26)
\]
The weak form of the KMS-condition, formulated as $EG(4)$ can be observed easily from the explicite formulae (2.8) for the corresponding Green functions.

Remarks
As it was demonstrated in [20] the multitime Euclidean Green functions of any $C^*$–(or $W^*$)–KMS structure obey similar properties $EG(1) \div EG(4)$ with the obvious modifications of the continuity properties $EG(2) (ii)$ and $EG(3)$. It can be checked using the basic results of [1, 2, 3, 4] that the Euclidean Green functions of Dilute Bose gases (and also of Dilute Fermi gases built over the CAR algebra over $\h\h$) in the regime considered by Ginibre [1] obey the system $EG(1) \div EG(5)$. The detailed study of arising modular structures (see below) are now under investigations. The Euclidean Green functions of the critical Bose gas also obey properties similar to $EG(1) \div EG(5i)$ and their restrictions to the Abelian sector (of the Weyl algebra) fulfill also $EG(6)$ (see below).

The complex subalgebra $\mathcal{A}(\h)$ of $\mathcal{W}(\h)$ generated by the elements $W(f)$, with $f = \overline{f}$ will be called an Abelian sector of $\mathcal{W}(\h)$ and the corresponding free Euclidean Green functions restricted to $\mathcal{A}(\h)$ will be denoted by $EAG_0$. For $-\frac{\beta}{2} \leq s_1 \leq \ldots \leq s_n \leq \frac{\beta}{2}$ we have the following formulae:

$$EAG_0((s_1, f_1), \ldots, (s_n, f_n)) = \prod_{1 \leq i \leq j \leq n} \exp \left( -\frac{1}{2} S_0^\beta (s_j - s_i, f_i \otimes f_n) \right)$$ (2.27)

where:

$$S_0^\beta (s, f_i \otimes f_n) = \int \hat{S}_0^\beta (s, p) \overline{f_i(-p)} f_j(p) \; dp;$$ (2.28)

$$\hat{S}_0^\beta (s, p) \equiv \frac{z^{s/\beta} e^{-s \xi(p)} + z^{1-s/\beta} e^{-(\beta-s) \xi(p)}}{1 - z e^{-\beta \xi(p)}}$$ (2.29)

The periodic extension of $\hat{S}_0^\beta (s, p)$ to the whole $\mathbb{R}$ shall be denoted by the same symbol. The fundamental properties of the free thermal kernels $S_0^\beta (s, x)$ are collected in the following Proposition.

Proposition 2.3
1. Let \( \hat{S}_0^\beta \) be the free thermal kernel (2.29) with \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2}, \) \( m > 0. \) Then for any \( 0 \leq s \leq \beta; \) \( z \) noncritical:

(i) \( 0 < S_0^\beta(s, \cdot) \in S(\mathbb{R}^d) \) if \( s \in (0, \beta) \)

(ii) \( S_0^\beta(0, \cdot) = S_0^\beta(\beta, \cdot) = C_0^\beta(\cdot) \) in \( \mathcal{D}'(\mathbb{R}^d) \) sense.

2. Let \( \mathcal{E}(p) = |p|, \) then for any \( 0 \leq s \leq \beta, \) \( 0 < z < 1 \)

(i) \( 0 < S_0^\beta(s, \cdot) \in L_1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d) \) if \( s \in (0, \beta) \)

(ii) \( S_0^\beta(0, \cdot) = S_0^\beta(\beta, \cdot) = C_0^\beta(\cdot) \) in \( \mathcal{D}'(\mathbb{R}^d) \) sense.

3. For \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2}, \) \( m \geq 0 \) the kernel \( S_0^\beta \) is stochastically positive on the space \( L_2(K_\beta \times \mathbb{R}^d), \) i.e. for any \( g_1, g_2 \in L_2(K_\beta); f_1, \ldots, f_n \in L_2(\mathbb{R}^d), C_1, \ldots, C_n \in C: \)

\[
\sum_{\alpha, \beta=1}^n C_\alpha \overline{C_\beta} S_0^\beta(g_\alpha \otimes f_\alpha | g_\beta \otimes f_\beta) \geq 0 \tag{2.30}
\]

where

\[
S_0^\beta(g \otimes f | g' \otimes f') = \int_0^\beta ds \int_0^\beta ds' g(s)g'(s') \int dx \int dy f(x)f(y)S_0^\beta(|s-s'|, x-y). \tag{2.31}
\]

4. For \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2}, \) \( m \geq 0 \) the kernel \( S_0^\beta \) is OS positive on the circle \( K_\beta, \) i.e. for any \( t_1, \ldots, t_n \) in \([0, \beta/2], f_1, \ldots, f_n \in L_2(\mathbb{R}^d), c_1, \ldots, c_n \in C: \)

\[
\sum_{\alpha, \beta} c_\alpha c_\beta \int dx \int dy S_0^\beta(t_\alpha + t_\beta | f_\alpha \otimes f_\beta) \geq 0 \tag{2.32}
\]

Proof:

From the assumption \( \sup_p z e^{-\beta \mathcal{E}(p)} < 1 \) it follows that:

\[
\hat{S}_0^\beta(s, p) = \sum_{n \geq 0} \hat{F}_n(s, p) \tag{2.33}
\]

where:

\[
\hat{F}_n(s, p) = z^{n+s/\beta} e^{-(\beta n+s)\mathcal{E}(p)} + z^{n+1-s/\beta} e^{-(\beta(n+1)-s)\mathcal{E}(p)} \tag{2.34}
\]
So, if \( \mathcal{E}(p) = p^2 \) or \( \mathcal{E}(p) = \sqrt{p^2 + m^2} \), \( m > 0 \) then \( F_n(s, p) \in S(\mathbb{R}^d) \) for each \( n \geq 1 \) and \( n = 0 \) if \( s \in (0, \beta) \). In this case also \( \sum_{n \geq 1} z^n \exp(-\beta n \mathcal{E}(p)) \in S(\mathbb{R}^d) \). Taking into account that

\[
\hat{S}_0^\beta(s, p) = \left( \sum_{n \geq 0} z^n e^{-\beta n \mathcal{E}(p)} \right) \left( z^{s/\beta} e^{-s \mathcal{E}(p)} + z^{1-s/\beta} e^{-(\beta-s) \mathcal{E}(p)} \right)
\]

it follows that also \( \hat{S}_0^\beta(s, p) \in S(\mathbb{R}^d) \) if \( s \in (0, \beta) \). Moreover \( S_0^\beta(s, x) > 0 \) for any \( x \in \mathbb{R}^d \).

Similarly, if \( \mathcal{E}(p) = |p| \) then we have:

\[
\hat{S}_0^\beta(s, p) = \left( \sum_{n \geq 0} z^n e^{-\beta n |p|} \right) \left( z^{s/\beta} e^{-s |p|} + z^{1-s/\beta} e^{-(\beta-s) |p|} \right)
\]

therefore from the continuity of Fourier transform and (2.4) we obtain

\[
S_0^\beta(s, x) = \sum_{n \geq 0} \frac{z^n c}{((\beta n + s)^2 + |x|^2)^{d+\frac{1}{2}}} \left( \frac{z^{s/\beta}}{(s^2 + |x|^2)^{d+\frac{1}{2}}} + \frac{z^{1-s/\beta}}{(\beta - s)^2 + |x|^2} \right)
\]

\[
= \sum_{n \geq 0} \frac{z^{n+\beta/\beta}}{((\beta n + s)^2 + |x|^2)^{d+\frac{1}{2}}} c + \sum_{n \geq 0} \frac{z^{n+1-s/\beta}}{((\beta (n+1) + s)^2 + |x|^2)^{d+\frac{1}{2}}} c.
\]

Because the above series are uniformly convergent on \( \mathbb{R}^d \) and define a continuous function with decay at least as \( \frac{1}{(s^2 + |x|^2)^{d+\frac{1}{2}}} \) for \( |x| \uparrow \infty \), which is integrable provided \( s > 0 \).

Although the claims 3 and 4 follow easily from a basic characterization theorem of KL [32] we present simple proofs of them for reader’s convenience. Expanding into the Fourier series the periodic function \( \hat{S}_0^\beta(s, p) \) we obtain:

\[
\hat{S}_0^\beta(s, p) = \sum_{n \in \mathbb{Z}} \left( (\mu \beta + \mathcal{E}(p))^2 + (2\pi n)^2 \right)^{-1} (2\beta (\mu \beta + \mathcal{E}(p))
\]

\[
\cdot \left( 1 - e^{-\beta (\mu + \mathcal{E}(p))} \right) \left( 1 - z e^{-\beta \mathcal{E}(p)} \right)^{-1} e^{i2\pi ns/\beta}.
\]
Because all the Fourier coefficients in the expansion (2.37) are positive the stochastic positivity (2.30) follows. The OS positivity of one-time Euclidean Green function is a general feature of all KMS systems as was demonstrated in [19]. The straightforward proof of 4) is as follows. Let

\[ \mathcal{G}_0 = (\mathcal{H}_0, \Omega_0, \pi_0, \alpha_t^0, \pi_0(W(h))'' \]

be the basic GNS $W^*$-KMS system of the free Bose gas. Then we can write:

\[
\sum_{\alpha, \beta} c_\alpha c_\beta S_0^\beta (s_\alpha + s_\beta | f_\alpha \otimes f_\beta) = \left| \sum_\alpha c_\alpha \alpha_\alpha (\pi_0(W(f_\alpha))\Omega_0) \right|^2 \\
\geq 0
\]

(2.38)

\[ \square \]

Remarks

1. Let $h_0^\mu$ be a nonnegative, selfadjoint generator of unitary group $U_0^t = e^{-it/\beta} e^{-itE(p)}$ acting in the space $\mathfrak{h} = L_2(\mathbb{R}^d)$ and let $dP^\mu$ be the corresponding spectral measure of $h_0^\mu$. Then defining the covariance operator

\[
\Gamma_0^\beta (s) \equiv \int_0^\infty \frac{dP^\mu(\lambda)}{1 - e^{-\beta \lambda}} (e^{-s\lambda} + e^{-(\beta-s)\lambda})
\]

(2.39)

acting in $\mathfrak{h}$ by definition:

\[
\langle f, \Gamma_0^\beta g \rangle \equiv \tilde{S}_0^\beta (s | f \otimes g)
\]

(2.40)

we see that the kernel $S_0^\beta (s | f \otimes g)$ belongs to the class of kernels considered in [32].

2. Let us observe that the periodic kernels $nS_\beta (s, p) \equiv F_n(s, p)$ for each $n$ also have the positivity properties stated in the points 3) and 4) of proposition 2.3. This leads to an interesting decomposition of the free thermal process $\xi_t^0$ defined below as a sum of independent OS-positive Gaussian processes $\xi_t^{0,n}$, which have covariances equal to $nS_\beta (s, p)$. This decomposition might be eventually used to develop a rigorous Renormalisation Group Analysis of interacting Boson Gases.
Proposition 2.4  The collection $^{E_A}G_0$ of the Euclidean Green functions of the free Bose Gas in the noncritical regime obeys the properties $EG(1)$–$EG(5)$ of proposition 2.2 and additionally:

$EG(6):$  (stochastic positivity)

for any: $S^k \in E_0^2, f^k = (f^k_1, \ldots, f^k_k): f^k_i = T_i \in L_2(\mathbb{R}^d),$

\[
\sum_{k,l} ^{E_A}G_{f^k_{\sim}, f^l_{\sim}}(S^k_{\sim}, S^l_{\sim}) \geq 0 \tag{2.41}
\]

Proof:

From assertion 3 of Prop 2.3 it follows by standard construction (see i.e. [28, 32]) that there exists a Gaussian process $(\xi^0_t)_{t \in K_\beta}$ indexed by $L_2(\mathbb{R}^d)$, with mean zero and the covariance given by $S_0^\beta(\tau, x)$. The r.h.s of (2.41) can be rewritten in terms of $(\xi^0_t)$ as

\[
E \left| \sum_{k} \prod_{l_k=1}^{n_k} \exp i \left< \xi^0_{s^k_{l_k}}; f^k_{l_k} \right> \right|^2 ,
\]

\[
\Box
\]

Having defined a system of Euclidean multitime Green functions with the properties listed in Proposition 2.2 we can apply the constructions of [20] to build certain $W^*$-KMS structures. The interesting aspect of the proposition below is that the system of Euclidean Green functions of the free Bose Gas restricted to $A(\h)$ already contains all information of the free Bose Gas.

Proposition 2.5  Let $\mathcal{E}(p) = p^2$ or $\mathcal{E}(p) = \sqrt{p^2 + m^2}$, $m \geq 0$ and let $z$ be noncritical. Then

1. There exists a unique (up to a unitary equivalence) $W^*$-KMS system $E_\mathcal{G}^r = (E_0^H, E_0^\Omega, E_0^\alpha, E_0^m)$ and a bounded $*$-representation $E_\pi_0$ of $\mathcal{W}(\h)$ such that:

   (i) $E_\pi_0(\mathcal{W}(\h)) \subseteq E_0^m$,

   (ii) the multitime Euclidean Green functions of $E_\mathcal{G}^r_0$ restricted to $E_\pi_0(\mathcal{W}(\h))$ coincide with $E_\mathcal{G}^r_0$. 

13
(iii) \[ E \mathbf{m}_0 = W^* \{ E^{\alpha_0}_1 (E^{\pi_0}(W(f_1))) \ldots E^{\alpha_0}_n (E^{\pi_0}(W(f_n))) \} , \]

2. There exists a unique (up to a unitary equivalence) $W^*$-KMS system $\mathcal{A}\mathcal{T} = (A\mathcal{H}_0, A\Omega_0, A^{\alpha_0}, A^{\mathbf{m}_0})$ and a bounded $^*$-representation $A^{\pi_0}$ of $\mathcal{A}(\hbar h)$ such that:

(i) $A^{\pi_0}(\mathcal{A}(\hbar h)) \subseteq A^{\mathbf{m}_0}$,

(ii) the multitime Euclidean Green functions of the system $\mathcal{A}\mathcal{T}_0$ restricted to $A^{\pi_0}(\mathcal{A}(\hbar h))$ coincide with $A^{\mathcal{G}}_0$.

(iii) $A^{\mathbf{m}_0} = W^* \{ A^{\alpha_0}_1 (A^{\pi_0}(W_1)) \ldots A^{\alpha_0}_n (A^{\pi_0}(W_n)) \} ,

for $W_1, \ldots, W_n \in \mathcal{A}(\hbar h)$.

3. Both systems $\mathcal{E}\mathcal{T}_0$ and $\mathcal{A}\mathcal{T}_0$ are unitarily equivalent to the GNS $W^*$ KMS system $\mathcal{A}\mathcal{T}_0 = (\mathcal{H}_0, \Omega_0, \alpha^{\pi_0}_0, \pi^{\pi_0}_0(\mathcal{W}(\hbar h)))''$.

Proof:

Step 1.

In the first step we apply in a sketchy way a general construction of [20] (see also [19]) to which we are referring for more details. Because in both cases the constructions of $\mathcal{E}\mathcal{T}_0$ and $\mathcal{A}\mathcal{T}_0$ are identical we shall restrict ourselves to the construction of $\mathcal{E}\mathcal{T}_0$ only.

Let $\tilde{V}^\beta$ be the free complex vector space built over the set $\{ (\tilde{W}^n, \tilde{s}^n) | \tilde{s}^n \in E^{\beta,+}_n \}$. Then we divide $\tilde{V}^\beta$ by the natural relations arising from the properties EG(1)(i), EG(1)(iv), EG(1)(v) and EG(1)(vi) obtaining a complex vector space $V^\beta$. The following sesquilinear form

\[
\left(\sum_{\alpha} c_{\alpha} (W^{n_{\alpha}}, s^{n_{\alpha}}); \sum_{\beta} d_{\beta} (W^{k_{\beta}}, s^{k_{\beta}}) \right) = \sum_{\alpha,\beta} c_{\alpha}d_{\beta} E G^{\circ}_{W^{n_{\alpha}}, W^{k_{\beta}}} (s^{n_{\alpha}*}, s^{k_{\beta}}) \]

defined on $V^\beta$ is nonnegative by EG(2). The corresponding Hilbert space will be denoted by $E\mathcal{H}_0$ and the corresponding classes of abstraction will be denoted by square brackets "[ ]".

Lifting the natural action $E^{\pi_0}$ of $\mathcal{W}(\hbar h)$ on $\tilde{V}^\beta$, defined by

\[ E^{\pi_0}(W)(\tilde{W}^n, \tilde{s}^n) = ((W, \tilde{W}^n); (0, \tilde{s}^n)) , \]

14
to the space $E\mathcal{H}_0$ we obtain a $*$-representation of $\mathcal{W}(h)$ in $E\mathcal{H}_0$ which is bounded because of $EG(3)$.

Lifting the local shift transformation given by $EG(1)(iii)$ into the space $E\mathcal{H}_0$ we obtain a uniquely determined selfadjoint generator $E\Omega_0$. Defining $E\Omega_0 = [(1, 0)] \in E\mathcal{H}_0$ we have that for any $[(W^n, s^n)] \in E\mathcal{H}_0$:

$$E\alpha_{t_n}^0(E\pi_0(W_1)) \cdots E\alpha_{t_1}^0(E\pi_0(W_n)) E\Omega_0 = [(W^n, s^n)].$$

Moreover, the vector valued maps:

$$E_n^{\beta +} \ni s^n \rightarrow \prod_{k=1}^n E\alpha_{t_k}^0(E\pi_0(W_k)) E\Omega \in \mathcal{H}^E$$

can be holomorphically extended to the tube $T^\beta_n$ being continuous on the boundary $\partial T^\beta_n$. In particular it can be proved (see [20]) that the vector $E\Omega_0$ is cyclic and separating for the $W^*$-closure $E\mathfrak{m}_0$ of the $*$-algebra generated by all products: $E\alpha_{t_n}^0(E\pi_0(W_1)) \cdots E\alpha_{t_1}^0(W_n))$ where $t_1, \ldots, t_n \in \mathbb{R}$; $W_1, \ldots, W_n \in \mathcal{W}(h)$. Thus we have sketched the construction and the proof that $E\mathfrak{q}_0 \equiv (E\mathcal{H}_0, E\Omega_0; E\alpha^0, E\mathfrak{m}_0)$ forms a $W^*$ KMS system. The Euclidean Green functions of the system $E\mathfrak{q}_0$ are equal to $E\mathfrak{q}_0$ by the very construction. Let $E\mathfrak{q}_0' \equiv (E\mathcal{H}_0', E\Omega_0'; E\alpha^0, E\mathfrak{m}_0')$ be another $W^*$ KMS system whose the Euclidean Green function coincide with $E\mathfrak{q}_0$ and such that $E\mathfrak{m}_0 \supset E\pi_0'(\mathcal{W}(h))$ for some $E\pi_0' \in \text{Rep}^*(\mathcal{W}(h), L(E\mathcal{H}_0'))$ and $E\mathfrak{m}_0' = W^*\{E\alpha^0(E\pi_0'(W_1)) \cdots E\alpha^0(E\pi_0'(W_n))\}$. Then the isometry

$$j : \quad E\alpha_{t_1}^0(E\pi_0(W_1)) \cdots E\alpha_{t_n}^0(E\pi_0(W_n)) E\Omega_0 \rightarrow \quad E\alpha_{t_1}^{0'}(E\pi_0'(W_1)) \cdots E\alpha_{t_n}^{0'}(E\pi_0'(W_n)) E\Omega_0'$$

can be extended to a unitary operator such that $j E\Omega_0 = E\Omega_0'$; $E\alpha_t^0 = j^{-1} E\alpha_t^{0'} j$; $E\mathfrak{m}_0' = j E\mathfrak{m}_0 j$.

**Step 2.**

In the second step we identify the $W^*$-KMS system $E\mathfrak{q}_0'$ with $\mathfrak{q}_0$. Although this identification follows from Section V of [20] we present straightforward
proof below. To start with let us define a linear space $\mathcal{D}^E$ generated by:

$$\{ \alpha_0^{\beta_0} (E_\pi f) \ldots \alpha_1^{\beta_1} (E_\pi f) \}$$

From the step 1. we know that $\mathcal{D}^E = E_{H_0}$ and for any $f_n \in L^2(\mathbb{R}^d)^n$ the map

$$E_\beta^{\beta_0} \ni s \mapsto T - \prod_{i=1}^n \alpha_0^{\beta_0} (W(f_i))$$

can be uniquely extended to a holomorphic, vector valued function on the tube $T_\beta^n$ and this extension gives also the holomorphic extension of the corresponding Green function.

Computing the r.h.s. of

$$\langle E_\pi (f) \alpha_0^{\beta_0} (E_\pi f) \ldots \alpha_1^{\beta_1} (E_\pi f) \rangle$$

we conclude that:

$$E_\pi (f) \alpha_0^{\beta_0} (E_\pi f) = E_\pi (W(z^{it/\beta} e^{itf}))$$

on a dense domain $\mathcal{D}^E$ and thus on $E_{H_0}$.

Defining a map,
we obtain a densely defined with a dense range isometry from $E\mathcal{H}_0$ to $\mathcal{H}_0$ which extends naturally to a unitary map $j_E$. From (2.45) we have:

$$j_E E^{t_0} j^{-1} = t_0, \quad j_E E^0 = 0 \quad \text{and} \quad E m_0 = j^{E^{-1}} F_0(W(h))'' j_E.$$

**Step 3.**

In the third step we identify the $W^*$-KMS system $\mathcal{A}_0$ with $\mathcal{C}_0$. The following lemma, whose proof is translated into the fourth step below plays a basic role.

**Lemma 2.6** Let $E(p) = p^2$ or $E(p) = \sqrt{p^2 + m^2}$, $m \geq 0$. Then the set of functions

$$V = \{ e^{itE(p)} f(p) \mid t \in \mathbb{R}; f = \mathcal{F} \in L_2(\mathbb{R}^d) \}$$

is $\mathbb{R}$-linearly dense in $L_2(\mathbb{R}^d)$.

It is because the Euclidean Green functions restricted to the abelian sector $A(h)$ of $\mathcal{W}(h)$ obey the properties EG(1)–EG(4) we can apply the construction presented in the step 1. obtaining again a $W^*$-KMS system $\mathcal{A}_0 = (A\mathcal{H}_0, A\Omega_0, A\alpha^0_t, A\Omega_0)$ where $A\Omega_0$ is the $W^*$-algebra generated by the operators $A\alpha^0_t (A\pi_0(W(f_1))) \cdots A\alpha^0_n (A\pi_0(W(f_n)))$, where $A\pi_0$ is the corresponding representation of $A(h)$ in $L(A\mathcal{H}_0)$ and all $f_i$ are real. From the cyclicity of $A\Omega_0$ under the action of $A\Omega_0$ it follows that the set of vectors

$$A\alpha^0_t (A\pi_0(W(f_1))) \cdots A\alpha^0_n (A\pi_0(W(f_n))) A\Omega_0$$

is linearly dense in $A\mathcal{H}_0$. Defining a map

$$j_A : A\alpha^0_t (A\pi_0(W(f_1))) \cdots A\alpha^0_n (A\pi_0(W(f_n))) A\Omega_0$$

$$\longrightarrow A\alpha^0_t (\pi_0(W(f_1))) \cdots A\alpha^0_n (\pi_0(W(f_n))) \Omega_0$$

$$\equiv \pi_0 \left( W(\sum_{\alpha=1}^n e^{it\alpha h^0} f_\alpha)\right) \Omega_0 \prod_{1 \leq \alpha < \beta \leq n} \exp \left\{ -i \sigma \left( e^{it\alpha h^0} f_\alpha; e^{it\beta h^0} f_\beta \right) \right\}$$

we see that it is an isometry with dense range because of Lemma 2.6. Moreover $j_A(A\Omega_0) = \Omega_0$. 

17
Computing
\[
\left( j_A \sum_{l=1}^n A_\alpha^0 \left( A_\pi_0(W(g_l)) \right) j_A^* \right) \prod_{k=1}^m \pi_0(W(e^{-i(S_k h)\mu}) f_k)) \Omega_0
\]
\[
= \prod_{l=1}^n \alpha^0_{it_l}(\pi_0(W(g_l))) \prod_{k=1}^m \alpha^0_{is_k}(\pi_0(W(f_k))) \Omega_0 \tag{2.48}
\]
we obtain
\[
j_A \left( \prod_{k=1}^m A_\alpha^0 \left( A_\pi_0(W(g_k)) \right) \right) j_A^* = \prod_{k=1}^m \alpha^0_{it_k}(\pi_0(W(g_k))) \tag{2.49}
\]
therefore applying Lemma 2.6 again we conclude that:
\[
j_A \left( A_\pi_0 \right) j_A^* = \pi_0(W(h))'' \tag{2.50}
\]
Let us observe also that the map
\[
A_\pi_0 : W \left( \sum_{\alpha} e^{it_\alpha h} f_\alpha \right) \rightarrow \prod_{1 \leq \alpha < \beta \leq n} \exp \{ i \sigma \left( e^{it_\alpha h} f_\alpha , e^{it_\beta h} f_\beta \right) \} \prod_{\alpha} A_\alpha^0 \left( A_\pi_0(W(f_\alpha)) \right) \tag{2.51}
\]
can be extended to representation of the full Weyl algebra $W(h)$ in $L(A H_0)$ and moreover the obtained representation extends $A_\pi_0'$. For this, let us observe that:
\[
A_\pi_0 \left( W \left( \sum_{\alpha} e^{it_\alpha h} f_\alpha \right) \right) \\
= \prod_{1 \leq \alpha < \beta \leq n} \exp \{ i \sigma \left( e^{it_\alpha h} f_\alpha , e^{it_\beta h} f_\beta \right) \} \prod_{\gamma} A_\alpha^0 \left( A_\pi_0(W(f_\gamma)) \right) \\
= \prod_{1 \leq \alpha < \beta \leq n} \exp \{ i \sigma \left( e^{it_\alpha h} f_\alpha , e^{it_\beta h} f_\beta \right) \} j_A^{-1} \left( \prod_{\gamma} \alpha^0_{it_\gamma}(\pi_0(W(f_\gamma))) \right) j_A \\
= j_A^{-1} \left( \pi_0(W(\sum_{\gamma} e^{it_\gamma h} f_\gamma)) \right) j_A , \tag{2.52}
\]
by using (2.49), and the fact that $\pi_0$ is a representation of $W(h)$. From lemma 2.6 we know that for any $g \in L_2(\mathbb{R}^d)$ there exists a sequence
\[
(t_1^\alpha, \ldots, t_n^\alpha), (f_1^\alpha, \ldots, f_n^\alpha) \in L_2(\mathbb{R}^d)
\]
such that $\sum_k \exp \{ it_k h^\mu \} f_k^\alpha \rightarrow g$ in $L^2(\mathbb{R}^d)$ sense.

Because $\pi_0$ is $L_2(\mathbb{R}^d)$ continuous representation of $\mathcal{W}(h)$ it follows that

$$
\lim_{\alpha \rightarrow \infty} \prod_{1 \leq \alpha < \alpha_n \leq n} \exp \{ i\sigma_{\alpha} \} \prod_{\gamma} A_{\alpha \gamma} (\lambda \pi_0(W(f_{\gamma}))) \equiv A_{\lambda} (W(g))
$$

exists in the weak sense. Now it is easy to check that $A_{\lambda}^\pi$ as defined in (2.53) really *-bounded representation of $\mathcal{W}(h)$ in $L(A^{\mathcal{H}_0})$ and such that $A_{\lambda}^\pi \pi_0 |A(h) = A_{\pi_0}$. □

**Step 4. proof of Lemma 2.6.**

The operator $e^{it\Delta}$ acts as $e^{it\Delta} f = (e^{it\hat{p}^2 \hat{f}})^\ast$, where $\ast$ and $\hat{\cdot}$ denote the Fourier transform and its inverse. Let us take $g \in C_c(\mathbb{R}^d)$ which is a dense subspace in $L^2(\mathbb{R}^d)$. Let $g_1(p) = \frac{1}{2i} [g(p) + g(-p)]$ and $g_2(p) = \frac{1}{2i} [g(p) - g(-p)]$ be hermitian parts of $g$. Because $g_1$ is the Fourier transform of a real valued function we may write

$$
\|g(p) - \sum_{k=1}^n \hat{f}_k(p) e^{it_k p^2} - \sum_{k=1}^n \hat{f}_k(p) e^{it_k p^2} - g_1(p)\|_{L^2} = \|ig_2(p) - i \sum_{k=1}^n \hat{f}_k(p) \sin(t_k p^2)\|_{L^2}
$$

so it is enough to show that for every $\varepsilon > 0$ there exist real valued functions $f_1, \ldots, f_n \in L^2(\mathbb{R}^d)$ and $t_1, \ldots, t_n \in \mathbb{R}$ such that

$$
\|g_2(p) - \sum_{k=1}^n \hat{f}_k(p) \sin(t_k p^2)\|_{L^2} < \varepsilon.
$$

Let $B$ denote a ball in $\mathbb{R}^d$ of radius $c > 0$ such that $\text{supp } g(p) \subset B$. Let $\hat{f}_k(p) = a_k(p) g_2(p)$, where $a_k(p) \in C_0(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $a_k(p) = a_k(p) = \frac{1}{2i} [g(p) - g(-p)]$.

19
It is clear that \( \hat{f}_k(p) \) is hermitian and belongs to \( L^2(\mathbb{R}^d) \). Then

\[
\| g_2(p) - \sum_{k=1}^{n} \hat{f}_k(p) \sin(t_k p^2) \|_{L^2} \leq \| g_2 \|_\infty \| 1 - \sum_{k=1}^{n} a_k(p) \sin(t_k p^2) \|_{L^2(B)}.
\]

(2.56)

Let us deform the constant function 1 to a function \( f_0 \in C(B) \) such that \( f_0(p) \geq 0 \ \forall p \in B \), \( f_0(0) = 0 \) and \( 1 - f_0 \|_{L^2(B)} < \varepsilon \), \( f_0(p) = f_0(p') \) if \( |p| = |p'| \). Then

\[
\| 1 - \sum_{k=1}^{n} a_k(p) \sin(t_k p^2) \|_{L^2(B)} \leq \varepsilon + \mu(B)^{\frac{1}{2}} \sup_{|p| \in [0,c]} |f_0(|p|) - \sum_{k=1}^{n} a_k(|p|) \sin(t_k p^2)|,
\]

(2.57)

where \( \mu(B) \) is the Lebesque measure of the ball \( B \).

We consider a real algebra generated by \( \sum_{k=1}^{n} a_k(|p|) \sin(t_k |p|^2) \) on \((0,c]\). It is clear that \( \sin(t |p|^2) \) separates points in \((0,c]\) and for every \( |p| \in (0,c] \) there exists \( t \in \mathbb{R} \) such that \( \sin t|p|^2 \neq 0 \). Because we may choose \( a(p) \) such that \( \alpha_{\|B\|} = 1 \) so our algebra separates points and nowhere vanishes in \((0,c]\). Thus applying Stone-Weierstrass theorem to \( C_0(0,c] \) we have that

\[
\sup_{|p| \in (0,c]} |f_0(|p|) - \sum_{k=1}^{n} a_k(|p|) \sin(t_k |p|^2)| < \varepsilon
\]

for some \( a_1, \ldots, a_n \) and \( t_1, \ldots, t_n \). Finally

\[
\| g_2(p) - \sum_{k=1}^{n} \hat{f}_k(p) \sin(t_k p^2) \|_{L^2} \leq \| g_2 \|_{\sup} (1 + \mu(B)^{\frac{1}{2}}) \varepsilon
\]

which proves the assertion for \( E(p) = p^2 \). The same proof works for \( E(p) = \sqrt{p^2 + m^2} \), \( m \geq 0 \).

\( \square \)

To exploit the stochastic positivity EG(6) of the system \( AEG \) and for the further development we shall introduce two basic concepts of the generalized thermal process and the generalized thermal random field.

If should be emphasized that these concepts are heavily inspired by the abstract theory developed by Klein and Landau in [25] (see also [32]).
Definition 2.7 Any generalized, periodic (with the period $\beta$) stochastic process $(\xi_t)_{t \in \mathbb{R}}$ with values in $\mathcal{D}'(\mathbb{R}^d)$ will be called a thermal process (with the temperature $\beta$) iff

Tp(1) the process $(\xi_t)_{t \in \mathbb{R}}$ is symmetric on $K_\beta$, i.e.
$$\forall -\beta/2 \leq \tau \leq \beta/2 \forall f \in \mathcal{D}(\mathbb{R}^d) \langle \xi_{\tau}, f \rangle = \langle \xi_{-\tau}, f \rangle$$ (in law) (2.58)

Tp(2) the process $(\xi_t)_{t \in \mathbb{R}}$ is (locally) homogeneous i.e.
$$\forall \tau, s \in K_\beta \forall f \in \mathcal{D}(\mathbb{R}^d) \langle \xi_{\tau+s}, f \rangle = \langle \xi_{\tau}, f \rangle$$ (in law) (2.59)

Tp(3) the process $(\xi_t)_{t \in \mathbb{R}}$ is OS-positive on $K_\beta$ i.e. for any bounded $F \in C_b(\mathbb{R}^n)$, any $\tau^n \in [0, \beta/2]^\times n$; $f^n \in \mathcal{D}(\mathbb{R}^d)^\times n$
$$0 \leq E F \left( \langle \xi_{-\tau^n_1}, f_1 \rangle, \ldots, \langle \xi_{-\tau^n_n}, f_n \rangle \right) F \left( \langle \xi_{\tau^n_1}, f_1 \rangle, \ldots, \langle \xi_{\tau^n_n}, f_n \rangle \right)$$ (2.60)

Tp(4) the moments:
$$E \left( \prod_{i=1}^n e^{i\langle \xi_{\tau_i}, f_i \rangle} \right) \equiv G_{f_1,\ldots,f_n}^{(\xi)}(\tau_1, \ldots, \tau_n)$$ (2.61)
are continuous in $\tau^n \in (K_\beta)^\times n$ and on $\mathcal{D}(\mathbb{R}^d)^\times n$

A thermal process $\xi$ is called Euclidean invariant if additionally:

Tp(5) the moments (2.61) are invariant under the action of the Euclidean Group $E(d)$ in $\mathcal{D}(\mathbb{R}^d)$.

A thermal process $(\xi_t)_{t \in \mathbb{R}}$ is called tempered iff the moments (2.61) are continuous on $S(\mathbb{R}^d)^\times n$, $L^p$-continuous iff the moments (2.61) are continuous on $L^p(\mathbb{R}^d)^\times n$, etc.

If $(\xi_t)_{t}$ is a generalized thermal process then its corresponding path space measure construction leads to the concept of the random generalized field.

Definition 2.8 Any generalized random field $\mu^\beta$ on $\mathcal{D}'(K_\beta \times \mathbb{R}^d)$ (i.e. any probabilistic, Borel cylindrical (PBC) measure) will be called generalized thermal random field iff
∀ \in C^\infty(K_{\beta}) \langle \phi; g \otimes f \rangle = \langle \phi; g \otimes f \rangle \quad \text{(in law } \mu^\beta) \quad (2.62)

where \( r(g \otimes f)(\tau, x) = g(-\tau)f(x) \).

∀ \in C^\infty(K_{\beta}) \langle \phi; t_s(g \otimes f) \rangle = \langle \phi; g \otimes f \rangle \quad \text{(in law } \mu^\beta) \quad (2.63)

for any \( s > 0 \) such that \( \text{supp } t_s(g) \subseteq [-\beta/2, \beta/2] \), where \( t_s(g)(\tau) \equiv g(t + s) \).

the field \( \mu^\beta \) inOS-positive on the circle \( K_{\beta} \), i.e. for any bounded cylindric function \( F \) based on \( (g_1 \otimes f_1, \ldots, g_n \otimes f_n) \), where \( g_i \in C^\infty[0, \beta/2] \) for all \( i \), \( f_i \in D(\mathbb{R}^d) \):

\[
0 \leq \mu^\beta \left( RF \left( \langle \phi, g_1 \otimes f_1 \rangle, \ldots, \langle \phi, g_n \otimes f_n \rangle \right) \right) F \left( \langle \phi, g_1 \otimes f_1 \rangle, \ldots, \langle \phi, g_n \otimes f_n \rangle \right) \quad (2.64)
\]

where:

\[
RF \left( \langle \phi, g_1 \otimes f_1 \rangle, \ldots, \langle \phi, g_n \otimes f_n \rangle \right) = F \left( \langle \phi, rg_1 \otimes f_1 \rangle, \ldots, \langle \phi, rg_n \otimes f_n \rangle \right) \quad (2.65)
\]

for any \( \tau \in K_{\beta} \) the random elements \( \langle \phi, \delta_\tau \otimes f \rangle \) (defined as a unique limits in \( L^p(d\mu^\beta) \) sense \( \lim_{\epsilon \downarrow 0} \langle \phi, \delta_\tau \otimes f \rangle \), for any mollifier \( \delta_\tau \rightarrow \delta_\tau \) exists and moreover the moments

\[
\mu^\beta \left( \prod_{i=1}^n \epsilon^{\langle \phi, \delta_\tau \otimes f_i \rangle} \right) \equiv G^{(\mu)}_{f_1, \ldots, f_n}(\tau_1, \ldots, \tau_n) \quad (2.66)
\]

are continuous in \( \tau^n \in K_{\beta}^{\times n} \); \( f^n \in D(\mathbb{R}^d)^{\times n} \).

a generalized random thermal field \( \mu \) is Euclidean invariant iff the moments \( G^{(\mu)} \) are invariant under the natural action of \( E(d) \) in \( D(\mathbb{R}^d) \).

Additionally: a generalized random thermal field \( \mu \) will be called tempered iff the moments \( (2.60) \) are tempered distributions, \( L^p \)-continuous iff the moments \( (2.60) \) are \( L^p \)-continuous, etc.
Proposition 2.9 1. Let \((\xi_t)_t\) be tempered thermal process with the temperature \(\beta\). There exists a unique (up to the unitary equivalence) W* KMS structure

\[
\mathcal{G}^\xi = \left( \mathcal{H}^\xi, \Omega^\xi; \alpha^\xi_t; \pi^\xi : \mathcal{A}(S(\mathbb{R}^d)) \rightarrow L(\mathcal{H}^\xi), \mathbf{m}^\xi \right)
\]

where

\[
\mathbf{m}^\xi = W^* - \{ \alpha^\xi_{t_1}(\pi^\xi(W(f_1))) \ldots \alpha^\xi_{t_n}(\pi^\xi(W(f_n))) \}
\]

with \(f_i = f_i \in S(\mathbb{R}^d)\) real, whose Euclidean Green functions restricted to \(\pi^\xi(\mathcal{A}(S(\mathbb{R}^d)))\) coincide with the moments \(G^\xi_{\tau_1, \ldots, \tau_n}\), i.e. for any \(-\beta/2 \leq \tau_1 \leq \ldots \leq \tau_n \leq \beta/2\):

\[
\langle \Omega^\xi; \alpha^\xi_{t_{\tau_n}}(\pi^\xi(W(f_n))) \ldots \alpha^\xi_{t_1}(\pi^\xi(W(f_1)))\Omega^\xi \rangle = G^\xi_{f_1, \ldots, f_n}(\tau_1, \ldots, \tau_n) = E e^{i\langle \xi_1, f_1 \rangle} \ldots e^{i\langle \xi_n, f_n \rangle}
\]

(2.67)

2. Let \(\mu\) be a tempered thermal field ( at the temperature \(\beta\)). There exists a unique (up to a unitary equivalence) W* KMS structure

\[
\mathcal{G}^{(\mu)} = \left( \mathcal{H}^{(\mu)}, \Omega^{(\mu)}; \alpha^{(\mu)}_t; \pi^{(\mu)} \in \text{Hom} (\mathcal{A}(S(\mathbb{R}^d)), L(\mathcal{H}^{(\mu)})); \mathbf{m}^{(\mu)} \right)
\]

where

\[
\mathbf{m}^{(\mu)} = W^* - \{ \alpha^{(\mu)}_{t_1}(\pi^{(\mu)}(W(f_1))) \ldots \alpha^{(\mu)}_{t_n}(\pi^{(\mu)}(W(f_n))) \}
\]

\(t_1, \ldots, t_n \in \mathbb{R}, \ f_1, \ldots, f_n \in S(\mathbb{R}^d); \ f_i = f_i \in \mathbb{F}_i\},

whose Euclidean Green function restricted to \(\pi^{(\mu)}(\mathcal{A}(S(\mathbb{R}^d)))\) coincide with \(G^{\mu}_{\tau_1, \ldots, \tau_n}\).

3. If the tempered random thermal fields \(\mu\) is the path space measure of a tempered process \((\xi_t)_t\) i.e. if:

\[
E e^{i\langle \xi_{\tau_1}, f_1 \rangle} \ldots e^{i\langle \xi_{\tau_n}, f_n \rangle} = \mu \left( e^{i\langle \phi_{\delta_{\tau_1}} \otimes f_1 \rangle} \ldots e^{i\langle \phi_{\delta_{\tau_n}} \otimes f_n \rangle} \right)
\]

(2.68)

for all \(\tau_1, \ldots, \tau_n \in K_\beta; \ f_1, \ldots, f_n \in S(\mathbb{R}^d)\) then the W*-KMS systems \(\mathcal{G}^{(\xi)}\) and \(\mathcal{G}^{(\mu)}\) coincide.
Proof:
Let \((\xi_t)_t\) be a given tempered thermal process at the temperature \(\beta > 0\). It follows from the definition 2.7 that the moments \(G_{f_1,\ldots,f_n}(\tau_1,\ldots,\tau_n)\) define on the abelian sector \(\mathcal{A}(S(\mathbb{R}^d))\) of the Weyl algebra \(\mathcal{W}(S(\mathbb{R}^d))\) a system of functions fulfilling EG(1)–EG(6) with modified EG(1)(ii):

\[
EG(1)(\text{ii}') : \text{the functionals } G_{\xi}^{(\xi)}(\tau_1,\ldots,\tau_n) : S(\mathbb{R}^d)^\times^n \ni (f_1,\ldots,f_n) \mapsto G_{f_1,\ldots,f_n}^{(\xi)}(\tau_1,\ldots,\tau_n)
\]

are continuous and \(\left|G_{f_1,\ldots,f_n}^{(\xi)}(\tau_1,\ldots,\tau_n)\right| \leq 1\)
and possible lack of EG(5)(ii). Also EG(3) should be properly modified. All these modifications however do not affect seriously the construction presented in the step 1 of the proposition 2.5. Proceeding analogously to step 1 of proposition 2.5 we can construct \(\mathcal{Q}^\xi\). Similarly we prove the existence of \(\mathcal{Q}^\mu\). The identification of \(\mathcal{Q}^\xi\) and \(\mathcal{Q}^\mu\) follows from 2.69 and the uniqueness part of (1) and (2).

\(\square\)

It follows from the results of [32], stochastic positivity EG(6) and the proposition 2.5 that the thermal structure of the free Bose Gas can be described fully in terms of the corresponding stochastic thermal structures

Proposition 2.10 Let \(E(p)\) be given by (2.4) and let \(0 < z\) be such that \(\sup_p z \exp\{-\beta E(p)\} < 1\). Then for any \(\beta > 0\):

1. There exists a unique (up to a stochastic equivalence) Gaussian thermal process \((\xi^0_t)_t\) with values in \(\mathcal{D}'(\mathbb{R}^d)\) such that:

\[
E \langle \xi^0_t, f \rangle = 0; \quad E \left( \langle \xi^0_t, f \rangle \langle \xi^0_t, g \rangle \right) = S^\beta_0(|t-t'|, f \otimes g)
\] (2.69)

The process \((\xi^0_t)_t\) is Euclidean invariant, ergodic and \(L^2\)-continuous.

2. There exists a unique (up to a stochastic equivalence) Gaussian generalized thermal random field \(\mu^\beta_0\) such that:

\[
\mu^\beta_0(\langle \phi, f \rangle) = 0, \quad \mu^\beta_0(\langle \phi, \delta_{\tau} \otimes f \rangle \langle \phi, \delta_{\tau'} f \rangle) = S^\beta_0(|\tau - \tau'|, f \otimes g).
\] (2.70)

The thermal field \(\mu^\beta_0\) is Euclidean invariant, ergodic and \(L^2\)-continuous.
3. The generalized random field $\mu^0_{\beta}$ can be identified with the path space measure of the process $(\xi^0_t)_{t \in \mathbb{R}}$ i.e. for any bounded, cylindric function $F$ with base $(\tau_1, f_1), \ldots, (\tau_n, f_n)$

$$EF ((\xi_{\tau_1}, f_1), \ldots, (\xi_{\tau_n}, f_n)) \equiv \mu^0_{\beta} (F ((\phi, \delta_{\tau_1} \otimes f_1), \ldots, (\phi, \delta_{\tau_n} \otimes f_n)))$$

(2.71)

4. Let $\nu^0_{\beta}$ be a Gaussian measure on $\mathcal{D}'(\mathbb{R}^d)$ with mean zero and the covariance given by:

$$\nu^0_{\beta} ((\phi, f) (\phi, g)) = C^0_{\beta} (f \otimes g)$$

(2.72)

Then the measure $\nu^0_{\beta}$ is the unique stationary measure of the process $(\xi^0_t)_{t \in \mathbb{R}}$ and $\nu^0_{\beta}$ is equal to the restriction of $\mu^0_{\beta}$ to the $\sigma$-algebra at $\tau = 0$, i.e. $\mu^0_{\beta}(\Sigma(0)) = \nu^0_{\beta}$, where

$$\Sigma(0) = \sigma \{ (\phi, \delta_0 \otimes f); f \in \mathcal{D}(\mathbb{R}^d) \}.$$ 

Moreover the measure $\nu^0_{\beta}$ is quasiinvariant under the translations by $\mathcal{D}'(\mathbb{R}^d)$.

Remarks

Other well known examples of the generalized thermal processes arise in the study of two-dimensional models of Euclidean (Quantum) Field Theory ([23, 33]) and also in the context of the Euclidean version of the Bisognano-Wichman theorem [33, 34]. Similar stochastic thermal structures on the abelian sectors of the corresponding algebras of observables do appear also in the context of (an)harmonic lattice crystals [21, 26, 27] and certain spin systems [24, 35].

The common problem of all these examples is to construct a modular structure on whole algebra of observables from arising stochastic thermal structures on the abelian sector. In the case of the Free Bose Gas the complete solution of this problem is given by proposition 2.73.

From the assumption $\sup_p |z \exp (-\beta \mathcal{E}(p))| < 1$ it follows that the operator $\left(1 - z \exp (-\beta \mathcal{E}(p))\right)^{-1}$ exists in $L^2(\mathbb{R}^d)$ and is bounded, strictly positive and self adjoint. Let $h^0_{\beta}(\mathbb{R}^d)$ be the metric completion of the space $\mathcal{D}(\mathbb{R}^d)$ equipped with the inner product

$$\langle f, g \rangle \equiv \int \bar{f}(x)(1 - ze^{-\beta \mathcal{E}(p)})^{-1}(x - y)g(y) \, dx \, dy.$$ 

(2.73)
From the simple estimates:

\[ \|f\|^2_{L^2(\mathbb{R}^d)} \leq \|f\|_\beta \leq \left( \inf_p (1 - ze^{-\beta E(p)}) \right)^{-1} \|f\|_{L^2(\mathbb{R}^d)} \]  

(2.74)

it follows that \( h^\beta \) is essentially equal to \( L_2(\mathbb{R}^d) \). Using the \( L^2(\mathbb{R}^d) \)-continuity of the process \( \xi^0_t \) and estimates (2.74) we can define a version \( \tilde{\xi}^0_t \) of \( \xi^0_t \) obtained by extension of the index space \( \mathcal{D}(\mathbb{R}^d) \) onto the space \( h^\beta \). The new process \( \tilde{\xi}^0_t \) is indexed by \( K_\beta \times h^\beta(\mathbb{R}^d) \). For any Borel subset \( I \subset K_\beta \) we denote by \( \Sigma(I) \) the smallest \( \sigma \)-algebra generated by \( \{(\tilde{\xi}^0_t, f) \mid t \in J, f \in h^\beta(\mathbb{R}^d)\} \). For any \( t, s \in K_\beta \) we will denote by \([t,s]\) the closed interval from \( t \) to \( s \) in the counterclockwise direction. The corresponding conditional expectations with respect to the \( \sigma \)-algebras \( \Sigma(t,s) \) will be denoted by \( \tilde{E}^0_{(t,s)} \) (resp. \( E^0_J \)).

**Proposition 2.11**

1. For any allowed form of \( E(p), z \) such that \( |ze^{-E(p)}| < 1 \) the corresponding free thermal process \( \tilde{\xi}^0_t \) has two-sided Markov property on \( K_\beta \) in the sense that:

\[ \tilde{E}^0_{(s,r)} \circ \tilde{E}^0_{(r,s)} = \tilde{E}^0_{(r,s)} \circ \tilde{E}^0_{(r,s)} \]  

(2.75)

2. Let \( \Xi(J) \equiv \sigma\{\phi(t, f) \mid t \in J, f \in h^\beta\} \) be the corresponding \( \sigma \)-algebras in \( B(\mathcal{D}(K_\beta \times \mathbb{R}^d)) \) and let \( \tilde{E}^0(J) \) denote the corresponding conditional expectation values. Then the free thermal random field \( \mu_0^\beta \) has the following two-sided Markov property on \( K_\beta \):

\[ \tilde{E}^0_{(s,r)} \circ \tilde{E}^0_{(r,s)} = \tilde{E}^0_{(r,s)} \circ \tilde{E}^0_{(r,s)} \]  

(2.76)

**Proof:**

It follows easily easily that the operator \( h^\mu = h_0 + \mu 1 \) is a nonnegative selfadjoint operator in \( h^\beta \) (on the same domain as in \( L^2(\mathbb{R}^d) \)). Moreover the covariance operator \( \Gamma^\beta_0(t) \) of the process \( \tilde{\xi}^0_t \) indexed by \( K_\beta \times h^\beta_r \) is given by:

\[ \Gamma^\beta_0(t) = e^{-th^\mu} + e^{-(\beta-t)h^\mu} \]  

(2.77)

Applying theorem 4.1 of \[32\] we conclude the proof of the first section. The second part follows easily by identification of \( \phi(t, x) \) with \( \xi_t(x) \) given in proposition \[2.10\] and the density of \( \mathcal{D}(\mathbb{R}^d) \) in the space \( h^\beta_r(\mathbb{R}^d) \).

\[ \Box \]
2.2 Local aspects (The case $\mathcal{E}(p) = p^2$)

Let $\Lambda \subset \mathbb{R}^d$ be a bounded region with a boundary $\partial \Lambda$ of a class at least $C^1$-piecewise. Then, for any $b \in C(\partial \Lambda)$ the selfadjoint extension $-\Delta^b_\Lambda$ of the symmetric operator $-\Delta$ defined on $C^\infty_0(\Lambda)$ can be constructed. It is well known that the arising semigroup $\{e^{-t\Delta^b_\Lambda}, t \geq 0\}$ is positivity preserving on $L^2(\Lambda)$ therefore there exists a stationary, with independent increments Markov process $\mathcal{B}^b_\Lambda(t)$ with values in $\overline{\Lambda}$ for which the kernel $K^b(t)$ of $e^{-t\Delta^b_\Lambda}$ plays the role of the transition function.

Let $\mathcal{W}_\Lambda$ be the local Weyl algebra built over the space $L^2(\Lambda)$ and let $\mathcal{W}^F_\Lambda$ be its Fock space realisation in the Fock-Bose space $\Gamma^{-1}(L^2(\Lambda))$. In particular we have

$$W^F_\Lambda(f) = e^{i[a^+_\Lambda(f)]} e^{i[a_\Lambda(f)]} \quad (2.78)$$

where $a_\Lambda$ and $a^+_\Lambda$ are standard annihilation and creation operators in $\Gamma^{-1}(L^2(\Lambda))$.

Let $P^{(\Lambda,b)}(d\lambda)$ be the spectral measure for the operator $-\Delta^b_\Lambda$. Then, we can define finite volume thermal state $\omega^b_0(\Lambda)$ on $\mathcal{W}^F_\Lambda$ by the formula

$$\omega^b_0(\Lambda, f) = \exp -\frac{1}{2} C^b_{\Lambda,0}(f) \quad (2.79)$$

where

$$C^b_{\Lambda,0}(f) \equiv \langle f | C^b_{\Lambda,0}(f) \rangle_{L^2(\Lambda)}; \quad (2.80)$$

$$\tilde{C}^b_{\Lambda,0}(f) = \int P^{(\Lambda,b)}(d\lambda) \frac{1 + z e^{-\beta \lambda}}{1 - z e^{-\beta \lambda}} \quad (2.81)$$

It is well known (see i.e. [17]) that for any monotonic sequence $(\Lambda_n)_n$ of bounded regions in $\mathbb{R}^d$ and with sufficiently regular boundaries $\partial \Lambda_n$ tending to $\mathbb{R}^d$ by inclusion and for any sequence $b_{\partial \Lambda_n} \in C(\partial \Lambda_n)$ we have the weak convergence: $\lim_{n \to \infty} \tilde{C}^b_{\Lambda_n,0} = C^b_{0,\Lambda,0}$ if $z \in (0,1)$. The corresponding GNS construction applied to $(\mathcal{W}^F_\Lambda(\cdot, \omega^b_0(\Lambda))$ leads again to $W^*$-KMS system $\mathcal{G}^{(\Lambda,b)} = (\mathcal{H}^{(\Lambda,b)}, \pi^{(\Lambda,b)}, \Omega^{(\Lambda,b)}, \alpha^{(\Lambda,b)}, \pi^{(\Lambda,b)}_0(W^F_\Lambda'))$ and the corresponding Green functions can again be easily computed and the analicity properties similar to those of $\mathcal{G}_0$ established. In particular the corresponding Euclidean Green functions $E^b \mathcal{G}_0(\Lambda, b_{\partial \Lambda})$ again fulfill the system of axioms $EG(1) - EG(4)$ and $EG(6)$, therefore the whole discussion from the sub-section 2.1 can be repeated with obvious modifications.
Lemma 2.12 Let $z = e^{-\beta \mu}$ be sufficiently small and let $(\Lambda_n)$ be a monotonic sequence of bounded convex regions in $\mathbb{R}^d$ with boundaries $\partial \Lambda_n$ of class at least $C^3$ and with mean curvatures uniformly bounded. Then for any choice of $b_n \in C(\partial \Lambda_n)$, any $f_1, \ldots, f_m \in L_2(\Lambda)$, $s \sim m \in T_\beta$ we have the convergence
\[
\lim_{n \to \infty} E G_0(\Lambda_n, b_n)((s_1, f_1), \ldots, (s_m, f_m)) = E G_0^0((s_1, f_1), \ldots, (s_m, f_m))
\] (2.82)

Proof:
The monotonicity in the boundary conditions:
If $b_1(x) \leq b_2(x)$ for all $x \in \partial \Lambda$, then
\[
K_t^{(\Lambda, b_1)}(x, y) \geq K_t^{(\Lambda, b_2)}(x, y)
\] (2.83)
for all $x, y \in \Lambda$, $t > 0$. Therefore
\[
\sup_{b \in C(\partial \Lambda)} |K_t^{(\Lambda, b)}(x, y) - K_t(x, y)| = |K_t^{(\Lambda, 0)}(x, y) - K_t(x, y)|
\] (2.84)
for all $t, x, y \in \Lambda$, where $K_t^{(\Lambda, 0)}$ is the kernel of the semigroup $\{e^{-t \Delta^N}(x, y), t \geq 0\}$, where $\Delta^N$ correspond to the Neumann boundary condition. By the (rough) estimate of [36] we have with our assumptions on $(\Lambda_n)$:
\[
|K_t^{(\Lambda_n, b^{\Lambda_n})}(x, y) - K_t(x, y)| \leq C e^{\lambda t} t^{-d/2} \exp \left\{ -c \left( \frac{d(x, \Lambda_n^c)^2 + d(y, \Lambda_n^c)^2}{4t} \right) \right\}
\] (2.85)
for all $x, y \in \Lambda$, $t \in \mathbb{R}$, where $C$, $c$, and $\lambda \geq 0$ are some constants.

It is due to the quasifree nature of the states $\omega_0^{(\Lambda, b)}$ that it is enough to consider the one time Green function only.

\[
\left| E G_0^0(\Lambda_n, b_n)((0, f_1), (s_1, f_2)) - E G_0^0((0, f_1), (s_1, f_2)) \right|
\]
\[
\leq \left| \exp \left( i\sigma(f_1, e^{i s_1 \mu}(\Lambda_n, b_n) f_2) \right) \right|
\]
\[
\left| S_0^\beta(s_1 | f_2 \otimes f_2) - e^{i\sigma(f_1, e^{i s_1 \mu} f_2) - \sigma(f_1, e^{i s_1 \mu} f_2)} S_0^\beta(s_1, f_1 \otimes f_2) \right|
\]
(2.86)

It is well known that $\lim_{n \to \infty} e^{it(-\Delta_{\Lambda_n} + \mu_1)} = e^{it(-\Delta + \mu_1)}$ strongly in $L^2(\mathbb{R}^d)$,
therefore we shall omit the symplectic factor in the last formula, concentrating attention on:

$$
|A_{n,b_n}S_{0}^{\beta}(s, f_1 \otimes f_2) - S_{0}^{\beta}(s, f_1 \otimes f_2)| \\
\leq \sum_{n \geq 0} z^{n+s/\beta} \int dx \, dy \, f_1(x)f_2(y) \left| K_{(\beta_n + s)}(x, y) - K_{(\beta_n + s)}(x, y) \right| \\
+ \sum_{n \geq 0} z^{n+1-s/\beta} \int dx \, dy \, f_1(x)f_2(y) \left| K_{(\beta_n, b_n)}(x, y) - K_{(\beta_{n+1} - s)}(x, y) \right|
$$

Therefore localizing firstly $f_1$, $f_2$ and taking into account (2.85) we obtain

$$
\lim_{n \to \infty} A_{n,b_n}S_{0}^{\beta}(s, f_1 \otimes f_2) = S_{0}^{\beta}(s, f_1 \otimes f_2)
$$

provided $e^{-\beta \mu} e^\lambda < 1$.

Remarks

The restriction $e^{-\beta \mu} e^\lambda < 1$ is by no doubts only an artifact of the rough estimate (2.85) used. It is natural to expect that actually this lemma is valid for all $0 < z < 1$. For a Dirichlet boundary condition the constant $\lambda$ can be taken equal 0 and this gives the result of the independence of the limiting Green functions of the Dirichlet boundary condition in the full noncritical interval $z \in (0, 1)$.

The finite volume, conditional thermal processes (resp. thermal random fields) will be denoted by $\xi_v^{(\Lambda, b, \partial \Lambda)}$ (resp. $\mu_v^{(\Lambda, b, \partial \Lambda)}$).

Having established properties EG(1) $\div$ EG(4) of the corresponding Euclidean Green functions $E \xi_0^{(\Lambda, b, \partial \Lambda)}$ (resp. $AE \xi_0^{(\Lambda, b, \partial \Lambda)}$) we can construct again three different a priori $W^*-KMS$ structures: $E \xi_0^{(\Lambda, b, \partial \Lambda)}$, $AE \xi_0^{(\Lambda, b, \partial \Lambda)}$, and the basic GNS system $G_0^{(\Lambda, b, \partial \Lambda)}$. It appears that all the claims of a properly modified proposition 2.3 are still valid and the proof is almost identical with the exception of the lemma 2.6 which is replaced by the lemma 2.13.

Let $\Lambda$ be a bounded, open and connected region in $\mathbb{R}^d$, $d \geq 2$ with a smooth boundary. Let us define $-\Delta^b_\Lambda(f) = -\Delta f$ for $f \in C^2_0(\Lambda)$, where $D(-\Delta^b_\Lambda)$ consists of those $f \in L^2(\Lambda)$ which satisfy

a) $f \in C^2(\Lambda)$,

29
b) \( \partial^n f(x) = b(x)f(x) \) for \( x \in \partial \Lambda \),

with \( \partial^n \) being normal inward derivative. It follows that \( -\Delta^b_\Lambda \) for \( b \in C^1(\partial \Lambda) \) is densely defined, symmetric and strongly positive. Let \( \hat{L}^b \) be the Friedrichs’ extension of \( -\Delta^b_\Lambda \) to a self-adjoint operator. Then at it is well known (see i.e. [37]) the spectrum of a selfadjoint \( L^b \) is purely discrete and all eigenfunctions of \( L^b \) are real valued. Moreover the semigroup \( e^{-tL^b_\Lambda} \) is of trace class.

It is well known (see i.e. [37]) that \( \hat{L}^b_\Lambda \) possesses real-valued eigenfunctions \( \{u_k\} \) associated with eigenvalues \( 0 > \lambda_1 \geq \lambda_2 \geq \ldots \). Moreover \( \{u_k\}_{k=1}^\infty \) form a complete set in \( L^2(\Lambda) \).

**Lemma 2.13** A linear space generated by functions \( e^{it\hat{L}^b}f \), where \( t \in \mathbb{R} \) and \( f = \overline{f} \), \( f \in L^2(\Lambda) \) is dense in \( L^2(\Lambda) \).

**Proof:**

It is enough to show that for every

\[
f = \sum_{k=1}^n z_k u_k, \quad z_k \in \mathcal{C}
\]

there exist \( t_0, t_1, \ldots, t_m \in \mathbb{R} \), \( f_0 = \overline{f}_0 \), \( f_1 = \overline{f}_1 \), \ldots, \( f_m = \overline{f}_m \) from \( L^2(\Omega) \) such that

\[
f = \sum_{j=0}^m e^{it_j\hat{L}^b}f_j.
\]

We exploit the fact that \( e^{it\hat{L}^b} = \sum_{k=1}^\infty e^{it\lambda_k} P_k \), where \( P_k \) is the one-dimensional projector onto \( u_k \).

Let \( z_k = a_k + ib_k \), \( a_k b_k \in \mathbb{R} \). Let us define \( t_0 = 0 \), \( f_0 = \sum_{k=1}^n a_k u_k \), \( m = 2n \), \( t_j = -t_{j+n} \) for \( j = 1, \ldots, n \) and

\[
f_j = \begin{cases} \frac{1}{2} b_j u_j & \text{for } j = 1, \ldots, n \\ -\frac{1}{2} b_{j-n} u_{j-n} & \text{for } j = n+1, \ldots, m. \end{cases}
\]

Then

\[
\sum_{j=0}^{2n} e^{it_j\hat{L}^b}f_j = \sum_{k=1}^n a_k u_k + \frac{1}{2} \sum_{j=1}^n b_j \left( e^{it_j\hat{L}^b} - e^{-it_j\hat{L}^b} \right) u_j
\]
but
\[ e^{it_j \hat{L}^b} - e^{-it_j \hat{L}^b} = 2i \sum_{k=1}^{\infty} (\sin t_j \lambda_k) P_k. \]

So by putting \( t_j = \frac{\pi}{2 \lambda_j} \), we obtain that
\[
\sum_{j=0}^{2n} e^{it_j \hat{L}^b} f_j = \sum_{k=1}^{n} a_k u_k + i \sum_{j=1}^{n} b_j u_j = \sum_{k=1}^{n} z_k u_k.
\]

\[ \square \]

In the sequel we shall need also the following Feynman-Kac formulae:

**Proposition 2.14** Let \( \mathcal{E}(p) = p^2 \), and let \( 0 < z < 1 \).

1. For any \( f = \mathcal{F} \in L_2(\Lambda) \), \( b \in C_+ (\partial \Lambda) \)
\[
\frac{\text{Tr}_{\Gamma^{-1}(L^2(\Lambda))} e^{i \phi_\lambda(f)} \Gamma^{-1} \left( e^{-\beta(\Delta_{\hat{A}}^b + \mu^1)} \right)}{\text{Tr}_{\Gamma^{-1}(L^2(\Lambda))} \left( \Gamma^{-1} \left( e^{-\beta(\Delta_{\hat{A}}^b + \mu^1)} \right) \right)}
\equiv \omega_0^{(\Lambda,b)}(W_F(f))
\equiv E e^{i \langle \xi^{(\Lambda,b)} , f \rangle} = \mu_0^{(\Lambda,b)}(e^{i \langle \phi , \delta_0 \otimes f \rangle})
\]
\[ (2.89) \]

2. For any \( -\beta/2 \leq \tau_1 \leq \ldots \leq \tau_n \leq \beta/2, f_1 = \mathcal{F}_1, \ldots, f_n = \mathcal{F}_n \in L_2(\Lambda) \)
\[
\frac{\text{Tr}_{\Gamma^{-1}(L^2(\Lambda))} \left( \alpha_{\tau_1}^{(\Lambda,b)}(E \pi_0(W_F(f_1))) \ldots E \alpha_{\tau_n}^{(\Lambda,b)}(E \pi_0(W_F(f_n))) \right) \Gamma^{-1} \left( e^{-\beta(\Delta_{\hat{A}}^b + \mu^1)} \right)}{\text{Tr}_{\Gamma^{-1}(L^2(\Lambda))} \left( \Gamma^{-1} \left( e^{-\beta(\Delta_{\hat{A}}^b + \mu^1)} \right) \right)}
\]
\[ = E \left( \prod_{i=1}^{n} e^{i \langle \xi^{(\Lambda,b)} , f_i \rangle} \right)
\]
\[ = \mu_0^{(\Lambda,b)} \left( \prod_{i=1}^{n} e^{i \langle \phi , \delta_{\tau_i} \otimes f_i \rangle} \right), \]
\[ (2.90) \]

3. For any sequence \( (\Lambda_m, b_{\partial \Lambda_m}) \) as in lemma 2.12, any \( -\beta/2 \leq \tau_1 \leq \ldots \leq \tau_n \leq \beta/2, f_1, \ldots, f_n \in L^2(\mathbb{R}^d) \) real and sufficiently small \( z \) limits
\[
\lim_{m \to \infty} E \left( \prod_{i=1}^{n} e^{i \langle \xi^{(\Lambda_m,b_m)} , f_i \rangle} \right)
\]
31
(resp. \( \lim_{m \to \infty} \mu_{0}^{(\Lambda_{m}, b_{\Lambda_{m}})} \left( \prod_{i=1}^{m} e^{i \langle \phi, \delta_{\tau_{i}} \otimes f_{i} \rangle} \right) \)),
exist and are equal to
\[ E \left( \prod_{i=1}^{m} e^{i \langle \epsilon_{i}^{0}, f_{i} \rangle} \right) \]
resp
\[ \mu_{0}^{\beta} \left( \prod_{i=1}^{m} e^{i \langle \phi, \delta_{\tau_{i}} \otimes f_{i} \rangle} \right). \]

3 Gentle perturbations of the free Bose Gas: Thermodynamic limits on the abelian sector

We shall study the thermodynamic limits of the multiplicative-like perturbations of the free thermal field \( \mu_{0}^{(\beta, \mu)} \) given by the following perturbations:
\[ \mu_{\Lambda, \epsilon}^{(\beta, \mu)} (d\phi) = Z_{\Lambda}^{-1} \exp W_{\Lambda}(\phi_{\epsilon}) \mu_{0}^{(\beta, \mu)} (d\phi) \] (3.1)
where the interactions \( W_{\Lambda}(\phi_{\epsilon}) \) will be of the following form:

(LGP) the local gentle perturbations:
\[ W_{\Lambda}^{L}(\phi_{\epsilon}) = \lambda \int d\rho(\alpha) \int_{0}^{\beta} d\tau \int_{\Lambda} : e^{i \alpha \phi_{\epsilon}(\tau, x)} : dx \] (3.2)
where:
\( : e^{i \alpha \phi_{\epsilon}(\tau, x)} : = \exp \frac{\alpha^{2}}{2} S_{\epsilon}^{\beta}(0, x) \exp i \alpha \phi_{\epsilon}(\tau, x) \)
\( d\rho \) is a complex, bounded measure with a compact support and such that \( d\rho(\alpha) = d\rho(-\alpha) \);
\( \phi_{\epsilon}(\tau, x) = (\phi \ast \chi_{\epsilon})(\tau, x) \), where \( (\chi_{\epsilon})_{\epsilon} > 0 \) is a positive mollifier i.e. \( 0 \leq \chi_{\epsilon} \in C_{c}^{\infty}(\mathbb{R}^{d}) \), with support of size smaller than \( \epsilon \) and such that \( \int_{\mathbb{R}^{d}} \chi_{\epsilon}(x) dx = 1 \);
\( \lambda \) is the strength of the perturbation.
(nLGP) the nonlocal gentle perturbations

\[
W^{nl}_\Lambda (\phi_\epsilon) = \lambda \int_0^\beta d\tau \int d\rho(\alpha) d\rho(\alpha') \\
\int dx \int dy : e^{i\alpha_\epsilon(\tau, x)} : V(x - y) : e^{i\alpha_\epsilon(\tau, y)} :
\]

(3.3)

where \( \lambda, d\rho, \phi_\epsilon \) are as in the local case, the kernel \( V \) is chosen to be \( L_1 \) integrable function.

**Lemma 3.1** For both choices (LGP) and (nLGP) the thermodynamic stability bound

\[
Z_\Lambda = \int d\mu_0^{(\beta, \mu)} \exp W_\Lambda (\phi_\epsilon) \leq \exp C \cdot |\Lambda|
\]

(3.4)

holds, where \( C \) is some constant depending on the details of the perturbations.

**Proof:**

We shall consider only the case (nLGP). By simple Gaussian calculations we obtain:

\[
\int d\mu_0^{(\beta, \mu)} (\phi) W^{nl}_\Lambda (\phi_\epsilon)^n \\
= \lambda^n \int_0^\beta d\tau_1^n \int d\rho(\alpha)_1^n \int d\rho(\alpha')_1^n \int dx_1^n \int dy_1^n \\
\prod_{i=1}^n V(x_i - y_i) \exp - \frac{1}{2} \sum_{i, j=1 \atop i \neq j}^n \alpha_i \alpha_j S^\beta_\epsilon (\tau_i - \tau_j, x_i - x_j) \\
\exp - \frac{1}{2} \sum_{i, j=1 \atop i \neq j}^n \alpha'_i \alpha'_j S^\beta_\epsilon (\tau_i - \tau_j, y_i - y_j) \exp - \frac{1}{2} \sum_{i, j=1}^n \alpha_i \alpha'_j S^\beta_\epsilon (\tau_i - \tau_j, x_i - y_j)
\]

(3.5)

Using the positive definitness of \( S^\beta_\epsilon \) we can estimate:

\[
\left| \int d\mu_0^{(\beta, \mu)} (\phi) W^{nl}_\Lambda (\phi_\epsilon)^n \right| = |\lambda|^n |\beta|^n \exp \left( 2n S^\beta_\epsilon (0) \right) \| V \|^n |\Lambda|^n
\]

(3.6)

which shows the bound \( \text{(3.4)} \) with

\[
C = |\lambda| \beta (\text{Var} \rho)^2 \exp \left( 2 S^\beta_\epsilon (0) \right) \| V \|_1.
\]

(3.7)
Moreover it follows that $Z_\Lambda$ are entire functions of the coupling constant $\lambda \in \mathcal{O}$.

The characteristic functionals of the perturbed measures $\mu^{(\beta,\mu)}_{\Lambda,\epsilon}$ can be written in the following forms

**LGP** case

$$
\mu^{(\beta,\mu)}_{\Lambda,\epsilon} \left( e^{i\langle \phi, g \otimes f \rangle} \right) = \exp \left( -\frac{1}{2} S_0^\beta (g \otimes f | g \otimes f) \right) \sum_{n \geq 0} \frac{1}{n!} \int_0^\beta d\tau_1^n \int d\rho |_1^n \prod_{i=1}^n \left[ e^{-i\alpha_i (g \otimes f) * S_0^\beta (\tau_i, x_i)} - 1 \right] \rho_{\Lambda,\epsilon}(\tau, x)_1^n
$$

where:

$$
\rho_{\Lambda,\epsilon}(\tau, \alpha, x)_1^n = \lambda^n \int d\mu^{(\beta,\mu)}_{\Lambda}(\phi) \prod_{l=1}^n : e^{i\alpha_l \phi_l (\tau_l, x_l)} : ,
$$

(3.8)

**nLGP** case

$$
\mu^{(\beta,\mu)}_{\Lambda,\epsilon} \left( e^{i\langle \phi, g \otimes f \rangle} \right) = \exp \left( -\frac{1}{2} S_0^\beta (g \otimes f | g \otimes f) \right) \sum_{n \geq 0} \frac{1}{n!} \int_0^\beta d\tau_1^n \int d\rho |_1^n \prod_{i=1}^n \left[ e^{-i\alpha_i (g \otimes f) * S_0^\beta (\tau_i, x_i) - \beta_i (g \otimes f) * S_0^\beta} - 1 \right] \sigma_{\Lambda,\epsilon}(\tau, (\alpha, x)_1^n, (\beta, y)_1^n)
$$

where:

$$
\sigma_{\Lambda,\epsilon}(\tau_1^n, (\alpha, x)_1^n, (\beta, y)_1^n) = \lambda^n \mu^{(\beta,\mu)}_{\Lambda} \left( \prod_{l=1}^n : e^{i\alpha_l \phi_l (\tau_l, x_l)} : \prod_{l=1}^n : e^{i\beta_l \phi_l (\tau_l, y_l)} : \right) \ 
$$

(3.10)

Employing the integration by parts formula we obtain the following equalities

(3.11)
\( \rho_{\Lambda,\varepsilon}(\tau, x)_1^n = \lambda^n \exp - \sum_{i=2}^n S^\beta_\varepsilon(\tau_1 - \tau_i|x_1 - x_i)\alpha_i\alpha_1 \)

\[
\mu^{(\beta,\mu)}_{\Lambda,\varepsilon} \left( \prod_{l=2}^n : e^{i\alpha_l\phi_\varepsilon(\tau_l, x_l)} : \exp \left\{ \lambda \int_0^\beta d\tau \int d\lambda \int dx \int d\rho(\alpha) \right\} \right.
\[
\left. \left[ e^{-\alpha_1 S_\varepsilon^\beta(\tau_1 - \tau_1|x_1 - x_1)} - 1 \right] : e^{i\alpha_1\phi_\varepsilon(\tau, x)} \right) \right) \quad (3.12)
\]

\( \sigma_{\Lambda,\varepsilon} \left( (\tau, \alpha, x)_1^n, (\tau, \beta, y)_1^n \right) \)

\[
\sigma_{\Lambda,\varepsilon} \left( (\tau, \alpha, x)_1^n, (\tau, \beta, y)_1^n \right) = \lambda^n \exp - \sum_{i=2}^n \alpha_1\alpha_i S_\varepsilon^\beta(\tau_1 - \tau_i|x_1 - x_i) \exp - \sum_{i=2}^n \beta_1\beta_i S_\varepsilon^\beta(\sigma_1 - \sigma_i|y_1 - y_i) \)
\[
\mu^{(\beta,\mu)}_{\Lambda,\varepsilon} \left( \prod_{l=2}^n : e^{i\alpha_l\phi_\varepsilon(\tau_l, x_l)} : \prod_{l=2}^n : e^{i\beta_l\phi_\varepsilon(\sigma_l, y_l)} : \exp \left\{ \lambda \int_0^\beta d\tau \int d\lambda \int dx \int d\lambda \int dy \right\} \right.
\[
\left. \left[ e^{-\alpha_1 S_\varepsilon^\beta(\tau_1 - \tau_1|x_1 - x_1)} e^{-\beta_1 S_\varepsilon^\beta(\tau_1 - \tau_1|y_1 - y_1)} - 1 \right] \right) \right) \quad (3.13)
\]

in which after an convergent expansion in powers of \( \lambda \) we recognize the well known \[22\] Kirkwood-Salsburg-like equalities that hold between the correlation functions. A straightforward application of the contraction map principle \[22\] or the methods of the dual pairs of Banach spaces \[38\] leads to the proof of the following proposition in the (LGP) case.

**Proposition 3.2 (LGP)**

1. For \(|\lambda| < \lambda_0(LGP)\), where

\[
\lambda_0(LGP) = \exp \left\{ -\alpha^2 S_\varepsilon(0, 0) - 1 \right\} C^{d-1}_\varepsilon, \quad (3.14)
\]
where

\[ C_\varepsilon^d \equiv \sup_{\alpha'} \int_0^\beta d\tau \int d|\lambda|(\alpha) \int dx |e^{\alpha' S_\varepsilon(\tau,x)} - 1|, \]

\[ \alpha^2_* = \sup\{ \alpha^2 \in \text{supp} \, d\lambda \}, \]

the unique thermodynamic limits

\[ \lim_{\Lambda^1 \uparrow \mathbb{R}^d} \rho_{\Lambda,\varepsilon}(\tau,\alpha,x)_1^n \equiv \rho_\varepsilon(\tau,\alpha,x)_1^n \]

exist in the sense of locally uniform convergence. The limiting correlation functions \( \rho_\varepsilon(\tau,\alpha,x)_1^n \) are continuous, translationally invariant and have cluster decomposition property. Moreover, they are analytic functions in \( \lambda \) for \( |\lambda| < \lambda_0(\text{LGP}) \).

2. Let

\[ \lambda \in \{ z \mid z^{-1} \notin \sigma_{\varepsilon}(K) \} \cap \{ |z| < \xi \} \]

where \( K \) is the corresponding infinite-volume KS-operator, \( \sigma_{\varepsilon}(K) \) is the spectrum of \( K \) in the corresponding Banach space \( B_\varepsilon \) (compare with [38, 39]).

Then for any such \( \lambda \) the unique thermodynamic limits

\[ \rho_\varepsilon(\tau,\alpha,x)_1^n = \lim_{\Lambda^1 \uparrow \mathbb{R}^d} \rho_{\Lambda,\varepsilon}(\tau,\alpha,x)_1^n \]

exist in the sense of locally uniform convergence and are analytic functions in \( \lambda \).

As a simple corollary we obtain:

**Proposition 3.3 (LGP)**

1. For \( \lambda \in \mathcal{C} \) as described in point 1. or 2. of Prop. [2.3] the weak limit \( d\mu_\varepsilon^\lambda \) of the measure \( d\mu_{\Lambda,\varepsilon}^\beta \) exists and the limiting measure \( d\mu_\varepsilon^\lambda \) is periodic in \( \beta \), symmetric on \( K_\beta \), OS-positive on \( K_\beta \). Moreover, \( d\mu_\varepsilon^\lambda \) is (weakly) analytic in \( \lambda \) perturbation of the free measure \( d\mu_0^\beta \).

2. For \( |\lambda| < \lambda_0(\text{LGP}) \) the limiting measure \( d\mu_\varepsilon^\lambda \) is translationally invariant with respect to the translations of \( \mathbb{R}^d \) and is ergodic under the action of this group.
3. For $\lambda$ as in 1. the characteristic functional of $d\mu^{\lambda}_{c}$ is given by the following formula

$$
\mu^{\lambda}_{c}(e^{i(\phi,g\otimes f)}) = \exp \left(-\frac{1}{2} S_0^{\beta}(g \otimes f|g \otimes f) \right)
\sum_{n \geq 0} \frac{1}{n!} \int d\rho(\alpha) \, d\tau \, dx \left| \prod_{l=1}^{n} [e^{-\alpha l S^{\beta}_{\epsilon}(\tau_l,x_l)} - 1] \right| \rho_{c}(\tau,\alpha,x)^{n} (3.15)
$$

A minor modification of the original analysis of the Kirkwood-Salsburg identities enables us to control also thermodynamic limits for nonlocal gentle perturbation (3.3) also.

Proposition 3.4 (nLGP) Let $W = (nGLP)$.

1. For $\lambda \in C : |\lambda| < \lambda_{0}(nLGP)$ where

$$
\lambda_{0}(nLGP) \equiv \exp \left(-2\alpha_{s}^{2} S_{\epsilon}(0,0) - 1 \right) \left(C^{nL}_{\epsilon} \right)^{-1},
$$

$$
C^{nL}_{\epsilon} \equiv \sup_{\gamma,\gamma'} \int_{0}^{\beta} d\tau \int d|\lambda(\alpha)| \int d|\lambda(\alpha')| \int dx \int dy V(x - y) \left| e^{-\alpha_{s} S_{\epsilon}(\tau,x)} e^{-\alpha'_{s} S_{\epsilon}(\tau,y)} - 1 \right|,
$$

$$
\alpha_{s} = \sup\{\alpha \in \text{supp} d\lambda\}
$$

$$
\lim d\mu^{\beta,\mu}_{\lambda,\epsilon} = d\mu^{\lambda}_{\epsilon} \text{ exists and the limiting measure } d\mu^{\lambda}_{\epsilon} \text{ is: periodic in } \beta, \text{ symmetric on } K_{\beta}, \text{ OS-positive on } K_{\beta}. \text{ Moreover, } d\mu^{\lambda}_{\epsilon} \text{ is (weakly) analitic in } \lambda \text{ perturbation of the free measure. The measure } d\mu^{\lambda}_{\epsilon} \text{ is } E(d) \text{ invariant and ergodic under the translations by } \mathbb{R}^{d}. \text{ The characteristic}
$$

37
functional of \( d\mu^\lambda_\epsilon \) is given by the following formula

\[
\mu^\lambda_\epsilon \left( e^{i(\phi \otimes g \otimes f)} \right) = \exp \left( -\frac{1}{2} C_0^\beta (g \otimes f | g \otimes f) \right) \\
\sum_{n \geq 0} \frac{1}{n!} \int d(\tau, x, \alpha)^n d(\tau', x', \alpha')^n \prod_{i=1}^n V(x'_i - x_i) \\
\prod_{i=1}^n \left( \exp \left( -\alpha_i S^\beta_\epsilon \ast (g \otimes f)(\tau_i, x_i) \right) \exp \left( -\alpha'_i S^\beta_\epsilon \ast (g \otimes f)(\tau'_i, x'_i) \right) - 1 \right) \\
\sigma^\lambda_\epsilon \left( \left( \tau, x, \alpha \right)^n_1; \left( \tau', x', \alpha' \right)^n_1 \right)
\]

where:

\[
\sigma^\lambda_\epsilon \left( \left( \tau, x, \alpha \right)^n_1; \left( \tau', x', \alpha' \right)^n_1 \right) \equiv \lim_{\Lambda \uparrow \mathbb{R}^d} \mu^\lambda_\epsilon \left( \prod_{i=1}^n : e^{i\alpha_i \phi_\epsilon (\tau_i, x_i)} : \prod_{i=1}^n : e^{i\alpha'_i \phi_\epsilon (\tau'_i, x'_i)} : \right) \\
= \mu^\lambda_\epsilon \left( \prod_{i=1}^n : e^{i\alpha_i \phi_\epsilon (\tau_i, x_i)} : \prod_{i=1}^n : e^{i\alpha'_i \phi_\epsilon (\tau'_i, x'_i)} : \right).
\]

(3.16)

In particular we have obtained the following functional integral representation of the corresponding multi-time Euclidean Green functions corresponding to the infinite-volume limit perturbations of the free Bose gas in the noncritical regime.

**Theorem 3.5** Let: \( V_\Lambda = (LGP) \) or \( V_\Lambda = (nLGP) \) and \( \lambda \in \mathfrak{C} \) be restricted as in 1. of Prop. 3.4 or 2. of Prop. 3.2 in (LGP) case. Then the Euclidean multitime Green functions on \( \mathcal{A}(\mathbf{h}) \) are given by the following functional in-
Some elementary albeit fundamental for the purposes of the present paper properties of the system are collected in the following proposition:

Proposition 3.6 Let \( \{ E_G^{\lambda,f_1,\ldots,f_n}(s_1,\ldots,s_n) \} \) be a collection of the Euclidean-multitime infinite volume Green functions constructed in Theorem 3.5. Then they can be extended by continuity to the Abelian sector \( A(h) \) of the Weyl algebra \( \mathcal{W}(h) \) and the continued Green functions denoted by the same symbol obey properties \( EG(1) \div EG(5)i. \)

Corollary 3.7 Let \( |\lambda| < \lambda_0(LGP) \) (for the case \( LPG \)) and \( |\lambda| < \lambda_0(nLGP) \) (for the case \( nLGP \)). Then the following perturbation expansions are con-
vergent:

\[ E G_{f_1, \ldots, f_n}^\lambda(s_1, \ldots, s_n) \] \( \overset{(LGP)}{=} \) \( \sum_{n \geq 0} \frac{\lambda^n}{n!} \int_{K_\beta \times \mathbb{R}^d \times \mathbb{R}^d} \cdots \int_{K_\beta \times \mathbb{R}^d \times \mathbb{R}^d} d\tau d\rho|1|^n \]

\[ \cdot \left\langle \prod_{l=1}^n e^{\phi \delta_{x_l} \otimes f_l} ; e^{i\alpha_1 \phi(x_1) \ldots} \right\rangle_{\beta,T}^T \]

(3.19)

where \( \langle \cdot ; \cdot ; \cdot \rangle_{\beta,T}^T \) denote the truncated expectation values with respect to the free gas measure \( d\mu_0^\beta \).

For a class of gentle perturbations of the free Bose gas stochastic structure another variety of the existence results can be established using the methods of [5, 40]. For this let us assume now that our perturbations are of the following forms

\[ (LGP)_e \quad W_\Lambda(\phi) = (3.2) \]

but now \( d\rho \) is an even bounded real measure, \( \lambda \geq 0 \) or

\[ (nLGP)_e \quad W_\Lambda(\phi) = (3.3) \]

where \( d\rho \) is also an even bounded real measure and \( V \in L_1(\mathbb{R}^d) \) is assumed to be pointwise nonnegative i.e. \( V(x) \geq 0 \) and \( \lambda \geq 0 \).

**Proposition 3.8** Let \( d\mu^\lambda_{\Lambda,\epsilon} \) be a locally perturbed free Bose gas measure by \((LGP)_e \) or \((nLGP)_e \) and let \( \lambda > 0 \). Then the following correlation inequalities of the Fröhlich-Park type are valid.

1. \( Z_{\Lambda_1 \cup \Lambda_2} \geq Z_{\Lambda_1} \cdot Z_{\Lambda_2} \), \hspace{1cm} (3.20)
2. \( \left\langle \phi^2(g \otimes f) ; \prod_{i=1}^n \cos \alpha_i \phi(x_i) \right\rangle_{\Lambda,\epsilon}^\lambda \leq 0 \), \hspace{1cm} (3.21)
3. \( \left\langle e^{i\phi(g \otimes f)} ; \prod_{i=1}^n \cos \alpha_i \phi(x_i) \right\rangle_{\Lambda,\epsilon}^\lambda \leq 0 \), \hspace{1cm} (3.22)
4. \( \left\langle e^{i\phi(g \otimes f)} ; \prod_{i=1}^n \cos \alpha_i \phi(x_i) \right\rangle_{\Lambda,\epsilon}^\lambda \geq 0 \), \hspace{1cm} (3.23)
5. \( \left\langle \prod_i \cos \alpha_i \phi(x_i) \prod_j \cos \beta_j \phi(y_j) \right\rangle_{\Lambda,\epsilon}^\lambda \geq 0 \). \hspace{1cm} (3.24)
**Proof:**

Basically the same as in [5] employing duplicate variable trick and the elementary trigonometric identities. 

□

**Theorem 3.9** Let us consider perturbation \((LGP)_e\) or \((nLGP)_e\) of the free Bose Gas thermal field \(d\mu_0^\beta\).

1. For any \(\lambda \geq 0\) the unique thermodynamic limit

\[
\lim_{\Lambda \uparrow \mathbb{R}} \mu_\Lambda^\lambda \left( \prod_{i=1}^n e^{i \langle \phi, \delta_{s_i} \otimes f_i \rangle} \right) \equiv \mu_\epsilon^\lambda \left( \prod_{i=1}^n e^{i \langle \phi, \delta_{s_i} \otimes f_i \rangle} \right)
\]

\[(3.25)\]

\[\equiv E_{f_1 \ldots f_n}^\lambda (s_1, \ldots, s_n) \text{ for } -\beta/2 \leq s_1 \leq \ldots \leq s_n \leq \beta/2\]

exists and the limiting Green functions obey all the properties \(EG(1) \div EG(5)(i)\).

2. In particular the following estimates hold:

(a) \[
\left| S_\lambda^2 (f \otimes g | f \otimes g) \right| \leq \frac{d^2}{d\alpha_1 d\alpha_2} E_{\alpha_1 g, \alpha_2 g}^\lambda (f, f)|_{\alpha_1 = 0} \leq S_0^\beta (f \otimes g | f \otimes g)
\]

\[(3.26)\]

(b) \[
\left| \mu_\epsilon^\lambda \left( \exp S \int_0^\beta d\tau f(\tau) \int dx g(x) \phi(\tau, x) \right) \right| \leq \exp \text{Re} S_0^\beta (f \otimes g | f \otimes g)
\]

\[(3.27)\]

(c) \[
\left| S_{\lambda}^{n,\beta} (f_1 \otimes g_2, \ldots, f_n \otimes g_n) \right| \equiv \mu_\epsilon^\lambda \left( \prod_{i=1}^n \langle \phi, f_i \otimes g_i \rangle \right) \leq \sigma(n)!^{\frac{1}{2}} \prod_{i=1}^n \left| S_0^\beta (f_i \otimes g_i | f_i \otimes g_i) \right|
\]

\[(3.28)\]
Proof:
From the correlation inequality (3.23) it follows that $\mu_\lambda^\Lambda (e^{i\phi(f \otimes g)})$ monotonously increase in the volume and that for real $t$ $\mu_\lambda^\Lambda (e^{t\phi(f \otimes g)})$ decrease as $\Lambda \uparrow \mathbb{R}^d$. This leads to the statement that the unique limit $\lim_\Lambda \mu_\lambda^\Lambda (e^{\zeta(f \otimes g)})$ exists and obeys the estimate (3.27). Then the estimates (3.28) follow by the application of the Cauchy integral formula and the analicity in $\zeta$ of $\mu_\lambda^\Lambda (e^{\zeta(f \otimes g)})$. Although the estimate (3.26) follows from (3.28) its independent proof follows easily from the correlation inequality (3.21) which says that $\mu_\lambda^\Lambda (\phi, f \otimes g)^2$ is monotonously decreasing in the volume.

Integrating by parts on the functional space $D'(K_\beta \times \mathbb{R}^d)$ with respect to the measure $d\mu_\lambda^\Lambda (\phi)$ the following formulae are obtained:

$$
E G_\lambda^\Lambda, f_1, ..., f_n (s_1, \ldots, s_n) = GLP e^{E G_0^\Lambda, f_1, ..., f_n (s_1, \ldots, s_n)}
$$

$$
\sum_{k \geq 0} \frac{\Lambda^k}{k!} \int_{K_\beta \times \mathbb{R}^d} d\tau d\lambda \left( \prod_{i=1}^n \left[ e^{-\sum_{j=1}^n \alpha_j S_j^2 (\delta_j \otimes f_j) - 1} \right] \right)
$$

(3.29)

From the correlation inequality (3.24) it follows that $\mu_\lambda^\Lambda \left( : \prod_{i=1}^n \cos \alpha_i \phi_e (\tau_i, x_i) : \right) \equiv C_\lambda^\Lambda (\alpha_1, \tau_i, x_i |_{1}^{n})$

(3.30)

monotonously increase in the volume $\Lambda$ and because they are uniformly bounded

$$
\left| C_\lambda^\Lambda (\tau_i, x_i |_{1}^{n}) \right| \leq \exp \frac{1}{2} \beta^2 n C^\beta_\epsilon (0),
$$

(3.31)

the unique thermodynamic limits $\lim_\Lambda C_\lambda^\Lambda \equiv C^\lambda$ exist pointwise on $(K_\beta \times \mathbb{R}^d)^{\otimes n}$. From this, the existence of pointwise limits

$$
\lim_\Lambda \mu_\lambda^\Lambda \left( \prod_{j=1}^n \phi_e (\tau_j, x_j) : \right) = \mu_\epsilon \left( \prod_{j=1}^n \phi_e (\tau_j, x_j) : \right)
$$

(3.32)
follows in the same way as demonstrated in [40] by the application of another correlation inequality (originally due to Pfister [41]) not listed in proposition 3.3 but formulated in [40] in a similar context. Finally the proven pointwise convergence is sharpened to the local uniform one by a standard application of the Mayer-Montrolle identities, see i.e. [38]. From the obtained convergence the following expression for the infinite-volume Euclidean Green functions \( E_{\lambda}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) \) follows easily from (3.29):

\[
E_{\lambda}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) = E_{\lambda}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) \\
\sum_{k \geq 0} \frac{1}{k!} \int_{\mathbb{R}^d \times \mathbb{R}^d} d\tau dx d\lambda(\alpha) \int_{\mathbb{R}^d} d\tau d\tau_1 \cdots d\tau_k \left( e^{-\sum_{j=1}^n \alpha_i S_{\beta}^* \delta_j \otimes f_j} - 1 \right) \\
\mu_\lambda \left( \prod_{i=1}^n e^{i \alpha_i \phi(\tau_i,x_i)} \right)
\]

(3.33)

The case of nLGP is analised in a similar way.

\[\square\]

Remarks

The existence and uniqueness of the thermodynamic limits for the Euclidean Green functions \( E_{\lambda}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) \) follows easily from the correlation inequality (3.23) and the uniform bound:

\[
\left| E_{\lambda}^{f_1,\ldots,f_n}(s_1,\ldots,s_n) \right| \leq 1.
\]

(3.34)

Using the methods based on the analysis of the corresponding Kirkwood-Salsburg identities one can study the gentle perturbations of the local, free, conditioned thermal fields described in the section 2.2.

For this goal let us consider a perturbation of the free, conditioned (by \( b_{b\Lambda} \in C(\partial \Lambda) \)), thermal field \( \mu_0^{(\Lambda,b_{b\Lambda})} \) of the form:

\[
\tilde{\mu}_{\Lambda,\epsilon}^{(\Lambda,b_{b\Lambda})}(d\Phi) = \tilde{Z}_{\Lambda,\epsilon}^{-1}(b_{b\Lambda}) \exp W_{\Lambda}(\Phi) \, d\mu_0^{(\Lambda,b_{b\Lambda})}(\Phi)
\]

(3.35)

where:

\[
\tilde{Z}_{\Lambda,\epsilon}(b_{b\Lambda}) = \mu_0^{(\Lambda,b_{b\Lambda})} (\exp W_{\Lambda})
\]

(3.36)

and \( W_{\Lambda}(\Phi) \) is given by (3.2) or (3.3).
Theorem 3.10  Let \((\Lambda_\alpha)\) be any arbitrary net of bounded subsets of \(\mathbb{R}^d\) with the boundaries of class at least \(C^3\)-piecewise. Additionally we shall require that the mean curvature of \(\partial \Lambda_\alpha\) is uniformly bounded in \(\alpha\). Let \((b_{\partial \Lambda_\alpha}^\alpha)\) be a sequence of continuous boundary conditions.

Then for \(|\lambda| < \lambda_0(LGP)\), if \(W_\Lambda = LGP\) (respectively \(|\lambda| < \lambda_0(nLGP)\), if \(W_\Lambda = nLGP\) the unique thermodynamic limits

\[
\lim_{\alpha} \tilde{\mu}_{\lambda, \epsilon}^{(\alpha, b_{\partial \Lambda_\alpha}^\alpha)} \equiv \tilde{\mu}_\epsilon^\lambda
\]

exists in the sense of weak convergence and moreover \(\tilde{\mu}_\epsilon^\lambda = \mu_\epsilon^\lambda\).

Proof:  

The method of the dual pair of Banach spaces as explained in [38] and applied in the similar situation in [39, 40] is applied.

\[\square\]

Remark

The method of [38, 39, 40] gives the existence and independence on the classical boundary conditions of the limiting thermal field \(\tilde{\mu}_\epsilon^\lambda\) in a larger set of \(\lambda\) (see also point 2 in Prop 3.3 above).

As a corollary we have the following result:

Corollary 3.11  Let \((\Lambda_\alpha)\), \((b_{\partial \Lambda_\alpha})\) be as in Theorem 3.10 and let \(\mathcal{MG}_\lambda(\Lambda_\alpha, b_{\partial \Lambda_\alpha})\) be the system of the Euclidean Green functions corresponding to the gentle perturbations of the local, conditioned, free \(W^*-\text{KMS}\) structure restricted to the Abelian sector \(\mathcal{A}(h_\lambda)\) of \(\mathcal{W}(h_\lambda)\). Then for \(\lambda\) as in Theorem 3.10 and \(0 < z < 1\) sufficiently small the unique thermodynamic limits of the corresponding Euclidean Green functions exist and are equal to those obtained in Theorem 3.3 and Theorem 3.9.

All the constructed in this section systems of limiting Euclidean Green functions obey properties \(EG(1) \div EG(5)(i)\) and corresponds to some generalized thermal processes.

Therefore the general reconstruction procedures of [20] applies, (see Prop 2.4), leading to certain \(W^*-\text{KMS}\) structures further analysis of the derived \(W^*-\text{KMS}\) structures is contained in the forthcoming papers.
4 Concluding Remarks

4.1

For the finite volume perturbations of the free thermal field $\mu^{(\beta,\mu)}_0$ the corresponding nonhomogeneous process $(\xi^{(\lambda,\lambda)}_t)_{t \in K_\beta}$ has two-sided Markov property on $K_\beta$ in the sense of Prop [2.11]. The interesting and important question is whether the homogenous limits $\Lambda \uparrow \mathbb{R}^d$ preserve the above Markov property. For a gentle perturbations of a class of lattice anharmonic crystals some results on the preservation of the two-sided Markov property in the thermodynamic limit have been established in [42]. A constructive route for the verification of the two-sided Markov property will be formulated below.

4.2

The notion of DLR equations for the gentle perturbations of the abelian sector of the free Bose Gas in the Euclidean region can be introduced. For this goal, let us denote by $\Pi(\Lambda^C)$ the orthogonal projector (in the space $\mathcal{H}^\beta_0 \equiv m.c. (C(K_\beta) \times D(\mathbb{R}^d); S_0^\beta)$) onto the subspace $\mathcal{H}^\beta_0(\Lambda^C) \equiv m.c. (C(K_\beta) \times C^\infty(\Lambda^C); S_0^\beta)$, for $\Lambda \subset \mathbb{R}^d$ open and bounded.

The free thermal kernel $S_0^\beta$ is then decomposed as:

$$ S_0^\beta = \Lambda^C S_0^\beta + \Lambda^C \Pi_0^\beta $$

where:

$$ \Lambda^C S_0^\beta \equiv S_0^\beta \circ (1 - \Pi(\Lambda^C)) ; \Lambda^C \Pi_0^\beta = S_0^\beta \circ \Pi(\Lambda^C) $$

Let $\mu^\Lambda_0^C$ be a Gaussian random field with the covariance given by $\Lambda^C S_0^\beta$. It is clear that the symmetricty and OS positivity on $K_\beta$ of the free conditioned Gaussian random field $\mu^\Lambda_0^C$ is preserved and moreover $\mu^\Lambda_0^C \rightarrow \mu_0^\beta$ weakly as $\Lambda \uparrow \mathbb{R}^d$.

Let $\Sigma^0(\Lambda^C)$ be the ($\mu^0$-complete) $\sigma$-algebra generated by the random elements of the form $\langle \Phi, f \rangle$, where $f \in \mathcal{H}_0^\beta(\Lambda^C)$. Then the conditional expectation values of the measure $\mu_0^\beta$ with respect to the $\sigma$-algebras $\Sigma^0(\Lambda^C)$ are given by:

$$ E_{\mu_0} \{ F | \Sigma^0(\Lambda^C) \} (\Psi) = \mu^\Lambda_0^C (F(\cdot + \Pi_{\Lambda^C}(\Psi)) $$

45
for: \( \mu_0 — \text{a.e. } \Psi \in \mathcal{D}'(K_\beta \times \mathbb{R}^d) \), where

\[
\langle \Pi^*_{\Lambda C}(\Psi), \ f > \equiv < \Psi, \ \Pi_{\Lambda C}(f) \rangle.
\]

(4.4)

The corresponding conditional expectation values of the perturbed measure are:

\[
E_{\mu_{\Lambda, \epsilon}} \{ F|\Sigma^0_{\Lambda C} \} (\Psi)
= \frac{\mu_{0}^\Lambda C (F(\cdot + \Pi^*_{\Lambda C}(\Psi)) \exp W_{\Lambda}(\cdot + \Pi^*_{\Lambda C}(\Psi))}{\mu_{0}^\Lambda C (\exp W_{\Lambda}(\cdot + \Pi^*_{\Lambda C}(\Psi))}
\]

(4.5)

for \( \mu_0 — \text{a.e. } \Psi \in \mathcal{D}'(K_\beta \times \mathbb{R}^d) \).

In analogy to [27] (see also [43, 44]) we define a classical thermal Gibbs measure corresponding to the gentle perturbation of the free Bose Gas as any probabilistic, cylindric Borel measure \( \mu \) on \( \mathcal{D}'(K_\beta \times \mathbb{R}^d) \) such that

\[
\mu \circ E_{\mu_{\Lambda, \epsilon}} \{ \Sigma(\Lambda^C) \} = \mu_{\text{DLR}}
\]

for any open bounded \( \Lambda \subset \mathbb{R}^d \).

It is evident that any solution of (DLR) defines a thermal random field in the sense of Def 2.8. Some results about the uniqueness of the solutions of (DLR) generalizing slightly Thm 3.10 shall be reported elsewhere (see also [39, 40]).

The introduced concept of the classical thermal Gibbs measure will be of particular interest in the case of polynomial perturbations where several solutions of the corresponding (DLR) equations may exist [18].

Using (DLR) equation the constructive approach to the problem of preservation of the two-sided Markov property on the circle \( K_\beta \) for the limiting thermal random field \( \mu^\lambda_{\epsilon} \) can be formulated. The idea is to show that for \( \mu^\lambda_{\epsilon} — \text{a.every } \Psi \in \mathcal{D}'(\mathbb{R}^d) \) the limits:

\[
\lim_{\Lambda \uparrow \mathbb{R}^d} E_{\mu^\lambda_{\epsilon}} \{ F|\Sigma^0([t, s]^C \times \Lambda^C) \} (\Psi)
\]

(where \( \Sigma([t, s]^C \times \Lambda^C) \) is the \( \sigma \)-algebras generated by the random elements \( \langle \Phi, \ g \otimes f \rangle \), with \( g \) supported on the segment \([t, s]^C \) and \( f \) supported in \( \Lambda^C \)) exist and are equal (\( \mu^\lambda_{\epsilon} — \text{a.e.} \)) to the conditional expectation values:

\[
E_{\mu^\lambda_{\epsilon}} \{ F|\Sigma([t, s]^C) \} (\psi)
\]

Details of the proof, that indeed, for small values of \( |\lambda| \) this is true, will be reported elsewhere [18].

46
4.3

For a bounded $\Lambda \subset \mathbb{R}^d$ the theory of bounded perturbations of the KMS structures (see i.e. [17], Ch. 4 and references therein) can be applied in the thermal representation enabling us to study the gentle perturbations on the whole Weyl algebra. It is proven in [18] that again the nonhomogeneous thermal process $(\xi_{t,\Lambda})_{t \in K_\beta}$ determines the corresponding $W^*$-KMS structure obtained from the corresponding GNS representation. The important problems of constructing the perturbed (euclidean-time) Green functions on the whole Weyl algebra $\mathcal{W}(\hbar)$ and the questions whether the corresponding homogeneous process $(\xi_{t})_{t \in K_\beta}$ determines them and also whether the limiting $W^*$-KMS structure on $\mathcal{W}(\hbar)$ forms a modular structure will be a topic of another paper in this series.

4.4

The Abelian sector of the free Bose critical gas can be described in the Euclidean region by certain nonergodic Gaussian generalized thermal process. Results complementary to those contained in the section 2 for the critical gas are obtained in [18], where also, thermodynamic limits of the gentle perturbations on the Abelian sector have been controlled by applications of the Fröhlich-Park correlation inequalities. The most interesting result of these investigations is that nonergodicity of the limiting, perturbed thermal process is preserved. Whether this is connected to the preservation of the Bose-Einstein condensate in the interacting system remains to be answered.

4.5

More general, unbounded perturbations (i.e. of polynomial type) will be described in an another paper of the planned series [18]. Standard tools of constructive Euclidean Quantum field theory like the high (and the low) temperature cluster expansions are used to study the corresponding perturbations of the free thermal structure on the Abelian sector.
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