Network Synchronization in a Noisy Environment with Time Delays: Fundamental Limits and Trade-Offs

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We study the effects of nonzero time delays in stochastic synchronization problems with linear couplings in an arbitrary network. Using the known exact threshold value from the theory of differential equations with delays, we provide the synchronizability threshold for an arbitrary network. Further, by constructing the scaling theory of the underlying fluctuations, we establish the absolute limit of synchronization efficiency in a noisy environment with uniform time delays, i.e., the minimum attainable value of the width of the synchronization landscape. Our results also have strong implications for optimization and trade-offs in network synchronization with delays.

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In network synchronization problems [1], individual units, represented by nodes in the network, attempt to adjust their local state variables (e.g., pace, load, orientation) in a decentralized fashion. They interact or communicate only with their local neighbors in the network, often with the intention to improve global performance. These interactions or couplings can be represented by directed or undirected, weighted or unweighted links. Applications of the corresponding models range from physics, biology, computer science to control theory, including synchronization problems in distributed computing [2], consensus, coordination and control in communication networks [3–6], flocking animals [7, 8], bursting neurons [9–11], and cooperative control of vehicle formation [12].

There has been a massive amount of research focusing on the efficiency and optimization of synchronization problems [1, 13–16] in various complex network topologies, including weighted [3, 17] and directed [6, 18, 19] networks. In this Letter, we study an aspect of stochastic synchronization problems which is present in all real communication, information, and computing networks [3–6, 20, 21], including neurobiological networks [10, 11]: the impact of time delays on synchronizability and on the breakdown of synchronization. The presence of time delays, however, will also present possible scenarios for trade-offs. Here we show that when synchronization networks are stressed by large delays, reducing local coordination effort will actually improve global coordination. Similarly, subtle results have also been found in neurobiological networks with the synchronizability efficiency exhibiting non-monotonic behavior as a function of the delay [10, 11].

For our study, we consider the simplest stochastic model with linear local relaxation, where network-connected agents locally adjust their state to closely match that of their neighbors (e.g., load, or task allocation) in an attempt to improve global performance. However, they react to the information or signal received from their neighbors with some time lag (as a result of finite processing, queuing, or transmission delays), motivating our study of the coupled stochastic equations of motion with delay,

\[ \partial_t h_i(t) = -\sum_{j=1}^{N} C_{ij} [h_i(t-\tau_{ij}) - h_j(t-\tau_{ij})] + \eta_i(t) . \] (1)

Here, \( h_i(t) \) is the generalized local state variable on node \( i \) and \( \eta_i(t) \) is a delta-correlated noise with zero mean and variance \( \langle \eta_i(t) \eta_j(t') \rangle = 2D \delta_{ij} \delta(t-t') \), where \( D \) is the noise intensity. \( C_{ij} = C_{ji} \geq 0 \) is the symmetric coupling strength \( (C_{ij} = W_{ij} A_{ij} \text{ in general weighted networks, where } A_{ij} \text{ is the adjacency matrix and } W_{ij} \text{ is the link weight}) \). \( \tau_{ij} \geq 0 \) is the time delay between two connected nodes \( i \) and \( j \). For initial conditions we use \( h_i(0) = 0 \) for \( t \leq 0 \). Eq. (1) is also referred to as the Edwards-Wilkinson process [22] on networks [3] with time-delays. Without the noise term, the above equation is often referred to as the consensus problem [2, 6] on the respective network.

The standard observable in stochastic synchronization problems, where relaxation competes with noise, is the width of the synchronization landscape [2, 3, 13, 16]

\[ \langle w^2(t) \rangle \equiv \left\langle \frac{1}{N} \sum_{i=1}^{N} [h_i(t) - \bar{h}(t)]^2 \right\rangle , \] (2)

where \( \bar{h}(t) = (1/N) \sum_{i=1}^{N} h_i(t) \) is the global average of the local state variables and \( \langle \ldots \rangle \) denotes an ensemble average over the noise. A network of \( N \) nodes is synchronizable if \( \langle w^2(\infty) \rangle < \infty \), i.e., if the width approaches a finite value in the \( t \to \infty \) limit. The smaller the width, the better the synchronization.

In the case of uniform delays \( \tau_{ij} \equiv \tau \), the focus of this
Laplacian, one can decompose the problem into Laplacian. In this case, by diagonalizing the network \( C_{ij} - C_{ij} \), is the symmetric network Laplacian. In this case, by diagonalizing the network Laplacian, one can decompose the problem into \( N \) independent modes

\[ \partial_t h_k(t) = -\lambda_k h_k(t - \tau) + \eta_k(t), \tag{4} \]

where \( \lambda_k, k = 0, 1, 2, \ldots, N - 1 \), are the eigenvalues of the network Laplacian and \( \langle \eta_k(t)\eta(t') \rangle = 2D\delta(t - t'). \) For a single-component (or connected) network, the Laplacian has a single zero mode (indexed by \( k=0 \)) with \( \lambda_0=0 \), while \( \lambda_k>0 \) for \( k \geq 1 \). Using the above eigenmode decomposition, the width of the synchronization landscape can be expressed as \( \langle w^2(t) \rangle = \langle 1/N \rangle \sum_{k=1}^{N-1} \langle \dot{h}^2_k(t) \rangle \). \[ \text{FIG. 1: Time series } \langle \dot{h}^2(t) \rangle \text{ for different delays, obtained by numerically integrating Eq. (6) and averaging over } 10^4 \text{ independent realization of the noise. Here, } \lambda = 1, D = 1, \text{ and } \Delta t = 0.01. \text{ The theoretical (continuous-time) threshold value of the delay } (\langle \dot{h}(\infty) \rangle \text{ to remain bounded}) \text{ is } \tau_c = \pi/2. \tag{25} \]

The solution of Eq. (7) with the largest real part governs the long-time temporal behavior of the respective mode (e.g., stability, approach to, or relaxation in the steady state). The condition for \( \langle \dot{h}^2(\infty) \rangle \) to remain finite is \( \text{Re}(\tau) < 0 \) for all \( \alpha \). As has been shown for Eq. (7), this inequality holds if \( \tau \lambda < \pi/2 \). In Fig. 1 we show the time-dependent width of the fluctuations associated with a single stochastic variable, obtained by numerically integrating Eq. (6) for a few characteristic cases. \[ \text{FIG. 2: Time series } \langle \dot{h}^2(t) \rangle \text{ for different delays, obtained by numerically integrating Eq. (6) and averaging over } 10^4 \text{ independent realization of the noise. Here, } \lambda = 1, D = 1, \text{ and } \Delta t = 0.01. \text{ The theoretical (continuous-time) threshold value of the delay } (\langle \dot{h}(\infty) \rangle \text{ to remain bounded}) \text{ is } \tau_c = \pi/2. \tag{25} \]

The above exact delay threshold for synchronizability has some immediate and profound consequences for unweighted networks. Here, the coupling matrix is identical to the adjacency matrix, \( C_{ij} = A_{ij} \), and the bounds and the scaling properties of the extreme eigenvalues of the network Laplacian are well known. In particular, \( N k_{\max}/(N - 1) \leq \lambda_{\max} \leq 2k_{\max} \) \[ \text{FIG. 3: Time series } \langle \dot{h}^2(t) \rangle \text{ for different delays, obtained by numerically integrating Eq. (6) and averaging over } 10^4 \text{ independent realization of the noise. Here, } \lambda = 1, D = 1, \text{ and } \Delta t = 0.01. \text{ The theoretical (continuous-time) threshold value of the delay } (\langle \dot{h}(\infty) \rangle \text{ to remain bounded}) \text{ is } \tau_c = \pi/2. \tag{25} \]

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The immediate message of the above result is rather in-
merically found that in the vicinity of λτ ≈ \frac{\pi}{2},
which can be associated with an arbitrary mode. In this regime one must have Re(\lambda τ) < 0 for all α, or equivalently, \tau \lambda < \pi/2. Defining a new variable z ≡ \tau z/\tau λ, the solutions for the scaled variable z can only depend on \lambda τ, z0 = z(\lambda τ), \alpha = 1, 2, . . . Thus, the solutions of the characteristic equation Eq. (4) must exhibit the scaling form s_\alpha = \tau^{-1} z_\alpha(\lambda τ), \alpha = 1, 2, . . . Substituting the above expression into Eq. (3) and tak-
ing the t → ∞ limit immediately yields the scaling form

\langle h^2(\infty) \rangle = D\tau f(\lambda τ). \tag{10}

Thus, for a single stochastic variable \hat{h}(t) governed by
the stochastic differential equation Eq. (1) (simple rel-
axation in a noisy environment with delay), plotting
\langle h^2(\infty) \rangle/\tau vs \lambda τ (for a fixed noise intensity D) should
yield full data collapse, as demonstrated in Fig. 2 [22].

While we do not have an analytic expression for the scal-
ing function, for small arguments it asymptotically
has to scale as f(x) ∝ 1/x to reproduce the exact limiting
case of zero delay, \langle h^2(\infty) \rangle/\tau ∝ D/\lambda. Further, we nu-
merically found that in the vicinity of \pi/2, it approxi-
ately diverges as (\pi/2 - x)^{-1}. The scaling function
f(x) is clearly non-monotonic; it exhibits a single min-
imum, at approximately x^* ≈ 0.73 with f = f(x^*) ≈ 3.1.

The immediate message of the above result is rather in-
teresting: For a single stochastic variable governed by
Eq. (5) with a nonzero delay, there is an optimal value of
the relaxation coefficient \lambda^* = x^*/\tau, at which point
the steady-state fluctuations attain their minimum value
\langle h^2(\infty) \rangle = D\tau f^* ≈ 3.1 D\tau. This is in stark contrast
with the zero-delay case where \langle h^2(\infty) \rangle = D/\lambda, i.e., there
the steady-state fluctuation is a monotonically decreasing
function of the relaxation coefficient.

In addition to gaining fundamental insights, construct-
ing the scaling function f(x) numerically with some ac-
ceptable precision of the single variable problem (Fig. 2
inset) also provides a method to obtain the steady-state
width of the network-coupled system: one can numeri-
cally diagonalize the Laplacian of the underlying network
and employ the scaling function f(x) to obtain the width,

\langle w^2(\infty) \rangle = \frac{1}{N} \sum_{k=1}^{N-1} \langle \hat{h}_k^2(\infty) \rangle = \frac{D\tau}{N} \sum_{k=1}^{N-1} f(\lambda_k τ). \tag{11}

Further, we can now extract the minimum attainable
width of the synchronization landscape in a noisy en-
vironment with uniform time delays. For a fixed \tau, each
term in Eq. (11) can be minimized by choosing \lambda_k = x^*/\tau
for all k ≥ 1. Then

\langle w^2(\infty) \rangle^* = \frac{N - 1}{N} D\tau f^* ≈ 3.1 D\tau \tag{12}

for large N. This number, the fundamental limit of syn-
chronization efficiency in a noisy environment with uni-
form time delays, can be used as a base-line value when
comparing networks from the viewpoint of synchroniza-
tion efficiency. Note that there is a trivial network which
realizes the optimal behavior: the fully connected graph
with identical coupling constants C_{ij} = x^*/N \tau for all i ≠ j.
(This network has N − 1 identical non-zero eigenvalues,
\lambda_k = x^*/\tau for all k ≥ 1.) In general, networks with a
narrow spectrum centered about \lambda^* = x^*/\tau shall perform
closer to optimal. How to construct such networks with
possible topological and cost constraints is a different and
challenging question which we will not pursue in detail
here, but we note that essentially the same problem arises
in the broader context of synchronization of generalized
dynamical systems [13, 16]. Recent methods tackling this
issue involve locally reweighting and/or removing links
from the networks to achieve optimal performance [19].

The essential non-monotonic feature of the scaling
function f(x) in Eq. (11) (including the potentially di-
verging contributions from large eigenvalues beyond the
threshold) presents various trade-off scenarios in network
synchronization problems with delays. As the simplest
and obvious application of the above results, consider
a network which is stressed by large delays beyond its
threshold, \tau \lambda_{\text{max}} > \pi/2 (so that the largest fluctuations
and the width are growing exponentially without bound).
Then even a suitably chosen uniform reduction of all cou-
lings \Delta C_{ij} = pC_{ij} (\lambda_k^* = p \lambda_k) with p < \pi/2\lambda_{\text{max}} \tau
will lead to the stabilization of the system, with a finite steady-state
width. In communication and computing networks, the
effective coupling strength C_{ij} can be controlled by the
frequency (or rate) of local synchronizations through the
respective link \lambda_{ij}. The above results then suggest that
when the system is beyond its stability threshold, syn-
chronizing sufficiently less frequently, can lead to stabili-
zation and better coordination. Figure 3 shows results
for the case when the communication neighborhood is
fixed, but the local synchronizations through the links
FIG. 3: Time dependent width for $\tau=1.2\pi/2\lambda_{\text{max}}$ for different values of the local synchronization rate $p$ on a fixed graph, obtained by the numerical integration of Eq. (1) with $\Delta t=0.005$ and $D=1.0$. The underlying network is a Barabási-Albert SF graph with $N=100$, $(k)\approx 6$, and $\lambda_{\text{max}}\approx 32$. The coupling terms in Eq. (1) are only performed with probability $p\leq 1$, while invoking the noise term at every time step. Indeed, reducing the local synchronization rate can improve global performance. In fact, even performing no local synchronizations at all ($p=0$) leads to a slower power-law divergence of the width with time, $\langle \Delta t^2(t) \rangle \approx 2D\Delta t$, as opposed to the exponential divergence governed by the largest eigenvalue(s) above the threshold.

In summary, we have obtained the delay threshold for the simplest stochastic synchronization problem with linear couplings in an arbitrary network. Further, by exploring and investigating the scaling properties of the fluctuations associated with the eigenmodes of the network Laplacian, we found the minimum attainable steady-state width of the synchronization landscape in any network. The non-monotonic feature of the scaling function governing the fluctuations can guide potential trade-offs and optimization in network synchronization. For systems with more general (non-linear) node dynamics, one can also expect that the synchronizability phase diagram will exhibit non-monotonic behavior as a function of the coupling strength and/or the delays $\tau$. In real communication and information networks, the delays $\tau_{ij}$ are not uniform, but are affected by the network neighborhood and spatial distance. We currently investigate the impact of heterogeneous delays on network synchronization.

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[1] A. Arenas et al., Phys. Rep. 469, 93 (2008).
[2] G. Korniss et al., Science 299, 677 (2003).
[3] G. Korniss, Phys. Rev. E 75, 051121 (2007).
[4] R. Johari and D. Kim Hong Tan, IEEE/ACM Trans. Networking 9, 818 (2001).
[5] R. Olfati-Saber and R.M. Murray, IEEE Trans. Automat. Contr. 49, 1520 (2004).
[6] R. Olfati-Saber, J.A. Fax, and R.M. Murray, Proc. IEEE 95, 215 (2007).
[7] T. Vicsek et al., Phys. Rev. Lett. 75, 1226 (1995).
[8] F. Cucker and S. Smale, IEEE Trans. Automat. Contr. 52, 852 (2007).
[9] M.R. Jovanović and F. Bullo, Proc. Natl. Acad. Sci. U.S.A. 107, 10342 (2010).
[10] T. Hogg and B.A. Huberman, IEEE Trans. on Sys., Man, and Cybernetics 21, 1325 (1991).
[11] M.G. Earl and S.H. Strogatz, Phys. Rev. E 67, 036204 (2003).
[12] S.F. Edwards and D.R. Wilkinson, Proc. R. Soc. London, Ser A 381, 17 (1982).
[13] R. Frisch and H. Holme, Econometrica 3, 225 (1935).
[14] N.D. Hayes, J. London Math. Soc. 2151, 226 (1950).
[15] The time discretization of Eq. (4), and in general, that of Eq. (4) naturally has its own effects on the stability of the underlying continuous-time system. Choosing $\Delta t<<\tau$ and $\Delta t<<1/\lambda_{\text{max}}$ will yield only small corrections to the behavior of the underlying continuous-time system.
[16] Note that this condition coincides with the convergence condition of the deterministic consensus problem [3, 4].
[17] W.N. Anderson and T.D. Morley, Lin. Multilin. Algebra 18, 141 (1985).
[18] A.-L. Barabási and R. Albert, Science 286, 509 (1999).
[19] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[20] S.N. Dorogovtsev and J.F.F. Mendes, Adv. in Phys. 51, 1079 (2002).