Properties of neutral and charged anyon fluids are examined, with the main focus on the question whether or not a charged anyon fluid exhibits a superconductivity at zero and finite temperature. Quantum mechanics of anyon fluids is precisely described by Chern-Simons gauge theory. The random phase approximation (RPA), the linearized self-consistent field method (SCF), and the hydrodynamic approach employed in the early analysis of anyon fluids are all equivalent. Relations and differences between neutral and charged anyon fluids are discussed. It is necessary to go beyond RPA and the linearized SCF, and possibly beyond the Hartree-Fock approximation, to correctly describe various phenomena such as the flux quantization, vortex formation, and phase transition.

1. Introduction

Is a charged anyon fluid a superconductor? Is there any difference in its behaviour from traditional superconductors described by Ginzburg-Landau-BCS theory? Can newly discovered high $T_c$ superconductors be anyon superconductors? These are the main questions addressed in this article.\cite{1,2,3,4,5,6,7,8,9,10} Since Laughlin suggested that a high $T_c$ superconductor may be viewed as an anyon fluid,\cite{11} an extensive study has been conducted by many authors by various methods.

After three years of investigation, we can now have a coherent assessment of the current understanding. A fair statement, at the moment, is that a charged anyon...
fluid seems to behave like a superconductor, but its properties have not been understood very well. We are not even sure if we have found a good approximate ground state of a charged anyon fluid, which should serve as an alternative to the BCS ground state of ordinary superconductors. The approximate ground state employed in the early investigation, characterized by a state of completely filled Landau levels, might miss many of important phenomena such as the flux quantization, Josephson effect, and vortex formation.

Anyons exist in Nature. Excited states (quasi particles) in fractional quantum Hall effect (FQHE) obey fractional statistics. Laughlin’s theory of FQHE implies that for a filling factor $\nu = p/q$, where $p$ and $q$ are coprime numbers, quasi particles have a statistics phase $\pm \pi/q$. The theory predicts a hierarchy structure in the Hall conductivity $\sigma_{xy}$ as the filling factor or magnetic field varies, which has been confirmed experimentally at multiple levels.

Where else are anyons? Three years ago Laughlin made a bold hypothesis that anyons are in newly discovered high $T_c$ superconductors. Where all these analyses looked quite different from each other, though reaching to the same conclusion, namely a superconductivity at $T = 0$. In the mean time the investigation has been extended in various directions, which includes analyses of
finite temperature, vortices, conductivity, $P$ and $T$ violation, higher order radiative corrections, and so force.\textsuperscript{34–58}

It has been recently shown\textsuperscript{44} that the RPA, self-consistent field method (SCF), and hydrodynamic approach yield the same results for many physical quantities such as the excitation spectrum and response function. In this paper we shall strengthen the statement. We show that RPA is exactly the same as the linearized version of SCF, and that the hydrodynamic approach describes the same physics in terms of the density and velocity field as SCF does in terms of the gauge fields. Hence all these three are equivalent.

Quantum mechanics of anyon systems is precisely described in terms of Chern-Simons gauge theory. The essence of anyon dynamics is contained in the Aharanov-Bohm effect with respect to Chern-Simons gauge fields. We shall establish the equivalence between the two descriptions in the following three sections. It will be seen that the language of Chern-Simons gauge theory facilitates and simplifies all the discussions of anyon fluids.

2. Anyons

Under the interchange of two identical particles, the Schrödinger wave function in quantum mechanics acquires a factor of either $+1$ or $-1$, depending on whether the particles are bosons or fermions. In two spatial dimensions there can be other possibility.\textsuperscript{12} The interchange of two particles, say $a$ and $b$, defines two paths $C_1$ and $C_2$ along which the particles $a$ and $b$ are transported to the original locations of $b$ and $a$, respectively. $C_1$ and $C_2$ together form an oriented closed loop. Pick the paths such that none of the other particles are inside the closed loop. Depending on whether the loop is oriented in a clockwise or counter-clockwise direction, the operation defines $P^-(a, b)$ or $P^+(a, b)$. (Fig. 1)

In three dimensional space there can be no distinction between $P^+(a, b)$ and $P^-(a, b)$ ($\equiv P(a, b)$), since $P^+$ can be continuously deformed to $P^-$. Hence $P(a, b)^2 = P_+(a, b)P_-(a, b) = 1$, and $P(a, b) = \pm 1$. In two spatial dimensions, however, one can have

$$P_{\pm}(a, b) \Phi(1, \cdots, q) = -e^{\pm i\theta_s} \Phi(1, \cdots, q) \quad (2.1)$$

where $\Phi(1, \cdots, q)$ is the Schrödinger wave function for a $q$-particle system. The minus sign in (2.1), retained for later convenience, may be absorbed in the definition of the statistical phase $\theta_s$. $\theta_s = 0$ or $\pi$ (mod $2\pi$) corresponds to fermions or bosons, respectively. Otherwise the statistics of the particles is in between. It is said that particles obey fractional statistics. Such particles are generically called anyons. It is easy to see that (2.1) satisfies, for instance, an operational identity (Fig. 2)

$$P_-(a, b)P_+(a, c)P_+(b, c) = P_+(a, c) \quad (2.2)$$

If the wave function is to be well-defined in the limit two of the coordinates $x_a$ and $x_b$ coincide, the identity (2.1) leads to

$$\Phi(1, \cdots, q) \bigg|_{x_a = x_b} = 0 \quad \text{for} \quad -e^{i\theta_s} \neq 1 \quad (2.3)$$
In other words, unless particles are bosons, the wave function must vanish when two coordinates coincide. Pauli’s exclusion principle applies to anyons.

Fig. 1 Interchange of two identical particles. The closed contour formed by $C_1$ and $C_2$ should not encircle any other particles.

Fig. 2 The identity (2.2).

A system of “free” anyons is described by the equation

$$i\frac{\partial}{\partial t}\Phi = H\Phi$$

$$H = \sum_{a=1}^{q} -\frac{\hbar^2}{2m} \nabla_a^2$$

(2.4)

where $\nabla_a = \partial / \partial x_a$. The equation (2.4) must be solved with the boundary condition (2.1). This system defines a neutral anyon fluid. We shall see below that except for the cases of bosons and fermions a neutral anyon fluid is not “free”. The energy of a many-anyon system is not the sum of single particle energies. An interaction is hidden in the nontrivial boundary condition (2.1).

It is most instructive to go over to a new gauge. We define

$$\Phi^f = \Omega_{\text{sing}} \Phi, \quad \Omega_{\text{sing}} = e^{i\omega}$$

$$\omega(x_1, \cdots, x_q) = \sum_{a<b} \frac{\theta_s}{\pi} \tan^{-1} \frac{x_{a2} - x_{b2}}{x_{a1} - x_{b1}}.$$ 

(2.5)
In terms of the new wave function, the equation and boundary condition become

\[
i \frac{\partial}{\partial t} \Phi^f = \sum_{a=1}^{q} -\frac{\hbar^2}{2m} \left( \nabla_a - i \mathbf{B}^{(a)}(\{x_b\}) \right)^2 \Phi^f
\]

\[
\mathbf{B}^{(a)}(\{x_b\}) = \nabla^2 \omega = -\frac{\theta_s}{\pi} \sum_{b \neq a} \epsilon_{jk} \frac{x_a - x_b}{(x_a - x_b)^2} - \theta_s \pi \sum_{b \neq a} \epsilon_{jk} \partial_k \ln |x_a - x_b|
\]  

\[P_{\pm}(a, b)\Phi^f(1, \cdots, q) = -\Phi^f(1, \cdots, q) \tag{2.6}\]

In the new representation the particles behave as fermions, but with a specific long range interaction described by \( \mathbf{B} \). \( \Phi^f \) is the Schrödinger wave function in the fermion representation.

The \( \mathbf{B}^{(a)}(\{x_b\}) \) term in (2.6) gives rise to two- and three-body interactions. It involves a velocity dependent potential. The interaction, which account for the anyon nature of the particles, can be interpreted as an Aharonov-Bohm effect or as a Chern-Simons gauge interaction, as we shall show in the subsequent sections. The gauge transformation potential \( \Omega_{\text{sing}} \) which connects the original \( \Phi \) and new \( \Phi^f \) is singular at \( x_a = x_b \). One advantage of working in the new gauge is that the wave function \( \Phi^f \) is a regular, single-valued function of the coordinates \( \{x_a\} \).

3. Aharonov-Bohm Effect

Suppose that there is a solenoid parallel to the \( x_3 \)-axis at \( x_1 = x_2 = 0 \) with a total magnetic flux \( \mu = \int dx_1 dx_2 B_3 \). Outside the solenoid there results a vector potential

\[
A_\phi = \frac{\mu}{2\pi r} , \quad A_r = 0 , \quad A^3 = 0 \tag{3.1}
\]

in cylindrical coordinates \( (x_1 = r \cos \phi, x_2 = r \sin \phi) \) or

\[
A^k = -\epsilon^{kl} \frac{\mu}{2\pi r^2} = \frac{\mu}{2\pi} \frac{\partial}{\partial x_k} \phi \quad (k = 1, 2). \tag{3.2}
\]

As far as \( E = B = 0 \), the motion of electrons outside the solenoid is not affected by the presence of the flux \( \mu \neq 0 \) in classical theory.

In quantum theory a non-vanishing gauge potential \( A_\mu \), which locally generates vanishing field strengths, but is not globally a pure gauge, affects the motion of electrons. Let us focus on the two-dimensional motion of electrons in the \( x_1-x_2 \) plane, supposing that a momentum in the \( x_3 \)-direction is zero. The Schrödinger equation outside the solenoid \( (r \geq R) \) is

\[
-\frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} - i \frac{e \mu}{2\pi \hbar c} \right)^2 \right\} \Phi_0 = E \Phi_0 . \tag{3.3}
\]
The general solution is
\[ \Phi_0 = \sum_{l=-\infty}^{\infty} e^{il\phi} \left\{ a_l J_{l+\alpha}(kr) + b_l J_{\alpha-l}(kr) \right\} \] (3.4)

where \( \alpha = \frac{e\mu}{2\pi\hbar c}, \) \( k^2 = 2mE/\hbar^2, \) and \( J_{\nu} \) is a Bessel function of fractional order \( \nu. \) The wave function is supposed to vanish at the boundary of the solenoid. The wave function and energy eigenvalue depend on \( \alpha, \) or on the magnetic flux \( \mu \) in a periodic fashion. It is called the Aharonov-Bohm effect.\(^{59}\)

A basic assumption in deriving (3.4) is that the wave function \( \Phi_0 \) is a single-valued function of \( r \) and \( \phi. \) Let us define a new wave function by \( \Phi_0(r, \phi) = e^{i\alpha\phi} \Phi_1(r, \phi). \) Then
\[ -\frac{\hbar^2}{2m} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right\} \Phi_1 = E \Phi_1 \] (3.5)

\[ \Phi_1(r, \phi + 2\pi) = e^{-i(e\mu/\hbar c)}\Phi_1(r, \phi) . \]

In this new gauge the electron wave function satisfies a free equation, but obey a non-trivial boundary condition upon making a trip around the solenoid. In other words, the Aharonov-Bohm effect is traded for the multi-valuedness of the wave function, the property anyons share.

Indeed, the analogy is exact. Now we imagine that particles (in two spatial dimensions) have both charge \( e \) and flux \( \mu, \) and that there exist only charge-flux interactions, but no charge-charge or flux-flux interactions. Each particle, say \( a, \) creates a vector potential
\[ A^k(x) = -e^{k j} \frac{\mu}{2\pi} \frac{x_j - x_{a j}}{|x - x_a|^2} \]

which is felt by other particles by the minimal coupling. Hence the Schrödinger equation is given by
\[ \sum_{a=1}^{n} -\frac{\hbar^2}{2m} \left( \nabla_a - i \frac{e}{\hbar c} A_a \right)^2 \Phi = E \Phi \]

\[ \frac{e}{\hbar c} A^i_a = -\frac{e\mu}{2\pi\hbar c} \sum_{b \neq a} e^{jk} \frac{(x_a - x_b)_k}{|x_a - x_b|^2} . \] (3.6)

This equation is exactly the same as (2.6) upon identifying
\[ \theta_s = \frac{e\mu}{2\hbar c} . \] (3.7)

It is recognized that the anyon interaction is nothing but an Aharonov-Bohm effect.\(^{60-61}\) Since only the product of \( e \) and \( \mu \) is relevant, one can phrase
\[ \text{anyon} = \begin{cases} \text{charge} : & 1 \\ \text{flux} : & 2\hbar c\theta_s \end{cases} . \] (3.8)
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It’s not exactly a Maxwell interaction, however, since there is no charge-charge interaction. We shall see in the next section that the Chern-Simons gauge theory precisely describes the anyon interaction.

We would like to note that Aharonov-Bohm effects have wider applications. Let us consider the motion of a particle in an arbitrary multiply-connected manifold, \( M \). We can imagine that the space itself has nontrivial topology like \( T^2 \times R^1 \) etc., or that the three-dimensional Euclidean space is obstructed by the presence of closed strings. Further we suppose that field strengths, or more specifically magnetic fields, \( F_{jk} = -\partial_j A^k + \partial_k A^j \), identically vanish in \( M \). Schrödinger equation is given by

\[
-\frac{\hbar^2}{2m} \left( \nabla - i \frac{e}{\hbar c} A \right)^2 \Phi_0 = E \Phi_0 \quad \text{in} \ M \quad (3.9)
\]

where \( \partial_j A^k - \partial_k A^j = 0 \). In general \( A_j \) is not gauge equivalent to \( A_j = 0 \) in a multiply-connected space.

We define a new wave function by

\[
\Phi_1(x; C) = \exp \left\{ -i \frac{e}{\hbar c} \int_{C(x_0, x)} dy \cdot A(y) \right\} \Phi_0(x) \quad (3.10)
\]

where \( C(x_0, x) \) starts at \( x_0 \) and ends at \( x \). For two paths, \( C_1(x_0, x) \) and \( C_2(x_0, x) \), continuously deformable to each other, \( \Phi_1 \) assumes the same value, since \( F_{jk} = 0 \);

\[
\Phi_1(x; C_1) = \Phi_1(x; C_2) \quad \text{if} \quad C_1(x_0, x) \sim C_2(x_0, x) \quad . \quad (3.11)
\]

Hence derivatives of \( \Phi_1 \) with respect to \( x \) is well defined.

\( \Phi_1 \) satisfies a free equation

\[
-\frac{\hbar^2}{2m} \nabla^2 \Phi_1 = E \Phi_1 \quad , \quad (3.12)
\]

but is not single-valued. Let \( \Gamma(x) \) denote a transport of \( x \) along a closed path \( \Gamma \). Then

\[
\Phi_1[\Gamma(x)] = W(\Gamma) \Phi_1(x) \quad , \quad (3.13)
\]

If \( \Gamma \) can be continuously shrunk to a point, then \( W = 1 \). \( W(\Gamma) \) depends on only homology of \( \Gamma \) with respect to \( M \). It is a non-integrable phase factor, often called a Wilson line integral in the particle physics literature. We have seen that the general Aharonov-Bohm problem is traded for a free system with nontrivial boundary conditions.

4. Chern-Simons Gauge Theory

Consider a quantum field theory described by a Lagrangian

\[
\mathcal{L}_0 = -\frac{N}{4\pi} \varepsilon^\mu\nu\rho a_\mu \partial_\nu a_\rho + i \psi^\dagger D_0 \psi - \frac{1}{2m} |D_k \psi|^2 \quad , \quad (4.1)
\]

\[
D_0 = \partial_0 + ia_0 \quad , \quad D_k = \partial_k - ia^k \quad .
\]
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\( a_\mu (a_0 = a^0, a_k = -a^k) \) is a gauge field whose motion is described by the Chern-Simons term \( \propto \varepsilon^{\mu \nu \rho} a_\mu \partial_\nu a_\rho \). (Refs. 63–70) \( \psi \) is a non-relativistic matter field, obeying either boson or fermion statistics. \( N \) determines the magnitude of the gauge coupling. A large \( |N| \) corresponds to a weak coupling, as can be seen by rescaling \( a_\mu \). We show that the system defined by (4.1) is equivalent to a neutral anyon fluid described by (2.6), and therefore by (2.4).

Euler equations derived from (4.1) are

\[
- \frac{N}{4\pi} \varepsilon^{\mu \nu \rho} f_{\nu \rho} = j^\mu \\
i \partial_0 \psi = \left\{ -\frac{1}{2m} D_k^2 + a_0 \right\} \psi
\]

where \( f_{\mu \nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \) and

\[
j^0 = \psi \dagger \psi \\
j^k = -\frac{i}{2m} \left( \psi \dagger D_k \psi - (D_k \psi) \dagger \psi \right)
\]

The strengths \( f_{\mu \nu} \) of the Chern-Simons gauge fields are determined by the current. There is no physical degree of freedom for the gauge field. Hence the Chern-Simons field can be eliminated in favor of the matter field.

It is convenient to take the radiation gauge \( \text{div} \mathbf{a} = 0 \). The Chern-Simons field equation in (4.2) becomes

\[
\frac{N}{2\pi} \Delta a^k = -\varepsilon^{kl} \partial_l j^0 \\
\frac{N}{2\pi} \Delta a_0 = \partial_1 j^2 - \partial_2 j^1
\]

With an appropriate boundary condition, the equations can be solved to express \( a_\mu \) in terms of \( j^\mu \).

In the case of a plane \( (R^2) \) the boundary condition at infinity is subtle. A safe and rigorous derivation is obtained on a torus \( (T^2) \). (Refs. 71–85) We quote the result on a torus, taking the infinite volume limit. The argument presented here is very close to those in ref. 70 and in ref. 85.

\[
a_0(x) = \int dy \ h_k(x-y) j^k(y) \\
a^i(x) = \tilde{a}^i(x) + \hat{a}^i(x) \\
\tilde{a}^i(x) = -\pi n_e \frac{e^{jk} x_k}{N} \left( \partial_1 \tilde{a}^2 - \partial_2 \tilde{a}^1 \right) = n_e \\
\hat{a}^i(x) = \int dy \ h_j(x-y) (j^0(y) - n_e)
\]

Here \( n_e \) is the average matter density, which generates \( \tilde{a}^i(x) \). \( h_j(x) \) is related to the two dimensional Green’s function \( G(x) = (2\pi)^{-1} \ln r \) by

\[
h_j(x) = -\frac{2\pi}{N} e^{jk} \partial_k G(x) = -\frac{e^{jk} x_k}{N r^2}
\]
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If a finite number of particles on an infinitely large plane \((R^2)\) are considered, one can set \(n_e = 0\) in the above formulas. In applying to the superconductivity it is convenient to deal with a system with a finite density on \(R^2\).

\(j^k(y)\) in the expression for \(a^0(x)\) in (4.5) contains \(a^k(y)\) which is expressed in terms of \(\psi\). Hence the Chern-Simons gauge fields are completely expressed in terms of the matter field. The substitution of (4.5) into (4.1) gives a Lagrangian which involves only \(\psi\) and \(\psi^\dagger\):

\[
\mathcal{L}_1 = i\psi^\dagger \dot{\psi} - \mathcal{H}_1
\]

\[
\mathcal{H}_1 = \frac{1}{2m}(D_k\psi)^\dagger(D_k\psi)
\]

(4.7)

where \(a^k(x)\) in \(D_k = \partial_k - ia^k\) is given by (4.5). As it stands, the resultant Hamiltonian \(H_1 = \int dx \mathcal{H}_1\) involves four- and six-fermi interactions. There arises an ambiguity in ordering operators. It can be shown that the system defined by the Hamiltonian with the ordering adopted in (4.7) is equivalent to the system described by the Schrödinger equation (2.6).

The Hamiltonian is not completely normal-ordered. It is given by

\[
H_1 = \frac{1}{2m} \int dx \ (\partial_k \psi^\dagger + i\psi^\dagger a^k)(\partial_k \psi - ia^k \psi)
\]

(4.8)

where

\[
H_1^{(1)} = \frac{1}{2m} \int dx \ (\bar{D}_k \psi)^\dagger(\bar{D}_k \psi)
\]

\[
H_1^{(2)} = \frac{i}{2m} \int dx dy \ h_k(x - y) \left\{ \psi^\dagger(x)(\psi^\dagger(y) - n_e)\bar{D}_k \psi(x) - (\bar{D}_k \psi)^\dagger(x)(\psi^\dagger(y) - n_e)\psi(x) \right\}
\]

\[
H_1^{(3)} = \frac{1}{2m} \int dx dy dz \ h_k(x - y)h_k(x - z) \times \psi^\dagger(x)(\psi^\dagger(y) - n_e)\psi^\dagger(z) - n_e)\psi(x)
\]

(4.9)

and \(\bar{D}_k \psi(x) = (\partial_k - ia^k(x))\psi(x)\). Upon making use of \(\int dy \ h_k(x - y) = 0\), one can write \(H_1^{(3)}\) as

\[
H_1^{(3)} = \frac{1}{2m} \int dx dy dz \ h_k(x - y)h_k(x - z)\psi^\dagger(x)(\psi^\dagger(y)\psi^\dagger(z)\psi(z)\psi(y)\psi(x)
\]

\[
+ \frac{1}{2m} \int dx dy \ [h_k(x - y)]^2\psi^\dagger(x)\psi^\dagger(y)\psi(y)\psi(x)
\]

(4.10)

The equation derived from \(H_1\) is

\[
i\dot{\psi}(x) = [\psi(x), H_1] = \mathcal{K}_0 \psi(x)
\]

\[
\mathcal{K}_0 = -\frac{1}{2m} D_k^2 + a_0(x) + g(x)
\]

(4.11)
where $a^k(x)$ in $D_k$ and $a_0(x)$ are given by (4.5) and
\[ g(x) = \frac{1}{2m} \int dy \left[ h_k(x - y)^2 \right] \psi^\dagger \psi(y) = \frac{1}{2mN^2} \int dy \frac{1}{(x - y)^2} \psi^\dagger \psi(y) \].

Eq. (4.11) differs from the classical Euler equation (4.2) by the $g(x)$ term. The additional term is important to establish the equivalence between the anyon quantum mechanics and Chern-Simons gauge theory.

Currents are given by
\[ J^0(x) = \psi^\dagger(x) \]
\[ J^k(x) = -\frac{i}{2m} \left\{ \psi^\dagger(x) \frac{\delta H_1}{\delta \nabla_k \psi}(x) - \frac{\delta H_1}{\delta \psi}(x) \right\} \]
\[ = -\frac{i}{2m} \left\{ \psi^\dagger(x) D_k \psi - (D_k \psi)^\dagger \psi \right\} \]
\[ = -\frac{i}{2m} \left\{ \psi^\dagger(x) \tilde{D}_k \psi(x) - (\tilde{D}_k \psi)^\dagger(x) \psi(x) \right\} \]
\[ - \frac{1}{m} \int dy \ h_k(x - y) \ \psi^\dagger(x) \psi^\dagger(y) \psi(y) \psi(x) \].

They are conserved: $\partial_0 J^0 + \nabla_k J^k = 0$.

The Schrödinger wave function $\Phi^f$ in quantum mechanics for a $q$-particle system is related to the field operator $\psi$ in (4.7) or (4.8) by
\[ \Phi^f(1, \ldots, q) = \langle 0 | \psi(1) \cdots \psi(q) | \Psi_q \rangle \].

Here $|0\rangle$ and $|\Psi_q\rangle$ are vacuum and $q$-particle states, respectively, and $\psi(a) = \psi(x_a)$ where $x_a = (t, x_a)$. For a system with a finite number of particles on a plane ($R^2$), one can put $n_c = 0$ and $\hat{a}^k(x) = 0$ in the above formulas. To obtain the Schrödinger equation, we differentiate $\Phi^f$ with respect to $t$ and make use of (4.11):
\[ \frac{\partial}{\partial t} \Phi^f(1, \ldots, q) = \sum_{a=1}^{q} \langle 0 | \psi(1) \cdots i \dot{\psi}(a) \cdots \psi(q) | \Psi_q \rangle \]
\[ = \sum_{a=1}^{q} \langle 0 | \psi(1) \cdots \mathcal{K} \psi(a) \cdots \psi(q) | \Psi_q \rangle \].

The following definitions and identities facilitate further manipulations. First we define
\[ \mathcal{K}_1(a, b) = \frac{1}{m} h_k(x_a - x_b) \{ \hat{a}^k(x_a) - \hat{a}^k(x_b) \} + 2g(x_a, x_b) \]
\[ V_2(a, b) = \frac{i}{m} (\nabla_k^a - \nabla_k^b) h_k(x_a - x_b) + \frac{1}{m} h_k(x_a - x_b)^2 \]
\[ V_3(a, b, c) = \frac{1}{m} \{ h_k(x_a - x_b) h_k(x_a - x_c) \]
\[ + h_k(x_b - x_a) h_k(x_b - x_c) + h_k(x_c - x_a) h_k(x_c - x_b) \} \].
Here
\[ g(x, y) = \frac{1}{2m} \int d\mathbf{z} \, h_k(x - \mathbf{z}) h_k(y - \mathbf{z}) \, \psi^\dagger(z) \psi(z) \] . (4.17)

\( K_1(a, b) \) is an operator, whereas \( V_2(a, b) \) and \( V_3(a, b, c) \) are c-number functions. These operators with \( K_0(a) = K_0(x_a) \) satisfy
\[ \psi(b) K_0(a) = \{ K_0(a) + K_1(a, b) + V_2(a, b) \} \psi(b) \]
\[ \psi(c) K_1(a, b) = \{ K_1(a, b) + V_3(a, b, c) \} \psi(c) \] (4.18)

and
\[ \langle 0 | K_0(a) \cdots = -\frac{1}{2m} (\nabla_k^a)^2 \, \langle 0 | \cdots \]
\[ \langle 0 | K_1(a, b) \cdots = 0 \] . (4.19)

Applications of (4.18) and (4.19) lead to
\[ \langle 0 | \psi(1) \cdots K_0 \psi(a) \cdots \psi(q) | \Psi_q \rangle \]
\[ = \langle 0 | \left\{ K_0(a) + \sum_{b=1}^{a-1} (K_1(a, b) + V_2(a, b)) + \sum_{b=2}^{a-1} \sum_{c=1}^{b-1} V_3(a, b, c) \right\} \times \psi(1) \cdots \psi(q) | \Psi_q \rangle \]
\[ = \left\{ -\frac{1}{2m} (\nabla_k^a)^2 + \sum_{b=1}^{a-1} V_2(a, b) + \sum_{b=2}^{a-1} \sum_{c=1}^{b-1} V_3(a, b, c) \right\} \Phi^f \] (4.20)

Therefore the Schrödinger equation for \( \Phi^f \) is given by
\[ i \frac{\partial}{\partial t} \Phi^f(1, \cdots, q) = H^{(q)} \Phi^f(1, \cdots, q) \] (4.21)

where
\[ H^{(q)} = -\frac{1}{2m} \sum_{a=1}^{q} (\nabla_k^a)^2 + \sum_{a=2}^{q} \sum_{b=1}^{a-1} V_2(a, b) + \sum_{a=3}^{q} \sum_{b=2}^{a-1} \sum_{c=1}^{b-1} V_3(a, b, c) \]
\[ = -\frac{1}{2m} \sum_{a=1}^{q} \left\{ \nabla_k^a - i \sum_{b \neq a} h_k(x_a - x_b) \right\}^2 \] . (4.22)

This is exactly Eq. (2.6), provided that
\[ \theta_\alpha = \frac{\pi}{N} \] (4.23)

and that \( \psi(x) \) satisfies anti-commutation relations. This establishes the equivalence of the anyon quantum mechanics and Chern-Simons gauge theory. The fermion representation is convenient to incorporate the Pauli principle (2.3) for anyons.
5. Charged anyon fluid

The anyon fluid described in the previous section is neutral. Anyons may be charged, interacting with each other electromagnetically. In the application to superconductivity, one needs to consider a charged anyon fluid.

We have in mind material which has a layered structure as in newly discovered high $T_c$ superconductors. The motion of electrons are mostly confined in two-dimensional layers. The probability of the hopping of electrons from one layer to adjacent layers is very small. In many high $T_c$ superconductors the resistivity of electrons in the direction perpendicular to CuO planes above $T_c$ is $10^2$ to $10^5$ times bigger than the in-plane resistivity. To the first approximation we may neglect the hopping interaction.\footnote{86}

We shall adopt the “holon” picture originally advocated by Anderson.\footnote{87−94} In this picture collective modes created by electron holes are spinless and charged. They are called holons, and are supposed to obey half-fermion statistics ($\theta_{\text{statistics}} = \pm \frac{1}{2}\pi$). The matter field denoted by $\psi(x)$ corresponds to holon excitations. In our language $\psi(x)$ satisfies anti-commutation relations, interacting through Chern-Simons gauge fields with the coefficient $N = \pm 2$ and through Maxwell fields with charge $e$.

The electromagnetic interaction is not two-dimensional, however. Certainly there is a Coulomb interactions among electrons in two different layers. Electromagnetic waves can propagate in three dimensional space.

In this article we consider two extreme limits. In one limit one can imagine an ultra-thin film which has a couple of, or just one, superconducting layers. Further we idealize the situation such that the only interaction among holons, other than the Chern-Simons or fractional statistics interaction, is the Coulomb interaction with a potential $1/r$. We call it the ultra-thin film approximation.

In the other limit we suppose material of an infinitely many layers (in the $x_1$-$x_2$ plane) which are evenly separated with a distance $d$. Further we suppose that (1) electromagnetic fields $E_3$, $B_1$, and $B_2$ identically vanish, and (2) all fields $E_1$, $E_2$, and $B_3$ are uniform in the $x_3$ direction. Consequently all physical quantities such as the expectation values of currents $\langle J^\mu \rangle$ are independent of $x_3$. One can mimic it by considering a system in the idealized two-dimensional space, suppressing the third coordinate $x_3$. It is called the two-dimensional approximation. Fluctuations of the $E_1$, $E_2$, and $B_3$ fields are retained, in addition to the Coulomb interaction.

We note that the two-dimensional approximation incorporates three-dimensional interactions in a specific way. Both the three-dimensional Coulomb interaction among electrons in distinct layers and the electron hopping between adjacent layers affect the three-dimensional motion of electrons or holons. When the electron or holon field is expanded in Fourier series in the $x_3$ direction with a momentum $k_3$, the two-dimensional approximation amounts to retaining only the $k_3 = 0$ component.

Real high $T_c$ superconductors lie somewhere between the two approximation. It is necessary and important to have more thorough examinations of effects of the three-dimensional motion. No such analysis is available at the moment.
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In the ultra-thin film approximation the Lagrangian is given by

\[
\mathcal{L}_{\text{film}}[\psi, \psi^\dagger, a_\mu; \phi, A_\mu^{\text{ext}}] = -\frac{N}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + i\psi^\dagger D_0\psi - \frac{1}{2m}|D_\mu\psi|^2 + \mathcal{L}^{3D}_{\text{Coulomb}},
\]

\[
\mathcal{L}^{3D}_{\text{Coulomb}} = -\frac{1}{2}\phi\nabla^2\phi + e\phi(\psi^\dagger\psi - n_e)
\]

\[
D_0 = \partial_0 + i(a_0 + eA_0^{\text{ext}}), \quad D_k = \partial_k - i(a_k + eA_k^{\text{ext}}).
\]

Here \(\phi(x)\) is an auxiliary field generating a \(1/r\) Coulomb potential. \(A_\mu^{\text{ext}}\) is an external electromagnetic field. \(\mathcal{L}_{\text{film}}\) is bilinear in \(\psi\) and \(\psi^\dagger\). After eliminating \(a_\mu\) and \(\phi\), one obtains a Hamiltonian solely in terms of \(\psi\) and \(\psi^\dagger\).

\[
H_{\text{film}} = \int \! d\mathbf{x} \left\{ \frac{1}{2m}(D_\mu\psi)(D^\mu\psi) + eA_0^{\text{ext}}\psi^\dagger\psi \right\} + \mathcal{H}^{3D}_{\text{Coulomb}} - \frac{1}{2}\int \! d\mathbf{x} d\mathbf{y} \psi^\dagger(x)\psi^\dagger(y) \frac{e^2}{|\mathbf{x} - \mathbf{y}|} \psi(y)\psi(x),
\]

and \(a_k\) in \(D_k\) is given by (4.5).

Equations of motion derived from \(\mathcal{L}_{\text{film}}\) are

\[
-\frac{N}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho} = j^\mu
\]

\[
\nabla^2\phi = e(j^0 - n_e)
\]

\[
i\partial_0 \psi = \left\{ -\frac{1}{2m}D_0^2 + a_0 + e(\phi + A_0^{\text{ext}}) \right\} \psi
\]

where \(j^\mu\) is given by (4.3) with the covariant derivatives in (5.1).

In the two-dimensional approximation the Lagrangian is given, instead, by

\[
\mathcal{L}_{2D}[\psi, \psi^\dagger, a_\mu, A_\mu]
\]

\[
= -\frac{1}{4}F_{\mu\nu}^2 - \frac{N}{4\pi}\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + en_e A_0 + i\psi^\dagger D_0\psi - \frac{1}{2m}|D_\mu\psi|^2,
\]

\[
D_0 = \partial_0 + i(a_0 + eA_0), \quad D_k = \partial_k - i(a_k + eA_k).
\]

Here the electromagnetic field \(A_\mu\) contains both external and dynamical fields: \(A_\mu = A_\mu^{\text{ext}} + A_\mu^{\text{dyn}}\). The corresponding Hamiltonian obtained by eliminating \(a_\mu\) is

\[
H_{2D} = \int \! d\mathbf{x} \left\{ \frac{1}{2m}(D_\mu\psi)(D^\mu\psi) + eA_0\psi^\dagger\psi + \frac{1}{2}(E_k^2 + B^2) \right\}
\]

where \(E_k = F_{0k}\) and \(B = -F_{12}\).

Equations of motion derived from (5.4) are

\[
-\frac{N}{4\pi}\varepsilon^{\mu\nu\rho}f_{\nu\rho} = j^\mu
\]

\[
\partial_\mu F^{\mu\nu} = e(j^\nu - en_e \delta^{0\nu})
\]

\[
i\partial_0 \psi = \left\{ -\frac{1}{2m}D_0^2 + (a_0 + eA_0) \right\} \psi
\]
Again \( j^\mu \) is given by (4.3) with \( D_k \) in (5.4).

The Lagrangian forms \( \mathcal{L}_0 \) in (4.1) (for neutral fluids) and \( \mathcal{L}_{\text{film}} / \mathcal{L}_{2D} \) (for charged fluids) are convenient to develop a perturbation scheme. They are bilinear in \( \psi \) or \( \psi^\dagger \), and in the charged case the gauge invariance can be easily implemented in the perturbation scheme. On the other hand, the Hamiltonian forms \( H_1 \) in (4.8) and \( H_{\text{film}} / H_{2D} \) has an advantage that they involve only physical fields, and are particularly suited to the evaluation of physical quantities beyond the perturbation scheme. We shall make use of both in subsequent sections.

As we shall see, there is a subtle but important difference between neutral and charged anyon fluids. It seems that charged fluids (with a neutralizing background charge) are more stable than neutral fluids.

6. Mean field ground state

We consider an anyon system with a finite density \( n_e \neq 0 \) on a plane. First we ask what would be the average Chern-Simons fields in the ground state. In the equation (4.2), (5.3) or (5.6), we replace the operator \( j^\mu \) by its expectation value \( \langle j^\mu \rangle \) in the ground state. We expect that \( \langle j^0 \rangle = n_e \) and \( \langle j^k \rangle = 0 \) and all Maxwell fields vanish. In both neural and charged fluids we have

\[
b = \frac{2\pi n_e}{N} \equiv b(0)
\]

and \( f_{0k} = 0 \). In other words, particles, or holons in high \( T_c \) superconductors, move in a uniform Chern-Simons magnetic field on the average.

In the mean field approximation all gauge fields in (4.2), (5.3), or (5.6) are replaced by the average fields. The equation for the field operator \( \psi \) is, in all cases,

\[
i\partial_0 \psi = -\frac{1}{2m} \vec{D}_k^2 \psi
\]

where the average vector potential is

\[
\vec{a}_k = \begin{cases} 
-\epsilon^{kj} \frac{n_e x_j}{N} = -\epsilon(N) \epsilon^{kj} \frac{x_j}{2l^2} & \text{in the symmetric gauge;} \\
-\delta^{kl} \frac{2\pi n_e x_2}{N} = -\epsilon(N) \delta^{kl} \frac{x_2}{l^2} & \text{in the Landau gauge.}
\end{cases}
\]

Here \( \epsilon(N) = +1 \) \((-1)\) for \( N > 0 \) \(< 0 \), and the magnetic length, \( l \), is defined by

\[
l^2 = \frac{|N|}{2\pi n_e}
\]

The corresponding one-particle Schrödinger equation

\[
-\frac{1}{2m} \vec{D}_k^2 u_\alpha(x) = \epsilon_\alpha u_\alpha(x)
\]
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is easily solved in either gauge. The energy spectrum is characterized by Landau levels:

\[ \alpha = (n, p) \quad n = 0, 1, 2, \cdots \]
\[ \epsilon_\alpha = \left( n + \frac{1}{2} \right) \frac{1}{ml^2} \equiv \epsilon_n \quad (6.5) \]

The first integer index \( n \) labels Landau levels. The second index \( p \) is either a momentum in the \( x_1 \)-direction in the Landau gauge, or an orbital angular momentum in the symmetric gauge.

In the symmetric gauge

\[ u_{sp}(r, \phi) = \left[ \frac{s!}{(s + |p|)!} \frac{1}{2\pi l^2} \right]^{1/2} e^{-i\epsilon(N)|p|\phi} \frac{u(|p|)}{e^{-w/2} L_s^{|p|}(w)} \quad w = \frac{r^2}{2l^2} \]
\[ \epsilon_{sp} = \left( s + \frac{1}{2} + \theta(-p)|p| \right) \frac{1}{ml^2} \quad (q = 0, 1, 2, \cdots \; \text{and} \; p \in \mathbb{Z}). \quad (6.6) \]

The Landau level index is \( n = s + \theta(-p)|p| \). Here \( L_s^\alpha(w) \) is the associated Laguerre polynomial

\[ L_s^\alpha(w) = \frac{1}{s!} w^{-\alpha} e^w \frac{d^s}{dw^s} (w^{s+\alpha}e^{-w}) \quad (6.7) \]

In the Landau gauge we impose a periodic boundary condition in the \( x_1 \)-direction: \( u_\alpha(x_1 + L_1, x_2) = u_\alpha(x_1, x_2) \). Then

\[ u_{np}^{\text{Landau}}(x) = \frac{1}{\sqrt{L_1}} e^{-ikx_1} v_n[(x_2 - \bar{x}_2)/l] \quad k = \frac{2\pi p}{L_1} \quad \bar{x}_2 = \epsilon(N) kl^2 \]
\[ \epsilon_{np} = \left( n + \frac{1}{2} \right) \frac{1}{ml^2} \quad (n = 0, 1, 2, \cdots \; \text{and} \; p \in \mathbb{Z}). \quad (6.8) \]

Here \( v_n(x) \) is related to the Hermite polynomial:

\[ v_n(x) = \frac{(-1)^n}{2^n/s! \pi^{1/2}(nl)^{1/2}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2} \]
\[ \int_{-\infty}^{\infty} dx \; v_n(x)v_m(x) = \delta_{nm} \quad (6.9) \]

\( u_{np}^{\text{Landau}}(x) \) is a plane wave in the \( x_1 \)-direction, but is localized around \( \bar{x}_2 \) in the \( x_2 \)-direction.

In a box \( 0 \leq x_2 \leq L_2 \), \( \bar{x}_2 \) must satisfy \( 0 \leq \bar{x}_2 \leq L_2 \), or equivalently \( 0 \leq p \leq L_1 L_2/2\pi l^2 \). Hence the number of states per area for each Landau level, \( n_L \), is given by

\[ n_L = \frac{1}{2\pi l^2} \quad (6.10) \]

Combining (6.10) with (6.3), one finds that the filling factor, \( \nu \), is given by

\[ \nu \equiv \frac{n_e}{n_L} = |N| \quad (6.11) \]
In other words, for an integer $N$, Landau levels are completely filled at least in the mean field approximation. It has to be stressed that this property holds irrespective of the density $n_e$. If the $\psi$ field has spin $\frac{1}{2}$ as electrons, the filling factor is given by $\nu = \frac{1}{2}|N|$ so that the complete filling of Landau levels holds for an even integer $N$, provided that the magnetic moment interaction is sufficiently small as in the usual cases. We are mostly interested in the case $|N| = 2$, corresponding to semions or half-fermions. There arises no qualitative change in physics properties, and we shall restrict ourselves in this article to spinless $\psi$.

We expand $\psi$ in terms of $\{u_\alpha(x)\}$:

$$\psi(x) = \sum_\alpha c_\alpha u_\alpha(x), \quad \{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}.$$  \hspace{1cm} (6.12)

The mean field ground state is given by, for $|N| = 2$,

$$|\Psi_{\text{mean}}\rangle = \prod_\alpha c^\dagger_\alpha |0\rangle$$  \hspace{1cm} (6.13)

$$G = \{\alpha = (n, p) ; \ n = 0, 1\}.$$  \hspace{1cm} (6.13)

The corresponding mean field energy is

$$E_{\text{mean}} = \langle \Psi_{\text{mean}} | H_1^{(1)} | \Psi_{\text{mean}} \rangle = \sum_\alpha \epsilon_\alpha.$$  \hspace{1cm} (6.14)

Here $H_1^{(1)}$ is given in (4.9). Again putting the system in a box in the Landau gauge, one finds the energy density to be

$$\varepsilon_{\text{mean}} = \frac{1}{L_1 L_2} \sum_{n=0,1} \sum_{p=1}^{L_1 L_2/2\pi^2} \left(n + \frac{1}{2}\right) \frac{1}{ml^2} = \frac{\pi n_e^2}{m}.$$  \hspace{1cm} (6.15)

We recall that the energy density of a free spinless fermion fluid is exactly $\pi n_e^2/m$. It is straightforward to check

$$\langle j^0(x) \rangle_{\text{mean}} = \langle \Psi_{\text{mean}} | \psi^\dagger \psi | \Psi_{\text{mean}} \rangle = \sum_{\alpha \in G} u_\alpha(x)^\dagger u_\alpha(x) = n_e$$

$$= n_e$$

$$\langle j^k(x) \rangle_{\text{mean}} = -\frac{i}{2m} \langle \Psi_{\text{mean}} | \{ \psi^\dagger \vec{D}_k \psi - (\vec{D}_k \psi)^\dagger \psi \} | \Psi_{\text{mean}} \rangle = \frac{i}{2m} \sum_{\alpha \in G} \{ u_\alpha^\dagger \vec{D}_k u_\alpha - (\vec{D}_k u_\alpha)^\dagger u_\alpha \}$$  \hspace{1cm} (6.16)

$$= 0$$
where $\overline{D}_k = \partial_k - i\bar{a}_k(x)$.

7. Hartree-Fock ground state

In this section we incorporate many-particle correlations in the Hartree-Fock (HF) approximation. The mean field approximation retains only $H_1^{(1)}$, defined in (4.9), in the total Hamiltonian $H_1$, (4.8). In the Hartree-Fock approximation “diagonal parts” of $H_1^{(2)}$ and $H_1^{(3)}$ in (4.9) are taken into account self-consistently, or equivalently to say, the ground state is determined to satisfy the Hartree-Fock equation.

It was shown by Hanna, Laughlin, and Fetter\footnote{28} that the Hartree-Fock ground state is exactly the same as the mean field ground state $|\Psi_{\text{mean}}\rangle$, (6.13).

$$|\Psi_{\text{HF}}\rangle = |\Psi_{\text{mean}}\rangle = \prod_{\alpha \in G} c_{\alpha}^\dagger |0\rangle .$$

However, it has a different energy.

$$E_{\text{HF}} = \langle H_1 \rangle_{\text{HF}} \equiv \langle \Psi_{\text{HF}} | H_1 | \Psi_{\text{HF}} \rangle \neq E_{\text{mean}} .$$

We first compute $E_{\text{HF}}$. The original computation of Hanna, Laughlin, and Fetter was given in the first quantized theory. We present the evaluation in the second quantized theory.

To facilitate the computations, we define the following sums.

$$f(x, y) = \sum_{\alpha \in G} u_\alpha(x)^* u_\alpha(y) = f(y, x)^*$$

$$f_k(x, y) = \sum_{\alpha \in G} u_\alpha(x)^* i\overline{D}_k u_\alpha(y) = i\overline{D}_k^\dagger f(x, y)$$

We need to evaluate expectation values of various products of $\psi$ and $\psi^\dagger$. We denote $\langle Q \rangle_{\text{HF}} = \langle \Psi_{\text{HF}} | Q | \Psi_{\text{HF}} \rangle$. Then

$$\langle \psi^\dagger(x) \psi(x) \rangle_{\text{HF}} = f(x, x)$$

$$\langle \psi^\dagger(x) \psi^\dagger(y) \psi(y) \psi(x) \rangle_{\text{HF}} = f(x, x) f(y, y) - f(x, y) f(y, x)$$

$$\langle \psi^\dagger(x) \psi^\dagger(y) \psi^\dagger(z) \psi(z) \psi(y) \psi(x) \rangle_{\text{HF}}$$

$$= f(x, x) f(y, y) f(z, z) + f(x, y) f(y, z) f(z, x) + f(x, z) f(y, x) f(z, y)$$

$$- f(x, x) f(y, y) f(z, z) - f(x, y) f(y, y) f(z, x) - f(x, z) f(y, x) f(z, y)$$

Secondly

$$i\langle \psi^\dagger(x) \overline{D}_k \psi(x) - (\overline{D}_k \psi(x))^\dagger \psi(x) \rangle_{\text{HF}} = f_k(x, x) + f_k(x, x)^*$$

$$i\langle \psi^\dagger(x) \psi^\dagger(y) \psi^\dagger(y) \overline{D}_k \psi(x) - (\overline{D}_k \psi(x))^\dagger \psi^\dagger(y) \psi(y) \psi(x) \rangle_{\text{HF}}$$

$$= \{ f_k(x, x) + f_k(x, x)^* \} f(y, y) - \{ f(x, y) f_k(y, x) + f(y, x) f_k(y, x)^* \} .$$
Making use of (4.9), (4.10), (7.4), and (7.5), one finds

\[ \mathcal{E}_{\text{HF}} = \frac{1}{\text{vol}} \langle (H_1^{(1)} + H_1^{(2)} + H_1^{(3)}) \rangle_{\text{HF}} = \mathcal{E}^{(1)} + \mathcal{E}^{(2)} + \mathcal{E}^{(3)} \]  

(7.6)

where \( \text{vol} \) is the volume. \( \mathcal{E}^{(1)} \) is the same as the mean field energy density

\[ \mathcal{E}^{(1)} = \mathcal{E}_{\text{mean}} = \frac{\pi n_e^2}{m} \]  

(7.7)

and

\[ \mathcal{E}^{(2)} = \frac{1}{\text{vol}} \frac{1}{2m} \int dxdy \ h_k(x - y) \left[ \{ f_k(x, x) + f_k(x, x)^* \} \{ f(y, y) - n_e \} - \{ f(x, y) f_k(y, x) + f(y, x) f_k(y, x)^* \} \right] \]

\[ \mathcal{E}^{(3)} = \frac{1}{\text{vol}} \frac{1}{2m} \int dxdydz \ h_k(x - y) h_k(x - z) \times \left[ 2f(x, y)f(y, z)f(z, x) - f(x, x)f(y, z)f(z, y) \right. \]

\[ - \left. 2|f(x, y)|^2 \{ f(z, z) - n_e \} + f(x, x) \{ f(y, y) - n_e \} \{ f(z, z) - n_e \} \right] \]

\[ + \frac{1}{\text{vol}} \frac{1}{2m} \int dxdy \ [h_k(x - y)]^2 \ {f(x, x)f(y, y) - |f(x, y)|^2} \]  

(7.8)

The quantities \( f(x, y) \) and \( f_k(x, y) \) in (7.3) depend on the gauge chosen. The symmetric and Landau gauge defined in (6.2) are related by

\[ \bar{a}_{\text{sym}}^k(x) = \bar{a}^k_{\text{Landau}} - \nabla_k \Lambda(x) \]

\[ \Lambda(x) = \epsilon(N) \frac{x_1 x_2}{2l^2} \]  

(7.9)

If \( u_{np}^{\text{sym}}(x) \) is a solution to the Schrödinger equation in the symmetric gauge, then \( e^{i\Lambda(x)} u_{sq}^{\text{sym}}(x) = u_{sq}^{\text{Landau}}(x) \) is a solution in the Landau gauge with the same energy eigenvalue. It is a linear combination of \( u_{np}^{\text{Landau}}(x) \) in (6.8). With a given Landau level (energy eigenvalue), the sets \( \{ u_{sq}^{\text{Landau}}(x) \} \) and \( \{ u_{np}^{\text{Landau}}(x) \} \) are related by a unitary transformation.

Hence, if the set \( G \) in (6.13) represents completely filled Landau levels as in the case under consideration, then

\[ \sum_{(s, q) \in G^{\text{sym}}} u_{sq}^{\text{Landau}}(x)^* u_{sq}^{\text{Landau}}(y) = \sum_{(n, p) \in G^{\text{Landau}}} u_{np}^{\text{Landau}}(x)^* u_{np}^{\text{Landau}}(y) \]

so that

\[ f(x, y)^{\text{sym}} = e^{i\{\Lambda(x) - \Lambda(y)\}} f(x, y)^{\text{Landau}} \]

\[ f_k(x, y)^{\text{sym}} = e^{i\{\Lambda(x) - \Lambda(y)\}} f_k(x, y)^{\text{Landau}} \]  

(7.10)
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Thanks to the relation (7.10), every term in (7.8) is separately gauge independent. It is easiest to evaluate \( f(x, y) \) in the Landau gauge. Making use of (6.8), one finds

\[
f(x, y)^\text{Landau} = \frac{1}{2\pi l} \sum_{n=0}^{\left| N \right| - 1} \sum_{p=-\infty}^{\infty} \frac{1}{\left| N \right|} \int_{-\infty}^{\infty} dk \ e^{ik(x_1 - y_1)} v_n[(x_2 - \bar{x}_2)/l] v_n[(y_2 - \bar{y}_2)/l] \]

Employing explicit forms of \( v_n \)'s and integrating over \( k \), one obtains

\[
N = \pm 1: \quad f(x, y)^\text{Landau} = n_e \cdot \exp \left\{ -\frac{(x - y)^2}{4l^2} \pm i (x_1 - y_1)(x_2 + y_2) \right\} \equiv f_0^\pm(x, y) \quad (7.12)
\]

\[
N = \pm 2: \quad f(x, y)^\text{Landau} = \left\{ 1 - \frac{(x - y)^2}{4l^2} \right\} \cdot f_0^\pm(x, y)
\]

Note that \( l^2 = |N|/2\pi n_e \). \( f_k(x, y) \) is obtained from (7.3).

\[
N = \pm 1: \quad f_1(x, y)^\text{Landau} = \frac{i}{2l^2} \cdot (x_1 - y_1 \mp i(x_2 - y_2)) \cdot f_0^\pm(x, y)
\]

\[
f_2(x, y)^\text{Landau} = \pm i f_1(x, y)^\text{Landau}
\]

\[
N = \pm 2: \quad f_k(x, y)^\text{Landau} = \left\{ 1 - \frac{(x - y)^2}{4l^2} \right\} \cdot f_k(x, y)^\text{Landau}_{N = \pm 1} \quad (7.13)
\]

\[+ \frac{i}{2l^2} (x_k - y_k) f_0^\pm(x, y)\]

We are ready to evaluate (7.8). We drop the superscripts ‘Landau’ in \( f(x, y) \) and \( f_k(x, y) \). Note that \( f(x, x) = n_e \) and \( f_k(x, x) = 0 \). Furthermore,

\[
h_k(x - y)f_k(x, y) = -\frac{1}{2l^2|N|} f(x, y) \quad . \quad (7.14)
\]

Therefore \( \mathcal{E}^{(2)} \) becomes

\[
\mathcal{E}^{(2)} = -\frac{1}{\text{vol}} \frac{1}{2|N|m l^2} \int dxdy \ |f(x, y)|^2 = \begin{cases} -\frac{\pi n_e^2}{m} \quad \text{for } N = \pm 1 \\ -\frac{\pi n_e^2}{4m} \quad \text{for } N = \pm 2 \end{cases} \quad (7.15)
\]

For \( \mathcal{E}^{(3)} \) we have

\[
\mathcal{E}^{(3)} = \frac{1}{\text{vol}} \frac{1}{2m} \int dxdydz \frac{1}{N^2} \frac{(x - y)(x - z)}{(x - y)^2(x - z)^2} \int dxdy \left[ 2f(x, y)f(y, z)f(z, x) - n_e \cdot |f(y, z)|^2 \right] \]

\[+ \frac{1}{\text{vol}} \frac{1}{2m} \int dxdy \frac{1}{(x - y)^2} \left\{ n_e^2 - |f(x, y)|^2 \right\}
\]

\[\equiv \left[ \mathcal{E}_1^{(3)} + \mathcal{E}_2^{(3)} \right] + \mathcal{E}_3^{(3)} \quad . \quad (7.16)
\]
As will be seen below, \( \mathcal{E}^{(3)}_2 \) and \( \mathcal{E}^{(3)}_3 \) have infra-red divergences, which cancel each other.

In evaluating \( \mathcal{E}^{(3)}_1 \), we rename \( y - x \) and \( z - x \) to be new \( y \) and \( z \), respectively. Then

\[
\mathcal{E}^{(3)}_1 = \frac{n^3}{N^2 m} \int dydz \frac{yz}{y^2 + z^2} g_N(y)g_N(z)g_N(y-z) \\
\times \exp \left\{ - \frac{1}{4l^2} (y^2 + z^2 + (y-z)^2) \pm \frac{i}{2l^2} (y_1 z_2 - y_2 z_1) \right\} \\
g_N(x) = \begin{cases} 
1 & \text{for } N = \pm 1 \\
1 - \frac{x^2}{4l^2} & \text{for } N = \pm 2
\end{cases}
\]  

(7.17)

We introduce polar coordinates by

\[
y_1 + iy_2 = \sqrt{2l} r e^{i\theta} , \quad z_1 + iz_2 = \sqrt{2l} \rho e^{i(\theta + \phi)} \, .
\]

The integrand is independent of \( \theta \), and \( yz/2l^2 = r \rho \cos \phi \) and \( (y_1 z_2 - y_2 z_1)/2l^2 = r \rho \sin \phi \). (7.17) is transformed to

\[
\mathcal{E}^{(3)}_1 = \frac{n^3}{N^2 m} \cdot 2\pi \cdot 2l^2 \int_0^\infty drd\rho \int_0^{2\pi} d\phi \cos \phi \cdot \exp \left\{ - r^2 - \rho^2 + r \rho e^{\pm i\phi} \right\} \\
\times \begin{cases} 
1 & \text{for } N = \pm 1 \\
(1 - \frac{1}{2} r^2)(1 - \frac{1}{2} \rho^2)(1 - \frac{1}{2} r^2 - \frac{1}{2} \rho^2 + r \rho \cos \phi) & \text{for } N = \pm 2
\end{cases}
\]

Integrating over \( \phi \), one finds

\[
\mathcal{E}^{(3)}_1 = \frac{4\pi l^2 n^3}{N^2 m} \int_0^\infty drd\rho \, e^{-r^2-\rho^2} \\
\times \begin{cases} 
\pi r \rho & \text{for } N = \pm 1 \\
\pi r(1 - \frac{1}{4} r^2)(1 - \frac{1}{4} \rho^2)(2 - \frac{1}{2} (r^2 + \rho^2) + \frac{1}{4} r^2 \rho^2) & \text{for } N = \pm 2
\end{cases}
\]  

(7.18)

Similarly, the second term, \( \mathcal{E}^{(3)}_2 \), in (7.16) becomes

\[
\mathcal{E}^{(3)}_2 = -\frac{n^3}{2mN^2} \int dydz \frac{yz}{y^2 + z^2} g_N(y-z)^2 e^{-(y-z)^2/2l^2} \\
= -\frac{n^3}{2mN^2} \int dydu \frac{y(y-u)}{y^2 + (y-u)^2} g_N(u)^2 e^{-u^2/2l^2}
\]  

(7.19)

where we have introduced \( u = y - z \). This time we define

\[
y_1 + iy_2 = \sqrt{2l} r e^{i\theta} , \quad u_1 + iu_2 = \sqrt{2l} \rho e^{i(\theta + \phi)} \, .
\]
Then
\[ E^{(3)}_2 = -\frac{2\pi l^2 n_e^3}{mN^2} \int dr \rho d\phi \frac{\rho}{r} \frac{r^2 - r \rho \cos \phi}{r^2 + \rho^2 - 2r \rho \cos \phi} g_N(u) e^{-r^2}. \] (7.20)

The \( \phi \)-integral gives
\[ \int_0^{2\pi} d\phi \frac{r^2 - r \rho \cos \phi}{r^2 + \rho^2 - 2r \rho \cos \phi} = 2\pi \theta(r - \rho). \]
The rest of the computation is straightforward. We find
\[ E^{(3)}_2 = -\frac{\pi n_e^2}{mN^2} \left\{ \int_0^\infty dr \frac{1 - e^{-r^2}}{r} + \frac{1}{4} \delta_{N,\pm 2} \right\}. \] (7.21)

The integral in (7.21) diverges in the upper limit.

The evaluation of \( E^{(3)}_3 \) is easy.
\[ E^{(3)}_3 = \frac{n_e^2}{2mN^2} \int dy \frac{1}{y^2} \left\{ 1 - g_N(y) e^{-y^2/2l^2} \right\} = \frac{n_e^2}{mN^2} \left\{ \int_0^\infty dr \frac{1 - e^{-r^2}}{r} + \frac{3}{8} \delta_{N,\pm 2} \right\}. \] (7.22)

The divergent integrals in (7.21) and (7.22) cancel each other.

Adding (7.18), (7.21), and (7.22), one finds
\[ E^{(3)} = \begin{cases} \frac{1}{2} \frac{\pi n_e^2}{m} & \text{for } N = \pm 1 \\ \frac{5}{32} \frac{\pi n_e^2}{m} & \text{for } N = \pm 2 \end{cases}. \] (7.23)

Finally combining (7.7), (7.15), and (7.23), we obtain
\[ E_{HF} = \begin{cases} \left( 1 - \frac{1}{4} + \frac{5}{32} \right) \frac{\pi n_e^2}{m} = \frac{1}{2} \frac{\pi n_e^2}{m} & \text{for } N = \pm 1 \\ \left( 1 - \frac{1}{4} + \frac{1}{2} \right) \frac{\pi n_e^2}{m} = \frac{9}{32} \frac{\pi n_e^2}{m} & \text{for } N = \pm 2 \end{cases}. \] (7.24)

The correction to the mean field energy is large for \( N = \pm 1 \), but is relatively small for \( N = \pm 2 \).

There are various ways of showing that \(|\Psi_{HF}\rangle\) in (7.1) is the Hartree-Fock ground state. The Hartree-Fock approximation amounts to finding an approximate ground state for a many-particle system in the form of a Slater determinant formed from one-particle wave functions. In the language of the second quantized theory
we expand the field operator \( \psi(x) \) in a complete orthonormal set \( \{ u_\alpha(x) \} \) to be yet determined, and write a trial ground state as

\[
\psi(x) = \sum c_\alpha u_\alpha(x) e^{-i\epsilon_\alpha HF t}
\]

\[
| \Psi_G \rangle = \prod_{\alpha \in G} c_\alpha^0 | 0 \rangle
\]

\[
\int dx \langle \psi^\dagger \psi(x) \rangle = n_e \cdot \text{vol}.
\] (7.25)

where \( \langle \mathcal{Q} \rangle = \langle \Psi_G | \mathcal{Q} | \Psi_G \rangle \). \( \{ u_\alpha(x) \} \) and the set \( G \) are determined to minimize the expectation value of the Hamiltonian \( \langle \mathcal{H} \rangle \).

In Eq. (4.11)

\[
i \dot{\psi}(x) = \left\{ -\frac{1}{2m} D_k^2 + a_0(x) + g(x) \right\} \psi(x)
\] (4.11)

the approximation amounts to retaining only diagonal pieces on the right hand side. For instance

\[
\psi^\dagger(y) \overline{D}_k \psi(y) \psi(x)
\]

\[
\rightarrow \langle \psi^\dagger(y) \overline{D}_k \psi(y) \rangle \psi(x) - \langle \psi^\dagger(y) \psi(x) \rangle \overline{D}_k \psi(y)
\]

\[
\psi^\dagger(y) \psi^\dagger(z) \psi(z) \psi(y) \psi(x)
\]

\[
\rightarrow \langle \psi^\dagger(y) \psi(y) \rangle \langle \psi^\dagger(z) \psi(z) \rangle \psi(x) - \langle \psi^\dagger(y) \psi(z) \rangle \langle \psi^\dagger(z) \psi(y) \rangle \psi(x)
\]

\[
- \langle \psi^\dagger(y) \psi(x) \rangle \langle \psi^\dagger(z) \psi(z) \rangle \psi(y) + \langle \psi^\dagger(y) \psi(z) \rangle \langle \psi^\dagger(z) \psi(x) \rangle \psi(y)
\]

\[
- \langle \psi^\dagger(y) \psi(y) \rangle \langle \psi^\dagger(z) \psi(x) \rangle \psi(z) + \langle \psi^\dagger(y) \psi(x) \rangle \langle \psi^\dagger(z) \psi(y) \rangle \psi(z).
\] (7.26)

The set \( \{ u_\alpha(x) ; \alpha \in G \} \) is determined to satisfy Eq. (4.11) with the above substitution made. The equation thus obtained is called the Hartree-Fock equation.

It was shown by Hanna, Laughlin, and Fetter\(^{28}\) that \(|N|\) completely filled Landau levels formed from the mean field eigenstates (6.6) or (6.8) satisfy the Hartree-Fock equation, and therefore \(| \Psi_{HF} \rangle \) in (7.1) is the Hartree-Fock ground state. The computation is similar to, but more complicated than, that of \( \langle H \rangle_{HF} \) presented above. Readers should refer to the original paper for details.

The Hartree-Fock equation can be written solely in terms of the function \( f(x, y) \) defined in (7.3). With the ansatz (7.25), the expectation value \( \langle \mathcal{H} \rangle \) can be viewed as a function of \( \langle \psi^\dagger(x) \psi(y) \rangle = f(x, y) \).

\[
\langle H \rangle = \mathcal{H}[f(x, y)] , \quad f(x, y)^* = f(y, x).
\] (7.27)

\( f(x, y) \) is determined by the extremum condition:

\[
\frac{\delta \mathcal{H}}{\delta f(x, y)} = 0 \quad \text{subject to} \quad \int dx \ f(x, x) = n_e \cdot \text{vol}.
\] (7.28)
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\( f(x, y) \) is determined by the equation up to an arbitrariness due to gauge degrees of freedom. Eq. (7.28) is equivalent to the Hartree-Fock equation.

8. RPA and SCF

There are approximation schemes which lie between the mean field and Hartree-Fock approximations, and are useful to investigate various physical quantities such as the excitation spectrum, current-current correlation functions, response to external perturbations, and so on. They are the random phase approximation (RPA), self-consistent field method (SCF), and hydrodynamic description. In the anyon model under consideration, all these three are equivalent to each other.

In a neutral anyon fluid RPA is formulated from the Hamiltonian \( H_1[\psi, \psi^\dagger] \) in (4.8), whereas SCF from the Lagrangian \( \mathcal{L}_0[\psi, \psi^\dagger, a_\mu] \) in (4.1). The difference lies in whether Chern-Simons fields are integrated out first, or retained. A similar statement is valid for a charged anyon fluid.

For definiteness we shall restrict ourselves to a neutral anyon fluid in this section. One can establish the diagram method, or Feynman rules, from the Hamiltonian \( H_1 \) in (4.9) defines a bare propagator for the \( \psi \)-field. It is a propagator in the mean field, depicted by a solid arrowed line. \( H_1^{(2)} \) and \( H_1^{(3)} \) define interaction vertices.

The two-body interaction generated by \( H_1^{(2)} \) is given by \( V_a \) in Fig. 3. A dashed line corresponds to \( h_k(x - y) \), representing virtual “propagation” of Chern-Simons fields. A crossed circle at the vertex \( x \) indicates the derivative factor \( iD_k \).

For the purpose of establishing Feynman rules, it is convenient to start with the normal-ordered form (4.10) for \( H_1^{(3)} \). It yields three- and two-body interactions \( V_b \) and \( V_c \) in Fig. 3, respectively. Both involve two dashed lines.

Fig. 3 Feynman rules derived from (4.8). The rules are simplified by retaining Chern-Simons fields as independent variables. See Fig. 7 in Section 10.

There are important rules resulting from the form of the Hamiltonian. Recall that \( \langle \psi^\dagger(x) \psi(x) \rangle = n_e \), \( \langle \psi^\dagger(x) D_k(x) \psi(x) \rangle = 0 \), and \( \int dx h_k(x) = 0 \). Therefore, contraction of two solid lines at the same vertex in \( V_a \) yield a vanishing result. Similarly,
contraction of two solid lines at either $y$ or $z$ vertex in $V_b$ yields a vanishing result. See Fig. 4a. However, contraction of two solid lines at the $x$ vertex in $V_b$, and at the $x$ and $y$ vertices in $V_c$ yield a non-vanishing result. See Fig. 4b.

Fig. 4  Feynman rules – constraints.

Let’s consider correlation functions of the currents $J^{\mu}(x)$ defined in (4.13). Note that $J^k(x)$ defines two- and four-point vertices. (See Fig. 5.) To all order in perturbation theory

$$\langle J^{\mu}(x) \rangle = n_e \delta^{\mu0} \quad .$$

(8.1)

Fig. 5  Currents $J^{\mu}(x)$.

For the correlation function for a neutral fluid

$$D^{\mu\nu}_n(x,y) = -i(T[\tilde{J}^{\mu}(x)\tilde{J}^{\nu}(y)]) \quad , \quad \tilde{J}^{\mu}(x) = J^{\mu}(x) - n_e \delta^{\mu0} \quad (8.2)$$

RPA constitutes in keeping only diagrams in which no dashed line is a part of closed loops involved.\textsuperscript{27–29} Typical diagrams involved are depicted in Fig. 6. In RPA we
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have only sums of bubble diagrams. Further, the four-point vertex in $J^k(x)$ does not enter at this level.

One can write down Schwinger-Dyson equations for $D_n^{\mu\nu}$, which was done by Fetter, Hanna, and Laughlin, and by Chen, Wilczek, Witten, and Halperin. Due to the presence of the off-diagonal couplings, the equation takes the form of a $3 \times 3$ matrix. In the following section we shall show an alternative method to evaluate $D_n^{\mu\nu}$.

The self-consistent field method starts with the field equation (4.2), iretaining the Chern-Simons gauge fields:

\begin{equation}
- \frac{N}{4\pi} \varepsilon^{\mu\nu\rho} f_{\nu\rho} = j^\mu \\
i \partial_0 \psi = \left\{ - \frac{1}{2m} D_k^2 + a_0 \right\} \psi .
\end{equation}

We suppose first that there is a consistent gauge field configuration $a_\mu(x)$ which is treated classically. Secondly, with this given $a_\mu(x)$ we determine a quantum mechanical state vector, $|\Psi[a_\mu]\rangle$, for the matter field $\psi(x)$, by solving the second equation of (4.2). Thirdly, we replace the current $j^\mu$ on the right side of the first equation of (4.2) by its expectation value:

\begin{equation}
- \frac{N}{4\pi} \varepsilon^{\mu\nu\rho} f_{\nu\rho} = (\Psi[a_\mu] | j^\mu | \Psi[a_\mu]) \equiv J^\mu[x; a_\mu] .
\end{equation}
Eq. (8.3) is solved for \( a_\mu(x) \) to find a self-consistent field configuration.

We have known that the configuration, (6.2), of the mean field ground state, which is the same as the Hartree-Fock ground state, solves Eq. (8.3). We are seeking for more general, \( x \)-dependent solutions. If a deviation of \( a_\mu(x) \) from (6.2) is small, one can employ a perturbation theory to find a solution.

For time-independent configurations the procedure is particularly simple, as was first performed by Hosotani and Chakravarty.\textsuperscript{32} The generalization to finite temperature is done by Randjbar-Daemi, Salam, and Strathdee,\textsuperscript{38} and by Hetrick, Hosotani, and Lee.\textsuperscript{44} Time-dependent configurations have been recently analysed by this method by Chakravarty.\textsuperscript{58}

Consider a time-independent, small fluctuation of Chern-Simons gauge fields, \( a^{(1)}_\mu(x) = a_\mu(x) - \bar{a}_\mu(x) \). We first solve the one-particle Schrödinger equation with this \( a_\mu(x) \):

\[
\left\{ -\frac{1}{2m} \partial^2_k + a_0(x) \right\} u_\alpha(x; a^{(1)}_\mu) = \epsilon_\alpha(a^{(1)}_\mu) \ u_\alpha(x; a^{(1)}_\mu) .
\] (8.4)

Both \( \epsilon_\alpha(a^{(1)}_\mu) \) and \( u_\alpha(x; a^{(1)}_\mu) \) are determined perturbatively. \( \{ u_\alpha(x; a^{(1)}_\mu) \} \) defines a complete, orthonormal basis, with which we expand \( \psi(x) \) as

\[
\psi(x) = \sum_\alpha c_\alpha(a^{(1)}_\mu) \ u_\alpha(x; a^{(1)}_\mu) , \quad \{ c_\alpha, c^\dagger_\beta \} = \delta_{\alpha\beta} .
\] (8.5)

So long as \( a^{(1)}_\mu(x) \) is sufficiently small and smooth, the spectrum retains the structure of Landau levels, although they are not degenerate any more in general. With the same set as \( G \) in (6.13), we define a state

\[
\Psi[a^{(1)}] = \prod_{\alpha \in G} c^\dagger_\alpha(a^{(1)}_\mu) \ |0\rangle ,
\] (8.6)

from which the current in (8.3) is determined as

\[
J^\mu[x; a_\mu] = n_e \ \delta^{\mu 0} - \int d^3y \ \Gamma^{\mu
\nu}(x, y) a^{(1)}_\nu(y) + \cdots .
\] (8.7)

At finite temperature \( T(=\beta^{-1}) \) we evaluate the matter part of the free energy with a given \( a_\mu(x) \) by

\[
e^{-\beta F[a_\mu]} = \text{Tr} \ e^{-\beta H_0[\psi, a_\mu]} \] (8.8)

where

\[
H_0[\psi, a_\mu] = \int dx \left\{ \frac{1}{2m} (D_k \psi)^\dagger(D_k \psi) + a_0 \psi^\dagger \psi \right\} .
\] (8.9)

The current is given by

\[
J^\nu[x; a_\mu] = \text{Tr} \ J^\nu(x) \ e^{\beta(F[a_\mu] - H_0[\psi, a_\mu])} = \text{Tr} \ \frac{\delta H_0}{\delta a^\dagger_\nu(x)} \ e^{\beta(F[a_\mu] - H_0[\psi, a_\mu])}
\]

\[
= \frac{\delta F[a_\mu]}{\delta a^\dagger_\nu(x)} ,
\] (8.10)
which leads to an expression similar to (8.7). Incorporation of electromagnetic interactions is straightforward.

In this section we have explained the two approximation methods, RPA and SCF. These two look quite different from each other. RPA is defined for Green’s functions for matter fields in terms of Feynman diagrams, whereas SCF is written in the form of gauge field equations. We shall show in the next section that these two are indeed the same, and are equivalent.

9. Path integral representation

A bridge between RPA and SCF becomes most transparent in the path integral formalism. One can deal with both neutral and charged anyon fluids at once. The key step is to consider the transition amplitude or partition function in the presence of external gauge potentials.

To simplify notations, we write as

\[ L_{\text{CS}}^0[a] = -\frac{N}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho , \quad L_{\text{EM}}^0[A] = -\frac{1}{4} F_{\mu\nu}^2 + e n_e A_0 \]  

(9.1)

where \( D_0 = \partial_0 + i a_0 \) and \( D_k = \partial_k - i a_k \). In terms of these definitions

\[ L_0[\psi, a] = L_{\text{CS}}^0[a] + L_f[\psi; a] \]

\[ L_{2D}[\psi, a, A] = L_{\text{CS}}^0[a] + L_{\text{EM}}^0[A] + L_f[\psi; a + eA] \]

Consider a charged anyon fluid described by (5.4) or (5.5). We introduce an external gauge potential \( a^\text{ext}_\mu(x) \) in a gauge invariant way, making a replacement

\[ L_f[\psi; a + eA] \rightarrow L_f[\psi; a + eA + a^\text{ext}] \]  

(9.2)

In other words, the external Lagrangian is given by

\[ L_{\text{ext}} = L_f[\psi; a + eA + a^\text{ext}] - L_f[\psi; a + eA] \]

\[ = -a^\text{ext}_0 j^0 + a^\text{ext}_k j^k - \frac{1}{2m} (a^\text{ext}_k)^2 j^0 \]  

(9.3)

where

\[
\begin{align*}
  j^0(x) &= \psi^\dagger \psi \\
  j^k(x) &= -\frac{i}{2m} (\psi^\dagger \nabla_k \psi - \nabla_k \psi^\dagger \cdot \psi) - \frac{1}{m} \psi^\dagger \psi \cdot (a_k + eA_k)
\end{align*}
\]

(9.4)

In the presence of an external potential a total gauge-invariant current is

\[
  j^\mu_{\text{tot}}(x) = -\frac{\delta}{\delta a^\text{ext}_\mu(x)} \int dy \ L_f[\psi; a + eA + a^\text{ext}](y) \\
  j^0_{\text{tot}} = j^0 , \quad j^k_{\text{tot}} = j^k - \frac{1}{m} a^k_{\text{ext}} \psi^\dagger \psi
\]

(9.5)
The transition amplitude at $T = 0$ is given, in the path integral representation, by

$$\exp \left( i I_{2D}[a^{\text{ext}}] \right) = \int \mathcal{D}A \mathcal{D}a \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ i \int dx \left( \mathcal{L}^{\text{CS}}_0[a] + \mathcal{L}^{\text{CS}}_g[a] \right. \right.$$  

$$\left. + \mathcal{L}^{\text{EM}}_0[A] + \mathcal{L}^{\text{EM}}_g[A] + \mathcal{L}_f[\psi; a + eA + a^{\text{ext}}] \right\}. \quad (9.6)$$

Here $\mathcal{L}^{\text{CS}}_g[a]$ and $\mathcal{L}^{\text{EM}}_g[A]$ are the gauge-fixing terms for $a_\mu$ and $A_\mu$, respectively, and shall be specified shortly. The initial and final state wave functions have been absorbed in the definition of the path integration measure. Their explicit forms are irrelevant to compute various quantities described below.

Several identities follow from (9.6). First

$$-\frac{\delta I_{2D}}{\delta a^{\text{ext}}_\mu(x)} = e^{-i I_{2D}} i \frac{\delta}{\delta a^{\text{ext}}_\mu(x)} e^{i I_{2D}} = \langle j^{\mu}_{\text{tot}}(x) \rangle_{a^{\text{ext}}}. \quad (9.7)$$

Recalling that $\langle j^{\mu}_{\text{tot}} \rangle_{a^{\text{ext}} = 0} = n_e \delta^{\mu 0}$, one finds a current dynamically induced to be

$$J^{\mu}_{\text{ind}}(x; a^{\text{ext}}) = \langle j^{\mu}_{\text{tot}}(x) \rangle_{a^{\text{ext}}} - n_e \delta^{\mu 0}$$  

$$= -\frac{\delta I_{2D}}{\delta a^{\text{ext}}_\mu(x)} - n_e \delta^{\mu 0}. \quad (9.8)$$

Similarly the second derivative leads to

$$e^{-i I_{2D}} i^2 \frac{\delta^2}{\delta a^{\text{ext}}_\mu(x) \delta a^{\text{ext}}_\nu(y)} e^{i I_{2D}}$$  

$$= \langle T[j^{\mu}_{\text{tot}}(x) j^{\nu}_{\text{tot}}(y)] \rangle_{a^{\text{ext}}} + i (1 - \delta^{\mu 0}) \delta^{\mu \nu} \frac{1}{m} \langle j^0(x) \rangle_{a^{\text{ext}}} \delta(x - y) \quad (9.9)$$

$$= -i \frac{\delta^2 I_{2D}}{\delta a^{\text{ext}}_\mu(x) \delta a^{\text{ext}}_\nu(y)} + \frac{\delta I_{2D}}{\delta a^{\text{ext}}_\mu(x)} \frac{\delta I_{2D}}{\delta a^{\text{ext}}_\nu(y)}$$

In the $a^{\text{ext}}_\mu(x) = 0$ limit one has

$$\langle T[j^{\mu}(x) j^{\nu}(y)] \rangle_{a^{\text{ext}} = 0} = -\delta^{\mu 0} \delta^{\nu 0} n_e^2$$  

$$= -i \frac{\delta^2 I_{2D}}{\delta a^{\text{ext}}_\mu(x) \delta a^{\text{ext}}_\nu(y)} \bigg|_{a^{\text{ext}} = 0} - i (1 - \delta^{\mu 0}) \delta^{\mu \nu} \frac{n_e}{m} \delta(x - y). \quad (9.10)$$

Formulas for a neutral anyon fluid are obtained from the expressions above, dropping electromagnetic fields $A_\mu$ entirely. The amplitude becomes

$$\exp \left( i n_a [a^{\text{ext}}] \right) = \int \mathcal{D}a \mathcal{D}\psi \mathcal{D}\bar{\psi}$$  

$$\times \exp \left\{ i \int dx \left( \mathcal{L}^{\text{CS}}_0[a] + \mathcal{L}^{\text{CS}}_g[a] + \mathcal{L}_f[\psi; a + a^{\text{ext}}] \right) \right\}. \quad (9.11)$$
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The induced current and correlation function are given by

\[
J^\mu_{\text{ind}}(x; a^\text{ext})_{\text{neutral}} = -\frac{\delta I_n}{\delta a^\text{ext}_\mu(x)} - n_e \delta^{\mu\nu} \\
\langle T[j^\mu(x)j^\nu(y)]\rangle_{a^\text{ext}=0}^{\text{neutral}} = -\delta^{\mu\nu} \delta^{\rho\sigma} \frac{\delta^2 I_n}{\delta a^\text{ext}_\mu(x)\delta a^\text{ext}_\nu(y)} \bigg|_{a^\text{ext}=0} - i(1 - \delta^{\mu\nu})\delta^{\rho\sigma} \frac{n_e}{m} \delta(x-y)
\]  

(9.12)

The first equation of (9.12) is related to (8.7) in SCF, whereas the second equation gives the correlation function (8.2) in RPA. Both are derived from the effective action \( I_n[a^\text{ext}] \) or \( I_{2D}[a^\text{ext}] \) for a neutral or charged anyon fluid, respectively. The connection between RPA and SCF is established by evaluating the effective action.

As for gauge-fixing, it is most convenient to take

\[
L_{CS}^{\text{f.g.}}[a] = \frac{1}{2\alpha} (\nabla_k a^k)^2 \\
L_{EM}^{\text{f.g.}}[A] = \frac{1}{2} (\partial_\mu A^\mu)^2
\]

(9.13)

\( L_{CS}^{\text{f.g.}}[a] \) gives the standard Lorentz covariant Feynman gauge for electromagnetic fields. We have retained the gauge parameter \( \alpha \) for Chern-Simons fields. For \( \alpha = 1 \), \( L_{CS}^{\text{f.g.}}[a] \) gives a spatial Feynman gauge. In the \( \alpha = 0 \) limit it reproduces the radiation gauge \( \nabla_k a^k = 0 \).

For Chern-Simons gauge fields we have

\[
L_0^{\text{CS}}[a] + L_{CS}^{\text{f.g.}}[a] = \frac{N}{2\pi} a_0^{(1)(0)} \delta^{(0)} - \frac{N}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu^{(1)} \partial_\nu a_\rho^{(1)} + \frac{1}{2\alpha} (\nabla_k a^{(1)k})^2 \\
= n_e a_0^{(1)} + \frac{1}{2} a_\mu^{(1)} (\Lambda_0 + \Lambda_{\text{f.g.}})^\mu_\nu a_\nu^{(1)}
\]

(9.14)

where the kernels \( \Lambda_0 \) and \( \Lambda_{\text{f.g.}} \) are given by

\[
\Lambda_0 = -\frac{N}{2\pi} \begin{pmatrix} 0 & -\partial_2 & +\partial_1 \\ +\partial_2 & 0 & -\partial_0 \\ -\partial_1 & +\partial_0 & 0 \end{pmatrix}, \quad \Lambda_{\text{f.g.}} = -\frac{1}{\alpha} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial_1^2 & \partial_1 \partial_2 \\ 0 & \partial_1 \partial_2 & \partial_2^2 \end{pmatrix}
\]

(9.15)

Note that \( a_\mu^{(1)} = (a_0^{(1)}, a_1^{(1)}, a_2^{(1)}) = (a^{(1)0}, -a^{(1)1}, -a^{(1)2}) \). Integration by parts has been made in the above formulas. The Chern-Simons term itself is singular, as \( det \Lambda_0 = 0 \). However the total kernel is regular:

\[
\Lambda = \Lambda_0 + \Lambda_{\text{f.g.}} \\
\quad det \Lambda = -\frac{1}{\alpha} \left( \frac{N}{2\pi} \right)^2 (\nabla)^2 \neq 0
\]

(9.16)

For electromagnetic fields we have

\[
L_0^{EM}[A] + L_{EM}^{\text{f.g.}}[A] = \frac{1}{2} A_\mu g^{\mu\nu} \partial_\nu A_\nu + en_e A_0
\]

(9.17)
where the metric is $g^{\mu\nu} = \text{diag}(1, -1, -1)$.

The integration over various fields in the formula (9.6) can be done in any order. Integrating first over the Chern-Simons fields $a_\mu(x)$ in the limit $\alpha = 0$ is equivalent to eliminating them to get the Hamiltonian (4.8) or (5.5). An alternative is to integrate the matter fields $\psi(x)$ and $\psi^\dagger(x)$ first, maintaining the symmetry of the CS and EM gauge couplings.

To see that the Hamiltonian (4.8) is reproduced in the neutral case by integrating $a_\mu$, we write

$$\frac{1}{2} a_\mu^{(1)} \Lambda^{\mu\nu} a_\nu^{(1)} + \mathcal{L}_f[\psi; a] = \frac{1}{2} a_\mu^{(1)} (\Lambda + \Xi)^{\mu\nu} a_\nu^{(1)} + a_\mu^{(1)} \tilde{j}^\mu + \mathcal{L}_f[\psi; a^{(0)}]$$

(9.18)

where

$$\Xi^{\mu\nu} = -(1 - \delta^{\mu0}) \delta^{\mu\nu} \frac{1}{m} \psi^\dagger \psi$$

$$\tilde{j}^0 = j^0 = \psi^\dagger \psi$$

$$\tilde{j}^k = -\frac{i}{2m} \left\{ \psi^\dagger \bar{D}_k \psi - (\bar{D}_k \psi)^\dagger \psi \right\}.$$  

(9.19)

Further we note that

$$\Lambda^{-1} = \frac{2\pi}{N} \frac{1}{\nabla^2} \begin{pmatrix} 0 & +\partial_2 & -\partial_1 \\ -\partial_2 & 0 & 0 \\ +\partial_1 & 0 & 0 \end{pmatrix} + O(\alpha)$$

(9.20)

$$(\Lambda + \Xi)^{-1} = \Lambda^{-1} + \Lambda^{-1} \Xi \Lambda^{-1} + \Lambda^{-1} \Xi \Lambda^{-1} \Xi \Lambda^{-1} + \cdots$$

In the $\alpha \to 0$ limit

$$\Lambda^{-1} \Xi \Lambda^{-1}|_{\alpha=0} = -\left( \frac{2\pi}{N} \right)^2 \frac{1}{m} \frac{\partial_k}{\nabla^2} j^0 \frac{\partial_k}{\nabla^2} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right)$$

$$\Lambda^{-1} \Xi \Lambda^{-1} \Xi \Lambda^{-1}|_{\alpha=0} = 0$$

etc.

so that the integration over $a_\mu^{(1)}$ yields a Lagrangian

$$\mathcal{L}_f[\psi; a^{(0)}] = \frac{1}{2} \tilde{j}^\mu \left( \Lambda + \Xi \right)^{-1} \mu \nu \tilde{j}^\nu$$

(9.21)

$$= \frac{1}{2m} \left( \frac{2\pi}{N} \right)^2 j^0 \frac{\partial_k}{\nabla^2} j^0 \frac{\partial_k}{\nabla^2} j^0 - \frac{2\pi}{N} \epsilon^{kl} \frac{\partial_k}{\nabla^2} j^0 \frac{\partial_l}{\nabla^2} j^0.$$  

Noticing that

$$\int dy \ h_j(x-y) j^0(y) = -\frac{2\pi}{N} \epsilon^{jk} \frac{\partial_k}{\nabla^2} j^0(x)$$

we observe that (9.21) yields the same Hamiltonian as in (4.8). The path integral formalism is equivalent to the operator formalism.
Neutral and Charged Anyon Fluids

10. RPA = linearized SCF

In practice it is more convenient to integrate the fermion fields $\psi$ and $\psi^\dagger$ first to evaluate $I_n[a^{\text{ext}}]$ in (9.12) or $I_{2D}[a^{\text{ext}}]$ in (9.6). Let us define

$$
\exp \{ iS_f[a] \} = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi \exp \left\{ i \int dx \mathcal{L}_f[\psi; a] \right\}. \tag{10.1}
$$

It gives the transition amplitude for the fermion fields in the presence of gauge fields $a_\mu(x)$. Then we have

$$
\exp \{ iI_n[a^{\text{ext}}] \} = \int \mathcal{D}a \exp \left\{ iS_f[a + a^{\text{ext}}] + i \int dx (\mathcal{L}_{0}^{\text{CS}}[a] + \mathcal{L}_{\text{g.t.}}^{\text{CS}}[a]) \right\}
$$

$$
\exp \{ iI_{2D}[a^{\text{ext}}] \} = \int \mathcal{D}A \exp \left\{ iI_n[eA + a^{\text{ext}}] + i \int dx (\mathcal{L}_{0}^{\text{EM}}[A] + \mathcal{L}_{\text{g.t.}}^{\text{EM}}[A]) \right\}. \tag{10.2}
$$

To evaluate $S_f[a]$ we decompose $\mathcal{L}_f[\psi; a]$ into the zeroth order and interaction parts:

$$
\mathcal{L}_f[\psi; a] = \mathcal{L}_f[\psi; a^{(0)}] + \mathcal{L}^{\text{int}}_f. \tag{10.3}
$$

Here $a^{(0)}_\mu(x)$ is the average field configuration given in (6.2) and

$$
\mathcal{L}^{\text{int}}_f = - a^{(1)}_0 \psi^\dagger \psi - a^{(1)}_k \frac{i}{2m} \left\{ \psi^\dagger \bar{D}_k \psi - (\bar{D}_k \psi)^\dagger \psi \right\} - (a^{(1)}_k)^2 \frac{1}{2m} \psi^\dagger \psi \tag{10.4}
$$

where $a^{(1)}_\mu = a_\mu - a^{(0)}_\mu$.

$\mathcal{L}_f[\psi; a^{(0)}]$ defines a propagator of the $\psi$ field in the background potential $a^{(0)}_\mu$, whereas $\mathcal{L}^{\text{int}}_f$ defines interaction vertices containing $a^{(1)}_\mu(x)$. The first and second terms in (10.4) give one gauge field leg, whereas the last term gives two legs, as depicted in Fig. 7.

$S_f[a]$ is nothing but the effective action for $a_\mu(x)$ generated by dynamics of $\psi$ and $\psi^\dagger$ fields. The standard diagram technique can be employed. Since $\psi$ and $\psi^\dagger$ are integrated, fermion lines must be closed. $S_f[a] - S_f[a^{(0)}]$ is the sum of connected diagrams. Further, since the interaction $\mathcal{L}^{\text{int}}_f$ is bilinear in $\psi$ and $\psi^\dagger$, diagrams thus generated are all one-loop. One can arrange them according to the number of legs.
of gauge field $a^{(1)}_{\mu}$ as in Fig. 8.

Fig. 7. Vertices generated by $L^{int}_J$ in (10.4). “0” and “k” at the ends of dashed lines indicate $a^{(1)}_0$ and $a^{(1)}_k$, respectively.

Fig. 8. The effective action $S_f[q]$.

Contributions coming from diagrams (a) and (b) in Fig. 8 are easy to evaluate.

\begin{align}
\text{diagram (a)} &= -a_0^{(1)} \langle \psi^\dagger \psi \rangle = -a_0^{(1)} n_e \\
\text{diagram (b)} &= -\frac{1}{2m} (a^{(1)}_k)^2 n_e.
\end{align}

Note that in the nonrelativistic system under consideration we always have $\langle \psi^\dagger \psi(x) \rangle$, instead of $\lim_{x \to y} (-1)\langle T[\psi(x)\psi^\dagger(y)] \rangle$, in the first-order perturbation. A similar diagram containing the second vertex in (10.4) vanishes, since $\langle j^k \rangle = 0$. 

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In general one has an expansion

\[
S_f[a] = S_f[a^{(0)}] - \int dx \, n_c \, a_0^{(1)}(x) + \int dx dy \, \frac{1}{2} a_\mu^{(1)}(x) \Gamma^{\mu\nu}(x - y) a_\nu^{(1)}(y) + \cdots.
\]  

(10.6)

In writing the \( \Gamma^{\mu\nu} \) term, we have employed the translation invariance of the zeroth-order system described by \( L_f[\psi, a^{(0)}] \). In the momentum space it becomes

\[
\int \frac{d\omega d\mathbf{q}}{(2\pi)^3} \, \frac{1}{2} a_\mu^{(1)}(-\omega, -\mathbf{q}) \Gamma^{\mu\nu}(\omega, \mathbf{q}) a_\nu^{(1)}(\omega, \mathbf{q}).
\]  

(10.7)

Contributions from diagrams (c), (d), and (e) to \( \Gamma^{\mu\nu} \) will be evaluated in the following sections.

The next step is to integrate over Chern-Simons fields \( a_\mu(x) \) in (10.2). Again a diagram method may be developed for the integral. Higher order terms in (10.6), namely terms involving three or more \( a_\mu^{(1)} \)'s, give “interaction” vertices.

By dropping all these higher order terms the system is linearized. We call it “the linear approximation”. The resulting integral is a simple Gaussian integral, whose evaluation is straightforward. As we shall see shortly, RPA and the linearized SCF are nothing but the linear approximation.

Let us consider a neutral anyon fluid. We observe

\[
S_f[a + a^{\text{ext}}] + \int dx \, (L_0^{\text{CS}}[a] + L_{\text{g.f.}}^{\text{CS}}[a])
\]

\[
= S_f[a^{(0)}] - \int n_c \, (a_0^{(1)} + a_0^{\text{ext}}) + \int \frac{1}{2} (a_\mu^{(1)} + a_\mu^{\text{ext}}) \Gamma^{\mu\nu}(a_\nu^{(1)} + a_\nu^{\text{ext}}) + \cdots
\]

\[
+ \left( \int n_c \, a_0^{(1)} + \int \frac{1}{2} a_\mu^{(1)} \Lambda^{\mu\nu} a_\nu^{(1)} \right)
\]

\[
= S_f[a^{(0)}] - \int n_c \, a_0^{\text{ext}}
\]

\[
+ \int \frac{1}{2} (a^{(1)} + a^{\text{ext}} \Gamma (\Lambda + \Gamma)^{-1}) (\Lambda + \Gamma) \{ a^{(1)} + (\Lambda + \Gamma)^{-1} \Gamma a^{\text{ext}} \}
\]

\[
- \int \frac{1}{2} a^{\text{ext}} \Gamma (\Lambda + \Gamma)^{-1} \Gamma a^{\text{ext}} + \int \frac{1}{2} a^{\text{ext}} \Gamma a^{\text{ext}}.
\]  

(10.8)

We have suppressed a measure \( dx \) or \( d\omega d\mathbf{q} \) in the expression. The integration over \( a_\mu^{(1)} \) immediately leads to

\[
I_n[a^{\text{ext}}] = - \int dx \, n_c \, a_0^{\text{ext}}(x) - \int dx dy \, \frac{1}{2} a_\mu^{\text{ext}}(x) Q_n^{\mu\nu}(x - y) a_\nu^{\text{ext}}(y) + \cdots
\]  

(10.9)

where

\[
Q_n = \Gamma (\Lambda + \Gamma)^{-1} \Gamma - \Gamma
\]

\[
= - \Gamma \frac{1}{1 + \Lambda^{-1} \Gamma} = - \frac{1}{1 + \Gamma \Lambda^{-1} \Gamma}
\]

(10.10)

\[
= - \Gamma + \Gamma \Lambda^{-1} \Gamma - \Gamma \Lambda^{-1} \Gamma \Lambda^{-1} \Gamma + \cdots
\]
\( \Lambda^{-1} \) represents a propagator for \( a^{(1)}_\mu \). Therefore \( Q_n \) has a diagram representation given in Fig. 9.

From (9.12) the \( jj \)-correlation function is given by

\[
\frac{1}{i} \left( \langle T[j^\mu(x)j^\nu(y)] \rangle^n_{\text{neutral}} - \delta^{\mu0}\delta^{\nu0} n_e^2 \right) = Q^{\mu\nu}_n(x-y) - (1 - \delta^{\mu0})\delta^{\mu\nu} \frac{n_e}{m} \delta(x-y) + \cdots .
\]

(10.11)

We recall that the contribution of diagram (b) in Fig. 8 to \( \Gamma \) is given by (10.5) so that

\[
-\Gamma^{(b)}_{jk} = \delta^{jk} \frac{n_e}{m} \delta(x-y) ,
\]

(10.12)

which is precisely the negative of the last term in (10.11). In a normal metal this is the end of cancellation. In anyon fluids something special happens. As shall be shown in section 12, diagram (e) in Fig. 8, in part, yields the same contribution as (10.12) with the opposite sign. Hence the last term in (10.11) survives.

The series generated in (10.11) with (10.10) substituted exactly reproduces the diagrams in Fig. 6 in Section 8. Hence

\[
D^{\mu\nu}_n(x,y)^{\text{RPA}} = D^{\mu\nu}_n(x,y)^{\text{linear}} = Q^{\mu\nu}_n(x-y) - (1 - \delta^{\mu0})\delta^{\mu\nu} \frac{n_e}{m} \delta(x-y) .
\]

(10.13)

RPA is equivalent to dropping, in \( I_n[a^{\text{ext}}] \), all terms cubic or higher-order in \( a^{\text{ext}}_\mu(x) \).
Neutral and Charged Anyon Fluids

How about SCF? We return to the expression (8.7). In SCF one determines the current in the presence of non-trivial gauge fields $a_\mu(x)$, which is nothing but evaluating $S_f[a]$ in (10.1). Since $j^\mu(x) = -\delta \int \mathcal{L}_f[\psi; a]/\delta a_\mu(x)$, one immediately finds

$$J^\mu[x; a]^{\text{SCF}} = e^{-iS_f[a]} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, j^\mu(x) e^{i \int dx \mathcal{L}_f}$$

$$= -\frac{\delta S_f[a]}{\delta a_\mu^{(1)}(x)}$$

(10.14)

$$= n_e \delta \sigma^0 - \int dy \, \Gamma^{\mu\nu}(x-y) a_\nu^{(1)}(y) + \cdots.$$  

We recognize that $\Gamma^{\mu\nu}$ appearing in (8.7) is the same as $\Gamma^{\mu\nu}$ defined in (10.6).

In SCF the current thus obtained is inserted into the Chern-Simons field equation (4.2), which is then solved to determine a self-consistent nontrivial field configuration. If higher-order terms in (10.14) are dropped, the resulting field equation becomes linear. Solving the equation is equivalent to performing a Gaussian integral in the path integral formalism, since the latter amounts to picking a stationary path of the action.

An alternative way of seeing this is to examine a response to an external field in SCF. Since $J^\mu_{\text{ext}} = \Lambda a_{\text{ext}}$, the field equation in SCF, with the gauge fixing term added, becomes

$$\Lambda a^{(1)} = J^\mu_{\text{ind}} + J^\mu_{\text{ext}} = (-\Gamma a^{(1)} + \cdots) + \Lambda a_{\text{ext}},$$

from which it follows that

$$J^\mu_{\text{ind linearized SCF}} = -\Gamma (\Lambda + \Gamma)^{-1} \Lambda a_{\text{ext}} = -\Gamma (1 + \Lambda^{-1}) (1 + \Lambda^{-1})^{-1} a_{\text{ext}}$$

$$= Q_n a_{\text{ext}}$$

(10.15)

$$= J^\mu_{\text{ind linear}}.$$  

We have stressed in the last equality that the expression is exactly what one obtains from the first equation of (9.12) combined with (10.9). The relations (10.13) and (10.15) together establish the equivalence between RPA and the linearized SCF.

The generalization to a charged anyon fluid is easy. In the two-dimensional approximation one needs to do one more integration over $A_\mu(x)$ in (10.2). With the expressions (10.9) and (9.17) inserted, the exponent of the integrand becomes

$$-n_e (eA_0 + a_0^{\text{ext}}) - \frac{1}{2} (eA + a^{\text{ext}}) Q_n (eA + a^{\text{ext}}) + \cdots + \frac{1}{2} A P A + en_e A_0$$

$$- \frac{1}{2} \{ A - a^{\text{ext}} eQ_n (P - e^2 Q_n)^{-1} (P - e^2 Q_n) \} (A - (P - e^2 Q_n)^{-1} eQ_n a^{\text{ext}})$$

$$- \frac{1}{2} a^{\text{ext}} eQ_n (P - e^2 Q_n)^{-1} eQ_n a^{\text{ext}} - \frac{1}{2} a^{\text{ext}} Q_n a^{\text{ext}} - n_e a_0^{\text{ext}} + \cdots$$

(10.16)

where

$$P^{\mu\nu} = g^{\mu\nu} \partial^2.$$  

(10.17)
Therefore we have

\[ I_{2D}[a^\text{ext}] = -\int dx \, n_e a^\text{ext}_0(x) - \int dx dy \, \frac{1}{2} a^\text{ext}_\mu(x) Q^\mu\nu(x-y) a^\text{ext}_\nu(y) + \cdots \] (10.18)

where

\[
Q_c = Q_n(e^{-2P} - Q_n)^{-1}Q_n + Q_n \\
= \frac{1}{1 - Q_n e^2P^{-1}} Q_n = \frac{1}{1 - e^2P^{-1}} Q_n \\
= Q_n + Q_n e^2P^{-1} Q_n + Q_n e^2P^{-1} Q_n e^2P^{-1} Q_n + \cdots
\] (10.19)

Combining (10.10) and (10.19), one finds

\[
Q_c = -\Gamma \frac{1}{1 + (\Lambda^{-1} + e^2P^{-1})\Gamma} = -\frac{1}{1 + \Gamma(\Lambda^{-1} + e^2P^{-1})} \Gamma \\
= -\Gamma + \Gamma \left(\frac{1}{\Lambda} + \frac{e^2}{P}\right) \Gamma - \Gamma \left(\frac{1}{\Lambda} + \frac{e^2}{P}\right) \Gamma \left(\frac{1}{\Lambda} + \frac{e^2}{P}\right) \Gamma + \cdots
\] (10.20)

The proper vertex \(-\Gamma\), which summarizes one-loop fermion interactions, is connected to the next one by a propagator of either Chern-Simons fields (\(\Lambda^{-1}\)) or electromagnetic fields (\(e^2P^{-1}\)). There is only one proper vertex, since both Chern-Simons and electromagnetic fields minimally couple to the fermions. The final expression for \(Q_c\) above is obvious from the viewpoint of the diagram method. (See Fig. 9.)

To summarize, in the linear approximation, which is equivalent to RPA and the linearized SCF,

\[
D^\mu\nu(x, y)^\text{linear} = Q^\mu\nu(x-y) - (1 - \delta^{\mu0})\delta^{\mu\nu} \frac{n_e}{m} \delta(x-y) \\
J^\text{linear}_{\text{ind}} = Q_c a^\text{ext}
\] (10.21)

11. Response function

As it stands from (10.13), (10.15), and (10.21), \(Q_n\) or \(Q_c\) determines a linear response to an external perturbation. It defines a response function. Location of poles in the response function \(Q_n\) or \(Q_c\) gives an energy spectrum of particle-hole excitations. In passing, location of poles in the fermion propagator yields a spectrum for fermionic excitations, which is quite different from that of particle-hole excitations in anyon fluids under consideration.

One can also examine a response to an external magnetic field in a charged anyon fluid, from which the existence or non-existence of a Meissner effect is checked. Conductivity or resistivity tensors can be computed from \(Q_n\) or \(Q_c\). Examining a response to external density perturbation \(J^\text{ext}_\text{ind}(x)\) gives information on muon spin relaxation. Many other consequences can be drawn from \(Q_n\) or \(Q_c\).
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We have seen in the previous section that $Q_n$ and $Q_c$ are determined by the proper vertex $\Gamma$ defined in (10.6). There are many restrictions resulting from the definition (10.6) and the conservation law.

First, from the definition we have $\Gamma_{\mu\nu}(x-y) = \Gamma_{\nu\mu}(-\omega, -q)$, or

$$\Gamma_{\mu\nu}(\omega, q) = \Gamma_{\nu\mu}(-\omega, -q). \quad (11.1)$$

Secondly, the gauge invariance or current conservation implies $\partial_\mu \Gamma_{\mu\nu}(x-y) = 0$ and $\partial_\nu \Gamma_{\mu\nu}(x-y) = 0$, or

$$q_\mu \Gamma_{\mu\nu}(\omega, q) = 0 = q_\nu \Gamma_{\mu\nu}(\omega, q) \quad (11.2)$$

where $q_\mu = (\omega, -q)$. Thirdly, the rotational invariance implies that

$$\Gamma_{00} = A$$
$$\Gamma^{0j} = q_j B + \epsilon_{jk} q_k C$$
$$\Gamma^j_0 = q_j B' + \epsilon_{jk} q_k C'$$
$$\Gamma^{jk} = \delta_{jk} D + \epsilon_{jk} E + q_j q_k F \quad (11.3)$$

where $A \sim F$ are functions of $\omega$ and $q = |q|$. Relations (11.1), (11.2), and (11.3) lead to a decomposition\textsuperscript{27,38}

$$\Gamma^{00}(\omega, q) = q^2 \Pi_0$$
$$\Gamma^{0j}(\omega, q) = \omega q_j \Pi_1 - i \epsilon_{jk} q_k \Pi_1$$
$$\Gamma^j_0(\omega, q) = \omega q_j \Pi_1 + i \epsilon_{jk} q_k \Pi_1$$
$$\Gamma^{jk}(\omega, q) = \delta_{jk} \omega^2 \Pi_2 + i \epsilon_{jk} \omega \Pi_1 - (q^2 \delta_{jk} - q_j q_k) \Pi_2 \quad (11.4)$$

where all $\Pi_j$’s are functions of $\omega^2$ and $q^2$ only. If the perturbative ground state is stable, $S_f[a]$ defined in (10.1) is real, and therefore from (10.6) $\Gamma^{\mu\nu}(x)^* = \Gamma^{\nu\mu}(x)$, or $\Gamma^{\mu\nu}(-\omega, -q)^* = \Gamma^{\nu\mu}(\omega, q)$. In other words, $\Pi_k$’s $(k = 0, 1, 2)$ are real: $\Pi_k^* = \Pi_k$.

In a frame $q = (q, 0)$,

$$\Gamma^{\mu\nu} = \begin{pmatrix}
q^2 \Pi_0 & \omega q \Pi_0 + iq \Pi_1 \\
\omega q \Pi_0 & \omega^2 \Pi_0 + i \omega \Pi_1 \\
-i q \Pi_1 & -i \omega \Pi_1 & \omega^2 \Pi_2 - q^2 \Pi_2
\end{pmatrix}. \quad (11.5)$$

To evaluate $Q_n$, we recall Eq. (10.10):

$$Q_n = \Gamma (\Lambda + \Gamma)^{-1} \Gamma - \Gamma \quad (11.6)$$

Take a frame in which $q = (q, 0)$. $\Gamma$ is given by (11.5), and

$$\Lambda = -\frac{N}{2\pi} \begin{pmatrix}
0 & 0 & iq \\
0 & 0 & i \omega \\
-q & -i \omega & 0
\end{pmatrix} + \frac{q^2}{\alpha} \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}. \quad (11.7)$$
Combining (11.5) and (11.7), one finds
\[
\Lambda + \Gamma = \begin{pmatrix}
q^2\Pi_0 & \omega q \Pi_0 & +iq\bar{\Pi}_1 \\
\omega q \Pi_0 & \omega^2 \Pi_0 + \alpha^{-1}q^2 & +i\omega \bar{\Pi}_1 \\
-iq\bar{\Pi}_1 & -i\omega \bar{\Pi}_1 & \bar{\Pi}_2
\end{pmatrix},
\]
(11.8)
\[
\bar{\Pi}_1 = \Pi_1 - \frac{N}{2\pi},
\]
\[
\bar{\Pi}_2 = \omega^2 \Pi_0 - q^2 \Pi_2.
\]
We need to evaluate \((\Lambda + \Gamma)^{-1}\). First we note
\[
det(\Lambda + \Gamma) = \frac{q^4}{\alpha}(\Pi_0 \bar{\Pi}_2 - \bar{\Pi}_1^2).
\]
(11.9)
A straightforward manipulation leads to
\[
(\Lambda + \Gamma)^{-1} = \frac{1}{q^2(\Pi_0 \bar{\Pi}_2 - \bar{\Pi}_1^2)} \begin{pmatrix}
\bar{\Pi}_2 & 0 & -iq\bar{\Pi}_1 \\
0 & 0 & 0 \\
+iq\Pi_1 & 0 & q^2\Pi_0
\end{pmatrix} + \frac{\alpha}{q^4} \begin{pmatrix}
\omega^2 & -\omega q & 0 \\
-\omega q & q^2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
(11.10)
Both \(\Lambda\) and \((\Lambda + \Gamma)^{-1}\) depend on \(\alpha\), but the product \(\Lambda (\Lambda + \Gamma)^{-1}\) or \((\Lambda + \Gamma)^{-1}\Lambda\) is independent of \(\alpha\). (We shall see shortly that the current conservation guarantees the \(\alpha\)-independence.) \(Q_n\) is given by
\[
Q_n = \left(\frac{N}{2\pi}\right)^2 \frac{1}{\Pi_0 \Pi_2 - \bar{\Pi}_1^2} \begin{pmatrix}
q^2\Pi_0 & \omega q \Pi_0 & -iq\bar{\Pi}_1 \\
\omega q \Pi_0 & \omega^2 \Pi_0 & -i\omega \bar{\Pi}_1 \\
+iq\bar{\Pi}_1 & +i\omega \bar{\Pi}_1 & \bar{\Pi}_2
\end{pmatrix}
\]
(11.11)
\[
- \frac{N}{2\pi} \begin{pmatrix}
0 & 0 & -iq \\
0 & 0 & -i\omega \\
+iq & +i\omega & 0
\end{pmatrix}.
\]
This expression was first given by Aronov and Mirlin.\(^{45}\)

For a charged fluid it is easiest and most convenient to evaluate \(Q_c\) in the form (10.20):
\[
Q_c = -\Gamma (1 + \Lambda_c^{-1})^{-1} = -(1 + \Gamma \Lambda_c^{-1})^{-1} \Gamma.
\]
(11.12)
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Here $P^{-1}$ is the propagator for electromagnetic fields. In the two-dimensional approximation it is given by (10.17). Its form in the ultra-thin film approximation can be deduced from (5.1).

\[
\frac{e^2}{P} = \begin{cases} 
\frac{e^2_{3D}}{q} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{(ultra-thin film approx.)} \\
\frac{e^2}{q^2 - \omega^2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \text{(two-dimensional approx.)}
\end{cases}
\]  

(11.13)

Here $e_{3D}$ is the coupling constant in three dimensions.

We evaluate $Q_c$ in the two-dimensional approximation. We introduce

\[
A = \frac{e^2}{q^2 - \omega^2} , \quad B = \frac{2\pi}{N} \frac{1}{q} .
\]

(11.14)

Then

\[
\Lambda_c^{-1} = \begin{pmatrix} A & 0 & -iB \\ 0 & -A & 0 \\ iB & 0 & -A \end{pmatrix} + \frac{\alpha}{q^4} \begin{pmatrix} \omega^2 & -\omega q & 0 \\ -\omega q & q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} .
\]

(11.15)

The substitution of (11.5) and (11.15) into (11.12) immediately confirms that the $\alpha$-dependence entirely drops in $Q_c$ (or $Q_n$), since both ($\Gamma^\mu$, $\Gamma^{1\nu}$) and ($\Gamma^{0\mu}$, $\Gamma^{1\mu}$) are proportional to ($q$, $\omega$). It is a consequence of the current conservation (11.2).

The computation of $Q_c$ is rather involved. We record intermediate steps for readers’ convenience. $\Lambda_c^{-1}\Gamma$ is found to be

\[
\Lambda_c^{-1}\Gamma = \begin{pmatrix} q(qA\Pi_0 - B\Pi_1) & \omega(qA\Pi_0 - B\Pi_1) & i(qA\Pi_1 - B\Pi_2) \\ -\omega qA\Pi_0 & -\omega^2 A\Pi_0 & -i\omega A\Pi_1 \\ iq(qB\Pi_0 + A\Pi_1) & i\omega(qB\Pi_0 + A\Pi_1) & -qB\Pi_1 - A\Pi_2 \end{pmatrix} .
\]

(11.16)

Further the determinant of $(1 + \Lambda_c^{-1}\Gamma)$ is evaluated to be

\[
\Delta_c \equiv \text{det} \ (1 + \Lambda_c^{-1}\Gamma)
\]

\[
= 1 + (q^2 - \omega^2)A\Pi_0 - 2qB\Pi_1 - A\Pi_2 \\
+ \{q^2(A^2 + B^2) - \omega^2A^2\}(\Pi_1^2 - \Pi_0\Pi_2)
\]

\[
= \left(\frac{2\pi}{N}\right)^2 (\Pi_1^2 - \Pi_0\Pi_2) + e^2\Pi_0 - \frac{e^2}{q^2 - \omega^2}\Pi_2 + \frac{e^4}{q^2 - \omega^2}(\Pi_1^2 - \Pi_0\Pi_2)
\]

(11.17)
Straightforward manipulations lead to

\[
(1 + \Lambda^{-1}_{c} \Gamma)^{-1} = \frac{1}{\Delta_{c}} \cdot (1 + \hat{M}_{1} + \hat{M}_{2})
\]

\[
\hat{M}_{1} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-iq^{2}B & -i\omega qB & (q^{2} - \omega^{2})A
\end{pmatrix} \Pi_{0} \\
+ \begin{pmatrix}
-qB & 0 & -iqA \\
0 & -qB & +i\omega A \\
-iqA & -i\omega A & -qB
\end{pmatrix} \Pi_{1} + \begin{pmatrix}
-A & 0 & iB \\
0 & -A & 0 \\
0 & 0 & 0
\end{pmatrix} \Pi_{2}
\]

\[
\hat{M}_{2} = \begin{pmatrix}
-\omega^{2} & -\omega q & 0 \\
\omega q & q^{2} & 0 \\
0 & 0 & 0
\end{pmatrix} A\Pi_{0} + \begin{pmatrix}
0 & \omega & 0 \\
0 & -q & 0 \\
0 & 0 & 0
\end{pmatrix} B\Pi_{1} \\
+ \begin{pmatrix}
-\omega^{2}A^{2} & -\omega q(A^{2} + B^{2}) & i\omega^{2}AB \\
\omega qA^{2} & q^{2}(A^{2} + B^{2}) & -i\omega qAB \\
0 & 0 & 0
\end{pmatrix} (\Pi_{1} - \Pi_{0}\Pi_{2})
\]

In computing \( \Gamma (1 + \Lambda^{-1}_{c} \Gamma)^{-1} \), the contribution from the \( \hat{M}_{2} \) part vanishes thanks to the current conservation \( \Gamma \hat{M}_{2} = 0 \). The final result takes a simple form.

\[
Q_{c} = -\frac{1}{\Delta_{c}} \left\{ \Gamma + \begin{pmatrix}
q^{2}A & \omega qA & -iq^{2}B \\
\omega qA & \omega^{2}A & -i\omega qB \\
iq^{2}B & i\omega qB & (\omega^{2} - q^{2})A
\end{pmatrix} \left( \Pi_{1}^{2} - \Pi_{0}\Pi_{2} \right) \right\}
\]

\[
= -\frac{1}{\Delta_{c}} \left\{ \Gamma + \begin{pmatrix}
\frac{e^{2}}{q^{2} - \omega^{2}} & q^{2} & \omega q & 0 \\
\omega q & \omega^{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} - e^{2} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \right\}
\]

\[
+ \frac{2\pi}{N} \begin{pmatrix}
0 & 0 & -iq \\
0 & 0 & -i\omega \\
iq & i\omega & 0
\end{pmatrix} \left( \Pi_{1}^{2} - \Pi_{0}\Pi_{2} \right) \right\}
\]

(11.19)

It is easy to check that the formula (11.19) reduces to (11.11) in the neutral limit \( e^{2} \to 0 \). Indeed one sees that

\[
\Delta_{c} \to \Delta_{n} = \left( \frac{2\pi}{N} \right)^{2} (\Pi_{1}^{2} - \Pi_{0}\Pi_{2})
\]

(11.20)

As we shall see, the difference between \( Q_{n} \) and \( Q_{c} \) is important in discussing superconductivity.
12. Evaluation of the kernel

We evaluate the kernel \( \Gamma_{\mu\nu}(\omega, q) \) defined in (10.7), or equivalently the invariant functions \( \Pi_a \)'s in (11.4). We need to evaluate the four diagrams (b), (c), (d), and (e) in Fig. 8. For \( q = (q, 0) \) we have

\[
q^2 \Pi_0 = \Gamma^{00}(\omega, q) = \Gamma^{(c)00}(\omega, q)
\]
\[
iq \Pi_1 = \Gamma^{02}(\omega, q) = \Gamma^{(d)02}(\omega, q)
\]
\[
\Pi_2 = \Gamma^{22}(\omega, q) = \Gamma^{(b,e)22}(\omega, q)
\]  

(12.1)

The diagram (b) has been already evaluated in (10.12):

\[
\Gamma^{(b)jk}(\omega, q) = -\frac{ne}{m} \delta^{jk} .
\]  

(12.2)

To evaluate other diagrams, we first examine the zeroth order propagator

\[
G(x, y) = -i \langle T[\psi(x)\psi^\dagger(y)] \rangle \left( i \frac{\partial}{\partial x_0} + \frac{1}{2m} \bar{D}_k \right) G(x, y) = \delta^3(x - y) .
\]  

(12.3)

Here the expectation value is taken with the ground state (6.13) and \( \psi(x) \) satisfies the mean field equation.

More explicitly

\[
G(x, y) = -i \left\{ \theta(x_0 - y_0) \sum_{\alpha \notin G} -\theta(y_0 - x_0) \sum_{\alpha \in G} \right\} u_\alpha(x) u_\alpha(y)^* e^{-i\epsilon_\alpha(x_0 - y_0)}
\]  

(12.4)

where \( u_\alpha(x) \) is given by either (6.6) or (6.8). In the Landau gauge

\[
G(x, y) = -i \left\{ \theta(x_0 - y_0) \sum_{n = |N|}^{\infty} -\theta(y_0 - x_0) \sum_{n = 0}^{\left|N\right|-1} \right\}
\]
\[
\times \frac{1}{2l_1} \sum_p e^{-2\pi ip(x_1 - y_1)/L_1} v_n \left( \frac{x_2 - \bar{x}_2}{l} \right) v_n \left( \frac{y_2 - \bar{y}_2}{l} \right)
\]  

(12.5)

\[
= e^{i\phi(x, y)} \cdot G_0(x - y)
\]

Here

\[
\phi(x, y) = -\epsilon(N) \frac{1}{2l_2} (x_1 - y_1)(x_2 + y_2)
\]
\[
G_0(x) = -i \left\{ \theta(x_0) \sum_{n = |N|}^{\infty} -\theta(-x_0) \sum_{n = 0}^{\left|N\right|-1} \right\} e^{-i\epsilon_n x_0}
\]
\[
\times \frac{1}{2\pi l_2} \int_{-\infty}^{\infty} dz e^{-izx_1/l} v_n[z - \bar{z}(x_2)] v_n[z + \bar{z}(x_2)]
\]  

(12.6)

\[
\bar{z}(x_2) = \epsilon(N) \frac{x_2}{2l} .
\]
We have introduced a new integration variable $z = (2\pi pl/L_1) - \epsilon(N)(x_2 + y_2)/2l$, taking the limit $L_1 \to \infty$. It should be noticed that the Green’s function $G(x, y)$ is not manifestly translation invariant, but is invariant up to a gauge transformation, due to the presence of a non-vanishing magnetic field. The Fourier transform of $G_0(x) = G_0(t, x)$ is given by

$$G_0(p) = G_0(\omega, p) = \int dt dx \ G_0(t, x) e^{i(\omega t - px)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{\omega - \epsilon_n + i\delta(n)} \int \frac{dx_2}{l} e^{-ip_2 x_2} v_n[p_1 l + \bar{z}(x_2)] v_n[p_1 l - \bar{z}(x_2)]$$

where $i\delta(n) = \begin{cases} +i & \text{for } n \geq |N| \\ -i & \text{for } n < |N| \end{cases}$.

We come back to evaluating the remaining diagrams. Recalling $\mathcal{L}_{int}^{(f)}$ in (10.4), we see that the diagram (c) in Fig. 8 yields

$$\frac{i^2}{2i} \int dx dy \ G(x, y) [-a_0(y)] G(y, x) [-a_0(x)]$$

$$= \frac{i}{2} \int dx dy \ a_0(x) G_0(x - y) G_0(y - x) a_0(y)$$

so that

$$\Gamma^{(c)00}(q) = i \int \frac{d^3p}{(2\pi)^3} G_0(p) G_0(p - q)$$.

The diagram (d) in Fig. 8 yields

$$\frac{1}{2} \int dx dy \ a_0(x) \Gamma^{(d)ij}(x, y) a^{(1)j}(y)$$

$$= \frac{i^2}{2i} \int dx dy \ [-a_0(x)] \left[ -\frac{i}{2m} a^{(1)j}(y) \right]$$

$$\times \left( G(x, y) \cdot [\partial^y_j - i\bar{\alpha}^j(y)] G(y, x) - [\partial^y_j + i\bar{\alpha}^j(y)] G(x, y) \cdot G(y, x) \right)$$.

With the aid of

$$[\partial^x_j - i\bar{\alpha}^j(x)] G(x, y) = +e^{i\phi(x,y)} D^-_j G_0(x - y)$$

$$[\partial^y_j + i\bar{\alpha}^j(y)] G(x, y) = -e^{i\phi(x,y)} D^+_j G_0(x - y)$$

$$D^\pm_0 G_0(x - y) = \left( \partial^x_j \mp i\epsilon(N) e^{ik \frac{xk - yk}{2l^2}} \right) G_0(x - y)$$

eq. (12.10) becomes

$$- \frac{1}{4m} \int dx dy \ a_0(x) a^{(1)j}(y)$$

$$\times \left\{ G_0(x - y) \cdot D^-_j G_0(y - x) + D^+_j G_0(x - y) \cdot G_0(y - x) \right\}$$.

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Further, since \( \omega \) encounter an integral to find \( \Pi_0 \) and therefore

\[
\Gamma^{(d)0j}(q) = -\frac{1}{2m} \int \frac{d^3p}{(2\pi)^3} \left\{ G_0(p) \cdot D_j^- G_0(p - q) + D_j^+ G_0(p) \cdot G_0(p - q) \right\} \tag{12.13}
\]

Similar manipulations lead to

\[
\frac{1}{2} \int dx dy \; a^{(1)j}(x) \Gamma^{(e)jk}(x, y) a^{(1)k}(y) = -\frac{i}{8m^2} \int dx dy \; a^{(1)j}(x) a^{(1)k}(y) \left\{ D_k^- G_0(y - x) \cdot D_j^- G_0(x - y) + D_j^+ G_0(y - x) \cdot D_k^+ G_0(x - y) + D_j^+ D_k^- G_0(y - x) \cdot G_0(x - y) + G_0(y - x) \cdot D_j^- D_k^+ G_0(x - y) \right\},
\]

and therefore

\[
\Gamma^{(e)jk}(q) = -\frac{i}{4m^2} \int \frac{d^3p}{(2\pi)^3} \left\{ D_k^- G_0(p) \cdot D_j^- G_0(p - q) + D_j^+ G_0(p) \cdot D_k^+ G_0(p - q) + D_j^+ D_k^- G_0(p) \cdot G_0(p - q) + G_0(p) \cdot D_j^- D_k^+ G_0(p - q) \right\}. \tag{12.15}
\]

We need to evaluate \( \Gamma^{(e)00}(q) \), \( \Gamma^{(d)02}(q) \), and \( \Gamma^{(e)22}(q) \) in the frame \( q = (q, 0) \) to find \( \Pi_0 \), \( \Pi_1 \), and \( \Pi_2 \). Upon inserting (12.7) into (12.9), (12.13), or (12.15), we encounter an \( \omega' \)-integral

\[
\frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \epsilon_n + i\delta(n)} \frac{1}{\omega' - \omega - \epsilon_m + i\delta(m)}
\]

\[
=\begin{cases}
\frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} & \text{for } n \geq |N|, m < |N| \\
\frac{1}{\epsilon_m - \epsilon_n + \omega - i\epsilon} & \text{for } n < |N|, m \geq |N| \\
0 & \text{otherwise}.
\end{cases} \tag{12.16}
\]

Further, since

\[
D_j^\pm G_0(p) = (ip_2 \pm \epsilon(N) \frac{1}{2l^2} \frac{\partial}{\partial p_1}) G_0(p)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{\omega - \epsilon_n + i\delta(n)} \int \frac{dx_2}{l} e^{-ip_2x_2} \times \left( \frac{\partial}{\partial x_2} \pm \epsilon(N) \frac{1}{2l^2} \frac{\partial}{\partial p_1} \right) \left\{ v_n(p_1l + \vec{z}) v_n(p_1l - \vec{z}) \right\},
\]

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Note that \( \Gamma^{(d)0j}(x, y) \) is a function of \( x - y \) only. Hence, in the Fourier space
one finds that

\[
D_2^\pm G_0(p) = \frac{\epsilon(N)}{l} \sum_{n=0}^{\infty} \frac{1}{\omega - \epsilon_n + i\delta(n)} \int \frac{dx_2}{l} e^{-ipz_2} \times \left\{ v_n^{(1)}(p_1l + \bar{z}) v_n(p_1l - \bar{z}) - v_n(p_1l + \bar{z}) v_n^{(1)}(p_1l - \bar{z}) \right\}.
\]

where \(v_n^{(p)}(z)\) is the \(p\)-derivative of \(v_n(z)\). Another integral which frequently appears is

\[
C_{nm}^{(p)}(a) \equiv \int_{-\infty}^{\infty} dx v_n^{(p)}(x) v_m(x - a) = (-1)^p \int_{-\infty}^{\infty} dx v_n(x) v_n^{(p)}(x - a) \ .
\]

It satisfies that

\[
C_{nm}^{(p)}(a) = (-1)^{n+m} C_{nm}^{(p)}(a) \\
C_{nm}^{(p)}(-a) = (-1)^{p+n+m} C_{nm}^{(p)}(a) .
\]

We start with computing \(\Pi_0\):

\[
q^2 \Pi_0 = \Gamma^{(c)00}(\omega, q, 0)
\]

\[
= i \int \frac{d\omega'd\mathbf{p}}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{1}{\omega' - \epsilon_n + i\delta(n)} \int \frac{dx}{l} e^{-ipz} v_n[p_1l + \bar{z}(x)] v_n[p_1l - \bar{z}(x)]
\]

\[
\times \sum_{m=0}^{\infty} \frac{1}{\omega' - \omega - \epsilon_m + i\delta(m)} \int \frac{dy}{l} e^{-ipz} v_m[p_1l - q\bar{z}(y)] v_m[p_1l - q\bar{z}(y)]
\]

\[
= \frac{1}{2\pi l^2} \left\{ \sum_{n=|N|}^{1-N-1} \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} + \sum_{m=|N|}^{1-N-1} \frac{1}{\epsilon_m - \epsilon_n - \omega - i\epsilon} \right\}
\]

\[
\times C_{nm}^{(0)}(ql)C_{nm}^{(0)}(ql).
\]

Therefore

\[
q^2 \Pi_0 = \frac{1}{2\pi l^2} \sum_{n=|N|}^{1-N-1} \sum_{m=0}^{1-N-1} \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon + \epsilon_n - \epsilon_m + \omega - i\epsilon} C_{nm}^{(0)}(ql)^2 .
\]

Evaluation of \(\Pi_1\) proceeds similarly. In view of of (12.13) and (12.17), one needs to make small modifications to (12.21)

\[
\times \frac{i \epsilon(N)}{2m l} C_{nm}^{(0)}(ql)^2 \Rightarrow 2C_{nm}^{(1)}(ql)C_{nm}^{(0)}(ql) .
\]
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to find

\[ q \Pi_1 = \frac{\epsilon(N)}{2\pi m l^3} \sum_{n=|N|}^{\infty} \sum_{m=0}^{|N|-1} \left\{ \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} + \frac{1}{\epsilon_n - \epsilon_m + \omega + i\epsilon} \right\} C_{nm}^{(1)}(ql) C_{nm}^{(0)}(ql). \]  \hfill (12.23)

To find \( \Pi_2 \), we first note that partial integrations in (12.15) lead to

\[ \Gamma^{(e)22}(\omega, q, 0) = -\frac{i}{m^2} \int \frac{d^3 p}{(2\pi)^3} D_2^- G_0(p) \cdot D_2^- G_0(p - q). \]  \hfill (12.24)

This time we modify (12.21) such that

\[ \times \left( -\frac{1}{m^2} \right) \left( \frac{\epsilon(N)}{l} \right)^2 \]  \hfill (12.25)

\[ C_{nm}^{(0)}(ql)^2 \implies - C_{nm}^{(1)}(ql)^2. \]

Since \( \Pi_2 = \Gamma^{(b)22}(\omega, q, 0) + \Gamma^{(e)22}(\omega, q, 0) \), we find that

\[ \Pi_2 = -\frac{n_e}{m} \]  \hfill (12.26)

\[ + \frac{1}{2\pi m^2 l^3} \sum_{n=|N|}^{\infty} \sum_{m=0}^{|N|-1} \left\{ \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} + \frac{1}{\epsilon_n - \epsilon_m + \omega + i\epsilon} \right\} C_{nm}^{(1)}(ql)^2. \]

(12.21), (12.23), and (12.26) are the results in the RPA and linearized SCF. These \( \Pi_k \)'s, through the formulas (10.15), (10.21), (11.11), and (11.19), determine the response to external perturbations. As is evident from the discussions in Sections 10 and 11, it describes a response to harmonic perturbations \( a_{ext}(x) \propto e^{ipx - i\omega t} \) introduced from \( t = -\infty \) to \( t = +\infty \). One may introduce a perturbation adiabatically from \( t = -\infty \) to the present (but not to \( t = +\infty \)). In this case the response function is related to the retarded, but not time-ordered, Green’s function of currents. This amounts to making a change \( \epsilon_n - \epsilon_m - \omega - i\epsilon \)

\[ \Rightarrow \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} + \frac{1}{\epsilon_n - \epsilon_m + \omega + i\epsilon} \]  \hfill (12.27)

in all formulas. Hence, for instance,

\[ q^2 \Pi_0^R = \frac{1}{2\pi l^2} \sum_{n=|N|}^{\infty} \sum_{m=0}^{|N|-1} \left\{ \frac{1}{\epsilon_n - \epsilon_m - \omega - i\epsilon} + \frac{1}{\epsilon_n - \epsilon_m + \omega + i\epsilon} \right\} C_{nm}^{(0)}(ql)^2. \]  \hfill (12.28)
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The $x$-integral in $C_{nm}^{(p)}(a)$, (12.18), can be done to yield

\[
C_{nm}^{(0)}(a) = \left( \frac{m!}{n!} \right)^{1/2} \left( \frac{a}{\sqrt{2}} \right)^{n-m} e^{-a^2/4} L_m^{-m}(a^2) \quad (n \geq m)
\]

\[
L_m^{-m}(z) = \frac{1}{m!} z^{m-n} e^z \frac{d^m}{dz^m} (z^n e^{-z})
\] (12.29)

\[
C_{nm}^{(1)}(a) = \frac{1}{\sqrt{2}} \left\{ -\sqrt{n+1} C_{n+1,m}^{(0)}(a) + \sqrt{n} C_{n-1,m}^{(0)}(a) \right\} \quad \text{etc.}
\]

$L_n^\alpha(x)$ is the Laguerre polynomial. This expression is useful to investigate the response function at finite $ql$.

For a small momentum $ql \ll 1$, one can expand $v_m(x-ql)$ in a Taylor series to find

\[
C_{nm}^{(p)}(a) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} d_{nm}^{k+p} a^k, \quad d_{nm}^k = \int_{-\infty}^{\infty} dx \, v_n(x) v_m^{(k)}(x),
\] (12.30)

where the coefficients $d_{nm}^k$’s are given by

\[
d_{nm}^0 = \delta_{n,m}
\]

\[
d_{nm}^1 = \frac{1}{\sqrt{2}} \left\{ \sqrt{m} \delta_{n,m-1} - \sqrt{m+1} \delta_{n,m+1} \right\}
\]

\[
d_{nm}^2 = \frac{1}{2} \left\{ \sqrt{m(m-1)} \delta_{n,m-2} - (2m+1) \delta_{n,m} + \sqrt{(m+1)(m+2)} \delta_{n,m+2} \right\}
\]

\[
d_{nm}^3 = \frac{1}{2\sqrt{2}} \left\{ \sqrt{m(m-1)(m-2)} \delta_{n,m-3} - 3m^{3/2} \delta_{n,m-1} + 3(m+1)^{3/2} \delta_{n,m+1} - \sqrt{(m+1)(m+2)(m+3)} \delta_{n,m+3} \right\}
\]

\[
d_{nm}^4 = \frac{1}{4} \left\{ \sqrt{m \cdots (m-3)} \delta_{n,m-4} - 2(2m-1) \sqrt{m(m-1)} \delta_{n,m-2} + 3(2m^2 + 2m + 1) \delta_{n,m} - 2(2m + 3) \sqrt{(m+1)(m+2)} \delta_{n,m+2} + \sqrt{(m+1) \cdots (m+4)} \delta_{n,m+4} \right\}
\]

\[
d_{nm}^5 = \frac{1}{4\sqrt{2}} \left\{ \sqrt{m \cdots (m-4)} \delta_{n,m-5} - 5(m-1)^{3/2} \sqrt{m(m-2)} \delta_{n,m-3} + 5(2m^2 + 1) \sqrt{m} \delta_{n,m-1} - 5(2m^2 + 4m + 3) \sqrt{m+1} \delta_{n,m+1} + 5(m+2)^{3/2} \sqrt{(m+1)(m+3)} \delta_{n,m+3} - \sqrt{(m+1) \cdots (m+5)} \delta_{n,m+5} \right\}
\]

\[
\cdots
\] (12.31)
In particular, for \( n > m \),

\[
\begin{align*}
C_{nm}^{(0)}(a)^2 &= a^2 \frac{n^2}{2} \delta_{n,m+1} + a^4 \left\{ - \frac{n^2}{4} \delta_{n,m+1} + \frac{n(n-1)}{16} \delta_{n,m+2} \right\} + \cdots, \\
C_{nm}^{(0)}(a) C_{nm}^{(1)}(a) &= a \frac{n^2}{2} \delta_{n,m+1} + a^3 \left\{ - \frac{n^2}{2} \delta_{n,m+1} + \frac{n(n-1)}{8} \delta_{n,m+2} \right\} + \cdots, \\
C_{nm}^{(1)}(a)^2 &= \frac{n}{2} \delta_{n,m+1} + a^2 \left\{ - \frac{3n^2}{4} \delta_{n,m+1} + \frac{n(n-1)}{4} \delta_{n,m+2} \right\} \\
&+ a^4 \left\{ \frac{n(37n^2 + 5)}{96} \delta_{n,m+1} - \frac{n(n-1)(2n-1)}{12} \delta_{n,m+2} \\
&+ \frac{n(n-1)(n-2)}{32} \delta_{n,m+3} \right\} + \cdots.
\end{align*}
\]

(12.32)

Applying (12.32) in (12.21), (12.23), and (12.26), one finds that for small frequency and momentum

\[
\begin{align*}
\Pi_0 &= \left( \frac{N}{2\pi} \right)^2 m \frac{\omega_c^2}{n_c} \left\{ 1 + \left( \frac{\omega}{\omega_c} \right)^2 \right\} - \frac{3}{8} |N| (ql)^2 + \cdots \\
\Pi_1 &= \frac{N}{2\pi} \left\{ 1 + \left( \frac{\omega}{\omega_c} \right)^2 \right\} - \frac{3}{4} |N| (ql)^2 + \cdots \\
\Pi_2 &= \frac{n_c}{m} \left\{ \left( \frac{\omega}{\omega_c} \right)^2 - |N| (ql)^2 + \cdots \right\} \\
l^2 &= \frac{|N|}{2\pi n_c} \omega_c^2, \quad \omega_c = \frac{1}{ml^2} = \frac{2\pi n_c}{|N|m}.
\end{align*}
\]

(12.33)

We also note that

\[
\begin{align*}
\Pi_2 &= \frac{1}{q^2} (\omega^2 \Pi_0 - \Pi_2) = \frac{N^2}{2\pi m} + \cdots.
\end{align*}
\]

(12.34)

There are two notable cancellations in the above formulas. First, in \( \Pi_1 \), the dominant term is exactly \( N/2\pi \) so that \( \Pi_1 = \Pi_1 - (N/2\pi) \) vanishes at \( q = \omega = 0 \). This fact is phrased in the literature that the bare Chern-Simons term is exactly cancelled by the one-loop correction. Secondly, \( \Pi_2 \) vanishes at \( q = \omega = 0 \). In other words, the first term (diagram (b)) in (12.26) is cancelled by the second term (diagram (e)). We have mentioned about it just below eq. (10.12).

\( \Pi(\omega, q = 0)'s \) can be evaluated in a closed form. It is straightforward to find

\[
\begin{align*}
\Pi_0(\omega, q = 0) &= \frac{|N|}{2\pi} \frac{\omega_c}{\omega_c^2 - \omega^2} \\
\Pi_1(\omega, q = 0) &= \frac{N}{2\pi} \frac{\omega_c^2}{\omega_c^2 - \omega^2} \\
\Pi_1(\omega, q = 0) &= \frac{N}{2\pi} \frac{\omega^2}{\omega_c^2 - \omega^2} \\
\Pi_2(\omega, q = 0) &= \frac{|N|}{2\pi} \frac{\omega_c \omega^2}{\omega_c^2 - \omega^2}.
\end{align*}
\]

(12.35)
Yutaka Hosotani

In the literature these kernels appear in different notations. For the sake of readers’ convenience we have summarized them in Table 1 below.

Table 1. The comparison of various references. Relations among $\Pi_k$’s, $\Gamma$, $Q_n$, and $Q_c$ are given by (11.4), (11.11), and (11.19). At $T \neq 0$, the notation $\Pi_k^E$ and $\Gamma_k^{\mu\nu}$ has been adopted in this article and in ref. 38.

| This article | ref. 27 | ref. 29 | ref. 49 | ref. 32 | ref. 44 | ref. 38 |
|--------------|--------|--------|-------|--------|--------|--------|
| $T = 0$      | $T = 0$| $T = 0$| $T \neq 0$| $T = 0$| $T \neq 0$| $T = 0$| $T \neq 0$|
| $\omega \neq 0$| $\omega \neq 0$| $\omega \neq 0$| $\omega = 0$| $\omega = 0$| $\omega \neq 0$| $\omega = 0$|

| $\Pi_0$ | $-\left(\frac{N}{2\pi}\right)^2 \frac{m}{n_e} \Sigma_0$ | $\Pi_0$ |
| $\Pi_1$ | $-\frac{N}{2\pi} \Sigma_1$ | $\Pi_1$ |
| $\Pi_2$ | $-\frac{n_e}{m} (1 + \Sigma_2)$ | $-\Pi_2$ |
| $\Gamma^{\mu\nu}$ | $-D_0^{\mu\nu} + \frac{n_e}{m} \delta^{\mu\nu}(1 - \delta^{\mu\nu})$ | $K^{\mu\nu}$ |
| $Q_n^{\mu\nu}$ | $e^{-2} K^{\mu\nu}$ | $e^{-2} R_n^{\mu\nu}$ |
| $Q_c^{\mu\nu}$ | $P_c^{\mu\nu}$ |

13. Phonons and plasmons

It follows from (10.11) or (10.21) that the location of poles of the response function $Q_n$ or $Q_c$ in the Fourier space determines the spectrum of excitations which couple to currents $J^\mu$. In SCF they appear as self-consistent configurations in the absence of external fields. Indeed,

$$Q_n^{-1} J_{\text{ind}} = a_{\text{ext}} = 0$$

or

$$Q_c^{-1} J_{\text{ind}} = a_{\text{ext}} = 0 \quad (13.1)$$

has a non-trivial solution ($J_{\text{ind}} \neq 0$) only at $(\omega, \mathbf{q})$ for which $\det Q_n^{-1} = 0$ or $\det Q_c^{-1} = 0$. 

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From (11.11), (11.17), (11.19), and (11.20) one finds that the location of poles are determined by

$$\Delta_n = \left(\frac{2\pi}{N}\right)^2 (\Pi_1^2 - \Pi_0 \Pi_2) = 0 \quad (13.2)$$

for a neutral anyon fluid, and by

$$\Delta_c = \left(\frac{2\pi}{N}\right)^2 (\Pi_1^2 - \Pi_0 \Pi_2) + e^2 \Pi_0 - \frac{e^2}{q^2 - \omega^2} \Pi_2 + \frac{e^4}{q^2 - \omega^2} (\Pi_1^2 - \Pi_0 \Pi_2) = 0 \quad (13.3)$$

for a charged anyon fluid.

Solving (13.2) for a small momentum, Fetter, Hanna, and Laughlin first showed\(^{27}\) that a neutral anyon fluid admits a phonon excitation and is a superfluid. Later the equation was numerically solved for a finite momentum in refs. (51) and (55). The spectrum in the charged case has been examined in ref. (58).

In this article we confine ourselves to the spectra at small momenta \(ql \ll 1\). For a neutral fluid one can employ the formula (12.33), as is justified a posteriori. Since

$$\Pi_0 = O(1)$$
$$\Pi_1 = \Pi_1 - \frac{N}{2\pi} = O(\omega^2, q^2)$$
$$\Pi_2 = O(\omega^2, q^2) \quad ,$$

Eq. (13.2) is solved for a small momentum by \(\Pi_2 = 0\), or

$$\omega^2 = c_s^2 q^2$$
$$c_s = \sqrt{|N| \omega_c} = \hbar \frac{\sqrt{2\pi N e}}{m} . \quad (13.4)$$

We have recovered \(\hbar\) in the last relation. It is a phonon excitation. The velocity \(c_s\) does not depend on \(N\) in this approximation (RPA, linearized SCF).

In the charged case \(\omega\) approaches a finite value as \(q \rightarrow 0\). To find \(\omega(q = 0)\), we insert (12.35) into (13.3). Many cancellations take place. One finds, at \(q = 0\),

$$\left(\frac{2\pi}{N}\right)^2 (\Pi_1^2 - \Pi_0 \Pi_2) = -\frac{\omega^2}{\omega_c^2 - \omega^2}$$
$$e^2 \Pi_0 + \frac{e^2}{\omega^2} \Pi_2 = 2 \frac{|N| e^2}{2\pi} \frac{\omega_c}{\omega_c^2 - \omega^2}$$
$$- \frac{e^4}{\omega^2} (\Pi_1^2 - \Pi_0 \Pi_2) = -\left(\frac{N e^2}{2\pi}\right)^2 \frac{\omega_c^2}{\omega^2(\omega_c^2 - \omega^2)}$$

Hence Eq. (13.3) reduces to a polynomial equation for \(\omega^2\):

$$\left(\frac{\omega^2}{\omega_c^2} - \frac{|N| e^2}{2\pi \omega_c}\right)^2 = 0 . \quad (13.5)$$
In other words, the dispersion relation is

\[ \omega(q = 0) = \sqrt{\frac{|N|e^2\omega_c}{2\pi}} = \sqrt{\frac{e^2n_e}{m}}. \] (13.6)

This is nothing but a plasmon, representing a plasma oscillation. There is only one solution for \( \omega \). [The two solutions in ref. (38) are the result of \( \Pi_k(\omega, 0) \)'s being approximated by \( \Pi_k(0, 0) \)'s.]

### 14. Hydrodynamic description

In the preceding sections we have integrated the matter field first, to obtain the effective theory for the gauge fields. The kernel \( \Gamma^{\mu\nu} \) which appears in the effective action (10.2) or (10.6) has an important physical meaning. It appears in (10.14) as the coefficient relating gauge field configurations to the induced current. In SCF, for a given gauge field configuration \( a_{\mu}(x) \) (in the neutral case),

\[
J^{\mu}(x) = \langle j^{\mu}(x) \rangle \\
= n_e \delta^{\mu 0} - \int dy \, \Gamma^{\mu \nu}(x-y) a^{(1)}_{\nu}(y) + \cdots,
\] (14.1)

with which the field equation

\[
-\frac{N}{4\pi} \varepsilon^{\mu \nu \rho} f_{\nu \rho} = J^{\mu}(x)
\] (14.2)

has been solved self-consistently.

One can read eq. (14.2) differently. It says that the Chern-Simons field strengths are nothing but the currents. The roles of the two equations (14.1) and (14.2) are interchanged. Now the latter gives a relation, while the former yields equations for the currents \( J^{\mu}(x) \).

The time component is the density field \( J^0(x) = n(x) \), while the spatial components define the velocity fields \( v(x) \) by \( J^k = n v^k \). The equations for \( n(x) \) and \( v(x) \) give the hydrodynamic description of the system. This is the viewpoint originally adopted by Wen and Zee. The advantage of this approach is that everything is expressed in terms of macroscopic physical quantities so that physics is clearest. In particular, as we shall see, it gives an interpretation of a phonon excitation as a breathing mode of a density wave, the picture first spelled out by Wen and Zee. Anyon fluids are unique systems in which the “microscopic” RPA or SCF is equivalent to the “macroscopic” hydrodynamic description.

Substitution of (11.4) into eq. (14.1) yields, in the linear approximation which drops \( O((J_{\text{ind}})^2) \),

\[
J^0 = n_e - i g_k f_{0k} \Pi_0 + b^{(1)}_1 \Pi_1, \\
J^k = -i \omega f_{0k} \Pi_0 + \epsilon^{kl} f_{0l} \Pi_1 - i \epsilon^{kl} q_l b^{(1)}_1 \Pi_2,
\] (14.3)
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where $b^{(1)} = -f^{(1)}_{12}$. At $T = 0$ all $\Pi(\omega, q)$’s have finite limits at $\omega = q = 0$ so that the right sides of (14.3) are expressed solely in terms of the field strengths $f_{\mu\nu}$. Hence, with the aid of the fundamental identity (14.2), eq. (14.1) gives a set of differential equations for $J^\mu(x)$. We remark that at finite temperature $\Pi_0$ develops a $1/q^2$ pole so that eq. (14.1) becomes an integral-differential equation. We shall come back to this point later.

Eq. (14.3) becomes

\[-\Pi_1 J^0_{\text{ind}} = \Pi_0 i(q_1 J^2_{\text{ind}} - q_2 J^1_{\text{ind}})\]
\[-\Pi_1 \epsilon^{jk} J^k_{\text{ind}} = -\Pi_0 i\omega J^j_{\text{ind}} + \Pi_2 i\epsilon^j J^0_{\text{ind}}\]

where $\Pi_1 = \Pi_1 - (N/2\pi)$. A solution to (14.4) exists only if $\Pi_1 (\Pi_1^2 - \Pi_0 \Pi_2) = 0$ where $\Pi_2 = \omega^2 \Pi_0 - q^2 \Pi_2$. $\Pi_1 = 0$ is not permissible, since the equations would imply $\Pi_2 = 0$ as well, which is incompatible. Hence we have

\[\Pi_1^2 - \Pi_0 \Pi_2 = 0\]

This is the same as eq. (13.2), $\Delta_n = 0$. As was shown in the previous section, it admits a phonon spectrum (13.4). We also note that eq. (14.4) contains the continuity equation. Indeed it follows from (14.4) that

\[\partial_0 J^0_{\text{ind}}(x) + c_s^2 \partial_x J^0_{\text{ind}}(x) = 0\]

with the continuity equation

\[\partial_0 J^0_{\text{ind}}(x) + \partial_x J^0_{\text{ind}}(x) = 0\]

The first of (14.6) says that there is no circulation (vorticity). The second of (14.6) and (14.7) give

\[c_s^2 \partial_x J^0_{\text{ind}}(x) = 0\]

implying a density wave.

It is easy to see that (14.6) results by linearizing the hydrodynamic equation. The Euler equation for an ideal fluid (with no viscosity and thermal conductivity) is

\[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{mn} \nabla P\]

(14.8)
where $P(x)$ is the pressure

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = -\left(\frac{\partial E}{\partial V}\right)_S .$$

We have evaluated the energy density $E/V$ at zero temperature in Sections 6 and 7. In the RPA and SCF

$$\text{SCF/RPA} : \quad E = V \frac{\pi n^2}{m} = \frac{\pi N_e^2}{m V} ,$$

where $N_e$ is the total anyon number, and in the Hartree-Fock approximation

$$\text{HF} : \quad E = \begin{cases} 
\frac{1}{2} \frac{\pi N_e^2}{m V} & \text{for } N = \pm 1 \\
\frac{29}{32} \frac{\pi N_e^2}{m V} & \text{for } N = \pm 2 .
\end{cases}$$

Hence

$$P = \frac{\pi n^2}{m} \quad \text{in RPA/SCF}$$

or

$$P = \begin{cases} 
\frac{1}{2} \frac{\pi n^2}{m} & \text{for } N = \pm 1 \\
\frac{29}{32} \frac{\pi n^2}{m} & \text{for } N = \pm 2 .
\end{cases}$$

Substituting (14.12) into (14.8) and keeping only linear terms in $n$ and $\nu$, one finds

$$\frac{\partial \nu}{\partial t} = -\frac{2\pi}{m^2} \nabla n$$

$$\Rightarrow \partial_0 J_k = \partial_0 (n \nu^k) \sim -\frac{2\pi n_e}{m^2} \partial_k J^0 ,$$

which is exactly the second equation in (14.6) with $c_s$ given by (13.4). This derivation demonstrates that in the Hartree-Fock approximation the sound velocity is modified to

$$c_s = \begin{cases} 
\frac{\hbar \sqrt{\pi n_e}}{m} & \text{for } N = \pm 1 \\
\frac{\sqrt{29} \hbar \sqrt{\pi n_e}}{4 m} & \text{for } N = \pm 2 .
\end{cases}$$

Crucial in the above argument is the fact that the energy density is given by (14.10) or (14.11), independent of the number density $n$. The energy density is proportional to $n^2$. In the Hartree-Fock language $|N|$ lowest Landau levels are completely filled even for slowly varying density $n(x)$. As $n(x)$ varies, the magnetic length $l$ also varies such that the particles precisely fill the space. If one looks at the motion of each particle (in the half classical picture), the Larmor orbit expands and shrinks periodically as the density changes. It breathes. Wen and Zee called it the breathing mode.
15. Effective theory

The effective theory of anyon fluids in terms of Chern-Simons and Maxwell fields is obtained by integrating the matter field $\psi(x)$ first. We have already encountered it in Sections 9 and 10. It is given by

$$L_{\text{eff}}(a,A) = L_{\text{CS}}^0 + L_{\text{EM}}^0 + L_F(a + eA)$$

(15.1)

where $L_{\text{CS}}^0(a)$ and $L_{\text{EM}}^0(A)$ are defined in (9.1):

$$L_{\text{CS}}^0[a] = -\frac{N}{4\pi} \epsilon_{\mu\nu\rho} a_{\mu} \partial_{\nu} a_{\rho}$$

$$L_{\text{EM}}^0[A] = -\frac{1}{4} F_{\mu\nu}^2 + e n_c A_0 .$$

$L_F(a + eA)$ summarizes the effect of the matter field, and is given by (10.6) with $a$ replaced by $a + eA$:

$$L_F(a + eA) = L_F^{(1)}(a + eA) + L_F^{(2)}(a + eA) + \cdots$$

(15.2)

In the linear approximation of SCF, or equivalently in RPA, higher order terms in $a^{(1)} + eA$ are neglected.

The kernel $\Gamma^{\mu\nu}$ has been evaluated in Section 12. If one is interested in physics at a large length scale, $\Gamma^{\mu\nu}$ can be expanded in a Taylor series in $\partial_{\mu}$. In this section we shall retain only the most dominant terms. From (12.33) and (12.34)

$$\Pi_0 = \frac{N^2 m}{4\pi^2 n_c}$$

$$\Pi_1 = \frac{N}{2\pi} \left( \frac{\sqrt{|N|} q l}{\sqrt{2\pi n_c}} \cdot q \ll 1 \right)$$

$$\Pi_2 = \frac{N^2}{2\pi m} \left( \frac{\omega}{\omega_c} = \frac{|N| m}{2\pi n_c} \cdot \omega \ll 1 \right).$$

(15.3)

We note that the linear approximation gets better for a larger $|N|$, but the low energy approximation (15.3) breaks down when $|N|$ becomes too large.

Since

$$L_{\text{CS}}(a) = n_c a_0 - \frac{N}{4\pi} \epsilon_{\mu\nu\rho} a_{\mu}^{(1)} \partial_{\nu} a_{\rho}^{(1)} ,$$

all linear terms in $L_{\text{eff}}(a,A)$ cancel each other. (11.4) and (15.3) immediately lead to

$$L_{\text{eff}}(a,A)^{T=0} = -\frac{N}{4\pi} \epsilon_{\mu\nu\rho} a_{\mu}^{(1)} \partial_{\nu} a_{\rho}^{(1)} - \frac{1}{4} F_{\mu\nu}^2 + L_F^{(2)}(a + eA)$$

$$L_F^{(2)}(a + eA) = \frac{N}{4\pi} \epsilon_{\mu\nu} (a_{\mu}^{(1)} + eA_{\mu}) \partial_{\nu} (a_{\rho}^{(1)} + eA_{\rho})$$

$$+ \frac{N^2 m}{8\pi^2 n_c} \left( f_{0j} + e F_{0j} \right)^2 - \frac{N^2}{4\pi m} (b^{(1)} + eB)^2 .$$

(15.4)
Or, noticing the cancellation of the bare Chern-Simons term, one may write

$$\mathcal{L}_{\text{eff}}(a, A)^{T=0} = -\frac{1}{4} F_{\mu\nu}^2 + eN \frac{\varepsilon^{\mu\nu\rho}}{8\pi} A_\mu (2 f_{1\nu(1)} + e F_{\nu})$$

$$+ \frac{N^2 m}{8\pi^2 n_e} (f_{0j} + e F_{0j})^2 - \frac{N^2}{4\pi m} (b^{(1)} + eB)^2 .$$

(15.5)

(15.4) or (15.5) is the effective theory of a charged anyon fluid, valid for slowly varying configurations. It replaces the Ginzburg-Landau (GL) free energy for BCS superconductors. Instead of the GL order parameter $\Psi_{\text{GL}}(x)$ we have Chern-Simons gauge fields $a_\mu^{(1)}(x)$. Higher order terms, namely terms cubic or quartic in $a^{(1)} + eA$, become important for large gauge field configurations such as vortices, but have not been evaluated so far.

The effective theory (15.4) or (15.5) was first derived by Hosotani and Chakravarty\(^{32}\) for static configurations at $T = 0$. There is an alternative way of writing an effective theory. Introducing a scalar field $\phi(x)$ in place of the Chern-Simons field $a_\mu^{(1)}(x)$, Chen et al. have written down\(^{29}\)

$$\mathcal{L}_{\text{eff}}(\phi, A) = -\frac{1}{4} F_{\mu\nu}^2 + g_1 F_{12}(\dot{\phi} - g_2 A_0)$$

$$+ \frac{1}{2} (\phi - g_2 A_0)^2 - \frac{1}{2} c_s^2 (\partial_\nu \phi - g_2 A_j)^2$$

(15.6)

with the sound velocity $c_s$ defined in (13.4). It has been known that in spite of different forms both (15.5) and (15.6) lead to the same predictions for many physical quantities. Similar effective Lagrangians have been written by Fradkin\(^{33}\) and by Banks and Lykken.\(^{36}\)

As we shall see in later sections, the effective theory $\mathcal{L}_{\text{eff}}(a, A)^{T=0}$ can be easily generalized to finite temperature. There we shall find that not only the coefficients in $\mathcal{L}_F^{(2)}(a + eA)$ become $T$-dependent, but also a new important term proportional to $(a_0 + eA_0)^2$ will turn up.

We notice that the effective theory $\mathcal{L}_{\text{eff}}(a, A)^{T=0}$ neatly summarizes the self-consistent field (SCF) method. Equations derived by taking variations over $a_\mu^{(1)}(x)$ and $A_\mu(x)$ are

$$-\frac{N}{4\pi} \varepsilon^{\mu\nu\rho} f_{\nu(1)} = J_{\text{ind}}^\mu$$

$$\partial_\nu F^{\nu\mu} = e J_{\text{ind}}^\mu$$

$$J_{\text{ind}}^\mu(x) = -\frac{\delta}{\delta a_\mu^{(1)}(x)} \left\{ \mathcal{L}_F^{(2)}(a + eA) + \cdots \right\}$$

$$= -\Gamma^{\mu\nu} (a_\nu^{(1)} + eA_\nu) + O[(a^{(1)} + eA)^2]$$

(15.7)
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The last equation which expresses the induced current \( J^\mu_{\text{ind}} \) in terms of the two gauge fields \( a_\mu \) and \( A_\mu \) may be viewed as a new London equation. The first and second equations lead to an identity

\[
-\frac{eN}{4\pi} \epsilon^{\mu\nu\rho} f^{(1)}_{\nu\rho} = \partial_\nu F^{\nu\mu},
\]  \hspace{1cm} (15.8)

or in the component form

\[
\frac{eN}{2\pi} b^{(1)} = \text{div} \, E \quad (E_k = F_{0k})
\]

\[
\frac{eN}{2\pi} f_{0k} = \epsilon^{kl} \partial_0 E_l + \partial_k B,
\]  \hspace{1cm} (15.9)

with which Chern-Simons fields may be eliminated. We note that the identity (15.8) or (15.9) is valid beyond the linear approximation. It is indeed a direct consequence resulting from the general structure of (15.1), the minimal gauge couplings.

For slowly varying configurations

\[
J^0_{\text{ind}}(x) = + \frac{N}{2\pi} (b^{(1)} + eB) - \frac{N^2m}{4\pi^2n_e} \partial_j (f_{0j} + eF_{0j}) + \cdots
\]

\[
J^k_{\text{ind}}(x) = + \frac{N}{2\pi} \epsilon^{kl} (f_{0l} + eF_{0l}) - \frac{N^2}{2\pi m} \epsilon^{kl} \partial_l(b^{(1)} + eB) + \cdots
\]  \hspace{1cm} (15.10)

With the aid of (15.9) we eliminate Chern-Simons fields to obtain

\[
eJ^0_{\text{ind}} = \left\{ \text{div} \, E \right\} - \frac{e^2N^2m}{4\pi^2n_e} \text{div} \, E
\]

\[
- \frac{Nm}{2\pi n_e} \partial_0 (\text{rot} \, E) + \frac{e^2N}{2\pi} (1 - \frac{m}{e^2n_e} \nabla^2) B + \cdots
\]

\[
eJ^k_{\text{ind}} = \left\{ - \partial_0 E_k + \epsilon^{kl} \partial_l B \right\} - \frac{e^2N^2}{2\pi m} \epsilon^{kl} \partial_l B
\]

\[
+ \frac{e^2N}{2\pi} \epsilon^{kl} E_l - \frac{N}{m} \epsilon^{kl} \partial_l \left( \text{div} \, E \right) + \cdots
\]  \hspace{1cm} (15.11)

where \( \text{rot} \, E = \partial_1 E_2 - \partial_2 E_1 \). Notice that the dominant terms in Eq. (15.10) represent an integer quantum Hall effect in the system.\(^{32}\)

Inserting (15.11) into the Maxwell equations

\[
\text{div} \, E = eJ^0_{\text{ind}}
\]

\[
- \partial_0 E_k + \epsilon^{kl} \partial_l B = eJ^k_{\text{ind}}
\]  \hspace{1cm} (15.12)

one recognizes that the terms in parenthesis \( \left\{ \right\} \) in (15.11) exactly cancell the left sides of the Maxwell equations. Crucial in this cancellation is the fact that the coefficient of the induced Chern-Simons term for \( a^{(1)}_\mu + eA_\mu \) in (15.4) is exactly the
negative of the coefficient of the bare Chern-Simons term for $a_\mu$. We thus arrive at equations

\[
\left(1 - \frac{m}{e^2 n_e} \nabla^2\right) B - \frac{Nm}{2\pi n_e} \text{div} \mathbf{E} - \frac{m}{e^2 n_e} \partial_0 \left(\text{rot} \mathbf{E}\right) = 0
\]

\[
E_k - \frac{2\pi}{e^2 m} \partial_k \left(\text{div} \mathbf{E}\right) - \frac{N}{m} \partial_k B = 0
\]

which describe electromagnetic fields in anyon fluids.

16. Meissner effect at $T = 0$

We examine a response of a charged anyon fluid to an external magnetic field. If the anyon fluid is a superconductor, a sufficiently small magnetic field must be expelled from the system. It must have a Meissner effect.

We shall show that it is indeed the case at least at $T = 0$ within RPA and SCF. There are two ways to demonstrate it, one in the real configuration space with an external magnetic field applied outside the body, and the other by introducing a test current of a $\delta$-function type in the middle of the body. The former approach corresponds to solving the Ginzburg-Landau equation in the BCS superconductors, whereas the latter to a linear response theory. We discuss both.

First we suppose that an anyon fluid occupies a half plane ($x_1 \geq 0$, $-\infty < x_2 < +\infty$) and an external magnetic field is applied such that $B_3 = B(x) = B_{\text{ext}}$ for $x_1 < 0$. The problem is to determine the magnetic field configuration $B(x) = B(x_1)$ inside the anyon fluid. One expects damping behaviour $B(x_1) \propto \exp(-x_1/\lambda)$ if the anyon fluid is a superconductor.

In this section we consider a “sufficiently small” $B_{\text{ext}}$. We expect that deviations of both Chern-Simons and Maxwell fields from the ground state values are small so that Eq. (15.13) may be employed. Together with the boundary condition $B(0) = B_{\text{ext}}$, the magnetic field $B(x_1)$ ($x_1 > 0$) inside the system is determined.

For configurations under consideration, Eq. (15.13) becomes

\[
\left(1 - \frac{m}{e^2 n_e} \partial^2\right) B - \frac{Nm}{2\pi n_e} \partial_1 E_1 = 0
\]

\[
E_1 - \frac{2\pi}{e^2 m} \partial_1^2 E_1 - \frac{N}{m} \partial_1 B = 0
\]

At this point one has to examine numerical values of various parameter. We give a summary of numerical values in Sections 18 and 20. It will be seen that to very good accuracy (16.1) is approximated by

\[
\left(1 - \frac{m}{e^2 n_e} \partial^2\right) B \sim 0
\]

\[
E_1 \sim 0
\]

\[
E_2 = 0
\]
Hence the solution is

\[ B(x_1) = B_{\text{ext}} e^{-x_1/\bar{\lambda}} \quad \text{for } x_1 > 0 \]

\[ \bar{\lambda} = \sqrt{\frac{m}{e^2 n_e}} . \quad (16.3) \]

The magnetic field is exponentially damped from the surface. The charged anyon fluid exhibits a Meissner effect at \( T = 0 \). The penetration depth coincides with the London penetration depth in BCS superconductors. The persistent current flows along the boundary.

Next we shall examine the same problem in the linear response theory. We imagine that a charged anyon fluid occupies the entire space in the \( x_1-x_2 \) plane. We introduce an external current of a \( \delta \)-function type at \( x_1 = 0 \):

\[ eJ^2_{\text{ext}}(x) = -2B_0 \delta(x_1) \]
\[ J^0_{\text{ext}}(x) = J^1_{\text{ext}}(x) = 0 \quad (16.4) \]

which generate an external magnetic field

\[ B_{\text{ext}}(x) = B_0 \epsilon(x_1) \quad . \quad (16.5) \]

In the momentum space

\[ eJ^2_{\text{ext}}(\omega, q) = -2B_0 \cdot (2\pi)^2 \delta(\omega)\delta(q_2) \]
\[ B_{\text{ext}}(\omega, q) = \frac{i}{q_1} eJ^2_{\text{ext}}(\omega, q) \]
\[ A^2_{\text{ext}}(\omega, q) = \frac{1}{q_1} eJ^2_{\text{ext}}(\omega, q) \quad , \quad A^0_{\text{ext}} = A^1_{\text{ext}} = 0 \quad . \quad (16.6) \]

The response of the system to an external perturbation is described by the response function \( Q_c \) determined in the preceeding sections. The relation to the induced current is given by (10.21):

\[ J^{\mu}_{\text{ind \ linear}} = -\Gamma^{\mu
u}(a^{(1)}_\nu + e A_\nu) = Q^{\mu\nu}_c eA^\nu_{\text{ext}} \quad . \quad (16.7) \]

For the configuration (16.6)

\[ J^0_{\text{ind}} = -Q^{02}_c eA^2_{\text{ext}} = -\frac{e^2}{q_1} Q^{02}_c J^2_{\text{ext}} \]
\[ J^1_{\text{ind}} = 0 \]
\[ J^2_{\text{ind}} = -Q^{22}_c eA^2_{\text{ext}} = -\frac{e^2}{q_1} Q^{22}_c J^2_{\text{ext}} \quad (16.8) \]
At this point we need to evaluate $Q^\mu_\nu$ for $\omega = 0$, $q_2 = 0$, and small $q_1 = q$. We notice that
\[
\Pi_0, \Pi_1, \Pi_2 = O(1) \\
\bar{\Pi}_1, \bar{\Pi}_2 = O(q^2) \\
\Pi_1^2 - \Pi_0 \bar{\Pi}_2 = O(1) \\
\bar{\Pi}_1^2 - \Pi_0 \bar{\Pi}_2 = O(q^2) .
\]
Hence in $\Delta_c$ in (11.17), only the second and fourth terms are relevant. Explicitly
\[
\Delta_c \sim \frac{e^4}{q^2} \left( \frac{N}{2\pi} \right)^2 \left\{ 1 + \bar{\lambda} q^2 - \frac{1}{2} |N|(ql)^2 + \frac{2\pi}{me^2} q^2 + O(q^4) \right\} \\
\sim \frac{e^4}{q^2} \left( \frac{N}{2\pi} \right)^2 (1 + \bar{\lambda} q^2) .
\]
(16.9)
In the second line we have suppressed numerically negligible terms. $\bar{\lambda}$ is given in (16.3).

From (11.19)
\[
Q^{22}_c = \frac{1}{\Delta_c} \left\{ -\bar{\Pi}_2 + e^2 (\Pi_1^2 - \Pi_0 \bar{\Pi}_2) \right\} \\
Q^{02}_c = \frac{iq}{\Delta_c} \left\{ -\Pi_1 + \frac{2\pi}{N} (\Pi_1^2 - \Pi_0 \bar{\Pi}_2) \right\} .
\]
It is straightforward to see that in our approximation
\[
Q^{22}_c \sim \frac{q^2}{e^2} \frac{1}{1 + \bar{\lambda} ^2 q^2} \\
Q^{02}_c \sim \frac{iN}{4e^4mc} \frac{q^2}{1 + \bar{\lambda} ^2 q^2} .
\]
(16.10)
The total current in the presence of the perturbation (16.4) becomes
\[
J^{2}_{tot} = J^{2}_{ind} + J^{2}_{ext} \\
= \left( 1 - \frac{e^2}{q_1^2} Q^{22}_c \right) J^{2}_{ext} \\
= \frac{\bar{\lambda} q_1^2}{1 + \bar{\lambda} ^2 q_1^2} J^{2}_{ext} .
\]
(16.11)
Notice that $J^{2}_{tot}$ vanishes at $q_1 = 0$, i.e. the external current is completely shielded.
The total magnetic field is given by
\[
B_{tot}(\omega, q) = \frac{i}{q_1} eJ^{2}_{tot} \\
= -2iB_0 \frac{\bar{\lambda} q_1}{1 + \bar{\lambda} ^2 q_1^2} \cdot (2\pi)^2 \delta(\omega)\delta(q_2)
\]
(16.12)
so that in the configuration space

\[ B(x) = B_0 \epsilon(x_1) e^{-|x_1|/\lambda} \]  
\[ (16.13) \]

We have reproduced the same result as in (16.3). The Meissner effect is complete at \( T = 0 \) for a sufficiently small external field.

For completeness we look at a response to an external static charge: \( J^0_{\text{ext}} \neq 0 \), \( J^k_{\text{ext}} = 0 \). One finds that, for \( \omega = 0 \),

\[ Q^0_{\text{c}} = -\frac{1}{\Delta_{\text{c}}} \left\{ q^2 \Pi_0 + e^2 (\Pi_1^2 - \Pi_0 \Pi_2) \right\} \sim -\frac{q^2}{e^2} \]  
\[ (16.14) \]

Noticing

\[ A^0_{\text{ext}} = \frac{1}{q^2} e J^0_{\text{ext}} \]

we find that

\[ J^0_{\text{ind}} = Q^0_{\text{c}} e A^0_{\text{ext}} = \frac{e^2}{q^2} Q^0_{\text{c}} J^0_{\text{ext}} \sim -J^0_{\text{ext}} \]  
\[ (16.15) \]

An external charge is completely shielded, as it should.

17. \( T \neq 0 \) – homogeneous fields

The behaviour of anyon fluids at finite temperature is particularly interesting to know. In a sense the behaviour of anyon fluids at zero temperature is very similar to that of conventional superfluids or superconductors. For instance, a charged anyon fluid exhibits a Meissner effect for sufficiently small magnetic fields with the same penetration depth as in BCS superconductors. It is very difficult to see an effect of the unique structure of the ground state, namely the complete filling of Landau levels in the Hartree-Fock approximation. (There is a tiny effect of \( P \)- and \( T \)-violation, which we shall briefly touch on in Section 23.)

What would happen at finite temperature? Does a charged anyon fluid behave quite differently from BCS superconductors? Is a charged anyon fluid really a “high” \( T_c \) superconductor? We shall show in this and following sections that there is an indication that a charged anyon fluid has \( T_c \) around 100 K, but not around 5 K or 1000 K. After all the most important feature of observed high \( T_c \) superconductors (cuprate superconductors) is that they have \( T_c \sim 50 – 100 \) K.

How is the order of a charged anyon fluid destroyed? In this respect it is suitable to consider effects of both finite magnetic fields and temperature.\(^{44}\) Recall that at zero temperature with no external fields Landau levels with respect to the Chern-Simons magnetic field are completely filled in the Hartree-Fock approximation. Particles, or holons, feel only the sum of Chern-Simons and Maxwell gauge fields. They interact with the gauge fields in the combination of \( a_\mu + eA_\mu \).

If an external uniform magnetic field \( B_{\text{ext}} \) is applied in the direction perpendicular to the two-dimensional plane, to the first approximation, particles feel the total magnetic field \( b_{\text{tot}} = b^{(0)} + eB_{\text{ext}} \), with which the Landau levels are not completely
filled any more. If the relative sign between $b^{(0)}$ and $eB_{\text{ext}}$ is negative, less states are available per Landau level so that some particles must be put in the higher energy level. On the other hand, if the relative sign is positive, more states are available so that there appear vacant states in the top filled level. Hence, if the complete filling were essential for superconductivity, an external magnetic field would destroy it.

Similarly, at finite temperature, the levels are not completely filled due to thermal excitations. Superconductivity should be destroyed at sufficiently high temperature.

The analysis at finite fields and temperature, however, is complicated by the plausible breakdown of the approximation in which homogeneous configurations are supposed. It is likely that a finite uniform external magnetic field creates vortices in anyon fluids, giving rise to inhomogeneous, though patterned, configurations. At finite temperature vortex-antivortex pair creation would become important.

Effects of vortex formation has been examined by Kitazawa and Murayama\textsuperscript{39} in the case of neutral anyon fluids. They have argued that vortices bring a stability to the superfluidity of neutral anyon fluids. At the moment the existence of real (electromagnetic) vortices in charged anyon fluids is yet to be established.

In this and following sections we shall examine effects of finite fields and temperature, ignoring contributions of vortices. This is a drastic approximation, which has to be improved in future. However, we shall find that even in this approximation anyon fluids exhibit interesting behaviour which is quite different from that of conventional (type I) superconductors. Some of the behaviour will be modified by the incorporation of vortices, but all this is certainly essential for full understanding of anyon fluids.

The first evaluation of finite temperature effects in the model under consideration was given by Randjbar-Daemi, Salam, and Strathdee.\textsuperscript{38} Hetrick, Hosotani, and Lee\textsuperscript{44} subsequently confirmed their result, discussing more physical implications with additional effects of finite fields. Later many authors, particularly, Fetter and Hanna,\textsuperscript{49} recovered the same result by different methods.

The evaluation consists of two parts. First thermodynamic quantities are evaluated for uniform field configurations at finite temperature, from which self-consistent uniform fields are determined. Secondly, inhomogeneous deviations of the fields from the self-consistent uniform configuration are incorporated in perturbation theory at finite temperature.

With a uniform magnetic fields $b^{(0)}$ and $B^{(0)}$, Landau levels are formed with respect to $b^{(0)} + eB^{(0)}$. In this section we suppress the superscript $^{(0)}$ to simplify the notation. It will be recovered when inhomogeneous configurations are examined in Section 19. The number of states per area per Landau level is

$$n_L = \frac{|b + eB|}{2\pi}$$

which defines the magnetic length

$$l(B)^2 = \frac{1}{|b + eB|}.$$
Energy levels are given by
\[ \epsilon_n(B) = \left( n + \frac{1}{2} \right) \epsilon, \quad \epsilon \equiv \frac{1}{m l(B)^2} \quad (n = 0, 1, 2, \ldots) . \] (17.3)

One of Chern-Simons equations implies that
\[ b = \frac{2\pi n_e}{N} \] (17.4)
is still valid. Therefore the filling factor as a whole is
\[ \nu = \frac{n_e}{n_L} = \frac{2\pi n_e \cdot l^2}{\left| \frac{Nb}{b + eB} \right|} \]
\[ = |N| \cdot \left| 1 + \frac{eB}{b} \right|^{-1} . \] (17.5)

For \(|eB/b| \ll 1\),
\[ \nu = |N| \cdot \left( 1 - \frac{N eB}{2\pi n_e} + \cdots \right) . \] (17.6)

We keep the above definition of \(\nu\) at \(T \neq 0\).

The distribution function \(\rho_{np}\) at level \(n\) with the second index \(p\) (defined in Section 6) is given by
\[ \rho_{np} = \frac{1}{e(\epsilon_n - \mu)/T + 1} \equiv \rho_n \] (17.7)
where \(\mu\) and \(T\) are the chemical potential and temperature, respectively. The chemical potential is fixed by the condition
\[ n_e = \frac{1}{V} \sum_{n,p} \rho_{np} . \] (17.8)

The summation over \(p\) gives \(V/2\pi l^2\), as is most easily seen in the Landau gauge. Hence
\[ \nu = 2\pi l^2 \cdot n_e = \sum_{n=0}^{\infty} \rho_n . \] (17.9)

The energy density \(E\) and entropy density \(S\) are given by
\[ \mathcal{E} = \frac{1}{V} \sum_{n,p} \epsilon_n \rho_n \]
\[ = \frac{2\pi n_e^2}{\nu^2 m} \sum_n \left( n + \frac{1}{2} \right) \rho_n \]
\[ S = -\frac{1}{V} \sum_{n,p} \left\{ \rho_n \ln \rho_n + (1 - \rho_n) \ln(1 - \rho_n) \right\} \]
\[ = -\frac{n_e}{\nu} \sum_n \left\{ \rho_n \ln \rho_n + (1 - \rho_n) \ln(1 - \rho_n) \right\} \] (17.10)
The free energy density $\mathcal{F}[B]$ is

$$\mathcal{F}[B] = \mathcal{E} - TS$$ \hspace{1cm} (17.11)

A good approximation to $\rho_n$ is obtained by examining numerical values of various parameters. As we shall see in the next section, with typical values of $m \sim 2m_e$ and $n_e \sim 2 \times 10^{14} \text{ cm}^{-2}$,

$$\epsilon = \frac{2\pi n_e}{|N|m} \sim \frac{2}{|N|} \cdot 2800 \text{ K}$$ \hspace{1cm} (17.12)

$$\frac{1}{e^b} = \frac{2\pi n_e}{|N|} \sim \frac{2}{|N|} \cdot 1200 \text{ T}$$

With this choice, $T_c$ will turn out about 100 K. In other words, with $N = \pm 2$ and for $T < 200 \text{ K}$ and $B < 30 \text{ T}$, the filling factor $\nu$ is very close to $|N|$ and thermal excitations are appreciable only to the $|N|$-th Landau level:

$$\rho_n \sim \begin{cases} 
1 & \text{for } n \leq |N| - 2 \\
0 & \text{for } n \geq |N| + 1 
\end{cases}$$ \hspace{1cm} (17.13)

Hence Eq. (17.9) becomes

$$x \equiv \nu - |N| = (\rho_{|N|-1} - 1) + \rho_{|N|}$$ \hspace{1cm} (17.14)

It is called the two-level approximation. The condition for its validity is

$$e^{-\epsilon/T} \ll 1$$ \hspace{1cm} (17.15)

Since

$$\rho_{|N|-1} = \frac{1}{e^{-\epsilon/2T}e^{(|N|\epsilon-\mu)/T} + 1} \quad \text{and} \quad \rho_{|N|} = \frac{1}{e^{+\epsilon/2T}e^{(|N|\epsilon-\mu)/T} + 1}$$

Eq. (17.14) is solved by

$$z(x,T)^\pm 1 = e^{\pm(|N|\epsilon-\mu)/T}$$

$$= \frac{1}{1 \pm x} \left( \sqrt{1 + x^2 \sinh^2 \frac{\epsilon}{2T} + x \cosh \frac{\epsilon}{2T}} \right)$$ \hspace{1cm} (17.16)

which determines the chemical potential $\mu$ with given $T$ and $B$. (Note that $\epsilon$ is a function of $B$.)

In particular,

$$\mu(T) = |N|\epsilon \quad \text{at } B = 0$$ \hspace{1cm} (17.17)

There is no $T$-dependence in the two-level approximation. The correction has been evaluated by Randjbar-Daemi et al. to be

$$\mu(B = 0) = |N|\epsilon - \frac{T}{2} e^{-|N|\epsilon/T} + \cdots$$ \hspace{1cm} (17.18)
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which is exponentially small.

We also remark that \( T \to 0 \) limit is singular, as can be seen from the presence of \( \sinh(\epsilon/2T) \) or \( \cosh(\epsilon/2T) \) in (17.16). At zero temperature for small \( |x| \)

\[
\mu(T = 0) = \begin{cases} 
(\lfloor N \rfloor - \frac{1}{2}) \epsilon & \text{for } x < 0 \\
(\lfloor N \rfloor + \frac{1}{2}) \epsilon & \text{for } x > 0 
\end{cases}
\]  

(17.19)

which also follows from the consideration of the Fermi level. Thermodynamic quantities at \( T = 0 \) have a singularity at \( x = 0 \). We shall come back to this point in the next section.

Let us define

\[
\bar{\rho} = \rho_{\lfloor N \rfloor} = \frac{1}{ze^{\epsilon/2T} + 1},
\]  

(17.20)

in terms of which the energy and entropy density in the two-level approximation are given by

\[
E = \frac{2\pi n_e^2}{\nu^2 m} \left\{ \frac{1}{2} |N|^2 + \left( |N| - \frac{1}{2} \right) x + \bar{\rho} \right\},
\]

\[
S = -\frac{n_e}{\nu} \left\{ \bar{\rho} \ln \bar{\rho} + (1 - \bar{\rho}) \ln(1 - \bar{\rho}) 
+ (\bar{\rho} - x) \ln(\bar{\rho} - x) + (1 - \bar{\rho} + x) \ln(1 - \bar{\rho} + x) \right\}
\]  

(17.21)

There are two parameters, \( x \) and \( T \). For experimentally available magnetic fields we always have \( |x| \ll 1 \). However, \( z(x, T) \) in (17.16) depends on \( T \) sensitively with small, finite \( x \). It is easy to see that

1) \( |x| \ll 1 \), \( |x| e^{\epsilon/2T} \ll 1 \):

\[
z = 1 \quad \rho_{\lfloor N \rfloor - 1} = \frac{1}{e^{-\epsilon/2T} + 1} \quad \rho_{\lfloor N \rfloor} = \frac{1}{e^{\epsilon/2T} + 1}
\]

2) \( |x| \ll 1 \), \( |x| e^{\epsilon/2T} \gg 1 \):

\[
x > 0 : \quad z = \frac{1}{x} e^{-\epsilon/2T} \quad \rho_{\lfloor N \rfloor - 1} = 1 \quad \rho_{\lfloor N \rfloor} = x
\]

\[
x < 0 : \quad z = -x e^{\epsilon/2T} \quad \rho_{\lfloor N \rfloor - 1} = 1 + x \quad \rho_{\lfloor N \rfloor} = 0
\]  

(17.22)

Note that with the numerical values (17.12) the border line defined by \( |x| e^{\epsilon/2T} = 1 \) is given by, for \( N = \pm 2 \),

\[
x = -6 \times 10^{-11} \quad B = 10^{-3} \quad T = 60 \text{ K}
\]

\[
x = -6 \times 10^{-9} \quad B = 10^{-1} \quad T = 74 \text{ K}
\]

\[
x = -8.3 \times 10^{-7} \quad B = 12.5 \quad T = 100 \text{ K}
\]

\[
x = -6 \times 10^{-6} \quad B = 10^{+3} \quad T = 146 \text{ K}
\]

\[
x = -6 \times 10^{-3} \quad B = 10^{+5} \quad T = 279 \text{ K}
\]  

(17.23)
It is interesting that the crossover takes place around 70 – 150 K for moderate magnetic fields. However, it should be borne in mind that the above result is in the approximation which ignores contributions of vortices.

For completeness we evaluate the specific heat (per volume) and pressure in the two-level approximation. From (17.21) one finds

$$C_v = \left( \frac{\partial E}{\partial T} \right)_V = \frac{2\pi n^2 e^2}{\nu^* m} \left( \frac{\partial \rho}{\partial T} \right)_V . \quad (17.24)$$

It follows from (17.16) and (17.20) that

$$\left( \frac{\partial \rho}{\partial T} \right)_V = \frac{\epsilon}{2T^2} \left( \frac{1}{z^{-1} e^{-\epsilon/2T} + 1} \right) \times \left\{ 1 - \frac{x \sinh(\epsilon/2T)}{1 + x^2 \sinh^2(\epsilon/2T)} \sinh^{-1/2} \right\} . \quad (17.25)$$

The expression for the pressure at finite $x$ is lengthy. The result for $x = 0$ is simple.

$$P = -\left( \frac{\partial F}{\partial V} \right)_T = \frac{2\pi n^2 e^2}{|N|^2 m} \left( \frac{1}{2} |N|^2 + \bar{\rho} \right) - \left( \frac{\partial \rho}{\partial T} \right)_T \left\{ \frac{2\pi n^2 e^2}{|N|^2 m} + \frac{2n_e T}{|N|} \ln \frac{\bar{\rho}}{1 - \bar{\rho}} \right\} . \quad (17.26)$$

Insertion of (17.20) with $z = 1$ shows that the two terms in the last parenthesis cancel each other. Hence

$$P = \frac{\pi n^2 e^2}{m} \left( 1 + \frac{2}{|N|^2} \bar{\rho} \right) \quad \text{at} \quad x = 0 . \quad (17.27)$$

There results only a tiny correction. However, this may be an artifact of the uniform field approximation. We shall see in Section 21 a sign of instability in a neutral anyon fluid.

18. de Haas – van Alphen effect in SCF

Charged anyon fluids have the structure of Landau levels, and therefore should exhibit a de Haas – van Alphen effect when external magnetic fields are applied. Of course, an implicit assumption is that the system remains uniform in the presence of uniform fields, which is probably not true even with a modest external field. Observed high $T_c$ superconductors are of type II, i.e. vortices are formed. Nevertheless, it is worthwhile to examine how the system responds to external fields in the uniform field approximation. We shall see that the Meissner effect at $T = 0$ for sufficiently small fields can be understood as a part of a de Haas – van Alphen
effect, and that an important departure from the Meissner effect of the BCS type results both at modest external fields and at finite temperature.\textsuperscript{44}

In the previous section we have evaluated in the two-level approximation the energy and entropy densities as functions of temperature $T$ and magnetic field $B$. In terms of the free energy density $F = \mathcal{E} - TS$, the magnetization is given by

$$M(T, B) = -\frac{\partial F(T, B)}{\partial B}. \quad (18.1)$$

$B$ is the total magnetic field (magnetic induction). The relation to an external field (thermodynamic field) $H$ is given by

$$B = H + M(T, B) \quad (18.2)$$

which defines a relation between $H$ and $B$.

Let us see first what happens in the $T \to 0$ limit in the uniform field approximation. The mean field energy density is easily computed to be

$$\mathcal{E}(0, B) = \frac{\pi n_e^2}{m} \left\{ 1 + \frac{(|N| - \nu)(\nu - |N| \pm 1)}{\nu^2} \right\} = \frac{\pi n_e^2}{m} \left\{ 1 + \frac{|eB|}{2\pi n_e} - |N|(|N| \mp 1) \left( \frac{eB}{2\pi n_e} \right)^2 \right\} \quad (18.3)$$

for \begin{align*}
|N| - 1 & \leq \nu \leq |N| \quad (NeB > 0), \\
|N| & \leq \nu \leq |N| + 1 \quad (NeB < 0).
\end{align*}

Notice the appearance of $|eB|$ in the expression. The energy density has a cusp at $B = 0$, as was first noticed by Chen et al.\textsuperscript{29} The magnetization is found to be

$$eB > 0 \quad M(0, B) = -\frac{en_e}{2m} \left\{ 1 - 2|N||N| \mp 1 \right\} \frac{eB}{2\pi n_e} \quad (18.4)$$

$$eB < 0 \quad M(0, B) = +\frac{en_e}{2m} \left\{ 1 + 2|N||N| \mp 1 \right\} \frac{eB}{2\pi n_e}.$$

It has a discontinuity at $B = 0$. The magnitude decreases from the value $M(0, 0^\pm)$ as $|B|$ increases. For magnetic fields available in laboratories, we always have $|eB/2\pi n_e| \ll 1$. Therefore, to good accuracy

$$M(0, B) = \pm\frac{en_e}{2m} \quad \text{for} \quad \begin{cases} eB > 0 \\ eB < 0 \end{cases} \quad (18.5)$$

This is nothing but a de Haas – van Alphen effect. (See Fig. 10.) The only difference from the standard one is that even in the absence of magnetic field ($B = \ldots$)
0), we have a Chern-Simons magnetic field \( b \neq 0 \) such that we are at the integer filling \( \nu = |N| \). The magnetization reaches its maximum \( |e| n_e / 2m \) at discontinuous points.

![Fig. 10](image)

What does this mean? When an external magnetic field \( H \) is applied in the
direction perpendicular to the plane, the solution to Eq. (18.2) is

\[
\begin{align*}
H &\leq -H'_{c} \quad B = H + H'_{c}, \quad M = +H'_{c}, \\
-H'_{c} \leq H \leq H'_{c} \quad B = 0, \quad M = -H, \\
H'_{c} \leq H &\quad B = H - H'_{c}, \quad M = -H'_{c},
\end{align*}
\]

where the critical field is given by

\[
H'_{c} = \frac{|e|n_{e}}{2m}.
\]

So long as \(|H| < H'_{c}\), there is no magnetic field \((B = 0)\) in bulk. It is a Meissner effect. However, if \(|H|\) exceeds \(H'_{c}\), a part of the external magnetic field penetrates inside the anyon fluid. (See Fig. 11.) As a consequence of the de Haas – van Alphen effect, there is the maximum for \(|M|\).

\[\text{Fig. 11} \quad H \text{ vs } B \text{ as } T \text{ varies.}\]

Of course, all of these results have been obtained in the uniform approximation. In reality the formation of vortices would invalidate the above picture. Then arises a question which one is smaller, \(H_{c1}\) or \(H'_{c}\).

So far all quantities are defined in the effective two-dimensional space. We have been supposing that three-dimensional material has a layered structure with interplanar spacing \(d \sim 5 \text{ Å}\). Two-dimensional quantities (denoted by \(\_\_d=2\)) are
related to three-dimensional quantities (denoted by \(d=3\)) by

\[
\left(\frac{e^2}{4\pi}\right)_{d=2} = \frac{1}{d} \alpha_{d=3} = \frac{1}{137} \frac{1}{d}
\]

\[
B_{d=2} = \sqrt{d} B_{d=3}
\]

\[
b_{d=2} = b_{d=3}
\]

\[
n^d_{e=2} = d n^d_{e=3}
\]

Hence the Chern-Simons magnetic field \(b\) in (17.4) and the critical field \(H'_c\) in (18.7) are given by

\[
\left(\frac{1}{e}\right)_{d=3} = \frac{1}{e_{d=3}} b_{d=2} = \frac{1}{N} \sqrt{\frac{\pi}{\alpha} n^d_{e=2}}
\]

\[
(H'_c)_{d=3} = \delta^{-1/2} (H'_c)_{d=2} = \frac{\sqrt{4\pi\alpha n^d_{e=3}}}{2m}
\]

The hole density \(n^d_{e=2}\) and \(n^d_{e=3}\), and the spacing \(d\) are measured directly. Typical values are

\[
n^d_{e=2} = (1 \sim 5) \times 10^{14} \text{ cm}^{-2}
\]

\[
n^d_{e=3} = (2 \sim 10) \times 10^{21} \text{ cm}^{-3}
\]

\[
d = 5 \text{ Å} = 5 \times 10^{-8} \text{ cm}
\]

We also note the conversion formulas:

\[
m_e = 5.1 \times 10^5 \text{ eV} = 2.6 \times 10^{10} \text{ cm}^{-1}
\]

\[
1 \text{ G} = 1.779 \times 10^8 \text{ cm}^{-2} = 6.903 \times 10^{-2} \text{ eV}^2
\]

\[
1 \text{ K} = 8.617 \times 10^{-5} \text{ eV} = 4.367 \text{ cm}^{-1}
\]

The effective mass \(m\) is not directly measurable. It can be fixed from observed values for the penetration depth at \(T = 0\) or \(T_c\). \((T_c\) is discussed in Section 22.) Recalling (16.3), one finds

\[
\lambda(T = 0) = \bar{\lambda} = \sqrt{\frac{m}{e^2_{d=2} n^d_{e=2}}} = \sqrt{\frac{m}{e^2_{d=3} n^d_{e=3}}} .
\]

Therefore

\[
m = 4\pi\alpha n^d_{e=3} \lambda(0)^2
\]

If one substitutes the values (18.10) and

\[
\lambda(0) = 1400 \text{ Å} \implies m = (1.4 \sim 6.9) \ m_e
\]

We shall see in Section 22 that

\[
T_c \sim 100 \text{ K} \implies m = (1 \sim 5) \ m_e
\]

(Indeed, the ratio \(n^d_{e=3}/m\) is related to \(\lambda(0)\) or \(T_c\).)
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Upon making use of (18.13), $H'_c$ can be written as

$$\begin{align*}
H'_c &= \frac{1}{4\sqrt{\pi\alpha}} \frac{1}{\lambda(0)^2} \\
&= 2 \cdot \frac{2\pi\hbar c}{2e} \cdot \frac{1}{4\pi\lambda(0)^2}.
\end{align*}$$

(18.16)

In the Ginzburg-Landau theory of conventional superconductors of type II $H_{c1}$ is approximately given by

$$H_{c1} \sim \frac{2\pi\hbar c}{2e} \cdot \frac{1}{4\pi\lambda^2} \cdot \ln \frac{\lambda}{\xi},$$

(18.17)

where $\xi$ is a coherence length. If one uses the GL parametrization to high $T_c$ superconductors, one typically finds $\lambda/\xi \approx 100$ so that $\ln(\lambda/\xi) \approx 4.6$. In other words,

$$\text{roughly} \quad H'_c \sim H_{c1}.$$  

(18.18)

The values for $b$ and $H'_c$ in three dimensions are

$$\begin{align*}
\frac{1}{e} b &= \frac{2}{N} \frac{n_e^{d=2}}{2 \times 10^{14} \text{ cm}^{-2}} \cdot 1.2 \times 10^7 \text{ G} \\
H'_c &= \frac{n_e^{d=3}}{4 \times 10^{21} \text{ cm}^{-3}} \frac{2m_e}{m} \cdot 66 \text{ G}.
\end{align*}$$

(18.19)

Or, with the aid of (18.16) one has

$$H'_c = \left( \frac{1400 \text{ Å}}{\lambda(0)} \right)^2 \cdot 47 \text{ G}.$$  

(18.20)

The Chern-Simons magnetic field is huge ($\sim 1000$ T), but $H'_c$ turns out to be in a modest range ($\sim 50$ G). The rough equality (18.18) suggests that vortices are formed in anyon fluids, and that the uniform magnetic field inside the fluid, $B$ in (18.6), may represent the average field over the vortex lattice.

The huge magnitude of $b$ might be related to $H_{c2}$ of high $T_c$ superconductors at $T = 0$, which is known to be much larger than 100 T. Related to the huge Chern-Simons magnetic field is the energy spacing in the Landau level.

$$\frac{1}{m l^2} = \frac{2\pi n_e^{d=2}}{|N|m} = \frac{2}{|N|} \frac{2m_e}{m} \frac{n_e^{d=2}}{2 \times 10^{14} \text{ cm}^{-2}} \cdot 2800 \text{ K} \ (0.24 \text{ eV}).$$

(18.21)

Generalization to finite temperature ($T \neq 0$) is straightforward. Magnetization $M(T, B)$ is determined by (18.1) with (17.21). Then, the $B$ vs $H$ relation is found from (18.2). We have given the result in Figs. 10 and 11.
As is seen from the figures, magnetization almost vanishes around 100 K. This is presumably related to the phenomenon observed in (17.23), and therefore need more elaboration by incorporating vortices.

19. \( T \neq 0 \) – inhomogeneous fields

The effective action or free energy at finite temperature can be evaluated in perturbation theory. One way is to write down the partition function in the path integral representation as we did in Section 9 for \( T = 0 \). This amounts to rotating a time \( t \) to an imaginary time \( \tau = it \) \((0 \leq \tau \leq \beta = 1/T)\) with appropriate boundary conditions on the fields imposed. In other words, we have Matsubara’s finite temperature Green’s functions in place of time-ordered Green’s functions at \( T = 0 \).

Most of the arguments in Sections 9 – 11 remain intact, provided that the frequency \( \omega \) is replaced by Matsubara frequency \( i\omega_n \). \([\omega_n = 2n\pi T \text{ or } (2n + 1)\pi T \text{ for bosons or fermions, respectively, where } n \text{ is an integer.}]\) In particular, the decomposition (11.4) of the kernel \( \Gamma \) in terms of \( \Pi \)'s, and the relation (11.11) or (11.19) between \( \Pi \)'s and the response function \( Q_n \) or \( Q_c \) are valid after the Wick rotation. However, one should note that all these are for \( \tau \)-ordered Matsubara’s Green’s functions so that to relate them to, for instance, a response function in real time, one need to make necessary transformations. There are also studies in the real-time formalism of finite temperature Green’s functions.\(^{54}\)

Again we integrate the fermion field \( \psi \) first to obtain the effective theory for Chern-Simons and Maxwell fields. We expand the gauge fields around constant magnetic fields. In the Landau gauge

\[
\begin{align*}
a^k(x) &= -b^{(0)} x_2 + a^{(1)k}(x) \\
A^k(x) &= -B^{(0)} x_2 + A^{(1)k}(x)
\end{align*}
\] (19.1)

where

\[
b^{(0)} = \frac{2\pi n_e}{N} \] (19.2)

and \( B^{(0)} \) is a constant background magnetic field. According to (17.1) – (17.5)

\[
e(N) \left( b^{(0)} + eB^{(0)} \right) = \frac{1}{l^2} = \frac{2\pi n_e}{\nu} .
\] (19.3)

The fermion part of the Hamiltonian is

\[
H_e[a + eA] = \int d\mathbf{x} \left\{ \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) + (a_0 + eA_0) \psi^\dagger \psi \right\}
\] (19.4)

where \( D_k = \partial_k - i(a^k + eA^k) \). We decompose it into the free and interaction parts:

\[
H_e[a + eA] = H_0 + H_{\text{int}} = H_0 + V_1 + V_2
\]

\[
H_0 = H_e[a^{(0)} + eA^{(0)}] = \int d\mathbf{x} \frac{1}{2m} (\overline{D}_k \psi)^\dagger (\overline{D}_k \psi)
\]

\[
V_1 = \int d\mathbf{x} \left( \frac{i}{2m} a^{(1)k}_q \{ \psi^\dagger \cdot \overline{D}_k \psi - (\overline{D}_k \psi)^\dagger \cdot \psi \} + a^{(1)k}_q \psi^\dagger \psi \right)
\]

\[
V_2 = \int d\mathbf{x} \frac{1}{2m} (a^{(1)k}_q)^2 \psi^\dagger \psi
\] (19.5)
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where

\[ \widetilde{D}_k = \partial_k - \delta_k^1 i \epsilon(N) \frac{x^2}{l^2}, \]

\[ a^{(1)0}_{\text{tot}} = a_0 + eA_0, \]

\[ a^{(1)k}_{\text{tot}} = a^{(1)k} + eA^{(1)k}. \]

The matter part of the free energy with given static \((t\)-independent) gauge field configurations is defined by

\[ e^{-\beta F_e[a+eA]} = \text{Tr}_{\text{canonical}} e^{-\beta H_e[a+eA]} \] (19.6)

In the previous section we have evaluated the zeroth order free energy:

\[ e^{-\beta F_0} = \text{Tr}_{\text{canonical}} e^{-\beta H_0}. \] (19.7)

In these formulas the trace is taken over states with a fixed number of particles, i.e. over a canonical distribution. One can consider, instead, a grand canonical distribution to define the thermodynamic potential \(\Omega_e\):

\[ e^{-\beta \Omega_e[a+eA]} = \text{Tr} e^{-\beta (H_e[a+eA]-\mu \hat{N})} \]

\[ \Omega_e = F_e - \mu N_e \] (19.8)

where

\[ \hat{N} = \int d\mathbf{x} \psi^\dagger \psi \]

\[ N_e = n_e V = \langle \hat{N} \rangle \]

\[ \langle Q \rangle = \text{Tr} Q e^{\beta (\Omega_e-H_e-\mu \hat{N})}. \] (19.9)

We have adopted notation \(\hat{N}\) for the number operator to distinguish it from the coefficient of the Chern-Simons term, \(N\).

The original computation of ref. (38) was performed for a grand canonical distribution. In view of the \(n_e\)-dependence of \(H_0\) through \(b^{(0)}\), a perturbation theory for a canonical distribution was employed in ref. (44). So long as macroscopic physical quantities are concerned, there arises no difference between the two. Even at the diagram level there is not much difference except for a minor change in the fermion propagator.

In this article we adopt a perturbation theory for a grand canonical distribution, which is summarized in the book of Abrikosov et al.. We outline the argument in the operator formalism, supplementing expressions in the path integral formalism.

We define finite temperature Heisenberg field operators by

\[ -\frac{\partial}{\partial \tau} \psi(\tau, \mathbf{x}) = \mathcal{M} \psi(\tau, \mathbf{x}) \quad \psi(0, \mathbf{x}) = \psi(\mathbf{x}) \]

\[ + \frac{\partial}{\partial \tau} \overline{\psi}(\tau, \mathbf{x}) = \mathcal{M} \overline{\psi}(\tau, \mathbf{x}) \quad \overline{\psi}(0, \mathbf{x}) = \overline{\psi}(\mathbf{x}) \] (19.10)

\[ \mathcal{M} = -\frac{1}{2m} D^2_k + a_0 + eA_0 - \mu \]
where $\psi(x)$ and $\psi^\dagger(x)$ are the operators in the Schrödinger representation.

If the Hamiltonian $H_e$ is $\tau$-independent, then the equations can be integrated as
\begin{align}
\psi(\tau,x) &= e^{\tau(H_e-\mu \hat{N})} \psi(x) e^{-\tau(H_e-\mu \hat{N})} \\
\bar{\psi}(\tau,x) &= e^{\tau(H_e-\mu \hat{N})} \psi^\dagger(x) e^{-\tau(H_e-\mu \hat{N})}
\end{align}
if $\frac{\partial}{\partial \tau} H_e = 0$. (19.11)

At this point we make one technical generalization. We allow that gauge fields $a^{(1)}_\mu$ and $A^{(1)}_\mu$ may depend on $\tau$, provided that they are periodic with a period $\beta$:
\begin{align}
a^{(1)}_\mu(\beta,x) + eA^{(1)}_\mu(\beta,x) &= a^{(1)}_\mu(0,x) + eA^{(1)}_\mu(0,x). 
\end{align}
(We assume that the zeroth order parts are $\tau$-independent.) This is a technical device which enables us to probe dynamical properties, namely time-dependent phenomena, of the system at finite temperature, through appropriate analytic continuation.

The transformation matrix for a general $\tau$-dependent $H_e$ is defined by
\begin{align}
U(\tau_2,\tau_1) &= T_\tau \exp \left\{ - \int_{\tau_1}^{\tau_2} d\tau \left( H_e(\tau) - \mu \hat{N} \right) \right\}, \quad (\tau_2 > \tau_1) \tag{19.13}
\end{align}
where $T_\tau$ indicates the $\tau$-ordering. The specification of the ordering is necessary, since $[H_e(\tau),H_e(\tau')] \neq 0$ for $\tau$-dependent gauge fields. By definition
\begin{align}
U(\tau_3,\tau_2) U(\tau_2,\tau_1) &= U(\tau_3,\tau_1) \quad (\tau_3 > \tau_2 > \tau_1). 
\end{align}(19.14)

Further $U(\tau) \equiv U(\tau,0)$ satisfies
\begin{align}
\frac{\partial}{\partial \tau} U(\tau) &= -(H_e(\tau) - \mu \hat{N}) U(\tau) \\
\frac{\partial}{\partial \tau} U(\tau)^{-1} &= +U(\tau)^{-1} (H_e(\tau) - \mu \hat{N}) \tag{19.15}
\end{align}

In terms of $U(\tau)$, Eq. (19.10) is integrated to yield
\begin{align}
\psi(\tau,x) &= U(\tau)^{-1} \psi(x) U(\tau) \\
\bar{\psi}(\tau,x) &= U(\tau)^{-1} \bar{\psi}(x) U(\tau). 
\end{align}(19.16)
Indeed,
\begin{align}
\left( - \frac{\partial}{\partial \tau} \right) U(\tau)^{-1} \psi(x) U(\tau) &= U(\tau)^{-1} [\psi(x),H_e(\tau) - \mu \hat{N}] U(\tau) \\
&= U(\tau)^{-1} \mathcal{M} \psi(x) U(\tau) \\
&= \mathcal{M} \psi(\tau,x). 
\end{align}

In the last equality we have made use of the fact that the differential operator $\mathcal{M}$ commutes with $U(\tau)$. 

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The next step is to introduce the interaction representation. With the free Hamiltonian $H_0$ in (19.5), field operators in the interaction representation are defined by

$$
\psi_{\text{int}}(\tau, x) = e^{\tau(H_0 - \mu \hat{N})} \psi(x) e^{-\tau(H_0 - \mu \hat{N})},
$$

$$
\bar{\psi}_{\text{int}}(\tau, x) = e^{\tau(H_0 - \mu \hat{N})} \bar{\psi}(x) e^{-\tau(H_0 - \mu \hat{N})}.
$$

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$$

$$
\bar{\psi}_{\text{int}}(\tau, x) = e^{\tau(H_0 - \mu \hat{N})} \bar{\psi}(x) e^{-\tau(H_0 - \mu \hat{N})}.
$$

The transformation matrix in the interaction representation is given by

$$
S(\tau) = T \tau \exp \left\{ - \int_0^\tau d\tau' \tilde{H}_{\text{int}}(\tau') \right\}
$$

$$
\tilde{H}_{\text{int}}(\tau) = e^{\tau(H_0 - \mu \hat{N})} H_{\text{int}} e^{-\tau(H_0 - \mu \hat{N})}
$$

$$
= H_{\text{int}}[\psi_{\text{int}}(\tau, x), \bar{\psi}_{\text{int}}(\tau, x); (a + eA)(\tau)]
$$

The fundamental operator identity is

$$
U(\tau) = e^{-\tau(H_0 - \mu \hat{N})} S(\tau).
$$

The diagram method is developed on the basis of (19.19). Let us denote

$$
e^{-\beta \Omega_0} = \text{Tr} e^{-\beta(H_0 - \mu \hat{N})}
$$

$$
\langle Q \rangle_0 = \text{Tr} Q e^{\beta(\Omega_0 - H_0 + \mu \hat{N})}.
$$

We define the Euclidean effective action $I_E[a + eA]$ by

$$
e^{-I_E[a + eA]} = \text{Tr} U(\beta).
$$

Taking a trace of (19.19), one finds

$$
I_E[a + eA] = \beta \Omega_0 - \ln \langle S(\beta) \rangle_0.
$$

The Bloch-De Dominicis theorem is applied to $\langle S(\beta) \rangle_0$, leading to

$$
I_E[a + eA] = \beta \Omega_0 - \left\{ \langle S(\beta) \rangle_e - 1 \right\}
$$

where the subscript $c$ indicates that only connected diagrams be taken into account.

For static gauge field configurations, one has

$$
\Omega_e = \frac{1}{\beta} I_E = \Omega_0 - \frac{1}{\beta} \left\{ \langle S(\beta) \rangle_e - 1 \right\}
$$

for static $a + eA$.

The path integral representation is obtained for $\text{Tr} U(\beta)$ in (19.21) by the standard technique:

$$
e^{-I_E[a + eA]} = \int_{\text{B.C.}} D\bar{\psi} D\psi \exp \left\{ - \int_0^\beta d\tau \int dx \bar{L}_e[a + eA] \right\}
$$

$$
\bar{L}_e = \bar{\psi} \psi + \frac{1}{2m} D_k^* \bar{\psi} D_k \psi + (a_0 + eA_0 - \mu) \bar{\psi} \psi
$$

B.C.: $\psi(\beta, x) = -\psi(0, x)$, $\bar{\psi}(\beta, x) = -\bar{\psi}(0, x)$
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As it stands, this expression is formally obtained from (10.1) by Wick-rotating the
time axis through 90 degrees and imposing the anti-periodic boundary condition
on the fermion fields. One should remember however that the rigorous derivation
follows from (19.21), and that the expression (19.24) is for grand canonical distri-
butions.

The free propagator for the fermion field $\psi$ is defined by

$$G(x, y; \tau_1 - \tau_2) = -\langle T_{\tau_1}[\psi(x, \tau_1) \bar{\psi}(y, \tau_2)] \rangle_0 \quad (0 < \tau_1, \tau_2 < \beta). \quad (19.25)$$

It is easy to see

$$G(x, y; \tau) = -G(x, y; \tau + \beta) \quad \text{for} \quad -\beta < \tau < 0. \quad (19.26)$$

In the Landau gauge, analogously to (12.5), we find

$$G(x, y; \tau) = \frac{1}{i l_1} \sum_{n, p} e^{-2\pi i p(x_1 - y_1)/l_1} v_n \left( \frac{x_2 - \bar{x}_2}{l} \right) e^{-2\pi i p(x_1 - y_1)/l_1} v_n \left( \frac{y_2 - \bar{y}_2}{l} \right)$$

$$\times e^{-(\epsilon_n - \mu)\tau} \left\{ \begin{array}{ll} \rho_n(\beta) - 1 & \text{for} \quad 0 < \tau < \beta \\
\rho_n(\beta) & \text{for} \quad -\beta < \tau < 0 \end{array} \right. \quad (19.27)$$

$$= e^{i\phi(x, y)}. G_0(x - y, \tau).$$

Here

$$\phi(x, y) = -\epsilon(N) \frac{1}{2l^2} (x_1 - y_1)(x_2 + y_2)$$

$$G_0(x, \tau) = \sum_n \frac{1}{2\pi l^2} \int_{-\infty}^{\infty} dz e^{-izx_1/l} v_n[z - \bar{z}(x_2)] v_n[z + \bar{z}(x_2)]$$

$$\times e^{-(\epsilon_n - \mu)\tau} \left\{ \begin{array}{ll} \rho_n(\beta) - 1 & \text{for} \quad 0 < \tau < \beta \\
\rho_n(\beta) & \text{for} \quad -\beta < \tau < 0 \end{array} \right. \quad (19.28)$$

$$\bar{z}(x_2) = \epsilon(N) \frac{x_2}{2l}.$$

As in the $T = 0$ case, the Green’s function $G(x, y; \tau)$ is not manifestly translation
invariant, but is invariant up to a gauge transformation.

In the Fourier space

$$G_0(x, \tau) = \frac{1}{\beta} \sum_r \int \frac{dp}{(2\pi)^2} G_0(\omega_r, p) e^{-i\omega_r \tau + ip \cdot x}$$

$$G_0(\omega_r, p) = \int_0^{\beta} d\tau \int dx \ G_0(x, \tau) e^{i(\omega_r \tau - p \cdot x)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{i\omega_r - \epsilon_n + \mu} \int \frac{dx_2}{l} e^{-ip_2x_2} v_n[p_1 + \bar{z}(x_2)] v_n[p_1 - \bar{z}(x_2)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{i\omega_r - \epsilon_n + \mu} \int \frac{dx_2}{l} e^{-ip_2x_2} v_n[p_1 + \bar{z}(x_2)] v_n[p_1 - \bar{z}(x_2)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{i\omega_r - \epsilon_n + \mu} \int \frac{dx_2}{l} e^{-ip_2x_2} v_n[p_1 + \bar{z}(x_2)] v_n[p_1 - \bar{z}(x_2)]$$

$$= \frac{(2r + 1)\pi}{\beta}. \quad (19.29)$$
We also note that
\[ \langle \bar{\psi}(x, \tau) \psi(x, \tau) \rangle_0 = G(x, x; 0^-) \]
\[ = \sum_n \frac{1}{2\pi l^2} \int_{-\infty}^{\infty} dz \, v_n(z)^2 \rho_n(\beta) = \frac{1}{2\pi l^2} \sum_n \rho_n \]
\[ = n_e. \]

Similarly
\[ \langle \bar{\psi} \cdot D_k \psi - \bar{D}_k \psi \cdot \psi \rangle = 0. \]

(19.30) and (19.31) appear in the first order perturbation.

As in (10.6), one can expand the Euclidean effective action \( I_E \) in a power series of \( a^{(1)} + eA^{(1)} \):
\[ I_E[a + eA] \]
\[ = \beta \Omega_0 + \int dx \, n_e(a_0 + eA_0)(x) \]
\[ - \int dx dy \, \frac{1}{2} (a + eA)^{(1)}(x) \Gamma_{E}^{\mu\nu}(x - y)(a + eA)^{(1)}(y) + \cdots \]
\[ = \beta \Omega_0 + \int dx \, n_e(a_0 + eA_0)(x) \]
\[ - \frac{1}{\beta} \sum_r \int \frac{dp}{(2\pi)^2} \frac{1}{2} (a + eA)^{(1)}(-p) \Gamma_{E}^{\mu\nu}(p)(a + eA)^{(1)}(p) + \cdots \]

where
\[ x = (\tau_1, x), \quad y = (\tau_2, y), \quad \int dx = \int_0^\beta d\tau \int dx \]
\[ p = (\omega_r, p), \quad \omega_r = \frac{2\pi r}{\beta}. \]

A relationship between \( \Gamma_{E}^{\mu\nu} \) and the response function at finite temperature can be found easily. Let us denote
\[ \bar{j}^0(\tau, x) = \bar{\psi}(\tau, x) \psi(\tau, x) \]
\[ \bar{j}^k(\tau, x) = -\frac{i}{2m} \left\{ \bar{\psi}(\tau, x) \cdot D_k \psi(\tau, x) - D_k \bar{\psi}(\tau, x) \cdot \psi(\tau, x) \right\}. \]

Then from (19.13) and (19.16) it follows that
\[ \frac{\delta \text{Tr} U(\beta)}{\delta a_0(\tau, x)} = -\text{Tr} U(\beta, \tau) j^0(x) U(\tau, 0) \]
\[ = -\text{Tr} U(\beta, 0) \bar{j}^0(\tau, x) \]
\[ = -\langle \bar{j}^0(\tau, x) \rangle \cdot \text{Tr} U(\beta) \]

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so that
\[
\frac{\delta I_E[a+eA]}{\delta a_0(\tau,x)} = +\langle j^0(\tau,x) \rangle \\
\frac{\delta I_E[a+eA]}{\delta a^k(\tau,x)} = -\langle j^k(\tau,x) \rangle .
\]
(19.34)

Obviously
\[
\langle j^\mu(\tau,x) \rangle = \langle j^\mu(x) \rangle \quad \text{for static } a+eA.
\]
(19.35)

We denote the thermal average of the induced current by
\[
\bar{J}_\text{ind}^\mu(\tau,x) = \langle j^\mu(\tau,x) \rangle - \delta_\mu^0 n_e.
\]
(19.36)

Making use of (19.32) and (19.34), one finds
\[
\bar{J}_\text{ind}^\mu(x) = -\int dy \Gamma^{\mu\nu}(x-y) (a+eA)_\nu^{(1)}(y) + \cdots,
\]
or in the Fourier space
\[
\bar{J}_\text{ind}^\mu(\omega_r,p) = -\Gamma^{\mu\nu}(\omega_r,p) (a+eA)_\nu^{(1)}(\omega_r,p) + \cdots.
\]
(19.37)

As in the zero temperature case, the self-consistent field approximation (SCF) is defined by Eq. (19.37) and the field equations for the gauge fields with the source replaced by the thermal average \( \langle j^\mu(\tau,x) \rangle \).

Of course, for general \( \tau \)-dependent field configurations, appropriate analytic continuation of the field equations is necessary. In this article we mostly restrict ourselves to physics for static configurations \( \omega_r = 0 \), for which field equations are the same as those at \( T = 0 \). In particular, the response functions \( Q_n \) and \( Q_c \) are defined in the same way as at \( T = 0 \), by introducing static external fields. In the linear approximation, in which higher order terms in (19.37) are neglected, one has

\[
\bar{J}_\text{ind}^\mu \text{ linear} = Q_n^{\mu\nu} a_\nu^{\text{ext}} \quad \text{or} \quad Q_c^{\mu\nu} a_\nu^{\text{ext}}.
\]
(19.38)

The relation among \( \Gamma, Q_n, \) and \( Q_c \) remains intact for \( \omega_r = 0 \). (The relation is valid even for \( \omega_r \neq 0 \) upon the substitution \( \omega \to i\omega_r \).)

We examine the current conservation:
\[
i \frac{\partial}{\partial \tau} j^0(\tau,x) = U(\tau)^{-1}i[H_e(\tau) - \mu \hat{N}, j^0(x)] U(\tau)
\]
\[= U(\tau)^{-1} \left( - \nabla_k j^k(x) \big|_{(a+eA)(\tau)} \right) U(\tau)
\]
\[= -\nabla_k \bar{j}^k(\tau,x) .
\]
Hence
\[
i \frac{\partial}{\partial \tau} j^0 + \nabla_k \bar{j}^k = 0.
\]
(19.39)
For the kernel $\Gamma^{\mu\nu}$ it implies that

$$i\omega_r \Gamma^{0\nu}_E - p_k \Gamma^{k\nu}_E = 0 = i\omega_r \Gamma^{0\mu}_E - p_k \Gamma^{\mu k}_E .$$

(19.40)

Other relations such as $\Gamma^{\mu\nu}(p) = \Gamma^{\mu\nu}(-p)$ remain intact. The decomposition of $\Gamma^{\mu\nu}_E$ is given by

$$
\Gamma^{00}_E(\omega_r, q) = q^2 \Pi^E_0 \\
\Gamma^{0j}_E(\omega_r, q) = +i\omega_r q_j \Pi^E_0 - i\epsilon_{jk} q_k \Pi^E_1 \\
\Gamma^{0k}_E(\omega_r, q) = +i\omega_r q_k \Pi^E_0 + i\epsilon_{jk} q_k \Pi^E_1 \\
\Gamma^{jk}_E(\omega_r, q) = -\delta_{jk} \omega_r^2 \Pi^E_0 - \epsilon_{jk} \omega_r \Pi^E_1 - (q^2 \delta_{jk} - q_j q_k) \Pi^E_2
$$

(19.41)

where all $\Pi_j$'s are functions of $\omega_r^2$ and $q^2$ only. In a frame $q = (q, 0)$

$$
\Gamma^{\mu\nu}_E = \begin{pmatrix}
q^2 \Pi^E_0 & i\omega_r q_0 \Pi^E_0 + iq \Pi^E_1 \\
i\omega_r q_1 \Pi^E_0 - \omega_r^2 \Pi^E_0 & -\omega_r \Pi^E_1 \\
-iq \Pi^E_1 + \omega_r \Pi^E_0 - \omega_r^2 \Pi^E_0 - q^2 \Pi^E_2
\end{pmatrix} .
$$

(19.42)

Mathematically

$$\Gamma^{\mu\nu}_E = \Gamma^{\mu\nu}(i\omega_r, q) , \quad \Pi^E_k = \Pi_k(-\omega_r^2, q^2) .$$

(19.43)

As in the $T = 0$ case, we need to evaluate one loop diagrams in Fig. 8 in Section 10 to find $\Gamma^{\mu\nu}_E$. Computations are completely parallel to those in Section 12. The diagram (a) yields the linear term in (19.32). The diagram (b) yields

$$\Gamma^{(b)jk}_E = -\delta_{jk} \frac{1}{m} \langle \bar{\psi} \psi \rangle_0 = -\delta_{jk} \frac{n_c}{m} .$$

(19.44)

For the diagrams (c), (d), and (e), the phase factor $\phi(x, y)$ in the propagator $G(x, y)$, (19.27), completely cancels. We have

$$
\Gamma^{(c)00}_E = -\frac{1}{\beta} \sum_s \int \frac{dp}{(2\pi)^2} G_0(p) G_0(p - q) \\
\Gamma^{(d)0j}_E = -\frac{i}{2m\beta} \sum_s \int \frac{dp}{(2\pi)^2} \left\{ G_0(p) \cdot D^-_j G_0(p - q) + D^+_j G_0(p) \cdot G_0(p - q) \right\} \\
\Gamma^{(e)jk}_E = \frac{1}{4m^2\beta} \sum_s \int \frac{dp}{(2\pi)^2} \left\{ D^-_k G_0(p) D^-_j G_0(p - q) + D^+_j G_0(p) D^+_k G_0(p - q) \\
+ D^+_j D^-_k G_0(p) \cdot G_0(p - q) + G_0(p) D^-_j D^+_k G_0(p - q) \right\}
$$

(19.45)
where
\[ p = (\omega_s, p) \quad , \quad \omega_s = \frac{2\pi(s + \frac{1}{2})}{\beta} \]
\[ q = (\omega_r, q) \quad , \quad \omega_r = \frac{2\pi r}{\beta} \]

\( G_0(\omega_s, p) \) is defined in (19.29). Without confusion we have adopted the same notation for the propagator as in the \( T = 0 \) case.

The only technical change to be made in comparison with the computations in Section 12 is the infinite sum over frequencies. Employing the formula
\[ \sum_{s = -\infty}^{\infty} g(s) = -\sum \text{poles } a_j \text{ of } g(z) \text{ Res } (\pi \cot \pi z \cdot g(z), a_j) \]
one easily finds that
\[ f(\omega_r; n, m) \equiv -\frac{1}{\beta} \sum_s \frac{1}{[i\omega_s - \epsilon_n + \mu][i(\omega_s - \omega_r) - \epsilon_m + \mu]} \]
\[ \beta \rho_n (1 - \rho_n) \quad \text{for } \omega_r = 0 \text{ and } n = m, \quad (19.46) \]
\[ -\frac{\rho_n - \rho_m}{\epsilon_n - \epsilon_m - i\omega_r} \quad \text{otherwise.} \]

\( \rho_n(\beta) \) is the distribution function for the \( n \)-th Landau level. This is the only place where \( \rho_n \) shows up in the computation of \( \Gamma^{\mu\nu}_{E} \). In other words, finite temperature effects in the linear approximation are contained solely in the discrete sum above.

We shall see that the diagonal component at zero frequency, \( n = m \) and \( \omega_r = 0 \), leads to unique behavior of anyon fluids at \( T \neq 0 \).

Working in the frame \( q = (q, 0) \), one finds \( \Pi^E_1 \)'s to be
\[ q^2 \Pi^E_0 = \Gamma^{00}_{E} = \frac{1}{2\pi l^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(\omega_r; n, m) C^{(0)}_{nm}(ql)^2 \]
\[ q \Pi^E_1 = -i \Gamma^{02}_{E} = \frac{\epsilon(N)}{2\pi ml^3} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(\omega_r; n, m) C^{(1)}_{nm}(ql) C^{(0)}_{nm}(ql) \]
\[ -\omega_r^2 \Pi^E_0 - q^2 \Pi^E_2 = \Gamma^{22}_{E} = -\frac{n_e}{m} + \frac{1}{2\pi ml^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(\omega_r; n, m) C^{(1)}_{nm}(ql)^2 \]  
\[ (19.47) \]

We are going to examine implications of the above result, particularly in the static case (\( \omega_r = 0 \)), in the following three sections.

20. Thermodynamic potential in inhomogeneous fields

In this section we first summarize the result in the previous section in the form of free energy or thermodynamic potential for slowly varying static gauge fields
configurations, from which it follows that the Meissner effect becomes partial at finite temperature at least in the self-consistent field method (SCF). Although it may be an artifact of the approximation which neglects vortices, the behavior found here is unique and seems essential for understanding properties of anyon fluids.

The infinite sum in (19.47) can be performed for static, slowly varying gauge field configurations. Let us define

\[
\Delta_p = \sum_{n=0}^{\infty} (n + \frac{1}{2})^p \rho_n
\]

\[
\delta_p = \sum_{n=0}^{\infty} (n + \frac{1}{2})^p \rho_n (1 - \rho_n)
\]

(20.1)

It follows from (17.5) and (17.9) that

\[
\Delta_0(T, B) = \nu
\]

\[
\Delta_0(T, 0) = |N|
\]

(20.2)

At \(T = 0\) and \(B = 0\), \(\rho_n = 0\) or 1 so that

\[
\Delta_0 = |N|, \quad \Delta_1 = \frac{N^2}{2}, \quad \Delta_2 = \frac{|N|(4N^2 - 1)}{12}, \quad \ldots
\]

\[
\delta_n = 0
\]

(20.3)

at \(T = 0, B = 0\).

In evaluating off-diagonal sums \((n \neq m)\) in (19.47), one also needs

\[
\sum_{n=0}^{\infty} I_n = S[I]
\]

where

\[
I_n
\]

\[
S[I]
\]

\[
(\rho_n - \rho_{n-1}) n
\]

\[
- \Delta_0
\]

\[
(\rho_n - \rho_{n-1}) n^2
\]

\[
- 2\Delta_1
\]

\[
(\rho_n - \rho_{n-1}) n^3
\]

\[
- 3\Delta_2 - \frac{5}{4}\Delta_0
\]

\[
(\rho_n - \rho_{n-2}) n(n - 1)
\]

\[
- 4\Delta_1
\]

\[
(\rho_n - \rho_{n-2}) n(n - 1)(2n - 1)
\]

\[
- 12\Delta_2 - 3\Delta_0
\]

\[
(\rho_n - \rho_{n-3}) n(n - 1)(n - 2)
\]

\[
- 9\Delta_2 - \frac{15}{4}\Delta_0
\]

(20.4)

With these preparations, we start to evaluate \(\Pi_0^E\) in (19.47) at \(\omega_r = 0\). Employing (19.46), one finds

\[
q^2 \Pi_0^E(0, q^2) = -\frac{\beta}{2\pi \lambda^2} \sum_{n=0}^{\infty} \rho_n (1 - \rho_n) C_{nn}(ql)^2 + \frac{\beta}{2\pi \lambda^2} \sum_{n=0}^{\infty} \rho_n (1 - \rho_n) C_{nn}(ql)^2
\]

(20.5)
where we have made use of $\epsilon_n = (n + \frac{1}{2})/(ml^2)$. For slowly varying configurations ($ql \ll 1$) the expansions (12.30) – (12.32) can be employed:

$$
q^2 \Pi_0^E(0, q^2) = -\frac{m}{\pi} \sum_{n>m} \rho_n - \rho_m \left\{ \frac{n}{2} \delta_{n,m+1} (ql)^2 + \frac{n^2}{4} \delta_{n,m+1} - \frac{n(n-1)}{16} \delta_{n,m+2} \right\} (ql)^4 + \cdots
$$

$$
+ \frac{\beta}{2\pi l^2} \sum_{n=0}^{\infty} \rho_n (1 - \rho_n) \left\{ 1 - \frac{2n+1}{4} (ql)^2 + \frac{2n^2+2n+1}{32} (ql)^4 + \cdots \right\}
$$

so that

$$
\Pi_0^E(0, q^2) = -\frac{\nu m}{4\pi^2 n_e} \left\{ \nu - \frac{3}{4} \Delta_1 (ql)^2 + \cdots \right\}
$$

$$
+ \frac{1}{q^2} \frac{\beta n_e}{\nu} \delta_0 - \frac{\beta}{2\pi} \left\{ \delta_1 - \left( \frac{3}{8} \delta_2 + \frac{1}{32} \delta_0 \right) (ql)^2 + \cdots \right\}
$$

(20.6)

$\Pi_1^E$ and $\Pi_2^E$ are similarly evaluated. One finds that for $\Pi_1^E$

$$
\Pi_1^E(0, q^2) = + \frac{\epsilon(N)}{2\pi} \left\{ \Delta_0 - \frac{3}{2} \Delta_1 (ql)^2 + \cdots \right\}
$$

$$
- \frac{\epsilon(N) \beta}{2\pi ml^2} \left\{ \delta_1 - \left( \frac{3}{4} \delta_2 + \frac{1}{16} \delta_0 \right) (ql)^2 + \cdots \right\}
$$

(20.7)

$$
= + \frac{\epsilon(N)}{2\pi} \left\{ \nu - \frac{3}{2} \Delta_1 (ql)^2 + \cdots \right\}
$$

$$
- \epsilon(N) \frac{\beta n_e}{m\nu} \left\{ \delta_1 - \left( \frac{3}{4} \delta_2 + \frac{1}{16} \delta_0 \right) (ql)^2 + \cdots \right\}
$$

For $\Pi_2^E$

$$
-q^2 \Pi_2^E(0, q^2) = \frac{n_e}{m} + \frac{1}{2\pi ml^2} \left\{ \Delta_0 - 2\Delta_1 (ql)^2 + \frac{3}{2} \Delta_2 + \frac{1}{8} \Delta_0 \right\} (ql)^4 + \cdots
$$

$$
+ \frac{\beta}{2\pi ml^2} \left\{ \delta_2 (ql)^2 - \frac{1}{2} \delta_3 + \frac{1}{8} \delta_1 \right\} (ql)^4 + \cdots
$$

so that

$$
\Pi_2^E(0, q^2) = + \frac{1}{2\pi m} \left\{ 2\Delta_1 - \left( \frac{3}{2} \Delta_2 + \frac{1}{8} \Delta_0 \right) (ql)^2 + \cdots \right\}
$$

$$
- \frac{\beta n_e}{m^2\nu} \left\{ \delta_2 - \left( \frac{1}{2} \delta_3 + \frac{1}{8} \delta_1 \right) (ql)^2 + \cdots \right\}
$$

(20.8)
Neutral and Charged Anyon Fluids

There are a few things to be recognized. $\Pi^E_0$ develops a pole $(1/q^2)$ at $T \neq 0$, which, as we shall see shortly, leads to a partial Meissner effect. We also argue in the next section that it determines the scale of the phase transition temperature, $T_c$. Secondly, the diamagnetic term $n_e/m$ in $q^2\Pi^E_2$ is cancelled by the induced term as at $T = 0$. No pole develops in $\Pi^E_2$. Thirdly $\Pi^E_1(0,0)$ determines the induced Chern-Simons term, which is not exactly $N/2\pi$ at $T \neq 0$. In other words, the cancellation between the bare and induced Chern-Simons terms is not exact. In some of the early literature in the anyon superconductivity it was said that the exact cancellation is essential for superconductivity, which, as we shall show, is rather misleading.

It follows from (19.32) and (19.41) that

$$\Omega_e[a,A] = \Omega_0[a,A] + \int dx \, n_e(a_0 + eA_0) - \int dx \left\{ \frac{1}{2} (a + eA)_0 q^2 \Pi^E_0 (a + eA)_0 - (a + eA)_0 \Pi^E_1 (b^{(1)} + eB^{(1)}) - \frac{1}{2} (b^{(1)} + eB^{(1)}) \Pi^E_2 (b^{(1)} + eB^{(1)}) \right\} + \cdots ,$$

where $\Pi^E_2(0,q^2) = \Pi^E_2(0,-\nabla^2)$. Insertion of (20.6) – (20.8) leads to

$$\Omega_e[a,A] = \Omega_0[a,A] + \int dx \left\{ n_e(a_0 + eA_0) - \frac{\beta n_e}{2\nu} \delta_0 (a_0 + eA_0)^2 - \left( \frac{\nu^2 m}{8\pi^2 n_e} - \frac{\beta}{4\pi} \delta_1 \right) (\nabla(a_0 + eA_0))^2 \right. \\
+ \left( \frac{3\nu^2 m}{64\pi^3 n_e} \Delta_1 - \frac{\nu \beta}{256\pi^3 n_e} (\delta_0 + 12\delta_2) \right) (\nabla^2 (a_0 + eA_0))^2 + \epsilon(N) \left( \frac{\nu}{2\pi} - \frac{\beta n_e}{m\nu} \delta_1 \right) (a_0 + eA_0) (b^{(1)} + eB^{(1)}) \\
- \epsilon(N) \left( \frac{3\nu}{8\pi^2 n_e} \Delta_1 - \frac{\beta}{32\pi m} (\delta_0 + 12\delta_2) \right) \nabla(a_0 + eA_0) \nabla(b^{(1)} + eB^{(1)}) \\
+ \left( \frac{1}{2\pi m} \Delta_1 - \frac{\beta n_e}{2m^2 \nu} \delta_2 \right) (b^{(1)} + eB^{(1)})^2 \\
- \left( \frac{\nu}{64\pi^2 mn_e} (\nu + 12\Delta_2) - \frac{\beta n_e}{32\pi m^2} (\delta_1 + 4\delta_3) \right) (\nabla(b^{(1)} + eB^{(1)}))^2 + \cdots \right\} .$$

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In applications at long wave length, dominant terms are given by
\[ \Omega_{\text{tot}}[a, A] = \Omega_e[a, A] + \int dx \left\{ \frac{1}{2} (F_{0k}^2 + B^2) - e n_e A_0 - \frac{N}{2\pi} a_0 b \right\} = (\text{const}) + \int dx \left\{ \frac{1}{2} (F_{0k}^2 + B^2) - \frac{N}{2\pi} a_0 b^{(1)} \right. 
- \frac{1}{2} \left( \frac{\nu}{2\pi} \right)^2 e^2 c_0 (a_0 + e A_0)^2 - \frac{\nu^2 m}{8\pi^2 n_e} (1 - c_1) (f_{0k} + e F_{0k})^2 
+ \epsilon(N) \frac{\nu}{2\pi} (1 - c_1) (a_0 + e A_0) (b^{(1)} + e B^{(1)}) 
+ \frac{\Delta_1}{2\pi m} (1 - c_2) (b^{(1)} + e B^{(1)})^2 + \cdots \right\} , \]
where dimensionless constants \( c_j(T,B)'s \) are defined by
\[ c_0 = \frac{4\pi^2 n_e \beta}{e^2 \nu^3} \delta_0 , \quad c_1 = \frac{2\pi n_e \beta}{m \nu^2} \delta_1 , \quad c_2 = \frac{\pi n_e \beta}{m \nu^2} \delta_2 . \]
Equations are given by
\[ \frac{\delta\Omega_{\text{tot}}}{\delta a_\mu(x)} = 0 = \frac{\delta\Omega_{\text{tot}}}{\delta A_\mu(x)} . \]

It is appropriate to examine numerical values of various coefficients. With the values \( m = 2m_e, n_e = 2 \times 10^{14} \text{ cm}^{-2} \), and \( d = 5\text{ Å} \),
\[ \frac{m e^2}{\pi^2 n_e} = 48 , \quad \frac{e^2}{\pi m} = 1.1 \times 10^{-5} , \quad \frac{\pi n_e}{m^2} = 2.4 \times 10^{-7} . \]
\( \delta_0 \) in \( e_k \) is approximately given by, for \( N = \pm 2 \),
\[ \delta_0 \sim \text{Max} \left( \frac{2e B^{(0)}(0)}{\delta_b(0)} , e^{-\epsilon/2T} \right) \sim \text{Max} \left( \frac{B^{(0)}(0)}{6 \times 10^6 \text{ G}} , 10^{-6 \cdot (100K/T)} \right) . \]
Other relevant coefficients are, for \( T = 100 \text{ K} \),
\[ \frac{\pi^2 n_e}{2 e^2 T} = 1.2 \times 10^6 , \quad \frac{\pi n_e}{2 m T} = 14 . \]

All \( c_k's \) at \( B^{(0)} = 0 \) are suppressed exponentially in the \( T \to 0 \) limit. \( c_1 \) and \( c_2 \) are negligible \( (\ll 1) \) for \( T < 200 \text{ K} \), whereas \( c_0 \) suddenly becomes large around \( T = 100 \text{ K} \). (See the discussion in Section 22, around (22.2).) The dominant finite temperature effect is contained in the \( c_0 \) term in (20.10), which represents an effect similar to the Debye screening in plasmas.

21. Partial Meissner effect in SCF

Anyon fluids have quite unusual behavior at \( T \neq 0 \). In this section we examine a response against static inhomogeneous external perturbations, both solving in real
configurations and looking at response functions.\textsuperscript{44} We write Eq. (20.12) in the form
\[
-\frac{N}{4\pi}\epsilon^{\mu\nu\rho}f_{\nu\rho} = J_{\text{ind}}^\mu + n_e\delta^{\mu\nu}f_{\nu\rho} = \frac{\delta\Omega_e}{\delta a_\mu(x)}.
\]
(21.1)

(Note that \(a_k = -a^k\).) The induced current \(J_{\text{ind}}^\mu(x)\) is given by
\[
J_{\text{ind}}^0(x) = \epsilon(N)\frac{\nu}{2\pi}(1 - c_1)(b^{(1)} + eB^{(1)})
\]
\[-\left(\frac{\nu}{2\pi}\right)^2 e^2c_0(a_0 + eA_0) - \frac{\nu^2m}{4\pi^2n_e}(1 - c_1)\partial_k(f_{0k} + eF_{0k}) + \cdots ,
\]
\[
J_{\text{ind}}^k(x) = \epsilon(N)\frac{\nu}{2\pi}(1 - c_1)e^2\partial_k(b^{(1)} + eB^{(1)}) + \cdots .
\]
(21.2)

We remark that at finite temperature not only field strengths but also the time component of the vector potentials, \(a_0 + eA_0\), appears in the expression for \(J_{\text{ind}}^0\) in (21.2).

The identities
\[
\frac{N}{2\pi}\epsilon b^{(1)} = \text{div } E ,
\]
\[
\frac{N}{2\pi}\epsilon f_{0k} = \partial_k B ,
\]
may be employed to eliminate the Chern-Simons fields. Note that the integration of the latter leads to
\[
\frac{N}{2\pi}\epsilon a_0(x) = -B(x) + \text{const} ,
\]
where the constant has to be determined with the aid of, for instance, the neutrality condition at one point in a given configuration. (To be precise, only the constant part of \(a_0 + eA_0\) is relevant in (21.2).) Substituting (21.3) and (21.4) into (21.2), one finds
\[
\epsilon J_{\text{ind}}^0 = (\text{const}) - \left(\frac{\nu e^2}{2\pi}\right)^2 c_0 A_0 + \left(\frac{\nu}{|N|} - (1 - c_1)\frac{\nu^2me^2}{4\pi^2n_e}\right)\text{div } E
\]
\[+ \epsilon(N)\frac{\nu e^2}{2\pi}\left\{\frac{1 + \frac{\nu}{|N|}c_0 - c_1}{B - \frac{\nu e^2}{2\pi}(1 - c_1)\frac{m}{e^2n_e}\nabla^2 B}\right\} ,
\]
\[
\epsilon J_{\text{ind}}^k = \left\{\frac{\nu}{|N|}(1 - c_1) - \frac{\Delta_1 e^2}{\pi m}(1 - c_2)\right\}e^2\partial_k B
\]
\[+ \epsilon(N)\frac{\nu e^2}{2\pi}e^2\left\{(1 - c_1)e_l - (1 - c_2)\frac{4\pi\Delta_1}{m\nu|N|}\partial_l(\text{div } E)\right\} .
\]
(21.5)

The equations in (21.5) correspond to the London equations in the conventional superconductors. Combined with the Maxwell equations, they determine electromagnetic fields inside anyon fluids. In general, however, one more equation, Eq. (18.2), has to be supplemented to fix the constant part of the magnetic field.
To illustrate the problem, we consider a configuration in which an anyon fluid occupies a half plane, say, \( x_1 > 0 \). We apply an uniform external magnetic field \( B_{\text{ext}} \) in the empty space \((x_1 < 0)\). The problem is to find \( B(x) = B(x_1) \) for \( x_1 > 0 \) with the boundary condition \( B(0) = B_{\text{ext}} \).

To extract the essence, we suppose that \( \epsilon B_{\text{ext}} \ll b^{(0)} \). To good accuracy one can approximate \( \nu = |N| \). With the aid of the numerical evaluation for various parameters given in Sections 18 and 20, one finds that the Maxwell equations become

\[
(1 + c_0)B - \frac{m}{e^2 n_e} \nabla^2 B - \frac{N e^2}{2\pi} c_0 A_0 + \frac{Nm}{2\pi n_e} \text{div} \mathbf{E} + (\text{const}) = 0 \\
- \frac{2\Delta_1}{Nm} \partial_t B + E_t - \frac{4\pi \Delta_1}{me^2 N^2} \partial_t (\text{div} \mathbf{E}) = 0
\]  

(21.6)

For the configuration under consideration one expects

\[ B(x) = B_{\text{in}} + (B_{\text{ext}} - B_{\text{in}}) e^{-x_1/\lambda_0} \quad (x_1 > 0) . \]  

(21.7)

\( B_{\text{in}} = B(+\infty) \) does not vanish at \( T \neq 0 \). It is determined by Eq. (18.2)

\[ B_{\text{in}} = B_{\text{ext}} + M(T, B_{\text{in}}) . \]  

(21.8)

It follows from (21.6) that \( E_2 = 0 \).

It is checked posterior that \( |E_1/(B - B_{\text{in}})| \ll 1 \) so that the first equation of (21.6) yields

\[ (1 + c_0)(B - B_{\text{in}}) - \frac{m}{e^2 n_e} \partial_t^2 (B - B_{\text{in}}) = 0 . \]

Hence the damping length \( \lambda_0 \) is approximately given by

\[ \lambda_0(T)^2 = \frac{1}{1 + c_0} \frac{m}{e^2 n_e} = \frac{\bar{\lambda}^2}{1 + c_0} \]  

(21.9)

and

\[
\frac{E_1(x_1)}{B(x_1) - B_{\text{in}}} \sim - \frac{N}{m \lambda_0} \sim 3 \times 10^{-6} (1 + c_0)^{1/2} .
\]  

(21.10)

Due to the non-vanishing \( B_{\text{in}} \), the damping length \( \lambda_0(T) \) should not be confused with the penetration depth, which measures how fast the magnetic field decreases in the material. One complication in the calculation is that Eq. (21.8) cannot be solved analytically at \( T \neq 0 \). We present the result of numerical evaluation in
As can be seen from the figure, $B_{\text{in}}$ is vanishingly small at low temperature, but starts to increase around 70 K and becomes almost equal to $B_{\text{ext}}$ around 100 K. Here we have only a partial Meissner effect, at least in SCF. The magnetic field configuration is not a simple exponential decay.

To avoid solving Eq. (21.8), one may apply a spatially alternating external magnetic field. In the linear response theory it is reduced to examining $B_{\text{ext}}(x_1) = B_0 \epsilon(x_1)$ applied to a system occupying the whole space as we did in Section 16 at $T = 0$. We are going to show that the partial Meissner effect is observed in the response function, too.

The equations to be solved are the same as in Section 16, with $\Pi_k$’s being replaced by $\Pi_k^{E_k}$’s. Employing the expansions (20.6), (20.7), and (20.8), and keeping dominant terms, one finds, instead of (16.9) and (16.10),

$$\Delta_c = \frac{e^4}{q^4} \left( \frac{N}{2\pi} \right)^2 (1 + c_0 + \bar{\lambda}^2 q^3)$$

$$Q_{c2}^{22} = \frac{q^2}{\epsilon^2} \frac{1}{1 + c_0 + \bar{\lambda}^2 q^2}.$$

As $T$ changes, the response function for a charged anyon fluid smoothly varies. However, its behavior is different from that in conventional superconductors.
Recalling $e J_{\text{ext}}^2(\omega, q) = -2 B_0 \cdot (2\pi)^2 \delta(\omega) \delta(q_2)$, one finds

$$J_{\text{tot}}^2 = J_{\text{ext}}^2 + J_{\text{ind}}^2 = \left(1 - \frac{e^2}{q_1^2} Q_{\xi}^{22}\right) J_{\text{ext}}^2$$

$$= \frac{c_0 + \lambda^{-2} q_1^2}{1 + c_0 + \lambda^{-2} q_1^2} J_{\text{ext}}^2.$$  \hspace{1cm} (21.12)

The external current is not completely cancelled by the induced current:

$$J_{\text{tot}}^2(q = 0) = \frac{c_0}{1 + c_0} J_{\text{ext}}^2(q = 0) \neq 0.$$  \hspace{1cm} (21.13)

The magnetic field is

$$B(q) = \frac{i}{q_1} J_{\text{tot}}^2$$

$$= i q_1 \left\{ \frac{c_0}{1 + c_0} - \frac{1}{q_1^2} + \frac{1}{1 + c_0} \frac{1}{q_1^2 + \lambda_0^{-2}} \right\} J_{\text{ext}}^2.$$  \hspace{1cm} (21.14)

A new pole develops at $q = 0$. In the configuration space

$$B(x) = B_0 \epsilon(x_1) \left\{ \frac{c_0}{1 + c_0} + \frac{1}{1 + c_0} e^{-|x_1|/\lambda_0} \right\}.$$  \hspace{1cm} (21.15)

At $T = 0$, $c_0 = 0$ so that the Meissner effect is complete. At $T \neq 0$, $c_0 \neq 0$, resulting a partial Meissner effect. As we have seen in the previous section, $c_0(T)$ suddenly becomes very large around $T = 100$ K. Therefore the Meissner effect effectively terminates around this temperature. In the approximation (SCF) in use, however, there does not result a phase transition. We shall argue in the next section that a phase transition should result if vortices are incorporated.

With the aid of (21.15) one can define an effective penetration depth, $\lambda_{\text{SCF}}$, which measures the rate of the change of the magnetic field. $\lambda_{\text{SCF}}(T; d)$ is related to the change of $B(x_1)$ over a distance $d$ by

$$e^{-d/\lambda_{\text{SCF}}} = \frac{B(d)}{B_0}.$$  \hspace{1cm} (21.16)

It depends on $d$. As a typical value we take $d = \bar{\lambda}$. Then

$$\frac{\bar{\lambda}}{\lambda_{\text{SCF}}} = - \ln \left\{ \frac{c_0}{1 + c_0} + \frac{1}{1 + c_0} e^{-(1+c_0)^{1/2}} \right\}.$$  \hspace{1cm} (21.17)

Approximately

$$\frac{\lambda_{\text{SCF}}}{\lambda} = \begin{cases} 1 + (e - 1.5)c_0 & \text{for } c_0 \ll 1, \\ c_0 & \text{for } c_0 \gg 1. \end{cases}$$  \hspace{1cm} (21.18)
The behavior of $\lambda_{\text{SCF}}(T)$ is depicted in Fig. 13.

As we have demonstrated, the dominant finite temperature effect is contained in $c_0(T)$. The cancellation of the bare Chern-Simons term by the induced one, for instance, is not exact at $T \neq 0$, since $c_1(T) \neq 0$. (See Eq. (20.10).) However, its effect is numerically negligible. For the Meissner effect, "$c_0$" is important. It makes the Meissner effect partial at $T \neq 0$.

Before closing the section, we briefly mention about the subtlety in neutral anyon fluids. The response function for a neutral anyon fluid at finite $T$ is given by

$$Q^{22}_n(q) \sim \frac{q^2}{\bar{c}_0 + (mq^2/n_e)} ,$$  \hspace{1cm} (21.19)

where

$$\bar{c}_0 = e^2 c_0 = \frac{4\pi^2 n_e}{\nu^3 T} \delta_0 .$$

It follows that the two limits, $q \to 0$ and $T \to 0$ do not commute with each other. It may reflect an instability in the system of neutral anyon fluids.

22. $T_c$

How large is $T_c$, if there is a phase transition? Having analysed properties of charged anyon fluids, we are in an awkward position. In the linearized SCF, or equivalently in RPA, we have seen no evidence for a phase transition, or more precisely, mathematical singularities in physical quantities, at finite temperature. For instance, we have seen in the previous section that the penetration depth $\lambda_{\text{SCF}}(T)$ rapidly increases around $T_c' \sim 100$ K, but never diverges.
We argue that this is an artifact of the approximation in use, and that in a full theory a charged anyon fluid should exhibit a phase transition around $T'_c$.

Crucially missing in the previous treatment is a vortex. It is missing, because the linearized version of the SCF equations (15.7) are linear in fields, and therefore do not admit a quantized flux. In terms of the effective theory obtained by integrating the fermion fields $\psi$, one needs to retain higher order terms, cubic, quartic $\cdots$ in the field $(a + eA)^{(1)}$. It is a challenging problem to show how a vortex solution comes out from such an effective theory.

Previously the quantization of vorticity in a neutral anyon fluid was examined by Hanna, Laughlin, and Fetter in the Hartree-Fock approximation. They showed that an elementary excitation has a vorticity given by (fundamental unit) $|N|$, although it has an infinitely large energy. Kitazawa and Murayama have examined effects of vortex-antivortex pair formation in a neutral anyon fluid at $T \neq 0$. They have contended that there is a Kosterlitz-Thouless phase transition at $T = \frac{1}{8} \epsilon$ for $|N| = 2$. The underlying assumption is that there are vortex pair excitations with logarithmic interactions.

Having vortex-antivortex pair excitations is one promising way of obtaining a phase transition in anyon fluids. Supposing abundant pair excitations, one still has to elaborate Kitazawa and Murayama’s argument for charged anyon fluids.

First of all interactions among vortices are not logarithmic at low temperature. The Meissner effect is operating so that interactions are exponentially suppressed at large distances. An energy of a single vortex, which is not known yet, must be a dominant factor at low $T$.

The situation becomes more complicated as temperature increases. As we have observed in the previous section, the Meissner effect effectively terminates around $T'_c$. There would be no screening of magnetic fields any more. The interaction among vortices becomes long-ranged, and the entropy factor becomes important. Whether or not this leads to a phase transition is a matter subject to future investigation. Any way there is not any trace of superconductivity well above $T'_c$. It is quite likely that $T_c$, which separates the superconducting and normal states, turns out around $T'_c$.

$T'_c$ signifies a temperature where $c_0(T)$ becomes large. From (20.11) to (20.15) one finds

$$ c_0(T) = \frac{\pi n_{e}^{d=2} d}{\alpha |N|^b T} \exp \left( - \frac{\pi n_{e}^{d=2}}{|N|mT} \right). \quad (22.1) $$

Numerically, for $N = \pm 2$,

$$ c_0(T) = \frac{121 K}{T} \frac{n_{e}^{d=2} d}{2 \times 10^{14} \text{cm}^{-2} 5 \lambda} \times \exp \left\{ -13.83 \left( \frac{100 K}{T} \frac{n_{e}^{d=2}}{2 \times 10^{14} \text{cm}^{-2}} \frac{2m_e}{m} - 1 \right) \right\}. \quad (22.2) $$

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Typical values are

\[
\begin{align*}
T & \quad 50 & 75 & 100 & 125 & 150 & 175 & 200 \\
\langle c_0(T) \rangle & \quad 2.4 \times 10^{-6} & 1.6 \times 10^{-2} & 1.2 & 15 & 81 & 260 & 610
\end{align*}
\]

Note that if the values of both \( n_e = 2 \) and \( m \) are doubled, the value of \( c_0 \) is also doubled.

The temperature dependence of \( c_0 \) is contolled by the exponential factor. The critically important value is the ratio \( n_e = 2/m \). With the given value in (22.2), \( c_0 \) suddenly becomes large around 120 K.

With the assumption \( T_c \sim T_c' \), we conclude

\[
T_c \sim \frac{2 n_e^{d=2}}{|N| 2 \times 10^{14} \text{cm}^{-2}} \frac{2m_e}{m} \times 120 \text{K} \quad (22.3)
\]

Of course, the effective mass \( m \) is very difficult to determine experimentally so that one more piece of information is necessary to predict \( T_c \). One way is to express \( m \) in terms of the penetration depth at \( T = 0 \), \( \lambda(0) = \lambda \), with the aid of (18.12) and (18.13). Then

\[
T_c \sim \frac{2}{|N|} \frac{d}{5 \AA} \left( \frac{1400 \AA}{\lambda(0)} \right)^2 \times 90 \text{K} \quad (22.4)
\]

We stress that the value \( T_c \sim 100 \text{K} \) is very natural in anyon fluids. The dependence \( T_c \propto n_e \) or \( \lambda(0)^{-2} \) has been observed in high \( T_c \) superconductors.100 This behavior, however, is not necessarily special to anyon superconductors.101

Although the discussion in this section is only plausible and further investigation is necessary, one might take (22.4) as a very encouraging result.

23. Other important issues

In this article we have analysed some of the basic problems in anyon fluids, attempting to summarize the first two years of the theory of anyon superconductivity. We have seen that various approaches are equivalent, leading to many interesting physics consequences. Nevertheless, it is appropriate to say that so far we have had only partial understandings of the full theory.

There are many important issues left over. We list them for readers’ convenience. For details readers should consult original papers and other review articles.

23.1. Beyond RPA and the linearized SCF

Going beyond RPA and the linearized SCF is important in many respects. RPA and the linearized SCF fail to predict a phase transition. It is essential to incorporate vortices in the theory.

Hanna, Laughlin, and Fetter28 examined various quantities in the Hartree-Fock approximation in the first quantized theory, as was described in Section 7. They have found that, for \( |N| = 2 \), the correction to the phonon spectrum in neutral anyon
fluids in the long wave length limit is relatively small (∼ 10 %). More recently Dai, Levy, Fetter, Hanna, and Laughlin\textsuperscript{55} have performed a diagram analysis at $T = 0$ equivalent to the Hartree-Fock approximation. In addition to recovering the old result, they have shown that there results an important modification in the behavior of the spectrum at short wave lengths. Furthermore $\sigma_{xy}$ vanishes for $N = \pm 1$ thanks to higher order radiative corrections. RPA and the linearized SCF predict a non-vanishing $\sigma_{xy}$ even for $N = \pm 1$, which is certainly wrong as fermions are merely converted to bosons, but not to genuine anyons.

Dai et al. have solved the Schwinger-Dyson equations numerically which involve about 100 different diagrams. They have noticed the importance of gauge invariance, and have observed many cancellations among various diagrams. Their Feynman rules are based on the Hamiltonian obtained after eliminating Chern-Simons fields. As we have recognized in Sections 9 – 11, keeping the Chern-Simons gauge fields as auxiliary fields greatly simplifies computations. Gauge invariance is easily implemented, and the notion of self-consistent fields (beyond the linear approximation) can be established.

23.2. Pair-correlation

Is there “Cooper” pairing in anyon superconductors? The Hartree-Fock ground state employed in the literature (see Sections 6 and 7) does not look like the BCS ground state. Is there a different kind of pairing, then? Is there an off-diagonal long range order? The answer has not been known for sure.

There is an indication for pairing in the $N = \pm 2$ theory, which, however, is quite different from the Cooper pairing. The unique feature of the Hartree-Fock ground state is that the complete filling of the Landau levels is achieved independent of the density $n_e$, provided that $N$ is an integer ($\neq 0$).

To be precise, suppose that each Landau level has $N_L$ available states so that the total particle number is $N_e = n_e \cdot (\text{vol}) = |N| \cdot N_L$. Assume that $|N| \geq 2$. If one tries to add or delete one particle to or from the system, one necessarily has to put the particle in the next level, or make a hole in the top filled level, in order to preserve the Landau level picture. In other words, the picture of the complete filling breaks down.

However, if a set of $|N|$ particles are added or deleted, one can still maintain the complete filling. One of the Chern-Simons field equations, $(N/2\pi) \cdot b = j^0$, implies that the increase (decrease) of the particle number leads to the increase (decrease) of the Chern-Simons magnetic field such that precisely one more (less) state is available in each Landau level.

The states with the particle number $N_e$ and $N_e \pm |N|$ are very much alike. In a macroscopic system $N_e \gg 1$, thermal fluctuations give $\Delta N_e \sim \sqrt{N_e}$. It is quite likely that the real ground state is not an eigenstate of the particle number, but is a coherent state:

$$\Psi_G(\theta) = \sum_{|kN| \leq \sqrt{N_e}} e^{ik\theta} \Psi_G(N_e + k|N|) \quad (23.1)$$
In particular, for $N = \pm 2$, the structure (23.1) is exactly the same as in the BCS theory.\textsuperscript{102}

It is not clear if (23.1) implies pairing in the $N = \pm 2$ theory. We should remember, however, that the structure of the coherent state is indispensable in understanding many phenomena in superconductivity.

### 23.3. Flux quantization

A magnetic flux is trapped by a superconducting ring. The flux takes quantized values in the unit of $2\pi \hbar c/2e$. Can the $N = \pm 2$ anyon theory explain this behavior? No convincing argument has been provided. Leggett\textsuperscript{48} has argued that the flux quantization is not achieved in anyon theory at least in the Hartree-Fock approximation.

One needs to show two things. First it must be shown that an energy is locally minimized when a flux takes a quantized value. Secondly, the energy barrier height between the states with no flux and with one unit of flux is proportional to the volume, but not the boundary area, of the superconducting ring.

### 23.4. Vortices

We have often mentioned in the preceding sections that the establishment of vortices in anyon superconductors is one of the major problems to be solved. The previous analysis by Fetter, Hanna, and Laughlin must be elaborated. Inclusion of electromagnetic interactions is essential to take account of the Meissner effect. Nonlinear terms in SCF must play an important role. At finite temperature vortices might lead to a phase transition, too.

### 23.5. The Josephson effect

The nature of the coherent state (23.1) is most important for the Josephson effect.\textsuperscript{102} A Josephson junction consists of two superconductors separated by a barrier. Electron tunneling through the barrier brings about phase coherence over the entire system. The energy is minimized by volume if the two phases, which characterize $\theta$'s in (23.1) of the two superconductors, coincide. The difference between the two phases should generate a current.

Experiments show that in order for anyon superconductivity to describe high $T_c$ superconductors, $N$ must be equal to $\pm 2$. It seems that a Josephson effect should exist even for a junction between a BCS superconductor and an anyon superconductor of $N = \pm 2$.

### 23.6. Interlayer couplings

High $T_c$ superconductors have the layered structure characterized by CuO planes. Our analysis has been performed in the effective two-dimensional theory obtained by the dimensional reduction. The implicit assumption was that the system is uniform in the direction perpendicular to the two-dimensional CuO planes.

Anyon theory has a parameter $N$, or the generated statistics phase $\theta_{\text{statistics}} = \pi/N$. Physics depends on $\exp(i\theta_{\text{statistics}})$. The theory with $N$ differs from that with
−N if |N| ≥ 2. The two theories are related by P (parity) and T (time reversal) transformations. The idea behind the degeneracy of the ground state is that P and T symmetry is spontaneously broken.

P and T invariant quantities such as the penetration depth and resistance do not depend on the sign of N. Moreover we have seen that even though the Chern-Simons magnetic field b/e is very large (∼ 1000 T), the dependence of the P- and T-odd magnetization $M(B)$ on the Maxwell magnetic field $B$ is symmetric to good accuracy, $M(−B) \sim M(B)$. The asymmetry arises only to the order $O(\epsilon B/b)$. (See Section 18.)

Nevertheless it is important to know how the sign of N is ordered among adjacent layers. Is it ordered ferromagnetically with the same sign (FM ordering), or antiferromagnetically with the alternate sign (AFM ordering)? Or, is it randomly distributed?

This problem has been examined by Rojo, Canright, and Leggett. Interactions among electrons in different layers fix a pattern of the ordering. There are two types of interactions. One is of a potential type, and the other is the hopping of electrons from one layer to adjacent ones.

The detailed examination in the case of potential interactions has been provided by the above authors both numerically and analytically. They have shown that T-invariant potential interactions always prefer the AFM ordering. The hopping interaction is expected to induce a Josephson effect and lead to the FM ordering. It is not clear which one is dominant.

23.7. P and T violation

Many experiments have been performed to check P or T violation in high $T_c$ superconductors. The result is confusing, but a fair statement is that so far there has been no solid evidence for P or T violation in high $T_c$ superconductors.

As explained above, P and T are ordered either ferromagnetically or antiferromagnetically among layers. Most of the experiments done so far measure P and T violation in bulk. Therefore, if material has the AFM (P-, T-) ordering, the effect cancels in bulk.

Experiments in this category are the electromagnetic wave polarization and Hall voltage. Originally proposed by Wen and Zee, the polarization experiment tries to measure the P, T violation effect, determining the transmission and reflection coefficients of injected polarized electromagnetic waves. The polarization vector is rotated in anyon superconductors. There is inconsistency among various experiments, however. We note that even though Wen and Zee argue that there must be appreciable effects in FM-ordered anyon superconductors, a really microscopic computation of the magnitude of the effect is still lacking. Also some criticisms have been provided on the interpretation of experimental results.

The Hall voltage experiment measures the temperature dependence of the transverse voltage (Hall voltage) when a current flows in a thin film. There is a microscopic calculation of the effect. Theory predicts a peak in the Hall voltage around $T_c$. The peak value predicted is, for a thin film of thickness 1000 Å with a
current $10^{-4}$ Amp, $V_{\text{Hall}} \sim 2 \times 10^{-7}$ Volt. It is inversely proportional to the thickness. The effect is tiny. Preliminary experiments have been performed. Due to the inhomogeneity of samples and also tiny temperature variation in the samples, no conclusion has been obtained concerning the existence or non-existence of the Hall voltage.$^{111,112}$

There is one experiment which measures $P, T$ violation in one layer, and therefore is sensitive even for AFM-ordered anyon superconductors. It is the muon spin relaxation experiment.$^{105}$ Injected muons are stopped in high $T_c$ superconductors. Since muons are charged, the distribution of electrons is deformed. In effect, the distribution of holons, or our $\psi$ particles, deviates from the uniform value. $\delta J^0(x) \neq 0$ results. In anyon superconductors it induces a current $J^k_{\text{ind}} \neq 0$, since $Q^{k0}_c \neq 0$. $J^k_{\text{ind}} \neq 0$ in turn generates Maxwell magnetic field $B^3_{\text{ind}}$, which is felt by muon spins. Muon spins start to precess, which can be observed experimentally.

Halperin, March-Russel, and Wilczek gave a plausible argument, predicting $B^3_{\text{ind}} \sim 10$ G. The experiment observed no effect.$^{113}$ We remark that the magnitude of the effect can be determined more microscopically from the knowledge of the response function $Q_c$ at both $T = 0$ and $T \neq 0$ without making the long wave length approximation.

23.8. Anyons in spin systems

In this article we have not discussed how anyon excitations arise in material, particularly in spin systems. We have started with the picture that there are excitations called “holons” which obey half-fermion ($N = \pm 2$) statistics.

The derivation of anyons, or fractional statistics, in realistic spin models for high $T_c$ material has been attempted by many authors.$^{88-94}$ The issue has not been settled yet, since the arguments involve many approximations for which justification is not clear. Readers are advised to read original papers. A closely related subject is the existence of the flux phase or chiral spin liquid state.

We note that Laughlin has given a spin model which has Laughlin’s wave function in the fractional quantum Hall effect as an exact ground state wave function.$^{90}$ This model provides the existence proof of anyons in spin models.

23.9. Variations of anyon models

We have analysed one particular anyon model, namely non-relativistic spinless fermions with the minimal Chern-Simons interaction. It is the simplest model of anyons, and is based on the holon picture of Anderson’s.

There are many variations. They are interesting in their own right. Historically Chern-Simons gauge theory was first analysed in relativistic field theory. Along this tradition Lykken, Sonnenschein, and Weiss have examined a relativistic anyon model, Dirac fields with Chern-Simons interactions.$^{36,41}$ They have argued that the neutral model retains a superfluidity to all orders at $T = 0$. In passing, Imai et al. have shown that the non-renormalization theorem for the induced Chern-Simons coefficient holds even in non-relativistic theory.$^{22}$ At finite temperature the relativistic model behaves in a fashion similar to, but not same as, the nonrelativistic
model. To discuss a Meissner effect, superconductivity etc. in condensed matter systems, one has to analyse nonrelativistic models. Based on non-vanishing finite temperature corrections to the induced Chern-Simons coefficient, Lykken et al. have incorrectly concluded that a superconductivity is lost at $T \neq 0$. As we have seen in Sections 20 and 21, the important finite temperature correction is the $c_0$ term, but not the $c_1$ term (Chern-Simons coefficient), in the non-relativistic theory.

We have supposed that particles (anyons) have a single component, i.e. they have the same coupling to the Chern-Simons fields. There might be two kinds of anyons, a half of them having the $+$ coupling and the other half having the $-$ coupling. Furthermore, in addition to the minimal gauge coupling to Chern-Simons fields, particles might have magnetic moment interactions. Such a model has been investigated.\textsuperscript{53}

So far we have started with fermion fields $\psi$. It is also possible to start with boson fields. It is not exactly the same as the fermion model, since a bose field can condensate by itself and there is no complete filling of Landau levels. As was pointed out by Boyanovsky et al., boson models have rich structures many of which need to be clarified further.\textsuperscript{69} It is also known that boson models are particularly useful to construct phenomenological theory of fractional quantum Hall effect.\textsuperscript{18–25}

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