Singular and tangent slit solutions to the Löwner equation

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Abstract. We consider the Löwner differential equation generating univalent maps of the unit disk (or of the upper half-plane) onto itself minus a single slit. We prove that the circular slits, tangent to the real axis are generated by Hölder continuous driving terms with exponent 1/3 in the Löwner equation. Singular solutions are described, and the critical value of the norm of driving terms generating quasisymmetric slits in the disk is obtained.

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1. Introduction

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) be the unit disk and \( T := \partial \mathbb{D} \). The famous Löwner equation was introduced in 1923 [3] in order to represent a dense subclass of the whole class of univalent conformal maps \( f(z) = z(1 + c_1 z + \ldots) \) in \( \mathbb{D} \) by the limit

\[
f(z) = \lim_{t \to \infty} e^t w(z,t), \quad z \in \mathbb{D},
\]

where \( w(z,t) = e^{-t}z(1 + c_1(t)z + \ldots) \) is a solution to the equation

\[
\frac{dw}{dt} = -w \frac{e^{iu(t)} + w}{e^{iu(t)} - w}, \quad w(z,0) \equiv z,
\]

with a continuous driving term \( u(t) \) on \( t \in [0, \infty) \), see [3] page 117]. All functions \( w(z,t) \) map \( \mathbb{D} \) onto \( \Omega(t) \subset \mathbb{D} \). If \( \Omega(t) = \mathbb{D} \setminus \gamma(t) \), where \( \gamma(t) \) is a Jordan curve in \( \mathbb{D} \) except one of its endpoints, then the driving term \( u(t) \) is uniquely defined and we call the corresponding map \( w \) a slit map. However, from 1947 [5] it is known that solutions to (1) with continuous \( u(t) \) may give non-slit maps, in particular, \( \Omega(t) \) can be a family of hyperbolically convex digons in \( \mathbb{D} \).

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Marshall and Rohde \cite{MarshallRohde} addressed the following question: \emph{Under which condition on the driving term \(u(t)\) the solution to (1) is a slit map?} Their result states that if \(u(t)\) is Lip(1/2) (Hölder continuous with exponent 1/2), and if for a certain constant \(C_D > 0\), the norm \(\|u\|_{1/2}\) is bounded \(\|u\|_{1/2} < C_D\), then the solution \(w\) is a slit map, and moreover, the Jordan arc \(\gamma(t)\) is a quasislit (a quasiconformal image of an interval within a Stolz angle).

As they also proved, a converse statement without the norm restriction holds. The absence of the norm restriction in the latter result is essential. On one hand, Kufarev’s example \cite{Kufarev} contains \(\|u\|_{1/2} = 3\sqrt{2}\), which means that \(C_D \leq 3\sqrt{2}\). On the other hand, Kager, Nienhuis, and Kadanoff \cite{KagerNienhuisKadanoff} constructed exact slit solutions to the half-plane version of the Löwner equation with arbitrary norms of the driving term.

Let us give here the half-plane version of the Löwner equation. Let \(H = \{z : \text{Im } z > 0\}\), \(\mathbb{R} = \partial \mathbb{H}\). The functions \(h(z, t)\), normalized near infinity by \(h(z, t) = z - 2t/z + b_{-2}(t)/z^2 + \ldots\), solving the equation

\[
\frac{dh}{dt} = -2\frac{h - \lambda(t)}{h}, \quad h(z, 0) \equiv z,
\]

where \(\lambda(t)\) is a real-valued continuous driving term, map \(\mathbb{H}\) onto a subdomain of \(\mathbb{H}\). The question about the slit mappings and the behaviour of the driving term \(\lambda(t)\) in the case of the half-plane \(\mathbb{H}\) was addressed by Lind \cite{Lind}. The techniques used by Marshall and Rohde carry over to prove a similar result in the case of the equation \cite{MarshallRohde}, see \cite{MarshallRohde} page 765]. Let us denote by \(C_H\) the corresponding bound for the norm \(\|\lambda\|_{1/2}\). The main result by Lind is the sharp bound, namely \(C_H = 4\).

In some papers, e.g., \cite{KagerNienhuisKadanoff, Lind}, the authors work with equations \cite{KagerNienhuisKadanoff, Lind} changing \((-)\) to \((+)\) in their right-hand sides, and with the mappings of slit domains onto \(\mathbb{D}\) or \(\mathbb{H}\). However, the results remain the same for both versions.

Marshall and Rohde \cite{MarshallRohde} remarked that there exist many examples of driving terms \(u(t)\) which are not Lip(1/2), but which generate slit solutions with simple arcs \(\gamma(t)\). In particular, if \(\gamma(t)\) is tangent to \(T\), then \(u(t)\) is never Lip(1/2).

Our result states that if \(\gamma(t)\) is a circular arc tangent to \(R\), then the driving term \(\lambda(t) \in \text{Lip}(1/3)\). Besides, we prove that \(C_D = C_H = 4\), and consider properties of singular solutions to the one-slit Löwner equation.

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2. Circular tangent slits

We shall work with the half-plane version of the Löwner equation and with the sign \((+)\) in the right-hand side, consequently with the maps of slit domains onto \(\mathbb{H}\).

We construct a mapping of the half-plane \(\mathbb{H}\) slit along a circular arc \(\gamma(t)\) of radius 1 centered on \(i\) onto \(\mathbb{H}\) starting at the origin directed, for example, positively. The inverse mapping we denote by \(z = f(w, t) = w - 2t/w + \ldots\). Then \(\zeta = 1/f(w, t)\) maps \(\mathbb{H}\) onto the lower half-plane slit along a ray co-directed with \(\mathbb{R}^+\) and having the distance 1/2 between them. Let \(\zeta_0\) be the tip of this ray. Applying the Christoffel-Schwarz formula we find \(f\) in
the form
\[ \frac{1}{f(w,t)} = \int_0^{1/w} \frac{(1 - \gamma w) \, dw}{(1 - \alpha w)^2 (1 - \beta w)} = \frac{\beta - \gamma}{(\alpha - \beta)^2} \log \frac{w - \alpha}{w - \beta} + \frac{\alpha - \gamma}{(\alpha - \beta) w - \alpha}. \]

(3)

where the branch of logarithm vanishes at infinity, and \( f(w,t) \) is expanded near infinity as
\[ f(w,t) = w - \frac{2t}{w} + \ldots \]

The latter expansion gives us two conditions: there is no constant term and the coefficient is \(-2t\) at \( w \), which implies \( \gamma = 2\alpha + \beta \) and \( \alpha(\alpha + 2\beta) = -6t \). The condition \( \text{Im} \, \zeta_0 = -1/2 \) yields
\[ \frac{-2\alpha}{(\alpha - \beta)^2} = \frac{1}{2\pi}. \]

Then, \( \beta = \alpha + 2\sqrt{-\alpha \pi} \), and \( \alpha(3\alpha + 4\sqrt{-\alpha \pi}) = -6t \). Considering the latter equation with respect to \( \alpha \) we expand the solution \( \alpha(t) \) in powers of \( t^{1/3} \). Hence,
\[ \alpha(t) = -\left( \frac{9}{4\pi} \right)^{1/3} t^{2/3} + A_2 t + A_3 t^{4/3} + \ldots \]

and
\[ \beta(t) = (12\pi)^{1/3} t^{1/3} + B_2 t^{2/3} + \ldots \]

Formula (3) in the expansion form regarding to \( 1/w \) gives
\[ \frac{\beta - \alpha}{2\pi} \frac{1}{w} + \frac{\beta^2 - \alpha^2}{4\pi} \frac{1}{w^2} + \ldots + \left( 1 + 2\frac{\alpha}{\beta} + 2\frac{\alpha^2}{\beta^2} + \ldots \right) \left( \frac{1}{w} + \frac{\alpha}{w^2} + \ldots \right) = \zeta. \]

(4)

Remember that this formula is obtained under the conditions \( \gamma = 2\alpha + \beta \) and \( (\alpha - \beta)^2 = 4\alpha \pi \). We substitute the expansions of \( \alpha(t) \) and \( \beta(t) \) in this formula and consider it as an equation for the implicit function \( w = h(z, t) \). Calculating coefficients \( B_2 \ldots B_4 \) in terms of \( A_2, \ldots, A_4 \), and verifying \( A_2 = -3/4\pi \) we come to the following expansion for \( h(z, t) \):
\[ w = h(z, t) = h\left( \frac{1}{\zeta}, t \right) = \frac{1}{\zeta} + 2\zeta t + \frac{3}{2}(12\pi)^{1/3} t^{4/3} + \ldots. \]

This version of the Löwner equation admits the form
\[ \frac{d}{dt} = \frac{2}{h - \lambda(t)}, \quad h(z, 0) \equiv z. \]

(5)

Being extended onto \( \mathbb{R} \setminus \lambda(0) \) the function \( h(z, t) \) satisfies the same equation. Let us consider \( h(z, t), \ z \in \mathbb{H} \setminus \lambda(0) \) with a singular point at \( \lambda(0) \), where \( \mathbb{H} \) is the closure of \( \mathbb{H} \). Then
\[ \lambda(t) = h(z, t) - \frac{2}{dh(z, t)/dt} = \lambda(0) + (12\pi)^{1/3} t^{4/3} + \ldots \]

about the point \( t = 0 \). Thus, the driving term \( \lambda(t) \) is \( \text{Lip}(1/3) \) about the point \( t = 0 \) and analytic for the rest of the points \( t \).
Remark 2.1. The radius of the circumference is not essential for the properties of $\lambda(t)$. Passing from $h(z,t)$ to the function $\frac{1}{r}h(rz,t)$ we recalculate the coefficients of the function $h(z,t)$ and the corresponding coefficients in the expansion of $\lambda(t)$ that depend continuously on $r$. Therefore, they stay within bounded intervals whenever $r$ ranges within the bounded interval.

Remark 2.2. In particular, the expansion for $h(z,t)$ reflects the Marshall and Rohde’s remark [4, page 765] that the tangent slits cannot be generated by driving terms from $\text{Lip}(1/2)$.

3. Singular solutions for slit images

Suppose that the Löwner equation (5) with driving term $\lambda(t)$ generates a map $h(z,t)$ from $\Omega(t) = \mathbb{H} \setminus \gamma(t)$ onto $\mathbb{H}$, where $\gamma(t)$ is a quasislit. Extending $h$ to the boundary $\partial\Omega(t)$ we obtain a correspondence between $\gamma(t)$ and a segment $I(t) \subset \mathbb{R}$, while the remaining boundary part $\mathbb{R} = \partial\Omega(t) \setminus \gamma(t)$ corresponds to $\mathbb{R} \setminus I(t)$. The latter mapping is described by solutions to the Cauchy problem for the differential equation (5) with the initial data $h(x,0) = x \in \mathbb{R} \setminus \lambda(0)$. The set $\{h(x,t) : x \in \mathbb{R} \setminus \lambda(0)\}$ gives $\mathbb{R} \setminus I(t)$, and $\lambda(t)$ does not catch $h(x,t)$ for all $t \geq 0$, see [2] for details.

The image $I(t)$ of $\gamma(t)$ can be also described by solutions $h(\lambda(0),t)$ to (5), but the initial data $h(\lambda(0),0) = \lambda(0)$ forces $h$ to be singular at $t = 0$ and to possess the following properties.

(i) There are two singular solutions $h^{-}(\lambda(0),t)$ and $h^{+}(\lambda(0),t)$ such that $I(t) = \{h^{-}(\lambda(0),t), h^{+}(\lambda(0),t)\}$.

(ii) $h^{\pm}(\lambda(0),t)$ are continuous for $t \geq 0$ and have continuous derivatives for all $t > 0$.

(iii) $h^{-}(\lambda(0),t)$ is strictly decreasing and $h^{+}(\lambda(0),t)$ is strictly increasing, so that $h^{-}(\lambda(0),t) < \lambda(t) < h^{+}(\lambda(0),t)$.

We will focus on studying the singularity character of $h^{\pm}$ at $t = 0$.

Theorem 3.1. Let the Löwner differential equation (5) with the driving term $\lambda \in \text{Lip}(1/2)$, $\|\lambda\|_{1/2} = c$, generate slit maps $h(z,t) : \mathbb{H} \setminus \gamma(t) \to \mathbb{H}$ where $\gamma(t)$ is a quasislit. Then $h^{+}(\lambda(0),t)$ satisfies the condition

$$\lim_{t \to 0^+} \sup_{t \leq 0} \frac{h^{+}(\lambda(0),t) - h^{+}(\lambda(0),0)}{\sqrt{t}} \leq \frac{c + \sqrt{c^2 + 16}}{2},$$

and this estimate is the best possible.

Proof. Assume without loss of generality that $h^{+}(\lambda(0),0) = \lambda(0) = 0$. Denote $\varphi(t) := h^{+}(\lambda(0),t)/\sqrt{t}$, $t > 0$. This function has a continuous derivative and satisfies the differential equation

$$t\varphi'(t) = \frac{2}{\varphi(t) - \lambda(t)/\sqrt{t}} - \frac{\varphi(t)}{2}.$$

This implies together with property (iii) that $\varphi'(t) > 0$ if

$$\frac{\lambda(t)}{\sqrt{t}} < \varphi(t) < \varphi_{1}(t) := \frac{\lambda(t)}{2\sqrt{t}} + \sqrt{\frac{\lambda^2(t)}{4t} + 4}.$$
Observe that $\varphi_1(t) \leq A := (c + \sqrt{c^2 + 16})/2$.

Suppose that $\lim_{t \to 0+} \sup \varphi(t) = B > A$, including the case $B = \infty$. Then there exists $t^* > 0$, such that $\varphi(t^*) > B - \epsilon > A$, for a certain $\epsilon > 0$. If $B = \infty$, then replace $B - \epsilon$ by $B' > A$. Therefore, $\varphi'(t^*) < 0$ and $\varphi(t)$ increases as $t$ runs from $t^*$ to 0. Thus, $\varphi(t) > B - \epsilon$ for all $t \in (0, t^*)$ and we obtain from (5) that

$$\frac{dh^+(\lambda(0), t)}{dt} \leq \frac{2}{\sqrt{t}(B - \epsilon - c)},$$

for such $t$. Integrating this inequality we get

$$h^+(\lambda(0), t) \leq \frac{4\sqrt{t}}{B - \epsilon - c} < \frac{4\sqrt{t}}{A - c},$$

that contradicts our supposition. This proves the estimate of Theorem 3.1.

In order to attain the equality sign in Theorem 3.1, one chooses $\lambda(t) = c\sqrt{t}$. Then $h^+(\lambda(0), t) = A\sqrt{t}$ solves equation (5) with singularity at $t = 0$. This completes the proof. 

\begin{remark}
Estimates similar to Theorem 3.1 hold for the other singular solution $h^-(\lambda(0), t)$.
\end{remark}

\begin{remark}
Let us compare Theorem 3.1 with the results from Section 2. The image of a circular arc $\gamma(t) \subset \mathbb{H}$ tangent to $\mathbb{R}$ is $I(t) = [h^-(\lambda(0), t), h^+(\lambda(0), t)]$, where $h^-(\lambda(0), t) = \alpha(t) = -(9/4\pi)^{1/3}t^{2/3} + \ldots$ and $h^+(\lambda(0), t) = \beta(t) = (12\pi)^{1/3}t^{1/3} + \ldots$, so that $h^-(\lambda(0), t) \in \text{Lip}(2/3)$ and $h^+(\lambda(0), t) \in \text{Lip}(1/3)$.
\end{remark}

\begin{remark}
Singular solutions to the differential equation (5) appear not only at $t = 0$ but at any other moment $\tau > 0$. More precisely, there exist two families $h^-(\gamma(\tau), t)$ and $h^+(\gamma(\tau), t)$, $\tau \geq 0$, $t \geq \tau$, of singular solutions to (5) that describe the image of arcs $\gamma(t)$, $t \geq \tau$ under map $h(z, t)$. They correspond to the initial data $h(\gamma(\tau), \tau) = \lambda(\tau)$ in (5) and satisfy the inequalities $h^-(\gamma(\tau), t) < \lambda(t) < h^+(\gamma(\tau), t)$, $t > \tau$. These two families of singular solutions have no common inner points and fill in the set

$$\{(x, t) : h^-(\lambda(0), t) \leq x \leq h^+(\lambda(0), t), 0 \leq t \leq t_0\},$$

for some $t_0$.
\end{remark}

4. Critical norm values for driving terms

In this section we discuss the results and techniques of Marshall and Rohde [4] and Lind [2]. The authors of [4] proved the existence of $C_\mathbb{D}$ such that driving terms $u(t) \in \text{Lip}(1/2)$ with $\|u\|_{1/2} < C_\mathbb{D}$ in [1] generate quasisymmetric slit maps. This result remains true for an absolute number $C_\mathbb{H}$ in the half-plane version of the Löwner differential equation (2), see e.g. [2].

Lind [2] claimed that the disk version [1] of the Löwner differential equation is ‘more challenging’, than the half-plane version [2]. Working with the half-plane version she showed that $C_\mathbb{H} = 4$. The key result is based on the fact that if $\lambda(t) \in \text{Lip}(1/2)$ in [2], and $h(x, t) = \lambda(t)$, say at $t = 1$, then $\Omega(t) = h(\mathbb{H}, t)$ is not a slit domain and $\|\lambda\|_{1/2} \geq 4$. Moreover, there is an example of $\lambda(t) = 4 - 4\sqrt{1-t}$ that yields $h(2, 1) = \lambda(1)$. Although...
there may be more obstacles for generating slit half-planes than that of the driving term $\lambda$
catching up some solution $h$ to [3]. Lind showed that this is basically the only obstacle. The
latter statement was proved by using techniques of [4].

We will modify here the main Lind’s reasonings so that they could be applied to the
disk version of the L"owner equation. After that it remains to refer to [4] and [2] to state
that $C_D$ also equals 4.

Suppose that slit disks $\Omega(t)$ correspond to $u \in \text{Lip}(1/2)$ in (11) with the sign ‘+’ in its
right-hand side instead of ‘−’. Then the maps $w(z, t)$ are extended continuously to $T \setminus \{e^{iu(0)}\}$. Let $z_0 \in T \setminus \{e^{iu(0)}\}$, and let $\alpha(t, \alpha_0) := \text{arg } w(z_0, t)$ be a solution to the following real-valued
initial value problem

$$\frac{d\alpha(t)}{dt} = \cot \frac{\alpha - u}{2}, \quad \alpha(0) = \alpha_0. \tag{6}$$

Similarly, suppose that slit half-planes $\Omega(t)$ correspond to $\lambda \in \text{Lip}(1/2)$ in (2) with the
sign ‘+’ in its right-hand side instead of ‘−’. Then the maps $h(z, t)$ are extended continuously
to $\mathbb{R} \setminus \lambda(0)$. Let $x_0 \in \mathbb{R} \setminus \lambda(0)$ and let $x(t, x_0) := h(x_0, t)$ be a solution to the following real-
valued initial value problem

$$\frac{dx(t)}{dt} = \frac{2}{x(t) - \lambda(t)}, \quad x(t_0) = x_0. \tag{7}$$

For all $t \geq 0$, $\tan((\alpha(t) - u(t))/2) \neq 0$ in (6), and $x(t) - \lambda(t) \neq 0$ in (7) (see [2] for the
half-plane version). Let us show a connection between the solutions $\alpha(t)$ to (6), and
$x(t)$ to (7), where the driving terms $u(t)$ and $\lambda(t)$ correspond to each other.

**Lemma 4.1.** Given $\lambda(t) \in \text{Lip}(1/2)$, there exists $u(t) \in \text{Lip}(1/2)$, such that equations (6) and
(7) have the same solutions. Conversely, given $u(t) \in \text{Lip}(1/2)$ there exists $\lambda(t) \in \text{Lip}(1/2)$,
such that equations (6) and (7) have the same solutions.

**Proof.** Given $\lambda(t) \in \text{Lip}(1/2)$, denote by $x(t, x_0)$ a solution to the initial value problem (7).
Then the solution $\alpha(t, \alpha_0)$ to the initial value problem (6) is equal to $x(t, \alpha_0)$ when

$$\tan \frac{\alpha - u}{2} = \frac{x - \lambda}{2},$$

and

$$x_0 = \lambda(0) + 2 \tan \frac{\alpha_0 - u(0)}{2}.$$ 

The function $u(t)$ is normalized by choosing

$$u(0) = x_0 - \arctan \frac{x_0 - \lambda(0)}{2}.$$ 

This condition makes $\alpha_0$ and $x_0$ equal. Hence, the first part of Lemma 4.1 is true if we put

$$u(t) = x(t, x_0) - 2 \arctan \frac{x(t, x_0) - \lambda(t)}{2}. \tag{8}$$

Obviously, (8) preserves the Lip(1/2) property.

Conversely, given $u(t) \in \text{Lip}(1/2)$, a solution $x(t, x_0)$ is equal to $\alpha(t, \alpha_0)$ when

$$\lambda(t) = \alpha(t, \alpha_0) - 2 \tan \frac{\alpha(t, \alpha_0) - u(t)}{2}. \tag{9}$$
Again (9) preserves the Lip($1/2$) property. This ends the proof.

Observe that in some extreme cases relations (8) or (9) preserve not only the Lipschitz class but also its norm. Lind [2] gave an example of the driving term $\lambda(t) = 4 - 4\sqrt{1 - t}$ in (7). It is easily verified that $x(t, 2) = 4 - 2\sqrt{1 - t}$. If $t = 1$, then $x(1, 2) = \lambda(1) = 4$, and $\lambda$ cannot generate slit half-plane at $t = 1$. This implies that $C_H \leq 4$. Going from (7) to (6) we use (8) to put

$$u(t) = x(t, 2) - 2\arctan \frac{x(t, 2) - \lambda(t)}{2} = 4 - 2\sqrt{1 - t} - 2\arctan \sqrt{1 - t}.$$  

From Lemma 4.1 we deduce that $\alpha(1, 2) = u(1)$. Hence $u$ cannot generate slit disk at $t = 1$, and $C_D \leq 4$.

Lemma 4.2. Let $u \in \text{Lip}(1/2)$ in (6) with $u(0) = 0$ and $\alpha_0 \in (0, \pi)$. Suppose that $\alpha(t)$ is a solution to (6) and $\alpha(1) = u(1)$. Then $\|u\|_{1/2} \geq 4$.

Proof. Observe that $\alpha(t)$ is increasing on $[0, 1]$, and $\alpha(t) - u(t) > 0$ on $(0, 1)$. Let $u \in \text{Lip}(1/2)$ in (3), and $\|u\|_{1/2} = c$. Then,

$$\alpha(t) - u(t) \leq \alpha(1) - u(1) + c\sqrt{1 - t} = c\sqrt{1 - t}. \tag{10}$$

Given $\epsilon > 0$, there exists $\delta > 0$, such that

$$\tan \frac{c\sqrt{1 - t}}{2} < \frac{c\sqrt{1 - t}}{2}(1 + \epsilon),$$

for $1 - \delta < t < 1$ and all $0 < c \leq 4$. We apply this inequality to (10) and obtain that

$$\frac{d\alpha}{dt} \geq c\frac{\sqrt{1 - t}}{2} \geq \frac{2}{c\sqrt{1 - t}(1 + \epsilon)}.$$  

Integrating gives that

$$\alpha(1) - \alpha(t) \geq \frac{4\sqrt{1 - t}}{c(1 + \epsilon)}.$$  

This allows us to improve (10) to

$$\alpha(t) - u(t) \leq \alpha(1) - \frac{4\sqrt{1 - t}}{c(1 + \epsilon)} - u(1) + c\sqrt{1 - t} = \left( c - \frac{4}{c(1 + \epsilon)} \right) \sqrt{1 - t}. \tag{11}$$

Repeating these iterations we get

$$\alpha(t) - u(t) \leq c_n \sqrt{1 - t},$$

where $c_0 = c$, $c_{n+1} = c - 4/[(1 + \epsilon)c_n]$, and $c_n > 0$. Let $g_n$ be recursively defined by (see Lind [2])

$$g_1(y) = y - \frac{4}{y}, \quad g_n(y) = y - \frac{4}{g_{n-1}(y)}, \quad n \geq 2.$$
It is easy to check that \( c_n < g_n((1 + \varepsilon)c) < (1 + \varepsilon)c_n \).

Lind [2] showed that \( g_n(y_n) = 0 \) for an increasing sequence \( \{y_n\} \), and \( g_{n+1}(y) \) is an increasing function from \( (y_n, \infty) \) to \( \mathbb{R} \). So \( c(1 + \varepsilon) > y_n \) for all \( n \), and it remains to apply Lind’s result [2] that \( \lim_{n \to \infty} y_n = 4 \). Hence, \( c \geq 4/(1+\varepsilon) \). The extremal estimate is obtained if \( \varepsilon \to 0 \) which leads to \( c \geq 4 \). This completes the proof. \( \square \)

Now Lind’s reasonings in [2] based on the techniques from [4] give a proof of the following statement.

**Proposition 4.1.** If \( u \in \text{Lip}(1/2) \) with \( \|u\|_{1/2} < 4 \), then the domains \( \Omega(t) \) generated by the Löwner differential equation (1) are disks with quasislits.

In other words, Proposition 4.1 states that \( C_D = C_H = 4 \).

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