(Mis-)handling gauge invariance in the theory of the quantum Hall effect III: The instanton vacuum and chiral edge physics

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Abstract

The concepts of an instanton vacuum and $\mathcal{F}$-invariance are being used to derive a complete effective theory of massless edge excitations in the quantum Hall effect. Our theory includes the effects of disorder and Coulomb interactions, as well as the coupling to electromagnetic fields and statistical gauge fields. The results are obtained by studying the strong coupling limit of a Finkelstein action, previously introduced for the purpose of unifying both integral and fractional quantum Hall regimes. We establish, for the first time, the fundamental relation between the instanton vacuum approach and the completely equivalent theory of chiral edge bosons. In this paper we limit the analysis to the integral regime. We show that our complete theory of edge dynamics can be used as an important tool to investigate longstanding problems such as long-range, smooth disorder and Coulomb interaction effects. We introduce a two dimensional network of chiral edge states and tunneling centers (saddlepoints) as a model for smooth disorder. This network is then used to derive a mean field theory of the conductances and we work out the characteristic temperature ($T$) scale at which the transport crosses over from mean field behavior at high $T$ to the critical behaviour plateau transitions at much lower $T$. The results explain the apparent lack of scaling which is usually seen in the transport data taken from arbitrary samples at finite $T$. Secondly, we address the problem of electron tunneling into the quantum Hall edge. We show that the tunneling density of states near the edge is affected by the combined effects of the Coulomb interactions and the smooth disorder in the bulk. We express the problem in terms of an effective Luttinger liquid with conductance parameter ($g$) equal to the filling fraction ($\nu$) of the Landau band. Hence, even in the integral regime our results for tunneling are completely non-Fermi liquid like, in sharp contrast to the predictions of single edge theories.

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I. INTRODUCTION

In problems of quantum transport symmetries play an important role. Recent advances in the theory of the quantum Hall effect primarily make use of electrodynamical gauge invariance as the fundamental symmetry of the strongly correlated electron gas. This symmetry permits one to proceed with a minimum of microscopic input. Applications of Chern-Simons theory have been largely based upon phenomenological arguments. These applications have provided a universal language for the fractional quantum Hall effect in which the various hierarchy schemes could be treated on equal footing.

Application of Chern-Simons theory has also led to the idea that many of the basic properties of incompressible quantum Hall states can be understood in terms of Luttinger liquid behaviour of the edge excitations. This non-Fermi liquid theory of edge excitations is now commonly used as a computational scheme for tunneling properties of different quantum Hall states as well as the thermodynamic properties of the fractionally charged quasiparticles. It is important to keep in mind, however, that unlike the the conductance parameters, physical quantities like the tunneling density of states do not necessarily follow the rules of incompressibility. The lack of a microscopic theory of the fractional quantum Hall effect has led to controversial issues regarding the definition of the Hall conductance (notably for those states that have edge channels of opposite chirality). Moreover, serious discrepancies have arisen between the predictions of the Luttinger liquid theory of edge excitations on the one hand and the experimental results on edge tunneling on the other.

This paper is the third in a series of recent articles in which we lay down the foundation for a microscopic theory of disordered compressible and incompressible states in the (fractional) quantum Hall regime. In previous papers (hereafter called I and II) we introduced an effective Finkelstein action for localisation and interaction effects. The Finkelstein action includes the topological concept of an instanton vacuum as well as the statistical (Chern-Simons) gauge fields. The inclusion of statistical gauge fields in the problem makes it possible to formulate a combined theory of composite fermions, localization and interaction effects. The results of weak coupling analyses (both perturbative and non-perturbative, i.e. instantons) can then be used to obtain a global scaling diagram for the conductances. The integral as well as the fractional quantum Hall regime are incorporated in this scaling diagram. In this work, we are primarily interested in the strong coupling limit of our action where the system has a gap in the density of states. This physical situation is the same as the one described by the Chern-Simons approach with one important exception: beside the Coulomb interactions we also deal from first principles with the effects of disorder.

One of the main objectives of this work is to derive microscopically a Luttinger liquid theory for edge excitations in the presence of disorder and electron-electron interactions. From our general, effective action point of view we can say that the physics of edge excitations has fundamental significance since it provides unique and invaluable information on the topological concept of an ‘instanton vacuum’ in strong coupling.

An additional important advancement is that we obtain for the first time the complete Luttinger liquid theory on the edge. We have the action for chiral edge bosons coupled to external electromagnetic fields. This theory can now be used to define the Hall conductance in a general, unambiguous manner by expressing the appearance of an ‘edge anomaly’ in
terms of Laughlin’s gauge argument.\footnote{13}

The details of the analysis of edge excitations are described in sections III and IV. This analysis is based, to a large extent, on the various new concepts which were introduced in [I] under the name of ‘$\mathcal{F}$-algebra’ and ‘$\mathcal{F}$-invariance’. Recall that in [II] we also studied these concepts but in the weak coupling regime. This paper therefore shows that $\mathcal{F}$-invariance retains its significance all the way down to the regime of strong coupling, where the massless excitations are confined to the edges of the sample. It is important to note that this is the first time that this symmetry is being demonstrated in the weak as well as the strong coupling regime.

The results of sections III and IV will serve as the starting point for a microscopic theory of edge excitations in the fractional quantum Hall effect. We shall limit ourselves here to the integer regime, since this already contains most of the difficulties. Extensions of our theory to include the fractional effect can be done by means of the statistical gauge fields. These will be reported elsewhere.

We shall begin by reviewing and extending the topological instanton vacuum approach to the $q$He, following the ordinary, free electron replica formalism in section II. In making the connection between topology and edge currents, we show that important aspects of the problem have previously been overlooked. In particular, we show that the massless excitations of the disordered edge states are obtained from the fluctuations about integer quantised topological charge (section A3). This important observation will serve as a starting point for most of the analyses in the remainder of this paper. Massless edge excitations appear in the theory only when the number of field components (‘replicas’) $N_r$ is sent to zero. For ordinary positive number of field components, $N_r > 0$, the theory always generates a mass gap at the edge and this, then, forces us to reinvestigate the physical meaning of topology in replica field theory. The present analysis revises our previously accumulated knowledge on the subject in at least two respects.

First we recognize that a direct relationship exists between the numerical value of the instanton parameter $\theta$ (or $\sigma_{xy}^0$, Ref.\footnote{10}) and the phenomenon of inter-channel scattering at the edge. Here the number of edge channels equals the number of fully occupied Landau levels, and the phrase “inter-channel scattering” refers to the effect of a random short-ranged potential.

Secondly, we review the earlier attempts toward establishing a general topological principle for quantisation of the Hall conductance which includes the effect of localisation of the bulk states. The theoretical subtleties of massless edge excitations in replica field theory turn out to have basic consequences for the quantisation phenomenon which now can be shown to occur in the theory with zero number of field components only! In ordinary applications of the instanton vacuum ($N_r > 0$) there are no massless excitations at the edge and the Hall conductance is not quantised. To date, this fundamental aspect of the problem has gone completely unnoticed.

In all our work sofar, we have substituted the phrase ‘electronic disorder’ for a white noise random potential. This was always done for technical reasons alone. However, it is well known that in real quantum Hall devices slowly varying potentials are often present.\footnote{10,11} Till now these have been in general difficult to handle. Our microscopic theory of the edge enables us to treat long-range potentials as well as electron-electron interactions. In this
paper we embark on solving two longstanding problems where smooth disorder and Coulomb interactions give rise to unexpected results. By addressing these problems we attack the core of the controversies that exist between the theory and experiments that presently span this subject.

The first problem we address is that of the plateau transitions. This we model as a percolating network of ‘edge states’ (equipotential contours) and widely separated ‘saddlepoints’. A large class of such systems is then ‘mapped’ onto the non-linear $\sigma$ model representation for localisation, and the main problem is to identify the length and energy scales of the ‘bare’ parameters, or the mean field conductances which together determine the renormalisation starting point, i.e. the point where scaling occurs first. This starting point can involve, in principle, arbitrarily large distances and arbitrarily small energies and this, obviously, complicates the observability of the critical behaviour of the Anderson (plateau) transitions. We argue that Coulomb interaction effects lead to a modified mean field theory of transport which is now observed in the experiments performed at finite temperatures. The chiral boson theory shall be used to actually compute the inelastic relaxation rate of the conducting electrons in the saddlepoint network. This, then, might conceivably be the explanation for the empirical fits of the transport data taken recently from presently available samples.

As the second typical example of long-ranged disorder effects we embark on the problem of electron tunneling into the quantum Hall edge. We show that the Coulomb interactions between the edge and the ‘localised’ bulk orbits dramatically differ from the predictions of theories which are based on isolated edges alone. Tunneling processes into the quantum Hall edge have, in fact, nothing to do with the quantisation of the Hall conductance or the ‘incompressibility’ statement which describe the non-equilibrium properties of the electron gas. We find that the tunneling density of states near the edge can be understood in terms of an effective edge theory which describes the equilibrium properties of the combined edge and bulk degrees of freedom. The Luttinger liquid parameter $g$ is related to the filling fraction $\nu$ of the bulk Landau level. This leads to a tunneling exponent which varies like $1/\nu$, in agreement with recent experimental data on the tunneling current, taken from samples in the fractional quantum Hall regime. This situation is dramatically different from what is expected while assuming an isolated edge, or in the case of short-ranged disorder which gives rise to scattering between different edge states.

In this paper and the one that follows we carefully re-examine the consequences of inter-channel edge scattering. We reproduce the completely different Kane-Fisher-Polchinsky scenario of tunneling exponents in the integral and fractional regimes from our strong coupling edge theory. However, we argue that both the assumptions (an isolated edge and short-ranged disorder or inter-channel scattering) are clearly incorrect since the problem is two dimensional and dominated by long-ranged potential fluctuations as well as interaction effects.

The organisation of this paper is as follows.

In Section I we introduce the problem in the language of the replica free electron theory. We briefly recall the instanton vacuum approach in (A1). The connection between topology and interchannel scattering between the chiral edge states is made in (A2). The most important result is contained in (A3) where we show that the fluctuations about the integral quantised topological charge describe the massless excitations of the disordered edge states.
In (A4) we analyse the large $N$ limit of the $CP^N$ theory for the purpose of illustration. In Section II.B we introduce a two dimensional network of chiral edge states as a model for the problem of long-ranged potential fluctuations. This is then used for mean field purposes and for demonstrating universality of the plateau transitions in (B1). In (B2) we extend the network approach to include interaction effects. A semiclassical theory of transport is introduced in order to explain the lack of scaling recently found in many (ordinary) quantum Hall devices at finite temperatures. (B3) contains several general remarks.

In Section III we present a detailed derivation of the complete chiral edge theory using the fermionic path integral.

In Section IV we make the fundamental connection between the instanton vacuum theory for the integral quantum Hall effect which is expressed in terms of the local field variables $Q_{\alpha \beta}^{pp'}$, where $\alpha, \beta = 1, \ldots, N_r$ are the replica indices and $p, p' = \pm 1$ are the indices denoting advanced/retarded waves. They can be represented as

$$Q = T^{-1} \Lambda T$$

with

$$\Lambda_{pp'}^{\alpha \beta} = \delta^{\alpha \beta} \delta_{pp'} \text{sgn}(p)$$

and $T$ a unitary matrix of size $2N_r \times 2N_r$. The complete action is given by

$$S[Q] = -\frac{1}{8} \sigma_{xx}^0 \int d^2x \text{ tr } (\nabla Q)^2 - \frac{1}{8} \sigma_{xy}^0 \int d^2x \text{ tr } \epsilon_{ij} Q \partial_i Q \partial_j Q + \rho^0 \eta \int d^2x \text{ tr } \Lambda Q.$$  

Here $\sigma_{ij}^0$ stands for the mean field conductances in units of $e^2/h$ (see figure 1), $\rho^0$ is the (exact) density of states at the Fermi energy and $\eta$ is the frequency. The second term in (2.2), proportional to the mean field Hall conductance ($\sigma_{xy}^0$), has remained one of the most difficult chapters in the theory of Anderson localisation in low dimensions. Most of the insight into the problem has come from weak coupling renormalisation theory (both perturbative and non-perturbative, i.e. instantons). In particular we mention the global scaling diagram of the conductances as well as the appearance of a critical fixed point in strong coupling regime. This fixed point theory predicts a massless (metallic) phase at the Landau band center as well as the following scaling result for the conductances

$$\sigma_{ij}(L, B) = g_{ij} \left( [L/\xi]^{1/\nu} \right)^{-\nu}; \quad \xi = |B - B^*|^{-\nu}$$

which cannot be obtained in any different way.

II. EDGE EXCITATIONS

A. Sigma model

Let us recall the instanton vacuum theory for the integral quantum Hall effect which is expressed in terms of the local field variables $Q_{\alpha \beta}^{pp'}$, where $\alpha, \beta = 1, \ldots, N_r$ are the replica indices and $p, p' = \pm 1$ are the indices denoting advanced/retarded waves. They can be represented as

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which cannot be obtained in any different way.
FIG. 1. Sketch of the mean field conductances for a short-range disorder potential. The inset is the strong field limit or quantum Hall regime. It also contains the renormalisation group flow lines which indicate how the mean field theory results change after successive length scale transformations. After Refs 8, 10 and 13.

Here, the function $g_{ij}(X)$ is a regular function of its argument, $B^*$ is the critical magnetic field strength and $\nu$ stand for the critical index for the localization length $\xi$. Following the experimental tests of (2.3) by H.P. Wei et al., extensive numerical work on the free electron gas has been performed and the quoted best value for the critical index is $\nu = 2.3$. To date, no exact (conformal) scheme for the critical indices exists. All that one can say at this time is that the field of exactly solvable models is not sufficiently developed to be able to handle the specific subtleties of topology and replica field theory. These subtleties are all well understood within the elaborate framework of weak coupling expansion techniques and the results were used to unfold and predict the entire singularity structure of the theory, notably (2.3).

In previous work we have shown that the theories of free and interacting electrons share the same basic features such as asymptotic freedom, instantons etc. The same scaling diagram for the conductances was obtained which means that (2.3) remains valid also when the Coulomb interactions are taken into account. This important result was conjectured but otherwise completely ununderstood at the time of the original experiments on criticality.

1. Strong coupling

In this paper we address the subtleties of replica field theory in an extremely important exactly solvable limit where $\rho_0 = \sigma_{xx}^0 = 0$ and where the Hall conductance is integer quantised ($\sigma_{xy} = m$). Physically this happens when the Fermi energy is located in a density of states
gap between adjacent Landau bands. In this \textit{strong coupling} limit massless excitations do exist at the edges of the system. Since several, basic aspects of the problem have previously gone unnoticed we shall proceed first within the free electron formalism of Eqs (2.1) and (2.2). We come back to the fermionic path integral in Sections III and IV.

For $m$ completely filled Landau levels the action becomes simply

$$S[Q] = -\frac{m}{2} \int d^2x \, \text{tr} \, \varepsilon_{ij} Q \partial_i Q \partial_j Q = -\frac{m}{2} \oint d\vec{x} \cdot \text{tr} \left( \Lambda T \nabla T^{-1} \right)$$

(2.4)

where the surface integral is taken over the sample’s edge. Recall that (2.4) is quantised according to

$$S[Q] = -2\pi im \cdot q[Q]$$

(2.5)

with $q$ the integer topological charge, provided that the $T$-matrix reduces to a $U(N_r) \times U(N_r)$ gauge at the edge.

Under these circumstances the sample edge has been contracted to a single point (spherical boundary conditions) and (2.3) is a realisation of the formal homotopy theory result $\pi_2(G/H) = \mathbb{Z}$ which states that the mapping of $Q$ onto the 2D plane is described by a set of integers $q$. It is natural to take the theory one step further and propose the quantisation of the charge $q(Q)$ as the topological principle in replica field theory which forces the Hall conductance ($m$) itself to be integer quantised. The idea has led to a consistent quantum theory of conductances that unifies a fundamental aspect of asymptotically free field theory (i.e. dynamic mass generation) with the quantum Hall effect. More specifically, it says that the conductances in Eq. (2.3) always scale toward $\sigma_{xx} = 0, \sigma_{xy} = m$ for $L$ large enough.

One can show\cite{8} that the $U(N_r) \times U(N_r)$ gauge condition at the edge is the replica field theory version of a static $U(1)$ gauge acting on the physical edge states. Such a $U(1)$ gauge implies that an integer number of edge levels has crossed the Fermi level. This level-crossing is necessarily induced by the averaging procedure over random potentials.

Nevertheless, it is somewhat disturbing to remark that the topological invariant in (2.2), as it was discovered originally in a microscopic derivation, is truly defined with \textit{free} boundary conditions and without any separation between edge and bulk degrees of freedom.\cite{8} So far, the precise significance of boundary conditions has remained obscure.

\textbf{2. Interchannel edge scattering}

In what follows, we show that the fluctuations about precisely quantised values for the topological charge represent, in fact, essential physics of the problem, since they describe the dynamics of (massless) edge excitations. In order to see this, we write $T$ as the product of a $U(N_r) \times U(N_r)$ gauge $U$ and a small fluctuation $t$,

$$T = Ut.$$ 

(2.6)

The action now becomes

$$S[Q] = -2\pi im \cdot q[U] - \frac{m}{2} \oint d\vec{x} \cdot \text{tr} \left( \Lambda t \nabla t^{-1} \right) + \rho_{\text{edge}} \eta \oint dx \, \text{tr} \, \Lambda Q$$

(2.7)
with $\rho_{\text{edge}}$ the density of edge states.

One way of identifying (2.7) as the effective theory of disordered chiral edge states is to redo the derivation, but now for the 1D system with Hamiltonian

$$H_{\text{edge}} = iv_d \partial_x + V(x)$$  \hspace{1cm} (2.8)

where $v_d$ is the drift velocity of the edge electrons and $V(x)$ the random potential.

It turns out that our initial guess (2.8) is correct only in the case $m = 1$ in (2.7). This problem is easily resolved once one realizes that $m$ really stands for the number of filled Landau levels, such that (2.8) should be replaced by a Hamiltonian for a total of $m$ edge channels. Hence, an obvious second guess would be

$$H_{\text{edge}} = \sum_{j=1}^{m} H_{\text{edge}}^{(j)}$$  \hspace{1cm} (2.9)

where $H_{\text{edge}}^{(j)}$ is the same for all $j$, i.e. each of the $m$ eigenstates experiences the same white noise potential $V(x)$, just as it appears in the original problem in two spatial dimensions. This, however, is not correct and the theory with general $m$, (2.7), necessarily requires inter-channel scattering to take place. We have to start from a matrix Hamiltonian

$$H_{\text{edge}}^{jj'} = iv_d \delta_{jj'} \partial_x + V_{jj'}(x)$$  \hspace{1cm} (2.10)

where $V$ is a Hermitian matrix. The matrix elements $V_{jj'}$ connect the edge channels $j, j'$ and are distributed with a weight

$$P[V] \propto \exp\{-\frac{1}{g} \int dx \ \text{tr} V^{\dagger} V\}.$$  \hspace{1cm} (2.11)

One can construct a generating function for the free particle Green’s functions as usual, according to

$$Z = \int \mathcal{D}[\bar{\psi}\psi] \int \mathcal{D}[V] P[V] \exp \sum_{p,\alpha,jj'} \int dx \ \bar{\psi}_p^\alpha \left[(\mu + ip\eta)\delta_{jj'} - H^{jj'}_{\text{edge}}\right] \psi_{p}^{\alpha,j'}.$$  \hspace{1cm} (2.12)

In appendix C we show that (2.12) and (2.7) describe the same physics in the limit of large distances.

3. Criticality at the edge

Next we work out the consequences of our one-dimensional theory in $t$ (2.7). Since the $t$ really is understood as a small fluctuation about unity, we can apply the standard rules of perturbation expansions. Specifically, $t$ may be parametrised as

$$t = \exp i \begin{pmatrix} 0 & V \\ V^\dagger & 0 \end{pmatrix}$$  \hspace{1cm} (2.13)

where the $V$ can be taken as $N_c^2$ independent complex field variables. (It can be checked that the Jacobian of this parametrisation is unity.) By expanding the $t$ as an infinite power
series in $V, V^\dagger$ one still is faced with a rather complicated field theory in general. However, a dramatic simplification occurs by letting the number of replica channels approach zero. The easiest way of dealing with this problem is by inserting background fields $t_0$; For instance, replace $t \to t \cdot t_0$,

$$
\oint d\vec{x} \cdot \text{tr} [\Lambda t_0 \nabla (t_0^{-1}t^{-1})]
\rightarrow \oint d\vec{x} \cdot \text{tr} [\Lambda t \nabla t^{-1}] + \oint d\vec{x} \cdot \text{tr} [\Lambda t_0 \nabla t_0^{-1}].
$$

(2.14)

The shift in the free energy due to the insertion of $t_0$ becomes

$$
\delta F[t_0] = \frac{m}{2} \oint d\vec{x} \cdot \text{tr} [t_0 \nabla t_0^{-1} \langle Q \rangle]
$$

(2.15)

where the expectation is with respect to the action (2.7). Notice that this quantity $\langle Q \rangle$ is the order parameter of the theory which in all respects is identical to the magnetisation in the language of the Heisenberg ferromagnet. Since, according to the Mermin-Wagner-Coleman theorem, a continuous symmetry cannot be spontaneously broken in low dimensions, we expect (2.15) to vanish and the theory to become massive for all positive integer values of $N_r$.

However, in the limit of $N_r \to 0$ this expectation becomes simply $\langle Q \rangle = \Lambda$, a well-known result in replica field theory. This result indicates that the quantity $m$ (the Hall conductance!) is unrenormalised. The same conclusion holds for the parameter $\rho^\beta \eta$ and, hence, we are dealing with a critical theory.

Next, we can stretch the argument a little further and look whether e.g. multifractal density fluctuations are predicted, as found in the localisation problem in $2+\varepsilon$ dimensions. It is important to keep in mind, however, that we are really dealing with chiral edge states for which backscattering processes are impossible. Since the action describes the system in the long wavelength limit, in which case the edge potential reduces to a sharp edge, we do not expect the quantities $m$ and $\rho_{\text{edge}}$ to be affected by the impurities. By the same token we do not expect the $m$ and $\rho_{\text{edge}}$ to fluctuate as one moves along the edge. Indeed, it is straightforward to show that an operator of the type $(\text{tr} \Lambda Q)^2$, which can be associated with the appearance of density fluctuations, is in fact redundant. This, then, is consistent with the chirality concept.

4. $CP^N$ model

In order to gain some more insight in the edge theory and to point out the importance of the replica limit $N_r \to 0$ we now consider the action (2.7) for nonzero, integer number of replicas. As we will see, for finite number of field components the long distance behaviour of the theory dramatically differs from the $N_r \to 0$ case: a mass is generated and therefore the parameters of the model flow to zero (i.e. zero response to background fields.) More specifically, we consider the large $N$ limit of the $CP^N$ model, obtained by working with $U(N+1)/U(1)^N$. Write $Q_{\alpha\beta} = 2(z^*_\alpha z_\beta - \delta_{\alpha\beta})$, then (2.7) becomes

$$
S[Q] = \oint dx \sum_\alpha \left(-mz_\alpha \partial_x z^*_\alpha + \rho_{\text{edge}} \eta_\alpha z_\alpha z^*_\alpha\right)
$$

(2.16)
where the traceless quantity \( \eta_\alpha (\sum_\alpha \eta_\alpha = 0) \) represents the symmetry breaking field.

The constraint \( \sum_\alpha |z_\alpha|^2 = 1 \) can be accounted for by introducing an extra multiplier field \( k(x) \),

\[
S[Q] = - \int dx \left[ \sum_\alpha z_\alpha(m\partial_x - \rho_{\text{edge}}\eta_\alpha)z_\alpha^* + ik(x)(1 - \sum_\alpha |z_\alpha|^2) \right].
\] (2.17)

Integration over the free fields \( z_\alpha, z_\alpha^* \) leads to the result

\[
S \rightarrow -i \int dx k(x) - N \text{Tr} \ln[m\partial_x - \rho_{\text{edge}}\eta_\alpha - ik(x)].
\] (2.18)

Notice that the multiplier field \( k \) acts like a gauge field. In order to define the theory at large momenta, we add a gauge invariant regulator field proportional to \( \text{tr} \left( \partial_x Q \right)^2 \) to the original theory (2.7). In \( CP^N \) language this amounts to adding a term \( z_\alpha^* D_x^2 z_\alpha \) to (2.16). The covariant derivative equals \( D_x = \partial_x - ia_x \), with \( a_x \) a \( U(1) \) gauge field. This leads to

\[
S \rightarrow -im \int dx \tilde{a}_x + S[\tilde{k}]
\] (2.20)

where

\[
S[\tilde{k}] = -im \int dx \tilde{k}(x) - \sum_\alpha \text{Tr} \ln \left[ -\frac{m}{4\mu} \partial_x^2 - im\tilde{k}(x) - \rho_{\text{edge}}\eta_\alpha \right]
\] (2.21)

The linear dependence on \( \tilde{a}_x \) indicates that gauge dependent correlations vanish identically. The \( S[\tilde{k}] \) on the other hand represents the non-linear sigma model in one dimension. It has a \( U(N) \) invariant saddlepoint \(-i\tilde{k}_{\text{sp}} = M \) according to

\[
1 = N \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left( \frac{m}{4\mu} q^2 + mM \right)^{-1} = \frac{N}{m} \left( M/\mu \right)^{-1/2}
\] (2.22)

or

\[
M/\mu = (N/m)^2.
\] (2.23)

Equation (2.23) implies that the response to a \( SU(N) \) background field insertion \( z_\alpha \rightarrow (t_0)_{\alpha\beta} z_\beta \) vanishes, which means that the effective parameters \( m' \) and \( \rho' \) are zero, as expected.

The results of this section merely illustrate that the physics of edge excitations is a very special property of the replica limit, but it has no general significance in replica field theory.
B. Plateau transitions revisited

In this section we show how the notion of critical edge states can be used in order to gain insight into the problem of ‘long ranged potential fluctuations’. This long standing problem, which is very difficult to handle within the formal non-linear sigma model methodology, plays an extremely important role experimentally. For instance, it has been stressed many times and at many places elsewhere that the plateau transitions as observed in the detailed experiments of H.P. Wei et al. are very difficult to observe in general in arbitrary samples, due to the presence of slowly varying potential fluctuations.

A slowly varying potential is the generic type of disorder in the standard GaAs heterostructure, which has historically led to semiclassical considerations (percolation picture) of delocalisation near the Landau band center. It is important to recognise that also our critical system is very sensitive to the presence of smooth potentials (or “inhomogeneities”) in the sample. For example, the critical magnetic field $B^*$ may be slowly varying throughout the system due to inhomogeneities in the electron density. This means that the scaling result is valid only up to a certain fixed value for $L$. Beyond this value the remaining “extended” states in the problem may be confined to the equipotential contours of the inhomogeneity potential, quite similar to the semiclassical picture of percolation.

It is generally difficult to obtain detailed knowledge on the various length- and energy scales that are involved in the cross-over problem between percolation and localization. In what follows, we present the simplest possible scenario for crossover that enables us to deal explicitly with such basic concepts as ‘mean field theory’ and ‘universality’ of the plateau transition.

1. Mean field theory

In order to fix the thought, we imagine the equipotential contours near half filling to form a large cluster (Fig. 2).

![Diagram](image)

**FIG. 2.** Backbone cluster as a network of saddlepoints. Shaded areas have $\nu=1$, white areas $\nu=0$. The arrows indicate the direction of the currents. a) Less than half filling; b) exactly half filling; c) filling fraction larger than one half.
Since the disconnected, closed contours do not contribute to the transport, we focus our attention to an infinite backbone cluster which we take as a regular 2D array of 'saddlepoints' and we disregard all the loose hanging, finite pieces (Fig 2b). The saddlepoints (the sites of the square lattice) are connected to one another by the disordered 1D chiral edge channels (links on the lattice). This network can alternatively be looked upon as a checkerboard with filling fractions alternating between the values \( \nu = 0 \) and \( \nu = 1 \). The kinetic part of the action for this system may be written in the form of (2.24)

\[
S[Q] = -\frac{1}{8} \int d^2 x \, m(\vec{x}) \text{tr} \, \varepsilon_{ij} Q \partial_i Q \partial_j Q
\]

with \( m(\vec{x}) = 0, 1 \) (Fig 2b). Using the parametrisation of (2.3) the action can also be written in the form (2.7) which is now solely defined on the links of the square lattice,

\[
S[Q] = -2\pi i \cdot q[U] - \frac{1}{2} \sum_i \int_i dx \, \text{tr} \, (\Lambda t_0 \partial_x t_0^{-1}) + \eta \rho_{\text{link}} \sum_i \int_i dx \, \text{tr} \, \Lambda Q
\]

Here, the sum is over all the black squares and the integrals are over the contours of the black squares.

Despite the fact that this action does not contain any dissipative \( \sigma_{xx} \) terms, it is easy enough to show that in the long wavelength limit, (2.24) reduces to the form of the sigma model action (2.2) with

\[
\sigma^0_{xx} = \frac{1}{2} ; \quad \sigma^0_{xy} = \frac{1}{2}
\]

The reason for this is contained in the fact that the saddlepoints act like scattering centers, which renders the system dissipative at large distances. In order to demonstrate this, all one needs to do is to follow up on (2.13) where the background field \( t_0 \) now represents the 'slow modes' that are kept. The \( t \) field variables are the 'fast modes' which contain all the wavelengths smaller than the lattice constant, i.e. the average distance between the saddlepoints, and which are integrated out. This leads to an effective action for each link according to

\[
S_{\text{link}}[t_0] = -\frac{1}{2} \int \text{link} dx \, \text{tr} \, (\langle Q \rangle t_0 \partial_x t_0^{-1}) + \frac{1}{8} \left\langle \int \text{link} dx \, \text{tr} \, (Qt_0 \partial_x t_0^{-1}) \right\rangle_{\text{conn}}
\]

\[
= -\frac{1}{2} \int \text{link} dx \, \text{tr} \, (\Lambda t_0 \partial_x t_0^{-1}) - \frac{\sigma^0_{xx}}{8} \int \text{link} dx \, \text{tr} \, (\partial_x Q_0)^2
\]

where \( Q_0 = t_0^{-1} \Lambda t_0 \) and \( \sigma^0_{xx} = L_0/2 \) is the 1D conductivity of a single channel of length \( L_0 \), a well-known result in the theory of pure metals. The subscript 'conn' indicates that only connected diagrams are taken. Next, by taking the sum over all links one can absorb the factor \( L_0 \) into the definition of a 2D integral,

\[
-\frac{1}{16} \sum_{\text{links}} L_0 \int \text{link} dx \, (\partial_x Q_0)^2 \longrightarrow -\frac{1}{16} \int d^2 x \, (\nabla Q_0)^2
\]

Here we only used the fact that the \( Q_0 \) field variable varies slowly over a distance \( L_0 \).

The first term in (2.27) can be handled in a similar way. For instance, it can be rewritten in the form of (2.24) with \( Q \) replaced by \( Q_0 \), which is then followed by taking the continuum limit according to
\[ -\frac{1}{2} \sum_{\text{links}} \int dx \, \text{tr} \left( \Lambda t_0 \partial_x t_0^{-1} \right) \longrightarrow -\frac{1}{8} \int d^2 x \, m(\vec{x}) \varepsilon_{ij} \text{tr} \, Q_0 \partial_i Q_0 \partial_j Q_0 \]

\[ \rightarrow -\frac{1}{16} \int d^2 x \, \varepsilon_{ij} \text{tr} \, Q_0 \partial_i Q_0 \partial_j Q_0 \]  \hspace{1cm} (2.29)

The result of (2.27–2.29) is identical to the statement made in (2.26). Notice that (2.26) is precisely the point where we expect the \( \sigma \) model action (2.2) to have a critical phase. Hence, we have established a direct connection between critical 1D edge states on the one hand and the 2D delocalisation transition of the band center on the other. It is important to stress that this connection has the following ingredients:

1. The infinite percolation cluster at the band center contains a finite density of saddlepoints. This translates into a finite density of scattering centers which, in turn, is responsible for making the sample diffusive (dissipative) at large distances.

2. The parameters \( \sigma_{0}^{xx}, \sigma_{0}^{xy} \) (2.26) constitute a mean field theory of the conductances which is valid for length scales \( L_0 \). In computing the numerical value of these parameters, we have taken the number of replicas a priori equal to zero.

Without going into further detail we mention the fact that the analysis can easily be generalized to more complicated situations. For example, the links between the saddlepoints need not be straight lines. They can be taken as arbitrarily complex, non-intersecting paths reflecting the highly ramified percolation contours (fig. 3). The same result (2.26) applies to all cases, indicating that the general result \( \sigma_{0}^{xx} = 1/2 \) actually stands for the quantized conductance in one dimension.

FIG. 3. Backbone cluster as in fig. 2b, but with highly ramified contours between saddle points (●).
Next, we wish to extend our mean field analysis \((2.20)\) to include also the energies away from the Landau band center. For this purpose we have to relate the range in energy \(W_0\) within which the equipotential contours form an infinite saddlepoint cluster to the total bandwidth \(W\) of the Landau band. It is understood that the phrase “saddlepoint” actually stands for those special points where two equipotential contours approach each other at a distance of the order of the magnetic length \(\ell_0\) or smaller. By assuming a simple quadratic form for the potential near saddle points we obtain the following estimate,

\[
W_0 \approx \left(\frac{\ell_0}{\lambda}\right)^2 W
\]

where \(\lambda\) is the characteristic correlation length of the random potential, which we have taken to be much larger than \(\ell_0\), and \(W\) equals the amplitude of the potential fluctuations. The sigma model theory or, equivalently, the scaling theory of localisation only applies to the (narrow) energy band \(W_0\) about the band center. For energies just outside \(W_0\) the network of saddlepoints is broken up into disconnected islands of size \(L_0 \times L_0\) (fig 2a and c). The absence of any quantum tunneling means that no correlation exists between the islands (they are represented by independent actions as long as one works within the free electron approach). In the language of the \(\sigma\) model, the situation is represented by putting \(\sigma_{xx} = 0\) but \(\sigma_{xy} = m = \text{integer}\). The latter follows from the long-ranged correlations which still exist near the edge and which can generally be expressed in terms of an integer number \(m\) of edge channels. In figure 4a we illustrate the behaviour of the density of states \(\rho\) and the conductances \(\sigma_{ij}^0\) as a function of energy \(\mu\) at zero temperature.

![Graphs showing the behaviour of \(\rho,\sigma_{xx}^0,\sigma_{xy}^0,\) and \(\sigma_{xy}^{max}\) as a function of \(\mu\).](image)

**FIG. 4.** Mean field theory for the lowest Landau level, with varying chemical potential \(\mu\).

(a) Smooth long-range disorder. (b) Short-range disorder (see text).
The sigma model conductance parameters $\sigma_{ij}^0$ can be expressed as a function of the dimensionless quantity $\Delta \mu / W_0$

$$\sigma_{ij}^0 = f_{ij}(\Delta \mu / W_0)$$

where $\Delta \mu$ is the energy relative to the Landau band center. The $f_{ij}$ are non-universal and generally depend on the microscopic details of the randomness. For comparison we have plotted the results of the more familiar theory of short-ranged scatterers (self-consistent Born approximation) in fig 4b. In this case, there is only a small difference between $W_0$ and $W$ due to the localized states in the gaussian tails of the Landau band.

An estimate for $L_0$ can be obtained as follows. Let $|\Delta \mu| \approx W_0$ denote the energies where the saddlepoint breaks up into disconnected equipotential contours of size $L_0 \times L_0$ (fig 2a,c). According to the semiclassical picture of percolation we can relate the typical cluster size $\xi_p$ to the energy $\Delta \mu$ according to

$$\xi_p \sim \lambda (\Delta \mu / W)^{-4/3}$$

where the critical index $4/3$ is the exponent for semiclassical localisation. By identifying the points $|\Delta \mu| = W_0$ and $\xi_p = L_0$ in (2.32) we obtain the estimate

$$L_0 \approx \ell_0 (\lambda / \ell_0)^{11/3} \quad (\lambda \gg \ell_0)$$

or, more generally,

$$W_n \approx \frac{\ell_n}{\lambda} W \quad ; \quad L_n \approx \ell_n (\lambda / \ell_n)^{11/3} \quad (\lambda \gg \ell_n).$$

The $\lambda$ is an adjustable parameter in the theory and it ranges between microscopic distances ($\ell_0 \approx 100\,\text{Å}$) and infinity.

2. Interaction effects

It is quite possible that $L_0$ (2.33) is many times larger than the micron regime which is the typical scale for inelastic processes at low temperatures. This means that the critical behavior (2.3) cannot be observed within the limitations of ordinary laboratory experiments. This, then, is the easiest and crudest explanation for the lack of scaling in many samples. As a first step toward a more quantitative understanding of transport at finite $T$, we come back to the distinction, made in the beginning, between the backbone cluster and the disconnected, ‘loose hanging’ pieces. Due to the electron-electron interactions, motion of the conducting electrons on the saddlepoint network is affected by the localised electrons. This may be expressed in terms of a relaxation time $\tau_{in}$ which is a characteristic time for equilibration between the conducting and localized electrons. Later on in this paper (Section V D) we shall address the problem of interaction effects and show that

$$1/\tau_{in} = \beta_1 T + \beta_2 T^2 + \cdots$$

at low temperatures. This expression is determined by the collection of ‘nearly saddlepoints’ where quantum tunneling is not possible but where the interactions between the conducting
and localized particles are strongest nevertheless. The importance of ‘nearly saddlepoints’
can be seen by comparing the wavefunctions at different energies close to the Landau band
center. What is a saddlepoint configuration at one energy may turn into a ‘nearly saddle-
point’ at another and vice versa. These abrupt changes in the configuration of the conducting
network at slightly different energies blur the distinction between saddlepoints and ‘nearly
saddlepoint’ configurations as far as finite temperatures are concerned. This means that the
relaxation time $\tau_{in}$ (2.35) determines an effective bandwidth $W_{\text{eff}} = W_0 + \tau_{in}^{-1}$ of states that
contribute to the conduction at finite temperatures. Eq. (2.31) is replaced by the expression

$$\sigma^0_{ij}(T) = f_{ij}(\Delta \mu/W_{\text{eff}}) = f_{ij}(\Delta \mu/[W_0 + \tau_{in}^{-1}]).$$

(2.36)

This is a characteristic feature of long-ranged potential fluctuations and it does not occur
in the problem of short-ranged scatterers.

To conclude this Section, we shall next estimate the range of validity of the result (2.36).
Write

$$v_d \tau_{in} = L_{in}, \quad v_d \approx 2\pi l_0^2 W/\lambda.$$  

(2.37)

The $L_{in}$ is the mean free path for drifting along the links of the lattice. We mentioned earlier
already that the actual path between two saddle points is arbitrarily convoluted and very
long. Let $L_t$ denote the actual path length between saddle points, then the criterion for
scaling is clearly given by

$$L_{in} > L_t.$$  

(2.38)

Next we use the ramification hypothesis in order to relate $L_t$ to the shortest distance
between saddlepoints ($L_0$). We obtain

$$L_t \propto L_0^\sigma$$

(2.39)

with $\sigma$ somewhere between 1 and 2. The criterion for scaling (2.38) now implies

$$\tau_{in}^{-1} < \left(\frac{l_0}{\lambda}\right)^{8\sigma/3} W_0 \ll W_0.$$  

(2.40)

This result indicates that (2.36) is very likely to be observed in the (many) samples that are
characterised by a smooth disorder potential.

The results of this Section are consistent with the recently reported empirical fitting of the transport data in the quantum Hall regime. Since we are necessarily operating with
an almost complete lack of knowledge on the microscopic details of sample disorder, it is
conceivable that other aspects of inhomogeneity especially in low mobility samples explain
the same thing.

3. Corrolaries

The subjects of critical edge states as well as long-ranged disorder have left several
conceptual questions that still need to be answered. For example, we have seen that short-
ranged disorder causes interchannel scattering between the chiral edge states. Since we
do not expect interchannel scattering to occur when the potential fluctuations are smooth (relative to the magnetic length), it is necessary to reinvestigate the meaning of instanton vacuum theory \((2.2)\) for \(\nu > 1\) \((\sigma_{xy}^0 > 1)\).

Scattering between multiple edge states is avoided by writing, instead of \((2.2)\)

\[
S_{\text{eff}}[Q^{(n)}] = \sum_{n=0}^{\infty} \left[ -\frac{1}{8}\sigma_{xx}^{(n)} \int d^2x \text{ tr } [\nabla Q^{(n)}]^2 - \frac{1}{8}\sigma_{xy}^{(n)} \int d^2x \text{ tr } \varepsilon_{ij} Q^{(n)} \partial_i Q^{(n)} \partial_j Q^{(n)} \\
+ \rho^{(n)} \eta \int d^2x \text{ tr } \Lambda Q^{(n)} \right]
\]  

(2.41)

where the sum runs over all the Landau levels \(n\). The \(Q^{(n)}\) stands for an independent field variable \(Q\) for each Landau level separately. The \(\sigma_{ij}^{(n)}\) are the \(n\)th Landau level contributions to the mean field conductances, which are now given by

\[
\sigma_{ij} = \sum_{n=0}^{\infty} \sigma_{ij}^{(n)}.
\]  

(2.42)

The \(\sigma_{ij}^{(n)}\) are all the same (Fig. 4a) except for an appropriate shift in energy. Since \(0 \leq \sigma_{xy}^{(n)} \leq 1\) for each \(n\), it is clear that \((2.41)\) is the appropriate generalisation of the theory (section II A) to include filling fractions larger than one. The theories of \((2.41)\) and \((2.2)\) are identical as far as the critical behaviour of the plateau transitions is concerned. Equation \((2.41)\) cannot, however, be used in the limit of small magnetic field, where the Landau levels partly or completely overlap. The details of crossover require a separate analysis.

Next there are the difficulties in establishing a topological principle for Hall quantisation, as mentioned in section II A 1. Since the previous arguments were actually based on an incomplete knowledge of edge effects, it is obviously important to redo the analysis and see whether the instanton vacuum approach is in fact free of ambiguities. For this purpose we shall follow up on the background field methodology which is known to generate the Kubo formulae for the conductances. We write

\[
\exp S_{\text{eff}}[t_0] := \int \mathcal{D}[Q] \exp \left( S_0[t_0^{-1}Qt_0] + \rho^0 \eta \text{Tr } \Lambda Q \right)
\]  

(2.43)

where

\[
S_0[Q] = -\frac{1}{8}\sigma_{xx}^0 \text{Tr } (\nabla Q)^2 - \frac{1}{8}\sigma_{xy}^0 \text{Tr } \varepsilon_{ij} Q \partial_i Q \partial_j Q.
\]  

(2.44)

Eq. \((2.43)\) defines an effective action for the fixed \(t_0\) matrix field. By making use of the local \(U(N_r) \times U(N_r)\) gauge invariance of the theory \((2.43)\), one can show that \(S_{\text{eff}}\) is of the same form as \(S_0\), i.e.

\[
S_{\text{eff}}[t_0] = -\frac{1}{8}\sigma_{xx} \text{Tr } (\nabla Q_0)^2 - \frac{1}{8}\sigma_{xy} \text{Tr } \varepsilon_{ij} Q_0 \partial_i Q_0 \partial_j Q_0
\]  

(2.45)

with \(Q_0 = t_0^{-1} \Lambda t_0\). The main problem next is to obtain explicit knowledge of the ‘effective’ parameters \(\sigma_{ij}\) in \((2.45)\), which now represent the (exact) Kubo expressions for the conductances. As long as one works with spherical boundary conditions on the matrix field \(Q\), the quantisation of the Hall conductance is readily established. All that one needs is in fact that the theory develops a mass gap in the limit of large distances. Under these circumstances
insertion of slowly varying, spherically symmetric background fields $t_0$ should be immaterial (as $\eta \to 0$) and this then necessarily implies that $\sigma_{xx} = 0$ and $\sigma_{xy} = \text{integer}$. For example, the explicit renormalisation group flows obtained from instanton calculations always indicate that the quantum Hall conditions $\sigma_{xx} = 0$, $\sigma_{xy} = \text{integer}$ are the stable (massive) fixed points of the theory for arbitrary number of field components $N_r$.

It is well known, however, that spherical boundary conditions are justified in the theory in weak coupling only, due to finite action requirements. Since there is no compelling argument which says that spherical boundary conditions should be retained all the way down to the strong coupling regime, one is clearly left with ambiguities in the instanton vacuum approach, at least as far as the quantisation of the Hall conductance is concerned.

Armed with the insight gained from edge excitations in the previous sections, we next apply the background field procedure to the theory, but now with free boundary conditions on $Q$, as it should be. For the special case where the Fermi energy lies in a density of states gap, (2.43) has already been worked out in detail (appendix C). A straightforward computation of $S_{\text{eff}}$ with $N_r$ put equal to zero leads to

$$S_{\text{eff}}[t_0] = -2\pi im q[Q_0] - \frac{m}{8\eta_{\text{edge}}} \oint (\partial_x Q_0)^2$$

where $q[Q_0] = \frac{1}{16\pi^2} \text{Tr} \, \varepsilon_{ij} Q_0 \partial_i Q_0 \partial_j Q_0$ and the contour integral is along the sample edge. The result naively diverges as $\eta \to 0$ and this clearly reflects the effect of the massless excitations in the problem. In the limit $\eta \to 0$, the field $t_0$ entering (2.46) is forced to obey the classical equations of motion (defined along the sample edge),

$$m \partial_t [Q_0, \Lambda] + \frac{m}{8\eta_{\text{edge}}} \partial_x [Q_0 \partial_x Q_0, \Lambda] = 0. \quad (2.47)$$

The solution $Q_0 = \text{constant}$ at the edge means that spherical boundary conditions are automatically generated by the background field procedure as $\eta \to 0$. The effect of $S_{\text{eff}}$ has now been reduced to a phase factor which is immaterial provided the Hall conductance precisely equals an integer. Physically, the phase factor describes an integer number of edge electrons which have crossed the Fermi level as a result of the background field insertion.

It is now clear that quantum Hall physics is a unique aspect of the theory with $N_r = 0$ number of field components. The same procedure can be repeated for the theory with $\sigma_{xx} \neq 0$, making use of the fact that a mass gap exists in the system of long wavelength excitations, i.e. a finite localisation length $\xi$. One expects (2.46) to be modified according to

$$S_{\text{eff}}[t_0] = -\frac{a_{xx}}{8} \text{Tr} \, (\nabla Q_0)^2 - 2\pi i \sigma_{xy} q[Q_0] - \frac{a_{xx}}{8\eta_{\text{edge}}} \oint (\partial_x Q_0)^2$$

where the $\sigma_{ij}$ represent the ‘conductances’,

$$\sigma_{xx} = f_{xx}(\eta \xi^2) \approx O(\eta^2); \quad \sigma_{xy} = f_{xy}(\eta \xi^2) \approx m + O(\eta^4) \quad (2.49)$$

and where $a_{xx} = m/2$ is the quantised 1D conductance of the chiral edge states. In the limit $\eta \to 0$ the $Q_0$ entering (2.48) is forced to obey not only the classical equations of motion on the edge (2.47), but also those arising from the bulk kinetic term in (2.48). The solutions are known as instantons and just as has happened before in the trivial example with a density of
states gap in the bulk, the effect of $S_{\text{eff}}$ reduces to that of a phase factor which is immaterial as long as $\sigma_{xy}=$integer. Therefore, the quantum Hall effect can be understood in terms of a continuous symmetry which is dynamically restored in the limit of large length scales.

In summary we can say that the quantisation phenomenon is a robust and fundamental property of the instanton vacuum theory with $N_r = 0$ field components. For any $N_r > 0$ the quantisation is neither robust nor fundamental, since it depends on how the theory is formulated, i.e. on the specific choice of boundary conditions. Finally, we mention that the frequency dependence of $\sigma_{xx}$ \cite{20} is a quite general result of Anderson localised states and it is consistent with the results of the perturbative renormalisation group. The $\eta$-dependence of the Hall conductance, on the other hand, can be obtained from the requirement that the response functions for localised bulk levels be local in time.

III. DERIVATION OF THE FULL EDGE THEORY

A. Preliminaries

1. Notation

Let us start by writing down the $Q$-field theory for disordered electrons in 2+1 dimensions in the presence of Coulomb interactions and external potentials, derived in \cite{1}

$$S[A, \tilde{Q}, \lambda] = -\frac{1}{2g} \text{Tr} \tilde{Q}^2 + \text{Tr} \ln[i\omega + i\hat{A}_0 + i\hat{\lambda} + \mu - \hat{\mathcal{H}} + i\tilde{Q}]$$

$$-\frac{1}{2} \beta \int d^2x d^2x' \lambda^1(x)U^{-1}(x,x')\lambda(x'). \quad (3.1)$$

The symbols appearing in this action have the following meaning: The $\tilde{Q}(\vec{x})$ is an infinite-dimensional matrix field with two replica indices and two Matsubara frequency indices. (In the derivation of the above action, it arises as a quadratic expression in the original electron field $\psi$; The saddle point is given by $\tilde{Q}^{\alpha\beta}_{nm} \propto \psi^\alpha_n \bar{\psi}^\beta_m$.) Upper Greek indices denote a replica channel, running from 1 to $N_r$, while lower latin indices stand for Matsubara frequencies, running from $-\infty$ to $\infty$. The matrix field $\tilde{Q}$ can be split into ‘transverse’ and ‘longitudinal’ components

$$\tilde{Q} = T^{-1}PT \quad P = P^\dagger \quad T \in SU(2\tilde{N}). \quad (3.2)$$

Here, $P$ has only block-diagonal components in frequency space (i.e. $P^{\alpha\beta}_{nm} \neq 0$ only for $\omega_m\omega_n > 0$) and $T$ is a unitary rotation. The size of the $\tilde{Q}$-matrix is given by $2\tilde{N}$, namely the number of replicas times the size of Matsubara frequency space. The matrix $\omega$ is unity in replica space, while in frequency space it is a diagonal containing the fermionic frequencies,

$$\omega^{\alpha\beta}_{nm} = \delta^{\alpha\beta}\delta_{nm}\omega_n \quad ; \quad \omega_n = \frac{2\pi}{\beta}(n + \frac{1}{2}) \quad (3.3)$$

with $\beta$ the inverse temperature. The symbol ‘Tr’ denotes a matrix trace as well as spatial integration. All spatial integrals are taken in the upper half plane $y > 0$. The sample edge is given by the line $y = 0$. The $U^{-1}(\vec{x},\vec{x}')$ is the matrix inverse of the Coulomb interaction $U(\vec{x},\vec{x}')$. $A_\mu$ is the external potential; $\lambda$ is the plasmon field. It is assumed that these fields
do not have a static ($n = 0$) component. The ‘hat’ notation ($\hat{}$) appearing in (3.1) is defined as follows

$$\hat{x} = \sum_{\alpha=1}^{N_r} \sum_{n=-\infty}^{\infty} x^\alpha_n \tilde{I}^\alpha_n$$

where $\tilde{I}^\alpha_n$ is the unity matrix in the $\alpha$’th replica channel, shifted by $n$ places in frequency space

$$(\tilde{I}^\alpha_n)^{\beta\gamma}_{kl} = \delta^{\alpha\beta} \delta^{\alpha\gamma} \delta_{k-l,n}. \quad (3.5)$$

The $H$ is the kinetic energy (differential) operator,

$$H = \frac{1}{2m_e} (\vec{\pi} - \vec{A}) \cdot (\vec{\pi} - \vec{A}) \quad \vec{\pi} = \frac{i}{\hbar} \nabla - \vec{A}_{st} \quad \vec{\pi} = -\frac{i}{\hbar} \nabla - \vec{A}_{st} \quad (3.6)$$

where $\vec{A}_{st}$ describes the static magnetic field according to $\nabla \times \vec{A}_{st} = B_{st}$.

2. Flux-charge composites

In order to describe the FQHE one also needs to include a statistical or Chern-Simons gauge field $a_\mu$ in (3.1) as follows

$$S[A, \tilde{Q}, \lambda] \rightarrow S[A + a, \tilde{Q}, \lambda] + \frac{i}{8\pi} \int d\tau d^2 x \varepsilon^{\mu\nu\sigma} a_\mu \partial_\nu a_\sigma, \quad (3.7)$$

with $2p$ an even integer denoting the number of elementary flux quanta $h/e$ attached to every electron. Note that in this procedure the zero-frequency components of all fields are to be treated at a mean field level. This amounts to adding an extra contribution $\vec{a}_{st}$ to the static part of the external vector potential $\vec{A}_{st}$, resulting in an effective magnetic field $B_{st} = \nabla \times (\vec{A}_{st} + \vec{a}_{st}) = B_{st} + 2pn_e h/e$, with $n_e$ the mean electron density. Jain’s composite fermion mapping is then implemented by integrating out the field $a_\mu$. In this paper, however, we only consider the integer quantum Hall effect; we deal with the fractional effect in a subsequent publication.

3. Landau gap

A theory for the edge is obtained by choosing the chemical potential $\mu$ approximately halfway between Landau energies, where the bulk density of states is virtually zero if the disorder is not too strong. The saddle point equation for $\tilde{Q}$ is given by

$$\tilde{Q}_{sp} \propto \rho T^{-1} \Lambda T \quad (3.8)$$

where $\rho$ is the density of states and $\Lambda$ is the matrix appearing in (2.1) but now with full frequency dependence

$$\Lambda^\alpha_{kl} = \delta^\alpha \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]_{kl}. \quad (3.9)$$
Since we are interested in the limit $\rho \to 0$, we may replace the full expression for $\tilde{Q}$ (3.2) by a much simpler one,
\[
\tilde{Q} \to \epsilon T^{-1} \Lambda T, \quad \epsilon \ll 1.
\] (3.10)

From detailed earlier work\[^{10}\] we know that (3.2) and (3.10) give rise to identical results as long as the bulk density of states $\rho$ can be safely taken to zero. However, in order to deal with the complications of $U(1)$ gauge invariance (section B), there is considerable advantage in working with the simplified expression (3.10), and we will refer to the details of more elaborate analyses only when necessary.

**B. Gauge invariance and truncation of frequency space**

The electromagnetic $U(1)$ gauge transformations in this theory are generated by the $\tilde{I}$-matrices. Multiplication of these matrices is very simple,
\[
\tilde{I}_n^\alpha \tilde{I}_m^\beta = \delta^{\alpha\beta} \tilde{I}_n^\alpha \delta, \quad \alpha, \beta = 0, 1, \ldots, \nu_n
\] (3.11)
and they form an abelian algebra. Gauge transformations are given by
\[
A_\mu \to A_\mu + \partial_\mu \chi \quad ; \quad \tilde{Q} \to e^{i\tilde{\chi}} \tilde{Q} e^{-i\hat{\chi}}
\] (3.12)
with $\chi^0_0 = 0$. The gauge invariance of (3.1) is easily checked by writing the transformed Tr ln in the form Tr ln($e^{-i\hat{\chi}} \cdot \cdot \cdot e^{i\hat{\chi}}$), noting that
\[
e^{-i\hat{\chi}} \omega e^{i\tilde{\chi}} = \omega - \hat{\partial}_0 \chi
\] (3.13)
and
\[
e^{-i\hat{\chi}}(\hat{\pi} - \hat{A} - \nabla \hat{\chi}) e^{i\hat{\chi}} = \hat{\pi} - \hat{A}
\] (3.14)
In order to facilitate the expansion of the Tr ln term in (3.1) we perform a gauge transformation that sets $A_0 + \lambda = 0$. Introducing the notation
\[
\tilde{W} = \exp \left( \sum_\alpha \sum_{n \neq 0} \frac{(A_0 + \lambda)^\alpha_n}{\nu_n} \tilde{I}_n^\alpha \right) \quad ; \quad \tilde{z}_n^\alpha = \tilde{A}_n^\alpha - i \frac{\nabla (A_0 + \lambda)^\alpha_n}{\nu_n}
\] (3.15)
and the gauge invariant quantity $\tilde{R} = \tilde{W} \tilde{Q} \tilde{W}^{-1}$, the Tr ln can be written as
\[
\text{Tr ln}[i \omega + \mu - \frac{1}{2 m_e} (\hat{\pi} - \hat{\tilde{z}}) \cdot (\hat{\pi} - \hat{\tilde{z}}) + i \hat{R}].
\] (3.16)

Notice that $\tilde{z}_n^\alpha = i (\partial_0 \tilde{A} - \nabla A_0)^\alpha_n / \nu_n$, from which it is clear that $\tilde{z}$ is also gauge invariant.

As was the case in [I], we have to impose a cutoff on the size of the Matsubara frequency space. Instead of being infinite, all matrices are now of size $2N'_\text{max} \times 2N'_\text{max}$ in frequency space. The Matsubara indices sit in the interval $(-N'_\text{max}, \ldots, N'_\text{max} - 1)$. 

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The truncated version of the $\tilde{I}$-matrices is denoted by $I^\alpha_n$. The ‘hat’ notation is now defined with respect to the truncated matrices $I^\alpha_n$. These no longer span an abelian algebra; instead their commutators are given by

$$
(I^\alpha_n I^\beta_m)^{\mu\nu}_{kl} = (\tilde{I}^\alpha_n \tilde{I}^\beta_m)^{\mu\nu}_{kl} g_{l+m} + \delta^{\alpha\beta}_{\mu\nu} \delta_{k-l,m+n} (g_{l+m} - g_{l+n})
$$

(3.17)

where $\delta^{\alpha\beta}_{\mu\nu}$ means that all replica indices have to be the same, and $g_i$ is a step function equal to one if $i \in \{-N'_\text{max}, \ldots, N'_\text{max} - 1\}$ and zero otherwise.

In order to retain some form of gauge invariance, a second cutoff $N_{\text{max}} \ll N'_\text{max}$ is introduced for the matrix field $T$. With the truncated $T$ we define the truncated equivalent of $\tilde{Q}$ (see figure 5),

$$
Q = T^{-1} \Lambda T.
$$

(3.18)

![FIG. 5. The truncated matrices $T$ and $Q$; Also is drawn the frequency band in which $\text{tr} \, I^\alpha_n Q \neq 0$](image)

It was shown in [1] that most of the problems caused by the change from (3.11) to (3.17) can be avoided by the introduction of the second cutoff. A remnant of the $U(1)$ symmetry is kept in this way: invariance of the action under the truncated equivalent of (3.12)

$$
A_\mu \to A_\mu + \partial_\mu \chi \quad ; \quad Q \to e^{i\tilde{\chi}} Q e^{-i\tilde{\chi}}.
$$

(3.19)

We do not have a full symmetry of the theory, since the integration measure $DQ$ is not invariant under (3.13). It can easily be checked that the ‘small’ form of $Q$, involving only small frequencies (fig. 5), is lost when a gauge transformation is applied. (Only in the limit $N_{\text{max}}, N'_\text{max} \to \infty$ full symmetry is regained. It is implicitly understood that this limit is taken after all calculations.) In order to be still able to work with a ‘small’ $Q$-matrix, we write

$$
A_\mu \to A_\mu = A_\mu - \partial_\mu \Omega
$$

(3.20)

where $\Omega$ is a gauge, chosen precisely such that $Q$ has the ‘small’ form. Observable quantities should not depend on $\Omega$; at a later stage, this statement will enable us to perform consistency checks and even to take shortcuts in certain calculations.

By taking the second cutoff $N_{\text{max}}$ we also restrict the interval of the frequency indices on $\lambda^\alpha_n$ and $(A_\mu)^\alpha_n$ to $n \in [-2N_{\text{max}} + 1, 2N_{\text{max}} - 1]$ (see fig. 5). This interval corresponds to $\text{tr} \, I^\alpha_n Q \neq 0$. Introducing the following abbreviations

$$
\mathcal{A}'_\mu = A_\mu + \delta_{\mu 0} \lambda \quad W = \exp \left[ \sum_n \Gamma_n^\alpha (A_0^\alpha)^{\alpha}_n / \nu_n \right] \quad R = W Q W^{-1}
$$

(3.21)
the action becomes (up to a constant)
\[
S[Q, \lambda, A] = -\frac{1}{2} \beta \int d^2x d^2x' \lambda^\dagger(\vec{x}) U^{-1}(\vec{x} - \vec{x}') \lambda(\vec{x}')
+ \text{Tr} \ln \left[ i \omega + \mu - \frac{1}{2m_e} (\vec{\pi} - \hat{\vec{z}}) \cdot (\vec{\pi} - \hat{\vec{z}}) + i \varepsilon R \right]
\] (3.22)
with \( \hat{\vec{z}} \) modified according to (3.20)

C. Expansion of the Tr ln

Let us look at the last term in (3.22), \( X := \text{Tr} \ln[i \omega + \mu - \mathcal{H}_z + i \varepsilon R] \). Introducing the notation
\[
D_{\omega} = TW^{-1} \omega WT^{-1} \quad ; \quad D_{\vec{z}} = TW^{-1} \left( \frac{1}{i} \nabla - \hat{\vec{z}} \right) WT^{-1}
\] (3.23)
(\text{where } D_{\vec{z}} \text{ is not a differential operator}) we can write
\[
X = \text{Tr} \ln[i D_{\omega} + \mu + i \varepsilon \Lambda - \frac{1}{2m_e} (\vec{\pi} \cdot \vec{\pi} + \vec{\pi} \cdot D_{\vec{z}} + D_{\vec{z}} \cdot \vec{\pi} + D_{\vec{z}}^2)].
\] (3.24)
Expansion to first order in \( D_{\omega} \) and \( D_{\vec{z}} \) yields
\[
X \approx \text{Tr} \ln G_0^{-1} + i \text{Tr} [G_0 D_{\omega} - \frac{1}{2m_e} \text{Tr} [G_0 \vec{\pi} \cdot D_{\vec{z}} + G_0 D_{\vec{z}} \cdot \vec{\pi}]]
\] (3.25)
where \( G_0 \) is the bare Green’s function \( \left[ \mu - \frac{1}{2m_e} \vec{\pi} \cdot \vec{\pi} + i \varepsilon \Lambda \right]^{-1} \). The Green’s function can be expressed in terms of the eigenfunctions \( \varphi_{n_j} \) of the bare Hamiltonian \( \mathcal{H}_0 = \frac{1}{2m_e} \vec{\pi} \cdot \vec{\pi} \),
\[
\langle x | G_0 | x \rangle = \sum_{n_j} \frac{|\varphi_{n_j}(x)|^2}{\mu - E_{n_j} + i \varepsilon \Lambda}
\] (3.26)
\[
\langle x | G_0 \vec{\pi} + \vec{\pi} G_0 | x \rangle = \sum_{n_j} \left[ \frac{\varphi_{n_j}^* \frac{1}{i} \nabla \varphi_{n_j} - \varphi_{n_j} \frac{1}{i} \nabla \varphi_{n_j}^* - 2 \varphi_{n_j}^* \varphi_{n_j} A}{2m_e(\mu - E_{n_j} + i \varepsilon \Lambda)} - \right]
\] (3.27)
Using the general relation \( \rho(x) = -\frac{1}{\pi} \text{Im} G^+(x, x) \) for the density of states at the Fermi energy \( \mu \), we get
\[
\langle x | G_0 | x \rangle = -i \pi \rho(x) \Lambda + c \mathbf{1} \quad ; \quad \langle x | G_0 \vec{\pi} + \vec{\pi} G_0 | x \rangle = -i \pi \vec{j}(x) \Lambda + c \vec{1},
\] (3.28)
where \( \vec{j}(x) \) is the current density per energy interval at the Fermi energy. The \( c \) and \( \vec{c} \) are real functions that disappear from the last two traces in (3.25). We can now write \( X \) in the form
\[
X \approx \text{Tr} \ln G_0^{-1} + \pi \int d^2x \rho(x) \text{tr} [\Lambda D_{\omega}] + i \pi \int d^2x \vec{j}(x) \cdot \text{tr} [\Lambda D_{\vec{z}}]
\] (3.29)
where we have used that $\frac{\partial f_{\text{edge}}}{\partial \mu} = \frac{m}{2\pi}$ with the plateau-center filling fraction $m = \frac{n_{\mu}}{B\epsilon}$ integer-valued, and $S_{\text{top}}$ is the topological action

$$S_{\text{top}}[R] = \frac{1}{8} \text{Tr} \, \varepsilon^{ij} R \partial_i R \partial_j R.$$  

Eq. (3.30), however, is not yet the complete answer. This can be seen from a different expansion procedure which can be followed in the special case where $T = 1$. In this case we have, instead of (3.24),

$$X \approx \text{Tr} \, \ln G_0^{-1} + \pi \rho_{\text{edge}} \int dx \, \text{tr} \, [\omega R] - i\frac{1}{2} m \int d\vec{x} \cdot \text{tr} \, [\hat{z} R] + m S_{\text{top}}[R]$$

(3.30)

where we have used that $\frac{\partial f_{\text{edge}}}{\partial \mu} = \frac{m}{2\pi}$ with the plateau-center filling fraction $m = \frac{n_{\mu}}{B\epsilon}$ integer-valued, and $S_{\text{top}}$ is the topological action

$$S_{\text{top}}[R] = \frac{1}{8} \text{Tr} \, \varepsilon^{ij} R \partial_i R \partial_j R.$$  

The 'polarisation operator' $\Pi_{ij}$ is given by

$$(\Pi_{ij})_{\alpha}^\alpha(x, x') = (\frac{1}{2m_e})^2 \text{tr} \, [G(x, x')(\pi_x + \pi_i) I_{\alpha} G(x', x)(\pi_j + \pi_i) I_{-\alpha}]$$

$$= (\frac{1}{2m_e})^2 \sum_k G_{k+n}(x, x')(\pi_i + \pi_i) G_k(x', x)(\pi_j + \pi_j)$$

(3.34)

The frequency sum can be split in two parts: (I) $k$ and $k+n$ have the same sign; (II) $k$ and $k+n$ have opposite signs. Case II has been done in great detail in the context of the SCBA. The conclusion is that (II) does not contribute either to $\sigma_{xx}$ or $\sigma_{xy}$ when $\mu$ is in a density of states gap. Case I, using the relation $\pi + \pi = -i 2m_e[G^{-1}, \vec{x}]$, gives rise to the familiar 'Streda' form for $\sigma_{xy}$. We arrive at the following expression,

$$X_2 \approx \text{Tr} \, \ln G^{-1} + \frac{1}{2} m \sum_{\alpha} \int d^2 x \, n \, z_{\alpha} \times \hat{z}_{-\alpha}.$$  

(3.35)

Now we have to find a match between the first order result (3.30) for $T \neq 1$ and the second order result (3.35) for $T = 1$. Up to a constant arising from the difference between $G_0$ and $G$, this match is given by

$$\text{Tr} \, \ln G^{-1} + \pi \rho_{\text{edge}} \int dx \, \text{tr} \, [\omega R] + m \left( \frac{1}{2} \varepsilon^{ij} \text{Tr} \, R[D_i, R][D_j, R] - i \frac{1}{2} \text{Tr} \, R \nabla \times \hat{z} \right)$$

$$= \text{Tr} \, \ln G^{-1} + \frac{m}{2\nu_d} \int dx \, \text{tr} \, R(\omega - iv_d \hat{z}_x) + m S_{\text{top}}[R] - \frac{im \beta}{4\pi} \int d^2 x \, \hat{z} \times \partial_0 \hat{z}$$

(3.36)
with \( v_d \) the electron drift velocity at the edge,

\[
v_d = \frac{m}{(2\pi \rho_{\text{edge}})}.
\]

Writing this result in terms of \( Q \) is nontrivial, since the I-matrices appearing in \( W \) are truncated and do not satisfy a \( U(1) \) algebra. As a consequence of the truncation procedure, quadratic terms in \( A' \) arise. Using the relations

\[
\begin{align*}
\text{tr} [\omega R] &= \text{tr} [\omega Q] + \text{tr} [\hat{A}_0 Q] - \frac{\beta}{2\pi} A_0^\dagger A_0' \\
\text{tr} [R \hat{z}_x] &= \text{tr} [Q \hat{A}_x] - \text{tr} Q \frac{\partial_x A'_0}{\partial_0} - \frac{\beta}{2\pi} A_0^\dagger A_0' + \frac{\beta}{2\pi} \frac{\partial_x A'_0}{\partial_0}
\end{align*}
\]

\[
\beta \int d^2 x \hat{z}^\dagger \times \partial_0 \hat{z} = -\sum_\alpha \int A'^\alpha \wedge d A'^\alpha - \beta \int dx [\partial_0 A_x - \partial_x A'_0]^\dagger A'_0
\]

which are a result of the peculiar “\( \mathcal{F} \)-algebra” structure (3.17), we obtain the following action

\[
S[Q, A, \lambda] = S_c[\lambda] + S_b[\lambda, A] + S_Q[Q, \lambda, A]
\]

\[
S_c = -\frac{1}{2} \int d^2 x d^2 x' \lambda^\dagger(x) U^{-1}(x - x') \lambda(x')
\]

\[
S_b = \frac{im}{4\pi} \sum_\alpha \int A'^\alpha \wedge d A'^\alpha - \frac{m\beta}{4\pi v_d} \int dx A'_0^\dagger A'_-
\]

\[
S_Q = \frac{m}{2\nu_d} \int dx \text{tr} Q(\omega + \hat{A}'_-) + mS_{\text{top}}[Q]
\]

The first term is the Coulomb energy contribution from the plasmon field; the \( S_b \) is a “boson” action (this adjective will become clear later on); the last expression, \( S_Q \), contains the action for the \( Q \) field and the coupling of \( Q \) with \( \lambda \) and \( A \). We have defined a ‘minus’ direction as follows,

\[
A'_- = A'_0 - i v_d A'_x
\]

reflecting the chirality inherent in the problem.

**IV. CHIRAL EDGE BOSONS**

Now we will show that the degrees of freedom contained in the matrix field \( Q \) can be translated into a theory of chiral edge bosons similar to the phenomenological theory for incompressible quantum Hall states.

**A. The noninteracting case**

In the case of free electrons, only the fields \( Q \) and \( A \) are present. In order to obtain an effective action for \( A_\mu \) we integrate out \( Q \). We parametrise \( T \) as in (2.13) and we obtain the tree level propagator
\[ \langle \text{tr} [\Gamma^\alpha_n Q(q)] \text{tr} [\Gamma^{-\alpha}_n Q(-q)] \rangle = \frac{\partial \nu_d}{2\pi m} \frac{\omega_n}{\omega_n + iv_d q}. \] (4.1)

This yields the result
\[
S = \frac{im}{4\pi} \left[ \sum_\alpha \int A^\alpha \wedge dA^\alpha + \beta \int dx \ E_x^\dagger \left( \partial^{-1}_- A_- + \Omega^{\text{res}} \right) \right] \] (4.2)

with the following meaning of the symbols.
\[
\partial_- = \partial_0 - iv_d \partial_x \] (4.3)

and the inverse \( \partial^{-1}_- \) is given by
\[
(\partial^{-1}_- F)(x,\tau) = \frac{1}{(2\pi)^2} \int dx' d\tau' \ F(x',\tau') \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \ \frac{\exp[ik(x-x')-i\omega(x-x')]}{-\omega + dvk} \\
= \frac{1}{2\pi} \int dx' d\tau' \ F(x',\tau') \left[ \frac{\Theta(\tau-\tau')}{\nu_d(\tau-\tau')-i(x-x')^2+\eta} + \frac{\Theta(\tau'-\tau)}{\nu_d(\tau-\tau')-i(x-x')^2-\eta} \right] \] (4.4)

with \( F \) an arbitrary function, \( \Theta \) the Heaviside step function and \( \eta \) a regulator. The operation \( \partial^{-1}_- \) does not completely commute with \( \partial_- \). On the one hand it is easily checked that
\[
\partial_- (\partial^{-1}_- F) = F, \]
but on the other hand we have
\[
\partial^{-1}_- (\partial_- F) = F - F^{\text{res}} \] (4.5)

with \( F^{\text{res}} \) defined as that part of \( F \) which satisfies \( \partial_- F = 0 \). Another property of this operation is
\[
\int dx d\tau \ F_1 (\partial^{-1}_- F_2) = - \int dx d\tau \ (\partial^{-1}_- F_1) F_2. \] (4.6)

The \( Q \)-integration can also be done by choosing \( \Omega \) in such a way that \( Q \) decouples from \( A_- \),
\[
\partial_- \Omega = A_-, \quad \Omega = \partial^{-1}_- A_- + \Omega^{\text{res}}, \] (4.7)
yielding exactly the same result (4.2). The action (4.2) can also be written as a path integral over \( m \) charge 1 bosons,
\[
S[A, \varphi_i] = \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_\alpha \int A^\alpha \wedge dA^\alpha - \beta \int dx \ (D_x \varphi_i^\dagger D_- \varphi_i - E_x^\dagger \varphi_i) \right], \] (4.8)

where the covariant derivative \( D \) is defined as \( D^\mu \varphi_i = \partial^\mu \varphi_i - A^\mu \). The zero-momentum part of each boson field has to be excluded from the path integral, since the action does not depend on it.

In order to make contact with Ref.1 we mention that (4.8) is equivalent to a Chern-Simons bulk theory with \( m \) gauge fields \( g_i \) that represent potentials for the electron currents, coupled to the external potentials \( A^\mu \),
\[
S[A, g_i] = \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_\alpha \int g_i^\alpha \wedge dg_i^\alpha + 2 \int g_i^\alpha \wedge dA^\alpha \right] \] (4.9)

where the \( g_i \) have the gauge fixing constraint \( (g_i)_{-}\text{edge} = 0 \). In appendix B we explicitly show how integration over the potentials \( g_i \) leads to the action (4.8).
B. The Coulomb case

1. Integration over $\lambda$ and $Q$

Now we look at the full action (3.39). In this expression the plasmon field $\lambda$ is contained in the following way

$$S_b[\mathcal{A}'] = S_b[\mathcal{A}] + \frac{im\beta}{2\pi v_d} \int d^2 x \lambda^\dagger B - \frac{m\beta}{4\pi v_d} \int dx (\lambda^\dagger \lambda + 2\lambda^\dagger \mathcal{A}_0)$$

$$S_Q[\mathcal{A}'] = S_Q[\mathcal{A}] + \frac{m}{2v_d} \int dx \text{tr} \dot{\lambda} Q.$$ 

Integrating out the plasmon field $\lambda$, we obtain an effective action for $Q$ coupled to $\mathcal{A}$, which we organize as follows

$$S = S_0[Q] + S_{\text{int}}[Q, \mathcal{A}] + S_{b}[\mathcal{A}] + S_{\text{flux}}[\mathcal{A}].$$

The first term is given by

$$S_0[Q] = m S_{\text{top}}[Q] + m \frac{\pi}{4\beta v_d} S_F[Q] - m \frac{\pi}{4\beta} \sum_{n, \alpha} \int dk_x \frac{1}{\text{waf}(k_x)} |\text{tr} I_n^\alpha Q|^2$$

with

$$S_F[Q] = \int dx \left[ \sum_{n, \alpha} \text{tr} I_n^\alpha Q \text{tr} I_n^\alpha Q + 4\text{tr} \eta Q \right]$$

the edge analogue of the $\mathcal{F}$-invariant Finkelstein action for the bulk [I] and

$$v_{\text{eff}}(k_x) := v_d + \frac{m}{\sqrt{2\pi}} U(k_x)$$

the “effective velocity”, where $U(k_x) = (2\pi)^{-1/2} \int dk_y U(\vec{k})$ is the Coulomb interaction on the edge. The last term in (4.12) is the edge version of the “Coulomb” term from [I]. Note that the Finkelstein and “Coulomb” terms together can be written as

$$\frac{m}{2v_d} \int dx \text{tr} \omega Q + \frac{m\pi}{4\beta v_d} \sum_{n, \alpha} \int dk_x \frac{\rho_{\text{edge}}}{\sqrt{2\pi} U^{-1}(k_x) + \rho_{\text{edge}}} |\text{tr} I_n^\alpha Q|^2$$

where the expression in front of the $|\text{tr} IQ|^2$ is just the 1D screened Coulomb interaction. The other terms in (4.11) are a coupling term

$$S_{\text{int}}[Q, \mathcal{A}] = \frac{m}{2} \int dk_x \frac{1}{\text{waf}(k_x)} \text{tr} Q \mathcal{A}_c^\text{eff},$$

a “boson” term

$$S_{b}[\mathcal{A}] = \frac{im}{4\pi} \sum_{\alpha} (\mathcal{A}_c^\text{eff})^\alpha \wedge d(\mathcal{A}_c^\text{eff})^\alpha - \frac{m\beta}{4\pi} \int dk_x \frac{1}{\text{waf}(k_x)} (\mathcal{A}_0^\text{eff})^\dagger \mathcal{A}_c^\text{eff},$$

and a flux-flux interaction term
Here we have introduced an ‘effective’ gauge field which contains a Coulomb correction to the scalar potential,
\[ \mathcal{A}_{\text{eff}} = \mathcal{A} ; \quad \mathcal{A}^\alpha_{\text{eff}}(\vec{x}) = \mathcal{A}^\alpha_0(\vec{x}) + \frac{i m}{2\pi} \int d^2x' U(\vec{x} - \vec{x}') B(\vec{x}'), \]
and an effective ‘minus’-direction denoted by the subscript ‘c’
\[ \partial_c := \partial_0 - i v^\text{eff} \partial_x ; \quad \mathcal{A}_c := \mathcal{A}^\alpha_0 - i v^\text{eff} \mathcal{A}^\alpha_x. \]
Comparing the result (4.11) with the free particle case (3.39 without \( \lambda \)) we see that the presence of the Coulomb interaction has the following effects:
- the appearance of the flux-flux interaction term \( S_{\text{flux}}[\mathcal{A}] \) and of the screened Coulomb interaction in \( S_0[Q] \).
- the replacements \( \mathcal{A}_0 \rightarrow \mathcal{A}_{\text{eff}}^0 \) and \( \mathcal{A}_- \rightarrow \mathcal{A}_{\text{eff}}^c \).
- the replacement \( v_d \rightarrow v^\text{eff}(k_x) \).
For what follows, it is convenient to rewrite the first three terms of (4.11) as
\[ S_0 + S_{\text{int}} + S_b = \frac{i m}{4\pi} \left[ \sum_\alpha \int (\mathcal{A}_{\text{eff}}^\alpha)^{\partial} d(\mathcal{A}_{\text{eff}}^{\alpha})^{\partial} - \beta \oint dx \mathcal{A}^{\dagger}_x \mathcal{A}_{\text{eff}}^{\alpha} + \Omega \right] + m S_{\text{top}}[Q] + \frac{m e_0}{4\pi} S_F[Q] \]
\[ - \frac{m e_0}{4\pi} \sum_\alpha \int d k_x | \vec{E}^\text{eff} |^2 \left| \frac{1}{2 \pi} \text{tr} I_{-n}^\alpha Q - \frac{\beta}{\pi} (\mathcal{A}_{\text{eff}}^{\alpha})_{n} \right|^2 \]
where, as in the bulk [I], the gauge field couples to \( Q \) only via the gauge non-invariant “Coulomb” term in (4.12). However, compared to the bulk case where the coupling results in the gauge invariant combination (\( \text{tr} I Q - \frac{\beta}{\pi} A_0 \)), the situation is more subtle in the edge case. The expression (\( \text{tr} I Q - \frac{\beta}{\pi} A_{\text{eff}}^c \)) appearing in (4.21) is, in fact, gauge variant, but this gauge variance is exactly what one needs to compensate for the edge contributions resulting from gauge transformations of the “boson” action \( S_b \) and the topological term. Therefore, the complete action \( (4.21 + S_{\text{flux}}) \) is fully gauge invariant.

We now proceed as in section [V.A] and integrate out the \( Q \) field. This is done in the same way as for the noninteracting case; either by doing it directly or by choosing \( \Omega \) such that \( Q \) decouples from \( \mathcal{A}_\mu \), i.e.
\[ \partial_c \Omega = \mathcal{A}^{\text{eff}}_c. \]
The only difference lies in the fact that we now work with effective quantities. The arguments about the ‘residual’ part of the electric field can again be applied, but now for the effective quantities (4.14, 4.19, 4.20). We then get the effective action for the external field \( \mathcal{A}_\mu \) in the presence of Coulomb interactions,
\[ S[\mathcal{A}] = \frac{i m}{4\pi} \left[ \sum_\alpha \int (\mathcal{A}_{\text{eff}}^\alpha)^{\partial} d(\mathcal{A}_{\text{eff}}^{\alpha})^{\partial} + \beta \oint dx (\partial_c^{-1} \mathcal{A}_{\text{eff}}^{\alpha} + \Omega_{\text{res}}^{\alpha})^{\dagger} \vec{E}_{\text{eff}}^{\alpha} \right] \]
\[ + \frac{\beta (m^2 \pi^2)}{4} \int d^2 x d^2 x' B^{\dagger}(\vec{x}) U(\vec{x} - \vec{x}') B(\vec{x}'). \]
Again, the difference with the free particle case is the appearance of a flux-flux term and various replacements by effective quantities.
2. Edge currents and Laughlin’s gauge argument

The action \( (4.23) \) contains complete information on the response of the system to external electromagnetic fields. We define the current as \( j^\mu(\vec{x}) = \delta S / \delta A_\mu(\vec{x}) \). In this way we find

\[
\begin{align*}
    j^0(\vec{x}) &= -\frac{i m}{2\pi} \left[ B - \delta(y) \partial_c^{-1} E_x \right] \\
    j^1(\vec{x}) &= -\frac{i m}{2\pi} \left[ E_y - \partial_y \int d^2x' \ U(\vec{x} - \vec{x}') j^0(\vec{x}') \right] - \frac{m n}{2\pi} \delta(y) \partial_c^{-1} E_x^\text{eff} \\
    j^2(\vec{x}) &= \frac{i m}{2\pi} \left[ E_x - \partial_x \int d^2x' \ U(\vec{x} - \vec{x}') j^0(\vec{x}') \right].
\end{align*}
\]

(4.24)

(4.25)

(4.26)

It is easily verified that \( \partial_\mu j^\mu = 0 \). The edge currents are obtained by taking only those terms that possess a delta function. On the edge we get

\[
\begin{align*}
    j^0_{\text{edge}} &= -\frac{i m}{2\pi} \partial_c^{-1} E_x^\text{eff} \\
    j^1_{\text{edge}} &= -i v_d \cdot j^0_{\text{edge}}.
\end{align*}
\]

(4.27)

(4.28)

This yields for the edge anomaly

\[
\partial_\mu j^\mu_{\text{edge}}(x) = -\frac{i m}{2\pi} \left[ E_x - \partial_x \int d^2x' \ U(x, \vec{x}') j^0(\vec{x}') \right].
\]

(4.29)

By applying Laughlin’s gauge argument \( \text{[10]} \) one can now directly relate the conductances defined by the bulk and by the edge. For example, let us do a linear response calculation for the case where \( N \) flux quanta \( h/e \) are created somewhere inside a hole in the sample. The charge \( q \) flowing from one edge into the other is found using \( (4.29) \),

\[
\frac{dq}{d\tau} = -i \oint \partial_\mu j^\mu_{\text{edge}} = \frac{m n}{2\pi} d\Phi / d\tau
\]

(4.30)

where \( \Phi \) is the total flux \( N \cdot h/e \) enclosed by the contour integral. This yields \( q = m \cdot N \), as it should.

3. Interacting chiral bosons

As was the case in the free electron situation, we can write the theory \( (4.23) \) as an edge boson coupled to the external field, exactly of the form \( (4.8) \), but now with effective quantities and an extra flux-flux term,

\[
S[A, \phi] = \frac{i}{\pi} \sum_{i=1}^{m} \left[ \sum_{\alpha} \int (A^\text{eff})^{\alpha} \wedge d(A^\text{eff})^{\alpha} - \beta \int dx \left( D_x \phi_i D^\text{eff}_c \phi_i - \phi_i^\dagger E_x^\text{eff} \right) \right] \\
+ \frac{\beta m}{2\pi} \int d^2x d^2x' \ B^\dagger(\vec{x}) U(\vec{x} - \vec{x}') B(\vec{x}').
\]

(4.31)

As in the noninteracting case, this result is equivalent to a Chern-Simons bulk theory of the form \( (4.9) \). In this case the action for the electron currents is given by
\[ S[A, g_i] = \frac{i}{4\pi} \sum_{\alpha} \sum_{i=1}^{m} \left[ - \int g_i^\alpha \wedge dg_i^\alpha + 2 \int g_i^\alpha \wedge d(A^{\text{eff}})^\alpha \right] \]
\[ + \frac{\beta}{2} \left( \frac{m}{2\pi} \right)^2 \int d^2x d^2x' \, B^\dagger(x)U(x-x')B(x') \]  
\[ + \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_{\alpha} \int A^\alpha \wedge dA^\alpha - \beta \int dx \left( D_{x_i}^\dagger \varphi_i^\dagger D_{-x}^\alpha \varphi_i - E_{x_i}^\dagger \varphi_i \right) \right]. \]  

(4.32)

with the gauge fixing conditions \( (g_i)_c|_{\text{edge}} = 0 \). In appendix B we show the equivalence.

It is very instructive to write (4.31) also in the following way

\[ S = - \frac{\beta}{2} \left( \frac{1}{2\pi} \right)^2 \sum_{i, j=1}^{m} \int d^2x d^2x' \, \left( \nabla \times \left( \theta(y) \tilde{D}_{\varphi_i}(x) \right)^\dagger \nabla' \times \left( \theta(y') \tilde{D}_{\varphi_j}(x') \right) \right) \]
\[ + \frac{i}{4\pi} \sum_{i=1}^{m} \left[ \sum_{\alpha} \int A^\alpha \wedge dA^\alpha - \beta \int dx \left( D_{x_i}^\dagger \varphi_i^\dagger D_{-x}^\alpha \varphi_i - E_{x_i}^\dagger \varphi_i \right) \right]. \]  

(4.33)

Notice that there are no effective quantities in this expression; the Coulomb interaction is completely contained in the first term. The charge density is given by \( \frac{m}{2\pi} \left[ B + \delta(y)m^{-1} \sum_i D_{x_i} \varphi_i \right] \). Notice also that we have written a two-dimensional integral containing \( \varphi_i \), even though the boson fields only exist on the edge. This is not a problem, since the \( \varphi_i \) only get evaluated at the edge.

C. Tunneling density of states

In [1] we expressed the one particle Green’s function which enters the tunneling density of states in terms of the matrix \( Q \) variable as follows

\[ \langle Q^{\alpha\alpha}(\tau_2, \tau_1, \vec{x}_0) \rangle := \sum_{n=-\infty}^{\infty} e^{i\nu_0(\tau_2-\tau_1)} \langle Q^{\alpha\alpha}_{nn}(\vec{x}_0) \rangle. \]  

(4.34)

The gauge transformation that in (4.21) decouples \( Q \) from \( A \) introduces into the path integral over (4.23) an extra factor

\[ \exp -i \left( \left[ A^{\text{eff}}_c / \partial_c \right]^\alpha(\tau_2, \vec{x}_0) - \left[ A^{\text{eff}}_c / \partial_c \right]^\alpha(\tau_1, \vec{x}_0) \right). \]  

(4.35)

When decoupling the quadratic edge term in \( A \) (4.23) with the use of boson fields, this factor translates to

\[ \exp -i \int_{\tau_1}^{\tau_2} d\tau \partial_\tau \varphi_j^\alpha(\tau, \vec{x}_0), \quad j = 1, \ldots, m \]  

(4.36)

in (4.31). The decoupling is not a unique procedure, since combinations of the boson fields \( \varphi_i \) can be chosen other than (4.36). However, the above form is the only one that yields the fermionic exponent for the expectation value \( \langle Q \rangle \).

\[ \left\langle \exp -i \int_{\tau_1}^{\tau_2} d\tau \partial_\tau \varphi_i^\alpha(\tau, \vec{x}_0) \right\rangle \propto (\tau_2 - \tau_1)^{-S} \quad S = 1. \]  

(4.37)

(See appendix A for the explicit calculation.) Notice that we would have had a serious problem at this point if we had not excluded the zero-momentum components of the \( \varphi_i^\alpha \) when we introduced these auxiliary fields. A redefinition of the integration measure, \( \int \mathcal{D}[\varphi + f] \), with \( \partial_x f(x, \tau) = 0 \), would yield a result depending on the arbitrary function \( f \).
V. LONG-RANGE DISORDER

A. Separation of edge channels

Long-range disorder can cause the edge states of different Landau levels to become spatially separated. A potential fluctuation at the edge can lift all states in such a way that new ‘edge’ states are created. (See Fig. 6) If the chemical potential lies between the shifted and unshifted energy of a Landau level, the edge states of this Landau level will be situated inside the sample, not on the outermost edge. If there are several potential jumps of this kind, all the edge channels can become separated. They can also start wandering into the interior of the sample.

We propose that ‘edge channel separation’ is the dominant effect of smooth potential fluctuations as opposed to ‘inter-channel scattering’ which only occurs when the potential changes abruptly. In this section we wish to embark on the problem of smooth potential fluctuations in the presence of the Coulomb interactions.

In order to fix the thought we imagine a quantum Hall sample with filling fraction \( \nu = 2 - \varepsilon \). Fig 7a illustrates the equipotential contours. We may distinguish between the localised (closed) orbitals in the bulk of the sample and the extended (chiral) edge states.

This picture leads us to the idea of describing the chiral bosons by one field \( \varphi(\vec{x}) \) that lives on all the ‘edges’ instead of independent fields for every edge. The action (4.31) then becomes

\[
S = \frac{i}{4\pi} \left[ \sum_{\alpha} \int n A^\alpha \wedge d A^\alpha - \beta \sum_{a=1}^{M} s_a \oint_{C_a} dx \left( D_x \varphi^\dagger [D_0 \varphi - is_a v_a D_x \varphi] - E_x^\dagger \varphi \right) \right]
- \frac{\beta^2}{(2\pi)^2} \int d^2 x d^2 x' U(\vec{x} - \vec{x}') \nabla' \times [n(\vec{x}) \vec{D} \varphi(\vec{x})] \nabla' \times [n(\vec{x}') \vec{D} \varphi(\vec{x}')] .
\]  

(5.1)
The \( n \) is a function of position labeling the ‘local’ filling fraction: outside the sample \( n(\vec{x}) \) is zero; going inward, it increases by one every time you cross an edge, until it reaches its bulk value \( m \). At the bulk orbitals, \( n(\vec{x}) \) jumps again. (In the case \( \nu = 2 - \varepsilon \), depicted in fig 7a, \( n(\vec{x}) = 1 \) inside the closed orbitals.)

Each edge is described by a contour labeled \( C_a \), with \( a = 1, \cdots, m \) for the edge states and \( a = m+1, \cdots, M \) for the closed bulk orbitals. The coordinate ‘\( x \)’, appearing in the edge terms, is defined on the contour and is taken in the positive (anticlockwise) direction. The symbol \( s_a \),

\[
 s = (+1, \cdots, +1, -1, \cdots, -1) 
\]  

incorporates the fact that the contours with \( a \leq m \) and \( a > m \) carry opposite current and charge densities. For simplicity we take the drift velocity \( v_d \) the same for all edges.

Integrating the boson field out yields the generalisation of (4.23),

\[
 S[A] = \frac{i}{4\pi} \left[ \sum_{a} \int n(A^{\text{eff}})^{\alpha} \wedge d(A^{\text{eff}})^{\alpha} + \beta \sum_{a} s_a \oint_{C_a} dx \left( \frac{\partial_x A^{\text{eff}}}{\partial_c} - A_x \right)^{\dagger} A^{\text{eff}} \right] 
\]  

\[
 + \frac{\beta}{2} \left( \frac{1}{2\pi} \right)^2 \int d^2x \int d^2x' \ n(\vec{x}) B^{\dagger}(\vec{x}) U(\vec{x} - \vec{x}') n(\vec{x}') B(\vec{x}). 
\]

The notation \( \partial_c \) (at contour \( C_a \)) now has the sign \( s_a \) in front of the velocity and contains Coulomb interactions with all contours instead of just \( C_a \) itself. The definition of the ‘effective’ potential \( A^{\text{eff}}_0 \) has also slightly changed,

\[
 A^{\text{eff}}_0(\vec{x}) = A_0(\vec{x}) + \frac{i}{2\pi} \int d^2x' \ U(\vec{x} - \vec{x}') n(\vec{x}') B(\vec{x}). 
\]

For completeness, in appendix D we also present the generalisation of the action \( S[Q, A] \) (4.11) for the case of separated edge channels.

Note that we are addressing the situation where the chemical potential is away from the narrow ‘percolation’ regime indicated by \( W_0 \) in Fig 4. We will next exploit the simplicity of our model and demonstrate that the Hall conductance and the tunneling density of edge states are fundamentally different quantities that correspond to completely different physical processes.

FIG. 7. (a) Plot of equipotential contours corresponding to filling fraction \( \nu = 2 - \varepsilon \). (b) Effective edge theory for filling fraction \( \nu = 2 - \varepsilon \). The dashed line represents the (anti)chiral contribution from the bulk orbitals.
B. Hall conductance

First, it is straightforward to generalise the results of section IV B 2 to include the separated edge channels and the bulk states into Laughlin’s flux argument. Differentiating the action (5.3) with respect to $A_\mu$, we obtain the generalised form of the currents (4.24 - 4.26),

$$j^0(\vec{x}) = \frac{i}{2\pi} \left[ n(\vec{x})B - \sum_{a=1}^{M} s_a \delta(\vec{x} \text{ on } C_a) \frac{E_{\text{eff}}}{\delta c} \right]$$  \hspace{1cm} (5.5)

$$j^i(\vec{x}) = -i \frac{n(\vec{x})}{2\pi} \varepsilon^{ij} \left[ E_j - \partial_j \int d^2x' \ U(\vec{x} - \vec{x}')j^0(\vec{x}') \right] - \frac{v_s}{2\pi} \sum_{a=1}^{M} \delta(\vec{x} \text{ on } C_a) E_{\text{eff}}(\vec{e}_a)_{i}$$  \hspace{1cm} (5.6)

where the vector $\vec{e}_a$ is tangent to the contour $C_a$ and points in the positive direction. Again it is easy to check that $\partial_\mu j^{\mu} = 0$, i.e. that current conservation is respected. The edge currents are given by

$$j^0_{\text{edge}}(C_a) = -\frac{i}{2\pi} s_a \frac{E_{\text{eff}}}{\delta c}$$  \hspace{1cm} (5.7)

$$j^x_{\text{edge}}(C_a) = -i s_a v_d \cdot j^0_{\text{edge}}(C_a).$$  \hspace{1cm} (5.8)

The edge anomaly applies to each bulk orbital and edge state separately,

$$\partial_\mu j^\mu_{\text{edge}}(C_a) = -\frac{i}{2\pi} s_a \left[ E_x - \partial_x \int d^2x' \ U(x, \vec{x}')j^0(\vec{x}') \right].$$  \hspace{1cm} (5.9)

As expected, the sign $s_a$ determines whether charge is transported into an edge or from an edge into the bulk. By repeating Laughlin’s flux argument it is now demonstrated explicitly that the localised bulk orbitals do not affect the transport of charge from one sample edge to the other, independent of the electron-electron interactions; taking (5.9) and performing the contour integral over $C_a$ we obtain the charge transported per unit of time from the $a$’th channel,

$$dQ_a/d\tau = -i \oint_{C_a} \partial_\mu j^\mu_{\text{edge}} = s_a \frac{1}{2\pi} d\Phi_a/d\tau$$  \hspace{1cm} (5.10)

where $\Phi_a$ is the magnetic flux enclosed by $C_a$. For $a > m$ this flux is obviously zero, since the localised bulk orbitals do not encircle the hole in the sample. This, then, shows that the Hall conductance is quantised (equal to $m$) independent of $\varepsilon$.

C. Tunneling density of states

Laughlin’s flux argument for the Hall conductance expresses the quantum Hall state as an exact ‘excited’ state of the system. Tunneling processes into the edge, on the other hand, are expressed in terms of eigenstates near the Fermi energy, i.e. the tunneling density of states, and due to the Coulomb interactions this quantity is sensitive to the presence of bulk orbitals.

We start from the action (5.4), omitting the replica indices for notational simplicity and putting $A_\mu = 0$,
\[ S = -\frac{i}{4\pi} \int d\tau \sum_{j=1}^{M} s_j \oint_{C_j} \partial_x \varphi (\partial_0 \varphi - i s_j v d \partial_x \varphi) \]
\[ - \frac{1}{8\pi^2} \int d\tau \sum_{j,j'=1}^{M} s_j s_{j'} \oint_{C_j} \oint_{C_{j'}} \partial_x \varphi U(x,x') \partial_x' \varphi. \]  
(5.11)

Following section IV C, (4.36), the one particle Green’s function can be written as follows
\[ G(\tau_2 - \tau_1) = \langle \exp -i[\varphi(\tau_2,x_0) - \varphi(\tau_1,x_0)] \rangle \]  
(5.12)

where \(x_0\) denotes a point on the edge contour \(C_1\).

The presence of the Coulomb interactions makes the computation of \(G\) a complicated two dimensional problem. Some procedure needs to be found which extracts the lowest energy excitations from (5.11). We follow the strategy of taking the boson fields as a two dimensional field variable and we then collect the terms with smallest momenta. This procedure is done in position space and we proceed by giving the details of a step by step analysis. The results for the tunneling exponents are given in the next Section (D) which also contains a brief summary in the end.

1. Gradient expansion

The interaction term in (5.11) can be written as a sum over area integrals,
\[ - \frac{1}{8\pi^2} \int d\tau \sum_{j,j'=1}^{M} \int_{C_j} \int_{C_{j'}} d^2 x d^2 x' s_j s_{j'} \partial_a \varphi(\vec{x}) U_{ab}(\vec{x} - \vec{x}') \partial_b \varphi(\vec{x}'). \]  
(5.13)

with
\[ U_{ab}(\vec{x} - \vec{x}') := \varepsilon_{ac} \varepsilon_{bd} \frac{\partial}{\partial x_c} \frac{\partial}{\partial x'_d} U(\vec{x} - \vec{x}'). \]  
(5.14)

Since we are only interested in the \(\varphi\) with the smallest momenta, we can make the replacement
\[ \sum_{j=m+1}^{M} \int_{C_j} d^2 x \rightarrow \Omega_f \int_{C_b} d^2 x. \]  
(5.15)

The \(\Omega_f\) stands for the fraction of the total area that is enclosed by all the bulk orbitals together. The contour \(C_b\) is not sharply defined and is located somewhere close to the edge (see Fig 7b). It encloses the region within which the bulk orbitals are contained. The joint Coulomb effects of the bulk orbitals will effectively be comprised on this contour. For the terms in (5.11) containing \(\partial_x \varphi \partial_0 \varphi\) we can write
\[ \sum_{j>m} \oint_{C_j} \partial_x \varphi \partial_0 \varphi = \sum_{j>m} \int_{C_j} d^2 x \nabla \times (\nabla \varphi \partial_0 \varphi) \rightarrow \Omega_f \int_{C_b} d^2 x \nabla \times (\nabla \varphi \partial_0 \varphi) = \Omega_f \oint_{C_b} \partial_x \varphi \partial_0 \varphi. \]  
(5.16)
The expression \[ \sum_{j>m} \oint_{C_j} \left( \partial_x \varphi \right)^2 \] averages out to \( \kappa \int_{C_b} d^2 x \left( \nabla \varphi \right)^2 \) with \( \kappa \) some positive constant related to the total length of all the bulk contours. If there are substantial stretches where a bulk orbital runs along the edge, interaction terms will arise, leading to a term \( \oint_{C_b} \left( \partial_x \varphi \right)^2 \).

Note that in doing the replacement (5.15) in (5.13), one also needs to introduce correction terms that compensate for the errors made when the separation \( |\vec{x} - \vec{x}'| \) is ‘small’ (of the order of the average size of the orbitals or less) and \( U_{ab} \) does not vary slowly. These corrections are of the form \( \int d^2 x \left( \nabla \varphi \right)^2 \).

Hence we define \( \Omega_f \) with \( \varepsilon \), so the fraction of the area occupied by bulk states is exactly the deviation from integer filling. We have written \( V_b(x, x') \) for the short-ranged interaction between two points on \( C_b \); The \( V_j(x, x') \) denotes the short-ranged interaction between a point \( x \) on \( C_j \) and \( x' \) on \( C_b \). The precise expression for \( V \) is unknown due to the fact that it has its origin in the twilight zone near the edge, where it is unclear whether a term contributes to the bulk or edge action.

Comparing this result (5.17) with (5.11), we see that the presence of the interacting bulk states effectively leads to the appearance of an additional (anti)chiral boson on the contour \( C_b \), an extra short-ranged interaction with this contour, and a lower dimensional left over bulk term \( f(\nabla \varphi)^2 \).

2. Effect of the bulk term

In order to be able to calculate the tunneling density of states (5.12) we need an effective theory for the edge degrees of freedom, and therefore we have to understand how they are affected by the left over bulk term. To this end, we are going to split bulk and edge degrees of freedom. We write the bulk term as \( \int_{C_b} (\nabla \Phi)^2 \), where \( \Phi \) represents the bulk degrees of freedom and is treated as an integration variable independent of \( \varphi \). To reflect the fact that it is actually an extension of \( \varphi \) into the bulk, we impose some boundary condition on \( \Phi \), for instance \( \Phi|_{\text{edge}} = \varphi \) or \( \partial_\perp \Phi|_{\text{edge}} = \partial_\perp \varphi \). (\( \partial_\perp \) is the derivative perpendicular to the contour.)

The effect of the bulk term on the edge theory is obtained by integrating out \( \Phi \), which leads to an effective action for the boundary conditions. Let us consider a general scenario and
impose the boundary conditions \( \Phi|_{\text{edge}} = \psi_0 \) and \( \partial_\perp \Phi|_{\text{edge}} = \psi_1 \), using constraint multipliers \( k_0 \) and \( k_1 \), respectively.

\[
e^{S_{\text{eff}}[\psi_0(x),\psi_1(x)]} = \int \mathcal{D}[\Phi(\vec{x})] \mathcal{D}[k_0(x)] \mathcal{D}[k_1(x)] \times \]
\[
\times \exp \left\{ i \oint k_0(\Phi - \psi_0) + i \oint k_1(\partial_\perp \Phi - \psi_1) - g \oint d^2x \ (\nabla \Phi)^2 \right\} .
\]

(5.18)

For notational simplicity we have omitted time dependence and the subscript \( C_b \) under all the integrals.

We first wish to integrate (5.18) over \( \Phi(\vec{x}) \) keeping \( k_0 \) and \( k_1 \) fixed. For this purpose we split \( \Phi \), which has free boundary values, into a bulk and an edge part by writing

\[
\Phi = \Phi_L + \hat{\Phi} \quad \partial_\perp \Phi|_{\text{edge}} = \partial_\perp \hat{\Phi}|_{\text{edge}} = 0 \quad (5.19)
\]

where \( \Phi_L \) satisfies Laplace’s equation

\[
\nabla^2 \Phi_L(\vec{x}) = 0 \quad (5.20)
\]

The \( \Phi_L(\vec{x}) \) is completely determined by \( \partial_\perp \Phi_L \) on the edge, which we now take as an independent edge degree of freedom denoted by \( E_1(x) \). Introducing the 2D Green’s function \( G \),

\[
G(\vec{x},\vec{x'}) = \frac{1}{2\pi} \ln |\vec{x} - \vec{x'}| \quad ; \quad \nabla^2 G(\vec{x},\vec{x'}) = \delta(\vec{x} - \vec{x'}),
\]

(5.21)

and using Green’s theorem, we solve Laplace’s equation and obtain for \( \Phi_L(\vec{x}) \)

\[
\Phi_L(\vec{x}) = -\oint dx' \left[ G(\vec{x},x')E_1(x') - \Phi_L(x') \frac{\partial G}{\partial y'}(x,y;x',0) \right] .
\]

(5.22)

This expression tells us that we need to now \( \Phi_L \) on the edge in order to evaluate \( \Phi_L \) in the bulk. Luckily, we do not need the full 2D \( \vec{x} \) dependence, since due to the splitting (5.19) \( \Phi_L \) will get evaluated at the edge only. Using a special property of the Green’s function (5.21), namely \( [\partial_y G](x,0;x',0) = 0 \), we can explicitly write \( \Phi_L \) on the edge as a function of \( E_1 \),

\[
\Phi_L(x) = -\oint dx' \ G(x,x')E_1(x').
\]

(5.23)

The action, written in terms of \( \hat{\Phi} \) and \( E_1 \), is now given by

\[
S = -g \oint (\nabla \hat{\Phi})^2 - g \oint E_1 GE_1 + \oint E_1 (2g \hat{\Phi} - igk_0)
\]
\[
+ i \oint k_0(\hat{\Phi} - \psi_0) + i \oint k_1(E_1 - \psi_1)
\]

(5.24)

where we have used the shorthand notation \( \oint AGB \) for the expression \( \oint dx \oint dx' A(x)G(x,x')B(x') \).

Integrating out \( \hat{\Phi} \) is now simply done by replacing \( \hat{\Phi} \) by its saddle point value. Varying the action with respect to \( \hat{\Phi} \), keeping \( E_1 \) fixed, we get the saddlepoint equation

\[
\nabla^2 \hat{\Phi} + \delta(y)[E_1 + \frac{i}{2g}k_0] = 0.
\]

(5.25)
Using the Green’s function’s property \[ \partial_y' G(x, 0; x', 0) = 0 \] again, we find the following solution on the edge

\[ \hat{\Phi}(x) = - \int dx' \ G(x, x')[E_1 + \frac{i}{2g} k_0] (x'). \] (5.26)

In substituting this solution into (5.24) we do not need the full 2D \( \vec{x} \)-dependence of \( \hat{\Phi}(\vec{x}) \), since we can write \( f(\nabla \hat{\Phi})^2 = - f \Phi \nabla^2 \Phi \) and \( \nabla^2 \Phi \) is an expression restricted to the edge. Substitution of (5.26) into (5.24) yields

\[ S = -2g \int E_1 G E_1 + \frac{1}{4g} \int k_0 G k_0 - i \int k_0 (\psi_0 + 2G E_1) + i \int k_1 (E_1 - \psi_1). \] (5.27)

Integrating out \( k_0 \) is straightforward and gives

\[ S = g \int (\psi_0 G^{-1} \psi_0 + 4 \psi_0 E_1 + 2E_1 G E_1) + i \int k_1 (E_1 - \psi_1). \] (5.28)

In the end we integrate out \( k_1 \), yielding the constraint \( E_1 = \psi_1 \). The final result for \( S_{\text{eff}}[\psi_0, \psi_1] \) becomes

\[ S_{\text{eff}}[\psi_0, \psi_1] = g \int (\psi_0 G^{-1} \psi_0 + 4 \psi_0 \psi_1 + 2 \psi_1 G \psi_1) \]
\[ = g \int (\psi_0, \psi_1) \begin{pmatrix} G^{-1} & 2 \\ 2 & 2G \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}. \] (5.29)

We are going to put \( \psi_0 = 0 \) in order to avoid double counting of \( (\partial_x \phi)^2 \) terms at the edge, and \( \psi_1 = \partial_\perp \phi \). The action (5.29) becomes

\[ S[\partial_\perp \phi] = 2g \int \partial_\perp \phi \ G \ \partial_\perp \phi. \] (5.30)

This edge term, derived from the interaction with the bulk orbitals, is seriously going to affect the tunneling exponent. A quick way to see this is as follows: on the contours \( C_1, \cdots, C_b \), the field \( \phi(x, y \text{ on } C_1) \) perpendicular derivatives. For the tunneling exponent, only \( \phi|_{C_1} \) is needed, so we can integrate out the perpendicular derivatives in (5.17 minus bulk term+5.30) to obtain an effective action for \( \phi \) on \( C_1 \). The dominant part of the 1D propagator for \( \partial_\perp \phi \) is given by \( G^{-1}(k) \propto |k| \), from which it follows that all terms introduced by the integration over \( \partial_\perp \phi \) are irrelevant. Higher powers of \( \partial_\perp \phi \) are even less relevant. Replacing all the \( \phi \) in (5.17) by \( \phi|_{C_1} \), we get a term \( \nu \int \partial_y \phi \partial_0 \phi \), leading to a tunneling exponent \( S = 1/\nu \) instead of the free particle result \( S = 1 \).

In the next section we are going to derive this result more formally, based on a consideration of the neutral modes in the theory where the edge channels are not spatially separated.

### 3. Demise of the neutral modes

In the long wavelength limit, the contours \( C_1, \cdots, C_b \) are lying so close together that we can effectively return to the picture where all the edge channels are sitting on top of
each other. We label the channels \( \varphi_1(x), \cdots, \varphi_m(x), \varphi_b(x) \). Let us for simplicity’s sake first consider the case \( \nu = 1 - \varepsilon \), where we just have the two fields \( \varphi_1 \) and \( \varphi_b \). In terms of these fields, the action \( (5.14) \), without the bulk term and the bulk effect \( (5.30) \), takes the form (again using abbreviated notation)

\[
S_0[\varphi_1, \varphi_b] = -\frac{1}{4\pi} \oint \left[ \partial_x \varphi_1 \partial_0 \varphi_1 - \varepsilon \partial_x \varphi_b \partial_0 \varphi_b \right] - \frac{1}{8\pi^2} \oint U[\partial_x \varphi_1 - \varepsilon \partial_x \varphi_b]^2 - \frac{1}{8\pi^2} \sum_{k,l=1,b} \oint V_{kl} \partial_x \varphi_k \partial_x \varphi_l.
\]

We have put all the short-range contributions into the \((m+1) \times (m+1)\) velocity matrix \( V \).

We next define a ‘charged mode’ \( \Gamma \) and a ‘neutral mode’ \( \gamma \) in such a way that only the charged mode feels the long-range part of the interaction,

\[
\Gamma = \frac{1}{\nu} (\varphi_1 - \varepsilon \varphi_b) \quad ; \quad \gamma = \varphi_1 - \varphi_b \\
\varphi_1 = \Gamma - \frac{\varepsilon}{\nu} \gamma \quad ; \quad \varphi_b = \Gamma - \frac{1}{\nu} \gamma.
\]

In the basis \((\Gamma, \gamma)\) the action \((5.31)\) becomes

\[
S_0[\Gamma, \gamma] = -\frac{1}{4\pi} \oint \left[ \nu \partial_x \Gamma \partial_0 \Gamma - \varepsilon \partial_x \gamma \partial_0 \gamma \right] - \frac{\nu}{8\pi^2} \oint \partial_x \Gamma \left( U + \nu \frac{\varepsilon}{\nu} \hat{V} \right) \partial_x \Gamma
\]

\[
-\frac{1}{8\pi^2} \oint \partial_x \Gamma \partial_x \gamma \hat{V} \left[ \frac{\partial_x \Gamma}{\partial_x \gamma} \right] \tag{5.33}
\]

where \( \hat{V} \) is the velocity matrix in the new basis.

The expression \( \partial_{\bot} \varphi \) in the theory for spatially separated channels is in the single-edge picture evidently equivalent to the neutral mode \( \gamma \propto \varphi_b - \varphi_1 \). The leftover bulk contribution \((5.30)\) therefore translates into an extra term involving the neutral mode,

\[
S_{\text{bulk}}[\gamma] = \text{constant} \cdot \oint \gamma \, G \, \gamma.
\]

The tunneling density of states is now expressed as

\[
\left\langle \exp -i \varphi_1 |_{\tau_1}^{\tau_2} \right\rangle \propto \int \mathcal{D}[\Gamma] \mathcal{D}[\gamma] \exp \left( -i (\Gamma - \frac{\varepsilon}{\nu} \gamma) |_{\tau_1}^{\tau_2} + S_0[\Gamma, \gamma] + S_{\text{bulk}}[\gamma] \right). \tag{5.35}
\]

If we perform the integration over \( \gamma \) first, we see that the ‘bulk’ part of the action yields the following contribution to the inverse propagator: \( G(k) \propto 1/|k| \), which is dominant at low momenta. The integration over \( \gamma \) yields \( \Gamma-\Gamma \) terms of order \( k^5 \hat{V}(k) \). These are clearly irrelevant. For the tunneling density of states we can write

\[
\left\langle \exp -i \varphi_1 |_{\tau_1}^{\tau_2} \right\rangle \propto \int \mathcal{D}[\Gamma] \exp \left( -i \Gamma |_{\tau_1}^{\tau_2} + S_{\text{eff}}[\Gamma] \right)
\]

\[
S_{\text{eff}}[\Gamma] = -\frac{\mu}{4\pi} \oint \partial_x \Gamma \partial_0 \Gamma - \frac{\nu^2}{8\pi^2} \oint \partial_x \Gamma (U + \nu^{-2} \hat{V}) \partial_x \Gamma. \tag{5.36}
\]

For small momenta the \( \hat{V} \) essentially reduces to a constant and we can use the results of appendix A, obtaining

\[
\left\langle \exp -i \varphi_1 |_{\tau_1}^{\tau_2} \right\rangle \propto (\tau_2 - \tau_1)^{-S} \quad ; \quad S = 1/\nu. \tag{5.37}
\]
4. The general case \( \nu = m - \varepsilon \)

The results for \( \nu = 1 - \varepsilon \) are easily generalised. From the ‘bulk’ channel \( \varphi_b \) and the edge channels \( \varphi_1, \ldots, \varphi_m \) we construct a charged mode \( \gamma_0 \) and \( m \) neutral modes \( \gamma_1, \ldots, \gamma_m \) as follows,

\[
\begin{align*}
\gamma_0 &= \frac{1}{\nu} \left( \sum_{k=1}^{m} \varphi_k - \varepsilon \varphi_b \right) \\
\gamma_a &= \frac{1}{a} \left( \sum_{k=1}^{a} \varphi_k - a \varphi_{a+1} \right) \quad a = 1, \ldots, m
\end{align*}
\]  
(5.38)

where we define \( \varphi_{m+1} \) as \( \varphi_b \). The neutral modes \( \gamma_1, \ldots, \gamma_{m-1} \) are the usual ones for a theory with \( m \) edges. They are mutually perpendicular and normal to the charged mode. The \( \varphi \)'s are expressed in terms of the \( \gamma \)'s as follows

\[
\begin{align*}
\varphi_b &= \gamma_0 - \frac{m}{\nu} \gamma_m \\
\varphi_k &= \gamma_0 - \frac{\varepsilon}{\nu} \gamma_m - (1 - \frac{1}{k}) \gamma_{k-1} + \sum_{a=k+1}^{m-1} \frac{1}{a} \gamma_a \quad k \leq m.
\end{align*}
\]  
(5.39)

Equation (5.31) is generalised to

\[
\begin{align*}
S[\varphi] &= -\frac{1}{4\pi} \oint \left[ \sum_{j=1}^{m} \partial_x \varphi_j \partial_0 \varphi_j - \varepsilon \partial_x \varphi_b \partial_0 \varphi_b \right] \\
&\quad - \frac{1}{8\pi^2} \oint U \left[ \sum_{j=1}^{m} \partial_x \varphi_j - \varepsilon \partial_x \varphi_b \right]^2 - \frac{1}{8\pi^2} \sum_{k,l=1}^{m+1} V_{kl} \partial_x \varphi_k \partial_x \varphi_l.
\end{align*}
\]  
(5.40)

Again, all the short-range contributions have been put into a velocity matrix \( V \), which now has dimension \((m+1) \times (m+1)\).

Writing (5.40) in terms of the \( \gamma \)-basis, we get

\[
\begin{align*}
S[\gamma] &= -\frac{1}{4\pi} \oint \left[ \nu \partial_x \gamma_0 \partial_0 \gamma_0 + \sum_{a=1}^{a+1} \partial_x \gamma_a \partial_0 \gamma_a - m \frac{\varepsilon}{\nu} \partial_x \gamma_m \partial_0 \gamma_m \right] \\
&\quad - \frac{\nu^2}{8\pi^2} \oint (\partial_x \gamma_0)^2 - \frac{1}{8\pi^2} \sum_{a,c=0}^{m} \oint \hat{V}_{ac} \partial_x \gamma_a \partial_x \gamma_c
\end{align*}
\]  
(5.41)

where \( \hat{V} \) is the velocity matrix in the basis of \( \gamma \)'s. The argument of [5.34 to 5.37] can be applied again, in a slightly modified form; the neutral modes are equivalent to \( \partial_n \varphi \) and higher derivatives. (A basis \( \hat{\gamma} \) can be found for the neutral modes in which \( \hat{\gamma}_n \) corresponds to the 1D lattice discretisation of \( \partial_n^1 \varphi \).) On dimensional grounds the propagator for the \( n \)'th normal derivative of \( \varphi \) has to be proportional to \( k^{2n-1} \), leading to irrelevant contributions. A more concrete way of making this statement would be to generalise the analysis presented in (5.18 to 5.29), including boundary conditions for the higher normal derivatives. However, that would also require us to take into account higher order terms in the \( \varphi \)-theory (5.17).
The resulting effective action for the charged mode $\gamma_0$ is of the form (5.36), with $\nu = m - \epsilon$.

We can summarise the results of section V as follows: We have seen that the Fermi liquid result $S = 1$ is obtained for the tunneling density of states (I) when the Coulomb interactions are omitted, or (II) when interactions are included but only short length scales are considered. An interacting theory for the lowest lying excitations, which are slowly varying field configurations, yields completely different results. The presence of bulk orbitals, interacting mutually and with the edge states, is effectively described by an extra edge channel with prefactor $-\epsilon$ plus a remnant of the interactions in the bulk of the form $\int (\nabla \varphi)^2$. The leftover bulk term serves to make all the neutral edge modes irrelevant, yielding an effective edge action for the one remaining, charged, mode. Due to the presence of the extra ‘bulk’ channel, the prefactor of this effective action $S[\Gamma]$ becomes $m - \epsilon = \nu$, which is a continuous parameter in sharp contrast to the integer quantised $m$.

D. Computation of $\tau_{in}$

We next return to the problem of the plateau transitions. Following section II B we expect that the transport at high temperatures is dominated by interactions between the conducting electrons on the backbone saddlepoint network and those on the disconnected pieces or clusters.

The fundamental quantity to compute is the characteristic time $\tau_{in}$ that is needed for the backbone electrons to equilibrate with the rest of the network. In order to set up a theory for relaxation, we consider the ‘nearly saddlepoints’ in the network, where tunneling is not possible but where the Coulomb forces nevertheless produce ‘sudden changes’ in the motion.
of the conducting electrons. Fig. 8 illustrates the interaction of the saddlepoint network with disconnected orbitals. The ‘nearly saddlepoints’ where the Coulomb forces are most effective are indicated by the shaded areas. We can model the situation by introducing a delta-function potential which acts in the small areas of the nearly saddlepoints only. The action can be written as

$$S_{\text{eff}}[\varphi] = S[\varphi_0] + \sum_i S[\varphi_i] - \sum_i \int d\tau \, \partial_x \varphi_0(\vec{a}_i) U_i \partial_x \varphi_i(\vec{a}_i)$$  \hspace{1cm} (5.42)

where $S[\varphi_0]$ is the action for the chiral boson field on a link of the saddlepoint network that we denote as the contour $C_0$,

$$S[\varphi_0] = \int d\tau \int dx \, \partial_x \varphi_0 \partial_- \varphi_0.$$  \hspace{1cm} (5.43)

This contour is taken to be very large or infinite. Similarly, we define chiral boson fields $\varphi_i$ on the disconnected but large contours $C_i$,

$$S[\varphi_i] = -\int d\tau \int dx \, \partial_x \varphi_i \partial_+ \varphi_i.$$  \hspace{1cm} (5.44)

The sum in the interaction term in (5.42) is over the discrete set of nearly saddlepoints $\vec{a}_i$ along the contour $C_0$ where the fields $\varphi_0$ and $\varphi_i$ interact with an appropriate, random strength $U_i$. This problem is in many ways quite similar to the problem of interacting edge channels with a randomly varying separation between them. We proceed along the same lines as in (18) and introduce a self energy $\Sigma$ for the density-density correlation of the field $\varphi_0$. If we denote the Fourier transforms of the propagators $\langle \partial_x \varphi_0(x,\tau) \partial_x \varphi_0(x',\tau') \rangle$ and $\langle \partial_x \varphi_i(x,\tau) \partial_x \varphi_i(x',\tau') \rangle$ (with $x, x'$ parametrising the positions on the contours $C_0, C_j$ respectively) as

$$D_0(\omega, q) = \frac{-iq}{i\omega - v_d q}; \quad D_j(\omega, q) = \frac{-iq}{i\omega + v_d q},$$  \hspace{1cm} (5.45)

then the introduction of a self energy takes the form

$$D_0(\omega, q) \rightarrow -i \frac{q}{i\omega - (v_d + \Sigma)q}.$$  \hspace{1cm} (5.46)

To lowest order in the interaction potential we may write

$$\Sigma(\omega) = \bar{z} U_j^2 \int dq \, D_j(\omega, q) = -\bar{z} U_j^2 |\omega|. \hspace{1cm} (5.47)$$

Here, the bar stands for the average over the random positions $\vec{a}_i$ along $C_0$ and $z$ is the linear density of saddlepoints. The result (5.47) can be used to obtain an expression for $1/\tau_n$, i.e. the imaginary part of the self energy as it appears in the electron Green’s function $G(\omega, q)$ as follows

$$1/\tau_n = \int d\omega dq \, \Sigma(\omega) G(\varepsilon - \omega, q). \hspace{1cm} (5.48)$$
The \( \tau_{\text{in}} \) determines the rate at which the electrons on the backbone cluster equilibrate with the rest of the electronic orbitals. We find \( \tau_{\text{in}}^{-1} \propto \varepsilon^2 \) or \( T^2 \) at finite temperatures. This admittedly crude approach toward electron relaxation can be improved in several ways. For example, as the most important correction to the self energy (5.47) we find the self-interacting orbitals as depicted in figure 8b. These corrections replace the momentum integral in (5.47) in the following way (in space-time notation)

\[
\int dq \ D_j(\omega, q) = \int d\tau \ e^{-i\omega(\tau - \tau')} D_j(0, 0; \tau - \tau') + \int d\tau_0 \int_0^L dx \int_x^L dy \ D_j(0, x; \tau - \tau_0) \tilde{U}_j \ D_j(y, L; \tau_0 - \tau') \tag{5.49}
\]

where \( x, y \) are the positions of the nearly saddlepoint where the self-interaction takes place. The integrals stand for the averaging over positions and all dimensional factors are absorbed into \( \tilde{U}_j \). The length of the orbital is given by \( L \) and boundary conditions \( x \equiv x + L \) and \( y \equiv y + L \) are understood. Equation (5.49) can be rewritten as a shift in the chemical potential,

\[
\int dq \ D_j(\omega, q) \rightarrow \int dq \ \frac{-iq}{i\omega - \delta\mu + \nu_d q} \quad \delta\mu = \tilde{U}_j. \tag{5.50}
\]

This leads to a modified self energy according to

\[
\Sigma(\omega) \rightarrow -\frac{i}{2\nu_d} \tilde{U}_j^2 (\omega + i\delta\mu) \text{sgn}(\omega). \tag{5.51}
\]

The shift \( \delta\mu \) can be translated into a shift in the expression for \( \tau_{\text{in}}^{-1} \) following

\[
\tau_{\text{in}}^{-1}(\varepsilon, \delta\mu) = \left(1 + i\delta\mu \frac{\partial}{\partial \varepsilon}\right) \tau_{\text{in}}^{-1}(\varepsilon). \tag{5.52}
\]

After the analytic continuation to real energies \( (i\varepsilon \rightarrow \varepsilon) \) has been performed, we obtain the final result \( \tau_{\text{in}}^{-1} \propto \varepsilon \) or \( \tau_{\text{in}}^{-1} \propto T \) at finite temperatures. More generally, we expect the equilibration rate to be given by a regular series expansion in powers of \( T \) which is dominated by the lowest order \( \tau_{\text{in}}^{-1} \propto T \) as \( T \) approaches absolute zero.

**VI. SUMMARY AND CONCLUSIONS**

We have shown that the massless edge excitations are a peculiar property of the instanton vacuum in the replica limit \( N_r = 0 \). This aspect of the problem has previously gone unnoticed. We have used the formalism of \( F \)-algebra introduced in our previous work and derived a complete theory of the edge. We have established the fundamental connection between the instanton vacuum and Chern-Simons gauge theory. Both theories have previously been studied independently and with different physical objectives. We have shown that our new approach to edge physics enables one to address several long standing problems of smooth disorder and interaction effects. We have pointed out that fundamental differences exist between tunneling at the edge and electron transport. Transport experiments inject electrons directly into edge states; these electrons do not get enough time to equilibrate with the rest of
the sample and are therefore effectively decoupled from the bulk. A tunneling measurement, however, probes eigenstates of the whole system, which involve not only edge electrons, but also localised bulk orbitals. Since tunneling processes do not probe the incompressibility of the electron gas, they are generally treated incorrectly by the theory of isolated edges. By taking into account the effect of Coulomb interactions between the edge and the localised bulk states, we have derived an effective edge theory that predicts a tunneling exponent $1/\nu$.

For the plateau transitions we have constructed a percolation model of interacting edges. We have shown how inelastic scattering at the “nearly saddle points” sets the temperature scale at which the transport coefficients cross over from mean field behaviour to critical scaling. This crossover can involve arbitrarily low temperatures and it explains the “lack of scaling” in the transport data taken from samples with long-range disorder at finite temperatures. Our mean field expression for the conductances agrees with recent empirical fits to transport data at plateau transitions.

The results of this paper serve as the basic starting point for a subsequent one where we extend the theory to include the statistical gauge fields and the fractional quantum Hall regime.

**Appendix A: One-dimensional propagator with Coulomb interaction**

In this appendix we calculate the correlation function $G(\tau,0)$ for the charged boson fields $\varphi_i$:

$$G(\tau, x) = \langle \varphi_i(\tau, x) \varphi_i(0, 0) \rangle \quad \tau > 0.$$  \(\text{(A1)}\)

In momentum and frequency space this correlator is given by (we omit the label $i$ since it is of no consequence)

$$\langle \varphi_a(k) \varphi_{-b}(-k') \rangle = \frac{2\pi i}{\beta} \frac{\delta_{ab} \delta(k - k')}{k^2 \omega_a + ikv^{\text{eff}}(k)}.$$  \(\text{(A2)}\)

We write the Coulomb interaction and the effective velocity $v^{\text{eff}}$ in the following form

$$U(k) = -c\sqrt{2\pi \ln(k/\Lambda)^2}; \quad v^{\text{eff}}(k) = -mc \ln(k/\Lambda D)^2$$  \(\text{(A3)}\)

where $c$ is a positive constant indicating the strength of the Coulomb interaction, $\Lambda$ is an ultraviolet cutoff and $D = \exp(v_d/2mc)$. We will only consider low momenta $|k| < \lambda \Lambda$, with $\lambda < 1$, so that we are well away from the point where the Hamiltonian becomes negative.

We take the Fourier transform of (A2) and change the frequency sum to an integral, writing $\sum_n = \frac{\partial}{2\pi} \int d\omega$,

$$\partial_\tau G(\tau, 0) = \frac{i}{2\pi} \int_{-\lambda \Lambda}^{\lambda \Lambda} dk \frac{v^{\text{eff}}(k)}{\omega + ikv^{\text{eff}}(k)} = -\int_{-\lambda \Lambda}^{\lambda \Lambda} dk \frac{v^{\text{eff}}(k)}{\Theta(-kv^{\text{eff}}(k))} e^{kv^{\text{eff}}(k)\tau}.$$  \(\text{(A4)}\)
The step function $\Theta(-k\epsilon^{\text{eff}})$ constrains the integration interval to $k<0$. We can split the last expression in (A4) into two parts, using $\ln k\,dk=d(k\ln k-k)$, and get

$$\partial_\tau G(\tau,0) = -\frac{1}{\tau}[1-(\frac{\lambda}{D})^{2mc\tau\Lambda}] - 2mc\Lambda D \int_0^{\lambda/D} du \exp[2mc\tau\Lambda D \cdot u \ln u]. \quad (A5)$$

The function $u \ln u$ is negative on the whole interval $(0, \lambda/D)$, since $\lambda/D<1$. If we now send the cutoff $\Lambda$ to infinity, the term with the integral in (A5) will go to zero as $1/\ln \Lambda$. The term $(\lambda/D)^{2mc\tau\Lambda}$ also vanishes, yielding the free particle result

$$G(\tau,0) = -\ln \tau + \text{constant}. \quad (A6)$$

**Appendix B: Chern-Simons action for bulk currents**

In this appendix we show that (4.8) is equivalent to the following bulk action:

$$S[A, g_i] = \frac{i}{4\pi} \sum_\alpha \sum_{i=1}^m \left[ -\int g_\alpha^i \wedge dg_\alpha^i + 2 \int g_\alpha^i \wedge dA_\alpha \right] \quad (B1)$$

with the condition $(g_\alpha^i)_{\text{edge}} = 0$ on the edge.

The $g_i$ are 2+1 dimensional potentials from which the electron current density $j$ for every Landau level can be found,

$$j_\mu^i \propto \varepsilon^{\mu\nu\lambda} \partial_\nu (g_i)_\lambda. \quad (B2)$$

Notice three important subtleties:

- The coupling of $g$ with the electromagnetic gauge field is of the form $g \wedge dA$ instead of the expected $A \wedge dg \propto j_\mu^A A^\mu$. These expressions differ by an edge term. The second form is not invariant under the gauge transformations $A \rightarrow A + d\chi$; the expression $dA$ on the other hand is manifestly gauge invariant.

- Putting an arbitrary spacetime component of $g$ zero on the edge ensures that the action is invariant under $g_\mu \rightarrow g_\mu + \partial_\mu \kappa$, a gauge transformation that does not affect the current density. Without such a condition, gauge invariance is broken at the edge.

- Because of the invariance under $g_\mu \rightarrow g_\mu + \partial_\mu \kappa$, a gauge fixing condition has to be specified for the path integration over $g$, for instance the Coulomb gauge $\nabla \cdot \vec{g} = 0$.

Let us now for simplicity drop the replica indices $\alpha$ and the Landau level index $i$ (effectively setting $m=1$).

Having taken the condition $g_\perp|_{\text{edge}} = 0$, the component $g_\perp$ in (B1) multiplies the following constraint:

$$\nabla \times (\vec{g} - \vec{A}) = 0. \quad (B3)$$

After integration over $g_\perp$, what remains of the action is
\[
\frac{i}{4\pi} \int d^2 x \left( -\vec{g} \times \partial_- \vec{g} + 2\vec{g} \times [\nabla A_- - \partial_- \vec{A}] \right)
\]

subject to the constraint (B3). The general solution of (B3) is given by

\[
\vec{g} = \vec{A} - \nabla \varphi
\]

with \(\varphi(x)\) a real scalar field which is now the only integration variable that is left. Substitution into (B4) yields an action where \(\varphi\) features only on the edge,

\[
S[\varphi, A] = \frac{i}{4\pi} \left[ A \wedge dA - \int dxd\tau \left( D_x \varphi D_- \varphi - \varphi E_x \right) \right].
\]

This is exactly of the form (4.8).

One may worry that the path integration over \(\varphi\) is ill-defined, because of the bulk degrees of freedom of \(\varphi\), which do not appear in (B6). However, \(\varphi\) inherits something from the gauge fixing condition of \(g\). This is most easily seen in the case of the Coulomb gauge; here, \(\varphi\) has to satisfy \(\nabla^2 \varphi = 0\). This means that the bulk degrees of freedom are completely determined by \(\varphi(x)\) at the edge (the well known case of Laplace’s equation with Dirichlet boundary conditions) and therefore aren’t independent integration variables.

One final remark on the boundary condition \(g_- = 0\): The Hamiltonian (density) corresponding to (B6) is given by \(v_d (D_x \varphi)^2\). It is not allowed to choose a velocity \(v_d < 0\), since this would lead to energies that are unbounded from below. In general, the boundary condition has to be taken in such a way that the velocity of the chiral bosons has the same sign as the prefactor multiplying \(\frac{i}{4\pi}\) in (B3), otherwise the integration over \(g\) is ill-defined on the edge.

### Appendix C: Inter-channel scattering at the edge

In this appendix we describe the various steps of the standard Q-field approach to (edge) disorder. For the general case of \(m\) chiral edge channels, one can differentiate between different types of disorder, depending on whether one allows inter-channel scattering or not. Although the different scattering potentials do not give rise to fundamentally different physical results, it is nevertheless important to define the ‘effective’ edge Hamiltonian (2.10) which gives rise to the same result (2.7) that was previously obtained for 2D electrons. Below we shall show that the following \(m\) channel model satisfies our requirements

\[
\mathcal{H}_{edge}^{kk'} = iv_d \delta_{kk'} \partial_x + V_{kk'}(x)
\]

where \(V\) is a hermitian random matrix and the elements \(V_{kk'}\) are distributed with a Gaussian weight

\[
P[V] = \exp\{-\frac{1}{g} \int dx \text{ tr } V^4 V\}.
\]

The indices \(k, k' = 1, \ldots, m\) label the edge channels. The form (C1) implies that single potential scattering, as described by the 2D Hamiltonian

\[
\mathcal{H}_{2D} = \frac{1}{2m} (\vec{p} - \vec{A})^2 + V(\vec{x}),
\]

is included.
does not naively translate into single potential scattering for the edge states as obtained by solving (C3) in the presence of an edge (infinite potential wall). Rather than that, one should allow for inter-channel scattering of the ‘pure’ eigenstates as in (C1) in order to reproduce the effect of dirt in the general 2D problem (C3).

We start from the following generating function for the averaged free particle propagators

$$Z = \int \mathcal{D} [\bar{\psi} \psi] \mathcal{D} [V] P[V] \exp \beta \sum_{p,\alpha,j,j'} \int dx \, \bar{\psi}_p^{\alpha,j} \left( (\mu + i\eta) \delta_{jj'} - \mathcal{H}_{\text{edge}}^{jj'} \right) \psi_{p}^{\alpha,j'}.$$  \hspace{1cm} (C4)

Integration over randomness and introduction of the matrix field $Q_{pp'}(x)$ by performing the Hubbard-Stratonovich trick leads to

$$Z = \int \mathcal{D} [Q] \exp \left\{ -\frac{1}{g} \text{Tr} Q^2 - m \text{Tr} \ln [\mu - iv_d \partial_x + iQ + \eta A] \right\}. \hspace{1cm} (C5)$$

Notice that the edge channel label is not present in the new field variable $Q$, but it is simply contained in an overall factor $m$. Notice also that the type of randomness as considered here has previously been introduced in a different context by the name of $N$-orbital scattering, where $N$ (here $m$) is commonly used for saddle point and large-$N$ expansion purposes.

We will next make use of the simple analytic properties of our 1D Hamiltonian and show that the saddlepoint technique yields, in fact, exact results for all $m$ and that therefore there is no need to rely on $m$ to be ‘large’. The stationary point equation for $Q$,

$$i [Q_{sp}]^{\alpha\beta}_{pp'} = \delta^{\alpha\beta} \delta_{pp'} (e_0 + (-1)^p i/\tau), \hspace{1cm} (C6)$$

can be written as

$$\frac{2}{g} (e_0 \pm i/\tau) \approx m \int_{-\infty}^{\infty} \frac{dq}{2\pi} [\mu - v_d q + e_0 \pm i/\tau \pm i\eta]^{-1} = im/v_d \hspace{1cm} (C7)$$

with the simple solution $e_0=0, \tau=2v_d/(mg)$. One may next replace the original $Q$-field by the following change of variables,

$$Q \rightarrow T^{-1} P T \rightarrow \frac{1}{\tau} T^{-1} \Lambda T =: \frac{1}{\tau} \tilde{Q}. \hspace{1cm} (C8)$$

Here, the $T \in SU(2N)$ are unitary rotations and the block-diagonal hermitian $P_{pp'}^{\alpha\beta} = \delta_{pp'} P_p^{\alpha\beta}$ represent the longitudinal components. Replacing $P$ by its saddlepoint value, as written in (C8), turns out to be an exact statement, valid for all $m$. The reason is contained in the fact that the fluctuations in $P$ are weighted by propagators with poles in either the positive or negative imaginary momentum plane. All the momentum integrals therefore sum up to zero, giving rise to a zero weight to all orders in the $P$-fluctuations. The replacement of (C8) is exact when inserted in the Tr ln.

Equation (C3) factorizes into

$$Z[Q] = Z[P] \cdot Z[T] \hspace{1cm} (C9)$$

$$Z[P] = \int \mathcal{D} [P] I[P] \exp \left\{ -\frac{1}{g} \text{Tr} P^2 \right\}$$

$$Z[T] = \int \mathcal{D} [T] \exp \{ -m \text{Tr} \ln [\mu - iv_d \partial_x + i\tau \Lambda + iB] \}$$
where all $T$-dependence is contained in the quantity $B$ according to

$$B = -v_d T \partial_x T^{-1} + \eta T \Lambda T^{-1} = : v_d T D_0 T^{-1}. \quad \text{(C10)}$$

Equation (C9) can be evaluated further, and to lowest few orders in an expansion in $B$ we obtain an effective action which can be written as

$$Z[T] = \int D[T] \exp S_{\text{eff}}[T]$$

$$S_{\text{eff}}[T] = \frac{m}{v_d} \text{Tr} \Lambda B(x) - \frac{m v_d}{v_d} \text{Tr} [B(x), \Lambda]^2 + \cdots \quad \text{(C11)}$$

$$= m \oint dx \text{tr} \Lambda T \partial_x T^{-1} + \frac{m v_d}{v_d} \eta \oint dx \text{tr} \Lambda \tilde{Q} - \frac{m v_d}{16} \oint dx \text{tr} [D_0, \tilde{Q}]^2. \quad \text{(C12)}$$

The coefficients appearing in (C12) all have a clear physical significance in the context of disordered edge states (see also the main text). In particular, $m$ stands for the quantised Hall conductance $\sigma_{xy}$; $m/v_d$ equals the total density of edge states $\rho_{\text{edge}}$. The quantity $m \tau v_d/2$ that appears in the higher dimensional operators is the 1D conductivity $\sigma_{xx}$ of $m$ channel edge states. Here, $\tau v_d$ is the linear dimension which sets the smallest wavelength for the $\tilde{Q}$ field variables and $m/2$ is the (quantised) conductance ($g_m$) of the wire.

We next extend the background field methodology of section II A and show that besides the parameters $\sigma_{xy} = m$ and $\rho_{\text{edge}} = m/v_d$ also the quantity $g_m = m/2$ is a length scale independent (or fixed point) parameter of the theory in the limit of zero number of replicas. For this purpose we make the shift

$$B \rightarrow B + T b_0 T^{-1}; \quad b_0 = v_d t_0 D_0 t_0^{-1} \quad \text{(C13)}$$

with $t_0$ a fixed background field. This shift will be exploited as follows in (C11),

$$S_{\text{eff}} \rightarrow \rho_{\text{edge}} \text{Tr} (B \Lambda + b_0 \tilde{Q}) - \frac{m v_d}{v_d} \text{Tr} [B + T b_0 T^{-1}, \Lambda]^2 + H_0^2 \text{Tr} \Lambda \tilde{Q}. \quad \text{(C14)}$$

Integration over the $T$ or $\tilde{Q}$ matrix field leads to an effective theory for the $b_0$ or $t_0$ background field with the additional field $H_0^2$ now fixing the new length scale or momentum cutoff. In the limit $H_0^2 \ll 1/(v_d \tau)$ the following result is obtained from a perturbative analysis of the $T$-fluctuations (section II A):

$$S_{\text{eff}}[t_0] = m \oint dx \text{tr} \Lambda t_0 \partial_x t_0^{-1} + \frac{m v_d}{v_d} \eta \oint dx \text{tr} \Lambda \tilde{Q}_0 - \frac{m v_d}{16 H_0^2} \oint dx \text{tr} [D_0, \tilde{Q}_0]^2 \quad \text{(C15)}$$

with $\tilde{Q}_0 := t_0^{-1} \Lambda t_0$.

In summary, we can say that the effective theory has precisely the same form as the original one (C12), with the same values of the parameters, except that the momentum cutoff $1/(v_d \tau)$ has been replaced by a new one, $H_0^2 \ll 1/(v_d \tau)$. Equation (C13) is strictly valid in the replica limit only, and for the same reasons as explained in section II A it is expected to hold to all orders in perturbation theory of the $T$-matrix.

The physical consequences of $N = 0$ replica field theory are dramatically different from what is naively expected on the basis of the Mermin-Wagner-Coleman theorem, which is applicable to any positive integer value of $N$. In this latter, general, case, one expects the
Appendix D: Action for $Q$ and $A$ on multiple edges

The generalisation of (4.11) is given by

$$S[Q, A] = \frac{\beta/2}{(2\pi)^2} \int d^2 x d^2 x' n(\vec{x}) B(\vec{x}) U(\vec{x} - \vec{x'}) n(\vec{x'})$$

$$+ \frac{i}{4\pi} \left[ \sum_{\alpha} \int n(A_{\alpha}^{\text{eff}}) \wedge d(A_{\alpha}^{\text{eff}}) + \beta \sum_{a=1}^M s_a \oint_{C_a} dx \left( A_{\alpha a}^{\text{eff}} A_0^{\text{eff}} - \frac{2\pi}{\beta} \text{tr} \hat{A}_{\alpha a} Q \right) \right]$$

$$+ \sum_{a=1}^M s_a S_{\text{top}}^{(a)}[Q] + \frac{\pi}{4\beta v_{\text{d}}} \sum_{a=1}^M S_{\text{F}}^{(a)}[Q]$$

$$- \frac{\pi}{4\beta} \sum_{a=1}^M \int dk_x \frac{1}{m_{a}(k_x)} \left| \text{tr} \Gamma_{\alpha}^a Q(k_x) - \frac{\beta}{\pi} (A_{\alpha 0}^{\text{eff}})^a_{-n}(k_x) \right|^2$$

$$+ \frac{1}{8\beta v_{\text{d}}} \sum_{a \neq b} s_a s_b \oint_{C_a} dx \oint_{C_b} dx' \sum_{n\alpha} \left| \text{tr} \Gamma_{\alpha}^a Q - \frac{\beta}{\pi} (A_{\alpha 0}^{\text{eff}})^a_{-n}(x) \times \right.$$  

$$\times U(x, x') \text{tr} \Gamma_{\alpha}^b Q - \frac{\beta}{\pi} (A_{\alpha 0}^{\text{eff}})^b_{-n}(x') \right|$$

$$\left. \right\} \text{(D1)}$$

where $U(x, x')$ denotes the full 2D Coulomb interaction. All terms except those quadratic in $Q$ arise by the obvious replacements $m \rightarrow n(\vec{x})$ and $m \oint \rightarrow \sum_a s_a \oint_{C_a}$ in (4.11). The terms quadratic in $Q$ can be understood as follows. In the generalised form of (3.39), the quadratic term in the plasmon field is given by

$$\int \lambda^\dagger U^{-1} \lambda$$

indicating that the propagator for $\lambda$ between two points on the same edge will be very different from the propagator between different edges. In the former case the propagator is proportional to $[U^{-1} + \frac{m}{\alpha_{\lambda}}(2\pi)^2]^{-1}$, which is exactly the form obtained by combining the Finkelstein term with (D1). In the latter case, the propagator is simply proportional to $U$. Finally, the signs can be understood by noticing that the coupling of the plasmon field to $Q$ is proportional to $\sum_a s_a \oint_{C_a} \text{tr} \hat{\lambda} Q$. 

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