Overlaps of partial Néel states and Bethe states

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Abstract. Partial Néel states are generalizations of the ordinary Néel (classical anti-ferromagnet) state that can have arbitrary integer spin. We study overlaps of these states with Bethe states. We first identify this overlap with a partial version of reflecting-boundary domain-wall partition function, and then derive various determinant representations for off-shell and on-shell Bethe states.

Keywords: algebraic structures of integrable models, integrable spin chains (vertex models), quantum integrability (Bethe Ansatz)
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1. Introduction

1.1. Overview

The Néel state is the simplest state with antiferromagnetic ordering, differing though from the true ground state of the antiferromagnetic XXX spin-\(1/2\) chain, which is more complicated. One may ask how close the Néel state is to the true eigenstate of the spin-chain Hamiltonian. The answer to this question is actually known, as the overlap of the Néel state with any given eigenstate can be explicitly calculated. As pointed out in [1], the overlap is related to the partition function of the six-vertex model on a rectangular lattice with reflecting boundary conditions. A determinant representation for this overlap was obtained by Tsuchiya [2]. Restricting Tsuchiya’s expression to Bethe eigenstates requires an extra step, and leads to a simpler, more compact determinant expression [3–5]. Applications of these results range from condensed-matter physics [3–7] to string theory [8] and algebraic combinatorics [9].

In this note, we generalize Tsuchiya’s result to the case of partition functions of six-vertex model configurations with domain-wall boundary conditions that are only partially reflecting. These partially reflecting domain-wall boundary conditions are related to Tsuchiya’s [2] in the same way that the partial domain-wall boundary conditions and partition function in [10] are related to Korepin’s domain-wall boundary conditions [11], and Izergin’s determinant expression for the corresponding partition function [12]. The on-shell version of the partition function that we study describes overlaps of the Bethe states with partial, or generalized Néel states [5, 8]. Our derivations closely follow those in [1–3].

1.2. Outline of content

In section 2, we recall basic definitions related to the XXX spin-\(1/2\) chain and the rational six-vertex model. In section 3, we recall the definition of Tsuchiya’s reflecting-boundary boundary conditions, and the corresponding partition function, then introduce partial versions thereof, which we subsequently calculate. In section 4, we compute the overlap of a partial Néel state and a parity-invariant highest-weight on-shell Bethe state. In section 5, we collect a number of remarks. and in appendix, we explain the reduction of the \([M \times M]\) determinant partition function to an \([M/2 \times M/2]\) one.

2. The XXX spin-\(1/2\) chain and the rational six-vertex model

We restrict our attention to the XXX spin-\(1/2\) chain of length \(L = 2N\) [13]. Each site carries an up-spin \(\uparrow\), or a down-spin \(\downarrow\). The Hamiltonian \(\mathcal{H}\) acts on \((\mathbb{C}^2)^\otimes L\) as

\[
\mathcal{H} = \sum_{l=1}^{2N}(1 - P_{l,l+1}),
\]

where \(P_{l,l+1}\) permutes the spins on two adjacent sites labeled \(l\) and \(l + 1\).
2.1. The $R$-matrix and the Yang–Baxter equation

The key object in the Bethe Ansatz solution of the XXX spin-$\frac{1}{2}$ chain is the $R$-matrix \cite{14}, represented in figure 1. The $R$-matrix, $R_{ab}(u)$, acts on the tensor product of two spins labelled $a$ and $b$, and depends on a complex spectral parameter $u$,

$$R_{ab}(u) = u + iP_{ab},$$

where $P_{ab}$ is the permutation operator. Most importantly, the $R$-matrix satisfies the Yang–Baxter equation,

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v),$$

represented in figure 2.

2.2. Notation and conventions

We will use the shorthand notation

$$y^{\pm} = y \pm \frac{i}{2}.$$  \hspace{1cm} (4)

The $R$-matrix acts from south to north or, for the vertical crossing, from south-east to north-west, as represented in figure 1. If each line carries a rapidity variable, the argument of the $R$-matrix is the difference of the two rapidity variables.

2.3. The $B$-operator

The states of the XXX spin-$\frac{1}{2}$ chain are generated by the $B$-operators. These operators are constructed by multiplying the $R$-matrices along the spin-chain threaded with a single auxiliary space,

$$B_y(x) = \left\langle t_a | R_{1,a}(x + y_1)R_{2,a}(x - y_2) \cdots R_{L-1,a}(x + y_{L-1})R_{L,a}(x - y_L) | t_a \right\rangle.$$  \hspace{1cm} (5)

\textbf{Figure 1.} The $R$-matrix and the boundary state.

\textbf{Figure 2.} The Yang–Baxter equation.
2.4. Inhomogeneity variables

The variables $y_k, k = 1, \ldots, L$, are the quantum-space or inhomogeneity variables. They play an important role in the intermediate steps of the derivations, but they are not necessary for diagonalizing the Hamiltonian (1). The inhomogeneity variables are common to all $B$-operators used to construct the state. In this note, we do not consider the most general inhomogeneity variables as in (5), but focus on parity-invariant Bethe states such that the inhomogeneity variables are paired, as in

$$B_y(x) = \langle \uparrow_a | R_{1,a}(x+y_1)R_{2,a}(x-y_1) \cdots R_{2N-1,a}(x+y_N)R_{2N,a}(x-y_N) | \uparrow_a \rangle.$$  

(6)

The restriction to paired inhomogeneity variables will be important later on, when we consider overlaps of Bethe states and the boundary states introduced in section 2.8.

2.5. On-shell Bethe states

The Bethe states of the XXX spin-1/2 chain are constructed by applying the $B$-operators on the ferromagnetic vacuum, $|0\rangle = |\uparrow \cdots \uparrow\rangle$, of the spin-chain,

$$|\mathbf{x}\rangle = B(x_M) \cdots B(x_1)|0\rangle,$$

(7)

where $B(x) = B_0(x)$, that is, the $B$-operator with all $y_a$-variables set to zero. For a Bethe state to be an eigenstate of the Hamiltonian, the rapidity variables $x_j$ must satisfy the Bethe equations,

$$\left( \frac{x_j^+}{x_j^-} \right)^{2N} = \prod_{k \neq j} \frac{x_j^+ - x_k^+}{x_j^- - x_k^-}. $$

(8)

As usual, we call Bethe states with rapidity variables subject to the Bethe equations on-shell states. Generic Bethe states, that are not eigenstates of the Hamiltonian, are referred to as off-shell states.

2.6. Partial Néel states

Given the definition of the Néel state, on an XXX spin-1/2 chain of length $L = 2N$,

$$|\text{Néel}\rangle = |\uparrow \downarrow \uparrow \cdots \uparrow\rangle + |\downarrow \uparrow \downarrow \cdots \downarrow\rangle,$$

(9)

and an integer $M$, such that $0 \leq M \leq N$, we define an $M$-partial Néel state as [5, 8]

$$|\text{Néel}_M\rangle = \sum_{l_1 < \cdots < l_M} |\cdots \uparrow_{l_1} \downarrow \cdots \uparrow_{l_M} \downarrow \cdots \downarrow_{l_M} \cdots \rangle.$$  

(10)

For $M = N$, the state has an equal number of up- and down-spins, and we recover the original Néel state, $|\text{Néel}_N\rangle = |\text{Néel}\rangle$.

2.7. Matrix product states

The matrix product state, MPS, is defined as

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\[ |MPS \rangle = \text{tr}_a (t_1^a | \uparrow \rangle \langle \uparrow | + t_2^a | \downarrow \rangle \langle \downarrow | ) \otimes \cdots \otimes (t_L^a | \uparrow \rangle \langle \uparrow | + t_L^a | \downarrow \rangle \langle \downarrow | ), \]

where \( t^a = \sigma_1 \) and \( t^a = \sigma_2 \). Following [8], all partial Néel states can be obtained from MPS by projecting on subspaces with a definite number of up- and down-spins. Denoting the projector on the state with \( M \) down-spins by \( P_M \), we have,

\[ |\text{Néel}_M \rangle = 2^M \left( \frac{1}{2} \right)^M P_M |MPS \rangle + S^- |\Psi \rangle, \]

where \( S^- \) is the spin-lowering operator. The last term does not contribute to the overlap with a Bethe state that is annihilated by \( S^+ \), such as the highest-weight on-shell Bethe states of the homogeneous XXX spin-\( \frac{1}{2} \) chain.

2.8. The boundary state

As noted in [1], the Néel state can be constructed from the boundary state associated with the diagonal reflection matrix. The overlap of a Néel state on a one-dimensional lattice of length \( L = 2N \) and a Bethe state that is not necessarily on-shell, characterised by \( N \) rapidity variables, is equal to the partition function of the six-vertex model on a rectangular lattice that has \( N \) horizontal lines and \( 2N \) vertical lines, with reflecting-boundary domain-wall boundary conditions. Following [2], the latter is an \([N \times N]\) determinant. We extend this construction by effectively allowing for non-diagonal scattering off the boundary. The latter boundary state reduces to a partial Néel state (10) for appropriate values of the variables. The boundary state is defined as

\[ \langle K_{ab}(u) \rangle = \langle \uparrow \downarrow \rangle(u^+ + \xi) + \langle \downarrow \uparrow \rangle(u^+ - \xi) + \langle \uparrow \uparrow \rangle\lambda u^+. \]

Although the boundary state is associated with two lattice lines, it depends on a single rapidity \( u \). The alternating inhomogeneity variables in (6) lead to one independent inhomogeneity variable for each boundary state. The variables \( \xi \) and \( \lambda \) are arbitrary, but fixed complex numbers.

The boundary state is the cross-channel representation of the reflection matrix [15, 16]. The most commonly used boundary state is the neutral one with \( \lambda = 0 \), which corresponds to the diagonal reflection [15, 16]. It is this diagonal reflection matrix that was used in the derivation of [2]. We extend this result by adding the last, two-spins-up term, allowed by the consistency conditions for integrable boundary scattering [17]. When representing the boundary state in terms of a diagram, we assume that it can only connect spins whose rapidity variables add to \( i \), as shown in figure 1.

2.9. The reflection equation

The boundary state obeys the reflection equation [15, 16], see figure 3, which, in our notation, takes the form

\[ \langle K_{23}(v) \otimes K_{12}(u) | R_{14}(u + v) R_{13}(u - v) = \langle K_{12}(u) \otimes K_{34}(v) | R_{23}(u + v) R_{24}(u - v). \]

One can, in principle, also add a term with two down-spins that has a weight \( \mu u^+ \), which would correspond to the most general rational solution of the reflection equation [17]. It would be interesting to generalize the computation of the overlaps and the partition functions defined below to this case as well.
2.10. The overlap of a partial Néel state and a parity-invariant highest-weight on-shell Bethe state

The object of our interest is the overlap

$$Z_{xy} = \langle K_{12}(y_1) \otimes \cdots \otimes K_{2N-1,2N}(y_N) | B_y(x_M) \cdots B_y(x_1) | 0 \rangle. \quad (15)$$

It depends on two sets of rapidity variables $\{x_j\}_{j=1,\ldots,M}$ and $\{y_k\}_{k=1,\ldots,N}$. We will not put further constraints on these variables at this point. The spectral parameters of the boundary state, as in figure 1, are correlated with the arguments of the $R$-matrices in (6). This is the reason for pairing them, instead of keeping them arbitrary.

The overlap of the on-shell Bethe states of the homogeneous spin-chain with the partial Néel states (10) can be obtained by setting $y_a = 0$, $\lambda = -2i$, and $\xi = \pm \frac{i}{2}$, and imposing the Bethe equations on the rapidity variables $x_j$,

$$\langle \text{Néel}_M | \mathbf{x} \rangle = (-i)^M \left( Z_{x|\lambda=-2i,\xi=\frac{i}{2}} + Z_{x|\lambda=-2i,\xi=-\frac{i}{2}} \right). \quad (16)$$

This follows from the structure of the boundary state section 2.8. Setting $\xi = \pm \frac{i}{2}$ leaves only two terms in the boundary state section 2.8 whose $N$-th tensor power then generates the sum of all partial Néel states. Since any given Bethe state has exactly $M$ down-spins, only the $M$-th Néel state can have a non-zero overlap with it.

To evaluate the overlaps (15) and (16), we proceed along the same lines as [1–3], introducing along the way modifications necessary to account for spin-non-preserving term in the boundary state, or equivalently, a non-diagonal term in the reflection matrix. The key step is to reformulate the problem in terms of a partition function of the rational six-vertex model on a rectangular lattice with a modified, or partial version of Tsuchiya’s reflecting-boundary domain-wall boundary conditions. It should also be possible to use the recursion relation that was derived using the algebraic Bethe ansatz in [18]. The Tsuchiya determinant is a solution of this recursion relation.

3. Partial reflecting-boundary domain-wall partition function

The overlap (15) can be represented as a partition function of the rational six-vertex model on the $[M \times 2N]$ rectangular lattice, as in figure 4, where the spin-chain sites are associated with the vertical lines and the horizontal lines represent the auxiliary spaces of the $B$-operators. The weights of the bulk vertices are the matrix elements of the

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Figure 3. The reflection equation for the boundary state.
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\[ R \text{-matrix, while the weights of the boundary vertices are the coefficients of the boundary state section 2.8, as in figure 5. The spins in the bulk are conserved in the sense that each vertex has two in- and two out-bound arrows, as in the left and middle columns of figure 5. The spins on the boundary, with reflecting boundary conditions at } \lambda = 0 \text{ are also conserved, as in the right column of figure 5. In this case, spin conservation implies that } M = N. \text{ The partition function on the resulting } [N \times 2N] \text{ lattice admits a determinant representation, as shown by Tsuchiya [2].} \]
In this note, we are interested in the more general case where the boundary conditions are partially reflecting, and the upper boundary can absorb an excess spin, thus allowing $M$ to be smaller than $N$. The resulting statistical mechanical system can be regarded as a degenerate version of Tsuchiya’s. The extra horizontal lines can be systematically eliminated by taking $(N - M)$ auxiliary-space rapidity variables, in the original system, to infinity, and renormalizing the partition function appropriately to obtain a finite result. The procedure is described in detail in [10], where the partition function of the six-vertex model with partial domain-wall boundary conditions was calculated by degenerating the partition function of the system with the domain-wall boundary conditions [11, 12]. We do not follow this route here. Instead, we follow the derivation of [2], while taking the more general structure of the boundary reflection matrix into account.

The derivation of [2] relies on a recursion relation which relates the partition functions on lattices of different sizes. The recursion relation is derived by $f$ parts of the configurations using suitable choices of some of the free variables, and the observation that the type-$a$ or type-$b$ weights in figure 5 vanish if the vertical and horizontal rapidity variables differ by $\pm \frac{1}{2}$. In this note, we generalize this recursion relation to accommodate the partially-reflecting boundary conditions at $\lambda \neq 0$. The partition function of the statistical mechanical system that we are interested in is completely specified by the following four conditions.

3.1. Condition 1

$Z_{xy}$ is symmetric in $\{x_j\}$, and separately in $\{y_a\}$. This follows from repeated application of the Yang–Baxter and reflection equations to the partition function. The symmetry in $\{x_j\}$ follows from commutativity of the $B$-operators, which is a consequence of the Yang–Baxter equation. The symmetry in $\{y_a\}$ can be proven by standard manipulations with the reflection equation, which we reproduce here for completeness. Multiplying both sides of the reflection equation (14) by

$$R_{1a}(x + u^-)R_{3a}(x + v^-)R_{4a}(x - v^+)R_{2a}(x - u^+),$$

and using the Yang–Baxter equation twice we get the equality depicted in figure 6,

$$\langle K_{2a}(v) \otimes K_{12}(u) | R_{3a}(x + v^-)R_{4a}(x - v^+)R_{1a}(x + u^-)R_{2a}(x - u^+)R_{14}(u + v)R_{13}(u - v) \rangle = \langle K_{12}(u) \otimes K_{5a}(v) | R_{1a}(x + u^-)R_{2a}(x - u^+)R_{3a}(x + v^-)R_{4a}(x - v^+)R_{23}(u + v)R_{24}(u - v) \rangle.$$

(17)
The process can be iterated to add an arbitrary number of horizontal lines. The resulting equality is an identity of two vectors in $(C^2)^4$. The next step is to project these vectors on the ground state $|\uparrow\uparrow\uparrow\uparrow\rangle$, in other words to specify all arrows at the bottom of the diagram in figure 7. By spin conservation, all vertices below the freezing line are of type $b$, and consequently produce a common scalar factor $R_{\uparrow\downarrow}(u-v)R_{\downarrow\uparrow}(u+v)$ on both sides of the equation. The remaining diagrams above the freezing line differ by the order of the vertical rapidity variables $u$ and $v$. The same procedure can be applied without much change to swap any two vertical rapidity variables on the $[M \times 2N]$ lattice.

3.2. Condition 2

$Z_{xy}$ is a polynomial of degree $(2N-1)$ in $x_j$. The variable $x_j$ appears in the Boltzmann weights associated with the $2N$ vertices on the $j$-th row of the partition function. Each type-$a$ and type-$b$ vertex contributes one power of $x_j$, but does not flip the horizontal spin, while a type-$c$ vertex flips the spin but does not depend on $x_j$. Since the spin flips at least once, each horizontal line has to contain at least one type-$c$ vertex. Configurations with one $c$-vertex and $(2N-1)$ vertices of $a$-type and $b$-type on the same line give a non-zero contribution to the statistical ensemble and, consequently, the partition function is a polynomial of degree $(2N-1)$ in $x_j$.

3.3. Condition 3

$Z_{xy}$ satisfies a recursion relation that equates the partition function on an $[M \times 2N]$ lattice to a partition function on a smaller $[(M-1) \times (2N-2)]$ lattice, as soon as one of the horizontal rapidity variables is set to $x_j = \pm y_a^\pm$. The recursion relation is derived by the freezing trick illustrated in figure 8.

First, using the Yang–Baxter and reflection equations, the $j$-th horizontal rapidity and the $a$-th vertical rapidity can be moved to the bottom-left corner of the lattice. Spin conservation leaves two possibilities for the bottom-left vertex, it is either type-$b$ or type-$c$, as follows from figure 5. If $x_j = -y_a^+$, the type-$b$ vertex has zero weight and the bottom-left corner of the partition function is then unambiguously determined. Once the corner weight is fixed, the vertices on the two left-most columns and the lower row are recursively determined by spin conservation. This freezes the lower and left edges of the partition function as in figure 8(a). Similarly, one can freeze the bottom-right

\[ \text{Figure 7.} \] Once the arrows on the entry lines of the diagram are specified, the spin conservation freezes all other spins in the lower part of the diagram, up to the freezing line shown in light blue. The vertices below the freezing line are all of $b$-type.
corner of the partition function by setting \( x_j = \pm y^+_a \). The freezing trick results in two recursion relations:

\[
Z_{xy}\left|_{x_j=\pm y^+_a} \right. = 2i y^+_a (\xi \pm y^+_a) \prod_{k \neq j} (x_k^2 - (y^+_a)^2) \prod_{b \neq a} ((y^+_a)^2 - y_b^2) Z_{\hat{x}_j, \hat{x}_j},
\]

where

\[
\hat{x}_j = \{x_1, \ldots, \hat{x}_j, \ldots, x_M\},
\]

and \( \hat{x}_j \) means that the variable \( x_j \) is omitted.

### 3.4. Condition 4

At \( M = 0 \),

\[
Z_{\emptyset y} = \lambda^N \prod_a y^+_a.
\]

The partition function with no horizontal lines is just a product of \( N \) non-reflective boundary weights.

### 3.5. Inhomogeneous \((N \times N)\) determinant partition function

The partition function vanishes for \( M > N \). For \( M \leq N \), it is completely determined by conditions 1, 2, 3 and 4, for any \( M \) and \( N \), since a polynomial of degree \((2N - 1)\) is completely fixed by its values at \( 2N \) distinct points. Condition 2, therefore, determines \( Z_{xy} \) as a function of \( x_j \). Eliminating \( \hat{x}'s \) one by one, we are left with \( Z_{\emptyset y} \), specified by condition 4.

The solution of the recursion relation that satisfies all four conditions can be represented in the determinant form.

\[
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\]

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\[ Z_{xy} = i^{N^2+3N+M} \left( \frac{\lambda}{2} \right)^{N-M} \prod_j (\xi + x_j) \prod_a (2y_a + i) \times \frac{\prod_{j,j} (x_j - y_a)^2 + \frac{1}{4})(x_j + y_a)^2 + \frac{1}{4})}{\prod_{j,j} (x_j^2 - x_k^2) \prod_{a,b} (y_a^2 - y_b^2)} \det M, \]  

(21)

where \( M \) is an \([N \times N]\) matrix,

\[ M_{ab} = \begin{cases} y_b^{2a-2}, & a = 1, \ldots, N - M \\ \frac{1}{((x_j - y_a)^2 + \frac{1}{4})(x_j + y_a)^2 + \frac{1}{4})}, & a = N - M + j, j = 1, \ldots, M. \end{cases} \]  

(22)

Checking the conditions 1–4 is straightforward. Symmetry in \( x_j \) and \( y_a \) is obvious. The poles of the prefactor at \( x_j = \pm x_k \), as well as at \( y_a = \pm y_b \), are cancelled by the zeros of the determinant. Hence, the partition function is a polynomial in each of the \( x_j \)'s. It is perhaps not immediately obvious why the degree of this polynomial is exactly \((2N - 1)\), but one can check that the expansion of \( \det M \) at \( x_j \to \infty \) starts with \( x_j^{-2(N-M)} \), because the lower-order terms are linear combinations of the first \((N - M)\) rows of the matrix \( M \). Checking the recursion relations is also easy, as the \((a, j)\) element of \( M \) develops a pole at \( x_j = \pm y_a^\pm \), which eliminates its \((N - M + j)\)-th row and \( a\)-th column.

The determinant representation (21) generalizes Tsuchiya formula [2], to which this expression reduces when \( M = N \). It represents the overlap as an \([N \times N]\) determinant, where \( N \) is half of the length of the spin chain. In the sequel, we derive a more compact representation in terms of an \([M \times M]\) determinant, where \( M \) is the number of magnons, which is general is smaller than \( N \). We also study the homogeneous limit when we set all the vertical rapidity variables to zero. We should stress that the expression (21) is valid off-shell, for any values of vertical and horizontal rapidity variables. Further, we study the on-shell limit of the partition function when the horizontal rapidity variables satisfy the Bethe equations and the state \(|x\rangle\) is an eigenstate of the Heisenberg Hamiltonian.

### 3.6. The homogeneous limit of the \((N \times N)\) determinant

Observing that the determinant in (21) scales as \( y^{M(N-1)} \), and using

\[ \det_{ab}(f_a(y_b^2)) \simeq \det_{ab} \left( \sum_{c=1}^{N} \frac{f^{(c-1)}(0)}{(c-1)!} y_b^{2c-2} \right) \]

\[ = \det_{ac} \left( \frac{f^{(c-1)}(0)}{(c-1)!} \right) \det_{bd}(y_a^{2c-2}) = i^{N(N-1)} \prod_{d < b} (y_a^2 - y_b^2) \det_{ac} \left( \frac{f^{(c-1)}(0)}{(c-1)!} \right), \]  

(23)

where \( \simeq \) means equality up to the leading order in \( y^2 \), as well as the expansion

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\[
\frac{1}{((x - y)^2 + \frac{1}{4})((x + y)^2 + \frac{1}{4})} = \frac{1}{2i\pi} \sum_{c=1}^{\infty} \left( \frac{1}{(x^-)^{2c}} - \frac{1}{(x^+)^{2c}} \right) y^{2c-2},
\]  

(24)

we find that

\[
Z_{x0} = i^{2N^2 - N + M^2 - M} \frac{X^{N-M}}{2^N} \prod_j (x_j + \xi) \det_{jk}((x_j)^{2k-2}(x_j^+)^{2N} - (x_j^{2k-2})(x_j^{2N})) \prod_{j<k}(x_j^2 - x_k^2).
\]

(25)

4. Overlap of a partial Néel state and an on-shell Bethe state

To compute the overlap of a partial Néel state with a Bethe state, according to (16), we

1. put the inhomogeneous \([N \times N]\) determinant of section 3.5 in a smaller, \([M \times M]\), still inhomogeneous form,

2. take the homogeneous limit of the \([M \times M]\) determinant,

3. put the auxiliary-space rapidity variables on-shell by imposing the Bethe equations,

then 4. reduce the size of the determinant to \([M \times M]\). The normalized overlap is given by the resulting formula divided by the Gaudin norm of the Bethe vector.

4.1. An inhomogeneous \((M \times M)\) determinant partition function

The \([N \times N]\) determinant (21) admits two, different but equivalent \([M \times M]\) representations,

\[
Z_{xy} = (-2i)^M \frac{X^{N-M}}{2^N} \prod_j (x_j + \xi) \prod_a y_a^+ 
\]

\[
\prod_{j<k} \frac{(x_j^+)^2 - (x_k^-)^2((x_j^-)^2 - (x_k^+)^2)}{(x_j^2 - x_k^2)((x_j^+)^2 - (x_k^-)^2)} \prod_{ja} (x_j^2 - y_a^2) 
\]

\[
\det_{jk} \left( \frac{1}{(x_j^+)^2 - (x_k^-)^2} \mp \frac{i\delta_{jk}}{2x_j} \prod_a \frac{(x_j^+)^2 - y_a^2}{(x_j^2 - y_a^2)} \prod_{l<x j} (x_j^+)^2 - (x_l^+)^2 \right). 
\]

(26)

The derivation of this result involves standard manipulations of rational sums [19–21], the details of which are presented in appendix. In this form, the determinant has almost no dependence on \(N\). We reproduce here, for completeness, the derivation in [3] of the on-shell overlap for the homogeneous spin chain, but all the next steps are mathematically the same as in the case of \(N = M\) considered in [3].

4.2. The homogeneous limit of the \((M \times M)\) determinant

Taking the homogeneous limit in the \([M \times M]\) determinant representation is straightforward and yields,

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\[ Z_{x|\lambda=-2i} = (-1)^N \prod_j (x_j + \xi)(x_j^*)^{2N} \prod_{j<k} \frac{((x_j^*)^2 - (x_k^*)^2) - ((x_j^*)^2 - (x_k^*)^2)}{(x_j^2 - x_k^2)((x_j^*)^2 - (x_k^*)^2)} \det B, \tag{27} \]

where \( B \) is an \([M \times M]\) matrix with matrix elements,

\[ B_{jk} = \frac{1}{(x_j^* - x_k^*)^2} \pm \frac{i\delta_{jk}}{2x_j} \prod_{l \neq j} \frac{(x_j^* - x_l^*)^2}{(x_j^* - x_l^*)^2} \tag{28} \]

4.3. The overlap of a partial Néel state and a generic on-shell Bethe state vanishes

Following [8], the MPS (11) is an eigenstate of the degree-3 conserved-charge operator \( H_3 \), with an \( H_3 \)-eigenvalue zero. Any on-shell Bethe state, \( \chi \), must be an eigenstate of \( H_3 \). Since the eigenvalue of \( H_3 \) is conserved, the overlap of the MPS with \( \chi \), can be non-zero only if the \( H_3 \)-eigenvalue of \( \chi \) is equal to that of the MPS, that is, also zero. For \( \chi \) to have an \( H_3 \)-eigenvalue zero, it must be parity-even, that is, the set of Bethe roots, \( \{ x_j \} \), of \( \chi \) must be invariant, as a set, under the parity transformation \( x_j \rightarrow -x_j \). Since the partial Néel states are components of the MPS, the same reasoning applies to the overlap of a partial Néel state and \( \chi \). The reason is that the latter is not an eigenstate of \( H_3 \), and the \( H_3 \) selection rule does not apply to its overlap with MPS and consequently with any of the Néel states.

Aside from the above reasoning, the technical reason for the vanishing of the overlap, in either (25) or (27), is not immediately obvious. However, it can be proven in the latter representation by noting that the matrix \( B \) in (28) has a zero eigenvalue if the \( x_j \) are Bethe roots. This can been seen as follows. Setting the rapidity variables \( x_j \) on-shell, the matrix \( B \) becomes,

\[ B_{jk} = \frac{1}{(x_j^* - x_k^*)^2} \pm \frac{i\delta_{jk}}{2x_j} \prod_{l \neq j} \frac{(x_j^* - x_l^*)^2}{(x_j^* - x_l^*)^2}. \tag{29} \]

The fact that this matrix is degenerate follows from the identity

\[ \sum_k \frac{\pm 2ix_k}{(x_j^* - x_k^*)^2} \prod_{l \neq k} \frac{(x_k^* - x_l^*)^2}{(x_k^* - x_l^*)^2} = \oint \frac{dz}{2\pi i} \frac{1}{((x_j^*)^2 - z)} \prod_{i=1}^M \frac{(z - (x_i^*)^2)}{(z - (x_i^*)^2)} = -1, \tag{30} \]

where the contour of integration encircles the poles of the integrand at \( z = (x_j^*)^2 \) counterclockwise, and the last equality is obtained by evaluating the residue at infinity. As a consequence of (30), the vector with components

\[ V_k = \pm 2ix_k \prod_{l \neq k} \frac{(x_l^* - x_k^*)^2}{(x_l^* - x_k^*)^2} \tag{31} \]

can be seen to be a zero eigenvector of \( B \),

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\]

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\[
\sum_k B_{jk} V_k = 0.
\]  
(32)

This implies that a partial Néel state has a zero overlap with a generic highest-weight on-shell Bethe state, where \textit{generic} here means a state which is \textit{not} invariant under parity transformation, \(x_j \rightarrow -x_j\).

### 4.4. The overlap of a partial Néel state and a parity-invariant on-shell Bethe state

If the set of rapidity variables \(\{x_j\}\) of the on-shell Bethe state \(\chi\) is invariant, as a set, under \(x_j \rightarrow -x_j\), then \(\chi\) is parity-invariant, its \(\mathcal{H}_3\)-eigenvalue is zero, and the argument of section 4.3 fails. More concretely, the rapidity variables form pairs of equal magnitudes but opposite signs,

\[
\{x_j\}_{j=1,\ldots,M} = \{u_j - u_j\}_{j=1,\ldots,M/2},
\]  
(33)

which leads to a pole in the prefactor of (27). This pole cancels the zero in the determinant which lead to the vanishing of the overlap in section 4.3. To compute the overlap, we resolve the 0/0 ambiguity, due to the zero and the pole, by shifting the rapidity variables slightly away from their parity-invariant values, by defining

\[
x_{rj} = ru_j + \varepsilon, \quad r = \pm 1, \quad j = 1, \ldots, \frac{M}{2},
\]  
(34)

calculate the overlap for small but finite \(\varepsilon\), then take the limit \(\varepsilon \rightarrow 0\). Some of the matrix elements of \(B\) in (28) diverge as \(\varepsilon \rightarrow 0\), but the resulting matrix is degenerate and we need to consider also the subleading, \(O(1)\) term in the determinant. In the following, it is convenient to factor out the diagonal matrix \(U\), with elements

\[
U_{rj, sk} = \frac{r}{2} \delta_{rs} \delta_{jk} u_j^r,
\]  
(35)

where \(r, s = \pm 1\), and \(j, k = 1, \ldots, \frac{M}{2}\). Defining

\[
\hat{B}_{rj, sk} = \frac{\delta_{jk}}{2\varepsilon} (\delta_{rs} + \delta_{r,-s}) + 2ru_j^{\mp r} \frac{1 - \delta_{jk} \delta_{r,-s}}{(u_j^{\mp r})^2 - (u_k^{\pm s})^2}
\]  
(37)

where we have used the Bethe equations, but only after expanding in \(\varepsilon\). The singular part of the matrix \(\hat{B}\) is proportional to the \(\{1 + \sigma^3\}\)-projector, and has a zero determinant, which why we need to keep the next \(O(1)\) term. To leading order in \(\varepsilon\), the matrix
\(\hat{B}\) has \(M/2\) large eigenvalues, with eigenvectors proportional to \([1, 1]^t\) and \(M/2\) small eigenvalues, with eigenvectors proportional to \([1, -1]^t\). Denoting the projections on these subspaces as

\[
\hat{B}^{L,S}_{jk} = \frac{1}{2} \hat{B}_{jk,sk} (\delta^{rs} \pm \delta^{r,-s}),
\]

where the large and small components are now \([M/2 \times M/2]\)-matrices, and taking into account that \(\hat{B}^L = \mathcal{O}(1/\varepsilon)\) and \(\hat{B}^S = \mathcal{O}(1)\), we have,

\[
\det \hat{B} = \det \hat{B}^L \det \hat{B}^S.
\]

The large component is

\[
\hat{B}^L_{jk} = \frac{\delta_{jk}}{\varepsilon}.
\]

To write the small component in compact form, we introduce the following notations,

\[
K^\pm_{jk} = \frac{2}{(u_j - u_k)^2 + 1} \pm \frac{2}{(u_j + u_k)^2 + 1}
\]

and

\[
G^\pm_{jk} = K^\pm_{jk} + \delta_{jk} \left( \frac{2N}{u_j^2 + \frac{1}{4}} - \sum_l K^+_{jl} \right).
\]

The small component is

\[
\hat{B}^S = \pm \frac{1}{2} G^+.
\]

Hence,

\[
\det U = (-1)^M 2^{-M} \prod_j \left( \frac{1}{u_j^2 + \frac{1}{4}} \right), \quad \det \hat{B}^L = \varepsilon^{-\frac{M}{2}}, \quad \det \hat{B}^S = \left( \mp \frac{1}{2} \right)^{\frac{M}{2}}.
\]

The \(\varepsilon^{-M/2}\) singularity of \(\det \hat{B}^L\) cancels the zero in the denominator of (27), and from (16) and (27), we get,

\[
\langle \text{Néel}_M | \mathbf{u} \rangle = 2 \left( \frac{i}{2} \right)^M \prod_j \left( \frac{u_j^2 + \frac{1}{4}}{u_j} \right)^{2N+1} \prod_{j<k} \frac{((u_j - u_k)^2 + 1)((u_j + u_k)^2 + 1)}{(u_j^2 - u_k^2)^2} \det G^+.
\]

Following [3], the determinant expression for the Gaudin norm of a parity-invariant on-shell Bethe state [11, 22], can be factorized in the form

\[
\langle \mathbf{u} | \mathbf{u} \rangle = \prod_j \left( \frac{u_j^2 + \frac{1}{4}}{u_j} \right)^{4N+1} \prod_{j<k} \frac{((u_j - u_k)^2 + 1)^2((u_j + u_k)^2 + 1)^2}{(u_j^2 - u_k^2)^4} \det G^+ \det G^-.
\]
The normalized overlap is given by

\[ \frac{\langle \text{Néel}_M | u \rangle}{\langle u | u \rangle^{\frac{1}{2}}} = 2 \left( \frac{i}{2} \right)^{\frac{M}{2}} \left( \prod_j \frac{u_j^2 + \frac{1}{4}}{u_j^2} \frac{\det G^+}{\det G^-} \right)^{\frac{1}{2}}. \] (47)

This formula was obtained in [3] for the Néel state with \( M = L/2 \). The derivation for arbitrary \( M \) follows from a symmetry argument and can be found in [4, 5]. Here, we rederive it by inspecting the partition function of the six-vertex model with partially reflecting domain-wall boundary conditions.

5. Comments

In [9], Kuperberg lists eight classes of domain-wall-type boundary conditions and partition functions. These include the original boundary conditions and partition function of Korepin and Izergin [11, 12] as well as Tsuchiya’s [2]. It is clear that the remaining six classes admit partial versions in parallel with those discussed in [10] and in this note.

The overlap studied in this note was used in [8, 23] to compute a class of one-point functions in a four-dimensional conformally-invariant supersymmetric gauge theory, in the presence of a defect. The formulation of the latter problem in terms of the six-vertex model with particular boundary conditions, may be useful in the sense that the six-vertex boundary states may have a direct physical meaning in the gauge theory. The boundary state in (2.8) is a building block of the generalized Néel states, and consequently of the MPS, which naturally appears in the weak-coupling gauge-theory calculations [8]. Identifying a similar building block in the D-brane boundary state that is related, at strong-coupling, to the defect via the AdS/CFT correspondence, would help in finding a fully non-perturbative construction, valid at any coupling.

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Appendix. The \((M \times M)\) determinant representation

To derive (26) from (21), we introduce two \([N \times N]\) matrices,
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\[ N_{ba}^\pm = \prod_{\alpha} \left( \frac{y_\alpha^2 - (x_\alpha^\pm)^2}{y_\alpha^2 - y_\alpha^2} \right) \times \begin{cases} y_b^{2a-2}, & a = 1, \ldots, N - M \\ \frac{1}{y_b^2 - (x_j^\pm)^2}, & a = N - M + j, \quad j = 1, \ldots, M. \end{cases} \]  
(A.1)

These matrices have the structure similar to (22), and while \( N^\pm \) are not exactly inverse to \( M \), the product \( MN^\pm \) is a rather simple matrix with a trivial determinant, as we shall see in the moment. We denote the product of \( M \) and \( N^\pm \) by \( I^\pm \):

\[ I^\pm = MN^\pm. \]  
(A.2)

The indices of \( I^\pm_{ad} \) naturally decompose in two sets, \( a = 1, \ldots, N - M \) and \( a = N - M + j \) with \( j = 1, \ldots, M \), as in (22) and (A.1). The matrix \( I^\pm \) therefore consists of four blocks, \( I^\pm_{ad}, I^\pm_{ak}, I^\pm_{jd}, \) and \( I^\pm_{jk} \), where the indices take values \( a, d = 1, \ldots, N - M \) and \( j, k = 1, \ldots, M \). We use the shorthand notation for the \((N - M + j)\)th index of \( I^\pm \) by simply omitting \( N - M \) in the label. The key observation is that \( I^\pm_{ak} \) is zero. Indeed, from (22), (A.1),

\[ I^\pm_{ak} = \sum_b y_b^{2a-2} \frac{\prod_{\ell \neq k} (y_\ell^2 - (x_j^\pm)^2)}{\prod_{c \neq b} (y_c^2 - y_c^2)} = \oint \frac{dz}{2\pi i} z^{a-1} \frac{\prod_{\ell \neq k} (z - (x_j^\pm)^2)}{\prod_{c} (z - y_c^2)}, \]  
(A.3)

where the contour of integration encircles the set of points \( \{ y_c^2 \} \) counterclockwise. But the integrand has no singularities outside the contour of integration. In particular the pole infinity vanishes because the integral behaves as \( z^{N-M-2} \), and \( 2 + N - M - a \) is always bigger than one. Therefore, \( I^\pm_{ak} = 0 \) and consequently the matrix \( I^\pm \), in the block form, is lower triangular,

\[ I^\pm = \begin{pmatrix} I^\pm_{ad} & 0 \\ I^\pm_{jd} & I^\pm_{jk} \end{pmatrix}. \]  
(A.4)

The other, non-zero components of \( I^\pm \) can be computed by the same trick. For the sake of calculating the determinant of \( I^\pm \), we only need its block-diagonal components, for which we have,

\[ I^\pm_{jk} = \sum_b \frac{1}{y_b^2 - (x_j^\pm)^2} \frac{\prod_{\ell \neq k} (y_\ell^2 - (x_j^\pm)^2)}{\prod_{c \neq b} (y_c^2 - y_c^2)} \]

\[ = \oint \frac{dz}{2\pi i} (z - (x_j^\pm)^2) \frac{1}{(z - (x_j^\pm)^2)} \frac{\prod_{\ell \neq k} (z - (x_j^\pm)^2)}{\prod_{c} (z - y_c^2)} \]

\[ = \pm \frac{1}{2ix_j} \left( \frac{\prod_{\ell \neq k} ((x_j^\pm)^2 - (x_j^\pm)^2)}{\prod_{c} ((x_j^\pm)^2 - y_c^2)} - \delta_{jk} \frac{\prod_{\ell \neq j} ((x_j^\pm)^2 - (x_j^\pm)^2)}{\prod_{c} ((x_j^\pm)^2 - y_c^2)} \right). \]  
(A.5)
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where the last equality is obtained by inflating the contour of integration and computing the residues at \( z = (x_j^\pm)^2 \) and \( z = (x_k^\pm)^2 \). The latter residue vanishes unless \( j = k \). For the \( ad \) components, we get

\[
\mathcal{I}_{ad} = \sum_b y_b^{2a+2d-4} \frac{\prod_c (y_b^2 - (x_j^\pm)^2)}{\prod_c (y_b^2 - y_c^2)} = -\text{res}_{z=\infty} z^{a+d-2} \frac{\prod_c (z - (x_j^\pm)^2)}{\prod_c (z - y_c^2)}. \tag{A.6}
\]

The residue on the right-hand-side vanishes for \( a + d < N - M \) and equals one for \( a + d = N - M \). The \([ (N - M) \times (N - M) ]\) matrix with elements \( \mathcal{I}_{ad} \) therefore has a triangular form

\[
\mathcal{I}_{ad} = \begin{pmatrix} 0 & \cdots & 1 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots \end{pmatrix}, \quad \det_{ad} \mathcal{I}_{ad} = (-1)^{(N-M-1)(N-M)/2}. \tag{A.7}
\]

As a consequence of (A.2),

\[
\det \mathcal{M} = \frac{\det \mathcal{I}^\pm}{\det \mathcal{N}^\pm} = (-1)^{(N-M-1)(N-M)/2} \frac{\det_{jk} \mathcal{I}_{jk}^\pm}{\det \mathcal{N}^\pm}. \tag{A.8}
\]

The denominator in this formula is a generalized Cauchy determinant that can be explicitly calculated

\[
\det \mathcal{N}^\pm = (-1)^{(N+1)M} \frac{\prod_{j<k} ((x_j^\pm)^2 - (x_k^\pm)^2)}{\prod_{a<b} (y_a^2 - y_b^2)}. \tag{A.9}\]

Collecting the pieces we get,

\[
\det \mathcal{M} = (-1)^{\frac{N(N-1)}{2} + M} \frac{\prod_{a<b} (y_a^2 - y_b^2) \prod_{j<k} ((x_j^\pm)^2 - (x_k^\pm)^2)(x_j^\pm)^2 - (x_k^\pm)^2)}{\prod_{a<b} (x_j^\pm)^2 - (x_k^\pm)^2} \times \det_{jk} \left\{ \frac{1}{(x_j^\pm)^2 - (x_k^\pm)^2} + \frac{i\delta_{jk}}{2x_j} \prod_a (x_j^\pm)^2 - y_a^2 \prod_{l<j} (x_j^\pm)^2 - (x_l^\pm)^2 \right\}. \tag{A.10}
\]

Equation (26) then follows from (21).

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