Second Octant Favored for Non-Maximal $\theta_{23}$

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Abstract

One of the most robust relationships predicted by binary tetrahedral ($T'$) flavor symmetry relates the reactor neutrino angle $\theta_{13}$ to the atmospheric neutrino angle $\theta_{23}$, independently of $\theta_{12}$. It has the form $\theta_{13} = \sqrt{2} \left| \frac{\pi}{4} - \theta_{23} \right|$. When this prediction first appeared in 2008, $\theta_{13}$ was consistent with zero and $\theta_{23}$ with $\pi/4$. Non-zero $\theta_{13}$ was established by Daya Bay in 2012. Non-zero $\left| \frac{\pi}{4} - \theta_{23} \right|$ is now favored by the NO$\nu$A experiment and, for $\theta_{23}$, the aforementioned $T'$ relation selects the second octant ($\theta_{23} > \pi/4$) over the first octant ($\theta_{23} < \pi/4$). This analysis initially assumes CP conservation in the lepton sector, but leptonic CP violation is discussed and it is shown that this specific $T'$ relationship is invariant.
1 Introduction

Among the 28 free parameters associated with the standard model of particle theory, the six mixing angles $\Theta_{ij}$ and $\theta_{ij}$ ($1 \leq i < j \leq 3$) associated respectively with the quarks $[1, 2]$ and the neutrinos $[3, 4]$ are the nearest to being calculable in a plausible theory. We have in mind the use of the binary tetrahedral group $T'$ as a family symmetry following the discussion in [5–9].

In the present article we shall invoke this family symmetry involving the binary tetrahedral group $T'$ which, unlike smaller discrete groups such as the unadorned tetrahedral group $T$, also known as $A_4$, has sufficient structure to accommodate and relate both quark and lepton mixing angles. In a previous study [7], a quite successful exact formula for the Cabibbo angle $\Theta_{12}$ was derived in a $(T' \times Z_2)$ model arriving at a Cabibbo angle value at the lowest order given by

$$\tan 2(\Theta_{12})_{T'} = \left(\frac{1}{3}(\sqrt{2})\right).$$

The three neutrino mixing angles $\theta_{12}, \theta_{23}, \theta_{13}$ have also been studied assiduously in a $T'$ context and the most robust prediction [8] is that

$$\theta_{13} = \sqrt{2} \left| \frac{\pi}{4} - \theta_{23} \right|$$

which interestingly links the non-zero value for $\theta_{13}$ to the departure of the atmospheric neutrino mixing angle $\theta_{23}$ from maximal mixing with $\theta_{23} = \pi/4$. This is the most definite such prediction from $T'$, independent of any further phenomenological input [2].

Subsequent to the first appearance [8] of Eq.(2), a further analysis in [9] found consistency of Eq.(2) with experiment, although at that time the RHS of Eq.(2) was empirically indistinguishable from zero so the comparison between theory and experiment was incomplete. In the present article, we study further the implications of Eq.(2).

Recent more accurate data from NOvA [12], Daya Bay [13] and Double Chooz [14] - especially the preference for nonmaximal $\theta_{23}$ from NOvA data - make possible this more detailed check.

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#2 A similar prediction from a different starting point appeared en passant in [10]. Eq.(2) was later derived in [11]
2 Experimental Data

The most suggestive data on nonmaximal $\theta_{23}$ comes from the measurement [12] of $\nu_{\mu}$ disappearance at NOvA as observed by the near and far detectors. This experiment finds a new value for the squared mass difference $\Delta m_{23}^2$ which is

$$\Delta m_{23}^2 = (2.52_{-0.18}^{+0.20}) \times 10^{-3} eV^2$$

(3)

The statistically-favored best fit of NOvA for a normal hierarchy (NH) give

$$\sin^2 \theta_{23} = 0.43 \text{ or } 0.60 \text{ (NH)}$$

(4)

while for an inverted hierarchy (IH) the NOvA best fits are

$$\sin^2 \theta_{23} = 0.44 \text{ or } 0.59 \text{ (IH)}$$

(5)

The first and second solutions in Eqs.(4) and (5) correspond to the first octant ($\theta_{23} < \pi/4$) and the second octant ($\theta_{23} > \pi/4$) respectively. In this note we attempt to discriminate between these two octants for $\theta_{23}$ by using the $T'$ relation, Eq.(2) above.

To carry this out, we need the best data on $\theta_{13}$ which are from two experiments [13,14]. From neutron capture on Hydrogen $H$ the result found at the Daya Bay reactor in Guangdong Province, China is

$$\sin^2 2\theta_{13} = 0.071 \pm 0.010 \text{ (nH)}$$

(6)

The best value found using the Double Chooz reactor experiment in Chooz, France from neutron capture on Gadolinium $Gd$ is

$$\sin^2 2\theta_{13} = 0.088 \pm 0.033 \text{ (nGd)}$$

(7)

When we convert Eqs.(6) and (7) to degrees for comparison with the non-maximality of $\theta_{23}$, we find for the central values in a form convenient to check Eq.(2)

$$\left(\frac{\theta_{13}}{\sqrt{2}}\right)_{nH} = 5.46^0$$

(8)

and

$$\left(\frac{\theta_{13}}{\sqrt{2}}\right)_{nGd} = 6.10^0$$

(9)

respectively.
Table 1: Values of $|\pi/4 - \theta_{23}|$ corresponding to the NOvA data.

| Hierarchy | $\sin^2 \theta_{23}$ | $\sin \theta_{23}$ | $\theta_{23}$ degrees | $|\pi/4 - \theta_{23}|$ degrees |
|-----------|----------------------|---------------------|-----------------------|-------------------------|
| NH        | 0.43                 | 0.66                | 40.97                 | 4.03                    |
| NH        | 0.60                 | 0.77                | 50.77                 | 5.77                    |
| IH        | 0.44                 | 0.66                | 41.55                 | 3.45                    |
| IH        | 0.59                 | 0.77                | 50.18                 | 5.18                    |

These results are compared to the nonmaximality of $\theta_{23}$ cited earlier in Eqs. (4) and (5) within the following table.

When we compare the values of $\theta_{13}/\sqrt{2}$ given in Eqs. (8) and (9) we note that only the second octant solution ($\theta_{23} > \pi/4$) is consistent. The first octant solution ($\theta_{23} < \pi/4$) is not. #3

#3Note that in [9] only the first octant for $\theta_{23}$ was considered because at that time the data were insufficiently accurate to discriminate between octants.
3 CP Violation

The above analysis assumed that no CP violation is associated with the neutrino mixing. The experimental situation about this is in a state of flux \[12,15–17\], but it is not premature to discuss how or whether non-vanishing \(\delta_{CP}\) could affect the \(T'\) relation, Eq.(2). The experimental papers \[12,15–17\] are consistent at 3\(\sigma\) with CP conservation, \(\delta_{CP} = 0\), but there is a hint from the data suggestive of a phase \(\delta_{CP} \simeq 3\pi/2\) at a level between 1\(\sigma\) and 2\(\sigma\) so let us re-do the \(T'\) analysis assuming general \(\delta_{CP} \neq 0\).

This requires us to re-examine the original derivation of Eq.(2) in the EFM paper \[8\]. We retain the convenient notation

\[
\theta_{ij} \equiv (\theta_{ij})_{TBM} + \epsilon_k
\]

(10)

where

\[
(\theta_{12})_{TBM} = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad (\theta_{23})_{TBM} = \left(\frac{\pi}{4}\right) \quad (\theta_{13})_{TBM} = 0
\]

(11)

are the Tri-Bi-Maximal (TBM) values. Our aim is to relate \(\epsilon_1\) and \(\epsilon_2\) by perturbation around the \(T'\) value for the Cabibbo angle \(\Theta_{12}\)

\[
\tan [2(\Theta_{12})_{T'}] = \left(\frac{\sqrt{2}}{3}\right)
\]

(12)

The PMNS mixing matrix, including non-zero \(\delta_{CP}\), is e.g \[18\]

\[
\begin{pmatrix}
-s_{12}s_{23} - c_{12}c_{23}s_{13} \exp(i\delta_{CP}) & -s_{12}c_{23} + c_{12}s_{23}s_{13} \exp(i\delta_{CP}) & c_{12}c_{13} \\
-c_{12}s_{23} + s_{12}c_{23}s_{13} \exp(i\delta_{CP}) & c_{12}c_{23} + s_{12}s_{23}s_{13} \exp(i\delta_{CP}) & s_{12}c_{13} \\
c_{23}c_{13} & -s_{23}c_{13} & s_{13} \exp(-i\delta_{CP})
\end{pmatrix}
\]

(13)

Substituting Eq.(11) into Eq.(13) gives, for any value of \(\delta_{CP}\),

\[
U_{PMNS}(TBM, \delta_{CP}) = \begin{pmatrix}
-\sqrt{\frac{1}{6}} - \sqrt{\frac{1}{6}} + \sqrt{\frac{2}{3}} \\
+ \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}} \\
+ \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{2}} 0
\end{pmatrix}
\]

(14)
Likewise, inserting Eq. (11) into Eq. (10) gives, independently of \( \delta_{CP} \) and assuming only that \( |\epsilon_i| << 1 \),

\[
\begin{align*}
    s_{12} &= \sqrt{\frac{1}{3}}(1 + \sqrt{2}\epsilon_3) \quad c_{12} = \sqrt{\frac{2}{3}}(1 - \epsilon_3/\sqrt{2}) \\
    s_{23} &= \sqrt{\frac{1}{2}}(1 + \epsilon_1) \quad c_{23} = \sqrt{\frac{1}{2}}(1 - \epsilon_1) \\
    s_{13} &= \epsilon_2 \quad c_{13} = 1
\end{align*}
\]  

(15)  

(16)  

(17)  

Using the small \( \epsilon_i \) approximation we may write

\[
U = U_{TBM} + \delta U_1 \epsilon_1 + \delta U_2 \epsilon_2 + \delta U_3 \epsilon_3
\]

(18)  

in which

\[
\delta U_1 = \begin{pmatrix}
    -\sqrt{\frac{1}{3}} + \sqrt{\frac{2}{3}} & 0 \\
    -\sqrt{\frac{1}{6}} + \sqrt{\frac{5}{6}} & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

(19)  

\[
\delta U_2 = \begin{pmatrix}
    -\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}} & 0 \\
    -\sqrt{\frac{1}{6}} + \sqrt{\frac{5}{6}} & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

(20)  

\[
\delta U_3 = \begin{pmatrix}
    -\sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} & 0 \\
    -\sqrt{\frac{1}{6}} - \sqrt{\frac{5}{6}} + \sqrt{\frac{3}{2}} & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

(21)  

For TBM mixing, before CP violation associated with non-vanishing \( \theta_{13} \), one has (with everything still real):

\[
(M_\nu)_{TBM} = U_{TBM}^T (M_\nu)_{diag} U_{TBM}
\]

(22)  

where

\[
(M_\nu)_{diag} = \begin{pmatrix}
    m_1 & 0 & 0 \\
    0 & m_2 & 0 \\
    0 & 0 & m_3
\end{pmatrix}
\]

(23)
This, in turn, yields a formula for \((M_\nu)_{TBM}\). Defining \(m_{12} = (m_1 - m_2)\) the symmetric matrix is

\[
(M_\nu) = \left( \frac{1}{6} \right) \begin{pmatrix}
    m_1 + m_2 + 3m_3 & m_1 + 2m_2 - 3m_3 & -2m_{12} \\
    m_1 + 2m_2 + 3m_3 & -2m_{12} & 4m_1 + 2m_2 \\
    4m_1 + 2m_2 & -2m_{12} & m_1 + m_2 + 3m_3
\end{pmatrix}
\]  

\[(24)\]

It follows from Eq.\((22)\) that the perturbation is given by (also using the notations of Eqs.\((19)\) to \((21)\))

\[
\delta(M_\nu)_{TBM} = \begin{pmatrix}
    \delta m_1 & 0 & 0 \\
    0 & \delta m_2 & 0 \\
    0 & 0 & \delta m_3
\end{pmatrix}
\]  

\[(25)\]

\[
\delta(M_\nu)_{TBM} = \delta U (M_\nu)_{TBM} U_T^{TBM} + U_{TBM} \delta M_\nu U_T^{TBM} + U_{TBM} (M_\nu)_{TBM} \delta U
\]  

\[(26)\]

At this stage, we must ensure that the \(T'\) calculation is suitably generalized to include CP-violating Yukawa couplings, meaning that the \(Y_i\) in the leptonic lagrangian

\[
\mathcal{L}_Y = \frac{1}{2} M_1 N_R^{(1)} N_R^{(1)} + M_{23} N_R^{(2)} N_R^{(3)}
\]

\[
+ Y_1 \left( L_L N_R^{(1)} H_3 \right)
\]

\[
+ Y_2 \left( L_L N_R^{(2)} H_3 \right)
\]

\[
+ Y_3 \left( L_L N_R^{(3)} H_3 \right)
\]  

\[(27)\]

are now complex, unlike in earlier \(T'\) discussion. The Majorana masses of the right-handed neutrinos are real.

Just to recall that, under the group \((T' \times Z_2)\) the Higgs doublet \(H_3\) is a \((3, +1)\). We shall not need the charged lepton mass terms which have been omitted in Eq.\((27)\). The VEV is \(< H_3 > = (V_1, V_2, V_3)\), with real \(V_i\) because CP is preserved by the vacuum.

The right-handed neutrinos \(N_R\) have a real mass matrix

\[
M_N = \begin{pmatrix}
    M_1 & 0 & 0 \\
    0 & 0 & M_{23} \\
    0 & M_{23} & 0
\end{pmatrix}
\]  

\[(28)\]
while the Dirac mass matrix is complex

\[ M_D = \begin{pmatrix} Y_1 V_1 & Y_2 V_3 & Y_3 V_2 \\ Y_1 V_3 & Y_2 V_2 & Y_3 V_1 \\ Y_1 V_2 & Y_2 V_1 & Y_3 V_3 \end{pmatrix} \]  

(29)

because of the \( Y_i \). Next we implement the see-saw mechanism \[19\]

\[ M_\nu = M_D M_N^{-1} M_D^\dagger \]  

(30)

and define auxiliary variables \( x_1 \equiv |Y_1|^2 / M_1 \) (real) and \( x_{23} \equiv Y_2^* Y_3 / M_{23} \) (complex) so that \( M_\nu \) becomes the necessarily symmetric matrix

\[
\begin{pmatrix}
  x_1 V_1^2 + 2x_{23} V_2 V_3 & x_1 V_1 V_3 + x_{23} (V_1^2 + V_1 V_3) & x_1 V_1 V_2 + x_{23} (V_2^2 + V_1 V_2) \\
  x_1 V_3^2 + x_{23} V_1 V_2 & x_1 V_2 V_3 + x_{23} (V_2^2 + V_2 V_3) & x_1 V_2^2 + 2x_{23} V_1 V_3
\end{pmatrix}
\]  

(31)

The next step is to compare the matrix (31) with the complex texture

\[ \delta M_{nu} = V_1' x_1 \begin{pmatrix} 2(-2a + b)y & a + (a - 4b)y & b + (2a + b)y \\ 2(a + by) & (-2a + b)(1 + y) & -4b + 2ay \end{pmatrix} \]  

(32)

which is obtained by using the perturbed VEV

\[ < H_3 > = V_1' (1, -2 + b, 1 + a) \]  

(33)

with \( a, b << 1 \) both real. The key observation is that only the (complex) parameter \( y = x_{23} / x_1 \) survives, despite the fact that the Yukawa couplings and the right-handed neutrino masses are all empirically unknown. It is now straightforward but tedious algebra to eliminate the unknown parameter \( y \) from Eq.(26) and Eq.(32) to obtain six independent equations, only one of which is relevant to our present discussion. It is

\[ \epsilon_2 = \sqrt{2} |\epsilon_1| \]  

(34)

so that, perhaps surprisingly, for arbitrary \( \delta_{CP} \) the \( T' \) formula,

\[ \theta_{13} = \sqrt{2} |\pi/4 - \theta_{23}| \]  

(35)

stated near the beginning as Eq.(2), first derived for \( \delta_{CP} = 0 \) in \[8\], remains invariant even in the presence CP violation with \( \delta_{CP} \neq 0 \).
4 Discussion

At first non-trivial order the prediction of $T'$ flavor symmetry for the PMNS angles $\theta_{ij}$ are

$$\tan^2 \theta_{12} = \left( \frac{1}{\sqrt{2}} \right); \quad \text{and} \quad \theta_{13} = \sqrt{2} \left| \frac{\pi}{4} - \theta_{23} \right|$$

and the $\theta_{12}$ prediction is consistent within less than $2\sigma$ of the latest data \cite{15} which averages to $\sin^2 \theta_{12} = 0.304 \pm 0.014$. The second formula in Eq.(36), for $\theta_{13}$, remains invariant in the presence of CP violation.

Our main result here is that combining the latest data on $\theta_{23}$ and $\theta_{13}$ with the $T'$ relation, Eq.(2), favors that the atmospheric neutrino angle $\theta_{23}$ lies in the second rather than the first octant, a distinction which cannot be made from the experimental data alone.

It will be very interesting to discover how future more precise measurements of the neutrino mixing angles remain consistent with these predictions.

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