An 4n-point Interpolation Formula for Certain Polynomials
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Abstract By using some techniques of the divided difference operators, we establish an 4n-point interpolation formula. Certain polynomials, such as Jackson’s \(8\phi_7\) terminating summation formula, are special cases of this formula. Based on Krattenthaler’s identity, we also give Jackson’s formula a determinantal interpretation.

1 Introduction

Recall that the \(i\)-th divided difference operator \(\partial_i\), acting on functions \(f(x_1, \ldots, x_n)\) of several variables, is defined by

\[
f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) \partial_i = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{(x_i - x_{i+1})}.
\]

To be more general, we introduce the operator \(c\partial_i\) which we call the \(i\)-th \(c\)-divided difference operator:

\[
f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) c\partial_i = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{(x_i - x_{i+1})(1 - c/x_i x_{i+1})}.
\]

(1.1)

Note that \(0\partial_i = \partial_i\).

Several properties of the \(c\)-divided difference operators will be given in next section, see Lemmas 2.1 to 2.5. Employing those lemmas, we obtain the main result of this paper.

Theorem 1.1 Given two sets of variables

\[
A = \{a_1, c/a_1, \ldots, a_n, c/a_n\}, \quad B = \{b_1, c/b_1, \ldots, b_n, c/b_n\},
\]

we have the following 4n-point interpolation formula for certain polynomials \(f(y)\) of degree 2n with symmetry \(y^{-n}f(y) = (c/y)^{-n}f(c/y)\) when \(c \neq 0\):

\[
f(y) = \frac{f(b_1)}{\prod_{i=1}^{n} (b_1 - a_i)(b_1 - c/a_i)} \prod_{i=1}^{n} (y - a_i)(y - c/a_i)
+ \frac{f(a_1)}{\prod_{i=1}^{n} (a_1 - b_i)(a_1 - c/b_i)} \prod_{i=1}^{n} (y - b_i)(y - c/b_i)
+ \sum_{j=1}^{n-1} C_j \cdot \prod_{i=1}^{j} (y - b_i)(y - c/b_i) \prod_{i=1}^{n-j} (y - a_i)(y - c/a_i),
\]

(1.2)
where
\[
C_j = \frac{f(b_1)b_1^{-j}}{\prod_{i=1}^{n-j+1}(b_1-a_i)(b_1-c/a_i)}e_1 \cdots e_j(b_{j+1} - a_{n-j+1})(1 - c/a_{n-j+1}b_{j+1}).
\]

Note that Theorem 1.1 leads to the following 2n-point interpolation formula given in [1] if \(c = 0\):
\[
f(y) = \frac{f(b_1)}{\prod_{i=1}^{n}(b_1-a_i)}\prod_{i=1}^{n}(y-a_i) + \frac{f(a_1)}{\prod_{i=1}^{n}(a_1-b_i)}\prod_{i=1}^{n}(y-b_i)
+ \sum_{j=1}^{n-1} \frac{f(b_1)}{\prod_{i=1}^{n-j+1}(b_1-a_i)}\partial_1 \cdots \partial_j(b_{j+1} - a_{n-j+1}) \cdot \prod_{i=1}^{j}(y-b_i) \prod_{i=1}^{n-j}(y-a_i).
\]

The symmetry \(y^{-n}f(y) = (c/y)^{-n}f(c/y)\) when \(c \neq 0\) implies that \(f(y)\) can be written as a product \(\prod_{i=1}^{n}(y-x_i)(c-x_iy)\). Considering the case
\[A = \{a, c/a, \ldots, aq^{1-n}, cq^{n-1}/a\}, \quad B = \{b, c/b, \ldots, bq^{1-n}, cq^{n-1}/b\}\]
and \(y = x_{n+1}\), one can check that Theorem 1.1 implies the following identity by expanding the determinant with respect to the last row:
\[
det \left( P_{n-j+1}(x_i, aq^{j-n})P_{n-j+1}(x_i, c/a)P_{n-j+1}(x_i, bq^{1-n})P_{n-j+1}(x_i, cq^{n-j+1}/b) \right)_{i,j=1}^{n+1}
= \prod_{1 \leq i < j \leq n+1} (x_i - x_j)(c - x_i x_j) b(1/2)^{n+1} q^{-(n+1)n(n-1)/3} \prod_{i=1}^{n+1} (a/b, cq^{2n+2-2i}/ab; q)_{i-1}, \quad (1.3)
\]
where \((a; q)_n\) is the q-shifted factorial defined by
\[(a; q)_0 = 1, \quad (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad n = 1, 2, \ldots\]
and we use \(P_n(a, b)\) called Cauchy polynomials [2] to denote \(a^n(b/a; q)_n\) for convenience.

After some rearrangement of (1.3), we may get Krattenthaler’s identity [5]:
\[
det \left( \frac{(ax_i, ac/x_i; q)_{n-j}}{(bx_i, bc/x_i; q)_{n-j}} \right)_{i, j=1}^{n}
= \prod_{1 \leq i < j \leq n} (x_j - x_i)(1 - c/x_i x_j) a^{(n)}(q^{n/2}) \prod_{i=1}^{n} \frac{(b/a, abcq^{2n-2i}; q)_{i-1}}{(bx_i, bc/x_i; q)_{n-1}}.
\]

Through the specializations \(f(y) = \prod_{i=1}^{n}(uy^{i-1} - y)\) and
\[A = \{a, c/a, \ldots, aq^{1-n}, cq^{n-1}/a\}, \quad B = \{b, c/b, \ldots, bq^{1-n}, cq^{n-1}/b\}\]
in Theorem 1.1 we have the following variation of Jackson’s \(\phi_7\) terminating formula.
Corollary 1.2  We have

\[
\prod_{i=1}^{n} (uq^{i-1} - y)(c - uq^{i-1}y)
= q^{\binom{n}{2}} \sum_{k=0}^{n} \binom{n}{k} \frac{P_{n-k}(b, uq^{n-1})P_k(a, uq^{n-k})P_{n-k}(u, c/b)P_k(c/a, u)}{P_n(b, a)(cq^{n-k-1}/ab; q)_{n-k}(cq^{2n-2k}/ab; q)_k} \\
\times P_{n-k}(y, aq^{k+1-n})P_{n-k}(y, c/a)P_k(y, bq^{1-n})P_k(y, cq^{n-k}/b),
\]

where \( \binom{n}{k} \) is the \( q \)-binomial coefficient defined by

\[
\binom{n}{k} = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.
\]

In section 3, we shall give Corollary 1.2 a different approach by considering a special case of (1.3). The key step of our approach is to evaluate the cofactor of each entry in the last row of the determinant in (1.3).

Note that if we write \( cq^{-1}/ab \) as \( a, y/a \) as \( b, c/au \) as \( c, c/ay \) as \( d \) and \( uq^{n-1}/b \) as \( e \) in the above corollary, we get Jackson’s \( 8 \phi_7 \) formula [3]:

\[
\frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n} = \sum_{k=0}^{n} \frac{(1 - aq^{2k})(a; q)_k(b; q)_k(c; q)_k(d; q)_k(e; q)_k(q^{-n}; q)_kq^{k}}{(1 - a)(q; q)_k(aq/b; q)_k(aq/c; q)_k(aq/d; q)_k(aq/e; q)_k(aq^{n+1}; q)_k}.
\]

2  The 4\( n \)-point interpolation formula

In this section, we shall focus our attention on the proof of Theorem 1.1. To this aim, we shall first introduce several elementary properties of the \( c \)-divided difference operators.

Lemma 2.1  \( c \)-divided difference operators satisfy the following Leibnitz rules:

\[
f(x_1)g(x_1)c\partial_1 = f(x_1)g(x_1)c\partial_1 + f(x_1)c\partial_1g(x_2)
\]

\[
f(x_1)g(x_1)c\partial_1 \cdots c\partial_n = \sum_{k=0}^{n} f(x_1)c\partial_1 \cdots c\partial_kg(x_{k+1})c\partial_{k+1} \cdots c\partial_n.
\]

Lemma 2.2  If \( f(x_1, x_2, \ldots, x_n) \) is a symmetric function of \( x_i, x_{i+1} \), then

\[
f(x_1, x_2, \ldots, x_n)c\partial_i = 0.
\]
Lemma 2.3 If $p_n(y)$ is a polynomial of degree $2n$ such that
\[ y^{-n}p_n(y) = (c/y)^{-n}p_n(c/y), \]
then we have
\[ y^{-n}p_n(y_1) \partial_1 \cdots \partial_m = \begin{cases} 0, & m > n, \\ 1, & m = n. \end{cases} \]  
(2.1)

Proof. Clearly, $y^{-n}p_n(y_1)$ can be rewritten as $\prod_{i=1}^n (y_1 - x_i)(1 - c/y_1 x_i)$. When $m = 1$ and $n = 0$ or $n = 1$, it is easy to verify that (2.1) holds. In view of Lemma 2.1, we can prove Lemma 2.3 by induction on the length of the operators.

Lemma 2.4 We have
\[ \prod_{k=1}^j (y_1 - b_k)(1 - c/y_1 b_k) \partial_1 \cdots \partial_i \bigg|_{y_k = b_k, 1 \leq k \leq i+1} = \begin{cases} 0, & j \neq i; \\ 1, & j = i. \end{cases} \]  
(2.2)

Proof. For $j \leq i$, Lemma 2.4 is a direct consequence of Lemma 2.3. For $j > i$, we have
\[ \prod_{k=1}^j (y_1 - b_k)(1 - c/y_1 b_k) \partial_1 \cdots \partial_i \bigg|_{y_k = b_k, 1 \leq k \leq i+1} = \prod_{k=2}^j (y_2 - b_k)(1 - c/y_2 b_k) \partial_2 \cdots \partial_i \bigg|_{y_k = b_k, 2 \leq k \leq i+1} = \cdots = \prod_{k=i+1}^j (y_{i+1} - b_k)(1 - c/y_{i+1} b_k) \bigg|_{y_k = b_k, i+1 \leq k \leq i+1} = 0. \]

We complete the proof.

Lemma 2.5 We have
\[ \prod_{k=1}^j \left( \frac{y_1 - b_k}{y_1 - a_k}(1 - c/y_1 b_k) \prod_{k=1}^{i-1} (y_1 - a_k)(1 - c/y_1 a_k) \partial_1 \cdots \partial_i \bigg|_{y_k = b_k, 1 \leq k \leq i+1} \right) = \begin{cases} 0, & j \neq i; \\ \frac{1}{(b_{j+1} - a_j)(1 - c/a_j b_{j+1})}, & j = i. \end{cases} \]  
(2.3)

Proof. For $j < i$, we have
\[ \prod_{k=1}^j \left( \frac{y_1 - b_k}{y_1 - a_k}(1 - c/y_1 b_k) \prod_{k=j+1}^{i-1} (y_1 - a_k)(1 - c/y_1 a_k) \partial_1 \cdots \partial_i \bigg|_{y_k = b_k, 1 \leq k \leq i+1} \right) = \sum_{l=0}^{i-j} \prod_{k=1}^j \left( \frac{y_1 - b_k}{y_1 - a_k}(1 - c/y_1 b_k) \partial_1 \cdots \partial_i \bigg|_{y_k = b_k, 1 \leq k \leq l+1} \right) \times \prod_{k=l+1}^{i-1} (y_{l+1} - a_k)(1 - c/y_{l+1} a_k) \partial_{l+1} \cdots \partial_i \bigg|_{y_k = b_k, l+1 \leq k \leq i+1}. \]
From Lemma 2.3 and Lemma 2.4 either the first product inside the sum or the second one vanishes, so does the sum.

For \( j \geq i \), we have
\[
\frac{\prod_{k=i}^{j}(y_1 - b_k)(1 - c/y_1 b_k)}{\prod_{k=i}^{j}(y_1 - a_k)(1 - c/y_1 a_k)} e_{\partial_1} \cdots e_{\partial_i} \bigg|_{y_k = b_k, 1 \leq k \leq i+1} 
= (y_1 - b_1)(1 - c/y_1 b_1) e_{\partial_1} \bigg|_{y_1 = b_1} \frac{\prod_{k=i}^{j}(y_2 - b_k)(1 - c/y_2 b_k)}{\prod_{k=i}^{j}(y_2 - a_k)(1 - c/y_2 a_k)} e_{\partial_2} \cdots e_{\partial_i} \bigg|_{y_k = b_k, 2 \leq k \leq i+1}
= \frac{\prod_{k=i}^{j}(y_2 - b_k)(1 - c/y_2 b_k)}{\prod_{k=i}^{j}(y_2 - a_k)(1 - c/y_2 a_k)} e_{\partial_2} \cdots e_{\partial_i} \bigg|_{y_k = b_k, 2 \leq k \leq i+1}
= \cdots = \frac{\prod_{k=i}^{j}(y_i - b_k)(1 - c/y_i b_k)}{\prod_{k=i}^{j}(y_i - a_k)(1 - c/y_i a_k)} e_{\partial_i} \bigg|_{y_k = b_k, i \leq k \leq i+1}
= \begin{cases} 
0, & j > i, \\
1/(b_{j+1} - a_j)(1 - c/a_j b_{j+1}), & j = i.
\end{cases}
\]

We complete the proof. \( \blacksquare \)

**Proof of Theorem 1.1**

Given a polynomial \( f(y) \) of degree \( 2n \) with symmetry \( y^{-n} f(y) = (c/y)^{-n} f(c/y) \), we assume that
\[
f(y) = \sum_{j=0}^{n} C_j \prod_{k=1}^{j} (y - b_k)(y - c/b_k) \prod_{k=j+1}^{n} (y - a_k)(y - c/a_k). \tag{2.4}
\]

Taking \( y = b_1 \) in (2.4), one has
\[
f(b_1) = C_0 \prod_{k=1}^{n} (b_1 - a_k)(b_1 - c/a_k).
\]

Therefore,
\[
C_0 = \frac{f(b_1)}{\prod_{k=1}^{n} (b_1 - a_k)(b_1 - c/a_k)}.
\]

Setting \( y = a_n \) in (2.4) leads to
\[
C_n = \frac{f(a_n)}{\prod_{k=1}^{n} (a_n - b_k)(a_n - c/b_k)}.
\]

Let \( g(y) = f(y)/ \prod_{k=1}^{n} (y - a_k)(y - c/a_k) \). Rewrite (2.4) as
\[
g(y) = g(b_1) + \sum_{j=1}^{n-1} C_j \frac{\prod_{k=1}^{j} (y - b_k)(1 - c/y b_k)}{\prod_{k=1}^{j} (y - a_k)(1 - c/y a_k)}
+ \frac{f(a_n)}{\prod_{i=1}^{n} (a_n - b_i)(a_n - c/b_i)} \frac{\prod_{i=1}^{n} (y - b_k)(1 - c/y b_k)}{\prod_{i=1}^{n} (y - a_k)(1 - c/y a_k)}.
\]
Multiplying both sides by \( \prod_{k=1}^{i-1} (y - a_k) (1 - c/ya_k) \), then applying the operator \( c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \), one has

\[
g(y_1) \prod_{k=1}^{i-1} (y_1 - a_k)(1 - c/y_1a_k) c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \bigg|_{y_k = b_k, 1 \leq k \leq i+1} \\
= \sum_{j=1}^{i-1} C_j \prod_{k=1}^{j} (y_1 - b_k)(1 - c/y_1b_k) \prod_{k=j+1}^{i-1} (y_1 - a_k)(1 - c/y_1a_k) c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \bigg|_{y_k = b_k, 1 \leq k \leq i+1} \\
+ \sum_{j=i}^{n-1} C_j \prod_{k=j+1}^{j+n-1} (y_1 - b_k)(1 - c/y_1b_k) c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \bigg|_{y_k = b_k, 1 \leq k \leq i+1}.
\]

By Lemma 2.5, we have

\[
g(y_1) \prod_{k=1}^{i-1} (y_1 - a_k)(1 - c/y_1a_k) c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \bigg|_{y_k = b_k, 1 \leq k \leq i+1} = \frac{C_i}{(b_i+1 - a_i)(1 - c/b_{i+1}a_i)}.
\]

Thus

\[
C_i = g(y_1) \prod_{k=1}^{i-1} (y_1 - a_k)(1 - c/y_1a_k) c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} \bigg|_{y_k = b_k, 1 \leq k \leq i+1} (b_i+1 - a_i)(1 - c/b_{i+1}a_i)
\]

\[
= \frac{f(b_1)b_1^{i-1}}{\prod_{k=1}^{n} (b_1 - a_k)(b_1 - c/ak)} c \frac{\partial}{\partial x} \cdots \frac{\partial}{\partial x} (b_{i+1} - a_i)(1 - c/b_{i+1}a_i).
\]

Replacing \( a_i \) by \( a_{n-i+1} \), we complete the proof.

3 Jackson’s \( \phi_7 \) terminating summation formula

Letting \( x_i = uq^{i-1} \) for \( 1 \leq i \leq n \) and \( x_{n+1} = y \), we shall show that (3.3) in this case is equivalent to Corollary 1.2. In other words, we shall give Jackson’s \( \phi_7 \) terminating summation formula a determinantal interpretation.

Our proofs in this section involve the following well-known symmetric functions. Given two sets of variables \( X \) and \( Y \), the \( i \)-th supersymmetric complete function \( h_i(X - Y) \) is defined by

\[
h_i(X - Y) = \left[ t^i \right] \frac{\prod_{y \in Y} (1 - yt)}{\prod_{x \in X} (1 - xt)} = \sum_{i=0}^{n} (-1)^i e_i(Y) h_{n-i}(X), \tag{3.1}
\]

where \( \left[ t^i \right] f(t) \) means the coefficient of \( t^i \) in \( f(t) \), \( e_i(X) \) and \( h_i(Y) \) are \( i \)-th elementary symmetric function and \( i \)-th complete symmetric function, respectively.
Expanding the determinant of (1.3) along the last row in the case \( x_i = uq^{i-1} \) for \( 1 \leq i \leq n \) and \( x_{n+1} = y \), we have

\[
\prod_{i=1}^{n} (uq^{i-1} - y)(c - uq^{i-1}y) b^{(n+1)} q^{-(n+1)n(n-1)/3}
\times \prod_{1 \leq i < j \leq n} (uq^{i-1} - uq^{j-1})(c - u^2q^{i+j-2}) \prod_{i=1}^{n+1} (a/b, cq^{2n+2i}/ab; q)_{i-1}
= \sum_{k=1}^{n+1} C_{n,k} P_{n-k+1}(y, aq^{k-n}) P_{n-k+1}(y, c/a) P_{k-1}(y, bq^{1-n}) P_{k-1}(y, cq^{n-k+1}/b),
\]

(3.2)

where \( C_{n,k} \) is the cofactor of the entry

\[
P_{n-k+1}(y, aq^{k-n}) P_{n-k+1}(y, c/a) P_{k-1}(y, bq^{1-n}) P_{k-1}(y, cq^{n-k+1}/b).
\]

It is easy to verify that \( C_{n,k} \) can be rewritten in terms of the supersymmetric complete functions:

\[
C_{n,k} = \prod_{1 \leq i < j \leq n} (uq^{j-1} - uq^{i-1}) \det(h_{2n-i+1}(U - Y_{j,k})),
\]

(3.3)

where the set

\[
Y_{j,k} = \begin{cases} 
\{a, c/a, \ldots, aq^{j-n}, cq^{n-j}/a, bq^{1-n}, cq^{n-1}/b, \ldots, bq^{j-n-1}, cq^{n-j+1}/b\}, & \text{if } 1 \leq j < k, \\
\{a, c/a, \ldots, aq^{j-n+1}, cq^{n-j}/a, bq^{1-n}, cq^{n-1}/b, \ldots, bq^{j-n}, cq^{n-j}/b\}, & \text{if } k \leq j \leq n.
\end{cases}
\]

For convenience, we denote by \( F_{n,k}(U, A, B) \) the determinant in (3.3). Now, in order to prove Corollary 1.2, we are left to evaluate these determinants \( F_{n,k}(U, A, B) \) for \( 1 \leq k \leq n + 1 \).

**Theorem 3.1** For \( 1 \leq k \leq n + 1 \), we have

\[
F_{n,k}(U, A, B) = \left[ \frac{n}{k-1} \right] q^{-\frac{n(n-1)(2n-1)}{6}} b^{(n)} \prod_{1 \leq i < j \leq n} (c - u^2q^{i+j-2})
\times P_{n-k+1}(b, uq^{n-1}) P_{k-1}(a, uq^{n-k+1}) P_{n-k+1}(u, c/b) P_{k-1}(c/a, u)
\times \prod_{i=1}^{n-1} (1 - aq^{i-1}/b)^{n-i} \prod_{j=0}^{n-1} \prod_{j \neq n-k+1} (1 - cq^{i+j-1}/ab).
\]

(3.4)

To make the proof clear, we shall first give two lemmas.

**Lemma 3.2** Let \( \{j_1, j_2, \ldots, j_n\} \) be a sequence of integers, and let \( X_1, \ldots, X_n \) and \( Y_1, \ldots, Y_n \) be sets of variables. The following relation holds

\[
\det \left( h_{j_k + k-l}(X_k - Y_k) \right)_{k,l=1}^{n} = \det \left( h_{j_k + k-l}(X_k - Y_k - D_{n-k}) \right)_{k,l=1}^{n},
\]
where \( D_0, D_1, \ldots, D_{n-1} \) are sets of indeterminates such that the cardinality of \( D_i \) is equal to or less than \( i \).

The second lemma is a special case of Theorem 3.1.

**Lemma 3.3** For \( 1 \leq k \leq n + 1 \), we have

\[
\det(e_{2n-i+1}(Y_{j,k}))^{n}_{i,j=1} = \left[ \begin{array}{c} n \\ k - 1 \end{array} \right] q^{(n-2)k} - \frac{n(n-1)(2n-1)}{6} \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{j=0}^{n-1} \prod_{i=1}^{n} (1 - cq^{i+j-1}/ab),
\]

(3.5)

where the sets of \( 2n \) variables \( Y_{j,k}, 1 \leq j \leq n \), are defined as above.

**Proof.** Obviously, \( e_{2n}(Y_{j,k}) = c^n \). We shall use induction on \( n \). When \( n = 1 \), we have

\[
\det(e_2(a,c/a)) = \det(e_2(b,c/b)) = c.
\]

Thus (3.5) is true for \( n = 1 \). Assume that (3.5) holds for \( 1 \leq m \leq n - 1 \), where \( n \geq 2 \). We now proceed to check that (3.5) is true for \( m = n \).

When \( k = 1 \), the substraction of two successive columns of the determinant gives

\[
e_i(Y_{j,1}) - e_i(Y_{j-1,1}) = q^{j-n}(b-a)(1-cq^{2n-2j}/ab)e_{i-1}(Y'_{j-1,1}),
\]

where \( Y'_{j-1,1} = Y_{j,1} \setminus \{bq^{j-n}, cq^{n-j}/b\} \).

It is easy to verify that

\[
\det(e_{2n-i+1}(Y_{j,1}))^{n}_{i,j=1} = \left[ \begin{array}{c} n \\ k - 1 \end{array} \right] q^{(n-2)k} - \frac{n(n-1)(2n-1)}{6} \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{j=0}^{n-1} \prod_{i=1}^{n} (1 - cq^{i+j-1}/ab),
\]

which is equal to the right side of (3.5). The case \( k = n + 1 \) is similar.

We now consider the case \( 2 \leq k \leq n \). According to

\[
e_i(Y_{k,k}) - e_i(Y_{k-1,k}) = q^{k-n}(b-a)(1-cq^{2n-2k}/ab)e_{i-1}(Y'_{k-1,k-1}) + q^{k-n-1}(b-a)(1-cq^{2n-2k+2}/ab)e_{i-1}(Y'_{k-1,k}),
\]

we have

\[
\det(e_{2n-i+1}(Y_{j,k})) = c^n q^{\binom{n}{2}}(b-a)^{n-1} \prod_{i=0}^{n-1} (1 - cq^{2i}/ab)
\]

\[
\times \left( \frac{\det(e_{2n-i+1}(Y'_{j,k-1}))q^{n-k}}{1 - cq^{2n-2k}/ab} + \frac{\det(e_{2n-i+1}(Y'_{j,k}))q^{n-k+1}}{1 - cq^{2n-2k+2}/ab} \right),
\]

(3.6)
where for $i = k - 1$ or $i = k$, we have

$$Y'_{j,i} = \begin{cases} Y_{j,k} \setminus \{aq^{j-n}, cq^{n-j}/a\}, & 1 \leq j < i, \\ Y_{j+1,k} \setminus \{bq^{j+1-n}, cq^{n-j-1}/b\}, & i \leq j \leq n - 1. \end{cases}$$

Our induction hypothesis implies that

$$\det(e_{2n-i-1}(Y'_{j,k})) = \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] c(n) q^{(n-k+1)(n-1)(n-2)(2n-3)/2} \times \prod_{i=1}^{n-2} (bq^{i} - aq^{i-1})^{n-i-1} \prod_{j=0}^{n-2} \prod_{i=j+1}^{n-1} (1 - cq^{i+j}/ab),$$

and

$$\det(e_{2n-i-1}(Y'_{j,k-1})) = \left[ \begin{array}{c} n-1 \\ k-2 \end{array} \right] c(n) q^{(n-k+1)(n-1)(n-2)(2n-3)/2} \times \prod_{i=1}^{n-2} (bq^{i} - aq^{i-1})^{n-i-1} \prod_{j=0}^{n-2} \prod_{i=j+1}^{n-1} (1 - cq^{i+j}/ab).$$

Therefore,

$$\det(e_{2n-i+1}(Y'_{j,k}))_{i,j=1}^{n} = \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] c(n+1) q^{(n-k+1)(n-1)(n-2)(2n-3)/2} \times \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{j=0}^{n-1} \prod_{i=j+1}^{n} (1 - cq^{i+j-1}/ab)$$

$$\times \left( \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] \frac{1 - cq^{2n-k}/ab}{1 - cq^{2n-2k+1}/ab} + \left[ \begin{array}{c} n-1 \\ k-2 \end{array} \right] \frac{1 - cq^{2n-k}/ab}{1 - cq^{2n-2k+1}/ab} q^{n-k+1} \right).$$

With the aid of the following recurrence

$$\left[ \begin{array}{c} n \\ k-1 \end{array} \right] = \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] + q^{n-k+1} \left[ \begin{array}{c} n-1 \\ k-2 \end{array} \right] = q^{k-1} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right] + \left[ \begin{array}{c} n-1 \\ k-2 \end{array} \right],$$

we complete the proof.

We are now ready to complete the proof of Theorem 3.1.

Proof of Theorem 3.1.

View $F_{n,k}(U, A, B)$ as a polynomial in $u$ of degree $n^2 + n$ with coefficients expressed in terms of the other variables. Applying Lemma 3.2, we first prove that $F_{n,k}(U, A, B)$ has $2n$ roots:

$$aq^{1-n}, \ldots, aq^{k-1-n}, cq^{2-k}/a, \ldots, c/a, bq^{k-2n+1}, \ldots, bq^{1-n}, c/b, \ldots, cq^{n-k}/b.$$
Let \( u = aq^{i-n} \) in \( F_{n,k}(U, A, B) \), where \( 1 \leq i \leq k-1 \). We take
\[
D_0 = \emptyset, \quad D_1 = \{aq\}, \ldots, D_{i-1} = \{aq, aq^2, \ldots, aq^{i-1}\},
\]
\[
D_i = \{aq^{i-n}, aq, aq^2, \ldots, aq^{i-1}\}, \ldots, D_{n-1} = \{aq^{i-n}, \ldots, aq^{-1}, aq, aq^2, \ldots, aq^{i-1}\},
\]
then apply Lemma 3.2.

Since \( e_k(X) = 0 \) if the cardinality of \( X \) is less than \( k \), \( F_{n,k}(U, A, B) \) can be transformed into a determinant whose \((i, j)\)-th entry is equal to 0 if
\[(i, j) \in \{(i, j) : 1 \leq j \leq k-1 \text{ and } 1 \leq i \leq n-k+2, \text{ or } k \leq j \leq n-1 \text{ and } 1 \leq i \leq n-j\}.\]

Thus \( F_{n,k}(U, A, B) \mid_{u=aq^{i-n}} = 0 \) for \( 1 \leq i \leq k-1 \).

The case \( u = cq^{-i}/a \), where \( 1 \leq i \leq k-1 \), is similar to the above if we take
\[
D_0 = \emptyset, \quad D_1 = \{cq^{-i}/a\}, \ldots, D_{i-1} = \{cq^{-i}/a, \ldots, cq^{-1}/a\},
\]
\[
D_i = \{cq^{-i}/a, \ldots, cq^{-1}/a, cq^{-i}/a\}, \ldots, D_{n-1} = \{cq^{-i}/a, \ldots, cq^{-1}/a, cq/a, \ldots, cq^{-i}/a\}.
\]

For the cases \( u = bq^{2-n-i} \) and \( u = cq^{i-1}/b \), where \( 1 \leq i \leq n-k+1 \), we take
\[
D_0 = \emptyset, \quad D_1 = \{bq^{-n}\}, \ldots, D_{i-1} = \{bq^{2-n-i}, \ldots, bq^{-n}\},
\]
\[
D_i = \{bq^{2-n-i}, \ldots, bq^{-n}, bq^{1-i}\}, \ldots, D_{n-1} = \{bq^{2-n-i}, \ldots, bq^{-n}, bq^{2-n}, \ldots, bq^{1-i}\}
\]
and
\[
D_0 = \emptyset, \quad D_1 = \{cq^{n}/b\}, \ldots, D_{i-1} = \{cq^{n}/b, \ldots, cq^{n+i-2}/b\},
\]
\[
D_i = \{cq^{n}/b, cq^n/b, \ldots, cq^{n+i-2}/b\}, \ldots, D_{n-1} = \{cq^{n-1}/b, \ldots, cq^{n-2}/b, cq^n/b, \ldots, cq^{n+i-2}/b\},
\]
respectively.

In view of Lemma 3.2, \( F_{n,k}(U, A, B) \) in both cases becomes a determinant whose \((i, j)\)-th entry is equal to 0 if
\[(i, j) \in \{(i, j) : 2 \leq j \leq k-1 \text{ and } 1 \leq i \leq j-1, \text{ or } k \leq j \leq n \text{ and } 1 \leq i \leq k\}.\]

Therefore \( F_{n,k}(U, A, B) \mid_{u=bq^{2-n-i}} = F_{n,k}(U, A, B) \mid_{u=cq^{i-1}/b} = 0 \) for \( 1 \leq i \leq n-k+1 \).

Secondly, we show that \( \prod_{1 \leq i < j \leq n} (c-u^2q^{i+j-2}) \) is a factor of \( F_{n,k}(U, A, B) \). This is because the determinant vanishes if we set \( x_i = c/x_j \) in (3.3) for each pair \( i, j \) where \( 1 \leq i < j \leq n \).

Based on the above, we may assume that
\[
F_{n,k}(U, A, B) = C \times \prod_{1 \leq i < j \leq n} (c-u^2q^{i+j-2})
\times P_{n-k+1}(b, uq^{n-1})P_{k-1}(a, uq^{n-k+1})P_{n-k+1}(u, c/b)P_{k-1}(c/a, u).
\]
To complete the proof, we need to determine $C$. Setting $u = 0$, then applying (3.5), we have

$$C = (-1)^{n-k+1} \frac{\det(e_{2n-i+1}(Y_{j,k}))_{i,j=1}^{n}}{c^{(n+1)/2} q^{(n-k+1)/2}} = \left[ \begin{array}{c} n \\ k-1 \end{array} \right] q^{-\frac{n(n-1)(2n-1)}{6}}$$

$$\times \prod_{i=1}^{n-1} (b - aq^{i-1})^{n-i} \prod_{j=0}^{n-1} \prod_{i=j+1}^{n} (1 - cq^{i+j-1}/ab),$$

as desired.

Putting (3.3) into (3.2), then divided both sides of (3.2) by

$$\prod_{1 \leq i < j \leq n} (uq^{i-1} - uq^{j-1})(c - u^2 q^{i+j-2})b^{(n+1)/2} q^{-(n+1)n(n-1)/3} \prod_{i=1}^{n+1} (a/b, cq^{2n+2-2i}/ab; q)_{i-1},$$

we complete the proof of Corollary 1.2.

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References

[1] S.H.L. Chen and A.M. Fu, A $2n$-point Interpolation Formula with Its Applications to $q$-Identities, Submitted.

[2] W.Y.C. Chen, A.M. Fu and B.Y. Zhang, The homogenous $q$-difference operator, Adv. Appl. Math., 31 (2003) 659–668.

[3] G. Gasper and M. Rahman, Basic Hypergeometric Series (second ed.), Encyclopedia Math. Appl., Vol. 96, Cambridge Univ. Press, Cambridge, 2004.

[4] A. Lascoux, Symmetric Functions and Combinatorial Operators on Polynomials, CBMS/AMS Lecture Notes, vol. 99, (2003).

[5] S.O. Warnaar, Summation and transformation formulas for elliptic hypergeometric series, Constr. Approx. 18 (2002) 479-502.