Entanglement structure: entanglement partitioning in multipartite systems and its experimental detection using optimizable witnesses

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Creating large-scale entanglement lies at the heart of many quantum information processing protocols and the investigation of fundamental physics. For multipartite quantum systems, it is crucial to identify not only the presence of entanglement, but also its detailed structure. This is because in a generic experimental situation with sufficiently many subsystems involved, the production of so-called genuine multipartite entanglement remains a formidable challenge. Consequently, focusing exclusively on the identification of this strongest type of entanglement may result in an all or nothing situation where some inherently quantum aspects of the resource are overlooked. On the contrary, even if the system is not genuinely multipartite entangled, there may still be many-body entanglement present in the system. An identification of the entanglement structure may thus provide us with a hint on where imperfections in the setup may occur, as well as where we can identify groups of subsystems that can still exhibit strong quantum-information-processing capabilities. However, there is no known efficient methods to identify the underlying entanglement structure. Here, we propose two complementary families of witnesses for the identification of such structures. They are based, respectively, on the detection of entanglement intactness and entanglement depth, each applicable to an arbitrary number of subsystems and whose evaluation requires only the implementation of solely two local measurements. Our method is also robust against noises and other imperfections, as reflected by our experimental implementation of these tools to verify the entanglement structure of five different eight-photon entangled states. In particular, we demonstrate how their entanglement structure can be precisely and systematically inferred from the experimental measurement of these witnesses. In achieving this goal, we also illustrate how the same set of data can be classically postprocessed to learn the most about the measured system.

I. INTRODUCTION

Entanglement [1], one of the defining features offered by quantum theory, is known to be an essential resource in many quantum information processing tasks, including quantum computing [2], quantum cryptography [3,4], quantum teleportation [5], and the reduction of communication complexity [6] via Bell nonlocality [7]. In the last decade, tremendous progress has been achieved in the experimental manipulation of small-scale multipartite entanglement using various physical systems [8–12]. Indeed, a long-term goal of quantum technology is to generate medium- and eventually large-scale quantum entanglement that realizes various quantum information processing tasks.

Along this spirit, several experiments have investigated entanglement in large-scale quantum systems involving hundreds (or more) atoms [10,13,15] or trapped ions [16]. However, experimentally producing large-scale genuine multipartite entanglement remains a formidable challenge owing to inevitable couplings to the environment. Consequently, an experimentally prepared $n$-partite state (for large enough $n$) typically contains only fewer-body entanglements that are segregated. To benchmark our technological progress towards the generation of large-scale genuine multipartite entanglement, it is thus essential to determine the corresponding entanglement depth [17], i.e., the extent to which the prepared state is many-body entangled. Likewise, to overcome imperfections in the preparation procedure, it would be crucial to identify the extent to which the entanglements produced are segregated, as captured by the nonseparability $\mathcal{I}$ of the state.

The identification of such entanglement structures is generally challenging, especially when full state reconstruction is infeasible. Still, the experimental preparation of a quantum resource generally follows some well-defined procedure with a well-defined target quantum state in mind. Moreover, even in the presence of experimental imperfections, such a priori knowledge of what to expect from the setup generally remains relevant. In this case, generalized entanglement witnesses (EW) [18,19] serve as powerful alternatives for retrieving information about the underlying entanglement structure. In gen-
eral, the experimental evaluation of an EW may require the measurement of local observables that depends on the number of subsystems involved. Nevertheless, entanglement can be witnessed by a constant number of local observables [20, 21], with two being the minimum since it is impossible to distinguish entangled states from fully separable states with only one local observable. Also worth noting is the fact that the majority of the theoretical tools developed for multipartite entanglement detection [22] have focused exclusively on the identification of genuine multipartite entanglement, thus rendering them irrelevant in identifying the subtle entanglement structure mentioned above.

In this work, we propose two families of EWs that can respectively certify the maximum number of segregations and the minimal extent of many-body entanglement present in an $n$-qubit system, each by the measurement of solely two local observables, i.e., the minimal possible in order to make any nontrivial conclusion. Importantly, each family of witnesses involve the same local measurement regardless of the number of subsystems present. They also do not depend on the extent of nonseparability or entanglement depth to be certified — this follows directly from the extent to which the respective witnesses are violated. As an illustration of how these witness fare in practice, we experimentally prepare several 8-photon quantum states and demonstrate how the measurement of these two families of EWs—which involves altogether the measurement of four distinct local observables—enable us to infer nontrivial information about the underlying entanglement structure.

II. ENTANGLEMENT STRUCTURE

Let $|\phi\rangle = \bigotimes_{i=1}^{m} |\psi_{G_{i}}\rangle$ be a quantum state of $n$ parties (subsystems) divided into $m$ disjoint subsets $\{G_{i}\}_{i=1,...,m}$, each of which is described by the genuinely multipartite entangled state $|\psi_{G_{i}}\rangle$. We say that $\{G_{i}\}_{i=1,...,m}$ fully specifies the entanglement structure of $|\phi\rangle$ as it identifies exactly all the entangled subsystems in the composite system. A partial specification of the entanglement structure can be achieved via its separability. Specifically, $|\phi\rangle$ is said to be $m$-separable [1] $(2 \leq m \leq n)$ as it can be written as the tensor product of a pure state $|\psi_{G_{i}}\rangle$ from $m$ disjoint subsets. The $m$-separability of a quantum state captures the notion of segregation, i.e., no physical interaction between any two subsystems from disjoint subsets is needed for the generation of $|\phi\rangle$. The larger the value of $m$, the more segregated $|\phi\rangle$ is. Conversely, the certification that a state is non-$m$-separable implies that $|\phi\rangle$ cannot be generated by segregating the subsystems into $m$ disjoint subsets and allowing arbitrary manipulations within each subset.

While the (non)-$m$-separability of $|\phi\rangle$ already provides us with important information about the entanglement structure of $|\phi\rangle$, it is not specifically meant to indicate the extent of many-body entanglement present in the system. To see this, note, for example, that the four-qubit states $|\chi\rangle_{ABC} \otimes |\zeta\rangle_{D}$ and $|\eta\rangle_{AB} \otimes |\tau\rangle_{CD}$ are both 2-separable, but the generation of the former may require three-body entanglement while the latter only require up to two-body entanglement. To this end, let us denote by $n_{i}$ the number of subsystems involved in the subset $G_{i}$ (note that $\sum_{i=1}^{m} n_{i} = n$). Then $|\phi\rangle$ is said to be $k$-producible [23] if the largest constituent of $|\phi\rangle$ involves at most $k$ parties, i.e., if $\max_{i} n_{i} \leq k$. In other words, a $k$-producible state requires at most $k$-body entanglement in its generation. Thus, the certification that a state is not $k$-producible implies that a higher level of many-body entanglement is required in its generation.

The $m$-separability and $k$-producibility of a general mixed state $\rho$ can be defined analogously: $\rho$ is $m$-separable (or $k$-producible) if it admits a convex decomposition in terms of $m$-separable ($k$-producible) pure states. Following Ref. [17], we say that $\rho$ has an entanglement depth of $k$ if it is $k$-producible but not $(k-1)$-producible. On the other hand, we say that a quantum state $\rho$ has an entanglement intactness of $m$ if it is $m$-separable but not $(m+1)$-separable. A genuinely $n$-partite entangled has an entanglement intactness (depth) of 1 ($n$), whereas a fully separable $n$-partite state has an entanglement intactness (depth) of $n$ (1). In particular, any quantum state that has an entanglement depth greater than 2 is conventionally said to contain multipartite (many-body) entanglement.

We are now in the position to introduce our witnesses for entanglement intactness. To certify the nonseparability and hence an upper bound on the entanglement intactness of a given quantum state, we introduce the following 2-parameter family of two-observable witnesses:

\[ W_{se}^{n}(\alpha) = \alpha M_{Z} + M_{X} \leq \sum_{n} \max\{\alpha, \frac{\alpha}{2^{n-1}} + 1\}, \]

where $\alpha \in (0, 2]$ is a free parameter, $M_{Z} = (|0\rangle\langle 0|)^{\otimes n} + (|1\rangle\langle 1|)^{\otimes n}$ and $M_{X} = \sigma_{x}^{\otimes n}$ are $n$-qubit observables, $\sigma_{x} = |0\rangle\langle 1| + |1\rangle\langle 0|$ is the Pauli $x$ matrix, $\{0\}, \{1\}$ are the computational basis states, $I_{n}$ is the $2^{n} \times 2^{n}$ identity matrix, and $m$-sep. in Eq. (1) signifies that the inequality holds true at the level of the expectation value for all $m$-separable states. In other words, for an arbitrary $n$-partite state $\rho$, if $\langle W_{se}^{n}(\alpha) \rangle_{\rho} > \max\{\alpha, \frac{\alpha}{2^{n-1}} + 1\}$, one certifies that $\rho$ has an entanglement intactness of $m-1$ or lower. To ease notation, we abbreviate $W_{se}^{n}(\alpha) = W_{n}$. For witnessing entanglement depth, inspired by Ref. [24], we introduce the following family of witnesses, which also involve only two local measurements but (possibly) in a basis different from those of $W_{se}^{n}(\alpha)$:

\[ W_{de}^{n}(\gamma) = \gamma \kappa^{n} A - A' k^{\text{prod.}} \leq I_{n} \sum_{n} \beta_{n,k}(\gamma) \]

where $\gamma \in (0, 2]$ is a free parameter, $A = (A_{+} + A_{-})^{\otimes n}$, and $A' = (A_{+})^{\otimes n}$ are $n$-partite $\pm 1$-valued observables, $A_{\pm}$ is a single-partite $\pm 1$-valued observable, $\kappa \neq 0$ (which holds for $A_{+} \neq A_{-}$) is a normalization constant, and
\[ \beta_{n,k}(\gamma) \] is the \( k \)-producible bound of the \( n \)-partite version of the witness \( W_{de}^n(\gamma) \). In Eq. [2], \( k \)-prod. signifies that the inequality holds true at the level of the expectation value for all \( n \)-partite \( k \)-producible states. In other words, for an arbitrary \( n \)-partite state \( \rho \), if \( \langle W_{de}^n(\gamma) \rangle_\rho > \beta_{n,k}(\gamma) \), one certifies that \( \rho \) has an entanglement depth of at least \( k + 1 \). For \( \gamma = 2 \) and if no further assumption (including the underlying Hilbert space dimension) is made on \( A_\pm \), it follows from Ref. [24] that \( \beta_{n,1} = 1, \beta_{n,2} = \sqrt{2}, \ldots \) for all \( n \geq 2 \). For specific choices of qubit observables \( A_\pm \), these bounds can be tightened to provide better noise robustness (see Sec. III) specifically our elaboration in page 4. Despite their simplistic form, the derivation of the bounds for the two families of witnesses is highly nontrivial and may serve as a basis for the derivation of other entanglement witnesses. For details, see Appendix A.

A few other remarks are now in order. First, in contrast with ordinary entanglement witnesses, we see from Eqs. [1] and [2] that the measured value for these witnesses is precisely the information that we need to provide further details about the underlying entanglement structure. Moreover, both families of witnesses involve a free positive parameter that may be optimized \( a \text{ posteriori} \) to identify the best possible upper (lower) bound on the entanglement intactness (depth) of \( \rho \). Finally, it is worth noting that these witnesses can be easily adapted to an arbitrary choice of local basis, i.e., even after applying an arbitrary local unitary transformation on each qubit, the \( m \)-separable bounds and the \( k \)-producible bounds of the transformed witnesses evidently remain unchanged. For example, via the local unitary transforma-
tion $I_2^{\otimes n-1} \otimes \sigma_z$, the witness $W_{se}^\alpha (\alpha)$ gets transformed to $W_{se}^\alpha (\alpha) = \alpha M_Z - M_X$, which complements $W_{se}^\alpha (\alpha)$ in detecting the many-body entanglement present in a larger set of quantum states, see Appendix [A1] for details.

### III. EXPERIMENTAL REALIZATION

Experimentally, we use the polarization degree of freedom to encode the qubit state $|H(V)\rangle = |0(1)\rangle$, where $H(V)$ denotes the horizontal (vertical) polarization. The experimental setup used to generate various entangled states is shown in Fig. [1](a). We first generate four pairs of maximally entangled states $1/\sqrt{2}(|H_i H_j\rangle + |V_i V_j\rangle)$ by shining an ultraviolet (UV) pulse successively on four BiB$_3$O$_6$ crystals [as shown in Fig. [1](b)] with $i, j$ denoting the path modes (more details are shown in Fig. [B1] of Appendix [B1]). Photons in path modes 2, 3, 6, and 7 are then injected into an interferometric network (IN), which consists of three polarization beam splitters (PBSs) with four input and output ports (as shown in Fig. 2). Each PBS is controlled by an individual lifting platform that can be set to either the up or down state. When a PBS is in the up state, it facilitates the interference of the two photons arriving at its two input ports. On the contrary, there is no interference between the incoming photons when the PBS is in the down state. With three independently controlled PBSs, one can construct eight interferometric geometries, which correspondingly lead to eight possible photonic entangled states that fall under five distinct entanglement structures.

Let $|GHZ_n\rangle = 1/\sqrt{2}(|H\rangle^{\otimes n} + |V\rangle^{\otimes n})$ denote an $n$-photon Greenberger-Horne-Zeilinger (GHZ) state. Then, in the absence of imperfection, the interferometer can thus be used to produce the five different entangled states is shown in Fig. [1](a). We first generate four pairs of maximally entangled states $1/\sqrt{2}(|H_1 H_2\rangle + |V_1 V_2\rangle)$ by shining an ultraviolet (UV) pulse successively on four BiB$_3$O$_6$ crystals [as shown in Fig. [1](b)] with $i, j$ denoting the path modes (more details are shown in Fig. [B1] of Appendix [B1]). Photons in path modes 2, 3, 6, and 7 are then injected into an interferometric network (IN), which consists of three polarization beam splitters (PBSs) with four input and output ports (as shown in Fig. 2). Each PBS is controlled by an individual lifting platform that can be set to either the up or down state. When a PBS is in the up state, it facilitates the interference of the two photons arriving at its two input ports. On the contrary, there is no interference between the incoming photons when the PBS is in the down state. With three independently controlled PBSs, one can construct eight interferometric geometries, which correspondingly lead to eight possible photonic entangled states that fall under five distinct entanglement structures.

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More details of the state preparation procedure are shown in Appendix [B].

Note that the geometries depicted in Fig. 2(c) and Fig. 2(d) produce essentially the same entangled state as $|G_{62}\rangle$ but differ in their path mode. Similarly, the entanglement produced in Fig. 2(f), Fig. 2(g), and Fig. 2(h) is essentially the same as that of $|G_{222}\rangle$. In our experiment, we choose the geometries in Figs. 2(a), 2(c), 2(e) and 2(h) for the preparation of five different entangled states. The generated 8-photon entanglement is detected and analyzed by eight witness analyzers in paths 1, 2’, 3’, 4, 5, 6’, 7’, and 8. As shown in Fig. 2(c), a witness analyzer consists of a quarter-wave plate (QWP), a half-wave plate (HWP), a PBS and two single-photon detectors.

In reality, there are always imperfections in the setup, and the entangled state produced is thus more aptly described by a density matrix $\rho$. For ease of comparison, in a setup used to produce the quantum state $|G_i\rangle$, we shall denote the actual quantum state produced by $\rho_i$. To determine the entanglement structure of $\rho_i$, we thus begin by measuring the expectation value of the observables $M_X = \sigma_x^{\otimes 8}$ and $M_Z = (|H\rangle\langle H|)^{\otimes 8} + (|V\rangle\langle V|)^{\otimes 8}$. Whenever the observed $\langle M_Z \rangle$ and $\langle M_Z \rangle$ violate the inequality [corresponding to Eq. (1) with $\alpha = 2$]

$$2\langle M_Z \rangle + \langle M_X \rangle \leq 2,$$

we can thus conclude that $\rho_i$ exhibits genuine eight-photon entanglement. As shown in Fig. 3(a), we observe that $\langle (M_Z, M_X) \rangle = (0.80(2), 0.63(4))$ on $\rho_8$ which violates Eq. (3), while the corresponding expectation values for $\rho_8, \rho_{62}, \rho_{44}, \rho_{22}, \rho_{222}$, as summarized in Table 1, satisfy Eq. (3). The results indicate that $\rho_8$ is genuinely eight-photon entangled, but the entanglement structure of the rest cannot be concluded from the witness of Eq. (3).

Note, however, that Eq. (3) only represents a specific case ($\alpha = m = 2$) of the family of witnesses considered in Eq. (1). Further nontrivial information on the entanglement structure, specifically the $m$-separability of $\rho_i$, can also be deduced from the measured value of $M_{Z\ell}$ and $M_X$. Specifically, by varying $\alpha \in (0, 2]$, one can identify the smallest value of $m = 2, 3, \ldots, 8$ whereby the witnesses of Eq. (1) are violated. This minimum value of $m$ then provides an upper bound of $m-1$ on the entanglement intactness of the measure system. To this end, it is worth noting that both $W_{se}^\alpha (\alpha), W_{se}^{\otimes 8}$ and their $m$-separable bounds are linear in $\alpha$. For any given value of $m$, the optimal choice of the free parameter $\alpha$ in Eq. (1) is obtained by setting $\alpha = \frac{m-1}{2}$, thereby giving $\alpha = \frac{m-1}{2} \leq 1$, e.g., $\alpha = 2, 4, 8$, and $16/15$, respectively, for $m = 2, 3, 4, 5$. Note that in each of these cases,

1 To decide if such a pretransformation is relevant, one may want to allow the measurement of an additional local observable, such as $\sigma_y$, by one of the parties (or an appropriate modification thereof) before measuring the witness operator itself.

2 Of course, for a very poorly prepared system, it could happen that for all $2 \leq m \leq n$, none of the witnesses from Eq. (1) is violated.
FIG. 2. **Interferometric geometries leading to different entanglements.** Here, $|\text{GHZ}_n\rangle$ is an example of a graph state and can be represented by a star graph. Such a representation makes its preparation procedure evident: Each node represents a photon prepared in the state $|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$ and each edge joining two nodes represents a controlled-$Z$ operation performed between the corresponding photons. The number inscribed in each node labels the path mode of the photons prepared in our experiment. The PBS is fixed on a lifting platform. By adjusting the lift height, we can switch the state of the PBS between up and down. For each PBS set to up, since we postselect the cases where photons exit from both output ports of the PBS, the count rate reduces approximately by half compared with the case when the PBS is set to down.

FIG. 3. **Experimental results for witnesses certifying genuine multipartite entanglement (GME), entanglement intactness and entanglement depth.** (a) GME revealed by the measurement of the witness of Eq. (3) via $\langle M_Z \rangle$ and $\langle M_X \rangle$. (b) Entanglement intactness revealed by the measurement of $\langle W^a_{\alpha}(\alpha) \rangle$ via $\langle M_Z \rangle$ and $\langle M_X \rangle$ for judicious choice of $\alpha$. (c) Entanglement depth revealed by the measurement of $\langle W^d_{\gamma}(\gamma) \rangle$ via $\langle A \rangle$ and $\langle A' \rangle$ for judicious choice of $\gamma$. For comparison, we have also included here with the red (blue) dot the theoretical value of the witness for an ideal $|G_{53}\rangle$ ($|G_{71}\rangle$) assuming the same set of measurements.

the value of $\alpha$ is precisely the $m$-separable bound given in Eq. (1). A direct comparison between the measured value
TABLE I. Summary of our experimental results for determining the entanglement intactness of the prepared state ρi.

| State  | m | α  | ⟨M_Z⟩ | ⟨M_X⟩ | ⟨W^k_e (α)⟩ | Ent. Intactness |
|--------|---|----|--------|--------|--------------|----------------|
| ρ8     | 2 | 2  | 0.80(2)| 0.63(4)| 2.23 (3)     | 1              |
| ρ62    | 3 | 4/3| 0.63(3)| 0.60(5)| 1.43(7)      | ≤ 2            |
| ρ44    | 3 | 4/3| 0.43(4)| 0.89(3)| 1.46(6)      | ≤ 2            |
| ρ222   | 4 | 8/7| 0.27(3)| 0.86(3)| 1.17(5)      | ≤ 3            |
| ρ2222  | 5 | 16/15| 0.18(2)| 0.09(2)| 1.09(3)      | ≤ 4            |
| (G71)  | 3 | 4/3| 0.5   | 1     | 5/3          | ≤ 2            |
| (G53)  | 3 | 4/3| 0.5   | 1     | 5/3          | ≤ 2            |

of W^k_e (α) and the various m-separable bounds then allows us to determine an upper bound on the entanglement intactness of the measured system.

A graphical illustration of these fine-tuned m-separability witnesses corresponding to the blue region in Fig. 2(a) is shown in Fig. 2(b). Once the observed expectation values (⟨M_Z⟩, ⟨M_X⟩) of ρi are found to lie in the (m−1)-separable region, it violates the witness of Eq. 1 for m-separability, thereby certifying that ρi has an entanglement intactness of m − 1 or lower. Equivalently, we see from Table 1 that the entanglement intactness of ρ62, ρ44, ρ222, and ρ2222 is upper bounded, respectively, by 2, 2, 3, and 4, which matches exactly with that of (G62), (G44), (G222), and (G2222).

While these bounds on entanglement intactness already shed some light on the underlying entanglement structure, they are not yet informative enough to suggest any specific entanglement structure associated with the measured system. To this end, we also measure the witnesses for entanglement depth given in Eq. 2 by measuring the expectation value of A and A′ for each of the prepared states. Specifically, based on the data that we have collected in measuring ⟨M_Z⟩ and ⟨M_X⟩ and the ansatz A_k = cos θ_k σ_x + sin θ_k σ_y (see Appendix A), a reasonably good choice of qubit observables appears to be those corresponding to θ_k = 3(1+θ) 8 (thereby giving k = cos 3 10), where σ_y = −i(|0⟩⟨1| + |1⟩⟨0|) is the Pauli y matrix.

The corresponding tightened k-producible bound β_k(γ) as a function of k and γ is given in the Appendix A.2. Our experimental results shown in Fig. 3(c), and summarized in Table 11 allow us to conclude a lower bound on entanglement depth of 4, 4, 3, 4, and 2, respectively, for the state ρ8, ρ62, ρ44, ρ222, and ρ2222. Evidently, only the measurements of ⟨W^k_e (γ)⟩ for ρ222 and ρ2222 reveal the expected entanglement depth, while the lower bound on entanglement depth obtained for the other states is clearly suboptimal. Our separate analysis shows that this is caused by the undesired noises in our experiment, specifically the higher-order emissions in SPDC and the mode mismatch of the interference. We analyze the decoherence induced by these two noises in Appendix B.2.

TABLE II. Summary of our experimental results for determining the entanglement depth of the prepared state ρi. The second and the third column give our choice of the free parameters k and γ in Eq. 2. The fourth column gives the value of the corresponding k-producible bound. The experimentally measured expectation values of A and A′ and ⟨W^k_e (γ)⟩ = γ^k (A − A′) are given in column 5-7. The last column gives our best lower bound on the entanglement depth of ρ, based on the measured value of W^k_e (γ). An entanglement depth of k means that the state requires at least k-body entanglement for its preparation. For comparison, we have also included here the theoretical values of the witness for an ideal |G53⟩ and |G71⟩ assuming the same set of measurements.

| State  | k  | γ   | ⟨β_k(γ)⟩ | ⟨A⟩  | ⟨A′⟩  | ⟨W^k_e (γ)⟩ | Ent. Depth |
|--------|----|-----|----------|------|-------|-------------|------------|
| ρ8     | 3  | 2   | 1.1699   | 0.54(9)| -0.57(9)| 1.32(15)    | ≥ 4        |
| ρ62    | 3  | 2   | 1.1699   | 0.73(5)| -0.27(8)| 1.29(8)     | ≥ 4        |
| ρ44    | 2  | 8/5 | 0.7904   | 0.76(3)| -0.07(5)| 0.91(7)     | ≥ 3        |
| ρ222   | 3  | 8/5 | 0.9137   | 0.84(3)| -0.02(6)| 0.95(7)     | ≥ 4        |
| ρ2222  | 1  | 2   | 0.8365   | 0.83(3)| 0.19(5)| 0.96(5)     | ≥ 2        |
| (G71)  | 6  | 2   | 1.8858   | 0.96(1)| -0.67(1)| 2.0106     | ≥ 7        |
| (G53)  | 4  | 2   | 1.3856   | 0.97(3)| -0.06(1)| 1.4164     | ≥ 5        |

Nevertheless, as we demonstrate below, the measurement results obtained from both witnesses are useful and complement each other nicely—at least in our setup—to suggest some minimal entanglement structure of the measured system. By minimal, we mean that the corresponding entanglement structure is compatible with all the empirical observation, and it is also not more entangled nor more complicated than necessary to explain these empirical observations. Thus, despite the fact that a general mixed state does not have a unique convex decomposition, we are only concerned with identifying a compatible entanglement structure that is not a convex mixture of different entanglement structures.

Coming back to the identification of a minimal entanglement structure associated with our setup, suppose that the IN is controlled by three binary random number generators, each of which determines the state of one of the PBSs. Thus, the IN is randomly set to be one of the geometries depicted in Fig. 2 thereby resulting in one of the corresponding entanglement structures. In the next two paragraphs, we show how to deduce the structure corresponding to |G44⟩ and |G2222⟩. The corresponding analysis for |G62⟩ and |G44⟩ is shown in Ap-
entanglement structure compatible with these observations. With the assumptions stated above, the first case of genuine multipartite entangled, we combine the respective pieces and color them the same way.

For example, since our measurement of $\langle W_{se}^5(\gamma) \rangle$ on $\rho_{422}$ witnesses an entanglement depth of 4 or more, at least four of the photons exhibit GME. Importantly, our measurement results of $\langle W_{se}^5(\alpha) \rangle$ also allow us to evaluate $\langle W_{se}^5(\alpha) \rangle$ among any $k$-partite subset of the 8 photons. Indeed, from the measured expectation values of the four-partite witness $W_{se}^5(\alpha = 2)$ [see Fig. 4(a)], only the four photons with path modes $\{5, 6, 7, 8\}$ seem to be genuinely-four-entangled. On the other hand, our measurement of $\langle W_{se}^8(\alpha = 8/7) \rangle$ concludes that $\rho_{422}$ is at most tripartisan. Combining this with the above observation suggests that $\rho_{12'3'4'}$ is a biseparable state. Thus, $\rho_{12'3'4'}$ can be either a tensor product of a genuinely three-photon entangled state and a single-photon state, or a tensor product of two two-photon entangled states. Our evaluation of $W_{se}^8(\alpha = 2)$ for all possible combinations of two photons from path modes $\{1, 2', 3', 4\}$ and 4 does not reveal any three-photon entanglement. The above observations, together with the assumptions stated above, lead us to conclude that $\rho_{422}$ shares the same (minimal) entanglement structure as $\langle H \rangle_{12'3'4'} \otimes \langle H \rangle_{12'56} \otimes \langle H \rangle_{23'} \otimes \langle H \rangle_{34'}$.

Similarly, for $\rho_{222}$, our measurement of $\langle W_{se}^8(\gamma) \rangle$ and $\langle W_{se}^8(\alpha) \rangle$ leads to the conclusion that $\rho_{222}$ involves at least 2-body entanglement while not being 5-separable. Various entanglement structures are compatible with these observations. However, from the observed values of $\langle W_{se}^2(\alpha = 2) \rangle$ for all possible two-photon combinations [see Fig. 4(c)], we see that the photon pairs from path modes $\{1, 2', 3', 4\}, \{5, 6\}$, and $\{7', 8\}$ are clearly entangled. Thus, with the assumptions stated above, the only entanglement structure compatible with these observations is that of $\langle H \rangle_{12'56} \otimes \langle H \rangle_{23'} \otimes \langle H \rangle_{34'} \otimes \langle H \rangle_{78'}$.

At this point, one may wonder whether our experimental setup is capable of generating 8-photon entangled states with other entanglement structures (such as those involving odd-party entangled states), and how our witnesses fare in those cases. Let us remark that our experimental setup is not limited to generating only the even-party-entangled state depicted in Fig. 2—it can, in principle be used to produce all quantum states of the form $\langle G_{11...m} \rangle$ (such as those containing only odd-party-entangled states). However, we did not experimentally prepare these other states as their generation (using our setup) involves heralding and thus a significantly lower count rate. As an example, to create an 8-photon entangled state $\langle G_{71} \rangle$ that is only 7-photon entangled, we can set two polarizers in paths 1 and 2 at $45^\circ$, which project $|+\rangle_{12'}$ on state $|+\rangle_{12'53}$. Then, by raising all three PBSs, the state $\langle G_{71} \rangle = \frac{1}{2}(\langle H \rangle_{12'}^\otimes + |V\rangle_{71}^\otimes)_{12'3'47'} \otimes (\langle H \rangle_{12'} + |V\rangle_{71})_1$ can be obtained. Likewise, to generate $\langle G_{53} \rangle$, we can set a polarizer between PBS$_1$ and PBS$_3$ at $45^\circ$ in the geometry shown in Fig. 2(a). Then, the state $\langle G_{53} \rangle = \frac{1}{2}(\langle H \rangle_{12'}^\otimes + |V\rangle_{71}^\otimes)_{12'3'47'} \otimes (\langle H \rangle_{12'}^\otimes + |V\rangle_{71}^\otimes)_{568'}$ can be generated.

To illustrate that our theoretical methods also work well in these cases, we have calculated $\langle M_Z \rangle$, $\langle M_X \rangle$, and $\langle A \rangle$, $\langle A' \rangle$ for $\langle G_{53} \rangle$ and $\langle G_{71} \rangle$, respectively. The results are given in Tables I and II and shown as blue and red dots in Fig. 3(b) and Fig. 3(c). Together, they confirm that $\langle G_{53} \rangle$ is at most a biseparable state with an entanglement depth of at least 5, and $\langle G_{71} \rangle$ is at most a biseparable state with an entanglement depth of at least 7. By systematically evaluating the value of the witnesses for the right number of parties over all possible choices of parties—as we did above—and finding out which subset of photons is genuinely multipartite entangled, we obtain a compatible entanglement structure, which coincides with that of the given state.

IV. CONCLUSION

In this work, we introduce the notion of an entanglement structure, which details not only the extent of many-body entanglement present but also their segregation among the various subsystems. Identifying the entanglement structure of an arbitrary multipartite quantum state, as with the certification of genuine multipartite entanglement, generally requires an exponential number of local measurements. Nonetheless, the retrieval of any partial information on the entanglement structure of an experimentally-prepared system is always welcome, as it provides diagnostic information on where imperfections in the setup may lie. Importantly, such information is often already available in the data collected for the measurement of entanglement witnesses, even if the measured value does not reveal genuine multipartite entanglement. Here, we propose two complementary families of witnesses capable of bounding, respectively, the entanglement intactness (i.e., nonseparability) and the entanglement depth of the measured system, thereby providing nontrivial information about the underlying entanglement structure.

Our scheme works for any number of parties and can be generalized to arbitrary dimensions [25]. In contrast with conventional entanglement verification schemes, our witnesses involve free parameters that can be varied a posteriori, thereby allowing us to optimize—in a similar spirit to Ref. [20]—the data collected to arrive at the strongest possible conclusion. Note also that the possibility to per-
Finally, it is worth noting that the entanglement in-tactness witnesses introduced in the current work have very recently been generalized [27] to the case of 1-dimensional cluster states. Since these states and GHZ states are both specific cases of a graph state, an open question that follows is whether these witnesses can be further generalized to cover a general graph state while maintaining their appealing feature of involving only two local observables (see also Ref. [20] in this regard). Given the importance of such states for one-way quantum computation [28], such a generalization may then be used to benchmark our progress towards the ultimate goal of demonstrating quantum supremacy.

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Appendix A: Theory

The structure of multipartite entanglement is much richer than the bipartite case. An \( n \)-partite pure state \(|\phi\rangle\) is said to be \( m \)-separable (\( 2 \leq m \leq n \)) if the \( n \) parties can be divided into \( m \) disjoint subsets \( \{G_i\}_{i=1}^m \) such that \(|\phi\rangle\) is the tensor product of a pure state \(|\psi_{G_i}\rangle\) from each of these subsets, i.e.,

\[
|\phi\rangle = \bigotimes_{i=1}^m |\psi_{G_i}\rangle.
\]

(A1)

The \( m \)-separability of a quantum state describes the extent of segregation. The larger the value of \( m \), the more segregated \(|\phi\rangle\) is. If \( m = n \), we refer to it as a fully separable state. Conversely, a non-\( m \)-separable state implies that it cannot be generated by segregating the subsystems into \( m \) disjoint subsets and allowing arbitrary local manipulations within each subset.

Though (non)-\( m \)-separability of \(|\phi\rangle\) already provides us with important information about the entanglement structure of \(|\phi\rangle\), it is still not enough. How many parties each disjoint subset contains is also part of the specification of its entanglement structure. Let us denote by \( n_i \) the number of subsystems involved in the subset \( G_i \) (note that \( \sum_{i=1}^m n_i = n \)); then \(|\phi\rangle\) is said to be \( k \)-producible if the largest constituent of \(|\phi\rangle\) involves at most \( k \) parties, i.e., if \( \max n_i = k \). Generating a \( k \)-producible state requires at most \( k \)-body entanglement and on the other hand, more than \( k \)-body entanglement is required in generating a not \( k \)-producible state.

For mixed state \( \rho \), the \( m \)-separability and \( k \)-producibility of a general mixed state \( \rho \) can be defined analogously: \( \rho \) is \( m \)-separable (or \( k \)-producible) if it admits a convex decomposition in terms of \( m \)-separable (\( k \)-producible) pure states. If a quantum state \( \rho \) is \( k \)-producible but not \((k-1)\)-producible, we say it has an entanglement depth of \( k \) \cite{17}. On the other hand, we say that a quantum state \( \rho \) has an entanglement integrality of \( m \) if it is \( m \)-separable but not \((m+1)\)-separable. A genuinely \( n \)-partite entangled has an entanglement integrality (depth) of \( 1 \) whereas a fully separable \( n \)-partite state has an entanglement integrality (depth) of \( n \) (1). In particular, any quantum state that has an entanglement depth greater than \( 2 \) is conventionally said to contain multipartite (many-body) entanglement.

1. A family of witnesses for non-\( m \)-separability with two local measurement settings

In this section, we introduce the following two-parameter family of two-observable witnesses:

\[
W^n_{se}(\alpha) = \alpha M_Z + M_X \overset{m-sep.}{\leq} \mathbb{I}_n \max \left\{ \alpha, \frac{\alpha}{2^{m-1}} + 1 \right\}, \quad \alpha \in (0, 2],
\]

(A2)

where \( M_Z = (|0\rangle\langle 0|)^{\otimes n} + (|1\rangle\langle 1|)^{\otimes n} \), \( M_X = \sigma_x^{\otimes n} \), \( \sigma_x \) is the Pauli \( x \) matrix, \{\{0\}, \{1\}\} are the computational basis states, \( \mathbb{I}_n \) is the \( 2^n \times 2^n \) identity matrix, and \( m \)-sep. in Eq. (A2) signifies that the inequality holds true at the level of the expectation value for all \( m \)-separable \( n \)-qubit states. In other words, for an arbitrary \( n \)-partite state \( \rho \), if \( \langle W^n_{se}(\alpha) \rangle_\rho > \max\{\alpha, \frac{\alpha}{2^{m-1}} + 1\} \), one certifies that \( \rho \) has an entanglement integrality of \( m - 1 \) or lower. Here, \( \alpha \) is a free positive parameter that may be varied to identify the best possible upper bound on the entanglement integrality of \( \rho \).

a. Family of genuine \( n \)-qubit entanglement witnesses

For the specific case of \( m = 2 \), the witness of Eq. (A2) reduces to one that can be used to certify genuine \( n \)-qubit entanglement.

**Theorem 1.** Let \( \rho \) be an arbitrary \( n \)-qubit biseparable state; then its expectation value for \( M_Z \) and \( M_X \) (and hence \( W^n_{se}(\alpha) \)) satisfy

\[
\alpha \langle M_Z \rangle_\rho + \langle M_X \rangle_\rho \overset{2-sep.}{\leq} \frac{\alpha}{2} + 1, \quad \alpha \in (0, 2].
\]

(A3)

To prove the theorem, let us denote by \( \vec{n} = \{n_1\}{n_2\} \) a partition of the \( n \) parties into a subset of \( n_1 \) parties and the complementary subset of \( n_2 = n - n_1 \) parties. The invariance of \( W^n_{se}(\alpha) \) with respect to an arbitrary permutation of subsystem Hilbert spaces implies that in determining the biseparable bound, i.e., the maximal quantum value of \( \langle W^n_{se}(\alpha) \rangle \) over all biseparable \( n \)-qubit states, the actual members of each subset \( G_1 \) and \( G_2 \) are irrelevant.

Without loss of generality, let us thus imagine that the first \( n_1 \) parties belong to \( G_1 \), and denote by \( S_{\vec{n}} \) the set of all \( n \)-qubit pure states that are biseparable with respect to this partitioning specified by \( \vec{n} \). We may then write the biseparable bound, i.e., the maximal value of the right-hand-side of Eq. (A3) as:

\[
\max_{\text{bisp.}} \langle W^n_{se}(\alpha) \rangle_\rho = \max_{\vec{n}} f_{\vec{n}} = \max_{\{n_1\}{n_2\}} f_{\{n_1\}{n_2\}},
\]

(A4)

where

\[
f_{\vec{n}} := \max_{|\phi\rangle \in S_{\vec{n}}} \langle W^n_{se}(\alpha) |\phi\rangle = \max_{|\phi\rangle \in S_{\vec{n}}} \text{Tr} \left[ \alpha M_Z + M_X |\phi\rangle \langle \phi| \right].
\]

(A5)

As noted above, \( W^n_{se}(\alpha) \) is permutational invariant, we thus have

\[
f_{\{n_1\}{n_2\}} = f_{\{n_2\}{n_1\}}
\]

(A6)
Moreover, it can be verified that the largest eigenvalue of the matrix corresponding to Eq. (A8) is again \(\lambda^+=\cos^{-1}\theta\), \(\varphi=0\), thereby giving a bipartite bound of \(f_{\hat{n}}\). For the more general family of witnesses for detecting non-\(m\)-separability (and hence an entanglement intactness of \(m-1\) or lower), we follow a very similar procedure as that adopted in the last section. Specifically, we first iteratively apply Lemma 1 and Eq. (A6) to show that determining the \(m\)-separable bound amounts to computing the maximal eigenvalue of \(M_b\).

**Proof.** From Lemma 1 we note that for arbitrary \(n_1, n_2\) such that \(n=n_1+n_2\), we have \(f_{\hat{n}}(\{n_1\}) = f_{\{n_1\}}\). Using Eq. (A6), \(f_{\hat{n}}(\{1\})\) can be rewritten as \(f_{\{1\}}\). Applying Lemma 1 again to \(f_{\{1\}}\), we thus find that the largest eigenvalue of \(M_b\) is maximized by setting \(\theta=\varphi=0\). The term in the square root is clearly maximized by setting \(\theta=\pi\), \(\varphi=0\), thereby giving a bipartite bound of \(f_{\{1\}}\).

Next, we present a key observation that allows one to simplify the maximization of Eq. (A4) over an arbitrary \((n_1+n_2)\)-qubit biseparable pure state to a maximization over an arbitrary \((n_1+1)\)-qubit biseparable pure state.

**Lemma 1.** The value of \(f_{\hat{n}}\) for the bipartition of \(n\) into \(n_1+n_2\) parties specified by \(\{n_1\}\{n_2\}\) is identical to the value of \(f_{\hat{n'}}\) for the bipartition of \(n'\) into \(n_1+1\) parties into \(n_1\{1\}\).

**Proof.** Let \(|\phi_a\rangle\) and \(|\phi_b\rangle\), respectively, be arbitrary \(n_1\)-qubit and \(n_2\)-qubit pure states. From the definition of \(f_{\hat{n}}\) given in Eq. (A5), one finds that

\[
\begin{align*}
\langle f_{\{n_1\}}\rangle_{n_2} &= \max_{|\phi_a\rangle,|\phi_b\rangle} \text{Tr}[(\alpha M_Z + M_X)|\phi_a\rangle|\phi_b\rangle], \\
\langle f_{\{n_1\}}\rangle_{n_2} &= \max_{|\phi_n\rangle} \text{Tr}[(\alpha M_Z + M_X)|\phi_n\rangle|\phi_n\rangle], \\
\langle f_{\{n_1\}}\rangle_{n_2} &= \max M_b,
\end{align*}
\]

where \(M_b\) is an observable defined on the remaining \(n_2\)-qubit space,

\[
M_b = \alpha x (|0\rangle\langle 0|) \otimes n_2 + \alpha y (|1\rangle\langle 1|) \otimes n_2 + \alpha z (\sigma_z) \otimes n_2,
\]

and it depends on \(|\phi_a\rangle\) via \(x = \langle \phi_a| (|0\rangle\langle 0|) \otimes n_1|\phi_a\rangle\), \(y = \langle \phi_a| (|1\rangle\langle 1|) \otimes n_1|\phi_a\rangle\), and \(z = \langle \phi_a| (\sigma_z) \otimes n_1|\phi_a\rangle\).

For any integer \(n_2 \geq 1\), \(M_b\) has the following generic (sparse) matrix representation:

\[
M_b = \begin{pmatrix} \alpha x & z \\ z & \alpha y \end{pmatrix}.
\]

Moreover, it can be verified that \(M_b\) has two \((2^{n_2}-1)\)-fold degenerate eigenvalues \(z\) and two nondegenerate eigenvalues \(\lambda_{\pm} = \frac{\alpha(x+y)}{2} \pm \sqrt{z + \alpha^2(x-y)^2}\). Since \(x, y\) are non-negative, it follows that \(z \leq \frac{\alpha(x+y)}{2} + \sqrt{\alpha^2(x-y)^2}\). Hence, the largest eigenvalue of \(M_b\) is necessarily \(\lambda_{\max} = \lambda_+\).

Importantly, as long as \(n_2 \geq 1\), the same conclusion holds regardless of the actual number of qubits involved in the definition of \(M_b\). In other words, while the size of \(M_b\) depend on \(n_2\), its largest eigenvalue \(\lambda_{\max}\) and hence \(f_{\hat{n'}}(\{1\})\) only depends on \(|\phi_a\rangle\) via \(x, y\) and \(z\). Consequently, the very same argument can be repeated in the computation of \(f_{\hat{n'}}\), with \(n' = \{n_1\}\{1\}\) to arrive at the conclusion that the largest eigenvalue of the matrix corresponding to Eq. (A8) is again \(\lambda_+\). Therefore, \(f_{\hat{n'}}\) for the bipartition specified by \(n' = \{n_1\}\{n_2\}\) coincides with \(f_{\hat{n'}}\) for the bipartition specified by \(n'' = \{n_1\}\{1\}\).  

Now, we are in the position to prove Theorem 1 by combining Lemma 1, Eq. (A6), and explicitly calculating the maximal eigenvalue of the resulting 2 \times 2 matrix.
To see that inequality (A13) holds, note from Eq. (A11) that this inequality is equivalent to:

\[
\frac{1}{2}(\alpha(x + y) + \sqrt{\alpha^2(x - y)^2 + 4z^2}) \leq \alpha, \\
\iff z^2 \leq \alpha^2(1-x)(1-y).
\] (A15)

From the non-negativity of \(1 - x, 1 - y\), the relationship between \(x, y, z\) given in Eq. (A12), and the assumption that \(\alpha \geq \frac{2^{m-1}}{2^m - 1}\), we see that proving Eq. (A13) for this interval of \(\alpha\) amounts to proving

\[
z^2 \leq \left(\frac{2^{m-1}}{2^m - 1}\right)^2 (1-x)(1-y),
\] (A16)

\[
\iff 2^{m-1}(2^{m-1} - 2)xy \leq 1 - x - y.
\]

For the convenience of subsequent discussions, let us define

\[
x_i := \cos^2 \theta_i, \quad y_i := \sin^2 \theta_i.
\] (A17)

Using the mathematical identity \(\prod_{i=1}^{m-1} (x_i + y_i) = 1\), we can now make both sides of the above inequality a degree \(2(m-1)\) homogeneous polynomial in the variables \(\{x_i, y_i\}_{i=1}^{m-1}\), namely,

\[
2^{m-1}(2^{m-1} - 2) \prod_{i=1}^{m-1} x_i y_i \leq \prod_{i=1}^{m-1} (x_i + y_i)
\]

\[
- \prod_{i \in H} x_i y_i \prod_{j \in H^c} \xi_j(\xi_j - \xi_j^2).\]

Combining these expressions while recalling from Eq. (A17) the non-negativity of \(x_i, y_i\), we then have

\[
g + \bar{g} - 2 \prod_{i=1}^{m-1} x_i y_i = \prod_{i \in H} x_i y_i \prod_{j \in H^c} (\xi_j - \xi_j^2) \geq 0,
\] (A20)

where the non-negativity of the overall expression follows from it being the square of some polynomial. To complete the proof, it suffices to note that for all \(g\) appearing in the RHS of Eq. (A15), \(\bar{g}\) also appears on the RHS as one of the \(2^{m-1}(2^{m-1} - 2)\) monomials. Thus, it follows from Eq. (A20) that the RHS-LHS of Eq. (A18) is indeed a SOS and thereby shows the validity of inequality (A18), as well as that of Eq. (A13). The proof of Eq. (A14) proceeds analogously to that given above.

Finally, to see that the \(m\)-separable bound given in Eq. (A13) is tight, it suffices to note that inequality (A13) is saturated when \(\sin^2 \theta_i = 1, \cos^2 \theta_i = 0\) (or \(\sin^2 \theta_i = 0, \cos^2 \theta_i = 1\)) for every \(i\). Similarly, the \(m\)-separable bound given in Eq. (A14) is tight as the corresponding bound is saturated when \(\sin^2 \theta_i = \cos^2 \theta_i = 1/2\) for every \(i\).

Note that the non-\(m\)-separability of a state also gives nontrivial information about its entanglement depth. For example, if a state \(\rho\) is 3-separable, then \(\rho\) is at least \(\left\lceil \frac{3}{7} \right\rceil\)-producible. Likewise, an \(m\)-separable state is at least \(\left\lceil \frac{m}{m} \right\rceil\)-producible. Thus, if the measured value of \(s = \langle M_Z + M_X \rangle\) for a state is such that \(\frac{3}{4} \geq s > \frac{5}{4}\), then the measured state is not 3-separable, but is bipartite; its entanglement depth is thus at least \(\left\lceil \frac{2}{2} \right\rceil\).
c. White-noise robustness of the non-m-separability
witnesses for GHZ states

In general, one may hope to apply our non-m-separability witnesses of Eq. \(A2\) to deduce some non-trivial lower bound on the entanglement intactness of any n-qubit (entangled) state prepared in the laboratory. In practice, however, as with any other entanglement witnesses, they are not without their limitations. For example, it is easy to verify that the entanglement present in the n-qubit \(W\) state cannot be certified at all by an evaluation of our witness \(\langle W^m_n(\alpha) \rangle\). Rather, our witness \(\langle W^m_n(\alpha) \rangle\) seems to be better suited for certain classes of states, such as the n-partite GHZ state \(\left| \text{GHZ}_n \right\rangle = \frac{1}{\sqrt{2}} \left( \left| 0 \right\rangle^\otimes n + \left| 1 \right\rangle^\otimes n \right)\), and their generalization (more on this below). In fact, the proposed witness with any parameter \(\alpha \in (0, 2]\) can detect the genuine n-partite entanglement present in these states.

To see how our witnesses fare in the presence of (white) noise, let us consider the mixed state

\[
\rho = (1 - p_{\text{noise}}) \left| \text{GHZ}_n \right\rangle \left\langle \text{GHZ}_n \right| + p_{\text{noise}} \frac{I_n}{2^n}.
\]  

(A21)

Evaluating \(\langle W^m_n(\alpha) \rangle\) against this state gives

\[
\alpha \langle M_Z \rangle + \langle M_X \rangle = (\alpha + 1)(1 - p_{\text{noise}}) + \alpha \frac{p_{\text{noise}}}{2^n - 1}.
\]  

(A22)

Comparing this with the maximal value of \(\langle W^m_n(\alpha) \rangle\) attainable by a biseparable state, i.e., \(\frac{\alpha}{2} + 1\), leads to the identification of

\[
p_{\text{noise}} < \frac{\alpha}{2 + (2 - 2^{2-n})\alpha}.
\]  

(A23)

as a genuine multipartite-entangled \(\rho\). Evidently, to maximize the right-hand-side of Eq. \(A23\), the optimal choice of \(\alpha \in (0, 2]\) is given by \(\alpha = 2\). This leads to

\[
p_{\text{noise}} < \frac{1}{3 - 2^{-n}},
\]  

(A24)

which tends to \(\frac{1}{3}\), i.e., less than \(\frac{1}{2}\)—the maximal noise tolerance achievable with the more well-known witness tailored for the GHZ state, \(W = \frac{1}{\sqrt{2}} \left| \text{GHZ}_n \right\rangle \left\langle \text{GHZ}_n \right|\).

When the noise parameter \(p_{\text{noise}}\) of \(\rho\) in Eq. \(A21\) increases, the state becomes more segregated, thus showing larger values of entanglement intactness. For the detection of the non-m-separability of these states, we compare instead the expectation value of Eq. \(A22\), with the \(m\)-separable bound of \(\frac{\alpha}{2^m - 1} + 1, \alpha\). Because of the linearity of these expressions in \(\alpha\), the optimal choice of \(\alpha\) takes place when they are equal, i.e., when \(\alpha = \frac{1}{1 - 2^{m-n-1}}\), thereby giving an \(m\)-separable bound of \(\frac{2^{m-1}}{2^{m-1} - 1}\). Solving for the corresponding threshold noise parameter shows that our witnesses reveal an upper bound on the entanglement intactness of \(m - 1\) for \(\rho\) whenever

\[
p_{\text{noise}} < \frac{2^m - 2}{2(2^m - 2^{m-n} - 1)}.
\]  

(A25)

d. White-noise robustness of the non-m-separability
witnesses for GHZ-like states

Having understood how our witnesses for non-m-separability work for GHZ states and their mixture with white noise, we now perform a similar analysis for the generalized GHZ state involving an arbitrary coherent superposition between \(\left| 0 \right\rangle^\otimes n\) and \(\left| 1 \right\rangle^\otimes n\). Specifically, let \(\left| \text{GHZ}_n(\theta, \phi) \right\rangle \equiv \cos \theta \left| 0 \right\rangle^\otimes n + e^{i\phi} \sin \theta \left| 1 \right\rangle^\otimes n\) where \(\theta \in (0, \frac{\pi}{2}]\) while \(\phi \in (0, 2\pi]\). For simplicity, our discussion here will focus mainly on the detection of genuine n-partite entanglement present in (the noisy version of) such states.

To this end, note that for \(\left| \text{GHZ}_n(\theta, \phi) \right\rangle\), the linear combination of expectation values [appearing in \(\langle W^m_n(\alpha) \rangle\) and \(\langle W^m_n(\alpha) \rangle\), see the last paragraph of Sec. II] gives

\[
\alpha \langle M_Z \rangle + \langle M_X \rangle = \alpha \pm \sin 2\theta \cos \phi.
\]  

(A26)

Clearly, if we take \(\alpha = 2\), then independent of the value of \(\theta \in (0, \frac{\pi}{2}]\) and except when \(\phi = \frac{\pi}{2}, \frac{3\pi}{2}\), the right-hand-side of the above expression—after maximizing over both signs \(\pm\)—always exceeds the biseparable bound of \(\alpha = 2\). Thus, the genuine multipartite entanglement of almost all \(\left| \text{GHZ}_n(\theta, \phi) \right\rangle\), except when \(\phi = \frac{\pi}{2}, \frac{3\pi}{2}\), can be certified via our nonseparability witnesses given in Eq. \(A2\). In fact, even the genuine multipartite entanglement present in the two remaining cases can be taken care of analogously by applying an appropriate local unitary transformation to \(W(\alpha)\). Specifically, for \(\phi = \frac{\pi}{2}, \frac{3\pi}{2}\), it suffices to apply the unitary transformation \(U^\otimes n \to (\frac{1}{\sqrt{2}} \left| 0 \right\rangle^\otimes n \pm \frac{1}{\sqrt{2}} \left| 1 \right\rangle^\otimes n)\) to \(W(\alpha)\), an evaluation of the resulting witness for \(\left| \text{GHZ}_n(\theta, \phi = \frac{\pi}{2}) \right\rangle\) then gives \(\alpha + \sin 2\theta\), which always exceeds the biseparable bound of \(\alpha\).

To determine the white-noise robustness of our witnesses against these generalized GHZ states, we consider

\[
\rho = (1 - p_{\text{noise}}) \left| \text{GHZ}_n(\theta, \phi) \right\rangle \left\langle \text{GHZ}_n(\theta, \phi) \right| + p_{\text{noise}} \frac{I_n}{2^n}.
\]  

(A27)

By a calculation similar to that presented in the last section [but now considering both \(\langle W^m_n(\alpha) \rangle\) and \(\langle W^m_n(\alpha) \rangle\)], one finds that the genuine n-partite entanglement present in these noisy versions of \(\left| \text{GHZ}_n(\theta, \phi) \right\rangle\) can always be certified as long as

\[
p_{\text{noise}} < \frac{\sin 2\theta |\cos \phi|}{2 + \sin 2\theta |\cos \phi| - 2^{-2-n}}.
\]  

(A28)

Likewise, it can be shown that as long as

\[
p_{\text{noise}} < \frac{2^n(2^m - 2) \sin 2\theta |\cos \phi|}{2^n(2^m - 2) \sin 2\theta |\cos \phi| + 2^m(2^n - 2)}.
\]  

(A29)

\[3\] The performance of these witnesses for some other experimentally inspired noise model can be found in Appendix B.
one could certify that the $\rho$ given in Eq. (A27) has an entanglement intactness upper bounded by $m - 1$.

2. A family of witnesses for non-$k$-producing with two local measurement settings

Our witnesses for entanglement depth have their origin in the family of device-independent (DI) witnesses for entanglement depth given in Ref. [24]:

$$T_n^k(\gamma) = \frac{\gamma}{2m} \sum_{\bar{x} \in \{0,1\}^n} E_n(\bar{x}) - E_n(\bar{1}) \leq S_{k}^{\text{Q}^n}(\gamma),$$

(A30)

where $\bar{x} = (x_1, x_2, \ldots, x_n)$ is an $n$-component vector describing the combination of measurement settings, $x_i \in \{0,1\}$, $E_n(\bar{x})$ is the $n$-partite full correlator (the expectation value of an $n$-partite $\pm 1$-valued outcome observable), and $S_{k}^{\text{Q}^n}(\gamma)$ is the maximal quantum value of $T_n^k(\gamma)$ attainable by any $k$-producible state. Some explicit values of these DI $k$-producible bounds (which hold for arbitrary dimensional $k$-producible states and arbitrary local $\pm 1$-valued observables) for the case of $\gamma = 2$ are [24]:

$$S_{1}^{\text{Q}^n} = 1, \quad S_{2}^{\text{Q}^n} = \sqrt{2}, \quad S_{3}^{\text{Q}^n} = \frac{3}{2}, \quad S_{4}^{\text{Q}^n} = 1.8428 \text{ etc.}$$

For an $n$-partite GHZ state, a good choice of local observables inspired by those of Ref. [24] is given by setting $A_{\pm} = \cos \theta_{\pm} \sigma_x + \sin \theta_{\pm} \sigma_y$, where $\theta_{\pm} \in \mathbb{R}$, while $A_-$ and $A_+$ are, respectively, the local observables for $x_i = 0$ and $x_i = 1$ (for all $i$). In particular, for the 8-partite states that we managed to produce experimentally, based on the measured values of $\langle M \rangle$ and $\langle M_X \rangle$, our offline numerical optimizations suggest that $\theta_{\pm} = \frac{3(1 \pm 8 \lambda)}{80}$ is a reasonably robust choice for witnessing the entanglement depth. Substituting these into the left-hand-side of Eq. (A30) and denoting the global Hermitian observables as $W_{de}$, i.e., $T_n^k = \text{tr}(\rho W_{de}^n)$, we then see that:

$$W_{de}^n(\gamma) = \gamma \left( A_+ + A_- \right) \otimes^n - A_+ \otimes^n = \gamma^{1/n} A - A_+ \otimes^n,$$

(A31)

where $A = \left( A_+ + A_- \right) \otimes^n$ is a $\pm 1$-valued Hermitian observable and $\kappa = \cos \frac{3}{10}$ is a normalization constant.

As Eq. (A30) holds for an arbitrary choice of local observables, we thus see that $\langle W_{de}^n(\gamma) \rangle \leq S_{k}^{\text{Q}^n}(\gamma)$ already represents a family of witnesses for entanglement depth. For the specific choice of observables given above, however, these $k$-producible bounds can be considerably tightened via numerical optimizations.

Specifically, our goal is to compute

$$\beta_{8,k}(\gamma) = \max_{k\text{-prod. } \rho} \text{tr} \left[ \rho W_{de}^n(\gamma) \right],$$

(A32)

i.e., to optimize the expectation value of $W_{de}^8$ over all possible 8-qubit $k$-producible states. A few simplifications can immediately be made. First, since the objective function $\text{tr} \left[ \rho W_{de}^8(\gamma) \right]$ is linear in $\rho$, there is no need to consider convex mixtures of $k$-producible 8-qubit states in the optimization. In other words, it suffices to consider $\rho = \otimes_{i=1}^m \rho_i$ where each $\rho_i = |\psi_i\rangle \langle \psi_i|$ is at most $k$-partite. Second, as $W_{de}^8$ is invariant under arbitrary permutation of parties, it suffices to consider one particular partitioning separating the 8 parties into $\lfloor \frac{8}{k} \rfloor$ groups of $k$ parties (possibly plus a remaining group of 8 mod $k$ parties).

Even with these simplifications, there is no straightforward way to determine the values of $\beta_{8,k}$, as the characterization of separable states—and, more generally $k$-producible quantum states—is a computationally difficult problem. Instead, we numerically determine some (matching) upper bound for the $k$-producible bound $\beta_{8,k}$ by employing (and generalizing) the idea of symmetric extension proposed in Ref. [24] to the present problem.

For example, in order to determine (an upper bound on) the 3-producible bound $\beta_{8,3}$, it suffices to consider $\rho = \rho_A \otimes \rho_B \otimes \rho_C$ where both $\rho_A$ and $\rho_B$ are three-qubit states and $\rho_C$ is a two-qubit state. Clearly, for all such states, there exists an $(n_1, n_2, n_3)$-copy symmetric extension $\rho$ (e.g., $\rho = \rho_A^{n_1} \otimes \rho_B^{n_2} \otimes \rho_C^{n_3}$) such that $\pi \rho \pi = \rho$ and $\text{tr}_{A \otimes B \otimes C}(\gamma^{n_1} \otimes \gamma^{n_2} \otimes \gamma^{n_3})\rho = \rho$ where $\pi$ is the projector onto the symmetric subspace of $n_1$ copies of A’s Hilbert space, $n_2$ copies of B’s Hilbert space, and $n_3$ copies of C’s Hilbert space while $\text{tr}_{A \otimes B \otimes C} = 1$ means a partial trace over $n_1 - 1$ copies of A’s Hilbert space etc. Therefore, a legitimate upper bound on $\beta_{8,3}$ can be obtained by solving the following semidefinite program:

$$\begin{array}{l}
\max \quad \text{tr} \left[ \rho W_{de}^n(\gamma) \right], \\
\text{s.t.} \quad \rho \geq 0, \quad \text{tr}(\rho) = 1, \\
\rho \geq 0, \quad \text{tr}(\rho) = 1, \quad \pi \rho \pi = \rho \\
\rho^{T_j} \geq 0 \quad \forall \; j \in \mathcal{I}
\end{array}$$

(A33)

where $\mathcal{O} \geq 0$ represents the positive-semidefinite requirement of $\mathcal{O}$, $\rho^{T_j}$ represents the partial transposition [39] of $\rho$ with respect to subsystem $j$ and $\mathcal{I}$ is the set of indices representing all possible combinations of varying number of copies of A, B, and C’s Hilbert spaces. Therefore, if $n_1 = n_2 = 1$ and $n_3 = 2$, the last line of Eq. (A33) represents the following set of constraints:

$$\rho^{T_A}, \quad \rho^{T_B}, \quad \rho^{T_{CC}}, \quad \rho^{T_C}, \quad \rho^{T_{AC}}, \quad \rho^{T_{BC}}.$$ 

(A34)

In this particular case, the upper bound on $\beta_{8,3}(\gamma = 2)$ that we obtained is 1.1699.

In general, the upper bounds that we obtained by solving Eq. (A33) are not necessarily tight, as the set of $\rho$ that we optimized over is generally a superset of the set of $k$-producible states. In our case, however, we could certify the tightness of these bounds by explicitly parametrizing a general 8-qubit $k$-producible pure state, a general dichotomic qubit observable (3 parameters for every such observable), and applying standard (but heuristic) algorithms to optimize Eq. (A32) over all these parameters 1000 times for each value of $k$. Our results for these optimizations are summarized in Table [III] and in Fig. [5].
The detailed experimental setup is shown in Fig. B1. A femtosecond pulse, which is with a duration of 130 fs, central wavelength of 780 nm, repetition rate of 80 MHz and power intensity of 3.8 W, is focused to an LBO crystal with a waist of 50 μm by a biconvex lens with focal length of 50 mm. As the instantaneous intensity of the focused pulse on LBO is extremely high, we add a Y-direction transition stage under the LBO, which moves 50 μm every 30 seconds to avoid destroying the LBO crystal. With such a device, we can observe a stable second harmonic generation (SHG) with an efficiency of 42.1%, i.e., generating ultraviolet pulses with an average power of 1.6W. The optical mode of the generated ultraviolet pulse disperses differently in the x direction and the y direction. To get a good Gaussian mode, we use two cylindrical lenses—one works for the x direction and the other works for the y direction—to reform the beam. After the reforming, the ultraviolet pulse is a Gaussian-like beam focused onto the first BiBO crystal with a beam waist of 170 μm. We use 0.6 mm BiB₃O₆ (BiBO) crystals cut at (111.4°, 55.1°) to generate entangled-photon pairs [11], in which case the two cones overlap along two lines separated by an angle of 6.9°. Compared to the traditional Beta barium borate (BBO) crystals [31], BiBO crystals with these cutting angles are expected to have a smaller spatial walk-off angle and a higher type-II second-order nonlinear coefficient [32]. Then, the entangled-photon pairs can be generated with a higher probability, and collected with a higher efficiency. The ultraviolet pulse is directed to the second BiBO crystal (BiBO₂), and refocused on the second BiBO₂ by a biconvex lens (f = 100mm) with a beam waist of 170μm. The same operations are performed successively on the ultraviolet pulse to make it shine on BiBO₃ and BiBO₄ with the same beam waist of 170μm. With this choice of beam waist and under such a pumping condition, SPDC on BiBO crystal takes place with a considerable generation rate and good collective efficiency. During our experiment, to suppress the higher-order emission rate in SPDC, we attenuated the average power of ultraviolet pulse to 500 mW, under which we observed 3×10⁸ two-fold coincidences in four entangled photon pairs. The average collection efficiency in modes 1-8 is 39% with maximal 42% and minimal 37%.

In our experiment, there are four active interferometers at minimum, and seven at most. To make sure that the interfered photons arrive at the polarization beamsplitter simultaneously, we add a motor-controlled prism in one arm of each interferometer [as shown in Fig. B1(b)]. The stepping motor moves the prism with a minimum step size of 1μm, which is good enough to find the biggest visibility in the interference (the coherence length in our case is 200μm). The seven motor-controlled prisms need to be controlled by a strict sequence. We divide them into three layers: The first layer contains four prisms, which are embedded in the four interferometer generating entangled photon pairs; the second layer contains two prism which are employed in the interferometer to generate four-photon entanglement; the last layer contains one prism in the interferometer to generate eight-photon entanglement. There is no order in controlling the prisms in the same layer, but the controlling order between different layers must follow: layer 1 → layer 2 → layer 3. When collecting data, all seven stepping motors need to be stable for dozens of hours simultaneously.

The three PBSs embedded in the interferometric geometry are attached in a lifting platform shown in Fig. B1(c). The lifting platform is a z direction transition stage with maximal tuning range of 25 mm, which is larger than our cube (12.7 mm). We emphasize that the prisms in the second and third layers should be adjusted accordingly when generating different entangle-

| k | γ | βₘ,k(γ) | Copies | Dimensions |
|---|---|---------|--------|-----------|
| 1 | 2 | 0.8365 | (1,1,1,2) | (4.4,4.4) |
| 2 | 2 | 1.0450 | (1,1,1,2) | (8.8,4.4) |
| 2 | 1.6 | 0.7904 | (1,1,1,2) | (16.16) |
| 3 | 2 | 1.1699 | (1,1,1,2) | (32.8) |
| 3 | 1.6 | 0.9137 | (1,1,1,2) | (64.4) |
| 4 | 2 | 1.3856 | (1,1) | (128.2) |
| 5 | 2 | 1.6357 | (1,1) | (128.2) |
| 6 | 2 | 1.8858 | (1,1) | (128.2) |
| 7 | 2 | 2.0578 | (1,1) | (128.2) |

TABLE III. Summary of numerically determined k-producible bounds βₘ,k(γ) for k = 1,2,⋯,7 and some specific values of γ. The last two columns give, respectively, the number of copies considered for each group and the Hilbert space dimension of each group in our computation of the (matching) upper bound on βₘ,k(γ).
The generated photon pairs have correlated polarization. In the ideal scenario, the polarization of these photon pairs is described by the maximally entangled two qubit state $|\Psi^{+}_{ij}\rangle = \frac{1}{\sqrt{2}}(|H_{o}V_{e}\rangle + |V_{o}H_{e}\rangle)_{ij}$, where the subscript o(e) represents the ordinary (extraordinary) component and $i,j$ denote the path label. Then, $|\Psi^{+}_{ij}\rangle$ is overlapped on PBS and becomes $|\Phi^{+}_{ij}\rangle = \frac{1}{\sqrt{2}}(|H_{o}H_{e}\rangle + |V_{o}V_{e}\rangle)_{ij}$. In order to get a better indistinguishability, we filter the photons with proper full width at half of the transmittance maximum (FWHM) depending on whether it is an o-component or an e-component light.

Specifically, in our experiment, the photons in path modes 1, 4, 6, 8 are o-component light and filtered by a narrow band filter with $\Delta_{\text{FWHM}} = 4.6$ nm. The photons in path modes $2',3',5',7'$, on the other hand, are e-component light and are filtered by a narrow band filter with $\Delta_{\text{FWHM}} = 2.8$ nm. With such filter settings, we observe an eight-fold coincidences of 70/h in creating $|G_{2222}\rangle$. Each interference on PBS$_{1}$, PBS$_{2}$ or PBS$_{3}$ will reduce half of the eight-fold coincidence due to postselecting probability. We experimentally observed that the eight-fold coincidences in creating $|G_{422}\rangle$, $|G_{44}\rangle$, $|G_{62}\rangle$, and $|G_{8}\rangle$ are 36/h, 20/h, 20/h, and 8/h, respectively. The total eight-fold coincidences we collected in measuring $\langle M_{Z}\rangle$, $\langle M_{X}\rangle$, $\langle A\rangle$, and $\langle A'\rangle$ are shown in Table B1.

### 2. Imperfections and noise model

The experimental imperfections are (mainly) caused by the higher-order emissions in SPDC and the mode mismatch of the interference when superposing photons on PBS to connect entangled photon pairs. The influence of these imperfections can be reflected by the interference...
visibility. We define the visibility for an experimentally generated state as

\[
v = \frac{\text{Target state} - \text{Noisy terms}}{\text{Target state} + \text{Noisy terms}}. \tag{B1}\]

For the entangled photon pair, the state can be written as

\[
\rho_2 = \frac{1 + v_1}{2} |\text{GHZ}_2\rangle \langle \text{GHZ}_2| + \frac{1 + v_2}{2} \frac{\mathbb{I}_2}{2^2}, \tag{B2}
\]

where \(v_1\) represents the visibility of the entangled photon pair. The imperfections mainly come from the higher-order emission in SPDC. Similarly, the four-photon GHZ state can be written as

\[
\rho_4 = \frac{1 + v_2}{2} \left( \frac{1 + v_1}{2} \right)^2 |\text{GHZ}_4\rangle \langle \text{GHZ}_4| + \left[ 1 - \frac{1 + v_2}{2} \left( \frac{1 + v_1}{2} \right)^2 \right] \frac{\mathbb{I}_4}{2^4}, \tag{B3}
\]

where \(v_2\) represents the visibility of interference on PBS.

As two entangled photon pairs are involved when generating \(\rho_4\), the factor \(\left( \frac{1 + v_1}{2} \right)^2\) is added. Generating an \(n\)-photon GHZ state requires \(n/2\) entangled photon pairs and \(n/2 - 1\) PBSs to connect them. Thus, an \(n\)-photon GHZ state, where \(n\) is an even number, can be written as

\[
\rho_n = \left( \frac{1 + v_2}{2} \right)^{\frac{n}{2} - 1} \left( \frac{1 + v_1}{2} \right)^{\frac{n}{2}} |\text{GHZ}_n\rangle \langle \text{GHZ}_n| + \left[ 1 - \left( \frac{1 + v_2}{2} \right)^{\frac{n}{2} - 1} \left( \frac{1 + v_1}{2} \right)^{\frac{n}{2}} \right] \frac{\mathbb{I}_n}{2^n}. \tag{B4}
\]
With this model, we calculate how the visibilities are related to our two witnesses. The calculations are shown in Fig. B2.

\[
\rho_n = (1 - \gamma_d^{(n)} - \gamma_w^{(n)})|\text{GHZ}_n\rangle\langle\text{GHZ}_n| + \frac{\gamma_d^{(n)}}{2} \left[ (|H\rangle\langle H|)^{\otimes n} + (|V\rangle\langle V|)^{\otimes n} \right] + \frac{\gamma_w^{(n)}}{2^n} |I_n\rangle\langle I_n|. \tag{B5}
\]

In Eq. (B5), the first term describes the contribution from a genuine \( n \)-photon GHZ entangled state. The second term accounts for the imperfection of interference, which occurs with a probability of \( \gamma_d^{(n)} \), where the polarized beam splitter does not superpose the photons from its two inputs. Experimentally, this is caused by the mode mismatch, including the mismatch of a narrowband filter, the misalignment of the beams' direction, and other imperfections. The last term in Eq. (B5) represents the higher-order emissions in SPDC processes, which is modeled by the white noise with corresponding probability \( \gamma_w^{(n)} \). The model we propose here is consistent with the observation that the expected value of \( M \) is much larger than that of \( W \), and photonic interferometer.

According to the model described in Eq. (B5), we can determine that the amount of noise our measurement of the witnesses \( \{W_{se}^n(n = 2)\} \) and \( \{W_{de}^n(\gamma = 2)\} \) may tolerate with respect to the states \( \rho_8, \rho_{62}, \rho_4, \rho_{44}, \rho_{222}, \rho_{2222} \) may be compared to the main contributions \( \gamma_d^{(6)} \) and \( \gamma_d^{(4)} \) from \( \rho_6 \) and \( \rho_4 \). The calculated results are shown in Fig. B2.

From Fig. B3, we observe that \( W_{se} \) can tolerate more noise than \( W_{de} \), so experimentally \( W_{se} \) witnesses more precisely than \( W_{de} \). We also estimate \( \gamma_d^{(n)} \) and \( \gamma_w^{(n)} \) of \( \rho_n \) by the measurements \( M_Z \) and \( M_X \), and mark them in Fig. B3 by a black circle. Note that \( \gamma_d^{(n)} \) and \( \gamma_w^{(n)} \) are related to \( \langle M_Z \rangle \) and \( \langle M_X \rangle \) by

\[
\langle M_Z \rangle = \text{tr}(M_Z \rho_n) = 1 - \frac{2^{n-1} - 1}{2^{n-1}} \gamma_w^{(n)}, \tag{B6}
\]

\[
\langle M_X \rangle = \text{tr}(M_X \rho_n) = 1 - \gamma_w^{(n)} - \gamma_d^{(n)}. \tag{B6}
\]

By measuring \( \langle M_Z \rangle \), \( \langle M_X \rangle \) and using Eq. (B6), we can calculate the values of \( \gamma_d^{(n)} \) and \( \gamma_w^{(n)} \).

For the state \( \rho_{2222} \), the white noise model fits very well, and there is no interference between independent SPDC processes; thus, we simplify the noise model of Eq. (B5) to consider solely the effect of white noise as

\[
\rho_2 = (1 - \gamma_w^{(2)})|\text{GHZ}_2\rangle\langle\text{GHZ}_2| + \frac{\gamma_w^{(2)}}{2} |I_2\rangle\langle I_2|. \tag{2}
\]

The imperfections can also be modeled by noise. For the experimentally prepared \( n \)-photon state \( \rho_n \) \( (n > 2) \) can be represented as follows.

3. Algorithmic procedure to deduce a minimal entanglement structure

In this section, we give a procedure to systematically deduce a minimal entanglement structure by using the results of \( \langle W_{se} \rangle \) and \( \langle W_{de} \rangle \). As shown in Fig. B5, we need to follow three steps to systematically deduce the underlying entanglement structure.

**Step 1:** For a given \( n \)-partite state \( \rho \), we check whether it is genuinely \( n \)-partite entangled or not by measuring the witness \( W_{se}(\alpha) \). If it is, the task is completed, otherwise, we proceed to step 2.

**Step 2:** The extent to which the state is \( (n) \)-separable for \( m > 1 \) can be analyzed by using the measured value obtained in step 1 and considering the different \( m \)-separable bounds given in Eq. (A2). Concurrently, we perform the measurement needed to evaluate \( \langle W_{de}(\gamma) \rangle \). As with the case of separability, a lower bound on the entanglement depth can be obtained by analyzing the measured value against the various \( k \)-producible bounds.

**Step 3:** Based on the results from step 2, we can conclude that the entanglement intactness and entanglement depth are, respectively, upper and lower bounded solv \( m \leq M \) and \( k \geq K \). From here, based on the data obtained during the measurement of \( \langle W_{se}^8(\alpha) \rangle \) and \( \langle W_{de}^8(\gamma) \rangle \), we may evaluate \( \langle W_{se}^{n'}(\alpha) \rangle \) for all combinations of \( n' \geq K \) parties to determine which among the \( n \) parties exhibit genuine \( K \)-photon (or more-partite) entanglement, and which exhibit less-partite entanglement.

---

4 Strictly speaking, this value of \( \gamma_w^{(2)} \) corresponds to that obtained by a better fitting of the experimental data to some \( |\text{GHZ}_2(\theta, \phi)\rangle \) (for \( \theta \neq \frac{\pi}{2} \)) state admixed with white noise.
FIG. B3. Calculation of the noise tolerance for our measurement of $\langle W_{de}^{8}(\alpha = 2) \rangle$ (top four plots) and $\langle W_{se}^{8}(\gamma = 2) \rangle$ (bottom four plots) with respect to $\rho_{44}$, $\rho_{22}$, $\rho_{42}$ and $\rho_{24}$ (from left to right). The straight line represents combinations of noise parameters where the values of the witnesses for $m$-separability and $k$-producibility (for appropriate values of $m$ and $k$, respectively) are saturated, see Eq. (B3). Here, $\gamma_{w}^{(8)}(\gamma_{d}^{(8)})$, $\gamma_{w}^{(6)}(\gamma_{d}^{(6)})$, $\gamma_{w}^{(4)}(\gamma_{d}^{(4)})$ denote the probability of white noise (decoherence noise) in preparing $|\text{GHZ}^{8}\rangle$, $|\text{GHZ}^{6}\rangle$ and $|\text{GHZ}^{4}\rangle$. The $\gamma_{w}^{(8)}$ are negligible compared to $\gamma_{w}^{(6)}$ and $\gamma_{w}^{(4)}$, so we do not consider them in the calculation of $\rho_{422}$, $\rho_{44}$ and $\rho_{42}$. For example, in the rightmost plots (for $\rho_{422}$), the straight line corresponds to the combination of $\gamma_{d}^{(8)}$ and $\gamma_{d}^{(4)}$ such that the 4-separable bound (upper plot) and the 3-producible (lower plot) bound are saturated. The black circle marks the noise parameters estimated from our measured value of $\langle M_{X} \rangle$ and $\langle M_{Z} \rangle$.

4. More experimental results for the deduction of a minimal entanglement structure

The entanglement structure can be deduced by employing the procedure described in Appendix B3. We show the results for $\rho_{422}$ and $\rho_{2422}$ in the main text. For state $\rho_{422}$, we show that photons in path mode $\{5, 6', 7, 8\}$ are four-photon entangled [Fig. 4(a) in the main text]. We omit the results of searching three-party GMEs in $\{1, 2', 3', 4\}$. Below, we show that our measurement of $\langle W_{se}^{3}(\alpha = 2) \rangle$ does not reveal any three-photon entanglement for any possible three-photon combination in $\{1, 2', 3', 4\}$. The results are shown in Fig. B6. We then search for two-party entanglement. The results are shown in Fig. 4(b) in the main text.

For state $\rho_{42}$, the measurement result $\langle W_{de}^{4}(\gamma) \rangle = 1.29 \pm 0.08$ indicates that there is at least 4-photon entanglement in $\rho_{42}$. So, we first try to identify the parties that exhibit this four-photon entanglement in $\rho_{42}$. As shown in Fig. B7(a), there are 7 four-photon combinations that violate the bipartite bound of $\langle W_{de}^{4}(\gamma) \rangle$, therefore indicating the presence of four-photon entanglement among these parties. However, the measured value of $\langle W_{de}^{4}(\gamma = 2) \rangle$ for the complementary set of parties does not reveal any four-photon entanglement. As $\langle W_{se}^{3}(\alpha = 4/3) \rangle = 1.43 \pm 0.07$ indicates that $m \leq 2$ for $\rho_{42}$, these results suggest that $\rho_{42}$ does not have the entanglement structure of $|G_{44}\rangle$. Similarly, the results in Fig. B7(b) suggest that $\rho_{42}$ does not have the entanglement structure of $|G_{35}\rangle$ either. Rather, an entanglement structure of $\rho_{42}$ that is compatible with our measurement results is that of $|G_{62}\rangle$ [Fig. B7(c)].

With the same procedure, according to the results $m \leq 2$, $k \geq 3$ (shown in the main text), we find that $\rho_{44}$ may not have the entanglement structure of $|G_{35}\rangle$. Instead, our results shown in Fig. B8 suggest that one possible entanglement structure of $\rho_{44}$ is that given by $|G_{44}\rangle$. 
**FIG. B5.** Procedure to deduce the underlying entanglement structure. Each orange box represents steps that only involve classical calculation (i.e., no consumption of any quantum resource is required). The blue box represents steps where both quantum resource and classical calculations are needed.

**FIG. B6.** Estimated value of the three-body entanglement witness $\langle W_{3e}^3(\alpha) \rangle$ for $\{1, 2', 3', 4\}$ of $\rho_{422}$ based on the data acquired during the measurements of $\langle W_{3e}^3(\alpha) \rangle$.

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FIG. B7. Results pertinent to the entanglement structure deduction for $\rho_{62}$

FIG. B8. Results pertinent to the entanglement structure deduction for $\rho_{44}$