Long-distance radiative corrections to the di-pion tau lepton decay

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We evaluate the model-dependent piece of $O(\alpha)$ long-distance radiative corrections to $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ decays by using a meson dominance model. We find that these corrections to the di-pion invariant mass spectrum are smaller than in previous calculations based on chiral perturbation theory. The corresponding correction to the photon inclusive rate is tiny ($-0.15\%$) but it can be of relevance when new measurements reach better precision.

1. INTRODUCTION

The decay $\tau^\pm \rightarrow \pi^\pm \pi^0 \nu_\tau$ ($\tau_{2\pi}$) is the dominant mode of $\tau$ lepton decays. Observables associated to this decay channel have been measured with great precision. The world average value of the $\tau_{2\pi}$ branching fraction has attained an accuracy of 0.5% \cite{1}. Similarly, the weak pion form factor has been measured with high precision, providing a valuable input to compute the dominant hadronic contribution to the muon anomalous magnetic moment \cite{2}. Further improvements in measurements of such observables are expected at B factories, where BaBar and Belle collaborations have recorded already about $10^9$ tau lepton decays \cite{3}. Therefore, present and future precision measurements of these observables demand the knowledge of radiative corrections for a correct comparison of theory and experiment.

On another hand, measurements of $\tau_{2\pi}$ decays provide a clean environment to test the Conserved Vector Current (CVC) hypothesis. As is well known, the two-pion production in tau decays and $e^+ e^-$ annihilations is driven by the isovector piece of the vector current: $\bar{q}q/2$, where $q = (\bar{u}, \bar{d})$. Thus, measurements of the pion form factor in these two reactions and of the $\tau_{2\pi}$ branching ratio can be used to provide one of the most precise tests of the CVC hypothesis. In order to perform a test of the CVC hypothesis at a few of per mille level, the isospin breaking effects must be included appropriately \cite{4}. The two identifiable sources of isospin breaking corrections in relating the $\tau_{2\pi}$ and $e^+ e^- \rightarrow \pi^+ \pi^-$ reactions are the mass difference of charged and neutral pions and, again, the radiative corrections.

The dominant piece of short-distance electroweak radiative corrections were computed long ago \cite{5}, and improvements that include resummation of dominant logs, subleading electroweak corrections \cite{6} and resummation of strong interactions corrections were subsequently incorporated \cite{7}. The effects of long-distance corrections to the hadronic spectrum of $\tau_{2\pi}$ decays were computed only recently in Refs. \cite{8,9} in the framework of Chiral Perturbation Theory supplemented with anomalous terms for the axial couplings \cite{10}. Since $\tau$ decays involve momentum transfers far from the chiral limit, an independent calculation of radiative corrections based on different model considerations is important. In this paper we evaluate the model-dependent contributions to long-distance radiative corrections in $\tau_{2\pi}$ decays based on a meson dominance model that was used in ref. \cite{11} to study the corresponding radiative tau decays ($\tau \rightarrow \pi\pi\nu\gamma$). We focus here on the long-distance radiative corrections of $O(\alpha)$ to the di-pion spectrum and to the decay rate of $\tau_{2\pi}$ decays.

2. LONG-DISTANCE CORRECTIONS TO THE DI-PION SPECTRUM

The hadronic spectrum of two-pions in tau decays, corrected by $O(\alpha)$ radiative corrections, is built out of three contributions:

$$\frac{d\Gamma(\tau_{2\pi}(t))}{dt} = \frac{d\Gamma^0}{dt} + \frac{d\Gamma^1_v}{dt} + \frac{d\Gamma^1_{r}}{dt}. \quad (1)$$

The superindexes in the r.h.s. terms denote the order in $\alpha$ and the subindex $v(r)$ refers to the virtual (real) corrections, while $t$ is the square of the two-pion invariant
mass. When we deal with the photon inclusive spectrum, integration over all energies of real photons should be done in the last term.

After regrouping the effects of all radiative corrections, the corrected spectrum can be rewritten as
\[
\frac{d\Gamma(\tau_{2\pi}(\gamma))}{dt} = \frac{G_F^2 m_\pi^2 S_{EW} |V_{ud}|^2}{384\pi^3} \beta_{\pi-}^2 \left( 1 - \frac{t}{m_\tau^2} \right)^2 \times \left( 1 + \frac{2t}{m_\pi^2} \right) |f_{+}(t)|^2 G_{EM}(t) ,
\]
(2)

where \( G_F \) denotes the Fermi constant, \(|V_{ud}| = 0.9740\) is the Cabibbo-Kobayashi-Maskawa \( ud \) matrix element, and \( \beta_{\pi-} \) is the pion velocity in the di-pion rest frame.

The weak form factor of the pion parametrizes the hadronic amplitude (an additional scalar form factor that is allowed by Lorentz covariance induces negligible small isospin breaking effects in eq. (2):)
\[
\langle \pi^+(q)\pi^0(q')|\bar{u}\gamma_{\mu}d|0 \rangle = \sqrt{2}f_{+}(t)(q - q')_\mu .
\]
(3)

Our numerical results in this paper were obtained using the pion form factor described in ref. \[11\]. If we use the form factor of ref. \[12\] we obtain almost identical numerical results.

The effects of short-distance corrections to the hadronic decay rate are encoded in \( S_{EW} = 1.0236 \pm 0.0003 \) \[7\], which includes the resummation of dominant logs, subleading corrections \[8\] and resummation of strong interactions effects \[9\]. The factor \( G_{EM}(t) \) introduced in eq. (2) contains all the effects of long-distance electromagnetic corrections of \( O(\alpha) \) and is the main focus of this paper. In terms of the rates appearing in eq. (1) it is defined as follows:
\[
G_{EM}(t) = 1 + \frac{d\Gamma_{\pi^0}^{1}}{dt} + \frac{d\Gamma_{\pi^+}^{1}}{dt} \frac{d\Gamma_{\pi^-}^{1}}{dt} .
\]
(4)

This function is finite in the limit of infrared photons since the divergent terms in virtual and real corrections cancel each other. \( G_{EM}(t) \) depends on the details of the model used to describe the interactions of photons with the hadrons. A useful theorem due to Burnett and Kroll and to Zakharov, Kondratyuk and Ponomarev \[13\], allows us to split the di-pion spectrum of radiative \( \tau_{2\pi} \) decays into its model-independent and model-dependent parts:
\[
\frac{d\Gamma_{\pi^0}^{1}}{dt} = \frac{d\Gamma_{\pi^0}^{1}(m.d.c)}{dt} + \frac{d\Gamma_{\pi^0}^{1}(m.d.d)}{dt} .
\]
(5)

The first term contains the infrared divergence in the photon energy that is necessary to cancel the corresponding divergent term in the virtual correction. It is model-independent in the sense that only the hadronic structure already present in the non-radiative decay contributes.

This term has its origin in the photons emitted off the charges of \( \tau^- \) and \( \pi^- \) as if they were taken as point particles.

The electromagnetic structure of hadrons and other model-dependent couplings of photons to hadrons enter into the second term of eq. (5), which is regular for infrared photons. In ref. \[9\] the model-dependent contributions were treated in the framework of Chiral Perturbation Theory supplemented with chiral anomalous terms. According to their results, the model-dependent terms affect the values of \( G_{EM}(t) \) by \(-0.5\%\) in most of the region of \( t \). As it was recognized in ref. \[9\], one should not expect that some of their model-dependent axial contributions remains adequate when they are extrapolated to large momenta as encountered in \( \tau \) decays. On the other hand, since the energy released to hadrons in \( \tau_{2\pi} \) decay can be as large as \( m_\tau \), some of the well established light resonances can be produced as intermediate states on their mass-shell. Given all these considerations, in ref. \[11\] we have used a meson dominance model to study the effects of model-dependent contributions in different observables associated with radiative \( \tau_{2\pi} \) decays. We have found that some of these observables exhibit important differences with respect to the results of ref. \[9\]. These differences arise from the model-dependent terms and are due essentially to the production and decay of the \( \omega(782) \) vector meson as an intermediate state in radiative tau decay. In the following we explore the consequences of such meson dominance model contributions to the model-dependent part of long-distance radiative corrections.

Using the decomposition shown in eq. (5), we can split the radiator function \( G_{EM}(t) \) as it was proposed in ref. \[9\]:
\[
G_{EM}(t) = G_{EM}^0(t) + G_{EM}^{m.d}(t) ,
\]
(6)

where the first term denotes the model-independent piece (tree-level, virtual and real), and the second term includes the model-dependent terms. In this paper we use the expression of \( G_{EM}^0(t) \) calculated in ref. \[9\]. We have checked that the piece in this term coming from model-independent bremsstrahlung is correct, and we have focused on the evaluation of the second term of eq. (6) based on the model discussed in our previous work \[17\].

In Figure 1 we have plotted our values for \( G_{EM}(t) \) (solid line). In the same plot we also display the values of \( G_{EM}^0(t) \) (short-dashed line) and the total \( G_{EM}(t) \) (long-dashed line) obtained in ref. \[9\]. The sharp increase of \( G_{EM}(t) \) close to threshold is due to the phase space suppression of the tree-level rate at those energies, the denominator of second term in eq. (4). This means that corrections can not be fully reliable for values of \( t \) very close to threshold. We observe that our model-dependent term differs from that obtained in ref. \[9\] for low and intermediate values of \( t \), finding the largest difference (of order 1%) for \( t \approx 1.5 \text{ GeV}^2 \).
contribution of the $\omega(782)$ intermediate state in radiative $\tau_{2\pi}$ decays (Figure 2.g, of ref. [11]). To confirm this, we plot in Figure 2 our results for $G_{EM}(t)$ obtained when we exclude the diagram containing the $\omega(782)$ vector meson (dotted line). It is interesting to see that when we exclude the contribution due to the $\omega(782)$ meson, our result coincides with the one obtained in ref. [9] (long-dashed line in Figure 2). Equivalently, if we exclude only the contributions involving an intermediate $a_1(1260)$ state (Figures 2.e, 2.f, 2.j and 2.k of ref. [11]), the values obtained for $G_{EM}(t)$ almost overlap the solid line in Figure 1.

The simple formula ($x = t/m_{\tau}^2$):

$$G_{EM}(t) = 1.107 - 1.326x + 5.667x^2 - 10.95x^3 + 9.735x^4 - 3.276x^5$$  \hspace{1cm} (7)

provides a good description of $G_{EM}(t)$ in the interval $0.18 \text{ GeV}^2 \leq t \leq 2.93 \text{ GeV}^2$. It differs numerically from the exact calculation by less than 0.3% for the interval under consideration. It can be useful to apply the radiative corrections to future data and/or theoretical predictions to the di-pion spectrum of photon inclusive $\tau_{2\pi}$ decays.

3. LONG-DISTANCE CORRECTIONS TO THE DECAY RATE

Radiative corrections to the decay rate of $\tau_{2\pi}$ are of interest for a precise comparison of theoretical and experimental branching ratios and also to provide a test of the CVC hypothesis at the few per mille level. This observable can be obtained by integrating eq. (1) over $t$. As usual, we define the radiative corrected decay rate as follows:

$$\Gamma_{\tau_{2\pi}(\gamma)} = \Gamma_{\tau_{2\pi}} \cdot (1 + \delta),$$  \hspace{1cm} (8)

where $\delta$ encodes the effects of long-distance corrections, while $\Gamma_{\tau_{2\pi}}$ corresponds to the integrated rate obtained when we set $G_{EM}(t) = 1$ in eq. (2).

If future measurements are able to discriminate real photons of energy larger than $\omega_0$, it is convenient to compute the radiative corrections $\delta(\omega < \omega_0)$ such that the corrected rate include photons of energies below this threshold. In Table I we show the long-distance corrections corresponding to different values of $\omega_0$.

The long-distance correction corresponding to the photon inclusive rate is shown in the last row of Table I. Just for comparison, we have evaluated this correction for the model of ref. [9] and displayed in the third column of Table I. The largest contribution to $\delta$ comes from the model-independent terms ($-0.32\%$). In our model, this correction is partially cancelled by the model-dependent contributions ($+0.17\%$). This is much smaller than the contribution of the hard-photon component of radiative corrections ($+0.8\%$) estimated in ref. [14].

| $\omega_0$ (MeV) | $\delta(\omega < \omega_0)$ in % | $\delta(\omega < \omega_0)$ in % |
|------------------|---------------------------------|---------------------------------|
| 300              | -0.31                           | -                               |
| 400              | -0.27                           | -                               |
| 500              | -0.23                           | -                               |
| 600              | -0.19                           | -                               |
| 700              | -0.16                           | -                               |
| 800              | -0.15                           | -                               |
| $\omega_{max}$   | -0.15                           | -0.38                           |

TABLE I: Long-distance radiative correction to the $\tau_{2\pi}$ decay rate when photons below the threshold energy $\omega_0$ are included.
4. CONCLUSIONS

The long-distance radiative corrections of $O(\alpha)$ to $\tau_{2\pi}$ decays in the framework of a vector dominance model, turns out to be smaller than the one calculated in ref. [9] using Chiral Perturbation theory. The corrections to the di-pion mass distribution are less than 1% in most of the relevant values of the $t$. In particular, the discrepancy between measurements of the pion electromagnetic form factors in $\tau$ lepton decays and $e^+e^−$ annihilations [2] cannot be explained by radiative corrections. The long-distance radiative correction to the photon inclusive decay rate of $\tau_{2\pi}$ ($−0.15\%$) is below the present experimental uncertainties ($±0.5\%$) but it can be of relevance for a test of the CVC hypothesis when improved measurements become available. From another perspective, we can conclude from this and previous calculations that long-distance corrections are well under control at a few per mille level.

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NOTE ADDED IN PROOFS

Model-dependent radiative corrections affect also the calculation of the two-pion vacuum polarization contribution to the muon anomalous magnetic moment ($a^{LO,\pi\pi}_\mu$) extracted from $\tau$ decay data (see for example [2]). The shift in $a^{LO,\pi\pi}_\mu$ produced by such corrections when compared to the uncorrected result is given by [3]:

$$\Delta a^{LO,\pi\pi}_\mu = \frac{1}{4\pi^3} \int_0^{m_\pi^2} dt K(t) \left( \frac{K(t) \frac{d\Gamma_{\pi\pi}(t)}{dt}}{K(t)} - 1 \right) \left( \frac{1}{G_{EM}(t)} - 1 \right)$$

$$= -3.7 \times 10^{-10}.$$

The kernel function $K(t)$ and the kinematical factors $K_{x\gamma}(t)$ can be found in reference [9]. Our result, shown in the equation above, is almost 4 times larger that the result reported in [3]. This difference can be traced back to the our model-dependent contribution involving the intermediate $\omega(782)$ vector meson (see discussion in section 2) and it was not considered in the calculation of ref. [9].

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