Constant rate linear interface depinning and self-organized criticality

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The precise determination of the universality classes in self-organized critical phenomena (SOC) is still an unsolved problem. Different SOC models like sandpile, linear interface depinning, and the Barkhausen effect have been investigated independently. In the present work we demonstrate that these models can all be mapped into a linear interface depinning model driven at constant rate. The model is found to belong to the universality class of constant force linear interface depinning above the depinning transition, with an upper critical dimension $d_c = 4$. Results are compared with numerical simulations, experiments, and previous theoretical works reported in the literature. In this way we demonstrate a precise connection between different SOC models which display the same universal behavior.

I. INTRODUCTION

About one decade ago Bak, Tang and Wiesenfeld (BTW) introduced the idea of self-organized criticality (SOC) to describe the critical behavior of a vast class of driven dissipative systems. The dynamics of SOC systems is characterized by long periods of quasi-equilibrium alternating with sudden rearrangements (avalanches) which may expand all the system. Avalanche dynamics has been observed in granular sandpiles, the Barkhausen effect, superconducting vortex piles, earthquakes, etc.

As usual the mean-field (MF) theory gave the first insight into SOC phenomena. We also count with some exact results for Abelian sandpile models, real space and momentum space renormalization group (RG) calculations, and singular diffusion equations. More recently the analogy with other non-equilibrium systems like systems with many absorbing states, directed percolation, and linear interface depinning (LID) has been exploited. This accumulated experience allowed a more precise formulation of SOC phenomena. It is now well known that the absence of control parameters, as it was believed at the early state, is not completely true. Some hidden parameters like the driving rate should be fine-tuned in order to reach the critical state.

However the precise determination of the universality classes in SOC phenomena is still an unsolved problem. For instance, it is not clear if deterministic and stochastic sandpile models belong to the same universality class. Some numerical simulations, real space RG calculations, and field theory pointed to the affirmative answer but other numerical simulations disagree. The problem becomes more difficult since in deterministic models multifractal scaling, instead of finite size scaling, is satisfied. On the other hand, some authors have pointed out the existence of universal behavior between sandpile and LID models. Paczuski and Boettcher mapped the random slope threshold sandpile model into a model of LID where the interface is pulled at one end. More recently we have extended this analogy to critical height models, which are mapped into a LID model driven at constant rate. In this case we have obtained the complete set of scaling exponents using previous RG analysis for constant force LID models. Lauritsen and Alava have also considered the analogy between critical height models and LID models. However in their work the way in which driving and dissipation take place are not well specified. The analysis is limited to a qualitative level and no rigorous determination of the scaling exponents was provided, thought they count with the previous work by us.

The way in which the interface is driven seems to be an important condition to obtain an ordinary depinning transition or SOC behavior. The interface may be driven either by extremal dynamics, or by constant force, or by constant rate. While constant force models has been extensively studied in the literature extremal dynamics and constant rate models are less known. However, in the last years, extremal dynamics and constant velocity models have gained more attention due to its relation with the theory of self-organized criticality (SOC). However, constant rate, and not constant force or constant velocity, is the predominant driving mechanism in experiments where avalanche dynamics is observed. Examples are found in granular sandpiles where the number of grains added to the system increases linearly with time, in Barkhausen and superconducting avalanches experiments where the applied magnetic field is increased at constant rate, and in earthquake dynamics where some tectonic plate dynamics gives a constant rate of stress increase leading to a stick-slip motion of other plates.

In the present work we investigate the existence of universality in different SOC models proposed in the literature, which can be mapped into a LID model driven at constant rate. The LID model driven at constant rate is taken here as the prototype of this class of SOC models because it allows some analytical treatment, which is very
difficult for instance in sandpile models. The existence of a
different driving mechanism does not change the univer-
salaty class; we show that the constant rate model belongs
to the universality class of constant force LID model, with an
upper critical dimension $d_c = 4$. Contrary to previous
works, our analysis is not limited to a qualitative level
but it is supported by a complete mapping of different
SOC models into a LID model. The complete set of scal-
ing exponents, including the avalanche exponents, are
computed using previous RG analysis developed for constant
force LID models. Part of these results are already
contained in \[25\].

The paper is organized as follows. In section II we
introduce the LID model driven at constant force and
review some of the results obtained from RG analysis.
The LID model driven at constant force has been already
investigated in the literature, we just present some im-
portant aspects which are determinant for the analysis
developed in the next section. In section III we consider
the constant rate LID model and illustrate how different
SOC models can be mapped into this model. The phase
diagram of this model is investigated in section IV. The
comparison of our results with experiments, numerical
simulations and other theoretical approaches proposed
in the literature is discussed in section V. Finally the
summary and conclusions are given in section VI.

\section*{II. CONSTANT FORCE LID MODELS}

There are many physical situations where one is in-
terested in an interface dynamics. Examples are found
in the domain wall dynamics \[41\], the displacement of
one fluid by another inside a porous media \[38,39\], con-
tact line depinning \[40\], and more \[41\]. In general a $d$-
dimensional self-affine interface, described by a single-
valued function $h(x,t)$, evolves in a $(d+1)$-dimensional
medium. Usually some kind of disorder affects the motion
of the interface leading to its roughening. Earlier studies \[12\] focus on time-independent uncorrelated dis-
order but most recent studies analyze the motion of inter-
faces under quenched disorder \[39,43,44\]. In the presence
of quenched disorder and constant force driving two uni-
versality classes has been found \[45\]. One is described by
the Kardar-Parisi-Zhang equation \[42\] with quenched
noise. In this case the interface is pinned by paths on
a directed percolation cluster of pinning sites \[44\]. The
second class is described by the Edwards-Wilkinson equa-
tion \[17\] with quenched noise, usually known as LID mod-
el.

In LID models the interface height satisfies the equa-
tion of motion

$$
\lambda \partial_t h = \Gamma \nabla^2 h + F + \eta(x,h),
$$

(1)

where $\partial_t$ denotes the partial time derivative and $\nabla$ is a $d$-dimensional Laplacian. $\lambda$ is a viscosity coefficient, $\Gamma$
the surface tension and $F$ is a constant force acting on
the interface. $\eta(x,h)$ is a random pinning force associ-
ated with the existence of random pinning centers in the
$(d+1)$-dimensional environment. In general it is assumed
that $\eta(x,h)$ is a Gaussian noise uncorrelated in the space
$x$, with zero mean and noise correlator

$$
\langle \eta(x,h)\eta(x',h') \rangle = \delta^d(x-x')\Delta(h-h').
$$

(2)

where $\Delta(-h) = \Delta(h)$ is a symmetric function, with a
fast decay to zero beyond the characteristic length of the
pinning centers.

The existence of pinning centers will carry as a conse-
quence that the interface will be not smooth. The rough-
ness of the interface is characterized by the height-height
correlation function

$$
\langle |y(x,t) - y(0,0)|^2 \rangle \sim |x|^{2\zeta}g(t/|x|^z),
$$

(3)

where $\zeta$ and $z$ are the roughness and dynamic scal-
ing exponents, respectively, and $g(x)$ a scaling function
with the asymptotic behaviors $g(x) \sim 1$ for $x \ll 1$ and
$g(x) \sim x^{2\zeta/z}$ for $x \gg 1$. The scaling exponents $z$
and $\zeta$ are related with the scaling behavior of the interface
fluctuations just at the critical state and therefore we ex-
pect that they are independent on the way the system is
driven out from the critical state.

A depinning transition takes place at certain critical
force $F_c$ determined by the disorder. For $F < F_c$ the
interface is pinned after certain finite time while for $F > F_c$
it moves with finite average velocity $v$. Above $F_c$ one
looks for a solution of eq. (1) as an expansion around
the flat co-moving interface $vt$, i.e. $h(x,t) = vt + y(x,t)$,
resulting the equation of motion for $y(x,t)$

$$
\lambda \partial_t y = \Gamma \nabla^2 y + F - \lambda v + \eta(x,vt+y).
$$

(4)

$v$ is obtained self-consistently using the constraint
$\langle y(x,t) \rangle = 0$. The noise term in eq. (4) makes this
equation non-linear requiring a RG analysis to determine the
scaling behavior in the neighborhood of the depinning
transition. This has been done using different approaches
by Natterman et al \[18\] and Narayan and Fisher \[19,50\].
The order parameter of the depinning transition is the
average interface velocity. For $F < F_c$ we have $v = 0$
while for $F > F_c$ the interface moves with the average
interface velocity $v \sim (F - F_c)^\beta$, with

$$
\beta = \nu(z - \zeta),
$$

(5)

where $\nu$ is the correlation length exponent.

The upper critical dimension is $d_c = 4$ and the scal-
ing exponents can be obtained through a functional RG
analysis. Below the upper critical dimension it results that

$$
\zeta = 4 - \frac{d}{3}, \quad z = 2 - \frac{2}{9}(4-d).
$$

(6)

Moreover, near the critical state the static susceptibil-
ity and correlation length diverges according to $\chi \sim$
\((F - F_c)^{-\gamma}\) and \(\xi \sim (F - F_c)^{-\nu}\), respectively, where the scaling exponents \(\gamma\) and \(\nu\) satisfy the scaling relation

\[ \frac{\gamma}{\nu} = 2. \]  

(7)

Alternative to the continuum equation (1) one may consider the following discrete model [33]. In a \(d + 1\) hypercubic discrete lattice each site is labeled by index \(i\) and height \(h_i\). Discrete time steps \(t = 0, 1, \ldots\) are taken. The force acting on each site and the evolution of the interface height \(h_i(t)\) are given by

\[ F_i = \sum_{nn} h_j - 2dh_i + F + \eta_i(h_i), \]

\[ h_i(t+1) - h_i(t) = \Theta(F_i), \]  

(8)

where \(\Theta(x)\) is the Heaviside unit step function, \(nn\) denotes that the sum runs over the \(2d\) nearest neighbors and \(\sum_{nn} h_j - 2dh_i\) is a discretized Laplacian. On each step \(t\) and at every site where \(F_i > 0 (\Theta(F_i) = 1)\) the interface is advanced in parallel by one unit. Thus, only at sites where \(F_i > 0\) the interface is active. The average interface velocity is thus given by (in non-dimensional variables)

\[ v = \rho_a, \]  

(9)

where \(\rho_a\) is the density of active sites. In the continuous representation this expression is equivalent to

\[ \langle \partial_t h(x, t) \rangle = \langle \rho_a(x, t) \rangle, \]  

(10)

where \(\rho_a(x, t)\) is the coarse-grained density of active sites.

LID models are Abelian, in the sense that the order in which sites advance is not important [2]. If a site \(i\) is active then it will transfer energy to its nearest neighbors, which at the same time may become active, and so on, an avalanche is generated. It is thus possible that at certain time step \(t\) there will be more that one active site. These sites will be updated in parallel according to the evolution rules described above. However, the process of toppling can never transform any active site, different from itself, in inactive and, therefore, the other active sites will remain active. On the other hand, the energy transferred to its neighbors is constant, independent of the total force at this site. Hence, the order in which these sites are updated is not important.

### III. CONSTANT RATE LID MODEL

In most systems which are expected to exhibit SOC the external field, instead of being constant, increases linearly with time. For instance, in sandpiles grains are usually added at constant rate so that the total number of grains added to the pile up to time \(t\) increases linearly with \(t\). A more evident example is found in magnetic noise measurements in ferromagnetic and superconductor materials where the applied magnetic field increases linearly with time. Hence, if we are trying to map any of these systems into a LID model we can not assume a constant force, which will not be in correspondence with the picture described above.

Motivated by this fact we propose a LID model where the force increases linearly with time. However, in order to reach an stationary state we must include a restoring force which balance the external field. If the force increases linearly with time \((F = ct)\) then after certain time it will overcome the critical force \(F_c\) and the interface will start moving with a velocity \(v(F = ct)\), which is time dependent. Hence the system could not reach a constant velocity stationary state. This problem can be solved adding a restoring force which balance the external driving and leads to an stationary state. This can be done in different ways. For instance, one may consider a local restoring force linear in \(h\) resulting

\[ F = ct - \epsilon h, \]  

(11)

In this case the interface is driven by an external force which increases at rate \(c\) and a restoring force which pull the interface to the substrate. The coefficient \(\epsilon\) is a measure of the strength of the restoring force. Now, suppose that at the initial state \(h(x, 0) = 0\). With increasing time the external force \(F(t) = ct\) will increase. At certain time \(F > F_c\) and the interface will start moving with an average velocity \(v\), i.e. \(\langle h \rangle = vt\). On the other hand, just when \(h\) is finite the restoring force will start pulling the interface to the substrate \(h = 0\). On average the restoring force will become \(F_R(t) = -\epsilon vt\). An stationary state will be obtained when the restoring and pinning forces balance the external driving force \((ct = F_c + \epsilon vt)\) resulting in the stationary state, the average interface velocity \(v = c/\epsilon\).

We can also have a model with \(\epsilon = 0\) but with the interface pinned at the boundary, i.e. \(h = 0\) at the boundary. In this case it is expected that the interface develop a parabolic profile so that the surface tension balance the action of the driving force.

Alternative, instead of the local restoring force in eq. (11) one may consider a global restoring force as follows

\[ F = ct - \epsilon \int \frac{dx'}{L^d} h(x', t), \]  

(12)

The qualitative analysis developed below will be also valid for this case. Moreover, we have recently shown that both restoring forces lead to the same critical behavior [3]. In the following we will only consider the local restoring force.

Next we proceed to show how different SOC systems can be mapped into a constant rate LID model.
A. The Barkhausen effect

An immediate realization of the equation of motion \([1]\) with a local force given either by eq. (13) or (12) is the dynamics of domain walls. It is well known that domain walls in ferromagnets move following an irregular motion in response to changes in an applied external magnetic field, leading to discrete jumps in the magnetization, a phenomenon known as the Barkhausen effect. In fact Urbach et al \([1]\) considered a model where the domain wall dynamics is described by a LID model, with either a local or global restoring force. More recently Cizeau et al \([6]\) have shown that the model by Urbach et al is only valid if long-range dipolar interactions are neglected, an approximation which is valid in soft magnetic materials \([52]\). Hence, the dynamics of a domain wall in a soft magnetic material can be described by a LID model driven at constant rate.

B. Sandpile models

In cellular automaton sandpile models a discrete or continuous variable \(z_i\), height or energy, is defined in a \(d\)-dimensional lattice. The dynamical evolution of \(z_i\) is defined by two evolution rules: adding and toppling. Different adding and toppling rules may be defined leading to different models \([4]\). In particular we consider the following rules

- adding: on each step each site receives a grain from the driving field with probability \(c\);
- toppling: if at certain site \(z_i > z_c = 2d - 1\) then \(z_i \rightarrow z_i - 2d - \epsilon + \eta_i\) and \(z_{nn} \rightarrow z_{nn} + 1\) at the 2\(d\) nearest neighbors;

\(\epsilon\) is the average dissipation rate per toppling event and \(\eta_i\) is a noise, which may have different origins. If \(\epsilon > 0\) the toppling rule is non-conservative and the model have bulk dissipation. On the contrary, if \(\epsilon = 0\) the toppling rule is conservative and one has to assume open boundary conditions to balance the input of grains from the external field. For instance, Chessa et al \([23]\) considered a non-conservative sandpile model where the toppling site loses its energy with probability \(p\) without transferring it to its neighbors. This corresponds to an average dissipation rate per toppling event \(\epsilon = 2dp\), while \(\eta_i\) will reflect the stochastic nature of the local dissipation. Another example is found in random threshold models \([22]\). In this case after each toppling event a new random threshold is assigned (a new critical slope in the model by Christensen et al \([22]\)), which is equivalent to introduce the noise \(\eta_i\). On the other hand, at a coarse-grained level the noise \(\eta_i\) will appear due to elimination of microscopic degrees of freedom. Hence, at this level, it will be present no matter if the model is stochastic or deterministic. In other words, we expect that deterministic and stochastic models belong to the same universality class.

In the original BTW model \(\epsilon = 0\) and dissipation takes place at the boundaries, while grains are added only after all sites become stable, i.e. \(z_i < z_c\) at all sites. However, it has been shown \([2, 8, 25, 53]\) that the BTW model and the model with bulk dissipation lead to the same critical behavior provided \(\epsilon \sim L^{-2}\), where \(L\) is the lattice size, and \(c \to 0\). In the limit \(c = 0^+\) we have separation of time scales between avalanche duration and energy addition, as assumed in the BTW model. Moreover, as discussed above, the noise \(\eta_i\) will appear at a coarse-grained level.

Paczuski and Boettcher \([24]\) noted that a particular critical slope sandpile model can be mapped into a LID model where \(h_i(t)\) is the number of toppling events at site \(i\) up to time \(t\). Their analysis was limited to one dimension but can be extended to larger dimensions and models with critical height rules \([23, 26]\). If at \(t = 0\) we have \(z_i = 0\) at all sites then, at time step \(t\), \(z_i\) and \(h_i\) are related via

\[
z_i = \sum_{nn} h_j - 2dh_i + ct - ch_i + \eta_i(h_i),
\]

where \(\sum_{nn} h_j\) gives the number of grains received from nearest neighbors, \(2dh_i\) the number of grains transferred to nearest neighbors, \(ct\) the number of grains received from the driving field, and \(−ch_i\) the number of dissipated grains. When a site topples \(h_i \to h_i + 1\) and therefore the interface profile \(h_i(t)\) will always advance in the positive direction. Since a site topples only when \(z_i > z_c\) then

\[
h_i(t + 1) - h_i(t) = \Theta(z_i - z_c),
\]

where \(\Theta(z)\) is the Heaviside function. Instead of follows the evolution of the variables \(z_i\), which are equivalent to the force in LID problems, one may follow the evolution of \(h_i\), which is equivalent to the interface height in LID models. Notice that the evolution equations \((13)\) and \((14)\) for sandpile models are equivalent to the discrete variant of LID models in eq. \((8)\), taking \(F_i = z_i - z_c\). Hence, after coarse-graining, we obtain an equation of motion for \(h(x, t)\) like \((1)\) with \(F\) given by eq. \((1)\). Actually there will be an additional term \(-z_c\) in the right hand side of eq. \((1)\) but it does not carry any changes in the critical behavior so that one can work without considering this term.

IV. PHASE DIAGRAM AND SCALING EXPONENTS

We have shown that different SOC models can be mapped into a LID model where the force increases at constant rate and a restoring force pulls the interface to the substrate. In this section we solve this model using previous results for the constant force variant. More precisely, we investigate the dynamics of an interface described by the equation of motion \((1)\) with \(F\) given by eq. \((1)\), i.e.

\[
\lambda \partial_t h = \Gamma \nabla^2 h + ct - ch + \eta(x, h).
\]
As we discussed above, in this case the interface will never be pinned but moves with a finite average velocity \( v \). It is thus expected that the dynamics will be similar to that observed in constant force models above the critical force. Based on this supposition we will perform a suitable change of reference in order to obtain an equation similar to that for the fluctuations around the average in constant force models, eq. (4).

### A. Case \( \epsilon = 0 \)

Let us first consider the case of boundary pinning which corresponds with \( \epsilon = 0 \) and \( h = 0 \) at the boundary. We look for a solution in the form

\[
h(x, t) = h_0(x, t) + y(x, t),
\]

so that when we substitute this expression in eq. (15) the constant rate force will be replaced by a constant force. This constraint will be satisfied taking \( h_0(x, t) \) as the solution of the problem

\[
\nabla^2 h_0 + ct = F.
\]

with the boundary condition \( h_0 = 0 \). The solution of this equation can be easily obtained assuming radial symmetry \( h_0 = h_0(r, t) \) with \( 0 < r < R \), where \( R \) is the system radius. It results that

\[
h_0(r, t) = \left(1 - \frac{r^2}{R^2}\right) \frac{ct - F}{2\Gamma} R^2.
\]

We have also analyzed the case of free boundary conditions \( dh_0(R, t)/dr = 0 \), resulting that eq. (17) have no solution. In this case the system will not reach an stationary state. This result supports our previous analysis about the need of a restoring force which balance the external driving of the interface.

Substituting eq. (16) in eq. (17) and taking the limits \( r \ll R \) and \( ct \gg F \) we obtain eq. (4) with

\[
v \approx \frac{\partial h_0}{\partial t} \approx \frac{cR^2}{2\Gamma d}.
\]

Notice that assuming \( r \ll R \) and \( ct \gg F \) we have approximated \( h \) by \( vt + y \) in the quenched noise \( \eta(x, h) \). However, the information contained in \( h_0 \) is not completely lost because we have the constant force \( F \) in the right hand side of eq. (4). These approximations will be valid for very long times (stationary state) and far from the boundary. \( v \) is then the average interface velocity in the stationary state, which depends on system size \( R \). When increasing system size the bulk increases faster than the surface. Hence, since the interface is only fixed at the boundary the concentration of points where the interface is fixed will decrease with increasing system size resulting on an increase of the interface velocity.

The constant force \( F \) has not been specified. It will be obtained self-consistently imposing the constraint \( \langle y(x, t) \rangle = 0 \). In constant force models \( F \) is a fixed parameter while \( v \) is obtained self-consistently from the equation of motion. However in the present model \( v \) is given by eq. (19) while \( F \) is the undetermined parameter. From the constant force variant it is known that a force \( F (F > F_c) \) gives an average interface velocity \( v \sim (F - F_c)^{\beta} \). Hence, in the constant rate model, to obtain a velocity \( v \) we should have the force \( F(v) = F_c + \text{const.} v^{1/\beta} \).

In this way, \( F(v) \) is obtained self-consistently imposing that the interface moves with an average velocity given by eq. (19). In spite of this difference both models have the same critical behavior.

To reach the critical force \( F_c \) we must fine-tune \( v \) to zero. For \( v > 0 \) there is a characteristic length \( \xi \sim v^{-\nu/\beta} \).

Now, the average interface velocity scales as \( v \sim cR^2 \) and therefore when we take the thermodynamic limit \( R \to \infty \) we must fine-tune \( c \) to zero to reach the critical state. If \( \xi \gg R \) then the system size \( R \) will be the only characteristic length and the system will be in a critical state. This condition is satisfied if \( c = O(R^{-2+\nu/\beta}) \) when \( R \to \infty \).

In the case of boundary pinning the average density of active sites, computed from eqs. (10), (16) and (18), is given by

\[
\langle \rho_a(r, t) \rangle = \left(1 - \frac{r^2}{R^2}\right) \frac{c}{2\Gamma} R^2.
\]

This expression is identical, except for some constant factors, to that obtained in the field theory by Vespignani et al [19]. It just reflects the balance between the driving force and the boundary pinning (the driving field and boundary dissipation in sandpile models). As one can see from eq. (21) the average density develops a parabolic profile, which is a consequence of the pinning at the boundary. This parabolic profile has been observed in recent numerical simulations by Barrat et al [53].

### B. Case \( \epsilon > 0 \)

Now we consider the case of a local restoring force. We again look for a solution in the form of eq. (4) which the same constraints, i.e. it replaces the constant rate force by a constant force \( F \). This constraint is satisfied if \( h_0(x, t) \) is the solution of the problem

\[
\nabla^2 h_0 - \epsilon h_0 + ct = F.
\]

with the corresponding boundary conditions. Independently of the boundary conditions assumed, the solution of this equation exists. If one assumes the fixed boundary condition then one will obtain a parabolic profile as in the \( \epsilon = 0 \) case. On the contrary, we will consider periodic or free boundary conditions. In this case we obtain
Substituting eq. (23) in eq. (13) and taking the limit \( vt \gg F/\epsilon \) we obtain

\[
\lambda \partial_t y = \Gamma \nabla^2 y - \epsilon y + F - \lambda v + \eta(x, vt + y).
\]

with

\[
v = \frac{c}{\epsilon}.
\]

Again assuming \( vt \gg h_0(x) \) we have approximated \( h \) by \( vt + y \) in the the quenched noise \( \eta(x, h) \). Eq. (24) is quite similar to that obtained for the fluctuations around the flat co-moving interface in constant force LID models, eq. (11). The only difference is found in the term \(-chy\), which accounts for the local restoring force with strength \( \epsilon \). However, in the limit \( \epsilon \to 0 \) the system will show the same critical behavior as in constant force models. The term \(-chy\) only affects the linear part of the bare propagator while non-linear terms, which are responsible for loop-corrections, remain identical. The critical exponents \( z \) and \( \zeta \) obtained from the RG analysis will thus be the same as those obtained for the constant force case. However the phase diagram will show a complex structure.

For \( v > 0 \) and \( \epsilon > 0 \) the system is driving out the critical state. Now in addition to the characteristic length \( \xi \sim \nu^{\nu/\beta} \) we have another characteristic length associated with the restoring force \( \xi_c \sim \epsilon^{-\nu_c} \), where \( \nu_c \) is calculated below. The phase diagram \((v, \epsilon)\) will have different regions depending on the ratio between these two characteristic lengths. We then define the characteristic velocity \( v_c \) as the velocity where these two characteristic lengths become identical. Thus taking \( \xi \sim \xi_c \) we obtain that \( v_c \sim \epsilon^{\zeta_c} \) with

\[
\beta_c = \frac{\nu_c}{\nu} (z - \zeta).
\]

In the region \( v > v_c \) (\( \xi > \xi_c \)) \( \xi \) is the only characteristic length, as in the case \( \epsilon = 0 \). However, we cannot say that the effect of the restoring force disappears because the average interface velocity depends on \( \epsilon \) (see eq. 23). There will be a complete equivalence if we take \( \epsilon \sim R^{-2} \). In other words, a model with bulk dissipation with \( v > v_c \) and \( \epsilon \sim R^{-2} \) is equivalent to a model without bulk dissipation and open boundaries. This equivalence was already pointed out by Vespignani et al using MF [12] and field theory approaches.

On the contrary, in the region \( v < v_c \) (\( \xi < \xi_c \)) \( \xi_c \) is the relevant characteristic length making the difference with the \( \epsilon = 0 \) case. In this region the static susceptibility, which characterizes the linear response of the system, is given by

\[
\chi(k, \epsilon) = 1/(\Gamma_{\text{eff}} k^2 + \epsilon).
\]

where \( \Gamma_{\text{eff}} \) includes loop corrections to \( \Gamma \). At the critical state \( \epsilon \to 0 \) we have \( \chi(k, 0) \sim k^{-2} \) as in the constant force case. On the other hand, for \( k \to 0 \) we obtain \( \chi = 1/\epsilon \) so that \( \chi \sim \epsilon^{-\gamma_c} \) with

\[
\gamma_c = 1.
\]

Eq. (27) can be thus written as

\[
\chi(k, \epsilon) \sim k^{-2} f(k \xi_c),
\]

where \( \xi_c \sim \epsilon^{-\nu_c} \). For \( k \xi_c \) large we should obtain \( \chi(k, \epsilon) \sim k^{-2} \) so that \( f(x) \sim 1 \) for large \( x \). On the other hand, for \( k \xi_c \) small we should now obtain \( \chi(k, \epsilon) \sim \epsilon^{-\nu_c} \) and therefore \( f(x) \sim x^{\nu_c/\nu} \) for \( x \) small with

\[
\frac{\nu_c}{\nu_c} = 2.
\]

From eqs. (28) and (31) we thus obtain

\[
\nu_c = 1/2.
\]

The exponents \( \gamma_c \) and \( \nu_c \) results different to the exponents \( \gamma \) and \( \nu \) but their ratio is the same, as one can see from eqs. (10) and (30). This will carry as a consequence that some exponents, the avalanche exponents for instance, measured in the region \( v < v_c \), can be extrapolated to the region \( v > v_c \).

Finally, in the case of a local restoring force the average density of active sites is given by

\[
\langle \rho_a(r, t) \rangle = \frac{c}{\epsilon}.
\]

Again this result is identical to the one obtained by Vespignani et al using a field theory for sandpile models. It reflects the balance between the driving and the local restoring forces (between driving and local dissipation in sandpile models). Moreover this flat profile has been observed by Barrat et al in numerical simulations of a sandpile model with local dissipation.

The constant rate LID model thus describe a wide variety of SOC systems, like the Barkhausen effect and sandpile models. It reproduces previous field theory predictions for sandpile models, which we now know are also valid for other models. Hence, the constant rate LID model provides a unifying point of view of SOC phenomena.

C. Avalanche exponents

In the preceding section we have shown that LID models driven either at constant force or rate belong to the same universality class. In particular they share the same roughness and dynamic scaling exponents, which can be estimated using previous RG calculations for the constant force LID model. Now we proceed to show how other scaling exponents can be obtained using \( \zeta \) and \( \zeta \). For instance, we are going to compute the avalanche exponents, which are often measured in numerical simulations of sandpile models.
Let $s$ be the avalanche size and $T$ its duration, which are distributed according to $P(s)$ and $P(T)$, respectively. Just at the critical state one expect that these distributions satisfy the power law behavior $P(s) \sim s^{-\tau_s}$ and $P(T) \sim T^{-\tau_t}$, where $\tau_s$ and $\tau_t$ are the avalanche distribution exponents, reflecting the unexistence of characteristic values for size and duration of the avalanches. However, for finite $\epsilon$ and $\nu$ characteristic cutoffs of avalanche size $s_\epsilon$ and duration $T_\epsilon$ will appear. These cutoffs will scale with the correlation length (either $\xi$ or $\xi_c$ depending on the model and on the region of the phase diagram) as $s_\epsilon \sim \xi^D$ and $T_\epsilon \sim \xi^\nu$, where $D$ is the avalanche fractal dimension. Near the critical state the distributions of avalanche size and duration will thus satisfy the scaling laws

$$P(s) \sim s^{-\tau_s} f_1(s/s_\epsilon), \quad P(T) \sim T^{-\tau_t} g_1(T/T_\epsilon),$$

where $f(x)$ and $g(x)$ are some cutoff functions with the asymptotic behaviors $f_1(x), g_1(x) \sim 1$ for $x \ll 1$ and $f_1(x), g_1(x) \ll 1$ for $x \gg 1$.

The exponents $\tau_s$, $\tau_t$, $D$ and $\nu$ are not all independent. Since $s \sim T^{\gamma/D}$ then the condition $\int ds P(s) = \int dT P(T)$ implies

$$(\tau_s - 1)D = (\tau_t - 1)\nu.$$  \tag{34}

Another scaling relation can be obtained taking into account that the short-wavelength static susceptibility scales as the mean avalanche size $\chi$. From eq. (33) we obtain $\langle s \rangle \sim \xi^{(2 - \tau_s)D}$ while $\chi \sim \xi^{\gamma/\nu} \sim \xi^\nu$. Thus, setting $\chi \sim \langle s \rangle$ we obtain

$$(2 - \tau_s)D = 2.$$  \tag{35}

Then from eqs. (34) and (35) it results that

$$\tau_s = 2 - \frac{2}{D}, \quad \tau_t = 1 + \frac{D - 2}{\nu}.$$  \tag{36}

Finally there is a scaling relation which relates the avalanche dimension exponent $D$ with the roughness exponent $\xi$. Below the upper critical dimension the avalanches are compact objects and therefore $s \sim \Delta h r^d$, where $\Delta h$ is the characteristic fluctuation of the interface during the avalanche and $r$ its characteristic linear extent in the $d$-dimensional substrate. Then, since $\Delta h \sim r^\zeta$ and $s \sim r^D$ one obtains

$$D = d + \zeta.$$  \tag{37}

Above the upper critical dimension the avalanches are no more compact and $D = d_c = 4$ [22], resulting the mean-field exponents $\tau_s = 3/2$ and $\tau_t = 2$.

These scaling relations are independent on the way the interface is driven. They have been obtained assuming that $\chi \sim \langle s \rangle$, $s \sim T^{D/\nu}$, there is a characteristic length $\xi$ and avalanches are compact objects (in the interface dynamics representation).

V. DISCUSSION

In this section we proceed to compare our results with experiments, numerical simulations and previous theoretical works in the literature.

The average density of active sites in the stationary state was found to be identical to the one obtained in the field theory by Vespignani et al. [14], which in principle was developed for sandpile models. Moreover, they predicted an upper critical dimension $d_c = 4$ in agreement with our result. Our analysis thus reveals that their approach is also valid for other SOC models, for instance for LID models driven at constant rate. The advantage of our approach is that it makes easier the use of momentum space RG analysis to obtain the scaling behavior in the neighborhood of the critical state. Using previous RG calculations for constant force LID models we compute the complete set of scaling exponents, including the avalanche exponents often measured in numerical simulations.

The upper critical dimension $d_c = 4$ is also consistent with previous RG analysis by Díaz-Guilera [10] of the corresponding Langevin equations for the height of sand columns $z_i$ (instead of $h_i$) in the BTW model. Their results are also consistent with a dynamic scaling exponent

$$z = \frac{d + 2}{3},$$  \tag{38}

an expression which was previously suggested by Zhang [55]. This dynamic scaling is however in contradiction with the one obtained here (see eq. (6)). In his formulation Díaz-Guilera considered a coarse graining-noise $\eta'(x,t)$. In the fast time scale (that of the evolution of the avalanches) they assumed a columnar noise uncorrelated in space, i.e.

$$\langle \eta'(x,t) \eta'(x',t') \rangle \sim \delta^d(x - x').$$  \tag{39}

On the contrary, in our approach $\eta(x,h)$ is a quenched noise, a new random variable is selected only when the interface at site $x$ advances, i.e. when the site $x$ is active. Our choice is more appropriate because it reflects the fact that the noise field is frozen in regions where there is no dynamical evolution, similar to the multiplicative noise assumed in absorbing-state phase transitions [15].

The estimate of the avalanche scaling exponents of sandpile models has been a very difficult task. In two dimensions we count with a real space RG approach by Pietronero et al [15], which predicts that undirected Abelian sandpile models, either deterministic or stochastic, belong to the same universality class. They also provided an estimate of the avalanche size exponent $\tau_s = 1.253$. On the other hand, Priezzhev et al [14] have obtained $\tau_s = 6/5 = 1.2$ in two dimensions, analyzing the structure of avalanches in the undirected Abelian sandpile model. Since the BTW and Manna model are both Abelian [13] this result will be valid for both models,
which are thus expected to belong to the same universality class. Moreover, this exponent is identical to the one obtained by De Menech et al. [30] for the BTW model, using multifractal rather than finite size scaling. These estimates can be compared with the one obtained within our approach. In two dimensions we obtain $\tau_s = \frac{\eta}{\nu} = \frac{5}{4}$ (see table I). This value is in very good agreement with the theoretical estimates reported above. The advantage of our approach is that we go beyond two dimensions and compute the scaling exponents in three dimensions. Moreover we have determined the upper critical dimension, $d_c = 4$ above which MF exponents are correct.

Numerical simulations of undirected sandpile models are generally performed with dissipation only at the boundary and assuming time scale separation. In this case the system self-organizes into a critical state where the system size is the only characteristic length, i.e. $\xi \sim L$. Moreover, it is observed that the susceptibility scales as $\chi \sim L^2$. This scale dependency has been demonstrated for Abelian models in one dimension [8] using the Abelian symmetry of the operators algebra. It is expected to be valid in any dimension as a consequence of conservation [12,19], understanding conservation as the balance between the input and output of grains from the system. This scaling dependence has been obtained here using the mapping of sandpile models to the constant rate LID model, and it is a consequence of the balance between the driving field and the restoring force.

The scaling exponents obtained from numerical simulations of constant force LID models, the Manna model (the prototype of stochastic sandpile model) and Barkhausen effect in soft-magnetic materials are shown in table I. Some experimental estimates for the Barkhausen effect measure in soft magnetic materials, where dipolar interactions can be neglected [7], are also displayed. Our estimates using eqs. (29) and (30) are also shown for comparison. The agreement is very good, specially in three dimensions where the RG calculations are expected to be more precise. Moreover, we can also observe that exponents obtained for the three different SOC models in two dimensions are very close, suggesting that they all belong to the same universality class. In this analysis we have not included numerical estimates for the avalanche exponents of the BTW model. Numerical simulations of this model [27,28] lead to contradictory results. This discrepancy may be associated with the existence of multifractal rather than finite size scaling [30].

This work does not cover the whole classes of SOC models. We have excluded, for instance, directed Abelian sandpile models, critical slope models [57] and non-linear interface depinning models [44]. In the case of directed Abelian sandpile models the critical exponents have been obtained using the Abelian symmetry of the algebra of operators [33]. In the case of critical slope models, inspired on the work by Paczuski and Boettcher [24], one can perform a similar mapping into a LID model [57]. Finally non-linear interface depinning models belongs to a different universality class described by the KPZ equation [42].

VI. SUMMARY AND CONCLUSIONS

We have shown that undirected Abelian sandpile models and the domain wall motion in magnetic materials where dipolar interactions are negligible can all be mapped into the constant rate LID model and therefore display the same universal behavior. The constant rate LID model was found to exhibit the same critical behavior as constant force LID models above the depinning transition. The scaling exponents were thus obtained using previous RG analysis for constant force LID models.

In this way we have shown the existence of universal behavior in a vast class of SOC models. We have proposed the LID driven at constant rate as the prototype of this universality class, because it allows the use of continuous approaches which are more treatable by analytical tools than the discrete analysis required in most cellular automaton SOC models.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
$d$ & Model & $\tau_s$ & $\tau_t$ & $\tau_z$ & $\nu$ & $D$ & Ref. \\
\hline
1 & LID & 1.42(3) & 2.25(1) & 0.78 & 2 & & [53] \\
& RG & 1 & 1 & $\frac{1}{4}$ & 1.33(1) & 2 & & \\
\hline
2 & LID & 1.29(2) & 1.58(4) & 2.75(2) & & & [35] \\
& Manna & 1.27(1) & 1.500 & 2.73(2) & & & [27] \\
& Manna & 1.273 & 1.50(1) & 2.73(2) & & & [54] \\
& BHN & 1.26(4) & 1.40(5) & & & & [7] \\
& BHE & 1.28(2) & 1.5(1) & & & & [4] \\
& RG & $\frac{1}{4} = 1.25$ & $\frac{1}{2} = 1.500$ & 1.43 & 1.56(2) & 2.67 & & \\
\hline
3 & LID & 3.34(1) & & & & & [48] \\
& Manna & 1.40 & 1.75 & 3.33 & & & [27] \\
& RG & $\frac{1}{4} = 1.4$ & $\frac{1}{2} = 1.75$ & 1.78(2) & 3.33 & & \\
\hline
\end{tabular}
\caption{Scaling exponents for constant force LID models (LID), the Manna $d$-state model (Manna), and those obtained from experiments (BHE) and numerical simulations (BHN) of the Barkhausen effect. Results obtained here using RG estimates are shown for comparison.}
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