The Frauchiger-Renner Gedanken Experiment:

an Interesting Laboratory for Exploring

Some Topics in Quantum Mechanics∗

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In a publication (Nature Comm. 3711, 9 (2018)), Daniela Frauchiger and Renato
Renner used a Wigner/friend gedanken experiment to argue that quantum mechanics
cannot describe complex systems involving measuring agents. They were able to produce
a contradictory statement starting with four statements about measurements performed
on an entangled spin system. These statements needed to be combined using the tran-
sitive property of logic: If A implies B and B implies C, then A implies C. However,
in combining successive statements for the Frauchiger-Renner gedanken experiment we
show that quantum mechanics does not obey transitivity and that this invalidates their
analysis. We also demonstrate that certain pairs of premises among the four statements
are logically incompatible, meaning that the statements cannot all be used at once. In
addition, to produce the contradiction, Frauchiger and Renner choose a particular run,
which they call the ‘OK’–’OK’ one. However, the restriction to this case invalidates three
of the four statements. Hence, there are three separate problems with logic in the 2018
Nature Communication publication. We also demonstrate the violation of the rules of
logic – including transitivity – in certain situations in quantum mechanics in general.

We use the Frauchiger-Renner gedanken experiment as a laboratory to explore a
number of topics in quantum mechanics including wavefunction logic, Wigner/friend ex-
periments, and the deduction of mathematical statements from knowledge of a wavefunc-
tion and obtain a number of interesting results. We show that Wigner/friend experiments
of the type used by Frauchiger and Renner are impossible if the Wigner measurements
are performed on macroscopic objects. They are possible on certain microscopic entities
but then the Wigner measurements are rendered “ordinary” (such as measurements on
a spin using a particular axis of quantization), in which case it is straightforward to
perform the Frauchiger-Renner experiment in a real laboratory setting.

Keywords: Quantum Logic; Foundations of Quantum Mechanics; Wigner/Friend Exper-
iments

1. Introduction

In an interesting paper[1] D. Frauchiger and R. Renner argue that quantum theory
cannot consistently describe the use of itself in the following sense: If quantum

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mechanics governs the agents involved in experiments then a gedanken experiment exists that leads to a contradiction. By agents, one means the experimentalists and their equipment that record the results of measurements on quantum systems. An experimentalist is able to note the outcome of an experiment through statements such as “I observed the spin of a spin-$\frac{1}{2}$ object to be up” (or down if that is the result of the measurement) where “up” and “down” refer to the direction of the spin in, say, the z-direction. The equipment of the experimentalist equally has this capacity by recording the result (be it up or down) in a database, for example.

Frauchiger and Renner make three reasonably-plausible assumptions about quantum mechanics. We refer the reader to the details of these assumptions in Reference 1 and, instead, in this Introduction describe them in general terms. The first assumption is the “quantum mechanical” one (Q). It says that the probability of an output of a measurement is given by the Born rule. Actually, reference 1 only needs a weaker version of (Q), namely, if the Born rule assigns a probability of 1 to a specific proposition, then an Agent can be certain of it. The second assumption (C) demands consistency in that if one agent comes to a conclusion about a statement or prediction, then another agent, when using the same theory, assumptions and information, will come to the same conclusion. Assumption (S) is that if an agent is certain of the outcome of a measurement or statement, then the agent cannot conclude something contradicting it. Assumption (S) may appear to be obvious, but the Contradictory Statement (see Section 3) that arises in the Frauchiger-Renner gedanken experiment is of the form “If I (Agent W) obtain a measurement of ‘−’ , then by using assumptions (Q) and (C), I can conclude that I should obtain a measurement of ‘+’”. Here, ‘−’ and ‘+’ are similar to the down and up states of a spin-$\frac{1}{2}$ object when the axis of spin quantization is in the x-direction. So, the Contradictory Statement makes use of (S).

However, there might be hidden assumptions in the analysis. Indeed, Frauchiger and Renner admit this: “Any no-go result ... is phrased within a particular framework that comes with a set of built-in assumptions. Hence it is always possible that a theory evades the conclusions of the no-go result by not fulfilling these implicit assumptions.” As an example of a harmless hidden assumption, the Frauchiger-Renner thought experiment assumes that during the period between two measurements quantum mechanics evolves in the usual manner respecting unitarity, an assumption that does not affect the generation of their Contradictory Statement. In unitary quantum mechanics (whose definition is given below) we verify the four basic “If ... then ...” statements in the Frauchiger and Renner publication in Section 3. However, we uncover an unstated assumption, namely, that the transitive property of logic needs to be applicable to statements about measurements and wavefunctions. Transitivity, that is, ((A ⇒ B) AND (B ⇒ C)) ⇒ (A ⇒ C), needs to be used three times to combine the four basic statements to generate the contradiction. We show that statements about measurements do not always obey the transitive property of logic, and that this invalidates the conclusions of the two authors. This means that the Frauchiger-Renner gedanken experiment does not
necessarily rule out the possibility that quantum mechanics can describe complex systems involving measuring agents.

The validity of the Frauchiger-Renner gedanken experiment has been challenged in two directions: whether the argument itself is incorrect or whether there are hidden assumptions in the argument. However, no publication has pointed out that the transitive property of logic might be invalid for Agent statements.

The Frauchiger-Renner argument involves an extended Wigner/friend gedanken experiment of two entangled spin- objects. Four measurements are conducted. The first two involve two different agents (the “friends” of the Wigner agents), each measuring the spin of one of the two spin- objects. Two more measurements are then conducted on the “friends” by the two Wigner agents. In Section 7, we point out that an overlooked agreement between a Wigner agent and the agent’s friend needs to be established prior to the start of the Frauchiger-Renner gedanken experiment. This agreement is of interest because it puts restrictions on the nature of Wigner/friend experiments. Indeed, the restrictions are so stringent as to rule out Wigner measurements of the type used in the Frauchiger-Renner gedanken experiment on macroscopic entities. We consider this to be one of the important results of our work. It is interesting that Heisenberg’s uncertainty principle plays a role in this. See Section 7. This restriction already casts doubt as to whether the Frauchiger-Renner gedanken experiment indicates that quantum mechanics cannot be extrapolated to complex systems.

We conduct the analysis within the framework of unitary quantum mechanics. An advantage to doing this is that one knows the form of the wavefunction at each stage of the Frauchiger-Renner gedanken experiment. In addition, it is quite easy to derive the four “If ... then ...” statements in reference as well as logical statements about the measurements.

In unitary quantum mechanics, measurement does not involve wavefunction collapse; the probability of an outcome is related to the absolute square of the wavefunction, and linearity and unitarity are strictly maintained even during a measuring process. There is a single universal wavefunction and one does not assign worlds to certain linear superpositions of this wavefunction; instead, in unitary quantum mechanics, the quantum mechanical interpretation of a situation is obtained by examining the wavefunction itself, and Sections and illustrate this.

Hence, unitary quantum mechanics is “standard” quantum mechanics without wavefunction collapse; However, unitarity and the absence of wavefunction collapse lead to some consequences for quantum measurement that many physicists may not be unaccustomed to, which are embodied in the following: The basic Measurement Rule is: If wavefunction collapse is not needed “to explain” an experimental result, then a single measuring event suffices to determine the state with certainty; If this is not the case, then the uncertainty of the quantum state is transferred to the measuring agent, multiple measurements are needed to determine the state, and an output reading indicating that the state is does not mean that the wavefunction
All the measurements in the Frauchiger-Renner gedanken experiment involve binary outcomes. Let us illustrate the Measurement Rule in unitary quantum mechanics for this case. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ indicate the two outcomes for a measurement, and think of them as the up and down spin of a spin-$\frac{1}{2}$ object. Let $A$ be an agent who is about to measure the spin. The word agent might include an experimentalist and her equipment. Let $\Psi_A$ be the agent’s wavefunction before the measurement takes place. Then there are two generic cases: (i) The initial state is not a superposition; it is either $|\uparrow\rangle$ or $|\downarrow\rangle$ (up to an overall phase), but it is unknown which of these two possibilities is happening. (ii) The initial state $S_0$ of the object to be measured is a superposition of up and down spin, that is, it is of the form $S_0 = a_\uparrow |\uparrow\rangle + a_\downarrow |\downarrow\rangle$ (with $|a_\uparrow|^2 + |a_\downarrow|^2 = 1$). In case (i), the schematic description of the process is

$$\Psi^A |\uparrow\rangle \rightarrow \Psi^A_\uparrow , \text{ if } S_0 = |\uparrow\rangle$$

$$\Psi^A |\downarrow\rangle \rightarrow \Psi^A_\downarrow , \text{ if } S_0 = |\downarrow\rangle.$$ 

(1)

Here, $\Psi^A_\uparrow$ and $\Psi^A_\downarrow$ are two different wavefunctions involving the quantum constituents of $A$ and the spin-$\frac{1}{2}$ object. In the schematic equations, the left-hand side (respectively, right-hand side) is the wavefunction before (respectively, after) the measurement is made. Case (i) corresponds to the situation when wavefunction collapse is not needed to explain the experimental result: If the spin is up, then it is measured to be up and no wavefunction collapse is needed; ditto for the situation when the spin is down. In case (ii), the schematic description in unitary quantum mechanics is

$$\Psi^A (a_\uparrow |\uparrow\rangle + a_\downarrow |\downarrow\rangle) \rightarrow a_\uparrow \Psi^A_\uparrow + a_\downarrow \Psi^A_\downarrow , \text{ case (ii)}.$$ 

(2)

Equation (2) follows from Eq.(1) and quantum-mechanical linearity, and linearity is a consequence of unitarity. If Agent $A$ is a machine with no thinking capability, then $\Psi^A_\uparrow$ in Eq.(2) is the same as $\Psi^A_\uparrow$ in Eq.(1) and its quantum constituents involve the coding of an up-spin output; likewise for $\Psi^A_\downarrow$. If Agent $A$ involves a human or entities with reasoning ability, and this is the case for the agents in the Frauchiger-Renner gedanken experiment, then the relation between the wavefunctions in Eqs.(2) and (1) depends on what $A$ knows about the initial state. If $A$ knows nothing, then the situation is the same as that of the pure machine case: the wavefunction components, $\Psi^A_\uparrow$ and $\Psi^A_\downarrow$, in equations (1) and (2) are equal. If $A$ knows that the initial situation is (i), then $\Psi^A_\uparrow$ contains a configuration of the human’s quantum constituents that embody the thought “I measured the spin to be up and so I know the initial wavefunction must have been $S_0 = |\uparrow\rangle$.” A similar thought holds with “up” replaced by “down” is contained in $\Psi^A_\downarrow$. If $A$ knows that the initial situation is (ii) and believes in unitary quantum mechanics, then $\Psi^A_\uparrow$ contains a configuration of the human’s quantum constituents that embodies the thought “I measured the spin to be up but I know that the current wavefunction must contain another component $\Psi^A_\downarrow$ in a linear superposition even though I cannot
be directly aware of its existence.” A similar statement arises for $\Psi_A^\downarrow$. In case (ii), even though $\Psi_A^\uparrow$ involves the observation of an output indicating up spin, Agent A cannot conclude that the spin is or was up. Indeed, this is correct because the spin was initially $a_\uparrow |\uparrow\rangle + a_\downarrow |\downarrow\rangle$ (and not just $|\uparrow\rangle$) and it is very unlikely to end up being proportional to $|\uparrow\rangle$ at any point during a physical measuring process. Furthermore, it takes multiple measurements beginning with the same $S_0 = a_\uparrow |\uparrow\rangle + a_\downarrow |\downarrow\rangle$ to determine information about the coefficients $a_\uparrow$ and $a_\downarrow$.

The agents in the Frauchiger-Renner gedanken experiment are not only informed of the initial wavefunction but also of the entire series of measurement steps. In relation to what was discussed in the previous paragraph, the analog is as follows: If Agent A was informed that $S_0$ was $a_\uparrow |\uparrow\rangle + a_\downarrow |\downarrow\rangle$, then, in this case, $\Psi_A^\uparrow$ contains a configuration of the human’s quantum constituents that embodies the thought “I measured the spin to be up but I know that the current wavefunction must contain another component $\Psi_A^\downarrow$ in a linear superposition even though I cannot be directly aware of its existence and that the form of this superposition is $a_\uparrow \Psi_A^\uparrow + a_\downarrow \Psi_A^\downarrow$, where $a_\uparrow$ and $a_\downarrow$ are the same coefficients as in $S_0$. ” Furthermore, another Agent B, having been given all the information about the initial state and experimental procedure can conclude using Assumption (C) that the form of the wavefunction after the measurement is $a_\uparrow \tilde{\Psi}_A^\uparrow + a_\downarrow \tilde{\Psi}_A^\downarrow$, where $\tilde{\Psi}_A^\uparrow$ and $\tilde{\Psi}_A^\downarrow$ are some wavefunctions that Agent B does not know in detail but which incorporate the same measurement statements as in $\Psi_A^\uparrow$ and $\Psi_A^\downarrow$: “I measured the spin to be up (or down) and/but I know that ... “.

Our notation differs somewhat from that of Frauchiger and Renner\footnote{The use of “bars” over objects is unchanged. However, we use “bra’s” and “ket’s” to denote discrete states, which happen to come in pairs for the Frauchiger-Renner gedanken experiment; so, we denote them with up and down spins ($|\uparrow\rangle$ and $|\downarrow\rangle$) to take advantage of the isomorphism with spin-$\frac{1}{2}$ objects. For states involving many degrees of freedom, we denote the wavefunction using the symbol $\Psi$. A subscript $M$ on a state indicates that it has been “measured” but $M$ can also stand for “message” because these states also embody statements such as the ones in quotes in the two previous paragraphs. The states $|\uparrow\rangle_S$ and $|\downarrow\rangle_S$ in reference\cite{1} are simply denoted by $|\uparrow\rangle$ and $|\downarrow\rangle$ in our paper. Our states $|\uparrow\rangle_R$ and $|\downarrow\rangle_R$ correspond to $|\text{heads}\rangle_R$ and $|\text{tails}\rangle_R$ in ref\cite{1} A discrete state associated with “ok” (respectively, “fail”) in ref\cite{1} is represented by a ‘−’ (respectively, a ‘+’) in our work (except our unbarred $|−\rangle$ corresponds to $−|\text{ok}\rangle_L$, that is, it differs by a minus sign). The measurement times $t_j$ in our paper are written as n:0j in reference\cite{1}.}
2. Wavefunction Representations of the Extended Wigner/Friend Experiment in Unitary Quantum Mechanics

The initial state $\Psi_0$ of the Frauchiger-Renner gedanken experiment can be taken to be

$$\Psi_0 = \frac{\psi^W \bar{\psi}^W \psi^F \bar{\psi}^F}{\sqrt{3}} (|\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|).$$

(3)

This state involves two qubits, which we take to be two spin-$\frac{1}{2}$ objects: an “unbarred” spin, for which the basis is $|\uparrow\rangle$ and $|\downarrow\rangle$, and a “barred” spin, for which the basis is $|\bar{\uparrow}\rangle$ and $|\bar{\downarrow}\rangle$. In both cases, the axis of quantization for the spin is taken to be in the positive $z$-direction.

The Gedanken experiment proceeds in four main measurement steps:

- In the first step, Agent $\bar{F}$ measures the spin of the barred spin-$\frac{1}{2}$ object in the $z$-direction at time $t_1$. This causes $\bar{\psi}^F |\bar{\uparrow}\rangle$ and $\bar{\psi}^F |\bar{\downarrow}\rangle$ to be replaced by $\tilde{\psi}^F_\uparrow$ and $\tilde{\psi}^F_\downarrow$ respectively. The wavefunction then becomes

$$\Psi_1 = \frac{\psi^W \bar{\psi}^W \psi^F \bar{\psi}^F}{\sqrt{3}} (|\uparrow\rangle_M \langle \downarrow| + |\downarrow\rangle_M \langle \uparrow| + |\downarrow\rangle_M \langle \downarrow|).$$

(4)

In Eq.(4), we have replaced $\tilde{\psi}^F_\uparrow$ and $\tilde{\psi}^F_\downarrow$ by $|\uparrow\rangle_M$ and $|\downarrow\rangle_M$ respectively. They can be considered to make up a discrete two-state system with a message associated with each. The justification for this is given in Section 7, which discusses some aspects of Wigner/friend experiments that are used in the Frauchiger-Renner gedanken experiment. By the way, this replacement does not affect any of the conclusions obtained in our paper including the issue with the transitivity property of logic. A reader who does not want to use this simplification can replace $|\uparrow\rangle_M$ and $|\downarrow\rangle_M$ with $\tilde{\psi}^F_\uparrow$ and $\tilde{\psi}^F_\downarrow$ in the equations below.

- In the second step, Agent $F$ measures the spin of the unbarred spin-$\frac{1}{2}$ object at time $t_2$, and the wavefunction becomes

$$\Psi_2 = \frac{\psi^W \bar{\psi}^W}{\sqrt{3}} (|\uparrow\rangle_M \langle \downarrow| + |\downarrow\rangle_M \langle \uparrow| + |\downarrow\rangle_M \langle \downarrow|).$$

(5)

The third step involves a Wigner measurement by Wigner Agent $\bar{W}$ of ‘measured’ barred spin in the $x$-direction. Here, we are thinking of $|\uparrow\rangle_M$ and $|\downarrow\rangle_M$ as the up and down $z$-components of a spin-$\frac{1}{2}$ system. More precisely, the measurement is

\[a]\text{In reference 1, the initial state is created differently: A quantum qubit of the form}\ \frac{1}{\sqrt{2}} (|\uparrow\rangle + e^{i\phi} |\downarrow\rangle) \text{is generated. Agent F measures this barred spin state. If the barred spin is up, then she sends the state} |\uparrow\rangle \text{to Agent F. If it is down, then she sends the state} (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \text{to Agent F. This procedure assumes that F is able to manipulate states easily. In particular, the overall phase of a wavefunction is not an observable and cannot be controlled, and so she cannot guarantee that} (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \text{as opposed to} e^{i\phi} (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2} \text{is sent. The resulting initial wavefunction would become} (|\uparrow\rangle \langle \downarrow| + e^{i\phi} |\uparrow\rangle \langle \downarrow| + |\downarrow\rangle \langle \downarrow|)/\sqrt{3}. \text{Put differently, Agent F cannot control the relative phase of the two terms when the procedure in reference 1 is used. Statement 2 below is not true unless} \phi = 0. \text{Hence, it is better to begin with Eq.(3).}\]
performed on the basis \(|\uparrow\rangle_M = (|\uparrow\rangle_M + |\downarrow\rangle_M)/\sqrt{2}\) (spin in the positive \(x\)-direction) and \(|\downarrow\rangle_M = (|\uparrow\rangle_M - |\downarrow\rangle_M)/\sqrt{2}\) (spin in the negative \(x\)-direction). Expressing \(|\uparrow\rangle_M\) as \(((|\uparrow\rangle_M + |\downarrow\rangle_M)/\sqrt{2}\) and \(|\downarrow\rangle_M\) as \(((|\uparrow\rangle_M - |\downarrow\rangle_M)/\sqrt{2}\), one arrives at

\[
\Psi_3 = \frac{\Psi^W}{\sqrt{6}} \left( (\bar{\Psi}^W_+ + \bar{\Psi}^W_-) |\downarrow\rangle_M + (\bar{\Psi}^W_+ - \bar{\Psi}^W_-) |\uparrow\rangle_M + (\bar{\Psi}^W_+ - \bar{\Psi}^W_-) |\downarrow\rangle_M \right),
\]  

(6)

as the wavefunction after Agent \(\bar{W}\) makes the measurement at time \(t_3\).

Finally, Agent \(W\) performs a similar measurement (that is, in the \(x\)-direction for ‘measured’ unbarred spin) as Agent \(\bar{W}\) but on the \(|\uparrow\rangle_M\) and \(|\downarrow\rangle_M\) with the result

\[
\Psi_4 = \frac{1}{\sqrt{12}} \left( (\bar{\Psi}^W_+ + \bar{\Psi}^W_-)(\bar{\Psi}^W_+ - \bar{\Psi}^W_-) + (\bar{\Psi}^W_+ - \bar{\Psi}^W_-)(\bar{\Psi}^W_+ - \bar{\Psi}^W_-) \right),
\]  

(7)

at the final time \(t_f = t_4\). The first term in Eq. (7) comes from the first term \(|\uparrow\rangle_M |\downarrow\rangle\) in Eq. (3), the second term from the second term \(|\downarrow\rangle_M |\uparrow\rangle\) in Eq. (3) and the third from the third one \(|\downarrow\rangle_M |\downarrow\rangle\). Some terms in Eq. (7) cancel among themselves (which can be considered a quantum interference effect) to give

\[
\Psi_4 = \frac{1}{\sqrt{12}} \left( 3\bar{\Psi}^W_+ \bar{\Psi}^W_- \bar{\Psi}^W_+ - \bar{\Psi}^W_- \bar{\Psi}^W_- \bar{\Psi}^W_+ - \bar{\Psi}^W_- \bar{\Psi}^W_- \bar{\Psi}^W_- \right).
\]  

(8)

Equations (5) - (8) are consistent with the results obtained in references [2] - [4] and [6] - [8] after one takes into account notational differences and/or the use of state-preserving measurements performed by the agents.

There is an additional step in which Agent \(W\) meets with Agent \(\bar{W}\) and they provide each other with their measurement results. This affects both of their wavefunctions yielding:

\[
\Psi_5 = \frac{1}{\sqrt{12}} \left( 3\bar{\Psi}^W_+ \bar{\Psi}^W_- \bar{\Psi}^W_+ - \bar{\Psi}^W_- \bar{\Psi}^W_- \bar{\Psi}^W_+ - \bar{\Psi}^W_- \bar{\Psi}^W_- \bar{\Psi}^W_- \right).
\]  

(9)

A subscript \(xy\) on a \(\Psi\) encodes the statement “Agent \(\bar{W}\) measured \(x\) at time \(t_3\) and Agent \(W\) measured \(y\) at time \(t_f\).” For example, \(\bar{\Psi}^W_{-+}\) involves “I, Agent \(\bar{W}\), measured ‘-’ at time \(t_3\) and subsequently met with Agent \(W\) and ‘learned’ that Agent \(W\) had measured ‘+’ at time \(t_f\).

3. The Frauchiger-Renner Argument

The wavefunctions in Section 2 encode measurement statements about themselves:

\[\Psi^A (|\uparrow\rangle + a_4 |\downarrow\rangle) \rightarrow a_4 \Psi^A_{++} |\uparrow\rangle + a_4 \Psi^A_{+-} |\downarrow\rangle,\]

where \(A\) is a “friend agent”, that is, \(F\) or \(\bar{F}\), or where \(A\) is a “Wigner agent”, that is, \(W\) or \(\bar{W}\) when \(\uparrow\) is replaced by \(+\) and \(\downarrow\) is replaced by \(-\). Note that the difference between this and Eq. (2) is that the states \(|\uparrow\rangle\) and \(|\downarrow\rangle\) remain unchanged after the measurement. One might call this type of measurement an observation: An agent observes the state but leaves it intact. The “measurement information statement” is still in \(\Psi^A_{++}\) and \(\Psi^A_{+-}\).
with the statement “Agent \( \bar{F} \) measured the barred spin to be up (that is, \(|\bar{\uparrow}\rangle_M\)) at time \( t_1 \)”,

\(|\uparrow\rangle_M\) with “Agent \( F \) measured the barred spin to be down at time \( t_1 \)”,
\(|\downarrow\rangle_M\) with “Agent \( F \) measured the unbarred spin to be up at time \( t_2 \)

\(\Psi^W_\bar{W}\) with “Agent \( W \) obtained a measurement of ‘−’ at time \( t_3 \)”,
\(\Psi^W_W\) with “Agent \( W \) obtained a measurement of ‘+’ at time \( t_4 \)” and
\(\Psi^\bar{W}_W\) with “Agent \( \bar{W} \) obtained a measurement of ‘−’ at time \( t_4 \)”.

The Frauchiger-Renner argument that quantum mechanics cannot consistently describe itself is based on four statements that can be derived from the wavefunction results in Section 2.

Statement 1: When the experiment is carried out multiple times, there is eventually a run in which Agent \( W \) measures ‘−’ and when he encounters Agent \( \bar{W} \) the latter informs the former that he has measured ‘−’. For this particular run, Agent \( W \) can say “If I (Agent \( W \)) measured ‘−’ at time \( t_4 \), then Agent \( \bar{W} \) measured ‘−’ at time \( t_3 \”).

Statement 2: If Agent \( \bar{W} \) measured ‘−’ at time \( t_3 \), then Agent \( F \) measured the unbarred spin to be up at time \( t_2 \).

Statement 3: If Agent \( F \) measured the unbarred spin to be up at time \( t_2 \) then Agent \( \bar{F} \) measured the barred spin to be down at time \( t_1 \).

Statement 4: If Agent \( \bar{F} \) measured the barred spin to be down at time \( t_1 \), then Agent \( W \) will measure ‘+’ at time \( t_4 \).

Using the transitive property of logic and combining Statements 1 to 4 in order for the run in which both Wigner agents get “minus” produces a contradiction: The “If ... then ...” logical statement begins with the premise “Agent \( W \) measured ‘−’ at time \( t_4 \)” and ends with the conclusion “Agent \( W \) will measure ‘+’ at time \( t_4 \)” which we call the Contradictory Statement. Notice that putting these statements together does not correspond to the order in which measurements take place. Frauchiger and Renner avoid this potential timing issue through the use of assumption (C). With this assumption, agents \( W \) and \( \bar{W} \) can deduce the four statements all at time \( t_4 \). Based on the Contradictory Statement, Frauchiger and Renner conclude that quantum mechanics cannot consistently describe itself.

Statement 1 follows from Eq.(8): there is a term \( -\frac{i}{\sqrt{12}} \bar{\Psi}^W \Psi^W \) in the final wavefunction meaning that 1/12th of the time Agent \( W \) measures ‘−’ and Agent \( \bar{W} \) measures ‘−’.

\(^c\)In unitary quantum mechanics, Statement 1 can be modified to “If, at time \( t_4 \), agents \( W \) and \( \bar{W} \) respectively obtain ‘−’ and ‘−’, then \( W \) obtained ‘+’.” This statement is trivially true. One also needs to note that the probability of them both measuring a ‘−’ result is non-zero. The advantage of this is that one does not have to repeatedly run the experiment until the “minus-minus” outcome happens. The Contradictory Statement is then changed to “If, at time \( t_4 \), agents \( W \) and \( \bar{W} \) respectively measure ‘−’ and ‘−’, then agent \( W \) can deduce that he will measure ‘+’.
Statement 2 follows from Eq. (6): If one looks at the term proportional to $\bar{\Psi}_W$, then the first and third terms involving $|\downarrow\rangle_M$ cancel leaving only the term $-\frac{1}{\sqrt{6}}\bar{\Psi}_W|\uparrow\rangle_M$.

Statement 3 follows from Eq. (5): The term proportional to $|\uparrow\rangle_M$ only involves $|\bar{\downarrow}\rangle_M$.

Statement 4 follows from Eqs. (4) and (7): If Agent $\bar{F}$ measured the barred spin to be down (that is, $|\bar{\downarrow}\rangle$) at time $t_1$, then the relevant part of the wavefunction consists of the second and third terms in Eq. (4). These two terms evolve to the second and third terms in Eq. (7) but the term proportional to $\bar{\Psi}_W$ cancels among them.

4. Quantum Logic and Mathematical Wavefunction Statements

In this section, we discuss the logical mathematical statements encoded in wavefunctions. Although individual mathematical statements have a physical analog in terms of experimental measurements, sets of mathematical statements do not necessarily have a physical analog in terms of a sequence of steps in an experiment, as will become clear below. Nevertheless, we feel that the purely mathematical results presented in this section provide insights into the Frauchiger-Renner gedanken experiment and the issue of violation of the transitive property of logic in quantum mechanics in general.

There are three basic rules:

(i) When a wavefunction is written as a linear superposition of several orthogonal component wavefunctions, then the logic statement involves logical disjunction and the OR symbol $\lor$.

(ii) When a wavefunction involves a product of several component wavefunctions, then the situation involves logical conjunction and the AND symbol $\land$.

(iii) The probability that a (normalized) state occurs is given by the absolute square of its coefficient in the wavefunction (the Born rule).

Let us illustrate these rules using the spin part of the wavefunction at time $t_0$ in Eq. (3):

$$\Psi = \frac{1}{\sqrt{3}}|\bar{\uparrow}\rangle_z|\downarrow\rangle_z + \frac{1}{\sqrt{3}}|\bar{\downarrow}\rangle_z|\uparrow\rangle_z + \frac{1}{\sqrt{3}}|\bar{\downarrow}\rangle_z|\downarrow\rangle_z + \frac{1}{\sqrt{3}}|\bar{\downarrow}\rangle_z|\downarrow\rangle_z.$$  \hspace{1cm} (10)

Rule (i) tells us that either the situation is $|\bar{\uparrow}\rangle_z|\downarrow\rangle_z$ OR $|\bar{\downarrow}\rangle_z|\uparrow\rangle_z$ OR $|\bar{\downarrow}\rangle_z|\downarrow\rangle_z$. This particular case not only involves logical disjunction but mutual exclusivity. If the wavefunction were only $|\bar{\uparrow}\rangle_z|\downarrow\rangle_z$, then Rule (ii) would tell us that the barred spin is up AND the unbarred spin is down. As a more complicated example, combining rules (i) and (ii), one arrives at the following mathematical statement from the

\cite{Footnote}

\footnote{Actually, the situation is even stronger than disjunction; it is exclusive disjunction (symbol XOR). The fact that linear superpositions of orthogonal wavefunctions involve exclusive disjunction is the reason why Schrödinger cats are non-problematic in unitary quantum mechanics.}
wavefunction in Eq. (10):

\begin{align}
\text{Either (barred spin is up AND the unbarred spin is down) OR} \\
\quad \text{(barred spin is down AND the unbarred spin is up) OR} \\
\quad \text{(barred spin is down AND the unbarred spin is down).}
\end{align}

One can also derive

Mathematical Statement 3:

If the unbarred spin is $|\uparrow\rangle_z$, then the barred spin is $|\bar{\downarrow}\rangle_z$, 

because only the middle term has the unbarred spin being up in Eq. (10).

Different logical statements derived from a wavefunction can be obtained by expanding the wavefunction in different ways. For example, the second and third terms in Eq. (10) combine to give

$$
\Psi = \frac{1}{\sqrt{3}} |\bar{\uparrow}\rangle_z |\downarrow\rangle_z + \sqrt{\frac{2}{3}} |\bar{\downarrow}\rangle_z |\uparrow\rangle_x ,
$$

where the $z$ and $x$ subscripts indicate the direction of spin quantization and where $|\bar{\uparrow}\rangle_x = (|\bar{\uparrow}\rangle_z + |\bar{\downarrow}\rangle_z)/\sqrt{2}$. From the second term, one deduces

Mathematical Statement 4:

If the barred spin is $|\bar{\downarrow}\rangle_z$, then the unbarred spin is $|\uparrow\rangle_x$. 

The violation of the transitive property of logic for mathematical statements derived from wavefunctions follows from Mathematical Statements 3 and 4: Let $A =$ “if the unbarred spin is $|\uparrow\rangle_z$”, $B =$ “the barred spin is $|\bar{\downarrow}\rangle_z$”, and $C =$ “the unbarred spin is $|\uparrow\rangle_x$”. Putting these two statements (that is, $A \Rightarrow B$ and $B \Rightarrow C$) together using the transitive property of logic yields

$$
\text{If the unbarred spin is } |\uparrow\rangle_z, \text{ then the unbarred spin is } |\uparrow\rangle_x,
$$

which is a contradiction. As we show below, the problem with transitivity for Mathematical Statements 3 and 4 in this paragraph is closely related to the problem with logic in combining Frauchiger-Renner’s measurement Statements 3 and 4, but with $|+\rangle_M$ playing the role of $|\uparrow\rangle_z$.

Rule (iii) tells us that the probability of barred spin being up AND unbarred spin being down is $1/3$ because the coefficient of $|\bar{\uparrow}\rangle_z |\downarrow\rangle_z$ is $1/\sqrt{3}$ in Eq. (10). Using Rule (iii), one can obtain mathematical wavefunction statements involving probabilities:

Mathematica Statement 4’:

If the barred spin is $|\bar{\downarrow}\rangle_z$, then

there is a 50% chance that unbarred spin is up ($|\uparrow\rangle_z$), and

there is a 50% chance that unbarred spin is down ($|\downarrow\rangle_z$).

This follows from the second and third terms in Eq. (10). Given that quantum mechanics is a theory of probability, it is natural for logical statements to involve
probabilities. Now, when Mathematical Statements 3 and 4' are combined using transitivity, they generate an invalid probabilistic statement:

If the unbarred spin is $|↑⟩_z$, then

- there is a 50% chance that unbarred spin is up ($|↑⟩_z$), and
- there is a 50% chance that unbarred spin is down ($|↓⟩_z$).

(17)

which is consistent with Eq.(15) because a spin in the up $x$-direction has a 50% chance of being up in the $z$-direction and a 50% chance of being down in the $z$-direction.

By expressing the wavefunction in Eq.(10), in various bases involving axes of spin quantization in the $z$ and $x$ directions, one can also obtain mathematic statements about wavefunctions analogous to Frauchiger-Renner measurement Statements 1 and 2. We leave this as an exercise for the reader. When all four statements are combined using logic, one can then rotate the unbarred spin to flip it completely (rather than rotating it 90° as is done in Eq.(15)). The chain of statements reads “If unbarred and barred spins are respectively $|↓⟩_x$ and $|↑⟩_x$, then the barred spin is $|↓⟩_x$.” “If the barred spin is $|↓⟩_x$, then the unbarred spin is $|↑⟩_z$.” “If the unbarred spin is $|↑⟩_z$, then the barred spin is $|↓⟩_z$.” and “If the barred spin is $|↓⟩_z$, then the unbarred spin is $|↑⟩_z$.” If these could be combined using the transitive property of logic, then one would obtain the Contradictory Mathematical Statement “If unbarred and barred spins are respectively $|↓⟩_x$ and $|↑⟩_x$, then the unbarred spin is $|↑⟩_x$.” The fact that there is a one-to-one correspondence between mathematical statements about the wavefunction in Eq.(10) and the Frauchiger-Renner measurement statements beginning with the wavefunction in Eq.(3) suggests that that problem with the transitive property of logic for mathematical statements derived from wavefunctions is likely to arise in the Frauchiger-Renner gedanken experiment, but this remains to be shown, and we show this below.

Summarizing, the transitive property of logic cannot always be used for mathematical statements derived from wavefunctions. The question arises as to whether the above discussion can be “translated” into statements about measurements.

Mathematical Statements 3 and 4 have physical equivalents involving measurement statements:

If the unbarred spin is measured to be $|↑⟩_z$, then the barred spin will be measured to be $|↓⟩_z$.

(18)

and

If the unbarred spin is measured to be $|↓⟩_z$, then the unbarred spin will be measured to be $|↑⟩_x$.

(19)

From the above, it might seem easy to create a physical experiment that generates a contradiction. This is not the case because the measurements occur at different times. If the measurement of the unbarred spin in Eq. (18) happens at time $t_1$ and that of the barred spin in Eqs. (18) and (19) at time $t_2$ then the second unbarred
spin measurement in Eq. (19) must necessarily happen after $t_2$ since the conclusion of the “If... then ...” statement in Eq. (19) is in the future. Suppose it happens at time $t_3$. Then combining Eqs. (18) and (19) using the transitive property of logic gives

$$\text{If the unbarred spin is measured to be } |\uparrow\rangle_z \text{ at time } t_1,$$

then the unbarred spin will be measured to be $|\uparrow\rangle_x$ at time $t_3$, \hfill (20)

which is not obviously a contradiction because the times are different. In addition, measurements can affect wavefunctions. The measurement of the unbarred spin at time $t_1$ may disturb it, changing it from $|\uparrow\rangle_z$ to some other value in an unpredictable way. Hence, the subsequent measurement of the unbarred spin at time $t_3$ could have a random relation to its value at time $t_1$. Frauchiger and Renner get around both these problems by using Wigner/friend measurements and Assumption (C). In doing so, the two authors avoid timing issues, but they did not avoid the problem of the use of the transitive property of logic in quantum mechanics.

Indeed, Statements 3 and 4 of the Frauchiger-Renner gedanken experiment only involve the second and third terms of Eq. (7). Let us focus on them and the measurements by agents F and Agent $\overline{F}$:

$$\Psi_0 = \frac{\Psi^F \overline{\Psi}^F}{\sqrt{2}} |\uparrow\rangle + |\downarrow\rangle \right), \hfill (21)$$

When Agent $\overline{F}$ makes her measurement at $t_1$, the wavefunction becomes

$$\Psi_1 = \frac{\Psi^F}{\sqrt{2}} (|\downarrow\rangle + |\downarrow\rangle \right), \hfill (22)$$

and when Agent F makes her measurement at $t_2$, it becomes

$$\Psi_2 = \frac{1}{\sqrt{2}} (|\uparrow\rangle \Psi^F + |\downarrow\rangle \overline{\Psi}^F). \hfill (23)$$

Now one has two valid statements, the original one of the Frauchiger-Renner gedanken experiment, which is,

Statement 3: If Agent F measured the unbarred spin to be up at time $t_2$ then Agent $\overline{F}$ measured the barred spin to be down at time $t_1$, and a modified version of Statement 4, namely,

Statement 4m: If Agent $\overline{F}$ measured the barred spin to be down at time $t_1$ then Agent F will measure at time $t_2$ the unbarred spin to be up with a probability of 50% and will measure it to be down with a probability of 50%.

The conclusion of Statement 3 is the premise of Statement 4m. Hence, if the transitive property of logic is valid for combining statements about measurements then Agent F can say, “If I measure unbarred spin to be up at time $t_2$ then by using (Q) and (C), I can conclude that I am not guaranteed to measure the unbarred spin to be up at time $t_2.” Since this is a contradiction, the premise of the statement, namely, the transitive property of logic is valid for combining statements about measurements, must be false. Note that this violation of transitivity occurs
for a microscopic system since spins are associated with microscopic objects. No measurements on the agents themselves are involved.

In logic, one can consider the situation in which two premises $P$ and $Q$ must be both satisfied. This is logical conjunction and is denoted by $(P \text{ AND } Q)$. For example, in Eq. (10), if $P = (\text{barred spin is up})$ and $Q = (\text{unbarred spin is down})$, then $(P \text{ AND } Q)$ means that both conditions hold and one is restricting the situation to the first term in Eq. (10). This is a valid use of conjunction in quantum mechanics.

Now consider, $P = (\text{unbarred spin is } |\uparrow\rangle)_{\text{z}}$ and $Q = (\text{unbarred spin is } (|\uparrow\rangle - |\downarrow\rangle)_{\text{z}}/\sqrt{2})$. These premises are incompatible and using them with conjunction is an invalid operation. One might think that $(P \text{ AND } Q)$ means (unbarred spin is $|\uparrow\rangle$ since $|\uparrow\rangle$ is common to both $P$ and $Q$. However, if one uses the $x$ axis of quantization, then $P = (\text{unbarred spin is } (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2})$ and $Q = (\text{unbarred spin is } |\downarrow\rangle_{\text{x}})$, and using the same faulty reasoning one might conclude that $(P \text{ AND } Q)$ means (unbarred spin is $|\downarrow\rangle_{\text{x}}$). In short, there is no way to define $(P \text{ AND } Q)$ for this situation.

A relevant generalization, which involves measurements, is the following: Suppose the wavefunction at time $t_0$ is

$$
\Psi_0 = \frac{\Psi_{\uparrow}}{\sqrt{2}} (\Psi^F_+ + \Psi^F_-) + \frac{\Psi_0^-}{\sqrt{2}} (\Psi^F_+ - \Psi^F_-),
$$

where $\Psi_0^+ \text{ and } \Psi_0^-$ do not involve unbarred spin and $|\Psi_0^+|^2 + |\Psi_0^-|^2 = 1$. Supposed that the evolution of $\Psi_0$ factorizes, that is, at time $t$,

$$
\Psi_t = \Psi_t^+ \Psi_t^W + \Psi_t^- \Psi_t^W,
$$

where $\Psi_0^+$ evolves to $\Psi_t^+ (|\uparrow\rangle_{\text{z}} + |\downarrow\rangle_{\text{z}})/\sqrt{2}$ evolves to $\Psi_t^W$, $\Psi_0^-$ evolves to $\Psi_t^-$, and $|\Psi_t^+|^2 - |\Psi_t^-|^2 = 1$. Suppose $P = (\text{Agent F measures unbarred spin to be up at time } t_0)$ and $Q = (\text{the measurement of Agent W is } '−' \text{ at time } t)$. Because of factorized evolution, premise $Q$ implies that the relevant part of the wavefunction is the second term in Eq. (24); It is proportional to $\Psi_t^W - \Psi_t^W$ and incompatible with premise $P$, which says that Agent F measured the spin to be up at time $t_0$. Hence, $(P \text{ AND } Q)$ has no logical sense. In the last part of Section 6 we show that certain pairs of premises in the Frauchinger-Renner argument are incompatible with logical conjunction.

5. Quantum Logic and Measurement Statements

The example associated with Eqs. (21) - (23) above involves going “backward and forward” in time. Is there an example in which one avoids this? Consider a system involving three spin-$\frac{1}{2}$ objects and three agents $A$, $B$ and $C$ who perform measurements on them. Use the following as the initial wavefunction:

$$
\Psi_0 = \Psi^A \Psi^B \Psi^C (\frac{1}{\sqrt{2}} |\uparrow\rangle_A |\uparrow\rangle_B |\uparrow\rangle_C + \frac{1}{\sqrt{2}} |\downarrow\rangle_A |\downarrow\rangle_B |\downarrow\rangle_C).
$$

We choose a minus sign in Q because it corresponds closer to the situation in reference
Agent A first measures A-spin at time $t_A$, then Agent B measures B-spin at time $t_B$ and finally Agent C measures C-spin at $t_C$. Here, $t_A < t_B < t_C$. After these measurements are made the structure of the wavefunction is

$$\Psi_f = \frac{1}{\sqrt{2}}\Psi_A^A\Psi_B^B\Psi_C^C + \frac{1}{\sqrt{2}}\Psi_A^A\Psi_B^B\Psi_C^C. \tag{27}$$

The agents A, B, and C can then get together to discuss their results. The following statements are derivable from Eq. (27) using the quantum logic rules (i)-(iii) and are true:

**Statement A:** If Agent A measures A-spin to be up at time $t_A$, then Agent B will measure B-spin to be up at time $t_B$.

**Statement T:** If Agent A measures A-spin to be up at time $t_A$, then Agent C will measure C-spin to be up at time $t_C$.

One also has:

**Statement B:** If Agent B measures B-spin to be up at time $t_B$, then Agent C at time $t_C$ will not necessarily measure C-spin to be up.

Indeed, if Agent B measures B-spin to be up at time $t_B$, then Agent C will measure C-spin to be up 50% of the time and down 50% of the time. If Statements A and B could be combined using the transitive property of logic, then one would obtain

**Statement F:** If Agent A measures A-spin to be up at time $t_A$, then it is not guaranteed that Agent C at time $t_C$ will measure C-spin to be up.

Statement F is false since it violates Statement T, the latter always being true. Therefore, the transitive property of logic can be violated in quantum mechanics concerning statements about measurements. Again, this is for a microscopic system since spins are involved. If there is any doubt about the above, Statements A, B and T can be verified in a real experiment; the difficult part is in generating the initial entangled spin state, but nowadays there are methods to handle this.

In standard logic, an “If ... then ...” statement cannot be “50% true”; If it is not always true, then it is considered false. However, given that the conclusions of the “If ... then ...” Statement F (when obtained using transitivity) and Statement B can be replaced by “Agent C will measure C-spin to be up 50% of the time and down 50% of the time”, one might characterize Statement F as being “50% true” and “50% false”. Reference 9 used the above example, but the coefficient of $|\uparrow\rangle_A|\uparrow\rangle_B|\uparrow\rangle_C$ in $\Psi_0$ was selected to be $\sqrt{0.5}$ while that of $|\downarrow\rangle_A|\uparrow\rangle_B|\downarrow\rangle_C$ was $\sqrt{0.5}$. With this change, Statement F is “90% false” and “10% true”. Obviously, one can

By performing a series of runs and collecting statistics, it can be verified that Statement F produces a false result among “50%” of the runs. It should be clear that standard logic is not the correct framework for dealing with statements about wavefunctions and measurements. In fuzzy logic, it is permissible to have statements that are “fractionally” true.
adjust the component coefficients to “increase” the “falsehood” of \( F \), but one cannot arrive at “100%” in this simple example. In addition, up to this point, all of the examples of violations of transitivity for statements about measurements involve using an “If ... then ...” statement in which one of the conclusions of a premise involves probabilities; It is quite natural and acceptable to have such statements since quantum mechanics is a probabilistic theory. In the Frauchiger-Renner gedanken experiment, none of Statements 1 - 4 involve probabilities; However, in the next section, we show that violations of transitivity and other rules of logic still arise.

Suppose that we have a wavefunction at time \( t = 0 \) of the form

\[
\Psi_0 = \Psi_a^0 + \Psi_b^0,
\]

(28)

where \( \Psi_a^0 \) and \( \Psi_b^0 \) are orthogonal. Suppose that one is trying to combine two “If ... then ...” statements using transitivity but the premise of the first “If ... then ...” statement involves a statement about \( \Psi_a^t \), while the premise of the second “If ... then ...” statement involves a statement about \( \Psi_a^t + \Psi_b^t \). Then, given that \( \Psi_a^t \) and \( \Psi_b^t \) are orthogonal and should represent mutually exclusive situations, there is likely to be a problem in the use of transitivity. This is the case in the above examples. It is accomplished by a “shift effect”: Let \( \Psi_0 = \Psi_a^0 + \Psi_b^0 = cu \ket{\uparrow}_A \ket{\uparrow}_B + cd \ket{\downarrow}_A \ket{\uparrow}_B \), where \( cu \) and \( cd \) are constants. The premise of “If \( \ket{\uparrow}_A \) then \( \ket{\uparrow}_B \)” involves only the first term of \( \Psi_0 \) (that is, \( \Psi_a^0 = cu \ket{\uparrow}_A \ket{\uparrow}_B \)), but the premise of “If \( \ket{\uparrow}_B \) then ...” involves both terms. Roughly speaking, the premise has “shifted” from the first term to both terms. The premise of \( \ket{\uparrow}_B \) can be true because of the second term and this can mean that the premise of “If \( \ket{\uparrow}_A \) then \( \ket{\uparrow}_B \)” can be false, (and, technically speaking, in logic this “If ... then ...” statement is considered to be true). Indeed, this is the origin of the violations of transitivity in the examples presented so far. Consider Statements A and B above. When the premise of B is true but the premise of A is false, A-spin is down and this corresponds exactly to the cases in which transitivity leads to a false result (that is, C-spin is down) in Statement F. The same thing happens when using the wavefunction in Eq. (21): unbarred spin corresponds to A-spin and barred spin to B-spin (but flipped).

Unitarity guarantees that the time evolutions of \( \Psi_a^t \) and \( \Psi_b^t \) can be independently evolved and that the two components remain orthogonal. This means that the future evolution of \( \Psi_a^t \) cannot depend on \( \Psi_b^t \), but when the use of transitivity produces a “shift effect”, this basic property of unitarity is violated. In such a case, the very use of logical transitivity is invalid. If \( \Psi_a^0 \) evolves to \( \Psi_a^T \) and \( \Psi_b^0 \) evolves to \( \Psi_b^T \), then there exist “If ... then ...” statements of the form “If A-spin is up at \( t = 0 \) then \( S_u \) at time \( T \)”, where \( S_u \) is a statement about \( \Psi_a^T \), and “If A-spin is down at \( t = 0 \) then \( S_d \) at time \( T \)”, where \( S_d \) is a statement about \( \Psi_b^T \). If \( S_u \) and \( S_d \) are mutually exclusive then the use of transitivity will be violated in a fraction of the cases:

\[
\text{violation fraction} = \frac{|cd|^2}{|cu|^2 + |cd|^2}.
\]

(29)

Since in the above examples \( cu = cd \), the “violation of transitivity is 50%”. In the example of reference 9, \( cd = \sqrt{0.9} \) and \( cu = \sqrt{0.1} \), and the “violation is 90%”. Even
if transitivity produces a valid statement, when the “shift effect” is present, this can be considered a coincidence. For example, if the C-spin of the second term in Eq. (26) is changed to be up, then the conclusions of statement B and F above are both changed to “Agent C at time $t_C$ will measure C-spin to be up”. However, the reason that Statement F is now true is because Agent C must always measure C-spin to be up. The situation corresponds to the discussion presented in the sentences just before Eq. (29) when $S_d = S_u$. In mathematical statements about wavefunctions, the “shift effect” can be traced to the reason why the incorrect result in Eq. (19) arises. In the next section, we show that the combining of Statements 1 and 2, of Statements 2 and 3 and of Statements 3 and 4 using transitivity in the Frauchiger-Renner gedanken experiment uses a “shift effect” and involves exactly the same structure discussed here with $c_u = c_d$. Hence, the logical statements obtained from them using transitivity cannot be true.

In addition to transitivity, there are other rules of logic that are violated in quantum mechanics. For example, in standard logic, if $P$ implies $R$, and $Q$ is any other condition, then $(P$ AND $Q)$ also implies $R$: $(P \Rightarrow R) \Rightarrow ((P$ AND $Q) \Rightarrow R)$. Return to the experiment associated with Eqs. (26) and (27), and let $P$ to be the premise of Statement B ($P$ = “Agent B measures B-spin to be up at time $t_B$”), let $R$ be the conclusion of Statement B ($R$ = “Agent C at time $t_C$ will not necessarily measure C-spin to be up”), and let $Q$ be the premise of Statement A ($Q$ = “Agent A measures A-spin to be up at time $t_A$”). Then $P$ AND $Q$ actually implies $S$ instead of $R$; The conclusion $S$ is “Agent C at time $t_C$ will measure C-spin to be up”.

We now illustrate the utility of mathematical statements about wavefunctions in unitary quantum mechanics by showing that all four measurement statements of the Frauchiger-Renner gedanken experiment can all be derived from the final state in Eq. (8), and knowledge of how the experiment was conducted, that is, Agent $\bar{F}$ first measured barred spin at time $t_1$, then Agent F measured unbarred spin at time $t_2$, etc. Note that $|\uparrow_M\rangle$ (respectively, $|\downarrow_M\rangle$) at time $t_2$ always evolves to $(\Psi^W + \Psi^W_\perp)/\sqrt{2}$ (respectively, $(\Psi^W_+ - \Psi^W^-)/\sqrt{2}$) at time $t_4$. The analogous statement is true for the barred states.

The first step is to rewrite Eq. (8) so that the $(\Psi^W_+ + \Psi^W_\perp)$ and $(\Psi^W_+ - \Psi^W^-)$ dependence is evident:

$$\Psi_4 = \frac{1}{\sqrt{12}}((2\Psi^W_+ (\Psi^W_+ - \Psi^W_-) + (\Psi^W_+ - \Psi^W_-)(\Psi^W_+ + \Psi^W_-))$$.

Statement 1 is a tautology: “If agents W and $\bar{W}$ respectively measured ‘=’ and ‘\neq’, then W measured ‘=’.” Now look at what multiplies $\Psi^W_+$ in Eq. (30). It is $(\Psi^W_+ + \Psi^W_-)$. Since it had to have evolved from $|\uparrow_M\rangle$, one derives Statement 2: “If Agent W obtained a measurement of ‘=’ at time $t_3$, then Agent F previously measured unbarred spin to be up.” Next look at the factor that multiplies what evolves from $|\uparrow_M\rangle$, namely $(\Psi^W_+ + \Psi^W_-)$. This factor is $(\Psi^W_+ - \Psi^W_-)$ and evolved from $|\downarrow_M\rangle$. Hence, one obtains Statement 3: “If Agent F measured unbarred spin up, then Agent $\bar{F}$ measured barred spin to be down.”
To obtain Statement 4, one needs to rewrite Eq. (8) so that the \((\bar{\Psi}_+^W - \bar{\Psi}_-^W)\) and \((\bar{\Psi}_-^W + \bar{\Psi}_-^W)\) dependence is evident:

\[\Psi_4 = \frac{1}{\sqrt{12}}((2(\bar{\Psi}_+^W - \bar{\Psi}_-^W)\Psi_+^W + (\bar{\Psi}_+^W + \bar{\Psi}_-^W)(\Psi_+^W - \Psi_+^W))\].

(31)

Barred down spin evolves to \((\bar{\Psi}_+^W - \bar{\Psi}_-^W)\) and it multiplies \(\Psi_+^W\) (the first term in Eq. (31), and so one obtains “If Agent \(\bar{F}\) previously measured barred spin down, then Agent \(W\) will obtain ‘+’ for his measurement,” which is Statement 4. The above shows that there is a close relation with the measurement statements in the Frauchiger-Renner gedanken experiment and the mathematical statements of Section 4, particularly those in the paragraph below Eq. (17), and that the reasons for the violation of transitivity are similar.

6. Issues with Logic in Unitary Quantum Mechanics with the Frauchiger-Renner Argument

Agent \(F\) can perform her measurement before Agent \(\bar{F}\) and the resulting wavefunction after both measurements remains the same. One can have \(t_1 \approx t_2\) or even have the two agents perform their measurements at the same time. Then Statements 3 and 4 are valid at the same time. So, one can take \(t_1 = t_2\) and use Eq. (31) as the starting point for the Frauchiger-Renner argument. Below, we often make this simplification.

Consider the logic involved in combining Statement 3 and Statement 4 in the logical chain that leads to the Contradictory Statement. Let \(P\) be the premise of Statement 3, that is, \(P = (\text{Agent } F \text{ measured the unbarred spin to be up at time } t_2)\). Let \(Q\) be the conclusion of Statement 3, which is also the premise of Statement 4. Here, \(Q = (\text{Agent } \bar{F} \text{ measured the barred spin to be down at time } t_1 = t_2)\). Finally, let \(R\) be the conclusion of Statement 4: \(R = (\text{Agent } W \text{ will measure ‘+’ at time } t_4)\). Statement 3 is \(P \Rightarrow Q\) and Statement 4 is \(Q \Rightarrow R\). Now, Statement 4 involves the second and third terms in the wavefunction at time \(t_2\) (those involving \(|\bar{\Psi}_-^M \rangle_M \langle \downarrow| \) and \(|\bar{\Psi}_-^M \rangle_M \langle \downarrow| \) in Eq. (5)) while Statement 3 involves the middle or second term (the one involving \(|\bar{\Psi}_+^M \rangle_M \langle \uparrow| \)). Let us just focus of this part of the wavefunction and its evolution to time \(t_4\):

\[\Psi_2 = \frac{\Psi_+^W \bar{\Psi}_+^W \sqrt{2}}{\sqrt{3}} (\cdots \sqrt{\frac{1}{2}} |\bar{\Psi}_+^M \rangle_M \langle \uparrow| + \sqrt{\frac{1}{2}} |\bar{\Psi}_-^M \rangle_M \langle \downarrow| ) \]



\[\Psi_4 = \sqrt{\frac{2}{3}} (\cdots + \frac{1}{2} \bar{\Psi}_+^W (\Psi_+^W + \Psi_+^W) + \frac{1}{2} \bar{\Psi}_+^W (\Psi_+^W - \Psi_+^W))\].

(32)

As an aside, it is also true that the temporal order does not matter for the two Wigner measurements at times \(t_3\) and \(t_4\); One can have \(t_3 < t_4, t_4 < t_3\) or \(t_3 = t_4\), and Statements 1-4 of the Frauchinger-Renner gedanken experiment are all still valid.
where $\Psi^W_\downarrow$ is an abbreviation for $(\Psi^W_+ - \Psi^W_-)/\sqrt{2}$. The first thing to note is that the “shift effect” is occurring, and so, given the results in Section 6 combining Statements 3 and 4 using transitivity is an invalid procedure. One can also “quantify” the violation of assuming $(P \Rightarrow Q) \text{ AND } (Q \Rightarrow R) \Rightarrow (P \Rightarrow R)$ using Eq. 29: $P \Rightarrow R$ should be “50% false”, which is easy to show.

The $|\uparrow\rangle_M |\uparrow\rangle_M$ term in Eq. (32) at time $t_2$ evolves to the $\Psi^W_\downarrow (\Psi^W_+ + \Psi^W_-)$ term at time $t_4$. Likewise, the last term in Eq. (32) at $t_2$ evolves to the last term at $t_4$ in $\Psi_4$. When the premise $Q$ (i.e., barred spin is measured to be down) of Statement 4 holds, both terms are relevant and a cancellation of $\Psi^W_-$ occurs in $\Psi_4$ thereby yielding the conclusion $R$ of Statement 4, that the probability of Agent W obtaining ‘+’ is 100%. However, if an “If ... then ...” statement involves premise $P$ (i.e., unbarred spin is measured to be up), then the relevant term is the $|\downarrow\rangle_M |\uparrow\rangle_M$ one. It evolves to something proportional to $(\Psi^W_+ + \Psi^W_-)$. So, if $P$ holds, that is, Agent F measured unbarred spin up at time $t_2$, then there is a 50% chance Agent W will obtain ‘+’ for his measurement and not 100%.

The conclusion of Statement 4 involves a delicate cancellation between two terms in Eq. (32) to eliminate the $\Psi^W_-$ dependence in $\Psi_4$. This can be considered a quantum interference effect. The best example of quantum interference is the double slit experiment. Imagine that $|\uparrow\rangle |\uparrow\rangle$ is associated with the wavefunction in the left-slit region and that $|\downarrow\rangle |\downarrow\rangle$ is associated with it in the region of the right slit.

---

**Fig. 1. Interference Effect Involved in Statement 4.**

![Detection Screen Diagram](image-url)
Suppose each evolves respectively to \( \bar{\Psi}_W \) \( \downarrow \) \( (\Psi_W^+ + \Psi_W^-) / \sqrt{2} \) and \( \bar{\Psi}_W \) \( \downarrow \) \( (\Psi_W^+ + \Psi_W^-) / \sqrt{2} \) when they reach a certain point on the detection screen. See Figure 1. If no attempt is made to detect whether the wavefunction goes through the left or right slit, then the quantum interference effect occurs, there is a “cancellation of the \( \bar{\Psi}_W \) amplitude”, and the screen will signal to Agent W a ‘+’ outcome. Now when premise \( P \) is operative, it means that Agent F has effectively “done something” to determine which slit the object went through. Indeed, she has determined that it went through the left slit because she has measured the unbarred spin to be up, which is associated with \( |\bar{\downarrow} \rangle |\uparrow \rangle \). This disturbs the quantum interference effect, the wavefunction will evolve to \( \bar{\Psi}_W \uparrow \) \( (\Psi_W^+ + \Psi_W^-) / \sqrt{2} \) at the screen, and the signal can no longer be guaranteed to be ‘+’. Half the time it will be ‘+’ and half the time it will be ‘−’.

Thus, using the analogy with the two-slit experiment, one understands physically why the transitive rule is violated in this case: When Statement 3 is combined with Statement 4, the resulting logical statement does not properly take into account the effect of the measurement performed by Agent F on the one by Agent W.

If the gedanken experiment involved only the terms displayed in Eq.(32), then one can derive the following two statements:

**Statement 3L**: If Agent F measures the unbarred spin to be up at time \( t_2 \), then Agent W at time \( t_4 \) will measure ‘+’ 50% of the time.

**Statement 3R**: If Agent F measures the unbarred spin to be down at time \( t_2 \), then Agent W at time \( t_4 \) will measure ‘+’ 50% of the time.

Now, in logic, if \( (L \Rightarrow Q) \) and \( (R \Rightarrow Q) \) are two valid “If ... then ...” statements, then one can conclude that \( (L \text{ OR } R) \Rightarrow Q \). Here, \( L \) and \( R \) are respectively the premises of Statements 3L and 3R, and \( Q = (\text{Agent W at time } t_4 \text{ will measure ‘+’ 50% of the time}) \), or one can use \( Q = (\text{Agent W at time } t_4 \text{ will not necessarily obtain a measurement of ‘+’}) \). However, the correct statement involving the premise \( (L \text{ OR } R) \) is \( (L \text{ OR } R) \Rightarrow Q' \), where \( Q' = (\text{Agent W at time } t_4 \text{ will measure ‘+’ with certainty}) \). This is just another example of how logic cannot be applied to statements about measurements, especially when quantum interference effects are involved.

When Statement 2 is combined with Statement 3 using transitivity, the result is “If Agent W measured ‘−’ at time \( t_3 \), then Agent F measured the barred spin to be down (\( |\bar{\downarrow} \rangle \)) at time \( t_1 \). However, we know from the methods used in the last two paragraphs of Section 5 that if Agent W measured ‘−’ at time \( t_3 \) then it had to have evolved from a term proportional to \( |\bar{\uparrow} \rangle_M - |\bar{\downarrow} \rangle_M \) in \( \Psi_1 \) at time \( t_1 \) and not something proportion to \( |\bar{\downarrow} \rangle_M \). In fact, one can show that it originated from \( (|\bar{\uparrow} \rangle_M - |\bar{\downarrow} \rangle_M) |\uparrow \rangle \) (up to a factor). Hence, combining Statement 2 with Statement 3 using transitivity generates an invalid logic statement. It is violated 50% of the time.

To analyze whether Statement 1 can be combined with Statement 2 using transitivity, one needs to consider the last two terms in \( \Psi_4 \) of Eq.(8). Recall that State-
Statement 1 can be expressed as “If, at time $t_4$, agents $W$ and $\bar{W}$ respectively measure $\langle - \rangle$ and $\langle = \rangle$, then $\bar{W}$ measured $\langle = \rangle$.” Hence, the premise of Statement 1 involves the last term in Eq.(8), whereas the conclusion of Statement 1, which is the premise of Statement 2, involves both the 3rd and 4th terms. A “shift effect” is present, and, not surprisingly, the logical statement generated using transitivity is “50% false”. The conclusion of Statement 2, namely that Agent $F$ measured the unbarred spin to be up at time $t_2$, uses a quantum interference effect similar to the one involved in combining Statements 3 and 4. In fact, the “wavefunction structures” and the “If ... then ...” statements for the two cases are isomorphic.

The “If ... then ...” Statements 1 through 4 in the Frauchiger-Renner gedanken experiment are all valid in unitary quantum mechanics when considered in isolation. In reference [1] a special run is selected, namely, the one in which agents $W$ and $\bar{W}$ measure $\langle - \rangle$ and $\langle = \rangle$. What happens when one considers the effect of imposing this? The answer is that one is restricting the wavefunction to the last term in Eq.(8), and Statements 2 through 4 become invalid, thereby ruining the argument in reference [1]. For example, the conclusion of Statement 3, which is Agent $\bar{F}$ measured the barred spin to be down at time $t_1$, is not a consequence of the premise “Agent $F$ measured the unbarred spin to be up at time $t_2$ AND $\bar{W}$ measures $\langle = \rangle$ at time $t_3$ AND Agent $W$ measures $\langle - \rangle$ at time $t_4$,” as one can verify. This is another example of the fact that the use of certain rules of logic do not always properly take into account the combined effects of measurements made by the agents. Intuitively, it is easy to understand why Statements 2 and 4 are rendered invalid when the ‘$\langle - \rangle$ - $\langle = \rangle$’ condition is imposed: The validity of both these statements depends on a perfect quantum interference cancellation between two terms in the wavefunction. Anything that disrupts the delicate cancellation will render the corresponding statement false. Consider Statement 2, for example. Its validity depends on a cancellation involving the second and third terms in Eq.(3) and the evolution of these two terms going forward in time. Refer to Figure 1. However, the constraint that agents $W$ and $\bar{W}$ respectively measure ‘$\langle - \rangle$’ and ‘$\langle = \rangle$’ affects all three terms in Eq.(8) and upsets the quantum interference cancellation. In fact, the constraint forces the initial wavefunction to be proportional to $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle - |\uparrow\rangle + |\downarrow\rangle$ instead of Eq.(8).

To generate the Contradictory Statement, the premises of Statements 1 through 4 must be all true at once. These premises are “Agent $W$ measures ‘$\langle - \rangle$’”, “$\bar{W}$ measures ‘$\langle = \rangle$”, etc. When Agent $W$ measures ‘$\langle - \rangle$’ at time $t_4$, it can be verified that Agent $\bar{F}$ had to have measured barred spin to be up at time $t_1$. The premise of Statement 4 is “Agent $\bar{F}$ measured the barred spin to be down at time $t_1$”. Hence, whenever the premise “Agent $W$ measures ‘$\langle - \rangle$’ at time $t_4” is true, the premise of Statement 4 is false, and vice versa. There are no instances when the premises of Statements 1 through 4 are all true at once. So, it is not surprising that, in incorrectly using the rules of logic for statements about measurements in the Frauchiger-Renner gedanken experiment, one can arrive at the logically contradictory state-
ment “If, at time $t_4$, agents W and $\bar{W}$ respectively measure ‘$-$’ and ‘$-$’, then agent $W$ can deduce that he will measure ‘$+$’.

Another way of analysing the previous paragraph is as follows: If the usual rules of logic hold, then $(A \Rightarrow B$ AND $B \Rightarrow C)$ $\Rightarrow (A$ AND $B) \Rightarrow C$. So, it should be true that $(P_1$ AND $P_2$ AND $P_3$ AND $P_4) \Rightarrow C_4$, where $P_i$ is the premise of Statement $i$ and $C_i$ is the conclusion of Statement 4. Hence, the premises must be all true at once. However, the situation is even worse: The premise of Statement 2 is “Agent $W$ measured ‘$-$’ at time $t_3$”. The premise of Statement 4 is “Agent $\bar{F}$ measured the barred spin to be down at time $t_1$”. Now from the analysis at the end of Section 5, we know that if Agent $W$ measured ‘$-$’ at time $t_3$ then the relevant part of the wavefunction had to have evolved from something proportional to $\bar{\Psi} \bar{\Psi} \uparrow \downarrow$ (which is the same as $|\uparrow\rangle_M - |\downarrow\rangle_M$) at time $t_1$. This means that we are in the situation described in the paragraph that contains Eqs. (24) and (25) except that barred spin is involved: $P_2$ and $P_4$ are conjunctually incompatible; It is illegitimate to have them appear in the same logical AND statement. One of the premises of Statement 1, namely that “Agent $W$ measures ‘$-$’ at time $t_4$”, is also conjunctually incompatible with the premise of Statement 3, namely that “Agent $\bar{F}$ measured the unbarred spin to be up at time $t_2$” for the same reason as the “barred” case.

7. The Frauchiger-Renner Wigner/Friend Measurements

In this section, we reveal a technical problem with the Wigner/friend measurements used in the Frauchiger-Renner gedanken experiment. The experiment makes use of two such measurements. Suppose that Agent $F$ measures a qubit, which we represent as the spin of a spin-$\frac{1}{2}$ object. Let $\Psi^F$ be the wavefunction of Agent $F$ before the measurement is made. The wavefunction $\Psi^F$ in general consists of many degrees of freedom – those of the experimentalist and those of her apparatus. When Agent $F$ measures the state $|\uparrow\rangle$, the wavefunction for $F$ changes: at a minimum, the apparatus records the up-spin result and the experimentalist notes in her brain that the spin was measured to be up. As explained in the Introduction, we denote the resulting wavefunction by $\Psi^F_{\uparrow}$. If the spin-$\frac{1}{2}$ object “survives”, its degrees of freedom are included in $\Psi^F_{\uparrow}$. In cases in which the qubit states are represented by the right and left polarizations of a photon and the photon is destroyed during the measurement, $\Psi^F_{\uparrow}$ does not include the qubit degree of freedom: The measurement process is still represented by $\Psi^F |\uparrow\rangle \to \Psi^F_{\uparrow}$. When Agent $F$ measures the state $|\downarrow\rangle$, statements similar to the above apply and the process is presented by $\Psi^F |\downarrow\rangle \to \Psi^F_{\downarrow}$.

A Wigner/friend measurement involves a new Agent $W$ who makes a measurement on the $\Psi^F_{\uparrow}$ and $\Psi^F_{\downarrow}$ states. In the Frauchiger-Renner gedanken experiment, Agent $W$ makes the measurement in the basis $\Psi^F_{\uparrow} = (\Psi^F_{\uparrow} + \Psi^F_{\downarrow})/\sqrt{2}$ and $\Psi^F_{\downarrow} = (\Psi^F_{\uparrow} - \Psi^F_{\downarrow})/\sqrt{2}$. If $\Psi^W$ is the wavefunction before the “Wigner” measurement is made, then, as in the case of Agent $F$ above, $\Psi^W$ is affected by the measurement and the process is represented by $\Psi^W \Psi^F_{\uparrow} \to \Psi^W_{\uparrow}$ and $\Psi^W \Psi^F_{\downarrow} \to \Psi^W_{\downarrow}$. The degrees
of freedom of $F$ are included in $\Psi^W_+$ and $\Psi^W_-$.

In the Frauchiger-Renner gedanken experiment, Agent $W$ has an almost impossible task in measuring the ‘$+$’ and ‘$-$’ states of such a complicated system. However, there is also a tremendous burden on Agent $F$. If Agent $F$ is a complicated object – and indeed up until now we have been assuming this since $F$ consists of the experimentalist and her equipment – then it is unlikely that the same $\Psi^F_+$ is produced each time $|\uparrow\rangle$ is measured. This is a problem for Agent $W$ because in the Wigner/friend experiment he is not allowed to examine $\Psi^F_+$ and $\Psi^F_\downarrow$. So, how can he measure $(\Psi^F_+ + \Psi^F_\downarrow)/\sqrt{2}$ and $(\Psi^F_+ - \Psi^F_\downarrow)/\sqrt{2}$ if he does not know what they are? The solution is that Agent $F$ must respond to the measurement of the spin in a predetermined known way to produce specific $\Psi^F_+$ and $\Psi^F_\downarrow$, and Agent $W$ must be informed of these “known” states at the start of the experiment and before the measurements are made. It is clearly impossible for Agent $F$ to produce a specified $\Psi^F_+$ or $\Psi^F_\downarrow$ given that Agent $F$ is such a complicated object. Among things, the center of mass coordinate of Agent $F$ is a continuous variable that cannot be precisely fixed because of the Heisenberg uncertainty principle.

To shed some light on the issue, consider replacing all the quantum degrees of freedom associated with Agent $F$ with a system of 100 qubits, with each qubit having two states: up and down. During the measuring process, these 100 qubits are “disturbed” randomly but the signal of the experimental outcome is encoded in the last qubit: If the spin was up (respectively, down) then the last qubit is up (respectively, down) after the measurement is performed. Now Agent $F$ and $W$ agree, for example, that $\Psi^F_+$ (respectively, $\Psi^F_\downarrow$) corresponds to all the qubits being up (respectively, down). Now when the experiment takes place Agent $F$ has the very, very difficult task of controlling how the experiment effects the first 99 bits. It is very unlikely that the qubits will all be up or all be down. Hence, almost all the time, Agent $W$ gets no signal when he tries to make his measurement. So, the “100-spin case” is difficult but still doable in principle. However, the situation is rendered impossible when one considers that among the enormous number of quantum degrees of freedom of Agent $F$, there are many – in fact most – which are continuous and not discrete.

A way around this problem is to have Agent $W$ interact only with a small “important” subset of the degrees of freedom of $F$. These degrees of freedom might include a qubit in a data base that recorded the reading as up or down (as in the previous paragraph) as well as data bytes providing the time of the measurement $t_m$ and statements such as “Agent $F$ knows that the measurement was up at time $t_m$” that are used in reference [1] and so on. In other words, if we write $\Psi^F = \Psi^F |S\rangle$ where $|S\rangle$ indicates a state associated with these “important” degrees of freedom, then we can have $\Psi^F |\uparrow\rangle = \Psi^F |S\rangle |\uparrow\rangle_M \rightarrow \Psi^F |S\rangle |\uparrow\rangle_M'$, where $|\uparrow\rangle_M$ indicates a specific state for the up case that provides the “measurement recording information.” For the down spin case, $\Psi^F |\downarrow\rangle \rightarrow \Psi^F |S\rangle |\downarrow\rangle_M$, where, again, $|\downarrow\rangle_M$ is some specific state. Agent $W$ can then be supplied in advance with $|\uparrow\rangle_M$ and $|\downarrow\rangle_M$. One
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simple realization of this is \( |S\rangle = |\uparrow\rangle_M \). One can then have \( |\uparrow\rangle_M \uparrow \rightarrow |\uparrow\rangle_M |\uparrow\rangle \) (= \( |\uparrow\rangle_M |\uparrow\rangle \)) and \( |\downarrow\rangle_M \downarrow \rightarrow |\downarrow\rangle_M |\uparrow\rangle \) (= \( |\downarrow\rangle_M |\uparrow\rangle \)) as the measurement process, in which case \( |S\rangle' = |\uparrow\rangle \).

If the initial state is \( |\uparrow\rangle \), for example, then \( \Psi_W \Psi_F |\uparrow\rangle = \Psi_W \Psi_F' |S\rangle |\uparrow\rangle \rightarrow \Psi_W \Psi_F' |S\rangle' |\uparrow\rangle = (\Psi_W |S\rangle' / \sqrt{2}) \Psi_W (\langle |\uparrow\rangle_M + |\downarrow\rangle_M \rangle / \sqrt{2} + \langle |\downarrow\rangle_M - |\uparrow\rangle_M \rangle / \sqrt{2}) = \Psi_{F'}' |S\rangle' (\Psi_W' + \Psi_W) / \sqrt{2} \). Likewise, \( \Psi_{F'}' |\downarrow\rangle \rightarrow \Psi_{F'}' |S\rangle' (\Psi_W' - \Psi_W) / \sqrt{2} \). It is easy to see that \( \Psi_{F'}' = \Psi_{F'} |S\rangle' \) plays no role in the Wigner/friend experiment: It is just an overall factor in the wavefunction from the time of the measurement by Agent F henceforth, and therefore can be ignored in the analysis. When this is done, \( |\uparrow\rangle_M \) and \( |\downarrow\rangle_M \) act like a qubit but with messages associated with them. This simplification justifies, for example, the replacement of a friend agent by photon polarizations as is done in reference \([20]\). Although there are many possibilities for agents F and W to agree in advance on what \( \Psi_F' \) and \( \Psi_F'' \) are, the result that these two states need to be replaced by two specific states, which we can call \( |\uparrow\rangle_M \) and \( |\downarrow\rangle_M \), is a general result. Agent W can still involve the many degrees of freedom of the human experimentalist and his equipment.

Note, when the procedure described in the previous two paragraphs is used, that the final wavefunction for the experimentalist must be the same whether the initial spin state is \( |\uparrow\rangle \) or \( |\downarrow\rangle \). This means that the experimentalist cannot be conscious of the experimental outcome. The definition of a measurement on a quantum system is not clearly defined. In its definition, one might require a human or intelligent being to be conscious of the outcome, in which case, a Wigner measurement of the type occurring in the Frauchiger and Renner gedanken experiment is impossible, given the results in the first four paragraphs of this section. Alternatively, the definition of a quantum measurement might only require the outcome to be “recorded” or “registered”, which means that it is not necessary for an intelligent being to be aware the experimental result. In this case, a Wigner measurement is possible but it must be made on an entity without a center of mass degree of freedom such as a spin, a photon polarization, a tensor product of these, et cetera. The Frauchiger and Renner experiment is therefore only possible if the states \( |\uparrow\rangle_M', |\downarrow\rangle_M', |\uparrow\rangle_M \) and \( |\downarrow\rangle_M \) are of this form. However, such states are necessarily microscopic. Hence, when the Wigner agents make their measurements, it is on microscopic entities, in which case, they are “ordinary” quantum measurements. Regardless of the problems with the transitive property of logic, the Frauchiger-Renner gedanken experiment cannot be making a statement about a macroscopic system.

If the measurements by Agents F and F are not considered measurements but recordings, then the subscripts ‘M’ in Sections 2 through 6 are misleading and should be replaced by ‘R’. For completeness, we provide a brief description on how the Frauchiger-Renner gedanken experiment is modified to take this into account. One needs to avoid saying that Agent F and Agent F make measurements on the barred and unbarred spins, since almost all but a few discrete quantum degrees of freedom of these two agents are affected. For example, the original first step, which
is “Agent F measures the spin of the barred spin-½ object in the z-direction at time \( t_1 \),” needs to be replaced by “An experimental procedure on the barred spin-½ object in the z-direction at time \( t_1 \) is performed and the outcome is recorded in another spin-½ object as \( \left| \uparrow \right>_R \) or \( \left| \downarrow \right>_R \).” A similar replacement occurs for step two. The subscript “M” is replaced by “R” on barred and unbarred spins in Sections 2 and 3. Statement 1 is unchanged but Statements 2 to 4 become:

Statement 2: If Agent W measured ‘-’ at time \( t_3 \), then the unbarred spin was recorded to be up at time \( t_2 \).

Statement 3: If the unbarred spin was recorded to be up at time \( t_2 \) then the barred spin was recorded to be down at time \( t_1 \).

Statement 4: If the barred spin was recorded to be down at time \( t_1 \), then Agent W will measure ‘+’ at time \( t_4 \).

If these four statements could be combined using the transitive property of logic, then one would still obtain the Contradictory Statement of Section 3.

It should be clear that any Wigner measurement on a linear combination of Agent F states involving all the degrees of freedom of a human and her equipment is, in general, impossible. For the case in which Agent F performs a measurement on a spin-½ object in the z-direction, this means that Agent W cannot perform a measurement using a basis of \( \cos \theta \Psi^F_{\uparrow} + \sin \theta \Psi^F_{\downarrow} \) and \( \sin \theta \Psi^F_{\uparrow} - \cos \theta \Psi^F_{\downarrow} \) for any \( \theta \) for which both \( \cos \theta \neq 0 \) and \( \sin \theta \neq 0 \). Hence, the only basis in which a Wigner agent can make a measurement on Agent F is one that is “aligned” with the basis that Agent F used. For the case of the spin-½ object discussed in this section, the basis is \( \Psi^F_{\uparrow} \) and \( \Psi^F_{\downarrow} \).

8. Discussion and Conclusions

In their work, Frauchiger and Renner concluded that quantum theory cannot be extrapolated to complex systems in a straightforward manner. They considered 10 interpretations/modifications of quantum mechanics and pointed out how each of them violates at least one of the three reasonable assumptions (Q), (C), and (S). Unitary quantum mechanics does not violate any of these assumptions (See Appendix A). However, the generation of a contradiction arises only if Assumption (S) is replaced by the stronger Assumption (L), which says that statements by agents concerning measurements of wavefunctions obey standard rules of logic. It is hard to argue that an interpretation or a modification of quantum mechanics should not obey Assumption (S). One might naively think that Assumption (L) should also hold. However, we have shown in this paper that this is not the case: statements about measurements cannot necessarily be combined to generate new statements using the standard rules of logic. This result also applies to the 10 interpretations/modifications of quantum mechanics considered in reference [1]. Once one understands this, there is nothing, in principle, ruling out quantum mechanics – and unitary quantum mechanics in particular – being able to govern complex and macroscopic systems. Indeed, unitary quantum mechanics does not
have an Einstein-Podolsky-Rosen paradox as was pointed out by Hugh Everett in his Ph.D. thesis, nor does unitary quantum field theory (the generalization of unitary quantum mechanics to include second-quantized processes) have a measurement problem.

Let us enumerate the explanations of Section 6 of why statements in the Frauchiger-Renner gedanken experiment cannot be combined using logic:

1. Combining Statements 1 and 2, Statements 2 and 3, as well as Statements 3 and 4 involve a “shift effect” (See Section 5), and this invalidates the use of transitivity for these pairs of statements. Eq. (29) provides a quantification of the violation, and it is 50% for each of the above three uses of transitivity.

2. The premise of the first “If ... then ...” statement is false in a certain fraction of the instances of the premise of the second “If ... then ...” statement. This fraction coincides with the result in Eq. (29). When the premise of the first statement is false while the premise of the second statement is true, a false result is generated from the “If ... then ...” statement obtained by combining the two “If ... then ...” statements using transitivity. This happens in combining Statements 1 and 2, Statements 2 and 3, and Statements 3 and 4.

3. Combining Statements 1 and 2 and Statements 3 and 4 involve a quantum interference effect. This interference effect is upset by the measurement associated with the premise of the first statement. In other words, the “If ... then ...” statement obtained by using transitivity does not properly take into account the effect of the measurement performed by the first agent on the measurement performed by the second agent.

4. All premises must be true at once to obtain a valid Contradictory Statement. One of these premises is that Agent W measures ‘−’. The premise of Statement 4 is that Agent Ĝ measures the unbarred spin to be down. It can be shown that when the premise that Agent W measures ‘−’ is true then the premise of Statement 4 is false, and when the premise of Statement 4 is true, the premise that Agent W measures ‘−’ is false.

5. Frauchiger and Renner run their experiment until both agents W and Ĝ respectively measure ‘−’ and ‘−’. It can be shown that restricting the run to this case renders Statements 2, 3 and 4 false.

6. In logic, \((A \Rightarrow B \land B \Rightarrow C) \Rightarrow ((A \land B) \Rightarrow C)\), however, in unitarity quantum mechanics for the cases involving Statements 1 - 4, \((A \land B) \Rightarrow C'\), where \(C'\) is a conclusion that is different from \(C\). This shows that this rule of logic is violated in the Frauchiger-Renner gedanken experiment. This is relevant for the generation of the contradictory statement because it is unclear whether one should use \(C\) or \(C'\); The argument in the Frauchiger-Renner publication needs to use \(C\) to generate the contradictory statement. However, in combining pairs of statements \(C'\) turns out to be the correct conclusion.

7. If one formulates the generation of the contradictory statement as \((P_1 \land P_2 \land P_3 \land P_4) \Rightarrow C_4\), where \(P_i\) is the premise of Statement \(i\)
and $C_4$ is the conclusion of Statement 4, then one finds that $P_2$ and $P_4$ are incompatible for use with logical conjunction. See the last paragraph of Section 6.

In the above, (1), (2), (3) and (6) involve the issue of combining two successive statements using transitivity and there is some overlap in the arguments showing that its use is not valid in the Frauchiger-Renner gedanken experiment. Items (4), (5), and (7) point out other problems in combining the four statement to produce the contradictory statement. In short, the usual rules of logic cannot be used on Statements 1 - 4 in the Frauchiger-Renner gedanken experiment.

At a minimum, our work has cleared up a misconception created by reference [1] that has already reached mainstream scientific media [17-19]. We have also obtained other important results. We developed the concept of quantum logic and used it to deduce physical and mathematical consequences from knowledge of a wavefunction. We have learned that one must be careful in using many of the "standard" rules of logic for statements about wavefunctions and measurements. Sections 4-6 shed light on why the violations of the rules of logic are expected in certain circumstances. In Section 7 we pointed out a restriction on Wigner/friend experiments. If this restriction is imposed, then the Wigner/friend measurement of the Frauchiger-Renner gedanken experiment becomes an ordinary quantum measurement, allowing the possibility of carrying out the experiment in a real laboratory setting.

The Frauchiger-Renner gedanken experiment is interesting in that it has forced us to think more deeply about quantum mechanics, quantum logic and Wigner/friend measurements.

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Appendix A. The Three Frauchiger-Renner Assumptions in Unitary Quantum Mechanics

Frauchiger and Renner make three assumptions about quantum mechanics. In this appendix, we argue that each of them holds in unitary quantum mechanics.

Assumption (Q) establishes that an agent can be certain that a given proposition holds whenever the quantum-mechanical Born rule assigns probability 1 to it. Unitary quantum mechanics is "very standard" in this regard because probabilities are based on the absolute square of the wavefunction. Frauchiger and Renner use (Q) to establish three of the four statements that combine to give the Contradictory Statement. Unitary quantum mechanics is able to obtain all of them.

Assumption (C) demands that if an Agent $A'$ establishes a statement using unitarity quantum mechanics then another Agent $A$ can be certain that the statement of $A'$ is valid. In unitary quantum mechanics for the extended Wigner/friend
gedanken experiment, all agents know the initial spin part of the wavefunction and the experimental procedure and therefore the overall structure of the wavefunction. For example, at time $t_4$, Agents W and $\bar{\text{W}}$ do not know that $\Psi_4$ is precisely as in Eq. (8), but they do know that the structure is

$$\Psi_4 = \frac{1}{\sqrt{12}} (3\Phi^W_+ \Phi_+^W - \Phi^W_- \Phi_+^W - \Phi^W_- \Phi_+^W - \Phi^W_+ \Phi_+^W),$$

(A.1)

where the $\Phi^A_a$ (with $a = +$ or $-$ and $A = W$ or $\bar{W}$) embody the same messages and statements as the $\Psi^A_a$ but they do not necessarily know the specific $\Psi^A_a$ wavefunctions. In the case of the wavefunction at time $t_2$, Agent W actually knows the spin part is of the form

$$\left(\frac{1}{\sqrt{3}}\left(\left|\bar{\uparrow}\right\rangle_M \left|\downarrow\right\rangle_M + \left|\bar{\downarrow}\right\rangle_M \left|\uparrow\right\rangle_M + \left|\bar{\downarrow}\right\rangle_M \left|\downarrow\right\rangle_M\right)\right),$$

(A.2)

where $\left|\bar{\uparrow}\right\rangle_M$ and $\left|\bar{\downarrow}\right\rangle_M$ embody the same messages and statements as $\left|\uparrow\right\rangle_M$ and $\left|\downarrow\right\rangle_M$. Agent W knows that the $\left|\uparrow\right\rangle_M$ and $\left|\downarrow\right\rangle_M$ in Eq. (A.2) are the same as the states in Eq. (5) because Agent W and Agent F have a prior agreement as to what the measured states will be, as explained in Section 7. Finally, note that in generating the Statements 1 - 4 of Sect. (3), one only needs the statement information associated with wavefunctions and not the detailed wavefunctions themselves. Hence, agents W and $\bar{W}$ can arrive at Statements 1 - 4 in unitary quantum mechanics. In reference 1, the statements are expressed somewhat differently than in Section 3. For example, Statement 4 in reference 1 is:

Statement 4: If I, Agent $\bar{\text{F}}$, measure the barred spin to be down at time $t_1$, then I am certain that Agent W will observe ‘+’ at time $t_4$.

In unitary quantum mechanics, an agent only needs to know the initial state and the experimental procedure (who measured what, at what time, and in what manner) to derive Statements 1 - 4. Unitary quantum mechanics is “powerful” in this regard. Indeed, in the paragraphs containing Eqs. (30) and (31), we showed that knowledge of the final state and the experimental procedure is also sufficient to derive Statements 1 - 4, a result that many might find remarkable. Thus, in unitary quantum it is not necessary to express the statements in the manner of reference 1 but one can do so if one wants to.

Assumption (S) says that “if Agent A is certain of the statement $x = v$ at time $t$ then he has to deny the statement $x \neq v$ at time $t$.” In unitary quantum mechanics, an agent can only be certain of the outcome of a measurement when wavefunction collapse is not needed to explain it. For example, if Agent A was trying to determine if a spin was up or down and the initial state $S_0$ to be measured was of the form $S'_0 \left|\uparrow\right\rangle$ where $S'_0$ is a wavefunction not involving the spin degree of freedom (we also include the case in which $S'_0$ is a phase), then this is similar to case (i) in Eq. (4) of the gedanken thought experiment. After the measurement, the wavefunction is $S'_0 \Psi^A_+ \Psi^A_+$ embodies the statement that Agent A measured the spin to be up and knows that it is up. He must deny that it is down in agreement with Assumption (S). On the other hand, if Agent A
was trying to determine if a spin was up and the initial state $S_0$ was of the form $a_↑ |↑⟩ + a_↓ |↓⟩$, where now $a_↑$ and $a_↓$ can be wavefunctions not involving the spin, then the situation is similar to case (ii) of the Introduction: After the measurement, the wavefunction is the result in Eq. (2) and, as explained in the Introduction when discussing the Measurement Rule, both $\Psi^{A}_↑$ and $\Psi^{A}_↓$ embody the concept that Agent A cannot be sure that the spin is up or down with certainty even though an output from an experimental device seems to be indicating a definite result for the spin state in $\Psi^{A}_↑$ and in $\Psi^{A}_↓$. After the measurement is made, the wavefunction associated with Agent A in case (ii) involves contributions from two different distributions (of Agent A’s quantum constituents): one associated with $\Psi^{A}_↑$ and one associated with $\Psi^{A}_↓$. These two embody different “statements” but common to both is the lack of certainty about the measurement when the initial state is $a_↑ |↑⟩ + a_↓ |↓⟩$. Basically, the situation is that the wavefunction for the quantum degrees of freedom in Agent A is in a linear superposition, and as such, there is uncertainty. So, for case (ii), Assumption (S) is satisfied because the premise (“Agent A is certain of the statement $x = v$ at time $t$”) is not satisfied. Alternatively, since the statement “Agent A is certain of the statement $x = v$ at time $t$” is a “classical” one but the situation is a “quantum” one, the premise of Assumption (S) can be considered as not making sense when Agent A is in a linear superposition as happens in case (ii).

Renner and Frauchiger use Assumption (S) in the following way: Four statements about wavefunction measurements are established in the following form: premise$_i$ implies conclusion$_i$ (where $i = 1$, 2, 3 or 4), and where conclusion$_i = \text{premise}_{i+1}$, for $i = 1$, 2, and 3). When these statements are combined assuming that the transitive property of logic is valid, they yield premise$_1$ implies conclusion$_4$. Premise$_1$ contains a statement of the form “Agent A measures $x$ to be $v$ at time $t$” and conclusion$_4$ contains a statement of the form “Agent A measures $x$ to be $v'$ at time $t'$” where $v' \neq v$. So, being cautious, Renner and Frauchiger require Assumption (S) so that premise$_1$ and conclusion$_4$ produce the Contradictory Statement. In unitary quantum mechanics, the above same four statements are derivable. However, the problem is not that premise$_1$ implies conclusion$_4$ is a contradiction; the problem is that the transitive property of logic does not always apply when combining statements about wavefunction measurements.

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h In the Wigner/friend gedanken experiment, Agent A is Agent W, $x$ is unbarred measured spin, $v = |−⟩_M$, $v' = |+⟩_M$ and $t = t_4$. 
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