CLASSICALITY FOR SMALL SLOPE OVERCONVERGENT AUTOMORPHIC FORMS ON SOME COMPACT PEL SHIMURA VARIETIES OF TYPE C

CHRISTIAN JOHANSSON

Abstract. We study the rigid cohomology of the ordinary locus in some compact PEL Shimura varieties of type C with values in automorphic local systems and use it to prove a small slope criterion for classicality of overconvergent Hecke eigenforms. This generalises the work of Coleman, and is a first step in an ongoing project to extend the cohomological approach to classicality to higher-dimensional Shimura varieties.

1. Introduction

A celebrated theorem of Coleman states that if \( f \) is an overconvergent modular form of weight \( k \geq 2 \) and tame level \( \Gamma_1(N) \) which is an eigenform for \( U_p \) with slope (i.e. the \( p \)-adic valuation of the eigenvalue) less than \( k-1 \), then \( f \) is in fact a (classical) modular form of weight \( k \) for the congruence subgroup \( \Gamma_1(N) \cap \Gamma_0(p) \). This theorem, usually referred to either as a “classicality theorem” or “control theorem”, generalized a previous result of Hida for ordinary \( p \)-adic modular forms. It is the key result needed for extending constructions on classical modular forms (such as construction of Galois representations) to overconvergent modular forms of finite slope by \( p \)-adic interpolation since it implies that classical forms are dense in Coleman families and on the Coleman-Mazur eigencurve. The Galois representations associated with finite slope overconvergent modular eigenforms were investigated by Kisin in [26], and it was shown that these Galois representations are trianguline at \( p \) and satisfy the Fontaine-Mazur conjecture.

In attempting to generalize Coleman’s geometric theory for \( p \)-adic interpolation of modular forms to other PEL Shimura varieties one quickly runs into two major obstacles; defining families and proving the analogue of the classicality criterion. Both problems seem hard, as Coleman’s methods do not generalize in an obvious way (using methods similar to those of Coleman, Kisin and Lai constructed one-dimensional families of Hilbert modular forms; this has recently been extended to the Siegel-Hilbert case by Mok and Tan [39]). Instead other methods of \( p \)-adic interpolation were developed (see e.g. [8], [11], [35], [14] and [50]), which have been applied with great success to the deformation theory of Galois representations.

Recently there has been much progress on also in the geometric theory, using methods that are very different to Coleman’s; see [4] for the construction of families and [42], [43] and [47] for classicality results. The method for proving classicality originates from work of Kassaei [23], building on previous work by Buzzard and Taylor [9] on the strong Artin conjecture for two-dimensional representations of \( Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \), and is in essence a geometric way of analytically continuing the overconvergent form to the whole modular curve (or more generally Shimura variety) of Iwahori level at \( p \). In particular it is entirely different from Coleman’s proof, which is cohomological in
nature, and instead requires a very explicit understanding of the geometry of the Shimura variety and the geometry of the $U_p$-correspondence.

In this paper, we revisit Coleman’s original method and generalize it to certain compact PEL Shimura varieties of type $C$, which are closely related to Hilbert modular varieties. For the exact definitions of objects and results mentioned in this introduction we refer to the main body of the text. To define our Shimura varieties, we start with a quaternion division algebra $B$ over a totally real field $F$ of degree $d$ over $\mathbb{Q}$. We fix a rational prime $p$ and assume that $B$ is split at all places above $p$ and also split at every real place of $F$. Such a $B$ then gives rise to a PEL data in a standard way, hence a reductive group $G^\times$ over $\mathbb{Q}$ and given an open compact subgroup $K \subseteq G^\times(\mathbb{A}_\mathbb{Q})$ we get an associated Shimura variety. For a special choice of $K = K_1(c, N)$, let us denote the corresponding Shimura variety by $X$.

It has potentially good reduction at $p$ and we may study the ordinary locus $X^\text{ord}_{\mathbb{Q}}$ in characteristic $p$ and its lift $X^\text{ord}_{\mathbb{Q}_p}$ inside the rigid analytification of the generic fibre $X_{\mathbb{Q}_p}$ of $X$. We may define and study spaces of classical (resp. overconvergent) automorphic forms on $X_{\mathbb{Q}_p}$ and $X^\text{ord}_{\mathbb{Q}_p}$ defined as sections (resp. sections overconvergent along the non-ordinary locus) of the appropriate sheaf (see section 3.1). These spaces carry actions of appropriately defined Hecke algebras, analogous to the situation for modular curves and Hilbert modular varieties. Decompose $p$ as $p = p_i^{e_i} \cdots p_r^{e_r}$ in $F$ and let $d_i = [F_{p_i} : \mathbb{Q}_p]$. By fixing embeddings $\mathbb{Q}_p \hookrightarrow \mathbb{C}$ and $\mathbb{Q}_p \hookrightarrow \mathbb{Q}$ we may index the weights of our automorphic forms by the embeddings $F \hookrightarrow \mathbb{Q}_p$; we label them $k_{ij}$ with $1 \leq i \leq r$, $1 \leq j \leq d_i$. We define a quantity $\lambda(k_1, ..., k_r)$ for integers $k_1, ..., k_r$ with $k_i \geq 2d_i$ by

$$\lambda(k_1, ..., k_r) = \inf_i ((k_i - 2d_i)\inf(1/2, 1/d_i))$$

Our main theorem is the following:

**Theorem.** 1) (Theorem 31(b)) Let $f$ be an overconvergent Hecke eigenform of weight $(k_1, ..., k_{d_1}, ...)$ $(k_{ij} \geq 2$ for all $i, j$) with $U_p$-slope less than $\inf(k_{ij} - 1, \lambda(k_1, ..., k_r))$ (here $p$ has valuation 1 and $k_i = \sum_j k_{ij}$). Then its system of Hecke eigenvalues comes from the $p$-stabilization of a classical form of level $K$.

2) (Theorem 33(b)) Let $F = \mathbb{Q}$. Assume that $f$ is an overconvergent Hecke eigenform of weight $k \geq 2$ with $U_p$-slope less than $k - 1$. Then its system of Hecke eigenvalues is classical of level $\Gamma_1(N) \cap \Gamma_0(pq_1 \cdots q_r)$ (where the $q_i \neq p$ are the primes where $B$ is ramified).

Let us briefly outline the contents of the paper. Section 2 is devoted to setting up the basic definitions of $B$, $G^\times$ and the Shimura varieties involved. We recall two different integral models (due to Deligne and Pappas [13] resp. Sasaki [44], the latter using ideas of Pappas and Rapoport on local models) and the algebraic representation theory of $G^\times$. In section 3 we define $p$-adic and overconvergent automorphic forms on $X$ using the automorphic vector bundles of Harris and Milne and define the Hecke operators acting on them. We give two definitions in particular of the $U_p$-operator and show that they agree. As in the theory for modular curves one of the definitions uses the canonical subgroup and therefore establishes a very direct link to the Frobenius morphism in characteristic $p$. A key construction in Coleman’s proof is that of a sheaf homomorphism

$$\theta = \theta^{k-1} : \omega^{2-k} \to \omega^k$$

As is no doubt well-known to experts, this is Faltings’s BGG complex [15] for the modular curve (and weight $k$). In sections 3.4 and 3.5 we give the analogues on $X$. In particular, this gives a
“theta map”

\[ \theta : \bigoplus_{i,j} H^0 \left( X_{rig}, W^\perp (k_{11}, \ldots, 2 - k_{ij}, \ldots, k_{rd}) \right) \rightarrow H^0 \left( X_{rig}, W^\perp (k_{11}, \ldots, k_{rd}) \right) \]

for weights with \( k_{ij} \geq 2 \) for all \( i \) and \( j \); here \( H^0 \left( X_{rig}, W^\perp (k_{11}', \ldots, k_{rd}') \right) \) denotes the spaces of overconvergent automorphic forms of weight \( (k_{11}', \ldots, k_{rd}') \).

Section 4 is the main part of the paper. We begin by recalling some notions from rigid cohomology and overconvergent de Rham cohomology, and define certain overconvergent \( F \)-isocrystals \( E_k \) that play a key role in the arguments, analogous to the sheaves \( H_k \) defined in §2 of [12]. In section 4.1 we prove the main comparison theorem, analogous to Theorem 5.4 of [12]. It identifies, in particular, the cokernel of \( \theta \) with the degree \( d \) rigid cohomology of \( E_k \) on \( X_{ord}^{PR} \), via Faltings’s BGG complex.

Section 4.2 proves the analogue of the crucial but innocent-looking Lemma 6.2 of op. cit., showing that forms of slope less than \( \inf (k_{ij} - 1) \) are not in image of \( \theta \) and hence that their system of Hecke eigenvalues occur in the cohomology of \( E_k \).

So far the arguments have made no essential use of any specific properties of our Shimura varieties; indeed the results and proofs would carry over for example to any compact PEL Shimura variety with nonvanishing Hasse invariant, or any PEL Shimura curve with nonvanishing Hasse invariant.

In section 4.3 we use the excision sequence to reduce the understanding of the degree \( d \) rigid cohomology of \( E_k \) on \( X_{ord}^{PR} \) to understanding the degree \( d \) cohomology on \( X_{ord}^{PR} \) and the degree \( d + 1 \) local cohomology on the complement. Here \( ^{PR} \) denotes that we are using the model of Sasaki (the "Pappas-Rapoport" model). The former is well understood, using comparison theorems between various cohomology theories, by the classical theory of automorphic forms (Matsushima’s formula).

We remark that this is where it is necessary to use the Pappas-Rapoport model; the rigid cohomology of the singular special fiber of the Deligne-Pappas model will most likely not agree with the de Rham cohomology of the generic fiber. To understand the local cohomology group we use information about the slopes of nonordinary abelian varieties for our moduli problem and some results of Kedlaya [25] to prove bounds for the Frobenius-slopes. The next section then translates these bounds into information about the \( U_p \)-operator, using the link between \( U_p \) and Frobenius given by the canonical subgroup, and deduces part 1) our main theorem above. Finally, for completeness, the last section gives a different treatment of the case \( F = \mathbb{Q} \) using a (somewhat simplified) version of Coleman’s dimension-counting argument, establishing part 2) of the main theorem (which is stronger than the special case \( F = \mathbb{Q} \) of part 1).

Let us make some remarks regarding our results. First of all, what we prove is that certain systems of Hecke eigenvalues are classical, rather than the stronger fact that the forms themselves are classical. This is the price we pay for working with Hecke modules and the flexibility they offer.

If one had some control on the dimension of the Hecke modules we work with (as Coleman has in [12]) or knew multiplicity one for overconvergent automorphic forms one could hope to recover the classicality of the forms themselves, but these results are not available in our setting (except when \( F = \mathbb{Q} \) where the first technique is available to us, see Remark 37). However, for applications to eigenvarieties and Galois representations this weakening is unimportant, as one passes directly to systems of Hecke eigenvalues anyway. As for optimality, the results of the paper are in general far from what is expected. On the automorphic side one would conjecture (by comparison with the theory for groups compact at infinity [35]) that an overconvergent eigenform of slope less than \( \inf (k_{ij} - 1) \) has a classical system of Hecke eigenvalues. Our theorem proves this for example when
there is only one prime above $p$ (i.e. $r = 1$), $d \geq 2$ and the weight is "not too parallel" (more precisely, under the condition that $\inf (k_{ij}) \leq (\sum k_{ij}/d) - 1$; note that $\inf (k_{ij}) \leq \sum k_{ij}/d$ always holds). However, as a vague rule, the bound gets worse as $r$ gets bigger. This may be compared with the bounds obtained in [42] in the unramified Hilbert setting, which are more uniform though not quite optimal. We should also mention, and are grateful to the referee for pointing out to us, that to state an optimal conjecture, one should look at the Galois side. Specifically, one should look at when trianguline representations are de Rham, which has been done by Nakamura [40]. One could also conjecture that any overconvergent eigenform not in the image of $\theta$ is classical (for modular curves this is Corollary 7.2.1 of [12]). We prove this in our case when $F = \mathbb{Q}$ and obtain a partial result in this direction (Theorem 31(a) ) when $F \neq \mathbb{Q}$, of which part 1) of the main theorem above is a corollary.

Next, let us discuss the possibility of extending the methods to other Shimura varieties. As mentioned above, everything up until section 4.3 generalizes e.g. to the case of compact PEL Shimura varieties with a nonvanishing Hasse invariant (or indeed an affine generalized ordinary locus), however everything after that depends, in its current form, heavily on the specifics of our moduli problem (in particular its "GL$_2$-nature"). We expect that the methods should extend (modulo some issues with the cusps, which we understand have been resolved) to prove the analogous result for overconvergent cusp forms on Hilbert modular varieties. We believe that there should also be a different, though more technical, way of completing the proof using (generalizations of) the results of Shin [45] and a comparison of trace formulas in $p$-adic (rigid) and $\ell$-adic (etale) cohomology, which should allow for a substantial generalization of our results. We are currently working out the details for some unitary Shimura varieties studied by Harris-Taylor [19] and Taylor-Yoshida [46] where the geometry is well documented. We believe that the techniques of rigid cohomology, in view of its direct relation to overconvergent automorphic forms, will be useful more generally in the theory of $p$-adic automorphic forms. As an example of this, let us mention that a central part of the spectacular recent announcement of Harris, Lan, Taylor and Thorne associating Galois representations to regular algebraic cuspidal automorphic representations of GL$_n$ is the realization of a certain system of Hecke eigenvalues coming from an Eisenstein series inside a rigid cohomology group analogous to the ones studied in this paper and by Coleman. Finally, we should mention that after this paper had been submitted for publication we were made aware of ongoing work of Tian and Xiao [48] on classicality for overconvergent Hilbert modular forms when $p$ is unramified in $F$. They make a detailed study of the Ekedahl-Oort stratification and obtain very complete results about the structure of the rigid cohomology of the ordinary locus as a Hecke module in order to deduce classicality for small slope overconvergent Hilbert modular forms using techniques similar to those of this paper.

1.1. Acknowledgments. The author would like to thank his PhD supervisor Kevin Buzzard for suggesting this problem and for his constant help and encouragement during every aspect of this project. He would also like to thank his second supervisor Toby Gee for valuable advice during the write-up, as well as Wansu Kim, Christopher Lazda, James Newton, Shu Sasaki and Teruyoshi Yoshida for many helpful discussions relating to this work, and Francesco Baldassarri and Bernard Le Stum for answering questions about rigid and overconvergent de Rham cohomology. The author wishes to thank the Engineering and Physical Sciences Research Council for supporting him throughout his doctoral studies. It is also a pleasure to thank the Fields Institute, where part of the write-up of this paper was done, for their support and hospitality as well as for the excellent working conditions provided. Finally the author wishes to thank the anonymous referee for correcting some
typos and for insightful comments. In the first version of this paper, \( p \) was assumed to be inert in \( F \). The author wishes to sincerely thank the referee for pointing out that the methods should extend to the case of \( p \) unramified, and for urging the author to investigate the general case.

2. The groups and the Shimura varieties

Throughout this article we fix a rational prime \( p \).

### 2.1. Groups and algebras.

Let \( F \) be a totally real field of degree \( d \) over \( \mathbb{Q} \), with ring of integers \( \mathcal{O}_F \) in which \( p \) splits as

\[
p = p_1^{e_1} \cdots p_r^{e_r}
\]

Write \( f_i \) for the inertia degree of \( p_i \) and put \( d_i = e_i f_i \). We let \( B \) denote a totally indefinite quaternion algebra over \( F \), which we in addition assume to be split at all \( p_i \) and a division algebra, i.e. not equal to \( M_{2|F} \). Denote by \( \mathcal{O}_B \) a maximal order of \( B \), which will be fixed throughout the paper. We will also fix an isomorphism \( \mathcal{O}_B \otimes_{\mathbb{Z}} \mathcal{O}_{F_p} \cong M_2(\mathcal{O}_{F_p}) \) (where \( F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p \)). Via the transpose this gives an isomorphism \( \mathcal{O}_{B^p} \otimes_{\mathbb{Z}} \mathcal{O}_{F_p} \cong M_2(\mathcal{O}_{F_p}) \). The group of invertible elements \( \mathcal{O}_B^\times \) is the \( \mathcal{O}_F \)-points of an algebraic group, and we denote by \( G \) the restriction of scalars of this group to \( \mathbb{Z} \), i.e. for any ring \( R \):

\[
G(R) = (\mathcal{O}_B \otimes_{\mathbb{Z}} R)^\times
\]

The reduced norm map \( \det : \mathcal{O}_B^\times \to \mathcal{O}_F^\times \) defines a homomorphism of algebraic groups \( \det : G \to R_{\mathbb{Z}} \mathcal{O}_F^\times \mathbb{G}_m \). We define an algebraic subgroup \( G^\ast \subseteq G \) by the Cartesian diagram

\[
\begin{array}{ccc}
G^\ast & \xrightarrow{\det} & G_{/\mathbb{Z}} \\
\downarrow & & \downarrow \\
\mathbb{G}_{m/\mathbb{Z}} & \xrightarrow{\det} & R_{\mathbb{Z}} \mathcal{O}_F^\times \mathbb{G}_{m/\mathcal{O}_F}
\end{array}
\]

where the lower horizontal map is the injection given on \( R \)-points by \( R^\times \to (\mathcal{O}_F \otimes_{\mathbb{Z}} R)^\times \), \( r \mapsto 1 \otimes r \). Note that the \( R \)-points of \( G^\ast \) are

\[
G^\ast(R) = \{ g \in (\mathcal{O}_B \otimes_{\mathbb{Z}} R) \mid \det(g) \in R^\times \}
\]

Let \( E \) be a finite extension of \( F \) that splits \( G \). We fix a Borel subgroup of \( G \) over \( E \) and by intersecting it with \( G^\ast \) one gets a Borel \( B^\ast \) of \( G^\ast \). We fix maximal tori \( T \) and \( T^\ast \) of \( G \) and \( G^\ast \) defined over \( E \). Since \( B \) is split at all \( p_i \), we note that

\[
G^\ast(\mathbb{Z}_p) = \{ g \in \text{GL}_2(\mathcal{O}_{F_p}) \mid \det(g) \in \mathbb{Z}_p^\times \}
\]

\[
G^\ast(\mathbb{Q}_p) = \{ g \in \text{GL}_2(F_p) \mid \det(g) \in \mathbb{Q}_p^\times \}
\]

Let us once and for all fix embeddings of \( \overline{\mathbb{Q}} \) into \( \mathbb{C} \) and \( \overline{\mathbb{Q}}_p \). This allows us to identify the archimedean places of \( F \) with the embeddings of \( F \) into \( \overline{\mathbb{Q}}_p \). We will enumerate them using pairs \((i,j)\) with \( 1 \leq i \leq r \) and \( 1 \leq j \leq d_i \) (here \( i \) is of course the same \( i \) as in \( p_i \)). The \( \mathbb{C} \)-points of \( G^\ast \) and \( T^\ast \) may then be described as follows

\[
G^\ast(\mathbb{C}) = \left\{ (g_{ij}) \in \prod_{i,j} \text{GL}_2(\mathbb{C}) \mid \det(g_{ij}) = \det(g_{i'j'}) \forall (i,j) \neq (i',j') \right\}
\]
\[ T^*(\mathbb{C}) = \{(g_{ij} \in G^*(\mathbb{C}) \mid g_{ij} \text{ diagonal } \forall (i, j)\} \]

The center of \( O_B \) is \( O_F \), hence the center of \( G \) is \( \text{Res}_{\mathbb{Q}^\times}^F \mathbb{G}_m \) and the center \( Z^* \) of \( G^* \) is \( \mathbb{G}_m \). We have (with the above description of \( G^*(\mathbb{C}) \))

\[ Z^*(\mathbb{C}) = \{(\lambda I)_{ij} \in G^*(\mathbb{C}) \mid \lambda \in \mathbb{C}^\times\} \]

The derived group of \( O_B^\times \) (as an algebraic group over \( O_F \)) consists of the elements of reduced norm 1. It follows that the derived subgroup of both \( G \) and \( G^* \) is the kernel of the reduced norm map \( \det \). As it is the same for both \( G \) and \( G^* \), we will denote it by \( G^{\text{der}} \). We have

\[ G^{\text{der}}(\mathbb{C}) = \prod_{i,j} \text{SL}_2(\mathbb{C}) \]

We fix a maximal torus \( T^{\text{der}} \) of \( G^{\text{der}} \) over \( E \) and make it so that

\[ T^{\text{der}}(\mathbb{C}) = T^*(\mathbb{C}) \cap G^{\text{der}}(\mathbb{C}) = \left\{ \left( \begin{array}{cc} a_{ij} & -1 \\ a_{ij} & 1 \end{array} \right) \in \prod_{i,j} \text{SL}_2(\mathbb{C}) \mid a_{ij} \in \mathbb{C}^\times \right\} \]

2.2. Representation theory of \( G^* \). In this section we describe the finite dimensional representation theory of \( G^* \) and its weights and central characters. As with any reductive group, its finite dimensional irreducible representations are given by a finite dimensional irreducible representation of its derived group together with a matching central character, where matching means that the representation and the central character must agree on the intersection between the derived group and the center.

Remark 1. The intersection of \( G^{\text{der}}(\mathbb{C}) \) and \( Z^*(\mathbb{C}) \) is \( \{\pm I\} = \left\{ \pm \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)_{ij} \in \prod_{i,j} \text{SL}_2(\mathbb{C}) \right\} \), so we need to check compatibility on the element \( -I \).

The representations of \( \text{SL}_2 \) are well known and gives us the following:

**Proposition 2.** The irreducible finite dimensional representations of \( G^{\text{der}}(\mathbb{C}) \) are parametrized by \( d \)-tuples of non-negative integers \((k_{11}, \ldots, k_{rd_r})\), corresponding to the representation

\[ \bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{d_{ij}}) \]

where \( S_{d_{ij}} \) is the representation given by projection \( G^{\text{der}}(\mathbb{C}) = \prod_{i,j} \text{SL}_2(\mathbb{C}) \to \text{SL}_2(\mathbb{C}) \) onto the \((i, j)\)-th factor together with the standard (left) representation of \( \text{SL}_2(\mathbb{C}) \) on \( \mathbb{C}^2 \). All these representations can be defined over any field extension of \( F \) that splits \( B \), in particular \( E \). The element \(-I \) acts on \( \bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{d_{ij}}) \) by \((-1)^{\sum k_{ij}} \).

Since \( Z^* \cong \mathbb{G}_m \), we deduce

**Corollary 3.** The irreducible finite dimensional representations of \( G^*(\mathbb{C}) \) are parametrized by \((d+1)\)-tuples of integers \((k_{11}, \ldots, k_{rd_r}, w)\), with \( k_{ij} \geq 0 \) for all \( i, j \) and \( w \equiv \sum k_{ij} \mod 2 \), and this corresponds to the representation

\[ \left( \bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{d_{ij}}) \right) \otimes \det^{(w-\sum k_{ij})/2} \]
Here, similar to before $S_{d_{ij}}$ is the representation given by projection $G^*(\mathbb{C}) \subseteq \prod_{i,j} \text{GL}_2(\mathbb{C}) \to \text{GL}_2(\mathbb{C})$ onto the $(i,j)$-th factor together with the standard (left) representation of $\text{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$, and $\det$ is the reduced norm character. $\bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{d_{ij}})$ corresponds to $(k_{11}, \ldots, k_{rd_r}, \sum k_{ij})$ and $\det$ corresponds to $(0, \ldots, 0, 2)$. As before, all representations can be defined over any extension of $F$ that splits $B$, in particular $E$.

We have a similar description of the characters of $T^*$ and $T^{\text{der}}$:

**Proposition 4.** The characters of $T^{\text{der}}(\mathbb{C})$ are parametrized by $d$-tuples of integers $(k_{11}, \ldots, k_{rd_r})$ and the characters of $T^*(\mathbb{C})$ are parametrized by $d + 1$-tuples $(k_{11}, \ldots, k_{rd_r}, w)$ of integers such that $w \equiv \sum k_{ij} \mod 2$. We will denote the corresponding characters by $\chi(k_{11}, \ldots, k_{rd_r})$ resp. $\chi(k_{11}, \ldots, k_{rd_r}, w)$.

Next we wish to describe a representation which will be important in what follows. This is the $Z$-representation, defined over $\mathbb{Z}$, given by the standard left action of $G^*(\mathbb{R}) \subseteq (O_B \otimes \mathbb{R})^\times$ on $O_B \otimes \mathbb{R}$, and we will denote it $S_d$. Over any extension that the $S_{d_{ij}}$ are defined over, it splits non-canonically as $S_d = \bigoplus_{i,j} (S_{d_{ij}} \oplus S_{d_{ij}})$. The representations we will be working with are certain summands of the symmetric powers $\text{Sym}^k(S_d)$ and certain of its subrepresentations. We have that

$$\text{Sym}^k(S_d) = \text{Sym}^k \left( \bigoplus_{i,j} (S_{d_{ij}} \oplus S_{d_{ij}}) \right) = \bigoplus_{(k_{11}, \ldots, k_{rd_r})} \bigotimes_i \text{Sym}^{k_i} \left( (S_{d_{ij}} \oplus S_{d_{ij}}) \right)$$

where the sum in the furthermost right hand side is taken over all $r$-tuples of non-negative integers $(k_1, \ldots, k_r)$ such that $\sum k_i = k$. Put $S_{d_i} = \bigoplus_{j} (S_{d_{ij}} \oplus S_{d_{ij}})$. The representations $\bigotimes_i \text{Sym}^{k_i} S_{d_i}$ are the representations that we will be interested. Note that they are defined over $\mathbb{Q}_p$. We have

$$\text{Sym}^{k_i}(S_{d_i}) = \bigoplus_{(k_{11}, \ldots, k_{id_i})} \bigotimes_j \text{Sym}^{k_{ij}} (S_{d_{ij}} \oplus S_{d_{ij}})$$

and

$$\text{Sym}^{k_{ij}}(S_{d_{ij}} \oplus S_{d_{ij}}) = \bigoplus_{0 \leq k_{ij} \leq k_{ij}} \left( \text{Sym}^{k_{ij}}(S_{d_{ij}}) \otimes \text{Sym}^{k_{ij} - k_{ij}}(S_{d_{ij}}) \right)$$

Moreover, we have

$$\text{Sym}^{k_{ij} - k_{ij}}(S_{d_{ij}}) \otimes \text{Sym}^{k_{ij}}(S_{d_{ij}}) = \bigoplus_{0 \leq a_{ij} \leq k_{ij}/2} \left( \text{Sym}^{k_{ij} - 2a_{ij}}(S_{d_{ij}}) \otimes \det^{a_{ij}} \right)$$

where the $a_{ij}$ are integers. Putting it together we have

$$\text{Sym}^{k_i}(S_{d_i}) = \bigoplus_{(k_{11}, \ldots, k_{id_i}, a_{11}, \ldots, a_{id_i})} \left( \bigotimes_j \text{Sym}^{k_{ij} - 2a_{ij}}(S_{d_{ij}}) \right) \otimes \det^\sum a_{ij}$$

with the $k_{ij}$ and $a_{ij}$ as above.
2.3. Shimura varieties defined by $G^\ast$ and their integral models. In this section we briefly recall some more or less well-known constructions, though as far as the author is aware, they are not explicitly stated in the literature when $p$ ramifies in $F$. When $p$ is unramified see e.g. [28], [38] and [29]. $B$ carries an involution $b \mapsto b^\ast$ of the first kind. Consider the opposite $\mathbb{Q}$-algebra $B^{op}$ with involution $b \mapsto b^\ast$, with the natural left action on $B$. Pick $\xi \in B$ such that $\xi^\ast = -\xi$, and define a $B^{op}$-involution on $B$ by $(x, y) = Tr_{F/q} Tr_{B/F}(x^\ast \xi y)$, where $Tr_{B/F}$ is the reduced trace and $Tr_{F/q}$ is the field trace. Together with the homomorphism $h : \mathbb{C} \rightarrow End_{B^{op} \otimes \mathbb{Q}}(B \otimes \mathbb{Q} \mathbb{R}) = M_2(F \otimes \mathbb{Q} \mathbb{R})$ given by

$$a + bi \mapsto \begin{pmatrix} 1 \otimes a & -1 \otimes b \\ 1 \otimes b & 1 \otimes a \end{pmatrix}$$

This defines a rational PEL-data of type C, and hence a Shimura datum whose group is $G^\ast$ acting on the disconnected Hermitian symmetric domain $(\mathcal{H}^+)^d \sqcup (\mathcal{H}^-)^d$, where $\mathcal{H}^+$ is the upper and $\mathcal{H}^-$ is the lower half plane. The associated Shimura varieties are moduli spaces for abelian varieties with extra structures and are defined over the reflex field $\mathbb{Q}$. For a given neat compact open subgroup $K \subseteq G^\ast(\mathbb{A}^\infty)$ we denote the corresponding Shimura variety by $Sh_K$. For primes not dividing the level or the discriminant of $F$ or $B$, the canonical models of $Sh_K$ have good reduction. In the more general case when $p$ is allowed to divide the discriminant of $F$ but not the discriminant of $B$, integral models may be constructed and studied by copying the methods of Deligne and Pappas [13] in the Hilbert case. We will now very briefly recall the construction of these integral models and a few of their properties.

Fix an open compact subgroup $K = K^p K_p \subseteq G^\ast(\mathbb{A}^\infty)$ such that $K_p = G^\ast(\mathbb{Z}_p)$, $K^p$ will be specified below and fix a fractional ideal $\mathfrak{c}$ of $F$ (without loss of generality coprime to $p$). We denote the totally positive elements of $\mathfrak{c}$ by $\mathfrak{c}^+$. Let $N \geq 5$ be an integer, coprime to $p$. Define a functor $\mathcal{X}^{DP}$ sending a locally Noetherian $\mathbb{Z}_p$-scheme $S$ to the set of isomorphism classes of quadruples $(A, \iota, \phi, \eta)$ where

1. $A/S$ is an abelian scheme of dimension $2d$
2. $\iota : O_{B^{op}} \rightarrow End_S(A)$ is a ring homomorphism
3. $\phi$ is an $O_F$-linear homomorphism of $\mathfrak{c}$ into the sheaf of symmetric homomorphisms $\lambda : A \rightarrow A^\vee$ satisfying $\iota(b)^\vee \circ \lambda = \lambda \circ \iota(b^\ast)$ (as quasi-isogenies) for all $b \in O_{B^{op}, (p)}$. We require that $\phi$ maps $\mathfrak{c}^+$ to polarizations, and that the map $A \otimes \mathfrak{c} \rightarrow A^\vee$ induced by $\phi$ is an isomorphism (the "Deligne-Pappas condition").
4. $\eta$ is an $O_{B^{op}}$-linear closed immersion $O_{B^{op}}/NO_{B^{op}} \rightarrow A[N]$ of group schemes.

By standard methods, this functor is represented by a projective scheme over $\mathbb{Z}_p$ which we also denote $\mathcal{X}^{DP}$. Properness is the only thing that differs from the Hilbert case. It follows (via the valuative criterion of properness) from the potentially good reduction of pairs $(A, \iota)$ over the fraction field of a discrete valuation ring (see the Proposition in §6 of [7], the proof there does not require the "Rapport condition" that is also assumed in their definition of an abelian scheme with an $O_{B^{op}}$-action). The generic fibre of $\mathcal{X}^{DP}$ is the canonical model of $Sh_K$ base changed to $\mathbb{Q}_p$; we will denote it by $X$. From now on, we will simply write $A$ for an isomorphism class of quadruples as above.

**Remark 5.** Assume that $A$ is a quadruple as above. It defines a principally polarized $p$-divisible group $A[p^\infty]$ of height $4d$ and dimension $2d$ with an action of $O_{B^{op}, p} = O_{B^{op}} \otimes \mathbb{Z}_p \cong M_2(O_{F, p})$. By Morita equivalence, this is equivalent to a principally polarized $p$-divisible group $G_A$ of height $2d$ and dimension $d$ with an action of $O_{F, p}$. The deformations of $G_A$ controls the local geometry of
the special fibre of $X^{DP}$ by Serre-Tate theory. This is identical to the situation in the Hilbert case, and we may hence use the local models of [13] to study the geometry of $X^{DP}$. In particular, the fibres of $X^{DP}$ are normal.

Let us now specify the tame level $K^p$ used above. It is analogous to the choice of "$\Gamma_1(\mathfrak{c},N)$"-level structure often made in the literature on overconvergent Hilbert modular forms. Let $c \in \mathbb{A}_{F}^\infty$ be a fixed representative of a double coset in $F^+_F \backslash \mathbb{A}_F^\infty / \hat{O}_F^\times$, where $F^+_F$ denotes the totally positive elements of $F^\times$ and $\hat{O}_F^\times = (O_F \otimes \mathbb{Z})^\times$. This $c$ corresponds to $\mathfrak{c}$ and is relatively prime to $p$. Define

$$K^G_1(N) = \left\{ g \in \text{GL}_2(\mathbb{A}_F^\infty) \mid g \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \text{ mod } N \right\}$$

Finally, we put

$$K_1(c,N) = G^*(A^\infty) \cap \left( \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} K^G_1(N) \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}^{-1} \right)$$

where the intersection takes place in $\text{GL}_2(\mathbb{A}_F^\infty)$. As $c$ and $N$ are prime to $p$, $K_1(c,N)_p = G^*(\mathbb{Z}_p)$. For $N$ as above $K_1(c,N)$ is neat, and we put $K = K_1(c,N)$. As $\det(K) = \mathbb{Z}^\times$, $X$ is geometrically connected. By the usual trick using Zariski’s connectedness principle, the special fiber of $X^{DP}$ is geometrically connected and hence geometrically irreducible by normality.

$X^{DP}$ is not smooth. We will need the fact that we can resolve the singularities of $X^{DP}$ after a ramified extension of valuation rings. This is identical to the situation in the Hilbert case as studied in [44] (following work of Pappas and Rapoport on local models) so we will be rather brief. The author wishes to thank Shu Sasaki for explaining his work to him. We will follow [44] closely in what follows. Let $L \subseteq \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ that contains the image of every embedding $F_i \rightarrow \overline{\mathbb{Q}}_p$, for every $i$, and let $O_L$ denote its ring of integers. Let $\pi_L$ denote a fixed uniformizer of $L$, and let $L^{ur}$ denote the maximal unramified subfield of $L$ (and similarly for other $p$-adic fields).

Before we give the new moduli problem we need some more notation. For each $i$, fix a uniformizer $\pi_i$ of $F_i$, satisfying an Eisenstein polynomial $E_i(u) \in O_{F_i^{ur}}[u]$. Moreover, we put $S_i = \text{Hom}_{\mathbb{Z}_p}(O_{F_i^{ur}}, O_{L^{ur}})$. For every $\sigma \in S_i$ we put $E_i(\sigma) = \sigma(E_i(u)) \in O_{L^{ur}}[u]$ and let $\{\sigma_1, ..., \sigma_e\}$ denote its set of of roots in $L$. Continuing, we denote by $\{\sigma(j)\}_{\sigma \in S_i, 1 \leq j \leq e_i}$ the $d_i$ embeddings of $F_i$ into $L$, where $\sigma(j)$ is defined $\sigma(j)|_{O_{F_i^{ur}}} = \sigma$ and that it maps $\pi_i$ to $\pi_{\sigma(j)}$. We have

$$O_{B^{ur}} \otimes \mathbb{Z} O_L \cong M_2(O_{F_i^{ur}}) \otimes_{\mathbb{Z}_p} O_L \cong M_2(O_F \otimes \mathbb{Z} O_L)$$

and

$$O_F \otimes \mathbb{Z} O_L \cong \bigoplus_i O_{F_i} \otimes_{\mathbb{Z}_p} O_L \cong \bigoplus_i \left( O_{F_i} \otimes_{O_{F_i^{ur}}} \left( O_{F_i^{ur}} \otimes_{\mathbb{Z}_p} O_L \right) \right) \cong \bigoplus_i \left( O_{F_i} \otimes_{O_{F_i^{ur}}} (\bigoplus_{\sigma \in S_i} O_L) \right)$$

Put $O_{i,\sigma} = O_{F_i} \otimes_{O_{F_i^{ur}}} O_L$, then we have that $O_{B^{ur}} \otimes \mathbb{Z} O_L \cong \bigoplus_{\sigma \in S_i} M_2(O_{i,\sigma})$. Let $A$ be an element of $X^{DP}(S)$. Then we get decompositions

$$H_1^{dR}(A/S) = \bigoplus_{\sigma \in S_i} H_1^{dR}(A/S)_{i,\sigma}$$

$$\text{Lie}(A'/S)^\vee = \bigoplus_{\sigma \in S_i} \text{Lie}(A'/S)^\vee_{i,\sigma}$$
where $H^d_{DR}(A/S)_{i,σ}$ is an $M_2(𝒪_{S} ⊗ 𝒪_{i,σ})$-module which is locally free of rank 4 as an $𝒪_{S} ⊗ 𝒪_{i,σ}$-module, and $\text{Lie}(A^V/S)_{i,σ}$ is an $M_2(𝒪_{S} ⊗ 𝒪_{i,σ})$-module that is, Zariski locally on $S$, a locally free direct summand of $H^d_{DR}(A/S)_{i,σ}$ of rank $2e_i$ as an $𝒪_{S}$-module.

We define a functor $X^{PR}$ from the category of locally Noetherian schemes over $𝒪_L$ to sets by letting, for $S$ a scheme over $𝒪_L$, $X^{PR}(S)$ be the set of isomorphism classes of data

$$(A, (F_{i,σ}(j))_{i,σ,j})$$

where

1. $A ∈ X^{DP}(S)$
2. For every $i$ and $σ ∈ S_i$, we have a filtration

$$0 = F_{i,σ}(0) ⊆ F_{i,σ}(1) ⊆ ... ⊆ F_{i,σ}(e_i) = \text{Lie}(A^V/S)_{i,σ}$$

of $M_2(𝒪_{S} ⊗ 𝒪_{i,σ})$-modules such that

(a) each $F_{i,σ}(j)$ is, Zariski locally on $S$, a direct summand of $\text{Lie}(A^V/S)_{i,σ}^\vee$ of rank $2j$ as an $𝒪_{S}$-module and

(b) on the quotient $F_{i,σ}(j)/F_{i,σ}(j − 1)$ ($j ≥ 1$), which is a locally free $𝒪_{S}$-module of rank $2$, $𝒪_{B_{σj}}$ acts via

$$𝒪_{B_{σj}} → M_2(𝒪_{F_{iσ}}) \cong M_2(𝒪_L) → M_2(𝒪_S)$$

Using Morita equivalence the proofs of [44] carry over verbatim and shows that the forgetful natural transformation $X^{PR} → X^{DP}$ (subscript denoting base change) is relatively representable by a projective morphism and hence that $X^{PR}$ is representable. We will denote the representing object by $X^{PR}$ as well. As in [44], $X^{PR}$ is smooth over $𝒪_L$ (this is proved using Grothendieck-Messing theory). Moreover, the morphism $X^{PR} → X^{DP}_{𝒪_L}$ is an isomorphism over the Rapoport locus (which coincides with the smooth locus of $X^{DP}_{𝒪_L}$), which includes the ordinary locus in the special fibre and the whole generic fibre. In particular, the generic fibre of $X^{PR}$ is $X_L$ and the fibres of $X^{PR}$ are geometrically connected.

Next we will add level structure at $p$. Define two subgroups $K_0(p), K_0^0(p)$ of $G^*(ℤ_p)$ by

$$K_0(p) = \left\{ g ∈ G^*(ℤ_p) \mid g ≡ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p \right\}$$

$$K_0^0(p) = \left\{ g ∈ G^*(ℤ_p) \mid g ≡ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \mod p \right\}$$

We let $Y$ resp. $Z$ be the base change of the canonical model of $Sh_{K^pK_0(p)}$ resp. $Sh_{K^pK_0^0(p)}$ to $ℚ_p$. $Y$ resp. $Z$ parametrize pairs $(A, H)$ resp. triples $(A, H_1, H_2)$, where $A$ is a point of $X$ and $H, H_1$ and $H_2$ are finite flat (in fact etale) $𝒪_{B_{σj}}$-stable subgroups of $A$ of rank $p^d$ which are killed by $p$ and isotropic with respect to the polarization. Moreover we require that $H_1 ∩ H_2 = 0$. The relative representability of these moduli problems over $X$ may be shown by standard methods (they are closed subschemes of Grassmannians). We have finite etale morphisms $Z → Y → X$ forgetting $H_1$ resp. $H_2$ resp. $H$. Since $\det(K^pK_0(p)) = \det(K^pK_0^0(p)) = ℤ^×$, $Y$ and $Z$ are geometrically connected.

Let $k_L$ denote the residue field of $L$. We will denote the special fibres of $X^{DP}$ over $𝔽_p, k_L$ resp. $𝔽_p$ by $X^{DP}_{𝔽_p}, X^{DP}_{k_L}$ resp. $X^{DP}_{𝔽_p}$, and similarly for $X^{PR}$.
Remark 6. We will use the notation \((A, \ldots)\) as above to denote points of the special and/or generic fibres of our moduli spaces; however we will also use the notation \(A^{DP}, A^{PR}, A, A^{DP}_{p}\) etc. (analogous to \(X^{DP}, X^{PR}, X, X^{DP}_{p}\) etc.) to denote the (abelian scheme associated with) the universal object over the appropriate moduli space. We hope there will be no confusion arising from this. Occasionally we will use the superscript \(\text{univ}\) to distinguish the universal object. We will denote the map from the universal object to the moduli space by \(\pi\) and the zero section of the universal object by \(e\); if there is need to identify which moduli space we are dealing we will use appropriate subscripts; we hope that no confusion will arise from this either.

3. Automorphic forms and Hecke operators

3.1. Automorphic vector bundles and automorphic forms. One way to define holomorphic automorphic forms is to use the automorphic vector bundle construction, as described e.g. in [37]. The theorem is the following, and only applies in characteristic 0. By abuse of notation, we also let \(\chi(k_{11}, \ldots, k_{rd}, w)\) denote the representation of \(B^r\) obtained from \(\chi(k_{11}, \ldots, k_{rd}, w)\) by letting the unipotent part of \(B^r\) act trivially.

**Theorem 7.** To any finite dimensional representation of \(B^r\) we may functorially associate a vector bundle on \(X\) such that equivariant maps between representations go to Hecke-equivariant \(\mathcal{O}_X\)-linear maps. To any finite dimensional representation of \(G^r\) we may functorially associate a vector bundle with an integrable connection. These bundles and maps are defined over the same fields as the representations and maps are (they are defined on the base change of canonical model to said field; we base change them to appropriate extensions of \(\mathbb{Q}_p\), and the construction respects direct sums and tensor operations, and the rank of the bundle is the dimension of the representation. We will denote by \(W(k_{11}, \ldots, k_{rd}, w)\) the line bundle associated to \(\chi(k_{11}, \ldots, k_{rd}, w)\) and by \(V(k_{11}, \ldots, k_{rd}, w)\) the vector bundle with connection associated to \((\bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{i,j})) \otimes \det(w - \sum k_{ij})/2\). The restriction \(\text{det}\) goes to the Tate twist \(\mathbb{Q}_p(1)\).

**Definition 8.** An automorphic form of weight \((k_{11}, \ldots, k_{rd}, w)\) and level \(K_1(c, N)\) is a global section of \(W(k_{11}, \ldots, k_{rd}, w)\) on \(X\) (and similarly, changing the level, for \(Y\) and \(Z\)).

The PEL datum is set up such that the standard representation \(Sd\) corresponds to \(H^1_{dR}(A/X)\), hence \(H^1_{dR}(A/X)\) corresponds to \(Sd^\vee\). \(Sd^\vee\), as a \(T^r\)-representation, is

\[
Sd^\vee = (\chi(1, 0, \ldots, 0, -1)^{\oplus 2} \oplus \cdots \oplus \chi(0, \ldots, 0, 1, -1)^{\oplus 2}) \oplus \\
\oplus (\chi(-1, 0, \ldots, 0, -1)^{\oplus 2} \oplus \cdots \oplus \chi(0, \ldots, 0, -1, -1)^{\oplus 2})
\]

Another bundle that will occur later is \(\Omega^1_X\). To start with, \(\Omega^1_X\) corresponds to the dual of the adjoint representation of \(B^r\) on \(\text{Lie}(G^r)/\text{Lie}(B^r) = \chi(2, 0, \ldots, 0, 0) \oplus \cdots \oplus \chi(0, \ldots, 0, 2, 0)\) (note the trivial central character). Therefore \(\Omega^1_X = \wedge^4 \Omega^1_X\) corresponds to \(\chi(2, \ldots, 2, 0)\).

**Remark 9.** 1) Let us briefly explain the relation between this and the perhaps more standard way of defining automorphic forms on \(X\), as in e.g. [22], from which part of this discussion is taken. This will also provide an integral structure to our sheaves of automorphic forms (at least after base change to \(L\)). Recall our identification of \(\mathcal{O}_{B^{op}} \otimes_{\mathbb{Z}} \mathbb{Z}_p\) with \(M_2(\mathcal{O}_F)\), and consider the two standard
orthogonal idempotents $e_1$ and $e_2$ in $M_2(O_{F_p})$. The sheaf $\pi_*\Omega^1_{A^{PR}/X^{PR}} = e^*\Omega^1_{A^{PR}/X^{PR}}$ injects into $H^1_dR(A^{PR}/X^{PR})$ and corresponds to

$$\chi(1, 0, ..., 0, -1)^{\oplus 2} \oplus ... \oplus \chi(0, ..., 0, 1, -1)^{\oplus 2}$$

on the generic fibre. $\pi_*\Omega^1_{A^{PR}/X^{PR}}$ inherits an action of $O_{B^{PR}}$ and carries a scalar action of $\mathbb{Z}_p$, hence has an action $O_{B^{PR}} \otimes \mathbb{Z}_p = M_2(O_{F_p})$. Taking the image of $e_2$ say (to be consistent with [22]), we obtain a sheaf $\omega = \omega_{A^{PR}/X^{PR}}$ which corresponds to

$$\chi(1, 0, ..., 0, -1) \oplus ... \oplus \chi(0, ..., 0, 1, -1)$$

on the generic fibre and still carries an action of $O_{F_p}$. Decomposing $\omega$ with respect to action of $O_{F_p}$, as in the Hilbert case, we obtain line bundles $\omega_{ij}$ corresponding to $\chi(0, ..., 1, 0, ..., 0, -1)$ on the generic fibre (the 1 in the $(i,j)$-th place), and automorphic forms of weight $(k_{11}, ..., k_{rd_1})$ are defined as global sections of $\bigotimes \omega_{ij}^{k_{ij}}$. Note that these correspond to our automorphic forms of weight $(k_{11}, ..., k_{rd_1}, - \sum k_{ij})$, or rather gives an integral structure to this space. We will see when we consider Hecke operators that, the way we are used to thinking about them, automorphic forms of weight $(k_{11}, ..., k_{rd_1})$ with their usual Hecke action corresponds to global sections of $\left( \bigotimes \omega_{ij}^{k_{ij}-2} \right) \otimes \Omega^1_{X^{PR}}$ (cf. [10] p. 258 for a similar remark in the Siegel case).

2) The central character is only important when we are considering Hecke operators; the bundles $W(k_{11}, ..., k_{rd_1}, w)$ are isomorphic for fixed $(k_{11}, ..., k_{rd_1})$ but varying $w$. Changing $w$ has the effect of scaling Hecke operators, which we will see and use explicitly later. Consequently, we will occasionally just refer to $(k_{11}, ..., k_{rd_1})$ as the weight and sometimes talk about “an automorphic form of weight $(k_{11}, ..., k_{rd_1})$", not specifying $w$, which we will refer to as “the central character". Sometimes we will include $w$ in the weight. We hope that this will not be confusing.

3) As the $W(k_{11}, ..., k_{rd_1}, w)$ are isomorphic for fixed $(k_{11}, ..., k_{rd_1})$ and varying $w$ by a canonical isomorphism (see Proposition 22) we may use this isomorphism to define an integral structure on $W(k_{11}, ..., k_{rd_1}, w)$ by transport of structure from $W(k_{11}, ..., k_{rd_1}, - \sum k_{ij})$.

### 3.2. Ordinary locus, canonical subgroups and overconvergent automorphic forms.

The Hasse invariant is defined as a section of $\left( \wedge^d e^*\Omega^1_{A^{PR}/X^{PR}} \right)^{(p-1)}$ (and can be defined more generally in this fashion for abelian schemes over arbitrary characteristic $p$ bases) on $X^{DP}_{F_p}$. The ordinary locus $X^{ord,DP}_{F_p}$ is the locus where the Hasse invariant does not vanish; its vanishing locus will be denoted $X^{ss,DP}_{F_p}$ (though it is not the supersingular locus except in some low dimensional cases, we hope this will not cause any confusion). $X^{ord,DP}_{F_p}$ is dense in $X^{DP}_{F_p}$ (as $X^{DP}_{F_p}$ is irreducible and $X^{ord,DP}_{F_p}$ is open). Moreover $X^{ord,DP}_{F_p}$ is smooth (see Remark 5). The Hodge bundle is ample (see e.g. [30] proof of Prop. 7.8) and hence $X^{ord,DP}_{F_p}$ is affine (it is the complement of the vanishing locus of a nonzero section of an ample line bundle on a projective variety). We may make the same definitions for $X^{PR}_{kL}$, giving us $X^{ord,PR}_{kL}$, an open dense affine subset of $X^{PR}_{kL}$, with complement $X^{ss,PR}_{kL}$. Since the map $X^{PR}_{kL} \rightarrow X^{DP}_{kL}$ is an isomorphism on the Rapoport locus, $X^{ord,PR}_{kL}$ is isomorphic to $X^{ord,DP}_{kL}$ and from now on we will drop the superscripts $PR$ or $DP$ from the ordinary locus.
Ultimately we will be interested in rigid-analytic phenomena. When we have a scheme $S/\mathbb{Q}_p$ (or over any extension of complete valued fields) we will let $S_{an}$ denote its Tate analytification, and whenever we have an scheme $S/\mathbb{Z}_p$ (or over any extension of complete valuation rings) we will let $S_{rig}$ denote the Raynaud generic fibre of the formal completion of $S$ along its special fibre. $S_{rig}$ carries a specialization map $s_p : S_{rig} \to S_{\mathbb{F}_p}$. When $S$ is the generic fiber of $S$ there is always an open immersion $S_{rig} \to S_{an}$ which is an isomorphism when $S$ is proper. These notions apply to $X$, $Y$ and $Z$ and their integral models when they exist. Inside $X_{an} = X_{rig}$, with respect to $s_p : X_{rig} \to X_{\mathbb{F}_p}$, we define $X_{rig}^{ord} = s_p^{-1}(X_{\mathbb{F}_p}^{ord})$ and $X_{rig}^{ss} = s_p^{-1}(X_{\mathbb{F}_p}^{ss,DP})$, the ordinary locus resp. non-ordinary locus in $X_{rig}$. Note that we could also have defined them using the Pappas-Rapoport model, but the result would be the same (after base change to $L$).

Let us briefly recall some well known facts about canonical subgroups. By Proposition 3.4 of [2] applied to the formal completion along the special fibers of $A^{DP} \to \mathcal{X}^{DP}$ we obtain a partial section $X_{rig}^{ord} \to Y_{rig}$, $A \mapsto (A, C_A)$ of the natural map $Y_{rig} \to X_{rig}$. $C_A$ is called the canonical subgroup of $A$. The image of this morphism will be denote by $Y_{rig}^{ord}$. By Theorem 3.5 of [2], the canonical subgroup overconverges to give a partial section $V \to Y_{rig}$ of $Y_{rig} \to X_{rig}$, where $V$ is some strict neighbourhood of $X_{rig}^{ord}$ in $X_{rig}$.

Remark 10. $Y_{rig}^{ord}$ is not the full ordinary locus in $Y$; it is the so-called ordinary-multiplicative locus. There are several ordinary loci in $Y_{rig}$. Somewhat ad hoc, we will define

$$Y_{ord} = \{(A, H) \in Y_{rig} \mid A \in X_{rig}^{ord}, H \cap C_A = 0\}$$

$Y_{ord}$ will only be used in an auxiliary role in the construction of the $U_p$-operator.

Next we will define $p$-adic and overconvergent automorphic forms. We will abuse notation and use $W(k_{11}, ..., k_{rd}, w)$ etc. to denote the analytification of those sheaves on $X_{rig}$ etc.

**Definition 11.** An element of $H^0(X_{rig}^{ord}, W(k_{11}, ..., k_{rd}, w))$ is called a $p$-adic automorphic form of weight $(k_{11}, ..., k_{rd}, w)$. An overconvergent automorphic form of weight $(k_{11}, ..., k_{rd}, w)$ is an element of

$$H^0((X_{rig}^{ord}, W(k_{11}, ..., k_{rd}, w))) = \lim_{\to} H^0(V, W(k_{11}, ..., k_{rd}, w))$$

where the direct limit is taken over any cofinal set of strict neighbourhoods of $X_{rig}^{ord}$ in $X_{rig}$. Note that by restriction we have an inclusion

$$H^0((X_{rig}^{ord}, W(k_{11}, ..., k_{rd}, w))) \subseteq H^0(X_{rig}^{ord}, W(k_{11}, ..., k_{rd}, w))$$

3.3. **Hecke operators and $U_p$.** We define Hecke operators for our Shimura varieties as in [28] section 6. For us a special role is played by the Hecke operator $U_p$, defined adelically on $Y$ by the double coset

$$K^p K_0(p) \left( \begin{array}{cc} p & 1 \\ & 1 \end{array} \right) K^p K_0(p)$$

or moduli theoretically by the correspondence

$$(p_1, p_2) : Z \to Y \times Y$$

where $p_1$ and $p_2$ are the two maps given by

$$p_1(A, H_1, H_2) = (A/H_2, A[p]/H_2)$$

$$p_2(A, H_1, H_2) = (A, H_1)$$
One also has the diamond operators \( (d) : X \to X \) for \( d \in \mathbb{Z} \) with \( d \) suitably coprime to \( K \) (we will only need the case \( d = p \)) defined by \( (d)(A) = A/A[d] \). Note that \( A \) and \( A/A[d] \) are isomorphic as abelian varieties.

From now on, in this section only, we will only work in the rigid analytic setting and therefore drop the “rig” from the notation in order to ease it. We wish to define operators on \( p \)-adic and overconvergent automorphic forms and so want to know that the \( U_p \)-correspondence restricts to \( Y^{\text{ord}} \). Let \( Z^{\text{ord}} = p^{-1}_2(Y^{\text{ord}}) \).

**Lemma 12.** \( p_1(Z^{\text{ord}}) \subseteq Y^{\text{ord}} \)

*Proof.* Let \( (A, H_1, H_2) \in Z^{\text{ord}} \). By definition \( (A, H_1) \in Y^{\text{ord}} \), so \( H_1 = C_A \). Therefore \( A[p]/H_2 = C_{A/H_2} \), hence \( p_1(A, H_1, H_2) = (A/H_2, A[p]/H_2) \in Y^{\text{ord}} \). \( \square \)

We may therefore restrict to get a correspondence

\[
(p_1, p_2) : Z^{\text{ord}} \to Y^{\text{ord}} \times Y^{\text{ord}}
\]

Using the isomorphism \( X^{\text{ord}} \cong Y^{\text{ord}} \) we may view this as a correspondence on \( X^{\text{ord}} \), and we may simplify \( Z^{\text{ord}} \) by noting that the forgetful map \( Z \to Y \) given by \( (A, H_1, H_2) \mapsto (A, H_2) \) identifies \( Z^{\text{ord}} \) with \( Y^{\text{ord}} = \{(A, H) \mid A \in X^{\text{ord}}, H \cap C_A = 0\} \) in \( Y \), so we get a \( U_p \)-correspondence

\[
(p_1, p_2) : Y^{\text{ord}} \to X^{\text{ord}} \times X^{\text{ord}}
\]

with

\[
p_1(A, H) = A/H \quad p_2(A, H) = A
\]

Next we wish to define another \( U_p \)-correspondence, call it \( \tilde{U}_p \), which will turn out to be isomorphic to \( U_p \). We have a map \( Fr : X^{\text{ord}} \to X^{\text{ord}} \) given by \( Fr(A) = A/C_A \). We denote it \( Fr \) because it is a lift of the relative Frobenius in the sense that

\[
\begin{array}{ccc}
X^{\text{ord}} & \xrightarrow{Fr} & X^{\text{ord}} \\
\downarrow \text{sp} & & \downarrow \text{sp} \\
X^{\text{ord}}_{\mathbb{F}_p} & \xrightarrow{Fr} & X^{\text{ord}}_{\mathbb{F}_p}
\end{array}
\]

commutes. This will be important when we consider rigid cohomology later. We define \( \tilde{U}_p \) as the correspondence

\[
(q_1, q_2) : X^{\text{ord}} \to X^{\text{ord}} \times X^{\text{ord}}
\]

where

\[
q_1 = id \quad q_2 = (p)^{-1} Fr
\]

**Lemma 13.** Define two morphisms \( \alpha : X^{\text{ord}} \to Y^{\text{ord}} \) and \( \beta : Y^{\text{ord}} \to X^{\text{ord}} \) by

\[
\alpha(A) = (A/C_A, A[p]/C_A) \\
\beta(A, H) = A/H
\]
Furthermore, define an automorphism $(p)_Y : Y_{ord} \to Y_{ord}$ by

$$(p)_Y(A, H) = \left( (p)(A), \left\{ a \in A \mid pa \in H \right\} / A[p] \right)$$

Then $\beta \alpha = (p)$ and $\alpha \beta = (p)_Y$, so $\beta$ defines an isomorphism $Y_{ord} \cong X_{ord}$.

**Proof.** We have (equalities as points in the moduli spaces)

$$\beta \alpha(A) = \beta(A/C_A, A[p]/C_A) = \frac{A/C_A}{A[p]/C_A} = \frac{A}{A[p]} = (p)A$$

and

$$\alpha \beta(A, H) = \alpha(A/H) = \left( \frac{A/H}{A[p]/H}, \left\{ a \in A \mid pa \in H \right\} / H \right) = (p)_Y(A, H)$$

where the last equality comes from noting that $A[p]/H$ is the canonical subgroup in $A/H$ and that the map $A/A[p] \to A$ induced by the $p$-power map on $A$ sends $\frac{\left\{ a \in A \mid pa \in H \right\}}{A[p]}$ to $H$. \qed

Finally we may prove

**Proposition 14.** $U_p \cong \hat{U}_p$

**Proof.** By the lemma we know that $X_{ord} \cong Y_{ord}$ via $\beta$, so it suffices to prove that $q_1 \beta = p_1$ and $q_2 \beta = p_2$. Now

$$q_1 \beta(A, H) = q_1(A/H) = A/H = p_1(A, H)$$

and

$$q_2 \beta(A, H) = q_2(A/H) = (p)^{-1} \left( \frac{A/H}{A[p]/H} \right) = (p)^{-1} \left( \frac{A}{A[p]} \right) = A = p_2(A, H)$$

\qed

We may therefore denote both correspondences by $U_p$. The description in terms of $Fr$ will prove useful in order to study the slopes of $U_p$.

It remains to extend $U_p$ to (small) strict neighbourhoods of $X_{ord}$. This can be done both from the more classical point of view, see [41] Prop. 4.8.5, or by the overconvergence of the canonical subgroup. In fact it is well known that the $U_p$-correspondence contracts strict neighbourhoods of the ordinary locus. This may be deduced for example by following [41] §1.2, defining the degree function on $Y_{rig}$ by pullback from $X(2d)$ (in the notation of [41] §1.2; although the setup there is for the Siegel modular variety for principally polarized abelian varieties, the arguments go through without change for Siegel modular varieties with polarization type of degree prime to $p$). Then, an argument as in the proof of Proposition 2.3.6 of [41] proves the desired contraction property. The correspondences hence induce compact operators on spaces of overconvergent automorphic forms.

**Remark 15.** 1) The Hecke correspondences away from $p$ preserve the ordinary locus. Hence, again using Prop. 4.8.5 of [41], these correspondences overconverge and define operators on overconvergent automorphic forms.

2) To properly let a correspondence $s = (s_1, s_2)$ act on automorphic forms of weight $(k_1, ..., k_d, w)$ one needs also to specify an isomorphism $s_1^* W(k_1, ..., k_d, w) \cong s_2^* W(k_1, ..., k_d, w)$. This is done in general by the theory of automorphic vector bundles. To study $p$-divisibility of $U_p$, it is preferable though to have some moduli-theoretic interpretation. It suffices to give such an isomorphism
for $\pi_{\text{univ}, \omega}^{1}(A, w)$ as corresponding to all sheaves of automorphic forms constructed from this data, and so we may describe automorphic forms as “functions” à la Katz defined on “points” $(A, \omega)$ with $\omega \in H^{0}(A, \Omega^{1}_{A})$. Thus, in order to describe the action of $U_{p}$ on automorphic forms we need to, given $(A, \omega)$ and $B = A/H \in U_{p}(A),$ functorially associate some $\omega' \in H^{0}(A/H, \Omega^{1}_{A/H})$. This is done by inverting the pullback of differentials along the isogeny $A \to A/H$.

For our second description of $U_{p}$ we may first of all ignore $(p)^{-1}$, as it only changes the level structure away from $p$. The natural map involved is then (a priori) the isogeny $B \to B/C_{B} = A$ and it would seem natural to use pullback of differentials along this isogeny. These definitions do not agree however, as the composition $B \to B/C_{B} = A \to A/H = B$ is multiplication by $p$ which induces multiplication by $p$ on differentials, so the two definitions disagree by a factor of $p$. As is standard, we choose the first definition, and modify the second by the appropriate factor of $p$. This corresponds geometrically to, rather than using $B \to A$, using its “dual” $A \to B$ (defined such that the composition both ways are multiplication by $p$, and related to the dual isogeny via our polarizations). More explicitly, one has $\tilde{U}_{p} = p^{-\sum k_{ij}}U_{p}$ on $H^{0,1}(X_{rig}^{ord}, W(k_{11}, \ldots, k_{rd, r}, -\sum k_{ij}))$ at first, and then scale so that $\tilde{U}_{p} = U_{p}$. Note that whereas the theory of automorphic vector bundles gives definitions of Hecke operators for all weights $(k_{11}, \ldots, k_{rd, r}, w)$, we make this moduli-theoretic definition a priori only for weights of the form $(k_{11}, \ldots, k_{rd, r}, -\sum k_{ij})$. For general central characters we scale appropriately to match the theory of automorphic vector bundles, cf. Proposition 22.

For the rest of the article we will let $\mathcal{H}_{K}$ denote the full Hecke algebra of $G^{*}(A^{\infty})$ with respect to the level $K$, and let $\mathcal{H}^{p}_{K}$ denote the full Hecke algebra of $G^{*}(A^{p,\infty})$ with respect to $K^{p}$. Later on when we consider eigenforms we will fix a commutative subalgebra $\mathcal{H}^{p} \subseteq \mathcal{H}^{p}_{K}$ (which is assumed to be full for primes $\ell \neq p$ for which $B$ is split and $K^{p}$ is maximal) and work with the (commutative) subalgebra $\mathcal{H} = \mathcal{H}^{p}[U_{p}, (p)] \subseteq \mathcal{H}_{K}$.

For future use we will define two other correspondences at $p$. The first is the Frobenius correspondence (or really morphism)

$$Fr : X^{ord} \to X^{ord} \times X^{ord}$$

with $Fr_{1} = Fr$ and $Fr_{2} = id$. The second is $T_{p}$:

$$T_{p} : Y \to X \times X$$

defined by $(T_{p})_{1}(A, H) = A/H$, $(T_{p})_{2}(A, H) = A$. The analytification of $T_{p}$ preserves the ordinary locus (as ordinarity is preserved by isogenies) and hence we may restrict, obtaining a correspondence on $X^{ord}$. As above both of these correspondences overconverge. Given $A \in X^{ord}$ and $\omega \in H^{0}(A, \Omega^{1}_{A})$, we have $Fr(A) = A/C_{A}$ and define a differential $\omega' \in H^{0}(A/C_{A}, \Omega^{1}_{A/C_{A}})$ by inverse pullback along $A \to A/C_{A}$. This makes $Fr$ act on automorphic forms by Remark 15. For $T_{p}$ the same discussion as for $U_{p}$ in Remark 15 applies to give the action on automorphic forms. We remark that, as correspondences, $T_{p} = U_{p} + Fr$ (see [31] section 1.6 for the definition of addition of correspondences) and with the conventions above $T_{p}$ and $U_{p} + Fr$ also induce the same actions on automorphic vector bundles.

3.4. BGG complexes for $G^{*}$. We wish to compute the BGG complex of the representation $Sym^{k-2d}(Sd)$, for $k \geq 2d$. For BGG complexes see [5] for the original paper and [21] for a recent detailed study. For our purpose, the theorem specialized to our situation is the following (the passage from semisimple to reductive Lie algebras merely consists of adding a central character):
Theorem 16. (BGG resolution) If \( V \) is the irreducible representation of the reductive Lie algebra \( g^* = \text{Lie}(G^*(\mathbb{C})) \) of dominant weight \( \lambda = (k_1, ..., k_d, w) \), then we have a resolution
\[
0 \to C^V_d \to ... \to C^V_0 \to V \to 0
\]
with \( C^V_r = \bigoplus_{w \in W^{(r)}} U(g^*) \otimes_{U(b^*)} \chi(w(\lambda + \rho) - \rho) \). The chain complex \( C^V_r \) is a quasi-isomorphic direct summand of the bar resolution \( D^V_r \) defined by \( D^V_r = U(g^*) \otimes_{U(b^*)} (\wedge^r(g^*/b^*) \otimes \mathbb{C} V) \), with \( b^* = \text{Lie}(B^*(\mathbb{C})) \).

Here \( W^{(r)} \) denotes the elements in the Weyl group of length \( r \). The Weyl group of \( G^*(\mathbb{C}) \) is the same as that for its derived group, hence isomorphic to \( \{ \pm 1 \}^d \), and an element \((\epsilon_{11}, ..., \epsilon_{rd})\) acts on a weight \((k_{11}, ..., k_{rd}, w)\) by \((\epsilon_{11}, ..., \epsilon_{rd}).(k_{11}, ..., k_{rd}, w) = (\epsilon_{11}k_{11}, ..., \epsilon_{rd}k_{rd}, w)\). The length of \((\epsilon_{11}, ..., \epsilon_{rd})\) is \# \{ \( i,j \) \mid \( \epsilon_{ij} = -1 \) \}. \( \rho \) denotes half the sum of the positive roots, which in our case is \((1, ..., 1, 0)\). The theorem assumes \( V \) irreducible; we may treat arbitrary semisimple representations by decomposing and taking direct sums (of course this decomposition may not be unique in general).

Recall from above our representations
\[
\bigotimes_i \text{Sym}^{k_i - 2a_i}(S_d)
\]
where the \( k_i \) are integers such that \( k_i \geq 2 \), and that
\[
\text{Sym}^{k_i - 2a_i}(S_d) = \bigoplus_{(k_{11}, ..., k_{rd}, a_{11}, ..., a_{rd})} \left( \bigotimes_j \text{Sym}^{k_{ij} - 2a_{ij}}(S_{d_{ij}}) \right) \otimes \text{det}^\Sigma a_{ij}
\]
and hence
\[
\bigotimes_i \text{Sym}^{k_i - 2a_i}(S_d) = \bigoplus_{(k_{11}, ..., k_{rd}, a_{11}, ..., a_{rd})} \left( \bigotimes_{i,j} \text{Sym}^{k_{ij} - 2a_{ij}}(S_{d_{ij}}) \right) \otimes \text{det}^\Sigma a_{ij}
\]
with \( (\bigotimes_{i,j} \text{Sym}^{(k_{ij} - 2) - 2a_{ij}}(S_{d_{ij}})) \otimes \text{det}^\Sigma a_{ij} \) irreducible of dominant weight
\[
(k_{11} - 2 - 2a_{11}, ..., k_{rd} - 2 - 2a_{rd}, k - 2d)
\]
The BGG complex of \( (\bigotimes_{i,j} \text{Sym}^{(k_{ij} - 2) - 2a_{ij}}(S_{d_{ij}})) \otimes \text{det}^\Sigma a_{ij} \) therefore has \( r \)-th term
\[
\bigoplus_{(\epsilon_{11}, ..., \epsilon_{rd})} U(g^*) \otimes_{U(b^*)} \chi(\epsilon_{11}(k_{11} - 1 - 2a_{11}) - 1, ..., \epsilon_{rd}(k_{rd} - 1 - 2a_{rd}) - 1, k - 2d)
\]
where the direct sum is taken over all \((\epsilon_{11}, ..., \epsilon_{rd})\) in \((W^{(r)})^d\). Note that \( \epsilon_{ij}(k_{ij} - 1 - 2a_{ij}) - 1 \) is \( k_{ij} - 2 - 2a_{ij} \) if \( \epsilon_{ij} = 1 \) and \(-k_{ij} + 2a_{ij} \) if \( \epsilon_{ij} = -1 \).

3.5. Dual BGG complexes for \( X \). The automorphic vector bundle construction produces, given the BGG complex of an irreducible representation \( V \), a complex of vector bundles and differential operators which is a quasi-isomorphic direct summand of the de Rham complex of the vector bundle with connection associated to \( V \) (see e.g. [15], [10] or [32]). Specialized to our situation, the theorem is:
Theorem 17. ([15] Thm 3, [10]) We have, associated to the irreducible representation of dominant weight \( \lambda = (k_1, ..., k_d, w) \), over \( \mathbb{T} \), a complex
\[
0 \to \mathcal{K}_0^\lambda \to \ldots \to \mathcal{K}_d^\lambda \to 0
\]
called the dual BGG complex, with \( \mathcal{K}_d^\lambda = \bigoplus_{w \in W(\tau)} W(w(\lambda + \rho) - \rho)^\vee \) on \( X \) where the maps are Hecke-equivariant differential operators, which is a quasi-isomorphic direct summand of the de Rham complex \( V(\lambda)^\vee \otimes_{\mathcal{O}_X} \Omega_X^1 \) of \( V(\lambda)^\vee \).

Here, as earlier and as will be the case in the rest of the article, \( V(\lambda) = V(k_{11}, ..., k_{rd}, w) \) denotes the vector bundle with connection associated to \( \bigotimes_{i,j} \text{Sym}^{k_{ij}}(S_{d_{ij}}) \otimes \det(w - \sum k_{ij})/2 \). As in the previous section, we may of course consider arbitrary semisimple representations by decomposing and taking direct sums. Thus we get BGG complexes of \( \text{Sym}^{k_{2d}}(H^1_{dR}(A/X)) \) resp. \( \bigotimes_i \text{Sym}^{k_{i} - 2d_i}(H^1_{dR}(A/X)_i) \) (associated with \( \text{Sym}^{k_{2d}}(S_{d^e}) \) resp. \( \bigotimes_i \text{Sym}^{k_{i} - 2d_i}(S_{d^e}_i) \)) that are direct summands of their respective de Rham complexes. Here we are using that the action of \( \mathcal{O}_{B^{Hor}} \otimes \mathbb{Q}_p = M_2(F_p) = \prod_i M_2(F_{p_i}) \) on \( H^1_{dR}(A/X) \) gives a decomposition
\[
H^1_{dR}(A/X) = \bigoplus_i H^1_{dR}(A/X)_i
\]
We have
\[
BGG \left( \bigotimes_i \text{Sym}^{k_{i} - 2d_i}(H^1_{dR}(A/X)_i) \right) = \bigoplus_{(k_{11}, ..., k_{rd}, a_{11}, ..., a_{rd})} BGG(V(k_{11} - 2a_{11} - 2, ..., k_{rd} - 2a_{rd}, -2, k - 2d)^\vee)
\]
and finally we note that the \( r \)-th term of the BGG complex of \( V(k_{11}, ..., k_{rd}, k - 2d)^\vee(- \sum a_{ij}) \) is
\[
\bigoplus_{(\epsilon_{11}, ..., \epsilon_{rd}, w)} W(\epsilon_{11}(k_{11} - 2a_{11} - 1) - 1, ..., \epsilon_{rd}(k_{rd} - 2a_{rd} - 1) - 1, k - 2d)^\vee
\]

4. Rigid and overconvergent de Rham cohomology

As references for rigid cohomology we will mainly use [33], but see also (for example) the papers [24], [25] for a slightly different and perhaps more concrete perspective, or the paper [34] for a site-theoretic framework paralleling that of crystalline cohomology. We are ultimately interested in the rigid cohomology groups of \( X^{ord}_{F_p} \) (and the overconvergent de Rham cohomology groups of \( X^{rig}_{ord} \)) with values in certain overconvergent \( F \)-isocrystals (or overconvergent differential modules), considered as Hecke modules and as \( F \)-isocrystals. Before we proceed, let us recall the notion of a frame from [33] (Def. 3.1.5).

Definition 18. Let \( K \) be a complete valued field, let \( \mathcal{V} \) be its valuation ring, and \( k \) its residue field. A \((K-)frame\) is a diagram
\[
S \hookrightarrow T \hookrightarrow P
\]
consisting of an open immersion of \( k \)-schemes \( S \hookrightarrow T \) and a closed immersion of the \( k \)-scheme \( T \) into a formal \( \mathcal{V} \)-scheme \( P \).
We will also write frames as $S \subseteq T \subseteq P$. Frames will be important later when we consider rigid cohomology. Morphisms of frames are simply commutative diagrams

$$
\begin{array}{ccc}
S & \longrightarrow & T' & \longrightarrow & P \\
\downarrow f & & \downarrow g & & \downarrow u \\
S' & \longrightarrow & T'' & \longrightarrow & P'
\end{array}
$$

where $f$ and $g$ are morphisms of $k$-schemes and $u$ is a morphism of formal $\mathcal{V}$-schemes ([33] Def. 3.1.6). The morphism is said to be quasi-compact if $u$ is quasi-compact ([33] Def. 3.2.1), and etale (resp. smooth) if $u$ is etale (resp. smooth) in a neighbourhood of $S$ (inside $P$) ([33] Def. 3.3.5). The morphism is said to be proper if $g$ is proper ([33] Def. 3.3.10). Given a morphism of frames as above, $u$ induces a morphism $u_K : P_{\text{rig}} \to P'_{\text{rig}}$ of rigid analytic varieties which maps $|S|_P$ into $|S'|_{P'}$.

To analyze the rigid cohomology groups of certain overconvergent isocrystals on $X_{\text{rig}}^{\text{ord}}$ we introduce the frames

- $A_{X_{\text{rig}}}^{\text{ord}} \subseteq A_{X_{\text{rig}}}^{\text{DP}} \subseteq \hat{A}_{X_{\text{rig}}}^{\text{DP}}$
- $X_{\text{rig}}^{\text{ord}} \subseteq X_{\text{rig}}^{\text{DP}} \subseteq \hat{X}_{\text{rig}}^{\text{DP}}$
- $A_{k_{\text{rig}}}^{\text{PR}} = A_{k_{\text{rig}}}^{\text{PR}} \subseteq \hat{A}_{k_{\text{rig}}}^{\text{PR}}$
- $X_{k_{\text{rig}}}^{\text{PR}} = X_{k_{\text{rig}}}^{\text{PR}} \subseteq \hat{X}_{k_{\text{rig}}}^{\text{PR}}$
- $X_{k_{\text{rig}}}^{ss,\text{PR}} = X_{k_{\text{rig}}}^{ss,\text{PR}} \subseteq \hat{X}_{k_{\text{rig}}}^{ss,\text{PR}}$

Note that there is a Cartesian map of frames from the first frame above to the second (this gives the definition of $A_{X_{\text{rig}}}^{\text{ord}}$) resp. from the third to the fourth coming from the map $A_{X_{\text{rig}}}^{\text{DP}} \to \hat{X}_{X_{\text{rig}}}^{\text{DP}}$ resp. $A_{k_{\text{rig}}}^{\text{PR}} \to \hat{X}_{k_{\text{rig}}}^{\text{PR}}$. We may use these frames to interpret overconvergent isocrystals (and rigid cohomology) on $X_{X_{\text{rig}}}^{\text{ord}}$ resp. $X_{X_{\text{rig}}}^{\text{PR}}$ as overconvergent (on $X_{\text{rig}}^{\text{ord}}$ resp. $X_{\text{rig}}^{\text{PR}}$) differential modules (and de Rham cohomology) on $X_{\text{rig}}$, since $X_{X_{\text{rig}}}^{\text{DP}}$ and $X_{X_{\text{rig}}}^{\text{PR}}$ are proper and $\hat{X}_{X_{\text{rig}}}^{\text{DP}}$ resp. $\hat{X}_{X_{\text{rig}}}^{\text{PR}}$ are smooth in a neighbourhood of $X_{\text{rig}}^{\text{ord}}$ resp. $X_{X_{\text{rig}}}^{\text{PR}}$ (this is Cor. 8.1.9 and Prop. 7.2.13 of [33]). We may also use lifts of Frobenius to calculate Frobenius actions (see [33] §8.3). It should be noted that functoriality is not as rigid as frames look like; given a frame $X \subseteq Y \subseteq P$ one does not need to lift morphisms to $P$, it is sufficient to lift them to a strict neighbourhood of $|X|_P$ (tube of $X$ inside $P$), see [33] Prop. 8.1.6 (see also [34], where this observation is built into the foundations).

Consider the universal abelian varieties $A_{X_{\text{rig}}}^{\text{ord}} \to X_{X_{\text{rig}}}^{\text{ord}}$ resp. $A_{k_{\text{rig}}}^{\text{PR}} \to X_{k_{\text{rig}}}^{\text{PR}}$. The relative rigid cohomology groups $H^1_{\text{rig}}(A_{X_{\text{rig}}}^{\text{ord}}/X_{X_{\text{rig}}}^{\text{ord}})$ resp. $H^1_{\text{rig}}(A_{k_{\text{rig}}}^{\text{PR}}/X_{k_{\text{rig}}}^{\text{PR}})$ are overconvergent $F$-isocrystals on $X_{X_{\text{rig}}}^{\text{ord}}$ resp. $X_{k_{\text{rig}}}^{\text{PR}}$ and its fibres over closed points are the contravariant Dieudonné module of the corresponding fibre of the universal abelian variety with its Frobenius action (see e.g. [49] Thm 4.1.4 for the relevant base change assertion). The morphism from the Frobenius pullback of $H^1_{\text{rig}}(A_{X_{\text{rig}}}^{\text{ord}}/X_{X_{\text{rig}}}^{\text{ord}})$ to $H^1_{\text{rig}}(A_{X_{\text{rig}}}^{\text{ord}}/X_{X_{\text{rig}}}^{\text{ord}})$ is given by pull back along the relative Frobenius of $A_{X_{\text{rig}}}^{\text{ord}}/X_{X_{\text{rig}}}^{\text{ord}}$ (this is the induced Frobenius structure on rigid cohomology; see also the remark by the end of section 2 of [12]). By [49] Thm 4.1.4 again, the restrictions of $H^1_{\text{rig}}(A_{k_{\text{rig}}}^{\text{PR}}/X_{k_{\text{rig}}}^{\text{PR}})$ to $X_{k_{\text{rig}}}^{\text{PR}}$ resp. $X_{k_{\text{rig}}}^{ss,\text{PR}}$ are $H^1_{\text{rig}}(A_{k_{\text{rig}}}^{\text{PR}}/X_{k_{\text{rig}}}^{\text{PR}})$ resp. $H^1_{\text{rig}}(A_{k_{\text{rig}}}^{ss,\text{PR}}/X_{k_{\text{rig}}}^{ss,\text{PR}})$ (where $A_{k_{\text{rig}}}^{ss,\text{PR}}$ denotes the restriction of $A_{k_{\text{rig}}}^{\text{PR}}$ to $X_{k_{\text{rig}}}^{\text{PR}}$ resp. $X_{k_{\text{rig}}}^{ss,\text{PR}}$). Since rigid cohomology commutes finite base extensions we have $H^1_{\text{rig}}(A_{X_{\text{rig}}}^{\text{ord}}/X_{X_{\text{rig}}}^{\text{ord}}) \otimes_{\mathbb{Q}_p} L = H^1_{\text{rig}}(A_{k_{\text{rig}}}^{\text{ord}}/X_{k_{\text{rig}}}^{\text{ord}})$ (as $F$-isocrystals). The rigid cohomology of certain
summands of symmetric powers of these overconvergent $F$-isocrystals will be our main object of study in this section.

Remark 19. 1) As we are using specific frames to compute rigid cohomology we will think of these rigid cohomology groups and overconvergent de Rham cohomology groups as “the same”; even though we write “$H_{rig}$” from now on we may occasionally want to think of these as overconvergent de Rham cohomology groups. Recall the Hecke algebras $\mathcal{H}_K$, $\mathcal{H}^p_K$, $\mathcal{H}^p$ and $\mathcal{H}$ introduced by the end of section 3.3. $\mathcal{H}_K^p$ acts as correspondences on $X^{\text{pr}}_p$ and $X^{\text{pr}}_K$ and both morphisms defining the correspondences are finite etale. Hence we get compatible actions on $X^{\text{ord}}_p$, $X^{\text{pr}}_k$ and $X^{rig}_k$ which preserve $X^{\text{ord}}_{k_0}$ and $X^{\text{ss,pr}}_{k_0}$ (as well as $X^{\text{ord}}_k$ and $X^{\text{ss}}_k$, by compatibility). At $p$ we will only consider $U_p$, $T_p$, $(\langle \rangle_p)_{\pm}$ and $Fr$. $Fr$ is a Frobenius lift for $X^{\text{ord}}_p$ and hence gives a concrete way of computing Frobenius actions on the relevant overconvergent $F$-isocrystals on $X^{\text{ord}}_p$. Furthermore, $Fr.U_p$, $(\langle \rangle_p)_{\pm}$ and $T_p$ define correspondences with both maps etale on $X^{\text{rig}}_p$ and $X^{\text{ord}}_p$ and will act on the relevant cohomology groups and spaces of automorphic forms on $X^{\text{rig}}_p$.

2) There is a point of concern of what the natural choice of base field is; when working with automorphic forms it is perhaps $\mathbb{Q}_p$, or a finite extension therefore when working with overconvergent $F$-isocrystals. In this section, when we consider schemes over $\mathbb{F}_p$, $k_L$ and $\overline{\mathbb{F}}_p$ respectively, our frames will be $\mathbb{Q}_p$-frames, $L$- and $\mathbb{C}_p$-frames respectively. We would therefore like to know that our constructions commute with the change of base field from a finite extension of $\mathbb{Q}_p$ to $\mathbb{C}_p$. Rigid cohomology (and coherent cohomology) commutes with a finite extension of base field. For $X^{\text{pr}}_{k_L}$ we may use rigid analytic GAGA and flat base change in the algebraic category. Finally, the base change assertion for $X^{\text{ss,pr}}_{k_L}$ follows from that for $X^{\text{ss,pr}}_{k_0}$ and $X^{\text{pr}}_{k_0}$ by the excision sequence and the functoriality of the base change morphism. Thus no real problem arises from changing base field. We hope that the reader will find it easy to determine which base field is appropriate throughout this section.

4.1. **Relation to overconvergent automorphic forms**. Given the Cartesian maps of frames

$$\left( A^{\text{ord}}_p \subseteq A^{\text{DP}}_p \subseteq \hat{A}^{\text{DP}} \right) \rightarrow \left( X^{\text{ord}}_p \subseteq X^{\text{DP}}_p \subseteq \hat{X}^{\text{DP}} \right)$$
and the fact that these both frames realize rigid cohomology, we deduce from the definition of rigid cohomology that the overconvergent resp. convergent $F$-isocrystal $H^*_\rig(A^{\ord}_p/X^{\ord}_p)$ resp. $H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p)$ is realized by the overconvergent resp. convergent de Rham cohomology $H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p)$. Since $A$ and $X$ are proper we have $H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p) = H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p)$ and by comparison between algebraic and rigid analytic de Rham cohomology (see e.g. [1] Thm. IV.4.1) we have $H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p) = H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p)$, hence $H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p) = (H^*_\rig(A^{\ord}_\mathbb{F}_p/X^{\ord}_\mathbb{F}_p))_{an}$ and similarly for its symmetric powers.

To simplify notation we will put

$$V^\dagger(k_{11}, ..., k_{rd_r}, w) = j^\dagger_{X^{\ord}_p}(V(k_{11}, ..., k_{rd_r}, w))_{an}$$

and

$$W^\dagger(k_{11}, ..., k_{rd_r}, w) = j^\dagger_{X^{\ord}_p}(W(k_{11}, ..., k_{rd_r}, w))_{an}$$

where $j$ denotes the open immersion $X^{\ord}_p \hookrightarrow X$. These are overconvergent sheaves on $X^{\ord}_p$ (see [33] section 5.1 for the definition of $j^\dagger$, it is probably easiest to use his Prop. 5.1.12 as the definition). We may replace $\mathbb{F}_p$ by $\mathbb{F}$ when the representation is defined over $\mathbb{Q}_p$. Applying analytification and $j^\dagger_{X^{\ord}_p}$ (both are exact functors) to our dual BGG complexes, we get overconvergent dual BGG complexes $K^{\dagger}_{(k_{11}, ..., k_{rd_r}, w)}$ on $X^{\ord}_p$ which are direct summands of corresponding the overconvergent de Rham complexes. Note that

$$H^0(X^{\ord}_p, W(k_{11}, ..., k_{rd_r}, w)) = H^0(X, W(k_{11}, ..., k_{rd_r}, w))$$

so the $W^\dagger(k_{11}, ..., k_{rd_r}, w)$ are the "sheaves of overconvergent automorphic forms". We now wish to interpret $H^0(X^{\ord}_p, V^\dagger(k_{11}, ..., k_{rd_r}, w))$ in terms of overconvergent automorphic forms. Since $X^{\ord}_p$ is affine, $X^{\ord}_p$ and its small strict neighbourhoods are quasi-Stein and hence

$$H^i(X^{\ord}_p, W^\dagger(k_{11}, ..., k_{rd_r}, w)) = 0$$

for $i \geq 1$ (coherent cohomology). From this we get the following theorem, which is the analogue of Theorem 5.4 of [12]:

**Theorem 20.** $H^i_{\rig}(X^{\ord}_p, V^\dagger(k_{11}, ..., k_{rd_r}, w))$ is equal to

$$h^i \left( \bigoplus_{(\epsilon) \in W^{d-1}} H^0(X^{\ord}_p, W^\dagger(\epsilon(k_{11} + 1) - 1, ..., \epsilon(k_{rd_r} + 1) - 1, w)) \right)$$

Here $h^i$ stands for "$i$-th cohomology of the complex". In particular, if we denote by $\theta(k_{11}, ..., k_{rd_r}, w)$ the map

$$\bigoplus_{(\epsilon) \in W^{d-1}} W^\dagger(\epsilon(k_{11} + 1) - 1, ..., \epsilon(k_{rd_r} + 1) - 1, w) \longrightarrow W^\dagger(k_{11} + 2, ..., k_{rd_r} + 2, -w)$$

and by abuse of notation also the induced map

$$\bigoplus_{(\epsilon) \in W^{d-1}} H^0(X^{\ord}_p, W^\dagger(\epsilon(k_{11} + 1) - 1, ..., \epsilon(k_{rd_r} + 1) - 1, w)) \longrightarrow$$
of global sections, then
\[ H^d_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, V^\dagger(k_{11}, \ldots, k_{rd_r}, w)\setminus) = \text{Coker } (k_{11}, \ldots, k_{rd_r}, w) \]

**Proof.** We have
\[ H^i_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, V^\dagger(k_{11}, \ldots, k_{rd_r}, w)\setminus) = H^d_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, V^\dagger(k_{11}, \ldots, k_{rd_r}, w)\setminus) = H^i(X_{\mathbb{F}_p}^{\text{ord}}, \mathcal{K}^{i, \bullet}_{k_{11}, \ldots, k_{rd_r}, w}) \]
where the first equality is by the definition of rigid cohomology and the second is by the quasi-isomorphism of the de Rham complex of \( V^\dagger(k_{11}, \ldots, k_{rd_r}, w)\setminus \) and \( \mathcal{K}^{i, \bullet}_{k_{11}, \ldots, k_{rd_r}, w} \). The vanishing
\[ H^i(X_{\mathbb{F}_p}^{\text{ord}}, V^\dagger(k_{11}, \ldots, k_{rd_r}, w)) = 0 \]
for \( i \geq 1 \) then gives the first statement by the hypercohomology spectral sequence. The last statement follows from the first and the definitions. \( \square \)

**Remark 21.** Note that all the previous equalities of cohomology groups are valid as equalities of Hecke modules (cf. Remark 19).

Now look at \( H^d_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, \text{Sym}^{k-2d} \left( H^1_{\text{rig}}(A_{\mathbb{F}_p}^{\text{ord}}/X_{\mathbb{F}_p}^{\text{ord}}) \right) ) \). It is an \( F \)-isocrystal over \( \mathbb{Q}_p \). It has a direct summand
\[ H^d_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, \bigotimes_i \text{Sym}^{k-2d_i} \left( H^1_{\text{rig}}(A_{\mathbb{F}_p}^{\text{ord}}/X_{\mathbb{F}_p}^{\text{ord}}) \right) ) \]
We have
\[ \bigotimes_i \text{Sym}^{k-2d_i} \left( H^1_{\text{rig}}(A_{\mathbb{F}_p}^{\text{ord}}/X_{\mathbb{F}_p}^{\text{ord}}) \right) = \bigoplus_{(k_{11}-2-2a_{11}, \ldots, k_{rd_r}-2-2a_{rd_r}, k-2d)\setminus} V(k_{11}-2-2a_{11}, \ldots, k_{rd_r}-2-2a_{rd_r}, k-2d)\setminus \]
and hence, letting \( \mathcal{E}_{k_{11}, \ldots, k_{rd_r}} = \bigotimes_i \text{Sym}^{k-2d_i} \left( H^1_{\text{rig}}(A_{\mathbb{F}_p}^{\text{ord}}/X_{\mathbb{F}_p}^{\text{ord}}) \right) \),
\[ H^d_{\text{rig}}(X_{\mathbb{F}_p}^{\text{ord}}, \mathcal{E}_{k_{11}, \ldots, k_{rd_r}}) = \bigoplus_{(k_{11}-2-2a_{11}, \ldots, k_{rd_r}-2-2a_{rd_r}, k-2d)\setminus} \text{Coker } (k_{11}-2-2a_{11}, \ldots, k_{rd_r}-2-2a_{rd_r}, k-2d) \]
Let us now fix \( (k_{11}, \ldots, k_{rd_r}) \) and \( (k_{11}, \ldots, k_{rd_r}) \) such that \( \sum_i k_i = k \), \( \sum_j k_{ij} = k_i \), and \( k_{ij} \geq 2 \) for all \( i \) and \( j \). One of the summands above is \( \text{Coker } (k_{11}-2-2a_{11}, \ldots, k_{rd_r}-2-2a_{rd_r}, k-2d) \) which is a quotient of \( H^0(X_{\mathbb{F}_p}, W^\dagger(k_{11}, \ldots, k_{rd_r}, k+2d)) \). This is the part of the cohomology we will be interested in.

### 4.2. Small slope criterion for occurring in the cohomology

Next, we need to know how to normalize the \( U_p \)-operator to achieve optimal \( p \)-integrality. This has been done by Hida in [20] in the general unramified situation and his method works for our \( U_p \)-operator as well, using the description as the trace of Frobenius (up to a diamond operator). We can formulate the result as:

**Proposition 22.** The \( U_p \)-operator is \( p \)-integral on \( H^0(X_{\mathbb{F}_p}, W^\dagger(k_{11}, \ldots, k_{rd_r}, -(\sum k_{ij}) + 2d)) \) and hence on \( H^0(X_{\mathbb{F}_p}, W^\dagger(k_{11}, \ldots, k_{rd_r}, -(\sum k_{ij}) + 2d)) \) (in the sense that its eigenvalues are \( p \)-integral) and has slope 0-eigenvectors on both these spaces. Moreover, shifting the central character up by 2 scales \( U_p \) by \( p^{-1} \).
Proof. As mentioned before the statement of the proposition, the first part follows by a standard calculation following Hida and the second part. We remark that this calculation is entirely analogous to the standard $q$-expansion calculation, using Serre-Tate coordinates instead of the Tate abelian variety. The proof of the second part is also by a standard calculation. Let us outline the argument. First, we prove the analogue statement over the complexes. Let $\Gamma = G^*(\mathbb{Q}) \cap K$ and let $h = \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix}$. Fix a weight $(k_{11}, \ldots, k_{rd}, w)$ and write $\chi = \chi(k_{11}, \ldots, k_{rd}, w)$. We may interpret automorphic forms of level $K$ and weight $(k_{11}, \ldots, k_{rd}, w)$ over $\mathbb{C}$ as functions

$$f : G^*(\mathbb{R}) \rightarrow \mathbb{C}$$

satisfying $f(\gamma g) = f(g)$ and $f(gk) = \chi(k)^{-1}f(g)$ for $\gamma \in \Gamma$ and $k \in K_{\infty}$, or equivalently as functions

$$\phi : G^*(\mathbb{A}) \rightarrow \mathbb{C}$$

such that $\phi(\gamma g) = \phi(g)$ and $\phi(gk) = \chi(k_{\infty})^{-1}\phi(g)$ for $\gamma \in G^*(\mathbb{Q})$ and $k \in K$ (plus analytic conditions that we will not need and therefore not go into). Given $f$, the associated $\phi$ is defined by $\phi(g) = f(g_{\infty})$. Note that we may describe local sections of $W(k_{11}, \ldots, k_{rd}, w)$ on $X(\mathbb{C})$ by the same equations, restricting the domain of $f$ to any open $U'$ which is the pullback of some analytic open $U$ under the natural map $G^*(\mathbb{R}) \rightarrow X(\mathbb{C})$. The adellic operator $U_p = [K \bar{h} K]$, which in the classical setting becomes $[\Gamma \bar{h}^{-1} \Gamma]$, acts as

$$(U_p f)(g) = \sum_i f(h^{-1}\gamma_i g)$$

for some (any) set $\gamma_1, \ldots, \gamma_r$ of coset representatives of $(\Gamma \cap h \Gamma h^{-1}) \backslash \Gamma$. Now consider changing the weight by a factor of $\det$, i.e. $(k_{11}, \ldots, k_{rd}, w)$ goes to $(k_{11}, \ldots, k_{rd}, w+2)$. There is an isomorphism of coherent sheaves

$$\varphi : W(k_{11}, \ldots, k_{rd}, w) \rightarrow W(k_{11}, \ldots, k_{rd}, w+2)$$

(which is valid over the reflex field) defined on local sections by

$$(\varphi(f))(g) = \det(g)^{-1}f(g)$$

Thus we see that

$$(U_p(\varphi(f)))(g) = \det(h^{-1})^{-1}\det(g)^{-1}\sum f(h\gamma_i g) = p \cdot (\varphi(U_p f))(g)$$

which is the result we wanted. Now as this identity holds analytically over $\mathbb{C}$, it also holds formally around every $\mathbb{C}$-point, hence formally around every $\overline{\mathbb{Q}}$-point, and hence rigid analytically in the ordinary locus by the principle of analytic continuation (the ordinary locus is connected, and contains $\overline{\mathbb{Q}}$-points).

Remark 23. 1) The choice $\phi(g) = f(g_{\infty})$ is nonstandard (but seems to the author to be a fairly natural choice). This is what forces $U_p$ to become $[\Gamma \bar{h}^{-1} \Gamma]$ in the classical setting; it differs from the usual choice using $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by a central factor of $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$, which reflects the fact that we didn’t throw in determinant factors in the equivalence $f \leftrightarrow \phi$.

2) We will also define the action of $Fr$ on $H^0(X_{rig}, W^+(k_{11}, \ldots, k_{rd}, w))$ by using the previously defined action on $H^0(X_{rig}, W^+(k_{11}, \ldots, k_{rd}, -\sum k_{ij} + 2d))$ and declaring that shifting the central character up by 2 scales $Fr$ by $p^{-1}$. This corresponds to the interpretation of the automorphic vector bundle of $det$ as the Tate twist $\mathbb{Q}_p(1)$. 

CLASSICALITY FOR SMALL SLOPE OVERCONVERGENT AUTOMORPHIC FORMS ON SOME COMPACT PEL SHIMURA VARIETIES OF TYPE C23
We may now prove the analogue of Lemma 6.3 of [12].

**Corollary 24.** Let $k_i \geq 2$ for all $i$. If $f \in H^0 (X_{rig}, W^\dagger (k_{11}, \ldots, k_{rd}, - (\sum k_{ij}) + 2d))$ is a $U_p$-eigenform of slope less than $\inf_i (k_{ij} - 1)$, then $f$ is not in the image of $\theta$.

**Proof.** Recall that $\theta$ is a $U_p$-equivariant map

$$\bigoplus_{i,j} H^0 (X_{rig}, W^\dagger (k_{11}, \ldots, 2 - k_{ij}, \ldots, k_{rd}, - (\sum k_{ij}) + 2d)) \to H^0 (X_{rig}, W^\dagger (k_{11}, \ldots, k_{rd}, - (\sum k_{ij}) + 2d))$$

Here the right hand side has the optimal $U_p$ whereas, by the previous Proposition, the optimal $U_p$ for weight $(k_{11}, \ldots, 2 - k_{ij}, \ldots, k_{rd})$ occurs with central character

$$- (2 - k_{ij} + \sum_{(i',j') \neq (i,j) \atop \ord k_{i',j'}} k_{i',j'}) + 2d = (\sum k_{i',j'}) + 2d + 2(k_{ij} - 1)$$

Thus $U_p$ acting on $H^0 (X_{rig}, W^\dagger (k_{11}, \ldots, 2 - k_{ij}, \ldots, k_{rd}, - (\sum k_{ij}) + 2d))$ has eigenvalues of valuation $\geq k_{ij} - 1$ by the previous Proposition. This proves the Corollary. \(\square\)

Thus, again for fixed $(k_{11}, \ldots, k_{rd})$ with $\sum_j k_{ij} = k_i$, $\sum_i k_i = k$, $H^d_{rig} (X_{k,\text{rig}}, \mathcal{E}_{k_1,\ldots,k_r})$ has a sub-Hecke module consisting of the overconvergent automorphic forms of weight $(k_{11}, \ldots, k_{rd}, -k + 2d)$ of $U_p$-slope $< \inf_i (k_{ij} - 1)$.

### 4.3. The excision sequence and a slope criterion

The next thing to do is to start analyzing $H^d_{rig} (X_{k,\text{rig}}, \mathcal{E}_{k_1,\ldots,k_r})$ using the formalism of rigid cohomology. To simplify notation we will write $\mathbb{k}$ for $(k_1,\ldots,k_r)$ and we continue to assume $k_{ij} \geq 2$ for all $i,j$. The excision sequence in rigid cohomology gives us a Frobenius-equivariant exact sequence

$$\ldots \to H^d_{rig} (X_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}}) \to H^d_{rig} (X^{\text{ord}}_{X_{k,\text{rig}}}, \mathcal{E}_{\mathbb{k}}) \to H^{d+1}_{X_{k,\text{rig}}, \text{rig}} (X^{PR}_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}}) \to \ldots$$

Here we have some knowledge of $H^d_{rig} (X_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}})$ as a Hecke module from comparison theorems and "classical" automorphic methods (Matsumura’s formula). The problematic term is the contribution from $H^{d+1}_{X_{k,\text{rig}}, \text{rig}} (X^{PR}_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}})$. We will deal with it by bounding its slopes. Before we do this we simplify it somewhat as follows:

**Proposition 25.** There is an isomorphism $H^{d+1}_{X_{k,\text{rig}}, \text{rig}} (X_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}}) \cong H^{d-1}_{rig} (X^{PR}_{k,\text{rig}}, \mathcal{E}_{\mathbb{k}}^c (d))$ which is Hecke and Frobenius-equivariant (where $(d)$ denotes a Tate twist by $d$).

**Proof.** This is Poincare duality, see [24] Thm 1.2.3 and also [25] section 2.1 or [33] Corollary 8.3.14 for the Frobenius-equivariant formulation. Hecke equivariance follows since the Hecke action is by correspondences. \(\square\)
We want to bound the range of the slopes of $H^{d-1}_{rig} \left( X^{ss,PR}_{k_L} , \mathcal{E}^{\vee}_{k} (d) \right)^{\vee}$. To do this we will use §6.7 of [25]. Since the fibre of $H_{rig}^{1}(A_{k_L}^P / X_{PR})$ at a closed point $x$ of $X_{k_L}$ is simply the rational Dieudonné module of $A_{k_L,x}$, we note that the slopes of $H_{rig}^{1}(A_{k_L} / X_{k_L})$ lie in $[0, 1]$ (the definition is in the second paragraph of section 6.7 of [25]; these slopes are “pointwise slopes”). However, more importantly for us:

**Proposition 26.** The slopes of $\mathcal{E}_{k}$ on $X^{ss,PR}_{k_L}$ are in $[\lambda, k - 2d - \lambda]$, where $\lambda = \inf_{i} ((k_i - 2d_i) \inf(1/2, 1/d_i))$.

*Proof.* Given a closed point $x$ of $X^{ss,PR}_{k_L}$, the corresponding abelian variety $A_x$ is isogenous over $\overline{\mathbb{F}}_p$ to the square of a non-ordinary abelian variety $A'$ with real multiplication by $F$ by the proof of Proposition 5.2 of [36] (the result as stated in [36] requires $F$ unramified at $p$, but this is not used in the proof of the particular fact we need). We may decompose the rational Dieudonné modules $D(A)$ and $D(A')$ according to primes above $p$ in $F$:

$$D(A) = \bigoplus_{i} D(A)_i$$

$$D(A') = \bigoplus_{i} D(A')_i$$

We have $D(A)_i = D(A')^{\otimes 2}_i$. Each $D(A')_i$ is a rank 2 rational Dieudonné module over $F_{p^i}$, coming from a rank 2 Dieudonné module over $\mathcal{O}_{F_{p^i}}$. The slopes of those may be calculated by comibing Theorem 5.2.1 [17], which does the unramified case, and Theorem 9.2 of [3], which does the totally ramified case (strictly speaking one should perhaps combine their methods, but this only amounts to changing notation in the proofs). The outcome is that $D(A')_i$ (and hence $D(A)_i$) has either two slopes $a/d_i$ and $(d_i - a)/d_i$ (with $a \in \mathbb{Z}$, $0 \leq a \leq d_i$) or a single slope $1/2$. If $A$, and hence $A'$, is non-ordinary, then there exists an $i$ such that the slopes of $D(A')_i$ are not 0 and 1. Define $\lambda(i) = \inf(1/2, 1/d_i)$, then the slopes of $D(A')_i$ are in the interval $[\lambda(i), 1 - \lambda(i)]$, and hence the slopes of

$$\bigotimes_{i} Sym^{k_i - 2d_i} D(A)_i$$

are in $[(k_i - 2d_i)\lambda(i), k - 2d - (k_i - 2d_i)\lambda(i)]$ (since slopes behave additively with respect to tensor operations). Thus we see that the slopes of $\mathcal{E}_{k}$ on $X^{ss,PR}_{k_L}$ are in $[\lambda, k - 2d - \lambda]$, where $\lambda = \inf_{i} ((k_i - 2d_i)\lambda(i))$, as desired.

Using this, we are ready to prove the main result of this section. Recall that a Tate twist by 1 decreases slopes by 1 and that dualizing sends a slope to its negative.

**Theorem 27.** The slopes of $H^{d-1}_{rig} \left( X^{ss,PR}_{k_L} , \mathcal{E}^{\vee}_{k} (d) \right)^{\vee}$ lie in $[\lambda + 1, k - d - \lambda]$.

*Proof.* By the previous Proposition the slopes of $\mathcal{E}_{k}$ on $X^{ss,PR}_{k_L}$ are in $[\lambda, k - 2d - \lambda]$, so by the remarks before this Theorem the slopes of $\mathcal{E}^{\vee}_{k} (d)$ are in $[\lambda + d - k, -\lambda - d]$.

Next we apply Theorem 6.7.1 of [25], a special case of which says that if $S$ is a proper separated scheme of finite type over $k_L$ of pure dimension $d - 1$ and $\mathcal{F}$ is an overconvergent $F$-isocrystal on $S$.
with slopes in \([r, s]\), then the slopes of \(H^{d-1}_{\text{rig}}(S, \mathcal{F})\) are in \([r, s + d - 1]\). In our situation this allows us conclude that the slopes of \(H^{d-1}_{\text{rig}}(X_{kL}^{ss, PR}, \mathcal{E}^\vee(d))\) are in \([\lambda + d - k, -\lambda - 1]\). Dualizing we see that the slopes of \(H^{d-1}_{\text{rig}}(X_{kL}^{ss, PR}, \mathcal{E}^\vee(d))\) lie in \([\lambda + 1, k - d - \lambda]\) as desired. \(\square\)

**Corollary 28.** The slopes of \(H^d_{\text{rig}}(X_{kL}^{ord}, \mathcal{E}_k)\) lie in \([0, k - d]\). Thus the part of cohomology with slopes in \([0, \lambda + 1) \cup (k - d - \lambda, k - d]\) lies in the image of \(H^d_{\text{rig}}(X_{kL}^{PR}, \mathcal{E}_k)\).

**Proof.** That the slopes of \(H^d_{\text{rig}}(X_{kL}^{ord}, \mathcal{E}_k)\) lie in \([0, k - d]\) follows from noting that the slopes of \(\mathcal{E}_k\) are in \([0, k - 2d]\) (since the slopes of \(H^d_{\text{rig}}(A_{kL}^{PR}/X_{kL}^{PR})\) are in \([0, 1]\)) and applying Theorem 6.7.1 of [25] (not the same special case as before, but the same if you replace “proper” by “smooth”). The second part then follows by the Theorem and the excision sequence, as the part of cohomology with slopes in \([0, \lambda + 1) \cup (k - d - \lambda, k - d]\) necessarily gets killed when mapped to \(H^d_{X_{kL}^{ss,PR},\text{rig}}(X_{kL}^{ord}, \mathcal{E}_k)\) and hence lies in the image of \(H^d_{\text{rig}}(X_{kL}^{ord}, \mathcal{E}_k)\). \(\square\)

### 4.4. Classicality for forms of small slope, the case of arbitrary \(d\).

Throughout this section we encourage the reader to keep part 2) of Remark 19 in mind. Recall the Frobenius correspondence \(F_r\) on \(X_{kL}^{ord}\) that we defined in section 3.3, and that it overconverges. Composing \(F_r\) with \(U_p\) in one way gives the correspondence

\[
r = (r_1, r_2) : X_{\text{rig}}^{\text{ord}} \rightarrow X_{\text{rig}}^{\text{ord}} \times X_{\text{rig}}^{\text{ord}}
\]

with \(r_1 = F_r\), \(r_2 = (p)^{-1} F_r\) (we define composition of correspondences as in [31] section 1.6). As \((p)^{\pm 1}\) commutes with the Frobenius morphism we rewrite this correspondence as the composition of \((p)\) with the correspondence

\[
r' = (r'_1, r'_2) : X_{\text{rig}}^{\text{ord}} \rightarrow X_{\text{rig}}^{\text{ord}} \times X_{\text{rig}}^{\text{ord}}
\]

with \(r'_1 = r'_2 = F_r\). Transferring differentials as for \(F_r\) and \(U_p\), we deduce that the action of \(r'\) on \(H^0(X_{\text{rig}}, W^1(k_{11}, ..., k_{rd}, -(\sum k_{ij}))\) is by \(p^{k+d}\) (\(p^k\) comes from the transfer of differentials, \(p^d\) is the degree of the morphism \(F_r\)) and hence acts on \(H^0(X_{\text{rig}}, W^1(k_{11}, ..., k_{rd}, -(\sum k_{ij}) + 2d))\) as \(p^{k-d}\) (by Prop. 22 and Rem. 23). Hence it acts on \(H^d_{\text{rig}}(X_{p_{\text{ss}, PR}}^{\text{ord}}, \mathcal{E}_k)\) by \(p^{k-d}\), and therefore \(r\) acts by \((p)^{p^{k-d}}\). Since \(H^d_{\text{rig}}(X_{p_{\text{ss}, PR}}^{\text{ord}}, \mathcal{E}_k)\) is finite-dimensional, one-sided inverses are two-sided inverses and we can conclude that

\[
(p)^{p^{k-d}} = \frac{1}{(p)^{p^{k-d}}}
\]

on \(H^d_{\text{rig}}(X_{p_{\text{ss}, PR}}^{\text{ord}}, \mathcal{E}_k)\). We may conclude that the slopes of \(U_p\) acting on \(H^d_{\text{rig}}(X_{p_{\text{ss}, PR}}^{\text{ord}}, \mathcal{E}_k)\) lie in \([0, k - d]\) (as the eigenvalues of \((p)\) are roots of unity), and we immediately deduce the following Lemma from Corollary 28:

**Lemma 29.** The part of \(H^d_{\text{rig}}(X_{kL}^{ord}, \mathcal{E}_k)\) with \(U_p\)-slope in \([0, \lambda) \cup (k - d - \lambda - 1, k - d]\) is in the image of \(H^d_{\text{rig}}(X_{kL}^{PR}, \mathcal{E}_k)\).

From this, our classicality criterion follows. Let us first state the following simple consequence of Matsushima’s formula:
Lemma 30. The Hecke module $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$ decomposes as a direct sum of Hecke modules of $K$-fixed vectors associated to automorphic representations of $G^*$.

Proof. The direct sum decomposition of $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$ reduces the question to the same assertion for the $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, V^1(k_{11}, ..., k_{rd}, w) \right)$. Since our Hecke operators are defined over $\mathbb{Q}$, by a sequence of comparison theorems/definitions (definition of rigid cohomology, complex and rigid analytic/algebraic comparison of de Rham cohomology and flat base change) we see that the Hecke modules $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, V^1(k_{11}, ..., k_{rd}, w) \right)$ and $H^d_{dR} \left( X(\mathbb{C}), V(k_{11}, ..., k_{rd}, w) \right)$ arise as base changes of the same Hecke module over $\mathbb{Q}$. We have Matsushima’s formula

$$H^d_{dR} \left( X(\mathbb{C}), V(k_{11}, ..., k_{rd}, w) \right) = \bigoplus_{\pi} m(\pi) \pi^K \otimes H^d(\mathfrak{g}^*, K_{dR}; \mathfrak{g}^*; \mathfrak{g}^*)$$

(the standard reference is [6] VII.5.2, see Thm 3.2 of [51] for the formulation above and some more details) where the summation is over all irreducible admissible representations of $G^*(\mathbb{A})$, $m(\pi)$ is the multiplicity of $\pi$ in the appropriate summand of $L^2(G^*(\mathbb{Q}) \backslash G^*(\mathbb{A}))$, $\pi^K$ is the $K$-fixed vectors of the finite part $\pi_f$ of $\pi$, $H^d(\mathfrak{g}^*, K_{dR}; \mathfrak{g}^*)$ is the $G^*$-cohomology with trivial Hecke action and $\pi(k_{11}, ..., k_{rd}, w) = (\bigotimes_{i,j} \text{Sym}^k \chi_i (S_{d_{ij}})) \otimes \text{det}^{(w-\sum k_{ij})/2}$. As $m(\pi).\dim H^d(\mathfrak{g}^*, K_{dR}; \mathfrak{g}^*) \otimes \pi(k_{11}, ..., k_{d}, w) = 0$ unless $\pi$ is the automorphic representation associated to some automorphic form of level $K$ and weight $(k_{11}, ..., k_{rd}, -w)$, the lemma follows. □

Theorem 31. a) Let $f$ be an overconvergent Hecke eigenform for $\mathcal{H}$, of weight $(k_{11}, ..., k_{rd})$, character $\chi$ for the diamond operators and with $U_p$-slope in $[0, \lambda) \cup (k-d-\lambda-1, k-d]$, and assume that it is not in the image of $\theta$. Then its system of Hecke eigenvalues for $\mathcal{H}$ comes from the $p$-stabilization of a classical form of level $K$.

b) Assume that $f$ is an overconvergent Hecke eigenform for $\mathcal{H}$, of weight $(k_{11}, ..., k_{rd})$, character $\chi$ for the diamond operators and with $U_p$-slope less than $\inf (k_{ij}-1, \lambda)$. Then its system of Hecke eigenvalues for $\mathcal{H}$ comes from the $p$-stabilization of a classical form of level $K$.

Proof. We look here at the direct summand $\text{coker} \theta(k_{11}-2, ..., k_{rd}-2, k-2d)$ of $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$. By Corollary 24 part b) follows directly from a), so we may focus on a). We assume that $f$ is not in the image of $\theta$, hence its system of Hecke eigenvalues outside $p$ occurs in $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$, and by Lemma 29 it comes from $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$. Lemma 30 now gives the theorem for $\mathcal{H}$. For $U_p$, note that the class of $f$ in $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$ is also an eigenvector for $Fr$ (by equation 4.1), hence for $T_p$ as $T_p = U_p + Fr$. Since $H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right) \rightarrow H^d_{rig} \left( X^{PR}_{\mathbb{F}_p}, \mathcal{E}_k \right)$ is equivariant for $T_p$, it follows that the $T_p$-eigenvalue of the class of $f$ is the $T_p$-eigenvalue of the associated classical form $g$ of level $K$, and that its $U_p$-eigenvalue satisfies the $p$-Hecke polynomial of $g$, as $U_p$ satisfies $x^2 - T_p x + \chi(p)p^{k-d}$. Hence the $U_p$-eigenvalue of $f$ agrees with that of a $p$-stabilization of $g$, which was what we wanted to prove. □
4.5. The case \( d = 1 \); the Hecke modules \( H^1_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \). For completeness we give a separate treatment of the case \( d = 1 \) in this subsection, where we can obtain better results by methods similar to those in [12]. We will drop the superscripts \( PR \) and \( DP \) since we are in an unramified case and the Pappas-Rapoport and Deligne-Pappas models agree. Let us first state Theorem 31 in the special case when \( d = 1 \). It is reminiscent of Gouvea’s original conjecture for overconvergent modular forms ([18], Conjecture 3):

**Theorem 32.** a) Assume that \( f \) is an overconvergent Hecke eigenform for \( \mathcal{H} \), of weight \( k \), character \( \chi \) for the diamond operators and assume that it is not in the image of \( \theta \). Then its system of Hecke eigenvalues for \( \mathcal{H} \) comes from the \( p \)-stabilization of a classical form of level \( K \).

b) Assume that \( f \) is an overconvergent Hecke eigenform for \( \mathcal{H} \), of weight \( k \), character \( \chi \) for the diamond operators and with \( U_p \)-slope not equal to \((k - 2)/2 \), and assume that it is not in the image of \( \theta \). Then its system of Hecke eigenvalues for \( \mathcal{H} \) comes from the \( p \)-stabilization of a classical form of level \( K \).

We will prove the following stronger theorem, which is a (slightly weaker) analogue of Corollary 7.2.1 of [12] (see Remark 37 for a strengthening of part b):

**Theorem 33.** a) Assume that \( f \) is an overconvergent Hecke eigenform for \( \mathcal{H} \), of weight \( k \), character \( \chi \) for the diamond operators and assume that it is not in the image of \( \theta \). Then its system of Hecke eigenvalues for \( \mathcal{H} \) is classical of level \( K^pK_0(p) \).

b) Assume that \( f \) is an overconvergent Hecke eigenform for \( \mathcal{H} \), of weight \( k \), character \( \chi \) for the diamond operators with \( U_p \)-slope less than \( k - 1 \). Then its system of Hecke eigenvalues is classical of level \( K^pK_0(p) \).

To do this we will aim directly at the cohomology groups \( H^1_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \) rather than interpreting them as summands of \( H^1_{\rig}(X^{\ord}_{p}, \text{Sym}^{k-2}(H^1_{\rig}(A_{p, \text{rig}}/X_p))) \). The excision sequence that we are interested in is then

\[
0 \to H^1_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \to H^1_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \to
\]

\[
H^2_{X^{\rig}_{p, \text{rig}}}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \to H^2_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \to 0
\]

where the first 0 is a local \( H^1 \) which vanishes by Poincaré duality (it corresponds to an \( H^1 \) on \( X^{\rig}_{p, \text{rig}} \), which is 0-dimensional) and the 0 at the end comes from the fact that \( X^{\ord}_{p, \text{rig}} \) is affine and 1-dimensional so any \( H^2_{\rig} \) vanishes. Rather than slopes we will analyze this using some dimension counting analogous to parts of [12] sections 5 and 6. The space \( H^1_{\rig}(X^{\ord}_{p}, V^\dagger(k - 2, k - 2)^\vee) \) looks (as a Hecke module) like two copies of the space of classical level \( K \) automorphic forms, by Matsushima’s formula. The Hecke-equivariant quotient map

\[
H^0(X, W^\dagger(k, -k + 2)) \to \text{Coker } \theta_{k-2,k-2} = H^1_{\rig}(X^{\ord}_{p, \rig}, V^\dagger(k - 2, k - 2)^\vee)
\]

injects the space of weight \( k \) level \( K^pK_0(p) \) classical \( p \)-new forms into \( H^1_{\rig}(X^{\ord}_{p, \rig}, V^\dagger(k - 2, k - 2)^\vee) \) (this follows from Cor. 24 since these \( p \)-new forms have slope \((k - 2)/2 \)). As they are \( p \)-new, they will not be in image of the map \( H^1_{\rig}(X^{\ord}_{p, \rig}, V^\dagger(k - 2, k - 2)^\vee) \to H^1_{\rig}(X^{\ord}_{p, \rig}, V^\dagger(k - 2, k - 2)^\vee) \) and hence the space of weight \( k \) level \( K^pK_0(p) \) classical \( p \)-new forms injects into \( H^2_{X^{\rig}_{p, \text{rig}}}(X^{\ord}_{p, \rig}, V^\dagger(k - 2, k - 2)^\vee) \).
Lemma 34. 1) Let \( k \geq 3 \). The space of weight \( k \) level \( K^pK_0(p) \) classical \( p \)-new forms has dimension \((k - 1)SS\), where \( SS \) is the number of supersingular points on \( X_{\overline{p}} \).

2) The space of weight 2 level \( K^pK_0(p) \) classical \( p \)-new forms has dimension \( SS - 1 \).

Proof. This is well known, we give a brief indication of the proof.

1) In general, one shows using the Kodaira-Spencer isomorphism and the Riemann-Roch theorem that for weight \( k \geq 3 \) and an arbitrary neat level \( K' \), the space of weight \( k \) and level \( K' \) classical automorphic forms has dimension \((k - 1)(g(X(K')) - 1)\) where \( g(X(K')) \) is the genus of the Shimura curve \( X(K') \) of level \( K' \). Let \( g \) denote the genus of \( X \). By looking at \( \overline{\mathcal{M}}_{g,0} \), one sees that the genus of \( Y \) is \( 2g + SS - 1 \). Since the dimension of the space of weight \( k \) level \( K^pK_0(p) \) classical \( p \)-old forms is twice that of the space weight \( k \) level \( K \) classical forms (each eigenform has two \( p \)-stabilizations), one gets the formula for the \( p \)-new forms.

2) Kodaira-Spencer shows that the space of weight 2 and level \( K' \) classical automorphic forms has dimension \( g(X(K')) \), hence the space of weight 2 level \( K^pK_0(p) \) classical \( p \)-new forms has dimension \((2g + SS - 1) - 2g = SS - 1\).

Lemma 35. \( \dim H^2_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) = SS(k - 1) \) for \( k \geq 2 \).

Proof. By Poincaré duality \( H^2_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) = H^0_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 4)) \) so since \( X_{\overline{p}} \) is \( SS \) points and \( V^\dagger(k - 2, k - 4) \) has rank \( k - 1 \), the formula follows.

The last ingredient of our dimension count is

Lemma 36. \( H^2_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) = 0 \) if \( k \geq 3 \), and the one-dimensional Hecke module corresponding to Tate twist by \(-1\) if \( k = 2 \).

Proof. This follows by Matsushima’s formula or other “classical methods” (e.g. degeneration of the BGG spectral sequence).

Adding up the dimensions in the previous lemmas we see that as Hecke modules,

\[ H^1_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) = H^0(Y, W(k, 2 - k)) \]

Here we are using that image of the injection \( H^1_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) \to H^1_{\text{rig}}(X_{\overline{p}}, V^\dagger(k - 2, k - 2)) \) is, as a Hecke module, the space of \( p \)-old forms inside \( H^0(Y, W(k, 2 - k)) \). For \( H^p \) and \( \langle p \rangle \) this follows from equivariance, for \( U_p \) this uses the same trick as in the proof of Theorem 31. Thus Theorem 33 follows.

Remark 37. The fact that these two are equal as Hecke modules does not mean that the composition \( H^0(Y, W(k, 2 - k)) \to H^0(X_{\text{rig}}, W^\dagger(k, 2 - k)) \to \text{coker} \theta(k - 2, k - 2) \) is an isomorphism. In the modular curve case, Coleman ([12]) shows the equality of Hecke modules as above but also that the composition above is not an isomorphism. However, by Corollary 24, the composition is an injection, and hence an isomorphism, on slope \( < k - 1 \) parts. This allows one to strengthen Theorem 33 b) to assert that \( f \) itself is classical (of level \( K^pK_0(p) \)). Corollary 7.2.1 of [12] asserts that the analogous strengthening of a) is true in the case of modular curve. However,
we cannot prove it by the same technique as we do not have $q$-expansions, and as a result do not know multiplicity 1 for overconvergent automorphic forms.

REFERENCES

[1] Andre, Y., Baldassarri, F. De Rham Cohomology of Differential Modules on Algebraic Varieties, Prepublication Institut de Mathematiques de Jussieu 184.
[2] Andreatta, F., Gasbarri, C.: The canonical subgroup for families of abelian varieties. Composition Math. 143(3), 566-602 (2007).
[3] Andreatta, F., Goren, E. Z.: Geometry of Hilbert modular varieties over totally ramified primes. Internat. Math. Res. Not. 33, 1785-1835 (2003).
[4] Andreatta, F., Iovita, A., Pilloni, V.: $p$-adic families of Siegel modular forms. Preprint, available at http://perso.ens-lyon.fr/vincent.pilloni/
[5] Bernstein, I. N., Gelfand, I. M., Gelfand, S. I.: Differential operators on the base affine space and a study of $g$-modules. In "Lie groups and their representations", Summer School of the Bolyai Janos Mathematical Society, (Budapest, 1971), Adam Hilger Ltd., London (1975).
[6] Borel, A., Wallach, N. Continuous cohomology, Discrete Subgroups and Representation theory of Reductive Groups. Second edition. Amer. Math. Soc. (1999).
[7] Boutot, J. F.: Varietes de Shimura: Le probleme de modules en inegalite caracteristique. In "Varietes de Shimura et fonctions L"., Publ. Math. Univ. Paris VII 6 (1979).
[8] Buzzard, K. M.: Eigenvarieties. In "L-functions and Galois representations", 59-120, London Math. Soc. Lecture Note Ser., 320, Cambridge Univ. Press (2007).
[9] Chai, C-L., Faltings, G.: Degenerations of Abelian varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 22, Springer-Verlag (1990).
[10] Chenevier, G.: Familles $p$-adiques de formes automorphes pour $GL(n)$. Journal fur die reine und angewandte Mathematik 570, 143-217, (2004).
[11] Coleman, R. F.: Classical and overconvergent modular forms. Inventiones Math. 124, 215-241 (1996).
[12] Deligne, P., Pappas, G.: Singularites des espaces de modules de Hilbert, en les caracteristiques divisant le discriminant. Compositio Math. 90, 59-79 (1994).
[13] Emerton, M.: On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms. Inventiones Math. 164, no. 1, 1-84 (2006).
[14] Faltings, G.: On the Cohomology of Locally Symmetric Hermitian Spaces. Lecture Notes in Mathematics 1029, 55-98 (1983).
[15] Goren, E. Z., Kassaei, P. L.: Canonical subgroups over Hilbert modular varieties. To appear in Journal fur die reine und angewandte Mathematik. Available at http://www.mth.kcl.ac.uk/~kassaei/research/files/cshmvy.pdf.
[16] Gouv?. F. Q.: Continuity properties of Modular forms, Elliptic Curves and related topics, CRM Proceedings and Lecture Notes, AMS 4, 85-99 (1994).
[17] Harris, M., Taylor, R. L.: The Geometry and Cohomology of Some Simple Shimura Varieties, Annals of Math. Studies 151, Princeton Univ. Press (2001).
[18] Ivasi?, K.: Points on Shimura varieties over finite fields. J. AMS 5, 735-783 (2005).
[19] Kottwitz. Points on Shimura varieties over finite fields. J. AMS 5, 373-444 (1992).
[29] Lan, K-W.: Arithmetic compactifications of PEL-type Shimura varieties. Ph.D. thesis, Harvard University (2008)
[30] Lan, K-W., Suh, J.: Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties. Duke Math. J. 161, no. 6, 1113-1170 (2012)
[31] Laumon, G.: Cohomology of Drinfel’d modular varieties I. Cambridge University Press (1996)
[32] Lan, K-W., Polo, P.: Dual BGG complexes for automorphic bundles. Preprint, available at http://www.math.princeton.edu/~klan/academic.html
[33] Le Stum, B.: Rigid cohomology, volume 172 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2007)
[34] Le Stum, B.: The overconvergent site. To appear in Memoire de la SMF (2012). Available at http://perso.univ-rennes1.fr/bernard.le-stum/Publications.html
[35] Loeffler, D.: Overconvergent algebraic automorphic forms. Proc. London Math. Soc. 102, no. 2, 193-228 (2011)
[36] Milne, J. S.: Points on Shimura varieties mod 
[37] Milne, J. S.: Canonical Models of (Mixed) Shimura varieties and Automorphic Vector Bundles. In “Automorphic Forms, Shimura Varieties, and L-functions”, (Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6-16, 1988), p. 283-414. Also available at http://www.jmilne.org/math/articles/
[38] Milne, J. S.: Introduction to Shimura varieties. In “Harmonie Analysis, the Trace Formula and Shimura Varieties” (James Arthur, Robert Kottwitz, Editors) AMS (2005) Also available at http://www.jmilne.org/math/articles/
[39] Mok, C-P., Tan, F.: Overconvergent family of Siegel-Hilbert modular forms. Preprint, available at https://www.math.mcmaster.ca/~cpmok/
[40] Nakamura, K.: Classification of split trianguline representations of 
[41] Pilloni, V.: Prolongement analytique sur les variétés de Siegel. Duke Math Journal, vol. 157, No. 1, 167-222 (2011)
[42] Pilloni, V., Stroh, B.: Surconvergence et classicite : le cas Hilbert. Preprint, available at http://perso.ens-lyon.fr/vincent.pilloni/
[43] Pilloni, V., Stroh, B.: Surconvergence et classicite : le cas deploye. Preprint, available at http://perso.ens-lyon.fr/vincent.pilloni/
[44] Sasaki, S.: Integral models of Hilbert modular varieties in the ramified case, deformations of modular Galois representation, and weight one forms. Preprint, available at http://www.cantabgold.net/users/s.sasaki.03/
[45] Shin, S. W.: On the cohomology of Rapoport-Zink spaces of EL-type, to appear in Amer. J. Math., available at http://math.mit.edu/~swshin/
[46] Taylor, R. L., Yoshida, T.: Compatibility of local and global Langlands correspondences. J. Amer. Math. Soc., 20-2, 467-493 (2007)
[47] Tian, Y.: Classicality of overconvergent Hilbert eigenforms: Case of quadratic residue degree. Preprint, available at http://arxiv.org/abs/1104.4583
[48] Tian, Y., Xiao, L.: p-adic cohomology and classicality of overconvergent Hilbert modular forms, available at http://math.uchicago.edu/~lxiao
[49] Tsuzuki, N.: On Base Change Theorem and Coherence in Rigid Cohomology. Documenta Mathematica, Extra vol., 89-198 (2003)
[50] Urban, E.: Eigenvarieties for Reductive Groups. Annals of Mathematics, vol. 174, no. 3, 1685-1784 (2011)
[51] Yoshida, T.: Betti Cohomology of Shimura varieties - the Matsushima formula. Notes, available at http://www.dpmms.cam.ac.uk/~ty245/2008_AGR_Fall/2008_agr_week2.pdf

Department of Mathematics, Imperial College London, London SW7 2AZ, UK
E-mail address: h.johansson09@imperial.ac.uk