Factored Notation for Interval I/O *

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Abstract

This note addresses the input and output of intervals in the sense of interval arithmetic and interval constraints. The most obvious, and so far most widely used notation, for intervals has drawbacks that we remedy with a new notation that we propose to call factored notation. It is more compact and allows one to find a good trade-off between interval width and ease of reading. We describe how such a trade-off can be based on the information yield (in the sense of information theory) of the last decimal shown.

1 Introduction

Once upon a time, it was a matter of professional ethics among computers never to write a meaningless decimal. Since then computers have become machines and thereby lost any form of ethics, professional or otherwise. The human computers of yore were helped in their ethical behaviour by the fact that it took effort to write spurious decimals. Now the situation is reversed: the lazy way is to use the default precision of the I/O library function. As a result it is common to see fifteen decimals, all but three of which are meaningless.

Of course interval arithmetic is not guilty of such negligence. After all, the very raison d’être of the subject is to be explicit about the precision of computed results. Yet, even interval arithmetic is plagued by phoney decimals, albeit in a more subtle way. Just as conventional computation often needs more care in the presentation of computational results, the most obvious interval notation with default precision needs improvement.

As a bounded interval has two bounds, say, $l$ and $u$, the most straightforward notation is something like $[l, u]$. Written like this, it may not be immediately obvious what is wrong with writing it that way. But when confronted with a real-life consequence of such notation, such as the

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1 As can be found on page 122 of [3].
statement that an unknown real $x$ belongs to

\[ [+0.6180339887498946804, +0.6180339887498950136], \]  

(1)

it becomes clear that this is not a practical notation.

This problem has been primarily addressed by Schulte, Zelov, Walster, and Chiriaev [6], who refer to the $[l, u]$ notation as the “conventional one”, and observe:

With conventional interval output, it is often difficult to determine the relative sharpness of interval results, . . . This is especially true when large amounts of interval data need to be examined.

1.1 The scanning problem

To start with, the notation exemplified by (1) requires one to scan both bounds to find the leftmost different digit. Only then does one have an idea of the width of the interval. In casting about for alternative notations, we note that an interval has more potentially useful attributes than the lower bound $l$ and the upper bound $u$. We could use the centre $c$, the radius $r$, or the diameter $d$. Any two independent attributes could be paired to give a notation for intervals. Of course, several such combinations are awkward. A reasonable alternative to pairing $l$ and $u$ is to combine $c$ with $r$. But then, in writing the pairs, we should drop the suggestive “[” and “]”. Thus we get $\langle c, r \rangle$. In this notation, for example, the interval (1) becomes:

\[ \langle 0.6180339887498946804, 0.6180339887498950136 \rangle. \]

(2)

The advantages of the $\langle c, r \rangle$ notation include solving the problem of having to scan for the leftmost differing digit. Also, it shows at a glance the width of the interval. While this is an improvement over (1), it is still awkward that instead of a long string of repeated digits, one now has a long string of zeros.

Factored notation as solution to the scanning problem

Now the problem with (1) is repeated occurrence of characters, much like the repeated occurrence of $a$ in the expression $ab + ac$. This analogous problem is solved by factoring, so that one gets $a(b + c)$. Consider now string concatenation as an infix multiplicative operation. That suggests “factoring” in (1) the longest initial sequence of common digits, so that

\begin{align*}
x \in [0.6180339887498946804, 0.6180339887498950136]\end{align*}

becomes

\begin{align*}
x \in 0.6180339887498946804[46804, 50136] \end{align*}

(3)

We call this the factored notation. It solves the scanning problem in a more compact way than the $\langle c, r \rangle$ notation.
1.2 The problem of useless digits

By solving the scanning problem, factored notation brings out the other problem: does one, in (3) for example, really want to know the width of an interval in a relative precision of $10^{-5}$? One should keep in mind that relaxing the relative precision in the width does not detract from the $10^{-14}$ relative precision in $x$ itself. If it is supremely important to preserve all the information that is available about $x$, then one should indeed preserve all the digits inside the brackets. But usually one does not want to keep all information: what matters is how much information the human reader can absorb in whatever limited time is available. By giving too much information about the width of the interval, it is usually the case that less information about $x$ is actually transferred. So one should present an approximation to the available information about $x$ that can be represented with fewer digits.

Now approximation is an ambiguous concept. Here we mean it in the sense of giving less information, under the constraint that what information remains truly is information, that is, a true statement. In this paper we assume that the purpose of using intervals at all is to obtain from a computation a true statement about the real-valued components of a solution. We assume that these statements say that a certain real belongs to an interval and that this interval is interpreted as a set of reals. However wide the interval $I$ is, there is nothing approximate about the statement $x \in I$ as such. Each bound, though a mere machine number, is also an unambiguously defined real. So $I$ is an unambiguously defined set. The relation $\in$ either holds or does not hold.

Computational results of the form $x \in I$ are made possible by correct rounding and by methods such as Interval Newton [2] or Interval Constraints [3] that take into account errors from all sources. Suppose that the $I$ in $x \in I$ is the raw computational result: an interval with binary floating-point numbers as bounds. For output, we need to represent it as a pair of decimal numerals in factored notation. In general, rounding is necessary in the transition between binary and decimal and care has to be taken to ensure that the rounding is outward.

Of course we want to keep the resulting interval as small as possible: the smaller, the more information about $x$. However, it may be that in the chosen output language for the interval (decimal numerals), the description of $I$ requires many numerals, in factored notation. Then we may prefer instead an output $x \in I'$, with $I \subseteq I'$, such that $I'$ has a simple representation. In that sense $I'$ is an approximation to $I$. But there is nothing approximate about the truth value of $I \subseteq I'$, and hence of $x \in I'$.

This suggests a sequence of intervals

$$I \subseteq I_1 \subseteq I_2 \subseteq I_3 \ldots$$

where each following interval requires fewer decimal digits than the one before.

As an illustration consider the interval that, in $[l, u]$ notation would appear as

$$[5.1268427635136, 5.1268472635136].$$

In the example the sequence $I, I_1, I_2, I_3, \ldots$, in factored notation, is as shown in Figure 1.2.
5.12684[27635136, 72635136] ⊆
5.12684[2763513, 7263514] ⊆
5.12684[276351, 726352] ⊆
5.12684[27635, 72636] ⊆
5.12684[2763, 7264] ⊆
5.12684[27, 8] ⊆

Figure 1: Successive approximations to an interval. Which is best?

2 Semantics of decimal numerals

We need decimal numerals to denote the bounds in an interval. Now, what exactly is meant by a given numeral? There seem to be two interpretations.

The first interpretation is appropriate in the context of physical measurements, where the convention is used that, in the absence of explicit qualification, all digits are significant. For example, when \( y = 0.123 \) is given as outcome of an experiment, the implication is that the fourth and later decimals may have any value. That is, no more is implied than \( 0.1230\infty \leq y \leq 0.1239\infty \) where \( \infty \) indicates infinite repetition of the preceding decimal.

This first interpretation has drastic consequences when we consider \( y^2 \), which cannot be given as \( 0.015129 \), because the last three digits are likely to be incorrect as the next digit after the ones given in 0.123 is not known.

One way of expressing this interpretation of \( y = 0.123 \) is to say that \( y \) belongs to a certain interval. It is natural to express this interval as \([a, b]\). But how to write \( a \) and \( b \)? If we write \([0.123, 0.124]\), then these numerals need to be interpreted in a different way.

Hence the second interpretation of a decimal numeral: the one that implies infinitely many zeros after the last decimal.

Thus, \( x = 0.d_1 \ldots d_k \) means either

\[
0.d_1 \ldots d_k 0\infty \leq x \leq 0.d_1 \ldots d_k 9\infty
\]

or

\[
x = 0.d_1 \ldots d_k 0\infty
\]

The first interpretation we call physics convention. It can be regarded as a restricted interval notation; one where only intervals can be denoted in which the bounds differ by one unit in the last decimal.

To implement our chosen semantics of intervals in general, we need to write decimals in the bounds. But for these the physics convention is not satisfactory; we need the latter
interpretation: bounds of intervals, in so far as they exist, are reals, not sets of reals. We call this the point convention.

3 How many digits for interval bounds?

According to Shannon’s theory of information, observations can reduce the amount of uncertainty about the value of an unknown quantity. The amount of information yielded by an observation is the decrease (if any) in the amount of uncertainty. Shannon argues that the amount of uncertainty is appropriately measured by the entropy of the probability distribution over the possible values. For a uniform distribution on a finite number of values, this reduces to the logarithm of the number of possible values. For an unknown real \( x \) that belongs to an interval we assume a uniform distribution over the interval and we take for the uncertainty the logarithm of the width of the interval.

The amount of information gained by resolving the choice between two equally probable alternatives is called the binary unit of information, or, the bit. This is widely used. Strictly analogously, but less widely used, is the decimal unit of information, or, the dit.

Before we look at intervals, let us evaluate, from an information-theoretic point of view, numerals interpreted according to the physics convention. There \( x = 0.123 \) means \( x \in [0.123, 0.124] \) with \( \log_{10} 0.001 = -3 \) as measure of uncertainty. Add one decimal, and the measure of uncertainty is reduced to \(-4\). The information yielded is one dit. At the same time, 0.123 leaves as uncertainty which the next more precise numeral is: 

\[ 0.1230 \text{ or } 0.1231 \text{ or } \ldots \text{ or } 0.1239. \]

These are ten possibilities, which are reduced to one by supplying the fourth decimal. Thus we see a perfect match between the expense (one dit) of adding a digit and the yield of information (one dit). Such a good match is rarely the case with interval notation.

In (4) we described how one can save on decimals while maintaining inclusion by widening the interval. To judge whether it is worthwhile to display all available decimals, let us determine the amount of information contained in the last decimal in each of the bounds for an interval.

Suppose all available decimals are as given in

\[ x \in 5.12684[27635136, 72635136]. \]

Let us consider the amount of information about \( x \) that is communicated by the successive digits within the brackets, starting with the second digit. If we only had one digit between the brackets, then we would write \( x \in 5.12684[2,8] \), with an uncertainty in \( x \) equal to \( \log_{10} 6 \times 10^{-6} = -6 + \log_{10} 6 \). The next digit gives \( x \in 5.12684[27,73] \), with an uncertainty of \( \log_{10} 46 \times 10^{-7} = -6 + \log_{10} 4.6 \). The decrease in uncertainty, that is the information, given by the last digit inside the brackets is \( \log_{10} 6 - \log_{10} 4.6 \), which is considerably less than 1.

Jumping to the final digit now: without it we have for the uncertainty in

\[ x \in 5.12684[27635136, 72635136] \]
an amount in dits equal to \( \log_{10} 4.500000 \times 10^{-6} \). If we had to do without that last digit, then all we can say is
\[
x \in 5.12684[2763513, 7263514]
\]
with an uncertainty equal to \(-\log_{10} 4.500001 \times 10^{-6}\).

Using the approximation that \( \log(1 + \alpha) \) is in the order of \( \alpha \) for small \( \alpha \), we find that the difference is in the order of \( 10^{-6} \) dits. This is the yield of writing the last digit in each bound. Ideally, the yield is one dit. With the physics convention for a decimal numeral, this maximum possible yield is guaranteed. As soon as we use the more flexible interval notation, we have to accept a lower yield. But accepting a yield as low as \( 10^{-6} \) dits is most situations far from optimal.

If one insists on the maximum yield possible with interval notation, one would write \( x \in 5.12684[2, 8] \), which is a perfectly good summary of what is known about \( x \). One may have sympathy for those who point out that the omitted decimals represent information that was gained at some, possibly considerable, computational cost. In that case, one would add one or two decimals, as in \( x \in 5.12684[276, 727] \). The next decimal will only add a negligible amount of information. Even though \( x \in 5.12684[27635136, 72635136] \) might represent the investment of huge computational resources, they would be wasted, as there just is hardly any more information than in the three-decimal version.

This is then our recommendation: use the factored notation to solve the scanning problem, and as default give two or three decimals inside the brackets.

### 4 The general case

In general one may have a bound written in scientific notation; that is, with an explicit power of ten. In case both bounds are so written, we propose to normalize one mantissa and to factor out its power of ten from both bounds. In the remaining bound, the decimal point is then adjusted so that its power of ten is zero, and can therefore be omitted.

For example,
\[
x \in [5.1268427635136 \times 10^2, 5.1268472635136 \times 10^3]
\]
becomes
\[
x \in [0.51268427635136, 5.1268472635136] \times 10^3
\]
Then the above considerations apply. They apply in the strongest way when both bounds have different normalized powers of ten; then we have the situation of a wide interval so that only few decimals should be used inside the brackets; \( x \in [0.51, 5.2] \times 10^3 \) seems plenty.

Suppose we write an interval as
\[
. d_0 \ldots d_{n-1}[e_0 \ldots e_k, f_0 \ldots f_k]
\]
then we have a width of about \( 10^{-(n+1)} \), neglecting the factor of \( \log(10) \). A unit in the last digits \( e_k \) and \( f_k \) can change the width by at most \( 10^{-(n+1+k)} \). Hence the width becomes about \( 10^{-(n+1)}(1+10^{-k}) \). Thus the uncertainty decreased by the last digit is about \( \log_{10}(1+10^{-k}) \), which is about \( 10^{-k} \). Hence our opinion that \( k \) equal to two or three is plenty.
5 Related work

Hansen [2], Hammer et al. [1], and Kearfott [5] opt for the straightforward \([l, u]\) notation. Hansen mostly presents bounds with few digits, but for instance on page 178 we find

\[ [0.192895419, 0.192895434], \]

demonstrating the problems addressed here.

The standard notation in the Numerica book [3] solves the scanning problem in an interesting way. It uses the idea of the \((c, r)\) notation, but writes instead \(c + [-r, +r]\). This variation has the advantage of not introducing new notation. The reason why we still prefer factored notation is clear from the \((c, r)\) example \([2]\), which, if rewritten as \(c + [-r, +r]\) becomes

\[ 0.6180339887498470 + [-0.0000000000000001666, +0.0000000000000001666] \]

Although it is attractive not to introduce special-purpose notation, there is so much redundancy here that the factored alternative:

\[ 0.6180339887498[46804, 50136] \]

seems worth the new notation.

Schulte et al. [6] describe single-number interval I/O as remedy for the scanning problem. This notation is used in several systems listed in [6]. Details differ between input and output. Here we sketch only the main features shared by single-number input and output. The three basic formats are \([d]\), \(d\), and \([d_1, d_2]\), where \(d, d_1, \) and \(d_2\) are decimal numerals. \([d]\) stands for the interval \([d, d]\). Suppose \(uld\) is the real number that is the value of the unit of the last decimal in \(d\), then the meaning of \(d\) in single-number interval notation is \([d - uld, d + uld]\). Finally, \([d_1, d_2]\) means just that; it appears that the tacit assumption is made of infinitely many zeros after the last digits of \(d_1\) and \(d_2\).

Single-number interval I/O can only solve the scanning problem by using the format \(d\) (single numeral without brackets). This implies a solution to the problem of useless digits, but more drastically than one might like. For example, 0.123[45678, 56789] would be rendered as 0.123. Thus the original interval is widened to 0.123[4, 6]. One might want to retain more information about the width of the interval. In factored notation one can choose among 0.123[4, 6], 0.123[45, 57], 0.123[456, 568], ....

In his interval system CLIP [4], Hickey has a form of what is called “single-number interval I/O” in [3], as well as several valuable distinctions not found elsewhere. Let \(uld\) be the real number that is the value of the unit of the last decimal in \(d\). The formats described by Hickey include \(d, d^*, \) and \(d \ldots\), where \(d\) is a decimal numeral.

\(d\) denotes the smallest interval that includes \(d\) and has binary floating numbers as bounds. \(d^*\) denotes \([d - uld, d + uld]\). \(d \ldots\) denotes \([d, d + uld]\). Finally, in CLIP, \(d^\#\) is used to denote, not an interval, but a binary floating-point number nearest \(d\). As in [3], the scanning problem is solved in CLIP. But the problem of useless digits is solved in the same inflexible way as in [3].
6 Conclusions

Interval methods are coming of age. When interval software was experimental, it didn’t matter whether interval output was easy to read. Now that the main technical challenges have been overcome, and we at least know how to ensure that the floating-point bounds include all reals that are possible values of the variable concerned, we need to turn our attention to small, mundane matters of housekeeping, which include taking care of the convenience of users. Factored notation seems to be an advance in this respect compared to alternatives described in the literature.

One may not agree with the assumptions made here for evaluating the utility of the last decimal used in specifying an interval bound. Information theory needs a probability distribution before it can assign an amount of information to an observation. The uniform distribution assumed here is not convincing. But it need not be: the amount of information in every next decimal declines so steeply, that most other assumptions would lead one to the same conclusion about a reasonable number of decimals inside the brackets of factored notation.

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References

[1] R. Hammer, M. Hocks, U. Kulisch, and D. Ratz. *C++ Toolbox for Verified Computing - Basic Numerical Problems*. Springer-Verlag, 1995.

[2] Eldon Hansen. *Global Optimization Using Interval Analysis*. Marcel Dekker, 1992.

[3] Pascal Van Hentenryck, Laurent Michel, and Yves Deville. *Numerica: A Modeling Language for Global Optimization*. MIT Press, 1997.

[4] Timothy J. Hickey. CLIP: A CLP(intervals) dialect for metalevel constraint solving. In *PADL2000*, pages 200–214. Springer-Verlag, 2000. Lecture Notes in Computer Science 1753.

[5] R. Baker Kearfott. *Rigorous Global Search: Continuous Problems*. Kluwer Academic Publishers, 1996. Nonconvex Optimization and Its Applications.

[6] Michael Schulte, Vitaly Zelov, G. William Walster, and Dmitri Chiriaev. Single-number interval I/O. In *Developments in Reliable Computing*, pages 141,148. Kluwer Academic Publishers, 1999.