A GENERALIZATION OF DIJKGRAAF-WITTEN THEORY

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ABSTRACT. The main purpose of this paper is to give a generalization of Dijkgraaf-Witten theory. For a given pairing of spectra \( \mu : E \otimes F \to G \), we construct a TQFT for \( E \)-oriented smooth manifolds using a representative of an \( F \)-cohomology class of the classifying space of a finite group. If \( E = H\mathbb{Z}, F = G = HU(1) \) and the pairing is induced by the \( \mathbb{Z} \)-module structure of \( U(1) \), then the TQFT reproduces Dijkgraaf-Witten theory. For the case that each of spectra \( E, F, G \) is given as the \( K \)-theory spectrum \( KU \), we further generalize our construction based on non-commutative settings.

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1. Introduction

The main goal of this paper is to construct a generalization of Dijkgraaf-Witten theory. Our strategy is to use a categorical group version of generalized (co)homology theory or $KK$-theory in order to construct a Lagrangian classical field theory which yields a TQFT via an integral.

Dijkgraaf-Witten invariant for $n$-manifolds is an invariant for oriented $n$-manifolds which is constructed from an ordinary cohomology class of the classifying space $B\Gamma$ of a finite group $\Gamma$. It is defined by taking an integral of a cobordism invariant for principal $\Gamma$-bundles [DW90], [Wak92], [FQ93], [Fre09], [DSFT09].

Dijkgraaf-Witten theory is a TQFT which extends the Dijkgraaf-Witten invariant: An ((n-1)+1)-dimensional TQFT consists of two assignments. It assigns a linear space to a closed (n-1)-manifolds and a linear map to an n-cobordism. We require that these assignments are compatible with each other in the sense that TQFT is given as a strong symmetric monoidal functor from n-cobordism category to a category of linear spaces [Ati88]. A functor $Z$ from a cobordism category to a linear category over a commutative field $F$ yields an invariant for top dimensional closed manifolds. The n-cobordism $\emptyset \to X^n \leftarrow \emptyset$ in the domain cobordism category determined by closed n-manifold corresponds to a morphism $Z(X^n) : Z(\emptyset) \to Z(\emptyset)$. Moreover if we have an isomorphism $Hom(Z(\emptyset), Z(\emptyset)) \cong F$, then $Z(X^n)$ produces a number under $I(X^n) \in F$. We say that the invariant $I(X^n)$ extends to a functor $Z$.

R. Dijkgraaf and E. Witten proposed a method to construct a (2+1)-dimensional Dijkgraaf-Witten theory starting from a cocycle of the classifying space $B\Gamma$ of a finite group $\Gamma$ [DW90]. M. Wakui constructed (2+1)-dimensional TQFT in a rigorous way based on Dijkgraaf-Witten’s idea [Wak92]. D. Freed and F. Quinn generalized Dijkgraaf-Witten theory to a higher dimensional TQFT starting from a cocycle of $B\Gamma$ [FQ93]. They also simplified the construction of the topological action by using a canonical integral of an n-cocycle on an closed oriented (n-1)-manifold, which is valued at some torsors. They refer to the topological action as a Lagrangian classical field theory. A. Sharma and A. A. Voronov reformulated Dijkgraaf-Witten theory using a categorical framework [SV17]. The canonical integral introduced by Freed and Quinn is well-understood under the categorical framework of Sharma and Voronov. G. Heuts and J. Lurie generalized Dijkgraaf-Witten theory focusing on the ambidexterity of the target category of local systems on spaces of connections [GH14].

We introduce a generalized Dijkgraaf-Witten invariant using a generalized cohomology class of $B\Gamma$, not necessarily an ordinary cohomology class. Some of main ingredients for the construction are a ring spectrum $E$, two spectra $F, G$, a pairing of spectra $\mu : E \wedge F \to G$, and an $F$-cohomology class of $[\alpha] \in F^n(B\Gamma)$. We define an invariant $Z_{\varphi, [\alpha]}$ for closed $E$-oriented $n$-manifolds where we explain $\varphi$ in subsection 3.1. It gives a generalization of Dijkgraaf-Witten invariant in the sense that our result is reduced to Dijkgraaf-Witten invariant if we consider $E = HZ, F = G = HU(1), \alpha$ the Eilenberg-Maclane spectra, and the pairing $HZ \wedge HU(1) \to HU(1)$ obtained from the module structure $Z \otimes U(1) \to U(1)$.

For a ring spectrum $E$, we define a cobordism category $\text{Cob}^E(n)_0$. We construct a TQFT $Z_{\varphi, \alpha}$ which is defined on the category $\text{Cob}^E(n)_0$ and extends the invariant $Z_{\varphi, [\alpha]}$. Here, $\alpha$ is a representative of the class $[\alpha] \in F^n(B\Gamma)$.

On the one hand, we also introduce another invariant $Z_{\varphi, [\beta]}$ for closed $KU$-oriented $n$-manifolds. Here, $[\beta]$ is a $KK$-theory class in $KK(A, C(B\Gamma) \otimes B)$ (see subsection 3.2) where $A, B$ are $C^\ast$-algebras. One of its advantage is that it becomes possible to cooperate with non-commutative settings. In addition, it gives a generalization of the previous invariant $Z_{\varphi, [\alpha]}$...
under the condition \( E = F = G = KU \). In fact, \( Z_p[a] \) coincides with \( Z_p[a] \) if \( A = S^n(= C_0(\mathbb{R}^n)), B = \mathbb{C} \) and the class \([a]\) corresponds to the class \([p]\) under the homomorphism \( KU^n(BT) \to KK(S^n, C(BT)) \).

We construct a TQFT \( Z_{\phi,\beta} \) which is defined on the category \( \text{Cob}^{KU}(n) \) and extends the invariant \( Z_{\phi,\beta} \). Here, \( \beta \) is a representative of the class \([\beta]\) \( \in KK(A, C(BT) \otimes B) \).

Since the invariants \( Z_{\phi,a} \) and \( Z_{\phi,\beta} \) coincide to each other under some conditions mentioned before, the dimension of linear spaces assigning to \((n-1)\) closed manifolds coincide to each other. In fact, \( Z(Y^{n-1} \times S^1) \) coincides with the dimension of the linear space \( Z(Y^{n-1}) \) if \( Z = Z_{\phi,a}, Z_{\phi,\beta} \) is defined on a closed \((n-1)\)-manifold \( Y^{n-1} \). Nonetheless, we do not have a natural isomorphism between \( Z_{\phi,a} \) and \( Z_{\phi,\beta} \). It seems necessary to construct a ‘map’ between classes of representatives of \( KU^n(BT) \) and \( KK(S^n, C(BT)) \) in an appropriate way, but we do not yet have such a ‘map’.

Note that the TQFT we construct is partial in some sense unless the generalized (co)homology theory satisfies some condition which ordinary (co)homology theory satisfies automatically. In fact, the object class of its domain cobordism category \( \text{Cob}^E(n)_0 \) consists of \((n-1)\)-manifolds \( Y^{n-1} \) such that \( E_n(Y^{n-1}) \cong 0 \) where \( n \) is the top dimension of manifolds we deal with. The partiality is mainly due to the following two problems. Firstly, we do not know what is the right way to take an integral of our classical field theories with respect to representatives of the fundamental class. For example, the homotopy fiber of the restriction functor \( [[X]]_E \to [[\partial X]]_E \) introduced in section 6 may have infinitely many components, so that the push-forward along the restriction functor is ill-defined in general. Secondly, the groupoid functor \( [[-]]_E \) is not gluable, i.e. the groupoids obtained by applying the functor \( [[-]]_E \) may not form a homotopy pull-back diagram even if the manifolds form a push-forward diagram in some sense (see subsection 6.3).

It is well-known that \( H_{n-k}(X; M) \cong 0 \) if \( X \) is a \( k \)-manifold where \( M \) is an abelian group. Hence, if \( E = HM \), the Eilenberg Maclane spectrum associated with the abelian group \( M \), \( \text{Cob}^E(n)_0 \) consists of any manifolds which are oriented under \( M \)-coefficients. In particular, a pairing of abelian groups \( M \otimes N \to P \) may be used to obtain a pairing \( \mu : HM \wedge HN \to HP \) which may yield a TQFT for manifolds which are oriented under \( M \)-coefficients. Note that if \( M = \mathbb{Z}/2\mathbb{Z} \), then a manifold which are oriented under \( M \)-coefficients is nothing but a manifold, which needs not be orientable in any sense.

We basically follow the approach of Sharma and Voronov. We formulate a categorical framework for generalized (co)homology theory and \( KK \)-theory. The categorical group version of singular (co)homology theory applied by A. Sharma and A. A. Voronov is due to A. del Río, J. Martínez-Moreno, E. M. Vitale [AdR05]. We introduce a categorical group version of generalized (co)homology theory and \( KK \)-theory based on a fundamental groupoid of certain simplicial sets. Sharma and Voronov constructed a Lagrangian classical field theory using the cap product associated with that categorical (co)homology theory. We use a pairing of generalized (co)homology theories or the Kasparov product of \( KK \)-theory to construct a Lagragian classical field theory.

In this paper, we use the notion of spectrum and smash product following Adams [Ada95], Switzer [Swi77]. \( E = \{E_n, s_n\} \) is a spectrum if \( E_n \) is a CW-complex and \( s_n : \Sigma E_n \to E_{n+1} \) is a CW-embedding for \( n \in \mathbb{Z} \). There are some natural equivalences under which the smash product is associative, commutative, and has the sphere spectrum as a unit up to coherent natural equivalence.

A. del Río, J. Martínez-Moreno, E. M. Vitale [AdR05] call by (co)homology categorical group the categorical group version of (co)homology theory induced by a chain complex of categorical groups. We follow the convention by calling by generalized (co)homology categorical group (resp. \( KK \) categorical group) the categorical group version of generalized (co)homology group (\( KK \)-group).
The organization of this paper is as follows. In section 2, we introduce some notations which appear very often throughout this paper. In section 3, we define generalized Dijkgraaf-Witten invariants. In section 4, we give the main results of this paper without precise meaning of notations. From section 5 to section 10, we construct a TQFT based on generalized (co)homology theory. In section 11, we introduce a symmetric categorical group version of generalized (co)homology theory. In section 12, we introduce a groupoid consisting of representatives of the fundamental class associated with an $E$-oriented manifold. In section 13, we prove some lemmas for construction of Lagrangian classical field theories based on abstract settings. In section 14, we construct a TQFT based on $KK$-theory. In section 15, we give computations of the TQFT’s for the untwisted case. In section 16, we give a remark about the symmetric monoidal functors $\hat{\phi}, \hat{\psi}$ which are used to construct the TQFT’s. In Appendix A, we prepare several notions and useful facts in category theory which is often used in this paper.

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2. Notations

- Let $X, Y$ be spaces. We denote by $Map(X, Y)$ the set of maps from $X$ to $Y$. We denote by $MAP(X, Y)$ the simplicial set whose $k$-skeleton $MAP(X, Y)_k$ is given by

\[ Map(X \times \triangle (k), Y), \]

where $\triangle (k)$ is the $k$-simplex. Then the face maps and degeneracy maps on simplices induce face maps and degeneracy maps of $MAP(X, Y)_k$’s. Similarly, we define a simplicial set $MAP_*(X, Y)$ for based spaces $X, Y$ where the $k$-skeleton is given by

\[ Map_*(X \wedge (\triangle (k))^+, Y). \]

- About bicategories, we follow the terminologies introduced in [Lei98]. We introduce following notations of several categories.
  - $Gpd$ is the 2-category of groupoids, functors and natural transformations.
  - $C^*$ is the 2-category whose objects are given as $C^*$-algebras and the morphism category $C^*(A, B)$ is given as the fundamental groupoid $\Pi_1HOM(A, B)$ for $C^*$-algebras $A, B$.
  - $CW_*$ is the 2-category whose objects are given as based CW-spaces and the morphism category $CW_*(X, Y)$ is given as the fundamental groupoid $\Pi_1MAP_*(X, Y)$ where $MAP_*(X, Y)$ is the simplicial set consisting of based maps from $X$ to $Y$. 

3.1. An invariant $Z_{φ,[α]}$ from generalized (co)homology theory. In this paper, we use the notion of spectrum following Adams [Ada95], Switzer [Swi17]. The notion of spectra is actually not that essential in this section since we only need the generalized (co)homology theories induced by spectra. Nonetheless, we use it in order to relate the invariants defined in this section with TQFT’s later.
Let $E$ be a ring spectrum. We apply an integral of an $n$-th $E$-cohomology class on an $E$-oriented $n$-manifold by using its $E$-orientation. We can consider more general setting. Suppose that we are given a pairing of spectra $\mu : E \wedge F \to G$ where $F, G$ are also spectra. The pairing $\mu$ induces a pairing $\langle \cdot , \cdot \rangle : E_\ast(X^n) \otimes F^n(X^n) \to G_0(\text{pt})$. Let $[\alpha] \in F^n(B\Gamma)$ be an $F$-cohomology class of the classifying space.

**Definition 3.1.** For an $E$-oriented closed $n$-manifold $X^n$ and a continuous map $f : X^n \to B\Gamma$, we define

$$S_{[\alpha]}(X^n, f) \overset{\text{def}}{=} \langle [X^n]_E, f_\ast([\alpha]) \rangle \in G_0(\text{pt}),$$

where $[X^n]_E \in E_n(X^n)$ is the fundamental class of the $E$-oriented manifold $X^n$.

We may observe that $S_{[\alpha]}(X^n, f)$ is a $G_0(\text{pt})$-valued cobordism invariant, i.e. we have

$$S_{[\alpha]}(X^n_0, f_0) = S_{[\alpha]}(X^n_1, f_1) \in G_0(S^0)$$

if $W^{n+1}$ is an $E$-oriented compact $(n+1)$-manifold such that $\partial W^{n+1} \cong (X^n_0 \sqcup X^n_1)$ as $E$-oriented manifolds, and $f_0 \sqcup f_1 \overset{g}{\approx} g \circ \partial W^{n+1}$ for a continuous map $g : W^{n+1} \to B\Gamma$.

We define Dijkgraaf-Witten invariant for closed $E$-oriented manifolds as follows.

**Definition 3.2.** Let $\varphi : G_0(\text{pt}) \to U(1)$ be a group homomorphism. For an $E$-oriented closed $n$-manifold $X^n$, we define

$$Z_{\varphi, [\alpha]}(X^n) \overset{\text{def}}{=} \sum_{f : X^n \to B\Gamma} \frac{\varphi(S_{[\alpha]}(X^n, f))}{\# \text{Aut}(f_\ast(ET))} \in \mathbb{C},$$

Here, $f$ runs on the set of homotopy classes of continuous maps. It is nothing but a finite sum since we have

$$[X^n, B\Gamma] \cong \text{Hom}(\pi_1(X^n), \Gamma), \quad X^n : \text{connected},$$

where $\pi_1(X^n)$ is finite since $X^n$ is a compact manifold [KLCS+69].

Then the cobordism invariance collapses since we are taking a sum to construct $Z_{\varphi, [\alpha]}(X^n)$. Nonetheless, the invariant $Z_{\varphi, [\alpha]}$ inherits some properties from $S_{[\alpha]}$. Note that $S_{[\alpha]}(X^n_0, f_0) + S_{[\alpha]}(X^n_1, f_1) = S_{[\alpha]}(X^n_0 \sqcup X^n_1, f_0 \sqcup f_1) \in G_0(S^0)$ due to the bilinearity of the pairing $\langle \cdot , \cdot \rangle : E_n(X^n) \times F^n(X^n) \to G_0(\text{pt})$. Due to this additivity, we obtain

$$Z_{\varphi, [\alpha]}(X^n_0) \cdot Z_{\varphi, [\alpha]}(X^n_1) = Z_{\varphi, [\alpha]}(X^n_0 \sqcup X^n_1) \in \mathbb{C}.$$}

It will be lifted to the symmetric monodromy of our TQFT functor.

### 3.2. An invariant $Z_{\varphi, [\alpha]}$ from $KK$-theory.

*K*-theory for $C^*$-algebras and analytic $K$-homology are generalized into a common framework called $KK$-theory [Bla98]. $KK$-theory is a bivariant functor $(A, B) \mapsto KK(A, B)$ from the category of $C^*$-algebras and homomorphisms to the category of abelian groups and homomorphisms, which is contravariant with respect to $A$ and covariant with respect to $B$.

We have a natural isomorphism $KU_k(Z, Z') \cong KK(C_0(Z, Z'), S^k)$ for a pair of finite CW complexes $(Z, Z')$ (Theorem 4 [Kas75]). Here the left hand side is the homology theory defined by the spectrum $KU$. If $Z'$ is a compact $KU$-oriented $k$-manifold, then its $KU$-orientation lives in the $KK$-group $KK(C(Z' \setminus \partial Z'), S^k)$. We denote it by $[Z', Z]_{akU}$. We use it to apply an integral to a $KK$-theory class of a $KU$-oriented manifold.

As a technical assumption, we fix a CW complex structure of the classifying space $B\Gamma$, whose $r$-skeleton $B\Gamma^{(r)}$ is compact and Hausdorff for every $r \in \mathbb{Z}^\geq 0$. Since a smooth $k$-manifold $Z^k$ has a $k$-dimensional CW complex structure (more strongly, a triangulation), the cellular approximation theorem induces an equivalence of groupoids,

$$\Pi_1 \text{MAP}(Z^k, B\Gamma^{(r)}) \to \Pi_1 \text{MAP}(Z^k, B\Gamma), \quad r \geq k + 2.$$
We fix a large \( r \geq n + 3 \) where \( n + 1 \) is the largest dimension of manifolds which we deal with in this paper. Let us denote by \( \Pi_1 \text{MAP}^f(Z^k, B\Gamma) \overset{\text{def}}{=} \Pi_1 \text{MAP}(Z^k, B\Gamma^{(r)}) \). For such \( r \), we denote
\[
\text{KK}(A, C(B\Gamma) \otimes B) \overset{\text{def}}{=} \text{KK}(A, C(B\Gamma^{(r)}) \otimes B).
\]
This assumption is necessary since the *-algebra formed by \( \mathbb{C} \)-valued continuous maps on \( B\Gamma \) does not induce a \( \mathbb{C}^* \)-algebra in general. It will be used again in section 13.

Let \( A, B \) be \( \mathbb{C}^* \)-algebras and let \( [\beta] \in \text{KK}(A, C(B\Gamma) \otimes B) \). We define an action functional for principal \( \Gamma \)-bundles over closed \( KU \)-oriented \( n \)-manifolds.

**Definition 3.3.** Let \( X^n \) be a closed \( KU \)-oriented \( n \)-manifold. For an object \( f \in \Pi_1 \text{MAP}^f(X^n, B\Gamma) \), we define
\[
S_{[\beta]}(X^n, f) \overset{\text{def}}{=} [X^n]_{\alpha \mathbb{K}} \otimes_{C(X^n)} f^*([\beta]) \in \text{KK}(A, S^n B)
\]
where \( \otimes_{C(X^n)} \) is the Kasparov product contracting \( C(X^n) \).

As the invariant \( S_{[\alpha]}(X^n, f) \) does, \( S_{[\beta]}(X^n, f) \) is a \( \text{KK}(C(X^n), S^n) \)-valued cobordism invariant.

We define Dijkgraaf-Witten invariant for closed \( KU \)-oriented manifolds as follows.

**Definition 3.4.** Let \( \psi : \text{KK}(A, S^n B) \rightarrow U(1) \) be a group homomorphism. Let \( X^n \) be a closed \( KU \)-oriented \( n \)-manifold. We define
\[
Z_{\psi,[\beta]}(X^n) \overset{\text{def}}{=} \sum_{[f] \in \mathbb{Z}} \psi(S_{[\beta]}(X^n, f)) \in \mathbb{C},
\]
Here, \( f \) runs on the set \( \pi_0(\text{MAP}^f(X^n, B\Gamma)) \overset{\text{def}}{=} [X^n, B\Gamma^{(r)}] \). It is also a finite sum due to the same reason explained in Definition 3.2.

Although the cobordism invariance collapses, the invariant \( Z_{\psi,[\beta]} \) inherits some properties from \( S_{[\beta]} \) as \( Z_{\psi,[\alpha]} \) does. See the final paragraph of subsection 3.1.

### 4. Main results

In this section, we give our main results without giving precise definition of notations. In subsection 4.1, we give one of our main theorems that the invariant \( Z_{\psi,[\alpha]} \) extends to a TQFT \( Z_{\psi,\alpha} \) where \( \varphi = \pi_1(\hat{\psi}) \). We give explanation that it yields the Dijkgraaf-Witten theory [FQ93] as its corollary. In subsection 4.2, we give our other main theorem that the invariant \( Z_{\psi,[\beta]} \) extends to a TQFT \( Z_{\psi,\beta} \). In subsection 4.3, we summarize their constructions.

#### 4.1. A TQFT \( Z_{\psi,\alpha} \) from generalized (co)homology theory
Let \( E \) be a ring spectrum again.

We denote by \( \text{Cob}^E(n)_0 \) a cobordism category of \( E \)-oriented smooth manifolds whose top dimension is \( n \in \mathbb{Z}_{\geq 0} \). Its object class consists of \((n-1)\)-dimensional closed \( E \)-oriented smooth manifolds \( Y^{n-1} \) such that
\[
E_n(Y^{n-1}) \cong 0.
\]

Here, \( E_n \) denotes the \( n \)-th \( E \)-homology group. The morphism class of the category \( \text{Cob}^E(n)_0 \) consists of \( n \)-dimensional smooth \( E \)-oriented cobordisms connecting such objects. Due to the additivity of \( E \)-homology with respect to disjoint union of spaces, the disjoint union of \( E \)-oriented manifolds induces a symmetric monoidal structure on the category \( \text{Cob}^E(n)_0 \).

In section 6, we construct a delooping of the abelian group \( G_0(\text{pt}) \) which is given by a symmetric categorical group \( \mathcal{S}(= \mathcal{H}(S^n, S^{n+1}; G)) \) such that
\[
\pi_1(\mathcal{S}) \cong G_0(\text{pt}),
\]
as abelian groups. Then our main theorem is stated as follows.

**Main theorem 1.** (see Corollary [10.3]) Let $\alpha$ be a representative of the $F$-cohomology class $[\alpha] \in F^n(B\Gamma)$. Let $\hat{\phi} : \mathcal{S} \to U(1)\text{Tor}$ be a symmetric monoidal functor such that $\pi_1(\hat{\phi}) = \phi : G_0(pt) \to U(1)$. The invariant $Z_{\phi,\alpha}$ for closed $E$-oriented $n$-manifolds extends to a strong symmetric monoidal functor,

$$Z_{\hat{\phi},\alpha} : (\text{Cob}^E(n)_0, \eta) \to (\text{Vect}, \otimes).$$  

(14)

It is a generalization of the Dijkgraaf-Witten theory [DW90], [FQ93]: We substitute the following spectra into $E,F,G$ in Theorem 8.9. 

(15) \[ E = H\mathbb{Z}, \quad F = HU(1), \quad G = HU(1). \]

Here $HM$ is the Eilenberg MacLane spectrum associated with an abelian group $M$. Then the $\mathbb{Z}$-module structure of $U(1)$ induces a pairing $\mu : H\mathbb{Z} \wedge HU(1) \to HU(1)$. Let $\varphi : U(1) \cong G_0(pt) = H_0(pt; U(1)) \to M = U(1)$ be the identity. Then it is lifted to an equivalence of symmetric categorical groups, $\varphi : \mathcal{S} \to M\text{Tor}$ where $M = U(1)$ (see section 15.6). Under these assumptions, the strong symmetric monoidal functor $Z_{\varphi,\alpha} : \text{Cob}^k(n)_0 \to \text{Vect}$ gives the Dijkgraaf-Witten theory. In fact, the cobordism category $\text{Cob}^k(n)_0$ is isomorphic to the cobordism category of oriented manifolds since any closed $k$-manifold $Z^k$ satisfies $H_{k+1}(Z^k; \mathbb{Z}) \cong 0$. In addition, it is obvious by definition that the invariant $Z_{\varphi,\alpha}$ gives the Dijkgraaf-Witten invariant associated with the ordinary cohomology class $[\alpha] \in H^n(B\Gamma; U(1))$.

4.2. A TQFT $Z_{\hat{\phi},\beta}$ from $KK$-theory. Recall that we denote by $\text{Cob}^{KU}(n)_0$ a cobordism category of $KU$-oriented smooth manifolds whose top dimension is $n \in \mathbb{Z}^>0$. Its object class consists of $(n-1)$-dimensional closed $KU$-oriented smooth manifolds $Y^{n-1}$ such that $K_n(Y^{n-1}) \cong 0$. 

(16) \[ K_n(Y^{n-1}) \]  

Here, $K_n(Y^{n-1})$ denotes the $n$-th $K$-homology group of $Y^{n-1}$ where $K_\ast$ denotes the analytic $K$-homology. Note that we have isomorphisms between analytic and topological $K$-homology theories for finite CW complexes. The morphism class of the category $\text{Cob}^{KU}(n)_0$ consists of $KU$-oriented smooth $n$-cobordisms connecting such objects. Due to the additivity of $K$-homology with respect to disjoint union of spaces, the disjoint union of manifolds induces a symmetric monoidal structure on the category $\text{Cob}^{KU}(n)_0$.

Let $A, B$ be $C^\ast$-algebras. In section 11 we construct a delooping of the abelian group $KK(A,S^nB)$ which is given as a symmetric categorical group $\mathcal{S}(= \mathcal{K}(A, S^{n+1}B))$ such that $\pi_1(\mathcal{S}) \cong KK(A, S^nB)$, 

(17) \[ \pi_1(\mathcal{S}) \cong KK(A, S^nB), \]  

as abelian groups.

**Main Theorem 2.** (see Corollary [14.3]) Let $A, B$ be $C^\ast$-algebras. Let $\beta$ be a representative of the $KK$-theory class $[\beta] \in KK(A, C(B\Gamma) \otimes B)$. Let $\hat{\phi} : \mathcal{S} \to U(1)\text{Tor}$ be a symmetric monoidal functor such that $\pi_1(\hat{\phi}) = \phi : KK(A, S^nB) \to U(1)$. The invariant $Z_{\phi,\beta}$ extends to a strong symmetric monoidal functor,

$$Z_{\hat{\phi},\beta} : (\text{Cob}^{KU}(n)_0, \eta) \to (\text{Vect}, \otimes).$$  

(18)

4.3. Summary of constructions. The construction of TQFT’s here is based on ‘finite path integral’ of our Lagrangian classical field theories. It is helpful to recall the definition of the invariant $Z = Z_{\varphi,\alpha}$ or $Z_{\phi,\beta}$. For each closed manifold $X^n$ which has an appropriate orientation with respect to some spectrum, we are given a set of finite gauge fields over which a Lagrangian classical field theory (or a topological action functional) $S = S_{\alpha}$ or $S_{\beta}$
is defined. Then \( Z(X^n) \) is defined via an integral of the functional \( S \) over finite gauge fields. Our strategy to construct TQFT’s is parallel to the construction of the invariant \( Z \). We extend the action functional \( S \) down to codimension one and apply an integral to \( \tilde{S} \).

In order to obtain an object with higher structures assigning to codimension-one manifolds, we start with a categorical framework of abelian groups, i.e. symmetric categorical groups. As generalized (co)homology groups or KK-theory groups are used to define the Lagrangian classical field theory for top dimensional closed manifolds, we use a categorical group version of generalized (co)homology theory or KK-theory to construct a Lagrangian classical field theory for codimension-one manifolds. Then values which the classical field theory takes live in a symmetric categorical group, not an abelian group. Let us write the symmetric categorical group as \( \mathcal{G} \) for a moment. On the other hand, it gives an extension of the previous classical field theory \( S \). In fact, any closed manifolds are considered as cobordism from the null set to itself so that they correspond to a morphism in the category \( \mathcal{G} \) via \( S \). Since the null sets correspond to the unit of \( \mathcal{G} \), the corresponding morphism yields a number in the automorphism group of the unit of \( \mathcal{G} \), which coincides with the value of \( S \).

We lift the generalized (co)homology groups and KK-theory groups to categorical groups as follows. Both of generalized (co)homology groups and KK-theory groups can be defined as a homotopy set \( \pi_0(Mor(a,b)) \) of a space \( Mor(a,b) \) for appropriate object \( a, b \), which are spectra for the former case and \( C^* \)-algebras for the latter case. Our categorical group version of generalized (co)homology groups and KK-theory groups are constructed on the fundamental groupoid \( \Pi_1 Mor(a,b) \).

The main feature of construction of classical field theory is that we use a natural homotopy to construct a morphism assigning to a cobordism. Let \( Y^{n-1}_0 \rightarrow X^n \leftarrow Y^{n-1}_1 \) be a \( n \)-cobordism of manifolds. It induces a continuous map \( Y^{n-1}_0 \uparrow Y^{n-1}_1 \rightarrow X^n \). We put its mapping cone as \( C \) which is defined by gluing basepoints of cones \( CY^{n-1}_i \) in the following space,

\[
CY^{n-1}_0 \bigcup_{Y^{n-1}_0 \rightarrow Y^{n-1}_1} X^n \bigcup_{Y^{n-1}_1 \rightarrow CY^{n-1}_1} CY^{n-1}_1.
\]

Here we denote by \( CZ \) a cone of a space \( Z \),

\[
(20) \quad CZ = Z \times [0, 1]/Z \times \{1\}.
\]

We define maps \( d_i : C \rightarrow \Sigma(Y^{n-1}_i) \) for \( i = 0, 1 \) where \( \Sigma \) denotes the suspension. \( d_0 \) collapses \( X^n \), \( CY^{n-1}_0 \) and assigns \( (y, s) \in \Sigma(Y^{n-1}_0) \) to \( (y, s) \in CY^{n-1}_0 \subset C \). \( d_1 \) collapses \( X^n \), \( CY^{n-1}_1 \) and assigns \( (y, s) \in \Sigma(Y^{n-1}_1) \) to \( (y, 1-s) \in CY^{n-1}_1 \subset C \). Then we have a homotopy \( \Phi_{01} : u_0 \circ d_0 \rightarrow u_1 \circ d_1 \):

\[
\begin{array}{ccc}
\Sigma(X^n) & \xrightarrow{u_0} & \Sigma(Y^{n-1}_0) \\
\Sigma(Y^{n-1}_1) & \xleftarrow{\Phi_{01}} & \Sigma(Y^{n-1}_1) \\
d_0 & \xleftarrow{\downarrow} & C \\
d_1 & \xrightarrow{\downarrow} & \end{array}
\]

Let us construct a homotopy in the above diagram. We define \( \Phi_{01}(x, t), (x, t) \in C \times [0, 1] \) by

- \( x = (y, s) \in CY^{n-1}_0 \rightarrow (y, (1-t)s + t) \in \Sigma(X^n) \)
- \( x \in X^n \rightarrow (x, t) \in \Sigma(X^n) \)
- \( x = (y, s) \in CY^{n-1}_1 \rightarrow (y, t(1-s)) \in \Sigma(X^n) \)

Now let us suppose that we are given a natural pairing \( A(Z) \times B(Z) \rightarrow C \) of groupoids for a space \( Z \) (see Definition 7.1). The canonical pairing associated with categorical group versions of generalized (co)homology theory and KK-theory are such examples. The pairing induces
following natural transformation between \( f_0 \circ g_0 \) and \( f_1 \circ g_1 \).

\[
\begin{array}{ccc}
A(\Sigma (Y_{0}^{n-1})^+) \times B(\Sigma (Y_{0}^{n-1})^+) & \xrightarrow{\psi_{01}} & A(\Sigma (Y_{1}^{n-1})^+) \times B(\Sigma (Y_{1}^{n-1})^+) \\
\downarrow f_0 \downarrow & & \downarrow f_1 \downarrow \\
A(C) \times B(\Sigma (X^n)^+) & \xrightarrow{g_0} & A(\Sigma (Y_{1}^{n-1})^+) \times B(\Sigma (X^n)^+).
\end{array}
\]

It plays a role of a precursor for our classical field theories since, in applications, a pair of a representative of fundamental class and a representative of cohomology class (given as a pull-back of a representative of cohomology class of a classifying space) lives in \( A(Z) \times B(W) \) where \( Z = \Sigma (Y_{0}^{n-1})^+, C, \Sigma (Y_{1}^{n-1})^+ \) and \( W = \Sigma (Y_{0}^{n-1})^+, \Sigma (X^n)^+, \Sigma (Y_{1}^{n-1})^+ \) respectively.

Although the above description is sufficient as an idea, here is a technical difficulty: we need to show the gluing property. If we are given cobordisms decorated with gauge fields, then the classical field theory assigns morphisms in the groupoid \( \mathcal{G} \) to them. We need to show that the composition of these morphisms coincides with the morphism assigned to the gauge field obtained by gluing them. Then we shall take one more ‘suspension’ of each manifolds in order to show the gluing property. Its abstract description is given in section \[7\] .

We apply an integral to the extend classical field theory \( S \) as the invariant \( Z = Z_{\phi, [a]} \) or \( Z_{\phi, [b]} \) is defined by applying an integral to the classical field theory \( S \) for top dimensional closed manifolds. Although the integral here is nothing but the ‘finite path integral’ \[\text{DSFT09}, \] we deal with it under our conventions where we use the terminology ‘push-forward’ instead in order to discuss reasons that the TQFT’s constructed here are partial in some sense.

5. GENERALIZED (CO)HOMOLOGY CATEGORICAL GROUP

In this section, we construct a categorical group version of generalized (co)homology theory as the fundamental groupoid of a space of morphisms between spectra where its 0-th homotopy set gives the generalized (co)homology group. The abelian group structure on generalized (co)homology group is lifted to the groupoid as a symmetric categorical group structure. As mentioned in subsection \[5.1\], we follow \[\text{Ada95}, \text{Swi17} \] with respect to the definitions of spectra and smash product.

We make use of a seemingly different approach to formulate generalized (co)homology theory from spectra. Let \( E \) be a spectrum. For based spaces \( X, Y \), we denote by \( HH(X, Y; E) \) the homotopy set of morphisms from the spectrum \( \Sigma^c X \) to the spectrum \( E \wedge \Sigma^c Y \) where \( \Sigma^c Z \) denote the suspension spectrum of a based space \( Z \). Then it becomes an abelian group due to the comultiplication on spheres. By definition, we have

\[
\begin{align*}
HH(X, S^n; E) & \cong \tilde{E}^n(X), \\
HH(S^n, Y; E) & \cong \tilde{E}^n(Y).
\end{align*}
\]

Here \( \tilde{E}^n, \tilde{E}_* \) are reduced \( E \)-cohomology and \( E \)-homology respectively. This formulation is not that essential for our purpose since \( HH(X, Y; E) \) is isomorphic to a (usual) generalized cohomology theory of \( X \) if we fix \( Y \). Nonetheless, it is more convenient when we deal with the pairing of generalized (co)homology theories induced by a pairing between spectra since we do not need to care about ‘degrees’ of (co)homology theory. It is rather ‘morphism-like’, especially if we consider a ring spectrum \( E \) then it is really possible to construct a category of based spaces and \( HH(X, Y; E) \) as a morphism set between \( X, Y \).

In subsection \[5.1\], we introduce a simplicial set consisting of morphisms between spectra, and discuss some properties of its fundamental groupoid. In subsection \[5.2\], we define a bivariant functor valued at symmetric categorical groups with canonical inverses. It will be written by \( \mathcal{H}(X, Y; E) \) where \( E \) is a spectrum and \( X, Y \) are based spaces. If we consider
a sphere as an one side of variables, it produces a categorical group version of (reduced) generalized (co)homology theory.

5.1. Preliminaries. We say that $E = \{E_n, s_n\}$ is a spectrum if $E_n$'s are based CW complexes and $s_n : \Sigma E_n \to E_{n+1}$'s are based CW-embeddings. For two spectra $E, F$, a map is a family of cellular maps $f_n : E_n \to F_n$ between CW-complexes which intertwines the structure maps $s_n$'s. For two spectra $E, F$, a morphism is an equivalence class of a pair $(f, E') : E \to F$ where $E' \subset E$ is cofinal subspectrum and $f : E' \to F$ is a map between spectra. We denote by $\text{Mor}(E, F)$ the set of morphisms from $E$ to $F$.

**Definition 5.1.** We define a simplicial set $\text{MOR}(E, F)$ for two spectra $E, F$. Its $n$-skeleton $\text{MOR}(E, F)_n$ is defined by

$$\text{MOR}(E, F)_n \overset{\text{def.}}{=} \text{Mor}(E \wedge (\Delta(n))^+, F).$$

Here $E \wedge (\Delta(n))^+ = \{E_n \wedge (\Delta(n))^+\}$ is the smash product of a spectrum and a based space. Then the face maps and degeneracy maps between simplices $\Delta(n)$ induce face maps and degeneracy maps between $\text{MOR}(E, F)_n$'s.

Let $E', E''$ be subspectra of a spectrum $E$, i.e. $E_n$ contains $E'_n, E''_n$ as subcomplexes. Then if any two maps $f' : E' \to F$ and $f'' : E'' \to F$ coincide on $E' \cap E''$, then they are glued to a map $f : E' \cup E'' \to F$. Hence, if $f' : E' \to F$ and $f'' : E'' \to F$ are morphisms such that they coincide on $E' \cap E''$ as morphisms, then they are glued to a unique morphism $f : E' \cup E'' \to F$ by using the assumption that the structure maps $s_n$'s are embeddings.

By previous discussion, we see that the simplicial set $\text{MOR}(E, F)$ satisfies the Kan condition [Kan55], [Kan56]. We can consider the fundamental groupoid of $\text{MOR}(E, F)$ which we denote by $\Pi_1 \text{MOR}(E, F)$.

Next, we lift the composition of morphisms to a simplicial map between $\text{MOR}(E, F)$'s:

**Definition 5.2.** We define a simplicial map,

$$\text{MOR}(E, F) \times \text{MOR}(F, G) \to \text{MOR}(E, G).$$

It suffices to define a map which is compatible with face maps and degeneracy maps,

$$\text{Mor}(E \wedge (\Delta(n))^+, F) \times \text{Mor}(F \wedge (\Delta(n))^+, G) \to \text{Mor}(E \wedge (\Delta(n))^+, G)$$

We define $h$ by taking compositions of the following maps where $f \in \text{Mor}(E \wedge (\Delta(n))^+, F)$, $g \in \text{Mor}(F \wedge (\Delta(n))^+, G)$:

1. $E \wedge (\Delta(n))^+ \overset{\Delta}{\to} E \wedge (\Delta(n) \times \Delta(n))^+$
2. $E \wedge (\Delta(n))^+ \overset{\Sigma}{\to} (E \wedge (\Delta(n))^+) \wedge (\Delta(n))^+$
3. $f : E \wedge (\Delta(n))^+ \to F \wedge (\Delta(n))^+$
4. $\overset{g}{\to} G$.

Since the composition simplicial map defined above satisfies the associativity strictly, it follows that \(26\) is natural with respect to $E, F, G$. For example, if we write by $*$ the one-point simplicial set, then $f \in \text{Mor}(E, F)$ induces a simplicial map $\overset{f}{\to} * \to \text{MOR}(E, F)$. Then we obtain a simplicial map from \(26\):

$$\text{MOR}(F, G) \to * \times \text{MOR}(F, G)$$

$$\overset{f}{\to} \text{MOR}(E, F) \times \text{MOR}(F, G)$$

$$\to \text{MOR}(E, G)$$
If we denote it by $f^* : \text{MOR}(F,G) \to \text{MOR}(E,G)$, then we have $(g \circ f)^* = f^* \circ g^*$ due to the associativity of the composition simplicial map. Therefore we obtain the following proposition.

**Proposition 5.3.** The composition simplicial map induces a functor between fundamental groupoids.

\[
\Pi_1\text{MOR}(E,F) \times \Pi_1\text{MOR}(F,G) \to \Pi_1\text{MOR}(E,G).
\]

This composition functor satisfies the associativity strictly, and it is natural with respect to $E,F,G$.

We have a bijection $\text{Mor}(E \vee E', F) \to \text{Mor}(E,F) \times \text{Mor}(E', F)$ which is determined by the restrictions. It is bijective since two morphisms from $E$ to $F$ and from $E'$ to $F$ are glued into a morphism from $E \vee E'$ to $F$ as we mentioned at the beginning of this subsection. It is possible to understand this property using our groupoids $\Pi_1\text{MOR}(E,F)$’s as follows.

**Proposition 5.4.** Let $E, E', F$ be spectra. Let us denote by $i : E \to E \vee E'$, $i' : E' \to E \vee E'$ the canonical inclusions. They induce an isomorphism of groupoids:

\[
\Pi_1\text{MOR}(E \vee E', F) \to \Pi_1\text{MOR}(E,F) \times \Pi_1\text{MOR}(E', F).
\]

**Proof.** The maps induced by applying $\pi_0, \pi_1$ to (37) are isomorphisms due to Proposition 1.14. [Rud98]. Hence (37) gives an equivalence of groupoids by Proposition [A,3]. Moreover, it is obvious that the functor induces a bijection between object classes. Therefore, (37) is an isomorphism of groupoids.

Let $\nu : S^n \to S^n \vee S^n$ be the usual comultiplication on $S^n$, i.e. the pinch map for $n \geq 1$. It induces a monoidal structure on the groupoid $\Pi_1\text{MOR}(S^n \wedge E, F)$. We denote by $\oplus : \Pi_1\text{MOR}(S^n \wedge E, F) \times \Pi_1\text{MOR}(S^n \wedge E, F) \to \Pi_1\text{MOR}(S^n \wedge E, F)$ the functor obtained by the following compositions:

\[
\Pi_1\text{MOR}(S^n \wedge E, F) \times \Pi_1\text{MOR}(S^n \wedge E, F) \cong \Pi_1\text{MOR}((S^n \wedge E) \vee (S^n \wedge E), F),
\]

\[
\cong \Pi_1\text{MOR}((S^n \vee S^n) \wedge E, F),
\]

\[
\overset{\nu^*}{\to} \Pi_1\text{MOR}(S^n \wedge E, F).
\]

The associator, unit with respect to this functor $\oplus$ are induced from the coalgebraic structure on the sphere $S^n$ associated with the comultiplication $\nu$.

**Proposition 5.5.** The monoidal structure of $\Pi_1\text{MOR}(S^n \wedge E, F)$ has a braiding if $n \geq 2$. Moreover, the braiding becomes a symmetry if $n \geq 3$.

As the homotopy group $\pi_n(X,x_0)$ has a group structure based on the loop product, the monoidal structure on the groupoid $\Pi_1\text{MOR}(S^n \wedge E, F)$ is a ‘group’, i.e. a categorical group. In fact, the above categorical group structure of the groupoid $\Pi_1\text{MOR}(S^n \wedge E, F)$ has canonical inverses since inverses are given by reversing loops. From now on, we regard $\Pi_1\text{MOR}(S^n \wedge E, F)$ as a symmetric categorical group.

**Proposition 5.6.** By Proposition 5.4 we have a symmetric monoidal functor whose underlying functor induces an isomorphism:

\[
\Pi_1\text{MOR}(S^n \wedge (E \vee E'), F) \to \Pi_1\text{MOR}(S^n \wedge E, F) \oplus \Pi_1\text{MOR}(S^n \wedge E', F)
\]

It gives a monoidal isomorphism if $n \geq 1$, and a braided isomorphism if $n \geq 2$.

**Proof.** It follows from the definition of the monoidal structure and its braiding for $n \geq 2$. In fact, the monoidal structure and its braiding are obtained from a coalgebraic structure of the sphere $S^n$ so that it is a symmetric monoidal functor preserving canonical inverses. Since it is an isomorphism, it completes our proof.
Let \( E, F, F' \) be spectra and \( n \geq 3 \). Denote by the canonical inclusions \( i : F \to F \vee F' \), \( i' : F' \to F \vee F' \). They induce a functor \( f \) by

\[
\Pi_1 \text{MOR}(S^n \wedge E, F) \oplus \Pi_1 \text{MOR}(S^n \wedge E, F') \to \Pi_1 \text{MOR}(S^n \wedge E, F \vee F')
\]

\[
(a, b) \mapsto i_* (a) \oplus i'_* (b)
\]

If \( p : F \vee F' \to F \) and \( p' : F \vee F' \to F' \) denotes the projections, we obtain a functor \( g \),

\[
\Pi_1 \text{MOR}(S^n \wedge E, F \vee F') \to \Pi_1 \text{MOR}(S^n \wedge E, F) \oplus \Pi_1 \text{MOR}(S^n \wedge E, F')
\]

\[
a \mapsto (p_* (a), p'_* (a))
\]

**Proposition 5.7.** The functors \( f, g \) are lifted to an adjoint equivalence of symmetric categorical groups with canonical inverses.

**Proof.** Since \( p \circ i = \text{id}_F \), \( p' \circ i' = \text{id}_{F'} \) and \( p \circ i' \) and \( p' \circ i \) are collapsing morphisms, \( g(f(a, b)) = f(i_* (a) \oplus i'_* (b)) = (p_* (i_* (a) \oplus i'_* (b)), p'_* (i_* (a) \oplus i'_* (b))) = (a \oplus (p \circ i')_* (b), (p' \circ i)_* (a) \oplus b) \cong (a, b) \). Hence we obtain a natural isomorphism \( \eta : g \circ f \cong \text{id} \) which is natural with respect to \( E, F, F' \).

On the one hand, \( f \) induces an equivalence of groupoids due to Proposition A.3 since its \( \pi_0, \pi_1 \) induce isomorphisms. Therefore, there exists a unique natural isomorphism \( \epsilon : \text{id} \cong f \circ g \) such that \( f, g, \eta, \epsilon \) gives an adjoint equivalence of groupoids.

The functor \( g \) is obviously a (strict) symmetric monoidal functor preserving canonical inverses due to definitions. Although it is not that obvious for the functor \( f \), the functor \( f \) is enhanced naturally to a symmetric monoidal functor preserving canonical inverses using the adjoint equivalence \( f, g, \eta, \epsilon \). Then \( f, g, \epsilon, \eta \) give an adjoint equivalence of symmetric categorical groups with canonical inverses.

\[\Box\]

5.2. **Construction.** In this subsection, we introduce a symmetric categorical group version of generalized (co)homology theory.

**Definition 5.8.** Let \( E \) be a spectrum. For based spaces \( X, Y \), we define a groupoid \( \mathcal{H}(X, Y; E) \) by

\[
\mathcal{H}(X, Y; E) \overset{\text{def}}{=} \Pi_1 \text{MOR}(S^\infty X \wedge E \wedge S^\infty Y).
\]

By definition, the 0-th homotopy set of the groupoid \( \mathcal{H}(X, Y; E) \) is isomorphic to the set \( HH(X, Y; E) \) introduced at the beginning of this section.

**Proposition 5.9.** We have a natural symmetric monoidal structure on the groupoid \( \mathcal{H}(X, Y; E) \).

**Proof.** It is a corollary of Proposition 5.5. \( \Box \)

The symmetric categorical group version of generalized (co)homology theory defined above is motivated by [SV17], [AdR05]. Sharma and Voronov used a symmetric categorical group version of ordinary (co)homology theory to construct the Dijkgraaf-Witten theory. Their categorical (co)homology group is based on [AdR05], i.e. it is induced from a chain complex of categorical groups. Although we use spectra to describe (co)homology theories, we have an equivalences between them where \( X \) is a space which is not based.

\[
\mathcal{H}(X^+, S^n; HM) \leftarrow \Pi_1 \text{MAP}(X, HM_n) \to H^n(X, M[0]).
\]

Here \( HM \) be the Eilenberg-MacLane spectrum associated with an abelian group \( M \), and \( H^n(X, M[0]) \) is the categorical group described in Sharma-Voronov [SV17]. The functor on the right side is determined by a universal element \( u \in H^n(HM_n, M[0]) \).

**Proposition 5.10.** The inclusions \( X \to X \vee X' \) and \( X \to X \vee X' \) induce the following symmetric monoidal isomorphism:

\[
\mathcal{H}(X \vee X', Y; E) \to \mathcal{H}(X, Y; E) \oplus \mathcal{H}(X', Y; E).
\]
Proof. It follows from Proposition \ref{proposition:adjunction}. \hfill \Box

**Proposition 5.11.** We have a symmetric monoidal adjoint equivalence which is natural with respect to $X, Y, Y'$:
\begin{equation}
\mathcal{H} \mathcal{C}(X, Y; E) \oplus \mathcal{H} \mathcal{C}(X, Y'; E) \to \mathcal{H} \mathcal{C}(X, Y \vee Y'; E).
\end{equation}

**Proof.** It follows from Proposition \ref{proposition:adjunction}. \hfill \Box

In this paragraph, let us fix a based space $X$. Let us denote by $F(Z)$ the groupoid $\mathcal{H} \mathcal{C}(X, Z; E)$ or the groupoid $\mathcal{H} \mathcal{C}(X, Z; E)$. From the above propositions, we obtain a natural transformation written by $U_{Z,Z'} : F(Z) \times F(Z') \to F(Z \vee Z')$; $(a, b) \mapsto U_{Z,Z'}(a, b)$. We have an ‘associator’ with respect to this binary operation, i.e. we are given an isomorphism in $F(Z \vee (Z' \vee Z'))$,
\begin{equation}
\Xi_{a,b,c} : U_{Z,Z',Z''}(U_{Z,Z'}(a, b), c) \to U_{Z,Z',Z''}(a, U_{Z',Z''}(b, c)).
\end{equation}
Here, $a \in F(Z), b \in F(Z'), c \in F(Z'')$. In fact, since $U_{Z,Z'}$ are monoidal functors in this case, the ‘associator’ is induced from the associator of the monoidal structure of $F(Z \vee (Z' \vee Z'))$. It satisfies ‘pentagon’ axiom in the sense that the following diagram commutes in the groupoid $F(Z \vee (Z' \vee Z''))$:
\[
\begin{array}{ccc}
U(U(U(a, b), c), d) & \xrightarrow{\Xi} & U(U(a, U(b, c)), d) \\
\downarrow{\Xi} & & \downarrow{\Xi} \\
U(U(a, b), U(c, d)) & \cong & U(a, U(U(b, c), d)) \\
\downarrow{\Xi} & & \downarrow{\Xi} \\
U(a, U(b, U(c, d))) & & \\
\end{array}
\]
Here, $a \in F(Z), b \in F(Z'), c \in F(Z''), d \in F(Z''')$.

By introducing the bivariant functor $\mathcal{H} \mathcal{C}(-, -; E)$, it is possible to consider the pairings of generalized (co)homology theory as ‘compositions’: Let us consider a pairing of spectra $\mu : E \wedge F \to G$ where $E, F, G$ are spectra where the smash product $E \wedge F$ follows Adams \cite{Ada}, Switzer \cite{Swi}. Via compositions of the following simplicial maps, we obtain a functor $\mathcal{H} \mathcal{C}(X, Y; E) \times \mathcal{H} \mathcal{C}(Y, Z; F) \to \mathcal{H} \mathcal{C}(X, Z; G)$:
\[
\begin{align*}
MOR(\Sigma^\infty X, E \wedge \Sigma^\infty Y) & \times \text{MOR}(\Sigma^\infty Y, F \wedge \Sigma^\infty Z) \\
\to \text{MOR}(\Sigma^\infty X, E \wedge \Sigma^\infty Y) & \times \text{MOR}(\Sigma^\infty Y, F \wedge \Sigma^\infty Z) \\
\to \text{MOR}(\Sigma^\infty X, E \wedge (F \wedge \Sigma^\infty Z)) & \\
\to \text{MOR}(\Sigma^\infty X, (E \wedge F) \wedge \Sigma^\infty Z) & \\
\to \text{MOR}(\Sigma^\infty X, G \wedge \Sigma^\infty Z)
\end{align*}
\]
If we apply the 0-th homotopy $\pi_0$ to the pairing $\mathcal{H} \mathcal{C}(X, Y; E) \times \mathcal{H} \mathcal{C}(Y, Z; F) \to \mathcal{H} \mathcal{C}(X, Z; G)$, then it induces a pairing which is reduced slant product, cap product, cup product, cross product of generalized (co)homology theories.

**Remark 5.12.** For a based map $f : X \to Y$, there is an associated long cofiber sequence $X \to Y \to C_f \to SX \to SY \to SC_f \to \cdots$, which is called Puppe sequence. If we applying generalized (co)homology theory to this sequence, then we obtain a long exact sequence which may be used to prove well-known long exact sequences.

On the other hand, there is lifted versions of long exact sequences given by Theorem 4.2 \cite{AdR}. They are called 2-exactness and relative 2-exactness where 2-exactness implies...
relative 2-exactness. We have a relative 2-exact sequence of our categorical (co)homology groups associated with Puppe sequence. Let \( f : X \to Y \) be a based map. The long cofiber sequence \( X \to Y \to C_f \to SX \to SY \to SC_f \to \cdots \) associated with \( f \) induces a relative 2-exact sequence:

\[
\begin{array}{ccccccc}
\mathcal{H}(Z,X) & \longrightarrow & \mathcal{H}(Z,Y) & \longrightarrow & \mathcal{H}(Z,C_f) & \longrightarrow & \mathcal{H}(Z,SX) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathcal{H}(X,Z) & \longleftarrow & \mathcal{H}(Y,Z) & \longleftarrow & \mathcal{H}(C_f,Z) & \longleftarrow & \mathcal{H}(SX,Z) & \longleftarrow & \cdots \\
\end{array}
\]

Its proof essentially comes from the (usual) exact sequence of generalized (co)homology groups.

6. A Groupoid Induced by \( E \)-Orientation

Given a ring spectrum \( E \), the \( E \)-orientation of an \( n \)-manifold is defined as an \( n \)-th \( E \)-homology class which gives the unit (or the minus unit) of the ring \( \tilde{E}_n(S^n) \cong \pi_0(E) \) if restricted to each chart. For the Eilenberg-Maclane spectrum \( E = H\mathbb{Z} \) associated with the group \( \mathbb{Z} \), it is equivalent with the classical orientation of manifolds.

If we consider the singular (co)homology theory, it is possible to take an ‘integral’ of a representative of \((m+1)\)-th cohomology class on an oriented \( m \)-manifold, but we should note that its value lives in a categorical group, not a group [FQ93]. The higher structure which naturally appears from the difference of their degrees leads to a Lagrangian classical field theory.

Sharma and Voronov interpret the ‘integral’ based on a categorical framework [SV17]. They realize the fundamental class of an oriented \( m \)-manifold as a full subgroupoid of (ordinary) categorical homology group, and ‘integral’ introduced by Freed, Quinn as the cap product ‘with’ that groupoid.

In this section, we introduce a groupoid containing representatives of the fundamental class associated with an \( E \)-oriented manifold.

6.1. Construction. For a ring spectrum \( E \) and an \( E \) oriented compact \( k \)-manifold \( Z^k \), we have a canonical homomorphism \( E_k(Z^k, \partial Z^k) \to E_{k-1}(Z^k) \) called the connecting homomorphism. It assigns the fundamental class of boundaries \( [\partial Z^k]_E \) to its fundamental class \( [Z^k]_E \). It is paraphrased that every representative of the class \( [Z^k]_E \) is assigned to a representative of the class \( [\partial Z^k]_E \) by

\[
\mathcal{H}(S^k, Z^k/\partial Z^k; E) \to \mathcal{H}(S^{k-1}, (\partial Z^k)^+; E).
\]

Since we do not know whether there is such a functor or not, we deal with this situation as follows. If we denote by \( C \) the mapping cone of the inclusion \( \partial Z^k \to Z^k \), then we have a well-defined functor via the collapsing map \( C \to S(\partial Z^k)^+ \):

\[
\mathcal{H}(S^k, C; E) \to \mathcal{H}(S^k, S(\partial Z^k)^+; E).
\]
We also have equivalences of groupoids \( \mathcal{H}(S^{k}, C; E) \to \mathcal{H}(S^{k}Z^{k}; E) \) and \( \mathcal{H}(S^{k-1}, (\partial X^{k})^{+}; E) \to \mathcal{H}(S^{k}, S(\partial Z^{k}); E) \). Due to these observations, it is convenient to deal with representatives of fundamental class \([Z^{k}]_{E}\) (resp. \([\partial Z^{k}]_{E}\)) in the groupoid \( \mathcal{H}(S^{k}, C; E) \) (resp. \( \mathcal{H}(S^{k}, S(\partial Z^{k}); E) \)).

It is a motivation for the definition of groupoids \([-][]_{E}\) in Definition 6.4.

**Definition 6.1.** Let \( \triangle(n) \) be defined by \( \{ (t_{0}, t_{1}, \cdots, t_{n}) \in \mathbb{R}^{n+1} | t_{i} \geq 0 \} \). We consider two 2-simplices \( \triangle(2) \). To distinguish them, we use the notation \( \triangle(2)_{0}, \triangle(2)_{1} \). We define \( \triangle(2)_{0} \triangle(2)_{1} \) as the space obtained by gluing the 0-face of \( \triangle(2)_{0} \) and the 1-face of \( \triangle(2)_{1} \). For \( t = (t_{0}, t_{1}) \in \triangle(1) \), we define \( h_{1}^{t} : \triangle(2)_{0} \triangle(2)_{1} \to \triangle(2) \) by gluing two affine maps \( f_{j} : \triangle(2)_{j} \to \triangle(2) \) where \( f_{0} \) is defined by \( f_{0}(e_{0}) = e_{0}, f_{0}(e_{1}) = e_{1}, \) and \( f_{0}(e_{2}) = t_{0}e_{0} + t_{1}e_{2} \). \( f_{1} \) is defined by \( f_{1}(e_{0}) = e_{1}, f_{1}(e_{1}) = e_{2}, \) and \( f_{1}(e_{2}) = t_{0}e_{0} + t_{1}e_{2} \). We consider the family of \( h_{1}^{t}, t \in \triangle(1) \) as the homotopy between maps \( h_{0}^{t} \) and \( h_{t}^{t} \).

Let us pick one more 2-simplex \( \triangle(2) \), and put it as \( \triangle(2)_{2} \). We define \( \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \) as the space obtained by gluing 0-face of \( \triangle(2)_{0} \) with the 1-face of \( \triangle(2)_{1} \), 0-face of \( \triangle(2)_{2} \), and the 1-face of \( \triangle(2)_{2} \) with the 1-face of \( \triangle(2)_{0} \). For \( t = (t_{0}, t_{1}, t_{2}) \in \triangle(2) \), we define \( h_{2}^{t} : \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \to \triangle(2) \) by gluing three affine maps \( g_{j} : \triangle(2)_{j} \to \triangle(2) \) where \( g_{0} \) is defined by \( g_{0}(e_{0}) = e_{0}, g_{0}(e_{1}) = e_{2} \) and \( g_{0}(e_{2}) = t_{0}e_{0} + t_{1}e_{1} + t_{2}e_{2} \). \( g_{1} \) is defined by \( g_{1}(e_{0}) = e_{0}, g_{1}(e_{1}) = e_{2} \) and \( g_{1}(e_{2}) = t_{0}e_{0} + t_{1}e_{1} + t_{2}e_{2} \).

Let us define \( p^{t} : \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \to \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \). \( p^{t} \) is determined by the identities \( \triangle(2)_{0} \triangle(2)_{1} \to \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \) and collapsing map on \( \triangle(2)_{2} \). \( p^{t} \) is determined by \( s_{2}^{t} : \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \to \triangle(2)_{0} \triangle(2)_{2} \triangle(2)_{1} \) and collapsing map on \( \triangle(2)_{1} \). Here we denote by \( s_{0}^{t} : \triangle(2) \to \triangle(2) \) the affine map determined by \( e_{k} \to e_{k} \) for \( k = 0, 1, 2 \). \( p^{t} \) is determined by identities (of underlying simplices) \( \triangle(2)_{1} \to \triangle(2)_{2} \triangle(2)_{1} \) and collapsing map on \( \triangle(2)_{2} \).

**Definition 6.2.** Let \( Z_{*} = (Z_{0} \to Z_{01} \to Z_{1}) \) be a cospan diagram of based maps between based spaces. We define \( R^{1}(Z_{*}) = (Z_{0} \leftarrow Z_{01} \to Z_{1}) \) as follows. Let \( K \) be a based space defined by

\[
K = \left( (\triangle(2)_{0}^{+} \triangle Z_{0}) \bigcup_{(\triangle(1))^{+} \triangle Z_{0}} (\triangle(1))^{+} \triangle Z_{0} \right) \bigcup_{(\triangle(1))^{+} \triangle Z_{1}} (\triangle(2)_{1}^{+} \triangle Z_{1}).
\]

Here, we use the 0-face map \( \triangle(1) \to \triangle(2)_{0} \) and the 1-face map \( \triangle(1) \to \triangle(2)_{1} \). Then we define \( Z_{01}^{t} \) by collapsing the following subspace of \( K \):

\[
(((\triangle(2)_{0}^{+} \triangle Z_{0}) \bigcup_{(\triangle(1))^{+} \triangle Z_{0}} (\triangle(1))^{+} \triangle Z_{0}) \bigcup_{(\triangle(1))^{+} \triangle Z_{1}} (\triangle(2)_{1}^{+} \triangle Z_{1}),
\]

where \( \triangle(n)_{k} \subset \triangle(n) \) denotes the horn obtained by discarding the \( k \)-th face. If we denote the quotient map by \( q : K \to Z_{01}^{t} \), then we define \( Z_{i}^{t} \) as the quotient space \( Z_{01}^{t}/q((\triangle(2)_{1-i}^{+} \triangle Z_{1-i}) \to Z_{1-i}) \) for \( i = 0, 1 \). Then we define the diagram \( \mathbb{R}^{1}(Z_{*}) \) using the quotient maps.

**Definition 6.3.** Let \( Z_{*} \) be the commutative diagram of base spaces,
Starting from this diagram, we define a commutative diagram $\mathcal{R}^2(Z_*)$ of based spaces,

\[
\begin{array}{ccc}
Z_0' & \xrightarrow{1} & (\Delta(2)_0)^+ \wedge Z_0 \\
\downarrow & & \downarrow \text{0-face} \\
\downarrow & & \downarrow \text{0-face} \\
Z_0' & \xrightarrow{0} & (\Delta(1)^+ \wedge Z_0_1) \\
\end{array}
\]

(48)

To begin with, we denote by $K$ the space obtained by gluing the following spaces:

\[
\begin{array}{ccc}
(\Delta(1)^+ \wedge Z_0_1) & \xrightarrow{1} & (\Delta(0)^+ \wedge Z_0_1) \\
\downarrow & & \downarrow \text{1-face} \\
(\Delta(1)^+ \wedge Z_0_1) & \xrightarrow{0} & (\Delta(0)^+ \wedge Z_0_1) \\
\end{array}
\]

(49)

Note that the arrows does not denote maps. For example, $(\Delta(1)^+ \wedge Z_0_1) \xrightarrow{0}$ means that we glue them along $(\Delta(1)^+ \wedge Z_0_1) \leftarrow (\Delta(1)^+ \wedge Z_0_1)$ $\xrightarrow{0} (\Delta(2)_0)^+ \wedge Z_0_1$. We define $Z_{012}$ by collapsing ‘boundary’ of $K$:

\[
\begin{array}{ccc}
(\tilde{c}_2(\Delta(2)_0))^+ \wedge Z_0 & \xrightarrow{1} & (\tilde{c}_1(\Delta(1)))^+ \wedge Z_0 \\
\downarrow & & \downarrow \text{1-face} \\
(\tilde{c}_2(\Delta(2)_0))^+ \wedge Z_0 & \xrightarrow{0} & (\tilde{c}_1(\Delta(1)))^+ \wedge Z_0 \\
\end{array}
\]

(50)

Here, $\tilde{c}_k(\Delta(n))$ denotes the k-face of the simplex $\Delta(n)$. Let us write the quotient map by $q : K \to Z_{012}$.

We define other $Z_*$’s by collapsing some subspaces of $Z_{012}$ as follows. We define $Z_{ij}$ by collapsing the image of $\{[t, x] \mid x \notin Z_i \text{ or } x \notin Z_j \}$ via $q : K \to Z_{012}$. Likewise, we define $Z_1$ by collapsing the image of $\{[t, x] \mid x \notin Z_i \}$ via $q$. Then the quotient maps between $Z_*$ induces a diagram $\mathcal{R}^3(Z_*)$.

From now on, we fix a ring spectrum $E$.

**Definition 6.4.** Let $Y^{n-1}$ be a closed E-oriented (n-1)-manifold. Its fundamental class is given as an E-homology class $[Y^{n-1}]_E \in E_{n-1}(Y^{n-1}) \cong \pi_0(\mathcal{H}(S^{n+1}, S^2Y^{n-1}; E))$. We define $[[Y^{n-1}]]_E$ as a full subgroupoid of the groupoid $\mathcal{H}(S^{n+1}, S^2Y^{n-1}; E)$ which contains representatives of the fundamental class $[Y^{n-1}]_E$.

As a next step, we consider an E-oriented n-cobordism $Y^{n-1}_0 \to X^n \to Y^{n-1}_1$. For simplicity, let $(Z_0 \to Z_{01} \to Z_1) = (Y^{n-1}_0)^+ \to (X^n)^+ \leftarrow (Y^{n-1}_1)^+$. Let us denote by $Z' = (Z_0' \leftarrow Z_{01}' \to Z_1')$ the diagram $\mathcal{R}^4(Z_*)$. Then we define $[[X^n]]_E$ by a full subgroupoid
of the groupoid $\mathcal{HC}(S^{n+1}, Z'_0; E)$ which contains representatives of the fundamental class $[X^n]_E \in E_0(X^n, \partial X^n) \cong \pi_0(\mathcal{HC}(S^{n+1}, S(X^n/\partial X^n); E)) \cong \pi_0(\mathcal{HC}(S^{n+1}, Z'_0; E))$.

Finally, we consider a commutative diagrams of inclusions between $E$-oriented compact manifolds.

\[
\begin{array}{c}
X^n_0 \\
\downarrow \\
W^{n+1} \\
\uparrow \\
X^n_2
\end{array}
\quad
\begin{array}{c}
Y^{n-1}_0 \\
\downarrow \\
X^n_2 \\
\uparrow \\
Y^{n-1}_2
\end{array}
\quad
\begin{array}{c}
Y^{n-1} \\
\downarrow \\
X^n_2 \\
\uparrow \\
Y^{n-1}
\end{array}
\quad
\begin{array}{c}
X^n_0 \\
\downarrow \\
W^{n+1} \\
\uparrow \\
X^n_2
\end{array}
\]

(51)

We assume further with respect to these manifolds: The boundaries of $\partial X^n_i = (-1)^{n-1} Y^{n-1} \cup (\partial X^n_i)$ as $E$-oriented manifolds. For the boundary $\partial W^{n+1}$ of $W^{n+1}$, we also assume that $\partial W^{n+1} = \bigcup_{i < j} X^n_{ij}$. By adding one extra point to each manifold, we obtain a diagram denoted by $Z_*$ in Figure 47. The $E$-orientation of the manifold $Z_{012} = W^{n+1}$ is given as $[W^{n+1}]_E \in E_{n+1}(W, \partial W) \cong \pi_0(\mathcal{HC}(S^{n+1}, Z_{012}; E))$. We define $[[W^{n+1}]]_E$ as a full subgroupoid of the groupoid $\mathcal{HC}(S^{n+1}, Z_{012}; E)$ which contains representatives of the fundamental class $[W^{n+1}]_E$.

Let us write $Z^k = Y^{n-1}, X^n, W^{n+1}$ for a moment. It is obvious that $\pi_0([[Z]]_E) \cong \ast$ and $\pi_1([[Z]]_E, a) \cong E_{k+1}(Z_*, \partial Z_*)$ for every $a \in [[Z]]_E$ by definitions. For example, for $E = HM$, the Eilenberg-Maclane spectrum associated with an abelian group $M$, the groupoid $[[Z]]_E$ is simply-connected.

6.2. Restriction to boundaries. If we are given an $E$-oriented $(n-1)$-manifold $Y^{n-1}$, then we are given a groupoid $[[Y^{n-1}]]_E$. Consider an $E$-oriented $n$-cobordism $Y^{n-1}_0 \to X^n \leftarrow Y^{n-1}$. We have a groupoid $[[X^n]]_E$. We construct functors $\partial_i^1 : [[X^n]]_E \to [[Y^{n-1}]]_E, i = 0, 1$ as follows. Again we put $Z_* = (Z_0 \to Z_{01} \to Z_1) = ((Y^{n-1}_0)^+ \to (X^1)^+ \leftarrow (Y^{n-1})^+)$ and then we obtain a span of based spaces $R(Z_*)[Z'_0 \leftarrow Z_{01} \to Z'_1]$. It induces a diagram of functors if $E = E$:

\[
\begin{array}{c}
\mathcal{HC}(S^{n+1}, Z'_0; E) \\
\downarrow \\
\mathcal{HC}(S^{n+1}, Z'_0; E) \to \mathcal{HC}(S^{n+1}, Z'_1; E)
\end{array}
\]

(52)

Recall that $[[Y^{n-1}]]_E$ is defined as a subgroupoid of the middle term in each diagram. Since $h^1_{c_0}$ induces $Z'_1 \cong S^2 Z_1$ and $h^1_{c_1}$ induces $Z'_0 \cong S^2 Z_0$, we obtain a diagram of group $\sigma \in [[X^n]]_E$, we simply write $\sigma|_{Y^{n-1}}$ the image of $\sigma$ via the functor $[[X^n]]_E \to [[Y^{n-1}]]_E$.

We construct the diagram (47) of based spaces by adding an extra point to the diagram (51) of manifolds. For that diagram (47), let us write $(W_0 \to W_{01} \to W_1) = Z_{i} \to Z_{ij} \leftarrow Z_{j}$ for $0 \leq i < j \leq 2$. If $\{i, j, k\} = \{0, 1, 2\}$, then $p^k$ in Definition 6.1 induces isomorphisms $R^k(W_*)_{01} \cong R^k(Z_*)_{ij}$ by definitions. Using the isomorphisms, we obtain three functors,

\[
\begin{array}{c}
[[X^n]]_E \\
\downarrow \\
[[W^{n+1}]]_E \\
\downarrow \\
[[X^{n*}]]_E
\end{array}
\]

(53)
Above all, if we are given a diagram (51) of manifolds, then we obtain the following diagram which is not necessarily commutative:

\[
\begin{array}{ccc}
[[Y^{n-1}_{0}]]_E & & [[]^nX_{01}]]_E \\
\downarrow & & \downarrow \quad \downarrow

[[W^{n+1}]]_E \\
[[Y^{n-1}]]_E & & [[]^nX_{12}]]_E
\end{array}
\]

(54)

In fact, it commutes up to natural isomorphisms: Let us denote by \(s_{bi_i^2} : \triangle(2) \to \triangle(2)\) the affine map determined by \(e_k \mapsto e_{i_k}\) for \(k = 0, 1, 2\). If \((i_0, i_1, i_2)\) is an even permutation, then \(s_{bi_i^2}\) is homotopic to the identity preserving its boundary. Hence, the functor \((s_{bi_i^2})_* : \mathcal{H}(S^{n+1}, \triangle(2)/\partial\triangle(2) \wedge Z; E) \to \mathcal{H}(S^{n+1}, \triangle(2)/\partial\triangle(2) \wedge Z; E)\) is naturally isomorphic to the identity for any based space \(Z\). By considering \(Z = Y^{n-1}_i\), we obtain the natural isomorphisms.

6.3. Gluability. In this subsection, we compare some features of functors \(\Pi_1 MAP(-; B\Gamma)\), \([[\ldots]]_E\) with respect to gluing manifolds. These two functors have much in common in the sense that they are compatible with appropriate homeomorphisms, reversing orientations, and disjoint union of manifolds. These properties play important roles in constructions of TQFT’s. However, \(\Pi_1 MAP(-; B\Gamma)\) is compatible with gluing manifolds whereas \([[\ldots]]_E\) is not as we will explain in this section.

To begin with, we consider the diagram (51) under some conditions as follows:

\[
\begin{array}{ccc}
Y^n_{01} & \to & X^n_{02} \\
\uparrow & & \downarrow \sim \\
X^n_{01} & \to & X^n_{02} \\
\downarrow & & \downarrow \sim \\
Y^n_{12} & \to & X^n_{12} \\
\end{array}
\]

(55)

The left bottom diagram forms a homotopy push-forward diagram,. In other words, the space obtained by gluing \(X^n_{01}, X^n_{12}\) along \(Y^n_{12}\) is homotopy equivalent with \(W^{n+1}\). Also, \((X^n_{02}, \partial X^n_{02}) \to (W^{n+1}, Y^n_{01} \cup Y^n_{12})\) induces a homotopy equivalence.
Put \( B(Z) = \Pi_1 MAP(Z; B\Gamma) \). Then we obtain the following diagram by applying \( B \) to the diagram \( (55) \).

\[
\begin{array}{cccc}
B(Y_0^{n-1}) & \quad \xrightarrow{\cong} \quad & B(X_0^n) \\
\downarrow & & \downarrow \\
B(W^{n+1}) & \quad \xrightarrow{\cong} \quad & B(Y_1^{n-1}) & \quad \xrightarrow{\cong} \quad & B(Y_2^{n-1}) \\
\downarrow & & \downarrow & & \downarrow \\
B(X_{01}^n) & \quad \xrightarrow{\cong} \quad & B(X_{02}^n) & \quad \xrightarrow{\cong} \quad & B(X_{12}^n)
\end{array}
\]

(56)

In this diagram, the left bottom diagram forms a homotopy pull-back diagram of groupoids and the functor \( B(W^{n+1}) \rightarrow B(X_{02}^n) \) gives a homotopy equivalence.

On the contrary, the functor \([[-]]_E\) does not have such glubility. The diagram \( (55) \) only induces the following diagram, i.e. the left bottom diagram does not form a homotopy pull-back diagram of groupoids in general.

\[
\begin{array}{cccc}
[[Y_0^n]]_E & \quad \xrightarrow{\cong} \quad & [[X_1^n]]_E & \quad \xrightarrow{\cong} \quad & [[X_2^n]]_E \\
\downarrow & & \downarrow & & \downarrow \\
[[Y_1^n]]_E & \quad \xrightarrow{\cong} \quad & [[W^n]]_E \\
\downarrow & & \downarrow & & \downarrow \\
[[X_{12}^n]]_E & \quad \xrightarrow{\cong} \quad & [[Y_2^n]]_E
\end{array}
\]

(57)

The left bottom diagram forms a homotopy pull-back diagram if and only if the following diagram forms a homotopy pull-back diagram.

\[
\begin{array}{ccc}
[[W^{n+1}]]_E & \xrightarrow{\cong} & [[Y_1^{n-1}]]_E \\
\downarrow & & \downarrow \Delta' \\
[[X_{01}^n X_{12}^n]]_E & \xrightarrow{\cong} & [[Y_1^{n-1} \circ Y_1^{n-1}]]_E
\end{array}
\]

(58)

In the following lemma, we denote by \( G[1] \) the groupoid of one object and \( g \in G \) as morphisms where composition is given by the group structure of \( G \).

**Lemma 6.5.** Let \( G, H \) be groups. For two functors \( \rho, \rho' : G[1] \rightarrow H[1] \), we have a homotopy limit of the following diagram is isomorphic to a groupoid \( G \backslash H \)

\[
\begin{array}{ccc}
H[1] & \xrightarrow{\Delta} & G[1] \\
\downarrow & & \downarrow \rho \times \rho' \\
G[1] & \xrightarrow{\rho \times \rho'} & H[1] \times H[1]
\end{array}
\]

where \( G \backslash H \) is the groupoid defined by the action of \( G \) on \( H \) by \( g \cdot h \overset{\text{def}}{=} \rho(g)h\rho'(g^{-1}) \) for \( g \in G, \ h \in H \). In particular, if \( H \) is abelian, then they are isomorphic to the homotopy fiber of the functor \( (\rho - \rho') : G[1] \rightarrow H[1] \).
By this lemma, it is straightforward to show that if the composition of canonical homomorphisms \( E_{n+1}(X_0^n, \partial X_0^n) \oplus E_{n+1}(X_1^n, \partial X_1^n) \to E_n(\partial X_0^n) \oplus E_n(\partial X_1^n) \to E_n(Y_1^{-1}) \) is not surjective, then the homotopy pull-back of \( [[X_0^n, X_1^n]]_E \to [[[Y_1^{-1}]]_E \to [[[Y_1^{-1}]]_E. \)

### 7. Some functors induced by a pairing

In this section, we prepare some abstract lemmas for construction of classical field theories in section \([8][13]\). For a 2-category \( \mathcal{C} = \text{CW}_* \) (resp. \( \mathcal{C}^* \)), generalized (co)homology theory (resp. \( KK \)-theory) induces a natural pairing (Definition \([7.1]\)). If we are given a natural pairing, then it is possible to construct a precursor of a Lagrangian classical field theory.

In subsection \([7.1]\) we construct some functors \( \mathcal{F}^i_{\mathcal{C}, \text{AB}} : \mathcal{F}^i_{\mathcal{C}, \text{AB}} \to \mathcal{N}(\mathcal{C}) \) for \( i = 0, 1, 2 \), which are the precursor of classical field theories. In subsection \([7.2]\) we show that the functors \( \mathcal{F}^i_{\mathcal{C}, \text{AB}} \) are improved to monoidal functors under additional assumptions for \( \mathcal{C} \) and the pairing. In subsection \([7.3]\) we further improve the functors \( \mathcal{F}^i_{\mathcal{C}, \text{AB}} \) to symmetric monoidal functors under more stronger assumptions for \( \mathcal{C} \) and the pairing.

#### 7.1. Construction

Let \( \mathcal{C} \) be a category. Let \( A : \mathcal{C} \to \text{Gpd} \) and \( B : \mathcal{C}^{op} \to \text{Gpd} \) be functors.

**Definition 7.1.** We say that a functor

\[
\langle \cdot, \cdot \rangle_X : A(X) \times B(X) \to C, \quad X \in \mathcal{C}
\]

gives a natural pairing if the following conditions hold. For a morphism \( f : X_0 \to X_1 \) in the category \( \mathcal{C} \), we have a natural isomorphism \( \Theta_f(a, b) : \langle f^*(a), b \rangle_{X_0} \to \langle a, f_*(b) \rangle_{X_1} \) where \( a \in A(X_1) \), \( b \in B(X_0) \) are objects. Equivalently, it can be described using the following diagram:

\[
\begin{array}{ccc}
A(X_0) \times B(X_0) & \xrightarrow{\Theta_i} & A(X_1) \times B(X_1) \\
\downarrow \langle \cdot \rangle_{X_0} & & \downarrow \langle \cdot \rangle_{X_1} \\
A(X_1) \times B(X_0) & \xrightarrow{\Theta_i} & A(X_1) \times B(X_1)
\end{array}
\]

We assume that the identity \( 1_X : X \to X \) in \( \mathcal{C} \) induces \( \Theta_{1_X}(a, b) = 1_{\langle a, b \rangle_X} \). For morphisms \( g : X_1 \to X_2 \) and \( f : X_0 \to X_1 \) in the category \( \mathcal{C} \), we assume that the diagram below commutes.

\[
\begin{array}{ccc}
\langle (g \circ f)_*(a), b \rangle_{X_0} & \xrightarrow{\Theta_{g \circ f}(a, b)} & \langle a, (g \circ f)^*(b) \rangle_{X_2} \\
\downarrow \Theta_{g*}(a, b) & & \downarrow \Theta_{f^*}(b) \\
\langle g_*(a), f^*(b) \rangle_{X_1} & \xrightarrow{\Theta_{f^*}(b)} & \langle a, f^*(b) \rangle_{X_1}
\end{array}
\]

Here, we denote by \( f_* = A(f) : A(X_0) \to A(X_1) \) and \( f^* = B(f) : B(X_1) \to B(X_0) \) the induced morphisms.

**Definition 7.2.** Let us use the previous notations. We define a category \( \mathcal{F}_{\mathcal{C}, \text{AB}} \). Its object class consists of \( (X, a, b) \) where \( X \) is an object of \( \mathcal{C} \), \( a \in A(X) \) and \( b \in B(X) \). \( (f, g, h) : (X, a, b) \to (X', a', b') \) is a morphism if \( f : X \to X' \) is a morphism in \( \mathcal{C} \), \( g : f_*(a) \to a' \) is a morphism in \( A(X) \) and \( h : b \to f^*(b') \) is a morphism in \( B(X') \).

We construct a functor \( \mathcal{F}_{\mathcal{C}, \text{AB}} : \mathcal{F}_{\mathcal{C}, \text{AB}} \to \mathcal{C} \) as follows. For an object \( (X, a, b) \in \mathcal{F}_{\mathcal{C}, \text{AB}} \), we set \( \mathcal{F}_{\mathcal{C}, \text{AB}}(X, a, b) = \langle a, b \rangle_X \in \mathcal{C} \). For a morphism \( (f, g, h) : (X, a, b) \to (X', a', b') \), we define \( \mathcal{F}_{\mathcal{C}, \text{AB}}(f, g, h) : \mathcal{F}_{\mathcal{C}, \text{AB}}(X, a, b) \to \mathcal{F}_{\mathcal{C}, \text{AB}}(X', a', b') \) by compositions of the morphisms below:

\[
\langle a, b \rangle_X \xrightarrow{\Theta} \langle a, f^*(b') \rangle_X \xrightarrow{\Theta} \langle f_*(a), b' \rangle_{X'} \xrightarrow{h} \langle a', b' \rangle_{X'}.
\]
**Lemma 7.3.** \( \mathcal{F}_{C,AB} : \mathcal{T}_{C,AB} \to \mathcal{C} \) is a functor.

**Proof.** For an object \((X, a, b) \in \mathcal{T}_{C,AB}\), let us denote by \(i\) the identity \((1_X, 1_a, 1_b) : (X, a, b) \to (X, a, b)\) in \(\mathcal{T}_{C,AB}\). Then \(\mathcal{F}_{C,AB}(1_X, 1_a, 1_b) \) is obtained by taking compositions of the following morphisms:

\[
\langle a, b \rangle_X \xrightarrow{\theta} \langle a, (1_X)^*(b) \rangle_X \xrightarrow{g} \langle a, b \rangle_X.
\]

By definitions, it coincides with the identity on \(\langle a, b \rangle_X\). Let \((f, g, h) : (X, a, b) \to (X', a', b')\) and \((f', g', h') : (X', a', b') \to (X'', a'', b'')\) be morphisms in the category \(\mathcal{T}_{C,AB}\). We show that \(\mathcal{F}_{C,AB}(f', g', h') \circ (f, g, h) = \mathcal{F}_{C,AB}(f, g, h) \circ \mathcal{F}_{C,AB}(f', g', h')\). By definition, we have \(\langle f', g', h' \circ (f, g, h) \rangle = \langle f' \circ f, g' \circ g, h' \circ h \rangle\).

Then our claim is proved using the following commutative diagram.

\[
\begin{array}{c}
\langle a, b \rangle_X \xrightarrow{f^*(b'\circ h)} \langle a, f^*(b') \rangle_X \xrightarrow{g} \langle a, b \rangle_X \\
\end{array}
\]

By far, we consider a category \(\mathcal{C}\) and a natural pairing to construct a functor \(\mathcal{F}_{C,AB}\). From now on, let \(\mathcal{C}\) be a 2-category. By considering the underlying category of \(\mathcal{C}\), the natural pairing is as before:

\[
\langle \cdot, \cdot \rangle_X : \mathcal{A}(X) \times \mathcal{B}(X) \to \mathcal{C}, X \in \mathcal{C}.
\]

For \(i = 0, 1, 2\), we construct a category \(\mathcal{C}^i\), functors \(\mathcal{A}_i : \mathcal{C}^i \to \mathbf{Gpd}, \mathcal{B}_i : \mathcal{C}_i^{op} \to \mathbf{Gpd}\) and a natural pairing \(\langle \cdot, \cdot \rangle_{X} : \mathcal{A}_i(X) \times \mathcal{B}_i(X) \to \mathbf{N}_i(C), X \in \mathcal{C}^i\).

The object class of \(\mathcal{C}^0\) consists of \((X, Y, \phi)\) where \(X, Y \in \mathcal{C}\) and \(\phi : Y \to X\) is an isomorphism in \(\mathcal{C}\). A morphism is given as a pair of morphisms for \(X, Y\)’s which intertwine \(\phi\)’s. Note that \(\mathbf{N}_0(\mathcal{C}) \equiv \mathcal{C}\). We set \(\mathcal{A}^0(X, Y, \phi) \overset{def}{=} \mathcal{A}(Y)\) and \(\mathcal{B}^0(X, Y, \phi) \overset{def}{=} \mathcal{B}(X)\). Let \(\langle a, b \rangle^0 \overset{def}{=} \langle \phi_0(a), b \rangle_X \in \mathcal{C}\) where \(Z = (X, Y, \phi)\). We write \((X, Y, \phi)\) as \(X \overset{\phi}{\to} Y\).

We construct a category \(\mathcal{C}^1\) from the 2-category \(\mathcal{C}\). Its object class consists of the following diagrams:

\[
\begin{array}{c}
\xymatrix{ & X_0 \ar@{~>}[dl]_{d_0} \ar@{~>}[dr]^{d_1} & \\
X_0 & & X_1 \ar@{~>}[ll]_{a_0} \ar@{~>}[rr]^{a_1} & & Y_0 \ar@{~>}[ll]_{u_0} \ar@{~>}[rr]^{u_1} & & Y_1 \ar@{~>}[ll]_{b_0} \ar@{~>}[rr]^{b_1}}
\end{array}
\]

i.e. \(X_0, Y_0\) are objects of \(\mathcal{C}\), \(\phi_0, u_0, d_0\)’s are 1-morphisms and \(\Phi_{01} : u_0 \circ \phi_0 \circ d_0 \to u_1 \circ \phi_1 \circ d_1\) is a 2-morphism in \(\mathcal{C}\). Let \(\phi'_i, u'_i, d'_i, \Phi'_{01}\) give another objects. Then \((f_{01}, f_0, f_1, g_{01})\) is a 1-morphism from \((\phi_i, u_i, d_i, \Phi_{01})\) to \((\phi'_i, u'_i, d'_i, \Phi'_{01})\) if \(f_0 : X_0 \to X'_0\)’s and \(g_{01} : Y_1 \to Y'_1\) are morphisms in \(\mathcal{C}\) which intertwine all of \(\phi_i, \phi'_i, u_i, d_i, u'_i, d'_i\)’s and \(\Phi_{01}, \Phi'_{01}\) coincide to each
other under the intertwiners. We set $\mathcal{A}^{1}(X) = \mathcal{A}(Y_{01})$ and $\mathcal{B}^{1}(X) = \mathcal{B}(X_{01})$ where $X \in \mathcal{C}^{1}$ denotes the diagram (66). We define $\langle \cdot, \cdot \rangle^{}_{\mathcal{X}} : \mathcal{A}^{1}(X) \times \mathcal{B}^{1}(X) \to N_{1}(\mathcal{C})$ as follows. Let $a \in \mathcal{A}^{1}(X) = \mathcal{A}(Y_{01})$ and $b \in \mathcal{B}^{1}(X) = \mathcal{B}(X_{01})$. Let us denote by $a_{i}$ the image of $a$ via $\mathcal{A}(Y_{01}) \to \mathcal{A}(Y_{i})$, similarly by $b_{i}$ the image of $b$ via $\mathcal{B}(X_{01}) \to \mathcal{B}(X_{i})$. Let $\langle x, y \rangle_{i}$ denote the image of $(x, y)$ by the pairing $\mathcal{A}(X) \times \mathcal{B}(Y) \to \mathcal{A}(X_{i}) \times \mathcal{B}(Y_{i})$. We define $\langle a, b \rangle^{}_{\mathcal{X}}$ as a morphism from $\langle a_{0}, b_{0} \rangle_{0}$ to $\langle a_{1}, b_{1} \rangle_{1}$ obtained from compositions of the following morphisms

$$
\langle a_{0}, b_{0} \rangle_{0} \xrightarrow{\theta_{a_{0}, b_{0}}} \langle a, (u_{0} \circ d_{0})^{*}(b) \rangle_{Y_{01}} \xrightarrow{\varphi_{01}} \langle a, (u_{1} \circ d_{1})^{*}(b) \rangle_{Y_{1}} \xrightarrow{\theta_{a_{1}, b_{1}}} \langle a_{1}, b_{1} \rangle_{1}
$$

**Remark 7.4.** Suppose that the following diagram in the groupoid $\mathcal{C}$ commutes for 1-morphisms $f, g : X_{0} \to X_{1}$ and a 2-morphism $\Phi : f \to g$:

$$
\langle f_{*}(a), b \rangle \xrightarrow{\theta(f, a, b)} \langle a, f^{*}(b) \rangle \xrightarrow{\Phi} \langle g_{*}(a), b \rangle \xrightarrow{\theta(g, a, b)} \langle a, g^{*}(b) \rangle
$$

Then it is equivalent to define it as

$$
\langle a_{0}, b_{0} \rangle_{0} \xrightarrow{\theta_{a_{0}, b_{0}}} \langle (u_{0} \circ d_{0})^{*}(a), b \rangle_{X_{01}} \xrightarrow{\phi_{01}} \langle (u_{1} \circ d_{1})^{*}(a), b \rangle_{X_{1}} \xrightarrow{\theta_{a_{1}, b_{1}}} \langle a_{1}, b_{1} \rangle_{X_{11}}
$$

due to the assumption (67).

Given a morphism between objects $(a, b), (a', b') \in \mathcal{A}^{1}(X) \times \mathcal{B}^{1}(X)$, we define morphisms $\langle a_{i}, b_{i} \rangle_{i} \to \langle a'_{i}, b'_{i} \rangle_{i}$ using the naturality of the pairing $\langle \cdot, \cdot \rangle^{}_{\mathcal{X}}$'s to satisfy the following commutative diagram.

$$
\langle a_{0}, b_{0} \rangle_{0} \xrightarrow{\langle a_{0}, b_{0} \rangle_{X}} \langle a_{1}, b_{1} \rangle_{1} \xrightarrow{\langle a'_{0}, b'_{0} \rangle_{X}} \langle a'_{1}, b'_{1} \rangle_{1}
$$

Then it gives a functor $\langle \cdot, \cdot \rangle^{}_{\mathcal{X}} : \mathcal{A}^{1}(X) \times \mathcal{B}^{1}(X) \to N_{1}(\mathcal{C})$.

We construct a category $\mathcal{C}^{2}$ as follows. An object of $\mathcal{C}^{2}$ consists of the following commutative diagrams in $\mathcal{C}$

$$
\begin{align*}
\begin{array}{ccc}
X_{0} & \xrightarrow{X_{01}} & X_{01} \\
\downarrow & & \downarrow \\
X_{02} & \xleftarrow{Y_{01}} & Y_{01} \\
\downarrow & & \downarrow \\
X_{1} & \xleftarrow{X_{12}} & X_{12} \\
\downarrow & & \downarrow \\
X_{012} & \xrightarrow{X_{012}} & Y_{012} \\
\downarrow & & \downarrow \\
Y_{12} & \xrightarrow{Y_{12}} & Y_{12} \\
\downarrow & & \downarrow \\
Z' & \xrightarrow{Z'} & Z'
\end{array}
\end{align*}
$$

with the following 2-morphisms for $0 \leq i < j \leq 2$ such that $\Phi_{12} \ast \Phi_{01} = \Phi_{02}$

$$
\begin{align*}
\begin{array}{ccc}
X_{ij} & \xrightarrow{X_{ij}} & X_{ij} \\
\downarrow & & \downarrow \\
Y_{ij} & \xrightarrow{Y_{ij}} & Y_{ij}
\end{array}
\end{align*}
$$
where the 1-morphisms in diagram (70) appear in diagram (69). We define \( A^2(X) \overset{\text{def}}{=} A(Y_{012}) \) and \( B^2(X) \overset{\text{def}}{=} B(X_{012}) \) where \( X \) denotes the diagram (69). Let us define a functor \( \langle \cdot, \cdot \rangle^2_{X} : A^2(X) \times B^2(X) \rightarrow N_2(C) \) as follows. Let \( a \in A^2(X) = A(Y_{012}) \) and \( b \in B(X) = B(X_{012}) \). We construct \( \langle a, b \rangle^2_{X} \in N_2(C) \) as the following diagram which commutes due to Proposition 7.5.

\[
\begin{array}{ccc}
\langle a_0, b_0 \rangle_0 & \xrightarrow{\langle a, b \rangle^1_{Y_{012}}} & \langle a_2, b_2 \rangle_2 \\
\downarrow & & \downarrow \\
\langle a_1, b_1 \rangle_1 & \xrightarrow{\langle a, b \rangle^1_{P_{12}(X)}} & \langle a_2, b_2 \rangle_2
\end{array}
\]

(71)

Here, \( \langle a, b \rangle^1_{P_{ij}(X)} \)'s are morphisms in \( C \) defined previously where \( P_{ij}(X) \) denotes the following object of \( C^1 \) induced by \( X \in C^2 \):

\[
\begin{array}{ccc}
X_{012} & \xrightarrow{\Phi_{ij}} & X_j \\
\phantom{\Phi_{ij}} & & \phantom{\Phi_{ij}} \\
Y_{012} & \xleftarrow{\Phi_{ij}} & Y_j
\end{array}
\]

(72)

For a morphism \( X \rightarrow X' \) in \( C \), we define \( \langle a, b \rangle^2_{X} \rightarrow \langle a, b \rangle^2_{X'} \), by the triple of \( \langle a, b \rangle^1_{P_{ij}(X)} \rangle_{i<j} \) where \( \langle \cdot, \cdot \rangle^1_{(-)} \) is defined before. Then we obtain a functor \( \langle \cdot, \cdot \rangle^2 : A^2(X) \times B^2(X) \rightarrow N_2(C) \).

**Proposition 7.5.** The diagram (71) in the groupoid \( C \) commutes:

**Proof.** By definitions of the object \( X \in C^2 \), we have \( \Phi_{12} \circ \Phi_{01} = \Phi_{02} \). It follows from the following computations:

\[
\begin{align*}
\langle a, b \rangle^1_{Z_{12}} \circ \langle a, b \rangle^1_{Z_{01}} &= (\Theta_{u_2})^{-1} \circ (\Phi_{12})_* \circ \Theta_{u_1} \circ (\Theta_{u_1})^{-1} \circ (\Phi_{01})_* \circ \Theta_{u_0} \\
&= (\Theta_{u_2})^{-1} \circ (\Phi_{12})_* \circ (\Phi_{01})_* \circ \Theta_{u_0} \\
&= \langle a, b \rangle^1_{Z_{02}}
\end{align*}
\]

\( \Box \)

**Definition 7.6.** We define a functor \( \mathcal{T}^i_{C,A,B} : \mathcal{T}^i_{C,A} \rightarrow N_i(C) \) for \( i = 0, 1, 2 \) by

\[
\mathcal{T}^i_{C,A,B} \overset{\text{def}}{=} \mathcal{T}^i_{C^i,A^i,B^i}.
\]

**Definition 7.7.** We introduce ‘face maps’ between \( \mathcal{T}^i_{C,A,B} \):

\[
\begin{array}{ccc}
\mathcal{T}^2_{C,A,B} & \xrightarrow{\mathcal{T}^1_{C,A,B}} & \mathcal{T}^1_{C,A,B} \\
\phantom{\mathcal{T}^0_{C,A,B}} & \xrightarrow{\mathcal{T}^1_{C,A,B}} & \phantom{\mathcal{T}^0_{C,A,B}}
\end{array}
\]

We define \( \mathcal{T}^i_{C,A,B} \), \( i = 0, 1 \) as a functor which assigns \( (X_i \overset{\Phi_i}{\rightarrow} Y_i, a_i, b_i) \in \mathcal{T}^0_{C,A,B} \) to \( (D, a, b) \in \mathcal{T}^1_{C,A,B} \) where \( D \) is the diagram (66). We define \( \mathcal{T}^1_{C,A,B} \), \( i = 0, 1, 2 \) as a functor which assigns \( (D', a_{ij}, b_{ij}) \in \mathcal{T}^1_{C,A,B} \) to \( (D'', a, b) \in \mathcal{T}^2_{C,A,B} \) where \( D', D'' \) are the diagrams (70), (69) respectively.

Note that the nerve of a category induces a simplicial set. The functor \( \mathcal{T}^i_{C,A,B} \) defined before is compatible with ‘face maps’, i.e. \( \mathcal{T}^i_{C,A,B} \) commutes strictly with ‘face maps’ of \( \mathcal{T}^i_{C,A,B} \) and \( N_i(C) \).
7.2. **Monoidality.** In this subsection, we prove that the functors $\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}$’s are lifted to a monoidal functor if the 2-category $\mathcal{C}$ has a monoidal structure with which the natural pairing $\langle \cdot, \cdot \rangle_{(-)}$ is compatible appropriately. We start with the following assumptions with respect to $\mathcal{C}$ and $\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}$.

**Assumption 7.8.** Let $(\mathcal{C}, \otimes)$ be a monoidal 2-category in the following sense. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a 2-functor which gives a monoidal structure to the underlying category of $\mathcal{C}$. For three 2-morphisms $\Phi_i : f_i \to g_i$ for $i = 0, 1, 2$, $(\Phi_0 \otimes \Phi_1) \otimes \Phi_2 : (f_0 \otimes f_1) \otimes f_2 \to (g_0 \otimes g_1) \otimes g_2$ and $\Phi_0 \otimes (\Phi_1 \otimes \Phi_2) : f_0 \otimes (f_1 \otimes f_2) \to g_0 \otimes (g_1 \otimes g_2)$ coincide to each other via the associator.

In our applications, we shall consider $(\mathcal{C}, \otimes) = (\text{Tor}_*, \vee)$ or $(\mathcal{C}^*, \oplus)$.

**Assumption 7.9.** Suppose that we have a lifted category structure on $\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}$ with respect to the projection $\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}} \to \mathcal{C}$; $(X, a, b) \mapsto X$, i.e. we are given a functor $\otimes : \mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}} \times \mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}} \to \mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}$ which gives a monoidal category structure on $\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}$ with some structure morphisms, and under which $(\mathcal{F}^i_{\mathcal{C}, \mathcal{A}, \mathcal{B}}, \otimes) \to (\mathcal{C}, \otimes)$ is a strict monoidal functor.

The assumption can be paraphrased as follows. Let us write the tensor product as

$$ (X, a, b) \otimes (X', a', b') = (X \otimes X', U^A_{X,X'}(a, a'), U^B_{X,X'}(b, b')). $$

Here, we have natural transformations $U^F_{X,X'} : F(X) \times F(X') \to F(X \otimes X')$, $X, X' \in \mathcal{C}$ where $F = A, B$. Also the assignment $(X, X') \mapsto U^F_{X,X'}$ is natural in the sense that the following diagram commutes for given 1-morphisms $f : X_0 \to X_1$ and $g : X'_0 \to X'_1$ in $\mathcal{C}$:

$$
\begin{array}{ccc}
F(X_0) \times F(X'_0) & \xrightarrow{U^F_{X_0,X'_0}} & F(X_0 \otimes X'_0) \\
F(f) \times F(g) \downarrow & & \downarrow F(\otimes) \\
F(X_1) \times F(X'_1) & \xrightarrow{U^F_{X_1,X'_1}} & F(X_1 \otimes X'_1)
\end{array}
$$

For simplicity of notations, we often abbreviate $U^F_{X,X'}$ by $U$ if there is no danger of confusion. In addition, we have an ‘associator’ for $(a, b) \mapsto U(a, b)$, i.e. we are given an isomorphism where we identify $(X \otimes X') \otimes X'' \cong X \otimes (X' \otimes X'')$:

$$ \Xi_{a,b,c} : U_{X \otimes X',X''}(U_{X,X'}(a, b), c) \cong U_{X,X' \otimes X''}(a, U_{X',X''}(b, c)) $$

More precisely, $\Xi$ is a natural isomorphism described using the following diagram:

$$
\begin{array}{ccc}
(F(X) \times F(X')) \times F(X'') & \xrightarrow{U} & F(X \otimes X') \times F(X'') \xrightarrow{U} F((X \otimes X') \otimes X'') \\
\downarrow & & \downarrow \\
F(X) \times (F(X') \times F(X'')) & \xrightarrow{U} & F(X) \times (F(X' \otimes X'') \xrightarrow{U} F(X \otimes (X' \otimes X''))
\end{array}
$$

We require a ‘pentagon’ axiom for $\Xi_{a,b,c}$ as follows where we freely use the associator with respect to $(\mathcal{C}, \otimes)$:

$$
\begin{array}{ccc}
U(U(U(a, b), c), d) & \xrightarrow{\Xi} & U(U(a, U(b, c)), d) \\
\downarrow & & \downarrow \\
U(U(a, b), U(c, d)) & \xrightarrow{\Xi} & U(a, U(U(b, c), d)) \\
\downarrow & & \downarrow \\
U(a, U(b, U(c, d))) & \xrightarrow{\Xi} & U(a, b, U(c, d)))
\end{array}
$$
We make use of the symmetric monoidal structure on $\mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}}$ to construct a symmetric monoidal category structure on $\mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}}^i$ for $i = 1, 2$. The natural transformation $U_{X,X'}^A : \mathcal{A}(X) \times \mathcal{A}(X') \to \mathcal{A}(X \otimes X')$ induces a natural transformation $U_{X,X'}^{A^i} : \mathcal{A}^i(X) \times \mathcal{A}^i(X') \to \mathcal{A}^i(X \otimes X')$ defined by

$$U_{Y_{012}, Y'_{012}}^A : \mathcal{A}(Y_{012}) \times \mathcal{A}(Y'_{012}) \to \mathcal{A}(Y_{012} \otimes Y'_{012}).$$

Similarly, we define $U_{X,X'}^{A^i}, U_{X,X'}^{B^i}$ for the case of $i = 0, 1, 2$. Then $\Xi_{a,b,c}^A, \Xi_{a,b,c}^B$ are induced from $\Xi_{a,b,c}^A, \Xi_{a,b,c}^B$ respectively. Then it gives a symmetric monoidal category structure on $\mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}}^i$ for $i = 1, 2$.

**Assumption 7.10.** We suppose that $\mathcal{C}$ is a monoidal groupoid and the functor $\mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}} : \mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}} \to \mathcal{C}$ is a monoidal functor.

This assumption is unpacked as follows. We are given a natural isomorphism,

$$\Psi_{a,b,a',b'} : \langle a, b \rangle_X \otimes \langle a', b' \rangle_{X'} \to \langle U_{X,X'}^A(a, a'), U_{X,X'}^B(b, b') \rangle_{X \otimes X'}.$$  

Since the structure map $\mathcal{T}(X, a, b) \otimes \mathcal{T}(X', a', b') \to \mathcal{T}(X \otimes X', (a, a'), (b, b'))$ is natural, the following diagram commutes.

\[
\begin{array}{ccc}
\langle f^*(a), b \rangle \otimes \langle g^*(a'), b' \rangle & \to & \langle a, f_*(b) \rangle \otimes \langle a', g_*(b') \rangle \\
\downarrow & & \downarrow \\
\langle U(f^*(a), g^*(a')), U(b, b') \rangle & \to & \langle U(a, a'), U(f_*(b), g_*(b')) \rangle
\end{array}
\]

(83)

Also the axiom which the structure morphism $\mathcal{T}(X, a, b) \otimes \mathcal{T}(X', a', b') \to \mathcal{T}(X \otimes X', (a, a'), (b, b'))$ satisfies induces the following commutative diagram. For simplicity, let us write $x = (a, b)$, $x' = (a', b')$ and $U(x, x') = (U_{X,X'}^A(a, a'), U_{X,X'}^B(b, b'))$. Then the following diagram in the groupoid $\mathcal{C}$ commutes where we freely use the associator of $(\mathcal{C}, \otimes)$ and $\Xi$ assumed before:

\[
\begin{array}{ccc}
\langle x \rangle_X \otimes \langle x' \rangle_{X'} \otimes \langle x'' \rangle_{X''} & \xrightarrow{\psi} & \langle U(x, x') \rangle_{X \otimes X'} \otimes \langle x'' \rangle_{X''} \\
\downarrow & & \downarrow \\
\langle x \rangle_X \otimes \langle x' \rangle_{X'} \otimes \langle x'' \rangle_{X''} & \xrightarrow{\psi} & \langle U(x, x') \rangle_{X \otimes X'} \otimes \langle x'' \rangle_{X''}
\end{array}
\]

In the remaining part of this subsection, we construct a symmetric monoidal functor whose underlying functor is $\mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}}^i : \mathcal{T}_{\mathcal{E},\mathcal{A},\mathcal{B}}^i \to \mathbf{N}(\mathcal{C})$. First of all, let us construct an isomorphism where $a \in \mathcal{A}^i(X), b \in \mathcal{B}^i(X), a' \in \mathcal{A}^i(X')$ and $b' \in \mathcal{B}^i(X')$:

$$\Psi_{a,b,a',b'}^{1} : \langle a, b \rangle^{1} \otimes \langle a', b' \rangle^{1} \to \langle U_{X,X'}^{A^i}(a, a'), U_{X,X'}^{B^i}(b, b') \rangle^{1}.$$  

Recall the definition of $\langle a, b \rangle^{1}_X$:

$$\langle a, b \rangle^{1}_X : \langle a_0, b_0 \rangle_0 \to \langle a_1, b_1 \rangle_1.$$
Hence it is necessary to construct a morphism from \((a_0, b_0) \otimes (a_0', b_0')_i\) to \(\langle U(a, a'), U(b, b')_0 \rangle_i\) for \(i = 0, 1\). Let us write \(X, X' \in C^1\) by

\[
\begin{array}{c}
X_0 \cong Y_0 \\
\downarrow \phi_0 \\
X_1 \cong Y_1 \\
\downarrow \phi_0 \\
X_0' \cong Y_0' \\
\downarrow \phi_0 \\
X_1' \cong Y_1'
\end{array}
\]

They \(X \otimes X' \in C^1\) is written as follows.

\[
\begin{array}{c}
X_0 \otimes X_0' \cong Y_0 \otimes Y_0' \\
\downarrow d_0 \otimes d_0 \\
X_1 \otimes X_1' \cong Y_1 \otimes Y_1'
\end{array}
\]

For \(i = 0, 1\) we define an isomorphism \(\tilde{\Psi}_i : \langle a_i, b_i \rangle_i \otimes \langle a_i', b_i' \rangle_i \rightarrow \langle U(a, a'), U(b, b')_i \rangle_i\) in the groupoid \(C\) obtained as a composition of the isomorphisms assumed before:

\[
\langle a_i, b_i \rangle_i \otimes \langle a_i', b_i' \rangle_i \\
\downarrow \\
\langle (u_i)_*(a_i), (d_i)_*(b_i) \rangle_X \otimes \langle (v_i)_*(a_i'), (c_i)_*(b_i') \rangle_{Y_i}
\]

\[
\langle U((u_i)_*(a_i), (v_i)_*(a_i')), U((d_i)_*(b_i), (c_i)_*(b_i')) \rangle_{X \otimes Y_i}
\]

\[
\langle (u_i \otimes v_i)_*(U(a, a')), (d_i \otimes c_i)_*(U(b, b')) \rangle_{X \otimes Y_i}
\]

\[
\langle U(a, a'), U(b, b')_i \rangle_i
\]

By Proposition 7.11 below, we obtain a natural transformation:

\[
\Psi^{1}_{a,b,a',b'} : \langle a, b \rangle^1 _X \otimes \langle a', b' \rangle^1 _{X'} \rightarrow \langle U^1_{X,X'}(a, a'), U^1_{X,X'}(b, b') \rangle^1 _X \otimes _{X'}
\]

It induces a morphism in \(C\) by definitions:

\[
\tilde{\mathcal{F}}^1_{C,A,B}(X, a, b) \otimes \tilde{\mathcal{F}}^1_{C,A,B}(X', a', b') \rightarrow \tilde{\mathcal{F}}^1_{C,A,B}(X \otimes X', U(a, a'), U(b, b')).
\]

It gives a natural transformation since the following diagram commutes for 1-morphisms \(f : X \rightarrow Y, g : X' \rightarrow Y'\) in \(C^1\) due to definition of \(\langle \cdot, \cdot \rangle^1_{(-)}\):

\[
\begin{array}{c}
\langle f_*(a), b \rangle^1_Y \otimes \langle g_*(a'), b' \rangle^1_{Y'} \\
\downarrow \\
\langle U(f_*(a), g_*(a')), U(b, b') \rangle^1_{Y \otimes Y'} \\
\downarrow \\
\langle (f \otimes g)_*(U(a, a')), U(b, b') \rangle^1_{Y \otimes Y'}
\end{array}
\]

\[
\begin{array}{c}
\langle a, f^*(b) \rangle^1_X \otimes \langle a', g^*(b') \rangle^1_{X'} \\
\downarrow \\
\langle U(a, a'), U(f_*(b), g_*(b')) \rangle^1_{X \otimes X'} \\
\downarrow \\
\langle U(a, a'), (f \otimes g)_*(U(b, b')) \rangle^1_{X \otimes X'}
\end{array}
\]
Proposition 7.11. Under the isomorphisms $\Psi$, the morphisms $\langle a, b \rangle_X \otimes \langle a, b \rangle_{X'}$ and $\langle a, b \rangle_{X \otimes X'}$ coincide with each other, i.e. the following diagram commutes.

$$
\begin{array}{c}
\langle a_0, b_0 \rangle_0 \otimes \langle a'_0, b'_0 \rangle_0 \\
\downarrow \\
\langle a, b \rangle_{X \otimes (a, b)_{X'}} \\
\downarrow \\
\langle a_1, b_1 \rangle_1 \otimes \langle a'_1, b'_1 \rangle_1 \\
\downarrow \\
\langle U(a, a'), U(b, b') \rangle_0 \\
\end{array}
$$

Proof. The assumption about the commutativity of the diagram $\Phi$ gives the following commutative diagram where $i = 0, 1$:

$$
\begin{array}{c}
\langle a, b \rangle_0 \otimes \langle a', b' \rangle_0 \\
\downarrow \\
\langle U(a, a'), U(b, b') \rangle_0 \\
\downarrow \\
\langle U(a, a'), (u_1 \circ d_i \otimes v_1 \circ c_i)_*(U(b, b')) \rangle_0 \\
\end{array}
$$

For $i = 0, j = 1$, we also have the following commutative diagram. The upper square diagram commutes due to the naturality of $\Phi$, and the lower square diagram commutes due to the naturality of the pairing $\langle , \rangle$.

$$
\begin{array}{c}
\langle a, (u_1 \circ d_i)_*(b) \rangle_0 \otimes \langle a', (v_1 \circ c_i)_*(b') \rangle_0 \\
\downarrow \\
\langle U(a, a') \otimes (u_i \circ d_j)_*(b), (v_i \circ c_j)_*(b') \rangle_0 \\
\downarrow \\
\langle U(a, a'), ((u_i \circ d_j)_*(b), (v_i \circ c_j)_*(b')) \rangle_0 \\
\downarrow \\
\langle U(a, a'), (u_1 \circ d_i \otimes v_1 \circ c_i)_*(U(b, b')) \rangle_0 \\
\end{array}
$$

Our claim is proven by combining those commutative diagrams. \qed

Lemma 7.12. The functor $\mathcal{F}_{\mathcal{L}, \mathcal{A}, \mathcal{B}} : \mathcal{T}_E \rightarrow \mathcal{N}_*(\mathbb{C})$ gives a monoidal functor.

Proof. The case of $i = 0$ is nothing but Assumption [7,10] so trivial. Consider the case of $i = 1$. We should verify that the natural transformation $\mathcal{F}(X) \otimes \mathcal{F}(X') \rightarrow \mathcal{F}(X \otimes X')$ in $\mathcal{F}$ preserves the associators. For simplicity, let us write $x = (a, b)$, $x' = (a', b')$ and $U(x, x') = (U_{X,X'}(a, a'), U_{X,X'}(b, b'))$ Recall the definition of $\Psi_{x,a,b,a',b'} = \Psi_{x,a,b,a',b'}$ in [83]. We show that the following diagram commutes.

$$
\begin{array}{c}
\langle x, x' \rangle_1 \otimes \langle x, x' \rangle_1 \\
\downarrow \\
\langle U(x, x') \rangle_1 \otimes \langle x, x' \rangle_1 \\
\downarrow \\
\langle U(U(x, x'), x, x') \rangle_1 \\
\end{array}
$$

If we denote by $x_i = (a_i, b_i)$ (and similarly $x'_i, x''_i$), then $\langle x \rangle_1$ is a morphism from $\langle x_0 \rangle_1$ to $\langle x_1 \rangle_1$. In order to prove the commutativity of the above diagram, it suffices to show that the
following diagram commutes. In fact, it holds due to Assumption 7.10
\[
\begin{align*}
\langle \langle x_i \rangle \rangle_i \otimes \langle \langle x'_i \rangle \rangle_i \otimes \langle \langle x''_i \rangle \rangle_i & \longrightarrow \langle \langle x_i \rangle \rangle_i \otimes \langle \langle x'_i \rangle \rangle_i \\
\downarrow \psi & \downarrow \psi \\
\langle U(x_i, x'_i) \rangle_i \otimes \langle x''_i \rangle_i & \longrightarrow \langle x_i \rangle_i \otimes \langle U(x'_i, x''_i) \rangle_i \\
\downarrow \psi & \downarrow \psi \\
\langle U(U(x_i, x'_i), x''_i) \rangle_i & \longrightarrow \langle U(x_i, U(x'_i, x''_i)) \rangle_i
\end{align*}
\]

In order to prove the case of \( i = 2 \), we again verify the commutative diagram above for \( \langle \langle \cdot \rangle \rangle_2 \) instead of \( \langle \langle \cdot \rangle \rangle_1 \). It commutes again due to Assumption 7.10.

7.3. Symmetric monoidality. In the previous subsection, we show that the functors \( F_i^j_{C, A, B} \) are lifted to symmetric monoidal functors. In this subsection, we improve the results under additional assumptions. The functors \( F_i^j_{C, A, B} \) are lifted to a symmetric monoidal functor if the 2-category \( C \) has a monoidal structure with which the natural pairing \( \langle \cdot, \cdot \rangle \) is compatible appropriately. We start with the following assumptions with respect to \( C \) and \( F_{C, A, B} \).

Assumption 7.13. Let \((C, \otimes)\) be a symmetric monoidal 2-category : \( \otimes : C \times C \to C \) is a 2-functor which gives a symmetric monoidal structure to the underlying category of \( C \). For three 2-morphisms \( \Phi_i : f_i \to g_i \) for \( i = 0, 1, 2 \), \( (\Phi_0 \otimes \Phi_1) \otimes \Phi_2 : (f_0 \otimes f_1) \otimes f_2 \to (g_0 \otimes g_1) \otimes g_2 \) and \( \Phi_0 \otimes (\Phi_1 \otimes \Phi_2) : f_0 \otimes (f_1 \otimes f_2) \to g_0 \otimes (g_1 \otimes g_2) \) coincide to each other via the associator. For two 2-morphisms \( \Phi_i : f_i \to g_i \) for \( i = 0, 1 \), \( \Phi_0 \otimes \Phi_1 : f_0 \otimes f_1 \to f_1 \otimes f_0 \) and \( \Phi_1 \otimes \Phi_0 : f_1 \otimes f_0 \to f_0 \otimes f_1 \) coincide to each other via the symmetry.

Assumption 7.14. Suppose that we have a lifted symmetric category structure on \( F_{C, A, B} \) with respect to the projection \( F_{C, A, B} \to \mathcal{C}; (X, a, b) \to X \), i.e. we are given a functor \( \otimes : F_{C, A, B} \times F_{C, A, B} \to F_{C, A, B} \) which gives a symmetric monoidal category structure on \( F_{C, A, B} \) with some structure morphisms, and under which \( (F_{C, A, B}, \otimes) \to (\mathcal{C}, \otimes) \) is a strict monoidal functor.

Assumption 7.15. We suppose that \( C \) is a symmetric monoidal groupoid and the functor \( F_{C, A, B} : F_{C, A, B} \to C \) is a symmetric monoidal functor.

Then by the discussion of subsection 7.2, the category \( F_{C, A, B} \) has a monoidal category structure naturally, and the functor \( F_{C, A, B} \) becomes a monoidal functor where \( i = 0, 1, 2 \). In this section, we improve the results using the symmetry of \( (\mathcal{C}, \otimes) \) and \( (F_{C, A, B}, \otimes) \) above. We are given a ‘symmetry’ where we identify \( X \otimes X' \cong X' \otimes X \) using the symmetry of \( (\mathcal{C}, \otimes) \):

\[
\Sigma_{a, a'} : U^F_{X, X'}(a, a') \to U^F_{X', X}(a', a)
\]

More precisely, \( \Sigma \) is a natural transformation described using the following diagram.

\[
\begin{array}{ccc}
F(X) \times F(X') & \xrightarrow{\psi} & F(X \otimes X') \\
\downarrow \Sigma & & \downarrow \\
F(X') \times F(X) & \xrightarrow{\psi} & F(X' \otimes X)
\end{array}
\]

We make use of it to construct

\[
\Sigma^F_{a, a'} : U^F_{X, X'}(a, a') \to U^F_{X', X}(a', a)
\]

where \( a \in F(X), a' \in F(X') \). We deal with \( F = A \) here, but our discussion is applied to the case of \( F = B \). Recall the definition of \( U^A_{X, X'} \). The natural transformation \( U^A_{X, X'} :
\( A^1(X) \times A^1(X') \rightarrow A^1(X \otimes X') \) was defined as
\[
U^A_{Y_{012}} : A(Y_{012}) \times A(Y'_{012}) \rightarrow A(Y_{012} \otimes Y'_{012}).
\]
We define (93) as the following natural transformation, which is the special case of (92).
\[
\Psi^A : A(Y_{012}) \times A(Y'_{012}) \rightarrow A(Y_{012} \otimes Y'_{012}).
\]

Lemma 7.16. \( T^i_{CAB} : T^i_{CAB} \rightarrow N_i(C) \) is a symmetric monoidal functor.

Proof. The case of \( i = 0 \) is nothing but Assumption 7.15, so trivial.

Consider the case of \( i = 1 \). We should verify that the natural transformation \( T^1(X, a, b) \otimes T^1(X', a', b') \rightarrow T^1(X \otimes X', U(a, a'), U(b, b')) \) in (99) preserves the symmetry. For simplicity, let us write \( x = (a, b) \), \( x' = (a', b') \) and \( U(x, x') = (U^A_{X,X'}(a, a'), U^B_{X,X'}(b, b')) \) Recall the definition of \( \Psi^1_{a,b,a',b'} = \Psi^1_{x,x'} \) in (88). We show that the following diagram commutes.
\[
\begin{array}{ccc}
\langle x \rangle_x^1 \otimes \langle x' \rangle_{x'}^1 & \rightarrow & \langle x' \rangle_{x'}^1 \otimes \langle x \rangle_x^1 \\
\downarrow & & \downarrow \\
\langle U(x, x') \rangle_{X \otimes X'} & \rightarrow & \langle U(x', x) \rangle_{X' \otimes X}
\end{array}
\]
If we denote by \( x_i = (a_i, b_i) \) and \( x'_i = (a'_i, b'_i) \), then \( \langle x \rangle_x^1 \) is a morphism from \( \langle x_0 \rangle_0 \) to \( \langle x_1 \rangle_1 \). In order to prove the commutativity of the above diagram, it suffices to show that the following diagram commutes. In fact, it holds due to Assumption 7.10
\[
\begin{array}{ccc}
\langle x_i \rangle_i \otimes \langle x'_i \rangle_i & \rightarrow & \langle x'_i \rangle_i \otimes \langle x_i \rangle_i \\
\downarrow & & \downarrow \\
\langle U(x_i, x'_i) \rangle_i & \rightarrow & \langle U(x'_i, x_i) \rangle_i
\end{array}
\]
In order to prove the case of \( i = 2 \), we again verify the commutative diagram above for \( \langle \rangle_{(-)}^2 \) instead of \( \langle \rangle_{(-)}^1 \). It commutes again due to Assumption 7.10. \( \square \)

8. Classical theory from generalized (co)homology

In this section, we introduce a Lagrangian classical field theory \( S_\alpha \) for \( E \)-oriented topological manifolds where \( E \) is a ring spectrum. It is possible to regard \( S_\alpha \) as an extension of the cobordism invariant \( S_{[\alpha]} \) in Definition 3.1 to closed \( (n-1) \) manifolds. Recall that the invariant \( S_{[\alpha]} \) depends on the \( F \)-cohomology class \([\alpha] \in F^n(B\Gamma)\). The system \( S_\alpha \) depends on \( \alpha \) which represents the class \([\alpha]\) in the following sense.

Assumption 8.1. \( \alpha \) is an object of the groupoid \( \mathcal{HC}(S^2 B\Gamma^+, S^{n+2}; F) \) which represents the class \([\alpha] \in F^n(B\Gamma) \cong \pi_0(\mathcal{HC}(S^2 B\Gamma^+, S^{n+2}; F)) \).

\( S_\alpha \) consists of three assignments. In particular, the second assignment of them, given as a symmetric monoidal functor, assigns a morphism in the groupoid \( \mathcal{HC}(S^n, S^{n+1}; G) \) to a triple of an \( E \)-oriented \( n \)-cobordism, a representatative of its fundamental class and a principal
\( \Gamma \)-bundle on itself. It recovers the cobordism invariant \( S_{[\omega]} \) in the following sense. If the target and the source of the \( n \)-cobordism are the null spaces, then the target and the source of its corresponding morphism is the unit of the categorical group \( \mathcal{H}(S_n, S^{n+1}; G) \), so that it is reduced to a number in an abelian group. The number coincides with the value computed via \( S_{[\omega]} \).

We assume that \( E, F, G \) are spectra, in particular \( E \) is a ring spectrum, and we are given a pairing of spectra \( \mu : E \wedge F \to G \).

**Definition 8.2.** We define three groupoids \( \mathcal{X}_0^\ell(n; o, \Gamma), \mathcal{X}_1^\ell(n; o, \Gamma), \mathcal{X}_2^\ell(n; o, \Gamma) \) for \( n \in \mathbb{Z}^{>0} \). The object class of \( \mathcal{X}_0^\ell(n; o, \Gamma) \) consists of triples \( (Y^{n-1}, \eta, g) \) such that

- \( Y^{n-1} \) is a closed \( E \)-oriented \((n-1)\)-manifold,
- \( \eta \) is an object of the groupoid \( \{[Y^{n-1}]\} \),
- \( g \in \Pi_1 \text{MAP}(Y^{n-1}, \overline{\text{B}^n}) \) is an object.

A triple \((u, v, \eta) : (Y^{n-1}_0, \eta_0, g_0) \to (Y^{n-1}_1, \eta_1, g_1)\) is a morphism if \( u : Y^{n-1}_0 \to Y^{n-1}_1 \) is an \( E \)-orientation preserving homeomorphism and \( v : u^*(\eta_0) \to \eta_1, w : g_0 \to u^*(g_1) \) are morphisms in groupoids \( \{[Y^{n-1}_j]\}_j \) and \( \Pi_1 \text{MAP}(Y^{n-1}_0, \overline{\text{B}^n}) \) respectively. The object class of \( \mathcal{X}_1^\ell(n; o, \Gamma) \) consists of \( (Y^{n-1}_0 \to X^n \to Y^{n-1}_1; \sigma, f) \) such that

- \( Y^{n-1}_0 \to X^n \to Y^{n-1}_1 \) is an \( E \)-oriented \( n \)-cobordism.
- \( \sigma \) is an object of the groupoid \( \{[X^n]\} \).
- \( f \in \Pi_1 \text{MAP}(X^n, \overline{\text{B}^n}) \) is an object.

The morphism in \( \mathcal{X}_1^\ell(n; o, \Gamma) \) is defined in a similar way as follows. A triple \((u, v, \eta) : (X^0_0, \sigma_0, f_0) \to (X^1_1, \sigma_1, f_1)\) is a morphism if \( u : X^0_0 \to X^1_1 \) is a homeomorphism preserving \( E \)-orientation and boundaries, and \( v : u^*(\sigma_0) \to \sigma_1, w : f_0 \to u^*(f_1) \) are morphisms in groupoids \( \{[X^0_j]\}_j \) and \( \Pi_1 \text{MAP}(X^0_n, \overline{\text{B}^n}) \) respectively where \( Y^{n-1}_j \)'s are omitted for brevity.

Finally, the object class of \( \mathcal{X}_2^\ell(n; o, \Gamma) \) consists of \((D, \xi, h)\) such that

- \( D \) is family of \( E \)-oriented manifolds given by the diagram \([57]\).
- \( \xi \) is an object of the groupoid \( \{[W^{n+1}]\}_e \).
- \( h \in \Pi_1 \text{MAP}(W^{n+1}, \overline{\text{B}^n}) \) is an object.

The definition of morphism in \( \mathcal{X}_2^\ell(n; o, \Gamma) \) is similar with those of \( \mathcal{X}_0^\ell(n; o, \Gamma) \), \( \mathcal{X}_1^\ell(n; o, \Gamma) \). \( Y^{n-1}_j \)'s and \( X^n \)'s are omitted for brevity. A triple \((u, v, \eta) : (W_0^{n+1}, \xi_0, h_0) \to (W_1^{n+1}, \xi_1, h_1)\) is a morphism if \( u : W_0^{n+1} \to W_1^{n+1} \) is an \( E \)-homeomorphism preserving \( E \)-orientation, boundaries and angles, and \( v : u^*(\xi_0) \to \xi_1, w : h_0 \to u^*(h_1) \) are morphisms in groupoids \( \{[W_1^{n+1}]\}_e \) and \( \Pi_1 \text{MAP}(W_0^{n+1}, \overline{\text{B}^n}) \) respectively.

Recall the ‘face maps’ defined in subsection [6.2]. Using them, we define ‘face maps’ between \( \mathcal{X}_i^\ell(n; o, \Gamma) \)'s.

**Definition 8.3.** We define \( c_i^\ell : \mathcal{X}_1^\ell(n; o, \Gamma) \to \mathcal{X}_0^\ell(n; o, \Gamma) ; (Y^{n-1}_0 \to X^n \to Y^{n-1}_1, f) \mapsto (Y^{n-1}_i, f|_{Y^{n-1}_i}) \) and a functor \( f_i^\ell : \mathcal{X}_2^\ell(n; o, \Gamma) \to \mathcal{X}_1^\ell(n; o, \Gamma) ; (D, h) \mapsto (Y^{n-1}_j \to X^n_j \to Y^{n-1}_k, h|_{X^n_k}) \) where \( D \) is the diagram \([57]\) and \( j < k, \{i, j, k\} = \{0, 1, 2\} \).

We introduce a symmetric monoidal structure on \( \mathcal{X}_0^\ell(n; o, \Gamma) \) (and on \( \mathcal{X}_1^\ell(n; o, \Gamma) \)) similarly for \( i = 1, 2 \) by

\[
(Y^{n-1}_0, \eta_0, g_0) \boxplus (Y^{n-1}_1, \eta_1, g_1) \overset{\text{def}}{=} (Y^{n-1}_0 \boxplus Y^{n-1}_1, \eta_0 \boxplus \eta_1, g_0 \boxplus g_1)
\]

where \( \eta_0 \boxplus \eta_1 \) is the image of \((\eta_0, \eta_1)\) via the functor

\[
\mathcal{H}(S^{n+1}, S^2(Y^{n-1}_0)^+; E) \times \mathcal{H}(S^{n+1}, S^2(Y^{n-1}_1)^+; E) \to \mathcal{H}(S^{n+1}, S^2(Y^{n-1}_0 \boxplus Y^{n-1}_1)^+; E).
\]

Then the ‘face maps’ \( c_i^\ell \)'s become symmetric monoidal functors.
Theorem 8.4. The cobordism invariant $S_{\alpha}$ in Definition 7.1 extends to a family of assignments $S_{\alpha} : \mathcal{X}_{\alpha}^E(n; o, \Gamma) \to \mathcal{N}_i(\mathcal{H}(S^n, S^{n+1}; G))$, $i = 0, 1, 2$ which are strong symmetric monoidal functors and compatible with the ‘face maps’ $\delta_i^\alpha$.

From now on, we prove the theorem by constructing the assignments $S_{\alpha}$’s. We substitute appropriate ones for $A, B, C$ in section 7.1. Let $A(Z) = \mathcal{H}(S^{n+1}, Z; E)$, $B(Z) = \mathcal{H}(Z, S^{n+2}; F)$ for a based space $Z$, and $C = \mathcal{H}(S^{n+1}, S^{n+2}; G)$. The pairing $\mu : E \wedge F \to G$ assumed at the beginning of this subsection induces a functor $\langle \cdot, \cdot \rangle$ on $Z$.

(99) $\mathcal{H}(S^{n+1}, Z; E) \times \mathcal{H}(Z, S^{n+2}, F) \to \mathcal{H}(S^{n+1}, S^{n+2}; G)$.

Proposition 8.5. The pairing gives a natural pairing in Definition 7.1

Proof. Let $Z_0, Z_1$ be based spaces. Then their suspension spectra induce a simplicial set $\text{MOR}(\Sigma^e Z_0, \Sigma^e Z_1)$. On the one hand, we have a simplicial set $\text{MAP}_*(Z_0, Z_1)$ (see section 2). Then the suspension of based maps between based spaces induces a simplicial map $\text{MAP}_*(Z_0, Z_1) \to \text{MOR}(\Sigma^e Z_0, \Sigma^e Z_1)$.

\[
\begin{align*}
\text{MOR}(\Sigma^e S^{n+1}, E \wedge \Sigma^e Z_0) \times \text{MAP}_*(Z_0, Z_1) \times \text{MOR}(\Sigma^e Z_1, F \wedge \Sigma^e S^{n+1}) & \to \text{MOR}(\Sigma^e S^{n+1}, E \wedge \Sigma^e Z_0) \times \text{MOR}(\Sigma^e Z_0, \Sigma^e Z_1) \times \text{MOR}(\Sigma^e Z_1, F \wedge \Sigma^e S^{n+1}) \\
& \to \text{MOR}(\Sigma^e S^{n+1}, E \wedge \Sigma^e Z_0) \times \text{MOR}(E \wedge \Sigma^e Z_0, E \wedge \Sigma^e Z_1) \times \text{MOR}(E \wedge \Sigma^e Z_1, (F \wedge \Sigma^e S^{n+1})) \\
& \to \text{MOR}(\Sigma^e S^{n+1}, E \wedge (F \wedge \Sigma^e S^{n+1}))
\end{align*}
\]

Here the final composition is well-defined since the composition simplicial map satisfies the strict associativity. Thus, we obtain a strict equality $\langle f_*(a), b \rangle_{Z_1} = \langle a, f_*(b) \rangle_{Z_0}$ where $f : Z_0 \to Z_1$ is a map, $a \in A(Z_0), b \in B(Z_1)$. 

Let us construct symmetric monoidal functors $\mathcal{T}_i^\alpha : \mathcal{X}_{\alpha}^E(n; o, \Gamma) \to \mathcal{J}_C^i_{\alpha}$, $i = 0, 1, 2$ where $\mathcal{C} = \text{CW}_s$ under the above assumptions.

For $(Y^{n-1}, \eta, g) \in \mathcal{X}_{\alpha}^E(n; o, \Gamma)$, we define $\mathcal{T}_0^\alpha (Y^{n-1}, \eta, g) = (Y^{n-1}, \eta, g^\alpha) \in \mathcal{J}_C^0_{\alpha}$.

In order to construct $\mathcal{T}_i^\alpha$, recall $h_1^i, t \in \Delta(1)$ in Definition 6.1. Let $(X^n, \sigma, f) \in \mathcal{X}_{\alpha}^E(n; o, \Gamma)$, where $X^n$ is a cobordism from $Y_0^{n-1}$ to $Y_1^{n-1}$. Put $(Z_0 \to Z_{01} \leftarrow Z_1) = (Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1})$, and let $Z_* = \mathcal{R}(Z_*)$. Via the homeomorphism $S^2 Z_i \cong Z'_j$, the family $h_1^i, t \in \Delta(1)$ induces the following homotopy.

\[
\begin{align*}
\Sigma^2 Z_{01} & \cong Z'_j \\
\Sigma^2 Z_1 & \cong Z'_j
\end{align*}
\]

We assign (Diagram 100), $\sigma, f^*(\alpha) \in \mathcal{J}_C^i_{\alpha}$ to $(X^n, \sigma, f) \in \mathcal{X}_{\alpha}^E(n; o, \Gamma)$.

In order to construct $\mathcal{T}_i^\alpha$, we make use of the family $h_1^2, t \in \Delta(2)$ in Definition 6.1. Let $(W^{n+1}, \xi, h) \in \mathcal{X}_{\alpha}^E(n; o, \Gamma)$ be an object. Here, $W^{n+1}$ omits for the diagram (51) of $E$-oriented manifolds. If we put the diagram as $Z_\ast$ as (47), then $Z_* = \mathcal{R}(Z_*)$ and the homeomorphisms $\Sigma^2 Z_i \cong Z'_i$ induces an object of $\mathcal{C}^2$ in subsection 7.1 as follows. Let $l(s) : s \in [0, 1]$ be the line segment connecting from $e_i \in \Delta(2)$ to $e_j \in \Delta(2)$. Using a homotopy $h^2_{l(s)} : \Lambda_3^3 \to \Delta(2)$, $s \in [0, 1]$, we construct a homotopy in diagram (101) where $i, j = 0, 1, 2$. In fact, the homotopy $h^2_{l(s)} : s \in [0, 1]$ induces a homotopy of based maps $(\Lambda_3^3 / \partial \Lambda_3^3) \wedge Z_{ij} \to (\Delta(2) / \partial \Delta(2)) \wedge Z_{ij} = \ldots$
$\Sigma^2 Z_i$. We take a composition with the based map $Z'_{ij} \to (\bigwedge_3^3 / \check{\bigwedge}_3^3) \wedge Z_{ij}$.

\begin{equation}
\begin{array}{ccc}
\Sigma^2 Z_i & \xrightarrow{u_i} & \Sigma^2 Z'_{ij} \\
\downarrow \phi_{ij} & & \downarrow \phi_{ij} \\
\Sigma^2 Z_j & \xrightarrow{u_j} & \Sigma^2 Z'_{ij}
\end{array}
\end{equation}

From $h_i^2$, $t \in \Delta(2)$, we obtain $\Phi_{12} \ast \Phi_{01} = \Phi_{02}$ if we take compositions of $\Sigma^2 Z_{ij} \to \Sigma^2 Z_{012}$ and $Z'_{012} \to Z'_{ij}$ to diagram (101). Thus it gives an object $O$ of $\mathfrak{C}^2$. Then we assign $(O, \xi, h^*(\alpha)) \in \mathcal{T}^2_{c,A,B}$ to the object $(W^{n+1}, \xi, h) \in \mathcal{X}^E(n; o, \Gamma)$.

These data satisfy the assumptions in subsection 7.1 so that we obtain a family of symmetric monoidal functors $\mathcal{T}^i_{c,A,B} : \mathcal{T}^i_{c,A,B} \to \mathbf{N}(\mathbb{G})$, $i = 0, 1, 2$. We define $S_{\alpha}$ as the functor obtained from a composition with the symmetric monoidal functor $\mathcal{T}^i_{c,A,B} : \mathcal{T}^i_{c,A,B} \to \mathbf{N}(\mathbb{G})$ and $\beta^i_{\alpha} : \mathcal{X}^E(n; o, \Gamma) \to \mathcal{T}^i_{c,A,B}$. By definition of $\beta^i_{\alpha}$, we are given obvious natural isomorphisms $\beta^i_{\alpha} \circ \beta^i_{\alpha} = \beta^i_{\alpha} \circ \beta^i_{\alpha}$. On the one hand, the functors $\mathcal{T}^i_{c,A,B}$’s commute strictly with ‘face maps’ as mentioned in the final paragraph in subsection 7.1. Hence the family of $S_{\alpha}$ is compatible with the ‘face maps’.

All that remain is to prove that the assignments extend the cobordism invariant $S_{[\alpha]}$. For an object $(Y_{0}^{n-1} \to X^n \leftarrow Y_{1}^{n-1}, \sigma, f) \in \mathcal{X}^E(n; o, \Gamma)$, suppose that $Y_{0}^{n-1} = Y_{1}^{n-1} = \emptyset$, i.e. the manifold $X^n$ is closed. Then we have isomorphisms $\langle \sigma|_{Y_{0}^{n-1}}, f^*(\alpha)|_{Y_{0}^{n-1}} \rangle \cong 0$, $\langle \sigma|_{Y_{1}^{n-1}}, f^*(\alpha)|_{Y_{1}^{n-1}} \rangle \cong 0$ in the groupoid $\mathcal{H}(S^n, S^{n+1}; G)$. Under these isomorphisms, the morphism $S_{\alpha}(X^n, \sigma, f) : \langle \sigma|_{Y_{0}^{n-1}}, f^*(\alpha)|_{Y_{0}^{n-1}} \rangle \to \langle \sigma|_{Y_{1}^{n-1}}, f^*(\alpha)|_{Y_{1}^{n-1}} \rangle$ induces an element in $\pi_1(\mathcal{H}(S^n, S^{n+1}; G)) \cong \tilde{G}_0(S^0)$. It coincides with $S_{[\alpha]}(X^n, f) \in \tilde{G}_0(S^0)$ due to the following Lemma 8.6. It completes our proof of Theorem 8.4.

**Lemma 8.6.** Let $Z_0 \to Z_0 \leftarrow Z_1$ be a diagram of based spaces. We denote by $Z_1 = \mathcal{R}^1(Z_*)$. Suppose that $Z_0 = Z_1 = pt$.

\begin{equation}
\begin{array}{ccc}
\Sigma^2 Z_0 & \xrightarrow{w_0} & \Sigma^2 Z_{01} \\
\downarrow \phi_{01} & & \downarrow \phi_{01} \\
\Sigma^2 Z_1 & \xrightarrow{w_1} & \Sigma^2 Z_{11}
\end{array}
\end{equation}

It induces a map $\chi : \Sigma Z_{01} \to \Sigma Z_{01}$. On the other hand, the diagram (102) is an object of $\mathfrak{C}_1$ where $\mathfrak{C} = \mathfrak{C}W_s$. Let $D$ be the object. For two objects $a \in A(Z'_{01}), b \in B(\Sigma^2 Z_0)$, the morphism $\mathcal{T}^1_{c,A,B}(D, a, b) : \langle a_0, b_0 \rangle_0 \to \langle a_1, b_1 \rangle_1$ induces a based loop in the groupoid $\mathcal{H}(S^n, S^{n+1}; G)$ since $a_1 \in A(Z_1) \cong *$ and $b_1 \in B(\Sigma^2 Z_1) \cong *$. Hence, we obtain a loop $[\mathcal{T}^1_{c,A,B}(D, a, b)] \in \pi_1(\mathcal{H}(S^n, S^{n+1}; G)) \cong \tilde{G}_0(S^0)$. We claim that the loop $[\mathcal{T}^1_{c,A,B}(D, a, b)]$ coincides with the homotopy class of $[\langle a, \chi^*(b) \rangle] \in \pi_0(\mathcal{H}(S^{n+1}, S^{n+1}; G)) \cong \tilde{G}_0(S^0)$.

**Proof.** The canonical simplicial map $\text{MOR}(S E^r, E^r) \to \Omega(\text{MOR}(E^r, E^r), 0)$ induces a map $\pi_0(\mathcal{H}(S^{n+1}, S^{n+1}; G)) \to \pi_1(\mathcal{H}(S^n, S^{n+1}; G))$. By definition, $[\langle a, \chi^*(b) \rangle]$ corresponds to $[\mathcal{T}^1_{c,A,B}(D, a, b)]$ under this isomorphism. Moreover the following diagram commutes by
definition.

\[
\pi_0(\mathcal{H}(S^{n+1}, S^{n+1}; G)) \xrightarrow{\cong} \pi_1(\mathcal{H}(S^n, S^{n+1}; G))
\]

(103)

\[
\hat{G}_0(S^0) \xrightarrow{\cong} \hat{G}_0(S^0)
\]

It completes our proof. \(\square\)

Recall the family of action functionals \(S_{p}\) described in Theorem 8.4. According to the definition of \(\mathcal{X}_i^{E}(n; \omega, \Gamma)\), it is obvious that the action functional \(S_{p}\) depends on not only ‘fields’ (classifying maps) but also representatives of fundamental classes. From now on, we construct an action functional which is independent of representatives of fundamental classes by restricting manifolds.

**Definition 8.7.** Let us define three groupoids \(\mathcal{Y}_0^{E}(n; \Gamma), \mathcal{Y}_1^{E}(n; \Gamma), \mathcal{Y}_2^{E}(n; \Gamma)\) for \(n \in \mathbb{Z}_{>0}\) instead of \(\mathcal{X}_0^{E}(n; \omega, \Gamma), \mathcal{X}_1^{E}(n; \omega, \Gamma), \mathcal{X}_2^{E}(n; \omega, \Gamma)\). Here, we only explain object classes of them where their morphism classes are defined in a similar way as the morphism classes of the groupoid \(\mathcal{X}_i^{KU}(n; \omega, \Gamma)\) is defined in Section 8. To begin with, the object class of \(\mathcal{Y}_0^{E}(n; \Gamma)\) consists of \((Y^{n-1}, g)\) such that

- \(Y^{n-1}\) is an \(E\)-oriented closed \((n-1)\)-manifold such that \(E_n(Y^{n-1}) \simeq 0\).
- \(g : Y^{n-1} \to B\Gamma\) is a continuous map.

Secondly, the object class of \(\mathcal{Y}_1^{E}(n; \Gamma)\) consists of \((Y^{n-1} \to X^n \leftarrow Y^{n-1}, f)\) such that

- \(Y^{n-1} \to X^n \leftarrow Y^{n-1}\) is an \(E\)-oriented \(n\)-cobordism such that \(E_n(Y^{n-1}) \simeq 0\) for \(i = 0, 1\), i.e. \(X^n\) is an \(n\)-cobordism from \(Y^{n-1}_0\) to \(Y^{n-1}_1\), \(X^n, Y^{n-1}_i\)'s are \(E\)-oriented, and \(\omega_0, \omega_1\) induce \(\partial X^n \simeq (-Y^{n-1}) \cup Y^{n-1}\) as \(E\)-oriented manifolds.
- \(f : X^n \to B\Gamma\) is a continuous map.

Thirdly, the object class of \(\mathcal{Y}_2^{E}(n; \Gamma)\) consists of \((D, h)\) such that

- \(D\) denotes the diagram (51) with \(E_n(Y^{n-1}) \simeq 0\) for \(i = 0, 1, 2\).
- \(h : W^{n+1} \to B\Gamma\) is a continuous map.

Each groupoid \(\mathcal{Y}_i^{E}(n; \Gamma)\) has a symmetric monoidal structure and an involution as \(\mathcal{X}_i^{E}(n; \omega, \Gamma)\) does. Also we can define ‘face maps’ \(\partial^k_i\) between \(\mathcal{Y}_i^{E}(n; \Gamma)\)'s, which is given as a symmetric monoidal functor.

**Assumption 8.8.** For an abelian group \(M\), let \(\hat{\phi} : (\mathcal{H}(S^n, S^{n+1}; G), \oplus) \to (M_{\text{Tor}}, \otimes)\) be a symmetric monoidal functor. Let us denote by \(\pi_1(\hat{\phi}) : G_0(\text{pt}) \cong \hat{G}_0(S^0) \cong \pi_1(\mathcal{H}(S^0, S^1; G)) \to \pi_1(M_{\text{Tor}}) = M\) the group homomorphism induced by the symmetric monoidal functor \(\hat{\phi}\) where we regard the unit with respect to the monoidal structures as the basepoint of each groupoid.

**Corollary 8.9.** The \(M\)-valued cobordism invariant \((X^n, f) \mapsto (\pi_1(\hat{\phi}))(S_{[\omega]}(X^n, f)) \in M\) extends to three assignments

\[
S_{\phi, \alpha} : \mathcal{Y}_i^{E}(n; \Gamma) \to N_i(M_{\text{Tor}}), \ i = 0, 1, 2,
\]

which are symmetric monoidal functors and compatible with the ‘face maps’ \(\partial^k_i\)'s.

**Proof.** We give an explanation what \(S_{\phi, \alpha}\)'s assigns to each object. Since we construct \(S_{\phi, \alpha}\)'s from \(S_{p}\)'s in a canonical way, corresponding morphisms are determined by natural transformations between \(S_{\alpha}\)'s.
Let \((Y^{n-1}, g) \in \mathcal{Y}_E^n(n; \Gamma)\) be an object. Then it induces a functor
\[
\hat{\varphi}(S_\alpha(Y^{n-1}, \eta, g)) : [Y^{n-1}]_E \to M\text{Tor}
\]
Since we have \(\pi_1([Y^{n-1}]_E) \cong E_n(Y^{n-1}) \cong 0\) by definition, this functor has a canonical limit in the category \(M\text{Tor}\). We define it as \(S_{\hat{\varphi}, \alpha}(Y^{n-1}, g) \in M\text{Tor}\).

Let \((Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1}, f) \in \mathcal{Y}_E^n(n; \Gamma)\). Then we are given the following \(M\)-equivariant map for \(\sigma \in [[X^n]]_E\):
\[
\omega_\sigma : \hat{\varphi}(S_\alpha(Y_0^{n-1}, \sigma|_{Y_0^{n-1}}, f^*(\alpha)|_{Y_0^{n-1}})) \to \hat{\varphi}(S_\alpha(Y_1^{n-1}, \sigma|_{Y_1^{n-1}}, f^*(\alpha)|_{Y_1^{n-1}})).
\]
If \(s = (s(\eta))_{\eta \in [[Y_0^n]]_E} \in \lim_{\eta \in [[Y_0^n]]_E} \varphi(\eta, (f|_{Y_0^n})^*(\alpha))\), we define \(t = (t(\eta))_{\eta \in [[Y_1^n]]_E} \in \lim_{\eta \in [[Y_1^n]]_E} \varphi(\eta, (f|_{Y_1^n})^*(\alpha))\) using the morphism (107). We put \(t = \omega_\sigma(s))\). Let \(\sigma' \in [[X^n]]_E\) be another object. Then we have a unique morphism \((p_1)_*(\sigma) \to (p_1)_*(\sigma)\) in the groupoid \([[Y_0^{n-1}]_E\) since \([[Y_1^{n-1}]_E\) is simply-connected due to the assumption \(\pi_1([[Y_0^{n-1}]_E) \cong E_n(Y^{n-1}) \cong 0\). Under the unique morphism, we have \(t((p_1)_*(\sigma)) \to t((p_1)_*(\sigma'))\). In other words, \((p_1)_*(\sigma) \to t((p_1)_*(\sigma))\) determines a partially defined section of \(S_{\hat{\varphi}, \alpha}(Y_0^{n-1}, f|_{Y_0^{n-1}})\). Since \([[Y_1^{n-1}]_E\) is simply-connected, the section extends uniquely to \([[Y_1^{n-1}]_E\). We define it the section \(s\) as \(S_{\hat{\varphi}, \alpha}(Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1}, f)\). Thus we obtain an \(M\)-equivariant map:
\[
S_{\hat{\varphi}, \alpha}(Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1}, f) : S_{\hat{\varphi}, \alpha}(Y_0^{n-1}, f|_{Y_0^{n-1}}) \to S_{\hat{\varphi}, \alpha}(Y_1^{n-1}, f|_{Y_1^{n-1}}).
\]
(109)
Let \((D, h) \in \mathcal{Y}_E^n(n; \Gamma)\). We define \(S_{\hat{\varphi}, \alpha}(D, h)\) as the following diagram:
\[
S_{\hat{\varphi}, \alpha}(Y_0^{n-1}, h|_{Y_0^{n-1}}) \xrightarrow{S_{\hat{\varphi}, \alpha}(X_0^{n-1}, h|_{X_0^{n-1}})} S_{\hat{\varphi}, \alpha}(Y_1^{n-1}, h|_{Y_1^{n-1}})
\]
Then the diagram (110) commutes due to Theorem 8.4.

These functors \(S_{\hat{\varphi}, \alpha}\)'s are symmetric monoidal since \(S_\alpha\)'s do. For example, since we have a natural isomorphism
\[
S_\alpha(Y_0^{n-1}, \eta_0, g_0) \oplus S_\alpha(Y_1^{n-1}, \eta_0, g_0) \cong S_\alpha(Y_0^{n-1} \sqcup Y_1^{n-1}, \eta_0 \sqcup \eta_1, g_0 \sqcup g_1)
\]
by Theorem 8.4, we are given an isomorphism
\[
S_{\hat{\varphi}, \alpha}(Y_0^{n-1}, \eta_0) \oplus S_{\hat{\varphi}, \alpha}(Y_1^{n-1}, \eta_1) \cong S_{\hat{\varphi}, \alpha}(Y_0^{n-1} \sqcup Y_1^{n-1}, \eta_0 \sqcup \eta_1)
\]
which also satisfies the axioms of symmetric monoidal functor. It completes our proof. \(\Box\)

9. SOME LEMMAS FOR PUSH-FORWARD

We shall take an integral of the classical field theory over the groupoid of classifying maps in order to construct a TQFT. In this section, we prepare some tools to deal with all of these ‘integrals’.

There is a notion of a cardinality of groupoids introduced by J. C. Baez and J. Dolan [BD01], which gives an invariant with respect to equivalence of groupoids. The notion cardinality for finite groupoids is generalized to finite categories, which is called the Euler characteristic of category [Lei08], [BL08]. We denote it \(\chi\) where \(\chi(\mathcal{G}) \in \mathbb{Q}\) where \(\mathcal{G}\) is a groupoid with finitely many components. It is characterized by the following axioms:

1. For groupoids \(\mathcal{G}, \mathcal{G}'\) with finitely many components, \(\mathcal{G} \simeq \mathcal{G}'\) implies that \(\chi(\mathcal{G}) = \chi(\mathcal{G}')\).
(2) For groupoids $\mathcal{G}, \mathcal{G}'$ with finitely many components,
\[ \chi(\mathcal{G} \sqcup \mathcal{G}') = \chi(\mathcal{G}) + \chi(\mathcal{G}'). \]

(3) For an arbitrary group $G$,
\[ \chi(G[1]) = \begin{cases} 1 & \text{if } \#G = +\infty, \\ 0 & \text{otherwise} \end{cases} \]

where if $\#G = +\infty$, we set $\chi(G[1]) = 0$.

Then the groupoid cardinality $\chi$ is used to define ‘integral’ of ‘functions’ over groupoids $\mathcal{G}$.

**Lemma 9.1.** Let $\mathcal{G}_0, \mathcal{G}_1$ be groupoids such that $\mathcal{G}_0$ has finitely many components and $\mathcal{G}_1$ is 0-connected. Let $f : \mathcal{G}_0 \to \mathcal{G}_1$ be a functor. Then we have $\chi(\mathcal{G}_0) = \chi(\mathcal{G}_1) \cdot \chi(F_f(a))$ for every object $a \in \mathcal{G}_1$ if $\chi = \chi$ and $F_f(a)$ has finitely many components. The notation $F_f(a)$ is the homotopy fiber over $a \in \mathcal{G}_1$ introduced in Definition A.5.

**Proof.** If $\#\pi_1(\mathcal{G}_1)$ is finite, then by Proposition A.7 we see that $\#\pi_1(\mathcal{G}_0), \#\pi_1(F_f(a))$ are finite. According to Proposition A.7 again, we obtain,
\[ \#\pi_1(F_f(a)) \cdot \#\pi_1(\mathcal{G}_0) \cdot \#\pi_0(\mathcal{G}) = \#\pi_1(\mathcal{G}_0) \cdot \#\pi_0(F_f(a)), \]

so that it proves $\chi(\mathcal{G}_0) = \chi(\mathcal{G}_1) \cdot \chi(F_f(a))$. If $\#\pi_1(\mathcal{G}_1)$ is infinite, then $\pi_1(\mathcal{G}_0)$ and $\#\pi_1(F_f(a))$ are infinite according to Proposition A.7. Hence we obtain $\chi(\mathcal{G}_0) = 0 = \chi(\mathcal{G}_1) \cdot \chi(F_f(a))$.  

**Definition 9.2.** Let $\mathcal{G}$ be a groupoid with finitely many components. For a linear space $V$ over $\mathbb{C}$, we define a map \( \sum_{\mathcal{G}} (-) \cdot \chi : \text{Map}(\pi_0(\mathcal{G}), V) \to V \) by
\[ \sum_{\mathcal{G}} f \cdot \chi \overset{\text{def}}{=} \sum_{[x] \in \pi_0(\mathcal{G})} f([x]) \cdot \chi([x]) \in V. \]

Here $[x]$ denotes the component groupoid which contains $x \in \mathcal{G}$.

**Definition 9.3.** Let $\mathcal{G}$ be a groupoid with finitely many components. We define a functor \( \lim_{\mathcal{G}} (\cdot) \cdot \chi : \text{Func}(\mathcal{G}, \text{Vect}^\times) \to \text{Vect}^\times \). For a functor $B : \mathcal{G} \to \text{Vect}^\times$, the underlying linear space of a hermitian linear space $\lim_{\mathcal{G}} (B \cdot \chi)$ is the limit $\lim_{\mathcal{G}} (i \circ B)$ where $i : \text{Vect}^\times \to \text{Vect}$ is the inclusion functor. For two sections $v, w \in \lim_{\mathcal{G}} (i \circ B)$, we define its inner product by
\[ \langle v, w \rangle \overset{\text{def}}{=} \sum_{[x] \in \pi_0(\mathcal{G})} \langle v(x), w(x) \rangle_{B(x)} \cdot \chi([x]) \in \mathbb{C}. \]

Here $[x]$ denotes the component groupoid which contains $x \in \mathcal{G}$.

**Definition 9.4.** Let $\mathcal{G}_0, \mathcal{G}_1$ be groupoids with finitely many components and $r \in \text{Func}(\mathcal{G}_0, \mathcal{G}_1), B \in \text{Func}(\mathcal{G}_1, \text{Vect}^\times)$. Suppose that any homotopy fiber $F_r(y)$ of the functor $r$ has finitely many components. We define a morphism $r_s : \lim_{\mathcal{G}_0} ((B \circ r) \cdot \chi) \to \lim_{\mathcal{G}_1} (B \cdot \chi)$ in $\text{Vect}$ i.e. a linear map (not in $\text{Vect}^\times$). Given $s \in \lim_{\mathcal{G}_0} ((B \circ r) \cdot \chi)$ and $y \in \mathcal{G}_1$, we define $(r_s(s))(y) \in B(y)$ by the following equation :
\[ (r_s(s))(y) \overset{\text{def}}{=} \sum_{F_r(y)} B_y \cdot \chi, y \in \mathcal{G}_1, \]

where $B_y \in \text{Map}(\pi_0(F_r(y)), B(y))$ is defined by $(x, \theta) \in F_r(y) \mapsto (B(\theta))(s(x)))$. It is well-defined due to the following lemma.

**Lemma 9.5.** (1) The map $B_y : \pi_0(F_r(y)) \to B(y)$ is well-defined
(2) If $\alpha : y_0 \to y_1$ is a morphism in $\mathcal{G}_1$, then we have $(B(\alpha))(r_s(s))(y_0)) = (r_s(s))(y_1)$. 
Proof. For the first claim, it suffices to show that \( \theta_\ast(s(x)) = \theta_\ast'(s(x')) \in B(y) \) if there is an isomorphism \( \beta : (x, \theta) \to (x', \theta') \). By the definition, \( \beta : x \to x' \) is a morphism in \( \mathcal{G}_0 \) such that \( \theta' \circ r(\beta) = \theta \) so that \( B(\theta)(s(x)) = B(\theta') \circ B(r(\beta))(s(x)) = B(\theta')(s(x')) \). The final equality holds since \( s \in \lim(F \circ r) \).

Let us prove the second claim. Note that \( \alpha : y_0 \to y_1 \) induces an isomorphism of groupoids \( F_r(y_0) \to F_r(y_1) \) defined as

\[
(x, \theta) \mapsto (x, \alpha \circ \theta)
\]

Then this morphism induces the commutative diagram.

\[
\begin{array}{ccc}
\pi_0(F_r(y_0)) & \xrightarrow{B_{\alpha}} & B(y_0) \\
\downarrow & & \downarrow \phi_{(\alpha)} \\
\pi_0(F_r(y_1)) & \xrightarrow{B_{\beta}} & B(y_1)
\end{array}
\]

It completes the proof. \( \square \)

**Definition 9.6.** Let \( \mathcal{G} \) be a groupoid with finitely many components and \( B_0, B_1 \in \text{Func}(\mathcal{G}, \text{Vect}^\times) \). We define a map \( \text{Nat}(B_0, B_1) \to \text{Vect}^\times(\lim_\mathcal{G}(B_0 \cdot \chi), \lim_\mathcal{G}(B_1 \cdot \chi)) ; \Phi \mapsto \Phi_\ast \) by

\[
(\Phi_\ast(s))(x) \overset{\text{def}}{=} (\Phi(x))(s(x)) \in B_1(x),
\]

where \( s \in \lim_\mathcal{G}(B_0 \cdot \chi), x \in \mathcal{G} \). Here, \( \text{Nat}(B_0, B_1) \) is the set of natural transformations from the functor \( B_0 \) to the functor \( B_1 \).

**Remark 9.7.** In Definition 9.4 we introduced the notation \( r_\ast \) for a functor \( r : \mathcal{G}_0 \to \mathcal{G}_1 \). On the other hand, in Definition 9.6 we introduced the notation \( \Phi_\ast \) for a natural transformation \( \Phi : B_0 \to B_1 \). These are quite different notions in that \( r_\ast \) depends on the groupoid cardinality \( \chi \) and \( \Phi_\ast \) does not.

We prove some lemmas using the above notations. These lemmas are important to prove some commutative diagrams which appear in construction of quantum theory.

**Lemma 9.8.** Let \( \mathcal{G} \) be a groupoid with finitely many components. Let \( B_0, B_1, B_2 \in \text{Func}(\mathcal{G}, \text{Vect}^\times) \) and \( \Phi : B_0 \to B_1, \Phi' : B_1 \to B_2 \) be natural transformations. Then the following diagram commutes:

\[
\begin{array}{ccc}
\lim_\mathcal{G}(B_0 \cdot \chi) & \xrightarrow{\Phi_\ast} & \lim_\mathcal{G}(B_1 \cdot \chi) \\
\downarrow (\Phi' \circ \Phi)_\ast & & \downarrow (\Phi')_\ast \\
\lim_\mathcal{G}(B_2 \cdot \chi)
\end{array}
\]

**Lemma 9.9.** Let \( \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2 \) be groupoids with finitely many components. Let \( \mathcal{G}_0 \overset{f}{\to} \mathcal{G}_1 \overset{g}{\to} \mathcal{G}_2 \) be a sequence of functors such that every homotopy fiber of \( f, g \) has finitely many components (hence so \( g \circ f \) does). For a functor \( B : \mathcal{G}_2 \to \text{Vect}^\times \), the following diagram of linear maps commutes:

\[
\begin{array}{ccc}
\lim_\mathcal{G}_0(B \circ g \circ f) \cdot \chi & \xrightarrow{f_\ast} & \lim_\mathcal{G}_1(B \circ g) \cdot \chi \\
\downarrow (g \circ f)_\ast & & \downarrow g_\ast \\
\lim_\mathcal{G}_2 B \cdot \chi
\end{array}
\]
Proof. Let \( v \in \lim_{g_0} (B \circ g \circ f) \cdot \chi \). According to definitions, we have the followings for an object \( c \in B_2 \):

\[
(g_* f_* v)(c) = \sum (B(\xi \circ g(\theta)))(v(a)) \cdot \chi([a, \theta]) \cdot \chi([b, \xi])
\]

\[
((g \circ f)_* v)(c) = \sum (B(\varepsilon))(v(a)) \cdot \chi([a, \varepsilon])
\]

where the first sum is taken for \([b, \xi] \in F_g(c), [a, \theta] \in F_f(b)\) and the second sum is taken for \([a, \varepsilon] \in F_{g\circ f}(c)\). Then by Lemma 9.1 and Proposition A.8, we obtain \((g_* f_* v)(c) = ((g \circ f)_* v)(c)\). Therefore, it completes the proof. \(\square\)

From Lemma 9.8 and Lemma 9.9, it is straightforward to obtain the following corollary.

**Corollary 9.10.** Let \( G_0, G_1, G_2 \) be groupoids with finitely many components. Consider functors \( B_j \in \text{Func}(G_j, \text{Vect}^\wedge) \) for \( j = 0, 1, 2 \), and \( r_0 \in \text{Func}(G_1, G_2) \), \( r_1 \in \text{Func}(G_0, G_2) \), \( r_2 \in \text{Func}(G_0, G_1) \). Let \( \Psi : r_0 \circ r_2 \to r_1 \) be a natural transformation. We also consider natural transformations given by the following diagram:

\[
\begin{array}{ccc}
G_0 & \xrightarrow{\phi_2} & G_1 \\
\downarrow{\phi_1} & & \downarrow{\phi_0} \\
\text{Vect} & \xrightarrow{\Psi} & \text{Vect} \\
\downarrow{i \circ B_2} & & \downarrow{i \circ B_2 \circ r_0} \\
G_2 & \xrightarrow{r_1} & G_1 \\
\end{array}
\]

Here \( i : \text{Vect}^\wedge \to \text{Vect} \) is the inclusion functor. Suppose that the following diagram in the category \( \text{Func}(G_0, \text{Vect}) \) commutes:

\[
\begin{array}{ccc}
i \circ B_0 & \xrightarrow{\phi_2} & i \circ B_1 \circ r_2 \\
\downarrow{\phi_1} & & \downarrow{\phi_0} \\
i \circ B_2 \circ r_1 & \xleftarrow{\Psi} & i \circ B_2 \circ r_0 \circ r_2
\end{array}
\]

Then the following diagram of linear maps commutes:

\[
\begin{array}{ccc}
\lim_{g_0} B_0 \cdot \chi & \xrightarrow{(r_2)_* \circ (\phi_2)_*} & \lim_{g_1} B_1 \cdot \chi \\
\downarrow{(r_1)_* \circ (\phi_1)_*} & & \downarrow{(r_0)_* \circ (\phi_0)_*} \\
\lim_{g_2} B_2 \cdot \chi & & \lim_{g_2} B_2 \cdot \chi
\end{array}
\]

In the following lemma, we give a condition for some push-forwards and pull-backs to commute.

**Lemma 9.11.** Let \( \mathcal{H}, \mathcal{H'}, \mathcal{G}, \mathcal{G'} \) be groupoids with finitely many components. Let \( B : \mathcal{G} \to \text{Vect}^\wedge \), \( B' : \mathcal{G'} \to \text{Vect}^\wedge \), \( C : \mathcal{H} \to \text{Vect}^\wedge \), \( C' : \mathcal{H'} \to \text{Vect}^\wedge \), \( \phi : \mathcal{H} \to \mathcal{G} \), \( \pi : \mathcal{H} \to \mathcal{H'} \), \( r' : \mathcal{H'} \to \mathcal{G} \), \( \Delta : \mathcal{G} \to \mathcal{G'} \) be functors. Let \( G : r' \circ \pi \to \Delta \circ \phi \) be a natural transformation. Let
A, A', S, T be natural transformations given by the following diagram:

Here $i : \text{Vect}^\times \to \text{Vect}$ is the inclusion functor. We suppose that

1. The natural transformation $G$ gives the following homotopy pull-back diagram:

2. Every homotopy fiber of $r'$ has finitely many components. We can deduce from the previous assumption that every homotopy fiber of $\phi$ has finitely many components.

3. The following diagram of natural transformations commutes:

Then the following diagram of linear homomorphisms commutes:

\[ \lim_{\to A} (C' \cdot \chi) \xrightarrow{A_u \cdot \phi_u} \lim_{\to B} (B \cdot \chi) \]

\[ \lim_{\to A} (C' \cdot \chi) \xrightarrow{A'_u \cdot \phi'_u} \lim_{\to B'} (B' \cdot \chi) \]

Proof. The proof follows from their definitions. Let $v \in \lim_{\to \mathcal{H}} (C' \cdot \chi)$, i.e. $v(P') \in C'(P')$ for $P' \in \mathcal{H}$ and $C'(f) : v(P'_0) \to v(P'_1)$ for a morphism $f : P'_0 \to P'_1$ in $\mathcal{H}$. For $Q' \in \mathcal{S}'$, we have $((A'_u \circ r'_u)(v))(Q') = \sum_{[P', \theta']} (B'(\theta')(A'(P'))(v(P')) \cdot \chi([P', \theta']))$ where the sum is taken for $[P', \theta'] \in \pi_0(F_{r'}(Q'))$. Hence, for $Q \in \mathcal{H}$, we have $((T_u \circ \Delta^* \circ A'_u \circ r'_u)(v))(Q) = \sum_{[P', \theta']} (T(Q))(S_{\pi}(A'(P'))(v(P')) \cdot \chi([P', \theta']))$ where the sum is taken for $[P', \theta'] \in \pi_0(F_{r'}(\Delta(Q)))$. By the first assumption, the canonical functor $F_{\phi}(Q) \to F_{r'}(\Delta(Q)) ; (P, \theta) \mapsto (\pi(P), \Delta(\theta) \circ G(P))$ induces an equivalence. Therefore, $((T_u \circ \Delta^* \circ A'_u \circ r'_u)(v))(Q)$ can be
represented as follows.

\[(T_\ast \circ \Delta^* \circ A'_\ast \circ r'_a)(v))(Q)\]

\[= \sum_{[P, \theta]} (T(Q))(B'(\Delta(\theta) \circ G(P))(A'(\pi(P)))(v(\pi(P)))) \cdot \chi([P, \theta])\]

\[= \sum_{[P, \theta]} (B(\theta) \circ T(\phi(P)) \circ B'(G(P)) \circ A'(\pi(P)))(v(\pi(P))) \cdot \chi([P, \theta])\]

\[= \sum_{[P, \theta]} (B(\theta) \circ A(P) \circ S(P))(v(\pi(P))) \cdot \chi([P, \theta])\]

On the other hand, \[((S_a \circ \pi^*)(v))(P) = (S(P))(v(\pi(P))) \in C(P)\) so that \[((A_\ast \circ \phi_\ast \circ S_\ast \circ \pi^*)(v))(Q) = \sum_{[P, \theta]} B(\theta)((A(P))(S(P))(v(\pi(P)))) \cdot \chi([P, \theta])\] where the sum is taken for \([P, \theta] \in \pi_0(F_G(Q))\). Above all, we obtain \[((T_\ast \circ \Delta^* \circ A'_\ast \circ r'_a)(v))(Q) = ((A_\ast \circ \phi_\ast \circ S_\ast \circ \pi^*)(v))(Q)\]. □

Let \(G_i \leftarrow G_{ij} \rightarrow G_j\) be a span of groupoids \(V_i : G_i \rightarrow \text{Vect}^\times\) be functors for \(i, j = 0, 1, 2, i < j\), and \(\Phi_{01} : V_0 \circ r_0 \rightarrow V_1 \circ r_1\) be a natural transformation. Then we obtain a linear map \(F_{ij} : \lim_{G_i} (V_i \cdot \chi) \rightarrow \lim_{G_j} (V_j \cdot \chi)\) from compositions of the following maps:

\[\text{(113)} \quad \lim_{G_i} (V_i \cdot \chi) \cong \lim_{G_i} (V_i \cdot \chi) \otimes \mathbb{C}\]

\[\rightarrow \lim_{G_i} (V_i \cdot \chi) \otimes \lim_{G_{ij}} (\mathbb{C} \cdot \chi)\]

\[\rightarrow \lim_{G_i} (V_0 \cdot \chi) \otimes \lim_{G_{ij}} \left(\frac{V_i \boxtimes V_j \cdot \chi}{V_j \cdot \chi}\right)\]

\[\cong \lim_{G_i} (V_i \cdot \chi) \otimes \lim_{G_{ij}} (V_i \cdot \chi) \otimes \lim_{G_j} (V_j \cdot \chi)\]

\[\rightarrow \lim_{G_j} (V_j \cdot \chi)\]

The map (114) is determined by \(1 \in \mathbb{C} \rightarrow (1_a)_{a \in G_j} \in \lim_{G_{ij}} (\mathbb{C} \cdot \chi)\). We use \((r_i \times r_j)_* \circ (\Phi_{ij})_* : \lim_{G_{ij}} (\mathbb{C} \cdot \chi) \rightarrow \lim_{G_i \times G_j} \left(\frac{V_i \boxtimes V_j}{V_j \cdot \chi}\right)\) to define (115). Under these notations, we have the following proposition.

**Proposition 9.12.** Suppose that we have the following commutative diagram of functors between groupoids.

\[
\begin{array}{ccc}
G_0 & \xrightarrow{ \cong } & G_0\\
\downarrow G_{01} & \nearrow & \downarrow G_{02}\\
G_1 & \xrightarrow{ \cong } & G_2\\
\downarrow G_{12} & & \downarrow G_{12}\\
G_2 & \xrightarrow{ \cong } & G_1\\
\end{array}
\]

If the functor \(G_{012} \rightarrow G_{02}\) induces an equivalence of categories and the left bottom commutative diagram with vertices \(G_1, G_{01}, G_{12}, G_{012}\) forms a homotopy pull-back diagram, then we have \(F_{12} \circ F_{01} = F_{02}\).
Proof. The assumption is paraphrased as following diagram:

\[
\begin{array}{ccc}
G_0 & \to & G_0 \times G_2 \\
\uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Theorem 10.1. Let \( \varphi \) be the homomorphism \( \pi_1(\hat{\varphi}) : G_0(pt) \cong \pi_1(\mathcal{H}H(S^n, S^{n+1}; G)) \to \pi_1(U(1)\text{Tor}) \cong U(1) \). The invariant \( Z_{\varphi,[\alpha]} \) in Definition 8.2 extends to a family of assignments,

\[
Z_{\varphi,\alpha} : X^E_i(n; o) \to N_i(\text{Vect}), \quad i = 0, 1, 2,
\]

which are symmetric monoidal functors and compatible with the ‘face maps’ \( \hat{\gamma}_i^k \)'s.

Proof. The assignments \( Z_{\varphi,\alpha} \)'s in the theorem are constructed through a linearization of the assignments \( S_{\varphi,\alpha} \)'s in Theorem 8.4 as follows. To begin with, we define a symmetric monoidal functor

\[
Z_{\varphi,\alpha} : X^E_0(n; o) \to N_0(\text{Vect}) \cong \text{Vect},
\]

by

\[
Z_{\varphi,\alpha}(Y^{n-1}, \eta) \overset{\text{def}}{=} \lim_{g} (S_{\varphi,\alpha}(Y^{n-1}, \eta, g) \cdot \chi) \in \text{Vect}.
\]

A morphism \((u, v) : (Y^{n-1}_0, \eta_0) \to (Y^{n-1}_1, \eta_1)\) in the groupoid \( X^E_i(n; o) \) induces a natural isomorphism \( S_{\varphi,\alpha}(Y^{n-1}, \eta_0, u^*(g)) \to S_{\varphi,\alpha}(Y^{n-1}, \eta_1, g), \ g \in \Pi_1\text{MAP}(Y^{n-1}_1, B\Gamma) \), which induces a unitary isomorphism which we denote by

\[
Z_{\varphi,\alpha}(u, v) : Z_{\varphi,\alpha}(Y^{n-1}_0, \eta_0) \to Z_{\varphi,\alpha}(Y^{n-1}_1, \eta_1).
\]

Then the assignment \( Z_{\varphi,\alpha} \) determines a functor.

As a next step, we define a symmetric monoidal functor

\[
Z_{\varphi,\alpha} : X^E_1(n; o) \to N_1(\text{Vect}),
\]

Let \((Y^{n-1}_0 \to X^n \leftarrow Y^{n-1}_1, \sigma)\) be an object of \( X^E_1(n; o) \). We set \( G_0 = \Pi_1\text{MAP}(Y^{n-1}_0, B\Gamma), \ G_1 = \Pi_1\text{MAP}(Y^{n-1}_1, B\Gamma) \), and \( G_{01} = \Pi_1\text{MAP}(X^n, B\Gamma) \)

\[
\Phi_{01} = S_{\varphi,\alpha}(Y^{n-1}_0 \to X^n \leftarrow Y^{n-1}_1, \sigma, f) : S_{\varphi,\alpha}(Y^{n-1}_0, \sigma|_{Y^{n-1}_0}, f|_{Y^{n-1}_0}) \to S_{\varphi,\alpha}(Y^{n-1}_1, \sigma|_{Y^{n-1}_1}, f|_{Y^{n-1}_1}).
\]

Then by applying the construction in the final paragraph of section 9 we obtain a linear map \( F_{01} \). We denote it by \( Z_{\varphi,\alpha}(Y^{n-1}_0 \to X^n \leftarrow Y^{n-1}_1, \sigma) : Z_{\varphi,\alpha}(Y^{n-1}_0, \sigma|_{Y^{n-1}_0}) \to Z_{\varphi,\alpha}(Y^{n-1}_1, \sigma|_{Y^{n-1}_1}) \) which is a 1-nerve of the category \( \text{Vect} \). Using the second assignment in Theorem 8.4, a morphism \((u, v) : (X^n, \sigma_0) \to (X^n, \sigma_1)\) in the groupoid \( X^E_i(n; o) \) induces

\[
S_{\varphi,\alpha}(u|_{Y^{n-1}_0}, v|_{Y^{n-1}_0}, 1_a \sigma(f)|_{Y^{n-1}_0}) \circ S_{\varphi,\alpha}(X^n, \sigma, u^*(f))
\]

\[
= S_{\varphi,\alpha}((X^n, \sigma, f) \circ S_{\varphi,\alpha}(u|_{Y^{n-1}_0}, v|_{Y^{n-1}_0}, 1_a \sigma(f)|_{Y^{n-1}_0}))
\]

It gives a morphism in \( N_1(\text{Vect}) \),

\[
Z_{\varphi,\alpha}(u, v) : Z_{\varphi,\alpha}(X^n, \sigma) \to Z_{\varphi,\alpha}(X^n, \sigma).
\]

Then the assignment \( Z_{\varphi,\alpha} \) gives a functor.

Finally, we construct a symmetric monoidal functor by applying Proposition 9.12

\[
Z_{\varphi,\alpha} : X^E_2(n; o) \to N_2(\text{Vect}).
\]

Let \((D, \xi)\) be an object of \( X^E_2(n; o) \). We set \( G_{ij} = \Pi_1\text{MAP}(Y^{n-1}_i, B\Gamma), \ G_{ij} = \Pi_1\text{MAP}(X^n_j, B\Gamma) \) and \( G_{012} = \Pi_1\text{MAP}(W^{n+1}, B\Gamma) \) in the setting of Proposition 9.12. We assign the triple of

\[
\left(Z_{\varphi,\alpha}(Y^{n-1}_i \to X^n_j \leftarrow Y^{n-1}_j, \xi|_{X^n_j}) \right)_{i < j}
\]

to \((D, \xi)\). Here is the reason we only consider the diagram \( D \) given as (55). Due to the condition of the diagram (55), these setting satisfies the assumption of Proposition 9.12. The triple is a 2-nerve of the category \( \text{Vect} \) from Proposition 9.12 so that it is well-defined. In
a similar way with the above construction of \( \text{(129)} \), we construct a morphism \( Z_{\hat{\varphi},\alpha}(u, v) : Z_{\hat{\varphi},\alpha}(W^{n+1}, \xi) \to Z_{\hat{\varphi},\alpha}(\hat{W}^{n+1}, \hat{\xi}) \) in the category \( N_2(\text{Vect}) \).

These three functors \( Z_{\hat{\varphi},\alpha} \) constructed before inherit symmetric monoidality from the three functors \( S_{\hat{\varphi},\alpha} \) in Theorem \( \text{8.4} \). Above all, it completes our proof. □

From now on, we give one of our main results by eliminating dependence on the extra data from \( Z_{\hat{\varphi},\alpha} \) in Theorem \( \text{10.1} \) by requiring some restrictions on manifolds. Let us define three groupoids \( \mathcal{Y}_0^E(n), \mathcal{Y}_1^E(n), \mathcal{Y}_2^E(n) \) for \( n \in \mathbb{Z}_{\geq 0} \). They are defined by eliminating the components of classifying maps in the definition of \( \mathcal{Y}_0^E(n; \Gamma), \mathcal{Y}_1^E(n; \Gamma), \mathcal{Y}_2^E(n; \Gamma) \). For the object class of \( \mathcal{Y}_0^E(n) \) consists of \( Y^{n-1} \) where \( Y^{n-1} \) is an \( E \)-oriented closed \((n-1)\)-manifold such that \( E_\alpha(Y^{n-1}) \cong 0 \). A morphism is given as an \( E \)-orientation preserving homeomorphism. Each groupoid \( \mathcal{Y}_i^E(n) \) has a symmetric monoidal structure as \( \mathcal{X}_E^E(n; o) \) does. Also we can define ‘face maps’ \( \hat{\varphi}_i \)'s between \( \mathcal{Y}_i^E(n) \)'s, which is given as a symmetric monoidal functor.

**Theorem 10.2.** Let \( \varphi \) be the homomorphism \( \pi_1(\hat{\varphi}) : G_0(\text{pt}) \cong \pi_1(\mathcal{H}(S^n, S^{n+1}; G)) \to \pi_1(U(1)\text{Tor}) \cong U(1) \). The invariant \( Z_{\varphi,\alpha} \) in Definition 3.2 extends to a family of assignments,

\[
(131) \quad Z_{\varphi,\alpha} : \mathcal{Y}_i^E(n) \to N_i(\text{Vect}), \quad i = 0, 1, 2,
\]

which are strong symmetric monoidal functors compatible with ‘face maps’ \( \hat{\varphi}_i \)'s.

**Proof.** We use the previous Theorem \( \text{10.1} \). For an \( E \)-oriented closed manifold \( Y^{n-1} \), we have a functor \([ [Y^{n-1}] ]_E \to \text{Vect} \) given by

\[
(132) \quad \eta \mapsto Z_{\varphi,\alpha}(Y^{n-1}, \eta)
\]

We define \( Z_{\varphi,\alpha}(Y^{n-1}) \equiv \lim_n Z_{\varphi,\alpha}(Y^{n-1}, \eta) \). Then the assignment \( Y^{n-1} \mapsto Z_{\varphi,\alpha}(Y^{n-1}) \) determines a functor \( Z_{\varphi,\alpha} : \mathcal{Y}_0^E(n) \to \text{Vect} \cong N_0(\text{Vect}) \).

In the same manner, we define \( Z_{\varphi,\alpha}(Y^{n-1} \to X^n \leftarrow Y^{n-1}) \in N_1(\text{Vect}) \) (i.e. a linear map) to satisfy the following equation

\[
(133) \quad w(\sigma'|_{Y^{n-1}}) = (Z_{\varphi,\alpha}(X^n, \sigma') (v(\sigma'|_{Y^{n-1}})) \in Z_{\varphi,\alpha}(X^n, \sigma'|_{Y^{n-1}}).
\]

where \( \eta_1 \in [ [Y^{n-1}] ]_E \mapsto w(\eta_1) \in Z_{\varphi,\alpha}(Y^{n-1}, \eta_1) \) is the section corresponding to \( \eta_0 \in [ [Y^{n-1}] ]_E \mapsto v(\eta_0) \in Z_{\varphi,\alpha}(Y^{n-1}, \eta_0) \). It is possible since \([ [X^n] ]_E \)'s are simply-connected and \([ [X^n] ]_E \) is connected. It determines a functor \( Z_{\varphi,\alpha} : \mathcal{Y}_1^E(n) \to N_1(\text{Vect}) \).

Finally we explain how to construct \( Z_{\varphi,\alpha} : \mathcal{Y}_2^E(n) \to N_2(\text{Vect}) \). For a diagram \( \text{(55)} \), we first construct the following diagram of linear maps.

\[
\begin{array}{ccc}
Z_{\varphi,\alpha}(Y_0^{n-1}) & \to & Z_{\varphi,\alpha}(Y_2^{n-1}) \\
Z_{\varphi,\alpha}(X_{02}) & \nearrow & Z_{\varphi,\alpha}(X_{01}) \\
Z_{\varphi,\alpha}(X_{00}) & \searrow & Z_{\varphi,\alpha}(X_{11})
\end{array}
\]

Since we have \( Z_{\varphi,\alpha}(X_{02}, \xi|_{X_{02}}) \circ Z_{\varphi,\alpha}(X_{01}, \xi|_{X_{01}}) = Z_{\varphi,\alpha}(X_{02}, \xi|_{X_{02}}), \xi \in [ [W^{n+1}] ]_E \) from Theorem \( \text{10.1} \) the above diagram commutes. Above all, it completes the proof. □

As a corollary of the previous theorem, we give a strong symmetric monoidal functor from a cobordism category to the category \( \text{Vect} \). We need to find an appropriate cobordism category. In order to define an appropriate cobordism category of \( E \)-oriented manifolds, we need to ensure that a cobordism obtained by gluing two cobordisms in that cobordism category lives in it again. Note that we consider topological manifolds whose underlying space is a CW-space until now in this section. We do not know whether two cobordisms
whose underlying spaces are CW-spaces are glued into a cobordism whose underlying space is a CW-space. Thus, for convenience, we restrict ourselves only to smooth manifolds.

We define a cobordism category \( \textbf{Cob}^E(n)_0 \) as follows. Until now, we consider topological manifolds, but we only deal with smooth manifolds for the rest of this section. Its object class consists of closed \( E \)-oriented smooth \((n-1)\)-manifolds \( Y^{n-1} \) such that \( E_n(Y^{n-1}) \cong 0 \). For such two objects \( Y_0^{n-1}, Y_1^{n-1} \), an \( E \) oriented cobordism \( Y_0^{n-1} \to X^n \hookrightarrow Y_1^{n-1} \) up to \( E \)-oriented diffeomorphism. Due to the long exact sequence associated with \( E \)-theory, it is possible to glue two \( E \)-oriented cobordisms along with some isomorphic components of boundaries:

\[
\begin{align*}
E_{n+1}(Y^{n-1} \times I, Y^{n-1} \times \partial I) & \cong \hat{E}_{n+1}(SY^{n-1}) \cong 0 \\
E_{n}(X^n, \partial X^n) & \\
E_{n}(X^n_0, \partial X^n_0) \oplus E_{n}(X^n_1, \partial X^n_1) & \\
E_{n}(Y^{n-1} \times I, Y^{n-1} \times \partial I)
\end{align*}
\]

We set the cylinder \( Y^{n-1} \times I \) as the identity for an object \( Y^{n-1} \). Smooth manifolds are glued into a smooth manifold up to diffeomorphism so that the category \( \textbf{Cob}^E(n)_0 \) is well-defined.

Since smooth manifolds are triangulable, they are CW-spaces so that we can apply Theorem 10.2. From Theorem 10.2, we obtain the following corollary.

**Corollary 10.3.** Let \( \varphi \) be the homomorphism \( \pi_1(\hat{\varphi}) : G_0(\text{pt}) \cong \pi_1(\mathcal{H}(S^n, S^{n+1}; G)) \to \pi_1(U(1)\text{Tor}) \cong U(1) \). The invariant \( Z_{\varphi,a} \) in Definition 3.2 extends to a strong symmetric monoidal functor:

\[
Z_{\hat{\varphi},a} : (\textbf{Cob}^E(n)_0, \text{11}) \to (\textbf{Vect}, \otimes).
\]

**Proof.** The functor \( Z_{\hat{\varphi},a} \) is constructed as follows. It assigns a hermitian space \( Z_{\hat{\varphi},a}(Y^{n-1}) \in \textbf{Vect} \) determined by the functor \( Z_{\hat{\varphi},a} : Y_0^E(n) \to N_0(\textbf{Vect}) \) in Theorem 10.2 to \( Y^{n-1} \in \textbf{Cob}^E(n)_0 \). It assigns a linear map \( Z_{\hat{\varphi},a}(Y_0^{n-1}) \to X^n \hookrightarrow Y_1^{n-1} \) determined by \( Z_{\hat{\varphi},a} : Y_1^E(n) \to N_1(\textbf{Vect}) \) in Theorem 10.2 to an isomorphism class of a cobordism \( Y_0^{n-1} \to X^n \hookrightarrow Y_1^{n-1} \). Due to the diagram (55), we see that these assignments give a functor since any composable morphisms in the category \( \textbf{Cob}^E(n)_0 \) can be denoted using the diagram (55). Finally, the functor \( Z_{\hat{\varphi},a} : \textbf{Cob}^E(n)_0 \to \textbf{Vect} \) constructed before is a strong symmetric monoidal since the family of functors \( Z_{\hat{\varphi},a} : Y_1^E(n) \to N_1(\textbf{Vect}) \) in Theorem 10.2 are strong symmetric monoidal and compatible with ‘face maps’ \( \partial_k^l \).

\[\square\]

11. **KK Categorical Group**

In this section, we construct a categorical group version of \( KK \)-theory as the fundamental groupoid of a space of quasi-homomorphisms between \( C^* \)-algebras where its 0-th homotopy set gives \( KK \)-theory up to isomorphism. The abelian group structure on \( KK \)-theory is lifted to that groupoid as a symmetric categorical group structure.

In subsection 11.1, we introduce two simplicial sets formed by homomorphisms of \( C^* \)-algebras and by quasi-homomorphisms of \( C^* \)-algebras. In subsection 11.2, we define a categorical group version of \( KK \)-theory as the fundamental groupoid of the simplicial set of quasi-homomorphisms.

11.1. **Preliminaries.** Let \( A, B \) be \( C^* \)-algebras. A map \( f : A \to B \) is a homomorphism if it is a \(*\)-homomorphism of \(*\)-algebras \( A, B \). We denote by \( \text{hom}(A, B) \) the set of homomorphisms from \( A \) to \( B \). In this section, we introduce a simplicial set structure ‘on’ the set of homomorphisms \( \text{hom}(A, B) \).
Definition 11.1. We define a simplicial set $\text{HOM}(A, B)$ for $C^*$-algebras $A, B$. For $n \in \mathbb{Z}_{\geq 0}$, we define a set of $n$-simplices of $\text{HOM}(A, B)$ as follows:

$$HOM(A, B)_n \overset{\text{def}}{=} \text{hom}(A, C(\Delta(n), B)).$$

Then the face maps and the degeneracy maps between simplices $\Delta(n)$’s induce a simplicial $C^*$-algebra $C(\Delta(n), B)$’s and a simplicial set $\text{hom}(A, C(\Delta^n, B)) = HOM(A, B)$.

Then the composition $\circ : \text{hom}(A, B) \times \text{hom}(B, C) \to \text{hom}(A, C)$ and the tensor product $\otimes : \text{hom}(A_0, A_1) \times \text{hom}(B_0, B_1) \to \text{hom}(A_0 \otimes B_0, A_1 \otimes B_1)$ are lifted as simplicial maps between $HOM(-, -)$. In other words, we have simplicial maps defined by the pointwise composition and the pointwise tensor product.

(139) $\circ : HOM(A, B) \times HOM(B, C) \to HOM(A, C),$

(140) $\otimes : HOM(A_0, A_1) \times HOM(B_0, B_1) \to HOM(A_0 \otimes B_0, A_1 \otimes B_1)$.

Here, the left hand side denotes the direct product of simplicial sets.

We introduce a notation for iterated Cuntz algebra $q^M A$ for a $C^*$-algebra $A$ following M. Joachim and S. Stolz [JS05].

Definition 11.2. For a $C^*$-algebra $A$ and a finite set $M$, we denote by $Q^M A$ a free product of copies of the $C^*$-algebra $A$ indexed by the power set of $M$:

$$\ast_{K \subseteq M} A.$$

The iterated Cuntz algebra $q^M A$ is defined as the ideal in $Q^M A$ generated by the elements

$$q^M (a) \overset{\text{def}}{=} \sum_{K \subseteq M} (-1)^{|K|} a^K \in Q^M A, a \in A.$$

Let us fix a countably infinite set $U$. We construct a direct system $(q^M A, \pi_{M,N}; M \subset U, z_M < +\infty)$ for a $C^*$-algebra $A$. Let $N \subset M$ be finite subsets in the set $U$. Let $\pi'_{M,N} : Q^M A \to Q^N A$ be the homomorphism induced by $a^K \mapsto a^K$ if $K \subseteq N$ and $a^K \mapsto 0$ otherwise. It induces a homomorphism $\pi_{M,N} : q^M A \to q^N A$.

On the one hand, using a rank-one projection $p \in \mathcal{K}$ we define a direct system $(\mathcal{K}^M \otimes B, p^{M,N} \otimes (-); M \subset U, z_M < +\infty)$. Here $\mathcal{K}$ is the $C^*$-algebra formed by compact operators on a fixed countable Hilbert space which is infinite dimensional. It determines a homomorphism $p^{M,N} \otimes (-) : \mathcal{K}^M \otimes B \to \mathcal{K}^N \otimes B$ where $p^{M,N}$ is the projection obtained by taking tensor products of copies of the projection $p$ indexed by the set $M \setminus N$.

The above direct systems induce a direct system $(HOM(q^M A, \mathcal{K}^M \otimes B), j_{M,N}; M \subset U, z_M < +\infty)$ where $j_{M,N} : HOM(q^N A, \mathcal{K}^N \otimes B) \to HOM(q^M A, \mathcal{K}^M \otimes B)$ is defined as follows:

$$HOM(q^N A, \mathcal{K}^N \otimes B) \xrightarrow{(p^{M,N} \otimes (-))} HOM(q^M A, \mathcal{K}^M \otimes B) \xrightarrow{\pi_{M,N}^*} HOM(q^M A, \mathcal{K}^M \otimes B).$$

Definition 11.3. For $C^*$-algebras $A, B$, we define a based simplicial set $Q(A, B)$ satisfying the Kan condition as:

$$Q(A, B) \overset{\text{def}}{=} \lim_{\rightarrow \mathcal{M}} HOM(q^M A, \mathcal{K}^M \otimes B).$$

It is well-defined since every direct system in the category of simplicial sets satisfying the Kan condition has a direct limit.
We define tensor product $\otimes : Q(A_0, A_1) \times Q(B_0, B_1) \to Q(A_0 \otimes B_0, A_1 \otimes B_1)$ as a simplicial map. Let $M, N$ be finite subsets of the set $U$. It is obtained from simplicial maps which is compatible with each direct system,

$$\otimes : HOM(q^M A_0, K^M \otimes A_1) \times HOM(q^N B_0, K^N \otimes B_1) \to HOM(q^{MIN} (A_0 \otimes B_0, K^{MIN} (A_1 \otimes B_1))$$

as compositions of the following simplicial maps:

(141) $\otimes : HOM(q^M A_0, K^M \otimes A_1) \times HOM(q^N B_0, K^N \otimes B_1)$

(142) $\otimes : HOM(q^M A_0 \otimes q^N B_0, (K^M \otimes A_1) \otimes (K^N \otimes B_1))$

(143) $\cong HOM(q^M A_0 \otimes q^N B_0, K^{MIN} \otimes B)$

(144) $\cong HOM(q^{MIN} (A_0 \otimes B_0), K^{MIN} \otimes B)$

Here the final simplicial map is induced by the canonical homomorphism $q^{MIN} (A_0 \otimes B_0) \to q^M A_0 \otimes q^N B_0$.

In a similar way, we construct a composition $\otimes_D : Q(A, D) \times Q(D, B) \to Q(A, B)$ as a simplicial map. For finite subsets $M, N$ in the set $U$, we define a simplicial map

$$\otimes_D : HOM(q^M A, K^M \otimes D) \times HOM(q^N D, K^N \otimes B) \to HOM(q^{MIN} A, K^{MIN} \otimes B)$$

as compositions of the following simplicial maps:

(145) $\otimes_D : HOM(q^M A, K^M \otimes D) \times HOM(q^N D, K^N \otimes B)$

(146) $q^N \times (K^M \otimes (-)) : HOM(q^N (q^M A), q^N (K^M \otimes D)) \times HOM(K^M \otimes q^N D, K^M \otimes (K^N \otimes B))$

(147) $\otimes_D : HOM(q^N (q^M A), K^M \otimes q^N D) \times HOM(K^M \otimes q^N D, K^M \otimes (K^N \otimes B))$

(148) $\otimes_D : HOM(q^N (q^M A), K^M \otimes (K^N \otimes B))$

(149) $\otimes_D : HOM(q^{MIN} A, K^{MIN} \otimes B)$

The second simplicial map $\chi^{MN}$ is induced by the canonical homomorphism $q^N (K^M \otimes D) \to K^M \otimes q^N D$.

11.2. Construction. In this section, we introduce a categorical group version of $KK$-theory. Its underlying groupoid $KK(A, B)$ is defined as the fundamental groupoid of the simplicial set $Q(A, B)$ for $C^*$-algebras $A, B$. Each element of $z \in U$ induces a categorical group structure on the groupoid where $U$ is used to define $Q(A, B)$. The composition and the tensor product are defined. We explain how $KK(A, B)$ and $KK(A, B)$ are related to each other.

In subsection 5.2, we introduce a categorical group version of generalized (co)homology theory. Its monoidal structure essentially comes from the fact that the pinch map $S^n \to S^n \vee S^n$ gives a comultiplication on $S^n$. In analogy to it, we shall introduce a monoidal structure on $KK(A, B)$ using the fact that a homomorphism $K \otimes K \to K$ gives a multiplication on $K$ in some sense (see Proposition 11.2). Here, $K$ is a $C^*$-algebra formed by compact operators on a fixed infinite dimensional separable Hilbert space.

The assignment $(A, B) \mapsto \Pi_1 HOM(A, B)$ for arbitrary $C^*$-algebras $A, B$ induces a 2-category $\Pi_1 C^*_k$. The 2-category $\Pi_1 C^*_k$ consists of the following data subject to the axioms of 2-categories:

1. The class of objects consists of $C^*$-algebras.
2. For two objects $A, B$, a groupoid $\Pi_1 HOM(A, B)$ is given, which is the collection of morphisms. $\Pi_1 HOM(A, B)$ denotes the fundamental groupoid of the simplicial set $HOM(A, B)$ satisfying the Kan condition.
(3) For three objects $A, B, D$, there is a functor which gives the composition:

$$\circ = \circ_{A,D,B} : \Pi_1\text{HOM}(A, D) \times \Pi_1\text{HOM}(D, B) \to \Pi_1\text{HOM}(A, B).$$

(4) For an object $A$, there is a simplicial map $1_A : \ast \to \Pi_1\text{HOM}(A, A)$ which gives the identity with respect to the above composition. Here, $\ast$ is a fixed one-point groupoid.

**Proposition 11.4.** Let $\mathcal{K}$ be the C*-algebra of compact operators on an infinite dimensional separable Hilbert space $\mathbf{H}$. Let us consider $\mathcal{K}$ as an object of $\Pi_1\text{C}^*_\mathbf{H}$. A unitary isomorphism $\mathbf{H} \oplus \mathbf{H} \cong \mathbf{H}$ of separable Hilbert spaces gives $\mathcal{K}$ a structure of symmetric monoid in $(\Pi_1\text{C}^*_\mathbf{H}, \oplus)$. Here $(\Pi_1\text{C}^*_\mathbf{H}, \oplus)$ is considered as the symmetric monoid in the $2$-category $2\text{Cat}$.

**Proof.** Let $w : \mathbf{H} \oplus \mathbf{H} \cong \mathbf{H}$ be a unitary isomorphism. It induces two unitary isomorphisms $w_{(1(12))}, w_{(1(23))} : \mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H} \to \mathbf{H}$ according to the order of applying $w : \mathbf{H} \oplus \mathbf{H} \to \mathbf{H}$, for example, $w_{(1(23))}$ is defined as follows:

$$\mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H} \stackrel{w \oplus idH}{\longrightarrow} \mathbf{H} \oplus \mathbf{H} \stackrel{w}{\longrightarrow} \mathbf{H}$$

Then there is a continuous path $\gamma(t)$ in the space of unitary isomorphisms from $\mathbf{H} \oplus \mathbf{H} \oplus \mathbf{H}$ to $\mathbf{H}$ under the norm topology such that $\gamma(0) = w_{(1(12))}$ and $\gamma(1) = w_{(1(23))}$.

Note that the unitary isomorphism $w$ induces a homomorphism:

$$\mu : \mathcal{K} \oplus \mathcal{K} \to M_2(\mathcal{K}) \cong \mathcal{K}(\mathbf{H} \oplus \mathbf{H}) \xrightarrow{\text{Adj}[w]} \mathcal{K}$$

Here $j(x, y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We consider the homomorphism $\mu$ as a morphism in the category $\Pi_1\text{C}^*_\mathbf{H}$. Then the continuous path $\gamma(t)$ chosen above gives an associator for $\mu$. In fact, $\gamma(t)$ gives a path in the simplicial set $\text{HOM}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}, \mathcal{K})$ which connects $\mu_{(1(12))}$ and $\mu_{(1(23))}$. It determines a morphism in $\Pi_1\text{HOM}(\mathcal{K} \oplus \mathcal{K} \oplus \mathcal{K}, \mathcal{K})$ which connects its objects $\mu_{(1(12))}$ and $\mu_{(1(23))}$. We denote it as $\alpha$. Note that $\alpha$ does not depend on the choice of $\gamma$ since for two different $\gamma(t)$, $\gamma'(t)$ they are homotopic to each other preserving their boundary values due to the Kuiper’s theorem. Then $\alpha : \mu_{(1(12))} \to \mu_{(1(23))}$ satisfies the pentagon identity due to Kuiper’s theorem again.

We consider the zero homomorphism $0 : 0 \to \mathcal{K}$ as a morphism $u : 0 \to \mathcal{K}$ in the $2$-category $\Pi_1\text{C}^*_\mathbf{H}$. From now on we show that $u$ gives a unit for $\mu$. We take a continuous path $\gamma'(t)$ in the space of isometries on $\mathbf{H}$ under the strong operator topology, from $id_\mathbf{H} : \mathbf{H} \to \mathbf{H}$ to the isometry $\overline{\mu_{(1(12))} \oplus 0} : \mathbf{H} \oplus \mathbf{H} \to \mathbf{H}$. We claim that $\gamma'(t)$ induces a right unit cancellation for $\mu$. In fact, via its adjoint homomorphism, $\gamma'(t)$ induces a path in the simplicial set $\text{HOM}(\mathcal{K}, \mathcal{K})$ which connects $id_\mathcal{K}$ to $\mu \circ (id_\mathcal{K} \oplus u)$. It determines a morphism in $\Pi_1\text{HOM}(\mathcal{K}, \mathcal{K})$ which connects its objects $id_\mathcal{K}$ to $\mu \circ (id_\mathcal{K} \oplus u)$. We denote the morphism as $r$. Similarly, we define a left cancellation $l$ which is a morphism in $\Pi_1\text{HOM}(\mathcal{K}, \mathcal{K})$ which connects its objects $id_\mathcal{K}$ to $\mu \circ (u \oplus id_\mathcal{K})$.

**Corollary 11.5.** A unitary isomorphism $\mathbf{H} \oplus \mathbf{H} \cong \mathbf{H}$ gives the groupoid $\Pi_1\text{HOM}(A, B \otimes \mathcal{K})$ symmetric monoidal groupoid structure naturally.

**Proof.** From the object $\mu \in \Pi_1\text{HOM}(\mathcal{K} \oplus \mathcal{K}, \mathcal{K})$, we obtain a functor:

$$\Pi_1\text{HOM}(A, B \otimes \mathcal{K}) \times \Pi_1\text{HOM}(A, B \otimes \mathcal{K}) \xrightarrow{\otimes} \Pi_1\text{HOM}(A, B \otimes (\mathcal{K} \oplus \mathcal{K}))$$

$$\xrightarrow{\text{Adj}[\mu]} \Pi_1\text{HOM}(A, B \otimes \mathcal{K})$$

Then $\alpha : \mu_{(1(12))} \to \mu_{(1(23))}$, $r : \mu \circ (id_\mathcal{K} \oplus u) \to id_\mathcal{K}$, $l : \mu \circ (u \oplus id_\mathcal{K}) \to id_\mathcal{K}$ in the proof of Proposition 11.4 induce a structure of symmetric monoidal groupoid on the groupoid $\Pi_1\text{HOM}(A, B \otimes \mathcal{K})$. 

$\square$
By far, we showed that the fundamental groupoid $\Pi_1\text{HOM}(A, B \otimes \mathcal{K})$ has a symmetric monoidal groupoid structure. From now on, we will show that if the C*-algebra $A$ is a C*-algebra associated with a C*-algebra, then $\Pi_1\text{HOM}(A, B \otimes \mathcal{K})$ becomes a symmetric categorical group.

**Definition 11.6.** If $(M, m)$ is a based set, then the C*-algebra $Q^MA$ in Definition [11.2] has a natural involution $\Delta$ defined as follows. Then for $K \subset M$, let us define homomorphisms $f_K : A \to QA$ by $a^K \mapsto a^{K \cup \{m\}}$ if $m \in K$ and $a^K \mapsto a^{K \cup \{m\}}$ if $m \notin K$. Then the family $f_K : K \subset M$ gives a homomorphism $Q^MA \to Q^MA$. We define its restriction to $q^MA$ and denote it by $\Delta$, which is obviously an involution.

**Lemma 11.7.** For a homomorphism $f \in \text{hom}(qA, B)$, the homomorphism obtained from the following compositions is homtopic to the zero homomorphism. In particular, we can take a canonical one as such homotopy.

$$ qA \Delta qA \oplus qA \xrightarrow{f \oplus (f \circ t)} B \oplus B \xrightarrow{\Delta} M_2(B) $$

**Proof.** The claim is equivalent with that $qA \to M_2(qA); x \mapsto \left( \begin{array}{cc} x & 0 \\ 0 & t(x) \end{array} \right)$ is homtopic to the zero homomorphism. For $t \in I$, homomorphisms $g : A \to M_2(QA); a \mapsto \left( \begin{array}{cc} a^0 & 0 \\ 0 & a^1 \end{array} \right)$ and $A \to M_2(QA); a \mapsto R(t) \left( \begin{array}{cc} a^0 & 0 \\ 0 & a^1 \end{array} \right) R(t)^*$ induce a homomorphism $h_t : QA \to M_2(QA)$ by the universality of $QA$. Here we set $R(t) = \left( \begin{array}{cc} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{array} \right)$. Moreover, we have $h_t(x) \in M_2(qA)$ for $x \in qA \subset QA$. By the continuity of $h_t$ with respect to $t$, $h_t : qA \to M_2(qA)$ induces $h_0 \simeq h_1$ where $h_1$ is the above homomorphism $g$ and $h_0 = 0$.

**Lemma 11.8.** The involution $t : qA \to qA$ and the homotopy from $j \circ (id_{qA} \oplus t) \circ \Delta_{qA} \simeq 0 : qA \to M_2(qA)$ in the proof of Lemma [11.7] give a canonical inverse over the symmetric monoidal groupoid $\Pi_1\text{HOM}(qA, \mathcal{K} \otimes B)$.

**Proof.** The homomorphism $t : qA \to qA$ induces a based map $\text{hom}(qA, C(I^n, \mathcal{K} \otimes B)) \to \text{hom}(qA, C(I^n, \mathcal{K} \otimes B))$ which is an involution for $n \geq 0$. Then it induces a based simplicial map $\tau : \text{HOM}(qA, \mathcal{K} \otimes B) \to \text{HOM}(qA, \mathcal{K} \otimes B)$ which is also an involution.

Let us denote by $h$ the homotopy from $j \circ (id_{qA} \oplus t) \circ \Delta_{qA} \simeq 0 : qA \to M_2(qA)$ in the proof of Lemma [11.7]. Then $h$ gives a morphism $l_\tau$ in $\Pi_1\text{HOM}(qA, M_2(qA))$. Consider the following composition:

$$ \Pi_1\text{HOM}(qA, M_2(qA)) \times \Pi_1\text{HOM}(M_2(qA), M_2(\mathcal{K} \otimes B)) \to \Pi_1\text{HOM}(qA, M_2(\mathcal{K} \otimes B)) $$

We obtain a natural isomorphism $\mu$ from $(j \circ (id_{qA} \oplus t) \circ \Delta_{qA})^*$ to the constant functor valued at the zero homomorphism. Since the composition $\mu \circ (1_{\text{HOM}(qA, \mathcal{K} \otimes B)} \times \tau) \circ \Delta_{\text{HOM}(qA, \mathcal{K} \otimes B)}$ coincides with the following compositions,

$$ \text{HOM}(qA, \mathcal{K} \otimes B) \xrightarrow{id_{M_2(C)} \otimes \text{id}_{\mathcal{K} \otimes B}} \text{HOM}(M_2(C) \otimes qA, M_2(C) \otimes \mathcal{K} \otimes B) $$

$$ \cong \text{HOM}(M_2(qA), M_2(\mathcal{K} \otimes B)) \xrightarrow{(j \circ (id_{qA} \oplus t) \circ \Delta_{qA})^*} \text{HOM}(qA, M_2(\mathcal{K} \otimes B)) $$

$$ (\text{Ad}(w))^* \text{HOM}(qA, \mathcal{K} \otimes B) $$

the natural isomorphism $\mu$ induces a natural isomorphism from $\mu \circ (1_{\text{HOM}(qA, \mathcal{K} \otimes B)} \times \tau) \circ \Delta_{\text{HOM}(qA, \mathcal{K} \otimes B)}$ to the constant functor valued at the zero homomorphism. It completes the proof. \( \square \)
Corollary 11.9. A based finite set \((M, m)\) induces a structure of symmetric categorical group on the fundamental groupoid \(\Pi A\) based finite set

\[\text{Corollary 11.9.} \quad \text{An element } z \text{ in } U \text{ naturally gives a symmetric categorical group structure the groupoid } \mathcal{KK}(A, B). \text{ Also it has canonical inverses.} \]

Proof. By applying Lemma 11.8 to \(HOM(q^M A, \mathcal{KK}(M \otimes B))\), the claim is proved. \(\square\)

Definition 11.10. We define a groupoid \(\mathcal{KK}(A, B)\) as follows:

\[\mathcal{KK}(A, B) \overset{\text{def}}{=} \Pi Q(A, B).\]

Proposition 11.11. An element \(z \in U\) naturally gives a symmetric categorical group structure the groupoid \(\mathcal{KK}(A, B)\). Also it has canonical inverses.

Proof. We have a simplicial isomorphism:

\[\lim_{(M, z)} HOM(q^M A, \mathcal{KK}(M \otimes B)) \rightarrow Q(A, B).\]

Here, \((M, z)\) in the left hand side runs on the directed set consisting of finite subsets of \(U\) based at \(z \in U\). We obtain a SCG structure on the fundamental groupoid \(\Pi Q(A, B)\) of the simplicial set \(Q(A, B)\). \(\square\)

Proposition 11.12. For a bijection of sets \(U \cup U \equiv U\), we have a functor natural with respect to \(C^*\)-algebras \(A, B, D\):

\[\otimes_D : \mathcal{KK}(A, D) \times \mathcal{KK}(D, B) \rightarrow \mathcal{KK}(A, B).\]

Proof. Recall that we constructed a natural pairing \(Q(A, D) \times Q(D, B) \rightarrow Q(A, B)\) in subsection 11.1 where \(A, B, C\) are \(C^*\)-algebras. By definitions it induces a functor \(\otimes_D : \mathcal{KK}(A, D) \times \mathcal{KK}(D, B) \rightarrow \mathcal{KK}(A, B)\) which is natural with respect to \(A, B, D\). \(\square\)

By definitions, we have a natural isomorphism for \(C^*\)-algebras \(A, B\):

\[\pi_n(HOM(A, B)) \cong [A, S^n B].\]

Here \([A, B]\) denotes the set of homotopy classes of homomorphisms from \(A\) to \(B\). Thus, considering the Cuntz picture of \(KK\)-theory, we obtain a natural isomorphisms of groups for \(C^*\)-algebras \(A, B\):

\[\pi_0(\mathcal{KK}(A, B)) \cong KK(A, B), \quad \pi_1(\mathcal{KK}(A, B)) \cong KK(A, SB).\]

Here, we consider the unit object of \(\mathcal{KK}(A, B)\) as the basepoint to take \(\pi_1(\mathcal{KK}(A, B))\), but it holds for arbitrary basepoints since \(\mathcal{KK}(A, B)\) has a symmetric categorical group structure. Under these isomorphisms, the pairing \(\otimes_D\) in Proposition 11.12 induces the Kasparov product \(\otimes_D : KK(A, D) \times KK(D, B) \rightarrow KK(A, B)\).

The following statements, we explain ‘bilinearity’ of the functor \(\mathcal{KK}(-, -)\) with respect to the direct sum of \(C^*\)-algebras. Proposition 11.13, 11.14 correspond to Proposition 5.6, 5.7 in generalized (co)homology settings respectively.

Proposition 11.13. Let \(A, B, B'\) be \(C^*\)-algebra. Let \(B \oplus B' \rightarrow B\) and \(B \oplus B' \rightarrow B'\) be the canonical projections. They induce an isomorphism of groupoids:

\[\mathcal{KK}(A, B \oplus B') \rightarrow \mathcal{KK}(A, B) \times \mathcal{KK}(A, B').\]

Proof. The maps induced by applying \(\pi_0, \pi_1\) to (37) are isomorphisms. Hence (150) gives an equivalence of groupoids by Proposition 11.3. Moreover, it is obvious that the functor induces a bijection between object classes by definition of \(\mathcal{KK}(-, -)\). Therefore, (150) is an isomorphism of groupoids. \(\square\)
Let $A, A', B$ be C*-algebras. Denote by the canonical inclusions $i : A \to A \oplus A'$, $i' : A' \to A \oplus A'$. They induce a functor $f$ by
\[
\mathcal{K}K(A, B) \oplus \mathcal{K}K(A', B) \to \mathcal{K}K(A \oplus A', B)
\]
\[
(a, b) \mapsto i^*(a) \oplus (i')^*(b)
\]
If $p : A \oplus A' \to A$ and $p' : A \oplus A' \to A'$ denotes the projections, we obtain a functor $g$,
\[
\mathcal{K}K(A \oplus A', B) \to \mathcal{K}K(A, B) \oplus \mathcal{K}K(A', B)
\]
\[
a \mapsto (p^*(a), (p')^*(a))
\]

**Proposition 11.14.** The functors $f, g$ are lifted to an adjoint equivalence of symmetric categorical groups with canonical inverses.

**Proof.** The proof proceeds in a parallel way to the proof of Proposition 5.7 \(\square\)

**Remark 11.15.** There is a Puppe exact sequence of $KK$-theory (see section 19\([\text{Bla98]})$). It is an exact sequence associated with two C*-algebras, a homomorphism of them and its cone. We also have a Puppe exact sequence on the symmetric categorical group version of $KK$-theory in a similar way with Remark 5.12.

12. A groupoid induced by $KU$-orientation

In this section, we introduce a groupoid containing representatives of the fundamental class associated with a $KU$-oriented manifold. It is an analogy to $[[-]]_{KU}$ defined in subsection 6.1. The groupoid $[[Z]]_{aKU}$ will be a subgroupoid of a $KK$ categorical group associated with a $KU$-oriented manifold $Z$, so that it is different from $[[Z]]_{KU}$. We define ‘restrictions to boundaries’ with respect to $[[-]]_{aKU}$ from subsection 6.2 by replacing $[[-]]_E$ with $[[-]]_{aKU}$. Then it is also possible to discuss the glauability of $[[-]]_{aKU}$ following subsection 6.3.

**Definition 11.12.** Let $E_* = (E_0 \leftarrow E_{01} \to E_1)$ be a cospan diagram of C*-algebras. We define $\mathcal{R}^1(E_*) = (E' \to E_0' \leftarrow E_1')$ as follows. Let $K$ be the subspace defined by
\[
K = C(\bigtriangleup(2)_0, E_0) \times_{C(\bigtriangleup(1),E_0)} C(\bigtriangleup(1), E_{01}) \times_{C(\bigtriangleup(1),E_1)} C(\bigtriangleup(2)_1, E_1).
\]
Here, we use the 0-face map $\bigtriangleup(1) \to \bigtriangleup(2)_0$ and the 1-face map $\bigtriangleup(1) \to \bigtriangleup(2)_1$. In other words, we set $(a, b, c) \in K \subset C(\bigtriangleup(2)_0, E_0) \times C(\bigtriangleup(1), E_{01}) \times C(\bigtriangleup(2)_1, E_1)$ if and only if
\[
a|_{\bigtriangleup(1)} = r_0(b), \quad c|_{\bigtriangleup(1)} = r_1(b)
\]
in $E_{01}$ if $b = c = 0$, and let $(a, b, c) \in E'_1$ if $a = b = 0$. Then the inclusion homomorphisms induce a diagram $E_0' \to E_{01}' \leftarrow E_1'$.

**Definition 11.12.** Let $E_\ast$ be the following commutative diagram of C*-algebras.

\[
\begin{align*}
E_0 & \quad E_0' \quad E_0'' \quad E_0''' \\
E_0'' & \quad E_0''' \\
E_0''' & \quad E_0''' \\
E_0' & \quad E_0'' \\
E_0 & \quad E_0'
\end{align*}
\]
Starting from this diagram, we construct a commutative diagram $E'_{\bullet} = \mathcal{R}^{\infty}(E_{\bullet})$ of $C^*$-algebras in a natural way.

![Diagram](image)

(154)

To begin with, we denote by $K$ the $C^*$-algebra obtained by gluing the following $C^*$-algebras:

![Diagram](image)

(155)

Note that the arrows do not denote homomorphisms. For example, $C(\triangle(1), E_{01}) \xrightarrow{0\text{-face}} C(\triangle(2)_0, E_0)$ means that we glue them along $C(\triangle(1), E_{01}) \leftarrow C(\triangle(1), E_0) \xrightarrow{0\text{-face}} C(\triangle(2)_0, E_0)$.

We define $E'_{012}$ as the kernel of the homomorphism from $K$ to ‘boundary’ of $K$:

![Diagram](image)

(156)

Here, $\partial_k(\triangle(n))$ denotes the $k$-face of the simplex $\triangle(n)$.

We denote an element of $E'_{012}$ by 7-tuples $(f_0, f_1, f_2, f_{01}, f_{12}, f_{02}, f_{012})$ such that $f_{\bullet} \in C(\triangle(3\setminus \bullet), E_{\bullet})$. Then we set $(f_{\bullet}) \in E'_{ij}$ if and only if $\bullet$ contains $k$ where $\{i, j, k\} = \{0, 1, 2\}$. Also we set $(f_{\bullet}) \in E'_{ij}$ if and only if $\bullet$ contains $k$ or $j$ where $\{i, j, k\} = \{0, 1, 2\}$.

We make use of the fact that analytic $K$-homology is naturally isomorphic to the $K$-theory obtained from the spectrum $KU$ [Kas75]. Then every compact $KU$-oriented manifold $Z^k$ whose underlying space is a CW-space has its fundamental class in its analytic $K$-homology,

$$\text{[}Z^k\text{]}_{aKU} \in K_k(Z^k).$$

(157)

Here, we denote by $K_*(-)$ the analytic $K$-homology theory. For example, of such $Z^k$, we have $\text{Spin}^c$ manifolds. Since smooth manifolds are triangulable, $Z^k$ is a CW-space. On the one hand, the $\text{Spin}^c$ structure of $Z^k$ induces an analytic $K$-homology class in $K_k(Z^k)$, which is also
called the fundamental class \([HR00]\). In fact, the \(K\)-homology class gives a \(KU\)-orientation for \(Z^n\) under the isomorphism between two \(K\)-homology theories mentioned before.

**Definition 12.3.** Let \(Y^{n-1}\) be a closed \(KU\)-oriented \((n-1)\)-manifold. Its fundamental class is given as a \(K\)-homology class \(\langle Y^{n-1} \rangle_{KU} \in K_{n-1}(Y^{n-1}) \cong \pi_0(\mathcal{K}(S^2C(Y^{n-1}), S^{n+1}))\). We define \(\langle Y^{n-1} \rangle_{aKU}\) as a full subgroupoid of the groupoid \(\mathcal{K}(S^2C(Y^{n-1}), S^{n+1})\) which contains representatives of the fundamental class \(\langle Y^{n-1} \rangle_{KU}\).

As a next step, we consider a \(KU\)-oriented \(n\)-cobordism \(Y^0 \rightarrow X^n \leftarrow Y_{1}^{n-1}\). It induces a span of \(C^*\)-algebras \(C(Y^{n-1}) \leftarrow C(X^n) \rightarrow C(Y_{1}^{n-1})\). For simplicity, let \((E_0 = E_{01} \rightarrow E_1) = (C(Y^{n-1}) \leftarrow C(X^n) \rightarrow C(Y_{1}^{n-1}))\). It induces a diagram \(E'_* = R^1(E'_*)\). We define \(\langle [X^n] \rangle_{aKU}\) as a full subgroupoid of the groupoid \(\mathcal{K}(E'_{01}, S^{n+1})\) which contains representatives of \([X^n]_{aKU}\). We again assume with respect to these manifolds: The boundaries of \(X_j^n\) maps' between \(X_k^n\). The \(C^*\)-algebras consisting of continuous maps over manifolds induces a commutative diagram \(E_*\) of \(C^*\)-algebras as the diagram (12.3). The \(KU\)-orientation of the manifold \(W_{n+1}\) is given as \(\langle [W_{n+1}] \rangle_{aKU} \in K_{n+1}(W_{n+1}; \partial W_{n+1}) \cong \pi_0(\mathcal{K}(E'_{012}, S^{n+1}))\). We define \(\langle [W_{n+1}] \rangle_{aKU}\) as a full subgroupoid of the groupoid \(\mathcal{K}(E'_{012}, S^{n+1})\) which contains representatives of the class \([W_{n+1}]_{aKU}\).

For \(Z^k = Y^{n-1}, X^n, W_{n+1}\), it is obvious that \(\pi_0(\langle [Z^k] \rangle_{aKU}) = \ast\) and \(\pi_1(\langle [Z^k] \rangle_{aKU}, a) \cong K_{k+1}(Z; \partial Z)\) for every \(a \in \langle [Z^k] \rangle_{aKU}\) by definitions.

13. **Classical theory from \(KK\)-theory**

In this section, we construct a Lagrangian classical field theory for \(KU\)-oriented manifolds. We extend the cobordism invariant \(S_{[\beta]}\) in Definition 3.3 to closed \((n-1)\)-manifolds in some sense. We obtain a system of ‘cobordism invariants’ \(S_{[\beta]}\). Recall that the invariant \(S_{[\beta]}\) depends on the \(KK\)-theory class \([\beta] \in KK(A, C(B\Gamma) \otimes B)\). The system \(S_{[\beta]}\) depends on a representative \(\beta\) of the class \([\beta]\), which is assumed as follows.

**Assumption 13.1.** \(\beta\) is an object of the groupoid \(\mathcal{K}(A, C(B\Gamma) \otimes B)\) which represents the class \([\beta] \in KK(A, C(B\Gamma) \otimes B) \cong \pi_0(\mathcal{K}(A, C(B\Gamma) \otimes B))\).

The system \(S_{[\beta]}\) consists of three assignments. The second assignment of them, given as a symmetric monoidal functor \(S_{[\beta]} : \mathcal{X}_{1}^{aKU}(n; \alpha, \Gamma) \rightarrow N_{1}(\mathcal{K}(A, S^{n+1}B))\), assigns a morphism in the groupoid \(\mathcal{K}(A, S^{n+1}B)\) to a triple of a \(KU\)-oriented \(n\)-cobordism, a representative of its fundamental class and a principal \(\Gamma\)-bundle on itself. It extends the cobordism invariant \(S_{[\beta]}\) in the following sense. If the target and the source of the \(n\)-cobordism are the null spaces, then the target and the source of its corresponding morphism is the unit of the categorical group \(\mathcal{K}(A, S^{n+1}B)\), so that it is reduced to a number in an abelian group \(\pi_1(\mathcal{K}(A, S^{n+1}B)) \cong KK(A, S^{n+1}B)\). The number coincides with the value computed via \(S_{[\beta]}\).

We give our results in a parallel way to section 8. To begin with, we define three groupoids \(\mathcal{X}_{0}^{aKU}(n; \alpha, \Gamma), \mathcal{X}_{1}^{aKU}(n; \alpha, \Gamma), \mathcal{X}_{2}^{aKU}(n; \alpha, \Gamma)\) for \(n \in \mathbb{Z}^{>0}\). These are defined by replacing \([[-]]_{E}\) with \([[-]]_{aKU}, \Pi_{1}MAP(-, B\Gamma)\) with \(\Pi_{1}MAP'(-, B\Gamma)\) in Definition 8.2. Then we obtain ‘face maps’ between \(\mathcal{X}_{i}^{aKU}(n; \alpha, \Gamma)\)’s as in Definition 8.3.

We introduce a symmetric monoidal structure on \(\mathcal{X}_{0}^{aKU}(n; \alpha, \Gamma)\) (and on \(\mathcal{X}_{i}^{aKU}(n; \alpha, \Gamma)\) similarly for \(i = 1, 2\)) by

\[
(158) \quad (Y_{0}^{n-1}, \eta_0, g_0) \amalg (Y_{1}^{n-1}, \eta_1, g_1) \overset{\text{def.}}{=} (Y_{0}^{n-1} \amalg Y_{1}^{n-1}, \eta_0 \amalg \eta_1, g_0 \amalg g_1)
\]
where \( \eta_0 \oplus \eta_1 \) is the image of \( (\eta_0, \eta_1) \) via the functor
\[
\mathcal{K}(S^2C(Y_0^{-1}, S^{n+1})) \times \mathcal{K}(S^2C(Y_1^{-1}, S^{n+1})) \\
\rightarrow \mathcal{K}(S^2C(Y_0^{-1}) \oplus S^2C(Y_1^{-1}, S^{n+1})) \\
\cong \mathcal{K}(S^2C(Y_0^{-1}, Y_1^{-1}, S^{n+1})).
\]

Then ‘face maps’ \( \partial_i^k \)'s become symmetric monoidal functors.

**Theorem 13.2.** The cobordism invariant \( S_\beta \) in Definition 3.3 extends to a family of assignments,
\[
S_\beta : \mathcal{X}_i^{iKU}(n; o, \Gamma) \rightarrow \mathcal{N}_i(\mathcal{K}(A, S^{n+1}B)), \ i = 0, 1, 2,
\]
which are symmetric monoidal functors and compatible with the ‘face maps’ \( \partial_i^k \)'s.

From now on, we prove the theorem by constructing \( S_\beta \). Let \( A(E) \overset{\text{def}}{=} \mathcal{K}(A, E \otimes B) \) and \( B(E) \overset{\text{def}}{=} \mathcal{K}(E, S^{n+1}) \) for a \( C^* \)-algebra \( E \). Let us construct symmetric monoidal functors \( \beta_\beta : \mathcal{X}_i^{iKU}(n; o, \Gamma) \rightarrow \mathcal{T}_{E,AB}^i, \ i = 0, 1, 2 \) where \( C = C^* \).

For \( (Y^{n-1}, \eta, g) \in \mathcal{X}_i^{iKU}(n; o, \Gamma) \), we define \( \beta_\beta(Y^{n-1}, \eta, g) \overset{\text{def}}{=} (Y^{n-1}, \eta, g^s(\beta)) \in \mathcal{T}_{E,AB}^0 \).

In order to construct \( \beta_\beta \), recall \( h_i^t, \ t \in \Delta(1) \) in Definition 6.1. Let \( (X^n, \sigma, f) \in \mathcal{X}_i^{iKU}(n; o, \Gamma) \), where \( X^n \) is a cobordism from \( Y_0^{-1} \) to \( Y_1^{-1} \). Put \( (E_0 \leftarrow E_1 \rightarrow E_i) = (C(Y_0^{-1}) \leftarrow C(X^n) \rightarrow C(Y_1^{-1})) \), and let \( E_i^\prime = \mathcal{R}^1(E_i) \). Via the homeomorphism \( S^2E_i \cong E_i' \), the family \( h_i^t, \ t \in \Delta(1) \) induces the following homotopy.
\[
\begin{array}{ccc}
S^2E_0 & \overset{\Phi_0}{\rightarrow} & S^2E_i' \\
\rightarrow & & \downarrow \\
E_0' & \rightarrow & S^2E_i \\
\end{array}
\]

We assign \( (\text{Diagram163}, \sigma, f^*(\alpha)) \in \mathcal{T}_{E,AB}^1 \) to \( (X^n, \sigma, f) \in \mathcal{X}_i^{iKU}(n; o, \Gamma) \).

In order to construct \( \beta_\beta \), we make use of the family \( h_i^t, \ t \in \Delta(2) \) in Definition 6.1. Let \( (W^{n+1}, \xi, h) \in \mathcal{X}_i^{iKU}(n; o, \Gamma) \) be an object. Here, \( W^{n+1} \) omits for the diagram (51) of \( KU \)-oriented manifolds. If we put the diagram as \( E_* \) as (153), then \( E_i^\prime = \mathcal{R}^2(E_i) \) and the isomorphisms \( S^2E_i \cong E_i' \) induces an object of \( \mathcal{C}^2 \) in subsection 7.1 as follows. Let \( l(s) ; s \in [0, 1] \) be the line segment connecting from \( e_i \in \Delta(2) \) to \( e_j \in \Delta(2) \). Using a homotopy \( h_l^2 \left/ \triangle(2) \right. \), \( s \in [0, 1] \), we construct a homotopy in the following diagram where \( i, j = 0, 1, 2 \).
\[
\begin{array}{ccc}
S^2E_{ij} & \overset{\Phi_{ij}}{\rightarrow} & S^2E_{i}' \\
\rightarrow & & \downarrow \\
E_{ij} & \rightarrow & S^2E_j \\
\end{array}
\]

From \( h_l^2, \ t \in \Delta(2) \), we obtain \( \Phi_{12} \ast \Phi_{01} = \Phi_{02} \) if we take compositions of \( S^2E_{012} \rightarrow S^2E_{ij} \) and \( E_{ij}' \rightarrow E_{012} \) to diagram (164). Thus it gives an object \( O \) of \( \mathcal{C}^2 \). Then we assign \( (O, \xi, h^s(\alpha)) \in \mathcal{T}_{E,AB}^2 \) to the object \( (W^{n+1}, \xi, h) \in \mathcal{X}_i^{iKU}(n; o, \Gamma) \).

We apply the results in section 7 by putting \( C \overset{\text{def}}{=} \mathcal{K}(A, S^{n+1}B) \). We use the Kasparov product:
\[
\mathcal{K}(A, E \otimes B) \times \mathcal{K}(E, S^{n+1}) \rightarrow \mathcal{K}(A, S^{n+1}B).
\]
It gives a natural pairing in Definition 7.1. These data satisfy the assumptions in subsection 1.1 so that we obtain a family of symmetric monoidal functors $\mathcal{T}_{\text{C},\text{AB}}^i : \mathcal{T}_{\text{C},\text{AB}}^i \to \mathbf{N}_i(\mathbf{C})$, $i = 0, 1, 2$. We define $S_\beta$ as the functor obtained from a composition with the symmetric monoidal functor $\mathcal{T}_{\text{C},\text{AB}}^i : \mathcal{T}_{\text{C},\text{AB}}^i \to \mathbf{N}_i(\mathbf{C})$ and $\beta_\beta : \mathcal{X}_\text{KU}(n; o, \Gamma) \to \mathcal{T}_{\text{C},\text{AB}}^i$.

All that remain is to prove that the assignments extend the cobordism invariant $S_\beta$. For an object $(Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1}, \sigma, f) \in \mathcal{X}_\text{KU}(n; o, \Gamma)$, suppose that $Y_0^{n-1} = Y_1^{n-1} = \emptyset$, i.e. the manifold $X^n$ is closed. Then we have isomorphisms $\langle \sigma|_{Y_0^{n-1}}, f^*(\beta)|_{Y_0^{n-1}} \rangle \cong 0$, $\langle \sigma|_{Y_1^{n-1}}, f^*(\beta)|_{Y_1^{n-1}} \rangle \cong 0$ in the groupoid $\mathcal{X}_\text{KU}(n, S^{n+1}B)$. Under these isomorphisms, the morphism $S_\beta(X^n, \sigma, f) : \langle \sigma|_{Y_0^{n-1}}, f^*(\beta)|_{Y_0^{n-1}} \rangle \to \langle \sigma|_{Y_1^{n-1}}, f^*(\beta)|_{Y_1^{n-1}} \rangle$ induces an element in $\pi_1(\mathcal{X}_\text{KU}(n, S^{n+1}B)) \cong KK(A, S^nB)$. It coincides with $S_\beta(X^n, f) \in KK(A, S^nB)$ due to the following Lemma 13.3. It completes our proof of Theorem 13.2.

**Lemma 13.3.** Let $E_0 \to E_{01} \to E_1$ be a diagram of based spaces. We denote by $E'_{\ast} = \mathcal{R}_1(E_{\ast})$. Suppose that $E_0 = E_1 = 0$, the zero $C^\ast$-algebra.

\[
\begin{array}{ccc}
S^2E_{01} & \overset{S^2E_{01}}{\longrightarrow} & E_{01} \\
\downarrow \cong & & \downarrow \cong \\
S^2E_1 & \rightarrow & E_{01} \\
\end{array}
\]

(166)

It induces a homomorphism $\chi : S^2E_{01} \to S^2E_{01}$. On the other hand, the diagram (166) is an object of $\mathcal{C}_1$ where $\mathcal{C} = C^\ast$. Let $D$ be the object. For two objects $a \in A(S^2E_{01})$, $b \in B(E'_{01})$, the morphism $\mathcal{T}_{\text{C},\text{AB}}^1(D, a, b) : (a_0, b_0) \to \langle a_1, b_1 \rangle$ induces a based loop in the groupoid $\mathcal{X}_\text{KU}(n, S^{n+1}B)$ since $a_i \in A(S^2E_i) \cong 
ast$ and $b_i \in B(E'_{01}) \cong 
ast$. Hence, we obtain a loop $\mathcal{T}_{\text{C},\text{AB}}^1(D, a, b) \in \pi_1(\mathcal{X}_\text{KU}(n, S^{n+1}B))$ isomorphically. We claim that the loop $\mathcal{T}_{\text{C},\text{AB}}^1(D, a, b)$ coincides with the homotopy class of $[\langle a, \chi^*(b) \rangle] \in \pi_0(\mathcal{X}_\text{KU}(n, S^{n+1}B))$.

**Proof.** The canonical simplicial map $HOM(D', S^D) \to \Omega(HOM(D', D''), 0)$ induces a map $\pi_0(\mathcal{X}_\text{KU}(A, S^nB)) \to \pi_1(\mathcal{X}_\text{KU}(A, S^{n+1}B))$.

By definition, $[\langle a, \chi^*(b) \rangle]$ corresponds to $[\mathcal{T}_{\text{C},\text{AB}}^1(D, a, b)]$ under this isomorphism. Moreover the following diagram commutes by definition.

\[
\begin{array}{ccc}
\pi_0(\mathcal{X}_\text{KU}(A, S^nB)) & \longrightarrow & \pi_1(\mathcal{X}_\text{KU}(A, S^{n+1}B)) \\
\downarrow \cong & & \downarrow \cong \\
\pi_0(\mathcal{X}_\text{KU}(A, S^nB)) & \longrightarrow & \pi_1(\mathcal{X}_\text{KU}(A, S^{n+1}B)) \\
\end{array}
\]

(167)

$KK(A, S^nB)$ $KK(A, S^{n+1}B)$

It completes our proof. \qed

Recall the family of action functionals $S_\beta$ described in Theorem 13.2. According to the definition of $\mathcal{X}_\text{KU}(n; o, \Gamma)$, it is obvious that the action functional $S_\beta$ depends on not only ‘fields’ (classifying maps) but also representatives of fundamental classes. From now on, we construct an action functional which is independent of representatives of fundamental classes by restricting manifolds.

We define three groupoids $\mathcal{Y}_{\text{KU}}^0(n; \Gamma)$, $\mathcal{Y}_{\text{KU}}^1(n; \Gamma)$, $\mathcal{Y}_{\text{KU}}^2(n; \Gamma)$ for $n \in \mathbb{Z}^{>0}$. They are defined following Definition 8.7 by replacing the condition $E_4(Y^{n-1}) \cong 0$ with the condition $K_n(Y^{n-1}) \cong 0$, and $\Pi_1\text{MAP}(-; B\Gamma)$ with $\Pi_1\text{MAP}^{\text{pt}}(-; B\Gamma)$. Then each groupoid $\mathcal{Y}_{\text{KU}}^i(n; \Gamma)$ has a symmetric monoidal structure from the disjoint union as $\mathcal{X}_\text{KU}(n; o, \Gamma)$ does. Also we can define ‘face maps’ $\partial_i$ between $\mathcal{Y}_{\text{KU}}^i(n; \Gamma)$’s, which is given as a symmetric monoidal functor.
Assumption 13.4. For an abelian group $M$, let $\hat{\psi} : (\mathbb{KK}(A, S^{n+1}B), \otimes) \to (M\text{Tor}, \otimes)$ be a symmetric monoidal functor. Let us denote by $\pi_1(\hat{\psi}) : \mathbb{KK}(A, S^nB) \cong \pi_1(\mathbb{KK}(A, S^{n+1}B)) \to \pi_1(M\text{Tor}) \cong M$ the group homomorphism induced by the symmetric monoidal functor $\hat{\psi}$ where we use the unit of each monoidal groupoid as a basepoint of each groupoid.

Corollary 13.5. The $M$-valued cobordism invariant $(X^n, f) \mapsto (\pi_1(\hat{\psi}))(S_{[\beta]}(X^n, f)) \in M$ extends to the assignments

$$S_{\hat{\psi}, \beta} : Y^i_{aKU}(n; \Gamma) \to N_i(M\text{Tor}), \quad i = 0, 1, 2,$$

which are symmetric monoidal functors and compatible with the ‘face maps’ $\hat{c}_i$’s.

**Proof.** The proof proceeds in a parallel way to the proof of Corollary 8.9. □

14. **Quantum Theory from $KK$-theory**

In this section, we construct a TQFT for $KU$-oriented manifolds starting from a representative $\beta$ of a $KK$-theory class of $BT$. The TQFT yields an invariant for closed $KU$-oriented $n$-manifolds, which coincides with the generalized Dijkgraaf-Witten invariant defined in subsection 3.2.

To begin with, let us define three groupoids $\mathcal{X}^i_{\beta KU}(n; o)$, $\mathcal{X}^i_{aKU}(n; o)$, $\mathcal{X}^i_{aKU}(n; o)$ for $n \in \mathbb{Z}_{>0}$. They are defined by eliminating the components of classifying maps in the definition of $\mathcal{X}^i_{\beta KU}(n; o, \Gamma)$, $\mathcal{X}^i_{aKU}(n; o, \Gamma)$, $\mathcal{X}^i_{aKU}(n; \overline{o}, \Gamma)$, but nonetheless we should note that we require some condition for $\mathcal{X}^i_{\beta KU}(n; o)$. It consists of $(D, \xi)$ such that $D$ is a diagram \([\mathbb{KU}]_{\beta KU}\) of $KU$-oriented manifolds and $\xi$ is an object of the groupoid $[[W^{n+1}]_{\beta KU}]$ as before.

We define ‘face maps’ as functors $\hat{c}_i^k : \mathcal{X}^i_{\beta KU}(n; o) \to \mathcal{X}^i_{aKU}(n; o)$ as in Definition 8.3, but we do not need to consider the classifying maps appearing in Definition 8.3. The disjoint union of manifolds induces a symmetric monoidal groupoid structure on each $\mathcal{X}^i_{\beta KU}(n; o)$. Then $\hat{c}_i^k$’s become symmetric monoidal functors in obvious ways.

**Theorem 14.1.** Let $\psi$ be the homomorphism $\pi_1(\hat{\psi}) : \mathbb{KK}(A, S^nB) \cong \pi_1(\mathbb{KK}(A, S^{n+1}B)) \to \pi_1(U(1)\text{Tor}) \cong U(1)$. The invariant $Z_{\hat{\psi}, [\beta]}$ in Definition 3.4 extends to a family of assignments

$$Z_{\hat{\psi}, [\beta]} : \mathcal{X}^i_{\beta KU}(n; o) \to N_i(\text{Vect}), \quad i = 0, 1, 2,$$

which are strong symmetric monoidal functors and compatible with the ‘face maps’ $\hat{c}_i^k$’s.

**Proof.** The assignments $Z_{\hat{\psi}, [\beta]}$’s in the theorem are constructed through a linearization of the assignments $S_{\hat{\psi}, [\beta]}$’s in Theorem 13.2. The proof proceeds in a parallel way to the proof of Theorem 10.1. □

From now on, we give one of our main results by eliminating dependence on the extra data from $Z_{\hat{\psi}, [\beta]}$ in Theorem 14.1 by requiring some restrictions on manifolds. Let us define three groupoids $Y^i_{\beta KU}(n)$, $Y^i_{aKU}(n)$, $Y^i_{aKU}(n)$ for $n \in \mathbb{Z}_{>0}$ instead of $\mathcal{X}^i_{\beta KU}(n; o)$, $\mathcal{X}^i_{aKU}(n; o)$, $\mathcal{X}^i_{aKU}(n; o)$. They are defined by eliminating the components of classifying maps in the definition of $Y^i_{\beta KU}(n; \overline{\Gamma})$, $Y^i_{aKU}(n; \Gamma)$, $Y^i_{aKU}(n; \Gamma)$. For example, the object class of $Y^i_{\beta KU}(n)$ consists of $Y^{n-1}$ where $Y^{n-1}$ is a closed $KU$-oriented $(n-1)$-manifold such that $K_n(Y^{n-1}) \cong 0$. A morphism is given as a $KU$-orientation preserving homeomorphism. Each groupoid $Y^i_{\beta KU}(n)$ has a symmetric monoidal structure as $\mathcal{X}^i_{\beta KU}(n; o)$ does. Also we can define ‘face maps’ $\hat{c}_i^k$ between $Y^i_{aKU}(n)$’s, which is given as a symmetric monoidal functor.

**Theorem 14.2.** Let $\psi$ be the homomorphism $\pi_1(\hat{\psi}) : \mathbb{KK}(A, S^nB) \cong \pi_1(\mathbb{KK}(A, S^{n+1}B)) \to \pi_1(U(1)\text{Tor}) \cong U(1)$. The invariant $Z_{\hat{\psi}, [\beta]}$ extends to a family of assignments,

$$Z_{\hat{\psi}, [\beta]} : Y^i_{aKU}(n) \to N_i(\text{Vect}), \quad i = 0, 1, 2,$$

which are strong symmetric monoidal functors.
Proof. It follows from Theorem 14.1. The proof proceeds in a parallel way to the proof of Theorem 10.2.

We define a cobordism category \( \text{Cob}^{KU}(n)_0 \) as follows. Its object class consists of closed \( KU \)-oriented smooth \((n-1)\)-manifolds \( Y^{n-1} \) such that its \( n \)-th \( K \)-homology group is trivial, i.e. \( K_n(Y^{n-1}) \cong 0 \). For such two objects \( Y_0^{n-1}, Y_1^{n-1} \), an \( KU \)-oriented \( n \)-cobordism \( Y_0^{n-1} \to X^n \leftarrow Y_1^{n-1} \) upto \( KU \)-orientation preserving diffeomorphism. We set the cylinder \( Y^{n-1} \times I \) as the identity for an object \( Y^{n-1} \). We set gluing cobordisms as the composition of the category \( \text{Cob}^{KU}(n)_0 \). The reason that we restrict ourselves to smooth manifolds is explained in the discussion above Corollary 10.3.

Corollary 14.3. Let \( \psi \) be the homomorphism \( \pi_1(\hat{\psi}) : KK(A,S^n B) \cong \pi_1(KK(A,S^{n+1} B)) \to \pi_1(\text{U}(1)\text{Tor}) \cong U(1) \). The invariant \( Z_{\psi,\beta} \) in Definition 3.2 extends to a strong symmetric monoidal functor:

\[
Z_{\hat{\psi},\beta} : (\text{Cob}^{KU}(n)_0,1) \to (\text{Vect}, \otimes).
\]

Proof. The proof proceeds in a parallel way to the proof of Corollary 10.3.

15. Untwisted theory

In this section, we give some computations for the simplest case : \( \alpha \cong 0 \) (resp. \( \beta \cong 0 \)). We first give results under the simplest case : \( \alpha = 0 \) (resp. \( \beta = 0 \)) strictly. If we have a strict equality \( \alpha = 0 \) (resp. \( \beta = 0 \)), then it is easy to compute \( Z_{\psi,\alpha} \) (resp. \( Z_{\hat{\psi},\beta} \)). After that, we explain the dependence of \( Z_{\psi,\alpha} \) (resp. \( Z_{\hat{\psi},\beta} \)) on \( \alpha \) (resp. \( \beta \)). We use the dependence to compute the theories for \( \alpha \cong 0 \) (resp. \( \beta \cong 0 \)).

Let \( \alpha \in H\mathcal{H}(S^2B\Gamma^+, S^{n+2}; E) \) be the unit of the monoidal groupoid \( H\mathcal{H}(S^2B\Gamma^+, S^{n+2}; E), \boxtimes \). i.e. it is a collapsing morphism between some spectra. Since \([\alpha] = 0 \in \text{F}^n(\text{B}G)\), it is obvious that \( S_{\alpha}(X^n,f) = 0 \in \tilde{G}_0(S^n) \). Hence, for a connected closed \( E \)-oriented \( n \)-manifold \( X^n \), we have

\[
Z_{\psi,\alpha}(X^n) = \frac{\sharp \text{Hom}(\pi_1(X^n), \Gamma)}{\sharp \Gamma}.
\]

Let us compute \( Z_{\psi,\alpha}(Y^{n-1}) \) for a closed \( E \)-oriented \((n-1)\)-manifold. In this case, \( S_{\alpha}(Y^{n-1}, \eta, g) \) is strictly the unit so that we have a natural isomorphism

\[
Z_{\psi,\alpha}(Y^{n-1}, \eta) \cong \bigoplus_{g:Y^{n-1} \to \text{B}G} \mathbb{C} \sharp \text{Aut}(g^*(\text{ET}))
\]

In particular, the dimension of \( Z_{\psi,\alpha}(Y^{n-1}, \eta) \) is given as \( \sharp \text{Hom}(\pi_1(Y^{n-1}), \Gamma) \) if \( Y^{n-1} \) is connected. In fact, it is the maximal dimension that \( Z_{\phi,\alpha}(Y^{n-1}, \eta) \) can have for all of \( \phi, \alpha \) by definitions.

Let us prove that the assignments \( \alpha \mapsto Z_{\phi,\alpha} \) and \( \beta \mapsto Z_{\hat{\psi},\beta} \) are functorial. Let \( \alpha_0, \alpha_1 \in H\mathcal{H}(S^2B\Gamma^+, S^{n+2}; E) \) be objects and \( I : \alpha_0 \to \alpha_1 \) be a morphism. Then we have a natural isomorphism \( S_I : S_{\alpha_0}(Y^{n-1}, \eta, g) \to S_{\alpha_1}(Y^{n-1}, \eta, g) \) due to the naturality of the pairing for categorical (co)homology groups. Then it induces natural isomorphisms such as \( S_{\phi,\alpha_0}(Y^{n-1}, \eta, g) \to S_{\phi,\alpha_1}(Y^{n-1}, \eta, g), Z_{\phi,\alpha_0}(Y^{n-1}, \eta) \to Z_{\phi,\alpha_1}(Y^{n-1}, \eta) \) and \( Z_{\phi,\alpha_0}(Y^{n-1}) \to Z_{\phi,\alpha_1}(Y^{n-1}) \). Hence if we are given \( \alpha \cong 0 \), then the TQFT’s we obtained are naturally computed into the above results. We can apply similar discussion to the TQFT \( Z_{\hat{\psi},\beta} \).

16. \( \hat{\phi}, \hat{\psi} \) from one-dimensional unitary representations

We give some remarks with respect to the symmetric monoidal functors \( \hat{\phi}, \hat{\psi} \) in Assumption 8.8, 13.3. We put \( \hat{\phi} = \hat{\phi}, \hat{\psi} \). Note that the symmetric monoidal functor \( \hat{\phi} \) always induces
a group homomorphism \( \pi_1(\hat{\phi}) : G \to M \) by applying the 1-st homotopy group of \( \hat{\phi} \) as mentioned in Assumption 8.8 [13,4]. Here, \( G \) is an abelian group \( \pi_1(\mathcal{H}(S^n, S^{n+1}; G)) \cong \tilde{G}_0(S^0) \) or \( \pi_1(\mathbb{K}\mathbb{K}(A, S^{n+1}B)) \cong \mathbb{K}K(A, S^nB) \). In this section, we give a sufficient condition that a group homomorphism \( \hat{\varphi} : G \to M \) naturally induces such a symmetric monoidal functor.

**Proposition 16.1.** Let \( \mathcal{H} \) be a symmetric categorical group. Set \( H \) as the automorphism group \( \mathcal{H}(1_{3\mathcal{L}}, 1_{3\mathcal{L}}) \) where \( 1_{3\mathcal{L}} \) is the unit of the symmetric categorical group \( \mathcal{H} \). If \( \mathcal{H} \) is 0-connected, then for an object \( 1_{3\mathcal{L}} \in \mathcal{H} \) the following assignments form a symmetric monoidal functor:

\[
T_{3\mathcal{L}} : \mathcal{H} \to H\text{Tor}
\]

\[
a \mapsto \mathcal{H}(1_{3\mathcal{L}}, a)
\]

\[
(a \xrightarrow{f} b) \mapsto (\mathcal{H}(1_{3\mathcal{L}}, a) \xrightarrow{f} \mathcal{H}(1_{3\mathcal{L}}, b)).
\]

**Proof.** The assignment form a functor \( \mathcal{H} \to \text{Set} \) obviously. Let us determine a \( H \)-torsor structure of \( T_{3\mathcal{L}}(a) = \mathcal{H}(1_{3\mathcal{L}}, a) \) for \( a \in \mathcal{H} \). Due to the monoidal structure of \( \mathcal{H} \), we have a map \( (g, x) \mapsto g \cdot x \) defined by the following compositions.

\[
H \times \mathcal{H}(1_{3\mathcal{L}}, a) = \mathcal{H}(1_{3\mathcal{L}}, 1_{3\mathcal{L}}) \times \mathcal{H}(1_{3\mathcal{L}}, a) \to \mathcal{H}(1_{3\mathcal{L}} \otimes 1_{3\mathcal{L}}, 1_{3\mathcal{L}} \otimes a) \cong \mathcal{H}(1_{3\mathcal{L}}, a)
\]

Then the coherence of structure morphisms of \( \mathcal{H} \) shows that the correspondence \( (g, x) \mapsto g \cdot x \) gives an action of \( H \) on the set \( T_{3\mathcal{L}}(a) \). We show that this action is free. If \( g \cdot x = x \), then we have the following commutative diagram in \( \mathcal{H} \):

\[
\begin{array}{ccc}
1_{3\mathcal{L}} & \xrightarrow{a} & 1_{3\mathcal{L}} \\
\approx & & \approx \\
1_{3\mathcal{L}} \otimes 1_{3\mathcal{L}} & \xrightarrow{id \otimes x} & 1_{3\mathcal{L}} \otimes a \\
\end{array}
\]

Hence we obtain \( g = id_{1_{3\mathcal{L}}} \in \mathcal{H}(1_{3\mathcal{L}}, 1_{3\mathcal{L}}) \) due to the coherence of structure morphisms.

On the one hand, since \( \mathcal{H}(1_{3\mathcal{L}}, a) \) is not empty, there exists an isomorphism \( \mathcal{H}(1_{3\mathcal{L}}, a) \cong \mathcal{H}(1_{3\mathcal{L}}, 1_{3\mathcal{L}}) \). Then since the action of \( H \) on \( \mathcal{H}(1_{3\mathcal{L}}, a) \cong \mathcal{H}(1_{3\mathcal{L}}, 1_{3\mathcal{L}}) \) is free, the \( H \)-set \( \mathcal{H}(1_{3\mathcal{L}}, a) \) is a \( H \)-torsor.

Finally, we have natural isomorphisms of \( H \)-torsors by the structure morphisms:

\[
\mathcal{H}(a_0, a_1) \otimes_H \mathcal{H}(b_0, b_1) \cong \mathcal{H}(a_0 \otimes b_0, a_1 \otimes b_1),
\]

where \( \otimes_H \) denotes the tensor product of \( H \)-torsors. Due to this natural isomorphism, \( T_{3\mathcal{L}} \) becomes a symmetric monoidal functor. \( \square \)

Let \( \mathfrak{S} \) be the symmetric categorical group \( \mathfrak{S}\mathfrak{K}(S^n, S^{n+1}; G) \) or \( \mathfrak{S}\mathfrak{K}(A, S^{n+1}B) \). As a corollary, we have a natural construction of a symmetric monoidal functor \( \hat{\varphi} : \mathfrak{S} \to M\text{Tor} \) starting form a group homomorphism \( \varphi : G \to M \).

**Corollary 16.2.** Let \( \varphi : G \to M \) be a group homomorphism. Suppose that the underlying groupoid of \( \mathfrak{S} \) is 0-connected, i.e. \( \text{HH}(S^n, S^{n+1}; G) \cong \tilde{G}_0(S^1) \cong 0 \) or \( \text{KK}(A, S^{n+1}B) \cong 0 \) respectively. Denote by \( \times_{\varphi} : G\text{Tor} \to M\text{Tor} \) the symmetric monoidal functor given by the associated bundle construction. Then \( \varphi \) induces a symmetric monoidal functor \( \hat{\varphi} : \mathfrak{S} \to M\text{Tor} \) via compositions of \( \times_{\varphi} \) and \( T_{3\mathcal{L}} \) in Proposition 16.1.

**A. SOME USEFUL FACTS IN CATEGORY THEORY**

For convenience of the reader, we prepare some facts in category theory which we often use throughout this paper. For more information in category theory, we refer the reader to the reference of [13].
Definition A.1 (π₀, π₁ of Groupoids). For a small groupoid \( \mathcal{G} \), we define \( \pi_0(\mathcal{G}) \) as the set of isomorphism classes of the groupoid \( \mathcal{G} \). For an object \( a \in \mathcal{G} \), we define \( \pi_1(\mathcal{G}, a) \) as \( \mathcal{G}(a, a) \), i.e. the automorphism group of \( a \in \mathcal{G} \).

Remark A.2. The combinatorial definitions of \( \pi_0, \pi_1 \) are isomorphic to the 0-th homotopy set and the 1-st homotopy group of the classifying space of the groupoid \( \mathcal{G} \) respectively since \( \mathcal{G} \) is a groupoid.

It is well-known that a functor induces an equivalence of categories if and only if it is essentially surjective and fully faithful. For groupoids, there is a more convenient way to verify an equivalence:

Proposition A.3. Let \( \mathcal{G}_0, \mathcal{G}_1 \) be groupoids and \( f : \mathcal{G}_0 \to \mathcal{G}_1 \) is a functor. Then the functor \( f \) induces an equivalence if and only if:

- The induced map \( \pi_0(f) : \pi_0(\mathcal{G}_0) \to \pi_0(\mathcal{G}_1) \) is bijective.
- For any \( x \in \mathcal{G}_0 \), the induced homomorphism \( \pi_1(f) : \pi_1(\mathcal{G}_0, x) \to \pi_1(\mathcal{G}_1, f(x)) \) is isomorphic.

In particular, if \( \mathcal{G}_0, \mathcal{G}_1 \) are categorical groups and \( f : \mathcal{G}_0 \to \mathcal{G}_1 \) is a monoidal functor, then \( f \) induces an equivalence of the underlying groupoids if and only if:

- The induced map \( \pi_0(f) : \pi_0(\mathcal{G}_0) \to \pi_0(\mathcal{G}_1) \) is bijective.
- The induced homomorphism \( \pi_1(f) : \pi_1(\mathcal{G}_0, 1) \to \pi_1(\mathcal{G}_1, 1) \) is isomorphic where 1 denotes the units of the categorical groups \( \mathcal{G}_0, \mathcal{G}_1 \).

This proposition is very useful in this paper. Since some functors between categorified algebraic topological objects induce isomorphisms on the abelian group level, so that they induce equivalences according to the proposition.

Proposition A.4. Let \( \mathcal{C} \) be a bicategory. Let \( a, b \) be two objects of \( \mathcal{C} \) and \( f : a \to b \) be an equivalence in \( \mathcal{C} \). Then \( f \) induces an adjoint equivalence.

If a (symmetric) monoidal functor induces an adjoint equivalence, then it induces a (symmetric) monoidal equivalence. On the one hand, every equivalence is enhanced to an adjoint equivalence by Proposition A.4. Hence, a symmetric monoidal functor between symmetric categorical groups induces a symmetric monoidal equivalence if and only if \( \pi_0, \pi_1 \) induce isomorphisms.

From now on, we introduce the notion of homotopy fiber of a functor between groupoids as follows. It is often used in Section 9.

Definition A.5 (Homotopy Fiber). Let \( f : \mathcal{G} \to \mathcal{G}' \) be a functor between groupoids. Then the homotopy fiber with respect to an object \( a \in \mathcal{G}' \) is defined as a groupoid \( F_f(a) \) such that:

- The class of objects consist of \( (b, \theta) \) where \( b \) is an object of \( \mathcal{G} \) and \( f(b) \to a \) is a morphism in \( \mathcal{G}' \).
- \( \beta : (b_0, \theta_0) \to (b_1, \theta_1) \) is a morphism when \( \beta : b_0 \to b_1 \) is a morphism in \( \mathcal{G} \) and \( \theta_1 \circ f(\beta) = \theta_0 \).

Then we have the following proposition:

Definition A.6. For an object \( (b, \theta) \in F_f(a) \), we define a based map \( \partial : \pi_1(\mathcal{G}', f(b)) = \mathcal{G}'(f(b), f(b)) \to \pi_0(F_f(a)) \) by the equation:

\[ \partial(\alpha) \overset{\text{def}}{=} [b, \theta \circ \alpha] \in \pi_0(F_f(a)), \quad \alpha \in \mathcal{G}'(f(b), f(b)) \]

Then we can show the following proposition according to definitions.
Let us apply the notations of the previous proposition to Definition A.5. We obtain $K$ of the diagram $\pi_1(\mathcal{G}_i, f(b))$

$$0 \to \pi_1(\mathcal{F}_f(a), (b, \theta)) \xrightarrow{\pi_1(f)} \pi_1(\mathcal{G}, b) \xrightarrow{\pi_1(f)} \pi_1(\mathcal{G}', f(b))$$

Here we consider $\pi_0(\mathcal{F}_f(a)), \pi_0(\mathcal{G}), \pi_0(\mathcal{G}')$ as based sets with basepoints $[b, \theta], [b], [f(b)]$ respectively.

**Proposition A.8.** Let $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2$ be groupoids and $\mathcal{G}_0 \xrightarrow{f} \mathcal{G}_1 \xrightarrow{g} \mathcal{G}_2$ be a sequence of functors. Consider a functor $h : \mathcal{F}_g(c) \to \mathcal{F}_g(c); (a, \epsilon) \mapsto (f(a), \epsilon)$ for an object $c \in \mathcal{G}_2$. For an object $b \in \mathcal{G}_1$, there is an isomorphism of groupoids:

$$\mathcal{F}_f(b) \xrightarrow{\cong} \mathcal{F}_h(b, \epsilon).$$

In other words, a sequence of functors $\mathcal{F}_f(b) \to \mathcal{F}_g(c) \xrightarrow{h} \mathcal{F}_g(c)$ forms a fibration.

We introduce the notion of homotopy pull-back diagram in the 2-category of groupoids.

**Definition A.9 (Homotopy Pull-back).** Let $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}$ be groupoids and $f_i : \mathcal{G}_i \to \mathcal{H}$ be functors for $i = 1, 2$. Let us denote by $\mathcal{C}$ the 2-category formed by quadruples $(X, g_1, g_2, \Phi)$ such that $g_1 : X \to \mathcal{G}_i$ are functors for $i = 1, 2$ and $\Phi : f_1 \circ g_1 \to f_2 \circ g_2$ is a natural isomorphism. For such objects $(X, g_1, g_2, \Phi), (X', g_1', g_2', \Phi')$ we set a morphism $(k, \phi_1, \phi_2) : (X, g_1, g_2, \Phi) \to (X', g_1', g_2', \Phi')$ as a functor $k : X \to X'$ and a natural isomorphism $\phi_i : g'_i \circ k \to g_i$ for $i = 1, 2$ which satisfies the following commutative diagram:

$$\begin{array}{ccc}
 f_1 \circ g'_1 \circ k & \xrightarrow{k} & f_2 \circ g'_2 \circ k \\
 \downarrow f_1 \circ \phi_1 & & \downarrow f_2 \circ \phi_2 \\
 f_1 \circ g_1 & \xrightarrow{\Phi} & f_2 \circ g_2
\end{array}$$

For such two morphisms $(k, \phi_1, \phi_2)$ and $(k', \phi'_1, \phi'_2)$, we set a 2-morphism $\Psi : (k, \phi_1, \phi_2) \to (k', \phi'_1, \phi'_2)$ if $\Psi : k \to k'$ is a natural isomorphism such that $\phi'_i \circ (g'_i \circ \Psi) = \phi_i$ for $i = 1, 2$.

We call a weak terminal object $X_\infty$ of the 2-category $\mathcal{C}$ as a homotopy pull-back of the diagram $\mathcal{G}_1 \xrightarrow{f_1} \mathcal{H} \xrightarrow{f_2} \mathcal{G}_2$. In other words, for every object $X \in \mathcal{C}$ there exists a morphism $X \to X_\infty$ which is unique up to a unique 2-isomorphism. Then we write as a diagram:

$$\begin{array}{ccc}
 X_\infty & \xrightarrow{s} & \mathcal{G}_2 \\
 \downarrow & & \downarrow \beta \\
 \mathcal{G}_1 & \xrightarrow{f_1} & \mathcal{H}
\end{array}$$

In the next statement, we construct a canonical homotopy pull-back.

**Proposition A.10.** Let $\mathcal{G}_1 \xrightarrow{f_1} \mathcal{H} \xrightarrow{f_2} \mathcal{G}_2$ be a sequence of functors between groupoids. Consider a groupoid $X_\infty$ formed by $(x_1, x_2, \xi)$ such that $x_i \in \mathcal{G}_i$ for $i = 1, 2$ and $\xi : f_1(x_1) \to f_2(x_2)$ is a morphism in $\mathcal{H}$. For objects $(x_1, x_2, \xi), (x'_1, x'_2, \xi')$, we set a morphism $(\eta_1, \eta_2) : (x_1, x_2, \xi) \to (x'_1, x'_2, \xi')$ to satisfy $\eta_1 : x_1 \to x'_1$ is a morphism in $\mathcal{G}_1$ for $i = 1, 2$ and $\xi' \circ f_1(\eta_1) = f(\eta_2) \circ \xi$. Then we claim that the groupoid $X_\infty$ gives a homotopy pull-back of the diagram $\mathcal{G}_1 \xrightarrow{f_1} \mathcal{H} \xrightarrow{f_2} \mathcal{G}_2$.

**Remark A.11.** Let us apply the notations of the previous proposition to Definition A.5. We obtain $\mathcal{K} = \mathcal{F}_f(a)$ for $\mathcal{G}_1 = \mathcal{G}, \mathcal{H} = \mathcal{G}'$ and $\mathcal{G}_2 = \{a\}$ (one-point groupoid) and $f_1 = f$ and $f_2 : \{a\} \to \mathcal{G}'$.  

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