Classical interactions of the instanton-dyons with antidyons

Rasmus Larsen and Edward Shuryak
Department of Physics and Astronomy, Stony Brook University, Stony Brook NY 11794-3800, USA

Instantons, also known as instanton-monopoles or instanton-quarks, are topological constituents of the instantons at nonzero temperature and holonomy. While the interaction between instanton-dyons have been calculated to one-loop order by a number of authors, that for dyon-antidyon pairs remains unknown even at the classical level. In this work we are filling this gap, by performing gradient flow calculations on a 3d lattice. We start with two separated and unmodified objects, following through the so called “streamline” set of configurations, till their collapse to perturbative fields.

I. INTRODUCTION

Instantons [1] are Euclidian 4-dimensional topological solitons of the Yang-Mills gauge fields known to be important ingredient of the gauge fields in the QCD vacuum, as well as finite-temperatures comparable to the critical one $T \sim T_c$. Chiral anomalies induce sermonic zero modes of instantons, which provide the so called ’t Hooft interaction between fermions, which explicitly violate $U_A(1)$ chiral symmetry. Furthermore, collectivization of instanton zero modes create the so called Zero Mode Zone of quasi-zero eigenstates, which break spontaneously the $SU(N_f)$ chiral symmetry. Although those states includes only tiny $(\sim 10^{-4})$ subset of all fermionic states in lattice numerical simulations, they are responsible for a significant fraction of hadronic masses. The so called Interacting Instanton Liquid Model (IILM) has been developed, including ’t Hooft interaction to all orders, for a review see [2].

The first step in generalization of instantons to finite temperatures was finding the so called “caloron” solutions, periodic in Matsubara time. The second [3, 4] – and much more nontrivial step was inclusion of the so called nonzero holonomy – nonzero mean value of the 4-th component of the gauge field $A_4 = v$. This reveals the substructure of the (anti)instanton: at nonzero $v$ it gets split into $N_c$ (number of colors) (anti)dyons, (anti)self-dual 3d solitons with nonzero (Euclidean) electric and magnetic charges. In this work we will focus on the simplest gauge group $SU(2)$: in it there is only one diagonal generator and thus one Abelian subgroup which remains unbroken by $v$. In total there are four instanton-dyons corresponding to the different possible combinations of electric and magnetic charges. By tradition the selfdual ones are called $M$ with charges $(e, m) = (+, +)$ and $L$ with charges $(e, m) = (-, -)$, the anti-selfdual antidyons are called $\bar{M}$, $(e, m) = (+, -)$ and $\bar{L}$, $(e, m) = (-, +)$.

In the last few years studies of the gauge field topology based on instanton-dyons has developed along two different but strongly related directions. One of them [5] starts with a very specific setting – supersymmetric theories on a $R^4 \otimes S^1$ where the circle is spatial and fermions are periodic – making coupling weak and topological effects to be exponentially small but under the theoretical control. Another [6, 7] study them in the pure gauge and QCD-like theories. Both groups recently argued that instanton-dyons are relevant for confinement, see [5, 8], and not only breaking of the chiral symmetries. Thus instanton-dyons seem to be crucially important for understanding of the nonperturbative QCD. Yet an understanding of the dyon ensemble can only be achieved if we first understand the forces acting between the pairs of such solitons.

There exists a principal difference between the (i) single duality sector (only self- or antiselfdual objects) with (ii) the interaction between self- or antiselfdual objects. Inside single duality class the celebrated Bogomolny inequality becomes equality, requiring the action of the configuration to be entirely determined by its global topological or magnetic charge. As it is well known, this eliminates interaction at the classical level and leads to the so called moduli space, well studied in the mathematical literature. For example, a problem of two monopoles/dyons has lead to the famous Atiyah-Hitchin manifold. The moduli space metric for those and related spaces can be calculated, and it provides the correct (parameter-independent) measure of integration over the collective variables. This metric is traditionally expressed via a determinant of a certain matrix. For the LM pair this bosonic matrix has been calculated by Diakonov et al [9]. It contains the Colomb-like interactions $O(1/r_{LM})$, supplemented by the “Debye screening” potential $O(r_{LM})$. Later Diakonov [10] conjectured a volume element as a determinant, combining the DGPS one with Gibbons-Manton approximation to Atiyah-Hitchin metric. This interaction has been included in the first numerical simulations [9].

The second case (ii) – the interaction of self and anti-selfdual objects – is however much more difficult to study. There is no classical BPS protection of the action and thus no moduli spaces or corresponding solutions. Such configurations can be mapped via the so called streamline one-parameter set of solutions defined by a condition that the driving force, whole nonzero, is tangent to the set. The practical way to generate them is to follow the gradient flow, starting from some initial ansatz, as was done for numerical solution for the double-well potential in [11]. For gauge field instantons and in the large-distance approximation this was done analytically in [12]. The instanton problem is intrinsically conformal,
which allowed to do a conformal transformation into a co-

central configuration, relating the gauge theory and the
double-well instantons. Using this method Verbaarschot
[13] found a quite accurate analytic approximation to the
instanton-antiinstanton streamline.

In this paper we, to our knowledge for the first time,
study the streamline configurations of the dyon-antidy

on. More specifically, we look at pairs with the same
electric but different magnetic charge, $MM, LL$. Unfortunately,
the finite $T$ problem at nonzero holonomy $v$ is
not conformal, there is no co-central case available here:
those can only be studied in the side-by-side setting. So,
the aim of the present paper is to generate such configu-

rations and study their properties. Their total action in
particular defines what one can calls “classical interac-
tion potential” between them. Note, that although this
interaction is parametrically larger than that following
from one loop moduli spaces or sermonic zero modes, it
has been unknown and thus not included in the first sim-
ulations [6]: we certainly plan to do so in subsequent
publications.

II. THE SETTING

A. Instanton-dyons and their superposition

We do not present here extensive introduction on the
configurations and their history, which can be found in
literature such as [10].

While the objects were originally discussed in the in-

teracting setting of the instanton – the $L, M$ pair in our
notations, the proper place to start is from independent
(well separated) solutions to the equations of motion for
pure SU(2) Yang-Mills theory. Let us just remind that
“Higgsing” the SU(2) gauge theory by nonzero VEV of
the “Higgs” field is at large $r$ along the unit radial vector
$A_r = \hat{r}_m \to v \hat{r}_m$. The solutions are

$$
A^a_4 = \pm \hat{r}_a \left( \frac{1}{r} - \frac{v}{\sinh(vr)} \right),
$$

$A^a_7 = \epsilon_{aij} \hat{r}_j \left( \frac{1}{r} - \frac{v}{\sinh(vr)} \right),
$$

where $+$ corresponds to the $M$ dyon and $-$ corresponds
to the $L$ dyon. $r$ is the length in position space. The $L$
and $L$ dyon are obtained by a replacement $v \to 2\pi T - v$.

Superposition of the dyons at nonzero holonomy is
nontrivial since holonomy should match not only in mag-

nitude but also in its direction in the color space. This
is achieved by the following four-step procedure:

(i) “combing”, or going to a gauge in which the “Higgs
field” $A_4$ of a dyon at large distances is the same in all
directions and for all objects
(ii) performing a time-dependent gauge transformation

which removes the holonomy $v \to 0$
(iii) superimposing the dyons $v \to 0$
(iv) making one more time-dependent gauge transforma-
tion, re-introducing $v$ back

(i) Description of the “combing” procedure can be
found in reviews on monopoles, e.g. in [10], so we just
remind the main formula (as there are misprints in this
reference). The gauge matrices are rotations which put a
radially directed unit vector into $\pm z$ direction. The plus
one is

$$
S_+(x) = \begin{pmatrix} 
\cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2})e^{-i\phi} \\
-\sin(\frac{\theta}{2})e^{i\phi} & \cos(\frac{\theta}{2}) 
\end{pmatrix},
$$

where $\theta$ and $\phi$ are polar angles of the current position $x$
point in respect with the center of the dyon. $S_-$ is obtained by setting $\theta \to \theta - \pi$. The matrix $\Omega = S_+\Omega_s$ is used in the general gauge transformation of
the gauge field

$$
\hat{A}_\mu \Rightarrow \hat{A}'_\mu = \Omega A_\mu \Omega^\dagger + i(\partial_\mu \Omega)\Omega^\dagger,
$$

which is expressed in a standard matrix-valued form

$$
\hat{A}_\mu = A^a_\mu \tau^a_2,
$$

where Pauli matrices divided by two are the SU(2) gen-
erators in standard normalization.

(ii) The next gauge rotation matrix depends on Eu-
clidene time and is

$$
\Omega_2 = \exp(-ix_4 v \frac{\tau_3}{2}),
$$

so the derivative term produces $-v$ and cancels the holon-
omy.

(iii) The rotated dyon and antidyon are simply added
together

$$
A = A^{dyon} + A^{antidyon}.
$$

(iv) now one has to perform a gauge rotation opposite
to that in point (ii) with $\Omega_3 = \Omega_2^\dagger$. Since these
rotations commute, they just cancel each other except for
the derivative term which puts back holonomy equal $v$ at
infinity. (If one would not perform steps (ii) and (iv) but
would naively do step (iii), the holonomy would be $2v$.)

Superimposing two dyons by such a procedure, a sum
of the correctly combed potentials, is the so called sum
anzats. Needless to say, it is only an approximate solution
at large separation between the solitons, used only as the
starting point in our studies. Before we put these configu-
rations on the lattice, as we detail shortly, we calculated
the corresponding fields strengths and currents in Carte-
sian coordinates, both in Maple and Mathematica[14].

As it is well known, a “combed” monopole or dyon
must possess the Dirac string, a singular gauge artifact
propagating one unit of magnetic flux from infinity to
the dyon center. By selecting appropriate gauge one can
direct the Dirac string to have arbitrary direction.
Superimposing into a sum of two dyons with different
directions of the Dirac string one gets non-equivalent
configurations: the interference of singular and regular
terms make the Dirac strings no longer invisible or pure
gauge artifact. (However, this is cured during the gradient
flow process, as we will discuss below.)

Two obvious extreme selections for the Dirac strings
are: (a) a “minimally connected dipole” when it goes
along the line connecting two dyon centers; and (b) a
“maximally disconnected” pair, in which the Dirac
strings go into the centers from two opposite directions, see Fig. (a) Under the gradient flow the former
is supposed to reach magnetically trivial configuration,
while the latter must relax to a pure gauge Dirac-string-like
state passing the flux through the system, from minus to
plus infinity. The former case seems to be simpler and
more natural to use: but our experience has shown the
opposite, that (b) generates smaller artifacts since the
Dirac strings interfere less with the gradient flow changes
between the two objects. So we will use case (b) as our
starting configurations below.

Since the sum ansatz is not the appropriate solution,
one finds certain artifacts. One is that the Dirac string is
now visible in the action plot. This is to be expected due
to interference of the singular pure gauge string with reg-
ular solution for the other dyon. Furthermore, a correct
smooth behavior at the center of each dyon is also vio-
lated, as well as a left-right symmetry between the dyon
and antidyon.

To cure some of the artifacts one may invent certain
improved profiles. For example multiplying the “Higgs"
component of the field by the factor

$$A_3^\pm \rightarrow A_3^\pm \frac{(x - X_M)^2(x - X_M^\perp)^2}{p^2 + (x - X_M)^2[p^2 + (x - X_M^\perp)^2]},$$

which forces the field to vanishes at the centers. How-
ever, the gradient flow procedure we use takes care of the
artifacts automatically, with results independent of these
artifacts in the starting configuration, so such improve-
ments are not needed.

### III. DIONS ON THE LATTICE

#### A. The gauge fields

On the lattice the representation of the gauge field is
given in terms of the so-called link variables

$$U_\mu(x) \equiv P e^{ig_0 \int_x^{x + \hat{e}_\mu} A_\nu(z)dz} = e^{ig_0 a A_\nu(x + \hat{e}_\mu)/2} + O(a^3) \tag{10}$$

and

$$U_{-\mu}(x) = U^{\dagger}_\mu(x - \hat{e}_\mu). \tag{11}$$

The simplest gauge invariant quantity we can build
using the gauge link is the plaquette

$$P_{\nu\mu}(x) = U_\mu(x)U_\nu(x + \hat{e}_\mu)U^{\dagger}_\mu(x + \hat{e}_\nu)U^{\dagger}_\nu(x) \tag{12}$$

and with the plaquette we can define a lattice gauge
action with the correct continuum limit: \( S = -\frac{1}{4} \int d^4 x F^{\mu\nu} F^{\mu\nu} \)

$$S = \frac{2N}{g_0^2} \sum_x \sum_{\mu < v} \left( 1 - \frac{1}{N} \text{Tr}[P_{\nu\mu}(x) + P^{\dagger}_{\mu\nu}(x)] \right). \tag{13}$$

#### B. The gradient flow

An important role in what follows is a color current

$$j^a_\mu = -\frac{\delta S}{\delta A^a_\mu} \big|_{A = A_{\text{ansatz}}} = (D^{b\mu}_v G^b_{v\mu}) \big|_{A = A_{\text{ansatz}}} \neq 0. \tag{7}$$

The current vanishes for extrema (solutions of the YM
equation, such as a single dyon), but it is nonzero for
dyon-antidyon configurations which we study. It has the
meaning of the force in the functional space showing the
direction towards a reduction of the action.

The gradient flow is a process, in a computer time \( \tau \),
thus the current would be the driving force in the next
sections.

The configuration can be decomposed into two parts,
the longitudinal and the transverse one, according to the
chosen ansatz plane. For all the points on the streamline
the current should be tangential to the solution, in other
words it should have

$$j^a_\mu(x) = 0. \tag{8}$$

Unfortunately it is not easy to find such streamline con-
dfigurations analytically. But we can do it numerically:
one should just start with two well-separated pseudopar-
ticles and follow the direction of the driving force, which
leads along the streamline downstream.

Introducing the fictitious time coordinate \( \tau \) we can
write the trajectory of the resulting gradient flow toward,
and then along, the streamline

$$\frac{dA^a_\mu}{d\tau} = -\frac{\delta S}{\delta A^a_\mu}. \tag{9}$$

In the following we will solve this equation putting the
two dyons on a lattice.
To visualize the gauge field it will be useful to plot the action density using
\[
s(x) = \frac{2N}{g_0^2} \left( 1 - \frac{1}{48N} \text{Tr} \left[ \sum_{\mu > \nu \in \pm 1} (P_{\mu \nu}(x) + P_{\mu \nu}^\dagger(x)) \right] \right)
\tag{14}
\]

We now have to translate eq. [9] into the lattice language.

In order to not have too rough corrections to the dyon configurations coming from \(O(a^2)\) terms in the definition of the links, we need \([aA_\mu] << 1\). Then going to the gauge where the \(M\) and \(\bar{M}\) dyons or the \(L\) and \(\bar{L}\) dyons has the same \(A_4\) field at infinity, the fields are no longer a good solutions to the equations of motion on the lattice. Therefore all the transformations explained in section [11] will be performed on the lattice instead. On the lattice the gauge transformation is
\[
U_\mu(x) \rightarrow \Omega(x)U_\mu(x)\Omega^\dagger(x + \hat{e}_\mu).
\tag{15}
\]
The sum ansatz says that we add up the dyons, which now becomes multiplication instead. We also do a gauge transformation in time first to put the asymptotic value of \(A_4\) to zero. This leaves an extra term when we add the two dyons given as the following element of the temporal gauge transformation
\[
\delta\Omega_i = \exp(ia\hat{\tau}_3 i/2).
\tag{16}
\]

Combining everything gives us the initial configuration as
\[
U_4(x) = \sum \Omega(x)U_{4,i}(x)\Omega^\dagger(x + \hat{e}_\mu),
\tag{17}
\]
\[
S_+(x)U_{1,4}(x)S^\dagger_+(x + \hat{e}_4)\delta\Omega_i S_-(x)U_{2,4}(x)S^\dagger_-(x + \hat{e}_4)
U_4(x) = \sum S_+(x)U_{1,4}(x)S^\dagger_+(x + \hat{e}_4)S_-(x)U_{2,4}(x)S^\dagger_-(x + \hat{e}_4),
\tag{18}
\]
where \(U_{1,\mu}(x)\) and \(U_{2,\mu}(x)\) are the links of the \(M\) and \(M\) or \(L\) and \(L\) dyon given in equation [11]. The difference between looking at \(M\) and \(M\) dyons and \(L\) and \(L\) dyons is which gauge transformation you use. For \(M\) you transform \(M\) with \(S_+\) and \(M\) with \(S_+\) such that \(A_4\) at infinity becomes \((v - \frac{1}{2})\hat{\tau}_3\). With \(LL\) you instead transform \(L\) with \(S_+\) and \(L\) with \(S_+\) such that \(A_4\) at infinity becomes \((v - 2\pi T + \frac{1}{2})\hat{\tau}_3\). In order to have the holonomy to be \(v\), an additional transformation is made, though this is done after the dyons have been set to zero at infinity and added up, and the results should therefore been invariant with respect to this transformation unlike the other transformations made before we add the dyons.

Varying the action with small rotations in \(SU(2)\), the current can be found as
\[
J_\mu(x) = \sum \text{Tr}[P_{\mu \nu}(x) - P_{\mu \nu}^\dagger(x)]
= \sum \text{Tr}[P_{\mu \nu}(x - \hat{e}_\nu) - P_{\mu \nu}^\dagger(x - \hat{e}_\nu)],
\tag{19}
\]
which is simply the imaginary part of all plaquettes that include \(U_\mu(x)\). This is scaled by \(dt\) which is the chosen size of one step in the cooling process. The 3 components of \(J_\mu\) is then calculated
\[
J_{1,\mu} = dt\text{Tr}[\tau_1 J_\mu(x)],
\tag{20}
\]
As can be seen, this definition depends on which of the 4 links the \(\tau_1\) is placed at. In order to get the correction to \(U_\mu(x)\) we need that \(J_\mu\) starts from \(U_\mu(x)\). The matrix used for cooling is calculated as
\[
L_\mu(x) = \sqrt{J^2_{1,\mu} + J^2_{2,\mu} + J^2_{3,\mu}}
\tag{21}
\]
\[
\theta_{1,\mu}(x) = J_{1,\mu}/L_\mu
\tag{22}
\]
\[
C_\mu(x) = \cos(L_\mu)I + \sin(L_\mu)\sum_i \theta_{1,\mu} \tau_i.
\tag{23}
\]
\(C_\mu(x)\) is used to change all the links as
\[
U_\mu(x) \rightarrow C_\mu(x)U_\mu(x).
\tag{24}
\]

B. Lattice details

Since \(M\bar{M}\) pairs and \(LL\) pairs are time independent (\(LL\) pairs are time independent in the gauge where the Higgs field is \((2\pi T - v))\), the lattice used is a \(N^3\) sized lattice.

The lattice size in natural units is \(40/v\) in each dimension. Lattice spacing is such that it has 64\(^3\) points. This might seem like a rough lattice since \(a = 0.625/v\). Note however, that the configurations before combing have sufficiently small \(A\) around the cores, thus we do have \([aA_\mu] \ll 1\). The combing may appear to produce large \(A \sim 1/a\), yet those are pure gauge and is not spoiling the action. Some higher gradient lattice corrections of course exist, but it has not been observed to be important.

The lattice used is not periodic, instead we hold the sides constant, i.e. don’t update with the current on the surface of our box.

On this setup the analytic solution of one dyon is stable at a value 5\% lower than the analytic value of \(4\pi v\) with an absolute value of electric and magnetic charge, calculated by Gaussian flux integrals near the box surface, of exactly 1.

Let us remind that the gauge action can be expressed in terms of the 3-dimensional action
\[
S = \frac{1}{g^2} \int_0^{1/T} dx_4 S_3 = \frac{S_3}{g^2 T}
\tag{25}
\]
which itself scales as \(S_3 \sim v\); thus the \(M\) dyon action is \(\sim v/T\). We do not care about \(T\) and the gauge coupling \(g\) since it is just an overall factor in the action, and work with the \(S_3\) itself. Furthermore, since our classical 3d theory is invariant under the transformation \(A_\mu \rightarrow vA_\mu\) and \(r \rightarrow vr\), the absolute units are unimportant and we can work with \(v = 1\).
Apart of the action and electric and magnetic fields, we also want to observe the Dirac string as the system evolves (cools). In the continuum the gauge transformations introduce a singularity in $A_\mu$, stretching from the dyon centers out to infinity. If one goes around it, the total phase should be $2\pi$. But as long as the phases of subsequent links are added together in the $\tau_3$-direction, using the inverse of the parametrization

$$U_\mu(x) = \cos(\phi)I + \sin(\phi)\sum_i \theta_{i,\mu}\tau_i$$

we do observe the famous $2\pi$ phase.

**IV. RESULTS**

**A. Qualitative features of the streamline**

Before we present our results in detail, we would like to give a brief overview of the findings, starting with a reminder of the streamline for the instanton-antiinstanton case. These configurations, either in quantum mechanical setting [11] or gauge fields [13], have the meaning of tunneling forth and back, with only finite time spent in the second well (valley). When this time goes to zero, there is no reason for the configuration itself to be different from zero (path or gauge fields). Thus the end of the streamline is expected to be made of weak (perturbative) fields. Those are supposed to already be treated in the harmonic approximation in perturbation series, and thus should not be double counted again in the nonperturbative sector: thus one uses a phenomenological “repulsive core” in the instanton liquid model.

The case at hand, with the instanton-dyons, is a bit different. Two charges – the magnetic and the topological ones – still add to zero and can annihilate each other, but there is one remaining – the electric charge, which adds to 2 units rather than annihilating each other. Those should of course be conserved and survive till the end of the gradient flow process, although in a perturbative form.

The gradient flow process was found to proceed via the following stages:

(i) near initiation: starting from relatively arbitrary ansatz one finds rapid disappearance of artifacts and convergence toward the streamline set

(ii) following the streamline itself. The action decrease at this stage is small and steady. The dyons basically approach each other, with relatively small deformations: thus the concept of an interaction potential between them makes sense at this stage

(iii) a metastable state at the streamline’s end: the action remains constant, evolution is very slow and consists of internal deformation of the dyons rather than further approach

(iv) rapid collapse into the perturbative fields plus some (pure gauge) remnants

We will detail properties of these stages below, for now restricting to general comments. One is the existence of the stage (iii) which has not been anticipated on general grounds. Since all configurations corresponding to it have the same action, one can perhaps lump all of them into a new class of states, corresponding to the same dyon-antidyon distance. Unlike the instanton-dyons themselves, such states have not yet been identified on the lattice.

Our other comment is the action value even at the end of the streamline is not that far from the sum of the two dyon masses. In other words, the classical interaction potential happens to be rather small numerically, a welcoming feature for statistical mechanics simulations.

Last but not least, we do observe the universality of the streamline. As expected, independent on the initial dyon separation we found that gradient flow proceeds through essentially the same set of configurations at stages (ii-iv). Thus one-parameter characterization of those is possible. A parameter we found most practical in this work is simply its lifetime – duration in our computer time $\tau$ needed for a particular configuration to reach a final collapse. (Of course, for statistical mechanics applications one better map that into some collective coordinate, such as the dyon separation, whatever way it can be defined).

**FIG. 2: Action for $v = 1$ as a function of computer time (in units of iterations of all links) for a separation $|r_M - r_{\bar{M}}| v = 0, 2.5, 5, 7.5, 10$ between the $M$ and $\bar{M}$ dyon from right to left in the graph. The action of two well separated dyons is 23.88.**

**B. The action**

We now show the results for a $M$ and $\bar{M}$ dyon separated by a distance (in natural units $1/v$) of the order 0 to 10 along the z-axis which is cooled using gradient flow. The action of an individual dyon on the lattice was found to be 11.94, 5% lower that the analytic value $4\pi$. This gives the action of 23.88 for two well separated dyons. Any action lower than this therefore is ascribed to an attractive binary potential between the dyons.
The gauge transformations on the initial configuration have been done such that each string goes away from the other dyon. This is done since the strings are not physical, but a remnant from the original configurations, which becomes visible because of the sum ansatz. By choosing to take the strings away from the other dyons, we seek to reduce the overlap.

The simulations of the streamline for different configurations starts out with a slightly higher action around that of 2 individual dyons, this then converge to an almost stable configuration, the Streamline. A typical action history is shown in Fig. 2 for the initial separation values from 0 to 10. If the separation is bigger than 5, the two cores of the dyons are seen to move towards each other and their action smoothly decreases. We show four important computer times in Fig. 3, where the action density is plotted along the z-axis (along the dyons separation) for at start separation of 10. We have also found the maximums of the action density and converted this into a plot of action vs separation as shown in Fig. 4 (a).

A fit has been attempted on this plot. We want the fit to reduce to the action of two well separated dyons, which is $S_3(\infty) = 23.88 \approx 8\pi$ in our calculations. We also require the fit to go as $1/r$ for large $r$ since the abelian Higgs field is long ranged (recall that electric repulsion is compensated by magnetic attraction). Last our results require that the potential levels off before the collapse. We therefore propose the following parameterization

$$V(r) = S_3(r) - S_3(\infty) \approx -\frac{A(rv - B)^2}{(rv)^3 + C}$$

The fitted function is shown together with the data in Fig. 4(a), for $A = 30.925, B = 0.9072, C = 15.795$.

For a dyon separation of the order of $4/v$ or smaller the streamline effectively ends. If one starts with a smaller separation, say $2.5/v$, one finds the same action density as that for larger separations. (So one should not use our parameterization for $rv < 4$, since there are no dyon-antidyon configurations with smaller separation, the “repulsive core” is not like the one shown in Fig. 4 (a), but is absolute.)

At the metastable configurations the peaks of the action density has stopped around a separation of slightly bigger than 4. From this point onward the only thing seen to change is a bit of electric charge leaking from the inner part around the dyons.

The metastable state at the end of the streamline is followed by a quick drop to zero, as can be better seen in Fig. 3. By the way, we found that the drop in the action is not immediate even for no separation between the dyons, as can be seen in Fig. 4(b).

C. The Higgs profile

Looking at the Higgs field ($A_4^\pm$), the collapse is seen as a loss of the barrier between the two minimums. The collapse of the barrier between the dyons has been observed to happen exactly when the action starts to drop.

It is observed how for two dyons we initially see two valleys which go from $v$ to around zero. At the collapse the "wall" between the two dyons collapse and it becomes a deeper valley, around -0.6. The thickness is dictated by
FIG. 5: Action density along the z axis in natural units for a separation \(|r_M - \bar{r}_M|v = 10\) between the center of the 2 dyons. The configuration with the maximums furthest from each other is the start configuration. After 3000 steps it has moved further towards the center. At 12000 steps the configuration has reached the metastable configuration with a separation between the maximums of around 4. At 13700 the configuration has collapsed around halfway, and will continue to shrink until the action is 0. Times are as shown in Fig. 2. the original distance between the two dyons. The Higgs field also goes quicker to its mean value \(v\) when you leave the valley.

For a separation of 5 the cooling only smooths the Higgs field slightly, while for a separation of 10 we also get a drop in the minimum while the configuration runs along the streamline. This means that while we observe the action of the streamline to be universal, the Higgs profile around the core of the dyons are not. This is of course happening because this quantity is not gauge invariant. In fact the endpoint of the gradient flow leaves a rather complicated, although zero action – and thus pure gauge – configuration.

D. The electric and magnetic charges

As we know the total charge of the setup should be 2 for electric charge and 0 for magnetic for the \(\hat{r}_3\) component. We observe the electric charge to be held in place for quite a time when the solution moves along the streamline, see Fig. 7. However when the configuration begin to collapse the electric charge quickly starts to move away, as is seen in the same Fig. after time \(t > 10000\). The dyon collapse is triggered by the very rapid annihilation of the magnetic charges of the dyons, as shown in Fig. 8. Thus it is clearly the dyon magnetic structure which is crucial for their individual existence.

E. The Dirac string

While the strings only appear as a result of the combing gauge transformations for individual dyons, they are always present for dyon-antidyon case we study. To look at the strings we, as mentioned in section III, evaluate the phase of the spatial loop \(\int A^3_m dx^m\) around a closed line around the string – such as a square. In Fig. 9(a) we plot the spatial loop phase along the z-axis at the beginning of the gradient flow process. The end of the flow for the same configurations is shown in Fig. 9(b).

For the smallest (square) loop used the phase takes a value close to 0 in between the two strings: there is no string there. Increasing the size of the loop, the phase get closer and closer to \(2\pi\), especially after the cooling has been done, where we observe that the phase goes quicker towards \(2\pi\). It means that the string is there at all times, but is just less concentrated in between the dyons. Note also, that if one shifts the position of the loop such that \(x = y = 0\) is not included, it will drop promptly from \(2\pi\) to 0. This however only happens for \(|z| > 2.5\) for the configurations used, where the separation between the dyons was 5.
In summary, our starting configurations possess singular Dirac strings from infinity to their centers: the flux in between appears in the form of dipole magnetic fields at small \( z \) region. At the end of the cooling, all magnetic fields are gone, and yet a version of the Dirac string for \( r |z| < 2.5 \) is still there. (Once again, recall that the residual surviving configuration has zero action and is a pure gauge.)

F. \( L \bar{L} \) pairs

For the \( L \bar{L} \) pairs the story is very much the same. The difference is that the original combing is done in such a way that the Higgs field points in the negative direction with a value of \( 2\pi T - v \). We of course need to make a time dependent gauge transformation to put the holonomy to \( v \), but the simulations them self are here done in the time independent gauge where the Higgs field of the \( L \bar{L} \) pair is \(- (2\pi T - v) \). Since nothing different from the \( M \bar{M} \) pairs happens for the charge and action we won’t show those graphs. More interesting is the Higgs field which after a time dependent gauge transformation at the beginning looks like in Fig. 10 (a).

It is seen how the valleys is now instead a mountain for the \( L \) and \( \bar{L} \) dyons. The \( A_4 \) field is completely time independent since the \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) component of \( A_4 \) are zero.

Then the configuration collapses into one big mountain, we do get a time dependent part in \( A_4 \) from the now non-zero values of the \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) component of \( A_4 \), which have gained a core as shown in Fig. 11. As for the \( A_3^2 \) component which is the mountain shown in Fig. 10 (b), it stays time independent.

The time dependence that \( A_4 \) do gain is only for the \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) component of \( A_4 \), since \( \exp(i\pi T x_4 \hat{\tau}_3^3) \hat{\tau}_{(1,2)} = \hat{\tau}_{(1,2)} \exp(-i\pi T x_4 \hat{\tau}_3^3) \), so we only have a mixing of the \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \) component of \( A_4 \) which mix with a time dependent phase.
FIG. 10: $A_2^4$ for the $L\bar{L}$ dyon pair along the separation of the two dyons (z-axis), in natural units, at the beginning (a) and end (b) of the simulation. The start value was $2\pi T - \nu = 1$ and the time dependent gauge transformation has been performed to make the Higgs field at infinity equal to 1. The separation between the dyons is 5.

FIG. 11: $A_1^4$ and $A_2^4$ along the separation of the $L$ and $\bar{L}$ dyon (z-axis), in natural units, after the cooling, before the extra time dependent gauge transformation has been done. The separation between the dyons is 5.

V. SUMMARY AND OUTLOOK

In summary, we performed the gradient flow studies of the instanton-dyon-antidyon configurations. We found that, after a brief period of initial relaxation, the process settle in to a rather universal “streamline” set of states, with steady and slow reduction of the action. We found that numerically the dyon-antidyon interaction is relatively small.

We observe one expected feature: below certain separation of the dyons the streamline no longer exist. We also found an unexpected feature: between the “streamline” set and the final collapse to perturbative (small action) fields we found a metastable state, which takes longer to reach the larger the initial separation between the dyon and the antidyon is. The action and the separation between the dyons remains about the same, independent of the initial conditions. The force is so small in this state that the gradient flow gets extremely slow and takes a considerable computer time.

We observe that an initial configuration of a $M(L)$ and a $M(\bar{L})$ which is magnetic neutral and have a total electric charge of 2(-2) with two separate valleys(mountains) in the Higgs field, will when cooled move slowly towards the center along the streamline until it reaches a metastable configuration at a separation slightly bigger than $4/\nu$. After certain time the magnetic charges annihilate each other and configuration rapidly collapses to a pure gauge configuration, while the electric charge flies away. For separations smaller than 4, they immediately fall into the very stable configuration until they collapse.

As shown for the $M\bar{M}$ pair, we observe the universality of the streamline. We have used this universality to make an effective potential dependent on the separation of the dyons, using the position of the maximums of the action density. The effective potential between a dyon and a antidyon was found to be of the order of $-20\%$ with respect to the total action. This result, shown in shown in Fig. 4 (a) and parameterized in (27), is the main result of the current work.

Speaking about future work, an obvious extension of the current work would be a calculation of the Dirac eigenvalue spectrum for the streamline configurations at hand. Indeed, as experience with the “instanton liquid” has shown [2], the main interaction in QCD-like theories is the fermion-induced one, which happens precisely between the two duality sectors. It is even more so in the case of QCD-like theories with many quark flavors $N_f \sim 10$, which is currently under active investigation by the lattice community. Recent work [7] had argued that this interaction leads to $LL$ clusters of small size $\sim 1/N_f$, a descendants of instanton-antinstanton molecules. The reader may also find in this paper discussion of a number of specific issues/observables, relating the dyonic picture of QCD topology at $T > T_c$ to various lattice data.

The next logical step (after deriving the interaction between the topological objects in question) is of course some study of the resulting statistical ensemble, improving on simulations done in [6].

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