Influence of bed material entrainment and non-Newtonian rheology on turbulent geophysical flows dynamics. Numerical study

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Abstract. This paper deals with the mathematical and numerical modeling of the propagation stage of geophysical gravity-driven flows, such as snow avalanches, mudflows, and rapid landslides. New mathematical models are presented which are based on full, not-depth-averaged equations of mechanics of continuous media. The models account for three important issues: non-Newtonian rheology of the moving material, entrainment of the bed material by the flow, and turbulence. The main objective is to investigate the effect of these three factors on the flow dynamics and on the value of the entrainment rate. To exclude the influence of many other factors, e.g., the complicated slope topography, only the motion down a long uniform slope with a constant inclination angle is studied numerically. Moreover, the entire flow from the front to the rear area was not modeled, but only its middle part where the flow is approximately uniform in length. One of the qualitative results is that in motion along homogeneous slope the mass entrainment increases the flow velocity and depth while the entrainment rate at large time tends to become constant which depends on the physical properties of the flow and the underlying material but not on the current values of the flow velocity and depth.

1. Introduction

This paper deals with mathematical and numerical modeling of dense geophysical gravity-driven mass flows, such as snow avalanches, mudflows, and rapid landslides. These flows can pose grave hazards to people and property. Knowledge of their dynamic parameters and run-out distance, i.e., the boundaries of the affected area, is needed for properly land planning in mountains and design of defense constructions. Numerical modeling may provide one of the tools to obtain this knowledge.

Mathematical models of dense geophysical flows can be divided into three groups according to the complexity level and the extent account for different flow characteristics. The simplest models treat the flow as a moving material point, i.e., the internal structure is not accounted for. In models of the second group the mass flow is a flow of a continuous medium but only the depth-averaged velocities are calculated besides the flow depth and width. Models of the third group describe the structure of the flow not only along its body, but also in a direction normal to the slope. They are based on full, not-depth-averaged equations of mechanics of continuous media. The development and use of such models have been launched recently [1-4]. This became possible due to the appearance of new measuring techniques and increase of the capabilities of computers. To construct these new models one should know 1) the equations which relate stresses and deformation characteristics inside the flow,
i.e., the rheological properties of the moving medium, 2) the equation which determines the entrainment of mass from the slope by the flow since dense gravity-driven flows always entrain bed material while descending steep slopes [5, 6], and 3) the equations which describe the characteristics of turbulence since most of large gravity-driven flows are turbulent.

The paper presents new models of the third group. The models take into account all three factors mentioned above. The first point is the formulation of the constitutive rheological relations for the flow material. These relations are different for flows of different physical nature: dry and wet dense snow avalanches, powder avalanches, mudflows, debris flows, volcanic lava, landslides, water flows etc. The simplest model for dense snow avalanches is the model of incompressible isotropic linear-viscous (Newtonian) fluids with high value of the viscosity coefficient [2, 7-9]. The components $\tau_{ij}$ of the viscous stresses tensor in this model are linear functions of the components of the strain rate tensor $e_{ij}$, and vanish at $e_{ij} = 0$: $\tau_{ij} = 2\mu e_{ij}$, $\mu$ being the viscosity coefficient. However, observations and measurements indicate the necessity to introduce more sophisticated models: measured velocity profiles differ from those for Newtonian fluids [9-14]. Besides, many geophysical flows, e.g., avalanches and landslides can stop at an incline, being at rest with non-zero tangential stress. The latter fact means that the flow material have a yield limit: it doesn’t move or moves without deformations as a rigid body until the stress intensity is less than the yield stress. To describe the rheological properties of the flow material here we use the so-called Herschel–Bulkley model, which can correspond to both linear and nonlinear viscous (power-law) fluids, as well as media with a yield stress, in particular, the Bingham fluid, by the appropriate choice of the coefficients. The Bingham and Herschel–Bulkley models have been proposed as possible rheological models for snow avalanches, mudflows, lava flows, and landslides, for example in [1-4, 11, 13]. The second point is including bed material entrainment. Field evidence shows that a flow’s volume can increase many fold as a consequence of entrainment [5, 6]. It is known that the size, speed, and destructive potential of geophysical mass flows are strongly influenced by entrainment of bed material. Here we model the effect of the basal entrainment assuming the following hypothesis [6, 9]: the entrainment occurs, when the shear stress on the flow bottom reaches the value of the bed material shear strength. The last point in constructing the mathematical model is the account for turbulence of the flow. We deal with the Reynolds’s averaged equations and flow parameters. To obtain the closed system of equations we employed the three-parameters turbulence model developed by Luschik, Paveliev and Yakubenko [15, 16] which was successfully applied to simulate turbulent flows in pipes and channels with account for heat transfer, mass exchange through porous walls and nonlinear temperature dependence of the fluid viscosity coefficient.

Simulations of flows with different rheological properties have been done by the code developed by the authors. The main objective was to investigate the effect of flow rheology, bed material entrainment and turbulence on the flow dynamics and on the value of the entrainment rate. To exclude the influence of many other factors, e.g., the complicated slope topography, only the motion down a long uniform slope with a constant inclination angle was studied. Moreover, the entire flow from the leading edge to the tail was not modeled, but only its middle part where the flow is approximately uniform in length. The statement of the problem and examples of simulations results are given in the next sections.

2. Statement of the problem
We consider the flow on an infinitely long uniform slope with a constant angle $\theta$. The fluid is incompressible, with a bulk density $\rho = \rho_0 = \text{const}$. The origin of the Cartesian coordinate system is taken on the free surface of the flow, the $x$ axis is parallel to the slope and the $z$ axis is normal to it and downward directed (Figure 1). The flow bottom $AB$ moves down if the entrainment of the bed material occurs.
The Reynolds-averaged flow parameters including the streamwise velocity depend only on \( z \) and time \( t \), while the flow depth \( h \) varies with time if the flow entrains the bed material. The function \( h(t) \) is to be determined. The Reynolds equation reads

\[
\frac{\partial \langle v_x \rangle}{\partial t} = g \sin \theta + \frac{1}{\rho} \frac{\partial T_{xz}}{\partial z} \quad \text{at} \quad 0 \leq z \leq h(t).
\]  

(1)

Here, \( T_{xz} = \langle \tau_{xz} \rangle + \rho T \) is the total shear stress, i.e., the sum of the averaged molecular stress \( \langle \tau_{xz} \rangle \) and the turbulent stress \( \rho T \), \( T = -\langle v'_x v'_z \rangle \). The primes denote the fluctuated velocity components, while the angular brackets stand for Reynolds averaging. Below we omit the angular brackets in denoting the average values. Boundary condition at the free surface \( z = 0 \) is \( T_{xz} = 0 \); at the bottom \( z = h(t) \) the no-slip condition is assumed: \( v_x = 0 \) at \( z = h(t) \). Besides, if the flow entrains the bed material, then, in accordance with the assumed hypothesis on the entrainment mechanism, an additional boundary condition at the bottom is set: \( |T_{xz}| = \tau_c \) at \( z = h(t) \), where \( \tau_c \) is the shear strength of the bed material. This additional condition serves to calculate the increasing flow depth.

The full system of equations consists of the Reynolds equation (1), rheological relation which determines the molecular stress \( \tau_{xz} \) and its average, and equations for turbulent stress and other turbulence characteristics. As it was mentioned in the Introduction, we assumed the Herschel–Bulkley models to approximate rheological relations for the moving media. For a laminar simple shear flow along the \( x \) axis \((v_x = v_x(z,t), v_y = v_z = 0)\) the Herschel–Bulkley equation is

\[
\text{if} \quad |\tau_{xz}| \leq \tau_0 \quad \text{then} \quad \frac{\partial v_x}{\partial z} = 0; \quad \text{if} \quad |\tau_{xz}| > \tau_0 \quad \text{then} \quad |\tau_{xz}| = \tau_0 + K \left| \frac{\partial v_x}{\partial z} \right|^n.
\]  

(2)

Here \( \tau_0 \) is the yield limit, \( K, n \) are coefficients called the consistency and the power-law index respectively. This equation and its generalisation for complex flows have been proposed to describe the behaviour of different non-Newtonian flows met in nature and in engineering practice. In particular, it corresponds to a Newtonian fluid if \( \tau_0 = 0, n = 1 \); to a power-law fluid if \( \tau_0 = 0, n \neq 1 \), and to a Bingham fluid if \( \tau_0 \neq 0, n = 1 \). Examples of the coefficients values are 1) \( \rho = 200\text{kg/m}^3 \), \( \tau_0 = 0, n = 1 \), \( K/\rho = 10^{-3}\text{m}^2/\text{s} \) (dry snow avalanche as a Newtonian flow [7]); 2) \( \rho = 500\text{kg/m}^3 \), \( \tau_0 / \rho = 2\text{m}^2/\text{s}^2 \), \( n = 2 \quad K/\rho = 89\cdot10^{-6}\text{m}^2 - \text{dense avalanche} [10] \); 3) \( \tau_0 / \rho = 1.5\text{m}^2/\text{s}^2 \), \( n = 0.33 \), \( K/\rho = 0.1\text{m}^2/\text{s}^{2-n} \) - mudflow, kaolin concentration >10% [11]; 4) \( \rho = 1130\text{kg/m}^3 \), \( \tau_0 = 10\text{Pa}, n = 0.8 \), \( K = 0.03\text{Pa} \cdot \text{s}^n \) - a suspension of mineral particles, concentration 3-5% [13].
For turbulent flows the problem arises in averaging the nonlinear rheological relation. We further assume that in a turbulent shear flow the average value of the molecular stress is given by the relation of the same structure as (2). This assumption is the simplest possible hypothesis until detailed experimental data about fluctuations of the viscosity coefficient appear.

The three-parameter turbulence model developed by Luschik, Paveliev and Yakubenko which we referred to as LPY model includes the differential equations for three parameters, namely, $T$ - the turbulent stress divided by the bulk density, $E = 0.5 \left( v_x' v_x' + v_y' v_y' + v_z' v_z' \right)$ - the turbulent energy density, and the parameter $\omega$ related to the turbulence scale $L$ by the formula $\omega = E / L^2$. The equations of the LPY model for the flow in consideration are

$$\frac{\partial E}{\partial t} = -\left( c_1 E + c_1 v \right) \frac{E}{L^2} + \frac{\partial}{\partial z} \left( D_\epsilon \frac{\partial E}{\partial z} \right) + T \frac{\partial v}{\partial z},$$

$$\frac{\partial T}{\partial t} = -\left( c_5 E + c_5 v \right) \frac{T}{E^2} + \frac{\partial}{\partial z} \left( D_\tau \frac{\partial T}{\partial z} \right) + c_7 E \frac{\partial v}{\partial z},$$

$$\frac{\partial \omega}{\partial t} = -\left( 2 c_6 E + 1.4 c_6 v f_\omega \right) \frac{\omega}{E^2} + \frac{\partial}{\partial z} \left( D_\omega \frac{\partial \omega}{\partial z} \right) + \left[ \frac{T}{E} + 2 c_4 \text{sign} \left( \frac{\partial v}{\partial z} \right) \right] \omega \frac{\partial v}{\partial z},$$

$$D_\phi = a_\phi \sqrt{EL} + \alpha_\phi v, \quad (\phi = E, T, \omega), \quad f_\omega = 1 - \frac{1}{2c_1} \left( \frac{L}{E} \frac{\partial E}{\partial z} \right)^2.$$

Here, $v = v_x$ and $v = \mu / \rho$ is the kinematic coefficient of effective molecular viscosity. The values of the dimensionless coefficients are as follows: $c = 0.3$, $c_1 = 5 \pi / 4$, $c_4 = 0.04$, $c_5 = 3c$, $c_6 = 9c_1$, $c_7 = 0.2$, $a_E = a_\omega = 0.06$, $a_T = a_E c_5 / c$, $a_\alpha = 1$, $a_\omega = 1.4$. These values of the coefficients were successfully used without any variations in simulations of different flows, including the cases in which the viscosity coefficient $\mu$ was not constant but was a function of the variable temperature [16]. However, the applicability of these values for non-Newtonian flows needs experimental justification.

The boundary conditions for $T, E, \omega$ were at the open surface $z = h(t)$: $T = 0$, $E = 0$, $\frac{\partial E}{\partial z} = 0$; at the open surface $z = 0$: $T = 0$, $\frac{\partial E}{\partial z} = 0$, $\frac{\partial \omega}{\partial z} = 0$.

3. Results of simulations

The computational code was developed on the base of an implicit finite difference scheme with iterations. Simulations of flows with different rheological properties moving with and without entrainment down slopes with different inclination angles and different strength of the bed material have been done [4]. Below only the results concerning the comparison of Newtonian and Bingham flows behavior are presented. The features of all other investigated flows are similar. The rheological relations for both flows are

$$\begin{align*}
&\text{if } |r_{zc}| < \tau_0, \quad \text{then } \frac{\partial v_z}{\partial z} = 0; \quad \text{if } |r_{zc}| \geq \tau_0, \quad \text{then } r_{zc} = \tau_0 + \mu \frac{\partial v_z}{\partial z}.
\end{align*}$$

Here $\tau_0$ is the yield stress (equal to zero for Newtonian fluid), $\mu = \text{const}$ - the viscosity coefficient. One of the typical features of Bingham flows is the appearance of the quasi-solid parts in the flow, where the fluid moves without deformations. Since according to the boundary condition at the upper surface $r_{zc} = 0$, there exists a layer $0 \leq z \leq r(t)$ adjacent to the flow surface where $|r_{zc}| \leq \tau_0$. This layer moves without deformation as a solid (Figure 2). This layer is referred to a plug layer. The
depth \( r(t) \) of the plug layer is not known in advance. It can be calculated by the condition
\[ |r_x| = r_0 \text{ at } z = r(t). \]

The effect of entrainment of the bed material is demonstrated on Figures 3, 4 for turbulent flows of a linearly viscous fluid. The calculations were performed with the following values of the input parameters: \( \tau_0 = 0, \ \tau_c / \rho = 1.5 \text{m}^2/\text{s}^2, \ \mu / \rho \equiv \nu = 0.001 \text{m}^2/\text{s}, \ h_0 = 1 \text{m} \) where \( h_0 \) is the initial flow depth.

**Figure 3.** Newtonian flow. (a) - velocity profiles 10 seconds after the start of entrainment, (b) - the average velocity vs. time. 1 - motion without mass entrainment, 2 - with entrainment.

**Figure 4.** Newtonian flow with mass entrainment. Dependence of (a) - the flow thickness and (b) - the entrainment rate on time.

The graphs in Figures 3 and 4 show that in motion down a long homogeneous slope with mass entrainment, the velocity increases. At large time the dependence of the flow thickness and maximum and average velocities on time tends to be linear, and the entrainment rate tends to a constant (Figure 4(b)). Thus, for large time the entrainment rate does not depend on the velocity and thickness of the flow. Such an asymptotic behaviour is observed if the thickness of the layer accessible for the
entrainment is sufficiently large. Otherwise, after the entire available layer is exhausted, the motion tends to become stationary, corresponding to the new value of the thickness.

Figure 5. a) Velocity profiles in flows with mass entrainment. b) The shape of the velocity profiles near the upper surface. (1) - Newtonian, (2) – Bingham (τ₀/ρ = 0.05 m²/s² ) fluid.

Figure 5(b) demonstrates a plug layer near the upper surface of the Bingham flow.

Figure 6. Influence of the yield limit value (a) - on the start of the bed material entrainment, (b) - on the magnitude of the entrainment rate. (1) - (4) - τ₀/ρ = 0, 0.01, 0.02, 0.05 m²/s².

Figure 6 (a) shows that the yield stress reduces the flow velocity, mass entrainment rate, and the entrainment starts later comparing to the linearly viscous flow. However, if the entrainment lasts for a long time the effect of the yield stress on the entrainment rate weakens (Figure 6 (b)).

4. Conclusion
This paper presents models for slope flows based on equations of continuum mechanics that are not averaged over depth. Models take into account the nonlinear rheological properties of moving media, the entrainment of the underlying material, and the turbulent nature of the motion. Numerical investigation of the effect of these three factors is performed for flow at long homogeneous slope. The main conclusions resulting from analysis of the simulations are as follows:

• By entrainment the underlying material, the velocity and thickness of the flow increase.
• When moving along homogeneous slope with entrainment, at large times from the start of the entrainment, regardless of the rheological properties of the flow, the velocity on the surface of the flow, the depth-averaged velocity and the flow depth grow linearly with time until exhausted the entire available mass.
• For all the models studied, the entrainment rate of the bottom material tends with time to a constant, the magnitude of which depends only on the slope angle and the physical properties of the flow and slope materials, but not on the current velocity or depth.
The presence of the yield point leads to a later start of entrainment, a lower flow velocity and an entrainment rate compared to flow of media that do not have a yield point.

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**References**

[1] Eglit M E and Yakubenko A E 2014 Numerical modeling of slope flows entraining bottom material *Cold Reg. Sci. Technol.* **108** 139–148

[2] Bovet E, Chiaia B and Preziosi L 2010 A new model for snow avalanche dynamics based on non-Newtonian fluids. *Meccanica* **45** 6 753-765

[3] Zaiko Yu S 2016 Numerical modeling of downslope flows of different rheology *Fluid Dyn.** **51**(4) 443 - 450

[4] Eglit M E and Yakubenko A E 2016 The effect of bed material entrainment and non-Newtonian rheology on dynamics of turbulent slope flows *Fluid Dyn.** **51** (3) 299–310

[5] Sovilla B, Burlando P and Bartelt P 2006 Field experiments and numerical modeling of mass entrainment in snow avalanches *J. Geophys. Res.** **111**(F03007)

[6] Hungr O, McDougall S and Bovis M 2005 Entrainment of material by debris flows. *Debris-Flow Hazards and Related Phenomena* ed Jakob M and Hungr O (Praxis-Springer, Berlin Heidelberg) 135-158.

[7] Dent J D and Lang T E 1980 Modeling of Snow Flow *J. Glaciology** **26**(94) 131

[8] Nishimura K and Maeno N 1989 Contribution of viscous forces to avalanche dynamics *Annals of Glaciology** **13** 202

[9] Issler D and Pastor Peréz M 2011 Interplay of entrainment and rheology in snow avalanches; a numerical study *Annals of Glaciology** **52**(58) 143-147

[10] Kern M F, Tiefenbacher J and McElwaine J 2004 The rheology of snow in large chute flows *Cold Reg. Sci. and Techn.** **39** 181–192.

[11] Coussot P 1997 Mudflow Reology and Dynamics (Balklema Publ., Rotterdam)

[12] Issler D 2003 Experimental information on snow avalanches. *Dynamic Response of granular and porous materials under large and catastrophic deformations*. Ed Hutter K and Kirchner N (Springer Belin) 109–160

[13] Chhadra R P and Richardson J F 2008 Non-Newtonian Flow and Applied Rheology: Engineering Applications (Elsevier Oxford) 518

[14] Naaim M Naaim-Bouvet F Faug T and Bouchet A. 2004 Dense snow avalanche modeling: flow, erosion, deposition and obstacle effects *Cold Reg. Sci. Technol.** **39** 193–204.

[15] Lushchik V G, Paveliev A A and Yakubenko A E 1978 Three-parameter model of shear turbulence *Fluid Dynamics** **13**(3) 350-362

[16] Leontiev A I, Lushchik V G and Yakubenko A E 2009 A heat-insulated permeable wall with suction in a compressible gas flow *Int. J. Heat Mass Transfer** **52**(17–18) 4001