Parametric description of the quantum measurement process

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Abstract – We present a description of the measurement process based on the parametric representation with environmental coherent states. This representation is specifically tailored for studying quantum systems whose environment needs being considered through the quantum-to-classical crossover. Focusing upon projective measures, and exploiting the connection between large-N quantum theories and the classical limit of related ones, we manage to push our description beyond the pre-measurement step. This allows us to show that the outcome production follows from a global-symmetry breaking, entailing the observed system’s state reduction, and that the statistical nature of the process is brought about, together with the Born’s rule, by the macroscopic character of the measuring apparatus.

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Introduction. – The measurement process plays a fundamental role in any physical theory, and indeed a crucial one in quantum mechanics. The set of works dedicated to the subject is almost uncountable and different descriptions of the process often imply different interpretation of quantum mechanics itself [1–7] (see, for instance, refs. [4,8,9] for comprehensive discussions and thorough bibliographies).

In this work, embodying the formal definition of the large-N limit of a quantum theory [10] into the parametric representation with environmental coherent states (PRECS) [11,12], we complement the description of the measurement process with the quantum-to-classical crossover of the apparatus only. We thus manage to understand how objective results emerge from the entangled states of a macroscopic measuring apparatus, and why their production is an inherently statistical phenomenon. The observed system’s state reduction and the Born’s rule are contextually obtained. It is to be noticed that environmental coherent states, together with the results of the above ref. [10], have already been used in relation with the quantum measurement process, e.g., in ref. [13] and related works; however, it is the original use of the PRECS that allows us to embody the symmetry breaking phenomenon and ultimately describe the outcome production.

Pre-measurement and parametric representation. – Let us first outline the formalism to which we will essentially refer in describing the quantum measurement process. Be Γ the observed quantum system (with Hilbert space $\mathcal{H}_\Gamma$), and Ξ the measuring apparatus (with Hilbert space $\mathcal{H}_\Xi$). The composite system $\Psi=\Gamma+\Xi$ is assumed isolated, i.e., $\rho_{\Psi}(t)=|\Psi(t)\rangle\langle\Psi(t)|$, at any time prior to the output production ($t<T$); moreover its state is taken separable before the measurement starts ($t\leq0$),

$$|\Psi(t\leq0)\rangle = |\Gamma\rangle \otimes |\Xi\rangle.$$  

(1)

Note that the validity of these assumptions should not be taken for granted, as extensively discussed in refs. [4,8].

Let us concentrate upon sharp observables, defined as—and identified with—projection operator valued measures on the real Borel space $\mathcal{H}_\Gamma$, or a subset of it [4]. Any such measure, $O_\Gamma$, determines a unique Hermitian operator $\hat{O}_\Gamma$ acting on $\mathcal{H}_\Gamma$; if the observable is further assumed (for the sake of clarity) discrete and non-degenerate, it is
\( \hat{O}_T = \sum_\gamma \omega_\gamma |\gamma\rangle \langle \gamma | \), and the \( \hat{O}_T \)-eigenvectors \( \{ |\gamma\rangle \} \) form an orthonormal basis for \( \mathcal{H}_T \). Further ingredients of a scheme designed for describing the measure of \( \mathcal{O}_T \) are i) a pointer observable \( \hat{P}_Z \) of \( \Xi \), ii) a pointer function correlating the value sets of \( \hat{P}_Z \) and \( \hat{O}_T \), iii) a measurement coupling between \( \Gamma \) and \( \Xi \), ultimately responsible for the transformation \( \rho_\Psi(0) \sum_y \rho_\Psi(t) \) occurring before the actual production of a specific output is obtained. It can be shown that a sufficient condition for a state transformation to qualify as a proper pre-measurement [4], is that \( \hat{V} \) be a trace-preserving linear mapping. When \( \hat{V} \) is further assumed to be unitary, the process coincides with the one first described by von Neumann [1], later generalized by several authors [14–17], and characterized by Ozawa [18] under the name of conventional measuring process. Assuming the pointer observable is sharp, and hence related to a Hermitian operator \( \hat{P}_Z \), the choice \( V = e^{-i\hat{H}_T} \) with

\[
\hat{H}_T = u \hat{O}_T \otimes \hat{O}_Z + \hat{\Gamma}_T \otimes \hat{H}_Z ,
\]

where \( \hat{O}_Z \) ideally is the operator conjugate to \( \hat{P}_Z \) and \( u \) is a coupling parameter, defines the standard model [4] for describing pre-measurements as dynamical processes; notice that \( \hat{H}_Z \) acts on \( \mathcal{H}_Z \), \( \hat{I}_Z \) is the identity operator on \( \mathcal{H}_Z \), and we have set \( h=1 \). In what follows we will specifically study the above standard model, taking the identity as the pointer function, for the sake of simplicity. Writing \( |\Gamma \rangle \) in eq. (1) on the basis of the \( \hat{O}_T \)-eigenstates, from eq. (2) it follows that

\[
|\Psi(t)\rangle = \sum_\gamma c_\gamma |\gamma\rangle \otimes |\Xi(t)\rangle ,
\]

(3)

at any time during the pre-measurement \((0 < t < T)\), with

\[
|\Xi(t)\rangle \equiv e^{-i\hat{H}_Z^2 t} |\Xi\rangle ,
\]

(4)

and

\[
\hat{H}_Z^2 \equiv u \omega_\gamma \hat{O}_Z + \hat{H}_Z .
\]

(5)

In the standard model the states \( |\Xi(t)\rangle \) are usually taken as eigenstates of \( \hat{P}_Z \), and the possibility of extracting information about \( \Gamma \) reporting on \( \Xi \), relies on the \( \gamma \)-dependence of \( |\Xi(t)\rangle \), i.e. on the dynamical entanglement generation induced by the coupling \( u \hat{O}_T \otimes \hat{O}_Z \) if, and only if, \( |\Xi\rangle \) is not a \( \hat{O}_Z \)-eigensate. In the usual case, we expect to be in a stationary state before the above coupling is switched on. Therefore, it is usually taken

\[
\hat{H}_Z |\Xi\rangle = E_0 |\Xi\rangle \quad \text{and} \quad [\hat{O}_Z, \hat{H}_Z] \neq 0 .
\]

(6)

The evolution described by eq. (3) can be studied by the PRECS [12], a method based on the use of generalized coherent states [19–21] for \( \Xi \), whose construction, as far as the model (2) is concerned, can be summarized as follows. Consider the operators \( \hat{O}_Z \) and \( \hat{H}_Z \); they will generally be elements of a Lie group \( G \), usually dubbed (environmental) dynamical group, and, in most physical situation, they also belong to the related Lie algebra \( \mathfrak{g} \) (in that they are linear combination of the group generators).

The arbitrary choice of a reference state \( |R\rangle \in \mathcal{H}_Z \) defines the subgroup \( F \) of the operators \( \hat{f} \) acting trivially on \( |R\rangle \), i.e. such that \( \hat{f}|R\rangle = e^{i\phi_R}|R\rangle \), and hence the environmental coset \( G/F \). The environmental coherent states (ECS) are the states

\[
|\Omega\rangle = \hat{\Omega}|R\rangle \quad \text{with} \quad \hat{\Omega} \in G/F .
\]

(7)

It is demonstrated [19–21] that coherent states \( |\Omega\rangle \) are in one-to-one correspondence with points \( \Omega \) on a differentiable manifold \( \mathcal{M} \). The construction of ECS entails the definition of an invariant (with respect to \( \hat{G} \)) measure \( d\rho(\Omega) \) on \( \mathcal{M} \), as well as of a metric tensor \( \mathfrak{m} \). Moreover, ECS form an over-complete set on \( \mathcal{H}_Z \), and provide an identity resolution in the form

\[
\hat{\mathfrak{m}}_Z = \int_{\mathcal{M}} d\rho(\Omega) |\Omega\rangle \langle \Omega| ,
\]

(8)

where \( \hat{\mathfrak{m}}_Z \) is the identity operator on \( \mathcal{H}_Z \). Getting back to the model (2), if \( \mathfrak{g} \) is semi-simple, referring to its Cartan basis\(^1\), and reminding condition (6), one recognizes that \( \hat{H}_Z^2 \) is in the canonical form with \( \hat{O}_Z \) Hermitian linear combination of shift-up and -down operators: This usually entails the choice of \( |R\rangle \) as the eigenstate of \( \hat{H}_Z \) which is annihilated by one of the above shift operators, a choice that naturally provides \( \mathcal{M} \) with a symplectic structure [22], i.e. \( \Omega = (z, \bar{z}) \) with \( z, \bar{z} \) canonical coordinates, and \( d\rho(\Omega) = \det(\mathfrak{m})dzd\bar{z} \). Consistently with conditions (6), we can set \( |\Xi\rangle = |R\rangle \), implying

\[
\langle \Xi| \hat{\mathfrak{m}}_Z |\Xi\rangle = 0 .
\]

(9)

Coherent states have peculiar dynamical properties, which are often summarized by the motto “once a coherent state, always a coherent state” [19]. In the specific case of a system ruled by the Hamiltonian (2), with initial state (1), these properties, complemented with the choice \( |R\rangle = |\Xi\rangle \), allow one to write [12]

\[
|\Xi(t)\rangle = e^{i\phi(t)} |\Xi(t_0)\rangle ,
\]

(10)

where \( |\Xi(t)\rangle \) is the coherent state corresponding to the point \( \Xi(t) \) along the trajectory on \( \mathcal{M} \) defined by the solution of the classical-like equations of motion

\[
\frac{d^2z}{dt^2} = \frac{\partial}{\partial \bar{z}} \hat{H}_Z^2(\Omega) ,
\]

(11)

with \( \hat{H}_Z^2(\Omega) = \langle \Omega| \hat{H}_Z^2 |\Omega\rangle \) and \( \Xi(0) = 0 \); as for the phase factor it is \( \phi(t) = \int_{t_0}^t \! dy/\langle \Xi(y)| \{ \hat{O}_Z, \hat{H}_Z \} |\Xi(y)\rangle \).

Once the ECS are constructed, any state of the composite system \( \Psi \) can be parametrically represented by formally splitting \( \Gamma \) and \( \Xi \) through the insertion of \( \hat{\mathfrak{m}}_Z \) in the
for all \( \{ \} \). Notice that, although such condition specifically concerns formative apparatus: in fact, defining the \( \chi \) measurement can be considered successfully concluded.

we will hereafter drop the \( \tau \) phase equal to 0. Equations (12)–(14) define the parametric representation with ECS of \( |\Psi(t)\rangle \). Notice that the dependence of the principal system’s pure states \( |\delta_i(\Omega)\rangle \) on \( \Omega \) is the signature that \( \Gamma \) and \( \Xi \) are entangled \([11]\). Moreover, due to \( \langle \Psi(t)|\Psi(t)\rangle=1 \), it is \( \int_M d\mu(\Omega)\chi^2_\gamma(\Omega)=1 \) at any time, which allows one to interpret \( \chi^2_\gamma(\Omega) \) as the normalized density distribution of ECS on the manifold \( M \) \([11,19]\).

Informative apparatus. – The essential goal of any pre-measurement is that of setting a cogent relation between elements of \( \{ |\gamma\rangle \}_{\mathcal{H}_\gamma} \) and pointer states \([23]\) of the measuring apparatus. Referring to eq. (3), this implies that

\[
|\psi_\gamma(\Omega)\rangle = \frac{1}{\chi_\gamma(\Omega)} \sum_\gamma c_\gamma (\Omega)|\Xi^\prime(\tau)\rangle |\gamma\rangle,
\]

(13)

|\psi_\gamma(\Omega)\rangle = \sum_\gamma |c_\gamma|^2 h_\gamma^2(\Omega), \quad h_\gamma^2(\Omega) = |(\Omega|\Xi^\prime(\tau))|^2,
\]

(14)

and we have set \( \chi_\gamma(\Omega) \) in \( R^+ \) by choosing its arbitrary phase equal to 0. Equations (12)–(14) define the parametric representation with ECS of \( |\Psi(t)\rangle \). Notice that the dependence of the principal system’s pure states \( |\delta_i(\Omega)\rangle \) on \( \Omega \) is the signature that \( \Gamma \) and \( \Xi \) are entangled \([11]\). Moreover, due to \( \langle \Psi(t)|\Psi(t)\rangle=1 \), it is \( \int_M d\mu(\Omega)\chi^2_\gamma(\Omega)=1 \) at any time, which allows one to interpret \( \chi^2_\gamma(\Omega) \) as the normalized density distribution of ECS on the manifold \( M \) \([11,19]\).

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\[
\text{different states } |\Xi^\prime(\tau)\rangle \text{ and } |\Xi^\prime(\tau)\rangle,
\]

(15)

\[
\text{produce different outcomes whenever } \gamma \neq \gamma',
\]

(16)

for \( \tau>\tau_d \), where \( \tau_d \) is the time when the pre-measurement can be considered successfully concluded. Condition (15), (16) can be translated \([24]\) into some property that \( \chi^2_\gamma(\Omega) \) must feature in order to describe an informative apparatus: in fact, defining the \( \varepsilon \)-support of each component \( h_\gamma^2(\Omega) \) as the region \( \mathcal{S}_\gamma \subseteq \mathcal{M} \) such that \( h_\gamma^2(\Omega) \propto \forall \Omega \in \mathcal{S}_\gamma \), with \( \varepsilon \) a small number in \( R^+ \), the request (15), (16) can be identified \([24]\) with the condition

\[
\mathcal{S}_\gamma \cap \mathcal{S}_{\gamma'} = \emptyset \; \text{for } \gamma \neq \gamma'.
\]

(17)

The one-to-one correspondence thus established between each function \( h_\gamma^2(\Omega) \) and the region \( \mathcal{S}_\gamma \) in \( \mathcal{M} \) allows one to write

\[
|\psi(\tau)\rangle = \sum_\gamma c_\gamma \int_{\mathcal{S}_\gamma} d\mu(\Omega) e^{i\varepsilon^2 h_\gamma^2(\Omega)} |\gamma\rangle \otimes |\Omega\rangle
\]

(18)

for all \( \tau \) in the time interval where condition (17) holds. Notice that, although such condition specifically concerns the apparatus, eq. (18) strictly implies that decoherence with respect to the basis \( \{ |\gamma\rangle \}_{\mathcal{H}_\gamma} \) has occurred \([24]\), and the time \( \tau_d \) is consistently recognized as a decoherence time for \( \Gamma \). For the sake of a lighter notation we will hereafter drop the \( \tau \) index in \( \mathcal{S}_\gamma \) understanding that, whenever \( \mathcal{S}_\gamma \) is referred to, it is \( \tau>\tau_d \). For a more transparent discussion, it is now worth considering two models with Hamiltonian of the form (2), that have been introduced as paradigmatic ones for describing the decoherence phenomenon, and extensively studied in various contexts \([8,25]\). They are described by

\[
\hat{H}_{qh} = \nu \hat{b}^\dagger \hat{b} + g \hat{\sigma} z \otimes (\hat{b} + \hat{b}^\dagger),
\]

(19)

\[
\hat{H}_{qs} = \hbar \hat{J}^z + \mu \hat{\sigma} z \otimes \hat{J}^x,
\]

(20)

where \( \hat{\sigma} z \) is the \( Z \)-Pauli matrix, \( \hat{b} \) and \( \hat{b}^\dagger \) are bosonic operators \( [\hat{b}, \hat{b}^\dagger]=1 \), and \( \hat{J}^z = \hat{J}^x = \hat{J}^y = i e^{\alpha^2 \hat{a}^\dagger, \hat{a}^\dagger} \). The coefficients \( g \) and \( \mu \) are the coupling parameters corresponding to \( u \) in eq. (2), while \( \nu \) and \( \hbar \) are the frequencies of the free environmental Hamiltonian in each model. The respective ECS are those usually referred to as field and spin coherent states, with \( |R\rangle \) such that \( \hat{b}|R\rangle=0 \) and \( \hat{J}^z |R\rangle=0 \), and manifold \( M \) the complex plane and the unit 2-sphere. The trajectories \( \Xi^\prime_\gamma \) defining the states \( |\Xi^\prime_\gamma\rangle \) in eqs. (13), (14), with \( \gamma=\pm \), can be explicitly determined and are shown in fig. 1 as black lines on the respective manifold. In the same figure, the distributions \( \chi^2_\gamma(\Omega) \equiv \text{det}(M)\chi^2_\gamma(\Omega) \) are plotted at various times: it strikes that, as time goes by, they acquire a multi-modal structure, with as many distinct modes as the number of different \( \gamma \) in the \( \Gamma \)-spectrum, thus visualizing how condition (17) is dynamically achieved. Moreover, considering \( \chi^2_\gamma(\Omega) \) for different values of \( g \) and \( J \), as in fig. 2, reveals that when such parameters increase, each \( \gamma \)-peak becomes more pronounced, and the corresponding support \( \mathcal{S}_\gamma \) consequently shrinks around \( \Xi^\prime_\gamma \). This evidence, that reflects the more general result \([10]\) briefly stated at point ii) of the next section, is clearly reminiscent of some sort of classical limit for \( \Xi \). The description of the process which is now available allows us to push the formalism towards a well-defined macroscopic limit for the measuring apparatus only, that leaves the principal system unaffected.

Large-\( N \) limit. – This section is based on a work by Yaffe \([10]\), dealing with the fundamental question “Can one find a classical system whose dynamics is equivalent to some \( N \rightarrow \infty \) limit of a given quantum theory?” where \( N \) is some measure of the number of dynamical variables. An extensive discussion of ref. \([10]\) goes beyond the scope of

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Fig. 1: (Colour on-line) Evolution of \( \chi^2_\gamma(\Omega) \) for the models (19) and (20), top and bottom panels, respectively, as time goes by from left to right. The black lines indicate the trajectories \( \Xi^\prime_\gamma \), with \( \gamma=\pm \).
this letter, but briefly retracing the reasoning underlying the results which are most relevant to get to our final goal, is necessary. In doing that we will try to keep contact with what we have described so far.

Given that:

- a classical theory, \(C\), is defined by a phase space \(\mathcal{C}\), a Poisson bracket, and a classical Hamiltonian \(H_{cl}\);

- a quantum theory, \(Q\), is defined by a Hilbert space, a Lie algebra, and a Hamiltonian \(H\) (and dynamical group \(G\)),

then:

i) Let \(Q_\kappa\) be a quantum theory characterized by some (quanticity) parameter \(\kappa\), assumed to take positive real values including the limiting \(\kappa=0\) one. This is the theory that describes the apparatus \(\Xi\) in the above sections, and one of the above conditions for all \(\kappa\) i.e., \(\kappa \neq 0\), depending on specific features of \(Q_N\). The relation between the two theories is established via their respective dynamical groups, \(G_N\) and \(G_\kappa\), by making them be different representations of the same algebra. Operators \(\hat{A}_\kappa\) and \(\hat{A}_N\) in the two different theories are also formally related (though it is not possible to explicitly express such relation in general).

\[ \lim_{N \to \infty} Q_N = \lim_{\kappa \to 0} Q_\kappa = \mathcal{C}, \]

ii) There exists a minimal set of conditions that \(Q_\kappa\) must fulfill to guarantee that its \(\kappa \to 0\) limit is a classical theory \(C\). Such conditions emerge in terms of coherent states for the dynamical group of the theory, \(G_\kappa\), and establish a one-to-one correspondence, \(\mathcal{M}_\kappa \to \mathcal{C}\), between points on the related manifold and on the phase space \(\mathcal{C}\) of the theory \(C\). These coherent states are the ECS defined in the previous section, and one of the above conditions implies

\[ h^2/\hbar (\Omega) \det(m) \xrightarrow{\kappa \to 0} \delta(\Omega - \Xi_\kappa^2), \tag{21} \]

as suggested in fig. 2.

iii) Let \(Q_N\) be a quantum many-body (field) theory with some global \((N)\)-symmetry \((X=O, Sp, U, \ldots)\), dynamical group \(G_N\), and related manifold \(\mathcal{M}_N\). This is the microscopic quantum theory that would exactly describe the apparatus \(\Xi\), were we able to determine the details of its internal interactions as well as of those between each one of its components and \(\Gamma\).

iv) Any such \(Q_N\) theory defines a \(Q_\kappa\) one, by this meaning that the latter can be explicitly defined from the former, such that \(\kappa=1/N\) or \(1/N^2\), depending on specific features of \(Q_N\). The relation between the two theories is given by:

\[ \mathcal{M}_N \to \mathcal{M}_\kappa \to \mathcal{C}, \]

Fig. 3: (Colour on-line) Graphical depiction of the relation between effective and microscopic theories, \(Q_\kappa\), \(Q_N\), giving rise to the same classical theory \(C\).

Macroscopic measuring apparatus. – Regarding the measurement process, Yaffe’s results teach us that the classical limit \((\kappa \to 0)\) of the effective theory \(Q_\kappa\) used for describing \(\Xi\) during the pre-measurement, can keep describing a macroscopic \((N \to \infty)\) apparatus; however, in order for this to be the case, an \((N)\)-invariant \(Q_N\) theory must actually underlie \(Q_\kappa\), being the one that would provide us with the exact, microscopic description of \(\Xi\), if we were able to deal with it in the large-\(N\) limit.

Fig. 2: (Colour on-line) \(\chi^2(\Omega)\) at a given time for the models (19) and (20) with different parameters \(g\) and \(J\), left and right panels, respectively. The black lines are as in fig. 1.
We can now get back to the PRECS treatment, to find that the theory \( Q_N \), with its related coherent states \( |\Omega\rangle \), is already well defined\(^2\), and the symbols \( \langle \Omega|A|\Omega\rangle \) are the Husimi functions \([19–21]\) of operators acting on \( \mathcal{H}_\Xi \). Therefore, from condition (17) and eq. (22), we find that to each \( S^\gamma_N \), and hence to each trajectory in \( \mathcal{M} \), there corresponds a set \( \{|\Omega_N\rangle\}_{\gamma} \) of classically equivalent coherent states of the microscopic, \( X(N) \)-invariant, quantum theory \( \hat{Q}_N \).

States belonging to the same set \( \{|\Omega_N\rangle\}_{\gamma} \) keep being distinct in the large-\( N \) limit, as this limit does not affect the transformations \( \hat{U}_{\tau,j} \) that relate them. Therefore, the set \( V^\gamma_N \) of points corresponding to \( \{|\Omega_N\rangle\}_{\gamma} \) in \( \mathcal{M}_N \) has a finite volume \( V^\gamma_N \) even if the apparatus becomes macroscopic.

States belonging to sets \( \{|\Omega_N\rangle\}_{\gamma} \) labeled by different \( \gamma \)'s are related by
\[
|\Omega_N(t)\rangle^{\gamma'} = U_N^{\gamma\gamma'}(t)|\Omega_N(t)\rangle^{\gamma}
\]
with
\[
U_N^{\gamma\gamma'}(t) \equiv e^{-i\hat{H}_N^\gamma t+i\hat{H}_N^\gamma'},
\]
which follows from the fact that evolutions defined by different \( \gamma \)'s have the same initial state. For \( t>\tau d \) they are not classically equivalent, implying \( V^\gamma_N \cap V^{\gamma'}_N = \emptyset \), but yet they have the same energy in the \( N \to \infty \) limit,
\[
\begin{align*}
\lim_{N \to \infty} \langle \Omega_N|H^2_{\gamma,N}|\Omega_N\rangle &= \Omega_{N\in V^\gamma_N} = \Omega_{N\in V^{\gamma'}_N} \quad (25) \\
\lim_{\kappa \to 0} \langle \Omega|H^2_{\kappa}|\Omega\rangle &= \Omega_{\in V^\gamma_N} = H_{cl}(\Xi^\gamma) = E_0, \quad (26)
\end{align*}
\]
as seen from eqs. (22), (21), (9), and (6).

The above analysis tells us that if one were to study the behaviour of a macroscopic measuring apparatus in terms of its microscopic quantum theory, despite not being able to do it exactly due to the large number of dynamical variables, yet one could extract information on \( \Xi \), for to do it exactly due to the large number of dynamical variables, yet one could extract information on \( \Xi \), for
\[
\Xi \to \infty
\]
cannot commute with any of the global transformations \( U_N^{\gamma\gamma'} \) in eqs. (23), (24), nor can it depend on the spectrum \( \omega_N \), it is \( \langle \Omega_N|\delta(\Omega_N)\rangle^{\gamma'} \delta(\Omega_N) \delta(\Omega_N)^{\gamma'} \) unless they both vanish. In other terms, a local perturbation on \( \Xi \) that is independent on the previous evolution of \( \Psi \), cannot cause the same energy lowering in states belonging to different sets of classically equivalent coherent states.

Therefore, only one \( \gamma_{out} \) will be selected,
\[
\bigcup_{\gamma} \{|\Omega_N\rangle\}_{\gamma} \longrightarrow \{\Omega_N\}^{\gamma_{out}}_{X(N)-SB}.
\]
Picking one specific set of classically equivalent coherent states, \( \{\Omega_N\}^{\gamma_{out}}_{X(N)-SB} \), ensures that all classical functions get their respective definite value \( A_{cl}(\Xi^{\gamma_{out}}) \) on the classical phase space \( C \) via eq. (22), and a classical behaviour unambiguously emerges for the measuring apparatus. In particular, the classical function corresponding to \( \hat{O}_\Xi \) will take the value \( O_{cl}(\Xi^{\gamma_{out}}) \), which will be the result of the measurement process.

Notice that the effective theory describing the apparatus during the pre-measurement is \( X(N) \)-symmetric if and only if the interaction between \( \Gamma \) and \( \Xi \) also features such symmetry; therefore, the \( X(N) \)-SB is only made possible by the outwards opening of \( \Psi \), \( i.e. \) by enlarging the system considered during the pre-measurement from \( \Psi = \Gamma + \Xi \) to \( W = \Psi + R \), where by \( R \) we mean the “rest of the world”. The action of \( R \) on \( \Xi \) can be controlled, just like in an actual measurement where it is triggered by the reading of the apparatus. It can also be completely random, in which case the \( X(N) \)-SB induces the emergence of classicality. Whatever the situation, such action is uncorrelated with the dynamics of \( \Psi \) before the symmetry breaking.

**Born’s rule.** – In order to understand what value of \( \gamma_{out} \) one should in principle expect, let us go back to our measuring apparatus after the pre-measurement is concluded but before the SB has occurred (\( \tau d < \tau < T \)): In the previous sections we have learned that its macroscopic \( (N \to \infty) \) behaviour follows from the features of the ensemble \( \{\{|\Omega_N\rangle\}_{\gamma},\{|\Omega_N\rangle\}_{\gamma},\ldots\} \) of degenerate coherent states, grouped into disjoint sets, each labeled by a specific \( \gamma \). States belonging to the same set correspond, as \( N \to \infty \) and at any time \( \tau > \tau d \), to the same point \( \Xi^\gamma \) in \( C \), and hence to the same possible outcome \( O_{cl}(\Xi^\gamma) \). On the other hand, due to degeneracy and to the fact that the perturbation causing the \( X(N) \)-SB is uncorrelated with the evolution of the overall system \( \Psi \), states belonging to the above ensemble are all equally likely. Therefore, the principles of statistical mechanics tell us that the probability \( p(\gamma_{out}) \) of the outcome \( O_{cl}(\Xi^{\gamma_{out}}) \) is proportional to the volume \( V^\gamma_N \) occupied by the set \( V^\gamma_X \) of representative points in \( \mathcal{M}_N \), at \( \tau = T \) and as \( N \to \infty \). Being left with the final problem of evaluating \( V^\gamma_N \), we consider the following:

Prior to the symmetry breaking, the two theories \( Q_N \) and \( Q_\kappa \), in their respective large-\( N \) and \( \kappa \to 0 \) limit, describe the same classical behaviour of \( \Xi \). Therefore, getting back
to $\Psi = \Gamma + \Xi$, for $\tau > \tau_d$ one has

$$\langle \Psi | \hat{H}_\gamma \otimes A | \Psi \rangle = \sum_\gamma \int_{S_\gamma} d\mu(\Omega) \chi^2(\Omega) \langle \Omega | A | \Omega \rangle$$

(28)

and it must be

$$\lim_{\kappa \to 0} \langle \Psi | \hat{H}_\gamma \otimes A | \Psi \rangle = \lim_{N \to \infty} \sum_\gamma \int_{V_N} d\mu(\Omega_N) \langle \Omega_N | A_N | \Omega_N \rangle$$

(29)

for any classical operator $A$ acting on $H_\Xi$, and for all initial states $|\Gamma\rangle$ (i.e. for all possible sets of $\{c_\gamma\}$ such that $\sum_\gamma |c_\gamma|^2 = 1$). Given eq. (22), this is seen to require

$$V_N^2 \propto \lim_{N \to \infty} \int_{V_N} d\mu(\Omega_N) = \lim_{\kappa \to 0} \int_{V_\gamma} d\mu(\Omega) \chi^2(\Omega) = |c_\gamma|^2,$$

(30)

which results in the Born’s rule

$$p(\gamma_{out}) = |c_\gamma_{out}|^2.$$

(32)

Finally notice that the selection entailed by eq. (27) trails behind itself that of the state $|\gamma_{out}\rangle$ for $\Gamma$, as seen from eqs. (13) and (21), thus realizing the reduction of the observed system’s quantum state.

**Conclusions.** – Let us briefly retrace the route that brought us from the initial separable state $|\Gamma \otimes \Xi\rangle$ to the production of an outcome, and the Born’s rule. We have considered projective measures and, referring to the standard model of unitary pre-measurements, analyzed their dynamics by means of some specific tools, namely generalized coherent states and the PRECS. After having expressed the necessity of decoherence as a condition on the distribution of quantum states relative to the apparatus, we have formally substantiated the intuitive relation between

1) the classical limit of the effective quantum theory used in the standard model for describing the apparatus by a limited number of dynamical variables, and

2) the large-$N$ limit of the microscopic theory that would exactly describe, were we able to handle it, a macroscopic quantum measuring apparatus.

This has enlightened that a global symmetry must characterize 2) if 1) is to be given a physical meaning: In fact, the breaking of such symmetry causes the breakdown of the effective quantum description of the apparatus adopted during the pre-measurement and contextually selects an objective outcome.

Our description can be tested by artificially causing the global-symmetry breaking, after decoherence has occurred in a pre-measurement process. To this respect, we are investigating the possibility of an experimental realization of the model (20) by a spin star, with the external ring playing the role of the apparatus, and the global symmetry being the one that ensures its total spin $J$ to be fixed during the pre-measurement.

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