A class of functionals possessing multiple global minima

Biagio Ricceri

Abstract. We get a new multiplicity result for gradient systems. Here is a very particular corollary: Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a smooth bounded domain and let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function, with $\Phi(0,0) = 0$, such that

$$\sup_{(u,v) \in \mathbb{R}^2} \frac{|\Phi_u(u,v)| + |\Phi_v(u,v)|}{1 + |u|^p + |v|^p} < +\infty$$

where $p > 0$, with $p = \frac{2}{n-2}$ when $n > 2$. Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$-\Delta u = (\alpha(x) \cos(\Phi(u,v)) - \beta(x) \sin(\Phi(u,v)))\Phi_u(u,v) \text{ in } \Omega$$

$$-\Delta v = (\alpha(x) \cos(\Phi(u,v)) - \beta(x) \sin(\Phi(u,v)))\Phi_v(u,v) \text{ in } \Omega$$

$$u = v = 0 \text{ on } \partial\Omega$$

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega) \times H^1_0(\Omega)$ of the functional

$$(u,v) \to \frac{1}{2} \left( \int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx \right)$$

$$-\int_\Omega (\alpha(x) \sin(\Phi(u(x),v(x))) + \beta(x) \cos(\Phi(u(x),v(x))))dx .$$

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1. Introduction

Let $S$ be a topological space. A function $g : S \to \mathbb{R}$ is said to be inf-compact if, for each $r \in \mathbb{R}$, the set $g^{-1}([-\infty, r])$ is compact.

If $Y$ is a real interval and $f : S \times Y \to \mathbb{R}$ is a function inf-compact and lower semicontinuous in $S$, and concave in $Y$, the occurrence of the strict minimax inequality

$$\sup_Y \inf_S f < \inf_S \sup_Y f$$

implies that the interior of the set $A$ of all $y \in Y$ for which $f(\cdot, y)$ has at least two local minima is non-empty. This fact was essentially shown in [4], giving then raise to an enormous number of subsequent applications to the multiplicity of solutions for nonlinear equations of variational nature (see [7] for an account up to 2010).

In [6] (see also [5]), we realized that, under the same assumptions as above, the occurrence of the strict minimax inequality also implies the existence of $\tilde{y} \in Y$ such that the function $f(\cdot, \tilde{y})$ has at least two global minima. It may happen that $\tilde{y}$ is unique and does not belong to the closure of $A$ (see Example 7 of [1]).

In [8] and [12], we extended the result of [6] to the case where $Y$ is an arbitrary convex set in a vector space. We also stress that such an extension is not possible for the result of [4]. We then started to build a network of applications of the results of [8] and [12] which touches several different topics: uniquely remotal sets in normed spaces ([8]); non-expansive operators ([9]); singular points ([10]); Kirchhoff-type problems ([11]); Lagrangian systems of relativistic oscillators ([13]); integral functional of the Calculus of Variations ([14]); non-cooperative gradient systems ([15]); variational inequalities ([16]).

The aim of this paper is to establish a further application within that network.

2. Results

The main abstract result is as follows:

**Theorem 2.1.** Let $X$ be a topological space, $(Y, \langle \cdot, \cdot \rangle)$ a real Hilbert space, $T \subseteq Y$ a convex set dense in $Y$ and $I : X \to \mathbb{R}$, $\varphi : X \to Y$ two functions such that, for each $y \in T$, the function $x \mapsto I(x) + \langle \varphi(x), y \rangle$ is lower semicontinuous and inf-compact. Moreover, assume that there exists a point $x_0 \in X$, with $\varphi(x_0) \neq 0$, such that

(a) $x_0$ is a global minimum of both functions $I$ and $\|\varphi(\cdot)\|$;

(b) $\inf_{x \in X} \langle \varphi(x), \varphi(x_0) \rangle < \|\varphi(x_0)\|^2$.

Then, for each convex set $S \subseteq T$ dense in $Y$, there exists $y^* \in S$ such that the function $x \mapsto I(x) + \langle \varphi(x), y^* \rangle$ has at least two global minima in $X$.

**Proof.** In view of (b), we can find $\tilde{x} \in X$ and $r > 0$ such that

$$I(\tilde{x}) + \frac{r}{\|\varphi(x_0)\|} \langle \varphi(\tilde{x}), \varphi(x_0) \rangle < I(x_0) + r\|\varphi(x_0)\|.$$  \hspace{1cm} (2.1)
Thanks to (a), we have
\[ I(x_0) + r\|\varphi(x_0)\| = \inf_{x \in X} (I(x) + r\|\varphi(x)\|). \] (2.2)

The function \( y \to \inf_{x \in X} (I(x) + \langle \varphi(x), y \rangle) \) is weakly upper semicontinuous, and so there exists \( \tilde{y} \in B_r \) such that
\[ \inf_{x \in X} (I(x) + \langle \varphi(x), \tilde{y} \rangle) = \sup_{y \in B_r} \inf_{x \in X} (I(x) + \langle \varphi(x), y \rangle), \quad (2.3) \]

\( B_r \) being the closed ball in \( X \), centered at 0, of radius \( r \). We distinguish two cases. First, assume that \( \tilde{y} \neq \tfrac{r\varphi(x_0)}{\|\varphi(x_0)\|}. \) As a consequence, taking into account that \( r\|\varphi(x_0)\| \) is the maximum of the restriction to \( B_r \) of the continuous linear functional \( \langle \varphi(x_0), \cdot \rangle \) (attained at the point \( \tfrac{r\varphi(x_0)}{\|\varphi(x_0)\|} \) only), we have
\[ \inf_{x \in X} (I(x) + \langle \varphi(x), \tilde{y} \rangle) \leq I(x_0) + \langle \varphi(x_0), \tilde{y} \rangle < I(x_0) + r\|\varphi(x_0)\|. \] (2.4)

Now, assume that \( \tilde{y} = \tfrac{r\varphi(x_0)}{\|\varphi(x_0)\|} \). In this case, due to (2.1), we have
\[
\inf_{x \in X} (I(x) + \langle \varphi(x), \tilde{y} \rangle) \leq I(\tilde{x}) + \langle \varphi(\tilde{x}), \tilde{y} \rangle = I(\tilde{x}) + \frac{r}{\|\varphi(x_0)\|} \langle \varphi(\tilde{x}), \varphi(x_0) \rangle < I(x_0) + r\|\varphi(x_0)\|. \] (2.5)

Therefore, from (2.2), (2.3), (2.4) and (2.5), it follows that
\[
\sup_{y \in B_r} \inf_{x \in X} (I(x) + \langle \varphi(x), y \rangle) < \inf_{x \in X} \sup_{y \in B_r} (I(x) + \langle \varphi(x), y \rangle). \] (2.6)

Now, let \( S \subseteq T \) be a convex set dense in \( Y \). By continuity, we clearly have
\[
\sup_{y \in B_r \cap S} \langle \varphi(x), y \rangle = \sup_{y \in B_r} \langle \varphi(x), y \rangle
\]
for all \( x \in X \). Therefore, in view of (2.6), we have
\[
\sup_{y \in B_r \cap S} \inf_{x \in X} (I(x) + \langle \varphi(x), y \rangle) \leq \sup_{y \in B_r} \inf_{x \in X} (I(x) + \langle \varphi(x), y \rangle) < \inf_{x \in X} \sup_{y \in B_r \cap S} (I(x) + \langle \varphi(x), y \rangle). \]

At this point, the conclusion follows directly applying Theorem 1.1 of [12] to the restriction of the function \((x, y) \to I(x) + \langle \varphi(x), y \rangle \) to \( X \times (B_r \cap S) \). \( \square \)

We now present an application of Theorem 2.1 to elliptic systems.

In the sequel, \( \Omega \subseteq \mathbb{R}^n \) \((n \geq 2)\) is a bounded domain with smooth boundary.

We denote by \( \mathcal{A} \) the class of all functions \( H : \Omega \times \mathbb{R}^2 \to \mathbb{R} \) which are measurable in \( \Omega \), \( C^1 \) in \( \mathbb{R}^2 \) and satisfy
\[
\sup_{(x,u,v) \in \Omega \times \mathbb{R}^2} \frac{|H_u(x,u,v)| + |H_v(x,u,v)|}{1 + |u|^p + |v|^p} < +\infty
\]
where \( p > 0 \), with \( p < \frac{n+2}{n-2} \) when \( n > 2 \).
Given $H \in \mathcal{A}$, we are interested in the problem
\[-\Delta u = H_u(x,u,v) \text{ in } \Omega \]
\[-\Delta v = H_v(x,u,v) \text{ in } \Omega \]
\[u = v = 0 \text{ on } \partial \Omega ,\]
$H_u$ (resp. $H_v$) denoting the derivative of $H$ with respect to $u$ (resp. $v$).

As usual, a weak solution of this problem is any $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that
\[
\int_{\Omega} \nabla u(x) \nabla \varphi(x) dx = \int_{\Omega} H_u(x,u(x),v(x)) \varphi(x) dx ,
\]
\[
\int_{\Omega} \nabla v(x) \nabla \psi(x) dx = \int_{\Omega} H_v(x,u(x),v(x)) \psi(x) dx
\]
for all $\varphi, \psi \in H^1_0(\Omega)$.

Define the functional $I_H : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ by
\[
I_H(u,v) = \frac{1}{2} \left( \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega} |\nabla v(x)|^2 dx \right) - \int_{\Omega} H(x,u(x),v(x)) dx
\]
for all $(u,v) \in H^1_0(\Omega) \times H^1_0(\Omega)$.

Since $H \in \mathcal{A}$, the functional $I_H$ is $C^1$ in $H^1_0(\Omega) \times H^1_0(\Omega)$ and its critical points are precisely the weak solutions of the problem. Moreover, due to the Sobolev embedding theorem, the functional $(u,v) \to \int_{\Omega} H(x,u(x),v(x))$ has a compact derivative and, as a consequence, it is sequentially weakly continuous in $H^1_0(\Omega) \times H^1_0(\Omega)$.

Also, we denote by $\lambda_1$ the first eigenvalue of the Dirichlet problem
\[-\Delta u = \lambda u \text{ in } \Omega \]
\[u = 0 \text{ on } \partial \Omega .\]

Our result is as follows:

**Theorem 2.2.** Let $F, G \in \mathcal{A}$, with $p = \frac{2}{n+2}$ when $n > 2$, and let $K \in \mathcal{A}$, with $K(x,0,0) = 0$ for all $x \in \Omega$, satisfy the following conditions:

(a) one has
\[
\lim_{s^2+t^2 \to +\infty} \frac{\sup_{x \in \Omega}(|F(x,s,t)| + |G(x,s,t)|)}{s^2+t^2} = 0 ;
\]

(b) there is $\eta \in \left(0, \frac{\lambda_1}{2}\right]$ such that
\[
K(x,s,t) \leq \eta(s^2+t^2)
\]
for all $x \in \Omega$, $s,t \in \mathbb{R}$ ;

(c) one has
\[
\text{meas}(\{x \in \Omega : 0 < |F(x,0,0)|^2 + |G(x,0,0)|^2\}) > 0 \quad (2.7)
\]
and
\[
|F(x,0,0)|^2 + |G(x,0,0)|^2 \leq |F(x,s,t)|^2 + |G(x,s,t)|^2 \quad (2.8)
\]
for all $x \in \Omega$, $s,t \in \mathbb{R}$ ;
Proof. We are going to apply Theorem 2.1, with the following choices: 

\[
\text{one has}
\]
\[
\text{for all } (\alpha, \beta) \in S \text{ such that the problem}
\]
\[
\text{has at least three weak solutions, two of which are global minima in } H_0^1(\Omega) \times H_0^1(\Omega)
\]
of the functional
\[
(u, v) \rightarrow \frac{1}{2} \left( \int_{\Omega} |\nabla u(x)|^2 dx + \int_{\Omega} |\nabla v(x)|^2 dx \right)
\]
\[
- \int_{\Omega} (\alpha(x)F(x, u(x), v(x)) + \beta(x)G(x, u(x), v(x)) + K(x, u(x), v(x))) dx .
\]

Then, for every convex set \( S \subseteq L^\infty(\Omega) \times L^\infty(\Omega) \) dense in \( L^2(\Omega) \times L^2(\Omega) \), there exists \((\alpha, \beta) \in S\) such that the problem
\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
u &= v = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

for all \((\alpha, \beta) \in S\) such that the problem
\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
\end{align*}
\]

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\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
\end{align*}
\]

for all \((\alpha, \beta) \in S\) such that the problem
\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
\end{align*}
\]

for all \((\alpha, \beta) \in S\) such that the problem
\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
\end{align*}
\]

for all \((\alpha, \beta) \in S\) such that the problem
\[
\begin{align*}
-\Delta u &= \alpha(x)F_u(x, u, v) + \beta(x)G_u(x, u, v) + K_u(x, u, v) \quad \text{in } \Omega \\
-\Delta v &= \alpha(x)F_v(x, u, v) + \beta(x)G_v(x, u, v) + K_v(x, u, v) \quad \text{in } \Omega \\
\end{align*}
\]
for all \((u, v) \in X\). In particular, since \(K(x, 0, 0) = 0\) in \(\Omega\) and \(\frac{\mu}{\lambda_1} < \frac{1}{2}\), from (2.9) we infer that \((0, 0)\) is a global minimum of \(I\) in \(X\). So, condition \((a)\) is satisfied. Now, let us verify condition \((b)\). To this end, set

\[
P(x, s, t) = F(x, 0, 0)F(x, s, t) + G(x, 0, 0)G(x, s, t) - |F(x, 0, 0)|^2 - |G(x, 0, 0)|^2
\]

for all \((x, s, t) \in \Omega \times \mathbb{R}^2\) and

\[
D = \left\{ x \in \Omega : \inf_{(s, t) \in \mathbb{R}^2} P(x, s, t) < 0 \right\}.
\]

By \((a_4)\), \(D\) has a positive measure. In view of the Scorza-Dragoni theorem, there exists a compact set \(C \subset D\), with positive measure, such that the restriction of \(P\) to \(C \times \mathbb{R}^2\) is continuous. Fix a point \(\tilde{x} \in C\) such that the intersection of \(C\) and any ball centered at \(\tilde{x}\) has a positive measure. Choose \(\tilde{s}, \tilde{t} \in \mathbb{R} \setminus \{0\}\) so that \(P(\tilde{x}, \tilde{s}, \tilde{t}) < 0\). By continuity, there is \(r > 0\) such that

\[
P(x, \tilde{s}, \tilde{t}) < 0
\]

for all \(x \in C \cap B(\tilde{x}, r)\). Set

\[
\gamma = \sup_{(x, s, t) \in \Omega \times [-|\tilde{s}|, |\tilde{s}|] \times [-|\tilde{t}|, |\tilde{t}|]} |P(x, t, s)|.
\]

Since \(F, G \in \mathcal{A}\), \(\gamma\) is finite. Now, choose an open set \(A\) such that

\[
C \cap B(\tilde{x}, r) \subset A \subset \Omega
\]

and

\[
\text{meas}(A \setminus (C \cap B(\tilde{x}, r))) < \frac{\int_{C \cap B(\tilde{x}, r)} P(x, \tilde{s}, \tilde{t})dx}{\gamma}. \quad (2.10)
\]

Finally, choose two functions \(\tilde{u}, \tilde{v} \in H^1_0(\Omega)\) such that

\[
\tilde{u}(x) = \tilde{s}, \quad \tilde{v}(x) = \tilde{t}
\]

for all \(x \in C \cap B(\tilde{x}, r)\),

\[
\tilde{u}(x) = \tilde{v}(x) = 0
\]

for all \(x \in \Omega \setminus A\) and

\[
|\tilde{u}(x)| \leq |\tilde{s}|, \quad |\tilde{v}(x)| \leq |\tilde{t}|
\]

for all \(x \in \Omega\). Then, taking (2.10) into account, we have

\[
\langle \varphi(\tilde{u}, \tilde{v}), \varphi(0, 0) \rangle_Y - \|\varphi(0, 0)\|_Y^2 = \int_{\Omega} P(x, \tilde{u}(x), \tilde{v}(x))dx
\]

\[
= \int_{C \cap B(\tilde{x}, r)} P(x, \tilde{s}, \tilde{t})dx + \int_{A \setminus (C \cap B(\tilde{x}, r))} P(x, \tilde{u}(x), \tilde{v}(x))dx
\]

\[
< \int_{C \cap B(\tilde{x}, r)} P(x, \tilde{s}, \tilde{t})dx + \gamma \text{meas}(A \setminus (C \cap B(\tilde{x}, r))) < 0.
\]

This shows that \((b)\) is satisfied. Finally, fix \(\alpha, \beta \in L^\infty(\Omega)\). Clearly, the function

\[
(x, s, t) \to \alpha(x)F(x, s, t) + \beta(x)F(x, s, t) + K(x, s, t)
\]
belongs to $\mathcal{A}$, and so the functional
\[(u,v) \rightarrow I(u,v) + \langle \varphi(u,v), (\alpha, \beta) \rangle_Y\]
is sequentially weakly lower semicontinuous in $X$. Let us show that it is coercive. Set
\[\theta = \max \left\{ \|\alpha\|_{L^\infty(\Omega)}, \|\beta\|_{L^\infty(\Omega)} \right\}\]
and fix $\epsilon > 0$ so that
\[\epsilon < \frac{1}{\theta} \left( \frac{\lambda_1}{2} - \eta \right). \tag{2.11}\]
By $(a_1)$, there is $c_\epsilon > 0$ such that
\[|F(x,s,t)| + |G(x,s,t)| \leq \epsilon(|s|^2 + |t|^2) + c_\epsilon\]
for all $(x,s,t) \in \Omega \times \mathbb{R}^2$. Then, for each $u, v \in H^1_0(\Omega)$, recalling (2.9), we have
\[I(u,v) + \langle \varphi(u,v), (\alpha, \beta) \rangle_Y \geq \left( \frac{1}{2} - \frac{\eta}{\lambda_1} \right) \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2)dx - \int_\Omega |\alpha(x)F(x,u(x),v(x)) + \beta(x)G(x,u(x),v(x))|dx \geq \left( \frac{1}{2} - \frac{\eta}{\lambda_1} - \frac{\theta \epsilon}{\lambda_1} \right) \int_\Omega (|\nabla u(x)|^2 + |\nabla v(x)|^2)dx - \theta c_\epsilon \text{meas}(\Omega).\]
Notice that, in view of (2.11), we have $\frac{1}{2} - \frac{\eta}{\lambda_1} - \frac{\theta \epsilon}{\lambda_1} > 0$, and so
\[
\lim_{\|(u,v)\|_X \to +\infty} (I(u,v) + \langle \varphi(u,v), (\alpha, \beta) \rangle_Y) = +\infty,
\]
as claimed.
In particular, this also implies that the functional $(u,v) \rightarrow I(u,v) + \langle \varphi(u,v), (\alpha, \beta) \rangle_Y$ is weakly lower semicontinuous, by the Eberlein-Smulian theorem. Thus, the assumptions of Theorem 2.1 are satisfied. Therefore, for each convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $H^1_0(\Omega) \times H^1_0(\Omega)$, there exists $(\alpha, \beta) \in S$, such that the functional
\[(u,v) \rightarrow \frac{1}{2} \left( \int_\Omega |\nabla u(x)|^2dx + \int_\Omega |\nabla v(x)|^2dx \right) - \int_\Omega (\alpha(x)F(x,u(x),v(x)) + \beta(x)G(x,u(x),v(x)) + K(x,u(x),v(x)))dx\]
has at least two global minima in $H^1_0(\Omega) \times H^1_0(\Omega)$. Finally, by Example 38.25 of [17], the same functional satisfies the Palais-Smale condition, and so it admits at least three critical points, in view of Corollary 1 of [3]. The proof is complete. □

Remark 2.3. We are not aware of known results close enough to Theorem 2.2 in order to do a proper comparison. This sentence also applies to the case of single equations, that is to say when $F, G, K$ depend on $x$ and $s$ only. For an account on elliptic systems, we refer to [2].
Among the various corollaries of Theorem 2.2, we wish to stress the following ones:

**Corollary 2.4.** Let $K \in A$, with $K(x,0,0) = 0$ for all $x \in \Omega$, satisfy condition $(a_2)$. Moreover, let $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be a non-constant $C^1$ function, with $\Phi(0,0) = 0$, belonging to $A$, with $p = \frac{2}{n-2}$ when $n > 2$.

Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$
-\Delta u = (\alpha(x) \cos(\Phi(u,v)) - \beta(x) \sin(\Phi(u,v))) \Phi_u(u,v) + K_u(x,u,v) \quad \text{in } \Omega
$$

$$
-\Delta v = (\alpha(x) \cos(\Phi(u,v)) - \beta(x) \sin(\Phi(u,v))) \Phi_v(u,v) + K_v(x,u,v) \quad \text{in } \Omega
$$

$$
u = v = 0 \quad \text{on } \partial \Omega
$$

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega) \times H^1_0(\Omega)$ of the functional

$$
(u,v) \to \frac{1}{2} \left( \int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |\nabla v(x)|^2 dx \right) - \int_\Omega (\alpha(x) \sin(\Phi(u(x),v(x))) + \beta(x) \cos(\Phi(u(x),v(x))) + K(x,u(x),v(x))) dx .
$$

**Proof.** It suffices to apply Theorem 2.2 to the functions $F,G : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$
F(s,t) = \sin(\Phi(s,t)) ,
$$

$$
G(s,t) = \cos(\Phi(s,t))
$$

for all $(s,t) \in \mathbb{R}^2$. \hfill \square

**Corollary 2.5.** Let $F,G : \mathbb{R} \to \mathbb{R}$ belong to $A$, with $p = \frac{2}{n-2}$ when $n > 2$. Moreover, assume that $F,G$ are twice differentiable at $0$ and that

$$
\lim_{|s| \to +\infty} \frac{|F(s)| + |G(s)|}{s^2} = 0 ,
$$

$$
0 < |F(0)|^2 + |G(0)|^2 = \inf_{s \in \mathbb{R}} (|F(s)|^2 + |G(s)|^2) ,
$$

$$
F''(0)F(0) + G''(0)G(0) < 0 . \quad (2.12)
$$

Then, for every convex set $S \subseteq L^\infty(\Omega) \times L^\infty(\Omega)$ dense in $L^2(\Omega) \times L^2(\Omega)$, there exists $(\alpha, \beta) \in S$ such that the problem

$$
-\Delta u = \alpha(x)F'(u) + \beta(x)G'(u) \quad \text{in } \Omega
$$

$$
u = 0 \quad \text{on } \partial \Omega
$$

has at least three weak solutions, two of which are global minima in $H^1_0(\Omega)$ of the functional

$$
\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega (\alpha(x)F(u(x)) + \beta(x)G(u(x))) dx .
$$
Proof. We apply Theorem 2.2 taking $K = 0$. Since 0 is a global minimum of the function $|F(\cdot)|^2 + |G(\cdot)|^2$, we have

$$F'(0)F(0) + G'(0)G(0) = 0$$

and so, in view of (2.12), 0 is a strict local maximum for the function $F(\cdot)F(0) + G(\cdot)G(0)$.

Hence, $(a_4)$ is satisfied and Theorem 2.2 gives the conclusion. □

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Biagio Ricceri
Department of Mathematics and Informatics,
University of Catania,
Viale A. Doria 6,
95125 Catania, Italy
e-mail: ricceri@dmi.unict.it