DEMAZURE OPERATORS VIA SHIFTED $q = 0$ AFFINE ALGEBRAS

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Abstract. In this article, we construct two families of idempotent operators/projection functors acting on the K-theory/derived category of coherent sheaves on $n$-step partial flag varieties. Those operators can be viewed as a generalization of the Demazure operators for the full flag variety, and the two families of idempotent operators generate respectively two algebras that are similar to the $q = 0$ Hecke algebra.

The main idea comes from the interpretation of the Demazure operators in terms of elements in the shifted $q = 0$ affine algebra. As a main result, we show that a categorical action of the shifted $q = 0$ affine algebra naturally induces actions of two variants of the $q = 0$ Hecke algebra on each weight category. Finally, we calculate the action of those operators on the basis given by the Kapranov exceptional collection.

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1. Introduction

In this article, we work over the field of complex numbers $\mathbb{C}$.

1.1. Background. Let $G$ be a complex semisimple algebraic group. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B \subset G$. We also fix a parabolic subgroup $B \subset P \subset G$ and denote by $W = N_G(T)/T$ the Weyl group.

The geometry of the full flag variety $G/B$ (or the partial flag variety $G/P$) plays an essential role in representation theory (and related areas). One of the important problems is to study their topological invariants, in particular, the singular cohomology $H^*(G/B)$ and the K-theory $K(G/B)$. Since both objects carry natural ring structures, the study of their ring structures is known as the Schubert calculus.
The $B$-orbits of $G/B$ form a stratification into Schubert cells. Their closures in $G/B$ are called the Schubert varieties. The class of the Schubert varieties and their structure sheaves form an additive basis for $H^*(G/B)$ and for $K(G/B)$, respectively. According to Borel [Bo], there is an identification of $H^*(G/B)$ (and thus $K(G/B)$) with the quotient of the polynomial ring by the ideal generated by $W$-invariant polynomials.

A central problem in the Schubert calculus was to find polynomials (later called Schubert/Grothendieck polynomials) representing the basis given by Schubert varieties and their structure sheaves in the Borel presentations of $H^*(G/B)$ and of $K(G/B)$, respectively. This problem was solved by Bernstein-Gelfand-Gelfand [BGG] for $H^*(G/B)$, and Demazure [D1], [D2] for $K(G/B)$. They introduced the so-called divided difference operators; in particular, the BGG operators act on $H^*(G/B)$; while the Demazure operators act on $K(G/B)$. The way to define the BGG/Demazure operators is by the fundamental constructions of push-pull operators $\pi^*_i \pi_{i*}$ where $\pi_i : G/B \to G/P_i$ is the natural projection, and $P_i$ is a minimal parabolic subgroup.

These operators satisfy the following essential properties: First, they generate algebras which are certain degeneration of the Hecke algebra of the Weyl group $W$. If one works with singular cohomology, these operators generate the nil-Hecke algebra. If one works instead with $K$-theory, they generate the $q=0$ Hecke algebra. Second, starting from a cohomology class of highest codimension (the Schubert class of a point), one obtains all Schubert classes by applying a succession of divided difference operators corresponding to simple roots. Similar results hold for $K$-theory.

After this, there is an abundance of generalizations and further developments, including Lusztig's [Lu] $q$-analogue of Demazure operators called the Demazure-Lusztig operators, an action of the affine Hecke algebra on the $G \times \mathbb{C}^*$-equivariant $K$-theory $K^{G \times \mathbb{C}^*}(G/B)$. The Demazure-Lusztig operators were used later by Kazhdan-Lusztig [KaLu] to construct representations of affine Hecke algebra in terms of equivariant $K$-theory of Springer fibres.

On the other hand, Kostant-Kumar [KoKu1], [KoKu2] to $T$-equivariant cohomology $H^*_T(G/B)$ and $T$-equivariant $K$-theory $K_T(G/B)$ for an arbitrary Kac-Moody group $G$. Since then, people tried to replace $H^*$ and $K$ by more general algebraic/complex oriented cohomology theory $h^*(G/B)$, e.g. Bressler-Evens [BrEv], [EB] and this leads to the definition of formal (affine) Demazure/Hecke algebras [HMSZ].

1.2. The problem. One natural generalization that we did not mention above is the construction of [BGG] and [D1], [D2] to more general partial flag varieties $G/P$, i.e., to construct BGG and Demazure operators acting on $H^*(G/P)$ and $K(G/P)$, respectively. Note that the projection $G/B \to G/P$ makes $H^*(G/P)$ and $K(G/P)$ submodules of $H^*(G/B)$ and $K(G/P)$, resp. However, they are not invariant under the action of divided difference operators. So we can not define the operators directly by restricting the actions of the BGG/Demazure operators to the partial flag varieties.

One solution for the construction of BGG and Demazure operators for $G/P$ was discovered during the study of $T$-equivariant cohomology $H^*_T(G/B)$. In an unpublished work, Knutson [Knu] observes that the left and right action of $W$ on $G/B$ leads to the left and right divided difference operators acting on $H^*_T(G/B)$. Moreover, the right divided difference operators coincide with the classical BGG operators and are only defined for the full flag variety $G/B$. 
On the other hand, the left divided difference operators can be defined for $T$-equivariant cohomology/K-theory of any partial flag variety. They appear in Brion [Br2] for the study of $T$-equivariant Chow/cohomology groups and also in Peterson’s note [Pet]. Later they were used to study $T$-equivariant cohomology of partial flag varieties, for example, the recursive formula for calculating equivariant Schubert class, see [Tym]. For more general $T$-equivariant oriented cohomology, see [ZZ]. Moreover, Mihalcea-Naruse-Su [MNS] lifted the left divided difference operators to $T$-equivariant quantum cohomology/K-theory; they also defined the left Demazure-Lusztig operators and studied the actions.

However, the action of left divided difference operators is trivial in the non-equivariant cohomology setting (which is pointed out in [MNS]). So the question of constructing BGG and Demazure operators for $G/P$ remains open if we do not work in the $T$-equivariant setting. In this article, we give a (partial) solution to this problem. More precisely, we construct operators that are Demazure-like, i.e. they are idempotent (and thus they are projections), acting on non-equivariant K-theory/derived category of all $n$-step partial flag varieties. We call it ”partial” for the reason that those operators do not satisfy the braid relations. The main tool we use is the categorical action of shifted $q = 0$ affine algebra on the bounded derived categories of coherent sheaves on partial flag varieties, which is constructed by the author [Hsu].

1.3. Main results. Now we explain the main result of this article, which is the construction of Demazure operators for partial flag varieties. We work on the type $A$ case, i.e. $G = \text{SL}_N(\mathbb{C})$. Then the full flag variety has the following description

$$G/B = \{0 = V_0 \subset V_1 \subset \ldots \subset V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for all } k\},$$

and similarly the partial flag variety

$$G/P_i = \{0 \subset V_1 \subset V_2 \subset \ldots V_{i-1} \subset V_{i+1} \subset \ldots V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for } k \neq i\}$$

where $P_i$ is a minimal parabolic subgroup for $1 \leq i \leq N - 1$.

Let $\pi_i : G/B \to G/P_i$ be the natural projection, which is a $\mathbb{P}^1$-fibration for all $1 \leq i \leq N - 1$. We have the Demazure operators $\delta_i := \pi_i^* \pi_i^* : K(G/B) \to K(G/B)$, and $\{\delta_i\}$ generate the $q = 0$ Hecke algebra, denoted by $H_N(0)$, acting on $K(G/B)$.

Next, let us briefly recall the categorical action of the shifted $q = 0$ affine algebra $\check{U}_{0,N}(L\mathfrak{sl}_n)$ on the bounded derived categories of coherent sheaves on partial flag varieties that constructed in [Hsu]. We use $\underline{k} \vdash N$ to denote a $n$-tuple positive integers $\underline{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$ such that $\sum_i k_i = N$. The $n$-step partial flag variety is given by

$$\text{Fl}_{\underline{k}}(\mathbb{C}^N) := \{V_* = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N) \mid \dim V_i/V_{i-1} = k_i \text{ for all } i\}.$$  

The generators of $\check{U}_{0,N}(L\mathfrak{sl}_n)$ are $e_i, r_1, f_i, s_1$ and they act on $\bigoplus_{\underline{k} \vdash N} D^b(\text{Fl}_{\underline{k}}(\mathbb{C}^N))$ by the following correspondence diagram

$$\begin{array}{c}
\text{Fl}_{\underline{k}}(\mathbb{C}^N) \\
\downarrow p_1 \\
W_i^1(\underline{k}) \\
\downarrow p_2 \\
\text{Fl}_{\underline{k}+\alpha_i}(\mathbb{C}^N)
\end{array}$$
where $p_1, p_2$ are the natural projections. Here
\[ W_i^1(k) := \{ (V_\bullet, V_\bullet') \in Fl_k(\mathbb{C}^N) \times Fl_{k+\alpha_i}(\mathbb{C}^N) \mid V_j = V_j' \text{ for } j \neq i, \ V_i' \subset V_i \} \]
and $\alpha_i = (0, 0, -1, 1, 0, 0, \ldots)$ is the simple root with the $-1$ is in the $i$-th position. The generators $e_{i,r}^1_k$ act on $\bigoplus_{k=0}^N D^b(\mathbb{C}^N)$ by lifting to the following functors
\[ E_{i,r}^1_k := p_{2*}(p_1^* \otimes (V_i/V_i')) : D^b(Fl_k(\mathbb{C}^N)) \to D^b(Fl_{k+\alpha_i}(\mathbb{C}^N)) \]
where we denote $V_i, V_i'$ to be the tautological bundles on $W_i^1(k)$ of rank $k_1 + \ldots + k_i$, $k_1 + \ldots + k_i - 1$ respectively, and thus $V_i/V_i'$ is the tautological line bundle. Similarly for the lift of $f_{i,s}^1_k$ to $f_{i,s}^1_k$. We refer the readers to Section 4 for details about the definition of $\mathcal{U}_{0,N}(\text{Ls}l_n)$ and its categorical action.

To generalize the construction of the Demazure operators $\delta_i$ from $G/B$ to $Fl_k(\mathbb{C}^N)$, our motivation comes from the interpretation of $\delta_i$ in terms of elements in the shifted $q = 0$ affine algebra. This was already done in [Hsu] where we construct a categorical action of $H_N(0)$ on $D^b(G/B)$ via using the categorical relations of the shifted $q = 0$ affine algebra. Let us recall such interpretation in loc. cit.. Using the above notation, we have $G/B = Fl(1,1,\ldots,1)(\mathbb{C}^N)$ and there is a categorical action of $\mathcal{U}_{0,N}(\text{Ls}l_n)$ on $D^b(G/B)$. Passing to the Grothendieck group, we get an action of $\mathcal{U}_{0,N}(\text{Ls}l_n)$ on $K(G/B)$.

We try to relate this action to the action of $H_N(0)$ on $K(G/B)$. Let $\alpha_3$ be the simple root $(0,0,-1,1,0,0,\ldots)$ where the $-1$ is in the $i$-th position for $1 \leq i \leq N-1$. Then we have
\[ G/P_i = Fl(1,1,\ldots,1)+\alpha_i(\mathbb{C}^N) = Fl(1,1,\ldots,1)-\alpha_i(\mathbb{C}^N), \]
and from the following diagram
\[
\begin{array}{ccc}
K(G/P_i = Fl(1,1,\ldots,1)-\alpha_i(\mathbb{C}^N)) & \xrightarrow{e_{i,r}} & K(G/B = Fl(1,1,\ldots,1)(\mathbb{C}^N)) \\
\xrightarrow{f_{i,s}} & & \xrightarrow{f_{i,s}} \\
& K(G/P_i = Fl(1,1,\ldots,1)+\alpha_i(\mathbb{C}^N))
\end{array}
\]
we can try to interpret the Demazure operators $\delta_i$ in terms of elements in $\mathcal{U}_{0,N}(\text{Ls}l_n)$.

From [Hsu], the Demazure operators $\delta_i$ (act on $K(G/B)$) can be written as
\[ \delta_i = e_{i,0} f_{i,1}(\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} = f_{i,0} e_{i,-1}(\psi_i^-)^{-1} 1_{(1,1,\ldots,1)} \quad (1.1) \]
where $\psi_i^+ 1_{(1,1,\ldots,1)}, \psi_i^- 1_{(1,1,\ldots,1)}$ are elements in $\mathcal{U}_{0,N}(\text{Ls}l_n)$ that act on $K(G/B)$ by multiplication of certain line bundles, see Theorem 1.3 for the definition.

Since we have the action of $\mathcal{U}_{0,N}(\text{Ls}l_n)$ on $\bigoplus_k D^b(Fl_k(\mathbb{C}^N))$, we try to generalize the above constructions to actions on $D^b(Fl_k(\mathbb{C}^N))$ and $K(Fl_k(\mathbb{C}^N))$. Motivated by (1.1), we define the operators
\[ \delta_i^\prime := e_{i,0} f_{i,k_i+1}(\psi_i^+)^{-1} 1_{\mathbb{C}^N}, \quad \delta_i^\prime := f_{i,0} e_{i,-k_i}(\psi_i^-)^{-1} 1_{\mathbb{C}^N} : K(Fl_k(\mathbb{C}^N)) \to K(Fl_k(\mathbb{C}^N)). \quad (1.2) \]
Note that in general we have $\delta_i^\prime \neq \delta_i^\prime$ for all $i$, and $\delta_i^\prime = \delta_i^\prime$ holds when we are in the full flag variety setting.

It turns out that they are idempotent operators, i.e. $(\delta_i^\prime)^2 = \delta_i^\prime$, $(\delta_i^\prime)^2 = \delta_i^\prime$. Thus they are our desire Demazure operators. However, the relations that $\delta_i^\prime$ and $\delta_i^\prime$ satisfy are not
the braid relations but pretty much close to them. By abusing of notations, we define an algebra $H'_n(0)$ abstractly with generators $\delta'_i$ and relations that $\delta'_i$ satisfy above. Similar an algebra $H''_n(0)$ for $\delta''_i$. See Definition 3.7 and 3.8 for details.

Then the following theorem is our main result. We state it as a more general and abstract version since the categorical action of $U_{0,N}(Lsl_n)$ can be defined for abstract categories.

**Theorem 1.1.** (Theorem 5.5) Given a categorical $U_{0,N}(Lsl_n)$ action on $K$. Each weight category $K(k)$ carries actions of $H'_n(0)$ and $H''_n(0)$, where the generators $\delta'_i \in H'_n(0)$ and $\delta''_i \in H''_n(0)$ act by the functors $E_{i,0}F_{i,k_1+1}(\psi^+)^{-1}1_k$ and $F_{i,0}E_{i,-k_1}(\psi^-)^{-1}1_k$ respectively.

**Remark 1.2.** It may be a bit unclear at first glance to see the naturality for the definition of $\delta'_i$ and $\delta''_i$ in (1.2). In fact, it comes from the natural construction of compositions of functor with its left/right adjoints and will be explained in section 5. Moreover, we expect it is similar to the construction of spherical twist/co-twist functors in [AT], [ST].

By Theorem 1.1 there is a categorical action of $U_{0,N}(Lsl_n)$ on $\bigoplus_k D^b(Fl_k(C^N))$. Thus, as a corollary, we have

**Corollary 1.3.** (Corollary 5.4 and 5.5) There exist categorical actions of $H'_n(0)$ and $H''_n(0)$ on $D^b(Fl_k(C^N))$ where the generators $\delta'_i$, $\delta''_i$ acting by convolutions of Fourier-Mukai kernels. Passing to the Grothendieck group, we obtain actions of $H'_n(0)$ and $H''_n(0)$ on $K(Fl_k(C^N))$.

Roughly speaking, the above construction can be summarized as the following diagram.

![Diagram](image)

Although we can not prove the braid relations in the actions of $H'_n(0)$ and $H''_n(0)$ in Theorem 1.1. Examples (see Section 6) suggest that the braid relations hold when they are of geometric origins, e.g., $Fl_k(C^N)$. Thus we propose the following conjecture.

**Conjecture 1.4.** (Conjecture 5.1) The actions of $H'_n(0)$ and $H''_n(0)$ on $K(Fl_k(C^N))$ in Corollary 1.3 are in fact the actions of the $q = 0$ Hecke algebra $H_n(0)$; i.e., the braid relations hold for the actions of $H'_n(0)$ and $H''_n(0)$ on $K(Fl_k(C^N))$.

Finally, like the classical BGG/Demazure operators act on the base given by the cohomological classes/structure sheaves of Schubert varieties (see Theorem 3.2), it is natural...
to find a suitable basis for $K(Fl_k(C^N))$ and calculate the action of the Demazure operators that constructed in [1.2]. The structure sheaves of Schubert varieties give one basis; however, it is pretty non-trivial to calculate the action. The second one is the exceptional collection given by Kapranov [Ka1], [Ka2]. We found that it is natural to calculate the action since we can write the exceptional objects in terms of elements in $\mathcal{U}_{0,N}(Ls_n)$ via using the Borel-Weil-Bott theorem.

We only mention the Grassmannian case $G(k,N)$ here. For general case, see Theorem 6.10. The Kapranov exceptional collection is given by $\{S_\lambda V\}$ where $V$ is the tautological rank $k$ bundle and $S_\lambda$ is the Schur functor associated to partitions $\lambda = (\lambda_1,\ldots,\lambda_k)$ with $0 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq N - k$. It gives the basis $\{[S_\lambda V]\}$ for $K(G(k,N))$. Then the action of $\delta'_1 = e_0f_{N-k}(\psi^-)^{-1}1_{(k,N-k)}$ is given by

\[
\delta'_1([S_\lambda V]) = \begin{cases} 
0 & \text{if } \lambda_1 = N - k \\
[S_\lambda V] & \text{if } 0 \leq \lambda_1 \leq N - k - 1,
\end{cases}
\]

Remark 1.6. Actually we also calculate the action of the other Demazure operator $\delta''_1 = f_0e_{-k}(\psi^-)1_{(k,N-k)}$ on the basis which are dual (as vector spaces) of the dual exceptional collection. It turns out that they are the eigenbasis for such Demazure operators in the Grassmannian case.

Remark 1.7. Since the Kapranov exceptional collection $\{S_\lambda V\}$ gives rise to a semiorthogonal decomposition of $\mathcal{D}^b(G(k,N))$, we believe that a categorical version of this theorem should provide an explicit example to Kuznetsov’s work (i.e., Theorem 7.1 in [Ku]) on projection functors of a semiorthogonal decomposition are kernel functors.

1.4. Further directions. We mention some remarks and possible further directions of the current work.

First, the author believes that the Demazure operators construct in this article can be extended to the equivariant setting, e.g., $K_T(G/P)$ or $K_G(G/P)$ and also for more general Kac-Moody groups $G$. However, the $q = 0$ Hecke-like algebras $H_q(0)$ and $H'_q(0)$ generated by the Demazure operators $\delta'_i$ and $\delta''_i$ are not the $q = 0$ Hecke-like algebra of the whole Weyl group (the symmetric group $S_N$ for our case). In our case, we have less operators $\delta_i$ with $1 \leq i \leq n - 1$ and $n \leq N$. We expect that our actions should be the right operators in the sense of [Knu] but correspond to the Weyl group $W_P$ of the parabolic subgroup.

Second, we also believe that such method or idea can be used to construct BGG-like operators acting on $H^*(G/P)$ without the need to use $T$-equivariant cohomology, and it would be interesting to see if we can write such operators as more general divided difference operators.

Since the work here is only for type $A$ case, one possible further direction is to generalize the current work to general type and study the actions on the exceptional collections for general type partial flag varieties constructed by Kuznetsov-Polishchuk [KuPol]. This also relates to the definition of shifted $q = 0$ affine algebras for more general types.

The other direction to study is the interactions between our Demazure operators and the left Demazure operators that were studied by [Br2], [Pet], [Knu], [Tym] and [MNS]. For
example, the commutativity of both actions and compute the actions on the basis given by Schubert varieties.

Finally, it is natural to ask whether we can extend the actions of $H'_n(0)$ and $H''_n(0)$ induced by the categorical action of $\check{U}_{0,N}(L\mathfrak{sl}_n)$ to affine version. Note that there are natural line bundles on the partial flag varieties $Fl_k(\mathbb{C}^N)$ that are given by determinant line bundles of the tautological bundles. However, a simple check tells us that we can not get the desire affine Hecke relations from the relations of the categorical actions.

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2. Preliminaries

In this section, we briefly recall the definition of exceptional objects/collections and Fourier-Mukai transformations that will be frequently used in the later sections. We mainly follow the book [Huy] for the definitions and details.

2.1. Exceptional collections and Fourier-Mukai transformations. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category.

**Definition 2.1.** An object $E \in \text{Ob}(\mathcal{D})$ is called exceptional if

$$\text{Hom}_{\mathcal{D}}(E, E[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0. \end{cases}$$

Then we define the notion of exceptional collections.

**Definition 2.2.** An ordered collection $\{E_1, \ldots, E_n\}$, where $E_i \in \text{Ob}(\mathcal{D})$ for all $1 \leq i \leq n$, is called an exceptional collection if each $E_i$ is exceptional and $\text{Hom}_{\mathcal{D}}(E_i, E_j[l]) = 0$ for all $i > j$ and $l \in \mathbb{Z}$.

The collection (sequence) is called strong exceptional if in addition

$$\text{Hom}_{\mathcal{D}}(E_i, E_j[l]) = 0 \text{ for all } i, j \text{ and } l \neq 0.$$

**Remark 2.3.** An exceptional collection $\{E_1, \ldots, E_n\}$ is called full if the smallest triangulated subcategory of $\mathcal{D}$ containing all the objects $E_1, \ldots, E_n$ is $\mathcal{D}$ itself.

Next, we define the notion of dual exceptional collection.

**Definition 2.4.** Let $\{E_1, \ldots, E_n\}$ be an exceptional collection on the triangulated category $\mathcal{D}$. An exceptional collection $\{F_1, \ldots, F_n\}$ of objects in $\mathcal{D}$ is called dual to $\{E_1, \ldots, E_n\}$ if

$$\text{Hom}_{\mathcal{D}}(F_i, E_i[l]) = \begin{cases} \mathbb{C} & \text{if } l = 0 \\ 0 & \text{if } l \neq 0 \end{cases}$$

and $\text{Hom}_{\mathcal{D}}(F_i, E_j[l]) = 0$ for $i \neq j$ and $l \in \mathbb{Z}$. 
Next, we introduce the language of Fourier-Mukai transformations. Let $X$ be a smooth and complex projective variety. We will work with the bounded derived category of coherent sheaves on $X$, which we always denote by $\mathcal{D}^b(X)$. Throughout this article, unless we explicitly mention, otherwise all functors between derived categories will be assumed to be derived. For example, we will write $f^*$ and $\otimes$ instead of $Rf^*$ and $\otimes^L$, respectively.

**Definition 2.5.** Let $X, Y$ be two smooth and complex projective varieties. A Fourier-Mukai kernel is an object $P$ of the bounded derived category of coherent sheaves on $X \times Y$. Given $P \in \mathcal{D}^b(X \times Y)$, we may define the associated Fourier-Mukai transform, which is the functor

$$\Phi_P : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$$

$$\mathcal{F} \mapsto \pi_2^*(\pi_1^*(\mathcal{F}) \otimes P)$$

where $\pi_1, \pi_2$ are natural projections from $X \times Y$ to $X, Y$ respectively.

We call $\Phi_P$ the Fourier-Mukai transform with (Fourier-Mukai) kernel $P$. For convenience, we will just write FM for Fourier-Mukai.

The first property of FM transforms is that they have left and right adjoints that are themselves FM transforms.

**Proposition 2.6** ([Huy] Proposition 5.9). For $\Phi_P : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is the FM transform with kernel $P$, define

$$P_L = P^\vee \otimes \pi_2^*\omega_Y[\dim Y], \quad P_R = P^\vee \otimes \pi_1^*\omega_X[\dim X].$$

Then

$$\Phi_{P_L} : \mathcal{D}^b(Y) \to \mathcal{D}^b(X), \quad \Phi_{P_R} : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$$

are the left and right adjoints of $\Phi_P$, respectively.

The second property is that the composition of FM transforms is also a FM transform.

**Proposition 2.7** ([Huy] Proposition 5.10). Let $X, Y, Z$ be smooth projective varieties over $\mathbb{C}$. Consider objects $P \in \mathcal{D}^b(X \times Y)$ and $Q \in \mathcal{D}^b(Y \times Z)$. They define FM transforms $\Phi_P : \mathcal{D}^b(X) \to \mathcal{D}^b(Y), \Phi_Q : \mathcal{D}^b(Y) \to \mathcal{D}^b(Z)$. We use $*$ to denote the operation for convolution, i.e.

$$Q * P := \pi_{13*}((\pi_{12}^*(P) \otimes \pi_{23}^*(Q)))$$

Then for $R = Q * P \in \mathcal{D}^b(X \times Z)$, we have $\Phi_Q \circ \Phi_P \cong \Phi_R$.

**Remark 2.8.** Moreover by [Huy] remark 5.11, we have $(Q * P)_L \cong (P)_L * (Q)_L$ and $(Q * P)_R \cong (P)_R * (Q)_R$.

3. **BGG-Demazure divided difference operators and their action**

In this section, we recall the well-known classical results about the divided difference operators and their actions. They were introduced by Bernstein-Gelfand-Gelfand [BGG] in cohomology and Demazure [D1], [D2] in K-theory. Those operators play important tools in the Schubert calculus, which studies the ring structure of the cohomology and K-theory of flag varieties.
3.1. General setup. First, we recall the general theory of linear algebraic groups and Schubert varieties. We refer to [Br2] for more details. Let $G$ be a complex semisimple algebraic group and assume that $G$ is connected and simply connected. Let $B \subset G$ be a Borel subgroup, and $G/B$ be the full flag variety. We fix a maximal torus $T \subset B \subset G$. Denoted by $W = N(T)/T$ the Weyl group and $l : W \to \mathbb{N}$ the associated length function.

The triple $(G, B, T)$ determine a root system $\Delta$ with positive roots $\Delta^+$. Let $\Sigma = \{\alpha_1, ..., \alpha_r\} \subset \Delta^+$ be the set of simple roots. For each simple root $\alpha_i$ we denote $s_i$ to be the associated simple reflection and $B \subset P_i \subset G$ to be the associated minimal parabolic subgroup.

The full flag variety $G/B$ is a homogeneous space and its $B$-orbits are the Schubert cells $C_w := BwB/B$ where $w \in W$. Their closure $X_w := \overline{C_w}$ are the Schubert varieties and $\dim_C X_w = l(w)$. There induces a partial order $\leq$ on the Weyl group $W$ called the Bruhat order, which is defined by $v \leq w$ if and only if $X_v \subset X_w$.

We denote $[X_w] \in H^{2c}(G/B, \mathbb{Z})$ to be the cohomological class, where $c = \dim_C G/B - \dim_C X_w$. Similarly, let $\mathcal{O}_{X_w}$ to be the structure sheaf of $X_w$ and denote $[\mathcal{O}_{X_w}]$ to be the class in the Grothendieck group $K(G/B)$. The following result is well-known.

**Proposition 3.1.** $\{[X_w]\}$ is a free additive basis for $H^{*}(G/B, \mathbb{Z})$ and $\{[\mathcal{O}_{X_w}]\}$ is a free additive basis for $K(G/B)$.

Next, we introduce the BGG-Demazure divided difference operators. For each $\alpha_i$ consider the natural projection $\pi_i : G/B \to G/P_i$. Then it induces pullback and pushforward both in cohomology and K-theory. More precisely, we have $\pi_i^* : H^*(G/P_i) \to H^*(G/B)$, $\pi_i^* : K(G/P_i) \to K(G/B)$, $\pi_i : H^*(G/B) \to H^*(G/P_i)$, and $\pi_i : K(G/B) \to K(G/P_i)$. Thus we have the operators $\pi_i^*, \pi_i$ both in cohomology and K-theory. To distinguish them, we use $\partial_i = \pi_i^* \pi_i : H^*(G/B) \to H^*(G/B)$ for the cohomology and $\delta_i = \pi_i^* \pi_i : K(G/B) \to K(G/B)$ for the K-theory. Then we have the following theorem.

**Theorem 3.2 ([BGG], [D1], [D2]).** We have

$$\partial_i^2 = 0, \quad \delta_i^2 = \delta_i,$$

for all $i$ and both $\partial_i$, $\delta_i$ satisfy the braid relations: i.e., if $s_is_j$ is of order $m_{ij}$ for $i \neq j$, then

$$\partial_i \partial_j \partial_i ..., = \partial_j \partial_i \partial_j ..., (m_{ij} \text{ factors}),$$

$$\delta_i \delta_j \delta_i ..., = \delta_j \delta_i \delta_j ..., (m_{ij} \text{ factors}).$$

Moreover, their actions on the corresponding basis given in Proposition 3.1 can be described by the following

$$\partial_i[X_w] = \begin{cases} [X_{ws_i}] & \text{if } ws_i > w \\ 0 & \text{otherwise} \end{cases} \quad \delta_i[\mathcal{O}_{X_w}] = \begin{cases} [\mathcal{O}_{X_{ws_i}}] & \text{if } ws_i > w \\ [\mathcal{O}_{X_w}] & \text{otherwise} \end{cases}$$

**Remark 3.3.** The algebra generated by $\partial_i$ is called the nil-Hecke algebra, which is denoted by $NH_r$. The algebra generated by $\delta_i$ is called the $q = 0$ Hecke algebra, which is denoted by $H_r(0)$.

**Remark 3.4.** The above result also holds for the equivariant setting. In particular, $T$-equivariant cohomology $H^*_T(G/B)$ and K-theory $K_T(G/B)$. However, since we will not touch this in the rest of this paper, we refer to [KoKu1], [KoKu2] for details.
3.2. Type $A$ case and the affine version. In this section, we focus on the type $A$ case where $G = \text{SL}_N(\mathbb{C})$. This will be helpful in the later sections where we relate the $q = 0$ affine Hecke algebra to the shifted $q = 0$ affine algebra.

Let $B \subset G$ be the subgroup of upper triangulated matrices and $T \subset G$ be the diagonal matrices. The Weyl group is the symmetric group $W = S_N$. Then the full flag variety $G/B$ can be described as the following space

$$G/B = \{0 \subset V_1 \subset V_2 \subset \ldots \subset V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for all } k\},$$

and similarly the partial flag varieties

$$G/P_i = \{0 \subset V_1 \subset V_2 \subset \ldots V_{i-1} \subset V_{i+1} \subset \ldots V_N = \mathbb{C}^N \mid \dim V_k = k \text{ for } k \neq i\}. \quad (3.2)$$

We denote $\mathcal{V}_i$ to be the tautological bundle of rank $i$ on $G/B$, and $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ the natural line bundles. Let $a_i = c_1(\mathcal{L}_i) \in H^2(G/B, \mathbb{Z})$ be the first Chern class of $\mathcal{L}_i$, and $b_i = [\mathcal{L}_i] \in K(G/B)$ be the class in the Grothendieck group. Then the cohomology $H^*(G/B)$ and K-theory $K(G/B)$ have the following description as quotient of polynomial rings.

**Proposition 3.5.**

$$H^*(G/B) = \mathbb{C}[a_1, \ldots, a_N]/I, \quad K(G/B) = \mathbb{C}[b_1^+, \ldots, b_N^+] / J$$

where $I$ is the ideal generated by $e_i(a_1, \ldots, a_N) = 0$ and $J$ is the ideal generated by $e_i(b_1, \ldots, b_N) = \binom{N}{k}$. Here $e_i$ is the $i$th elementary symmetric polynomials.

The natural projection $\pi_i : G/B \to G/P_i$ is a $\mathbb{P}^1$-fibration for all $i$. Under the presentations in Proposition 3.5, the BGG-Demazure operators $\partial_i$ and $\delta_i$ have the following explicit descriptions.

$$\partial_i = \frac{1 - s_i}{a_{i+1} - a_i}, \quad \delta_i = \frac{b_{i+1} - b_is_i}{b_{i+1} - b_i}, \quad (3.3)$$

where $s_i$ is the simple reflection that permute $a_i$, $a_{i+1}$ or $b_i$, $b_{i+1}$.

Beside the operators $\partial_i$, $\delta_i$, we have more operators in this presentation. By abusing notations, we still denote $a_i : H^*(G/B) \to H^*(G/B)$ and $b_i : K(G/B) \to K(G/B)$ to be the map that given by multiplication by $a_i$ and $b_i$, respectively. Note that $b_i$ is invertible here.

With the help of (3.3), we can easily verify the following relations.

$$a_ia_j = a_ja_i \text{ for all } i,j, \quad \partial_ia_j = a_ia_j \partial_i \text{ if } j \neq i, i + 1, \quad (3.4)$$
$$a_i\partial_i = \partial_i a_i - 1, \quad a_{i+1}\partial_i = b_i\partial_i, \quad (3.5)$$
$$b_ib_j = b_jb_i \text{ for all } i,j, \quad b_i\delta_j = b_j\delta_i \text{ if } j \neq i, i + 1, \quad (3.6)$$
$$b_{i+1}\delta_i = \delta_ib_i + b_{i+1}, \quad b_{i+1}\delta_i = b_i\delta_i - b_{i+1}. \quad (3.7)$$

With all the above works, we have the following theorem.

**Theorem 3.6.** There is an action of the nil-affine Hecke algebra, denoted by $\mathcal{N}H_N$, on $H^*(G/B)$ via the generators $\{\partial_i, a_j\}$ such that $\{\partial_i\}$ generate the nil-Hecke algebra $NH_N$ and plus the extra relations (3.4) and (3.5).

There is an action of the $q = 0$ affine Hecke algebra, denoted by $\mathcal{H}_N(0)$, on $K(G/B)$ via the generators $\{\delta_i, b_j\}$ such that $\{\delta_i\}$ generate the $q = 0$ Hecke algebra $\mathcal{H}_N(0)$ and plus the extra relations (3.6) and (3.7).
3.3. Two variants of the $q = 0$ Hecke algebra. In this section, we introduce two variants of the $q = 0$ Hecke algebra $H_N(0)$, which will be used in later sections.

We define the first variant of $H_N(0)$

**Definition 3.7.** We define $H_N'(0)$ to be the algebra generated by $\delta_i'$ where $1 \leq i \leq N - 1$ such that they satisfy the following relations

\[
\begin{align*}
(\delta_i')^2 &= \delta_i', \\
\delta_i' \delta_j' &= \delta_j' \delta_i' \text{ for all } |i - j| \geq 2, \\
\delta_i' \delta_{i+1}' \delta_i' &= \delta_i' \delta_{i+1}' \delta_i' = \delta_{i+1}' \delta_i' \delta_{i+1}' .
\end{align*}
\]

Then the second variant is given by

**Definition 3.8.** We define $H_N''(0)$ to be the algebra generated by $\delta_i''$ where $1 \leq i \leq N - 1$ such that they satisfy the following relations

\[
\begin{align*}
(\delta_i'')^2 &= \delta_i'', \\
\delta_i'' \delta_j'' &= \delta_j'' \delta_i'' \text{ for all } |i - j| \geq 2, \\
\delta_i'' \delta_{i+1}'' \delta_i'' &= \delta_i'' \delta_{i+1}'' \delta_i'' = \delta_{i+1}'' \delta_i'' \delta_{i+1}'' .
\end{align*}
\]

It is easy to see that we have two natural surjective algebra homomorphisms

\[
\begin{align*}
\rho_1 : H_N'(0) &\rightarrow H_N(0), \quad \rho_1(\delta_i') = \delta_i' \forall i, \\
\rho_2 : H_N''(0) &\rightarrow H_N(0), \quad \rho_2(\delta_i'') = \delta_i'' \forall i,
\end{align*}
\]

and the kernels are given by

\[
\begin{align*}
\ker(\rho_1) &= \langle \delta_{i+1}' \delta_i' \delta_{i+1}'(\delta_i' - 1) \rangle_{1 \leq i \leq N-1}, \\
\ker(\rho_2) &= \langle \delta_i'' \delta_i'' \delta_{i+1}''(\delta_i'' - 1) \rangle_{1 \leq i \leq N-1},
\end{align*}
\]

where we use $\langle \rangle$ to denote the ideal generated by specific elements.

Thus we have the following isomorphisms

\[
H_N(0) \cong H_N'(0)/\langle \delta_{i+1}' \delta_i' \delta_{i+1}'(\delta_i' - 1) \rangle_{1 \leq i \leq N-1} \cong H_N''(0)/\langle \delta_i'' \delta_i'' \delta_{i+1}''(\delta_i'' - 1) \rangle_{1 \leq i \leq N-1}.
\]

4. Shifted $q = 0$ affine algebras and its categorical action

In this section, we recall the definitions of the shifted $q = 0$ affine algebra $\mathcal{U}_{0,N}(\mathfrak{L}t_n)$ and its categorical action. We also mention the result where there is a categorical action on the bounded derived categories of coherent sheaves on partial flag varieties.

Before we go to the detailed setup and the definition of the shifted $q = 0$ affine algebra, we should give the readers some background for it. The main motivation comes from the study of categorical action and categorification of the quantum group $\mathcal{U}_q(\mathfrak{g})$ for $\mathfrak{g}$ a semisimple or Kac-Moody Lie algebra.

We restrict to the case $\mathfrak{g} = \mathfrak{sl}_2$ and keep in mind that all the results can be naturally generalized to the $\mathfrak{sl}_n$ case. Based on the work [BLM], Beilinson-Lusztig-MacPherson give a geometric model for $\mathcal{U}_q(\mathfrak{sl}_2)$. This can be used to construct categorical $\mathcal{U}_q(\mathfrak{sl}_2)$-action.
The weight space $V_\lambda$ is replaced by the weight category $\mathcal{C}(\lambda) = D^b \text{Con}(\mathbb{G}(k,N))$, which is the bounded derived category of constructible sheaves on $\mathbb{G}(k,N)$ with $\lambda = N - 2k$. The generators $e$, $f$ act on them by using the following correspondence diagram

$$Fl(k-1,k) = \{ 0 \leq 1 \subset V' \subset V \subset \mathbb{C}^N \}$$

where $Fl(k-1,k)$ is the 3-step partial flag variety and $p_1$, $p_2$ are the natural projections. Then we define $E := p_{2*}p_1^*$ and a similar functor $F$ in the opposite direction that can be viewed as lift of $e$ and $f$, respectively. The functors $E$, $F$ satisfy the defining relations of $U_q(\mathfrak{sl}_2)$, i.e., we have

$$EF|_{\mathcal{C}(\lambda)} \cong FE|_{\mathcal{C}(\lambda)} \bigoplus_{[\lambda]} \text{Id}_{\mathcal{C}(\lambda)}$$

if $\lambda \geq 0$,

similarly for $\lambda \leq 0$. Here $[n] := q^{-n+1} + q^{-n+3} + \ldots + q^{n-3} + q^{n-1}$ is the quantum integer and $\bigoplus_{[\lambda]}$ denotes a graded direct sum. For example $\bigoplus_{[2]} f = f[1] \bigoplus f[-1]$, where $[1]$ is the homological shift in the derived category that uses to keep track of the power of the variable $q$.

Motivating by the above construction, now we consider the weight categories to be $D^b(\mathbb{G}(k,N))$, i.e., we replace the constructible sheaves by coherent sheaves. Let $\mathcal{V}$, $\mathcal{V}'$ to be the tautological bundles on $Fl(k-1,k)$ of rank $k$, $k-1$ respectively. Then there is a natural line bundle $\mathcal{V}/\mathcal{V}'$ on $Fl(k-1,k)$. Using the same correspondence (4.1), instead of just pulling back and pushing forward directly, we have an extra twist by the line bundles $(\mathcal{V}/\mathcal{V}')^r$ where $r \in \mathbb{Z}$. So we have the functors

$$E_r := p_{2*}(p_1^* \otimes (\mathcal{V}/\mathcal{V}')^r) : D^b(\mathbb{G}(k,N)) \to D^b(\mathbb{G}(k-1,N))$$

and similarly $F_s$ where $r, s \in \mathbb{Z}$.

The shifted $q = 0$ affine algebra arises naturally from the studies of the $L\mathfrak{sl}_2 := \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}]$-like algebra acting on $\bigoplus_k D^b(\mathbb{G}(k,N))$, where $e \otimes t^r$ and $f \otimes t^s$ acting via the functors $E_r$, $F_s$ respectively for $r, s \in \mathbb{Z}$. In [Hsu], the author is the first one to study this action in detail.

After decategorifying (pass to the K-theory), we obtain an algebra with loop generators $e_r$, $f_s$ and relations that look similar to the shifted quantum affine algebra defined in [FT]. There is a variable $q$ in their definition that stands for the $q$-deformation of quantum affine algebra. From the geometric aspect, it comes from the $C^*$-action. In our case, we do not have a natural $C^*$-action on Grassmannians, so we do not have a variable like $q$.

We call the resulting algebra shifted $q = 0$ affine algebra. First, the "$q = 0$" part in the name comes from the reason that some of the relations can be obtained from the relations in shifted quantum affine algebra by taking $q = 0$ directly. Second, the "shifted" comes from the fact that the commutator relations between $e_r$ and $f_s$ vanishing within certain range of $r + s$. Finally, even though the algebra is given by the loop generators $e_r$, $f_s$, but the representation $\bigoplus_k K(\mathbb{G}(k,N))$ we study is finite dimensional. Since affine algebras
are central extensions of loop algebras, and central extension acts trivially on the finite dimensional representations, we still use "affine algebra" in the name.

4.1. **Shifted q = 0 affine algebras.** In this section, we define the shifted $q = 0$ affine algebras. By imitating the presentation in [FT] (the so-called Levedorskii type presentation), the presentation we use here is by finite numbers of generators and defining relations. In [Hsu], we also give a conjectural presentation defined by generating series and conjecture that the two presentations are equivalent, see conjecture A.2. in loc. cit.

Similarly to the dot version $\mathcal{U}_q(sl_2)$ of $\mathcal{U}_q(sl_2)$ that introduced in [BLM], the shifted $q = 0$ affine algebras we introduce below is also an idempotent version. This means that we replace the identity by the direct sum of a system of projectors, one for each element of the weight lattices. They are orthogonal idempotents for approximating the unit element. We refer to part IV in [Lu1] for details of such modification.

Throughout the rest of this article, we fix a positive integer $N \geq 2$. Let

$$C(n, N) := \{ \underline{k} = (k_1, ..., k_n) \in \mathbb{N}^n \mid k_1 + ... + k_n = N \}.$$ 

We regard each $\underline{k}$ as a weight for $\mathfrak{sl}_n$ via the identification of the weight lattice of $\mathfrak{sl}_n$ with the quotient $\mathbb{Z}^n/(1,1,...,1)$. We choose the simple root $\alpha_i$ to be $(0,0,-1,1,0,0) \ldots$ where the $-1$ is in the $i$-th position for $1 \leq i \leq n - 1$. Finally, we denote $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$ to be the standard pairing. Then we introduce the shifted $q = 0$ affine algebra for $\mathfrak{sl}_n$, which is defined by using finite generators and relations.

**Definition 4.1.** We define the shifted $q=0$ affine algebra, denote by $\hat{U}_{0,N}(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}$-algebra generated by

$$\bigcup_{\underline{k} \in C(n,N)} \{ 1_{\underline{k}}, \ e_{i,r}1_{\underline{k}}, \ f_{i,s}1_{\underline{k}}, \ (\psi^+_i)^{\pm 1}1_{\underline{k}}, \ (\psi^-_i)^{\pm 1}1_{\underline{k}}, \ h_{i,\pm 1}1_{\underline{k}} \mid 1 \leq i \leq n \}$$

with the following relations with the following relations

$$1_{\underline{k}}1_{\underline{l}} = \delta_{\underline{k},\underline{l}}1_{\underline{k}}, \ e_{i,r}1_{\underline{k}} = 1_{\underline{k}+\alpha_i}e_{i,r}, \ f_{i,r}1_{\underline{k}} = 1_{\underline{k}-\alpha_i}f_{i,r}, \ (\psi^+_i)^{\pm 1}1_{\underline{k}} = 1_{\underline{k}}(\psi^+_i)^{\pm 1}, \ h_{i,\pm 1}1_{\underline{k}} = 1_{\underline{k}}h_{i,\pm 1}, \ \ \ (U01)$$

$$\{ (\psi^+_i)^{\pm 1}1_{\underline{k}}, (\psi^-_i)^{\pm 1}1_{\underline{k}}, h_{i,\pm 1}1_{\underline{k}} \mid 1 \leq i \leq n, \underline{k} \in C(n,N) \} \text{ pairwise commute,} \ \ (U02)$$

$$\langle \psi^+_i, \psi^-_j \rangle^{\mp 1}1_{\underline{k}} = \delta_{\underline{k},\underline{l}}1_{\underline{l}}, \langle \psi^-_i, \psi^-_j \rangle^{\mp 1}1_{\underline{k}} = \delta_{\underline{k},\underline{l}}1_{\underline{l}}, \langle \psi^+_i, \psi^-_j \rangle^{\mp 1}1_{\underline{k}} = \delta_{\underline{k},\underline{l}}1_{\underline{l}}, \ \ (U03)$$

$$e_{i,r}e_{j,s}1_{\underline{k}} = \begin{cases} -e_{i,s+1}e_{i,r-1}1_{\underline{k}} & \text{if } j = i \\ e_{i+1,s}e_{i,r}1_{\underline{k}} - e_{i+1,s-1}e_{i,r+1}1_{\underline{k}} & \text{if } j = i + 1 \\ e_{i,s+1}e_{i-1,r-1}1_{\underline{k}} - e_{i-1,s-1}e_{i,r+1}1_{\underline{k}} & \text{if } j = i - 1 \\ e_{j,s}e_{i,r}1_{\underline{k}} & \text{if } |i - j| \geq 2 \end{cases}, \ \ (U04)$$

$$f_{i,r}f_{j,s}1_{\underline{k}} = \begin{cases} -f_{i,s-1}f_{i,r+1}1_{\underline{k}} & \text{if } j = i \\ f_{i-1,s}f_{i,r}1_{\underline{k}} - f_{i-1,s+1}f_{i,r-1}1_{\underline{k}} & \text{if } j = i + 1 \\ f_{i,s}f_{i-1,r}1_{\underline{k}} - f_{i-1,s+1}f_{i,r-1}1_{\underline{k}} & \text{if } j = i - 1 \\ f_{j,s}f_{i,r}1_{\underline{k}} & \text{if } |i - j| \geq 2 \end{cases}. \ \ (U05)$$
\[ \psi_i^+ e_{j,r} 1_{k}^1 = \begin{cases} 
- e_{i,r+1} \psi_i^+ 1_{k}^1 & \text{if } j = i \\
- e_{i+1,r} \psi_i^+ 1_{k}^1 & \text{if } j = i + 1 \\
e_{i-1,r} \psi_i^+ 1_{k}^1 & \text{if } j = i - 1 \\
e_{j,r} \psi_i^+ 1_{k}^1 & \text{if } |i-j| \geq 2 \end{cases} \]

\[ \psi_i^- e_{j,r} 1_{k}^1 = \begin{cases} 
- e_{i,r+1} \psi_i^- 1_{k}^1 & \text{if } j = i \\
e_{i+1,r} \psi_i^- 1_{k}^1 & \text{if } j = i + 1 \\
e_{i-1,r} \psi_i^- 1_{k}^1 & \text{if } j = i - 1 \\
e_{j,r} \psi_i^- 1_{k}^1 & \text{if } |i-j| \geq 2 \end{cases} \] (U06)

\[ \psi_i^+ f_{j,r} 1_{k}^1 = \begin{cases} 
- f_{i,r-1} \psi_i^+ 1_{k}^1 & \text{if } j = i \\
f_{i+1,r} \psi_i^+ 1_{k}^1 & \text{if } j = i + 1 \\
f_{i-1,r} \psi_i^+ 1_{k}^1 & \text{if } j = i - 1 \\
f_{j,r} \psi_i^+ 1_{k}^1 & \text{if } |i-j| \geq 2 \end{cases} \]

\[ \psi_i^- f_{j,r} 1_{k}^1 = \begin{cases} 
- f_{i,r-1} \psi_i^- 1_{k}^1 & \text{if } j = i \\
f_{i+1,r} \psi_i^- 1_{k}^1 & \text{if } j = i + 1 \\
f_{i-1,r} \psi_i^- 1_{k}^1 & \text{if } j = i - 1 \\
f_{j,r} \psi_i^- 1_{k}^1 & \text{if } |i-j| \geq 2 \end{cases} \] (U07)

\[ [h_{i,\pm 1}, e_{j,r}] 1_{k}^1 = \begin{cases} 
0 & \text{if } i = j \\
e_{i+1,r} 1_{k}^1 & \text{if } j = i + 1 \\
e_{i-1,r} 1_{k}^1 & \text{if } j = i - 1 \\
0 & \text{if } |i-j| \geq 2 \end{cases} \]

\[ [h_{i,\pm 1}, f_{j,r}] 1_{k}^1 = \begin{cases} 
f_{i+1,r} 1_{k}^1 & \text{if } j = i + 1 \\
f_{i-1,r} 1_{k}^1 & \text{if } j = i - 1 \\
0 & \text{if } |i-j| \geq 2 \end{cases} \] (U08)

for any 1 \leq i, j \leq n - 1 and r, s such that the above relations make sense.

4.2. **Categorical \( \mathcal{U}_{0,N}(L_{\mathbb{C}^n}) \) action.** In this section, we recall the definition of the categorical action for \( q = 0 \) affine algebra that defined in [Hsu]. However, since we will not use any of the categorical relations that involve the elements \( h_{i,\pm 1} 1_{k}^1 \) in the rest of this article, we do not present them here. We refer the readers to Definition 3.1. in loc. cit. for a full definition of the categorical action.

**Definition 4.2.** A categorical \( \mathcal{U}_{0,N}(L_{\mathbb{C}^n}) \) action consists of a target 2-category \( \mathcal{K} \), which is triangulated, \( \mathbb{C} \)-linear and idempotent complete. The objects in \( \mathcal{K} \) are

\[ \text{Ob}(\mathcal{K}) = \{ \mathcal{K}(k) \mid k \in C(n, N) \} \]

where each \( \mathcal{K}(k) \) is a triangulated category, and \( \text{Hom}(\mathcal{K}(k), \mathcal{K}(l)) \) is also a triangulated category for all \( k, l \in C(n, N) \).

The 1-morphisms are are given by the following:

\[ 1_{k}^1, E_{i,r} 1_{k}^1 = 1_{k}^1 + \alpha_i, E_{i,r}, F_{i,s} 1_{k}^1 = 1_{k}^1 - \alpha_i, F_{i,s}, (\psi_i^\pm)^{\pm 1} 1_{k}^1 = 1_{k}^1 (\psi_i^\pm)^{\pm 1}, \]
where \(1 \leq i \leq n - 1, \ -k_i - 1 \leq r \leq 0, 0 \leq s \leq k_i + 1\). Here \(1_k\) is the identity functor of \(K(k)\). On this data we impose the following conditions.

1. The space of maps between any two 1-morphisms is finite dimensional.
2. If \(\alpha = \alpha_i\) or \(\alpha = \alpha_i + \alpha_j\) for some \(i, j\) with \(\langle \alpha_i, \alpha_j \rangle = -1\), then \(1_k^{\alpha_i} = 0\) for \(r \gg 0\) or \(r \ll 0\).
3. Suppose \(i \neq j\). If \(1_k^{\alpha_i}\) and \(1_k^{\alpha_i + \alpha_j}\) are nonzero, then \(1_k\) and \(1_k^{\alpha_i + \alpha_j}\) are also nonzero.
4. The left and right adjoints of \(E_{i,r}\) and \(F_{i,s}\) are given by conjugation of \(\psi_i^{\pm}\) up to homological shifts. More precisely,
   a. \((E_{i,r}1_k)^R \cong 1_k^{\psi_i^+ r + 1}E_{i,k_i + 1,2}(\psi_i^+)^{-r - 2}[-r - 1]\) for all \(1 \leq i \leq n - 1\),
   b. \((E_{i,r}1_k)^L \cong 1_k^{\psi_i^- r + k_i^{1}}E_{i,k_i + 1}(\psi_i^-)^{-r - k_i[r + k]}\) for all \(1 \leq i \leq n - 1\),
   c. \((F_{i,s}1_k)^R \cong 1_k^{\psi_i^+ s + 1}E_{i - k_i - 2}(\psi_i^-)^{-s - 2}[s - 1]\) for all \(1 \leq i \leq n - 1\),
   d. \((F_{i,s}1_k)^L \cong 1_k^{\psi_i^- s + k_i + 1 - 1}E_{i}(\psi_i^-)^{s - k_i + 1}[-s + k_i + 1]\) for all \(1 \leq i \leq n - 1\).
5. 
   \((\psi_i^+)_{\pm 1}^{\pm 1}(\psi_i\)\)^{\pm 1}\(1_k^{\pm 1}\) \(1_k\) for all \(i, j\),
   \((\psi_i^+)_{\pm 1}^{\pm 1}(\psi_i\)\)^{\pm 1}\(1_k\) \(1_k\) for all \(i\).
6. The relations between \(E_{i,r}\), \(E_{j,s}\) are given by the following
   a. We have
   \[
   E_{i,r+1}E_{s,1_k} \cong \begin{cases} 
   E_{s+1}E_{i,r}1_k[-1] & \text{if } r - s \geq 1 \\
   0 & \text{if } r = s \\
   E_{s+1}E_{i,r}1_k[1] & \text{if } r - s \leq -1.
   \end{cases}
   \]
   b. \(E_{i,r}\), \(E_{i+1, r}\) are related by the following exact triangle
   \[
   E_{i+1, r}E_{i,r+1}1_k \rightarrow E_{i+1, s+1}E_{i,r}1_k \rightarrow E_{i,r}E_{i+1, s+1}1_k.
   \]
   c. We have
   \[
   E_{i,r}E_{j,s}1_k \cong E_{j,s}E_{i,r}1_k \text{ if } |i - j| \geq 2.
   \]
7. The relations between \(F_{i,r}\), \(F_{j,s}\) are given by the following
   a. We have
   \[
   F_{i,r}F_{s,1_k} \cong \begin{cases} 
   F_{s+1}F_{i,r}1_k[1] & \text{if } r - s \geq 1 \\
   0 & \text{if } r = s \\
   F_{s+1}F_{i,r}1_k[-1] & \text{if } r - s \leq -1.
   \end{cases}
   \]
   b. \(F_{i,r}\), \(F_{i+1, r}\) are related by the following exact triangles
   \[
   F_{i+1, r}F_{i,r+1}1_k \rightarrow F_{i,r}F_{i+1, s+1}1_k \rightarrow F_{i+1, s+1}F_{i,r}1_k.
   \]
   c. We have
   \[
   F_{i,r}F_{j,s}1_k \cong F_{j,s}F_{i,r}1_k \text{ if } |i - j| \geq 2.
   \]
8. The relations between \(E_{i,r}\), \(\psi_i^\pm\) are given by the following
   a. For \(i = j\), we have
   \[
   \psi_i^\pm E_{i,r}1_k \cong E_{i,r+1}1_k^\pm[1].
   \]
For $|i - j| = 1$, we have the following

\[ \Psi^\pm K_{i+1,r} \cong E_{i+1,r}^{-1} \Psi^\pm K_i, \]
\[ \Psi^\pm K_{i+1,r} \cong E_{i+1,r} \Psi^\pm K_i. \]

For $|i - j| \geq 2$, we have

\[ \Psi^\pm K_{i,r} \cong E_{i,r} \Psi^\pm K_i. \]

The relations between $F_{i,r}$, $\Psi^\pm_j$ are given by the following

(a) For $i = j$, we have

\[ \Psi^\pm F_{i,r} K_i \cong F_{i,r}^{-1} \Psi^\pm K_i. \]

(b) For $|i - j| = 1$, we have the following

\[ \Psi^\pm F_{i+1,s} K_i \cong F_{i+1,s} \Psi^\pm K_i, \]
\[ \Psi^\pm F_{i+1,s} K_i \cong F_{i+1,s} \Psi^\pm K_i. \]

(c) For $|i - j| \geq 2$, we have

\[ \Psi^\pm F_{i,r} K_i \cong F_{i,r} \Psi^\pm K_i. \]

(10) If $i \neq j$, then $E_{i,r} F_{j,s} K_i \cong F_{j,s} E_{i,r} K_i$ for all $r, s \in \mathbb{Z}$.

(11) For $E_{i,r} F_{j,s} K_i, F_{i,s} E_{i,r} K_i \in \text{Hom}(\mathcal{K}(k), \mathcal{K}(k))$, they are related by exact triangles, more precisely,

(a) $F_{i,r} E_{i,r} K_i \rightarrow E_{i,r} F_{i,s} K_i \rightarrow \Psi^\pm K_i$ if $r + s = k_i + 1$,

(b) $E_{i,r} F_{i,s} K_i \rightarrow F_{i,s} E_{i,r} K_i \rightarrow \Psi^\pm K_i$ if $r + s = k_i$,

(c) $F_{i,s} E_{i,r} K_i \cong E_{i,r} F_{i,s} K_i$ if $- k_i + 1 \leq r + s \leq k_i + 1 - 1$,

for all $r, s$ that make the above conditions make sense, and the isomorphisms between functors appear in every condition are abstractly defined, i.e., we do not specify any 2-morphisms that induce those isomorphisms.

4.3. Geometric example. In this section, we mention the main theorem (Theorem 5.2 in [Hsu]), which says that there is a categorical action of $\mathcal{U}_{0,N}(L\mathfrak{s}l_n)$ on the bounded derived categories of coherent sheaves on partial flag varieties.

For each $K \in C(n, N)$, we define the partial flag variety

\[ FL_k(C^N) := \{ V_i = (0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N) \mid \dim V_i / V_{i-1} = k_i \text{ for all } i \}. \] (4.2)

We denote $Y(k) = FL_k(C^N)$ and $\mathcal{D}^b(Y(k))$ to be the bounded derived categories of coherent sheaves on $Y(k)$. Those will be the objects $\mathcal{K}(k)$ of the triangulated 2-category $\mathcal{K}$ in Definition 4.2. On $Y(k)$ we denote $V_i$ to be the tautological bundle whose fibre over a point $(0 = V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^N)$ is $V_i$.

To define those 1-morphisms $E_{i,r} K_i, F_{i,s} K_i, H_{i, \pm 1} K_i, (\Psi^\pm) K_i$, we use the language of FM transforms, which means we will define them by using FM kernels. However, since the FM kernels for $H_{i, \pm 1} K_i$ will not be used in the rest of this article, we only mention the FM kernels for $E_{i,r} K_i, F_{i,s} K_i$ and $(\Psi^\pm) K_i$ for simplicity.
We define correspondences $W_i^1(\mathbf{k}) \subset Y(\mathbf{k}) \times Y(\mathbf{k} + \alpha_i)$ by

$$W_i^1(\mathbf{k}) := \{(V_i, V'_i) \in Y(\mathbf{k}) \times Y(\mathbf{k} + \alpha_i) \mid V_j = V'_j \text{ for } j \neq i, \ V'_i \subset V_i\},$$

then we have the natural line bundle $\mathcal{V}_i/\mathcal{V}'_i$ on $W_i^1(\mathbf{k})$, where $\mathcal{V}_i, \mathcal{V}'_i$ are the tautological bundle on $W_i^1(\mathbf{k})$ of rank $k_1 + \ldots + k_i$, $k_1 + \ldots + k_i - 1$, respectively. We also have the transpose correspondence $^T W_i^1(\mathbf{k}) \subset Y(\mathbf{k} + \alpha_i) \times Y(\mathbf{k})$. Let $\iota(\mathbf{k}) : W_i^1(\mathbf{k}) \hookrightarrow Y(\mathbf{k}) \times Y(\mathbf{k} + \alpha_i)$, $^T \iota(\mathbf{k}) : ^T W_i^1(\mathbf{k}) \hookrightarrow Y(\mathbf{k} + \alpha_i) \times Y(\mathbf{k})$ be the inclusions and $\Delta(\mathbf{k}) : Y(\mathbf{k}) \rightarrow Y(\mathbf{k}) \times Y(\mathbf{k})$ be the diagonal map. Then we have the following theorem.

**Theorem 4.3** (Theorem 5.2 [Hsu]). Let $\mathcal{K}$ be the triangulated 2-categories whose nonzero objects are $\mathcal{K}(\mathbf{k}) = \mathcal{D}^b(Y(\mathbf{k}))$ where $\mathbf{k} \in C(n, N)$. The 1-morphisms $\mathcal{E}_{i,s} : \mathcal{1}_\mathbf{k}, \mathcal{1}_\mathbf{k} \mathcal{F}_{i,s}, (\Psi^+_i)^{\pm 1}1_\mathbf{k}$ and $(\Psi^-_i)^{\pm 1}1_\mathbf{k}$ are FM transformations with kernels given by

$$\mathcal{E}_{i,s} : \iota(\mathbf{k}), \mathcal{1}_\mathbf{k} \mathcal{F}_{i,s} := ^T \iota(\mathbf{k}), (\Psi^+_i)^{\pm 1}1_\mathbf{k} := \Delta(\mathbf{k}) \det(\mathcal{V}_i/\mathcal{V}_j)^{\pm 1}[\pm (1 - k_i)],$$

respectively, and the 2-morphisms are maps between kernels. Then this gives a categorical $\mathcal{U}_{0,N}(L\mathfrak{s}_{1n})$ action.

5. Demazure operators via $\mathcal{U}_{0,N}(L\mathfrak{s}_{1n})$

In this section, we prove the main result of this article; i.e., a categorical action of the shifted $q = 0$ affine algebra induces actions of $H'_n(0)$ and $H''_n(0)$ on each weight category. In particular, this leads to constructing Demazure operators for more general $n$-step partial flag varieties.

5.1. The full flag variety. We begin with the main motivation, i.e., interpretation of the Demazure operators $\delta_i$ in terms of elements in the shifted $q = 0$ affine algebra. Note that in the notation (1.2), we have the type $A$ full flag variety $G/B = Fl_{(1,1,\ldots,1)}(\mathbb{C}^N)$. Similarly, for the partial flag varieties $G/P_i$ given in (3.2), we have $G/P_i = Fl_{(1,1,\ldots,1)+\alpha_i}(\mathbb{C}^N) = Fl_{(1,1,\ldots,1)-\alpha_i}(\mathbb{C}^N)$. Here $\alpha_i$ is the simple root $(0,0,\ldots,0,-1,0,0,\ldots)$ where the $-1$ is in the $i$-th position for $1 \leq i \leq N - 1$.

Recall from Theorem 3.9 that there is an action of the $q = 0$ affine Hecke algebra $\mathcal{H}_N(0)$ on the $K$-theory of the full flag variety $K(G/B)$ via the generators $\{\delta_i, b_j\}$ that defined in Section 3. This action can be lifted to the bounded derived category of coherent sheaves $\mathcal{D}^b(G/B)$, and we denote the lift of $\delta_i$ and $b_j$ to the derived category level by the functors $\mathcal{T}_i$ and $\mathcal{X}_j$, respectively.

Here we recall the details of Section 6 from [Hsu]. We will use the language of FM transforms from Section 2 to define the functors $\mathcal{T}_i$ and $\mathcal{X}_j$. We have $\mathcal{T}_i \cong \Phi_{\mathcal{T}_i}$ is a FM transform with kernel $\mathcal{T}_i = \mathcal{O}_{G/B \times G/P_i}G/B$, which is the structure sheaf of the Bott-Samelson variety $G/B \times G/P_i G/B$. Similarly, $\mathcal{X}_j \cong \Phi_{\mathcal{X}_j}$ is a FM transform with kernel $\mathcal{X}_j = \Delta_+(\mathcal{V}_j/\mathcal{V}_{j-1}) \in \mathcal{D}^b(G/B \times G/B)$, where $\Delta : G/B \rightarrow G/B \times G/B$ is the diagonal...
map. Note that $X_j$ is invertible with its inverse functor $X_j^{-1}$ given by the FM kernel $X_j^{-1} = \Delta_s(V_j/V_{j-1}) \in D^b(G/B \times G/B)$.

To verify that the functor $T_i$, $X_j$ defined above give an action of $\mathcal{H}_N(0)$ on $D^b(G/B)$. From proposition 2.4, the composition of FM transformations is again a FM transformation, so it suffices to verify the convolution of FM kernels. The next Theorem is the second main result in [Hsu].

**Theorem 5.1.** (Theorem 6.3 in [Hsu]) There is a categorical action of the $q = 0$ affine Hecke algebra $\mathcal{H}_N(0)$ on $D^b(G/B)$. More precisely, if we define the FM kernels $T_i = O_{G/B \times G/B}$ and $X_j = \Delta_s(V_j/V_{j-1})$ for $1 \leq i \leq N - 1$, $1 \leq j \leq N$, then we have the following categorical relations

$$T_i \star T_j \cong T_i, \quad (5.1)$$

$$T_i \star T_j \cong T_j \star T_i \text{ if } |i - j| \geq 2, \quad (5.2)$$

$$T_i \star T_j \star T_i \cong T_j \star T_i \star T_j \text{ if } |i - j| = 1, \quad (5.3)$$

$$X_i \star X_i^{-1} \cong X_i^{-1} \star X_i \cong O_\Delta, \quad (5.4)$$

$$X_j \star X_j \cong X_j \star X_i \text{ for all } i, j, \quad (5.5)$$

$$T_i \star X_j \cong X_j \star T_i \text{ if } j \neq i, i + 1, \quad (5.6)$$

We have the following exact triangles in $D^b(G/B \times G/B)$

$$T_i \star X_i \rightarrow X_{i+1} \star T_i \rightarrow X_{i+1}, \quad (5.7)$$

$$X_i \star T_i \rightarrow T_i \star X_{i+1} \rightarrow X_{i+1}. \quad (5.8)$$

In loc. cit. we do not prove this by direct calculations of the convolution of kernels. Instead, we use the categorical action of shifted $q = 0$ affine algebras, more precisely Theorem 4.3 to help us prove it. This serves as our main motivation to construct more general Demazure operators in the next subsection.

In the full flag variety case we have $n = N$, so Theorem 4.3 tells us that there is a categorical action of $\mathcal{U}_{0,N}(L\mathfrak{s}\mathfrak{l}_N)$ on $D^b(G/B)$. The proof of Theorem 5.1 can be achieved by writing the generators $\delta_i, b_j$ (and their FM kernels $T_i, X_j$) in terms of the elements of $\mathcal{U}_{0,N}(L\mathfrak{s}\mathfrak{l}_N)$ (and the corresponding FM kernels). Then all the relations that we need to verify are a consequence of the relations in the definition of categorical action.

We will explain this in a bit more detail. From Section 6 of [Hsu], we have the following isomorphisms of FM kernels

$$T_i \cong (F_{i,0} \star F_{i,0})1_{(1,1,\ldots,1)} \cong (F_{i,0} \star F_{i,0})1_{(1,1,\ldots,1)} \quad (5.9)$$

$$\cong (F_{i,0} \star F_{i,1} \star (\Psi^+)^{-1})1_{(1,1,\ldots,1)} \cong (F_{i,0} \star F_{i,1} \star (\Psi^+)^{-1})1_{(1,1,\ldots,1)}. \quad (5.10)$$

for all $1 \leq i \leq N - 1$. Also, it is easy to see that $X_i \cong \Psi_i^+1_{(1,1,\ldots,1)} \cong (\Psi_i^-)^{-1}1_{(1,1,\ldots,1)}$ for all $1 \leq i \leq N$.

From conditions (11)(a)(b), we have the following two exact triangles

$$(F_{i,1} \star F_{i,0})1_{(1,1,\ldots,1)} \rightarrow (F_{i,0} \star F_{i,1})1_{(1,1,\ldots,1)} \rightarrow \Psi_i^+1_{(1,1,\ldots,1)}, \quad (5.11)$$

$$(\Psi_i \star F_{i,0})1_{(1,1,\ldots,1)} \rightarrow (F_{i,0} \star \Psi_i)1_{(1,1,\ldots,1)} \rightarrow \Psi_i^-1_{(1,1,\ldots,1)}. \quad (5.12)$$
An easy check tells us that
\[(E_{i,0} \ast F_{i,1})\mathbf{1}_{(1,1,\ldots,1)} \cong (E_{i,0} \ast F_{i,0} \ast \Psi_i^+)\mathbf{1}_{(1,1,\ldots,1)} \cong T_i \ast \mathcal{K}_{i+1},\]
\[(F_{i,1} \ast E_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \cong ((\Psi_i^-)^{-1} \ast F_{i,0} \ast E_{i,0})\mathbf{1}_{(1,1,\ldots,1)} \cong \mathcal{K}_i \ast T_i.\]
Thus (5.11) is equivalent to the affine Hecke relation (5.7). A similar check tells us that (5.12) is equivalent to the affine Hecke relation (5.8).

For the rest relations, most of them from (5.1) to (5.6) are easy to verify. The difficult one is the braid relation (5.3). In [Hsu], we prove the braid relation by using relations in (5.12) is equivalent to the affine Hecke relation (5.7).

We use (5.15) and (5.17) to define \(\mathcal{T}_i\mathbf{1}_{(1,1,\ldots,1)}\) and \(\mathcal{T}''_i\mathbf{1}_{(1,1,\ldots,1)}\), respectively. The reason will be explained later. We define
\[\mathcal{T}'_i\mathbf{1}_{(1,1,\ldots,1)} := E_{i,0}F_{i,1}(\Psi_i^+)^{-1}\mathbf{1}_{(1,1,\ldots,1)}.\]
Then we calculate
\[
(T'_i)^2 1_{(1,1,\ldots,1)} = E_{i,0} F_{i,1}(\psi_i^+)^{-1} E_{i,0} F_{i,1}(\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} \\
\approx E_{i,0} F_{i,1} E_{i,-1}(\psi_i^+)^{-1} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)}[1] \\
\approx E_{i,0} (\psi_i)(\psi_i^+)^{-1} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)}[1 - 1] \\
\approx E_{i,0} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} = T'_i 1_{(1,1,\ldots,1)}
\]  
(5.19)

where the first isomorphism comes from condition (8)(a) in Definition 4.12. So \( T'_i 1_{(1,1,\ldots,1)} := E_{i,0} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} \) is our desired projection functor.

On the other hand, we define
\[
T''_i 1_{(1,1,\ldots,1)} := F_{i,0} E_{i,-1}(\psi_i^-)^{-1} 1_{(1,1,\ldots,1)},
\]
(5.20)

and an easy check tells us that it satisfies \((T''_i)^2 1_{(1,1,\ldots,1)} \cong T''_i 1_{(1,1,\ldots,1)}. \) Thus \( T''_i 1_{(1,1,\ldots,1)} \) is also a desired projection functor.

**Remark 5.2.** Note that even though we have the isomorphism (5.10), this does NOT imply that \( E_{i,0} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} \cong F_{i,0} E_{i,-1}(\psi_i^-)^{-1} 1_{(1,1,\ldots,1)} \) since the categorical action now is on an abstract category \( \mathcal{K} \). It holds only on the geometric example \( \mathcal{K}((1,1,\ldots,1)) = D^b(G/B) \).

If we define \( T'_i 1_{(1,1,\ldots,1)} \) or \( T''_i 1_{(1,1,\ldots,1)} \) to be the functor \( E_{i,0} F_{i,0} 1_{(1,1,\ldots,1)} \cong F_{i,0} E_{i,0} 1_{(1,1,\ldots,1)} \), then a simple check gives us that
\[
E_{i,0} F_{i,0} E_{i,0} F_{i,0} 1_{(1,1,\ldots,1)} \cong E_{i,0} (F_{i,0} E_{i,0} 1_{(1,1,\ldots,1)} - \alpha_i) F_{i,0} 1_{(1,1,\ldots,1)} \cong E_{i,0} \psi_i^+ F_{i,0} 1_{(1,1,\ldots,1)}[-1]
\]
which tells us that this is not a suitable definition for the projection functor.

From condition (11)(a)(b) in Definition 4.12, we have the following two exact triangles involving functors
\[
\begin{align*}
F_{i,1} E_{i,0} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} & \to T'_i 1_{(1,1,\ldots,1)} = E_{i,0} F_{i,1} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)} \to 1_{(1,1,\ldots,1)}, \\
E_{i,-1} F_{i,0} (\psi_i^-)^{-1} 1_{(1,1,\ldots,1)} & \to T''_i 1_{(1,1,\ldots,1)} = F_{i,0} E_{i,-1} (\psi_i^-)^{-1} 1_{(1,1,\ldots,1)} \to 1_{(1,1,\ldots,1)}.
\end{align*}
\]
(5.21) (5.22)

After decategorifying, both exact triangles give us elements in the Grothendieck group \( K(\mathcal{K}((1,1,\ldots,1)), \mathcal{K}((1,1,\ldots,1))) \)
\[
\begin{align*}
[F_{i,1} E_{i,0} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)}] & = [T'_i 1_{(1,1,\ldots,1)}] - [1_{(1,1,\ldots,1)}], \\
[E_{i,-1} F_{i,0} (\psi_i^-)^{-1} 1_{(1,1,\ldots,1)}] & = [T''_i 1_{(1,1,\ldots,1)}] - [1_{(1,1,\ldots,1)}].
\end{align*}
\]

We define
\[
\begin{align*}
S'_i 1_{(1,1,\ldots,1)} & := F_{i,1} E_{i,0} (\psi_i^+)^{-1} 1_{(1,1,\ldots,1)}[1], \\
S''_i 1_{(1,1,\ldots,1)} & := E_{i,-1} F_{i,0} (\psi_i^-)^{-1} 1_{(1,1,\ldots,1)}[1],
\end{align*}
\]
(5.23) (5.24)

then the idempotent property of \( T'_i 1_{(1,1,\ldots,1)} \) and \( T''_i 1_{(1,1,\ldots,1)} \) implies that both \( S'_i 1_{(1,1,\ldots,1)} \) and \( S''_i 1_{(1,1,\ldots,1)} \) are also projection functors. Indeed, by using categorical action and similar calculation as (5.19), we can show that
\[
(S'_i)^2 1_{(1,1,\ldots,1)} \cong S'_i 1_{(1,1,\ldots,1)}, \quad (S''_i)^2 1_{(1,1,\ldots,1)} \cong S''_i 1_{(1,1,\ldots,1)}.
\]

Now, although the definitions (5.18) and (5.20) for our projection functors \( T'_i 1_{(1,1,\ldots,1)} \) and \( T''_i 1_{(1,1,\ldots,1)} \) are a bit not so natural, we explain the reason for using (5.15) and (5.17)
to define them. Note that by condition (4) in Definition 4.2, $E$ and $F_i$, $s$ both have left and right adjoints. Using conditions (8)(a) and (9)(a) in Definition 4.2 and a simple check tells us that

$$
E_{i,0}1_{(1,1,\ldots,1)} - \alpha_i (E_{i,0}1_{(1,1,\ldots,1)} - \alpha_i)^R \cong E_{i,0}F_i,1(\psi^+_i)^{-1}1_{(1,1,\ldots,1)},
$$

$$
F_{i,0}1_{(1,1,\ldots,1)} + \alpha_i (F_{i,0}1_{(1,1,\ldots,1)} + \alpha_i)^R \cong F_{i,0}E_{i,1}(\psi^-_i)^{-1}1_{(1,1,\ldots,1)}.
$$

Thus we have

$$
T'_i1_{(1,1,\ldots,1)} \cong E_{i,0}1_{(1,1,\ldots,1)} - \alpha_i (E_{i,0}1_{(1,1,\ldots,1)} - \alpha_i)^R, 
$$

$$
T''_i1_{(1,1,\ldots,1)} \cong F_{i,0}1_{(1,1,\ldots,1)} + \alpha_i (F_{i,0}1_{(1,1,\ldots,1)} + \alpha_i)^R. 
$$

Similarly, the definition of $S'_i1_{(1,1,\ldots,1)}$ and $S''_i1_{(1,1,\ldots,1)}$ also comes from such construction, i.e.

$$
S'_i1_{(1,1,\ldots,1)} := F_{i,0}E_{i,0}(\psi^+_i)^{-1}1_{(1,1,\ldots,1)}[1] \cong \psi^+_i (E_{i,0}1_{(1,1,\ldots,1)})^RE_{i,0}(\psi^+_i)^{-1}1_{(1,1,\ldots,1)}, 
$$

$$
S''_i1_{(1,1,\ldots,1)} := E_{i,1}F_{i,0}(\psi^-_i)^{-1}1_{(1,1,\ldots,1)}[1] \cong \psi^-_i (F_{i,0}1_{(1,1,\ldots,1)})^RF_{i,0}(\psi^-_i)^{-1}1_{(1,1,\ldots,1)}. 
$$

So the naturality of the definition comes from the construction of composition of functors with their left/right adjoints.

5.2.2. The general case $n < N$. Now, we move to the general case (in the geometric example, from the full flag variety to the $n$-step partial flag varieties); i.e., we have an abstract categorical $\mathcal{U}_0N(L\mathfrak{s}l_n)$ action on $\mathcal{K}$ with $n < N$. Like the study for the full flag variety case, we have the following picture like (5.13).

$$
\begin{array}{c}
\mathcal{K}(\bar{k} - \alpha_i) \xleftarrow{E_{i,r}} \mathcal{K}(\bar{k}) \xrightarrow{F_{i,s}} \mathcal{K}(\bar{k} + \alpha_i) \\
\end{array}
$$

To construct projection functors $T'_i1_{\bar{k}}$ and $T''_i1_{\bar{k}}$ for $\mathcal{K}(\bar{k})$, since $E_{i,r}$ and $F_{i,s}$ both admit left/right adjoints, we imitate the natural construction of the projection functors $T'_i1_{(1,1,\ldots,1)}$ and $T''_i1_{(1,1,\ldots,1)}$ for $\mathcal{K}(1,1,\ldots,1)$ in (5.25) and (5.26) and generalize to the following

$$
E_{i,0}1_{\bar{k} - \alpha_i} (E_{i,0}1_{\bar{k} - \alpha_i})^R \cong E_{i,0}F_{i,k_{i+1}}(\psi^+_i)^{-1}1_{\bar{k}},
$$

$$
F_{i,0}1_{\bar{k} + \alpha_i} (F_{i,0}1_{\bar{k} + \alpha_i})^R \cong F_{i,0}E_{i,k_{i+1}}(\psi^-_i)^{-1}1_{\bar{k}}.
$$

Thus we define

$$
T'_i1_{\bar{k}} := E_{i,0}F_{i,k_{i+1}}(\psi^+_i)^{-1}1_{\bar{k}}, 
$$

$$
T''_i1_{\bar{k}} := F_{i,0}E_{i,k_{i+1}}(\psi^-_i)^{-1}1_{\bar{k}}. 
$$

Similarly, from we define

$$
S'_i1_{\bar{k}} := F_{i,k_{i+1}}E_{i,0}(\psi^+_i)^{-1}1_{\bar{k}}[1] \cong \psi^+_i (E_{i,0}1_{\bar{k}})^RE_{i,0}(\psi^+_i)^{-1}1_{\bar{k}},
$$

$$
S''_i1_{\bar{k}} := E_{i,-k_{i}}F_{i,0}(\psi^-_i)^{-1}1_{\bar{k}}[1] \cong \psi^-_i (F_{i,0}1_{\bar{k}})^RF_{i,0}(\psi^-_i)^{-1}1_{\bar{k}}.
$$

The main result of this article (see Theorem 5.3 below) is to study the four functors $T'_i1_{\bar{k}}$, $T''_i1_{\bar{k}}$, $S'_i1_{\bar{k}}$, and $S''_i1_{\bar{k}}$ defined from (5.29) to (5.32) acting on the abstract category $\mathcal{K}(\bar{k})$. 
Before we state the main result, we discuss the similarity between the constructions of the four functors and spherical twist/co-twist. Note that from conditions (11)(a)(b) in Definition 4.2, we have the following two exact triangles like (5.21) and (5.22)

\[ T'_i 1_k = E_{i,0} F_{i,k+1} (\psi^+)^{-1} 1_k \to 1_k \to S'_i 1_k = F_{i,k+1} E_{i,0} (\psi^+)^{-1} 1_k[1], \quad (5.33) \]
\[ T''_i 1_k = F_{i,0} E_{i,-k} (\psi^-)^{-1} 1_k \to 1_k \to S''_i 1_k = E_{i,-k} F_{i,0} (\psi^-)^{-1} 1_k[1]. \quad (5.34) \]

Using the adjoint conditions (4)(a), (4)(c) in Definition 4.2, we calculate the following Hom spaces of natural transformations

\[
\text{Hom}(E_{i,0} 1_k, E_{i,0} 1_k) \cong \text{Hom}(1_k, (E_{i,0} 1_k)^R E_{i,0} 1_k) \cong \text{Hom}(1_k, \psi^+ F_{i,k+1} + 2(\psi^+)^{-2} E_{i,0} 1_k[-1]) \\
\cong \text{Hom}(1_k, (\psi^+)^{-1} F_{i,k+1} E_{i,0} 1_k[1]),
\]
\[
\text{Hom}(E_{i,0} 1_{k-\alpha_i}, E_{i,0} 1_{k-\alpha_i}) \cong \text{Hom}(1_k, E_{i,0} (E_{i,0} 1_{k-\alpha_i})^R) \cong \text{Hom}(E_{i,0} \psi^+ F_{i,k+1} - 1 1_k[-1], 1_k) \\
\cong \text{Hom}(E_{i,0} F_{i,k+1} (\psi^+)^{-1} 1_k[1], 1_k),
\]
\[
\text{Hom}(F_{i,0} 1_k, F_{i,0} 1_k) \cong \text{Hom}(1_k, (F_{i,0} 1_k)^R F_{i,0} 1_k) \cong \text{Hom}(1_k, \psi^- E_{i,-k-2} (\psi^-)^{-2} F_{i,0} 1_k[-1]) \\
\cong \text{Hom}(1_k, (\psi^-)^{-1} E_{i,-k} F_{i,0} 1_k[1]),
\]
\[
\text{Hom}(F_{i,0} 1_{k+\alpha_i}, F_{i,0} 1_{k+\alpha_i}) \cong \text{Hom}(1_k, (F_{i,0} 1_{k+\alpha_i})^R) \cong \text{Hom}(F_{i,0} \psi^- E_{i,-k-1} (\psi^-)^{-2} 1_k[-1], 1_k) \\
\cong \text{Hom}(F_{i,0} E_{i,-k} (\psi^-)^{-1} 1_k, 1_k). \quad (5.35)
\]

Then we can get the unit and counit natural transformations

\[
\eta : 1_k \to \psi^+ (E_{i,0} 1_k)^R E_{i,0} (\psi^+)^{-1} 1_k \cong F_{i,k+1} E_{i,0} (\psi^+)^{-1} 1_k[1] = S'_i 1_k
\]
\[
\eta' : 1_k \to \psi^- (F_{i,0} 1_k)^R F_{i,0} (\psi^-)^{-1} 1_k \cong E_{i,-k} F_{i,0} (\psi^-)^{-1} 1_k[1] = S''_i 1_k
\]
\[
\epsilon : E_{i,0} (E_{i,0} 1_{k-\alpha_i})^R 1_k \cong E_{i,0} F_{i,k+1} (\psi^+)^{-1} 1_k = T'_i 1_k \to 1_k
\]
\[
\epsilon' : F_{i,0} (F_{i,0} 1_{k+\alpha_i})^R 1_k \cong F_{i,0} E_{i,-k} (\psi^-)^{-1} 1_k = T''_i 1_k \to 1_k
\]

that correspond to the identity natural transformations under the above adjunctions (5.35).

We expect that the morphisms for those exact triangles in (5.33) and (5.34) should come from the units and counits \(\epsilon, \epsilon', \delta, \delta'\). We would like to address this in the future when it comes to the study of the higher categorical structure of \(\mathcal{U}_{0,N}(Lsl_n)\).
Assume it is true from now, i.e., the morphisms in those exact triangles really comes from the units and counits. Then we obtain

\[ T'_1 \mathbf{k} := E_{i,0} F_{i, k_1+1} (\Psi^+)^{-1} \mathbf{k} = \text{Cone}(1_{\mathbf{k}} \to F_{i, k_1+1} E_{i,0} (\Psi^+)^{-1} \mathbf{k}[1][1][-1] \]
\[ = \text{Cone}(\mathbf{k} \to \Psi^+ E_{i,0} \mathbf{k} R E_{i,0} (\Psi^+)^{-1} \mathbf{k}[1][-1]) = \text{Cone}(\eta)[1], \]
\[ T''_1 \mathbf{k} := F_{i,0} E_{i,-k_1} (\Psi^-)^{-1} \mathbf{k} = \text{Cone}(1_{\mathbf{k}} \to E_{i,-k_1} F_{i,0} (\Psi^-)^{-1} \mathbf{k}[1][1][-1] \]
\[ = \text{Cone}(\mathbf{k} \to \Psi^- F_{i,0} \mathbf{k} R F_{i,0} (\Psi^-)^{-1} \mathbf{k}[1][-1]) = \text{Cone}(\eta'')[-1], \]
\[ S'_1 \mathbf{k} := F_{i, k_1+1} E_{i,0} (\Psi^+)^{-1} \mathbf{k}[1] = \text{Cone}(E_{i,0} F_{i, k_1+1} (\Psi^+)^{-1} \mathbf{k} \to \mathbf{k}) \]
\[ = \text{Cone}(1_{\mathbf{k}} E_{i,0} (E_{i,0} \mathbf{k} - \alpha_i)^R \to \mathbf{k}) = \text{Cone}(\epsilon), \]
\[ S''_1 \mathbf{k} := E_{i,-k_1} F_{i,0} (\Psi^-)^{-1} \mathbf{k}[1] = \text{Cone}(F_{i,0} E_{i,-k_1} (\Psi^-)^{-1} \mathbf{k} \to \mathbf{k}) \]
\[ = \text{Cone}(1_{\mathbf{k}} F_{i,0} (F_{i,0} \mathbf{k} + \alpha_i)^R \to \mathbf{k}) = \text{Cone}(\epsilon'). \]

The above discussion somehow tells us that \( T'_1 \mathbf{k}, T''_1 \mathbf{k}, S'_1 \mathbf{k} \) and \( S''_1 \mathbf{k} \) look like spherical twist/co-twist functors that developed in [A], [AT] and [ST]. However, they are not spherical twist/co-twist functors since we will prove that all of them are idempotent in the following theorem. The following is the main result of this article.

**Theorem 5.3.** Given a categorical \( \mathcal{U}_0 N(L\mathfrak{sl}_h) \) action on \( K \). Then the 4 functors \( T'_1 \mathbf{k}, S'_1 \mathbf{k}, T''_1 \mathbf{k}, S''_1 \mathbf{k} \) defined by (5.29), (5.30), (5.31), (5.32) satisfy the following properties.

1. There exist exact triangles in \( \text{Hom}(K(\mathbf{k}), K(\mathbf{k})) \)

\[ T'_1 \mathbf{k} \to \mathbf{k} \to S'_1 \mathbf{k}, \]
\[ T''_1 \mathbf{k} \to \mathbf{k} \to S''_1 \mathbf{k}. \]

2. All of them are projection functors; more precisely we have

\[ (T'_1)^2 \mathbf{k} \cong T'_1 \mathbf{k}, \quad (T''_1)^2 \mathbf{k} \cong T''_1 \mathbf{k}, \]
\[ (S'_1)^2 \mathbf{k} \cong S'_1 \mathbf{k}, \quad (S''_1)^2 \mathbf{k} \cong S''_1 \mathbf{k}. \]

3. Mutually orthogonal property

\[ T'_1 S'_1 \mathbf{k} \cong S'_1 T'_1 \mathbf{k} \cong 0, \]
\[ T''_1 S''_1 \mathbf{k} \cong S''_1 T''_1 \mathbf{k} \cong 0. \]

4. For \( |i - j| \geq 2 \), we have

\[ T'_1 T'_1 \mathbf{k} \cong T'_1 T''_1 \mathbf{k}, \quad S'_1 S'_1 \mathbf{k} \cong S'_1 S''_1 \mathbf{k}, \]
\[ T''_1 T''_1 \mathbf{k} \cong T''_1 T''_1 \mathbf{k}, \quad S''_1 S''_1 \mathbf{k} \cong S''_1 S''_1 \mathbf{k}. \]
(5) We have the following vanishing results
\[ S_i''T_i''S_i''1_k = S_i''T_i''S_i''1_k = T_i''S_i''1_k = S_i''T_i''S_i''1_k = 0, \]
\[ T_i''S_i''1_k = T_i''S_i''1_k = S_i''T_i''S_i''1_k = 0. \]

(6) We have the following exact triangles in \( \text{Hom}(K(k), K(k)) \):
\[ T_i''1_k = T_i''1_k \rightarrow T_i''1_k \rightarrow T_i''1_k \rightarrow T_i''1_k. \]
\[ S_i''1_k = S_i''1_k \rightarrow S_i''1_k \rightarrow S_i''1_k \rightarrow S_i''1_k. \]
\[ T_i''1_k = T_i''1_k \rightarrow T_i''1_k \rightarrow T_i''1_k \rightarrow T_i''1_k. \]
\[ S_i''1_k = S_i''1_k \rightarrow S_i''1_k \rightarrow S_i''1_k \rightarrow S_i''1_k. \]

In particular, this implies that the weight category \( K(k) \) carries actions of \( H_i'(0) \) and \( H_i''(0) \) where the generators \( \delta_i' \in H_i'(0), \delta_i'' \in H_i''(0) \) act on \( K(k) \) via \( T_i''1_k \), \( T_i''1_k \) respectively.

**Proof.** Before we give the proof, we want to mention that there are many cases to prove for each property. However, the argument to prove them are all similar. Thus we will only prove one or two of them and leave the rest to the readers. Also, since we will keep using Definition 4.2 to help us prove the properties, when we mention the conditions, we always mean those from Definition 4.2.

For (1), this is just a direct consequence of conditions (11)(a)(b). For (4), it is a direct consequence of conditions (6)(c), (7)(c), (8)(c), (9)(c), and (10).

For (2), we prove the cases for \( T_i''1_k \) and \( S_i''1_k \). By condition (8)(a), we have
\[ (T_i''1_k)^21_k = E_{i,0}F_{i,k_{i+1}}(\Psi_i^+)^{-1}E_{i,0}F_{i,k_{i+1}}(\Psi_i^+)^{-1}1_k = E_{i,0}F_{i,k_{i+1}}E_{i,-1}1_k = E_{i,0}F_{i,k_{i+1}}E_{i,-1}1_k. \]

From condition (11)(a), we have the following exact triangle
\[ F_{i,k_{i+1}}E_{i,-1}1_k \rightarrow \psi_{i,1_k} \rightarrow \psi_{i,1_k}, \]
so we obtain the following exact triangle
\[ (T_i''1_k)^21_k = E_{i,0}F_{i,k_{i+1}}E_{i,-1}1_k \rightarrow \psi_{i,1_k} \rightarrow \psi_{i,1_k} \rightarrow (T_i''1_k)^21_k. \]

By condition (6)(a), we get \( E_{i,0}E_{i,-1}F_{i,k_{i+1}}1_k = (\psi_{i}^+)^{-1}F_{i,k_{i+1}}1_k = (\psi_{i}^+)^{-1}1_k \). This implies that \( (T_i''1_k)^21_k \approx 0. \)

Similarly, by condition (9)(a) we have
\[ (S_i''1_k)^21_k = F_{i,k_{i+1}}E_{i,0}(\Psi_i^+)^{-1}F_{i,k_{i+1}}E_{i,0}(\Psi_i^+)^{-1}1_k = F_{i,k_{i+1}}E_{i,0}F_{i,k_{i+1}}1_k \rightarrow \psi_{i,1_k} \rightarrow \psi_{i,1_k} \rightarrow (S_i''1_k)^21_k. \]

From condition (11)(a), we have the following exact triangle
\[ F_{i,k_{i+1}}E_{i,0}1_k \rightarrow \psi_{i,1_k} \rightarrow \psi_{i,1_k}. \]
so we obtain the following exact triangle

\[ F_{i,k_{i+1}+1} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} E_{i,0}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \to (S'_{i+1})^2 1_{\mathbb{A}} \cong F_{i,k_{i+1}+1} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} E_{i,0}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \]

\[ \to F_{i,k_{i+1}+1} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} E_{i,0}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong F_{i,k_{i+1}+1} E_{i,0}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] = S'_{i+1} 1_{\mathbb{A}}. \]

By condition (7)(a), we get

\[ F_{i,k_{i+1}+1} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} E_{i,0}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong 0, \quad \text{so} \quad \text{(5.36).} \]

Next, we only prove \( T'_i S'_{i+1} 1_{\mathbb{A}} \cong S'_{i+1} T'_i 1_{\mathbb{A}} \cong 0 \) in (3). By conditions (7)(a) and (9)(a) we have

\[ T'_i S'_{i+1} 1_{\mathbb{A}} \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong 0, \]

similarly by conditions (6)(a) and (8)(a) we have

\[ S'_{i+1} T'_i 1_{\mathbb{A}} \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong 0. \]

Next, we prove \( S'_{i+1} S'_{i+1} T'_i 1_{\mathbb{A}} \cong 0 \) and \( T'_i T'_{i+1} S'_{i+1} 1_{\mathbb{A}} \cong 0 \) for (4). By definition and conditions (8)(a)(b), (9)(a)(b), we have

\[ S'_{i+1} T'_i 1_{\mathbb{A}} \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong 0, \]

\[ T'_i T'_{i+1} 1_{\mathbb{A}} \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong E_{i,0} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i}(\psi_i^+)^{-1} 1_{\mathbb{A}}[1] \cong 0. \]

By condition (10), \( (5.36) \) becomes

\[ F_{i+1,k_{i+2}} F_{i,k_{i+1}} E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \]

\[ \text{applying } E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \text{ and using condition (6)(a), we get} \]

\[ E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \]

\[ \text{in particular, for } r = 0, \text{ we have } E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong 0 \text{ (by condition (6)(a))} \]

\[ \text{so } (5.37) \text{ equal to } 0 \text{ and thus } S'_{i+1} S'_{i+1} T'_i 1_{\mathbb{A}} \cong 0. \]

Similarly, by definition and conditions (8)(a)(b), (9)(a)(b), and (10)

\[ T'_i T'_{i+1} S'_{i+1} 1_{\mathbb{A}} \]

\[ \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \cong E_{i,0} 1_{\mathbb{A}+\alpha_i+1} \]

\[ \text{so } (5.37) \text{ equal to } 0 \text{ and thus } S'_{i+1} S'_{i+1} T'_i 1_{\mathbb{A}} \cong 0. \]
From condition (7)(b) with \( s = k_{i+2} - 1, r = k_{i+1} \), we have the following exact triangle
\[
F_{i,k_{i+1}+1}F_{i+1,k_{i+2}-1}1_{k_{i+1}+\alpha_i} \rightarrow F_{i,k_{i+1}}F_{i+1,k_{i+2}}1_{k_{i+1}+\alpha_i} \rightarrow F_{i+1,k_{i+2}}F_{i,k_{i+1}}1_{k_{i+1}+\alpha_i},
\]
applying \( F_{i,k_{i+1}+1}1_{k_{i+1}+\alpha_i+1} \) and \( F_{i+1,k_{i+2}}1_{k_{i+1}+\alpha_i+1} \) we get
\[
F_{i,k_{i+1}+1}F_{i+1,k_{i+2}-1}1_{k_{i+1}+\alpha_i+1} \rightarrow F_{i,k_{i+1}}F_{i+1,k_{i+2}}1_{k_{i+1}+\alpha_i+1} \rightarrow F_{i+1,k_{i+2}}F_{i,k_{i+1}}1_{k_{i+1}+\alpha_i+1}.
\]

Using condition (7)(a), we get
\[
F_{i,k_{i+1}+1}F_{i+1,k_{i+2}-1}1_{k_{i+1}+\alpha_i+1} \Rightarrow 0,
\]
so \( F_{i,k_{i+1}+1}F_{i+1,k_{i+2}}1_{k_{i+1}+\alpha_i+1} \Rightarrow 0 \) and (5.38) equal to 0. Thus
\[
T_1^{\prime}T_1^{\prime\prime}S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

Using property (1) and (3), we can show that
\[
S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

From (4), we have \( S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0 \). Applying \( T_1^{\prime}T_1^{\prime\prime}1_{k_{i+1}} \) to the exact triangle
\[
T_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 1_{k_{i+1}} \Rightarrow S_1^{\prime\prime}1_{k_{i+1}},
\]
we get
\[
T_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 1_{k_{i+1}} \Rightarrow S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

Finally, we prove there exists exact triangle
\[
T_1^{\prime\prime}T_1^{\prime\prime\prime}1_{k_{i+1}} \Rightarrow T_1^{\prime\prime}T_1^{\prime\prime\prime}1_{k_{i+1}} \Rightarrow T_1^{\prime\prime}T_1^{\prime\prime\prime}S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

which implies that
\[
T_1^{\prime\prime}T_1^{\prime\prime\prime}1_{k_{i+1}} \Rightarrow T_1^{\prime\prime}T_1^{\prime\prime\prime}S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

Thus (5.39) becomes
\[
T_1^{\prime\prime}T_1^{\prime\prime\prime}1_{k_{i+1}} \Rightarrow T_1^{\prime\prime}T_1^{\prime\prime\prime}T_1^{\prime\prime\prime}1_{k_{i+1}} \Rightarrow T_1^{\prime\prime}T_1^{\prime\prime\prime}T_1^{\prime\prime\prime}S_1^{\prime\prime}1_{k_{i+1}} \Rightarrow 0.
\]

We state some corollaries of this theorem. The first is the application to the geometric setting where the weight categories are derived categories of coherent sheaves on partial flag varieties.

**Corollary 5.4.** We have categorical actions of \( H_1^\prime(0) \) and \( H_1^\prime(0) \) on \( \mathcal{D}(F_{i,k_{i+1}}(C)) \) for all \( k_{i+1} \), where the generators \( \delta_1, \delta_2' \) act on \( \mathcal{D}(F_{i,k_{i+1}}(C)) \) by FM transforms with the following FM kernels
\[
T_1^{\prime}1_{k_{i+1}} := E_{i,0} \ast F_{i,k_{i+1}} \ast (\Psi_1^+)^{-1} 1_{k_{i+1}} \in \mathcal{D}(F_{i,k_{i+1}}(C)),
\]
\[
S_1^{\prime}1_{k_{i+1}} := E_{i,0} \ast F_{i,k_{i+1}} \ast (\Psi_1^+)^{-1} 1_{k_{i+1}} \in \mathcal{D}(F_{i,k_{i+1}}(C)),
\]
respectively and \( 1_{k_{i+1}} - \delta_1, 1_{k_{i+1}} - \delta_2' \) act by FM transforms with the following FM kernels
\[
T_1^{\prime\prime}1_{k_{i+1}} := E_{i,0} \ast F_{i,k_{i+1}} \ast (\Psi_1^+)^{-1} 1_{k_{i+1}} \in \mathcal{D}(F_{i,k_{i+1}}(C)),
\]
\[
S_1^{\prime\prime}1_{k_{i+1}} := E_{i,0} \ast F_{i,k_{i+1}} \ast (\Psi_1^+)^{-1} 1_{k_{i+1}} \in \mathcal{D}(F_{i,k_{i+1}}(C)),
\]
respectively. More precisely, under the operation of convolution, the FM kernels $T'_i 1_{\underline{k}}$, $T''_i 1_{\underline{k}}$, $S'_i 1_{\underline{k}}$, and $S''_i 1_{\underline{k}}$ satisfy properties (1) to (6) in Theorem 5.3.

**Proof.** This is a direct consequence of Theorem 4.3 and Theorem 5.3. □

Next, we decategorify the above results and pass it to the Grothendieck group $\mathcal{K}(Fl_{\underline{k}}(\mathbb{C}^N))$. This means that we have an action of the shifted $q = 0$ affine algebra $\mathcal{U}_{0,N}(\mathcal{L}m_{\underline{n}})$ on $\bigoplus_{\underline{k}} \mathcal{K}(Fl_{\underline{k}}(\mathbb{C}^N))$. For example, the generators $e_{i,r} 1_{\underline{k}} : K(Fl_{\underline{k}}(\mathbb{C}^N)) \to K(Fl_{\underline{k}+\alpha_i}(\mathbb{C}^N))$ act by the following correspondence

$$
\begin{array}{c}
W^1_1(k) \\
F_{\underline{k}}(\mathbb{C}^N) \\
F_{\underline{k}+\alpha_i}(\mathbb{C}^N)
\end{array}
\xleftarrow{p_1} \xrightarrow{p_2}
$$

where $e_{i,r} 1_{\underline{k}} = p_2*(p_1^* (|V_i/V_i'|)^r)$ and $W^1_1(k)$ is defined in (4.3). Similarly for other generators. Thus we obtain

**Corollary 5.5.** There are actions of $H'_n(0)$ and $H''_n(0)$ on $K(Fl_{\underline{k}}(\mathbb{C}^N))$ for all $\underline{k}$, where the generators $\delta'_i$ and $\delta''_i$ act on $K(Fl_{\underline{k}}(\mathbb{C}^N))$ by the following operators $e_{i,0} f_{i,k,i+1} (\psi_i^+)^{-1} 1_{\underline{k}}$ and $f_{i,0} e_{i,-k,i} (\psi_i^-)^{-1} 1_{\underline{k}}$, respectively. More precisely, we have $\delta'_i$ and $\delta''_i$ satisfy the relations in Definition 3.7 and 3.8, respectively.

**Proof.** Since $S'_i 1_{\underline{k}}$ and $S''_i 1_{\underline{k}}$ are FM kernels for the lifting of $1_{\underline{k}} - \delta'_i$ and $1_{\underline{k}} - \delta''_i$ respectively, the relations can be easily seen from Theorem 5.3 and Corollary 5.3. □

Finally, we formulate a conjecture which says that the actions we constructed in Corollaries 5.3, 5.4, 5.5 indeed satisfy the braid relations.

**Conjecture 5.6.** The actions of $H'_n(0)$ and $H''_n(0)$ in Corollary 5.3, 5.4, 5.5 are in fact the actions of the $q = 0$ Hecke algebra $H_n(0)$; i.e. the braid relations are satisfy. This is equivalent to the vanishing of the correction terms $\mathcal{T}'_{i+1} * \mathcal{T}'_i * \mathcal{T}'_{i+1} * \mathcal{S}'_i 1_{\underline{k}}$ and $\mathcal{T}''_{i+1} * \mathcal{T}''_i * \mathcal{T}''_{i+1} * \mathcal{S}''_i 1_{\underline{k}}$.

**Remark 5.7.** We give another proof of the categorical braid relations, i.e. (5.3) in Theorem 5.1 for the action of the $q = 0$ Hecke algebra.

For the full flag variety case $G/B = Fl_{(1,1,...,1)}(\mathbb{C}^N)$, we have

$$
\mathcal{T}'_i 1_{(1,1,...,1)} = \mathcal{E}_{i,0} * \mathcal{F}_{i,1} * (\psi_i^+)^{-1} 1_{(1,1,...,1)} = \mathcal{F}_{i,0} * \mathcal{E}_{i,1} * (\psi_i^-)^{-1} 1_{(1,1,...,1)} = \mathcal{T}''_i 1_{(1,1,...,1)}.
$$

Since $\mathcal{T}'_i 1_{(1,1,...,1)} = \mathcal{T}''_i 1_{(1,1,...,1)} = \mathcal{T}_i 1_{(1,1,...,1)}$, by properties (1) and (5) in Theorem 5.3, we obtain that $\mathcal{S}'_i 1_{(1,1,...,1)} = \mathcal{S}''_i 1_{(1,1,...,1)}$ and

$$
\mathcal{T}'_{i+1} * \mathcal{T}'_i * \mathcal{T}'_{i+1} * \mathcal{S}'_i 1_{\underline{k}} = \mathcal{T}''_{i+1} * \mathcal{T}''_i * \mathcal{T}''_{i+1} * \mathcal{S}''_i 1_{(1,1,...,1)} = 0,
$$

which tells us that the correction terms in (6) are zero and implies the braid relations in the $q = 0$ Hecke algebra $H_N(0)$. 

In this section, we study more about the action of the Demazure operators defined in Section 5 on the Grothendieck group of partial flag variety $K(Fl_k(\mathbb{C}^N))$. In particular, since there is a natural basis on $K(Fl_k(\mathbb{C}^N))$ given by Kapranov exceptional collection, we calculate the action of those Demazure operators on them.

6.1. The Kapranov exceptional collection. In this section, we recall the exceptional collection which constructed by Kapranov \cite{Ka1}, \cite{Ka2} for $D^b(Fl_k(\mathbb{C}^N))$.

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a non-increasing sequence of positive integers. We can represent $\lambda$ as a Young diagram with $n$ rows, aligned on the left, such that the $i$th row has exactly $\lambda_i$ cells. The size of $\lambda$, denoted by $|\lambda|$, is the number $|\lambda| = \sum_{i=1}^{n} \lambda_i$. The transpose diagram $\lambda^*$ is obtained by exchanging rows and columns of $\lambda$.

For a Young diagram $\lambda$, we define the notion of its associated Schur functor. For more details about Schur functors, we refer the readers to Chapter 4 and chapter 6 \cite{FH}.

Definition 6.1. Let $n \geq 1$ be a positive integer and $\lambda = (\lambda_1, ..., \lambda_n)$ be a sequence of non-increasing positive integers. The Schur functor $S_\lambda$ associated to $\lambda$ is defined as a functor $S_\lambda : \text{Vect}_\mathbb{C} \to \text{Vect}_\mathbb{C}$ such that for any vector space $V$, $S_\lambda V$ coincides with the image of the Young symmetrizer $c_\lambda$ in the space of tensors of $V$ of rank $n$: i.e., $S_\lambda V = \text{Im}(c_{\lambda}|_{V^{\otimes n}})$.

First, for the case where $n = 2$, the partial flag varieties are the Grassmannians $G(k, N) = \{0 \subset V \subset \mathbb{C}^N \mid \dim_\mathbb{C} V = k\}$ of $k$-dimensional subspaces in $\mathbb{C}^N$.

Denote $V$ to be the tautological rank $k$ bundle on $G(k, N)$ and $\mathbb{C}^N/V$ to be the tautological rank $N-k$ quotient bundle. For non-negative integers $a$, $b \geq 0$, we denote by $P(a, b)$ the set of Young diagrams $\lambda$ such that $\lambda_1 \leq a$ and $\lambda_{b+1} = 0$. Then we have the following theorem.

Theorem 6.2 (\cite{Ka1}). The following collection of sheaves $$R_{(k,N-k)} = \{S_\lambda V \mid \lambda \in P(N-k,k)\}$$ is a strong full exceptional collection in $D^b(G(k,N))$. Its dual exceptional collection is given by $$R'_{(k,N-k)} = \{S_\mu \mathbb{C}^N/V[-|\mu|] \mid \mu \in P(k,N-k)\}.$$ This can be generalized to the partial flag variety $Fl_k(\mathbb{C}^N)$, which is the following theorem.

Theorem 6.3 (\cite{Ka2}). The following collection of sheaves $$R_k = \{ \bigotimes_{i=1}^{n-1} S_{\lambda_i} V_i \mid \lambda_i \in P(k_{i+1} + \ldots + k_i) \text{ for } 1 \leq i \leq n-1\}$$ is a strong full exceptional collection in $D^b(Fl_k(\mathbb{C}^N))$. 
Remark 6.4. Note that for the partial flag varieties, we do not have the dual exceptional collection in Theorem 6.2. It was pointed out in [Ka2] that there are objects satisfy the Definition 2.4 for dual exceptional collection; however, they do have higher Ext’s between collection in Theorem 6.2. It was pointed out in [Ka2] that there are objects satisfy the Definition 2.4 for dual exceptional collection; however, they do have higher Ext’s between themselves.

6.2. The action. From Theorem 6.3 we have the full exceptional collection \( R_\mathbb{K} \) for \( \mathcal{D}^b(Fl_\mathbb{K}(\mathbb{C}^N)) \). Thus we obtain a basis \([R_\mathbb{K}]\) for \( K(Fl_\mathbb{K}(\mathbb{C}^N)) \). Moreover, when \( n = 2 \) and the n-step partial flag varieties are the Grassmannians, Theorem 6.2 tells us that we have a basis \([R_{(k,N-k)}]\) for \( K(G(k,N)) \) and its dual basis \([R'_{(k,N-k)}]\) with respect to the bilinear Euler form \( \sum_{i\in \mathbb{Z}} (-1)^i \text{dim}_\mathbb{C} \text{Ext}^i(\cdot,\cdot) \).

Since we have actions of \( H_\eta^*(0) \) and \( H_\eta^*(0) \) on \( K(Fl_\mathbb{K}(\mathbb{C}^N)) \) from Corollary 5.5 it is natural to calculate the action of the generators on the basis \([R_\mathbb{K}]\). Note that the generators \( \delta'_i, \delta''_i \) act on \( K(Fl_\mathbb{K}(\mathbb{C}^N)) \) by

\[
\delta'_i = e_{i,0} f_{i,k_i+1}(\psi_i^+)^{-1} 1_{\mathbb{K}}, \quad (6.1) \\
\delta''_i = f_{i,0} e_{i,-k_i}(\psi_i^-)^{-1} 1_{\mathbb{K}}, \quad (6.2)
\]

where \( \delta'_i = [T'_i 1_{\mathbb{K}}] \) and \( \delta''_i = [T''_i 1_{\mathbb{K}}] \) are the decategorification of the functors \( T'_i 1_{\mathbb{K}} \) and \( T''_i 1_{\mathbb{K}} \) that given by the FM kernels \( T'_i 1_{\mathbb{K}} = \mathcal{E}_{i,0} \ast \mathcal{F}_{i,k_i+1} \ast (\psi_i^+)^{-1} 1_{\mathbb{K}} \) and \( T''_i 1_{\mathbb{K}} = \mathcal{F}_{i,0} \ast \mathcal{E}_{i,-k_i} \ast (\psi_i^-)^{-1} 1_{\mathbb{K}} \) respectively. By Theorem 5.3 we have \( f_{i,k_i+1} e_{i,0}(\psi_i^+)^{-1} = \delta'_i \) and \( e_{i,-k_i} f_{i,0}(\psi_i^-)^{-1} = \delta''_i \). Thus it suffices to know the actions of \( \delta'_i \) and \( \delta''_i \).

To calculate the actions of \( \delta'_i \) and \( \delta''_i \), since we have a categorical action \( \mathcal{U}_{0,N}(\text{Lat}_\eta) \) on \( \bigoplus_\mathbb{K} \mathcal{D}^b(Fl_\mathbb{K}(\mathbb{C}^N)) \) by Theorem 4.3, the key observation we use is to interpret the Kapranov exceptional collection in terms of the categorical action via convolution of FM kernels. More precisely, by using the Borel-Weil-Bott Theorem, in the Grassmannians case, we have

**Lemma 6.5.** Let \( \lambda = (\lambda_1, ..., \lambda_k) \in P(N-k,k) \) and \( \mu = (\mu_1, ..., \mu_{N-k}) \in P(k,N-k) \). Then

\[
\mathcal{S}_\lambda \mathcal{V} \cong \mathcal{F}_{\lambda_1} \ast \cdots \ast \mathcal{F}_{\lambda_k} 1_{(0,N)} \in \mathcal{D}^b(\mathbb{G}(0,N) \times \mathbb{G}(k,N)), \\
\mathcal{S}_\mu (\mathbb{C}^N/\mathcal{V})^\vee \cong \mathcal{E}_{-\mu_1} \ast \cdots \ast \mathcal{E}_{-\mu_{N-k}} 1_{(N,0)} \in \mathcal{D}^b(\mathbb{G}(N,N) \times \mathbb{G}(k,N)).
\]

This can be generalized to the n-step partial flag varieties case, which is the following.

**Lemma 6.6.** Let \( \lambda_i = (\lambda_{i,1}, ..., \lambda_{i,k_i+\ldots+k_i}) \in P(k_{i+1}, k_{i+\ldots+k_i}) \) for all \( 1 \leq i \leq n-1 \). Then we have

\[
\bigotimes_{i=1}^{n-1} \mathbb{S}_{\lambda_i} \mathcal{V}_i \cong \mathcal{F}_{i,\lambda_{i_1}} \ast \cdots \ast \mathcal{F}_{i,\lambda_{i_{k_i}}} 1_{\eta} \in \mathcal{D}^b(Fl_\mathbb{K}(\mathbb{C}^N) \times Fl_\mathbb{K}(\mathbb{C}^N))
\]

where \( \mathcal{F}_{i,\lambda_i} := \mathcal{F}_{i,\lambda_{i_1}} \ast \cdots \ast \mathcal{F}_{i,\lambda_{i_{k_i}}} \) and \( \eta := (0,0, \ldots, 0, N) \) is the highest weight.

Thus passing to the Grothendieck group, the basis given by Kapranov exceptional collection can be described in terms of the elements in the shifted \( q = 0 \) affine algebra. For example in the Grassmannian case \( \mathbb{S}_\lambda \mathcal{V} = f_{\lambda_1} \cdots f_{\lambda_k} 1_{(0,N)} \in K(G(k,N)) \). Moreover, we can use the relations in the shifted \( q = 0 \) affine algebra defined in Definition 4.3 to help us calculate the action of \( \delta_i \) and \( \delta''_i \) defined by (6.1) and (6.2) on them.
When \( n = 2 \) (so \( i = 1 \)), we have two bases \([R_{k,N-k}]\) and \([R'_{k,N-k}]\). We found that (see Theorem 6.7) it is easy to calculate the action of \( \delta'_1 \) on \([R_{k,N-k}]\) and \( \delta''_1 \) on the dual (as vector space) of elements \([R'_{k,N-k}]\), but complicated to calculate on the other direction.

For general \( n \)-step partial flag varieties, since we do not have the dual exceptional collection for \([R_k]\) and from the calculation of the Grassmannian case, we will only calculate the action of \( \delta''_1 \) on \([R_k]\).

First, we consider the Grassmannian \( \mathbb{G}(k,N) \) case. In this case \( i = 1 \) and the Demazure operators are \( \delta'_1 = e_0 f_{N-k}(\psi^+)^{-1} l_{k,N-k} \) and \( \delta''_1 = f_0 e_{k-1}(\psi^-)^{-1} l_{k,N-k} \). Then we have the following result.

**Theorem 6.7.** For \( \lambda = (\lambda_1, \ldots, \lambda_k) \in P(N-k,k) \) and \( \mu = (\mu_1, \ldots \mu_{N-k}) \in P(k,N-k) \) we have

\[
\delta'_1([\mathcal{S}_\lambda \mathcal{V}]) = \begin{cases} 
0 & \text{if } \lambda_1 = N-k, \\
[\mathcal{S}_\lambda \mathcal{V}] & \text{if } 0 \leq \lambda_1 \leq N-k-1,
\end{cases}
\]

\[
\delta''_1([\mathcal{S}_\mu (\mathcal{C}/\mathcal{V})]) = \begin{cases} 
0 & \text{if } \mu_1 = k, \\
[\mathcal{S}_\mu (\mathcal{C}/\mathcal{V})] & \text{if } 0 \leq \mu_1 \leq k-1.
\end{cases}
\]

**Proof.** We prove the case for \( \delta'_1 \) only, the proof for \( \delta''_1 \) is similar.

By Lemma 6.5 and relation (U07) in Definition 4.1 we have

\[
\delta'_1([\mathcal{S}_\lambda \mathcal{V}]) = \delta'_1(f_{\lambda_1} \cdots f_{\lambda_k} l_{1,0}(0,N)) = e_0 f_{N-k}(\psi^+)^{-1} f_{\lambda_1} \cdots f_{\lambda_k} l_{1,0}(0,N)
= (-1)^k e_0 f_{N-k} f_{\lambda_1+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N).
\]

Note that since \( \lambda \in P(N-k,k) \), we have \( 0 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq N-k \). So if \( \lambda_1 = N-k \), then by relation (U05) we have \( \delta'_1([\mathcal{S}_\lambda \mathcal{V}]) = 0 \).

Otherwise we have \( 0 \leq \lambda_1 \leq N-k-1 \) and by relation (U07) we have \( e_0 f_{N-k} l_{1,0}(0,N) = f_{N-k} e_0 l_{1,0}(0,N-k) + \psi^+ 1_{(k,N-k)} \). Thus

\[
(-1)^k e_0 f_{N-k} f_{\lambda_1+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N)
= (-1)^k f_{N-k} e_0 f_{\lambda_1+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N) + (-1)^k \psi^+ f_{\lambda_1+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N).
\]

By relation (U07) we have \( (-1)^k \psi^+ f_{\lambda_1+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N) = f_{\lambda_1} \cdots f_{\lambda_k} l_{1,0}(0,N) \). Since \( 0 \leq \lambda_1 \leq N-k-1 \), we get \( 1 \leq \lambda_k+1 \leq \ldots \leq \lambda_1+1 \leq N-k \) and thus

\[
e_0 f_{\lambda_1+1} l_{1,N-k-1}(0,N-k+1) f_{\lambda_k+1}(\psi^+)^{-1} l_{1,0}(0,N) = f_{\lambda_1+1} e_0 f_{\lambda_2+1} \cdots f_{\lambda_k+1}(\psi^+)^{-1} l_{1,N-k+1}
= \ldots = f_{\lambda_1+1} \cdots f_{\lambda_k+1} e_0 (\psi^+)^{-1} l_{1,0}(0,N) = 0.
\]

Combining these we obtain \( \delta'_1([\mathcal{S}_\lambda \mathcal{V}]) = [\mathcal{S}_\lambda \mathcal{V}] \). \( \square \)

**Remark 6.8.** Note that \( (\delta'_1)^2 = \delta'_1 \) and \( (\delta''_1)^2 = \delta''_1 \). The above Theorem implies that the Kapranov exceptional collections for Grassmannians are the eigenbasis (with eigenvalues 0 and 1) for such Demazure operators.

**Remark 6.9.** Since the Kapranov exceptional collection \( \{\mathcal{S}_\lambda \mathcal{V}\} \) gives rise to a semiorthogonal decomposition of \( D^b(G(k,N)) \), we believe that a categorical version of Theorem 6.7 should provide an explicit example to Kuznetsov’s work (i.e., Theorem 7.1 in [Ku]) on projection functors of a semiorthogonal decomposition are kernel functors.

We move to the \( n \)-step partial flag variety \( Fl_k(\mathbb{C}^N) \) case, where \( k = (k_1, \ldots, k_n) \) with \( k_i \geq 1 \) for all \( i \) and \( \sum_{i=1}^n k_i = N \). However, unlike the Grassmannian case, this time, some
of the elements in \([R_k]\) are not eigenbasis for our Demazure operators \(\delta_i'\). We also find that the statement/formulation of the result we obtain is similar to the one prove by J. Tymoczko for left divided difference operators, i.e., Lemma 4.4 and Corollary 4.5. in [Tym].

**Theorem 6.10.** Let \(\lambda_j = (\lambda_{j,1}, \ldots, \lambda_{j,k_1+\ldots+k_j}) \in P(k_{j+1}, k_1 + \ldots + k_j)\) for all \(1 \leq j \leq n-1\). Then we have

\[
\delta_i' \left( \prod_{j=1}^{n-1} [S_{\lambda_j} V_j] \right) = \begin{cases} 
\prod_{j=1}^{n-1} [S_{\lambda_j} V_j] - f_{i,k_i+1}^{-1} k_{i+1} (\sigma) & \text{if } \lambda_{i,1} = k_{i+1}, \\
\prod_{j=1}^{n-1} [S_{\lambda_j} V_j] & \text{if } 0 \leq \lambda_{i,1} \leq k_{i+1} - 1.
\end{cases}
\]

where

\[
\sigma = [S_{\lambda_1} V_1'] \ldots [S_{\lambda_{i-1}} V_{i-1}'] [S_{\lambda_i} V_i'] [S_{\lambda_{i+1}} V_{i+1}'] \ldots [S_{\lambda_{n-1}} V_{n-1}] \in K(Fl_{k+\alpha_i}(C^N))
\]

with \(\lambda_i' = (\lambda_{i,2}, \ldots, \lambda_{i,k_1+\ldots+k_i}) \in P(k_{i+1} + 1, k_1 + \ldots + k_i - 1)\).

Before we give the proof, we have to assume the following assumption.

**Assumption 1.** Conjecture A.2. in [Hsu] holds.

Roughly speaking, the conjecture says that the presentation of \(\mathcal{U}_{0,N}(\mathfrak{sl}_n)\) by finite numbers of generators and relations in Definition 6.1 is equivalent to another presentation which is given by generating series. The reason is that the elements \(f_{i,s} \mathbb{1}_k\) with \(s < k \in C(n, N)\) and \(s \leq -1\), which are not defined in Definition 6.1, will appear in the calculation of the action. But those elements are defined in the presentation by generating series (which includes all the loop generators). We will mention the assumption in the proof when we use it.

**Proof.** To simplify the notations, we denote \(f_{j,\lambda_j} = f_{j,\lambda_{j,1}} \ldots f_{j,\lambda_{j,k_1+\ldots+k_j}}\) for all \(1 \leq j \leq n-1\). Also we define

\[
\xi_i := \sum_{m=i}^{n-1} (k_1 + \ldots + k_m) \alpha_m
\]

for all \(1 \leq i \leq n-1\). Thus we obtain \((0, \ldots, 0, k_1 + \ldots + k_i, k_{i+1}, \ldots, k_n) = \eta - \xi_i\) for all \(1 \leq i \leq n-1\).

Using Lemma 6.6 and relations (U05), (U07), (U09) we have

\[
\delta_i' \left( \prod_{j=1}^{n-1} [S_{\lambda_j} V_j] \right) = e_{i,0} f_{i,k_i+1}^{-1} (\psi_i^+)^{-1} 1_k f_{i,\lambda_1} \ldots f_{i,\lambda_{k-\lambda_1+1} - 1} \eta
\]

\[
= f_{i,\lambda_1} \ldots e_{i,0} f_{i,k_i+1} f_{i-1,\lambda_i-1} (\psi_i^+)^{-1} \eta - \xi_i f_{i,\lambda_i} \ldots f_{i-n+1,\lambda_{n-1} - 1} \eta
\]

\[
= (-1)^{k_1 + \ldots + k_i} f_{i,\lambda_1} \ldots e_{i,0} f_{i,k_i+1} f_{i-1,\lambda_i-1} f_{i,\lambda_i+1} \ldots f_{i,\lambda_i+\ldots+k_i} (\psi_i^+)^{-1} \ldots f_{i-n+1,\lambda_{n-1} - 1} \eta.
\]

We have to calculate \(f_{i,k_i+1} f_{i-1,\lambda_i-1} \eta - \xi_i\) by using relations (U05) since \(f_{i,r}\) and \(f_{i-1,s}\) do not commute with each other. In particular, we have \(f_{i,r} f_{i-1,s} = f_{i-1,s} f_{i,r} - f_{i-1,s+1} f_{i,r-1}\).
By a direct calculation and using the Assumption [1], we obtain
\[ f_{i,k_i+1} f_{i-1,\lambda_i-1} 1_{\eta - \xi_i} = f_{i,k_i+1} f_{i-1,\lambda_i-1,1} \cdots f_{i-1,\lambda_i-1,k_i+\cdots+k_i-1} 1_{\eta - \xi_i} \]
\[ = f_{i-1,\lambda_i-1,1} \cdots f_{i-1,\lambda_i-1,k_i+\cdots+k_i-1} 1_{\eta - \xi_i} + \sum_{r=1}^{k_i+\cdots+k_i-1} f_{i-1,\lambda_i-1,r+1} \cdots f_{i-1,\lambda_i-1,k_i+\cdots+k_i-1} 1_{\eta - \xi_i} + \cdots \]
\[ + (-1)^{k_i+\cdots+k_i-1} f_{i-1,\lambda_i-1,1} \cdots f_{i-1,\lambda_i-1,k_i+\cdots+k_i-1} f_{i,\lambda_i+1-k_i} 1_{\eta - \xi_i}. \quad (6.4) \]

Since \( e_i,0 \) commutes with \( f_{i-1,a} \) by relation (U09), to know (6.3) we have to calculate
\[ (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,k_i+1} - m f_{i,\lambda_i+1} \cdots f_{i,\lambda_i+1+k_i-1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} \]
for all \( 0 \leq m \leq k_i \).

For \( m = 0 \), this reduces to the Grassmannian (or \( \mathfrak{sl}_2 \)) case, and we have
\[ (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,k_i+1} f_{i,\lambda_i+1} \cdots f_{i,\lambda_i+1+k_i-1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = \begin{cases} 0 & \text{if } \lambda_i,1 = k_i+1, \\ f_{i,\lambda_i} 1_{\eta - \xi_i+1} & \text{if } 0 \leq \lambda_i,1 \leq k_i+1-1. \end{cases} \quad (6.6) \]

For \( 1 \leq m \leq k_i \), by relation (U09) we have
\[ e_i,0 f_{i,k_i+1} - m 1_{\eta - \xi_i} = f_{i,k_i+1} - m e_i,0 1_{\eta - \xi_i}, \]
since \( k_i+1 - k_i - \cdots - k_i+1-m \leq k_i+1-1 \) and \( k_i+k_i+1 \geq 1 \) implies \( -k_i - \cdots - k_i+1 \).

So to know (6.5) we have to calculate
\[ (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,\lambda_i+1+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} \]
Note that \( 0 \leq \lambda_i,1+k_i \leq \cdots \leq \lambda_i,1 \leq k_i+1 \). If \( \lambda_i,1 \leq k_i+1-1 \), then \( 0 \leq \lambda_i,1+k_i \leq \cdots \leq \lambda_i,1 \leq k_i+1-1 \) and by relation (U09) we get
\[ (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,\lambda_i+1+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,\lambda_i+2+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = \cdots = (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,\lambda_i+1+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = 0. \]

Otherwise \( \lambda_i,1 = k_i+1 \) and by the same argument in the proof of Theorem 6.7 we have
\[ (-1)^{k_i+\cdots+k_i} e_i,0 f_{i,\lambda_i+1+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = (-1)^{k_i+\cdots+k_i} f_{i,\lambda_i+2+k_i+1} (\psi_i^+)^{-1} 1_{\eta - \xi_i+1} = (-1)^{k_i+\cdots+k_i} f_{i,\lambda_i+2+k_i+1} 1_{\eta - \xi_i+1}. \]
Thus we obtain that when $1 \leq m \leq k_1 + \ldots + k_{i-1}$

\[
(-1)^{k_1 + \ldots + k_i} e_{i,0} f_{i,k_{i+1}} \ldots f_{i-1,\lambda_i-1,1} \ldots f_{i-1,\lambda_i,1,k_1+\ldots+k_{i-1}} f_{i,k_{i+1}+1} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} (\psi_i^+)^{-1} 1_{\eta-\xi+i+1}
\]

\[
= \begin{cases} 
-f_{i,k_{i+1}+m} f_{i,\lambda_i,1} \ldots f_{i,k_{i+1}+1} f_{i,\lambda_i,1} f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} 1_{\eta-\xi+i+1} & \text{if } \lambda_i,1 = k_{i+1}, \\
0 & \text{if } 0 \leq \lambda_i,1 \leq k_{i+1} - 1.
\end{cases} 
\tag{6.7}
\]

By (6.4) and (6.7) we get that when $\lambda_i,1 = k_{i+1}$

\[
(-1)^{k_1 + \ldots + k_i} e_{i,0} f_{i,k_{i+1}} \ldots f_{i-1,\lambda_i-1,1} \ldots f_{i-1,\lambda_i,1,k_1+\ldots+k_{i-1}} f_{i,k_{i+1}+1} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} (\psi_i^+)^{-1} 1_{\eta-\xi+i+1}
\]

\[
= \sum_{r=1}^{k_{i+1}} f_{i-1,\lambda_i-1,1} \ldots f_{i-1,\lambda_i-1,r+1} \ldots f_{i-1,\lambda_i,1,k_1+\ldots+k_{i-1}} f_{i,k_{i+1}-1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1} f_{i,k_{i+1}+1} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} 1_{\eta-\xi+i+1}
\]

\[+ \ldots
\]

\[+ (-1)^{k_1 + \ldots + k_{i-1}+1} f_{i-1,\lambda_i-1,1} \ldots f_{i-1,\lambda_i-1,k_1+\ldots+k_{i-1}+1} f_{i,k_{i+1}-k_1-\ldots-k_{i-1}} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} 1_{\eta-\xi+i+1}
\]

Combing with (6.6) we get

\[
(-1)^{k_1 + \ldots + k_i} e_{i,0} f_{i,k_{i+1}} \ldots f_{i-1,\lambda_i-1,1} \ldots f_{i-1,\lambda_i,1,k_1+\ldots+k_{i-1}} (\psi_i^+)^{-1} 1_{\eta-\xi+i+1}
\]

\[
= \begin{cases} 
(f_{i-1,\lambda_i-1} f_{i,\lambda_i} - f_{i,k_{i+1}} f_{i-1,\lambda_i,1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1} f_{i,k_{i+1}+1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}}) 1_{\eta-\xi+i+1} & \text{if } \lambda_i,1 = k_{i+1}, \\
f_{i-1,\lambda_i-1} f_{i,\lambda_i} 1_{\eta-\xi+i+1} & \text{if } 0 \leq \lambda_i,1 \leq k_{i+1} - 1.
\end{cases} 
\]

As a result, from (6.3) we conclude that

\[
\delta_i' \left( \prod_{j=1}^{n-1} [S_{\lambda_j} V_j] \right)
\]

\[
= \begin{cases} 
\left( f_{i,\lambda_i} \ldots f_{n-1,\lambda_i-1} 1_{\eta} - f_{i,k_{i+1}} f_{i-1,\lambda_i,1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1} f_{i,k_{i+1}+1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} f_{i-1,\lambda_i-1} f_{i,\lambda_i} \right) 1_{\eta-\xi+i+1} & \text{if } \lambda_i,1 = k_{i+1}, \\
f_{i,\lambda_i} \ldots f_{n-1,\lambda_i-1} 1_{\eta} & \text{if } 0 \leq \lambda_i,1 \leq k_{i+1} - 1.
\end{cases} 
\]

\[
= \begin{cases} 
\left( f_{i,\lambda_i} \ldots f_{n-1,\lambda_i-1} 1_{\eta} - f_{i,k_{i+1}} f_{i-1,\lambda_i,1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1} f_{i,k_{i+1}+1} f_{i,\lambda_i,2} \ldots f_{i,\lambda_i,1,k_1+\ldots+k_{i-1}} f_{i-1,\lambda_i-1} f_{i,\lambda_i} \right) 1_{\eta-\xi+i+1} & \text{if } \lambda_i,1 = k_{i+1}, \\
f_{i,\lambda_i} \ldots f_{n-1,\lambda_i-1} 1_{\eta} & \text{if } 0 \leq \lambda_i,1 \leq k_{i+1} - 1.
\end{cases} 
\]

\[
= \left( \prod_{j=1}^{n-1} [S_{\lambda_j} V_j] - f_{i,k_{i+1}} \sigma \right) 1_{\eta-\xi+i+1} \text{ if } \lambda_i,1 = k_{i+1},
\]

\[
= \left( \prod_{j=1}^{n-1} [S_{\lambda_j} V_j] - f_{i,k_{i+1}} \sigma \right) \text{ if } 0 \leq \lambda_i,1 \leq k_{i+1} - 1.
\]

where

\[
\sigma = [S_{\lambda_1} V_1'] \ldots [S_{\lambda_{i-1}} V_{i-1}'][S_{\lambda_i} V_i'][S_{\lambda_{i+1}} V_{i+1}'][S_{\lambda_{i+1}} V_{i+1}'][S_{\lambda_{n-1}} V_{n-1}] = [S_{\lambda_1} V_1'][S_{\lambda_2} V_2'][S_{\lambda_3} V_3]'[S_{\lambda_4} V_4] \ldots [S_{\lambda_{n-1}} V_{n-1}]
\]

which proves the result.

\[\square\]
6.3. Calculations of braid relations for a simple example. From Theorem 6.3 we have a full exceptional collection $R_k$ for $D^b(Fl_k(C^N))$, so $[R_k]$ gives us a basis for $K(Fl_k(C^N))$. Also from Theorem 6.10 we know that how the generators $\delta'_i \in H'_n(0)$ acting on the basis elements. In this section, we calculate an example to see the braid relations for the action of $H'_n(0)$ on $K(Fl_k(C^N))$.

Example 6.11. Consider the partial flag variety

$$Fl_{(1,2,1)}(C^4) = \{0 \subset V_1 \subset V_2 \subset C^4 \mid \dim V_1 = 1, \dim V_2/V_1 = 2\}.$$  

Then we have the full exceptional collection $R_{(1,2,1)} = \{S_{\lambda_1}V_1 \otimes S_{\lambda_2}V_2\}$ for $D^b(Fl_{(1,2,1)}(C^N))$ where $\lambda_1 \in P(2,1)$ and $\lambda_2 \in P(1,3)$. Thus a basis $\{[S_{\lambda_1}V_1][S_{\lambda_2}V_2]\}$ for $K(Fl_{(1,2,1)}(C^N))$. For the action of $H'_2(0)$ on $K(Fl_{(1,2,1)}(C^N))$, we have two generators $\delta'_1 = \epsilon_1 f_{1,2}(\psi_1^+)^{-1} 1(1,2,1)$ and $\delta'_2 = \epsilon_2 f_{2,1}(\psi_2^+)^{-1} 1(1,2,1)$. We try to verify the braid relation by check that the two linear maps $\delta'_1\delta'_2\delta'_1$, $\delta'_2\delta'_1\delta'_2$ agree on the basis elements.

From above, we have

$$[S_{\lambda_1}V_1][S_{\lambda_2}V_2] = f_{1\lambda_1} f_{2,\lambda_2,1} f_{2,\lambda_2,2} f_{2,\lambda_2,3} 1(0,0,4),$$

where $\lambda_1 \in P(2,1)$, $\lambda_2 = (\lambda_{2,1}, \lambda_{2,2}, \lambda_{2,3}) \in P(1,3)$. By Theorem 6.10 it is easy to see that

$$\delta'_1([S_{\lambda_1}V_1][S_{\lambda_2}V_2]) = \begin{cases} 0 & \text{if } \lambda_1 = 2, \\ [S_{\lambda_1}V_1][S_{\lambda_2}V_2] & \text{if } 0 \leq \lambda_1 \leq 1. \end{cases} \quad (6.8)$$

Similarly, we have

$$\delta'_2([S_{\lambda_1}V_1][S_{\lambda_2}V_2]) = \begin{cases} [S_{\lambda_1}V_1][S_{\lambda_2}V_2] - f_{2,1}([S_{\lambda_1}V_1][S_{(\lambda_{2,1},\lambda_{2,2},\lambda_{2,3})}V_2')] & \text{if } \lambda_{2,1} = 1, \\ [S_{\lambda_1}V_1][S_{\lambda_2}V_2] & \text{if } \lambda_{2,1} = 0. \end{cases} \quad (6.9)$$

Note that since $\lambda_2 = (\lambda_{2,1}, \lambda_{2,2}, \lambda_{2,3}) \in P(1,3)$, $\lambda_{2,1} = 0$ implies that $\lambda_2 = (0,0,0)$. We try to simplify the term

$$[S_{\lambda_1}V_1][S_{\lambda_2}V_2] - f_{2,1}([S_{\lambda_1}V_1][S_{(\lambda_{2,1},\lambda_{2,2},\lambda_{2,3})}V_2']) = f_{1\lambda_1} f_{2,1} f_{2,\lambda_2,2} f_{\lambda_2,3} 1(0,0,4) - f_{2,1} f_{1\lambda_1} f_{2,\lambda_2,2} f_{\lambda_2,3} 1(0,0,4). \quad (6.10)$$

From relation (U05), we have $f_{2,1} f_{1\lambda_1} = f_{1\lambda_1} f_{2,1} - f_{1\lambda_1+1} f_{2,0}$, so (6.10) becomes $f_{1\lambda_1+1} f_{2,0} f_{\lambda_2,2} f_{\lambda_2,3} 1(0,0,4)$.

Since $(\lambda_{2,1}, \lambda_{2,2}, \lambda_{2,3}) \in P(1,3)$, we have $0 \leq \lambda_{2,3} \leq \lambda_{2,2} \leq 1$, thus

$$f_{1\lambda_1+1} f_{2,0} f_{\lambda_2,2} f_{\lambda_2,3} 1(0,0,4) = \begin{cases} 0 & \text{if } \lambda_{2,2} = 1 \\ [S_{\lambda_1+1}V_1] & \text{if } \lambda_{2,2} = 0 \end{cases}$$

So (6.9) becomes

$$\delta'_2([S_{\lambda_1}V_1][S_{\lambda_2}V_2]) = \begin{cases} 0 & \text{if } \lambda_2 = (1,1,0), (1,1,1), \\ [S_{\lambda_1+1}V_1] & \text{if } \lambda_2 = (1,0,0), \\ [S_{\lambda_1}V_1] & \text{if } \lambda_2 = (0,0,0). \end{cases} \quad (6.11)$$
Combining (6.8) and (6.9), we have

\[
\delta'_2 \delta'_1([S_{\lambda_1} \mathcal{V}_1][S_{\lambda_2} \mathcal{V}_2]) = \begin{cases} 
0 & \text{if } \lambda_1 = 2, \text{ or } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (1,1,0), (1,1,1), \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (0,0,0), \\
[S_{\lambda_1+1} \mathcal{V}_1] & \text{if } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (1,0,0).
\end{cases}
\]

Thus to calculate all the cases for \(\delta'_1 \delta'_2 \delta'_1([S_{\lambda_1} \mathcal{V}_1][S_{\lambda_2} \mathcal{V}_2])\), it remains to calculate \(\delta'_1([S_{\lambda_1+1} \mathcal{V}_1])\) for \(0 \leq \lambda_1 \leq 1\). This can be done by using (6.8), which implies that

\[
\delta'_1([S_{\lambda_1+1} \mathcal{V}_1]) = \begin{cases} 
0 & \text{if } \lambda_1 = 1 \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } \lambda_1 = 0.
\end{cases}
\]

Hence we get

\[
\delta'_1 \delta'_2 \delta'_1([S_{\lambda_1} \mathcal{V}_1][S_{\lambda_2} \mathcal{V}_2]) = \begin{cases} 
0 & \text{if } \lambda_1 = 2, \text{ or } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (1,1,0), (1,1,1), \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (0,0,0), \\
0 & \text{if } \lambda_1 = 1, \text{ } \lambda_2 = (1,0,0), \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } \lambda_1 = 0, \text{ } \lambda_2 = (1,0,0).
\end{cases}
\] (6.12)

On the other hand, we have

\[
\delta'_1 \delta'_2([S_{\lambda_1} \mathcal{V}_1][S_{\lambda_2} \mathcal{V}_2]) = \begin{cases} 
0 & \text{if } \lambda_2 = (1,1,0), (1,1,1), \\
\delta'_1([S_{\lambda_1+1} \mathcal{V}_1]) & \text{if } \lambda_2 = (1,0,0), \\
\delta'_1([S_{\lambda_1} \mathcal{V}_1]) & \text{if } \lambda_2 = (0,0,0).
\end{cases}
\]

\[
= \begin{cases} 
0 & \text{if } 0 \leq \lambda_1 \leq 2, \text{ } \lambda_2 = (1,1,0), (1,1,1), \\
0 & \text{if } \lambda_1 = 1, \text{ } \lambda_2 = (1,0,0), \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } \lambda_1 = 0, \text{ } \lambda_2 = (1,0,0), \\
\delta'_1([S_{\lambda} \mathcal{V}_1]) & \text{if } \lambda_1 = 2, \text{ } \lambda_2 = (0,0,0), \\
0 & \text{if } \lambda_1 = 2, \text{ } \lambda_2 = (0,0,0), \\
[S_{\lambda_1} \mathcal{V}_1] & \text{if } 0 \leq \lambda_1 \leq 1, \text{ } \lambda_2 = (0,0,0).
\end{cases}
\]

So we have to calculate \(\delta'_1([S_{\lambda} \mathcal{V}_1]) = e_{1,0} f_{1,2} e_{1,2}(\psi_1^+)^{-1} f_{1,3} f_{2,0} f_{2,2} f_{2,1}(0,0,4)\). Using \(e_{1,0} f_{1,2} e_{1,2}(1,2,1) = f_{1,2} e_{1,0} e_{1,2} + \psi_1^+ h_{1,1} (1,2,1)\) we have

\[
\delta'_1([S_{\lambda} \mathcal{V}_1]) = f_{1,2} e_{1,0} (\psi_1^+)^{-1} f_{1,3} f_{2,0} f_{2,2} f_{2,1}(0,0,4) + f_{1,3} f_{2,0} f_{2,0} f_{2,1}(0,0,4)
\]

\[
= -f_{1,2} e_{1,0} (\psi_1^+)^{-1} f_{2,0} f_{2,0} f_{2,0} f_{2,1}(0,0,4) + f_{1,3} f_{2,0} f_{2,0} f_{2,1}(0,0,4). 
\] (6.13)

Using relation (6.9), we have \(e_{1,0} f_{1,4} e_{1,0}(0,3,1) = f_{1,4} e_{1,0} e_{1,0} + \psi_1^+ h_{1,1} (0,3,1)\). So (6.13) becomes

\[
-f_{1,2} f_{1,4} e_{1,0} (\psi_1^+)^{-1} f_{2,0} f_{2,0} f_{2,0} f_{2,1}(0,0,4) - f_{1,2} h_{1,1} f_{2,0} f_{2,0} f_{2,1}(0,0,4) + f_{1,3} f_{2,0} f_{2,0} f_{2,1}(0,0,4).
\]
Since $e_{1,r}$ commutes with $f_{2,s}$, we have $-f_{1,2}f_{1,4}e_{1,0}(\psi_1^+)^{-1}f_{2,0}f_{2,0}f_{2,0}1(0,0,4) = 0$. Using relation \(U05\), we have $h_{1,1}f_{2,0} = f_{2,0}h_{1,1} + f_{2,1}$. So

\[
f_{1,2}h_{1,1}f_{2,0}f_{2,0}f_{2,0}1(0,0,4) = f_{1,2}f_{2,1}f_{2,0}f_{2,0}1(0,0,4) + f_{1,2}f_{2,0}h_{1,1}f_{2,0}f_{2,0}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}h_{1,1}f_{2,0}1(0,0,4)
\]

\[
= f_{1,2}f_{2,1}f_{2,0}f_{2,0}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}h_{1,1}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}h_{1,1}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}h_{1,1}1(0,0,4)
\]

(by relation \(U05\)).

By definition, we have $h_{1,1}1(0,0,4) = (\psi_1^+)^{-1}(e_{1,0}f_{1,1} - f_{1,1}e_{1,0})1(0,0,4) = 0$. Thus $f_{1,2}h_{1,1}f_{2,0}f_{2,0}f_{2,0}1(0,0,4) = f_{1,2}f_{2,1}f_{2,0}f_{2,0}1(0,0,4)$ and

\[
\delta'_1([S_3\mathcal{V}_1]) = -f_{1,2}f_{2,1}f_{2,0}f_{2,0}1(0,0,4) + f_{1,2}f_{2,0}f_{2,0}1(0,0,4) = -[S_2\mathcal{V}_1][S_{(1,0,0)}\mathcal{V}_2] + [S_3\mathcal{V}_1]
\]

Finally, to calculate $\delta'_2\delta'_2([S_{\lambda_1}\mathcal{V}_1][S_{\lambda_2}\mathcal{V}_2])$, by (5.11) we have

\[
\delta'_2\delta'_1([S_3\mathcal{V}_1]) = \delta'_2(-[S_2\mathcal{V}_1][S_{(1,0,0)}\mathcal{V}_2] + [S_3\mathcal{V}_1]) = -[S_3\mathcal{V}_1] + [S_3\mathcal{V}_1] = 0,
\]

since we also have $\delta'_2([S_{\lambda_1}\mathcal{V}_1]) = [S_{\lambda_1}\mathcal{V}_1]$ for all $0 \leq \lambda_1 \leq 1$, we conclude that

\[
\delta'_2\delta'_1([S_{\lambda_1}\mathcal{V}_1][S_{\lambda_2}\mathcal{V}_2]) = \begin{cases} 0 & \text{if } 0 \leq \lambda_1 \leq 2, \lambda_2 = (1, 1, 0), (1, 1, 1), \\ 0 & \text{if } \lambda_1 = 1, \lambda_2 = (1, 0, 0), \\ [S_1\mathcal{V}_1] & \text{if } \lambda_1 = 0, \lambda_2 = (1, 0, 0), \\ 0 & \text{if } \lambda_1 = 2, \lambda_2 = (1, 0, 0), \\ 0 & \text{if } \lambda_1 = 2, \lambda_2 = (0, 0, 0), \\ [S_{\lambda_1}\mathcal{V}_1] & \text{if } 0 \leq \lambda_1 \leq 1, \lambda_2 = (0, 0, 0), 
\end{cases}
\]

which agree with $\delta'_1\delta'_2([S_{\lambda_1}\mathcal{V}_1][S_{\lambda_2}\mathcal{V}_2])$ from (6.12). Hence the braid relation holds.

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