An extensive operational law for monotone functions of LR fuzzy intervals with applications to fuzzy optimization

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Abstract
The operational law proposed by Zhou et al. (J Intell Fuzzy Syst 30(1): 71–87, 2016) contributes to developing fuzzy arithmetic, while its applicable conditions are confined to strictly monotone functions and regular LR fuzzy numbers, which are hindering their operational law from dealing with more general cases, such as problems formulated as monotone functions and problems with fuzzy variables represented as fuzzy intervals (e.g., trapezoidal fuzzy numbers). In order to handle such cases we generalize the operational law of Zhou et al. in both the monotonicity of function and fuzzy variables in this paper and then apply the extensive operational law to the cases with monotone (but not necessarily strictly monotone) functions with regard to regular LR fuzzy intervals (LR-FIs) (of which regular LR fuzzy numbers are special cases). Specifically, we derive the computational formulae for expected values (EVs) of LR-FIs and monotone functions with regard to regular LR-FIs, respectively. On the other hand, we develop a solution scheme to dispose of fuzzy optimization problems with regular LR-FIs, in which a fuzzy programming is converted to a deterministic equivalent one and a newly devised solution algorithm is utilized to get the deterministic programming solved. The numerical experiments are conducted using our proposed solution scheme and the traditional fuzzy simulation-based genetic algorithm in the context of a purchasing planning problem. Computational results show that our method is much more efficient, yielding high-quality solutions.

Keywords Fuzzy interval · Monotone function · Operational law · Expected value · Fuzzy programming

1 Introduction

In real-life cases, uncertainty on input parameters involved in some optimization problems is inevitable due to the unpredictable natural factors. In this regard, fuzzy set theory initiated by Zadeh (1965) as one of the popular ways coping with uncertainties is applied to practical optimization process of various research fields, such as supply chain management (Ke et al. 2018), transportation (Büyüközkan et al. 2018), and finance investment (Stoklasa et al. 2021). Under the fuzzy circumstance, uncertain parameters are commonly assigned to be fuzzy variables, in which fuzzy numbers and fuzzy intervals (Dubois and Prade 1988) (or flat fuzzy numbers (Dubois and Prade 1979)) are two frequently used types. The main difference between them is that the modal value of fuzzy number is a point value, while the set of modal values of fuzzy interval is an interval. From the mathematical viewpoint, we can consider fuzzy numbers as a particular situation of fuzzy intervals. Therefore, the emphasis of research in this paper is placed on fuzzy intervals.

With regard to the parametric representation of fuzzy interval, Dubois and Prade (1979, 1988) defined a well-known L-R representation, in which L and R are shape functions representing the left and right sides of membership function, respectively. From another point of view, Goetschel and Voxman (1986) proposed an equivalent representation named as L-U representation in accordance with the lower and upper branches, which define the two endpoints of an α-cut. Stefanini and Guerra (2017) suggested ACF-representation to describe a fuzzy interval by using a new defined average cumulative function on the basis of the possibility theory. It should be mentioned that the typical L-R representation in Dubois and Prade (1988) is utilized to delineate fuzzy interval...
in this paper, and the corresponding fuzzy interval is termed as LR-FI accordingly.

Fuzzy arithmetic plays an essential role in processing fuzzy variables, which has attracted interests of many researchers. Zadeh (1975) initially extended the common arithmetic operations for real numbers to fuzzy intervals via the proposed extension principle on the basis of a triangular norm (t-norm). As for the t-norm-based arithmetic operations, an important feature is to offer a way for controlling the rise on uncertainty in the process of computations and avoid variables shifting off their most vital values simultaneously. However, the practical use of Zadeh’s extension principle is a little complicated owing to the involved non-linear operators. Subsequently, Dubois and Prade (1979) proposed some analytical calculations including the basic arithmetic addition, subtraction, multiplication, and division among LR-FIs together with some properties. Meanwhile, there is some other literature focusing on algebraic operations of LR-FIs. Hwang and Lee (2001) studied the sum for LR-FIs in accordance with a given nilpotent t-norm. For the sake of preserving the shapes of fuzzy intervals during the practical computation, some shape-preserving operations on fuzzy intervals with sigmoid and bell-shaped membership functions (Dombi and Gyöorbiro 2006; Hong 2007) were investigated. With the same aim, Mako (2012) constructed the real vector space of LR-FIs, and then presented the algebraic forms and the associated application. Based upon the definition of unrestricted LR-FI (Kaur and Kumar 2012), Kaur and Kumar (2013) presented the product of unconstrained LR-FIs, thereby formulating a Mehar’s method to deal with the linear programming problems. Recently, Abbasi and Allahviranloo (2021) proposed new fuzzy arithmetic operations on LR-type flat fuzzy numbers based on the transmission average. Additionally, some arithmetic operations on a specific type of fuzzy intervals such as trapezoidal fuzzy numbers (Shakeel et al. 2019a, b), and pseudo-octagonal fuzzy numbers (Abbasi and Allahviranloo 2019) were also discussed.

As a particular kind of LR-FIs, LR fuzzy numbers have got quite a few attentions because of the good interpretability and easy performing for usual operations since they were introduced by Dubois and Prade (1978). So far there have been many relevant studies on arithmetic operations of LR fuzzy numbers (see, e.g., Ban et al. 2016; Garg 2018; Garg and Ansha 2018; Ghanbari et al. 2022). In particular, on account of the credibility measure pioneered by Liu (2002), Zhou et al. (2016) proposed an operational law targeting at strictly monotone functions with regard to LR fuzzy numbers. Based on this, a crispy solution framework for the fuzzy programming was formulated, which reduces the computation complexity a lot. Given the effectiveness of the operational law in Zhou et al. (2016), it has been gradually employed to handle different optimization problems.

For example, Wang et al. (2018) developed a revised hybrid intelligent algorithm to solve a green-fuzzy vehicle routing problem. Fang et al. (2020) devised an analytical method to tackle a newly established fuzzy quality function deployment model for product design in multi-segment markets. Besides, the research findings in Zhou et al. (2016) were also used for diagnosis of prostate cancer (Kar and Majumder 2017), reliability analysis (Dutta 2019), preventive maintenance scheduling problem (Zhong et al. 2019; Wang et al. 2020), location problem (Soltanpour et al. 2019; Yang et al. 2019), and so on.

From the review of existing research on the fuzzy arithmetic, the majority of research studied arithmetic calculations on fuzzy variables and presented useful fuzzy arithmetic operations, while Zhou et al. (2016) focused on handling functions with fuzzy variables and proposed the operational law for computing the inverse credibility distributions (ICDs). Nonetheless, their operational law aims at strictly monotone functions with regard to regular LR fuzzy numbers. As we know, many optimization problems (e.g., classical newsvendor problem), in practice, cannot be modeled using strictly monotone functions, and there are some problems where LR-FIs (e.g., trapezoidal fuzzy numbers) defined by Liu et al. (2020) are more appropriate to represent fuzzy variables. In such cases, the operational law in Zhou et al. (2016) is unable to come into use. Therefore, it is necessary and valuable to make an extensive study for Zhou et al. (2016). The purpose of this paper is to propose an extensive operational law based on the one proposed by Zhou et al. (2016) so that more fuzzy optimization problems modeled using monotone (but not necessarily strictly monotone) functions and regular LR fuzzy intervals (LR-FIs) can be handled.

The main contributions of this paper to the field of fuzzy arithmetics and fuzzy optimization are fourfold.

1. We propose the ICD of an LR-FI based on credibility measure, which is a generalization for the ICD of regular LR fuzzy number defined in Zhou et al. (2016), and verify two equivalent conditions of regular LR-FIs.

2. We present an extensive operational law on monotone functions with regard to regular LR-FIs, which generalizes the operational law in Zhou et al. (2016) from both the function monotonicity and the type of fuzzy variables. Concretely, the strictly monotone functions are extended to be monotone (but not necessarily strictly monotone) functions, and the regular LR fuzzy numbers are generalized to regular LR-FIs such as trapezoidal fuzzy numbers.

3. We develop calculation formulas for EVs of LR-FIs and monotone functions with regard to regular LR-FIs based on the extensive operational law. In accordance with the calculation formulas, the EVs of monotone functions of
regular LR-FIs can be derived directly by means of the corresponding ICDs.

4. We construct a solution strategy including a newly devised heuristic algorithm with a new effective simulation for the fuzzy chance-constrained programming (CCP) with monotone objective and constraint functions regarding regular LR-FIs. Then we illustrate the better performances of our method on both solution accuracy and efficiency in comparison with another traditional heuristic algorithm through a purchasing planning problem.

The remaining of this paper is organized as follows. Section 2 recalls some fundamental notions regarding the LR-FI, defines its ICD in the light of the credibility distribution, and then derives the equivalent conditions of regular LR-FIs. In Sect. 3, we explore the property of monotone functions with regard to regular LR-FIs, propose a new operational law and then discuss the EVs of LR-FIs and monotone functions in regard to regular LR-FIs. In Sect. 4, a solution strategy for the fuzzy CCP is formulated based on the new operational law. To exhibit the effectiveness of our strategy, some numerical experiments are implemented by using our method and a traditional heuristic method, respectively, in the context of a purchasing planning problem. Finally, the main findings are concluded in Sect. 5. The conceptual framework of our study is demonstrated in Fig. 1.

2 LR fuzzy interval and its inverse credibility distribution

In this section, some elementary conceptions in relation to LR-FIs and the credibility distribution of fuzzy variable are reviewed first. We subsequently define the ICD of an LR-FI, and derive its mathematical expression. After that, we introduce the definition of the regular LR-FI and prove its two necessary and sufficient conditions.

2.1 LR fuzzy interval and its credibility distribution

The well-known LR-FI was initially proposed by Dubois and Prade (1988), in which \(L\) and \(R\) are the decreasing left and right shape functions from \([0, \infty) \rightarrow [0, 1]\) with \(L(0) = 1\) and \(R(0) = 1\), respectively. The LR-FI is the most general class of fuzzy intervals, and the LR fuzzy number can be seen as a special case of LR-RI with unique modal value. The LR-FI is also a kind of fuzzy parameters commonly and frequently used and covers most of fuzzy parameters used in the fuzzy optimization. Different from other fuzzy intervals, the LR-FI is one type of unimodal fuzzy intervals and its corresponding membership function can be expressed by two decreasing shape functions \(L\) and \(R\) with four parameters, in which the left side of membership function is monotonically increasing and right side of membership function is monotonically decreasing.

**Definition 1** (Dubois and Prade 1988) A fuzzy interval \(\tilde{M}\) defined on universal set of real numbers \(\mathbb{R}\) is said to be an LR-FI if it has the membership function with shape functions \(L, R\) and four parameters \(\xi, \tau, \rho > 0, \sigma > 0\) as

\[
v(t) = \begin{cases} L\left(\frac{t-\xi}{\rho}\right), & \text{if } \xi < t < \tau \\ 1, & \text{if } \xi \leq t \leq \tau \\ R\left(\frac{t-\tau}{\sigma}\right), & \text{if } t > \tau, \end{cases}
\]

where \([\xi, \tau]\) is the core of \(\tilde{M}\), \(\xi\) and \(\tau\) are respectively, called the lower and upper modal values, \(\rho\) and \(\sigma\) are respectively called the left and right spreads. More briefly, the fuzzy interval \(\tilde{M}\) is denoted by \((\xi, \tau, \rho, \sigma)_{LR}\).

**Remark 1** If \(\rho\) and \(\sigma\) are both equal to 0, that is, \(\tilde{M} = (\xi, \tau, 0, 0)_{LR}\) for all \(L\) and \(R\), then \(\tilde{M}\) degenerates into a real interval as \([\xi, \tau]\). If the core of \(\tilde{M}\) is a real number \(\xi\) (i.e., \(\xi = \tau\)), then \(\tilde{M}\) degrades into an LR fuzzy number denoted by \((\xi, \rho, \sigma)_{LR}\). Furthermore, when \(\tilde{M} = (\xi, 0, 0)_{LR}\) for all \(L\) and \(R\), \(\tilde{M}\) is just a real number \(\xi \in \mathbb{R}\).

To measure fuzzy events in the fuzzy world, Zadeh (1978) suggested the possibility measure. However, it lacks self-duality. To overcome this deficiency, Liu and Liu (2002) defined the credibility measure based on the possibility measure and proved its self-duality. A self-dual measure is needed in this study as only when the measure satisfies self-duality can we use inverse distribution to draw some important inferences about fuzzy arithmetic. Hence the credibility measure is adopted in this paper rather than the possibility measure.

Suppose that \(\xi\) is a fuzzy variable with membership function \(v\) and \(t\) is a real number. The credibility of fuzzy event \(\{\xi \leq t\}\) is defined by Liu and Liu (2002)

\[
\text{Cr}[\xi \leq t] = \frac{1}{2} \left(\sup_{y \leq t} v(y) + 1 - \sup_{y > t} v(y)\right). \tag{2}
\]

To describe a fuzzy variable, credibility distribution as a carrier of incomplete information of this variable is defined by Liu (2004) as follows.

**Definition 2** (Liu 2004) If \(\xi\) is a fuzzy variable, then its credibility distribution \(\psi : \mathbb{R} \rightarrow [0, 1]\) is defined by

\[
\psi(t) = \text{Cr}[\xi \leq t]. \tag{3}
\]

In accordance with the mathematical properties of credibility measure, it is known that the credibility distribution \(\psi\) is non-decreasing on \(\mathbb{R}\), in which \(\psi(-\infty) = 0\) and \(\psi(+\infty) = 1\).
As for the LR-FI $\tilde{M} = (\zeta, \bar{\zeta}, \rho, \sigma)_{LR}$ with the membership function $\nu$ in Eq. (1), on account of Eqs. (2)-(3), its credibility distribution can be derived as

$$\psi(t) = \begin{cases} 
0.5 L \left( \frac{\zeta - t}{\rho} \right), & \text{if } t < \zeta \\
0.5, & \text{if } \zeta \leq t \leq \bar{\zeta} \\
1 - 0.5 R \left( \frac{t - \bar{\zeta}}{\sigma} \right), & \text{if } t > \bar{\zeta}.
\end{cases}$$

(4)

**Example 1** If an LR-FI $(\zeta, \bar{\zeta}, \rho, \sigma)_{LR}$ has the shape functions $L(t) = R(t) = \max\{0, 1 - t\}$, then it is called trapezoidal fuzzy number denoted by $T(\zeta, \bar{\zeta}, \rho, \sigma)_{LR}$ with the membership function

$$\nu(t) = \begin{cases} 
\frac{t + \rho - \zeta}{\rho}, & \text{if } \zeta - \rho \leq t < \zeta \\
1, & \text{if } \zeta \leq t \leq \bar{\zeta} \\
\frac{\bar{\zeta} + \sigma - t}{\sigma}, & \text{if } \bar{\zeta} < t \leq \bar{\zeta} + \sigma \\
0, & \text{otherwise}
\end{cases}$$

and credibility distribution

$$\psi(t) = \begin{cases} 
0, & \text{if } t < \zeta - \rho \\
\frac{t + \rho - \zeta}{2\rho}, & \text{if } \zeta - \rho \leq t < \zeta \\
0.5, & \text{if } \zeta \leq t \leq \bar{\zeta} \\
\frac{t + \sigma - \bar{\zeta}}{2\sigma}, & \text{if } \bar{\zeta} < t \leq \bar{\zeta} + \sigma \\
1, & \text{if } t > \bar{\zeta} + \sigma
\end{cases}$$

as depicted in Fig. 2a, b, respectively.

**Example 2** If an LR-FI $(\zeta, \bar{\zeta}, \rho, \sigma)_{LR}$ has the shape functions $L(t) = R(t) = \max\{0, 1 - t^2\}$, denoted by $A(\zeta, \bar{\zeta}, \rho, \sigma)_{LR}$, then it has the membership function

$$\nu(t) = \begin{cases} 
1 - \frac{(\zeta - t)^2}{\rho^2}, & \text{if } \zeta - \rho \leq t < \zeta \\
1, & \text{if } \zeta \leq t \leq \bar{\zeta} \\
1 - \frac{(t - \bar{\zeta})^2}{\sigma^2}, & \text{if } \bar{\zeta} < t \leq \bar{\zeta} + \sigma \\
0, & \text{otherwise}
\end{cases}$$

$$\psi(t) = \begin{cases} 
0, & \text{if } t < \zeta - \rho \\
\frac{t + \rho - \zeta}{2\rho}, & \text{if } \zeta - \rho \leq t < \zeta \\
0.5, & \text{if } \zeta \leq t \leq \bar{\zeta} \\
\frac{t + \sigma - \bar{\zeta}}{2\sigma}, & \text{if } \bar{\zeta} < t \leq \bar{\zeta} + \sigma \\
1, & \text{if } t > \bar{\zeta} + \sigma
\end{cases}$$
Fig. 2 The membership function and credibility distribution of $T(\xi, \bar{\xi}, \rho, \sigma)_{LR}$ in Example 1

\[
\psi(t) = \begin{cases} 
0, & \text{if } t < \xi - \rho \\
0.5 - \frac{(t - \xi)^2}{2\rho^2}, & \text{if } \xi - \rho \leq t < \xi \\
0.5, & \text{if } \xi \leq t \leq \bar{\xi} \\
0.5 + \frac{(t - \bar{\xi})^2}{2\sigma^2}, & \text{if } \bar{\xi} < t \leq \bar{\xi} + \sigma \\
1, & \text{if } t > \bar{\xi} + \sigma,
\end{cases}
\]

as depicted in Fig. 3a, b, respectively.

**Example 3** If an LR-FI $(\xi, 2, 4, 2, 4)_{LR}$ has the shape functions $L(t) = \max\{1 - t, 0\}$ and $R(t) = \max\{1 - t^2, 0\}$, denoted by $B(\xi, \bar{\xi}, \rho, \sigma)_{LR}$, then it has the membership function

\[
v(t) = \begin{cases} 
\frac{t + \rho - \xi}{\rho}, & \text{if } \xi - \rho \leq t < \xi \\
1, & \text{if } \xi \leq t \leq \bar{\xi} \\
1 - \frac{(t - \bar{\xi})^2}{\sigma^2}, & \text{if } \bar{\xi} < t \leq \bar{\xi} + \sigma \\
0, & \text{otherwise}
\end{cases}
\]

and the credibility distribution

\[
\psi(t) = \begin{cases} 
0, & \text{if } t < \xi - \rho \\
0.5 - \frac{(t - \xi)^2}{2\rho^2}, & \text{if } \xi - \rho \leq t < \xi \\
0.5, & \text{if } \xi \leq t \leq \bar{\xi} \\
0.5 + \frac{(t - \bar{\xi})^2}{2\sigma^2}, & \text{if } \bar{\xi} < t \leq \bar{\xi} + \sigma \\
1, & \text{if } t > \bar{\xi} + \sigma,
\end{cases}
\]

as depicted in Fig. 4a, b, respectively.

**Example 4** If an LR-FI $(2, 4, 2, 4)_{LR}$ has the following shape functions

\[
\begin{align*}
L(t) &= \begin{cases} 
-0.8t + 1, & \text{if } 0 \leq t \leq 0.25 \\
0.8, & \text{if } 0.25 < t \leq 0.5 \\
-1.6t + 1.6, & \text{if } 0.5 < t \leq 1 \\
0, & \text{otherwise},
\end{cases} \\
R(t) &= \begin{cases} 
0.3, & \text{if } 0 \leq t < 1 \\
0.6, & \text{if } 1 \leq t < 2 \\
1, & \text{if } 2 \leq t \leq 4 \\
-0.2t + 1.8, & \text{if } 4 < t \leq 5 \\
0.8, & \text{if } 5 < t \leq 6 \\
-0.4t + 3.2, & \text{if } 6 \leq t \leq 8 \\
0, & \text{otherwise}
\end{cases}
\end{align*}
\]

then it can be deduced that it has the membership function
Fig. 3 The membership function and credibility distribution of \( A(\xi, \bar{\xi}, \rho, \sigma)_{LR} \) in Example 2

Fig. 4 The membership function and credibility distribution of \( B(\xi, \bar{\xi}, \rho, \sigma)_{LR} \) in Example 3

and the credibility distribution

\[
\psi(t) = \begin{cases} 
0, & \text{if } t \leq 0 \\
0.15, & \text{if } 0 < t < 1 \\
0.3, & \text{if } 1 \leq t < 2 \\
0.5, & \text{if } 2 \leq t < 4 \\
0.1t + 0.1, & \text{if } 4 \leq t < 5 \\
0.6, & \text{if } 5 \leq t < 6 \\
0.2t - 0.6, & \text{if } 6 \leq t < 8 \\
1, & \text{if } t \geq 8,
\end{cases}
\]

as depicted in Fig. 5a, b, respectively.

2.2 Inverse credibility distribution of LR fuzzy interval

For our purpose, the ICD of an LR-FI is defined as below, which will play an important role then.

Definition 3 Let \( \xi \) be an LR-FI. A multi-valued function \( F : [0, 1] \rightarrow \mathbb{R} \) is called the ICD of \( \xi \) if

\[
\text{Cr}\{\xi \leq f_\delta\} = \bar{\delta}, \quad \delta \in [0, 1],
\]

where \( f_\delta \in \{t|t = F(\delta)\} \) and \( \bar{\delta} = \sup\{\gamma|F(\gamma) = F(\delta)\} \).

Remark 2 For simplicity, \( F(\delta) \) is denoted by \( \psi^{-1}(\delta) \), which differs from the inverse function of \( \psi(t) \).
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Theorem 1 Let \( \psi(t) \) be the credibility distribution of an LR-FI \( \zeta \), and \( D_\psi \) be the domain of values of \( \psi(t) \). Then the ICD of \( \zeta \) is deduced as

\[
\psi^{-1}(\delta) =\begin{cases} 
\sup\{t|\psi(t) = 0\}, & \text{if } \delta = 0 \\
\{t|\psi(t) = \delta\}, & \text{if } 0 < \delta < 1 \\
\inf\{t|\psi(t) \geq \delta\}, & \text{if } 0 < \delta < 1 \\
\inf\{t|\psi(t) = 1\}, & \text{if } \delta = 1.
\end{cases}
\]

(6)

Proof The proof is provided in Appendix B.

Example 5 The ICD of \( T(\zeta, \bar{\zeta}, \rho, \sigma)_{LR} \) in Example 1 is (see Fig. 6)

\[
\psi^{-1}(\delta) =\begin{cases} 
2\rho \delta + \zeta - \rho, & \text{if } 0 \leq \delta < 0.5 \\
\{\zeta, \bar{\zeta}\}, & \text{if } \delta = 0.5 \\
2\sigma \delta + \bar{\zeta} - \sigma, & \text{if } 0.5 < \delta \leq 1.
\end{cases}
\]

Example 6 The ICD of \( A(\zeta, \bar{\zeta}, \rho, \sigma)_{LR} \) in Example 2 is (see Fig. 7)

\[
\psi^{-1}(\delta) =\begin{cases} 
\zeta - \rho \sqrt{1 - 2\delta}, & \text{if } 0 \leq \delta < 0.5 \\
\{\zeta, \bar{\zeta}\}, & \text{if } \delta = 0.5 \\
\bar{\zeta} + \sigma \sqrt{2\delta - 1}, & \text{if } 0.5 < \delta \leq 1.
\end{cases}
\]

Example 7 The ICD of \( B(\zeta, \bar{\zeta}, \rho, \sigma)_{LR} \) in Example 3 is (see Fig. 8)

\[
\psi^{-1}(\delta) =\begin{cases} 
2\rho \delta + \zeta - \rho, & \text{if } 0 \leq \delta < 0.5 \\
\{\zeta, \bar{\zeta}\}, & \text{if } \delta = 0.5 \\
\bar{\zeta} + \sigma \sqrt{2\delta - 1}, & \text{if } 0.5 < \delta \leq 1.
\end{cases}
\]
Example 8 The ICD of the LR-FI \((2, 4, 2, 4)_{LR}\) in Example 4 is (see Fig. 9)

\[
\psi^{-1}(\delta) = \begin{cases} 
(-\infty, 0], & \text{if } \delta = 0 \\
0, & \text{if } 0 < \delta < 0.15 \\
(0, 1), & \text{if } \delta = 0.15 \\
1, & \text{if } 0.15 < \delta < 0.3 \\
[1, 2), & \text{if } \delta = 0.3 \\
2, & \text{if } 0.3 < \delta < 0.5 \\
[2, 4), & \text{if } \delta = 0.5 \\
10\delta - 1, & \text{if } 0.5 \leq \delta < 0.6 \\
[5, 6), & \text{if } \delta = 0.6 \\
5\delta + 3, & \text{if } 0.6 \leq \delta < 1 \\
[8, +\infty), & \text{if } \delta = 1.
\end{cases}
\]

2.3 Regular LR fuzzy interval

It is worth noting that the credibility distributions \(\psi(t)\) (or the ICDs \(\psi^{-1}(\delta)\)) of LR-FIs in Examples 1-3 (or in Examples 5-7) are continuous and strictly increasing on the domain \(|t| 0 < \psi(t) < 0.5\) or \(0.5 < \psi(t) < 1\). For the sake of describing such kind of LR-FIs, we first introduce the definition of regular LR-FI proposed in Liu et al. (2020) and then verify two equivalent conditions.

Definition 4 (Liu et al. 2020) If the shape functions \(L\) and \(R\) of an LR-FI \(\xi\) are continuous and strictly decreasing on the domains \(|t| 0 < L(t) < 1\) and \(|t| 0 < R(t) < 1\), respectively, then the LR-FI is regular.

Theorem 2 An LR-FI is regular if and only if it satisfies any one of the following conditions:

(i) The credibility distribution \(\psi(t)\) is continuous on the domain \(|t| 0 < \psi(t) < 1\) and strictly increasing on the domain \(|t| 0 < \psi(t) < 0.5\) or \(0.5 < \psi(t) < 1\);
(ii) The ICD \(\psi^{-1}(\delta)\) is continuous on \((0, 1)\) and strictly increasing on \((0, 0.5) \cup (0.5, 1)\).

\[\psi^{-1}(\delta) = \begin{cases} 
L - \rho, & \text{if } 0 \leq \delta < 0.5 \\
\xi, & \text{if } \delta = 0.5 \\
L - R^{-1}(2\delta) - 1, & \text{if } 0.5 \leq \delta \leq 1.
\end{cases} \]

3 Operational law and expected value for monotone function of regular LR fuzzy intervals

This section first recalls the the concept of monotonicity of functions, and discusses the property of credibility distributions of monotone functions with regard to regular LR-FIs. On this basis, we put forward a novel operational law on monotone function with regard to regular LR-FIs and further explore its EV.

3.1 Monotone function of regular LR fuzzy intervals

At present the monotonicity of a function is defined by many scholars from various perspectives. The definitions of monotone function and strictly monotone function proposed by Liu et al. (2016) and Liu (2010) respectively are employed in this paper.
**Definition 5** (Liu et al. 2016; Liu 2010) A real-valued function \( f(t_1, t_2, \ldots, t_n) \) is called monotone function if it is increasing regarding \( t_1, t_2, \ldots, t_k \) and decreasing regarding \( t_{k+1}, t_{k+2}, \ldots, t_n \), that is,

\[
f(t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) \geq f(s_1, \ldots, s_k, s_{k+1}, \ldots, s_n)
\]

holds for any \( t_i \geq s_i \) for \( i = 1, 2, \ldots, k \) and \( t_i \leq s_i \) for \( i = k + 1, \ldots, n \). Moreover, if the function \( f(t_1, t_2, \ldots, t_n) \) satisfies

\[
f(t_1, \ldots, t_k, t_{k+1}, \ldots, t_n) > f(s_1, \ldots, s_k, s_{k+1}, \ldots, s_n)
\]

for any \( t_i > s_i \) for \( i = 1, 2, \ldots, k \) and \( t_i < s_i \) for \( i = k + 1, \ldots, n \), then it is said to be strictly monotone.

**Remark 3** A real-valued function \( f(t_1, t_2, \ldots, t_n) \) is called increasing (decreasing) function regarding \( t_1, t_2, \ldots, t_n \) if \( f(t_1, t_2, \ldots, t_n) \geq f(s_1, s_2, \ldots, s_n) \) holds for any \( t_i \geq s_i (t_i \leq s_i) \) for \( i = 1, 2, \ldots, n \). Moreover, if the function \( f(t_1, t_2, \ldots, t_n) \) satisfies \( f(t_1, t_2, \ldots, t_n) > f(s_1, s_2, \ldots, s_n) \) for any \( t_i > s_i (t_i < s_i) \) for \( i = 1, 2, \ldots, n \), then it is called strictly increasing (decreasing) function.

**Example 9** The following functions are strictly monotone,

\[
f(t_1, t_2) = t_1 - t_2,
\]

\[
f(t_1, t_2) = t_1 \times t_2, \quad t_1, t_2 > 0.
\]

**Example 10** The following functions are monotone but not strictly monotone,

\[
f(t_1, t_2) = a \lor t_1 - b \land t_2,
\]

\[
f(t_1, t_2) = a \lor t_1/(b \land t_2), \quad a \lor t_1, b \land t_2 > 0.
\]

In some practical optimization problems, the objective functions of the formulated optimization models are usually monotone but not strictly monotone, such as the well-known newsvendor problem, inventory problem, project scheduling problem, etc. Considering the generality of monotone functions in practical applications, we will proceed to analyze the property of continuous and monotone (but not necessarily strictly monotone) functions of regular LR-FIs.

**Theorem 4** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent regular LR-FIs and \( f : \mathbb{R}^n \to \mathbb{R} \) a continuous and monotone function. Then \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) is an LR-FI.

**Proof** The proof is provided in Appendix C. \( \square \)

**Example 11** Let \( \xi_1 \sim T(2, 4, 2, 3)_{LR} \). The function \( f_1 \)

\[
f_1(t) = \begin{cases} 
  t - 1, & \text{if } t < 3 \\
  2, & \text{if } 3 \leq t \leq 5 \\
  t - 3, & \text{if } t > 5
\end{cases}
\]

is increasing but not strictly increasing. Then the credibility distribution of \( f_1(\xi_1) \) is obtained as follows

\[
\psi(t) = \begin{cases} 
  0, & \text{if } t < -1 \\
  t + 1, & \text{if } -1 \leq t < 1 \\
  \frac{4}{5}, & \text{if } 0 \leq t \leq 4 \\
  \frac{6}{7}, & \text{if } 2 < t \leq 4 \\
  1, & \text{if } t > 4,
\end{cases}
\]

as depicted in Fig. 10. It can be concluded from Fig. 10 that \( f_1(\xi_1) \) is not a regular LR-FI.

### 3.2 Operational law

Based on the extensive applications of monotone functions and regular LR-FIs in optimization problems, a new operational law is proposed in this subsection, which can be considered as an extension to the one developed in Zhou et al. (2016).

**Theorem 5** Let \( \xi_1, \xi_2, \ldots, \xi_n \) be independent regular LR-FIs with ICD \( \psi_1^{-1}, \psi_2^{-1}, \ldots, \psi_n^{-1} \), respectively. If the continuous function \( f(t_1, t_2, \ldots, t_n) \) is increasing regarding \( t_1, t_2, \ldots, t_k \) and decreasing regarding \( t_{k+1}, t_{k+2}, \ldots, t_n \), then \( \xi = f(\xi_1, \xi_2, \ldots, \xi_n) \) is an LR-FI with the ICD

\[
\psi^{-1}(\delta) = (\psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1 - \delta), \ldots, \psi_n^{-1}(1 - \delta)).
\]

**Proof** The proof is provided in Appendix D. \( \square \)

**Remark 4** It should be noted that the operational law proposed by Zhou et al. (2016) is used to compute the ICDs of strictly monotone functions regarding regular LR fuzzy numbers. However, the new operational law provides a convenient and powerful approach to computing the ICDs of monotone (but not strictly monotone) functions regarding regular LR-FIs, which are unable to be obtained by the operational law in Zhou et al. (2016).

**Example 12** Let \( \xi_1 \) be a trapezoidal fuzzy number denoted by \( T(2, 4, 2, 3)_{LR} \) with the credibility distribution \( \psi_1 \). Then the ICD of \( f_1(\xi_1) \) with \( f_1 \) defined in Eq. (8) is deduced from Theorem 5 as

\[
\psi_{f_1}^{-1}(\delta) = f_1(\psi_1^{-1}(\delta)) = \begin{cases} 
  4\delta - 1, & \text{if } 0 \leq \delta \leq \frac{1}{2} \\
  [1, 2], & \text{if } \delta = \frac{1}{2} \\
  2, & \text{if } \frac{1}{2} < \delta \leq \frac{2}{3} \\
  6\delta - 2, & \text{if } \frac{2}{3} < \delta \leq 1,
\end{cases}
\]

\[\square\]
Example 13  Let \( \zeta_2 \) be a trapezoidal fuzzy number denoted by \( T(1, 2, 2, 1) \) with the credibility distribution \( \psi_2 \). Then by using the operational law in Theorem 5, it is easy to deduce the ICD of \( f_2(\zeta_2) \) where the function \( f_2 \) is defined as

\[
f_2(t) = 5 - 2t \vee 1.
\]

(11)

Since \( f_2 \) is decreasing, in the light of Theorem 5, the ICD of \( f_2(\zeta_2) \) is derived as (see Fig. 12)

\[
\psi_2^{-1}(\delta) = f_2(\psi_2^{-1}(1 - \delta))
\]

\[
= \begin{cases} 
4\delta - 1, & \text{if } 0 \leq \delta < \frac{1}{2} \\
1, & \text{if } \delta = \frac{1}{2} \\
8\delta - 1, & \text{if } \frac{1}{2} \leq \delta < \frac{5}{8} \\
4, & \text{if } \frac{5}{8} \leq \delta \leq 1.
\end{cases}
\]

(12)

as depicted in Fig. 11.

Example 14  Let \( \zeta_1 \) and \( \zeta_2 \) be two independent trapezoidal fuzzy numbers denoted by \( T(2, 4, 2, 3) \) with the ICDs \( \psi_1^{-1} \) and \( \psi_2^{-1} \), respectively. As the function \( f(\zeta_1, \zeta_2) = f_1(\zeta_1) + f_2(\zeta_2) \) with \( f_1 \) and \( f_2 \) defined in Eqs. (8) and (11), respectively, is increasing regarding \( \zeta_1 \) and decreasing regarding \( \zeta_2 \), in accordance with Theorem 5, the ICD of \( \zeta = f(\zeta_1, \zeta_2) = f_1(\zeta_1) + f_2(\zeta_2) \) is obtained as

\[
\psi^{-1}(\delta) = f_1(\psi_1^{-1}(\delta)) + f_2(\psi_2^{-1}(1 - \delta)).
\]

Then, on account of Eqs. (10) and (12), we can obtain that

\[
\psi^{-1}(\delta) = \begin{cases} 
8\delta - 2, & \text{if } 0 \leq \delta < \frac{1}{2} \\
[2, 5], & \text{if } \delta = \frac{1}{2} \\
8\delta + 1, & \text{if } \frac{1}{2} < \delta < \frac{5}{8} \\
6, & \text{if } \frac{5}{8} < \delta \leq \frac{2}{3} \\
6\delta + 2, & \text{if } \frac{2}{3} < \delta \leq 1.
\end{cases}
\]

(13)

as depicted in Fig. 13.

3.3 Expected value

Expected value (EV) is the mean value of all possible values of a fuzzy variable in the sense of fuzzy measure. Based on the EV of a fuzzy variable defined by Liu and Liu (2002), a calculation formula of the EV of an LR-FI is presented.
Theorem 7 If the EV of the LR-FI $\zeta = f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is increasing regarding $t_1, t_2, \ldots, t_k$ and decreasing regarding $t_{k+1}, t_{k+2}, \ldots, t_n$, then the EV of the LR-FI $\xi = f(\xi_1, \xi_2, \ldots, \xi_d)$ is

$$E[\xi] = \int_0^1 f(\psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1-\delta), \ldots, \psi_n^{-1}(1-\delta))d\delta.$$ (14)

**Proof** On the basis of Theorem 5, we can derive that the ICD of $\zeta = f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is $f(\psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1-\delta), \ldots, \psi_n^{-1}(1-\delta))$. Then by using Theorem 6, we obtain Eq. (14). □

**Example 16** Let us consider the EV of $\zeta = f(\zeta_1, \zeta_2) = f_1(\zeta_1) + f_2(\zeta_2)$ in Example 13, whose ICD is $f_1(\psi_1^{-1}(\delta)) + f_2(\psi_2^{-1}(1-\delta))$ in Eq. (13). Then according to Theorem 7, the EV of $\zeta = f(\zeta_1, \zeta_2)$ is

$$E[\xi] = \int_0^1 \left(f_1(\psi_1^{-1}(\delta)) + f_2(\psi_2^{-1}(1-\delta))\right)d\delta$$

$$= \int_0^{\frac{1}{2}} (8\delta - 2)d\delta + \int_{\frac{1}{2}}^1 (8\delta + 2)d\delta$$

$$+ \int_{\frac{1}{2}}^1 6d\delta + \int_{\frac{1}{2}}^1 (6\delta + 2)d\delta$$

$$= \frac{157}{48}.$$

### 4 Fuzzy programming

Fuzzy programming is a type of mathematical models to address optimization problems involving fuzzy parameters, which has been studied by many researchers from different points of view (see Liu 1998; Liu and Iwamura 1998a,b; Liu and Liu 2002; Zhou et al. 2016). In this section, we discuss the fuzzy CCP model in Zhou et al. (2016) containing monotone but not necessarily strictly monotone objective and constraint functions with regular LR-FIs and then develop a solution framework.

#### 4.1 Fuzzy CCP model and its equivalent crisp model

Suppose that $t$ is a decision vector, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ is an $n$-dimensional fuzzy vector, $f(t, \xi)$ is the objective function, and $h_v(t, \xi)$ is the constraint function for $v = 1, 2, \ldots, w$. Owing to the fuzziness of the objective function $f(t, \xi)$, it is hard to be minimized directly. As an alternative way, it is quite natural to minimize its EV, i.e., $E[f(t, \xi)]$. In addition, as to the fuzzy constraints $h_v(t, \xi) \leq 0, v = 1, 2, \ldots, w$, since there is no deterministic feasible set defined by them, Liu and Iwamura (1998a) suggested that it should be desirable that the solutions satisfy the fuzzy constraints at predetermined
confident levels \( \delta_1, \delta_2, \ldots, \delta_w \), that is,
\[
\text{Cr}[h_v(t, \xi) \leq 0] \geq \delta_v, \quad v = 1, 2, \ldots, w.
\]

In this way, a fuzzy CCP model to minimize the EV of objective function under a series of chance constraints was constructed by Zhou et al. (2016) as
\[
\begin{aligned}
& \min \ E[f(t, \xi)] \\
& \text{subject to:} \\
& \text{Cr}[h_v(t, \xi) \leq 0] \geq \delta_v, \quad v = 1, 2, \ldots, w.
\end{aligned}
\tag{15}
\]

When fuzzy parameters in the fuzzy CCP model (15) are regular LR-FIs, and the objective and constraint functions are both continuous and monotone with regard to these fuzzy parameters, model (15) can be converted to a deterministic counterpart, which is verified in the following theorem.

**Theorem 8** Assume that the function \( f(t, \xi_1, \xi_2, \ldots, \xi_n) \) is continuous and increasing regarding \( \xi_1, \xi_2, \ldots, \xi_k \) and decreasing regarding \( \xi_{k+1}, \xi_{k+2}, \ldots, \xi_w \), and the function \( h_v(t, \xi_1, \xi_2, \ldots, \xi_n) \) is increasing regarding \( \xi_1, \xi_2, \ldots, \xi_{k_v} \) and decreasing regarding \( \xi_{k_v+1}, \xi_{k_v+2}, \ldots, \xi_w \) for \( v = 1, 2, \ldots, w \). If \( \xi_1, \xi_2, \ldots, \xi_n \) are independent regular LR-FIs, then model (15) can be converted to the following crisp equivalent one
\[
\begin{aligned}
& \min \int_0^1 f(t, \psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1-\delta), \\
& \quad \ldots, \psi_n^{-1}(1-\delta))d\delta \\
& \text{subject to:} \\
& h_v(t, \psi_1^{-1}(\delta_v), \ldots, \psi_k^{-1}(\delta_v), \psi_{k+1}^{-1}(1-\delta_v), \\
& \quad \ldots, \psi_n^{-1}(1-\delta_v)) \leq 0, \\
& v = 1, 2, \ldots, w,
\end{aligned}
\tag{16}
\]

where \( \psi_i^{-1} \) is the ICD of \( \xi_i \) for \( i = 1, 2, \ldots, n \).

**Proof** The proof is provided in Appendix F.

\[ \square \]

### 4.2 Solution methods

It is worth noting that there exists an integral in the objective function of model (16), which means that the fuzzy model (15) cannot be solved directly by well-developed software packages after translation. In order to solve model (15), Liu (2002) designed a fuzzy simulation-based genetic algorithm, called hybrid intelligent algorithm (HIA), by integrating a stochastic discretization algorithm (SDA) into a classical genetic algorithm. However, Li (2015) and Liu et al. (2020) pointed out that SDA has poor performance both on accuracy and computational time over simulating the EV. Liu et al. (2020) subsequently proposed a numerical-integral based algorithm, but it is not applicable to monotone but not strictly monotone functions with regard to regular LR-FIs. Thus this paper proposes a new numerical integration algorithm (NIA) to fill the gap.

With regard to the basic principle of NIA for simulating \( E[f(t, \xi)] \), on account of Theorem 8, we know that \( E[f(t, \xi)] \) is an integration of function \( f(t, \psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1-\delta), \ldots, \psi_n^{-1}(1-\delta)) \). Based on the definition of definite integral in mathematics, we partition the closed interval \([0, 1]\) into \( S \) equal parts and take the value of the right of each equal part as the integration variable, that is, \( \delta = s/S \) for \( s = 1, 2, \ldots, S \). When the number of integration points, \( S \), is set to be sufficiently large, we can obtain that
\[
E[f(t, \xi)] \approx \sum_{s=1}^S f(t, \psi_1^{-1}(s/S), \ldots, \psi_n^{-1}(1-s/S)), \psi_{k+1}^{-1}(1-s/S), \ldots, \psi_n^{-1}(1-s/S))/S.
\]

The NIA is given as Algorithm 1.

**Algorithm 1:** (NIA)

**Step 1.** Initialize the number of integration points \( S \); \( E = 0 \); \( s = 1 \).

**Step 2.** Calculate \( \gamma_i = \psi_i^{-1}(s/S) \) for each \( 1 \leq i \leq k \) and \( \gamma_i = \psi_i^{-1}(1-s/S) \) for \( k+1 \leq i \leq n \) based on Eq. (7).

**Step 3.** Update \( E = E + f(t, \gamma_1, \gamma_2, \ldots, \gamma_n)/S \) and \( s = s + 1 \).

**Step 4.** If \( s \leq S \), go to Step 2. Otherwise, return the \( E \).

To illustrate the performance of NIA on accuracy and efficiency, comparisons between NIA and SDA for simulating the EV of monotone functions over some numerical experiments of two examples are conducted.

**Example 17** Consider two continuous and monotone functions \( f_1(\xi_1, \xi_2) = 5 \wedge \xi_1 + 1 \vee \xi_2 \) and \( f_2(\xi_1, \xi_2) = (5 \wedge \xi_1)/(1 \vee \xi_2) \) regarding two fuzzy intervals, which are set to regular LR-FIs appearing in Examples 1-3 successively as listed in Table 1.

**Example 18** Consider two more complex monotone functions \( f_3(\xi_1, \xi_2, \xi_3, \xi_4) = (5 \wedge \xi_1)/(1 \vee \xi_2) + (4 \vee \xi_3)/(3 \wedge \xi_4) \) and \( f_4(\xi_1, \xi_2, \xi_3, \xi_4) = (5 \wedge \xi_1)/(1 \vee \xi_2) \times (4 \vee \xi_3)/(3 \wedge \xi_4) \) regarding four fuzzy intervals, which are also set to three kinds of regular LR-FIs as listed in Table 1.

For each case, after using SDA and NIA (5000 points for simulation) to calculate E vs of the functions, respectively, the experimental results of two examples covering exact value, simulation value and running time are all listed.
in Tables 2 and 3, respectively. To facilitate comparing the differences between the simulation values obtained by two algorithms and the exact value obtained based on the extensive operation law, a parameter named Error is introduced, which is derived from the formula $\frac{|\text{simulation value} - \text{exact value}|}{\text{exact value}} \times 100\%$.

From Tables 2 and 3, it can be seen that there are slight differences over errors (i.e., $\leq 0.05\%$) between the EVs obtained by NIA and the exact values, and the errors are still small as the number of fuzzy intervals and the complexity of monotone functions increase. When it comes to the function $f_4$, however, the largest error between the EVs derived by SDA and the exact values is up to 12.82%. In addition, as the number of fuzzy intervals and the complexity of monotone functions increase, the errors may further increase. Therefore, NIA is more reliable and stable in terms of the accuracy of solutions compared with SDA. On the other hand, the running time of NIA can be negligible, while the running time of SDA is more than 120 times slower than NIA’s. Overall, NIA outperforms SDA and is probably able to get an accurate value in a relatively short time.

Based on the above analyses, we embed NIA used for simulating $E[f(t, \xi)]$ into a classical genetic algorithm, thereby formulating a new algorithm (NIA-GA) to dispose of model (16), whose performance will be compared with HIA algorithm and evaluated on a set of numerical experiments from a purchasing planning problem in the following section.

### 4.3 Numerical example

Provided that there is a dealer selling $n$ types of products, he would like to determine the optimal order quantity to satisfy customer demands for products with the aim of maximizing the total profit. In order to have a better understanding for this problem, some assumptions are given as follows and some relevant notations are shown in Table 4 where the parameter values are summarized in Table 5.

#### Assumption

1. The customer demands are uncertain and characterized by regular LR-FIs.
2. Any leftover inventory can be salvaged at a unit value, which is lower than the selling price.
3. The total cost of purchasing products from supplier is not more than budget.

According to the assumptions and notations, the total procurement cost, total opportunity loss and the total profit are

\[
C(t) = \sum_{i=1}^{n} c_i t_i,
\]

\[
S(t, \xi) = \sum_{i=1}^{n} s_i (\xi_i - \xi_i \wedge t_i)
\]

and

\[
\Pi(t, \xi) = \sum_{i=1}^{n} \left[ (p_i - v_i)(\xi_i \wedge t_i) + (v_i - c_i)t_i \right],
\]

respectively.

Assuming that the total budget on the procurement is \(C^0\) and the biggest opportunity loss the dealer can undertake is \(S^0\). \(C^0\) is mainly determined based on the dealer’s financial performance, and \(S^0\) is subjectively determined by the decision maker according to his risk-averse attitude and the available budget \(C^0\). Then it follows from the idea of fuzzy CCP model (15) that a fuzzy CCP model for this problem is constructed as

\[
\begin{align*}
\max & \quad E \left[ \sum_{i=1}^{n} [(p_i - v_i)(\xi_i \wedge t_i) + (v_i - c_i)t_i] \right] \\
\text{subject to:} & \quad \sum_{i=1}^{n} c_i t_i \leq C^0 \\
& \quad \text{Cr} \left\{ \sum_{i=1}^{n} s_i (\xi_i - \xi_i \wedge t_i) \right\} \geq S^0 \\
& \quad t_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{align*}
\]

(17)

where \(\psi_i^{-1}\) is the ICD of \(\xi_i\), which can be derived from Theorem 1.

Afterward, 40 test problems are generated by increasing \(C^0\) from 61000 to 70000 with an increase of 1000 and decreasing \(S^0\) from 6500 to 2000 with a decrease of 500 simultaneously under the confidence level fixed at 0.6, 0.7, 0.8 and 0.9, respectively. For each problem, HIA and NIA-GA (5000 points for simulation and 300 generations in genetic algorithm) are run accordingly to solve the models (17) and (18), respectively. Considering the randomness of results obtained by metaheuristic algorithms, we implement each test problem for 10 times and then select the optimal solution with the best target value as the final solution. Then the optimal solutions, the corresponding target values, \(E[\Pi(t^*, \xi)]\), and the average time for running 10 times are shown in Table 6. Moreover, in accordance with the poor performance of SDA in HIA as illustrated in Sect. 4.2, a new EV of profit, \(E[\Pi(t^*, \xi)]^*\), is computed by substituting the optimal solution acquired by HIA into NIA, which is listed in the last column of Table 6. So, it makes sense to judge the quality of the optimal solutions obtained by HIA and NIA-GA by comparing \(E[\Pi(t^*, \xi)]^*\) with \(E[\Pi(t^*, \xi)]\) in last third column of Table 6. Furthermore, for the sake of visualizing the differences better, the target values, \(E[\Pi(t^*, \xi)]^*\) obtained by two solution methods and \(E[\Pi(t^*, \xi)]^*\) in Table 6 are plotted in Fig. 14a–d.

From Table 6, it can be seen that NIA-GA has an outstanding advantage over HIA in terms of running time. Concretely, the running time of NIA-GA is almost one hundred times faster than that of HIA. The reason is that the fuzzy simulation (SDA) in HIA for the EV of profit and chance constraint of opportunity loss is time-consuming. In the meantime, as for the quality of solutions found by two methods, we can conclude that the solutions derived
| No. | $\delta_0$ | $C^0$ | $S^0$ | $E[\Pi(t^*, \zeta)]$ | $E[\Pi(t^*, \zeta)]$ | Running time (s) |
|-----|-----------|-------|-------|-----------------|-----------------|------------------|
| 1   | 0.9       | 61000 | 6500  | 33366.09        | 33548.43        | 15.21            |
| 2   | 0.9       | 62000 | 6000  | 33207.61        | 33850.47        | 15.23            |
| 3   | 0.9       | 63000 | 5500  | 33502.66        | 34088.54        | 15.60            |
| 4   | 0.9       | 64000 | 5000  | 33349.17        | 34280.58        | 16.16            |
| 5   | 0.9       | 65000 | 4500  | 33890.26        | 34542.78        | 16.60            |
| 6   | 0.9       | 66000 | 4000  | 34354.63        | 34515.35        | 16.72            |
| 7   | 0.9       | 67000 | 3500  | 34731.30        | 34525.38        | 16.75            |
| 8   | 0.9       | 68000 | 3000  | 34745.76        | 34525.23        | 16.88            |
| 9   | 0.9       | 70000 | 2500  | 34772.15        | 34525.39        | 16.81            |
| 10  | 0.8       | 61000 | 6500  | 33280.12        | 33560.44        | 15.41            |
| 11  | 0.8       | 62000 | 6000  | 33478.67        | 33876.97        | 15.44            |
| 12  | 0.8       | 63000 | 5500  | 33790.93        | 34050.07        | 15.52            |
| 13  | 0.8       | 64000 | 5000  | 33907.88        | 34256.08        | 15.54            |
| 14  | 0.8       | 65000 | 4500  | 34602.57        | 34417.79        | 15.52            |
| 15  | 0.8       | 66000 | 4000  | 34748.72        | 34515.35        | 16.37            |
| 16  | 0.8       | 67000 | 3500  | 34760.56        | 34525.38        | 16.40            |
| 17  | 0.8       | 68000 | 3000  | 34719.65        | 34525.38        | 16.71            |
| 18  | 0.8       | 69000 | 2500  | 34750.27        | 34525.23        | 16.51            |
| 19  | 0.8       | 61000 | 6500  | 33321.03        | 33595.46        | 15.36            |
| 20  | 0.7       | 62000 | 6000  | 33934.34        | 33845.53        | 15.44            |
| 21  | 0.7       | 63000 | 5500  | 34325.94        | 34088.54        | 15.47            |
| 22  | 0.7       | 64000 | 5000  | 34449.53        | 34275.55        | 15.99            |
| 23  | 0.7       | 65000 | 4500  | 34623.54        | 34428.12        | 16.15            |
| 24  | 0.7       | 66000 | 4000  | 34644.35        | 34524.37        | 16.62            |
| 25  | 0.7       | 67000 | 3500  | 34754.97        | 34525.38        | 16.54            |
| 26  | 0.7       | 68000 | 3000  | 34739.31        | 34525.38        | 16.57            |
| 27  | 0.7       | 69000 | 2500  | 34755.82        | 34525.21        | 16.37            |
| 28  | 0.7       | 61000 | 6500  | 33324.75        | 33419.17        | 15.49            |
| 29  | 0.6       | 62000 | 6000  | 33949.51        | 33823.66        | 15.47            |
| 30  | 0.6       | 63000 | 5500  | 34013.85        | 34050.07        | 15.57            |
| 31  | 0.6       | 64000 | 5000  | 34440.24        | 34268.57        | 15.89            |
| 32  | 0.6       | 65000 | 4500  | 34553.59        | 34448.29        | 15.84            |
| 33  | 0.6       | 66000 | 4000  | 34749.81        | 34503.84        | 16.58            |
| 34  | 0.6       | 67000 | 3500  | 34738.98        | 34525.38        | 16.51            |
| 35  | 0.6       | 68000 | 3000  | 34744.23        | 34525.23        | 16.44            |
| 36  | 0.6       | 69000 | 2500  | 34703.76        | 34525.21        | 16.53            |
| 37  | 0.6       | 70000 | 2000  | 34764.36        | 34525.39        | 16.33            |
by NIA-GA are all better than those obtained by HIA, since it can be observed from Fig. 14a–d that $E[\Pi(t^*, \xi)]$ derived by NIA-GA are all larger than $E[\Pi(t^*, \xi)^s]$, and the maximum relative deviation (calculated by formula $|E[\Pi(t^*, \xi)] - E[\Pi(t^*, \xi)^s]/E[\Pi(t^*, \xi)^s] \times 100\%)$ is up to 3.12% for No.4 test problem. Moreover, we can also see the obvious difference between $E[\Pi(t^*, \xi)]$ from HIA and $E[\Pi(t^*, \xi)^s]$ recalculated by NIA, and the maximum relative deviation (calculated by formula $|E[\Pi(t^*, \xi)] - E[\Pi(t^*, \xi)^s]/E[\Pi(t^*, \xi)^s] \times 100\%)$ reaches 0.81% for No.10 test problem. The reason is that NIA-GA reduces the fuzzy simulation on constraint function, and NIA for the computation of EV of total profit $E[\Pi(t^*, \xi)]$ is more precise than SDA, which has been proved in Examples 17 and 18. All in all, compared with HIA, NIA-GA not only has excellent performance on running time but also can obtain a better target value.

In order to see the impact of parameter $\delta_0$ on the expected profit, the sensitivity analyses of parameter $\delta_0$ with the total budget on the procurement $C^0$ under a fixed $S^0$ at 3000 and
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Fig. 15  Sensitivity analysis of parameter \( \delta_0 \) on the expected profit

with the biggest opportunity loss the dealer can undertake \( S^0 \) under a fixed \( C^0 \) at 60000 are conducted separately, which are visualized in Fig. 15a, b, respectively. From Fig. 15a, b, we can see that the expected profit decreases with \( \delta_0 \) but the pace of declines becomes slower as \( C^0 \) and \( S^0 \) increase. For the fixed \( C^0 \) and \( S^0 \), the increase of \( \delta_0 \) means that more products with less procurement cost but high opportunity loss need to be purchased to meet opportunity loss constraint, but those products have lower profit rate, which leads to the reduction of order quantity for other products with higher profit rate due to the limited budget. Thus the expected profit decreases with \( \delta_0 \). If \( C^0 \) or \( S^0 \) increases, more products with high profit rate can be purchased, and thus the expected profit decreases slowly with \( \delta_0 \).

In addition, it is clear that the expected profit increases with \( C^0 \) since more products can be purchased to meet customer demands as the \( C^0 \) increases. However, as the \( C^0 \) further increases, the dealer cannot increase the order quantity exceeding customer demands. In such a situation, the expected profit has no change. Similarly, the increase of \( S^0 \) can also improve the expected profit since more products with high profit rate can be purchased. For the higher \( S^0 \), the opportunity loss constraint is not bound and has no impact on the expected profit.

In summary, this section discusses fuzzy CCP involving monotone objective and constraint functions of regular LR-RIs and the performances of two solution methods. One is the HIA for solving the fuzzy CCP model. As to the second one, it is proved that the fuzzy CCP can be translated into crisp one based on the operational law and then NIA-GA is designed to handle the crisp model. The main difference between two methods is that HIA employs fuzzy simulation to handle the objective and constraint functions while our method utilizes NIA to deal with the objective function and reduces the fuzzy simulation on constraint function. Subsequently, two methods are used to deal with a fuzzy purchasing planning problem and their performances are evaluated by a set of numerical experiments. The experimental results demonstrate that NIA-GA not only has excellent performance on running time but also can obtain a better target value compared with HIA. Finally, sensitivity analysis for some parameters are carried out. In particular, the parameter \( \delta_0 \) has negative impact on the expected profit and its influence decreases with the increase of \( C^0 \) and \( S^0 \), while \( C^0 \) and \( S^0 \) have positive impact on the expected profit and their influences fade if \( C^0 \) or \( S^0 \) is large enough.

5 Conclusion

Fuzzy arithmetic is of great importance as an advanced tool to be used in fuzzy optimization and control theory. In this research field, Zhou et al. (2016) proposed an operational law to exactly calculate the ICDs of strictly monotone functions regarding regular LR fuzzy numbers, which facilitates the development of fuzzy arithmetic both in theory and application. Although the operational law is rather useful to handle many fuzzy optimization problems, restrictions on the strictly monotone functions and regular LR fuzzy numbers block its applications to some problems modeled by
monotone functions with LR fuzzy intervals, such as the classical newsvendor problem with fuzzy demands represented by trapezoidal fuzzy numbers. Thus, this paper aims at generalizing the operational law in Zhou et al. (2016) and exploring the generalized operational law’s applications to fuzzy arithmetic and fuzzy optimization problems.

The main findings of this study are summarized as follows. First, the ICD of an LR-FI in view of the credibility measure was defined and accordingly its calculation formula was suggested. Following that, some equivalent conditions of the regular LR-FIs were proved. Next, an extensive operational law on exactly calculating ICDs of monotone functions with regard to regular LR-FIs was proposed. Then an equivalent formula for calculating the EVs of monotone functions was proposed. Subsequently, a solution strategy for the fuzzy CCP with monotone functions of regular LR-FIs was formulated, where the fuzzy model was translated into a crisp equivalent one first and then a new heuristic algorithm called NIA-GA which integrates NIA with a standard genetic algorithm was devised. Finally, we used a purchasing planning problem to illustrate the performance of proposed solution method by comparing with HIA over a set of numerical experiments. The computational results revealed that our method outperforms HIA in both solution accuracy and efficiency. In summary, this paper made a contribution to fuzzy arithmetics and fuzzy optimization problems.

Even so this paper still has some limitations may open opportunities for future research. First, we just applied the theoretical findings to solve a fuzzy purchasing planning problem. As a matter of fact, the proposed extensive operational law is a general approach able to handle different fuzzy optimization problems involving monotone objective and constraint functions concerning regular LR-FIs. In practice, there are many such type of optimization problems, e.g., simultaneous delivery and pickup problem, project scheduling problem and reliability optimization problem under fuzzy environment, so the theoretical findings can be further applied to deal with those fuzzy optimization problems in future. Second, we suggested a new solution method to address the fuzzy optimization involving monotone function with respect to a special kind of fuzzy variables called regular LR-FIs. Apart from LR-FIs, there also exist other types of fuzzy variables, such as type-2 fuzzy numbers, intuitionistic fuzzy numbers and hesitant fuzzy numbers, which are also used by many scholars to represent the uncertainty in some situations. Future research can further extend the study to those kinds of fuzzy variables in theory so that more optimization problems can be worked out easily. Third, we devised a new heuristic algorithm called NIA-GA which integrates NIA with a standard genetic algorithms. Nevertheless, the genetic algorithm is not the only choice and many classical algorithms, such as simulated annealing algorithm, evolutionary algorithm, particle swarm optimization, etc., may be considered as an alternative. Future research can integrate NIA with other heuristic algorithms so as to handle fuzzy optimization problems effectively.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest to this work.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

**Appendix A The abbreviations**

**Appendix B The proof of Theorem 1**

**Proof** For the case of , we have , thus Eq. (5) holds.

For any and , it is obvious that there should be at least one point makes , that is, . It is not difficult to find that . Equation (5) is proved.

If and , in the light of Eq. (6), we can get . Let . Then we have . Equation (5) holds.

If , Eq. (5) is proved.

In the light of Definition 3, is the ICD of .
Appendix C The proof of Theorem 4

Proof For simplicity, we just consider the situation of \( n = 2 \). Assume that

\[
\zeta = f(\xi_1, \xi_2),
\]

where \( f(\xi_1, \xi_2) \) is increasing for \( \xi_1 \) and decreasing for \( \xi_2 \). In addition, suppose that

\[
f(t_1, t_2) = s_0, \quad \forall (t_1, t_2) \in \{(t_1, t_2)|\delta \leq R_1(\frac{t_1 - t_2}{\rho_1}) \leq \bar{\delta}, \delta \leq L_2(\frac{t_2 - t_1}{\rho_2}) \leq \bar{\delta}\}
\]

where \( s_0 \) is a constant, \( 0 < \bar{\delta} < \delta < 1 \).

According to Zadeh’s extensive principal, we know that \( \nu(s) = \sup\{\nu_1(t_1) \land \nu_2(t_2) | f(t_1, t_2) = s\} \). Since \( f(\xi_1, \xi_2) \) is increasing for \( \xi_1 \) and decreasing for \( \xi_2 \), and \( \xi_1 \) and \( \xi_2 \) are regular LR-FIs, in view of Definition 1, we obtain that

\[
\nu(s)_{f(t_1, t_2) = s} = L_1\left(\frac{t_1 - t_2}{\rho_1}\right) = R_2\left(\frac{t_2 - t_1}{\rho_2}\right).
\]

And \( \forall t_1 \in \{t_1|L_1(\frac{t_1 - t_2}{\rho_1}) \in (0, 1)\}, t_2 \in \{t_2|R_2(\frac{t_2 - t_1}{\rho_2}) \in (0, 1)\},
\]

\[
\nu(s)_{f(t_1, t_2) = s} = R_1\left(\frac{t_1 - \xi_1}{\sigma_1}\right) = L_2\left(\frac{t_2 - \xi_2}{\sigma_2}\right).
\]

Besides, it is easily known that \( \nu(s_0)_{f(t_1, t_2) = s_0} = \bar{\delta} \) for \( (t_1, t_2) \in \{(t_1, t_2)|\bar{\delta} \leq R_1(\frac{t_1 - t_2}{\rho_1}) \leq \bar{\delta}, \delta \leq L_2(\frac{t_2 - t_1}{\rho_2}) \leq \bar{\delta}\}. 

Based on the above analysis and Definition 1, we can get that \( \zeta = f(\xi_1, \xi_2) \) is an LR-FI. 

\[\square\]

Appendix D The proof of Theorem 5

Proof According to Theorem 4, it is easily known that \( \zeta \) is an LR-FI. Now we verify that Eq. (9) holds. For simplicity, we just verify the situation of \( k = 1 \) and \( n = 2 \). Assume that

\[
\zeta = f(\xi_1, \xi_2),
\]

where \( f \) is increasing for \( \xi_1 \) and decreasing for \( \xi_2 \). In addition, suppose that

\[
\psi^{-1}(\delta) = f\left(\psi_1^{-1}(\delta), \psi_2^{-1}(1 - \delta)\right),
\]

where \( \psi_1^{-1} \) and \( \psi_2^{-1} \) are the ICDs of \( \xi_1 \) and \( \xi_2 \), respectively. For each \( \delta \in [0, 1] \), it is defined that

\[
\bar{\delta} = \sup \left\{ \gamma | f\left(\psi_1^{-1}(\gamma), \psi_2^{-1}(1 - \gamma)\right) = f\left(\psi_1^{-1}(\delta), \psi_2^{-1}(1 - \delta)\right) \right\},
\]

which means that

\[
f\left(\psi_1^{-1}(\delta), \psi_2^{-1}(1 - \delta)\right) = f\left(\psi_1^{-1}(\bar{\delta}), \psi_2^{-1}(1 - \bar{\delta})\right).
\]

We know that \( \psi_1^{-1}(\delta) \) and \( \psi_2^{-1}(1 - \delta) \) are both intervals or both points, in which points can be considered as a special kind of intervals. Therefore, we only prove the case of intervals. The case of points can be verified similarly.

In view of Eq. (20), it is obvious that both \( \psi_1^{-1}(\bar{\delta}) \) and \( \psi_2^{-1}(1 - \bar{\delta}) \) are also intervals. Then we can attain that, for \( \forall t_1 \in \psi_1^{-1}(\bar{\delta}) \) and \( \forall t_2 \in \psi_2^{-1}(1 - \bar{\delta}) \),

\[
f(t_1, t_2) \in \psi^{-1}(\bar{\delta}) = \psi^{-1}(\delta).
\]

For one thing, considering that \( f \) is increasing for \( \xi_1 \) and decreasing for \( \xi_2 \), it can be deduced that

\[
\xi_1 \leq t_1 \text{ and } \xi_2 \geq t_2 \Rightarrow f(\xi_1, \xi_2) \leq f(t_1, t_2),
\]

which means that

\[
\{\xi_1 \leq t_1\} \cap \{\xi_2 \geq t_2\} \subseteq \{f(\xi_1, \xi_2) \leq f(t_1, t_2)\}. \tag{21}
\]

In accordance with the increase of the credibility measure \( Cr \), we can obtain

\[
Cr\{\xi \leq f(t_1, t_2)\} \geq Cr\{\{\xi_1 \leq t_1\} \cap \{\xi_2 \geq t_2\}\}.
\]
Then it can be attained that
\[
\begin{align*}
\text{Cr} \{ \{ \zeta_1 \leq t_1 \} \cap \{ \zeta_2 \geq t_2 \} \} \\
= \text{Cr} \{ \{ \zeta_1 \leq t_1 \} \wedge \text{Cr} \{ \zeta_2 \geq t_2 \} \} \\
= \bar{\delta} \wedge \bar{\delta} = \overline{\delta}.
\end{align*}
\] (22)

In accordance with Eqs. (21) and (22), we get
\[
\text{Cr} \{ \{ \zeta \leq f (t_1, t_2) \} \} \geq \overline{\delta}, \quad \forall f (t_1, t_2) \in \psi^{-1}(\delta). \quad (23)
\]

For another thing, since \( f \) is increasing for \( \zeta_1 \) and decreasing for \( \zeta_2 \), it can be deduced that
\[
f (\zeta_1, \zeta_2) \leq f (t_1, t_2) \Rightarrow \zeta_1 \leq t_1 \text{ or } \zeta_2 \geq t_2. \quad (24)
\]

Following from Eq. (24), we can get
\[
\{ f (\zeta_1, \zeta_2) \} \leq \{ f (t_1, t_2) \} \subseteq \{ \zeta_1 \leq t_1 \} \cup \{ \zeta_2 \geq t_2 \}.
\]

In terms of the increase of the credibility measure \( \text{Cr} \), it can be attained that
\[
\text{Cr} \{ \{ \zeta \leq f (t_1, t_2) \} \} \subseteq \text{Cr} \{ \{ \zeta_1 \leq t_1 \} \cup \{ \zeta_2 \geq t_2 \} \}.
\] (25)

Then it can be derived that
\[
\text{Cr} \{ \{ \zeta_1 \leq t_1 \} \cup \{ \zeta_2 \geq t_2 \} \} \\
= \text{Cr} \{ \{ \zeta_1 \leq t_1 \} \lor \text{Cr} \{ \zeta_2 \geq t_2 \} \} \\
= \bar{\delta} \lor \bar{\delta} = \overline{\delta}.
\] (26)

In view of Eqs. (25) and (26), we have
\[
\text{Cr} \{ \{ \zeta \leq f (t_1, t_2) \} \} \leq \overline{\delta}, \quad \forall f (t_1, t_2) \in \psi^{-1}(\delta). \quad (27)
\]

Finally, combing Eqs. (23) and (27), we can get
\[
\text{Cr} \{ \{ \zeta \leq f_\delta \} \} = \overline{\delta}, \quad \forall f (t_1, t_2) = f_\delta \in \psi^{-1}(\delta),
\]

where \( \overline{\delta} = \sup \{ \gamma \mid \psi^{-1}(\gamma) = \psi^{-1}(\delta) \} \) holds in accordance with Eq. (19). In terms of Definition 3, it can be known that
\[
\psi^{-1}(\delta) = f (\psi_1^{-1}(\delta), \psi_2^{-1}(1 - \delta)) \text{ is just the ICD of } \zeta = f (\zeta_1, \zeta_2).
\]

By taking \( \delta \) to replace \( \psi (t) \) and \( \psi^{-1}(\delta) \) to replace \( t \), then it can be derived that
\[
E[\zeta] = \int_{0}^{1} \psi^{-1}(\delta)d\delta.
\]

\section*{Appendix E The proof of Theorem 6}

\textbf{Proof} Following from Definition 6, we can get that
\[
E[\zeta] = \int_{0}^{+\infty} \text{Cr}[\zeta \geq t]dt - \int_{0}^{+\infty} \text{Cr}[\zeta \leq t]dt \\
= \int_{0}^{+\infty} (1 - \psi (t)) dt - \int_{-\infty}^{+\infty} \psi (t)dt \\
= \int_{0}^{+\infty} t \psi (t)dt.
\]

\section*{Appendix F The proof of Theorem 8}

\textbf{Proof} In the light of Theorem 7, it is deduced that
\[
E[f (t, \zeta)] = \int_{0}^{1} f (t, \psi_1^{-1}(\delta), \ldots, \psi_k^{-1}(\delta), \psi_{k+1}^{-1}(1 - \delta))d\delta.
\]

In view of Theorem 5, the ICD of \( h_v (t, \zeta_1, \zeta_2, \ldots, \zeta_n) \) is derived as
\[
\phi_v^{-1}(\delta) = h_v (t, \psi_1^{-1}(\delta), \ldots, \psi_{k_v}^{-1}(\delta), \psi_{k_v+1}(1 - \delta), \ldots, \psi_n^{-1}(1 - \delta))
\]

It is not hard to find that \( \text{Cr} \{ h_v (t, \zeta_1, \zeta_2, \ldots, \zeta_n) \leq 0 \} \geq \delta \) holds if and only if \( \phi_v^{-1}(\delta) \leq 0 \). Especially, when \( \phi_v^{-1}(0.5) \) is not unique, \( \text{Cr} \{ h_v (t, \zeta_1, \zeta_2, \ldots, \zeta_n) \leq 0 \} \geq 0.5 \) holds if only if and if \( \inf \phi_v^{-1}(0.5) \leq 0 \).

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