A CHARACTERIZATION OF QUADRIC CONSTANT MEAN CURVATURE HYPERSURFACES OF SPHERES

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Dedicated to the memory of Professor Luis J. Alías-Pérez

Abstract. Let \( \phi : M \rightarrow S^{n+1} \subset \mathbb{R}^{n+2} \) be an immersion of a complete \( n \)-dimensional oriented manifold. For any \( v \in \mathbb{R}^{n+2} \), let us denote by \( \ell_v : M \rightarrow \mathbb{R} \) the function given by \( \ell_v(x) = \langle \phi(x), v \rangle \) and by \( f_v : M \rightarrow \mathbb{R} \), the function given by \( f_v(x) = \langle \nu(x), v \rangle \), where \( \nu : M \rightarrow S^n \) is a Gauss map. We will prove that if \( M \) has constant mean curvature, and, for some \( v \neq 0 \) and some real number \( \lambda \), we have that \( \ell_v = \lambda f_v \), then, \( \phi(M) \) is either a totally umbilical sphere or a Clifford hypersurface. As an application, we will use this result to prove that the weak stability index of any compact constant mean curvature hypersurface \( M^n \) in \( S^{n+1} \) which is neither totally umbilical nor a Clifford hypersurface and has constant scalar curvature is greater than or equal to \( 2n + 4 \).

1. Introduction

Let \( \phi : M \rightarrow S^{n+1} \subset \mathbb{R}^{n+2} \) be an immersion of a complete \( n \)-dimensional oriented manifold. For every \( x \in M \) we will denote by \( T_x M \) the tangent space of \( M \) at \( x \). Sometimes, specially when we are dealing with local aspects of \( M \), we will identify \( M \) with the set \( \phi(M) \subset \mathbb{R}^{n+2} \), and the space \( T_x M \) with the linear subspace \( d\phi_x(T_x M) \) of \( \mathbb{R}^{n+2} \). Let us denote by \( \nu : M \rightarrow S^{n+1} \subset \mathbb{R}^{n+2} \), a normal unit vector field along \( M \), i.e., for every \( x \in M \), \( \nu(x) \) is perpendicular to the vector \( x \) and to the vector space \( T_x M \). The shape operator \( A_x : T_x M \rightarrow T_x M \), is given by \( A_x(v) = -d\nu_x(v) = -\beta'(0) \) where \( \beta(t) = \nu(\alpha(t)) \) and \( \alpha(t) \) is any smooth curve in \( M \) such that \( \alpha(0) = x \) and \( \alpha'(0) = v \). It can be shown that the linear map \( A_x : T_x M \rightarrow T_x M \) is symmetric, therefore it has \( n \) real eigenvalues \( \kappa_1(x), \ldots, \kappa_n(x) \). These eigenvalues are known as the principal curvatures of \( M \) at \( x \). The mean curvature of \( M \) at \( x \) is the average of the principal curvatures,

\[
H(x) = \frac{\kappa_1(x) + \cdots + \kappa_n(x)}{n},
\]

and the norm square of the shape operator is defined by the equation

\[
\|A\|^2(x) = \text{trace}(A^2_x) = \kappa_1^2(x) + \cdots + \kappa_n^2(x).
\]

2000 Mathematics Subject Classification. Primary 53C42, Secondary 53A10.
Key words and phrases. constant mean curvature, Clifford hypersurface, stability operator, first eigenvalue.

L.J. Alías was partially supported by MEC project MTM2007-64504, and Fundación Séneca project 04540/GERM/06, Spain. This research is a result of the activity developed within the framework of the Programme in Support of Excellence Groups of the Región de Murcia, Spain, by Fundación Séneca, Regional Agency for Science and Technology (Regional Plan for Science and Technology 2007-2010).

A. Brasil Jr. was partially supported by CNPq, Brazil, 306626/2007-1.
1.1. Examples: Totally umbilical spheres and Clifford hypersurfaces. In this section we will describe two families of examples that are related with the main result of this paper.

**Example 1.** Let \( v \in \mathbb{R}^{n+2} \) be a fixed unit vector and let a real number \( c \) with \( |c| < 1 \).

Let us define

\[
S^n(v, c) = \{ x \in \mathbb{R}^{n+2} : \langle x, v \rangle = c \}.
\]

Clearly, \( S^n(v, c) \) is a hypersurface of \( S^{n+1} \). In this case the map \( \nu : S^n(v, c) \to S^{n+1} \) given by

\[
\nu(x) = \frac{1}{\sqrt{1 - c^2}} (v - cx)
\]

is a normal unit vector field along \( S^n(v, c) \). Therefore, for every \( x \in S^n(v, c) \) the shape operator \( A_x \) is the map \( c(1 - c^2)^{-\frac{1}{2}} I \), where \( I \) is the identity map, and

\[
\kappa_1(x) = \cdots = \kappa_n(x) = \frac{c}{\sqrt{1 - c^2}}
\]

for all \( x \in S^n(v, c) \). It is not difficult to show that these examples are the only totally umbilical complete hypersurfaces of \( S^{n+1} \). In this case

\[
H = \frac{c}{\sqrt{1 - c^2}} \quad \text{and} \quad \|A\|^2 = \frac{nc^2}{1 - c^2}
\]

are both constant on \( S^n(v, c) \).

**Example 2.** Given any integer \( k \in \{1, \ldots, n-1\} \) and any real number \( r \in (0, 1) \), let us define \( \ell = n - k \) and

\[
M_k(r) = \{(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} : \| x \|^2 = r^2 \quad \text{and} \quad \| y \|^2 = 1 - r^2 \}
\]

\[
= S^k(r) \times S^{n-k}(\sqrt{1 - r^2}) \subset S^{n+1}.
\]

It is not difficult to see that for any \( (x, y) \in M_k(r) \) one gets

\[
T_{(x,y)}M_k(r) = \{(v, w) \in \mathbb{R}^{k+1} \times \mathbb{R}^{n-k} : \langle x, v \rangle = 0 \quad \text{and} \quad \langle w, y \rangle = 0 \}
\]

Therefore, the map \( \nu : M_k(r) \to S^{n+1} \) given by

\[
\nu(x, y) = \left( \frac{\sqrt{1 - r^2}}{r} x, -\frac{r}{\sqrt{1 - r^2}} y \right)
\]

defines a normal unit vector field along \( M_k(r) \), i.e. it is a Gauss map on \( M_k(r) \). Notice that the vectors in \( T_{(x,y)}M_k(r) \) of the form \( (v, 0) \) define a \( k \) dimensional space. A direct computation, using the expression for \( \nu \), gives us that if \( (v, 0) \in T_{(x,y)}M_k(r) \), then

\[
A_{(x,y)}(v, 0) = -\frac{\sqrt{1 - r^2}}{r}(v, 0).
\]

Therefore \( -\sqrt{1 - r^2}/r \) is an eigenvalue of \( A_{(x,y)} \) with multiplicity \( k \). In the same way we can show that \( r/\sqrt{1 - r^2} \) is an eigenvalue of \( A_{(x,y)} \) with multiplicity \( \ell \). Therefore, the principal curvatures of \( M_k(r) \) are given by

\[
\kappa_1(x, y) = \cdots = \kappa_k(x, y) = -\frac{\sqrt{1 - r^2}}{r}, \quad \kappa_{k+1}(x, y) = \cdots = \kappa_n(x, y) = \frac{r}{\sqrt{1 - r^2}}
\]

and we also have that

\[
H = \frac{nr^2 - k}{nr\sqrt{1 - r^2}} \quad \text{and} \quad \|A\|^2 = \frac{k}{r^2} + \frac{n-k}{1-r^2} - n
\]
are both constant. Hypersurfaces that, up to a rigid motion, are equal to \(M_k(r)\) for some \(k\) and \(r\), are called Clifford hypersurfaces.

1.2. Two families of geometric functions on hypersurfaces in spheres.

Given a fixed vector \(v \in \mathbb{R}^{n+2}\), let us define the functions \(\ell_v : M \to \mathbb{R}\) and \(f_v : M \to \mathbb{R}\) by \(\ell_v(x) = \langle \phi(x), v \rangle\) and \(f_v(x) = \langle \nu(x), v \rangle\), where \(\nu : M \to S^{n+1}\) is a Gauss map. When we consider all possible \(v \in \mathbb{R}^{n+2}\) we obtain the families

\[
V_1 = \{\ell_v : v \in \mathbb{R}^{n+2}\} \quad \text{and} \quad V_2 = \{f_v : v \in \mathbb{R}^{n+2}\}.
\]

These two families are very useful in the study of the spectrum of important elliptic operators defined on \(M\) like the Laplacian and the stability operator. For example, in [10] and [11], Solomon computed the whole spectrum for the Laplace operator of every minimal isoparametric hypersurface of degree 3 in spheres using \(f_v\). Indeed, it is not difficult to prove that if for some compact hypersurface \(M^n\) in \(S^{n+1}\), we have that either \(\dim(V_1) < n+2\) or \(\dim(V_2) < n+2\), then \(M = S^n(v, 0)\) for some unit vector \(v \in \mathbb{R}^{n+2}\), [8, Lemma 3.1].

If we take \(c \neq 0\), and we consider the example \(S^n(v, c)\) we observe that if \(w \in \mathbb{R}^{n+2}\) is a vector perpendicular to the vector \(v\), then

\[
f_w = -\frac{c}{\sqrt{1-c^2}} \ell_w.
\]

We also have this kind of relation between the function \(f_w\) and the function \(\ell_w\) in the Clifford hypersurfaces; more precisely, if we consider the example \(M_k(r)\) and we take \(w = (w_1, \ldots, w_{k+1}, 0, \ldots, 0) \in \mathbb{R}^{n+2}\) then we have that

\[
f_w = \frac{\sqrt{1-r^2}}{r} \ell_w.
\]

Also, if we take \(w = (0, \ldots, 0, w_{k+2}, \ldots, w_{n+2}) \in \mathbb{R}^{n+2}\), then, we have that

\[
f_w = -\frac{r}{\sqrt{1-r^2}} \ell_w.
\]

In this paper we will prove that these two examples are the only hypersurfaces with constant mean curvature in \(S^{n+1}\) where the relation \(f_w = \lambda \ell_w\), for some non-zero vector \(w \in \mathbb{R}^{n+2}\), is possible. More precisely, we will prove the following result.

**Theorem 3.** Let \(\phi : M \to S^{n+1} \subset \mathbb{R}^{n+2}\) be an immersion with constant mean curvature of a complete \(n\)-dimensional oriented manifold. If for some non-zero vector \(v \neq 0\) and some real number \(\lambda\), we have that \(\ell_v = \lambda f_v\), then, \(\phi(M)\) is either a totally umbilical sphere or a Clifford hypersurface.

Recall that constant mean curvature hypersurfaces in \(S^{n+1}\) are characterized as critical points of the area functional restricted to variations that preserve a certain volume function. As is well-known, the Jacobi operator of this variational problem is given by \(J = \Delta + \|A\|^2 + n\), with associated quadratic form given by

\[
Q(f) = -\int_M fJf
\]

and acting on the space

\[
C^\infty_T(M) = \{f \in C^\infty(M) : \int_M f = 0\}.
\]
Precisely, the restriction \( \int_M f = 0 \) means that the variation associated to \( f \) is volume preserving.

In contrast to the case of minimal hypersurfaces, in the case of hypersurfaces with constant mean curvature one can consider two different eigenvalue problems: the usual Dirichlet problem, associated with the quadratic form \( Q \) acting on the whole space of smooth functions on \( M^n \), and the so called twisted Dirichlet problem, associated with the same quadratic form \( Q \), but restricted to the subspace of smooth functions satisfying the additional condition \( \int_M f = 0 \). Similarly, there are two different notions of stability and index, the strong stability and strong index, denoted by \( \text{Ind}(M) \) and associated to the usual Dirichlet problem, and the weak stability and weak index, denoted by \( \text{Ind}_T(M) \) and associated to the twisted Dirichlet problem. Specifically, the strong index of the hypersurface is characterized as

\[
\text{Ind}(M) = \max \{ \text{dim} V : V \subseteq C^\infty(M), \quad Q(f) < 0 \quad \text{for every} \quad f \in V \},
\]

and \( M \) is called strongly stable if and only if \( \text{Ind}(M) = 0 \). On the other hand, the weak stability index of \( M^n \) is characterized by

\[
\text{Ind}_T(M) = \max \{ \text{dim} V : V \subseteq C^\infty_T(M), \quad Q(f) < 0 \quad \text{for every} \quad f \in V \},
\]

and \( M \) is called weakly stable if and only if \( \text{Ind}_T(M) = 0 \). From a geometrical point of view, the weak index is more natural than the strong index. However, from an analytical point of view, the strong index is more natural and easier to use (for further details, see [1]).

As an application of our Theorem 3 we will prove that the weak stability index of a compact constant mean curvature hypersurface \( M^n \) in \( S^{n+1} \) with constant scalar curvature must be greater than or equal to \( 2n + 4 \) whenever \( M^n \) is neither a totally umbilical sphere nor a Clifford hypersurface (see Theorem 9). This result complements the one obtained in [2] where the authors showed that the weak index of a compact constant mean curvature hypersurface \( M^n \) in \( S^{n+1} \) which is not totally umbilical and has constant scalar curvature is greater than or equal to \( n + 2 \), with equality if and only if \( M^n \) is a Clifford hypersurface \( M_k(r) = S^k(r) \times S^{n-k}(\sqrt{1-r^2}) \) with radius \( \sqrt{k/(n+2)} \leq r \leq \sqrt{(k+2)/(n+2)} \). At this respect, it is worth pointing out that the weak stability index of the Clifford hypersurfaces \( M_k(r) \) depends on \( r \), reaching its minimum value \( n + 2 \) when \( \sqrt{k/(n+2)} \leq r \leq \sqrt{(k+2)/(n+2)} \), and converging to \( +\infty \) as \( r \) converges to 0 or 1 (see [2] Section 3 for further details).

2. Preliminaries and auxiliary results

Let us start this section by computing the gradient of the functions \( \ell_v \) and \( f_v \). For any fixed vector in \( \mathbb{R}^{n+2} \), let us define the tangent vector field \( v^\top : M \to \mathbb{R}^{n+2} \) by

\[
v^\top(x) = v - \ell_v(x)x - f_v(x)\nu(x) \quad \text{for all} \quad x \in M,
\]

where, as in the previous section, \( \nu : M \to \mathbb{R}^{n+2} \) is a Gauss map. Clearly, \( v^\top \) is a tangent vector field on \( M \) because \( \langle v^\top(x), x \rangle = 0 \) and \( \langle v^\top(x), \nu(x) \rangle = 0 \) for every \( x \in M \). More precisely, \( v^\top(x) \) is the orthogonal projection of the vector \( v \) on \( T_x M \).

**Proposition 4.** If \( M^n \) is a smooth hypersurface of \( S^{n+1} \) and \( A \) denotes its shape operator with respect to the unit normal vector field \( \nu : M \to \mathbb{R}^{n+2} \) then, the gradient of the functions \( \ell_v \) and \( f_v \) are given by:

\[
\nabla \ell_v = v^\top, \quad \nabla f_v = -A(v^\top).
\]
Proof. For any vector \( w \in T_xM \), let \( \alpha : (-\varepsilon, \varepsilon) \to M \) be a curve such that \( \alpha(0) = x \) and \( \alpha'(0) = w \). Notice that

\[
\frac{d}{dt} \langle \alpha(t), v \rangle \bigg|_{t=0} = \frac{d}{dt} \langle \alpha'(t), v \rangle \bigg|_{t=0} = \langle \alpha''(0), v \rangle = \langle w, v^\top(x) \rangle.
\]

Since the equality above holds true for every \( w \in T_xM \) and \( v^\top(x) \in T_xM \), then, \( \nabla \ell_v(x) = v^\top(x) \). For the function \( f_v \), we have

\[
\frac{d}{dt} f_v(w) = \frac{d}{dt} \langle \alpha(t), v \rangle \bigg|_{t=0} = \langle \alpha'(0), v \rangle = \langle w, A((v'(0)), v) \rangle = -\langle A(w), v^\top(x) \rangle = \langle w, -A(v^\top(x)) \rangle.
\]

Therefore, \( \nabla f_v(x) = -A(v^\top(x)) \).

We also have the following expressions for the Laplacian of the functions \( \ell_v \) and \( f_v \).

**Proposition 5.** If \( M^n \) is a smooth hypersurface of \( \mathbb{S}^{n+1} \) with constant mean curvature \( H \), and \( A \) denotes the shape operator with respect to the unit normal vector field \( \nu : M \to \mathbb{R}^{n+2} \) then, the Laplacian of the functions \( \ell_v \) and \( f_v \) are given by:

\[
\Delta \ell_v = -n\ell_v + nH f_v, \quad \Delta f_v = -\|A\|^2 f_v + nH \ell_v.
\]

**Proof.** For any vector \( w \in T_xM \), we have

\[
\nabla_w \nabla \ell_v = \nabla_w v^\top = -\ell_v(x)w + f_v(x)A_x(w),
\]

where \( \nabla \) denotes here the intrinsic derivative on \( M \). Let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_xM \). Then, the Laplacian of \( \ell_v \) at the point \( x \) is given by

\[
\Delta \ell_v(x) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla \ell_v, e_i \rangle = -n\ell_v(x) + \text{tr}(A_x) f_v(x) = -n\ell_v(x) + nH f_v(x).
\]

On the other hand, using Codazzi equation we also have that

\[
\nabla_w \nabla f_v = -\nabla_w (A(v^\top)) = -\langle \nabla_w A(v^\top(x)) - A_x(\nabla_w v^\top) \rangle = -\langle \nabla_{v^\top(x)} A(w) + \ell_v(x) A_x(w) - f_v(x) A_x^2(w) \rangle.
\]

Therefore

\[
\Delta f_v(x) = \sum_{i=1}^n \langle \nabla_{e_i} \nabla f_v, e_i \rangle = -\sum_{i=1}^n \langle (\nabla_{v^\top(x)} A)(e_i), e_i \rangle + nH \ell_v(x) - \|A\|^2 f_v(x) = -n\langle v^\top, \nabla H(x) \rangle + nH \ell_v(x) - \|A\|^2 f_v(x) = nH \ell_v(x) - \|A\|^2 f_v(x),
\]

since the mean curvature \( H \) is constant.

The following two lemmas will be used in the proof of our main theorem. The first one is an elementary geometric lemma whose proof is left to the reader.

**Lemma 6.** Let \( M^n \) be a smooth hypersurface of \( \mathbb{S}^{n+1} \) and let \( \alpha : I \subset \mathbb{R} \to M \) be a regular curve such that

\[
\alpha''(t) = f(t)\alpha'(t) + \eta(t)
\]

where \( f(t) \) and \( \eta(t) \) are smooth functions of \( t \).
where \( f : I \to \mathbb{R} \) is a smooth function and \( \eta : I \to \mathbb{R}^{n+2} \) is a normal vector field along \( \alpha \), i.e. \( \eta(t) \) is orthogonal to \( T_{\alpha(t)}M \). If \( s = s(t) \) is the arc-length parameter for \( \alpha \), then \( \beta(s) = \alpha(t(s)) \) satisfies that \( \beta''(s) \) is a normal vector field along \( \beta \), i.e. \( \beta \) is a geodesic in \( M \).

The other one is an algebraic lemma.

**Lemma 7.** If \( p_k(X) = b_1X + c_1, \ldots, p_k(X) = b_kX + c_k \) are \( k \) polynomials of degree 1, \( k \geq 2 \), with the property that \( c_i/b_i \neq c_j/b_j \) whenever \( i \neq j \), then, the polynomials

\[
q_i = \Pi_{j=1,j \neq i}^k p_j
\]

are linearly independent. Moreover, an equation of the form

\[
\frac{a_1}{p_1(X)} + \cdots + \frac{a_k}{p_k(X)} = d
\]

with \( a_i \) and \( d \) real numbers, can not hold true unless all the \( a_i \)'s and \( d \) are zero.

**Proof.** By the condition on the numbers \( c_j/b_j \) we have that at \( X_i = -c_i/b_i \) every polynomial \( q_j \), except the polynomial \( q_i \), vanishes. Therefore, if there exists constants \( \alpha_i \) such that

\[
\alpha_1 q_1(X) + \cdots + \alpha_k q_k(X) = 0
\]

then, taking \( X = X_i \) we get that \( \alpha_i = 0 \) for every \( i \). Therefore, the polynomials \( q_i \)'s are linearly independent. On the other hand, notice that the second equation in the lemma can be written as

\[
a_1 q_1(X) + \cdots + a_k q_k(X) = dR(X)
\]

where \( R \) is a polynomial of degree \( k \). Since the expression on the left of the last equation is a polynomial of degree \( k - 1 \), we obtain that the constant on the right hand side must be zero. Then the second part of the lemma follows by the independence of the polynomials \( q_i \)'s. \( \square \)

### 3. Proof of Theorem 3

We are now ready to give our main argument and prove Theorem 3. Since most of the arguments are local and the thesis of the theorem is on \( \phi(M) \) and not on \( M \), we will identify \( M \) with \( \phi(M) \) and \( T_xM \) with \( T_{\phi(x)}M \). By multiplying the equation \( \ell_v = \lambda f_v \) by an appropriated constant we may assume that \( |v| = 1 \). We will also assume that \( \ell_v \) is not constant, otherwise \( \phi(M) \subset S^n(v,c) \) for some \( c \), which implies, using the completeness of \( M \), that \( \phi(M) = S^n(v,c) \).

Notice that, since \( \ell_v \) is not constant, then \( \lambda \neq 0 \). Taking the gradient in both sides of the expression \( \ell_v = \lambda f_v \) we obtain that

\[
A(v^\top(x)) = -\lambda^{-1}v^\top(x)
\]

at every point \( x \in M \).

**Step 1: The integral curves of \( v^\top \) in \( M \) are Euclidean circles.** Let us take a point \( x \in M \) such that \( \nabla \ell_v(x) = v^\top(x) \) does not vanish. Let \( \alpha_x(t) \) be the integral curve of the vector field \( v^\top \) such that \( \alpha_x(0) = x \). Since

\[
\alpha_x'(t) = v^\top(\alpha_x(t)) = v - \ell_v(\alpha_x(t))\alpha_x(t) - f_v(\alpha_x(t))\nu(\alpha_x(t)) \\
= v - \ell_v(\alpha_x(t))(\alpha_x(t) + \lambda^{-1}\nu(\alpha_x(t)))
\]

we get that

\[
A(v^\top(\alpha_x(t))) = -\lambda^{-1}v^\top(\alpha_x(t)) = \ell_v(\alpha_x(t))\nu(\alpha_x(t))
\]

and

\[
A(v^\top(\alpha_x(t))) = -\lambda^{-1}v^\top(\alpha_x(t)) = \ell_v(\alpha_x(t))\nu(\alpha_x(t))
\]

which implies that \( v^\top(\alpha_x(t)) \) is a normal vector field along \( \alpha_x(t) \). Therefore, \( \alpha_x(t) \) is a geodesic in \( M \).
More precisely, if we define
\[ w(7) \]
and
\[ \eta(3) \]
From this last expression we obtain that
\[ y(2) \]
Step 2: The intersection \( N = M \cap S^o(v, 0) \) is non-empty. Let us compute \( \beta_x(0) \) and \( \beta''_x(0) \) in order to obtain an explicit expression for \( \beta_x(s) \). From the definition of \( \beta_x \) we have that \( \beta_x(0) = x \) and
\[
(\beta''_x)(s) + (1 + \lambda^{-2})\beta'_x(s) = 0.
\]
More precisely, if we define \( \omega = \sqrt{1 + \lambda^{-2}} > 0 \), then
\[
\beta'_x(s) = \beta'_x(0) \cos(\omega s) + w^{-1}\beta''_x(0) \sin(\omega s),
\]
and
\[
(5) \quad \beta_x(s) = w^{-1}\beta'_x(0) \sin(\omega s) - w^{-2}\beta''_x(0) \cos(\omega s) + \beta_x(0) + w^{-2}\beta''_x(0).
\]

Let us define \( a = \ell_v(x) \), and \( b = \sqrt{w^2 - a^2} \). By (7) we have that \( b > 0 \), because \( \nabla \ell_v(x) = v^\top(x) \neq 0 \). With this notation, we obtain that \( |v^\top(x)|^2 = 1 - w^2a^2 = w^2b^2 \), and

\[
\langle \beta_x(0), v \rangle = \ell_v(x) = a \\
\langle \beta_x'(0), v \rangle = \left( \frac{v^\top(x)}{|v^\top(x)|}, v \right) = \left( \frac{v^\top(x)}{|v^\top(x)|}, v^\top(x) \right) = |v^\top(x)| = \sqrt{1 - w^2a^2} = wb \\
\langle \beta_x''(0), v \rangle = \langle -\beta_x(0) - \lambda^{-1}v(\beta_x(0)), v \rangle = -a - \lambda^{-2}a = -w^2a,
\]

where we have used (11) to derive the last equation. Now, using these equations jointly with (5) we get that

\[
\ell_v(\beta_x(s)) = \langle \beta_x(s), v \rangle = a \cos(ws) + b \sin(ws).
\]

Notice that \((wa)^2 + (wb)^2 = 1\) with \(wb > 0\). Therefore for some \(s_1 \in (-\frac{\pi}{2w}, \frac{\pi}{2w})\) we have

\[-wa = \sin(ws_1) \quad \text{and} \quad wb = \cos(ws_1),\]

so that

\[
\ell_v(\beta_x(s)) = a \cos(ws) + b \sin(ws) = w^{-1} \sin(ws - ws_1).
\]

Notice that when \(s\) moves from 0 to \(s_1\), we have that \(\ell_v(\beta_x(s))\) never reaches the values \(\pm w^{-1}\), therefore by (7) \(v^\top(\beta_x(s)) \neq 0\) and all these \(\beta_x(s)\) belong to the integral curve of the vector field \(v^\top\). In particular, \(\ell_v(\beta_x(s_1)) = 0\) and \(v^\top(\beta_x(s_1)) = v \neq 0\). This argument shows that

\[
N = \ell_v^{-1}(0) = \{ y \in M : \ell_v(y) = 0 \}
\]

is not empty. Observe that if we were assuming that \(M\) were compact instead of complete, the fact that \(N = \ell_v^{-1}(0)\) is not empty would have followed from the fact that the function \(\ell_v\) must reach its maximum value and a minimum value on \(M\), and the fact that necessarily these values must be \(\pm w^{-1}\), since \(\nabla \ell_v = v^\top\) must vanish at its critical points. From now on we will assume that the \(x\) that we were considering before is an element in \(N\), i.e, we will assume that \(a = 0\), and therefore \(b = w^{-1}\) and \(s_1 = 0\).

**Step 3:** The intersection \(N = M \cap S^n(v, 0)\) as a hypersurface of \(M\) and as a hypersurface of \(S^n(v, 0)\). Clearly the set \(N \subset M^n\) is an \((n - 1)\)-dimensional manifold because 0 is a regular value of the function \(\ell_v\) on \(M\). Moreover, for every \(x \in N\) we have that \(\nabla \ell_v(x) = v^\top(x) = v\) is a constant vector, and therefore \(N\) is a totally geodesic hypersurface of \(M\). Notice that for every \(x \in N\) we have that \(v \in T_xM\) and \(A_x(v) = -\lambda^{-1}v\). Therefore we can take vectors \(v_1, \ldots, v_{n-1}\) in \(T_xM\), all of them orthogonal to \(v\), such that \(A_x(v_i) = \lambda_i(x)v_i\). Since the vectors \(v_i\)'s are perpendicular to \(v = \nabla \ell_v(x)\), they form a basis for \(T_xN\). On the other hand, notice that \(N\) is also a hypersurface of the unit \(n\)-dimensional sphere \(S^n(v, 0)\), and that for every \(x \in N\), \(\nu(x)\) gives a unit vector field normal to \(N\) in \(S^n(v, 0)\) (see Figure 1).
Step 4: Computation of the principal curvatures of $v^\top$. Under the assumption that $x \in N$, we obtain from (3) that 
\[ \beta'_x(0) = v. \]
Therefore, from (4) and (5) we get the following expression for $\beta_x(s)$,
\[ \beta_x(s) = w^{-1} \sin(ws)v + w^{-2}(\cos(ws) - 1)(x + \lambda^{-1}v(x)) + x. \]
By differentiating two times this equation, and using the equation (4), we obtain the following expression,
\[ \nu(\beta_x(s)) = \lambda w \sin(ws)v + \lambda \cos(ws)(x + \lambda^{-1}v(x)) - \lambda \beta_x(s). \]
Recall that, if $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, then
\[ |\tau_x(\beta_x(s))| < w^{-1} \quad \text{and} \quad v^\top(\beta_x(s)) \neq 0. \]
Observe that if $\gamma(t)$ is a smooth curve in $N$ such that $\gamma(0) = x$ and $\gamma'(0) = v_i$, then by (3), we have that the curve 
\[ \gamma_x(t) = \beta_x(\gamma(t)) = \beta_x(\gamma(t)) = w^{-1} \sin(ws)v + w^{-2}(\cos(ws) - 1)(\gamma(t) + \lambda^{-1}v(\gamma(t))) + \gamma(t) \]
is a curve on $M$ such that $\gamma_x(0) = \beta_x(s)$. A direct computation shows that 
\[ \gamma'_x(0) = w^{-2}(\cos(ws) - 1)(v_i - \lambda^{-1}\lambda_i(x)v_i) + v_i = \mu_i(x)v_i, \]
where
\[ \mu_i(x) = \frac{\lambda(\lambda - \lambda_i(x))\cos(ws) + (1 + \lambda \lambda_i(x))}{1 + \lambda^2}. \]
The computation above shows us that the vectors $v_i$’s are all elements in $T_{\beta_x(s)}M$ for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$. Actually, it follows directly from (11) that if $\mu_i(x) \neq 0$ then $v_i = \gamma'_x(0)/\mu_i(x) \in T_{\beta_x(s)}M$; hence by a continuity argument, since the equation $\mu_i(x) = 0$ has finitely many solutions on $(-\frac{\pi}{2w}, \frac{\pi}{2w})$, we conclude that $v_i \in T_{\beta_x(s)}M$ for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$.
Recall that, by (10) and (11), $-\lambda^{-1}$ is a principal curvature at the point $\beta_x(s)$, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, with associated principal direction in the direction of $v^\top(\beta_x(s)) \neq 0$. Let us compute now the other $n - 1$ principal curvatures of $M$ at
the point $\beta_x(s)$. Since $\gamma(t) \in N$ for every $t$, then the expression (9) holds true when replacing $x$ by $\gamma(t)$ and then we have that
\[ \nu(\gamma_t(t)) = \nu(\beta_s(t)(s)) = \lambda w \sin(ws) v + \lambda \cos(ws) (\gamma(t) + \lambda^{-1} \nu(\gamma(t))) - \lambda \gamma_t(s). \]
Differentiating this equation with respect to $t$ at $t = 0$ and using (11), we get that
\[ A_{\beta_x(s)}(\gamma_s(0)) = \mu_i(x) A_{\beta_x(s)}(v_i) = -\nu(\gamma_s(0)) = (\lambda_i(x) - \lambda) \cos(ws) v_i + \lambda \mu_i(x) v_i. \]
That is,
\[ A_{\beta_x(s)}(v_i) = \left( \lambda + \frac{(\lambda_i(x) - \lambda)(1 + \lambda^2) \cos(ws)}{\lambda(\lambda - \lambda_i)(x)} \cos(ws) + (1 + \lambda \lambda_i(x)) \right) v_i \]
Therefore, we get the following expression for the other $n - 1$ principal curvatures at $\beta_x(s)$,
\[ \lambda_i(\beta_x(s)) = \lambda + \frac{(\lambda_i(x) - \lambda)(1 + \lambda^2) \cos(ws)}{\lambda(\lambda - \lambda_i(x)) \cos(ws) + (1 + \lambda \lambda_i(x))} \]
(12)
\[ = -\lambda^{-1} + \frac{(1 + \lambda^2)(\lambda^{-1} + \lambda_i(x))}{\lambda(\lambda - \lambda_i(x)) \cos(ws) + (1 + \lambda \lambda_i(x))}. \]
Notice that, as it is supposed to be, when $s = 0$, i.e at the point $x$, the expression (12) above reduces to $\lambda_i(x)$. Also notice that if $\lambda_i(x) = -\lambda^{-1}$ then, the expression (12) reduces to $-\lambda^{-1}$ for every $s$.

**Step 5:** $M$ is isoparametric with at most two distinct principal curvatures. Now, we will use the hypothesis on the mean curvature of $M$. By (12), for every point $x \in N$ and every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$ we have that
\[ nH = nH(\beta_x(s)) = -\lambda^{-1} + \sum_{i=1}^{n-1} \lambda_i(\beta_x(s)) \]
\[ = -n\lambda^{-1} + (1 + \lambda^2) \sum_{i=1}^{n-1} \lambda^{-1} + \lambda_i(x) \]
\[ = \frac{n(H + \lambda^{-1})}{1 + \lambda^2}. \]
That is,
\[ \sum_{i=1}^{n-1} \frac{\lambda^{-1} + \lambda_i(x)}{\lambda(\lambda - \lambda_i(x)) \cos(ws) + (1 + \lambda \lambda_i(x))} = \frac{n(H + \lambda^{-1})}{1 + \lambda^2}. \]
(13)
For every $x \in N$, let
\[ I_1(x) = \{ i \in \{1, \ldots, n - 1\} : \lambda_i(x) = -\lambda^{-1} \}, \]
\[ I_2(x) = \{ i \in \{1, \ldots, n - 1\} : \lambda_i(x) = \lambda \}, \]
\[ I_3(x) = \{1, \ldots, n - 1\} \setminus (I_1(x) \cup I_2(x)). \]
Then (13) can be written as
\[ \sum_{i \in I_3(x)} \frac{\lambda^{-1} + \lambda_i(x)}{\lambda(\lambda - \lambda_i(x)) \cos(ws) + (1 + \lambda \lambda_i(x))} = d(x) \]
(14)
where
\[ d(x) = \frac{n(H + \lambda^{-1}) - n_2(x)(\lambda + \lambda^{-1})}{1 + \lambda^2}. \]
and $n_i(x) = \text{card}(I_i(x))$. We claim that $I_3(x) = \emptyset$. Otherwise, for every $i \in I_3(x)$ let $a_i(x) = \lambda^{-1} + \lambda_i(x) \neq 0$, $b_i(x) = \lambda(\lambda - \lambda_i(x)) \neq 0$, and $c_i(x) = 1 + \lambda \lambda_i(x) \neq 0$. 

Thus, equation (14) means that, for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$, $\cos(ws)$ is a root of the polynomial equation on $X$

\[
\sum_{i \in I_3(x)} \frac{a_i(x)}{b_i(x)X + c_i(x)} = d(x).
\]

If $\lambda_i(x) = \lambda_j(x)$ for every $i, j \in I_3(x)$ (in particular, if $n_3(x) = 1$), then (15) becomes

\[
\frac{n_3(x)a_i(x)}{b_i(x)X + c_i(x)} = d(x),
\]

which can hold only if $a_i(x) = d(x) = 0$. But this is a contradiction because $a_i(x) \neq 0$. Therefore, we can decompose

\[
I_3(x) = \bigcup_{i=1}^{k} J_i(x), \quad k \geq 2,
\]

with $\lambda_{j_1}(x) = \lambda_{j_2}(x)$ if and only if $j_1, j_2 \in J_i(x)$ for some $i$. In that case, let $\lambda_i(x) = \lambda_j(x)$ for every $j \in J_i(x)$, and (15) becomes

\[
\sum_{i=1}^{k} \frac{m_i(x)a_i(x)}{b_i(x)X + c_i(x)} = d(x)
\]

with $m_i(x) = \text{card}(J_i(x)) \geq 1$, $m_i(x)a_i(x) \neq 0$. But this contradicts our Lemma 7 because

\[
\frac{c_i(x)}{b_i(x)} = \frac{1 + \lambda\lambda_i(x)}{\lambda(\lambda - \lambda_i(x))} \neq \frac{1 + \lambda\lambda_j(x)}{\lambda(\lambda - \lambda_j(x))} = \frac{c_j(x)}{b_j(x)}
\]

for every $i \neq j$, $1 \leq i, j \leq k$.

Summing up, $I_3(x) = \emptyset$ for every $x \in N$, which means that all the principal curvatures of $M$ at the points of $N$ are constant and they are equal to either $-\lambda^{-1}$ or $\lambda$. From the expression (12), the same happens along the geodesics $\beta_x(s)$ for every $s \in (-\frac{\pi}{2w}, \frac{\pi}{2w})$. Taking into account that every point of $M$ which is not a critical point of $\ell_0$, can be reached through a geodesic $\beta_x(s)$, we conclude that the principal curvatures of $M$ are constant on the whole $M$ and they are equal to either $-\lambda^{-1}$ or $\lambda$. That is, $M$ is a complete isoparametric hypersurface of $S^{n+1}$ with at most two distinct principal curvatures, and from the well known rigidity result by Cartan [3] (see also [6, Chaper 3]) we conclude that $M$ is either a totally umbilical sphere (in the case that all its principal curvatures are equal to $-\lambda^{-1}$) or it is either Clifford hypersurface of the form $M_k(r) = S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ with radius $0 < r < 1$ (in the case that the principal curvatures take both values).

This finishes the proof of Theorem 3.

Let us exhibit an example that shows that the condition on the mean curvature to be constant is necessary in the previous result.

**Example 8.** Let $e_1 = (1,0,\ldots,0) \in \mathbb{R}^{n+1}$ and $c = 4/5$. From Example 1 we know that the principal curvatures of $S^{n-1}(e_1, c) \subset S^n$ are all equal to $4/3$. By perturbing $S^{n-1}(e_1, c)$ we can find a hypersurface $N \subset S^n$ whose mean curvature is not constant and such that all its principal curvatures $\lambda_i$ satisfy that

\[
1 < \lambda_i(x) < 2 \quad \text{for every } x \in N \text{ and } i = 1, \ldots, n-1
\]
Let $M^n = S^1 \times N$ and $\phi : M \to S^{n+1} \subset \mathbb{R}^{n+2}$ the map given by
\[
\phi((\cos s, \sin s), x) = \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \frac{1}{2}(x + \nu(x)) \cos(\sqrt{2}s) + \frac{1}{2}(x - \nu(x)) \right),
\]
where $x \in N \subset S^n \subset \mathbb{R}^{n+1}$ denotes the points in $N$ and $\nu : N \to S^n \subset \mathbb{R}^{n+1}$ is a Gauss map of $N$. In particular, $(x, \nu(x)) = 0$.

Let $\frac{\partial}{\partial s} = (-\sin s, \cos s)$ and let $v_1, \ldots, v_{n-1}$ be a basis of $T_x N$ such that $-d\nu_x(v_i) = \lambda_i(x) v_i$. Notice that $\frac{\partial}{\partial s} = ((-\sin s, \cos s), 0) \in \mathbb{R}^{n+3}$ and $\bar{v}_1 = (0, 0, v_1), \ldots, \bar{v}_{n-2} = (0, 0, v_{n-2})$ form a basis for the tangent space of $M$ at $p = ((\cos s, \sin s), x)$. A direct computation shows that
\[
d\phi_p(\frac{\partial}{\partial s}) = (\cos(\sqrt{2}s), -\frac{1}{\sqrt{2}}(x + \nu(x)) \sin(\sqrt{2}s))
\]
and
\[
d\phi_p(\bar{v}_i) = \frac{1}{2}(0, ((1 - \lambda_i(x)) \cos(\sqrt{2}s) + 1 + \lambda_i(x)) v_i).
\]
By (17), the expression $(1 - \lambda_i(x)) \cos(\sqrt{2}s) + (1 + \lambda_i(x))$ never vanishes, therefore $\phi$ is an immersion. Moreover, it is easy to check that $\tilde{\nu} : M \to S^{n+1} \subset \mathbb{R}^{n+2}$ given by
\[
\tilde{\nu}(p) = \left( \frac{1}{\sqrt{2}} \sin(\sqrt{2}s), \frac{1}{2}(x + \nu(x)) \cos(\sqrt{2}s) - \frac{1}{2}(x - \nu(x)) \right)
\]
is a Gauss map on $M$. Using the expression for $\phi$ and for $\tilde{\nu}$ we get that $\ell_v = f_v$ for $v = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2}$.

4. Stability Index of Hypersurfaces with Constant Mean Curvature

In this section, and as an application of our Theorem 3 we will prove that the weak stability index of a compact constant mean curvature hypersurface $M^n$ in $S^{n+1}$ with constant scalar curvature must be greater than or equal to $2n + 4$ whenever $M^n$ is neither a totally umbilical sphere nor a Clifford hypersurface. Recall that constant mean curvature hypersurfaces in $S^{n+1}$ are critical points of the area functional restricted to variations that preserve a certain volume function. The Jacobi operator of this variational problem is given by $J = \Delta + \|A\|^2 + n$, with associated quadratic form given by
\[
Q(f) = -\int_M f J f
\]
and acting on the space
\[
C_+^\infty(M) = \{ f \in C^\infty(M) : \int_M f = 0 \}.
\]
Precisely, the restriction $\int_M f = 0$ means that the variation associated to $f$ is volume preserving. The weak stability index of the hypersurface, denoted here by $\text{Ind}_T(M)$, is characterized by
\[
\text{Ind}_T(M) = \max\{ \dim V : V \subseteq C_+^\infty(M), \quad Q(f) < 0 \quad \text{for every } f \in V \},
\]
and $M$ is called weakly stable if and only if $\text{Ind}_T(M) = 0$ (see [1] for further details).

In [3], Barbosa, do Carmo and Eschenburg characterized the totally umbilical spheres as the only compact weakly stable constant mean curvature hypersurfaces in $S^{n+1}$. In [2] the authors have recently showed that the weak index of a compact constant mean curvature hypersurface $M^n$ in $S^{n+1}$ which is not totally umbilical and has constant scalar curvature is greater than or equal to $n + 2$, with equality
if and only if $M^n$ is a Clifford hypersurface $M_k(r) = S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ with radius $r \leq \sqrt{(k+2)/(n+2)}$. Here we will complement this result by showing the following.

**Theorem 9.** Let $M^n$ be a compact orientable hypersurface immersed into the Euclidean sphere $\mathbb{S}^{n+1}$ with constant mean curvature. If $M$ has constant scalar curvature and $M$ is neither a Clifford nor an umbilical hypersurface, then the weak stability index of $M$ is greater than or equal to $2n+4$.

**Proof.** The condition on the scalar curvature implies that, $\|A\|^2$ is constant. Let us first consider the case where $H = 0$. Since $M^n$ is not totally umbilical (i.e., totally geodesic), then $\|A\|^2 > 0$. Even more, since $M$ is not a minimal Clifford hypersurface we have that $\|A\|^2 > n$, by a classical result due to [9] and [5, 7] (see [11] Theorem 6). By Proposition 5 we have that the functions $\ell_v$ and $f_v$ are eigenfunctions of the Laplacian with positive eigenvalues $n$ and $\|A\|^2 > n$, respectively (observe that with our criterion, a real number $\lambda$ is an eigenvalue of $\Delta$ if and only if $\Delta u + \lambda u = 0$ for some smooth function $u \in C^\infty(M)$, $u \not\equiv 0$). In particular, the functions $\ell_v$ and $f_v$ satisfy the condition $\int_M f = 0$, and they also satisfy $J(\ell_v) = \|A\|^2 \ell_v$ and $Jf_v = nf_v$. That is, they are also eigenfunctions of $J$ with negative eigenvalues $-\|A\|^2$ and $-n$, respectively. Let

$$V_1 = \{\ell_v : v \in \mathbb{R}^{n+2}\} \text{ and } V_2 = \{f_v : v \in \mathbb{R}^{n+2}\}.$$  

Then,

$$\text{Ind}_T(M) \geq \text{dim}(V_1 \oplus V_2) = \text{dim}V_1 + \text{dim}V_2,$$

where the last equality is due to the fact that $V_1$ and $V_2$ are $L^2$-orthogonal subspaces, because they are eigenspaces of $\Delta$ associated to different eigenvalues. Finally, as pointed out in Subsection 1.2 we also know that if either $\text{dim}V_1 < n+2$ or $\text{dim}V_2 < n+2$, then $M$ must be a totally geodesic sphere (see [8 Lemma 3.1]). Therefore, in our case we have $\dim V_1 = \dim V_2 = n+2$, and by (18) we conclude that $\text{Ind}_T(M) \geq 2n+4$.

We will now consider the case $H \neq 0$. By Cauchy-Schwarz inequality we have that $\|A\|^2 \geq nH^2$, and equality only occurs if $M$ is totally umbilical. In this case, following our ideas in [2], we will work with test functions of the form $\ell_v - \alpha \pm f_v$, where

$$\alpha \pm = \frac{\|A\|^2 - n \pm \sqrt{D}}{2nH} \text{ with } D = (\|A\|^2 - n)^2 + 4n^2H^2 > 0.$$  

Let

$$U_+ = \{\ell_v - \alpha_+ f_v : v \in \mathbb{R}^{n+2}\} \text{ and } U_- = \{\ell_v - \alpha_- f_v : v \in \mathbb{R}^{n+2}\}.$$  

Then, by Proposition 5 we have that $\Delta u + \mu_\pm u = 0$ for every $u \in U_\pm$, where

$$0 < \mu_- = \frac{n + \|A\|^2 - \sqrt{D}}{2} < \mu_+ = \frac{n + \|A\|^2 + \sqrt{D}}{2},$$

and, therefore, $Ju + \lambda_\pm u = 0$ for every $u \in U_\pm$, with

$$\lambda_- = \frac{-(n + \|A\|^2) - \sqrt{D}}{2} < \lambda_+ = \frac{-(n + \|A\|^2) + \sqrt{D}}{2} < 0$$

(for the details, see [2] Section 4). In particular, functions belonging to $U_\pm$ also satisfy the condition $\int_M f = 0$, and

$$\text{Ind}_T(M) \geq \text{dim}(U_+ \oplus U_-) = \text{dim}U_+ + \text{dim}U_-.$$
Finally, since $M$ is neither a totally umbilical sphere nor a Clifford hypersurface, our Theorem 3 implies that $\dim U_+ = \dim U_- = n + 2$, and by (19) we conclude that $\text{Ind}_T(M) \geq 2n + 4$.

\[\square\]

Acknowledgements

The authors would like to thank the referee for valuable suggestions which improved the paper.

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