SYMMETRICALLY FACTORIZABLE GROUPS AND SET-THEORETICAL SOLUTIONS OF THE PENTAGON EQUATION

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Abstract. The notion of a symmetrically factorizable Lie group is introduced. It is shown that each symmetrically factorizable Lie group is related to a set-theoretical solution of the pentagon equation. Each simple Lie group (after a certain Abelian extension) is symmetrically factorizable.

1. Introduction

Let $V$ be a vector space. We say that a linear operator $S \in \text{End}(V \otimes V)$ satisfies the pentagon equation, if

$$S_{12}S_{13}S_{23} = S_{23}S_{12}, \quad S_{ij} \in \text{End}(V \otimes V \otimes V) \quad (1)$$

where $S_{ij}$ acts as $S$ on $i$-th and $j$-th factors in the tensor product and leaves unchanged vectors in the remaining factor. It is clear that the invertibility of $S$ is a necessary property of solutions to the pentagon equation.

Let $M$ be a set. A set-theoretical solution to the pentagon equation is an invertible mapping $s : M \times M \to M \times M \quad (2)$

which satisfies the “reversed” pentagon equation with respect to compositions of transformations of $M \times M \times M$

$$s_{23} \circ s_{13} \circ s_{12} = s_{12} \circ s_{23} \quad (3)$$

with analogous meaning of subscripts as above. For each transformation $s$ of $M \times M$ one associates its pull-back $S \equiv s^\ast$, i.e. the linear operator in the space of complex valued functions on $M \times M$,

$$Sf(x, y) = f(s(x, y)). \quad (4)$$

It is clear that if $s$ is a set-theoretical solution to the pentagon equation, its pull-back $S$ satisfies the operator pentagon equation with appropriate definition for the tensor product of infinite dimensional vector spaces.

The Heisenberg double of a Hopf algebra [4] is a natural algebraic framework for operator pentagon equation as its canonical element can be interpreted as a solution of the pentagon equation in a certain algebraic sense, for example in the context of $C^\ast$-algebras [1]. It can also be interpreted as the pentagon equation for the associativity constraint in a semisimple monoidal category generated by tensor powers of a single simple object $X$ with $\text{Hom}(X \otimes^2 X) \simeq V$.

In this paper we study set-theoretical solutions to the pentagon equation when $M$ is an algebraic manifold. In this case, the pull-back solution to the pentagon equation acts on $C(M \times M)$, the space of algebraic functions on $M \times M$. So, the operator $S$ is defined when $s$ makes sense on a Zariski open subset of $M \times M$. We introduce the notion of a symmetrically factorizable Lie group and show that it provides us with set-theoretical solutions to the pentagon equation. This paper is one of the several papers in which we will study topological field theories for simple
complex Lie groups. In a follow-up paper we will use the results presented here to construct corresponding representations of the modular group of a punctured surface.

In Section 2 we analyze set-theoretical solutions to the pentagon equation. In Section 3 we define the notion of symmetrically factorizable groups and give basic examples. Solutions of the pentagon equation related to symmetrically factorizable Lie groups are described in Section 4.

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2. Set-theoretical solutions of the pentagon equation

2.1. The mapping $\rho$. Let $s$ be a transformation (invertible mapping) of $M \times M$. Define binary operations $\ast$ and $\cdot$ as

\[ s(x, y) = (x \cdot y, x \ast y). \]  

(5)

First, we derive the equations to these operations which ensure the pentagon equation for $s$.

**Proposition 1 ([3]).** The mapping $s$ is a set-theoretical solution to the pentagon equation if and only if the following equations hold

\[ (x \cdot y) \cdot z = x \cdot (y \cdot z) \]  

(6)

\[ (x \ast y) \cdot ((x \cdot y) \ast z) = x \ast (y \cdot z) \]  

(7)

\[ (x \ast y) \ast ((x \cdot y) \ast z) = y \ast z \]  

(8)

and for any pair $x, y \in M$ there exists a unique pair $u, z \in M$ such that

\[ u \cdot z = x, \quad u \ast z = y. \]  

(9)

**Example 1.** If the set $M$ is a group with respect to dot-operation, $x \cdot y \equiv xy$, then the only solution to the system (7)—(8) is given by $x \ast y = y$ [3]. To prove this statement, notice that invertibility of $s$ is equivalent to uniqueness of the pair $u, z \in M$ in eqns (9). Using the group structure, we can solve the first of these equations for $u$ and substitute the result into the second. Thus, for any pair $x, y \in M$ there exists a unique $z \in M$ such that

\[ xz^{-1} \ast z = y. \]  

(10)

Specifying $y = z^{-1}$ in eqn (7) we have

\[ (x \ast z^{-1})(xz^{-1} \ast z) = x \ast 1, \]  

which (when considered at $z = 1$) implies that

\[ x \ast 1 = 1 \]  

(11)

and thus $x^{-1} \ast z = (x \ast z^{-1})^{-1}$. Combining this with uniqueness of $z$ for any $x, y$ in eqn (10), we conclude that the mapping

\[ y \mapsto x \ast y \]  

(12)

is a bijection for any $x \in M$. Equation (8) with $y = 1$ reads

\[ (x \ast 1) \ast (x \ast z) = 1 \ast z, \]  

or taking into account eqn (11), we have

\[ 1 \ast (x \ast z) = 1 \ast z. \]  

Since the mapping $z \mapsto 1 \ast z$ is a bijection as a special case of eqn (12), we equivalently have

\[ x \ast z = z \]  

Remark 1. As is observed in [2], in the case when $M$ is a group, the pull-back of the corresponding set-theoretical solution of the pentagon equation

$$s(x, y) = (xy, y)$$

is nothing else but the faithful realization of the canonical element of the Heisenberg double of the group algebra of $M$.

As one can see from this example, the essential part of the proof that $x \ast y = y$ is the assumption that $x \ast y$ is defined for $x = 1$. If we assume that this operation is defined only on an open dense subset of $M$ which does not include 1, then (under reasonable assumptions about this dense subset) we can still have solutions to the pentagon equation in some class of functions.

In this paper we will focus on the case when the set $M$ is an algebraic group. We will assume that the star operation is either an algebraic mapping defined on a Zariski open subset of $M \times M$ or some birational correspondence. Then, equations (7)–(8) are expected to hold on a Zariski open subset of $M \times M \times M$.

Example 2. Let $M = \mathbb{C}^* \times \mathbb{C}$ be group of upper triangular matrices of the form

$$
\begin{pmatrix}
x_1 & x_2 \\
0 & 1
\end{pmatrix}
$$

We will also write $x = (x_1, x_2)$ for elements of this group. In this notation the matrix multiplication is

$$x \cdot y = (x_1 y_1, x_1 y_2 + x_2).$$

It was shown in [3] that this operation together with the operation

$$x \ast y = (y_1 x_2 (x_1 y_2 + x_2)^{-1}, y_2 (x_1 y_2 + x_2)^{-1})$$

defined on a Zariski open subset of $M \times M$ satisfies equations (6)–(8).

Notice that this operation differs from $x \ast y = y$ and the reason is that it is not defined for $x = 1$ and, therefore, arguments used in the previous example do not apply.

The following proposition gives a construction of the star operation if the dot operation comes from a group with a special structure.

**Proposition 2** ([3]). Let $M$ be an algebraic group. Assume that it is equipped with invertible mapping (or birational correspondence)

$$\rho: M \to M$$

defined on a Zariski open subset of $M$ satisfying the condition that

$$\sigma(x) \equiv (\rho(y))^{-1} \rho(\rho(x)(\rho(xy))^{-1})$$

is independent of $y$ on a Zariski open subset of $M \times M$. Then, operations

$$x \cdot y = xy, \quad x \ast y = \rho(x)(\rho(xy))^{-1}$$

define a set-theoretical solution of the pentagon equation.

**Proof.** Equations (6)–(8) can be checked straightforwardly. The inverse mapping to $s$ defined by (16) has the form

$$(x, y) \mapsto (\rho^{-1}(y \rho(x)), (\rho^{-1}(y \rho(x)))^{-1}x).$$

**Corollary 1.** If $\rho(x)$ satisfies the conditions of Proposition 2 then so does $\tilde{\rho}(x) = \rho(x)a$ for any $a \in M$ and they define one and the same solution of the pentagon equation.
This corollary demonstrates that if the mapping $\rho$ exists it is not unique. Fixing the equivalence class of $\rho$, one can choose having in mind some additional properties. In the simplest case of Example 1 one can choose $\rho(x) = x^{-1}$. In other examples we will have $\rho$ of order three and will use it later in a construction of the representation of the mapping class group of a surface.

**Example 3.** In Example 2 the order three mapping $\rho$ defined as

$$\rho(x) = (x_2 x_1^{-1}, x_1^{-1})$$

satisfies the conditions of Proposition 2 with $\sigma(x) = \rho^{-1}(x)$.

**Example 4.** Let us fix rational numbers $s_1, s_2, s_3$ such that

$$s_1 + s_2 + s_3 = 0, \quad s_2(s_3 - s_1) \neq 0$$

and define

$$M \equiv \{ x = (x_0, x_1, x_2, x_3) \in \mathbb{C}^4 \}$$

with dot-mapping

$$x \cdot y = (x_0 y_0, x_1 y_0^{s_2}, y_1 + x_2 y_0^{s_1} + y_2 + x_1 y_3 y_0^{s_2} x_3 y_0^{s_3} + y_3), \quad s_{ij} \equiv s_i - s_j$$

given by the multiplication rule of triangular $3 \times 3$-matrices

$$x \mapsto \begin{pmatrix} x_0^{s_1} & x_0^{s_1} x_1 & x_0^{s_1} x_2 \\ 0 & x_0^{s_2} & x_0^{s_2} x_3 \\ 0 & 0 & x_0^{s_3} \end{pmatrix}$$

The following birational correspondence

$$\rho(x) = \left( (x_2 x_4)^{1/(s_2 s_3)}, x_0^{s_2} x_1 x_4^{-1}, x_0^{s_3} x_4^{-1}, x_0^{s_2} x_3 x_2^{-1} \right)$$

where $x_4 \equiv x_1 x_3 - x_2$, is of order three and satisfies eqn (15) with $\sigma(x) = \rho^{-1}(x)$.

### 2.2. Set-theoretical solutions and group operations.

If $s$ is a set-theoretical solution to the pentagon equation then so is the mapping $\bar{s} = s_{21}^{-1}$. Using this fact, we define two more binary operations on $M$:

$$\bar{s}(x, y) \equiv (x \odot y, x \oplus y)$$

Clearly, the pentagon equation implies that these operations also satisfy relations (6)—(8) with replacements

$$\cdot \rightarrow \odot, \quad * \rightarrow \oplus$$

In particular, $\odot$ is an associative operation. Besides that, from the definition of $\bar{s}$ it follows that

$$\bar{s}(x \odot y, x \cdot y) = y, \quad (x \odot y) \odot (x \cdot y) = x$$

and

$$\bar{s}(x \oplus y, x \odot y) = y, \quad (x \oplus y) \odot (x \odot y) = x.$$
Theorem 1. Let \( G, s, \tilde{s} \) be as above. Mapping \( s \) is a set-theoretical solution of the pentagon equation if and only if

\[
j(x \cdot y) = j(x) \cdot j(k(x) \cdot j(y)), \quad k = i \circ j \circ i \tag{22}
\]

on a Zariski open subset of \( G \times G \). Furthermore, eqn (22) implies that the mapping

\[
\rho = i \circ j \tag{23}
\]

defined on a Zariski open subset of \( G \) is of order three and satisfies all conditions of Proposition 2 with \( \sigma(x) = \rho^{-1}(x) \).

Proof. To simplify formulae we will omit the dot in denoting the group multiplication in \((G, \cdot, i)\) as well as the composition symbol for mappings \( i \) and \( j \).

Suppose eqn (22) holds. Using the same equation in its right hand side we obtain the consistency condition

\[
j(xy) = j(x)jiji(x)j(y) \tag{24}
\]

which is equivalent to

\[
k = iji = jij. \tag{24}
\]

This implies that mapping \( \rho \), defined in eqn (23), is of third order. Expression (15) is evaluated as follows

\[
ip(y)\rho(p(x)\rho(xy)) = j(y)\rho(iji(x)j(xy)) = j(y)\rho(jiji(x)j(y)) = ji(x) = \rho^{-1}(x). \tag{25}
\]

Thus, all conditions of Proposition 2 are satisfied with \( \sigma(x) = \rho^{-1}(x) \).

Assume now that \( s \) is a set-theoretical solution of the pentagon equation. The first of eqns (20) can be solved for the \(*\)-operation:

\[
x \ast y = y \circ j(xy). \tag{26}
\]

Substituting it into (7), we obtain

\[
(y \circ j(j(x)i(z)))(z \circ x) = (yz) \circ x, \tag{27}
\]

where we have replaced \( x \to j(x)i(yz) \). Substituting here \( x \to j(x) \), then multiplying from the right by \( x \) with respect to \( \circ \), and replacing consequently \( y \to y \circ (xi(z)) \) and \( z \to z \circ x \), we obtain

\[
(y \circ (xi(z) \circ x))(z \circ x) = (yz) \circ x \tag{28}
\]

where we have also exchanged the left and right hand sides. Comparing eqns (25) and (26) we obtain the relation

\[
j(j(x)i(z)) = xi(z \circ x) \tag{29}
\]

which can be solved for the operation \( \circ \):

\[
x \circ y = ij(j(y)i(x))y. \tag{30}
\]

The symmetry \((\cdot, i) \leftrightarrow (\circ, j)\) leads to the equation

\[
xy = ji(i(y) \circ j(x)) \circ y, \tag{31}
\]

which after using eqn (27) gives

\[
xy = ji(i(j(xy)i(x)) \circ y = j(i(x)j(xy)) \circ y = ij(j(y)i(j(x)j(xy)))y. \tag{32}
\]

Solving this for \( j(xy) \), we come to equation (22).
Example 5. Let $M = \mathbb{C}^* \times \mathbb{C}$ with the dot and star operations being given as in Example 2. The $\odot$ operation can be found from the equation (20):

$$x \odot y = (x_2 y_1 + x_1, x_2 y_2).$$

Clearly, this operation is equivalent to the matrix multiplication of $2 \times 2$ matrices

$$\begin{pmatrix} 1 & 0 \\ x_1 & x_2 \end{pmatrix}.$$

The inverse for the multiplications $\cdot$ is

$$i(x) = \left(1/x_1, -x_2/x_1\right).$$

The operation $j$ exists and is given by

$$j(x) = (-x_1/x_2, 1/x_2).$$

If we include the line $(0, y)$ in $G$ and remove the line $(x, 0)$ then this operation becomes the inverse for the associative operation $\odot$ with the unit $(0, 1)$.

Composition of $i$ and $j$ operations give

$$k(x) = iji(x) = jij(x) = (x_2, x_1),$$

while the third order mapping

$$\rho(x) = i(j(x)) = (-x_2/x_1, 1/x_1)$$

differs from that of Example 3, see eqn (17), in the sign of the first component. This is equivalent to taking the composition with the automorphism $x_i \rightarrow -x_i$.

**Corollary 2.** Eqn (22) implies that the group $(G, \cdot, i)$ is birationally equivalent to "almost a group" $(G, \odot, j)$. The equivalence is given by the mapping $k = iji$:

$$x \odot y = k(k(x)k(y)).$$

**Proof.** Indeed, let us rewrite eqn (27) by using (22)

$$x \odot y = i(yj(kj(y)ji(x)))y = i(kj(y)ji(x)) = iji(iji(x)ikj(y)) = k(k(x)k(y)).$$

Intertwining of the inverse operation follows from eqn (24): $j = iki = kik$. 

**Corollary 3.** Operations $i$ and $j$ realize the action of the permutation group $S_3$ on $M$.

**Proof.** It is clear that $i^2 = j^2 = 1$ which together with eqn (24) imply that $(ij)^3 = 1$. Thus, elements $i$ and $j$ generate a group which is isomorphic to $S_3$. 

**Remark 2.** This $S_3$ symmetry reflects the tetrahedral symmetry of the corresponding set-theoretical solution of the pentagon equation.

3. **Symmetric factorization in a Lie group**

Here we introduce the notion of symmetric factorization in a Lie group.

**Definition 1.** A Lie group $G$ with the Lie algebra $\mathfrak{g}$ is called symmetrically factorizable if

- $\mathfrak{g}$ as a vector space is isomorphic to the direct sum of its Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, i.e. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$.
- there exists an element $\theta \in G$ which conjugates Lie subgroups $G_\pm$ corresponding to Lie subalgebras $\mathfrak{g}_\pm$, i.e. $\theta G_- \theta = G_+ G_-.$

The element $\theta$ is called conjugating element. In a symmetrically factorizable Lie group there exists an open dense neighborhood $G' \subset G$ of the unit element such that for any $g \in G'$ there exist $g_\pm, \bar{g}_\pm \in G_\pm$ such that

$$g = g_+ g_-^{-1} = \bar{g}_-^{-1} \bar{g}_+.$$
Example 6. Let $G = GL(2N, \mathbb{C})$. Any element of this Lie group can be regarded as a block matrix of the form

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad \det(g) \neq 0, \quad g_{ij} \in \text{Mat}(N, \mathbb{C}).$$

Subgroups

$$G_+ = \{ g \in G | g_{21} = 0, g_{22} = 1 \}, \quad G_- = \{ g \in G | g_{12} = 0, g_{11} = 1 \}$$

are conjugate to each other

$$\theta G_- = G_+ \theta, \quad \theta \equiv \begin{pmatrix} 0 & b \\ b^{-1} & 0 \end{pmatrix}, \quad b \in GL(N, \mathbb{C}).$$

Thus, Lie group $GL(2N, \mathbb{C})$ is symmetrically factorizable.

Let $G' \subset G$ be the subset of all elements $g$ such that

$$g_{22}, g_{11} \in GL(N, \mathbb{C}).$$

Every $g \in G'$ has unique factorization:

$$g = g_+ g_-^{-1} = \bar{g}_-^{-1} \bar{g}_+,\,$$

where

$$g_+ = \begin{pmatrix} g_{11} - g_{12} g_{22}^{-1} g_{21} & g_{12} g_{22}^{-1} \\ 0 & 1 \end{pmatrix}, \quad g_-^{-1} = \begin{pmatrix} 1 & 0 \\ g_{21} & g_{22} \end{pmatrix}$$

and

$$\bar{g}_-^{-1} = \begin{pmatrix} 1 & 0 \\ g_{21} g_{11}^{-1} g_{22} & g_{22} - g_{21} g_{11}^{-1} g_{12} \end{pmatrix}, \quad \bar{g}_+ = \begin{pmatrix} g_{11} & g_{12} \\ 0 & 1 \end{pmatrix}.$$

Thus, we obtain explicit description of the symmetrical factorization of $GL(2N, \mathbb{C})$.

Example 7. Let $G$ be a finite dimensional complex algebraic simple Lie group with fixed Borel subgroup $B \in G$. On a Zariski open subset $G' \subset G$ we have the Gauss decomposition

$$G' = N_+ H N_- \cap N_- H N_+,$$

where $N_{\pm}$ are nilpotent in the Borel subgroups $B_{\pm} = H N_{\pm}$ ($B_+ \equiv B$). The Weyl group $W$ of $G$ can be naturally identified with the quotient group

$$W = N(H)/H, \quad N(H) = \{ g \in G : g H g^{-1} = H \}.$$

Let $w_0 \in W$ be the longest element in $W$ and $\{ h_i \}_{i=1}^r, r = \text{rank}(G)$, be a basis in the Cartan subalgebra $\mathfrak{h} \in \mathfrak{g} = \text{Lie}(G)$ generated by simple roots. We choose an enumeration of simple roots which for root system of type $A_n, D_n, E_6$ corresponds.
to the Cartan matrices

\[
A_n: \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & 2 & -1 & \ldots & 0 \\
0 & -1 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 2
\end{pmatrix}
\]

\[
D_n: \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 2 & 0 \\
0 & 0 & 0 & \ldots & -1 & 0 & 2
\end{pmatrix}
\]

\[
E_6: \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & -1 & 0 \\
0 & 0 & -1 & 2 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 2 -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

The longest element of the Weyl group acts on \( h_i \) as:

\[ w_0(h_i) = -h_{\tau(i)}, \]

where \( \tau \) is an automorphism of order 2 of the set of vertices of the Dynkin diagram of the Lie algebra \( \mathfrak{g} \). The automorphism \( \tau \) is non-trivial only for root systems of the type \( A_n, D_n, E_6 \) where it acts as follows,

\[
A_n: \tau(i) = n + 1 - i, \quad i = 1, \ldots, n;
\]

\[
D_n: \begin{cases} 
  n & \text{if } i = n - 1, \\
  i & \text{otherwise}; \\
  6 & \text{if } i = 1, \\
  1 & \text{if } i = 6, \\
  i & \text{otherwise};
\end{cases}
\]

\[
E_6: \begin{cases} 
  6 & \text{if } i = 1, \\
  1 & \text{if } i = 6, \\
  i & \text{otherwise};
\end{cases}
\]

Consider the decomposition of the Cartan subalgebra \( \mathfrak{h} \) of the Lie algebra \( \mathfrak{g} \) into a direct sum

\[ \mathfrak{g} \supset \mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}' \oplus \mathfrak{h}'', \]

where \( \mathfrak{h}_0 \) is the subspace on which \( w_0 \) acts as \(-1\), while \( w_0(\mathfrak{h}') = \mathfrak{h}'' \). The following table gives the dimensions of the subspaces:

| \( \mathfrak{g} \) | \( \dim(\mathfrak{h}_0) \) | \( \dim(\mathfrak{h}') = \dim(\mathfrak{h}'') \) |
|-----------------|-----------------|-----------------|
| \( X_n \neq A_n, D_n, E_6 \) | \( n \) | \( 0 \) |
| \( A_{2k} \) | \( 0 \) | \( k \) |
| \( A_{2k+1} \) | \( 1 \) | \( k \) |
| \( D_n \) | \( n - 2 \) | \( 1 \) |
| \( E_6 \) | \( 2 \) | \( 2 \) |

Denote by \( H_0, H', H'' \) the subgroups of the Cartan subgroup \( H \) corresponding to \( \mathfrak{h}_0, \mathfrak{h}', \mathfrak{h}'' \), respectively, and consider a Lie group\(^1\) \( D = G \times H_0 \) together with its

\(^1\)In the case of \( A_{2k} \) we take just \( D = G. \)
subgroups
\[
D_+ = \{(h_0h'f, h_0) \mid h' \in H', \ h_0 \in H_0, \ f \in N_+\},
\]
\[
D_- = \{(h_0^{-1}h''f, h_0) \mid h'' \in H'', \ h_0 \in H_0, \ f \in N_-\}.
\]
Let \(\bar{w}_0 \in N(H) \subset G\) be a representative of the longest element \(w_0 \in W\) of the Weyl group. Set \(\theta = (\bar{w}_0, 1)\). We have
\[
\theta(h_0^{-1}h''f, h_0)\theta^{-1} = (h_0w_0(h'')\bar{w}_0f\bar{w}_0^{-1}, h_0) \in D_+.
\]
Thus, the triple \((D, D_-, D_+)\) determines a factorizable Lie group with the conjugating element \(\theta\). Notice that we can always choose \(\bar{w}_0\) such that \(\bar{w}_0^2\) is central. For this choice \(\theta^2\) is central in \(D\). If necessary, taking quotient with respect to a center of \(G\), we can always assume that \(\theta^2 = 1\).

4. Symmetrical factorization and the pentagon equation

Now we can relate symmetrically factorizable Lie groups to set-theoretical solutions of the pentagon equation. Our basic examples are symmetrically factorizable Lie groups related to simple algebraic Lie groups. In this case we know (see examples above) that we can assume that the conjugating element \(\theta\) can be chosen such that \(\theta^2 = 1\). In fact, this choice can be made under more general assumptions.

**Proposition 3.** If element \(\theta^2\) is uniquely left factorizable,
\[
\theta^2 = (\theta^2)_+ (\theta^2)^{-1}_-
\]
then
\[
\tilde{\theta} \equiv (\theta^2)^{-1}_+ \theta
\]
is also a conjugating element and it is unipotent.

**Proof.** It is clear that \(\tilde{\theta}\) conjugates \(G_\pm\)
\[
\tilde{\theta}G_- \tilde{\theta}^{-1} = (\theta^2)^{-1}_+ \theta G_- \theta^{-1} (\theta^2)^{-1}_+ = (\theta^2)^{-1}_+ G_+(\theta^2)_+ = G_+.
\]
The unipotency of \(\tilde{\theta}\) follows from the equalities:
\[
\tilde{\theta}^2 = (\theta^2)^{-1}_+ \theta (\theta^2)_+^{-1}_+ \theta = (\theta^2)_+^{-1} \theta^2 \theta^{-1} (\theta^2)_+^{-1}_+ \theta = (\theta^2)_+^{-1} \theta^{-1} (\theta^2)_+^{-1}_+ \theta
\]
\[
= (\theta^2)^{-1}_+ \theta (\theta^2)_+^{-1}_- \theta = (\theta^2)^{-1}_+ \theta (\theta^2)^{-1}_+ \theta
\]
\[
= (\theta^2)^{-1}_+ \theta (\theta^2)^{-1}_+ \theta
\]
Notice that uniqueness of factorization of \(\theta^2\) is essential here. \(\square\)

**Lemma 1.** If both \(\theta\) and \(\theta^{-1}\) are conjugating elements, then for any \(g \in G\) the following relations hold:
\[
(\theta g^{-1}\theta^{-1})_\pm = \theta g_\mp \theta^{-1}.
\]

**Proof.** In this case we have simultaneously \(\theta G_\pm = G_\pm \theta\), so that for any \(g \in G\)
\[
\theta g^{-1}\theta^{-1} = \theta g_\mp \theta^{-1} (\theta g_\pm \theta^{-1})^{-1}.
\]
\(\square\)

The next theorem can be regarded as the main result of the paper.

**Theorem 2.** (A) Let \(G\) be a symmetrically factorizable algebraic group. Then the mapping
\[
\rho(x) \equiv (\theta x)_+^{-1}, \quad \rho^{-1}(x) = (\theta^{-1} x^{-1})_+
\]
defined on a Zariski open subset of \(G_+\), satisfies the conditions of Proposition 2.

(B) If furthermore \(\theta^2 = 1\), then \(j(x) \equiv (\theta x)_+\) is involutive and satisfies eqn (22).
(C) With any involutive solution \( j \) of eqn (22) there exists an associated symmetrically factorized “almost a group”.

**Proof.** (A) Evaluating (15) we have

\[
(r(y))^{-1}p(x(r(xy))^{-1}) = (\sigma y)_+ ((\sigma x)_+^{-1}(\sigma xy)_+)^{-1} = (\sigma y)_+ ((\sigma x)_+^{-1}x^{-1}\sigma^{-1})^{-1}
\]

which is independent of \( y \).

(B) Mapping \( j \) is involutive:

\[
j \circ j(x) = (\theta(\theta x)_+)_+ = (\theta^2 x)_+ = (x)_+ = x
\]

while eqn (22) is checked as follows

\[
j(xy) = (\theta xy)_+ = ((\theta x)_+ (\theta x)_+^{-1})_+ = (\theta x)_+ ((\theta x)_+^{-1}\theta(\theta y)_+)_+
\]

\[
= j(x)j(\theta(\theta x)_+^{-1}\theta j(y)) = j(x)j((k(x)j(y))
\]

where by using Lemma 1

\[
k(x) = \theta(\theta x)_+^{-1}\theta = (\theta(\theta x)_+^{-1}\theta)_+^{-1} = (\theta x)_+^{-1} = iji(x)
\]

(C) “Almost a group” \( G \) is \((M \times M) \cup \theta\) as a set with the following multiplication table

\[
(x_1, x_2)(y_1, y_2) \equiv (x_1 j(x_2 j(y_1)), k(k(x_2)j(y_1)))
\]

\[
\theta(x_1, x_2) = (j(x_1), k(x_1)j(x_2))
\]

where \( k \equiv iji \). The unit element and the inverse operation are correspondingly

\[
1 = (1, 1), \quad (x_1, x_2)^{-1} = (ik(k(x_1)x_2), ij(x_1 j(x_2)))
\]

The subgroups \( G_\pm \) are

\[
G_+ = \{(x, 1) | x \in M\}, \quad G_- = \{(1, x) | x \in M\}
\]

with the conjugating element \( \theta \):

\[
\theta(1, x) = (x, 1)\theta,
\]

and the factorization formula

\[
(x_1, x_2) = (x_1, 1)(1, x_2).
\]

**Corollary 4.** If \( \theta^2 \) is central, then \( \sigma(x) = \rho^{-1}(x) \) and \( \rho \) is of order three.

**Proof.** In this case we have simultaneous conjugations \( \theta G_\pm = G_\mp \theta \) so that

\[
\rho \circ \rho(x) = (\theta(\theta x)_+^{-1})_+^{-1} = (\theta(\theta x)_+^{-1}x^{-1}\theta^{-1})_+^{-1}
\]

\[
= (\theta(\theta x)_+^{-1}\theta^{-1}x^{-1}\theta^{-1})_+^{-1} = (\theta(\theta x)_+^{-1}\theta^{-1})_+^{-1} = (\theta(\theta x)_+^{-1}\theta^{-1})_+ = \sigma(x).
\]

Using Lemma 1, we have

\[
\sigma(x) = (\theta(\theta x)_+^{-1}\theta^{-1})_+ = (\theta x^{-1}\theta^{-2})_+ = (\theta^{-1}x^{-1})_+ = \rho^{-1}(x)
\]

Thus, \( \rho^3(x) = x \).

**Example 8.** Consider the Lie group \( G = GL(2, \mathbb{C}) \) with the symmetrical factorization (see example 6) determined by subgroups

\[
G_+ = \left\{(x_1 \begin{smallmatrix} x_2 & x_2 \\ 0 & 1 \end{smallmatrix}) | x_1 \in \mathbb{C}^*, \; x_2 \in \mathbb{C}\right\}
\]
\[ G_+ = \left\{ \begin{pmatrix} 1 & 0 \\ x_1 & x_2 \end{pmatrix} \mid x_1 \in \mathbb{C}, x_2 \in \mathbb{C}^* \right\} \]

with the conjugating element
\[ \theta = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, \quad bc \in \mathbb{C}^*. \]

It is easy to verify that if we parameterize \( g \in G_+ \) by coordinates \( x_1, x_2 \) as above, then
\[ \rho(x) = (\theta g)_+^{-1} = \begin{pmatrix} -x_2/(bx_1) & 1/(cx_1) \\ 0 & 1 \end{pmatrix} \]

which at \( c = -b = 1 \) reproduces eqn (17). Element \( \theta \) is unipotent when \( bc = 1 \), and we have
\[ j(x) = (\theta g)_+ = \begin{pmatrix} -bx_1x_2^{-1} & b^2x_2^{-1} \\ 0 & 1 \end{pmatrix} \]

which at \( b = 1 \) reproduces Example 5.

**Example 9.** Let \( G = SL(3, \mathbb{C}) \) and \( s_1, s_2, s_3 \) be as in Example 4. A symmetric factorization is defined by the subgroups

\[ G_+ = \left\{ \begin{pmatrix} x_0^{s_1} & x_0^{s_2}x_1 & x_0^{s_2}x_2 \\ 0 & x_0^{s_2} & x_0^{s_2}x_3 \\ 0 & 0 & x_0^{s_3} \end{pmatrix} \mid (x_0, x_1, x_2, x_3) \in \mathbb{C}^* \times \mathbb{C}^3 \right\} \]

\[ (31) \]

\[ G_- = \left\{ \begin{pmatrix} y_0^{s_1} & 0 & 0 \\ y_0^{s_2}y_3 & y_0^{s_2} & 0 \\ y_0^{s_3}y_1 & y_0^{s_3} & y_0^{s_3} \end{pmatrix} \mid (y_0, y_1, y_2, y_3) \in \mathbb{C}^* \times \mathbb{C}^3 \right\} \]

\[ (32) \]

On a Zariski open subset of \( G \) given by \( g_{33} \bar{g}_{11} \neq 0 \), \( \bar{g} = g^{-1} \), there exists a unique factorization
\[ g = \|g_{ij}\|_{i,j=1}^3 = g + g^{-1} \]

with
\[ (g_+)^{-1} \leftrightarrow (g_{11}^{s_1}g_{33}^{s_1})^{1/(s_2s_3)}, \quad \bar{g}_{12} \bar{g}_{13} \bar{g}_{23} - \bar{g}_{33} \]

and
\[ (g_-)^{-1} \leftrightarrow \frac{g_{33} g_{13}^{s_1} (g_{33}^{s_1})^{1/(s_2s_3)}}{g_{11}}, \quad \frac{g_{31}}{g_{33}} - \frac{\bar{g}_{21}}{g_{11}} \]

Here four-component vectors parametrize matrices \( (g_\pm)^{-1} \) according to eqns (31) and (32). For any \( a, b \in \mathbb{C}^* \) the element
\[ \theta = \begin{pmatrix} 0 & 0 & b \\ 0 & -a/b & 0 \\ 1/a & 0 & 0 \end{pmatrix} \]

is conjugating. It is unipotent if \( a = b \).

Set \( x_4 = x_1x_3 - x_2 \), where \( (x_0, x_1, x_2, x_3) \) are coordinates on \( G_+ \) (as in (33)), then the inverse mapping \( i(x) \) has the form
\[ i(x) = (x_0^{-1}, -x_0^{s_1}x_1, x_0^{s_3}x_4, -x_0^{s_2}x_3) \]

The birational correspondence \( \rho(x) = (\theta x)_+^{-1} \) is given by
\[ \rho(x) = \left( (x_2/a)^{s_3}(x_4/b)^{s_1})^{1/(s_2s_3)}, a^{-1}b^2x_0^{s_2}x_1x_4^{-1}, abx_0^{s_3}x_4^{-1}, a^2b^{-1}x_0^{s_1}x_3x_2^{-1} \right), \]
which at $a = b = 1$ reproduces eqn (18). When $a = b$, the correspondence $j(x) = (\theta x)_+$ is of order two and can be written as a composition $j = iki = kik$, where

$$k(x) = \left( x_0(b^2x_2x_4^{s_1})^{1/(s_2x_3)}, -bx_1x_2^{-1}, b^2x_2^{-1}, -bx_3x_4^{-1} \right).$$

Thus, according to Proposition 2, on the group $G_+$ we have the group multiplication $\cdot$ and the binary operation given by the birational correspondence $\ast$ defined by the factorizable structure on $G_+$. Therefore, for each symmetrically factorizable algebraic Lie group we have a set-theoretical solution to the pentagon equation.

Finally, one should notice that by varying $\theta$, one can have a family of set-theoretical solutions to the pentagon equation. Let $h \in G$ be an element such that $hG_+ = G_+h$. Then, $\theta h$ is again a conjugating element and, therefore, defines a $\ast$ operation on $G_+$. This gives a family of solutions to the pentagon equation defined by a symmetrically factorizable Lie group. We have such families in Examples 8 and 9.

We have chosen the algebro-geometrical setting, because it is the case when, in a forthcoming paper we will use our results for studying the moduli space of flat $G$-connections. Similar analysis can be done in other topological settings. We will not do it here.

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