Preparing two-atom entangled state in a cavity and probing it via quantum nondemolition measurement

D. Z. Rossatto and C. J. Villas-Boas
Departamento de Física, Universidade Federal de São Carlos, CEP 13565-905, São Carlos, SP, Brazil

We propose a probabilistic scheme to prepare a maximally entangled state between a pair of two-level atoms inside a leaking cavity, without requiring precise time-controlling of the system evolution and initial atomic state. We show that the steady state of this dissipative system is a mixture of two parts: either the atoms being in their ground state or in a maximally entangled one. Then, by applying a weak probe field on the cavity mode we are able to distinguish those states without disturbing the atomic system, i.e., performing a quantum nondemolition measurement via the cavity transmission. In this scheme, one has nonzero cavity transmission only when the atomic system is in an entangled state so that a single click in the detector is enough to ensure that the atoms are in an maximally entangled state. Our scheme relies on an interference effect as it happens in electromagnetically induced transparency phenomenon so that it works out even in the limit of decay rate of the cavity mode much stronger than the atom-field coupling.

The preparation and manipulation of entangled states have attracted much interest in last years, as they do not have a classical counterpart. These states are key ingredients for quantum nonlocality tests [11–14] and play an important role in achieving tasks of quantum computation and communication [2], such as quantum cryptography [3], computers [4] and teleportation [5]. Entangled states can be prepared either directly by coherent control of unitary dynamics [9], as consequence of measurements [7], or even using a dissipative process [8]. Recently, preparing quantum systems in an entangled state by dissipative schemes has been actively studied since the noise, which is always present in the experiments, can be used as a resource for entanglement generation, avoiding the usual destructive effect on the quantum system coherence owing to the system-environment interaction.

On the other hand, entanglement quantifiers, such as concurrence [9] and negativity [10], are not physical observables, i.e., there are no directly measurable observables, until now, to describe the entanglement of a given arbitrary quantum state. In general, it is necessary to perform the quantum state tomography to calculate these entanglement quantifiers, perturbing the state of the system, although some interesting methods have been recently proposed to construct direct observables related to entanglement [11–15]. Whereas the authors in Refs. [11–14] can determine the entanglement when few copies of the quantum system are available, in Ref. [15] the authors do this by introducing a probe atom that performs a single experimental run, \( \rho_{ss} \) shows us that the atomic system can be either in an uncorrelated state or in a maximally entangled one. In this way, if we are able to distinguish both states without disturbing the atomic system, then we will be able to prepare it in a maximally entangled state. In fact, we can do this by employing a weak probe field on the atom-cavity system. As we show bellow, when the atomic system is in an uncorrelated state the cavity transmission goes to zero, contrary to the maximum transmission which happens only when the atomic system is in the maximally entangled state. So, a single click on the detector works out as a witness of the entanglement generation of the atomic system. On the other hand, we also show that, if we have a unknown mixed state \( \rho \) (between the states \( |G⟩ \) and \( |D⟩ \)), the average transmission of the atom-cavity system is exactly equal to the concurrence of the state \( \rho \), thus providing a direct method to measure the degree of entanglement of the atomic system.

\( \text{Model:} \) Consider a pair of identical two-level atoms \( (|g⟩)_j = \text{ground state, } |e⟩_j = \text{excited state}) \) coupled resonantly with a cavity mode with coupling strength \( g \), modeled by Tavis-Cummings Hamiltonian \( (\hbar = 1) \)

\[ H = \omega a^\dagger a + \frac{\omega}{2} S_z + g (a S_+ + a^\dagger S_-), \]

where the cavity mode and the atomic transition are at frequency \( \omega \). The operators \( S_z = \sum_{j=1}^{2} \sigma_z^j \) and \( S_{\pm} = \sum_{j=1}^{2} \sigma_{\pm}^j \) are the collective spin operators [17] with \( \sigma_z^j = (\sigma_z^g \pm i \sigma_z^e) / 2 \) and \( \sigma_{\pm}^j = \sigma_x^j \pm \sigma_y^j \) being the Pauli operators for each atom; \( a (a^\dagger) \) is the annihilation (creation) operator of the cavity field. Assuming a leaking cavity at zero temperature, the dynamics of this system is governed by the master equation [18]

\[ \dot{\rho} = -i [H, \rho] + \kappa (2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a), \]
with $\kappa$ being the dissipation rate of the cavity mode. The proposed experimental setup is shown in Fig. 1(a).

![Experimental Setup Diagram]

**FIG. 1:** (color online). (a) Experimental setup. A pair of two-level atoms inside a leaking cavity. Once the system reaches the steady state the weak probe field is switched on and the cavity transmission is monitored. (b) Energy level diagram of the whole system considering the decay rates and the probe field.

The spectrum of the system, i.e., the allowed states of it, is given by the dressed states of $H$

\[
\begin{align*}
|G, 0\rangle &= |G\rangle \otimes |0\rangle_c, \\
|\pm\rangle &= \frac{1}{\sqrt{2}} (|B\rangle \otimes |0\rangle_c \pm |G\rangle \otimes |1\rangle_c), \\
|D, n\rangle &= |D\rangle \otimes |n\rangle_c,
\end{align*}
\]

with energies $-\omega$, $\pm\sqrt{2}g$, and $n\omega$, respectively, where

\[
\begin{align*}
|G\rangle &= |g\rangle_1 \otimes |g\rangle_2, \\
|B\rangle &= (|g\rangle_1 \otimes |e\rangle_2 + |e\rangle_1 \otimes |g\rangle_2) / \sqrt{2}, \\
|D\rangle &= (|g\rangle_1 \otimes |e\rangle_2 - |e\rangle_1 \otimes |g\rangle_2) / \sqrt{2},
\end{align*}
\]

and $|n\rangle_c$ is the cavity mode state in the Fock basis, with $n = 0, 1$. Here we are considering only the ground, first and second excited eigenstates of our system once we are interested in the steady state which is a mixture of those eigenstates. Besides, our scheme requires a weak probe field which also keeps the cavity field with up to one photon as we will explain bellow.

The damping of the cavity mode can promote transitions between the eigenstates of the system whose rates can be obtained through the Fermi golden rule [19]. As we are considering only the cavity decay, the transition rate from a higher energy state $|i\rangle$ to a lower one $|f\rangle$ is given by $\Gamma_{i\rightarrow f} = \kappa |\langle f | a | i \rangle|^2$. When we take into account the eigenstates of our system it is easy to see that we have two independent subspaces: \{|$G, 0\rangle \rightarrow |-\rangle$ or $|+\rangle$\} and \{|$D, n\rangle$\}, i.e., there is no transitions between states which belongs to distinct subspaces. Therefore, the nonzero transition rates are $\Gamma_{G,0\rightarrow G,0} = \kappa/2$ and $\Gamma_{D, n+1\rightarrow D, n} = \kappa$. In the Fig. 1(b) is depicted the energy level diagram of the whole system considering the decay rates and the probe field (frequency $\omega_p$) that will be introduced later.

Owing to the existence of two independent subspaces, for any general initial state, the steady state of the system is a mixture between the lowest energy eigenstates of each subspace, i.e.,

\[
\rho_{ss} = (1 - P) |G, 0\rangle \langle G, 0| + P |D, 0\rangle \langle D, 0|,
\]

with $P = Tr[\rho(0) |D\rangle \langle D|]$ being the projection of the initial state on the dark state $|D\rangle$. This result can be obtained directly from the Eq. 4 for $t \rightarrow \infty$ ($\beta = 0$). It is important to emphasize that we are not considering atomic damping as it destroys the entanglement in the steady state so that $\rho(t \rightarrow \infty) \rightarrow |G, 0\rangle \langle G, 0|$ for any initial state. As a real two level system always has a spontaneous decay $\gamma$, our results are valid in a time window defined by $\gamma t \lesssim 1$ and $g^2 t / \kappa \gtrsim 1$ so that we must have $g^2 / \kappa \gtrsim \gamma$ [20].

From the point of view of a single experimental run, we can see from the $\rho_{ss}$ that the system can be either in the atomic ground state $|G\rangle$ or in the entangled state $|D\rangle$ with probabilities $1 - P$ and $P$, respectively. So, there is a probability of having the atoms in a maximally entangled state. However, a direct measurement of the atoms would destroy such entangled state. To circumvent this problem, we must be able to nondestructively measure our atomic system. We can do that by probing our system with a weak probe field which allow us to distinguish the atomic states ($|G\rangle$ or $|D\rangle$) through the cavity transmission without disturbing the atomic system.

To nondestructively measure the system, firstly we must wait until the system reach its steady state. Then, we apply a weak probe field on the cavity, whose Hamiltonian is described by

\[
H_p = \varepsilon (a e^{i\omega_p t} + a^\dagger e^{-i\omega_p t}),
\]

with $\varepsilon \ll g$. Here, $\varepsilon$ and $\omega_p$ are the strength and the frequency of the probe field, respectively.

In order to understand how this probe field can provide us information about the atomic state, firstly we will consider the resonant case, i.e., $\omega_p = \omega$. If the system is in the $|D, 0\rangle$ state, we can see from the Fig. 1(b) that the probe field is able to promote the transition $|D, 0\rangle \leftrightarrow |D, 1\rangle$. However, as $|D\rangle$ is a dark state, it is decoupled from the cavity mode so that the system behaves as an empty cavity case ($g = 0$). In this case, the asymptotic cavity field state is a coherent field $|\alpha\rangle_c = e^{-|\alpha|^2/2} (|0\rangle_c + \alpha |1\rangle_c + ...)$, with $\alpha = -i\varepsilon/\kappa$. 
Then, for very weak probe field ($\varepsilon \ll \kappa$) the steady state of the atom-field system will be given by

$$|\psi\rangle_{ss}^{D} \approx |D\rangle \otimes \left[\left(1 - \frac{\varepsilon^2}{2 \kappa^2}\right)|0\rangle_c - \frac{\varepsilon}{\kappa}|1\rangle_c\right]. \quad (7)$$

On the other hand, if the system is in the $|G,0\rangle$ state, the weak probe field could \textit{a priori} induce two off-resonant transitions: $|G,0\rangle \leftrightarrow |-\rangle$ and $|G,0\rangle \leftrightarrow |+\rangle$, with detuning between the frequencies of the probe and atom-field system given by $\sqrt{2g}$. However, when $\omega_p = \omega$, the probe field does not introduce any photon into the cavity in the stationary regime, no matter the value of the atom field coupling $g$ (bellow we explain this point in more detail). Then, when the system is in the $|G,0\rangle$ state, the probe field is not able to introduce any excitation in the system so that the steady state of the system is

$$|\psi\rangle_{ss}^{G} \approx |G,0\rangle . \quad (8)$$

Therefore, the normalized transmission of the cavity, $T = \langle a^\dagger a \rangle / \langle |\varepsilon|/\kappa \rangle^2$, is useful to provide us information about the atomic steady state [4]. i.e., we have $T = 1$ when the atoms are in the maximally entangled state and $T = 0$ when the atoms are in a separable state. So, in the stationary regime, after applying a weak probe field on the system, the transmission works out as a nondestructive measurement of the atomic state, allowing us to know whether the system is in a maximally entangled state or not. Besides, our system does not require a high efficiency photon detector since a single click is enough to discriminate between the two atomic states present in the steady state [4]. If we are interested in preparing an entangled state, we can simply monitor the transmission in the time interval $\kappa/g^2 \ll t < 1/\gamma$: any click on the detector within this time window projects the system in the maximally entangled state. If no click is registered, then we must reset the system and start the experiment again.

The total transmission is expected to be maximum when the atoms are in the dark state $|D\rangle$ because in this state the atomic system is decoupled from the cavity mode so that the atom-field system behaves as an empty cavity case ($g = 0$). However, when the system is in the $|G,0\rangle$ state, the transmission is expected to be zero (or close to zero). The origin of this zero transmission could be in the detuning between the weak probe field and the atom-field system: the two transitions $|G,0\rangle \leftrightarrow |-\rangle$ and $|G,0\rangle \leftrightarrow |+\rangle$ are coupled by the probe field, but with detuning $-\sqrt{2g}$ and $\sqrt{2g}$, respectively. As both states $|\pm\rangle$ have decay rates $\Gamma_{\pm \rightarrow G,0} = \kappa/2$, one can see that, for $\sqrt{2g} \gg \kappa/2$, the probe field is very out of resonance with the atom-field system and then it is expected an absorption close to zero. If this is the case, one could argue that our system only works in the strong coupling regime. However, our scheme is also valid for weak atom-field coupling $g$, as the real reason why there is no transmission from the cavity is that our system has two absorption channels $|G,0\rangle \leftrightarrow |-\rangle$ and $|G,0\rangle \leftrightarrow |+\rangle$ which destructively interfere producing zero absorption in the resonant case $\omega_p = \omega$, analogously to the phenomenon of electromagnetically induced transparency [21]. In our case the probe field is reflected by the cavity mirror owing to this destructive interference [22]. The Fig. 2 shows the cavity transmission as a function of the detuning between the probe field and the cavity mode, $\Delta_p = \omega_p - \omega$, considering the atom-field coupling $g = 0.1\kappa$ and the strength of the probe field $\varepsilon = 0.1g$. Then, our scheme works out for any value of $g$. However, the smallest the $g$ the longest the time to the system reach the steady state since it is proportional to $\kappa/g^2$.

To simulate an experiment, we employed numerical simulations using the quantum jump approach (also called the quantum trajectories method) [23]. It is shown in Fig. 3 a single trajectory simulating a single run of an experiment for the case when $\rho_{ss} \rightarrow |D,0\rangle$ [Fig. 3(a)] and when $\rho_{ss} \rightarrow |G,0\rangle$ [Fig. 3(b)]. In these simulations we used $g = 0.5\kappa$, $\varepsilon = 0.05g$ and $\rho(0) = |\phi(0)\rangle \langle \phi(0)|$, with $|\phi(0)\rangle = |g\rangle_1 \otimes |e\rangle_2 \otimes |0\rangle_c$. Here, the quantification of the degree of entanglement is done through the Wootters’ concurrence $C$ [24]. As we can see in Fig. 3 when the atoms are in the maximally entangled (separable) state, the transmission of the probe field in the monitoring region is maximum (zero). This figure also helps us to see the evolution of a single trajectory of the system: in $t = 0$ we have the preparation of the initial state $\rho(0)$, followed by the stabilization of the system; then we switch on the probe field, which requires a second stabilization time; finally we have the monitoring region where the atomic state is nondestructively measured.

As we could see so far, to be able to generate the maximally entangled state it is required that the initial atomic

![FIG. 2: (color online). Cavity transmission versus detuning between the probe and the cavity field considering $g = 0.1\kappa$ and $\varepsilon = 0.1g$. We observe that, even for $\sqrt{2g} \gtrless \kappa/2$, the cavity transmission is close to zero for $\Delta_p = 0$ when $\rho_{ss} \rightarrow |G,0\rangle$ (solid line). The dashed line represents the cavity transmission when $\rho_{ss} \rightarrow |D,0\rangle$ (empty cavity-like).](image-url)
state $\rho(0)$ has a nonzero projection on the dark state $|D\rangle$. This can be done in different ways, for example: i) if we are able to address the atoms individually, then one can prepare the initial state $|g_1\rangle \otimes |e_2\rangle \otimes |0\rangle_c$; ii) if we are not able to address the atoms individually, then one can apply an incoherent field on both atoms simultaneously so that one can prepare a completely mixed state $\rho(0) = \frac{1}{2} |0\rangle_c \langle 0|$, with $1_a$ being the identity atomic matrix $(1_a = (|g\rangle \langle g| + |e\rangle \langle e|)\otimes (|g\rangle \langle g| + |e\rangle \langle e|))$. In the first case, the projection on the dark state is $P = 1/2$ while in the second one $P = 1/4$, which are the probabilities of preparing the atoms in a maximally entangled state.

Direct measurement of the concurrence: besides using our scheme as a source to produce maximally entangled states, our scheme can also be used as a direct method to measure the concurrence of the atoms. As we explained above, for any initial state, the steady state of the atom-field system is given by the Eq. 5, which is a mixture between a completely separable state and a maximally entangled one. By applying a weak probe field, the steady state turns out to be

$$\rho_{ss} \rightarrow \widetilde{\rho}_{ss} \approx (1 - P) |\psi\rangle^G_{ss} \langle \psi| + P |\psi\rangle^D_{ss} \langle \psi|.$$  \hfill (9)

with $|\psi\rangle^G_{ss}$ and $|\psi\rangle^D_{ss}$ belonging to distinct subspaces. For this state, the average transmission is $T(\widetilde{\rho}_{ss}) = P$. However, the concurrence of the atomic state is also $C[Tr_c(\widetilde{\rho}_{ss})] = P$, where $Tr_c$ means the trace over the cavity mode variables. Therefore, we see that the transmission of the atom-field system $T(\widetilde{\rho}_{ss})$ is exactly the degree of entanglement (concurrence) between the two atoms. In this way, our scheme works out as a direct method to measure of the concurrence of the atomic steady state, without requiring any tomographic reconstruction of the atomic density matrix.

In conclusion, we have shown a probabilistic scheme to prepare a maximally entangled state between a pair of two-level atoms inside a leaking cavity, and how to probe this atomic steady state, without perturbing it, via the cavity transmission. From the point of view of a single run of the experiment, we have seen that if the atomic system is in an entangled state, then there will be a nonzero cavity transmission. On the other hand, if the system is in an uncorrelated state, the cavity transmission goes to zero. In this way, a single click in the detector is enough to ensure that the atoms are in an maximally entangled state. We have also seen that our scheme works out as a direct method to measure of the concurrence of the atomic steady state, without requiring any tomographic reconstruction of the atomic density matrix.

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[24] $C = \max \{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ with $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4$ the square roots of the eigenvalues, in decreasing order, of the matrix $R = \rho (\sigma^z_1 \otimes \sigma^z_2) \rho^* (\sigma^z_1 \otimes \sigma^z_2)$. Here $\rho^*$ denotes the complex conjugation of the matrix $\rho$ in the basis \{\ket{gg}, \ket{ge}, \ket{eg}, \ket{ee}\}. 