Topological and metric recurrence
for general Markov chains

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The paper is dedicated to the memory of Robert Adolfovich Minlos.

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Abstract

Using ideas borrowed from topological dynamics and ergodic theory we introduce topological and metric versions of the recurrence property for general Markov chains. The main question of interest here is how large is the set of recurrent points. We show that under some mild technical assumptions the set of non recurrent points is of zero reference measure. Necessary and sufficient conditions for a reference measure $m$ (which needs not to be dynamically invariant) to satisfy this property are obtained. These results are new even in the purely deterministic setting.

1 Introduction

The aim of this work is to analyze a circle of questions related to the notion of recurrence in general Markov chains. Being well known in two very different subfields of random systems: lattice random walks and ergodic theory of continuous selfmaps, the recurrence property is next to being neglected in general theory of Markov chains (perhaps except for a few notable exceptions which we will discuss in detail).

There are two known approaches to the definition of a recurrent point: topological and metric (defined in very different contexts). In order to introduce their analogies for general Markov chains let us recall basic definitions.

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**Definition 1** By an inhomogeneous Markov chain one means a random process $\xi_t : (\Omega, \mathcal{F}, P) \rightarrow (X, \mathcal{B}, m)$ acting on a Borel $(X, \mathcal{B})$ space with a finite reference measure $m$ (which needs not to coincide with the distribution of the process $\xi_t$). This process is completely defined by a family of transition probabilities $Q_t^s(x, A) := P(\xi_{s+t} \in A | \xi_s = x), \ A \in \mathcal{B}$.

If the chain is homogeneous, i.e. the transition probabilities do not depend on $s$ we drop the lower index and write $Q_t^s(x, A) \equiv Q_t(x, A)$.

**Definition 2** By the $t$-preimage with $t \geq 0$ of a set $B \in \mathcal{B}$ under the action of the homogeneous Markov chain $\xi_t$ we call the set of points $Q^{-t}(B) := \{x \in X : Q_t(x, B) > 0\}$.

In other words this is the set of initial points of trajectories which reach the set $B$ at time $t$ with positive probability.

Now we are ready to return to the notion of recurrence. Observe that since the phase space is equipped with the Borel $\sigma$-algebra, it is equipped with the corresponding topology as well. We start from the notion of the topological recurrence well known in the field of topological dynamics (see e.g. [3]).

**Definition 3** A point $x \in X$ is called topologically recurrent if for any open neighborhood $U \ni x$ for each $s$ there exists an (arbitrary large) $t = t(x, U, s)$ such that $Q_t^s(x, U) > 0$ (i.e. a trajectory eventually returns to $U$ with positive probability).

**Definition 4** A point $x \in X$ is called metrically recurrent if for any set $V \ni x$ of positive $m$-measure and any $s$ there exists an (arbitrary large) $t = t(x, V, s)$ such that $Q_t^s(x, V) > 0$ (i.e. a trajectory eventually returns to $V$ with positive probability).

The last definition is our modification of the metric version of the recurrence property proposed by T.E. Harris in [5] in order to get reasonably general assumptions guaranteeing the existence of an invariant measure. In fact, Harris used this property only in the case when the reference measure $m$ is invariant with respect to the process. Another weak point of Harris’ approach is that whence a point $x$ is metrically recurrent, the corresponding trajectory (realization of the process) needs to visit any set of positive measure with positive probability, which looks way too excessive.

The 3d approach to the recurrence notion is related to ergodic theory of deterministic dynamical systems, where it is well known and studied, but again only in the case when the measure $m$ is dynamically invariant. The probabilistic version may be formulated as follows.
**Definition 5** A point $x \in X$ is called *Poincare recurrent* with respect to a $\mathcal{B}$-measurable set $A \ni x$ if for each $s$ there exists an (arbitrary large) $t = t(x, A, s)$ such that $Q^t_s(x, A) > 0$ (i.e. a trajectory eventually returns to $A$ with positive probability).

Comparing these definitions with their deterministic counterparts (see e.g. [8, 2]) or to the notions of recurrence and transience well studied for the case of countable Markov chains (see e.g. [3, 7, 9, 10]) one is tempted to make the conditions stronger assuming that the corresponding events take place with probability one (instead of just being positive). Unfortunately, as we will show in section 6 the results we are looking for do not hold under these stronger assumptions. To some extent this may explain why these properties were not studied earlier.

The main question of interest for us is to find how large is the set of recurrent points?

The celebrated Poincare recurrence theorem (see e.g. [2]) states that for a measurable discrete time dynamical system $(T, X)$ and any measurable set $A$ the set of non Poincare recurrent points is of zero measure with respect to any $T$-invariant measure. There is a number of obstacles which one needs to overcome to generalize this result for a general Markov chain: continuous time, inhomogeneity, and more important a principal absence of invariant measures (stationary distributions) for inhomogeneous Markov chains. Here we formulate our main result in this direction only for the discrete time case ($t \in \mathbb{Z}$) leaving the more complicated continuous time version for Section 6.

**Theorem 1** Let $m$ be a finite measure on $(X, \mathcal{B})$ and let $Q^t_s$ does not depend on $s$. Then the property that for each set $A \in \mathcal{B}$ the set of Poincare recurrent points in $A$ is of full $m$-measure (i.e. its complement in $A$ is of zero $m$-measure) is equivalent to:

$$\sum_{n \geq 1} m(Q^{-n}(A) \cap A) = \infty \quad \forall A \in \mathcal{B} : m(A) > 0. \quad (1)$$

An important observation here is that if $Q^t_s$ does depend on $s$, then the direct statement in Theorem 1 (that (1) implies the abundance of Poincare recurrent points) remains correct, but the inverse one may fail (see Section 5).

The situation with other types of recurrence is much more subtle, in particular a very “good” Markov chain, for which all points are topologically recurrent with respect to the “standard” topology, may change drastically

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1Note that non translation-invariant infinite volume Gibbs measures might be still present in this setting.
when one chooses a different topology instead. Under this new topology all points might become non-recurrent, e.g. irrational circle rotation with the discrete topology.

The following result gives conditions for the topological recurrence of “typical” points. Here and in the sequel we assume that the measurable space \((X, \mathcal{B})\) is equipped with a topology \(\mathcal{T}\), which is compatible with the \(\sigma\)-algebra \(\mathcal{B}\).

**Theorem 2** Let \((X, \mathcal{T})\) be a compact metric space and the finite reference measure \(m\) satisfy the property \((\mathfrak{1})\). Then \(m\)-almost every point \(x \in X\) is topologically and metrically recurrent.

It is natural to ask what can we say about recurrent points if the property \((\mathfrak{1})\) does not hold. As we already noted if there is a dynamical invariant measure \(\mu\) then one can easily construct from it a reference measure \(m\) satisfying \((\mathfrak{1})\). For example, any finite measure absolutely continuous with respect to \(\mu\). Therefore to answer to this question one needs to consider systems without invariant measures. This will be done in section 6 where among other results we will demonstrate that a measurable Markov chain with a compact phase space may have no recurrent points.

It is worth noting that the results formulated in Theorems 1 and 2 are new even in the purely deterministic setting.

The paper is organized as follows. In the next section we review a few definitions related to Markov chains which are necessary for the further analysis. The sections 3 and 4 are dedicated to the proof of Theorems 1 and 2 respectively. Then in section 5 we study connections between the condition \((\mathfrak{1})\) and other “conservativity” type properties. One of interesting moments here is a striking difference between forward and backward in time statistics. Finally in section 6 we discuss some generalizatins. Namely we show that a stronger version of the recurrence property plays a very different role than the one defined above; discuss recurrence in the absence of invariant measures; give continuous time versions of our main results; and finally analyze their connection to yet to be proven stochastic version of the so called multiple recurrence problem.

## 2 Preliminaries

The transition probabilities \(Q_t^x(\cdot, \cdot)\) satisfy the following standard properties:

- For fixed \(s, t, x\) the function \(Q_t^x(x, \cdot)\) is a probability measure on the \(\sigma\)-algebra \(\mathcal{B}\).
• For fixed $s, t, A$ the function $Q_t^s(\cdot, A)$ is $\mathcal{B}$-measurable.

• For $t = 0$ \( Q_t^s(x, A) = \delta_x(A). \)

• For each $s, 0 \leq t \leq t'$ and $A \in \mathcal{B}$ we have

\[
Q_{t'}^s(x, A) = \int_X Q_{t}^s(x, dy) Q_{t'}^{-t}(y, A).
\]

The process $\xi_t$ induces the action on measures:

\[
Q_t^s \mu(A) := \int Q_t^s(x, A) \, d\mu(x)
\]

and the action on functions:

\[
Q_t^s \phi(x) := \int \phi(y) Q_t^s(x, dy).
\]

In particular, the well known Feller property in terms of the action on functions means that $Q_t^s : C^0 \to C^0 \ \forall s, t \geq 0$.

**Definition 6** A Borel measure $\mu$ is said to be invarient or stationary for the Markov chain $\xi_t$ if it is a solution to the equation

\[
Q_t^s \mu = \mu \ \forall s, t.
\]

In the discrete time setting $t \in \mathbb{Z}$ and we write $Q(\cdot, \cdot) \equiv Q^1(\cdot, \cdot)$.

**Definition 7** We say that a discrete time homogeneous Markov chain on a measurable space $(X, \mathcal{B})$ defined by the transition probabilities $Q(\cdot, \cdot)$ is deterministic if there exists a measurable self-map $T : X \to X$ such that

\[
Q(x, A) := \begin{cases} 
1 & \text{if } Tx \in A \\
0 & \text{otherwise.}
\end{cases}
\]

The deterministic nature of a Markov chain may be characterized also in terms of the action on measures, namely that the image of any Dirac measure is again a Dirac measure, i.e.

\[
\forall x \in X \ \ \exists y \in X : Q \delta_x := \delta_y.
\]

By the properties of the transition probabilities, if $B \in \mathcal{B}$, then the $t$-preimage $Q^{-t}(B)$ is measurable as well $\forall t \geq 0$. In the deterministic case the map $T$ acts on measures as follows:

\[
T \mu(A) := \mu(T^{-1}A).
\]
Therefore one might expect that a similar property holds true for a general Markov chain, i.e.

$$Q^t m(B) = m(Q^{-t}(B)).$$

The following trivial example demonstrates that this is absolutely not the case and it emphasizes that some care is necessary when one is trying to apply arguments well known in the deterministic dynamical systems theory for a general Markov chain.

**Example 1** Let $\xi_n$ be a discrete time 2-state Markov chain with the transition probability matrix $(q_{ij})$ with $0 < q_{11} < q_{22} < 1/2$, and let $A$ stand for the 1st state.

Then $Q^{-1}(A) = X$, and for a given nontrivial distribution $m$ on $X$ we have

$$m(Q^{-1}(A)) = m(X) \neq Qm(A) = q_{11}m(A) + q_{21}m(X \setminus A).$$

**Definition 8** We say that a Markov chain $\xi_t$ satisfies the Poincare recurrence property with respect to a measure $\mu$ and use the notation $\text{PRP}(\mu)$ for this, if the set of Poincare recurrent points in each set $A \in B$ is of full $m$-measure.

In these terms Theorem 1 claims that the Poincare recurrence property is equivalent to the property (1).

3 **Proof of Theorem 1**

First we prove that the property (1) implies that the set of Poincare recurrent points is of full $m$-measure. For a given set $A \in B$ with $m(A) > 0$ denote by $\hat{A}$ the set of non Poincare recurrent points in it, i.e.

$$\hat{A} := \{ x \in A : Q^t(x, A) = 0 \ \forall t > 0 \}.$$

Assume from the contrary that $m(\hat{A}) > 0$.

In the deterministic setting when the Markov chain is defined by a measurable self-map $T : X \to X$ one may use the assumption

$$\sum_{n \geq 1} m(T^{-n}A) = \infty,$$

which is even weaker than (1), and argue that for any pair of different moments of time $t \neq s$ the corresponding preimages of the set $\hat{A}$ are disjoint. Indeed, for

$$x \in Q^{-t}(\hat{A}) \equiv T^{-t}(\hat{A})$$
the forward “orbit” of $x$ visits the set $A \supseteq \hat{A}$ last time at the moment $t$, i.e. $Q^s(x, A) = 0$ if $t > s$.

Therefore

$$m(X) \geq m(\bigcup_{n \geq 1} Q^{-n}((\hat{A})) = \sum_{n \geq 1} m(Q^{-n}(\hat{A})).$$

It remains to notice that the measure $m$ is finite, but the right hand side is equal to infinity by the assumption (1). We came to the contradiction.

Unfortunately for a general Markov chain the preimages of the set $\hat{A}$ need not to be disjoint. To demonstrate this consider an example.

**Example 2** Let $\xi_n$ be a discrete time 3-state Markov chain with the transition probability matrix

$$
\begin{pmatrix}
0 & 0 & 1 \\
1/2 & 1/2 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

and let the reference measure $m$ be uniform on the phase space.

The set $A := \{1\}$, consisting of the 1st state of this Markov chain, is obviously non-recurrent, since this state returns back with probability zero. On the other hand, all its $t$-preimages for $t \geq 1$ coincide with the 2nd state, hence they are not disjoint. Additionally,

$$\sum_{n \geq 1} m(Q^{-n}(A)) = \sum_{n \geq 1} m(Q^{-n}(\{2\})) = \infty,$$

since the $m$-measure of the 2nd state is equal to $1/3$.

Therefore to study a general Markov chain we need to use a slightly more complex assumption (1) to deal with the analysis of the set of non Poincare recurrent points $\hat{A}$, for which

$$m(Q^{-n}(\hat{A}) \cap \hat{A}) \leq m(Q^{-n}(\hat{A}) \cap A) = 0.$$ 

Thus the property (1) fails and we came to the contradiction, which implies that the set of non Poincare recurrent points $\hat{A}$ should have zero $m$-measure.

To prove the claim in the inverse direction we need to demonstrate that the Poincare recurrence property implies the divergence of the sum in (1). Denote the partial sum

$$S(N) := \sum_{n=1}^{N} m(Q^{-n}(A) \cap A).$$
and assume from the contrary that

\[ S := \limsup_{N \to \infty} S(N) < \infty. \]

Then for each \( \varepsilon > 0 \) there exists \( N_\varepsilon < \infty \) such that

\[ 0 \leq S - S(N_\varepsilon) < \varepsilon. \tag{2} \]

On the other hand, by PRP\((m)\) for each \( N < \infty \)

\[ m(x \in A : Q^n(x, A) > 0, \ n \geq N) > m(A)/2, \]

which contradicts to (3) since all addends in \( S(N) \) are nonnegative.

Theorem 1 is proven. \( \square \)

## 4 Proof of Theorem 2

In this Section we again consider only homogeneous Markov chains.

In fact we will prove Theorem 2 in a bit more general setting, namely instead of compactness of the set \( X \) we assume only that the topological space \((X, \mathcal{T})\) has a countable base \( \{\beta_i\}_{i \in \mathbb{Z}_+} \) and that \( \mu \) is a finite \( \sigma \)-additive measure on the measurable space \((X, \mathcal{B})\), where the topology \( \mathcal{T} \) is compatible with the \( \sigma \)-algebra \( \mathcal{B} \). This idea was proposed originally by my student A. Zhevnerchuk in the analysis of the purely deterministic dynamics with an invariant measure \( m \). Here we use it for a general homogeneous Markov dynamics and a reference measure \( m \) which needs not to be dynamically invariant.

**Lemma 1** Consider a family of binary valued measurable functionals \( \phi_B : X \to \{0, 1\} \) indexed by measurable sets \( B \in \mathcal{B} \), such that

\[ \{x \in X : \phi_B(x) = 1\} \subseteq B \ \forall B \in \mathcal{B}. \]

Then \( \forall B \in \mathcal{T} \)

\[ \mu(x \in X : \phi_B(x) = 0) \leq \sum_i \mu(x \in X : \phi_{\beta_i}(x) = 0). \]

\[ \text{See also a close approach in } \text{http://planetmath.org/proofofpoincarerecurrencetheorem2}. \]
Proof. For each open set $\beta$ from the base of topology introduce the set
\[ \alpha_i := \{ x \in X : \phi_\beta(x) = 0 \}. \]
Then due to the definition of the topological base and the $\sigma$-additivity of the measure $m$ we immediately have
\[ \mu(x \in X : \phi_B(x) = 0) \leq \sum_i \mu(\alpha_i). \]
\hfill \Box

Using this result we will prove Theorem 2 as follows. Let $x \in X$ and let $B$ be its any open neighborhood. Since $B$ is a union of some elements from the countable base $\{\beta_i\}$ there exists an element $\beta_i(x)$ such that $x \in \beta_i(x) \subseteq B$.

Denote the functionals $\phi_A$ to be equal to 1 if $Q^n(x, A) > 0$ for some $n \in \mathbb{Z}_+$ and to 0 otherwise. Then the sets $\alpha_i$ defined in the proof of Lemma 1 are the subsets of the base sets $\beta_i$ whose points never return to $\beta_i$ under dynamics. Thus $m(\alpha_i) = 0 \ \forall i$ by the assumption on the PRP($m$). By Lemma 1 we get
\[ \mu(x \in X : \phi_B(x) = 0) \leq \sum_i \mu(x \in X : \phi_{\beta_i}(x) = 0) = 0. \]
Therefore the measure of topologically non-recurrent points in $X$ is zero.

To prove that the measure of metrically non-recurrent points is also zero one applies the argument above to a set of positive $m$-measure instead of the open neighborhood.

Theorem 2 is proven. \hfill \Box

5 Discussion

It is easy to see that whence the reference measure $m$ is dynamically invariant or is absolutely continuous with respect to an invariant measure, then the assumption (I) holds true automatically. The aim of this section is to study the connections of a somewhat unusual property (I) with other assumptions of similar type, in particular with the assumptions related to the conservativity property in dynamical systems theory.

5.1 Time inversion

With respect to a given topology $\mathcal{T}$ the definition of the $t$-preimage (with $t > 0$) of a set $B$ may be reformulated as follows
\[ Q^{-t}(B) := \{ y \in X : Q^t(y, U(B)) > 0 \ \forall U(B) \in \mathcal{T}, \ B \subseteq U(B) \}. \]
Similarly one defines a $t$-image having in mind the action on sets in the “positive” time direction.

**Definition 9** By the $t$-image with $t \geq 0$ of a measurable set $A \in \mathcal{B}$ we mean

$$Q^t(A) := \cup_{x \in A}\{ y \in X : Q^t(x, U(y)) > 0 \ \forall U(y) \in \mathcal{T}, \ y \in U(y) \}. \quad (4)$$

As we already noted in Section I in the inhomogeneous case (i.e. if $Q^t_s$ does depend on $s$) the direct statement in Theorem I remains correct, but the inverse one fails. Namely, it is possible that the sum in (1) is infinite for all $s$, but

$$\limsup_{t \to \infty} P(\xi_{s+t} \in A | \xi_s \in A) = 0,$$

for some $s$, i.e. there is no recurrence. The reason for this is that despite the “chain of (pre)images” of the set $A$ inevitably intersects itself an infinite number of times, the original set needs not to be included to the intersections (see Fig. 1).

![Diagram](image)

**Figure 1:** Intersection of images

A counterexample may be constructed as a finite state discrete time Markov chain with 4 states, whose graph of transitions is given by the following diagram:

$$x_3 \leftarrow x_3 \leftarrow x_0 \leftarrow x_1 \leftrightarrow x_2. \quad (5)$$

To be more precise, we consider a deterministic version of the process.
Example 3 Let $X := \{x_i\}_{i=0}^3$, $x_i \neq x_j$ for $i \neq j$, and we have a family of one-to-one maps $T_s : X \to X$ indexed by $s \in \mathbb{Z}$. Denote the only image of a state $x \in X$ at time $s$ by $T_s^1 x$ and its unique pre-image by $T_s^{-1} x$. Then we set

$$A := x_0 = T_1^{-1} x_1, \quad T_2^{-1} x_1 = x_2, \quad T_3^{-1} x_2 = x_1, \quad T_1^1 x_0 = x_3, \quad T_i^1 x_3 = x_3 \forall i.$$  

Clearly the graph of transitions defined by this construction satisfies the diagram (5). This shows that for the measure $m$ uniformly distributed on $X$ and the state $x_0$ the relation (6) holds true, but this state is non-recurrent.

5.2 Conservativity

Let us discuss the connection of a somewhat unusual property (1) with more "classical" notions related to the conservativity of a system.

First of all using the forward image of a set consider a version of (1) in the opposite time direction:

$$\sum_{n \geq 0} m(Q^n(A) \cap A) = \infty \quad \forall A \in \mathcal{B} : m(A) > 0$$  

and a very similar property formulated in terms of the action on measures rather than on sets:

$$\sum_{n \geq 1} Q^n m(A) = \infty \quad \forall A \in \mathcal{B} : m(A) > 0.$$  

Consider also two more properties often associated with the (forward/backward) conservativity in dynamical systems theory:

$$Q^1(A) \subseteq A \in \mathcal{B} \implies m(A \setminus Q^1(A)) = 0,$$  

$$Q^{-1}(A) \subseteq A \in \mathcal{B} \implies m(A \setminus Q^{-1}(A)) = 0.$$  

In the first of these properties we either assume that $Q^1(B) \subseteq \mathcal{B}$ (i.e. 1-images of all measurable sets are measurable), or have to consider the inner measure $m_*$ instead of $m$ in the right hand side of the implication.

Theorem 3 (a) (1) is equivalent to (8), (b) (6) does not imply $PTP(m)$, (b') $PTP(m)$ does not imply (7), (c) (9) implies (8).

Proof. We start with the direct statement in (a). Let $Q^1(A) \subseteq A \in \mathcal{B}$ and $m(A) > 0$ (otherwise the statement is obvious). Assume that $m(A) > m(Q^1(A))$. Then $m(B := A \setminus Q^1(A)) > 0$. Therefore

$$\sum_{n \geq 0} m(Q^{-n} B) = \infty.$$  

11
by (1). On the other hand, by the construction
\[ Q^k(B) \cap B = \emptyset \quad \forall k \in \mathbb{Z}_+. \]
Hence \( Q^{-k}(B) \cap Q^{-n}(B) = \emptyset \) if \( k \neq n \). Thus
\[ m(X) \geq m(\cap_{n \geq 0} Q^{-n}(B)) = \sum_{n \geq 0} m(Q^{-n}(B)) = \infty \]
by (1). We came to the contradiction.

It remains to consider the case \( Q^1(A) \setminus A \neq \emptyset \). In this case we set \( B := A \) and repeat the previous argument.

To prove (a) in the inverse direction consider the case \( m(A) > 0, Q^1(A) \subseteq A \) and \( m(A \setminus Q^1(A)) = 0 \). Observe that \( Q^1(A) \subseteq A \) implies that \( Q^{-1}(A) \supseteq A \) and hence \( Q^{-n}(A) \supseteq A \ \forall n \in \mathbb{Z}_+ \). Therefore
\[ \sum_{n \geq 0} m(Q^{-n}(A)) \geq \sum_{n \geq 0} m(A) = \infty, \]
which proves validity of (1).

To demonstrate (b) consider the following example.

**Example 4** Let a Markov chain be defined on a compact space \( X \) equipped with a finite reference measure \( m \) and divided into two disjoint parts \( X_1 \) and \( X_2 \) such that \( m(X_i) > 0 \ \forall i \). The one-step transition probabilities are defined by the relation:
\[ Q(x, B) := m(B \cap X_2)/m(X_2). \]

Then for each set \( A \subseteq X_1 \) with \( m(A) > 0 \) the property (6) holds true, while no point from the set \( A \) can return back by dynamics and hence \( \text{PTP}(m) \) is violated.

The claim (b’) will be proven in Section 6 during the analysis of Example 5 (see Lemma 4).

(c) The property (9) implies that for each backward invariant set the restriction of the \( m \)-measure to the set \( A \) is preserved under dynamics. Therefore for such sets \( \text{PTP}(m) \) is obviously satisfied. On the other hand, from the proof of Theorem 1 it follows that the assumptions (1) and \( m(A) > 0 \) taken together imply the existence of a backward invariant set of positive \( m \)-measure. \( \Box \)

Observe that neither of the properties \( Q^1(A) \subseteq A \) and \( Q^{-1}(A) \subseteq A \) imply another one.

12
6 Generalizations

6.1 Strong recurrence

As we already noted in the Introduction, comparing our main results with their deterministic counterparts or with results on recurrence and transience well studied for the case of countable Markov chains (see e.g. [3, 7, 9, 10]) one is tempted to make considerably stronger statements assuming that the corresponding events take place with probability one (instead of just being positive). To be precise let us give a definition.

**Definition 10** A point \( x \in X \) is called strongly Poincare recurrent with respect to a \( \mathcal{B} \)-measurable set \( A \ni x \) if there exists an (arbitrary large) \( t = t(x, A) \) such that \( Q^t(x, A) = 1 \).

Consider an example.

**Example 5** Let a Markov chain be defined on a compact space \( X \) equipped with a finite reference measure \( m \) and divided into two disjoint parts \( X_1 \) and \( X_2 \) such that \( m(X_i) > 0 \ \forall i \). The one-step transition probabilities are defined by the relation:

\[
Q(x, B) := \begin{cases} 
\frac{m(B \cap X_1)}{2m(X_1)} + \frac{m(B \cap X_2)}{2m(X_2)} & \text{if } x \in X_1 \\
\frac{m(B \cap X_2)}{m(X_2)} & \text{if } x \in X_2.
\end{cases}
\]

The idea under this example is that the action of \( Q \) on measures splits them into two parts: one part remains confined in \( X_1 \), while another part is moved to \( X_2 \) and never returns back. Observe that without some special assumptions of mixing type this situation is more or less generic.

**Lemma 2** Let a measurable set \( A \) belongs to \( X_i \) for some \( i \in \{1, 2\} \), then each point \( x \in A \) is Poincare recurrent, but for any set \( B \subseteq X_1 \) there are no strongly Poincare recurrent points.

**Proof.** The first part of the claim follows from the fact that if \( A \subseteq X_1 \), then

\[
Q(x, A) = \frac{m(A)}{2m(X_1)} > 0,
\]

while if \( A \subseteq X_2 \), then

\[
Q(x, A) = \frac{m(A)}{m(X_2)} > 0.
\]

The second part is a consequence of the fact that no point from the set \( X_1 \) can return back under dynamics with probability one. \( \square \)
Corollary 3  In the example PRP(m) holds true, but the similar statement for the strong recurrence fails.

Lemma 4  For a given measurable set \( A \subset X_1 \) with \( m(A) > 0 \)
\[
\sum_{n \geq 1} Q^n m(A) < \infty.
\]

The proof follows from the direct computation. This result finishes the proof of part (b') of Theorem 3 since it demonstrates that PRP(m) does not imply (7).

Stronger versions of other types of recurrence may be formulated similarly to the Poincare recurrence.

Theorem 2 claims that under very mild assumptions the set of (weakly) recurrent points is of full reference measure. On the other hand, for the Markov chain in Example 5 the \( m \)-measure of the set of strongly recurrent points (both topological and metric) is equal to \( m(X_2) < m(X) \).

6.2 Recurrence in the absence of invariant measures

Let us answer the question, what happens to recurrence when the corresponding Markov chain has no invariant measures. The simplest example of this type has a deterministic nature and we assume the standard Borel \( \sigma \)-algebra and the corresponding topology.

Example 6  Let \( X := [0, 1] \) and
\[
T_x := \begin{cases} x^2 & \text{if } x > 0 \\ 1 & \text{otherwise.} \end{cases}
\]

It is easy to see that the dynamical system defined by the Example 6 has no invariant measures. We refer the reader to [1], where more interesting examples of systems without invariant measures are studied from the point of view of Birkhoff type averaging.

Lemma 5  Let \( m \) be the Lebesgue measure on \( X := [0, 1] \). Then for each set \( A \) containing a neighborhood of the origin all points in \( A \) are Poincare recurrent. The set of (both topological and metrical) recurrent points coincides with the only point at the origin.

In this example despite the absence of invariant measures the recurrent point is still present. To demonstrate that the situation may be even worse we use the map from Example 6 as a building block in the following construction.
Example 7 Let $X := [0, 1]$ and

$$Tx := \begin{cases} 
\frac{4}{5} & \text{if } x = 0 \\
x^2 & \text{if } 0 < x < 1/2 \\
1 - T(1 - x) & \text{otherwise}.
\end{cases} \quad (11)$$

The graph of this map is depicted on Fig. 2.

![Figure 2: Map with no recurrent points](image)

Lemma 6 The map $T$ has no invariant measures and no topological or metrical recurrent points.

Proof. The 1st claim is an easy consequence of the fact that all points in $X$ converge under dynamics either to 0 or to 1. Thus the only candidates for the role of recurrent points are exactly these two points, however under dynamics each of them is mapped to the basin of attraction of the different point. \qed

6.3 Continuous time

Consider now a continuous time version of Theorem 1.

Theorem 4 Let $m$ be a finite measure on $(X, \mathcal{B})$ and let $Q^t_s$ does not depend on $s$. Then $PRP(m)$ is equivalent to the existence of a constant $\gamma > 0$ such that

$$\sum_{n \geq 1} m(Q^{-n\gamma}(A) \cap A) = \infty \quad \forall A \in \mathcal{B} : m(A) > 0. \quad (12)$$
The proof of this result follows from the time discretization with time step $\gamma$ of the continuous Markov chain under study. This gives a discrete time Markov chain $\tilde{\xi}_n$ with one-step transition probabilities $\tilde{Q}(\cdot, \cdot) := Q(\cdot, \cdot)$. By Theorem 1 the property PRP($m$) is equivalent to

$$\sum_{n \geq 1} m(\tilde{Q}^{-n}(A) \cap A) = \infty \quad \forall A \in \mathcal{B} : m(A) > 0,$$

which immediately implies the claim of Theorem 4.

In Theorem 4 we assume the lattice structure of the set of moments of time $\{n\gamma\}_{n \in \mathbb{Z}}$ at which we observe our Markov chain. Can we use instead an arbitrary sequence of moments of time $t_1 < t_2 < \ldots < t_n < \ldots$ growing to infinity? Of course, one can make the time discretization with respect to these moments, but the corresponding discrete time Markov chain will no longer be homogeneous.

It is worth noting that since the proof of Theorem 2 does not use the discrete structure of time, it remains valid in the continuous time setting as well.

### 6.4 Multiple recurrence

Furstenberg Theorem asserts that if $T$ is a measure preserving bijection of a measure space $(X, \mathcal{B}, \mu)$ with $\mu(X) < \infty$ and if $A \in \mathcal{B}$ with $\mu(A) > 0$ and $k \geq 2$ is any integer, then there exists $n$ with

$$\mu(A \cap T^n A \cap T^{2n} A \ldots T^{(k-1)n} A) > 0.$$  

For $k = 2$, this is the Poincare recurrence property, and follows readily from the fact that $T$ is measure preserving and $\mu(X) < \infty$. For $k > 2$ the proof of this result is quite nontrivial and despite of a large number of publications (see e.g. [6] and further references therein) claiming that this result holds true for a general measure preserving dynamical system (rather than a bijection as in the original Furstenberg publication), we were able to find a complete proof only for the invertible case. No results in this direction are known for general Markov processes.

Nevertheless we expect that a probabilistic version of this result should hold true under the assumptions of Theorem 1, namely that for any measurable set $A \in \mathcal{B}$ with $m(A) > 0$ and any integer $k \geq 2$ there exists a positive integer $n$ such that

$$m(\cap_{j=0}^{k-1} Q^{-jn}(A)) > 0.$$
Since the methods used to prove the original result due to Furstenberg (which were further developed by his numerous followers) are not applicable in the context of general Markov chains (in particular, due to the non-invertibility of the dynamics) a different approach needs to be developed. It is worth noting that the original proof is quite involved and complicated.

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