UPGRADING SUBORDINATION PROPERTIES IN FREE PROBABILITY

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Abstract. The existence of Voiculescu’s subordination functions in the context of non-tracial operator-valued $C^*$-probability spaces has been established using analytic function theory methods. We use a matrix construction to show that the subordination functions thus obtained also satisfy an appropriately modified form of subordination for conditional expectations.

1. Introduction

Suppose that $(A, \tau)$ is a tracial $W^*$-probability space. That is, $A$ is a von Neumann algebra, and $\tau : A \to \mathbb{C}$ is a faithful normal trace mapping the unit of $A$ to the complex number one. Let $x_1, x_2 \in A$ be two freely independent, selfadjoint elements. Voiculescu [7] showed that, at least generically, there exists an analytic selfmap $\omega$ of the complex upper half-plane $\mathbb{H}$ satisfying the identity

$$\tau \left( (\lambda - (x_1 + x_2))^{-1} \right) = \tau \left( (\omega(\lambda) - x_1)^{-1} \right), \quad \lambda \in \mathbb{H}. \quad (1.1)$$

The genericity hypothesis was subsequently removed by Biane [4], who showed that $\omega$ satisfies a stronger condition than (1.1). In order to state this condition, we denote by $Y$ the von Neumann algebra generated by $x_1$, that is, $Y = \{x_1\}''$, and we let $E_Y : A \to Y$ be the trace-preserving conditional expectation. Then

$$E_Y[(\lambda - (x_1 + x_2))^{-1}] = (\omega(\lambda) - x_1)^{-1}, \quad \lambda \in \mathbb{H}, \quad (1.2)$$

and this, of course, implies (1.1) upon applying $\tau$. Since $E_Y$ is just the orthogonal projection in the scalar product induced by $\tau$, (1.2) is equivalent to the following relation that involves just $\tau$:

$$\tau \left( y(\lambda - (x_1 + x_2))^{-1} \right) = \tau \left( y(\omega(\lambda) - x_1)^{-1} \right), \quad \lambda \in \mathbb{H}, y \in Y. \quad (1.3)$$

Subsequently, Voiculescu [9] discovered an underlying algebraic structure that allows for a conceptually simple proof of Biane’s extension and, indeed, for a much more general result. Namely, suppose that in addition to the probability space $(A, \tau)$, we consider a von Neumann subalgebra $B \subset A$ containing the unit of $A$, and denote by $E_B : A \to B$ the trace-preserving conditional expectation. Let $x_1, x_2 \in A$ be two selfadjoint elements of $A$ that are free relative to $E_B$ (or, more simply, $E_B$-free). Denote by $\mathbb{H}(B)$ the collection of those $b \in B$ that satisfy $\Im b := \frac{b - b^*}{2i} \geq \varepsilon$ for some $\varepsilon > 0$. Finally, let $B\langle x_1 \rangle$ be the von Neumann algebra generated by $B$ and $x_1$, and let $E_B\langle x_1 \rangle : A \to B\langle x_1 \rangle$ denote the trace-preserving conditional expectation. Then there exists an analytic function $\omega : \mathbb{H}(B) \to \mathbb{H}(B)$ such that

$$E_B\langle x_1 \rangle \left[ (b - (x_1 + x_2))^{-1} \right] = (\omega(b) - x_1)^{-1}, \quad b \in B. \quad (1.3)$$

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Like (1.2), this identity can be rewritten as

\[ \tau \left( y(b - (x_1 + x_2)^{-1}) \right) = \tau \left( y(\omega(b) - x_1)^{-1} \right), \quad b \in B, y \in B(x_1). \]

Using the fact that \(E_B\) is trace-preserving, (1.4) is also seen to be equivalent to

\[ \mathbb{E}_B \left[ y(b - (x_1 + x_2))^{-1} \right] = \mathbb{E}_B \left[ y(\omega(b) - x_1)^{-1} \right], \quad b \in B, y \in B(x_1). \]

As seen in [2], the existence of subordination functions can be proved using methods of Banach space analyticity. In fact [2] provides the following analog of Voiculescu’s original result from [7] that does not require the presence of a trace and it applies to operator-valued \(C^*\)-probability spaces. The statement uses the notation

\[ F_{x_1 + x_2}(b) = F_{x_1}(\omega_1(b)) = F_{x_2}(\omega_2(b)) = \omega_1(b) + \omega_2(b) - b \]

for every \(b \in H(B)\).

Theorem 1.1. Let \(A\) be a unital \(C^*\)-algebra, let \(B \subset A\) be a unital sub-\(C^*\)-algebra containing the unit of \(A\), and let \(E : A \to B\) be a faithful, completely positive conditional expectation. Given selfadjoint elements \(x_1, x_2 \in A\) that are \(E\)-free, there exist unique analytic functions \(\omega_1, \omega_2 : H(B) \to H(B)\) such that

\[ F_{x_1 + x_2}(b) = F_{x_1}(\omega_1(b)) = F_{x_2}(\omega_2(b)) = \omega_1(b) + \omega_2(b) - b \]

for every \(b \in H(B)\).

The purpose of this note is to extend the subordination result of [9] to this general context. Of course, there may not exist a conditional expectation onto \(B(x_1)\), so (1.3) may not make sense, but (1.5) does make sense, and it is this equation that we extend to the general context in Theorem 3.1. Note that this result is somewhat stronger than that of [9] in the tracial case because we allow for the conditional expectation onto an algebra that is possibly larger than \(B(x_1)\). This improvement can however be obtained already with the methods of [9]. We actually prove an even more general result (Theorem 2.1) that applies in the original context of operator-valued free probability [8]. In this result, \(A\) is just a Banach algebra and \(E : A \to B\) is a continuous conditional expectation. In this general context, the subordination functions \(\omega_j(b)\) are only defined for invertible elements \(b \in B\) for which \(|b^{-1}|\) is sufficiently small. Theorem 3.1 follows from the Banach algebra result by analytic continuation.

The idea for this work arose from the observation that, in the context of the study of bi-free additive convolution [1], a method based on analytic functions and expansion to operator matrices allows for the recovery of Biane’s result. This led to the extension of the operator-valued subordination result of [9] to the more general context described in Theorem 3.1, again using purely analytic elementary methods.

2. Subordination in Banach algebraic probability spaces

Let \((A, B, E)\) be a Banach algebraic probability space. That is, \(A\) is a complex, unital Banach algebra, \(B\) is a closed subalgebra containing the unit of \(A\), and \(E : A \to B\) is a continuous conditional expectation. The concept of \(B\)-freeness with respect to \(E\), which we refer to more simply as \(E\)-freeness, was introduced in [8]. It was also shown in [8] that the calculation of (the symmetric parts of) free convolutions of \(B\)-valued distributions follows the same pattern first discovered in [6] when \(B = \mathbb{C}\). We recall briefly the relevant notation. Given a random variable,
that is, an element $x \in \mathcal{A}$, and given $b \in \mathcal{B}$, we write
\[
G_x(b) = \mathbb{E}[(b - x)^{-1}], \\
F_x(b) = G_x(b)^{-1}, \\
\tilde{G}_x(b) = G_x(b^{-1}) = b + \sum_{n=1}^{\infty} \mathbb{E}[b(xb)^n],
\]
whenever the quantities on the right-hand side make sense. In particular, $\tilde{G}_x$ is defined for $\|b\| < 1/\|x\|$ and it is an analytic function. Moreover, the derivative of $\tilde{G}_x$ at $b = 0$ is the identity map of $\mathcal{B}$. It follows that $\tilde{G}_x$ maps conformally a neighborhood of $b = 0$ onto another such neighborhood, and thus has an inverse function denoted $\tilde{K}_x$. Observe also that
\[
\tilde{G}_x(b) = b \left(1 + \sum_{n=1}^{\infty} \mathbb{E}[(xb)^n]\right) = b(1 + O(\|b\|)),
\]
and thus, provided $\|b\|$ is sufficiently small, $b$ is invertible if and only if $\tilde{G}_x(b)$ is invertible. It follows that both $\tilde{G}_x$ and $\tilde{K}_x$ map invertible elements to invertible elements when restricted to a sufficiently small neighborhood of $0 \in \mathcal{B}$. Finally, define
\[
R_x(b) = \tilde{K}_x(b)^{-1} - b^{-1}
\]
for $b \in \mathcal{B}$ such that $\|b\|$ is sufficiently small. (As in [3], we reserve the exponent $-1$ for inverses in the algebra $\mathcal{A}$.) The function $R_x$ continues analytically to a neighborhood of $b = 0$.

Suppose now that $x_1, x_2 \in \mathcal{A}$ are $\mathbb{E}$-free, and set $x = x_1 + x_2$. Then it is shown in [3] (see the end of Section 4) that
\[
R_x(b) = R_{x_1}(b) + R_{x_2}(b)
\]
for $b \in \mathcal{B}$ such that $\|b\|$ is sufficiently small. We define now
\[
\omega_j(b) = (\tilde{K}_x(\tilde{G}_x(b^{-1})))^{-1}, \quad j = 1, 2,
\]
for invertible elements $b \in \mathcal{B}$ such that $\|b^{-1}\|$ is sufficiently small. These functions are analytic for such values of $b$, and the equalities
\[
F_x(b) = F_{x_1}(\omega_1(b)) = F_{x_2}(\omega_2(b)) = \omega_1(b) + \omega_2(b) - b
\]
hold throughout their domain of definition (see [3] for the scalar-valued case). The functions $\omega_j$ will be referred to as the subordination functions associated to the variables $x_1$ and $x_2$. (These subordination functions are most useful when they can be defined on a standard domain, as it happens when we work with selfadjoint variables in a $C^*$-probability space—see Section [3] below.)

We are now ready to prove the main result of this section. The special case $y = 1$ follows, of course, from (2.2).

**Theorem 2.1.** Let $(\mathcal{A}, \mathcal{B}, \mathbb{E})$ be a Banach algebraic probability space, and let $x_1, x_2$ and $y$ be elements of $\mathcal{A}$ such that $x_2$ is $\mathbb{E}$-free from $\{x_1, y\}$. Denote by $\omega_1$ and $\omega_2$ the subordination functions associated to $x_1$ and $x_2$, and set $x = x_1 + x_2$. Then we have
\[
\mathbb{E}[(y(b - x)^{-1})] = \mathbb{E}[y(\omega_1(b) - x_1)^{-1}] \quad \text{and} \quad \mathbb{E}[(b - x)^{-1}y] = \mathbb{E}[(\omega_1(b) - x_1)^{-1}y]
\]
for every invertible $b \in \mathcal{B}$ such that $\|b^{-1}\|$ is sufficiently small.
Proof. Consider the probability space \((A_2, B_2, \mathbb{E}_2)\), where \(A_2\) is the Banach algebra consisting of \(2 \times 2\) matrices of the form
\[
\begin{bmatrix}
    a_{1,1} & 0 \\
    a_{2,1} & a_{2,2}
\end{bmatrix}, \quad a_{1,1}, a_{2,1}, a_{2,2} \in A,
\]
and \(B_2\) is the subalgebra consisting of the matrices whose entries belong to \(B\). The norm of such a matrix can be taken to be the operator norm when the matrix is viewed as a left multiplication operator on \(A \oplus A\) endowed with the \(\ell^1\) norm. The conditional expectation \(\mathbb{E}_2\) is simply \(\mathbb{E}\) applied entrywise. The elements
\[
X_1 = \begin{bmatrix} x_1 & 0 \\ y & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} x_1 & 0 \\ 0 & 0 \end{bmatrix}
\]
are \(\mathbb{E}_2\)-free (this is a straightforward verification, but can as well be deduced from [5] or from [8, Corollary 3.7]). We denote by \(X\) their sum, and we denote by \(\Omega_j(B)\), \(j = 1, 2\), the corresponding subordination functions, defined for invertible matrices \(B \in B_2\) such that \(\|B^{-1}\| < \epsilon\) for some \(\epsilon > 0\).

We first show that the matrices \(\Omega_j(B)\) have the form
\[
\Omega_j(B) = \begin{bmatrix}
    \omega_j(b_{1,1}) & 0 \\
    g_j(B) & b_{2,2}
\end{bmatrix}, \quad B = \begin{bmatrix} b_{1,1} & 0 \\ b_{2,1} & b_{2,2} \end{bmatrix},
\]
for some analytic functions \(g_1, g_2\). To do this, we use (2.1), so we calculate
\[
(B - X_1)^{-1} = \begin{bmatrix}
    (b_{1,1} - x_1)^{-1} & 0 \\
    -b_{2,2}(b_{2,1} - x_1)(b_{1,1} - x_1)^{-1} & b_{2,2}
\end{bmatrix},
\]
(2.3)
\[
G_{X_1}(B) = \begin{bmatrix}
    G_{x_1}(b_{1,1}) & 0 \\
    -b_{2,2}E[(b_{2,1} - y)(b_{1,1} - x_1)^{-1}] & b_{2,2}
\end{bmatrix},
\]
and
\[
\tilde{G}_{X_1}(B) = G_{X_1}(B^{-1}) = \begin{bmatrix}
    \tilde{G}_{x_1}(b_{1,1}) & 0 \\
    E[(b_{2,1}b_{1,1}^{-1} + b_{2,2}y)(b_{1,1}^{-1} - x_1)^{-1}] & b_{2,2}
\end{bmatrix}.
\]
The (1,1) entry of this matrix only depends on \(b_{1,1}\), and therefore
\[
\tilde{K}_{X_1}(B) = \begin{bmatrix}
    \tilde{K}_{x_1}(b_{1,1}^{-1}) & 0 \\
    h_1(B) & b_{2,2}
\end{bmatrix},
\]
for some analytic function \(h_1\). Replacing \(x_1\) by \(x = x_1 + x_2\), we similarly obtain
\[
\tilde{G}_X(B) = \begin{bmatrix}
    \tilde{G}_x(b_{1,1}^{-1}) & 0 \\
    h(B) & b_{2,2}
\end{bmatrix},
\]
and therefore, according to (2.1),
\[
\Omega_1(B) = \tilde{K}_{X_1}(\tilde{G}_X(B^{-1}))^{-1} = \begin{bmatrix}
    \tilde{K}_{x_1}(\tilde{G}_x(b_{1,1}^{-1})) & 0 \\
    h_1(\tilde{G}_X(B^{-1})) & b_{2,2}^{-1}
\end{bmatrix}^{-1}
\]
\[
= \begin{bmatrix}
    \omega_1(b_{1,1}) & 0 \\
    -b_{2,2}h_1(\tilde{G}_X(B^{-1}))\omega_1(b_{1,1}) & b_{2,2}
\end{bmatrix}.
\]
The argument for \(\Omega_2\) is obtained upon replacing \(x_1\) by \(x_2\) and \(y\) by zero.
Fix $t \in \mathbb{R}_+$ and an invertible $b \in \mathcal{B}$ such that $t > 1/\varepsilon$ and $\|b^{-1}\| < \varepsilon$. We use the subordination equation (2.2) with $X_j$ in place of $x_j$ and with $B$ in place of $b$, where

$$B = \begin{bmatrix} b & 0 \\ 0 & t \end{bmatrix}.$$ 

As seen above, for this particular form of $B$ we have

$$\Omega_j(B) = \begin{bmatrix} \omega_j(b) & 0 \\ \beta_j & t \end{bmatrix}, \quad j = 1, 2.$$ 

We proceed to determine $\beta_j$. The calculations above, and one more inversion, yield

$$G_X(B) = \begin{bmatrix} G_x(b) & 0 \\ -t^{-1}\mathbb{E}[y(b-x)^{-1}] & t^{-1} \end{bmatrix}, \quad F_X(B) = \begin{bmatrix} F_x(b) & 0 \\ -\mathbb{E}[y(b-x)^{-1}]F_x(b) & t \end{bmatrix}.$$ 

Formula (2.3) gives

$$G_{X_1}(\Omega_1(B)) = \begin{bmatrix} G_{x_1}(\omega_1(b)) & 0 \\ -t^{-1}\mathbb{E}[(\beta_1 - y)(\omega_1(b) - x_1)^{-1}] & t^{-1} \end{bmatrix},$$

and thus

$$F_{X_1}(\Omega_1(B)) = \begin{bmatrix} F_{x_1}(\omega_1(b)) & 0 \\ \mathbb{E}[(\beta_1 - y)(\omega_1(b) - x_1)^{-1}]F_{x_1}(\omega_1(b)) & t \end{bmatrix}.$$ 

Replacing $x_1$ by $x_2$ and $y$ by zero we obtain

$$G_{X_2}(\Omega_2(B)) = \begin{bmatrix} G_{x_2}(\omega_2(b)) & 0 \\ -t^{-1}\mathbb{E}[\beta_2(\omega_2(b) - x_2)^{-1}] & t^{-1} \end{bmatrix},$$

and

$$F_{X_2}(\Omega_2(B)) = \begin{bmatrix} F_{x_2}(\omega_2(b)) & 0 \\ \mathbb{E}[\beta_2(\omega_2(b) - x_2)^{-1}]F_{x_2}(\omega_2(b)) & t \end{bmatrix}.$$ 

The $(2, 1)$ entry in the equality $F_X(B) = F_{X_1}(\Omega_1(B))$ amounts to

$$(2.4) \quad -\mathbb{E}[y(b-x)^{-1}]F_x(b) = \mathbb{E}[(\beta_1 - y)(\omega_1(b) - x_1)^{-1}]F_{x_1}(\omega_1(b))$$

$$= \mathbb{E}[\beta_2(\omega_2(b) - x_2)^{-1}]F_{x_2}(\omega_2(b)),$$

and the last term is

$$\mathbb{E}[\beta_2(\omega_2(b) - x_2)^{-1}]F_{x_2}(\omega_2(b)) = \beta_2G_{x_2}(\omega_2(b))F_{x_2}(\omega_2(b)) = \beta_2.$$ 

Thus $\beta_2 = -\mathbb{E}[y(b-x)^{-1}]F_x(b)$. On the other hand, the $(2, 1)$ entry in the equality $\Omega_1(b) + \Omega_2(b) - b = F_X(b)$ yields

$$\beta_1 + \beta_2 = -\mathbb{E}[y(b-x)^{-1}]F_x(b),$$

showing that $\beta_1 = 0$. The equality $\mathbb{E}[y(b-x)^{-1}] = \mathbb{E}[y(\omega_1(b) - x_1)^{-1}]$ follows now from (2.4). The proof of the second identity in the statement is similar, using upper triangular $2 \times 2$ matrices.
3. Subordination in $C^*$-probability spaces

Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued $C^*$-probability space. Thus, $\mathcal{A}$ is a unital $C^*$-algebra, $\mathcal{B} \subset \mathcal{A}$ is a $C^*$-subalgebra containing the unit of $\mathcal{A}$, and $E : \mathcal{A} \to \mathcal{B}$ is a completely positive conditional expectation. We denote by $\mathbb{H}(\mathcal{B}) \subset \mathcal{B}$ the set consisting of those elements $b \in \mathcal{B}$ whose imaginary part $\Im b = (b - b^*)/2i$ is nonnegative and invertible.

Suppose now that $x_1, x_2 \in \mathcal{A}$ are two selfadjoint elements that are $E$-free, and set $x = x_1 + x_2$. It was shown in [2] that there exist unique analytic functions $\omega_1, \omega_2 : \mathbb{H}(\mathcal{B}) \to \mathbb{H}(\mathcal{B})$ such that

$$F_x(b) = F_{x_1}(\omega_1(b)) = F_{x_2}(\omega_2(b)) = \omega_1(b) + \omega_2(b) - b, \quad b \in \mathbb{H}(\mathcal{B}).$$

These functions are, of course, analytic continuations of the subordination functions considered in Section 2. The following statement follows from Theorem 2.1 by analytic continuation.

**Theorem 3.1.** Let $(\mathcal{A}, \mathcal{B}, E)$ be an operator valued $C^*$-probability space. Consider elements $x_1, x_2, y \in \mathcal{A}$ such that $x_1$ and $x_2$ are selfadjoint, and $x_2$ is $E$-free from $\{x_1, y\}$. Then the subordination function $\omega_1$ satisfies

$$E[y(b - x^{-1})] = E[y(\omega_1(b) - x_1)^{-1}] \quad \text{and} \quad E[(b - x)^{-1}y] = E[(\omega_1(b) - x_1)^{-1}y],$$

for every $b \in \mathbb{H}^+(\mathcal{B})$.

The result applies, for instance, to elements $y$ in the closure of the algebra $\mathcal{B}(x_1)$ generated by $\mathcal{B}$ and $x_1$. To see how this result relates to [9], consider an arbitrary state $\varphi$ on $\mathcal{B}$, and construct its extension $\psi = \varphi \circ E$. Then Theorem 3.1 simply states that $(b - x)^{-1} - (\omega_1(b) - x_1)^{-1}$ is orthogonal to $y^*$ in the scalar product induced by $\psi$. In other words, if $\mathcal{Y}$ is any $C^*$-algebra algebra containing $\mathcal{B}(x_1)$, and if $\mathcal{Y}$ is $E$-free from $x_2$, then the orthogonal projection of $(b - x)^{-1}$ onto the closure of $\mathcal{Y}$ (in the scalar product induced by $\psi$) is precisely $(\omega_1(b) - x_1)^{-1}$. In particular, this orthogonal projection belongs to the closure of $\mathcal{B}(x_1)$.

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