On a Batalin–Vilkovisky operator generating higher Koszul brackets on differential forms

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Abstract
We introduce a formal ℏ-differential operator Δ that generates higher Koszul brackets on the algebra of (pseudo)differential forms on a \( P_\infty \)-manifold. Such an operator was first mentioned by Khudaverdian and Voronov in arXiv:1808.10049. (This operator is an analogue of the Koszul–Brylinski boundary operator \( \partial_P \) which defines Poisson homology for an ordinary Poisson structure.) Here, we introduce \( \Delta = \Delta_P \) by a different method and establish its properties. We show that this BV type operator generating higher Koszul brackets can be included in a one-parameter family of BV type formal ℏ-differential operators, which can be understood as a quantization of the cotangent \( L_\infty \)-bialgebroid. We obtain symmetric description on both \( \Pi TM \) and \( \Pi T^*M \). For the purpose of the above, we develop in detail a theory of formal ℏ-differential operators and also of operators acting on densities on dual vector bundles. In particular, we have a statement about operators that can be seen as a quantization of the Mackenzie–Xu canonical diffeomorphism. Another interesting feature is that we are able to introduce a grading, not a filtration, on our algebras of operators. When operators act on objects on vector bundles, we obtain a bi-grading.

Keywords BV operator · higher Koszul brackets · \( P_\infty \) structure · quantum \( L_\infty \) bialgebroid · formal ℏ-differential operator · quantum Mackenzie-Xu transformation

Mathematics Subject Classification 53D17 · 58C50 · 16S32 · 53Z05
1 Introduction

Higher Koszul brackets generalize the classical (binary) Koszul bracket on forms on a Poisson manifold. They are defined on the algebra of pseudodifferential forms on a supermanifold with a $P_\infty$- or homotopy Poisson structure. Their construction was introduced in 2008 by Khudaverdian and Voronov [17].

It has been noticed by Voronov [19, 37, 40] that a Lie algebroid structure and similar structures like $L_\infty$-algebroids can be seen as ideal objects which “manifest” themselves in different equivalent ways on “neighbor” vector bundles obtained from each other by dualization and parity reversion functor $\Pi$.

From this viewpoint, the Koszul bracket is a manifestation on $\Pi TM$ of the Lie algebroid structure on the cotangent bundle $T^*M$ induced by a Poisson structure on $M$ and is the corresponding Lie–Schouten bracket, see Mackenzie [27, Ch.10]. An equivalent manifestation on $\Pi T^*M$ of the same Lie algebroid structure is the Poisson–Lichnerowicz differential $d_P: \mathfrak{A}^k(M) \to \mathfrak{A}^{k+1}(M)$. (Here, $\mathfrak{A}^k(M)$ denotes multivector fields of degree $k$ on a manifold $M$.)

In the similar way, higher Koszul brackets are one of the equivalent manifestations of the $L_\infty$-algebroid structure on the cotangent bundle of a $P_\infty$-manifold.

It was shown by Koszul [24] that classical Koszul bracket can be generated by a second-order “Batalin–Vilkovisky (BV) type” differential operator $\partial_P$, known as the Koszul–Brylinski differential. (It defines Poisson homology dual to Poisson cohomology [8].) In the appendix to the recent paper [19], it was briefly indicated how Koszul’s construction can be extended to the “higher” case.

In the present letter, we introduce a BV type operator generating higher Koszul brackets by a different method and establish its properties. It is a formal $\hbar$-differential operator of (in general) infinite order. We develop a theory of such operators. Using this operator, we can consider “quantum higher Koszul brackets” depending on parameter $\hbar$.

Further, recall that in the classical situation of a Poisson manifold, the cotangent Lie algebroid is actually a Lie bialgebroid (if one takes into account the tautological Lie algebroid structure of $TM$). This also generalizes to the higher case as an $L_\infty$-bialgebroid [41, 43]. Here, we show that the BV operator generating higher Koszul brackets can be included in a one-parameter family of formal $\hbar$-differential operators which we can interpret as a structure of a “quantum $L_\infty$-bialgebroid”.

To keep the symmetry between the manifestations existing on the classical level, on the quantum level we need to introduce into consideration densities of different weights.

The structure of the letter is as follows. In Sect. 2, we recall the classical Koszul bracket for a Poisson manifold and the BV operator generating it. Then, we recall the construction of the sequence of “higher Koszul brackets” for a homotopy Poisson or $P_\infty$-structure on a supermanifold due to Khudaverdian–Voronov.

In Sect. 3, we define (following [19]) an operator serving as an analogue of Koszul’s second-order BV type operator, for higher Koszul brackets. It looks superficially simi-
lar to the classical operator. However, it is, in general, a differential operator of infinite order. More precisely, it is a formal $\hbar$-differential operator. We need the theory of such operators, which is interesting also for broader purposes.

We develop such a theory in Sect. 3.2. In particular, we show that any such an operator (of an infinite order from the conventional viewpoint) has a principal symbol, which is a well-defined formal function on $T^*M$. Planck’s constant $\hbar$ is treated as a formal variable and one of the generators of our algebra. This makes it possible to introduce a grading, rather than a filtration, in the space of our operators. For a formal $\hbar$-differential operator, we consider the sequences of “quantum” and “classical” brackets (first introduced in [38]) and prove that the master Hamiltonian for the sequence of classical brackets is exactly the principal symbol.

Using the results above, we construct formal $\hbar$-differential operators that can be seen as quantizations of the classical Hamiltonians arising in the definition of higher Koszul brackets.

This immediately gives the following main statement: There exists an odd formal $\hbar$-differential operator $\Delta_P$ acting on pseudodifferential forms and generating higher Koszul brackets and which also obeys $\Delta_P^2 = 0$ (“a BV operator for higher Koszul brackets”). Moreover, we show that the constructed BV operator can be combined with the operator $-i\hbar d$ (where $d$ is the de Rham differential\(^2\)) into a one-parameter family of BV operators. We interpret it as a “quantization” of the structure of the cotangent $L_\infty$-bialgebroid of a $P_\infty$-manifold, in the manifestation on the bundle $\Pi TM$. We then look into a similar quantization in the dual manifestation on the bundle $\Pi T^*M$. It turns out that for a natural construction of a BV operator on $\Pi T^*M$ there is an obstruction, which is the modular class of a given $P_\infty$-structure. The formula can be remedied by a correction term, but we lose the symmetry between the manifestations on $\Pi TM$ and $\Pi T^*M$ that exist classically.

In Sect. 4, we are concerned with developing the desired symmetric picture. For that, we have to depart from operators acting on functions on a supermanifold and consider the more general case of operators acting on densities of various weights. As the goal is to obtain BV operators for $L_\infty$-bialgebroids in dual manifestations (note that we do not stick to one formal definition of what an $L_\infty$-bialgebroid is, since it looks to be still a rather open question, but of course there is some general idea), we need to develop a theory of operators for dual bundles.

We use the fiberwise Fourier transform discovered by Voronov and Zorich [44] for the purposes of supermanifold integration theory. We use an $\hbar$-version of it for fiberwise densities.

And we also introduce a new $\mathbb{Z} \times \mathbb{Z}$-grading for $\hbar$-differential operators on a vector bundle. (This $\mathbb{Z} \times \mathbb{Z}$-grading of operators can be seen as a quantum analogue of the graded manifold approach to the double vector bundle structure of the cotangent of a vector bundle as in [39,40].) We prove that the fiberwise $\hbar$-Fourier transform induces anti-isomorphisms of algebras of formal $\hbar$-differential operators on dual vector bundles acting on densities of suitable weights. We also show that the $\mathbb{Z} \times \mathbb{Z}$-grading undergoes a mirror reflection. In particular, this gives a quantum version of the fundamental Mackenzie–Xu canonical antisymplectomorphism [28] (which is induced in the limit

\(^2\) The purpose of inserting the factor $-i\hbar$ is to have an $\hbar$-differential operator.
We also show that the fiberwise $h$-Fourier transform preserves the class of “quantum pullbacks” of Th. Voronov \[42\], which are particular type of Fourier integral operators, with an explicit description of the action on phase functions. In Sect. 4.3, we return from general theory to the concrete situation of our interest. We again consider the cotangent $L_\infty$-bialgebroid for a $P_\infty$-manifold. The difference with the approach in Sect. 3 is that now we construct BV type operators acting on half-densities, not on functions. (Worth mentioning that in the actual physical BV setup, the canonical odd Laplacian due to Khudaverdian \[13\] acts on half-densities.) This way we achieve a complete symmetry in the manifestations on $\Pi T^*M$ and $\Pi TM$. The modular class of a $P_\infty$-structure that was popping up in another approach, here has a different role; it seems to be an obstruction to a “quantum lift” of the anchor (such a lift would be given by an operator of “quantum pullback” type mentioned above).

**Notation.** Here, “forms” and “multivector fields” on ordinary or super manifold, are understood as functions on the supermanifolds $\Pi_1 T^*M$ and $\Pi_1 TM$, respectively. Hence, our “forms” are actually pseudodifferential forms in the supercase and inhomogeneous differential forms in the ordinary case, similar for multivector fields.

By $\Omega(M)$ we denote the algebra of forms, by $A(M)$ the algebra of multivector fields, and by $A_k(M)$ multivectors of degree $k$ or $k$-vectors. All super formulas are written for quantities that are homogeneous in the sense of parity, which is denoted by the tilde over a symbol, e.g., $\tilde{x} = 0, 1$ for $x$ even or odd. If $x^a$ are local coordinates on a (super)manifold, then forms and multivector fields are functions of the variables $x^a$, $dx^a$ and $x^a, x^*_a$, respectively. The variables $dx^a$ and $x^*_a$ have parities opposite to those of the corresponding coordinates: $\tilde{dx}^a = \tilde{x}^*_a = \tilde{a} + 1$, where $\tilde{a} = x^a$. Under a change of coordinates, the variables $x^*_a$ transforms in the same way as the partial derivatives $\partial_a$. The canonical Poisson bracket (even) on $T^*M$ for a supermanifold $M$ is denoted $\{ - , - \}$, while the canonical Schouten bracket on $\Pi T^*M$ (odd) is denoted $[ [ - , - ] ]$.

## 2 Recollection of higher Koszul brackets

### 2.1 Binary Koszul bracket

The classical (binary) Koszul bracket on differential forms on a Poisson manifold was introduced by Koszul \[24\]; the case of 1-forms was known earlier, in particular, in the context of integrable systems. (See more in \[22\].) Below are the main formulas (which we give in a version generalized to supermanifolds, see \[17\]). Let $\{ f , g \}_P$ denote the Poisson bracket corresponding to a Poisson bivector $P = \frac{1}{2} P^{ab}(x) x^*_b x^*_a$, then the Koszul bracket on forms denoted $\{ \omega , \sigma \}_P$ is given by

\[
[f , g]_P := 0, \quad [df , g]_P := \{ f , g \}_P, \quad [df , dg]_P := (-1)^{\tilde{f}} d\{ f , g \}_P \quad (1)
\]
together with the symmetry, linearity and Leibniz conditions:

\[
[c\omega, \sigma]_P = (-1)^{\hat{c}} c[\omega, \sigma]_P ,
\]

\[
[\omega, \sigma]_P = (-1)^{\hat{c}\sigma} [\sigma, \omega]_P ,
\]

\[
[\omega, \sigma \tau]_P = [\omega, \sigma]_P \tau + (-1)^{(\hat{c}\sigma+1)} \sigma[\omega, \tau]_P ,
\]

which imply the Jacobi identity in the form

\[
[\omega, [\sigma, \tau]_P]_P = (-1)^{\hat{c}\sigma+1} [[\omega, \sigma]_P, \tau]_P + (-1)^{(\hat{c}\sigma+1)} \sigma[\omega, [\tau, \sigma]_P]_P .
\] (5)

(The conditions (2), (3), (4), (5) together mean that \(\Omega(M)\) is a “Schouten” or “odd Poisson” algebra under the Koszul bracket. Kosmann-Schwarzbach [21] refers to the Koszul bracket as “Koszul-Schouten bracket”.) Note that the sign conventions followed here may be different from other sources; we chose those convenient for generalization to higher brackets. The Koszul bracket also satisfies

\[
d[\omega, \sigma]_P = -[d\omega, \sigma]_P + (-1)^{\hat{c}\sigma+1} [\omega, d\sigma]_P .
\] (6)

The derivation property (6) expresses the fact that \(T^* M\) is actually a Lie bialgebroid (see [27] and references therein).

Koszul showed in [24] that the bracket \([\omega, \sigma]_P\) can be obtained from an odd second-order differential operator \(\Delta\),

\[
\Delta = \partial_P := [d, i(P)]
\] (7)
on the algebra \(\Omega(M)\) by the formula

\[
\Delta(\omega \sigma) = \Delta(\omega) \sigma + (-1)^{\hat{c}\omega} \omega \Delta(\sigma) + [\omega, \sigma]_P
\] (8)

Here, \(i(P)\) is the operator of the interior product with the bivector \(P\). It decreases the degrees of forms by two; hence, its commutator with \(d\) is of degree \(-1\). The operator \(\partial_P\) is often referred to as the “Koszul–Brylinski differential.” It plays the role of the boundary operator in the definition of Poisson homology [8], a dual notion to Poisson cohomology defined by the Lichnerowicz differential \(d_P\) (similarly with the relation between Lie algebra homology and Lie algebra cohomology).

Equation (8) is similar in form to the analogous equation relating the canonical Schouten bracket on the algebra of multivector fields \(\mathfrak{A}(M)\) and the divergence operator \(\delta = \delta_\rho\) on multivector fields (defined using a choice of a volume element \(\rho\)),

\[
\delta(T S) = \delta(T) \sigma + (-1)^{\hat{T} \sigma} T \delta(S) + \|T, S\| ,
\] (9)

which is classically known (see, e.g., [20]).
Although we are used to thinking about divergence as a first-order operator, from the viewpoint of supermanifolds $\delta$ is an odd differential operator of order two,

$$\delta T = (-1)^{\bar{a}} \frac{1}{\rho(x)} \frac{\partial}{\partial x^a} \left( \rho(x) \frac{\partial T}{\partial x^*_a} \right)$$

(10)

for a multivector field $T = T(x, x^*)$ (see, e.g., [36, Ch.5]). Since their appearance in quantum field theory [3, 4], odd differential operators generating an odd bracket by a formula like (8) or (9) are referred to as “Batalin–Vilkovisky type operators”. (Geometric meaning of these operators as odd analogues of the ordinary Laplacians was first understood by H. Khudaverdian [12] and A.S. Schwarz [29]. See also [14–16, 23].)

Y. Kosmann-Schwarzbach [23] showed that the Koszul operator $\Delta$ is a BV operator in the narrow sense, i.e., it is the odd Laplacian constructed from a bracket and a volume element, where the bracket is the Koszul bracket and the volume element is the canonical invariant volume element on $\Pi T M$.

In the next subsection, we recall a “higher analogue” of the classical Koszul bracket (when a Poisson structure is replaced by a $P_\infty$-structure) and in Sect. 3 we shall introduce and study an analogue for a higher setting of Koszul’s BV operator. Unlike the second-order Batalin–Vilkovisky operators related to binary brackets, this will be a formal differential operator of infinite order. Binary Koszul bracket can be (and in fact originally was) defined on an ordinary manifold. As for higher Koszul brackets, for the theory to be nontrivial one really needs supermanifolds.

2.2 Higher Koszul brackets for a $P_\infty$-structure

Let $M$ be a supermanifold. A homotopy Poisson structure or $P_\infty$-structure is given on $M$ by an even function on $\Pi T^* M$, denote it $P$, which satisfies the equation $[[P, P]] = 0$. (We only need its infinite jet at $M \subset \Pi T^* M$, i.e., the power expansion in the variables $x^*_a$.) It generates the sequence of higher Poisson brackets on functions on $M$ by Th. Voronov’s higher derived bracket formulas [38],

$$\{f_1, \ldots, f_n\}_P := [[\ldots [[P, f_1]], \ldots, f_n]]|_M$$

(11)

(the restriction on $M$ at the right-hand side simply means setting $x^*_a = 0$). Here $n = 0, 1, 2, 3 \ldots$ The brackets (11) have alternating parities, in particular, the binary bracket is even. Condition $[[P, P]] = 0$ is equivalent to the sequence of “higher Jacobi identities” for the brackets (11).

Notice that $P_\infty$-structures are not some abstract generalization. They can naturally arise in ordinary Poisson geometry. Cattaneo and Felder [9] discovered $P_\infty$-structures from deformations of coisotropic submanifolds of an ordinary Poisson manifold.

Analogue of the Koszul bracket for a $P_\infty$-structure was found by H. Khudaverdian and Th. Voronov [17]. It is a sequence of odd symmetric brackets on the algebra $\Omega(M)$, notation $[\omega_1, \ldots, \omega_n]_P$, where $\omega_i \in \Omega(M)$ and $n = 0, 1, 2, 3, \ldots$, defined
on functions and exact 1-forms by the equations

\[
\begin{align*}
[d f_1, \ldots, d f_{n-1}, f_n]_P &:= \{ f_1, \ldots, f_n \}_P, \\
[d f_1, d f_2, \ldots, d f_n]_P &:= (-1)^{\frac{n(n-1)}{2}} d\{ f_1, f_2, \ldots, f_n \}_P,
\end{align*}
\]

(12)

and for all other combinations the brackets are zero, and then extended to the whole algebra $\Omega(M)$ by the Leibniz rule as multiderivations. The resulting brackets are called higher Koszul brackets. They also showed that the higher Jacobi identities for Poisson brackets giving $P_\infty$-structure imply the higher Jacobi identities for the higher Koszul brackets, so they form what is known as an $S_\infty$-structure on the algebra $\Omega(M)$. Note that a $P_\infty$-structure is a generalization of an even Poisson structure, while an $S_\infty$-structure is a generalization of an odd Poisson structure. In particular, brackets in the $P_\infty$ case are antisymmetric and of alternating parities, while in the $S_\infty$ case they are symmetric and all odd.

In other words, the higher Koszul brackets for a $P_\infty$-structure on $M$ are an $S_\infty$-structure for the supermanifold $\Pi T M$. They are the Lie–Schouten brackets corresponding to an $L_\infty$-algebroid structure in the cotangent bundle $T^* M$ induced by a $P_\infty$-structure on $M$.

Any $S_\infty$-structure on a supermanifold is given [38] by an odd function $H$ on its cotangent bundle satisfying $[H, H] = 0$ (an odd master Hamiltonian). This master Hamiltonian $H_P \in C^\infty(T^*(\Pi T M))$ for higher Koszul brackets is constructed as follows [17,19]. We take the Hamiltonian lift of the homological vector on $\Pi T^* M$ and apply to it the natural antisymplectomorphism $T^*(\Pi T M) \cong T^*(\Pi T^* M)$. This vector field $d_P \in \text{Vect}(\Pi T^* M)$, which is the analogue of the Lichnerowicz differential, is given by $d_P := [[P, -]]$. One may now note that the canonical Schouten bracket on $\Pi T^* M$ is itself derived from a master Hamiltonian, which is the invariant quadratic function $D^* \in T^*(\Pi T^* M)$,

\[
D^* = (-1)^{\frac{n(n-1)}{2}} \pi^a p_a,
\]

(13)

where $p_a, \pi^a$ are the conjugate momenta for the variables $x^a, x_a^*$. The function $D^*$ on $T^*(\Pi T^* M)$ corresponds under the identification $T^*(\Pi T M) \cong T^*(\Pi T^* M)$ to the function $D = dx^a p_a$ on $T^*(\Pi T M)$, the lift the de Rham differential. Therefore, vector field $d_P \in \text{Vect}(\Pi T^* M)$ lifts to the Hamiltonian

\[
H_P^* = \{ D^*, P \}.
\]

(14)

By using the identification $T^*(\Pi T M) \cong T^*(\Pi T^* M)$, which preserves the canonical Poisson bracket on the cotangent bundle up to a sign, we arrive at

\[
H_P = \{ D, P^* \}.
\]

(15)

and this is the sought-for master Hamiltonian for the higher Koszul brackets. Here, $P^* \in C^\infty(T^*(\Pi T M))$ is obtained from $P = P(x, x^*)$ by substituting $x^*_a = \pi_a$. It is
also possible to write down an explicit formula for $H_P$,

$$H_P = dx^a \frac{\partial P}{\partial x^a}(x, \pi) + (-1)^{\hat{a}} \frac{\partial P}{\partial \pi^a}(x, \pi) p_a$$

(16)

(see [17]), but we do not need it.

**Remark 2.1** We are using many times the natural diffeomorphism $T^*E \cong T^*(E^*)$ for an arbitrary vector bundle $E$ preserving the canonical Poisson brackets up to a sign, which was discovered in [28] and is called the Mackenzie–Xu transformation. See its generalization to the super case and an odd version in [37,40].

### 3 Generating BV operator

#### 3.1 Construction of the operator

Let $M$ be a supermanifold endowed with a $P_\infty$-structure. Our goal is to prove that the following formula gives an operator generating the higher Koszul brackets on $\Omega(M)$:

$$\Delta_P := [d, \hat{P}].$$

(17)

Here, $P$ is the Poisson tensor specifying a $P_\infty$-structure on $M$. Recall that it is an even function on $\Pi T^*M$ satisfying $\{P, P\} = 0$. Notation $\hat{P}$ has the following meaning. For an arbitrary multivector field $P = P(x, x^*)$,

$$\hat{P} := P \left( x, \frac{\hbar}{i} \frac{\partial}{\partial dx} \right).$$

(18)

It is a “vertical” formal $\hbar$-differential operator (see the next subsection) on $\Omega(M)$ canonically corresponding to $P$. (It is very close to the standard notion of the interior product $i(P)$ defined for multivectors of fixed degree, from which it differs by the factor of $(\hbar/i)^k$, where $k$ is the degree.) A quick way of defining $\hat{P}$ is by the Berezin integral

$$(\hat{P} \omega)(x, dx) = \int D(x^*) D(dx') e^{i\pi(dx - dx')x^*} P(x, x^*) \omega(x, dx')$$

(19)

(see more in Sect. 4.1). Integration in (19) over $\Pi T_x^*M \times \Pi T_x M$. Here, notation such as $D(x^*)$ means the coordinate volume element normalized by the factor arising in the inverse Fourier transform.

The exact statement is as follows.

**Theorem 3.1** The operator $\Delta_P$ defined by (17) is a formal $\hbar$-differential operator on the algebra $\Omega(M)$. The sequence of classical brackets generated by $\Delta_P$ is the higher Koszul brackets corresponding to a $P_\infty$-structure given by $P$. 
In other words, we claim that

$$[\omega_1, \ldots, \omega_n]_P = \lim_{\hbar \to 0} (-i\hbar)^{-n}[\ldots [\Delta_P, \omega_1], \ldots, \omega_n](1)$$

(20)

for all \( n = 0, 1, 2, \ldots \) and \( \omega_i \in \Omega(M) \). At the right-hand side, we identify a differential form \( \omega \) with the operator of multiplication by \( \omega \) on the algebra \( \Omega(M) \), and the result is evaluated at \( 1 \in \Omega(M) \).

**Example 3.1** Suppose our \( P_\infty \)-structure is an ordinary Poisson structure, i.e., \( P = \frac{1}{2} P_{ab}(x)x^*_b x^*_a \) is a bivector field. Then,

$$\hat{P} = -\frac{\hbar^2}{2} P_{ab}(x) \frac{\partial}{\partial x^b} \frac{\partial}{\partial x^a}$$

differs by the factor of \(-\hbar^2\) from \( i(P) \). Hence, \( \Delta_P = [d, \hat{P}] = -\hbar^2[d, i(P)] = -\hbar^2 \partial_P \) where \( \partial_P \) is defined by (7). Note that since \( d(1) = i(P)(1) = 0 \), we have \( \Delta_P(1) = 0 \). Consider the right-hand side of (20). Since here \( \hat{P} \) is a second-order differential operator on the algebra \( \Omega(M) \), the commutator \([\ldots [\Delta_P, \omega_1], \ldots, \omega_n]\) gives zero for \( n > 2 \). For \( n = 0, 1, 2 \). For \( n = 0 \), we have \( \Delta_P(1) = 0 \). For \( n = 1 \), we have \([\Delta_P, \omega](1) = \Delta_P(\omega) - (-1)^{\tilde{\omega}_1} \omega \Delta_P(1) = \Delta_P(\omega) - \hbar^2 \partial_P(\omega) \).

Hence,

$$(-i\hbar)^{-1}[\Delta_P, \omega](1) = -i\hbar \partial_P(\omega)$$

and in the limit \( \hbar \to 0 \) we get zero. Finally, for \( n = 2 \), we obtain

$$(-i\hbar)^{-2}[\Delta_P, \omega_1]\omega_2)(1) = [[\partial_P, \omega_1], \omega_2](1)$$

$$= \partial_P(\omega_1 \omega_2) - \partial_P(\omega_1) \omega_2 - (-1)^{\tilde{\omega}_1} \omega_1 \partial_P(\omega_2) = \omega_1, \omega_2]_P,$$

as claimed.

Hence, we conclude that the operator \( \Delta_P \) given by (17) indeed generalizes the Koszul–Brylinski operator \( \partial_P \) given by (7) and formula (20) generalizes Koszul’s formula (8).

**Example 3.2** A slight extension of the previous example can be obtained by allowing a linear term in a \( P_\infty \)-structure: \( P = P_1 + P_2 \), where \( P_1 = P^a(x)x^*_a \) and \( P_2 = \frac{1}{2} P_{ab}(x)x^*_b x^*_a \). It is convenient to introduce a vector field \( Q \in \text{Vect}(M) \) by \( Q = Q^a(x) \partial_a, Q^a = -P^a \). Then, \( Q \) is homological and the \( P_\infty \)-structure given by \( P \) is a differential Poisson structure in the sense that \( P_2 \) defines a usual (binary) Poisson bracket and the homological field \( Q \) is a derivation of the bracket. The operator \( \Delta_P \) corresponding to such \( P \) will have the form \( \Delta_P = [d, \hat{P}] = -i\hbar L_Q - \hbar^2 \partial_{P_2}, \) where \( \partial_{P_2} \) is the Koszul–Brylinski operator and \( L_Q \) is the (odd) operator of the Lie derivative along \( Q \). For the brackets generated by \( \Delta_P \), we get again \( \Delta_P(1) = 0 \), so the 0-bracket is zero; for the 1-bracket,

$$(-i\hbar)^{-1}[\Delta_P, \omega](1) = L_Q(\omega) - i\hbar \partial_P(\omega),$$
so for $\hbar \to 0$,

$$[\omega]_P = L_Q(\omega),$$

and for the 2-bracket, one can see that we obtain

$$[\omega_1, \omega_2]_P = [\omega_1, \omega_2]_{P_2},$$

the Koszul bracket for the Poisson structure $P_2$.

A $P_{\infty}$-structure $P = P_1 + P_2$ of Example 3.2 is the simplest case of the structure given by the Cattaneo-Felder construction [9] on the supermanifold $\Pi E^*$, where $E$ is a vector bundle equipped with an ordinary Poisson structure $\bar{P}$ such that the zero section is coisotropic: If $\bar{P}$ has only terms of degrees $-1$ and $0$ in fiber coordinates on $E$, then on $\Pi E^*$ this gives a homological vector field of degree $+1$ (making $E^*$ a Lie algebroid) and a compatible Poisson bracket of degree $0$.

Formula (17) with a sketch of a brute-force proof was suggested in the appendix to [19]. Here, we prove Theorem 3.1 in a conceptual way. The underlying idea is that the operator (17) should be seen as a “quantization” of the cotangent $L_\infty$-algebroid. We will come back to the main claim in Sect. 3.3. Before that we need to develop some general theory, which also has independent interest.

### 3.2 Formal $\hbar$-differential operators and their symbols

Recall $\hbar$-differential operators. They can be defined algebraically for an arbitrary commutative (super)algebra over a ring of formal power series in $-i\hbar$ or for a module over such an algebra. Basically, they correspond to a concept well known in the theory of linear partial differential operators (see, e.g., Shubin [32]). Such are also the examples arising in quantum mechanics.

Here, Planck’s constant is treated as a formal parameter and not as some small number. So the algebraic definition is as follows [42]. An operator is called $\hbar$-differential of order $n$ (meaning $\leq n$) if its commutator with the multiplication operator by an element of the algebra is $-i\hbar$ times an $\hbar$-differential operator of order $n - 1$, and the operators of negative order are zero. In particular, every $\hbar$-differential operator of order $n$ is a differential operator of order $n$ in the usual sense. Informally, for a partial differential operator $L$ written in local coordinates, the condition that it is $\hbar$-differential means that every partial derivative occurring in $L$ carries the factor of $-i\hbar$, and the coefficients can themselves depend on $\hbar$.

In other words, in a local chart the algebra of $\hbar$-differential operators is generated by arbitrary functions $f = f(x)$ and the momentum operators $\hat{p}_a = -i\hbar \partial / \partial x^a$. There is the “Heisenberg commutation relation”

$$[\hat{p}_a, f] = -i\hbar \frac{\partial f}{\partial x^a}. \quad (21)$$

Here, functions are allowed to be formal power series in $\hbar$ (with non-negative powers). Our operators act on scalar functions, but everything generalizes to operators acting
on sections of a vector bundle over a manifold or supermanifold, and we will use it later.

A formal $h$-differential operator is defined as a formal expression (a non-commutative formal power series)

$$L = L^0(x) + L^a(x) \hat{p}_a + L^{a_1 a_2}(x) \hat{p}_{a_1} \hat{p}_{a_2} + \ldots$$  \hfill (22)

where the coefficients are smooth functions that are also formal power series in $h$. As we shall explain, such sums are invariant under changes of variables. We shall also show how they can be seen as actual operators. But before that, we briefly return to $h$-differential operators, i.e., finite sums.

The commutation relation (21) is homogeneous in $\hat{p}_a$ and $\hbar$ taken together. We introduce a grading in the algebra of $h$-differential operators by defining the total degree of such an operator as the degree in $\hat{p}_a$ and $\hbar$. It is a grading, not a filtration. Elements of the so-obtained graded algebra are $h$-differential operators whose coefficients are polynomials in $\hbar$. Call them $h$-differential operators of finite type. From (21), it is clear that the total degree of an operator of finite type does not depend on its presentation as a non-commutative polynomial in the momentum operators $\hat{p}_a$ with coefficients in the algebra of functions.

**Lemma 3.1** The total degree of an $h$-differential operator of finite type does not depend on a choice of coordinates.

**Proof** The operator $\hat{p}_a$ has the transformation law

$$\hat{p}_a = \frac{\partial x^{a'}}{\partial x^a} \hat{p}_{a'},$$

which is homogeneous, assuming the changes of coordinates not depend on $\hbar$. For a product $\hat{p}_{a_1} \ldots \hat{p}_{a_k}$, we obtain

$$\hat{p}_{a_1} \ldots \hat{p}_{a_k} = \frac{\partial x^{a_1'}}{\partial x^{a_1}} \hat{p}_{a_1'} \ldots \frac{\partial x^{a_k'}}{\partial x^{a_k}} \hat{p}_{a_k'},$$

where it remains to move all the “new” momentum operators to the right of the coefficients. By induction we see that\(^3\)

$$\hat{p}_{a_1} \ldots \hat{p}_{a_k} = \frac{\partial x^{a_1'}}{\partial x^{a_1}} \ldots \frac{\partial x^{a_k'}}{\partial x^{a_k}} \hat{p}_{a_1'} \ldots \hat{p}_{a_k'} + (-i\hbar) R_{k-1} + (-i\hbar)^2 R_{k-2} + \ldots + (-i\hbar)^k R_0,$$

where each $R_s$ is a linear combination of products of exactly $s$ operators $\hat{p}_{a'}$ with some coefficients not depending on $\hbar$ at the left. The claim immediately follows. \(\Box\)

\(^3\) For simplicity, we write them without signs as if in purely even case, but everything holds in the general super case.
Now if we have a formal series such as (22), a formal power series in both $\hat{\rho}$ and $\hbar$, we can re-arrange the summation into an infinite sum over the total degree $n = 0, 1, 2, \ldots$, so that for each $n$ there will be only a finite number of terms:

$$L = \sum_{k=0}^{+\infty} L^{a_1^{r_1} \ldots a_k^{r_k}} (x) \hat{p}_{a_1} \ldots \hat{p}_{a_k} = \sum_{k=0}^{+\infty} \sum_{r=0}^{+\infty} (-i \hbar)^r L^{a_1^{r_1} \ldots a_k^{r_k}} (x) \hat{p}_{a_1} \ldots \hat{p}_{a_k}$$

$$\equiv \sum_{n=0}^{+\infty} \left( L_0^{a_1^{r_1} \ldots a_n^{r_n}} (x) \hat{p}_{a_1} \ldots \hat{p}_{a_n} + (-i \hbar) L_1^{a_1^{r_1} \ldots a_n^{r_n-1}} (x) \hat{p}_{a_1} \ldots \hat{p}_{a_{n-1}} + \ldots + (-i \hbar)^n L_n^0 (x) \right)$$

$$\equiv \sum_{n=0}^{+\infty} L^n [n].$$

(23)

Here, $L^n [n]$ is a finite-type $\hbar$-differential operator of total degree $n$, the component of total degree $n$ of an operator $L$. Hence, formal $\hbar$-differential operators make an algebra, which is the formal completion of the graded algebra of $\hbar$-differential operators of finite type.

**Lemma 3.2**

1. A formal $\hbar$-differential operator $L$ gives rise to a formal power series

$$L \pmod{\hbar} = \sum_{n=0}^{+\infty} L_0^{a_1^{r_1} \ldots a_n^{r_n}} (x) \ p_{a_1} \ldots p_{a_n},$$

(24)

which is a well-defined formal function on $T^* M$, $p_a$ is identified with $\hat{p}_a \pmod{\hbar}$;

2. There is a well-defined action of formal $\hbar$-differential operators on functions which are formal power series in $\hbar$ and this action respects grading;

3. There is a well-defined action of formal $\hbar$-differential operators on functions of the form $e^{\hat{\rho} / \hbar} g(x)$, where $g(x)$ is a formal power series in $\hbar$, which gives products of $e^{\hat{\rho} / \hbar} g(x)$ with formal power series in both $\hbar$ and $\lambda$.

**Proof** In the proof of Lemma 3.2, we observed the transformation law of a typical summand of a formal $\hbar$-differential operator. Modulo $\hbar$, it is the same as for the corresponding monomial in the variables $p_a$. This proves part 1. For part 2, if $L = \sum_{n=0}^{+\infty} L^n [n]$ as in (23) and $f = \sum_{n=0}^{+\infty} (-i \hbar)^n f_n$ is the expansion of a function $f \in C^\infty (M) [[\hbar]]$, then

$$L (f) = \sum_{n=0}^{+\infty} \sum_{r+s=n} (-i \hbar)^s L^r (f_s)$$

and it remains to observe that

$$L^r (f_s) = L_0^{a_1^{r_1} \ldots a_r} \hat{p}_{a_1} \ldots \hat{p}_{a_r} (f) + (-i \hbar) L_1^{a_1^{r_1} \ldots a_{r-1}} \hat{p}_{a_1} \ldots \hat{p}_{a_{r-1}} (f) + \ldots + (-i \hbar)^s L^0 f \times (-i \hbar)^r \left( L_0^{a_1^{r_1} \ldots a_r} \partial_{a_1} \ldots \partial_{a_r} f + L_1^{a_1^{r_1} \ldots a_{r-1}} \partial_{a_1} \ldots \partial_{a_{r-1}} f + \ldots + L^0 f \right),$$

so it is of degree $r$ in $\hbar$. The action of an operator $L^r$ of total degree $r$ on a function of degree $s$ in $\hbar$ gives a function of degree $r + s$ in $\hbar$. As for part 3, consider the action
of \( \hat{p}_a \) on a function of the form \( f e^{\frac{i}{\hbar} \lambda g} \); we obtain

\[
\hat{p}_a \left( f e^{\frac{i}{\hbar} \lambda g} \right) = (-i \hbar \partial_a f + \lambda f \partial_a g) e^{\frac{i}{\hbar} \lambda g}.
\]

Similarly,

\[
\hat{p}_a \hat{p}_b \left( f e^{\frac{i}{\hbar} \lambda g} \right) = \left( (-i \hbar)^2 \partial_a \partial_b f + (-i \hbar) \lambda (\partial_a f \partial_b g + \partial_b f \partial_a g) \right) e^{\frac{i}{\hbar} \lambda g}.
\]

By induction we can see that always \( L^{[n]} \left( f e^{\frac{i}{\hbar} \lambda g} \right) = P_{\hbar, \lambda} e^{\frac{i}{\hbar} \lambda g} \), where \( P_{\hbar, \lambda} \) is a homogeneous polynomial of total degree \( n \) in \( \hbar \) and \( \lambda \) whose term of degree \( r \) in \( \hbar \) is polynomial of degree \( n - r \) in partial derivatives of \( g \) of order \( \leq r + 1 \) and linear in partial derivatives of \( f \).

Instead of writing an object modulo \( \hbar \), we also write \( \lim_{\hbar \to 0} \).

The formal function on \( T^*M \) defined by formula (24) is called the principal symbol of a formal \( \hbar \)-differential operator \( L \) and will be denoted \( \sigma(L) \),

\[
\sigma(L) := \sum_{n=0}^{+\infty} L_0^{a_1 \ldots a_n}(x) p_{a_1} \ldots p_{a_n},
\]

if \( L \) is given by (22), (23). It is different from a (coordinate-dependent) full symbol of \( L \) obtained by formally replacing \( \hat{p}_a \) by \( p_a \) in the expansion (22) without setting \( \hbar \) to zero. Also this principal symbol is different from the principal symbol of a differential operator of order \( \leq n \) (which is a homogeneous polynomial of degree \( n \) corresponding to the top order derivatives).

**Lemma 3.3** For formal \( \hbar \)-differential operators,

\[
\sigma(AB) = \sigma(A)\sigma(B)
\]

(hence \( \sigma(\{A, B\}) = 0 \) for all \( A, B \)). The commutator \( \{A, B\} \) is always divisible by \( \hbar \) and

\[
\sigma(i\hbar^{-1}\{A, B\}) = \{\sigma(A), \sigma(B)\},
\]

where at the right-hand side there is the Poisson bracket on \( T^*M \).

**Proof** Formula (26) is obvious from the definition of principal symbol and the rules of multiplication of formal power series. Since the product of functions on \( T^*M \) is commutative, \( \sigma(\{A, B\}) = [\sigma(A), \sigma(B)] = 0 \). To prove formula (27), since both commutator of operators and Poisson bracket on \( T^*M \) satisfy the Leibniz identity, it is sufficient to check it on the generators such as \( \hat{p}_a \) and \( f(x) \), for which it becomes obvious. \( \square \)

The following definitions were introduced by Th. Voronov [38] as a modification of the construction of Koszul [24]. (See Remark 3.1.)
Definition 3.1 For an operator $L$ on an algebra,
\[
\{f_1, \ldots, f_n\}_{\text{L}, \hbar} := (-i\hbar)^{-n} \ldots [L, f_1], \ldots, f_n \} (1)
\] (28)
is the quantum $n$-bracket and
\[
\{f_1, \ldots, f_n\}_{L} := (-i\hbar)^{-n} \ldots [L, f_1], \ldots, f_n \} (1) \pmod{\hbar}
\] (29)
is the classical $n$-bracket generated by $L$. Here, $n = 0, 1, 2, 3, \ldots$

Here, $f_i$ are functions on a supermanifold or elements of an abstract commutative superalgebra. One has to assume that an $n$-fold commutator $\ldots [L, f_1], \ldots, f_n \}$ in the above formulas is divisible by $(-i\hbar)^n$. In particular, this makes sense for formal $\hbar$-differential operators on supermanifolds as defined here.

Example 3.3 (0-, 1- and 2-brackets) The quantum 0-bracket is simply
\[
\{\emptyset\}_{\text{L}, \hbar} = L(1)
\] (30)
for the quantum 1-bracket take $[L, f](1) = L(f1) - (-1)^{\hat{L}\hat{f}} fL(1) = L(f) - L(1)f$, hence
\[
\{f\}_{\text{L}, \hbar} = i\hbar^{-1}(L(f) - L(1)f)
\] (31)
similarly, for the 2-bracket one has
\[
\{f, g\}_{L, \hbar} = -\hbar^{-2}(L(fg) - L(f)g - (-1)^{\hat{L}\hat{f}} fL(g) + L(1)fg)
\] (32)

Quantum brackets are themselves (formal) differential operators in each argument (but not $\hbar$-differential); moreover, it is known [38] that for any $n$ the $n$-bracket generates the $(n + 1)$-bracket as a “quantum correction” to the Leibniz rule:
\[
\{f_1, \ldots, f_{n-1}, fg\}_{L, \hbar} = \{f_1, \ldots, f_{n-1}, f\}_{L, \hbar} g + (-1)^g \{f_1, \ldots, f_{n-1}, g\}_{L, \hbar} f
\] (33)
where $(-1)^g = (-1)^{\hat{L} + f_1 + \ldots + f_{n-1}}\hat{f}$. Modulo $\hbar$ the extra term disappears, and the resulting classical brackets become multiderivations. Hence, they must correspond to a Hamiltonian.

Theorem 3.2 Let $L$ be a formal $\hbar$-differential operator. The Hamiltonian $H$ for the classical brackets generated by $L$ is the principal symbol of $L$, $H = \sigma(L)$.

Proof We need to prove the identity
\[
(-i\hbar)^{-n} \ldots [L, f_1], \ldots, f_n \} (1) \pmod{\hbar} = \{\ldots [H, f_1], \ldots, f_n\}_{M}
\] (34)
where \( H = \sigma(L) \in C^\infty(T^*M) \), \( f_i \in C^\infty(M) \), and the brackets at the right-hand side are the Poisson brackets on \( T^*M \). Indeed, we observe that for an arbitrary formal \( \hbar \)-differential operator \( A \),

\[
A(1) \pmod{\hbar} = \sigma(A)_{|M}
\]

(application to 1 gives the free term of the operator \( A \), which modulo \( \hbar \) is the zeroth term in the expansion of the principal symbol). Hence, by induction,

\[
(-i\hbar)^{-n} \left[ \ldots [L, f_1], \ldots, f_n \right] (1) \pmod{\hbar} = \sigma \left( i\hbar^{-1} \left[ i\hbar^{-1} \left[ \ldots i\hbar^{-1} [L, f_1], \ldots, f_{n-1} \right], f_n \right] \right)_{|M}
\]

where we used formula (27).

Let a formal \( \hbar \)-differential operator \( \Delta \) be odd and satisfy \( \Delta^2 = 0 \). Since \( \Delta^2 = \frac{1}{2} [\Delta, \Delta] \), from Theorem 3.2 and Lemma 3.3 it follows that the corresponding odd Hamiltonian \( H := \sigma(\Delta) \) satisfies \( \{H, H\} = 0 \) and therefore, the classical brackets (29), with \( L = \Delta \), form an \( S_\infty \)-algebra. Moreover, \( \Delta^2 = 0 \) implies (in fact, is equivalent to) that the quantum brackets (28) generated by \( \Delta \) satisfy themselves the higher Jacobi identities and so form an \( L_\infty \)-algebra [38], though they are no longer multiderivations of the associative product and satisfy instead relation (33). Such a structure introduced in [38] is called in [42, §5] an \( S_{\infty, \hbar} \)-algebra.

**Remark 3.1** (on history and terminology) A sequence of multilinear operations \( \Phi^n_L \) for an operator \( L \) on a graded-commutative algebra was first introduced by Koszul [24]. They are basically (28) without division by \( (-i\hbar)^n \). Koszul himself was mostly interested in the case of an odd second-order operator \( \Delta \). He established an identity linking the failure of Jacobi for \( \Phi^3_\Delta \) with \( \Phi^3_\Delta \) and \( \Phi^3_\Delta \) and basically meaning that if \( \Delta^2 = 0 \), then the operation \( \Phi^3_\Delta \) satisfies Jacobi identity up to a chain homotopy with \( \Phi^3_\Delta \) as the homotopy operator. Hence, if \( \Delta \) is of second order and \( \Delta^2 = 0 \), it generates an odd Poisson (=Schouten or Gerstenhaber) bracket. Such operators later became known as Batalin–Vilkovisky (BV) operators. A graded commutative algebra with a BV operator is called a BV algebra.

In this paper, we use the name “Batalin–Vilkovisky operator” in a broader sense including operators of higher order. The study of an analogue of BV algebras based on a higher-order operator \( \Delta \) such that \( \Delta^2 = 0 \) was initiated by O. Kravchenko in a seminal paper [25]. She called the obtained structure a BV\( _{\infty} \)-algebra. Kravchenko noticed that the condition \( \Delta^2 = 0 \) is equivalent to the sequence of higher Jacobi identities for the sequence of brackets \( \Phi^n_L \) [25, Prop. 2], so that BV\( _{\infty} \) implies \( L_{\infty} \).

(Note that there are also more general notions of a homotopy Gerstenhaber [34] and homotopy BV algebras [33], which we do not need here.) In [38], Th. Voronov put forward a general algebraic mechanism leading to \( L_{\infty} \)-algebras, for which Koszul’s construction of brackets and Kravchenko’s theorem are a particular example. The
modification of Koszul’s definition by the factor of \((-i\hbar)^n\) was suggested in [38] to obtain a deformation of an \(S_\infty\)-structure. We took the notion of \(S_\infty, h\)-algebras from [42]. It is very close to \(BV_\infty\)-algebras in the sense of Kravchenko, but the difference is that it is based on \(h\)-differential operators, which is essential for our purposes.

### 3.3 Main statement. “Quantum” and “classical” higher Koszul brackets

We can apply the above considerations to the situation where on a supermanifold \(M\) there is an \(S_\infty\)-structure specified by an odd Hamiltonian \(H\) satisfying the “classical master equation” \(\{H, H\} = 0\). If there is an odd formal \(\hbar\)-differential operator satisfying \(\Delta^2 = 0\) such that the odd brackets on \(M\) coincide with the classical brackets generated by \(\Delta\), the operator \(\Delta\) is called a Batalin–Vilkovisky operator for a given \(S_\infty\)-structure. (See also the remark above.) By Theorem 3.2, then \(\sigma(\Delta) = H\). Hence finding \(\Delta\) for a given \(S_\infty\)-structure, i.e., lifting it to an \(S_\infty, h\)-structure, is a “quantization problem” and \(\Delta\) is not unique (since an operator \(\Delta\) contains more data than its principal symbol).

Now return to our particular problem. Our goal is to find a Batalin–Vilkovisky operator for the higher Koszul brackets on \(\Omega_1(M) = C_\infty(\Pi T^* M)\) induced by a \(P_\infty\)-structure on \(M\).

Recall from Sect. 2.2 that the odd master Hamiltonian for the higher Koszul brackets is

\[
H_p = \{ D, P^* \},
\]

where \(D = dx^a p_a\) and \(P^* \in C_\infty(T^*(\Pi T^* M))\) is obtained from \(P \in C_\infty(\Pi T^* M)\) \(\subset C_\infty(T^*(\Pi T^* M))\) by the Mackenzie–Xu transformation. See formulas (14), (15). In local coordinates, if \(P = P(x, x^*)\), then \(P^* = P(x, \pi)\), with \(\pi_a\) being the momenta canonically conjugate with \(dx^a\).

Both Hamiltonians \(D\) and \(P^*\) have natural quantizations.

**Example 3.4** The operator \(-i\hbar d = -i\hbar dx^a \frac{\partial}{\partial x^a}\) is a quantization of the Hamiltonian \(D = dx^a p_a\), i.e., \(\sigma(-i\hbar d) = D\).

**Example 3.5** For an arbitrary function \(P \in C_\infty(\Pi T^* M)\), the operator \(\hat{P} = P(x, -i\hbar \frac{\partial}{\partial x^a})\) is a quantization of the Hamiltonian \(P^* = P(x, \pi)\), i.e., \(\sigma(\hat{P}) = P^*\).

The following statement is a generalization of Cartan’s identity.

**Lemma 3.4** For arbitrary \(T, S \in C_\infty(\Pi T^* M)\),

\[
[[d, \hat{T}], \hat{S}] = [\hat{T}, S]\hat{\gamma}.
\]

**Proof** Direct calculation using formula (19). Or, alternatively, choose a volume element \(\rho\) and take fiberwise \(\hbar\)-Fourier transform of the left-hand side of (36) and obtain \([[\delta, \hat{T}], \hat{S}]\) where \(\delta = \delta_\rho\) is the divergence operator on multivector fields. But since \(\delta\) is a differential operator of second order, \([[\delta, \hat{T}], \hat{S}]\) is a differential operator of order zero, i.e., a multivector field. Then, \([[\delta, \hat{T}], \hat{S}] = [[\delta, \hat{T}], S](1) = \delta(T S) - \delta(T) S - (-1)^{\hat{T}} T \delta(S) = [\hat{T}, S]\). \(\square\)
Now everything is ready for the main statement.

**Theorem 3.3** (a stronger version of Theorem 3.1) The operator $\Delta_P = [d, \hat{P}]$ is a Batalin–Vilkovisky operator for the $S_\infty$-structure on $\Pi T M$ induced by a $P_\infty$-structure on $M$ (i.e., for higher Koszul brackets).

**Proof** There are two statements: that $\Delta_P$ indeed generates the Koszul brackets (as the classical brackets) and that $\Delta_P^2 = 0$. For the first statement, consider the principal symbol of $\Delta_P$. By Lemma 3.3,

$$\sigma(\Delta_P) = \sigma([d, \hat{P}]) = \sigma((-i\hbar)^{-1} [-i\hbar d, \hat{P}]) = \{\sigma(-i\hbar d), \sigma(\hat{P})\} = \{D, P^*\}.$$

(Note that $d$ is not an $\hbar$-differential operator, so one cannot mistakenly decide that $\sigma([d, \hat{P}])$ is zero!) For the second statement, consider $\Delta_P^2$. We have

$$[[d, \hat{P}], \hat{P}] = 0$$

(since by Lemma 3.4, the left-hand side is $[[P, P]]$). By applying the commutator with $d$, we obtain

$$0 = [d[d, \hat{P}], \hat{P}] = \pm[[d, \hat{P}], [d, \hat{P}]]$$

(since $d^2 = 0$). But this is exactly $[\Delta_P, \Delta_P] \equiv 2\Delta_P^2 = 0$.

Note that we also obtain quantum Koszul brackets as the quantum brackets generated by $\Delta_P$,

$$[\omega_1, \ldots, \omega_n]_{P, \hbar} := (-i\hbar)^{-n} \ldots [\Delta_P, \omega_1], \ldots, \omega_n](1). \quad (37)$$

This is a useful notion even for the ordinary Poisson case.

**Example 3.6** Let $P$ be a bivector field defining an ordinary Poisson structure on a supermanifold $M$. From calculations in Example 3.1, $\Delta_P = -\hbar^2 \partial_P$ (where $\partial_P$ is the Koszul–Brylinski operator) and we can see that

$$[\emptyset]_{P, \hbar} = 0,$$

$$[\omega]_{P, \hbar} = -i\hbar \partial_P(\omega),$$

$$[\omega_1, \omega_2]_{P, \hbar} = [\omega_1, \omega_2]_P.$$

Quantum and classical Koszul 2-brackets coincide because $\Delta_P$ is of second order. All the higher brackets are zero. In particular, from the quantum viewpoint, the Koszul–Brylinski operator (with the factor of $-i\hbar$) is itself part of the sequence of brackets and the known derivation property [24]

$$\partial_P[\omega_1, \omega_2]_P = -[\partial_P \omega_1, \omega_2]_P + (-1)^{\omega_1 + 1} [\omega_1, \partial_P \omega_2]_P$$

becomes part of higher Jacobi identities for quantum Koszul brackets.
3.4 “Quantum cotangent $L_\infty$-bialgebroid”

Construction of the BV operator $\Delta P$ generating the higher Koszul brackets on $\Omega(M) = C^\infty(\Gamma TM)$ can be interpreted as a “quantization” of the cotangent $L_\infty$-algebroid structure. We will see how to extend that to a “quantization” of an $L_\infty$-bialgebroid.

The cotangent $L_\infty$-algebroid is indeed an $L_\infty$-bialgebroid. In the manifestation on $\Pi TM$, that means that $D$, the master Hamiltonian for the de Rham differential, and $H_P$, the master Hamiltonian for the higher Koszul brackets, make a commuting pair.

This is equivalent to the odd Hamiltonian

$$D_t = D + th_P$$

(39)

depending on parameter $t \in \mathbb{R}$ satisfying $[D_t, D_t] = 0$ for all $t$. In terms of the brackets, this is the derivation property for $d$ and all the higher Koszul brackets.

This lifts to the “quantum level” as follows.

**Theorem 3.4** For every $t$, the formal $\hbar$-differential operator

$$\hat{D}_t = -i\hbar d + t\Delta P.$$  

(40)

is a Batalin–Vilkovisky operator which is a quantum lift of the Hamiltonian $D_t = D + tH_P$. The operator $\hat{D}_t$ can be also written as

$$\hat{D}_t = e^{-\frac{i}{\hbar}t\hat{P}}(-i\hbar d)e^{\frac{i}{\hbar}t\hat{P}}.$$  

(41)

**Proof** It is clear that $\sigma(\Delta_t) = D_t$, so $\hat{D}_t$ is a quantization of the master Hamiltonian $D_t$. We need to show that $\hat{D}_t^2 = 0$. Indeed, $d^2 = 0$ and we know that $\Delta_P^2 = 0$, so we need $[d, \Delta_P] = 0$. But $[d, \Delta_P] = [d, [d, \hat{P}]] = (ad d)^2(\hat{P}) = 0$. Finally, we need to establish the identity

$$-i\hbar d + t\Delta P = e^{-\frac{i}{\hbar}t\hat{P}}(-i\hbar d)e^{\frac{i}{\hbar}t\hat{P}}.$$  

Indeed,

$$e^{-\frac{i}{\hbar}t\hat{P}}(-i\hbar d)e^{\frac{i}{\hbar}t\hat{P}} = e^{-\frac{i}{\hbar}t(ad \hat{P})(-i\hbar d)} = -i\hbar d - \frac{i}{\hbar}t(ad \hat{P})(-i\hbar d)$$

$$= -i\hbar d - t(ad \hat{P})(d)$$

$$= -i\hbar d - t[\hat{P}, d] = -i\hbar d + t[d, \hat{P}] = -i\hbar d + t\Delta P$$

because

$$(ad \hat{P})^2(d) = [\hat{P}, [\hat{P}, d]] = \pm \{P, P\} = 0.$$  

\[\square\]

We considered the cotangent $L_\infty$-bialgebroid in the manifestation on $\Pi TM$ and for it constructed a quantization. Let us see how this can be done in the dual picture on $\Pi T^* M$ and how these pictures will be related “on the quantum level.”
On $\Pi T^*M$, the roles of brackets and homological vector field are swapped compared to $\Pi TM$: Instead of $d$, there is the canonical Schouten bracket; and instead of the higher Koszul brackets, there is the Lichnerowicz differential $d_P$. A Batalin–Vilkovisky operator for the Schouten bracket is $-\hbar^2 \delta$, where $\delta = \delta_\rho$ is the divergence operator constructed with the help of some volume element $\rho$ on $M$. Hence, the operator

$$ -\hbar^2 \delta + t(-i\hbar)d_P $$

(42)

seems a natural choice for a Batalin–Vilkovisky operator on $\Pi T^*M$ for the $L_\infty$-bialgebroid structure. However, (42) does not work because this operator does not in general square to zero. Indeed, although $\delta^2 = 0$ and $d_P^2 = 0$, we have for $[\delta, d_P]$

$$ [\delta, d_P](T) = \delta [P, T] + \|P, \delta(T)] = -\|\delta(P), T\| - \|P, \delta(T]\| + \|P, \delta(T)\| $$

$$ = -\|\delta(P), T\|. $$

Hence, unless $\delta(P)$ is zero, the operators $\delta$ and $d_P$ do not commute and $(-\hbar^2 \delta + t(-i\hbar)d_P)^2 \neq 0$. One recognizes in $\delta(P)$ a representative of the modular class of a $P_\infty$-structure: a cohomology class $[\delta(P)] \in H^*(\mathfrak{A}(M), d_P)$ defined with the help of a volume element $\rho$ but independent of a choice of $\rho$. So a different choice of $\rho$ does not solve the problem if the modular class $[\delta(P)]$ is nonzero.

**Remark 3.2** The modular class of a $P_\infty$-structure $[\delta(P)] \in H^*(\mathfrak{A}(M), d_P)$ is directly analogous to the constructions for ordinary Poisson manifolds [45], Lie algebroids [10] and $Q$-manifolds [26,39].

The situation can be remedied by taking on $\Pi T^*M$ the operator

$$ \hat{D}_P := -i\hbar(d_P + \delta(P)) $$

(43)

instead of $-i\hbar d_P$. This does not change the principal symbol. The term $-i\hbar \delta(P)$ is a “quantum correction.”

**Lemma 3.5** One can express

$$ \hat{D}_P = -i\hbar[\delta, P]. $$

(44)

The operator $\hat{D}_P$ has square zero.

**Proof** We have $[\delta, P](T) = \delta(P) T - P \delta(T) = \delta(P) T + P \delta(T) + \|P, T\| - P \delta(T) = \delta(P) T + \|P, T\| = (d_P + \delta(P))(T)$. Now, $(d_P + \delta(P))^2 = d_P^2 + (\delta(P))^2 + [d_P, \delta(P)] = [d_P, \delta(P)] + d_P(\delta(P)) = \|P, \delta(P)\| = \pm \delta \|P, P\| = 0. \square$

**Theorem 3.5** For every $t$, the operator

$$ \hat{D}_t^* = -\hbar^2 \delta + t\hat{D}_P $$

(45)

is a Batalin–Vilkovisky operator for the $L_\infty$-bialgebroid structure on $\Pi T^*M$. The operator $\hat{D}_t^*$ can be also written as

$$ \hat{D}_t^* = e^{-\frac{t}{\hbar^2}P}(-\hbar^2 \delta)e^{\frac{t}{\hbar^2}P}. $$

(46)
Proof The principal symbol of $\hat{D}_t^*$ is

$$\sigma(\hat{D}_t^*) = \sigma(-\hbar^2 \delta) - t \sigma(i \hbar d_P + i \hbar \delta(P)) = \sigma(-\hbar^2 \delta) - t \sigma(i \hbar d_P) = D_t^* H_P^*.$$ 

as claimed. For $(\hat{D}_t^*)^2$, we have

$$(\hat{D}_t^*)^2 = (-\hbar^2 \delta + t \hat{D}_t) = -\hbar^2 t [\delta, \hat{D}_t] = 0.$$

Finally,

$$e^{-\frac{i}{\hbar} P (-\hbar^2 \delta)}e^{\frac{i}{\hbar} t P} = e^{-\frac{i}{\hbar} t \text{ad} P (-\hbar^2 \delta)} = -\hbar^2 \delta - \frac{i}{\hbar} t \text{ad} P (-\hbar^2 \delta) = \hat{D}_t^*.$$

Classical description of the cotangent $L_\infty$-bialgebroid structure is symmetric with respect to $\Pi T M$ and $\Pi T^* M$, and one can be obtained from another by the Mackenzie–Xu transformation.

On the level of the constructed BV operators, our construction loses this symmetry. To amend this, we will introduce operators acting on half-densities instead of functions.

4 “Symmetric theory”

In this section, we first develop a general theory of operators acting on densities for dual vector bundles. Here, the main results are Theorems 4.1 and 4.2. Secondly, we explain how to get brackets on functions from an operator acting on a one-dimensional module (such as half-densities). Finally, we arrive to Theorem 4.3, which gives the desired fully symmetric (in both manifestations on $\Pi T M$ and $\Pi T^* M$) description of the quantum cotangent $L_\infty$-bialgebroid.

4.1 Dualization for operators on vector bundles

Our goal here is to develop convenient tools for working with operators on dual vector bundles such as $\Pi T M$ and $\Pi T^* M$. It will be based on fiberwise $\hbar$-Fourier transform that we will introduce below.

Let $E \to M$ be a vector bundle, $E$ and $M$ are (super)manifolds. Informally, we want to describe the dual space $(C^\infty(E))'$ to the space of functions $C^\infty(E)$ in terms of geometric objects on the dual bundle $E^* \to M$. It is convenient to work in a slightly more general setting, namely to consider densities instead of functions.

On a vector bundle $E$ consider densities of weight $(\lambda, \mu)$ as objects of the form

$$f(x,u)Dx^\lambda Du^\mu$$

in local coordinates, where $x^a$, $u^i$ are coordinates on the base and the fiber respectively. Here, by $Dx$ we denote the (Berezin) coordinate volume element which transforms
according to the formula $Dx = (Dx/Dx')Dx'$, where $Dx/Dx' = \text{Ber}(\partial x/\partial x')$, and similarly for $Du$.\(^4\) Denote this space of densities $\text{Dens}_{\lambda,\mu}(E)$. In particular, $\text{Dens}_{\lambda,\lambda}(E) = \text{Dens}_{\lambda}(E)$, where $\text{Dens}_{\lambda}(E)$ is the usual space of $\lambda$-densities on a supermanifold $E$. (Unlike [30] we will be considering operators on densities of fixed weight, not on the algebra of densities.)

Let $f(x, u)Dx^\lambda Du^\mu \in \text{Dens}_{\lambda,\mu}(E)$. Introduce its fiberwise $h$-Fourier transform by:

$$F[f(x, u)Dx^\lambda Du^\mu] = \left( \int_E e^{-\frac{i}{\hbar}u^i w_i} f(x, u) Du \right) Dx^\lambda Dw^{1-\mu}.$$ \hspace{1cm} (47)

Here, $w_i$ are coordinates in the fiber of $E^*$ such that the bilinear form $u^i w_i$ is invariant. In (47), we use the identification $Du^{\mu-1} = Dw^{1-\mu}$. It establishes an isomorphism

$$F: \text{Dens}_{\lambda,\mu}(E) \rightarrow \text{Dens}_{\lambda,1-\mu}(E^*),$$ \hspace{1cm} (48)

which holds for all $\lambda$ and $\mu$. Together with the natural identification $(\text{Dens}_{\lambda,\mu}(E))' \cong \text{Dens}_{1-\lambda,1-\mu}(E)$ (here the dual of a functional space can mean its "smooth part" or we can simply agree provisionally not to keep track of the smoothness), this gives rise to an isomorphism, which we denote by the same letter $F$,

$$F: (\text{Dens}_{\lambda,\mu}(E))' \rightarrow \text{Dens}_{1-\lambda,\mu}(E^*).$$ \hspace{1cm} (49)

(here we applied (48) for $1 - \lambda, 1 - \mu$ instead of $\lambda, \mu$). To put it differently, the isomorphism (49) is equivalent to a non-degenerate pairing

$$\text{Dens}_{\lambda,\mu}(E) \times \text{Dens}_{1-\lambda,\mu}(E^*) \rightarrow \mathbb{C}$$ \hspace{1cm} (50)

given by the integral

$$\int_E e^{-\frac{i}{\hbar}u^i w_i} f(x, u) g(x, w) Du Dw Dx.$$ \hspace{1cm} (51)

Here, $f(x, u)Dx^\lambda Du^\mu \in \text{Dens}_{\lambda,\mu}(E)$ and $g(x, w)Dx^{1-\lambda} Dw^{\mu} \in \text{Dens}_{1-\lambda,\mu}(E^*)$.

The fiberwise $h$-Fourier transform (47) or the pairing that it provides (51) makes it possible to consider the formal duals to linear maps between functions or densities on vector bundles as linear maps between the suitable densities on the dual bundles: If

$$L: \text{Dens}_{\lambda,\mu}(E_1) \rightarrow \text{Dens}_{\lambda,\mu}(E_2),$$ \hspace{1cm} (52)

\(^4\) On supermanifolds there are more types of orientation conditions because different combinations of signs in $\text{Ber}_{\alpha,\beta} J := (\text{sgn det } J_{00})\alpha (\text{sgn det } J_{11})^\beta \text{Ber } J$, where $J$ is the Jacobi matrix of a change of coordinates, give different analogues of det $J$ and $| \text{det } J |$ of the ordinary case. Change of variables in Berezin integral includes $\text{Ber}_{1,0} J$, hence $\text{det } J_{00} > 0$ is the orientability condition required for integration of densities of the form $f(x)dx$ over a supermanifold. Integration of pseudodifferential forms $\omega(x, dx)$ requires a different condition, $\text{Ber } J > 0$. See [36]. There are the corresponding types of densities whose transformation laws include powers of $\text{Ber}_{\alpha,\beta} J$. In the case of super fiber bundles, the number of orientation and density types becomes even larger. These distinctions are not relevant for our purposes, so we completely ignore them and in particular will write $Dx^\lambda$ etc., instead of a more refined notation.
then
\[ L^*: \text{Dens}_{1-\lambda,\mu}(E^*_2) \to \text{Dens}_{1-\lambda,\mu}(E^*_1). \] (53)

If \( L' \) stands for the usual formal dual treated as a map
\[ \text{Dens}_{1,1-\mu}(E_2) \cong (\text{Dens}_{\lambda,\mu}(E_2))^\vee \to (\text{Dens}_{\lambda,\mu}(E_1))^\vee \cong \text{Dens}_{1-\lambda,1-\mu}(E_1), \] (54)
then
\[ L^* = F_1 \circ L' \circ F_2^{-1}, \] (55)
where \( F_1 = F_{E_1} \) and \( F_2 = F_{E_2} \) are the Fourier transforms for \( E_1 \) and \( E_2 \), respectively.

One particular useful case if that of half-densities, where an \( L: \text{Dens}_{1/2}(E_2) \) induces the \( L^*: \text{Dens}_{1/2}(E^*_2) \to \text{Dens}_{1/2}(E^*_1) \). Another option is to make use of a volume element for the base \( M \) in order to identify “base \( \lambda \)-densities” with “base 0-densities.” This would allow to introduce a \( \rho \)-dependent dual \( L^*_\rho \), where \( \rho \in \text{Vol}(M) \) is a chosen volume element. If, for example,
\[ L: \mathcal{C}^\infty(E_1) \to \mathcal{C}^\infty(E_2), \] (56)
then
\[ L^*_\rho = \rho^{-1} \circ L^* \circ \rho: \mathcal{C}^\infty(E^*_2) \to \mathcal{C}^\infty(E^*_1). \] (57)

We will apply these constructions to \( \hbar \)-differential operators on vector bundles and also to some special Fourier integral type operators (see below).

Before doing that, consider gradings. Any object on a vector bundle has a natural \( \mathbb{Z} \)-grading, which following [37] we call weight and denote \( \mathbf{w} \). So on \( E \), with fiber coordinates \( u^i \), we have \( \mathbf{w}(u^i) = +1 \) and \( \mathbf{w}(\partial/\partial u^i) = -1 \). Also, if \( w_i \) are the corresponding dual fiber coordinates for \( E^* \), then \( \mathbf{w}(w_i) = -1 \). (This is counting weights relative \( E \), \( \mathbf{w} = \mathbf{w}_E \). Of course, for \( E^* \) considered on its own, \( \mathbf{w}_E = -\mathbf{w}_{E^*} \).)

Dealing with densities, one needs to take into account the weights of coordinate volume elements. If we have \( n \) even and \( m \) odd variables among \( u^i \), then
\[ \mathbf{w}(Du) = n - m. \]
Respectively, if we need to use \( \delta \)-functions, then
\[ \mathbf{w}(\delta(u)) = -n + m. \]

The integral symbol \( \int \) has weight zero. The fiberwise \( \hbar \)-Fourier transform (47) and the pairing (50) behave nicely with respect to grading.

**Lemma 4.1** *The fiberwise \( \hbar \)-Fourier transform*
\[ F: \text{Dens}_{\lambda,\mu}(E) \to \text{Dens}_{\lambda,1-\mu}(E^*) \] (58)
*preserves grading given by \( \mathbf{w}_E \). Formula (50) gives a non-degenerate pairing of elements of weight \( \alpha \) in \( \text{Dens}_{\lambda,\mu}(E) \) with elements of weight \(-\alpha \) in \( \text{Dens}_{1-\lambda,\mu}(E^*) \).*
Proof} Note that weights of densities need not be integral: for elements of $\mathcal{Dens}_{\lambda,\mu}(E)$ they take values in $\mathbb{Z} + \mu(n-m)$, where $\dim E_x = n|E|$. So $w(f(x,u)Dx^k Du^\ell) = \#u^i + \mu(n-m)$. One immediately checks that $w(F[f(x,u)Dx^k Du^\ell]) = \#u^i + n - m + (1 - \mu)(-n + m) = \#u^i + \mu(n-m) = w(f(x,u)Dx^k Du^\ell)$. (Note that $w(u^i w_i) = 0$.) Similarly for the pairing given by (50), if $\#u^i(f) = k$, then for the integral to be non-zero, it should be that $\#w_i(g) = k$, i.e., $w_E(g) = -k$. So the subspace of elements of weight $k + \mu(n-m)$ in $\mathcal{Dens}_{\lambda,\mu}(E)$ is non-degenerately paired with the subspace of elements of weight $-k + \mu(-n+m)$ in $\mathcal{Dens}_{1-\lambda,\mu}(E^*)$.

Denote by $\text{DO}_\hbar(\mathcal{Dens}_{\lambda,\mu}(E))$ the algebra of finite type $\hbar$-differential operators on $(\lambda, \mu)$-densities on $E$ with fiberwise-polynomial coefficients.

There are two natural gradings defined on elements of this algebra: one by total degree of operator (see Sect. 3.2) and another by weight coming from the vector bundle structure of $E$. We can write them as

\[
\deg_E(L) = \#\hat{p}_a + \#\hat{p}_i + \#\hbar \quad (59)
\]

\[
w_E(L) = \#u^i - \#\hat{p}_i \quad (60)
\]

(where # denotes the degree in given variables). It is possible to introduce another invariant grading as the sum $\deg_E^* := \deg_E + w_E$, or

\[
\deg_E^*(L) = \#\hat{p}_a + \#u^i + \#\hbar \quad (61)
\]

We can consider the algebra $\text{DO}_\hbar(\mathcal{Dens}_{\lambda,\mu}(E))$ as bi-graded by $(\deg_E, \deg_E^*)$. Note that $\deg_E, \deg_E^* \geq 0$. Define now the algebra $\widehat{\text{DO}}_h(\mathcal{Dens}_{\lambda,\mu}(E))$ as the formal completion of the bi-graded algebra $\text{DO}_h(\mathcal{Dens}_{\lambda,\mu}(E))$. Elements of $\widehat{\text{DO}}_h(\mathcal{Dens}_{\lambda,\mu}(E))$ are formal $\hbar$-differential operators with fiberwise formal coefficients. We simply call them “formal $\hbar$-differential operators” on $E$.

**Theorem 4.1** Taking dual $L \mapsto L^*$ is an anti-isomorphism of algebras

\[
\widehat{\text{DO}}_h(\mathcal{Dens}_{\lambda,\mu}(E)) \rightarrow \widehat{\text{DO}}_h(\mathcal{Dens}_{1-\lambda,\mu}(E^*)) ,
\]

\[
(L_1 L_2)^* = (-1)^{\hat{L}_1 \hat{L}_2} L_2^* L_1^* .
\]

which maps the subalgebra $\text{DO}_h(\mathcal{Dens}_{\lambda,\mu}(E))$ on the subalgebra $\text{DO}_h(\mathcal{Dens}_{1-\lambda,\mu}(E^*))$ and swaps the bi-grading: for every $L \in \text{DO}_h(\mathcal{Dens}_{\lambda,\mu}(E))$

\[
\deg_{E^*}(L^*) = \deg_E^*(L) \quad \text{and} \quad \deg_{E^*}(L^*) = \deg_E(L) .
\]

In the limit $\hbar \to 0$, the map $L \mapsto L^*$ induces an algebra isomorphism

\[
C^\infty(T^* E) \to C^\infty(T^* (E^*))
\]

---

5 We attach the subscript $E$ to stress the relation with the bundle $E$. 

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which is the pull-back by the Mackenzie–Xu antisymplectomorphism $T^*(E^*) \to T^*E$.

**Proof.** The fact that it is an anti-isomorphism follows from the definition, as $(L_1 L_2)' = (-1)^{L_1 L_2} L_2' L_1'$. To prove the rest, one needs to see the images of the local generators.

By integration by parts, one gets $(\partial / \partial x^a)^* = -\partial / \partial x^a$, $(\partial / \partial u^i)^* = \frac{i}{\hbar} w_i$, $(u^i)^* = (-1)^i \frac{\hbar}{\pi} \partial / \partial w_i$. In other words,

$$
(\hat{p}_a)^* = -\hat{p}_a, \quad (\hat{p}_i)^* = w_i, \quad (u^i)^* = \hat{p}_i (-1)^i
$$

and since $(f(x))^* = f(x)$, it is an anti-isomorphism of $C^\infty(M)$-algebras. The statements concerning gradings follow immediately.

The relation with the Mackenzie–Xu transformation $T^*(E^*) \to T^*E$ is clear from the above formulas (as, without hats, they give the coordinate expression of this map). Indeed, since $L \mapsto L^*$ is an anti-isomorphism of $C^\infty(M)$-algebras, it remains so modulo $\hbar$. Hence, the induced map (65) is an isomorphism of commutative $C^\infty(M)$-algebras, so it must arise from some diffeomorphism $T^*(E^*) \to T^*E$ over $M$. That it coincides with the Mackenzie–Xu diffeomorphism, follows from (66). Finally, since the map of algebras of formal $\hbar$-differential operators is an anti-isomorphism, it takes the commutator of operators to the negative of the commutator. Modulo $\hbar \to 0$ this gives that the Poisson bracket on $T^*E$ is mapped to the negative of the Poisson bracket on $T^*(E^*)$. (So we in particular recover the antisymplectomorphism property of the Mackenzie–Xu map, which is of course clear directly.)

**Remark 4.1** In the classical limit, the bi-grading by $(\deg_E, \deg_{E^*})$ becomes the bi-grading associated with the double vector bundle structure:

$$
T^*E \cong T^*(E^*) \longrightarrow E^* \quad \downarrow \quad \downarrow
$$

$$
\begin{array}{ccc}
E & \longrightarrow & M,
\end{array}
$$

$$
\deg_E = \#p_a + \#p_i = \#p_a + \#w_i, \quad \text{and} \quad \deg_{E^*} = \#p_a + \#p^i = \#p_a + \#u^i.
$$

As mentioned, the “most symmetric” picture is obtained for half-densities, $\lambda = \mu = 1/2$. An example particularly interesting for us in this paper is that of $E = \Pi TM$. Then $D(x, dx)$ is an invariant volume element, so on $\Pi TM$ one can identify densities of any weight with just functions, i.e., pseudodifferential forms on $M$. On the other hand, for $\Pi T^*M$, the volume element $D(x, x^*)$ transforms as $(Dx)^2$. Hence, half-densities on $\Pi T^*M$ can be identified with multivector densities on $M$, i.e., (pseudo)integral forms on $M$. They can be written as $\sigma(x, x^*)dx$. The pairing (51) takes the form

$$
\int_{\Pi TM} D(x, dx) e^{-\frac{i}{\hbar} dx^a x^a} \omega(x, dx) \sigma(x, x^*).
$$

Here, $\omega(x, dx) \in \Omega(M) \cong \text{Dens}_{1/2}(\Pi TM)$ and $\sigma(x, x^*)dx \in \mathcal{A}(M, \text{Vol} M) \cong \text{Dens}_{1/2}(\Pi T^*M)$.
**Example 4.1** We have for operators on the algebra $\Omega(M)$, from the pairing (67), $(\partial/\partial x^a)^* = -\partial/\partial x^a$, $(dx^a)^* = -\frac{\hbar}{i}\partial/\partial x^a(-1)^a$ and $(\partial/\partial dx^a)^* = \frac{\hbar}{i}x^a$. We obtain, in particular,

$$(i\hbar\partial)^* = -\hbar^2\delta$$

(68)

where

$$\delta = (-1)^a \frac{\partial}{\partial x} \frac{\partial}{\partial x^a}$$

(69)

and

$$P \left( x, \frac{\hbar}{i} \frac{\partial}{\partial dx} \right)^* = P(x, x^*).$$

(70)

(Note that here unlike Sect. 3, $\delta$ is an operator on multivector densities and does not require a choice of a volume form on $M$ for its definition.)

Consider now an operator $L: \text{Dens}_{\lambda,\mu}(E_2) \to \text{Dens}_{\lambda,\mu}(E_1)$, $L: f_2(x, u_2)dx^\lambda Du_2^\mu \mapsto f_1(x, u_1)Dx^\lambda Du_1^\mu$, expressed by an integral formula

$$f_1(x, u_1) = \int Du_2 Dw_2 e^{\frac{i}{\hbar}(S(u_1|w_2) - u_2w_2)} f_2(x, u_2).$$

(71)

Here, we use $u_1^i$ and $u_2^g$ for fiber coordinates for $E_1$ and $E_2$, $w_1 = (w_1i)$ and $w_2 = (w_2\alpha)$ for fiber coordinates in $E_1^*$ and $E_2^*$. The function $S(u_1|w_2)$ in the phase of the exponential is called “generating function.” It is a formal power series in $w_2$. (It has a non-trivial transformation law under changes of coordinates that guarantees the invariance of the integral transform.) Such integral operators are a particular case of the operators introduced in [42] and interpreted there as certain “quantum pullbacks”. In particular, they include ordinary pullbacks. See also [31].

**Example 4.2** Suppose $\Phi: E_1 \to E_2$ is a morphism of vector bundles over $M$, $u_2^g = u_1^i \Phi_1^g(x)$. Let $S(u_1|w_2) := u_1^i \Phi_1^g(x)w_2\alpha$. Then, it is easy to see that the integral operator (71) with such a generating function $S$ is the pullback of functions, $L = \Phi^*: \mathcal{C}\infty(E_2) \to \mathcal{C}\infty(E_1)$.

**Theorem 4.2** If $L: \text{Dens}_{\lambda,\mu}(E_2) \to \text{Dens}_{\lambda,\mu}(E_1)$ is an operator of the form (71), then its dual $L^*: \text{Dens}_{1-\lambda,\mu}(E_1^*) \to \text{Dens}_{1-\lambda,\mu}(E_2^*)$ is an operator of the same form given by the integral formula

$$g_2(x, w_2) = \int Dw_1 Du_1 e^{\frac{i}{\hbar}(S^*(w_2|u_1) - u_1w_1)} g_1(x, u_1),$$

(72)

where $S^*(w_2|u_1) = S(u_1|w_2)$.

**Proof** Directly, we need to prove the equality

$$\int Dx Du_1 Dw_1 e^{-\frac{i}{\hbar}u_1w_1} f_1(x, u_1)g_1(x, w_1)$$

$$= \int Dx Du_2 Dw_2 e^{-\frac{i}{\hbar}u_2w_2} f_2(x, u_2)g_2(x, w_2)$$

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for arbitrary \( f_2(x, u_2) \) and \( g_1(x, w_1) \), where \( f_1(x, u_1) \) and \( g_2(x, w_2) \) are given by formulas (71) and (72). By substituting, we arrive at the identity. \( \square \)

Theorem 4.2 is a “quantum analogue” of Theorem 8 in [42].

**Example 4.3** Continuing in the setup of Example 4.2, we see that the dual to the pullback by a vector bundle morphism \( \Phi \) over \( M \) is the pullback by the dual morphism \( \Phi^* \). Hence, the presented construction makes it possible to extend the notion of the dual operator to more general situations that naturally occur. For example, to the case of a nonlinear fiberwise map between vector bundles, a situation typical for \( L_\infty \)-algebroids.

**Remark 4.2** The fact that Fourier transform for odd variables acts like “Hodge star operator” was observed in early years of supergeometry. See Berezin [5]; Voronov–Zorich [44], also [36]. Fourier transform of geometric objects on vector bundles was considered in [35,44]. In [44], a non-standard \( \mathbb{Z} \)-grading of pseudodifferential forms on a vector bundle was introduced (from modern viewpoint this extra grading comes from the double vector bundle structure of \( \Pi T E \)), and it was shown that fiberwise Fourier transform of forms preserves that grading. This is analogous to our Lemma 4.1. That the divergence operator \( \delta \) on multivector densities is “dual” to the de Rham differential \( d \) is a classical fact in differential geometry. It reappeared in supergeometry in Bernstein–Leites construction of integral forms [6,7] and in connection with Batalin–Vilkovisky formalism [13,18]. Novel in this subsection are Theorems 4.1 and 4.2, as well as bi-grading \((\deg E, \deg E^*)\) of formal \( \hbar \)-differential operators on a vector bundle \( E \).

**Remark 4.3** Since we treat \( \hbar \) as a formal parameter, one may ask in which sense oscillating exponentials in \( \hbar \)-Fourier transform, as well as in the integral operators (71), are understood. These exponentials with \( \hbar \) in the denominator can be defined as formal symbols satisfying natural properties, as explained in [42]. An axiomatic theory of formal oscillatory integrals was developed by A. Karabegov [11].

### 4.2 More on BV operators and brackets

In Section 3, we considered brackets generated by an operator acting on a commutative algebra, with the standard example of the algebra of functions on a supermanifold. Now, we need operators defined on a (locally) one-dimensional free module, an example of which is the case of operators acting on densities of a fixed weight. Particularly interesting for us is the case of half-densities.

Let \( S \) be such a module over an algebra \( A \). Let \( \sigma \) be some basis element of \( S \). Consider an arbitrary operator \( L: S \to S \). In concrete examples, it will be a formal \( \hbar \)-differential operator. We can define an operator \( L_\sigma: A \to A \) depending on a choice of \( \sigma \), by

\[
L_\sigma(f) := \sigma^{-1} L(\sigma f).
\]  

(73)

Consider quantum and classical brackets on \( A \) generated by \( L_\sigma \) by (28) and (29). Denote them \( \{ f_1, \ldots, f_n \}_{L, \sigma, \hbar} \) and \( \{ f_1, \ldots, f_n \}_{L, \sigma} \).
Lemma 4.2  For any $n$,
\[
\{ f_1, \ldots, f_n \}_{L, \sigma, \hbar} = (\hbar)^{-n} \sigma^{-1} \cdots [L, f_1], \ldots, f_n \}_L (\sigma) \tag{74}
\]
\[
\{ f_1, \ldots, f_n \}_{L, \sigma} = (\hbar)^{-n} \sigma^{-1} \cdots [L, f_1], \ldots, f_n \}_L (\sigma) \pmod{\hbar} \tag{75}
\]

\textbf{Proof}  Observe that $[L_\sigma, f] = [L, f]_\sigma$ and then apply induction. \hfill \Box

A natural question is about the dependence of the constructed brackets on a choice of $\sigma$. Suppose $\sigma' = e^g \sigma$. We can apply a method from [42] to compare the homological vector fields on the algebra $A$ for the brackets corresponding to $\sigma$ and $\sigma'$. For brackets “without dash,”
\[
f \mapsto f + \varepsilon e^{-i \hbar f} L_\sigma (e^{i \hbar f})
\]
and for brackets with dash,
\[
f \mapsto f + \varepsilon e^{-i \hbar f} L_{\sigma'} (e^{i \hbar f}) = f + \varepsilon e^{-g} e^{-i \hbar f} L_\sigma (e^g e^{i \hbar f}) = f + \varepsilon e^{-i (f + \hbar g)} L_\sigma (e^{i (f + \hbar g)}).
\]

In other words, the “new” homological field is obtained by the shift of argument by a fixed function $\frac{\hbar}{i} g$. This is an $L_\infty$-isomorphism which is the identity modulo $\hbar$. In particular, classical brackets do not depend on a choice of $\sigma$.

4.3 Application to cotangent $L_\infty$-bialgebroid. Discussion

We come back to the construction of a Batalin–Vilkovisky operator for the cotangent $L_\infty$-algebroid (for a given $P_\infty$-structure). As we know, it is actually an $L_\infty$-bialgebroid. The difference with Sect. 3.4 is that we use operators acting on half-densities (instead of functions).

\textbf{Theorem 4.3}  The Hamiltonians defining the cotangent $L_\infty$-bialgebroid structure lift to mutually dual Batalin–Vilkovisky operators on $\Pi T^* M$ and $\Pi T M$:
\[
e^{-i \hbar P} \circ (-\hbar^2 \delta) \circ e^{i \hbar P} \tag{76}
\]
(acting on multivector densities on $M$) and
\[
e^{-i \hbar \hat{P}} \circ (-i \hbar d) \circ e^{i \hbar \hat{P}} \tag{77}
\]
(acting on forms). Here, $P = P(x, x^*)$ and $\hat{P} = P(x, -i \hbar \frac{\partial}{\partial x})$ are as before.

Half-densities on $\Pi T^* M$ are multivector densities on $M$ and half-densities on $\Pi T M$ because of canonical volume element can be identified with functions, i.e., forms on $M$.

We do not discuss the most general and “abstract” notion of $L_\infty$-bialgebroid or it quantum version. See different approaches and analysis in Bashkirov and A. Voronov [1,2] and Th. Voronov [42,43]. One thing that we would like to note, is the
role of the modular class (which was appearing in Sect. 3 when we used an asymmetric picture). In the description based on half-densities, we expect it to arise as an obstruction to a “quantum lift” of the anchor. If such an obstruction vanishes, there are dual descriptions of the quantum anchor on $\Pi T^* M$ and $\Pi TM$ given by mutually dual integral operators as in Sect. 4.1.

Let me elaborate this point. It is convenient to speak about general $L_\infty$-algebroids. If an $L_\infty$-algebroid structure is given in a vector bundle $E \to M$, its manifestation on $\Pi E$ is just a homological vector field $Q \in \text{Vect}(\Pi E)$. Higher anchors as multilinear maps from $E$ to $TM$ — part of an $L_\infty$-algebroid structure — combine into a single nonlinear fiberwise $Q$-map $\Pi E \to \Pi TM$. See, e.g., [42, Lem. 1]. In the dual manifestation on $\Pi E^*$, an $L_\infty$-algebroid structure in $E$ becomes a collection of higher Lie-Schouten brackets, i.e., an $S_\infty$-structure on $\Pi E^*$. The “dual anchor” then is a thick $S_\infty$-morphism $\Pi T^* M \to \Pi E^*$ in Voronov’s sense [41] inducing an $L_\infty$-morphism $C^\infty(\Pi E^*) \to C^\infty(\Pi T^* M)$ [19,41,42]. (The particular case of $E = T^* M$ and an $L_\infty$-morphism between the higher Koszul brackets and the canonical Schouten bracket was a problem posed by Khudaverdian-Voronov in [17] and solved in the above-cited works.) We see the quantum version as follows. A quantum $L_\infty$-algebroid structure in $E$ is given by an odd $\hbar$-differential operator $\hat{H}$ on $\text{Dens}_{1/2}(\Pi E)$, of total degree $\deg \hat{H} = +1$ (see Sect. 4.1, where $\hat{H}^2 = 0$. Its classical limit, i.e., the principal symbol, gives a usual $L_\infty$-algebroid structure in $E$. A quantum anchor has to be defined as an integral operator of type (71), $\Omega(M) \to \text{Dens}_{1/2}(\Pi E)$ intertwining $-i \hbar d$ and $\hat{H}$. In the dual description, a dual quantum anchor will be an integral operator $\text{Dens}_{1/2}(\Pi E^*) \to \mathcal{A}(M)$ intertwining $\hat{H}^*$ and $-\hbar^2 \delta$. This is a lifting of the standard classical picture. We expect the modular class of $E$ play a role for such a lifting. We hope to explore this and the “bi--” case elsewhere.

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