Optimal estimates and joint measurement uncertainty relations

Michael J.W. Hall
Theoretical Physics, IAS, Canberra ACT 0200, Australia
mjh105@rsphysse.anu.edu.au

Abstract
Often, one would like to determine some observable \( A \), but can only measure some (hopefully related) observable \( M \). This can arise, for example, in quantum eavesdropping, or when the research lab budget isn’t large enough for a 100\% efficient photodetector. It also arises whenever one tries to jointly determine two complementary observables \( A \) and \( B \), via some measurement \( M \).

This raises three natural questions:
(i) what is the best possible estimate of \( A \) from \( M \) ?
(ii) how ‘noisy’ is such an estimate ?
(iii) are there any universally valid uncertainty relations for joint estimates ?

Quite general answers, and applications to heterodyne detection and EPR joint measurements, are briefly reviewed.

* Poster presentation for the Australian Institute of Physics 16\textsuperscript{th} Biennial Congress (ANU, Canberra, 31 Jan - 4 Feb 2005)
1. Thought experiment

Would like to determine observable $A$, but can only measure observable $M$.

*What is the best possible estimate of $A$ from the measurement result $M=m$?*

\[ \psi \rightarrow \boxed{\text{Measure} \atop M} \rightarrow \boxed{\text{Estimate} \atop A} \rightarrow A_{est} = ? \]

This question arises, for example in

- Quantum eavesdropping
- Joint measurements of two quantum observables $A$ and $B$ (e.g., position and momentum)
- Classical estimation theory
- Non-ideal lab equipment (?)
2. How good is a given estimate?

If \( f(m) \) denotes the estimate of \( A \) from measurement result \( M=m \), then the estimate is equivalent to measuring the observable

\[
A_f = f(M) = \sum_m f(m) |m> <m|.
\]

Hence can decompose any estimate as\(^{1,2}\)

\[
A_f = A + N_f,
\]

i.e.,

\[
\text{estimate} = \text{signal} + \text{noise},
\]

where \( N_f \) is the \textit{noise operator} associated with the estimate.

The \textit{inaccuracy in the estimate} is defined to be the \textit{rms noise}:

\[
\varepsilon(A_f)^2 := <N_f^2>,
\]

which vanishes for a \textit{perfect} estimate.
3. Optimal estimate $\equiv$ smallest noise

It may be shown that the noise has the fundamental lower bound $^{3-5}$

$$
\varepsilon(A_f)^2 \geq \sum_m \left| \text{Im} \frac{\langle m|A|\psi\rangle}{\langle m|\psi\rangle} \right|^2 ,
$$

and hence that $^{5}$

incompatibility $\Rightarrow$ noise

(i.e., $\varepsilon(A_f)>0$ for non-commuting $A$ and $M$).

The lower bound in (1) corresponds to the optimal estimate of $A$, which is given, for measurement result $M=m$ on state $\psi$, by $^{3-5}$

$$
A_{opt} = \sum_m \text{Re} \frac{\langle m|A|\psi\rangle}{\langle m|\psi\rangle} |m\rangle <m|.
$$

One finds a spread vs noise tradeoff $^{5}$:

$$(\Delta A_{opt})^2 + \varepsilon(A_{opt})^2 = (\Delta A)^2 ,$$

i.e., a geometric uncertainty relation.
4. Aside: examples of optimal estimates

**Momentum:** Writing \( \psi = Re^{iS/\hbar} \), the optimal estimate of momentum, from a position measurement result \( X = x \), follows as

\[
P_{opt}(x) = \nabla S .
\]

This estimate achieves the lower bound in (1), which can be rewritten as an *exact uncertainty relation* \(^3\):

\[
\delta X \varphi(P_{opt}) = \hbar / 2 .
\]

This implies, and is far stronger than, the Heisenberg uncertainty relation \( \Delta X \Delta P \geq \hbar / 2 \).

**Energy:** The optimal estimate for energy is

\[
E_{opt}(x) = |\nabla S|^2/(2m) + V + Q,
\]

where \( Q = -\hbar^2 \nabla^2 R/(8mR) \) is the so-called quantum potential.
5. Joint measurements

All measurements are joint measurements!

Why? - the information gained from any measurement $M$ can always be used to make estimates of any two observables $A$ and $B$, via

$$A_f = f(M) \quad \text{and} \quad B_g = g(M).$$

Example: Heterodyne detection

The statistics of heterodyne detection are given by the Husimi Q-function

$$Q(\alpha) = |<\alpha|\psi>|^2/\pi. \quad (\alpha = \alpha_1 + i\alpha_2)$$

A standard joint estimate of the quadratures $X=(a+a^\dagger)/2$, $Y=(a-a^\dagger)/2i$, is then:

$$X_{est}=\alpha_1, \quad Y_{est}=\alpha_2, \quad \Rightarrow \quad \Delta X_{est} \Delta Y_{est} \geq 1/2.$$

The optimal joint estimate is 4 times better!:

$$X_{opt}=\alpha_1 + \frac{1}{4} \partial_1 \ln Q, \quad Y_{opt}=\alpha_2 + \frac{1}{4} \partial_2 \ln Q,$$

$$\Rightarrow \quad \Delta X_{opt} \Delta Y_{opt} \geq 1/8.$$
6. A universal joint measurement uncertainty relation

Let $A_f$ and $B_g$ denote any two estimates of observables $A$ and $B$, from some measurement $M$ on state $\psi$. One then has the joint uncertainty relation

$$\Delta A_f \varepsilon(B_g) + \varepsilon(A_f) \Delta B_g + \varepsilon(A_f) \varepsilon(B_g) \geq \frac{1}{2} |<[A,B]>| . \quad (2)$$

This relation is the long-looking for universal quantification of complementarity:

For two incompatible observables $A$ and $B$, there is no joint estimate having both zero spread and zero noise.

**Special case: unbiased estimates**
If a measurement $M$ yields estimates of $A$ and $B$ which are on average equal to $<A>$ and $<B>$, for all states $\psi$, then

$$\varepsilon(A_f) \varepsilon(B_g) \geq \frac{1}{2} |<[A,B]>| ,$$

i.e., unbiased estimates of incompatible observables cannot be arbitrarily accurate.
7. Example: EPR estimates and continuous variable teleportation

In continuous variable teleportation, Alice and Bob ideally share a perfect EPR state:

\[ \psi_{EPR}(x,x') = \delta(x-x'-a) e^{ib(x+x')/2\hbar} \]

But such states are unphysical: in practice they must use the approximate EPR state

\[ \psi = K \exp\left[-(x-x'-a)^2/4\sigma^2 - \tau^2(x+x')^2/4\hbar^2\right] e^{ib(x+x')/2\hbar} \]

with the almost perfect correlations:

\[ \langle X-X' \rangle = a, \quad \text{Var} (X-X') = \sigma^2 << 1, \]
\[ \langle P+P' \rangle = b, \quad \text{Var} (P+P') = \tau^2 << 1. \]

Now, if Alice transmits a measurement result \( P=p \) to Bob, then what is the best estimate Bob can make for \( P' \)?

\[ P'_{\text{est}} = b-p \quad \times \quad P'_{\text{opt}} = \frac{\hbar^2(b-p) + \sigma^2 \tau^2 p}{\hbar^2 + \sigma^2 \tau^2} \quad \checkmark \]

\[ \varepsilon(P'_{\text{opt}})/\varepsilon(P'_{\text{est}}) = \left(1+\sigma^2 \tau^2/\hbar^2\right)^{-1/2} < 1 \]

⇒ optimal estimates can improve the teleportation protocol - and achieve the fundamental lower bound in (2).
8. Generalisations: POMs and density operators

All of the main results above can be generalised to the case where

- the measurement $M$ is described by a probability operator measure (POM), i.e., by a set of positive operators $\{M_m\}$ with $\sum M_m = 1$ (eg, $M_m = |m><m|$).

- the state of the system prior to measurement is described by a density operator $\rho$ (eg, $\rho = |\psi><\psi|$).

The geometric and joint measurement uncertainty relations remain unchanged, and the lower bound (1) for noise, and the formula for the optimal estimate take the respective forms

$$
\varepsilon(A_f)^2 \geq \sum_m \frac{|tr[\rho(AM_m - M_mA)]|^2}{4 \text{tr}[\rho M_m]},
$$

$$
A_{opt} = \sum_m \frac{tr[\rho(AM_m + M_mA)]}{2 \text{tr}[\rho M_m]} |m><m|. 
$$
9. Summary

Any estimate of an observable $A$ from some measurement $M$ can be decomposed as

\[ \text{estimate} = \text{signal} + \text{noise} . \]

The noise cannot vanish if $A$ and $M$ are incompatible:

\[ \text{incompatibility} \Rightarrow \text{noise} . \]

There is a **geometric uncertainty relation** for optimal estimates, reflecting a **fundamental trade-off between the spread and the noise of an optimal estimate**:

\[ (\Delta A_{opt})^2 + \varepsilon(A_{opt})^2 = (\Delta A)^2 \]

There is a **universal joint measurement uncertainty relation**, valid for the estimates of any two observables $A$ and $B$ from any measurement process:

\[ \Delta A_f \varepsilon(B_g) + \varepsilon(A_f) \Delta B_g + \varepsilon(A_f) \varepsilon(B_g) \geq \frac{1}{2} |<[A,B]>| \]
References

(NB: material for this poster is largely drawn from Ref. 5, where many more details and examples may be found)

1. E. Arthurs and J.L. Kelly, Bell Syst. Tech. J. 44 (1965) 725

2. D.M. Appleby, Int. J. Theor. Phys. 37 (1998) 1491

3. M.J.W. Hall, Phys. Rev. A 64 (2001) 052103

4. L.M. Johansen, Phys. Lett. A 322 (2004) 298

5. M.J.W. Hall, Phys. Rev. A 69 (2004) 052113

6. M. Ozawa, Phys. Lett. A 320 (2004) 367

7. W.P. Bowen et al., Phys. Rev. A 67, (2003) 032302

8. C.M. Caves & K. Wodkiewicz, eprint quant-ph/0409063