Confederated Learning: Federated Learning with Decentralized Edge Servers

Bin Wang, Jun Fang, Hongbin Li, Fellow, IEEE, Xiaojun Yuan, and Qing Ling

Abstract—Federated learning (FL) is an emerging machine learning paradigm that allows to accomplish model training without aggregating data at a central server. Most studies on FL consider a centralized framework, in which a single server is endowed with a central authority to coordinate a number of devices to perform model training in an iterative manner. Due to stringent communication and bandwidth constraints, such a centralized framework has limited scalability as the number of devices grows. To address this issue, in this paper, we propose a Confederated Learning (CFL) framework. The proposed CFL consists of multiple servers, in which each server is connected with an individual set of devices as in the conventional FL framework, and decentralized collaboration is leveraged among servers to make full use of the data dispersed throughout the network. We develop an alternating direction method of multipliers (ADMM) algorithm for CFL. The proposed algorithm employs a random scheduling policy which randomly selects a subset of devices to access their respective servers at each iteration, thus alleviating the need of uploading a huge amount of information from devices to servers. Theoretical analysis is presented to justify the proposed method. Numerical results show that the proposed method can converge to a decent solution significantly faster than gradient-based FL algorithms, thus boasting a substantial advantage in terms of communication efficiency.

Index terms—Confederated learning, ADMM, random scheduling.

I. INTRODUCTION

In recent years, the rapid development of machine learning has gained much attention in both the academia and the industry. The tremendous success of machine learning is inseparable from the help of huge data sets. Most conventional machine learning algorithms are implemented in a centralized manner, requiring the training data to be collected and processed in a central node. However, securely aggregating heterogeneous data dispersed over various data sources or organizations is a non-trivial task. Processing the huge amount of data in a centralized fashion also poses significant challenges for the data server. The challenges concurrently arise from a privacy-protecting perspective. In some data-sensitive areas such as the health care and financial services, the confidentiality of users’ data is of great concern and should be protected. In such cases, sending users’ data to a centralized node may not be allowed.

Federated learning (FL) [1] is a new paradigm that enables model training without gathering data at a central server. Such a merit makes it amiable for data-intensive and privacy-sensitive machine learning applications. So far most studies [1]–[10] focus on a centralized FL framework, in which there is a central server and a number of spatially distributed devices (users). The server is bidirectionally connected to each user which holds the data. To accomplish model training, FL employs a computation-then-aggregation strategy. Specifically, in each iteration, the central server first distributes the global model to each user. Based on the global model, each user updates its local model using its local data. The updated local model is then uploaded to the server. At last, the server fuses the local models to obtain a new global model. During this training process, the data are preserved locally and only the training model is exchanged, thus circumventing the need of gathering the data from users to the central server.

Nevertheless, FL still faces challenges from both theoretical and practical aspects. One fundamental problem of the single-server FL system is poor scalability. Note that FL may operate in a wireless edge network where the communication resource is severely constrained. Due to limited bandwidth, at each iteration only a small subset of users can be selected to interact with the server, which leads to a low efficiency and also calls for a judiciously designed scheduling policy [11]–[13]. A line of research to address the scalability issue is decentralized FL [14]–[23], which has attracted much interest due to their enhanced scalability as well as its strengthened robustness to server failures. Typically, decentralized FL is implemented on a decentralized network consisting of a number of nodes. The decentralized network does not have a global coordinator; instead, all nodes are connected in a peer-to-peer manner. In these works, the nodes are assumed to be the data-holders and thus the decentralized network forms a D2D (Device-to-Device) network. Nevertheless, such a fully decentralized setting may not fit in well with the current wireless edge network.

Another major challenge of FL is excessively high communication overhead caused by frequent information exchange between the server and the users. In many practical scenarios, communication is much more costly than computation. It is, therefore, of vital importance to reduce the communication overhead for FL. Many existing studies [6]–[10], [22], [24]–[26] employ gradient descent or proximal type of methods to perform training. These methods require a massive amount...
of information exchanges because gradient descent (with decreasing stepsizes) requires a large number of iterations to converge. To relieve this issue, some works [7]–[10], [24]–[26] suggest to run multiple iterations of local gradient descent between adjacent aggregation steps. However, recent studies [27] find that setting the number of local iterations too large may have an unfavorable impact on the convergence speed. Recently, more advanced optimization algorithms [2]–[4], [28] are employed in FL. These works are mainly based on the ADMM (alternating direction method of multipliers) algorithm, which decomposes the original problem into a number of subproblems. In general, ADMM type of algorithms require only a small number of iterations to converge, thus having the potential to substantially reduce the communication cost. Nevertheless, none of these algorithms can be nontrivially extended to the CFL framework considered in this work.

In this paper, we introduce a multi-server based FL framework, whereby the servers form a decentralized network while each server is connected to an individual set of edge devices. Such a framework is a union of sovereign servers united for the purpose of learning a global model, and thus is referred to as confederated learning (CFL). CFL can better address the scalability issue than the centralized one. Meanwhile, it does not involve complex network management required by the D2D network. Note that it is reasonable to assume the servers to work in a decentralized manner since there may not be a global center to coordinate these servers. In addition, the intelligent nature of 5G and 6G networks calls for extensive and flexible self-organizations of local or trans-regional cooperations. We note that confederated learning was introduced in [30] as a term to characterize FL with “vertically separated” data, e.g., different data types (lab tests, diagnosis, medications, treatments, etc.) of a given patient are located at different locations and cannot be easily matched with each other. Although using the same term, the meaning of CFL in this work is totally different from that of [30].

Within this framework, we develop an efficient ADMM-based CFL algorithm. The proposed ADMM algorithm is characterized with two distinctive features. Firstly, to alleviate the need of uploading a huge amount of information from massive distributed devices to each server, a random scheduling policy is employed, whereby each device, at each iteration, is randomly activated with a small probability and participates in the training process. Secondly, considering the fact that subproblems of ADMM may not have a closed-form solution, the proposed ADMM allows the subproblem to be solved up to a certain accuracy. Theoretical analysis reveals that the proposed algorithm enjoys a sublinear convergence rate. Numerical results show that the proposed method can converge to a decent solution significantly faster (i.e. with much fewer communication rounds) than those gradient-based CFL algorithms, thus presenting a substantial advantage in terms of communication efficiency.

The rest of this paper is organized as follows. Some preliminaries on convex functions are first introduced in Section II. Then in Section III, we present a CFL framework and formulate the CFL problem. A new ADMM algorithm is proposed in Section IV. The convergence result of the proposed algorithm and its proof are provided in Section V and Section VI, respectively. Simulations results are provided in Section VII followed by concluding remarks in Section VIII.

II. Preliminaries

A. Properties of Convex Functions

The subgradient of a convex function $f$ is denoted as $\partial f$. If $f$ is continuously differentiable, then we have $\partial f = \nabla f$. For a convex function $f$, it always holds that

$$f(x) \geq f(y) + \langle \partial f(y), x - y \rangle, \forall x, y,$$

$$f\left(\sum_{t=1}^{T} \delta_t x_t\right) \leq \sum_{t=1}^{T} \delta_t f(x_t), \text{ if } \sum_{t=1}^{T} \delta_t = 1 \text{ and } \delta_t \geq 0, \forall x_t.$$  \hspace{1cm} (1)

where the second inequality is known as the Jensen’s inequality. A function $f$ is said to be $\mu$-strongly convex if it satisfies

$$f(x) \geq f(y) + \langle \partial f(y), x - y \rangle + \frac{\mu}{2} ||x - y||_2^2, \forall x, y.$$  \hspace{1cm} (2)

B. Commonly Used Inequalities

Given a triple of arbitrary vectors $x, y, z$, it holds

$$2\langle x - y, x - z \rangle = ||x - y||_2^2 + ||x - z||_2^2 - ||y - z||_2^2.$$  \hspace{1cm} (3)

Meanwhile, for $\forall x, y$, it holds

$$2\langle x, y \rangle \leq \omega ||x||_2^2 + \omega^{-1} ||y||_2^2, \forall \omega > 0.$$  \hspace{1cm} (4)

III. Confederated Learning

A. CFL Framework

We consider a CFL framework consisting of $l$ edge servers (ESs), in which the $i$th ES is connected to $|S_i|$ edge devices (i.e. users) which hold the data. Here $S_i$ represents the set of users served by the $i$th ES and $|S_i|$ is the cardinality of $S_i$. Let $u_{ij}$ denote the $j$th user served by the $i$th ES. It is assumed that the sets of users served by different ESs are disjoint. Each ES can communicate with its own users, while communications among users are not allowed. Also, ESs form a decentralized network that can be abstracted as a graph $G = \{V, E\}$, in which there is no global coordinator and each ES is only allowed to communicate with its neighboring ESs. Clearly, the conventional single ES-based FL framework is a special case of the CFL framework (see Fig. 2). The CFL framework also covers the centralized multi-ES framework as a special case, where the ESs form a star-type communication network.

The CFL framework is different from the peer-to-peer FL [14]–[17]. The CFL system is more suitable for applications residing on wireless edge networks while the peer-to-peer FL is more suitable for D2D networks. A recent work [31] proposed an in-network acceleration scheme by appointing a portion of nodes to be the (virtual) local fusion centers. However, the communication pattern still follows a fully decentralized manner.
B. Problem Formulation

Consider the following optimization problem:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^I f_i(x) = \sum_{i=1}^I \sum_{j=1}^{\vert S_i \vert} f_{ij}(x; D_{ij})$$

(5)

where $f_i(x) = \sum_{j=1}^{\vert S_i \vert} f_{ij}(x; D_{ij})$ is a convex, proper and lower semi-continuous function held by user $u_{ij}$ and $D_{ij}$ represents the local data set stored at user $u_{ij}$. For a learning task, the variable $x \in \mathbb{R}^n$ represents the global model parameter vector that is to be learned. The function $f_{ij}$ is referred to as the local loss function. If $l = 1$, then (5) degenerates into the standard FL problem. By introducing a set of auxiliary variables $\{y_i\}$, we can reformulate (5) into the following problem:

$$\min_{\{x_{ij}\}, \{y_i\}} \sum_{i=1}^I \sum_{j=1}^{\vert S_i \vert} f_{ij}(x_{ij}; D_{ij})$$

s.t. $x_{ij} = y_i, \forall i \in \{1, \ldots, I\}, \forall j \in S_i,$

$$y_1 = y_2 = \cdots = y_I.$$  

(6)

where $x_{ij} \in \mathbb{R}^n$ is the local variable held by user $u_{ij}$ and $y_i \in \mathbb{R}^n$ is the local variable held by the $i$th ES. In (6), the first equality constraint, i.e., $x_{ij} = y_i$, forces the consistency between the $i$th ES’s local variable and those of its users. The second constraint forces the local variables of the ESs to be equal to each other. Clearly, (6) is essentially the same as (5). Nevertheless, (6) cannot be solved in a decentralized manner since tackling the second constraint demands centralized operations. To circumvent this obstacle, we resort to solving the following equivalent problem:

CFL Optimization:

$$\min_{\{x_{ij}\}, \{y_i\}} \sum_{i=1}^I \sum_{j=1}^{\vert S_i \vert} f_{ij}(x_{ij}; D_{ij})$$

s.t. $x_{ij} = y_i, \forall i \in \{1, \ldots, I\}, \forall j \in S_i,$

$$A y = 0.$$  

(7)

where $y \in \mathbb{R}^n$ denotes the vertical stack of $y_i$s, i.e., $y \triangleq [y_1; y_2; \cdots; y_I]$. $A \triangleq A_{in} \otimes I_n$, $A_{in}$ is the incidence matrix of the graph $G$, $\otimes$ denotes Kronecker product and $I_n$ is an $n \times n$ identity matrix. It is well-known that $A y = 0 \iff y_1 = y_2 = \cdots = y_I$. Thus (7) is also equivalent to (5). Notably, previous research [32]–[34] on decentralized optimization has paved a way on how to handle the second constraint in a decentralized manner. To ease subsequent expositions, hereafter we omit $D_{ij}$ in $f_{ij}$.  

C. Communication Bottleneck and Random Scheduling

In our proposed CFL framework, there exists two types of data transmissions, namely, user-to-ES (U2E) communications and ES-to-ES (E2E) communications. Generally, for CFL, the communication bottleneck lies in the U2E communications. This is because each ES may be assigned with a large number of users. Thus sending the local update from each user to its associated ES consumes a significant amount of communication resource and meanwhile may incur a high latency. To overcome this difficulty, in our algorithm, we randomly choose a small subset of users at each iteration to communicate with its ES. Specifically, each user is assigned a same probability $\alpha$, and is independently activated with probability $\alpha$ at each iteration to report its local update to its associated ES. This user selection policy is termed as a random scheduling policy. Such a policy allows each user to have the same chance to access its associated ES. Meanwhile, at each iteration only a small number of users are activated to access ESs, which enables the algorithm to operate under stringent communication and delay constraints.

IV. PROPOSED ALGORITHM

In this section, we propose a new ADMM algorithm that can accommodate the CFL framework. Some discussions are then provided to shed some insight into the proposed algorithm.

A. Algorithm Development

To facilitate subsequent expositions, we first introduce the following notations:

$$x \triangleq [x_1; \cdots; x_I], x_i \triangleq [x_{i1}; \cdots; x_{i|S_i|}], \lambda \triangleq [\lambda_1; \cdots; \lambda_I],$$

$$\lambda_i \triangleq [\lambda_{i1}; \cdots; \lambda_{i|S_i|}], \partial f(x) \triangleq \partial f_1(x_1); \cdots; \partial f_I(x_I)],$$

$$\partial f_i(x_i) \triangleq \partial f_{i1}(x_{i1}); \cdots; \partial f_{i|S_i|}(x_{i|S_i|}), \bar{\lambda} \triangleq [\bar{\lambda}_1; \cdots; \bar{\lambda}_I],$$

$$y \triangleq [y_1; \cdots; y_I], H = \text{bldig}\{H_1; \cdots; H_I\},$$

(8)

where $H_i \in \mathbb{R}^{n \times n|S_i|}$ is a matrix obtained by concatenating $|S_i|$ identity matrices of size $n \times n$.

The augmented Lagrangian function of (7) is given as

$$L_A(\{\{x_{ij}, \lambda_{ij}\}_{j=1}^{|S_i|}, y_i\}_{i=1}^I, \beta) = \sum_{i=1}^I \sum_{j=1}^{|S_i|} (f_{ij}(x_{ij}) + \langle \lambda_{ij}, x_{ij} - y_i \rangle + \frac{\beta}{2} \|x_{ij} - y_i\|^2_2) + \langle \beta, Ay \rangle + \frac{\beta}{2} \|Ay\|^2_2$$

(9)
where \( \{\lambda_{ij}\} \) and \( \beta \) are Lagrangian multipliers, \( \sigma_1 \) and \( \sigma_2 \) are man-crafted parameters. Based on \( L_A \), we can easily deduce a standard ADMM algorithm as follows:

\[
x_{ij}^{k+1} = \arg \min_{x_{ij}} f_i(x_{ij}) + \frac{\sigma_1}{2} \|x_{ij} - y^k + \lambda_{ij}^k\|_2^2,
\]

\[
y^{k+1} = \arg \min_y \sum_{i=1}^l \sum_{j=1}^{|S_i|} \left( \frac{\sigma_2}{2} \|x_{ij}^k - y_i + \sigma_1^{-1} \lambda_{ij}^k\|_2^2 \right) + \frac{\sigma_1}{2} \|Ay + \sigma_2^{-1} \beta^k\|_2^2,
\]

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \sigma_1 (x_{ij}^{k+1} - y_{ij}^k), \quad \forall i, \forall j \in S_i,
\]

\[
\beta^{k+1} = \beta^k + \sigma_2 Ay^{k+1}.
\]

(10)

Nevertheless, the above algorithm can not fulfill our needs since, firstly, this algorithm requires all users to participate in the \( x_{ij}^{k+1} \)-update and send their local updates to their respective ESs, which incurs a prohibitively high communication cost. Secondly, the algorithm demands an exact solution of the \( x_{ij}^{k+1} \)-subproblem. This is a stringent requirement since obtaining the exact solution of an optimization problem might be computationally expensive. Thirdly, the \( y^{k+1} \)-subproblem can not be solved in a decentralized manner since \( \|Ay\|_2^2 \) is a nonseparable term.

To address the above difficulties, we propose a new ADMM algorithm, which is summarized in Algorithm 1. Specifically, in each iteration of Algorithm 1 only a subset of users are selected (with probability \( \alpha \)) to participate in the \( x_{ij}^{k+1} \)-update, thus avoiding the need of data transmissions from every user to its ES. Meanwhile, Algorithm 1 allows the \( x_{ij}^{k+1} \)-subproblem to be solved up to an \( \epsilon \)-accuracy instead of solving it exactly. Lastly, in the proposed algorithm, we use a judiciously designed extra proximal term such that the \( y^{k+1} \)-subproblem can be solved in a decentralized manner.

With the notations defined in (6), the update of \( \lambda_{ij}^{k+1} \) and \( \lambda_{ij}^{k+1} \) in Algorithm 1 can be compactly written as

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \sigma_1 (x_{ij}^{k+1} - H^T y^{k+1}),
\]

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \alpha (\lambda_{ij}^{k+1} - \lambda_{ij}^k).
\]

(11)

The \( y^{k+1} \)-subproblem, \( k < \bar{k} \), can also be compactly written as

\[
y^{k+1} = \arg \min_y \frac{\sigma_2}{2} \|x^{k+1} - H^T y + \lambda_{ij}^k\|_2^2 + \frac{\sigma_1}{2} \|Ay + \lambda_{ij}^k\|_2^2 + \frac{\sigma_1}{2} \|y - y^k\|_{\alpha^{-1}, p}^2.
\]

(12)

**B. Training Process and Communication Efficiency**

1) **Training Process:** At each iteration of Algorithm 1 the \( i \)th ES first distributes \( y_i^k \) to its associated users. Then the selected users update their local models by solving the \( x_{ij}^{k+1} \)-subproblem, \( \forall j \in I_k^{k+1} \), where \( I_k^{k+1} \) denotes the index set of the users selected by ES \( i \) at the \( (k+1) \)th iteration. After the local update, the selected users upload \( x_{ij}^{k+1} \) to its associated ES. Then the ESs collaboratively solve the \( y^{k+1} \)-subproblem through local information exchange. As will be shown later, the \( y^{k+1} \)-subproblem admits a closed-form solution. Solving the \( y^{k+1} \)-subproblem only needs to exchange information among neighboring ESs once, which does not incur additional latency and communication costs. It should be noted that solving the \( y^{k+1} \)-subproblem also involves the local model parameters of those unselected users. Nevertheless, since we have \( x_{ij}^{k+1} = x_{ij}^k \) for those \( j \notin I_k^{k+1} \), we can use the model parameters obtained in the previous iteration for these unselected users. For this purpose, each ES can build a history database to store its users’ model parameters obtained in the previous iteration. At last, the update of \( \lambda_{ij}^{k+1} \) can be conducted locally at each user.

2) **Communication Overhead Analysis:** At the \( (k+1) \)th iteration, each ES needs to broadcast its local variable \( y_i^k \) to its users, and each selected user uploads its local variable \( x_{ij}^{k+1} \) to its associated ES. Since each user is selected with a same probability \( \alpha \), the average number of users that participate the uplink U2E transmission at each iteration is \( \alpha \sum_{i=1}^l |S_i| \). As for the E2E communication, each ES needs to communicate with its one-hop neighboring ESs only once at each iteration. Overall, in an average sense, the total number of messages

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**Algorithm 1 CFL-ADMM**

**Inputs:** parameters \( \sigma_1 \) and \( \sigma_2 \), the activation probability \( \alpha \) and the maximum number of iterations \( \bar{k} \). All initial vectors are set to 0.  

While \( (k+1) \leq \bar{k} \) do

1. **User selection:** Each user has a probability of \( \alpha \) to be selected. The index set of the users selected by ES \( i \) in the \( (k+1) \)th iteration is denoted as \( I_k^{k+1} \).

2. **Users solve:**

\[
x_{ij}^{k+1} = \arg \min_{x_{ij}} f_i(x_{ij}) + \frac{\sigma_1}{2} \|x_{ij} - y_i^k + \lambda_{ij}^k\|_2^2,
\]

\[
y^{k+1} = \arg \min_y \sum_{i=1}^l \sum_{j=1}^{|S_i|} \left( \frac{\sigma_2}{2} \|x_{ij}^{k+1} - y_i + \sigma_1^{-1} \lambda_{ij}^k\|_2^2 \right) + \frac{\sigma_1}{2} \|Ay + \sigma_2^{-1} \beta^k\|_2^2,
\]

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \sigma_1 (x_{ij}^{k+1} - y_{ij}^k), \quad \forall i, \forall j \in S_i,
\]

\[
\beta^{k+1} = \beta^k + \sigma_2 Ay^{k+1}.
\]

(13)

3. **Model upload:** Selected users upload their local variables to ES \( i \).

4. **ESs solve:**

\[
y^{k+1} = \arg \min_y \sum_{i=1}^l \sum_{j=1}^{|S_i|} \left( \frac{\sigma_2}{2} \|x_{ij}^{k+1} - y_i + \sigma_1^{-1} \lambda_{ij}^k\|_2^2 \right) + \frac{\sigma_1}{2} \|Ay + \sigma_2^{-1} \beta^k\|_2^2 + \frac{\sigma_1}{2} \|y - y^k\|_{\alpha^{-1}, p}, \quad k + 1 < \bar{k},
\]

\[
y^k = \arg \min_y \sum_{i=1}^l \sum_{j=1}^{|S_i|} \frac{\sigma_2}{2} \|x_{ij}^k - y_i + \sigma_1^{-1} \lambda_{ij}^k\|_2^2 + \frac{\sigma_1}{2} \|Ay + \lambda_{ij}^k\|_2^2 + \frac{\sigma_1}{2} \|y - y^k\|_{\alpha^{-1}, p}, \quad k + 1 = \bar{k},
\]

\[
\beta^{k+1} = \beta^k + \sigma_2 Ay^{k+1}, \quad \text{if } (k+1) < \bar{k},
\]

\[
\beta^k = \beta^{k+1} + \frac{\sigma_2}{\alpha} Ay^k, \quad \text{if } (k+1) = \bar{k}.
\]

(14)

5. **Model download:** Each ES broadcasts its local variable to its serving users.

6. **Users update:**

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \sigma_1 (x_{ij}^{k+1} - y_{ij}^{k+1}), \quad \forall i, \forall j \in S_i,
\]

\[
\lambda_{ij}^{k+1} = \lambda_{ij}^k + \alpha (\lambda_{ij}^{k+1} - \lambda_{ij}^k), \quad \forall i, \forall j \in S_i.
\]

(15)

End While; Outputs: \( x_{ij}^{k+1} \).
that are exchanged between ESs and between ESs and users is up to \(2l + \alpha \sum_{i=1}^l |S_i|\) at each iteration.

C. Implementations and Discussions

1) Implementations of the \(x_{ij}^{k+1}\)-subproblem: In the \(x_{ij}^{k+1}\)-subproblem, the notation \(\approx\) means that this problem is solved up to an \(\epsilon^{k+1}\)-accuracy, i.e. the gradient of the objective function satisfies \(\|\tau_{ij}^{k+1}\|_2 \leq \epsilon^{k+1}\), where

\[
\tau_{ij}^{k+1} = \delta f_{ij}(x_{ij}^{k+1}) + \lambda_{ij}^{k+1} + \sigma_1(x_{ij}^{k+1} - y_i^{k+1})
\]  

(16)

Such a metric can be conveniently evaluated as the gradient descent-based method is commonly used in solving the local subproblem. Note that the \(\epsilon\)-accuracy is widely used in existing literatures, e.g., [3]. If \(\epsilon\) is set to 0, then the subproblem should be solved exactly.

2) Implementations of the \(y_{k+1}\)-subproblem: In the \(y_{k+1}\)-subproblem, an extra proximal term is added to enable the decentralized implementation. Here \(P \in \mathbb{R}^{n \times n}\) is chosen to be

\[
\alpha^{-1} P = D - A^T A,
\]  

(17)

where \(D \in \mathbb{R}^{n \times n}\) is a diagonal matrix whose choice will be elaborated in Section V-A. It can be readily verified that the \(y_{k+1}\)-subproblem, \(k + 1 < \bar{k}\), i.e., \([12]\), admits a closed-form solution given as

\[
y_{k+1} = \left[ (\alpha \sigma_1 H H^T + \sigma_2 D)^{-1} (\alpha \sigma_1 H (x_k^{k+1} - H^T y_i^k)) \right]_j + \sigma_2 A^T \beta^k + \sigma_2 (D - A^T A) y_i^k
\]  

(18)

Note that the term \(A^T \beta^k\) in \([13]\) can not be directly computed because we do not have access to \(A^T\). Nevertheless, we can unfold \(H^k\) and \(A^T \beta^k\) to obtain

\[
H^k \beta^k = H (\lambda^{k+1} - \lambda^{k-1}) = H (\lambda^{k+1} - \lambda^*)
\]  

and

\[
A^T \beta^k = A^T (\beta^{k+1} - \sigma_2 A y_i^k)
\]

where \(\lambda^k\) is obtained by setting \(\lambda = 0\) and \(\beta = 0\), respectively. Substituting \([19]\) into \([18]\) yields

\[
y_{k+1} = (\alpha \sigma_1 H H^T + \sigma_2 D)^{-1} \left[ (\alpha \sigma_1 H (x_k^{k+1} + H^T y_i^k)) - \sum_{j=2}^k \sigma_2 A^T A y_j^k + \sigma_2 (D - A^T A) y_i^k \right]
\]  

(19)

Note that \(H H^T\) is a diagonal matrix. Also, recall that \(H x_{ij}^{k+1} = [H_1 x_{ij}^{k+1}; \ldots; H_{\bar{k}} x_{ij}^{k+1}]\), the vector \(H x_{ij}^{k+1}\) can be obtained by the \(\bar{k}\)th ES through the model parameter upload step. Meanwhile, \(A^T A y_i^k\) only involves information exchange among neighboring ESs. Therefore by letting each ES sending its local variable \(y_i^k\) to its neighboring ESs, \(y_i^{k+1}\) can be easily calculated at each ES.

It should be mentioned that if \(k + 1 = \bar{k}\), then the \(y_{k+1}\)-subproblem can not be solved in a decentralized manner.

Nevertheless, this is inconsequential because we only need to acquire \(x_i^k\) in the last iteration. The \(y_i^k\)-subproblem listed in Algorithm [1] is only for an analysis purpose.

3) \(\lambda_{ij}^{k+1}\)-update: Observe that the update of \(\lambda_{ij}^{k+1}\) in \([10]\) is replaced by a two-step update. In the first step, we calculate \(\lambda_{ij}^{k+1}\) in a way similar to \([10]\). Afterwards, an over-relaxation step, i.e., \(\lambda_{ij}^{k+1} = \lambda_{ij}^{k+1} + \alpha (\lambda_{ij}^{k+1} - L_i^*)\), is conducted to obtain \(\lambda_{ij}^{k+1}\). Breaking the standard update into such a two-step procedure is essential to the global convergence of the proposed algorithm. Here are some intuitions. At the \((k + 1)\)th iteration, only a subset of users are selected to update \(x_{ij}^{k+1}\). However, all users, including those are selected or unselected, are required to update \(\lambda_{ij}^{k+1}\). Thus there exists an imbalance between the update of the primal and that of the dual variables. To guarantee the convergence of the algorithm, an over-relaxation step with an inertia of \(\alpha\) is included to constrain the speed of the dual update since the over-relaxation step forces \(\lambda_{ij}^{k+1}\) to be close to \(\lambda_i^*\).

V. CONVERGENCE ANALYSIS

In this section, we provide a theoretical justification for our proposed ADMM algorithm. Our main results are summarized as follows.

Theorem I: Denote \(\{x_{ij}^*, y_{ij}^*\}_{i=1}^\infty\) as the optimal solution to the problem \([7]\), where \(x_{ij}^* = [x_{i1}^*; x_{i2}^*; \ldots; x_{i|S_i|^*}]\). At each iteration each user is selected/activated with probability \(\mu\). The maximum number of iterations is set to \(\bar{k}\). In addition, it is assumed that \(f_{ij}\) is \(\mu\)-strongly convex (see \([2]\)) and the \(x_{ij}^*\)-subproblem is solved up to an \(\epsilon^{k+1}\)-accuracy. If \(P\) is chosen such that

\[
P \succ \left( \frac{1}{\alpha^2} - 1 \right) \sqrt{\mu} H H^T - \frac{\alpha}{2} A^T A,
\]  

(21)

then the sequence generated by Algorithm [1] satisfies

\[
E[\|i=1 \sum_{i=1}^l (f_i(x_{ij}^* - f_i(x_{ij}^*)))\|] \leq \frac{\bar{c}_0}{1 + \alpha (k - 1)} + \frac{\bar{c}_0}{2\mu k},
\]  

(22)

and

\[
E[\|i=1 \sum_{i=1}^l \|x_{avg,i} - H^T y_{avg}^k\|^2 + \|Ay_{avg}^k\|^2]\]

\[
\leq \frac{\bar{c}_0}{\psi^2 (1 + \alpha (k - 1))} + \frac{\sum_{i=1}^l (\epsilon^2 \sum_{i=1}^l |S_i|)}{2\mu k},
\]  

(23)

where the expectation is taken over all possible realizations due to the random user selection, and

\[
x_{avg,i} = \sum_{t=1}^k \delta^t x_{i}^t, \quad y_{avg}^k = \sum_{t=1}^k \delta^t y_{i}^t,
\]

\[
\delta^k = (1 + \alpha (k - 1))^{-1}, \quad \delta^t = \alpha^t \delta^k, \quad 1 \leq t \leq k - 1,
\]

\[
\psi = \min \left\{ \{\|\lambda^*\|_2 + \xi\}^\gamma_{i=1}, \|\beta^*\|_2 + \xi \right\},
\]

(24)

in which \(\lambda^*\) and \(\beta^*\) are the optimal dual variables, \(\xi\) is a small positive scalar and \(\bar{c}_0\) is a constant.

A. Discussions

Note that the first term on the left-hand side of \([23]\), i.e. \(\|x_{avg,i} - H^T y_{avg}^k\|^2\), measures the discrepancy between the \(i\)th ES’s local variable and the local variables of its
serving users. The second term, i.e., \(\|Ay\|_2\), measures the discrepancy between different ESs’ local variables. If the sum of these two quantities is zero, it means that the proposed algorithm achieves a consensus in which all nodes’ (including ESs and users) local model parameters are equal to each other.

To gain insight into our result, we now turn to the terms on the right-hand side of (22) and (23). We see that the first term approaches 0 as \(k\) increases. The second term is an error term which is dependent on \(\epsilon^t\). Suppose we set \(\epsilon^t = 0, \forall t\), which means that the \(x^{(t)}_{ij}\)-subproblem is solved exactly. In this case, the second term vanishes and our proposed algorithm will eventually achieve consensus and obtain the optimal solution as \(k \to \infty\).

Nevertheless, in practice, it may be computationally expensive to find the exact solution of the \(x^{(t)}_{ij}\)-subproblem. Consider the case where \(\{\epsilon^t\}\) is a non-zero sequence. If \(\epsilon^t\) is fixed as a constant scalar, say \(\epsilon\), then the right-hand side of (22) and (23) is a function of \(\mu\) and \(\epsilon\). Recall that the value \(\mu\) is used to quantify the curviness of \(f_{ij}\). Specifically, a larger \(\mu\) indicates a more curvy \(f_{ij}\), and for a fixed \(\epsilon\), a more curvy function \(f_{ij}\) means that \(x^{(k+1)}_{ij}\) is more close to the optimal solution of the subproblem. Hence a larger \(\mu\) results in a smaller error. Although the second term on the right-hand side of (22) and (23) cannot be removed for a nonzero \(\epsilon\), our simulation results show that for a reasonable value of \(\epsilon\), our proposed algorithm can achieve an accurate solution close enough to the optimal one. Instead of choosing a fixed \(\epsilon\), an alternative is to employ a sequence \(\{\epsilon^t\}\) with decreasing values of \(\epsilon^t\). One option is to let \(\{\epsilon^{(t)}_1\}^{+\infty}_{t=2}\) be a summable sequence, say \(\epsilon^{(t)}_1 = t^{-2}\). For such a choice, \(\sum_t=1(\epsilon^t)^2\) is a finite number and thus the error term in (22) and (23) tends to 0 as \(k\) increases.

We now discuss the design of the matrix \(P\). As discussed in (17), in order to achieve decentralized implementation, \(P\) should satisfy \(\alpha^{-1}P = D - A^2A\), where \(D\) is a diagonal matrix. Moreover, as stated in Theorem 1, \(P\) should also satisfy the condition (21). Combining these two conditions leads to

\[
D \triangleright \frac{1}{\alpha} \left( \frac{1}{\alpha^2} - 1 \right) \frac{\sigma_2}{\sigma_2} HH^T + \frac{3}{4}A^T A.
\]  

(25)

To satisfy the above condition, we write \(D = D_{in} \otimes I_n\), where \(D_{in} \in \mathbb{R}^{l \times l}\) is a diagonal matrix. Note that \(H = H_{dig} \otimes I_n\), where \(H_{dig} \triangleq \text{bld} \{h_1; \ldots; h_l\}\) and \(h_i\) is an all-one row vector of size \(|S_i|\). Also, we have \(A \triangleq A_{in} \otimes I_n\). Thus (25) can be equivalently written as

\[
\left( D_{in} - \frac{1}{\alpha} \left( \frac{1}{\alpha^2} - 1 \right) \frac{\sigma_2}{\sigma_2} H_{dig} H_{dig}^T - \frac{3}{4} A_{in}^T A_{in} \right) \otimes I_n \triangleright 0 \Rightarrow D_{in} - \frac{1}{\alpha} \left( \frac{1}{\alpha^2} - 1 \right) \frac{\sigma_2}{\sigma_2} H_{dig} H_{dig}^T - \frac{3}{4} A_{in}^T A_{in} \otimes I_n \triangleright 0
\]  

(26)

Observe that \(H_{dig} H_{dig}^T \in \mathbb{R}^{l \times l}\) is a diagonal matrix with its \(i\)th diagonal element being \(|S_i|\). On the other hand, since \(A_{in}\) is the incidence matrix of the graph \(G\), \(A^T_{in} A_{in}\) is the Laplacian matrix of the graph \(G\). Let \(D_{L}\) be a diagonal matrix whose diagonal elements equal those of \(A^T_{in} A_{in}\). We have \(2D_{L} \triangleright A^T_{in} A_{in}\). Hence it can be readily verified that the matrix \(D\) defined as

\[
D = \left( \frac{1}{\alpha} \left( \frac{1}{\alpha^2} - 1 \right) \frac{\sigma_2}{\sigma_2} H_{dig} H_{dig}^T + \frac{3}{2} D_{L} \right) \otimes I_n
\]  

(27)

satisfies the condition (25).

In the following, we provide a proof of Theorem 1. We first define a function that will be frequently used:

\[
F^{(t)}_{(G, \alpha)} \triangleq (x^t - x^{(t)})^T G (x^t - x^{(t-1)})
\]  

(28)

where \(G\) is an arbitrary positive semidefinite matrix. Also, we introduce the following inequalities that will be used in our proof. Regarding (7), according to (2.1) in [55], we know that the following variational inequality holds for \(\forall x_i, y_i\):

\[
\sum_{i=1}^{l} (f_i(x_i) - f_i(x^*_i) + \langle \lambda^*_i, x_i - H_i^T y_i \rangle) + \langle \beta^*, Ay \rangle \geq 0,
\]  

(29)

where \(\lambda^*_i\) and \(\beta^*\) are the optimal dual variables. Employing the Cauchy-Schwarz inequality, we further have

\[
\sum_{i=1}^{l} (f_i(x_i) - f_i(x^*_i) + \|\lambda^*_i\|_2 \|x_i - H_i^T y_i\|_2) + \|\beta^*\|_2 \|Ay\|_2 \geq 0.
\]  

(30)

VI. PROOF OF THEOREM 1

The proof of Theorem 1 consists of three parts. In the first part, we establish an inequality (48). Then in the second part, based on (48), we obtain an inequality (52) that is close to our final results, except that the values of \(\{\lambda_i\}\) and \(\beta\) remain to be determined. At last, by assigning appropriate values for \(\{\lambda_i\}\) and \(\beta\) we obtain the desired results.

A. Part 1

1) The \(y^{k+1}\)-subproblem: Invoking the notations in (8), the \(y^{k+1}\)-subproblem, \(k + 1 < \bar{k}\), can be compactly written as

\[
y^{k+1} = \arg \min_y \frac{\alpha}{2} \|x^{k+1} - H^T y + \frac{\lambda}{\alpha} \|_2^2 + \frac{\gamma}{\alpha} \|Ay + \frac{\gamma}{\alpha} \|_2^2 + \frac{\gamma}{\alpha} \|y - y^{k}\|_2^2 - P \|y^{k+1} - y^k\|_2^2.
\]  

(31)

Taking the gradient of the objective function and set it to 0 yields

\[
0 = H(-\lambda^k + \alpha \sigma_1 (H^T y^{k+1} - x^{k+1})) + A^T (\beta^{k+1} + \sigma_2 Ay^{k+1}) + \frac{\gamma}{\alpha} P(y^{k+1} - y^k)
\]  

(32)

\[
\text{where } (a) \text{ is due to the update rule of } \lambda^{k+1} \text{ and } \beta^{k+1}, \text{ while (b) is due to the update rule of } \lambda^{k+1}. \text{ Analogously, if } k+1 = \bar{k}, \text{ it holds}
\]

\[
0 = -H \lambda^{k+1} + A^T \beta^{k+1} + \sigma_2 P(y^{\bar{k}} - y^{\bar{k}-1}).
\]  

(33)

Multiplying \(y^{\bar{k}} - y^{k+1}\) (resp. \(y^{k} - y^{k}\)) to both sides of (32) (resp. (33)) yields

\[
0 = (y^{\bar{k}} - y^{k+1}) (\alpha H \lambda^{k+1} + \alpha A^T \beta^{k+1} + \sigma_2 P(y^{k+1} - y^{k}) - \bar{k}, k+1 < \bar{k}),
\]

\[
0 = (y^{\bar{k}} - y^{k+1}) (-H \lambda^{k} + A^T \beta^{k} + \sigma_2 P(y^{k} - y^{k-1})),
\]  

(34)

\[
k+1 = \bar{k}.
\]
Additionally, according to the $\beta^{k+1}$-update in (14), we have
\[0 = (\beta - \beta^{k+1})T(\sigma^2 - (\beta^{k+1} - \beta^k) - A_y^{k+1}), \quad k + 1 < \bar{k},
0 = (\beta - \beta^k)T(\alpha \sigma^2 - (\beta^k - \beta^{k-1}) - A_y^k), \quad k + 1 = \bar{k}.
\] (35)
where $\beta$ is an arbitrary vector of the same dimension as $\beta^{k+1}$.

Summing (34) and (35) yields
\[0 = V^k + \alpha \sum_{t=1}^{k-1} V^t + \sigma_2 \sum_{t=1}^{k} F_t^t(p, y) + \sigma_2 \sum_{t=1}^{k} (\beta - \beta^t)^T(\beta^t - \beta^{t-1}),
\] (36)
where $F_t^t(p, y)$ is defined in (28) and
\[V^k \equiv (y^k - y^{k-1})T( - H_k A_T T^k) - (\beta - \beta^{k-1}T A_y^k),
V^t \equiv (y^t - y^{t-1})T( - H_k A_T T^k) - (\beta - \beta^{t-1}T A_y^t),
\] for $i < \bar{k}$.

2) The $x^{k+1}$-subproblem: Note that the $x^{k+1}$-subproblem is solved up to an $\epsilon^{k+1}$ accuracy, which means that
\[
\|\tau^{k+1}_{ij}\|_2 \leq \epsilon^{k+1} \quad \forall j \in I_{k+1}
\] (38)
where
\[
\tau^{k+1}_{ij} = \partial f_{ij}(x^{k+1}) + \lambda^k_{ij} + \sigma_1(x^{k+1}_{ij} - y^k_{ij})
\] (39)
Based on (38), we can arrive at the following inequality (see Appendix A)
\[0 \leq (\alpha - 1)(F^k + M^k + G^k) + \mathbb{E}_{a^{k+1}}[F^{k+1} + M^{k+1} + (1 - \alpha)G_i^{k+1} + T_i^{k+1} + \frac{\alpha |S| (k^{k+1})}{2\mu} |\{a^i_t\}]
\] (40)
where $\{a^i_t\}$ is used to represent $\{(a^i_t)_{i=1}^{k+1}\}_{t=1}^l$, $a^{k+1} \triangleq \hat{a}^{k+1} \otimes I_n$, $\hat{a}^{k+1} \in \mathbb{R}^{|S|}$ is a random binary vector with its $j$th element $a_{ij}^{k+1}$ equal to 1 if user $a_{ij}$ is selected, and 0 if otherwise, and $F^k_t \triangleq f_t(x^k_t) - f_t(x^*_{ij})$, $M^k_t \triangleq (x^k_t - x^*_{ij})^T \lambda^k_{ij}$,
\[G^k_t \triangleq \sigma_1(x^*_{ij} - x^k_t)^T (x^k_t - H^k_{ij} y^k),
\[
\tau^{k+1}_{ij} \triangleq \sigma_1(x^*_{ij} - x^{k+1}_{ij})^T H^k_{ij} (y^k - y^{k-1}).
\] (41)
Regarding the conditional expectation, we have
\[\mathbb{E}_x[f(x, y) | y] \geq \mathbb{E}_x[f(x, y)] \geq 0
\] if $f(x, y) \geq 0$ and
\[\mathbb{E}_x[f(x, y)] \geq \mathbb{E}_x[f(x, y)] dx \geq 0
\] (42)
Applying the above formula to (40) and summing the resulting inequalities for all $i$, we have
\[0 \leq \mathbb{E}_\{(a^i_t)_{i=1}^{k+1}\}_{t=1}^l \left[ \sum_{i=1}^l \left( (\alpha - 1)(F^k + M^k) + F^k_t + M^k_t + (1 - \alpha)(G^k_t - G_i^{k+1}) + T_i^{k+1} + \frac{\alpha |S| (k^{k+1})}{2\mu} \right) \right], \quad k + 1 \leq \bar{k},
\] (43)
Hereafter we omit the subscript in $\mathbb{E}$ for the sake of simplicity. Summing the above inequality for all $k + 1$s (up to $\bar{k}$) yields
\[0 \leq (\alpha - 1)(C_0 + C_1^k + \mathbb{E}[F^k + M^k + \alpha \sum_{t=1}^{k-1} (F_t^t + M_t^t) + (1 - \alpha)G^k_t + \sum_{t=1}^{k} T_t^t])
\] (44)
where $C_0 \triangleq (\alpha - 1)(F^k + M^k + G^k)$ and $C_1^k \triangleq \frac{\alpha |S| (k^{k+1})}{2\mu}$.

Note that (44) is due to the elimination of the repeated terms in the summation, (b) has invoked the fact that $M_t^k + (1 - \alpha)G^k_t = (x^k_t - x^*_{ij})^T \lambda^k_{ij}$ (since $\lambda^k_{ij} \leq -\lambda^k_{ij} + \alpha (\lambda^k_{ij} - \lambda^{k-1})$ and $\sigma_1(x^*_{ij} - H^k_{ij} y^k)$ $\lambda^k_{ij} - \lambda^{k-1}$), (c) has used the fact that (44-3) is 0 and (44-2) is 0, and (d) is a simple reorganization of the terms.

3) Combining: Summing up (44) for all $i, 1 \leq i \leq l$, and then summing the resulting inequality with (46) yields
\[0 \leq \sum_{t=1}^l \left( \mathbb{E}[A_t^k + \alpha \sum_{i=1}^{k-1} A_t^i + \sum_{t=1}^{k} T_t^i + (44-4)] + C_0^0 + C_1^k \right) + 4 \mu^2
\] (45)
\[= A^k + \alpha \sum_{t=1}^{k-1} A^i + \sum_{t=1}^{k} C_t^i + R
\] (46)
where $A_t^k \triangleq \mathbb{E}[V^k + \sum_{i=1}^{k-1} A^i], A_t^i \triangleq \mathbb{E}[V^t + \sum_{i=1}^{k-1} A^i], t < \bar{k}$, and $R$ represents the rest of the terms. According to the derivations attached in Appendix A, it holds $R \leq \tilde{C}^0(\{\lambda^i\}, \beta)$, where
\[\tilde{C}^0(\{\lambda^i\}, \beta) \triangleq \left( \sum_{i=1}^l \left( \frac{\alpha_1}{2} ||H^k_{ij} y^k - y^*_{ij}||^2_2 + \frac{\alpha_2}{2} ||\lambda^k_{ij} - \lambda_i||^2_2 \right) \right) + \frac{\alpha_3}{2} ||\beta - \beta^k||^2_2 + \alpha_4 \mu^2 ||A^0||^2_2 + \sum_{t=1}^{k} C_t^0
\] (47)
is a function of $\{\lambda^i\}$ and $\beta$. As such, (46) implies that
\[0 \leq A^k + \alpha \sum_{t=1}^{k-1} A^i + \sum_{t=1}^{l} C_t^i + \tilde{C}^0(\{\lambda^i\}, \beta)
\] (48)

B. Part II

Eliminating the repeated terms in $A^i$ (also using the fact that $x_i^* - H^k_{ij} y^*_{ij} = 0$ and $Ay^* - 0$), it can be derived that
\[A^k = \mathbb{E}[\sum_{i=1}^l (f_i(x^*_{ij}) - f_i(x^*_{ij}) - \langle \lambda^i, x^*_{ij} - H^k_{ij} y^*_{ij} \rangle -]
\]
\[ \langle \beta, Ay \rangle, 1 \leq t \leq k. \] 

Substituting the right hand side of (49) into (48) yields

\[
\sum_{i=1}^{l} C_{i} + \tilde{C}^{0}\left(\{\lambda_{i}\}, \beta\right) \geq -A \theta - \alpha \sum_{i=1}^{k-1} A^{t} \left( f_{i}(x_{i}^*) - f_{i}(x_{i}^*) + \alpha \sum_{i=1}^{k-1} (f_{i}(x_{i}^*) - f_{i}(x_{i}^*)) \right)
\]

\[
+ \sum_{i=1}^{l} (\lambda_{i} x_{i} - H_{i}^{T} y_{i}^*) + \langle \beta, Ay \rangle + \alpha \left( \sum_{i=1}^{l} (\lambda_{i} x_{i}^* - H_{i}^{T} y_{i}^*) \right) + \alpha \sum_{i=1}^{k-1} \langle \beta, Ay \rangle
\]

\[
\geq (1+\alpha(\tilde{k} - 1)) \mathbb{E} \left[ \sum_{i=1}^{l} (f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*)) \right] + \mathbb{E} \left[ \sum_{i=1}^{l} (\lambda_{i} x_{avg,i}^* - H_{i}^{T} y_{avg,i}^*) + \langle \beta, Ay \rangle \right]
\]

(50)

where \( \tilde{x}_{avg,i} \triangleq \sum_{t=1}^{k} \delta x_{i}^t, y_{avg,i} \triangleq \sum_{t=1}^{k} \delta y_{i}^t, \delta = (1 + \alpha(\tilde{k} - 1))^{-1}, \delta = \alpha \delta t, \tilde{k} < k, \) and (a) has invoked Jensen's inequality (1) in the following manner:

\[
(\tilde{S})[1] \geq (1+\alpha(\tilde{k} - 1)) (\sum_{t=1}^{l} f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*))
\]

(51)

Multiplying \((1+\alpha(\tilde{k} - 1))^{-1}\) to both sides of (50) leads to

\[
\leq \frac{\tilde{C}^{0}((\lambda_{i}), \beta)}{1+\alpha(\tilde{k} - 1)} + \frac{1}{2\alpha} \sum_{i=1}^{k} (\lambda_{i})^{2} \sum_{i=1}^{l} |S_{i}| |
\]

(52)

where we have invoked the definition of \( C_{i} \) that is given below (44).

C. Part III

To obtain our final result, let \( \lambda_{i} \) and \( \beta \) be chosen as

\[
\lambda_{i} = 2(\| \lambda_{i} \|_{2} + \xi) \cdot \frac{x_{avg,i}^* - H_{i}^{T} y_{avg,i}^*}{\| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2}},
\]

\[
\beta = 2(\| \beta \|_{2} + \xi) \cdot \frac{A y_{avg}^{*}}{\| A y_{avg}^{*} \|_{2}}
\]

(53)

where \( \xi \) is a positive scalar. Substituting (53) into both sides of (52) yields

\[
\mathbb{E} \left[ \sum_{i=1}^{l} (f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*)) \right] + 2(\| \beta \|_{2} + \xi) \| Ay_{avg}^{*} \|_{2}
\]

\[
+ 2 \sum_{i=1}^{l} (\| \lambda_{i} \|_{2} + \xi) \| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2}
\]

\[
\leq \frac{\tilde{C}^{0}}{1+\alpha(\tilde{k} - 1)} + \frac{1}{2\alpha} \sum_{i=1}^{k} (\lambda_{i})^{2} \sum_{i=1}^{l} |S_{i}|
\]

(54)

where \( \tilde{C}^{0} \) is an upper bound of \( \tilde{C}^{0}(\{\lambda_{i}\}, \beta) \) (recall that \( \tilde{C}^{0}(\{\lambda_{i}\}, \beta) \) is finite since \( \lambda_{i} \) and \( \beta \) have finite length).

Regarding (54)-1, we have

\[
\geq \sum_{i=1}^{l} (f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*))
\]

(55)

\[
\geq - \sum_{i=1}^{l} (f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*))
\]

(56)

\[
\mathbb{E} \left[ \sum_{i=1}^{l} f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*) \right] \geq 2(\| \beta \|_{2} + \xi) \| Ay_{avg}^{*} \|_{2}
\]

\[
+ 2 \sum_{i=1}^{l} (\| \lambda_{i} \|_{2} + \xi) \| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2}
\]

\[
\leq \frac{\tilde{C}^{0}}{1+\alpha(\tilde{k} - 1)} + \frac{1}{2\alpha} \sum_{i=1}^{k} (\lambda_{i})^{2} \sum_{i=1}^{l} |S_{i}|
\]

(54)

which means that

\[
(\text{54})-1 \geq \left\| \sum_{i=1}^{l} (f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*)) \right\|
\]

(57)

Combining (54) and (54) yields

\[
\mathbb{E} \left[ \left\| \sum_{i=1}^{l} f_{i}(x_{avg,i}^*) - f_{i}(x_{i}^*) \right\| \right] \leq (\text{54})-2
\]

(58)

Additionally, we have

\[
(\text{54})-1 \geq \sum_{i=1}^{l} (\| \lambda_{i} \|_{2} + \xi) \| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2} + \| \beta \|_{2} + \xi \| Ay_{avg}^{*} \|_{2}
\]

\[
\geq \psi \left( \sum_{i=1}^{l} (\| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2} + \| Ay_{avg}^{*} \|_{2}) \right)
\]

(59)

where \( \psi \triangleq \min \{ \{ \| \lambda_{i} \|_{2} + \xi \}_{i=1}, \| \beta \|_{2} + \xi \}. \)

Combining (59) and (54) leads to

\[
\mathbb{E} \left[ \sum_{i=1}^{l} \| x_{avg,i}^* - H_{i}^{T} y_{avg,i}^* \|_{2} + \| Ay_{avg}^{*} \|_{2} \right]
\]

\[
\leq \psi^{-1}(\text{54})-2
\]

(60)

Note that (58) and (60) are exactly the results in Theorem 1. Our proof is completed here.

VII. SIMULATION RESULTS

In this section, we provide simulation results to illustrate the performance of the proposed ADMM algorithm (abbreviated as CFL-ADMM). To demonstrate the efficiency of the algorithm, we compare it with the GT-SAGA (gradient tracking-stochastic average gradient) method [56] and the D-SGD (decentralized stochastic gradient descent) method [22]. We first discuss the setup of our experiments and the implementation details of respective algorithms.

A. Setup

1) Experimental Setup: In our experiments, the CFL network consists of \( l = 20 \) ESs and 1000 users. We assume that each ES serves \( S_{i} = 50 \) users. The communication network of ESs is depicted in Fig. 2. We consider an \( \ell_{2} \)-regularized logistic regression problem:

\[
\min_{x \in \mathbb{R}^{d}} \sum_{i=1}^{l} S_{i} f_{ij}(x),
\]

(61)

where \( f_{ij}(x) = g_{ij}(x) + h_{ij}(x), g_{ij}(x) = \frac{1}{2} \| x \|_{2}^{2}, \kappa = 0.01, \) and

\[
h_{ij}(x) = \sum_{j=1}^{W_{ij}} \left( -y_{ij,j} \cdot \log((1 + e^{-w_{ij,j} \cdot x}))^{-1} - \right.
\]

\[
\left. w_{ij,j} \right)
\]

Fig. 2. Topology of the communication network of ESs.
\[(1 - y_{ij,j'}) \cdot \log \left( 1 - \left( 1 + e^{-\omega_{ij,j'} \cdot x} \right)^{-1} \right) \]

in which \(\{\omega_{ij,j'} \in \mathbb{R}^n, y_{ij,j'} \in \{0, 1\}\}\) is the \(j'\)th training sample stored at user \(u_{ij}\). Note that \(f_{ij}\) is strongly convex and its gradient is Lipschitz continuous.

Our experiments are based on the Credit 1 dataset\(^4\), which consists of 30000 real data samples. Each sample includes 24 entries, in which the first 23 entries along with a bias value 1 constitute \(\omega_{ij,j'} \in \mathbb{R}^{24}\) in (62) and the last entry is the corresponding binary label \(y_{ij,j'}\). We randomly choose 20000 samples for training and each user is assigned with 20 samples. For the proposed CFL-ADMM, the \(x_{ij}^{k+1}\)-subproblem is solved via a simple gradient descent method. Since the gradient of the objective function in the \(x_{ij}^{k+1}\)-subproblem is Lipschitz continuous, the gradient descent method is guaranteed to converge to the optimal solution provided that the stepsize is appropriately selected. The initial point of the gradient descent method for solving the \(x_{ij}^{k+1}\)-subproblem is chosen to be the solution obtained in the last iteration, i.e. \(x_{ij}^k\).

2) Implementations of GT-SAGA and D-SGD: Note that both GT-SAGA and D-SGD were originally developed for D2D networks. Nevertheless, they can be easily adapted to the considered CFL framework. Take GT-SAGA as an example. The GT-SAGA aims to solve problems of the same form as (5). In GT-SAGA, it is assumed that each data-holder holds a local objective function \(f_i(x) = \sum_{j=1}^{S_i} f_{ij}(x; D_{ij})\), where \(D_{ij}\) represents the data set corresponding to the loss function \(f_{ij}\). The GT-SAGA assumes that there is no user and the data-holders collaboratively solve (5). In each iteration, each data-holder randomly selects a portion of \(f_{ij}\) to update the local model, followed by an information exchange between the data-holders to enforce the consensus among local variables. We can adapt the GT-SAGA to our CFL framework by distributing \(f_{ij}\) and \(D_{ij}\) to user \(u_{ij}\). In such a setting, each user first downloads the model vector, say \(y_{ij}^{k+1}\), from the ES, followed by the computation of the gradient of \(f_{ij}\) at \(y_{ij}^{k+1}\), and then uploads the gradient vector to its associated ES for aggregation. The D-SGD method can be adapted to our CFL framework in a similar way.

Note that when adapting those decentralized stochastic gradient-based methods to the CFL framework, only a single gradient descent step is allowed to be performed at each iteration. Those methods which perform multiple rounds of gradient descent at each iteration\(^7\)–\(^10\), \(^24\)–\(^26\) are not applicable. This is because for those decentralized stochastic gradient-based methods, each user is required to upload the gradient of \(f_{ij}\) to its associated ES. More specifically, suppose the user \(u_{ij}\) receives a model parameter vector \(y_{ij}^{k+1}\) from the \(i\)th ES at the \((k+1)\)th iteration. Then the gradient of \(f_{ij}\) should be computed at the point \(y_{ij}^{k+1}\). Performing multiple steps of gradient descent at each user and then reporting the final gradient will lead to incorrect results.

B. Results on \(\ell_2\)-Regularized Logistic Regression

To evaluate the performance of the proposed method, the following metric is introduced, namely, an optimality gap \(d_k\):

\[d_k \triangleq \frac{1}{\|x^*\|^2_2} \sum_{i=1}^{l} \sum_{j=1}^{S_i} \|x_{ij}^k - x^*\|^2_2,\]  

where \(x_{ij}^k\) is the solution obtained at the \(k\)th iteration, and \(x^*\) is the optimal solution of the problem. Note that the optimality metric is defined by using the instantaneous output \(x_{ij}^k\) instead of the time average defined in Theorem 1. This is because the time average is overly pessimistic and leads to a relatively slow convergence speed.

Fig. 3 plots the optimality gap of the proposed CFL-ADMM vs. the number of iterations under different selection probabilities \(\alpha\) and different error tolerances \(\epsilon\). Results are averaged over 100 independent runs, with users randomly selected for each run and each iteration. Clearly, when using a nonzero \(\epsilon\), the algorithm does not converge to the true solution \(x^*\). Instead, it converges to a neighborhood of \(x^*\). From Fig. 3 it can be observed that the converged point is closer to \(x^*\) when a smaller \(\epsilon\) is employed. In addition, it is observed that a larger user selection probability \(\alpha\) leads to a faster convergence speed. Nevertheless, the performance improvement becomes insignificant as the selection probability exceeds \(\alpha > 0.3\). Since the average amount of communication overhead grows linearly with \(\alpha\), it is better to choose a moderate value of \(\alpha\) to strike a reasonable balance between the performance and the communication cost.

In Fig. 4 we evaluate the performance of the proposed algorithm under different values of error tolerance \(\epsilon\). The user selection probability is set to \(\alpha = 0.3\). We see that the choice of \(\epsilon\) does not affect the convergence speed of the proposed algorithm, which is in consistent with the results reported in Theorem 1.

Next, we compare the performance of our proposed algorithm with GT-SAGA and D-SGD. The parameters of respective algorithms are tuned to achieve the best performance. For our proposed algorithm, instead of using a fixed \(\epsilon\), we employ a decreasing error tolerance sequence \(\{\epsilon^k\}\) to ensure that it converges to the optimal solution. More specifically, we set \(\epsilon^k = \frac{\epsilon^{k-1}}{1+k}\). Fig. 5 plots the optimality gap of respective algorithms vs. the number of iterations. With a same \(\alpha\), all three algorithms have the same per-iteration communication cost. It can be observed that the proposed CFL-ADMM converges much faster than the other two stochastic gradient-based algorithms, which implies that the proposed algorithm can attain a solution of a same quality with much fewer rounds of communication, and thus achieves a higher communication efficiency.

We would like to point out that the improved communication efficiency of the proposed algorithm comes at the expense of involving more computations at users. Specifically, for GT-SAGA and D-SGD, each user only needs to compute the gradient of its local objective function once at each iteration, while for the proposed algorithm, each user needs to solve a subproblem up to a certain accuracy, which usually requires several or tens of iterations of gradient descent. Nevertheless, nowadays the computing power of mobile devices such as smartphones has increased to an impressive level. In contrast,
as the information are usually transmitted wirelessly from users to ESs, communications are more expensive and power-consuming than computations. In addition, more rounds of communications result in a higher latency, which is also a critical factor that should be considered in FL applications. In fact, since the initial point of the \( \alpha_{ij}^{k+1} \) subproblem of CFL-ADMM is chosen as \( x_{ij}^{k} \), it only takes several iterations of gradient descent (except for the first few tens of ADMM iterations) to reach the specified accuracy. Therefore the disadvantage of the proposed algorithm on the computational aspect is not that significant.

**VIII. CONCLUSIONS**

In this paper, we introduced a hybrid centralized and decentralized FL framework (referred to as CFL) to enhance the scalability of FL. The framework consists of multiple servers, in which each server serves an individual set of devices as in the conventional FL framework, and multiple servers form a decentralized network. An ADMM algorithm was developed within such a hybrid framework. The proposed ADMM randomly selects each user with a certain probability at each iteration, thus alleviating the heavy communication burden caused by the interaction between the servers and the users. Moreover, the proposed ADMM allows the subproblem to be inexactly solved at each user, making it amiable for machine learning applications. Our theoretical analysis showed that the proposed ADMM enjoys a \( O(1/k) \) convergence rate. Numerical results were provided to illustrate the effectiveness and superiority of the proposed ADMM.

**APPENDIX A**

**FROM (38) TO (40)**

Since (39) holds for \( \forall j \in I_{i}^{k+1} \), we can compactly rewrite it as

\[
\alpha_{ij}^{k+1} \odot \tau_{i}^{k+1} = \alpha_{ij}^{k+1} \odot (\partial f_{i}(x_{ij}^{k+1}) + \lambda_{ij}^{k} + \sigma_{ij}(x_{ij}^{k+1} - H_{i}^{T}y_{ij}^{k}))
\]

where \( x_{i}, \lambda_{i}, H_{i} \) and \( y_{ij} \) are defined in (38), \( \odot \) represents element-wise product, \( \tau_{i} \trianglerighteq \sum_{l=1}^{I_{i}}[\tau_{i1}|\tau_{i2}|\cdots|\tau_{i|S_{i}|}] \), \( \partial f_{i}(x_{i}) \triangleq [\partial f_{i1}(x_{i1}); \cdots; \partial f_{i|S_{i}|}(x_{i|S_{i}|})] \), \( \alpha_{ij}^{k+1} \triangleq \alpha_{ij}^{k+1} \odot 1_{n} \) and \( \hat{a}_{ij}^{k+1} \in \mathbb{R}^{[S_{i}]} \) is a random binary vector with its \( j \)th element \( \hat{a}_{ij}^{k+1} \) equal to 1 if user \( u_{ij} \) is selected while equal to 0 otherwise. Note that the \( j \)th element of \( \alpha_{ij}^{k+1} \) has a probability of \( \alpha \) (resp. \( 1 - \alpha \)) to be equal to 1 (resp. 0). Multiplying \( \alpha_{ij}^{k+1} \odot (x_{i}^{k} - x_{i}^{k+1}) \) to both sides of (64) and then taking the expectation of the resulting equality yields

\[
E_{\alpha_{ij}^{k+1}} \left[ (\alpha_{ij}^{k+1} \odot (x_{i}^{k} - x_{i}^{k+1}))^{T} \tau_{i}^{k+1} \right] \{\alpha_{ij}^{k+1}\}
\]

\[
= E_{\alpha_{ij}^{k+1}} \left[ (\alpha_{ij}^{k+1} \odot (x_{i}^{k} - x_{i}^{k+1}))^{T} \left( \partial f_{i}(x_{ij}^{k+1}) + \lambda_{ij}^{k} + \sigma_{ij}(x_{ij}^{k+1} - H_{i}^{T}y_{ij}^{k}) \right) \right] \{\alpha_{ij}^{k+1}\},
\]

where \( \{\alpha_{ij}^{k+1}\} \) is used to represent \( \{\alpha_{ij}^{k+1}\}_{i=1}^{k} \) and the above equality comes from the fact that \( (\alpha_{ij}^{k+1} \odot x)^{T}(\alpha_{ij}^{k+1} \odot y) = (\alpha_{ij}^{k+1} \odot x)^{T} y, \forall x, y \). Clearly, taking an expectation w.r.t.
\(a_i^{k+1}\) is equivalent to taking an expectation w.r.t. \(\{\hat{a}_{ij}^{k+1}\}\).

Note that the expectation in (65) is taken only w.r.t. \(a_i^{k+1}\) instead of all random vectors because the randomness of other random vectors, say \(x_i^{k+1}\), originates in that of \(a_i^{k+1}\). Next, we separately upper bound the terms on the right hand side of (65).

1) Bounding the first term: Consider the first term in (65), we have

\[
\begin{align*}
\mathbb{E}_{a_i^{k+1}}[(a_i^{k+1} \circ (\sigma_* - x_i^{k+1}))^T \partial f_i(x_i^{k+1})] & \{a_i^j\} \\
\leq & \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) \right. \\
& \left. - f_i(x_i^{k+1}) - \frac{1}{2} \| x_i^{k+1} - \hat{x}_i^{k+1} \|^2 \right] \{a_i^j\} \\
= & \alpha \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) + \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) \right. \\
& \left. - f_i(x_i^{k+1}) - \frac{1}{2} \| x_i^{k+1} - \hat{x}_i^{k+1} \|^2 \right] \{a_i^j\} \\
= & (\alpha - 1) \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) - \frac{1}{2} \| x_i^{k+1} - \hat{x}_i^{k+1} \|^2 \{a_i^j\}
\end{align*}
\]

(66)

where (a) has invoked (2). (b) is because

\[
\mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) \right] \{a_i^j\} = \alpha \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right)
\]

and (c) is due to

\[
f_i(x_i^{k}) - f_i(x_i^{k+1}) + \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) \right] \{a_i^j\}
\]

\[
\begin{array}{c}
\begin{align*}
= & f_i(x_i^{k}) - f_i(x_i^{k+1}) + \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) \right] \{a_i^j\} \\
= & \mathbb{E}_{a_i^{k+1}} \left[ f_i(x_i^{k}) - f_i(x_i^{k+1}) \right] \{a_i^j\},
\end{align*}
\end{array}
\]

(68)

in which the first equality is because \(x_i^{k+1} = x_i^0\) when \(\hat{a}_{ij}^{k+1} = 0\). Thus we have

\[
\sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) = \sum_{j=1}^{\mid S_i \mid} \left( f_i(x_i^{k}) - f_i(x_i^{k+1}) \right) = f_i(x_i^{k}) - f_i(x_i^{k+1})
\]

(69)

Note that the expectation in the second line of (68) can not be removed since \(x_i^{k+1}\) is a random vector determined by \(a_i^{k+1}\).

2) Bounding the rest terms: Regarding these terms, we have

\[
\begin{align*}
\mathbb{E}_{a_i^{k+1}} \left[ (a_i^{k+1} \circ (\sigma_* - x_i^{k+1}))^T (\lambda_i^k + \sigma_i(x_i^{k+1} - H_i^T y_i^{k})) \right] \{a_i^j\} \\
= \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( (x_i^{k} - x_i^{k+1})^T \lambda_i^j + (x_i^{k} - x_i^{k+1})^T \lambda_i^j \right) \\
+ \sigma_i(x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - H_i^T y_i^{k}) \right] \{a_i^j\} \\
= \alpha \left( x_i^{k} - x_i^{k+1} \right)^T \lambda_i^k + \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \left( (x_i^{k} - x_i^{k+1})^T \lambda_i^j \right) \\
+ \sigma_i(x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - H_i^T y_i^{k}) \right] \{a_i^j\} \\
= (\alpha - 1) \left( x_i^{k} - x_i^{k+1} \right)^T (x_i^{k+1} - H_i^T y_i^{k}) + \sigma_i \mathbb{E}_{a_i^{k+1}} \left[ (x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - H_i^T y_i^{k}) \right] \{a_i^j\} \\
+ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} \sigma_i \left( x_i^{k} - x_i^{k+1} \right)^T (x_i^{k+1} - H_i^T y_i^{k}) \{a_i^j\}
\end{align*}
\]

(70)

where (a) is because \(\mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} (x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - H_i^T y_i^{k}) \right] \{a_i^j\} = \alpha \left( x_i^{k} - x_i^{k+1} \right)^T \lambda_i^k\), \(\theta\) and \((c)\) have invoked the same logic as in (68). Regardting \([90]-1\) and \([90]-2\), we have

\[
\begin{align*}
\mathbb{E}_{a_i^{k+1}} \left[ (x_i^{k} - x_i^{k+1})^T (\lambda_i^k + \sigma_i(x_i^{k+1} - H_i^T y_i^{k})) \right] \{a_i^j\} \\
= \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} (x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - H_i^T y_i^{k}) \right] \{a_i^j\} \\
= \alpha \left( x_i^{k} - x_i^{k+1} \right)^T (\lambda_i^k + \sigma_i H_i^T (y_i^{k+1} - y_i^{k})) \{a_i^j\} \\
= \alpha \left( x_i^{k} - x_i^{k+1} \right)^T (\lambda_i^k + \sigma_i H_i^T (y_i^{k+1} - y_i^{k})) \{a_i^j\}
\end{align*}
\]

(71)

where \(\lambda_i^{k+1}\) is defined in (15), \((a)\) and \((c)\) have invoked (15), (b) is because

\[
\begin{align*}
\left( x_i^{k} - x_i^{k+1} \right)^T (x_i^{k} - x_i^{k+1}) + \\
\sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} (x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - x_i^{k}) = 0
\end{align*}
\]

(72)

since \(x_i^{k+1} = x_i^k\), \(\forall i \in T^{k+1}\), and \((d)\) is due to

\[
\lambda_i^k - \lambda_i^{k+1} (\alpha) = -\alpha (\lambda_i^k - \lambda_i^{k+1}) (\alpha) = -\alpha \sigma_i (x_i^{k+1} - H_i^T y_i^{k+1})
\]

in which \((e)\) and \((f)\) come from the second line and the first line of (11), respectively. Substituting (66), (70) and (71) into (65) yields

\[
0 \leq (\alpha - 1)(F_i^k + M_i^k + G_i^k) + \mathbb{E}_{a_i^{k+1}} \left[ \sum_{j=1}^{\mid S_i \mid} \hat{a}_{ij}^{k+1} (x_i^{k} - x_i^{k+1})^T (x_i^{k+1} - x_i^{k}) \right] \{a_i^j\}
\]

(73)
where $F_i^k \triangleq f_i(x_i^*) - f_i(x_i^k)$, $M_i^k \triangleq (x_i^* - x_i^k)^T \lambda_i^k$,
$G_i^k \triangleq \sigma_1 (x_i^* - x_i^k)^T (x_i^* - H_i^T y_i^*)$, 
$T_i^{k+1} \triangleq \sigma_1 (x_i^* - x_i^{k+1})^T H_i^T (y_i^{k+1} - y_i^-)^T$,
and the left hand side of (65) has been moved to the right hand side of (73), i.e., the first term in (73)-1. Having
(73)-1, we have
\[
[73]-1 \leq \sum_{i=1}^{\lceil \hat{t} \rceil} \hat{a}_{ij}^k \left( \frac{\nu_i}{2} \| x_i^* - x_i^{k+1} \|_2^2 + \frac{\nu_i}{2} \| \tau_{ij}^k \|_2^2 \right) - \frac{\nu_i}{2} \| x_i^* - x_i^{k+1} \|_2^2 \leq \frac{(\nu_i + 1) \sum_{i=1}^{\lceil \hat{t} \rceil} \hat{a}_{ij}^k}{2} \left( \frac{\nu_i}{2} \| x_i^* - x_i^{k+1} \|_2^2 \right),
\]
where (a) has invoked (4) and (b) is because $\| \tau_{ij}^k \|_2 \leq \epsilon_{k+1}$, see (16). Substituting (75) into (73) and also using the fact that $\mathbb{E}_{a_{ij}^k} \left[ \sum_{i=1}^{\lceil \hat{t} \rceil} \hat{a}_{ij}^k \right] = \alpha |S_i|$ yields the desired result.

**APPENDIX B**

**Proving $R \leq C^0(\{\lambda_i\}, \beta)$**

First notice that
\[
R = \sum_{i=1}^{n} \left( C_i^0 + \mathbb{E} \left[ \sum_{k=1}^{\hat{t}} T_i^k + \frac{1}{\sigma_1} (\lambda_i^k - \lambda_i) (T_i^k - \lambda_i^k) \right] + \mathbb{E} \left[ \sum_{k=1}^{\hat{t}} (\lambda_i^k - \lambda_i) (T_i^k - \lambda_i^k) \right] \right) + \sigma_2 \sum_{i=1}^{n} F_{i}^T (\rho, y) + \frac{\sigma_2}{\alpha_2} \sum_{i=1}^{n} \sum_{k=1}^{\hat{t}} (\beta - \beta^k)^T (\beta - \beta^k). 
\]
We then separately upper bound some of the terms in $R$ to prove the claim.

1) **Bounding $T_i^k$:** Regarding this term, first notice that
\[
\lambda_i^k \triangleq \lambda_i^k - \sigma_1 (x_i^* - H_i^T y_i^*), \quad x_i^* - H_i^T (y_i^* - y_i^k) = 0
\]
where (a) comes from (15). Thus we have
\[
T_i^k = - (\lambda_i^k - \lambda_i^{k-1})^T H_i^T (y_i^k - y_i^{k-1}) + \sigma_1 (H_i^T (y_i^* - y_i^k)) - \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2
\]
\[
\leq \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2 - \frac{\alpha_2}{\alpha_2} \| H_i^T (y_i^* - y_i^{k-1}) \|_2^2 
\]
(78)

2) **Bounding $T_i^t$, $t \leq \hat{k}$:** Similar to (77), we can deduce that
\[
x_i^* - x_i^t = - \frac{1}{\alpha_2} (\lambda_i^t - \lambda_i^{t-1}) + H_i^T (y_i^t - y_i^k), \quad t \leq \hat{k}
\]
where the inequality is due to (15) as well as the fact that $x_i^* - H_i^T y_i^* = 0$. Substituting the right hand side of (79) into $T_i^t$, we have
\[
T_i^t = - \frac{1}{\alpha_2} (\lambda_i^t - \lambda_i^{t-1})^T H_i^T (y_i^t - y_i^{t-1}) + \sigma_1 (H_i^T (y_i^t - y_i^k)) - \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^t - y_i^{k-1}) \|_2^2
\]
\[
\leq \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^t - y_i^{t-1}) \|_2^2 - \frac{\alpha_2}{\alpha_2} \| H_i^T (y_i^t - y_i^k) \|_2^2 
\]
(79)

where (a) has invoked (4) and (b) is because $\| \tau_{ij}^k \|_2 \leq \epsilon_{k+1}$, see (16). Substituting (78) into (73) and also using the fact that $\mathbb{E}_{a_{ij}^k} \left[ \sum_{i=1}^{\lceil \hat{t} \rceil} \hat{a}_{ij}^k \right] = \alpha |S_i|$ yields the desired result.

3) **Bounding the rest terms:** Regarding these terms, by (3) we have
\[
(-\lambda_i^k - \lambda_i) \leq -0.5(\| \lambda_i^k - \lambda_i \|_2^2 + \| \lambda_i^{k-1} - \lambda_i^k \|_2^2 - \| \lambda_i^{k-1} - \lambda_i \|_2^2)
\]
(80)

4) **Combining:** Substituting (78), (80) and (81) into $R$, we have
\[
R \leq \sum_{i=1}^{n} C_i^0 + \mathbb{E} \left[ \sum_{i=1}^{n} \left( \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2 - \| H_i^T (y_i^* - y_i^{k-1}) \|_2^2 \right) + \frac{\alpha_2}{\sigma_2} \| \lambda_i^k - \lambda_i \|_2^2 - \| \lambda_i^{k-1} - \lambda_i \|_2^2 \right] 
+ \frac{\alpha_2}{\alpha_2} \sum_{i=1}^{n} \sum_{k=1}^{\hat{t}} (\beta - \beta^k)^T (\beta - \beta^k).
\]
(82)

Eliminating the repeated terms in the right hand side of (82) leads to
\[
R \leq \sum_{i=1}^{n} C_i^0 + \mathbb{E} \left[ \sum_{i=1}^{n} \left( \frac{\alpha_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2 - \| H_i^T (y_i^* - y_i^{k-1}) \|_2^2 \right) + \frac{\alpha_2}{\sigma_2} \| \lambda_i^k - \lambda_i \|_2^2 - \frac{\alpha_2}{\sigma_2} \| \lambda_i^{k-1} - \lambda_i \|_2^2 \right] 
\]
(83)

where the second inequality is obtained by defining
\[
C_i^0(\{\lambda_i\}, \beta) \triangleq \sum_{i=1}^{n} C_i^0 + \left( \sum_{i=1}^{n} \frac{\alpha_2}{\sigma_2} \| H_i^T (y_i^* - y_i^0) \|_2^2 + \frac{1}{\sigma_2} \| \lambda_i^0 - \lambda_i \|_2^2 \| H_i^T (y_i^* - y_i^0) \|_2^2 \right), 
\]
(84)

where (a) and (b) are obtained similarly as in (78).

4) **Combining:** Substituting (78), (80) and (81) into $R$, we have
\[
R \leq \sum_{i=1}^{n} C_i^0 + \mathbb{E} \left[ \sum_{i=1}^{n} \left( \frac{\sigma_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2 - \| H_i^T (y_i^* - y_i^{k-1}) \|_2^2 \right) + \frac{\alpha_2}{\sigma_2} \| \lambda_i^k - \lambda_i \|_2^2 - \frac{\alpha_2}{\sigma_2} \| \lambda_i^{k-1} - \lambda_i \|_2^2 \right] 
\]
(82)

Eliminating the repeated terms in the right hand side of (82) leads to
\[
R \leq \sum_{i=1}^{n} C_i^0 + \mathbb{E} \left[ \sum_{i=1}^{n} \left( \frac{\alpha_2}{\alpha_2} \| H_i^T (y_i^* - y_i^k) \|_2^2 - \| H_i^T (y_i^* - y_i^{k-1}) \|_2^2 \right) + \frac{\alpha_2}{\sigma_2} \| \lambda_i^k - \lambda_i \|_2^2 - \frac{\alpha_2}{\sigma_2} \| \lambda_i^{k-1} - \lambda_i \|_2^2 \right] 
\]
(83)

where the second inequality is obtained by defining
\[
C_i^0(\{\lambda_i\}, \beta) \triangleq \sum_{i=1}^{n} C_i^0 + \left( \sum_{i=1}^{n} \frac{\alpha_2}{\sigma_2} \| H_i^T (y_i^* - y_i^0) \|_2^2 + \frac{1}{\sigma_2} \| \lambda_i^0 - \lambda_i \|_2^2 \| H_i^T (y_i^* - y_i^0) \|_2^2 \right), 
\]
(84)

where (a) and (b) are obtained similarly as in (78).
\[ - \sum_{t=1}^{k} \left( \frac{\alpha}{2} \| y_t - y_{t-1} \|^2_P + \frac{\alpha}{\sigma^2} \| \beta_t - \beta_{t-1} \|^2_2 \right) \] (84)

and also by omitting some negative terms in the right hand side of the first inequality. Next, we separately bound the terms in \( C^2 \). According to the definitions of \( H_t \) and \( H \), i.e. (8), it holds

\[
= \left( \frac{\alpha}{2} - \frac{\sigma^2}{2} \right) \| H_0^T ( y_1 - y_{-1} ) \|_2^2.
\] (85)

Meanwhile, regarding \( \sum_{t=1}^{k-1} \| \beta_t - \beta_{t-1} \|^2_2 \) we have

\[
= \frac{\alpha}{2} \sum_{t=1}^{k-1} \| A y_t - A y_{t-1} \|^2_2 - \frac{\sigma^2}{2} \| A y_0 \|^2_2
\]

\[
\geq \frac{\alpha}{2} \sum_{t=1}^{k-1} \| A (y_t - y_{t-1}) \|^2_2 - \frac{\sigma^2}{2} \| A y_0 \|^2_2
\]

\[ \sum_{t=1}^{k-1} (\| A y_t \|^2_2 + \| A y_{t-1} \|^2_2) - \frac{\sigma^2}{2} \| A y_0 \|^2_2 \]

where \((a)\) comes from (14) and \((b)\) is due to the fact that \( x + y \leq 2 \| x \|_2 + 2 \| y \|_2, \forall x, y. \) Substituting (85) and (86) into \( C^2 \) yields

\[ C^2 \leq \left( \frac{\alpha}{2} - \frac{\sigma^2}{2} \right) \| y - y_{-1} \|^2_P + \frac{\alpha}{\sigma^2} \| \beta - \beta_{-1} \|^2_2 \]

\[ + \left( \frac{\alpha}{2} - \frac{\sigma^2}{2} \right) \sum_{t=1}^{k} \| H_0^T ( y_t - y_{t-1} ) \|_2^2 - \frac{\sigma^2}{2} \sum_{t=1}^{k-1} \| y_t - y_{t-1} \|_2^2 \]

\[ \leq \frac{\alpha}{2} \| A y_0 \|^2_2 \]

which is the second inequality is due to the condition imposed on \( P \), i.e., (21). Substituting (87) into (83), and defining \( C^0(\{ \lambda_1 \}, \beta) \triangleq C^0(\{ \lambda_2 \}, \beta) + \frac{\alpha}{\sigma^2} \| A y_0 \|_2^2 \), we obtain the desired result.

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