Dynamic valid models for the conservative
Hénon-Heiles system

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Abstract. In this work the discretization of the Hénon-Heiles system obtained by applying the Monaco and Normand-Cyrot method is investigated. In order to obtain dynamically valid models, several approaches covering from the choice of terms in the difference equation originated from the discretization process to the increase of the discretization order are analyzed. As a conclusion it is shown that discretized models that preserve both the symmetry and the stability of their continuous counterpart can be obtained, even for large discretization steps.

1. Introduction

When dealing with the solution of conservative nonlinear differential equations, several problems such as energy loss and symmetry break can occur. In order to avoid such problems as much as possible, several numerical integration methods can be found in the literature \cite{1}. Although the primary objective of these methods is the solution itself, a possible and interesting by-product is a difference equation (a discretized model) that hopefully reproduces the same behavior as the one generated by the original differential equations.

In \cite{2} Mickens proposes a discretization scheme that it is claimed to be robust to numerical instabilities. Such instabilities may occur when difference equations do not meet restrictions imposed by the differential equations: conservation of energy, monotonicity, boundedness, and positivity, just to mention a few. A different discretization scheme was proposed by Monaco and Normand-Cyrot \cite{3}, which was studied in \cite{4} in the context of dynamical systems that can exhibit chaotic behavior.

The authors of \cite{5, 6} have successfully obtained discretized models for a dissipative system using the aforementioned method. The models found there are topologically equivalent to the original continuous system, except for a small displacement in the parameter space. However, for conservative systems, \cite{7} shows that the symmetry is not conserved and numerical instabilities may occur when the discretization step increases, whether for the discretization scheme proposed by Mickens or for the one proposed by Monaco and Normand-Cyrot when only low order parameter is considered.

The purpose of this work is to find valid discretized models using the discretization scheme proposed by Monaco and Normand-Cyrot. Here the discretization order is no longer as low as in the previous cases. The example used throughout the paper is the Hénon-Heiles system and the main objective is to define high-order discretized models that exhibit the same behavior as the original counterpart and therefore conserve the energy and symmetry of the solution even for large discretization steps.
The work is organized as follows. Section 2 describes the Hénon-Heiles system. Section 3 briefly introduces the discretization scheme proposed by Monaco and Normand-Cyrot. Then, a few remarks in section 4 about energy conservation in discrete systems are given. Section 5 shows results of the discretization scheme proposed by Monaco and Normand-Cyrot applied to the Hénon-Heiles system. Finally, the conclusions are presented in section 6.

2. The Hénon-Heiles system
The Hénon-Heiles system was firstly devised as a model to describe planetary motion, during large time intervals, inside a gravitational potential $U_0$, in a galaxy with cylindrical symmetry [1, 8, 9]. Nowadays, this well known nonlinear conservative system is widely studied [1, 7] in the realm of dynamical systems. After an order reduction, [1, 7] shows that the Hamiltonian function with two degrees of freedom is:

$$H(x, y, u, v) = \frac{1}{2}(u^2 + v^2) + U(x, y)$$ (1)

where $U(x, y)$, the potential function, is given by:

$$U(x, y) = \frac{1}{2}(x^2 + y^2 - \frac{2}{3}y^3) + x^2y$$ (2)

After the derivation of the equations of motion, the Hénon-Heiles system can be described by the following four ordinary differential equations:

$$\begin{align*}
\dot{x} &= u \\
\dot{y} &= v \\
\dot{u} &= -x - 2xy \\
\dot{v} &= -x^2 + y^2 - y
\end{align*}$$ (3)

For the initial condition $(x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)$ used in [7, 9] the energy is equal to $H = 0.128546999$, which means that the system is behaving chaotically.

The Hénon-Heiles system was simulated with the aforementioned condition using the implicit Runge-Kutta method with coefficients given by the 6th order Gauss-Legendre method, a symplectic integrator with area preserving [1] - and integration step $\tau = 0.001s$. The Poincaré section is shown in Figure 1. In this chaotic tangle, there are several islands, which are associated to orbits of periods 1 to 5. Symmetry, a characteristic of conservative systems, occurs for $v_n = 0$.

When the sampling frequency $f_s$ is higher than twice the highest significative frequency $f_{max}$ in the power spectrum, the Nyquist criterion states that sufficient information is available for a trustworthy description of the dynamics. This value works as an upper limit for the discretization time step of any discretization scheme, so $f_s \geq 2f_{max} \Rightarrow \frac{T_s}{f_s} = h_{max} \leq \frac{1}{2f_{max}}$. As long as the Nyquist criterium is satisfied, numerical instabilities are avoided and discretized models generate solutions which are equivalent to the one of the original system except for a small displacement in the parameter space [5, 6, 7]. For the variable $x$, Fourier analysis (Figure 2) shows that the maximum frequency is $f_{max} \approx 0.75Hz$. Therefore, the maximum discretization step must be less than $h_{max} = 0.67s$ to satisfy the Nyquist criterium.

3. Monaco and Normand-Cyrot discretization scheme
Consider the dynamical system:

$$\dot{x} = f(x)$$ (4)

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ are the state variables, $f(x) = (f_1(x), \ldots, f_m(x))$ are analytic functions. The derivative of $x$ with respect to time is denoted by $\dot{x}$.
Figure 1: Poincaré section of the Hénon-Heiles system defined by
\[ P = \{ (y_n, u_n, v_n) \in \mathbb{R}^3 \mid x_n = 0 \} \] with initial condition \((x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)\). The continuous system was simulated using the implicit Runge-Kutta method with 6th order Gauss-Legendre coefficients \([1]\), and integration step \(\tau = 0.001\text{s}\).

Figure 2: Frequency spectrum of the variable \(x\) of the Hénon-Heiles system, with initial conditions \((x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)\).

The discrete model of eq (4) is given by:
\[ x_{k+1} = g(x_k, h) \] (5)
where \(x_k \in \mathbb{R}^m\) are discrete state variables at time \(t = t_0 + kh\), and \(h\) is the discretization step. Since discrete models derived from a set of differential equations have an extra parameter, the discretization step \(h\), the parameter space to be considered is larger than the continuous counterpart. The sole choice of \(h\) may lead to numerical instabilities \([7]\).

In \([4, 7]\) it is shown that the discretization, originally proposed by Monaco e Normand-Cyrot \([3]\), can be accomplished by the Lie exponential expansion of Eq. (4), as follows:
\[ x_{k+1} = \exp (h L_f (x_k)) = x_k + \sum_{n=1}^{\eta} \frac{h^n}{n!} L^n_f (x_k) \] (6)
where \(\eta\) is the expansion order. The Lie derivative is given by:
\[ L_f (x_k) = \sum_{j=1}^{m} f_j \frac{\partial x}{\partial x_j} \] (7)
where \(f_j\) represents the \(j\)th component in the vector field. Higher derivative orders can be calculated recursively by:
\[ L^n_f (x_k) = L_f \left( L^{n-1}_f (x_k) \right) \] (8)

Equation (6) can be rewritten as follows:
\[ x_{k+1} = x_k + h f + \frac{h^2}{2} J(f) f + \frac{h^3}{6} J(J(f) f) f + \cdots \] (9)
where \(J(\cdot)\) is the Jacobian matrix of the argument. Note that the above equation should be truncated to avoid an excessive amount of terms and thus making computational simulations
feasible. Besides, when $\eta \to \infty$, terms tend to zero which means that no information is added to the model output due to the lack of computational precision.

$\eta = 1$ results in the well-known explicit Euler method that preserves the number and location of the fixed-points of the original differential equations as shown in [4]. For $\eta \geq 2$, the location of the original system fixed-points is also preserved, but spurious fixed-points are added. The number and location of the spurious fixed points depend on the discretization order $\eta$ and the discretization step $h$, respectively. It is worth noticing that spurious fixed points are a characteristic of the discretization scheme.

4. Energy analysis of discrete models

4.1. Hamiltonian Function

The energy is given by the hamiltonian function. The energy value of the discrete model is related to the model structure and the discretization step $h$. Energy mean and standard deviation of the discrete model simulation can be used to measure if the energy is conserved along the simulation trajectory for different discretization steps and are defined as follows:

$$\mu_H = \frac{1}{N+1} \times \sum_{k=0}^{N} H(x_k)$$  \hspace{1cm} (10)

$$\sigma_H = \sqrt{\frac{1}{N} \times \sum_{k=0}^{N} (H(x_k) - \mu_H)^2}$$  \hspace{1cm} (11)

In order to check whether the discretized model can reproduce the dynamic motion of the continuous system, $\mu_H$ must be equal to the energy defined for the continuous system, and $\sigma_H$ must be equal to 0 in order to guarantee that $\mu_H$ is constant $\forall k$.

As observed in [7], but also inferred from [10], the energy conservation for discretized models can be checked using the following relation

$$H(x_{k+1}) - H(x_k) = 0$$  \hspace{1cm} (12)

Thus, considering the simulation given by a discretized model at instants $k = (0, 1, ..., N)$, the following additional conditions must be derived in order to check the energy conservation along the trajectory

$$\mathcal{H}_1 = \frac{1}{Nh} \times \sum_{k=0}^{N-1} (H(x_{k+1}) - H(x_k)) = \frac{1}{Nh} \times (H(x_N) - H(x_0)) \to 0$$  \hspace{1cm} (13)

and also:

$$\mathcal{H}_2 = \frac{1}{Nh} \times \sum_{k=0}^{N-1} |H(x_{k+1}) - H(x_k)| \to 0$$  \hspace{1cm} (14)

where $\mathcal{H}_1$ is the mean of the total energy variation and $\mathcal{H}_2$ is the mean of the absolute energy variation, both normalized in a simulation period of 1s.

4.2. Jacobian Matrix

For the discrete system of Eq. (6), the jacobian matrix is given by:

$$J_{\eta}(x_k, h) = \frac{\partial g}{\partial x_k}$$  \hspace{1cm} (15)
where, for each element \((i, j)\), one have:

\[
(J_\eta)_{ij} = \mathcal{O}(h^n) \quad i \neq j
\]

\[
(J_\eta)_{ij} = 1 + \mathcal{O}(h^n) \quad i = j
\]  

(16)

and \(\mathcal{O}(h^n)\) is a polynomial of \(h\) of \(n^{th}\) order, without the constant term. Therefore, when \(h \to 0\), \(J_\eta\) tends to an identity matrix and \(x_{k+1} \to x_k\).

The volume conservation in the context of discrete systems is accomplished by [11]:

\[
\mathcal{L}(x_k, h) = \prod_{s=1}^{m} \lambda_s = 1
\]  

(17)

where \(\lambda_s\) is an eigenvalue of the jacobian matrix \(J_\eta\), Eq. (15).

The eigenvalue product mean and standard deviation are another measurement of interest and are defined as follows

\[
\mu_\mathcal{L} = \frac{1}{N+1} \times \sum_{k=0}^{N} \mathcal{L}(x_k, h)
\]

\[
\sigma_\mathcal{L} = \sqrt{\frac{1}{N} \times \sum_{k=0}^{N} (\mathcal{L}(x_k, h) - \mu_\mathcal{L})^2}
\]  

(18)

If the above product is very close to 1 for a discretized model then the energy is said to be conserved.

An equivalent analysis, based on the jacobian matrix trace, was proposed in [7]:

\[
tr\{J_\eta\} = m + \mathcal{O}(h^n) \approx m
\]  

(20)

Similarly to the case of eigenvalue product, if the trace is practically equal to the system dimension, the energy is conserved and, therefore, approximately constant to small \(h\) values.

Another interesting measure, which will be used latter on, is the trace increment when discretization order increases:

\[
\Delta tr\{J_\eta\} = tr\{J_\eta\} - tr\{J_{\eta-1}\}
\]  

(21)

5. Results: Hénon-Heiles discretization

5.1. Mickens discretization scheme

Using Mickens discretization scheme [2] and the following tranformation

\[
\begin{align*}
1^{st} \text{ Eq.} & : (x_k, y_k, u_k, v_k) \mapsto (x_k, y_k, u_{k+1}, v_{k+1}) \\
2^{nd} \text{ Eq.} & : (x_k, y_k, u_k, v_k) \mapsto (x_k, y_{k+1}, u_{k+1}, v_k) \\
3^{rd} \text{ Eq.} & : (x_k, y_k, u_k, v_k) \mapsto (x_{k+1}, y_k, u_k, v_k) \\
4^{th} \text{ Eq.} & : (x_k, y_k, u_k, v_k) \mapsto (x_k, y_{k+1}, u_k, v_k)
\end{align*}
\]  

(22)

results in the discrete model (also used in [7])

\[
\begin{align*}
x_{k+1} &= x_k + h u_{k+1} \\
y_{k+1} &= y_k + h v_{k+1} \\
u_{k+1} &= u_k + h (-x_k - 2x_k y_k) \\
v_{k+1} &= v_k + h (-y_k + y_k^2 - x_k^2)
\end{align*}
\]  

(23)
Figure 3: Poincaré section of the Hénon-Heiles system, defined by \( P \equiv \{ (y_n, u_n, v_n) \in \mathbb{R}^3 \mid x_n = 0 \} \), using the method proposed by Mickens, with initial condition \((x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)\) and last simulation instant \(N \approx 120000\). 

Simulating Eq. (23) with \( h < 0.3s \), therefore \( h < h_{\text{max}} \), generates valid solutions as shown in [7]. Poincaré section of the Hénon-Heiles discretized model with \( h = 0.02s \), shown in Figure 3a, is practically the same as the one generated by the original continuous system (Figure 1). 

When the discretization step increases, the symmetry is broken due to the impossibility to estimate the derivatives and the variables at the same time and thus breaking the time invariance in the discretized form. The Poincaré section of the discretized model with \( h = 0.67s \), the maximum step size suggested by the Nyquist criteria, is shown in Figure 3b. Note that the symmetry is not conserved, as already observed in [7].

The energy indexes calculated under the simulation conditions given above are presented at Table 1. Energy mean is closer to the original energy value for small \( h \)s, but, as \( h \) increases, the energy increases as well which suggests that the model starts to behave like it would somehow stretch indefinitely. \( H_1 \) shows that the change in total energy slowly increases as \( h \) increases. In addition to that, it can be noticed the variance of this change also increases with \( h \).

### 5.2. Monaco and Normand-Cyrot discretization scheme

In this section several discretized models were obtained using the Monaco e Normand-Cyrot method. Each one was simulated with different discretization steps.

The simulation of the 3\(^{rd}\) order discretized model, eq (6) with \( \eta = 3 \), generates a Poincaré section that is similar to the one generated by the original continuous system for small discretization steps (\( h \leq 0.02s \)). The symmetry and the chaotic islands of the original system are preserved. For larger step sizes (\( h > 0.02s \)), numerical instabilities start to occur, resulting in Poincaré sections with points converging to the fixed-point located at the origin. This fact, observed in [7], is explained by analyzing the energy. As the step size increases, the energy decays, as shown in Figure 4, resulting in points closer to the origin. It can be also observed that as the step size increases, the eigenvalue product of the jacobian matrix is not preserved, as shown in Figure 5. This also indicates that the energy is not conserved.

The energy indexes calculated for the 3\(^{rd}\) order discretized model with \( h = 0.01s \) and \( N \approx 6.0 \times 10^6 \) are listed at Table 1. The energy mean, \( \mu_H = 0.12811895 \), differs the original energy only on the 4\(^{th}\) decimal digit. The energy mean for higher \( h \) values is smaller, thus, the original energy is not conserved. The total energy variation and the absolute energy variation within a 1s period are small, but a large variation is seen for higher \( h \) values, demonstrating that the model structure is not sufficient to conserve the energy for higher \( h \) values.

Authors of [7] added/deleted terms in Eq. (6) in order to force the jacobian matrix trace to be equal to 4, no matter the choice of the step size value. In [7] it was also checked that the
The jacobian matrix trace of the 2\textsuperscript{nd} order discretized model was equal to $4 - 2h^2$ and the one of the 3\textsuperscript{rd} order was equal to $4 - 2h^2$. To accomplish this, 2\textsuperscript{nd} order term from the 3\textsuperscript{rd} order discretized model was eliminated. Unfortunately simulations show that these results are not satisfactory, because they lead the model to instability.

The addition of a single 3\textsuperscript{rd} order term to the Euler approximation is not enough to produce a valid model. The same reasoning to avoid the stability problem and symmetry loss was applied to the discretized models. The jacobian matrix trace variation was studied for each increment in the discretization order, using Eq. (21). A further attempt to solve the problem was to add higher order terms considering only the jacobian matrix trace. The 3\textsuperscript{rd} order discretized model was chosen because it can successfully reproduce the original Hénon-Heiles system for $h < 0.02s$. 6\textsuperscript{th} and 7\textsuperscript{th} order terms were added to this model since their low contribution to the trace. The resulting model with the chosen terms was simulated and presented better results than the plain 3\textsuperscript{rd} order model. The model can successfully reproduce the Poincaré section with step sizes up to $h \approx 0.4s$.

The Normand-Cyrot method was again applied, without the choice of terms, and the 9\textsuperscript{th} order discretized model was simulated. The 9\textsuperscript{th} order model proved to be much more robust than the ones shown previously. For discretization steps less then $h \approx 0.5s$, the Poincaré section is very close to the section of the original continuous system.

With the purpose of checking the robustness of the 9\textsuperscript{th} order discretization, new sections in the Poincaré section were generated as a function of the step $h$. New sections at $v_n = 0$ were achieved by getting points of the Poincaré section in which $-0.01 \leq v_k \leq 0.01$ was verified. Same reasoning was made for the Poincaré section at $y_n = 0$. Such graphics for the 9\textsuperscript{th} order discretized model, without chosen terms, as well as the stationary energy and eigenvalue product, are shown in Figures 6, 7 and 8.

Region I of Figure 6 shows that for $h < 0.5s$, the islands of the Poincaré section are the same as the original continuous system. Region I of the energy plot (Figure 7) shows that the energy is practically constant, resulting in an attractor similar to the original one. The Poincaré section for $h = 0.5s$ is shown at Figure 9a. For $h > 0.5s$, islands begin to cease to exist (region II of Figure 6). Region II of the energy plot (Figure 7) points to the beginning of a more considerable energy loss, resulting in slightly smaller attractors. Even though, at $h = 0.67s$, when the energy is equal to 0.122 (5% of the initial energy), the model still has a solution topologically equivalent to the original system one. For region III of the energy plot (Figure 7), $h > 0.67s$, spurious behavior and numerical instabilities begin to occur, as expected, because the discretization step is above the maximum allowed by the Nyquist criteria. For $0.67s < h \leq 0.87s$, some details of
Figure 6: Cuts in the Poincaré section of the Hénon-Heiles system for the 9th order discretized model, using the method proposed by Monaco e Normand-Cyrot, versus the discretization step \( h \). The attractor of the model at region (I) is similar to the attractor of the continuous system \((0 < h \leq 0.5 s)\); (II) slightly smaller than the attractor of the continuous system \((0.5 s < h \leq 0.67 s)\); (III) without a few details and with some numerical instabilities, due to the simulation with step sizes higher than the one limited by the Nyquist Criteria \((0.67 s < h \leq 0.87 s)\); and (IV) predominancy of numerical instabilities \((h > 0.87 s)\).

The Poincaré section are lost, but the symmetry still remains, as seen in the Poincaré section at \( h = 0.87 s \) (Figure 9b). Only after \( h \approx 0.87 s \) that numerical instabilities take over. This can be inferred by several abrupt changes in the Poincaré section (region IV of Figure 6).

Eigenvalue product of the jacobian matrix slightly decreases as the discretization step increases, as shown in Figure 8. But this characteristic is much smoother than the one of the 3rd order model.

The energy indexes calculated for the 9th order discretized model are listed in Table 1. The energy mean calculated in the simulation with this model and \( h = 0.50 s \) is closer to the original energy than the simulation with the 3rd order model and \( h = 0.01 s \). The total energy variation and the absolute energy variation within a period of 1 sec are of the same order of magnitude. This reassures that the 9th order model structure is better for energy conservation for higher \( h \) values than the 3rd order one. As the step size \( h \) increases, the energy variation increases as well.

A few remarks must be made for the simulation with \( h = 0.87 s \) (Figure 9b): i) points are agglomerated in well defined regions, ii) the Poincaré section lacks details, and iii) the presence of aliasing phenomenon. This is explained by the frequency analysis of variable \( x \). Figure 10a
Figure 9: Poincaré section of the Hénon-Heiles system, defined by $P \equiv \{(y_n, u_n, v_n) \in \mathbb{R}^3 \mid x_n = 0\}$, using the method proposed by Monaco and Normand-Cyrot, $\eta = 9$, initial condition $(x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)$ and last simulation instant $N \approx 120000$. $(a) h = 0.50 \text{s}$ $(b) h = 0.87 \text{s}$

Figure 10: Frequencies of the variable $x$ of the Hénon-Heiles system, with initial conditions $(x_0 = 0.000, y_0 = 0.670, u_0 = 0.093, v_0 = 0.000)$, simulation of the 9th order discretized model. $(a) h = 0.50 \text{s}$ $(b) h = 0.67 \text{s}$

Table 1: Energy indexes calculated for the discrete models.

| Model       | $h$ (s) | $\mu_H$     | $\sigma_H$  | $\mathcal{H}_1$ | $\mathcal{H}_2$ |
|-------------|---------|-------------|-------------|-----------------|-----------------|
| Mickens     | 0.01    | 0.12852172  | $3.656872 \times 10^{-4}$ | $5.588389 \times 10^{-9}$ | $6.436041 \times 10^{-4}$ |
| Mickens     | 0.02    | 0.12852644  | $7.284395 \times 10^{-4}$ | $8.847790 \times 10^{-9}$ | $1.297851 \times 10^{-4}$ |
| Mickens     | 0.05    | 0.13267830  | $1.894329 \times 10^{-2}$ | $2.536240 \times 10^{-8}$ | $3.208169 \times 10^{-3}$ |
| Mickens     | 0.67    | 0.13651811  | $2.639399 \times 10^{-2}$ | $4.064695 \times 10^{-8}$ | $4.268835 \times 10^{-4}$ |
| Nor.-Cyr. $\eta = 3$ | 0.01 | 0.12811895  | $2.322869 \times 10^{-4}$ | $1.333164 \times 10^{-9}$ | $1.360052 \times 10^{-4}$ |
| Nor.-Cyr. $\eta = 9$ | 0.50 | 0.12869485  | $1.019699 \times 10^{-3}$ | $2.895236 \times 10^{-9}$ | $4.990294 \times 10^{-8}$ |
| Nor.-Cyr. $\eta = 9$ | 0.67 | 0.12932747  | $5.262554 \times 10^{-4}$ | $1.503759 \times 10^{-8}$ | $7.073433 \times 10^{-7}$ |
| Nor.-Cyr. $\eta = 9$ | 0.87 | 0.12264905  | $3.211799 \times 10^{-3}$ | $9.568887 \times 10^{-8}$ | $7.150463 \times 10^{-7}$ |

shows that there is no border effect for frequencies near $\pi$ at $h = 0.50$. $h = 0.67$ (Figure 10b) is the upper bound of the discretization step. Thus, higher frequencies begin to overlap for $h > 0.67$.

6. Conclusion

In the context of the analysis of nonlinear conservative systems, the discretization scheme proposed by Monaco and Normand-Cyrot achieved better results and shown to be more robust than the one proposed by Mickens as the discretization step increases.

The maximum discretization step for the simulations using the discretization scheme proposed by Mickens is $h \approx 0.3 \text{s}$. For such a range of values the Poincaré section generated from the iteration of the discretized models is close to the original continuous counterpart. $h \approx 0.85 \text{s}$ was
achieved before the trajectories were ejected to infinity [7]. The symmetry is not kept as the step size increases.

As for the discretization scheme proposed by Monaco e Normand-Cyrot the maximum discretization step was found to be \( h \approx 0.5s \). For the 9th order model the Poincaré section is close to the original. The maximum step size provided by the Nyquist criteria, \( h \approx 0.67s \), produced a Poincaré section that resembles the original one but with an displacement in the parameter space. For \( 0.67s < h \leq 0.87s \), the Poincaré sections lacked details due to border effect - overlapping of higher frequencies. For \( h > 0.87s \) the discretized models no longer generate Poincaré sections that resemble the original continuous Poincaré section. The symmetry of the conservative system was preserved in all simulations, even for discretization steps higher than the limit imposed by Nyquist criterium.

Simulations with different model structures showed that low order terms are necessary for the model stability. The 3rd order model managed to reproduce the Poincaré section for \( h < 0.02s \). Even if the simulation was performed with a higher order model, the higher order terms would not be representative, because \( h \) is too small. But removing the 2nd order term makes the 3rd order model to unstable. As \( h \) increases, higher order terms become more representative and cannot be disregarded, and therefore they are crucial for long simulations. These higher order terms introduce tiny increments that are responsible for the success of long simulations for such non-linear systems.

Sensitivity to initial conditions is a well known problem in non-linear systems [12]. An analogous reasoning can be applied to the discretization method proposed by Monaco and Normand-Cyrot. The initial conditions are the same, but the model structure is slightly different as the discretization order increases. Numerical instabilities appear when the model structure cannot reproduce the original continuous system for long simulations at a certain \( h \). In this case, it is necessary to add higher order terms to improve the model approximation.

In order to verify whether the discretized system can reproduce the continuous system behavior, the eigenvalue product of the jacobian matrix was used as a measure of energy conservation. If it is closer to one, the system is closer to conserve energy.

Finally, the discretization scheme proposed by Monaco and Normand-Cyrot has an advantage to be a straightforward and simple method that provides robust discretized models. The only necessary step is to determine the desired order to obtain the difference equations; no further study of the system is required. The disadvantage is that the increase in the discretized model order means more computational effort. For conservative systems, it was shown that method can generate valid models that preserves the symmetry for any value of the discretization step.

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