DIRICHLET’S TYPE OF TWISTED EULERIAN POLYNOMIALS IN CONNECTION WITH TWISTED EULERIAN-$L$-FUNCTION

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Abstract. In the present paper, we effect Dirichlet’s type of twisted Eulerian polynomials by using $p$-adic fermionic $q$-integral on the $p$-adic integer ring. Also, we introduce some new interesting identities for them. As a result of them, by using contour integral on the generating function of Dirichlet’s type of twisted Eulerian polynomials and so we define twisted Eulerian-$L$-function which interpolates of Dirichlet’s type of Eulerian polynomials at negative integers which we state in this paper.

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1. Introduction

As it is well-known, Eulerian polynomials, $A_n(x)$, are given by means of the following exponential generating function:

$$e^{A(x)t} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{1-x}{e^{e(t-1)-x}}$$

where $A^n(x) := A_n(x)$, symbolically. Eulerian polynomials can find via the following recurrence relation:

$$(A(t) + (t-1))^n - tA_n(t) = \begin{cases} 1-t & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, \end{cases}$$

(for details, see [2], [13] and [22]).

Imagine that $p$ be a fixed odd prime number. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The $p$-adic absolute value is defined by

$$|p|_p = \frac{1}{p}.$$ 

Let $q$ be an indeterminate with $|q - 1|_p < 1$. Thus, we give definition of $q$-integer of $x$, which is defined by

$$[x]_q = \frac{1-q^x}{1-q}$$
and
$$[x]_{-q} = \frac{1-(-q)^x}{1+q},$$

where we note that $\lim_{q \to 1} [x]_q = x$ (see [1-26]).
Let $UD \left( \mathbb{Z}_p \right)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For a positive integer $d$ with $(d, p) = 1$, let

$$X = X_d = \lim_{n \to \infty} \mathbb{Z}/dp^n \mathbb{Z} = \bigcup_{a=0}^{dp-1} (a + dp\mathbb{Z}_p)$$

with

$$a + dp^n \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^n$ and let $\sigma : X \to \mathbb{Z}_p$ be the transformation introduced by the inverse limit of the natural transformation $\mathbb{Z}/dp^n \mathbb{Z} \to \mathbb{Z}/p^n \mathbb{Z}$.

If $f$ is a function on $\mathbb{Z}_p$, then we will utilize the same notation to indicate the function $f \circ \sigma$.

For a continuous function $f : X \to \mathbb{C}_p$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by T. Kim in [3] and [4], as follows:

$$(3) \quad I_q(f) = \int_X f(\nu) d\mu_q(\nu) = \int_{\mathbb{Z}_p} f(\nu) d\mu_q(\nu) = \lim_{n \to \infty} \frac{1}{[p^n]_q} \sum_{\nu=0}^{p^n-1} q^\nu f(\nu).$$

The bosonic integral is considered as the bosonic limit $q \to 1$, $I_1(f) = \lim_{q \to 1} I_q(f)$. In [8], [9] and [10], similarly, the $p$-adic fermionic integration on $\mathbb{Z}_p$ is given by Kim as follows:

$$(4) \quad I_{-q}(f) = \lim_{q \to -q} I_q(f) = \int_{\mathbb{Z}_p} f(\nu) d\mu_{-q}(\nu).$$

By (4), we have the following well-known integral equation:

$$(5) \quad q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l)$$

here let $f_n(x)$ be a translation with $f_n(x) := f(x + n)$. By (5), we readily derive the following

$$(6) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0).$$

Replacing $q$ by $q^{-1}$ in (6), we easily derive the following

$$(7) \quad I_{-q^{-1}}(f_1) + qI_{-q^{-1}}(f) = [2]_q f(0).$$

Recently, Kim et al. [2] is considered $f(x) = e^{-x(1+q)t}$ in (7), then they gave Witt’s formula of Eulerian polynomials as follows:

$$I_{-q^{-1}}(x^n) = \frac{(-1)^n}{(1+q)^n} A_n(-q).$$

In previous paper [22], Araci, Acikgoz and Gao are introduced generating function of Dirichlet’s type of Eulerian polynomials as

$$(8) \quad \sum_{n=0}^{\infty} A_{n,\chi}(-q) \frac{t^n}{n!} = [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \chi(l) \frac{e^{-(1+q)t}}{e^{-(1+q)t} + q^d}.$$
Also, they gave Witt’s formula for Dirichlet’s type of Eulerian polynomials, as follows:

\[ I_{-q^{-1}} (\chi (x) x^n) = \frac{(-1)^n}{(1+q)^n} A_{n, \chi} (-q). \]

In this paper, we will also consider Dirichlet’s type of twisted Eulerian polynomials. By applying Mellin transformation to generating function of Dirichlet’s type of twisted Eulerian polynomials, we will describe twisted Eulerian-\(L\)-function. Next, by utilizing from Cauchy-Residue theorem, we will see that twisted Eulerian-\(L\)-function is related to Dirichlet’s type of twisted Eulerian polynomials at negative integers.

2. On the Dirichlet’s type of twisted Eulerian polynomials

By using (5), it is easy to see that

\[ I_{-q^{-1}} (f_d) + q^d I_{-q^{-1}} (f) = [2^q] \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} f (l). \]

Let \( C_p^n = \{ \zeta \mid \zeta^{p^n} = 1 \} \) be the cyclic group of order \( p^n \), and let

\[ T_p = \lim_{n \to \infty} C_{p^n} = C_{p^n} = \bigcup_{n \geq 0} C_{p^n}, \]

we want to note that \( T_p \) is locally constant space.

Let \( \chi \) be a Dirichlet’s character of conductor \( d (= \text{odd}) \) and \( \zeta \in T_p \), then, taking \( f(x) = \zeta^x \chi (x) e^{-x(1+q)t} \) in (10), then it is equality to

\[
I_{-q^{-1}} \left( \zeta^x + \chi (x) e^{-x(1+q)t} \right) + q^d I_{-q^{-1}} \left( \zeta^x \chi (x) e^{-x(1+q)t} \right)
\]

\[ = [2^q] \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \chi (l) e^{-(1+q)t} \zeta^l. \]

From this, we easily see that

\[ I_{-q^{-1}} (\zeta^x \chi (x) e^{-x(1+q)t}) = [2^q] \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \zeta^l \chi (l) \frac{e^{-(1+q)t}}{\zeta e^{-d(1+q)t} + q^d}. \]

Now, we give definition of generating function of Dirichlet’s type of twisted Eulerian polynomials as follows:

**Definition 1.** Let \( G_{q, \zeta} (t \mid \chi) = \sum_{n=0}^{\infty} A_{n, \chi, \zeta} (-q) \frac{t^n}{n!} \) and \( \zeta \in T_p \). Then, we define twisted Dirichlet’s type of Eulerian polynomials by means of the following generating function:

\[ G_{q, \zeta} (t \mid \chi) = [2^q] \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \zeta^l \chi (l) \frac{e^{-(1+q)t}}{\zeta e^{-d(1+q)t} + q^d}. \]

By considering the above definition, we readily derive the following corollary.

**Corollary 2.1.** For any \( \zeta \in T_p \) and \( n \in \mathbb{N}^* \), then we have

\[ A_{n, \chi, \zeta} (-q) = \text{coefficient of } \frac{t^n}{n!} \text{ of } [2^q] \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \zeta^l \chi (l) \frac{e^{-(1+q)t}}{\zeta e^{-d(1+q)t} + q^d}. \]
By using (11) and Definition 1, we procure the following theorem which is the Witt’s formula for Dirichlet’s type of twisted Eulerian polynomials.

**Theorem 2.2.** The following equality

\[
I_{-q^{-1}} (\zeta^x \chi (x) x^n) = \int_{\mathbb{Z}_p} \zeta^x \chi (x) x^n d\mu_{-q^{-1}} (x) = \frac{(-1)^n}{(1 + q)^n} A_{n, \chi, \zeta} (-q)
\]

holds true.

By using Definition 1, becomes

\[
\sum_{n=0}^{\infty} A_{n, \chi, \zeta} (-q) \frac{t^n}{n!} = [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \zeta^l \chi (l) \frac{e^{-l(1+q)t}}{\zeta^d e^{-d(1+q)t} + q^d}
\]

\[
= [2]_q \sum_{l=0}^{d-1} (-1)^l q^{d-l+1} \zeta^l \chi (l) e^{-l(1+q)t} \sum_{m=0}^{\infty} (-1)^m \zeta^m q^{-md} e^{-md(1+q)t}
\]

\[
= q [2]_q \sum_{m=0}^{\infty} \sum_{l=0}^{d-1} (-1)^l (-1)^m \zeta^m \chi (l + md) q^{-l(m+md)} \zeta^{l+md} e^{-(l+md)(1+q)t}
\]

Thus, we get the following theorem.

**Theorem 2.3.** For \(\zeta \in T_p\), then we have

\[
G_{q, \zeta} (t \mid \chi) = \sum_{n=0}^{\infty} A_{n, \chi, \zeta} (-q) \frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty} (-1)^m \zeta^m \chi (m) e^{-m(1+q)t}
\]

\[
\frac{q}{(1 + q)^{n+1}} A_{n, \chi, \zeta} (-q) = \sum_{m=1}^{\infty} (-1)^m \zeta^m \chi (m) m^n.
\]

From expressions of (12) and (14), we easily effect the following corollary:

**Corollary 2.5.** For \(\zeta \in T_p\) and \(n \in \mathbb{N}\), then we procure the following

\[
\lim_{n \to \infty} \sum_{x=0}^{n^{n-1}} \frac{(-1)^x \zeta^x \chi (x) x^n}{q^x} = 2q^2 \sum_{m=1}^{\infty} (-1)^m \zeta^m \chi (m) m^n.
\]

Now, we want to show multiplication theorem for Dirichlet’s type of twisted Eulerian polynomials by using \(p\)-adic fermionic \(q\)-integral on \(\mathbb{Z}_p\), as follows:
For each \( \zeta \in \mathcal{T}_p \) and \( n \in \mathbb{N}^* \), so we have

\[
I_{-q^{-1}}(\zeta^x \chi(x) x^n) = \int_{\mathbb{Z}_p} \zeta^x \chi(x) x^n d\mu_{-q^{-1}}(x)
\]

\[
= \lim_{m \to \infty} \frac{1}{[dp]^m_{-q^{-1}}} \sum_{x=0}^{dp^{m-1}-1} (-1)^x \zeta^x \chi(x) x^n q^{-x}
\]

\[
= \frac{a^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \zeta^a \chi(a) q^{-a} \int_{\mathbb{Z}_p} \left( \frac{a}{d} + x \right)^n \zeta^dx d\mu_{-q^{-1}}(x).
\]

Then, we can express Dirichlet’s type of twisted Eulerian polynomials in terms of \( p \)-adic fermionic \( q \)-integral on \( \mathbb{Z}_p \), as follows:

**Theorem 2.6.** The following

\[
(15) \quad \frac{(-1)^n}{(1+q)^n} A_{n, \chi, \zeta} (-q) = \frac{d^n}{[d]_{-q^{-1}}} \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \int_{\mathbb{Z}_p} \left( \frac{a}{d} + x \right)^n \zeta^dx d\mu_{-q^{-1}}(x)
\]

is true.

Equation (15) seems to be interesting for evaluating at \( q = 1 \), so we see that

\[
(16) \quad \frac{(-1)^n}{2^n} A_{n, \chi, \zeta} (-1) = d^n \sum_{a=0}^{d-1} (-1)^a \chi(a) q^{-a} \int_{\mathbb{Z}_p} \left( \frac{a}{d} + x \right)^n \zeta^dx d\mu_{-1}(x).
\]

Next, with the help of Rim and Kim’s paper \[12\], we can write the following:

\[
(17) \quad E_{n, \zeta}(x) = \int_{\mathbb{Z}_p} \zeta^y (x+y)^n d\mu_{-1}(y)
\]

where \( \zeta \in \mathcal{T}_p \) and taking \( x = 0 \) in the above equation, we have \( E_{n, \zeta}(0) := E_{n, \zeta} \), which is defined via the following generating function:

\[
(18) \quad \sum_{n=0}^{\infty} E_{n, \zeta} \frac{t^n}{n!} = \frac{2 \varphi d e^d t}{\zeta^d e^d t + 1}, \quad |t| < \frac{\pi}{d}.
\]

By expressions of (16), (17) and (18), we state the following corollary.

**Corollary 2.7.** For any \( \zeta \in \mathcal{T}_p \) and \( n \in \mathbb{N}^* \), then we obtain

\[
A_{n, \chi, \zeta} (-1) = (-2d)^n \sum_{a=0}^{d-1} (-1)^a \chi(a) \zeta^a E_{n, \zeta, \frac{a}{d}}.
\]
3. On the twisted Eulerian-$L$-function in $\mathbb{C}$

The objective of this part is to describe twisted Eulerian-$L$-function by applying Mellin transformation to the generating function of Dirichlet’s type of twisted Eulerian polynomials, which seem to be interpolation function of Dirichlet’s type of twisted Eulerian polynomials at negative integers by using Cauchy-Residue theorem. We want to determine that these functions are very useful in Number Theory, Complex Analysis and Mathematical Physics, $p$-adic analysis and other areas cf. [1], [4], [5], [7], [11], [14], [15], [17], [18], [20], [24], [25] and [26]. Thus, by (13), for $s \in \mathbb{C}$, we consider the following integral:

\[
\mathcal{L}_{E,\zeta}(s, \chi) = \frac{\int_{0}^{\infty} t^{s-1} \mathcal{G}_{\zeta}(t \mid \chi) \, dt}{\int_{0}^{\infty} t^{s-1} e^{-t} \, dt}.
\]

By (19), we can derive the following:

\[
\mathcal{L}_{E,\zeta}(s, \chi) = q \left[ 2 \right]_{q}^{\infty} \sum_{m=0}^{\infty} (-1)^{m} \chi(m) \zeta^{-m} q^{-m} \left\{ \frac{\int_{0}^{\infty} t^{s-1} e^{-m(1+q)t} \, dt}{\int_{0}^{\infty} t^{s-1} e^{-t} \, dt} \right\}.
\]

On the other hand, we easily see that

\[
\mathcal{L}_{E,\zeta}(s, \chi) = \sum_{n=0}^{\infty} A_{n,\chi,\zeta}(-q) \left( \frac{\int_{0}^{\infty} t^{s-n+1} \, dt}{\int_{0}^{\infty} t^{s-1} e^{-t} \, dt} \right).
\]

So, we give definition of twisted Eulerian $L$-function as follows:

**Definition 2.** Let $\zeta \in \mathbb{T}_{p}$ and $s \in \mathbb{C}$, then we define the following

\[
\mathcal{L}_{E,\zeta}(s, \chi) = \frac{q}{(1+q)^{s-1}} \sum_{m=1}^{\infty} (-1)^{m} \chi(m) \zeta^{m} q^{m} m^{s}.
\]

By using (19) and (20), then, we derive the following which seems to be interesting and useful for improving of Complex Analysis and Theory of Analytic Numbers.

**Theorem 3.1.** The following equality holds true:

\[
\mathcal{L}_{E,\zeta}(-n, \chi) = (-1)^{n} A_{n,\chi,\zeta}(-q).
\]

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