A NEW GAP FOR CMC BIHARMONIC HYPERSURFACES IN EUCLIDEAN SPHERES

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Abstract. In this note we improve a gap result concerning the range of the mean curvature of complete CMC proper-biharmonic hypersurfaces in unit Euclidean spheres.

1. Introduction

The study of biharmonic maps is of great interest for many mathematicians, especially for those working in Differential Geometry. As suggested by J. Eells and J. H. Sampson (see [12,13]), or J. Eells and L. Lemaire (see [14]), a biharmonic map \( \varphi : M \to N \) between two Riemannian manifolds is a critical point of the bienergy functional

\[
E_2 : C^\infty (M, N) \to \mathbb{R}, \quad E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 \, dv,
\]

where \( M \) is compact and \( \tau(\varphi) = \text{trace} \nabla d\varphi \) is the tension field associated to \( \varphi \). In 1986, G.-Y. Jiang (see [20,21]) proved that the biharmonic maps are characterized by the vanishing of their bitension fields, where the bitension field associated to a map \( \varphi \) is given by

\[
\tau_2(\varphi) = -\Delta \tau(\varphi) - \text{trace} R^N (d\varphi(\cdot), \tau(\varphi)) \, d\varphi(\cdot).
\]

The nonlinear fourth order elliptic equation \( \tau_2(\varphi) = 0 \) is called the biharmonic equation.

Trivially, any harmonic map is biharmonic, so we will focus on the study of proper-biharmonic maps, i.e., the biharmonic maps which are not harmonic. A biharmonic submanifold of \( N^n \) is an isometric immersion \( \varphi : M^m \to N^n \) which is also a biharmonic map. Sometimes, throughout this paper, we will indicate a submanifold as \( M^m \) rather than mentioning the isometric immersion \( \varphi \).

When the curvature of the ambient manifold is non-positive, with just one exception (see [30]), we have only non-existence results, i.e., biharmonicity implies harmonicity (minimality). On the other hand, in spaces of positive curvature, especially in Euclidean spheres, we have many examples and classification results for proper-biharmonic submanifolds (see, for example [3,15,16,28,29]).

A particular case is given by the study of biharmonic hypersurfaces in the unit Euclidean sphere \( \mathbb{S}^{m+1} \). A very interesting situation is the case when the mean curvature \( f \) is a non-zero constant because, with this assumption, the hypersurface \( M^m \) is proper-biharmonic if and only if \( |A|^2 = m \), where \( A \) is the shape operator. Since the minimal, i.e., \( f = 0 \), hypersurfaces with \( |A|^2 = m \) were already classified by S. S. Chern, M. do Carmo and S. Kobayashi in their famous paper [9], the study of
proper-biharmonic hypersurfaces in Euclidean spheres with \( f \) constant, i.e., CMC, can be seen as a natural generalization of the above mentioned classical problem.

The only known examples of proper-biharmonic hypersurfaces in \( S^{m+1} \) are open parts of the small hypersphere of radius \( 1/\sqrt{2} \), i.e., \( S^m (1/\sqrt{2}) \), and of the Clifford tori \( S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \), with \( m_1 \neq m_2 \) and \( m_1 + m_2 = m \) (see [5, 21]).

Moreover, it was proved that, under various additional geometric assumptions, the proper-biharmonic hypersurfaces have to be the above ones or (at least) they must be CMC. Consequently, the following two conjectures have been proposed in 2008 (see [1]):

**Conjecture 1. (C1)** Any proper-biharmonic hypersurface in \( S^{m+1} \) is either an open part of \( S^m (1/\sqrt{2}) \), or an open part of \( S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2}) \), with \( m_1 \neq m_2 \) and \( m_1 + m_2 = m \).

**Conjecture 2. (C2)** Any proper-biharmonic submanifold in \( S^{m+1} \) is CMC.

We recall that, when the ambient space is the Euclidean space \( \mathbb{R}^n \), the famous Chen Conjecture remains unsolved. The conjecture says: any biharmonic submanifold in \( \mathbb{R}^n \) is minimal (see [6]). Since any CMC biharmonic submanifold in \( \mathbb{R}^n \) is minimal (see [11]), the Chen Conjecture can be reformulated in a weaker form:

**Weak version of Chen Conjecture.** Any biharmonic submanifold in \( \mathbb{R}^n \) is CMC.

Thus, conjecture C2 can be seen as an extension of the Weak version of Chen Conjecture to Euclidean spheres.

Obviously, C2 (for hypersurfaces) is weaker than C1, but to directly prove the first conjecture seems to be quite a complicated task. Conjecture C2 can be seen as an intermediate step for proving C1. However, even if C2 will be proved, the proof of C1 will still be a real challenge, and the additional hypothesis of compactness of the hypersurface does not simplify it.

Until now, C1 was proved only when \( m = 2 \) (see [5]). Also, it was proved in several particular cases, imposing additional geometric hypotheses on the biharmonic hypersurface: \( m = 3 \) and the hypersurface \( M \) is complete (see [2, 28]), or \( M^m \) has at most two distinct principal curvatures at any point (see [1]), or \( M^m \) is isoparametric (see [18, 19]), or \( M^m \) is CMC and has non-positive sectional curvature (see [28]), or \( M^m \) is compact and belongs to a hemisphere (see [32]), etc.

As it is not clear how C2 could imply C1, we propose an intermediary objective.

**Open Problem.** Let \( M^m \) be a CMC proper-biharmonic hypersurface in \( S^{m+1} \). Then, the set of all possible values of the mean curvature is discreet and, more precisely,

\[
f \in \left\{ \frac{m - 2r}{m} \mid r \in \mathbb{N}, 0 \leq r \leq s^* \right\},
\]

where \( s^* = s - 1 \), if \( m = 2s \), and \( s^* = s \), if \( m = 2s + 1 \).

Finally, we should prove that, if \( f = (m - 2r)/m \), then \( M^m \) must be an open part of \( S^r (1/\sqrt{2}) \times S^{m-r} (1/\sqrt{2}) \).

We mention here that there is a deep link between the proof of C1 knowing that C2 is true and the Generalized Chern Conjecture, as we will explain below.

First, inspired by the well-known Chern Conjecture concerning compact minimal hypersurfaces and its generalization to CMC hypersurfaces (see, for example [10, 22, 31]), we can state the following

**Generalized Chern Conjecture.** Let \( M^m \) be a CMC hypersurface in \( S^{m+1} \) with constant squared norm of the shape operator. Then, \( M \) is isoparametric.

Further, as a non-minimal CMC hypersurfaces in \( S^{m+1} \) is proper-biharmonic if and only if \( |A|^2 = m \), if the Generalized Chern Conjecture and C2 will be proved
to be true, then our C1 will follow immediately using the results in \cite{18, 19}. Therefore, the proof of C1 under the CMC assumption can be seen as a special case of Generalized Chern Conjecture.

However, the Generalized Chern Conjecture seems very difficult to be proved in its full generality, so we think that there are more chances to prove it under the additional hypothesis $|A|^2 = m$, i.e., to prove C1, assuming that C2 is true.

The main result of this paper gives a partial answer to the above Open Problem showing that

$$f \in (0, \gamma) \cup \left\{ \frac{m-2}{m} \right\} \cup \{1\},$$

where $\gamma$ is a real constant depending only on $m$ and

$$\frac{m-3}{m} < \gamma < \frac{m-2}{m}.$$

**Conventions and notations.** In this paper, the Riemannian metrics are indicated by $\langle \cdot, \cdot \rangle$. We assume that all manifolds are connected and oriented, and we use the following sign conventions for the rough Laplacian acting on sections of $\varphi^{-1}(TN)$ and for the curvature tensor field, respectively:

$$\Delta \varphi = -\text{trace} (\nabla^2 \varphi - \nabla^2 \varphi)$$

and

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z.$$  

For an (oriented) hypersurface $\varphi : M^m \to N^{m+1}$ we label the principal curvatures of $M$ such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m.$$

2. PROPER-BIHARMONIC HYPERSURFACES WITH CONSTANT MEAN CURVATURE

Let $M^m$ be a hypersurface in the unit Euclidean sphere $S^{n+1}$. For simplicity, we assume that $M$ is oriented. We recall that $A = A_\eta$ denotes the shape operator of $M$ and $B$ is the associated second fundamental form, $B(X,Y) = \langle A(X), Y \rangle \eta$, where $\eta$ is the unit normal vector field, globally defined on $M$. In this paper, we consider the normalized version for the mean curvature function, i.e., $f = (\text{trace} A) / m$. When $M$ is a non-minimal CMC hypersurface, i.e., $f$ is a non-zero constant, we can consider $\eta = H / |H|$ and so $f = |H|$, where $H$ is the mean curvature vector field.

Now, we recall a first result that supports the above Open Problem.

**Theorem 2.1** (\cite{27, 28}). Let $\varphi : M^m \to S^n$ be a CMC proper-biharmonic submanifold. Then $f \in (0, 1]$, and $f = 1$ if and only if $\varphi$ induces a minimal immersion of $M$ into $S^{n-1} \left( \frac{1}{\sqrt{2}} \right) \subset S^n$.

We also recall

**Theorem 2.2** (\cite{4}). Let $\varphi : M^m \to S^n$ be a proper-biharmonic immersion with parallel mean curvature vector field. Assume that $m > 2$ and $f \in (0, 1)$. Then $f \in (0, (m-2)/m]$, and $f = (m-2)/m$ if and only if locally $\varphi(M)$ is an open part of a standard product

$$S^1 \left( \frac{1}{\sqrt{2}} \right) \times M_1 \subset S^n,$$

where $M_1$ is a minimal embedded submanifold of $S^{n-2} \left( \frac{1}{\sqrt{2}} \right)$. Moreover, if $M$ is complete, then the above decomposition of $\varphi(M)$ holds globally, where $M_1$ is a complete minimal submanifold of $S^{n-2} \left( \frac{1}{\sqrt{2}} \right)$.
As a corollary of Theorem 2.2, we get a second result confirming the Open Problem.

**Corollary 2.3 ([4]).** Let $\varphi : M^m \to S^{m+1}$ be a CMC proper-biharmonic hypersurface with $m > 2$. Then $f \in (0, (m - 2)/m] \cup \{1\}$. Moreover, $f = 1$ if and only if $\varphi(M)$ is an open subset of the small hypersphere $S^m(1/\sqrt{2})$, and $f = (m-2)/m$ if and only if $\varphi(M)$ is an open subset of the standard product $S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

The same result of Theorem 2.2 was proved, independently and in the same time, in [33].

We mention that Corollary 2.3 can be reobtained, under the additional assumption $M^m$ compact, from the following known result.

**Theorem 2.4 ([7, 35]).** Let $M^m$ be a compact non-minimal CMC hypersurface in $S^{m+1}$. If $|A|^2 \leq \alpha$, then $M$ is either an umbilical sphere, or a Clifford torus.

Here,

$$\alpha = \alpha(m, f) = m + \frac{m^3}{2(m-1)f^2} - \frac{m(m-2)}{2(m-1)} \sqrt{m^2 f^4 + 4(m-1)f^2}. \tag{2.1}$$

Indeed, if $M^m$ is a compact CMC proper-biharmonic hypersurface in $S^{m+1}$, i.e., $|A|^2 = m$, the hypothesis $|A|^2 \leq \alpha$ in the above theorem is equivalent to $f \geq (m-2)/m$. Thus, taking into account Theorem 2.1, we get that a compact CMC proper-biharmonic hypersurface satisfying $|A|^2 \leq \alpha$ must obey $f \in [(m-2)/m, 1]$. Therefore, when $f$ belongs to the above interval, from Theorem 2.4 we get that $M$ is either an umbilical sphere, or a Clifford torus. It is known that the only biharmonic umbilical hypersurface is $S^m(1/\sqrt{2})$, and so $f = 1$, and the only biharmonic Clifford torus is $S^{m_1}(1/\sqrt{2}) \times S^{m_2}(1/\sqrt{2})$, where $1 \leq m_1 \leq m - 1$, $m_1 \neq m_2$ and $m_1 + m_2 = m$ (see [2, 21]). In the later case, we can assume $m_1 < m_2$ and get $f = (m_2 - m_1)/m$. Further, as by hypothesis $f$ belongs to the interval $[(m-2)/m, 1]$, the only possibility is that $m_1 = 1$, and therefore $f = (m-2)/m$.

Our main result is an improvement of Corollary 2.3. We show that there is a larger gap for $f$ than $((m-2)/m, 1)$. More precisely, considering $m \geq 4$ and denoting

$$\gamma = (m-2) \sqrt{\frac{m-1}{m^2(m-1) + B_m(B_m + m^2)}}, \tag{2.2}$$

where

$$B_m = \begin{cases} 0.2, & 4 \leq m \leq 42 \\ 0.199, & 43 \leq m \leq 65 \\ 0.198, & 66 \leq m \leq 149 \\ 0.197, & m \geq 150 \end{cases}, \tag{2.3}$$

we have

**Theorem 2.5.** Let $M^m$ be a complete CMC proper-biharmonic hypersurface in $S^{m+1}$. If $m \geq 4$ and the mean curvature $f \in [\gamma, (m-2)/m]$, then $f = (m-2)/m$ and $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

**Remark 2.6.** The real constant $\gamma$ satisfies

$$\frac{m-3}{m} < \gamma < \frac{m-2}{m}, \quad m \geq 4.$$
A direct consequence of Theorem 2.5 is the next result, which gives the new gap for $f$.

**Corollary 2.7.** Let $M^m$ be a complete CMC proper-biharmonic hypersurface in $S^{m+1}$, with $m \geq 4$. Then

$$f \in (0, \gamma) \cup \left\{ \frac{m-2}{m} \right\} \cup \{1\}.$$  

Moreover, $f = (m-2)/m$ if and only if $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$, and $f = 1$ if and only if $M = S^m(1/\sqrt{2})$.

We note that Theorem 2.5 can be deduced from Theorem 1 in [17] taking $|A|^2 = m$, $f \in [\gamma, (m-2)/m]$, but more precise values of $B_m$. We will provide a slightly simpler proof than the original one, using the properties of biharmonic hypersurfaces.

### 3. The proof of Theorem 2.5

In the first part of the proof we will give some algebraic results that will be very helpful. They will mainly involve certain quantities denoted by $\mu_i$, with $i \in \{1, 2, \ldots, m\}$, $\phi$, $\eta$ and $\sigma$. At the beginning, we will justify why we introduce them.

We recall that when $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$, the principal curvatures are constant and given by

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = \lambda_3 = \cdots = \lambda_m = 1 = \frac{mf - \lambda_1}{m-1}.$$

For a hypersurface $M^m$ in $S^{m+1}$ satisfying the hypotheses of Theorem 2.5, at a certain step, the term

$$\text{trace } A^3 = \sum_{i=1}^{m} \lambda_i^3$$

will appear. For our objective, it will be more convenient to replace $\lambda_i$ by

$$\lambda_i = \sqrt{m(1-f^2)}\mu_i + f.$$

Now, the advantage of using $\mu_i$’s is that

$$(3.1) \quad \sum_{i=1}^{m} \mu_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} \mu_i^2 = 1.$$

We note that, in the particular case when $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$, we have

$$\mu_1 = -\sqrt{\frac{m-1}{m}} \quad \text{and} \quad \mu_2 = \mu_3 = \cdots = \mu_m = \frac{1}{\sqrt{m(m-1)}} = -\frac{\mu_1}{m-1}.$$

We also need to consider on the hypersurface $M^m$ three functions $\phi$, $\eta$ and $\sigma$ given by

$$(3.2) \quad \phi = \sum_{i=1}^{m} \mu_i^3 + \frac{m-2}{\sqrt{m(m-1)}}, \quad \eta = \sqrt{\frac{m}{m-1}}\mu_1 + 1, \quad \sigma = \sqrt{\sum_{i=2}^{m} \left( \mu_i + \frac{\mu_1}{m-1} \right)^2}.$$

**Remark 3.1.** According to Okumura Lemma (see [25]) we have

$$0 \leq \phi \leq \frac{2(m-2)}{\sqrt{m(m-1)}}.$$
As we will see in the following, $\phi$ is indeed nonnegative (Lemma 3.2) and in Lemma 3.4 we will impose an upper bound of $\phi$ which is less than $2(m-2)/\sqrt{m(m-1)}$.

We mention that, in the particular case, when $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$, we get

$$\phi = \eta = \sigma = 0.$$  

Fixing arbitrarily a point of a hypersurface $M^m$ in $S^{m+1}$ that satisfies the hypotheses of Theorem 2.5, the above functions become, obviously, real numbers, and in the following we will give three algebraic lemmas. The first two lemmas are obtained in [17]. Their proofs are elementary but skilful and we do not present them here. Concerning the third lemma, we mention that it is originated in a result in [17], but our statement is more accurate and the proof is different.

**Lemma 3.2 ([17]).** If $m \geq 3$, the real numbers $\phi$, $\eta$ and $\sigma$ satisfy

$$\frac{\sqrt{m(m-1)}}{m-2}\phi \geq \eta \geq \frac{\sigma^2}{2}$$

and

$$\phi \sqrt{m(m-1)} \geq \eta \left[3m - 3(m+1)\eta - 2\sigma \sqrt{m(m-1)}\right].$$

In order to state the next lemma, we consider a positive number defined by

$$\alpha_0 = \alpha - mf^2 =$$

$$= m + \frac{m^3}{2(m-1)}f^2 - \frac{m(m-2)}{2(m-1)} \sqrt{m^2f^4 + 4(m-1)f^2 - mf^2}.$$  

**Lemma 3.3 ([17]).** If $m \geq 3$, then the following equality holds

$$(m-2)f \sqrt{\frac{m}{m-1}} \alpha_0 = m \left(f^2 + 1\right) - \alpha_0.$$  

From the definitions of $\sigma$ and $\eta$ we get a link between the difference $\mu_2 - \mu_1$ and a quantity which contains $\sigma$, $\eta$ and $m$. This we will be useful to prove the third lemma.

$$\mu_2 - \mu_1 \geq \left(1 - \eta - \sigma \sqrt{\frac{m-1}{m}}\right) \sqrt{\frac{m}{m-1}}.$$  

Now, using the real constant $B_m$ given in (2.3), which, clearly, depends on $m$, we can state the following lemma.

**Lemma 3.4.** Let $m \geq 4$. If

$$\phi \leq \frac{B_m}{2} \sqrt{\frac{m}{m-1}},$$

then $2\sigma + 3\eta < 3/4$ and

$$\mu_2 - \mu_1 > \frac{2}{3 - 3^{-1/10}} \sqrt{\frac{m}{m-1}}.$$  

**Proof.** First, using (3.3), (3.8) and the hypothesis $m \geq 4$, we get that $\eta \leq B_m$.

Then, using (3.4) and (3.8), we obtain

$$3\eta - 3 \left(1 + \frac{1}{m}\right) \eta^2 - 2\eta \sigma \sqrt{1 - \frac{1}{m}} - \frac{B_m}{2} \leq 0.$$
Further, from (3.1) and (3.2), it is easy to see that $\sigma = \sqrt{2\eta - \eta^2}$, and $\eta, \sigma \in [0, 1]$. Previously, we have seen that $\eta \leq B_m$, so, in fact $\eta \in [0, B_m]$ and $\sigma \in [0, \sqrt{2B_m - B_m^2}]$.

Therefore, the above inequality can be rewritten as

$$3\eta - 3 \left(1 + \frac{1}{m}\right) \eta^2 - 2\eta \sqrt{1 - \frac{1}{m} \sqrt{2\eta^2 - \eta^2}} - \frac{B_m^2}{2} \leq 0.$$

As $m > 0$, it follows that $-2\sqrt{1 - 1/m} > -2$, and then, from (3.10), we get

$$3\eta - 3 \left(1 + \frac{1}{m}\right) \eta^2 - 2\eta \sqrt{2\eta^2 - \eta^2} - \frac{B_m^2}{2} < 0.$$

Further, we continue with an argument which allows us to motivate the values of $B_m$ given in (2.3).

Since $m \geq 4$, it follows that $-3(1 + 1/m) \geq -15/4$, and from (3.11) we obtain

$$3\eta - \frac{15}{4} \eta^2 - 2\eta \sqrt{2\eta^2 - \eta^2} - \frac{B_m^2}{2} < 0.$$

Let $B_m = 0.2$. Now, as $\eta \leq B_m$, $\eta \in [0, 0.2]$ and from (3.12), using the program Mathematica, we get the maximum range of $\eta$, i.e., $\eta \in (0, 0.0446008)$. Next, from $\sigma = \sqrt{2\eta - \eta^2}$, we also get the maximum range of $\sigma$, i.e. $\sigma \in (0, 0.295317)$.

Clearly, $2\sigma + 3\eta < 3/4$.

Further, in order to prove (3.9), taking into account (3.7), it is enough to have

$$1 - \eta - \sigma \sqrt{\frac{m - 1}{m}} > \frac{2}{3 - 3^{1/10}}.$$

From the maximum ranges of $\eta$ and $\sigma$ (which depend on the assumption $m \geq 4$ and the chosen value of $B_m$), using again Mathematica, we can see that (3.13) holds only when $m \leq 22$.

So, if $m \in [4, 22]$ and $B_m = 0.2$, inequality (3.9) holds.

We continue the proof, assuming now that $m \geq 23$ and $B_m$ has the same value, i.e. $B_m = 0.2$. The argument will be similar, but since the minimum value of $m$ changed, also the maximum range of $\eta$ and $\sigma$ would change.

More precisely, as $m \geq 23$, it follows that $-3(1 + 1/m) \geq -72/23$, and from (3.11) we get

$$3\eta - \frac{72}{23} \eta^2 - 2\eta \sqrt{2\eta^2 - \eta^2} - \frac{B_m^2}{2} < 0.$$

Since $B_m = 0.2$, using Mathematica, from (3.14), we obtain $\eta \in (0, 0.043933)$ and $\sigma \in (0, 0.29315)$.

Clearly, $\eta$ and $\sigma$ belonging to these new intervals also satisfy $2\sigma + 3\eta < 3/4$ and imposing again (3.13), it follows, this time, that $m \leq 39$.

So, if $m \in [23, 39]$ and $B_m = 0.2$, we also proved that inequality (3.9) holds.

The process will continue in the same way. Considering $m \geq 40$ and $B_m = 0.2$, from (3.11), we obtain

$$3\eta - \frac{123}{40} \eta^2 - 2\eta \sqrt{2\eta^2 - \eta^2} - \frac{B_m^2}{2} < 0,$$

so $\eta \in (0, 0.043875)$ and $\sigma \in (0, 0.292962)$. Thus $2\sigma + 3\eta < 3/4$ and imposing (3.13), it follows that $m \leq 42$.

So, if $m \in [40, 42]$ and $B_m = 0.2$, the inequality (3.9) holds.
Trying to continue the proof with the same argument, we assume now $m \geq 43$ and $B_m = 0.2$. In this case, we can see that (3.13) does not hold, so we cannot motivate that (3.9) holds. For this reason, we need to decrease the value of $B_m$.

Let $m \geq 43$ and $B_m = 0.199$. In the same way, first we get that $m \leq 62$ and then, supposing $m \geq 63$ and $B_m = 0.199$, we obtain $m \leq 65$. If we continue with $m \geq 66$ and $B_m = 0.199$, (3.13) is not valid and we cannot conclude. Thus, we change again the value of $B_m$.

If we considering $m \geq 66$ and $B_m = 0.198$, following the same steps as the above, we can conclude that for $m \leq 149$, $2\sigma + 3\eta < 3/4$ and the inequality (3.9) holds. Instead, we cannot conclude that (3.9) holds with this value of $B_m$ and $m \geq 150$.

Finally, if we assume $m \geq 150$ and $B_m = 0.197$ we obtain that $2\sigma + 3\eta < 3/4$ and (3.13) is valid for any $m \geq 150$. Therefore, the conclusion of the lemma is true.

\[ \Box \]

Our purpose is to show that the functions $\phi$, $\eta$ and $\sigma$ vanish on the hypersurface $\varphi : M^m \to \mathbb{S}^{m+1}$ satisfying the hypotheses of Theorem 2.5, as they do when $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$. In order to achieve this, we will prove that the smallest distinct principal curvature has constant multiplicity 1, so $\lambda_1$ is smooth on $M$. Then, we compute and estimate $\Delta \lambda_1$ and further, using Omori-Yau maximum principle, we conclude that the three functions vanish on $M$. In fact, we will show that $A$ is parallel and from here, using the properties of biharmonic hypersurfaces we obtain that $M = S^1(1/\sqrt{2}) \times S^{m-1}(1/\sqrt{2})$.

We continue our proof recalling that the principal curvature functions $\lambda_i$’s are continuous on $M$ and the set of the points where the number of distinct principal curvatures is locally constant, denoted by $M_A$, is an open and dense subset in $M$. On a non-empty connected component of $M_A$, which is open in $M_A$ and also in $M$, the number of distinct principal curvatures is constant. Therefore, the multiplicities of distinct principal curvatures are constant and thus, on a connected component of $M_A$, the functions $\lambda_i$’s are smooth and the shape operator $A$ is locally smoothly diagonalizable.

In order to prove that $\lambda_1$ has constant multiplicity 1, we will employ Lemma 3.4, showing that $\mu_1 < \mu_2$. For this purpose, we recall the following Simons’ type formula, valid on $M^m$, that holds for any CMC hypersurface in $\mathbb{S}^{m+1}$,

\[
(3.16) \quad \frac{1}{2} \Delta |A|^2 = -|\nabla A|^2 + m^2 f^2 + |A|^2 \left(|A|^2 - m\right) - mf \text{trace} A^3.
\]

Formula (3.16) was obtained by K. Nomizu in [24].

Next, we define the positive numbers

\[
m_0 = m - mf^2 \quad \text{and} \quad \delta = \frac{m}{m - 1} f^2 B_m,
\]

that will be useful for our formulas. We note that the hypotheses of Theorem 2.5 are equivalent to

\[
f \leq \frac{m - 2}{m} \Leftrightarrow m \geq \alpha \Leftrightarrow \alpha_0 \leq m_0
\]

and

\[
f \geq \gamma \Leftrightarrow m \leq \alpha + \delta \Leftrightarrow m_0 \leq \alpha_0 + \delta,
\]

where $\alpha$ is defined in (2.1) and $\alpha_0$ in (3.5).
Now, using the biharmonicity hypothesis, i.e., $|A|^2 = m$, from (3.16) we obtain, on $M$,

$$|\nabla A|^2 = m^2 f^2 - m f \text{trace } A^3 =$$

$$= m^2 f^2 - m f \sum_{i=1}^{m} \lambda_i^3,$$

which can be easily rewritten as

$$|\nabla A|^2 = m_0^2 - m \left( f^2 + 1 \right) m_0 - m f \sqrt{m_0^3 \sum_{i=1}^{m} \mu_i^3}.$$

Using the definition of $\phi$ and the above expression of $|\nabla A|^2$, we get, on $M$,

$$m f \phi \sqrt{m_0^3} = |\nabla A|^2 + m f \phi \sqrt{m_0^3},$$

In order to fulfill the hypothesis of Lemma 3.4 and since

$$(3.18) \quad m f \phi \sqrt{m_0^3} \leq |\nabla A|^2 + m f \phi \sqrt{m_0^3},$$

the next step consists in finding a convenient upper bound for the term in the right hand side of (3.17).

First, as $m_0 \leq \alpha_0 + \delta$, it is easy to see that $\sqrt{m_0} < \sqrt{\alpha_0} + \delta / (2 \sqrt{\alpha_0})$, and then, using these inequalities and (3.6), we get

$$(3.19) \quad m_0 - m \left( f^2 + 1 \right) m_0 \leq m_0 - m \left( f^2 + 1 \right) m_0 + \delta + \frac{m - 2}{2 \sqrt{\alpha_0}} \delta f \sqrt{\frac{m}{m - 1}}.$$ 

Clearly, from the definition of $m_0$, we have $m_0 < m$ and therefore $\alpha_0 < m$. Now, using the definition of $\delta$, the relation (3.6) and the inequality $\alpha_0 - m < 0$, we obtain

$$\delta < \frac{(m - 2) B_m f \sqrt{m}}{m - 1} \alpha_0 <$$

$$< B_m f \sqrt{\frac{m}{m - 1}},$$

and, then

$$\frac{m - 2}{2 \sqrt{\alpha_0}} \delta f \sqrt{\frac{m}{m - 1}} < \frac{m - 2}{m - 1} \frac{B_m}{2} f^2 <$$

$$< \frac{B_m}{2} f^2.$$ 

If we consider again (3.6) and the inequality $\alpha_0 - m < 0$, we get

$$(3.21) \quad \frac{m - 2}{2 \sqrt{\alpha_0}} \delta f \sqrt{\frac{m}{m - 1}} < \frac{(m - 2) B_m}{2} f \sqrt{\frac{m}{m - 1}} \alpha_0.$$ 

Using (3.19), (3.20), (3.21) and then, $\alpha_0 \leq m_0$, one obtains

$$m_0 - m \left( f^2 + 1 \right) + (m - 2) f \sqrt{\frac{m}{m - 1}} m_0 < \frac{m B_m}{2} f \sqrt{\frac{m}{m - 1}} \leq$$

$$\leq \frac{m B_m}{2} f \sqrt{\frac{m}{m - 1}} m_0.$$ 

(3.22)
We note that, from (3.17), (3.18) and (3.22), we have on $M$

$$mf\phi\sqrt{m_0^3} \leq |\nabla A|^2 + mf\phi\sqrt{m_0^3} < \frac{mB_m}{2}f\sqrt{\frac{m}{m-1}}m_0^3,$$

and therefore, on $M$,

$$\phi < \frac{B_m}{2}\sqrt{\frac{m}{m-1}}.$$

Now, we can apply Lemma 3.4 and achieve $\mu_2 > \mu_1$, which is equivalent to $\lambda_2 > \lambda_1$ on $M^m$. Therefore, since the smallest principal curvature $\lambda_1$ of $M$ has (constant) multiplicity 1 on $M$, it follows that it is smooth on $M$ and there exists a local smooth unit vector field $E_1$ such that $A(E_1) = \lambda_1 E_1$ (see [23]). Then, we can find a local expression of $\Delta \lambda_1$ but, in order to work with, it is more convenient to fix arbitrarily a point $p$ and consider $\{e_1, \ldots, e_m\}$ an orthonormal basis which diagonalize the shape operator $A$ such that $e_1 = E_1(p)$. Let $b_{ijk} = \langle (\nabla A)(e_i, e_j), e_k \rangle$ be the components of the totally symmetric tensor $\langle (\nabla A)(\cdot, \cdot), \cdot \rangle$. It was shown in [17] that, at $p$,

$$\Delta \lambda_1 = mf + (|A|^2 - m)\lambda_1 - mf\lambda_1^2 - 2\sum_{i=1}^{m} \frac{b_{i1k}^2}{\lambda_1 - \lambda_k}.$$

Now, since $M^m$ is proper-biharmonic and therefore $|A|^2 = m$, the above equation (that holds for any CMC hypersurface in $S^{m+1}$) becomes

$$(3.23) \quad \Delta \lambda_1 = mf - mf\lambda_1^2 - 2\sum_{i=1}^{m} \frac{b_{i1k}^2}{\lambda_1 - \lambda_k}.$$

Using the link between $\lambda_i$’s and $\mu_i$’s and between $\mu_1$ and $\eta$, we can rewrite the above expression as

$$\Delta \lambda_1 = mf - mf\left[(\eta - 1)\sqrt{\frac{m-1}{m}m_0} + f\right]^2 - \frac{2}{\sqrt{m_0}}\sum_{i=1}^{m} \frac{b_{i1k}^2}{\mu_1 - \mu_k}.$$

By a straightforward computation, combining in a suitable way the terms from the right hand side of the above equation, and using the definition of $m_0$, one gets, at $p$

$$\Delta \lambda_1 = -\eta\sqrt{m_0}\left[\eta - 2(m - 1)f\sqrt{m_0} + 2f^2\sqrt{m(m-1)}\right] -$$

$$-\left[m_0 - m(f^2 + 1) + (m - 2)f\sqrt{\frac{m}{m-1}m_0}\right]\sqrt{\frac{m-1}{m}m_0} -$$

$$-\frac{2}{\sqrt{m_0}}\sum_{i=1}^{m} \frac{b_{i1k}^2}{\mu_1 - \mu_k}.$$
We notice that the second squared parenthesis can be replaced by \( (|\nabla A|^2 + m f \phi \sqrt{m_0^2})/m_0 \) from (3.17), and we can rewrite the above equation as

\[
\Delta \lambda_1 = \frac{\eta \sqrt{m_0}}{m_0} \left\{ -(\eta - 2)(m - 1) f \sqrt{m_0} + \left[ m_0 - m (f^2 + 1) \right] \sqrt{\frac{m - 1}{m}} \right\} - \sqrt{\frac{m - 1}{mm_0}} |\nabla A|^2 - m_0 \phi \sqrt{m(m - 1)} - \frac{2}{\sqrt{m_0}} \sum_{k \geq 2} \frac{b_{i1}^2}{\mu_1 - \mu_k}.
\]

(3.24)

Further, in order to obtain \( \Delta \lambda_1 < 0 \), we will find certain convenient upper bounds for some terms in the right hand side of (3.24).

First, it is easy to see that

\[
\Theta := -(\eta - 2)(m - 1) f \sqrt{m_0} + \left[ m_0 - m (f^2 + 1) \right] \sqrt{\frac{m - 1}{m}} < 2(m - 1) f \sqrt{m_0} + \left[ m_0 - m (f^2 + 1) \right] \sqrt{\frac{m - 1}{m}}.
\]

Second, using \( m_0 \leq \alpha_0 + \delta, \sqrt{m_0} < \sqrt{\alpha_0} + \delta/(2\sqrt{\alpha_0}) \) and

\[
f^2 + 1 = \frac{1}{m} \left( (m - 2) f \sqrt{\frac{m}{m - 1}} \alpha_0 + \alpha_0 \right),
\]

obtained from Lemma 3.3, we get

\[
\Theta < m f \sqrt{\alpha_0} + \frac{m - 1}{\sqrt{\alpha_0}} \delta f + \delta \sqrt{\frac{m - 1}{m}}.
\]

Now, we apply inequality (3.20) twice. First, one obtains

\[
\delta \sqrt{\frac{m - 1}{m}} < B_m f \sqrt{\alpha_0},
\]

and then, it quickly follows that \( f < \sqrt{\alpha_0(m - 1)/m} \).

Next, using the later inequality and, again, (3.20), one obtains

\[
\frac{m - 1}{\sqrt{\alpha_0}} \delta f < (m - 1) B_m f \sqrt{\alpha_0}.
\]

Therefore,

\[
\Theta < m f \sqrt{\alpha_0} + (m - 1) B_m f \sqrt{\alpha_0} + B_m f \sqrt{\alpha_0} = m (1 + B_m) f \sqrt{\alpha_0}.
\]

Moreover, since \( \alpha_0 \leq m_0 \) and \( 1 + B_m \leq 6/5 \) (from (2.3)), we get

(3.25)

\[
\Theta < \frac{6}{5} m f \sqrt{m_0}.
\]

We continue our argument by finding an appropriate upper bound for the last term of (3.24). Taking into account Lemma 3.4 and the expression of \( |\nabla A|^2 \) with
respect to $b_{ijk}$'s, it follows that, at $p$,

\[
\sum_{i=1}^{m} \frac{b_{i1k}^2}{\mu_1 - \mu_k} \geq \sum_{i=1}^{m} \frac{b_{i1k}^2}{\mu_1 - \mu_2} > \frac{3 - 3^{-10}}{2} \sqrt{\frac{m-1}{m}} \sum_{i=1}^{m} b_{i1k}^2 =
\]

\[
= -\frac{1 - 3^{-11}}{2} \sqrt{\frac{m-1}{m}} \left( 3 \sum_{i=1}^{m} b_{i1k}^2 \right) \geq -\frac{1 - 3^{-11}}{2} \sqrt{\frac{m-1}{m}} \left( 3 \sum_{i=1}^{m} b_{i1k}^2 + b_{111}^2 + \sum_{i,j,k \geq 2} b_{ijk}^2 \right) =
\]

\[
= -\frac{1 - 3^{-11}}{2} \sqrt{\frac{m-1}{m}} |\nabla A|^2. \tag{3.26}
\]

Now, from (3.24), (3.25) and (3.26), we achieve, at $p$,

\[
\Delta \lambda_1 < \frac{6}{5} m m_0 \eta f - 3^{-11} \sqrt{\frac{m-1}{m m_0}} |\nabla A|^2 - m_0 \phi f \sqrt{m(m-1)}. \tag{3.27}
\]

We are not yet able to conclude that $\Delta \lambda_1 < 0$ on $M$, so we need a better estimation. We can continue our process, with the following algebraic remarks: $\sqrt{m(m-1)} < m+1$, for any positive integer $m$, $3\eta + 2\sigma < 3/4$, from Lemma 3.4, and $9m/4 - 3/4 > 2m$, for any $m > 3$. Then, using these inequalities and (3.4), we obtain

\[
\phi \sqrt{m(m-1)} > \eta [3m - 3(m+1)\eta - 2(m+1)\sigma] = \eta [3m - (m+1)(3\eta + 2\sigma)] > \eta \left[ 3m - \frac{3}{4}(m+1) \right] = \eta \left( \frac{9}{4}m - \frac{3}{4} \right) > 2m\eta,
\]

and, therefore

\[
\frac{6}{5} m m_0 \eta f < \frac{3}{5} m_0 \phi f \sqrt{m(m-1)}.
\]

From (3.27) and the above estimation, we conclude with

\[
\Delta \lambda_1 < -3^{-11} \sqrt{\frac{m-1}{m m_0}} |\nabla A|^2 - \frac{2}{5} m_0 \phi f \sqrt{m(m-1)}. \tag{3.28}
\]

Therefore, at $p$, we have $\Delta \lambda_1 < 0$. As the point $p$ was arbitrarily fixed, we conclude that $\Delta \lambda_1 < 0$ on $M$.

Further, since $|A|^2 = m$, we obtain that the Ricci curvature of $M$ is bounded below,

\[
\text{Ricci}(X, X) \geq -2m(m-1), \quad X \in C(TM).
\]
Knowing also that $M$ is complete, we can apply the Omori-Yau maximum principle (see, for example [8, 26, 34]) and obtain that there exists a sequence of points $\{p_k\}_{k \in \mathbb{N}} \subset M$ such that

$$\Delta \lambda_1(p_k) > -\frac{1}{k}.$$ 

But we have seen that $\Delta \lambda_1 < 0$ at any point of $M$, so, in particular, $(\Delta \lambda_1)(p_k) < 0$. Therefore

$$\lim_{k \to \infty} (\Delta \lambda_1)(p_k) = 0,$$

and, moreover, from (3.28), we deduce

$$\lim_{k \to \infty} |\nabla A|^2(p_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \phi(p_k) = 0.$$ 

Now, using the fact that the quantity $|\nabla A|^2 + m f \sqrt{m_0^2}$ is constant (see (3.17)), it follows that

$$|\nabla A| \equiv 0 \quad \text{and} \quad \phi \equiv 0.$$ 

Further, on the one hand, from (3.17) we get

$$m_0 - m (f^2 + 1) + (m - 2) f \sqrt{\frac{m}{m - 1}} m_0 = 0,$$

and on the other hand, from (3.6), we have

$$\alpha_0 - m (f^2 + 1) + (m - 2) f \sqrt{\frac{m}{m - 1}} \alpha_0 = 0.$$ 

Therefore, $\alpha_0 = m_0$ and then

$$f = \frac{m - 2}{m}.$$ 

Finally, from $\phi \equiv 0$, we see that we have equality in the Okumura Lemma (in the left-hand side), see [25], and so

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = \lambda_3 = \cdots = \lambda_m = 1.$$ 

Thus, $M$ is the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^{m-1}(1/\sqrt{2})$.

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