Hochschild Cohomology of Some Finite Category Algebras as Simplicial Cohomology

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Abstract—By a result of Gerstenhaber and Schack, the simplicial cohomology ring \( H^\bullet(C, k) \) of a poset \( C \) is isomorphic to the Hochschild cohomology ring \( HH^\bullet(kC) \) of the category algebra \( kC \), where the poset is viewed as a category and \( k \) is a field. Extending results of Mishchenko, under certain assumptions on a category \( C \), we construct a category \( D \) and a graded \( k \)-linear isomorphism \( HH^\bullet(kC) \cong H^\bullet(D, k) \). Interpreting the degree one cohomology, we also show how the \( k \)-space of derivations on \( kC \) graded by some semigroup corresponds to the \( k \)-space of characters on \( D \).

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1. INTRODUCTION

Let \( k \) be a field, and let \( C \) be a finite category, i.e., a category in which the objects form a finite set and every set of morphisms is finite. By a result of Gerstenhaber and Schack [1], if \( C \) is a poset, then the simplicial cohomology \( H^\bullet(C, k) \) is isomorphic to the Hochschild cohomology ring \( HH^\bullet(kC) \) of the category algebra \( kC \). The proof of this result is simplified in [2], with the notion of \( R \)-relative Hochschild cohomology for subrings \( R \) of \( kC \). By a recent result of Mishchenko [3, Theorem 1], if \( C \) is a group, then \( HH^\bullet(kC) \) is isomorphic to the cohomology \( H^\bullet(BGr, k) \) of the classifying space of some groupoid \( Gr \). The proof of this result describes an explicit \( k \)-linear isomorphism. In [4] Fei Xu constructs a counterexample to a conjecture of Snashall and Solberg [5] by studying a split surjective ring homomorphism \( HH^\bullet(kC) \to H^\bullet(BC, k) \). The natural question which arises is: for which categories can the Hochschild cohomology of \( kC \) be explicitly identified with the singular cohomology of the classifying space of some category? Owing to the well-known isomorphism, \( H^\bullet(C, k) \cong H^\bullet(BC, k) \) (see [6, §5]), this is equivalent to asking for the existence of a category \( D \) such that \( HH^\bullet(kC) \) is isomorphic to the simplicial cohomology \( H^\bullet(D, k) \).

In this paper, we replace the category \( F(C) \) used in [4] with the adjoint category \( F^{ad}(C) \) generalizing the explicit construction of the groupoid \( Gr \) in [3]. We denote the set of objects in \( C \) by \( Ob(C) \) and the set of morphisms in \( C \) by \( Mor(C) \).

Let \( x_1, x_2 \in Ob(C) \). The category \( F^{ad}(C) \) has the subset \( EndC \) of endomorphisms in \( Mor(C) \) as objects; the morphisms are defined in the following way. A morphism from \( a \in End_C(x_1) \) to \( b \in End_C(x_2) \) in \( F^{ad}(C) \) is a morphism \( g \in Hom_C(x_1, x_2) \) making the following diagram commutative:

\[
\begin{array}{ccc}
  x_1 & \xrightarrow{g} & x_2 \\
  a \downarrow & & b \downarrow \\
  x_1 & \xrightarrow{g} & x_2
\end{array}
\]

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If $C$ is a group, then $F^{\text{ad}}(C)$ is isomorphic to the groupoid $Gr$ constructed in [3]; if $C$ is a poset, then $F^{\text{ad}}(C)$ is isomorphic to $C$.

To extend the construction in [3], we impose certain conditions on $C$. The category $C$ is said to be left deterministic if for any $b \in \text{End}_C(x_2)$ and $g \in \text{Hom}_C(x_1, x_2)$ there exists an $a \in \text{End}_C(x_1)$ such that the diagram (1) commutes. In a similar way, the category $C$ is said to be right deterministic if for any $a \in \text{End}_C(x_1)$ and $g \in \text{Hom}_C(x_1, x_2)$ there exists a $b \in \text{End}_C(x_2)$ such that the diagram (1) commutes. We say that $C$ is deterministic if it is both left and right deterministic. A category $C$ is left (respectively, right) cancellative if the composition of morphisms has the left (respectively, right) cancellation property. We say that $C$ is cancellative if it is both left and right cancellative. The notion of cancellative categories needed for our results is not new. Lawson and Wallis use it to generalize some of their results in [7] from monoids to categories. There is a natural right action of the monoid $\text{End}_C(x_1)$ on the set $\text{Hom}_C(x_1, x_2)$ given by composition,

$$\text{Hom}_C(x_1, x_2) \times \text{End}_C(x_1) \rightarrow \text{Hom}_C(x_1, x_2), \quad (g, a) \mapsto g \circ a.$$ 

We say that $C$ is $rr$-transitive if for any objects $x_1, x_2 \in \text{Ob}C$ the right action of the monoid $\text{End}_C(x_1)$ on the set $\text{Hom}_C(x_1, x_2)$ is transitive; see [8, Section 2.1] for more details about actions of monoids on sets.

We state our first result.

**Theorem 1.** If $C$ is a finite deterministic cancellative category, then there exists a surjective morphism of graded $k$-vector spaces $T^\bullet : \text{HH}^\bullet(kC) \rightarrow \text{H}^\bullet(F^{\text{ad}}(C), k)$ between the Hochschild cohomology of the category algebra $kC$ and the simplicial cohomology of the category $F^{\text{ad}}(C)$. Moreover, if $C$ is $rr$-transitive, then $T^\bullet$ is an isomorphism.

The map $T^\bullet$ is induced from an explicitly constructed cochain map and unifies the constructions in [1] and [3] in our context. Moreover, one can readily describe a right inverse of $T^\bullet$, again, induced from an explicit cochain map (Propositions 6 and 7). Examples of finite deterministic cancellative $rr$-transitive categories are groupoids and posets (Subsec. 2.3); hence we obtain the following corollary.

**Corollary 1.** If $C$ is a finite groupoid or a finite poset, then the Hochschild cohomology of the category algebra $kC$ is isomorphic to the simplicial cohomology of the category $F^{\text{ad}}(C)$.

The isomorphism $\text{HH}^\bullet(kC) \simeq \text{H}^\bullet(C, k)$ for a poset $C$ does not hold for general $C$ (see [6, §5] for more details). For certain EI categories (every endomorphism is an isomorphism), which are called amalgams of groups and posets, see [9, Section 4], Lodder describes $\text{H}^\bullet(C, k)$ as a direct summand of $\text{HH}^\bullet(kC)$.

In Example 1, we describe an EI category $C$ for which the map $T^\bullet$ is an isomorphism but $C$ is neither a groupoid nor a poset. The category $C$ is not even an amalgam of groups and posets.

The results in [3] build on ideas developed by Arutyunov and Mishchenko in [10], where they show that $\text{HH}^1(kC) \cong H^1(BF^{\text{ad}}(C), k)$ in the case where $C$ is a group. By Theorem 1, there exists an isomorphism of the first cohomology groups $\text{HH}^1(kC) \cong H^1(F^{\text{ad}}(C), k)$. However, the method in [10] relies on a description of the derivations of $kC$ in terms of characters on $F^{\text{ad}}(C)$ (see Sec. 5) and the fact that $\text{Der}(kC)/\text{Im}(kC) \cong \text{HH}^1(kC)$, where $\text{Der}(kC)$ is the algebra of derivations of $kC$ and $\text{Im}(kC)$ is the algebra of inner derivations. To extend the description in [10], in Subsec. 5.1, we introduce a semigroup with zero denoted by $S$ and explain how $kC$ is graded by $S$. We denote by $\text{Der}^{gr}(kC)$ the $k$-vector space of all $S$-graded derivations. Our second result extends the characterization of $\text{Der}(kC)$ in [10] to group algebras as follows.

**Theorem 2.** Let $C$ be a finite $rr$-transitive deterministic cancellative category. There exists a $k$-linear isomorphism $T : \text{Der}^{gr}(kC) \rightarrow \text{Char}(F^{\text{ad}}(C))$ between graded derivations on $kC$ and characters on the category $F^{\text{ad}}(C)$. 
Note that the above results can be given for any small category \( C \) if we use a condition of finiteness of supports for the cochains appearing in the above cohomology. Such generalizations are possible whenever the right finiteness conditions are imposed. For example, this is should also be possible for FC-groups as in [11]. In this paper, we assume that \( C \) is finite. We also point out that the notion of character was generalized to \( n \)-categories in [12], where similar results were obtained.

In [1], Gerstenhaber and Shack show how the simplicial cohomology of a locally finite simplicial complex can be interpreted as the Hochschild cohomology of some associative unital \( k \)-algebra. It is known that the simplicial cohomology agrees with the singular cohomology (which is better adapted to theory than to computation) of topological spaces which can be triangulated. It is possible to compute simplicial cohomology of a simplicial complex efficiently; hence it is better suited for calculations. In Theorem 1, we describe certain finite categories that have Hochschild cohomology isomorphic (as graded \( k \)-vector space) to the simplicial cohomology of some other categories with trivial coefficients. The Hochschild cohomology is equipped with a very rich structure: it is a graded commutative algebra via the cup product; it has a graded Lie bracket of degree \(-1\); with this bracket, the Hochschild cohomology becomes a Gerstenhaber algebra. Recently, the second author introduced a Beilinson–Drinfeld structure (BD-structure) in group cohomology [13]. He showed that the Hochschild cohomology \( \text{HH}^\bullet(F_3C_3) \) of the cyclic 3 group algebra is a BD-algebra. BD-algebras are very similar to BV-algebras (Batalin–Vilkovisky), which have been intensively studied for Hochschild cohomology of various algebras. It would be interesting to investigate the extent to which the maps \( T^\bullet \) and \( X^\bullet \) can transport (preserve) all these structures between Hochschild cohomology and simplicial cohomology.

The paper is organized as follows: in Sec. 2 we recall some facts on Hochschild cohomology and cohomology of the nerve of a small category. Subsec. 2.3 discusses the concepts of deterministic, cancellative, and rr-transitive categories, which are needed in Sec. 3 to generalize the construction in [3]. The proof of Theorem 1 is given in Sec. 4. In Sec. 5 we describe characters on the category \( F^{\text{ad}}(C) \) and give the proof of Theorem 2.

## 2. PRELIMINARIES

For an object \( x \) in \( C \), we denote by \( \text{id}_x \) the identity endomorphism on \( x \). For a morphism \( a : x \to y \) in \( C \), we denote by \( s(a) \) the source \( x \) of \( a \) and by \( t(a) \) the target \( y \) of \( a \). For any category \( C \) and any field \( k \), the category algebra \( kC \) is the \( k \)-vector space with basis the morphisms in \( C \) and with product defined by

\[
ab = \begin{cases} 
  a \circ b & \text{if } s(b) = t(a) \\
  0 & \text{otherwise.}
\end{cases}
\]

We will use the notation \( \otimes \) for \( \otimes_k \), and if \( A, B \) are \( k \)-vector spaces, by \( \text{Hom}_k(A, B) \) we denote the \( k \)-vector space of all \( k \)-linear maps from \( A \) to \( B \).

### 2.1. Hochschild Cohomology Complex

The Hochschild cochain complex [14, Definition 1.1.13], for the algebra \( kC \) is

\[
\begin{array}{cccccc}
0 & C^0(kC) & \overset{\partial^0}{\rightarrow} & C^1(kC) & \overset{\partial^1}{\rightarrow} & \cdots \overset{\partial^{m-1}}{\rightarrow} C^m(kC) & \overset{\partial^m}{\rightarrow} C^{m+1}(kC) & \overset{\partial^{m+1}}{\rightarrow} \cdots
\end{array}
\]

Here \( C^0(kC) = kC \) and for \( m \geq 1 \),

\[
C^m(kC) = \{ f : kC \otimes \cdots \otimes kC \rightarrow kC \mid f \text{ is } k\text{-linear} \}.
\]

For any \( f \in C^m(kC) \) and any \( a_1, \ldots, a_{m+1} \in kC \), we have

\[
\begin{align*}
\partial^m(f)(a_1 \otimes \cdots \otimes a_{m+1}) &= a_1 f(a_2 \otimes \cdots \otimes a_{m+1}) \\
&\quad + \sum_{j=1}^{m} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{m+1}) \\
&\quad + (-1)^{m+1} f(a_1 \otimes \cdots \otimes a_m) a_{m+1}
\end{align*}
\]
if \( m \geq 1 \), and \( \partial^0(f)(a) = af - fa \) for any \( f \in C^0(kC) \). It is well known that

\[
\HH^0(kC) = \text{Ker}(\partial^0) = Z(kC)
\]

and

\[
\HH^m(C) := \text{Ker}(\partial^m)/\text{Im}(\partial^{m-1})
\]

for \( m \geq 1 \).

Since \( C \) is finite, any \( k \)-linear map \( f \in C^m(kC) \) is represented by a matrix

\[
(f^h_{g_1, \ldots, g_m})_{g_1, \ldots, g_m, h \in \text{Mor}(C)}
\]

with respect to the basis \( \text{Mor}(C) \) of \( kC \):

\[
f(g_1 \otimes \cdots \otimes g_m) = \sum_{h \in \text{Mor}(C)} f^h_{g_1, \ldots, g_m} h
\]

for all \( g_1, \ldots, g_m \in \text{Mor}(C) \). In particular, the differential operator \( \partial^m \) applied to \( f \) is determined by \( \partial^m(f)(g_1 \otimes \cdots \otimes g_{m+1}) \), which equals

\[
\sum_{h \in \text{Mor}(C)} \left( f^h_{g_2, \ldots, g_{m+1}} g_1 h + \sum_{j=1}^{m} (-1)^j f^h_{g_1, \ldots, g_j g_{j+1}, \ldots, g_m} h + (-1)^m f^h_{g_1, \ldots, g_m} h g_{m+1} \right).
\]

(2)

2.2. Simplicial Cohomology

The simplicial cohomology \( H^\bullet(C, k) \) is the cohomology \( H^\bullet(BC, k) \) of the nerve of \( C \); see [6, §5]. The nerve of \( C \) is

\[
NC = \{ \sigma : a_0 \xrightarrow{g_0} a_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{m-2}} a_{m-1} \xrightarrow{g_{m-1}} a_m \mid \sigma \text{ is an } m \text{-chain of morphisms in } C \}
\]

The subset of \( m \)-chains is denoted by \( NC_m \), and the 0-chains are \( NC_0 = \text{Ob}(C) \). The simplicial homology with coefficients in \( k \) is obtained using the simplicial chain complex

\[
\cdots \xrightarrow{\delta_{m-1}} kNC_{m+1} \xrightarrow{\delta_m} kNC_m \xrightarrow{\delta_{m-1}} \cdots \xrightarrow{\delta_1} kNC_1 \xrightarrow{\delta_0} kNC_0 \xrightarrow{0}
\]

Here \( kNC_m \) is the \( k \)-vector space with basis \( NC_m \), and

\[
\delta_m = \sum_{i=0}^{m+1} (-1)^i d^m_i,
\]

where \( d^m_i : NC_{m+1} \to NC_m \) is the well-known \( i \)-th face map (extended by linearity).

One obtains the cochain complex for the calculation of the simplicial cohomology as

\[
\cdots \xrightarrow{\delta_{m-1}} C^0(kNC, k) \xrightarrow{\delta^0} \cdots \xrightarrow{\delta^m} C^m(kNC, k) \xrightarrow{\delta^m} C^{m+1}(kNC, k) \xrightarrow{\delta^{m+1}} \cdots
\]

where \( C^m(kNC, k) = \text{Hom}_k(kNC_m, k) \) and

\[
\delta^m : \text{Hom}_k(kNC_m, k) \to \text{Hom}_k(kNC_{m+1}, k)
\]

is given by \( \delta^m(f) = f \circ \delta_m \).
We begin with some properties of deterministic cancellative categories.

Lemma 1. Let \( x_1, x_2, x_3 \in \text{Ob}(C) \), \( f, g \in \text{Hom}_C(x_1, x_2) \), and \( h \in \text{Hom}_C(x_2, x_3) \).

(a) If \( C \) is a left deterministic and left cancellative category, then there exists a unique \( a \in \text{End}_C(x_1) \) such that the diagram (1) commutes.

(b) If \( C \) is a right deterministic and right cancellative category, then there exists a unique \( b \in \text{End}_C(x_2) \) such that the diagram (1) commutes.

(c) Let \( C \) be a deterministic cancellative category. For any \( g \in \text{Mor}_C(x_1, x_2) \) there exists a monoid isomorphism \( \varphi_g : \text{End}_C(x_1) \to \text{End}_C(x_2) \) such that \( g \circ a = \varphi_g(a) \circ g \) with inverse \( \varphi^{-1}_g \) satisfying \( b \circ g = g \circ \varphi^{-1}_g(b) \) for any \( a \in \text{End}_C(x_1) \) and \( b \in \text{End}_C(x_2) \). Moreover, we have \( \varphi_{ho} = \varphi_h \circ \varphi_g \).

Proof. Claims (a) and (b) follow from the definition, using the cancellative properties. For (c), using statement (b), we define \( \varphi_g(a) \) to be the unique element in \( \text{End}_C(x_2) \), such that \( g \circ a = \varphi_g(a) \circ g \). For two elements \( a_1, a_2 \in \text{End}_C(x) \), we have
\[
\varphi_g(a_1 \circ a_2) \circ g = g \circ (a_1 \circ a_2) = (g \circ a_1) \circ a_2 = \varphi_g(a_1) \circ a_2 = \varphi_g(a_1) \circ g \circ (a_2) \circ g;
\]

hence, since \( C \) is right cancellative, we obtain \( \varphi_g(a_1 \circ a_2) = \varphi_g(a_1) \circ \varphi_g(a_2) \). In a similar way, we can construct the morphism of monoids \( \varphi^{-1}_g : \text{End}_C(x_2) \to \text{End}_C(x_1) \). It is easy to verify that \( \varphi^{-1}_g \) is an inverse of \( \varphi_g \); the last statement is similar.

A left cancellative and rr-transitive category has the following characterization.

Proposition 1. Let \( C \) be a left cancellative category. The category \( C \) is rr-transitive if and only if, for any \( x_1, x_2 \in \text{Ob}(C) \) and any \( f, g \in \text{Hom}_C(x_1, x_2) \), there exists a unique \( a \in \text{End}_C(x_1) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
x_1 & \xrightarrow{g} & x_2 \\
| & f & | \\
x_1 & \xrightarrow{a} & \\
\end{array}
\]

In other words, for any \( x_1, x_2 \in \text{Ob}(C) \) and \( g \in \text{Hom}_C(x_1, x_2) \) we have \( \text{Hom}_C(x_1, x_2) = g \circ \text{End}_C(x_1) \).

The category \( F^{ad}(C) \) can be viewed as a subcategory of the arrow category \( \text{Arr}(C) \), the category with objects \( \text{Mor}(C) \), and morphisms \( (f, g) : a \to b \) (with \( a, b \in \text{Mor}(C) \)) for each commutative diagram as in

\[
\begin{array}{ccc}
x_1 & \xrightarrow{g} & x_2 \\
| & f & | \\
x_1 & \xrightarrow{a} & y_1 \\
\end{array}
\]

Note that one obtains \( \text{Arr}(C) \) by reversing one arrow in the definition of Quillen’s category of factorizations \( F(C) \) [15]. A main ingredient for the proof in [4] is the construction of a split surjective ring homomorphism \( HH^\bullet(kC) \to H^\bullet(BF(C), k) \). In this paper, we replace \( F(C) \) with \( F^{ad}(C) \), generalizing the explicit construction of the groupoid \( Gr \) in [3].
Proposition 2. Let \( C \) be a cancellative category. The category \( C \) is right deterministic and rr-transitive if and only if, for any \( b \in \text{End}_C(x_1) \) and \( g, f \in \text{Hom}_C(x_1, x_2) \), there exists a unique \( a \in \text{End}_C(x_1) \) such that the following diagram commutes:

\[
\begin{array}{c}
 x_1 \quad g \\
 a \quad x_2 \\
 f \quad b \\
 x_1 \quad f \\
\end{array}
\]

(3)

In a similar way, the category \( C \) is left deterministic and rr-transitive if and only if, for any \( b \in \text{End}_C(x_2) \) and \( g, f \in \text{Hom}_C(x_1, x_2) \), there exists a unique \( a \in \text{End}_C(x_2) \) such that the above diagram commutes.

Proof. Choose \( a \in \text{End}_C(x_1) \) and \( g, f \in \text{Hom}_C(x_1, x_2) \). Since \( C \) is rr-transitive and right cancellative, there exists a unique \( a' \in \text{End}_C(x_1) \) such that \( g = f \circ a' \). Since \( C \) is right deterministic and right cancellative, there exists a unique \( b \) such that \( f \circ (a' \circ a) = b \circ f \). Thus, we have a unique \( b \) such that \( g \circ a = b \circ f \). The other direction also follows. The second claim follows with a similar argument.

Proposition 3. A groupoid is cancellative, deterministic, and rr-transitive.

Proof. Let \( C \) be a groupoid. Since any morphism \( f \in \text{Mor}(C) \) has an inverse \( f^{-1} \in \text{Mor}(C) \), one easily checks that \( C \) is cancellative. Consider the commutative diagram (3). Since \( g \) is invertible, the relation \( g \circ a = b \circ g \) implies \( b = g \circ a \circ g^{-1} \). A similar argument shows that \( a \) is uniquely determined by \( b \); hence \( C \) is deterministic and rr-transitive.

Proposition 4. A poset is cancellative, deterministic, and rr-transitive.

Proof. Let \((C, \leq)\) be a poset, and let \( g, h, f \) be morphisms such that \( g \circ h \) and \( g \circ f \) are morphisms. Then \( s(h) \geq t(h) = s(g) \geq t(g) \) and \( s(f) \geq t(f) = s(g) \geq t(g) \). If \( g \circ h = g \circ f \), then \( s(g \circ h) = s(h) = s(f) = s(g \circ f) \); hence \( f \) and \( g \) are the unique morphisms between \( s(h) = s(f) \) and \( s(g) \). Thus, \( C \) is left cancellative. A similar argument shows that \( C \) is also right cancellative. Consider the commutative diagram (3). Since the only endomorphisms in \( C \) are the identity morphisms, we have \( a = \text{id}_{x_1} \) and necessarily \( b = \text{id}_{x_2} \). Hence \( C \) is deterministic and rr-transitive.

Further, we give the promised example of a category that satisfies the assumptions needed in our main results but which is neither a groupoid nor a poset.

Example 1. Let \( C \) be the category with two objects (denoted \( x_1 \) and \( x_2 \)), with morphisms given by \( \text{Mor}(C) = \{ \text{id}_{x_1}, a, \text{id}_{x_2}, b, \varphi, \psi \} \), and with the compositions of morphisms

\[
a \circ a = \text{id}_{x_1}, b \circ b = \text{id}_{x_2}, \varphi \circ a = \psi, \psi \circ a = \varphi, b \circ \varphi = \psi, b \circ \psi = \varphi.
\]

It is an easy exercise to verify that \( C \) is a deterministic, cancellative, rr-transitive finite category.

3. GRADED MORPHISMS

3.1. The Graded Morphism \( T^* \)

For any \( m \geq 0 \), we define \( k \)-linear maps

\[
T^m : C^m(kC) \to C^m(kNF^{ad}(C), k).
\]

A \( k \)-linear map \( X \in C^m(kC) \) is determined by its values on a basis of \((kC)^{\otimes m}\). If \( m \geq 1 \), then, for each \( g_0, \ldots, g_{m-1} \in \text{Mor}(C) \), we have the following unique decomposition:

\[
X(g_{m-1} \otimes \cdots \otimes g_0) = \sum_{h \in \text{Mor}(C)} X^h_{g_{m-1}, \ldots, g_0} h.
\]
Next, we define $T^m(X) \in C^m(kNF_{\text{ad}}(C), k)$ on the basis $NF_{\text{ad}}(C)_m$ of $kNF_{\text{ad}}(C)_m$ as in

$$
T^m(X) = \left( \begin{array}{c}
0 \\
0 \\
0 \\
x_0 \\
x_1 \\
x_m \\
\cdots \\
x_{m-1} \\
x_{m-2} \\
\vdots \\
x_1 \\
x_0 \\
\end{array} \right) = \left( \begin{array}{c}
X_{g_0 g_1 \cdots g_{m-1}} \to X \\
X_{a_0 a_1 a_m} \to X \\
X_{x_0 x_1 x_m} \to X \\
x_{m-2} x_{m-1} x_m \\
x_{g_0 g_1 \cdots g_m} \to X \\
\vvdots \\
x_{g_0 g_1 \cdots g_m} \to X \\
x_{g_0 g_1 \cdots g_m} \to X \\
x_{g_0 g_1 \cdots g_m} \to X \\
\end{array} \right)
$$

If $m = 0$, then $C^0(kC) = kC$ and $X = \sum_{h \in \text{Mor}(C)} X^h h$. In this case, $T^0$ is given by

$$
T^0(X) = \left( \begin{array}{c}
X_{0} \\
X_{a_0} \\
X_{x_0 x_0} \\
\cdots \\
X_{g_0 g_0} \\
\vvdots \\
X_{g_0 g_0} \\
X_{g_0 g_0} \\
X_{g_0 g_0} \\
\end{array} \right) = X_{0}.
$$

Extending $T^m(X)$ by linearity, we obtain a graded morphism $T^\bullet = (T^m)_{m \geq 0}$ between the cochain complexes $C^\bullet(kC)$ and $C^\bullet(kNF_{\text{ad}}(C), k)$.

**Proposition 5.** If $C$ is a finite cancellative category, then $T^{m+1} \circ \partial^m = (-1)^{m+1} \sigma^m \circ T^m$.

**Proof.** We follow the argument in [3, §3.1] and show that, for any $X \in C^m(kC)$, we have

$$
T^{m+1}(\partial^m X)(\sigma) = (-1)^m \sigma^m (T^m X)(\sigma)
$$

for all $(m+1)$-chains $\sigma$.

The expression $T^{m+1}(\partial^m X)(\sigma)$ is the coefficient of $h' = g_m g_{m-1} \cdots g_1 g_0 a_0$ in $(\partial^m X)(g_m \otimes \cdots \otimes g_0)$, which, by (2), is equal to

$$
\sum_{h \in \text{Mor}(C)} \left( X_{g_{m-1} \cdots g_0}^h g_m h + \sum_{j=0}^{m-1} (-1)^{m-j} X_{g_{m-1} \cdots g_0}^h g_{m-j+1} g_j g_{j+1} \cdots g_0 h + (-1)^m X_{g_{m-1} \cdots g_0}^h g_{m-1} g_0 h \right).
$$

Since $\text{Mor}(C)$ is a basis for $kC$, it follows that the coefficient of $h'$ is

$$
\sum_{h \in \text{Mor}(C)} X_{g_{m-1} \cdots g_0}^h + \sum_{j=0}^{m-1} (-1)^j X_{g_{m-1} \cdots g_0}^h g_{m-j+1} + (-1)^m \sum_{h \in \text{Mor}(C)} X_{g_{m-1} \cdots g_0}^h.
$$

It follows from the commutativity of the diagram $\sigma$ that the equation $h g_0 = h'$ admits the solution $h_1 = g_m \cdots g_0 a_1$. If $h$ is another solution to $h g_0 = h'$, then $h_1 g_0 = h_2 g_0$, and since $C$ is right cancellative, we have $h_1 = h_2$. For the equation $g_m h = h'$, since $C$ is left cancellative, we have the unique solution $h = g_{m-1} \cdots g_0 a_0$. Hence $T^{m+1}(\partial^m X)(\sigma)$ is equal to

$$
\frac{X_{g_{m-1} \cdots g_0}^h g_m h + \sum_{j=0}^{m-1} (-1)^j X_{g_{m-1} \cdots g_0}^h g_{m-j+1} + (-1)^m \sum_{h \in \text{Mor}(C)} X_{g_{m-1} \cdots g_0}^h}{T^m(X) \circ \partial^{m+1} \sigma + (-1)^{m+1} \sum_{j=1}^{m} X_{g_{m-1} \cdots g_0}^h g_{m-j+1} + (-1)^m X_{g_{m-1} \cdots g_0}^h}.
$$

From the description of $\delta^m$ in Subsec. 2.2, this also equals $(-1)^m \sigma^m (T^m X)(\sigma)$. This concludes the proof.

For clarity, we treat the case of $m = 0$ separately. If $X \in C^0(kC)$, then $T^1(\partial^0 X)(\sigma)$ is the coefficient of $g_0 a_0$ in

$$
(\partial^0 X)(g_0) = g_0 X X_{g_0} = \sum_{h \in \text{Mor}(C)} X_h g_0 h = X_h h_{g_0}.
$$
If $g_0h = g_0 a_0$, then $h = a_0$, and if $hg_0 = g_0 a_0$, then $h = a_1$. Hence $T^1(\partial^0 X)(\sigma) = X^{a_0} - X^{a_1}$. On the other hand, $X^{a_0} = T^0(X) \circ d_1^g$ and $X^{a_1} = T^0(X) \circ d_0^g$. Thus, $T^1(\partial^0 X)(\sigma) = -\delta^1(T^m X)(\sigma)$. 

3.2. A Section $\mathcal{X}^\bullet$ for $T^\bullet$

In this subsection, we assume that $\mathcal{C}$ is right deterministic and right cancellative. With these assumptions, we can describe a section $\mathcal{X}^\bullet : C^\bullet(k N F^{\text{ad}}(\mathcal{C}), k) \to C^\bullet(k \mathcal{C})$ for the map $T^\bullet$. The $k$-linear map $T \in C^m(k N F^{\text{ad}}(\mathcal{C}), k)$ is determined by its values on $N F^{\text{ad}}(\mathcal{C})_m$. For each $m$-chain

$$\sigma = \left( \begin{array}{c}
\sigma_0 \\
\sigma_1 \\
\vdots \\
\sigma_n \\
\sigma_m 
\end{array} \right) \in N F^{\text{ad}}(\mathcal{C})_m$$

by Lemma 1 (b), any sequence $g_{m-1}, \ldots, g_0, a_0$ of composable morphisms in $\mathcal{C}$, as in (4), uniquely determines the morphisms $a_1, \ldots, a_m$. Hence we adopt the notation

$$T_{g_m \cdots g_1 g_0 a_0}^{g_m-1 \cdots g_1} := T(\sigma)$$

if $m \geq 1$ and $T_{a_0} := T(\sigma)$ if $m = 0$. If $m \geq 1$, we define $\mathcal{X}^m : C^m(k N F^{\text{ad}}(\mathcal{C}), k) \to C^m(k \mathcal{C})$ by

$$\mathcal{X}^m(T)(g_{m-1} \otimes \cdots \otimes g_0) = \sum_{a_0 \in \text{End}_C(s(g_0))} T_{g_m \cdots g_1 g_0 a_0}^{g_m-1 \cdots g_1} g_{m-1} g_{m-2} \cdots g_2 g_1 g_0 a_0$$

for any $g_0, \ldots, g_{m-1} \in \text{Mor}(\mathcal{C})$. If $m = 0$, we let $\mathcal{X}^0(T) = \sum_{a_0 \in \text{End}(\mathcal{C})} T_{a_0} a_0$. For all $m \geq 0$, we extend $\mathcal{X}^m(T)$ by linearity. If the composition $g_{m-1} \circ g_{m-2} \cdots g_2 \circ g_1 \circ g_0 \circ a_0$ does not exist, then the element $T_{g_m \cdots g_1 g_0 a_0}^{g_m-1 \cdots g_1}$ is not defined; however, in this case $g_{m-1} g_{m-2} \cdots g_2 g_1 g_0 a_0$ is zero in $k \mathcal{C}$; i.e., the value of $\mathcal{X}^m(T)$ on noncomposable morphisms is zero.

Proposition 6. If $\mathcal{C}$ is a finite deterministic cancellative category, then

$$\mathcal{X}^{m+1} \circ \delta^m = (-1)^{m+1} \delta^m \circ \mathcal{X}^m.$$

Proof. For $m \geq 0$, we show that

$$(-1)^{m+1} \mathcal{X}^{m+1}(\partial^m T)(g_m \otimes \cdots \otimes g_0) = \partial^m(\mathcal{X}^m T)(g_m \otimes \cdots \otimes g_0)$$

for any $T \in C^m(k N F^{\text{ad}}(\mathcal{C}), k)$ and for all $g_0, \ldots, g_m \in \text{Mor}(\mathcal{C})$. The right-hand side is

$$g_m(\mathcal{X}^m T)(g_{m-1} \otimes \cdots \otimes g_0) + \sum_{j=0}^{m-1} (-1)^{m-j}(\mathcal{X}^m T)(\cdots \otimes g_{j+1} g_j \otimes \cdots)$$

$$+ (-1)^{m+1}(\mathcal{X}^m T)(g_m \otimes \cdots \otimes g_1) g_0.$$

By the definition of $\mathcal{X}^m$, if $g_{m-1} \circ \cdots \circ g_0$ exists, then $(\mathcal{X}^m T)(g_{m-1} \otimes \cdots \otimes g_0) \in \text{Mor}_C(s(g_0), t(g_{m-1}));$ hence if $g_m \circ \cdots \circ g_0$ does not exist, then $t(g_{m-1}) \neq s(g_m)$ and, consequently,

$$g_m(\mathcal{X}^m T)(g_{m-1} \otimes \cdots \otimes g_0) = 0.$$

A similar argument shows that if $g_m \circ \cdots \circ g_0$ does not exist, then all other terms in the above sum are zero. In this case, the left-hand side of (5) is zero as well. Assume that $g_m \circ \cdots \circ g_0$ exists. By the definition of $\mathcal{X}^m$, the right-hand side of (5) further equals

$$g_m \sum_{a_0 \in \text{End}_C(s(g_0))} T^{g_m-1 \cdots g_0 a_0}_{g_m \cdots g_1 a_1} g_{m-1} \cdots g_0 a_0 + \sum_{j=0}^{m-1} (-1)^{m-j} \sum_{a_0 \in \text{End}_C(s(g_0))} T^{g_m-1 \cdots g_1 a_1}_{g_m \cdots g_1 a_1} g_{m-1} \cdots g_1 a_1 g_0.$$
Since $C$ is deterministic, by Lemma 1 (c) we have a monoid isomorphism
\[ \varphi_{g_0} : \text{End}_C(s(g_0)) \to \text{End}_C(s(g_1)) \]
such that $\varphi_{g_0}(a_0) \circ g_0 = g_0 \circ a_0$ for all $a_0 \in \text{End}_C(s(g_0))$. Hence
\[ \sum_{a_1 \in \text{End}_C(s(g_1))} T^{g_{m_1}}_{g_{m_1}^*} \cdot g_{m_1} \cdot g_{a_1} g_0 = \sum_{a_0 \in \text{End}_C(s(g_0))} T^{g_{m_2}}_{g_{m_2}^*} \cdot \varphi_{g_0}(a_0) \cdot g_{m_2} \cdot g_1 g_0 a_0, \]
which equals $\lambda^{m+1}(T \circ d_{m+1}^m)(g_m \otimes \cdots \otimes g_0)$. We also have
\[ \sum_{a_0 \in \text{End}_C(s(g_0))} T^{g_{m_1}}_{g_{m_1}^*} \cdot g_{m_1} \cdot g_0 a_0 = \lambda^{m+1}(T \circ d_{m-j}^m)(g_m \otimes \cdots \otimes g_0). \]
After this, for $0 \leq j \leq m - 1$, we obtain
\[ \sum_{a_0 \in \text{End}_C(s(g_0))} T^{g_{m_1}}_{g_{m_1}^*} \cdot g_{m_1} \cdot g_0 a_0 = \lambda^{m+1}(T \circ d_{m-j}^m)(g_m \otimes \cdots \otimes g_0). \]
Since
\[ \sum_{j=0}^{m-1} (-1)^{m-j} \lambda^{m+1}(T \circ d_{m-j}^m) = (-1)^m \sum_{i=1}^m (-1)^i \lambda^{m+1}(T \circ d_i^m), \]
we see that the claim follows from the description of $\delta^m$ in Subsec. 2.2.

For clarity, we treat the case of $m = 0$ separately. If $T \in \mathcal{C}(kN\text{F}^{ad}(C), k)$ then, by definition,
\[ \mathcal{X}^0(T) = \sum_{a_0 \in \text{End}(C)} T^{a_0} a_0. \]
Thus,
\[ (\partial^0 \circ \mathcal{X}^0)(T)(g_0) = \sum_{a_0 \in \text{End}(C)} T^{a_0} (g_0 a_0 - a_0 g_0). \]
Hence
\[ (\partial^0 \circ \mathcal{X}^0)(T)(g_0) = \sum_{a_0 \in \text{End}_C(s(g_0))} T^{a_0} g_0 a_0 - \sum_{a_1 \in \text{End}_C(t(g_0))} T^{a_1} a_1 g_0 \]
\[ = \sum_{a_0 \in \text{End}_C(s(g_0))} T^{a_0} g_0 a_0 - \sum_{a_0 \in \text{End}_C(s(g_0))} T^{\varphi_{g_0}(a_0)} \varphi_{g_0}(a_0) g_0 \]
for the isomorphism $\varphi_{g_0} : \text{End}_C(s(g_0)) \to \text{End}_C(t(g_0))$ given in Lemma 1 (c). On the other hand,
\[ \mathcal{X}^1(\partial^0 T)(g_0) = \sum_{a_0 \in \text{End}_C(s(g_0))} (\partial^0 T)^{g_0 a_0} g_0 a_0 \]
for any $g_0 \in kC$, where, by definition, $(\partial^0 T)^{g_0 a_0} = (\partial^0 T)(\sigma)$ with $\sigma$ as in (4) and $m = 1$. Since $(\partial^0 T)(\sigma) = T^{a_1} - T^{a_0}$, where $T^{a_1} = T \circ d_0^1(\sigma)$ and $T^{a_0} = T \circ d_0^1(\sigma)$, it follows that $\mathcal{X}^1(\partial^0 T)(g_0)$ is equal to
\[ \sum_{a_0 \in \text{End}_C(s(g_0))} (T^{\varphi_{g_0}(a_0)} - T^{a_0}) g_0 a_0 = \sum_{a_0 \in \text{End}_C(s(g_0))} T^{\varphi_{g_0}(a_0)} g_0 a_0 - \sum_{a_0 \in \text{End}_C(s(g_0))} T^{a_0} g_0 a_0. \]
Since $g_0 a_0 = \varphi(a_0) g_0$, we obtain $\partial^0 \circ \mathcal{X}^0(T)(g_0) = -\mathcal{X}^1(\partial^0 T)(g_0)$. \hfill \Box

**Proposition 7.** Let $C$ be a right deterministic cancellative category. The map $\mathcal{X}^\bullet$ is a $k$-linear right inverse of $T^\bullet$.

**Proof.** Let us show that
\[ (T^m \circ \mathcal{X}^m)(T) = T \]
for all \( T \in \mathcal{C}^m(k NF^{ad}(C), k) \) and for all \( m \geq 0 \). For each \( \sigma \) as in Fig. 4, \( T^m(\mathcal{X}^m(T))(\sigma) \) is the coefficient of \( g_{m-1}g_{m-2} \cdots g_2g_1g_0a_0 \) in \( \mathcal{X}^m(T)(g_{m-1}, \ldots, g_0) \). Since \( C \) is left cancellative, it follows that the equation

\[
g_{m-1}g_{m-2} \cdots g_1g_0x = g_{m-1}g_{m-2} \cdots g_1g_0a_0
\]

has the unique solution \( x = a_0 \). If \( m = 0 \), then the above equation is \( x = a_0 \) and we do not need \( C \) to be left cancellative in this case. Hence

\[
T^m(\mathcal{X}^m(T))(\sigma) = T_{g_{m-1}, \ldots, g_0} = T(\sigma),
\]

and the claim follows. \( \square \)

4. RELATIVE HOCHSCHILD COHOMOLOGY

The notion of relative Hochschild cohomology introduced in [2] simplifies the comparison of Hochschild cohomology and simplicial cohomology done in [1]. By [2, Theorem 1.2], if \( R \) is a separable subalgebra \([14, \text{Definition } 4.1.7]\) of \( kC \), then the \( R \)-relative Hochschild cohomology of \( kC \) is isomorphic to the Hochschild cohomology of \( kC \). For our purpose, we choose \( R \) to be the subalgebra \( kC^{id} \) of \( kC \) generated by the set of all identity morphisms on each object.

**Lemma 2.** The algebra \( kC^{id} \) is separable.

**Proof.** The algebra \( R = kC^{id} \) is separable if and only if there exists an idempotent \( e = \sum_i u_i \otimes v_i \) in the enveloping algebra \( R \otimes R^{op} \) of \( R \) such that \( \sum_i u_i v_i = 1 \) and \( (r \otimes 1)e = (1 \otimes r)e \). One checks that \( e = \sum_{x \in \text{Ob}(C)} 1_x \otimes 1_x \) satisfies this property. \( \square \)

As in [2, §1], the \( kC^{id} \)-relative Hochschild cohomology of \( kC \) is the cohomology of the subcomplex of \( kC^{id} \)-relative cochains \( C^*(kC, kC^{id}) \subseteq C^*(kC) \). The \( kC^{id} \)-relative \( m \)-cochain group \( C^m(kC, kC^{id}) \) consists of \( k \)-linear functions \( f \) satisfying

- \( f(a g_1 \otimes \cdots \otimes g_m) = af(g_1 \otimes \cdots \otimes g_m) \),
- \( f(g_1 \otimes \cdots \otimes g_m a) = f(g_1 \otimes \cdots \otimes g_m)a \),
- \( f(g_1 \otimes \cdots \otimes g_i a \otimes g_{i+1} \otimes \cdots \otimes g_m) = f(g_1 \otimes \cdots \otimes g_i \otimes ag_{i+1} \otimes \cdots \otimes g_m) \) \( 1 \leq i \leq m - 1 \)

for all \( g_1, \ldots, g_m \in \text{Mor}(C) \) and any \( a \in kC^{id} \). Note that

\[
C^0(kC, kC^{id}) = C^0(kC) = kC.
\]

**Proposition 8.** If \( m \geq 1 \), then an element \( f \in C^m(kC) \) lies in \( C^m(kC, kC^{id}) \) if and only if, for any \( g_1, \ldots, g_m \in \text{Mor}(C) \), one has

- \( f(g_1 \otimes \cdots \otimes g_m) = 0 \) if \( t(g_{i+1}) \neq s(g_i) \) for some \( 1 \leq i \leq m - 1 \).
- \( f(g_1 \otimes \cdots \otimes g_m) \in k \{ \text{Hom}_C(s(g_m), t(g_1)) \} \) otherwise.

**Proof.** Let \( f \) be a \( kC^{id} \)-relative \( m \)-cochain. For any \( g_1, \ldots, g_m \in \text{Mor}(C) \), we have

\[
f(g_1 \otimes \cdots \otimes g_m) = f(\text{id}_{(g_2)}g_1 \otimes \cdots \otimes g_m) = \text{id}_{(g_1)}f(g_1 \otimes \cdots \otimes g_m) = \text{id}_{(g_1)}f(g_1 \otimes \cdots \otimes g_m).
\]

Since \( f(g_1 \otimes \cdots \otimes g_m) = \sum_{h \in \text{Mor}(C)} f^h \cdot g_1 \otimes \cdots \otimes g_m h \), it follows that the nonzero terms in this sum are \( f_{g_1 \otimes \cdots \otimes g_m}^h \) with \( t(h) = t(g_1) \). A similar argument shows that \( s(h) = s(g_m) \); hence

\[
f(g_1 \otimes \cdots \otimes g_m) = \sum_{h \in \text{Hom}_C(s(g_m), t(g_1))} f_{g_1 \otimes \cdots \otimes g_m}^h.
\]
Let \( i \in \{1, \ldots, m-1\} \). Moreover,

\[
f(g_1 \otimes \cdots \otimes g_i \otimes g_{i+1} \otimes \cdots \otimes g_m) = f(g_1 \otimes \cdots \otimes g_i \otimes \text{id}_{t(g_{i+1})} g_{i+1} \otimes \cdots \otimes g_m)
\]

Thus, if \( s(g_i) \neq t(g_{i+1}) \), then \( g_i \text{id}_{t(g_{i+1})} = 0 \), in which case \( f(g_1 \otimes \cdots \otimes g_i \otimes g_{i+1} \cdots \otimes g_m) = 0 \). Hence a \( kC^{\text{id}} \)-relative \( m \)-cochain has the indicated properties. Conversely, if \( f \) has the indicated properties, one checks that it is \( kC^{\text{id}} \)-relative using the fact that \( kC^{\text{id}} \) is generated by \( \{\text{id}_x : x \in \text{Ob}(C)\} \).

\[\square\]

**Corollary 2.** Let \( C \) be a left cancellative rr-transitive category. Then \( f \in C^m(kC, kC^{\text{id}}) \) if and only if

\[
f(g_{m-1} \otimes \cdots \otimes g_0) = \sum_{a \in \text{End}_C(s(g_0))} f^{g_{m-1}g_{m-2} \cdots g_0a}_{g_{m-1}, \ldots, g_0}^a
\]

for any \( g_0, \ldots, g_{m-1} \in \text{Mor}(C) \).

**Proof.** By Proposition 8, for any \( f \in C^m(kC, kC^{\text{id}}) \) and any \( g_0, \ldots, g_{m-1} \in \text{Mor}(C) \) we have

\[
f(g_{m-1} \otimes \cdots \otimes g_0) = \sum_{h \in \text{Hom}_C(s(g_0), t(g_{m-1}))} f^h_{g_{m-1}, \ldots, g_0}
\]

where the sum is zero if \( t(g_i) \neq s(g_{i+1}) \) for some \( 0 \leq i \leq m-2 \). Since \( C \) is rr-transitive and left cancellative, the claim follows by Proposition 1, with \( g := g_{m-1} \circ g_{m-2} \circ \cdots \circ g_1 \circ g_0 \) and \( f := h \).

\[\square\]

**Proof of Theorem 1.** For a small category \( C \), we have (see [6, §5])

\[
\text{H}^\bullet(F^\text{ad}(C), k) \cong \text{H}^\bullet(B F^\text{ad}(C)) \cong \text{H}^\bullet(C^\bullet(kN F^\text{ad}(C), k), \delta^\bullet).
\]

If we modify the differential by a nonzero constant, \( \alpha^m = (-1)^{m+1} \delta^m \), for any nonnegative integer \( m \), the cohomology does not change, \( \text{H}^\bullet(F^\text{ad}(C), k) \cong \text{H}^\bullet(C^\bullet(kN F^\text{ad}(C), k), \alpha^\bullet) \), but now \( T^\bullet \) becomes a cochain map if \( C \) is cancellative (by Proposition 5) and \( X^\bullet \) becomes a cochain map if \( C \) is also deterministic (by Proposition 6). With Proposition 7, the map \( T^\bullet \) is a surjective cochain map if \( C \) is deterministic and cancellative; hence, it induces a surjective homomorphism in cohomology.

Since \( kC^{\text{id}} \) is a separable subalgebra of \( kC \) (by Lemma 2), as in [2, §1], \( \text{H}^\bullet(kC) \) is isomorphic to the \( kC^{\text{id}} \)-relative Hochschild cohomology. By [2, Theorem 1.2], the inclusion of complexes \( C^\bullet(kC, kC^{\text{id}}) \rightarrow C^\bullet(kC) \) induces an isomorphism in cohomology. That is, we can replace the cohomology of the complex \( C^\bullet(kC) \) by the cohomology of its subcomplex \( C^\bullet(kC, kC^{\text{id}}) \) consisting of \( kC^{\text{id}} \)-relative cochains.

Further, assume that \( C \) is rr-transitive. Note that, by the definition of \( X^\bullet \) and Corollary 2, we have \( \text{Im}(X^\bullet) \subset C^\bullet(kC, kC^{\text{id}}) \). As in the proof of Proposition 7, one checks that \( X^\bullet \) is a \( k \)-linear right inverse of the restriction \( T^\bullet \) of \( T^\bullet \) to \( C^\bullet(kC, kC^{\text{id}}) \).

Moreover, for \( X \in C^m(kC, kC^{\text{id}}) \) and \( g_0, \ldots, g_{m-1} \in \text{Mor}(C) \) we obtain

\[
X^m(T^m(X))(g_{m-1} \otimes \cdots \otimes g_0) = \sum_{a \in \text{End}_C(s(g_0))} (T^m(X))^{g_{m-1} \cdots g_0a}_{g_{m-1}, \ldots, g_0} g_{m-1} \cdots g_0a
\]

where the last equality is true by Corollary 2. Hence \( X^\bullet \) is also a left inverse for \( T^\bullet \).

\[\square\]
5. DERIVATIONS ON $k\mathcal{C}$ AND CHARACTERS ON $F^{ad}(\mathcal{C})$

5.1. Graded Derivations

Any category $\mathcal{C}$ gives rise to a semigroup with zero, which we denote by $S$. The set $S$ is given by

$$S := \{(x_1, x_2) \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{C})|\text{Hom}_\mathcal{C}(x_1, x_2) \neq \emptyset\} \cup \{(0, 0)\}.$$

The operation on $S$ is

$$(x_1, x_2)(x_3, x_4) = \begin{cases} (x_1, x_4) & \text{if } x_2 = x_3, \\ 0 & \text{if } x_2 \neq x_3. \end{cases}$$

The algebra $k\mathcal{C}$ is graded by $S$ with $k\mathcal{C} = \bigoplus_{(x,y) \in S} R_{(x,y)}$, where $R_{(x_1,x_2)} = k[\text{Hom}_\mathcal{C}(x_1, x_2)]$ if $(x_1, x_2) \in S \setminus \{(0, 0)\}$ and $R_{(0,0)} = 0$.

A linear endomorphism $\phi \in \text{End}_k(k\mathcal{C})$ is said to be $S$-graded if $\phi$ preserves the grading by $S$; i.e., $\phi(R_{(x_1,x_2)}) \subseteq R_{(x_1,x_2)}$, for any $(x_1, x_2) \in S$. We denote by $\text{End}^{gr}(k\mathcal{C})$ the $k$-subspace of all $S$-graded endomorphisms and by $\text{Der}^{gr}(k\mathcal{C})$ the $k$-subspace of all $S$-graded derivations of $k\mathcal{C}$. Since $\mathcal{C}^1(k\mathcal{C}) = \text{End}_k(k\mathcal{C})$, it follows from Proposition 8 that $\text{End}^{gr}(k\mathcal{C}) = \mathcal{C}^1(k\mathcal{C}, k\mathcal{C}^{id})$; i.e., $S$-graded endomorphisms are $k\mathcal{C}$-relative 1-cochain maps.

5.2. Characters on $F^{ad}(\mathcal{C})$

One can extend the notion of character used in [10, §3] for the group case to any finite category $\mathcal{D}$ as follows. Denote by $k\text{Mor}(\mathcal{D})$ the $k$-vector space of all set maps from $\text{Mor}(\mathcal{D})$ to $k$. A set map $T \in k\text{Mor}(\mathcal{D})$ is called a character if

$$T(\eta \circ \zeta) = T(\eta) + T(\zeta)$$

for all morphisms $\zeta, \eta \in \text{Mor}(\mathcal{D})$ such that $s(\eta) = t(\zeta)$. We denote the $k$-subspace in $k\text{Mor}(\mathcal{D})$ consisting of all characters on $\mathcal{D}$ by $\text{Char}(\mathcal{D})$.

Here we are interested in the case where $\mathcal{D} = F^{ad}(\mathcal{C})$ and we identify $\mathcal{C}^1(k\mathcal{C}, F^{ad}(\mathcal{C}))$ with $k\text{Mor}(F^{ad}(\mathcal{C}))$. Recall that, for two morphisms $\zeta, \eta$ in $F^{ad}(\mathcal{C})$ such that $s(\eta) = t(\zeta)$, their composition is as in

$$\zeta = \begin{pmatrix} x_1 & g & x_2 \\ a & \downarrow b & \downarrow c \\ x_1 & g & x_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} x_2 & f & x_3 \\ b & \downarrow c & \downarrow d \\ x_2 & f & x_3 \end{pmatrix}, \quad \eta \circ \zeta = \begin{pmatrix} x_1 & f \circ g & x_3 \\ a & \downarrow b & \downarrow c \\ x_1 & f \circ g & x_3 \end{pmatrix}. \tag{6}$$

In Sec. 3 we have defined the map $T^1 : \text{End}_k(k\mathcal{C}) \to k\text{Mor}(F^{ad}(\mathcal{C}))$. For any $X \in \text{End}_k(k\mathcal{C})$, we have a unique decomposition $X \gamma = \sum_{h \in \text{Mor}(\mathcal{C})} X_h^\gamma h$ for all $g \in \text{Mor}(\mathcal{C})$, and $T^1(X)$ is defined by $T^1(X)(\sigma) = X_\gamma^\sigma$ for all 1-simplices $\sigma$ as in (1). If $\mathcal{C}$ is right deterministic and right cancellative, we have defined $\mathcal{X}^1 : k\text{Mor}(F^{ad}(\mathcal{C})) \to \text{End}_k(k\mathcal{C})$ by

$$T \mapsto \mathcal{X}^1(T), \quad \mathcal{X}^1(T)(g) = \sum_{a \in \text{End}_{\mathcal{C}}(s(g))} T_g^\gamma a \circ a$$

for all $g \in k\mathcal{C}$, where $T_g^\gamma a$ is the value $T(\sigma)$ with $\sigma$ as in (1).

**Proof of Theorem 2.** We adopt the notation $T := T^1|_{\text{Der}^{gr}(k\mathcal{C})}$ for the restriction of $T^1$ to $\text{Der}^{gr}(k\mathcal{C})$ and $\mathcal{X} := \mathcal{X}^1|_{\text{Char}(F^{ad}(\mathcal{C}))}$ for the restriction of $\mathcal{X}^1$ to $\text{Char}(F^{ad}(\mathcal{C}))$.

First, we show that $\text{Im}(T) \subseteq \text{Char}(F^{ad}(\mathcal{C}))$; i.e., we show that, for $X \in \text{Der}^{gr}(k\mathcal{C})$ and any composable morphisms $\zeta, \eta \in \text{Mor}(F^{ad}(\mathcal{C}))$ as in (6), we have

$$T(X)(\eta \circ \zeta) = T(X)(\zeta) + T(X)(\eta).$$

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By the definition of $T$, $\zeta$, and $\eta$, this is equivalent to
\[
X^\text{fogoa}_{fog} = X^\text{goa}_{g} + X^\text{fob}_{f}.
\]
Since $\mathcal{C}$ is rr-transitive and left cancellative, by Corollary 2, for any $h \in \text{Mor}(\mathcal{C})$, we have
\[
X(h) = \sum_{d \in \text{End}_{\mathcal{C}}(s(h))} X_{h}^{\text{hod}} \circ d.
\]
Since $X$ is a derivation, we have $X(fg) = X(gf) + fX(g)$, which is equivalent to
\[
\sum_{d \in \text{End}_{\mathcal{C}}(s(g))} X^\text{fogod}_{fog} f \circ g \circ d = \sum_{d' \in \text{End}_{\mathcal{C}}(s(f))} X^\text{fod}_{f} f \circ d' \circ g + \sum_{d \in \text{End}_{\mathcal{C}}(s(g))} X^\text{fogod}_{f} f \circ g \circ d.
\]
Since $\mathcal{C}$ is deterministic and cancellative, by Lemma 1 (c) there exists an isomorphism of monoids $\varphi_{g}^{-1} : \text{End}_{\mathcal{C}}(t(g)) \to \text{End}_{\mathcal{C}}(s(g))$ such that $d' \circ g = g \circ \varphi_{g}^{-1}(d')$. Hence
\[
\sum_{d \in \text{End}_{\mathcal{C}}(s(g))} X^\text{fogod}_{fog} f \circ g \circ d = \sum_{d' \in \text{End}_{\mathcal{C}}(s(f))} X^\text{fod}_{f} f \circ g \circ \varphi_{g}^{-1}(d') + \sum_{d \in \text{End}_{\mathcal{C}}(s(g))} X^\text{fogod}_{f} f \circ g \circ d.
\]
Since $\mathcal{C}$ is cancellative, we have $f \circ g \circ d = f \circ g \circ a$ if and only if $d = a$; $\varphi_{g}^{-1}(d') = a$ if and only if $d' = b$. Hence the coefficient of $f \circ g \circ a$ is
\[
X^\text{fogoa}_{fog} = X^\text{fob}_{f} + X^\text{goa}_{g}.
\]
This shows that $T$ maps graded derivations to characters.

By the definition of $\mathcal{X}$, we have $\text{Im}(\mathcal{X}) \subseteq \text{End}^{k}\{kC\}$. So it suffices to show that $\text{Im}(\mathcal{X}) \subseteq \text{Der}(kC)$; i.e., we show that, for $T \in \text{Char}(\text{Fad}(\mathcal{C}))$ and any $f, g \in \text{Mor}(\mathcal{C})$, we have
\[
\mathcal{X}(T)(fg) = \mathcal{X}(T)(g) + f\mathcal{X}(T)(g).
\]
(7)
By the definition of $\mathcal{X}$, for any $h \in \text{Mor}(\mathcal{C})$ we have
\[
\mathcal{X}(T)(h) \subseteq k[\text{End}_{k}(s(h), t(h))];
\]
hence if $s(f) \neq t(g)$, both sides of (7) are zero. If $s(f) = t(g), (7)$ is equivalent to
\[
\sum_{a \in \text{End}_{\mathcal{C}}(s(fg))} T^{\text{fogoa}}_{fog} f \circ g \circ a = \sum_{b \in \text{End}_{\mathcal{C}}(s(f))} T^{\text{fob}}_{f} f \circ b \circ g + \sum_{a \in \text{End}_{\mathcal{C}}(s(g))} T^{\text{goa}}_{g} f \circ g \circ a.
\]
Since $\mathcal{C}$ is deterministic and cancellative, it follows by Lemma 1 (c) that there exists a monoid isomorphism $\varphi_{g} : \text{End}_{\mathcal{C}}(s(g)) \to \text{End}_{\mathcal{C}}(s(f))$ satisfying
\[
\varphi_{g} \circ g = \varphi_{g}(a) \circ g = g \circ a.
\]
Hence (7) is equivalent to
\[
\sum_{a \in \text{End}_{\mathcal{C}}(s(g))} T^{\text{fogoa}}_{fog} f \circ g \circ a = \sum_{a \in \text{End}_{\mathcal{C}}(s(g))} \left( T^{f \circ \varphi_{g}}_{f} + T^{\text{goa}}_{g} \right) f \circ g \circ a;
\]
that is, (7) is equivalent to
\[
T^{\text{fogoa}}_{fog} = T^{f \circ \varphi_{g}}_{f} + T^{\text{goa}}_{g}
\]
(8)
for all $a \in \text{End}_{\mathcal{C}}(s(g))$. For any $a \in \text{End}_{\mathcal{C}}(s(g))$, we have
\[
T^{\text{goa}}_{g} = T(\alpha), T^{f \circ \varphi_{g}}_{f} = T(\beta), T^{\text{fogoa} \circ f_{og}}(a) = T(\gamma),
\]
where $\alpha, \beta,$ and $\gamma$ are as in the following diagrams:

$\alpha = \begin{pmatrix} s(g) & g & t(g) \\ a & \varphi_{g}(a) & t(g) \end{pmatrix}$
$\beta = \begin{pmatrix} t(g) & f & t(f) \\ \varphi_{g}(a) & \varphi_{f}(\varphi_{g}(a)) & t(f) \end{pmatrix}$
$\gamma = \begin{pmatrix} s(g) & f_{og} & t(f) \\ a & \varphi_{f_{og}}(a) & t(f) \end{pmatrix}$
Applying Lemma 1 (c), we obtain $\gamma = \beta \circ \alpha$. Thus, equation (8) is true, because $T$ is a character. In other words, we have $k$-linear maps

$$T : \text{Der}^{gr}(kC) \to \text{Char}(F^{\text{ad}}(C)), \quad \mathcal{X} : \text{Char}(F^{\text{ad}}(C)) \to \text{Der}^{gr}(kC).$$

The fact that $T$ is surjective with section $\mathcal{X}$ can be verified using the same arguments as in Proposition 7. To see that this map is an isomorphism, we verify that $\mathcal{X}$ is also a left inverse of $T$. We have

$$\mathcal{X}(T(X))(g) = \sum_{a \in \text{End}_C(s(g))} (T(X))^{g \circ a} g \circ a = \sum_{a \in \text{End}_C(s(g))} X^{g \circ a} g \circ a = X(g),$$

where the first equality follows from the definition of $\mathcal{X}$, and the last equality is true by Corollary 2. □

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