Explicit expressions for Šapovalov elements in Type A.

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December 6, 2022

Abstract

We give explicit expressions for Šapovalov elements in Type A Lie algebras and superalgebras. Explicit expressions were already given in [Mus17] Section 9, using non-commutative determinants, and in fact our first main results, Theorems 2.3 and 2.6 can be viewed as complete expansions of these determinants. But we give new proofs, which seem easier because they avoid induction and cofactor expansion. We also describe Šapovalov elements for \( gl(m, n) \) with respect to an arbitrary Borel subalgebra in Theorem 3.7 and interpret Šapovalov elements in Type A as determinants of Hessenberg matrices in Theorems 4.5 and 4.9. The exact form of the explicit expressions depends on an ordering on the set of positive roots, and Hessenberg matrices are useful in changing the ordering.

Having explicit expressions for Šapovalov elements allows us to give easy proofs of several results in representation theory, see Section 6 and Subsection 4.6.

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1 Definition and significance of Šapovalov elements.

Let \( \mathfrak{g} \) be a simple Lie algebra or a contragredient Lie superalgebra with set of simple and positive roots \( \Pi \) and \( \Delta^+ \) respectively. In the Type A case we consider the cases \( \mathfrak{gl}(m) \) and \( \mathfrak{gl}(m, n) \). Fix a positive root \( \eta \) and a positive integer \( m \). If \( \eta \) is isotropic, assume \( m = 1 \), and if \( \eta \) is odd non-isotropic, assume that \( m \) is odd. Let \( \mathcal{P}(m \eta) \) be the set of partitions of \( m \eta \), see [Mus12] Remark 8.4.3, and also (8.4.4) for the notation \( e_{-\pi} \) below. Then let \( \pi^0 \in \mathcal{P}(m \eta) \) be the unique partition of \( m \eta \) such that

*Research partly supported by NSA Grant H98230-12-1-0249, and Simons Foundation grant 318264.
\( \pi^0(\alpha) = 0 \) if \( \alpha \in \Delta^+ \setminus \Pi \). We say that \( \theta = \theta_{\eta, \underline{m}} \in U(b^-)^{-m\eta} \) is a Šapovalov element for the pair \((\eta, m)\) if it has the form

\[
\theta = \sum_{\pi \in P(m\eta)} e_{-\pi} H_{\pi}, \tag{1.1}
\]

where

\[
H_{\pi} \in U(h), \quad H_{\pi^0} = 1, \tag{1.2}
\]

and

\[
e_{\alpha} \theta \in U(g) (h_\eta + \rho(h_\eta) - m(\eta, \eta)/2) + U(g)n^+, \text{ for all } \alpha \in \Delta^+. \tag{1.3}
\]

For a semisimple Lie algebra, the existence of such elements was shown by Šapovalov, Šap72 Lemma 1. Consider the hyperplane in \( h^* \) given by

\[
\mathcal{H}_{\eta, \underline{m}} = \{ \lambda \in h^* | (\lambda + \rho, \eta) = m(\eta, \eta)/2 \}. \tag{1.4}
\]

From (1.3), the Šapovalov element \( \theta_{\eta, \underline{m}} \) has the important property that if \( \lambda \in \mathcal{H}_{\eta, \underline{m}} \) then \( \theta_{\eta, \underline{m}} v_\lambda \) is a highest weight vector of weight \( \lambda - m\eta \) in \( M(\lambda) \). The normalization condition \( H_{\pi^0} = 1 \) guarantees that \( \theta_{\eta, \underline{m}} v_\lambda \) is never zero. Hence \( x_{\lambda-m\eta} \rightarrow x_{\theta_{\eta, \underline{m}} v_\lambda} \) defines a non-zero map between Verma modules \( M(\lambda - m\eta) \rightarrow M(\lambda) \). In the semisimple case, all maps between Verma are composites of these maps, Hum08 Theorem 5.1.

Šapovalov elements have also appeared in a number of situations in representation theory, usually in Type A. Though not given this name, they appear in the work of Carter and Lusztig CL74. Indeed determinants similar to those in our Theorem 4.5 were introduced in CL74 Equation (5), and our Theorem 5.2 may be viewed as a version of CL74 Theorem 2.7. Carter and Lusztig use their result to study tensor powers of the defining representation of \( GL(V) \), and homomorphisms between Weyl modules in positive characteristic, see also CPS80 and Fra88. Later Carter Car87 used Šapovalov elements to construct raising and lowering operators for \( \mathfrak{sl}(n, \mathbb{C}) \), see also Bru98, Car95. In Car87, these operators are used to construct orthogonal bases for non-integral Verma modules, and all finite dimensional modules for \( \mathfrak{sl}(n, \mathbb{C}) \).

From the results in Section 5 it is only really necessary to consider the case \( m = 1 \), so to simplify notation we set \( \mathcal{H}_\eta = \mathcal{H}_{\eta, 1} \), and denote a Šapovalov element for the pair \((\eta, 1)\) by \( \theta_\eta \). In addition in the Type A case, it is only necessary to consider the highest root (resp. highest odd root) of \( \mathfrak{gl}(m) \) (resp. \( \mathfrak{gl}(m, n) \)), since any positive root of the same type is the highest root of some general linear subalgebra in the Lie algebra (resp. superalgebra) case. These two observations greatly simplify our computations. If \( g = \mathfrak{gl}(m, n) \) and we use the distinguished Borel subalgebra, then a Šapovalov element for an even root is just a Šapovalov element for one of the summands of \( g_0 \). We do not consider what happens to Šapovalov elements for an even root when the Borel subalgebra is changed, but see Mus17 Section 4.

Let \( h \) be a Cartan subalgebra of \( g \) and \( W \) the Weyl group. Unless otherwise stated.
we use the distinguished Borel subalgebra of \( \mathfrak{g} \) and denote the corresponding set of simple and positive roots by \( \Pi \) and \( \Delta^+ \) respectively. The even and odd roots are \( \Delta_0 \) and \( \Delta_1 \), and we set \( \Delta_i^+ = \Delta_i \cap \Delta^+ \), \( \rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha \) for \( i = 0, 1 \) and \( \rho = \rho_0 - \rho_1 \). The dot action of \( W \) on \( \mathfrak{h}^* \) is defined by \( w \cdot \lambda = w(\lambda + \rho) - \rho \) for \( w \in W \) and \( \lambda \in \mathfrak{h}^* \).

For a basic classical simple Lie superalgebra \( \mathfrak{g} \) not of type A, the behaviour of Šapovalov elements is rather subtle. Let \( \Pi \) be the distinguished or anti-distinguished set of roots and let \( W_{\text{nonisotropic}} \) (resp. \( W_{\text{even}} \)) be the subgroup of \( W \) generated by the reflections \( s_\alpha \), where \( \alpha \) is simple and nonisotropic (resp even). Then in \( \mathfrak{Mus17} \) Šapovalov elements \( \theta_{\gamma,m} \) are shown to exist under suitable conditions on \( m \) if \( \gamma \) is in the \( W_{\text{nonisotropic}} \) or \( W_{\text{even}} \) orbit of a simple root. However even for \( \mathfrak{g} = \mathfrak{osp}(3,2) \) the latter condition does not always hold. In the orthosymplectic case, we can gain some insight into the structure of Verma modules by considering the effect of changing the Borel subalgebra on Šapovalov elements. \( \mathfrak{Mus17} \) Section 4. But complications can arise. Consider \( \mathfrak{osp}(3,2) \), which seems to be a good test case for these kind of issues. Changing the Borel subalgebra leads to a map Verma modules \( \mathcal{M}(s,\gamma \cdot \lambda) \longrightarrow \mathcal{M}(\lambda) \) which is non-zero and not injective with \( \gamma \) an even root. \( \mathfrak{Mus12} \) Example 9.3.4. Verma modules for \( \mathfrak{osp}(3,2) \) can have 8 non-isomorphic composition factors \( \mathfrak{Mas13} \), and 20 or 23 submodules \( \mathfrak{Mus21} \).

This paper replaces Sections 9 and 10 of the unpublished manuscript \( \mathfrak{Mus17} \). Earlier sections will be replaced by \( \mathfrak{Mus23} \) and \( \mathfrak{Mus23b} \). I thank Jon Brundan for suggesting that Šapovalov elements could be used to prove Theorem 6.5. I am also grateful to the referee for many helpful comments.

### 2 The Type A Case.

The main results of this Section are Theorems 2.3 and 2.6. Let \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) be the simple roots of \( \mathfrak{gl}(m) \) for \( 1 \leq i \leq m - 1 \). For the superalgebra case we will also need the simple roots \( \gamma_i = \delta_i - \delta_{i+1} + 0 \oplus \mathfrak{gl}(n) \) for \( 1 \leq i \leq n - 1 \), and \( \beta = \epsilon_m - \delta_1 \). Set

\[
\begin{align*}
e_{\alpha_i} &= e_{i,i+1}, & e_\beta &= e_{m,m+1}, & e_\gamma &= e_{m+j,m+j+1}, \\
h_{\alpha_i} &= e_{i,i} - e_{i+1,i+1}, & h_\beta &= e_{m,m} + e_{m+1,m+1}, & h_\gamma &= e_{m+j+1,m+j+1} - e_{m+j,m+j}.
\end{align*}
\]

For any simple root \( \sigma \), let \( e_{-\sigma} \) be the transpose of \( e_\sigma \). As remarked above, we can assume that \( \eta = \epsilon_1 - \epsilon_m \) in the \( \mathfrak{gl}(m) \) case, and \( \eta = \epsilon_1 - \delta_n \) in the case of \( \mathfrak{gl}(m,n) \). The general case follows from this by relabelling the indices. Let \( (\ , \ ) = (\ , \ )_{m,n} \) be the bilinear form on \( \mathfrak{h}^* \) defined by

\[
(\epsilon_i, \epsilon_j)_{m,n} = -(\delta_i, \delta_j)_{m,n} = \delta_{i,j}
\]

for all relevant indices \( i, j \).

**Remark 2.1.** (a) For a non-isotropic root \( \alpha \), set \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \). From (2.2), it follows that \( (\alpha, \alpha) = 2 \), and \( \alpha^\vee = \alpha \) (resp. \( (\gamma, \gamma) = -2 \) and \( \gamma^\vee = -\gamma \)) for a simple root \( \alpha \) of \( \mathfrak{gl}(m) \oplus 0 \) (resp. \( 0 \oplus \mathfrak{gl}(n) \)). In the latter case, it is not possible
to choose elements $e = e_\gamma, f = e_{-\gamma}$ and $h = h_\gamma = [e_\gamma, e_{-\gamma}]$, such that the elements of $e, f, h$ satisfy the usual defining relations for $\mathfrak{sl}(2)$, see for example [Mus12] (A.4.2). With the above choices for $\gamma$ we have $[e_\gamma, e_{-\gamma}] = -h_\gamma$.

For $\alpha \in \mathfrak{h}^*$, let $h_\alpha \in \mathfrak{h}$ be the unique element such that $(\alpha, \beta) = \beta(h_\alpha)$ for all $\beta \in \mathfrak{h}^*$. If $N$ is a positive integer, we set $[N] = \{1, 2, \ldots, N\}$. For $k \in [m - 1]$, let

$$
\sigma_k = \epsilon_1 - \epsilon_{k+1}, \quad h_k = h_{\sigma_k} + (\rho, \sigma_k) - 1. 
$$

(2.3)

Lemma 2.2. (a) If $\alpha$ is a simple root of $\mathfrak{g} = \mathfrak{gl}(m, n)$ then $2(\rho, \alpha) = (\alpha, \alpha)$.

(b) For $k \in [m - 1]$, $h_k - h_{k-1} = h_{\alpha_k} + 1$ if $k > 1$, and $h_1 = h_{\alpha_1}$.

Proof. (a) is well known, see for example [Mus12] Corollary 8.5.4. Then (b) follows from (2.3).

Now consider a strictly descending sequence of integers

$$
I = \{i_0, i_1, \ldots, i_s, i_{s+1}\}. 
$$

(2.4)

If $I$ is a singleton set, then set $f_I = 1$. Otherwise define $f_I \in U(n^-)$ by

$$
f_I = e_{i_0,i_1}e_{i_1,i_2}e_{i_2,i_3}\ldots e_{i_s,i_{s+1}}. 
$$

(2.5)

2.1 The case of $\mathfrak{gl}(m)$.

Let

$$
\mathbb{I} = \{I \subseteq [m]|1, m \in I\}, 
$$

(2.6)

and for $I \in \mathbb{I}$, define

$$
r(I) = \{s - 1|s \in \bar{I}\} 
$$

(2.7)

where $\bar{I}$ be the complement of $I$ in $\mathbb{I}$. Recall $h_i$ from (2.3) and define

$$
H_J = \prod_{i \in r(J)} h_i. 
$$

(2.8)

Theorem 2.3. Let $\eta = \epsilon_1 - \epsilon_m$, and suppose $v$ is a highest weight vector in the $\mathfrak{g}$-Verma module $M(\lambda)$ of weight $\lambda$, and set

$$
\Theta_\eta = \sum_{J \subseteq \mathbb{I}} f_J H_J. 
$$

(2.9)

Then $e_{\alpha_k}\Theta_\eta v = 0$ for $k \in [m - 2]$. If $(\lambda + \rho, \eta) = 1$, Then $\Theta_\eta v$ is a $\mathfrak{g}$-highest weight vector, and so $\Theta_\eta$ is a Šapovalov element for the pair $(\eta, 1)$. 

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Proof. Clearly sets in \( \mathcal{I} \) correspond to partitions of \( \eta \), and (1.1) holds. Furthermore the partition \( \pi^0 \) corresponds to \( [m] \in \mathcal{I} \) and \( e_{\pi^0} = f_{[m]} \). Since \( H_{[m]} = 1 \), (1.2) holds. We show that \( \sum_{J \subseteq I} f_J H_J \) satisfies Equation (1.3) in the definition of a Sapovalov element. Given a simple root \( \alpha_k = e_k - e_{k+1} \), to avoid double subscripts we sometimes set \( \alpha = \alpha_k \). Thus \( e_\alpha = e_{k,k+1} \). The goal is to show that if \( v \) is a highest weight vector with weight \( \lambda \in \mathcal{H}_\eta \), then \( e_\alpha \sum_{J \subseteq I} f_J H_J v = 0 \). Consider the set
\[
\mathcal{S}_k = \{ I \subseteq \mathcal{I} | k, k+1 \in I \}.
\]
Suppose first \( I \in \mathcal{S}_k \). In this case \( e_{-\alpha} \) is a factor of \( f_I \). There is a unique \( g \) such that \( k + 1 = i_g \) and \( k = i_{g+1} \). With \( I \) as in (2.4), \( i_0 = m \) and \( i_{s+1} = 1 \), we set
\[
I_1 = \{m,i_1,\ldots ,i_{g-1},k+1\} \text{ and } I_2 = \{k,i_{g+2},\ldots ,i_s,1\}.
\]
Then \( I \) is a disjoint union \( I = I_1 \cup I_2 \), and we have
\[
e_\alpha f_I v = f_{I_1} h_\alpha f_{I_2} v = \begin{cases} f_{I_1} h_\alpha v & \text{if } I_2 = \{1\} \text{ is a singleton} \\ f_{I_1} f_{I_2} (h_\alpha + 1)v & \text{otherwise.} \end{cases}
\]
Define
\[
I^+ = I \setminus \{k\}, \text{ and } I^- = I \setminus \{k+1\}.
\]
If \( k = m - 1 \), then \( I^- \not\subseteq \mathcal{I} \) and if \( k = 1 \), then \( I^+ \not\subseteq \mathcal{I} \). We set \( f_{I^-} = 0 \) or \( f_{I^+} = 0 \) in these cases. It is easy to check the following result.

Lemma 2.4. We have

(a) \( e_\alpha f_{I^+} v = f_{I_1} f_{I_2} v \) and \( e_\alpha f_{I^-} v = -f_{I_1} f_{I_2} v \).

(b) \( r(I^+) = r(I) \cup \{k-1\} \) and \( r(I^-) = r(I) \cup \{k\} \).

(c) If \( J \subseteq \mathcal{I} \) and \( e_\alpha f_J v \in f_{I_1} f_{I_2} U(h)v \), then
\[
J = I, I^+ \text{ or } I^-.
\]

If \( I^+ \) or \( I^- \) is not a subset of \( \mathcal{I} \), they should be left out in the above equation.

From (b) in the Lemma and the definition of \( H_J \), we have
\[
H_{I^+} = h_{k-1} H_I \text{ and } H_{I^-} = h_k H_I.
\]

Now we apply \( e_\alpha \) to the sum of terms in (2.3) corresponding to \( I, I^+ \) and \( I^- \). Suppose that \( k \in [m-2] \). For any highest weight vector \( v \), we have using (2.13), (2.11) and Lemma 2.4
\[
e_\alpha (f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = \begin{cases} f_{I_1} f_{I_2} (h_{\alpha_1} - h_1) H_I v & \text{if } k = 1 \\ f_{I_1} f_{I_2} (h_{\alpha_k} + 1 + h_{k-1} - h_k) H_I v & \text{if } k > 1. \end{cases}
\]
and this is zero by Lemma 2.2 provided that \( k \neq m - 1 \). Now if \( J \) does not contain \( k \) or \( k + 1 \), then \( e_\alpha f_I H_J v = 0 \). Otherwise there exists a unique set \( I \in S_k \) such that \( J = I, I^+ \) or \( I^- \). Thus we have using (2.14),

\[
e_\alpha \sum_{J \subseteq I} f_J H_J v = e_\alpha \sum_{I \in S_k} (f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = 0.
\]

This proves the first statement in the Theorem. Now suppose \( k = m - 1 \), so \( f_{I^-} = 0 \), and \( \lambda \in \mathcal{H}_\eta \). For \( I \in S_{m-1} \), we have \( e_\alpha f_I v = h_\alpha f_I v = f_J (h_\alpha + 1) v \) where \( J = I \setminus \{m\} \). Set \( I^+ = I \setminus \{m-1\} \). Then \( e_\alpha f_{I^+} v = f_J v \). Now by Equation (2.13), \( H_{I^+} = h_{m-2} H_I \). Thus using (2.3) and the facts that \( \sigma_{m-2} \) is a sum of \( m - 2 \) simple roots, and \( \alpha_{m-1} + \sigma_{m-2} = \eta \), we have

\[
e_\alpha (f_I H_I + f_{I^+} H_{I^+}) v = f_I H_I (h_{\alpha_{m-1}} + 1 + h_{m-2}) v = f_I H_I (h_{\alpha_{m-1}} + h_{\sigma_{m-2}} + m - 2) v = f_I H_I (h_\eta + m - 2) v.
\]

(2.15)

Now since \( \lambda \in \mathcal{H}_\eta \) and \( (\rho, \eta) = m - 1 \), it follows that \( (\lambda, \eta) = 2 - m \). Thus the expression in (2.15) is zero. \( \square \)

**Remark 2.5.** There is a simple method that can be used to determine the linear factors \( h_k \) of \( H_I \). Consider the case that \( I = [m] \) is as large as possible. Then \( H_I = 1 \). If \( \alpha = \alpha_k \), for \( k \in [m-2] \) and \( k > 1 \), consider the condition, compare (2.14),

\[
0 = e_\alpha (f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = f_I f_{I^+} (h_\alpha + 1 + h_{k-1} - h_k) v.
\]

(2.16)

This gives a recurrence relation on the coefficients \( h_k \). Furthermore when \( k = 1 \), there are only two terms in the above equation, and this gives \( h_1 \). This remark can be applied to other situations, in particular we used it to develop the proof of Theorem 3.7, see Lemma 3.5. The recurrence in (2.16) is obtained by working from left to right along the Dynkin diagram. We could have instead worked from right to left, and obtained an different expression \( \sum_{J \subseteq I} f_J H'_J \) for the Sapovalov element \( \theta_\eta \).

(Recall that by [Mus17] Theorem 2.1, Šapovalov elements are only unique modulo a left ideal.) However for \( \lambda \in \mathcal{H}_\eta \), \( \sum_{J \subseteq I} f_J H_J(\lambda) = \sum_{J \subseteq I} f_J H'_J(\lambda) \). Thus both expressions give rise to the same highest weight vector in \( M(\lambda) \). In the super case, Theorem 2.6 we work from both ends of the diagram towards the unique grey node. In all the proofs the assumption \( \lambda \in \mathcal{H}_\eta \) is used only in the last step.

### 2.2 The case of \( \mathfrak{gl}(m, n) \).

For \( i \in [n - 1] \), set \( \tau_i = \delta_i - \delta_n \). Let \( \mathbb{I} = \{ I \subseteq [m + n]|1, m + n \in I \} \). We consider \( I \in \mathbb{I} \) as a strictly descending sequence of integers as in Equation (2.4), and define the corresponding element \( f_I \in U(\mathfrak{n}^-) \) as in (2.5). We use the set \( \mathbb{I} \) to
index an expression for the Šapovalov element $\theta_\eta \in U(b^-)$ analogous to that given in Theorem 2.3. For $k \in [m + n - 1]$, consider the set

$$\mathcal{S}_k = \{ I \subseteq \bar{I} | k, k + 1 \in I \}.$$  

Define as before,

$$r(I) = \{ s - 1 | s \in \bar{I} \}$$  \hspace{1cm} (2.17)

where $\bar{I}$ be the complement of $I$ in $\bar{I}$. Also let $I^\pm$ be as in (2.12). If $k = m + n - 1$, then $I^- \notin \bar{I}$ and if $k = 1$, then $I^+ \notin \bar{I}$. As before, set $f_{I^-} = 0$ or $f_{I^+} = 0$ in these cases. Next set

$$h_i = \begin{cases} 
h_{\sigma_i} + (\rho, \sigma_i) - 1 & \text{if } i \in [m - 1] \\
h_{\tau_{i+1-m}} + (\rho, \tau_{i+1-m}) & \text{if } m \leq i \leq m + n - 2. \end{cases}$$  \hspace{1cm} (2.18)

Then we have an analog of Lemma 2.2 (b),

$$h_{m+j-1} - h_{m+j} = h_{\gamma_j} - 1 \text{ for } j \in [n-2].$$  \hspace{1cm} (2.19)

If we set $h_{m+n-1} = 0$, then (2.19) holds also for $j = n - 1$. Now define

$$H_J = \prod_{i \in r(J)} h_i.$$  

**Theorem 2.6.** Suppose $\eta = \epsilon_1 - \delta_n$, $v$ is a highest weight vector in the $\mathfrak{g}$-Verma module $M(\lambda)$ of weight $\lambda$, and set

$$\Theta_\eta = \sum_{J \subseteq \bar{I}} f_J H_J.$$  

Then $\Theta_\eta v$ is a $\mathfrak{g}_0$-highest weight vector. If $(\lambda + \rho, \eta) = 0$, then $\Theta_\eta v$ is a $\mathfrak{g}$-highest weight vector, and so $\Theta_\eta$ is a Šapovalov element for the pair $(\eta, 1)$.

**Proof.** This follows the same general pattern as the proof of Theorem 2.3 but there are several differences resulting from Remark 2.1, so we give some details. Suppose $I \in \mathcal{S}_k$. First if $\alpha = \alpha_k$ for $k \in [m - 1]$ the proof of (2.14) shows that

$$e_\alpha(f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = 0.$$  

Next suppose that $k = m + j$ for $j \in [n - 1]$, and set $\alpha = \gamma_j$, see (2.1). Instead of (2.10), we set

$$I_1 = \{m + n, i_1, \ldots, i_{g-1}, k + 1\} \text{ and } I_2 = \{k, i_{g+2}, \ldots, i_s, 1\}.$$  \hspace{1cm} (2.20)

Note that $e_{-\alpha}$ is a factor of $f_I$. Thus from Remark 2.1, we have instead of (2.11),

$$e_\alpha f_I v = -f_{I_1} h_{\alpha} f_{I_2} v = f_{I_1} f_{I_2} (1 - h_{\alpha}) v.$$  \hspace{1cm} (2.21)

By (2.13), we have $H_{I^+} = h_{m+j-1} H_I$ and $H_{I^-} = h_{m+j} H_I$. Note that Lemma 2.4 still holds in this situation. Now in place of (2.14) we have using (2.21),

$$e_\alpha(f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = f_{I_1} f_{I_2} (1 - h_{\gamma_j} + h_{m+j-1} - h_{m+j}) H_I v$$  \hspace{1cm} (2.22)
and this is zero by (2.19). This proves the first statement of the Theorem.

Finally suppose that \( k = m \) so that \( \alpha \) is the odd simple root and \( e_\alpha = e_{m,m+1} \). Fix \( I \in \mathcal{S}_m \). Then \( f_{I^+} \) (resp. \( f_{I^-} \)) contains the odd root vector \( e_{m+1,i_{y+2}} \) (resp. \( e_{i_{y-1},m} \)) as a factor. Hence

\[
e_\alpha f_{I^+} H_{I^+} v = f_{I_1} f_{I_2} h_{m-1} H_I v \quad \text{and} \quad e_\alpha f_{I^-} H_{I^-} v = f_{I_1} f_{I_2} h_m H_I v.
\]

On the other hand

\[
e_\alpha f_I H_I v = f_{I_1} h_\alpha f_{I_2} H_I v = f_{I_1} f_{I_2} (1 + h_\alpha) H_I v.
\]

Therefore by (2.18), we have for \( \lambda \in \mathcal{H}_\eta \),

\[
e_\alpha (f_I H_I + f_{I^+} H_{I^+} + f_{I^-} H_{I^-}) v = f_{I_1} f_{I_2} (1 + h_\alpha + h_{m-1} + h_m) H_I v
= f_{I_1} f_{I_2} (h_\alpha + h_{\sigma_{m-1}} + h_{\tau_1} + (\rho, \sigma_{m-1} + \tau_1)) H_I v.
= f_{I_1} f_{I_2} (h_\eta + (\rho, \eta)) H_I v.
= 0.
\]

\[\square\]

3 Šapovalov elements for arbitrary Borel subalgebras.

Let \( b^{\text{dist}} \) denote the distinguished Borel subalgebra of \( \mathfrak{gl}(m,n) \). We denote the set of Borel subalgebras with the same even part as \( b^{\text{dist}} \) by \( \mathcal{B} \). For \( b \in \mathcal{B} \) and \( \eta \) a positive root of both \( b^{\text{dist}} \) and \( b \) we describe the Šapovalov elements with respect to \( b \). In [Mus12] 3.3, the set \( \mathcal{B} \) is described in terms of shuffles. Here the notation is slightly different. We write a permutation \( \sigma \) of the set \( \|m+n\| := \{1, 2, \ldots, m, 1', 2', \ldots, n'\} \) in one-line notation as

\[\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(m), \sigma(1'), \sigma(2'), \ldots, \sigma(n')).\]

The last \( n \) entries of \( \|m+n\| \) are called \textit{primed}, the others \textit{unprimed}. The shuffle condition is that both \( 1, 2, \ldots, m \) and \( 1', 2', \ldots, n' \) are subsequences of \( \sigma \). A useful way to think about this is by using the Dynkin-Kac diagram of \( b \), see [Mus12] section 3.4. The simple roots \( \alpha_1, \ldots, \alpha_{m+n-1} \) are used to label the nodes from left to right. Then augment the diagram by adding the entries of \( \sigma \), in order above the diagram between the nodes, and at both ends. Without loss we assume that \( \eta = \epsilon_1 - \delta_n \) is the longest odd positive root of \( g = \mathfrak{gl}(m,n) \), since any odd root which is positive in both \( b^{\text{dist}} \) and \( b \) is the longest root in a possibly smaller Type A superalgebra. This assumption implies that \( \sigma(1) = 1 \) and \( \sigma(n') = n' \). From the diagram we can immediately read off the simple root corresponding to each node. Each simple root \( \alpha_i \) lies between a pair of neighbors in \( \sigma \), and there are 4 possibilities. The rule is

if the neighbors of \( \alpha_i \) are \( (a, a+1), (a, b'), (a', b), (b', b+1) \),
then the root \( \alpha_i \) is respectively \( e_a - e_{a+1}, e_a - \delta_b, \delta_a - e_b, \delta_b - \delta_{b+1} \).
Example 3.1. We give an example of an augmented Dynkin-Kac diagram when \( m = 4, n = 3 \).

\[
\begin{array}{cccccc}
1 & 1' & 2 & 3 & 4 & 2' & 3' \\
\otimes & \otimes & \circ & \circ & \otimes & \circ & \circ
\end{array}
\]

\( \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \)

In this example \( \sigma = (1, 1', 2, 3, 4, 2', 3') \) and

\[ \alpha_1 = \epsilon_1 - \delta_1, \alpha_2 = \delta_1 - \epsilon_2, \alpha_3 = \epsilon_2 - \epsilon_3, \alpha_4 = \epsilon_3 - \epsilon_4, \alpha_5 = \epsilon_4 - \delta_2, \alpha_6 = \delta_2 - \delta_3. \]

In a similar way we can read off any positive root from the diagram, and it follows that \( \sum_{k=1}^{m+n-1} \alpha_k = \eta \). Under the isomorphism \( h^* \to \mathfrak{h} \), suppose \( \alpha_k \) maps to \( h_k \).

We deduce that

\[
\sum_{k=1}^{m+n-1} h_k = h_\eta. \tag{3.1}
\]

Let \( \mathbb{I} = \{ I \subseteq \parallel m + n \parallel | 1, n' \in I \} \). The shuffle \( \sigma \) induces a decreasing order on the set \( \parallel m + n \parallel \). Explicitly

\[ \sigma(n') = n' > \sigma(n - 1') > \ldots > \sigma(2) > \sigma(1) = 1. \]

and for \( I \in \mathbb{I}, J \subseteq I \), we use this order to form the products \( f_J \) by analogy with Equation (2.5).

Now fix a simple root \( \alpha = \alpha_k \) and let \( i(k) \) (resp. \( j(k) \)) be the number of grey nodes to the left (resp. right) of \( \alpha \) in the diagram. If \( a \) and \( b \) are the left and right neighbors of \( \alpha \) respectively, define

\[ S_\alpha = \{ I \in \mathbb{I} | a, b \in I \}, \]

and note that \( a \) and \( b \) can be primed or unprimed numbers. For \( I \in \mathbb{I} \) we can write

\[ I = \{ n', \ldots, c, b, a, *, \ldots, 1 \} \tag{3.2} \]

as a disjoint union, \( I_1 \cup I_2 \), where

\[ I_1 = \{ n', \ldots, b \} \quad \text{and} \quad I_2 = \{ a, \ldots, 1 \}. \tag{3.3} \]

The bilinear form \( (,)_m,n \) on \( \mathfrak{h}^* \) defined in (2.2) induces an isomorphism \( \mathfrak{h}^* \to \mathfrak{h} \) with \( \alpha \mapsto h_\alpha \). On the other hand there is a bilinear form on \( \mathfrak{h} \) induced by the supertrace \( \text{Str} \), that is \( (h, h') \mapsto \text{Str}(hh') \). These are related by

\[ (\alpha, \beta) = \text{Str}(h_\alpha h_\beta). \tag{3.4} \]

We need an explicit expression for \( h_k = h_{\alpha_k} \), in terms of the neighbors of \( \alpha_k = \alpha \).

For \( \alpha \) is even, we already know the answer,

\[ h_k = (-1)^{i(k)}(e_{aa} - e_{bb}). \tag{3.5} \]

Indeed \( \alpha \) is a root of \( \mathfrak{gl}(m) \oplus 0 \) if \( i(k) \) is even, and a root of \( 0 \oplus \mathfrak{gl}(n) \) if \( i(k) \) is odd.

Thus (3.5) follows from Remark 2.1.
Lemma 3.2. We have

(a) 

\[ [e_\alpha, e_{-\alpha}] = (-1)^i(k) h_k. \]  

(3.6)

(b) If \( \alpha \) is odd, then

\[ h_k = (-1)^i(k)(e_{aa} + e_{bb}). \]  

(3.7)

Proof. (a) This is an easy computation. (b) It seems that the easiest way to proceed is to take (3.7) as the definition, and then show that (3.4) holds. Consider the Cartan matrix \((\alpha_k, \alpha_j)\). Suppose first that \( k \neq 1, m + n - 1 \). The row indexed by \( \alpha_k \) has consecutive entries \(-1, 0, 1\) (resp. \( 1, 0, -1 \)) if \( i(k) \) is even (resp. odd) with 0 on the main diagonal. All other entries in this row are zero. If \( k = 1 \) (resp. \( k = m + n - 1 \)), the first (resp. last) entry in the above sequences must be deleted, since 0 is on the main diagonal. We show that the same holds for the Gram matrix of the bilinear form on \( \mathfrak{h} \) induced by the supertrace form, \((h, h') \mapsto \text{Str}(hh')\). For example if \( i(k) \) is even, then \( a \) is unprimed. If \( \alpha_k \) is even, then \( h_{k-1} = e_{aa} - e_{aa} \) where * is the entry in \( I \) to the right of \( a \), see (3.2). Then \( \text{Str}(h_{k-1}h_k) = -1 \). If \( \alpha_k \) is odd, then \( h_{k-1} = (-1)^i(k-1)(e_{aa} + e_{aa}) \) and \( i(k-1) \) is odd. Again we obtain \( \text{Str}(h_{k-1}h_k) = -1 \). Similarly \( \text{Str}(h_k h_{k+1}) = 1 \). We leave the cases where \( i(k) \) is odd, \( k = 1 \) or \( k = m + n - 1 \) to the reader.  

Note also that

\[ (e_{aa} \pm e_{bb})f_{I_2} = f_{I_2}(e_{aa} \pm e_{bb} + 1) \] if \( I_2 \) is not a singleton.  

(3.8)

Lemma 3.3. Suppose that \( v \) is a highest weight vector, and \( I \in S_\alpha \). Then

(a) If \( \alpha \) is even we have

\[
\begin{align*}
    e_\alpha f_I v &= f_I [e_\alpha, e_{-\alpha}] f_I v \nonumber \\
    &= \begin{cases}
    (-1)^i(k) f_I h_k v & \text{if } I_2 = \{1\} \text{ is a singleton} \\
    f_I f_{I_2} (1 + (-1)^i(k) h_k) v & \text{otherwise.}
    \end{cases}
\end{align*}
\]

(3.9)

(b) If \( \alpha \) is odd we have

\[
\begin{align*}
    e_\alpha f_I v &= (-1)^i(k) f_I [e_\alpha, e_{-\alpha}] f_I v \\
    &= \begin{cases}
    f_I h_k v & \text{if } I_2 = \{1\} \text{ is a singleton} \\
    f_I f_{I_2} (h_k + (-1)^i(k)) v & \text{otherwise.}
    \end{cases}
\end{align*}
\]

(3.10)

Proof. (a) The first equality follows since \( e_\alpha \) commutes with \( f_{I_1} \) and then the second follows from (3.5) and (3.8).

(b) If \( \alpha \) is odd then \( e_\alpha \) anticommutes with \( j(k) \) factors in \( f_{I_1} \) and commutes with the others. Note that the parity of \( j(k) \) depends only on the first and last entries in \( I_1 \). This gives the first equality. The condition that \( \sigma(1) = 1 \) and \( \sigma(n') = n' \) implies that the total number of odd simple roots is odd, so if \( \alpha \) is odd, then \( i(k) + j(k) \) is even. Hence the second equality holds by Lemma 3.2 (a) and (3.8).
Next for $I \in S_{\alpha}$, set
\[ I^+ = I \backslash \{a\}, \quad \text{and} \quad I^- = I \backslash \{b\}. \]  
(3.11)

If $I^\pm \notin \mathbb{I}$, then as before, we set $f_{I^\pm} = 0$. Then we have an analog of Lemma [2.4]

**Lemma 3.4.** We have 

(a) If $\alpha$ is even, then $e_\alpha f_{I^+} v = f_{I_1} f_{I_2} v$ and $e_\alpha f_{I^-} v = -f_{I_1} f_{I_2} v$.

(b) If $\alpha$ is odd, then
\[ e_\alpha f_{I^+} v = -(1)^{(k+1)} f_{I_1} f_{I_2} v, \]  
(3.12)

and
\[ e_\alpha f_{I^-} v = -(1)^{(k)} f_{I_1} f_{I_2} v = -(1)^{(k)} f_{I_1} f_{I_2} v. \]  
(3.13)

Furthermore Lemma [2.4] (c) holds.

**Proof.** (a) If $\alpha$ is even, then “even rules” apply to the commutator of $e_\alpha$ and any root vector which is a factor of $f_{I^\pm}$. Thus we get the same result as Lemma [2.4] (a).

(b) We prove (3.12). Define
\[ I_1^- = \{n', \ldots, c\} \quad \text{and} \quad I_2^- = \{a, \ldots, 1\} = I_2 \]  
(3.14)

Then $f_{I^-} = f_{I_1^-} e_{ca} f_{I_2^-}$, so
\[ e_\alpha f_{I^-} v = -(1)^{(k+1)} f_{I_1^-} e_{ca} f_{I_2^-} v \]
\[ = \begin{cases} 
-(1)^{(k+1)} f_{I_1} f_{I_2} v & \text{if } \alpha_{k+1} \text{ is even} \\
-(1)^{(k+1)+1} f_{I_1} f_{I_2} v & \text{if } \alpha_{k+1} \text{ is odd.}
\end{cases} \]

The first equality holds since there are exactly $j(k+1)$ factors in $f_{I_1^-}$ that are odd root vectors. For the second, note that if $\alpha_{k+1}$ is even (resp. odd), then $e_{ca}$ is an odd (resp. even) root vector, and hence $[e_{ab}, e_{ca}] = e_{cb}$ (resp. $[e_{ab}, e_{ca}] = -e_{cb}$). Now if $\alpha_{k+1}$ is even, then mod 2, $j(k+1) \equiv i(k+1) + 1$, while if $\alpha_{k+1}$ is odd, then $j(k+1) + 1 \equiv i(k+1) + 1$. This gives (3.12).

The proof of (3.13) is similar to (3.12) but easier. Note that, if
\[ I_1^+ = \{n', \ldots, b\} = I_1 \quad \text{and} \quad I_2^+ = \{*, \ldots, 1\}, \]  
(3.15)

we have $f_{I^+} v = f_{I_1^+} e_{bs} f_{I_2^+} v$. The last statement in the Lemma is easy to prove. \( \square \)

Recall $h_k$ defined by (3.5) or (3.7), and set $s_k = h_1 + \ldots + h_k = s_{k-1} + h_k$. Next let
\[ \ell(k) = |i| \in \{[k-1] | \alpha_i \text{ is a root of } 0 \oplus \mathfrak{gl}(n)\} \quad \text{and} \quad \bar{i}(k) = (1 - (1)^{(k)})/2. \]  
Thus $\bar{i}(k) \in \{0, 1\}$, and $\bar{i}(k) \equiv i(k) \mod 2$. Set $d_k = \ell(k) + \bar{i}(k)$. Thus $d_1 = 0$. 

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Lemma 3.5. If \( k \in [m + n - 2] \), we have the following recurrence,

\[
d_{k+1} = d_k - (-1)^i(k+1).
\]

Proof. We have to show that

\[
\ell(k + 1) + \tilde{i}(k + 1) = \ell(k) + \tilde{i}(k) - (-1)^i(k+1).
\] (3.16)

If \( \alpha_k \) is even, then \( i(k + 1) = i(k) \). Thus (3.16) becomes

\[
\ell(k + 1) = \ell(k) - (-1)^i(k).
\] (3.17)

Now if \( i(k) \) is even (resp. odd), then \( \alpha_k \) is a root of \( \mathfrak{gl}(m) \oplus 0 \) (resp. \( 0 \oplus \mathfrak{gl}(n) \)) so \( \ell(k + 1) = \ell(k) - 1 \) (resp. \( \ell(k + 1) = \ell(k) + 1 \)). This gives (3.17).

If \( \alpha_k \) is odd, then \( \ell(k + 1) = \ell(k) \), and \( i(k + 1) = i(k) + 1 \), so we have to show

\[
\tilde{i}(k + 1) = \tilde{i}(k) + (-1)^i(k).
\]

This follows easily by considering the cases, \( i(k) \) even and odd separately. \( \square \)

Lemma 3.6. \( d_{m+n-1} = - (\rho, \eta) \).

Proof. Let \( r \) (resp. \( s \)) be the total number of roots in the Dynkin-Kac diagram belonging to \( 0 \oplus \mathfrak{gl}(n) \) (resp. \( \mathfrak{gl}(m) \oplus 0 \)). Thus \( (\rho, \eta) = - r + s \) by Remark 2.1 and Lemma 2.2 (a). If \( \alpha = \alpha_{m+n-1} \) is even, then it is a root of \( 0 \oplus \mathfrak{gl}(n) \). Thus \( \ell(m + n - 1) = r - 1 - s \). Also \( i(m + n - 1) = m + n - 1 = r - s \). If \( \alpha \) is odd, then \( i(m + n - 1) \) is even, so \( d_{m+n-1} = \ell(m + n - 1) = r - s \). This gives the result. \( \square \)

If \( (a, b) \) are the neighbors of \( \alpha_k \), and \( 1 < k < m + n - 1 \), set

\[
h_a = (-1)^i(k)(s_{k-1} - d_{k-1}) \quad \text{and} \quad h_b = (-1)^i(k+1)(s_k - d_k).
\] (3.18)

If \( k = 1 \) (resp. \( k = m + n - 1 \)), then \( I^+ \notin I \) (resp. \( I^- \notin I \)), but we still define \( h_b \) (resp. \( h_a \)) as in (3.18). Note that \( b \) is also the left neighbor of \( \alpha_{k+1} \), and Equation (3.18) is consistent with this fact. Also (3.18) is very convenient for the proof of the main result because of (3.19) below. However we need another ingredient to define the coefficients \( \Omega_I \). If \( e \) is the right neighbor of \( \alpha_j \), then set \( r(e) = j \). Then define

\[
t_e = (-1)^{i(r(e)+1)}(s_{r(e)} - d_{r(e)}),
\]

and \( \Omega_I = \prod_{e \in \bar{I}} t_e \), where \( \bar{I} \) is the complement of \( I \) in \( \|m + n\| \). For example \( r(a) = k - 1, r(b) = k \), so by (3.18) \( t_a = h_a \) and \( t_b = h_b \). If \( (a, b) \) are as above, it follows from (3.11) that

\[
\Omega_{I^+} = h_a \Omega_I \quad \text{and} \quad \Omega_{I^-} = h_b \Omega_I.
\] (3.19)
Theorem 3.7. Set

$$\Theta_\eta = \sum_{J \subseteq I} f_I H_J.$$  

Then $\Theta_\eta$ is a Šapovalov element $\theta_\eta$ for the pair $(\eta, 1)$ for the Borel subalgebra $b$.

Proof. Assume first that $k \neq 1, m + n - 1$. If $\alpha$ is even then, $i(k+1) = i(k)$. Hence by (3.9), (3.19) and Lemma 3.4 (a), then (3.18) and Lemma 3.5

$$e_\alpha(f_1 H_I + f_I H_{I+} + f_I H_{I-}) v = f_1 f_I (1 + (-1)^{i(k)} h_k + h_{\hat{a}} - h_{\hat{b}}) H_I v$$

$$= (-1)^{i(k)} f_1 f_I (h_k + s_{k-1} - s_k) H_I v$$

$$+ f_1 f_I (1 - (-1)^{i(k)} d_{k-1} + (-1)^{i(k)} d_k) H_I v$$

$$= 0. \quad (3.20)$$

If $k = 1$, then $I_2$ is a singleton and $d_1 = i(2) = 0$, so by (3.9), (3.13) and (3.18),

$$e_\alpha(f_1 H_I + f_I H_{I-}) v = f_1 (h_1 - h_{\hat{b}}) H_I v$$

$$= f_1 (h_1 - s_1 + d_1) H_I v$$

$$= 0.$$  

Now assume $\alpha$ is odd and $k \neq 1, m + n - 1$. Then, by Lemma 3.3 (b), (3.13) and (3.12), (3.19), then (3.18) and Lemma 3.5

$$e_\alpha(f_1 H_I + f_I H_{I+} + f_I H_{I-}) v = f_1 f_I (h_k + (-1)^{i(k)} + (-1)^{i(k)} h_{\hat{a}} - (-1)^{i(k+1)} h_{\hat{b}}) H_I v$$

$$= f_1 f_I ((h_k + s_{k-1} - s_k) H_I v$$

$$+ f_1 f_I ((-1)^{i(k)} - d_{k-1} + d_k) H_I v$$

$$= 0. \quad (3.21)$$

If $k = 1$, then by (3.10), (3.12), (3.18) and (3.19),

$$e_\alpha(f_1 H_I + f_I H_{I-}) v = f_1 (h_1 - (-1)^{i(k+1)} h_{\hat{b}}) H_I v$$

$$= f_1 ((h_1 - (s_1 - d_1)) H_I v$$

$$= 0.$$  

Now suppose that $k = m + n - 1$, and that $v$ has weight $\lambda \in \mathcal{H}_\eta$. If $\alpha$ is even, then $i(k)$ is odd. In the sequence of equalities below the first follows from (3.9) and (3.19), the second from (3.18). Then the third follows from (3.1) and Lemma 3.5 the fourth from Lemma 3.6 and the final equality holds since $\lambda \in \mathcal{H}_\eta$.

$$e_\alpha(f_1 H_I + f_I H_{I+}) v = f_I (1 - h_k + h_{\hat{a}}) H_I v$$

$$= f_I (1 - h_k - s_{k-1} + d_{k-1}) H_I v$$

$$= -f_I (h_\eta - d_{m+n-1}) H_I v$$

$$= -f_I (h_\eta + (\rho, \eta)) H_I v = 0.$$
Suppose $\alpha$ is odd. Then $i(k)$ is even. For the first two equalities below we use (3.10), (3.13), (3.19) and (3.18). The remainder of the proof is similar to the case where $\alpha$ is even.

\[
e^\alpha(f_I H_I + f_{I+} H_{I+}) v = f_{I2}((-1)^{i(k)} + h_k + (-1)^{i(k)} h_{\hat{a}}) H_I.
= f_{I2}(1 + h_k + s_{m+n-2} - d_{m+n-2}) H_I
= f_{I2}(h_{\eta} - d_{m+n-1}) H_I
= f_{I2}(h_{\eta} + (\rho, \eta)) H_I v = 0.
\]

\[\square\]

4 Šapovalov elements as determinants of Hessenberg matrices.

4.1 Hessenberg Matrices.

An $n \times n$ matrix $B = (b_{ij})$ is \textit{(upper) Hessenberg of order} $n$ if $b_{ij} = 0$ unless $i \leq j + 1$. The only assumption on the Hessenberg matrix $B$ is that the entries on the subdiagonal commute with all other entries. We define a noncommutative determinant of the $n \times n$ matrix $B = (b_{ij})$, working from left to right, by

\[
\rightarrow \det(B) = \sum_{w \in S_n} \text{sign}(w)b_{w(1),1} \ldots b_{w(n),n}, \tag{4.1}
\]

Cofactor expansions of $\rightarrow \det(B)$ are valid as long as the overall order of the terms is unchanged.

\textbf{Lemma 4.1.} Suppose that $B$ is Hessenberg of order $n$.

(a) For a fixed $q \in [n - 1]$, let $T = -b_{q+1,q}$. Then

\[
\rightarrow \det B = T \rightarrow \det B'' + \rightarrow \det B', \tag{4.2}
\]

where $B'$ is obtained from $B$ by setting $T = 0$, and $B''$ is obtained from $B$ by deleting the row and column containing $T$.

(b) The matrix $B'$ is block upper triangular, with two diagonal blocks which are upper Hessenberg of order $q$ and $n - q$.

(c) The matrix $B''$ is upper Hessenberg of order $n - 1$. Also any term in the expression \textbf{(4.1)} for $\rightarrow \det(B)$ which contains a factor of the form $b_{iq}$ or $b_{q+1j}$ cannot occur in $\rightarrow \det B''$.

\textbf{Proof.} Part (a) follows by separating the products in \textbf{(4.1)} that contain $T$ from those that do not. Note that $T$ commutes with all entries in $B$, and that the order of all other factors of the products is unchanged. The rest is easy. \[\square\]
We consider determinants of Hessenberg matrices with entries on or above the diagonal from \( n^- \) and subdiagonal entries which commute with all other matrix entries. Explicit expressions in \( U(b^-) \) for Šapovalov elements can be obtained as complete expansions of suitable determinants of this kind. There are significant differences in the complete expansions depending on the ordering of entries in the matrices. We consider two orderings on the set of positive roots, and explain the relationships between the determinants without reference to Šapovalov elements. First consider the matrix with entries in \( U(gl(m)) \), or any \( U(gl(r, s)) \) with \( r + s = m \).

\[
D = \begin{bmatrix}
    e_{m,m-1} & e_{m,m-2} & \cdots & e_{m,2} & e_{m,1} \\
    -a_{m-2} & e_{m-1,m-2} & \cdots & e_{m-1,2} & e_{m-1,1} \\
    0 & -a_{m-3} & \cdots & e_{m-2,2} & e_{m-2,1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & -a_1 & e_{2,1}
\end{bmatrix}.
\]

Note that in (4.3) the entries from the same row from \( n^- \) have the form \( e_{i,\ast} \) for some fixed \( i \). Another possibility is to require that all entries from the same row have the form \( e_{\ast,i} \). Thus consider the matrix, where \( c_i = a_i + 1 \),

\[
E = \begin{bmatrix}
    e_{2,1} & e_{3,1} & \cdots & e_{m-1,1} & e_{m,1} \\
    -c_1 & e_{3,2} & \cdots & e_{m-1,2} & e_{m,2} \\
    0 & -c_2 & \cdots & e_{m-1,3} & e_{m,3} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & \cdots & -c_{m-2} & e_{m,m-1}
\end{bmatrix}.
\]

The next result is used in the proof of Theorem 4.15.

**Proposition 4.2.** We have \( \overrightarrow{\det D} = \overrightarrow{\det E} \)

**Proof.** This is proved using cofactor expansion and induction. Rather than giving full details, an example might be more helpful here. It is easy to see how the example generalizes. In the general case, we need a commutation relation for cofactors of \( E \). \( \square \)

**Lemma 4.3.** Let \( E^1, E^2 \) be the cofactors of entry \( e_{m,m-1} \) and \( -c_{m-2} \) in \( E \) respectively. Then \( \overrightarrow{\det E^1, e_{m,m-1}} = -\overrightarrow{\det E^2} \).

**Proof.** Note that \( E^1 \) is obtained from \( E^2 \) by replacing the entries \( e_{m,i} \) with \( e_{m-1,i} \) for \( i \in [m-2] \). Since \( i < m - 1 \), it is impossible for both \( e_{m-1,i} \) and \( e_{m,m-1} \) to be odd. Thus \( [e_{m-1,i}, e_{m,m-1}] = -e_{m,i} \), the result follows. \( \square \)

**Example 4.4.** Let

\[
E = \begin{bmatrix}
    e_{2,1} & e_{3,1} & e_{4,1} \\
    -c_1 & e_{3,2} & e_{4,2} \\
    0 & -c_2 & e_{4,3}
\end{bmatrix}.
\]
By cofactor expansion along the last row,
\[ \overrightarrow{\det E} = \overrightarrow{\det E^1 e_{4,3} + c_2 \overrightarrow{\det E^2}}. \]

where
\[ E^1 = \begin{bmatrix} e_{2,1} & e_{3,1} \\ -c_1 & e_{3,2} \end{bmatrix} \quad \text{and} \quad E^2 = \begin{bmatrix} e_{2,1} & e_{4,1} \\ -c_1 & e_{4,2} \end{bmatrix}. \]

Next set
\[ D^1 = \begin{bmatrix} e_{3,2} & e_{3,1} \\ -a_1 & e_{2,1} \end{bmatrix} \quad \text{and} \quad D^2 = \begin{bmatrix} e_{4,2} & e_{4,1} \\ -a_1 & e_{2,1} \end{bmatrix}. \]

Now by Lemma 4.3,
\[ [\overrightarrow{\det E^1}, e_{4,3}] = -\overrightarrow{\det E^2}, \]
so
\[ \overrightarrow{\det E} = e_{4,3} \overrightarrow{\det E^1} + (c_2 - 1) \overrightarrow{\det E^2} \]
\[ = e_{4,3} \overrightarrow{\det D^1} + a_2 \overrightarrow{\det D^2} \tag{4.5} \]

using induction for the second equality. This is the cofactor expansion of \( \overrightarrow{\det D} \) down the first column.

4.2 Application to Šapovalov elements in Type A.

Expansions of Šapovalov elements in Type A were already given in [Mus17] Section 9, using determinants of a certain Hessenberg matrices. However it seems unlikely that this method will generalize, because the determinant of Hessenberg matrix of order \( m \) has \( 2^m - 1 \) terms, and outside of Type A, the number of partitions of a positive root is rarely a power of 2. Nevertheless the use of Hessenberg matrices gives more insight in the Type A case. In particular they can be used to give expressions for Šapovalov elements using different orderings on the set of positive roots.

Consider the Lie algebra \( \mathfrak{gl}(m) \) with simple roots \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) for \( i \in [m - 1] \). Let \( \eta' = \epsilon_1 - \epsilon_m \) and \( \mu \in \mathcal{H}_{\eta'} \). We show that \( \theta_{\eta'} \) can be expressed as a determinant of a certain Hessenberg matrix. Recall the elements
\[ h_i = h_{\sigma_i} + (\rho, \sigma_i) - 1 \]
from (2.3), for \( i \in [m - 1] \). Then let \( D^m(\mu) \) be the matrix \( D \) from (4.3) with
\[ a_i = h_i(\mu) = (\mu + \rho, \sigma_i) - 1. \tag{4.6} \]

**Theorem 4.5.** The Šapovalov element for \( \eta' \) satisfies
\[ \theta_{\eta'}(\mu) = \overrightarrow{\det D^m(\mu)} \]
for all \( \mu \in \mathcal{H}_{\eta'} \).
We give two proofs of Theorem 4.5. A comparison of the two approaches is necessary to prove Proposition 5.2. Define $H_J$ from (2.8). In the first we show that the complete expansion of the determinant $\overrightarrow{\det} D^m(\mu)$ is the evaluation of the element $\Theta_{\eta'}$ from Theorem 2.3 at $\mu \in \mathcal{H}_{\eta'}$. In other words we show
\[ \overrightarrow{\det} D^m(\mu) = \sum_{J \subseteq I} f_J H_J(\mu). \] (4.7)

In the second we use induction on $m$, cofactor expansion and the fundamental Lemma 4.8 below. This Lemma which is [Mus12] Lemma 9.4.3, is the basis for the proof of the existence of Sapovalov elements in the general case. Essentially the same proof is given in [Hum08] Section 4.13.

**Theorem 4.5: First Proof.** Consider the matrix $D^m(\mu)$ from Theorem 4.5. We obtain the complete expansion of $\overrightarrow{\det} D^m(\mu)$. Let $I$ be as in (2.6). Each term in the complete expansion is obtained by choosing a non-zero product of elements from each column, with each row occurring exactly once. The product of the chosen elements lying above the subdiagonal has the form $f_I$ for some $I \in \mathcal{I}$. The proof of (4.7) is completed by the following Lemma. □

**Lemma 4.6.** The product $a_I$ of subdiagonal terms accompanying $f_I$ is given by
\[ a_I = \prod_{i \in r(I)} a_i = \prod_{i \in r(I)} h_i(\mu) = H_I(\mu). \] (4.8)

**Proof.** From the form of the matrix $D$ in (4.3) we see that $e_{j,i}$ is a factor of $f_I$ iff $j \in I$ iff $-a_{j-1}$ a factor of $a_I$. Thus the product of subdiagonal terms accompanying $f_I$ must be
\[ \pm \prod_{i \in r(I)} a_i = \pm \prod_{i \in r(I)} h_i(\mu). \] (4.9)

Next we make a remark about the (non-commutative) determinant of a Hessenberg matrix $B$ of order $n$. For a general $n \times n$ matrix the complete expansion of the determinant is indexed by permutations $w$ from the symmetric group of degree $n$. In Equation (4.1) the corresponding term is zero unless $w(i) \geq i - 1$ for $2 \leq i \leq n$. Now it is easy to see the following.

**Lemma 4.7.** Assume that in the expression for $\overrightarrow{\det} (B)$ given in (4.1), the term indexed by $w$ is nonzero. Let $C = \{|i|w(i) = i - 1\}$. Then $\text{sign}(w) = |(-1)^{|C|}|$.

**Proof.** Each $i \in C$ corresponds to an inversion $(i, i - 1)$ in $w$. There are no other inversions. □

This is fortunate for us because the set $C$ corresponds to the elements on the subdiagonal of $B$ and these all have the form $-a_j$ for some $j$. This means that (4.9) can be replaced with (4.8) as in the statement of Lemma 4.6. □
Theorem 4.5: Second Proof. The Šapovalov element $\theta_\eta$ can be constructed inductively using the next Lemma.

Lemma 4.8. Suppose that $\alpha$ is a simple root and set $\mu = s_\alpha \cdot \lambda$, $\eta' = s_\alpha \eta$. Assume that

(a) $p = (\mu + \rho, \alpha^\vee) \in \mathbb{N}\setminus\{0\}$ and $q = (\eta, \alpha^\vee) \in \mathbb{N}\setminus\{0\}$

(b) $\mu \in H_{\eta'}$, and consequently $\lambda \in H_{\eta}$.

Then the evaluation of the Šapovalov elements $\theta_{\eta'}, \theta_\eta$ at $\mu$ and $\lambda$ satisfy

$$e_{-\alpha}^{p+q} \theta_{\eta'}(\mu) = \theta_\eta(\lambda) e_{-\alpha}^p.$$ (4.10)

Proof. See the sources cited above. \qed

Consider the Lie algebra $\mathfrak{gl}(m+1)$ with simple roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [m]$, and set $\sigma_i = \epsilon_1 - \epsilon_{i+1}$. Let $\alpha = \alpha_m$, $\eta' = \epsilon_1 - \epsilon_m$ and $\eta = \epsilon_1 - \epsilon_{m+1} = s_\alpha \eta'$. Suppose $\alpha, \mu, \eta$ satisfy the hypotheses of Lemma 4.8. Then $\lambda = s_\alpha \cdot \mu \in H_{\eta}$. Thus since $(\lambda + \rho, \alpha^\vee) = -p$, it follows that

$$1 = (\lambda + \rho, \eta) = (\lambda + \rho, \sigma_{m-1} + \alpha) = -p + (\lambda + \rho, \sigma_{m-1}),$$

which implies that $p = (\lambda + \rho, \sigma_{m-1}) - 1 := a_{m-1}$. If $i \in [m-2]$, then $(\sigma_i, \alpha) = 0$, it follows that the $a_i$ as defined in (4.6) satisfy

$$a_i = (\lambda + \rho, \sigma_i) - 1,$$ (4.11)

which means that $D^m(\mu)$ fits into the lower right corner of $D^{m+1}(\lambda)$. Then $D^{m+1}(\lambda)$ is obtained from $D^m(\mu)$ by first adding a column on the left, whose only non-zero entry is $-a_{m-1}$ in the first row, and then adding a row so that $e_{m+1,i}$ is directly above $e_{m,i}$ for $i \in [m-1]$, and with $e_{m+1,m}$ as first entry.

Next introduce two matrices $D_1, D_2$ by

$$D_1 = \begin{bmatrix} e_{m+1,m} & 0 & \ldots & 0 \\ 0 & D^m(\mu) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e_{m+1,1} \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 & \ldots & e_{m+1,1} \\ e_{m+1,m} & \cdots & e_{m+1,m-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{m+1,m} & \cdots & e_{m+1,m-1} & 0 \end{bmatrix}. \quad (4.12)$$

Observe that $e_{m+1,m}$ commutes with all entries in $D^m(\mu)$ except for those in the first row, and that

$$e_{m+1,m}^{p+1} e_{m,i} = (e_{m+1,m} e_{m,i} + p e_{m+1,i}) e_{m+1,m}^p. \quad (4.13)$$

This implies that
$e^{p+1}_{m+1,m} \det D_m(\mu) = (\det D_1 + \det D_2)e^{p}_{m+1,m}.$ \hspace{1cm} (4.14)

In (4.14), $D_2$ arises from the second term in the factor $(e_{m+1,m}e_{m,i} + p e_{m+1,i})$ from Equation (4.13), and similarly $D_1$ corresponds to the first term. But $\det D_1 + \det D_2$ is just the cofactor expansion of $\det D_{m+1}(\lambda)$ down the first column. By induction

$$\theta_\eta'(\mu) = \det D_m(\mu).$$

Now comparing (4.10) and (4.14), and using the fact that $e_{-\alpha}$ is not a zero divisor in $U(g)$ it follows that

$$\theta_\eta(\lambda) = \det D^{m+1}_m(\lambda).$$

□

We record how Šapovalov elements can be expressed using non-commutative determinants in the case of $g = gl(m,n)$. First define

$$B_i = \begin{cases} (\lambda + \rho, \sigma_i) - 1 & \text{for } i \in [m - 1], \\ (\lambda + \rho, \tau_{i+1-m}) & \text{for } m \leq i \leq m+n-2. \end{cases}$$ \hspace{1cm} (4.15)

With $h_i$ as in (2.18), we have

$$h_i(\lambda) = B_i \text{ for all } i \in [m + n - 2].$$ \hspace{1cm} (4.16)

Next consider the determinant

$$A(\lambda) = \begin{bmatrix} e_{m+n,m+n-1} & e_{m+n,m+n-2} & \cdots & \cdots & \cdots & e_{m+n,2} & e_{m+n,1} \\ -B_{m+n-2} & e_{m+n-1,m+n-2} & \cdots & \cdots & \cdots & e_{m+n-2,2} & e_{m+n-2,1} \\ 0 & -B_{m+n-3} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -B_m & e_{m+1,m} & \cdots & \cdots & e_{m+1,1} \\ 0 & 0 & -B_{m-1} & e_{m,m-1} & \cdots & \cdots & e_{m,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & -B_1 & e_{2,1} \end{bmatrix},$$ \hspace{1cm} (4.17)

Recall the expression for the Šapovalov element $\theta_\eta$ from Theorem 2.6.

**Theorem 4.9.** We have

$$\det A(\lambda) = \sum_{J \subseteq \mathfrak{i}} f_J H_J(\lambda).$$

Thus $\det A(\lambda)$ is the evaluation of the Šapovalov element $\theta_\eta$ at $\lambda \in \mathcal{H}_\eta$.

**Proof.** This follows from (4.16), Theorem 2.6 and the argument of Theorem 4.5 showing that $f_J$ is correctly paired with $H_J$ in the above sum. \hspace{1cm}□
4.3 Šapovalov elements for arbitrary odd roots.

Assume $\gamma$ is an odd root. Although as we remarked in Section [1] to compute Šapovalov element $\theta_{\gamma}$ we can assume $\gamma$ is the highest odd root of $\mathfrak{gl}(m, n)$, we later apply our results when this is not the case. Thus it will be convienent to have and explicit expressions for a Šapovalov elements for an arbitrary odd root.

Let $t = m + 1 - r$. We need the Šapovalov element for $\gamma = \epsilon_{r} - \delta_{s}$ in $U(\mathfrak{gl}(m, n))$. Let $\mathfrak{g}'$ be the subalgebra $\mathfrak{g} = \mathfrak{gl}(m, n)$ with rows and columns indexed by the set \{r, ..., m + s\}. Note that $\mathfrak{g}'$ and $\mathfrak{g}$ both share the same odd simple root vector $e_{\beta} = e_{m,m+1}$. Also $\mathfrak{g}' \cong \mathfrak{gl}(t, s)$, and $\gamma$ is the longest odd positive root for the subalgebra $\mathfrak{g}'$. The Šapovalov element $\theta_{\gamma}$ is easily found using the matrix

$$A^{r,s}(\lambda) = \begin{bmatrix} e_{m+s,m+s-1} & e_{m+s,m+s-2} & \cdots & e_{m+s,r+1} & e_{m+s,r} \\ -A_{m+s-2} & e_{m+s-1,m+s-2} & \cdots & e_{m+s-1,r+1} & e_{m+s-1,r} \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & 0 & \cdots & -A_{r+1} & e_{r+2,r+1} \\ 0 & 0 & \cdots & 0 & e_{r+2,r} & e_{r+1,r} \end{bmatrix} \quad (4.18)$$

Note that all entries in the above matrix belong to the subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. Set

$$\sigma_{i} = \epsilon_{r} - \epsilon_{r+i}, \text{ and } \omega_{i} = \delta_{i+1-r} - \delta_{s}. \quad (4.19)$$

Then we define $A_{i}$ by

$$A_{i+r-1} = \begin{cases} (\lambda + \rho, \sigma_{i}) - 1 & \text{for } i \in [t - 1], \\ (\lambda + \rho, \omega_{i}) & \text{for } t \leq i \leq t + s - 2. \end{cases} \quad (4.20)$$

We have $(\rho, \alpha) = 1$ or $-1$, respectively if $\alpha$ is a simple root of $\mathfrak{gl}(t) \oplus 0$ or $0 \oplus \mathfrak{gl}(s)$ and $(\rho, \beta) = 0$ for the unique simple odd root, see Remark 2.1. These are the only properties of $\rho$ we need, so there is no need to introduce the analog of $\rho$ for $\mathfrak{gl}(t, s)$. Let

$$\mathbb{I} = \{I \subseteq \{r, r + 1, \ldots, m + s\} | r, m + s \in I\}. \quad (4.21)$$

Then for $J \in \mathbb{I}$, define $f_{J, r}(J)$ as usual. Define $H_{i+r-1} \in U(\mathfrak{h})$ by

$$H_{i+r-1} = \begin{cases} h_{\sigma_{i}} + \rho(h_{\sigma_{i}}) - 1 & \text{for } i \in [t - 1], \\ h_{\omega_{i}} + \rho(h_{\omega_{i}}) & \text{for } t \leq i \leq t + s - 2. \end{cases} \quad (4.22)$$

Then let $H_{J} = \prod_{i \in r(J)} H_{i}$, and note that by (4.20) and (4.22), we have by

$$H_{i+r-1}(\lambda) = A_{i+r-1} \text{ for } i \in [t + s - 2]. \quad (4.23)$$

By Theorem 4.9, $\det A^{r,s}(\lambda)$ is the evaluation of the Šapovalov element $\theta_{\gamma}$ at $\lambda \in \mathcal{H}_{\gamma}$. We use the same procedure as before to obtain the complete expansion of $\det A^{r,s}(\lambda)$ and thus the Šapovalov element $\theta_{\gamma} \in U(\mathfrak{gl}(t, s)) \subseteq U(\mathfrak{gl}(m, n))$. 

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Theorem 4.10. The Šapovalov element $\theta_\gamma$ for the root $\gamma = \epsilon_r - \delta_s$ in $U(\mathfrak{gl}(m, n))$ is given by

$$\theta_\gamma = \sum_{J \subseteq I} f_J H_J.$$  \hspace{1cm} (4.24)

Proof. This follows from the proof of Theorem 2.6. \qed

Corollary 4.11. For $\lambda \in \mathcal{H}_\gamma$, $\theta_\gamma(\lambda) = \rightarrow \det \mathbb{A}_{r,s}(\lambda)$.

Proof. By Equation (4.23),

$$\sum_{J \subseteq I} f_J H_J(\lambda) = \sum_{J \subseteq I} f_J \prod_{i \in r(J)} A_i$$

and the argument in Theorem 4.5 shows that in the complete expansion of $\rightarrow \det \mathbb{A}_{r,s}(\lambda)$, the coefficient of $f_J$ is $\prod_{i \in r(J)} A_i$. \qed

4.4 More Expansions for Šapovalov elements.

We consider two new orders on root vectors, in one the unique odd root vector appears last in each product $f_J$. In the other they appear first. Applications of each are given in Subsection 4.6.

4.4.1 Putting odd root vectors last.

With $I$ as in (4.21), we give another expansion of the Šapovalov element $\theta_\gamma$ using a different order on positive order roots. With this new order for each $I \subseteq \mathfrak{I}$, the unique odd root vector appears last. Because of this requirement, the definition of $f_I$ is not as easily described using an ordering on $I$. Thus consider $I$ as an unordered set

$$I = \{m + s, i_1, i_2, \ldots, i_g, j_1, \ldots, j_h\}$$  \hspace{1cm} (4.25)

where $m + s > i_1 > i_2 > \ldots > i_g \geq m + 1$ and $r = j_1 < \ldots < j_h \leq m$. Then we define

$$f_I = e_{m+s,i_1} e_{i_1,i_2} \ldots e_{i_{g-1},i_g} e_{j_2,j_1} \ldots e_{j_h,j_{h-1}} e_{i_r,j_h}.$$  \hspace{1cm} (4.26)

Note that the odd root vector $e_{i_r,j_h}$ is the last entry in $f_I$. The definition of $H_I$ is also changed. Instead of (4.22), we define $H_{i+r-1} \in U(\mathfrak{h})$ by

$$H_{i+r-1} = \left\{ \begin{array}{ll}
    h_{\sigma_i} + \rho(h_{\sigma_i}) & \text{for } i \in [t-1] \\
    h_{\omega_i} + \rho(h_{\omega_i}) & \text{for } t \leq i \leq t + s - 2.
\end{array} \right.$$  \hspace{1cm} (4.27)

Recall the definition of $A_i$ from (4.20) and set $C_i = A_i + 1$ for all $i$. Then

$$H_{i+r-1}(\lambda) = \left\{ \begin{array}{ll}
    C_{i+r-1} & \text{for } i \in [t-1] \\
    A_{i+r-1} & \text{for } t \leq i \leq t + s - 2.
\end{array} \right.$$  \hspace{1cm} (4.28)

Define $r(I)$ as usual, then set $H_I = \prod_{i \in r(I)} H_i$. Then we have
Theorem 4.12. With the above notation, the Šapovalov element $\theta_\gamma$ for the root $\gamma = \epsilon_r - \delta_s$ in $U(\mathfrak{gl}(m,n))$ is given by

$$\theta_\gamma = \sum_{I \subseteq \mathbb{I}} f_I H_I. \quad (4.29)$$

Proof. Recall the matrix $A^{r,s}_t(\lambda)$ defined in (4.18). The entries in $A^{r,s}_t(\lambda)$ belong to the subalgebra $\mathfrak{g}' \cong \mathfrak{gl}(t,s)$. Note that the entries in the last $t$ columns and first $s$ rows of $A^{r,s}_t(\lambda)$ are precisely the odd negative root vectors of $\mathfrak{g}'$. We use the following cofactor procedure. Begin with a cofactor expansion of $A^{r,s}_t(\lambda)$ down the first column. Then use a cofactor expansion of the resulting cofactors down their first columns. Repeat this a total of $s-1$ times. Note that every time we use cofactor expansion on a minor, the two resulting cofactors have one fewer row containing odd elements. Thus at the end of the procedure, all the remaining minors contain only odd root vectors in the first row, and all root vectors in this row are odd.

Next we use Proposition 4.2 to replace each minor by a matrix containing only odd vectors in the last column. Then the result follows easily after some bookkeeping. Let $K = \{K \subseteq \{m+1, \ldots, m+s\} | m+s \in K\}$. For $K \in \mathbb{K}$, put the entries of $K$ is descending order and define $\bar{f}_K$ as in (2.3). Denote the complement of $K$ in $\{m+1, \ldots, m+s\}$ by $\bar{K}$, and set $s(K) = \{p-1 | p \in \bar{K}\}$. The smallest element of $\bar{K}$ is $m+j$ for some $j \in [s]$, and we set $j(K) = j$ in this situation. Next we describe the minors that can arise at the end of the procedure. First let

$$\mathfrak{F}(\lambda) = \begin{bmatrix} -A_{m-1} & e_{m,m-1} & e_{m,m-2} & \cdots & e_{m,r+1} & e_{m,r} \\ 0 & -A_{m-2} & e_{m-1,m-2} & \cdots & \cdots & e_{m-1,r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & -A_{r+1} & e_{r+2,r+1} & e_{r+2,r} \\ 0 & 0 & 0 & 0 & -A_r & e_{r+1,r} \end{bmatrix}, \quad (4.30)$$

and note that $\mathfrak{F}(\lambda)$ fits into the bottom right corner of $A^{r,s}_t(\lambda)$. Now let $\mathfrak{F}^{(j)}(\lambda)$ be the matrix obtained from $\mathfrak{F}(\lambda)$ obtained by adjoining, as the first row the vector

$$(e_{m+j,m}, e_{m+j,m-1}, \ldots, e_{m+j,r+1}, e_{m+j,r}). \quad (4.31)$$

Note that all entries in (4.31) are odd, while all other entries in $\mathfrak{F}^{(j)}(\lambda)$ not on the subdiagonal belong to $\mathfrak{gl}(m) \oplus 0$. Then the result of the cofactor procedure is the following.

Lemma 4.13. If $\lambda \in \mathcal{H}_\gamma$, then

$$\theta_\gamma(\lambda) = \sum_{K \in \mathbb{K}} \bar{f}_K \prod_{k \in s(K)} A_k \overrightarrow{\det}(\mathfrak{F}^{(j(K))}(\lambda))$$

Proof. In each step of the cofactor procedure we choose either an element from $\mathfrak{n}^-$ or a subdiagonal element of $A^{r,s}_t(\lambda)$. In column $i$, the entries in $\mathfrak{n}^-$ all have the form $e_{s,m+s-i-1}$ and the subdiagonal element is $-A_{m+s-i-1}$. We choose an entry from $\mathfrak{n}^-$.
from this column iff \( m + s - i \in K \). Otherwise we choose \(-A_{m+s-i-1}\), which agrees with \( m + s - i - 1 \in s(K)\). Now in the complete expansion of \( \mathcal{K}^{r,s}(\lambda)\), \( f_K \) is the initial part of various products of terms \( f_I \) with \( I \in \mathcal{I} \). If \( j = j(K)\), then the last factor in \( f_K \) has the form \( e_{\ast,m+j} \). Then to get a valid product the next term should be \( e_{m+j,\ast} \), which means that the minor associated to \( K \) at the end of the cofactor procedure is \( \mathcal{P}(j(K))(\lambda) \).

Now let \( G^{(j)}(\lambda) \) be the matrix

\[
G^{(j)}(\lambda) = \begin{bmatrix}
  e_{r+1,r} & e_{r+2,r} & e_{r+3,r} & \cdots & e_{m,r} & e_{m+j,r} \\
  -C_r & e_{r+2,r+1} & e_{r+2,r+1} & \cdots & e_{m,r+1} & e_{m+j,r+1} \\
  0 & -C_{r+1} & e_{r+4,r+2} & \cdots & \cdots & e_{m+j,r+2} \\
  \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \cdots & -C_{m-2} & e_{m,m-1} & e_{m+j,m-1} \\
  0 & 0 & 0 & 0 & -C_{m-1} & e_{m+j,m}
\end{bmatrix}.
\]

(4.32)

Then by Proposition [4.12], \( \det(\mathcal{P}(j)(\lambda)) = \det(G^{(j)}(\lambda)) \). Hence

\[
\theta_\gamma(\lambda) = \sum_{K \in \mathcal{K}} f_K \prod_{k \in s(K)} A_k \det(G^{(j(K))}(\lambda)).
\]

Let \( \mathcal{J}(j) = \{ J \subseteq \{ r, r+1, \ldots, m, m+j \} \mid r, m+j \in J \} \). It is clear that in the complete expansion of the determinant \( \det(G^{(j)}(\lambda)) \), we encounter only terms \( f_J \), for some \( J \in \mathcal{J}(j) \) ordered as in the last part of \([4.26]\), and that all such \( J \) occur. To complete the proof we need a slightly different cofactor procedure than before. We start with a cofactor expansion along the last row, which we call row 1, and continue upwards, numbering rows from bottom to top. In row \( i \) the entries from \( \mathfrak{n}^+ \) have the form \( e_{\ast,m+1-i} \) and the subdiagonal entry is \(-C_{m-i}\). Now \( m + 1 - i \in J \) iff we choose an element from \( \mathfrak{n}^+ \) from this row. Otherwise we choose \(-C_{m-i}\), which agrees with \( m-i \in r(J) \). Now define \( t(J) \) using the complement of \( J \) in \( \mathcal{J}(j) \). Then we have

\[
\det(G^{(j)}(\lambda)) = \sum_{J \in \mathcal{J}(j)} f_J \prod_{i \in t(J)} C_i.
\]

Thus

\[
\theta_\gamma(\lambda) = \sum_{K \in \mathcal{K}} f_K \sum_{J \in \mathcal{J}(j(K))} f_J \prod_{i \in s(K)} A_i \prod_{i \in t(J)} C_i.
\]

It is clear that for any \( K \in \mathcal{K} \) and \( J \in \mathcal{J}(j) \) with \( j = j(K) \), we have \( f_K f_J \in U(\mathfrak{n}^-)^{-\gamma} \) and the terms are correctly ordered so \( f_K f_J = f_I \) where \( I = K \cup J \in \mathcal{I} \). We have \( r(I) = s(K) \cup t(J) \). Now to complete the proof of Theorem [4.12] we need to show that

\[
\prod_{i \in r(I)} H_i(\lambda) = \prod_{i \in s(K)} A_i \prod_{i \in t(J)} C_i.
\]

This follows from \([4.28]\).
4.4.2 Putting odd root vectors first.

If $I$ is the set in (4.25) define
\[ f_I = e_{i_g,j_h}e_{j_2,j_1} \cdots e_{j_h,j_{h-1}}e_{m+s,i_1}e_{i_1,i_2} \cdots e_{i_g-1,i_g}. \] (4.33)

Next instead of (4.27), we define $H_{i+r-1} \in U(\mathfrak{h})$ by
\[ H_{i+r-1} = \begin{cases} 
  h_{\sigma_i} + \rho(h_{\sigma_i}) - 1 & \text{for } i \in [t-1] \\
  h_{\omega_i} + \rho(h_{\omega_i}) + 1 & \text{for } t \leq i \leq t + s - 2.
\end{cases} \] (4.34)

Note that in place of (4.28), we now have
\[ H_{i+r-1}(\lambda) = \begin{cases} 
  A_{i+r-1} & \text{for } i \in [t-1] \\
  C_{i+r-1} & \text{for } t \leq i \leq t + s - 2.
\end{cases} \] (4.35)

**Theorem 4.14.** With the above notation, the Šapovalov element $\theta_\gamma$ for the root $\gamma = \epsilon_r - \delta_s$ in $U(\mathfrak{gl}(m,n))$ is given by
\[ \theta_\gamma = \sum_{I \subseteq I} f_I H_I. \] (4.36)

**Proof.** This is similar to the proof of Theorem 4.12 so we give fewer details. We use a cofactor expansion of $A^{r,s}(\lambda)$ along the last $t-1$ rows working from the bottom up. The subdiagonal entries that emerge in these steps are the $A_{i+r-1}$ for $i \in [t-1]$. At the end of these steps, the unexpanded cofactors have last column consisting of odd root vectors and no other odd root vectors. Then we use Proposition 4.2 to replace each of these cofactors with a matrix containing only odd vectors in the first row, and subdiagonal entries $C_{i+r-1}$ for $t \leq i \leq t + s - 2$. The proof concludes as before. \qed

4.5 Partial Expansions of Šapovalov elements.

We obtain two expansions of the Šapovalov element for the root $\gamma = \epsilon_r - \delta_s$ in (4.29) under some additional assumptions. These expansions correspond to partial expansions of the determinant $\det A^{r,s}(\lambda)$, obtained by separating out the terms containing certain subdiagonal entries, whence the title of this subsection. The main results are Theorems 4.15 and 4.16. Motivation for these results is given in Subsection 4.6.

The additional assumptions are

**Case 1** We assume that $(\lambda + \rho, \gamma) = (\lambda + \rho, \gamma') = 0$, where $\gamma' = s_\alpha \gamma = \gamma - \alpha$ for some positive root $\alpha$. This means that $(\lambda + \rho, \alpha^\vee) = 0$ and by [Mus17] Theorem 5.9 we have, in this situation $\theta_\gamma v_{\lambda} = \theta_{\alpha_1} \theta_{\gamma'} v_{\lambda}$. Theorem 4.15 below (the main result in Case 1) could be considered as a refinement of this equation when $\lambda$ is replaced by $\tilde{\lambda} = \lambda + T_\xi$.

**Case 2** We suppose $\alpha_1, \alpha_2$ (resp. $\gamma_1, \gamma_2$) are distinct positive even (resp. odd) roots such that
\[ \alpha_1 + \gamma_1 + \alpha_2 = \gamma_2. \] (4.37)
This means that $\alpha_2^\gamma \equiv \alpha_2^\gamma \mod Z \gamma_1 + Z \gamma_2$. If $\gamma_1, \gamma_2 \in X$ there are significant consequences for the structure of the modules $M^X(\lambda)$ from \[Mus17\] Theorem 1.10, see \[4.59\] and factors of the corresponding Šapovalov determinant $\det F^X_\eta$.

We disregard the notation for $\alpha_i, \gamma_i$ introduced at the start of Section 2. Then for Case 1 we assume that

$$\gamma = \epsilon_r - \delta_s, \alpha = \epsilon_r - \epsilon_\ell \text{ and } \gamma' = s_\alpha \gamma = \epsilon_\ell - \delta_s.$$  \hspace{1cm} (4.38)

In Case 2, we may assume that

$$\alpha_1 = \epsilon_r - \epsilon_\ell, \alpha_2 = \delta_k - \delta_s \text{ and } \gamma_1 = \epsilon_\ell - \delta_k, \gamma_2 = \epsilon_r - \delta_s.$$ \hspace{1cm} (4.39)

**4.5.1 Case 1.**

We assume that Case 1 and \[4.38\] hold. Before we state the main result there is another issue to deal with. We need to use a different ordering on positive roots. To do this we introduce the matrix

$$B^{\gamma,s}(\lambda) = \begin{bmatrix}
e_{r+1,r} & e_{r+2,r} & \cdots & \cdots & e_{m+s-1,r} & e_{m+s,r} \\
-C_r & e_{r+2,r+1} & \cdots & \cdots & e_{m+s-1,r+1} & e_{m+s,r+1} \\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & -C_{m+s-1} & e_{m+s-1,m+s-2} & e_{m+s,m+s-2} \\
0 & 0 & \cdots & -C_{m+s-2} & e_{m+s,m+s-1} & e_{m+s,m+s-2} \end{bmatrix}.$$ \hspace{1cm} (4.40)

By Proposition \[4.2\] and \[4.18\], we have

$$\overrightarrow{\det} A^{\gamma,s}(\lambda) = \overrightarrow{\det} B^{\gamma,s}(\lambda).$$ \hspace{1cm} (4.41)

At this point we bring $\alpha = \epsilon_r - \epsilon_\ell = \sigma_{\ell-r}$ and $T = H_{\ell-1}$ into the story by noting that $(\ell - r) + (r - 1) = \ell - 1$, so by Equations \[4.20\] and \[4.28\], we have

$$T(\lambda) = C_{\ell-1} = (\lambda + \rho, \alpha_1).$$ \hspace{1cm} (4.42)

With $I$ as in \[4.21\], we define

$$I_\ell = \{ I \subseteq I | \ell - 1 \in r(I) \}, \quad I^\ell = \{ I \subseteq I | \ell - 1 \notin r(I) \}.$$  

Note that $I \subseteq I^\ell$ if and only if $\ell \in I$. Also $T = H_{\ell-1}$ is a factor of $H_J$ iff $J \subseteq I_\ell$. For $J \subseteq I_\ell$, define $H'_J = H_J/T$.

**Theorem 4.15.** If $\theta_\gamma$ is as in Theorem \[4.10\], then for $\lambda \in H_\gamma$, we have

$$\theta_\gamma v_\lambda = (\theta_{\alpha_1} \theta_\gamma + \sum_{J \subseteq I_\ell} f_J H'_J T)v_\lambda.$$ \hspace{1cm} (4.43)
Proof. We have
\[ \theta_\gamma = \sum_{J \subseteq \mathbb{I}^\ell} f_J H_J + \sum_{J \subseteq \mathbb{I}_\ell} f_J H_J'T. \]  
(4.44)

It is useful to think about this proof in terms of non-commutative determinants. Thus we set \( B = B_{r,s}(\lambda) \) as in (4.40), and recall that \( T(\lambda) = C_{\ell-1} \) by (4.42). By Corollary 4.11 and (4.41) we have \( \theta_\gamma(\lambda) = \overrightarrow{\text{det}}B \). From (4.2),
\[ \overrightarrow{\text{det}}B = \overrightarrow{\text{det}}B' + T(\lambda)\overrightarrow{\text{det}}B'', \]  
(4.45)

where \( B' \) and \( B'' \) are obtained from \( B \) by setting \( T(\lambda) = 0 \), and by deleting the row and column containing \( T(\lambda) \) respectively. Hence \( B'' \) is the cofactor of entry \( T(\lambda) \) in \( B \).

Also any term in the expression (4.1) for \( \overrightarrow{\text{det}}(B) \) which contains a factor of the form \( e_{s,\ell}, e_{\ell,s} \), cannot occur in \( \overrightarrow{\text{det}}B'' \). This means that \( \overrightarrow{\text{det}}B'' \) belong to \( U(\mathfrak{g}) \) where \( \mathfrak{g} \) the subalgebra isomorphic to \( \mathfrak{gl}(m-1,n) \) and with rows and columns indexed by the set \( \mathbb{I}(\ell) = \{ r, \ldots, \ell, \ldots, m+s \} \). As mentioned above \( J \subseteq \mathbb{I}_\ell \) if and only if \( \ell \not\in J \).

Thus the second term on the right of (4.45) equals the evaluation of the second term on the right of (4.44) at \( \lambda \). Note that when \( T(\lambda) = 0 \), \( \theta_\gamma v_\lambda = \theta_v \alpha_1 \theta_v' v_\lambda \), as already mentioned, so the proof is complete.

4.5.2 Case 2.

We assume that Case 1 and (4.39) hold. The proof of our main result depends on Theorem 4.12. Hence we define \( f_I, A_i, H_i \) as in (4.26), (4.20) and (4.27) respectively. The relation between these elements is given by (4.28). Recall \( \alpha_1 = \epsilon_r - \epsilon_\ell = \sigma_{\ell-r} \) and \( \alpha_2 = \delta_k - \delta_s \) and we have
\[ A_{\ell-1} = (\lambda + \rho, \alpha_1) - 1 \text{ and } A_{m+k-1} = (\lambda + \rho, \alpha_2). \]  
(4.46)

It may help to observe that in (4.19) \( \omega_i = \delta_k - \delta_s = \alpha_2 \), when \( i = k + t - 1 \), and thus \( i + r - 1 = m + k - 1 \) in (4.28).

We treat
\[ T = H_{\ell-1} \text{ and } S = -H_{m+k-1} \]  
(4.47)

as indeterminates which can be evaluated on any \( \lambda \in \mathcal{H}_\gamma \). By Equation (4.28), we have
\[ T(\lambda) = C_{\ell-1} = (\lambda + \rho, \alpha_1') \text{ and } S(\lambda) = -A_{m+k-1} = (\lambda + \rho, \alpha_2'). \]  
(4.48)

For each subset \( J \) in (4.29) we are interested in when \( S \) or \( T \), from (4.47), or both are factors of \( H_J \). Note that
\[ T \] is a factor of \( H_J \) iff \( \ell - 1 \in r(J) \), iff \( J \in \mathbb{I}(\alpha_2) \) or \( \mathbb{I}(\alpha_1, \alpha_2) \)
\[ S \] is a factor of \( H_J \) iff \( m + k - 1 \in r(J) \) iff \( J \in \mathbb{I}(\alpha_1) \) or \( \mathbb{I}(\alpha_1, \alpha_2) \).  
(4.49)
Observe that $\mathbb{I}(\emptyset) = \{ J \subseteq \ell - 1, m + k - 1 \not\in r(J) \}$, $\mathbb{I}(\alpha_1) = \{ J \subseteq \ell - 1 \not\in r(J), m + k - 1 \in r(J) \}$, $\mathbb{I}(\alpha_2) = \{ J \subseteq m + k - 1 \not\in r(J), \ell - 1 \in r(J) \}$, $\mathbb{I}(\alpha_1, \alpha_2) = \{ J \subseteq \ell - 1, m + k - 1 \in r(J) \}$. The notation is inspired by Equation (4.48). Alternatively gives $\theta$. Proof. We break the sum in Theorem 4.29 into 4 pieces corresponding to the disjoint union of the above 4 sets. Next for $J$ a subset of $\mathbb{I}(\alpha_1), \mathbb{I}(\alpha_2), \mathbb{I}(\alpha_1, \alpha_2)$ elements $H_J^{(1)}$, $H_J^{(2)}$, $H_J^{(3)}$ respectively by

$$H_J^{(1)} = H_J/S, \ H_J^{(2)} = H_J/T, \ H_J^{(3)} = H_J/ST. \quad (4.52)$$

By (4.49) these are all elements of $U(h)$. Theorem 4.16. The Šapovalov element $\theta_\gamma$ for the root $\gamma = \epsilon_r - \delta_s$ in satisfies

$$\theta_\gamma = \theta_{\alpha_1} \theta_{\alpha_2} \theta_{\gamma_1} - \sum_{J \in \mathbb{I}(\alpha_1)} f_J H_J^{(1)} S + \sum_{J \in \mathbb{I}(\alpha_2)} f_J H_J^{(2)} T - \sum_{J \in \mathbb{I}(\alpha_1, \alpha_2)} f_J H_J^{(3)} ST. \quad (4.53)$$

Proof. We break the sum in Theorem 4.29 into 4 pieces corresponding to the disjoint union of the sets in (4.51). Then we have to show that $\theta_{\alpha_1} \theta_{\alpha_2} \theta_{\gamma_1} = \sum_{J \in \mathbb{I}(\emptyset)} f_J H_J$. Recall that $T(\lambda) = C_{\ell-1}$ and $S(\lambda) = -A_{m+k-1}$. We first consider what happens when these are zero. Since $C_{\ell-1} = 0$, $\mathbb{B}^{r,s}(\lambda)$ is block upper triangular, and this gives $\det \mathbb{B}^{r,s}(\lambda) = \theta_{\alpha_1} \det \mathbb{B}^{\ell,s}(\lambda) = \theta_{\alpha_1} \det \mathbb{A}^{\ell,s}(\lambda)$. Then $A_{m+k-1} = 0$ implies that $\mathbb{A}^{\ell,s}(\lambda)$ is block upper triangular, and we deduce that $\theta_\gamma = \theta_{\alpha_1} \theta_{\alpha_2} \theta_{\gamma_1}$. Now $J \in \mathbb{I}(\emptyset)$ means that $\ell - 1, m + k - 1 \not\in r(J)$, and thus neither of $S$ or $T$ is a factor of $H_J$. So the sum $\sum_{J \in \mathbb{I}(\emptyset)} f_J H_J$ is independent of $T(\lambda)$ and $S(\lambda)$. It follows that $\theta_{\alpha_1} \theta_{\alpha_2} \theta_{\gamma_1} = \sum_{J \in \mathbb{I}(\emptyset)} f_J H_J$ as required.

Remark 4.17. From the proof of Theorem 4.15 that the term $\sum_{J \in \mathbb{I}(\emptyset)} f_J H_J'$ in is the Šapovalov element for the longest root $\tilde{\gamma}$ of the subalgebra $\mathfrak{g}$ of $\mathfrak{g}$. To see this observe that with $\det B''$ as in the proof, we have $\theta_{\tilde{\gamma}}(\lambda) = \det B''$. We can make a similar remark about the last three sums on the right of (4.53). To do this we need the index sets, cf. (4.51),

$$\mathbb{I}[\alpha_1] = \{ r, r + 1, \ldots, \tilde{\ell}, \ldots, m + s \},$$

$$\mathbb{I}[\alpha_2] = \{ r, r + 1, \ldots, m + k, \ldots, m + s \}$$

$$\mathbb{I}[\alpha_1, \alpha_2] = \{ r, r + 1, \ldots, \tilde{\ell}, \ldots, m + k, \ldots, m + s \}. \quad (4.54)$$

The set $\mathbb{I}[\alpha_1]$ indexes the rows and columns of the subalgebra $\mathfrak{g} \cong \mathfrak{gl}(m - 1, n)$ as in Case 1. Define subalgebras $\mathfrak{g} \cong \mathfrak{gl}(m, n - 1)$ and $\mathfrak{g} \cong \mathfrak{gl}(m - 1, n - 1)$, similarly, using the sets $\mathbb{I}[\alpha_2]$ and $\mathbb{I}[\alpha_1, \alpha_2]$ respectively to index the rows and columns. Then in Equation (4.53) from Theorem 4.16 the coefficients of $S, T$ and $ST$ are the Šapovalov elements for the longest odd root of the subalgebras $\mathfrak{g}, \mathfrak{g}$ and $\mathfrak{g}$ respectively. More details on this approach were given in [Mus17]. Since we do not need this extra information about Theorems 4.15 or 4.16 we give no further details here.
Example 4.18. Let \( g = gl(2, 2) \) and using the notation of \([2.1]\), set \( \alpha = \alpha_1 \) and \( \gamma = \gamma_1 \). We also need
\[
e^{-\alpha - \beta} = e_{31}, \quad e^{-\beta - \gamma} = e_{42}, \quad e^{-\alpha - \beta - \gamma} = e_{41}.
\]
We find the Šapovalov elements for the roots \( \alpha + \beta, \beta + \gamma \) and \( \alpha + \beta + \gamma \) using Theorem [2.6]
\[
\theta_{\alpha + \beta} = \quad e^{-\beta}e^{-\alpha} + e^{-\alpha - \beta}h_\alpha \\
= \quad e^{-\alpha}e^{-\beta} + e^{-\alpha - \beta}(h_\alpha + 1).
\]
\[
\theta_{\beta + \gamma} = \quad e^{-\gamma}e^{-\beta} + e^{-\beta - \gamma}(h_\gamma - 1) \\
= \quad e^{-\beta}e^{-\gamma} + e^{-\beta - \gamma}h_\gamma.
\]
We give four expressions for \( \theta_{\alpha + \beta + \gamma} \). The first using Theorem [2.6] the second using Theorem [4.12] or Theorem [4.16] the third using Theorem [4.14] and the fourth by expanding the determinant of the matrix in [4.40].
\[
\theta_{\alpha + \beta + \gamma} = \quad e^{-\gamma}e^{-\beta}e^{-\alpha} + e^{-\gamma}e^{-\alpha - \beta}h_\alpha + e^{-\beta - \gamma}e^{-\alpha}(h_\gamma - 1) + e^{-\alpha - \beta - \gamma}h_\alpha(h_\gamma - 1) \\
= \quad e^{-\gamma}e^{-\alpha}e^{-\beta} + e^{-\gamma}e^{-\alpha - \beta}(h_\alpha + 1) + e^{-\alpha}e^{-\beta - \gamma}(h_\gamma - 1) + e^{-\alpha - \beta - \gamma}(h_\alpha + 1)(h_\gamma - 1) \\
= \quad e^{-\beta}e^{-\gamma}e^{-\alpha} + e^{-\alpha - \beta}e^{-\gamma}h_\alpha + e^{-\beta - \gamma}e^{-\alpha}h_\gamma + e^{-\alpha - \beta - \gamma}h_\alpha h_\gamma \\
= \quad e^{-\alpha}e^{-\beta}e^{-\gamma} + e^{-\alpha - \beta}e^{-\gamma}(h_\alpha + 1) + e^{-\alpha}e^{-\beta - \gamma}h_\gamma + e^{-\alpha - \beta - \gamma}(h_\alpha + 1)h_\gamma.
\]
Using the relations in \( U(g) \), the right sides of the above expressions are easily seen to be equal.

4.6 Motivation for the Expansions.

Let \( X \) be an orthogonal set of positive isotropic roots. Suppose \( \langle \lambda + \rho, \gamma \rangle = 0 \) for all \( \gamma \in X \). In \([Mus17]\), see also \([Mus23b]\) we constructed some highest weight modules \( M^X(\lambda) \) with highest weight \( \lambda \), and character \( e^\lambda p_X \), where \( p_X \) is a partition function that counts partitions not involving roots in \( X \). The definition of these modules is as follows. Let \( T \) be an indeterminate and set \( A = k[T], B = k(T) \). For \( R = A \) or \( B \), set \( U(g)_R = U(g) \otimes_k R \). Now choose \( \xi \in h^* \) subject to certain conditions which are spelled out in the two cases below, but otherwise generic, set \( \tilde{\lambda} = \lambda + T\xi \), and consider the Verma module \( M(\tilde{\lambda})_B \) over \( U(g)_B \) with highest weight \( \tilde{\lambda} \). Then set
\[
M^X(\tilde{\lambda})_B = M(\tilde{\lambda})_B / \sum_{\gamma \in X} U(g)_B \theta_\gamma v_\gamma.
\]
Next let \( M^X(\tilde{\lambda})_A \) be the \( U(g)_A \)-submodule of \( M^X(\tilde{\lambda})_B \) generated by the highest weight vector and define
\[
M^X(\lambda) = M^X(\tilde{\lambda})_A / TM^X(\tilde{\lambda})_A.
\]
In \([Mus17]\) Section 11, we evaluate the Šapovalov determinant \( \det F^X_\eta \) and give a Jantzen sum formula for these modules. A difficulty which does not arise in the
case of Verma modules [Mus12] Theorem 10.2.5, is that there is no natural $A$-basis for the weight space $M^X(\tilde{\lambda})_\Lambda$ indexed by partitions. To surmount this difficulty we use related determinant $\det G^X_\eta$ of a matrix $G^X_\eta$ with rows and columns indexed by elements $e_{-\pi v_\lambda}^\vee$, with $\pi$ a partition of $\eta$ as in the classical case. The leading term of $\det G^X_\eta$ is easy to compute. In Theorem 11.1 of [Mus17] we determine the relationship between the leading terms of the two determinants $\det G^X_\eta$ and $\det F^X_\eta$, and evaluate the latter. The comparison of leading terms relies on Equations (4.60) and (4.61) below, which are used repeatedly to improve the $A$-basis used to calculate $\det G^X_\eta$. Below we refer to Cases 1 and 2 from the previous Subsection.

Case 1: Let $T = T$ be as in (4.42). Suppose $(\lambda + \rho, \gamma) = (\lambda + \rho, \gamma') = 0$, where $\gamma' = s_\alpha \gamma = \gamma - \alpha$ for some positive root $\alpha$. Then $(\lambda + \rho, \alpha^\vee) = 0$. Choose $\xi \in \mathfrak h^*$ so that $(\xi, \gamma) = (\xi, \gamma') = 0 = (\xi, \alpha^\vee)$ and with $\xi$ otherwise generic. Then set $\tilde{\lambda} = \lambda + T \xi$. Using the notation of (4.43), consider the elements of $M(\tilde{\lambda})$ and (4.61) below, which are used repeatedly to improve the $A$-basis.

$$p = \theta_{\alpha,1} \theta_{\gamma} v_\lambda \quad \text{and} \quad q = \sum_{J \subseteq \ell} f_J H^T_J v_\lambda,$$

we have the relation $\theta_{\gamma} v_\lambda = p + q T$ in $M(\tilde{\lambda})_A$. Thus if $\gamma \in X$, we have in $M^X(\tilde{\lambda})_A$.

$$p + q T = 0. \quad (4.60)$$

This is a key step in the proof of Theorem 11.4 of [Mus17].

Case 2: Recall by (4.48) that $T(\lambda) = (\lambda + \rho, \alpha^\vee_1)$ and $S(\lambda) = (\lambda + \rho, \alpha^\vee_2)$. Set $S = T = T + 1$. Thus if $T = 0$, we have $(\lambda + \rho, \alpha^\vee_1) = (\lambda + \rho, \alpha^\vee_2) = 1$ which is the assumption of Theorem 11.8 of [Mus17]. Assume that $\lambda \in \mathcal H_{\gamma_1} \cap \mathcal H_{\gamma_2}$. Now choose $\xi \in \mathfrak h^*$ so that $(\xi, \gamma_1) = (\xi, \gamma_2) = 0$, and $(\xi, \alpha^\vee_1) = (\xi, \alpha^\vee_2) = 1$ and with $\xi$ otherwise generic. Then set $\tilde{\lambda} = \lambda + T \xi$, and consider the Verma module $M(\tilde{\lambda})$ over $B$ with highest weight $\tilde{\lambda}$. Let $M$ be the factor module of $M(\tilde{\lambda})_A$ defined by setting $\theta_{\gamma_1} v_\lambda = \theta_{\gamma_2} v_\lambda = 0$. Then $(\lambda + \rho, \alpha^\vee_1) = (\lambda + \rho, \alpha^\vee_2) = T + 1$. Hence in the notation of Theorem 4.16 we have $S = T = T + 1$. Thus since $T + 1$ is not a zero divisor in $B$, if $\theta_{\gamma_1} v_\lambda = \theta_{\gamma_2} v_\lambda = 0$, then (4.53) yields the following relation in $M$

$$0 = [ \sum_{J \in I(\alpha_1)} f_J H^1_J ] - [ \sum_{J \in I(\alpha_2)} f_J H^2_J ] + [ \sum_{J \in I(\alpha_1, \alpha_2)} f_J H^3_J (T + 1) ] v_\lambda \quad (4.61)$$

This gives a generalization of Equation (B.20) from [Mus17] and is a key step in the proof of Theorem 11.8 of [Mus17].

Finally Theorem 4.14 can be used to give a new proof of a result of Serganova, [Ser96] Theorem 5.5, see Theorem 6.5. For the proof it is important that root vectors are ordered so that the unique odd root vector in each $f_J$ is the first factor. For the proof of Theorem 6.4 the unique odd root vector needs to be the last factor in each $f_J$. 

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5 Powers of Šapovalov elements.

Powers of Šapovalov elements are dealt with in detail in Section 5.1 of [Mus17]. The method, which works for all non-exceptional algebras, depends on Lemma 4.8, which is the basis of the original construction of Šapovalov elements from [šap72]. We briefly review the details. First write the root η in the form η = wβ with β a simple root and w ∈ W. Next by induction on the length of the w, we construct θ_η ∈ U(n^-)^-η such that θ_ην_λ is a highest weight vector for M(λ) whenever λ ∈ H_η, and the coefficient of e_{-ρ^o} in θ_η equals 1, see (1.2). Then we fix a (non-unique) lifting of θ_η to an element Θ_η ∈ U(b^-)^-η = U(n^-)^-η ⊗ U(η) such that Θ_ην_λ = Θ_η(λ)ν_λ = θ_ην_λ, for all λ ∈ H_η.

For an isotropic root η, if λ ∈ H_η, then also λ − η ∈ H_η, and by induction on the length of the w we have

**Proposition 5.1.** Let η be an odd positive root of gl(m,n), and define Θ_η as in Theorem 2.6. If λ ∈ H_η, then Θ^2_ην_λ = 0.

**Proof.** Combine Theorem 5.1 from [Mus17] and Theorem 2.6.

For a non-isotropic root η, the situation is more delicate, because it is no longer true that λ ∈ H_η,p implies λ − η ∈ H_η,p, and thus we need to evaluate Θ_η,p at a point that is not in H_η,p. To resolve this issue, a uniform inductive construction of Θ_η is given in [Mus17] and [Mus23b]. This depends on a specific choice of β and a shortest length expression for w. Then Θ_η,p is a Šapovalov element for the pair (η,p), by [Mus17] Theorem 5.8.

Needless to say, this method of proof is completely different from the direct approach used in most of this paper. However the two approaches are brought together by the two proofs of Theorem 4.5, the second of which uses the same choice of β as in [Mus17]. From this we can deduce the following

**Proposition 5.2.** Let η be a positive root of gl(m), and define Θ_η as in Theorem 2.3. Then, for p ≥ 1, Θ^p_η is the Šapovalov element for the pair (η,p). In other words if λ ∈ H_η,p, we have

Θ^p_η(λ) = θ_η,p(λ).

**Proof.** This follows from the above discussion and Theorem 2.3.

The above result may be viewed as a version of [CL74] Theorem 2.7.

6 Survival of Šapovalov elements in factor modules.

Let ν_λ be a highest weight vector in a Verma module M(λ) with highest weight λ, and suppose γ is an odd root with (λ + ρ, γ) = 0. We are interested in the condition
that the image of \( \theta_\gamma v_\lambda \) is non-zero in a factor module \( M \) of \( M(\lambda) \). If this is the case, we might say that \( \theta_\gamma v_\lambda \) survives in \( M \). Throughout this section we assume that \( g = \mathfrak{g}(m,n) \). Šapovalov elements and Verma modules are defined using the distinguished Borel subalgebra.

### 6.1 Independence of Šapovalov elements.

Given \( \lambda \in h^* \) set

\[
B(\lambda) = \{ \gamma \in \Delta_+^\perp \mid (\lambda + \rho, \gamma) = 0 \}.
\]

If \( \gamma, \gamma' \) are positive non-orthogonal isotropic roots, then \( \gamma' = s_\alpha \gamma \) for some even root \( \alpha \). Assume that

\[
(\gamma, \alpha^\vee) = 1 \text{ and } (\lambda + \rho, \alpha^\vee) = 0 \quad (6.1)
\]

In this situation the next result relates Šapovalov elements for \( \gamma, \gamma' \) and \( \alpha \).

**Theorem 6.1.** Let \( v_\lambda \) be a highest weight vector in \( M(\lambda) \) and set \( \gamma' = s_\alpha \gamma \). If \( (\lambda + \rho, \alpha^\vee) = 0 \) and \( \gamma' \in B(\lambda) \) we have

\[
\theta_\gamma v_\lambda = \theta_{\alpha,1} \theta_{\gamma'} v_\lambda. \quad (6.2)
\]

**Proof.** This is a special case of [Mus17] Theorem 5.9 (a). Note that by (6.1) and the hypothesis, \( \gamma \in B(\lambda) \).

Define a “Bruhat order” \( \leq \) on \( B(\lambda) \) as follows. First write \( \gamma' \prec \gamma \) if \( \gamma - \gamma' \) is a positive even root \( \alpha \), \( (\lambda + \rho, \alpha^\vee) = 0 \) and \( s_\alpha \gamma = \gamma' \). The relation \( \leq \) is the transitive extension of \( \prec \).

Thus \( \gamma' \leq \gamma \) if there is a sequence of elements \( \gamma = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma' \) such that \( \gamma_i \prec \gamma_{i-1} \) for \( i \in [n] \). Then we say that \( \gamma \) is \( \lambda \)-minimal if \( \gamma' \leq \gamma \) with \( \gamma' \in B(\lambda) \) implies that \( \gamma' = \gamma \). For \( \gamma \in B(\lambda) \) set \( B(\lambda)^{-}\gamma = B(\lambda) \setminus \{\gamma\} \). We say \( \gamma \) is independent at \( \lambda \) if

\[
\theta_\gamma v_\lambda \notin \sum_{\gamma' \in B(\lambda)^{-}\gamma} U(g)\theta_{\gamma'} v_\lambda.
\]

**Proposition 6.2.** If \( \gamma' \prec \gamma \) with \( \gamma' \in B(\lambda) \), then \( \theta_\gamma v_\lambda \in U(g)\theta_{\gamma'} v_\lambda \).

**Proof.** By hypothesis (6.1) holds. Thus the result follows from Theorem 6.1.

By the Proposition, if we are interested in the independence of the Šapovalov elements \( \theta_\gamma \) for distinct isotropic roots, it suffices to study only \( \lambda \)-minimal roots \( \gamma \).

We order the positive roots of \( g \) as in Theorem 4.12, thus for each \( \pi \in \overline{P}(\gamma) \) the odd root vector is the last factor of \( e_{-\pi} \), that is we have \( e_{-\pi} \in U(n^-_0)n^-_1 \).

**Lemma 6.3.** If \( \gamma \) is \( \lambda \)-minimal, then \( e_{-\gamma} v_\lambda \) occurs with non-zero coefficient in \( \theta_\gamma v_\lambda \).

**Proof.** Assume \( \gamma = \epsilon_r - \delta_s \). Then if \( \alpha = \epsilon_r - \epsilon_i \) with \( r < i \), or \( \alpha = \delta_j - \delta_s \) with \( j < s \) we have \( (\lambda + \rho, \alpha^\vee) \neq 0 \), since \( \gamma \) is \( \lambda \)-minimal. Thus by Equation (4.27) and Theorem 4.12 the coefficient of \( e_{-\gamma} v_\lambda \) in \( \theta_\gamma v_\lambda \) is non-zero.
**Theorem 6.4.** The isotropic root $\gamma$ is independent at $\lambda$ if and only if $\gamma$ is $\lambda$-minimal.

**Proof.** Set $B = B(\lambda)^{-\gamma}$. If $\gamma$ is not $\lambda$-minimal then $\gamma$ is not independent at $\lambda$ by Proposition 6.2. Suppose that $\gamma$ is $\lambda$-minimal and

$$\theta_\gamma v_{\lambda} \in \sum_{\gamma' \in B} U(\mathfrak{g}) \theta_{\gamma'} v_{\lambda} = \sum_{\gamma' \in B^+} U(\mathfrak{g}^-) \theta_{\gamma'} v_{\lambda}.$$ 

Hence by Lemma 6.3 there are $a_\pi \in \mathbb{k}$ such that

$$e^{-\gamma} v_{\lambda} = \sum_{\pi \in \mathbf{P}(\gamma) : e^{-\pi} \neq e^{-\gamma}} a_\pi e^{-\pi} v_{\lambda}.$$ 

This contradicts the fact that the set $\{e^{-\pi} v_{\lambda} | \pi \in \mathbf{P}(\gamma)\}$ is a basis for $M(\lambda)^{\lambda-\gamma}$. \qed

6.2 Survival of Šapovalov elements in Kac modules.

We have $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, where $\mathfrak{g}_1^+$ (resp. $\mathfrak{g}_1^-$) is the set of block upper (resp. lower) triangular matrices. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices, and set $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1^+$. Next let

$$P^+ = \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha^\vee) \in \mathbb{Z}, (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Delta_0^+\}$$

For $\lambda \in P^+$, let $L^0(\lambda)$ be the (finite dimensional) simple $\mathfrak{g}_0$-module with highest weight $\lambda$. Then $L^0(\lambda)$ is naturally a $\mathfrak{p}$-module and we define the Kac module $K(\lambda)$ by

$$K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L^0(\lambda).$$

By a corollary to the PBW Theorem, see [Mus12] Corollary 6.1.5, we have as a vector space

$$K(\lambda) = \Lambda(\mathfrak{g}_1^-) \otimes L^0(\lambda).$$

The next result is well-known. Indeed two methods of proof are given in Theorem 4.37 of [Bru03]. The second of these is based on Theorem 5.5 in [Ser96]. We give a short proof using Theorem 4.14. Assume that the roots are ordered as in Theorem 4.14, that is with the odd root vector first.

**Theorem 6.5.** Let $\gamma = \epsilon_r - \delta_s$, and suppose $\lambda$ and $\lambda - \gamma$ belong to $P^+$ and $(\lambda + \rho, \gamma) = 0$. Let the highest weight vector $v_{\lambda}$ be the highest weight vector in $K(\lambda)$ of weight $\lambda$.

Then

(a) There is a non-zero map of $\mathfrak{g}$-modules $K(\lambda - \gamma) \to K(\lambda)$, sending $xv_{\lambda-\gamma}$ to $x^0_\gamma v_{\lambda}$.

(b) $$[K(\lambda) : L(\lambda - \gamma)] \neq 0.$$
Proof. Let $\theta_\gamma$ be the in the Šapovalov element for the root $\gamma$. Because of the way the roots are ordered, applying the right side of (4.36) to the highest weight vector $v_\lambda \in K(\lambda)$ yields

$$\theta_\gamma v_\lambda \in \mathfrak{g}_1^- \otimes L^0(\lambda) \subseteq \Lambda \mathfrak{g}_1^- \otimes L^0(\lambda).$$

Therefore to prove (a) it suffices to show that the coefficient of $e_{-\gamma}v_\lambda = e_{-\gamma} \otimes v_\lambda$ in $\theta_\gamma v_\lambda$ is nonzero. Now the term $e_{-\gamma}$ arises in the sum (4.36) when $I \in \mathcal{I}$ is as small as possible, that is $I = \{m + s, r\}$. In this case we have, by (4.19) and (4.34)

$$H_I = \prod_{k=1}^{m-r} (h_{\epsilon_r - \epsilon_{r+k}} + \rho(h_{\epsilon_r - \epsilon_{r+k}} - 1) \prod_{j=1}^{s-1} (h_{\delta_j - \delta_s} + \rho(h_{\delta_j - \delta_s} + 1).$$

Thus the coefficient of $e_{-\gamma}v_\lambda$ is

$$H_I(\lambda) = \prod_{k=1}^{m-r} ((\lambda + \rho, \epsilon_r - \epsilon_{r+k}) - 1) \prod_{j=1}^{s-1} ((\lambda + \rho, \delta_j - \delta_s) + 1).$$

Since $\lambda \in P^+$, the only factors in this product that could be zero arise when $k = 1$ or $j = s - 1$. Let $\sigma = \epsilon_r - \epsilon_{r+1}$ and $\tau = \delta_{s-1} - \delta_s$. Then the only factors that could be zero are

$$(\lambda + \rho, \sigma) - 1 = (\lambda, \sigma) \text{ and } (\lambda + \rho, \tau) + 1 = (\lambda, \tau).$$

However $(\gamma, \sigma^\vee) = (\gamma, \tau^\vee) = 1$. So if this happens, then $\lambda - \gamma, \sigma^\vee) = -1$ or $(\lambda - \gamma, \tau^\vee) = -1$, and $\lambda - \gamma \notin P^+$. The claim in (b) follows immediately from (a). \qed

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