Moments of Poisson stochastic integrals with random integrands

Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

May 10, 2014

Abstract

We show that the moment of order \( n \) of the Poisson stochastic integral of a random process \((u_x)_{x \in X}\) over a metric space \( X \) is given by

\[
E \left[ (\int_X u_x(\omega) d\omega)^n \right] = \sum_{P_1, \ldots, P_k} E \left[ \int_{X^k} \varepsilon_{s_1}^+(u_{s_1}^{P_1}) \cdots u_{s_k}^{P_k}) \sigma(ds_1) \cdots \sigma(ds_k) \right],
\]

(0.1)

where the sum runs over all partitions \( P_1 \cup \cdots \cup P_k \) of \( \{1, \ldots, n\} \), \(|P_i|\) denotes the cardinality of \( P_i \), and \( \varepsilon_{s_k}^+ \) is the operator that acts by addition of points at \( s_1, \ldots, s_k \) to Poisson configurations. This formula recovers known results in case \((u(x))_{x \in X}\) is a deterministic function on \( X \).

Key words: Poisson stochastic integrals, moment identities, Skorohod integral.

Mathematics Subject Classification (2010): 60G57, 60G55, 60H07.

1 Introduction

Let \( \Omega^X \) denote the configuration space on a \( \sigma \)-compact metric space \( X \) with Borel \( \sigma \)-algebra \( B(X) \) and a \( \sigma \)-finite diffuse measure \( \sigma \), i.e. \( \Omega^X \) is the space of at most countable and locally finite subsets of \( X \), defined as

\[
\Omega^X = \{ \omega = (x_i)_{i=1,\ldots,N} \subset X, \; x_i \neq x_j \; \forall i \neq j, \; N \in \mathbb{N} \cup \{\infty\} \}.
\]
Each element $\omega$ of $\Omega^X$ is identified with the Radon point measure 

$$
\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i},
$$

where $\epsilon_x$ denotes the Dirac measure at $x \in X$ and $\omega(X) \in \mathbb{N} \cup \{\infty\}$ represents the cardinality of $\omega(X)$. The space $\Omega^X$ is endowed with the Poisson probability measure $\pi_\sigma$ on $X$ such that for all compact disjoint subsets $A_1, \ldots, A_n$ of $X$, $n \geq 1$, the mapping

$$
\omega \mapsto (\omega(A_1), \ldots, \omega(A_n))
$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective intensities $\sigma(A_1), \ldots, \sigma(A_n)$.

In [1] the moment formula

$$
E\left[ \left( \int_X f(x)\omega(dx) \right)^n \right] = n! \sum_{r_1+2r_2+\cdots+n r_n = n} \prod_{k=1}^{n} \frac{1}{(k!)^{r_k} r_k!} \left( \int_X f^k(x)\sigma(dx) \right)^{r_k}
$$

(1.1)

has been proved for $f : X \to \mathbb{R}$ a deterministic sufficiently integrable function. The proof of [1] relies on the Lévy-Khintchine representation of the Laplace transform of $\int_X u(x)\omega(dx)$, and this result can also be recovered under a different combinatorial interpretation by the Faà di Bruno formula, cf. e.g. § 2.4 of [5], from the relation

$$
E\left[ \left( \int_X u(x)\omega(dx) \right)^n \right] = \sum_{P_1, \ldots, P_n} \int_X u|P_1|(x)\sigma(dx) \cdots \int_X u|P_n|(x)\sigma(dx),
$$

(1.2)

between the moments and the cumulants $\kappa_n = \int_X u^n(x)\sigma(dx)$, $n \geq 1$, of $\int_X u(x)\omega(dx)$, where the sum runs over all partitions $P_1 \cup \cdots \cup P_n$ of $\{1, \ldots, n\}$.

Recently, (1.1) has been applied to control the $p$-variation and the number of crossings of fractional Poisson and shot noise processes with deterministic kernels, cf. [3].

In this paper we extend the above formula (1.1) to random integrands. Namely, we state that given $u : \Omega^X \times X \to \mathbb{R}$ a sufficiently integrable random process we have

$$
E\left[ \left( \int_X u(x)\omega(dx) \right)^n \right] = \sum_{P_1, \ldots, P_n} E\left[ \int_{X^k} \varepsilon^+_g ( u|P_1| \cdots u|P_n| ) \sigma(ds_1) \cdots \sigma(ds_k) \right],
$$

(1.3)
cf. Proposition 3.1 below, where the sum runs over all (disjoint) partitions \( P_1 \cup \cdots \cup P_k \) of \( \{1, \ldots, n\} \), \( k = 1, \ldots, n \), and \( |P| \) denotes the cardinal of \( P \subset \{1, \ldots, n\} \). In (1.3), \( \varepsilon^+_s \) is the addition operator defined on any random variable \( F : \Omega^X \to \mathbb{R} \) by
\[
\varepsilon^+_{s_k} F(\omega) = F(\omega \cup \{s_1, \ldots, s_k\}), \quad \omega \in \Omega^X, \quad s_1, \ldots, s_k \in X,
\]
where
\[
s_k = (s_1, \ldots, s_k) \in X^k, \quad k \geq 1.
\]
As expected, \((u(x))_{x \in X}\) is a deterministic function we have
\[
\varepsilon^+_s u(s_i) = u(s_i), \quad 1 \leq i \leq k,
\]
in which case (1.3) recovers (1.2).

Examples

In the case of second order moments, (1.3) yields
\[
E \left[ \left( \int_X u_x(\omega) \omega(dx) \right)^2 \right] = E \left[ \int_X \varepsilon^+_s u^2 \sigma(ds) \right] + E \left[ \int_{X^2} \varepsilon^+_s \varepsilon^+_t (u_{s_1} u_{s_2}) \sigma(ds_1) \sigma(ds_2) \right],
\]
which, if \( \lambda := \sigma(X) < \infty \), recovers
\[
E \left[ (\omega(X))^6 \right] = E \left[ \left( \int_X u_x(\omega) \omega(dx) \right)^3 \right]
\]
\[= E \left[ \int_X \varepsilon^+_s (\omega(X))^2 \sigma(ds) \right] + E \left[ \int_{X^2} \varepsilon^+_s \varepsilon^+_t (\omega(X))^2 \sigma(ds_1) \sigma(ds_2) \right]
\]
\[= \lambda E \left[ (\omega(X) + 1)^3 \right] + \lambda^2 E \left[ (\omega(X) + 2)^3 \right]
\]
\[= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4
\]
\[= B_4(\lambda),
\]
by taking \( u_x(\omega) = \omega(X) \), \( x \in X \), where \( B_4 \) is the Bell polynomial of order 4, cf. (4.3) below.

Concerning third order moments, (1.3) shows that
\[
E \left[ \left( \int_X u_x(\omega) \omega(dx) \right)^3 \right] = E \left[ \int_X \varepsilon^+_s u^3 \sigma(ds) \right] + 3E \left[ \int_{X^2} \varepsilon^+_s \varepsilon^+_t (|u_{s_1}|^2 u_{s_2}) \sigma(ds_1) \sigma(ds_2) \right]
\]
3
\[ + E \left[ \int_{X^3} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ \varepsilon_{s_3}^+ (u_{s_1} u_{s_2} u_{s_3}) \sigma(ds_1) \sigma(ds_2) \sigma(ds_3) \right], \]

which, if \( \lambda := \sigma(X) < \infty \), and taking \( u_x(\omega) = \omega(X), x \in X \), yields

\[
E [(\omega(X))^6] = E \left[ \left( \int_X u_x(\omega) \omega(dx) \right)^3 \right]
\]

\[
= E \left[ \int_X \varepsilon_{s_1}^+ (\omega(X))^3 \sigma(ds) \right] + 3E \left[ \int_{X^2} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ ((\omega(X))^2 \omega(X)) \sigma(ds_1) \sigma(ds_2) \right]
\]

\[
+ E \left[ \int_X \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ \varepsilon_{s_3}^+ (\omega(X))^3 \sigma(ds_1) \sigma(ds_2) \sigma(ds_3) \right]
\]

\[
= \lambda E \left[ (\omega(X) + 1)^3 \right] + 3\lambda^2 E \left[ (\omega(X) + 2)^2 (\omega(X) + 2) \right]
\]

\[
+ \lambda^3 E \left[ (\omega(X) + 3)^3 \right]
\]

\[
= \lambda + 31\lambda^2 + 90\lambda^3 + 65\lambda^4 + 15\lambda^5 + \lambda^6
\]

\[
= B_6(\lambda),
\]

where \( B_6 \) is the Bell polynomial of order 6.

We proceed as follows. In Section 2 we rewrite a result of [10] on the moments of compensated Poisson-Skorohod integrals in the language of set partitions. In Section 3 we deduce formulas for non-compensated integrals of random integrands by binomial inversion. In the case of deterministic integrands and indicator functions, in Section 4 we recover and extend known relations between the moments of the Poisson distribution and Stirling numbers.

The moment identities in this paper are stated for bounded random variables \( F \) and processes \( u \) with compact support, however they can be extended by assuming suitable conditions ensuring that the right hand side of the formula is finite.

## 2 Poisson-Skorohod integrals

Our proof of moment identities relies on the Skorohod integral operator \( \delta \) which is defined on any measurable process \( u : \Omega^X \times X \rightarrow \mathbb{R} \) by the expression

\[
\delta(u) = \int_X u_x(\omega \setminus x) \omega(dx) - \int_X u_x(\omega) \sigma(dx), \quad (2.1)
\]
provided $E\left[ \int_X |u_x(\omega)|\sigma(dx) \right] < \infty$, cf. Corollary 1 of [7]. In (2.1), $\omega \setminus x$ denotes the configuration $\omega \in \Omega^X$ after removal of the point $x$ in case $x \in \omega$.

We start with a moment identity for compensated Poisson-Skorohod integrals, obtained by rewriting Theorem 5.1 of [10] using set partitions. By saying that $u : \Omega^X \times X \to \mathbb{R}$ has a compact support in $X$ we mean that there exists a compact subset $K$ of $X$ such that $u_x(\omega) = 0$ for all $\omega \in \Omega^X$ and $x \in X \setminus K$.

**Proposition 2.1** Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable and let $u : \Omega^X \times X \to \mathbb{R}$ be a bounded process with compact support in $X$. For all $n \geq 0$ we have

$$E[\delta(u)^n F] = \sum_{c=0}^{n} (-1)^c \binom{n}{c} \sum_{k=0}^{n-c} \sum_{l_1, \ldots, l_k = n-c \atop l_1, \ldots, l_k \geq 1} N_{k_c} E\left[ \int_{X^{k+c}} \varepsilon_{s_k}^+ F \prod_{p=1}^{k+c} \varepsilon_{s_p}^+ u_{s_p}^{l_p} d\sigma^{k+c}(s_{k+c}) \right],$$

where $d\sigma^b(s_b) = \sigma(ds_1) \cdots \sigma(ds_b)$, $\mathcal{L}_k = (l_1, \ldots, l_k)$, and $N_{k_c}$ is the number of partitions of a set of $l_1 + \cdots + l_k$ elements into $k$ subsets of lengths $l_1, \ldots, l_k \geq 1$.

**Proof.** The proof of this formula relies on the identity

$$E[\delta(u)^n F] = \sum_{k=0}^{n} \sum_{b=k}^{n} (-1)^{b-k} \sum_{l_1 + \cdots + l_k = n-(b-k) \atop l_1, \ldots, l_k \geq 1 \atop l_{k+1}, \ldots, l_b = 1} C_{k,b} E\left[ \int_{X^{b+c}} \varepsilon_{s_b}^+ F \prod_{p=1}^{b} \varepsilon_{s_p}^+ u_{s_p}^{l_p} d\sigma^b(s_b) \right]$$

(2.2)

for the moments of the compensated Poisson-Skorohod integral $\delta(u)$, cf. Theorem 5.1 of [10] and Theorem 1 of [8], where

$$C_{k,b} = \sum_{0=r_{c+1} < \cdots < r_0 = k+c+1}^{c} \prod_{q=0}^{r_q + q - c - 1} \prod_{p=r_q+1+q-c+1}^{r_q} l_1 + \cdots + l_p + q - 1 \left( l_1 + \cdots + l_{p-1} + q \right).$$

(2.3)

Next we note that $C_{k,b}$ defined in (2.3) represents the number of partitions of a set of $n = l_1 + \cdots + l_k + c$ elements into $k$ subsets of lengths $l_1, \ldots, l_k$ and $c$ singletons, hence when $l_1 + \cdots + l_k = n - c$ we have

$$C_{k,b} = \binom{n}{c} N_{k_c},$$
since $\mathcal{N}_{\delta_k}$ is the number of partitions of a set of $l_1 + \cdots + l_k = n - c$ elements into $k$ subsets of lengths $l_1, \ldots, l_k$. Hence we have

$$E[\delta(u)^n F] = \sum_{k=0}^{n} \sum_{b=k}^{n-c} (-1)^{b-k} \sum_{l_1+\cdots+l_b=n-(b-k)}^{k} C_{\delta k, b} E \left[ \int_{X^b} \varepsilon_{s_k}^+ F \prod_{p=1}^{b} \varepsilon_{s_k \setminus s_p}^{l_p} u_{s_p}^{l_p} d\sigma^b(s_b) \right]$$

$$= \sum_{c=0}^{n} \sum_{k=0}^{n-c} (-1)^c \sum_{l_1+\cdots+l_k=n-c}^{k+c} C_{\delta k, k+c} E \left[ \int_{X^{k+c}} \varepsilon_{s_k}^+ F \prod_{p=1}^{k+c} \varepsilon_{s_k \setminus s_p}^{l_p} u_{s_p}^{l_p} d\sigma^{k+c}(s_{k+c}) \right]$$

$$= \sum_{c=0}^{n} (-1)^c \binom{n}{c} \sum_{k=0}^{n} \sum_{l_1+\cdots+l_k=n-c}^{k+c} \mathcal{N}_{\delta k} E \left[ \int_{X^{k+c}} \varepsilon_{s_k}^+ F \prod_{p=1}^{k+c} \varepsilon_{s_k \setminus s_p}^{l_p} u_{s_p}^{l_p} d\sigma^{k+c}(s_{k+c}) \right].$$

The proof of (2.2) relies on the duality relation

$$E[(DF, u)_{L^2(X)}] = E[F\delta(u)], \quad (2.4)$$

between $\delta$ and the finite difference gradient

$$D_x F(\omega) = \varepsilon_x^+ F(\omega) - F(\omega), \quad \omega \in \Omega^X, \quad x \in X, \quad (2.5)$$

for all $F$ and $u$ in the respective closed $L^2$ domains $\text{Dom}(\delta) \subset L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)$ and $\text{Dom}(D) \subset L^2(\Omega^X, \pi_\sigma)$ of $D$ and $\delta$, cf. [10] and references therein.

### 3 Pathwise integrals

The next Proposition 3.1 is the main result of this paper, it yields (1.3) and follows directly by binomial inversion of Proposition 2.1. We state its proof due to the additional presence of expectations.

**Proposition 3.1** Let $F : \Omega^X \to \mathbb{R}$ be a bounded random variable, and let $u : \Omega^X \times X \to \mathbb{R}$ be a bounded random process with compact support in $X$. For all $n \geq 0$ we have

$$E \left[ F \left( \int_X u_x(\omega)(dx) \right)^n \right] = \sum_{P_1, \ldots, P_k} E \left[ \int_X \varepsilon_{s_k}^+ (F u_{P_1}^{P_1} \cdots u_{s_k}^{P_k}) \sigma(ds_1) \cdots \sigma(ds_k) \right].$$
Proof. We have, applying Proposition 2.1 at the rank \( n - i \) to the process \( \varepsilon^+ u = (\varepsilon^+ u_x)_{x \in X} \),
\[
E \left[ F \left( \int_X u_x(\omega) \omega(dx) \right)^n \right] = E \left[ F \left( \delta(\varepsilon^+ u) + \int_{X} \varepsilon^+_x u_x \sigma(dx) \right)^n \right] 
= \sum_{i=0}^{n} \binom{n}{i} E \left[ F(\delta(\varepsilon^+ u))^{n-i} \left( \int_{X} \varepsilon^+_x u_x \sigma(dx) \right)^i \right] 
= \sum_{i=0}^{n} \binom{n}{i} \sum_{c=0}^{n-i} (-1)^c \binom{n-i}{c} \sum_{k=0}^{n-i-c} \mathcal{N}_{\mathcal{L}_k} E \left[ \int_{X^{k}} \varepsilon^+_{\mathcal{L}_k} \left( F \left( \int_{X} \varepsilon^+_x u_x \sigma(dx) \right)^i \left( \int_{X} \varepsilon^+_x u_x \sigma(dx) \right)^{a-i} \right) \prod_{p=1}^{k} \varepsilon^+_p u_p \sigma^k(s_k) \right] 
= \sum_{a=0}^{n} \binom{n}{a} \sum_{k=0}^{n} \sum_{i=0}^{a} (-1)^{a-i} \binom{n}{a} \sum_{k=0}^{n-a} \mathcal{N}_{\mathcal{L}_k} E \left[ \int_{X^{k}} \varepsilon^+_{\mathcal{L}_k} \left( F \left( \int_{X} \varepsilon^+_x u_x \sigma(dx) \right)^{a} \right) \varepsilon^+_k \prod_{p=1}^{k} u_p \sigma^k(s_k) \right] 
= \sum_{k=0}^{n} \sum_{l_1+\ldots+l_k=n-a} \mathcal{N}_{\mathcal{L}_k} E \left[ \int_{X^{k}} \varepsilon^+_{\mathcal{L}_k} \left( F \prod_{p=1}^{k} u_p \right) \sigma^k(s_k) \right],
\] (3.1)
since \( \sum_{i=0}^{a} (-1)^{a-i} \binom{a}{i} = 1_{\{a=0\}} \) with the convention \( 0^0 = 1 \).

When \( u : X \to \mathbb{R} \) is a deterministic function, Proposition 2.1 yields
\[
E \left[ F \left( \int_X u(x) \omega(dx) \right)^n \right] = \sum_{P_1,\ldots,P_a} \int_{X^n} u_{s_{1}}^{P_1} \ldots u_{s_{a}}^{P_a} E \left[ (\varepsilon^+_X F) \sigma(ds_1) \ldots \sigma(ds_a) \right],
\]
which recovers (1.1) by taking \( F = 1 \), and
\[
\text{Cov} \left( F, \left( \int_X u(x) \omega(dx) \right)^m \right) = \sum_{P_1,\ldots,P_a} \int_{X^n} u_{s_{1}}^{P_1} \ldots u_{s_{a}}^{P_a} E \left[ (\varepsilon^+_X F - F) \sigma(ds_1) \ldots \sigma(ds_a) \right].
\]
By (3.1), Proposition 3.1 also rewrites for compensated integrals as follows.

**Proposition 3.2** Let \( F : \Omega^X \to \mathbb{R} \) be a bounded random variable, and let \( u : \Omega^X \times X \to \mathbb{R} \) be a bounded random process with compact support in \( X \). For all \( n \geq 0 \) we have

\[
E \left[ F \left( \int_X u_x(\omega)(\omega(dx) - \sigma(dx)) \right)^n \right] = \sum_{c=0}^n (-1)^c \binom{n}{c} \sum_{P_1, \ldots, P_c \subset \{1, \ldots, n-c\}} E \left[ \int_{X^{a_1}} \varepsilon_{a_1}^+ \left( F \left( \int_X u_x(\omega)\sigma(dx) \right)^c u_{s_1}^{[P_1]} \cdots u_{s_n}^{[P_n]} \right) d\sigma^a(s_n) \right].
\]

**Proof.** By (3.1) we have

\[
E \left[ F \left( \int_X u_x(\omega)(\omega(dx) - \sigma(dx)) \right)^n \right] = \sum_{c=0}^n (-1)^c \binom{n}{c} \left( \int_X u_x(\omega)\sigma(dx) \right)^{n-c} \sum_{a=0}^n \sum_{l_1 + \cdots + l_a = n-c} \cdot \cdot \cdot \sum_{l_a+1, \ldots, l_a+c = 1} E \left[ \int_{X^{a+c}} \varepsilon_{a+c}^+ \left( F \prod_{p=1}^{a+c} u_{s_p}^{[P_p]} \right) d\sigma^{a+c}(s_{a+c}) \right].
\]

The next proposition specializes the above result to the case of deterministic integrands. We note that it can also be obtained independently as in (1.2) from the relation between the moments and the cumulants \( \kappa_1 = 0, \kappa_n = \int_X u^n(x)\sigma(dx), n \geq 2, \) of \( \int_X u(x)(\omega(dx) - \sigma(dx)) \).

**Proposition 3.3** Let \( f : X \to \mathbb{R} \) be a bounded deterministic function with compact support on \( X \). For all \( n \geq 1 \) we have

\[
E \left[ \left( \int_X f(x)(\omega(dx) - \sigma(dx)) \right)^n \right] = \sum_{P_1, \ldots, P_k} \int_X f^{[P_1]}(x)\sigma(dx) \cdots \int_X f^{[P_k]}(x)\sigma(dx),
\]

where the sum runs over all partitions \( P_1 \cup \cdots \cup P_k \) of \( \{1, \ldots, n\} \) of size at least 2.
Proof. By Proposition 3.1 and binomial inversion we have

\[ E \left[ \left( \int_X f(x) \omega(dx) - \int_X f(x) \sigma(dx) \right)^n \right] \]

\[ = \sum_{c=0}^{n} (-1)^c \binom{n}{c} \left( \int_X f(x) \sigma(dx) \right)^c E \left[ \left( \int_X f(x) \omega(dx) \right)^{n-c} \right] \]

\[ = \sum_{c=0}^{n} (-1)^c \binom{n}{c} \left( \int_X f(x) \sigma(dx) \right)^c \sum_{l_1+\ldots+l_a = n-c} N_{2a} \prod_{p=1}^{a} f_i^p(s_p) d\sigma^a(s_a) \]

\[ = \sum_{c=0}^{n} (-1)^c \binom{n}{c} \sum_{k=0}^{n-c} \binom{n-c}{k} \left( \int_X f(x) \sigma(dx) \right)^k \sum_{l_1+\ldots+l_a = n-c-k} N_{2a} \prod_{p=1}^{a} f_i^p(s_p) d\sigma^a(s_a) \]

\[ = \sum_{b=0}^{n} \binom{n}{b} \sum_{c=0}^{b} (-1)^c \binom{b}{c} \left( \int_X f(x) \sigma(dx) \right)^b \sum_{l_1+\ldots+l_a = n-b} N_{2a} \prod_{p=1}^{a} f_i^p(s_p) d\sigma^a(s_a) \]

\[ = \sum_{b=0}^{n} \binom{n}{b} \sum_{c=0}^{b} (-1)^c \binom{b}{c} \left( \int_X f(x) \sigma(dx) \right)^b \sum_{l_1+\ldots+l_a = n-b} N_{2a} \prod_{p=1}^{a} f_i^p(s_p) d\sigma^a(s_a) \]

\[ = \sum_{l_1+\ldots+l_a = n} N_{2a} \prod_{p=1}^{a} f_i^p(s_p) d\sigma^a(s_a). \]

\[ \square \]

4 Indicator functions and polynomials

When \( u(x) = 1_A(x) \) is a deterministic indicator function with \( A \in \mathcal{B}(X) \),

\[ Z := \int_X u(x) \omega(dx) = \int_X 1_A(x) \omega(dx) = \omega(A) \]

is a Poisson random variable with intensity \( \lambda = \sigma(A) < \infty \), and Proposition 3.1 yields the following corollary.

Corollary 4.1 Let \( F : \Omega^X \to \mathbb{R} \) be a bounded random variable. We have

\[ E[FZ^n] = \sum_{k=0}^{n} S(n, k) \int_{A^k} E[\varepsilon_{s_k}^+ F] \sigma(ds_1) \cdots \sigma(ds_k), \quad n \in \mathbb{N}. \]
where $S(n, k)$ denotes the Stirling number of the second kind, i.e. the number of ways to partition a set of $n$ objects into $k$ non-empty subsets.

Proof. By Proposition 3.1 we have

$$E[FZ^n] = \sum_{P_1, \ldots, P_k} E\left[\int_{A^k} \varepsilon_{s_k} F \sigma(ds_1) \cdots \sigma(ds_k)\right]$$

$$= \sum_{k=0}^{n} \sum_{l_1 + \cdots + l_k = n \atop l_1, \ldots, l_k \geq 1} N\varepsilon_{s_k} \int_{A^k} E\left[\varepsilon_{s_k} F\right] \sigma(ds_1) \cdots \sigma(ds_k),$$

and it remains to note that

$$S(n, k) = \sum_{l_1, \ldots, l_k} N\varepsilon_{l_1, \ldots, l_k}, \quad 0 \leq k \leq n.$$

□

As a consequence of Relation (4.1) we find

$$\text{Cov}(F, Z^n) = \sum_{k=0}^{n} S(n, k) \int_{A^k} E\left[\varepsilon_{s_k} F - F\right] \sigma(ds_1) \cdots \sigma(ds_k), \quad n \in \mathbb{N},$$

and

$$E[F e^{tZ}] = \sum_{k=0}^{\infty} \frac{1}{k!} (e^t - 1)^k \int_{A^k} E\left[\varepsilon_{s_k} F\right] \sigma(ds_1) \cdots \sigma(ds_k), \quad (4.2)$$

using e.g. Relation (3) page 2 of [2]. Relation (4.2) also recovers the decomposition of the Fourier transform (also called $U$-transform) on the Poisson space, cf. e.g. Proposition 3.2 of [4].

When $F$ has the form $F = f(Z)$ with $f : \mathbb{N} \to \mathbb{R}$, Relation (4.1) also yields an extended Chen-Stein identity (see e.g. Lemma 3.3.3 of [6]), as

$$E[Z^n f(Z)] = \sum_{k=0}^{n} \lambda^k S(n, k) E[f(Z + k)],$$

and

$$E[f(Z) e^{tZ}] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (e^t - 1)^k E[f(Z + k)].$$
In particular, in case $F = 1$, (4.1) corresponds to the classical relation

$$E[Z^n] = B_n(\lambda), \quad n \in \mathbb{N},$$

between the Poisson moments and the Bell polynomials

$$B_n(\lambda) = \sum_{k=0}^{n} \lambda^k S(n, k), \quad n \in \mathbb{N},$$

(4.3)

cf. e.g. Proposition 3.3.2 of [6] and references therein. The comparison of (4.1) and (4.3) yields the relation

$$S(n, k) = n! \sum_{r_1+2r_2+\cdots+n r_n = n \atop r_1+2r_2+\cdots+r_n = k} \prod_{k=1}^{n} \frac{1}{(k!)^{r_k} r_k!} = n! \sum_{r_1+2r_2+\cdots+(n-k+1)r_{n-k+1} = n \atop r_1+2r_2+\cdots+r_{n-k+1} = k} \prod_{k=1}^{n} \frac{1}{(k!)^{r_k} r_k!},$$

cf. e.g. Proposition 2.3.4 of [6].

Similarly, Proposition 3.3 applied to $u(x) = 1_{A}(x)$ shows that

$$E[(Z - \lambda)^n] = \sum_{k=0}^{n} \lambda^k S_2(n, k), \quad n \in \mathbb{N},$$

which recovers the fact that the centered moments of a Poisson random variable can be written using the number $S_2(n, k)$ of partitions of a set of size $n$ into $k$ non-singleton subsets, cf. [9] and Proposition 3.3.6 of [6].

References

[1] B. Bassan and E. Bona. Moments of stochastic processes governed by Poisson random measures. Comment. Math. Univ. Carolin., 31(2):337–343, 1990.

[2] M. Bernstein and N. J. A. Sloane. Some canonical sequences of integers. Linear Algebra Appl., 226/228:57–72, 1995.

[3] H. Biermé, Y. Demichel, and A. Estrade. Fractional Poisson field on a finite set. Preprint hal-00597722, 2011.

[4] Y. Ito. Generalized Poisson functionals. Probab. Theory Related Fields, 77:1–28, 1988.

[5] E. Lukacs. Characteristic functions. Hafner Publishing Co., New York, 1970. Second edition, revised and enlarged.

[6] G. Peccati and M. Taqqu. Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation. Bocconi & Springer Series. Springer, 2011.
[7] J. Picard. Formules de dualité sur l’espace de Poisson. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(4):509–548, 1996.

[8] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.

[9] N. Privault. Generalized Bell polynomials and the combinatorics of Poisson central moments. *Electron. J. Combin.*, 18(1):Research Paper 54, 10, 2011.

[10] N. Privault. Invariance of Poisson measures under random transformations. Preprint arXiv:1004.2588v3, to appear in Ann. Inst. H. Poincaré Probab. Statist., 2011.