ON HOMOCLINIC POINTS, RECURRENCES AND CHAIN RECURRENCES OF VOLUME-PRESERVING DIFFEOMORPHISMS WITHOUT GENERICITY

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Abstract. Let $M$ be a manifold with a volume form $\omega$ and $f : M \to M$ be a diffeomorphism of class $C^1$ that preserves $\omega$. In this paper, we do not assume $f$ is $C^1$-generic. We have two main themes in the paper: (1) the chain recurrence; (2) relations among recurrence points, homoclinic points, shadowability and hyperbolicity. For (1) (without assuming $M$ is compact), we have the theorem: if $f$ is Lagrange stable, then $M$ is a chain recurrent set. If $M$ is compact, then the Lagrange-stability is automatic. For (2) (assuming the compactness of $M$), we prove some various implications among notions, such as: (i) the $C^1$-stable shadowability equals to the hyperbolicity of $M$; (ii) if a point $p \in M$ has a recurrence point in the unstable manifold $W^u(p, f)$ and there is no homoclinic point of $p$, then $f$ is nonshadowable; (iii) if $f$ has the shadowing property and $p$ has a recurrence point in $W^u(p, f)$, then the recurrent point is in the limit set of homoclinic points of $p$.

1. Introduction

An important part of the study of Dynamical system is to understand the structures of the stable manifolds (and the unstable manifolds), attractors and chain recurrent sets of the dynamical system. In order to grasp the geometric structure in Dynamical system, the above concepts have played an important role in the field bearing the useful properties.

Our purpose of this paper is to elucidate the relation between the set of homoclinic points and the set of recurrence points, and to know the existence of the chain recurrence points. In the both themes, we want to maintain the volume-preserving (or symplectic) dynamics, but to drop the genericity assumption. Since the symplectic diffeomorphisms are automatically volume-preserving, our results in the paper is applicable to the symplectic dynamics as well.

In this paper, we follow the two basic definitions of attractors and chain recurrences due to Conley [Co]. Note that even before Conley’s definition,
other definitions of attractor can be found in several papers (see, for reference, [Mi]) while the equivalence of those definitions partly remain as conjecture.

By a theorem of Moriyasu [Mo], one can see that if the chain recurrent set $CR(f)$ is $C^1$-stably shadowable, then it is hyperbolic (see [WGW]). In [NN], Nakayama and Noda proved that for a minimal flow on a closed orientable 3-manifold, a chain recurrent set of the induced projective flow is the whole projectivized bundle. Recently, it is proved in [WGW] that if a chain component is $C^1$-stably shadowable, then the chain component is hyperbolic. Other results for the $C^1$-stable shadowing properties can be found in [LMS] and [Sa].

We obtain our first result of this article as follows.

**Theorem 3.18.** Let $M$ be a compact manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. Assume that every chain component has a hyperbolic periodic point. If every chain component is $C^1$-stably shadowable, then $M$ is hyperbolic. Furthermore, the converse is also true.

By the diffeomorphisms throughout this paper, we always mean the $C^1$-diffeomorphisms.

From Conley’s theorem, we get easily the Chain recurrence theorem on the compact manifolds with volume-preserving diffeomorphisms, i.e., $M$ is chain recurrent for $f$. Furthermore, in fact, $M$ is nonwandering for $f$ by the Nonwandering theorem. However, unfortunately, the same statement for the chain recurrence does not hold for the non-compact manifolds with volume-preserving diffeomorphisms. In the non-compact case, we may consider a canonical notion, Lagrange stability, and using this concept, we prove the following theorem.

**Theorem 4.10.** Let $M$ be a manifold with a volume form $\omega$, and $f$ be a Lagrange-stable volume-preserving diffeomorphism on $M$. Then, $M$ is strongly chain recurrent for $f$, i.e., every point of $M$ is strongly chain recurrent with respect to $f$.

The above theorem follows from a stronger claim, Proposition 4.9. The proposition says that if the assumptions in Theorem 4.10 without the Lagrange-stability assumption hold, then almost everywhere in $U - A$ should have the unbounded orbit, where $A$ is an attractor and $U$ is an attractor block of $A$. I.e., the set of points of $U - A$ with bounded orbits is of measure 0.

Once focusing on dynamics on symplectic geometry, so called Hamiltonian dynamics, one realizes that the study of homoclinic points sits in a weighty place, which dates back to a century ago (see [BGW]). More generally, Poincaré raised a question: *Do transverse homoclinic points occur generically?* It is interpreted as whether transverse homoclinic points are
dense in both stable and unstable manifolds. He conjectured that the answer is yes. Xia [Xi1] answered positively for the Poincaré question on arbitrary compact manifolds with a generic volume-preserving diffeomorphism. His results in [Xi1] generalize Takens’ result [Ta]: in the generic volume-preserving dynamics on a compact manifold (it is called a discrete conservative system in [Ta]; Takens also treats Hamiltonian flows, a continuous conservative system), for a hyperbolic periodic point, the set of all homoclinic points of the hyperbolic periodic point are dense in both the stable and unstable manifolds.

In [Xi2], for the compact surface case (with volume-preserving diffeomorphisms), Xia shows even that the union of the stable manifolds (unstable manifolds) of hyperbolic periodic points are dense in the whole manifold.

Furthermore, it is proved that the existence of a residual subset $\mathcal{R}$ in the set of volume-preserving diffeomorphisms such that for any $f \in \mathcal{R}$ and any hyperbolic periodic point of $f$, the set of corresponding homoclinic points is dense in $M$ (SX).

There is a parallel question with the above discussion on homoclinic points: how generically do recurrence points occur? It turns out, due to Oliveira [Ol2], that the answer is that for a generic $C^1$-diffeomorphism preserving $\omega$, the positive (negative) recurrence points form a dense subset in the stable manifolds (unstable manifolds) of hyperbolic period points.

Lastly, in this paper, we prove the following theorem.

**Theorem 5.2** Let $M$ be a compact manifold with a volume form $\omega$ and $f : M \to M$ be a volume-preserving diffeomorphism. If $p$ has a recurrence point in $W^u(p, f)$ and there is no homoclinic point of $p$, then $f$ is nonshadowable.

In [BGW], for $C^1$-generic diffeomorphisms, it is proven that every chain recurrent class that has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with $\dim E^c = 1$ either is an isolated hyperbolic periodic orbit, or is accumulated by non-trivial homoclinic classes. For the recent investigations of the theory of volume-preserving or symplectic dynamics, see, for instance, [SX, XZ].

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2. **Definitions and Notations**

We fix the notations and definitions used throughout the paper.

Let $M$ be an $n$-dimensional differentiable manifold with a metric $d$, and $f : M \to M$ be a $C^1$-diffeomorphism. A volume form $\omega$ on $M$ is a nowhere vanishing $n$-form on $M$. A symplectic form $\omega$ on $M$ is a nowhere degenerate 2-form on $M$. Here, the non-degeneracy of $\omega$ is the same as its $(n/2)$-times wedge product $\omega^\wedge = \omega \wedge \cdots \wedge \omega$ defines a volume form on $M$. Thus, when we
say a symplectic form, \( n \) is assumed to be even. Integration along the subsets of \( M \) defines a Lebesgue measure \( m \). Indeed, by the para-compactness of \( M \), locally \( m \) is written as a product of a \( C^1 \)-function and the standard Lebesgue measure on \( \mathbb{R}^n \) (via the \( C^1 \)-transition). This clarifies a Lebesgue measurable subset of \( M \), a countable union of Lebesgue measurable subsets of \( \mathbb{R}^n \) (via the \( C^1 \)-transition). Thanks to the well-known theory of Lebesgue measures and Borel measures, one guarantees any compact subset of \( M \) is Lebesgue measurable and is of finite measure. By the compactness, the closed balls (with finite radii) are of finite measure, as well.

If one says \( f \) preserves \( \omega \), this means \( f^* \omega = \omega \). When \( \omega \) is a symplectic form, the \( \omega \)-preservation implies the volume-preservation. The volume-preservation of \( f \) amounts to the measure-preservation. In the case, for a Lebesgue measurable subset \( N \subset M \), we have \( m(N) = m(f(N)) \).

For \( p \in M \), the stable and unstable manifolds are defined by

\[
W^s(p, f) = \{ u \in X : d(f^n(u), f^n(p)) \to 0 \text{ as } n \to \infty \},
\]

\[
W^u(p, f) = \{ u \in X : d(f^{-n}(u), f^{-n}(p)) \to 0 \text{ as } n \to \infty \},
\]

respectively.

**Definition 2.1.** A sequence \((x_i)_{i \in \mathbb{Z}}\) in \( M \) is a \( \delta \)-pseudo-orbit of \( f \) if for all \( i \in \mathbb{Z} \),

\[
d(x_{i+1}, f(x_i)) < \delta.
\]

Given \( \epsilon > 0 \), a pseudo-orbit \((x_i)\) is \( \epsilon \)-shadowed by an actual orbit \((f^i(x))_{i \in \mathbb{Z}}\) of \( f \) if for all \( i \in \mathbb{Z} \),

\[
d(x_i, f^i(x)) < \epsilon.
\]

The diffeomorphism \( f \) has the **shadowing property** if for every \( \epsilon > 0 \), there exists some \( \delta > 0 \) such that every \( \delta \)-pseudo-orbit is \( \epsilon \)-shadowed by some actual orbit of \( f \). In contrast, a diffeomorphism \( f \) is nonshadowable if there is some \( \epsilon > 0 \) such that for every \( \delta > 0 \), there is some \( \delta \)-pseudo-orbit which is not \( \epsilon \)-shadowed by any actual orbit in \( M \).

**Definition 2.2.** A linear map of \( \mathbb{R}^n \) is called **hyperbolic** if it admits \( n \) eigenvectors (counted with multiplicities) and all the eigenvalues have absolute values different from one. A fixed point \( p \) of a differentiable map \( f \) is called **hyperbolic** if the tangent map \( Df_p \) is hyperbolic. A periodic point \( p \) of \( f \) with period \( n \) is called **hyperbolic periodic** for \( f \) if \( D(f^n)_p : T_p M \to T_p M \) is hyperbolic. We call the orbit of such \( p \) a **hyperbolic periodic orbit**.

**Definition 2.3.** A point \( q \) is called **homoclinic** if it is in the intersection of the stable manifold \( W^s(p, f) \) and the unstable manifold \( W^u(p, f) \) of the same hyperbolic periodic point \( p \) and \( p \neq q \).
Definition 2.4. A compact $f$-invariant set $\Lambda$ is called hyperbolic if the tangent bundle $T_\Lambda M$ has a continuous $Df$-invariant splitting $E \oplus F$ and there exist constants $C > 0$, $0 < \lambda < 1$ such that $\|Df^n|_{E(x)}\| \leq C\lambda^n$ and $\|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 0$.

The following definitions do not require the base space $X$ is a manifold.

Definition 2.5. Let $f$ be a homeomorphism of a compact metric space $X$ and $p \in X$. We define the omega-limit set of $p$, denoted as $\omega(p, f)$, by

$$\{q \in M : q = \lim_{j \to \infty} f^{n_j}(p) \text{ for some sequence of integer } n_j \to \infty\}.$$ 

The set $\omega(p, f)$ is closed, nonempty and $f$-invariant, i.e., $f(\omega(p, f)) = \omega(p, f)$.

Definition 2.6. A point $q \in X$ is (positively) recurrent or omega-recurrent with respect to $f$ if $q \in \omega(q, f)$.

3. $C^1$-Stably Shadowing Theorem

In this section, on a compact manifold $M$ with a volume-preserving diffeomorphism, we study chain recurrence sets and attractors. And, we prove that if $M$ is $C^1$-stably shadowable, then $M$ is hyperbolic. I.e., we can state that the notion of hyperbolicity is expressed in terms of the shadowing property with the conditions for robustness in volume-preserving diffeomorphisms. Therefore, the notions of hyperbolicity and $C^1$-stable shadowability coincide.

Firstly, we consider a Chain Recurrence Theorem and Nonwandering Theorem on compact manifolds and then show that on the connected compact manifolds $M$ with volume-preserving diffeomorphisms, $M$ is chain transitive (thanks to a theorem of Bonatti and et al).

Let $M$ be a compact manifold and $\omega$ be a volume form. Let $Diff^1_\omega(M)$ be the set of volume-preserving $C^1$-diffeomorphisms of $M$. Bonatti and et al. [BC] proved that for $C^1$-generic diffeomorphisms, the limit set, the non-wandering set and the chain recurrence set coincide, and that they are the closure of the set of hyperbolic periodic points. Also they showed that (assuming that the manifold is connected) there is a residual subset of $Diff^1_\omega(M)$ consisting of transitive diffeomorphisms.

The definitions below need not assume the base space is a manifold. Let $(X, d)$ be a metric space and $f : X \to X$ be a continuous map.

Definition 3.1. Let $\varepsilon > 0$. Then a nonempty open subset $U$ of $X$ is called $\varepsilon$-absorbing if $U$ contains the $\varepsilon$-ball centered at $f(x)$ for each $x$ in $U$. $U$ is absorbing if it is $\varepsilon$-absorbing for some $\varepsilon > 0$. In an equivalent description, $f$ maps $U$ in a uniform distance $\varepsilon$ into its interior.
Definition 3.2. Let $U$ be a nonempty absorbing subset of $X$. The closed set

$$A = \bigcap_{n \geq 0} f^n(U)$$

is called the attractor-like set determined by $U$. In the compact case, $A$ is usually referred to simply as the attractor determined by $U$. The open set $\cup_{n \geq 0} f^{-n}(U)$ is the set of all points whose omega-limit sets are contained in $A$ and it is called the basin of $A$ relative to $U$, denoted by $B(A;U)$. It is well known that, if $X$ is compact, $B(A;U)$ is independent of the choice of absorbing open sets $U$.

Definition 3.3. A point $x$ in $X$ is nonwandering with respect to $f$ if any open neighborhood $U$ of $x$, there exists a positive integer $N$ such that $f^N(U) \cap U \neq \emptyset$. The set of all nonwandering points of $f$ is denoted by $NW(f)$.

The nonwandering set $NW(f)$ is closed and positively invariant. If, in particular, $f$ is bijective, then $f(NW(f)) = NW(f)$. In the compact space, the nonwandering set is nonempty.

Definition 3.4. An $\varepsilon$-chain (= $\varepsilon$-pseudo-orbit) from $x_0$ to $x_n$ for $f$ is a sequence $x_0, x_1, \cdots, x_n$ with the property that $d(f(x_i), x_{i+1}) < \varepsilon$ for $0 \leq j < n$. A point $x$ is a chain recurrence point if for every $\varepsilon > 0$, there exists an $\varepsilon$-pseudo-orbit with the starting and ending points at $x$, that is, there exist $n \in \mathbb{Z}_+$ and $x_0, x_1, \cdots, x_n \in X$ such that $x = x_0, x_n$ and $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \cdots, n - 1$. The chain recurrence set $CR(f)$ of $f$ is the set of all chain recurrence points, i.e.,

$$CR(f) = \{ p \in X | \text{there exists an } \varepsilon\text{-chain from } p \text{ to itself, for every } \varepsilon > 0 \}.$$ 

It is obvious that the chain recurrence set $CR(f)$ is closed. This set was defined by C. C. Conley [Co]. He showed that in case $X$ is a compact metric space, $CR(f)$ is globally determined by the attractors and the basin of attractors. The following theorem is Conley’s characterization of the chain recurrence set for continuous maps:

Lemma 3.5. (Conley’s Theorem [Co]) Let $X$ be a compact metric space and $f : X \to X$ be continuous. Then the chain recurrence set of $f$ is the complement of the union of the set $B(A) - A$, as $A$ runs over the collection of attractors of $f$; here $B(A)$ denotes the basin of attractor $A$, that is,

$$X - CR(f) = \bigcup_{A \text{ attractor}} (B(A) - A).$$

The volume-preserving condition obstructs the existence of attractors, and hence the existence of non-chain recurrences, as below.

Proposition 3.6. Let $M$ be a compact manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. Then, $M$ is a chain recurrence for $f$, i.e., every point of $M$ is a chain recurrence for $f$. 

Proof. By the definition of the volume-preserving diffeomorphisms, there does not exist an absorbing set, because any subset is of finite measure. Thus, there does not exist an attractor. Therefore the right side of the equation in Lemma 3.3 is empty. Hence $M = CR(f)$, i.e., $M$ is chain recurrent for $f$. □

As one deals with non-compact manifolds, the definitions around attractors should be corrected, which we do in the next section. However, in the volume-preserving dynamics on non-compact manifolds with a volume form $\omega$, in general, we do not guarantee the parallel consequence of the theorem 3.6 (Example 4.3).

Remark 3.7. The following theorem exhibits that a compact manifold $M$ with a volume form $\omega$ and a volume-preserving diffeomorphism $f$ is non-wandering. It is easy to see that the nonwandering set is contained in the chain recurrence set, that is, $NW(f) \subseteq CR(f)$. Using these facts, we directly say that the above $M$ is chain recurrent.

Theorem 3.8. Let $M$ be a compact manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. Then $M = NW(f)$, i.e., every point of $M$ is nonwandering.

Proof. Assume that the conclusion is contrary. Thus there exists a wandering point $p$ of $M$. Then we have some neighborhood $U$ of $p$ satisfying that $f^n(U) \cap U = \emptyset$ for all positive integer $n$. In fact, the positive orbit of $U$ under $f$ is composed the disjoint union of each iteration of $U$ under $f$. If the union is not disjoint, then there are positive integers $n_1, n_2 (n_1 > n_2)$ such that $f^{n_1}(U) \cap f^{n_2}(U) \neq \emptyset$. Then we obtain that $f^{n_1-n_2}(U) \cap U \neq \emptyset$, which is contradiction. Therefore $M$ contains the disjoint union $\bigcup_{n \geq 0} f^n(U)$ and so

$$m(M) \geq \sum_{n \geq 0} m(f^n(U)) = \sum_{n \geq 0} m(U) = \infty.$$ 

Hence the compactness of $M$ implies a contrary. This completes the proof. □

From Theorem 3.8 in the volume-preserving version, we can obtain a stronger concept, say the notion of nonwandering. However, nevertheless, we mainly treat the notion of chain recurrence in the rest of this section because we have useful tools to develop arguments.

Moriyasu [Mo] proved that the interior of the set of elements $f$ in $Diff^1(M)$ with the property that the restriction of $f$ to the nonwandering set has the shadowing property, is included in $F(M)$. Here, $F(M)$ is the set of elements $f \in Diff^1(M)$ such that there exists a $C^1$-neighborhood $U(f)$ of $f$ satisfying that for every element $g$ of $U(f)$, all periodic points of $g$ are hyperbolic. It is inferred from [WGW, pp.3] and [We] that $CR(f)$ is $C^1$-stably shadowable if and only if $CR(f)$ is hyperbolic. Therefore, we obtain the following theorem.
Theorem 3.9. Let $M$ be a compact manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. Then, $M$ is $C^1$-stably shadowable if and only if $M$ is hyperbolic.

Proof. This follows from Theorem 3.6 and the result in [Mo] (see [WGW, pp.2–3]). □

We use (and have used implicitly) the uniform metric on the space $\text{Diff}^1_\omega(M)$ of volume-preserving diffeomorphisms, given by $d_U(f, g) := \sup_{x \in M} d(f(x), g(x))$.

A subset $R$ is called residual if it contains a countable intersection of open and dense subsets. If there exists a residual set $R$ in $\text{Diff}^1_\omega(M)$ such that any $g$ in $R$ possesses the same property $P$, then we call the property $P$ generic. Such a residual set is called a generic set.

Bonatti and Crovisier [BC] proved that, generically, a volume-preserving diffeomorphism is transitive in a compact, connected manifold with a volume-preserving diffeomorphism.

Theorem 3.10. (Theorem 1.3. in [BC]) Suppose that $M$ is a compact connected manifold with a volume form $\omega$. Then, there exists a residual set $\mathcal{G}_\omega$ in $\text{Diff}^1_\omega(M)$ consisting of transitive diffeomorphisms.

Let $f$ be a diffeomorphism on $M$. An invariant subset $A$ of $M$ is chain transitive if for every two points $p, q$ in $A$ and for every $\varepsilon > 0$, there exists an $\varepsilon$-pseudo-orbit with the starting point $p$ and the end point $q$, that is, a finite sequence $p = x_0, x_1, \ldots, x_n = q$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i$. The sequence $\{x_0, x_1, \ldots, x_n\}$ is called an $\varepsilon$-chain in $A$ connecting $p$ and $q$.

Theorem 3.11. Let $M$ be a connected and compact manifold with a volume form $\omega$, and $f$ be a volume-preserving diffeomorphism on $M$. Then $M$ is chain transitive for $f$.

Proof. Firstly we denote the transitive residual set in Theorem 3.10 by $R$. For every $\varepsilon > 0$, there exists a positive number $\delta$ with $\delta < \varepsilon$ such that

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \varepsilon.$$ 

From the Baire category theorem, we can pick a volume-preserving mapping $g$ in $R$ such that $d_U(f, g) = \sup_{x \in M} d(f(x), g(x)) < \frac{\delta}{2}$. By the definition of $R$, $g$ is transitive, i.e., there is a point $p_0$ in $M$ whose orbit closure is the whole manifold $M$. For every point $p, q$, since the orbit of $p_0$ is dense, there are positive integers $m, n$ such that $g^m(p_0) \in B(p, \frac{\delta}{2})$ and $g^n(p_0) \in B(q, \frac{\delta}{2})$. We may assume $n > m$. Now we consider the sequence $\{p, f^m(p_0), f^{m+2}(p_0), \ldots, f^{n-1}(p_0), q\}$. Thus, by the uniform continuity of $f$, the above sequence is an $\varepsilon$-chain in $M$ connecting $p$ and $q$. This completes the proof. □

Remark 3.12. It is clear that every point of the chain transitive set is a chain recurrence. Thus, in the connected case, not using the theorem 3.6 we are also able to say that $M$ is chain recurrent for $f$. 
Next we state the case of symplectic manifolds.

**Remark 3.13.** Let $M$ be a connected and compact manifold and $\omega$ be a symplectic form defined on $M$ and $\text{Symp}_1^\omega(M)$ the set of symplectic $C^1$-diffeomorphisms of $M$. [ABC] shows that $C^1$-generic symplectic diffeomorphisms of a compact manifold are transitive, that is, the transitive symplectic diffeomorphisms form a residual subset $G$ of $\text{Symp}_1^\omega(M)$.

The following theorem also generalizes a previous result in [BC] assuring the density of transitive diffeomorphisms in $\text{Diff}_1^\omega(M)$.

**Theorem 3.14.** ([ABC, Theorem 1]) If $(M, \omega)$ is a compact connected symplectic manifold. The set $G$ of transitive symplectic diffeomorphisms forms a dense $G_\delta$-set in $\text{Symp}_1^\omega(M)$. Moreover, the subset $G' \subseteq G$ of diffeomorphisms with the unique homoclinic classes $M$, forms a dense $G_\delta$-set in $\text{Symp}_1^\omega(M)$.

**Remark 3.15.** By the similar method of the proof of Theorem 3.11 we can say that the compact connected symplectic manifold is also chain transitive.

Let $f$ be a diffeomorphism on a compact manifold $M$. Let $\Lambda \subseteq M$ be a closed $f$-invariant set. We say the subset $\Lambda$ of $M$ is locally maximal in $U$ if there exists a compact neighborhood $U$ of $\Lambda$ such that $\cap_{n \in \mathbb{Z}} f^n(U) = \Lambda$.

**Definition 3.16.** A subset $\Lambda$ of $M$ is $C^1$-stably shadowable if $\Lambda$ is locally maximal in some compact neighborhood $U$ and there exists a $C^1$-neighborhood $U(f)$ of $f$ such that for any $g \in U(f)$, $g|\Lambda_g$ has the shadowing property where $\Lambda_g = \cap_{n \in \mathbb{Z}} g^n(U)$.

The above set $\Lambda_g$ is called the continuation of $\Lambda$. Specially, if $\Lambda = M$, we call simply $f$ is $C^1$-stably shadowable.

Let $p, q$ be points in $M$. Then, for every $\varepsilon > 0$, we can consider an equivalence relation $R$ on $CR(f)$ as follows. $pRq$ means there exist both an $\varepsilon$-chain from $p$ to $q$ and an $\varepsilon$-chain from $q$ to $p$. Then we call the equivalence classes the chain recurrence classes or the chain transitive components of $f$, simply call the chain components. The components are compact invariant sets and cannot be decomposed into two disjoint compact invariant sets. Denote $C_f(p)$ the chain component of $f$ that contains $p$. It is natural that $C_f(p)$ is $C^1$-stably shadowable if there is a $C^1$-neighborhood $U$ of $f$ such that for every $g \in U$, $C_g(p_g)$ has the shadowing property, where $p_g$ is the continuation of $p$.

The following theorem by Wen et.al. is useful.

**Theorem 3.17.** [WGW] Let $p$ be a hyperbolic periodic point of $f$. If $C_f(p)$ is $C^1$-stably shadowable, then $C_f(p)$ is hyperbolic.
Theorem 3.18. Let $M$ be a compact manifold with a volume form $\omega$ and $f$ be a volume-preserving diffeomorphism on $M$. Assume that every chain component has a hyperbolic periodic point. If every chain component is $C^1$-stably shadowable, then $M$ is hyperbolic. Furthermore, the converse is also true.

Proof. Since the number of connected components of $M$ is finite, there exists $n \in \mathbb{Z}_{>0}$ such that $f^n$ maps each component to itself. I.e., the restriction of $f^n$ to any connected component is also a volume-preserving diffeomorphism on the component. From Theorem 3.11, the connected component of compact manifold $M$ is chain transitive for $f^n$. By the invariance, the connected component becomes a chain component for $f^n$. Note that every chain component of $f^n$ is included in some chain component of $f$. Thus we can also say that every chain component of $f$ is some union of connected components of $M$. Then every chain component of $f$ is clopen (=closed and open) in $M$. By the assumption of this theorem and Theorem 3.17, the chain component is also hyperbolic with respect to $f$. Since $M$ is chain recurrent, it consists of finite union of some clopen chain components which are hyperbolic. Then we conclude the first implication. The converse follows directly from the definitions. □

Remark 3.19. In general, the number of the chain components of a compact manifold is infinite. However, from the proof of the above theorem, in the volume-preserving case, we can say that the number of the components of $M$ is just finite. More precisely, $M$ is composed of finitely many chain transitive components and every component is also composed of finitely many connected components.

4. Chain recurrences on Non-compact manifolds

We start with some necessary notions which are different from those for the compact manifolds. We assume that $(X,d)$ is a metric space and $f : X \to X$ is a homeomorphism. We define

$$\mathcal{P} = \text{the set of } \mathbb{R}^+\text{-valued continuous functions on } X.$$

Definition 4.1. A nonempty open subset $U$ of $X$ is weakly absorbing for $f$ if there exists a mapping $\varepsilon \in \mathcal{P}$ such that the closed ball $B_{\epsilon(f(x))}(f(x)) \subseteq U$ for each $x \in U$. When $U$ is weakly absorbing, the set

$$A = \bigcap_{n \geq 0} f^n(U)$$

is the weak attractor determined by $U$. If $\varepsilon \in \mathcal{P}$, then $x_0, x_1, \ldots, x_n$ is an $\varepsilon(x)$-chain if $d(f(x_j), x_{j+1}) < \varepsilon(f(x_j))$ for $0 \leq j < n - 1$. The number $n$ is called the length of the $\varepsilon(x)$-chain. A point $p$ is strongly chain recurrent
for \( f \) if for every \( \varepsilon \in \mathcal{P} \), there exist an \( \varepsilon(x) \)-chain of length at least 1 that begins and ends at \( p \). We denote by
\[
CR^+(f) = \text{the set of all strong chain recurrence points of } f.
\]

In this section, for brevity, an attractor always means a weak attractor and also an attractor block or an absorbing set means a weakly absorbing set. It is easily proved that \( U \) is weakly absorbing if and only if \( f(U) \subseteq U \).

**Definition 4.2.** In the case of non-compact spaces, we define the \textit{basin of an attractor} \( A \) relative to \( U \), \( B(A;U) \) as the open set \( \bigcup_{n \geq 0} f^{-n}(U) \). Every point of \( B(A;U) \) has the omega-limit sets contained in \( A \). When \( X \) is a compact space, \( B(A;U) \) is independent of \( U \) while it is not true for non-compact manifolds. Therefore, we define the \textit{extended basin} \( B(A) \) of \( A \) by the union of the set \( B(A;U) \) as \( U \) runs over all the absorbing sets that determine \( A \).

Let us get back to Proposition 3.6 for a while. The example below exhibits the failure of the theorem in the volume-preserving dynamics over non-compact manifolds.

**Example 4.3.** Let \( M = \mathbb{R}^2 \) and \( f : M \to M \) given by \( f(x, y) = (x + 1, y) \).
Let \( \omega \) be a volume form (equivalently, a symplectic form) by \( \omega = dx \wedge dy \).
Then, it is clear that \( f \) preserves \( \omega \). Let
\[
U_n = \{(x, y) \in M \mid y < \frac{-1}{x - n}, \ x < n\} \cup \{(x, y) \in M \mid x \geq n\}.
\]
Since \( f(U_n) = U_{n+1} \), we can easily check that \( U_0 \) is an attractor block for the translation \( f \) and
\[
A = \{(x, y) \mid y \leq 0\}
\]
is the attractor. Whilst, no point of \( M \) is a (strong) chain recurrence for \( f \).

The following theorem by Hurley is a generalized version of Conley’s theorem. In [Hu1, Hu2], Hurley investigates the extent to which Conley’s Theorem can be salvaged, and generalized Conley’s theorem from dynamical systems on compact metric spaces to those on locally compact metric spaces. Thus, his theorem is applicable to an arbitrary manifold.

**Theorem 4.4.** [Hu1, Hu2] If \( X \) is a locally compact metric space and \( f : X \to X \) is continuous, then the strong chain recurrence set \( CR^+(f) \) of \( f \) is the complement of the union of the set \( B(A) - A \), as \( A \) runs over the collection of weak attractors of \( f \). I.e.,
\[
X - CR^+(f) = \bigcup_{A:\text{weak attractor}} (B(A) - A).
\]

**Remark 4.5.** A topological dynamical system \((X, f)\) is called \textit{minimal} if the orbit of every point \( x \in X \) is dense in \( X \). Let \( f : X \to X \) be a continuous map on a metric space \( X \), and \((X, f)\) be a minimal dynamical system. From the definition of the minimality, \( M \) is strongly chain recurrent for \( f \), i.e., every point of \( M \) is strongly chain recurrent for \( f \).
Now we have a generalization of Proposition 3.6 to the case of compact attractors.

**Theorem 4.6.** Let $M$ be a manifold (not necessarily compact) with a volume form $\omega$, and $f$ be a volume-preserving diffeomorphism on $M$. If every attractor of $M$ is compact, then $M$ is strongly chain recurrent for $f$.

**Proof.** Assume that there exists an element $p$ of $M$ that is not strongly chain recurrent with respect to $f$. By Hurley’s theorem, there is an attractor $A$ and a basin $B(A)$ of $A$ such that $p \in B(A) - A$. Since $A$ is compact, we may assume that there is an attractor block whose closure is compact. From now on, the remaining part of the proof is parallel with the proof of Proposition 3.6 for the compact manifolds.

The following proposition (and its corollary) shows the invariance of the attractors and the boundaries.

**Proposition 4.7.** Let $f$ be a homeomorphism on a metric space $X$, $U$ be an attractor block, and $A$ be a weak attractor depending on the attractor block. If a point $x$ is in $U - A$, then the intersection of the (positive) $f$-orbit of $x$ and the attractor is empty.

**Proof.** Let $O_f^+(x)$ be the (positive) $f$-orbit of $x$. Suppose $O_f^+(x) \cap A \neq \emptyset$, then there exists a nonnegative integer $k$ such that $f^k(x) \in A$, that is, $f^k(x) \in \cap_{n \geq 0} f^n(U)$. Note that, $f(U) = \overline{f(U)}$. Thus $x \in f^{n-k}(U)$ for all $n \geq 1$ and so $x \in A$ by the shrinking property. This is a contradiction, which completes the proof.

When $f : X \to X$ is continuous, by the definition, it is easily shown that an attractor is positively $f$-invariant. If $f$ is a homeomorphism, an attractor $A$ is $f$-invariant, i.e., $f(A) = A$. Indeed, if $f(A) \neq A$ then there is an element $x$ in $A - f(A)$. From the definitions, $f^{-1}(x) \in U - A$, where $U$ is an associated attractor block. Then, Proposition 4.7 we must meet a contradiction. Hence an attractor is invariant.

**Corollary 4.8.** Let $f$ be a homeomorphism on a locally compact metric space $M$. Then the boundary of every weak attractor is positively $f$-invariant, that is, $f(\partial A) \subseteq \partial A$ for every weak attractor $A$.

**Proof.** Suppose the contrary of the conclusion. Then by the above statement, we may assume that there exists a boundary point $x$ satisfying $f(x)$ is in the interior of $A$. From the local compactness, we can choose compact neighborhood $C$ of $f(x)$ such that $f^{-1}(C)$ is also a compact neighborhood of $x$. Then, we are able to pick a point in $U - A$ where $U$ is an associated attractor block which corresponds to an interior point of the attractor. By Proposition 4.7 it is a contradiction.

Now we embark on the main proposition and theorem for the chain recurrences on the non-compact cases. The first proposition tells us that the
points near an attractor with bounded orbits form a measure 0 set, in the volume-preserving dynamics. Recall Example 4.3 which shows every orbit is unbounded.

**Proposition 4.9.** Let $M$ be a manifold (not necessarily compact) with a volume form $\omega$. Let $f$ be a volume-preserving diffeomorphism on $M$. Let $A$ be any attractor and $U$ be an associated attractor block. Then, the complement in $U - A$ of the set of points $p \in U - A$ with unbounded orbits is of measure 0.

**Proof.** Let $p \in U - A$ and $K \subset U - A$ be a compact neighborhood of $p$ with a finite measure $c > 0$. Let us fix any point $x_0 \in M$. Let $B_l(x_0)$ be the closed ball of the radius $l$ centered at $x_0 \in M$ (where $l \in \mathbb{Z}_+$). Let us define

$$K_r = \{ q \in K | f^k(q) \notin B_r(x_0) \text{ for some positive integer } k \}$$

It suffices to show

$$L = \bigcap_{r \in \mathbb{Z}_+} K_r$$

is of measure $c$, because $L$ is the set of points of $K$ with unbounded orbits and $m(L) = m(K)$ implies the statement of the proposition. Note that

$$K - K_r = \{ q \in K | f^k(q) \in B_r(x_0) \text{ for all } k \in \mathbb{Z}_+ \} = \bigcap_{k \in \mathbb{Z}_+} \{ q \in K | f^k(q) \in B_r(x_0) \}$$

and thus $K - K_r$ is measurable as $\{ q \in K | f^k(q) \in B_r(x_0) \}$ is measurable for each $k \in \mathbb{Z}_+$. Therefore, $L$ is measurable as well.

Let us observe that

$$m((f^k(U) - f^k(A)) \cap B_r(x_0)) \to 0$$

as $k \to \infty$. Indeed, Lebesgue’s dominated convergence theorem assures it from the following:

(a) definition of attractors (i.e., $\cap_k f^k(U) = A$),
(b) the $f$-invariance of $A$,
(c) $B_r(x_0)$ is of finite measure.

Note that $\{ q \in K | f^k(q) \in B_r(x_0) \} = f^{-k}(B_r(x_0)) \cap K$. Thus, we have

$$m(\{ q \in K | f^k(q) \in B_r(x_0) \}) = m(f^{-k}(B_r(x_0)) \cap K) = m((B_r(x_0)) \cap f^k(K))$$

where the latter equality is due to the measure-preservation of $f$. Because of the inclusion $f^k(K) \subset f^k(U) - f^k(A)$ and (4.4), we obtain

$$m(\{ q \in K | f^k(q) \in B_r(x_0) \}) \to 0$$

as $k \to \infty$ by Lebesgue’s dominated convergence theorem. Therefore, for each $r$, we have $m(K - K_r) = 0$, equivalently, $m(K_r) = m(K) - m(K - K_r) =$
By applying Lebesgue’s dominated convergence theorem to (4.3), we obtain $m(L) = c$, as desired. □

For $p \in M$, we denote by $O^+_f(p) = \{ f^n(p) | n \geq 0 \}$ and we define that $K^+(p) = O^+_f(p)$. $M$ is said to be Lagrange-stable for $f$ if for $p \in M$, $K^+(p)$ is compact. Since we are working on a metric space, the Lagrange stability amounts to $O^+_f(p)$ is bounded.

**Theorem 4.10.** Let $M$ be a manifold with a volume form $\omega$, and $f$ be a Lagrange-stable volume-preserving diffeomorphism on $M$. Then, $M$ is strongly chain recurrent for $f$, i.e., every point of $M$ is strongly chain recurrent with respect to $f$.

**Proof.** This follows from Hurley’s theorem (Theorem 4.4) for locally compact spaces. To prove the theorem, as was shown in Theorem 4.2, the nonexistence of attractors should be guaranteed. On the contrary, suppose that a nonempty attractor $A$ exists. By the above proposition, almost every point of $U - A$ has an unbounded orbit, where $U$ is an attractor block of $A$. This contradicts to our assumption of the Lagrange stability. □

## 5. Nonshadowability and Recurrence

In this section, we show the theorems about the nonshadowability and recurrence on compact manifolds.

**Lemma 5.1.** Let $p \in M$ be a hyperbolic periodic point. If for $x \in W^s(p, f)$, the positive orbit starting from $x$ is $\epsilon$-shadowed by an actual orbit for all sufficiently small $\epsilon > 0$, then the actual orbit is in the stable manifold.

**Proof.** The lemma is proven in a small neighborhood near $p$; the problem is linearized as follows. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a hyperbolic linear map. Let $x$ lie in the direct sum $\Sigma$ of eigenspaces with eigenvalues of absolute values $< 1$. Because the positive orbit of any point $y$ outside $\Sigma$ diverges, the positive orbit of $x$ cannot be $\epsilon$-shadowed by the positive orbit of $y$. □

With the above lemma, we can prove the following :

**Theorem 5.2.** Let $M$ be a compact manifold with a volume form $\omega$ and $f : M \to M$ be a volume-preserving diffeomorphism. If $W^u(p, f)$ has a recurrence point and there is no homoclinic point of $p$, then $f$ is nonshadowable.

**Proof.** Let $q$ be a recurrence point in $W^u(p, f)$. Then, there exists a point $x_0$ in $W^s(p, f)$ such that $x_0$ is in the omega-limit set of $q$ (see [XiI, Proof of Theorem 1]). To show nonshadowability, it suffices to find a $\delta$-pseudo-orbit for each $\delta > 0$ which is not $\epsilon$-shadowed for some $\epsilon > 0$. First, select a point $f^{n_0}(q)$ in the positive orbit of $q$ such that the distance between $f^{n_0}(q)$ and $x_0$ is less than $\delta$. Then, we consider the following $\delta$-pseudo-orbit $(y_i)$ defined as
(1) for $i < n_0 - 1, y_i = f^i(q)$,
(2) for $i \geq n_0, y_i = f^{i-n_0}(x_0)$.

By applying Lemma 5.1 respectively to the positive orbit and the negative orbit of $(y_i)$, it is easily obtained that there exists a positive number $\varepsilon$ such that the $\delta$-pseudo-orbit $(y_i)$ cannot be $\varepsilon$-shadowed by any actual orbit because $p$ has no homoclinic point. □

Because by [Xi1], a hyperbolic periodic point has a homoclinic point for a generic volume-preserving diffeomorphism, the above theorem seems uninteresting. But, there is an interesting example in [Ol1, p. 655] which admits a recurrence point but has no homoclinic point. Hence, our theorem is not vacuous and guarantees that the example is nonshadowable [Ol3].

With the similar arguments, we can obtain a result for the case when there is a homoclinic point under the shadowing property.

**Theorem 5.3.** Let $M$ be a compact manifold with a volume form $\omega$ and $f : M \to M$ be a volume-preserving diffeomorphism. Let $p$ be a hyperbolic periodic point. Assume that $f$ has the shadowing property. If there is a recurrence point in $W^u(p, f)$, then the recurrence point lies in the limit set of homoclinic points of $p$.

**Proof.** Let $q$ be the recurrence point. We need to find homoclinic points converging to $q$. For each $n \in \mathbb{Z}_+$, we can construct a $1/n$-pseudo-orbit by the exactly same method in the proof of the above theorem. Then, by the shadowability of $f$, the pseudo-orbit is $\varepsilon$-shadowed by an actual orbit for every sufficiently small $\varepsilon > 0$. Especially, we may assume that $\varepsilon$ is less than $1/n$. By applying Lemma 5.1 twice as in the proof of Theorem 5.2, the actual orbits are all homoclinic points. Hence, we can find a homoclinic point in the ball centered at $q$ of radius $1/n$. □

By this result, we can obtain the following directly.

**Corollary 5.4.** Let $M$ be a compact manifold with a volume form $\omega$ and $f : M \to M$ be a volume-preserving diffeomorphism. Let $p$ be a hyperbolic periodic point. If $f$ has the shadowing property and the recurrence points are dense in $W^u(p, f)$, then the homoclinic points are also dense in $W^u(p, f)$. □

In [Ol2], it is proved that the recurrence points of a generic volume-preserving diffeomorphism on a compact manifold $M$ with volume form $\omega$, are dense in an unstable manifold. Xia [Xi1] also showed generically hyperbolic periodic point $p$ of $f$ has a homoclinic point, and moreover, the homoclinic points of $p$ are dense in both the stable manifold and the unstable manifold of $p$. By the corollary above, without genericity, we obtain a connection between the recurrence points and homoclinic points under the shadowability condition.
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