THE REDUCIBILITY OF AN AIRY OPERATOR

LOTFI SAIDANE

Abstract. We show that the determinant $\nabla(d, \alpha)$, which seems to be not considered in the past, is not zero. As an application of this result we prove that the Setoyanagi operator $S_{p,q} = \partial^2 - (ax^p + bx^q)$ is irreducible over $\mathbb{C}[x][\partial]$.

1. Introduction

It is well known (see [D, L-R]) that is the operator $\partial^2 - q$, $q \in \mathbb{C}[x]$ is reducible in $\mathbb{C}(x)[\partial]$ if and only if the Ricatti equation $u' + u^2 = q$ has a solution in $\mathbb{C}(x)$. We propose a more manageable criterion using the determining factors properties of the operator. We can easily prove that an Airy operator $L = \sum_{i=0}^{n} a_i \partial^i + Q_m(x)$, of bidegree $(n, m)$, $n \leq m$, and $\{ \int R_i(x^{1/n}) dx, i = 1, \ldots, n \}$ as set of determining factor is reducible in $\mathcal{D}_K$ with a right factor of order 1 if and only if, $n$ divides $m$ and there exist $i$ in $\{1, \ldots, n\}$ such that the differential equation $L(\partial + R_i)(u) = 0$ has a polynomial solution. As an application, we prove that the Setoyanagi operator $S_{p,q} = \partial^2 - (ax^p + bx^q)$ (see [D, L-R], Example 4) is irreducible over $\mathbb{C}[x][\partial]$. For this purpose, we show that the determinant $\nabla(d, \alpha) = |c_{i,j}|_{1 \leq i,j \leq d+1}$, defined in section 4, which seems to be not considered in the past, is not zero.

2. Determining Factors

Let $k$ be an algebraically closed field of characteristic zero, $K$ is the quotient field of the ring of polynomials $R = k[x]$, $\partial = \frac{d}{dx}$ is the derivation of $K$, $\mathcal{D} = R[\partial] = k[x, \partial]$ is the Weyl algebra over $k$ and $\mathcal{D}_K = K[\partial]$ is the set of differential operators with coefficients in $K$, so $\mathcal{D}_K$ is an associative noncommutative $k$-algebra. We denote, also, by $\partial$ the extension of $\partial$ to the Picard-Vessio extensions of $K$, $V$ an $R$-module of rank $n$, and $\nabla_\partial$ a contraction by $\partial$ of a connection on $V$, i.e. a $k$-linear map on $V$ satisfying:

$$\nabla_\partial (av) = (\partial a) v + a \nabla_\partial (v), \quad a \in R, \quad v \in V.$$

We define a structure of left $\mathcal{D}$-module on $V$ by setting:

$$\left( \sum_{i=0}^{n} a_i \partial^i \right) v = \sum_{i=0}^{n} a_i \nabla_\partial^i (v), \quad a_i \in R, \quad v \in V.$$ 

Inversely, if $V$ is a left $\mathcal{D}$-module of finite rank as an $R$-module, than we can define a connection on it by putting $\nabla (x) = \partial (x)$.

An operator $L \in \mathcal{D}$ is said to be monic if its leading coefficient, with respect to $\partial$ is one. If $L$ is monic operator than $\mathcal{D}/DL$ is a free $R$-module of finite rank.

Date: February 2010.
1991 Mathematics Subject Classification. [2000] Primary 12 H 05, 33C 10, 11C20, 15B36.
Key words and phrases. Airy operator, Setoyanagi operator, Determining factor, Differential equation, Determinant.
We say that a $D$-module $V$ is cyclic if there exist a monic operator $L$ in $D$ such that $V$ is isomorphic, as a $D$-module to, $D/LL$. By scalar extension we can define $V_K = K \otimes_R V$, thus $V_K$ is a $K$-vector space of dimension the rank of $V$ as a free $R$-module. As above, we define a connection on $V_K$, denoted by $\nabla_K$, or simply $\nabla$, as follows:

$$\nabla (a \otimes v) = (\partial a) \otimes v + a \otimes \nabla (v), \quad a \in K, \; v \in V.$$ 

thereby $\nabla$ define a structure of $D_K$-module on $V_K$.

**Definition 1.** Two operators of $D$ (resp. $D_K$) are said equivalent if their corresponding $D$-modules are equivalent.

The following properties (which we can find a proof in [Sg], § 2) will be useful for the rest of this paper.

**Proposition 1.** Two monic operators $L_1$ and $L_2$ of $D_K$ are equivalent if and only if there exists $L_3$ in $D_K$ having no common factor on the right with $L_1$ and $L_2$ in $D_K$ such that $L_2 \circ L_3 = L_4 \circ L_1$.

Let $\bar{K}$ be a Picard-Vessiot extension of $K$, containing the Picard-Vessiot extension of $L_1$ and $L_2$. We denote by $V_1$, $V_2$ the respective solutions spaces of of $L_1(y) = 0$ and $L_2(y) = 0$. Let $G$ be the differential Galois group of $\bar{K}$. Then, the operators $L_1$ and $L_2$ are equivalent if and only if their corresponding $G$-modules $V_1$ and $V_2$ are isomorphic (the isomorphism is given by the natural action on $V_1 \subset \bar{K}$. The operator $L_3$ can be chosen such that its order is strictly lower than that of $L_1$, see [Sg], Lemma 2.5.

**Definition 2.** An element of $D_K$ is called reducible (resp. completely reducible) if it decomposes into a product of at least two factors of order $\geq 1$ (resp. of order 1).

An operator of $D$ may be reducible in $D_K$ without being on $D$ (see [Be] § 3.). Let $\bar{K} = k((1/x))$ be the field of meromorphic formal series, near the infinity, with coefficients in $k$ ($k$ is an algebraically closed field of characteristic zero), equipped with its usual derivation $\partial = \frac{d}{dx}$ and its valuation $1/x$-adic $v$. Let $\bar{K}$ be the algebraic closure of $K$, then the valuation and the derivation of $\bar{K}$ extends uniquely to $\bar{K}$. For example if $a \in k[x]$ is a polynomial of degree $\deg(a)$, $v(a) = -\deg(a)$.

More generally, we will put $\deg(a) = -v(a)$ for all $a \in \bar{K}$. Let

$$L = \partial^n + \sum_{i=0}^{n-1} a_i \partial^i, \quad a_i \in \bar{K}$$

be a differential operator with coefficients in $\bar{K}$. The theorem and Hukuhara Turrittin (see [I]) shows the existence of a basis $(u_1, ..., u_n)$ of solutions of the form:

$$u_i(x) = (\exp K_i(x))(1/x)^{\lambda_i} v_i(1/x), \quad i = 1, ..., n,$$

where $K_i(x)$ is a polynomial in $x^{1/q}$ (for some integer $q$) without constant term, $\lambda_i$ is an element of $k$, $(1/x)^{\lambda_i}$ is a solution (in a Picard-Vessiot extension of $\bar{K}$) of the differential equation $y' = -(\lambda_i/x)y$ and

$$v_i(x) = \sum_{j=0}^{n_i} v_{i,j} (1/x)(\text{Log}x)^j,$$
where
\[ v_{i,j}(x) = \sum_{k=0}^{\infty} v_{i,j,k} x^{-k/q} \in \bar{K}. \]

**Definition 3.** The polynomials \( K_i, i = 1, \ldots, n \) are called the determining factors of the operator \( L \). We say that \( L \) is of simple characteristics, if for every pair \((i, j)\) of distinct indices in \( \{1, 2, \ldots, n\} \), we have \( \deg(K_i - K_j) = \deg(K_i) \).

**Definition 4.** Let \( n, m \in \mathbb{N}, n \neq 0 \), \( P_n, Q_m \) two polynomials in \( k[x] \) of degree \( n \) and \( m \) and \( \partial = \frac{d}{dx} \). The operator \( L = P_n(\partial) + Q_m(x) \) is called an Airy operator of bidegree \( (n, m) \).

Airy operator generalize the classical Airy equation \( y'' - xy = 0 \) (where \( n = 2 \) and \( m = 1 \)). Their study was initiated by N. Katz \[K\], in order to calculate the differential Galois group. Katz shows that this calculation is reduced precisely to questions of reducibility and self duality for the operator \( L \).

**Lemma 1.** The characteristics of an Airy operator \( L = P_n(\partial) + Q_m(x) \) of bidegree \( (n, m) \) is simple and the determining factors \( K_i \) are polynomials in \( x^{1/n} \) without constant terms. Their derivatives \( K'_i = R_i \) are, in the case \( m \geq n \), solutions to the inequality
\[ \deg(P_n(R) + Q_m(x)) \leq \frac{nm - m - n}{n}. \]
where \( R \in \bar{K} \) is a polynomial in \( x^{1/n} \). \[?] \n
**Proof.** See \[S\], Lemma 1.2.1 page 525. \( \square \)

**Definition 5.** The Fourier transform of an operator \( L = \sum_{i=0}^{N} a_i(x) \partial^i, a_i \in R \) of \( D \), relatively to \( \partial \), is the operator \( \mathcal{F}(L) = \sum_{i=0}^{N} a_i(\partial)(-x)^i \).

**Remark 1.** The Fourier transform \( \mathcal{F} \) is a \( k \)-linear bijection from \( D \) to itself. We have
\[ \mathcal{F}^2(L) = [-1]^t L = \sum_{i=0}^{N} a_i(-x)(-\partial)^i. \]

**Proposition 2.** \( L \in D \) is reducible (in \( D \)) if and only if its Fourier transform \( \mathcal{F}(L) \) is reducible.

An operator \( L \in D \) is called biunitary if its leading coefficient relatively to \( \partial \) and \( x \) are unity of \( R \). The following result improve the previous.

**Proposition 3.** A biunitary operator \( L \in D \) is reducible (in \( \mathcal{D}K \) if and only if its Fourier transform \( \mathcal{F}(L) \) is reducible in \( \mathcal{D}K \).

**Proof.** See \[S\], Proposition 1.3.1. \( \square \)

### 3. Reducibility

Katz ([Ka], page 26), has proved that an Airy operator of bidegree \( (n, m) \), where \( n \) and \( m \) are coprime, is irreducible in \( D_K \). The following result improve the preview.
Theorem 1. An Airy operator \( L = \sum_{i=0}^{n} a_i \partial^i + Q_m(x) \), of bidegree \((n, m)\), \( n \leq m \), and \( \{ \int R_i (x^{1/n}) \, dx, \, i = 1, \ldots, n \} \) as set of determining factor is reducible in \( D_K \) with a right factor of order 1 if and only if the following two two conditions are satisfied:

1) \( n \) divides \( m \),
2) there exist \( i \) in \( \{1, \ldots, n\} \) such that the differential equation \( L(\partial + R_i)(u) = 0 \) has a polynomial solution.

Proof. See [S] Proposition 2.1.1.

If \( L = \sum_{i=0}^{n} a_i \partial^i + Q_m(x) \), then the adjoint operator of \( L \), denoted \( L' \), is defined by

\[ L' = \sum_{i=0}^{n} (-\partial)^i a_i + Q_m(x). \]

With the same hypothesis as the theorem, we can easily prove that \( L \) is reducible in \( D_K \) with a left factor of order 1 if and only if \( n \) divides \( m \) and there exist \( i \) in \( \{1, \ldots, n\} \) such that the differential equation \( L'(\partial - R_i)(u) = 0 \) has a polynomial solution.

It is well known (see [D, L-R]) that the operator \( \partial^2 - q, \, q \in \mathbb{C}[x] \) is reducible in \( \mathbb{C}(x)[\partial] \) if and only if the Ricatti equation \( u' + u^2 = q \) has a solution in \( \mathbb{C}(x) \). We propose a more manageable criterion using the determining factor properties of the operator.

4. Application

Let \( p, q \in \mathbb{Q}, \, q < p, \, a, b \in \mathbb{C}, \, ab \neq 0 \). Let \( S_{p,q} = \partial^2 - (ax^p + bx^q) \) the Setoyanagi operator. If we assume that the operator \( S_{p,q} \) is reducible in \( \mathbb{C}(x)[\partial] \), then Theorem 1 implies that \( p \) is an even integer. We assume \( p = 2m \). The determining factors \( \int R \) of \( S_{p,q} \), according to Lemma ??, are given by:

\[ R = \varepsilon \sqrt{a} x^m \sum_{i=0}^{r} \left( \frac{1}{2} \right)_{i} (b/a)^i x^{i(q-2m)} \]

where \( r = E \left( \frac{m}{2m-q} \right) \) is the integral part of \( \frac{m}{2m-q} \) and \( \varepsilon = \pm 1 \). Using Theorem 1 we deduce that there exists \( d \in \mathbb{N} \) such that if \( d_{m-1} \) denote the coefficient of \( x^{m-1} \) in \( R^2 - [ax^{2m} + bx^q] \) so we obtain

\[ \frac{d_{m-1}}{(2d + m) \sqrt{a}} = \varepsilon, \text{ with } \varepsilon = \pm 1. \]

We therefore find the conditions cited by [D, L-R], namely, \( \frac{m+1}{2m-q} \) is a natural number \( s \geq 1 \) (in fact \( (r+1)(q-2m) + 2m \) is equal to \( m - 1 \)) and condition

\[ d_{m+1} = 2a \left( \frac{1}{2} \right)_{s} (b/a)^s = \varepsilon (2d + m) \sqrt{a}, \]

we can write as follows: there exist \( d \in \mathbb{N} \) such that

\[ \varepsilon \sqrt{a} \left( \frac{1}{2} \right)_{s} (b/a)^s - \frac{m}{2} = d. \]

Setoyanagi (cited by [D, L-R], Example 4) gave a necessary and sufficient conditions of reducibility in the case \( \frac{m+1}{2m-q} = s \leq 2 \). For \( s = 2 \), we can show that this case reduces to the case \( m = q = 1 \).
As an application, we will consider \( m = 2 \) and \( q = 3 \) (so \( s = 3 \)), we show the following result:

**Proposition 4.** The Setoyanagi operator \( S_{4,3} = \partial^2 - (ax^4 + bx^3) \), \( a, b \in \mathbb{C} \), \( a \) or \( b \neq 0 \), is irreducible in \( \mathbb{C}(x)[\partial] \).

**Proof.** If \( a \) or \( b = 0 \), it is easy to verify that \( S_{4,3} \) is irreducible. For the following, we assume that \( ab \neq 0 \). The change of variable \( x - \frac{b}{4a} \) preserves the reducibility properties of the operator \( S_{4,3} \). Let \( S \) be the operator obtained after the change of variable. We have

\[
S = \partial^2 - Q,
\]

where

\[
Q = ax^4 - \frac{3b^2}{8a}x^2 + \frac{b^3}{8a^2}x - \frac{3b^4}{4a^3}.
\]

The determining factors of \( S \) are \( \int R \) with

\[
R = \varepsilon \sqrt{a} \left[ x^2 - \frac{3b^2}{16a^2} \right], \quad \varepsilon = \pm 1.
\]

The operator \( S^R = S (\partial + R) \) is equal to \( \partial^2 + 2R\partial + [R^2 + R' - Q] \). If \( S \) is reducible than the differential equation \( S^R (u) = 0 \) have a polynomial solution. However, there exist \( d \in \mathbb{N} \) such that

\[
\frac{b^3}{16a^{3/2}} = d + 1.
\]

As above we can suppose \( s = 1 \). If necessary, we change the argument of \( \sqrt{a} \). We put \( \alpha = \frac{2a}{b} \), consequently

\[
\sqrt{a} = 2 (d + 1) \alpha^3
\]

and

\[
S^R = \partial^2 + \left[ 4(d + 1) \alpha^3 x^2 - 3(d + 1) \alpha \right] \partial + \left[-4d(d + 1) \alpha^3 x + 3(d + 1)^2 \alpha^2 \right]
\]

The differential equation \( S^R (u) = 0 \) have a polynomial solution of degree \( d \) if and only if \( S^R \), considered as a linear operator on \( \mathbb{C}_d [x] \) is not an injection. This is equivalent to determinant \( \nabla (d, \alpha) = |c_{i,j}|_{1 \leq i, j \leq d+1} \), defined by

\[
c_{i,j} = \begin{cases} 
3(d + 1)^2 \alpha^2, & \text{if } j = i \\
-4(d - i + 2) (d + 1) \alpha^3, & \text{if } j = i - 1 \\
-3i (d + 1) \alpha, & \text{if } j = i + 1 \\
i. (i + 1), & \text{if } j = i + 2 \\
0, & \text{else}
\end{cases}
\]

is zero. \( \square \)

**Lemma 2.** \( \nabla (d, \alpha) \neq 0 \).

**Proof.** \( \nabla (d, \alpha) \) is a polynomial in two variables \( d \) and \( \alpha \), homogeneous of degree \( 2(d + 1) \) compared to \( \alpha \). Therefore

\[
\nabla (d, \alpha) = \mu (d) \alpha^{2(d+1)}.
\]

It suffices to show that \( \mu (d) \neq 0 \). For this purpose, we note that if \( d \) is an even integer then \( \mu (d) \) is congruent to 1 modulo 2 and \( \mu (d) \) is not zero. Assume, for
the rest that $d$ is an odd integer. Let $(a_{i,j})$ the matrix obtained from $(c_{i,j})$ after putting $\alpha = 1$. Then

$$|a_{i,j}| = \mu(d).$$

We consider the order $p$ determinants extracted from $(a_{i,j})$ as follows:

$$\nabla_p = |a_{i,j}|_{d-p+2 \leq i,j \leq d+1},$$

for example

$$\nabla_2 = \begin{vmatrix} 3(d+1)^2 & -3d(d+1) \\ -4(d+1) & 3(d+1)^2 \end{vmatrix} = 3(d+1)^2 (3d^2 + 2d + 3)$$

and

$$\nabla_0 = 1, \quad \nabla_{d+1} = \mu(d).$$

Consequently, after developing $\nabla_{d+1}$ with respect to its first colon, we obtain the recurrence relation

$$\nabla_{p+1} = (d+1)^2 [3\nabla_p - 12\lambda_{p-1}\nabla_{p-1} + 16\lambda_{p-1}\lambda_{p-2}\nabla_{p-2}],$$

for $p = 2, \ldots, d$, and

$$\lambda_k = (k+1)(d-k).$$

We propose to prove that, for $p \in \{1, \ldots, d-1\}$

$$\nabla_{p+1} > 4\lambda_p\nabla_p.$$

For the rest we put $y = 3(d+1)^2$, then

$$\nabla_{p+1} = (d+1)^2 \left[3\nabla_p - 12\lambda_{p-1}\nabla_{p-1} + 16\lambda_{p-1}\lambda_{p-2}\nabla_{p-2}\right].$$

We assume that there exists an integer $q \in \{1, \ldots, d-1\}$ such that

$$\nabla_{q+1} \leq 4\lambda_q\nabla_q.$$

Let $p = \inf \{q; \nabla_q \leq 4\lambda_q\nabla_q\}$. The system \ref{eq:4.2} leads that $p \geq 4$.

By writing

$$\nabla_p > 4\lambda_{p-1}\nabla_{p-1}$$

and

$$\nabla_{p+1} \leq 4\lambda_p\nabla_p,$$

we can deduce the following relation

$$(y - 4\lambda_p)\nabla_p \leq 4\lambda_{p-1}y \left(\nabla_{p-1} - \frac{4}{3}\lambda_{p-2}\nabla_{p-2}\right),$$

where

$$\nabla_p \leq \frac{4\lambda_{p-1}y}{y - 4\lambda_p} \left(\nabla_{p-1} - \frac{4}{3}\lambda_{p-2}\nabla_{p-2}\right).$$

The relations

$$\nabla_p = y \left[\nabla_{p-1} - 4\lambda_{p-2}\nabla_{p-2} + \frac{16}{3}\lambda_{p-2}\lambda_{p-3}\nabla_{p-3}\right].$$
Replacing the sign of \( h \)

\[ (4.4) \quad y \left(1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p}\right) \nabla_{p-1} \leq 4\lambda_{p-2}y \left[1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p}\right] \nabla_{p-2} - \frac{16}{3} \lambda_{p-2} \lambda_{p-3} y \nabla_{p-3}. \]

Replacing \( \nabla_{p-1} \) by its expression in the inequality (4.4) we obtain

\[ y^2 \left[ 1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p} \right] \nabla_{p-2} - \frac{16}{3} y \lambda_{p-2} \lambda_{p-3} y \nabla_{p-3} \]

\[ \nabla_{p-3} - \frac{16}{3} y^2 \left[1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p}\right] \lambda_{p-3} \lambda_{p-4} y \nabla_{p-4}. \]

From the definition of \( p \), we deduce that \( \nabla_{p-3} > 4\lambda_{p-4} \nabla_{p-4} \), and the relationship (4.5) then leads

\[ y \left(1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p}\right) \nabla_{p-2} - \frac{16}{3} y \lambda_{p-2} \lambda_{p-3} y \nabla_{p-3} \]

\[ < 4\lambda_{p-3} y \left[ \frac{2}{3} y \left[1 - \frac{4\lambda_{p-1}}{y - 4\lambda_p}\right] - \frac{4}{3} \lambda_{p-2} \right] \nabla_{p-3}. \]

Similarly, the relationships

\[ \nabla_{p-2} > 4\lambda_{p-3} \nabla_{p-3} \]

and (4.6) leads

\[ y (y - 4\lambda_p + 4\lambda_{p-1} + 4\lambda_{p-2} (y - 4\lambda_p)) - 12\lambda_{p-2} \left(y - 4\lambda_p - \frac{4}{3} \lambda_{p-1}\right) < 0. \]

Thereby, we have

\[ (4.7) \quad y^2 - y (4\lambda_p + 4\lambda_{p-1} + 8\lambda_{p-2}) + 32\lambda_p \lambda_{p-2} + 16\lambda_{p-1} \lambda_{p-2} < 0 \]

We put, for \( p \in \{2, \ldots, d-1\} \),

\[ h(p) = y^2 - y (4\lambda_p + 4\lambda_{p-1} + 8\lambda_{p-2}) + 32\lambda_p \lambda_{p-2} + 16\lambda_{p-1} \lambda_{p-2} \]

we differentiate the function \( h \) and we replace replace \( \lambda_k \) by \((k+1)(d-k)\), thus

\[ h'(p) = 8 \left(d - p + 3\right) (4\lambda_p + 2\lambda_{p-1} - y) + 4 \left(d - 2p + 1\right) (8\lambda_{p-2} - y) \]

\[ + 4 \left(d - 2p + 1\right) (4\lambda_{p-2} - y). \]

For \( k \in \{0, 1, \ldots, d-1\} \) the function \( \lambda_k \) varies between \( d \) and \( \frac{(d+1)^2}{2} \). As \( y = 3(d+1)^2 \), the sign of \( h' \) is strictly positive if \( p > \frac{d+1}{2} \) is strictly negative if \( p < \frac{d+1}{2} \) (as \( d \) is an odd integer \( d \), view the beginning of the prof, and \( p \) is an integer). We calculate \( h\left(\frac{d-1}{2}\right), h\left(\frac{d+1}{2}\right) \) and \( h\left(\frac{d+3}{2}\right) \). We find

\[ h\left(\frac{d-1}{2}\right) = 56d^2 112d + 120, \]

\[ h\left(\frac{d+1}{2}\right) = 16d^2 32d + 48, \]

\[ h\left(\frac{d+3}{2}\right) = 24 \left(d + 1\right)^2. \]
For $p \in \{2, \ldots, d-1\}$, the least value of $h$ is $h\left(\frac{d+1}{2}\right)$ which is strictly greater than zero, which contradicts inequality 4.7. Therefore,

$$\nabla_{p+1} > 4\lambda_p \nabla_p, \quad p \in \{1, \ldots, d-1\}.$$ 

By putting $p = d$ in (4.1) and $p = d - 1$ in (4.7), we obtain

$$\nabla_{d+1} = y \left[ \nabla_d - 4 \lambda_{d-1} \nabla_{d-1} + \frac{16}{3} \lambda_{d-1} \lambda_{d-2} \nabla_{d-2} \right]$$

and

$$\nabla_d > 4 \lambda_{d-1} \nabla_{d-1},$$

which leads to $\nabla_{d+1} > \frac{16}{3} y \lambda_{d-1} \lambda_{d-2} \nabla_{d-2}$, and over $p$ and $\nabla_{d+1}$ is strictly positive, which achieve the proof.

\[\Box\]

References

[1] D. Bertrand, Extensions de D-modules et groupes de Galois différentiels, Springer L.N, N° 1454, p. 125-141.

[2] A. Duval - M. Loday-Richaud, Kovacic’s algorithm and its application to some families of special functions, AAECC, 3. 1992, p. 211-246.

[3] E. L. Ince, Ordinary differential equations London 1927.

[4] N. Katz, On the calculation of some differential Galois groups, Invent. Math. 87 (1987) p 13 - 61.

[5] L. Saidane, Propriétés algébriques des opérateurs d’Airy de petit ordre; Ann. Fac. Sc. Toulouse, 6ème série, tome 9, n 3(2000), p. 519-550.

[6] M. Singer, Testing reducibility of linear differential operators : A group theoretic perspective. AAECC. (1995).

Département de Mathématiques, Faculté des Sciences de Tunis, Université de Tunis-El Manar, Campus universitaire, 2092, El-Manar, Tunis, TUNISIA

E-mail address: lotfi.saidane@fst.rnu.tn.