Current and charge distributions
of the fractional quantum Hall liquids with edges

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Abstract

An effective Chern-Simons theory for the quantum Hall states with edges is studied by treating the edge and bulk properties in a unified fashion. An exact steady-state solution is obtained for a half-plane geometry using the Wiener-Hopf method. For a Hall bar with finite width, it is proved that the charge and current distributions do not have a diverging singularity. It is shown that there exists only a single mode even for the hierarchical states, and the mode is not localized exponentially near the edges. Thus this result differs from the edge picture in which electrons are treated as strictly one dimensional chiral Luttinger liquids.

74.10.-d, 73.20.Dx, 73.40.Hm
I. INTRODUCTION

We consider the large scale physics of the fractional quantum Hall (FQH) liquids with boundaries. Our analysis is based on the effective Chern-Simons gauge field theory \[1-10\].

In an approach initiated by Wen \[11\] (see also Ref. \[9\]), which is different from ours, a one-dimensional edge action is added to the original action in order to assure the gauge invariance for the Chern-Simons gauge field for restricted geometries. This has been followed by a number of authors \[5,12–15\] who tried to explain the FQH effect based only on the 1d theory (edge picture) \[16,17\]. It has an attractive feature, if correct, that the celebrated Tomonaga-Luttinger liquids are realized at the edges of FQH systems \[18-20\]. In those theories, however, apparently the global properties like distributions of current, charge and electromagnetic fields are assumed to be insignificant. Thus one of the main consequences is that the charges and currents are localized near the edges.

Recently Nagaosa and Kohmoto \[21\] examined boundary conditions for FQH liquids and it was shown that the gauge invariance is not violated if one imposes a physically suitable boundary condition. Thus one needs not to modify the original action as was done by Wen. In this way, it was possible to study the edge and bulk properties on an equal footing. They showed the fractional quantization of Hall conductance by considering both edge and bulk.

We begin with the composite fermion picture of the FQH effect in Sec. II. Then the Chern-Simons effective field theory to describe the composite fermions and the boundary condition are introduced in Sec. III.

Following the line of Ref. \[21\] the charge and current distributions of the FQH liquids in steady states are discussed in Sec. IV. It is shown that the effect of the interaction can not be treated perturbatively from the non-interacting situations. The approach we are taking for this correlated electron problem is totally non-perturative. The following two geometries are considered: (i) half-plane (subsection A) and (ii) Hall bar (subsection B).

(i) An exact solution is obtained using the Wiener-Hopf method. This method was used, in the same geometry, by Thouless \[29\] to solve the equations of MacDonald, Rice and
Brinkman [30] for the non-interacting integer quantum Hall liquid.

(ii) The asymptotic behavior near the edges are obtained using the theorems of the singular integral equations (Hilbert transformation). It is shown that there exists only a single mode even for the hierarchical states. This mode is not localized exponentially along the boundaries and our results are not consistent with the edge picture of the FQH effect.

It is also shown that the charge distribution is finite over the sample including the edges. This is indispensable to obtain a well-defined theory, since the electron number density must be positive. If the charge density has singularities [30], one cannot avoid negative divergence in the electron number density which leads to an ill-defined theory.

II. COMPOSITE FERMION PICTURE OF FRACTIONAL HALL LIQUIDS

Filling factor is defined by

\[ \nu = \frac{N_e}{N_\phi} \]  \hspace{1cm} (2.1)

where \( N_e \) is the number of electrons and \( N_\phi \) is the number of flux quanta. For \( \nu = 1 \) the first Landau level is totally filled and the higher levels are all empty. Let us see why Hall liquids with inverse filling factor equal to an odd integer might be special. Recall that in this case the background magnetic field contains an odd number of magnetic flux quanta per electron.

The first argument we cite is due to Jain. [22,23] Imagine an adiabatic process in which we somehow move some of the flux quanta so that \( p \) units of flux are attached to each electron. For \( p \) even, the additional Dirac-Aharonov-Bohm phase associated with moving one electron around another is \( e^{i\pi p} = 1 \) and so the statistics of the electron is unchanged. The electrons are now moving in a reduced magnetic field \( B_{\text{eff}} = B - 2\pi pn \) where \( n \) is the number density of electrons. (Note that in our convention, the unit of flux is \( 2\pi \).) The filling factor has been increased to \( \nu_{\text{eff}} \), given by \( \nu_{\text{eff}}^{-1} = (B - 2\pi pn)/2\pi n = \nu^{-1} - p \). For \( \nu_{\text{eff}} = m \) an integer, we have \( \nu^{-1} = p + m^{-1} \) and
\[ \nu = \frac{m}{mp + 1}. \]  

Thus, fractional Hall systems with \( \nu = m/(mp + 1) \) (\( p \) even) may be adiabatically changed into an integer Hall system with filling factor \( m \), as was also emphasized by Greiter and Wilczek [24]. Note that this argument gives us more than we had hoped for. The case we wanted to understand, with \( \nu^{-1} \) an odd integer, is obtained for \( m = 1 \).

In this way a Hall liquid with \( \nu = m/(mp + 1) \) with \( m \) an integer and \( p \) an even integer, is related to the integer Hall liquid. We may thus want to argue that since the integer Hall liquid is incompressible the fractional Hall liquid is also incompressible.

Another argument, historically earlier than Jain’s argument, is due to Zhang, Hansson, and Kivelson [1]. Imagine attaching all of the flux to the electrons. Thus, each electron gets attached to it an odd number of flux quanta. By our preceding argument, the electrons with the attached flux quanta become bosons. We now have bosons moving in the absence of a background magnetic field. Since bosons can condense according to Bose and Einstein, we conclude that a Hall liquid with inverse filling factor equal to an odd integer is energetically favored.

## III. EFFECTIVE FIELD THEORY

The situation described in Sec. II above may be effectively represented by the Chern-Simons gauge field theory coupled with external electromagnetic potential. The Chern-Simon term plays a role of attaching fluxes to electrons. By solving the equation of motion and the Maxwell equation consistency, we can obtain the charge density, current and potential profiles.

The effective Chern-Simons gauge Lagrangian density in the dual representation [23–27] is

\[
\mathcal{L} = \frac{1}{4\pi} \sum_{I,J} K_{IJ} \varepsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{J\lambda} - \sum_f \left( \frac{1}{2\pi} A_\mu \varepsilon^{\mu\nu\lambda} \partial_\nu a_{f\lambda} + \frac{1}{g} f_{I\mu\nu} f_I^{\mu\nu} \right),
\]  

(3.1)
where $a_{I\mu}$ is the Chern-Simons gauge field, the integer-valued symmetric matrix $K$ is written $K = I + pC$, where $I$ is the $m \times m$ identity matrix and $C$ is the $m \times m$ matrix in which every element is unity, namely,

$$C = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \vdots & \\
& \vdots & \ddots & \\
1 & \cdots & 1 
\end{pmatrix}.$$  

The matrix $K$ specifies the coupling among the Chern-Simons gauge fields and is also related to the filling factor by

$$\nu = \sum_{IJ} (K^{-1})_{IJ}$$

of FQH liquid [28]. The Maxwell term $(1/g) f_{I\mu\nu} f^{\mu\nu}_I$ in (3.1) is explicitly written as $(2/g)[c^2 f_{Ixy} f_{I0x}^2 - f_{I0x}^2 - f_{I0y}^2]$, where $g$ is the coupling constant and $c$ is the velocity of the Bogoliubov mode [2]. The vector potential $A_\mu (\mu = 0, x, y)$ of the electromagnetic field is coupled to the $\mu$-th component of the charge current density which is given by

$$J_\mu = \sum_{I=1}^m J_I^\mu,$$  

where the contribution from $I$-th conserved current densities is

$$J_I^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_{I\lambda}.$$  

Note that the vector potential for the constant external magnetic field $B_0$ has been already taken into account in the structure of the $K$ matrix, and is not included in $A_\mu$. Similarly $a_{I\mu}$ and the density $J_I^0$ are measured from their average values in the following discussion.

The Lagrangian density is integrated over a sample $S$. On the boundary $\partial S$ we impose

$$\sum_{\alpha=x,y} J_I^n a_\alpha |_{\partial S} = 0,$$  

where $\vec{n} = (n_x, n_y)$ is the unit vector normal to the boundary. This boundary condition simply expresses the physical condition that the current can not flow through the boundary
∂S. Since it is a physical requirement, it is obviously invariant with respect to a gauge transformation \( a_{I \mu} \to a_{I \mu} + \partial_{\nu} \phi_I \). A remarkable fact is that the Chern-Simons term in Lagrangian density (3.1) is also gauge invariant after integration over \( S \) with the boundary condition (3.4).

**IV. CHARGE AND CURRENT DISTRIBUTION**

We consider a FQH liquid with filling \( \nu = m/(mp + 1) \) where \( m \) is an integer and \( p \) is an even integer. To study steady-state distributions of charges and currents, we need the equation of motion derived from the effective action for the Lagrangian density (3.1). It reads

\[
\sum_{J} K_{IJ} J_{J}^{\mu} = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_{\nu} A_{\lambda} + 4 \partial_{\nu} f_{I}^{\mu\nu} / g
\]

which is expressed in terms of the current density only and is gauge invariant. It is explicitly written

\[
\begin{bmatrix}
K & -8\pi g^{-1} \partial_{t} & -8\pi c^{2} g^{-1} \partial_{y} \\
8\pi g^{-1} \partial_{t} & K & 8\pi c^{2} g^{-1} \partial_{x} \\
-8\pi g^{-1} \partial_{y} & 8\pi g^{-1} \partial_{x} & K
\end{bmatrix}
\begin{bmatrix}
J^{x} \\
J^{y} \\
J^{0}
\end{bmatrix}
= \frac{1}{2\pi}
\begin{bmatrix}
(\partial_{y} A_{0} - \partial_{t} A_{y}) q \\
-(\partial_{x} A_{0} - \partial_{t} A_{x}) q \\
B q
\end{bmatrix},
\]

where \( J^{\mu} = [J^{\mu}_{1}, \cdots , J^{\mu}_{m}] \) and \( q = [1, \cdots, 1] \) are vectors with \( m \) components.

Let us investigate the stationary distributions of potential and charge in a Hall bar which is uniform in the \( y \)-direction, then one may put \( \partial_{t} J_{I}^{\mu} = \partial_{t} A_{I} = 0 \) in (4.2). From the homogeneity of the system in the \( y \)-direction, impose \( \partial_{y} J_{I}^{\mu} = \partial_{y} A_{I} = 0 \). Thus we have

\[
K J^{y} + \frac{8\pi c^{2}}{g} \partial_{x} J^{0} = -\frac{1}{2\pi} \partial_{x} A_{0} q,
\]

\[
\frac{8\pi}{g} \partial_{y} J^{y} + K J^{0} = 0.
\]

Combining these, we have an equation for \( J^{0} \):

\[
K^{2} J^{0} - \left( \frac{8\pi c}{g} \right)^{2} \partial_{x}^{2} J^{0} = \frac{4}{g} \partial_{x}^{2} A_{0} q.
\]
Here the potential $A_0$ is connected to the charge current $J^0(x) \equiv \sum_I J^0_I(x)$ through

$$A_0(x) = -\xi \int_{L_1}^{L_2} dx' \ln |x - x'| J^0(x')$$

(4.6)

where $\xi$ is a constant having the dimension of velocity $[\text{L}T]$. Once $J^0$ is determined, the current $J^y$ is obtained from (4.3) or (4.4).

To diagonalize these matrix equations, introduce an orthogonal matrix

$$U = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \cdots & 0 & -\frac{1}{\sqrt{2}} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \\
1/\sqrt{m} & 1/\sqrt{m} & \cdots & 1/\sqrt{m} & 1/\sqrt{m}
\end{pmatrix}.$$ 

The matrix $C$ is diagonalized as

$$UCU^{-1} = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & m
\end{pmatrix}.$$ 

Then (4.3) becomes decoupled equations

$$(1 + mp)^2 J^0_I(x) - \left(\frac{8\pi c}{g}\right)^2 \partial_x^2 J^0_I(x) = \frac{4m}{g} \partial_x^2 A_0(x),$$

(4.7)

$$(J^0_I - J^0_m) - \left(\frac{8\pi c}{g}\right)^2 \partial_x^2 (J^0_I - J^0_m) = 0 \quad \text{for } I = 1, \cdots, m - 1.$$ 

(4.8)

The densities $J^0_I - J^0_m$ ($I = 1, \cdots, m - 1$) which are orthogonal to $J^0(x)$ and sometimes called as “neutral modes” are unphysical degrees of freedom, since they do not have electromagnetic couplings.

Therefore there is only a single mode even for the hierarchical FQH states and it is not exponentially localized near the edge as shown below. This result is in sharp contrast with the claims of Wen [12] and Macdonald [17] that there exist a number of edge branches in the hierarchical FQH liquids.
By solving (4.7), all the long-range behavior of the electronic density and current can be obtained. We have two characteristic length scales

$$\lambda_1 = \frac{8\pi c}{g(1 + mp)},$$  

and

$$\lambda_2 = \frac{4\pi m\xi}{g(1 + mp)^2}.$$

These two scales should be much larger than the magnetic length scale since our starting point is the long-range effective theory described by the Lagrangian density (3.1). The length scale $\lambda_1$ appears as the localization length of an edge mode in [21]. This edge mode, however, does not exist when the Hall conductance is quantized, i.e. when the longitudinal voltage drop is zero. In what follows, we will denote the ratio of the two scales as

$$\eta = \left(\frac{\lambda_1}{\lambda_2}\right)^2 = \left(\frac{2c}{\nu\gamma}\right)^2,$$

and study the effects of the parameter $\eta$.

In the Lagrangian density (3.1), the particle-particle repulsive interaction is represented by the Maxwell term (more specifically, by the spatial part $c^2 f_{Ixy}$). Thus if $\eta = \lambda_1/\lambda_2 = 0$, (4.6) and (4.7) represent a non-interacting case. Remarkably these equations with $\eta = 0$ are essentially identical to those of MacDonald, Rice and Brinkman [30] obtained to study the charge and potential profiles in the integer quantum Hall effect. It is quite unexpected since our method is based on the effective field theory and is completely different from theirs.

A. Half-plane

The Wiener-Hopf method is used to obtain the charge, current and potential profiles of the FQH liquids of an infinitely wide Hall bar with a single edge located at $x = 0$. For the integer quantum Hall liquid i.e. $\eta = 0$, Thouless [29] analytically solved the equations of MacDonald et al. by the Wiener-Hopf method in the same geometry. It is shown, however, that the nature of the solutions are completely different for the two cases: $\eta = 0$ and $\eta \neq 0$. 

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So the effect of the interaction cannot be treated perturbatively from the non-interacting situations ($\eta = 0$). The approach we are taking for this correlated electron problem is totally non-perturbative.

In the present geometry, (4.6) and (4.7) are written

\[ J_0(x) - \lambda_1^2 \partial_x^2 J_0(x) = \frac{\lambda_2}{\pi \xi} \partial_x^2 A_0(x) \quad (0 \leq x < \infty), \tag{4.12} \]

\[ A_0(x) = -\xi \int_0^\infty dx' \ln |x - x'| J_0(x'). \tag{4.13} \]

To apply the Wiener-Hopf technique, we separate $A_0(x)$ as

\[ A_0^+(x) = \begin{cases} A_0(x) & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \tag{4.14} \]

\[ A_0^-(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ A_0(x) & \text{for } x < 0. \end{cases} \tag{4.15} \]

Extend (4.12) to the region $x < 0$ as

\[ J_0(x) - \lambda_1^2 \partial_x^2 J_0(x) = \frac{\lambda_2}{\pi \xi} \partial_x^2 (A_0^+(x) + \theta(-x)(A_0^+(0) + A_0^{++}(0)x)) \tag{4.16} \]

\[ = \frac{\lambda_2}{\pi \xi} (\partial_x^2 A_0^+(x) - \delta'(x) A_0^+(0) - \delta(x) A_0^{++}(0)x) \quad (-\infty < x < \infty), \tag{4.17} \]

where $\theta(x)$ is the step function: $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$, and prime denotes differentiation with respect to $x$.

Introduce Fourier transforms of these functions as

\[ f(k) = \int_{-\infty}^{\infty} dx J_0(x)e^{ikx}, \tag{4.18} \]

\[ g^\pm(x) = \int_{-\infty}^{\infty} dx A_0^\pm(x)e^{ikx}. \tag{4.19} \]

Note $f(k)$ and $g^+(k)$ converge in the upper complex $k$ plane and $g^-(k)$ converges in the lower complex $k$ plane. Regularize the Fourier transformation of the logarithm by multiplying a dumping factor $e^{-a|x|}$ as

\[ \int_{-\infty}^{\infty} dx e^{-a|x|} \ln |x|e^{i k x} = \frac{-2}{k^2 + a^2} \left[ a C + \frac{1}{2} a \ln(k^2 + a^2) + k \tan^{-1} \frac{k}{a} \right], \tag{4.20} \]
where $\gamma = \lim_{m \to \infty} \left( \sum_{n=1}^{m} \frac{1}{n} - \ln m \right)$ is Euler’s constant. The r.h.s. becomes $-\frac{\pi}{|k|}$ if $k \neq 0$ in the limit $a \to 0$ and it diverges if $k = 0$. By discarding this divergence, we have regularized equations

$$f(k) + \lambda_1^2 k^2 f(k) = \frac{\lambda_2}{\pi \xi} (-k^2 g_+(k) - A_0^+(0) + iA_0^+(0)k), \quad (4.21)$$

$$g_+(k) + g_-(k) = \xi \frac{\pi}{|k|} f(k). \quad (4.22)$$

Eliminating $g_+(k)$, we have

$$f(k) \left(1 + \lambda_2 |k| + \lambda_1^2 k^2\right) = \frac{\lambda_2}{\pi \xi} (k^2 g_-(k) - A_0^+(0) + iA_0^+(0)k). \quad (4.23)$$

In order to analyze (4.23) we consider factorization of the analytic function

$$1 + |z| + \eta z^2 = \frac{N(z)}{D(z)}, \quad (4.24)$$

where $|z|$ is defined by $|z| = z$ if $\text{Im}(z) \geq 0$ and $|z| = -z$ if $\text{Im}(z) < 0$; $N(z)$ is analytic on the complex plane except for $\{z = iy | y \geq 0\}$ and $D(z)$ is analytic except for $\{z = -iy | y \geq 0\}$.

Solutions for $D(z)$ and $N(z)$ are

$$D(z) = \eta^{-1/2} \exp \left( \frac{1}{\pi} \int_0^\infty idy \frac{y \tan^{-1} \frac{y}{1-\eta y}}{iy + z} \right), \quad (4.25)$$

$$N(z) = \eta^{1/2} \exp \left( -\frac{1}{\pi} \int_0^\infty idy \frac{y \tan^{-1} \frac{y}{1-\eta y}}{iy - z} \right). \quad (4.26)$$

where a branch of $\tan^{-1} y$ is chosen as $0 < \tan^{-1} y < \pi$. On the real axis ($z = k \in \mathbb{R}$), $D(k)$ can be written

$$D(k) = (1 + |k| + \eta k^2)^{-1/2} e^{i\varphi(k; \eta)}, \quad (4.27)$$

where

$$\varphi(k; \eta) = \frac{k}{\pi} \int_0^\infty dy \frac{\tan^{-1} \frac{y}{y^2 + k^2} - \pi/2}{y^2 + k^2} + \frac{\pi}{2} \left( \theta(k) - 1/2 \right) \quad \text{for } \eta = 0,$$

$$= \frac{k}{\pi} \int_0^\infty dy \frac{\tan^{-1} \frac{y}{y^2 + k^2} - \pi}{y^2 + k^2} + \pi \left( \theta(k) - 1/2 \right) \quad \text{for } \eta > 0. \quad (4.28)$$

Note that the phase factor $\varphi(k; \eta)$ has a singularity at $\eta = 0$. If $\eta = 0$, $\varphi(k; \eta)$ varies from $-\pi/4$ to $\pi/4$. For $\eta > 0$, however, it varies from $-\pi/2$ to $\pi/2$. 


By substituting $z = \lambda_2 k$ to (1.24) we obtain

$$
(1 + \lambda_2 |k| + \lambda_1^2 k^2) = \frac{N(\lambda_2 k)}{D(\lambda_2 k)},
$$

where $D(\lambda_2 k)$ converges in the upper half complex $k$-plane and $N(\lambda_2 k)$ converges in the lower half complex $k$-plane. Rewrite (4.23) as

$$
f(k)D(\lambda_2 k) = \frac{\lambda_2}{\pi^2} (k^2 g_2(k) - A'(0) + iA(0)k)
$$

then both sides of the equation must be well-defined. Therefore they are entire functions which is a constant. Thus we have a solution

$$
f(k) = const. D(\lambda_2 k). \quad (4.29)
$$

Now we study the asymptotic behavior of $J^0(x)$ for $\eta > 0$. The differential equation for $\varphi(k; \eta)$ is

$$
\frac{d\varphi(k; \eta)}{dk} = \frac{\beta_+ \log \alpha_+ k^2}{2\pi(\alpha_+ k^2 - 1)} + \frac{\beta_- \log \alpha_- k^2}{2\pi(\alpha_- k^2 - 1)},
$$

where

$$
\alpha_\pm = \frac{2\eta^2}{1 - 2\eta \pm \sqrt{1 - 4\eta}},
\beta_\pm = \frac{\eta(\pm 1 \mp 4\eta + \sqrt{1 - 4\eta})}{\sqrt{1 - 4\eta}(1 - 2\eta \pm \sqrt{1 - 4\eta})}.
$$

From this, we obtain the asymptotic forms of $\varphi(k; \eta)$ as

$$
\varphi(k; \eta) \sim -\frac{1}{2\pi} \left( \beta_+ \log \alpha_+ + \beta_- \log \alpha_- \right) k - \frac{1}{\pi} k(\log k - 1) + \cdots \quad (k \sim 0),
$$

$$
\sim \frac{\pi}{2} - \frac{1}{2\pi} \left( \frac{\beta_+}{\alpha_+} \log \alpha_+ + \frac{\beta_-}{\alpha_-} \log \alpha_- \right) \frac{1}{k} + \frac{1}{\eta\pi} \frac{1}{k} (\log \frac{1}{k} - 1) + \cdots \quad (k \sim \infty),
$$

To study the behavior of $J^0$ on the edge, we estimate

$$
\int_\infty^c dk f(k) \sim \int_0^c dk (1 + k + \eta k^2)^{-1/2} (c_1 k + c_2 k(\log k - 1))
$$

$$
+ \int_c^\infty dk (1 + k + \eta k^2)^{-1/2} (c_1 \frac{1}{k} + c_2 \frac{1}{k} (\log \frac{1}{k} - 1))
$$

$$
\sim \int_0^c dk (1 - \frac{1}{2} k)(c_1 k + c_2 k(\log k - 1)) \quad (4.31)
$$

$$
+ \int_c^\infty dk \eta^{-1/2} k^{-1} (c_1 \frac{1}{k} + c_2 \frac{1}{k} (\log \frac{1}{k} - 1))
$$

$$
< \infty.
$$
Here we used (4.27), (4.29) and the constants $c_i$, $c_i'$ are obtained by combining $\alpha_\pm$ and $\beta_\pm$ suitably. From this, it is expected that the charge density takes a finite value at the edge ($x = 0$). In other words, $J(x)$ has no divergence if $\eta > 0$. Since we have the Maxwell term in the effective Lagrangian which partially comes from Coulomb repulsion between electrons, it is a natural consequence. Furthermore, we can obtain the fourier transform of $f(k)$, using approximations $(1 + k + \eta k^2)^{-1/2} \sim \eta^{-1/2} k^{-1}$ and $\varphi(k; \eta) \sim \pi/2 - \theta(-k)\pi - c^2/k$, as

$$
\int_{-\infty}^{\infty} dk f(k)e^{-ikx} \sim \int_0^{\infty} dk k^{-1} \sin(kx + \frac{c^2}{k})
= \text{const.} J_0(2cx^{1/2}) \quad (x \geq 0),
= 0 \quad (x < 0),
$$

(4.32)

where $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots$ is the zeroth Bessel function of the first kind.

From these and the results of Thouless [29], we have

$$
J^0(x) \sim x^{-1/2} \quad \eta = 0,
\sim 1 - \text{const.}x \quad \eta \neq 0.
$$

(4.33)

This behavior is shown in Fig. 1. It can be shown from (4.4) that

$$
\frac{8\pi}{g} \partial_x J^y + (1 + mp)J^0 = 0.
$$

(4.34)

Thus the asymptotic behavior of the current distribution near the edge is

$$
J^y(x) \sim -x^{1/2} + \text{const.} \quad \eta = 0,
\sim -x + \text{const.} \quad \eta \neq 0.
$$

(4.35)

**B. Hall bar**

For a Hall bar with finite width $[-L, L]$, although time-reversal symmetry ($T$) is broken due to the existence of the magnetic field, the system has $TP$ symmetry where $P$ is parity. It leads to $J^0(x) = -J^0(-x)$ and $A_0(x) = -A_0(-x)$. Since the integral in (4.6) has a finite
interval, the Fourier transformation method used in the last subsection for the half plane does not work well. Thus we use another powerful method of Hilbert transformation. In what follows, we rescale $x$ such that the interval $[-L, L]$ becomes $[-1, 1]$. Thus we should rescale the parameters as $\lambda_i/L$. We will write these rescaled parameters as $\lambda_i$ for simplicity.

The derivative of (4.6) is represented by the Hilbert transformation as

$$\partial_x A_0(x) = \xi \text{p.v.} \int_{-1}^{1} dy \frac{J^0(y)}{y-x} = \pi \xi \mathcal{T}_x[J^0(y)],$$

(4.36)

where $\mathcal{T}_x$ denotes the Hilbert transformation

$$\mathcal{T}_x[f(y)] \equiv \text{p.v.} \int_{-1}^{1} \frac{dy}{\pi} \frac{f(y)}{y-x}.$$

(4.37)

Here p.v. $\int dy$ denotes the principal value integral. From (4.6), (4.7) and (4.36) the equations for currents $J^0(x)$, and $J^y(x)$ are

$$J^0(x) - \lambda_1^2 \partial_x^2 J^0(x) = \lambda_2 \partial_x \mathcal{T}_x[J^0(y)],$$

(4.38)

$$\frac{\lambda_1}{c} J^y(x) = -\lambda_1^2 \partial_x J^0(x) - \lambda_2 \mathcal{T}_x[J^0(y)].$$

(4.39)

i) $\lambda_1 = 0 (\eta = 0)$: Suppose that the charge distribution $J^0(x)$ has singularities at the edges $x = \pm 1$, as it has in the half-plane Hall bar. The singularities must be integrable, namely, $J^0(x) \sim (1 \pm x)^{-\alpha}$ with $0 < \alpha < 1$. It can be shown from Theorem II in Appendix that if $\alpha \neq 1/2$, the r.h.s. of (4.38) has singularities $\sim (1 \pm x)^{-\alpha-1}$ which leads to a contradiction. Thus the singularity is not allowed except $\alpha = 1/2$. The factor $\cot(\alpha \pi)$ in Theorem II vanishes if $\alpha = 1/2$ and the above argument against the existence of singularities does not hold.

**Proposition I.** If a solution of (4.38) with $\lambda_1 = 0 (\eta = 0)$ has singularities at the edges $(x = \pm 1)$, the power of the singularity must be $-1/2$.

Note that this singularity $-1/2$ coincides with the result for the half-plane (see (4.33) and [29]).

The inverse operation of the Hilbert transformation (Theorem III in Appendix), gives a systematic expansion of $J^0(x)$ for $\lambda_2 > 1$ as
\[ J^0(x) = \sum_{n=0}^{\infty} \lambda_2^{-n} j_n(x), \quad (4.40) \]

where \( j(x) \)'s satisfy the relation

\[
0 \leftarrow \frac{\partial_x T_x}{j_0(x)} \leftarrow \frac{\partial_x T_x}{j_1(x)} \leftarrow \frac{\partial_x T_x}{\cdots}.
\]

At the edges \( j_0(x) \) has singularities of power \(-1/2\) and \( j_n(\pm1) = 0 \) for \( n \geq 1 \). Theorems I and III in Appendix give the explicit solutions for \( j_0 \) and \( j_1 \) as

\[
\begin{align*}
  j_0(x) & = -\frac{x}{\sqrt{1-x^2}}, \\
  j_1(x) & = -\frac{1}{\pi} \sqrt{1-x^2} \log \left( \frac{1-x}{1+x} \right).
\end{align*}
\]

Using \((4.39)\), we have

\[
\begin{align*}
  \frac{8\pi}{g(1+mp)} J^0(x) & = \lambda_2 \left( 1 + \frac{1}{\lambda_2} \left( \frac{2}{\pi} - \sqrt{1-x^2} \right) \right) + O(\lambda_2^{-1}), \\
  \frac{1}{\pi \xi} A_0(x) & = -\left( \frac{2 + \pi \lambda_2}{\pi \lambda_2} x - \frac{x}{\sqrt{1-x^2}} - \frac{\sin^{-1} x}{2\lambda_2} \right) + O(\lambda_2^{-2}).
\end{align*}
\]

The current distribution \((4.43)\) is symmetric. It is contrasted with the edge picture in which the currents flow in the opposite directions at the two edges. The density, current and potential profiles thus obtained is plotted in Fig. 2.

ii) \( \lambda_1 > 0 \) (\( \eta > 0 \)): In this case the singularities of the charge distribution at the edges is suppressed due to the second derivative term in the r.h.s. of \((4.38)\).

If \( J^0(x) \) has power singularities \( \sim (1 \pm x)^{-\alpha} \) \((0 < \alpha < 1)\) at the edges, the l.h.s. of \((4.38)\) has singularities of power \(-\alpha - 2\). On the other hand the r.h.s. of \((4.38)\) has singularities of power \(-\alpha - 1\). Thus power singularities are forbidden.

If \( J^0(x) \) has a logarithmic singularities \( J^0(x) \sim \log(1-x) - \log(1+x) \) (which we will call as a “simple” logarithmic singularity), one can prove that \((4.38)\) does not hold since

\[
\partial_x T_x [\log(1-y) - \log(1+y)] = -\frac{2 \log(1-x) - \log(1+x)}{1-x^2}.
\]
For another logarithmic singularities like higher power of logarithm, for example, to find explicit Hilbert transformations becomes more cumbersome. However, it is expected that

$$\partial_x T_x[\text{logarithmic singularity}] \sim \frac{\text{logarithmic singularity}}{1 - x^2},$$

from a power counting argument. The second derivative of this logarithmic singularity gives a singularity with power $-2$. Then $J^0(x)$ can not have any logarithmic singularities which is a contradiction. Thus we assert

**Proposition II.** A solution of (4.38) with $\lambda_1 > 0$ ($\eta > 0$) has neither a power singularity nor a (simple) logarithmic singularity at the edges ($x = \pm 1$).

Since the Coulomb interaction repels particles each other, a divergent singularity at an edge is physically unacceptable. This observation is consistent with Proposition II. Thus we claim that the charge density is finite at the edges. Note that there is a singularity at the edge in the noninteracting case $\eta = 0$ (**29**). This is caused by the absence of particle repulsion and disappears once an interaction is taken into account.

In order to obtain $J^0(x)$ we expand it in powers of $\lambda_2$ as

$$J^0(x) = \sum_{n=0}^{\infty} \lambda_2^n j_n(x).$$

By substituting this into (4.38), we obtain

$$j_0(x) \equiv -\frac{\sinh(x/\lambda_1)}{\sinh(1/\lambda_1)}, \tag{4.45}$$

and

$$j_1(x) - \lambda_1^2 \partial_x^2 j_1(x) = \partial_x T_x[j_0(y)]. \tag{4.46}$$

The solution of the inhomogenous differential equation (4.46) is

$$j_1(x) = \frac{1}{8\pi \lambda_1^3 \sinh(1/\lambda_1)} \times$$
$$\times \left[ \frac{\sinh(x/\lambda_1)}{\sinh(1/\lambda_1)} \right] \left( (-2 - \lambda_1 e^{-2/\lambda_1} + 2\lambda_1) e^{-1/\lambda_1} \text{Ei} \left( \frac{2}{\lambda_1} \right) \right)$$
\[+(2 - \lambda_1 e^{2/\lambda_1} + 2\lambda_1)e^{1/\lambda_1}Ei\left(\frac{-2}{\lambda_1}\right)\]
\[+2(-\lambda_1 \cosh(1/\lambda_1) + 2\sinh(1/\lambda_1))\operatorname{Chi}\left(\frac{2}{\lambda_1}\right)\]
\[-2(-\lambda_1 \sinh(1/\lambda_1) + 2\cosh(1/\lambda_1))\operatorname{Shi}\left(\frac{2}{\lambda_1}\right)\}
\[+\left\{(-2 - \lambda_1 e^{-2/\lambda_1} + 2\lambda_1)\left(e^{x/\lambda_1}Ei\left(\frac{1-x}{\lambda_1}\right) - e^{-x/\lambda_1}Ei\left(\frac{1+x}{\lambda_1}\right)\right)\right\}
\[+(2 - \lambda_1 e^{2/\lambda_1} + 2\lambda_1)\left(e^{-x/\lambda_1}Ei\left(-\frac{1+x}{\lambda_1}\right) - e^{x/\lambda_1}Ei\left(-\frac{1-x}{\lambda_1}\right)\right)\}
\[+2\left\{(-\lambda_1 \cosh(x/\lambda_1) + 2x \sinh(x/\lambda_1))\left(\operatorname{Chi}\left(\frac{1-x}{\lambda_1}\right) - \operatorname{Chi}\left(\frac{1+x}{\lambda_1}\right)\right)\right\}
\[+(-\lambda_1 \sinh(x/\lambda_1) + 2x \cosh(x/\lambda_1))\left(\operatorname{Shi}\left(\frac{1-x}{\lambda_1}\right) + \operatorname{Shi}\left(\frac{1+x}{\lambda_1}\right)\right)\}\right], \quad (4.47)

where \(\operatorname{Ei}(x)\), \(\operatorname{Shi}(x)\) and \(\operatorname{Chi}(x)\) are the exponential, hyperbolic sine and hyperbolic cosine integrals, respectively, which are explicitly written

\[\operatorname{Ei}(x) = -\text{p.v.} \int_{-x}^{\infty} \frac{e^{-t}}{t} dt, \quad (4.49)\]
\[\operatorname{Shi}(x) = \int_{0}^{x} \frac{\sinh t}{t} dt, \quad (4.50)\]
\[\operatorname{Chi}(x) = \gamma + \ln x + \int_{0}^{x} \frac{\cosh t - 1}{t} dt, \quad (4.51)\]

and we have used the Hilbert transformation

\[\mathcal{T}_x[-\frac{\sinh(y/\lambda_1)}{\sinh(1/\lambda_1)}] \]
\[= -\frac{\sinh(x/\lambda_1)}{\pi \sinh(1/\lambda_1)} \left(\operatorname{Chi}\left(\frac{1-x}{\lambda_1}\right) - \operatorname{Chi}\left(\frac{1+x}{\lambda_1}\right)\right) \]
\[-\frac{\cosh(x/\lambda_1)}{\pi \sinh(1/\lambda_1)} \left(\operatorname{Shi}\left(\frac{1-x}{\lambda_1}\right) + \operatorname{Shi}\left(\frac{1+x}{\lambda_1}\right)\right). \quad (4.52)\]

Using (4.6) and (1.39), we have

\[\frac{8\pi}{g(1+mp)} J^y(x) = -\lambda_1^2 \partial_x (j_0(x) + \lambda_2 j_1(x)) - \lambda_2 \mathcal{T}_x[j_0(y)] + \mathcal{O}(\lambda_2^2), \quad (4.53)\]
\[\frac{1}{\xi} A_0(x) = -\text{p.v.} \int_{-1}^{1} dx' \log |x' - x| (j_0(x') + \lambda_2 j_1(x')) + \mathcal{O}(\lambda_2^2). \quad (4.54)\]

Note that the current density \(J^y\) in (4.53) has no divergence at the edges and it is again symmetric. It is contrasted with the edge picture in which the currents flow in the opposite directions at the two edges. The lowest order approximation of \(A_0(x)\) is obtained by using
\[-\text{p.v.} \int_{-1}^{1} dy \log |y - x| j_0(y) \]
\[= \frac{\lambda_1}{\sinh(1/\lambda_1)} \times \]
\[\times \left\{ -\cosh(x/\lambda_1) \left( \text{Chi} \left( \frac{1-x}{\lambda_1} \right) - \text{Chi} \left( \frac{1+x}{\lambda_1} \right) \right) \right. \]
\[\left. - \sinh(x/\lambda_1) \left( \text{Shi} \left( \frac{1-x}{\lambda_1} \right) - \text{Shi} \left( \frac{1+x}{\lambda_1} \right) \right) \right\} + \cosh(1/\lambda_1) \log \left( \frac{1-x}{1+x} \right) \right\}. \quad (4.55)\]

The electron density is plotted in Fig. 3 (a) in the lowest order approximation \((\lambda_2 = 0)\) (4.45) and (b) with the correction (4.48). The current distribution is plotted in Fig. 4 (a) in the lowest order approximation \((\lambda_2 = 0)\) and (b) with the correction (4.53).

Note that in the lowest order approximation, both the electron density and the current is localized near the edges exponentially with the localization length \(\lambda_1\). This behavior, however, completely disappears again once the corrections are taken into account as seen from Fig. 3 (b) and Fig. 4 (b) where the bulk currents do not vanish.

The potential is plotted in Fig. 5. Note that in the lowest order approximation, \(A_0(x)\) has a maximum and a minimum near the edges in contrast to the non-interacting case \(\eta = 0\) (4.44). These maximum and minimum of \(A_0(x)\) survive even if the next leading contribution is taken into account as seen from Fig. 5 (b).

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APPENDIX A: THEOREMS OF THE SINGULAR INTEGRAL EQUATIONS

The following Theorems I, II and III [31] are used in the main text.

Theorem I.
\[ T_x \left[ \left( \frac{1 - y}{1 + y} \right)^\alpha \right] = \cot(\alpha \pi) \left( \frac{1 - x}{1 + x} \right)^\alpha - \frac{1}{\sin(\alpha \pi)} \]  

(A1)

for \(-1 < \alpha < 1\).

**Theorem II.** Let \(f(x)\) be an \(L_p\)-function \((p > 1)\) which in a small neighborhood \((-1, -1 + \delta)\) \((\delta > 0)\) of the point \(x = -1\) can be written in the form

\[ f(x) = A(1 + x)^{-\alpha} + g(x) \quad (0 \leq \alpha < 1), \]

where \(A\) is a constant, \(g(x)\) vanishes at \(x = -1\) and satisfies (uniformly) a Lipschitz condition of positive order \(\epsilon\), i.e.

\[ |g(x) - g(x_0)| < K|x - x_0|^\epsilon. \]

Then the Hilbert transform of \(f(x)\) has the asymptotic representation

\[ T_x[f(y)] = A \cot(\alpha \pi)(1 + x)^{-\alpha} + O(1) \quad (x \to -1), \]

if \(0 < \alpha < 1\), and the asymptotic representation

\[ T_x[f(y)] = -\frac{A}{\pi} \log(1 + x) + O(1) \quad (x \to -1), \]

if \(\alpha = 0\).

If the point \(x = -1\) is replaced by the point \(x = +1\), then all remains the same except that \(\cot(\alpha \pi)\) is changed into \(-\cot(\alpha \pi)\) and \(-\log(1 + x)\) is changed into \(+\log(1 - x)\).

**Theorem III.** If a given function \(f(x)\) belongs to the class \(L_{4/3+\epsilon}\) for sufficiently small \(\epsilon > 0\), the equation

\[ f(x) = T_x[\phi(y)], \]  

(A2)

has the solution

\[ \phi(x) = \frac{C}{\sqrt{1 - x^2}} - \text{p.v.} \int_{-1}^{1} \frac{dy}{\pi} \sqrt{1 - y^2} \frac{f(y)}{1 - x^2 y - x}, \]

(A3)

where \(C\) is an arbitrary constant.
REFERENCES

[1] S.C. Zhang, T.H. Hansson, and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989).

[2] S.C. Zhang, Int. J. Mod. Phys. B6, 25 (1992) and references therein.

[3] A. Zee, in Field Theory, Topology and Condensed Matter Physics, Proceedings of the Ninth Chris Engelgrecht Summer School in Theoretical Physics, ed. H.B. Geyer, (Springer-Verlag Berlin Heidelberg, 1995) and references therein.

[4] X.G. Wen and A. Zee, Nucl. Phys., (suppl.) B15, 135 (1990).

[5] B. Blok and X.G. Wen, Phys. Rev. B42, 8133 (1990); Phys. Rev. B 43, 8337 (1991).

[6] X.G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).

[7] X. G. Wen and A. Zee, Phys. Rev. B 44, 274 (1991).

[8] Z.F. Ezawa and A. Iwasaki, Phys. Rev. B 43, 2637 (1991).

[9] J. Fröhlich and T. Kerler, Nucl. Phys. B354, 369 (1991).

[10] J. Fröhlich and A. Zee, Nucl. Phys. B364, 517 (1991).

[11] X.G. Wen, Phys. Rev. B40, 7387 (1989); Int. J. Mod. Phys. B2, 239 (1990).

[12] X.G. Wen, Phys. Rev. B41, 12838 (1990); Phys. Rev. B43, 11025 (1991); Phys. Rev. Lett. 64, 2206 (1990).

[13] D.H. Lee and X.G. Wen, Phys. Rev. Lett. 66, 1765 (1991).

[14] C.L. Kane, M.P.A. Fisher, and J. Polchinski, Phys. Rev. Lett. 72, 4129 (1994).

[15] F.D.M. Haldane, Phys. Rev. Lett. 74, 2090 (1995).

[16] M. Büttiker, Phys. Rev. B38, 9375 (1988).

[17] A.H. MacDonald, Phys. Rev. Lett. 64, 220 (1990).

[18] C.L. Kane and M.P.A. Fisher, Phys. Rev. Lett. 68, 1220 (1992).
[19] A. Furusaki and N. Nagaosa, Phys. Rev. B47, 3827 (1993).

[20] K. Moon, H. Yi, C.L. Kane, S.M. Girvin, and M.P.A. Fisher, Phys. Rev. Lett. 71, 4381 (1993).

[21] N. Nagaosa and M. Kohmoto, Phys. Rev. Lett. 75, 4294 (1995).

[22] J.K. Jain, Phys. Rev. Lett. 63, 199 (1989).

[23] J.K. Jain, Phys. Rev. B40, 8079 (1989).

[24] M. Greiter and F. Wilczek, Mod. Phys. Lett. B4, 1063 (1990).

[25] X.G. Wen and A. Zee, Phys. Rev. Lett. 62, 1937 (1989).

[26] M.P.A. Fisher and D.H. Lee, Phys. Rev. Lett. 63, 903 (1989); D.H. Lee and M.P.A. Fisher, Int. J. Mod. Phys. B5, 2675 (1991).

[27] X.G. Wen and A. Zee, Phys. Rev. B41, 240, (1990); Int. J. Mod. Phys. B4,437 (1990).

[28] X.G. Wen and A. Zee, Phys. Rev. B46, 2290 (1992).

[29] D.J. Thouless, J. Phys. C18, 6211 (1985).

[30] A.H. MacDonald, T.M. Rice, and W.F. Brinkman, Phys. Rev. B28, 3648 (1983)

[31] F.G. Tricomi, *Integral Equations*, Pure and Applied Mathematics Volume V, (International Publishers, INC., New York, 1957)
FIGURES

1 The electron density $J^0(x)$ for the half-plane. (a) Non-interacting case $\eta = 0$ in the log-log scale. (b) $\eta = 0.1, 0.5, 1.0, 1.5$ and $2.0$.

2 The density profiles for the Hall bar in the non-interacting case $\lambda_1 = 0$, $\lambda_2 = 1.2$. (a) Electron density $J^0(x)$ which has singularities with power $-1/2$ at the edges $x = \pm 1$. (b) Current $\frac{8\pi}{g(1+mp)} J^y(x)$. Note that it is symmetric. (c) Potential $\frac{1}{\pi \xi} A_0(x)$. Note that it is a monotonic function and the electric field is in the same direction in the sample.

3 The electron density $J^0(x)$ for the Hall bar. (a) $\lambda_1 = 0.1$ and $\lambda_2 = 0$. (b) $\lambda_1 = 0.1$ and $\lambda_1 = 0.15$.

4 The current density $\frac{8\pi}{g(1+mp)} J^y(x)$ for the Hall bar. (a) $\lambda_1 = 0.1$ and $\lambda_2 = 0$. (b) $\lambda_1 = 0.1$ and $\lambda_1 = 0.15$.

5 The potential $\frac{1}{\xi} A_0(x)$ for the Hall bar. (a) $\lambda_1 = 0.1$ and $\lambda_2 = 0$, (b) $\lambda_1 = 0.1$ and $\lambda_1 = 0.15$. Note that they have a maximum and a minimum near the edges. Thus the electric field is not in the same direction in the sample and vanishes at these points.