Propagations of massive graviton in the deformed Hořava-Lifshitz gravity

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Abstract

We study massive graviton propagations of scalar, vector, and tensor modes in the deformed Hořava-Lifshitz gravity by introducing Lorentz-violating mass term. It turns out that vector and tensor modes are massively propagating on the Minkowski spacetime background. However, adding the mass term does not cure a ghost instability in the Hořava scalar.

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1 Introduction

Recently Hořava has proposed a renormalizable theory of gravity at a Lifshitz point [1], which may be regarded as a UV complete candidate for general relativity. Very recently, the Hořava-Lifshitz gravity theory has been intensively investigated in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24], its cosmological applications in [25, 26], and its black hole solutions in [27, 28].

There are four versions of Hořava-Lifshitz gravity in the literature: with/without the detailed balance condition and with/without the projectability condition [29]. Hořava has originally proposed the projectability condition with/without the detailed balance condition. We mention that the IR vacuum of this theory is anti de Sitter (AdS) spacetimes. Hence, it is interesting to take a limit of the theory, which leads to a Minkowski vacuum in the IR sector. To this end, one may modify the theory by including $\mu^4 R$ and then, take the $\Lambda_W \to 0$ limit [15]. This deformed Hořava-Lifshitz gravity does not alter the UV properties of the theory, but it changes the IR properties from AdS vacuum to Minkowski vacuum. Hence, in order to see propagations of fields on the Minkowski spacetime background, we consider the deformed Hořava-Lifshitz gravity without the detailed balance condition.

Concerning the projectability condition, its role should be dealt with carefully to identify the propagation of Hořava scalar on the Minkowski background. Actually, there exists a close relation between projectability and scalar degrees of freedom. The projectability condition requires that the perturbation $A$ of the lapse function $N$ depends only on time, thus $A = A(t)$. This implies that the $A$-perturbation is not a Lagrange multiplier (field) but a time-dependent parameter. This is the key of the theory.

An urgent issue of the deformed Hořava-Lifshitz gravity is to answer to the question of whether it can accommodate the Hořava scalar $\psi$, in addition to two degrees of freedom (DOF) for a massless graviton. We would like to mention two relevant works. The authors [18] have shown that without the projectability condition, the Hořava scalar $\psi$ is related to a scalar degree of freedom appeared in the massless limit of a massive graviton. This is reminiscent of Fierz-Pauli massive gravity [30] in which the longitudinal scalar becomes strongly coupled as $m \to 0$, leading to the vDVZ discontinuity [31]. They argued that perturbative general relativity cannot be reproduced in the IR-limit of deformed Hořava-Lifshitz gravity because of the strong coupling problem. With the projectability condition, on the other hand, the authors [19] have argued that $\psi$ is propagating around the Minkowski space but it has a negative kinetic term, showing a ghost mode. Moreover, it was found that the Hořava scalar is a ghost if the sound speed squared is positive [23].

In order to understand better the problems arising when one modified the gravity in
the Lorentz-invariant way, it was instructive to consider first the Lorentz-invariant massive gravity by adding the Fierz-Pauli mass term. However, this term is not suitable for studying scalar propagations under the projectability condition. We remind the reader that the deformed Hořava-Lifshitz gravity is a Lorentz-violating gravity. Hence the Lorentz-violating mass terms are more attractive to study the issue on the propagation of Hořava scalar in the deformed Hořava-Lifshitz gravity.

In this work, we investigate massive graviton propagations of scalar, vector, and tensor modes in the deformed Hořava-Lifshitz gravity under the projectability condition by introducing Lorentz-violating mass terms.

2 Deformed Hořava-Lifshitz gravity

First of all, we introduce the ADM formalism where the metric is parameterized as

\[ ds^2_{ADM} = -N^2 dt^2 + g_{ij} \left( dx^i - N^i dt \right) \left( dx^j - N^j dt \right), \]  

(1)

Then, the Einstein-Hilbert action can be expressed as

\[ S^{EH} = \frac{1}{16\pi G} \int d^4 x \sqrt{g} N \left( K_{ij} K^{ij} - K^2 + \mathcal{R} \right), \]  

(2)

where \( G \) is Newton’s constant and extrinsic curvature \( K_{ij} \) takes the form

\[ K_{ij} = \frac{1}{2N} \left( \partial_i N_j - \partial_j N_i \right). \]  

(3)

Here, a dot denotes a derivative with respect to \( t \) (\( \dot{\cdot} = \frac{\partial}{\partial t} \)).

On the other hand, a deformed action of the Hořava-Lifshitz gravity is given by \[15\]

\[ S^{dHL} = \int dt d^3 x \left( \mathcal{L}_0 + \sqrt{g} N \mu^4 R + \mathcal{L}_1 \right), \]  

(4)

\[ \mathcal{L}_0 = \sqrt{g} N \left\{ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda_W R - 3 \Lambda_W^2)}{8(1 - 3\lambda)} \right\}, \]  

(5)

\[ \mathcal{L}_1 = \sqrt{g} N \left\{ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{\eta^4} \left( C_{ij} - \frac{\mu^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu^2}{2} R^{ij} \right) \right\}. \]  

(6)

Here \( C_{ij} \) is the Cotton tensor defined by

\[ C^{ij} = \epsilon^{ik\ell} \nabla_k \left( R_{j\ell} - \frac{1}{4} R \delta_{j\ell} \right), \]  

(7)

which is obtained from the variation of gravitational Chern-Simons term with coupling \( 1/\eta^2 \).

The full equations of motion were derived in \[25\] and \[27\], but we do not write them due to the length. Taking a limit of \( \Lambda_W \to 0 \) in \( \mathcal{L}_0 + \sqrt{g} N \mu^4 R \), we obtain the Einstein-Hilbert action with \( \lambda \) \[15\]

\[ S^{EH\lambda} \equiv \int dt d^3 x \tilde{\mathcal{L}}_0 = \int dt d^3 x \sqrt{g} N \left[ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \mu^4 R \right]. \]  

(8)
Comparing Eq. (8) with general relativity (2), the speed of light and Newton's constant are given by
\[ c^2 = \frac{\kappa^2 \mu^4}{2}, \quad G = \frac{\kappa^2}{32\pi c}, \quad \lambda = 1. \] (9)
Since we consider the \( z = 3 \) Hořava-Lifshitz gravity, scaling dimensions are \([t] = -3, [x] = -1, [\kappa] = 0, [\mu] = 1, \) and \([c] = 2.\) Even though the scaling dimensions are relevant to the UV properties, these are also necessary to define the linearized theory of \( z = 3 \) Hořava-Lifshitz gravity consistently. The reason is that we have to keep the same dimensions six for all terms, although couplings of the kinetic term \((2/\kappa^2)\) and the sixth order derivatives \((\kappa^2/2\eta^4)\) are dimensionless. To see the UV property of power-counting renormalizability, it is better to switch from the \( c = 1 \) units to (9) units that impose the scaling dimensions. Switching back to \( c = 1 \) units leads to the case that is more suitable for discussing the IR properties of strong coupling problem and vDVZ discontinuity.

The deformed Lagrangian which is relevant to our study takes the form [15]

\[ \tilde{\mathcal{L}} = \tilde{\mathcal{L}}_0 + \mathcal{L}_1 \] (10)

\[ = \sqrt{g} N \left[ \frac{2}{\kappa^2} (K_{ij}K_{ij} - \lambda K^2) + \mu^4 \left( R + \frac{1}{2\omega} \frac{4\lambda - 1}{3\lambda - 1} R^2 - \frac{2}{\omega} R_{ij}R_{ij} \right) \right. \] (11)

\[ + \frac{\kappa^2 \mu^2}{2\eta^2} \epsilon^{ijk} R_{il} \nabla_j R^l_k - \frac{\kappa^2}{2\eta^4} C_{ij} C_{ij} \right] \] (12)

where a characterized parameter \( \omega \) is given by
\[ \omega = \frac{16 \mu^2}{\kappa^2} = \frac{16\sqrt{2} c}{\kappa^3}. \] (13)

Actually, the Lagrangian (11) is enough to describe scalar and vector propagations because (12) from the Cotton tensor contributes to the tensor propagations only. For \( \lambda = 1, \) taking the IR-limit of \( \omega \to \infty \) while keeping \( c^2 = 1 \) fixed is equivalent to recovering the Einstein gravity. Explicitly, this limit implies \( \kappa^2 \to 0(\mu^4 \sim \kappa^{-2} \to \infty) \) which means that the kinetic term and curvature term \( \mu^4 R \) dominate over all higher order curvature terms. The deformed Lagrangian (10) can be redefined to be
\[ \tilde{\mathcal{L}} = \mathcal{L}_K + \mathcal{L}_V, \] (14)

where \( \mathcal{L}_K (\mathcal{L}_V) \) denote the kinetic (potential) Lagrangian.

We wish to consider perturbations of the metric around Minkowski spacetimes, which is a solution to the full theory (10)
\[ g_{ij} = \delta_{ij} + \eta h_{ij}, \quad N = 1 + \eta n, \quad N_i = \eta n_i, \] (15)

where a dimensionless coupling constant \( \eta \) from gravitational Chern-Simons term is included to define the perturbation. The inclusion of \( \eta \) makes sense because the noninteracting limit
corresponds to sending \( \eta \to 0 \) while keeping the ratio \( \gamma = \kappa/\eta \) fixed \[1\]. This in turn provides the IR-limit of \( \kappa \to 0 (\omega \to \infty) \). For \( \lambda = 1 \), this limit yields a one-parameter family of free-field fixed points parameterized by \( \gamma \).

At quadratic order the action \( S \) turns out to be

\[
S_{2EH\lambda}^2 = \eta^2 \int dt d^3 x \left\{ 1 \kappa_2 \left[ \frac{1}{2} h_{ij}^2 - \frac{\lambda}{2} h_i^2 + (\partial_i n_j)^2 + (1 - 2\lambda)(\partial \cdot n)^2 - 2\partial_i n_j (h_{ij} - \lambda \delta_{ij}) \right] \\
+ \frac{\mu^4}{2} \left[ - \frac{1}{2} (\partial_k h_{ij})^2 + \frac{1}{2} (\partial_i h)^2 + (\partial_i h_{ij})^2 - \partial_i h_{ij} \partial_j h + 2n(\partial_i \partial_j h_{ij} - \partial^2 h) \right] \right\}
\]

(16)

with \( h = h_{ii} \). A general Lorentz-violating (LV) mass term is given by \[32, 33\]

\[
S_{2LV}^2 = \frac{\eta^2}{2\kappa^2} \int dt d^3 x \left\{ 4m_0^2 n^2 + 2m_1^2 n_i^2 - \tilde{m}_2^2 h_{ij}^2 + \tilde{m}_3^2 h^2 + 4\tilde{m}_4^2 nh \right\}.
\]

(17)

As was pointed out in \[34\], it provides various phases of mass gravity in general relativity. In this work, we add Eq. (17) to the linearized theory of deformed Hořava-Lifshitz gravity to investigate the instability of Hořava scalar and strong coupling problem. In this work, we choose the case of \( m_0 = 0 \) and \( \tilde{m}_4 = 0 \) because the lapse parameter \( n(t) \) enters these terms. At this stage, we would like to mention that for generic backgrounds, the case of \( m_1 = 0 \) has provided a well-defined case in bigravity and massive gravity \[33, 35\]. Also, the generic case could be well behaved in generic backgrounds \[35\]. We compare (17) with the Lorentz-invariant Fierz-Pauli mass term as \[36\]

\[
S_{2FP}^2 = \frac{\eta^2}{2\kappa^2} \int dt d^3 x \left\{ - m^2 h_{\mu\nu} h^{\mu\nu} + m^2 (h_{\mu\nu})^2 \right\}.
\]

(18)

In order to analyze physical propagations thoroughly, it is convenient to use the cosmological decomposition in terms of scalar, vector, and tensor modes under spatial rotations \( SO(3) \) \[37\]

\[
n = -\frac{1}{2} A, \\
n_i = (\partial_i B + V_i), \\
h_{ij} = (\psi \delta_{ij} + \partial_i \partial_j E + 2\partial_i F_j + t_{ij}),
\]

(19)

where the conditions of \( \partial^i F_i = \partial^i V_i = \partial^i t_{ij} = t_{ii} = 0 \) are imposed. The last two conditions mean that \( t_{ij} \) is a transverse and traceless tensor in three spatial dimensions. Using this decomposition, the scalar modes \( (A, B, \psi, E) \), the vector modes \( (V_i, F_i) \), and the tensor modes \( (t_{ij}) \) decouple completely from each other. These all amount to 10 degrees of freedom for a symmetric tensor in four dimensions.

Before proceeding, let us check dimensions. This is a necessary step to obtain a consistently massive linearized theory. We observe that \( [n] = 0, \ [n_i] = 2, \) and \( [h_{ij}] = 0 \), which
imply $[A] = 0$, $[B] = 1$, $[V_i] = 2$, $[\psi] = 0$, $[E] = -2$, $[F_i] = -1$, and $[t_{ij}] = 0$. Also, the masses take scaling dimensions: $[m_1^2] = 2$ and $[\tilde{m}_2^2] = [\tilde{m}_3^2] = [\tilde{m}_4^2] = 6$. In order to find the true mass with dimension 1, we redefine mass squares as

$$m_i^2 = c^2 m_i^2,$$  \hspace{1cm} \text{for} \hspace{1cm} i = 2, 3, 4 \hspace{1cm} \text{(20)}$$

which implies that

$$[m_2^2] = [m_3^2] = [m_4^2] = 2.$$  \hspace{1cm} \text{(21)}$$

The Fierz-Pauli mass term is recovered when all masses are equal except for $m_0$ as

$$m_1^2 = m_2^2 = m_3^2 = m_4^2 = m^2; \hspace{1cm} m_0 = 0.$$  \hspace{1cm} \text{(22)}$$

Hence, the Fierz-Pauli mass term is not suitable for studying massive scalar propagations with projectability condition because the latter condition implies that $n(t)$ is not a field and thus, it requires $m_0^2 = 0$.

The bilinear action is obtained by substituting (19) into the quadratic action (16) as

$$S_2^{EH\lambda} = \frac{1}{2\gamma^2} \int dt d^3 x \left\{ \left[ 3(1-3\lambda)\psi^2 + 2\partial_i w_j \partial^i w^j - 4\left( (1-3\lambda)\dot{\psi} + (1-\lambda)\partial^2 \dot{E} \right) \partial^2 B \right. \\
+ 4(1-\lambda)(\partial^2 B)^2 + 2(1-3\lambda)\psi \partial^2 \dot{E} + (1-\lambda)(\partial^2 \dot{E})^2 + t_{ij} t^{ij} \left. \right] + c^2 \left( 2\partial_k \psi \partial^k \psi + 4A \partial^2 \psi - \partial_k t_{ij} \partial^k t^{ij} \right) \right\}.$$  \hspace{1cm} \text{(23)}$$

with $\gamma^2 = \kappa^2/\eta^2$ and $w_i = V_i - \dot{F}_i$. We would like to point out the coupling of $\frac{1}{2\gamma^2}$ in the front of the quadratic action because we have chosen the perturbations (15). The higher order action obtained from $L_1$ takes the form

$$S_2^1 = \frac{\kappa^2 \mu^2 \eta^2}{8} \int dt d^3 x \left\{ -\frac{1-\lambda}{2(1-3\lambda)} \psi \partial^4 \psi - \frac{1}{4} t_{ij} \partial^4 t^{ij} + \frac{1}{\mu^2 \eta^4} (\partial^4 \partial^4 + \partial^4 \partial^k t^t) \right\}.$$  \hspace{1cm} \text{(24)}$$

We find that two modes of scalar $\psi$ and tensor $t_{ij}$ exist in the higher order action only, missing vector modes. Since the spatial slice is conformally flat, the vanishing Cotton tensor and the absence of six derivative term result in the scalar sector. Also, the Cotton tensor does not contribute to vector modes ($V_i, \dot{F}_i$). The vectors are decoupled completely from bilinear terms of the potential $L_V$. This is because the vector belongs to gauge degrees of freedom in the massless graviton theory, while it has 2 DOF in the massive graviton theory. Hence, the disappearance of vector is natural for the massless theory of $z = 3$ Hořava-Lifshitz gravity.

Now we are in a position to discuss the diffeomorphism in the $z = 3$ Hořava-Lifshitz gravity. Since the anisotropic scaling of temporal and spatial coordinates ($t \to b^z t, x^i \to$...
bx^i), the time coordinate t plays a privileged role. Hence, the spacetime symmetry is smaller than the full diffeomorphism (Diff) in the general relativity [38]. The quadratic action of $S_{EH}^{EHA} + S_1^2$ should be invariant under the “foliation-preserving” diffeomorphism (FDiff) whose transformation is given by

$$t \rightarrow \tilde{t} = t + \epsilon^0(t), \quad x^i \rightarrow \tilde{x}^i = x^i + \epsilon^i(t, \mathbf{x}).$$

(25)

Using the notation of $\epsilon^\mu = (\epsilon^0, \epsilon^i)$ and $\epsilon_\nu = \eta_{\nu\mu} \epsilon^\mu$, the perturbation of metric transforms as

$$\delta g_{\mu\nu} \rightarrow \delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu.$$  

(26)

Further, making a decomposition $\epsilon^i$ into a scalar $\xi$ and a pure vector $\zeta^i$ as $\epsilon^i = \partial^i \xi + \zeta^i$ with $\partial_\xi \xi = 0$, one finds the transformation for scalars

$$A(t) \rightarrow \tilde{A}(t) = A(t) - 2 \dot{\xi}(t), \quad \psi \rightarrow \tilde{\psi} = \psi, \quad B \rightarrow \tilde{B} = B + \xi, \quad E \rightarrow \tilde{E} = E + 2 \xi.$$  

(27)

On the other hand, the vector and the tensor take the forms

$$V_i \rightarrow \tilde{V}_i = V_i + \dot{\zeta}_i, \quad F_i \rightarrow \tilde{F}_i = F_i + \zeta_i, \quad t_{ij} \rightarrow \tilde{t}_{ij} = t_{ij}.$$ 

(28)

Considering scaling dimensions of $[\epsilon^0] = -3$ and $[\epsilon^i] = -1$, we have $[\xi] = -2$ and $[\zeta^i] = -1$. For the FDiff transformations, gauge invariant combinations are

$$t_{ij}, \quad w_i = V_i - \tilde{F}_i,$$  

(29)

for tensor and vector modes, respectively and

$$\psi, \quad \Pi = 2B - \tilde{E}$$ 

(30)

for two scalar modes. At this stage, we note scaling dimensions: $[w_i] = 2$ and $[\Pi] = 1$.

Let us express the quadratic action (23) in terms of gauge-invariant quantities as (20)

$$S_2^{EH} = \frac{1}{2c^2} \int dt d^3x \left\{ 3(1 - 3\lambda)\ddot{\psi}^2 - 2w_i \triangle w^i - 2(1 - 3\lambda)\dot{\psi} \triangle \Pi + (1 - \lambda)(\triangle \Pi)^2 \right. 
+ \left. \dot{t}_{ij} \dot{t}^{ij} \right\} + c^2 \left( -2\psi \triangle \psi + 4A(t) \triangle \psi + t_{ij} \triangle t^{ij} \right)$$

(31)

with the spatial Laplacian $\triangle = \partial^2$. We note that $S_1^1$ in (21) contains only $\psi$ and $t_{ij}$, which are gauge-invariant quantities. We emphasize that “$A(t)$” leaves a gauge-dependent quantity alone. Thus, it seems that if $A(t) \neq 0$, one does not obtain the gauge-invariant quadratic action $S_2^{EH}$. However, “$4A(t) \triangle \psi$” is a surface term and thus, we drop it from studying the propagations of massive graviton.
Finally, the LV mass term \( S^{LV}_2 \) leads to
\[
S^{LV}_2 = \frac{1}{2\gamma^2} \int dt d^3x \left[ 2m^2_1 \left( V^2_i + (\partial_i B)^2 \right) - \tilde{m}^2_2 \left( t_{ij} t^{ij} + 2(\partial_i F_j)^2 + (\partial_i \partial_j E)^2 + 2\psi \partial^2 E + 3\psi^2 \right) 
+ \tilde{m}^2_3 \left( \partial^2 E + 3\psi \right)^2 \right].
\] (32)
which is not invariant under FDiff transformations because we could not express whole terms in terms of gauge-invariant quantities.

3 Massive tensor and vector propagations

Before proceeding, we conjecture that out of the 5 DOF of the massive graviton, 2 of these are expressed as transverse and traceless tensor modes \( t_{ij} \), 2 of these are expressed as transverse vector modes \( F_i \), and the remaining one is from Hořava scalar \( \psi \).

3.1 Tensor modes

The field equation for tensors is given by
\[
\ddot{t}_{ij} - C^2 \triangle t_{ij} + C^2 m^2_2 t_{ij} + \frac{2C^2}{\omega'} \Delta^2 t_{ij} - \frac{\kappa^4 \mu}{4\eta^2} \epsilon_{ilm}\partial^l \Delta^2 t_{jm} - \frac{\kappa^4}{4\eta^4} \triangle^3 t_{ij} = 0.
\] (33)
The requirement that these modes are not tachyonic gives the stability condition
\[
m^2_2 \geq 0.
\] (34)
In the absence of mass, these modes describe the chiral primordial gravitational waves [24,39]. These circularly polarized modes are possible because the Cotton tensor \( C_{ij} \) is present, making parity violation. In the presence of a mass term, it may describe massive chiral gravitational waves.

3.2 Vector modes

It is clear from Eqs.(23) and (32) that \( V_i \) enters the action without temporal derivatives, that is, it is a non-dynamical field in the massless theory. A massive vector Lagrangian takes the form
\[
\mathcal{L}^v = \frac{1}{\gamma^2} \left[ -w_i \Delta w^i + m^2_1 V_i^2 - \tilde{m}^2_2 (\partial_i F_j)^2 \right]
\] (35)
with \( w_i = V_i - \dot{F}_i \). It is clear that in the absence of mass terms, \( w_i \) is a nonpropagating vector mode. We integrate \( V_i \) out using the field equation obtained by varying action with respect to \( V_i \)
\[
\Delta (V_i - \dot{F}_i) - m^2_1 V_i = 0
\] (36)
which implies
\[ V_i = \frac{\Delta}{\Delta - m_1^2} \dot{F}_i. \tag{37} \]

Plugging this expression into Eq.(35) leads to be
\[ \mathcal{L}^v = \frac{1}{\gamma^2} \left[ \frac{\Delta m_1^2}{\Delta - m_1^2} \dot{F}_i^2 + \tilde{m}_2^2 F_i \Delta F_i \right]. \tag{38} \]

Considering \( \Delta < 0 \), the time kinetic term is always positive. In this case, we introduce a canonical vector field \( \tilde{F}_i \) to obtain a canonical action as
\[ F_i = \frac{\gamma}{m_1} \sqrt{\frac{\Delta - m_1^2}{2 \Delta}} \tilde{F}_i \propto \frac{1}{m_1 M_{Pl}} \sqrt{\frac{\Delta - m_1^2}{2 \Delta}} \tilde{F}_i \tag{39} \]
in the \( c = 1 \) units. Then, the Lagrangian \( \mathcal{L}^v \) takes the canonical form
\[ \mathcal{L}^v_c = \frac{1}{2} \left[ \dot{\tilde{F}}_i^2 - \frac{m_2^2}{m_1} (\partial_i \tilde{F}_j)^2 - m_2^2 \tilde{F}_j^2 \right]. \tag{40} \]

Now let us discuss the strong coupling issue. In order to discuss the strong coupling problem, we first note that
\[ \frac{1}{8\pi G} = \frac{4c}{\kappa^2} \equiv M_{Pl}, \tag{41} \]
which leads to an important relation between \( \gamma \) and Planck mass scale \( M_{Pl} \)
\[ \gamma = \frac{2\sqrt{c}}{\eta M_{Pl}} \propto \frac{1}{M_{Pl}} \tag{42} \]
in the \( c = 1 \) units. Considering the relation Eq.(39), the original vector field is proportional to \( (m M_{Pl})^{-1} \) and from Eq.(35), the gauge-invariant combination \( w_i \) takes the form
\[ w_i \propto \frac{m}{M_{Pl}} \tilde{F}_i \tag{43} \]
which show that vector modes at small \( m \) is precisely the same as in the Fierz-Pauli case.

The analysis in Ref.[40] suggests the strong coupling occurs at \( E \sim \sqrt{m M_{Pl}} \), which is a high scale. Its equation of motion is given by
\[ \ddot{\tilde{F}}_i - \frac{m_2^2}{m_1^2} \Delta \tilde{F}_j + m_2^2 \tilde{F}_i = 0. \tag{44} \]
The above leads to the dispersion relation
\[ p_0^2 = \frac{m_2^2}{m_1} p^2 + m_2^2, \tag{45} \]

where
\[ \frac{\partial}{\partial (ct)} \equiv \hat{t} \rightarrow -p_0, \quad \frac{\partial}{\partial x^i} \rightarrow -p_i. \tag{46} \]
For \( m_1^2 > 0 \) and \( m_2^2 > 0 \), it is obvious that there is no ghosts.

In the Fierz-Pauli case of \( m_1^2 = m_2^2 \), the massive vector equation reduces to
\[ (\Box - m^2) \tilde{F}_i = 0 \tag{47} \]
which represents a massive vector with two degrees of freedom. Here \( \Box = -\partial_0^2 + \Delta. \)
4 Massive scalar propagations

It turned out that for $\frac{1}{3} < \lambda < 1$, there is ghost instability for the Hořava scalar [24]. Thus, our primary concern is to investigate whether adding a LV mass term can cure this instability. The scalar Lagrangian composed of $\psi$, $B$, and $E$ takes the form

$$L_s = \frac{1}{2\gamma^2} \left[ -3(3\lambda - 1)\dot{\psi}^2 + 2(3\lambda - 1)\psi \Delta (2B - \dot{E}) - (\lambda - 1)\left(\Delta (2B - \dot{E})\right)^2 - 2c^2\psi \Delta \psi - \frac{(1 - \lambda)}{2(3\lambda - 1)} \frac{4c^2}{\omega} \psi \Delta^2 \psi - 2m_1^2 B \Delta B - \tilde{m}_2^2 \left( E \Delta^2 E + 2\psi \Delta E + 3\psi^2 \right) + \tilde{m}_3^2 \left( \Delta E + 3\psi \right)^2 \right].$$ \hspace{1cm} (48)

Variations with respect to $B$ and $E$ lead to

$$(3\lambda - 1)\ddot{\psi} + (\lambda - 1) \Delta \dot{E} - 2(\lambda - 1) \Delta B - m_1^2 B = 0,$$ \hspace{1cm} (49)

$$(3\lambda - 1)\ddot{\psi} - (\lambda - 1) \Delta (2\dot{B} - \dot{E}) + (\tilde{m}_3^2 - \tilde{m}_2^2) \Delta E + (3\tilde{m}_3^2 - \tilde{m}_2^2) \psi = 0$$ \hspace{1cm} (50)

which show complicated relations between three fields. In this case, the diagonalization process seems to a formidable task. Hence, we consider three cases of massless, $B = 0$ and $E = 0$.

4.1 Massless case and Strong coupling problem

In the massless case, Eqs. (49) and (50) reduces to a single relation

$$\Delta \Pi = \frac{1 - 3\lambda}{1 - \lambda} \psi$$ \hspace{1cm} (51)

with $\Pi = 2B - \dot{E}$. Substituting this into Eq. (48) with $m_1^2 = m_2^2 = m_3^2 = 0$, one finds the Hořava Lagrangian for $\psi$

$$L_{H}^s = \frac{1}{2\gamma^2} \left[ \frac{2(1 - 3\lambda)}{1 - \lambda} \dot{\psi}^2 - 2c^2 \psi \Delta \psi - \frac{1 - \lambda}{2(1 - 3\lambda)} \frac{4c^2}{\omega} \psi \Delta^2 \psi \right].$$ \hspace{1cm} (52)

We note that the case of $E = 0$-gauge leads to Eq. (52) exactly. Here, it is obvious that for $\frac{1}{3} < \lambda < 1$, the time kinetic term becomes negative and thus, the Hořava scalar suffers from the ghost instability. In addition, comparing it with the tensor Lagrangian indicates that the second term is opposite and the third term is consistent with the tensor term. The dispersion relation is given by

$$p_0^2 = -\frac{1 - \lambda}{2(1 - 3\lambda)}p^2 + \frac{(1 - \lambda)^2}{2(1 - 3\lambda)^2} \frac{4}{\omega} p^4.$$ \hspace{1cm} (53)
Now let us mention the strong coupling problem. Introducing the sound speed squared $c^2_\psi$ as
\[ c^2_\psi = \frac{1 - \lambda}{3\lambda - 1}, \] (54)
the Hořava Lagrangian can be rewritten to be
\[
\mathcal{L}^s_H = -\frac{c^2_\psi}{\sqrt{2}} \left[ \frac{1}{c^2_\psi} (\psi')^2 + \psi \Delta \psi - \frac{c^2_\psi}{\omega} \frac{\psi}{\Delta} \psi' \right] \] (55)
which is exactly the same form as in Ref.[41] for $\gamma^2 = c^2 = 1$ and ignoring the fourth order derivative term. For $1/3 < \lambda < 1$, $c^2_\psi > 0$ but the time kinetic term is negative definite (ghost).

On the other hand, if the Hořava scalar $\psi$ is not a ghost, then it is unstable because of $c^2_\psi < 0$ [23]. Considering the canonical scalar $\tilde{\psi}$ with $c^2 = 1$
\[
\psi = \frac{\gamma|c_\psi|}{\sqrt{2}} \tilde{\psi}, \] (56)
the above scalar Lagrangian takes the canonical form
\[
\mathcal{L}^c_H = \left[ \frac{1}{2} (\tilde{\psi}')^2 - \frac{c^2_\psi}{2} \tilde{\psi} \Delta \tilde{\psi} + \frac{c^4_\psi}{2\omega} \tilde{\psi} \Delta^2 \tilde{\psi} \right]. \] (57)
In this case, we need to take into account the last term of Eq. [55] to address the fate of the instability. Here the time scale of the instability is at least $\frac{2^2}{|c_\psi|} \propto \frac{1}{M_{Pl}}$. In order not to have the instability within the age of the universe ($1/H_0$ with the present Hubble parameter $H_0$), one needs to have $|c_\psi| \sim H_0/M_{Pl}$ which means that $|c_\psi| \rightarrow 0$ ($\lambda \rightarrow 1$) or the UV scale of the theory is very low [23]. It is known that if $c^2_\psi$ is small, the higher order interactions (for example, cubic interactions of $\psi$) become increasingly important. For simplicity, let us consider the scalar sector $n_i = \partial_i B$ and $h_{ij} = e^\psi \delta_{ij}$ with $E = 0$-gauge. Then, the constraint takes the form
\[
\Delta B = \frac{c^2_\psi}{2(1 - \lambda)} \dot{\psi} = -\frac{1}{2c^2_\psi} \dot{\psi} = -\frac{c}{2c^2_\psi} \psi'. \] (58)
The third order Lagrangian is given by [41]
\[
\mathcal{L}^3_s \sim \frac{c^2}{\gamma^2} \left[ \psi \partial_i \psi \partial^i \psi - \frac{3}{c^2_\psi} \psi (\psi')^2 + \frac{3}{c^2_\psi} \left( \partial_i \partial_j B \partial^i \partial^j B - (\Delta B)^2 \right) - \frac{2}{c^2_\psi} \Delta B \partial_i \psi \partial^i B \right]. \] (59)
Plugging Eq. [58] into $\mathcal{L}^3_s$ leads to
\[
\mathcal{L}^3_s \sim \frac{c^2}{\gamma^2} \left[ \psi \partial_i \psi \partial^i \psi - \frac{3}{c^2_\psi} \psi (\psi')^2 + \frac{3}{8c^2_\psi} \left( \partial_i \partial_j \psi \partial^i \partial^j \psi \right) - \frac{2}{c^2_\psi} \Delta \partial_i \psi \partial^i \psi \right]. \] (60)
Finally, using [56], we have the canonical third order Lagrangian [42]
\[
\mathcal{L}^c_s \sim \frac{1}{2\sqrt{2}} \frac{c^3_\psi}{M_{Pl}} \tilde{\psi} \partial_i \tilde{\psi} \partial^i \tilde{\psi} - \frac{3}{M_{Pl}} \tilde{\psi} (\psi'')^2 + \frac{3}{8c^2_\psi M_{Pl}} \left( \partial_i \partial_j \tilde{\psi} \partial^i \partial^j \tilde{\psi} \right) - \frac{2}{c^2_\psi M_{Pl}} \tilde{\psi}' \partial_i \tilde{\psi} \partial^i \tilde{\psi}'. \] (61)
We observe that the last two terms scale as \((c_{\psi}M_{Pl})^{-1}\) and thus, the Hořava scalar becomes strong coupled for \(c_{\psi} \to 0(\lambda \to 1)\). Importantly, we note that all terms which blow up in that limit come from the kinetic Lagrangian \(L_K\) in Eq.\((14)\). This means that the potential terms cannot cure the strong coupling problem.

4.2 \(B=0\)-gauge case

In this case, Eq.\((50)\) leads to

\[
(\lambda - 1) \Delta \dddot{E} + (\bar{m}_3^2 - \bar{m}_2^2) \Delta E + (3\lambda - 1) \dddot{\psi} + (3\bar{m}_3^2 - \bar{m}_2^2) \psi = 0
\]  

(62)

which seems to be difficult to express \(E\) in terms of \(\psi\). For \(m_2^2 = m_3^2 \equiv m^2\) case, the above relation leads to a rather simple one

\[
\Delta \dddot{E} = \frac{1 - 3\lambda}{1 - \lambda} \dddot{\psi} + \frac{2c^2 m^2}{1 - \lambda} \psi.
\]  

(63)

However, it is not easy to derive a relation without derivative from \((63)\). Hence, we could not express \(4c^2 m^2 \psi \Delta E\) in Eq.\((48)\) in terms of \(\psi\).

4.3 \(E=0\)-gauge case

In this case, we require \(m_3^2 = 0\) for the consistency. For \(m_3^2 \neq 0\), the mass term of \(\psi\) takes \(3c^2(3m_3^2 - m_2^2)\psi^2\), which induces a tachyonic mass for \(3m_3^2 > m_2^2\). The relation between \(B\) and \(\psi\) takes the form

\[
B = \frac{(3\lambda - 1) \dot{\psi}}{2(\lambda - 1) \Delta + m_1^2}.
\]  

(64)

The \(m_1^2 = 0\) case leads to the well-known relation of \(B = 2\dot{\psi}/m_1\). Substituting this into Eq.\((58)\), we have the Lagrangian

\[
\mathcal{L}_{E=0}^B = \frac{1}{2\gamma^2} \left[ \dddot{\psi} \left(3(1 - 3\lambda) + \frac{4(1 - 3\lambda)^2}{2(\lambda - 1) \Delta + m_1^2} \right) \dot{\psi} - \psi \left(2c^2 \Delta + \frac{1 - \lambda}{2(1 - 3\lambda)} \frac{4c^2 \Delta^2}{\omega} + 3c^2 m_2^2 \right) \right].
\]  

(65)

It is clear that \(\mathcal{L}_{E=0}^B\) with \(m_1^2 = m_2^2 = 0\) recovers \(\mathcal{L}_H^r\) in Eq.\((52)\). As is shown in Table 1, the mass term contributes to negative term in the time kinetic terms. Hence, adding the mass term \((m_1^2 \Delta /2)\) does not change the ghost instability for \(\frac{1}{3} < \lambda < 1\) and further, it may induces the ghost instability even for \(\lambda > 1\). The latter case is obviously free from the ghost instability for the massless case. We show that even if a Lorentz violating mass term is introduced at the quadratic level, it could not cure the instability which is present in the massless case of the Hořava-Lifshitz gravity. The same mass of \(m_1^2 = m_2^2\) does not resolve the instability issue.
Table 1: Signs for time kinetic terms for massive Hořava scalar $\psi$ with $\Delta < 0$.

|                   | $1/3 < \lambda < 1$ | $\lambda > 1$ |
|-------------------|----------------------|---------------|
| $3(1 - 3\lambda)$ | $-$                  | $-$           |
| $(\lambda - 1)\Delta^2$ | $-$                  | $+$           |
| $\frac{m_1^2 \Delta}{2}$ | $-$                  | $-$           |
| $\frac{2(1 - 3\lambda)}{1 - \lambda}(m_1^2 = 0)$ | $-$                  | $+$           |

5 Discussions

In order to understand better the problems arising when one attempts to modify the gravity in the Lorentz-violating way, we have studied massive propagations of scalar, vector, and tensor modes in the deformed Hořava-Lifshitz gravity by introducing Lorentz-violating mass term. In this approach, we did choose a gauge of $E = 0$ to study massive scalar propagations. We have found that tensor modes $t_{ij}$ and vector modes $\tilde{F}_i$ are propagating in the Minkowski spacetimes for both mass terms (17) and (18). However, the propagation of Hořava scalar $\psi$ is not well defined because it still has the ghost instability.

We remark that there exists a strong coupling problem for an interacting theory of $z = 3$ Hořava-Lifshitz gravity beyond the linearized theory [19, 23, 41, 42]. In this case, the Hořava scalar is a ghost if the sound speed squared is positive. In order to make the scalar graviton healthy, the sound speed squared must be negative but it is inevitably unstable. Thus, one way to avoid this is to choose the case that the sound speed squared is close to zero, which implies $\lambda \rightarrow 1$. However, in the small sound speed limit, the cubic interactions blows up which means that they are important at very low energies. This invalidates any linearized analysis and any predictability is lost due to unsuppressed loop corrections.

Consequently, the Hořava scalar is still unstable by including a Lorentz-violating mass term. This implies that the mass terms do not regularize the bad behavior of the Hořava scalar in the $z = 3$ Hořava-Lifshitz gravity which was discussed in [23].

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References

[1] P. Horava, Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].
[2] P. Horava, JHEP 0903, 020 (2009) [arXiv:0812.4287 [hep-th]].
[3] M. Visser, Phys. Rev. D 80, 025011 (2009) [arXiv:0902.0590 [hep-th]].
[4] P. Horava, Phys. Rev. Lett. 102, 161301 (2009) [arXiv:0902.3657 [hep-th]].
[5] A. Volovich and C. Wen, JHEP 0905, 087 (2009) [arXiv:0903.2455 [hep-th]].
[6] J. Kluson, JHEP 0907, 079 (2009) [arXiv:0904.1343 [hep-th]].
[7] H. Nikolic, [arXiv:0904.3412] [hep-th].
[8] H. Nastase, [arXiv:0904.3604] [hep-th].
[9] K. I. Izawa, [arXiv:0904.3593] [hep-th].
[10] G. E. Volovik, JETP Lett. 89, 525 (2009) [arXiv:0904.4113 [gr-qc]].
[11] T. P. Sotiriou, M. Visser and S. Weinfurtner, Phys. Rev. Lett. 102, 251601 (2009) [arXiv:0904.4464 [hep-th]].
[12] B. Chen and Q. G. Huang, Phys. Lett. B 683, 108 (2010) [arXiv:0904.4565 [hep-th]].
[13] R. G. Cai, B. Hu and H. B. Zhang, Phys. Rev. D 80, 041501 (2009) [arXiv:0905.0255 [hep-th]].
[14] T. Nishioka, Class. Quant. Grav. 26, 242001 (2009) [arXiv:0905.0473 [hep-th]].
[15] A. Kehagias and K. Sfetsos, Phys. Lett. B 678, 123 (2009) [arXiv:0905.0477 [hep-th]].
[16] D. Orlando and S. Reffert, Class. Quant. Grav. 26, 155021 (2009) [arXiv:0905.0301 [hep-th]].
[17] R. A. Konoplya, Phys. Lett. B 679, 499 (2009) [arXiv:0905.1523 [hep-th]].
[18] C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, JHEP 0908, 070 (2009) [arXiv:0905.2579 [hep-th]].
[19] T. P. Sotiriou, M. Visser and S. Weinfurtner, JHEP 0910, 033 (2009) [arXiv:0905.2798 [hep-th]].
[20] Y. W. Kim, H. W. Lee and Y. S. Myung, Phys. Lett. B 682, 246 (2009) [arXiv:0905.3423 [hep-th]].
[21] G. Calcagni, arXiv:0905.3740 [hep-th].

[22] M. Sakamoto, Phys. Rev. D 79, 124038 (2009) arXiv:0905.4326 [hep-th]].

[23] D. Blas, O. Pujolas and S. Sibiryakov, JHEP 0910, 029 (2009) arXiv:0906.3046 [hep-th]].

[24] C. Bogdanos and E. N. Saridakis, arXiv:0907.1636 [hep-th].

[25] E. Kiritsis and G. Kofinas, Nucl. Phys. B 821, 467 (2009) arXiv:0904.1334 [hep-th]].

[26] G. Calcagni, JHEP 0909, 112 (2009) arXiv:0904.0829 [hep-th]];
T. Takahashi and J. Soda, Phys. Rev. Lett. 102, 231301 (2009) arXiv:0904.0554 [hep-th]];
S. Mukohyama, JCAP 0906, 001 (2009) arXiv:0904.2190 [hep-th]];
R. Brandenberger, Phys. Rev. D 80, 043516 (2009) arXiv:0904.2835 [hep-th]];
Y. S. Piao, Phys. Lett. B 681, 1 (2009) arXiv:0904.4117 [hep-th]];
X. Gao, arXiv:0904.4187 [hep-th];
B. Chen, S. Pi and J. Z. Tang, JCAP 0908, 007 (2009) arXiv:0905.2300 [hep-th];
E. N. Saridakis, arXiv:0905.3532 [hep-th];
S. Mukohyama, Phys. Rev. D 80, 064005 (2009) arXiv:0905.3563 [hep-th];
X. Gao, Y. Wang, R. Brandenberger and A. Riotto, arXiv:0905.3821 [hep-th];
M. Minamitsuji, arXiv:0905.3892 [astro-ph.CO];
A. Wang and Y. Wu, JCAP 0907, 012 (2009) arXiv:0905.4117 [hep-th];
S. Nojiri and S. D. Odintsov, arXiv:0905.4213 [hep-th].

[27] H. Lu, J. Mei and C. N. Pope, Phys. Rev. Lett. 103, 091301 (2009) arXiv:0904.1595 [hep-th]].

[28] R. G. Cai, L. M. Cao and N. Ohta, Phys. Rev. D 80, 024003 (2009) arXiv:0904.3670 [hep-th];
R. G. Cai, Y. Liu and Y. W. Sun, JHEP 0906, 010 (2009) arXiv:0904.4104 [hep-th];
E. O. Colgain and H. Yavartanoo, JHEP 0908, 021 (2009) arXiv:0904.4357 [hep-th];
Y. S. Myung and Y. W. Kim, arXiv:0905.0179 [hep-th];
R. G. Cai, L. M. Cao and N. Ohta, Phys. Lett. B 679, 504 (2009) arXiv:0905.0751 [hep-th];
A. Ghodsi, [arXiv:0905.0836] [hep-th];
Y. S. Myung, Phys. Lett. B 678, 127 (2009) [arXiv:0905.0957] [hep-th];
S. Chen and J. Jing, [arXiv:0905.1409] [gr-qc];
S. b. Chen and J. l. Jing, Phys. Rev. D 80, 024036 (2009) [arXiv:0905.2055] [gr-qc];
J. Chen and Y. Wang, [arXiv:0905.2786] [gr-qc];
M. i. Park, M. i. Park, JHEP 0909, 123 (2009) [arXiv:0905.4480] [hep-th];
M. Botta-Cantcheff, N. Grandi and M. Sturla, [arXiv:0906.0582] [hep-th].

[29] S. Mukohyama, JCAP 0909, 005 (2009) [arXiv:0906.5069] [hep-th].
[30] A. Aubert, Phys. Rev. D 69, 087502 (2004) [arXiv:hep-th/0312246].

[31] H. van Dam and M. J. G. Veltman, Nucl. Phys. B 22, 397 (1970); V. I. Zakharov,
JETP Lett. 12, 312 (1970) [Pisma Zh. Eksp. Teor. Fiz. 12, 447 (1970)].

[32] V. A. Rubakov, [arXiv:hep-th/0407104].

[33] V. A. Rubakov and P. G. Tinyakov, Phys. Usp. 51, 759 (2008) [arXiv:0802.4379] [hep-
th]].

[34] S. L. Dubovsky, JHEP 0410, 076 (2004) [arXiv:hep-th/0409124].

[35] D. Blas, D. Comelli, F. Nesti and L. Pilo, Phys. Rev. D 80, 044025 (2009)
[arXiv:0905.1699] [hep-th]].

[36] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A 173, 211 (1939).

[37] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203
(1992).

[38] E. Alvarez, D. Blas, J. Garriga and E. Verdaguer, Nucl. Phys. B 756, 148 (2006)
[arXiv:hep-th/0606019];
D. Blas, [arXiv:0809.3744] [hep-th];
E. D. Skvortsov and M. A. Vasiliev, Phys. Lett. B 664, 301 (2008)
[arXiv:hep-th/0701278];
J. J. van der Bij, H. van Dam and Y. J. Ng, Physica 116A, 307 (1982).

[39] Y. S. Myung, Phys. Lett. B 681, 81 (2009) [arXiv:0909.2075] [hep-th]].

[40] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305, 96 (2003)
[arXiv:hep-th/0210184].
[41] K. Koyama and F. Arroja, [arXiv:0910.1998] [hep-th].

[42] A. Papazoglou and T. P. Sotiriou, [arXiv:0911.1299] [hep-th].