Testing for unit roots based on sample autocovariances

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Abstract

We propose a new unit-root test for a stationary null hypothesis \( H_0 \) against a unit-root alternative \( H_1 \). Our approach is nonparametric as \( H_0 \) only assumes that the process concerned is \( I(0) \) without specifying any parametric forms. The new test is based on the fact that the sample autocovariance function (ACVF) converges to the finite population ACVF for an \( I(0) \) process while it diverges to infinity for a process with unit-roots. Therefore the new test rejects \( H_0 \) for the large values of the sample ACVF. To address the technical challenge ‘how large is large’, we split the sample and establish an appropriate normal approximation for the null-distribution of the test statistic. The substantial discriminative power of the new test statistic is rooted from the fact that it takes finite value under \( H_0 \) and diverges to infinity under \( H_1 \). This allows us to truncate the critical values of the test to make it with the asymptotic power one. It also alleviates the loss of power due to the sample-splitting. The test is implemented in a user-friendly R-function.

Keywords: Autocovariance, Integrated processes, Normal approximation, Power-one test, Sample-splitting.

1 Introduction

Models with unit-root are frequently used for modeling nonstationary time series. The importance of the unit-root concept stems from the fact that many economic, financial, business and social-domain data exhibit segmented trend-like or random wandering phenomena. While the random-walk-like behavior of stock prices was notified and recorded much earlier by, for example, Jules Regnault, a French broker, in 1863 and then by Louis Bachelier in his 1900 PhD thesis, the development of statistical inference for unit-roots only started in late 1970s. Nevertheless the
The literature on unit-root tests by now is immense and diverse. We only state a selection of some important developments below, which naturally leads to a new test to be presented in this paper.

The Dickey-Fuller tests (Dickey & Fuller, 1979, 1981) dealt with Gaussian random walks with independent error terms. Effort to relax the condition of independent Gaussian errors leads to, among others, the augmented Dickey-Fuller (ADF) tests (Said & Dickey, 1984; Elliott et al., 1996) which replace the error term by an autoregressive process, the Phillips-Perron test (Phillips, 1987; Phillips & Perron, 1988) which estimates the long-run variance of the error process non-parametrically. The ADF tests are further extended for dealing with structural breaks in trend (Zivot & Andrews, 1992), long memory processes (Robinson, 1994), seasonal unit roots (Chan & Wei, 1988; Hylleberg et al., 1990), bootstrap unit-root tests (Paparoditis & Politis, 2005), nonstationary volatility (Cavaliere & Taylor, 2007), panel data (Pesaran, 2007), and local stationary processes (Rho & Shao, 2019). We refer to survey papers Stock (1994) and Phillips & Xiao (1998), and monographs Hatanaka (1996) and Maddala & Kim (1998) for further references.

The Dickey-Fuller tests and their variants are based on the regression of a time series on its first lag in which the existence of unit root is postulated as a null hypothesis in the form of the regression coefficient being equal to one. This null hypothesis is tested against a stationary alternative that the regression coefficient is smaller than one. This setting leads to innate indecisive inference for ascertaining the existence of unit-roots, as a statistical test is incapable in accepting null hypothesis. To make the assertion of unit-roots on a firmer ground, Kwiatkowski et al. (1992) adopted a different approach: the proposed KPSS test considers a stationary null hypothesis against a unit-root alternative. It is based on a plausible representation for possible nonstationary time series in which a unit-root is represented as an additive random walk component. Then under null the variance of the random walk component is zero. The KPSS test is the one-sided Lagrange multiplier test for testing the variance to be zero against greater than zero.

In spite of the many exciting developments stated above, testing for the existence of unit roots remains as a challenge in time series analysis, as most available methods suffer from the lack of accurate size control and poor power. In this paper we propose a new test, based on a radically different idea from the existing approaches. Our setting is similar in spirit to the KPSS test as we test for stationary null hypothesis \( H_0 \) against a unit-root alternative \( H_1 \). However our approach is nonparametric as \( H_0 \) only assumes that the process concerned is \( I(0) \) without specifying any parametric forms. The new test is based on the simple fact that under \( H_0 \) the sample autocovariance function (ACVF) converges to the finite population ACVF while under \( H_1 \) it diverges to infinity. Therefore we can reject \( H_0 \) for large (absolute) values of the sample ACVF. To address the technical challenge ‘how large is large’, we split the sample and establish an appropriate normal approximation for the null-distribution of the test statistic. Note that our
sample ACVF based test statistic offers substantial discriminative power as it takes finite value under $H_0$ or diverges to infinity under $H_1$. This allows us to truncate the critical values determined by the normal approximation to make the test with the asymptotic power one. Furthermore, it also alleviates the loss of power due to the sample-splitting as it outperforms the KPSS test in the power comparison in simulation. Another advantage of the new method is that it has a remarkable discriminate power to tell the difference between, for example, a random walk and an AR(1) with the autoregressive coefficient close to (but still smaller than) one, for which most the available unit-root tests, including the KPSS method, suffer from weak discriminate power. Admittedly the new test is technically sophisticated, which, we argue, is inevitable in order to gain improvement over the existing methods. Nevertheless to make it user-friendly, we have developed an R-function \texttt{ur.test} in the package \texttt{HDTSA} which implements the test in an automatic manner.

## 2 Main results

### 2.1 A power-one test

A time series \{\(Y_t\)\} is said to be \(I(0)\), denoted by \(Y_t \sim I(0)\), if \(E(Y_t) \equiv \mu, \ E(Y_t^2) < \infty, \ \gamma(k) \equiv \text{Cov}(Y_{t+k}, Y_t)\), and \(\sum_{k=0}^{\infty} |\gamma(k)| < \infty\). Let \(\nabla Y_t = Y_t - Y_{t-1}\), \(\nabla^0 Y_t = Y_t\) and \(\nabla^d Y_t = \nabla(\nabla^{d-1} Y_t)\) for any integer \(d \geq 1\). \(\{Y_t\}\) is said to be \(I(d)\), denoted by \(Y_t \sim I(d)\), if \(\{\nabla^d Y_t\}\) is \(I(0)\) and \(\{\nabla^{d-1} Y_t\}\) is not \(I(0)\). An \(I(d)\) process is also called a unit-root process with the integration order \(d\). With the observations \(\{Y_t\}_{t=1}^{n}\), we are interested in testing the hypotheses

\[ H_0: Y_t \sim I(0) \quad \text{versus} \quad H_1: Y_t \sim I(d) \quad \text{for some integer} \ d \geq 1. \]  

Write \(\bar{Y} = n^{-1} \sum_{t=1}^{n} Y_t\) and denote the sample ACVF at lag \(k\) by \(\hat{\gamma}(k) = n^{-1} \sum_{t=1}^{n-k} (Y_{t+k} - \bar{Y})(Y_t - \bar{Y})\), which is consistent estimator for \(\gamma(k)\) under \(H_0\). Proposition \(\text{[1]}\) indicates that \(\hat{\gamma}(k)\) diverges to infinity under \(H_1\). Thus we can reject \(H_0\) for large values of \(|\hat{\gamma}(k)|\). When \(Y_t \sim I(d)\), the Wold’s decomposition for the purely non-deterministic \(I(0)\) process admits

\[ \nabla^d Y_t = \mu_d + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \]  

where \(\mu_d = E(\nabla^d Y_t)\) is a constant, \(\psi_0 = 1\), and \(\{\epsilon_t\}\) is a white noise sequence.

**Proposition 1.** Let \(Y_t\) satisfy \((2)\), with independent \(\epsilon_t \sim (0, \sigma_\epsilon^2)\) and \(\sum_{j=1}^{\infty} j|\psi_j| < \infty\). Write \(a = \sum_{j=0}^{\infty} \psi_j\) and \(V_{d-1}(t) = F_{d-1}(t) - \int_0^1 F_{d-1}(t) \, dt\) with the scalar multi-fold integrated Brownian motion \(F_{d-1}(t)\) defined recursively as \(F_j(t) = \int_0^1 F_{j-1}(x) \, dx\) for any \(j \geq 1\) and the standard
Brownian motion $F_0(t)$. For any given integer $k \geq 0$, as $n \to \infty$, it holds that (i) $n^{-(2d-1)}\hat{\gamma}(k) \to a^2\sigma_d^2 \int_0^1 V_d^2(t) \, dt$ in distribution if $\mu_d = 0$, and (ii) $n^{-2d}\hat{\gamma}(k) \to \phi_d,k\mu_d^2$ in probability if $\mu_d \neq 0$, where $\phi_d,k > 0$ is a bounded constant only depending on $d$ and $k$.

By Proposition 1 one may consider to reject $H_0$ for the large values of $T_{\text{naive}} = \sum_{k=0}^{K_0} |\hat{\gamma}(k)|^2$ with a prescribed integer $K_0 \geq 0$, as $T_{\text{naive}}$ converges to $\sum_{k=0}^{K_0} |\gamma(k)|^2 < \infty$ under $H_0$. Unfortunately, there are two obstacles preventing using $T_{\text{naive}}$: (i) to determine the critical values one has to derive the null-distribution of $a_n\{T_{\text{naive}} - \sum_{k=0}^{K_0} |\gamma(k)|^2\}$ with some $a_n \to \infty$, (ii) one needs a consistent estimator for $\sum_{k=0}^{K_0} |\gamma(k)|^2$ under $H_0$, which is not readily available as we do not know if $H_0$ holds or not in practice. To overcome these two obstacles, we implement the idea of ‘data splitting’. Let $N = \lfloor n/2 \rfloor$. Define $\hat{\gamma}_1(k) = N^{-1} \sum_{t=1}^{N-k} (Y_{t+k} - \bar{Y})(Y_t - \bar{Y})$ and $\hat{\gamma}_2(k) = N^{-1} \sum_{t=N-k+1}^{N} (Y_{t+k} - \bar{Y})(Y_t - \bar{Y})$. The test statistic for (1) is defined as

$$T_n = \sum_{k=0}^{K_0} |\hat{\gamma}_2(k)|^2,$$

where $K_0 \geq 0$ is a prescribed integer which controls the amount of information from different time lags to be used. Although our theory allows $K_0$ diverging with sample size $n$, the simulation results reported in §3 indicate that the finite sample performance of the test is robust with respect to the different values of $K_0$ and it works well even with small $K_0$.

Formally we reject $H_0$ at the significance level $\phi \in (0, 1)$ if $T_n > \text{cv}_\phi$, where $\text{cv}_\phi$ is the critical value satisfying $\text{pr}(T_n > \text{cv}_\phi) \to \phi$. As we will see in 3, $\{\hat{\gamma}_1(k)\}_{k=0}^{K_0}$ are used to determine the critical value $\text{cv}_\phi$. One obvious concern for splitting the sample into two halves is the loss in testing power. However the fact that $T_n$ takes finite values under $H_0$ and it diverges to infinity under $H_1$ implies that $T_n$ has a strong discriminant power to tell apart $H_1$ from $H_0$, which is enough to sustain the adequate power in comparison to that of, for example, the KPSS test. Our simulation results indicate that the sample-splitting works well even for sample size $n = 80$. Under $H_0$, write $y_{t+k} = 2\{(Y_t - \mu)(Y_{t+k} - \mu) - \gamma(k)\} \text{sgn}(k + t - N + 1/2)$. For $\ell \geq 1$, define $B^2 = E\{\sum_{t=1}^{\ell} Q_t\}^2$ with $Q_t = \sum_{k=0}^{K_0} \xi_{t,k}$ and $\xi_{t,k} = 2y_{t+k}\gamma(k)$. The following regularity conditions are now in order. See the supplementary material for the discussion of their validity.

**Condition 1.** Under $H_0$, $\max_{1 \leq t \leq n} E(|Y_t|^{2s_1}) \leq c_1$ for two constants $s_1 \in (2, 3]$ and $c_1 > 0$.

**Condition 2.** Under $H_0$, $\{Y_t\}$ is $\alpha$-mixing with $\alpha(\tau) = \sup_{A \in \mathcal{F}_t, B \in \mathcal{F}_{t+\tau}} |\text{pr}(AB) - \text{pr}(A)\text{pr}(B)| \leq c_2\tau^{-\beta_1}$ for any $\tau \geq 1$, where $\mathcal{F}_{t-\infty}$ and $\mathcal{F}_{t+\infty}$ denote the $\sigma$-fields generated by $\{Y_u\}_{u \leq t}$ and $\{Y_u\}_{u \geq t+\tau}$, respectively, $c_2 > 0$ and $\beta_1 > 2(s_1 - 1)s_1/(s_1 - 2)^2$ are two constants with $s_1$ specified in Condition 1.

**Condition 3.** Under $H_0$, there is a constant $c_3 > 0$ such that $B^2 \geq c_3\ell$ for any $\ell \geq 1$. 


Theorem 1. Let $H_0$ hold with Conditions\(1,3\) being satisfied, and $K_0 = o\{n^{\xi(\beta,s_1)}\}$ with $\xi(\beta,s_1) = \min[(s_1 - 2)/(4s_1), (\beta - 1)(s_1 - 2)/(2\beta + 2s_1)]$, where $s_1$ and $\beta$ are specified, respectively, in Conditions\(1,2\) and $\beta = \beta_1(s_1 - 2)^2/(2s_1(s_1 - 1))$. Then, as $n \to \infty$,

$$\sup_{u > 0} \left| \Pr\left\{ \sqrt{n} T_n > u + \sqrt{n} \sum_{k=0}^{K_0} |\hat{\gamma}_1(k)|^2 \right\} - 1 + \Phi\left( \frac{2Nu}{B_{2N-K_0} \sqrt{n}} \right) \right| \to 0.$$  

One may select the critical value as $cv_{\phi,\text{naive}} = z_{1-\phi} \hat{B}_{2N-K_0} / (2N) + \sum_{k=0}^{K_0} |\hat{\gamma}_1(k)|^2$, where $z_{1-\phi}$ is the $(1 - \phi)$-quantile of $\mathcal{N}(0,1)$ and $\hat{B}_{2N-K_0}$ is an estimate of $B_{2N-K_0}$ satisfying the condition $\hat{B}_{2N-K_0} / B_{2N-K_0} \to 1$ in probability under $H_0$ (see §2.3 and Theorem 3), as then the rejection probability of the test under $H_0$ converges to $\phi$. Unfortunately $\sum_{k=0}^{K_0} |\hat{\gamma}_1(k)|^2$ diverges to infinity under $H_1$. This causes substantial power loss. To rectify this defect, we apply here the truncation idea as in §2.3 of Chang et al. (2017). More precisely we set the critical value as

$$cv_{\phi} = cv_{\phi,\text{naive}} \cdot I(T) + \kappa_n \cdot I(T^c),$$  

where $\kappa_n = 0.1 \times \log N$ with $N = \lfloor n/2 \rfloor$, and the event $T$ satisfies conditions $\Pr_{H_0}(T) \to 1$ and $\Pr_{H_1}(T^c) \to 1$ as $n \to \infty$. Note that $\Pr_{H_0}(cv_{\phi} = cv_{\phi,\text{naive}}) \to 1$ and $\Pr_{H_1}(cv_{\phi} = \kappa_n) \to 1$. The former one makes the rejection probability of the proposed test under $H_0$ converges to the nominal level $\phi$. Proposition 1 shows that $\Pr_{H_1}\{|\hat{\gamma}_2(0)|^2 > \kappa_n\} \to 1$. Due to $T_n \geq |\hat{\gamma}_2(0)|^2$, we have $\Pr_{H_1}(T_n > \kappa_n) \to 1$, which entails that the proposed test has power one asymptotically. We will state in §2.2 how to specify a qualified event $T$.

Theorem 2. Let $cv_{\phi}$ be defined by (3) with $T$ satisfying $\Pr_{H_0}(T) \to 1$ and $\Pr_{H_1}(T^c) \to 1$, and $\hat{B}_{2N-K_0} / B_{2N-K_0} \to 1$ in probability under $H_0$, as $n \to \infty$. Then it holds that (i) $\Pr_{H_0}(T_n > cv_{\phi}) \to \phi$ if the conditions of Theorem 1 hold, and (ii) $\Pr_{H_1}(T_n > cv_{\phi}) \to 1$ if $Y_t$ satisfies (2) with independent $\epsilon_t \sim (0, \sigma_t^2)$ and $\sum_{j=1}^{\infty} j |\psi_j| < \infty$.

2.2 Determining the event $T$ in (3)

The critical value $cv_{\phi}$ defined in (3) depends on event $T$ critically. Let $X_t = \nabla Y_t$ and $\hat{\gamma}_x(k) = (n-1)^{-1} \sum_{t=2}^{n-k} (X_{t+k} - \bar{X})(X_t - \bar{X})$ for $k \geq 0$, where $\bar{X} = (n-1)^{-1} \sum_{t=2}^{n} X_t$. To avoid the effect of the innovation variance $\sigma_t^2$, we consider the ratio $R = \{\hat{\gamma}(0) + \hat{\gamma}(1)\} / \{\hat{\gamma}_x(0) + \hat{\gamma}_x(1)\}$. Notice that $R = O_p(1)$ under $H_0$, and $\Pr_{H_1}(R \geq C_* N^{3/5}) \to 1$ for any fixed constant $C_* > 0$. We define $T$ in (3) as follows:

$$T = \{R < C_* N^{3/5}\}.$$  

(4)

To use $T$ with finite samples, $C_*$ must be specified according to the underlying process.
Proposition 2. Let \( Y_t \sim I(1) \) satisfy (2) with independent \( \epsilon_t \sim (0, \sigma^2) \) and \( \sum_{j=1}^{\infty} j |\psi_j| < \infty \). Write
\[
\eta = \sum_{j=0}^{\infty} \psi_j^2 + \sum_{j=0}^{\infty} \psi_j \psi_{j+1}.
\]
As \( n \to \infty \), we have (i) \( n^{-1} R \to 2a^2 \eta^{-1} \int_0^1 V_0^2(t) \, dt \) in distribution if \( \mu_1 = 0 \), where \( a \) and \( V_0(t) \) are defined in Proposition 1 and (ii) \( n^{-2} R \to 6^{-1} \sigma^2 \eta^{-1} \mu_1^2 \) in probability if \( \mu_1 \neq 0 \).

Proposition 2 shows that \( R \) with \( \mu_1 \neq 0 \) diverges faster than that with \( \mu_1 = 0 \). Thus for any given \( C_* > 0 \) the requirement \( \Pr[H_t(T^c)] \to 1 \) is satisfied more readily with \( \mu_1 \neq 0 \). Hence we focus on the cases with \( \mu_1 = 0 \) only. Recall \( X_t = \nabla Y_t = \mu_1 + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} \). Then \( a^2 \eta^{-1} = \lambda^{-1}(1 + \rho)^{-1} \), where \( \rho = (\sum_{j=0}^{\infty} \psi_j^2)^{-1} \sum_{j=0}^{\infty} \psi_j \psi_{j+1} \) is the first order autocorrelation coefficient, and \( \lambda = \sigma^2_{2L}/\sigma^2_{L} \) with the short-run variance \( \sigma^2_{2} = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \) and the long-run variance \( \sigma^2_{L} = \sigma^2 (\sum_{j=0}^{\infty} \psi_j)^2 \). Write \( \hat{\sigma}^2_{2} = \hat{\gamma}(0) \). Applying the estimation method for the long-run variance suggested in (2.3) we can obtain \( \hat{\sigma}^2_{L} \), the kernel-type estimate of \( \sigma^2_{L} \), based on \( \{X_t - \bar{X}\}_{t=2}^{n} \). Then we can estimate \( \lambda \) and \( \rho \) by \( \hat{\lambda} = \hat{\sigma}^2_{2} / \hat{\sigma}^2_{L} \) and \( \hat{\rho} = \hat{\gamma}(1) / \hat{\gamma}(0) \). As \( E\{\int_0^1 V_0^2(t) \, dt\} = 1/6 \), we now specify the model-dependent constant \( C_* \) in (4) as
\[
C_* = 2c_\kappa / \{\hat{\lambda}(1 + \hat{\rho})\}
\]
for some constant \( c_\kappa > 1/6 \). Our extensive simulation results indicate that this specification of \( C_* \) with \( c_\kappa \in [0.45, 0.65] \) works well across a variety of models.

Though the above specification was derived for \( Y_t \sim I(1) \), our simulation results indicate that it also works well for \( I(2) \) processes. Note that testing \( I(0) \) against \( I(d) \) with \( d > 1 \) is easier than that with \( d = 1 \), as the autocovariances are of the order at least \( n^{d-1} \) for \( I(d) \) processes. So the difference between the values of \( T_n \) under \( H_1 \) and those under \( H_0 \) increases as \( d \) increases.

2.3 Estimation of \( B^2_{2N-K_0} \)

Write \( m = 2N - K_0 \). Recall \( Q_t = \sum_{k=0}^{K_0} \xi_{t,k} \) with \( \xi_{t,k} = 2y_{t,k} \gamma(k) \). Let \( V_m \) be the long-run variance of the sequence \( \{Q_t\}_{t=1}^{n} \). We then have \( B^2_{2N-K_0} = mV_m \). Define \( \tilde{Q}_t = \sum_{k=0}^{K_0} \tilde{\xi}_{t,k} \) with \( \tilde{\xi}_{t,k} = 2y_{t,k} \hat{\gamma}(k) \), where \( \hat{\gamma}(k) = 2(1)_{(Y_t - \bar{Y})(Y_{t+k} - \bar{Y}) - \hat{\gamma}(k)} \). Write \( \tilde{G}_{j} = m^{-1} \sum_{t=j+1}^{m} \tilde{Q}_t \tilde{Q}_{t-j} \) if \( j \geq 0 \) and \( \tilde{G}_{j} = m^{-1} \sum_{t=-j+1}^{m} \tilde{Q}_{t+j} \tilde{Q}_t \) otherwise. We can estimate \( V_m \) by \( \hat{V}_m = m^{-1} \sum_{j=-m+1}^{m} K(j/b_m) \hat{G}_j \) with a kernel \( K(\cdot) \) and bandwidth \( b_m \). Let
\[
\hat{B}_{2N-K_0} = (m \hat{V}_m)^{1/2}.
\]
Andrews (1991) found that the Quadratic Spectral kernel is optimal for such estimation. We suggest using this kernel in practice by calling function \texttt{1rvar} from the R-package \texttt{sandwich} with the default bandwidth specified in the function. To state the required asymptotic property for \( \hat{B}_{2N-K_0} \) with general kernels, we need following regularity conditions.
Condition 4. The kernel function $K(\cdot) : \mathbb{R} \rightarrow [-1, 1]$ is continuously differentiable on $\mathbb{R}$ and satisfies conditions: (i) $K(0) = 1$, (ii) $K(x) = K(-x)$ for any $x \in \mathbb{R}$, and (iii) $\int_{-\infty}^{\infty} |K(x)| \, dx < \infty$. Let $K_s = K_0 + 2$ satisfying $K_s^{13} \log K_s = o(n^{1-2/s_2})$ with $s_2$ specified in Condition 5. The bandwidth $b_m \rightarrow \infty$ as $n \rightarrow \infty$ satisfies $b_m = o\{n^{1/2-1/s_2}(K_s^5 \log K_s)^{-1/2}\}$ and $K_s^4 = o(b_m)$.

Condition 5. Under $H_0$, $\max_{1 \leq t \leq n} E(\{|Y_t|^2\}) \leq c_4$ for two constants $s_2 > 4$ and $c_4 > 0$, and the $\alpha$-mixing coefficients $\{\alpha(\tau)\}_{\tau \geq 1}$ satisfy $\alpha(\tau) \leq c_5 \tau^{-\beta_2}$ for two constants $c_5 > 0$ and $\beta_2 > \max\{2s_2/(s_2 - 2), s_2/(s_2 - 4)\}$, where $\alpha(\tau)$ is defined in Condition 2.

Theorem 3. Let Conditions 4 and 5 hold. Then as $n \rightarrow \infty$, $\hat{B}_{2N-K_0}/B_{2N-K_0} \rightarrow 1$ in probability under $H_0$.

2.4 Implementation of the test

Based on §2.2 and §2.3, Algorithm 1 outlines the steps to perform our test which includes two tuning parameters. The algorithm is implemented in an R-function `ur.test` in the package HDTSA available at CRAN. To perform the test using function `ur.test`, one merely needs to input time series $\{Y_t\}_{t=1}^n$ and nominal level $\phi$. The package sets the default value $c_\kappa = 0.55$ and returns the five testing results for $K_0 = 0, 1, \ldots, 4$ respectively. One can also set $(c_\kappa, K_0)$ subjectively. We recommend to use $c_\kappa \in \{0.45, 0.65\}$ and $K_0 \in \{0, 1, 2, 3, 4\}$.

Input: Time series $\{Y_t\}_{t=1}^n$, nominal level $\phi$, and two (optional) tuning parameters $(c_\kappa, K_0)$.

Step 1. Compute $\hat{\gamma}(k)$, $\hat{\gamma}_1(k)$, $\hat{\gamma}_2(k)$ and $\hat{\gamma}_x(k)$. Put $\hat{\rho} = \hat{\gamma}_x(1)/\hat{\gamma}_x(0)$.

Step 2. Call function `lrvar` from the R-package `sandwich` (with the default bandwidth in the function) to compute the long-run variances of $\{\hat{Q}_t\}$ and $\{X_t\}$, denoted by $\hat{V}_{2N-K_0}$ and $\hat{\sigma}_L^2$, respectively, where $\hat{Q}_t$ is defined in §2.3. Put $\hat{\lambda} = \hat{\gamma}_x(0)/\hat{\sigma}_L^2$.

Step 3. Calculate the test statistic $T_n = \sum_{k=0}^{K_0} \hat{\gamma}_2(k)^2$, and the critical value $cv_{\phi}$ as in (3) with $\hat{B}_{2N-K_0} = (2N - K_0)^{1/2}\hat{V}^{1/2}_{2N-K_0}$ and $T$ given in (4) for $C_\kappa$ specified in (5).

Step 4. Reject $H_0$ if $T_n > cv_{\phi}$.

Algorithm 1: Sample ACVF-based unit-root test.

To illustrate the robustness with respect to the choice of $(c_\kappa, K_0)$, we apply our test to the 14 US annual economic time series [Nelson & Plosser, 1982] that were often used for testing unit-roots in the literature; leading to the exactly same results with $c_\kappa \in \{0.45, 0.55, 0.65\}$ and $K_0 \in \{0, 1, 2, 3, 4\}$ for each of the 14 time series. See the details in the supplementary material.
3 Simulation study

We illustrate the finite sample properties of our test $T_n$ by simulation with $K_0 \in \{0, 1, 2, 3, 4\}$ and $c_\kappa \in \{0.45, 0.55, 0.65\}$. We also consider $T_n$ with the untruncated critical value $cv_{\phi, \text{naïve}}$, i.e. $c_\kappa = \infty$ in [5]. Hualde & Robinson [2011] proposed the pseudo MLE $\hat{d}$ for the integration order $d$ in the ARFIMA models that can be used to construct a $t$-statistic $\hat{d}/sd(\hat{d})$ for $H_0 : d = 0$ versus $H_1 : d \geq 1$. We call it HR test that rejects $H_0$ if $\hat{d}/sd(\hat{d}) > z_{1-\phi}$, where $z_{1-\phi}$ is the $(1-\phi)$-quantile of $\mathcal{N}(0, 1)$. For comparison, we also include the KPSS test (Kwiatkowski et al., 1992) and the HR test in our experiments. We set $N = 40, 70, 100$ and repeat each setting 2000 times. To examine the rejection probability of the tests under $H_0$, we consider three models:

Model 1. $Y_t = \rho Y_{t-1} + \epsilon_t$.
Model 2. $Y_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_2 \epsilon_{t-2}$.
Model 3. $Y_t - \rho_1 Y_{t-1} - \rho_2 Y_{t-2} = \epsilon_t + 0.5 \epsilon_{t-1} + 0.3 \epsilon_{t-2}$.

To examine the rejection probability of the tests under $H_1$, we consider the following models:

Model 4. $\nabla Y_t = Z_t, Z_t = \rho Z_{t-1} + \epsilon_t$.
Model 5. $\nabla Y_t = Z_t, Z_t = \epsilon_t + \phi_1 \epsilon_{t-1} + \phi_2 \epsilon_{t-1}$.
Model 6. $\nabla Y_t = Z_t, Z_t - \rho_1 Z_{t-1} - \rho_2 Z_{t-2} = \epsilon_t + 0.5 \epsilon_{t-1} + 0.3 \epsilon_{t-1}$.
Model 7. $\nabla^2 Y_t = Z_t, Z_t = \epsilon_t + \phi_1 \epsilon_{t} + \phi_2 \epsilon_{t-1}$.

Unless specified otherwise, we always assume that $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2)$ independently with $\sigma_\epsilon^2 = 1$ or 2, and set the nominal level $\phi = 5\%$. The results with different $(c_\kappa, K_0)$ are similar; indicating once again that our test is robust with respect to the choice of $(c_\kappa, K_0)$. We only list the results with $K_0 = 0$ and $\sigma_\epsilon^2 = 1$ in Table [1] and report other results in the supplementary material. We also consider the cases $\epsilon_t \sim t(2)$ and $t(5)$, and report the results in the supplementary material.

Overall the rejection probabilities of our test under $H_0$ are close to the nominal level $\phi = 5\%$ especially with large $n$ ($N = 100$). The performance of our test is stable across different models with different parameters, different $K_0$ and different innovation distributions, while that of the KPSS test and the HR test vary and are adequate only for some settings. Table [1] indicates that our test works well for Model 1 with both positive and negative $\rho$, while the KPSS test and the HR test perform poorly when $\rho < 0$, and even worse when $\rho > 0$. The KPSS test and the HR test completely fail when $\rho = 0.9$, as the rejection probabilities are at least 46.7%. This is due to the fact that when $\rho$ is close to 1, the KPSS test and the HR test have difficulties in distinguishing it from 1 which is unit-root. See also Table 3 of Kwiatkowski et al. (1992). Our test does not suffer from this closeness to 1, as for which the order of the magnitude of ACVF matters. Our test
Table 1: The rejection probabilities (%) of the proposed test $T_n$ with $K_0 = 0$ and $c_\kappa = 0.45, 0.55, 0.65, \infty$, the KPSS test and the HR test. The nominal level is 5%.

| $\rho$ | $N$ | $0.45$ | $0.55$ | $0.65$ | KPSS | HR | $\rho$ | $N$ | $0.45$ | $0.55$ | $0.65$ | KPSS | HR |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 40 | 6.0 | 6.0 | 6.0 | 10.4 | 5.7 | 0.5 | 40 | 11.7 | 94.2 | 88.4 | 84.0 | 84.2 | 96.4 |
| 70 | 6.9 | 6.9 | 6.9 | 10.1 | 7.0 | 70 | 11.7 | 96.5 | 92.9 | 88.4 | 90.9 | 99.8 |
| 100 | 6.1 | 6.1 | 6.1 | 10.2 | 8.4 | 100 | 11.3 | 98.0 | 95.5 | 92.2 | 95.5 | 100 |
| 0.9 | 40 | 7.2 | 41.9 | 30.0 | 20.3 | 51.2 | 46.8 | 0.9 | 40 | 13.1 | 99.2 | 97.3 | 94.6 | 91.1 | 98.9 |
| 70 | 7.8 | 23.7 | 14.6 | 10.4 | 46.7 | 58.8 | 70 | 14.8 | 99.8 | 99.1 | 97.9 | 95.3 | 100 |
| 100 | 8.5 | 12.7 | 9.4 | 8.6 | 49.2 | 61.1 | 100 | 16.4 | 99.9 | 99.5 | 99.1 | 97.2 | 100 |
| $-0.5$ | 40 | 7.4 | 7.4 | 7.4 | 7.4 | 1.8 | 0.1 | $-0.5$ | 40 | 5.6 | 82.2 | 75.1 | 67.6 | 81.5 | 99.7 |
| 70 | 6.9 | 6.9 | 6.9 | 6.9 | 2.5 | 0.2 | 70 | 6.3 | 92.1 | 86.1 | 80.0 | 90.1 | 100 |
| 100 | 6.4 | 6.4 | 6.4 | 6.4 | 1.8 | 0.3 | 100 | 5.8 | 94.2 | 89.5 | 85.2 | 94.5 | 100 |
| Model 1 | Model 2 | Model 3 | Model 4 | Model 5 | Model 6 | Model 7 |

(for $\phi_1, \phi_2$) $N$ $0.45$ $0.55$ $0.65$ KPSS HR (for $\phi_1, \phi_2$) $N$ $0.45$ $0.55$ $0.65$ KPSS HR

$55$ $0$ $65$ $KPSS$ $HR$ $\rho_1, \rho_2$ $N$ $0.45$ $0.55$ $0.65$ KPSS HR (for $\phi_1, \phi_2$) $N$ $0.45$ $0.55$ $0.65$ KPSS HR

$45$ $0$ $45$ $0$ $45$ $0$

$55$ $0$ $65$ $KPSS$ $HR$

and the KPSS test work well for Model 2 while the HR test is too conservative. For Model 3, the rejection probabilities of our test and the HR test are close to 5% while the KPSS test does not work as its rejection probabilities range from 16.6% to 26.2%. Our test with $c_\kappa = \infty$ has no power which shows that the truncation step for the critical value in (3) is necessary. The KPSS test has impressive power due to the fact that it has a tendency to overestimate the rejection probability under $H_0$, leading to inflated power. Nevertheless our test displays greater power in most cases in comparison to the KPSS test. The HR test has good power in Models 4 and 5 while it performs poorly in Model 6. The power one property of our test is observable in the simulation as the rejection probability tends to 1 when $N$ increases. Comparing the results of Models 5 and 7, we found that our tests show off the power one property more distinctly as our test statistic has more discriminate power between $I(2)$ and $I(0)$ than that between $I(1)$ and $I(0)$. 9
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Supplementary material

Supplementary material available at Biometrika online includes all the technical proofs and some additional numerical results.

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