Non existence and strong ill-posedness in $C^k$ and Sobolev spaces for SQG

Diego Córdoba* and Luis Martínez-Zoroa†

Instituto de Ciencias Matemáticas CSIC-UAM-UCM-UC3M

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Abstract

We construct solutions in $\mathbb{R}^2$ with finite energy of the surface quasi-geostrophic equations (SQG) that initially are in $C^k$ ($k \geq 2$) but that are not in $C^k$ for $t > 0$. We prove a similar result also for $H^s$ in the range $s \in \left(\frac{3}{2}, 2\right)$. Moreover, we prove strong ill-posedness in the critical space $H^2$.

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*deg@icmat.es
†luis.martinez@icmat.es
1 Introduction

We say a function $\theta(x, t) : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is a solution to the SQG equation with initial conditions $\theta(x, 0) = \theta_0(x)$ if the equation

$$\frac{\partial \theta}{\partial t} + v_1 \frac{\partial \theta}{\partial x_1} + v_2 \frac{\partial \theta}{\partial x_2} = 0 \quad (1)$$

is fulfilled for every $x \in \mathbb{R}^2$ and the derivatives exists for every $x \in \mathbb{R}^2$. The velocity field $v = (v_1, v_2)$ is defined by

$$v_1 = -\frac{\partial}{\partial x_2} \Lambda^{-1} \theta = -\mathcal{R}_2 \theta$$

$$v_2 = \frac{\partial}{\partial x_1} \Lambda^{-1} \theta = \mathcal{R}_1 \theta$$

where $\mathcal{R}_j$ are the Riesz transforms in 2 dimensions, with the integral expression

$$\mathcal{R}_j \theta = \frac{\Gamma(3/2)}{\pi^{1/2}} \text{P.V.} \int_{\mathbb{R}^2} \frac{(x_j - y_j) \theta(y)}{|x - y|^3} dy_1 dy_2$$

for $j = 1, 2$. We denote $\Lambda^\alpha f \equiv (-\Delta)^{\alpha/2} f$ by the Fourier transform $\hat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi)$.

This model arises in a geophysical fluid dynamics context (see [18] and [27]) and its mathematical analysis was initially treated by Constantin, Majda and Tabak in [10] motivated by the number of traits it shares with 3-D incompressible Euler system, where they already established local existence in $H^s$ (see also [11] for bounded domains) and in the case of $C^k, \alpha$ ($k \geq 1$ and $1 > \alpha > 0$) see [31] by Wu. In the critical Sobolev space $H^2$ Chae and Wu [8] proved local existence for a logarithmic inviscid regularization of SQG (see also [20]). Finite time formation of singularities for smooth initial data with finite energy remains an open problem for both SQG and 3-D incompressible Euler equations.

Due to incompressibility and the transport structure of SQG the $L^p$ ($1 \leq p \leq \infty$) norms of the scalar $\theta$ and the $L^2$ norm of the velocity field $v = (v_1, v_2)$ (kinetic energy) are conserved quantities of the system (1) for sufficiently regular solutions. Global existence of weak solutions in $L^2$ was proven by Resnick in [28] (see also [12] in the case of bounded domains) and extended by Marchand in [25] to the class of initial data in $L^p$ with $p > \frac{3}{2}$. However non-uniqueness of weak solutions was obtained by Buckmaster, Shkoller and Vicol in [4] for solutions in $\Lambda^{-1} \theta \in C^2_t C^3_x$ with $\frac{1}{3} < \beta < \frac{2}{3}$ and $\sigma < \frac{1}{2}$. One of the main objectives of this paper is to construct solutions in $\mathbb{R}^2$ of SQG that initially are in $C^k \cap L^2$ ($k \geq 2$) but are not in $C^k$ for $t > 0$. Note that if we consider a velocity field $v(\theta) = \nabla \perp \Lambda^{-\gamma} \perp A(\theta)$ with $\gamma > 0$, then we have local existence in $C^k$ for [11]. We also prove strong ill-posedness in $H^s$ for critical and supercritical spaces in the range $s \in (\frac{3}{2}, 2]$. Moreover we construct solutions that are initially in $H^s$ for $s \in (\frac{3}{2}, 2)$ but are not in $H^s$ for $t > 0$, and that are unique in a certain sense that we will specify later. For the SQG equation, there were no strong ill-posedness results in $H^s$ and $C^k$ prior to the ones obtained in this paper. There are ill-posedness results for active scalars with more singular velocities obtained by Kukavica, Vicol and Wang in [21] and, in the case of SQG, in [15] Elgindi and Masmoudi a mild ill-posedness result is obtained for perturbations of a stationary solution. This, however, does not imply mild or strong ill-posedness for SQG. For more details about this as well as the specific definitions of mild and strong ill-posedness, see subsection 1.4 below. A few days after our result appeared on the arXiv, Jeong and Kim [19] posted an article on the arXiv with a similar result to the one we have for the critical space $H^2$.

There are some remarkable results regarding norm growth in the periodic setting for SQG. Kiselev and Nazarov [22] showed that there exists initial conditions with arbitrarily small norm in $H^s$ ($s \geq 11$) that become large after a long period of time. Recently, He and Kiselev proved in [17] an exponential in time growth for the $C^2$ norm

$$\sup_{t \leq T} |\nabla^2 \theta|_{L^\infty} \geq \exp \gamma T \quad \text{for} \gamma(\theta_0) > 0.$$
On the other hand, numerical simulations suggested the existence of solutions with very fast growth of $|\nabla \theta|$ starting with a smooth profile by a collapsing hyperbolic saddle scenario (see [10], [26] and [9]). Such a scenario cannot develop a singularity as shown analytically in [13] and [14], where a double exponential bound on $|\nabla \theta|$ is obtained. A different blow-up scenario was proposed in [29] where the fast growth of $|\nabla \theta|$ is associated to a cascade of filament instabilities.

1.1 The main theorems

In this paper we prove the following results:

**Theorem 1.1.** *(Strong ill-posedness in $C^k$)* For any $c_0 > 0$, $M > 0$, $2 \leq k \in \mathbb{N}$ and $t_* > 0$, we can find a function $\theta_0(x) \in H^{k+\frac{3}{4}} \cap C^k$ with $||\theta_0(x)||_{C^k} \leq c_0$ such that the unique solution $\theta(x,t) \in H^{k+\frac{3}{4}}$ to the SQG equation (7) with initial conditions $\theta_0(x)$ satisfies $||\theta(x,t_*)||_{C^k} \geq M c_0$.

**Theorem 1.2.** *(Non existence in $C^k$)* Given $c_0 > 0$, $t_* > 0$ and $2 \leq k \in \mathbb{N}$, there are initial conditions $\theta_0 \in H^{k+1/8} \cap C^k$ for the SQG equation (7) such that $||\theta_0||_{C^k} \leq c_0$ and the unique solution $\theta(x,t) \in H^{k+1/8}$ exists and satisfies that $||\theta(x,t)||_{C^k} = \infty$ for all $t \in (0, t_*]$.

In fact, for the initial conditions given by theorem 1.2, there cannot be a solution $\theta(x,t) \in L^\infty_t L^2_x$ to (1) with those initial conditions and $||\theta(x,t)||_{C^k} \leq M(t)$, $M(t) : \mathbb{R}_+ \to \mathbb{R}_+$, even if we allow for $|M(t)|_\infty = \infty$. For more details, see remark 2 after theorem 1.2.

**Theorem 1.3.** *(Strong ill-posedness in $H^s$)* For any $c_0 > 0$, $M > 0$, $s \in \left(\frac{3}{2}, 2\right]$ and $t_* > 0$, we can find a $H^2$ function $\theta_0(x)$ with $||\theta_0(x)||_{H^s} \leq c_0$ such that the only solution $\theta(x,t) \in H^3$, with $\beta(s) > 2$ to the SQG equation (7) with initial conditions $\theta_0(x)$ satisfies $||\theta(x,t_*)||_{H^s} \geq M c_0$.

**Remark 1.** The purpose of this paper is not to obtain the optimal range of Sobolev spaces in which strong ill-posedness is achieved. There are refinements to the methods used in theorem 1.3 that would allow us to decrease the lower bound in the interval of ill-posedness.

**Theorem 1.4.** *(Non existence in $H^s$ in the supercritical case)* For any $t_*, c_0 > 0$ and $s \in \left(\frac{3}{2}, 2\right]$ we can find initial conditions $\theta_0(x)$ with $||\theta_0(x)||_{H^s} \leq c_0$ such that there exists a solution $\theta(x,t)$ to (7) with $\theta(x,0) = \theta_0(x)$ satisfying $||\theta(x,t)||_{H^s} = \infty$ for all $t \in (0, t_*]$. Furthermore, it is the only solution with initial conditions $\theta_0(x)$ such that $\theta(x,t) \in L^\infty_t C^0_x \cap L^2_t L^2_x$ ($0 < \alpha_1 < \frac{1}{2}$) with the property that $||\theta(x,t)||_{H^2} \leq M(t) (1 < \alpha_2 \leq \frac{1}{2})$ for some function $M(t)$.

**Theorem 1.5.** *(Non uniform existence in $H^2$)* For any $c_0 > 0$ there exist initial conditions $\theta(x,0)$ with $||\theta(x,0)||_{H^2} \leq c_0$ such that any solution $\theta(x,t)$ to (7) satisfies

$$\text{ess-sup}_{t \in [0,\epsilon]} ||\theta(x,t)||_{H^2} = \infty$$

for any $\epsilon > 0$.

The proof of theorems 1.4 and 1.5 can be adapted to work in the critical spaces $W^{1+\frac{2}{p},p}$, $p \in (1,\infty]$, but we will not go into detail since that is not the goal of the paper. For more information regarding the necessary changes to adapt the proof for these cases, see remark 5 after theorem 1.4.

1.2 The strategy of the proof

Ill posedness in critical spaces for the incompressible Euler equations was already considered in papers by Bourgain and Li (see [2] and [1]) obtaining strong ill-posedness for the velocity in the 2D and 3D Euler equations in $C^k$, $k \geq 1$ and for the vorticity in the space $H^{d/2}$ (with $d$ the dimension). In fact, they obtained stronger results, in [1] they obtain a velocity $u$ satisfying that, for $0 < t_0 \leq 1$

$$\text{ess-sup}_{0 < t < t_0} ||u(t,\cdot)||_{C^0} = \infty,$$

$$||u(0,\cdot)||_{C^0} \leq c_0$$

and in [2] the vorticity $\omega$ satisfies

$$\text{ess-sup}_{0 < t < t_0} ||\omega(t,\cdot)||_{H^{d/2}} = \infty.$$
namely to prove strong ill-posedness in fulfils the evolution equation with an appropriate small source term (for a more precise definition pseudo-solution of SQG. We say that a function \( \bar{\theta} \) is a pseudo-solution to the SQG equation if it fulfills the evolution equation with an appropriate small source term (for a more precise definition see section 2.2 below). Namely to prove strong ill-posedness in \( C^K \) we will use the following family of pseudo-solutions in the time interval \( t \in [0, T] \)

\[
\bar{\theta}_{\lambda,J,N}(r,\alpha,t) := C_{\lambda}(r) + \lambda f_2(N^{1/2}(r-1) + 1) \sum_{j=1}^{J} \sin(Nj\alpha - \lambda t N j^2 \omega(f_1) - \lambda C_0 t - \delta j),
\]

where \((r,\alpha)\) are the polar coordinates, \( f_1 \) are smooth compactly supported radial functions, \( v_\alpha(f_1) \) is the angular velocity generated by the function \( f_1 \), the parameters fulfil \( \lambda, J, N \in (\mathbb{R}_+, \mathbb{N}, \mathbb{N}) \) and \( C_0 \) is a constant that arises from the velocity operator. This \( \bar{\theta}_{\lambda,J,N} \) fulfills the evolution equation

\[
\frac{\partial \bar{\theta}_{\lambda,J,N}}{\partial t} + \frac{\partial v_\alpha(\lambda f_1)}{\partial \alpha} = \lambda C_0 H(\bar{\theta}_{\lambda,J,N}) = 0
\]

where \( H \) is the Hilbert transform with respect to the \( \alpha \) variable. The ill-posedness arises from the unboundedness of the operator \( H \) in the \( C^K \cap L^2 \) spaces. Note however that the appearance of an unbounded operator in our evolution equation does not imply directly ill-posedness, since for example in the Burger-Hilbert’s equation

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + H(f) = 0,
\]

although the \( L^\infty \) norm has a fast growth (see [5]) as long as the solution is \( C^{1,\delta} \), Bressan and Nguyen [3] proved the surprising result of global existence in \( L^2 \cap L^\infty \).

We denote \( \theta_{\lambda,J,N}(r,\alpha,t) \) to be the unique \( H^{k+\frac{1}{2}} \) solution of (1) satisfying initially

\[
\theta_{\lambda,J,N}(r,\alpha,0) = \bar{\theta}_{\lambda,J,N}(r,\alpha,0).
\]

We will prove that that for sufficiently large \( N \) we have

\[
||\theta_{\lambda,J,N}(r,\alpha,t) - \bar{\theta}_{\lambda,J,N}(r,\alpha,t)||_{H^K} \leq C N^{-\frac{1}{2} + a(k)}
\]

with \( a(k) > 0 \) and the constant \( C \) depends on the parameters \( \lambda, J, k, T \). With this bound and the properties of the pseudo-solution we obtain

\[
||\theta_{\lambda,J,N}(r,\alpha,t)||_{C^K} \geq \tilde{C} \lambda^2 \ln(J)t
\]

where \( \tilde{C} \) is a universal constant.

Once we have solutions with arbitrary large growth in norm we prove non existence of solutions in \( C^K \) by considering the following initial conditions

\[
\theta(x,0) = \sum_{n \in \mathbb{N}} T_R u(\bar{\theta}_{\lambda_n,J_n,N_n}(x,0))
\]

with \( T_R(f(x_1,x_2)) = f(x_1 + R, x_2) \). By choosing appropriately the parameters \( \lambda_n \in \mathbb{R}_+, (K_n)_n \in \mathbb{N}, (N_n)_n \in \mathbb{N} \) and \( (R_n)_n \in \mathbb{N} \) we can show that the unique solution \( \theta(x,t) \in H^{k+\frac{1}{2}} \) with this initial data will leave \( C^K \) instantly. In particular the solution \( \theta(x,t) \) is not in \( C^K \) for any time \( t \in (0,T] \).
In the case of strong ill-posedness in Sobolev spaces, theorem 1.3, we will use a similar strategy in the range below the critical exponent $s = 2$, although the proofs are more involved since we do not have any existence result for the supercritical Sobolev space $s$. However, in the critical case (theorem 1.5) it is not clear that a suitable pseudo-solution could be constructed by perturbing a radial solution. In order to overcome this obstacle we need a different strategy. In this case our initial data is similar to the one consider in [2] with the following expression

$$\theta_{c,J,b}(x,0) = \sum_{j=1}^{J} c f(b^{-j}r)b^j \sin(2\alpha_j)$$

where the radial function $0 < f \in C^\infty$ has $\text{supp}(f) \in \left[\frac{1}{2}, \frac{3}{2}\right]$, $c > 0$ and $J \in \mathbb{N}$. The main difficulty when considering this type of initial conditions is that the usual energy estimates only give existence for a short time interval which does not provide enough growth in $H^2$. To obtain improved time intervals of existence we decompose our solution as a sum of pseudo-solutions with initial conditions

$$c f(b^{-j}r)b^j \sin(2\alpha_j)$$

for $j = 1, \ldots, J$. To finish the proof we perturb this solution with a small $H^2$ function localized around the origin that will experience very large norm growth.

The paper is organized as follows. First in section 2 we prove strong ill-posedness and non existence for the space $C^k$. In section 3 we show strong ill-posedness and non existence for Sobolev spaces in the supercritical case. Finally in section 4 we prove strong ill-posedness for the critical $H^2$ space.

1.3 Notation

In this paper we will consider functions $f(x) : \mathbb{R}^2 \to \mathbb{R}$ in $C^k$ with $k$ a positive integer and $H^s$ with $s$ a positive real number. These spaces allow many different equivalent norms, but we will specifically use

$$\|f(x)\|_{C^k} = \sum_{i=0}^{k} \sum_{j=0}^{i} \left\| \frac{\partial^i f(x)}{\partial x_1^{i-j} x_2^j} \right\|_{L^\infty}$$

and for $H^s$, when $s$ is a positive integer we will use

$$\|f(x)\|_{H^s} = \sum_{i=0}^{s} \sum_{j=0}^{i} \left\| \frac{\partial^i f(x)}{\partial x_1^{i-j} x_2^j} \right\|_{L^2},$$

where the derivative is understood in the weak sense.

For $s$ non integer, the standard way of defining the norm is by

$$\|f(x)\|_{H^s} = \|\mathcal{F}^{-1} \left( (1 + |\xi|^2)^s \mathcal{F} f \right) \|_{L^2},$$

where $\mathcal{F}$ is the Fourier transform. We will not require to use this definition to compute the norm in these spaces through this paper. For $s$ a positive integer, we will sometimes write

$$\|f(x)1_A\|_{H^s},$$

where $1_A$ is the characteristic function in the set $A$. This is slightly an abuse of notation since the function $f(x)1_A$ may not be in $H^s$, but we will use this as a more compact notation to write

$$\sum_{i=0}^{s} \sum_{j=0}^{i} \int_{A} \left( \frac{\partial^i f(x)}{\partial x_1^{i-j} x_2^j} \right)^2 dx \frac{1}{2}.$$
We will work both in normal cartesian coordinates and in polar coordinates, using the change of variables \( x_1 = r \cos(\alpha), x_2 = r \sin(\alpha) \). We will sometimes define a function in the space \((x_1, x_2)\) \( f(x) \) and then refer to \( f(r, \alpha) \) (or vice versa), and this is an abuse of notation since we should actually write, if \( F(r, \alpha) \) is the change of variables that takes us from \((r, \alpha)\) to \((x_1, x_2)\), \( f(F(r, \alpha)) \). Furthermore, given a function in polar coordinates, we define

\[
\| f(r, \alpha) \|_{H^s} := \| f(F(r, \alpha)) \|_{H^s},
\]

\[
\| f(r, \alpha) \|_{C^k} := \| f(F(r, \alpha)) \|_{C^k}.
\]

For two sets \( A_1, A_2 \), we will use \( d(A_1, A_2) \) to refer to the distance between the sets.

### 1.4 Ill-posedness

Since we will be dealing with ill-posedness through this paper, it is important that we clarify exactly what we mean by mild and strong ill-posedness, specially since one can give similar (but not necessarily equivalent) definitions of these concepts. Through this paper we will use the same definition as in [13], that is

**Definition 1.** Given spaces \( X, Y \) with \( Y \) continuously embedded in \( X \) and an evolution equation

\[
\frac{\partial f(x, t)}{\partial t} = G(f(x, t))
\]

\( f(x, 0) = f_0(x) \)

we say that the evolution equation is mildly ill-posed if we can find \( f_\epsilon(x) \in X \) such that there exists a unique solution \( f_\epsilon(x, t) \) in \( L^\infty([0, \epsilon]; Y) \) to our evolution equation with initial conditions \( f_\epsilon(x) \) such that \( \| f_\epsilon(x) \|_X \leq \epsilon \) but there exists a time \( t \in (0, \epsilon) \) such that \( \| f_\epsilon(x) \|_X \geq c \) with \( c > 0 \) some constant independent of \( \epsilon \). Furthermore, if we can take \( \epsilon = \frac{1}{k} \), then we say that the problem is strongly ill-posed.

What these notions tell us about the evolution equation is that it is not well behaved in the space \( X \). More precisely, mild ill-posedness tells us that the solution map is not continuous with respect to the initial conditions, and strong ill-posedness shows both that and arbitrarily fast norm growth, which could potentially lead to an instantaneous blow up, and therefore, to non-existence of solutions.

Although strong and mild ill-posedness are related, and in fact in some situations they are equivalent (for example, if your evolution equation has appropriate scaling properties), one does not imply the other. In fact, if we consider a radial function \( f(r) = r^{k+\gamma}g(r) \) with \( g(r) \) a \( C^\infty \) function such that \( g(r) = 1 \) if \( r \in [0, 1] \), \( g(r) = 0 \) if \( r \geq 2 \), we have that the evolution equation for perturbations of \( f(r) \) for SQG, which is

\[
\frac{\partial \theta_{pert}}{\partial t} + u(\theta_{pert}) \cdot \nabla \theta_{pert} + u(f(r)) \cdot \nabla \theta_{pert} + u(\theta_{pert}) \cdot \nabla f(r) = 0
\]

is mildly ill-posed in \( C^{k,\gamma} \) but not strongly ill-posed in \( C^{k,\gamma} \). The mild ill-posedness is easy to obtain by noting that, if \( v(\theta_{pert}(x,0))(x = 0) = (a, b) \neq (0, 0) \), then, by using that \( \theta_{pert}(x, t) = w(x, t) - f(r) \), with \( w(x, t) \) the solution to SQG with initial conditions \( \theta_{pert}(x, 0) + f(r) \), we get that

\[
\lim_{t \to 0^+} \left( \lim_{h \to 0^+} \frac{\partial^k \theta_{pert,0}(x = 0)}{\partial x_1^k} - \frac{\partial^k \theta_{pert,0}(x = h)}{\partial x_1^k} \right) = \prod_{i=1}^{k} (i + \gamma),
\]
and thus
\[ \lim_{t \to 0^+} \frac{||{θ}_\text{pert}(x,t)||_{C^k}}{t} \geq \prod_{i=1}^{k}(i + \gamma) \]
and since this can be obtained independently of the norm of \( {θ}_\text{pert}(x,0) \), we obtain mild ill-posedness. But since we know that solutions of SQG in \( C^k \) will obtain would remain the same if we were to change \( v \) with \( a \) stationary smooth radial solution. In contrast, there are previous results ([6] and [7]) where the perturbation of a radial function led to global \( C^k \) solutions respectively.

In this section we will use the following expression of the velocity field
\[ v(θ(x)) = \frac{Γ(3/2)}{π^{3/2}} \cdot PV. \int_{R^2} \frac{(x - y)^{-1}θ(y)}{|x - y|^3} dy_1 dy_2 \]
with \( v = (v_1, v_2) \) and for a vector \( (a, b) \) we define \( (a, b)^\perp := (-b, a) \).

We will omit the constant on the outside of the integral from now on, since all the results we will obtain would remain the same if we were to change \( \frac{Γ(3/2)}{π^{3/2}} \) for an arbitrary (non-zero) constant.

**Lemma 2.1.** Given natural numbers \( n, N \) and a \( L^\infty \) function \( g_N(r) : [0, \infty) \to R \) with support in \( (1 - \frac{3}{2}\sqrt{2}, 1 + \frac{3}{2}\sqrt{2}) \) we have that, for \( θ(r, α) = g_N(r)sin(Nα) \), there exists a constant \( C \) (depending on \( n \)) such that, for \( N \) big enough and \( r \in [1 - \frac{3}{2}, 1 + \frac{3}{2}] \)

\[ |v_r(θ(\cdot))(r, α)| - cos(Nα) \int_{R^2 × [-π, π]} \frac{r^2 α' g_N(r + h)sin(Nα')}{|h^2 + r^2(α')^2|^{3/2}} dh \]

\[ \leq C||g_N||_{L^\infty} N^{-1/2} \]

Analogously, for \( θ(r, α) = g_N(r)cos(Nα) \) we have that

\[ |v_r(θ(\cdot))(r, α) + sin(Nα) \int_{R^2 × [-π, π]} \frac{r^2 α' g_N(r + h)sin(Nα')}{|h^2 + r^2(α')^2|^{3/2}} dh \]

\[ \leq C||g_N(r)||_{L^\infty} N^{-1/2} \]
Analogously if with \(A\) to where we have used trigonometric identities and eliminated the terms that are odd with respect to \(g\). Proof.

Before we get into the proof, a couple of comments need to be made. First, \(v_r\), refers to the radial component of the velocity at a given point, that is to say, if we call \(\hat{r}\) to the unitary vector in the direction of \(x\) then

\[
v_r(\theta(\cdot,\cdot))(x) = P.V. \int_{\mathbb{R}^2} \hat{r} \frac{(x-y) \cdot \theta(y)}{|x-y|^3} dy_1 dy_2.
\]

However, the expression obtained in lemma 2.1 requires us to work in polar coordinates. Therefore, considering a generic function \(f(r) \sin(ka)\) and making the usual changes of variables \((x_1, x_2) = r(\cos(\alpha), \sin(\alpha)), (y_1, y_2) = r'(\cos(\alpha'), \sin(\alpha'))\) we obtain

\[
v_r(\theta(\cdot, \cdot, \cdot))(r, \alpha) = P.V. \int_{\mathbb{R}^2} (r')^2 \frac{(\cos(\alpha) \sin(\alpha') - \sin(\alpha) \cos(\alpha')) f(r') \sin(ka')}{[(r \cos(\alpha) - r' \cos(\alpha'))^2 + (r \sin(\alpha) - r' \sin(\alpha'))^2]^3/2} d\alpha' d\alpha'
\]

\[
= P.V. \int_{\mathbb{R}^4} (r')^2 \frac{\sin(\alpha' - \alpha)}{[(r - r')^2 + 2rr'(1 - \cos(\alpha - \alpha'))]^3/2} f(r') \sin(ka') d\alpha' d\alpha'
\]

\[
= \cos(ka) P.V. \int_{\mathbb{R}^4} (r')^2 \frac{\sin(\alpha' - \alpha) f(r') \sin(ka' - ka)}{[(r - r')^2 + 2rr'(1 - \cos(\alpha - \alpha'))]^3/2} d\alpha' d\alpha'
\]

\[
= \cos(ka) P.V. \int_{\mathbb{R}^4} (r + h)^2 \frac{\sin(\alpha') f(r + h) \sin(ka')}{[h^2 + 2(r + h)r(1 - \cos(\alpha'))]^3/2} d\alpha' dh,
\]

where we have used trigonometric identities and eliminated the terms that are odd with respect to \(\alpha' - \alpha\). Note that in the last line we have relabeled \(\alpha' - \alpha\) as \(\alpha'\) for a more compact notation. Analogously if \(\theta(r, \alpha) = f(r) \cos(ka)\) we obtain

\[
v_r(\theta(\cdot, \cdot, \cdot))(r, \alpha) = -\sin(ka) \int_{\mathbb{R}^4} (r + h)^2 \frac{\sin(\alpha') f(r + h) \sin(ka')}{[h^2 + 2(r + h)r(1 - \cos(\alpha'))]^3/2} d\alpha' dh.
\]

With this, we are now ready to start the proof of lemma 2.1.

Proof. We need to find bounds for

\[
\int_{\mathbb{R}^4} \frac{r^2 \alpha' g_N(r + h) \sin(N\alpha')}{[h^2 + r^2(\alpha')^2]^3/2} d\alpha' dh - \int_{\mathbb{R}^4} \frac{(r + h)^2 \sin(\alpha') g_N(r + h) \sin(N\alpha')}{[h^2 + 2(r + h)r(1 - \cos(\alpha'))]^3/2} d\alpha' dh
\]

with \(g_N(r)\) satisfying our hypothesis. We will first focus on

\[
I_A := \int_{A} \frac{r^2 \alpha' g_N(r + h) \sin(N\alpha')}{[h^2 + r^2(\alpha')^2]^3/2} d\alpha' dh - \int_{A} \frac{(r + h)^2 \sin(\alpha') g_N(r + h) \sin(N\alpha')}{[h^2 + 2(r + h)r(1 - \cos(\alpha'))]^3/2} d\alpha' dh
\]

with \(A := [-2N^{-1/2}, 2N^{-1/2}] \times [-2N^{-1/2}, 2N^{-1/2}]\). This is accomplished in several steps. It should be noted that the constant \(C\) may depend on \(n\) and it may change through the proof, as it is the name we use for a generic constant that is independent of \(N\) and \(g\).
Step 1:

\[
| \int_A (r + h)^2 (\sin(\alpha') - \alpha') g_N(r + h) \sin(N\alpha') \frac{\alpha'}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} \, d\alpha' \, dh |
\leq C \int_A (r + h)^2 \frac{|\alpha'|^3 g_N(r + h)}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} \, d\alpha' \, dh
\leq C \int_A |g_N(r + h)| \, d\alpha' \, dh
\leq C N^{-1} \|g_N\|_{L^\infty}
\]

Step 2: Defining

\[
F(r, h, \alpha') := \frac{1}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} - \frac{1}{|h^2 + (r + h)\alpha'|^3} \, d\alpha' \, dh
\]

we estimate the following integral by

\[
| \int_A (r + h)^2 \alpha' g_N(r + h) \sin(N\alpha') F(r, h, \alpha') \, d\alpha' \, dh |
\leq C \int_A |\alpha'| |g_N(r + h)| \frac{(\alpha')^4}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} \, d\alpha' \, dh
\leq C \int_A |g_N(r + h)| \, d\alpha' \, dh
\leq C N^{-1} \|g_N\|_{L^\infty}
\]

Step 3:

\[
| \int_A (r + h)^2 - r^2 \alpha' g_N(r + h) \sin(N\alpha') \frac{\alpha'}{|h^2 + (r + h)\alpha'|^2|^{3/2}} \, d\alpha' \, dh |
\leq C \int_A \frac{|\alpha'| |g_N(r + h)|}{|h^2 + (r + h)\alpha'|^2|^{3/2}} \, d\alpha' \, dh
\leq C \int_A |g_N(r + h)| \, d\alpha' \, dh
\leq C N^{-1/2} \|g_N\|_{L^\infty}
\]

Combining all these three steps we conclude

\[
| \int_A \frac{r^2 \alpha' g_N(r + h) \sin(N\alpha')}{|h^2 + r(r + h)\alpha'|^{3/2}} \, d\alpha' \, dh - \int_A \frac{(r + h)^2 \sin(\alpha') g_N(r + h) \sin(N\alpha')}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} \, d\alpha' \, dh |
\leq C \|g_N\|_{L^\infty} N^{-1/2},
\]

and to bound the contribution of the integral in \( A \) we also need

\[
| \int_A \frac{\alpha' g_N(r + h) \sin(N\alpha')}{|h^2 + (r + h)\alpha'|^{3/2}} \, d\alpha' \, dh - \frac{r^2 \alpha' g_N(r + h) \sin(N\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha' \, dh |
\leq C \int_A |\alpha'| |g_N(r + h)| \frac{(\alpha')^2|h|}{|h^2 + \alpha'|(\alpha')^2|^{5/2}} \, d\alpha' \, dh
\leq C \int_A |g_N(r + h)| \frac{|h|}{|h^2 + 2(\alpha')^2|^{5/2}} \, d\alpha' \, dh
\leq C N^{-1/2} \|g_N\|_{L^\infty}
\]
Therefore adding and subtracting
\[ \int_A \frac{r^2 \alpha' g_N(r + h) \sin(Nn\alpha')}{|h^2 + (r + h)r(\alpha')^2|^{3/2}} \]
to (3) we obtain that
\[ I_A \leq C||g_N||_{L^\infty} N^{-1/2}. \]

Finally, we need to deal with the integral outside of A. First we bound the following integral
\[ \int_{\mathbb{R} \times [-\pi, \pi] \setminus A} \frac{r^2 \alpha' g_N(r + h) \sin(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} \, dh \]
\[ = 2 \int_{[-2N^{1/2}, 2N^{-1/2}]} \int_{[2N^{-1/2}, \pi]} \frac{r^2 \alpha' g_N(r + h) \sin(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} \, dh \, d\alpha'. \]

To do this we compute, fixed arbitrary h and r, the integral over an interval of the form \( \alpha \in \left[k \frac{2\pi}{Nn}, \frac{2\pi}{Nn} + (k + 1) \frac{2\pi}{Nn} - \frac{2\pi}{Nn}\right] \) (which we will denote \([\alpha_k, \alpha_{k+1})\)). Note that it has the length of the period of \( \sin(Nn\alpha) \) and that \( \sin(Nn\alpha) \) is an even function around the point \( k \frac{2\pi}{Nn} + \frac{\pi}{2Nn} \).

If we define
\[ H(\alpha', h, r) := \frac{\alpha'}{|h^2 + r^2(\alpha')^2|^{3/2}} \]
we have that
\[ \int_{[\alpha_k, \alpha_{k+1}]} \sin(Nn\alpha') \frac{\alpha'}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha' \]
\[ = \int_{[\alpha_k, \alpha_{k+1}]} \sin(Nn\alpha') \left[ H(\frac{\alpha_k + \alpha_{k+1}}{2}, h, r) + \frac{\partial H(\frac{\alpha_k + \alpha_{k+1}}{2}, h, r)}{\partial \alpha'} (\alpha' - \frac{\alpha_k + \alpha_{k+1}}{2}) + \frac{\partial^2 H(\alpha', h, r)}{\partial \alpha'^2} \frac{1}{2} (\alpha' - \frac{\alpha_k + \alpha_{k+1}}{2})^2 \right] \, d\alpha' \]
\[ \leq C \int_{[\alpha_k, \alpha_{k+1}]} \left| \sin(Nn\alpha') \right| (\alpha' - \frac{\alpha_k + \alpha_{k+1}}{2})^2 \frac{1}{|h^2 + r^2\alpha_k^2|^{3/2}} \, d\alpha' \]
\[ \leq C \left( \frac{2\pi}{Nn} \right)^3 \frac{1}{|h^2 + r^2\alpha_k^2|}, \]

where we have used a second degree taylor expansion around \( \frac{\alpha_k + \alpha_{k+1}}{2} \) for \( H \), and \( c(\alpha') \) is where we need to evaluate the second derivative to actually obtain an equality. Now, adding over all the intervals \([\alpha_k, \alpha_{k+1}]\) with \( \pi - \frac{k\pi}{2N} \geq \alpha_k \geq 2N^{-1/2} \), we get the upper bound
\[ \sum_{\alpha_k \geq 2N^{-1/2}} \frac{2\pi}{Nn} \left( \frac{2\pi}{Nn} \right)^3 \frac{1}{|h^2 + r^2\alpha_k^2|^{3/2}} \leq \sum_{k \geq k_{N/2}} C \left( \frac{2\pi}{Nn} \right)^3 \frac{1}{|h^2 + r^2\alpha_k^2|^{3/2}} \]
\[ \leq C \left( \frac{2\pi}{Nn} \right)^3 \int_{\frac{k_{N/2}}{Nn} - 1}^{\frac{k_{N/2}}{Nn}} \frac{1}{|h^2 + r^2(x\frac{2\pi}{Nn} - \frac{Nn}{2r})^2|^{3/2}} \, dx \]
\[ \leq C \left( \frac{2\pi}{Nn} \right)^3 \int_{\frac{k_{N/2}}{Nn} - 2}^{\frac{k_{N/2}}{Nn} - 1} \frac{1}{|h^2 + (r\frac{2\pi}{Nn})^2|^{3/2}} \, dx \]
\[ \leq C \left( \frac{2\pi}{Nn} \right)^3 \left( \frac{2\pi}{Nn} \right)^{-4} \left( \frac{2\pi}{N^3/2n} \right) \leq C N^{-1/2}, \]
where we took $N$ big to pass from the third to the fourth line. The only contribution missing now from the integral in the $\alpha'$ variable, if we call $\alpha_{k_0}$ the smallest $\alpha_k$ such that $\alpha_k \geq 2N^{-1/2}$ and $\alpha_\infty$ the biggest one with $\pi \geq \alpha_\infty$, is

$$
\int_{[2N^{-1/2},\alpha_{k_0}] \cup [\alpha_\infty, \pi]} \sin(Nn\alpha') \frac{\alpha'}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha',
$$

but

$$
| \int_{2N^{-1/2}}^{\alpha_{k_0}} \sin(Nn\alpha') \frac{\alpha'}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha' | \leq C,
$$

$$
| \int_{\alpha_\infty}^{\pi} \sin(Nn\alpha') \frac{\alpha'}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha' | \leq \frac{C}{N}.
$$

Combining all three contributions and integrating with respect to $h$ we get

$$
|2 \int_{[-2N^{1/2}, 2N^{-1/2}]} \int_{[2N^{-1/2}, \pi]} \frac{r^2 \alpha' g_N(r + h) \sin(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} \, d\alpha' \, dh |
\leq \int_{[-2N^{1/2}, 2N^{-1/2}]} C|g_N(r + h)| \, dh \leq C||g_N||_{L^\infty} N^{-1/2}.
$$

The term

$$
| \int_{\mathbb{R} \times [-\pi, \pi] \setminus A} (r + h)^2 \frac{\sin(\alpha') g_N(r + h) \sin(Nn\alpha')}{|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}} \, d\alpha' \, dh |
$$

is bounded in a similar fashion, integrating first with respect to $\alpha'$ in intervals of the form $[\alpha_k, \alpha_{k+1}]$ and then bounding by brute force the parts that are not covered exactly by said intervals, and with that we would be done.

Now that we have a manageable expression for the radial velocity we are ready to compute it explicitly (with some error) for some special kind of functions.

**Lemma 2.2.** Given natural numbers $n$, $N$ and a $C^2$ function $g_N(\cdot) : \mathbb{R} \to \mathbb{R}$ with support in the interval $(1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2})$ satisfying $||g_N||_{C^i} \leq MN^{1/2}$ for $i = 0, 1, 2$ there exists a constant $C_0 \neq 0$ (independent of $N$, $n$ and $g_N$) such that for $\tilde{x} \in [1 - N^{-\frac{1}{2}}, 1 + N^{-\frac{1}{2}}]$

$$
|C_0 g_N(\tilde{x}) - \int_{\mathbb{R} \times [-\pi, \pi]} g_N(\tilde{x} + h_1) \frac{\sin(Nn h_2)h_2}{(h_1^2 + h_2^2)^{3/2}} \, dh_1 dh_2 |
\leq CMN^{-1/2},
$$

with $C$ depending on $n$.

**Proof.** The strategy of this proof is to first show that

$$
| \int_{\mathbb{R} \times [-\pi, \pi]} (g_N(\tilde{x} + h_1) - g_N(\tilde{x})) \frac{\sin(Nn h_2)h_2}{(h_1^2 + h_2^2)^{3/2}} \, dh_1 dh_2 |
\leq CMN^{-1/2},
$$

and then prove that

$$
I_{N,n} := \int_{\mathbb{R} \times [-\pi, \pi]} \frac{\sin(Nn h_2)}{(h_1^2 + h_2^2)^{3/2}} \, dh_1 dh_2
$$
is a Cauchy series with respect to \( N \), satisfying
\[
|I_{N_1,n} - I_{N_2,n}| \leq C \sup(N_1, N_2)^{-1/2}
\]
with \( C \) depending on \( n \).

Combining both of these results and taking
\[
C_0 = \lim_{N \to \infty} I_{N,n}
\]
we obtain (7), and we only need to check that \( C_0 \) is different from zero and independent of \( n \).

We start by obtaining bound (5), noting that, by parity
\[
\text{we start by fixing some } h_1 \text{ and obtaining bounds for the integral with respect to } h_2. \text{ This is done as in lemma 2.1, dividing in periods of length } \frac{2N}{N_n} \text{ starting at } \frac{\pi}{2N}, \text{ and approximating } \frac{h_2}{h_1 + h_2} \text{ by its second order Taylor expansion, since the first two orders will cancel. That way, for the interval with } h_2 \in [k \frac{2N}{N_n} + \frac{\pi}{2N}, (k + 1) \frac{2N}{N_n} + \frac{\pi}{2N}] \text{ we obtain the bound}
\]
\[
|\int_{[2N^{-1/2},2N^{-1/2}]} (g_N(\tilde{x} + h_1) - g_N(\tilde{x}))sin(Nnh_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}}dh_1dh_2| = |\int_{[0,2N^{-1/2}]} (g_N(\tilde{x} + h_1) + g_N(\tilde{x} - h_1) - 2g_N(\tilde{x}))sin(Nnh_2)h_2 \frac{h_2}{(h_1^2 + h_2^2)^{3/2}}dh_1dh_2|.
\]

We start by fixing some \( h_1 \) and obtaining bounds for the integral with respect to \( h_2 \). This is done as in lemma 2.1, dividing in periods of length \( \frac{2N}{N_n} \) starting at \( \frac{\pi}{2N} \), and approximating \( \frac{h_2}{h_1 + h_2} \) by its second order Taylor expansion, since the first two orders will cancel. That way, for the interval with \( h_2 \in [k \frac{2N}{N_n} + \frac{\pi}{2N}, (k + 1) \frac{2N}{N_n} + \frac{\pi}{2N}] \) we obtain the bound
\[
|\int_{[k \frac{2N}{N_n} + \frac{\pi}{2N}, (k + 1) \frac{2N}{N_n} + \frac{\pi}{2N}]} \sin(Nnh_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}}dh_2| \leq C \left( \frac{2\pi}{N_n} \right)^{3} \left( \frac{1}{(h_1^2 + (\frac{2\pi}{N_n})^2)^2} \right).
\]

We can add periods contained in the interval \([0,2N^{-1/2}]\) and, if we denote by \( k_\infty = k_\infty(N,n) \) the biggest integer \( k \) such that \( (k + 1) \frac{2N}{N_n} + \frac{\pi}{2N} \leq 2N^{-1/2} \), we get that
\[
\begin{align*}
|\int_{\frac{2\pi}{N_n}}^{(k+1)\frac{2\pi}{N_n} + \frac{\pi}{2N}} \sin(Nnh_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}}dh_2| &\leq \sum_{k=1}^{k_\infty} C \left( \frac{2\pi}{N_n} \right)^{3} \left( \frac{1}{(h_1^2 + (\frac{2\pi}{N_n})^2)^2} \right) \leq C \left( \frac{2\pi}{N_n} \right)^{3} \int_{0}^{\frac{2\pi}{N_n}} \frac{1}{(h_1^2 + (\frac{2\pi}{N_n})^2)^2}dh_1d2x \leq C \left( \frac{2\pi}{N_n} \right)^{3} \int_{0}^{\frac{2\pi}{N_n}} \frac{1}{(h_1^2 + (\frac{2\pi}{N_n})^2)^2}dh_1d2x \leq C \left( \frac{2\pi}{N_n} \right)^{2} \frac{1}{h_1^2}.
\end{align*}
\]

This allows us to bound the contribution when \( h_1 \geq \frac{2\pi}{N_n} \) by dividing it in three parts:

1) If \( h_2 \leq \frac{5\pi}{2N} \),
\[
|\int_{\frac{2\pi}{N_n}}^{\frac{2\pi}{N_n}} \frac{g_N(\tilde{x} + h_1) + g_N(\tilde{x} - h_1) - 2g_N(\tilde{x})sin(Nnh_2)h_2}{(h_1^2 + h_2^2)^{3/2}}dh_1dh_2| \leq C \left( \frac{5\pi}{2N} \right)^{2} \left( \frac{1}{h_1^2}N \right) \leq C MN^{-1/2}.
\]
2) If \( \frac{\pi}{2Nn} \leq h_2 \leq (k_\infty + 1) \frac{2\pi}{Nn} + \frac{\pi}{2Nn} \):

\[
\left| \int_{2N \pi \pm \frac{\pi}{2Nn}}^{2N \pi \pm \frac{2\pi}{Nn}} (g_N(\bar{x} + h_1) + g_N(\bar{x} - h_1) - 2g_N(\bar{x})) \frac{\sin(Nnh_2)h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 \right| dh_1 \leq CMN^{-1}\log(N).
\]

3) If \( (k_\infty + 1) \frac{2\pi}{Nn} + \frac{\pi}{2Nn} \leq h_2 \leq 2N^{-1/2} \):

\[
\left| \int_{2N \pi \pm \frac{\pi}{2Nn}}^{2N \pi \pm \frac{2\pi}{Nn}} (g_N(\bar{x} + h_1) + g_N(\bar{x} - h_1) - 2g_N(\bar{x})) \frac{\sin(Nnh_2)h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 \right| dh_1 \leq CMN^{-\frac{2}{3}}.
\]

Finally, we bound the error when \( h_1 \leq \frac{\pi}{2Nn} \):

1) If \( |h_2| \leq 2N^{-1/2} \):

\[
\left| \int_{0}^{2N \pi \pm \frac{\pi}{2Nn}} \int_{0}^{2N \pi \pm \frac{2\pi}{Nn}} \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 \right| dh_1 \leq CMN^{-1/2}.
\]

2) If \( |h_2| \geq 2N^{-1/2} \):

\[
\left| \int_{0}^{2N \pi \pm \frac{\pi}{2Nn}} \int_{2N \pi \pm \frac{\pi}{2Nn}} \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 \right| dh_1 \leq CMN^{-1}.
\]

Combining all these bounds we obtain (5). Therefore, we have that it is enough to prove that

\[
|C_0 g_N(\bar{x}) - \int_{\mathbb{R} \times [-\pi, \pi]} g_N(\bar{x}) \sin(Nnh_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} h_1 dh_2| dh_1 \leq CMN^{-1/2},
\]

which is equivalent to study the behaviour of \( I_{N,n} \), defined as in (6).

To obtain the properties of \( I_{N,n} \), we start by transforming the integral with a change of variables

\[
h_1 = Nnh_1, \quad h_2 = Nnh_2,
\]

although we will relabel \( h_1, h_2 \) as \( h_1, h_2 \) to simplify the notation.

\[
g_N(\bar{x}) \int_{[-2N^{-1/2}, 2N^{-1/2}] \times [-\pi, \pi]} \sin(Nnh_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_1 dh_2
\]

\[
= g_N(\bar{x}) \int_{[-2N^{1/2}, 2N^{1/2}] \times [-\pi, \pi]} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_1 dh_2.
\]
If we compare the integral for different values of $N$, $N_1 \geq N_2$ we get

$$I_{N_1,n} - I_{N_2,n} = \int_{A \cup B} \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_1 dh_2$$

with

$$A = [-2nN_2^{1/2}, 2nN_2^{1/2}] \times [N_2 n\pi, N_1 n\pi] \cup [-2nN_2^{1/2}, 2nN_2^{1/2}] \times [-N_1 n\pi, -N_2 n\pi],$$

$$B = [2nN_2^{1/2}, 2nN_1^{1/2}] \times [-nN_1 \pi, nN_1 \pi] \cup [-2nN_1^{1/2}, -2nN_2^{1/2}] \times [-nN_1 \pi, nN_1 \pi].$$

To get an estimate for the integral on $A$ we use symmetry to focus on $h_2 > 0$ and separate the integral into three parts, $h_2 \in [2\pi k_0 + \frac{\pi}{2}, 2\pi (k_\infty + 1) + \frac{\pi}{2}]$ (with $k_0 = k_0(N_2,n)$ the smallest integer with $2\pi k_0 + \frac{\pi}{2} \geq N_2 n\pi$ and $k_\infty = k_\infty(N_1,n)$ the biggest one such that $(k_\infty + 1)2\pi + \frac{\pi}{2} \leq N_1 n\pi$), $h_2 \in [N_2 n\pi, 2\pi k_0 + \frac{\pi}{2}]$ and $h_2 \in [(k_\infty + 1)2\pi + \frac{\pi}{2}, N_1 n\pi]$, and we estimate each part separately:

1) If $h_2 \in [2\pi k_0 + \frac{\pi}{2}, (k_\infty + 1)2\pi + \frac{\pi}{2}]$

$$\left| \int_{-2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2\pi k_0 + \frac{\pi}{2}}^{2\pi (k_\infty + 1) + \frac{\pi}{2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right| \leq \int_{-2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2\pi k_0 + \frac{\pi}{2}}^{2\pi (k_\infty + 1) + \frac{\pi}{2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \leq C \int_{-2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2\pi k_0 + \frac{\pi}{2}}^{2\pi (k_\infty + 1) + \frac{\pi}{2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \leq C \frac{1}{N_2^{3/2} n^2}.$$  

2) If $h_2 \in [N_2 n\pi, 2\pi k_0 + \frac{\pi}{2}]$

$$\left| \int_{-2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{N_2 n\pi}^{2\pi k_0 + \frac{\pi}{2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right| \leq C \frac{1}{N_2^{3/2} n^2}.$$

3) If $h_2 \in [(k_\infty + 1)2\pi + \frac{\pi}{2}, N_1 n\pi]$

$$\left| \int_{-2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{N_1 n\pi}^{(k_\infty + 1)2\pi + \frac{\pi}{2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right| \leq C \frac{1}{N_2^{3/2} n^2}.$$

For the integration in $B$ we use a similar trick, using parity to consider only $h_2 \geq 0$ and separating in the parts $h_2 \leq \frac{\pi k}{2}, \frac{\pi k}{2} \leq h_2 \leq 2\pi (k_\infty + 1) + \frac{\pi}{2}$ and $2\pi (k_\infty + 1) + \frac{\pi}{2} \leq h_2 \leq N_1 n\pi$, with $k_\infty = k_\infty(N_1,n)$ the biggest integer such that $(k_\infty + 1)2\pi + \frac{\pi}{2} \leq N_1 n\pi$: 1) If $\frac{\pi k}{2} \leq h_2 \leq \frac{\pi}{2}$,
\[2\pi(k_\infty + 1) + \frac{\pi}{2}\]

\[\left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2(\pi k_\infty + 1) + \frac{\pi}{2}}^{2\pi(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right|\]

\[\leq \left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \sum_{k=1}^{k_\infty} \frac{1}{(h_1^2 + (k(2\pi))2)^2} dh_1 \right| \leq C \left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \sum_{k=1}^{k_\infty} \frac{1}{(h_1 + k2\pi)^4} dh_1 \right|\]

\[\leq C \left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \frac{1}{(h_1 + x2\pi)^4} dx dh_1 \right| \leq C \left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \frac{1}{h_1^4} dh_1 \right|\]

\[\leq \frac{C}{N_2^{\pi}}\]

2) If \( h_2 \leq \frac{\pi}{2} \)

\[\left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_0^{2nN_2^{1/2}} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right| \leq \frac{C}{N_2^{\pi}}\]

3) If \( 2\pi(k_\infty + 1) + \frac{\pi}{2} \leq h_2 \leq N_1\pi \)

\[\left| \int_{2nN_2^{1/2}}^{2nN_1^{1/2}} \int_{2(\pi k_\infty + 1) + \frac{\pi}{2}}^{N_1\pi} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1 \right| \leq \frac{C}{N_2^{\pi}}\]

Putting together the estimates in the regions A and B we have that \( \lim_{N \to \infty} I_{N,n} = C_0(n) \), and that \( |C_0 - I_{N,n}| \leq C N^{-1/2} \). The only thing left to do is to prove that \( C_0 \) is indeed different from 0 and independent of \( n \).

To prove that \( C_0(n) \) is actually independent of \( n \), it is enough to prove that, for two arbitrary integers \( n_1, n_2 \),

\[\lim_{N \to \infty} I_{N,n_1} - I_{N,n_2} = 0.\]

The proof is equivalent to that of (1), so we will omit it.

To prove that \( C_0 \neq 0 \), we start by focusing on the integral with respect to \( h_2 \) for any fixed \( h_1 \) on an interval of the form \([-K\pi, K\pi]\) with \( K \in \mathbb{N} \)

\[\int_{-K\pi}^{K\pi} \sin(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{1/2}} dh_2\]

\[= \int_{-K\pi}^{K\pi} \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_2 - \left[ \sin(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} \right]_{h_2=-K\pi}^{h_2=-K\pi}\]

\[= \int_{-K\pi}^{K\pi} \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_2 = 2 \int_{[0,K\pi]} \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_2;\]

and we can use this property to compute the integral in \([-2nN_1^{1/2}, 2nN_1^{1/2}] \times [-K\pi, K\pi]\) as
\[
\int_{-2n^{1/2}}^{2n^{1/2}} \int_0^K \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_2 dh_1 = \int_{-2n^{1/2}}^{2n^{1/2}} \int_0^K \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_1 dh_2
\]

\[
= \int_0^K \cos(h_2) \int_{-2n^{1/2}}^{2n^{1/2}} \frac{1}{(x^2 + 1)^{1/2}} dx dh_2 = 2 \int_0^K \cos(h_2) \log\left(\frac{2n^{1/2}}{h_2} + (1 + \frac{4n^2}{h_2^2})^{1/2}\right) dh_2
\]

\[
+ 2 \int_0^K \cos(h_2) \log\left(\frac{4n^{1/2}}{h_2}\right) dh_2.
\]

And we can evaluate the last line by checking the two integrals separately

\[
\int_0^K \cos(h_2) \log\left(\frac{2n^{1/2}}{h_2} + (1 + \frac{4n^2}{h_2^2})^{1/2}\right) dh_2 = - \int_0^K \cos(h_2) \log(h_2) dh_2
\]

\[
= - \left[ \log(x) \sin(x) - Si(x) \right]_0^K = Si(K) > 0,
\]

where \(Si(x) \equiv \int_0^x \frac{\sin(t)}{t} dt\) is the sine integral function, and

\[
| \int_0^K \cos(h_2) \log\left(\frac{2n^{1/2}}{h_2} + (1 + \frac{4n^2}{h_2^2})^{1/2}\right) dh_2 | 
\]

\[
= C \frac{K^3}{N}. 
\]

Furthermore, we can bound the integral outside of the interval \(h_2 \in [-K \pi, K \pi]\). The particular way we divide the integral depends on the parity of \(K\) and \(Nn\). Here we will obtain the bounds in the case \(K\) even and \(Nn\) odd, the other cases being analogous:

\[
\int_{-2n^{1/2}}^{2n^{1/2}} \int_0^{n\pi} \cos(h_2) \frac{h_2}{(h_1^2 + h_2^2)^{3/2}} dh_2 dh_1
\]

\[
\leq \int_{-2n^{1/2}}^{2n^{1/2}} \sum_{k=0}^{n-1} \frac{1}{(h_1^2 + (2\pi k)^2)^{3/2}} dh_1 + \int_{-2n^{1/2}}^{2n^{1/2}} \int_{(2n-1)\pi}^{n\pi} \frac{1}{h_1^2 + h_2^2} dh_2 dh_1
\]

\[
\leq C \int_0^{2n^{1/2}} \sum_{k=0}^{n-1} \frac{1}{(h_1 + 2\pi k)^2} dh_1 + \frac{C}{N^2}
\]

\[
\leq C \int_0^{2n^{1/2}} \frac{1}{(h_1 + 2\pi (\frac{K}{2} - 1))^2} dh_1 + \frac{C}{N^2} \leq \left( \frac{C}{K - 2} \right)^2 + \frac{C}{N^2}.
\]

Combining all these together we get that, for any \(K \leq nN\)

\[
\int_{-2n^{1/2}}^{2n^{1/2}} \int_0^{n\pi} \cos(h_2) \frac{1}{(h_1^2 + h_2^2)^{1/2}} dh_2 dh_1
\]

\[
\geq Si(K\pi) - \left( \frac{C}{K - 2} \right)^2 + \frac{C}{N^2}.
\]
and by taking \( K \) big enough so that \( \frac{\sin(K \pi)}{2} - \left( \frac{C}{K - \frac{1}{2}} \right)^2 > 0 \) and then \( N \) big enough so that 
\[
\frac{\sin(K \pi)}{2} - \frac{C}{K^3} - \frac{C}{N^2} > 0
\]
we are done. \( \square \)

We can now combine both lemmas to obtain

**Lemma 2.3.** Given natural numbers \( n, N \) and a \( C^2 \) function \( g_N(\cdot) : \mathbb{R} \to \mathbb{R} \) with support in the interval \( (1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2}) \) and \( ||g_N||_{C^2} \leq MN^{1/2} \) for \( i = 0, 1, 2 \), we have that there exists a constant \( C_0 \neq 0 \) such that, for \( r \in (1 - N^{-1/2}, 1 + N^{-1/2}) \),
\[
|v_r(g_N(r)\sin(Nn\alpha)) - C_0 \cos(Nn\alpha)g_N(r)| \leq CMN^{-1/2}
\]
with \( C \) depending on \( n \) but not on \( N \) or \( g \).

Analogously, we have that
\[
|v_r(g_N(r)\cos(Nn\alpha)) + C_0 \sin(Nn\alpha)g_N(r)| \leq CMN^{-1/2}
\]
with \( C \) depending only on \( n \).

**Proof.** We already know by lemma 2.1 that
\[
|v_r(g_N(r)\sin(Nn\alpha)) - \cos(Nn\alpha)\int_{\mathbb{R} \times [-\pi, \pi]} \frac{r^2 \alpha' g_N(r + h)\sin(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} dh \, d\alpha'| \leq C||g_N(r)||_{L^\infty} N^{-1/2}
\]
and, by a change of variables we have that
\[
\int_{\mathbb{R} \times [-\pi, \pi]} \frac{r^2 \alpha' g_N(r + h)\sin(Nn\alpha')}{|h^2 + r^2(\alpha')^2|^{3/2}} dh \, d\alpha' = \int_{\mathbb{R} \times [-\pi, \pi]} \frac{\alpha' g_N(r + hr)\sin(Nn\alpha')}{|h^2 + \alpha^2|^{3/2}} dh \, d\alpha'.
\]

But, for any fixed \( r \in [1/2, 3/2] \), we have \( ||g_N(r + rh)||_{C^1} \leq 2^i ||g_N(r + h)||_{C^i} \) and thus applying lemma 2.2 we get
\[
|C_0 g_N(r) - \int_{\mathbb{R} \times [-\pi, \pi]} \frac{\alpha' g_N(r + hr)\sin(Nn\alpha')}{|h^2 + \alpha^2|^{3/2}} dh \, d\alpha'| \leq 2CMN^{-1/2},
\]
and combining (11) and (12) we get the desired result.

We omit the proof of (10) since it is completely analogous to the previous result. \( \square \)

All these results will allow us to compute locally the radial velocity with a small error, but we would like to also have decay as we go far away from \( r = 1 \). For that we have the following lemma.

**Lemma 2.4.** Given a \( L^\infty \) function \( g_N(\cdot) : \mathbb{R} \to \mathbb{R} \) with support in the interval \( (1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2}) \), and let \( \theta \) be defined as
\[
\theta(r, \alpha) := \sin(Nn\alpha)g_N(r)
\]
with \( N, n \) natural numbers.

Then there is a constant \( C \) such that, if \( N \) is big enough and \( 1/2 > |r - 1| \geq N^{-1/2} \) or \( r \geq 3/2 \), we have
\[
|v_r(\theta)(r, \alpha)| \leq \frac{C ||g_N||_{L^\infty}}{N^{n/2}|r - 1|^2}.
\]
Proof. To estimate $|v_r(\theta)(r, \alpha)|$ we will use expression \(2\) and therefore we need to find upper bounds for

$$
|R \times [\pi, \pi], (r + h)^2 \frac{\sin(\alpha') g_N(r + h) \sin(Nna')}{h^2 + 2(r + h)r(1 - \cos(\alpha'))} \, d\alpha' \, dh.
$$

Let us fix $h$ such that $r + h \in (1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2})$ and with $r \geq 1/2$. Using that \(\int_{\frac{\pi}{N}}^{(l+1)\frac{2\pi}{N}} \sin(Nna') \, da = 0\) and a degree one Taylor expansion around $\alpha' = k\frac{2\pi}{N} + \frac{r}{N}$ for $|h^2 + 2(r + h)r(1 - \cos(\alpha'))|^{3/2}$ we can bound the integral over a single period

$$
\left| \int_{\frac{k\pi}{N}}^{\frac{(k+1)\pi}{N}} \frac{\sin(\alpha') \sin(Nna')}{h^2 + 2(r + h)r(1 - \cos(\alpha'))} \, d\alpha' \right|
\leq \int_{\frac{k\pi}{N}}^{\frac{(k+1)\pi}{N}} C \frac{1}{(Nn)^2 |h + ck\frac{2\pi}{N}|^3} \, dx
\leq \frac{C}{(Nn)^2 |h + c\frac{2\pi}{N}|^3},
$$

with $c$ small and $C$ big, where we used that $r + h, r \geq 1/2$ and that there exists $c > 0$ such that $\frac{1}{h^2}(1 - \cos(\alpha')) \geq (\alpha')^2$ if $\alpha' \in [\pi, \pi]$. Adding over all the relevant periods we obtain

$$
\sum_{k=0}^{nN} \frac{C}{(Nn)^2 |h + ck\frac{2\pi}{N}|^3} \leq \int_{-1}^{Nn} \frac{C}{(Nn)^2 |h + cx\frac{2\pi}{N}|^3} \, dx
\leq \frac{C}{Nn |h - \frac{2\pi c}{N}|^3} \leq \frac{C}{Nn h^2}.
$$

Furthermore, since the support of $g_N (r)$ lies in $(1 - \frac{N^{-1/2}}{2}, 1 + \frac{N^{-1/2}}{2})$ and $|r - 1| > N^{-1/2}$ we have that $|h| \geq \frac{|r - 1|}{2}$, so, by integrating in $h$ we get

$$
\int R \frac{Nn}{h^2} \frac{|g_N (r+h)|}{h} \, dh
\leq \int_{r+h-1 \in (-\frac{N^{-1/2}}{2}, \frac{N^{-1/2}}{2})} \frac{C}{Nn |r - 1|^2 \|g\|_{L^\infty}} \, dh \leq \frac{C}{N^{3/2} n |r - 1|^2 \|g\|_{L^\infty}}.
$$

\[\square\]

2.2 The pseudo-solution method for ill-posedness in $C^k$

We say that a function $\tilde{\theta}$ is a pseudo-solution to the SQG equation if it fulfills that

$$
\frac{\partial \tilde{\theta}}{\partial t} + v_1(\tilde{\theta}) \frac{\partial \tilde{\theta}}{\partial x_1} + v_2(\tilde{\theta}) \frac{\partial \tilde{\theta}}{\partial x_2} + F(x, t) = 0
$$

$$
v_1(\tilde{\theta}) = \frac{\partial}{\partial x_2} (-\Delta)^{1/2} \tilde{\theta} = -R_2 \tilde{\theta}
$$

$$
v_2(\tilde{\theta}) = \frac{\partial}{\partial x_1} (-\Delta)^{1/2} \tilde{\theta} = R_1 \tilde{\theta}
$$

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\[ \bar{\theta}(x, 0) = \theta_0(x), \]

for some \( F(x, t) \). Obviously, this definition is not restrictive at all, since you can get essentially anything by choosing the right \( F(x, t) \). We will, however, try and use the term pseudo-solution only for functions where \( F(x, t) \) is small in a suitable norm.

With this in mind we are ready to discuss the initial conditions we will be considering. Namely, in polar coordinates we will work with initial conditions of the form

\[
\lambda(f_1(r) + f_2(N^{1/2}(r - 1) + 1) \sum_{k=1}^{K} \frac{\sin(N\alpha)}{N^2 k^3})
\]

with \( N \) and \( K \) natural numbers, \( \lambda > 0 \) and where \( f_1 \) and \( f_2 \) satisfy the following conditions:

- Both \( f_1(r) \) and \( f_2(r) \) are \( C^\infty \) functions.
- \( f_2(r) \) has its support contained in the interval \((1/2, 3/2)\) and \( f_1 \) has its support in \((1/2, 3/2) \cup (M_1, M_2)\) with some \( M_1, M_2 \) big.
- \( \frac{\partial f_1(r)}{\partial r} = 1 \) in \((3/4, 5/4)\).
- \( f_2(r) = 1 \) in \((3/4, 5/4)\).
- \( \frac{\partial^k v_\alpha(f_1)}{\partial r^k}(r = 1) = 0 \) when \( r = 1, k = 1, 2 \), where \( v_\alpha(f_1) \) is the velocity produced by \( f_1 \) in the angular direction.

We will use these pseudo-solutions to prove ill-posedness in \( C^2 \), and at the end of this section we will explain how to extend the proof to \( C^k, k > 2 \).

It is not obvious that the properties we require for \( f_1 \) can be obtained, so we need the following lemma.

**Lemma 2.5.** There exists a \( C^\infty \) compactly supported function \( g(.) : [0, \infty) \to \mathbb{R} \) with support in \((2, \infty)\) such that \( \frac{\partial^k g((f_1)^{1/3}(r))(r = 1)}{\partial r^k} = a_i \) with \( i = 1, 2 \) and \( a_i \) arbitrary.

**Proof.** We start by considering a \( C^\infty \) function \( h(x) : \mathbb{R} \to \mathbb{R} \) which is positive, with support in \((-1/2, 1/2)\) and \( \int h dx = 1 \). We define the family of functions

\[
f_{n_1, n_2}(r) := n_1 h(n_1(r - n_2)),
\]

with \( n_2 \geq n_1 \geq 2, n_1, n_2 \in \mathbb{N} \). These functions are \( C^\infty \) for any \( n_1, n_2 \), and have their support in the interval \( (n_2 - \frac{1}{2n_2}, n_2 + \frac{1}{2n_2}) \). Now let’s consider the associated family of vectors

\[
V = \cup V_{n_1, n_2},
\]

with

\[
V_{n_1, n_2} := \left( \frac{\partial v_\alpha(f_{n_1, n_2})}{\partial r}(r = 1), \frac{\partial^2 v_\alpha(f_{n_1, n_2})}{\partial r^2}(r = 1) \right).
\]

Note that to prove our lemma it is sufficient to show that this family is in fact a base of the space \( \mathbb{R}^2 \). Before we can actually prove that this is the case, we need to find expressions for \( V_{n_1, n_2} \). For our purposes it is enough to compute \( \lambda_{n_1, n_2} V_{n_1, n_2} \) since this vectors will span the same space as long as \( \lambda_{n_1, n_2} \neq 0 \).
Furthermore, we have that

\[ v_\alpha(\ldots) = P.V. \int_{\mathbb{R}^2} \frac{\hat{x} \cdot (x-y) \hat{y}}{|x-y|^4} \, dx \, dy = P.V. \int_{\mathbb{R}^2} \frac{\hat{x} \cdot (x-y) \hat{y}}{|x-y|^4} \, dx \, dy \]

\[ = P.V. \int_{\mathbb{R}_+ \times [-\pi, \pi]} r' \left( f(r') - f(r) \right) \frac{r - r' \cos(\alpha) - \alpha}{|r^2 + (r')^2 - 2rr' \cos(\alpha)|^{3/2}} \, da' \, dr' \]

and, integrating with respect to \( r \), we get that

\[ f(r, n_1, n_2) \]

hence, we will be considering functions with support in \((2, \infty)\), after relabeling \( \alpha - \alpha' \) as \( \alpha' \) we end up with the expression

\[ P.V. \int_{2}^{\infty} \int_{-\pi}^{\pi} r' \frac{r - r' \cos(\alpha')}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^{3/2}} (f(r') - f(r)) \, dr' \, da'. \]

Furthermore, if we write

\[ F(r, r', \alpha) := r' \frac{r - r' \cos(\alpha')}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^{3/2}} \]

for \( r = 1 \) we can use differentiation under the integral sign and obtain

\[ \frac{\partial^j v_{\alpha}(f(\cdot))}{\partial r^j}(r = 1) = \int_{(2, \infty) \times [-\pi, \pi]} \frac{\partial^j F}{\partial r^j}(r, r', \alpha')(r = 1) f(r') \, dr' \, da'. \]

But for \( f = f_{n_1, n_2} \) we have that

\[ \left| \int_{(2, \infty) \times [-\pi, \pi]} \frac{\partial^j F}{\partial r^j}(r, r', \alpha') f_{n_1, n_2}(r') \, da' \, dr' - \int_{[-\pi, \pi]} \frac{\partial^j F}{\partial r^j}(r, n_2, \alpha') \, da' \right| \leq \frac{C}{n_1}, \]

with \( C \) depending on \( r \) and, in particular, since \( \text{span}(V) \) is a closed set, by taking \( \lim_{n_1 \to \infty} V_{n_1, n_2} \) we get that

\[ (\int_{[-\pi, \pi]} \frac{\partial F}{\partial r}(1, n_2, \alpha') \, da', \int_{[-\pi, \pi]} \frac{\partial^2 F}{\partial r^2}(1, n_2, \alpha') \, da') \in \text{span}(V) \]

Furthermore, we have that

\[ \frac{\partial F}{\partial r}(r, r', \alpha')(r = 1) = r' \left( \frac{1}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^{3/2}} - \frac{3(r - r' \cos(\alpha'))^2}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^{5/2}} \right)(r = 1) \]

and, integrating with respect to \( \alpha' \) we get

\[ \int_{[-\pi, \pi]} \frac{\partial F}{\partial r}(r, n_2, \alpha')(r = 1) \, d\alpha' = -\frac{\pi}{(r')^2} \left( 1 + O\left( \frac{1}{r'} \right) \right). \]

With the second derivative we obtain

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\[
\frac{\partial^2 F}{\partial r^2}(r', r, \alpha')(r = 1) = -r' \left( \frac{9(r - r' \cos(\alpha'))}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^5/2} - \frac{15(r - r' \cos(\alpha'))^3}{|r^2 + (r')^2 - 2rr' \cos(\alpha')|^{17/2}} \right) (r = 1).
\]

Now before we get into more details regarding this value, note that
\[
r'(1 + \left( \frac{9(r - r' \cos(\alpha'))}{|1 + (r')^2 - 2rr' \cos(\alpha')|^5/2} - \frac{15(1 - r' \cos(\alpha'))^3}{|1 + (r')^2 - 2rr' \cos(\alpha')|^{17/2}} \right) (r = 1) \leq \frac{1}{(r')^3}.
\]
Therefore, we have that
\[
\left( \frac{1}{(n_2)^2} + O(1/n_2^3), O(1/n_2^3) \right) \in \text{span}(V),
\]
and again since \( \text{span}(V) \) is a closed set, the vector \((1, 0)\) belongs to \(\text{span}(V)\). Now we only need to prove that there exists a point \( r' \) such that
\[
\int_{[-\pi, \pi]} -r' \left( \frac{9(1 - r' \cos(\alpha'))}{|1 + (r')^2 - 2rr' \cos(\alpha')|^5/2} - \frac{15(1 - r' \cos(\alpha'))^3}{|1 + (r')^2 - 2rr' \cos(\alpha')|^{17/2}} \right) d\alpha' \neq 0,
\]
so that we can find a vector \( V_{n_1, n_2} \) of the form \((a, b)\) with \(b \neq 0\). But, for example, using that, for \( \delta > 0 \) and \( r' \) big
\[
\frac{1}{(1 + (r')^2 - 2rr' \cos(\alpha'))^\delta} + \frac{1}{(1 + (r')^2 + 2rr' \cos(\alpha'))^\delta} - \frac{2}{(1 + (r')^2)^\delta} \leq \frac{C}{(r')^{2(\delta + 1)}}
\]
once can check that
\[
\int_{[-\pi, \pi]} -r' \left( \frac{9(1 - r' \cos(\alpha'))}{|1 + (r')^2 - 2rr' \cos(\alpha')|^5/2} - \frac{15(1 - r' \cos(\alpha'))^3}{|1 + (r')^2 - 2rr' \cos(\alpha')|^{17/2}} \right) d\alpha' = \frac{C}{(r')^{2}} + O(1/(r')^3)
\]
with \( C \neq 0 \), and taking \( r' \) big enough we are done. \(\blacksquare\)

Therefore, to obtain \( f_1 \) with the desired properties, we first consider a radial \( C^\infty \) function with support in \((\frac{1}{2}, \frac{3}{2})\) and derivative 1 in \((\frac{3}{4}, \frac{5}{4})\) and then add a \( C^\infty \) function with support in \([2, M]\) that cancels the derivatives of the velocity at \( r = 1 \), and such a function exists thanks to lemma \ref{lemma:existence}.

Once we choose specific \( f_1 \) and \( f_2 \), this family of initial conditions has some useful properties that we will use later. First, for any fixed \( K \) and \( \lambda \) our initial conditions are bounded in \( H^{2+1/4} \) independently of the choice of \( N \). Furthermore, the \( C^2 \) norm is bounded for any fixed \( \lambda \) independently of both \( N \) and \( K \), and can be taken as small as we want by taking \( \lambda \) small.

For any such initial conditions, we consider the associated pseudo-solution
\[
\tilde{\theta}_{\lambda, K, N}(r, \alpha, t) := \lambda(f_1(r) + f_2(N^{-1/2}(r - 1) + 1) \sum_{k=1}^{K} \frac{\sin(Nk\alpha \cdot \lambda Nk\alpha \cdot f_1) - \lambda C_0 t)}{N^{2k^3}}, \quad (14)
\]
with \( C_0 \) the constant from lemma \ref{lemma:estimate} and \ref{lemma:derivative}. We don’t add subindexes for \( f_1 \) and \( f_2 \) since we consider them fixed from now on. Furthermore, the constants appearing in most results will also depend on \( f_1 \) and \( f_2 \), but since we consider them fixed we will not mention this.

This function for \( N \geq 4 \) satisfies
\[
\frac{\partial \tilde{\theta}_{\lambda, K, N}(r, \alpha, t)}{\partial t} + \frac{\partial \tilde{\theta}_{\lambda, K, N}}{\partial \alpha} \frac{v_\alpha}{r} + \frac{\partial \lambda f_1}{\partial r} v_r(\tilde{\theta}_{\lambda, K, N}) = 0 \quad (15)
\]
with
\[ \bar{v}_r(f(r)\sin(k\alpha + g(r))) = C_0 f(r)\sin(k\alpha + g(r) + \frac{\pi}{2}) \]
if \( k \neq 0 \), and \( \bar{v}_r(f(r)) = 0 \). Note that, for arbitrary fixed \( T \), these functions satisfy that \( \|\bar{\theta}_{\lambda,K,N}\|_{H^{2+1/4}} \leq C\lambda K \), with \( C \) depending only on \( T \).

Furthermore, we can rewrite (13) as
\[
\frac{\partial \bar{\theta}_{\lambda,K,N}(r, \alpha, t)}{\partial t} + \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial \alpha} v_\alpha(\bar{\theta}_{\lambda,K,N}) + \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial r} v_r(\bar{\theta}_{\lambda,K,N}) + \frac{\partial \lambda f_1}{\partial r} (\bar{v}_r(\bar{\theta}_{\lambda,K,N}) - v_r(\bar{\theta}_{\lambda,K,N})) = 0
\]

Therefore \( \bar{\theta} \) is a pseudo-solution with source term
\[ F_{\lambda,K,N}(x, t) = \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial \alpha} (\lambda f_1 - \bar{\theta}_{\lambda,K,N}) + \frac{\partial (\lambda f_1 - \bar{\theta}_{\lambda,K,N})}{\partial r} v_r(\bar{\theta}_{\lambda,K,N}) + \frac{\partial \lambda f_1}{\partial r} (\bar{v}_r(\bar{\theta}_{\lambda,K,N}) - v_r(\bar{\theta}_{\lambda,K,N})). \]

Next we would like to prove that this source term is, indeed, small enough to obtain the desired results. We start by proving bounds on \( L^2 \) and \( H^3 \) for \( F_{\lambda,K,N}(x, t) \).

**Lemma 2.6.** For \( t \in [0, T] \) and a pseudo-solution \( \bar{\theta}_{\lambda,K,N} \) as in (14) the source term \( F_{\lambda,K,N}(x, t) \) satisfies
\[ ||F_{\lambda,K,N}(x, t)||_{L^2} \leq CN^{-(2+3/4)} \]
with \( C \) depending on \( K, \lambda \) and \( T \).

**Proof.** We start bounding the term \( \frac{\partial \lambda f_1}{\partial r} (\bar{v}_r(\bar{\theta}_{\lambda,K,N}) - v_r(\bar{\theta}_{\lambda,K,N})) \). First we decompose each function
\[
\frac{\sin(Nk\alpha - \lambda N\bar{v}_\alpha(f_1)/r - \lambda C_0 t)}{N^2 k^3} = \frac{\sin(Nk\alpha) \cos(\lambda N\bar{v}_\alpha(f_1)/r + \lambda C_0 t) - \cos(Nk\alpha) \sin(\lambda N\bar{v}_\alpha(f_1)/r + \lambda C_0 t)}{N^2 k^3}
\]
and using that \( \frac{\partial \bar{v}_\alpha(f_1)}{\partial r}(r = 1) = 0 \) for \( k = 1, 2 \), then for \( r \in (1 - 2N^{-1/2}, 1 + 2N^{-1/2}) \) we have that
\[ \left| \frac{\partial \bar{v}_\alpha(f_1)}{\partial r} \right| \leq \frac{C}{N} \]
and thus
\[ ||\frac{\partial \cos(\lambda Nk\bar{v}_\alpha(f_1)/r + \lambda C_0 t)}{\partial r}||_{L^\infty} \leq C \]
\[ ||\frac{\partial \sin(\lambda Nk\bar{v}_\alpha(f_1)/r + \lambda C_0 t)}{\partial r}||_{L^\infty} \leq C. \]

Therefore, we can directly apply lemma 2.3 to obtain
\[|v_r(f_2(N^{1/2}(r - 1) + 1) + \frac{\sin(N\kappa_0)\cos(\lambda v_0(f_1) + \lambda C_0 t)}{N^2k^3}) - \bar{v}_r(f_2(N^{1/2}(r - 1) + 1) + \frac{\sin(N\kappa_0)\cos(\lambda v_0(f_1) + \lambda C_0 t)}{N^2k^3})| \leq \frac{C}{N^{5/2}k^3},\]

\[|v_r(f_2(N^{1/2}(r - 1) + 1) + \frac{\cos(N\kappa_0)\sin(\lambda v_0(f_1) + \lambda C_0 t)}{N^2k^3}) - \bar{v}_r(f_2(N^{1/2}(r - 1) + 1) + \frac{\cos(N\kappa_0)\sin(\lambda v_0(f_1) + \lambda C_0 t)}{N^2k^3})| \leq \frac{C}{N^{5/2}k^3}.

With this we can estimate
\[
\int_{1-N^{-1/2}}^{1+N^{-1/2}} \int_{-\pi}^{\pi} \frac{\partial f_1}{\partial r} (v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))^2 \, dr \, d\theta \leq \left(\frac{\|\partial f_1\|_{L^\infty}}{\partial r}\right)^2 \frac{C}{N^{5+1/2}}.
\]

For \(r \in (1/2, 1 - N^{-1/2}) \cup (1 + N^{-1/2}, \infty)\), we use that \(\bar{v}\) is zero in those points and lemma 2.4 to obtain
\[
\int_{1/2}^{1-N^{-1/2}} \int_{-\pi}^{\pi} \frac{\partial f_1}{\partial r} (v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))^2 \, dr \, d\theta \leq C(\|\partial f_1\|_{L^\infty})^2 \int_{1/2}^{1-N^{-1/2}} \int_{-\pi}^{\pi} \left(\frac{\|f_2\|_{L^\infty}}{N^{3/2}|r-1|^2}\right)^2 \, dr \, d\theta \leq \frac{C}{N^{5+1/2}}(\|\partial f_1\|_{L^\infty})^2(\|f_2\|_{L^\infty})^2.
\]

and similarly
\[
\int_{1+N^{-1/2}}^{\infty} \int_{-\pi}^{\pi} \frac{\partial f_1}{\partial r} (v_r(\bar{\theta}) - \bar{v}_r(\bar{\theta}))^2 \, dr \, d\theta \leq \frac{C}{N^{5+1/2}}(\|\partial f_1\|_{L^\infty})^2(\|f_2\|_{L^\infty})^2.
\]

Combining all of these inequalities we get
\[
\left\|\frac{\partial f_1}{\partial r} (v_r(\bar{\theta}) - \bar{v}_r(\bar{\theta}))\right\|_{L^2} \leq \frac{C}{N^{2+1/4}}
\]

with \(C\) depending on \(\lambda, K\) and \(T\).

For the term \(\frac{\partial (f_1 - \bar{\theta}_{\lambda,K,N})}{\partial r}\) we simply use \(\left\|\frac{\partial (f_1 - \bar{\theta}_{\lambda,K,N})}{\partial r}\right\|_{L^\infty} \leq \frac{C}{N}\) and
\[
\left\|\frac{\partial (f_1 - \bar{\theta}_{\lambda,K,N})}{\partial r}\right\|_{L^2} \leq \frac{C}{N^{2+1/4}}
\]

so
\[
\left\|\frac{\partial (f_1 - \bar{\theta}_{\lambda,K,N})}{\partial \alpha}\right\|_{L^2} \leq \frac{C}{N^{2+1/4}}.
\]

Similarly for \(\frac{\partial (f_1 - \bar{\theta}_{\lambda,K,N})}{\partial r} v_r(\bar{\theta}_{\lambda,K,N})\) we have that
\[
\left\|v_r(\bar{\theta}_{\lambda,K,N})\right\|_{L^2} \leq \frac{C}{N^{2+1/4}} \quad \text{and} \quad \left\|\frac{\partial (\bar{\theta}_{\lambda,K,N} - f_1)}{\partial r}\right\|_{L^\infty} \leq \frac{C}{N}.
\]
Lemma 2.7. For $t \in [0, T]$, given a pseudo-solution $\bar{\theta}_{\lambda,K,N}$ as in [13] the source term $F_{\lambda,K,N}(x,t)$ satisfies

$$\|F_{\lambda,K,N}(x,t)\|_{H^3} \leq CN^{3/4}$$

with $C$ depending on $K$, $\lambda$, and $T$.

Proof. To prove this we will use that, given the product of two functions, we have

$$\|fg\|_{H^3} \leq C(||f||_{L^\infty} ||g||_{H^3} + ||f||_{C^1} ||g||_{H^2} + ||f||_{C^2} ||g||_{H^1} + ||f||_{C^3} ||g||_{L^2})$$

Furthermore, for the pseudo-solutions considered, we have that $||\bar{\theta}_{\lambda,K,N} - \lambda f_1||_{C^k} \leq CN^{k-2-1/4}$, $||\bar{\theta}_{\lambda,K,N} - \lambda f_1||_{H^k} \leq CN^{k-2-1/4}$, $||\lambda f_1||_{C^k} \leq C$ with the constants $C$ depending on $k$, $\lambda$ and $K$.

Therefore we have that, using the bounds for the support of $\bar{\theta}_{\lambda,K,N}$

$$\|\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r} v_r(\bar{\theta}_{\lambda,K,N})\|_{H^3} \leq C(||\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{L^\infty} ||v_r(\bar{\theta}_{\lambda,K,N})||_{H^3} + ||\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{C^1} ||v_r(\bar{\theta}_{\lambda,K,N})||_{H^2}$$

analogously

$$\|\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r} v_r(\bar{\theta}_{\lambda,K,N})\|_{H^3} \leq C(||\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{L^\infty} ||v_r(\bar{\theta}_{\lambda,K,N})||_{H^3} + ||\frac{\partial (\bar{\theta}_{\lambda,K,N} - \lambda f_1)}{\partial r}||_{C^1} ||v_r(\bar{\theta}_{\lambda,K,N})||_{H^2}$$

and finally

$$\|\frac{\partial f_1}{\partial r}(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))\|_{H^3} \leq C(||\frac{\partial f_1}{\partial r}||_{L^\infty} ||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^3} + ||\frac{\partial f_1}{\partial r}||_{C^1} ||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^2}$$

$$+ ||\frac{\partial f_1}{\partial r}||_{C^2} ||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{H^1} + ||\frac{\partial f_1}{\partial r}||_{C^3} ||(v_r(\bar{\theta}_{\lambda,K,N}) - \bar{v}_r(\bar{\theta}_{\lambda,K,N}))||_{L^2} \leq CN^{3/4}$$
We can combine these two lemmas and use the interpolation inequality for Sobolev spaces to obtain that

$$||F||_{H^{2+1/4}} \leq C(N^{-(2+3/4)^{1/4}}(N^{3/4})^{3/4}) \leq CN^{-1/8}.$$  

With this, we are ready to study how the real solution behaves. If we define

$$\Theta_{\lambda,K,N} = \theta_{\lambda,K,N} - \bar{\theta}_{\lambda,K,N},$$

with $\theta_{\lambda,K,N}$ the only $H^{2+1/4}$ solution to the SQG equation with the same initial conditions as $\bar{\theta}_{\lambda,K,N}$, we have that

$$\Theta_{\lambda,K,N}(x, t) = 0,$$

and we have the following results regarding the evolution of $\Theta_{\lambda,K,N}$.

**Lemma 2.8.** Let $\Theta_{\lambda,K,N}$ defined as in (16), then if $\theta_{\lambda,K,N}$ exists for $t \in [0, T]$, we have that

$$||\Theta_{\lambda,K,N}(x, t)||_{L^2} \leq \frac{Ct}{N^{(2+3/4)}}$$

with $C$ depending on $\lambda$, $K$ and $T$.

**Proof.** We start by noting that

$$\frac{\partial}{\partial t} ||\Theta_{\lambda,K,N}||_{L^2}^2 = - \int_{\mathbb{R}^2} \Theta_{\lambda,K,N}
\left((v_1(\Theta_{\lambda,K,N}) + v_1(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1} + (v_2(\Theta_{\lambda,K,N}) + v_2(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_2}
+ v_1(\Theta_{\lambda,K,N}) \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N}) \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial x_2} - F_{\lambda,K,N}(x, t)\right) dx,$$

but, by incompressibility we have that

$$\int_{\mathbb{R}^2} \Theta_{\lambda,K,N}
\left((v_1(\Theta_{\lambda,K,N}) + v_1(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1} + (v_2(\Theta_{\lambda,K,N}) + v_2(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_2}
+ v_1(\Theta_{\lambda,K,N}) \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N}) \frac{\partial \bar{\theta}_{\lambda,K,N}}{\partial x_2} - F_{\lambda,K,N}(x, t)\right) dx = 0,$$

and therefore we get that

$$\frac{\partial}{\partial t} ||\Theta_{\lambda,K,N}||_{L^2}^2 \leq \int_{\mathbb{R}^2} \Theta_{\lambda,K,N}
\left((v_1(\Theta_{\lambda,K,N}) + v_1(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1} + (v_2(\Theta_{\lambda,K,N}) + v_2(\bar{\theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_2}
+ F_{\lambda,K,N}(x, t)\right) dx \leq \frac{C}{N^{(2+3/4)}},$$

and using that $||F_{\lambda,K,N}||_{L^2} \leq \frac{C}{N^{(2+3/4)}}$, $||\bar{\theta}_{\lambda,K,N}||_{C^1} \leq C$ and integrating we get that
\[ ||\Theta_{\lambda,K,N}||_{L^2} \leq \frac{C(e^{Ct} - 1)}{N^{(2+3/4)}}. \]

**Lemma 2.9.** Let \( \Theta_{\lambda,K,N} \) defined as in (16), then for \( N \) big enough, \( \theta_{\lambda,K,N} \) exists for \( t \in [0,T] \) and

\[ ||\Theta_{\lambda,K,N}(x,t)||_{H^{2+1/4}} \leq \frac{Ct}{N^{1/8}} \]

with \( C \) depending on \( \lambda, K \) and \( T \).

**Proof.** It is enough to prove that

\[ ||D^{2+1/4}\Theta_{\lambda,K,N}||_{L^2} \leq \frac{Ct}{N^{1/8}} \]

since \( ||f||_{H^s} \leq C(||D^s f||_{L^2} + ||f||_{L^2}) \) with \( D^s = (-\Delta)^{s/2} \) and we already have the result \( ||\Theta_{\lambda,K,N}||_{L^2} \leq \frac{Ct}{N^{(2+3/4)}}. \)

We will use the following result found in [24].

**Lemma 2.10.** Let \( s > 0 \). Then for any \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 = s \), and any \( f, g \in S(\mathbb{R}^2) \), the following holds:

\[ ||D^s(fg) - \sum_{|k| \leq s_1} \frac{1}{k!}\partial^k fD^{s-k}g - \sum_{|j| \leq s_2} \frac{1}{j!}\partial^j gD^s f||_{L^2} \leq C||D^{s_1} f||_{L^2}||D^{s_2} g||_{BMO} \quad (17) \]

where \( \mathbf{j} \) and \( \mathbf{k} \) are multi-indexes, \( \partial^\mathbf{j} = \frac{\partial}{\partial x_{1}^{j_1} \partial x_{2}^{j_2}} \), \( \partial^\mathbf{j}_\xi = \frac{\partial}{\partial \xi_{1}^{j_1} \partial \xi_{2}^{j_2}} \) and \( D^{s_\mathbf{j}} \) is defined using

\[ D^{s_\mathbf{j}}(\xi) = D^{s_1}(\xi) f(\xi) \]

\[ D^{s_\mathbf{j}}(\xi) = i^{-|\mathbf{j}|} \hat{\partial}^\mathbf{j}_\xi(\xi^{|\mathbf{s}|}). \]

Although this result is for functions in the Schwartz space \( S \), since we only consider compactly supported functions we can apply it to functions in \( H^s \). We will consider \( s = 2 + 1/4 \), although we will just write \( s \) for compactness of notation.

Then

\[ \frac{\partial}{\partial t} ||D^s\Theta_{\lambda,K,N}||_{L^2}^2 = - \int_{\mathbb{R}^2} D^s \Theta_{\lambda,K,N} \]

\[ D^s \left( (v_1(\Theta_{\lambda,K,N}) + v_1(\bar{\Theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1} + (v_2(\Theta_{\lambda,K,N}) + v_2(\bar{\Theta}_{\lambda,K,N})) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_2} \right) \]

\[ + v_1(\Theta_{\lambda,K,N}) \frac{\partial \bar{\Theta}_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N}) \frac{\partial \bar{\Theta}_{\lambda,K,N}}{\partial x_2} + F_{\lambda,K,N}(x,t) \] \( dx. \)

We will focus for now on

\[ \int_{\mathbb{R}^2} D^s \Theta_{\lambda,K,N}D^s \left( (v_1(\Theta_{\lambda,K,N}) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N}) \frac{\partial \Theta_{\lambda,K,N}}{\partial x_2} \right) dx. \]

Applying (17) with \( s_2 = 1 \), \( g = v_i(\theta_{\lambda,K,N}) \), \( f = \frac{\partial \Theta_{\lambda,K,N}}{\partial x_i} \), \( i = 1, 2 \) we would get that...
have that

\[
(D^{s}\Theta_{\lambda,K,N}, D^{s}(fg) - \sum_{|k| \leq s_1} \frac{1}{k!} \partial^{j} f D^{s} j g - \sum_{|k| \leq s_2} \frac{1}{k!} \partial^{k} g D^{s} k f)_{L^2}
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|D^{s}f\|_{L^2}\|D^{s}g\|_{BMO}
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|\bar{\theta}_{\lambda,K,N}\|_{H^s}\|\Theta_{\lambda,K,N}\|_{H^s}.
\]

Furthermore we have that

\[
(D^{s}\Theta_{\lambda,K,N}, D^{s}(\frac{\partial \Theta_{\lambda,K,N}}{\partial x_1}) v_i (\bar{\theta}_{\lambda,K,N}) + D^{s}(\frac{\partial \Theta_{\lambda,K,N}}{\partial x_2}) v_2 (\bar{\theta}_{\lambda,K,N}))_{L^2}
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_1} [D^{s}\Theta_{\lambda,K,N}]^2 v_1 (\bar{\theta}_{\lambda,K,N}) + \frac{\partial}{\partial x_2} [D^{s}\Theta_{\lambda,K,N}]^2 v_2 (\bar{\theta}_{\lambda,K,N}) dx = 0
\]

and, for \(i = 1, 2\), using that the operators \(D^{s-c}\) are continuous from \(H^a\) to \(H^{a-s+c}\), we have the following three estimates

1) \[
|\|D^{s}\Theta_{\lambda,K,N}, \sum_{|k| = 1} \frac{1}{k!} \partial^{k} v_1 (\bar{\theta}_{\lambda,K,N}) D^{s} k \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1})|_{L^2}|
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|v_1 (\bar{\theta}_{\lambda,K,N})\|_{H^{s+c}}\|\Theta_{\lambda,K,N}\|_{H^s}
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|\bar{\theta}_{\lambda,K,N}\|_{H^s}\|\Theta_{\lambda,K,N}\|_{H^s}.
\]

2) \[
|\|D^{s}\Theta_{\lambda,K,N}, \sum_{|j| = 1} \frac{1}{j!} \partial^{j} \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1}) D^{s} j v_i (\bar{\theta}_{\lambda,K,N})|_{L^2}|
\]

\[
\leq C\sum_{|j| = 1} \|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|\partial^{j} \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1})\|_{L^2/(3-s)}\|D^{s} j v_i (\bar{\theta}_{\lambda,K,N})\|_{L^2/(r-2)}
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|\Theta_{\lambda,K,N}\|_{H^s}\|\bar{\theta}_{\lambda,K,N}\|_{H^s}.
\]

3) \[
|\|D^{s}\Theta_{\lambda,K,N}, \frac{\partial \Theta_{\lambda,K,N}}{\partial x_1}) D^{s} v_i (\bar{\theta}_{\lambda,K,N})|_{L^2}|
\]

\[
\leq C\|D^{s}\Theta_{\lambda,K,N}\|_{L^2}\|\Theta_{\lambda,K,N}\|_{H^s}\|\bar{\theta}_{\lambda,K,N}\|_{H^s}.
\]

Most of the other terms are bounded in a similar way without any complication, although a comment needs to be made about bounding the terms

\[
\int_{\mathbb{R}^2} D^{s}(\Theta_{\lambda,K,N}) \left( v_1 (\Theta_{\lambda,K,N}) \frac{\partial D^{s} \bar{\theta}_{\lambda,K,N}}{\partial x_1} + v_2 (\Theta_{\lambda,K,N}) \frac{\partial D^{s} \bar{\theta}_{\lambda,K,N}}{\partial x_2} \right) dx.
\]

At first glance one could think that, since we are considering \(\bar{\theta}_{\lambda,K,N}\) bounded in \(H^{2+1/4}\) but not in higher order spaces, we could have a problem bounding this integral. However, we actually have that

\[
\|\frac{\partial D^{s} \bar{\theta}_{\lambda,K,N}}{\partial x_i}\|_{L^\infty} \leq C\|D^{s} \bar{\theta}_{\lambda,K,N}\|_{H^{2+s}} \leq C\|\bar{\theta}_{\lambda,K,N}\|_{H^{2+1/4}} \leq C N^{2^+}.
\]
and thus

\[ \left| \int_{\mathbb{R}^2} D^s(\Theta_{\lambda,K,N}) \left( v_1(\Theta_{\lambda,K,N}) \frac{\partial D^s \Phi_{\lambda,K,N}}{\partial x_1} + v_2(\Theta_{\lambda,K,N}) \frac{\partial D^s \Phi_{\lambda,K,N}}{\partial x_2} \right) dx \right| \]

\[ \leq C T \| D^s(\Theta_{\lambda,K,N}) \|_{L^2} N^{-3/4 + \epsilon} \leq C T \| D^s(\Theta_{\lambda,K,N}) \|_{L^2} N^{-1/8}, \]

and combining all of this together plus similar bounds for the other terms, and using

\[ \| \theta_{\lambda,K,N} \|_{H^s} \leq C, \quad \| F_{\lambda,K,N} \|_{H^s} \leq C N^{-1/8} \]

with \( C \) depending on \( \lambda, K \) and \( T \), we get

\[ \frac{\partial}{\partial t} \| D^s \Theta_{\lambda,K,N} \|_{L^2}^2 \leq \| D^s \Theta_{\lambda,K,N} \|_{L^2}(C N^{-1/8} + C \| \Theta_{\lambda,K,N} \|_{H^s} + C \| \theta_{\lambda,K,N} \|_{H^s}^2), \]

which gives us, using

\[ \| \Theta_{\lambda,K,N} \|_{H^s} \leq C (\| \Theta_{\lambda,K,N} \|_{L^2} + \| D^s \Theta_{\lambda,K,N} \|_{L^2}) \leq C (\| D^s \Theta_{\lambda,K,N} \|_{L^2} + N^{-2 + 3/4}) \]

that

\[ \frac{\partial}{\partial t} \| D^s \Theta_{\lambda,K,N} \|_{L^2} \leq (C N^{-1/8} + C \| D^s \Theta_{\lambda,K,N} \|_{L^2} + C \| D^s \Theta_{\lambda,K,N} \|_{L^2}^2). \]

Now, we restrict ourselves to \([0, T_\ast]\), with \( T_\ast \) the smallest time such that \( \| D^s \Theta_{\lambda,K,N} \|_{L^2} \leq 1 \) (or \( T \) if \( T_\ast \) is bigger than \( T \) or it does not exist). Integrating for those times we get

\[ \| D^s \Theta_{\lambda,K,N} \|_{L^2} \leq \frac{C(e^{C_t} - 1)}{N^{1/8}}, \]

and since for \( N \) big enough we have that \( T \leq T_\ast \) we are done.

\[ \square \]

Now we are finally prepared to prove strong ill-posedness in \( C^2 \) for the SQG equation.

**Theorem 2.11.** For any \( c_0 > 0, M > 0 \) and \( t_\ast > 0 \), we can find a \( C^2 \cap H^{2+1/4} \) function \( \theta_0(x) \) with \( \| \theta_0(x) \|_c^2 \leq c_0 \) such that the only solution \( \theta(x,t) \in H^{2+1/4} \) to the SQG problem \([1]\) with initial conditions \( \theta_0(x) \) that satisfy \( \| \theta(x,t_\ast) \|_{C^2} \geq M c_0 \).

**Proof.** We will prove this by constructing a solution with the desired properties. We fix arbitrary \( c_0 > 0, M > 0 \) and \( t_\ast \), and consider the pseudo-solutions \( \theta_{\lambda,K,N} \). First, note that, for any \( N, K \) natural numbers, for \( \lambda > 0 \) small enough our family of pseudo-solutions has a small initial norm in \( C^2 \), so we consider \( \lambda = \lambda_0 \) small so that \( \| \theta_{\lambda_0,K,N}(x,0) \|_{C^2} \leq c_0 \) for all \( K, N \) natural and such that \( |\lambda_0 C_0 t_\ast| \leq \frac{\epsilon}{2} \).

These pseudo-solutions fulfill that, at time \( t \), for \( \alpha = \lambda_0 f_2 t_\star \)

\[ \left| \frac{\partial^2 \theta_{\lambda_0,K,N}(x,t)}{\partial \alpha^2} \right| = |\lambda_0 f_2(N^{1/2}(r - 1) + 1) \sum_{k=1}^{K} \sin(Nk\alpha - \lambda_0 N \frac{v_s(f_1)}{\bar{r}} - \lambda_0 C_0 t)| \]

\[ = |\lambda_0 f_2(N^{1/2}(r - 1) + 1) \sum_{k=1}^{K} \sin(-\lambda_0 C_0 t)| \]

\[ \geq |\lambda_0 |f_2(N^{1/2}(r - 1) + 1)| \ln(K) |\sin(-\lambda_0 C_0 t)|. \]
Furthermore, we can find \( c > 0 \) small such that, for \( \alpha \in \left[ \lambda_0 t \frac{v_0(t)}{r}, c \frac{2 \pi}{N K} \right] \), we have
\[
\left| \frac{\partial^2 \tilde{\theta}_{\lambda_0, K, N}(x, t)}{\partial \alpha^2} \right| \geq \lambda_0 \frac{|f_2(N^{1/2}(r - 1) + 1)|\ln(K)|\sin(-\lambda_0 C_0 t)|}{2}.
\]

Therefore by using that \( f(r) = 1 \) if \( r \in (3/4, 5/4) \) and defining
\[
B = \bigcup_{j \in \mathbb{N}} \left[ \frac{2 \pi}{N} + \lambda_0 t \frac{v_0(t)}{r} - c \frac{2 \pi}{N K}, \frac{2 \pi}{N} + \lambda_0 t \frac{v_0(t)}{r} + c \frac{2 \pi}{N K} \right]
\]
and \( A = [1 - \frac{N-1/2}{4}, 1 + \frac{N-1/2}{4}] \), then
\[
\int_A \int_B \left( \frac{1}{\rho^2} \left| \frac{\partial^2 \tilde{\theta}_{\lambda_0, K, N}}{\partial \alpha^2} \right| \right)^2 \, d\rho dr \geq \frac{\lambda_0^2 \ln(K)^2}{4(1 + N - \frac{1}{4})^2} |A||B|\sin(-\lambda_0 C_0 t)|^2,
\]  
(18)

with \(|A|, |B|\) the length of \( A \) and \( B \) respectively. We now consider \( K \) big enough such that
\[
\lambda_0 \ln(K)|\sin(-\lambda_0 C_0 t_*)| \geq 16 M c_0,
\]
and thus, for \( N \) big
\[
\int_A \int_B \frac{1}{\rho^2} \left( \frac{\partial^2 \tilde{\theta}_{\lambda_0, K, N}}{\partial \alpha^2} \right)^2 \, d\rho dr \geq 16 M^2 c_0^2 |A||B|.
\]  
(19)

Now, we can use lemmas \( \ref{2.4} \) and \( \ref{2.5} \) plus the interpolation inequality for Sobolev spaces to obtain that, for \( N \) big enough, we have
\[
\| \Theta_{\lambda_0, K, N} \|_{H^2} \leq C N^{a-1/4}
\]
for some \( a > 0 \) which can be computed explicitly but whose particular value is not relevant for this proof. With this we have that the solution \( \theta_{\lambda_0, K, N} \) satisfies that, at \( t = t_* \)
\[
\left( \int_A \int_B \frac{1}{\rho^2} \left( \frac{\partial^2 \theta_{\lambda_0, K, N}}{\partial \alpha^2} \right)^2 \, d\rho dr \right)^{1/2} \geq 4 M c_0 |A|^{1/2} |B|^{1/2} - C t_* N^{-a-1/4}
\]
where we used that there is a constant \( C \) such that if \( S \subset \left\{ \frac{1}{2} \leq |x| \leq \frac{3}{2} \right\} \) then
\[
\| \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \alpha^2} |S| \|_{L^2} \leq C \| g1 |S| \|_{H^2} \leq C \| g |C^2 \|_{H^2}.
\]  
(20)

But \(|A||B| \geq C N^{-1/2} \), so, taking \( N \) big enough we get
\[
\left( \int_A \int_B \frac{1}{\rho^2} \left( \frac{\partial^2 \Theta_{\lambda_0, K, N}}{\partial \alpha^2} \right)^2 \, d\rho dr \right)^{1/2} \geq 3 M c_0 |A|^{1/2} |B|^{1/2}.
\]

But, if \( S \subset \left\{ \frac{1}{2} \leq |x| \leq \frac{3}{2} \right\} \) then
\[
\sup_{x \in S} \left| \frac{1}{\rho^2} \frac{\partial^2 g}{\partial \alpha^2} \right| \leq 2 \| g1 |S| \|_{C^2} \leq 2 \| g \|_{C^2},
\]  
(21)
Proof. We consider a family of pseudo-solutions to the SQG equation (14) such that \( \|\theta_0\|_{C^2} \leq c_0 \) and the only solution \( \theta(x, t) \in H^{2+\frac{1}{8}} \) with \( \theta(x, 0) = \theta_0(x) \) satisfies that there exists a \( t_* > 0 \) with \( \|\theta(x, t)\|_{C^2} = \infty \) for all \( t \) in the interval \( (0, t_*) \).

Remark 2. We can actually prove that, for the initial conditions \( \theta_0(x) \) obtained in theorem 2.12, there is no solution in \( L^\infty_t L^2_x \) such that \( \theta(x, t) \in C^2 \) for \( t \) in some small time interval (even if we allow \( \text{ess-sup} \sup_{t \in [0, t]} \|\theta(x, t)\|_{C^2} = \infty \)), since, if we call \( \theta_1(x, t) \) the solution found in theorem 2.12 and \( \theta_2(x, t) \) the new solution belonging pointwise in time to \( C^2 \) for a small time interval, we can obtain the bound

\[
\frac{\partial \|\theta_2(x, t) - \theta_1(x, t)\|_{L^2}}{\partial t} \leq C \|\theta_2(x, t) - \theta_1(x, t)\|_{L^2}
\]

which implies that \( \|\theta_2(x, t) - \theta_1(x, t)\|_{L^2} = 0 \).

Remark 3. The value of \( t_* \) can be made arbitrarily big if wanted with very small adjustments on the proof, but for simplicity we provide the proof without worrying about the specific value of \( t_* \).

Proof. We consider a family of pseudo-solutions to the SQG equation

\[
\bar{\theta}_n(x, t) = \tilde{\theta}_{\lambda_n, K_n, N_n}(x, t)
\]

for \( n \in \mathbb{N} \), with \( \tilde{\theta}_{\lambda_n, K_n, N_n} \) defined as in (13). Although \( \bar{\theta}_n \) depends on the choice of \( \lambda_n, K_n \) and \( N_n \), we do not write the dependence explicitly to get a more compact notation. We start by fixing \( \lambda_n \) satisfying

\[
\lambda_n \leq 2^{-n},
\]

and such that \( \|\bar{\theta}_n(x, 0)\|_{C^2} \leq c_0 \) independently of the choice of \( K_n \) and \( N_n \).

Note that this already tells us that for any fixed arbitrary \( T \), if \( 0 \leq t \leq T \) then

\[
\|\bar{\theta}_n(x, t)\|_{H^{2+\frac{1}{8}}} \leq C 2^{-n} (\frac{K_n}{N_n^{1/8}} + 1)
\]

with \( C \) depending on \( T \). We will only consider \( N_n^{1/8} \geq K_n \), so that \( \|\bar{\theta}_n(x, t)\|_{H^{2+\frac{1}{8}}} \leq C 2^{-n} \). We fix now \( K_n \) so that \( \lambda_n^{1/4} \ln(K_n) \geq 16n \). Note that, then, as seen in the proof of theorem 2.11 we have that there is a set \( S_n = S_{\lambda_n, K_n, N_n, t} \) (see (13)) with measure \( |S_n| \geq \frac{8n}{K_n N_n^{1/8}} > 0 \) such that the function \( \bar{\theta}_n(x, t) \) fulfills that

\[
\frac{1}{r^2} \frac{\partial^2 \bar{\theta}_n(x, t)}{\partial \alpha^2} 1_{S_n} \|_{L^2} \geq 4n \frac{|S_n|^{1/2} \sin(\lambda_n C_0 t)}{\lambda_n}.
\]

(22)

Let us consider now the initial conditions

\[
(\int_B \int_B \frac{1}{r^2} \left( \frac{\partial^2 \bar{\theta}_{\lambda_0, K, N}}{\partial \alpha^2} \right)^2 \, dx \, dr)^{1/2} \leq 2 |A|^{1/2} |B|^{1/2} \|\bar{\theta}_{\lambda_0, K, N}\|_{C^2},
\]

and thus

\[
\|\bar{\theta}_{\lambda_0, K, N}\|_{C^2} \geq \frac{3M c_0}{2}.
\]
$$\theta((\lambda_n)_{n \in \mathbb{N}}, (K_n)_{n \in \mathbb{N}}, (N_n)_{n \in \mathbb{N}}, (R_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} T_{R_n}(\hat{\theta}_n(x,0))$$

with $T_R(f(x_1, x_2)) = f(x_1 + R, x_2)$, with $R_n$ yet to be fixed. We will refer to these initial conditions simply as $\theta(x,0)$ and to the unique $H^{2+\frac{1}{2}}$ solution to the SQG equation with initial conditions $\theta(x,0)$, as $\theta(x,t)$ for a more compact notation, keeping in mind that the function depends on multiple parameters. Since $||\theta(x,0)||_{H^{2+1/8}} \leq C2^{-n}$ we have that $||\theta(x,0)||_{H^{2+1/8}} \leq C$, and thus we can use the a priori bounds to assure the existence of $\theta(x,t)$ for some time interval $[0, t_{\text{crit}}]$ and also $||\theta(x,t)||_{H^{2+1/8}} \leq C$ for some big $C$ for $t \in [0, \frac{t_{\text{crit}}}{4}]$. This also tells us that, in particular, $||v_j(\theta)||_{L^\infty} \leq v_{\text{max}}$ for some big constant $v_{\text{max}}$ for $t \in [0, \frac{t_{\text{crit}}}{4}]$ and $j = 1, 2$.

We restrict ourselves now to study the interval $t \in [0, t_{\text{crit}}]$ with

$$t_{\text{crit}} = \min\left(\frac{t_{\text{crit}}}{2}, \frac{\pi}{\sup_n(\lambda_n)|C_0|2}\right).$$

By construction, $\hat{\theta}_n(x,0)$ is contained in a ball of a certain radius $D$. Then, if we consider $R_n = R_{n-1} + 2D + 4v_{\text{max}}t_{\text{crit}} + D_n + D_{n-1}$ with $D_n, D_{n-1} > 0$, we have that

$$d(\text{supp}(1_{B_{D+2v_{\text{max}}t_{\text{crit}}}}(-R_n,0)\theta(x,t)), \text{supp}(\hat{\theta}(x,t) - 1_{B_{D+2v_{\text{max}}t_{\text{crit}}}}(-R_n,0)\theta(x,t))) > D_n$$

and

$$\hat{\theta}_n(x,t) := \theta(x,t)1_{B_{D+2v_{\text{max}}t_{\text{crit}}}}(-R_n,0)$$

is a pseudo-solution fulfilling

$$\frac{\partial \hat{\theta}_n}{\partial t} + v_1(\hat{\theta}_n)\frac{\partial \hat{\theta}_n}{\partial x_1} + v_2(\hat{\theta}_n)\frac{\partial \hat{\theta}_n}{\partial x_2} + \tilde{F}_n = 0$$

$$v_1(\hat{\theta}_n) = -\theta \Lambda^{-1}\hat{\theta}_n = -\mathcal{R}_2\theta$$

$$v_2(\hat{\theta}_n) = \frac{\partial}{\partial x_2}\Lambda^{-1}\hat{\theta}_n = \mathcal{R}_1\theta$$

$$\tilde{F}_n = v_1(\theta - 
\hat{\theta}_n)\frac{\partial \hat{\theta}_n}{\partial x_1} + v_2(\theta - \hat{\theta}_n)\frac{\partial \hat{\theta}_n}{\partial x_2}$$

$$\hat{\theta}_n(x,0) = \theta(x,0)1_{B_{D+2v_{\text{max}}t_{\text{crit}}}}(-R_n,0).$$

If we now define $\Theta_n := \hat{\theta}_n - T_{R_n}(\hat{\theta}_n)$ we get

$$\frac{\partial \Theta_n}{\partial t} + v_1(\Theta_n)\frac{\partial \Theta_n}{\partial x_1} + v_2(\Theta_n)\frac{\partial \Theta_n}{\partial x_2} + v_1(\hat{\theta}_n)\frac{\partial T_{R_n}(\hat{\theta}_n)}{\partial x_1} + v_2(\hat{\theta}_n)\frac{\partial T_{R_n}(\hat{\theta}_n)}{\partial x_2}$$

$$+ v_1(\hat{\theta}_n)\frac{\partial \Theta_n}{\partial x_1} + v_2(\hat{\theta}_n)\frac{\partial \Theta_n}{\partial x_2} - T_{R_n}F_{\lambda_n,K_n,N_n}(x,t) + \tilde{F}_n = 0,$$

with $F_{\lambda_n,K_n,N_n}$ the source term of our pseudo-solution $\hat{\theta}_n = \hat{\theta}_{\lambda_n,K_n,N_n}$ and therefore satisfying the bounds given by lemmas 2.8 and 2.9

$$||F_{\lambda_n,K_n,N_n}||_{L^2} \leq C \frac{1}{N_n^{2+3/4}}$$

and

$$||F_{\lambda_n,K_n,N_n}||_{H^{2+1/4}} \leq C \frac{1}{N_n^{1/8}}.$$
It is easy to prove that
\[ ||v_1(\theta - \tilde{\theta}_n)1_{\text{supp}(\tilde{\theta}_n)}||_{L^\infty} \leq \frac{C}{(D_n)^2} \]
and in fact
\[ ||v_1(\theta - \tilde{\theta}_n)1_{\text{supp}(\tilde{\theta}_n)}||_{C^k} \leq \frac{C}{(D_n)^2} \tag{24} \]
since
\[ d(\text{supp}(\tilde{\theta}_n), \text{supp}(\theta - \tilde{\theta}_n)) \geq D_n. \]

Taking, for example, \( D_n = N_n^{-\frac{2+3/4}{4}} \) to obtain that \( ||\tilde{F}_n||_{L^2} \leq \frac{C}{N_n^{2+\frac{3}{2}}} \) we can argue as in lemma 2.8 to get that
\[ ||\Theta_n||_{L^2} \leq \frac{Ct}{N_n^{2+3/4}} \]
for all \( t \in [0, t_{\text{crit}}] \). We can also estimate \( ||\Theta_n||_{H^{2+1/8}} \) as in lemma 2.9, being the only difference that now we have the extra term \( \tilde{F}_n \). Therefore, it is enough to obtain bounds for
\[ \int_{\mathbb{R}^2} D^s(\Theta_n)(v_1(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)\frac{\partial \tilde{\theta}_n}{\partial x_2})dx_1dx_2 \]
with \( s = 2 + 1/8 \).

Using lemma 2.10 in the same way as we did in lemma 2.9 we can decompose this integral in several terms that are easy to bound using (24) plus the term
\[ \int_{\mathbb{R}^2} D^s(\Theta_n)(v_1(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_2})dx_1dx_2 \]
which is, in principle, too irregular to be bounded. However, using incompressibility and \( \Theta_n = \tilde{\theta}_n - T_{R_n}(\tilde{\theta}_n) \) we get
\[
\begin{align*}
\int_{\mathbb{R}^2} D^s(\Theta_n)(v_1(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_1} &+ v_2(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_2})dx_1dx_2 | \\
&= \int_{\mathbb{R}^2} D^s(T_{R_n}(\tilde{\theta}_n))(v_1(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)D^s\frac{\partial \tilde{\theta}_n}{\partial x_2})dx_1dx_2 | \\
&= \int_{\mathbb{R}^2} D^s(\tilde{\theta}_n)(v_1(\theta - \tilde{\theta}_n)D^s\frac{\partial T_{R_n}(\tilde{\theta}_n)}{\partial x_1} + v_2(\theta - \tilde{\theta}_n)D^s\frac{\partial T_{R_n}(\tilde{\theta}_n)}{\partial x_2})dx_1dx_2 | \\
&\leq ||D^s\tilde{\theta}_n||_{L^2} \frac{C}{N_n^{2+\frac{3}{2}}} N_n^2 \leq \frac{C}{N_n^3}.
\end{align*}
\]

Therefore, as in lemma 2.9 we get
\[ ||\Theta_n||_{H^{2+1/8}} \leq \frac{Ct}{N_n^{1/8}}. \]

This combined with the \( L^2 \) norm and using the interpolation inequality for Sobolev spaces gives us
\[ ||\Theta_n||_{H^2} \leq \frac{Ct}{N_n^{11/34}} = \frac{Ct}{N_n^{1/4+a}}, \]
with \( a > 0 \), for all \( t \in [0, t_{\text{crit}}] \).
But, this means that, if we consider the polar coordinates around the point \((-R_n, 0)\), which we will call \((r R_n, \alpha R_n)\), and using (20)

\[
\|\frac{1}{r R_n} \frac{\partial^2 \theta(x, t)}{\partial \alpha^2} T_{R_n}(1_{S_n})\|_{L^2} \\
\geq \|T_{R_n} \left(\frac{1}{r} \frac{\partial^2 \tilde{\theta}(x, t)}{\partial \alpha^2} 1_{S_n}\right)\|_{L^2} - \|\tilde{\theta}_n - T_{R_n}(\tilde{\theta}_n(x, t))\|_{H^2} \\
\geq 4n|S_n|^{1/2} |\sin(\lambda_n C t)| - \frac{C t}{N_1^{1/4+a}}
\]

but, using \(C_0 t \lambda_n \leq \frac{\pi}{2}\), \(|S_n| \geq C K_n^{-1} N_n^{-1/2}\) and taking \(N_n\) big enough we get

\[
\|T_{R_n}(1_{S_n}) \frac{1}{r R_n} \frac{\partial^2 \theta(x, t)}{\partial \alpha^2}\|_{L^2} \geq cnt |S_n|^{1/2}
\]

for some small constant \(c\).

But then

\[
\|T_{R_n}(1_{S_n}) \frac{1}{r R_n} \frac{\partial^2 \tilde{\theta}_n(x, t)}{\partial \alpha^2}\|_{L^2} \\
\leq \|T_{R_n} \tilde{\theta}_n(x, t)\|_{C^2} |S_n|^{1/2}
\]

and thus \(\|T_{R_n}(1_{S_n}) \theta(x, t)\|_{C^2} \geq cnt\) and we are done since we can do this for every \(n\).

Both results in this section can be obtained in \(C^m\) for \(m \geq 2\), using exactly the same method. To do it we consider pseudo-solutions of the form

\[
\lambda(f_1(r) + f_2(N^{1/2}(r - 1) + 1) \sum_{k=1}^K \frac{\sin(N k \alpha)}{N^{m_{k^{m+1}}}}).
\]

The proof follows exactly the same method, only this time we have that the associated source terms \(F_{\lambda, K, N}\) of these pseudo-solutions fulfill \(\|F_{\lambda, K, N}\|_{L^2} \leq \frac{CN}{m_{k^{m+1}}}, \|F_{\lambda, K, N}\|_{H^2} \leq C N^{k-m-1/4}\), which gives us, by taking \(K\) big an using the interpolation inequality that

\[
\|F_{\lambda, K, N}\|_{H^{m+\delta}} \leq C N^{-\frac{1}{4}+\delta}
\]

for \(\delta > 0\) arbitrary.

Note also that analogous expressions as (20) and (21) exists for higher order derivatives in \(\alpha\), albeit with different constants.

### 3 Strong ill-posedness and non existence in supercritical Sobolev spaces

#### 3.1 Pseudo-solutions for \(H^s\)

The proof for ill-posedness in supercritical Sobolev spaces follows a very similar strategy as before. We find an appropriate pseudo-solution with the desired properties, we find bounds for the source term and then we obtain bounds for the difference between the real solution and the pseudo-solution. This time, we will consider pseudo-solutions of the form

\[
\tilde{\theta}(r, \alpha, t) = f_1(r) + f_2(r) \frac{\sin(N \alpha - N t \frac{u_{\alpha}(r)}{r})}{N^\beta} r_0^{\beta}
\]
with \( f_1, f_2 \) compactly supported \( C^\infty \) functions, \( r_0 > 0 \) and \( v_\alpha(f_1(r)) \) is the angular velocity generated by the function \( f_1(r) \).

The choice of \( f_1, f_2 \) and \( r_0 \) will depend on the specific behaviour we want our pseudo-solutions to have. Before we start to specify how we choose them and how we will label the pseudo-solutions, we need the following technical lemma.

**Lemma 3.1.** For any \( \beta \in (\frac{3}{2}, 2) \) and \( K, c > 0 \), there exists a \( C^\infty \) radial function \( f_1(r) : \mathbb{R}_+ \times [0, 2\pi] \to \mathbb{R} \), with support in some \([a_1, a_2] \times [0, 2\pi]\), \( 0 < a_1 < a_2 \) depending on \( K, c \) and \( \beta \) such that \( \|f_1(r)\|_{H^\beta} \leq c \), and \( \frac{\partial v_\alpha(f_1(r))}{\partial r}(r = \frac{a_2}{a_1}) \geq \frac{\beta}{a_2} \).

**Proof.** By lemma \( 2.5 \) we can find a \( C^\infty \) function \( g(r) : \mathbb{R}_+ \times [0, 2\pi] \to \mathbb{R} \) with support in \( r \in [2, M] \) such that \( \frac{\partial v_\alpha(g(r))}{\partial r}(r = 1) = 1 \). If we consider now the functions

\[
g_{\lambda_1, \lambda_2}(r) := \frac{g(\lambda_1 r)}{\lambda_2 \lambda_1^{\beta - 1}}, \quad \lambda_1, \lambda_2 > 1
\]

we have (for example using the interpolation inequalities for Sobolev spaces) that

\[
\|g_{\lambda_1, \lambda_2}(r)\|_{H^\beta} \leq C \frac{\lambda_1}{\lambda_2}
\]

with \( C \) depending on \( \|g(r)\|_{H^2} \).

Furthermore, \( v_\alpha(f_1(r)) = v_\alpha(f(r))(r) \), \( \frac{\partial v_\alpha(f_1(r))}{\partial r}(r) = \lambda \frac{\partial v_\alpha(f(r))}{\partial r}(r) \), so

\[
\left. \frac{\partial v_\alpha(g_{\lambda_1, \lambda_2}(r))}{\partial r} \right|_{r = 1} = \lambda_1^{1-\beta} \frac{\lambda_2^{1-\beta}}{\lambda_1 \lambda_2} = \lambda_1^{1-\beta} - \beta \frac{\lambda_2^{1-\beta}}{\lambda_1 \lambda_2}.
\]

Therefore it is enough to take \( g_{\lambda_1, \lambda_2} \) with \( \lambda_2 \) big enough so that \( \frac{\lambda_1^{1-\beta}}{\lambda_2} \leq c \) (\( C \) the constant in \( 25 \)) and then \( \lambda_1 \) big enough so that \( \lambda_2^{1-\beta} \geq K \) and \( g_{\lambda_1, \lambda_2} \) with \( a_1 = \frac{\lambda_1}{\lambda_2}, a_2 = \frac{\lambda_1}{\lambda_1} \) will have all the properties desired.

From now on we consider \( \beta \) a fixed value in the interval \( (\frac{3}{2}, 2) \). The family of pseudo-solutions we consider to obtain ill-posedness in \( H^\beta \) is, for \( N \in \mathbb{N} \)

\[
\bar{\theta}_{N,c,K}(r, \alpha, t) = f_{1,c,K}(r) + f_{2,c,K}(r) r^\beta \sin(N\alpha - N t \frac{v_\alpha(f_1(r))}{N^\beta})
\]

with \( f_{1,c,K} \) the function given by lemma \( 3.1 \) for the specific values of \( c \) and \( K \) considered and \( r_{c,K} = \frac{\lambda_1}{\lambda_2} \) given by the lemma. By continuity, we have that there exists an interval \([r_{c,K} - \epsilon, r_{c,K} + \epsilon]\) such that if \( \bar{r} \in [r_{c,K} - \epsilon, r_{c,K} + \epsilon] \) then

\[
\left. \frac{\partial v_\alpha(f_{1,c,K}(\bar{r}))}{\partial r} \right|_{r = \bar{r}} \geq K \frac{2 r}{r^2 - \frac{\lambda_1}{\lambda_2}}.
\]

We take \( f_{2,c,K} \) to be a \( C^\infty \) function with support in \([r_{c,K} - \epsilon, r_{c,K} + \epsilon] \cap [r_{c,K}, 2 r_{c,K}] \) and fulfilling \( \|f_{2,c,K}\|_{L^2} = c \).

These pseudo-solutions fulfil the evolution equation

\[
\frac{\partial \bar{\theta}_{N,c,K}}{\partial t} + \frac{v_\alpha(f_{1,c,K}(\bar{r}))}{r} \frac{\partial \bar{\theta}_{N,c,K}}{\partial \alpha} = 0
\]

and therefore they are pseudo-solutions with source term

\[
F_{N,c,K} = -(\frac{v_\alpha(f_{1,c,K}(\bar{r}))}{r} \frac{\partial \bar{\theta}_{N,c,K}}{\partial \alpha} - v_\alpha(f_{1,c,K}(\bar{r})) \frac{\partial \bar{\theta}_{N,c,K}}{\partial r})
\]

(28)
Next we need to obtain bounds for our source term. To do this, we start with a lemma analogous to lemma 2.4.

**Lemma 3.2.** Given a $L^\infty$ function $g(.) : \mathbb{R} \to \mathbb{R}$ with support in the interval $(a,b)$ then if we define $g_N$ as

$$g_N(r, \alpha) := \sin(N\alpha + \alpha_0)\tilde{g}_N(r)$$

with $N$ a natural number, then there is a constant $C$ depending on $(a,b)$ such that if $r > b$, then

$$|v_r(g_N)(r, \alpha)| \leq \frac{C\|g_N\|_{L^\infty}}{N|r - b|^2}.$$  

Furthermore, we have that if $\|\tilde{g}_N\|_{C^i} \leq MN^i$ for $i = 0, 1, ..., m$, then

$$\left|\frac{\partial^m v_r(g_N)}{\partial x_1^{m-1} \partial x_2}(r, \alpha)\right| \leq \frac{CMN^{m-1}}{|r - b|^2},$$

with $C$ depending on $(a,b)$ and $m$.

**Proof.** The proof for the decay of the velocity it is analogous to that of lemma 2.4. As for the higher derivatives, using that

$$v_r(w) = \cos(\alpha(x))v_1(w) + \sin(\alpha(x))v_2(w),$$

one can obtain that

$$\left|\frac{\partial^m v_r(g_N)}{\partial x_1^{m-1} \partial x_2}(r, \alpha)\right|$$

$$\leq \left|v_r\left(\frac{\partial^m g_N}{\partial x_1^{m-1} \partial x_2}(r, \alpha)\right)(r, \alpha)\right|$$

$$+ C \sum_{i=0}^{m-1} \sum_{j=0}^{i} \left|\frac{\partial^i v_1(g_N)}{\partial x_1^{i-j} \partial x_2^{j}}(r, \alpha)\right|$$

$$+ C \sum_{i=0}^{m-1} \sum_{j=0}^{i} \left|\frac{\partial^i v_1(g_N)}{\partial x_1^{i-j} \partial x_2^{j}}(r, \alpha)\right|$$

with $C$ depending on $m, a$ and $b$, and using the decay for $v_r$, and

$$|v_1(w)(x)| \leq C\frac{||w||_{L^1}}{|d(x, \text{supp}(w))|^2}$$

$$|v_2(w)(x)| \leq C\frac{||w||_{L^1}}{|d(x, \text{supp}(w))|^2}$$

we are done.

With this, we are now ready to obtain the bounds for our source term.

**Lemma 3.3.** For $t \in [0,T]$ and a pseudo-solution $\bar{\theta}_{N,c,K}$ as in (26) then the source term $F_{N,c,K}(x, t)$ as in (28) satisfies

$$\|F_{N,c,K}(x, t)\|_{L^2} \leq CN^{-(2\beta-1)}$$

with $C$ depending on $c$, $K$ and $T$. 

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Proof. In order to obtain the desired estimate we divide the source term in several parts. First we have
\[
||v_\alpha(\tilde{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot)) \partial_\alpha \tilde{\theta}_{N,c,K} ||_{L^2} \leq C ||v_\alpha(\tilde{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot)) ||_{L^2} ||\partial_\alpha \tilde{\theta}_{N,c,K} ||_{L^\infty} \leq \frac{C}{N^{2\beta-1}}
\]
and analogously
\[
||v_r(\tilde{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot)) \partial_r \tilde{\theta}_{N,c,K} ||_{L^2} \leq ||v_r(\tilde{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot)) ||_{L^2} ||\partial_r \tilde{\theta}_{N,c,K} ||_{L^\infty} \leq \frac{C}{N^{2\beta-1}}.
\]

Finally, by using that \(\text{supp}(f_{1,c,K}) \in [2r_{c,k}, a_2]\) (see lemma 3.1 and the definition of the pseudo-solution \(\tilde{\theta}_{N,c,K}\), \(\text{supp}(f_{2,c,K}) \in [\frac{1}{2}r_{c,k}, 3r_{c,k}]\) and together with lemma 3.2 we have
\[
||v_r(\tilde{\theta}_{N,c,K}(\cdot) - f_{1,c,K}(\cdot)) \partial f_{1,c,K}(r) ||_{L^2} \leq \left( \int_{2r_{c,k}}^{a_2} \frac{C}{N^{2r_{c,k}+2\beta}}(r - \frac{3r_{c,k}}{2})^\frac{1}{2} dr \right)^{1/2} \leq \frac{C}{N^{1+\beta}}.
\]

Combining all three bounds we obtain the desired result. \(\square\)

Lemma 3.4. For \(t \in [0,T]\) and a pseudo-solution \(\tilde{\theta}_{N,c,K}\) as in (26) then the source term \(F_{N,c,K}(x,t)\) as in (28) satisfies, for \(k \in \mathbb{N}\)
\[
||F_{N,c,K}(x,t)||_{H^k} \leq \frac{C}{N^{2\beta-1-k}}
\]
with \(C\) depending on \(k, c, K\) and \(T\).

Proof. We separate the source term in three different parts:
1) Using the properties of the support of \(\tilde{\theta}_{N,c,K}\)
\[
||\frac{\partial \tilde{\theta}_{N,c,K} v_\alpha(f_{1,c,K} - \tilde{\theta}_{N,c,K})}{r} ||_{H^k} \leq C \sum_{i=0}^{k} ||\frac{\partial \tilde{\theta}_{N,c,K}}{\partial \alpha}||_{C^i} ||v_\alpha(f_{1,c,K} - \tilde{\theta}_{N,c,K})||_{H^{k-i}} \leq \frac{C}{N^{2\beta-1-k}},
\]
2)
\[
||\frac{\partial(\tilde{\theta}_{N,c,K} - f_{1,c,K})}{\partial r} v_r(\tilde{\theta}_{N,c,K}) ||_{H^k} \leq C \sum_{i=0}^{k} ||\frac{\partial(\tilde{\theta}_{N,c,K} - f_{1,c,K})}{\partial r}||_{C^i} ||v_r(\tilde{\theta}_{N,c,K})||_{H^{k-i}} \leq \frac{C}{N^{2\beta-1-k}},
\]
3) To bound \( \| \frac{\partial f_{1,c,K}}{\partial r} v_r(\theta_{N,c,K}) \|_{H^k} \), we just apply lemma 3.2 as in (29) to obtain
\[
\| \frac{\partial f_{1,c,K}}{\partial r} v_r(\theta_{N,c,K}) \|_{H^k} \leq \sum_{i=0}^{k} \sum_{j=0}^{i} \| \frac{\partial^i (\frac{\partial f_{1,c,K}}{\partial r}) v_r(\theta_{N,c,K})}{\partial x_1^j \partial x_2^i} \|_{L^2} \leq \frac{C}{N^{k+1-k}} \leq \frac{C}{N^{2k-1-k}}.
\]

And applying the interpolation inequality for Sobolev spaces (with \( L^2 \) and for example \( H^3 \)) we obtain the following corollary:

**Corollary 3.4.1.** For \( t \in [0, T] \) and a pseudo-solution \( \theta_{N,c,K} \) as in (26) then the source term \( F_{N,c,K}(x,t) \) as in (28) satisfies
\[
\| F_{N,c,K}(x,t) \|_{H^{\beta+\frac{1}{2}}} \leq C N^{-\beta-\frac{5}{2}}
\]
with \( C \) depending on \( c, K, T \).

Now, as in last section, we define \( \theta_{N,c,K}(x,t) \) to be the unique \( H^{\beta+\frac{1}{2}} \) solution to (1) with initial conditions \( \theta_{N,c,K}(x,0) = \bar{\theta}_{N,c,K}(x,0) \), and we denote
\[
\Theta_{N,c,K} := \theta_{N,c,K} - \bar{\theta}_{N,c,K}.
\]

The next step now is to find bounds for \( \Theta_{N,c,K} \).

**Lemma 3.5.** Let \( \Theta_{\lambda,K,N} \) defined as in (30), then if \( \theta_{\lambda,K,N} \) exists for \( t \in [0, T] \), we have that
\[
\| \Theta_{N,c,K}(x,t) \|_{L^2} \leq \frac{Ct}{N^{(2\beta-1)}}
\]
with \( C \) depending on \( \lambda, K \) and \( T \).

**Proof.** As in the proof of lemma 2.8 we obtain the equation
\[
\frac{\partial}{\partial t} \left( \frac{\| \Theta_{N,c,K} \|_{L^2}^2}{2} \right) \leq \int_{\mathbb{R}^2} \Theta_{N,c,K} \left( v_1(\Theta_{N,c,K}) \frac{\partial \theta_{N,c,K}}{\partial x_1} + v_2(\Theta_{N,c,K}) \frac{\partial \theta_{N,c,K}}{\partial x_2} + F_{N,c,K}(x,t) \right) dx
\]
\[
\leq \| \Theta_{N,c,K} \|_{L^2} \left( \| \Theta_{N,c,K} \|_{L^2} \| \theta_{N,c,K} \|_{C^1} + \| F_{N,c,K}(x,t) \|_{L^2} \right).
\]

By using that \( \| F_{N,c,K} \|_{L^2} \leq \frac{C}{N^{(2\beta-1)}} \), \( \| \theta_{\lambda,K,N} \|_{C^1} \leq C \) and integrating it follows
\[
\| \Theta_{N,c,K} \|_{L^2} \leq \frac{C(\text{e}^{Ct} - 1)}{N^{(2\beta-1)}}.
\]

Before obtaining the bounds for the higher order norms of \( \Theta_{N,c,K} \) we need a couple of technical lemmas:

**Lemma 3.6.** Given a \( C^1 \) function \( h(x) : \mathbb{R}^2 \rightarrow \mathbb{R} \) with \( \| h \|_{L^\infty} \leq M \), \( \| h \|_{C^1} \leq MN \) and \( a \in (0,1) \), then there exists a constant \( C \) depending on \( a \) such that
\[
\| (\Delta)^{a/2}(h(x)) \|_{L^\infty} \leq CMN^a.
\]
Proof. Using the integral expression from the fractional Laplacian
\[
(-\Delta)^{\alpha/2}(h)(x) = C \int_{\mathbb{R}^d} \frac{(h(x) - h(z))}{|x - z|^{2+\alpha}} dz
\]
and dividing the integral in two parts depending on the value of $|x - z|$ we get
\[
\int_{|x - z| \geq \frac{T}{2}} \frac{(h(x) - h(z))}{|x - z|^{2+\alpha}} dz \leq CN^{\alpha}||h||_{L^\infty} = CMN^{\alpha}
\]
\[
\int_{|x - z| \leq \frac{T}{2}} \frac{(h(x) - h(z))}{|x - z|^{2+\alpha}} dz \leq CN^{\alpha-1}||h||_{C^1} = CMN^{\alpha}
\]
and we are done.

Lemma 3.7. Given a $C^1$ function $h : \mathbb{R}^2 \to \mathbb{R}$ with $||h||_{L^\infty} \leq M$, $||h||_{C^1} \leq MN$ and with support in the set $[-R, R]^2$ for some $R$, we have that there exists a constant $C$ depending on $R$ such that for $i = 1, 2$

\[
||v_i(h(x))||_{L^\infty} \leq CMI\log(N).
\]

Furthermore, if $||h||_{C^n} \leq M$, $||h||_{C^{n+1}} \leq MN$ for some natural number $n$ we also have that, for $i = 1, 2$, $k = 0, 2, \ldots, n$

\[
||\partial^k v_i(h(x))||_{L^\infty} \leq CMI\log(N).
\]

Proof. The proof of the first part is the same as in lemma 3.6 but using the kernel for $v_i$ instead of the one for $(-\Delta)^{\alpha/2}$. For the second part we just need to use that, for sufficiently regular functions we have that

\[
\frac{\partial v_i(h(x))}{\partial x_j} = v_i \left( \frac{\partial h(x)}{\partial x_j} \right).
\]

Lemma 3.8. Let $\Theta_{N,c,K}$ defined as in (37), then we have that, for $N$ large, $\theta_{N,c,K}$ exists for $t \in [0, T]$ and

\[
||\Theta_{N,c,K}(x, t)||_{H^{\beta+\frac{1}{2}}} \leq \frac{Ct}{N^{3-\frac{2}{\beta}}}
\]

with $C$ depending on $\lambda$, $K$ and $T$.

Proof. The proof is very similar to that of lemma 2.4. We will prove the inequality for the time interval $[0, T^*]$ with $T^*$ the smallest time fulfilling $||\Theta_{N,c,K}(x, t)||_{H^{\beta+\frac{1}{2}}} = \log(N)N^{-(\beta+\frac{1}{2})}$, (we can just consider $t \in [0, T]$ directly if $T^* > T$ or if it does not exists) but note that this is enough since then we can take $N$ big enough so that $T^* \geq T$. Note also that, since we have local existence, obtaining this bound also allows us to ensure that we have existence for the times considered.

First we have that, for $s = \beta + \frac{1}{2}$

\[
\frac{\partial}{\partial t} \frac{||D^s \Theta_{N,c,K}||_{L^2}^2}{2} = - \int_{\mathbb{R}^d} D^s \Theta_{N,c,K}
\]

\[
D^s \left( (v_1(\Theta_{N,c,K}) + v_1(\theta_{N,c,K})) \frac{\partial \Theta_{N,c,K}}{\partial x_1} + (v_2(\Theta_{N,c,K}) + v_2(\theta_{N,c,K})) \frac{\partial \Theta_{N,c,K}}{\partial x_2} \right)
\]

\[
+ v_1(\Theta_{N,c,K}) \frac{\partial \theta_{N,c,K}}{\partial x_1} + v_2(\Theta_{N,c,K}) \frac{\partial \theta_{N,c,K}}{\partial x_2} + F_{N,c,K}(x, t) dx.
\]

We start bounding

\[
\int_{\mathbb{R}^d} D^s \Theta_{N,c,K} D^s \left( (v_1(\theta_{N,c,K}) \frac{\partial \Theta_{N,c,K}}{\partial x_1} + v_2(\theta_{N,c,K}) \frac{\partial \Theta_{N,c,K}}{\partial x_2} \right) dx.
\]
Applying lemma [2,10] with $s_1 = s - 1$, $s_2 = 1$, $f = v_i(\bar{\theta}_{N,c,K})$, $g = \frac{\partial \Theta_{N,c,K}}{\partial x_i}$, $i = 1, 2$ we would get that

\[
(D^s \Theta_{N,c,K}, D^s (fg) - \sum_{|k| \leq s_1} \frac{1}{k!} \partial^k f D^s k g - \sum_{|j| \leq s_2} \frac{1}{j!} \partial^j g D^s j f)_{L^2} \leq C ||D^s \Theta_{N,c,K}||_{L^2} ||D^{s_1} f||_{BMO} ||D^{s_2} g||_{L^2} \leq C ||D^s \Theta_{N,c,K}||_{L^2} ||\Theta_{N,c,K}||_{H^s},
\]

where we used $||D^{s_1} v_i(\bar{\theta}_{N,c,K})||_{L^\infty} \leq C$. Furthermore we have that

\[
(D^s \Theta_{N,c,K}, D^s (\frac{\partial \Theta_{N,c,K}}{\partial x_1}) v_1(\bar{\theta}_{N,c,K}) + D^s (\frac{\partial \Theta_{N,c,K}}{\partial x_2}) v_2(\bar{\theta}_{N,c,K}))_{L^2} = \frac{1}{2} \int_{\mathbb{R}^2} \frac{\partial}{\partial x_1} (D^s \Theta_{N,c,K})^2 v_1(\bar{\theta}_{N,c,K}) + \frac{\partial}{\partial x_2} (D^s \Theta_{N,c,K})^2 v_2(\bar{\theta}_{N,c,K}) dx = 0
\]

and, for $i = 1, 2$, using that the operators $D^s k$ are continuous from $H^a$ to $H^{a-s+k}$,

\[
|\langle D^s \Theta_{N,c,K}, \sum_{|k| = 1} \frac{1}{k!} \partial^k v_i(\bar{\theta}_{N,c,K}) D^s k \frac{\partial \lambda_{N,c,K}}{\partial x_1} \rangle_{L^2} | \leq C ||D^s \Theta_{N,c,K}||_{L^2} ||v_i(\bar{\theta}_{N,c,K})||_{C^1} ||\Theta_{N,c,K}||_{H^s} \leq C ||D^s \Theta_{N,c,K}||_{L^2} ||\Theta_{N,c,K}||_{H^s}
\]

where we used $||v_i(\bar{\theta}_{N,c,K} - f_{1,c,K})||_{C^1} \leq C log(N) N^{\beta - 1}$ (consequence of lemma [5.7]) and $||v_i(f_{1,c,K})||_{C^1} \leq C$.

We also have

\[
|\langle D^s \Theta_{\lambda,K,N}, \sum_{|j| = 1} \frac{1}{j!} \partial^j \frac{\partial \lambda_{K,N}}{\partial x_i} D^s j v_i(\bar{\lambda}_{K,N}) \rangle_{L^2} | \leq C \sum_{|j| = 1} ||D^s \Theta_{\lambda,K,N}||_{L^2} ||\frac{1}{j!} \partial^j \frac{\partial \lambda_{K,N}}{\partial x_i}||_{L^2} ||D^s j v_i(\bar{\lambda}_{K,N})||_{L^\infty} \leq C \sum_{|j| = 1} ||D^s \Theta_{\lambda,K,N}||_{L^2} ||\Theta_{\lambda,K,N}||_{H^s} ||D^{s-2} \partial^j v_i(\bar{\lambda}_{K,N})||_{L^\infty} \leq C ||D^s \Theta_{\lambda,K,N}||_{L^2} ||\Theta_{\lambda,K,N}||_{H^s} (\frac{NN^{s-2} log(N)}{N^\beta} + C) \leq C ||D^s \Theta_{\lambda,K,N}||_{L^2} ||\Theta_{\lambda,K,N}||_{H^s},
\]

where we used lemmas [5.6] and [5.7] the expression for $D^s j$ and the bounds for the derivatives of $\bar{\theta}_{N,c,K}$.

The last part to bound from the term with $v_i(\bar{\theta}_{N,c,K})$ is, for $i = 1, 2$

\[
|\langle D^s \Theta_{N,c,K}, \frac{\partial \Theta_{N,c,K}}{\partial x_i} D^s v_i(\bar{\theta}_{N,c,K}) \rangle_{L^2} | \leq C ||D^s \Theta_{N,c,K}||_{L^2} ||\Theta_{N,c,K}||_{H^s} ||D^{s-2} v_i(\Delta \bar{\theta}_{N,c,K})||_{L^\infty} \leq C ||D^s \Theta_{N,c,K}||_{L^2} N^{-\beta} \sqrt{\frac{CN^{s-2} log(N) N^2}{N^\beta}} \leq C ||D^s \Theta_{N,c,K}||_{L^2} N^{-\beta} N^{\frac{1}{2}} \log(N) \leq C ||D^s \Theta_{N,c,K}||_{L^2} N^{-\frac{1}{2}},
\]

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where we used that, for the times considered, using lemma 3.5 and the interpolation inequality we have \( \|\Theta_{N,c,K}\|_{H^s} \leq C N^{-\beta/2} \) (the bound is actually better, but this is enough).

The rest of the terms not depending on \( F_{N,c,K} \) are bounded in a similar fashion, and using \( \|\overline{\theta}_{\lambda,K,N}\|_{H^s} \leq C, \|F_{N,c,K}\|_{H^s} \leq C N^{-\beta - \frac{1}{2}} \) with \( C \) depending on \( c, K \) and \( T \), we get

\[
\frac{\partial}{\partial t} \|D^s \Theta_{N,c,K}\|_{L^2}^2 \leq \|D^s \Theta_{N,c,K}\|_{L^2}(CN^{-(\beta + \frac{1}{2})} + C\|\Theta_{N,c,K}\|_{H^s} + C\|\Theta_{N,c,K}\|_{H^s}^2)
\]

which gives us, using

\[
\|\Theta_{N,c,K}\|_{H^s} \leq C(\|\Theta_{N,c,K}\|_{L^2} + \|D^s \Theta_{N,c,K}\|_{L^2}) \leq C(\|D^s \Theta_{N,c,K}\|_{L^2} + N^{-(2\beta - 1)})
\]

that

\[
\frac{\partial}{\partial t} \|D^s \Theta_{N,c,K}\|_{L^2} \leq (CN^{-(\beta + \frac{1}{2})} + C\|D^s \Theta_{N,c,K}\|_{L^2} + C\|D^s \Theta_{\lambda,K,N}\|_{L^2}^2),
\]

and using \( \|D^s \Theta_{\lambda,K,N}\|_{L^2} \leq \log(N)N^{-(\beta + \frac{1}{2})} \) and integrating we get

\[
\|D^s \Theta_{N,c,K}\|_{L^2} \leq \frac{C(e^{Ct} - 1)}{N^{\beta - \frac{1}{2}}}
\]

\[
\square
\]

### 3.2 Strong ill-posedness in supercritical Sobolev spaces

Now we are ready to prove strong ill-posedness in supercritical Sobolev spaces:

**Theorem 3.9. (Strong ill-posedness in \( H^\beta \))** For any \( c_0 > 0 \), \( M > 1 \), \( \beta \in (\frac{3}{2}, 2) \) and \( t_* > 0 \), we can find a \( H^{\beta + \frac{1}{2}} \) function \( \theta_0(x) \) with \( \|\theta_0(x)\|_{H^\beta} \leq c_0 \) such that the unique solution \( \theta(x,t) \) in \( H^{\beta + \frac{1}{2}} \) to the SQG equation (1) with initial conditions \( \theta_0(x) \) is such that \( \|\theta(x,t_*)\|_{H^\beta} \geq M c_0 \).

**Proof.** First we prove a bound for the pseudo-solution \( \overline{\theta}_{N,c,K} \) defined in (23). More precisely

\[
\|f_{2,c,K}(r) \frac{r^\beta \sin(N\alpha)}{N^\beta} \|_{L^2} \leq \frac{C r^\beta_{c,k}}{N^\beta},
\]

and

\[
\|f_{2,c,K}(r) \frac{r^\beta \sin(N\alpha)}{N^\beta} \|_{H^s} \leq \frac{C r^\beta_{c,k} - 2}{N^\beta - 2},
\]

which in combination with the interpolation inequality for Sobolev spaces and the bounds for \( f_{1,c,K} \) gives us

\[
\|\overline{\theta}_{N,c,K}(x,0)\|_{H^\beta} \leq C_1 e
\]

with \( C_1 \) depending only on \( \beta \).

Furthermore, at time \( t \) we have that our pseudo-solution fulfills

\[
\|\overline{\theta}_{N,c,K}(x,t) - f_{1,c,K}\|_{L^2} \leq \frac{C r^\beta_{c,k}}{N^\beta}
\]

and we can find the lower bound for the \( H^1 \) norm of \( \overline{\theta}_{N,c,K} - f_{1,c,K} \) by using

\[
\frac{\partial (\overline{\theta}_{N,c,K} - f_{1,c,K})}{\partial x_1} = \cos(\alpha) \frac{\partial (\overline{\theta}_{N,c,K} - f_{1,c,K})}{\partial r} - \frac{\sin(\alpha)}{r} \frac{\partial (\overline{\theta}_{N,c,K} - f_{1,c,K})}{\partial x}
\]

and integrating we get

\[
\frac{\partial (\overline{\theta}_{N,c,K} - f_{1,c,K})}{\partial t} \leq \frac{C e^{Ct} - 1}{N^{\beta - \frac{1}{2}}}
\]

\[
\square
\]
which gives us, after some trigonometric manipulations and using (27) that, for $N$ large

$$\|\bar{\theta}_{N,c,K}(x,t) - f_{1,c,K}\|_{H^s} \geq C \frac{c t^{K^2/2}}{N^{3/2}}$$

with $C$ a constant.

Furthermore, since $\text{supp}(\bar{\theta}_{N,c,K} - f_{1,c,K}) \cap \text{supp}(f_{1,c,K}) = \emptyset$ we have that

$$\|\bar{\theta}_{N,c,K}(x,t)\|_{H^s} \geq \|\bar{\theta}_{N,c,K}(x,t) - f_{1,c,K}\|_{H^s} \geq C \frac{c t^{K^2/2}}{N^{3/2}}$$

for sufficiently large $N$. On the other hand the interpolation inequality gives us

$$\|\bar{\theta}_{N,c,K}(x,t)\|_{H^s} \leq \|\bar{\theta}_{N,c,K}(x,t)\|_{L^2}^1 \|\bar{\theta}_{N,c,K}(x,t)\|_{L^2}^{\delta-1} \|\bar{\theta}_{N,c,K}(x,t)\|_{L^2}^2$$

and using our bounds for $\|\bar{\theta}_{N,c,K}(x,t)\|_{L^2}$ and $\|\bar{\theta}_{N,c,K}(x,t)\|_{H^s}$ we get

$$\|\bar{\theta}_{N,c,K}(x,t)\|_{H^s} \geq C_2 c K^\delta t^\beta$$

with $C_2$ depending only on $\beta$. Therefore, by choosing $c, K$ appropriately we have that, for all $N$ big enough,

$$\|\bar{\theta}_{N,c,K}(x,0)\|_{H^s} \leq c_0$$

$$\|\bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^s} \geq 2M c_0.$$  

Now, considering the solution $\bar{\theta}_{N,c,K}$ of (11) with initial conditions $\bar{\theta}_{N,c,K}(x,0)$, we know that

$$\|\bar{\theta}_{N,c,K}(x,0)\|_{H^s} \leq c_0,$$

and, using lemma 3.8 if $N$ large,

$$\|\bar{\theta}_{N,c,K}(x,t^\nu) - \bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^s} \leq \|\bar{\theta}_{N,c,K}(x,t^\nu) - \bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^{s+\frac{1}{2}}} \leq \frac{c t^{\nu}}{N^{3/2}}$$

and by taking $N$ big enough we can conclude

$$\|\bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^s} \geq \|\bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^s} - \|\bar{\theta}_{N,c,K}(x,t^\nu) - \bar{\theta}_{N,c,K}(x,t^\nu)\|_{H^s} \geq M c_0.$$

\hfill \Box

### 3.3 Non existence in supercritical Sobolev spaces

In this section we prove the following theorem:

**Theorem 3.10.** (Non existence in $H^3$ in the supercritical case) For any $t_0$, $c_0 > 0$ and $\beta \in (\frac{5}{2}, 2)$ we can find initial conditions $\theta_0(x)$, with $\|\theta_0(x)\|_{H^s} \leq c_0$ such that there exists a solution $\theta(x,t)$ to (11) with $\theta(x,0) = \theta_0(x)$ satisfying $\|\theta(x,t)\|_{H^s} = \infty$ for all $t \in (0, t_0]$. Furthermore, it is the only solution with initial conditions $\theta_0(x)$ that satisfy $\theta(x,t) \in L^\infty_\infty C^1_{t} \cap L^\infty_\infty L^2_{x} (0 < \gamma_1 < \frac{5}{2})$ with the property that $\|\theta(x,t)\|_{H^{s+\frac{1}{2}}} \leq M(t) \left( 1 \leq \frac{\gamma_1}{2} \leq \frac{\beta}{2} \right)$ for some function $M(t)$.

**Remark 4.** $M(t)$ is not necessarily a bounded function, so this rules out the existence of solutions in $\theta(x,t) \in L^\infty_\infty C^1_{t} \cap L^\infty_\infty L^2_{x}$ such that $\theta(x,t) \in H^3$ for $t \in (0, t^\nu]$ with any $0 < t^\nu < t_0$.

**Proof.** Let’s first note some of the properties that the pseudo-solutions $\bar{\theta}_{N,c,K}$ (for some fixed $\beta$) have:
Lemma 3.11. Let $\tilde{\theta}_{N,c,K}(x,t)$ be $C^\infty$ for all $t \in [0,t_0]$, with $||\tilde{\theta}_{N,c,K}(x,t)||_{C^s} \leq CcN^{k-\beta}$, $||\tilde{\theta}_{N,c,K}(x,t)||_{H^s} \leq CcN^{k-\beta}$ for any natural $k \geq 2$, with the constant $C$ depending on $k$, $K$ and $t_0$. Also, for $eta > s \geq 0$ we have $||\tilde{\theta}_{N,c,K}(x,t)||_{H^s} \leq C_1cN^{3-\beta} + C_2c$ with $C_1$ depending on $K$, $s$ and $t_0$ and $C_2$ a constant.

For $N$ large we have the lower bound $||\tilde{\theta}_{N,c,K}(x,t)||_{H^\beta} \geq Cct^\beta K^\beta$ with $C$ a constant.

$\tilde{\theta}_{N,c,K}(x,t)$ is supported in the ball of radius $M$ centered at zero $B_M(0)$ for some $M$ independent of the values of the parameters.

Furthermore, we have the following result.

**Lemma 3.11.** Let $\tilde{\theta}_{N,c,K}$ with $\tilde{\theta}_{N,c,K}(x,0) = \tilde{\theta}_{N,c,K}(x,0)$ and satisfying the equation

$$
\frac{\partial \tilde{\theta}_{N,c,K}}{\partial t} + (v_1(\tilde{\theta}_{N,c,K}) + v_{1,ext}^{N,c,K}) \frac{\partial \tilde{\theta}_{N,c,K}}{\partial x_1} + (v_2(\tilde{\theta}_{N,c,K}) + v_{2,ext}^{N,c,K}) \frac{\partial \tilde{\theta}_{N,c,K}}{\partial x_2} = 0
$$

with

$$
\frac{\partial v_{2,ext}^{N,c,K}}{\partial x_2} = -\frac{\partial v_{1,ext}^{N,c,K}}{\partial x_1}
$$

and

$$
||v_{i,ext}^{N,c,K}||_{C^3} \leq CN^{-3}
$$

with $C$ depending on $c$ and $K$.

Then for any $T > 0$ we have that if $N$ is big enough, then for $t \in [0,T]$ there exists a unique $\tilde{\theta}_{N,c,K}(x,t) \in H^{2+\frac{s}{2}}$ and

$$
||\tilde{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)||_{L^2} \leq CtN^{-(2\beta-1)}
$$

$$
||\tilde{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)||_{H^{\beta+\frac{s}{2}}} \leq CtN^{-(\beta+\frac{s}{2})}
$$

with $C$ depending on $c$, $K$ and $T$.

The local well posedness is straightforward since $v_{i,ext}^{N,c,K}$ for $i = 1, 2$ are $C^3$. As for the error bounds, they are obtained in the same way as in lemmas 3.5 and 3.8 i.e. studying the evolution equation for $\tilde{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)$ now with new terms depending on $v_{i,ext}^{N,c,K}$ and $\frac{\partial \tilde{\theta}_{N,c,K}(x,t)}{\partial x_i}$. These terms, however, are easily bounded by writing

$$
\tilde{\theta}_{N,c,K}(x,t) = (\tilde{\theta}_{N,c,K}(x,t) - \tilde{\theta}_{N,c,K}(x,t)) + \tilde{\theta}_{N,c,K}(x,t)
$$

and using our bounds for $v_{i,ext}^{N,c,K}$ and $\tilde{\theta}_{N,c,K}(x,t)$.

This new lemma tells us that our pseudo-solutions as in (26) stay close to other pseudo-solutions that have the same initial conditions and an error term in the velocity if that term is small enough. Now, to obtain the initial conditions that will produce instantaneous loss of regularity, we consider

$$
\theta(x,0) = \sum_{j=1}^{\infty} T_{R_j}(\tilde{\theta}_{N_j,c_j,K_j}(x,0)),
$$

with $T_{R}(f(x_1,x_2)) = f(x_1 + R, x_2)$, and $R_j$ yet to be fixed.

We will refer to the solution of (11) with this initial conditions and $H^{\frac{s}{2}}$ regularity (if it exists) as $\theta(x,t)$, keeping in mind that it depends on the values for $R_j$, $N_j$, $c_j$, $K_j$, with $j \in \mathbb{N}$.

We start by fixing $c_j$ and $K_j$ with the following properties:

1) $||\tilde{\theta}_{N_j,c_j,K_j}(x,0)||_{H^{s}} \leq c_0 2^{-j}$, $||\tilde{\theta}_{N_j,c_j,K_j}(x,0)||_{L^1} \leq c_0 2^{-j}$. \hspace{1cm} (31)

2) If $N_j$ large enough then $||\tilde{\theta}_{N_j,c_j,K_j}(x,t)||_{H^{s}} \geq t c_0 2^j$ \hspace{1cm} (32)

and $||\tilde{\theta}_{N_j,c_j,K_j}(x,t)||_{H^{\frac{s}{2}}} \leq c_0 2^{-j}$
for $t \in [0, t_0]$.

This gives us a bound for the velocity generated by $\sum_{j=1}^{\infty} T_{R_j}(\bar{\theta}_{N_j,c_j,K_j}(x,t))$, which we will call $v_{\text{max}}$.

As for $R_j$, we will consider $R_j = R_{j-1} + D_j + D_{j-1}$, $R_0 = 0$, and we will take $D_j = j^4 N_j^4 + 2M + 8e_{\text{max}} t_0$.

Now, we say that a sequence $(w_j(x,t))_{j \in \mathbb{N}}$ is in the space $W_{(N_j)_{j \in \mathbb{N}},c_0}$ if it satisfies the following four conditions:

1) $w_j(x,t) \in H^{3+\frac{1}{2}}$ for $t \in [0, t_0]$.

2) $w_j(x,t)$ satisfy

$$\frac{\partial w_j(x,t)}{\partial t} = (v_1(w_j) + v_{1,\text{ext}}^j(x,t)) \frac{\partial w_j}{\partial x_1} - (v_2(w_j) + v_{2,\text{ext}}^j(x,t)) \frac{\partial w_j(x,t)}{\partial x_2}$$

with $v_{1,\text{ext}}^j$ fulfilling

$$||v_{1,\text{ext}}^j||_{C^\alpha} \leq \frac{C_0}{j^4 N_j^4}$$

and

$$\frac{\partial v_{1,\text{ext}}^j}{\partial x_1} = - \frac{\partial v_{2,\text{ext}}^j}{\partial x_2}$$

3) $\sum_{j=1}^{\infty} ||v_i(w_j(x,t))||_{L^\infty} \leq 2v_{\text{max}}$ for $i = 1, 2$.

4) $\sum_{j=1}^{\infty} ||w_j(x,t)||_{L^2} < \infty$ for $t \in [0, t_0]$.

The space $W_{(N_j)_{j \in \mathbb{N}},c_0}$ is a complete metric space with the norm

$$||(w_j)_{j \in \mathbb{N}}||_W := \sup_{j \in \mathbb{N}} \sup_{i=1,2} \sup_{t \in [0, t_0]} j^2 N_j^3 ||v_{i,\text{ext}}^j(x,t)||_{C^\alpha}.$$
Note that \(|v_i^0_{j,ext}(w_j)\) \(\in C^\alpha\) \(\leq \frac{C}{N_0^0}\), and thus \(|v_i^0_{j,ext}(w_j)\) \(\in C^\alpha\) \(\leq \frac{C}{N_0^0}\) if \(N_0\) is large. Furthermore, if \(x \in B_{4v_{\text{max}}+M}(-R_{j_0}, 0)\), then
\[
v_i^0_{j,ext}(w_j) = v_i(\sum_{j=1}^\infty w_j) - w_{j_0}
\]  
and, since, \(\text{supp}(\theta_{j_0}(x, t)) \subset B_{4v_{\text{max}}+M}(-R_{j_0}, 0)\), we could actually use \(\text{(35)}\) as our definition of \(v_i^0_{j,ext}(w_j)\) without changing anything.

This allows us to define the operator \(G\) over a sequence \(w\) in the space \(W_{(N_j) \in \mathbb{N}, C_0}\) as
\[
G(w) = (G^i(w))_{j \in \mathbb{N},}
\]
with \(G^i(w)(x, t)\) the only \(H^{3+\frac{1}{2}}\) function for \(t \in [0, t_0]\) satisfying
\[
\frac{\partial G^i(w)}{\partial t} + (v_1(G^i(w)) + v_1^i_{j,ext}(w)) \frac{\partial G^i(w)}{\partial x_1} + (v_2(G^i(w)) + v_2^j_{j,ext}(w)) \frac{\partial G^i(w)}{\partial x_2} = 0,
\]
\[G^i(w)(x, 0) = T_{R_j}(\bar{\theta}_{N_j, c_j, K_j}(x, 0)).\]

The operator \(G\) maps (for \((N_j)_{j \in \mathbb{N}}\) large) \(W_{(N_j)_{j \in \mathbb{N}}, C_0}\) to \(W_{(N_j)_{j \in \mathbb{N}, C_0}}\) and actually, if we can find a point \(w \in W_{(N_j)_{j \in \mathbb{N}, C_0}}\) such that \(G(w) = w\), then
\[
\theta(x, t) = \sum_{j=1}^\infty w_j(x, t)
\]
is a solution to \(\text{(1)}\) with initial conditions
\[
\theta(x, 0) = \sum_{j=1}^\infty T_{R_j}(\bar{\theta}_{N_j, c_j, K_j}(x, 0)).
\]

If we now consider two sequences \(w^1 = (w^1_{j})_{j \in \mathbb{N}}, w^2 = (w^2_{j})_{j \in \mathbb{N}}\) \(\in W_{(N_j)_{j \in \mathbb{N}}, C_0}\) and we define \(||w^1 - w^2||_{L^2} = \sup_{t \in [0, t_0]} \sum_{j=1}^\infty ||w^1_j - w^2_j||_{L^2}||\), we can compute \(||G(w^1) - G(w^2)||_{L^2}||\). By defining \(\bar{w}_j = G^i(w^1) - G^i(w^2)\), since it fulfills the evolution equation
\[
\frac{\partial \bar{w}_j}{\partial t} = -\frac{\partial (G^i_{j,ext}(w^1))}{\partial x_1} \; v_1(\bar{w}_j) - \frac{\partial \bar{w}_j}{\partial x_1} \; v_1(G^i(w^2)) - \frac{\partial (G^i_{j,ext}(w^1))}{\partial x_2} \; v_2(\bar{w}_j) - \frac{\partial \bar{w}_j}{\partial x_2} \; v_2(G^i(w^2)) - \frac{\partial (G^i_{j,ext}(w^1 - w^2))}{\partial x_1} \; \bar{v}^i_{1,ext}(w^2) - \frac{\partial \bar{w}_j}{\partial x_1} \; \bar{v}^i_{1,ext}(w^2) - \frac{\partial (G^i_{j,ext}(w^1 - w^2))}{\partial x_2} \; \bar{v}^i_{2,ext}(w^2) - \frac{\partial \bar{w}_j}{\partial x_2} \; \bar{v}^i_{2,ext}(w^2).
\]

This gives us a bound for the evolution of the \(L^2\) norm of \(\bar{w}_j\)
\[
\frac{\partial ||\bar{w}_j||_{L^2}}{\partial t} \leq C ||G^i(w^1)||_{C^\alpha} ||\bar{w}_j||_{L^2} + 2 ||G^i(w^1)||_{C^\alpha} ||w^1 - w^2||_{W} + \frac{C}{N_0^3}.\]
But for $N_j$ large we can bound $||G'(w^1)||_{C^1}$ by some constant $\bar{C}_j$ using (54), and thus we obtain, for $t \in [0, t_0]$

$$||\tilde{w}_j(x,t)||_{L^2} \leq \bar{C}_j(\epsilon C_{t_0} - 1) \frac{||w^1 - w^2||_{W}}{j^4 N_j^3}$$

and for $N_j$ large

$$||\tilde{w}_j(x,t)||_{L^2} \leq \epsilon \frac{||w^1 - w^2||_{W}}{j^4}$$

with $\epsilon$ as small as we want. Adding over all $j$ we obtain, for $t \in [0, t_0]$

$$||G(w^1)(x,t) - G(w^2)(x,t)||_{L^2} \leq C\epsilon||w^1 - w^2||_{W}.$$ 

But we have that

$$||v_{i,ext}^j(G(w^1)) - v_{i,ext}^j(G(w^2))||_{C^0} j^4 N_j^3 \leq \frac{C}{N_j} \sum_{j=1}^{\infty} ||G(w^1) - G(w^2)||_{L^2}$$

and thus for $(N_j)_{j \in \mathbb{N}}$ big enough

$$||G(w^1) - G(w^2)||_{W} \leq \epsilon||w^1 - w^2||_{W}$$

with $\epsilon$ arbitrarily small and in particular the map $G$ is a contraction. Furthermore the set $W((N_j)_{j \in \mathbb{N}}, C_0)$ is not empty, since it includes at least the point where $v_{i,ext} = 0$ when $(N_j)_{j \in \mathbb{N}}$ is large, and therefore, using the Banach point fixed theorem there exists $G(w) = w \in W((N_j)_{j \in \mathbb{N}}, C_0)$. But as we pointed out earlier that implies that $w$ is a solution to (11) with initial conditions

$$\theta(x,0) = \sum_{j=1}^{\infty} T_{R_j}(\theta_{N_j, c_j, K_j}(x,0)).$$

Properties (61), (62) and (64) finish the proof that a solution with the desired properties of theorem 3.10 exists.

For uniqueness in the space mentioned we call $\theta_1(x,t)$ the solution we constructed above and assume the existence of another solution $\theta_2(x,t) \in L^\infty_1C_2^1 \cap L^\infty_1L^2_2$ (0 < $\gamma_1 < \frac{1}{2}$) with the property that $||\theta_1(x,t)||_{H^{\gamma_2}} \leq M(t)$ (1 < $\gamma_2 \leq \frac{1}{4}$) for some function $M(t)$. In particular (since it is in $L^\infty_1C_2^1$), there exists a certain constant $v_{2, max}$ such that $||v_1(\theta_2)||_{L^\infty} \leq v_{2, max}$. We start by studying the uniqueness for $t \in [0, \min(t^*, t_0)]$ with $t^* v_{2, max} = 4 t_0 v_{max}$. In particular, we have that $supp(\theta_2(x,t)) \subset \cup_{j \in \mathbb{N}} T_{R_j}(B_{t_0 v_{max} + M}(0))$. We define

$$\theta_1^1(x,t) = 1_{B_{t_0 v_{max} + M(-R_j,0)}} \theta_1(x,t)$$

$$\theta_2^1(x,t) = 1_{B_{t_0 v_{max} + M(-R_j,0)}} \theta_2(x,t).$$

If we define $\Theta^1 := \theta_2^1 - \theta_1^1, \Theta := \theta_2 - \theta_1$, we get
\[
\frac{\partial \Theta_j}{\partial t} = -\frac{\partial \theta_j}{\partial x_1}v_1(\Theta^j) - \frac{\partial \Theta_j}{\partial x_1}v_1(\Theta^j) - \frac{\partial \Theta_j}{\partial x_2}v_2(\Theta^j) \\
- \frac{\partial \theta_j}{\partial x_1}v_1(\theta_1^j) - \frac{\partial \Theta_j}{\partial x_2}v_2(\theta_1^j) - \frac{\partial \theta_j}{\partial x_1}v_1(\Theta - \Theta^j) - \frac{\partial \Theta_j}{\partial x_2}v_2(\Theta - \Theta^j) \\
- \frac{\partial \theta_j}{\partial x_2}v_2(\Theta - \Theta^j) - \frac{\partial \Theta_j}{\partial x_1}v_2(\Theta - \Theta^j) \\
= \frac{\partial \Theta_j}{\partial x_1}(\theta_1 - \theta_1^j) - \frac{\partial \Theta_j}{\partial x_2}(\theta_1 - \theta_1^j)
\]

which give us

\[
\frac{\partial ||\Theta_j||_{L^2}}{\partial t} \leq C||\theta_j||_{C^1}||\Theta_j||_{L^2} + C||\theta_j||_{C^1}||\Theta||_{L^2} \leq C 4 N_j^4
\]

and by taking \(N_j\) big we get

\[
||\Theta_j||_{L^2} \leq \epsilon ||\Theta||_{L^2}
\]

and adding over all \(j\) and taking \(\epsilon\) small

\[
||\Theta||_{L^2} \leq \frac{||\Theta||_{L^2}}{2}
\]

and thus \(||\Theta||_{L^2}\) for \(t \in [0, t^*]\). Iterating the argument allows us to prove \(||\Theta||_{L^2} = 0\) for \(t \in [0, t_0]\).

\[\square\]

### 4 Strong ill-posedness in the critical Sobolev space \(H^2\)

For this section, we will consider solutions of (1) that are in layers around zero, each one closer to the origin, so that the exterior layers effect over the inner layers will give us (in the limit) an evolution system of the form

\[
\frac{\partial \bar{\theta}}{\partial t} + (v_1(\bar{\theta}) + K(t)x_1)\frac{\partial \bar{\theta}}{\partial x_1} + (v_2(\bar{\theta}) - K(t)x_2)\frac{\partial \bar{\theta}}{\partial x_2} = 0
\]

\[
v_1 = -\frac{\partial}{\partial x_2}\Lambda^{-1}\bar{\theta} = -\mathcal{R}_2 \bar{\theta}
\]

\[
v_2 = \frac{\partial}{\partial x_1}\Lambda^{-1}\bar{\theta} = \mathcal{R}_1 \bar{\theta}
\]

\[
\bar{\theta}(x, 0) = \theta_0(x).
\]

But first we need to obtain an expression for \(\frac{\partial v_i(\theta)(0)}{\partial x_j} (i, j = 1, 2)\) for \(\theta\) with support far away from 0. We consider first \(i = 1\). We have

\[
v_1(\theta) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} \frac{-x_2 + y_2 \theta(y)}{|x - y|^3} dy_1 dy_2.
\]

For \(\theta\) with support far away from \(x = 0\) we can just differentiate under the integral sign and when we evaluate at \(x = 0\) this yields
\[
\frac{\partial v_1(\theta)}{\partial x_1}(x = 0) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} -3y_1 \frac{y_2 \theta(y)}{|y|^3} dy_1 dy_2,
\]
\[
\frac{\partial v_1(\theta)}{\partial x_2}(x = 0) = \frac{\Gamma(3/2)}{\pi^{3/2}} P.V. \int_{\mathbb{R}^2} -3y_2 \theta(y) + \frac{\theta(y)}{|y|^3} dy_1 dy_2.
\]

We will consider \(\theta(x_1, x_2)\) satisfying \(\theta(-x_1, x_2) = -\theta(x_1, x_2), \theta(x_1, -x_2) = -\theta(x_1, x_2)\), so

\[
\frac{\partial v_1(\theta)}{\partial x_1}(x = 0) = 0.
\]

If we take a look at the expression for \(\frac{\partial v_1(\theta)}{\partial x_1}\) in polar coordinates and combining all the constant into a certain \(C_0 > 0\) we obtain

\[
\frac{\partial v_1(\theta)}{\partial x_1}(x = 0) = -C_0 P.V. \int_{\mathbb{R}^+ \times [0, \pi/2]} \frac{\sin(2\alpha') \theta(r', \alpha')}{(r')^2} dr' d\alpha'.
\]

The expressions for \(v_2\) are obtained the same way and in fact we have

\[
\frac{\partial v_2(\theta)}{\partial x_1}(x = 0) = 0,
\]
\[
\frac{\partial v_2(\theta)}{\partial x_2}(x = 0) = C_0 P.V. \int_{\mathbb{R}^+ \times [0, \pi/2]} \frac{\sin(2\alpha') \theta(r', \alpha')}{(r')^2} dr' d\alpha'.
\]

Analogously, the second derivatives of \(v_i\) all vanish.

We will be interested in studying the evolution of initial conditions of the form

\[
\sum_{j=1}^J f(b^{-j}r) b^j \sin(2\alpha)
\]

for \(f(r)\) a positive \(C^\infty\) function with compact support and \(\frac{1}{2} > b > 0\). More precisely, we would like to study the behaviour of the unique \(H^4\) solution with said initial conditions when \(b\) tends to zero. One could think that we can just check the evolution of each of the terms \(f(b^{-j}r) b^j \sin(2\alpha)\) and then add them together, hoping that the interaction between them gets small as \(b \to 0\). However this is not true, and we get an interaction depending on \(\frac{\partial \theta}{\partial t}\). To get specific results, we fix some positive radial function \(f\) in \(C^\infty\) with \(supp(f) \subset \{ r \in [1/2, 3/2]\}\) and \(||f(r)\sin(2\alpha)||_{H^4} = 1\). We define \(\theta_{c,J,b}\) as the unique \(H^4\) solution of

\[
\frac{\partial \theta_{c,J,b}}{\partial t} + v_1(\theta_{c,J,b}) \frac{\partial \theta_{c,J,b}}{\partial x_1} + v_2(\theta_{c,J,b}) \frac{\partial \theta_{c,J,b}}{\partial x_2} = 0
\]

with

\[
v_1(\theta_{c,J,b}) = -\frac{\partial}{\partial x_2} \Lambda \theta_{c,J,b} = -\mathcal{R}_2 \theta_{c,J,b}
\]
\[
v_2(\theta_{c,J,b}) = \frac{\partial}{\partial x_1} \Lambda \theta_{c,J,b} = \mathcal{R}_1 \theta_{c,J,b}
\]

\[
\theta_{c,J,b}(x, 0) = \sum_{j=1}^J c \frac{f(b^{-j}r) b^j \sin(2\alpha)}{j}, \quad \frac{1}{2} > b > 0.
\]

Note that the odd symmetry is preserved in time.

A few comments need to be made regarding the properties of the transformation \(h(r, \alpha) \to \frac{h(\lambda r, \alpha)}{\lambda}\) (or equivalently \(h(x) \to \frac{h(\lambda x)}{\lambda}\)). We have that

- If \(\lambda > 1\), then \(\|h(\lambda r, \alpha)\|_{H^2} \leq \|h(r, \alpha)\|_{H^2}\).
If \( h(r, \alpha, t) \) is a solution to (1) with initial conditions \( h(r, \alpha, 0) \), then \( \frac{h(r, \alpha, t)}{\lambda} \) is a solution to (1) with initial conditions \( \frac{h(r, \alpha, 0)}{\lambda} \).

For \( i = 1, 2, j = 1, 2 \), we have:

\[
\frac{\partial v_i(h(\cdot, \cdot), r, \alpha)}{\partial x_j}(r, \alpha) = \frac{1}{2} v_i(h(\cdot, \cdot))(r, \alpha),
\]

\[
\frac{\partial v_i(h(\cdot, \cdot), r, \alpha)}{\partial x_j}(r, \alpha) = \frac{\partial v_i(h(\cdot, \cdot), r, \alpha)}{\partial x_j}(r, \alpha).
\]

The initial conditions in (36) fulfill that, taking \( c \) small and \( J \) big, they have an arbitrarily small \( H^2 \) norm and an arbitrarily big value of \( |\frac{\partial v_i(h(\cdot, \cdot), 0, t)}{\partial x_j}(r, \alpha)| \). If \( |\frac{\partial v_i(h(\cdot, \cdot), 0, t)}{\partial x_j}(r, \alpha)| \) remained big for a long enough time and \( \theta \) remained sufficiently regular during that time, we could then use a small perturbation around \( x = 0 \) to obtain a big growth in some \( H^s \) norm.

The main problem here is that we cannot assure existence for sufficiently long times using just the a priori bounds, so we need some extra machinery to be able to work with these solutions. For that we consider \( C \) the constant fulfilling that, for any \( H^4 \) solution of SQG (1) we have:

\[
\frac{\partial}{\partial t}||\theta(x, t)||_{H^4} \leq C ||\theta(x, t)||_{H^4}^2
\]

and, fixed constants \( t_0, K > 0 \), we define \( t_{\text{cr}, K, c, J, b}^t \) as the biggest time fulfilling that, for all times \( t \) satisfying \( t \geq t_0 \geq t \geq 0 \) we have:

- \( t \leq t_0 \).
- If \( x \in [\frac{1}{2} b^n, \frac{3}{2} b^n] \) for \( 1 \leq n \leq J \), then \( \phi_{c, J, b}(x, t) \in [b^{a+\frac{1}{2}}, b^{a-\frac{1}{2}}] \), with \( \phi_{c, J, b}(x, t) \) the flow given by:
  \[
  \frac{d\phi_{c, J, b}(x, t)}{dt} = v(\phi_{c, J, b}(x, t)).
  \]
- \( ||b^{-j}\theta_{c, J, b}(b^{j} x, t)||_{H^4} \leq \frac{1}{t_0 C} \) for \( 1 \leq j \leq J \).
- \( \int_0^t |\frac{\partial v_i(h(\cdot, \cdot), 0, s)}{\partial x_j}(0, s)|ds \leq K \).

Let us make a few remark on these conditions. First, due to the odd symmetry of the solution and the initial conditions, \( \frac{\partial v_i(h(\cdot, \cdot), 0, t)}{\partial x_j}(r, \alpha) \) is always negative and thus:

\[
\int_0^t |\frac{\partial v_i(h(\cdot, \cdot), 0, s)}{\partial x_j}(0, s)|ds
\]

is a monotone function with respect to \( t \). Note also that we can check that the norm:

\[
||b^j \theta_{c, J, b}(b^j x, t)||_{H^4}
\]

is continuous in time by checking the evolution equation for it and using that \( \theta_{c, J, b} \) exists locally in time. Also, depending on the choice of parameters \( t_{\text{cr}, K, c, J, b}^t \) could not exist ( the second and third condition may no bet satisfied for \( t = 0 \) ), so we will only consider \( c < \frac{1}{C t_0} \) and \( b < 2^{-8} \) to avoid that. Finally, if we only consider the typical a priori bounds, the second and third conditions could make \( t_{\text{cr}, K, c, J, b}^t \) tend to zero as we make \( b \) small, which would be a problem for our purposes. However, we have the following lemma.

**Lemma 4.1.** Fixed \( t_0, K, c \) and \( J \) fulfilling \( c < \frac{c_{\text{min}}}{C t_0} \) and \( K > \max(1, t_0) \), we have that, if \( b \) is small enough, then the unique \( H^4 \) solution \( \theta_{c, J, b} \) with initial conditions as in (36) satisfies:

\[
||b^j \theta_{c, J, b}(b^j x, t)||_{H^4} \leq \frac{1}{t_0 C}
\]

for \( 1 \leq j \leq J \), \( t \in [0, t_{\text{cr}, K, c, J, b}^t] \) and if \( x \in [b^{a+\frac{1}{2}}, b^{a-\frac{1}{2}}] \) then \( \phi_{c, J, b}(x, t) \in (b^{a+\frac{1}{2}}, b^{a-\frac{1}{2}}) \) if \( 0 \leq t \leq t_{\text{cr}, K, c, J, b}^t \).
Proof. Before we get into the proof, we need to define

\[
k_n(t) := \left| \frac{\partial v_1(\theta_{c,J,b}^1 [0,\infty))}{\partial x_1} (0, t) \right|,
\]

\[
K_n(t) := \int_0^t k_n(s)ds.
\]

We will study the evolution of \( \theta_j := \theta_{c,J,b}^1 [0,\infty) (r) \) (these functions obviously depend on \( c, J \) and \( b \), but we will omit this dependence to obtain a more compact notation). These functions satisfy the evolution equation

\[
\frac{\partial \theta_j}{\partial t} + \frac{\partial \theta_j}{\partial x_1} + v_2(\theta_j) \frac{\partial \theta_j}{\partial x_2} + v_2(\theta_{c,J,b} - \theta_j) \frac{\partial \theta_j}{\partial x_2} + v_1(\theta_{c,J,b} - \theta_j) \frac{\partial \theta_j}{\partial x_1} = 0.
\]

Furthermore, we have that \( \theta'_j (x,t) = b^{-1} \theta_j (b^j x, t) \) fulfills the evolution equation

\[
\frac{\partial \theta'_j}{\partial t} + v_1(\theta'_j) \frac{\partial \theta'_j}{\partial x_1} + v_2(\theta'_j) \frac{\partial \theta'_j}{\partial x_2} + v_2(\theta'_{c,J,b} - \theta'_j) \frac{\partial \theta'_j}{\partial x_2} + v_1(\theta'_{c,J,b} - \theta'_j) \frac{\partial \theta'_j}{\partial x_1} = 0, \tag{38}
\]

with \( \theta'_{c,J,b} (x,t) := b^{-1} \theta_{c,J,b} (b^j x, t) \).

We want to obtain suitable bounds for the terms depending on \( \theta'_{c,J,b} - \theta'_j \). To do this we decompose \( \theta'_{c,J,b} - \theta'_j \) as

\[
\theta'_{c,J,b} - \theta'_j = \theta'_{+,j} + \theta'_{-,j}
\]

with \( \theta'_{+,j} = (\theta'_{c,J,b} - \theta'_j)1_{[1,\infty)} (r) \) and \( \theta'_{-,j} = (\theta'_{c,J,b} - \theta'_j)1_{[0,1]} (r) \).

But \( \theta'_{-,j} \) satisfies that \( ||\theta'_{-,j}||_{L^1} \leq Cb^3 \), \( d(supp(\theta'_j), supp(\theta'_{-,j})) \geq \frac{b^{-3}}{2} \) which gives us, if we define \( v_i^{-,j} (x) := v_1(\theta'_{-,j}) (x) \)

\[
||v_i^{-,j} (x)1_{supp(\theta'_j)}||_{C^4} \leq Cb^{-3/2}.
\]

For the term depending on \( \theta'_{+,j} \), we use that, for \( k \geq 1 \)

\[
||\theta'_{c,J,b}^1 [0,\infty) [b^{-k+\frac{1}{2}}, b^{-k+\frac{1}{2}}] ||_{L^1} \leq Cb^{-3k}
\]

\[
d(supp(\theta'_{c,J,b}^1 [b^{-k+\frac{1}{2}}, b^{-k+\frac{1}{2}}]), supp(\theta'_j)) \geq \frac{b^{-k+\frac{1}{2}}}{2}
\]

which gives us, after adding the contributions for all the \( k \)

\[
|\frac{\partial^2 v_1(\theta'_{+,j}) (x)}{\partial^2 x_1 \partial^2 x_2} (x)| \leq Cb^{\frac{1}{2}}.
\]

Therefore, using a second order Taylor expansion for the velocity we obtain that, for \( |x| \leq b^{-\frac{1}{2}} \)

\[
v_1(\theta'_{+,j}) = k_{j-1} (t) x_1 + v_1^{+,j,\text{error}} (x),
\]

with \( ||v_1^{+,j,\text{error}} (x)1_{[b^{\frac{1}{2}}, b^{-\frac{1}{2}}]} (r) ||_{L^1} \leq Cb^{\frac{1}{2}} \). Furthermore by computing the derivatives of \( v_1(\theta'_{+,j}) \) we actually obtain \( ||v_1^{+,j,\text{error}} (x)1_{[b^{\frac{1}{2}}, b^{-\frac{1}{2}}]} (r) ||_{C^4} \leq Cb^{\frac{1}{2}} \).

Analogously, we have

\[
v_2(\theta'_{+,j}) = -k_{j-1} (t) x_2 + v_2^{+,j,\text{error}} (x),
\]

with \( ||v_2^{+,j,\text{error}} (x)1_{[b^{\frac{1}{2}}, b^{-\frac{1}{2}}]} (r) ||_{C^4} \leq Cb^{\frac{1}{2}} \). Writing \( v_i^{+,j,\text{error}} := v_1^{+,j,\text{error}} (x) + v_i^{-,j} (x) \), we get that (38) is equivalent to
\[ \frac{\partial \theta_j^i}{\partial t} + (v_1(\theta_j^i) + v_1^{\text{error}} + k_{j-1} x_1) \frac{\partial \theta_j^i}{\partial x_1} + (v_2(\theta_j^i) + v_2^{\text{error}} - k_{j-1} x_2) \frac{\partial \theta_j^i}{\partial x_2} = 0, \]

with \( \|v_1^{\text{error}}\|_{C^4} \leq C \beta \) To obtain the evolution of the \( H^4 \) norm, we note that, with our definition of \( H^4 \) norm

\[ \frac{\partial}{\partial t} \| \theta_j^i \|_{H^4} = \sum_{i=0}^4 \sum_{j=0}^4 \frac{\partial}{\partial t} \| \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \|_{L^2}^2 \]

and

\[ \frac{\partial}{\partial t} \| \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \|_{L^2}^2 \]

\[ = 2 \left( \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} (v_1(\theta_j^i) + v_1^{\text{error}} + k_{j-1} x_1) \frac{\partial \theta_j^i}{\partial x_1} + (v_2(\theta_j^i) + v_2^{\text{error}} - k_{j-1} x_2) \frac{\partial \theta_j^i}{\partial x_2} \right)_{L^2}. \]

But using \( \|v_1^{\text{error}}\|_{C^4} \leq C \beta \) and incompressibility we get, for \( i = 0, 1, ..., 4, j = 0, ..., i \)

\[ \left| \left( \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \right)^2 \right| \leq C \beta \| \theta_j^i \|_{H^4}^2, \]

and

\[ \left| \left( \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \right)^2 \right| \leq \left| \left( \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \right)^2 \right| \leq C \beta \| \theta_j^i \|_{H^4}^2, \]

which gives us, by adding all the terms and including the contribution from the terms depending on \( v_1(\theta_j^i) \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \) and \( v_2(\theta_j^i) \frac{\partial \theta_j^i}{\partial x_1 \partial x_2} \)

\[ \frac{\partial}{\partial t} \| \theta_j^i \|_{H^4} = \frac{\partial}{\partial t} \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4} \]

\[ \leq \left( 4k_{j-1} + C \beta \right) \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4} + C \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4}^2, \]

with \( C \) given by \( \text{(37)} \).

Using that, by hypothesis

\[ \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4} \leq \frac{1}{lo C}, \]

\[ \| b^i \theta_{c,j,b}(b^{-j} x, 0) 1_{[b^i, b^{-j}]}(r) \|_{H^4} \leq c \]

and integrating \( \text{(39)} \) we get

\[ \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4} \leq ce^{4K_{j-1} t} \]

and using \( K_{j-1}(t) \leq K \), and taking \( b \) small enough

\[ \| b^i \theta_{c,j,b}(b^{-j} x, t) 1_{[b^i, b^{-j}]}(r) \|_{H^4} \leq ce^{6K} < \frac{1}{Clo}, \]

which gives us the first property we wanted.
As for the bounds for $\phi_{c,j,b}(x, t)$, we again work in the equivalent problem with $\theta'_{c,j,b}$ and note that we just proved that

$$|v_{j}(\theta'_{c,j,b})(x)1_{[b^{+}, b^{-}]}| \leq (k_{j}(t) + Cb^{+})|x| + |v_{j}(\theta'_{j})(x)|,$$

and since $|v_{j}(\theta'_{j})| \leq \min(C, C|x|)$ (by using our bounds in $H^{4}$ plus $v_{j}(\theta'_{j})(x) = 0$), integrating in time we have that, for $b$ small, the particles under that flow starting in $[\frac{k_{j}}{2}, \frac{k_{j}}{2}]$ will stay in $(e^{-C}, e^{C}) \subset (b^{+}, b^{-})$, with $C$ depending on $K$ and $t_{0}$ and we are done by returning to the original problem.

Note that last lemma tells us that for $b$ small enough, at $t = t_{0}, K, c, J$, either $t = t_{0}$ or $\int_{t_{0}}^{t_{0+}} \frac{\partial_{u}(\theta_{c,j,b})}{\partial_{u}}(0, s)|ds = K$. Our next goal is to prove that, if the right conditions are met, we will actually have $\int_{t_{0}}^{t_{0+}} \frac{\partial_{u}(\theta_{c,j,b})}{\partial_{u}}(0, s)|ds = K$.

**Lemma 4.2.** For fixed $t_{0}, K$ and $c$ fulfilling $c < \frac{e^{-ak}}{Ct_{0}}$ and $K > max(1, t_{0})$, we can find $J$ and $b$ such that at time $t = t_{0+}$, $K, c, J, b$ we have that $\int_{t_{0}}^{t_{0+}} \frac{\partial_{u}(\theta_{c,j,b})}{\partial_{u}}(0, s)|ds = K$.

**Proof.** We start by studying the trajectories of particles with $|x| \in [b^{+}, b^{-}]$.

In the proof of lemma 4.1 we obtained that, for $|x| \in [b^{+}, b^{-}]$,

$$v_{1}(\theta'_{c,j,b}) = v(\theta'_{j}) + v_{1}^{error}(x) + k_{j-1}(t)x_{1}$$

$$v_{2}(\theta'_{c,j,b}) = v(\theta'_{j}) + v_{1}^{error}(x) - k_{j-1}(t)x_{2}$$

(\text{let us remember that here $\theta'_{j}$ actually depends on $c, J$ and $b$ but we omit it}), with $||v_{1}^{error}(x)||_{C^{2}} \leq C_{1}b^{+}$ for $i = 1, 2$, with $C_{1}$ depending on $c, J$ and $||v(\theta'_{j})||_{C^{1}} \leq C_{2}$ with $C_{2}$ depending on $t_{0}$.

By returning to the original problem, we get that, for $|x| \in [b^{+}, b^{-}]$

$$v_{1}(\theta_{c,j,b}) = v(\theta_{j}) + v_{1}^{error,j}(x) + k_{j-1}(t)x_{1}$$

$$v_{2}(\theta_{c,j,b}) = v(\theta_{j}) + v_{2}^{error,j}(x) - k_{j-1}(t)x_{2}$$

with $||v_{i}^{error,j}||_{C^{1}} \leq Cb^{+}$ and $||v(\theta_{j})||_{C^{1}} \leq C_{2}$ with $C_{2}$ depending on $t_{0}$.

We are interested in studying the $\phi$ associated to this problem in polar coordinates for particles starting in $(r, a) \in ([\frac{k_{j}}{2}, \frac{k_{j}}{2}], [0, 2\pi])$. We study separately the evolution of the radial coordinate and of the angular coordinate for simplicity.

For the radial coordinate, if we call $\phi_{r}(r_{0}, a_{0}, t)$ the flow associated to (41) that gives us the radial coordinate of the particle that was initially in $(r_{0}, a_{0})$, we have that

$$\frac{\phi_{r}(r_{0}, a_{0}, t)}{r_{0}} \leq e^{K_{1j}k_{j-1}(s)|ds + C_{1}b^{+}t + C_{2}t \leq e^{K_{1j}k_{j-1}(s)|ds + C_{1}b^{+}t + C_{2}t}.$$

As for the change in the angular coordinate, we are interested in finding bounds for how fast a particle can approach the lines $a = i\frac{\pi}{2}$, $i = 0, 1, 2, 3$. All four cases are equivalent, so we will consider $i = 0$. We have that

$$v_{a}(r, 0, t) = 0$$

and, since for $i = 1, 2 ||\frac{\partial_{a}}{\partial_{a}}||_{C^{1}} \leq C(|k_{j-1}| + C_{1}b + C_{2})$ (with $C$ a universal positive constant) we get, defining $\phi_{a}$ similarly as we did with $\phi_{r}(r_{0}, a_{0}, t)$, we have

$$\frac{\phi_{a}(r_{0}, a_{0}, t)}{a_{0}} \geq e^{-C_{1j}k_{j-1}(s)|ds + C_{1}b^{+}t + C_{2}t} \geq e^{-C(K + C_{1}b^{+}t + C_{2}t)}.$$
Now we are ready to obtain bounds for
\[
\int_{t_0;K,c,J,b}^{t_{\text{crit}};K,c,J,b} \frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)ds.
\]
Since the transformation
\[
\theta_{c,J,b}(x) \to \frac{\theta_{c,J,b}(\lambda x)}{\lambda}
\]
does not change the value of \(\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)\) and by linearity, we have that, for \(s = 0\) we can compute
\[
\frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(x = 0, t = 0) = \sum_{j=1}^{J} c_j \frac{\partial v_1(f(r)\sin(2\alpha))}{\partial x_1}(x = 0) = C(\sum_{j=1}^{J} c_j) \geq C\ln(J).
\]
For times \(t > 0\), writing for the flow map \(\phi_{c,J,b}(x,t) = (\phi_{1,c,J,b}(x,t), \phi_{2,c,J,b}(x,t))\)
\[
\left|\frac{\partial v_1(\theta_{c,J,b}(r,\alpha,t))}{\partial x_1}\right| = C \int_{\mathbb{R}^2_+} y_1 y_2 \frac{\theta_{c,J,b}(y,t)}{|y|^5} dy_1 dy_2
\]
\[
= C \int_{\mathbb{R}^2_+} \phi_{1,c,J,b}(\tilde{y},t) \frac{\phi_{2,c,J,b}(\tilde{y},t)\theta_{c,J,b}(\tilde{y},0)}{|\phi_{c,J,b}(\tilde{y},t)|^5} d\tilde{y}_1 d\tilde{y}_2
\]
\[
= C \int_{\mathbb{R}^2_+} \phi_{1,c,J,b}(\tilde{y},t) \frac{\phi_{2,c,J,b}(\tilde{y},t)\theta_{c,J,b}(\tilde{y},0)}{|\phi_{c,J,b}(\tilde{y},t)|^5} \frac{\phi_{c,J,b}(y_0)}{y_1 y_2} \frac{|\tilde{y}|^5}{|\tilde{y}|^5} d\tilde{y}_1 d\tilde{y}_2
\]
with \(C\) a constant, but (passing to polar coordinates to obtain the bound more easily)
\[
\phi_{c,J,b}(x,t) \frac{\phi_{2,c,J,b}(x,t)}{|\phi_{c,J,b}(x,t)|^5} \geq e^{-C(K+c_1b_1^2+ct)}
\]
for some \(C\), and thus
\[
\left|\frac{\partial v_1(\theta_{c,J,b}(r,\alpha,t))}{\partial x_1}\right| \geq Ce^{-C(K+c_1b_1^2+ct)} \int_{\mathbb{R}^2_+} \frac{\tilde{y}_1 \tilde{y}_2 \theta_{c,J,b}(\tilde{y},0)}{|\tilde{y}|^5} d\tilde{y}_1 d\tilde{y}_2
\]
and integrating in time
\[
\int_{t_0;K,c,J,b}^{t_{\text{crit}};K,c,J,b} \frac{\partial v_1(\theta_{c,J,b})}{\partial x_1}(0,s)ds \geq t_{t_0;K,c,J,b}^{t_{\text{crit}};K,c,J,b} C\ln(J)e^{-C(K+c_1b_1^2+t_0+c_2t_0)}
\]
To finish our prove, we just fix some \(K, t_0\) and \(c\) fulfilling our hypothesis, we take \(J\) big enough so that
\[ t_0 C \ln(J)e^{-C(K+C_2 t_0)} > K + 1 \]

and then take \( b \) small enough so that using lemma 4.2 either \( t_0 = t_{\text{crit},t_0,c,J,b}^{\text{crit}} \) or

\[ \int_{0}^{t_{\text{crit},t_0,c,J,b}} \left| \frac{\partial v_1(\theta_{c,J,b})}{\partial x_1} \right|(0,s) ds = K \]

and such that

\[ t_0 C \ln(J)e^{-C(K+C_1 \hat{\theta}_t + C_2 t_0)} > K. \]

The result then follows by contradiction, since if we assume \( t_0 = t_{\text{crit},t_0,c,J,b}^{\text{crit}} \) we obtain

\[ \int_{0}^{t_0} \left| \frac{\partial v_1(\theta_{c,J,b})}{\partial x_1} \right|(0,s) ds \geq t_0 C \ln(J)e^{-C(K+C_1 \hat{\theta}_t + C_2 t_0)} > K. \]

\[ \square \]

**Corollary 4.2.1.** There are initial conditions \( \theta_{\text{initial}} \in H^4 \) with \( \| \theta_{\text{initial}} \|_{H^2} \leq \bar{c} \) such that there exists \( 0 < t_{K,t_0,\bar{c}}^{\text{crit}} \leq t_0 \) and a solution \( \theta_{K,t_0,\bar{c}}(x,t) \) to (7) with \( \theta_{\text{initial}} \) as initial conditions fulfilling

\[ \int_{0}^{t_{K,t_0,\bar{c}}^{\text{crit}}} \frac{\partial v_1(\theta_{K,t_0,\bar{c}})}{\partial x_1}(0,s) ds = -K, \]

\[ \| \theta_{K,t_0,\bar{c}}(x,t) \|_{H^4} \leq M_{K,t_0,\bar{c}}. \]

Furthermore we have \( \text{supp}(\theta_{\text{initial}}) \subset \{ r \in (a_1, \frac{3}{2}) \} \), \( \text{supp}(\theta_{K,t_0,\bar{c}}(x,t)) \subset \{ r \in (a_1, a_2) \} \) with \( a_1, a_2 \) depending on \( K, t_0 \) and \( \bar{c} \).

**Proof.** The initial conditions and solution are the ones obtained in lemma 4.2, we only need to note that \( \| \theta_{c,J,0} \|_{H^2} = c(\sum_{j=1}^{J} \frac{1}{2^j})^{\frac{1}{2}} \leq Cc \), and thus we need to take \( Cc \leq \bar{c} \) and then apply lemma 4.2. As for the condition regarding the support, we just need to use that since the solution remains in \( H^4 \) the velocity is \( C^4 \) and that the velocity at \((x_1, x_2) = (0,0)\) is zero and thus particles can only approach the origin exponentially fast.

\[ \square \]

**Theorem 4.3.** For any \( c_0 > 0 \), \( M > 1 \) and \( t_* > 0 \), we can find a \( H^{2+\frac{3}{2}} \) function \( \theta_0(x) \) with \( \| \theta_0(x) \|_{H^{2+\frac{3}{2}}} \leq c_0 \) such that the only solution \( \theta(x,t) \in H^{2+\frac{3}{2}} \) to the SQG equation (7) with initial conditions \( \theta_0(x) \) is such that there exists \( t \leq t^* \) with \( \| \theta(x,t) \|_{H^2} \geq M_{c_0} \).

**Proof.** We consider the pseudo-solution

\[ \theta_{M,t^*,c_0,N} = \theta_{K=4M,t_0=t^*,\bar{c} = \frac{c_0}{2}}(x,t) \]

\[ + \frac{c_0}{4} g_1(e^{G(t)} N_0^2 x_1) g_2(e^{-G(t)} N_0^2 x_2) \frac{\sin(e^{G(t)} N x_1)}{N^2} \]

with \( \theta_{K,t_0,\bar{c}} \) given by corollary 4.2.1 with \( \bar{c} = \frac{c_0}{2}, t_0 = t^* \) and \( K = 4M \),

\[ G(t) = - \int_{0}^{t} \frac{\partial v_1(\theta_{K,t_0,\bar{c}})}{\partial x_1}(0,s) ds \]

and \( g_1(x_1), g_2(x_2) \) \( C^\infty \) functions with support in \([-1,1]\) and \( \| g_i \|_{L^2} = 1 \). We will define

\[ f_{M,t^*,c_0}^1(x,t) := \theta_{K=4M,t_0=t^*,\bar{c} = \frac{c_0}{2}}(x,t) \]

\[ f_{c_0,N}^2(x,t) := \frac{c_0}{4} g_1(e^{G(t)} N_0^2 x_1) g_2(e^{-G(t)} N_0^2 x_2) \frac{\sin(e^{G(t)} N x_1)}{N^2} \]

for a more compact notation.

These pseudo-solutions have the following properties:
• For $N$ large, $||\tilde{\theta}_{M,t^*,c_0,N}(t = 0)||_{H^2} \leq c_0$.

• There exists a $t_{crit} \leq t^*$ (given by corollary [1.2.1]) such that, for $N$ large, we have

\[ ||\tilde{\theta}_{M,t^*,c_0,N}(t = t_{crit})||_{H^2} \geq \frac{c_0}{16} e^{16M} > c_0 e^M \]

where we used that, since $g_1, g_2 \in C^1$ and have compact support, for $\lambda > 0$

\[ \lim_{N \to \infty} ||N^\frac{\lambda}{2} g_1(\lambda N^\frac{\lambda}{2} x_1)g_2(\lambda N^\frac{\lambda}{2} x_2) \sin(\lambda N x_1)||_{L^2} = \frac{1}{\sqrt{2}} ||g(x_1)||_{L^2}. \]

Furthermore they fulfill the evolution equation

\[
\frac{\partial \tilde{\theta}_{M,t^*,c_0,N}}{\partial t} + v_1(f_{M,t^*,c_0}^1) \frac{\partial f_{M,t^*,c_0}^1}{\partial x_1} + v_2(f_{M,t^*,c_0}^1) \frac{\partial f_{M,t^*,c_0}^2}{\partial x_2} + x_1 \frac{\partial v_1(f_{M,t^*,c_0}^1)}{\partial x_1} + x_2 \frac{\partial v_2(f_{M,t^*,c_0}^1)}{\partial x_2} = 0
\]

and thus it is a pseudo-solution with source term

\[
F_{M,t^*,c_0,N}(x,t) = F_{M,t^*,c_0,N}^1(x,t) + F_{M,t^*,c_0,N}^2(x,t) + F_{M,t^*,c_0,N}^3(x,t),
\]

\[
F_{M,t^*,c_0,N}^1(x,t) := -(v_1(f_{c_0,N}^2) \frac{\partial f_{c_0,N}^2}{\partial x_1} + v_2(f_{c_0,N}^2) \frac{\partial f_{c_0,N}^2}{\partial x_2})
\]

\[
F_{M,t^*,c_0,N}^2(x,t) := -(v_1(f_{c_0,N}^1) \frac{\partial f_{M,t^*,c_0}^1}{\partial x_1} + v_2(f_{M,t^*,c_0}^1) \frac{\partial f_{M,t^*,c_0}^2}{\partial x_2})
\]

\[
F_{M,t^*,c_0,N}^3(x,t) := (x_1 \frac{\partial v_1(f_{M,t^*,c_0}^1)}{\partial x_1} - v_1(f_{M,t^*,c_0}^1)) \frac{\partial f_{c_0,N}^2}{\partial x_1}
\]

As usual we want to find bounds for the source term for $t \in [0,t_{crit}]$. For $F_{M,t^*,c_0,N}(x,t)$ it is easy to obtain that

\[
||F_{M,t^*,c_0,N}^1(x,t)||_{L^2} \leq C N^{-\frac{3}{2}}, \quad ||F_{M,t^*,c_0,N}^1(x,t)||_{H^3} \leq C N^{-\frac{1}{2}}
\]

with $C$ depending on $M$ and $c_0$.

For $F_{M,t^*,c_0,N}^2(x,t)$, using that $||f_{c_0,N}^2||_{L^1} \leq C N^{-\frac{3}{4}}$ and that the support of $f_{M,t^*,c_0}^1$ lies away from 0, we get

\[
||F_{M,t^*,c_0,N}^2(x,t)||_{L^2} \leq C N^{-\frac{3}{2}}, \quad ||F_{M,t^*,c_0,N}^2(x,t)||_{H^3} \leq C N^{-\frac{1}{2}}
\]

with $C$ depending on $M, t^*$ and $c_0$.

Finally, for $F_{M,t^*,c_0,N}^3(x,t)$, using that, for $i = 1,2$

\[
x_i \frac{\partial v_i(f_{M,t^*,c_0}^i)}{\partial x_i} - v_i(f_{M,t^*,c_0}^i)
\]

vanishes to second order around 0, that the third derivatives of $v_i(f_{M,t^*,c_0}^i)$ are bounded around 0, and $supp(f_{c_0,N}^2) \subset [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}] \times [-N^{-\frac{1}{2}}, N^{-\frac{1}{2}}]$, we get
\[ ||F_{M,t^*,c_0,N}^3||_{L^2} \leq CN^{-\frac{7}{2}}, \quad ||F_{M,t^*,c_0,N}^4||_{H^{\frac{1}{2}}} \leq CN^{-\frac{7}{4}}, \]

with \( C \) depending on \( M, t^* \) and \( c_0 \).

With all this combined and using the interpolation inequality, we get

\[ ||F_{M,t^*,c_0,N}||_{L^2} \leq CN^{-\frac{5}{4}}, \quad ||F_{M,t^*,c_0,N}||_{H^{\frac{1}{2}+\frac{1}{4}}} \leq CN^{-\frac{5}{4}}. \]

This allows us to obtain, in a similar way as in lemmas 2.8, 2.9, 3.5 and 3.8 that, if \( \theta_{M,t^*,c_0,N}(x,t) \) is the solution to (1) with \( \theta_{M,t^*,c_0,N}(x,0) = \theta_{M,t^*,c_0,N}(x,0) \) then

\[ ||\theta_{M,t^*,c_0,N}(x,t) - \theta_{M,t^*,c_0,N}(x,t)||_{H^{\frac{1}{2}+\frac{1}{4}}} \leq CT^{-\frac{1}{4}} \]

and this combined with the properties of \( \theta_{M,t^*,c_0,N}(x,t) \) finishes the proof.

\[ \square \]

**Theorem 4.4.** For any \( c_0 > 0 \) there exist initial conditions \( \theta(x,0) \) with \( ||\theta(x,0)||_{H^2} \leq c_0 \) such that any solution \( \theta(x,t) \) to (1) satisfies

\[ \text{ess-sup}_{t \in [0,\epsilon]} ||\theta(x,t)||_{H^2} = \infty \]

for any \( \epsilon > 0 \).

**Proof.** We start by fixing some arbitrary \( c_0 > 0 \) and defining

\[ \tilde{\theta}_{n,R,N}(x,t) := T_R(\tilde{\theta}_{M=4^n,t^*=2^{-n},c_0=2^{-n},N}), \]

with \( \tilde{\theta}_{M,t^*,c_0,N} \) as in (12) and \( T_R(f(x_1,x_2)) = f(x_1 + R, x_2) \). We will also refer to the first time when

\[ ||\tilde{\theta}_{n,R,N}(x,t)||_{H^2} \geq 2^n \]

(which we already know exists and is smaller than \( 2^{-n} \)) as \( t_{\text{crit},n} \).

We will study the initial conditions

\[ \theta(x,0) = \sum_{n=1}^{\infty} \tilde{\theta}_{n,R_n,N_n}(x,0), \quad (43) \]

which fulfill \( ||\theta(x,0)||_{H^2} \leq c_0 \) if each \( N_n \) is big enough, and we will prove by contradiction that if we choose appropriately \((R_n)_{n \in \mathbb{N}}\) and \((N_n)_{n \in \mathbb{N}}\) there cannot exists a solution \( \theta(x,t) \) with this initial conditions that satisfies

\[ \text{ess-sup}_{t \in [0,\epsilon]} ||\theta(x,t)||_{H^2} \leq P \quad (44) \]

for some \( \epsilon, P \). Note also that \( \tilde{\theta}_{n,R_n,N_n}(x,0) \) is supported in \( B_{\frac{1}{R_n}}(-R_n,0) \). We can assume that our \( L^2 \) norm is conserved, since this will be true if equation (44) holds (for the time intervals that we will consider). We will assume without loss of generality that \( \epsilon \leq 1 \), and we define \( v_{\text{max}} \) as the maximum velocity that a function \( f \) with \( ||f||_{H^2} \leq 1, ||f||_{L^2} \leq ||\theta(x,0)||_{L^2} \) can produce. With this in mind, we write

\[ R_n = D_n + D_{n+1} + 4v_{\text{max}}2^{n-1} + R_{n-1} + 3 \]

with \( D_n = N_n^2 \) and we will prove that, if \( N_n \) is big enough, then any solution to (11) with initial conditions (43) will satisfy

\[ \text{ess-sup}_{t \in [0,2^{-n}]} ||\theta(x,t)||_{H^2} \geq 2^{n-1}. \quad (45) \]

Note that with this definition of \( R_n \), we have, for any \( i \neq n \) that

\[ d(\text{supp}(T_{R_n}(\tilde{\theta}_{n,R_n,N_n}(x,0))), \text{supp}(T_{R_i}(\tilde{\theta}_{i,R_i,N_i}(x,0)))) \geq 4v_{\text{max}}2^{n-1} + D_n \]
For this, we focus on the evolution of
\[ \theta_n(x, t) := 1_{B_{D_n+2^n max^2 n-1+1}}(\cdot, -R_n, 0) \theta(x, t) \]
and we will assume that
\[ \text{ess-sup}_{t \in [0, 2^{-n}]} \| \theta(x, t) \|_{H^2} < 2^{n-1}. \] (46)

Then if \( t \in [0, 2^{-n}] \), we have that \( \theta_n(x, t) \) will fulfil the evolution equation
\[
\frac{\partial \theta_n}{\partial t} + (v_1(\theta_n) + v_1(\theta - \theta_n)) \frac{\partial \theta_n}{\partial x_1} + (v_2(\theta_n) + v_2(\theta - \theta_n)) \frac{\partial \theta_n}{\partial x_2} = 0
\]
and \( \| v_i(\theta - \theta_n) 1_{B_{max^2 n}(R_n)} \|_{L^\infty} \leq C N_n^{-4} \) since \( d(\text{supp}(\theta - \theta_n), \text{supp}(\theta_n)) \geq N_n^2 \).

But then we can argue as in lemmas 3.5, 3.8 and 3.11 to show that, for \( t \in [0, t_{\text{crit}, n}] \), if \( N_n \) is large, we have that
\[ \| \theta_n(x, t) - T_{R_n}(\tilde{\theta}_{n,R_n,N_n}(x, t)) \|_{H^{2+\frac{1}{p}}} \leq C N_n^{-\frac{2}{p}}. \]

Since for some \( t_{\text{crit}, n} \in [0, 2^{-n}] \) we have that
\[ \| T_{R_n}(\tilde{\theta}_{n,R_n,N_n}(x, t_{\text{crit}, n})) \|_{H^2} \geq 2^n, \]
and the \( H^2 \) norm of \( T_{R_n}(\tilde{\theta}_{n,R_n,N_n}(x, t)) \) is continuous in time, we arrive to a contradiction by taking \( N_n \) big enough and repeating this argument for each \( n \in \mathbb{N} \).

**Remark 5.** The proof can be adapted to work in the critical spaces \( W^{1+\frac{2}{p}, p} \) for \( p \in (1, \infty] \). For this, note that it is easy to obtain a version of corollary 4.2.1 but with small \( W^{1+\frac{2}{p}, p} \), since the function
\[
\sum_{j=1}^{J} \frac{c(b^{-j}r b_j \sin(2\alpha))}{j}
\]
has a \( W^{1+\frac{2}{p}, p} \) norm as small as we want by taking \( c \) small. As for the perturbation, we need to consider
\[ \lambda g_1(N^b x_1) g_2(N^b x_2) \frac{\sin(N x_1)}{N^{1+a}}, \]
with \( a = a(p), b = b(p) \geq 0 \) values that keep the norm \( W^{1+\frac{2}{p}, p} \) bounded (but not tending to zero) as \( N \to \infty \) (for example, in \( W^{1, \infty} \) we consider \( a = 0 \)) and \( \lambda > 0 \). Taking \( b = \frac{1}{2} \) and arguing as in theorems 4.3 and 4.4 allows us to obtain ill-posedness for a wide range of \( p \), but we need to include some refinements to obtain the result for all \( p \in (1, \infty] \). Namely, approximations for the velocity similar to those obtained in lemma 2.5 are needed and we have to include one extra time dependent term in the pseudo-solution.

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