The Dirac equation in Kerr-Taub-NUT spacetime

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Received 18 June 2013, in final form 5 July 2013
Published 30 July 2013
Online at stacks.iop.org/CQG/30/175005

Abstract
We study the Dirac equation in the Kerr–Taub–NUT spacetime. We use Boyer–Lindquist coordinates and separate the resulting equations into radial and angular parts. We obtain some exact analytical solutions of the angular equations for some special cases. We also obtain the radial wave equations with an effective potential. Finally, we discuss the potentials by plotting them as a function of radial distance in a physically acceptable region.

PACS numbers: 04.62.+v, 95.30.Sf

(Some figures may appear in colour only in the online journal)

1. Introduction

The Kerr–Taub–NUT (Newman–Unti–Tamburino) spacetime is obtained by introducing an extra non-trivial magnetic mass parameter called the ‘gravitomagnetic monopole moment’ in the Kerr metric. It describes the spacetime of a localized, stationary and axially symmetric object [1]. The solution contains three physical parameters: the gravitational mass which is also called the gravitoelectric charge, the gravitomagnetic mass that is also identified as the NUT charge and the rotation parameter that is the angular speed per unit mass. The NUT charge produces an asymptotically non-flat spacetime in contrast to the Kerr geometry that is asymptotically flat [2]. Although the Kerr–Taub–NUT spacetime has no curvature singularities, there exist conical singularities on the axis of symmetry [3]. One can get rid of conical singularities by taking a periodicity condition over the time coordinate. But, this leads to the emergence of closed time-like curves in the spacetime. This means that, in contrast to the Kerr solution interpreted as a regular rotating black hole, the Kerr–Taub–NUT solution cannot be identified as a regular black hole solution due to its singularity structure. An alternative physical interpretation of the Kerr–Taub–NUT spacetime can be found in [4] where the NUT metric is interpreted as a semi-infinite massless source of angular momentum. Despite the fact that the Kerr–Taub–NUT solution has some unpleasing properties, it is vastly studied for exploring various physical phenomena in general relativity due to its asymptotically non-flat spacetime structure [5–9].
In this paper, we study the Dirac equation in the Kerr–Taub–NUT spacetime. The Dirac equation has been extensively examined in various gravitational spacetimes including the Schwarzschild geometry [10], Kerr spacetime [11–14], Taub–NUT geometry [15], Kerr–Newman–AdS black hole background [16], four-dimensional constant-curvature black hole spacetime [17], rotating Bertotti–Robinson geometry [18], four-dimensional Nutku helicoid spacetime [19] and open universe geometry [20]. In some of these background spacetimes, some exact analytical solutions of massive and massless Dirac equations have been presented [15, 17–20]. In [16], spectral properties of the Dirac Hamiltonian are given. However, in the background of the rotating Kerr spacetime, exact solutions of the Dirac equation have been obtained only for some special values of the parameters [13, 14]. In [13], the series solutions of the angular Dirac equation have been given, while in [14], angular solutions have been presented by using the spectral decomposition method in which the angular wavefunctions are expanded in terms of spheroidal harmonics. By this method, a three-term recursion relation is achieved and eigenvalues of the angular equations are solved.

On the other hand, in almost all these works, the separability of the Dirac equation has also been discussed. Separability was first discovered in Hamilton–Jacobi and relativistic wave equations through the pioneering works of Carter [21, 22]. The separability has been shown to be closely related to the existence of second-order Stäckel–Killing tensors [22]. Later on, the separability of Hamilton–Jacobi and relativistic field equations has been extended to higher dimensional spacetimes in which the Stäckel–Killing tensors are also given explicitly (see [23–25] and the references therein). The separability of the Dirac equation however was first noticed by Chandrasekhar [11, 12]. Later, it was shown that the separability of the Dirac equation has also been connected with the existence of a second-order Killing–Yano tensor [26]. In close connection with these tensors, the Dirac equation has been proved to be separable in general vacuum type-D spacetimes [27]. In addition, the separability of the Dirac equation has been investigated in spherically symmetric spacetimes [28]. In [29], the separability of the Dirac equation in the Kerr–Newman geometry has been explicitly shown by using Boyer–Lindquist coordinates. In a recent work, the authors have demonstrated the separability of the massive Dirac equation in AdS–Kerr–Taub–NUT spacetimes [30].

In this work, we obtain the set of equations by employing an axially symmetric ansatz for the Dirac spinor. The equations obtained are separated into radial and angular parts with appropriate substitutions of spinor fields. We try to solve angular equations exactly. But unfortunately, we are unable to obtain exact analytical solutions to general angular equations for all physical parameters. Under some restrictions implemented on the separation constant, we present some exact solutions of the equations with and without gravitomagnetic mass and rotation parameters. Indeed, they can be solved exactly in terms of hypergeometric functions for the cases where the mass of the Dirac particle is equal to or twice the frequency of the spinor wavefunction. In the final part, radial equations are discussed. With some transformations on the dependent and independent field variables, wave equations with an effective potential barrier are obtained. To understand the physical behavior of the potentials, they are plotted with changing frequency and gravitomagnetic mass parameter in the physically acceptable regions.

The organization of the paper is as follows. In section 2, we present the general form of the Dirac equation in exterior forms. In section 3, we obtain the Dirac equation in the Kerr–Taub–NUT spacetime. In the subsections, we discuss the separability of the equations and obtain the angular and radial equations. Next, we find some exact analytical solutions of the angular equations. Finally, we study the radial wave equations and examine the behavior of the potential barriers that emerge in the transformed radial equations. We end up with some comments and conclusions.
2. Dirac equation in the four-dimensional spacetime

We consider a four-dimensional spacetime manifold $M$ equipped with a Lorentzian metric $g$ with a signature $(-, +, +, +)$ and a metric compatible connection $\nabla$. We assume that our spacetime manifold has a spin structure group $Spin_+(3, 1)$. It is known that the fundamental group of the Lorentzian group $SO_+(3, 1)$ is $\mathbb{Z}_2$, so that it has a universal covering group of $Spin_+(3, 1)$ that is the multiplicative subgroup of the complex Clifford algebra $\mathbb{C}l_{3,1}$.

In exterior forms, the Dirac equation can be written as [31]

$$* \gamma \wedge D\psi + \mu \psi * 1 = 0,$$

where $\gamma$ is the $\mathbb{C}l_{3,1}$-valued 1-form $\gamma = \gamma^a e_a$. We choose the units such that $c = 1$ and $h = 1$. Here $*$ denotes Hodge-star operator and $\mu$ is the mass of the particle. $[e_a]$s are the orthonormal co-frame 1-forms such that the metric $g = \eta_{ab} e^a \otimes e^b$. $\psi$ represents the $\mathbb{C}^4$-valued Dirac spinor whose covariant exterior derivative can be written as

$$D\psi = d\psi + \frac{1}{2} \sigma^{ab} \omega_{ab} \psi,$$

where $\sigma^{ab} = \frac{1}{2} [\gamma^a, \gamma^b]$ and $[\gamma^a]$s satisfy the relations

$$[\gamma^a, \gamma^b] = (\gamma^a \gamma^b + \gamma^b \gamma^a) = 2\eta^{ab} I_{4\times4}.$$

$\omega_{ab}$ are the connection 1-forms that satisfy the Cartan structure equations

$$de^a + \omega^a_{\,b} \wedge e^b = T^a$$

where $T^a$ denotes the torsion 2-form and metric compatibility implies that $\omega_{ab} = -\omega_{ba}$.

Since $\mathbb{C}l_{3,1}$ is isomorphic to $M_4(\mathbb{C})$ that is the set of $4 \times 4$ complex matrices, we can choose the representation

$$\gamma^0 = i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^1 = i \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix},$$

$$\gamma^2 = i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = i \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix},$$

where $\sigma^i$ are the Pauli spin matrices and $I$ is the $2 \times 2$ identity matrix.

3. Dirac equation in the Kerr–Taub–NUT spacetime

In this section, we examine the Dirac equation in the Kerr–Taub–NUT spacetime. In Boyer–Lindquist coordinates, the Kerr–Taub–NUT spacetime can be described by the metric with an asymptotically non-flat structure,

$$g = \frac{\Delta}{\Sigma} (dt - \chi \, d\varphi)^2 + \Sigma \left( \frac{d\varphi^2}{\Delta} + d\theta^2 \right) + \frac{\sin^2 \theta}{\Sigma} \left( a \, dt - (r^2 + \ell^2 + a^2) \, d\varphi \right)^2,$$

(3.1)

where

$$\Sigma = r^2 + (\ell + a \cos \theta)^2,$$

$$\Delta = r^2 - 2Mr + a^2 - \ell^2,$$

and

$$\chi = a \sin^2 \theta - 2\ell \cos \theta.$$

Here, $M$ is a parameter related to the physical mass of the gravitational source. $a$ is associated with its angular momentum per unit mass and $\ell$ denotes the gravitomagnetic monopole moment of the source. For the metric (3.1), we choose the co-frame 1-forms

$$e^0 = \left( \frac{\Delta}{\Sigma} \right)^{1/2} (dt - \chi \, d\varphi), \quad e^1 = \left( \frac{\Sigma}{\Delta} \right)^{1/2} \, dr,$$

$$e^2 = \Sigma^{1/2} \, d\theta, \quad e^3 = \frac{\sin \theta}{\Sigma^{1/2}} \left( a \, dt - (r^2 + \ell^2 + a^2) \, d\varphi \right).$$

(3.2)
We consider that the spacetime is Levi-Civita (torsion-free) such that the connection 1-forms can be determined from the equation
\[ \text{de}^\alpha + \omega^\alpha_b \wedge \text{e}^b = 0, \]
which has a unique solution
\[ \omega^\alpha_b = \frac{1}{2} (e^i \epsilon^{ib} \text{d}e_i + e_b \text{d}e^i - e^i \text{d}e_b). \tag{3.3} \]

Here \( e_i = t_i \), are inner-product operators that satisfy \( e_i e^i = \delta^i_0 \). From equation (3.3), we can determine connection 1-forms:
\[ \begin{align*}
\omega^0_1 &= A_2 \text{d}t + A_3 \text{d}\varphi, \\
\omega^0_2 &= B_2 \text{d}\varphi, \\
\omega^0_3 &= C_1 \text{d}r + C_2 \text{d}\theta, \\
\omega^1_2 &= E_1 \text{d}r + E_2 \text{d}\theta, \\
\omega^1_3 &= G_3 \text{d}\varphi,
\end{align*} \tag{3.4} \]

where
\[ \begin{align*}
A_2 &= \frac{Mr^2 + 2r^2 r + 2\ell a \cos \theta - M (\ell + a \cos \theta)^2}{\Sigma}, \\
A_3 &= \frac{\chi (m - r) \Sigma - 2r^2 r - 2Mr^2}{\Sigma}, \\
B_3 &= -\frac{\Delta^{1/2} \sin \theta (\ell + a \cos \theta)}{\Sigma}, \\
C_1 &= \frac{a r \sin \theta}{\Sigma \Delta^{1/2}}, \\
C_2 &= -\frac{\Delta^{1/2} (\ell + a \cos \theta)}{\Sigma}, \\
E_1 &= -\frac{a r \sin \theta (\ell + a \cos \theta)}{\Sigma \Delta^{1/2}}, \\
E_2 &= -\frac{r \Delta^{1/2} \sin \theta}{\Sigma}, \\
G_3 &= \frac{r \sin \theta \Delta^{1/2}}{\Sigma}, \\
K_3 &= \frac{\cos \theta (r^2 + \ell^2 + a^2) (\Sigma + 2\ell (\ell + a \cos \theta) ) + \chi (2Mr + 2\ell^2) (\ell + a \cos \theta)}{\Sigma^2}. \tag{3.5}
\end{align*} \]

Since the spacetime is axially symmetric, we can take
\[ \psi = e^{-i\omega \ell r} e^{im \varphi} \begin{pmatrix} \psi_1 (r, \theta) \\ \psi_2 (r, \theta) \\ \psi_3 (r, \theta) \\ \psi_4 (r, \theta) \end{pmatrix}, \tag{3.6} \]
where \( m \) denotes the azimuthal quantum number. Then we substitute (3.4)–(3.6) into the Dirac equation (2.1) and obtain the following equations:
\[ \begin{align*}
\left( \frac{\omega \alpha_1 + m \alpha_3}{\Delta^{1/2} \sin \theta} \right) \psi_3 &= \frac{i \Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_4}{\partial r} - \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_4}{\partial \theta} + \frac{i}{2} (\delta_2 + \delta_4) \psi_4 \\
&+ \frac{1}{2} (\delta_1 - \delta_3) \psi_4 + \mu \psi_1 = 0, \\
\left( -\omega \alpha_3 + m \alpha_4 \right) \psi_4 &= \frac{i \Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_3}{\partial r} + \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_3}{\partial \theta} + \frac{i}{2} (\delta_4 - \delta_2) \psi_3 \\
&+ \frac{1}{2} (\delta_1 + \delta_3) \psi_3 + \mu \psi_2 = 0, \tag{3.7}
\end{align*} \]
\[ \begin{align*}
\left( -\omega \alpha_1 + m \alpha_4 \right) \psi_1 &= \frac{i \Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_2}{\partial r} + \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_2}{\partial \theta} + \frac{i}{2} (\delta_2 - \delta_4) \psi_2 \\
&+ \frac{1}{2} (\delta_1 + \delta_3) \psi_2 + \mu \psi_3 = 0. \tag{3.8}
\end{align*} \]
We then simplify the resulting equations and finally obtain

\[
\left( \frac{\omega a_1 + ma_2}{\Delta^{1/2} \sin \theta} \right) \psi_2 + \frac{i \Delta^{1/2}}{\Sigma^{1/2}} \frac{\partial \psi_1}{\partial r} - \frac{1}{\Sigma^{1/2}} \frac{\partial \psi_1}{\partial \theta} - \frac{i}{2} (\delta_2 + \delta_4) \psi_1 + \frac{1}{2} (\delta_1 - \delta_3) \psi_1 + \mu \psi_4 = 0,
\]

where

\[
\alpha_1 = \frac{\chi \Delta^{1/2} - (r^2 + a^2 + \ell^2) \sin \theta}{\Sigma^{1/2}}, \quad \alpha_2 = \frac{a \sin \theta - \Delta^{1/2}}{\Sigma^{1/2}},
\]

\[
\alpha_3 = \frac{\chi \Delta^{1/2} + (r^2 + a^2 + \ell^2) \sin \theta}{\Sigma^{1/2}}, \quad \alpha_4 = \frac{a \sin \theta + \Delta^{1/2}}{\Sigma^{1/2}},
\]

and

\[
\delta_1 = - \frac{(l + a \cos \theta) \Delta^{1/2}}{\Sigma^{3/2}}, \quad \delta_2 = \frac{ar \sin \theta}{\Sigma^{3/2}},
\]

\[
\delta_3 = \frac{\cos \theta - a \sin^2 \theta (l + a \cos \theta)}{\sin \theta \Sigma^{3/2}}, \quad \delta_4 = - \frac{(\Sigma(r - M) + r \Delta)}{\Delta^{1/2} \Sigma^{3/2}}.
\]

### 3.1. Separability of the equations

Although in [27] the Dirac equation is proven to be separable in the Carter class of type-D vacuum spacetimes which our Kerr-Taub-NUT background also belongs to, and the existence of a second-order Killing-Yano tensor implies the separability of the Dirac equation, for completeness of the work and also for instructional purposes, it would be useful to discuss and explicitly illustrate the separability work by employing Boyer-Lindquist coordinates. For that purpose, we add and subtract equations (3.7)–(3.10), and define

\[
F_1 = i (r - i(\ell + a \cos \theta))^{1/2} (\psi_1 + \psi_2),
\]

\[
F_2 = -i (r - i(\ell + a \cos \theta))^{1/2} (\psi_2 - \psi_1),
\]

\[
F_3 = (r + i(\ell + a \cos \theta))^{1/2} (\psi_3 + \psi_4),
\]

\[
F_4 = (r + i(\ell + a \cos \theta))^{1/2} (\psi_4 - \psi_3).
\]

We then simplify the resulting equations and finally obtain

\[
\left\{ \frac{-ma - \omega (r^2 + \ell^2 + a^2)}{\Delta^{1/2}} - \mathcal{D} \right\} F_3 + \left\{ \frac{m - \omega \chi}{\sin \theta} - \mathcal{L} \right\} F_4 - \mu (ir - (\ell + a \cos \theta)) F_1 = 0,
\]

\[
\left\{ \frac{-m + \omega \chi}{\sin \theta} - \mathcal{L} \right\} F_3 + \left\{ -ma + \omega (r^2 + \ell^2 + a^2) \right\} \frac{1}{\Delta^{1/2}} - \mathcal{D} \right\} F_4 - \mu (ir - (\ell + a \cos \theta)) F_2 = 0,
\]

\[
\left\{ \frac{ma - \omega (r^2 + \ell^2 + a^2)}{\Delta^{1/2}} + \mathcal{D} \right\} F_1 + \left\{ \frac{m - \omega \chi}{\sin \theta} - \mathcal{L} \right\} F_2 + \mu (ir + (\ell + a \cos \theta)) F_3 = 0,
\]

\[
\left\{ \frac{m - \omega \chi}{\sin \theta} + \mathcal{L} \right\} F_1 + \left\{ \frac{ma - \omega (r^2 + \ell^2 + a^2)}{\Delta^{1/2}} - \mathcal{D} \right\} F_2 - \mu (ir + (\ell + a \cos \theta)) F_4 = 0,
\]
With the ansatz above, the equations take the following forms:

\[
\mathcal{D} = \frac{i}{2} \left( \frac{r - M}{\Delta^{1/2}} + \Delta^{1/2} \frac{\partial}{\partial r} \right), \quad \mathcal{L} = \frac{1}{2} \cot \theta + \frac{\partial}{\partial \theta}.
\]

Equations (3.12)–(3.15) imply the separability ansatz

\[
\begin{align*}
F_1 &= R_1(r) S_1(\theta), \\
F_2 &= R_2(r) S_2(\theta), \\
F_3 &= R_2(r) S_1(\theta), \\
F_4 &= R_1(r) S_2(\theta).
\end{align*}
\]

(3.16)

With the ansatz above, the equations take the following forms:

\[
\begin{align*}
\left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D}\right] R_2(r) - i\mu r R_1(r) &= S_1(\theta) \\
\left[\frac{m - \omega \chi}{\sin \theta} - \mathcal{L}\right] S_2(\theta) + \mu (\ell + a \cos \theta) S_1(\theta) &= R_1(r) = 0, \\
\end{align*}
\]

(3.17)

\[
\begin{align*}
\left[\frac{-m + \omega \chi}{\sin \theta} - \mathcal{L}\right] S_1(\theta) + \mu (\ell + a \cos \theta) S_2(\theta) &= R_2(r) \\
\left[\frac{-ma + \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D}\right] R_1(r) - i\mu r R_2(r) &= S_2(\theta) = 0, \\
\end{align*}
\]

(3.18)

\[
\begin{align*}
\left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D}\right] R_1(r) + i\mu r R_2(r) &= S_1(\theta) \\
\left[\frac{m - \omega \chi}{\sin \theta} - \mathcal{L}\right] S_2(\theta) + \mu (\ell + a \cos \theta) S_1(\theta) &= R_2(r) = 0, \\
\end{align*}
\]

(3.19)

\[
\begin{align*}
\left[\frac{m - \omega \chi}{\sin \theta} + \mathcal{L}\right] S_1(\theta) - \mu (\ell + a \cos \theta) S_2(\theta) &= R_1(r) \\
\left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D}\right] R_2(r) - i\mu r R_1(r) &= S_2(\theta) = 0. \\
\end{align*}
\]

(3.20)

These equations further imply that

\[
\begin{align*}
\lambda_1 R_1(r) &= \left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D}\right] R_2(r) - i\mu r R_1(r), \\
\lambda_2 R_2(r) &= \left[\frac{-ma + \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - \mathcal{D}\right] R_1(r) - i\mu r R_2(r), \\
\lambda_3 R_2(r) &= \left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D}\right] R_1(r) + i\mu r R_2(r), \\
\lambda_4 R_1(r) &= \left[\frac{ma - \omega (a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + \mathcal{D}\right] R_2(r) - i\mu r R_1(r),
\end{align*}
\]

(3.21)–(3.24)

and

\[
\begin{align*}
\lambda_1 S_1(\theta) &= \left[\frac{-m + \omega \chi}{\sin \theta} + \mathcal{L}\right] S_2(\theta) - \mu (\ell + a \cos \theta) S_1(\theta), \\
\end{align*}
\]

(3.25)
Then, taking $x$ and (3.21)–(3.28), we choose $\lambda_1 = \lambda_3 = \lambda_4 = \lambda$ and $\lambda_2 = -\lambda$. Then, we obtain the following independent radial and angular equations:

$$
\lambda R_1(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} - D \right\} R_2(r) - i\mu r R_1(r), \tag{3.29}
$$

$$
\lambda R_2(r) = \left\{ \frac{ma - \omega(a^2 + r^2 + \ell^2)}{\Delta^{1/2}} + D \right\} R_1(r) + i\mu r R_2(r), \tag{3.30}
$$

and

$$
\lambda S_1(\theta) = \left\{ \frac{\omega x - m}{\sin \theta} - L \right\} S_2(\theta) + \mu(\ell + a \cos \theta) S_2(\theta), \tag{3.31}
$$

$$
\lambda S_2(\theta) = \left\{ \frac{\omega x - m}{\sin \theta} + L \right\} S_2(\theta) - \mu(\ell + a \cos \theta) S_1(\theta). \tag{3.32}
$$

### 3.2. Angular equations

Angular equations (3.31) and (3.32) can be arranged as

$$
\frac{dS_1}{d\theta} + \left\{ \left( \frac{1}{2} + 2a \ell \right) \cot \theta - a \omega \sin \theta + \frac{m}{\sin \theta} \right\} S_1(\theta) = (\mu(\ell + a \cos \theta) - \lambda) S_2(\theta),
$$

and

$$
\frac{dS_2}{d\theta} + \left\{ \left( \frac{1}{2} - 2a \ell \right) \cot \theta + a \omega \sin \theta - \frac{m}{\sin \theta} \right\} S_2(\theta) = (\mu(\ell + a \cos \theta) + \lambda) S_1(\theta).
$$

At this stage, we affect the transformation

$$
S_1 = \cos \left( \frac{\theta}{2} \right) T_1 + \sin \left( \frac{\theta}{2} \right) T_2,
$$

$$
S_2 = -\sin \left( \frac{\theta}{2} \right) T_1 + \cos \left( \frac{\theta}{2} \right) T_2.
$$

Then, taking $x = \cos \theta$ and redefining $T_1 = T_1$ and $T_2 = T_2$, one can easily see that (3.33) and (3.34) satisfy the following second-order differential equations:

$$
(1 - x^2) \frac{d^2 T_+}{dx^2} + M_+ \frac{dT_+}{dx} + N_+ T_+ = 0 \tag{3.36}
$$

where

$$
M_+ = \frac{\ell(2\omega - \mu)(1 - x^2) - 2a(\mu - \omega)x(1 - x^2)}{\left( \frac{1}{2} + \lambda \right) - m} - \ell(2\omega - \mu)x + a\omega + (a\mu - a\omega)x^2 - 2x \tag{3.37}
$$

and

$$
N_+ = \frac{1}{(1 - x^2) + M_+ \frac{dT_+}{dx}}.
$$
and
\[ N_\pm = -\frac{(m \pm \frac{1}{2})^2 + 4\omega \ell (m \pm \frac{1}{2})x + 4\omega^2 \ell^2}{1 - x^2} + (\pm \ell(\mu - 2\omega) + (2\omega^2 - \mu^2)2\ell \omega)x + (\pm 2 - \mu a - \omega \mu a - \omega a)x^2 + (4\omega^2 - \mu^2)\ell^2 \mp (\mu - \omega)a - \lambda^2 \omega^2 + 2ma\omega + \lambda(\lambda + 1) + \left(\frac{1}{2} \pm m\right)x \pm 2\omega \ell \pm (\ell(\mu - 2\omega) + a(\mu - \omega)x)(1 - x^2) \right) \times (2a(\mu - \omega)x - (2\omega - \mu)\ell). \] (3.38)

Now we investigate exact solutions to equations (3.36) for some special cases.

(i) \( \ell = 0, a = 0 \). In that case, equations (3.36) take the simple form,
\[
\frac{d}{dx} \left( (1 - x^2) \frac{dT_\pm}{dx} \right) + \left( \tilde{\lambda}(\tilde{\lambda} + 1) - \frac{(m \pm \frac{1}{2})^2}{1 - x^2} \right) T_\pm = 0. \] (3.39)

In general, for generic values of \( \lambda \), the solutions to those equations can be expressed in terms of the associated Legendre functions \( P_{\nu}^{\pm} \), with \( \nu = m = \pm \frac{1}{2} \). When \( \lambda \) and \( \nu = m \) are integers (\( \nu_a \) being even), solutions describe the associated Legendre polynomials.

(ii) \( \ell = 0, a \neq 0 \) and \( \mu = \omega \). For that special case, equations (3.36) can be simplified to
\[
\frac{d}{dx} \left( (1 - x^2) \frac{dT_\pm}{dx} \right) + \left( \tilde{\lambda}(\tilde{\lambda} + 1) - \frac{(m \pm \frac{1}{2})^2}{1 - x^2} \right) T_\pm = 0, \] (3.40)

where
\[ \tilde{\lambda}(\tilde{\lambda} + 1) = \lambda(\lambda + 1) + 2ma\omega - a^2 \omega^2. \] (3.41)

Again the solutions are the same as in case (i), except that \( \lambda \) is replaced by \( \tilde{\lambda} \).

(iii) \( \ell = 0, a \neq 0 \), \( \mu \neq \omega \). In that case, equations (3.36) can be reduced to
\[
(1 - x^2) \frac{d^2 T_\pm}{dx^2} + (Cx^4 + D_\pm x^2 + E_\pm) T_\pm = 0, \] (3.42)

where we restrict the eigenvalue
\[ \lambda = \pm a\mu \mp m - \frac{1}{2}. \] (3.43)

Here,
\[ C = a^2 \mu^2 - a^2 \omega^2, \]
\[ D_\pm = -a^2 \mu^2 + 2a^2 \omega^2 \mp m \mp a\omega - \lambda^2 - 2a\omega m - \frac{1}{2}, \] (3.44)
\[ E_\pm = -(m \pm \frac{1}{2})^3 \mp a\omega + \lambda^2 + 2ma\omega - a^2 \omega^2 - \frac{1}{2}. \]

Equations of the type (3.42) have exact analytical solutions for \( C = 0, D_\pm = 0 \) or \( C = 0, D_\pm \neq 0 \). However, in our case, since \( \mu \neq \omega, C \neq 0 \). In fact, when \( \lambda \) is further restricted to be \( \lambda = \pm a\mu \) (which corresponds to \( m = \mp \frac{1}{2} \)), equations (3.42) have simple analytical solutions. When the mass of the particle is greater than the frequency of the spinor wave (\( \mu > \omega \)), the solutions describe periodic waves. However, if the mass of the particle is smaller than the frequency of the spinor wave (\( \mu < \omega \)), the solutions are exponential.
(iv) $\ell \neq 0$, $a = 0$ and $\mu = 2\omega$. In that case, equations (3.36) can be arranged as
\[ (x^2 - 1) \frac{d^2T_\pm}{dx^2} + 2x(x^2 - 1) \frac{dT_\pm}{dx} + \left( C_\pm + D_\pm x - \lambda(\lambda + 1)x^2 \right)T_\pm = 0, \] (3.45)
where
\[ C_\pm = \lambda(\lambda + 1) - \left( m \pm \frac{1}{2} \right)^2 - 4\omega^2 \ell^2, \quad D_\pm = -4a\ell \left( m \pm \frac{1}{2} \right). \]

Under the transformation
\[ \xi = \frac{1}{2}(1 - x), \quad Y_\pm = (x + 1)^{-p_\pm} (1 - x)^{-q_\pm} T_\pm, \] (3.46)
the equation satisfied by $Y_\pm$ takes the form
\[ \xi (\xi - 1) \frac{d^2Y_\pm}{d\xi^2} + \left[ \xi (\alpha_\pm + \beta_\pm + 1) - \gamma_\pm \xi \right] \frac{dY_\pm}{d\xi} + \alpha_\pm \beta_\pm Y_\pm = 0, \] (3.47)
whose solution is given by the hypergeometric function
\[ Y_\pm(\xi) = F(\alpha_\pm, \beta_\pm, \gamma_\pm; \xi). \] (3.48)

Hence, the solutions are
\[ T_\pm(x) = (1 + x)^{p_\pm} (1 - x)^{q_\pm} F(\alpha_\pm, \beta_\pm, \gamma_\pm; \frac{1}{2}(1 - x)). \] (3.49)

Here,
\[ q_\pm^2 = \frac{1}{4} \left( m \pm \frac{1}{2} + 2a\ell \right)^2, \] (3.50)
\[ p_\pm^2 = \frac{1}{4} \left( m \pm \frac{1}{2} - 2a\ell \right)^2. \] (3.51)

$\alpha_\pm, \beta_\pm$ and $\gamma_\pm$ can be obtained from
\[ \alpha_\pm + \beta_\pm + 1 = 2(p_\pm + q_\pm + 1), \]
\[ \alpha_\pm \beta_\pm = (p_\pm + q_\pm)^2 - p_\pm + q_\pm + 2(p_\pm + q_\pm) - \lambda(\lambda + 1), \] (3.52)
\[ \gamma_\pm = 2q_\pm + 1. \]

(v) $\ell \neq 0$, $a = 0$ and $\mu \neq 2\omega$. In that case, equations (3.36) reduce to
\[ (1 - x^2) \frac{d^2T_\pm}{dx^2} + \left( -2x + \frac{\ell(2\omega - \mu)(1 - x^2)}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x} \right) \frac{dT_\pm}{dx} \]
\[ + \left( -\left( m \pm \frac{1}{2} \right)^2 + 4a\ell \left( m \pm \frac{1}{2} \right) x + 4\omega^2 \ell^2 \right) \frac{1}{1 - x^2} \]
\[ + 4\omega^2 - \mu^2 \ell^2 + \left( \frac{1}{2} \pm m \right) x \pm 2a\ell \pm \ell(\mu - 2\omega)(1 - x^2) \frac{1}{(\mp(\frac{1}{2} + \lambda) - m) - \ell(2\omega - \mu)x} (\mu - 2\omega) \ell \right) \]
\[ \times T_\pm = 0. \] (3.53)

As we have done in case (iii), we simplify (3.53) by taking the constraints
\[ \mu \ell - 2a\ell = \left( \frac{1}{2} + \lambda \right) \pm m \] (3.54)
in the equations satisfied by $T_+$ and $T_-$. respectively. With these restrictions, equations (3.53) satisfied for $T_+$ and $T_-$ reduce to
\[ (1 - x^2) \frac{d^2T_+}{dx^2} + (1 - x)(1 - x^2) \frac{dT_+}{dx} + (\tilde{C}_+ x^2 + \tilde{D}_+ x + \tilde{E}_+) T_+ = 0 \] (3.55)
\[(1 - x^2) \frac{d^2 T_+}{dx^2} + (1 + x)(1 - x^2) \frac{dT_+}{dx} + (\ddot{C}_- x^2 + \ddot{D}_- x + \ddot{E}_-) T_+ = 0, \quad (3.56)\]

respectively, where
\[
\ddot{C}_+ = \ell(\mu - 2\omega)(4\omega + 2m \pm 1) - (m \pm \frac{1}{2})^2, \\
\ddot{D}_+ = -4\omega \ell(m \pm 1) - (m \pm \frac{1}{2}), \\
\ddot{E}_+ = 4\omega \ell(\omega + \mu \pm m) - (\frac{1}{2} \pm m)(2\mu + 1).
\]

Equations of the types (3.55) and (3.56) seem harder to obtain for exact analytical solutions. However, with \(\lambda = \mu \ell\), both equations take the following simple forms:
\[
(1 + x)^2 \frac{d^2 T_+}{dx^2} + (1 + x) \frac{dT_+}{dx} - \left( m + \frac{1}{2} \right)^2 T_+ = 0, \quad (3.58)
\]

and
\[
(x - 1)^2 \frac{d^2 T_-}{dx^2} + (x - 1) \frac{dT_-}{dx} - \left( m - \frac{1}{2} \right)^2 T_- = 0, \quad (3.59)
\]

whose solutions are given by
\[
T_+(x) = c_1 (1 + x)^{(m + \frac{1}{2})} + c_2 (1 + x)^{-(m + \frac{1}{2})}, \quad (3.60)
\]

\[
T_-(x) = d_1 (1 - x)^{(m - \frac{1}{2})} + d_2 (1 - x)^{-(m - \frac{1}{2})}, \quad (3.61)
\]

where \(c_1, c_2, d_1, d_2\) are real constants.

(vi) \(\ell \neq 0, a \neq 0\) and \(\mu = 2\omega\). In that case, equations (3.36) take the following form:
\[
(1 - x^2) \frac{d^2 T_+}{dx^2} - \left( 2x + \frac{2a_0 \omega x (1 - x^2)}{\left( \frac{1}{2} + \lambda \right) - m + a_0 + a_0 x^2 \right) \frac{dT_+}{dx} \\
+ \left( -(m \pm \frac{1}{2})^2 + 4\omega \ell (m \pm \frac{1}{2}) x + 4\omega^2 \ell^2 \right) x - \lambda \left( \frac{\lambda}{2} + 1 \right) \\
- 4\ell a_0^2 x + \left( \pm 2 - 3a_0 \psi \right) a_0 x^2 + a_0 - \lambda^2 \omega^2 + 2ma_0 \\
+ 2a_0 \psi \left( \frac{\lambda}{2} \pm m \right) x \pm 2 a_0 \psi \pm a_0 x (1 - x^2) \right) \frac{dT_+}{dx} T_+ = 0. \quad (3.62)
\]

Under the constraints
\[
2a_0 = m = \left( \frac{1}{2} + \lambda \right), \quad (3.63)
\]
equations (3.62) can be simplified as
\[
(1 - x^2)^2 \frac{d^2 T_+}{dx^2} + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) T_+ = 0 \quad (3.64)
\]

and
\[
(1 - x^2)^2 \frac{d^2 T_-}{dx^2} + (\bar{a}_0 + \bar{a}_1 x + \bar{a}_2 x^2 + \bar{a}_3 x^3 + \bar{a}_4 x^4) T_- = 0, \quad (3.65)
\]
respectively, where
\[
\begin{align*}
\lambda_0 &= 3a^2\omega^2 - 4\omega^2\ell^2 - (m + \frac{1}{2})(1 + 2\omega), \\
\lambda_1 &= -4\ell\omega(m + \frac{1}{2} + a\omega), \\
\lambda_2 &= -6a^2\omega^2 + (m + \frac{1}{2})(2a\omega - m - \frac{1}{2}), \\
\lambda_3 &= 4a\ell\omega^2, \\
\lambda_4 &= 3\omega^2\ell^2,
\end{align*}
\]
and
\[
\begin{align*}
\tilde{\lambda}_0 &= 3a^2\omega^2 - 4\omega^2\ell^2 + (m - \frac{1}{2})(1 - 2\omega), \\
\tilde{\lambda}_1 &= -4\ell\omega(m + a\omega - \frac{1}{2}), \\
\tilde{\lambda}_2 &= -6a^2\omega^2 + (m - \frac{1}{2})(2a\omega - m + \frac{1}{2}), \\
\tilde{\lambda}_3 &= 4a\ell\omega^2, \\
\tilde{\lambda}_4 &= 3\omega^2\ell^2.
\end{align*}
\]
In order to obtain an exact analytical solution, equation (3.64) can further be simplified when \(\lambda = 2a\omega + 1\) (which corresponds to \(m = -\frac{3}{2}\)) and \(\ell = \frac{1}{2}\omega\) as
\[
(1 + x)^2 \frac{d^2 T_+}{dx^2} + (3a^2\omega^2 + 2a\omega + 3a^2\omega^2 x)T_+ = 0.
\]
Then the solution for \(T_+\) can be written as
\[
T_+(x) = e^{i\sqrt{3}\omega\ell x}\Phi(\tilde{\lambda} + 1, 2, z),
\]
with \(z = -2i\sqrt{3}\omega\ell(x + 1)\) and \(\tilde{\lambda} = -\frac{1}{\sqrt{3}}\). Similarly (3.65) takes a simpler form when \(\lambda = 2a\omega - 1\) (which corresponds to \(m = \frac{1}{2}\)) and \(\ell = \frac{1}{2}\omega\) as
\[
(1 - x)^2 \frac{d^2 T_-}{dx^2} + (3a^2\omega^2 - 2a\omega - 3a^2\omega^2 x)T_- = 0,
\]
whose solution can be given as
\[
T_-(x) = e^{-i\sqrt{3}\omega\ell x}\zeta(\tilde{\lambda} + 1, 2, \xi),
\]
with \(\zeta = -2i\sqrt{3}\omega\ell(x - 1)\). For both \(T_+\) and \(T_-\), the solutions describe oscillating wave solutions with non-uniform amplitudes. The details of the derivation of the solutions are given in the appendix.

3.3. Radial equations

Radial equations (3.34), (3.35) can be rearranged as
\[
\begin{align*}
\frac{dR_1}{dr} + \left(\frac{r - M}{\Delta} - i\left(\frac{ma - \omega a^2 + r^2 + \ell^2}{\Delta}\right) \right) R_1 &= -\frac{2}{\sqrt{\Delta}} (\mu r + i\kappa) R_2, \\
\frac{dR_2}{dr} + \left(\frac{r - M}{\Delta} + i\left(\frac{ma - \omega a^2 + r^2 + \ell^2}{\Delta}\right) \right) R_2 &= -\frac{2}{\sqrt{\Delta}} (\mu r - i\kappa) R_1.
\end{align*}
\]
To obtain the radial equations in the form of a wave equation, we follow the method applied in Chandrasekhar’s book [12], but with \(\ell \neq 0\). Hence, we consider the transformations
\[
P_1 = i\Delta^{1/2} R_1, \quad P_2 = \Delta^{1/2} R_2,
\]
(3.74)
Let $\Omega = r^2 + a^2 + \ell^2 - \frac{ma}{\omega}$. With these transformations, equations (3.72) and (3.73) take the forms
\[
\frac{dP_1}{dr} + 2i\omega\Omega P_1 = -\frac{2i(\mu r + i\lambda)}{\Delta^{1/2}} P_2, \quad (3.75)
\]
\[
\frac{dP_2}{dr} - 2i\omega\Omega P_2 = \frac{2(\mu r - i\lambda)}{\Delta^{1/2}} P_1, \quad (3.76)
\]
Let
\[
\frac{du}{dr} = \frac{\Omega}{\Delta}, \quad \beta^2 = M^2 - a^2 + \ell^2. \quad (3.77)
\]
Then, in terms of the new independent variable $u$, we obtain
\[
\frac{dP_1}{du} + 2i\omega P_1 = -\frac{2i(\mu r + i\lambda)}{\Omega} \Delta^{1/2} P_2, \quad (3.78)
\]
\[
\frac{dP_2}{du} - 2i\omega P_2 = \frac{2(\mu r - i\lambda)}{\Omega} \Delta^{1/2} P_1, \quad (3.79)
\]
where
\[
u = r + \frac{2Mr_+ + 2\ell^2 - \frac{ma}{\omega}}{2\beta} \ln \left( \frac{r - r_+}{r_+} \right) - \frac{2Mr_- + 2\ell^2 - \frac{ma}{\omega}}{2\beta} \ln \left( \frac{r - r_-}{r_-} \right), \quad (r > r_+) \quad (3.80)
\]
for $M^2 > a^2 - \ell^2$. Here, $r_+ = M + \beta$ and $r_- = M - \beta$. Relation (3.80) is single-valued for $r > r_+$ if the inequality
\[
r_+^2 + a^2 + \ell^2 - \frac{ma}{\omega} > 0 \quad (3.81)
\]
is satisfied. This requires that in the frequency range
\[
\omega_2 < \omega < \omega_1, \quad (3.82)
\]
where
\[
\omega_2 = \frac{ma}{2Mr_+ + 2\ell^2}, \quad \omega_1 = \frac{ma}{\ell^2 + a^2}, \quad m > 0, \quad (3.83)
\]
u should become single-valued in $r$ in the region where $r > r_+$. It is seen that as $r \to \infty, u \to \infty$ and as $r \to (r_+)^+, u \to -\infty$. For completeness, we also note that for the critical case when $M^2 = a^2 - \ell^2$, $u$ becomes
\[
u = r - \frac{2(\ell^2 + 2M^2 - \frac{ma}{\omega})}{r - M} + 2M \ln(r - M), \quad (r > M). \quad (3.84)
\]
Concentrating on the case $M^2 > a^2 - \ell^2$, let us make another transformation:
\[
\vartheta = \arctan \left( \frac{\mu r}{\lambda} \right). \quad (3.85)
\]
With the new definitions
\[
P_1 = \phi_1 e^{-i\vartheta}, \quad P_2 = \phi_2 e^{i\vartheta}, \quad (3.86)
\]
equations (3.78) and (3.79) take the forms
\[
\frac{d\phi_1}{du} + 2i\omega \left( 1 - \frac{\lambda \mu \Delta}{4\omega(\lambda^2 + \mu^2 r^2)\Omega} \right) \phi_1 = 2\sqrt{\lambda^2 + \mu^2 r^2} \Omega \Delta^{1/2} \phi_2, \quad (3.87)
\]
\[
\frac{d\phi_2}{du} - 2i\omega \left( 1 - \frac{\lambda \mu \Delta}{4\omega(\lambda^2 + \mu^2 r^2)\Omega} \right) \phi_2 = 2\sqrt{\lambda^2 + \mu^2 r^2} \Omega \Delta^{1/2} \phi_1. \quad (3.88)
\]
Redefining the independent variable as

$$
\hat{u} = u - \frac{1}{4\omega} \arctan \left( \frac{\mu}{\lambda} \right),
$$

(3.89)
equations (3.87) and (3.88) can be simplified as

$$
\frac{d\phi_1}{d\hat{u}} + 2i\omega\phi_1 = W\phi_2,
$$

(3.90)

$$
\frac{d\phi_2}{d\hat{u}} - 2i\omega\phi_2 = W\phi_1,
$$

(3.91)
where

$$
W = \frac{2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2}}{(\lambda^2 + \mu^2 r^2)\Omega - \frac{4\mu\Delta}{4\omega}}.
$$

(3.92)
By further defining $Z_1 = \phi_1 + \phi_2$ and $Z_2 = \phi_1 - \phi_2$, equations (3.90) and (3.91) can be rewritten as

$$
\frac{dZ_1}{d\hat{u}} - WZ_1 = -2i\omega Z_2,
$$

(3.93)

$$
\frac{dZ_2}{d\hat{u}} + WZ_2 = -2i\omega Z_1.
$$

(3.94)
From equations (3.93) and (3.94), we obtain one-dimensional wave equations

$$
\frac{d^2Z_1}{d\hat{u}^2} + 4\omega^2 Z_1 = V_+ Z_1,
$$

(3.95)

$$
\frac{d^2Z_2}{d\hat{u}^2} + 4\omega^2 Z_2 = V_- Z_2,
$$

(3.96)
where the effective potentials

$$
V_\pm = W^2 \pm \frac{dW}{d\hat{u}}.
$$

(3.97)
We calculate the potentials as

$$
V_\pm (r) = \frac{2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2}}{r^2}\left[ 2(\lambda^2 + \mu^2 r^2)^{3/2} \Delta^{1/2} \pm 3\mu^2 r\Delta \mp (\lambda^2 + \mu^2 r^2) (r - M)\right]
$$

$$
\mp \frac{\Delta}{I} \left( 2\mu^2\Omega r + 2(\lambda^2 + \mu^2 r^2) r - \frac{\lambda\mu(r - M)}{2\omega} \right),
$$

(3.98)
where

$$
I = (\lambda^2 + \mu^2 r^2) \Omega - \frac{\lambda\mu \Delta}{4\omega}.
$$

(3.99)
We see that the effective potentials depend on the gravitomagnetic monopole moment $\ell$ via the functions $\Delta$ and $\Omega$, where the $\ell = 0$ case is discussed in [12]. We also report that, for $\mu = 0$, the potentials take the simple form

$$
V_\pm (r) = \frac{2\Delta^{1/2}\lambda}{\Omega^2} \left[ 2\lambda\Delta^{1/2} \pm (r - M) \mp \frac{2\Delta r}{\Omega} \right].
$$

(3.100)
4. Discussion

To see the asymptotic behavior of the potentials and the radial solutions and to expose the effect of the gravitomagnetic monopole moment, we can expand the potentials up to order $O(\ell^2)$. At this order, for the massive case (i.e. for $\mu \neq 0$), the potentials behave as

$$V_{\pm} \simeq 4\mu^2 - 8\mu^2 M \frac{1}{r} + \eta_{\pm} \frac{1}{r^2} + O\left(\frac{1}{r^3}\right),$$

where

$$\eta_{\pm} = \frac{2}{\mu}(\lambda + 4\mu^2 ma + 2x^2\omega - 2\mu^2 a^2\omega - 6\mu^2 \ell^2 \omega \pm M\mu\omega).$$

Here, the first term corresponds to the constant value of the potential at the asymptotic infinity. The second term represents the monopole-type (or Coulomb-type) potential, while the third term exhibits a dipole-type potential. As can be seen from the asymptotic expansion of the potentials, the effect of the NUT charge in the massive case appears in a dipole-type potential at the leading order. In the massless case (i.e. for $\mu = 0$), the potentials simply take the form

$$V_{\pm} \simeq 4\lambda^2 \frac{1}{r^2}$$

up to order $O(\ell^2)$. With the asymptotic form of the potentials given above, the radial equations (3.95) and (3.96) take the following forms:

$$\frac{d^2Z_1}{dr^2} + 4\omega^2 Z_1 = \left(4\mu^2 - 8\mu^2 M \frac{1}{r} + \eta_{+} \frac{1}{r^2}\right)Z_1,$$

$$\frac{d^2Z_2}{dr^2} + 4\omega^2 Z_2 = \left(4\mu^2 - 8\mu^2 M \frac{1}{r} + \eta_{-} \frac{1}{r^2}\right)Z_2,$$

whose solutions can be given by

$$Z_{1,2} = r^s e^{2i\sqrt{\omega^2 - \mu^2}r} \Phi(\tilde{c}_{\pm}, \tilde{d}_{\pm}; \xi),$$

in terms of confluent hypergeometric functions $\Phi(\tilde{c}_{\pm}, \tilde{d}_{\pm}; \xi)$, where $+$ corresponds to the solution for $Z_1$, while $-$ corresponds to the solution for $Z_2$. We also consider that $\omega > \mu$. Here,

$$s_{\pm} = \frac{1 + \sqrt{1 + 4 \eta_{\pm}}}{2},$$

and

$$\xi = -4i\sqrt{\omega^2 - \mu^2}r, \quad \tilde{c}_{\pm} = s_{\pm} - \frac{2i\mu^2 M}{\sqrt{\omega^2 - \mu^2}}, \quad \tilde{d}_{\pm} = 2s_{\pm}.\quad (4.8)$$

For the physically acceptable solutions, the inequality

$$1 + 4\eta_{\pm} \geq 0\quad (4.9)$$

should also be imposed. At the asymptotic infinity ($r \to \infty$), the behavior of the solutions (4.6) can be represented as

$$Z_{1,2} \sim a_1 e^{ip(r)} + a_2 e^{-ip(r)} + (b_1 e^{ip(r)} + b_2 e^{-ip(r)}) \frac{1}{r} + (c_1 e^{ip(r) + \frac{\pi}{2}} + c_2 e^{ip(r) - \frac{\pi}{2}}) \frac{1}{r^2} + O\left(\frac{1}{r^3}\right),$$

where

$$p(r) = \frac{\pi}{2} s_{\pm} - 2 \left(\sqrt{\omega^2 - \mu^2}r + \frac{\mu^2 M}{\sqrt{\omega^2 - \mu^2}} \ln(4\sqrt{\omega^2 - \mu^2}r)\right).$$

(4.10)
and $a_1, a_2, b_1, b_2, c_1$ and $c_2$ are constants coming from the asymptotic expansion of $\Phi(\vec{c}_\pm, \vec{d}_\pm; \xi)$. The asymptotic behavior (4.10) represents incident and reflected planar-type waves plus incident and reflected spherical-type waves at infinity.

Interestingly, in the critical case when $\omega = \mu$, radial wave equations accept the solution

$$Z_{1,2} = r^{1/2} J_v(\beta \sqrt{r}),$$

where $v_\pm = 1 - 2s_\pm$ and $\beta = 4\mu \sqrt{2M}$, and $J_v(\beta \sqrt{r})$ represents Bessel functions of order $v$.

It would also be interesting to observe the behavior of the potentials graphically. From expressions (3.98) and (3.100), it is obvious that the potentials become singular in the massive case ($\mu \neq 0$) when $I = 0$, and in the massless case ($\mu = 0$) when $\Omega = 0$. Moreover, they possess local extrema when $\frac{dv}{dr} = 0$, which leads to a very complicated algebraic equation to solve for the extremum distance of $r$. However, in order to understand the physical behavior of the potentials $V_\pm$ in the physical region $r > r_+$, we make two-dimensional and three-dimensional plots of the potentials for massive particles. It is also clear that the potentials depend on the physical parameters $a, M, m$ and $\ell$ implicitly in the metric functions. In all plots, we take the physical parameters $M = 1, \lambda = 1, m = 0.5$ and $a = 0.95$.

As can be seen in the two-dimensional graphs, the potentials are plotted as a function of radial distance $r$. Figures 1 and 2 describe the effective potentials $V_\pm$ for massive particles with rest mass $\mu = 0.12$ such that $\mu < \omega$. In the first graph, also taking the frequency constant ($\omega = 0.2$), we examine the effect of the NUT parameter $\ell$ by obtaining potential curves for some specific values of the gravitomagnetic monopole moment $\ell$. We see that, for sufficiently small values of $\ell$ including $\ell = 0$, potentials have sharp peaks in the physical region $r > r_+$. When the NUT parameter $\ell = 0$, the peak is seen to be maximum. It is also observed that while $\ell$ increases, the sharpness of the peaks decreases. While the peaks become smaller, the potentials still have some maxima. The peaks tend to disappear after a specific value of the NUT parameter. This means that, for small values of the NUT parameter, a massive Dirac particle moving in the region $r_+ < r < \infty$ may encounter sharp potential
barriers resulting in the decrease of its kinetic energy; but for sufficiently large values of the gravitomagnetic monopole moment, it may advance in the same region even without encountering any peaks. Potentials become bounded regardless of the value of $\ell$ and approach a constant value in the sufficiently large values of $r$ (or $r \to \infty$). In the second graph, we keep the gravitomagnetic monopole moment $\ell$ fixed ($\ell = 0.4$). In that case, we investigate the behavior of the potentials by obtaining potential curves for some specific values of the frequency $\omega$ that can take values in the range $(3.82)$ for $\hat{u}$ or $u$ to be single-valued. Again, one can clearly see that potentials have some local maxima in the low frequencies and the peaks are observed. While the frequency increases, the peaks again disappear as in figure 1 and potentials behave similarly in the sufficiently large distances. We also remark that, in the massless case ($\mu = 0$), the two-dimensional plots of the behavior of the potentials are similar to those in figures 1 and 2.
To observe the effect of the NUT parameter $\ell$ explicitly, we also realize three-dimensional plots of the potentials with respect to the NUT parameter $\ell$ and the radial distance $r$. As can be seen from the three-dimensional graphs 3 and 4, we observe a three-dimensional peak for small values of gravitomagnetic moment. As the value of the NUT parameter and radial distances increases, potentials level off. Again, in the massless case, the three-dimensional plots of the behavior of the potentials are similar to those in figures 3 and 4.

5. Conclusion

In this work, we examine the Dirac equation in a four-dimensional Kerr–Taub–NUT spacetime described by the physical parameters: the mass $M$, angular speed $a$ per unit mass and gravitomagnetic monopole moment $\ell$. By taking an axially symmetric ansatz for the spinor field, we obtain massive Dirac equations. By using Boyer–Lindquist coordinates, we explicitly work out the separability of the equations into radial and angular parts. We obtain angular and radial equations for arbitrary $\ell$. We find some exact solutions to the angular equations with and without the gravitomagnetic monopole moment $\ell$ and the rotation parameter $a$. We see that, for the massive Dirac equation, when the mass of the particle is equal to or twice the frequency of the spinor wavefunction, some angular solutions can be represented in terms of hypergeometric functions.

We also discuss the radial equations and obtain a wave equation with an effective potential. We obtain the asymptotic expansion of the potentials to observe the effect of the NUT parameter. We have seen that the effect of the NUT charge manifests itself in a dipole-type potential at the leading order. With the asymptotic form of the potentials, we obtain the solutions of the radial wave equations. We have seen that the radial wavefunctions physically represent plane wave-type and spherical wave-type solutions. Moreover, to realize the physical interpretations of the potentials, we make two- and three-dimensional plots of them. From the plots, it can be seen that the peak values of the potential barriers decrease and the potential curves level off while the NUT charge $\ell$ increases.

We believe that in order to better understand the physical significance of the NUT charge, a Dirac Hamiltonian should be constructed for the Dirac equation [31] and e.g. the effect of the NUT parameter on the neutrino oscillations can be examined. For future work, it can be further suggested that, by using a similar spectral method to that presented in [14], angular solutions can be represented in terms of spheroidal harmonics and eigenvalues can be solved.
numerically. Finally, we remark that one can study the massive Dirac equation for a charged Dirac particle in the background of the Kerr–Newman–Taub–NUT spacetime as well. These are devoted to future research.

Acknowledgment

We would like to thank the anonymous referee whose useful suggestions and comments led us to improve our paper.

Appendix. Derivation of the solutions of (3.68) and (3.70)

Equations (3.68) and (3.70) are of the type

\[(c_2 x + b_2) \frac{d^2 y}{dx^2} + (c_1 x + b_1) \frac{dy}{dx} + (c_0 x + b_0) y = 0,\]  

(A.1)

with \(c_1 = 0, b_1 = 0\). Under the transformation [32]

\[y = e^{\alpha z}, \quad z = \frac{1}{\Lambda} (x - \mu),\]  

(A.2)

the equation satisfied by \(\Omega(z)\) takes the form

\[z \frac{d^2 \Omega}{dz^2} + (\bar{b} - z) \frac{d\Omega}{dz} - \bar{a} \Omega = 0,\]  

(A.3)

where

\[\bar{a} = \frac{b_2 k^2 + b_0}{2 c_2 k}, \quad \bar{b} = 0.\]  

(A.4)

Here, \(\mu = -\frac{b_0}{b_2}\) and \(\Lambda = -\frac{1}{c_2}\). \(k\) can be calculated from \(c_2 k^2 + c_0 = 0\). A particular solution of A.3 (with \(\bar{b} = 0\)) can be written in terms of confluent hypergeometric function as

\[\Omega(z) = z \Phi(\bar{a} + 1, 2, z).\]  

(A.5)

In our case, for \(T_+\),

\[c_2 = 1, \quad b_2 = 1, \quad c_0 = 3 a^2 \omega^2, \quad b_0 = 3 a^2 \omega^2 + 2a \omega.\]

So the solution for \(T_+\) can be written as

\[T_+(x) = e^{i \sqrt{3} \omega \mu x} \Phi(\bar{a} + 1, 2, z),\]

with \(z = -2i \sqrt{3} \omega \mu (x + 1)\) and \(\bar{a} = -\frac{1}{\sqrt{3}}\). On the other hand, for \(T_-\),

\[c_2 = -1, \quad b_2 = 1, \quad c_0 = -3 a^2 \omega^2, \quad b_0 = 3 a^2 \omega^2 - 2a \omega.\]

In that case, the solution for \(T_-\) can be written as

\[T_-(x) = e^{-i \sqrt{3} \omega \mu x} \Phi(\bar{a} + 1, 2, \xi),\]

with \(\xi = -2i \sqrt{3} \omega \mu (x - 1)\).
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