Detecting a stochastic background of gravitational waves by correlating $n$ detectors

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Abstract
We discuss the optimal detection strategy for a stochastic background of gravitational waves in the case $n$ detectors are available. In the literature so far, only two cases have been considered: 2- and $n$-point correlators. We generalize these analyses to $m$-point correlators (with $m < n$) built out of the $n$ detector signals, obtaining the result that the optimal choice is to combine 2-point correlators. Correlating $n$ detectors in this optimal way will improve the (suitably defined) signal-to-noise ratio with respect to the $n = 2$ case by a factor equal to the fourth root of $n(n - 1)/2$. Finally, we give an estimation of how this could improve the sensitivity for a network of multi-mode spherical antennas.

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(Some figures in this article are in colour only in the electronic version)

1. Correlation of two detectors
As is well known [1], the sensitivity to a stochastic background signal can be greatly enhanced by correlating the output of two detectors. To show how this works it is useful to consider the cross correlation $S_{12}$ [2] between two detector outputs $S_1$ and $S_2$, defined by

$$S_{12} \equiv \int_{-T/2}^{T/2} dt \int_{-T/2}^{T/2} dt' S_1(t)S_2(t')Q(t-t') = \int_{-\infty}^{\infty} df \tilde{S}_1(f)\tilde{S}_2(f)\tilde{Q}(f),$$

(1)

where the filter function $Q(t)$ has been introduced. The cross correlation $S_{12}$ depends only on the time difference $t - t'$ as stationarity in both the signal and the noise is assumed. In the last equality, the Fourier transform of the signal and the limit $T \to \infty$ have been taken. For any finite $T$, $S_{12}$ is made of the sum of statistically independent random variables involving $\tilde{S}_1(f)$ and $\tilde{S}_2(f')$, which are correlated only over a frequency range $|f - f'| < 1/T$. Thus, as $S_{12}$ is the product of random variables, it is a random variable itself and it can be approximated...
by a Gaussian variable by virtue of the central limit theorem, even in the case of narrow band detectors, provided that $T$ is much larger than the inverse of the bandwidth. The same will be true in the case of the product of more than two random variables that will be considered later.

The outputs of two detectors can be split as $S_{1,2} = s_{1,2} + N_{1,2}$, $s_i$ the physical signal and $N_i$ the noise. The signal-to-noise ratio for the correlation of the two detectors at our disposal (this redundant notation will be useful later, where $m$-point correlators out of $n$ detectors will be considered) is given by

$$[\text{SNR}(2;2)]^2 = \frac{\langle S_{12} \rangle}{\sigma_{12}^2} = \frac{\langle S_{12} \rangle}{\left(\langle S_{12}^2 \rangle - \langle S_{12} \rangle^2\right)^{1/2}} = \frac{\langle s_1 s_2 \rangle}{\sqrt{\langle N_1^2 \rangle \langle N_2^2 \rangle}},$$

where $\langle S_{12} \rangle$ and $\sigma_{12}$ are, respectively, the average and the square root of the variance of the cross correlation. We have adopted the convention which makes the signal-to-noise ratio proportional to the metric perturbation $h$, so that in our notation $\text{SNR} \propto h$, as in [3], differently from [2] where $\text{SNR} \propto h^2$. To obtain the last equality in (2), we have made the basic assumptions that we will never drop throughout this paper: both the signal and the noise are Gaussian, they are statistically independent, stationary and with zero mean, $N_i \gg s_i$ and finally the noises of different detectors are completely uncorrelated.

The filter function $Q(t)$ appearing in (1) can be freely chosen in order to maximize the signal-to-noise ratio. The best choice is obtained in the standard way by imposing the functional variation of (1) with respect to $Q(t)$ equal to zero and solving it for $Q(t)$. To write down the explicit form of the filter function, it is necessary to introduce some further quantity. The signal can be usefully written as

$$s_i(t_i, x_i) = \int_{-\infty}^{\infty} df_i \int d\Omega_i \ h_A(f_i, \Omega_i) \exp(2\pi i f_i (t_i - \Omega_i x_i)) F^A(\Omega_i),$$

where $h_A$ is the Fourier transform of the metric perturbation with polarization $A$ and $F^A$ is the pattern function of the detectors, which encodes the information on its angular sensitivity, $x_i$ is the position of the $i$th detector and $\Omega_i$ is the wave arrival direction. Given the stochastic nature of the signal, the 2-point correlator (ensemble average of the Fourier components) of the metric perturbation can be parametrized as

$$\langle h(f_1, \Omega_1) h(f_2, \Omega_2) \rangle = \delta(f_1 + f_2) \frac{1}{4\pi} \delta^3(\Omega_1, \Omega_2) \frac{1}{2} S_h(f_1),$$

where the spectral function $S_h$ has been introduced. Analogously, a noise spectral function $S_{N,i}$ for the $i$th detector can be defined through

$$\langle N_i(f_1) N_j(f_2) \rangle = \delta_{ij} \delta(f_1 + f_2) \frac{1}{2} S_{N,i}(f_1).$$

The filter function which maximizes the signal-to-noise ratio is

$$Q(f) \propto \frac{S_h(f) \Gamma(f, x_{12})}{S_{N,1}(f) S_{N,2}(f)},$$

where the overlap function $\Gamma$ has been introduced. Its definition involves the relative distance and orientation of the two detectors

$$\Gamma(f_i, x_{ab}) = \frac{1}{4\pi} \int d^3 \Omega \sum_A F_A^{(a)}(\Omega) F_A^{(b)}(\Omega) \exp(2\pi i f_i \Omega(x_a - x_b)).$$

$F_A^{(a,b)}(\Omega)$ being the pattern function of the detector at site $a, b$ for a wave coming from direction $\Omega$. Inserting the optimal filter function (6) in (1) and (2), the explicit form of the signal-to-noise ratio for the correlation of two detectors is obtained

$$\text{SNR}(2, 2) = \left( T \int_{-\infty}^{\infty} df \Gamma^2(f, x_{12}) \frac{S_h^2(f)}{S_{N,1}(f) S_{N,2}(f)} \right)^{1/4},$$

(8)
Detecting a stochastic background of gravitational waves by correlating \( n \) detectors which gains in the case of two identical detectors with respect to the single detector case, as is well known, a factor roughly equal to \((T / \Delta f)^{1/4}\) multiplied by the overlap function, \( T \) being the experiment time and \( \Delta f \) the bandwidth.

2. Correlation of \( n \) detectors

One might now ask what can be gained by the correlation of several such detectors. A partial answer is obtained by generalizing (8) to the case of \( 2n \) detectors (the number of detectors must be even for the correlator not to vanish) [2]

\[
\text{SNR}(2n|2n) = \left( \frac{\langle S_1 \cdots S_{2n} \rangle}{\sqrt{\langle (S_1^2 \cdots S_{2n}^2) \rangle}} \right)^{1/2} = T^{1/4} \left( \int df_1 \cdots \int df_n \left( \prod_{i=1}^n S_h(f_i) \Gamma(f_i, x_{i,n+i}) \right)^2 + \text{perm} \right)^{1/4n},
\]

where, in our notation, \( \text{SNR}(i|j) \) is the signal-to-noise ratio given by \( \text{SNR}(i|j) = \langle S_{ij} \rangle^2 / \sigma_{ij}^2 \)

\[
\text{SNR}(i|j) = \left( \frac{\langle S_{ij} \rangle}{\sigma_{ij}} \right)^2, \quad \text{SNR}(i|j) = \left( \frac{\langle S_{ij} \rangle}{\sigma_{ij}} \right)^2. \tag{11}
\]

We now show that there exists a better way to treat data obtained from \( 2n \) detectors, as out of \( 2n \) detectors, \( 2m \)-correlators can be considered, for any \( m < n \). For \( m = 1 \), we can follow the analysis of [1] or [2] and consider all the possible pairs taken out of \( 2n \) detectors. For each detector pair, a mean value and a variance can be defined as usual

\[
\bar{S}_{ij} \equiv \langle S_{ij} \rangle = \bar{S}_2, \quad \sigma_{ij}^2 = \langle S_{ij}^2 \rangle - \bar{S}_{ij}^2, \tag{12}
\]

where the optimal filter function has been normalized so as to make the theoretical mean \( \langle S_{ij} \rangle = \bar{S}_2 \) equal for every pair. A \( \text{SNR}(i, j|2) \) of the type (8) can thus be assigned to each pair

\[
\text{SNR}(i, j|2)^2 = \frac{\bar{S}_{ij}}{\sigma_{ij}}. \tag{13}
\]

The best way to gather the information from all the pairings is to take a weighted average with weights \( \lambda_{ij} \)

\[
S_2 \equiv \frac{\sum_{i<j} \lambda_{ij} S_{ij}}{\sum_{i<j} \lambda_{ij}}, \tag{14}
\]

whose variance is

\[
\sigma_{S_2}^2 \equiv \langle S_2^2 \rangle - \langle S_2 \rangle^2 = \frac{\sum_{i<j} \lambda_{ij}^2 \sigma_{ij}^2}{\left( \sum_{i<j} \lambda_{ij} \right)^2}. 
\]
which is justified by the large noise approximation we are using that allows us to neglect non-diagonal terms like $\sigma_{ij}\sigma_{kl}$ (for $\{i, j\} \neq \{k, l\}$) compared to $\sigma_{ij}^2$. The signal-to-noise ratio obtained by combining the $2n$ detector outputs in pairs in this way is given by

$$\text{[SNR}(2|2n)]^4 = \frac{(S_{ij})^2}{\sigma_{ij}^2} = \frac{\left(\sum_{i<j} \lambda_{ij}\bar{S}_{ij}\right)^2}{\sum_{i<j} \lambda_{ij}^2 \sigma_{ij}^2}. \quad (15)$$

The best signal-to-noise ratio is obtained by choosing $\lambda_{ij} \propto \sigma_{ij}^{-2}$ (which correspond to weighing less the more noisy data) and it is

$$\text{[SNR}(2|n)]^4 = \sum_{i<j} \frac{S_{ij}^2}{\sigma_{ij}^2} = \sum_{i<j} \text{[SNR}(i, j|2)]^4,$$

where we have dropped the unnecessary hypothesis of the number of detectors being even. The optimal signal-to-noise ratio is thus given by the sum of terms like (8) (to the fourth power); note that we recover the time dependence of (9): $\text{SNR}(2|n) \propto T^{1/4} \ [1]$. For $n$ detectors with equal noise level, data collection time and overlap functions, we have

$$\text{SNR}(2|n) \propto [n(n - 1)]^{1/4}. \quad (17)$$

We now generalize the analysis of the combination of 2-point correlators to the case of $2m$-point correlators. Analogously to (14), we can define a linear combination $S_{2n}$ of the $(n)!/(2m)! (n - 2m)! 2m$-point correlators $S_{i_{1} \cdots i_{2m}}$ that it is possible to build out of $n$ detectors. Defining a signal-to-noise ratio of the type (9)

$$\text{SNR}(i_{1} \cdots i_{2m}|2m)\]^{2m} = \frac{\{S_{i_{1} \cdots i_{2m}}\}}{\sigma_{i_{1} \cdots i_{2m}}} = \frac{S_{2m}}{\sigma_{i_{1} \cdots i_{2m}}},$$

as a natural generalization of (13), we are led to consider the combination of the $2m$-correlators analogous to (14)

$$S_{2m} = \frac{\sum_{i_{1} < \cdots < i_{2m}} \lambda_{i_{1} \cdots i_{2m}} \bar{S}_{2m}}{\sum_{i_{1} < \cdots < i_{2m}} \lambda_{i_{1} \cdots i_{2m}}} , \quad \sigma_{2m}^2 = \frac{\sum_{i_{1} < \cdots < i_{2m}} \lambda_{i_{1} \cdots i_{2m}}^2 \sigma_{i_{1} \cdots i_{2m}}^2}{(\sum_{i_{1} < \cdots < i_{2m}} \lambda_{i_{1} \cdots i_{2m}})^2},$$

(with $i_{k} \in \{1 \cdots 2n\}$) so that the signal-to-noise ratio for $2m$-correlators can be written, for the optimal choice of weights $\lambda_{i_{1} \cdots i_{2m}} \propto \sigma_{i_{1} \cdots i_{2m}}^{-2}$, as

$$\text{[SNR}(2m|2n)]^{4m} = \frac{(S_{2m})^2}{\sigma_{2m}^2} = \sum_{i_{1} < \cdots < i_{2m}} \frac{\{S_{i_{1} \cdots i_{2m}}\}^2}{\sigma_{i_{1} \cdots i_{2m}}^2} = \sum_{i_{1} < \cdots < i_{2m}} \text{[SNR}(i_{1} \cdots i_{2m}|2m)]^{4m} \quad (18).$$

Each of the terms in the sum in the most rhs of (18) is on its own the sum of $(2m - 1)!$!! terms as shown in (9).

For equal noises, observation times and overlap functions, the scaling of the signal-to-noise ratio with respect to the number of detectors $n$, and with the order of the correlator $2m$, is given by

$$\text{SNR}(2m|n) \propto \left[(2m - 1)! \times \left(\begin{array}{c} n \\ 2m \end{array}\right)\right]^{4m}, \quad (19)$$

where the first factor comes from the number of contributions in each $\text{SNR}(i_{1} \cdots i_{2m}|2m)$ and the binomial coefficient from the possible choices of $2m$-ple out of $n$ detectors.

For any fixed $n$, the maximum is obtained always for $m = 1$ implying that the optimal signal-to-noise ratio is obtained by combining the detectors in pairs as in (16). In particular for a network made of a large number of detectors the signal-to-noise is expected to scale with the square root of the number of detectors as in (17).
3. Stochastic background

To parametrize conveniently the detector sensitivity to a stochastic background, it is useful to introduce the normalized spectral energy density of gravitational waves $\Omega_{gw}(f)$ defined as follows:

$$\Omega_{gw}(f) = \frac{1}{\rho_c} \frac{d\rho_{gw}(f)}{d\ln f},$$

where $\rho_c = (3h_{100}^2 H_0^2) / (8\pi G_N)$ is the critical energy density of the universe ($h_{100} H_0$ is the Hubble constant and $H_0 \equiv 100$ Mpc$^{-1}$ km s$^{-1}$) and $\rho_{gw}$ is the Fourier transform of the energy density in a gravitational wave. In terms of the spectral function $S_h$ introduced in (4), $\rho_{gw}$ can be written as

$$\frac{d\rho_{gw}(f)}{d\ln f} = \frac{\pi}{2G_N} f^3 S_h(f).$$

Using this formula it is possible to rewrite (8) in terms of $\Omega_{gw}$

$$[\text{SNR}(2\mid n)]^2 = \sqrt{\frac{2T}{3h_{100}^2 H_0^2}} \int_0^\infty df \frac{\Gamma^2(f, x_{1,2}) \Omega_{gw}^2(f)}{f^6 S_{n,1} S_{n,2}} \bigg[ \int_0^\infty df \frac{\Gamma^2(f, x_{1,2}) \Omega_{gw}^2(f)}{f^6 S_{n,1} S_{n,2}} \bigg]^{1/2},$$

which can be used in (16) to express the SNR$(2\mid n)$ as a function of $\Omega_{gw}$. The SNR necessary to claim detection can be computed once a false alarm rate $\alpha$ and a false dismissal rate $1 - \gamma$ are specified ($\gamma$ is called the detection rate), according to the Neymann–Pearson detection criterion (see [2] for the definition of false alarm and false dismissal rates). A natural choice is $\alpha = 5\%$ and $\gamma = 95\%$.

Even if it is useful in principle to correlate as many detectors as possible, in practice the number of high sensitivity detectors available with non-negligible overlap function is not too large. For instance, in [2] it is shown that with the current light interferometers it is possible to detect a stochastic background of gravitational waves provided that their normalized energy density satisfies the bound

$$h_{100}^2 \Omega_{gw}^{95\%,5\%} \gtrsim 6.5 \times 10^{-6},$$

for $\gamma = 95\%$ and $\alpha = 5\%$, constant $\Omega_{gw}$ and an observation period of 4 months. This bound is obtained by correlating the two LIGOs, VIRGO and GEO600, but a numerically similar one is obtained by correlating just the two LIGOs.

From the phenomenological point of view it is interesting to note that the total gravitational wave energy stored in a stochastic background cannot exceed the bound

$$\Omega_{gw} \lesssim 6 \times 10^{-6},$$

surprisingly numerically close to (23), otherwise the universe would expand too rapidly in the epoch of primordial nucleosynthesis, thus spoiling the beautiful agreement between theory and observation of the primordial abundance of light elements. The nucleosynthesis epoch corresponds to 10 s after the big bang or to a temperature of the order of a few MeV.

This limit does not apply to a background of gravitational waves produced after the nucleosynthesis epoch. There exist indeed astrophysical sources which may produce a continuous stochastic signal in the phenomenologically interesting frequency range. For instance rapidly rotating young neutron stars could be the source of gravitational radiation with an amplitude of $h_{100}^2 \Omega_{gw} \sim 10^{-8}$ for the frequency range 0.7–1 kHz [4].

A system which is able of producing gravitational wave to an observable level is represented by cusps and kinks in cosmic string network [5]. For some range of the parameters entering the description of the cosmic strings the energy of gravitational wave $\Omega_{gw}$ could be as high as $10^{-6}$ and it could be produced either before or after the nucleosynthesis epoch.
4. Network of antennas

We have shown in section 2 that with $n$ detectors available, the best strategy to detect a stochastic background is to correlate them in pairs, and in the case that noise levels and overlap functions have equal values the signal-to-noise ratio increases as $n^{1/2}$ for large $n$.

Let us now turn our attention to a case in which it is important to correlate a large number of detectors to increase the detector sensitivity.

An interesting case can be realized by multi-mode detectors like spherical antennas [6] as they have five quadrupolar modes which couple to a gravitational wave with generic incident direction. The correlation of such modes among several antennas can be considered, thus increasing the number of effective detectors available.

Anyway it has to be considered that modes of the same sphere cannot be correlated among themselves, as their noises are correlated and most of the mode pairs have negligible values of the overlap functions. Figure 1 shows the overlap reduction function for different pairs of modes. Denoting by $m$ the integer number labelling different quadrupolar modes ($-2 \leq m \leq 2$), the five overlap functions in the figure are obtained by correlating the $m = 0$ mode of a sphere and the five modes of a second sphere located at 100 km (the quantization axes have been chosen parallel to each other in order to maximize the sum of the overlap functions).

This makes the statistics a bit different from in (17), so that for a set of $n$ spherical antennas the analogue of (16) is

$$[\text{SNR}_{\text{sphere}}(2|n)]^4 = \sum_{i < j \leq n} \sum_{m, m' = 1}^5 [\text{SNR}(i, m; j, m'|2)]^4,$$

(25)
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Figure 2. Overlap reduction function, as a function of the frequency, between the $m = 0$ mode of a sphere at lat. 52.2° N, long. 4.5° E (quantization axis along the local meridian) and the five quadrupolar modes (labelled 20, 1c, 1s, 2c, 2s after [6]) of a second sphere located at 100 km (lat. 52.2° N, long. 6.0° E, quantization axis at 45° with respect to the local meridian).

where the indices $i, j$ run over different detectors and $m, m'$ run over the quadrupolar modes. If all the modes are equally noisy it is clear from figure 1 that every mode of the first sphere can be effectively correlated with just one or two modes of the second sphere, as it has almost vanishing overlap with the others. The situation does not improve by choosing a different orientation for the quantization axis of different spheres, as an example for an angle of 45° as shown in figure 2.

At the moment, the only operating spherical antenna is miniGRAIL (and another one will soon start working, the Brazilian Gravitational Wave Detector Mario Schenberg [8]), thus to obtain some numbers let us consider the present noise spectral function of miniGRAIL [7], whose diameter is 68 cm. It can presently reach a strain sensitivity $h_c \equiv \sqrt{S_n} \sim 10^{-20}$ Hz$^{-1/2}$, it has a resonant frequency of 2.9 kHz and a bandwidth of about 230 Hz. For these figures one obtains, for a single pair of detectors,

$$h_{100}^{2} = 4\pi^{2} \int_{0}^{\infty} \frac{df}{\Omega_{gw}} \frac{f^{2}}{S_{N}^{2}(f)} \sum_{i<j} \sum_{m,m'} \Gamma^{2}(f, x_{i,j})$$

for an observation time of 1 year. Using a set of identical spheres, equation (25) can be explicited as

$$[\text{SNR}_{\text{sphere}}(2|n)]^{4} = \frac{4\pi^{2}}{3H_{0}^{2}} \sum_{i<j} \sum_{m,m'} \Gamma^{2}(f, x_{i,j})$$

for the detector pair $i-j$, which is understood to depend also on $m$ and $m'$. The situation can be considerably improved by using larger spheres, with a consequent lower resonant frequency. We can estimate, for instance, that slightly improving the sensitivity to $h_c \simeq 10^{-21}$ Hz$^{-1/2}$,
a resonant frequency at, say, 300 Hz (and bandwidth 100 Hz) one can reach

\[ h_{100}^2 \Omega_{gw}^{0.5\%, 5\%} \simeq 4 \times 10^{-4} \times \left[ \frac{n(n - 1)}{2} \right]^{-1/2}, \]  

(28)

which is obtained by inverting (27) for a constant \( \Omega_{gw} \), thus allowing us to obtain \( \Omega_{gw} \simeq 3 \times 10^{-5} \) in the experiment bandwidth for a set of \( n = 10 \) detectors. We note that it is important to have a not too high resonant frequency for detector correlation, as overlap functions go to zero at \( f \gtrsim \frac{1}{\pi L} \), \( L \) being the detector separation. This is still far from light interferometers, see equation (23) and the phenomenological bound given by equation (24), but the effect of the multiple correlation is quantitatively important, so it is not excluded that once higher sensitivity can be achieved the correlation effect will be crucial for detection.

The mechanism of sensitivity enhancement will actually not work if a sphere is correlated with an interferometer as only one mode of the sphere can effectively be correlated with an interferometer. This can be seen in figure 3, which shows the overlap function between VIRGO and the five quadrupolar modes of a hypothetical sphere placed in Rome. A sphere like the one with the characteristics leading to (28) has a narrower bandwidth than an interferometer, but similar sensitivity in its bandwidth, so correlating a sphere with an interferometer would lead to a result equal to (28) but for the factor in square brackets, as one cannot take advantage of the correlation of several modes of the same sphere: correlation with an interferometer would just add one more single-mode detector.

5. Conclusion

We have analysed the utility of considering multiple detector correlators to detect a stochastic background of gravitational waves. The main result of the paper is the demonstration...
that the best way to correlate the outputs of different detectors is in pairs, no matter how large the number of detectors, instead of taking $m$-correlators with $m > 2$. Finally, as a potentially interesting application of this result we applied this strategy to a set of identical spheres, showing that correlation of several pairs of detectors is important in increasing the sensitivity.

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References

[1] Christensen N 1992 *Phys. Rev. D* **46** 5250
[2] Allen B and Romano J D 1999 *Phys. Rev. D* **59** 102001
[3] Maggiore M 2000 *Phys. Rep.* **331** 283
[4] Ferrari V, Matarrese S and Schneider R 1999 *Mon. Not. R. Astron. Soc.* **303** 258
[5] Damour T and Vilenkin A 2005 *Phys. Rev. D* **71** 063510
[6] Zhou C Z and Michelson P F 1995 *Phys. Rev. D* **51** 2517
[7] de Waard A et al 2005 *Class. Quantum Grav.* **22** S215
[8] Aguiar O D et al 2005 *Class. Quantum Grav.* **22** S209