A Phragmén-Lindelöf principle for slice regular functions

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1 Introduction

The celebrated 100-year old Phragmén-Lindelöf theorem [15,16] is a far reaching extension of the maximum modulus theorem for holomorphic functions that in its simplest form can be stated as follows:

**Theorem 1.1.** Let $\Omega \subset \mathbb{C}$ be a simply connected domain whose boundary contains the point at infinity. If $f$ is a bounded holomorphic function on $\Omega$ and $\limsup_{z \to z_0} |f(z)| \leq M$ at each finite boundary point $z_0$, then $|f(z)| \leq M$ for all $z \in \Omega$.

The term Phragmén-Lindelöf also applies to a number of variations of this result, which guarantee a bound for holomorphic functions, when conditions are known on their growth. The two most famous variations deal with functions which are holomorphic in an angle or in a strip, and can be stated as follows (see, for instance, [2, 13] as well as [1, 8]).

**Theorem 1.2.** Let $f$ be a holomorphic function on an angle $\Omega$ of opening $\pi$. Suppose $f$ is continuous up to the boundary and such that, for some $\rho < \alpha$, $|f(z)| \leq \exp(|z|^\rho)$ asymptotically. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial \Omega$ then $|f| \leq M$ in $\Omega$.

**Theorem 1.3.** Let $f$ be a holomorphic function on a strip $\Omega$ of width $2\gamma$, continuous up to the boundary. Suppose that $|f(z)| \leq N \exp(e^{k|z|})$ in $\Omega$ for some positive constants $N$ and $k < \frac{\pi}{2\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial \Omega$ then $|f| \leq M$ in $\Omega$. 

In some recent papers [9, 10, 11, 12] there has been a resurgence of interest in Phragmén-Lindelöf type theorems. Specifically, [9, 10] consider solutions of suitable partial differential equations while [11, 12] deal with the case of functions of a hypercomplex variable. In the present article we obtain the analogs of theorems 1.1, 1.2 and 1.3 for slice regular functions, a class of functions of a quaternionic variable introduced in [6, 7] and studied in subsequent papers (for a survey, see [5]).

In section 2 we will provide the necessary background about the theory of slice regular quaternionic functions. In section 3 we give direct proofs of quaternionic analogs of theorems 1.1 and 1.2. Finally, in section 4 we use a different approach, which exploits the intrinsic nature of slice regular functions, to extend our results.

Acknowledgements. The first two authors are grateful to Chapman University for the hospitality during the preparation of this paper.

2 Preliminaries

Let $\mathbb{H}$ denote the skew field of quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the $x_i$ are real, and $i, j, k$ are such that

\[ i^2 = j^2 = k^2 = -1, \]

\[ ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \]

We set

\[ \text{Re}(q) = x_0, \quad \text{Im}(q) = ix_1 + jx_2 + kx_3, \quad |q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}. \]

$\text{Re}(q)$, $\text{Im}(q)$ and $|q|$ are called the real part, the imaginary part and the module of $q$, respectively. The quaternion

\[ \bar{q} = \text{Re}(q) - \text{Im}(q) = x_0 - ix_1 - jx_2 - kx_3 \]

is called the conjugate of $q$ and satisfies

\[ |q| = \sqrt{\bar{q}q} = \sqrt{q\bar{q}}. \]

The inverse of any element $q \neq 0$ is given by

\[ q^{-1} = \frac{\bar{q}}{|q|^2}. \]

We denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.

\[ \mathbb{S} = \{q = ix_1 + jx_2 + kx_3 : x_1^2 + x_2^2 + x_3^2 = 1\} \]

so that every quaternion $q$ which is not real (i.e. with $\text{Im}(q) \neq 0$) can be written as $q = x + Iy$ for $x = \text{Re}(q), y = |\text{Im}(q)|$ and $I = \frac{\text{Im}(q)}{|\text{Im}(q)|} \in \mathbb{S}$. Also, if we set $r = |q|$ then $q = re^{I\vartheta}$ for some $I \in \mathbb{S}$ and $\vartheta \in \mathbb{R}$.
Beginning with the seminal papers of Fueter [3, 4], many mathematicians have developed theories of holomorphicity in the quaternionic setting (for an overview, see the introduction of [5]). More recently, in [6, 7], the authors proposed a new notion of holomorphicity (called \textit{slice regularity}) for quaternion-valued functions of a quaternionic variable. Unlike Fueter’s, this theory includes the polynomials and the power series of the quaternionic variable $q$ of the type $\sum_{n \geq 0} q^n a_n$, with coefficients $a_n \in \mathbb{H}$. Furthermore, analogs of most of the fundamental properties of holomorphic functions of one complex variable can be proven in this new setting (see also [5] and references therein).

**Definition 2.1.** Let $\Omega$ be an open set in $\mathbb{H}$. A function $f : \Omega \to \mathbb{H}$ is said to be \textit{slice regular} if, for every $I \in \mathbb{S}$, its restriction $f_I$ to the complex line $L_I = \mathbb{R} + RI$ passing through the origin and containing 1 and $I$ satisfies

$$\overline{\partial}_I f(x + yI) := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f_I(x + yI) = 0,$$

in $\Omega_I = \Omega \cap L_I$.

The following result is a key tool in the study of slice regular functions, and it will be used extensively in section [9].

**Lemma 2.2** (Splitting Lemma). If $f$ is a slice regular function on an open set $\Omega$ then, for every $I \in \mathbb{S}$ and every $J \perp I$ in $\mathbb{S}$, there exist two holomorphic functions $F, G : \Omega_I \to L_I$ such that

$$f_I(z) = F(z) + G(z)J$$

for all $z \in \Omega_I$.

We now identify a class of domains that naturally qualify as domains of definition of regular functions.

**Definition 2.3.** Let $\Omega$ be a domain in $\mathbb{H}$. We say that $\Omega$ is a slice domain if $\Omega \cap \mathbb{R}$ is non empty and if $\Omega_I = \Omega \cap L_I$ is a domain in $L_I$ for all $I \in \mathbb{S}$.

Indeed, an analog of the identity principle holds for slice regular functions on slice domains.

**Theorem 2.4.** Let $f, g : \Omega \to \mathbb{H}$ be slice regular functions on a slice domain $\Omega$. If $f$ and $g$ coincide in $T \subseteq \Omega$ and if there exists $I \in \mathbb{S}$ such that $T_I = T \cap L_I$ has an accumulation point in $\Omega_I$, then $f$ and $g$ coincide in $\Omega$.

Furthermore, analogs of the two classic statements of the maximum modulus principle hold.

**Theorem 2.5.** Let $f : \Omega \to \mathbb{H}$ be a slice regular function on a slice domain $\Omega$. If $|f|$ has a relative maximum point in $\Omega$, then $f$ is constant in $\Omega$.

**Corollary 2.6.** Let $f : \Omega \to \mathbb{H}$ be a slice regular function on a bounded slice domain $\Omega$. If $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial \Omega$ then $|f| \leq M$ in $\Omega$. 

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Due to the non-commutativity of $\mathbb{H}$, pointwise multiplication and composition do not preserve slice regularity in general. Nevertheless, slice regularity is preserved for the following class of functions.

**Definition 2.7.** Let $f : \Omega \to \mathbb{H}$ be a slice regular function. We say that $f$ is a slice preserving function if $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$.

**Proposition 2.8.** Let $f, g : \Omega \to \mathbb{H}$ be slice regular functions. If $f$ is a slice preserving function then the product $f \cdot g$ is slice regular.

**Proposition 2.9.** Let $f : \Omega \to \Omega' \subseteq \mathbb{H}$ and $g : \Omega' \to \mathbb{H}$ be slice regular functions. If $f$ is a slice preserving function then the composition $g \circ f$ is slice regular.

### 3 The Phragmén-Lindelöf principle

In this section we will give a direct proof of the Phragmén-Lindelöf principle for slice regular functions defined on suitable domains $\Omega$ in the quaternionic space $\mathbb{H}$. We will also study the special case in which $\Omega$ is a cone.

As in the complex case, the quaternionic Phragmén-Lindelöf principle generalizes the maximum modulus principle (2.6) to unbounded domains. Let $\hat{\mathbb{H}} = \mathbb{H} \cup \{\infty\}$ denote the Alexandroff compactification of $\mathbb{H}$. We define the extended boundary $\partial_{\infty} \Omega$ of any $\Omega \subseteq \hat{\mathbb{H}}$ to be the boundary of the closure of $\Omega$ in $\hat{\mathbb{H}}$. As customary, we will denote by $\partial \Omega = \partial_{\infty} \Omega \ \{\infty\}$ the finite boundary of $\Omega$.

**Theorem 3.1 (Phragmén-Lindelöf principle).** Let $\Omega \subset \mathbb{H}$ be a domain whose extended boundary contains the point at infinity and suppose that there exist a real point $t \in \mathbb{R} \cap \Omega$ such that $\Omega \setminus (-\infty, t]$ (or $\Omega \setminus [t, +\infty)$) is a slice domain. If $f$ is a bounded slice regular function on $\Omega$ and $\limsup_{q \to q_0} |f(q)| \leq M$ for all $q_0 \in \partial \Omega$, then $|f(q)| \leq M$ for all $q \in \Omega$.

**Proof.** Since $q \mapsto q+t$ and $q \mapsto -q$ are slice preserving functions, by proposition 2.9 we can assume that $t = 0$ and $\Omega \setminus (-\infty, 0]$ is a slice domain. Choose $r > 0$ such that the closure of $B_r = B(0, r)$ is contained in $\Omega$ and let $\omega_r(q) = q^{-1}r$ for $q \neq 0$. Notice that $|\omega_r| < 1$ in $\mathbb{H} \setminus B_r$, that $|\omega_r| = 1$ on $\partial B_r$, and that $\omega_r$ is a slice preserving regular function.

For all $q \in \mathbb{H} \setminus (-\infty, 0]$, define the principal logarithm of $q$ as

$$Log(q) = \ln |q| + \arccos \left( \frac{Re(q)}{|q|} \right) \frac{Im(q)}{|Im(q)|}.$$ 

Notice that the principal logarithm is a slice regular function and it is slice preserving. By proposition 2.9 setting $\omega_r^q(q) := e^{iLog_q(q)}$ for all $q \in \mathbb{H} \setminus (-\infty, 0]$ defines a slice regular function. Finally, by proposition 2.8 the product $\omega_r^q f$ is a slice regular function on $\Omega' = \Omega \setminus (\overline{B_r} \cup (-\infty, r))$, which by hypothesis is a slice domain when $r$ is sufficiently small. The behavior of $|\omega_r^q f|$ on the extended boundary $\partial_{\infty} \Omega' = \{\infty\} \cup \partial \Omega \cup \partial B_r \cup (\Omega \cap (-\infty, r))$ is the following:
1. \( \limsup_{q \to \infty} |\omega^\delta f(q)| = \limsup_{q \to \infty} |f(q)| \frac{r^\delta}{|q|} = 0, \)
2. \( \limsup_{q \to q_0} |\omega^\delta f(q)| < \limsup_{q \to q_0} |f(q)| \leq M \) for all \( q_0 \in \partial \Omega, \)
3. \( \limsup_{q \to q_0} |\omega^\delta f(q)| = |f(q_0)| \leq \max_{\partial B_r} |f| =: M_r \) for all \( q_0 \in \partial B_r, \)
4. \( \limsup_{q \to q_0} |\omega^\delta f(q)| \leq \sup_{\Omega \cap (-\infty, r]} \frac{r^\delta}{|q|} |f(q)| =: N \) for all \( q_0 \in \Omega \cap (-\infty, r]. \)

Let us prove that \( N \) is finite. Choose \( \{a_n\}_{n \in \mathbb{N}} \subset \overline{\Omega} \cap (-\infty, r] \) such that \( \lim_{n \to \infty} \frac{r^\delta}{|a_n|} |f(a_n)| = N. \) If \( N = 0 \), there is nothing to prove. Otherwise, by point 1, \( \{a_n\}_{n \in \mathbb{N}} \) must be bounded. By possibly extracting a subsequence, we may suppose \( \{a_n\}_{n \in \mathbb{N}} \) to converge to some \( q_0 \in \overline{\Omega} \cap (-\infty, r]. \) If \( q_0 \in \partial \Omega \) then \( N \leq M \) by hypothesis. Else \( q_0 \in \Omega \) and \( N = \frac{r^\delta}{|q_0|} |f(q_0)|. \)

As a consequence of points 1-4, \( \limsup_{q \to q_0} |\omega^\delta f| \leq \max\{M, M_r, N\} \) for all \( q_0 \in \partial \infty \Omega' \) and, by an easy application of the maximum modulus principle \( 2.6 \) \( |\omega^\delta f| \leq \max\{M, M_r, N\} \) in \( \Omega'. \)

Now let us prove that \( N \leq \max\{M, M_r\} \). Suppose by contradiction that the opposite inequality holds. In particular \( N > M \) and (as we explained above) there exists \( q_0 \in \Omega \cap (-\infty, r] \) such that \( N = \frac{r^\delta}{|q_0|} |f(q_0)|. \) In a ball \( B(q_0, \varepsilon) \) contained in \( \Omega \setminus \overline{B_r} \), we define a new branch of logarithm \( \log \) by letting

\[
\log(q) = \ln |q| + \left[ \arccos \left( \frac{\text{Re}(q)}{|q|} \right) - \pi \right] \frac{\text{Im}(q)}{|\text{Im}(q)|}.
\]

As before, the function \( g = e^{i\log} f \) is slice regular in \( B(q_0, \varepsilon) \) and \( |g(q)| = \frac{r^\delta}{|q|} |f(q)| \) for all \( q \in B(q_0, \varepsilon). \) As a consequence, \( |g(q_0)| \geq |g(q_0)| \) for all \( q \in B(q_0, \varepsilon). \) Indeed:

1. for all \( q \in (q_0 - \varepsilon, q_0 + \varepsilon), \) \( |g(q)| \leq \sup_{\Omega \cap (-\infty, r]} \frac{r^\delta}{|q|} |f(q)| = N = |g(q_0)|; \)
2. for all \( q \in B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon), \) we proved \( |g(q)| = |\omega^\delta(q)f(q)| = \frac{r^\delta}{|q|} |f(q)| \leq \max\{M, M_r, N\} = N = |g(q_0)|. \)

Hence \( |g| \) has a maximum at \( q_0 \) and \( g \) must be constant. Therefore \( |\omega^\delta f| = |g| \equiv N \) in \( B(q_0, \varepsilon) \setminus (q_0 - \varepsilon, q_0 + \varepsilon). \) In particular \( \omega^\delta f, \) which is a slice regular function on the slice domain \( \Omega', \) has an interior maximum point. As before, the maximum modulus principle \( 2.6 \) yields that \( \omega^\delta f \) must be constant. As a consequence, there exists a constant \( c \) such that \( f(q) = q^\delta c \) in \( \Omega', \) a contradiction with the hypothesis that \( f \) is bounded.

So far, we proved that \( |\omega_r^\delta| \leq \max\{M, M_r\} \) in \( \Omega \setminus \overline{B_r}. \) We deduce that

\[
|f| \leq \frac{\max\{M, M_r\}}{\omega^\delta}
\]

in \( \Omega \setminus \overline{B_r} \) and letting \( \delta \to 0^+ \) we conclude that \( |f| \leq \max\{M, M_r\} \) in \( \Omega \setminus \overline{B_r}. \)
If we let \( r \to 0^+ \) we obtain \( |f(q)| \leq \max\{M, |f(0)|\} \) for all \( q \in \Omega \setminus \{0\} \), hence for all \( q \in \Omega \). Finally, we prove that \( |f(0)| \leq M \): if it were not so, then \( |f| \) would have a maximum at 0, a contradiction by the maximum modulus principle.

We now tackle the case in which \( \Omega \) is the circular cone
\[
C(\varphi) = \{re^{i\theta} : r > 0, |\theta| < \varphi/2, I \in S\}
\]
for some \( \varphi < 2\pi \). Such cones certainly satisfy the hypotheses of theorem 3.1. Moreover, we can prove that it is not necessary to suppose that \( f \) is bounded as long as the opening \( \varphi \) of the cone is suitably related to the growth order of \( f \), defined as follows. If \( f \) is a slice regular function on \( \Omega = C(\varphi) \), continuous up to the boundary, we set
\[
M_f(r, \Omega) = \max\{|f(q)| : q \in \overline{\Omega}, |q| = r\}
\]
and define the order \( \rho \) of \( f \) as
\[
\rho = \limsup_{r \to +\infty} \frac{\ln^+ \ln^+ M_f(r, \Omega)}{\ln r}.
\]

**Theorem 3.2** (Phragmén-Lindelöf principle for circular cones). Let \( f \) be a slice regular function in \( C(\frac{\pi}{\alpha}) \), continuous up to the boundary. Suppose the order \( \rho \) of \( f \) to be strictly less than \( \alpha \). If there exists an \( M \geq 0 \) such that \( |f| \leq M \) in \( \partial C(\frac{\pi}{\alpha}) \) then \( |f| \leq M \) in \( C(\frac{\pi}{\alpha}) \).

**Proof.** Choose \( \gamma \) such that \( \rho < \gamma < \alpha \). For \( q \in C\left(\frac{\pi}{\alpha}\right) \) we define \( \omega(q) = e^{q^\gamma} \) with \( q^\gamma = e^{\gamma \operatorname{Log}(q)} \) (where \( \operatorname{Log}(q) \) is the principal logarithm of \( q \)). For all \( \delta > 0 \) we set \( \omega^\delta(q) = e^{-\delta q^\gamma} \) and we have that
\[
|\omega^\delta(re^{i\theta})| = e^{-\delta r^\gamma \cos(\gamma \theta)}.
\]
For \( -\frac{\pi}{\alpha} < \theta < \frac{\pi}{\alpha} \) and \( \rho < \rho_1 < \gamma \) the following holds asymptotically:
\[
|\omega^\delta(re^{i\theta}) f(re^{i\theta})| < e^{\rho_1 - \delta r^\gamma \cos(\gamma \theta)}.
\]
Since \( \gamma < \alpha \), we have \( -\frac{\pi}{\alpha} < \gamma \theta < \frac{\pi}{\alpha} \) so that \( \cos(\gamma \theta) > 0 \). Since \( \rho_1 < \gamma \) we conclude that in \( C\left(\frac{\pi}{\alpha}\right) \)
\[
\lim_{q \to \infty} |\omega^\delta(q) f(q)| = 0.
\]
Since for all \( q \in \partial C\left(\frac{\pi}{\alpha}\right) \) we have \( |\omega^\delta(q) f(q)| < |f(q)| \leq M \), we conclude that \( \lim_{q \to q_0} |\omega^\delta(q) f(q)| \leq M \) for all \( q_0 \in \partial_{\infty} C\left(\frac{\pi}{\alpha}\right) \). Applying the maximum modulus principle we get \( |\omega^\delta f| \leq M \) in \( C\left(\frac{\pi}{\alpha}\right) \). The inequality \( |f| \leq M \omega^{-\rho}, \) which holds for all \( \delta > 0 \), yields that \( |f| \leq M \) in \( C\left(\frac{\pi}{\alpha}\right) \).

In the next section we will offer an alternative proof extending theorem 3.2 to a larger class of domains.
4 A slicewise approach

The proofs we gave in the previous section are intrinsic to the quaternionic setting. However, in the theory of slice regular functions it is often possible to use a different technique. Specifically (see e.g. [7]), one can apply the splitting lemma 2.2 to reduce to the case of holomorphic functions of one complex variable. In this section, we employ this technique to give different proofs of theorems 3.1 and 3.2. In fact, this allows slight variations of the hypotheses. We also prove two other results, which we were unable to obtain using a direct approach.

Theorem 4.1. Let \( \Omega \subset \mathbb{H} \) be a domain whose extended boundary contains the point at infinity and such that, for all \( I \in \mathbb{S} \), \( \Omega_I = \Omega \cap L_I \) is simply connected. If \( f \) is a bounded slice regular function on \( \Omega \) and \( \limsup_{q \to q_0} |f(q)| \leq M \) for all \( q_0 \in \partial \Omega \), then \( |f(q)| \leq M \) for all \( q \in \Omega \).

Proof. Suppose that there exists \( p \in \Omega \) such that \( |f(p)| > M \). Possibly multiplying \( f \) by the constant \( \frac{f(p)}{|f(p)|} \), we may suppose \( f(p) > 0 \). Let \( I \in \mathbb{S} \) be such that \( p \in L_I \), choose \( J \in \mathbb{S} \) such that \( J \perp I \) and let \( F, G : \Omega_I = \Omega \cap L_I \to L_I \) be holomorphic functions such that \( f_I = F + GJ \) (see lemma 2.2). Then \( f_J(p) = F(p) \). On the other hand, since \( |F| \leq |f_I| \leq M \) in \( \partial \Omega_I \subset \partial \Omega \) and \( |F| \leq |f_J| \) is bounded in \( \Omega_I \), we must have \( |F| \leq M \) in \( \Omega_I \) by the complex Phragmén-Lindelöf principle [1.4].

Remark 4.2. We required \( \Omega_I \) to be simply connected, as in the classic (complex) Phragmén-Lindelöf principle [1.4]. Notice however that this hypothesis can be weakened (see exercise 1 in [8]).

A similar proof allows us to extend theorem 3.2 to a larger class of domains.

Definition 4.3. We call a slice domain \( \Omega \subset \mathbb{H} \) an angular domain if, for all \( I \in \mathbb{S} \), \( \Omega_I \) is an angle \( \{re^{i(\zeta_I + \varphi)} : r > 0, |\varphi| < \varphi_I/2 \} \) for some \( \zeta_I, \varphi_I \) with \( \zeta_I \in \mathbb{R}, 0 < \varphi_I < 2\pi \). The opening of \( \Omega \) is defined to be \( \sup_{I \in \mathbb{S}} |\varphi_I| \).

The following proposition shows, once again, the surprising geometrical properties of the quaternions.

Proposition 4.4. Let \( \Omega \) be an open subset of \( \mathbb{H} \) such that, for all \( I \in \mathbb{S} \), \( \Omega_I \) is an angle \( \{re^{i(\zeta_I + \varphi)} : r > 0, |\varphi| < \varphi_I/2 \} \). If \( I \mapsto \zeta_I \) and \( I \mapsto \varphi_I \) are continuous in \( \mathbb{S} \) then \( \Omega \) is automatically a slice domain.

Proof. In order to prove our assertion it suffices to show that at least one slice \( \Omega_I \) contains a real half line. Notice that, for all \( I \in \mathbb{S} \), \( \Omega_I = \Omega_{-I} \). Hence \( \varphi_I = \varphi_{-I} \) and \( \zeta_I = 2k\pi - \zeta_{-I} \) for some \( k \in \mathbb{Z} \). Since

\[
I \mapsto (\zeta_I, \varphi_I)
\]

is a continuous function from \( \mathbb{S} \cong S^2 \) to \( \mathbb{R}^2 \), by the Borsuk-Ulam theorem (see Corollary 9.3 in [14]) there exist two antipodal points of \( \mathbb{S} \) having the same image. In particular, there exists a \( J \in \mathbb{S} \) such that \( \zeta_J = \zeta_{-J} \) and we conclude that \( \zeta_J = k\pi \) for some \( k \in \mathbb{Z} \). \( \square \)
The order of a slice regular function $f$ on an angular domain $\Omega$ is defined by equations (1) and (2) as in the case of circular cones.

**Theorem 4.5.** Let $f$ be a slice regular function on an angular domain $\Omega$ of opening $\frac{\pi}{p}$. Suppose $f$ is continuous up to the boundary and has order $\rho < \alpha$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial \Omega$ then $|f| \leq M$ in $\Omega$.

The proof of theorem 4.5 is completely analogous to that of theorem 4.1 and it makes use of the Phragmén-Lindelöf principle for complex angles 1.2. As in the complex case, the hypothesis that $\rho < \alpha$ cannot be weakened. Indeed, we have the following.

**Example 4.6.** We can define a slice regular function $f$ of order $\rho > 0$ on $C \left( \frac{\pi}{p} \right)$ by setting $f(q) = e^{q^\rho}$ where $q^\rho = e^{\rho \log(q)}$. We notice that, for all $q = re^{\pm I \frac{\pi}{p}} \in \partial C \left( \frac{\pi}{p} \right)$, $|f(q)| = |\exp(r^\rho e^{\pm I \frac{\pi}{p}})| = |\exp(\pm r^\rho)| = 1$, while the function $f$ is unbounded in $C \left( \frac{\pi}{p} \right)$.

Nevertheless, as in the complex case, when a function $f$ has order $\rho$ in angular domain of opening $\frac{\pi}{p}$ we can control the growth of $f$ in terms of its type $\sigma$, defined as

$$
\sigma = \limsup_{r \to +\infty} \frac{\ln^+ M_f(r, \Omega)}{r^\rho}.
$$

(3)

**Theorem 4.7.** Let $\Omega$ be an angular domain with $\Omega_I = \{re^{I(\zeta_I + \vartheta)} : r > 0, |\vartheta| < \varphi_I/2\}$ for all $I \in S$. Let $f$ be a slice regular function of order $\rho$ and type $\sigma$ on $\Omega$, continuous up to the boundary. If the opening of $\Omega$ is not greater than $\frac{\pi}{p}$ and $|f|$ is bounded by $M$ in $\partial \Omega$, then for all $I \in S$

$$
|f(re^{I(\zeta_I + \vartheta)})| \leq M e^{\sigma r^\rho \cos(\rho \vartheta)}
$$

(4)

for $r > 0$ and $|\vartheta| < \varphi_I/2$.

**Proof.** Suppose that there exists $p = re^{I(\zeta_I + \vartheta)} \in \Omega_I$ such that

$$
|f(p)| > M e^{\sigma r^\rho \cos(\rho \vartheta)}.
$$

As in the proof of theorem 4.1 we may suppose $f(p) > 0$. Choosing $J \perp I$ and holomorphic functions $F, G : \Omega_I \to L_I$ such that $f_I = F + GJ$, we have $f_I(p) = F(p)$. Now, $|F| \leq |f_I| \leq M$ in $\partial \Omega_I$ and $F$ has order less than or equal to $\rho$ and type less than or equal to $\sigma$ in $\Omega_I$. By theorem 22 in [13], we conclude $|F(p)| \leq M e^{\sigma r^\rho \cos(\rho \vartheta)}$, a contradiction.

An analogous proof allows us to derive the quaternionic version of theorem 4.7.

**Definition 4.8.** We call a slice domain $\Omega \subset S$ a strip domain if, for all $I \in S$, there exist a line $\ell_I$ in $L_I$ and a positive real number $\gamma_I$ such that $\Omega_I$ is the strip $\{z \in L_I : |z - \ell_I| < \gamma_I/2\}$. The width of $\Omega$ is defined to be $\sup_{I \in S} |\gamma_I|$.
Theorem 4.9. Let $f$ be a slice regular function on a strip domain $\Omega$ of width $\gamma$, continuous up to the boundary. Suppose that $|f(q)| \leq N \exp(e^{k|q|})$ in $\Omega$ for some positive constants $N$ and $k < \frac{\pi}{\gamma}$. If there exists an $M \geq 0$ such that $|f| \leq M$ in $\partial \Omega$ then $|f| \leq M$ in $\Omega$.

The slicewise approach adopted in this section has an important bearing for entire functions, i.e. for slice regular functions on $\Omega = \mathbb{H}$. Indeed, if we define the order and the type of the entire function $f$ by equations (1), (2) and (3), then the quaternionic Liouville theorem proven in [7] generalizes as follows.

Theorem 4.10. Let $f$ be a quaternionic entire function of first order at most, having type 0. In other words, for all $\varepsilon > 0$ we suppose $|f(q)| < e^{\varepsilon|q|}$ when $|q|$ is large enough. If, for some $I \in \mathbb{S}$, the plane $L_I$ contains a line on which $|f|$ is bounded then $f$ is constant.

Proof. Let $J \in \mathbb{S}$ be orthogonal to $I$ and let $F, G : L_I \rightarrow L_I$ be holomorphic functions such that $f_I = F + GJ$. By the corollary to theorem 22 in [13], $F$ and $G$ are constant. Hence $f_I$ is constant and, by the identity principle 2.4, $f$ must be constant too. \qed

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