ON THE NUMBER OF POINTS OF ALGEBRAIC SETS OVER FINITE FIELDS

GILLES LACHAUD AND ROBERT ROLLAND

Abstract. We determine upper bounds on the number of rational points of an affine or projective algebraic set defined over an extension of a finite field by a system of polynomial equations, including the case where the algebraic set is not defined over the finite field by itself. A special attention is given to irreducible but not absolutely irreducible algebraic sets, which satisfy better bounds. We study the case of complete intersections, for which we give a decomposition, coarser than the decomposition in irreducible components, but more directly related to the polynomials defining the algebraic set. We describe families of algebraic sets having the maximum number of rational points in the affine case, and a large number of points in the projective case.

Nous déterminons des majorations du nombre de points d’un ensemble algébrique affine ou projectif, défini sur une extension d’un corps fini par un système d’équations polynomiales, y compris dans le cas où l’ensemble algébrique n’est pas défini sur le corps fini lui-même. Une attention particulière est portée aux ensemble algébriques irréductibles mais non absolument irréductibles, pour lesquels nous obtenons de meilleures bornes. Nous étudions le cas des intersections complètes, pour lesquelles nous construisons une décomposition moins fine que la décomposition en composantes irréductibles, mais plus directement liée aux polynômes qui définissent l’ensemble algébrique. Enfin, nous construisons des familles d’ensembles algébriques atteignant le nombre maximum de points rationnels dans le cas affine, et comportant de nombreux points dans le cas projectif.

Introduction

Let \(X\) be an algebraic subset of the affine or projective space, defined over an extension of a given finite field \(F_q\). Our purpose is to give several bounds on the maximum number of points of \(X\), with coordinates in \(F_q\) (unless explicitly stated, we do not assume that \(X\) is defined over \(F_q\)). These bounds are expressed in terms of the degree of \(X\), and they are obtained by applying various versions of Bézout’s Theorem. Hence, the notions of degree and cumulative degree are essential tools in our computations, and they are introduced in Section 1. We establish a general upper bound in Section 2 (Theorem 2.1). We improve this general bound if \(X\) is irrational, that is, not defined over \(F_q\), by introducing the greatest \(k\)-closed subset in \(X\). Surprisingly, we obtain in this case a bound of order \(q^\dim X - 1\) (Corollary 2.11). In Section 3 we assume that \(X\) is relatively irreducible, and study the decomposition of \(X\) in absolutely irreducible components of \(X\). This
decomposition leads to a better upper bound (Corollary 3.6) than the general upper bound given in Section 2, and also to a bound of order $q^{\dim X - 1}$. We also show that the set of rational points of $X$ is contained in the singular locus of $X$, and moreover $X(k) = \emptyset$ if $X$ is normal (Proposition 3.8). We assume in section 4 that $X$ is an (ideal-theoretic) complete intersection, for which an exact formula for the degree is known. We describe a decomposition of $X$ directly related to a system $(f_1, \ldots, f_r)$ of polynomials defining $X$, namely the coarse decomposition (Proposition 4.5). The decomposition in irreducible components is finer than the coarse decomposition, but the later can be explicitly constructed from $(f_1, \ldots, f_r)$. This leads to an upper bound on the number of rational points of $X$, improving the general upper bound if every polynomial among $(f_1, \ldots, f_r)$ is relatively irreducible, but at least one is not absolutely irreducible (Proposition 4.3). In Section 5 we construct a family of affine algebraic sets over $\mathbb{F}_q$ (the tubular sets) reaching the general upper bound given in Section 2. The corresponding projective family has also a large number of points but does not reach the the general upper bound (Theorem 5.1). It is worthwhile to precise that our results generalize and improve those previously obtained in the case of hypersurfaces defined over $\mathbb{F}_q$, for which the best bounds are given in [15] in the affine case, and in [20] and [22] in the projective case. Also note that some of our methods can be seen as similar, although in a more explicit and precise way, to the general approach of Heath-Brown in [16, Th. 3].

1. The cumulative degree

Let $k$ be a field and $K$ the algebraic closure of $k$. We are interested in the solutions in the affine space $k^n$ of a system

$$f_i(T_1, \ldots, T_n) = 0 \quad (1 \leq i \leq r)$$

with $f_i \in K[T_1, \ldots, T_n]$, and $r \leq n$. We are also concerned about solutions in the projective space $\mathbb{P}^n(k)$ of a system

$$f_i(T_0, \ldots, T_n) = 0 \quad (1 \leq i \leq r)$$

with homogeneous polynomials $f_i \in K[T_0, \ldots, T_n]$. These systems define respectively a $K$-algebraic subset $X$ in the affine space $A^n = K^n$ and in the projective space $\mathbb{P}^n = \mathbb{P}^n(K)$. If $a$ is an ideal of $K[T_1, \ldots, T_n]$, the subset of zeros of $a$ is denoted by $V(a)$. Hence, $X = V(a)$ where $a$ is the ideal generated by $f_1, \ldots, f_r$. If $S$ is a subset of $A^n$ or $\mathbb{P}^n$, the ideal of $S$, denoted by $I(S)$, is the radical ideal of polynomials vanishing on $S$. Hence, $I(X)$ is the radical $\sqrt{a}$ of $a$.

Let $Z_1, \ldots, Z_t$ be the irreducible components of $X$, in such a way that

$$X = Z_1 \cup \cdots \cup Z_t.$$ 

We put $m = \dim X = \max_{1 \leq i \leq t} \dim Z_i$. Then $m \geq n - r$ since, by the Generalized Principal Ideal Theorem [4] Ch. VIII, §3, Prop. 4:

$$\min_{1 \leq i \leq t} \dim Z_i \geq n - r.$$ 

We denote by $\deg X$ the (usual) degree of $X$, for which we refer to Fulton [9], Harris [12], and Hartshorne [13]. Recall that if $X$ is of dimension $m$, then $\deg X$ is equal to $|X \cap L|$ for almost all linear varieties $L$ of complementary dimension $n - m$. For
0 \leq l \leq m$, put
\[ c-deg_l X = \sum_{\text{dim } Z_i = l, 1 \leq i \leq t} \deg Z_i. \]
The cumulative degree of $X$ (Heintz [14], Burgisser [5]) is
\[ c-deg X = \sum_{l=0}^{m} c-deg_l X = \sum_{i=1}^{t} \deg Z_i. \]

Since [9, Ex. 2.5.2(b)] or [13, Prop. 7.6(b)]:
\[ \deg(X) = c-deg_m(X), \]
we have
\[ \deg(X) \leq c-deg(X), \]
with equality if and only if $X$ is equidimensional of dimension $m$ (i.e. every irreducible component of $X$ has dimension $m$).

There are many ways to state Bézout’s Theorem. The more general one is the Main Theorem and its refined version, see Fulton [9, Th. 12.3 and Ex. 12.3.1] and Vogel [23, Th. 2.1 and Cor. 2.26]. We use here three variants of Bézout’s Theorem which are the more appropriate for our purposes, namely, Theorems 1.1 [2.3] and 4.1. Although they can be undoubtedly deduced from the general theory, we give in each case specific references for these statements.

**Theorem 1.1** (Bézout’s Theorem, cumulative degree). *Let $Z$ be an algebraic subset, and $H_1, \ldots, H_r$ a sequence of hypersurfaces in $\mathbb{A}^n$ or $\mathbb{P}^n$. Then*
\[ c-deg(Z \cap H_1 \cap \ldots \cap H_r) \leq c-deg(Z) \prod_{i=1}^{r} \deg(H_i). \]

*Proof.* See Heintz [14, Th. 1], Burgisser & al. [5, Prop. 8.28]. □

Theorem 1.1 shows that if $X$ is given by (1) or (2), by taking for $Z$ the whole space, and for $H_i$ the hypersurface $V(f_i)$, then
\[ c-deg(X) \leq \prod_{i=1}^{r} \deg(f_i). \]

**Example 1.2.** Consider the couple of polynomials
\[ f_1(T_1, T_2) = T_2(T_2 - 1), \quad f_2(T_1, T_2) = T_1 T_2, \]
and let $X = V(f_1, f_2)$. Then $X = Z_1 \cup Z_2$, where $Z_1$ is the line $T_2 = 0$ and $Z_2$ is the point $(0,1)$. Hence,
\[ \deg X = 1, \quad c-deg X = 2, \quad (\deg f_1)(\deg f_2) = 4. \]

**2. Bounds for algebraic sets**

**2.1. General case.** Here $k = \mathbb{F}_q$ is the field with $q$ elements, and $K = \overline{\mathbb{F}}_q$.

**Theorem 2.1.** *Let $X$ be a $K$-algebraic set of dimension $m$ in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$). If $X$ is affine, then*
\[ |X \cap \mathbb{F}_q^n| \leq \sum_{l=0}^{m} c-deg_l(X) q^l \leq c-deg(X) q^m. \]
If $X$ is projective, then
\[ |X \cap \mathbb{P}^n(F_q)| \leq \sum_{l=0}^{m} c\text{-deg}_i(X)\pi_l \leq c\text{-deg}(X)\pi_m, \]
where we have put $\pi_n = |\mathbb{P}^n(F_q)| = q^n + \cdots + 1$ for $n \geq 0$.

With the help of (3) we get

**Corollary 2.2.** $X$ be a $K$-algebraic set of dimension $m$ in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$), which is the zero set of a family of polynomials $(f_1, \ldots, f_r)$. Let $d_i = \text{deg } f_i$. Then
\[ |X \cap \mathbb{F}_q^n| \leq d_1 \ldots d_r q^m, \text{ resp. } |X \cap \mathbb{P}^n(F_q)| \leq d_1 \ldots d_r \pi_m. \]

If we are only interested in the points of $X \cap \mathbb{F}_q^n$, one can replace in Corollary 2.2 the polynomials $f_i$ by their reduction modulo the ideal generated by $T_{q_1} - T_1, \ldots, T_{q_n} - T_n$, in such a way that $\text{deg } f_i \leq (q - 1)$.

If $X$ is not defined over $F_q$, the bound of Theorem 2.1 can be rough: see Corollary 2.11.

The following proposition is a particular case of Theorem 2.1, but implies immediately this theorem. We define a $k$-variety as a $k$-irreducible algebraic set.

**Proposition 2.3.** If $X$ is an affine (resp. projective) $K$-subvariety in $\mathbb{A}^n$ (resp. $\mathbb{P}^n$), then
\[ |X \cap \mathbb{F}_q^n| \leq (\text{deg } X)q^m, \text{ resp. } |X \cap \mathbb{P}^n(F_q)| \leq (\text{deg } X)\pi_m. \]

**Proof of Theorem 2.1.** We assume that $X$ is affine, the argument being similar if $X$ is projective. Let $Z_1, \ldots, Z_t$ be the irreducible components of $X$. Since
\[ |X \cap \mathbb{F}_q^n| \leq \sum_{l=0}^{m} \sum_{\dim Z_i = l} |Z_i \cap \mathbb{F}_q^n|, \]
we have, by Prop. 2.3
\[ |X \cap \mathbb{F}_q^n| \leq \sum_{l=0}^{m} \sum_{\dim Z_i = l} \text{deg}(Z_i)q^l = \sum_{l=0}^{m} c\text{-deg}_i(X)q^l. \]

It remains to prove Proposition 2.3. If $X$ is defined over $\mathbb{F}_q$, these results are proved in [18, Prop. 2.3] and [10, Prop. 12.1] (the hypothesis of equidimensionality must be added in the statements), and also in [8, Lemma 3.1]. We provide here a complete proof for reader’s convenience.

A $K$-subvariety $X \subset \mathbb{A}^n$ (resp. $\mathbb{P}^n$) is called nondegenerate if it does not lie in any hyperplane, i. e. if the $K$-linear subvariety generated by $X$ is equal to the whole space. We show in the next result that enumeration problems for the number of points of general $K$-subvarieties can be reduced to nondegenerate ones, whether they are rational over $\mathbb{F}_q$ or not.

**Lemma 2.4.** Let $X$ be a $K$-subvariety in $\mathbb{P}^n$, which is not a linear variety. There is a projection $\mathbb{P}^n \rightarrow \mathbb{P}^r$ inducing an isomorphism of $X$ onto a nondegenerate $K$-subvariety $X' \subset \mathbb{P}^r$, and
\[ \text{deg } X' = \text{deg } X, \quad |X \cap \mathbb{P}^n(\mathbb{F}_q)| \leq |X' \cap \mathbb{P}^r(\mathbb{F}_q)|. \]
The same result holds for a $K$-subvariety in an affine space.
Proof. If $X$ is nondegenerate there is nothing to prove. Suppose that $X$ is included in a hyperplane $H \subset \mathbb{P}^n$. We can, and will, assume that $H$ is given by
\[ T_n = l(T_0, \ldots, T_{n-1}), \]
with a linear form $l$ with coefficients in $K$. Let
\[ \varphi(T_0 : \ldots : T_n) = (T_0 : \ldots : T_{n-1}) \]
be the projection from $(0 : \ldots : 0 : 1)$. The inverse map $\psi : \mathbb{P}^{n-1} \rightarrow H$ of the restriction of $\varphi$ to $H$ is
\[ \psi : (T_0 : \ldots : T_{n-1}) \mapsto (T_0 : \ldots : T_{n-1} : l(T_0, \ldots, T_{n-1})). \]
If $X$ is defined by
\[ f_i(T_0, \ldots, T_n) = 0 \quad (1 \leq i \leq r), \]
then $X' \subset \mathbb{P}^{n-1}$ is defined by
\[ f_i(T_1, \ldots, T_{n-1}, l(T_1, \ldots, T_{n-1})) = 0 \quad (1 \leq i \leq r), \]
and $\varphi$ defines a $K$-isomorphism from $X$ onto $X'$. Now $\deg X' = \deg X$ because $\varphi$ is a projection \[12\] Ex. 18.16, p. 234, and since $\varphi$ maps $\mathbb{P}^n(\mathbb{F}_q)$ onto $\mathbb{P}^{n-1}(\mathbb{F}_q)$, we have $|X \cap \mathbb{P}^n(\mathbb{F}_q)| \leq |X' \cap \mathbb{P}^{n-1}(\mathbb{F}_q)|$. If $X'$ is nondegenerate, we are done. In the opposite, we repeat the construction, and this comes to an end by infinite descent. The proof is the same for subvarieties in affine spaces.

Remark 2.5. Since, a priori, $\psi$ does not map $\mathbb{P}^{n-1}(\mathbb{F}_q)$ onto $\mathbb{P}^n(\mathbb{F}_q)$, equality has no reason for being in the proposition.

Lemma 2.6. If $X$ is a nondegenerate subvariety of dimension $m$ in $\mathbb{A}^n$ (resp. in $\mathbb{P}^n$), if $H$ is a hyperplane, and if $X \cap H \neq \emptyset$, then $X$ and $H$ intersect properly, that is, $X \cap H$ is equidimensional of dimension $m - 1$. Moreover, $\deg(X \cap H) = \deg(X)$.

Proof. See Hartshorne \[13\] Ex. I.1.8] or Harris \[12\] Ex. 11.6]. About the degree, see \[13\] Prop. I.7.6(d)].

Proof of Proposition 2.3. By induction on $m$. If $m = 0$ then
\[ |X(\mathbb{F}_q)| \leq |X(\mathbb{F}_q)| = \deg X. \]
The proposition is obvious if $X$ is a linear variety. Otherwise, we can assume by Lemma 2.4 that $X$ is nondegenerate. If $H$ is a hyperplane, then $X \cap H$ is equidimensional of dimension $m - 1$ and of degree $d$ by Lemma 2.6. We denote by $Z_1, \ldots, Z_t$ the irreducible components of $X \cap H$.

a) The proof is straightforward if $X$ is affine. By the induction hypothesis,
\[ |Z_i \cap \mathbb{F}_q^n| \leq \deg(Z_i)q^{m-1}. \]
and
\[ |X \cap H \cap \mathbb{F}_q^n| \leq \sum_{i=1}^t \deg(Z_i)q^{m-1} = \deg(X)q^{m-1}. \]
Denote now by $H_c$ the hyperplane $x_1 = c$. If $c \in \mathbb{F}_q$, we deduce from the preceding inequality
\[ |X \cap H_c \cap \mathbb{F}_q^n| \leq \deg(X)q^{m-1}. \]
Since
\[ X = \bigcup_{c \in \mathbb{F}_q} [X \cap H_c], \]
we get the result is proved if $X$ is affine.

b) Assume $X$ projective. Let $\mathbb{G}_{n-1}$ be the variety of hyperplanes in $\mathbb{P}^n$, and $T$ the incidence correspondence \cite{[
\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\b
is an isomorphism. Then 
\[ b = b_kB, \quad b_k = b \cap A, \]
and \( b \) is the \textit{smallest ideal of} \( B \) \textit{containing} \( a \) \textit{defined over} \( k \).

\textbf{Remark 2.7.} A family of generators of \( b_k \) can be deduced from a set of generators \( \{f_1, \ldots, f_r\} \) of \( a \) in the following way. The family 
\[ f_1, \ldots, f_r, f_1^\sigma, \ldots, f_r^\sigma, \ldots \]
generates \( b \). Let \( (\xi_1, \ldots, \xi_s) \) a basis of \( k' \) over \( k \). There is a unique family \( (g_{ij}) \) of polynomials in \( k[T_1, \ldots, T_n] \) such that 
\[ f_i^\sigma(T_1, \ldots, T_n) = \sum_{j=1}^s \xi_j^\sigma g_{ij}(T_1, \ldots, T_n), \quad 1 \leq i \leq r, \sigma \in G. \]
Let \( (\eta_1, \ldots, \eta_s) \) be the basis of \( k' \) dual to \( (\xi_1, \ldots, \xi_s) \), in such a way that
\[ \text{Tr}_{k'/k}(\xi_i\xi_j) = \sum_{\sigma \in G} \xi_j^\sigma \eta_j^\sigma = \delta_{ij}, \quad 1 \leq i, j \leq s. \]
Then
\[ g_{ij}(T_1, \ldots, T_n) = \sum_{\sigma \in G} \eta_j^\sigma f_i^\sigma(T_1, \ldots, T_n), \quad 1 \leq i, j \leq s. \]
The two formulas \([9]\) and \((7)\) show that \( (g_{ij}) \) is a family of generators of \( b \) in \( k[T_1, \ldots, T_n] \), that is, a family of generators of \( b_k \).

Let \( X = V(a) \) and \( X^\sigma = V(a^\sigma) \). We put
\[ Q_{k'/k}(X) = V(b_k) = \bigcap_{\sigma \in G} X^\sigma. \]
The algebraic subset \( Q_{k'/k}(X) \) depends only on \( X \) and not of \( a \) : it is the \textit{greatest} \( k \)-closed subset in \( X \). If \( l \) = \( \tau(b) \) is the ideal of \( Q_{k'/k}(X) \), then \( l_k = \sigma \cap A \) is a radical ideal, and \( l = l_kB \) [11] p. 22–24. Moreover \( \text{codim} \ Q_{k'/k}(X) \leq s \ \text{codim} \ X \).

\textbf{Lemma 2.8.} Let \( X \) be an algebraic subset of \( \mathbb{A}^n \), defined over a Galois extension \( k' \) of \( k \), and \( Y = Q_{k'/k}(X) \). Then
\[ Y(k) = X \cap k^n. \]
\textbf{Proof.} If \( x \in X \cap k^n \), then \( f(x) = 0 \) for every \( f \in a \) and every \( \sigma \in G \), hence, \( f(x) = 0 \) for every \( f \in b \), hence, \( f \in Y(k^n) \). The reciprocal is immediate. \( \square \)

\textbf{Theorem 2.9 (Bézout’s Theorem, general version).} Let there be given \( r \) equidi- mensional subvarieties \( X_1, \ldots, X_r \) of \( \mathbb{P}^n \). Then
\[ \text{deg}(X_1 \cap \ldots \cap X_r) \leq \text{deg}(X_1) \ldots \text{deg}(X_r). \]
\textbf{Proof.} Fulton [9] 8.4.6 and Ex. 12.3.1, Vogel [23] Cor. 2.26]. \( \square \)

Recall that the set of fields of definition of an algebraic subset of \( \mathbb{A}^n \) or \( \mathbb{P}^n \) has a smallest element [11] 4.8.11.

\textbf{Proposition 2.10.} Let \( X \) be an absolutely irreducible algebraic subset of \( \mathbb{A}^n \), and \( k' \) the smallest field of definition of \( X \). Let \( Y = Q_{k'/k}(X) \), and assume that \( s = [k' : k] > 1 \). Then
\[ \dim Y \leq \dim X - 1, \quad \text{deg} \ Y \leq (\text{deg} \ X)^s. \]
Proof. Recall that $Y \subset X$. If $\dim Y = \dim X$, then $Y$ contains an irreducible component of $X$. Since $X$ is irreducible, we find that $X = Y$ and $X$ is defined over $k$, contrary to the hypotheses. Since $\deg X^\sigma = \deg X$, we get the last inequality by Theorem 2.9. □

As a consequence of Lemma 2.8 and Proposition 2.10, the following holds:

**Corollary 2.11.** Assume $k = \mathbb{F}_q$. Let $X$ be an absolutely irreducible algebraic subset of $\mathbb{A}^n$, of dimension $m$, and $k'$ the smallest field of definition of $X$. Let $Y = Q_{k'/k}(X)$, and assume $s = [k': k] > 1$. Then

$$|X \cap \mathbb{F}_q^n| \leq (\deg Y)^{q^{m-1}} \leq (\deg X)^s q^{m-1}.$$  

This bound is better than the usual one if $q \geq (\deg X)^s - 1$. It is worthwhile to specify that in this corollary, the integer $s$ depends on $q$. Analogous results hold for subsets of $\mathbb{P}^n$ (substitute $\pi_{m-1}$ to $q^{m-1}$ in Corollary 2.11).

**Example 2.12.** Let $k$ be a field of characteristic $\neq 3$. We assume that 2 is not a square in $k$, and that 1 has three cube roots in $k$. Let $k' = k(\sqrt[3]{2})$. Define

$$f(T_1, T_2, T_3) = T_1^3 - T_2^3 + \sqrt{2}(T_3^3 - T_2^3),$$

and let $C$ the plane projective cubic defined over $k'$ with equation $f = 0$. One checks that $C$ is nonsingular, hence, $C$ is absolutely irreducible. If

$$g_1(T_1, T_2, T_3) = T_1^3 - T_2^3, \quad g_2(T_1, T_2, T_3) = T_3^3 - T_2^3,$$

then

$$f(T_1, T_2, T_3) = g_1(T_1, T_2, T_3) + \sqrt{2}g_2(T_1, T_2, T_3),$$

and the algebraic subset $Y$ of $\mathbb{P}^2$ defined by

$$g_1(T_1, T_2, T_3) = 0, \quad g_2(T_1, T_2, T_3) = 0$$

is of dimension 0. Here, $s = 2$ and there are $(\deg C)^2 = 9$ points in $Y(k) = C \cap \mathbb{P}^2(k)$, namely the points $(1 : \eta : \zeta)$, where $\eta$ and $\zeta$ runs over the three cubic roots of 1.

**Remark 2.13.** We are merely concerned here by upper bounds on the number of points of algebraic sets. But it is worthwhile to recall the lower bounds obtained with the help of the Chevalley-Warning Theorem. Let $(f_1, \ldots, f_r)$ be a sequence of polynomials in $k[T_1, \ldots, T_n]$, of respective degrees $d_1, \ldots, d_r$, and write $d = d_1 + \cdots + d_r$. Let $X$ be the set of zeros of these polynomials in $\mathbb{A}^n$. Then Warning [24] proved that if $d \leq n$ and $X(k) \neq \emptyset$, then

$$|X(k)| \geq q^{n-d}.$$ 

See [16] for a discussion and improvements of this result.

3. Relatively irreducible sets

Here $k$ is a field and $K = \bar{k}$.
3.1. Decomposition in absolute components.

Theorem 3.1. Let $X$ be a $k$-irreducible subset in $\mathbb{A}^n$ or $\mathbb{P}^n$, $k(X)$ the field of rational functions on $X$, and $k'$ the (relative) algebraic closure of $k$ in $k(X)$. Assume that $k'$ is a Galois extension of $k$ and let $G = \text{Gal}(k'/k)$. Let $Z$ be the set of absolutely irreducible components of $X$ and $Z \in Z$.

(i) The smallest field of definition of every element of $Z$ is equal to $k'$, and $Z$ has $[k':k]$ elements.

(ii) Let $p$ the ideal of $X$ in $A = k[T_1, \ldots, T_n]$, and $\mathfrak{p}$ the ideal of $Z$ in $B = k'[T_1, \ldots, T_n]$. Then

$$pB = \bigcap_{\sigma \in G} \mathfrak{p}^\sigma.$$

Proof. The statement (i) is proved in EGA in a more general setting, see [11, Prop. 4.5.10]. Let $\mathfrak{p}_i$ the ideal of $Z_i$ in $B = k'[T_1, \ldots, T_n]$. Then

$$X = \bigcup_{i=1}^s Z_i, \quad pB = \bigcap_{i=1}^s \mathfrak{p}_i.$$

Let $\Pi = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. The group $G$ operates transitively on $\Pi$ [2, Ch. V, §2, Th. 2], and even simply transitively on $\Pi$, since $|G| = |\Pi| = s$, according to (i). The result follows. □

Theorem 3.2. Let $X$ be a $k$-irreducible subset in $\mathbb{A}^n$ or $\mathbb{P}^n$, $k(X)$ the field of rational functions on $X$, and $k'$ the (relative) algebraic closure of $k$ in $k(X)$. Assume that $k'$ is a Galois extension of $k$ and let $G = \text{Gal}(k'/k)$. Let $Z$ be the set of absolutely irreducible components of $X$. Then:

(i) $X$ is equidimensional.

(ii) $G$ operates simply transitively on $Z$ : if we choose $Z \in Z$, then

$$X = \bigcup_{\sigma \in G} Z^\sigma.$$

(iii) For any $Z \in Z$ one has $\deg X = [k':k] \deg Z$.

Proof. The statement (i) is proved in EGA in a more general setting, see [11, Prop. 5.2.1]. The statement (ii) follows from Theorem 3.1(ii). By the results of section 3.

Let $\deg X = \sum_{i=1}^s \deg Z_i$;

but by (ii) all the $Z_i$ of $Z$ have the same degree, this proves (iii). □

Remark 3.3. Denote by $S_{k'/k}(Z)$ the Zariski $k$-closure of $Z$. Then Theorem 3.2(ii) means that $X = S_{k'/k}(Z)$. With the help of Lemma 2.8 we get

$$Q_{k'/k}(Z) \subset Z \subset S_{k'/k}(Z),$$

and the preceding results show that this is true for every Zariski closed subset $Z$ of $\mathbb{A}^n$ or $\mathbb{P}^n$, provided that the smallest field $k'$ of definition of $Z$ is a Galois extension of $k$. One could say that $Q_{k'/k}(Z)$ is “inscribed” in $Z$, and that $S_{k'/k}(Z)$ is “escribed” to $Z$. Moreover,

$$Q_{k'/k}(Z)(k) = Z \cap k^n = S_{k'/k}(Z)(k).$$
Example 3.4. In Example 2.12 denote by σ the automorphism of k' such that \((\sqrt{2})^\sigma = -\sqrt{2}\). Let
\[
h(T_1, T_2, T_3) = f(T_1, T_2, T_3) f'(T_1, T_2, T_3) = (T_1^3 - T_2^3)^2 - 2(T_1^3 - T_2^3)^2
\]
Then h is irreducible in k[T_1, T_2, T_3], and the plane curve \(D\) defined over k with equation \(h = 0\) is k-irreducible. Then
\[
\sqrt{2} = \frac{T_1^3 - T_2^3}{T_3^3 - T_2^3}
\]
and k' is the algebraic closure of k in \(k(D)\). The absolutely irreducible components of \(D\) are the curves \(C\) and \(C\sigma\), defined over k'. The 9 points of \(C(k)\) are the points of \(C \cap C\sigma\); they are the singular points of \(D\).

Example 3.5. [17 Ex. 2.6]. Let \(k = \mathbb{F}_q\), \(k' = \mathbb{F}_{q^n}\), \(G = \text{Gal}(k'/k)\) and let \(\alpha\) be a generator of \(k'/k\). Define
\[
g(T_0, T_1, \ldots, T_n) = T_1 + \alpha T_2 + \alpha^{n-1} T_n,
\]
and put
\[
f = \prod_{\sigma \in G} g^\sigma.
\]
The hypersurface \(X\) with equation \(f = 0\) is defined over \(k\) and \(k\)-irreducible, of degree \(n\), and \(|X(k)| = 1\), with one point \((1 : 0 : \cdots : 0) \in \mathbb{P}^n\). The algebraic closure of \(k\) in \(K(\mathbb{F}_q)\) is equal to \(k'\), and if \(Z\) is the hyperplane with equation \(g = 0\), the hypersurface \(X\) is the union of the hyperplanes \(Z^\sigma\).

Corollary 3.6. Assume \(k = \mathbb{F}_q\). Let \(X\) be a \(k\)-irreducible subset of dimension \(m\) in \(\mathbb{P}^n\), and \(k'\) the algebraic closure of \(k\) in \(k(X)\). Let \(s' = [k' : k]\).

(i) We have
\[
|X(\mathbb{F}_q)| \leq \frac{\deg X}{s'} \pi_m.
\]
(ii) If \(s' > 1\), then
\[
|X(\mathbb{F}_q)| \leq \left(\frac{\deg X}{s'}\right)^s \pi_{m-1}.
\]
Proof: Observe that \(X(\mathbb{F}_q) = Z \cap \mathbb{F}_q\), where \(Z\) is an absolutely irreducible component of \(X\), and apply successively Theorem 2.1 and Corollary 2.11 to \(Z\).

Example 3.7. In Example 2.12 denote by σ the automorphism of k' such that \((\sqrt{2})^\sigma = -\sqrt{2}\). Let
\[
h(X, Y, Z) = f(X, Y, Z) f'(X, Y, Z) = (X^3 - Y^3)^2 - 2(Z^3 - Y^3)^2
\]
Then h is irreducible in k[X, Y, Z], and the plane curve \(D\) defined over k with equation \(h = 0\) is k-irreducible. Then
\[
\sqrt{2} = \frac{X^3 - Y^3}{Z^3 - Y^3}
\]
and k' is the algebraic closure of k in \(k(D)\), with \(s = [k' : k] = 2\). The absolutely irreducible components of \(D\) are the curves \(C\) and \(C\sigma\), defined over k'. Indeed, the points of \(D(k)\) are the points of \(C \cap C\sigma\), and they are the singular points of \(D\). If \(k = \mathbb{F}_q\), then the inequality of Corollary 3.6(ii) is an equality:
\[
|D(\mathbb{F}_q)| = \left(\frac{\deg D}{s}\right)^s = 9.
\]
3.2. Normal and rational points. We transpose here to finite fields a remark of Serre about relatively irreducible varieties defined in characteristic 0 [21, p. 20].

Let \( k \) be any field, and \( X \) a \( k \)-irreducible subset of dimension \( m \) in \( \mathbb{A}^n \) or \( \mathbb{P}^n \) defined over \( k \). Recall that \( X \) is normal at the point \( x \in X \) if the local ring \( \mathcal{O}_x \subset k(X) \) is integrally closed. The algebraic subset \( X \) is normal if it is normal at every point. Let \( m_x \) be the maximal ideal of \( \mathcal{O}_x \), and \( \kappa(x) = \mathcal{O}_x/m_x \) the residual field. If \( L \) is an extension of \( k \), Any point \( x \in X(L) \) defines an injective homomorphism \( \kappa(x) \to L \). If \( X \) is normal at \( x \), then \( \kappa(x) \) contains the relative algebraic closure \( k' \) of \( k \) in \( k(X) \). Therefore \( k' \subset L \) if \( x \) is a normal point of \( X \) and \( x \in X(L) \).

Assume now that \( X \) is not absolutely irreducible. Then \( [k' : k] > 1 \), and no point \( x \in X(k) \) can be normal. Since a nonsingular point is normal, the set of points which are not normal are contained in the algebraic set \( \text{Sing} X \) of singular points of \( X \), which is of codimension \( \geq 1 \). Therefore \( X(k) \subset \text{Sing} X \), and actually \( X(k) = (\text{Sing} X)(k) \) since \( \text{Sing} X \subset X \).

Recall that if \( Z_1, \ldots, Z_s \) are the irreducible components of \( X \), then
\[
\text{Sing} X = \bigcup_{i=1}^s \text{Sing} Z_i \cup \bigcup_{i \neq j} Z_i \cap Z_j.
\]

Since \( Z_i \cap Z_j \geq \dim X - r \), we have \( \dim \text{Sing} X \geq \dim X - r \).

**Proposition 3.8.** Let \( X \) be a \( k \)-irreducible subset in \( \mathbb{A}^n \) or \( \mathbb{P}^n \) defined over \( k \), which is not absolutely irreducible. Then \( X(k) = (\text{Sing} X)(k) \). Moreover \( X(k) = \emptyset \) if \( X \) is normal. If \( k = \mathbb{F}_q \), then
\[
|X(k)| \leq (\deg \text{Sing} X) \pi_{m-1}.
\]

**Example 3.9.** Let \( X \) be a complete intersection in \( \mathbb{A}^n \) or \( \mathbb{P}^n \). If \( \dim \text{Sing} X \leq \dim X - 2 \), then \( X \) is normal by Serre’s criterion [11, Th. 5.8.6], and \( X(k) = \emptyset \).

**Example 3.10.** In Example [7,7] we saw that the \( k \)-rational points of \( D \) are exactly the 9 singular points of \( D \), hence, \( D(k) = \text{Sing} D \) in this case.

4. Complete intersections

4.1. Notation. Let \( X \) be an algebraic subset of \( \mathbb{P}^n \) of dimension \( m = n - r \).

- \( X \) is a set-theoretic (s.t.) complete intersection if \( X = V(a) \), where \( a \) is generated by \( r \) homogeneous polynomials in \( K[T_0, \ldots, T_n] \).
- \( X \) is an ideal-theoretic (i.t.) complete intersection if moreover \( a \) is a radical ideal, that is, if the ideal of \( X \) can be generated by \( r \) homogeneous polynomials.

One defines in the same way complete intersections in \( \mathbb{A}^n \). An s.t. complete intersection is equidimensional, by the Generalized Principal Ideal Theorem (section [1]). The cumulative and the usual degree coincide, and
\[
c\deg X = \deg X = \sum_{i=1}^t \deg Z_i,
\]
where \( Z_1, \ldots, Z_t \) are the irreducible components of \( X \). Let \( X \) be an i.t. complete intersection, \( (f_1, \ldots, f_r) \) the ideal of \( X \), and \( d_i = \deg f_i \). Since the numerator of the Hilbert series of \( X \) equals [7, Th. III-83]
\[
(1 - T^{d_1})(1 - T^{d_2}) \ldots (1 - T^{d_r}),
\]
the family \((d_1, \ldots, d_r)\), which is called the multidegree of \(X\), depends only of \(X\) and not of the system of generators \(f_1, \ldots, f_r\).

**Theorem 4.1 (Bézout’s Theorem, complete intersections).** Let \(X\) be an i.t. complete intersection with multidegree \((d_1, \ldots, d_r)\). Then

\[
\deg X = \prod_{i=1}^r d_i.
\]

**Proof.** Eisenbud-Harris [7, Th. III-71] (using schemes), Vogel [23, §1.35] (using regular sequences). □

### 4.2. Coarse decomposition of complete intersections.

We shall now describe a decomposition of complete intersections, explicitly constructed from the system of equations defining \(X\), which can be used as a substitute for the decomposition in irreducible components.

We first give a lemma from an unpublished paper of 1994 by the first author, stated without proof in [22] and [19]. If \(g \in B\), the polynomial

\[
N_{k'/k}(g) = \prod_{\sigma \in G} g^\sigma \in A
\]

is called a norm polynomial.

**Lemma 4.2.** Let \(f \in A\) be a \(k\)-irreducible polynomial, and \(k'\) the algebraic closure of \(k\) in the field of fractions of \(A/(f)\). Assume that \(k'\) is a Galois extension of \(k\) and let \(G = \text{Gal}(k'/k)\). Then there is an absolutely irreducible polynomial \(g \in B\) such that

\[
f = N_{k'/k}(g),
\]

and \(\deg f = [k' : k] \deg g\).

**Proof.** Let \(X\) be the hypersurface in \(\mathbb{A}^n\) defined by \(f\). With the notation of Lemma 3.1 and its proof, \(fB = pB\) is generated by \(f\). In the same way, \(\mathfrak{P}\) is a principal ideal by (i). Let \(g\) be a generator of \(\mathfrak{P}\). From Theorem 3.1 (ii) we deduce that

\[
f = c \prod_{\sigma \in G} g^\sigma,
\]

with \(c \in B\). But \(g = \prod g^\sigma\) is in \(A\), since it is invariant under \(G\), and consequently \(c \in A\). But \(c \in k^\times\), since \(f\) is \(k\)-irreducible. If we choose now an element \(\lambda \in k'\) of norm \(1/c\), and substitute \(\lambda g\) to \(g\), we get the result. □

Let \(f_1, \ldots, f_r\) be a sequence of \(k\)-irreducible homogeneous polynomials in \(A = k[T_0, \ldots, T_n]\), of respective degrees \(d_1, \ldots, d_r\), such that the algebraic subset of \(\mathbb{P}^n\)

\[
X = V(f_1, \ldots, f_r) = \bigcap_{i=1}^r H_i, \quad \text{with} \quad H_i = V(f_i),
\]

is defined over \(k\), of dimension \(m = n - r\). In other words, \(X\) is a set-theoretic complete intersection, hence, equidimensional. For \(i \leq r\), let \(k_i\) be the algebraic closure of \(k\) in the field \(k(H_i)\) of rational functions on \(H_i\). Assume \(k_i\) is a finite Galois extension of \(k\), and put \([k_i : k] = s_i\). By Lemma 4.2 there is an absolutely
irreducible homogeneous polynomial \( g_i \in k_i[T_0, \ldots, T_n] \) of degree \( \deg g_i = e_i \) such that
\[
f_i = \prod_{\sigma \in \Gamma_i} g_i^\sigma, \quad e_i = \frac{d_i}{s_i},
\]
with \( \Gamma_i = \text{Gal}(k_i/k) \). The smallest field of definition of
\[
Y = V(g_1, \ldots, g_r) = \bigcap_{i=1}^r G_i, \quad \text{with} \quad G_i = V(g_i),
\]
is the composite extension \( k_0 \) of the fields \( k_i \). Then \( k_0 \) is Galois, and
\[
s_0 = [k_0 : k] \leq s, \quad \text{with} \quad s = s_1 \ldots s_r.
\]
If \( k = \mathbb{F}_q \), then \( s_0 = \text{lcm}(s_1, \ldots, s_r) \).

**Proposition 4.3.** Let \( X \) be a set theoretical (s.t.) complete intersection. With the
preceding notation and hypotheses:

(i) The subset \( Y \) is a s.t. complete intersection of dimension \( m \), and its smallest
field of definition is \( k_0 \).

(ii) Let \( Z \) be the set of absolutely irreducible components of \( X \). There is a subset
\( \tilde{Z}(0) \subset Z \) such that
\[
Y = \bigcup_{Z \in \tilde{Z}(0)} Z.
\]

(iii) We have
\[
\deg Y \leq e_1 \ldots e_r = \frac{d_1 \ldots d_r}{s}.
\]

(iv) We have
\[
X(k) = Y \cap \mathbb{P}^n(k).
\]

(v) If \( k = \mathbb{F}_q \), then
\[
|X(k)| \leq (\deg Y) \pi_m.
\]

A similar result holds for algebraic subsets of \( \mathbb{A}^n \).

**Proof.** By the Generalized Principal Ideal Theorem (section 1), \( \dim Y \geq m \). But
\( Y \subset X \), hence \( \dim Y = m \), and \( Y \) is a s.t. complete intersection; this proves (i).
Since \( Y \) is equidimensional of dimension \( m \) and \( Y \subset X \), we get (ii). One deduce (iii)
from (ii). The formula (iv) is immediate, and (v) follows from (iv) and Theorem 2.1. \( \square \)

**Remark 4.4.** From Corollary 2.11 and by putting \( W = Q_{k_0/k}(Y) \), we have
\[
|X(k)| \leq (\deg W) \pi_{m-1} \leq (\deg Y)^{s_0} \pi_{m-1}.
\]

Since
\[
H_i = \bigcup_{\sigma \in \Gamma_i} G_i^\sigma,
\]
we have, by putting \( \Gamma = \Gamma_1 \times \cdots \times \Gamma_r \),
\[
X = \bigcap_{i=1}^r \bigcup_{\sigma \in \Gamma_i} G_i^\sigma = \bigcup_{a \in \Gamma} \bigcap_{i=1}^r G_i^{a(i)},
\]
with \( a = (a(1), \ldots, a(r)) \in \Gamma \). Define
\[
Y^{(a)} = \bigcap_{i=1}^r G_i^{a(i)}.
\]
Proposition 4.5. The subset $Y^{(a)}$ is a s.t. complete intersection of dimension $m$, and its smallest field of definition is $k_0$. We have

$$X = \bigcup_{a \in \Gamma} Y^{(a)}.$$  

For every $a \in \Gamma$, there is a subset $Z(a) \subset Z$ such that

$$Y^{(a)} = \bigcup_{Z \in Z(a)} Z.$$  

The covering (8) of $X$ is called the coarse decomposition of $X$.

If $X$ is a $k$-irreducible complete intersection, it can be interesting to compare the coarse decomposition of $X$ with its decomposition in irreducible components detailed in Theorem 3.2. This can be performed under suitable conditions; for instance, we have the following result.

Proposition 4.6. With the preceding notation, assume that $X$ is a $k$-irreducible i.t. complete intersection over $k$, and that $Y$ is an i.t. complete intersection with field of definition $k_0$. Let $s = s_1 \ldots s_r$ as above, and $k'$ the smallest field of definition of the irreducible components of $X$. Then:

(i) The family $\{Z(a)\}_{a \in \Gamma}$ is a partition of $Z$ into $s$ subsets with $s'/s$ elements, and $k_0 \subset k'$.

(ii) We have

$$\deg Y = e_1 \ldots e_r = \frac{\deg X}{s}.$$  

(iii) The coarse decomposition of $X$ is irredundant.

Proof. The subset $Y^{(a)}$ is defined over $k'$ by (8), hence, $k_0 \subset k'$. If we choose $Z \in Z$, then

$$\deg Y^{(a)} = |Z(a)| \deg Z,$$

since all the elements of $Z$ have the same degree by Theorem 3.2. By Theorem 4.1, we have

$$\deg Y^{(a)} = e_1 \ldots e_r, \quad \deg X = d_1 \ldots d_r = s \deg Y^{(a)},$$

hence,

$$\deg X = s \deg Y^{(a)} = s \deg Z.$$

and this proves (iii). Since $|Z| = s' = [k' : k]$ and $\deg X = s' \deg Z$, we see that $s' = cs$. This proves (ii), from which we see that no $Y^{(a)}$ can be dropped in the coarse decomposition, whence (iii). \qed

If $Y$ is irreducible, i.e., $c = 1$, then the coarse decomposition is identical to the decomposition of $X$ in irreducible components. Otherwise the covering of $X$ by irreducible components refines the coarse one.

Remark 4.7. If $k = \mathbb{F}_q$ and $c > 1$, the bound

$$|X(k)| \leq \frac{\deg X}{s'} \pi_m$$

of Corollary 3.6 is better than the bound

$$|X(k)| \leq \frac{\deg X}{s} \pi_m$$

of Proposition 4.3(iv). If $c = 1$, they are identical.
Remark 4.8. If the polynomials defining $X$ are $K$-irreducible, Proposition 4.3 does not bring anything new. But if at least one of these polynomials is relatively irreducible, that is, $\mathbb{F}_q$-irreducible but not $K$-irreducible, then some of the $c_i$ are $\geq 2$, and the bound of Proposition 4.3 is better than the one of Corollary 2.2.

5. Tubular sets

We give in this section examples of algebraic sets with many points. The following construction generalizes a construction of Serre [20], corresponding to the case of codimension $r = 1$. Take a sequence $d_1, \ldots, d_r$ of integers $\geq 1$, and choose a family

$$a_{i,j} \in \mathbb{F}_q, \quad i \in \{1, \ldots, r\}, \quad j \in \{1, \ldots, d_i\},$$

where we assume, for every $i$, that $a_{i,j} \neq a_{i,k}$ if $j \neq k$. Note that this condition implies $d_i \leq q$ for $1 \leq i \leq r$. Denote by $a_{i,j}$ the principal ideal $(T_i - a_{i,j}T_0)$ of $\mathbb{F}_q[T_0, \ldots, T_n]$, and by $H_{i,j} = V(a_{i,j})$ the corresponding hyperplane of $\mathbb{P}^n$. Let

$$D = \prod_{i=1}^r \{1, \ldots, d_i\}.$$

If $J = (j(1), \ldots, j(r)) \in D$, we put

$$p_J = (T_1 - a_{1,j(1)}T_0, T_2 - a_{2,j(2)}T_0, \ldots, T_r - a_{r,j(r)}T_0), \quad Y_J = V(p_J).$$

We have

$$p_J = \sum_{i=1}^r a_{i,j(i)}, \quad Y_J = \bigcap_{i=1}^r H_{i,j(i)}.$$  

The projective linear variety $Y_J$ is defined by $r$ linearly independent forms, and hence, of dimension $n - r$. Observe that if $J$ and $K$ are in $D$ and if $J \neq K$, then $j(i) \neq k(i)$ for some $i$, hence $a_{i,j(i)} \neq a_{i,k(i)}$, and $H_{i,j(i)} \cap H_{i,k(i)}$ is the linear variety $X_0$ of dimension $n - r - 1$ with ideal

$$p_0 = (T_0, T_1, T_2, \ldots, T_r).$$

This proves that $Y_J \cap Y_K = X_0$. The tubular set $X$ defined by the family $(a_{i,j})$ is the algebraic subset of $\mathbb{P}^n$ which is the union of the varieties $Y_J$ (if $n = 3$, $r = 2$, the subsets $Y_J$ can be see as “tubes”, whence the name “tubular set”). If $a$ is the ideal of $X$, then

$$a = \bigcap_{J \in D} p_J, \quad \text{and} \quad X = V(a) = \bigcup_{J \in D} Y_J.$$

The irreducible components of $X$ are the linear varieties $Y_J$, and $\dim X = n - r$. By distributivity of union over intersection,

$$a = \bigcap_{J \in D} \left( \sum_{i=1}^r a_{i,j(i)} \right) = \sum_{i=1}^r \left( \bigcap_{j=1}^{d_i} a_{i,j} \right).$$

Similarly,

$$X = \bigcup_{J \in D} \left( \bigcap_{i=1}^r H_{i,j(i)} \right) = \bigcap_{i=1}^r \left( \bigcup_{j=1}^{d_i} H_{i,j} \right).$$
For $1 \leq i \leq r$, the ideal

$$a_i = \bigcap_{j=1}^{d_i} a_{i,j}$$

is principal, since $a_i = (f_i)$, where

$$f_i(T_0, T_1, \cdots, T_n) = \prod_{j=1}^{d_i} (T_i - a_{i,j}T_0),$$

and $X_i = V(a_i)$ is an hypersurface of degree $d_i$:

$$X_i = \bigcup_{j=1}^{d_i} H_{i,j}.$$ 

Now $a = a_1 + \cdots + a_r$, and

$$X = V(a) = \bigcap_{i=1}^{r} X_i.$$

Then $X$ is an i.t. complete intersection, and

$$\deg X = \sum_{J \in D} \deg Y_J = |D| = \prod_{i=1}^{r} d_i.$$ 

We write $A^n = \mathbb{P}^n \setminus H_0$, where $H_0$ is the hyperplane $T_0 = 0$. The linear variety $Y_J$ is the disjoint union of the affine variety $Y'_J = Y_J \cap A^n$ and of $X_0 = Y_J \cap H_0$. The tubular set $X$ and its affine part $X' = X \cap A^n$ are obtained as disjoint unions

$$X = X_0 \amalg X' \subset \mathbb{P}^n, \quad X' = \coprod_{J \in D} Y'_J \subset A^n.$$ 

The enumeration of points of a tubular set is as follows:

**Theorem 5.1.** Let $(d_1, \ldots, d_r) \in \mathbb{N}^r$ with $1 \leq d_i \leq q$ for any $i$ in $\{1, \cdots, r\}$. The tubular set $X \subset \mathbb{P}^n$ defined above is an i.t. complete intersection, defined over $\mathbb{F}_q$, of dimension $m = n - r$, multidegree $(d_1, \ldots, d_r)$, and degree $d = d_1 \cdots d_r \leq q^r$.

(i) The affine algebraic subset $X'$ satisfies

$$|X'(\mathbb{F}_q)| = dq^m.$$

(ii) The projective algebraic subset $X$ satisfies

$$|X(\mathbb{F}_q)| = dq^m + \pi_{m-1} = d\pi_m - (d-1)\pi_{m-1}.$$ 

**Proof.** We only have to prove (i) and (ii). Apply (10): since $|Y'_J(\mathbb{F}_q)| = q^m$ for any $J \in D$, we get (i), and we prove (ii) by observing that $|X_0(\mathbb{F}_q)| = \pi_{m-1}$. 

Theorem 5.1(i) shows that the bound of Corollary 2.2 is attained in the affine case. The projective case is different: we do not know any examples of i.t. complete intersections in $\mathbb{P}^n$ reaching the bound of Corollary 2.2. Hence, we ask:

**Question 5.2.** What is the value of

$$M_{m,d}(q) = \max_{X} |X(\mathbb{F}_q)|,$$

when $X$ runs over projective i.t. complete intersections of dim. $m$ and degree $d$?
By the preceding, we know that

$$dq^n + \pi_{n-1} \leq M_{m,d}(q) \leq dq^n + d\pi_{n-1}.$$  

In small codimension, the first author put forward in [10, Conj. 12.2]:

**Conjecture 5.3.** If $X \subset \mathbb{P}^n$ is a projective algebraic set defined over $\mathbb{F}_q$ of dimension $m \geq n/2$ and degree $d \leq q + 1$ which is a i.t. complete intersection, then

$$|X(\mathbb{F}_q)| \leq d\pi_m - (d-1)\pi_{2m-n} = d(\pi_m - \pi_{2m-n}) + \pi_{2m-n}.$$  

Conjecture 5.3 has just been proved by Couvreur [6]. We shall be content here to specify that there are two cases where it is easy to verify this conjecture:

— The conjecture is true if $X$ is of codimension 1. This is Serre’s inequality [20]: if $X$ is an hypersurface of dimension $m$ and of degree $d \leq q + 1$, then

$$|X(\mathbb{F}_q)| \leq dq^n + \pi_{m-1}.$$  

— The conjecture is also true if $X$ is a union of linear varieties of the same dimension. Precisely, assume that $X$ is the union of $d$ linear varieties $Y_1, \ldots, Y_d$ of dimension $m \geq n/2$. We prove the inequality by induction on $d$. Write $Y_i(\mathbb{F}_q) = Y_i(1 \leq i \leq d)$ for brevity. If $d = 1$ then

$$|Y_1| = \pi_m = (\pi_m - \pi_{2n-m}) + \pi_{2n-m},$$  

and the assertion is true. Now if $Y_1$ and $Y_2$ are two linear varieties of dimension $m$, then $\dim Y_1 \cap Y_2 \geq 2m - n$. Hence, for $d > 1$,

$$|Y_d \cap (Y_1 \cup \cdots \cup Y_{d-1})| \geq \pi_{2m-n}.$$  

Now note that

$$|Y_1 \cup \cdots \cup Y_d| = |Y_1 \cup \cdots \cup Y_{d-1}| + |Y_d| - |Y_d \cap (Y_1 \cup \cdots \cup Y_{d-1})|.$$  

If we apply the induction hypothesis we get

$$|Y_1 \cup \cdots \cup Y_d| \leq (d-1)(\pi_m - \pi_{2m-n}) + \pi_{2m-n} + \pi_m - \pi_{2m-n} = d(\pi_m - \pi_{2m-n}) + \pi_{2m-n},$$  

which proves the desired inequality.

**References**

[1] A. Borel. *Linear Algebraic Groups.* Second edition. Graduate Texts in Mathematics, 126. Springer, Berlin, 1991.

[2] N. Bourbaki. *Commutative Algebra.* Springer, Berlin, 1989.

[3] N. Bourbaki. *Algebra II. Chapters 4–7.* Springer, Berlin, 2003.

[4] N. Bourbaki. *Algèbre commutative, chapitres 8 et 9.* Springer, Berlin, 1998.

[5] P. Burgisser, M. Clausen, and A. Shokrollahi. *Algebraic Complexity Theory.* Springer, Berlin, 1997.

[6] A. Couvreur. Letter to the Authors, June 17, 2014.

[7] D. Eisenbud and J. Harris. *The Geometry of Schemes.* Graduate Texts in Mathematics, 197. Springer, Berlin, 2000.

[8] M. D. Fried, D. Haran, and M. Jarden. Effective Counting of the Points of Definable Sets over Finite Fields. *Israel Journal of Mathematics*, 85(1-3):103–133, 1994.

[9] W. Fulton. *Intersection Theory.* Second edition. Ergebnisse der Mathematik Und Ihrer Grenzgebiete, 3. Folge. Band 2. Springer, Berlin, 1998.

[10] S. Ghorpade and G. Lachaud. Étale Cohomology, Lefschetz Theorems and Number of Points of Singular Varieties over Finite Fields. *Moscow Mathematical Journal*, 2(3):589–631, 2002. Corrigenda and addenda: *Moscow Mathematical Journal*, 9(2):431–438, 2009.
[11] A. Grothendieck and J. Dieudonné. Éléments de Géométrie Algébrique, Ch. IV, Seconde Partie, volume 24. Publ. Math. I.H.E.S., 1965.
[12] J. Harris. Algebraic Geometry: A First Course. Graduate Texts in Mathematics, 133. Springer, Berlin, 1992.
[13] R. Hartshorne. Algebraic Geometry. Graduate Texts in Mathematics, 52. Springer, Berlin, 1977.
[14] J. Heintz. Definability and Fast Quantifier Elimination in Algebraically Closed Fields. Theoretical Computer Science, 24:239–277, 1983.
[15] T. Kasami, S. Lin, and W. Peterson. New Generalizations of the Reed-Muller Codes Part I: Primitive Codes. IEEE Transactions on Information Theory, IT-14(2):189–199, 1968.
[16] D.R. Khis-Braun [Heath-Brown]. On Chevalley-Warning theorems. Uspekhi Mat. Nauk, 66, no 2(398):223–232, 2011. engl. tr. in Russian Math. Surveys, 66, no 2, 427–436, 2011.
[17] J. Kollár. Looking for Rational Curves on Cubic Hypersurfaces. Higher-dimensional geometry over finite fields, 92–122, IOS Press, Amsterdam, 2008.
[18] G. Lachaud. Number of Points of Plane Sections and Linear Codes Defined on Algebraic Varieties. In Arithmetic, Geometry and Coding Theory, 77–104. de Gruyter, 1996.
[19] R. Rolland. Number of Points of Non-Absolutely Irreducible Hypersurfaces. In Algebraic Geometry and its Applications. Proceedings of the first SAGA Conference, 2007, Papeete, 481–487. Number Theory and Its Applications, 5. World Scientific, Hackensack, 2008.
[20] J.-P. Serre. Lettre à M. Tsfasman. In Journées arithmétiques, Luminy, 1989, Astérisque No 198–200, 351–353. Société Mathématique de France, 1992. = Œuvres, Vol. IV, No 155, 240–242, Springer, Berlin, 2000.
[21] J.-P. Serre. Topics in Galois theory. Second edition. A. K. Peters, Wellesley, 2008.
[22] A.B. Sørensen. Projective Reed-Muller Codes. IEEE Transactions on Information Theory, IT-37(6):1567–1576, 1991.
[23] W. Vogel. Lectures on Results on Bezout’s Theorem. Lectures on Mathematics, 74. TIFR, Bombay and Springer, Berlin, 1984.
[24] E. Warning. Bemerkung zur vorstehenden Arbeit von Herrn Chevalley. Abh. Abh. Math. Semin. Hamb. Univ., 11:76–83, 1935.

Gilles Lachaud, Université d’Aix-Marseille, Institut de Mathématiques de Marseille, Luminy case 907, F13288 Marseille cedex 9, France
E-mail address: gilles.lachaud@univ-amu.fr

Robert Rolland, Université d’Aix-Marseille, Institut de Mathématiques de Marseille, Luminy case 907, F13288 Marseille cedex 9, France
E-mail address: robert.rolland@acrypta.fr