Onset of the Limit Cycle and Universal Three-Body Parameter in Efimov Physics

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The Efimov effect is the only experimentally realized universal phenomenon that exhibits the renormalization-group limit cycle with the three-body parameter parametrizing a family of universality classes. Recent experiments in ultracold atoms have unexpectedly revealed that the three-body parameter itself is universal when measured in units of an effective range. By performing an exact functional renormalization-group analysis with various finite-range interaction potentials, we demonstrate that the onset of the renormalization-group flow into the limit cycle is universal, regardless of short-range details, which connects the missing link between the two universalities of the Efimov physics. A close connection between the topological property of the limit cycle and few-body physics is also delineated.

In the early 1970s, V. Efimov predicted a counterintuitive quantum phenomenon, in which resonantly interacting three bosons form an infinite series of three-body bound states even if the interaction is too weak to support a two-body bound state [1]. The Efimov effect emerges in a wide range of systems including identical bosons [1], mass-imbalanced fermions [2, 3], particles in mixed dimension [4], nucleons [5], magnons [6], and macromolecules such as DNAs [7]. Besides its universality, the Efimov spectrum shows a discrete scale invariance, where the energy eigenvalues of the trimers are related to one another by a universal scaling factor of 22.7^2. This peculiar property provides a unique example of a renormalization-group (RG) limit cycle [8], which refers to a periodic behavior of a RG flow and had been elusive until the emergence of the Efimov effect. Because of the universality and uniqueness, the Efimov effect has been extensively studied in various fields of physics such as atomic, chemical, nuclear, and particle physics. In particular, experimental observations of the Efimov effect in ultracold atoms [9–14] have given an enormous impetus to the development of Efimov physics.

Among the crucial discoveries in recent ultracold atom experiments is the universality in the three-body parameter κ^* (or equivalently the scattering length a_− at the triatomic resonance) [15–17], which sets the energy scale of the lowest-lying Efimov state (see Fig. 1). While low-energy two-body observables are universally described by the s-wave scattering length a, the existence of Efimov states leads to an additional dependence of the low-energy three-body observables on the three-body parameter, which encapsulates short-range details of the three-body physics and had therefore been considered to be non-universal. Recent experiments in ultracold atoms, however, have revealed that a_− takes on almost the same value when measured in units of the van der Waals length r_{vdW} for various atomic species, different internal states, and different Feschbach resonances, suggesting some underlying physics that makes such an agreement possible. Recently, it has been suggested [18–20] that systems other than the atomic van der Waals systems such as nucleons 

![Energy spectrum of three-identical bosons with resonant interaction (not to scale). The abscissa shows the inverse s-wave scattering length a_− and the ordinate shows the square root of the energy eigenvalues. The Efimov states are related to one another by the universal scaling factor of 22.7^2. This peculiar property provides a unique example of a renormalization-group (RG) limit cycle, which refers to a periodic behavior of a RG flow and had been elusive until the emergence of the Efimov effect.](image-url)

\[ \kappa^* = \frac{a}{\kappa} \]

\[ \kappa^* \approx 22.7^2 \]

\[ a_\pm \approx \frac{\kappa}{\kappa^*} \]

\[ \kappa \]

\[ a \neq 0 \]

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\[ \kappa \]

\[ a \neq 0 \]
In contrast, since we are interested in the universality of the three-body parameter, in which a finite-range nature of the interaction plays a crucial role \[18-20, 27\], we have performed a functional renormalization-group (FRG) analysis for different Hamiltonians with various finite-range interactions. We have obtained an exact RG flow of the three-body coupling constant, which is defined as a dimensionless particle-dimer scattering amplitude (see Fig. 2). We have found that, in contrast with the zero-range model, the RG flow starts at a point away from the limit cycle and exhibits characteristic behaviour that depends on the short-range details of each individual interaction potential; however, in the infrared regime, the flow begins to show the limit-cycle behavior, the onset of which is found to give the universal value of the three-body parameter \(\kappa^* r_{\text{eff}} = 0.49\). The onset is evaluated as the RG scale at which the first divergence of the coupling constant occurs. By using the universal relation between \(\kappa^* \) and \(a_-=19\), we obtain \(a_-= -4.3 r_{\text{eff}}\) in excellent agreement with the experimental results \[13-17\]. We thus identify the universality of the onset of the limit cycle with that of the three-body parameter. We note that the non-perturbative nature of the FRG have played a decisive role in revealing this relation, since we have to deal with a diverging coupling constant which the perturbative Wilsonian RG cannot deal with. It is striking that the geometrical property (i.e., the onset point) of the limit cycle can be related to the universality of the three-body parameter in the Efimov physics. This observation can be further generalized to the topological constraint of the limit-cycle behavior on the relationship between an Efimov state and its four-body companions as we discuss later.

We now present our theoretical framework for obtaining these results. To perform an exact RG analysis on finite-range interactions, we use a simple microscopic model which accurately reproduces pair correlations of model interaction potentials and can be solved exactly for three particles. We use a separable-potential model whose interaction Hamiltonian is written in the form of a projection operator \(\hat{V}_f = \xi |\chi\rangle \langle \chi|\), which retains the simplicity of the zero-range (delta-function) interaction \(\hat{V}_z = g |\mathbf{r}\rangle \langle \mathbf{r}|\). The microscopic action for identical bosons is then written as
\[
S[\psi, \psi^*] = \int_Q \psi^*(Q)(i q^0 + q^2 - \mu)\psi(Q) + \frac{\xi}{4} \int_{Q_1 Q_2 Q_3 Q_4} \delta(Q_1 + Q_2 - Q_1' - Q_2') \chi^* \left( \frac{q_1' - q_2'}{2} \right) \chi \left( \frac{q_2 - q_1}{2} \right) \times \psi^*(Q_1') \psi^*(Q_2') \psi(Q_2) \psi(Q_1),
\]
where \(Q\) denotes the four momentum consisting of Matsubara frequency \(\nu_0\) and momentum \(\mathbf{q}, \mu\) is the chemical potential, \(\chi(q) := \langle q | \chi \rangle\) in momentum representation, \(\psi\) denotes the bosonic field, and \(\int_Q = \int \frac{d^3q d^3q'}{(2\pi)^3}\). Throughout this Letter, we employ the units \(\hbar = k_B = 2m = 1\), where \(k_B\) is the Boltzmann constant and \(m\) is the mass of the particle.

We can choose an appropriate \(|\chi\rangle\) of \(\hat{V}_f\) so that \(\hat{V}_f\) reproduces the low-energy pair correlation, including a nonzero range, of model potentials. This approximation can be systematically developed with arbitrarily high accuracy by adding another projection term to \(\hat{V}_f\) \[20\]. The construction procedure of \(|\chi\rangle\) is described in Refs. \[28-27\]. Despite its simplicity, \(\chi(q)\) reproduces two-body observables, including phase shifts and bound-state energies, of exact model potentials with high accuracy. Here we use four different types of the separable models: van der Waals, Yukawa, infinite square-well, and Gaussian, which are available in Refs. \[21-27\].

Based on this model, we perform an exact RG analysis based on FRG, which provides a non-perturbative...
RG scheme dealing with strongly correlated situations as Efimov physics. We start from the Wetterich equation [29]:

\[ \partial_k \Gamma_k[\psi, \psi^*] = \frac{1}{2} \text{Tr} \delta_k \ln \left( \frac{\delta^2 \Gamma_k}{\delta \psi(q) \delta \psi^*(q)} + R_k(q) \right), \]  

(2)

where \( \Gamma_k \) is the one-particle irreducible (1PI) effective action of the scale-dependent action \( S_k = S + \int Q R_k(q) \psi^*(Q) \psi(Q) \) and reduces in the ultraviolet limit \( k = \Lambda \) to the microscopic action \( S \) and in the infrared limit \( k = 0 \) to the usual effective action \( \Gamma \), defined as the Legendre transform of the Schwinger functional. The symbol Tr implies the sum over momenta, Matsubara frequencies, and internal indices. The symbol \( \delta_k \) acts only on the Litim’s optimized regulator [30] \( R_k(q) := (k^2 - q^2) \Theta(k^2 - q^2) \), where \( \Theta \) is the unit-step function. To deal with the RG flow of the three-body coupling constant, we perform a vertex expansion [31] of Eq. (2) with respect to the field variables to derive the RG equations for 1PI vertices:

\[ \Gamma_k = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int_{K_1, \ldots, K_n} \Gamma_k^{(2n)}(K_1, \ldots, K_n; K_n', \ldots, K_1') \times \delta(K_1 + \cdots + K_n - K_n' - \cdots - K_1') \times \psi^*(K_1) \cdots \psi^*(K_n) \psi(K_n') \cdots \psi(K_1'), \]  

(3)

where \( \Gamma_k^{(2n)} \) is the \( 2n \)-th-order 1PI vertex, which represents the correlation of \( n \) particles. Since we are interested only in the three-body physics, we have only to consider terms up to \( n = 3 \). Indeed, the exact RG flow equations are closed up to \( n = 3 \) since in the vacuum limit (i.e. the limits of diverging inverse temperature \( \beta \to \infty \) and the vanishing number density of particles \( n \to 0 \)), the physics of four or more number of particles does not affect the three-body physics [28]. In other words, in the vacuum limit, the diagrams containing particle-hole loops vanish because of the infinitely large chemical potential, which leads to decoupling of higher-order vertices from lower-order vertices, allowing an exact treatment of the RG equations.

We first consider one- and two-body sectors, which renormalize the three-body coupling constant. The exact RG equations in the vacuum limit for one- and two-body sectors are depicted in Figs. 3(a) and 3(b), respectively. Noting the ultraviolet boundary condition \( \Gamma_k = S \) \( (k = \Lambda) \), we find that the one-body sector is given as

\[ \Gamma_k^{(2)}(P) = i p^0 + p^2 - \mu, \]  

(4)

which is consistent with the fact that the self-energy correction is absent in the particle vacuum. Because of the separate dependence on the relative momentum of the separable model, the two-body sector can be decomposed into the total-momentum and the relative-momentum parts as depicted in Fig. 3(c), providing an analytical solution as follows:

\[ \frac{1}{\Gamma_k^{(4)}}(P; P_1, P_2) = \chi^*(P_2) \Gamma_k^S(P) \chi(P_1), \]  

(5)

\[ \frac{1}{\Gamma_k^S(P)} = \frac{1}{16 \pi a} - \frac{1}{2} \int \frac{d^3 l}{(2\pi)^3} \left[ i p^0 + \frac{p^2}{2} - 2 \mu + R_k(\frac{p}{2} + 1) + R_k(\frac{-p}{2} - 1) \right] \chi(l)^2, \]  

(6)

where \( a \) is the \( s \)-wave scattering length.

The three-body sector can then be solved numerically based on these analytical expressions. Following a similar procedure as in Ref. [32], we decompose the six-point 1PI vertex as described in Fig. 4. The exact RG flow equation for the three-body sector can then be analytically integrated with respect to \( k \) and composed into a simple form as depicted in Fig. 4. We note that in the infrared limit \( k = 0 \), this integrated RG equation reduces to the Skornyakov-Ter-Martirosyan equation [33] for the separable model. Since we are only interested in the spatially isotropic \( s \)-wave component, which is relevant for Efimov physics, we make a projection onto \( T_k(p, q) := \int d^3 p_q T_k(p_{\text{onshell}} = 3 \mu; p, q) \), and define the dimensionless three-body coupling constant \( g_3 \) as a rescaled particle-dimer scattering amplitude as

\[ g_3 := k^2 T_k(p = 0, q = 0). \]  

(7)
constant four-body physics has shown that the four-body coupling induces a universal scaling [35, 36]. We may relate this universally appear associated with one Efimov state existing as [35, 36]. We suggest that such a nontrivial topology of the limit cycle may support the robustness of the number of bound states against a continuous change of the Hamiltonian.

By solving the exact RG equation of the three-body coupling constant for the four different types of inter-particle interaction numerically, we obtain Fig. 2. We can see that the RG flows for the four different potentials show the interaction-dependent behavior at high energy; however, in the infrared regime, flows converge to the limit cycle. The onset point of the limit cycle is evaluated as $k \alpha_{\text{eff}} = 0.49(4)$, in excellent agreement with the universal three-body parameter $\alpha_{-}/r_{\text{eff}} = -0.43$. This observation suggests that the universality of the three-body parameter can be understood from the RG point of view as the universality of the onset of the limit cycle, which provides the first example relating the geometrical aspect of the limit cycle to the universal property of few-body physics. Our result also suggests that the three-body parameter can be regarded as the energy scale below which the discrete scale invariance of the system (including not only the Efimov states but also the periodic momentum dependence of the scattering observables) emerges.

In this Letter, we have connected the missing link between two universalities of Efimov physics, namely the universal discrete scaling of the energy spectrum and the universal three-body parameter, by demonstrating that the renormalization-group limit cycle starts at the same point, regardless of short-range details. An intriguing extension of the present work is to relate topological aspects of the limit cycle with universal properties of few-body physics. For example, when four identical bosons interact via a resonant interaction, two four-body bound states universally appear associated with one Efimov state exhibiting a universal scaling $\frac{\alpha_{-}}{r_{\text{eff}}}$. We may relate this universal four-body bound states with a topological aspect of RG limit cycle. The previous RG analysis of four-body physics has shown that the four-body coupling constant $g_4$ forms a closed RG limit cycle which is solely induced by the limit cycle of the three-body coupling constant $g_3$ [37]. From this result we suggest that if we constitute a torus of the $g_3 - g_4$ space by enclosing the space periodically, the closed limit cycle winds twice on the torus as schematically illustrated in Fig. 4. Since the winding number of the limit cycle on the torus is topological, we may conclude that the number of four-body bound states is a topological winding number irrespective of the details of inter-particle interactions. This may afford a fundamental example that relates a topological property of a limit cycle to a universal property of few-body physics with scaling violation.

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FIG. 5. Schematic illustration of the limit cycle on the space of the three- and four-body coupling constants $g_3$ and $g_4$. A torus of $g_3 - g_4$ space can be constructed by enclosing the space periodically. The limit cycle winds twice in the $g_4$ direction while it winds once in the $g_3$ direction. This reflects the fact that each Efimov state is associated with two four-body bound states $g_3, g_4$. We suggest that such a nontrivial topology of the limit cycle may support the robustness of the number of bound states against a continuous change of the Hamiltonian.

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