Application of M.Riesz potentials for solving a 3D inverse problem of acoustic sounding

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Abstract. An inverse coefficient problem for time-dependent 3D wave equation is under consideration. We recover a spatially varying coefficient of this equation knowing special time integrals of the wave field in an observation domain. The inverse problem has applications to the acoustic sounding, medical imaging, etc. We reduce the inverse problem to a new linear 3D Fredholm integral equation of the first kind in which the integral operator has the form of well-known M.Riesz potentials. The equation has a unique solution for a considered class of coefficients. Assuming a special scheme for recording the data of the inverse problem, we present and substantiate a numerical algorithm of solving this integral equation. The algorithm does not require significant computational resources and a long solution time. It is based on the use of fast Fourier transform. Typical results of solving 3D inverse problem in question on a personal computer for simulated data demonstrate high capabilities of the proposed algorithm.

1. Introduction

We consider the wave equation with homogeneous initial conditions:

\[
\begin{align*}
\frac{1}{c^2(x)} u_{tt}(x,t) &= \Delta u(x,t) - g(t) \varphi(x), \quad x \in \mathbb{R}^3, \ t > 0 \\
u(x,0) &= u_t(x,0) = 0, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3.
\end{align*}
\]

For this equation, the coefficient inverse problem is usually formulated as follows: knowing the wave field \( u(x,t) \) in a domain \( Y \subset \mathbb{R}^3 \) and the source functions \( g(t), \varphi(x), x \in D \), to find the function \( c(x) \) in a known domain \( X, X \cap Y = \emptyset, X \cap D = \emptyset \), if \( c(x) \) is a known constant \( c_0 \) outside the region \( X \). A detailed statement of the inverse problem with an indication of the spaces used is given below. Such a task, as established in a number of works (see, e.g. [1, 2, 3]), requires for its solution significant computing resources (supercomputer, long calculation time, etc.). However, there are alternative statements of such a problem, in which special integral functionals of the function \( u(x,t) \) are used as data. For example, in [4], a three-dimensional linear integral equation of the first kind was obtained for finding the function \( \xi(x) = \frac{1}{c_0^2} - \frac{1}{c^2(x)} \) associated with \( c(x) \). The right-hand side of this equation contains the function \( V_2(x) = \int_0^\infty t^2 u(x,t) dt, \ x \in Y \) that can be calculated from the scattered wave field or directly measured. However, the solution of the integral equation may not be unique, and it was suggested in [4] to seek the normal solution. In the present report, the uniqueness problem is eliminated by considering a new integral equation.
2. A new equation for finding the function $c(x)$

Our goal is to obtain the basic integral equation with a unique solution for finding the function $c(x)$. We assume that the following conditions are satisfied. 1) $u(x, t) \in C^{2,2}(\mathbb{R}^3 \times [0, \infty))$; 2) the integrals $V_0(x) = \int_0^\infty u(x, t)dt$, $V_2(x) = \int_0^\infty t^2 u(x, t)dt$ converge for all $x \in \mathbb{R}^3$, and the functions $V_0(x)$, $V_2(x)$ are regular at infinity ($|x| \to \infty$); 3) the equalities

$$
\int_0^\infty \Delta u(x, t)dt = \Delta \left( \int_0^\infty u(x, t)dt \right),
\int_0^\infty t^2 \Delta u(x, t)dt = \Delta \left( \int_0^\infty t^2 u(x, t)dt \right),
$$

are valid; 4) $\lim_{t \to \infty} t u(x, t) = \lim_{t \to \infty} t^2 u(t, x) = 0$, $\forall x \in \mathbb{R}^3$; 5) the auxiliary function $\xi(x) = \frac{1}{c_0} - c^{-1}(x)$ is continuous in $\mathbb{R}^3$ and compactly supported in the bounded domain $X$; the function $\varphi(x) \in C^1(\mathbb{R}^3)$ is positive and compactly supported in a bounded domain $D$; $g(t) \in C[0, +\infty)$; the integrals $A_0 = \int_0^\infty g(t)dt$, $A_2 = \int_0^\infty t^2 g(t)dt$ converge, and $A_0 \neq 0$.

Similar conditions were used in the article [4], in which their realizability is discussed.

Integrating the equation (1) with respect to the argument $t$ under conditions 1) - 5), we find:

$$
\Delta V_0(x) = A_0 \varphi(x),
$$

and therefore the regular at infinity solution of this equation, that is the function

$$
V_0(x) = -\frac{A_0}{4\pi} \int_D \frac{\varphi(x')dx'}{|x-x'|} \in C^2(\mathbb{R}^3),
$$

can be precomputed. Thereafter, it is assumed to be known. Moreover, we further assume that the function $V_2(x) = \int_0^\infty t^2 u(x, t)dt$ and its analogue $V_2^{(0)}(x) = \int_0^\infty t^2 u^{(0)}(x, t)dt$ for the case when there are no scatterers in the region $X$ can be measured or calculated in the region $Y$.

Introducing the finite function $\zeta(x) = V_0(x)\left(\frac{1}{c_0} - \frac{1}{c^{-1}(x)}\right)$ associated with $c(x)$, we can further obtain from (1) the equality

$$
\int_X \zeta(x')dx' = w(x), \ x \in Y,
$$

with $w(x) = \frac{1}{2}b(\lambda) \left\{ (-\Delta)^{\frac{3}{2}}(V_2(x) - V_2^{(0)}(x)) \right\}, b(\lambda) = 2^{3-\lambda} \pi^{3/2} \Gamma \left( \frac{3-\lambda}{2} \right) \Gamma^{-1} \left( \frac{1}{2} \right)$. The function $w(x) \in L_2(Y)$ can be found by using the functions $V_2(x), V_2^{(0)}(x) \in L_2(X)$. How to do this will be discussed in Sec.3. Then the relation (2) is the required integral equation for $\zeta(x)$. To the left in this equation, there is an integral, which is called the M.Riss potential. As shown by M. Riss in 1938 [5], the solution of such an equation is unique for $1 < \lambda < 3$ (and for some other $\lambda$). Especially convenient is the case $\lambda = 2$, in which

$$
\int_X \frac{\zeta(x')dx'}{|x-x'|^2} = w(x), \ x \in Y; w(x) = \pi^2 (-\Delta)^{\frac{1}{2}}[V_2(x) - V_2^{(0)}(x)].
$$

Below we will solve this particular integral equation.

3. Data for the inverse problem

The initial data for the inverse problem (3) are the function $V_2(x) = \int_0^\infty t^2 u(x, t)dt$, $x \in Y$, and its analogue $V_2^{(0)}(x)$. For the inverse problem of acoustic sounding, their finding is associated with a special accumulation of information about the sound pressure $u(x, t)$ at the registration points. To solve the equation (3), preliminary data processing is required, namely, the calculation of the function $w(x) = \pi^2 (-\Delta)^{\frac{1}{2}}[V_2(x) - V_2^{(0)}(x)]$. Now we describe a special algorithm for calculating it. The algorithm uses the system of eigenfunctions $\Psi_k(x)$ and the eigenvalues $\{\lambda_k\}$ of the Laplace operator for the domain $Y$ that can be found from the boundary value problem

$$
\left\{ \begin{array}{l}
\Delta \Psi(x) + \lambda \Psi(x) = 0, \ x \in Y \\
\Psi(x)|_{\partial Y} = 0.
\end{array} \right.
$$
Algorithm 1 (for computing the function $w(x)$ in the domain $Y$).

1. Find a function $\tilde{u}(x) \in L_2(\mathbb{R}^3)$ satisfying the condition $\tilde{u}(x)_{|\partial Y} = u(x)_{|\partial Y}$ and "sufficiently fast" decreasing as $|x| \to \infty$.

2. Calculate the function $\tilde{U}(x) = (-\Delta)^{\frac{1}{2}} \tilde{u}(x) = F^{-1}(|\omega|F(\tilde{u})(\omega))$ by the use of the direct and inverse Fourier transforms $F(\cdot)(\omega)$, $F^{-1}(\cdot)(x)$.

3. Find the expansion of the function $v(x) = u(x) - \tilde{u}(x)$ with respect to the basis $\{\Psi_k(x)\}$: $v(x) = \sum_k C_k \Psi_k(x)$ in the domain $Y$.

4. Calculate the function $W(x) = (-\Delta)^{\frac{1}{2}} v(x) = \sum_k C_k \sqrt{\lambda_k} \Psi_k(x)$, $x \in Y$. Here, the regularization of this ill-posed problem may be necessary.

5. Find the function $w(x) = W(x) + \tilde{U}(x)$, $x \in Y$.

4. Scheme for obtaining data of the inverse problem and a special form of the basic equation

In what follows, we take for the coordinates $x = (x_1, x_2, x_3)$ the notation $x, y, z$. Then for the geometric scheme of solving the inverse problem in a plane layer (see Fig. 1), that is for $X \subset \mathbb{R}^2_{xy} \times [h, H]$, $Y = \mathbb{R}^2_{xy} \times [l, L]$, $H < l$, the equation (3) can be written in the form

$$\int_h^H d' = \int_{\mathbb{R}^2_{xy}} K(x - x', y - y', z - z') \zeta(x', y', z') dx' dy' = w(x, y, z),$$

$(x, y) \in \mathbb{R}^2_{xy}$, $z \in [l, L]$, with the kernel $K(x, y, z) = (x^2 + y^2 + z^2)^{-2}$. From the assumptions about the function $c(x, y, z)$ and the form of the functions $K, w$, the inclusions $\zeta \in L_2(X)$, $w \in L_2(Y)$ can be derived, as well as the inclusions $\zeta(x', y', z') \in L_2(\mathbb{R}^2_{xy})$, $w(x, y, z) \in L_2(\mathbb{R}^2_{xy})$ for every admissible $z, z'$. The following relations also hold: $||K(x, y, z - z')||_{L_2(\mathbb{R}^2_{xyz})} < \infty$, $\forall z, z' \in [l, L]$, $x, y \in [h, H]$. Then, using the two-dimensional Fourier transforms of the functions $K, \zeta$ and $w$ with respect to the variables $(x, y)$, i.e. the values $\hat{K}(\omega_1, \omega_2, z)$, $\hat{\zeta}(\omega_1, \omega_2, z), \hat{w}(\omega_1, \omega_2, z)$, we obtain by the convolution theorem the family of one-dimensional integral equations of the first kind:

$$\int_h^H \hat{K}(\omega_1, \omega_2, z - z') \hat{\zeta}(\omega_1, \omega_2, z') dz' = \hat{w}(\omega_1, \omega_2, z), \quad z \in [l, L].$$

(4)

To solve these equations, we first need to find $w(x)$ using Algorithm 1, and then calculate $\hat{w}(\omega_1, \omega_2, z)$.

5. Method for solving the integral equations

Equations (4) can be represented in the operator form $A_0(\omega_1, \omega_2)\hat{\zeta} = \hat{w}(\omega_1, \omega_2, z)$ with linear bounded integral operators $A_0(\omega_1, \omega_2)\hat{\zeta} = \int_h^H \hat{K}(\omega_1, \omega_2, z - z') \hat{\zeta}(\omega_1, \omega_2, z' ) dz'$ acting from $L_2[h, H]$ to $L_2[l, L]$. These equations belong to the class of linear operator equations $A_0 \varphi = \psi$, where $\varphi \in Z$ is an unknown, $\psi \in U$ is data, and $Z, U$ are Hilbert spaces. At present, many stable methods (regularizing algorithms, RAs) are worked out for finding solutions of linear operator equations in Hilbert spaces (see, e.g. [6, 8]). Suppose that a parametric family of operators $R_\alpha(A_0, \psi) : U \to Z$ represents one of these RAs. We assume that we have at our disposal approximate data of the problem $A_0 \varphi = \psi$ with a known level of error $\delta$, i.e. instead of $\psi$ the element $\psi_\delta \in U$ is given for which the estimate $||\psi - \psi_\delta||_U \leq \delta$ holds. Then, with the proper choice of the parameter $\alpha = \alpha(\delta)$, the convergence $||\varphi_\delta - \varphi||_Z \to 0$ as $\delta \to 0$ is provided for approximate solutions $\varphi_\delta = R_\alpha(\delta)(A_0, \psi_\delta).$ Sufficient conditions for such convergence (see [6]) are the regularity conditions for elements $\varphi_\delta$: $\lim_{\delta \to 0} ||\varphi_\delta||_Z \leq ||\varphi||_Z$, $\lim_{\delta \to 0} ||A_0 \varphi_\delta - \psi_\delta||_U = 0$. 

Theorem 1. Assume that the Cauchy data $u(x)_{|\partial Y} = v(x)$ are given in every layer $x \in \mathbb{R}^2_{xy} \times [l, L]$ where $H < l$. Then for $x \in \mathbb{R}^2_{xy} \times [h, H]$, we have $w(x) = (\Delta)^{\frac{1}{2}} (u - v)_{|\partial Y}$.
Now we formulate the Algorithm 2 of solving the inverse problem, assuming that \( w(x) \) is already found by Algorithm 1.

1) Computation of two-dimensional Fourier transforms \( \tilde{K}(\omega_1, \omega_2, z) \) for each \( z \in [h, H] \), and \( \tilde{w}(\omega_1, \omega_2, z) \) for each \( z \in [l, L] \).

2) Finding the approximate solution of the integral equation (4), \( \tilde{\zeta}_\delta(\omega_1, \omega_2, z) \), for each \( (\omega_1, \omega_2) \) with the aid of an RA that ensures the fulfillment of the regularity conditions.

3) Calculation of the approximate solution \( \zeta_\delta(x, y, z) \) for each \( z \in [h, H] \) using the two-dimensional inverse Fourier transforms \( F^{-1}: \zeta_\delta(x, y, z) = F^{-1}[\zeta_\delta(\omega_1, \omega_2, z)](x, y) \).

4) Finding the function \( \xi_\delta(x, y, z) = \zeta_\delta(x, y, z)/V_0(x, y, z) \) that is an approximation to \( \xi(x, y, z) \), and, further, an approximate determination of the coefficient \( c(x, y, z) \) from the equality \( \xi(x, y, z) = \frac{1}{c_0} - \frac{1}{c^2(x, y, z)} \).

We can prove the following assertion.

**Theorem 1.** Algorithm 2 ensures the convergence of \( \|\zeta_\delta(x, y, z) - \tilde{\zeta}(x, y, z)\|_{L_2(X)} \to 0 \) of approximate solutions to the exact solution of the inverse problem, \( \tilde{\zeta}(x, y, z) \), as \( \delta \to 0 \).

The regularity conditions required in Sec. 5) for the used RA are fulfilled, for example, for Tikhonov regularization with a choice of the parameter by the discrepancy principle or by the generalized discrepancy principle [6]. Similar property has well-known TSVD method (see, e.g. [7, 8]) with the appropriate choice of the regularization parameter.

6. The finite-dimensional approximation of the problem and the numerical implementation of Algorithm 2

We replace the spaces \( \mathbb{R}^3 \) and \( \mathbb{R}_{xy}^2 \) by the domains \( \Pi = [-r, r] \times [-r, r] \times [-r, r] \) and \( \Pi_{xy} = [-r, r] \times [-r, r] \) respectively with \( r \) large enough and carry out an approximation of the equations (4) in \( \Pi \) by a finite-difference method. To this end, we introduce uniform grids with respect to \( x, \omega_1 \in [-r, r] \) and \( y, \omega_2 \in [-r, r] \) of size \( N \), as well as grids for \( z, z': \{ z_i \} \in [l, L], \{ z'_j \} \in [h, H] \) of size \( M \) and \( M' \), accordingly.

Using the fast Fourier transform in part 1 of the Algorithm 2 for calculating the discrete analogs of the functions \( \tilde{K}(\omega_1, \omega_2, z), \tilde{w}(\omega_1, \omega_2, z) \), we obtain \( N^2 \) systems of linear algebraic equations (SLAEs) for \( \tilde{\zeta} \):

\[
A_0^{(m)}\tilde{\zeta}^{(m)} = \tilde{w}^{(m)}, \quad m = 1, \ldots, N^2,
\]

with the matrices \( A_0^{(m)} = [\nu_{ij}\tilde{K}(\omega_1^{(m)}, \omega_2^{(m)}, z_i - z'_j)] \) and the right-hand sides \( \tilde{w}^{(m)} = \tilde{w}(\omega_1^{(m)}, \omega_2^{(m)}, z_i) \). Here \( \omega_1^{(m)}, \omega_2^{(m)} \) are points of the grid with respect to \( (\omega_1, \omega_2) \) enumerated by a single index \( m \), and \( \nu_{ij} \) are quadrature coefficients.

Systems (5) should be solved with the help of RAs that guarantee the regularity conditions. For these purposes, we used several regularizers: Tikhonov and iterated Tikhonov algorithms, the TSVD method and other. Numerical experiments showed that the best results when solving systems (5) gives the TSVD method. These results are presented in the next section.

7. Numerical experiments

For the experimental scheme shown in Fig. 1, we define the model solution in the form

\[
\xi(x, y, z) = a_1 \exp(-x^2 - 2y^2) + a_2 \exp[-3(x + 4)^2 - (y - 5)^2 +
\quad + (x + 4)(y - 5)] + a_3 \exp{-0.9 [(x - 4)^2 - (y + 4)^2 + (x - 4)(y + 4)]},
\]

with \( a_1 = 16 \max \{0.16 - (z - 1.5)^2, 0\}, \quad a_2 = 13 \max \{0.04 - (z - 1.3)^2, 0\}, \quad a_3 = 8 \max \{0.04 - (z - 1.7)^2, 0\} \) and \((x, y, z) \in X = [-10, 10] \times [-10, 10] \times [1, 2] \). The region of registration of the scattered wave field is \( Y = [-10, 10] \times [-10, 10] \times [6, 7] \). We use a \( \delta \)-like source.
Figure 1. Scheme for data measuring ‘in a layer’ $Y$ with $X = R$. Asterisks show conditional positions of the field sources.

$g(t)\varphi(x)$ with position $(0,0,5)$ assuming also that $g(t) = \exp(-t)$ and $c_0 = 1$. Next, we calculate the model functions $w(x)$ and $u(x)$ from the equalities

$$w(x) = \int_X \frac{\xi(x')V_0(x')dx'}{|x - x'|^2}, \quad u(x) = \frac{1}{2\pi^4} \int_{R^3} \frac{w(x')dx'}{|x - x'|^2}, \quad x \in Y,$$

by applying the discrete Fourier transform (algorithmically, fast Fourier transform) on fine grids $N = M = M' = 2048$. Then we form the matrices $A_0^{(m)}$ of the systems of linear equations (5) and the right-hand sides $\tilde{w}^{(m)}$ on coarser grids ($N = 128 - 512, M = M' = 51 - 100$). We solve the systems (5) by the proposed algorithm and find the function $\xi(x) = \frac{\xi(x)}{V_0(x)}$. Finally, we compute from the approximately found function $\xi(x)$ the so-called contrast $\mu = \frac{c_0^2}{c^2(x)} = 1 - c_0^2\xi(x)$.

In Fig.2, the exact contrasts $\mu_{\text{exact}} = \frac{c_0^2}{c^2(x)}$ (above) and found approximations $\mu_{\text{appr}}$ (below) are shown in pairs for qualitative comparison for several values of $z \in [1, 2]$. For quantitative comparison of the contrasts $\mu_{\text{exact}}$ and $\mu_{\text{appr}}$, we use the relative error for each $z \in [1, 2]$,

$$\Delta_C(z) = \frac{\|\mu_{\text{appr}}(x, y, z) - \mu_{\text{exact}}(x, y, z)\|_{C(R^2_{xy})}}{\|\mu_{\text{exact}}(x, y, z)\|_{C(R^2_{xy})}}.$$
This dependence is shown in Fig. 3 for exact and perturbed data. The solid line represents the value $\Delta C(z)$ for the unperturbed data $u(x)$, and the dotted line depicts the same for data with random point-wise perturbation of the order $10^{-8}$. Calculations were carried out in MATLAB on a PC with an Intel (R) Core (TM) i7-7700 CPU 3.60 GHz, 16 GB RAM (without parallelization).

8. The speed of the algorithm

Since we are talking about the creation of a fast method for solving our inverse problem, we point out some temporal characteristics of proposed algorithms. The dimensions of grids in the variables $z, z'$, that is, the numbers $M$ and $M'$, determine the speed of solving the one-dimensional integral equation (4) at fixed $\omega_1, \omega_2$, or what is the same, solving the SLAE (5) for a fixed $m$. The corresponding solution time, $t_0(M, M')$, varies a little in passing from one of equations (5) to another. This time is controlled by the desired resolution of the algorithm in the variable $z$. Actually, we have the estimate $t(N, M, M') \approx t_0(M, M') \cdot N^2$ for full solution time of the inverse problem for chosen grids, and the number $N$ is controlled here by the required resolution in $x, y$. Figure 4 presents the dependence $t(N) = t(N, M, M')$ calculated on the same computer for fixed $M = 51, M' = 51$ and for different $N$. In particular, the time to solve the three-dimensional inverse problem for $N = 512$ is less than 10 minutes. Data preparation, that is calculation of $w(x)$, takes about 3 minutes.

9. Conclusions

1) The 3D inverse coefficient problem for the wave equation that arises in the modeling of acoustic sounding can be solved numerically faster if its data are not the time dependencies of the scattered field $u(x, t)$ in a region of space $Y$, but some integrals of these dependencies in time. Possible integrals of such a kind are the functions $V_2(x)$, $V_2^{(0)}(x)$ or their partial derivatives.

2) For this type of data recorded in a plane layer, a numerical algorithm is proposed that allows to solve the inverse problem on a personal computer for sufficiently fine grids in a relatively short time, without the use of supercomputer systems. A similar algorithm can be proposed for other schemes of data acquisition.

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