Absolutely continuous spectrum for random Schrödinger operators on tree-strips of finite cone type.

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A tree-strip of finite cone type of width $m$ is the product graph of a finite graph $G$ with $m$ vertices and a rooted tree of finite cone type.

The product graph of two graphs $G_1$ and $G_2$ is defined as follows: $(x_1, y_1)$ and $(x_2, y_2)$ are connected by an edge, iff either $x_1 = x_2$ and $y_1$ and $y_2$ are connected by an edge, or $y_1 = y_2$ and $x_1$ and $x_2$ are connected by an edge.
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A tree of finite cone type can be constructed from an \( s \times s \) substitution matrix \( S \) with non-negative integer entries, as follows: Each vertex has a label \( p \in \{1, \ldots, s\} \). A vertex of label \( p \) has \( S_{p,q} \) children of label \( q \). Except for the root, a vertex of label \( p \) has \( \sum_q S_{p,q} + 1 \) neighbors. The tree is determined by the label of the root, if the label is \( p \) we call the tree \( T(p) \).

The cone of descendents of a vertex with label \( p \) is equal to \( T(p) \). Hence, there are only finitely many different types of cones of descendents and therefore the tree is called of 'finite cone type'.
We consider the Hilbert space 
\[ \ell^2(\mathbb{T}^{(p)} \times \mathbb{G}) = \ell^2(\mathbb{T}^{(p)}) \otimes \mathbb{C}^m = \ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m). \] An element 
\[ u \in \ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m) \] is considered as a function 
\[ u : \mathbb{T}^{(p)} \rightarrow \mathbb{C}^m \] with 
\[ \sum_{x \in \mathbb{T}^{(p)}} \|u(x)\|^2 < \infty. \]

On \( \ell^2(\mathbb{T}^{(p)}, \mathbb{C}^m) \) we consider the random Schrödinger operators 
\[ (H_\lambda u)(x) = \sum_{y : d(x,y) = 1} u(y) + Au(x) + \lambda V(x) u(x). \]
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\( V(x) \in \text{Sym}(m) \) for \( x \in B \) are i.i.d. random, real symmetric matrices with common distribution \( \mu \). We assume that all mixed moments of \( \mu \) exist. We may think of \( V(x) \) as random matrix-valued potential.

\( \lambda \in \mathbb{R} \) is the disorder and supposed to be small.

One may write 
\[ H_\lambda = \Delta \otimes 1 + 1 \otimes A + \lambda \bigoplus_{x \in T(p)} V(x). \]
Assumptions

We need the following assumptions on the substitution matrix:

\[ \| S \| < \min\{ S_{p, p}, p \}^{2} \]

With some weaker assumptions, Keller, Lenz and Warzel showed that the spectrum of the adjacency operator on \( T(p) \) is purely a.c., independent on \( p \) and consists of finitely many intervals. Let us call this set \( \Sigma \).
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- For each $p, q$ there is a natural number $n$ such that $(S^n)_{p,q}$ is not zero. This means, that each tree $\mathbb{T}^{(p)}$ has vertices of each label $q \in \{1, \ldots, s\}$.
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- With some weaker assumptions, Keller, Lenz and Warzel showed that the spectrum of the adjacency operator on $\mathbb{T}^{(p)}$ is purely a.c., independant on $p$ and consists of finitely many intervals. Let us call this set $\Sigma$. 
We denote the eigenvalues of the free vertical operator $A$ by $a_1 \leq a_2 \leq \ldots \leq a_m$.

There is an orthogonal matrix $O \in O(m)$ such that $O^t A O = \text{diag}(a_1, \ldots, a_m) = A_d$.

Let $Uu(x) = Ou(x)$ then $U$ is unitary and

$$U^* H_\lambda U = \Delta \otimes 1 + 1 \otimes A_d + \lambda \bigoplus_{x \in \mathbb{B}} O^t V(x)O.$$
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Hence by changing the distribution $\mu$ to $O^t \mu O$ we may assume w.l.o.g. that $A = A_d$ is diagonal.

The free operator can be written as direct sum

$$H_0 = \Delta \otimes 1 + 1 \otimes A_d = \bigoplus_{k=1}^m \Delta + a_k$$

and has a.c. band spectrum $\bigcup_k (\Sigma + a_k)$.
Assume $S$ and $A$ are such that $I_{A,S} = \bigcap_k (\Sigma + a_k)$ has non-empty interior.
(For fixed $A$ and $S$ there exists a $b_0$ such that for $b > b_0$ the set $I_{A,bS}$ has non-empty interior).

**Theorem (S. 2012)**

Under the assumptions above, there exists a dense open subset $I \subset I_{A,S}$ and an open neighborhood $U$ of $\{0\} \times \mathbb{R}$ in $\mathbb{R}^2$ such that one obtains the following:

The spectrum of $H_\lambda$ in $U_\lambda = \{E : (\lambda, E) \in U\}$ is purely absolutely continuous with probability one.
Some previous results:

- Klein (1994, 1996) a.c. spectrum and ballistic wave spreading for Anderson model on Bethe lattice (regular tree)
- Aizenman, Sims, Warzel (2006), different proof for ac spectrum
- Froese, Hasler, Spitzer (2007), different proof for ac spectrum
- Keller, Lenz, Warzel (2010), ac spectrum for Anderson model on trees of finite cone type
- Froese, Halasan, Hasler (2010), ac spectrum for Bethe strip of width 2
- Klein, S. (2011) a.c. spectrum and ballistic behavior for Bethe strip of arbitrary finite width
- Aizenman, Warzel (2011), a.c. spectrum for Anderson model on Bethe lattice for small disorder in bigger region; absence of mobility edge for small disorder; ballistic behavior in this regime
The \( m \times m \) Green’s matrix function of \( H_\lambda \) is given by

\[
G_\lambda (x, y; z)_{k,l} = \langle x, k | (H_\lambda - z)^{-1} | y, l \rangle
\]

for \( x, y \in \mathbb{T}^{(p)}, \ k, l \in \mathbb{G} \) and \( z = E + i \eta \) with \( E \in \mathbb{R}, \ \eta > 0 \).
• The \( m \times m \) Green’s matrix function of \( H_\lambda \) is given by

\[
[G_\lambda(x, y; z)]_{k,l} = \langle x, k | (H_\lambda - z)^{-1} | y, l \rangle
\]

for \( x, y \in \mathbb{T}^{(p)}, \ k, l \in G \) and \( z = E + i\eta \) with \( E \in \mathbb{R}, \ \eta > 0 \).

• We define the integrated, averaged density of states by

\[
N_\lambda(E) = \frac{1}{m} \mathbb{E} \left( \sum_{k=1}^{m} \langle 0, k | \chi_{(-\infty, E]}(H_\lambda) | 0, k \rangle \right)
\]

for the root \( 0 \in \mathbb{T}^{(p)} \).
The $m \times m$ Green's matrix function of $H_\lambda$ is given by

$$[G_\lambda(x, y; z)]_{k,l} = \langle x, k | (H_\lambda - z)^{-1} | y, l \rangle$$

for $x, y \in \mathbb{T}^p(p)$, $k, l \in \mathbb{G}$ and $z = E + i \eta$ with $E \in \mathbb{R}$, $\eta > 0$.

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for the root $0 \in \mathbb{T}^p(p)$.

**Theorem**

There is an open neighborhood $U$ of $\{0\} \times I$ as above, such that the following hold: The integrated, averaged density of states is absolutely continuous in $U_\lambda$, its density is strictly positive in $U_\lambda$ and it depends continuously on $(\lambda, E) \in U$. 

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The Theorems above follow from the following:

**Theorem**

There is a dense open subset \( I \subset I_{A,S} \) and an open neighborhood \( U \) of \( \{0\} \times I \) in \( \mathbb{R}^2 \), such that for all \( x \in \mathbb{T}^{(p)} \) the continuous functions

\[
(\lambda, E', \eta) \in U \times (0, \infty) \quad \rightarrow \quad \mathbb{E}(|G_\lambda(x,x;E' + i\eta)|^2)
\]

and

\[
(\lambda, E', \eta) \in U \times (0, \infty) \quad \rightarrow \quad \mathbb{E}(G_\lambda(x,x;E' + i\eta))
\]

have continuous extensions to \( U \times [0, \infty) \).
Idea of Proof

- Let $G$ be a symmetric $m \times m$ matrix with positive imaginary part.
- On the set of pairs of positive semi-definite matrices $M$ we define the exponential decaying functions

$$f_G(M_+, M_-) = \exp(i \text{Tr}(GM_+ - G^* M_-)).$$

- Let $G^{(p)}_{\lambda}(z)$ denote the matrix Green’s function at the root of $T^{(p)}$ for the random operator $H_{\lambda}$ and define

$$\xi^{(p)}_{\lambda, E, \eta} = \mathbb{E}\left(f_{G^{(p)}_{\lambda}}(E + i\eta)\right).$$

- There is a certain Banach space $\mathcal{K}$, such that $\left[\xi^{(p)}_{\lambda, E, \eta}\right]_p$ is in this Banach space for $\text{Im}(z) > 0$ and it converges in $\mathcal{K}$ for $\lambda = 0$, $E \in \tilde{I}_{A, S}$ and $\eta \downarrow 0$. 
Moreover, there is a continuous function $F : \mathbb{R}^2 \times (0, \infty) \times \mathcal{K} \to \mathcal{K}$, Frechet differentiable with respect to the element in $\mathcal{K}$, such that

$$F\left(\lambda, z, [\xi^{(p)}_{\lambda,z}]_p\right) = 0.$$ 

Use the implicit function Theorem at $\lambda = 0$, $E \in I \subset I_{A,S}$ to show that $[\xi^{(p)}_{\lambda,E+i\eta}]_p$ extends continuously to $(\lambda, E, \eta) \in U \times [0, \infty)$ where $U$ is an open neighborhood of $I$ which in turn is a dense open subset of the interior of $I_{A,S}$.

Everything follows from these continuous extensions.