When Superconductivity Crosses Over: From BCS to BEC

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(Dated: October 10, 2023)

New developments in superconductivity, particularly through unexpected and often astonishing forms of superconducting materials, continue to excite the community and stimulate theory. It is now becoming clear that there are two distinct platforms for superconductivity through natural and synthetic materials. Indeed, the latter category has greatly expanded in the last decade or so, with the discoveries of new forms of superfluidity in artificial heterostructures and the exploitation of proximitization. The former category continues to surprise through the Fe-based pnictides and chalcogenides, and nickelates as well as others. It is the goal of this review to present this two-pronged investigation into superconductors, with a focus on those which we have come to understand belong somewhere between the BCS and Bose-Einstein condensation (BEC) regimes. We characterize in detail the nature of this “crossover” superconductivity, which is to be distinguished from crossover superfluidity in atomic Fermi gases. In the process, we address the multiple ways of promoting a system out of the BCS and into the BCS-BEC crossover regime within the context of concrete experimental realizations. These involve natural materials, such as organic conductors, as well as artificial, mostly two-dimensional materials, such as magic-angle twisted bilayer and trilayer graphene, or gate-controlled devices, as well as one-layer and interfacial superconducting films. This work should be viewed as a celebration of BCS theory by showing that even though this theory was initially implemented with the special case of weak correlations in mind, it can in a very natural way be extended to treat the case of these more exotic strongly correlated superconductors.

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VII. Application of BCS-BEC Crossover in the Literature
There has been a recent explosion of papers addressing the concept of BCS–Bose-Einstein condensation (BEC) crossover in superconductors. Besides the well-known applications to ultracold atomic Fermi gases and high-temperature cuprate superconductors [1–3], examples of systems exhibiting BCS-BEC crossover-like characteristics include: iron-based superconductors, organic superconductors, magic-angle twisted bilayer (MATBG) and trilayer graphene (MATTG), gate-controlled two-dimensional devices, interfacial superconductivity, and magnetoexcitonic condensates in graphene heterostructures. BCS-BEC crossover theory is a theory in which one contemplates that the attraction (of unspecified origin) which causes the pairing that drives superconductivity is stronger than in conventional materials. As a result, fermion pairs form at higher temperature before they Bose condense at the superfluid transition temperature $T_s$, much as in a Bose superfluid. This is in contrast to the well-established theory of Bardeen, Cooper, and Schrieffer (BCS), where pairing and condensation occur at precisely the same temperature.

This Review article is written to summarize what has been observed in these candidate two-dimensional (2D) and three-dimensional (3D) superconductors which are somewhere between BCS and BEC. We will discuss how these reported phenomena relate to BCS-BEC crossover, paying special attention to 2D materials where there seems to be a surprisingly large number of examples. In the process we present a theoretical understanding at general temperatures.

Lest there be any confusion at the start, throughout this paper what we mean by “BCS-BEC crossover” is not the onset or proximity to the BEC as defined by some, but an intermediate regime between BCS and BEC, where a significant departure from strict BCS theory is apparent. It should also be stressed that what is being discussed here pertains to the theoretical “machinery” of superconductivity rather than the microscopic pairing mechanism.

We will begin the discussion of BCS-BEC crossover by following the original discovery papers [4, 5] which focus on a particular choice of ground state, namely that having the form originally introduced in BCS theory. While there is a body of literature on alternative approaches to BCS-BEC crossover in the solid state, (some of which is reviewed here), we will focus mainly on this so called “BCS-Leggett” ground state and its finite temperature implications [6] rather than on variants which have ground states that are incompletely characterized and less well understood.

The appreciation of this broader applicability of BCS theory and its straightforward extension to a form of Bose condensation underlines how remarkable the original contribution of Bardeen, Cooper, and Schrieffer was. It should be noted that their discovery has provided support and a crucial framework for multiple Nobel prizes besides their own. One could argue that these number of the order of 10 or so including nuclear and particle physics as well. In this way, the recognition of its even greater generality is particularly significant.

This recognition can be credited to two physicists: A. J. Leggett [4] and D. M. Eagles [5]. Leggett’s contribution was motivated by the discovery of a BCS-like triplet-pairing state in the neutral superfluid helium-3. He emphasized that this form of fermionic superfluidity has features which are clearly distinct from conventional superconductors: here the Cooper pairs have complex degrees of freedom. Moreover, the underlying attraction which leads to superconductivity in this neutral system must derive from a quite distinct pairing mechanism [7].

In making his claims, Leggett pointed to the sweeping gen-
Figure 1. Contrasting behavior of the 3D, s-wave BCS-BEC crossover phase diagram for (a) superconductors, as in the attractive Hubbard model, and (b) Fermi gases, with contact interactions and free particle dispersion. Note the contrasting behavior in the BEC regime where $T_c$ approaches either zero (a) or a finite number (b). Also important is the ubiquitous dome shape in the solid-state system. The minimum or shoulder in both $T_c$ curves marks a transition to a different physical regime, as it corresponds to the onset of a bosonic superfluid, with $\mu = 0$. Important to stress here is that the crossover regime begins at the point where the two temperature scales ($T^*$ corresponding to the opening of a pairing gap) and $T_c$ become distinct. Microscopic units for the superconducting case are provided in Fig. 12(a) in a later section of the paper.

Figure 1.

The greater applicability is exhibited by self consistently adjusting the variational parameters $u_k$ and $v_k$ as one varies the strength of the attractive interaction. This accommodates a continuous evolution from weak to strong pairing. One can replace $u_k$ and $v_k$ by more experimentally relevant parameters: the fermionic chemical potential $\mu$ along with the zero temperature fermionic excitation gap parameter

$$\Delta_0 \equiv \Delta(T = 0).$$

These are two important parameters that we will refer to throughout this Review. Notably, the wavefunction $\Psi_{\text{BCS}}$ supports a smooth transition between a BCS phase having large pair size and small $\Delta_0$ along with chemical potential equal to the non-interacting Fermi energy, $E_F$, and a BEC-like phase having small pair size. In this latter case $\Delta_0$ is large (comparable to or larger than $E_F$) while $\mu$ is negative.

It should be emphasized that this BEC phase is specific to the ground-state fermionic wave function and need not represent that of a true weakly interacting Bose gas. Importantly, within a generalized BCS framework it is relatively straightforward to address finite temperatures above and below $T_c$ [6]; this is, in part, a consequence of the fact that the pairing formalism is closely related to an exactly solvable many-body problem [8].

In a related way, Eagles [5] also made groundbreaking observations. He should be credited with emphasizing the concept of “pairing without superconductivity”. This preformed-pair normal-state scenario is at the heart of BCS-BEC crossover theory, once one leaves the BCS regime. He should be additionally credited with drawing attention to the possibility that superconductivity in lightly doped semiconductors may be considered to be described by a form of BCS-BEC crossover. Indeed, we will see in this Review that there is currently renewed interest in these low carrier density superconductors.

A. Early theoretical work: Extending BCS-BEC crossover theory to finite temperatures

In 1985, Nozières and Schmitt-Rink (NSR) began to think about going beyond the ground state and including the effects of finite temperature. They wrote a famous paper [9] that brought attention back to the earlier work by Eagles and Leggett and presented an in-depth discussion of the ground

earality of the BCS ground state:

$$\Psi_{\text{BCS}} = \Pi_k \left( u_k + v_k \alpha_{k, \uparrow} \alpha_{-k, \downarrow}^\dagger \right) |0\rangle,$$

where $\alpha_{k, \uparrow} \alpha_{-k, \downarrow}^\dagger$ creates a pair of fermions with opposite spin and opposite momentum, $k$ and $-k$, from the vacuum ($|0\rangle$).

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state of Eq. (1). Moreover, they suggested an approach for computing the transition temperature $T_c$. It should be noted however that the extrapolated ground state associated with the NSR’s finite temperature theory is different [10] from the expression $\Psi^{\text{BCS}}$ in Eq. (1). Importantly, the NSR paper was the first to emphasize that BCS-BEC crossover theory in a solid-state lattice system assumes a character in the strong-coupling BEC regime quite different from that of a Fermi gas.

The schematic plot in Fig. 1 relates to this observation. It compares the phase diagram for BCS-BEC crossover in (a) a lattice as contrasted with (b) a Fermi gas. A central difference arises from the kinetic energy degrees of freedom associated with the motion of fermions in periodic solids as distinct from their motion in free space. The most striking consequence is that in a solid which is in the BEC regime, $T_c$ can become arbitrarily small as the pairing strength increases. Indeed, we emphasize this distinction in the present paper, as it bears on the relevance (or lack thereof) of the ultracold atomic Fermi gas superfluids to the solid-state superconductors we discuss here.

Related work in the form of a review was written by Micnas and co-workers in 1990 [11] addressing superconductors in the BEC-like or strong-attraction limit. In their approach a local pairing scenario was adopted, rather like treating a hard core Bose gas on a lattice. The emphasis was on clarifying the various alternative phases which compete with superconductivity. Subsequently, the finite-temperature theory of the NSR paper was followed by work from Sá de Melo, Randeria, and Engelbrecht [12], which provided a functional-integral reformation.

Around the same time, and in collaboration with Trivedi and others [13], these researchers presented a series of papers using Monte Carlo simulation (QMC) techniques to address normal-state features of the attractive 2D Hubbard model. This was thought to be relevant to high-temperature superconductivity and its anomalous “pseudogap” phase. This phase corresponds to an above-$T_c$ or “normal” state in which there is a gap for fermionic excitations. In their work it was presumed that the pseudogap is associated with pairing in the absence of condensation 1.

The onset temperature for such a normal state gap is called $T^*$. Although there are a number of competing explanations, understanding the origin of this pseudogap which shows up in thermodynamics and transport [14], has been a central focus in the cuprate field. We emphasize that the pseudogap, as well as the distinct temperature scales $T^* \neq T_c$, play an important role in general BCS-BEC crossover physics and will be discussed in more detail throughout this paper; they are also depicted in the schematic comparison plot in Fig. 1.

B. BCS-BEC crossover in cold-atom experimental research

Because these systems constitute ideal laboratories for investigating the phenomena of BCS-BEC crossover (albeit in a Fermi gas), it is useful to next summarize the groundbreaking achievements beginning around 2003 which reported Fermi condensates in trapped atoms. Condensation was initially observed [15, 16] at strong coupling in the BEC regime (where $\mu < 0$) and shortly thereafter [17, 18] at intermediate coupling (in a “unitary” gas, where the chemical potential was positive). These experiments should be recognized by the solid-state physics community as a true “tour de force”. Researchers managed to surmount multiple challenges stemming from the fact that the atomic gases are charge neutral, they are confined to inaccessible traps, and moreover there is no direct way of measuring their temperature.

As a result, in the first few generations of experiments, “proof” of superfluidity was established indirectly through magnetic field sweeps. These sweeps make use of a Feshbach resonance to take a gas in the more fermionic regime and quickly change the magnetic field thereby projecting the system onto the strong-pairing regime. In this limit a bimodality in the density profiles of the fermion pairs, with a narrow central peak on top of a broad distribution, reveals the presence of a condensate along with thermally excited pairs. Over the next year or two, subsequent experiments made claims for superfluidity through measurements of the specific heat [19] and later, it was quite spectacularly established, through direct observation of quantized vortices [20].

With increasing understanding of these Fermi gas superfluids, the community then focused on additional probes such as transport [21, 22] and additional complexities associated with spin-imbalanced or polarized gases [23, 24] (very much like superconductors in magnetic fields) as well as in optical lattices [25]. Along these lines, there were interesting accompanying theoretical contributions [26, 27] as well as those which contemplated even more exotic (spin-orbit coupled and topological) phases [28–31]. Also notable were the contrasts with solid-state superconductors centered around low viscosity or “perfect fluids” [32, 33] in the Fermi gases and “bad metals” [34, 35] associated with highly resistive transport as in the cuprate superconductors.

The collective contribution of the dedicated experimental groups who met the challenge of finding and characterizing these Fermi condensates deserves enormous respect. Among the groups were those of Jin [15, 17, 36], Ketterle [18, 37], Grimm [16, 38], Thomas [39, 40], Hulet [41, 42], and Salomon [43].

Among the first theorists [44] to apply BCS-BEC crossover theory to the cold gases were Y. Ohashi and A. Griffin who implemented the theory of Nozieres and Schmitt-Rink [9]. This was followed by work from our group [45] which, shortly before the 2003 discovery, called attention to the expected importance of a pseudogap in these cold gases. This, in turn, helped motivate experimental efforts beginning with early observations of possible pseudogap signatures [16] using radio frequency (RF) spectroscopy [46]. Later research by Jin and her colleagues [36] introduced a rather ingenious analogue of

1They noted their particular numerics supported the interpretation of the pseudogap (or equivalently a normal-state excitation gap) as a “spin gap” in which the charge degrees of freedom did not equally participate.
angle resolved photoemission spectroscopy (ARPES) to investigate the pseudogap in more detail. These experiments have been revisited more recently by removing some of the trap complications, using a so-called “box” trap [47], where pseudogap effects appear more prominent.

In addition to this pseudogap focus, substantial effort was devoted to the unitary gas, intermediate between BCS and BEC, where the scattering length becomes infinite. Here precise numbers for thermodynamic features, variables in the equation of state, and special inter-relationships [48–50] provided a series of challenges to test the numerical accuracy of different BCS-BEC crossover theories.

C. Hamiltonian and interpretation of the ground state wavefunction

All discussions of detailed theory will be deferred to later sections in the paper, but for the purposes of an overview we next introduce the underlying Hamiltonian. As in all superconductors, it is assumed that electrons are paired in the superconducting phase. This pairing arises from an attractive interaction. In strict BCS theory, pairing takes place only between electrons with opposite momenta \((k, -k)\). More generally, in BCS-BEC crossover theory we consider pairing between \(k + q/2\) and \(-k + q/2\), where \(q\) can be arbitrary, but generally small. This pairing physics is described by the following Hamiltonian:

\[
\mathcal{H} = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a^\_\{\sigma\} + \sum_{kk'q} V_{kk'} a^\dagger_{k\sigma} \frac{1}{2} a^\dagger_{-k+q\sigma} a^\dagger_{-k-q\sigma} a_{-k+q\sigma} a_{-k-q\sigma} + \frac{1}{2} a_{k'\sigma} a^\dagger_{k'\sigma} + \frac{1}{2} a_{k'\sigma} a^\dagger_{k'\sigma},
\]  

(2)

where \(a^\dagger_{k\sigma}\) creates a particle in the momentum state \(k\) with spin \(\sigma\), and \(\epsilon_k\) is the kinetic energy dispersion. We assume a separable potential \(V_{kk'} = U \phi_k \phi_{k'}^\dagger\), where \(U = -|U|\) is the attractive coupling strength; the momentum dependent function \(\phi_k\) will determine the symmetry of the order parameter. For a contact potential or on-site interactions, \(\phi_k = 1\), whereas for \(d\)-wave cuprate superconductors, \(\phi_k = \cos k_x = \cos k_y\). To avoid this notational complexity here we will drop \(\phi_k\), and set the volume to unity in free space. Similarly we chose the lattice constant to be 1.

In Eq. (2) we have assumed spin-singlet pairing, which is relevant for both simple \(s\)-wave and \(d\)-wave superconductors. We do not make any assumptions throughout this paper about the origin or the detailed nature of the interaction, other than that it is attractive. The energy dispersion \(\epsilon_k\) can be associated either with a lattice or a free gas. We generally consider only a one-band model (with the exception of Section V E where band topology plays a role), but this Hamiltonian can be extended to include more bands and a finite range of interaction. For the \(s\)-wave case on a lattice, the interaction \(V_{kk'}\) in Eq. (2) corresponds to an attractive Hubbard model with on-site interactions. We have found that the effect of a finite range is generally not qualitatively important in the context of BCS-BEC crossover. In the \(d\)-wave case, \(V_{kk'}\) is in general nonlocal in real space and should be regarded as an approximation to the actual pairing interaction in real materials.

It is important, however, to note that when we refer to finite \(q\) pairing here this does not refer to condensed Larkin-Ovchinnikov [51] or Fulde-Ferrell [52] phases but rather to non-condensed or thermally excited pair states. These are to be distinguished from condensed pairs having zero center of mass momentum and lower energy. We stress that BCS-BEC crossover deals with superconductors that have strong pairing or strong “glue”. This characterizes the interaction term \(V_{kk'} = U \phi_k \phi_{k'}^\dagger\), in the Hamiltonian where it is assumed that \(|U|\) is not small compared to the kinetic energy. As a result of large \(|U|\), pairing and condensation will take place at different temperatures. In particular, at the superconducting transition temperature \(T_c\) there will be a finite number of non-condensed pairs present.

Note that \(\mathcal{H}\) in Eq. (2) is a many-body Hamiltonian and there are many ways of solving it. In this paper, and as in the literature [4], we base our solution on a variational ground state of the BCS form that was presented in Eq. (1). By contrast with strict BCS theory we allow the attractive interaction to be arbitrarily strong, assuming this does not change the generic form of the variational wavefunction \(\Psi_{BCS}\). We emphasize that \(\Psi_{BCS}\) is not an exact solution of Eq. (2), but rather an approximation which presumes that the system does not make large excursions from BCS theory no matter how strong the attraction is. Throughout this paper we adopt this particular version of BCS-BEC crossover theory and, unless indicated otherwise, all equations we present in this review are based on this particular ground state and its finite temperature implications.

We stress the advantage of this approach to BCS-BEC crossover theory is that we are dealing with a known ground state. This preserves the fundamental way we have come to understand superconductivity. Another advantage of the BCS wavefunction is that these Cooper pairs form an essentially ideal gas. One can see this from the form of the BCS wave function of Eq. (1), which can be rewritten as \(\Psi_{BCS} \propto e^{i b_0^\dagger}|0\rangle\) with the composite bosonic operator \(b_0^\dagger = \sum_k \tau_{k\uparrow}/\sqrt{N}\), hence \(\Psi_{BCS} = \sum_k \psi_k a^\dagger_{k\uparrow}\). Thus, this condensate corresponds to a ground state containing bosons which interact directly with the fermions and only indirectly with each other [53, 54]. This makes for a simpler and more solvable many-body problem [8].

One could contemplate other ground states with a structure different from the Gaussian-like \(\Psi_{BCS}\), in which one has a composite bosonic operator in the exponent that involves four or more fermionic creation operators [49]. Such approaches can be viewed as more equivalent to a weakly interacting theory of bosons: Bogoliubov theory. But such a more complicated theory is not necessarily an improvement as Bogoliubov theory for bosons is known not to be appropriate to temperatures near \(T_c\), or even well above \(T = 0\), as it is strictly a low temperature theory.

Nevertheless, the known weaknesses of the BCS-Leggett approach should be clarified at this point. In particular, such an approach leads to inaccuracies in numerical values of thermodynamic parameters associated with the unitary gas. One
can in part attribute this to the approximate treatment of the particle-hole channel for BCS-based theories which focus primarily on the particle-particle channel. This is evident, for example through the Bertsch parameter appearing as the ground state fermionic chemical potential ratio, $\mu/E_F$, of the unitary Fermi gas. This is found experimentally [50] to be around 0.37, whereas in the BCS ground state this parameter is equal [55] to 0.59.

D. Kadanoff and Martin interpretation: BCS theory as a Bose condensation of electron pairs

Knowing the ground state still leaves the challenge of how to introduce finite temperature effects. At this stage, to gain further physical insight into BCS-BEC crossover theory, it is useful first to revisit an approach due to Schafroth [56]. Two years before the BCS ground state of Eq. (1) was ever proposed, Schafroth suggested a more expanded interpretation of superconductivity. He argued that superconductivity could be thought of as associated with Bose condensation of an ideal charged Bose gas. While most in the community view his scheme as appropriate to the extreme BEC, often called the "local pair limit", here we wish to think about this approach to fermionic superconductivity more generally, for all systems just beyond the strict BCS limit.

Schafroth argued that condensation sets in at the transition temperature $T_c$ where there are preformed electron pairs. The expression for this temperature, following that of an ideal Fermi gas. This is found experimentally [50] to be around

$T_c = \left( \frac{2\pi}{\zeta(3/2)^2/3} \right) \frac{\rho_\text{B}^{2/3}(T_c)}{M_\text{B}(T_c)}$, \hspace{1cm} (3)

(where $C = [\zeta(3/2)]^{2/3}$ with $\zeta(3/2) \approx 2.61$, where $\zeta$ is the Riemann zeta function. Throughout this Review we set $\hbar = k_B = 1$, unless indicated otherwise.) The parameters $n_\text{B}$ and $M_\text{B}$ represent the (3D) number density and mass of the bosons. We should view these as yet unspecified bosons as representing fermionic pair degrees of freedom so that

$n_\text{B} \equiv n_\text{pair} \quad \text{and} \quad M_b \equiv M_\text{pair}$. \hspace{1cm} (4)

Note that, at the time of the BCS discovery, there was some resistance to Schafroth’s notion that his approach had anything in common with BCS theory. The key point which Schafroth emphasized is that there must be a form of Bose condensation embedded in superconductivity theory and this boson inevitably involves a pair of electrons.

Schafroth’s work introduces an important question: what kind of out-of-condensate boson or preformed pair is in fact compatible with BCS theory? The answer to this query would allow us to compute the transition temperature, after establishing a precise meaning for $n_\text{pair}$ and $M_\text{pair}$. Presumably because his work predated BCS theory, Schafroth did not ascribe any complexity to these quantities which we now think must depend on both temperature and attractive interaction strength. Importantly, because of the latter, we inevitably have to deal with BCS-BEC crossover physics.

The challenge to quantitatively characterize these out-of-condensate pairs at general temperatures $T$ was met in an important paper by Kadanoff and Martin [6]. Just as Eagles [5] and Leggett [4] recognized the greater generality of the BCS ground state wavefunction, Kadanoff and Martin provided key insights into the finite temperature physics of BCS theory. Their work was based on a systematic study of the coupled equations of motion. This established how to characterize the non-condensed pairs associated with BCS theory (through their propagator or "t-matrix").

Kadanoff and Martin made an important observation which related to the Schafroth picture. They state “Below the transition temperature... a nonperturbative, stable solution involving a Bose condensation of pairs can be derived within the pair correlation approximation... which [approximation] is identical with the one proposed by BCS. ... that the superconducting transition is a Bose condensation phenomenon [was] originally proposed by Schafroth [and co-workers].”

From their work one infers that the BCS gap equation can be reinterpreted as a BEC condition requiring that the non-condensed pairs have zero chemical potential (that is, are gapless) at every $T \leq T_c$. This Hugenholtz-Pines [57] constraint is a generalization, as well, of the familiar Thouless condition [58]. While in strict BCS theory, all preformed pairs at the onset of the superconducting transition should be viewed as virtual, it is reasonable to presume that once one enters the BCS-BEC crossover regime, these pairs are no longer virtual at the onset and their number and mass at general $T$ can be quantified according to the prescription of Kadanoff and Martin.

The work we summarize here should be differentiated from other approaches to BCS-BEC crossover, such as that of Nozières and Schmitt-Rink and others [9, 12, 44, 59]. Their finite-temperature analysis was presumably designed to accommodate some of the physics of bosonic Bogoliubov theory for the fermion pairs. In the NSR picture, which involves more strongly interacting composite bosons than would be associated with a BCS-like ground state, the bosonic degrees of freedom are described [9] as: “A bound pair [which] is a collective mode of the superfluid... $T_c$ thus results from thermal excitation of collective modes”. Their scenario can be compared with other work [60, 61] which addresses the extreme BEC regime and investigates the nature of that fermionic ground-state wave function associated with a composite-boson Bogoliubov picture (including Lee-Huang-Yang corrections).

E. Mechanisms for driving BCS-BEC crossover

An important aim of this Review is to communicate in physical terms what BCS-BEC crossover is and what it is not. More specifically we ask: how do we know when a superconductor is promoted out of the BCS regime and what are typical mechanisms for promoting it?

It is useful to establish the variables that quantify the size of the deviation from BCS. One of these, the ratio $T^*/T_c$ has already emerged. When this ratio exceeds unity the superconductor may no longer be in the BCS regime. Here, as defined
previously, \( T^* \) corresponds to that temperature at which a gap opens in the fermionic excitation spectrum, while \( T_c \) corresponds to the temperature for condensation. Strong pairing is not uniquely implied by large \( T^*/T_c \), as there can be other mechanisms for this spectral gap opening; it is, however, a necessary consequence.

By contrast the presence of a large zero-temperature gap to \( E_F \) ratio (\( \Delta_0/E_F \)) is more unambiguously suggestive of a system which has been promoted out of the BCS regime. Finally there is a third, equally important parameter which quantifies the deviation from BCS theory. This corresponds to the size of the Ginzburg-Landau (GL) coherence length which we define more precisely later in this subsection. When this is anomalously small the system may be driven away from the BCS regime.

What then are the mechanisms which are responsible for driving a superconductor out of the BCS regime? We identify 3 main mechanisms: low dimensionality, strong attraction, and low electronic energy scales.

We begin with the issue of low dimensionality which is known to naturally introduce distinct energy scales \( T^* \) and \( T_{\text{BKT}} \). Notably, as stated by Kosterlitz [62] “The onset of superconductivity in 2D . . . requires a pre-existing condensate or pairing of electrons.” One can understand this by noting that the underlying physical picture characterizing the onset of two dimensional superconductivity (or the Berezinskii-Kosterlitz-Thouless (BKT) superconducting state [63, 64]) assumes the separation of energy scales: phase coherence cannot occur until a pairing amplitude is established.

An equally important aspect of superconductivity in 2D is that there is stronger tendency to pair. In particular, in the low density limit where there is a quadratic band dispersion near the conduction band bottom, it follows that there is no critical value of the pairing interaction which is required to form two-body bound states. This is in contrast to the situation in 3D. Hence the “pairing glue” in a 2D superconductor need not be anomalously strong to promote the system into the BCS-BEC crossover regime. These observations may explain why there are many 2D examples in the recent BCS-BEC crossover literature.

Figure 2 represents a summary figure which depicts schematically the three distinct ways for a superconductor to be in the crossover regime. Plotted on the vertical axis is \( T^*/T_c \) (or for the two-dimensional system \( T^*/T_{\text{BKT}} \)). The horizontal axis indicates the strength of the dimensionless attractive interaction in units of a characteristic electronic energy scale \( E_{\text{kin}} \).

This figure shows that a relatively weaker attraction is needed to promote 2D superconductors into the crossover regime, where \( T^*/T_c \) deviates significantly from unity. The figure is characteristic of the moderately low-density limit. There are two other ways of obtaining values for \( T^*/T_c \) in excess of unity. There is the possibility of a very strong pairing “glue”, i.e., associated with anomalously large \( |U| \). We might speculate this stronger pairing scenario applies, if at all, to the cuprate superconductors. Finally, an important alternative for arriving at the strong-pairing regime is when the characteristic electronic energy scales (called \( E_{\text{kin}} \)) become anomalously small. This can occur through flat bands (because of small hopping, called \( t \), or small bandwidth) or through low electronic densities which reduce \( E_F \). We will see in this Review that both two dimensionality and/or small electronic energy scales are likely responsible for the many recent observations of BCS-BEC crossover of superconductivity.

The fact that there is no critical value of the pairing required to form bound states in a moderately low density 2D superconductor, also serves to interpret the illustrations to the left of the curves in Fig. 2. These are schematic representations of the number of pairs (or pair density, \( n_{\text{pair}} \)) in the 2D sheets or 3D volumes at the onset of the transition. For the same fixed attractive interaction, these schematic figures emphasize that in 2D there is a significantly higher density of pairs at \( T_{\text{BKT}} \) than for the analogue in the 3D system.

We end this discussion by introducing the GL coherence length which provides a quantifiable measure of where a superconductor is within the BCS-BEC spectrum. This is based on a calculation of \( T_{\text{BKT}} \) rather similar to the Schafroth like result in Eq. (3) but here for the 2D limit. This analysis is abbreviated here, by way of a summary, and later discussed in more detail in Sec. V.

We approach the BKT state from the high-temperature side and, thus, we will use the methodology advocated by the cold-atom community [65–67], where in atomic Bose gases one finds some of the most convincing evidence for a Kosterlitz-Thouless state. Although originally much of this literature was focused on BKT for bosonic superfluids, by extension to fermionic superconductors and superfluids, one can deduce
that this transition temperature roughly scales as \(^2\)

\[ T_{\text{BKT}} \sim \frac{n_B(T_{\text{BKT}})}{M_B(T_{\text{BKT}})}, \]

(5)

where, again, these as yet unspecified bosons with (2D) number density \(n_B\) and mass \(M_B\) represent pair degrees of freedom as defined in Eq. (4). It is important to note that a fraction involving the same temperature-dependent terms \(n_{\text{pair}}(T)\) and \(M_{\text{pair}}(T)\) enters in both the 2D and 3D expressions for the transition temperature. Here the omitted prefactor represents a slightly more complicated term which will be discussed later in the context of Eq. (23).

These Schafroth-like expressions for the transition temperatures in 2D and 3D (Eqs. (3) and (5)) then provide a simple expression for the important superconducting Ginzburg-Landau coherence length, \(\xi_{\text{coh}}\); this is given by \([69, 70]\)

\[ \frac{\hbar^2}{2M_{\text{pair}}(\xi_{\text{coh}}^0)^2} = k_B T_c, \]

where we have restored the Planck constant \(\hbar\) and Boltzmann constant \(k_B\). As a result \(\xi_{\text{coh}}^0\) depends only on the pair density \(n_{\text{pair}}\) (presumed at the onset of the transition). Importantly this coherence length reveals the location of a given system within the BCS-BEC crossover:

\[ k_F \xi_{\text{coh}}^0 \propto \left(\frac{n}{n_{\text{pair}}}\right)^{1/2} \]

for the two dimensional case. Here \(k_F\) reflects the total particle density, \(n\), and a similar expression (with the exponent of 1/3) can be obtained in the 3D case as well. Since the number of pairs at \(T_c\) varies from essentially 0 in the BCS limit to \(n/2\) in the BEC case, this provides a measure of where a given superconductor is within the BCS-BEC spectrum. Fortunately, this GL coherence length is rather widely discussed in the experimental literature. It is accessible through the magnetic-field response of charged superconductors.

## II. Overview of BCS-BEC Crossover

### A. Signatures of BCS-BEC crossover

Since the concept of BCS-BEC crossover is sometimes interpreted in different ways in the literature it is important to emphasize what we associate with the term “crossover” in this Review. We consider here solid state superconductors (as distinct from atomic Fermi gases) which are promoted out of the strict BCS regime through moderately strong pairing interactions (or through a combination of the mechanisms discussed in Section 1E). These interactions, in turn, lead to emerging bosonic degrees of freedom which coexist with a well defined Fermi surface. With ever increasing interaction strength the bosonic component will eventually become dominant leading to a disappearance of the fermiology; here the system enters the BEC regime. It is still an open question whether a BEC phase (with its attendant very low transition temperatures) has ever been observed in a solid state system. While some \([71]\) have identified crossover with the onset of the BEC limit, in this paper, we adhere to the conventional definition of “BCS-BEC crossover” emphasizing the associated new and interesting properties, which are distinct from those observed in either the BEC or BCS regime.

There are a number of signatures of BCS-BEC crossover, some of which we discussed in the previous section and which we more precisely quantify here. Many of these features can have multiple interpretations. While the first three criteria in the list below are necessary conditions, a conclusion in support of the appropriateness of a BCS-BEC crossover for a particular superconductor often comes from the preponderance of evidence, rather than from any “smoking gun”, single signature in this list. One observes:

1. Large values of normalized zero-temperature pairing gap \(\Delta_0/E_F\), from \(\approx 0.1 - 1.0\).
2. The presence of a normal-state gap (or pseudogap) with onset at \(T^*/T_c \gtrsim 1.2\).
3. A moderately short coherence length which should be no longer than \(k_F \xi_{\text{coh}}^0 \sim 30\).
4. Enhanced superconducting fluctuation-like behavior, particularly in the response to a magnetic field (such as the Nernst effect and diamagnetic susceptibility), well above \(T_c\).
5. A precursor downturn \([14, 72]\) in the temperature dependence of the resistivity around the gap onset temperature \(T^*\).
6. The presence of bosonic (or pair) degrees of freedom above the transition. The pairing gap and the bosonic degrees of freedom are indeed two sides of the same coin, although the latter are more difficult to identify.
7. BCS mean-field-like relations which characterize the ratio of the ground-state excitation gap, \(\Delta_0\), and the pairing onset temperature, \(T^*\).
8. Two distinct energy gaps. In contrast to strict BCS theory, in the crossover regime, the gap associated with coherent superconducting phenomena which set in at \(T_c\) is distinct from that associated with bosonic or pair excitations which appear in the vicinity of \(T^*\).
9. Normal-state experimental observations such as shot noise \([73]\) which are indications of \(2e\) charge carriers.
10. The observation of BCS-like “back bending” \([74]\) of the electronic band dispersion in the vicinity of but above \(T_c\).

### B. Analogies with an ideal Bose gas

What is essential is that the treatment of the BCS-BEC crossover which we present here be compatible with

\(^2\)The proportionality constant between \(T_{\text{BKT}}\) and \(n_B/M_B\) in Eq. (5) has an additional double-logarithmic dependence \([68]\) on \(n_B\), which is very weak.
generalized-BCS physics, both in the ground state as well as at all temperatures $T \leq T_c$. Unlike in strict BCS theory, in the crossover regime, bosonic degrees of freedom or preformed pairs will appear at the onset of condensation. Their number progressively increases as the system evolves from BCS to BEC. These normal state pairs are associated with an excitation gap (or “pseudogap”) in the fermionic spectrum and in BCS-BEC crossover this is present at the onset of condensation: $\Delta(T_c) \neq 0$. The gap size increases continuously starting at nearly 0 in the BCS regime. The excited pair states involve a combination of two fermions associated with momenta $k + q/2$ and $-k + q/2$ where, specifically, the pair momentum $q$ is non-zero. Preformed pairs are necessarily distinct from condensed pairs, for which $q = 0$ and whose energy is lower.

To understand these preformed pairs we present a simple figure based on a rather close analogy to an ideal Bose gas. The upper row of Fig. 3 is a schematic representation of the temperature evolution of a BCS-BEC crossover superfluid. This shows that as temperature decreases below an onset temperature, called $T^*$ say, the new form of quasiparticle or excitation appears. These non-condensed pairs are represented by dashed circles in red. At this same temperature a pairing gap or pseudogap is present which reflects the fact that there must be an input of energy to create fermionic excitations by breaking pairs. As temperature further decreases to just above $T_c$, the number of these preformed pairs increases. Note that, the figure shows that there are also a number of unpaired fermions at the transition. The ratio of the boson to fermion number continuously increases from BCS to BEC. In the BCS limit the number of pairs at $T_c$ is essentially zero, while in the BEC limit that number approaches $n/2$.

Below $T_c$, condensed pairs (solid circles in blue) appear. As temperature is lowered further, non-condensed pairs gradually, (and at $T = 0$ completely), convert to the condensed state. There are no non-condensed pairs in the BCS-like ground state. Importantly, strict BCS theory is the special case where $T^* = T_c$ and concomitantly where the number of non-condensed bosons becomes arbitrarily small at any temperature $T$. This signals that there is essentially no pairing-related gap in the fermionic excitation energy spectrum at $T_c$.

C. Contrasting the present pair-fluctuation and phase-fluctuation scenarios

We emphasize that this pair fluctuation picture of BCS-BEC crossover is not the same as the phase fluctuation [75] scenario. There are similarities, but the contrast has been emphasized previously by Emery and Kivelson [75] who describe the phase-fluctuation scenario as follows: “Our discussion attributes the properties of high-temperature superconductors to the low superfluid density … and not to a short in-plane coherence length and a crossover to real-space pairing”.

The most significant differences would appear, then, to be attached to the driving mechanisms (small superfluid density vs strong attraction) behind the observed exotic normal states, as well as the pair “size” or in-plane coherence length. This can help experimentalists distinguish between the so-called phase fluctuation picture and BCS-BEC crossover. A small coherence length or the observation of concomitant, moderately large $\Delta_0/E_F$ similarly lends support to the crossover scenario.

To compare these two scenarios we turn back to Fig. 3. In this figure, the pair-fluctuation or BCS-BEC crossover picture in the upper panel is to be associated with a new type of paired quasi-particle (excited pair states) whereas the phase fluctuation scenario in the lower panel relates to more collective behavior. In this collective behavior low carrier density is associated with poor screening which is then responsible for small phase stiffness. As a further point of contrast, it should be stressed that all parameters pertaining to the fermionic sector ($\Delta_0$, $T^*$, etc) are essentially absent in the phase fluctuation scenario, as this theory is an effective low-energy description of the bosonic degrees of freedom once the fermions are integrated out.

At the same time, the deep BEC limit of the BCS-BEC
crossover scenario, where the fermions are essentially absent at $T_c$, will have features in common with the phase fluctuation scenario. Similarly in 2D where fluctuation effects become more pronounced the differences between the two approaches become more subtle, despite the fact that these bosonic regimes are driven by strong pairing “glue” rather than low carrier density.

Finally, we emphasize that phase fluctuations themselves will be present in the (usually narrow) critical region of temperature near $T_c$ of all superconductors, once one includes beyond-mean-field effects, which are not addressed in this Review.

**D. Quantitative summary of the present theory**

It should not be surprising that accompanying the two forms of (red, blue) quasi-particles in the upper panel of Fig. 3 are two different forms of fermionic excitation gaps: $\Delta_{pg}$ and $\Delta_{sc}$. One can think of these as representing the contributions from non-condensed and condensed pairs, respectively. Indeed, their squares will turn out to be proportional to the number density of these two types of pairs.

As shown in Fig. 4, the two contributions to $\Delta^2$, called $\Delta_{pg}^2$ and $\Delta_{sc}^2$, play a similar role to their respective counterparts in the ideal Bose-gas scenario. This latter theory considers a decomposition of the total number of bosonic particles, $N_B$, into condensed and excited contributions (upper panel) with the analogue decomposition for a fermionic superfluid (lower panel) which involves the square of the pairing gap $\Delta^2$ as a function of temperature $T$, $T_{BEC}$ and $T_{BCS-BEC}$ are the respective transition temperatures. This figure shows that the two gap contributions to $\Delta^2$, called $\Delta_{pg}^2$ and $\Delta_{sc}^2$, are closely analogous to their counterparts in the ideal Bose gas. Indicated schematically is how to arrive at the respective transition temperatures associated with the intersection of the “excited” curve with either the total boson number curve (black line in the top panel) or the total $\Delta^2$ curve (black line in the bottom), which marks the onset of the condensate contribution.

In this way, not only can one directly derive the Schafroth [56] expression shown in Eq. (3), but one can write this same equation in a way which is more familiar from the perspective of BCS theory. In strict BCS theory, $T_c$ is obtained from

$$1 = (-U) \sum_{\mathbf{k}} \frac{1 - 2 f(|\xi_{\mathbf{k}}|)}{2|\xi_{\mathbf{k}}|^2},$$

where $U < 0$ and $f(x) = 1/(e^{x/T} + 1)$ is the Fermi-Dirac distribution function. Here $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ is the bare fermion dispersion measured from $\mu$. It will be shown that, in the present BCS-BEC crossover theory, we have a similar expression for the determination of $T_c$:

$$1 = (-U) \sum_{\mathbf{k}} \frac{1 - 2 f(\tilde{E}_{\mathbf{k}})}{2\tilde{E}_{\mathbf{k}}} \bigg|_{T=T_c},$$

where $\tilde{E}_{\mathbf{k}} \equiv \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2(T_c)}$.  

![Figure 4](image-url)
Thus, the central change from strict BCS theory (aside from a self-consistent readjustment of the fermionic chemical potential [4]) is that $T_c$ is determined in the presence of a finite excitation gap, $\Delta(T_c)$. Solving for $T_c$ involves finding the point of separation between $\Delta^2_{\text{BCS}}(T)$ and the mean-field gap $\Delta^2_{\text{pg}}(T)$ as a function of decreasing temperature, as shown in the bottom panel of Fig. 4.

We now have two different equations, Eq. (9) and the Schafroth expression in Eq. (3), which determine the transition temperature in the BCS-BEC crossover theory, and both are intuitively quite reasonable. What is satisfying is to find that these two equations are equivalent, provided one properly computes the number of pairs and their mass. Thus, this meets the goal of connecting a Schafroth-like approach to a more microscopic approach based on BCS theory. Schafroth’s expression for $T_c$ in this extended form is appropriate throughout the crossover, once the system has emerged from the BCS limit so that $\Delta(T_c)$ is no longer strictly zero.

### E. Qualitative summary of BCS-BEC crossover

Before going into more technical details of the present BCS-BEC crossover theory, as will be addressed in Section III, we now consider some of the more obvious questions which can be raised at this point. One of the first issues which arises is to clarify what is generic about BCS-BEC crossover theories. We note that BCS-BEC crossover theory belongs to the class of theories of strong coupling superconductors. While there are a number of others in this class what is essential is that this particular form of strong coupling superconductivity is driven by charge $2e$ Cooper pairing. This differs from some of the alternative types of strongly correlated superconductors: spinon-holon pairing [76], kinetic energy driven superconductivity [77], superconductivity strongly coupled to antiferromagnetism (“SO(5)”) [78] and fractionalized electron superconductivity [79].

Moreover, within the BCS-BEC crossover class there is a number of variants, some of which will be briefly reviewed in Sec. II F. Generically a BCS-BEC crossover theory of superconductivity represents an interpolation scheme between weak and strong coupling forms of $2e$ pairing-governed superconductivity. In the weak coupling limit the fermions within a pair are very loosely associated whereas in the strong coupling limit they become tightly bound. In between the two extremes, there is generally a smooth crossover. In all theories of the BEC regime in a lattice, the fermionic chemical potential lies below the bottom of the (non-interacting) conduction band. These generic features are illustrated in Fig. 1(a) which indicates how the transition temperature and pairing onset temperatures smoothly vary between the fermionic and bosonic regimes.

There are, however, a number of features which are not generic in the family of BCS-BEC crossover theories. For example, not all theories reproduce BCS theory in the weak coupling limit. Indeed, even the “BEC” limit has many different interpretations. Some would argue that the BEC limit should be that of a true weakly interacting Bose system. Alternatively, in the present theory it is argued to be distinctly different as this state is characterized through its fermionic properties, even though a Fermi surface is no longer present. In such a BEC limit, for example, the fermionic pairing gap parameter is large and temperature independent well above and below $T_c$. Among other features which are not generic is the presence in the intermediate coupling regime of a pseudogap, which is indicated in Fig. 1. This pseudogap appears in some crossover theories [80], but not in others [81, 82].

More precisely the pseudogap corresponds to a gap in the fermionic excitation spectrum, which has a smooth onset at $T^* > T_c$. The pseudogap we consider here enters into the theoretical framework as a distinct parameter $\Delta_{\text{pg}}$ and is more apparent [83]; in other approaches [80] it is only indirectly seen to be present through the behavior of the fermionic spectral function. It reflects the fact that electrons are starting to pair up at $T^*$ and that breaking the pairs in order to create fermions will cost a (gap) energy. There is no true ordering or broken symmetry which takes place at $T^*$, only the onset of bosonic (pair) degrees of freedom. Because of the pseudogap, superconductivity at $T_c$ will occur in the presence of a finite fermionic excitation gap $\Delta(T_c)$.

Additionally, we argue that these pseudogap effects persist below $T_c$ as they reflect the contribution of non-condensed pairs which are continuously converting to the condensate as temperature is lowered towards the ground state. Below $T_c$ there is the additional energy gap deriving from the order parameter, $\Delta_{\text{sc}}$. It is often difficult to disentangle these two gap parameters, which reflect the energies that must be input to break the non-condensed and condensed pairs, and for many purposes they contribute additively in quadrature. Importantly, the pseudogap is not associated with superconducting coherence and is not responsible for Meissner or Josephson effects.

More concretely, this energy gap appears in both the charge and spin channels and more generally in thermodynamics and transport in many respects similar to the way the below-$T_c$ superconducting gap shows up in BCS theory. It enters, however, as a slightly rounded or smeared gap structure in normal state tunneling, and photoemission and leads to a gentle onset of a decrease in entropy with decreasing $T$. Importantly, it does not correspond to a true zero of the fermionic spectral function but rather to a depression which appears at energies around the chemical potential due to a finite lifetime of the non-condensed pairs.

In the present approach, to a good approximation (see Eqs. (16a) and (16b) below) the electron spectral function $A(\omega, k)$ depends on a self energy of the form [84, 85]

$$
\Sigma(\omega, k) = \frac{\Delta^2_{\text{pg}}}{\omega + \xi_{-k} + i\gamma} + \frac{\Delta^2_{\text{sc}}}{\omega + \xi_{-k}},
$$

which contains both gap parameters (in the $s$-wave case). Note the presence of a phenomenological factor $i\gamma$ which reflects the fact that the non-condensed pairing has a finite lifetime or is meta-stable. Its magnitude is not particularly important. Indeed, in the normal state this expression is associated with a phenomenology widely used for the cuprates and introduced by M.R. Norman and collaborators in their analysis of
ARPES data [86].

Additionally the pseudogap can be detected indirectly through bosonic contributions which emerge as a result of the pairing of fermions. These are generally associated with familiar fluctuation transport signatures, as, for example, seen in a downturn in the dc resistivity around $T^*$. In this Review we aim to connect the BCS-BEC crossover scenario to experiments. There is a challenge here because the fundamental tuning parameter $|U|$ of the BCS-BEC crossover is not accessible. This is in contrast to the Fermi gases where the interaction strength can be directly measured through a scattering length. What is most important is that it can be reasonably straightforward to replace the attractive interaction parameter which always appears in traditional BCS-BEC crossover calculations on a lattice in favor of measurable variables. This imposes a requirement on lattice crossover theories: a broad range of phenomena must be able to be addressed, enabling connections to multiple experiments. The phenomena of interest involve parameters which scale directly through bosonic contributions which emerge as a result of the ARPES data [86].

| Fundamentals tuning parameter |
|-------------------------------|

These are, for example, $T^*/T_c$, $\Delta_0/E_F$ and $k_F\xi_{coh}$. How to make sense of experimental observations is the final important issue we consider in this qualitative summary section. In particular, one needs to determine whether there are experimentally verifiable conditions surrounding the applicability of BCS-BEC crossover. We identify qualitative trends which are seen through important correlations. These involve the fact that increases in $\Delta_0/E_F$ should be associated with increases in $T^*/T_c$, and that decreases in the coherence length, through $k_F\xi_{coh}$, should be correlated with increases in $T^*/T_c$. In this Review these correlations are represented in a more quantitative fashion by detailed predictive curves. These are shown in a number of plots as in Figs. 10, 12, 13, 15, and most importantly, in Figs. 36 and 40, for example. Related issues have come up in experimental studies, as seen for example in Fig. 19. To address specific experiments, these predicted associations, of course, have to be tested carefully by changing an internal variable such as pressure or possibly doping within the same superconducting family.

**F. Other theoretical approaches: addressing BCS-BEC crossover on lattices**

As emphasized in the Introduction, this Review primarily focuses on one particular theoretical approach to BCS-BEC crossover based on the ground state of Eq. (1). Nevertheless, for the sake of completeness, it is useful to give an overview of some alternative theoretical schemes in the literature which are particularly relevant to solid state systems.

We first note that there is significantly less literature on BCS-BEC crossover theory in solid-state superconductors as compared to that in the Fermi gases. For these atomic systems this extensive effort has been largely driven by experimental discoveries. Review articles are available which summarize different variations [83, 87] of a “t-matrix approach” to BCS-BEC crossover theory at finite temperature. Key aspects of these comparisons will be briefly discussed in Sec. III D, albeit with an emphasis on applications to solid state systems. Among the reviews are those from our own group [1], from the Camerino group [80], and the Munich group [88], as well as extensive overviews from Randeria and Taylor [3] and Bloch [89] and co-workers. What hasn’t been as thoroughly reviewed is the next generation research on crossover effects associated with superconductors in the solid state. Notable is a nice overview from Loktev and co-workers [90], which covers early work through 2001.

As discussed earlier, a key property which characterizes superconducting fermions on a lattice is that $T_c$ approaches zero in the extreme BEC limit. Important here is the fact that in superconductors the kinetic degrees of freedom are associated with the fermions. The “composite bosons” do not directly hop, even in the BEC regime, as a consequence of the assumed form for the Hamiltonian in Eq. (2). It is useful to return to Fig. 1 to address this issue. This asymptotic behavior sets in once $\mu$ becomes negative or equivalently falls below the band bottom; it is associated with the onset of the shoulders in Fig. 1. This is reasonably straightforward to understand. The hopping of pairs requires the individual hopping of fermions, and, when two fermions are tightly glued together, this hopping is highly suppressed, leading to the asymptotic behavior seen in Fig. 1(a).

More quantitatively, as shown by Nozières and Schmitt-Rink, this suppressed hopping of pairs varies as $t^2/|U|$, where $t$ is the fermionic hopping matrix element and $|U|$ is the magnitude of the attractive interaction. Interestingly, these authors noted [9] that their particular “continuum model . . . provides an accurate description of the two (BCS-BEC) limits but [leads to] a failure for a lattice gas”. In hindsight, this is probably an unduly negative assessment, but perhaps it bears on the rather small body of literature applying NSR theory to solid state superconductors.

Most of the overall features in the lattice phase diagram, such as those shown in Fig. 5 (panels (a) through (c), including this $t^2/|U|$ asymptote, can be obtained from different BCS-BEC crossover theories. These involve the t-matrix approximation (TMA) based approaches (of which there are three main categories [83, 87] briefly discussed in Sec. III D), dynamical mean field theory (DMFT) [91–99], quantum Monte Carlo simulations [100], functional renormalization group [101], as well as others. Among these, the TMA approach is principally analytical and, thus, provides more intuition about the relevant physical processes behind the crossover, making it the primary theoretical tool to be discussed in this Review.

We can understand why there is a relatively smaller body of analytical literature on lattice BCS-BEC crossover theories as compared to the Fermi gases as due in part to the fact that many of the sophisticated and insightful field theory techniques, such as large-$\mathcal{N}$ and $\epsilon$-expansions [103–110], are not directly adaptable to lattice systems. In the following we will summarize some of the DMFT and QMC studies, highlighting a few prototypical phase diagrams shown in Fig. 5, which reflect a spectrum of different approaches in the literature. To begin, we note that Sewer, Zotos and Beck [100] have provided a very useful study of 3D comparative crossover ap-
proaches which yield the phase diagrams shown in Fig. 5(c). These are in many ways similar to their 2D analogues (see Fig. 5(b) for Monte Carlo-based results).

DMFT studies of the attractive Hubbard model (addressing either the ground state or the normal state) have been presented by Keller et al [111], Garg et al [112], Capone et al [113], and Bauer et al [97]. Some example phase diagrams [92] are presented in Fig. 5(a). In DMFT the attractive Hubbard model is mapped to an impurity problem on a lattice, which typically has a dimension that is effectively infinite. In this infinite-dimension limit, the fermionic self energy associated with pairing becomes a function only of frequency. As a result computing the self energy can be reduced to self-consistently solving a local impurity problem, for which one can generally resort to various numerical methods. The advantage of DMFT is that it can capture local dynamical quantum fluctuations non-perturbatively, which can be important for a quantitative accounting of the quasiparticle spectral function at intermediate coupling $|U| \sim 8t$. On the other hand, DMFT is exact only in infinite dimensions because it ignores both spatial fluctuations beyond mean field level as well as dimensional fluctuations. Therefore, the DMFT results need to be interpreted with care when making a quantitative comparison to other approaches in three or two dimensions.

Keller et al [111] have provided an interesting DMFT study of the normal phase of the attractive Hubbard model showing that it is a Fermi liquid at weak coupling but consists of bound pairs and pseudogap physics at strong coupling. Perhaps surprisingly, the crossover between these two normal states may not be smooth at temperatures lower than $T_c$, when the superconductivity is suppressed. There are indications at these very low temperatures that in this form of DMFT, a first order transition occurs in the attractive Hubbard model between a thermally excited Fermi liquid state and a thermally excited transition occurs in the attractive Hubbard model between a very low temperatures that in this form of DMFT, a first order perconductivity is suppressed. There are indications at these

Figure 5(b) shows a Monte Carlo result for $T_c$ or $T_{\text{BKT}}$ for an attractive Hubbard model on a 2D square lattice with nearest neighbor hopping [102]. At a generic electron filling level, the overall shape of $T_{\text{BKT}}$ vs $|U|/t$ curve looks rather similar to its 3D counterpart as shown in Fig. 5(c).

It is notable, that in two dimensions, it is more straightforward to arrive at a mean-field-level understanding of $T_{\text{BKT}}$ varying from BCS to BEC when the lattice is away from half-filling. An illustrative example [114] is based on calculations of the superfluid density or helicity modulus where one treats crossover effects at the mean field level. This can be done either within the attractive Hubbard model or within its repulsive counterpart, obtained by a particle-hole transformation on the bipartite lattice. The $T_{\text{BKT}}$ results calculated in this way are quite similar to those shown in Fig. 5(b).

For completeness, it is useful to highlight some additional literature contributions which address the physics of BCS-BEC crossover for fermions on a lattice. Closely related to the NSR theory (which has been mostly applied to the Fermi gas state) is work by Wallington et al [115, 116] who studied lattice crossover theory using a functional integral formalism, including Gaussian fluctuations. Their focus was on the effects of varying the symmetry of the order parameter within the extended attractive Hubbard model. Similarly Tamaki and co-workers [117] also addressed NSR theory on a lattice providing an interesting comparison with other t-matrix theories.

It is useful also to summarize additional miscellaneous references which may be of interest to the reader. Zero temperature approaches mainly based on the BCS-like ground state wavefunction in Eq. (1) are addressed by Pistolesi and Nozieres [118], by Herbut [119], by Andrenacci et al [120], and by Voleko and Quader [121]. Similarly relevant to topics in the present paper are observations about the contrast between s- and d-wave superconductors [90], where it has been noted that in the d-wave case moderate densities and large coupling suppress the BEC region of the phase diagram leading to a premature disappearance of the superfluid phase deep inside the fermionic regime [122].

Finally, by way of a digest of the more analytical theories of the crossover (for the gas as well as lattice) we note that in describing BCS-BEC crossover effects, it is tempting to introduce features of Bose superfluidity. As in Bogoliubov theory this includes more direct interaction effects between pairs of fermions or bosons. In doing so, one is saddled however with theoretical obstacles as finite temperature effects are much more difficult to include properly in Bose superfluids than in the BCS (fermionic) case. In strict BCS theory the
entire temperature range is accessible, where by contrast, in the Bose case one is restricted to the low temperature regime. As a consequence in many BCS-BEC crossover approaches one can encounter unphysical effects which are inherited from problems in theories of Bose gases [123, 124]. Among these are first order jumps in thermodynamic properties at $T_c$ and violations [125] of the Hugenholtz-Pines constraint [57].

III. DETAILED MICROSCOPIC THEORY OF 3D BCS-BEC CROSSOVER SUPERCONDUCTIVITY AT $T \neq 0$

Section II D provided a brief but reasonably complete summary of results from the current formalism. In this section we present additional details for the interested reader.

A. Characterizing the bosons embedded in BCS theory

Here we determine how to microscopically and quantitatively understand the non-condensed bosons of the BCS approach using a slightly different language [1] from that of Kadanoff and Martin. We present the theory for the $s$-wave case, while the application to $d$-wave superconductivity can be found elsewhere [126]. We build on a centrally important observation: at any temperature in which there is a condensate, the non-condensed bosons which are in equilibrium with the condensate must have a vanishing chemical potential:

$$\mu_{\text{pair}} = 0 \quad \text{for} \quad T \leq T_c. \quad (11)$$

This statement is equivalent to the famous Hugenholtz-Pines theorem [57]. How do we guarantee that the chemical potential is zero? BCS provides us with an important temperature-dependent self-consistency condition known as the gap equation, valid for all $T \leq T_c$. This gap equation is

$$0 = \frac{1}{U} + \sum_{k} \left( \frac{1 - 2f(E_k)}{2E_k} \right), \quad (12)$$

where $E_k = \sqrt{\xi_k^2 + \Delta^2}$ and $\Delta$ is the temperature-dependent pairing gap.

We argue that Eq. (12) should be incorporated in one way or another to arrive at an understanding of pair excitations. This leads us to constrain the form of the pair propagator $t(q)$ (or more precisely the t-matrix) for the non-condensed pairs to satisfy

$$t^{-1}(q = 0) \propto \mu_{\text{pair}} = 0, \quad T \leq T_c. \quad (13)$$

Indeed, Thouless has argued that a divergence of a sum of “ladder” diagrams (within a pair propagator) is to be associated with the BCS transition temperature. Here we assert that this Thouless condition can be extended to characterize the full temperature-dependent gap equation for all $T \leq T_c$, not just the transition region. This constraint leads to a pair propagator of the form

$$t^{-1}(q) = \sum_k G(k)G_0(q - k) + U^{-1}, \quad (14)$$

whose diagrammatic representation is shown in Fig. 6. In the above equation $G_0(k) = (i\omega_n - \xi_k)^{-1}$ and $G(k) = \left[ G_0^{-1}(k) - \Sigma(k) \right]^{-1}$, corresponding to the bare and dressed fermionic Green’s functions with $\Sigma(k) = -\Delta^2 G_0(-k)$. We define $k = (i\omega_n, k)$ and $q = (i\Omega, q)$ as two four-vectors with $\omega_n = (2n + 1)\pi T$ and $\Omega = 2l\pi T$, and $\sum_k$ is a short-hand notation for $\sum_{n\Omega} \sum_k$, with $\{n, l\} \in Z$.

It is important in Eq. (14) to properly define the fermionic chemical potential $\mu$. In this way one avoids unphysical effects which stem from the fact that the t-matrix of BCS theory takes an asymmetric form, involving different spin states pertaining to dressed and bare Green’s functions. If care is not taken [127], such calculations may lead incorrectly to the presumption that there is a Fermi surface mismatch for the two spin states and, consequently, regions of unstable superconductivity.

Importantly, Kadanoff and Martin [6, 128] arrived at the same conclusion concerning the presence of both dressed and bare Green’s functions. As stated by Kadanoff and Martin: “This asymmetry … has led several people to surmise that the symmetrical equation … solved in the same approximation would be more accurate. This surmise is not correct...”.

B. Determining the pair mass $M_{\text{pair}}$ and the non-condensed pair number density $n_{\text{pair}}$ for $T \leq T_c$

The fundamental quantities which determine the transition temperatures [1] in Eqs. (3) and (5) require we determine $n_{\text{pair}}$ and $M_{\text{pair}}$. We argue that both of these must depend on the BCS gap $\Delta$. In general t-matrix theories the self energy is given by a convolution between a Green’s function and the t-matrix. Here this self energy due to non-condensed pairs takes the form

$$\Sigma_{\text{pg}}(k) = \sum_{q \neq 0} t(q)G_0(q - k). \quad (15)$$

Note that the $q = 0$ component of $t(q)$ (which corresponds to the condensate) is necessarily excluded in the summation above. To proceed further one adopts the so-called “pseudo-gap (pg) approximation”. This was motivated originally by detailed numerical work [129, 130]. It should be emphasized that it is appropriate at all $T$ below $T_c$. It also applies for a restricted set of temperatures in the vicinity of but slightly above the transition [129, 130] where $|\mu_{\text{pair}}|$ is very small. Since

---

3A more systematic and first principles derivation of this t-matrix can be found using Eqs. (2.3-2.4), (2.7-2.8), (2.7’-2.8’) and (2.10) in Ref. [6]

4To be consistent this requires setting ReE(kp) = 0, so that Hartree-like terms in the diagonal part of the self energy are absorbed into the chemical potential. Here kp is the wavevector on the Fermi surface.
\[|\mu_{\text{pair}}| \approx 0, t(q) \text{ is strongly peaked about } q = 0, \text{ so that the self energy can be approximated by}\]

\[
\Sigma_{\text{pg}}(k) \approx -\Delta_{\text{pg}}^2 G_0(-k), \tag{16a}
\]

with \( \Delta_{\text{pg}}^2 = -\sum_{q \neq 0} t(q), \quad T \lesssim T_c. \tag{16b} \)

We emphasize that the above two equations constitute the central approximation made (for the sake of numerical simplicity [129]) in the present theoretical framework. The other crucial approximation is the adoption of Eq. (1) as the essential starting point.

We are now in a position to compute the pair mass and number density. After analytical continuation, \( i \Omega \rightarrow \Omega + i0^+ \), we expand the (inverse) t-matrix for small argument \( q \) to find

\[
t(\Omega, q) = \frac{Z^{-1}}{\Omega - i\Gamma_0,q + \mu_{\text{pair}} + i\Gamma_{\Omega,q}}, \tag{17}
\]

where \( Z \) is a frequency- and momentum-independent proportionality coefficient; the pair mass is contained in the pair dispersion \( \Omega_q = q^2/(2M_{\text{pair}}) \); the last term in the denominator \( i\Gamma_{\Omega,q} \) is frequency dependent and describes the finite lifetime of the non-condensed pairs due to decay into the two-fermion continuum. Defining the propagator for the non-condensed pairs as \( Zt(\Omega, q) \) and neglecting the generally small dissipative term \( i\Gamma_{\Omega,q} \), one can obtain the non-condensed pair density as

\[
n_{\text{pair}} = \sum_q b(\Omega_q) = Z\Delta_{\text{pg}}^2, \tag{18}
\]

which is naturally temperature dependent. Here, \( b(x) = 1/(e^{x/T} - 1) \) is the Bose-Einstein distribution function.

We have asserted above that the total fermionic gap \( \Delta^2 = \Delta_{\text{sc}}^2 + \Delta_{\text{pg}}^2 \). To complete the arguments we now show that this derives from two self energy contributions — from the condensate (sc) and the non-condensate (pg):

\[
\Sigma(k) = \sum_q t(q)G_0(-k + q) = \Sigma_{\text{sc}}(k) + \Sigma_{\text{pg}}(k). \tag{19}
\]

Here, \( \Sigma_{\text{sc}} \) comes from the Dirac delta function piece of \( t(q) \) at \( q = 0 \), i.e., \( t_{\text{sc}} \equiv t(q = 0) = -(\Delta_{\text{sc}}^2/T)\delta(q) \). Using Eq. (16a), we then obtain

\[
\Sigma(k) \approx -(\Delta_{\text{sc}}^2 + \Delta_{\text{pg}}^2)G_0(-k) \equiv -\Delta^2 G_0(-k). \tag{20}
\]

In this way, Eq. (6) results and we have \( \Delta^2 = \Delta_{\text{sc}}^2 + \Delta_{\text{pg}}^2 \).

5In quasi-2D, one may expand the pair dispersion as \( \Omega_q = q_{\parallel}^2/(2M_{\text{pair,||}}) + q_{\perp}^2/(2M_{\text{pair,\perp}}) \), where the subscripts \(||\) and \(\perp\) denote in-plane and out-of-plane components, respectively. Away from the long wavelength limit on a lattice, one can use a Bloch band dispersion instead of a simple parabola. An \( \Omega^6 \) term may be added to the \( t^{-1}(q) \) expansion for better quantitative accuracy.

6A systematic derivation following Kadanoff and Martin can be found in Ref. [126], or more directly from Eqs. (2.3-2.4), (2.7-2.8), (2.7’-2.8’) and (2.10) in Ref. [6].

![Figure 6](Image)

**Figure 6.** The pair propagator of Kadanoff and Martin [6]. \( U \) is the attractive interaction; \( G_0 \) and \( G \) are the bare and dressed fermionic Green's functions respectively.

**C. Establishing the form of \( T_c \)**

We approach \( T_c \) from high temperatures, where \( \Delta_{\text{pg}}^2 = \Delta^2 \) and \( \mu_{\text{pair}} < 0 \). As \( T \) decreases, \( \mu_{\text{pair}} \) increases, and Eq. (18) will be satisfied under the condition \( \Delta_{\text{pg}}^2 = \Delta^2 \), at \( T \geq T_c \). The transition temperature \( T_c \) is reached when this is no longer possible; at this temperature \( \Delta_{\text{pg}}^2 \) can not accommodate the value of \( \Delta^2 \), so that an additional contribution \( \Delta_{\text{sc}}^2 \) is needed. This occurs when \( \mu_{\text{pair}} \), as a function of decreasing \( T \), first reaches zero in Eq. (18), from which one recovers a Schafroth-like expression for \( T_c \):

\[
T_c = \left( \frac{2\pi}{c} \right) \frac{n_{\text{pair}}(T_c)}{M_{\text{pair}}(T_c)}, \tag{21}
\]

as was anticipated in Eq. (3). While it was not recognized in the original Schafroth calculations, on the right-hand side of Eq. (21), both \( n_{\text{pair}} \) and \( M_{\text{pair}} \) depend on \( \Delta^2 \), and are therefore functions of \( T \). Below \( T_c \), Eq. (18) is valid with \( \mu_{\text{pair}} = 0 \) and \( \Delta_{\text{pg}}^2 < \Delta^2 \).

**D. Alternative t-matrix approaches to BCS-BEC crossover**

From Fig. 6 or equivalently Eq. (14) one can see that, within the BCS ground state based t-matrix approach to BCS-BEC crossover, an asymmetric combination of dressed \( (G) \) and bare \( (G_0) \) Green's functions enters the definition of the t-matrix or pair propagator. The connection between this particular combination and BCS theory was first identified by Kadanoff and Martin [6]. It should be emphasized that this asymmetric combination is a specific consequence of the equation of motion approach of Kadanoff and Martin.
earlier in the context of the cuprates [133–135] and this is associated with two dressed Green’s functions. It is also known as the Luttinger-Ward formalism [81] or Galitskii-Feynman theory [136]. This Φ-derivable theory does possess an appealing simplicity as it readily satisfies conservation laws but this particular t-matrix theory will not satisfy the equations of motion, e.g., those derived by Kadanoff and Martin [6], as prescribed by the Hamiltonian.

Comparisons among these different t-matrix schemes have been extensively discussed in the literature [83]. Here, we give a brief but critical summary, noting that it is useful to discuss the comparisons first in the context of the Fermi gases and then turn to the lattice case. While the differences among different schemes might seem to be rather technical and therefore possibly minor, they have led to significantly different qualitative physics. Among these is the fact that the transition at $T_c$ is first order [59, 132, 137, 138] in the standard NSR based approaches as well as in the SCTA scheme. This leads to unwanted features in the Fermi gas density profiles [139] and temperature dependence of the superfluid density [137]. The interested reader can consult other references [140, 141] which address other worries about the NSR approach. Some additional concerns about the SCTA scheme are the failure to satisfy the Hugenholtz-Pines gapless condition [125]. In this context it was also noted by Haussmann et al that [81] “a simple pseudogap ansatz for the spectral function [86] is not consistent with our results ... we do not observe a strong suppression of the spectral weight near the chemical potential.” More generally, there is some controversy in the Fermi gas literature [82, 88, 134, 136] about the presence or absence of a (pseudo)gap in this SCTA approach. Finally, we note that the principal weakness of the BCS-Leggett approach is that it focuses on the pairing channel while embedding all Hartree-like effects in the (effective) chemical potential. This leads to numerical discrepancies of some significance, particularly for the unitary Fermi gas.

In the lattice case an on-site attractive Hubbard Hamiltonian provides a prototypical model for studying BCS-BEC crossover in the literature. While in many ways t-matrix schemes involving all fully dressed Green’s functions [117, 142, 143] would seem to be more complete, in this model, the nature of the (pseudo)gap and whether it exists both above and below $T_c$ continues to be debated in the lattice context as well [133–135, 144, 145]. Indeed, a rather complete study of the associated excitation spectrum [146] for a conserving SCTA formalism shows multiple, complex excitation branches.

A useful reference to consult [117] presents comparative $T_c$ calculations for SCTA schemes along with the NSR approach and with DMFT. Here one sees that the transition temperatures in the NSR scheme are significantly higher (particularly in the asymptotic regime at large $|U|$) and this is attributed to the fact that this approach may tend to underestimate the effects of an indirect repulsion between fermion pairs. All t-matrix approaches, in some sense, ignore the effects of direct repulsion [11], but indirect effects appear via the interactions with the fermions. These observations may bear on Haussmann’s observation [147] that the approach to the BEC asymptote in the Fermi gas case should be from below and not above, as found for example by NSR.

IV. QUANTITATIVE IMPLICATIONS FOR 3D CROSSOVER SUPERCONDUCTORS

A. Two-gap physics present in BCS-BEC crossover

It is important to understand the necessity of having two distinct energy gaps in BCS-BEC crossover physics. These were illustrated in Fig. 4. The recognition of these two distinct gaps is an issue which bears on some of the interesting candidate materials that are claimed to exhibit BCS-BEC crossover, as we discuss in this Review.

Indeed, one of the central ways in which these two gap contributions are manifested has to do with the distinction between two classes of experiments: these are associated with phenomena which reflect superfluid coherence and those which reflect an excitation or pairing gap. The superfluid density $n_s$ [126, 148] provides a useful example, as it necessarily vanishes when coherence is destroyed or equivalently when $\Delta_{\text{sc}} = 0$. But, notably, it also depends on the total fermionic excitation gap $\Delta$ through the quasiparticle energy $E_k$:

$$
\frac{n_s}{m} = \frac{2}{3} \sum_k \left( \frac{\partial \xi_k}{\partial k} \right)^2 \frac{\Delta^2}{E_k^2} \left[ 1 - \frac{2f(E_k)}{E_k} \right] + \frac{\partial f(E_k)}{\partial E_k}
$$

for an isotropic $s$-wave superconductor in 3D with fermion mass $m$.

Similarly, it has been argued that Andreev scattering appears to measure the gap associated with the order parameter as distinct from conventional quasi-particle tunneling which measures the full pairing gap, $\Delta$. This has been claimed for the cuprates [149] as well as for twisted bi-layer graphene [150].

It is useful at this point to emphasize the fact that even though the bosonic degrees of freedom may be viewed as “quasi-ideal” within this generalized BCS framework, in contrast to an ideal Bose gas this does not compromise the existence of stable superfluidity. Superconductivity is stable in this framework as it is to be associated with the underlying fermionic degrees of freedom.

This analysis of the superfluid density provides a general template for other experiments which reflect true long-range order in a superconductor. Its low $T$ behavior has often been used to distinguish superconductors of different pairing symmetry, such as $s$- versus $d$-wave. We end by noting that this intrinsic two-gap behavior appears to have no natural counterpart in other preformed-pair scenarios (e.g., the phase-fluctuation approach) for the pseudogap.

B. Contrasting BCS-BEC crossover in $s$- and $d$-wave superconductors

A crucial feature of BCS-BEC crossover in superconductors (in either 2D or 3D) to be emphasized throughout this Review is that the canonical plots of the phase diagram (based on
the Fermi gases) do not capture the physics of superconductivity in the solid state. For the latter, as shown in Fig. 1(a), one finds $T_c$ follows a superconducting dome as a function of variable interaction strength, within the fermionic regime. Thus, one should not infer, as is often the case, that for solid-state superconductors in the BEC there is a large and maximal transition temperature.

Figure 7 provides more quantitative details on the key energy scale parameters which enter BCS-BEC crossover for the $s$-wave lattice case of Fig. 1(a). The figure indicates the behavior of $\Delta$ and $\mu$ at $T_c$ in units of a characteristic electronic scale (in this case corresponding to the half bandwidth). These energies are plotted as a function of varying attractive interaction strength, normalized to the half bandwidth $W = 6t$, where $t$ is the hopping matrix element. Also plotted here is the important parameter $n_{\text{pair}}$ which corresponds to the number density of pairs at the onset of the transition (normalized by $n/2$, as determined from Eq. (18)).

In particular, one can glean from the plot of $n_{\text{pair}}$ that the BEC or $\mu = 0$ transition is associated with the absence of fermions so that only pairs are present ($n_{\text{pair}} = n/2$). More generally, one can view the function $n_{\text{pair}}$ as a kind of theoretical “dial” informing about where a given system is within the crossover. Tuning the dial provides access to the counterpart values of $\mu$ and $\Delta$ at $T_c$. When $n_{\text{pair}}$ is essentially zero this corresponds to the BCS case and when $n_{\text{pair}} \approx n/2$ one enters the BEC regime.

The crossover behavior for a $d$-wave superconductor is generally different [122] and some aspects are additionally discussed in Appendix B. For definiteness, we consider here the symmetry to be of the form $d_{x^2-y^2}$, which is relevant to the cuprate superconductors. The central contrasting feature is the termination of $d$-wave superconductivity well before the BEC regime is entered. This is found at all but very low electron densities and derives principally from the fact that $d$-wave pairs have a more extended size. As a result a pair-pair repulsive interaction which is always present [11] is sufficiently strong so that it inhibits pair hopping and pairs become localized. And, importantly, this happens in the fermionic regime, well away from where $\mu < 0$. Consequently, in the $d$-wave case, the BEC limit cannot generally be accessed [122]. This important effect is illustrated in the phase diagram shown in Fig. 8.

What this implies more concretely is that the $d$-wave system undergoes a transition at moderately strong attraction, where $T_c \rightarrow 0$. Here superconductivity continuously disappears, albeit in the presence of a finite pairing gap $\Delta$ or finite $T^*$. This has features which are suggestive of the widely discussed “Cooper-pair insulator” [151–153] or a pair density wave alternative [154]. This form of pair localization pertains to a clean system and represents a different mechanism, deriving from strong intra-pair attraction and strong inter-pair repulsion, which inhibits pair hopping. This same localization has also been observed in cases where the band filling is high in $s$-wave superconductors, as well as in 2D systems. In these instances it provides an interesting comparison, but is not to be associated with strong disorder effects which are known to drive a superconductor-insulator transition in superconducting films [151, 152, 155, 156].
Figure 9 provides more quantitative details on the characteristic energy scale parameters which enter BCS-BEC crossover for this \(d\)-wave lattice case [122]. Plotted here is the behavior of \(\Delta\) and \(\mu\) at \(T_c\) as a function of varying attractive interaction. Also indicated is the number of pairs (derived from Eq. (18)), \(n_{\text{pair}}\), at the onset of the transition.

C. The interplay of conventional fluctuations and BCS-BEC crossover physics: Normal-state transport

The question of how conventional superconducting fluctuations relate to BCS-BEC crossover physics continues to be raised in the literature. In this regard it is interesting to note that the treatment of preformed pairs presented here is closely related to self-consistent theories of fluctuation superconductivity. In particular, it represents a natural extension to arbitrarily strong attraction of time-dependent Ginzburg-Landau-based transport theory [157] when the quartic terms in this free-energy expansion are treated in a self-consistent Hartree-level approximation [128, 157–159]. This observation suggests that there is a continuous variation, associated with an enhancement of many transport fluctuation signatures, as the coupling varies from weak to strong.

To address these issues more quantitatively, we note that dominating transport in these more strongly correlated superconductors [69, 70, 72] is the fact that there are now two distinct temperature scales which control “fluctuation” effects: \(T_c\) and \(T^*\). Transport is complicated additionally by the fact that there are two types of quasiparticles: fermions which experience the gap onset at \(T^*\) at which point they generally become less conducting, and bosons whose presence is expected to increase conductivity at temperatures somewhat below \(T^*\). These two types of quasiparticles are represented schematically in the upper row of Fig. 3.

The fermionic contribution has been discussed [160, 161] in some detail both above and below \(T_c\). The more familiar fluctuation contributions to bosonic transport derive from the Aslamazov-Larkin [162] diagrams and are associated with a small pair chemical potential, \(\mu_{\text{pair}}(T)\), which is found in the immediate vicinity of \(T_c\). In conventional superconductors, \(\mu_{\text{pair}}\) depends only on \(T_c\), but in the presence of more stable preformed pairs one expects that \(T^*\) will play an important role. It is at this higher temperature that the pair density vanishes; consequently, fluctuation effects are expected to have some presence even at temperatures as high as \(T^*\).

The above discussion leads one to conclude that, for more strongly coupled superconductors, the nature of “fluctuation” effects associated with \(T^*\) in transport requires that one establish the relative size of the contributions from the fermionic and bosonic channels; as we have seen these generally introduce opposite temperature dependencies in their conduction properties. Their relative size depends on their relative scattering times.

Central to this comparison is the fact that the resistivity downturn, a canonical signature of the pseudogap onset at \(T^*\), is frequently associated with the concomitant and rather ubiquitous large normal-state resistivity. This “bad-metal” behavior [34, 72] reflects a suppressed fermionic conduction channel. Importantly, bad metallicity allows the bosonic conducting channel to become more prominent and, for example, leads to a boson-related downturn near \(T^*\) in the resistivity which would otherwise be obscured by gap effects in the fermionic spectrum.

We will see later in this Review examples of transport signatures which are viewed as indicative of the presence of BCS-BEC crossover physics. In addition to a resistivity downturn, these include enhanced diamagnetism and Nernst signatures, albeit not all uniquely pointing to a BCS-BEC crossover scenario.

D. Relation between BCS-BEC crossover and the Uemura plots

In an interesting series of papers, Y. Uemura [163, 164] has used muon spin resonance (\(\mu\)SR) experiments to establish a classification scheme for superconducting materials. This classification, in effect, distinguishes so-called “exotic” superconductors from conventional superconductors. The \(\mu\)SR relaxation rates in these experiments effectively measure the London penetration depth which in turn reflects the ratio of the number of superfluid electrons \(n_s\) to their effective mass \(m^*\). Notably, at sufficiently low temperatures, these same two quantities help to determine an effective Fermi temperature.

Uemura used this analysis to suggest that “unconventional” superconductors are characterized by the proportionality \(T_c \propto T_F\), where \(T_F = E_F/k_B\) is the Fermi temperature. This observation, which follows from plots of the transition temperature versus muon-spin relaxation rate, has led many to believe that a dependence on a single parameter \(T_F\) is suggestive of a Bose-condensation description of exotic superconductors. Underlying this inference is the behavior of the Fermi-gas phase diagram as shown, for example, in Fig 1(b), where the asymptotic
It will be useful, thus, in this Review to show how to arrive at a more discriminating procedure, inspired to some extent by Fig. 10. We will do so here, focusing on 2D superconductors in the form of plots of \( \Delta_0/E_F \) versus \( T^*/T_{\text{BKT}} \). First, however, one has to have a better understanding of 2D superconductivity.

Such an analysis emphasizes that, for an arbitrary superconductor, more relevant for establishing that a crossover picture is applicable is showing the presence of distinct energy scales \( T^* \) and \( T_c \). This is a necessary but not sufficient requirement. In the crossover scenario there must be simultaneously present a moderately large value for \( \Delta_0/E_F \). In this way the Uemura plots have elucidated a useful classification scheme, but we emphasize that one should be cautious about inferring too strong a connection to BCS-BEC crossover.

It will be useful, thus, in this Review to show how to arrive at a more discriminating procedure, inspired to some extent by Fig. 10. We will do so here, focusing on 2D superconductors in the form of plots of \( \Delta_0/E_F \) versus \( T^*/T_{\text{BKT}} \). First, however, one has to have a better understanding of 2D superconductivity.

**V. BCS-BEC CROSSOVER PHYSICS IN THE 2D LIMIT**

**A. Overview of 2D theory**

In two dimensions there is no true condensation with off-diagonal long range order. More quantitatively, in the language of a \( t \)-matrix approach to BCS-BEC crossover, the chemical potential for pairs \( \mu_{\text{pair}} \) never reaches zero; this is effectively a consequence of the Thouless criterion which provides a constraint on the \( t \)-matrix. A subtle issue which is pertinent here and in the following discussion is that a fermionic system in either 2D or 3D involves in some sense non-interacting bosons, but these non-interacting pairs nevertheless support superconductivity only because they interact indirectly through their underlying fermionic nature.

In this Review we build on the cold-atom literature to address the BKT phase transition \([63, 64]\). This focuses on the approach from the high-temperature side and on bosonic degrees of freedom or bosonic "quasi-condensation" (associated with algebraic rather than long range order). The onset of this transition can be equivalently described as that of the onset of vortex-pair binding and unbinding as in the original BKT papers; in this context the role of superfluid phase stiffness is more apparent.

From the bosonic perspective, the BKT transition occurs when the de Broglie wavelength is large and comparable to the inter-pair separation, similar to a BEC transition in 3D. More precisely, this transition arises when the temperature-dependent bosonic phase-space density reaches a critical value as was independently established in famous papers by Hohenberg and Fisher \([68]\) and also by Popov \([166]\). This leads to

\[
T_{\text{BKT}} = \left( \frac{2\pi}{D_{\text{crit}}^\text{pair}} \right) \frac{n_{\text{pair}}(T_{\text{BKT}})}{M_{\text{pair}}(T_{\text{BKT}})}, \tag{23}
\]

where \( D_{\text{crit}}^\text{pair} \) is the critical phase space density which is essentially a constant and will be specified shortly. Importantly, here we have replaced the number density and mass of true bosons appearing in the standard expression (Eq. (5)) by their counterpart values for a composite-boson (or fermion-pair) system. In this way we see that the pair density and pair mass play a similar role as in the 3D superfluid transition in Eq. (21).

Note that, since \( n_{\text{pair}}(T) \) is temperature dependent and disappears at \( T^* \), there is a significant difference between BKT behavior in Bose and Fermi superfluids. That is, the latter will be implicitly dependent on the two distinct temperature scales \( T^* \) and \( T_{\text{BKT}} \). Since \( T_{\text{BKT}} \leq T^* \), the physical implications of these two scales become apparent only when studying the BKT transition, as we do here, by approaching the transition from the normal state.

The most detailed numerical analysis of 2D atomic-gas condensates focuses on the Bose gas in the weakly interacting limit and provides \([67]\) results for the critical value \( T_{\text{pair}}^\text{crit} \) which is given by

\[
D_{\text{pair}}^\text{crit} = \ln(C/\bar{g}), \tag{24}
\]
where \( \tilde{g} \) is a dimensionless coupling constant reflecting the effective repulsive interaction between pairs. Importantly, the constant

\[
C \approx 380
\]

(25)

has been established [67] from Monte-Carlo studies. We note that the \( \tilde{g} \) in Eq. (24) is, in principle, dependent on the bosonic pair density, as shown in Ref. [68]. However, this dependence is logarithmic, and therefore weak, and can be neglected for most purposes because of the large constant \( C \). This normal-state approach to the BKT transition, using the phase space density, has been supported by numerous experimental studies on atomic Bose gases [65, 167, 168].

It is useful to compare this with the more familiar expression [169] for the same \( T_{\text{BKT}} \) in a superconductor, when approached from the low-temperature superfluid side. This provides a complementary interpretation.

\[
T_{\text{BKT}} = \frac{\pi}{2} \rho_s(T_{\text{BKT}}) = \frac{\pi}{8} \left( \frac{n_s}{m} \right) (T_{\text{BKT}}),
\]

(26)

where one introduces the temperature-dependent superfluid phase stiffness \( \rho_s(T) \), evaluated at \( T_{\text{BKT}} \), instead of the total pair density as in the formula of Eq. (23). In this equation, \( n_s \) and \( m \) are the superfluid density and effective mass of fermions. To connect Eq. (23) to Eq. (26), one replaces \( D_{\text{pair}}^{\text{crit}} \) with 4 and converts from pairs to fermions, following Halperin and Nelson [170].

It should be noted that there is a practical difficulty in using either of these formulations. We need phenomenological input to arrive at \( \tilde{g} \) in Eq. (24). Whereas to apply Eq. (26), one must approximate \( \rho_s(T) \) by a suitably chosen (generally mean-field) expression 7.

B. Procedure for determining \( T_{\text{BKT}} \) in the Fermi gases

The Heidelberg cold-atom group [171] has claimed that the fits for their 2D Fermi gas data find [171, 172] a range of values for \( D_{\text{pair}}^{\text{crit}} = 4.9 - 6.45 \). These values are close to but somewhat different from values for atomic Bose gases, where the range is about 6 - 10. In general, \( D_{\text{pair}}^{\text{crit}} \) depends on the non-universal boson-boson interaction strength \( \tilde{g} \), about which one has no precise knowledge. However, a relatively small value of \( \tilde{g} \) is presumed in the theoretical framework [67, 173], representing an effectively weakly interacting gas. This would be expected in a BCS ground state of composite bosons, as the bosonic degrees of freedom enter this wave function in a quasi-ideal manner. For the analysis in this Review, we adopt the value \( D_{\text{pair}}^{\text{crit}} = 4.9 \), which turns out to best fit [172] the data on Fermi gases 8.

Figure 11. (a) Comparison of theory [174] and experiment for quasi-condensation phase diagram of the strongly interacting 2D Fermi gas. The color variations reflect the normalized momentum distribution of pairs at low momentum \( q \), \( N_q/N \) which is used to quantify the quasi-condensate fraction [172]. (b) Theory results (with a trap included) [174]. Here the color variations similarly refer to the pair momentum distribution at low \( q \). The estimated onset of the superfluid transition which derives from a rather abrupt change in \( N_q/N \) is indicated by the black solid line in both panels, dashed for experiment and solid for theory. The white dashed line in panel (a) is a theoretical estimate for the BKT transition from Ref. [175].

Therefore, based on experiments [172] in Fermi gases, the 2D BKT superconducting transition is thus interpreted as a “quasi-condensation” of preformed Cooper pairs. For application to 2D superconductors, more generally, the BKT transition temperature is presumed to be:

\[
\frac{n_{\text{pair}}(T_{\text{BKT}})}{M_{\text{pair}}(T_{\text{BKT}})} = \left( \frac{4.9}{2\pi} \right) T_{\text{BKT}} \quad \text{in 2D.}
\]

(27)

Experiments from the Heidelberg group [171, 172] on a strongly interacting 2D Fermi gas use the momentum distribution to establish the presence of a quasi condensate. This is based on magnetic field sweeps which, through a Feshbach resonance, convert pairs to deeply bound molecules. As shown in the upper panel in Fig. 11, in this way one obtains a plot of the quasi-condensation transition temperature as a function of scattering length or equivalently variable interaction strength. Importantly, these measurements show BKT signatures. An overlay of theory and experiment is shown in the top panel of Fig. 11, while the bottom panel represents only the theory [174]. It should be noted that there are claims [176] that the maximum observed experimentally, which goes beyond \( T_F/8 \), could be an artifact of coupling to a third dimension in the trap, although this issue which pertains exclusively to the 2D Fermi gas, has not been settled.

\[\text{7This excludes using the present t-matrix theory, more precisely the 2D counterpart of Eq. (22), where the superfluid density } n_s \text{ is necessarily zero in 2D, reflecting the fact that simple bosonic condensation with long range order cannot occur.}\]

\[\text{8It should be noted that this best fit case does presume a larger value of } \tilde{g} \text{ than would be expected for the weakly interacting case [171].}\]
Subsequent experiments on the 2D gas [177, 178] extended these measurements on trapped gases to accommodate a box potential. Here an alternative methodology was used to obtain the momentum distribution. These studies presented more direct measurements of superfluidity, as distinct from pair quasi-condensation. Determination of one particular critical temperature in the BCS regime yielded consistency with the experiments of the Heidelberg group as a check.

C. Quantitative description of BCS-BEC crossover in 2D and comparison with 3D

Equation (27) is adopted along with the results of Sec. III B for \( n_{\text{pair}} \) and \( M_{\text{pair}} \) to characterize \( T_{\text{BKT}} \) and other features of 2D superconductors. Figure 12 presents a comparison of transition temperatures, pairing onset temperatures, pair size [179], gap size and coherence length in both two and three dimensions for the s-wave case. In panel (a) one sees the presence of a dome-like structure reflecting BCS-BEC crossover in the solid state, which should be evident for \( T_c \). Determination of one particular critical temperature, as distinct from pair quasi-condensation, requires measurements of superfluidity, as distinct from pair quasi-condensation. These studies presented more direct measurements of superfluidity, as distinct from pair quasi-condensation. Determination of one particular critical temperature in the BCS regime yielded consistency with the experiments of the Heidelberg group as a check.

The inset in Fig. 12(a) quantifies the important effect of two dimensionality which was presented earlier in the schematic plot shown in Fig. 2. This inset, representing moderately low filling \( n = 0.1 \) per unit cell, shows that the deviation from BCS behavior (associated with \( T_\ast /T_c \) substantially above 1.0) occurs at significantly smaller attraction for 2D as compared with 3D superconductors.

We turn now to Figs. 12(b) and 12(c) which are the basis for further experimentally relevant studies. The main plots in these two figures represent a natural extension of the Tallon-Uemura scaling in Fig. 10, but for the case of s-wave pairing in both two and three dimensions. They show that the ratio of the two distinct temperature scales \( T_\ast /T_c \) or \( T_\ast /T_{\text{BKT}} \) (which are, in principle, measurable), is correlated with the magnitude of the \( T \approx 0 \) value of \( \Delta /E_F \) (which is also measurable).

The inset in Fig. 12(b) shows how the zero-temperature pair size, \( \xi_0 \), varies as the system crosses out of the BCS regime. Representing this crossover in the figure is \( T_\ast /T_c \), chosen as the horizontal axis. The pair size is a reasonably good indicator of when the system is promoted out of the BCS regime. However, it can be inferred from Fig. 12(c), (where the BEC onsets are marked) that it does not display features at the onset of the BEC; rather the pair size decreases continuously toward zero as this limit is approached. Interestingly, in 2D the pair sizes for equivalent \( T_\ast /T_c \) are significantly larger than in the 3D case.

Finally, it should be emphasized that the pair size (which is less accessible experimentally) and the coherence length represent important but distinct length scales. The “bare” Ginzburg-Landau coherence length can be most readily obtained experimentally from the measured slope of the upper critical field \( H_{c2} \) vs temperature \( T \) plot

\[
\frac{dH_{c2}}{dT} |_{T=T_c} = -\frac{\Phi_0}{2\pi \xi_0^2 T_c} \text{ with } \Phi_0 = \frac{\hbar c}{|2e|},
\]

where \( h = 2\pi \hbar \). Here, the slope is evaluated at the zero field \( T_c \), and \( \xi_0^2 \) is theoretically given by [69, 70]

\[
\xi_0^2 = \frac{\hbar}{\sqrt{2M_{\text{pair}}(\hbar T_c)}}, \quad (28)
\]

This quantity times the Fermi wave-vector is plotted in the inset in Fig. 12(c). From an experimental point of view there may be some advantage to measuring and evaluating \( \xi_0^2 \) in a somewhat different way, just above \( T_c \) in the normal state [180] as here one avoids the rather challenging determination of \( T_c(H) \) which corresponds to a magnetic field broadened transition.

The coherence length has a distinct physical interpretation when we make use of the expressions for the transition temperatures in Eqs. (3) and (23). First, define \( k_F \) in terms of the free and isotropic electron dispersion, so that \( k_F \equiv (3\pi/2n)^{1/3} \) or \((2\pi n)^{1/2}\) in three and two dimensions respectively where we use the same symbol \( n \) to refer to the appropriate particle density. It follows, then, that \( k_F \xi_0^2 \) evaluated near the transition temperature depends only on the normalized pair density, \( n_{\text{pair}}/n \). This leads to

\[
k_F \xi_0^2 = 1.6(n/n_{\text{pair}})^{1/2} \quad (29)
\]

and

\[
k_F \xi_0^2 = 1.2(n/n_{\text{pair}})^{1/3} \quad (30)
\]

for 2D and 3D respectively.

We note that the above equations are relatively easy to understand physically. The coherence length is a length scale representing the effective separation between pairs. We find here, not surprisingly for only weakly interacting pairs, that it relates to the density of pairs. This is distinct from the pair size. In BCS theory there are almost no pairs present at \( T_c \) and the length which represents their average separation is necessarily very long. As pairing becomes stronger more pairs form and their separation becomes shorter. On a lattice, in the BEC regime their separation is bounded from below by the characteristic lattice spacing and \( \xi_0^2 \) approaches an asymptote set by the inter-particle distance as the system varies from BCS to BEC.

Moreover, from plots of \( n_{\text{pair}}/n \) such as those in Fig. 7, one sees that \( k_F \xi_0^2 \) allows a very useful and direct monitoring of the location of a system between the BCS and BEC limits. Notably, \( k_F \xi_0^2 \) reaches a finite lower bound at the onset of the BEC, given by \( k_F \xi_0^2 \approx 2.2 \) for 2D and 1.5 for 3D (for the case of s-wave superconductors). Importantly, that these saturation numbers are of order unity is consistent with what has been anticipated by the experimental community [181].

We end this section with Fig. 12(c) which presents a “zoomed out” view of the main figure in Fig. 12(b). This provides information about where one should expect the onset of
the BEC. Importantly, the BEC regime appears to be associated with very large values of $T^*/T_c$. In this way, one might expect the BEC limit to be rather inaccessible.

D. Low carrier density in BCS-BEC crossover

In this subsection we wish to clarify what one should expect when the carrier density is dramatically reduced in a lattice superconductor. For definiteness we will consider only two dimensional systems here and presume that “low density” corresponds well below 1/4 filled bands, say $n < 0.1$.

The notion that low carrier density promotes a system out of the BCS regime dates back to Eagles [5]. Indeed in the literature it has been stressed [74] that when the band is nearly empty it requires only a small change in the attractive interaction to push the fermionic chemical potential below the conduction band bottom; hence the BEC regime is more accessible at low $n$.

What is not so clear is whether low $n$ alone can increase the magnitude of $T_c$ (or $T_{\text{BKT}}$) or not. Also of interest is determining whether or not at low densities the nature of the underlying lattice dispersion becomes irrelevant. If so, this would mean that the low-density system could be treated as a Fermi gas.

In the phase fluctuation approach (of Sec. II C) low density plays a rather dominant role [75]. While this scenario has been developed primarily for the cuprates, it can be considered in a broader context, much as the BCS-BEC crossover scenario is viewed as more generally applicable. Indeed, one might wonder if the two scenarios converge in the low carrier density limit. We find that they do not.

In the phase fluctuation scenario it is emphasized that low carrier density is associated with both poor screening and small phase stiffness or low superfluid density. Small phase stiffness, in turn, means that classical phase fluctuations of the superconducting order parameter become more prominent. These fluctuations necessarily lead to a more extensive (in temperature) “critical regime”.

To address to what extent this scenario is to be distinguished from the low carrier density limit in BCS-BEC crossover it is
Figure 14. $T_{\text{BKT}}$ as a function of $U$ on a semi-log scale for variable carrier density, showing a nearly universal shape but with a dramatically decreasing magnitude of the transition temperature. The model and dispersion are the same as in Fig. 13. The small dips in this figure are associated with the crossover to the BEC regime after which the canonical $t^2/|U|$ dependence is found for the transition temperature. This dependence is a lattice effect, which persists even in the zero carrier density limit. Here $W$ is half the bandwidth.

It is evident that, as expected, low density makes the BEC limit more accessible, as it occurs for smaller attraction strength.

An important message is contained in Fig. 13(b). This shows that $\Delta_0/E_F$ remains remarkably large at low and relatively high densities for the same $T^*/T_{\text{BKT}}$. Thus pairing remains strong and because of the large size of $\Delta_0/E_F$ (and small size of $k_{\text{F},c}^\text{coh}$, which is not shown), even in the very low carrier density limit, it should be possible to distinguish BCS-BEC crossover from a phase fluctuation scenario. It should be stressed, however, that the phase fluctuation approach does not address fermionic degrees of freedom, so that, strictly speaking, the pairing gap is irrelevant.

Figure 14 presents a plot of the normalized transition temperature $T_{\text{BKT}}/W$ as a function of normalized interaction strength for variable density. One sees that all curves assume a fairly universal shape, but there is a dramatic reduction in the size of the transition temperature as the density is decreased. One can glean from these observations a notable trend. In both the case of changing dimensionality from three to two and changing carrier density from moderate to very low, it follows that the superconductor is more readily promoted out of the strict BCS regime. But at the same time, the transition temperatures are significantly reduced.

Another important observation from Fig. 14 is that the effect of the underlying lattice structure is always present in the BEC regime of the $T_{\text{BKT}}$ phase diagram [182]. In particular, the $T_{\text{BKT}} \sim t^2/\langle U \rangle$ asymptote at large $|U|$ persists all the way to the zero carrier density limit, so that a Fermi gas description of the phase diagram is not applicable. At the same time, interestingly, Fig. 13(b) indicates $\Delta_0/E_F$ approaches its counterpart value for a Fermi gas. This occurs at extremely low densities but still in the BCS-BEC crossover regime, where the strength of $|U|$ is such that the fermionic chemical potential $\mu$ remains positive.

The small size of $T_{\text{BKT}}$ found here for BCS-BEC crossover at low density, should not be surprising also from the perspective of the phase fluctuation scenario, as the transition temperature, even in 3D, is governed by the small superfluid density. But it is interesting to note that there are instances in the literature when low carrier density is found to be associated with an increase in the transition temperatures [183]. This would seem to require that the pairing mechanism is assisted by low density. Although this is highly speculative, one might suspect that when this occurs Coulomb interactions are driving the pairing and not undermining it.

E. Topology and quantum geometry in BCS-BEC crossover

In this Review, we will see that current experimental candidates for BCS-BEC crossover tend to have values of $T^*/T_c$ of the order of 2 or 3, and corresponding values of $\Delta_0/E_F$ on the order of 0.5. From Fig. 12(c), one might infer that these are not likely to be in the BEC regime. There is, however, an exception having to do with flat-band, topological systems. These may be relevant to the recent discovery of 2D superconductivity in MATBG and MATTG where there are claims [181, 185, 186] that these flat-band systems are somewhere between BCS and BEC (MATBG) or even beyond, that is, within the BEC regime (MATTG).

Experimentally, when twist angles in these graphene systems are associated with very flat bands, this seems to correlate with the highest transition temperatures. There is, however, a subtle and important feature here. In flat-band superconductors, pair hopping, like single-particle hopping, is also suppressed [184, 187, 188]. As a consequence, the pair mass $M_{\text{pair}}$ becomes large and the superfluid stiffness is small. This would lead to a vanishing $T_{\text{BKT}}$ in the extremely flat-band limit, were it not for multi-band/multi-orbital effects. Moreover, it has been emphasized [188] that these latter inter-band contributions (which work to decrease the pair mass) can be amplified in the presence of nontrivial normal-state band topology. This occurs through so-called quantum geometric effects.

Such multiband effects have been incorporated into a 2D s-wave BCS-BEC framework [184] where a phase diagram with the usual superconducting dome is found, as shown in Fig. 15(a). The model topological Hamiltonian yields two bands, whose conduction bandwidth is much smaller than the inter-band energy separation. The calculated phase diagram
resembles that obtained from Monte Carlo results [189] using the same model Hamiltonian. Importantly, this phase diagram can be used to extract the ratio $T^*/T_{BKT}$ along with the number of bosons $n_{pair}/n$, as shown in Fig. 15(b); both these variables are plotted as a function of renormalized interaction strength. The quantity $n_{pair}$ provides a ready indication of where the BEC sets in, as here $n_{pair}$ first reaches $n/2$.

At the transition point to the BEC regime (indicated by the arrows), the interaction strength $U$ is on the order of the entire conduction band width. Correspondingly, $\Delta_0/E_F \sim 3$ as shown in Fig. 15(c), which is not so different from the single band result in Fig. 12(c). On the other hand, because of quantum geometry, $T_{BKT}$ is substantially enhanced by inter-band effects while $T^*$ is almost unaffected, leading to a smaller and physically more accessible value of $T^*/T_{BKT} \sim 5$. This behavior is summarized in Fig. 15(c), where the BEC onset point is indicated by the arrow. This provides a counterpart plot of Fig. 12(c) but here for a multi-band, topological case. We note that the value of $\Delta_0/E_F$ at the BEC onset is non-universal. For a topological band structure with an extremely flat conduction band [184], $\Delta_0/E_F$ can be as large as 30.

The above contrast leads us to the interesting conclusion that in the presence of flat bands and non-trivial band topology a BEC phase can potentially become more accessible, as it leads to a moderate size for $T^*/T_{BKT}$. We emphasize that these effects derive from the participation of more than one band in the superconductivity and note, for completeness, that there are other, rather different approaches in the literature which also treat BCS-BEC phenomena in multi-band systems both analytically [190, 191] and numerically [192].

### VI. STRONGLY DISORDERED CONVENTIONAL FILMS: TWO ENERGY SCALES AND A PSEUDOGAP

We return to Fig. 12(b) noting that this figure presents a unique signature of 2D pseudogap effects associated with a strong-pairing mechanism. It may seem surprising, but strong disorder can lead to similar pseudogap effects in 2D superconducting films [193]. However, the parameters governing these dirty thin films are very different from those indicated in Fig. 12(b). In understanding the origin of this other pseudogap, it is important to recall that 2D superconductors have a propensity for manifesting a separation of the two energy scales $T^*$ and $T_{BKT}$ which can be thought of as corresponding to the onset temperatures for amplitude and phase coherence, respectively. As an important signature, those conventional superconducting films in which the two temperature scales are well separated due to disorder [193, 194] will have rather small values of $\Delta_0/E_F$.

While the distinctions between the two scenarios for a pseudogap (strong pairing and strong disorder) should be obvious, a number of phenomenological similarities are rather striking. Most notable among these are the reported observations of charge $2e$ pairs [195, 196], the contrasting behavior of Andreev and conventional tunneling [150, 197], and the observations of boson or pair localization [122, 153]. The behavior found rather generically for a highly disordered 2D superconductor is illustrated in Fig. 16 which represents an experimentally determined phase diagram [198] with temperature on the vertical axis and disorder measured through $k_F l$ on the horizontal axis. Here $l$ is the electron mean-free path. In Fig. 16, the superconducting state is shown in orange, the pseudogap state in red, and the normal-state metal in white. Also indicated are the temperatures $T^*$ and $T_c = T_{BKT}$.

There are three demarcated regions. At very small disorder (region I) a pseudogap is absent and $T^* \approx T_c$, while as disorder increases (region II), $T^*$ separates from $T_c$ and is relatively independent of the disorder strength, while the transition temperature (which is more sensitive to the undermining of coherence) rapidly decreases. Finally in region III, $T_c$ vanishes although there are indications that pairing persists. The two temperatures become distinct at a critical value of $k_F l$.

These experiments on NbN are reasonably generic and similar observations have been made for TiN and InO$_x$ as well.
Figure 16. Experimental temperature scales as a function of mean-free path $k_F$ in disordered NbN films [198]. The value of $k_F$ is determined from resistance and Hall-coefficient measurements at $T = 285$ K. With increasing disorder, or sufficiently small $k_F$, a pseudogap (PG) phase appears associated with $T^* \neq T_c$ in region II, while in region III, $T_c$ is zero although pairing likely persists in this insulating phase.

Figure 17. Experimental behavior of characteristic temperatures $T^* \equiv T_{\Delta}$ and $T_c \equiv T_{BKT}$ as a function of the thickness $d$ of Pb films [194]. “ML” stands for monolayer. A more extensive analysis of the resistivity (see text) suggests that the evident pseudogap effects, here and in the previous figure, are likely associated with high disorder, rather than strong pairing correlations.

Figure 18. $V$-$I$ isotherms on a log-log plot associated with the Pb films in the previous figure [194]. Each curve is labeled by its temperature (in units of Kelvin) and a straightforward analysis identifies $T_{BKT}$ with the $V \propto I^3$ black line. One sees that the $V$-$I$ characteristics display a continuous evolution towards Ohmic behavior as the temperature is raised to the pairing onset temperature $T^*$, here identified to be 7 K (the $V \sim I$ black line) for a Pb film of a particular fixed thickness.

Figure 19. $V$-$I$ isotherms on a log-log plot associated with the Pb films in the previous figure [194]. Each curve is labeled by its temperature (in units of Kelvin) and a straightforward analysis identifies $T_{BKT}$ with the $V \propto I^3$ black line. One sees that the $V$-$I$ characteristics display a continuous evolution towards Ohmic behavior as the temperature is raised to the pairing onset temperature $T^*$, here identified to be 7 K (the $V \sim I$ black line) for a Pb film of a particular fixed thickness.

where [193] the authors claim that a pseudogap appears to be present which reflects the existence of paired electrons above $T_{BKT}$. Importantly, this pseudogap is found to be continuously and directly transformed into a superconducting gap below the transition.

An interesting set of parallel experiments [194] shown in Fig. 17 was performed on Pb films by a group at Tsinghua University, who determined the experimental phase diagram obtained by studying crystalline and atomically flat Pb films, now as a function of variable thickness. In Fig. 17, temperature appears on the vertical axis and increasing thickness on the horizontal axis. Here the superconducting state is shown in green, the “fluctuating” or pseudogap state in blue (where non-superconducting Cooper pairs are said to exist) and the normal-state metal in yellow. The solid symbols represent superconducting or phase-coherent order, as determined by transport with an onset at $T_{\varphi} \equiv T_{BKT}$; the open symbols represent the pairing transition ($T_{\Delta} \equiv T^*$), which is established by tunneling spectroscopy.

From Fig. 17 one can infer that the pairing temperature remains nearly constant with variable thickness, while the coherence temperature is strongly depressed. This appears to suggest that disorder may be playing a role, as supported by the sheet resistance data measured by the same group.

It is reasonably well established that, quite generally, $T_{BKT}$ decreases with decreasing thickness in 2D films [200], although there is no consensus on the extent to which disorder is the only relevant mechanism. The central point, then, is that pairs form at higher temperatures than those at which they exhibit superfluidity. Equivalently, at $T_{\varphi}$, while phase coherence is destroyed, the superconducting gap remains non-zero. Note that for Pb, the two characteristic temperatures merge in the 3D regime, as is the hallmark of a “conventional” weak-coupling bulk superconductor.

A key finding of the Tsinghua group [194] pertains to the voltage-current ($V$-$I$) characteristics which provide an alternative method for deducing the pairing onset temperature $T^*$.

---

[9] Since $T^*$ essentially represents a mean-field transition temperature of an $s$-wave superconductor, this should satisfy Anderson’s theorem [199] of disordered superconductivity; $T^*$ is thus expected to remain relatively robust in the presence of disorder that does not break time-reversal symmetry, provided that the effective pairing interaction is not strongly affected by localization effects.
We emphasize that this “short-cut” procedure should be applicable to all 2D superconductors. More precisely, the authors have shown that $V$-$I$ plots of this type can be used to simultaneously measure the two important energy scales $T^*$ and $T_{\text{BKT}}$. This is illustrated in Fig. 18 where voltage-current plots are presented for a range of different temperatures in one particular Pb thin film.

More specifically, it is well known [170] that estimates based on $V$-$I$ curves allow one to determine the BKT transition which occurs when the condition $V \propto I^\alpha$ is satisfied with a particular value of $\alpha = 3$. Importantly, the authors in the Pb experiments [194] have pointed out that one can also obtain the pairing onset temperature, $T^*$, from $V$-$I$ plots. This is associated with the recovery of fully Ohmic behavior shown in Fig. 18 by the $V \propto I$ black line.

While this observation could seem intuitively obvious, the authors have made the last point more convincing by accompanying their analysis with more direct measurements of the pairing gap through scanning tunneling microscopy (STM) experiments which yield $\Delta(T)$ from which they infer $T^*$. We caution by noting that one should take care in establishing the “Ohmic recovery” temperature as it involves the behavior of the entire $V$-$I$ curve, for an extended range of $I$ above the critical current.

VII. APPLICATION OF BCS-BEC CROSSOVER IN THE LITERATURE (BEYOND FERMI GASES)

In this section we present summaries of experimental observations concerning candidate systems for BCS-BEC crossover. We will show that the majority of the candidates appear consistent with this scenario, as they possess all or most of the (first three) discriminating properties listed in Sec. II A. These correspond to (i) the observation of large $\Delta_0/E_F$, (ii) the presence of a normal-state pseudogap so that $T^*$ is significantly above 1.0 and (iii) a moderately short coherence length, $k_F\xi_{\text{coh}}$. Also reported in a few cases is the observation of enhanced superconducting fluctuation-like behavior in the normal state, particularly in the response to a magnetic field [201, 202].

Notably, however, what is missing in a number of cases (particularly for the organic superconductors [203] and the two twisted magic angle graphene systems) is information about how the very important temperature scale $T^*$ varies across their respective $T_c$ domes. We note that in strictly 2D systems this appears to be reasonably accessible should there be future measurements of the $V$-$I$ characteristics. This capability was discussed in Sec. VI, based on the Ohmic recovery temperature which effectively yields $T^*$.

Overall, what seems to be nearly universally observed in these candidate BCS-BEC crossover superconductors is a large magnitude for $\Delta_0/E_F$ and a relatively small size for the Ginzburg-Landau coherence length $k_F\xi_{\text{coh}}$. The focus on this last quantity serves to emphasize the striking contrast with the Fermi gases, where this coherence length is not as readily accessible.

Connections to more specific aspects of BCS-BEC crossover theory are presented in Sec. IX via a summary figure (Fig. 36) for all the candidate materials in 2D. Unlike the Fermi gases, where the magnitude of the attractive (Hubbard-like) interaction can be quantified, here one has to circumvent this parameter. As will be shown in the plot we address correlations of $T^*/T_{\text{BKT}}$ and $\Delta_0/E_F$ instead. Moreover, a related plot which focuses on commonalities between the graphene and cuprate families is Fig. 40. While in Fig. 36 a simple tight binding bandstructure is used for all candidate materials, we argue in the spirit of this Review, the specific details of the bandstructure are viewed as less important than distinguishing between $s$- and $d$-wave pairing symmetries, or 2D and 3D systems or addressing some of the more universal features of the crossover.

A. BCS-BEC crossover in the 2D organic conductors

Over the years there have been observations that a class of quasi-2D organic superconductors based on the BEDT-TTF molecule, of the type $\kappa$-(BEDT-TTF)$_2$X, might have something in common with the high temperature superconductors [204]. Here X is an inorganic anion and $\kappa$ denotes a particular packing arrangement in the crystal. The basic unit here is a dimer consisting of two BEDT-TTF molecules stacked on top of one another. Upon binding with the anion the dimer provides one electron to the anion leaving behind a mobile hole.

The similarity with the cuprates has been based on the observations [205–207] of competing metallic, insulating, superconducting and antiferromagnetic states in the phase diagram which is generally plotted as a function of pressure. As the pressure decreases (presumably in analogy to a decrease in doping in the cuprates) the properties of the molecular conductor (and its superconducting phase) deviate progressively from those of a typical metal (and BCS superconductor). Conversely with an increase in pressure the behavior appears more conventional.

Of some interest is the case where X involves HgBr (more particularly one studies $\kappa$-(BEDT-TTF)$_2$Hg$_2$S$_8$Br$_6$) as the “parent” compound in these systems which seems to exhibit features of a quantum spin-liquid [203, 208]. This quantum spin liquid is associated with a frustrated spin configuration, often modeled theoretically [209] by a triangular Hubbard lattice. Notably [205–207] with varying pressure this particular class of organic superconductors exhibits possible $d_{x^2-y^2}$ ordering, and transition temperatures as high as $7 \sim 10$ K, with possibly pseudogap behavior for $T > T_c$. One also sees an unexpectedly large slope for $dH_{c2}/dT$ near $T_c$ in both fields in parallel with and perpendicular to the two-dimensional conducting layers. There is also a very wide region of fluctuating superconductivity above $T_c$, along with a large superconducting energy gap.

What is particularly relevant to the present Review is that recent studies have more quantitatively addressed pressure variations in $\kappa$-(BEDT-TTF)$_2$Hg$_2$S$_8$Br$_6$ in the context of BEC-BCS crossover. It is presumed that pressure works to enhance the itinerant nature of electrons through the increase
of the transfer integral $t$ between molecular orbitals, leading to a pressure dependent bandstructure. Thus, one might imagine in the context of Fig. 12 that variable pressure could cause a variation in $T_c$ through the generic phase diagram parameter $|U|/t$; as $t$ increases the dimensionless interaction strength decreases, thus promoting a system closer to the BCS regime.

Indeed, this is what is observed in Fig. 19. Of considerable interest in this figure are the combined plots of the in-plane coherence length $k_F\xi_0^{\text{coh}}$ and superconducting transition temperatures in $\kappa$-(BEDT-TTF)$_2$Hg$_2$Br$_4$. Here $k_F$ is determined by the Hall coefficient. If we make the assumption that pressure scales inversely with the attractive interaction strength, (although ideally one should have access to $T^*$ to ascertain this) the $T_c$ dome with overlain coherence length provides a rather ideal prototype for BCS-BEC crossover in the solid state.

![Figure 19. Pressure dependence of the measured $[203]$ in-plane coherence length $k_F\xi_0^{\text{coh}}$ and superconducting transition temperatures in $\kappa$-(BEDT-TTF)$_2$Hg$_2$Br$_4$. Here $k_F$ is determined by the Hall coefficient. If we make the assumption that pressure scales inversely with the attractive interaction strength, (although ideally one should have access to $T^*$ to ascertain this) the $T_c$ dome with overlain coherence length provides a rather ideal prototype for BCS-BEC crossover in the solid state.](image)

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periments have provided evidence for $d$-wave pairing as well as a pseudogap.

We turn next to the Nernst studies [203] in the HgBr system. In the strong pairing regime, quite generally, as $T_c$ is approached the Nernst coefficient acquires [72] a large positive (magnetic field dependent) value which peaks within the superconducting state and subsequently falls below. From Fig. 20 it can be seen that the Nernst coefficient becomes anomalously large well above the transition temperature, for those pressures which place the molecular superconductor closest to the BEC end of the spectrum. This enhancement of the standard Aslamazov-Larkin (AL) contribution is expected [72]. This reflects the fact that the non-condensed pairs have a more extended temperature region where the chemical potential of the pairs, $|\mu_{\text{pair}}|$ (which governs the size of the AL contribution), becomes small. Such an enhancement becomes more pronounced as the system deviates progressively from the BCS regime.

In summary, these studies of $\kappa$-(BEDT-TTF)$_2$Hg$_2$Br$_4$ seem to suggest a welcome convergence between different schools of thought for treating strongly correlated superconductors through the concept of “Mott-driven BCS-BEC crossover”. In the context of the cuprates both the “doped Mott insulator” [76] and the “BCS-BEC” scenarios have been widely discussed. It would appear in this organic superconductor system that both aspects are combined: Mott physics may well provide the source of the pairing mechanism, while BCS-BEC crossover is relevant to the machinery.

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B. BCS-BEC crossover in the iron chalcogenides

Considerable attention has been paid to superconducting properties of the iron chalcogenides [211–217] where there
appears to be growing evidence that FeSe and isovalent substituted FeSe$_{1-x}$S$_x$ and FeSe$_{1-x}$Te$_x$ may be in the BCS-BEC crossover regime. These systems, in which the characteristic electronic energy scales are anomalously low, appear to exhibit strong pairing effects, but not because of two dimensionality or because the pairing “glue” is itself anomalously large on an absolute scale. Rather, the attractive interaction is large when compared to the characteristic very low Fermi energies. Also present, and possibly relevant are nematic effects [216, 218] associated with broken rotational symmetry (but preserved translational symmetry). FeSe is a layered anisotropic material; it is also a compensated semi-metal, with roughly equal densities of electron and hole carriers. This leads to both electron and hole pockets and a more complicated scenario for BCS-BEC crossover.

Adding to the support for a BCS-BEC crossover picture is the fact that in the iron chalcogenides [216] the characteristic Fermi energies and zero-temperature gap magnitudes are comparable. STM and STS experiments indicate gap sizes of the order of $\Delta_1 \approx 3.5$ meV and $\Delta_2 \approx 2.5$ meV for the two bands. From this it follows that the ratio of the pairing gaps to transition temperatures ($T_c \approx 9$ K) in FeSe are large, of the order of $2\Delta_1/k_BT_c \approx 9$ and $2\Delta_2/k_BT_c \approx 6.5$, well beyond the BCS value of 3.5. The Fermi energies associated with the two nearly cylindrical Fermi surface sheets are anomalously small, of the order of $E_F \approx 10 \sim 20$ meV for the hole-like Fermi surface [216]. Important this leads to estimates of $T_c/E_F \approx 0.04 \sim 0.08$. This analysis has led many to conclude that these superconductors are well outside the strict BCS regime.

ARPES experiments [218] on bulk FeSe show that rather than the characteristic back-bending associated with conventional BCS superconductors, there is instead a flat dispersion near $k = 0$, which appears to be more typical of the crossover regime. This flat-band feature is even more enhanced with the addition of sulfur.

Of considerable importance is the characteristic correlation length extracted from magnetic field data [212] which is argued to be small, of the order of $k_Bc_{coh} \approx 1 - 4$. One can deduce from these numbers that FeSe superconductors are most likely not in the BCS regime. One should also compare with earlier theoretical estimates [Fig. 12(c)] of $k_Bc_{coh}$ which found a BEC saturation value of approximately 2 to 3. We caution, however, that complementary diagnostic information comes from vortex imaging using STM. This derives from the sub-gap fermionic states which are inside the vortex core. The observation of Friedel-like oscillations [215, 219] suggests that fermionic degrees of freedom are still present in bulk FeSe and thus these superconductors are not yet in the BEC regime.

Also notable is that there are enhanced superconducting fluctuation effects [211] in FeSe. This enables identification of a characteristic temperature $T^*$ where, in particular, diamagnetism sets in. Figure 21 presents a plot of this “unprecedented, giant” diamagnetic response. The inset serves to emphasize the key point that the diamagnetic fluctuation regime in FeSe is considerably wider than predicted from the conventional fluctuation theory of Aslamazov and Larkin (AL) theory [220], showing a very extended range of fluctuations.

Figure 21. Diamagnetic magnetization response [211] in bulk FeSe as a function of temperature at different values of the applied magnetic field $H$. The inset presents a comparison of the diamagnetic susceptibility $\chi_{diam}$ with the predictions of Aslamazov-Larkin (AL) theory [220], showing a very extended range of fluctuations.

There has also been a focus on crossover from BCS to BEC in a slightly different iron chalcogenide [222] Fe$_{1+y}$Se$_x$Te$_{1-x}$ where chemically doping the carrier concentration, through decreasing $y$, introduces an increased ratio of $\Delta_0/E_F$, where $E_F$ can be as small as a few meV. Here, for example, there are claims based on figures such as Fig. 22 that as $\Delta_0/E_F$ increases, the dispersion in the ARPES evolves from the characteristic back-bending behavior seen in the BCS regime to a BEC-like signature with a gap minimum at $k = 0$.

All of this would make a nice illustration of superconductivity in the intermediate and even strong coupling regime were it not for the fact that STM/STS experiments do not support the existence of a spectroscopic pseudogap [216] in this class of compounds. Understanding this behavior is still a work in progress; it can be speculated that the multiband character of the iron chalcogenides may be relevant here. Issues such as inter-band pairing may also be playing an important role.

\footnote{There are complications in this analysis due to the vicinity of a heavy $d_{z^2}$ band which may affect the interpretation}
Figure 22. ARPES signatures in Fe$_{1+y}$Se$_x$Te$_{1-x}$ where chemically doping the carrier concentration is through decreasing $y$. Shown in (A-C) are ARPES spectra \[222\] for three samples in order of decreasing $y$ from left to right. The green dashed line is the best fit to the data. The three lower panels (D-F) are theory plots using parabolic band dispersion and other model parameters.

C. BCS-BEC crossover in interfacial superconductivity

A great deal of excitement has been generated recently in studies of interfacial superconductivity \[217, 223–236\], particularly involving the iron chalcogenide FeSe. Here one sees an unexpected and dramatic enhancement of the pairing onset temperature \[236\] in interfacial monolayer FeSe. While the early literature \[229, 235, 237\] did not often distinguish this pairing gap onset from that of coherent superconductivity, it is now becoming clear that this system is associated with a large pseudogap, as well as a sizeable BKT transition temperature.

Indeed, it was discovered in 2012 that one-unit-cell-thick (1UC) FeSe grown on SrTiO$_3$ exhibits a gap \[238\] which survives up to $60 \sim 70$ K. This remarkable gap onset temperature is one order of magnitude higher than the $T_c$ of bulk FeSe, and it has inspired an enormous effort to reveal the mechanism driving the interfacial enhancement. Due to the extreme air sensitivity, it has been challenging to perform traditional resistivity measurements. FeTe-capping or \textit{in situ} transport measurements have made it possible to characterize the $T_c$ from the resistivity transition. Among these measurements, except for a singular study which reported a $T_c$ of 109 K, all other transport studies reported a resistivity onset associated with coherent superconductivity at $T < 45$ K.

Recent work by one of the coauthors \[239\] combined \textit{in situ} ARPES and \textit{in situ} transport measurements to simultaneously characterize the spectroscopic and resistive transitions (Fig. 23). The former is sensitive to the presence of a pseudogap which can be associated with pairing while the latter probes superconductivity. The bandstructure of the 1UC FeSe is somewhat simpler than in the bulk system. Only electron-like Fermi surfaces are identified by ARPES near the Brillouin zone corners, with a Fermi energy $E_F \approx 60$ meV \[240\]. An excitation gap $\Delta \approx 15$ meV is observed at 12 K and persists up to 73 K. This leads to a ratio of $\Delta/E_F$ of the order of 0.25. The coherence length from vortex mapping is about 2 nm \[241\], which suggests $k_F\xi_{coh} \approx 4$. This places 1UC FeSe/SrTiO$_3$ firmly in the BCS-BEC crossover regime, but not yet in the BEC.

A second example of interfacial superconductors which has been interpreted in terms of a possible BCS-BEC crossover \[195\] scenario corresponds to a superconductor formed within the conducting 2D interface between two band insulators, LaAlO$_3$ and SrTiO$_3$. This belongs to the class of superconductors with anomalously low carrier density. Indeed, it is argued that this 2D superconductor is similar in many ways to the behavior in 3D doped SrTiO$_3$, and also has features of the high $T_c$ copper oxides. The phase diagram \[223\] shown in Fig. 24 is analogous to the cuprates in many ways; additionally there are claims of preformed pairs in both. In the two cases the gap onset temperature does not follow $T_c$ in the underdoped region but increases with charge carrier depletion.

This heterostructural system is particularly useful as it can be tuned continuously through gating. There is a superconducting dome along with a pairing gap $\Delta$, which survives up to $T^* \approx 500$ mK \[223\] for the 2D carrier density $n \sim 0.02$ per unit cell. At $T = 0$, $\Delta_0 \approx 65 \mu$eV. Moreover, with decreasing temperature, the pseudogap $\Delta_{pg}$ evolves smoothly into the pairing gap within the superconducting phase. Also support-
Figure 23. Combined ARPES and transport studies [239] on 1UC FeSe/SrTiO$_3$ showing (a) ARPES data near the M point of the Brillouin zone taken at 12K. (b) Extracted values of the gap $\Delta$ and spectral weights $\delta_{SW}$ at the Fermi level as a function of temperature. (c) Resistivity measurements. (d) Voltage-current relationship.

Figure 24. Interface superconductivity in LaAlO$_3$-SrTiO$_3$ (shown in red) which is tuned with an electric gate field. The figure [223] represents a comparison between high-$T_c$ cuprate superconductors and the n-doped interface superconductors. In this figure the horizontal axis is the carrier density per unit cell. SC: superconducting; AFM: antiferromagnetic. The end point of the LaAlO$_3$-SrTiO$_3$ SC dome on the underdoped side is a quantum critical point that separates the superconducting from an insulating phase [226].

The pairing-onset interpretation of $T^*$ is that the ratio of $\Delta_0$ to $T^*$ remains close to the BCS prediction; at more general temperatures the pairing gap follows the BCS-like mean-field temperature dependence.

Using an atomic force microscope (AFM) tip, the Levy group [225] was able to draw single-electron transistors on the LaAlO$_3$/SrTiO$_3$ interface. Importantly, this enabled observation of perovskite pairs which range up to 900 mK, well above the transition temperature which ranges between 200 $\sim$ 300 mK.

These temperature scales, however, pose some concerns about interpreting the nature of interfacial superconductivity in LaAlO$_3$/SrTiO$_3$. The Fermi energies of various $t_{2g}$ bands have been characterized by soft X-ray ARPES [242] and found to be around 50 meV for the $d_{xy}$ orbital band [243, 244] $^{11}$, which leads to a rather small ratio of $\Delta_0/E_F \sim 10^{-3}$.

This observation, indicative of a more BCS-like system, appears incompatible with a strong pairing crossover scenario. Even more persuasive of this incompatibility is the additional fact that the measured coherence length is large, of the order of 30 $\sim$ 70 nm [246], leading to $k_F\xi_0 \approx 30 \sim 70$. This is based on previous estimates in the literature for $k_F \approx 0.1$ Å$^{-1}$ [244].

At the same time this interfacial system does not obviously fit into the class of highly disordered superconductors discussed in Sec. VI as for example the behavior of $T^*$ is different from that observed for other materials in this class. Thus we are left to conclude that, at this stage, it is difficult to firmly categorize the nature of interfacial superconductivity in LaAlO$_3$/SrTiO$_3$, but a strong pairing mechanism does not seem to be operative. Possibly relevant to these observations are theoretical calculations [247], albeit for 3D s-wave systems, which reveal that disorder induced superconductor-insulator quantum phase transitions can occur in the BCS regime; here the superconducting order is destroyed, leading to an insulating phase which is caused by a residual pseudogap.

D. BCS-BEC crossover in magic-angle twisted bilayer and trilayer graphene

There is growing support that MATBG [185] as well as MATTG [181, 186] superconductors exhibit BCS-BEC crossover features. Notably, these are very clean systems, associated with a BKT transition. One piece of cited evidence is based on the relatively large values of $T_{BKT}/T_F$. These were reported in the initial groundbreaking paper [185] as well as $^{11}$We note that in the literature it is still being debated whether the $d_{xy}$ orbital actively participates in the superconductivity or not (see, for example [245]). Using the $d_{x^2-y^2}/d_{xy}$ orbital band for $E_F$ would lead to a relatively larger $\Delta_0/E_F \sim 0.05$. Our choice of the $d_{xy}$ band for $E_F$ is based on the consistency between the estimated $\Delta_0/E_F$ and $k_F\xi_0$.
Figure 25. Superconducting properties of MATBG and MATTG. (a) Phase diagram of hole doped MATBG superconductors (SC). The electron filling factor $\nu = 4n/n_s$, where $n$ is the carrier density defined by the applied gate voltage and $n_s$ is the corresponding $n$ when the lower four-fold degenerate Moire flat band is fully filled. In this diagram, a very large pseudogap regime, indicated in light blue, is determined by combining conventional STM and point-contact Andreev tunneling spectroscopy [150]. (b) Gap size $\Delta$ versus the gate voltage $V_{\text{Gate}}$ (and the filling factor $\nu$) for MATTG. The $\Delta$ is measured from conventional STM tunneling at low temperatures. The data points are extracted from the separation between coherence peaks at the half-way point (black squares) and from a nodal gap fit (red dots) [186]. In the green and violet regions the $dI/dV$ curve exhibits a V shape and a U shape, respectively. (c) The $T - \nu$ phase diagram of MATTG at displacement field $D/\epsilon_0 = -0.5$ V nm$^{-1}$, along with the curves of the interparticle distance $d \equiv d_{\text{particle}}$ and the coherence length $\xi_{\text{GL}}$ [181]. Here $d_{\text{particle}} = 1/\sqrt{n^*}$, where $n^*$ is the effective carrier density that can be deduced from quantum oscillation and Hall density measurements. Note that $n^*$ is different from the density $n$. (d) Replotting of the $\xi_{\text{GL}}$ data from (c) in terms of the product $k_F\xi_{\text{GL}}$. To convert $n^*$ to $k_F$ we have used $k_F = (2\pi n^*)^{1/2}$ [150]. The blue dashed line shows the expected $k_F\xi_{\text{GL}}$ value when $n_{\text{pair}}$ saturates to $n^*/2$. (e) The product $k_F\xi_{\text{GL}}$ calculated theoretically as a function of $T^* / T_{\text{BKT}}$ for a 2D s-wave superconductor. In the theoretical calculation, $n^*$ is the same as $n$.

In subsequent works [150, 186, 248]. Such estimates are, in turn, based on $V$-$I$ plots which allow one to determine the BKT transition that occurs when $V = I^{\alpha}$ with a specific value of $\alpha = 3$. As a caution we note that the ratio $T_{\text{BKT}} / T_F$ should not be viewed as a proxy for the fraction of electrons involved in superconductivity; in the BEC regime, this parameter becomes very small.

More recent tunneling experiments [150] (which are summarized in Fig.25(a)) on MATBG help to make the association with BCS-BEC crossover stronger; they have presented clearer indications of an extensive pseudogap phase in the phase diagram, as can be seen from the figure. These STM experiments suggest [150] an anomalously large value for the ratio $2\Delta_0/(k_B T_{\text{BKT}}) \approx 25$, which can be viewed as representative of strong pseudogap effects, equivalently associated with large $T^*/T_{\text{BKT}}$. Adding support to a BCS-BEC crossover scenario is the presence of another much smaller energy-gap scale associated with point-contact Andreev tunneling which is only present in the ordered phase where there is phase coherence.

The results from this STM tunneling [150] provide a value for $\Delta_0 \approx 1.4$ meV in MATBG. In an earlier section we pointed out that $V$-$I$ measurements in 2D films can be used [194] for estimates of $T^*$. One can infer from these data [185] that $T^* = 3 \sim 5$ K, which is obtained from the Ohmic recovery temperatures.$^{12}$ This should be compared with the transition temperature $T_{\text{BKT}} \approx 1$ K and the Fermi energy of the bilayer system which is estimated to be $T_F \approx 20$ K [185]. The resulting relatively large ratios of $T^*/T_{\text{BKT}}$ and $\Delta_0 / E_F$ suggest that MATBG is a superconductor in the intermediate BCS-BEC crossover regime.

Indeed, based on the claims [150] that MATBG has some similarities with high $T_c$ superconductors, it is striking to observe similar characteristic numbers in Fig.40 (Appendix C) for the underdoped cuprates and (both) twisted graphene families of superconductors. This figure addresses this similarity in a more quantitative way.

The situation for MATTG appears to be somewhat clearer.

$^{12}$Ideally one could arrive at more accurate numbers by making systematic $V$-$I$ plots over finely separated temperature intervals in order to more precisely establish the temperature for the Ohmic recovery, corresponding to $T^*$. 
and provides more quantitative information. Some pertinent results [181, 186] are summarized in Fig. 25, where panels (c) and (d) address very useful coherence-length experiments [181] based on the magnetic-field dependence of the superconducting transition temperature. Fig. 25(c) shows this published data for $\xi_{0}^{\text{coh}}$ as well as the inter-particle distance $d$ as a function of the band filling factor $\nu$, along with the transition temperature $T_{\text{BKT}}$. It should be noted that the error bars are large here and that this is indicative of the experimental challenges which are encountered when deducing the coherence length using the finite field resistivity. Particularly in 2D and extreme type-II superconductors, this necessarily leads to very broad transitions making it difficult to establish $T_{c}(H)$ without incorporating a fairly arbitrary standard for determining where the transition is located.

The experimentally observed dimensionless product $k_{F}\xi_{0}^{\text{coh}}$ (Fig. 25(d)) \textsuperscript{13} can be compared with theory in Fig. 25(e), where $k_{F}\xi_{0}^{\text{coh}}$ is plotted as a function of $T^{*}/T_{c}$ (This is similar to the inset in Fig. 12(c)). We note that the plot in Fig. 25(d) and the theory plot in Fig. 25(e) are for different horizontal axis variables; however, a direct association of the two would allow one to relate the important ratio $T^{*}/T_{c}$ with the filling factor $\nu$, hence completing the $T^{*}/T_{c}$ versus $\nu$ phase diagram. From the data in the figures it follows that the relevant parameters for $\nu \gtrsim -2.5$ suggest that MATTG also belongs in the intermediate BCS-BEC crossover regime.

Recent tunneling experiments [186] provide additional important quantitative information about MATTG with a focus on the gap energy scale as plotted in Fig. 25(b) as a function of $\nu$. These studies indicate $T^{*} = 7$ K at the $\nu$ value where the gap is maximum. Additional parameters are: $T_{\text{BKT}} \approx 2.25$ K [181] with the estimated Fermi temperature given by $T_{F} \approx 30$ K.

Overall, there appears to be compatibility between the $\xi_{0}^{\text{coh}}$ data from the MIT group and pairing gap experiments [186] shown in Fig. 25(b). Making use of the estimates of $E_{F}$ based on quantum oscillation experiments [181] it follows that the ratio $\Delta_{0}/E_{F}$ exhibits a similar trend as $\xi_{0}^{\text{coh}}$, changing from more BCS-like behavior at $\nu \approx -3$ to characteristic crossover behavior at $\nu \approx -2.2$. We note that interpretations of these tunneling experiments [186] have suggested that the BCS regime is reached around the upper half of the $T_{\text{BKT}}$ dome at $\nu \gtrsim -2.5$, although it is not straightforward to reconcile a BEC phase with the presence of coherence peaks in the tunneling data.

Finally, it should additionally be noted that the theory plot of the coherence length in Fig. 25(e) is for the $s$-wave case, while the experimental data seem to suggest a nodal form of superconductivity. Some aspects of crossover theory for an anisotropic gap symmetry have been addressed in this Review (in Sect. IV B)\textsuperscript{14}, but one might additionally expect that other ingredients such as flat energy bands and quantum geometry (discussed in Sect. V E) may play an important role as well in reaching an ultimate understanding of BCS-BEC crossover for MATBG and MATTG.

E. BCS-BEC crossover for 2D gated semiconductors

There has been recent interest [183, 249, 250] in a group of layered nitrides, Li$_{x}$ZrNCl, which are intrinsically semiconductors and exhibit superconductivity through Li-intercalated doping. These experiments impose control of the carrier density by use of ionic gating which provides access to very low carrier density systems that are otherwise inaccessible. Concomitantly the varying carrier number enables a tuning of the weakly- to strongly-coupled superconducting regimes by controlling both the carrier density and simultaneously a dimensional crossover from anisotropic-3D to 2D. Both tunneling and resistivity measurements [183] yield systematic information about the detailed phase diagram of this system.

The phase diagram [183], shown in Fig. 26, indicates a pronounced pseudogap regime established from $dI/dV$ measurements. This is particularly notable at low carrier densities, where the system is more two dimensional. In particular, at extreme underdoping $T_{\text{BKT}}$ shows a maximum of 19 K. In the most underdoped sample probed, $\Delta_{0}/E_{F} \approx 0.3$, $T_{\text{BKT}}/T_{F} \approx 0.12$, and $T^{*}$ is roughly 3$T_{\text{BKT}}$.

A very nice summary [183] of experimental observations is presented in Fig. 26 as a plot in terms of $T/T_{F}$ vs $\Delta_{0}/E_{F}$ with data points indicating $T_{\text{BKT}}$ and $T^{*}$. The pseudogap and associated $T^{*}$ were found to be largest when the carrier number was lowest. Here, for these large gap systems (which are in the strong-coupling limit) one finds the smallest coherence length, $k_{F}\xi_{0}^{\text{coh}} \approx 3$, as obtained from the upper critical fields. This suggests a system which may be close to but not yet in the BEC regime. In the opposite regime, at the highest electron doping regime one recovers more characteristic BCS behavior with $T_{\text{BKT}} \approx T^{*}$. We conclude that all of this adds up to a body of evidence which lends reasonably strong support to a BCS-BEC crossover description of these ionic gated superconductors.

F. Magnetoexciton condensates with BCS-BEC crossover

The concept of condensation based on particle-hole pairs [54, 251, 252] should be thought of as a very natural extension of particle-particle pairing in superconductors. Indeed one usually invokes the same ground-state wave function as in Eq. (1) modified by the replacement of one of the electron operators with a hole operator and presuming that the two are associated with different bands. This subject has generated considerable excitement as one could conceive of such condensation as taking place at very high temperatures. There are a number of subtle features, however, as the electrons and holes need to be sufficiently well separated so as to avoid recombination. Their number and effective masses also need to

\textsuperscript{13} Note that the band degeneracy used in the conversion here is 2, not the naive 4. As supported by experiments, the spin-valley 4-fold degeneracy is broken to 2 at $-3 < \nu \lesssim -2$.

\textsuperscript{14} In the single band $d$-wave case, the counterpart of the curve in Fig. 25(e) looks qualitatively similar at very low density but will not reach BEC until a much larger $T^{*}/T_{\text{BKT}}$. No BEC is found at high densities.
be equivalent, otherwise pairing can be impeded as this system behaves like a superfluid with population or mass imbalance.

An important configuration for arriving at exciton condensation involves quantum Hall fluids [253, 254], as was first implemented by Eisenstein et al. in a GaAs/AlGaAs heterostructure. Here two thin GaAs layers are separated by the AlGaAs spacer layer, which serves to mitigate electron-hole recombination processes. Because each layer forms a 2D electron gas, their energies are quantized into Landau levels (LL). These bilayer quantum Hall systems have the potential to realize novel quantum states that have no analog in a single layer. A relevant parameter for characterizing such states is \( d/\ell_B \), where \( d \) is the inter-layer spacing and \( \ell_B = \sqrt{\hbar/eB} \) is the magnetic length.

There has been a focus [253] on the interlayer coherent state observed in the zero or small interlayer tunneling limit and at total electron filling fraction \( \nu_{\text{tot}} = \nu_1 + \nu_2 = 1/2 + 1/2 = 1 \). Here, the electron filling fraction, \( \nu_i = n_i(2\pi \hbar^2 B) \), is defined for each individual layer with \( n_i \) the electron density of the \( i \)-th layer. Important questions such as whether there is a quantum phase transition separating the large and small \( d/\ell_B \) limits have been raised [255–258], although recently [259–262] there has been the suggestion that the evolution of the state from the large to small \( d/\ell_B \) might be understood as a crossover of BCS to BEC condensation of magneto-excitons.

This picture can be understood in terms of Jain’s composite fermions (CF) [262], where a CF can be roughly viewed as the original electron attached to two magnetic flux quanta \((\hbar/e)\). In the extreme \( d \to \infty \) limit, the two layers decouple and each of them has a LL filling fraction \( \nu = 1/2 \) which can be described by a metallic state [263] of either electron-like or, equivalently, hole-like CFs with well defined Fermi surfaces.

At finite \( d \) one can then consider electron- and hole-like CFs from the two different layers forming inter-layer Cooper pairs, i.e., magnetoexcitons. Importantly, it is reasonable to assume that their effective masses are equal near \( \nu = 1/2 \), due to an approximate particle-hole symmetry. The pair formation is driven by an inter-layer attraction, \( U \) which is derived from the original interlayer Coulomb interaction between electrons and holes, whose magnitude is \( |U| \approx V_{\text{inter}} = e^2/(\epsilon d) \), where \( \epsilon \) is the background dielectric constant. At the same time the parameter \( E_{\text{kin}} \), which represents the kinetic energy of a partially filled Landau state is set by the intralayer Coulomb repulsion [263].

In this way the important ratio \( |U|/E_{\text{kin}} \approx \ell_B/d \), which sets the scale of a BCS-BEC crossover can be tuned experimentally by varying either \( d \) or \( \ell_B \). Large \( d \) or high magnetic fields corresponds to the BCS-like limit, while the more BEC regime is present at small \( d \) or low magnetic fields (see Fig. 27). This BCS-BEC crossover picture is supported by recent measurements on graphene double-layer heterostructures [259, 265]. Compared to the GaAs/GaAlAs double-layer experiments, this graphene bilayer system has an additional advantage as it allows the two graphene layers to be separated by a thin hexagonal boron nitride layer, which prohibits direct interlayer tunneling without introducing disorder.

Because the magnetoexcitons are neutral and cannot be probed in traditional electronic transport, two unconventional designs for resistance measurements have been employed to experimentally probe the magnetoexciton superfluidity via “counterflow” and “drag” experiments [253]. Figure 27 presents a summary of the results from these measurements for the double-layer graphene system [259].

In the counterflow configuration electric currents in the two layers are of the same magnitude but flow in opposite directions. The absence of dissipation due to “superfluidity” is associated with a vanishing \( R_{xx} \), which measures the longitudinal resistance. These experiments serve to determine the transition temperature \( T_c \) (solid black line) in Fig. 27(a).

A striking signature of magneto-excitonic superfluidity is a quantized Hall drag resistance at low temperature in the

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15When \( d \ll \ell_B \), the inter-layer interaction is actually governed by \( e^2/(\epsilon d) \), not \( e^2/(\epsilon \ell_B) \). It should also be noted that the actual inter-layer interaction between CFs is not the same as \( V_{\text{inter}} \). Instead, it is mediated by an emergent Chern-Simons gauge field that makes the renormalized interaction highly frequency dependent [256, 263, 264]. Here, we ignore these complications.
“drag” configuration. Here the electric current is fed only to one layer, while the Hall voltage drops are measured in both layers, from which one can define the usual Hall resistance, \( R_{xy} \), for the current-driving layer. One can also define a Hall drag resistance, \( R_{xy}^{\text{drag}} \), for the passive layer.

Both \( R_{xy} \) and \( R_{xy}^{\text{drag}} \) are expected to be quantized to the same value \( h/(e^2\nu_{\text{tot}}) \) at low \( T \). As \( T \) increases above \( T_c \), \( R_{xy}^{\text{drag}} \) decreases monotonically. In Ref. [259] the important temperature scale, \( T^* \), is defined as the point where \( R_{xy}^{\text{drag}} \) drops to below 5\% of \( h/(e^2\nu_{\text{tot}}) \). This \( T^* \) is plotted in Fig. 27(a) as the dashed black line. It is reasonable to associate the residual \( R_{xy}^{\text{drag}} \) at high temperatures with incoherent pair correlations between electron- and hole-like CFs. In this way one interprets \( T^* \) as the onset of electron-hole CF pair formation. While there are some uncertainties in the definition of \( T^* \), a clear separation of the two temperature scales, \( T_c \) and \( T^* \), is apparent from Fig. 27(a), which is to be compared to the schematic phase diagram sketched in Fig. 27(b).

What is not as clear is whether at the lowest applied magnetic field \( B \approx 5 \) T the system has reached the BEC regime, as suggested by the schematic figure. In comparing with a prototypical example of BCS-BEC crossover, as in the 2D electron gas it is useful to establish the magnitude of the effective \( \Delta_0/F \), which would be expected to become arbitrarily large in a more traditional BEC superconductor. On the other hand, exact diagonalization studies [260] show that for the bilayer magnetoexciton system \( \Delta_0 \lesssim F \). This contrast highlights some of the key differences between traditional superconductors and the magnetoexciton bilayer that one needs to bear in mind in the interpretation of the phenomenology. It is clear that quantification of the exact behavior of \( T_c/F \), and other quantities characteristic of BCS-BEC crossover, for the entire range of \( d/\ell_B \) from \( \infty \) to 0 requires further work, both theoretical and experimental.

One might speculate that, since one defining feature of the BEC regime is the disappearance of Fermi surfaces, a potentially useful future experiment is to directly probe the Fermi surface of CFs at \( T_c < T < T^* \) for small \( d/\ell_B \), using geometric resonance techniques as employed in the determination of the Fermi wave vector of CFs for the single layer \( \nu = 1/2 \) state [266]. Achieving a number of these goals seems promising given the high tunability of the bilayer graphene heterostructure, as demonstrated by the new generation of experiments [259, 265].

Figure 27. BCS-BEC crossover for magnetoexcitons [259]. The color coding in panel (a) is associated with the temperature derivatives of the measured longitudinal resistance in the counterflow configuration. Hall drag and counterflow resistances are used respectively to arrive at the dashed line, representing the pairing onset temperature \( T^* \) as a function of effective attraction/kinetic energy ratio (through the magnetic field \( B \)), and to infer \( T_c \) shown by the solid black line. (b) Schematic phase diagram expected for a magnetoexciton condensate. \( T_{\text{pair}} \) is the same as \( T^* \).

VIII. APPLICATION TO THE CUPRATES

A. Support for and counter-arguments against BCS-BEC crossover in the cuprates

The question of whether a BCS-BEC scenario is relevant to the cuprates is, like all aspects of the cuprate literature, a highly controversial one. Despite this controversy it is useful to let the reader independently judge; thus, here near the end of this Review article we discuss what the implications are of such a theory for the cuprates. We address aspects that are both consistent and inconsistent with the data.

There are claims in the literature that the cuprates are somehow between BCS and BEC. We cite some of these here.

- From A. J. Leggett [267]: “The small size of the cuprate pairs puts us in the intermediate regime of the so-called BCS-BEC crossover.”
- From G. Sawatzky and colleagues [268]: “High-\( T_c \) superconductors cannot be considered as classical BCS superconductors, but rather are smoothly evolving from BEC into the BCS regime.”

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16 Rescaling the measured \( T_c \) of the top panel by \( T_{\text{N}} \), which can be estimated as \( e^2/(\epsilon \ell_B) \), and plotting the obtained \( T_c/T_{\text{N}} \) as a function of \( B \) shows that this ratio has not passed the point where it starts to decrease with decreasing \( B \) even at \( B \approx 5 \) T. It suggests that the system may still be in the crossover regime, not yet into the BEC. If we compare this trend of \( T_c/T_{\text{N}} \) to that for a 2D electron gas in Fig. 11.
• From I. Bozovic and J. Levy [195]: “We show the likely existence of preformed pairs in the cuprates . . . The existence of preformed pairs is a necessary but not sufficient condition for BEC or for BCS-BEC crossover to occur. Indeed, since Fermi surfaces have been mapped out . . . this favors a picture in which pairing is relatively strong, pre-formed pairs first appear at $T > T_c$ . . . but copper oxides are still on the BCS side of the crossover.”

• From Y. Uemura [163]: “Combining universal correlations . . . and pseudogap behavior in the underdoped region, we obtain a picture to describe superconductivity in cuprate systems in evolution from Bose-Einstein to BCS condensation.”

It should be noted that even if BCS-BEC crossover theory plays a role in the cuprate superconductors this will not address or elucidate a number of important issues which characterize their behavior and need to be understood in an ultimate theory. Among these are [76] the pairing mechanism, which remains unknown; also challenging is arriving at an understanding of the “strange metal” behavior including the linear temperature dependence of the resistivity which is, indeed, very widespread among other strongly correlated superconductors [269]. Another puzzle is the distinct change observed in carrier concentration which seems to correlate with the presence of a pseudogap [202]. This appears consistent with recent ARPES claims [270] that the pseudogap suddenly collapses at a fixed hole concentration. As emphasized throughout, this Review will focus on strongly correlated superfluids other than the Fermi gases and cuprates.

Nevertheless, in the interest of completeness it is useful to list some of the issues that have been raised to challenge the relevance of BCS-BEC crossover theory for the cuprates. Examples are the following:

1. Current cuprate experiments show no signs of a chemical potential $\mu$ which is near or below the band bottom, as might be expected in the BEC regime. This would show up in photoemission experiments.

2. $T_c$ and $T^*$ are observed to vary inversely in the underdoped regime. Some have argued that if $T^*$ were related to preformed pairs, then as pairing becomes stronger both $T_c$ and $T^*$ would tend to increase together.

3. One finds [271] that a number of (but not all) superconducting fluctuation phenomena appear only in the immediate vicinity of $T_c$, well away from the pseudogap onset temperature $T^*$.

4. There are multiple signatures of “a nodal-antinodal dichotomy” [272], corresponding to a different behavior of the $d$-wave energy gap along the nodal and antinodal directions. This is widely interpreted to mean that rather than preformed pairs, another (unspecified) ordering must be responsible for the pseudogap which is mostly confined to the anti-nodes.

5. There are ARPES experiments [273] which indicate that at higher temperatures in the normal state, but well below $T^*$, the fermionic dispersion shows disagreement with the characteristic energy dispersion associated with BCS-like quasi-particles.

6. There are other indications [274] of additional ordering associated with the pseudogap phase, quite possibly with an onset associated with its boundary [275, 276].

7. There are claims [277] suggesting that quantum critical behavior is present so that $T^*$ actually vanishes beneath the superconducting dome; this is inconsistent with a BCS-BEC picture in which $T^*$ is necessarily larger than $T_c$.

Of this list of 7, the last two seem to be most challenging for the BCS-BEC crossover scenario, while the first 5 are not necessarily so, as will be discussed in this section and the Appendices. Attributing the cuprate pseudogap to preformed pairs as distinguished from a competing order parameter is admittedly highly controversial and this should not be viewed as a central component of this Review which is focused principally on non-cuprate superconductors. Nevertheless, for completeness, it is useful to present the predictions concerning the cuprates which derive from one particular pre-formed-pair scenario – a BCS-BEC crossover perspective – and leave the reader to make his/her own assessments. The discussion presented here and in Appendices B and C should be viewed as a catalogue summary of some relevant theory literature. The interested reader can consult the cited papers to obtain more details.

B. Experimental evidence that BCS-BEC crossover may be relevant to the cuprates

All indications are that, if this scenario is relevant to the cuprates, these superconductors are on the BCS side and well away from BEC. This is consistent with the claims in a recent paper [71], although these authors adopted a different definition of “crossover” associating it with proximity to a BEC. Indeed, there are several experiments which stand out as providing among the strongest support for a BCS-BEC-crossover-like description of the copper oxides.

ARPES measurements [74] reveal a Bogoliubov-like dispersion in part of the Brillouin zone which is away from the nodal Fermi-arc region. Importantly this is observed somewhat above $T_c$, as shown in Fig. 28. It is highly unlikely, and indeed inconsistent with the theory we are discussing (see Eq. (16) which bears on point 5 in Sec. VIII A), that this Bogoliubov dispersion continues up to much higher temperatures, near the onset of the pseudogap. Indeed, there are studies [273] that suggest this characteristic back-bending dispersion is absent well below $T^*$. But in the normal state, not too far from $T_c$, these experiments [74] provide indications that the presence of a pseudogap is associated with the same fermionic quasi-particles as are found in the ordered phase.

In a similar vein a smooth evolution of the measured ARPES excitation gap around the antinodes as temperature
Figure 28. Experimental pseudogap ARPES data [74] showing back-bending of the dispersion in the normal state (b), suggestively similar to that in the superconducting state (a).

is varied from above to below $T_c$ lends some support to the crossover picture.

An additional, conceptually simple experiment involves STM studies which compare the ratio of the zero-temperature pairing gap to $T^*$. This ratio appears to be very close to the expected mean-field result [278, 279]. This associates the ratio of $\Delta_0$ and $T^*$ in an analogous fashion as for the BCS prediction of $\Delta_0$ and $T_c$, and for a $d$-wave case.

There are additional classes of experiments which constitute less direct support, but which are worthy of note and will be discussed in this section. These involve (i) recent shot-noise measurements [73] which provide a more direct and quantitative signature of pairing above $T_c$. Through pair contributions to tunneling these shot-noise experiments indicate [73] that "pairs of charge $2e$ are present in large portions of the parameter space dominated by the pseudogap." We caution here, however, that evidence [196] of $2e$ pairing may be found in the pseudogap phase of highly disordered, presumably weakly coupled 2D superconductors. In this way, $2e$ pairing is a necessary but not sufficient effect to establish BCS-BEC crossover.

(ii) Also relevant is the two-gap dichotomy [268, 272] in which there are distinctive temperature dependencies of the ARPES- or STM-associated gaps in the nodal and anti-nodal regions. In the BCS-BEC crossover scenario this two-gap behavior derives from the simultaneous presence of condensed and non-condensed pairs.

(iii) Additionally, an observed downturn [14] in the DC resistivity near or below $T^*$ seems most naturally to be associated with the contribution from bosonic transport or from preformed pairs. Indeed this small downturn feature is often used as the canonical signature of $T^*$.

(iv) Lending some support to the crossover picture is the behavior of the Ginzburg-Landau coherence length in the cuprates, which is still not firmly established, as it turns out to be quite difficult to measure due to vortex liquid effects. Some indications of behavior rather similar to the organic 2D superconductor [203] is Figure 14(a) in Ref. [180], measured above $T_c$ in the normal state.

(v) Finally, there is a notable similarity between many properties of a single layer cuprate material and that found for its counterpart in bulk systems [280]; this would seem to be compatible with the similarity contained in Eqs. (3) and (5).

We will discuss some of these experiments in the following subsections.

C. The spectral function: distinguishing condensed and non-condensed pairs

We first address the so-called "two-gap dichotomy" [268, 272] which pertains to the behavior of the spectral function where it should be clear that $d$-wave pairing plays an important role. In the BCS-BEC crossover scenario [1] the fermionic self energy which is measured in the spectral function has two contributions from non-condensed (pg) and con-
“Bad-metal” behavior is important here as the small conductivity in the fermionic channel enables the bosonic downturn in the resistivity to be more evident. Panels (a)–(c) and (e): representative spectral function $A(\omega = 0, k)$ for temperatures (a) $T/t = 0.11$, (b) 0.15, (c) 0.18, and (e) 0.23. Here, $T_c/t = 0.11$ and $T^*/t = 0.23$, where $t$ is the nearest neighbor hopping integral. In panel (d) black dots are experimental data for an underdoped Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$ [282]. The solid and dashed lines are theoretical fits. Blue solid line: calculated total $\rho_{xx}$. Red dashed (dark-green dashed) line: fermionic (bosonic) contribution to $\rho_{xx}$.

This same spectral function appeared earlier as Eq. (10), but here we emphasize the momentum dependence associated with non-$s$-wave pairing, and, as in customary, we add an additional phenomenological lifetime $\Gamma_0$ deriving from incoherent, single-particle scattering processes. It might be noted that because of these two components, this BCS-BEC crossover scheme has Green’s functions which are similar to those in a highly regarded cuprate theory often called “YRZ” theory [283] after the authors Yang, Rice, and Zhang. In the BCS-BEC crossover scenario one finds Fermi arcs whereas YRZ incorporates Fermi pockets [284].

In the normal state, a form [285] similar to Eq. (31) was shown [86] to provide a reasonably good fit to ARPES data and insights into the Fermi arcs [286]. How do the Fermi arcs originate? One should note that the non-condensed pairs have finite lifetime, in contrast to the condensate. This is particularly important for the case of $d$-wave pairing. If we consider cooling from above to below $T_c$, we see that the onset of the condensate gap $\Delta_{sc}$ in the fermionic spectral function is more dramatic in the nodal region where there is no normal state background gap already present. By contrast, in the antinodal region the onset of $\Delta_{sc}$ on top of a large $\Delta_{pg}$ has very little impact. Thus, as illustrated below, it is the temperature dependence of the nodal gap which reflects the onset of the ordered state.

More quantitatively [281, 285], one defines the spectral (or ARPES) gap as one-half the peak to peak separation in the spectral function. Figure 29 illustrates the temperature evolution of the spectral function for $\varphi = 9^\circ$ (close to the antinodes in Fig. 31) and $\varphi = 36^\circ$ (close to the nodes) at varying $T/T_c$ from top to bottom. Above $T_c$ (top panel) the well understood behavior [74, 287, 288] sets the stage for the normal phase which underlies the superconducting state in the next two panels. At this temperature, $T/T_c = 1.1$, one sees Fermi arcs in the Brillouin zone. Here the spectral function is gapless on the Fermi surface near the nodal direction while it is gapped in the vicinity of the anti-nodal direction. The Fermi arcs derive from the presence of a temperature independent broadening term $\gamma$ in $\Sigma_{pg}$. When $T$ is slightly below $T_c$ (middle panel), a dip in the spectral function at $\varphi = 36^\circ$ suddenly appears at $\omega = 0$. At this $\varphi$ the underlying normal state is gapless so that the onset of the additional component of the self energy via $\Sigma_{sc}$ with long-lived pairs leads to the opening of a spectral gap.

By contrast, the presence of this order parameter is not responsible for the gap near the anti-nodes ($\varphi = 9^\circ$), which,
D. Transport in the cuprates

That the cuprates are highly resistive or bad metals [34] is important for understanding their transport properties. This is what allows the boson-related downturn for transport at $T^*$ in the resistivity, a canonical signature of the pseudogap onset [14], to become evident (see Fig. 30). This would otherwise be obscured by gap effects in the fermionic spectrum. The fits to the longitudinal DC resistivity shown in Fig. 30 are based on a phenomenological model [72] for the pair chemical potential ($\mu_{\text{pair}}$) which incorporates the standard fluctuation behavior within the critical regime, $\mu_{\text{pair}} \approx (8/\pi)(T_c - T)$, along with the natural interpolation of higher temperature effects which associate $T^*$ with the temperature at which the number of pairs must necessarily vanish. This leads to a consolidated form:

$$\mu_{\text{pair}} = \frac{8}{\pi} (T^* - T_c) \ln \frac{T^* - T}{T_c},$$

(32)

This form for $\mu_{\text{pair}}$ leads to fits to the resistivity, $\rho(T)$, and its downturn in Fig. 30 which are not unreasonable; also emphasized here is the presence of “Fermi arcs” which additionally help to reveal bosonic transport by suppressing the gap in the fermionic spectrum. With the same parameters one can arrive at some understanding of the Nernst effect [72]. However there are problematic issues concerning the Hall coefficient [72, 290] and the thermopower, which affect essentially all theoretical attempts to understand these cuprate data and make a direct comparison between theory and experiment difficult.

Indeed, there is a sizeable literature dealing with the Hall coefficient in the underdoped regime [291–299]. Among the most serious problems is that the measured $\sigma_{xy}$ appears to be not as singular near $T_c$ as is predicted by Gaussian fluctuation theories, where the expected singularity is stronger than in $\sigma_{xx}$. This is presumably associated with the experimental observation that $R_{\text{H}} \propto \rho_{\text{xy}}$ starts to drop with decreasing $T$ at $T$ slightly above $T_c$ [293, 295] and can even change its sign as $T$ decreases towards $T_c$.

Similarly, the normal state thermopower in underdoped cuprates [300–304] (at $T \approx T^*$) is positive in the experiments for the samples with the largest pseudogap. This is opposite to the usual band structure predictions, and also opposite to the sign of the Hall coefficient. Given these problems for the thermopower and Hall coefficients, comparisons between experiments are best addressed in the case of the Nernst coefficient.

E. Quantifying the Fermi arcs

Understanding and quantifying the Fermi arcs has become an important issue in the cuprates. In addition to ARPES experiments the existence of Fermi arcs appears to have been independently established in STM data as well [305, 306]. The right panel of Fig. 31 presents gaps extracted from ARPES data [289] for a moderately underdoped sample. The three different curves correspond to three different temperatures with the legend the same as that in the left panel (representing the results of theory). Importantly one sees a pronounced temperature dependence in the behavior of the ARPES spectral gap for the nodal region (near $45^\circ$) as compared with the antinodal region (near 0 and $90^\circ$), where there is virtually no $T$ dependence. The left panel presents the counterpart theoretically predicted behavior, which exhibits some similarities.

Figure 32 addresses the temperature dependence of the Fermi arcs and their sharp collapse [285] from above to below $T_c$. Note that here it is assumed (for simplicity) that the broadening parameter $\gamma$ is temperature independent as the non-condensed pairs which persist below $T_c$ continue to be distinguished from the condensate there. Plotted is the percentage of arc length as a function of $T^*/T^*$ and for different doping concentrations from the optimal to the underdoped regime. There is a clear universality seen in the normal state, in both theory and experiment [307] (shown in the inset).

F. Behavior of the finite-$\omega$ conductivity

There is a substantial interest [308, 309] in the complex ac conductivity $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$ in the cuprates, notably both in the optical regime and at THz frequencies. These experiments are particularly useful as they can reveal important information about low-energy excitations and charge dynamics. Both gapped fermions and non-condensed Cooper pairs can contribute to $\sigma(\omega)$. In theory work summarized here only the fermionic contributions were considered and this might reasonably be viewed as a shortcoming.

A key feature of the in-plane $\sigma_{1}(\omega)$ is its two component nature consisting of a “coherent” Drude-like low-$\omega$ feature followed by an approximately $T$-independent mid-infrared (MIR) peak [308, 310, 311]. This is illustrated in Fig. 33. As stated in Ref. [310]: “The two component conductivity extends to the pseudogap boundary in the phase diagram... Moreover a softening of the mid-infrared band with doping resembles the decrease of the pseudogap temperature $T^{*}$. Also of importance is the fact [312] that “high $T_c$ materials are in the clean limit” and also that “…the MIR feature is seen above and below $T_c$.” Thus, it appears that this MIR feature is not associated with disordered superconductivity and related momentum non-conserving processes, but rather it is due to the unconventional nature of the finite-frequency response [308]. Within the crossover scenario, the presence of non-condensed pairs both above and below $T_c$ yields [160] a mid-infrared peak. This peak occurs around the energy needed to break pairs and thereby create conducting fermions. Its position is doping dependent, and only weakly temperature de-
Figure 31. Inferred ARPES gaps as a function of $k$ in one quadrant of the Brillouin zone. Fermi arcs (associated with $d$-wave pairing) appear on the Fermi surface near the nodal direction around $\varphi = 45^\circ$. Comparison of theory [281] on the left with experiment [289] on the right.

Figure 32. ARPES comparisons in cuprates showing collapse of the Fermi arcs at the superconducting transition. The figure compares experimental [307] data points with theoretical curves [285]. Here $T^*_e$ is the experimental $T^*$ determined by ARPES data.

Figure 33. ARPES arc length $L_{\text{arc}}$ as a function of $T/T^*_e$ for different doping levels $x$.

Figure 34 right and left panels, respectively, show the theoretical prediction [161] and experimental behavior [309] found for the imaginary part of the THz conductivity, $\sigma_2(\omega)$. With decreasing temperature, at roughly $T_c$, $\sigma_2$ shows a sharp upturn at low $\omega$, of the form $\sigma_2 \propto \nu_s/\omega$, where $\nu_s$ is the superfluid density. The low-$\omega$ contribution above $T_c$ is of interest to the extent that it may reflect the presence of dynamical superfluid correlations. This is shown in the insets which present an expanded view of the temperature dependencies near $T_c$. Both theory and experiment show that the nesting of the $\sigma_2$ versus $T$ curves switches order above $T_c$. It should be stressed that for this particular class of experiments the contribution from preformed pairs does not extend to very high temperatures. Indeed, here the effects are confined to temperatures in the vicinity of $T_c$, well below $T^*$. This is in contrast to other fluctuation experiments. It is notable, however, that the experimental data shows a more pronounced normal-state contribution than found in theory.

G. Precursor diamagnetism

The normal-state diamagnetic susceptibility in cuprates has also been widely discussed [201]. Here, by contrast with the discussion surrounding $\sigma(\omega)$ above, the interest is focused on the bosonic contributions. In conventional fluctuation theory [220] the diamagnetic susceptibility, $\chi_{\text{dia}}$, in the vicinity of $T \approx T_c$ can be relatively large as it scales (in three dimensions) with the inverse of $T - T_c$. What happens in BCS-BEC crossover theory as a consequence of the presence of a pseudogap? In a BCS-BEC crossover scenario $\chi_{\text{dia}}$ now scales [70] as $1/|\mu_{\text{pair}}|$ and, as can be seen from Eq. (32), the principal effect is that the inverse pair chemical potential remains appreciable now for an extended range of temperatures well beyond the critical region around $T_c$ and strictly vanishing only at $T^*$.

This, in turn, suggests that there are fluctuation contributions to the diamagnetism at relatively higher temperatures than generally observed in conventional superconductors. It should be noted, however, that the visibility of fluctuation diamagnetism depends on other background, generally paramagnetic, contributions, which are often difficult to quantify. A more detailed analysis leads to the results in Fig. 35 which shows a comparison between experiment [314] and theory [70].

H. Other applications of BCS-BEC crossover: Features of the non-Fermi liquid

By way of completeness we end by including several other literature contributions which address BCS-BEC crossover theory in cuprates but for which there are no direct back-to-back experimental comparisons. These involve studies of how the non-Fermi liquid pseudogap state is reflected in quasi-particle-interference (QPI) experiments [315] based on STM...
probes, and how it is reflected in quantum oscillations [316]. In particular, it is found that the observation of a QPI pattern consistent with the so-called [317] “octet model” is a direct signature of coherent superconducting order. It appears from theory that the QPI pattern in the pseudo-gap state [315] is distinctly different from that in the superconducting phase.

IX. CONCLUSIONS

A. Summary

This Review article has been written in response to the large and relatively recent experimental literature on strongly correlated superconductors which are thought to exhibit BCS-BEC crossover phenomena. Many of these derive from artificial materials such as magic-angle twisted bilayer and tri-layer graphene, quantum Hall bi-layers, or ionic-gate tuned semiconductors, as well as single unit cell and interfacial superconducting films. Also exciting are naturally grown superconductors, such as the Fe chalcogenides and the organic superconductor $\kappa$-(BEDT-TTF)$_2$Hg$_2$BO$_3$.

Because of the widespread interest, it is important to establish more precisely what BCS-BEC crossover theory is and what it is not. We have done so in this Review and in the process clarified distinctions between the Fermi gas and solid-state superconductors, between two and three-dimensional materials, between $s$- and $d$-wave order parameter symmetries and we have established distinguishing features of the BEC phase.

More generally, in this paper and in the context of different experiments, we addressed the three distinct ways of promot-
The inter-dependencies are generally robust to variations in the interaction strength parameter. We have emphasized that “domes” and “pseudogaps” are ubiquitous for crossover systems in periodic lattices.

The narrative arc of this Review is encapsulated through the evolution from Fig. 1 to the next figure we now discuss, Fig. 36. Figure 1 introduced the concept of BCS-BEC crossover by raising the question of how to treat superconductivity in the presence of progressively stronger attractive interaction strengths. Notably, in contrast to the cold Fermi gases, solid state experiments have little access to this interaction strength parameter.

Fig. 36, which represents a summary of many of the various 2D superconducting materials discussed in this Review, allows us to compare crossover theory and experiment. This is made possible by effectively representing the dimensionless interaction strength parameter in BCS-BEC crossover theory through dimensionless ratios of physically accessible parameters, such as $T^*/T_{BKT}$ and $\Delta_0/E_F$. One could similarly consider $k_F\xi_{coh}$ in counterpart plots. All of these are strongly inter-connected and, importantly, the figure indicates that their inter-dependencies are generally robust to variations in the pairing symmetry (here from s-wave to d-wave).

Plotted on the vertical axis in a logarithmic scale is $\Delta_0/E_F$, where $\Delta_0$ is the zero-temperature excitation gap, while on the horizontal axis in a linear scale is $T^*/T_{BKT}$ for two-dimensional superconductors. The upper (black) and lower (blue) theory curves are for s- and d-wave pairing symmetries, respectively. The data points come from the lithium-intercalated nitride films [183], from one unit cell FeSe on strontium titanate [239] and from magic-angle twisted bilayer as well as trilayer graphene [150, 181, 185, 186].

Two additional data sets are associated with strongly disordered Pb films [194] and from the interface superconductor LaAlO$_3$/SrTiO$_3$ [195]; the latter system does not fall into any simple category. In this plot, because of their small $\Delta_0/E_F$ ratios, both are clearly distinct from BCS-BEC crossover candidate materials. A comparison of theory and experiment in this replotting, thus, serves to highlight the distinction between strong pairing and strong disorder. In this way, the figure serves as a template for helping identify BCS-BEC crossover systems. The existence of a pseudogap (through the deviation of $T^*/T_c$ from unity), as well as observations of $2e$ pairing appear insufficient.

Additionally, we have addressed the question of under what circumstances should one expect to reach the Bose-Einstein-condensation limit for a solid-state superconductor. In general, in this regime, rather than a very large transition temperature, one finds very small magnitudes of $T_c$ or $T_{BKT}$. This point is often missed in the literature because the standard for the BCS-BEC crossover phase diagrams is based on Fermi-gas physics where the BEC asymptote is large. This distinction is emphasized in Fig. 1 of this Review.

In the BEC regime, all signs of a Fermi surface have disappeared. Thus far, we are not able to report any unambiguous evidence that candidate systems have reached the BEC limit. Some signatures of the BEC we invoked earlier are that in this limit the character of the states within vortex cores [219] is distinctly different. Similarly, in this limit, coherence peaks in the quasiparticle tunneling characteristics will be absent. Theoretical indications are that a Bose condensed Fermi superconductor can occur when either $T^*/T_{BKT}$ is much larger, say of the order of 10, accompanied by more conventional values of $\Delta_0/E_F$ or alternatively with $\Delta_0/E_F$ of the order of 10 or more, accompanied by more conventional values of $T^*/T_{BKT}$. The latter relates to the interesting scenario in which superconductivity occurs in the presence of very flat energy bands with nontrivial band-topology and quantum geometry.

It is important to stress that to establish a given supercon-
B. Outlook

More generally in looking toward the future, we are poised at the beginning of an extremely exciting era where the development of synthetic superconductors by design seems limitless. It is clear that tunable 2D superconductors (such as MATBG [150, 185], MATTG [181, 186], Li$_x$ZrNCl [183] etc.) will be crucial to realize superconductivity in the strong coupling regime. The coupling strength and Fermi energy can be dramatically and precisely tuned by twisting, gating, and doping, which provides the best platform to observe BCS-BEC crossover physics and to compare with theory.

Importantly, the present review can serve as a blueprint for future experimental endeavors, as it establishes concrete, experimentally falsifiable criteria to determine whether a given superconductor is in the BCS-BEC crossover regime. A singular observation of only the pseudogap phase, or pairing above $T_c$ no longer suffices. Future experimental studies will need to combine measurements of $\Delta_0/E_F$, $T^*/T_c$, and $T_c$ or $T_{\text{BKT}}$ to place candidate materials on Fig. 36. Critical tests will be to perform these measurements with a continuous tuning parameter (gating, doping, twisting, or isovalent substitution), to enable the comparison between theory and experiment in an extended region of Fig. 36. An example of such very complete studies is the work summarized here on Li$_x$ZrNCl [183].

It should be noted that other tunable 2D superconductors such as twisted transition metal dichalcogenides can also host flat bands [318, 319], and should be viewed as future candidates for superconductivity in the BCS-BEC crossover regime. It has also been predicted that nonequilibrium optical driving on twisted bilayer graphene can induce flat-band behavior associated with an effective Floquet Hamiltonian [320]; this provides a nonequilibrium route towards the strong-coupling limit. The implications of the BCS-BEC crossover scenario in the general nonequilibrium context will be important to address. Ultrafast spectroscopic experi-
ments should more generally be explored to characterize this nonequilibrium band structure engineering and its potentially new forms of superconductivity.

Table I. Experimental data collected for Fig. 36. Here we identify the low temperature gap with $\Delta_0$. For Li$_x$ZrNCl different rows are for different carrier densities.

| Materials           | $T_{\text{BKT}}$ | $T^*$ | $\Delta_0$ | $E_F$ | $T^*/T_c$ | $\Delta_0/E_F$ |
|---------------------|------------------|------|------------|-------|-----------|----------------|
| (FeSe)$_1$/STO      | 38 K             | 72 K | 15 meV     | 60 meV| 1.89      | 0.25           |
| (Pb)$_2$/Si         | 2.4 K            | 6.9 K| 0.35 meV   | 380 meV| 2.9       | 0.001          |
| (001) LAO/STO       | 100 mK           | 500 mK| 65 $\mu$eV| 47 meV| 5         | 0.001          |
| Li$_x$ZrNCl         | 0.031 $T_F$     | 0.055 $T_F$ | – | – | 1.78 | 0.067 |
|                     | 0.061 $T_F$     | 0.13 $T_F$ | – | – | 2.1 | 0.18 |
|                     | 0.088 $T_F$     | 0.20 $T_F$ | – | – | 2.25 | 0.26 |
|                     | 0.097 $T_F$     | 0.24 $T_F$ | – | – | 2.45 | 0.27 |
|                     | 0.10 $T_F$      | 0.30 $T_F$ | – | – | 2.84 | 0.31 |
|                     | 0.12 $T_F$      | 0.35 $T_F$ | – | – | 3.0 | 0.36 |
| MATBG               | 1.0 K            | 4 K  | 1.4 meV   | 20 K  | 4         | 0.8            |
| MATTG               | 2.25 K           | 7 K  | 1.6 meV   | 32 K  | 3.1       | 0.58           |

Additionally, the study of high-$T_c$ Fe-based superconductors will lead to new opportunities and challenges to explore the connection between the BCS-BEC crossover physics, high-$T_c$ superconductivity, and topological superconductivity. It is worth noting that the disparity between the transport $T_c$ ($\sim$ 40 K) and the spectroscopic $T^*$ ($\sim$ 70 K) has been a fundamental issue undermining further progress on monolayer FeSe/SrTiO$_3$ systems [239]. This review can serve as the starting point to systematically explore crossover physics for understanding this remarkable 2D high-$T_c$ superconductor. A systematic tuning experiment using gating, doping, or Se:Te substitution will need to be performed. Importantly, with a specific Se:Te ratio $x:1-x$ between $x = 0.45$ and $x = 0.55$ the FeTe$_{1-x}$Se$_x$ bulk system exhibits a non-trivial topology with a superconducting topological surface state [321]. It remains to understand what the role of this topology is in the crossover physics.

Among new theoretical challenges, BCS-BEC crossover theories of superconductivity will need to accommodate the effect of magnetic fields, which will complete understanding of the canonical superconducting phase diagrams. What is the nature of the non-condensed pairs in the presence of a magnetic field [322]? How does condensation proceed when the dimensions of the system are effectively reduced by the presence of Landau levels [56, 323] and how does one understand the dynamics of vortices [324] from BCS to BEC? Conceptually related is the central and difficult issue: how to generalize the Bogoliubov de Gennes equations to the crossover situation at finite temperature. This would enable other important calculations, for example, describing Andreev tunneling, effects of proximitization and addressing the vast number of situations which involve spatially dependent superconductivity. It is notably a difficult problem as one needs to incorporate two distinct types of (now spatially dependent) gaps, associated with condensed and non-condensed pairs.

In a discipline, where theory and experiment work hand-in-hand, it should be clear that the multiple experimental platforms described in this section, collectively present enormous opportunities for future theoretical developments. In the process they enhance our understanding of this generalized BCS theory in a deeper and much broader sense.

X. ACKNOWLEDGMENTS

We would like to thank our various collaborators who contributed to this work and to our understanding over the years: B. Jankó, I. Kosztin, J. Maly, J. Stajic, A. Iyengar, S. Tan, C.-C. Chien, Y. He, H. Guo, D. Wulin, P. Scherpelz, B. Anderson, C.-T. Wu, X. Wang, K. M. Shen, and D. G. Schلوم. This work was partially (K. L., Z. W.) supported by Department of Energy (DE-SC0019216). Q. C. was supported by the Innovation Program for Quantum Science and Technology (Grant No. 2021ZD0301904). R. B. was supported by the Department of Physics and Astronomy, Dartmouth College. S.-L. Y. acknowledges the support by the U.S. National Science Foundation through Grant 2145373.

Appendix A: Experimental Data for 2D Superconductors

In this Appendix, we present in Table I the data collected for Fig. 36 from various sources. In this table, if $T_{\text{BKT}}$ is not available, we use the corresponding $T_c$. The abbreviations are: (FeSe)$_1$/STO = monolayer FeSe grown on the SrTiO$_3$ substrate, (Pb)$_2$/Si = 4-monolayer Pb film grown on the Si substrate, (001) LAO/STO = (001)-oriented LaAlO$_3$/SrTiO$_3$ in-
terface, MATBG = magic-angle twisted bilayer graphene, and MATTG = magic-angle twisted trilayer graphene.

The sources of the data are as follows: for (FeSe)$_1$/STO, $\{T_{BKT}, T^*\}$ are taken from Ref. [239], and $\{\Delta_0, E_F\}$ from Ref. [240]. For (Pb)$_n$/Si the data for $\{T_{BKT}, T^*\}$ are from Ref. [194]. To estimate $\Delta_0/E_F$ we use Ref. [325], where the sample used is actually a monolayer Pb film on Si substrate, (Pb)$_n$/Si. We do not expect $\Delta_0/E_F$ to differ much between (Pb)$_n$/Si and (Pb)$_1$/Si.

The data for Li$_x$ZrNCl are taken from Ref. [183]. For (001)LAO/STO we use Ref. [244] for $T_{BKT}$, Ref. [223] for $\{T^*, \Delta_0\}$, and Ref. [243, 244] for $E_F$. In this system we have used the $d_{xy}$ orbital band to arrive at $E_F$, and the data collected all roughly correspond to the same gating voltage $V_g \approx -100$ V.

The values of $\{T_{BKT}, T^*, E_F\}$ for MATBG are taken from Ref. [185] for a twist angle $\theta \approx 1.05^\circ$. Here, $T^*$ is estimated from the Ohmic recovery point from the V-I characteristic measurement. $\Delta_0$ is obtained from Ref. [150] which is appropriate to a very close but slightly different twist angle $\theta \approx 1.01^\circ$ system.

For MATTG we use Ref. [181] for $T_{BKT}$ and Ref. [186] for $\{T^*, \Delta_0\}$. The value of $E_F$ is estimated by Stevan Nadji-Perge and provided through a private communication.

Fig. 37(a) plots a $d$-wave phase diagram at a hole concentration $p = 1 - n = 0.15$ as a function of attractive cou-

Figure 37. (a) $T_c - U$ phase diagram for a d-wave superconductor with electron density $n = 0.85$ on a quasi-2D square lattice. The energy dispersion is $\epsilon_k = (4t + 4t' + 2t_0) - 2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y - 2t_0 \cos k_z$ with $t' = -0.3t$ and $t_0/t = 0.01$. All energies are normalized by $W = 4t$. The pairing gap is $\Delta_k = \Delta \varphi_k$ with $\varphi_k = \cos k_x - \cos k_y$. (b) Temperature dependencies of the chemical potential $\mu$ and the extrapolated normal state $\mu_{N}$, for interaction strength $U/W = -0.45$, corresponding to the vertical dotted line in (a). Emphasizing the small variations in $\mu$, here $\mu$ changes by $-0.5\%$ from $T = 0$ to the pairing onset $T^*$, and $(\mu - \mu_0)/\mu_{NS}$ is found to be $3.8\%$ at $T = 0$.

Appendix B: General BCS-BEC Crossover Theory for D-wave Case Near Half Filling

In this appendix we present additional details about BCS-BEC crossover theory in the d-wave case, focusing on the region around half filling in the electron band. The results here are presumed to be generally appropriate to nodal superconductors in this half-filled regime where (as discussed in the text) a BEC is not accessible. In the next Appendix C we make contact with some aspects of cuprate experiments, but it is important not to confuse the phenomenological appendix with the more precise predictions we present here.

For definiteness we look at a typical band structure which happens to be used for cuprates (but otherwise is of no particular consequence). We take $\epsilon_k = (4t + 4t' + 2t_0) - 2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y - 2t_0 \cos k_z$ with $t' / t = -0.3$. This band structure is more complicated than that used in the main text of the paper (for both $s$- and $d$-wave systems), as it has a Van Hove singularity which is prominent for the band fillings we address. This is found to affect some properties of the crossover.

The goal of this appendix is to present the general behavior of the $T^*$ and $T_c$ phase diagrams, and the associated properties of the chemical potential. The latter is useful to establish because it can, in principle be measured. Moreover, the size of the fermionic chemical potential is often viewed as a measure of where a given system is in the crossover spectrum. By contrast, we emphasize here, unlike in the Fermi gases, how improbable it is to find a solid state superconductor anywhere in proximity to a BEC. As discussed in the main text there are better indicators of crossover physics than found in $\mu$, through the behavior, for example, of $T^*/T_c$ and the coherence length. Indicated are representative values of $T^*$ and $T_c$. In the next figure, the solid line in Fig. 37(b) serves to

\[ T_c - U \text{ phase diagram} \]

\[ \Delta_k = \Delta \varphi_k, \varphi_k = \cos k_x - \cos k_y \]

\[ \mu_{N}, \mu \text{ for } U/W = -0.45 \]

\[ \mu - \mu_0/\mu_{NS} \text{ at } T = 0 \]

\[ T^* \text{ and } T_c \text{ phase diagrams} \]

\[ T^*/T_c \text{ and coherence length} \]
characterize the behavior of the self-consistently determined fermionic chemical potential $\mu(T)$ for this particular interaction strength, as a function of temperature $T$. The dashed line indicates the counterpart value in the extrapolated normal state, $\mu_0(T)$, obtained by turning off the attraction. A crucial point follows by comparing Figs. 37(a) and 37(b) where we see that, although, there is an appreciable separation between $T^*$ and $T_c$ the chemical potential differs only slightly from its normal state value.

Figure 38 presents results for a range of hole concentrations, near half filling. For reasons which will become clear later, we choose $T^*/T_c$ to be 4.7 to illustrate the behavior for a slightly lower hole doping $p = 0.1$, while $T^*/T_c = 1.05$ for a system with higher doping corresponding to $p = 0.25$. These two cases respectively show the effects of increasing and decreasing the size of the pseudogap.

Table II summarizes some central findings. Here we tabulate results for all three hole doping levels, $p = \{0.1, 0.15, 0.25\}$, including the behavior of the chemical potentials. This table presents the ratios of the zero temperature chemical potential $\mu$ to their normal state counterparts. The difference from unity is small and in the most extreme case, still less than 10%. From this comparison, one might view these systems as conventional BCS superconductors, but it should be stressed that they all belong to the BCS-BEC crossover regime as $T_c$ and $T^*$ are quite distinct.

Table II. Table showing changes in chemical potential associated with different values of $T^*/T_c$. Here $W = 4t$.

| hole doping ($p$) | $T^*/T_c$ | $|U|/W$ | $\mu(T = 0)/\mu_0(T = 0)$ |
|-------------------|----------|-------|-----------------------------|
| $p = 0.10$        | 4.73     | 1.06  | 1.09                        |
| $p = 0.15$        | 2.03     | 0.45  | 1.04                        |
| $p = 0.25$        | 1.05     | 0.095 | 1.003                       |

Figure 38. $T_c - U$ phase diagrams for quasi-2D d-wave superconductors with the same energy dispersion as in Fig. 37, computed for different electron densities $n = 1 - p$, where $p$ is the hole doping. The symbols indicate where a given system (represented by the $n$ value and $T^*/T_c$) is located in the corresponding experimental phase diagram [272]. For clarity, here we show the $T^*$ line for $n = 0.75$ only.

Figure 39. Experimental cuprate phase diagram, taken from Ref. [272].

### Appendix C: Implications of the Cuprate Phase Diagram and Relation to Twisted Graphene Family

Whether any of the above discussion is relevant to the cuprates cannot be unequivocally established. But it is useful to explore what the consequences are if we assume the values of $n$ and $T^*/T_c$ chosen above and then establish the implications of this $d$-wave BCS-BEC crossover. Indeed, the correspondence between both of these parameters can be seen to be reasonably compatible with the cuprate phase diagram, which is shown in Fig. 39 [272]. This compatibility of the parameter set, of course, depends on assuming that the measured $T^*$ is related to pairing.

We stress that there are complexities concerning this phase diagram which are still not fully settled. Among these is the observation of a second characteristic temperature $T^*_0$ which is not shown in the plot. This temperature is typically about 20% above $T_c$, although significantly below $T^*$ for heavily underdoped cuprates; one might speculate that this is associated with the onset of a more extended fluctuation regime where bosonic transport, derived from quasi-stable pre-formed pairs near condensation, is significant. Here we focus only on the presumed gap opening temperature $T^*$ plotted above. We stress that there is no unanimity about whether one should associate the experimental $T^*$ with pairing or an alternative energy scale, for example, deriving from possible ordering (e.g., $d$-density wave [326]) or fluctuations in the particle-hole channel.

We view the ratio $T^*/T_c$ and corresponding density as input parameters. However, one test of the applicability of this theory comes from establishing the corresponding size of the electronic energy scales needed to match the size of the measured $T_c$ and $T^*$, say in Kelvin. At issue is the hopping matrix elements $t$ which determine the bandwidth and Fermi energy for each cuprate with a different hole concentration.

One might estimate that $T_c/T_F$ is around 0.1 in the underdoped cuprates, as is confirmed in Table III where we present...
Table III. Key parameters for hole doped cuprates. In some sense these are near weak coupling which reflects the fact that the cuprate $T^*/T_c$ are not very large except at extreme underdoping. Here $\Delta_0 = 2\Delta$, which is the zero temperature spectral gap $|\Delta_k| = |\Delta(\cos k_x - \cos k_y)|$ at $(k_x, k_y) = (\pi, 0)$ as measured in ARPES.

| hole doping ($p$) | $T^*$ (K) | $T_c$ (K) | $T^*/T_c$ | $t$ (meV) | $2\Delta_0/k_BT_c$ | $T_F$ (K) | $T_c/T_F$ | $|U|$ (meV) |
|------------------|-----------|-----------|-----------|-----------|-------------------|-----------|-----------|-----------|
| $p = 0.10$       | 260       | 55        | 4.73      | 22.7      | 2.59              | 502       | 0.11      | 96.4      |
| $p = 0.15$       | 190       | 93        | 2.03      | 46.6      | 9.85              | 975       | 0.095     | 84.0      |
| $p = 0.25$       | 32        | 30.6      | 1.05      | 130       | 4.28              | 2466      | 0.012     | 49.3      |

We note that the values for $T_F$ shown appear to be slightly smaller, but not by orders of magnitude, than those presented by Uemura [163]. As yet this remains an unsettled issue.

More specific cuprate parameters are presented in Table III which indicates the (only) adjustable parameter, $t$, in the fifth column of the table. It should be noted that this fitting suggests that the effective bandwidths will have to decrease as the system becomes more underdoped. Moreover, the attractive interaction $U$ appears to become stronger as the insulator is approached. This should have some consequences for the origin of the pairing “glue”.

We take note of recent work applying BCS-BEC crossover theory to the cuprates [327]. Here it was suggested that the cuprates with a “magic” ratio of $2\Delta_0/T_c = 6.5$ can be identified with the unitary point in a three dimensional cold Fermi gas. This unitary point relates to the location of an isolated two-body bound state. However, as emphasized in this review, the superconducting phase diagrams of solid-state superconductors and Fermi gases are quite different, making such an identification difficult to support. In particular, from Table III it follows that even at optimal doping $p = 0.15$, we have $2\Delta_0/T_c = 9.85$, which is, indeed, also consistent with numbers obtained from photoemission experiments [328]. This value is larger than 6.5 and it follows that, on the basis of the analysis of the chemical potential (Table II), such systems are far from the BEC as well.

We end this paper with a figure (Fig. 40) consolidating the results in the above table with those in Figs. 10 and 36. This presents a combination of the key parameters associated with both MATBG and MATTG and a collection of counterfactual data on the hole doped cuprates. Indeed, one can see that the two graphene points are sandwiched between the two most underdoped cuprates ($p = 0.10$ and $p = 0.15$). While it has been conjectured [150] that MATBG bears a striking similarity to the cuprates, the figure has presented some quantitative evidence in support of this point.
could be extrapolated down to $T = 0$.

$\Delta$ — Fermionic excitation gap
$\Delta_n$ — Superconducting/superfluid order parameter
$\Delta_{\text{BCS}}$ — Mean-field gap obtained from BCS theory.
$\Delta_0 \equiv \Delta(T = 0)$ — Zero temperature gap

Four-vector $k \equiv (\omega_n, k)$, $\Sigma_k = T \sum_n \sum_k$, where $\omega_n = (2n + 1)\pi k_B T/h$ is the odd (fermionic) Matsubara frequency, with $n \in \mathbb{Z}$.

Four-vector $q \equiv (i\Omega_l, q)$, $\Sigma_q = T \sum_l \sum_q$, where $\Omega_l = 2\pi k_B T/h$ is the even (bosonic) Matsubara frequency, with $l \in \mathbb{Z}$.

$f(x) = 1/(e^{x/k_B T} + 1)$ — Fermi distribution function
$b(x) = 1/(e^{x/k_B T} - 1)$ — Bose distribution function

$G(k)$, $G_0(k)$ — Full and bare Green’s functions for fermions
$\Sigma(k)$ — Self energy of fermions
$\Sigma_{\text{sc}}(k)$ — Superconducting self energy of fermions
$\Sigma_{\text{ph}}(k)$ — Pseudogap self energy of fermions

$\chi(q)$ — Pair susceptibility
$t(q)$ — $t$-matrix

$U_c$ — Critical interaction strength at which the two-body scattering length diverges in free space, or more generally, the strength at which a bound state starts to emerge.

$V_{kk'} = U \phi_k \phi_{k'}$ — Separable pairing interaction, with strength $U < 0$ and the symmetry factor $\phi_k$. For a contact potential or the attractive Hubbard model, $\phi_k = 1$; for the cuprates, $\phi_k = \cos k_x - \cos k_y$.

$E_{\text{kin}}$ — Characteristic kinetic energy scale, can be taken to be half band width at moderate density or $E_F$ at low density.

$\epsilon_{k} = k^2/2m$ — Bare fermion dispersion in free space, with $h = 1$.

$\epsilon_{k} = 2t(2 - \cos k_x - \cos k_y) + 4t'(1 - \cos k_x \cos k_y) + 2t_z(1 - \cos k_z)$ — Bare fermion dispersion in a quasi-2D square lattice, where $t$ and $t'$ are the nearest- and next-nearest-neighbor in-plane hopping integral, respectively, and $t_z$ is the out-of-plane hopping integral. Here the lattice constants have been set to unity, $a = b = c = 1$.

$\xi_k = \xi_k - \mu$ — Bare fermion dispersion measured from the chemical potential.

$E_{\text{B}}$ — Bogoliubov quasiparticle dispersion

$\psi_{k}^2 = \frac{1}{2}(1 + \xi_k/E_k)$, $\psi_{k}^2 = \frac{1}{2}(1 - \xi_k/E_k)$ — Coherence factors as given in BCS theory.

$\Psi_{\text{BCS}}$ — Ground state BCS wavefunction

$n$ — Fermion number density

$p = 1 - n$ or $x = 1 - n$ — Hole doping concentration (in the cuprates)

$n_{\text{pair}}$ — Fermion pair or boson number density

$M_{\text{pair}}$ — Effective mass of fermion pairs or bosons

$N_\pi/N$ — Quasi-condensate fraction (in 2D Fermi gas experiment)

$\rho_s$ — Superfluid phase stiffness, having dimension $[n]/[m]$.

$\sigma_s$ — $s$-wave inter-fermion scattering length

$\sigma_{\text{3D}}$ — $2D$ $s$-wave inter-fermion scattering length

$d \equiv d_{\text{particle}}$ — Intercell distance (in MATBG and MATTG)

$\xi_{\text{coh}}$ — GL coherence length

$\xi_0$ — Pair size

$H_{\text{c2}}$ — Upper critical field

$B$ — Magnetic field strength

$\ell_B = \sqrt{\hbar/|eB|}$ — Magnetic length

$\Phi_0$ — Flux quantum

$\rho_{\text{xx}}$ — Longitudinal resistivity

$\rho_{\text{xy}}$ — Transverse resistivity

$R_{\text{xx}}, R_{\text{xy}}$ — Hall, transverse resistance

$R_{\text{counter}}$ — Counter flow resistance measured in double-layer quantum Hall systems

$\rho_{\text{drag}}$ — Hall drag resistance

$\sigma_1, \sigma_2$ — Real and imaginary parts of the conductivity $\sigma(\omega)$

$\chi_{\text{dia}}$ — Diamagnetic susceptibility

$M_{\text{dia}}$ — Diamagnetic response in magnetization

$\Delta_{\text{crit}}$ — Critical value associated with the phase space density of pairs for TBK transition.

$1/T_1$ — Nuclear spin-lattice relaxation rate

$V_{g}$, $V_{\text{gate}}$ — Gating voltage

$\nu$ — Electronic band filling factor (in MATBG and MATTG)

$\theta$ — Twist angle (in MATBG and MATTG)

We always refer to the absolute value when we refer to the interaction parameter $U$ as increasing or decreasing.

### 2 Convention for units

Throughout this Review, we use the convention for units where it is not explicitly spelled out:

$h = k_B = c = 1$.

In numerics, we set the volume to unity, and $E_F = T_F = k_B = 2m = 1$ for the free space cases, which leads to $n = 1/3\pi^2$ in 3D.

For the lattice cases, we take the half bandwidth $W = zt = 1$ and lattice constants $a = b = c = 1$. In a simple (quasi-)2D square or 3D cubic lattice, $n = 1$ at half filling.

Our fermionic chemical potential $\mu$ is measured with respect to the bottom of the non-interacting energy band, such that $\epsilon_{k=0} = 0$. This leads to (i) $\mu = E_F$ in the non-interacting limit at $T = 0$, and (ii) $\mu$ changes sign when the system crosses the boundary between fermionic and bosonic regimes.

### 3 Abbreviations

3D — Three dimensions
2D — Two dimensions
1UC — One unit cell (thickness)
AL — Aslamazov-Larkin (theory)
AFM — Antiferromagnet (or atomic force microscope)
ARPES — Angle-resolved photoemission spectroscopy
BCS — Bardeen-Cooper-Schrieffer (theory)
BEC — Bose-Einstein condensation
BKT — Berezinskii-Kosterlitz-Thouless (transition)
BSCCO, Bi2212 — Bi$_2$Sr$_2$CaCu$_2$O$_{8+\delta}$

$\sigma_1, \sigma_2$ — Real and imaginary parts of the conductivity $\sigma(\omega)$

$\chi_{\text{dia}}$ — Diamagnetic susceptibility

$M_{\text{dia}}$ — Diamagnetic response in magnetization

$\Delta_{\text{crit}}$ — Critical value associated with the phase space density of pairs for TBK transition.

$1/T_1$ — Nuclear spin-lattice relaxation rate

$V_{g}$, $V_{\text{gate}}$ — Gating voltage

$\nu$ — Electronic band filling factor (in MATBG and MATTG)
CF — Composite fermion
DC — Direct current
DMFT — Dynamical mean field theory
GL — Ginzburg-Landau (theory)
GP — Gross-Pitaevskii (equation)
LAO/STO — LaAlO$_3$/SrTiO$_3$ (interface)
LSCO, La$_2$14 — La$_{1-x}$Sr$_x$CuO$_4$
LL — Landau levels
MIR — Mid-infrared (conductivity)
MATBG (MATTG) — Magic angle twisted bilayer (trilayer) graphene
meV — milli-electron volts
NMR — Nuclear magnetic resonance
NSR — Nozières and Schmitt-Rink
OD — Overdoped (cuprates)
PG — Pseudogap
QMC — Quantum Monte Carlo (simulations)
QPI — Quasi-particle interference
RF — Radio frequency (spectroscopy)
RPA — Random phase approximation
TDGL — Time-dependent Ginzburg-Landau (theory)
TMA — $t$-matrix approximation
SC — Superconductor
SCTA — Self-consistent $t$-matrix approximation
SI — Superconductor-insulator (transition)
SIN — Superconductor-insulator-normal metal (tunneling junction)
STM/STS — Scan tunneling microscopy/spectroscopy
UD — Underdoped (cuprates)
YBCO — YBa$_2$Cu$_3$O$_{7-\delta}$
Y123 — Y$_{0.8}$Ca$_{0.2}$Ba$_2$Cu$_3$O$_{7-\delta}$
YRZ — Yang, Rice, Zhang (theory)
µSR — Muon spin resonance/rotation/relaxation

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