Analytical solvability of the two-axis countertwisting spin squeezing Hamiltonian

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There is currently much interest in the two-axis countertwisting spin squeezing Hamiltonian suggested originally by Kitagawa and Ueda, since it is useful for interferometry and metrology. No analytical solution valid for arbitrary spin values seems to be available. In this article we systematically consider the issue of the analytical solvability of this Hamiltonian for various specific spin values. We show that the spin squeezing dynamics can be considered to be analytically solved for angular momentum values up to 21/2, i.e. for 21 spin half particles. We also identify the properties of the system responsible for yielding analytical solutions for much higher spin values than based on naive expectations. Our work is relevant for analytic characterization of squeezing experiments with low spin values, and semi-analytic modeling of higher values of spins.

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I. INTRODUCTION

The two-axis countertwisting spin-squeezing Hamiltonian \( \hat{H}_{TA} \) was originally proposed by Kitagawa and Ueda \[1,2\]. In Eq. (1) \( J_+ \) and \( J_- \) are the angular momentum raising and lowering operators \[3\], \( i = \sqrt{-1} \) and \( \chi \) is a constant. The quantity \( J \) may refer to a real angular momentum or a pseudospin \( J = N/2 \) representing the collective squeezing of \( N \) spin half systems \[4\]. The Hamiltonian \( \hat{H}_{TA} \) yields maximal squeezing, with a squeezing angle independent of system size or evolution time. Experimental implementation has not yet been achieved, while a number of theoretical proposals have been put forward \[6–12\].

A general analytic solution to the Hamiltonian \( \hat{H}_{TA} \) for arbitrary angular momentum does not seem to be available \[13,14\]. Solutions to the dynamics for low values of spin are of interest to experiments with trapped ions \[15\] and quantum magnets \[16\], for example. In Ref. \[13\] a bound of spin 3/2 was stated as the maximum angular momentum value for which analytic solutions can be found. We presume this was based on the fact that the Hamiltonian matrix in that case is of dimension \( 2 \times 3 \times 2 + 1 = 4 \), leading to a characteristic polynomial of quartic order, which is the highest degree for which algebraic solutions can generally be found \[17\]. In Ref. \[13\] the existence of analytical solutions (with the additional presence of an external field) up to spin 10 was reported and explicit expressions were provided for spin 1. No explanation was given of the unexpected fact that solutions for spins much larger than 3/2 were found, nor was the maximum value of spin for which analytical solutions could be found supplied.

In the present article we systematically analyze the question of analytical solvability of the two-axis countertwisting spin squeezing Hamiltonian of Eq. (1). We extend the bound for solvability to spin 21/2, i.e. 21 spin half particles (we note that in Ref. \[13\] spin 10 corresponds to 10 spin half particles). We point out that critical roles in the solvability of the model for large spin values are played by the chiral symmetry and sparsity of the Hamiltonian matrix. Our approach requires only the use of matrix representations of the angular momentum operators and the evaluation the time evolution operator \[17\]. Our results may be useful for experiments with small number of spins.

II. SOME PROPERTIES OF \( \hat{H}_{TA} \)

In this section we go over some properties of the Hamiltonian \( \hat{H}_{TA} \) of Eq. (1) which are relevant to the discussion of analytical solvability. First, it can be seen readily by using

\[
J_+ = J_1^\dagger, 
\]

that \( \hat{H}_{TA} \) is Hermitian, implying that its eigenvalues are real.

Further, by using

\[
J_{\pm} = J_x \pm i J_y, 
\]

we can rewrite

\[
\hat{H}_{TA} = \chi (J_x J_y + J_y J_x). 
\]

Now let us consider a rotation by the angle \( \pi \) about the \( J_y \) axis. This rotation leaves \( J_y \) unaffected, but reverses the sign of \( J_x \), i.e.

\[
e^{-i \pi J_y} \hat{H}_{TA} e^{i \pi J_y} = -\hat{H}_{TA},
\]
which can be written as the anticommutation relation
\[ \{ H_{TA}, e^{i\pi J_y} \} = 0. \]
(6)

This relation implies that \( e^{i\pi J_y} \) is a chiral symmetry of
\( H_{TA} \). In practical terms, the implication of the anti-
commutation is that eigenvalues of \( H_{TA} \) occur as signed
pairs \( \pm \lambda_1, \pm \lambda_2, \ldots \). (A brief proof is provided in the Ap-
pendix for the reader’s convenience). Therefore if \( H_{TA} \) can be represented by an even dimensional matrix, then
the characteristic polynomial of \( H_{TA} \)
\[ P(\lambda) = |H_{TA} - \lambda I| = 0, \]
is even in \( \lambda \), where \( I \) is the unit matrix of the same dimen-
sion as \( H_{TA} \). If instead \( H_{TA} \) is of odd dimension, then
its characteristic polynomial is \( \lambda \) times a polynomial even
in \( \lambda \). In this case there is necessarily a zero eigenvalue,
and all other eigenvalues are signed pairs. Specific examples
will be given below. As will be verified with these
examples, the property of chiral symmetry contributes
to giving a simple form to the characteristic poly-
nomials for even large spins, and making them analytically
solvable. For completeness we note that the Hamiltonian
considered by the authors of Ref. [13] is
\[ H_f = H_{TA} + \Omega J_z, \]
where \( \Omega \) represents an external field along the \( z \) axis,
also possesses the chiral symmetry indicated above. This
partly explains the solvability of the squeezing model \( H_f \)
analytically for up to spin 10.

Finally, by using the matrix elements in the basis \([j, m]\)
where \( J_z \) is diagonal
\[ (j, m'|J_z|j, m) = \sqrt{j(j+1) - m(m' \pm 1)}, \]
it can be readily verified that \( H_{TA} \) is a rather sparse ma-
trix, i.e. most of its elements are zero. This feature also
contributes to simplifying the form of the characteristic
polynomial.

III. THE TIME EVOLUTION OPERATOR

We now consider the time evolution operator
\[ U_{TA} = e^{-iH_{TA} t}. \]
(10)

Since \( U_{TA} \) determines the spin dynamics completely, the
model is analytically solvable if a matrix representation
for \( U_{TA} \) can be found, with all entries determined ana-
lytically. A straightforward way to implement this is to
diagonalize \( H_{TA} \). If the eigenvalues of \( H_{TA} \) can be found
analytically, the diagonal and analytic form of \( U_{TA} \) fol-
lows. However, the calculation of expectation values then
requires the relevant initial state to be rotated by the unit-
ary matrix that diagonalizes \( H_{TA} \). As these matrices can be
quite cumbersome and require the determination and
careful handling of the eigenvectors of \( H_{TA} \), we follow
instead an equivalent procedure that only deals with the
eigenvalues of \( H_{TA} \), but avoids referring to its eigen-
vectors.

Our approach is to expand the right hand side of
Eq. (10) in a Taylor series. The termination of that series
is actually guaranteed since \( H_{TA} \) is a finite dimensional
matrix. This guarantee comes from the Cayley-Hamilton
theorem, which states that every square matrix satisfies
its own characteristic equation, which in turn implies that
any power of the \( H_{TA} \) exists in terms of the matrix powers \((H_{TA})^0, (H_{TA})^1, \ldots, (H_{TA})^{2J} \), where
\( J \) is the associated spin. The entries of the terminat-
ing matrix representation of \( U_{TA} \), it turns out, can be
written as functions of the eigenvalues of \( H_{TA} \), see
below. Therefore, if the eigenvalues can be calculated ana-
lytically, then the spin dynamics can be found exactly.

For reference, we quote the explicit expression for \( U_{TA} \)
in the case where \( H_{TA} \) possesses nondegenerate eigen-
values \( \lambda_k, k = 1, 2, \ldots \) (the degenerate case can be handled
via some straightforward modifications) \[ 12 \]
\[ U_{TA} = \sum_{k=1}^{2J+1} e^{-i\lambda_k t} \prod_{n=1, n \neq k}^{2J+1} \left( \frac{H_{TA} - \lambda_n}{\lambda_k - \lambda_n} \right). \]
(11)

IV. RESULTS

A. Solvability

In Table 1, we show for various spins the characteristic
polynomial \( P(\lambda) \) of \( H_{TA} \). We make some comments on
the entries in this table. For \( J = 1/2 \), the eigenvalues are
both zero, and it can be verified that
\[ H_{TA}(J = 1/2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \]
(12)
consistent with the observation, first made by Kitagawa
and Ueda, that a single spin half particle cannot be
squeezed.\[ 2 \]

Generally for half-integer spin, \( P(\lambda) \) is even in \( \lambda \) and
some of the eigenvalues are degenerate. In Table 1, we
have indicated which spin values lead to degeneracy in
the eigenvalues of \( H_{TA} \), so that the appropriate pro-
cedure may be used for obtaining \( U_{TA} \). Generally, if
any factor in the characteristic polynomial repeats, then
there is degeneracy in the energy spectrum. To make this
identification rigorous, we have calculated the discrimi-
nant of \( P(\lambda) \), which returns a zero value if degeneracy is
can be considered as being polynomials of degree 5 in \(\lambda\), etc. Starting from the initial state

\[ as \] degree 12 in \(\lambda\), thus yielding a semianalytic solution for any spin value.

While the roots of such factors cannot be found algebraically, they can be stated in terms of hypergeometric functions \[19\]. However, for \(J = 11\), there is a factor of degree 12 in \(\lambda\) (i.e. of degree 6 in \(\lambda^2\)). While the roots of polynomials of degree 6 and higher can be found in terms of modular functions (for example), they involve transcendental functions, and we will consider them not to be of closed form and therefore analytically unsolvable \[20\]. We note that for \(J > 21/2\) the polynomial roots can be found numerically and inserted in Eq. (11), thus yielding a semianalytic solution for any spin value.

\[ J_+ = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

(13)

\[ J_- = J_+^\dagger, \] and

\[ H_{TA} = i\chi \begin{pmatrix}
0 & 0 & -\sqrt{6} & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 \\
\sqrt{6} & 0 & 0 & 0 & -\sqrt{6} & 0 \\
0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 & 0 & 0
\end{pmatrix}, \]

(14)

The eigenvalues of \(H_{TA}\) can be read off from Table 1, as \(0, \pm 3, \pm 2\sqrt{3}\). The eigenvectors are \((-1, 0, i\sqrt{2}, 0, 1)/\sqrt{2}, (-1, 0, -i\sqrt{2}, 0, 1)/\sqrt{2}, (0, i, 0, 1, 0)/\sqrt{2}, (0, -i, 0, 1, 0)/\sqrt{2}, (1, 0, 0, 0, 1)/\sqrt{2}\) in no specific order. The time evolution operator is

\[ U_{TA} = \\
\begin{pmatrix}
\cos^2(\sqrt{3}\chi t) & 0 & -\sin(2\sqrt{3}\chi t)/2 & 0 & \sin^2(\sqrt{3}\chi t) \\
0 & \cos(3\chi t) & 0 & -\sin(3\chi t) & 0 \\
\sin(2\sqrt{3}\chi t)/2 & 0 & \cos(2\sqrt{3}\chi t) & 0 & -\sin(2\sqrt{3}\chi t)/2 \\
0 & \sin(3\chi t) & 0 & \cos(3\chi t) & 0 \\
\sin^2(\sqrt{3}\chi t) & 0 & \sin(2\sqrt{3}\chi t)/2 & 0 & \cos^2(\sqrt{3}\chi t)
\end{pmatrix}, \]

(15)

The time evolution of any operator \(O\) is given by \(O(t) = U_{TA}^{-1}OU_{TA}\). Using this relation we can find the time-evolved quantities \(J_y(t), J_y^2(t)\), etc., and variances such as

\[(\Delta J_y(t))^2 = \langle J_y^2(t) \rangle - \langle J_y(t) \rangle^2, \]

(16)

e etc. Starting from the initial state

\[ |i\rangle = e^{-i\frac{J_y}{2}} |j = 2, m = 2\rangle = \frac{1}{2}\left| \begin{array}{c}
\frac{1}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}
\end{array} \right|, \]

(17)

i.e. the stretched state along \(z\) rotated by \(90^\circ\) about the \(y\) axis \[2\], we find the squeezing parameters following

\[ \xi_y = \sqrt{\frac{\langle (\Delta J_y(t))^2 \rangle}{\langle J_y(t) \rangle^2}}, \]

\[ = \frac{\sqrt{2}(17 - 6 \cos 6\chi t - 6 \cos 2\sqrt{3}\chi t + 3 \cos 4\sqrt{3}\chi t)^{1/2}}{[\cos 3\chi t(1 + 3 \cos 2\sqrt{3}\chi t + \sqrt{3} \sin 3\chi t \sin 2\sqrt{3}\chi t)]}, \]

(18)

and

B. Spin squeezing

To compactly illustrate our results, we present the details for \(J = 2\) (4 spin half particles), a case for which there seem to be no explicit results in the literature. In this instance, the matrix representations are given by

Wineland et al. \[21, 22\]


\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{\(J\)} & \textbf{\(P(\lambda)\)} & \textbf{Degenerate} \\
\hline
1/2 & \(\lambda^2\) & Yes \\
1 & \(\lambda(1 - \lambda^2)\) & No \\
3/2 & \((\lambda^2 - 3)^2\) & Yes \\
2 & \(-\lambda(\lambda^2 - 3)(\lambda^2 - 12)\) & No \\
5/2 & \(\lambda^2(\lambda^2 - 28)^2\) & Yes \\
3 & \(-\lambda(\lambda^2 - 60)(\lambda^2 - 6\lambda - 15)(\lambda^2 + 6\lambda - 15)\) & No \\
7/2 & \((\lambda^4 - 126\lambda^2 + 945)^2\) & Yes \\
4 & \(-\lambda(\lambda^2 - 28)(\lambda^2 - 208)\times(\lambda^2 + 10\lambda - 63)(\lambda^2 - 10\lambda - 63)\) & No \\
9/2 & \(\lambda^2(\lambda^4 - 396\lambda^2 + 19008)^2\) & Yes \\
5 & \(-\lambda(\lambda^2 - 108)(\lambda^2 - 528)\times(\lambda^6 - 651\lambda^4 + 65619\lambda^2 - 455625)\) & No \\
11/2 & \((\lambda^6 - 1001\lambda^4 + 172315\lambda^2 - 2338875)^2\) & Yes \\
6 & \(\lambda(\lambda^2 - 336)(\lambda^4 - 1176\lambda^2 + 55440)\times(\lambda^6 - 1491\lambda^4 + 421155\lambda^2 - 12006225)\) & No \\
13/2 & \(\lambda^2(\lambda^6 - 2184\lambda^4 + 1012752\lambda^2 - 74794752)^2\) & Yes \\
7 & \(-\lambda(\lambda^2 - 784)(\lambda^4 - 22963)^2 + 353808)\times(\lambda^8 - 3108\lambda^6 + 2236710\lambda^4 - 328692196\lambda^2 + 3773030625)\) & No \\
15/2 & \((\lambda^6 - 4284\lambda^6 + 4488102\lambda^4 - 1062230652\lambda^2 + 22347950625)^2\) & Yes \\
8 & \(-\lambda(\lambda^4 + 6624\lambda^2 + 1900800)(\lambda^4 + 16704\lambda^2 + 28753920)\times(\lambda^8 + 23184\lambda^6 + 138054240\lambda^4 + 204233529600\lambda^2 + 33886369440000)\) & No \\
17/2 & \(\lambda^2(\lambda^8 - 7752\lambda^6 + 16263696\lambda^4 - 9531032320\lambda^2 + 995361177600)^2\) & No \\
9 & \(-\lambda(\lambda^4 - 7056\lambda^2 + 6441984)(\lambda^4 - 3096\lambda^2 + 668304)\times(\lambda^8 + 10197\lambda^6 + 5340359\lambda^4 - 25878927978\lambda^2 + 5213177173701\lambda^2 - 88322827390625)\) & No \\
19/2 & \((\lambda^10 - 13167\lambda^8 + 5064028\lambda^6 - 6276422286\lambda^4 + 1962735976789\lambda^2 - 584689432201875)^2\) & Yes \\
10 & \(-\lambda(\lambda^4 - 5456\lambda^2 + 3165184)(\lambda^6 - 11936\lambda^4 + 20438704\lambda^2 - 203148000)\times(\lambda^8 + 167971\lambda^6 + 848699994\lambda^4 - 45169193178\lambda^2 + 68747106284901\lambda^2 - 3870591128105625)\) & No \\
21/2 & \(\lambda^2(\lambda^10 - 21252\lambda^8 + 140008176\lambda^6 - 329460868800\lambda^4 + 241815611520000\lambda^2 - 33685691719680000)^2\) & Yes \\
11 & \(-\lambda(\lambda^4 - 8976\lambda^2 + 10644480)(\lambda^6 - 17556\lambda^4 + 55226160\lambda^2 - 15437822400)\times(\lambda^8 + 265930\lambda^6 + 225185103\lambda^4 - 712278892116\lambda^2 + 768687668937315\lambda^4 - 202420859545362150\lambda^2 + 4712996874211250625)\) & No \\
\hline
\end{tabular}
\caption{Characteristic Polynomials of \(H_{T.A}/\chi\)}
\end{table}

The parameter \(\xi_z\) is plotted in Fig. 2. Squeezing in the \(z\) quadrature can be seen for two short intervals in the diagram.

For completeness we mention that correlations between the various spin components, which are relevant to squeezing \cite{1}, can also be found analytically for this solvable model. For example, plotted in Fig. 3 is the

\(\xi_z = \sqrt{\frac{1}{\langle J_z(t) \rangle}} \langle J_x(t) \rangle,\)

\[
\xi_z = \frac{2(7 - 3\cos 4\sqrt{3}\chi t - 6\chi t + \sqrt{3}\sin 2\sqrt{3}\chi t)^2)_{1/2}}{\cos 3\chi t(1 + 3\cos 2\sqrt{3}\chi t + \sqrt{3}\sin 3\chi t \sin 2\sqrt{3}\chi t)}.
\]

Squeezing occurs when the squeezing parameter is less than 1. The parameter \(\xi_y\) is plotted versus time in Fig. 1.

There is no squeezing in the \(y\) quadrature for the duration shown. The parameter \(\xi_x\) is plotted in Fig. 1.
FIG. 1: (Color online) The squeezing parameter $\xi_y$ [Eq. (18)] as a function of the dimensionless time $\chi t$ for $J = 2$.

FIG. 2: (Color online) The squeezing parameter $\xi_z$ [Eq. (19)] as a function of the dimensionless time $\chi t$ for $J = 2$.

FIG. 3: (Color online) The correlation $\langle J_x J_z + J_z J_x \rangle$ [Eq. (20)] as a function of the dimensionless time $\chi t$ for $J = 2$.

V. CONCLUSION

We have shown that the two-axis counter-twisting spin squeezing Hamiltonian can be solved analytically for up to angular momentum $21/2$. We have discussed the properties of the Hamiltonian that lead to such a high degree of solvability. From our results the axis of optimum squeezing can be found readily. Our methods can also be used to find useful quantities such as entanglement measures, in closed form. Future work will investigate the effects of decoherence on the solutions. We would like to thank K. Hazzard for stimulating discussions.

VI. APPENDIX

In this Appendix we show that the anticommutation of Eq. (13) implies the pairing of eigenvalues of $H_{TA}$. Consider an eigenvector $\psi_+$ of $H_{TA}$ with an eigenvalue $\lambda$, i.e.

$$H_{TA}\psi_+ = \lambda \psi_+,$$  \hspace{1cm} (21)

Multiplying from the left by $e^{i\pi J_y}$ and using the anticommutation of Eq. (13), we find the left-hand side of Eq. (21) reads

$$e^{i\pi J_y} H_{TA} \psi_+ = -H_{TA} e^{i\pi J_y} \psi_+,$$  \hspace{1cm} (22)

while the right-hand side reads

$$e^{i\pi J_y} (\lambda \psi_+) = \lambda (e^{i\pi J_y} \psi_+).$$  \hspace{1cm} (23)

Equating the right hand sides of Eqs. (22) and (23), we arrive at

$$H_{TA} (e^{i\pi J_y} \psi_+) = -\lambda (e^{i\pi J_y} \psi_+),$$  \hspace{1cm} (24)

which implies that

$$\psi_- = e^{i\pi J_y} \psi_+,$$  \hspace{1cm} (25)

is an eigenfunction of $H_{TA}$ with an eigenvalue of $-\lambda$. Thus the anticommutation of the operator $e^{i\pi J_y}$ with $H_{TA}$ leads to the $\pm \lambda$ pairing of eigenvalues in the spectrum of $H_{TA}$.\]
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