Joint Bayesian Treatment of Poisson and Gaussian Experiments
in a Chi-squared Statistic

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Abstract

Bayesian Poisson probability distributions for $\bar{n}$ can be analytically converted into equivalent chi-squared distributions. These can then be combined with other Gaussian or Bayesian Poisson distributions to make a total chi-squared distribution. This allows the usual treatment of chi-squared contours but now with both Poisson and Gaussian statistics experiments. This is illustrated with the case of neutrino oscillations.
I. INTRODUCTION

In analyzing the joint probability for mutual experimental results or for parameters, often a number of Poisson statistics experiments with a low number of events may be mixed with Gaussian experiments with high numbers of events. It is desirable to combine both types in a way to maintain the simplicity of a chi-squared distribution for all of the experiments. In this paper we show a simple mathematical identity between the Bayesian Poisson distribution for the average and an associated chi-squared distribution that allows us to accomplish this. We then apply this to the case of neutrino oscillation experiments with few events requiring a Poisson treatment and find the form of the addition to $\chi^2$ from the Poisson experiments to combine with Gaussian treated experiments to form a combined $\chi^2$ to study the oscillation and mixing parameters. Having achieved the general result of including Poisson experiments with Gaussian experiments, we then solve the simplest analytical cases for linear parameter dependences in the appendices.

In section 2 we review the method for joining two chi-squared distributions into a joint chi-squared distribution. In section 3 we review using Bayes’ theorem to find the Bayesian Poisson distribution for the average. In section 4 we show the exact equivalence of the Bayesian Poisson distribution for the average to a chi-squared distribution. We also show the domain of accuracy when a background is present. In section 5 we derive the joint probability distribution for combining a single Bayesian Poisson distribution for the average with a chi-squared distribution. In section 6 we then use the results of section 2 to combine in general the Bayesian Poisson distributions for averages with chi-squared distributions from Gaussian distributions. In section 7 we apply the method to the analysis of neutrino oscillation experiments with small numbers of events. In section 8 we present our conclusions.

Several appendices complete the necessary tools with expanded probability tables. Others solve the simplest analytic cases for contributions linear in the physical parameters. Appendix A reviews the comparison of the integrated probability of the Bayesian Poisson distribution for the average with the classical Poisson sum which is often used. Appendix
B gives a table of two-sided confidence level limits for the Bayesian Poisson average for a single experiment. Appendix C gives a table of chi-squared confidence levels which are useful for the joint distribution. Appendix D gives the solution for the minimum chi-squared for the case that the means only depend linearly on the parameters in both the Poisson and Gaussian distributions. Appendix E gives the most probable value and limits for a single linear parameter in the combination of one Poisson experiment with one Gaussian experiment. Appendix F examines the consistency of converting Poisson to chi-squared distributions in the case of combining two Poisson distributions whose averages depend on one linear parameter.

II. METHOD OF JOINING TWO CHI-SQUARED DISTRIBUTIONS

First we show the result that will allow us to join Poisson distributions for the averages when we relate them to chi-squared distributions. We show that the chi-squared distributions convolute to form a joint chi-squared distribution.

The basic chi-squared distribution with N degrees of freedom is

$$ f_N(\chi^2) = \frac{(\chi^2)^{N/2-1}e^{-\chi^2/2}}{2^{N/2}\Gamma(N/2)}, \quad (1) $$

with norm

$$ 1 = \int_0^\infty d\chi^2 f_N(\chi^2). \quad (2) $$

The convolution integral for combining two chi-squared distributions for $N_1$ and $N_2$ to produce a joint chi-squared distribution is

$$ f_N(\chi^2) = \int_0^{\chi^2} d\chi_1^2 f_{N_1}(\chi_1^2) f_{N_2}(\chi^2 - \chi_1^2) \quad (3) $$

where $\chi_2^2$ is replaced by $(\chi^2 - \chi_1^2)$. By substituting chi-squared distributions in the above, and changing variable the integration variable to $t = \chi_1^2/\chi^2$, we get

$$ f_N(\chi^2) = \frac{1}{2^{(N_1+N_2)/2}\Gamma(N_2/2)\Gamma(N_2/2)} e^{-\chi^2/2(N_1+N_2)/2} (\chi^2)^{(N_1+N_2)/2-1} \int_0^1 dt \ t^{N_1/2-1}(1-t)^{N_2/2-1}. \quad (4) $$
Using the formula for the $t$ integral, which is a beta function equal to $\Gamma(N_1/2)\Gamma(N_2/2)/\Gamma((N_1+N_2)/2)$, one sees that the result $f_N(\chi^2)$ is the chi-squared distribution function for $N = N_1 + N_2$. (The analogous formula for joining two Poisson distributions, with averages $\bar{n}_1$ and $\bar{n}_2$ to produce $n_t$ total events is

$$P(n_t; \bar{n}_t) = \sum_{n_1=0}^{n_t} P(n_1; \bar{n}_1)P(n_t - n_1; \bar{n}_2), \tag{5}$$

where $\bar{n}_t = \bar{n}_1 + \bar{n}_2$.)

III. POISSON DISTRIBUTION AND BAYES THEOREM FOR LIMITING $\bar{N}$

According to Bayes’ Theorem \cite{1, 2, 3}, the probability for a given “theoretical parameter average” $\bar{n}$ given an observed number of events $n$, $P(\bar{n}; n)$, is proportional to the probability of observing $n$ events from a Poisson distribution with an average number of events $\bar{n}$, or $P(n; \bar{n})$ \cite{4}. The latter is

$$P(n; \bar{n}) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}. \tag{6}$$

The probability distribution for $\bar{n}$, $P(\bar{n}; n)$, is proportional to this \cite{4}, subject to the normalization condition that the probability for all possible $\bar{n}$ should integrate to unity

$$\int_0^\infty d\bar{n}P(\bar{n}; n) = 1. \tag{7}$$

This is satisfied by the formula for $P(n; \bar{n})$ without further renormalization, since the integral is seen to be the form for $\Gamma(n+1)/n! = 1$. Thus we have the normalized distribution for $\bar{n}$ which we call the Bayesian Poisson distribution for the average \cite{4}.

$$P(\bar{n}; n) = \frac{\bar{n}^n e^{-\bar{n}}}{n!}. \tag{8}$$

IV. CONNECTION OF THE BAYESIAN POISSON DISTRIBUTION FOR THE AVERAGE TO A CHI-SQUARED DISTRIBUTION

We will show a mapping of the variables $(\bar{n}, n)$ from a Bayesian Poisson distribution for the average to $(\chi^2, N)$ for a chi-squared distribution that keeps the identical probability
distribution and integration of the Poisson distribution, but is now in a chis-squared form. This may be used by itself using usual chi-squared probabilities and contours, or included with other chi-squared joined experiments by the convolution integral in section 2.

The chi-squared distribution to be integrated over \(d\chi^2\) for \(N\) degrees of freedom is

\[
f_N(\chi^2) = \frac{1}{2\Gamma(N/2)} e^{-\chi^2/2} \left(\frac{\chi^2}{2}\right)^{N/2 - 1}.
\]

(9)

This is identical to the \(\bar{n}\) distribution to be integrated over \(\bar{n}\)

\[
P(\bar{n}; n) = \frac{1}{n!} e^{-\bar{n} \bar{n}^n}
\]

(10)

with the identification of

\[
\bar{n} = \frac{\chi^2}{2}, \quad \text{or} \quad \chi^2 = 2\bar{n},
\]

(11)

and

\[
n = N/2 - 1, \quad \text{or} \quad N = 2(n + 1).
\]

(12)

The equivalency of the two forms is noted in the Particle Data Group article on statistics [7], but they do not use it to merge experiments into a chi-squared distribution. The identity includes the integrals over ranges of probabilities in \(\bar{n}\) or equivalently in \(\chi^2\) using

\[
d\bar{n} = \frac{1}{2} d\chi^2
\]

(13)

Thus a Poisson with \(n\) events now counts mathematically as a chi-squared distribution with \(N = 2n + 2\) degrees of freedom.

If the prior probability is of a logarithmic, power law preserving form preferred by statisticians, \(P(\bar{n}) = 1/\bar{n}\), then the normalized Bayesian Poisson distribution for the average is directly seen to be the same as that for the uniform prior for \(n - 1\) events, \(P(\bar{n}; n - 1)\) [4].

Since the Poisson form was the only requirement for the above connection between Bayesian Poisson and chi-squared distributions, the results still hold for the logarithmic prior, but with \(n\) replaced by \(n - 1\), so that \(N_{P-log} = 2n\).
For cases with an unknown mean signal number of events $\bar{n}_S$ plus an exact known background average $\bar{B}$, the Bayesian Poisson distribution for the mean $(\bar{n}_S + \bar{B})$ when $n_T$ events are observed is [8]

$$P(\bar{n}; n_T) = \frac{(\bar{n}_S + \bar{B})^{n_T} e^{-(\bar{n}_S + \bar{B})}}{\Gamma(n_T + 1, \bar{B})},$$

(14)

where $\Gamma(n_T + 1, \bar{B})$ is the incomplete Gamma function. This results from the normalization over only non-negative values of $\bar{n}_S$. However, this factor could ruin the simple convolution properties on which this paper is based. In cases where $\bar{B}$ is small and $n_T$ a few events, this correction is small and the $\Gamma(n_T + 1, \bar{B})$ can be replaced by $n_T!$ with little error, and the simple formulas of this paper can again be used with $n = n_T$ and $\bar{n} = \bar{n}_T = \bar{n}_S + \bar{B}$. To see when this occurs we note that

$$\Gamma(n_T + 1, \bar{B}) = n_T! \left(1 + \frac{\bar{B}}{2!} + \frac{B^2}{2!^2} + \ldots + \frac{B^{n_T}}{n_T!}\right) e^{-\bar{B}}.$$

(15)

For small $\bar{B}$ the above correction factor to $n_T!$ has leading term $(1 - \bar{B}^{n_T+1}/(n_T+1)!)$, giving hope of its being small if $n_T$ is not very small and $\bar{B}$ is. One way to state this is to give the value of $\bar{B}$ for each $n_T$ at which the correction factor becomes a given value. The following Table I gives the values at which the correction factor becomes 5% and 1%.

| $n_T$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-------|---|---|---|---|---|---|---|---|---|---|----|
| 5%    | 0.05 | 0.36 | 0.82 | 1.37 | 1.97 | 2.61 | 3.29 | 3.98 | 4.70 | 5.43 | 6.17 |
| 1%    | 0.01 | 0.15 | 0.44 | 0.82 | 1.28 | 1.79 | 2.33 | 2.91 | 3.51 | 4.13 | 4.77 |

V. DERIVATION OF JOINT PROBABILITY FOR A BAYESIAN POISSON DISTRIBUTION FOR THE AVERAGE AND A CHI-SQUARED DISTRIBUTION

Here we demonstrate the derivation of the product probability for the case of one Poisson distribution for the average with a chi-squared distribution for $\chi^2_G$ with $N_G$ degrees of freedom formed either from Gaussians or from joint Gaussian and Poisson distributions. The integrated product probability is

6
\[ 1 = \int_0^\infty d\bar{n}P(\bar{n}; n) \int_0^\infty d\chi^2_G f_{N_G}(\chi^2_G) \]  

We convert the integral over the average \( \bar{n} \) to the variable \( \chi^2_P = 2\bar{n} \) and rewrite using section 4

\[ d\bar{n}P(\bar{n}; n) = f_{N_P}(\chi^2_P) d\chi^2_P \]  

with \( N_P = 2n + 2 \). Into the new integral we now introduce the total \( \chi^2 \) by inserting

\[ 1 = \int_0^\infty d\chi^2 \delta(\chi^2 - \chi^2_P - \chi^2_G) \]  

and use this to do the \( d\chi^2 \) integral, which limits \( \chi^2_P \leq \chi^2 \) and gives

\[ 1 = \int_0^\infty d\chi^2 \int_0^{\chi^2} d\chi^2_P f_{N_P}(\chi^2_P) f_{N_G}(\chi^2 - \chi^2_P). \]  

By the chi-squared convolution integral, the second integral is \( f_{N_P+N_G}(\chi^2) \), which is the resultant probability distribution for this case, with \( \chi^2 = \chi^2_P + \chi^2_G \). The result is an exact joint chi-squared probability combining a Poisson experiment with a chi-squared distribution from previously combined experiments.

**VI. MERGING BAYESIAN POISSON AND CHI-SQUARED DISTRIBUTIONS**

Now that we have a \( f_N(\chi^2) \) distribution Eq.(9) that is equivalent to a Bayesian Poisson parameter distribution in value and in its probability integral, we can merge this (independent of its origin) with other chi-squared distributions using Eq. (3), the convolution, to obtain the final \( \chi^2 \) distribution.

The results can now be used, for example, in finding \( \chi^2 \) contours corresponding to various confidence levels. We must remember that a single Poisson experiment with a uniform prior now counts as \( N = 2(n + 1) \) degrees of freedom, where \( n \) is the number of observed events in the Poisson distribution. While this sounds counter-intuitive, we recall that the form of the \( \chi^2 \) distribution that we are using also has \( \chi^2 \) replaced by \( 2\bar{n} \), and with the above replacements, \( \chi^2 \) per degree of freedom \( N \) or \( \chi^2/N = 2\bar{n}/(2(n + 1)) \), approaches 1 at large \( n \) since \( n \) is within \( \sqrt{n} \) of \( \bar{n} \).
If $M_P$ is the number of Poisson experiments with $n_i$ events in the $i$’th experiment, we associate with each $N_i = 2n_i + 2$ degrees of freedom. We call the associated theoretical Poisson averages $\bar{n}_i$. The total Poisson degrees of freedom becomes

$$N_P = \sum_{i=1}^{M_P} N_i = \sum_{i=1}^{M_P} (2n_i + 2). \quad (19)$$

With the alternate choice of a logarithmic prior, $N_{P-log} = \sum_{i=1}^{M_P} 2n_i$. We now convolute the Poisson distributions for the average in the chi-squared forms, Eqs. (8-17) with the chi-squared distribution of $N_G$ Gaussian experimental degrees of freedom which have a chi-squared $\chi^2_G$. The result will use the joint chi-square

$$\chi^2_{PG} = 2\sum_{i=1}^{M_P} \bar{n}_i + \chi^2_G. \quad (20)$$

From successive convolutions in Eq. (3), the combined chi-squared distribution for the Poisson plus Gaussian distributions is finally

$$f_{(N_G+N_P)}(\chi^2_{PG}). \quad (21)$$

We emphasize that these results are an exact treatment, not involving large $n$ or other approximations. As in the standard treatment, if $N_{par}$ is the number of parameters that are being fitted, then the number of degrees of freedom is $dof = N = N_G + N_P - N_{par}$. In Appendix B we show how the $\chi^2$ limits at various confidence levels for two-sided distributions are related to Poisson sums. In Appendix B we give an expanded Table II that can be used for two-sided $\chi^2$ limits at given confidence levels. In Appendix C an expanded table for single-sided $\chi^2$ values or $\chi^2$ contours for $N$ up to 25 corresponding to various confidence levels. In the respective appendices we also give Mathematica programs to be used for larger $N$ or other confidence levels.

This method has been applied in analyzing the constraints of many experiments on new flavor changing neutral current models of CP violation in $B$ meson decay asymmetries [9]. There, all experiments have a Gaussian distribution, except for an experiment [10] where one event has been seen in $K^+ \rightarrow \pi^+\nu\bar{\nu}$ and is treated with an additional $\chi^2_P = 2\bar{n}$ and
adding four degrees of freedom. In that case, \( \bar{n} \) is a function of the down quark mixing matrix elements as are the other experiments. That analysis also provides an example of the sensitivity to the choice of a uniform or logarithmic prior probability distribution. With the uniform prior, the total number of degrees of freedom is seven, and the chi-squared limits are at 8.2, 12.0, and 14.3 for 1-\( \sigma \), 90\% (1.64-\( \sigma \)), and 2-\( \sigma \) confidence levels, respectively. With the logarithmic prior, the total number of degrees of freedom decreases by two to five, and the chi-squared limits are at 5.89, 9.24, and 11.3, for the 1-\( \sigma \), 90\%, and 2-\( \sigma \) confidence levels, respectively. The chi-squared per degree of freedom ratios stay within 10\% of each other between the two cases. However, use of the logarithmic prior does move the contours in by two to three units or about 1/2 of a standard deviation, and thus gives tighter bounds.

Parenthetically we add that in the limit of large \( n \) and \( \bar{n} \), just as the Poisson distribution becomes a Gaussian, so does the equivalent chi-squared distribution. The chi-squared distribution in Eq. (9) becomes

\[
G(\xi, \sigma) = \frac{e^{-\xi^2/2\sigma^2}}{\sqrt{2\pi\sigma}},
\]

where in our variables \( \sigma = \bar{n}^{1/2} = (\chi^2/2)^{1/2} \), \( \xi = n - \bar{n} = (N/2 - 1) - \chi^2/2 \), and \( d\chi^2 = -2d\xi \).

Since \( |\xi| \) is confined to the order of \( \sigma \) for large \( n \) and \( \bar{n} \), the difference \( |N/2 - 1 - \chi^2/2| \) is confined to the order of \( \sqrt{\chi^2/2} \) or \( \sqrt{N/2} \) for large \( N \) and \( \chi^2 \).

VII. NEUTRINO OSCILLATION EXPERIMENTS

Here we shall see that using the combined Poisson method for small numbers of events per bin leads to a result which considers only the total number of events in a single Poisson distribution, and makes the two methods identical.

A. Appearance Neutrino Oscillation Experiments

For example, we consider a \( \nu_\mu \rightarrow \nu_e \) appearance experiment. Let \( n_i^0 \) be the number of expected \( \nu_\mu \) in the i’th bin at energy \( E_i \), \( n_i \) the number of observed events in that bin, and \( b_i \)
the known background in that bin. With the two neutrino oscillation formula, the average number of electrons in that bin will be

$$\bar{n}_i = n^0_i \sin^2 (2\theta) \sin^2 (1.27\delta m^2 L/E_i).$$  \hspace{1cm} (23)

By the method of expressing Bayesian Poisson’s in the chi-squared formalism, we get the total chi-squared as a linear sum of expected events for each bin from Eq. 14, if the $b_i$ are sufficiently small

$$\chi^2 = \sum_i (2\bar{n}_i + 2b_i).$$  \hspace{1cm} (24)

Also, the number of degrees of freedom is twice the total number of observed events $n$ when using the logarithmic prior

$$N_{P-log} = 2 \sum n_i \equiv 2n.$$  \hspace{1cm} (25)

The sum of background events is denoted by $B = \sum b_i$.

With small bin size $\Delta E_i$, $n^0_i = (dn/dE)\Delta E_i$, and the sum of the expected number of events at full mixing can be converted into an integral

$$n^0(\delta m^2) \equiv \int dE \frac{dn}{dE} \sin^2 \left( \frac{1.27\delta m^2 L}{E} \right).$$  \hspace{1cm} (26)

So we now have a binning independent form for $\chi^2$ from the sum over bins

$$\chi^2 = 2 \sin^2 (2\theta)n_0(\delta m^2) + 2B$$  \hspace{1cm} (27)

and a binning independent number of degrees of freedom $N = 2n$. The probability distribution is now

$$f_N(\chi^2) = f_{2n}(2 \sin^2 (2\theta)n_0(\delta m^2) + 2B).$$  \hspace{1cm} (28)

We set 90% CL limits using a one-sided CL if there is no signal, and a two-sided CL if there is a signal. For the one-sided CL limit, the average background $\bar{B}$ has to be less than or equal to 0.05 events for the $n_T = 0$ Poisson to be accurately normalized.
Going backwards from a chi-squared distribution to its equivalent Poisson distribution, this chi-squared result is equivalent to a Bayesian Poisson distribution for the average with

\[ \bar{n} = \chi^2/2 = \sin^2(2\theta)n_0(\delta m^2) + B, \]  

(29)

with \( n \) events observed. This is the same as the usual approach of grouping all events into one bin of the total number of events, which is used if there are few events. As in the case of Eq. 14, \( B \) must be small enough not to significantly affect the normalization.

**B. Disappearance Neutrino Oscillation Experiments**

For a disappearance experiment, the expected number of events per bin is

\[ \bar{n}_i = n^0_i(1 - \sin^2(2\theta)\sin^2(1.27\delta m^2 L/E_i)). \]  

(30)

Using the same sums as for the appearance experiment, and defining the sum of the coefficients of the 1 term or the total number of expected neutrino events without oscillation as

\[ n^0 = \int dE \frac{dn}{dE}, \]  

(31)

we have the probability distribution

\[ f_N(\chi^2) = f_{2n}(2n^0 - 2\sin^2(2\theta)n^0(\delta m^2) + 2B). \]  

(32)

If the total number of events is large enough to use a Gaussian approximation, these are then the same results as using a single Gaussian in the usual method for comparing the total number of events with and without oscillation. But even with a limited total number of events, the formulas above with a chi-squared distribution are an improvement over a Gaussian, as long as the background \( b_i \) are small enough in each bin.
C. General Comments on Oscillation Results

What we have achieved is that for a small number of events, we have found the $\chi^2_P$ for the Poisson neutrino appearance and disappearance experiments, Eqs. (27) and (32), respectively, that can be added to $\chi^2$ from other Poisson or Gaussian neutrino experiments to determine neutrino oscillation and mixing parameters using standard $\chi^2$ methods with $2n$ extra degrees of freedom as in the logarithmic prior case. The drawback is that the result is equivalent to a comprehensive bin in energy containing all events. When the number of particles per each energy bin becomes significant, it is better to use a Gaussian for each bin to derive information contained in the detailed energy spectrum.

D. One-Sided Chi-squared Limits on Oscillation

We find a contour in the $(\sin^2(2\theta), \delta m^2)$ plane where for the probability distribution $f_{2n}(\chi^2)$ the amount of probability contained in the major part is the confidence level $CL$. The appropriate one-sided chi-squared limits $\chi^2_{CL^+}(2n)$ for $n$ observed events and $N_P = 2(n + 1)$ for a uniform prior or $N_{P-log} = 2n$ for a logarithmic prior are found in Table III of Appendix C.

1. Appearance Experiment

In practice, for each $\delta m^2$ we find the value of $\sin^2(2\theta)$ such that the bound becomes an equality

$$2 \sin^2(2\theta) n_0(\delta m^2) + 2B \leq \chi^2_{CL^+}(2n).$$

$CL^+$ means that for a 95% CL limit, only 5% is left off of the upper part of the distribution. The excluded region is where the left-hand-side is larger than the chi-squared upper CL limit, giving an upper bound on $\sin^2(2\theta)$. 
2. Disappearance Experiment

Here, the excluded region is where the left-hand-side is smaller than the chi-squared lower CL limit, and the allowed region is

\[ 2n^0 - 2\sin^2(2\theta)n^0(\delta m^2) + 2B \geq \chi^2_{CL}(2n), \] (34)

which again restricts the result with an upper bound on \( \sin^2(2\theta) \).

E. Large \( n \) Gaussian Approximation

While the previous results were accurate for small \( b_i \), for large \( n \) we may use the approximation that the chi-squared distribution resembles a Gaussian distribution near its peak

\[ f_{2n}(\chi^2) \approx \frac{1}{\sqrt{2\pi n}} \exp\left(\frac{-(n - \chi^2/2)^2}{2n}\right). \] (35)

A one-sided 95% CL limit which leaves 5% on one side, is at the same deviation (in \( \chi^2/2 \)) from the center of the Gaussian as the usual two-sided 90% CL limit which leaves 5% on both sides. This occurs at

\[ |n - \chi^2/2| = 1.64\sigma = 1.64\sqrt{n}. \] (36)

This yields the chi-squared limits below. Since the multiplier term of \( \sin^2(2\theta) \) can average to a half or be less than that, values of \( \sin^2(2\theta) \) greater than one can be reached in these limits, and they must be cut off at one.

For appearance experiments, the two sided 90% CL limits are

\[ \sin^2(2\theta) \leq \frac{n + 1.64\sqrt{n} - B}{n^0(\delta m^2)}. \] (37)

For disappearance experiments, the two sided 90% CL limits are

\[ \sin^2(2\theta) \leq \frac{n^0 + B - n + 1.64\sqrt{n}}{n^0(\delta m^2)}. \] (38)

For one-sided 90% CL limits we use 1.28\( \sigma \).
VIII. CONCLUSIONS

In conclusion, we have shown how the simplicity of the Bayesian \( \chi^2 \) analysis can be exactly extended to include experiments with a small number of events which are described by a Bayesian Poisson distribution for the average. This precise analytic treatment (provided the background is small) is useful since it uses the simple chi-squared treatment for all experiments, even if some experiments have too few events to be a standard Gaussian. We have provided useful tables for the method by extending them to larger \( n \) to accompany the larger number of degrees of freedom used. We have analyzed neutrino oscillation experiments and showed how the analytic combination of Poisson bins through the equivalent chi-squared distributions leads to the standard Poisson result for the total number of events. However, using the equivalence to a chi-squared distribution, we have found the appropriate \( \chi^2_P \) to add to the \( \chi^2 \) from other experiments to use standard \( \chi^2 \) methods.

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APPENDIX A: COMPARISON WITH OTHER FORMULAS USED FOR POISSON PARAMETER LIMITS

For completeness we include here some properties of and a comparison between the classical (or frequentist) and Bayesian Poisson limits on \( \bar{n} \). The methods are given full discussion by R. D. Cousins in Ref. 1. The classical Poisson parameter distribution used for the upper \( \bar{n} \) limit is to sum the Poisson distributions \( P(n; \bar{n}) \) from \( n + 1 \) events to infinity, when the number of observed events is \( n \), and use it as the probability for \( \bar{n} \) when \( \bar{n} \) is greater than \( n \). We show that the Bayesian Poisson parameter distribution Eq. (8)
integrated from zero to a cutoff $n_c$ agrees with the above formulation. First we do the integrated probability for $\bar{n}$ from $n_c$ to infinity by integrating $e^{-\bar{n}}$ by parts

$$I(n_c; n) = \int_{n_c}^{\infty} d\bar{n} \frac{\bar{n}^n}{n!} e^{-\bar{n}}$$  \hspace{1cm} (A1)

$$= \left[ \frac{\bar{n}^n}{n!} (-e^{-\bar{n}}) \right]_{n_c}^{\infty} + \int_{n_c}^{\infty} d\bar{n} \frac{\bar{n}^{n-1}}{(n-1)!} e^{-\bar{n}}$$  \hspace{1cm} (A2)

$$= P(n; n_c) + I(n_c; n-1).$$  \hspace{1cm} (A3)

Continued integration by parts shows that the integral over a semi-infinite interval beginning at $n_c$ of the Bayesian Poisson parameter distribution is

$$I(n_c; n) = P(n; n_c) + P(n-1; n_c) + \ldots + P(0; n_c).$$  \hspace{1cm} (A4)

The two methods are now seen to be equivalent using $\bar{n} = n_c$ and the fact that the Poisson terms sum to 1

$$\sum_{n'=n+1}^{\infty} P(n'; \bar{n}) = 1 - \sum_{n'=0}^{n} P(n'; \bar{n})$$  \hspace{1cm} (A5)

$$= 1 - I(\bar{n}; n)$$  \hspace{1cm} (A6)

$$= \int_{0}^{n_c} d\bar{n} \frac{\bar{n}^n}{n!} e^{-\bar{n}}.$$  \hspace{1cm} (A7)

from Eqs. (7) and (A3).

For $n$ the number of observed events, the rule for the “1-σ” upper limit on $n_c$ is to find $n_c^+$ such that 84% of the time there would be greater than $n$ events. Since “1-σ” means 32% is outside the central region, 16% should occur on one side. Thus the sum from $n + 1$ to infinity is set equal to 0.84

$$\sum_{n'=n+1}^{\infty} P(n'; n_c^+) = \int_{0}^{n_c^+} d\bar{n} \frac{\bar{n}^n}{n!} e^{-\bar{n}} = 1 - I(n_c^+, n) = 0.84$$  \hspace{1cm} (A8)

from Eq. (A6). So for the upper “1-σ” limit, $n_c^+$, both the Bayesian result of setting the integral of the Poisson distribution for the average in Eq.(A7) equal to 0.84 and the sum of higher $n$ agree.

For the lower 1-σ, the classical rule of setting the sum from 0 to $n - 1$ equal to 0.84 to determine $n_c^-$ (or the sum from $n$ to inf set to 0.16) gives
\[ \sum_{n'=0}^{n-1} P(n'; n^-) = I(n^-; n-1) = 0.84. \] (A9)

This is not the same as setting the integral of the Bayesian Poisson distribution for the average from 0 to \( n^- \) equal to 0.16

\[ 1 - I(n^-; n) = \int_0^{n^-} \frac{\bar{n}^n}{n!} e^{-\bar{n}} = 0.16 \quad \text{or} \quad I(n^-; n) = 0.84 \] (A10)

from Eq. (A7). To see the difference, we note from Eq. (A3)

\[ I(n^-; n) = P(n; n^-) + I(n^-; n-1). \] (A11)

With the prior chosen to be \( 1/\bar{n} \), the lower limits agree but not the upper limit.

**APPENDIX B: TABLE OF BAYESIAN POISSON CENTRAL LIMITS FOR THE AVERAGE AND TWO-SIDED CHI-SQUARED LIMITS**

The Bayesian Poisson average central interval limits with uniform prior are the upper or lower \( n_c^\pm \) limits as in Eq. (A8) or Eq. (A10) beyond which the confidence level is below a given value. This is in analogy with the \( \bar{x} \pm \sigma \) one \( \sigma \) limits in a single Gaussian distribution, where half of the excluded intervals on each side are used in the integral limits (0.16 on each side for 1\( \sigma \)). The following Table II covers lower and upper limits out to 3\( \sigma \), and for \( n = 0 \) to \( n = 24 \).

Comparing Eq. A1 with the results of section 4 we have the relation between the Poisson integral over the average and the equivalent chi-squared integral at a given confidence level, say \( CL^+ \)

\[ I(n_c^+; n) = \int_{n_c^+}^\infty \frac{\bar{n}^n}{n!} e^{-\bar{n}} = \int_{(\chi_2^2)^+}^\infty d\chi^2 f_N(\chi^2) = CL^+, \] (B1)

with \( (\chi_2^2)^+ = 2n_c^+ \) and \( N = 2(n+1) \). For the lower confidence level limits

\[ 1 - I(n_c^-; n) = \int_0^{n_c^-} \frac{\bar{n}^n}{n!} e^{-\bar{n}} = \int_0^{(\chi_2^2)^-} d\chi^2 f_N(\chi^2) = CL^- . \] (B2)

So in both cases, we can get the \( \chi^2 \) limits from Table II also by using
\( (\chi^2)^{\pm} = 2n_c^{\pm}. \) \hspace{1cm} (B3)

Table II was produced from the following Mathematica program (except for the n column), which can be used to extend the table as needed. It also shows the actual confidence levels used for the various column designations in the program.

\[
<< \text{Statistics'ContinuousDistributions'}
\]

\[
\text{cl} = \{0.0013499, 0.01, 0.0227501, 0.1, 0.158655, 0.5, 0.841345, 0.9, 0.9772500, 0.99, 0.9996500\}
\]

\[
\text{navgtable} := \text{N[Table[0.5*Quantile[ChiSquareDistribution[k],cl[[i]]],\{k,4,50,2\},\{i,1,11\}],4]}
\]

\[
\text{TeXForm[navgtable//TableForm].}
\]

For \( n = 0 \) events observed, the one-sided confidence interval upper bounds are meaningful as opposed to two-sided intervals. The upper limits of intervals starting from zero which contain 0.6827, 0.90, 0.95, 0.9545, 0.99, and 0.9973 probability are 1.15, 2.30, 3.00, 3.09, 4.61, and 5.9, respectively. G. J. Feldman and R. D. Cousins use an approach which carefully covers both single and double-sided cases [3].
| n  | -3σ | 0.01  | -2σ | 0.1   | -1σ | 0.5   | 1σ   | 0.9   | 2σ   | 0.99  | 3σ   |
|----|------|-------|-----|-------|-----|-------|------|-------|------|-------|------|
| 1  | 0.05288 | 0.1486 | 0.2301 | 0.5318 | 0.7082 | 1.678 | 3.300 | 3.890 | 5.683 | 6.638 | 10.39 |
| 2  | 0.2117 | 0.4360 | 0.5963 | 1.102  | 1.367  | 2.674 | 4.638 | 5.322 | 7.348 | 8.406 | 12.47 |
| 3  | 0.4653 | 0.8232 | 1.058  | 1.745  | 2.086  | 3.672 | 5.918 | 6.681 | 8.902 | 10.05 | 14.38 |
| 4  | 0.7919 | 1.279  | 1.583  | 2.433  | 2.840  | 4.671 | 7.163 | 7.994 | 10.39 | 11.60 | 16.18 |
| 5  | 1.175  | 1.785  | 2.153  | 3.152  | 3.620  | 5.670 | 8.382 | 9.275 | 11.82 | 13.11 | 17.90 |
| 6  | 1.603  | 2.330  | 2.758  | 3.895  | 4.419  | 6.670 | 9.584 | 10.53 | 13.22 | 14.57 | 19.56 |
| 7  | 2.068  | 2.906  | 3.391  | 4.656  | 5.232  | 7.669 | 10.77 | 11.77 | 14.59 | 16.00 | 21.17 |
| 8  | 2.563  | 3.507  | 4.046  | 5.432  | 6.057  | 8.669 | 11.95 | 12.99 | 15.94 | 17.40 | 22.75 |
| 9  | 3.084  | 4.130  | 4.719  | 6.221  | 6.891  | 9.669 | 13.11 | 14.21 | 17.27 | 18.78 | 24.30 |
| 10 | 3.628  | 4.771  | 5.409  | 7.021  | 7.734  | 10.67 | 14.27 | 15.41 | 18.58 | 20.14 | 25.82 |
| 11 | 4.191  | 5.428  | 6.113  | 7.829  | 8.585  | 11.67 | 15.42 | 16.60 | 19.87 | 21.49 | 27.32 |
| 12 | 4.772  | 6.099  | 6.828  | 8.646  | 9.441  | 12.67 | 16.56 | 17.78 | 21.16 | 22.82 | 28.80 |
| 13 | 5.367  | 6.782  | 7.555  | 9.470  | 10.30  | 13.67 | 17.70 | 18.96 | 22.43 | 24.14 | 30.26 |
| 14 | 5.977  | 7.477  | 8.291  | 10.30  | 11.17  | 14.67 | 18.83 | 20.13 | 23.70 | 25.45 | 31.70 |
| 15 | 6.599  | 8.181  | 9.036  | 11.14  | 12.04  | 15.67 | 19.96 | 21.29 | 24.95 | 26.74 | 33.13 |
| 16 | 7.233  | 8.895  | 9.789  | 11.98  | 12.92  | 16.67 | 21.08 | 22.45 | 26.20 | 28.03 | 34.55 |
| 17 | 7.877  | 9.616  | 10.55  | 12.82  | 13.80  | 17.67 | 22.20 | 23.61 | 27.44 | 29.31 | 35.95 |
| 18 | 8.530  | 10.35  | 11.32  | 13.67  | 14.68  | 18.67 | 23.32 | 24.76 | 28.68 | 30.58 | 37.34 |
| 19 | 9.193  | 11.08  | 12.09  | 14.53  | 15.57  | 19.67 | 24.44 | 25.90 | 29.90 | 31.85 | 38.72 |
| 20 | 9.863  | 11.83  | 12.87  | 15.38  | 16.45  | 20.67 | 25.55 | 27.05 | 31.13 | 33.10 | 40.10 |
| 21 | 10.54  | 12.57  | 13.65  | 16.24  | 17.35  | 21.67 | 26.66 | 28.18 | 32.34 | 34.35 | 41.46 |
| 22 | 11.23  | 13.33  | 14.44  | 17.11  | 18.24  | 22.67 | 27.76 | 29.32 | 33.55 | 35.60 | 42.82 |
| 23 | 11.92  | 14.09  | 15.23  | 17.97  | 19.14  | 23.67 | 28.87 | 30.45 | 34.76 | 36.84 | 44.17 |
| 24 | 12.62  | 14.85  | 16.03  | 18.84  | 20.03  | 24.67 | 29.97 | 31.58 | 35.96 | 38.08 | 45.51 |

Table II: Bayesian Poisson Central Limits for the Averages $n_c^-$ and $n_c^+$
APPENDIX C: TABLE FOR CHI-SQUARED VALUES AT VARIOUS
CONFIDENCE LEVELS

Since the joint method for \( n \) events requires \( \chi^2 \) for \( N = 2(n + 1) + N_G - N_{par} \) for a uniform prior, or for \( N = 2n + N_G - N_{par} \) for a logarithmic prior, both of which can be large, we give here a table of chi-squared values for various confidence levels for large \( N \) up to 25, and a program with which one can generate further limits.

In the following table, \( N \) is the number of degrees of freedom, and the designations of 1, 2, and 3 \( \sigma \) correspond to 1-CL of 0.682689, 0.954500, and 0.997300, respectively. The Mathematica program used to generate the table is

\[
\text{\textless\textless Statistics'}\text{\textquoteleft\textquoteleft ContinuousDistributions'}
\]
\[
\text{cl} = \{0.682689, 0.9, 0.954500, 0.99, 0.997300\}
\]
\[
\text{cstable := N[Table[Quantile[ChiSquareDistribution[k], cl[[i]]], \{k, 1, 25\}, \{i, 1, 5\}], 4]}
\]
\[
\text{TeXForm[cstable//TableForm].}
\]
| N  | 1σ  | 0.90 | 2σ  | 0.99 | 3σ  |
|----|-----|------|-----|------|-----|
| 1  | 1.00| 2.706| 4.000| 6.635| 9.000|
| 2  | 2.296| 4.605| 6.180| 9.210| 11.83|
| 3  | 3.527| 6.251| 8.025| 11.34| 14.16|
| 4  | 4.719| 7.779| 9.716| 13.28| 16.25|
| 5  | 5.888| 9.236| 11.31| 15.09| 18.21|
| 6  | 7.038| 10.64| 12.85| 16.81| 20.06|
| 7  | 8.176| 12.02| 14.34| 18.48| 21.85|
| 8  | 9.304| 13.36| 15.79| 20.09| 23.57|
| 9  | 10.42| 14.68| 17.21| 21.67| 25.26|
|10  | 11.54| 15.99| 18.61| 23.21| 26.90|
|11  | 12.64| 17.28| 19.99| 24.72| 28.51|
|12  | 13.74| 18.55| 21.35| 26.22| 30.10|
|13  | 14.84| 19.81| 22.69| 27.69| 31.66|
|14  | 15.94| 21.06| 24.03| 29.14| 33.20|
|15  | 17.03| 22.31| 25.34| 30.58| 34.71|
|16  | 18.11| 23.54| 26.65| 32.00| 36.22|
|17  | 19.20| 24.77| 27.95| 33.41| 37.70|
|18  | 20.28| 25.99| 29.24| 34.81| 39.17|
|19  | 21.36| 27.20| 30.52| 36.19| 40.63|
|20  | 22.44| 28.41| 31.80| 37.57| 42.08|
|21  | 23.51| 29.62| 33.07| 38.93| 43.52|
|22  | 24.59| 30.81| 34.33| 40.29| 44.94|
|23  | 25.66| 32.01| 35.58| 41.64| 46.36|
|24  | 26.73| 33.20| 36.83| 42.98| 47.76|
|25  | 27.80| 34.38| 38.07| 44.31| 49.16|
APPENDIX D: SOLUTION FOR CHI-SQUARED EXPANSION ABOUT THE MINIMUM FOR THE LINEAR PARAMETER DEPENDENCE CASE

For the case where the theoretical values for the mean in the Gaussian and Poisson distributions are linear in parameters to be fitted, the minimum of $\chi^2$ and its quadratic expansion about the minimum can be found analytically using the same method as for pure Gaussian distributions [12,13]. While this may prove useful, in the usage here, however, the maximal probability of the $\chi^2$ distribution is not at the minimum $\chi^2$, but at $\chi^2 \approx n$.

In the method of expressing Poisson distributions for the average as $\chi^2$ distributions in this paper, the final $\chi^2_{GP}$ is

\[
\chi^2_G = \sum_{i=1}^{N_G} \left( \frac{y_i - F_i(\alpha)}{\sigma_i^2} \right)^2, \quad \text{and}
\]

\[
\chi^2_{GP} = \chi^2_G + 2 \sum_{\ell=1}^{N_P} \bar{n}_\ell(\alpha), \quad \text{(D2)}
\]

where $\alpha$ is the set of $k$ parameters $\alpha_m$. The experiments described by $(y_i, F_i)$ can even be totally different, and the $F_i$ and $\bar{n}_\ell$ are assumed to be linearly expandable in the parameters $\alpha_m$

\[
F_i(\alpha) = \sum_{n=1}^{k} \alpha_n f_{in}, \quad \text{and}
\]

\[
\bar{n}_\ell(\alpha) = \sum_{j=1}^{k} n_{\ell j} \alpha_j. \quad \text{(D4)}
\]

Minimizing $\chi^2_{GP}$ with respect to each $\alpha_m$ gives rise to the vector $g$ and matrix $V^{-1}$ with components

\[
g_m = \sum_{i=1}^{N_G} y_i \frac{f_{im}}{\sigma_i^2} - \sum_{\ell=1}^{N_P} n_{\ell m}, \quad \text{and}
\]

\[
V^{-1}_{mn} = \sum_{i=1}^{N_G} \frac{f_{im} f_{in}}{\sigma_i^2}. \quad \text{(D6)}
\]

Using the inverse matrix $V$, the values of the parameters that give the minimum $\chi^2_{GP}$ are given by

\[
\hat{\alpha} = Vg, \quad \text{(D7)}
\]
APPENDIX E: SOLUTION OF ONE BAYESIAN POISSON DISTRIBUTION WITH ONE GAUSSIAN DISTRIBUTION AND ONE LINEAR PARAMETER

We present here the solution for the single linear parameter case with one Bayesian Poisson and one Gaussian distribution. For the unknown parameter \( a \), we have the theoretical relations \( \bar{n} = ac_P \) for the Poisson average, and \( \bar{x} = ac \) with known standard deviation \( \sigma \) for the Gaussian average, where coefficients \( c_P \) and \( c \) are given, and \( n \) and \( x \) are the results of the respective experiments. Then

\[
\chi^2_{PG} = 2\bar{n} + (x - \bar{x})^2/\sigma^2 \tag{E1}
\]

With one parameter to be fitted, the number of joint degrees of freedom with the equivalent chi-squared method with a uniform prior is \( N = 2n + 2 + 1 - 1 = 2n + 2 \) where one degree of freedom is cancelled by the one parameter. For the logarithmic prior, \( N = 2n + 1 - 1 = 2n \), which gives tighter \( \chi^2 \) limits.

The minimum of \( \chi^2_{PG} \) occurs at

\[
\bar{a}c = x - \sigma^2c_P/c \tag{E2}
\]

giving the minimum chi-squared

\[
\chi^2_{min} = 2x(c_P/c) - \sigma^2(c_P/c)^2. \tag{E3}
\]

When \( \chi^2_{PG} \) is set equal to a certain upper limit boundary at \( \chi^2_{lim} \), there are bounds on the range of \( a \) given by
\[ a_{\text{lim}}^\pm = \bar{a} \pm \frac{\sigma}{\sqrt{c}} \sqrt{\chi^2_{\text{lim}} - \chi^2_{\text{min}}}. \]  

(E4)

For physical reasons we may want \( \bar{a} \) to be positive when \( c_P \) and \( c \) are positive. Looking at \( \bar{x} = \bar{a}c \) above, we see that \( \bar{x} \) and \( \bar{a} \) are positive when \( x\bar{x}/\sigma^2 \geq \bar{n} \). In order to use a Gaussian, we expect at least a 3-\( \sigma \) separation of the peak from zero, or \( x/\sigma \geq 3 \) and \( \bar{x}/\sigma \geq 3 \). Thus for \( \bar{n} \leq 9 \), this method works and \( \bar{a} \geq 0 \). For \( \bar{n} \geq 9 \), \( \bar{n}/\sqrt{\bar{n}} \geq 3 \) and we can start using a Gaussian instead of a Poisson for the \( \bar{n} \) experiment. The same reasoning follows through if for example we require a 5-\( \sigma \) separation from zero to use a Gaussian.

**APPENDIX F: TWO POISSON DISTRIBUTIONS WITH ONE LINEAR PARAMETER**

We approach this problem both from Bayes theorem directly, and from converting the Bayesian Poisson distributions to chi-squared distributions as proposed in this paper. For the latter we then merge the chi-squared distributions to a single chi-square distribution for the linear parameter and then convert that back to a joint Poisson distribution, to compare to the direct approach. For the case of the logarithmic prior we find consistency.

The averages of the experiments are theoretically given by the parameter \( a \) with respective known coefficients \( \bar{n}_1 = ac_1 \) and \( \bar{n}_2 = ac_2 \). The direct Bayesian result is proportional to the probability for observing the experimental values \( n_1 \) and \( n_2 \) given a value of \( a \)

\[
\text{Prob}(a; n_1, n_2) = P(n_1; ac_1)P(n_2; ac_2)P(a)/(P(n_1)P(n_2)) \\
\propto (ac_1)^{n_1}(ac_2)^{n_2}e^{-ac_1}e^{-ac_2}P(a) \\
\propto (a(c_1 + c_2))^{(n_1+n_2)}e^{-a(c_1+c_2)}P(a) \\
\propto P(a(c_1 + c_2); n_1 + n_2)P(a). \tag{F1}
\]

For the uniform prior, \( P(a) = 1 \), the normalized result is \( P(a(c_1 + c_2); n_1 + n_2) \), integrating over \( da(c_1 + c_2) \). For the logarithmic prior with \( P(a) = 1/a \), the normalized result is the same as the uniform prior with total \( n \) lowered by 1, or \( P(a(c_1 + c_2); n_1 + n_2 - 1) \), integrating over \( da(c_1 + c_2) \).
If we now start with the method in this paper, we take the joint Bayesian result as the product of the Bayesian Poisson for each experiment as if they were independent, \( P(ac_1; n_1)P(ac_2; n_2) \) times either the uniform prior \( d\bar{n}_1d\bar{n}_2 \) or the logarithmic prior \( d\bar{n}_1d\bar{n}_2/(\bar{n}_1\bar{n}_2) \). The logarithmic prior is equivalent to \( P(ac_1; n_1 - 1)P(ac_2; n_2 - 1) \) with a uniform prior. Converting the uniform case to chi-squared distributions gives the convolution of the product \( f_{2n_1+2}(2ac_1)f_{2n_2+2}(2ac_2) \) leading to \( f_{2n_1+2n_2+4}(2ac_1 + 2ac_2) \). Converting this back to a Poisson distribution for the average gives \( P(ac_1 + ac_2; n_1 + n_2 + 1) \) for the uniform prior, which is inconsistent with the direct uniform Bayesian result in the previous paragraph. For the logarithmic prior, converting to chi-squared distributions gives the convolution of the product \( f_{2n_1}(2ac_1)f_{2n_2}(2ac_2) \) which is \( f_{2n_1+2n_2}(2ac_1 + 2ac_2) \). Converting this back to a Poisson distribution for the average gives

\[
P(ac_1 + ac_2; n_1 + n_2 - 1) \propto P(ac_1 + ac_2; n_1 + n_2) \frac{da(c_1 + c_2)}{a(c_1 + c_2)},
\]

which is consistent with the direct Bayesian result for the logarithmic prior in the previous paragraph.

In the combined form as a single Bayesian Poisson distribution for the average, both upper and lower limits on \( a \) for a given central confidence interval can be found using the table in appendix B. The case where no events were observed in either experiment can also be dealt with using one-sided bounds, which are also given in appendix B.
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[4] This is similar to the approach used by S. Baker and R. D. Cousins, Nucl. Inst. and Meth. 221 437, (1984), except that they use a likelihood function method which is the ratio of the Poisson distribution to the Poisson distribution with the true mean.

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[12] For the pure Gaussian case, see for example Jon Mathews and R. L. Walker, Mathematical Methods of Physics, Second Edition, Section 14-7, Addison-Wesley (1970).
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