FUNDAMENTAL SOLUTION OF KINETIC FOKKER-PLANCK OPERATOR 
WITH ANISOTROPIC NONLOCAL DISSIPATIVITY

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Abstract. By using the probability approach (the Malliavin calculus), we prove the existence of smooth fundamental solutions for degenerate kinetic Fokker-Planck equation with anisotropic nonlocal dissipativity, where the dissipative term is the generator of an anisotropic Lévy process and the drift term is allowed to be cubic growth.

1. Introduction and Main Result

Consider the following second order stochastic differential equation (SDE) in $\mathbb{R}^d$:

$$\frac{d^2X_t}{dt^2} = -\nabla V(X_t) + \frac{dW_t}{dt} - \frac{dX_t}{dt}, \quad X_0 = x,$$ (1.1)

where $V(x) : \mathbb{R}^d \to \mathbb{R}_+$ is a smooth function, and $(W_t)_{t \geq 0}$ is a $d$-dimensional standard Brownian motion. In phase space $\mathbb{R}^d \times \mathbb{R}^d$, the position and velocity vector field $(X_t, \dot{X}_t)$ solves the following degenerate SDE:

$$\begin{align*}
&\left\{ \begin{array}{ll}
    dX_t = \dot{X}_t dt, & X_0 = x, \\
    d\dot{X}_t = -\nabla V(X_t) dt - \dot{X}_t dt + dW_t, & \dot{X}_0 = v.
\end{array} \right.
\end{align*}$$ (1.2)

The celebrated Hörmander’s hypoellipticity theorem tells us that $(X_t, \dot{X}_t)$ admits a smooth density $\rho_{x,v}(t, x', v')$ (cf. [11, 16, 19, 18, 20]). Moreover, by Itô’s formula, one knows that $\rho_{t,v}(t, x', v')$ solves the following kinetic Fokker-Planck equation:

$$\partial_t \rho - v' \cdot \nabla_x \rho + \nabla V \cdot \nabla_v \rho = \Delta_v \rho - \text{div}(v' \rho).$$

It is easy to check that the equilibrium of this equation is given by

$$\rho_{\infty}(x, v) := \exp[-H(x, v)], \quad \text{where} \quad H(x, v) := \frac{|v|^2}{2} + V(x).$$

The rate of convergence to the equilibrium for the kinetic Fokker-Planck equation has been deeply studied in [8, 10, 27, 9], etc. Moreover, the stochastic flow property of SDE (1.2) was proven in [11, 30].

In this work, we shall consider equation (1.1) with Brownian motion $(W_t)_{t \geq 0}$ replaced by a Lévy process $(L_t)_{t \geq 0}$ (for example, the cylindrical $\alpha$-stable process). More generally, we consider the following stochastic Hamiltonian system driven by Lévy process:

$$\begin{align*}
&\left\{ \begin{array}{ll}
    dX_t = b_1(X_t, \dot{X}_t) dt, & X_0 = x, \\
    d\dot{X}_t = b_2(X_t, \dot{X}_t) dt + dL_t, & \dot{X}_0 = v,
\end{array} \right.
\end{align*}$$ (1.3)

where $b = (b_1, b_2)$ is a smooth vector field on phase space $\mathbb{R}^d \times \mathbb{R}^d$. The background about stochastic Hamiltonian system and related Fokker-Planck equation is refereed to [25]. From the microscopic viewpoint, stochastic equation (1.3) can be considered that the motion of particles is perturbed by a “discontinuous” stochastic force. We want to study the regularizing effect of Lévy noise to the system. It is well known that there are a lot of works devoting to the study of smooth densities for SDEs with jumps (see [6, 5, 21, 26, 12, 7, 14, 3], etc.). Nevertheless,
most of these works required that the jump noise is non-degenerate, and the main arguments
are based upon developing an analogue of the Malliavin calculus for jump diffusions.

The main goal of the present paper is to prove that under some assumptions on \( b \) and \( L_t \),
the solution \((X_t, \dot{X}_t)\) of SDE (1.3) still has a smooth density. When \( b \) has bounded derivatives
of all orders, \( (\nabla_s b_1)(\nabla_s b_1)^\top \) is uniform positive with respect to \((x, v)\), and \( L_t \) is an isotropic
\( \alpha \)-stable process, the smoothness of \( \rho \) was proved in [32]. However, in real model such as
stochastic oscillators, the nonlinear term \( b \) is usually non-Lipschitz, and the Lévy noise may be
anisotropic as that each component of \( L_t \) is independent.

Below, we first describe the noise \( L_t \) following [14]. Let \((L_t)_{t \geq 0}\) be a \( d \)-dimensional Lévy
process with the following form (called subordinated Brownian motion):
\[
L_t := W_{S_t} = \left( W_{S_t}^1, \cdots, W_{S_t}^d \right),
\]
where \( S_t = (S_t^1, \cdots, S_t^d) \) is an independent \( d \)-dimensional \( \mathbb{R}_+^d \)-valued Lévy process with
characteristic triple \((\vartheta, 0, \nu_S)\), more precisely, its Laplace transform is given by
\[
\mathbb{E}(e^{-z \cdot S_t}) = \exp \left\{ -t \vartheta \cdot z + \int_{\mathbb{R}_+^d} (e^{-z u} - 1) \nu_S(du) \right\},
\]
where \( \vartheta \in \mathbb{R}_+^d \) and the Lévy measure \( \nu_S \) satisfies
\[
\int_{\mathbb{R}_+^d} (1 \wedge |u|) \nu_S(du) < \infty.
\]
In particular, each component \( S_t^i \) is a subordinator (cf. [4, 24]). By easy calculations, one can
see that the characteristic function of \( L_t \) is given by
\[
\mathbb{E}e^{i z \cdot L_t} = \exp \left\{ -t \sum_k \vartheta_k z_k |z_k|^2 + t \int_{\mathbb{R}_+^d} (e^{iz y} - 1 - iz \cdot y 1_{|y| 
\]
\[
|y| < 1) \nu_L(du) \right\},
\]
where \( \nu_L \) is the Lévy measure given by
\[
\nu_L(\Gamma) = \int_{\mathbb{R}_+^d} \left( \int_{\Gamma} \left( \frac{(2 \pi)^{-d/2}}{1 \cdot \cdots \cdot u_d} e^{-\frac{y_1^2}{2u_1^2} - \cdots - \frac{y_d^2}{2u_d^2}} dy_1 \cdots dy_d \right) \nu_S(du_1, \cdots, du_d) \right)\nu_S(du).
\]
Here we use the convention that if \( u_i = 0 \) for some \( i \), then the inner integral is calculated
with respect to the degenerate Gaussian distribution. In particular, \( \nu_L \) may not be absolutely
continuous with respect to the Lebesgue measure. Obviously, \( \nu_L \) is a symmetric measure.

Now we state the main assumptions on \((\vartheta, \nu_S)\) and \( b\):

\((H^1_{nu_S})\) Let \( \phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be defined by
\[
\phi(\varepsilon) := \min_{i=1,\ldots,d} \left( \vartheta_i + \frac{1}{\varepsilon} \int_{|u| < \varepsilon} u_i \nu_S(du) \right).
\]
We assume that for some \( \theta \in (0, 1], \)
\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{\theta-1} \phi(\varepsilon) > 0.
\]

\((H^2_{nu_S})\) We assume that for any \( p > 0, \)
\[
\int_{|u| > 1} e^{p |u|} \nu_S(du) < \infty,
\]
which, by [24, p.159, Theorem 25.3], is equivalent to
\[
\mathbb{E}e^{\rho S_1} < \infty.
\]
(H₉) Assume that there exists a Lyapunov function \( H : \mathbb{R}^d_x \times \mathbb{R}^d \rightarrow \mathbb{R}_+ \) with
\[
|\nabla_x H|^2 \leq C_1 H, \quad |\nabla^2_x H| \leq C_2,
\]
and such that for any \( m \in \{0\} \cup \mathbb{N} \) and some \( q_m > 0 \),
\[
b \cdot \nabla H \leq C_3 H, \quad |\nabla^m b| \leq C_m(H^{q_m} + 1),
\]
where \( q_1 \in [0, \frac{1}{2}] \). Moreover,
\[
|\nabla_x b| + |\nabla^2_x b| + |\nabla^3 b| \leq C_4,
\]
and for any row vector \( u \in \mathbb{R}^d \),
\[
|u \nabla_x b(x, v)|^2 \geq C_5 |u|^2.
\]

**Remark 1.1.** Let \((S^i)_{i=1,\ldots,d}\) be independent \( \alpha_i \)-stable subordinators, where \( \alpha_i \in (0, 1) \). It is easy to check that \((1.8)\) holds with \( \theta = \min(\alpha_1, \ldots, \alpha_d) \).

**Remark 1.2.** In the case of equation \((1.1)\), we can take
\[
H(x, v) = \frac{1}{2}|v|^2 + V(x),
\]
where \( V \in C^\infty(\mathbb{R}^d; \mathbb{R}_+) \) satisfies that for any \( m \in \mathbb{N} \) and some \( q_m > 0 \),
\[
|\nabla^m V(x)| \leq C_m(V(x)^{q_m} + 1),
\]
with \( q_2 = \frac{1}{2} \). In particular, \( V(x) = |x|^4 \) satisfies this assumption. Since for any \( p > 1 \), compared with \((1.10)\), it holds in general that (cf. \cite{24} p.168, Theorem 26.1)
\[
\mathbb{E}e^{S^p} = \infty,
\]
we have to require \( q_1 \in [0, \frac{1}{2}] \) in \((1.12)\) (see Theorem 3.7 below).

The main result of this paper is:

**Theorem 1.3.** Under \((H_1^1)\), \((H_1^2)\) and \((H_9)\), there exists a smooth density \( \rho_{x,v}(t, x', v') \) to SDE \((1.3)\) with bounded derivatives of all orders with respect to \( x', v' \) and such that
\[
\partial_t \rho = \mathcal{L}_v \rho + \text{div}_v(b \rho), \quad r > 0,
\]
with \( \rho_{x,v}(0, x', v') = \delta_{x,v}(x', v') \), where
\[
\mathcal{L}_v f(v) = \text{P.V.} \int_{\mathbb{R}^d} (f(v + y) - f(v))\nu_1(dy) + \frac{1}{2} \sum_k (\partial^2_k f)(v) \theta_k,
\]
where P.V. stands for the Cauchy principal value. Moreover, there exist constants \( \beta_1, \beta_2, \beta_3 > 0 \) only depending on \( d, \theta \) and a positive continuous function \( (x, v) \mapsto C_{x,v} \) such that for all \((t, (x', v'), (x, v)) \in (0, 1] \times (\mathbb{R}^d_x \times \mathbb{R}^d)^2\),
\[
\rho_{x,v}(t, x', v') \leq C_{x,v} \left( t^{-\beta_1} \left( 1 + \frac{t^{\beta_2}}{|x - x'| + |v - v'|^{\beta_3}} \right) \right).
\]

**Remark 1.4.** If \( b \in C^\infty_b(\mathbb{R}^d) \), then the above \( C_{x,v} \) can be constant. In this case, if one only requires the existence of smooth density, then assumption \((H_1^2)\) can be dropped by using the same argument as in \cite{32} Section 3.3.
In order to prove this theorem, by taking regular conditional expectations with respect to $S_i$, we shall regard the solution of SDE (1.3) as a Wiener functional, and then use the classical Malliavin calculus to prove Theorem 1.3. Such an idea was first used by Léandre [17], and then in [14, 31]. We also mention that a derivative formula of Bismut type and the Harnack inequality for SDEs driven by $\alpha$-stable processes were derived in [31] and [28] following the same idea. It is quite interesting to have an analytic proof of Theorem 1.3. It should be noticed that the Lévy measure $\nu_L$ could be very singular. This leads to that the symbol of operator $L_v$ may not be $C^1$-continuous differentiable on $\mathbb{R}^d \setminus \{0\}$. Thus, the classical pseudo-differential operator theory seems not applicable (cf. [11]). Below, we list some open questions for further studies:

- Can we prove the same result for multiplicative Lévy noise?
- Is it possible to remove the assumptions $q_1 \in [0, \frac{1}{2}]$ in (1.12) and $(H^\nu_S)$?
- Is there a stationary distribution for stochastic Hamiltonian system (1.3)? If yes, how about the rate of convergence as $t \to \infty$?

This work is organized as follows: In Section 2, we prepare some notations and lemmas for later use. In particular, a Norris’ type lemma is proven. In Section 3, we prove some exponential moment estimate about the SDE driven by $W_S$ and with polynomial growth coefficients. In Section 4, we calculate the Malliavin covariance matrix for the solution of SDE as a Wiener functional. In Section 5, we prove the smoothness of distributional density of a degenerate SDE driven by $W_S$. Meanwhile, we conclude the proof of Theorem 1.3. Before concluding this introduction, we collect some notations or conventions for later use.

- Write $\mathbb{R}^d_+ = [0, \infty)^d$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.
- The inner product in Euclidean space is denoted by $\langle x, y \rangle$ or $x \cdot y$.
- For a vector $x = (x_1, \cdots, x_d)$, we write $|x| := \left( \sum_i |x_i|^2 \right)^{1/2} \sim \sum_i |x_i|$.
- $C^0_\nu(\mathbb{R}^d)$: The space of all smooth functions with compact support.
- $S(\mathbb{R}^d)$: The Schwartz space of rapidly decreasing smooth functions.
- $C^0_p(\mathbb{R}^d)$: The space of all smooth bounded functions with bounded derivatives of all orders.
- $C^p_0(\mathbb{R}^d)$: The space of all smooth functions, which together with the derivatives of all orders are at most polynomial growth.
- The asterisk $*$ denotes the transpose of a matrix or a column vector, or the dual operator.
- $\nabla$ denotes the gradient operator, and $D$ the Malliavin derivative operator.
- $C$ with or without index will denotes an unimportant positive constant.

2. Preliminaries

We first introduce the canonical space of subordinated Brownian motion $W_S$. Let $(\mathcal{W}, \mathcal{H}, \mu_\mathcal{W})$ be the classical Wiener space, i.e., $\mathcal{W}$ is the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$ with vanishing values at starting point 0, $\mathcal{H} \subset \mathcal{W}$ is the Cameron-Martin space consisting of all absolutely continuous functions with square integrable derivatives, $\mu_\mathcal{W}$ is the Wiener measure so that the coordinate process

$$W_t(w) := w_t$$

is a $d$-dimensional standard Brownian motion. Let $\mathcal{S}$ be the space of all càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d_+$ with $\ell_0 = 0$, where each component is increasing. Suppose that $\mathcal{S}$ is endowed with the
Skorohod metric and the probability measure \( \mu_S \) so that the coordinate process
\[
S_t(\ell) := \ell_t = (\ell^1_t, \cdots, \ell^d_t)
\]
is a \( d \)-dimensional Lévy process with Laplace transform (1.4). Consider the following product probability space
\[
(\Omega, \mathcal{F}, P) := \left( \mathcal{M} \times \mathcal{S}, \mathcal{B}(\mathcal{M}) \times \mathcal{B}(\mathcal{S}), \mu_M \times \mu_S \right),
\]
and define for \((w, \ell) \in \mathcal{M} \times \mathcal{S},\)
\[
L_\ell(w, \ell) := w_\ell := \left( w_1(\ell_1(t)), \cdots, w_d(\ell_d(t)) \right).
\]
Then \((L_\ell)_{\ell \geq 0}\) is a Lévy process with characteristic function (1.5). We use the following filtration:
\[
\mathcal{F}_t := \sigma\{W_{S_s}, S_s : s \leq t\}.
\]
Clearly, for \( t > s, W_{S_s} - W_{S_t} \) and \( S_t - S_s \) are independent of \( \mathcal{F}_s \).

2.1. An exponential estimate of \( S_t \). The following estimate of exponential type about \( S_t \) will play an important role in the proof of Theorem 1.3

**Lemma 2.1.** Let \( f : \mathbb{R}_+ \to \mathbb{R}_+^d \) be a continuous \( \mathcal{F}_t \)-adapted process. For any \( R, \varepsilon, \delta > 0 \), we have
\[
P \left\{ \int_0^{\tau_R} f_s \cdot dS_s \leq \varepsilon; \int_0^{\tau_R} |f_s| ds > \delta \right\} \leq e^{1 - \phi(\varepsilon/R; \delta/\varepsilon)},
\]
where \( \tau_R := \inf\{t \geq 0 : |f_t| > R\} \) and \( \phi \) is defined by (1.7).

**Proof.** For \( \lambda > 0 \), set
\[
g^\lambda_s := \int_{\mathbb{R}_+^d} (1 - e^{-\lambda f_s \cdot u}) \nu_S(du)
\]
and
\[
M^\lambda_t := -\lambda \int_0^t f_s \cdot dS_s + \lambda \int_0^t f_s \cdot \vartheta ds + \int_0^t g^\lambda_s ds.
\]
Let \( \mu(t, du) \) be the Poisson random measure associated with \( S_t \), i.e.,
\[
\mu(t, \Gamma) := \sum_{i \in \mathbb{N}} 1_{\Gamma}(S_{\tau_i} - S_{\tau_i^-}), \quad \Gamma \in \mathcal{B}(\mathbb{R}_+^d),
\]
Let \( \tilde{\mu}(t, du) \) be the compensated Poisson random measure of \( \mu(t, du) \), i.e.,
\[
\tilde{\mu}(t, du) = \mu(t, du) - tv_S(du).
\]
Then, by Lévy-Itô’s decomposition (cf. [24]), we can write
\[
S_t = t \left( \vartheta + \int_{|u| \leq 1} u \nu_S(du) \right) + \int_{|u| \leq 1} u \tilde{\mu}(t, du) + \int_{|u| > 1} u \mu(t, du), \tag{2.1}
\]
and so,
\[
\int_0^t f_s \cdot dS_s = \int_0^t f_s \cdot \left( \vartheta + \int_{|u| \leq 1} u \nu_S(du) \right) ds
\]
\[
+ \int_0^t \int_{|u| \leq 1} f_s \cdot u \tilde{\mu}(ds, du) + \int_0^t \int_{|u| > 1} f_s \cdot u \mu(ds, du).
\]
By Itô’s formula (cf. [2]), we have
\[
e^{M^\lambda_t} = 1 + \int_{\mathbb{R}_+^d} e^{M^\lambda_t} \{ e^{-\lambda f_s \cdot u} - 1 \} \tilde{\mu}(ds, du). \tag{2.2}
\]
Since for any \( x > 0 \),
\[
1 - e^{-x} \leq 1 \land x,
\]
we have

$$g_s^4 \leq \int_{\mathbb{R}^d} (1 \wedge (\lambda f_s \cdot u))\nu_S(du)$$

and

$$M_{t\wedge R}^4 \leq \lambda \int_0^{t\wedge R} f_s \cdot \theta ds + \int_0^{t\wedge R} g_s^4 ds \leq tR|\theta| + t \int_{\mathbb{R}^d} (1 \wedge (\lambda R|u|))\nu_S(du).$$

Hence, by (2.2) we have

$$\mathbb{E}e^{M_{t\wedge R}^4} = 1.$$

On the other hand, since for any $\kappa \in (0, 1)$ and $x \leq -\log k$,

$$1 - e^{-x} \geq \kappa x,$$

letting $\kappa = \frac{1}{e}$, we have for $s \leq \tau_R$,

$$\lambda f_s \cdot \theta + g_s^4 \geq \lambda f_s \cdot \theta + \int_{|u| \leq \frac{1}{\lambda R}} (1 - e^{-\lambda f_s \cdot u})\nu_S(du)$$

$$\geq \lambda f_s \cdot \theta + \frac{1}{e} \int_{|u| \leq \frac{1}{\lambda R}} (\lambda f_s \cdot u)\nu_S(du)$$

$$= \lambda f_s \cdot \left(\theta + \frac{1}{e} \int_{|u| \leq \frac{1}{\lambda R}} u\nu_S(du)\right)$$

$$\geq \lambda \phi(1/(\lambda R))|f_s|,$$

where $\phi$ is defined by (1.7). Thus,

$$\left\{ \int_0^{t\wedge R} f_s \cdot dS_s \leq \varepsilon; \int_0^{t\wedge R} |f_s| ds > \delta \right\}$$

$$\subset \left\{ e^{M_{t\wedge R}^4} \geq e^{-\lambda \phi(1/(\lambda R))|f_s| + \int_0^{t\wedge R} (\lambda f_s \cdot \theta + g_s^4)ds}; \int_0^{t\wedge R} (\lambda f_s \cdot \theta + g_s^4)ds \geq \lambda \phi(1/(\lambda R))\delta \right\}$$

$$\subset \left\{ e^{M_{t\wedge R}^4} \geq e^{-\lambda \phi(1/(\lambda R))\delta} \right\},$$

which then implies the result by Chebyshev’s inequality and letting $\lambda = \frac{1}{e}$. \qed

2.2. A Norris’ type lemma. Let $N(t, dy)$ be the Poisson random measure associated with $L_t = W_S$, i.e.,

$$N(t, \Gamma) = \sum_{s \leq t} 1_{\Gamma}(L_s - L_{s-}), \quad \Gamma \in \mathcal{B}(\mathbb{R}^d).$$

Let $\tilde{N}(t, dy)$ be the compensated Poisson random measure of $N(t, dy)$, i.e.,

$$\tilde{N}(t, dy) = N(t, dy) - t\nu_L(dy),$$

where $\nu_L$ is the Lévy measure of $L_t$ given by (1.6). By Lévy-Itô’s decomposition, we have

$$L_t = W_S = W_{\theta t} + \int_{|u| \leq 1} y\tilde{N}(t, dy) + \int_{|u| > 1} yN(t, dy),$$

where we have used that for any $0 < r < R < \infty$,

$$\int_{r < |u| < R} y\nu_L(dy) = 0.$$

Notice that $(W_{\theta t})_{t \geq 0}$, $\left( \int_{|u| \leq 1} y\tilde{N}(t, dy) \right)_{t \geq 0}$ and $\left( \int_{|u| > 1} yN(t, dy) \right)_{t \geq 0}$ are independent.

Recall the following result about the exponential estimate of martingales (cf. [20] p.352, (A.5) and [7] Lemma 1).
Lemma 2.2. Let $\delta, R, \eta, T > 0$.

(i) Let $M_t$ be a continuous square integrable martingale, then

$$P \left\{ \sup_{s \in [0, T]} |M_s| \geq \delta; \langle M \rangle_T < \eta \right\} \leq 2 \exp \left\{ -\frac{\delta^2}{2\eta} \right\}.$$  

(ii) Let $f(y)$ be a bounded $\mathcal{F}_t$-predictable process with bound $R$, then

$$P \left\{ \sup_{s \in [0, T]} \left| \int_0^T f(s) \tilde{N}(ds, dy) \right| \geq \delta, \int_0^T |f(s)|^2 \nu_L(dy)ds < \eta \right\} \leq 2 \exp \left( -\frac{\delta^2}{2(R\delta + \eta)} \right).$$

The following lemma is contained in the proof of Norris’ lemma (cf. [20, p.137] and [31]).

Lemma 2.3. For $T > 0$, let $f$ be a bounded measurable $\mathbb{R}^d$-valued function on $[0, T]$. Assume that for some $\varepsilon < T$ and $x \in \mathbb{R}^d$,

$$\int_0^T \left| x + \int_0^t f_s ds \right|^2 dt \leq \varepsilon^3. \tag{2.4}$$

Then we have

$$\sup_{t \in [0, T]} \left| \int_0^t f_s ds \right| \leq 2(1 + ||f||_\infty)\varepsilon.$$  

We now prove the following Norris’ type lemma (cf. [19, 20, 7, 31]).

Lemma 2.4. Let $Y_t = y + \int_0^t \beta_s ds$ be an $\mathbb{R}^d$-valued process, where $\beta_t$ takes the following form:

$$\beta_t = \beta_0 + \int_0^t \gamma_s ds + \int_0^t Q_s dW_{\theta_s} + \int_0^t \int_{\mathbb{R}^d} g_s(y) \tilde{N}(ds, dy),$$

where $\gamma_t : \mathbb{R}_+ \to \mathbb{R}^d$, $Q_t : \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ and $g_t(y) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are three left continuous $\mathcal{F}_t$-adapted processes. Suppose that for some left continuous $\mathcal{F}_t$-adapted $\mathbb{R}_+^d$-valued process $\alpha_t$,

$$|g_t(y)| \leq \alpha_t(1 \wedge |y|). \tag{2.5}$$

Then there exists a constant $C \geq 1$ such that for any $t \in (0, 1)$, $\delta \in (0, \frac{1}{t})$, $\varepsilon \in (0, t^3)$ and $R \geq 1$,

$$P \left\{ \tau_R > t, \int_0^t |Y_s|^2 ds < \varepsilon, \int_0^t |\beta_s|^2 ds \geq 9R^2\varepsilon^\delta \right\} \leq 4 \exp \left\{ -\frac{\varepsilon^\delta - \varepsilon}{CR^4} \right\}, \tag{2.6}$$

where

$$\tau_R := \inf \left\{ t \geq 0 : |\beta_t| + |\gamma_t| + |Q_t| + \alpha_t > R \right\}.$$

Proof. Let us define

$$h_t := \int_0^t \beta_s ds, \quad M^c := \int_0^t \langle h_s, Q_s dW_{\theta_s} \rangle, \quad M^d := \int_0^t \int_{\mathbb{R}^d} \langle h_s, g_s(y) \rangle \tilde{N}(ds, dy),$$

and

$$E_1 := \left\{ \int_0^t |Y_s|^2 ds < \varepsilon \right\}, \quad E_2 := \left\{ \sup_{s \in [0, t]} |h_s| \leq 4R\varepsilon^\frac{1}{2} \right\},$$

$$E_3 := \left\{ \langle M^c \rangle_t \leq C_0 R^4 \varepsilon^\frac{3}{2} \right\}, \quad E_4 := \left\{ \sup_{s \in [0, t]} |M^c_s| \leq \frac{\varepsilon^\frac{3}{2}}{2} \right\},$$

$$E_5 := \left\{ \langle M^d \rangle_t \leq C_1 R^4 \varepsilon^\frac{3}{2} \right\}, \quad E_6 := \left\{ \sup_{s \in [0, t]} |M^d_s| \leq \frac{\varepsilon^\frac{3}{2}}{2} \right\},$$

$$E_7 := \left\{ \int_0^t |\beta_s|^2 ds < 9R^2\varepsilon^\delta \right\}.$$
where $C_0$ and $C_1$ are two constants determined below.

First of all, by Lemma 2.3, one sees that for $\varepsilon < T^3$,

$$
\{\tau_R > t\} \cap E_1 \subset \{\tau_R > t\} \cap E_2 \subset \{\tau_R > t\} \cap E_3 \cap E_5,
$$

where the second inclusion is due to

$$
\langle M^c \rangle_t = \int_0^t |\langle h_s, Q_s \rangle|^2 \, ds \leq (4R^2)^2 |\theta|^2 \varepsilon^{\frac{\varepsilon}{R}} =: C_0 R^4 \varepsilon^{\frac{\varepsilon}{R}}
$$

and

$$
\langle M^d \rangle_t = \int_0^t \int_{\mathbb{R}^d} |\langle h_s, g_s(y) \rangle|^2 \nu_L(dy) \, ds \leq (4R^2)^2 R^2 \left( \int_{\mathbb{R}^d} 1 \wedge |y|^2 \nu_L(dy) \right) \varepsilon^{\frac{\varepsilon}{R}} =: C_1 R^4 \varepsilon^{\frac{\varepsilon}{R}}.
$$

On the other hand, by integration by parts formula, we have

$$
\int_0^t |\beta_s|^2 \, ds = \int_0^t \langle \beta_s, dh_s \rangle = \langle \beta_t, h_t \rangle - \int_0^t \langle h_s, \gamma_s \rangle \, dt - M^c_t - M^d_t.
$$

From this, one sees that on $\{\tau_R > t\} \cap E_2 \cap E_4$,

$$
\int_0^t |\beta_s|^2 \, ds \leq 4R^2 \varepsilon^{\frac{\varepsilon}{R}} (1 + t) + \varepsilon^\delta \leq (8R^2 + 1) \varepsilon^\delta \leq 9R^2 \varepsilon^\delta.
$$

This means that

$$
\{\tau_R > t\} \cap E_2 \cap E_4 \cap E_6 \subset \{\tau_R > t\} \cap E_7,
$$

which together with (2.7) gives

$$
\{\tau_R > t\} \cap E_1 \cap E_7 \subset \{\tau_R > t\} \cap E_1 \cap (E_5^c \cup E_6^c)
$$

$$
\subset \left( \{\tau_R > t\} \cap E_3 \cap E_5 \right) \cup \left( \{\tau_R > t\} \cap E_5 \cap E_2 \cap E_6 \right).
$$

Thus, by Lemma 2.2 we have

$$
P(\{\tau_R > t\} \cap E_1 \cap E_7) \leq P(E_3 \cap E_5) + P(\{\tau_R > t\} \cap E_2 \cap E_5 \cap E_6)
$$

$$
\leq 2 \exp \left\{ \frac{\varepsilon^\delta}{8C_0 R^4} \right\} + 2 \exp \left\{ - \frac{\varepsilon^\delta}{8(4R^2 \varepsilon^{\frac{\varepsilon}{R}} + C_1 R^4 \varepsilon^{\frac{\varepsilon}{R}})} \right\}
$$

$$
\leq 2 \exp \left\{ - \frac{\varepsilon^\delta}{8C_0 R^4} \right\} + 2 \exp \left\{ - \frac{\varepsilon^\delta}{8(4 + C_1) R^4} \right\},
$$

and (2.6) follows by choosing $C := 8(C_0 \vee (4 + C_1))$. \hfill \square

### 2.3. Malliavin’s calculus

In this subsection we recall some basic notions and facts about the Malliavin calculus (cf. [15, 18, 20]). Let $\mathbb{U}$ be a real separable Hilbert space. Let $\mathcal{C}(\mathbb{U})$ be the class of all $\mathbb{U}$-valued smooth cylindrical functionals on $\Omega$ with the form:

$$
F = \sum_{i=1}^m f_i(W(h_1), \cdots, W(h_n)) u_i,
$$

where $f_i \in C^\infty_p(\mathbb{R}^n)$, $u_i \in \mathbb{U}$, $h_1, \cdots, h_n \in \mathbb{H}$ and

$$
W(h) = \int_0^\infty h_s \, dW_s.
$$

The Malliavin derivative of $F$ is defined by

$$
DF := \sum_{i=1}^m \sum_{j=1}^n (\partial_j f_i)(W(h_1), \cdots, W(h_n)) u_i \otimes h_j \in \mathbb{U} \otimes \mathbb{H}.
$$
By an iteration argument, for any \( k \in \mathbb{N} \), the higher order Malliavin derivative \( D^k F \) of \( F \) can be defined as a random variable in \( U \otimes \mathbb{H}^\otimes k \). It is well known that the operator \((D^k, \mathcal{C}(U))\) is closable from \( L^p(\Omega; U) \) to \( L^p(\Omega; U \otimes \mathbb{H}^\otimes k) \) for each \( p > 1 \) (cf. [20, p.26, Proposition 1.2.1]). For every \( p > 1 \) and \( k \in \mathbb{N} \), we introduce a norm on \( \mathcal{C}(U) \) by

\[
\|F\|_{k,p} := \left( \mathbb{E}[|F|^p] + \sum_{l=1}^k \mathbb{E}\left(\|D^l F\|_{L^p}^p\right)\right)^{\frac{1}{p}}.
\]

The Wiener-Sobolev space \( \mathbb{D}^{k,p}(U) \) is defined as the closure of \( \mathcal{C}(U) \) with respect to the above norm. Below we shall simply write

\[
\mathbb{D}^\infty(U) := \cap_{m \in \mathbb{N}, p \geq 1} \mathbb{D}^{m,p}(U)
\]

and

\[
\mathbb{D}^{k,p} := \mathbb{D}^{k,p}(\mathbb{R}^d), \quad \mathbb{D}^\infty := \mathbb{D}^\infty(\mathbb{R}^d).
\]

The dual operator \( D^* \) of \( D \) (also called divergence operator) is defined by

\[
\mathbb{E}(DF, U)_H = \mathbb{E}(FD^* U), \quad U \in \text{Dom}(D^*) = \mathbb{D}^{1,2}(\mathbb{H}).
\]

The following Meyer’s inequality holds (cf. [20, p.75, Proposition 1.5.4]). For any \( p > 1 \) and \( U \in \mathbb{D}^{1,p}(\mathbb{H}) \),

\[
\|D^* U\|_p \leq C_p \|U\|_{1,p}.
\]  

(2.8)

Let \( F = (F_1, \cdots, F_d) \) be a random vector in \( \mathbb{D}^{1,2} \). The Malliavin covariance matrix of \( F \) is defined by

\[
(S_{ij}) := \langle DF^i, DF^j \rangle_H.
\]

The following theorem about the criterion that a random vector admits a smooth density in the Malliavin calculus can be found in [20] p.100-103].

**Theorem 2.5.** Assume that \( F = (F_1, \cdots, F_d) \in \mathbb{D}^\infty \) is a smooth Wiener functional and satisfies that for all \( p \geq 2 \),

\[
\mathbb{E}[\left(\det \Sigma F\right)^{-p}] < \infty.
\]

Let \( G \in \mathbb{D}^\infty \) and \( \varphi \in C^\infty_p(\mathbb{R}^d) \). Then for any multi-index \( \alpha = (\alpha_1, \cdots, \alpha_m) \in \{1, 2, \cdots, d\}^m \),

\[
\mathbb{E}[\partial_\alpha \varphi(F) G] = \mathbb{E}[\varphi(F) H_\alpha(F, G)],
\]

where \( \partial_\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_m} \), and \( H_\alpha(F, G) \) are recursively defined by

\[
H(i)(F, G) := \sum_j D^j(G(S_{ij})^{-1}DF^i),
\]

\[
H_\alpha(F, G) := H_{(\alpha_m)}(F, H_{(\alpha_{m-1}), \cdots, \alpha_1})(F, G).
\]

As a consequence, for any \( p \geq 1 \), there exists \( p_1, p_2, p_3 > 1 \) and \( n_1, n_2 \in \mathbb{N} \) such that

\[
\|H_\alpha(F, G)\|_p \leq C(\|\partial_\alpha \varphi\|_{L^p}^{1/p_1} \|DF\|_{m,p_2} \|G\|_{m,p_3}).
\]

(2.10)

In particular, the law of \( F \) possesses an infinitely differentiable density \( \rho \in \mathcal{S}(\mathbb{R}^d) \).

About the estimate of the density, we recall the following result from Kusuoka-Stroock [15, Theorem 1.28].

**Theorem 2.6.** In the situation of Theorem 2.5, for any \( q > d \), there exists a constant \( C = C(q, d) > 0 \) such that for any \( \psi \in C^\infty(\mathbb{R}^d) \),

\[
\sup_{y \in \mathbb{R}^d} |\psi(y) \rho(y)| \leq C \|\psi(F)\|_q^{1-q} \left(\sum_i \|H(i)(F, 1)\|_q\right)^{d/q},
\]

(2.11)
provided that the right hand side is finite.

3. Exponential moment estimate for SDEs driven by $W_S$.

In this section, we mainly prove some estimates about the exponential moments for the solutions of SDEs with non-Lipschitz coefficients. Consider the following SDE driven by $W_S$:

$$dX_t = b(X_t)dt + AdW_S, \quad X_0 = x,$$  \hspace{1cm} (3.1)

where $b: \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function and $A = (a_{ij})$ is a constant $d \times d$ matrix.

Recall that a $C^2$-function $H : \mathbb{R}^d \to \mathbb{R}^+$ is called a Lyapunov function if

$$\lim_{|x| \to \infty} H(x) = \infty.$$ \hspace{1cm} (3.2)

We assume that for some Lyapunov function $H$ and $k_1, k_2, k_3 \geq 0$,

$$b(x) \cdot \nabla H(x) \leq k_1 H(x),$$  \hspace{1cm} (3.3)

and for all $k = 1, \cdots, d$,

$$\left| \sum_i \partial_i H(x) a_{ik} \right|^2 \leq k_2 H(x),$$ \hspace{1cm} (3.4)

$$\sum_{ij} \partial_i \partial_j H(x) a_{ik} a_{jk} \leq k_3.$$ \hspace{1cm} (3.5)

Moreover, we also assume the following local Lipschitz condition: for any $R > 0$ and all $x, y \in \mathbb{R}^d$ with $H(x), H(y) \leq R$,

$$|b(x) - b(y)| \leq C_R |x - y|.$$ \hspace{1cm} (3.6)

It should be noticed that stochastic differential equation (3.1) can not be solved by using $Y_t = X_t - AW_S$, to transform (3.1) into an ordinary differential equation with time-dependent coefficients, since the above conditions are not invariant under this transform. Moreover, a direct application of Itô’s formula seems not work because of the nonlocal feature of Lévy processes.

The main aim of this section is to prove the following estimate.

**Theorem 3.1.** Assume (3.3)-(3.6). For any initial value $x \in \mathbb{R}^d$, there exists a unique càdlàg $\mathcal{F}_t$-adapted process $t \mapsto X_t$ solving equation (3.1), and for all $t \geq 0$,

$$\mathbb{E} \left[ \exp \left( \frac{2 \sup_{s \in [0,t]} H(X_s)}{e^{k_1|S_s|} + 1} \right) \right] \leq C_{k_2,k_3} e^{H(x)}.$$ \hspace{1cm} (3.7)

Moreover, if we also assume $(H^2_{\ell_1})$, then for any $p \geq 1$ and $t \geq 0$,

$$\mathbb{E} \left[ \exp \left( p \sup_{s \in [0,t]} H(X_s)^\ell \right) \right] \leq C_{k_1,k_2,k_3,p,\ell} e^{H(x)}.$$ \hspace{1cm} (3.8)

**Proof.** First of all, by (3.6) it is standard to prove the existence and uniqueness of local solutions for equation (3.1). Our main aim is to prove the a priori estimate (3.7). We shall use the approximation argument as used in (31).

For fixed $\ell \in \mathbb{S}$, consider the following SDE driven by discontinuous martingale $W_\ell$:

$$dX^\ell_t = b(X^\ell_t)dt + AdW^\ell_t, \quad X^\ell_0 = x.$$ \hspace{1cm} (3.9)

Clearly, it suffices to prove that there exists a unique càdlàg function $t \mapsto X^\ell_t$ solving equation (3.9), and for all $t \geq 0$,

$$\mathbb{E} \left[ \exp \left( \frac{2 \sup_{s \in [0,t]} H(X^\ell_s)}{e^{k_1|\ell_s|} + 1} \right) \right] \leq C_{k_2,k_3} e^{H(x)}.$$ \hspace{1cm} (3.10)
Below, for the simplicity of notations, we drop the superscripts “$\ell$”, and divide the proof into four steps.

**(Step 1).** Let us first consider the case that each component of $\ell$ is absolutely continuous and strictly increasing. By Itô’s formula, (3.3) and (3.5), we have

$$e^{-\kappa \ell s}H(X_s) = H(x) + \int_0^s e^{-\kappa \ell t}(b \cdot \nabla H - \kappa_1 H)(X_t)dt + \int_0^s e^{-\kappa \ell t}\langle \nabla H(X_t), \text{Ad}W_t \rangle$$

$$+ \frac{1}{2} \sum_{i,j,k} \int_0^s e^{-\kappa \ell t} \partial_i \partial_j H(X_t)a_{ik}a_{jk}dt \ell^k_s$$

$$\leq H(x) + \int_0^s e^{-\kappa \ell t}\langle \nabla H(X_t), \text{Ad}W_t \rangle + \frac{K_3}{2} |\ell|.$$  \hspace{1cm} (3.11)

For $R > 0$, define the stopping time

$$\tau_R := \inf\{t \geq 0 : |X_t| \geq R\}.$$

Taking expectations for both sides of (3.11), we obtain that for all $t > 0$,

$$\mathbb{E}\left(e^{-\kappa \ell (\tau_R \land t)}H(X_{\tau_R \land t})\right) \leq H(x) + \frac{K_3}{2} |\ell|.$$  \hspace{1cm} This implies by (3.2) that

$$\lim_{R \to \infty} \tau_R = \infty. \hspace{1cm} (3.12)$$

**(Step 2).** Write for $\lambda > 0$,

$$M^\lambda_t := \lambda \int_0^t e^{-\kappa \ell s}\langle \nabla H(X_s), \text{Ad}W_s \rangle.$$  \hspace{1cm}

Then by (3.11), we have

$$\exp\{\lambda e^{-\kappa \ell t}H(X_t)\} \leq \exp\left\{\lambda H(x) + \frac{K_3}{2} |\ell|\right\} \exp\{M^\lambda_t\}. \hspace{1cm} (3.13)$$

Notice that $t \mapsto M^\lambda_t$ is a continuous local martingale with covariance

$$\langle M^\lambda \rangle_t := \lambda^2 \sum_k \int_0^t \left|e^{-\kappa \ell s} \sum_{i,j} \partial_i H(X_t)a_{ik}a_{jk}\right|^2 \ell^k_s \leq \lambda^2 \kappa_2 G_t |\ell|,$$  \hspace{1cm} (3.14)

where

$$G_t := \sup_{s \in [0,t]} (e^{-\kappa \ell s}H(X_s)).$$  \hspace{1cm} (3.15)

By Novikov’s criterion (cf. [23]), one knows that

$$t \mapsto \exp\left\{M^\lambda_{t \land \tau_R} - \frac{1}{2} \langle M^\lambda \rangle_{t \land \tau_R}\right\}$$

is a continuous exponential martingale, and by Doob’s inequality about positive submartingales and Hölder’s inequality,

$$\mathbb{E}\exp\left\{\sup_{s \in [0,t]} M^\lambda_{s \land \tau_R}\right\} \leq 2 \left(\mathbb{E}\exp\left\{2M^\lambda_{t \land \tau_R}\right\}\right)^{\frac{1}{2}}$$

$$\leq 2 \left(\mathbb{E}\exp\left\{M^\lambda_{t \land \tau_R} - \frac{1}{2} \langle M^\lambda \rangle_{t \land \tau_R}\right\}\right)^{\frac{1}{2}} \left(\mathbb{E}\exp\left\{8\langle M^\lambda \rangle_{t \land \tau_R}\right\}\right)^{\frac{1}{2}}$$

$$= 2 \left(\mathbb{E}\exp\left\{8\langle M^\lambda \rangle_{t \land \tau_R}\right\}\right)^{\frac{1}{2}}.$$

Recalling (3.15) and by (3.13) and (3.14), we have

$$\mathbb{E}\exp\left\{\lambda G_{t \land \tau_R}\right\} \leq \exp\left\{\lambda H(x) + \frac{K_3}{2} |\ell|\right\} \mathbb{E}\exp\left\{\sup_{s \in [0,t]} M^\lambda_{s \land \tau_R}\right\}$$

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\[ \leq 2 \exp \left\{ \lambda H(x) + \frac{\lambda x}{2} |\ell_s| \right\} \left( \mathbb{E} \left\{ 8 \lambda^2 \kappa_2 G_{\lambda, x} |\ell_s| \right\} \right)^{\frac{1}{2}}. \]

Thus, if one takes \( \lambda = \frac{1}{8\kappa_2 |\ell_s|+1} \), then
\[
\mathbb{E} \exp \left\{ \frac{G_{\lambda, x}}{8\kappa_2 |\ell_s| + 1} \right\} \leq 2^\frac{\lambda}{4} \exp \left\{ \frac{H(x)}{6(\kappa_2 |\ell_s| + 1)} + \frac{\kappa_3}{12\kappa_2} \right\} \leq C_{\kappa_2, \kappa_3} e^{H(x)}. \]

Finally, by Fatou’s lemma and (3.12), letting \( R \to \infty \), we get
\[
\mathbb{E} \exp \left\{ \frac{\sup_{s \in [0, t]} H(X_s)}{8e^{\varepsilon s}(\kappa_2 |\ell_s| + 1)} \right\} \leq \mathbb{E} \exp \left\{ \sup_{s \in [0, t]} \left( e^{-\varepsilon s} H(X_s) \right) \right\} \leq C_{\kappa_2, \kappa_3} e^{H(x)}. \tag{3.16} \]

**Step 3.** For general \( \ell \in \mathbb{S} \), let us define the Stelkov’s average of \( \ell \) by
\[
\ell^n_t := n \int_0^{t/n} \ell_s ds + \frac{t}{n} = \int_0^{t} \ell_{t+s/n} ds + \frac{t}{n}. \]

It is clear that \( t \mapsto \ell^n_t \) is absolutely continuous and strictly increasing. Moreover, for each \( t > 0 \),
\[
\ell^n_t \downarrow \ell_t. \tag{17.17} \]

By (3.16) one has the following uniform estimate:
\[
\mathbb{E} \exp \left\{ \frac{\sup_{s \in [0, t]} H(X^n_s)}{8e^{\varepsilon s}(\kappa_2 |\ell^n_s| + 1)} \right\} \leq C_{\kappa_2, \kappa_3} e^{H(x)}. \tag{3.18} \]

If we define
\[
\tau^n_{R_1} := \inf \left\{ t \geq 0 : H(X^n_t) \geq R_1 \right\}
\]
and
\[
\tau_{R_2} := \inf \left\{ t \geq 0 : H(X^n_t) \geq R_2 \right\},
\]
then by (3.6) and equation (3.9), we have for \( t < \tau^n_{R_1} \wedge \tau_{R_2} \),
\[
|X^n_t - X^n_t| \leq \int_0^t |b(X^n_s) - b(X^n_t)| ds + \|A\| \cdot |W^n_{\ell_t} - W_{\ell_t}|
\leq C_{R_1 \wedge R_2} \int_0^t |X^n_s - X^n_t| ds + \|A\| \cdot |W^n_{\ell_t} - W_{\ell_t}|,
\]
which yields by Gronwall’s inequality that
\[
|X^n_t - X^n_t| \leq \|A\| \cdot |W^n_t - W_{\ell_t}| + \exp \{ C_{R_1 \wedge R_2} \|A\| \} \int_0^t |W^n_{\ell_t} - W_{\ell_t}| ds.
\]

Now, for any \( \varepsilon > 0 \), by Chebyshev’s inequality and (3.18), we have
\[
P \left( |X^n_t - X^n_t| > \varepsilon, t < \tau_{R_2} \right) \leq P \left( t \geq \tau^n_{R_1} \right) + P \left( |X^n_t - X^n_t| > \varepsilon, t < \tau^n_{R_1} \wedge \tau_{R_2} \right)
\leq P \left( \sup_{s \in [0, t]} H(X^n_s) \geq R_1 \right) + P \left( \|A\| \cdot |W^n_{\ell_t} - W_{\ell_t}| > \varepsilon \right) + \frac{1}{2}
\leq \frac{1}{R_1} \sup_{n} \mathbb{E} \left( \sup_{s \in [0, t]} H(X^n_s) \right) + \frac{2\|A\|}{\varepsilon} \mathbb{E} |W^n_{\ell_t} - W_{\ell_t}|
\leq \frac{1}{R_1} \mathbb{E} \left( \sup_{s \in [0, t]} H(X^n_s) \right) + \frac{2\|A\|}{\varepsilon} \mathbb{E} |W^n_{\ell_t} - W_{\ell_t}|
\leq \frac{C}{R_1} + \frac{2\|A\|}{\varepsilon} |\ell^n_t - \ell_t|^\frac{1}{2} + \frac{2\|C_{R_1 \wedge R_2} \|A\|}{\varepsilon} \int_0^t |\ell^n_{s} - \ell_s|^\frac{1}{2} ds,
\]
which tends to zero by (3.17) as \( n \to \infty \) and \( R_1 \to \infty \).

Let \( \mathbb{Q} \) be the set of all rational numbers. By a diagonalization argument, there exists a common subsequence \( n_m \) and a null set \( N \) such that for all \( \omega \notin N \) and \( t \in \mathbb{Q} \cap [0, \tau_{R_2}(\omega)] \),

\[
\lim_{m \to \infty} |X_{t}^{n_m}(\omega) - X_{t}(\omega)| = 0.
\]

Thus, by Fatou’s lemma and (3.18), we obtain

\[
\mathbb{E} \exp \left\{ \sup_{s \in [0, t \wedge \tau_{R_2}]} H(X_s^t) \right\} = \mathbb{E} \exp \left\{ \sup_{s \in [0, t \wedge \tau_{R_2}]} \frac{H(X_s^t)}{8e^{{\kappa}_{2}^2} + 1} \right\} 
\]

\[
= \mathbb{E} \exp \left\{ \sup_{s \in [0, t \wedge \tau_{R_2}]} \frac{H(X_s^t)}{8e^{{\kappa}_{2}^2} + 1} \right\} 
\]

\[
\leq \liminf_{m \to \infty} \mathbb{E} \exp \left\{ \sup_{s \in [0, t \wedge \tau_{R_2}]} \frac{H(X_s^t)}{8e^{{\kappa}_{2}^2} + 1} \right\} 
\]

\[
\leq C_{k_2, k_3} \exp \{H(x)\}.
\]

Finally, letting \( R_2 \to \infty \), we obtain (3.10).

(Step 4). As for (3.8), by Young’s inequality we have

\[
pH^\frac{1}{2}(X_s) \leq \frac{H(X_s)}{16e^{{\kappa}_{2}^2} + 1} + C_p(\kappa_2|S_s| + 1).
\]

By Hölder’s inequality and (3.7), we have

\[
\mathbb{E} \left[ \exp \left\{ p \sup_{s \in [0, t]} H(X_s) \right\} \right] \leq C_{k_2, k_3} e^{H(x)} \left( \mathbb{E} e^{2C_p(\kappa_2|S_s| + 1)} \right)^\frac{1}{2} \leq C_{k_1, k_2, k_3, p, \alpha} e^{H(x)},
\]

where the second inequality is due to (H^2).

\[\square\]

4. MALLIAVIN COVARIANCE MATRIX

In the sequel, in addition to (3.3)-(3.5), we also assume that for any \( m \in \mathbb{N}_0 \) and some \( q_m \geq 0 \),

\[
|\nabla^m b(x)| \leq C(H(x))^{q_m} + 1,
\]

where

\[
q_1 \in \left[0, \frac{1}{2}\right].
\]

By Theorem 3.1, it is easy to see that

\[
w \mapsto X_t(x, w, \ell) \in \mathcal{D}^{\infty}(\mathbb{R}^d),
\]

\[
x \mapsto X_t(x, w, \ell) \in C^{\infty}(\mathbb{R}^d).
\]

Let \( J_t = J_t(x) = \nabla X_t(x) \) be the derivative matrix of \( X_t(x) \) with respect to \( x \). Then \( J_t \) satisfies

\[
J_t = I + \int_0^t \nabla b(X_s) \cdot J_s ds.
\]

Let \( K_t \) be the inverse matrix of \( J_t \). Then \( K_t \) satisfies

\[
K_t = I - \int_0^t K_s \cdot \nabla b(X_s) ds.
\]

We prepare the following basic estimates for later use.

Lemma 4.1. Assume (H^2). For \( x \in \mathbb{R}^d \), let \( X_t(x) \) be the solution of SDE (3.1).
(i) For any $p \geq 1$, there exists a constant $C_p > 0$ such that for all $t \in [0, 1]$,
\[
\mathbb{E} \left( \sup_{s \in [0,t]} |S_s|^p \right) \leq C_p t. \tag{4.4}
\]

(ii) There exists a constant $C_\varepsilon > 0$ such that for all $t \in [0, 1]$ and $\varepsilon > 0$,
\[
P \left\{ \sup_{s \in [0,t]} |X_s(x) - x| > \varepsilon \right\} \leq \frac{C_\varepsilon t}{\varepsilon^2}. \tag{4.5}
\]

(iii) For any $p \geq 2$, there exists a constant $C_{p,\varepsilon} > 0$ such that for all $t \in [0, 1]$,
\[
\mathbb{E} \left( \sup_{s \in [0,t]} |J_s(x) - l|^p \right) + \mathbb{E} \left( \sup_{s \in [0,t]} |K_s(x) - l|^p \right) \leq C_{p,\varepsilon} t^p. \tag{4.6}
\]

(iv) For any $p \geq 2$ and $m, k \in \mathbb{N}_0$ with $m + k \geq 1$, there exists a constant $C_{p,m,k,\varepsilon} > 0$ such that for all $t \in [0, 1]$,
\[
\mathbb{E} \left( \sup_{s \in [0,t]} \|D^m \nabla^k X_s(x)\|_{\text{lip}}^p \right) \leq \begin{cases} 1, & m = 0, k = 1; \\ t, & m = 1, k = 0; \\ t^p, & m + k \geq 2. \end{cases} \tag{4.7}
\]

Proof. (i) By (2.1) and (H^2_{xy}), we can write
\[
S_t = t \left( \theta + \int_{\mathbb{R}^d_+} \nu_S (du) \right) + \int_{\mathbb{R}^d_+} u\tilde{\nu}(t, du) =: t\theta' + \int_{\mathbb{R}^d_+} u\tilde{\nu}(t, du).
\]
By Itô’s formula, we have
\[
|S_t|^p = p \int_0^t |S_s|^{p-2} \langle S_s, \theta' \rangle ds + \int_0^t \int_{\mathbb{R}^d_+} (|S_s - u|^p - |S_s - | s - | u|^p) \tilde{\nu}(ds, du) + \int_0^t \int_{\mathbb{R}^d_+} (|S_s - u|^p - |S_s - | s - | u|^p) \nu_S (du) ds.
\]
Taking expectations and by Young’s inequality, we obtain
\[
\mathbb{E} |S_t|^p \leq p |\theta'| \int_0^t \mathbb{E} |S_s|^{p-1} ds + p \int_0^t \int_{\mathbb{R}^d_+} |u| \left( \langle | S_s - | u | \rangle + | u |^{p-1} + | S_s - | u |^{p-1} \right) \nu_S (du) ds \leq C_p \int_0^t \mathbb{E} |S_s|^{p-1} ds + C_p t, \]
which then gives the estimate (4.4) by Gronwall’s inequality and that each component of $S_t$ is increasing.

(ii) Noticing that
\[
\sup_{s \in [0,t]} |X_s(x) - x| \leq \int_0^t |b(X_s(x))| ds + \sup_{s \in [0,t]} |W_s|,
\]
by Chebyshev’s inequality, we have
\[
P \left\{ \sup_{s \in [0,t]} |X_s(x) - x| > \varepsilon \right\} \leq \frac{2}{\varepsilon^2} \left( \int_0^t \mathbb{E} |b(X_s(x))|^2 ds + \mathbb{E} \left( \sup_{s \in [0,t]} |W_s|^2 \right) \right) \leq C \varepsilon^2 \left( t \int_0^t \langle \mathbb{E} |H(X_s(x))| \rangle ds + \mathbb{E} |S_1| \right),
\]
which yields (4.5) by (3.8) and (4.4).
Let

\[ |J_t - I| \leq \int_0^t |\nabla b(X_s)| \cdot |J_s - I| \, ds + \int_0^t |\nabla b(X_s)| \, ds. \]

which yields by Gronwall’s inequality that

\[ |J_t - I| \leq \left( \exp \left( \int_0^t |\nabla b(X_s)| \, ds \right) + 1 \right) \int_0^t |\nabla b(X_s)| \, ds. \]

By (4.1) with \( q_1 \in [0, \frac{1}{2}] \) and (3.8), we obtain (4.6).

(iv) Notice that for \( h \in \mathbb{H} \),

\[ D_hX_t = \int_0^t \nabla b(X_s)D_hX_s \, ds + Ah_{S_t}. \]  

(4.8)

Let \( \{h^n, n \in \mathbb{N}\} \) be an orthonormal basis of \( \mathbb{H} \). Then

\[
\|DX_t\|_{\mathbb{H}} = \left\{ \sum_n |D_{h^n}X_t|^2 \right\}^{1/2} \leq \int_0^t |\nabla b(X_s)| \cdot \|DX_s\|_{\mathbb{H}} \, ds + \left( \sum_n |Ah^n_{S_t}|^2 \right)^{1/2}.
\]

By Gronwall’s inequality and (4.10) below, we obtain

\[ \|DX_t\|_{\mathbb{H}} \leq \|A\| \cdot |S_t|^{1/2} + \exp \left( \int_0^t |\nabla b(X_s)| \, ds \right) \int_0^t \|A\| \cdot |S_s|^{1/2} \, ds, \]

which, by (3.8), (4.1) with \( q_1 \in [0, \frac{1}{2}] \), Hölder’s inequality and (i), then gives (4.7) for \( m = 1 \) and \( k = 0 \). For the general \( m \) and \( k \), it follows by similar calculations and induction method.

Remark 4.2. From the above proof, it is easy to see that if \( b \in C^\infty_b(\mathbb{R}^d) \), then \( C_x \) in (4.5), \( C_{p,x} \) in (4.6) and \( C_{p,m,k,x} \) in (4.7) can be independent of \( x \in \mathbb{R}^d \).

We need the following simple formula about the change of variables (cf. [14]).

Lemma 4.3. Let \( f : \mathbb{R}_+ \to \mathbb{R} \) be a bounded measurable function, and \( h : \mathbb{R}_+ \to \mathbb{R} \) an absolutely continuous function with integrable derivative. Given a càdlàg increasing function \( \ell \), we have

\[ \int_0^t f_s \, dh_{\ell_s} = \int_0^{\ell_t} f_{\ell^{-1}_s} h_s \, ds, \]  

(4.9)

where \( \ell^{-1}_t := \inf\{s \geq 0 : \ell_s > t\} \).

Proof. By definition, it is easy to see that

\[ \ell^{-1}_t > a \Rightarrow t \geq \ell_a, \]

and

\[ \ell^{-1}_t \leq a \Rightarrow t \leq \ell_a. \]

Thus, for \( 0 \leq a < b \leq t \) we have

\[ (\ell_a, \ell_b) \subset [s : \ell^{-1}_s \in (a, b)] \subset [\ell_a, \ell_b]. \]

Hence,

\[ \int_0^{\ell^{-1}_t} 1_{(a,b)}(\ell^{-1}_s) h_s \, ds = \int_0^{\ell_t} 1_{(\ell_a, \ell_b)}(s) h_s \, ds = h_{\ell_b} - h_{\ell_a} = \int_0^{\ell_t} 1_{(a,b)}(s) dh_{\ell_s}. \]

In particular, (4.9) holds for step functions. For general bounded measurable \( f \), it follows by a monotone class argument. \( \Box \)
Lemma 4.4. Let $f, g : \mathbb{R}^+ \to \mathbb{R}^d$ be two bounded measurable functions, and \( \{h^n, n \in \mathbb{N} \} \) an orthonormal basis of $\mathbb{H}$. We have
\[
\sum_n \left( \int_0^t f_s \cdot dh^n_t \right) \left( \int_0^t g_s \cdot dh^n_t \right) = \sum_k \int_0^t f^k_s g^k_s dt^k.
\] (4.10)

Proof. If we define
\[
\hat{f}^k_s := 1_{[0,t]}(s) f^k_t, \quad s \geq 0,
\]
and let $\hat{f}_s = (\hat{f}^1_s, \ldots, \hat{f}^d_s)$, then by formula (4.9), we have
\[
\int_0^t f_s \cdot dh^n_t = \int_0^\infty \hat{f}^n_s \cdot \hat{h}^n_s ds.
\]
Thus, by Parsavel’s equality, the left hand side of (4.10) equals to
\[
\sum_n \left( \int_0^\infty \hat{f}^n_s \cdot \hat{h}^n_s ds \right) \left( \int_0^\infty \hat{g}^n_s \cdot \hat{h}^n_s ds \right) = \int_0^\infty \hat{f}_s \cdot \hat{g}_s ds = \sum_k \int_0^\infty f^k_t g^k_t dt^k,
\]
which then gives (4.10) by (4.9) again. □

The following lemma originally appeared in the proof of [14, Theorem 3.3].

Lemma 4.5. Let \((\Sigma_t(x))_{ij} := (DX^i_t(x), DX^j_t(x))_{\mathbb{H}} \) be the Malliavin covariance matrix of $X_t(x)$. We have
\[
\Sigma_t(x) = J_t(x) \left( \sum_k J_t(x) K_t(x)a_i(K_t(x)a_i)^* dS^k_s \right) (J_t(x))^*.
\] (4.11)

Proof. By (4.2), (4.8) and the variation of constant formula, we have
\[
D_h X_t(x) = \int_0^t J_t(x) Adh^1_s.
\]
Let $\{h^n, n \in \mathbb{N} \}$ be an orthonormal basis of $\mathbb{H}$. Then,
\[
\Sigma_t(x) = \sum_n (D_{h^n}X_t(x) \cdot (D_{h^n}X_t(x))^*) = \sum_n \left( \int_0^t J_t(x) K_t(x)a_i dS^k_s \right) \cdot \left( \int_0^t J_t(x) K_t(x)a_i dS^k_s \right)^*,
\]
which in turn gives the formula (4.11) by (4.10). □

5. Proof of Main Theorem

In this section we consider the following SDE
\[
X_t = x + \int_0^t b(X_s) ds + AW_{S^t} \underbrace{\text{(2.3)}}_{2.3} + \int_0^t b(X_s) ds + AW_{S^t} + \int_{\mathbb{R}^d} AyN(t, dy),
\] (5.1)
where, in addition to \((\mathbf{H}^1_{\gamma'})\) and \((\mathbf{H}^2_{\gamma})\), we assume that $b \in C^\infty(\mathbb{R}^d)$ and \((\mathbf{H}^1_{\gamma})\). For some Lyapunov function $H$ and $\kappa_1, \kappa_2, \kappa_3 > 0$,\n\[
b(x) \cdot \nabla H(x) \leq \kappa_1 H(x), \quad \left| \sum_i \partial_i H(x)a_i \right|^2 \leq \kappa_2 H(x), \quad \sum_{ij} \partial_i \partial_j H(x)a_i a_j \leq \kappa_3, \quad (5.2)
\]
and for any $m \in \mathbb{N}_0$, there is a $q_m \geq 0$ such that
\[
|\nabla^m b(x)| \leq C(H(x)^{q_m} + 1),
\] (5.3)
where $q_1 \in [0, \frac{1}{2}]$.  

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(H^2_p) For some \(\kappa, \kappa_5, \kappa_6 > 0\),

\[
|\nabla b(x + Ay) - \nabla b(x)| \leq \kappa_4 (1 \wedge |y|), \tag{5.4}
\]

\[
|\nabla b(x + Ay) - \nabla b(x) - Ay \cdot \nabla^2 b(x)| \leq \kappa_5 |y|^2, \tag{5.5}
\]

\[
\inf_{x \in \mathbb{R}^d} \inf_{|\theta| = 1} \left( |uA|^2 + |u \nabla b(x)|^2 \right) = \kappa_6 > 0. \tag{5.6}
\]

The following estimate is the key part for proving the smoothness of density of \(X_t(x)\).

**Lemma 5.1.** Let \(\theta\) be given by (1.8). For any \(p > 1\) and \(x \in \mathbb{R}^d\), there exists a constant \(C = C(p, d, \theta, x) > 0\) such that for all \(t \in (0, 1]\),

\[
\|\det \Sigma_t(x)\|^{-1} \leq Ct^{-\frac{2d}{\theta}}.
\]

Moreover, if for all \(m \in \mathbb{N}, q_m = 0\) in (5.3), then the above constant \(C\) can be independent of \(x\).

**Proof.** We divide the proof into four steps.

**(Step 1).** Set

\[Y_i := uK_iA, \quad \beta_i := uK_i\nabla b(X_i)A, \quad Q_i := uK_i\nabla^2 b(X_i)A,\]

\[\gamma_i := uK_i \left( \sum b^i \cdot \partial_i \nabla bA - (\nabla b)^2 A + \frac{1}{2} \sum_{ijk} (\partial_i \partial_j \nabla b)a_{i}a_{jk}\gamma_k \right)(X_i)
+ uK_i \int_{\mathbb{R}^d} \left( \nabla b(X_i + Ay) - \nabla b(X_i) - Ay \cdot \nabla^2 b(X_i) \right) Ay \nu(dy),\]

\[g_i(y) := uK_i(\nabla b(X_{i-} + Ay) - \nabla b(X_{i-}))A.\]

By equations (4.3), (5.1) and Itô’s formula, one sees that

\[Y_t = aA + \int_0^t \beta_i ds,\]

and

\[\beta_t = a \nabla b(x)A + \int_0^t \gamma_i ds + \int_0^t Q_i ds + \int_0^t \int_{\mathbb{R}^d} g_i(y) \tilde{N}(ds, dy).\]

By (5.3)-(5.7), it is easy to see that

\[|g_i(y)| \leq C|K_i|(1 \wedge |y|),\]

and for some \(q > 0\),

\[|\beta_t| + |\gamma_i| + |Q_i| \leq C|K_i|(H(X)^{q} + 1).\]

**(Step 2).** For \(R \geq 1\), define the stopping times

\[\tau_R := \inf \left\{ t \geq 0 : |K_i| \geq R, H(X) \geq R \right\},\]

and

\[\tau_0 := \inf \left\{ s \geq 0 : |K_i - s| \geq \frac{1}{2} \right\}.\]

For \(\eta \in (0, 1)\), set

\[E^\varepsilon_t := \left\{ \int_0^t |uK_iA|^2 ds < \varepsilon^\eta \right\},\]

and for \(\delta \in (0, \frac{1}{2})\) and \(R \geq 1\),

\[F_t^{\varepsilon, R} := \left\{ \int_0^t |uK_i \nabla b(X_i)A|^2 ds < 9R^2 \varepsilon^{\eta} \right\}.\]
By Lemma 2.4, there is a constant $C_1 > 0$ such that for all $0 < \varepsilon < t^{3/4} \leq 1$ and $R \geq 1$,
\[
P(E_t^e \cap \{t < \tau_R\}) = P(E_t^e \cap (F_t^{e,R}) \cap \{t < \tau_R\}) + P(E_t^e \cap F_t^{e,R} \cap \{t < \tau_R\})
\leq 4 \exp \left\{ -\frac{\varepsilon^{\gamma(\delta - \frac{1}{2})}}{CR^4} \right\} + P(E_t^e \cap F_t^{e,R} \cap \{\tau_0 \geq \varepsilon^{\delta_0}\}) + P(\tau_0 < \varepsilon^{\delta_0}).
\]

On the other hand, by (5.6) we have
\[
E_t^e \cap F_t^{e,R} \subset \left\{ \int_0^\tau (|u_KA|^2 + |Ks\nabla b(X_s)A|^2)\,ds < \varepsilon^{\gamma} + 9R^2 \varepsilon^{\delta} \right\}
\subset \left\{ \int_0^\tau \frac{|u_KA|^2 + |Ks\nabla b(X_s)A|^2}{|u_K|^2} |u_K|^2 \,ds < 10R^2 \varepsilon^{\delta} \right\}
\subset \left\{ \kappa_6 \int_0^\tau |u_K|^2 \,ds < 10R^2 \varepsilon^{\delta} \right\}.
\]

Since for any $|u| = 1$ and $s \in [0, \tau_0]$,
\[
|u_K| \geq 1 - |K_s - I| \geq \frac{1}{2},
\]
it is easy to see that for any $\varepsilon < (\frac{\kappa_6 t}{20R^2})^{\frac{1}{\delta}}$,
\[
E_t^e \cap F_t^{e,R} \cap \{\tau_0 \geq \varepsilon^{\delta_0}\} = \emptyset.
\]

Hence, for $\eta \in (0, 1)$, $\delta \in (0, \frac{1}{2})$, $R \geq 1$ and $\varepsilon < (\frac{\kappa_6 t}{20R^2})^{\frac{1}{\delta}}$,
\[
P(E_t^e \cap \{t < \tau_R\}) \leq 4 \exp \left\{ -\frac{\varepsilon^{\gamma(\delta - \frac{1}{2})}}{CR^4} \right\} + P(\tau_0 < \varepsilon^{\delta_0}). \tag{5.7}
\]

(Step 3). Now, by Lemma 2.1 and (5.7), we have
\[
P\left\{ \int_0^\tau \sum_k |u_Ka_k|^2 \,dS_k^e \leq \varepsilon \right\} \leq P\left\{ \int_0^\tau \sum_k |u_Ka_k|^2 \,dS_k^e \leq \varepsilon, \int_0^\tau |u_KA|^2 \,ds \geq \varepsilon^{\gamma}; t < \tau_R \right\}
+ P\left\{ \int_0^\tau |u_KA|^2 \,ds < \varepsilon^{\gamma}; t < \tau_R \right\} + P(\tau_R \leq t)
\leq \exp \left\{ 1 - \frac{\phi(\varepsilon/R)\varepsilon^{\gamma}}{\varepsilon} \right\} + 4 \exp \left\{ -\frac{\varepsilon^{\gamma(\delta - \frac{1}{2})}}{CR^4} \right\}
+ P(\tau_0 < \varepsilon^{\delta_0}) + P(\tau_R \leq t), \tag{5.8}
\]
where
\[
\eta \in (0, 1), \quad \delta \in (0, \frac{1}{2}), \quad R \geq 1, \quad \varepsilon < (\frac{\kappa_6 t}{20R^2})^{\frac{1}{\delta}}.
\]
For any $p > 1$, by Chebyshev’s inequality and (4.6), we have
\[
P(\tau_0 < \varepsilon^{\delta_0}) = P\left\{ \sup_{s \in [0, \varepsilon^{\delta_0}]} |K_s - I| \geq \frac{1}{2} \right\} \leq 2^p \mathbb{E}\left( \sup_{s \in [0, \varepsilon^{\delta_0}]} |K_s - I|^p \right) \leq C \varepsilon^{\delta_0 p}, \tag{5.9}
\]
and by (5.8) and (4.6),
\[
P(\tau_R \leq t) \leq \frac{1}{R^p} \mathbb{E}\left( \sup_{s \in [0, t]} \left| K_s + H(X_s) \right|^p \right) \leq \frac{C}{R^p}. \tag{5.10}
\]
Let θ be given by (1.8). If we choose
\[ \eta = \frac{\theta}{2}, \quad \delta = \frac{1}{6}, \quad R = e^{-\frac{\theta}{6}}, \]
then by (5.8), (5.9) and (5.10), we get for any \( t \in (0, 1) \), \( p \geq 1 \) and \( \varepsilon \in \left(0, \left(\frac{6}{20}\right)^{\frac{1}{p}}\right)\),
\[
P(\int_0^t \sum_\kappa |uK_\kappa a_\kappa|^2 dS_\kappa \leq \varepsilon) \leq Ce^p. \tag{5.11} \]

(Step 4). Let
\[
\xi_t := \inf_{|\kappa|=1} \int_0^t |uK_\kappa a_\kappa|^2 dS_\kappa.
\]
Since \( S_t \) has finite moments of all orders, by (4.7) and a compact argument (see [20, p. 133, Lemma 2.3.1] for more details), for any \( p \geq 1 \), there exists a constant \( C_0 = C_0(p, d, \kappa_0) > 0 \) such that for all \( \varepsilon \in (0, C_0^{\frac{1}{p}}) \),
\[
P(\xi_t \leq \varepsilon) \leq C_p \varepsilon^p. \]
Hence, for all \( t \in (0, 1) \),
\[
\mathbb{E}(\xi_t^{-p}) = p \int_0^\infty \lambda^{p-1} P(\xi_t^{-1} \geq \lambda) d\lambda \leq p \int_0^\infty \lambda^{p-1} d\lambda + C_\rho \int_0^\infty \lambda^{p-2} d\lambda \leq C_p \rho^{\frac{2}{p}}.
\]
The desired estimate now follows by (4.6) and noticing that the smallest eigenvalue of a real symmetric matrix \( M \) is less than \((\det M)^{\frac{1}{2}}\).

We are now in a position to prove the following main result of this paper.

**Theorem 5.2.** Under \((\mathcal{H}^1_{\kappa_2}), (\mathcal{H}^2_{\kappa_2}), (\mathcal{H}_0^1)\) and \((\mathcal{H}_0^2)\), the solution \( X_t(x) \) of SDE (5.7) admits a smooth density \( \rho(t, x, y) \) as a function on \((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\). Moreover, we have the following conclusions:

(i) For each \( t > 0 \) and \( x \in \mathbb{R}^d \), \( \rho(t, x, \cdot) \in C(\mathbb{R}^d) \) and solves the following Fokker-Planck equation:
\[
\partial_t \rho = \mathcal{L} \rho(t, x, \cdot) + \text{div}(bp(t, x, \cdot)), \tag{5.12}
\]
where
\[
\mathcal{L} f(y) = \text{P.V.} \int_{\mathbb{R}^d} (f(y + Az) - f(y)) \nu_L(dz) + \frac{1}{2} \sum_{i,j,k} (\partial_i \partial_j f)(y)a_{ik}a_{jk}g_k.
\]

(ii) There exist constants \( \beta_1, \beta_2, \beta_3 > 0 \) only depending on \( d \) and \( \theta \) such that for all \( (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \),
\[
\rho(t, x, y) \leq C_x \left( \Gamma^{\beta_1} \left( \frac{\beta_2}{|x - y|^\beta_3} \right) \right), \tag{5.13}
\]
where \( C_x \) continuously depends on \( x \).

(iii) Suppose that \( b \in C^0_b(\mathbb{R}^d) \), then \( C_x \) in (5.13) can be independent of \( x \).

*Proof.* For \( k, m \in \mathbb{N} \), by the chain rule, we have
\[
\nabla^k \mathbb{E}(\nabla^m f)(X_t(x)) = \sum_{j=1}^k \mathbb{E}(\nabla^{m+j} f)(X_t(x)) G_j(\nabla X_t(x), \ldots, \nabla^k X_t(x)),
\]
where \( G_j \) are the gradient of the function \( g_j \).
where \( \{G_j, j = 1, \cdots , k\} \) are real polynomial functions. By Theorem 2.5 and Lemmas 4.1 5.1 one finds that there exist \( \gamma_{k,m} > 0 \) and \( C > 0 \) such that for all \( t \in (0, 1) \),

\[
|\nabla^k \mathbb{E}(\nabla^m f)(X_t(x))| \leq C\|f\|_{\infty} t^{-\gamma_{k,m}}.
\]

(5.14)

Now, by Sobolev’s embedding theorem (see [20, pp.102-103]), for each \( (t, x) \in (0, \infty) \times \mathbb{R}^d \), there exists a smooth density \( \rho(t, x, \cdot) \in \mathcal{S}(\mathbb{R}^d) \). Moreover,

\[
(x, y) \mapsto \rho(t, x, y) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d).
\]

By Itô’s formula, one sees that \( \rho \) satisfies equation (5.12). The smoothness of \( \rho(t, x, y) \) with respect to the time variable \( t \) follows by equation (5.12) and the standard bootstrap argument.

Now we use a trick of Kusuoka and Stroock [10] to prove (5.13). Let \( \chi : \mathbb{R}^d \to [0, 1] \) be a smooth cutoff function with \( \chi(y) = 0 \) for \( |y| < \frac{\varepsilon}{4} \) and \( \chi(y) = 1 \) for \( |y| > \frac{\varepsilon}{4} \). For \( \varepsilon > 0 \), define

\[
\chi_\varepsilon(y) := \chi(y/\varepsilon), \quad \chi_0(y) = 1.
\]

By (2.11), we have for any \( q > d \),

\[
\|\chi_\varepsilon(\cdot - x) \rho(t, x, \cdot)\|_\infty \leq C\|\chi_\varepsilon(X_t(x) - x)\|_{q}^{-(1 - \frac{d}{q})} \left( \sum_{i} \|H_{(i)}(X_t(x), 1)\|_q \right)^{(1 - \frac{d}{q})} \times \left( \sum_{i} \|H_{(i)}(X_t(x), \chi_\varepsilon(X_t(x) - x))\|_q \right)^{\frac{d}{q}}.
\]

(5.15)

By (2.9), (2.8) and Hölder’s inequality, we have

\[
\|H_{(i)}(X_t(x), \chi_\varepsilon(X_t(x) - x))\|_q \leq \|\nabla \chi_\varepsilon(X_t(x) - x)\| \cdot \|\Sigma_t^{-1}\| \cdot \|D_2 X_t\|_q^2 \|D_1 X_t\|_{q_1} \|D_3 X_t\|_{q_2} \|D_4 X_t\|_{q_3} \|D_x X_t\|_{q_4} \leq C\|\chi_\varepsilon(X_t(x) - x)\|_q \|\Sigma_t^{-1}\|_{q_2} \|D_2 X_t\|_q^2 \|D_3 X_t\|_{q_2} \|D_4 X_t\|_{q_2} \|D_x X_t\|_{q_2} \|D_x X_t\|_{q_2}.
\]

where \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3}. \) Let \( \text{adj}(\Sigma_t) \) be the adjugate matrix of \( \Sigma_t \). Observing that

\[
\Sigma_t^{-1} = \det(\Sigma_t)^{-1} \text{adj}(\Sigma_t), \quad D_2 \Sigma_t^{-1} = \Sigma_t^{-1} D_2 \Sigma_t^{-1},
\]

by the definition of \( \text{adj}(\Sigma_t) \), we have

\[
\|\Sigma_t^{-1}\|_{q_2} \leq \|\det(\Sigma_t)^{-1}\|_{2q_2} \|\text{adj}(\Sigma_t)\|_{2q_2} \leq C\|\det(\Sigma_t)^{-1}\|_{2q_2} \|D_2 X_t\|_{q_2}^{2(d-1)} q_2,
\]

and

\[
\|D_2 \Sigma_t^{-1}\|_{q_2} \leq \|\det(\Sigma_t)^{-1}\|_{8q_2} \|\text{adj}(\Sigma_t)\|_{8q_2} \|D_2 X_t\|_{8q_2} \|D_4 X_t\|_{1,4q_2} \leq C\|\det(\Sigma_t)^{-1}\|_{8q_2} \|D_2 X_t\|_{16q_2}^{(d-1)} \|D_4 X_t\|_{1,4q_2}^2.
\]

On the other hand, by (2.5), we have

\[
\|\chi_\varepsilon(X_t(x) - x)\|_{q_1} \leq P\left\{ |X_t(x) - x| \geq \frac{\varepsilon}{2} \right\}^{1/q_1} \leq C\frac{t^{1/q_1}}{\varepsilon^{2/q_1}}.
\]

and also

\[
\|\nabla \chi_\varepsilon(X_t(x) - x)\|_{q_1} \leq \frac{\|\nabla \chi\|_{\infty}}{\varepsilon} P\left\{ |X_t(x) - x| \geq \frac{\varepsilon}{2} \right\}^{1/q_1} \leq C\frac{t^{1/q_1}}{\varepsilon^{1+2/q_1}}.
\]
Combining the above calculations and by Lemma \(5.1\) and \((4.7)\), we obtain
\[
\|H_{(i)}(X_t(x), \chi_t(x) - x))\|_q \leq C \frac{t^{1/\eta_1}}{\epsilon^{1+2/\eta_1}} \cdot t^{\frac{4d}{q\eta_1}} \cdot t^{\frac{1}{q_1} + \frac{1}{\eta_1}} + C \frac{t^{1/\eta_1}}{\epsilon^{2/\eta_1}} \left( t^{\frac{4d}{q\eta_1}} \cdot t^{\frac{1}{q_1} + \frac{1}{\eta_1}} + t^{\frac{2d}{q\eta_1}} \cdot t^{\frac{1}{q_1} + \frac{1}{\eta_1}} \right).
\]
Similarly, we also have
\[
\|H_{(i)}(X_t(x), 1)\|_q \leq C \left( t^{\frac{4d}{q\eta_1}} \cdot t^{\frac{1}{q_1} + \frac{1}{\eta_1}} + t^{\frac{2d}{q\eta_1}} \cdot t^{\frac{1}{q_1} + \frac{1}{\eta_1}} \right).
\]
In \((5.15)\), taking \(\epsilon = 0\) and \(\epsilon = |x - y|\) separately, by careful choices of parameters, we obtain (ii). As for (iii), it follows by Remark \(4.2\) \(\Box\)

Now we can see that Theorem \(1.3\) is an easy application of Theorem \(5.2\).

**Proof of Theorem \(1.3\)**: In the situation of Theorem \(1.3\) we set
\[
A := \begin{pmatrix} 0, 0 \\ 0, t \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\]
By \((1.11)\) and \((1.12)\), it is easy to see that \((5.2)\) and \((5.3)\) hold. By \((1.13)\) and \((1.14)\), one can see that \((5.4)\), \((5.5)\) and \((5.6)\) hold.

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