QUANTUM SUSY OPERADS

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Abstract. In a recent paper, we described a lifting of coordinate rings of groups, loops, quantum groups, etc. to the categoric setup of operads. In most examples of that paper, these rings are non–commutative.

Quantum physics of the XX–th century added one more, quite nontrivial degree of freedom: coordinates might become fermionic. In their classical version, the fermionic coordinates anti–commute, and the resulting rings are called supersymmetric, or SUSY, ones.

In this paper, we try to lift operads involving fermionic coordinates to quantum operads. We have to restrict ourselves by lifting operads of supersymmetric rings. We also show that 1D supersymmetric algebras have an operad structure, and we analyze their symmetries, through their relation to Adinkra graphs, dessins and codes.

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0. Introduction

The lifting of classical structures to quantum ones generally goes through an initial stage that we call categorification: basic data and axioms of the structure we deal with become categories, functors, and commutative diagrams.

For this reason—which may seem a purely technical one—this paper turns out to be on the crossroads of two vast and differently motivated research domains: the general survey of the first one can be found in [Ke]; as for the second one, it can be found in [KaSch]. How disjoint they might seem to researchers, a reader can guess by simply comparing the lists of References, in these two surveys. The key references appear either in one of them, or in another, but rarely in both.

We have chosen [KaSch] as our main source of basic information.

The following four papers focus on the central objects of our current study:

- Operads and generalised operads: [BoMa];
- Operads of moduli spaces of algebraic curves: [GeKa];
- Quantum operads of moduli spaces of algebraic curves: [CoMaMar3];
- Generalised operads of moduli superspaces of stable SUSY curves: [KeMaWu].

keywords: monoidal categories, operads, super–symmetry.
Briefly, the goal of this paper is to present a combination of quantisation and supersymmetry. For further preparatory reading, we can recommend to a reader the following references \cite{MaPe, FeKaPo, HoKrStT, WuYau}.

The sequence, in which we explain and briefly discuss the potential setup of quantum SUSY modular operads, is motivated by the presentation of quantum operads, not involving fermionic coordinates, in \cite{CoMaMar3, HoKrStT}, and the survey \cite{Sm}. See also the introductory parts of \cite{BoMa} and \cite{MaVa}.

At the end of this paper, we discuss another source of quantum SUSY operads: $\mathbb{F}_2$-codes and Adinkras and investigate their hidden symmetries.

1. Categorifications and enrichments: the simplest examples

All our definitions and constructions below will refer to sets from a fixed universe $\mathcal{U}$: see \cite{KaSch}, Sec. 1.1, pp. 10–11. In particular, we will use basic notions of the language of categories as they are presented in \cite{KaSch}, in Sec. 1.2, and further on.

Somehow, this foundational book does not stress the importance (also historical importance) of the language of structures à la Bourbaki. So, we start this article with an attempt to formalise and generalise interrelationships between these two notions.

On the one hand, we discuss transitions from structures to categories, which we will call categorifications. On the other hand, we discuss transitions from a category to “structured objects of category”, which then form another category, being an enrichment of the initial one (or of its part).

Let us start with basic examples.

1.1. Monoids and monoidal categories. A monoid $M$, as it is defined on p.13 of \cite{KaSch}, is a set, endowed with binary multiplication $m : M \times M \to M$, satisfying the associativity law. It might also have left, resp. right, identities. If there is one element $1$, which is simultaneously left and right, it is called simply identity.

Recall, that for two $\mathcal{U}$-sets $X,Y$, their direct product $X \times Y$ is the $\mathcal{U}$-set, whose elements are ordered pairs $(x,y)$ with $x \in X, y \in Y$. If we write $(xy)$ in place of $m(x,y)$, the associativity condition can be written in the standard way: $((xy)z) = (x(yz))$. (Exterior brackets are usually omitted, but they become necessary for a linear ordering of iterated operation).

Categorification of monoids leads to the definition of a monoidal category (cf. \cite{Sm}, Sec. 2.2, 2.3, and \cite{KaSch}, Sec. 4.2, where monoidal categories are called tensor ones).
Definition 1.1.1. A monoidal category $\mathcal{T}$ is a category whose set of objects is endowed with associative multiplication and identity, functorial with respect to morphisms in this category.

More precisely, a monoidal structure on $\mathcal{T}$ is given by the following data, which we call structural functors: binary multiplication $\otimes$ of objects, with identity object $1$ and natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C), \quad \rho_A : A \otimes 1 \to A, \quad \lambda_A : 1 \otimes A \to A.$$ 

See the diagram (4.2.1) on p.96 of [KaSch] for a detailed description of the categorification of associativity law.

We refer to this set of data as structural functors.

1.2. Commutative monoids and symmetric monoidal categories. A monoid $(M, m)$ is commutative, if $m(x, y) = m(y, x)$ for all $x, y \in M$.

Respectively, the list of structural functors in the definition of a symmetric monoidal category includes additional twist isomorphisms $\tau_{A,B} : A \otimes B \to B \otimes A$, with $\tau_{A,B} \tau_{B,A} = \text{id}_{A \otimes B}$, satisfying axioms of compatibility with associativity, expressed by several commutative diagrams.

1.3. Magmas, comagmas, bimagmas in symmetric monoidal categories. The basic data for a magma: an object $A$ with multiplication morphism $\nabla : A \otimes A \to A$.

The basic data for a comagma: an object $A$ with comultiplication morphism $\Delta : A \to A \otimes A$.

The basic data for a bimagma: a triple $(A, \nabla, \Delta)$ as above such that the "bimagma diagram" ((2.4) [Sm], p.49) commutes.

$(\text{Co, bi})$–magmas in a symmetric monoidal category $\mathbf{V}$ are themselves objects of respective categories. Morphisms of them are those morphisms in $\mathbf{V}$, ([Sm], Def. 2.3), which are compatible with the respective basic data.

1.4. Commutative/cocommutative magmas and comagmas in symmetric monoidal categories. ([Sm], Def.2.3). Basically, these properties mean the compatibility with corresponding structural functors.

1.5. Monoids, comonoids, bimonoids, and Hopf algebras in symmetric monoidal categories. ([Sm], Def. 2.7). They are essentially $(\text{co, bi})$–magmas with additional $(\text{co,bi})$–associativity restrictions.
1.6. **Quantum quasigroups.** ([Sm], Sec. 3.1). A quantum quasigroup \((A, \∇, \Delta)\) is a bimagma, for which both left composite and right composite morphisms are invertible:

\[
A \otimes A \xrightarrow{\Delta \otimes \id_A} A \otimes A \otimes A \xrightarrow{\id_A \otimes \∇} A \otimes A,
\]

\[
A \otimes A \xrightarrow{id \otimes \Delta} A \otimes A \otimes A \xrightarrow{\∇ \otimes \id_A} A \otimes A.
\]

These morphisms are sometimes called *fusion operators* or *Galois operators*.

1.7. **Quantum loops.** A quantum loop in \(V\) is a biunital bigmagma \((A, \∇, \Delta, \eta, \varepsilon)\) such that \((A, \∇, \Delta)\) is a quantum quasigroup. (The notation \((\eta, \varepsilon)\) is introduced in [Sm16], Sec. 2.1).

1.8. **Functoriality.** ([Sm], Prop. 3.4). Any symmetric monoidal functor

\[
F : (V, \otimes, 1_V) \to (W, \otimes, 1_W)
\]

sends quantum quasigroups (resp. quantum loops) in \(V\) to quantum quasigroups (resp. quantum loops) in \(W\).

1.9. **Magmas etc. in the categories of sets with direct product.** According to [Sm], beginning of Sec. 3.3, in such categories comultiplication in a counital comagma is always the respective diagonal embedding. As a corollary, we see that quantum loops and counital quantum quasigroups in such categories are cocommutative and coassociative.

As a result, we see, that in such a category, counital quantum quasigroups are equivalent to classical quasigroups, and quantum loops are equivalent to classical loops ([Sm], Prop. 3.11).

### 2. Monoidal categories of operads

#### 2.1. Graphs and their categories.

Our basic definition of graphs as quadruples \((F, V, \partial, j)\) and their categories is explained in [BoMa], Sec. 1.1, p.251. There \(F\), resp. \(V\), are called the sets of *flags*, resp. *vertices*, and structure maps \(\partial\), resp. \(j\) are called *boundary maps*, resp. *involutions*. Usually one flag is a pair consisting of flag as such, and a *label*, that should be defined separately.

**Geometric realization** of a graph is the quotient set of the disjoint union of semi–intervals \((0, 1/2]\) labeled with flags of this graph, modulo equivalence relation, in which 0–points of a flag is glued to 1/2 of another flag, if these flags are related by the boundary relation, or structure involution.

Depending on the context and/or type of labelling of \(\tau\), elements of \(F_{\tau}\) might be called *flags*, *leaves*, *tails* ... In the study of magmatic operad ([ChCorGi]) and the relevant binary trees, vertices of the relevant corollas are called *nodes*,
non-root flags are called *left child*, *right child* etc. We will try to attach all such “heteronyms” to our basic terminology of [BoMa].

Below the most typical labeling of our graphs will be (see details in [BoMa], Sec. 1.3.2 a) and 1.3.2 e), pp. 257–259):

(i) **Orientation.**

(ii) **Cyclic labeling.**

To give an orientation and cyclic labeling of corolla is essentially the same as to define it as a *planar* graph: corolla, embedded into an oriented real affine plane, with labeling compatible with its orientation.

Graphs endowed with various labelings form categories, upon which the operation of disjoint union $\sqcup$ defines a monoidal structure: see [BoMa], Sec. 1.2.4, pp. 254–255. Our central objects of study are initially defined only for connected graphs. Therefore, introducing this monoidal product, we must first take care of “empty” (or partially empty) graphs and explain details of their functoriality. The paper [BoMa] is interspersed with subsections directly or indirectly motivated by this necessity.

For the purposes of this paper, the most important graphs are labelled *trees* and *forests* – disjoint unions of trees, forming “*selva selvaggia e aspra e forte*”.

### 2.2. Operads and categories of operads.

(See [BoMa], Sec. 1.6, p.262). We recall here the first definition of operads in [BoMa], 1.6 (I), and morphisms of operads as in [BoMa], Sec. 1.6.1.

First of all, we fix a symmetric monoidal category of labelled graphs $\Gamma$ with disjoint union $\sqcup$ defines a monoidal structure: see [BoMa], Sec. 1.2.4, pp. 254–255. Our central objects of study are initially defined only for connected graphs. Therefore, introducing this monoidal product, we must first take care of “empty” (or partially empty) graphs and explain details of their functoriality. The paper [BoMa] is interspersed with subsections directly or indirectly motivated by this necessity.

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#### (i) An operad is a tensor functor between two monoidal categories $A : (\Gamma, \sqcup) \to (\mathcal{G}, \otimes)$ that sends any grafting morphism to an isomorphism.

#### (ii) A morphism between two operads is a functor morphism.

Denote this category of operads by $\Gamma\mathcal{GOPER}$.

### 2.3. Operads and collections as symmetric monoidal categories.

Following [BoMa], Sec. 1.8, we will introduce now the monoidal “white product” of two operads $A, B : (\Gamma, \sqcup) \to (\mathcal{G}, \otimes)$ by the formula

$$A \circ B(\sigma) := A(\sigma) \otimes B(\sigma)$$

extended to morphisms in a straightforward way.

Clearly, $(\Gamma\mathcal{GOPER}, \circ)$ is a symmetric monoidal category.
An important related notion is that of \textit{collection}. Starting with $\Gamma$ as above, denote by $\Gamma \text{COR}$ its subcategory, whose objects are corollas in $\Gamma$, and morphisms between them are isomorphisms.

Combining it with the ground category $(\mathcal{G}, \otimes)$ as above, we can introduce the category $\Gamma \mathcal{G} \text{COLL}$ of $\Gamma \mathcal{G}$–collections: its objects are functors $A_1 : \Gamma \text{COR} \rightarrow \mathcal{G}$, and morphisms are natural transformations between these functors.

The restriction of white product $\circ$ to $\Gamma \mathcal{G} \text{COLL}$ defines on it the structure of symmetric monoidal category. If $(\mathcal{G}, \otimes)$ has an identity object $1$, then the collection $1_{\text{coll}}$ sending each corolla to $1$ and each isomorphism of corollas to the identical isomorphism of $1$, is the identity collection.

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\textbf{2.4. Operads as monoids.} We briefly describe here a construction by B. Vallette \cite{Val}, reproduced in \cite{BoMa}, Appendix A, Subsection 5.

We will have to use here a stronger labeling of graphs in $\Gamma$ than just orientation. Besides orientation, connected objects of $\Gamma$ must admit a continuous real–valued function such that it decreases whenever one moves in the direction of orientation along each flag. Such graphs are called \textit{directed ones} (see \cite{BoMa}, Sec. 1.3.2 b).

A graph $\tau$ is called \textit{two–level graph}, if it is oriented, and if there exists a partition of its vertices $V_\tau = V_1^\tau \sqcup V_2^\tau$ with the following properties:

(i) \textit{Tails at $V_1^\tau$ are all inputs of $\tau$, and tails at $V_2^\tau$ are all outputs of $\tau$.}

(ii) \textit{All edges in $E_\tau$ go from $V_1^\tau$ to $V_2^\tau$.}

For any two $\Gamma \mathcal{G}$–collections $A^1, A^2$ define their product as

$$(A^2 \boxtimes_c A^1)(\sigma) := \text{colim}(\otimes_{v \in V_2^\tau} A^1(\tau_v)) \otimes (\otimes_{v \in V_2^\tau} A^2(\tau_v)).$$

Here colim is taken over the category of morphisms from two level graphs to $\sigma$.

\textbf{Theorem 2.4.1.} The product $\boxtimes_c$ is a monoidal structure on collections, and operads are monoids in the respective monoidal category.

See \cite{Val} and \cite{BoMa}.

\textbf{2.4.2 Freely generated operads} For any $\Gamma \mathcal{G}$–collection $A_1$ one can define another collection $\mathcal{F}(A_1)$ together with a canonical structure of operad on it, and for any operad $A$ each morphism of collections $A_1 \rightarrow A$ extends to a morphism of operads $f_A : \mathcal{F}(A_1) \rightarrow A$.

We can imagine $\mathcal{F}(A_1)$ as the operad freely generated by the collection $A_1$. 
2.5. Comonoids in operadic setup. We will now introduce a category $OP$ of operads given together with their presentations ([BoMa], Sec. 2.4). We start with $\Gamma$ and $\mathcal{G}$ as above.

One object of $OP$ is a family $(A, A_1, i_A)$, where $A$ is a $\Gamma\mathcal{G}$–operad, $A_1$ is a $\Gamma\mathcal{G}$–collection, such that $f_A : F(A_1) \to A$ is surjective.

Define on $OP$ a product $\odot$ by the formula
\[
(A, A_1, i_A) \odot (B, B_1, i_B) = (C, C_1, i_C),
\]
in which $C_1 := A_1 \odot B_1$ (cf. 2.3 above), $C :=$ the minimal suboperad, containing the image $(i_A \circ i_B)(A_1 \circ B_1) \subset A \circ B$, and $i_C$ is the restriction of $I_A \circ i_B$ on $A_1 \circ B_1$.

**Theorem 2.5.1.** (See [BoMa], Sec. 2.4).

(i) The product $\odot$ defines on $OP$ a structure of symmetric monoidal category.

(ii) The category $OP$ is endowed with the functor of inner cohomomorphisms $\text{cohom}_{OP} : OP^{\text{op}} \times OP \to OP$

so that we can identify, functorially with respect to all arguments,

\[
\text{Hom}_{OP}(A, C \odot B) = \text{Hom}_{OP}(\text{cohom}_{OP}(A, B), C)
\]

(iii) Therefore, one can define canonical coassociative comultiplication morphisms

\[
\Delta_{A,B,C} : \text{cohom}_{OP}(A, C) \to \text{cohom}_{OP}(A, B) \odot \text{cohom}_{OP}(B, C).
\]

**Corollary 2.5.2.** For any $A$, the coendomorphism operad $\text{coend}_{OP}A := \text{cohom}_{OP}(A, A)$

is a comagma in the sense of 1.3 above.

2.6. The magmatic operad. (See [ChCorGi]). Below we give a brief survey of some definitions and results from [ChCorGi], sometimes slightly changing terminology and notation.

Here objects of our basic symmetric monoidal category $(\Gamma, \sqcup)$ will be disjoint unions of oriented trees with the following additional labeling: *for each tree, its outcoming flags (or leaves) are cyclically ordered.* Corollas in it are one–vertex graphs with one root and at least two leaves. Connected objects can be obtained from a union of disjoint corollas by grafting each root of a corolla to one of leaves of another corolla. Morphisms are compatible with labeling.

An algebra over magmatic operad is a family $(\mathcal{A}, *)$ consisting of a set $\mathcal{A}$ with binary composition law $*: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

Thus, corollas in the magmatic category correspond to products

\[
(x_1 * ((x_2) * \ldots ((x_n)))\ldots),
\]
and generally, connected graphs in it correspond to monomials of generic arguments with all possible arrangements of brackets.

2.7. Quasigroup monomials and planar trees. Monomials that can be obtained by iteration of binary multiplication \( \ast \) as in (0.1) correspond to planar trees: see 2.1 above for discussion of planar corollas. Below, discussing quasigroups in general, and Moufang loops in particular, we will consider connected planar trees and quasigroup monomials as encoding each other in this way.

3. Categories of quadratic data and their relationships to supersymmetry

The exposition in this section starts with structures, considered in \([\text{Ma}, \text{MaVa}]\), and proceeds to their categorifications and (partial) enrichments.

3.1. Quadratic data. According to Sec. 2 of \([\text{MaVa}]\), one object of this category is a pair \((V, R)\), consisting of a finite-dimensional \(\mathbb{Z}\)-graded vector space \(V\) (over a field of characteristic \(\neq 2\)), and a subspace \(R \subset V^\otimes 2\).

A morphism \(f : (V, R) \to (W, S)\) is a morphism of graded vector spaces \(f : V \to W\), such that \(f^\otimes 2(R) \subset S\).

3.1.1. Quadratic data in a symmetric monoidal category. Let \(\mathcal{T}\) be a symmetric monoidal category.

The respective enrichment of the category of quadratic data has—as its objects—pairs \((V, R)\), where now \(V\) is an object of \(\mathcal{T}\), and \(R\) is a subobject of \(V^\otimes 2\). A morphism \(f : (V, R) \to (W, S)\), is a morphism \(f : V \to W\) in \(\mathcal{T}\), satisfying the same restriction, as above, in 3.1.

3.2. A SUSY categorification of symmetric monoidal categories. The data is defined as above, and equipped with a functor “sign change” \(c : \mathcal{V} \to \mathcal{V}\) such that \(c \circ c = \text{Id}_\mathcal{V}\).

Moreover, we require \(c(1) = 1\).

For example, in the category of \(\mathbb{Z}_2\)-graded vector spaces, \(c\) acts as sign change on the oddly graded vectors.

This allows us to imitate constructions, using \(\mathbb{Z}_2\)-gradings, when we pass from a commutative setup to the supercommutative one.

3.3. SUSY geometry. The main constructions, upon which we will focus further, are based upon systems of notions developed in \([\text{De}]\) and then specialised in the domain of stable SUSY families of curves. We omit this general setup. An interested reader can turn to other references: see \([\text{BrHePo}]\).
4. Operads of moduli spaces of SUSY curves

This section starts with a brief exposition of the recent work [KeMaWu], whose main goal is a construction of the extension of the classical modular operad, taking into account enrichments of curves to supercurves.

More precisely, stable degenerations in families of classical curves, describing strata in moduli spaces, involve gluing pairs of smooth points into double points, or nodes.

When we pass from curves to supercurves, the stability restrictions become considerably stricter, and as was shown in [De], only two types of nodes do not break superstability: Neveu–Schwarz ones and Ramond ones.

Finally, as we proceed to quantisation, additional difficulties arise: modular operads in the style of [GeKa] can be constructed only from superspaces parametrising SUSY curves of genus zero. The main reason for it can be intuitively described as follows: whenever we work with the usual operadic maps for stacks of supermoduli of higher genus, the usual operadic morphisms lose information about the relevant spinor bundles, and therefore cannot combine to a full operad. For details, see [KeMaWu], Sec. 4.1.

The existence of the quantum genus zero SUSY operad, restricted to Neveu–Schwarz and Ramond punctures, is the principal new result of this paper.

4.1. Category of graphs, encoding SUSY curves. Start with choosing of $k_{NS} \in \mathbb{Z}_{\geq 0}$ and $k_{R} \in 2\mathbb{Z}_{\geq 0}$.

According to the Sec. 2.3 of [KeMaWu], one object of this category is a $(k_{NS}, k_{R})$-SUSY (or simply a SUSY–graph) $\tau = (F_{\tau}, V_{\tau}, \partial_{\tau}, j_{\tau})$, endowed with

(i) a genus labeling $g_{\tau} : V_{\tau} \rightarrow \mathbb{Z}_{\geq 0}$;

(ii) a puncture colouring $c_{\tau} : F_{\tau} \rightarrow \{NS, R\}$, such that each vertex $v \in V_{\tau}$ the number of adjacent to this vertex flags with color $R$ is even;

(iii) Two separate labelings of $NS$–tails and $R$–tails, that are bijections

$$l_{\tau,NS} : \{1, \ldots, k_{NS}\} \rightarrow T_{\tau,NS} := F_{\tau,NS} \cap T_{\tau},$$

$$l_{\tau,R} : \{1, \ldots, k_{R}\} \rightarrow T_{\tau,R} := F_{\tau,R} \cap T_{\tau}.$$

When these structures are chosen, then the sets of flags, tails, and edges, adjacent to each vertex $v \in V_{\tau}$, acquire colours $NS$ (Neveu–Schwarz) or $R$ (Ramond), and are called respectively.

A morphism in this category is a morphism of respective graphs in the sense of [BoMa], compatible with the genus labelings and colourings of flags. Some tails of the same colour can be grafted, and some pairs of the tails of the same colour can be virtually contracted.
4.2. **SUSY curves with punctures.** The notions of SUSY curves, their families, morphisms, etc. are natural extensions of the respective notions in the formalism of schemes, starting with a replacement of structure sheaves of commutative rings of schemes by structure sheaves of $\mathbb{Z}_2$ graded supercommutative rings of superschemes.

Skipping these foundational preparations, we pass to the description of relevant notions of SUSY curves with punctures, their families, and notions of stability, from [De] and [FeKaPo]:

**Definition 4.2.1.** A family of SUSY curves with $k_{\text{NS}}$ Neveu–Schwarz punctures and $k_{\text{R}}$ Ramond punctures over the base $B$ consists of the following data:

(i) A smooth proper morphism of superschemes $\pi : M \to B$, whose generic fibres have relative dimension $1|1$.

(ii) A sequence of sections $s_i : B \to M$, $i = 1, \ldots, k_{\text{NS}}$, such that on each fibre of $\pi$ the reductions of $s_i$ and $s_j$ are different for $i \neq j$.

These reductions are called Neveu–Schwarz punctures.

(iii) A sequence of components $r_j$, $j = 1, \ldots, k_{\text{R}}$ of an unramified effective Cartier divisor $\mathcal{R}$ of codimension $0|1$ of degree $k_{\text{R}}$.

These components are called Ramond punctures.

(iv) The line bundle $\mathcal{D}$, a subbundle of the tangent bundle $\mathcal{T}_M$ of rank $0|1$, such that the commutator of vector fields induces an isomorphism

$$\mathcal{D} \otimes \mathcal{D} \to (\mathcal{T}_M/\mathcal{D})(-\mathcal{R}).$$

4.3. **Stability.** Consider a family of SUSY curves with punctures over $B$ as above. It is called (super)stable, if it satisfies the following restrictions:

(i) $M$ is a proper, flat and relatively Cohen–Macaulay superscheme over $B$.

(ii) $M$ contains an open fibrewise dense subset $U$ containing all sections $s_i$ and $r_j$ and such, that $U/B$ is smooth of relative dimension $1|1$.

(iii) The reduction $M_{\text{red}} \to B_{\text{red}}$ is a stable family of marked curves.

4.4. **Dual graphs of stable SUSY families.** The dual graph $\tau$, associated with a stable family of SUSY curves (as in Def. 4.2.1 and Sec. 4.3 such that $B_{\text{red}}$ is a point) is defined as follows:

(i) The set of its vertices $V_\tau$ is identified with the set of irreducible components of $M$.

(ii) Flags of $\tau$ with boundary $v$ are identified with special points on the respective irreducible component of $M$. 
(iii) The involution $j_\tau$ changes places of halves of edges with one boundary $v$. In particular, it distinguishes punctures: those belonging to different components of a stable curve and those corresponding to double points of one component.

(iv) The genus labeling marks irreducible components by their genera.

(v) The type of a puncture breaks the whole set of flags into two disjoint subsets: $F_\tau = F_{\tau,NS} \cup F_{\tau,R}$.

(vi) The labelings $l_{\tau,NS}$, resp. $l_{\tau,R}$, mark flags, corresponding to respective punctures.

4.5. Moduli stacks of stable SUSY families. Generally, the functor of stable SUSY families of curves of genus $g$ with a fixed dual graph is represented by a smooth and proper Deligne–Mumford superstack $\overline{M}_{g,kNS,kR}$: this was proved in [FeKaPo].

The families of curves with at least one Neveu–Schwarz node, resp. at least one Ramond node, are represented by the boundary Cartier divisor $\Delta_{NS}$, resp. $\Delta_R$.

A fundamental role in the construction of operadic compositions for moduli stacks is played by gluing punctures of stable SUSY curves.

An attempt to lift all operadic morphisms from the classical to the SUSY setup was made in [KeMaWu], Sec. 4.

In this final part of our study we draw attention of the reader to the fact, that if we want to have these liftings and canonical morphisms among them to be unique, so that we get an actual operad on the level of SUSY moduli spaces, we have to restrict ourselves by genus zero and Neveu–Schwarz and Ramond punctures.

4.6. SUSY modular operad breaking down. The breaking of higher genus morphisms is explained in [KeMaWu], Sec. 4.1: any kind of gluing requires looking at higher general components: see equation (4.1.2), (4.1.3), (4.1.4).

However, if we consider only genus zero components and gluing of Neveu–Schwarz, resp. Ramond, punctures, the same diagrams become united into one operad.

5. Operadic structure of supersymmetry algebras

In Section 5 of [CoMaMar3], it was shown that certain classes of classical and quantum codes carry the structure of an algebra over a version of the little squares operad. In this section we show that the set of representations of 1D supersymmetry algebras carries the structure of an operad.
We first recall some general facts about 1D supersymmetry algebras and their description in terms of Adinkra graphs and linear codes. We then use the description in terms of codes to introduce the operadic structure.

5.1. Supersymmetry algebras and codes. In the setting of supersymmetric quantum mechanics, with a 1-dimensional space-time with time coordinate $t$ and a zero-dimensional space, the $(1|N)$ Superalgebras are generated by operators $Q_1, Q_2, \ldots, Q_N$, which give the supersymmetry generators, and $H = i\partial_t$, subject to the commutation and anticommutation relations:

\[
\left[ Q_k, H \right] = 0 \quad \text{and} \quad \{ Q_k, Q_\ell \} = 2\delta_{k\ell}H,
\]

where $\delta_{k\ell}$ stands for the Kronecker delta.

Representations of these algebras on bosonic and fermionic fields, $\{ \phi_1, \ldots, \phi_m \}$ and $\{ \psi_1, \ldots, \psi_m \}$, respectively, are of the following form:

\[
Q_k \phi_a = c \partial_t^\lambda \psi_b,
\]

\[
Q_k \psi_b = \frac{i}{c} \partial_t^{1-\lambda} \phi_a,
\]

with $c \in \{-1, 1\}$ and $\lambda \in \{0, 1\}$.

A graphical method for classifying these supersymmetry algebras representations was introduced by Faux and Gates, (see [FaGa]), in terms of decorated bipartite graphs with additional structure, called Adinkras. The geometry of Adinkras, their relation to a class of Grothendieck’s dessins d’enfant, and their classification in terms of linear codes were developed in [DFGHIL1], and [DIKLM1, DIKLM2].

Summarizing the correspondence between 1D SUSY algebras and Adinkras, we have the following setting, [FaGa].

Let $A$ be a finite graph with no looping edges and no parallel edges. Let $V(A)$ and $E(A)$ be the sets of vertices and edges. Such a graph is an $N$-dimensional chromotopology if it satisfies the following properties:

(a) $A$ is $N$-regular and bipartite, with vertices in the bipartition colored white and black respectively;

(b) The edges in $E(A)$ are colored by $N$ colors, labelled by $\{1, 2, \ldots, N\}$, with every vertex incident to exactly one edge of each color;

(c) For any pair $i \neq j$ of colors the edges in $E(A)$ labelled with colors $i$ and $j$ form a disjoint union of 4-cycles.

A ranking of the graph $A$ is a partial ordering of $V(A)$ determined by a function $h : V(A) \to \mathbb{Z}$. A dashing of the graph $A$ is a function $d : E(A) \to \mathbb{F}_2$ that assigns to each edge value 0 (for solid) or value 1 (for dashed). The graph $A$ is well dashed if all the 2-colored 4-cycles have odd-dashing, namely have an odd number of dashed edges.
An Adinkra is a well-dashed, $N$-dimensional chromotopology, endowed with a ranking where all the white vertices have even ranking and all the black vertices have odd ranking.

In the classification of \cite{FaGa}, the white vertices correspond to the boson fields and their time derivatives and the black vertices to the fermionic fields and their time derivatives. There is an edge between a pair of vertices whenever the corresponding fields are related by one of the relations (2) or (3). The edge is oriented from the white to the black vertex if $\lambda = 0$ and viceversa if $\lambda = 1$. The edge is dashed if $c = -1$ and solid if $c = 1$.

We refer the reader to \cite{FaGa} and \cite{DFGHIL1,DFGHIL2,DFGHILM,DIKLML} for more details about this classification result.

Recall that a binary linear code $L \subset F_2^N$ is even if every code word $c \in L$ has even weight $w(c) = \# \{ w_i = 1 \} \in 2\mathbb{Z}_{\geq 0}$ and it is doubly even if every code word $c \in L$ has weight that is divisible by 4. Theorem 4.4 of \cite{DFGHIL2} shows that Adinkras are classified by doubly-even binary linear codes, in the sense that every Adinkra $A$ is obtained as a quotient $A = F_2^N/L$, for $L$ a doubly-even binary linear code and all the Adinkra structure (that is bipartition, ranking, well-dashing) is determined by this description.

**Proposition 5.1.1.** For a fixed $N \in \mathbb{N}$, consider the set $\mathcal{D}_N$ of all doubly even linear codes $L \subset F_2^N$,

$$\mathcal{D}_N := \{ L \subset F_2^N \mid L \text{ doubly even linear code} \}.$$

Then, the collection $\{ \mathcal{D}_N \}_{N \geq 1}$ forms a non-unital operad under the composition

$$\gamma : \mathcal{D}_N \times \mathcal{D}_{k_1} \times \cdots \times \mathcal{D}_{k_N} \to \mathcal{D}_{k_1 + \cdots + k_N},$$

given by

$$\gamma(L; L_1, \ldots, L_N) = \left\{ c \in F_2^{k_1 + \cdots + k_N} \mid c = (c^{(1)})_{c_1} \ldots, (c^{(N)})_{c_N} \text{ with } \right.$$  
$$\left. (c^{(1)}, \ldots, c^{(N)}) \in L, \text{ and } c_i \in L_i \right\}.$$  

**Proof.** The code $\gamma(L; L_1, \ldots, L_N) \subset F_2^{k_1 + \cdots + k_N}$ is a linear code since it is a direct sum of copies of the linear codes $L_i$ corresponding to the subspaces $F_2^{r_i}$, with $i$ such that $c^{(i)} = 1$. It is also doubly even since $w(c) = \sum_{c^{(i)} = 1} w(c_i)$ and each code $L_i$ is doubly even so each $w(c_i)$ is divisible by 4, hence $w(c)$ is also divisible by 4 for all $c \in \gamma(L; L_1, \ldots, L_N)$.

The composition (1) satisfies the associativity condition given by the identities

$$\gamma(\gamma(L^{(N)}; L^{(k_1)}, \ldots, L^{(k_N)}); L^{(r_1,1)}, \ldots, L^{(r_1,k_1)}, \ldots, L^{(r_N,1)}, \ldots, L^{(r_N,k_N)}) =$$  
$$\gamma(L^{(N)}; \gamma(L^{(k_1)}, L^{(r_1,1)}, \ldots, L^{(r_1,k_1)}), \ldots, \gamma(L^{(k_N)}, L^{(r_N,1)}, \ldots, L^{(r_N,k_N)})).$$
for \( L^{(N)} \in \mathcal{R}_N \), \( L^{(k_i)} \in \mathcal{R}_{k_i} \), \( i = 1, \ldots, N \), and \( L^{(r_i,\ell_i)} \in \mathcal{R}_{r_i,\ell_i} \) with \( \ell_i = 1, \ldots, k_i \). It also satisfies the symmetry conditions given by

\[
\gamma(L; L_{\sigma^{-1}(1)}, \ldots, L_{\sigma^{-1}(N)}) = \tilde{\sigma}(\gamma(L; L_1, \ldots, L_N)),
\]

where on the right-hand-side \( \tilde{\sigma} \in \Sigma_{k_1+\cdots+k_N} \) is the permutation that splits the set of indices into blocks of \( k_i \) indices and permutes the blocks by \( \sigma \), and

\[
\gamma(L; \sigma_1(L_1), \ldots, \sigma_N(L_N)) = \tilde{\sigma}(\gamma(L; L_1, \ldots, L_N)),
\]

where on the right-hand-side \( \tilde{\sigma} \in \Sigma_{k_1+\cdots+k_N} \) is the permutation that acts on the \( i \)-th block of \( k_i \) indices as the permutation \( \sigma_i \).

However, the operad is non-unital since \( \mathcal{R}_1 \) only consists of the trivial subspace of \( \mathbb{F}_2 \) and that does not satisfy the unital conditions \( \gamma(L; 1, \ldots, 1) = L = \gamma(1; L) \).

Note that it is a non-unital operad in the stronger sense of [Markl], since the composition (4) is obtained from insertion operations

\[
o_i : \mathcal{D}_N \times \mathcal{D}_M \to \mathcal{D}_{N+M-1}
\]

of the form

\[
L \circ_i L' = \{ c \in \mathbb{F}_2^{N+M-1} \mid c = (c^{(1)}, \ldots, c^{(i-1)}, c^{(i)} c', c^{(i+1)}, \ldots, c^{(N)}) \text{ with } (c^{(1)}, \ldots, c^{(N)}) \in L \text{ and } c' \in L' \} ,
\]

with

\[
\gamma(L; L_1, \ldots, L_N) = (\cdots ((L \circ_N L_N) \circ_{N-1} L_{N-1}) \cdots \circ_1 L_1). \]

□

**Corollary 5.1.1.** Let \( \mathcal{R}_N \) be the set of representations of the form (2) or (3) of a \((1|N)\) SUSY algebra (1). Then, the \( \{\mathcal{R}_N\}_{N \geq 1} \) form a non-unital operad.

**Proof.** We use the classification of [FaGa] in terms of Adinkras and the result of Theorem 4.4 of [DFGHL12] recalled above, to describe equivalently the set \( \mathcal{R}_N \) in terms of doubly even linear codes in \( \mathbb{F}_2^N \). Thus, the operadic composition

\[
\gamma : \mathcal{R}_N \times \mathcal{R}_{k_1} \times \cdots \times \mathcal{R}_{k_N} \to \mathcal{R}_{k_1+\cdots+k_N}
\]

associated to a set \( (A, A_1, \ldots, A_N) \) of representations, identified with the corresponding Adinkras \( A = \mathbb{F}_2^N / L \), \( A_i = \mathbb{F}_2^{k_i} / L_i \), \( i = 1, \ldots, N \), with \( L \) and the \( L_i \) doubly even linear codes, the representation determined by the Adinkra

\[
\gamma(A; A_1, \ldots, A_N) := \mathbb{F}_2^{k_1+\cdots+k_N} / \gamma(L; L_1, \ldots, L_N),
\]

where \( \gamma(L; L_1, \ldots, L_N) \) is the operadic composition of Proposition 5.1.1 □
Hidden symmetries of Adinkras are investigated in this section. It is proved that Adinkras are invariant under Moufang loop symmetries (see \cite{Sm} for an introduction to quasigroups, loops and Moufang loops). These symmetries lead to exploring a very specific class of dessins, corresponding to the chromotopologies of Adinkras (see \cite{DIKL1} for an exposition on the relation between dessins and chromotopologies).

In what follows, all our definitions and constructions rely on Sec. 6 of our previous paper \cite{CoMaMar2} concerning Moufang patterns.

6.1. **Code loops and central extensions.** Sec. II.3, \cite{Che} presents a generalisation of extension theory defined initially for groups, for the case of loops and quasigroups. This topic has become a popular method, in more recent years \cite{NaSt}. We will be considering in what follows the extension of a doubly even code.

A loop $\mathcal{L}$ is called a code loop of $L$ if $\mathcal{L}$ has a central subgroup $Z$ of order 2 such that $\mathcal{L}/Z \cong L$, as an elementary abelian 2-group and comes equipped with some additional properties. As depicted in Def.2.4 \cite{Hsu} (and in a more general context Prop. 3.1. \cite{NaSt}), one needs to define three functions, $\alpha : L \rightarrow Z, \phi : L \times L \rightarrow Z$ and $\psi : L \times L \times L \rightarrow Z$ which play a central role in the definition of the multiplication operation for the code loop and also in the definition of the (central) extension of the (normal) subgroup $Z$ by $L$.

Consider a doubly even binary code $L \subset F_2^N$ and let $Z$ be a group of order 2. Then, by using Thm. 2.5 \cite{Hsu} a code loop of a doubly even code exists and is unique up to isomorphism. It is important to note that code loops are certain Moufang loop extensions of doubly even binary codes.

So, supposing that $L$ is a binary linear code in $F_2^N$ which is doubly even, then one can associate a Moufang loop $\mathcal{L}$ to it such that it satisfies the following exact sequence:

$$1 \rightarrow Z \rightarrow L \rightarrow L \rightarrow 1,$$

with $L$ a doubly even binary code, $\mathcal{L}$ the code loop of $L$ (Moufang loop) and $Z$ is a group of order 2 which is a central normal subgroup of $\mathcal{L}$. One has that $L$ is isomorphic to the factor loop $\mathcal{L}/Z$.

We prove the following statement.

**Theorem 6.1.1.** Every Adinkra, interpreted as the quotient $F_2^N/L$ (with $L$ being a doubly-even binary linear code) is invariant under a Moufang loop factor (i.e. a Moufang loop $\mathcal{L}$ quotiented by a finite group $Z$), where $Z$ and $\mathcal{L}$ satisfy the following exact sequence:

$$1 \rightarrow Z \rightarrow \mathcal{L} \rightarrow L \rightarrow 1,$$
with:
- \( L \) a doubly even binary code
- \( \mathcal{L} \) a Moufang loop
- \( Z \) a normal subgroup.

\textbf{Proof.} We apply the discussion above. We have that \( L \) is a doubly even binary code, and by Thm. 2.5 [Hsu] we have that a code loop of a doubly even code exists and is unique (up to isomorphism). Using the construction depicted above, we can proceed to defining the following short exact sequence

\[ 1 \to Z \to \mathcal{L} \to L \to 1, \]

where \( L \) is a doubly even binary code, \( \mathcal{L} \) is a Moufang loop and \( Z \) is a normal subgroup.

Given that Adinkras can be identified to the quotient \( F^N_2/L \) and that \( L \cong \mathcal{L}/Z \) this implies now that \( F^N_2 \) is quotiented by a Moufang loop. So, in other words, Adinkras are invariant under Moufang loop symmetries. □

6.2. \textbf{Adinkras, graphs and dessins.} From Sec. 5.1 we have that Adinkras form a special class of directed decorated graphs, being \( N \)-regular, \( N \)-edged colored bi-partite graphs, where:
- vertices are colored black or white and correspond to particles. White vertices correspond to the real bosonic component fields; black vertices correspond to real fermionic component fields.
- edges correspond to the supersymmetry generators, so that to each vertex \( N \) tails are attached.

\textbf{Proposition 6.2.1.} \textit{Let} \( A \) \textit{be the graph of an Adinkra. Then, this graph is invariant under Moufang loop symmetries.}

\textbf{Proof.} Previously we proved in Thm. 6.1.1 that every Adinkra is invariant under Moufang loop symmetries. The Moufang loop \( \mathcal{L} \) follows from the exact sequence from formula (5) and is constructed in a unique way from a given doubly even binary code \( L \). So, since the graph reproduces the relations within the Adinkra it follows naturally that the graph \( A \) is invariant under Moufang loop symmetries. □

[DIKLM] implies that, by forgetting all decorations of the Adinkra graphs: dashing, orientation, weights of vertices, (etc) one obtains a coarse version called the chromotopology of the Adinkra. This object encapsulates the topology of an Adinkra together with the vertex bipartition (coloring each vertex black or white) and the edge \( N \)-coloring. One can canonically realise these graphs as Grothendieck’s dessins d’enfants. The \( N \)-regular \( N \)-coloring of edges gives a cyclic ordering of the edges at each vertex of the graph based on their color, providing
6.3. Symmetries of Adinkras’ dessins. In this section the geometries of the
dessins, corresponding to the chromotopologies of Adinkras are investigated. The
symmetries of those special dessins have repercussions in relation to problems
concerning actions of the absolute Galois group $G_{\mathbb{Q}}$ and relates to [CoMaMar1].
This result appears also as a geometric counterpart of the recent result showing
that the Grothendieck–Teichmüller group $GT$ has dihedral symmetry relations
(see [Co]).

From Sec. 6.2 it followed that chromotopologies of Adinkras can be identified to
dessins. But dessins can interpreted as being in bijection with transitive subgroups
of the symmetric group $S_n$, generated by 2 elements (for all $n \geq 2$) up to conjugacy
(in cases where the dessin corresponds to a degree $n$ map). This follows from a
very geometric construction, which for the sake of clarity, we recall.

Consider the projective line minus three points $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The relation from
dessins to degree $n$, finite étale coverings of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is well known. Take a
point $x \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its fiber, the set $\{x_1, \cdots, x_n\}$. Then the topological
loops originating from the point $x$ and going around 0, 1 and $\infty$ generate permu-
tations of the points $\{x_1, \cdots, x_n\}$ denoted respectively $\sigma_0, \sigma_1, \sigma_\infty$ and satisfying
$\sigma_0 \sigma_1 \sigma_\infty = 1$. The permutation $\sigma_1$ is of order 2 and the permutations $\sigma_0, \sigma_1$ gen-
erate a subgroup of the symmetric group $S_n$ which is transitive, in case the covering
is connected.

So, to a dessin we can attach a triplet $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle \in (S_n)^3$ where:
- $\sigma_0 \sigma_1 \sigma_\infty = id_{S_n}$
- $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ is a transitive subgroup of $S_n$.

In short, denoting by $F_2$ the free group generated by two elements, one can thus
identify those objects $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle$ with some group homomorphisms $\tilde{\phi} : F_2 \to S_n$, for which the group $\tilde{\phi}(F_2)$ acts transitively on $\{1, 2, \cdots, n\}$. The equivalence
class of these triples forms the dessins; whereas isomorphism classes form the
monodromy group of the corresponding dessin.

Let us go back to the setting of Sec. 5.1 (i.e. $N$-regular bipartite graphs with $2m$
vertices). Let $D$ be a dessin with $n(=Nm)$ edges. Then, we can uniquely associate
to it (up to unique isomorphism) a finite set of size $n$ equipped with a pair of
permutations $\hat{\sigma}, \hat{\alpha} \in S_n$ acting transitively on the finite set which consists of the
set of edges of $D$ and $\hat{\sigma}, \hat{\alpha}$ are the generators of the transitive group corresponding
to the Dessin $D$.

**Proposition 6.3.1.** The dessins corresponding to chromotopologies of Adinkras
(being $N$-regular bipartite graphs having a number $n$ of edges and $2m$ of vertices)
can be uniquely associate—up to unique isomorphism—to a finite set of size $n,$
equipped with a pair of permutations \( \sigma, \alpha \in S_n \), where \( \sigma \) and \( \alpha \) are both decomposed into \( m \) cycles of length \( N \).

**Proof.** Indeed, the graph ought to be an \( N \)-regular bipartite graph (with \( m \) black vertices and \( m \) white vertices). The finite set corresponds to the set of (labelled) edges of the chromotopology graph. By definition of the dessin \( \sigma \) and \( \alpha \) correspond to a decomposition into cycles, where each cycle is a rotation about a black vertex (resp. white) of the finite set. Since we have that the graph is \( N \)-regular and bipartite with \( 2m \) vertices we have \( m \) blocks of cycles of length \( N \) defining \( \sigma \) and \( \alpha \) respectively.

**Proposition 6.3.2.** Consider the subcategory of dessins corresponding to Adinkras. Consider a dessin \( \mathcal{D} \) (\( N \)-regular bipartite graph on \( 2m \) vertices). Then, there exists a partition \( V_1 \) and \( V_2 \) of the set \( \{1, \cdots, Nm\} \) into \( m \) subsets of size \( N \) defining a specific dessin for which its automorphism group fixing the bipartite blocks (i.e. fixing the blocks of white and respectively the black vertices set-wise) is \( S_{N2m} \).

**Proof.** Consider pair of partitions (say \( V_1 \) and \( V_2 \)) of the set \( \{1, \cdots, mN\} \) into \( m \) subsets of size \( N \). This encodes an \( N \)-regular bipartite graph. Then, there exists partitions \( V_1 \) and \( V_2 \) of \( \{1, \cdots, mN\} \) such that: the automorphism group fixing the bipartite blocks (i.e. fixing the blocks of white and respectively the black vertices set-wise) is \( S_{N2m} \). This follows from the Lem 2.3 in [Ja].

The following proposition highlights the peculiar symmetries of the Adinkra in relation to the dessins.

**Proposition 6.3.3.** Let us consider an Adinkra \( F_N^2 / L \), where \( L \) is a doubly even binary code and the corresponding dessin \( \mathcal{D} \) to the Adinkra. Then, \( \mathcal{D} \) is a graph which is invariant under a symmetry group \( \mathcal{G} \) such that \( \mathcal{G} \supseteq L \), where \( L \) is the Moufang loop obtained by the extension defined in Eqn. (5) of the doubly even binary code \( L \).

**Proof.** Indeed, the dessin corresponding to the Adinkra \( F_N^2 / L \) encapsulates the topology of an Adinkra. Somehow, it remains a coarse version of the graph \( A \) associated to the Adinkra (because we “forget” the dashing of edges as well as weights of the vertices). In Prop. 6.2.1 we showed that the graph \( A \) carries the symmetries of the Moufang loop \( \mathcal{L} \) associated to its Adinkra. The coarse version (the dessin) therefore inherits the Moufang loop symmetries. However, given that the dashing and weighing of vertices are gone, one can gain more symmetries than in the case of \( A \). Therefore, the dessin is invariant under the symmetries of a group \( \mathcal{G} \) which contains as a subloop the Moufang loop \( \mathcal{L} \).
Note that Adinkras with chromotopology can be associated to an $N$-cube. The $N$-cube has the hypercube symmetry but in particular inherits the dihedral $D_{2N}$ symmetry also.

**Corollary 6.3.1.** Consider an Adinkra $F^{N}_{2}/L$. Then, it has the structure of the quotient of an $N$-hypercube by a Moufang loop and inherits dihedral symmetries, quotiented by relations given by $L$.

**Proof.** The proof follows from the fact that every connected Adinkra chromotopology is isomorphic to a quotient of a colored $N$-dimensional cube by the code of the chromotopology and from the natural symmetries of the hypercube.

Finally, since the $N$-hypercube has the symmetry of the wreath product of $S_{2} \wr S_{N}$ and $S_{N}$ has as a subgroup the dihedral group $D_{2N}$, we can say that this $N$-cube inherits the symmetries of the dihedral $D_{2N}$ group. So, the Adinkra (and chromotopology) have the symmetry of a dihedral $D_{2N}$ group quotiented by the relations given by $L$. $\square$

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**References**

[BoMa] D. Borisov, Yu. Manin, *Generalized operads and their inner cohomomorhisms*, In: Geometry and Dynamics of Groups and Spaces (In memory of Aleksander Reznikov). Ed. by M. Kapranov et al. Progress in Math., vol. 265 (2008), Birkhäuser, Boston, pp. 247–308. arXiv:math.CT/0609748.

[BrHePo] U. Bruzzo, D. Hernández Ruipérez, A. Polischuk, *Notes on fundamental algebraic supergeometry. Hilbert and Picard superschemes*, [arXiv:2008.00700 [math.AG]].

[Co] N. Combe, *Dihedral symmetries of the Grothendieck–Teichmüller group*, To appear.

[Che] O. Chein, *Examples and methods of construction*, Chapter II in Quasigroups and Loops: Theory and Applications (O. Chein, H. O. Pflugfelder, J. D. H. Smith, eds.). Heldermann Verlag, Berlin (1990), pp. 27–93.

[ChCorGi] C. Chenavier, C. Cordero, S. Giraudo, *Quotients of the magmatic operad: lattice structures and convergent rewrite systems*, arXiv:1809.05083v2.

[CoMaMar1] N. Combe, Yu. Manin., M. Marcolli, *Dessins for modular operad and Grothendieck–Teichmüller group*, “Topology and Geometry A Collection of Essays Dedicated to Vladimir G. Turaev”, European Mathematical Society (2021), arXiv:math.AG/2006.13663.

[CoMaMar2] N. C. Combe, Yu. Manin, M. Marcolli, *Moufang patterns and geometry of information*, To be published in the Collection, dedicated to Don Zagier, in Pure and Applied Math. Quarterly, arXiv:2107.07480, 42 pp.
[CoMaMar3] N. C. Combe, Yu. Manin, M. Marcolli, Quantum operads, to be published in the “C. N. Yang at 100” volume, arXiv:2112.15237, 34 pp.

[De] P. Deligne. Lettre à Manin, Princeton. URL: https://publications.ias.edu/sites/default/files/lettre-a-manin-1987-09-25.pdf

[DFGHIL1] C.F. Doran, M.G. Faux, S.J. Gates, Jr., T. Hubsch, K.M. Iga, G.D. Landweber, On graph-theoretic identifications of Adinkras, supersymmetry representations and superfields, Int. J. Mod. Phys. A 22 (2007), pp. 869–930.

[DFGHIL2] C.F. Doran, M.G. Faux, S.J. Gates, Jr., T. Hubsch, K.M. Iga, G.D. Landweber, Relating doubly-even error-correcting codes, graphs, and irreducible representations of N-supersymmetry, in “Discrete and computational mathematics”, pp.53–71, Nova Sci. Publ., (2008).

[DFGHILM] C.F. Doran, M.G. Faux, S.J. Gates, Jr., T. Hubsch, K.M. Iga, G.D. Landweber, R.L. Miller, Codes and supersymmetry in one dimension, Adv. Theor. Math. Phys., Vol.15 (2011) N.6, pp. 1909–1970.

[DIKLM1] C.F. Doran, K. Iga, J. Kostiuk, G. Landweber, S. Méndez-Diez, Geometrization of N-extended 1-dimensional supersymmetry algebras, I, Adv. Theor. Math. Phys. 19 (2015), no. 5, pp. 1043–1113.

[DIKLM2] C.F. Doran, K. Iga, J. Kostiuk, S. Méndez-Diez, Geometrization of N-extended 1-dimensional supersymmetry algebras, II, arXiv:1610.00983

[FaGa] M. Faux, S.J. Gates, Jr., Adinkras: a graphical technology for supersymmetric representation theory, Phys. Rev. D 71(3) (2005), 065002.

[FeKaPo] G. Felder, D. Kazhdan, A. Polishchuk, The moduli space of stable supercurves and its canonical line bundle, arXiv:2006.13271v2 [math.AG].

[GeKa] E. Getzler, M. Kapranov, Modular operads, In: Compositio Math. (1998), 110:1, pp. 65–125.

[HoKrStT] H. Hohnhold, M. Kreck, St. Stolz, P. Teichner, Differential forms and 0–dimensional supersymmetric field theories, Quantum Topology 2 (2011), pp. 1–41.

[Hsu] T. Hsu, Explicit constructions of code loops as centrally twisted products, Math. Proc. Camb. Phil. Soc. 128 (2000), pp. 223–232

[Ja] J.P. James, Partition actions of symmetric groups and regular bipartite graphs, Bull. London Math. Soc. 38 (2006), pp. 224–232

[KaSch] M. Kashiwara, P. Schapira, Categories and sheaves, Springer Verlag (2006), pp. x + 497.

[KeMaWu] E. Kessler, Yu. Manin, Y. Wu, Moduli spaces of SUSY curves and their operads, arXiv:2202.10321 [mathAG], 21 pp.

[Ke] B. Keller, Introduction to A_infinity–algebras and modules, In ”Homology, Homotopy and Applications” 3 (2001), pp.1–35 (electronic), arXiv:math/9910179v2, 31 pp.

[Ma] Yu. Manin, Quantum groups and non–commutative geometry, Publ. de CRM, Université de Montréal (1988), 91 pp.

[MaPe] Yu. Manin, I. Penkov, The formalism of left and right connections on supermanifolds, In ”Lectures on Supermanifolds, Geometrical Methods and Conformal Groups”, World Scientific (1989), pp. 3–13.

[MaVa] Yu. Manin, B. Vallette, Monoidal structures on the categories of quadratic data, Documenta Mathematica 25 (2020), pp. 1653 – 1712, arXiv:1902.03778
[Markl] M. Markl, *Operads and PROPs*, Handbook of algebra. Vol. 5, 87–140, Elsevier/North-Holland, 2008.

[NaSt] P. T. Nagy, K. Strambach, *Schreier loops*, Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 3, 759–786

[Sm] J. D. H. Smith, *Quantum quasigroups and loops*, Journ. of Algebra 456 (2016), pp. 46–75.

[Val] B. Vallette, *A Koszul duality for PROPs*, [arXiv:math0411542v3](http://arxiv.org/abs/math0411542), 78 pp.

[WuYau] Y. Wu, Sh.-T. Yau, *Comparison theorems of phylogenetic spaces and the moduli spaces of curves*, [arXiv:2006.04319](http://arxiv.org/abs/2006.04319) [math.AG].

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