The graviton background at early times in string cosmology

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We discuss some peculiar properties of the stochastic graviton background predicted by string cosmology. At Planckian times, for the values of the parameters of the model which are more interesting for the detection in gravitational wave experiments, the number density of gravitons is parametrically large compared to Planck density, and is peaked at small energies. The large parameter is related to the duration of the string phase and is a characteristic of string cosmology. The typical interaction time is parametrically larger than the Planck or string time. Therefore the shape of the graviton spectrum is not distorted by thermal effects in the Planck era and can carry informations on the pre-big bang phase suggested by string cosmology.

In recent works, Gasperini and Veneziano have developed a cosmological model which is based on the low energy action of string theory (see also [1,2]). Besides its theoretical interest, the model has important phenomenological consequences: the mechanism of amplification of vacuum fluctuations produces a stochastic graviton background which, for some values of the parameters of the model, is close to the planned sensitivity of the LIGO and VIRGO experiments, and within the sensitivity at which the advanced LIGO detector aims [3]. Electromagnetic fluctuations are also amplified, contrarily to what happens in inflationary cosmology, and might provide the seed for the generation of galactic magnetic fields [14].

The model has no initial singularity, but rather starts from the string perturbative vacuum at \( t \rightarrow -\infty \). Its evolution at this stage is governed by the string effective action at lowest order both in the string constant \( a' \) and in the loop corrections, which are governed by \( g^2 = e^{\phi} \), where \( \phi \) is the dilaton field. The curvature grows until, at a value \( \eta = \eta_s \) of conformal time, it becomes of the order of the string scale. At this point higher orders in \( a' \) in the effective action become important (while \( g^2 \) corrections can be still small), and the Hubble constant \( H \) approaches a constant value \( H_s \) [13]. In this phase (‘string phase’) the scale factor \( a(\eta) \) and the dilaton \( \phi(\eta) \), in the so called ‘string frame’, evolve as

\[
a(\eta) = -\frac{1}{H_s \eta}, \quad \phi(\eta) = \text{const.} - 2\beta \log \frac{\eta}{\eta_s^2}. \tag{1}
\]

Here \( \beta \) is a dimensionless constant whose value, as the value of \( H_s \), is in principle fixed by the \( a' \) corrections. This phase lasts up to some conformal time \( \eta = \eta_1 \). The corresponding value of cosmic time \( t_1 \) will be of the order of the Planck time \( t_{\text{Pl}} = l_{\text{Pl}}/c \) or of the string time \( \lambda_s/c \), where \( l_{\text{Pl}} \) is the Planck length and \( \lambda_s \) the string length (we set \( h = c = 1 \) in the following). The distinction between \( \lambda_s \) and \( l_{\text{Pl}} \) will not be important for our discussion. At \( \eta = \eta_1 \) the coupling \( g^2 = \exp(\phi) \) becomes large and other effects (\( g^2 \) corrections, non perturbative dilaton potential) come into play. The mechanism which should mediate the transition to the standard FRW cosmology with frozen dilaton (graceful exit problem [14]) is not yet well understood although a number of ideas have been suggested recently [14].

The model therefore has two dimensionfull parameters \( \eta_s, \eta_1 \), with \( \eta_s, \eta_1 < 0 \) and \( |\eta_1| < |\eta_s| \), and a dimensionless parameter \( \beta \). Denoting by \( k \) the comoving wave number of a graviton and by \( f \) its frequency red-shifted at the present epoch, we can define two new parameters \( f_1, f_s \) with dimension of frequency from \( k|\eta_1| = f/f_1, k|\eta_s| = f/f_s \), which we will use in place of \( \eta_1, \eta_s \). Using \( 2\pi f = k/a(t_{\text{pres}}) \), where \( t_{\text{pres}} \) is the present value of cosmic time, and eliminating \( \eta_1 \) from eq. (1), we have

\[
k|\eta_1| = \frac{2\pi f}{H_s} \frac{a(t_{\text{pres}})}{a(t_1)}, \tag{2}
\]

which shows that \( f_1 \) is just \( H_s/(2\pi) \), red-shifted from the Planck time \( t_1 \) to the present time. The value of \( 1/H_s \) is of the order of the string length \( \lambda_s \), times a numerical constant of order one which is fixed by the \( a' \) corrections [13], and therefore the value of \( f_1 \) can be estimated [14] to be on the order of \( 40 \) GHz. The other parameter \( f_s \) instead can range anywhere between \( 0 < f_s < f_1 \). From eq. (1), \( f_1/f_s = a(\eta_1)/a(\eta_s) \); the value of this ratio therefore depends on when the string phase begins, \( \eta_s \), and on when it ends, and therefore depends both on the \( a' \) corrections and on the mechanism which implements the graceful exit. In the absence of a detailed knowledge of these effects, we must consider \( f_1/f_s \) as a free parameter of the model with \( 1 < f_1/f_s < \infty \). In particular, this parameter can be large, and has no counterpart in standard cosmology.
ogy. The relic graviton spectrum also depends on the constant $\beta$ introduced in eq. (4), or rather on the combination $\mu = |\beta - (3/2)|$, which takes values in the range $0 < \mu \leq 3/2$. From the phenomenological point of view, an especially interesting situation is realized when $\mu$ is very close or equal to $3/2$ and $f_1/f_s$ is very large, at least on the order of $10^8$, since in this case the signal will be close to the planned sensitivity of ground based interferometers as the LIGO and VIRGO experiments. If $\mu$ is very close to $3/2$ the experimental bound coming from msec pulsar gives the constraint $f_1/f_s < 10^{17}$, otherwise there is basically no upper bound to $f_1/f_s$. As we will see below, the existence of a possible parametrically large quantity $f_1/f_s$ has interesting theoretical consequences.

Recent work has been focused on the energy density of the stochastic graviton background, which is the quantity of direct experimental relevance. Some interesting considerations can however be made considering instead the number density of gravitons. The number of gravitons per unit volume at present time is given by
\[
N_0 = \frac{4\pi}{(2\pi)^3} \int df f^2 n_f ,
\]
where $n_f$ is the number of gravitons per cell of the phase space and $f$ is the physical frequency observed today. The number density $N$ at Planck time $t_1$ is obtained from $N_0$ multiplying it by $(a(t_{\text{pres}})/a(t_1))^3$. Using the value of $n_f$ computed in ref. 10 in the low- and high-frequency limit, and expressing the result in term of the energy $E$ at Planck time, $E = 2\pi fa(t_{\text{pres}})/a(t_1)$, we obtain
\[
\frac{dN}{d\log E} \sim \frac{1}{\lambda_s^3} \times \left\{ \begin{array}{l}
\dfrac{c_1}{\lambda_s^2} \left( \dfrac{E}{\lambda_s} \log \dfrac{E}{\lambda_s} \right)^2 \dfrac{\lambda_s^2}{f_{\text{max}}} = \dfrac{f_s}{f_{\text{max}}} \ll 1 , \\
\dfrac{c_2}{\lambda_s^2} \left( \dfrac{E}{\lambda_s} \right)^{2-2\mu} \dfrac{\lambda_s^2}{f_{\text{max}}} = \dfrac{f_s}{f_{\text{max}}} < 1 .
\end{array} \right.
\]
(4)

The spectrum has a cutoff at $E \simeq H_s$. The numerical constants $c_1$, $c_2$ and the behavior in the intermediate frequency region can be obtained from ref. 13.

\[
c_1(\mu) = \frac{(2\mu - 1)^2(2\mu \alpha - 1 + \alpha)^2}{256\pi^3 \alpha^2 \mu^2} , \quad \alpha = 1/(1 + \sqrt{3})
\]
\[
c_2(\mu) = \frac{1}{64\pi^3} 2^{2\mu} (2^\mu - 1)^2 \Gamma^2(\mu) .
\]
(5)

For values of $\mu$ not very close to zero $c_1$, $c_2$ are small numbers: for instance $c_2(1.5) \simeq 0.012$ and $c_2(1) \simeq 0.002$. Similarly $c_1(\mu)$ has typical values $O(10^{-3})$.

From eq. (4) we see that the spectrum of graviton number density at the Planckian era, per unit logarithmic interval of energy, has a completely different behavior depending on whether $0 < \mu \leq 1$ or $1 < \mu \leq 3/2$. In the former case it is a monotonically growing function of the energy up to the cutoff energy $E \sim H_s$, and its value at the maximum is of order $c_2/\lambda_s^3$. In the latter case it has a maximum at $E \sim (f_s/f_1)H_s$ and then decreases. Fig. 1 shows the behavior of $\lambda_s^3 dN/d\log E$ for $\mu = 1.5$ and $\mu = 0.75$, for $f_1/f_s = 10^2$. The data for $\mu = 0.75$ have been multiplied by $10^3$ in order to plot them on the same scale as $\mu = 1.5$. For this value of $f_1/f_s$ the height of the peak is still smaller than one. For $f_1/f_s = 10^8$ the value at the peak becomes of order $10^6$.

We have used the analytical results of ref. 13 for the energy dependence in the whole region.

The value at the peak for $\mu > 1$ is
\[
\left( \frac{dN}{d\log E} \right)_{\text{max}} \sim \frac{c_2}{\lambda_s^3} \left( \frac{f_1}{f_s} \right)^{2\mu-2} .
\]
(6)

and is much larger than $1/\lambda_s^3$ if $f_1 \gg f_s$. The integrated density of gravitons for $\mu$ generic is
\[
N = \int d\log E \frac{dN}{d\log E} \sim \frac{1}{\lambda_s^3} \left[ c_1 + c_2 \left( \frac{f_1}{f_s} \right)^{2\mu-2} \right] .
\]
(7)

We see that if $\mu < 1$ we have about $10^{-3}$ graviton in a string volume $\lambda_s^3$, while if $\mu > 1$ and $f_1/f_s$ is sufficiently large, we have a parametrically large density.

It is interesting to ask whether graviton-graviton interactions at the Planckian era are strong enough to establish thermal equilibrium. The condition for thermalization is that the typical interaction time $\tau$ be much smaller than the expansion rate $1/H_s$. The average interaction time for a graviton of energy $E$ can be estimated from
\[
\frac{1}{\tau} = \int d\log E' \sigma(E, E') \frac{dN(E')}{d\log E'} .
\]
(8)

Here $\sigma(E, E')$ is the cross section for scattering of two gravitons with energies $E, E'$, and is of the order of

\[
\int d\log E' \sigma(E, E') \frac{dN(E')}{d\log E'} ....
\]

\[
\int d\log E' \sigma(E, E') \frac{dN(E')}{d\log E'} ....
\]
\[ \sigma(E, E') \sim G^2 s \sim l_{\text{Pl}}^2 (E E' / m_{\text{Pl}}^2), \]
where \( m_{\text{Pl}} \) the Planck mass, \( G \) the gravitational constant and \( s \) the square of the c.m. energy. Using eq. (4) we find separately the contribution to the integral from the scattering on soft and on hard gravitons,
\[ \frac{1}{\tau_{\text{soft}}} \sim E \left( \frac{f_1}{f_s} \right)^{2\mu - 3}, \quad \frac{1}{\tau_{\text{hard}}} \sim E \] (9)
so that for \( f_1 / f_s \) large the main contribution comes from scattering on hard gravitons and \( 1 / \tau \sim E \). In the case \( \mu < 1 \) most gravitons have a Planckian energy and therefore the typical value of \( \tau \) is of order one in Planck or string units, and it is therefore impossible to draw definite conclusions, since the establishing of thermal equilibrium depends on the balance between numbers which we are unable to compute. Of course, one might observe that the overall factors \( c_1(\mu), c_2(\mu) \) are numerically small and consider this as an indication that the gravitons do not thermalize, but such a conclusion would not be reliable. For \( \mu > 1 \) and \( f_1 / f_s \) very large, the situation is different and much more interesting. Almost all gravitons have a small energy, \( E \sim f_s / f_1 \) in Planck units, and therefore the typical interaction time \( \tau \sim E^{-1} \sim (f_1 / f_s) l_{\text{Pl}} \) is parametrically larger than \( 1 / H_s \sim \lambda_s \) (note also that the use of the low energy cross section is more justified in this case). We can therefore draw the conclusion that thermal equilibrium is not reached, and graviton-graviton interactions at the Planck era are negligible. This has an important phenomenological consequence. It implies that the relic graviton spectrum is not distorted by interactions during the Planck epoch, and therefore graviton-graviton interactions do not mask the signal coming from the pre-big bang era.

We conclude stressing that at the Planckian epoch the relic graviton background for \( 1 < \mu \leq 3 / 2 \) and \( f_1 / f_s \gg 1 \) is in a rather remarkable state. Its number density is huge, being parametrically larger than one in Planck units. However, it is made of parametrically soft gravitons, and the competition between these two effects results in a very weakly interacting system. The typical de Broglie wavelength associated with thermal motion is \( \lambda_T \sim 1 / E \gg l_{\text{Pl}} \) while the average spacing between gravitons is \( d \sim N^{-1 / 3} \ll l_{\text{Pl}} \), and therefore \( \lambda_T \gg d \); this suggests that this system is appropriately described by a collective wave-function describing the state of the gravitational field rather than in terms of single gravitons. Coherent quantum effects are therefore probably important at this stage. In the formal limit \( f_1 / f_s \rightarrow \infty \) the total number of gravitons diverges, see eq. (6), and almost all of them accumulate in the peak at the energy \( E_{\text{peak}} \sim (f_s / f_1) H_s \rightarrow 0 \). This fact, togheter with the condition \( \lambda_T \gg d \), shows an interesting analogy with Bose-Einstein condensation. The analogy is however only partial, because we are not dealing with thermal distributions.

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[1] G. Veneziano, Phys. Lett. B265 (1991) 287.
[2] M. Gasperini and G. Veneziano, Astropart. Phys. 1 (1993) 317; Mod. Phys. Lett. A8 (1993) 3701; Phys. Rev. D50 (1994) 2519.
[3] G. Veneziano, “String Cosmology: Basic Ideas and General Results”, in “String Gravity and Physics at the Planck Energy Scale”, N. Sanchez and A. Zichichi eds., Kluwer Publ., pag 285.
[4] A. Tseytlin and C. Vafa, Nucl. Phys. B372 (1992) 443; B. Campbell, A. Linde and K. Olive, Nucl. Phys. B355 (1991) 146; E. Copeland, A. Lahiri and D. Wands, Phys. Rev. D50 (1994) 4868.
[5] L.P. Grishchuk, Sov. Phys. JETP, 40 (1975) 409.
[6] A. Starobinski, JETP Lett. 30 (1979) 682; V. Rubakov, M. Sazhin and A. Veryaskin, Phys. Lett. 115B (1982) 189; R. Fabbri and M.D. Pollock, Phys. Lett. 125B (1983) 445; L. Abbott and M. Wise, Nucl. Phys. B244 (1984) 541; L. Abbott and D. Harari, Nucl. Phys. B264 (1986) 487; B. Allen, Phys. Rev. D37 (1988) 2078.
[7] R. Brustein, M. Gasperini, M. Giovannini and G. Veneziano, Phys. Lett. B361 (1995) 45.
[8] R. Brustein, M. Gasperini and G. Veneziano, “Peak and End Point of the Relic Graviton Background in String Cosmology”, preprint CERN-TH/96-37, hep-th/9604084.
[9] R. Brustein, “Spectrum of Cosmic Gravitational Wave Background”, preprint BGU-PH-96/08, hep-th/9604155, proceedings of the “International Conference on Gravitational Waves: Sources and Detectors”, Cascina (Pisa), Italy, March 19-23, 1996.
[10] B. Allen and R. Brustein, “Detecting Relic Gravitational Radiation from String Cosmology with LIGO”, arXiv:gr-qc/9609013.
[11] M. Gasperini, M. Giovannini and G. Veneziano, Phys. Rev. Lett. 75 (1995) 3796.
[12] M. Gasperini, M. Maggiore and G. Veneziano, “Towards a non-singular pre-big bang cosmology”, CERN-TH/96-267, hep-th/9611039.
[13] R. Brustein and G. Veneziano, Phys. Lett. B329 (1994) 429.
[14] N. Kaloper, R. Madden and K. Olive, Nucl. Phys. B452
(1995) 677.

[14] M. Gasperini, J. Maharana and G. Veneziano, Nucl. Phys. B472 (1996) 349;
    S. Rey, Phys. Rev. Lett. 77 (1996) 1929;
    M. Gasperini and G. Veneziano, Phys. Lett. B387 (1996) 715.

[15] A. Buonanno, M. Maggiore and C. Ungarelli, “Spectrum of relic gravitational waves in string cosmology”, IFUP-TH 25/96, gr-qc/9605072, Phys. Rev. D, in press.

[16] Y. Zeldovich and I. Novikov, Relativistic Astrophysics, The University of Chicago Press, 1971, chap. 7.2.