Sturm – Liouville integrable operators, generated by
generalized Dunkl operators

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Abstract. Based on the generalized intertwining relation of Dunkl operator with operator of
differentiation, eigenfunctions for certain Sturm – Liouville operators were obtained.

1. Introduction

Dunkl differential-difference operators were introduced in paper [1]. The study of these operators
showed their significant connection with various branches of mathematics. For example, they
are closely connected with second-order differential operators, namely with Sturm – Liouville
operators theory. (See e.g. [2, 3] and works cited therein).

Paper [4] considers the problem of Dunkl generalized differential-difference operator
classification (similar operators were also considered in [5])

\[ \nabla_{\kappa} = \frac{d}{dx} - \kappa(x) s \]  

based on a following intertwining property

\[ \nabla_{\kappa} V = V \frac{d}{dx} \]  

Here \( x \in \mathbb{R}, \frac{d}{dx} \) – differentiation operator for \( \mathbb{R} \), \( s \) – inversion operator for \( \mathbb{R} \) \( (sf(x) = f(-x)) \), \( \kappa(x) \) – multiplication operator for a sufficiently differentiable nonvanishable even or
odd function. \( V \) – a linear differential-difference operator which allows intertwining property
\( (1.2) \) in Cherednik algebra \( A = \langle 1, x, \frac{d}{dx}, s \rangle \):

\[ [1, x] = \left[ 1, \frac{d}{dx} \right] = [1, s] = 0, \]

\[ \left[ \frac{d}{dx}, x \right] = 1, [x, s] = 2x s, \left[ s, \frac{d}{dx} \right] = 2 s \frac{d}{dx}. \]

Let us consider the task of Dunkl operators classification (1.1) in the case when \( V \) belongs to
the extension of \( A^* \) algebra \( A \) with the help of pseudodifferential operator \( \frac{d^{-1}}{dx^{-1}} \) [6]:

\[ A^* = \left\langle A, \frac{d^{-1}}{dx^{-1}} \right\rangle, \]
Thus, the coefficients in the operators $A$ in algebra congruence if and only if $D$ then

\[
\begin{bmatrix}
1, \frac{d^{-1}}{dx^{-1}} \\
\frac{d^{-1}}{dx^{-1}}, \frac{d}{dx}
\end{bmatrix} = 0, \begin{bmatrix}
s, \frac{d^{-1}}{dx^{-1}} \\
\frac{d^{-1}}{dx^{-1}}, x\frac{d^{-1}}{dx^{-1}}
\end{bmatrix} = 2s \frac{d^{-1}}{dx^{-1}}, \begin{bmatrix}
x, \frac{d^{-1}}{dx^{-1}}
\end{bmatrix} = \frac{d^{-2}}{dx^{-2}}.
\]

It is required to establish, at which $\varphi(x)$ and $V \in A^*$ is fulfilled (1.2).

This requirement allows to obtain additional data on integrable operators of Sturm – Liouville type, which are in this case the constraint of Dunkl operators squares for the subspace of even $\mathcal{F}_+$ or odd $\mathcal{F}_-$ functions on $\mathbb{R}$:

\[
\nabla^2_x = \frac{d^2}{dx^2} - \varphi(x) - \varphi'(x)s,
\]

\[
\mathcal{L}_+ = \nabla^2_{x|\mathcal{F}_+} = \frac{d^2}{dx^2} - [\varphi(x) + \varphi'(x)], \quad \mathcal{L}_- = \nabla^2_{x|\mathcal{F}_-} = \frac{d^2}{dx^2} - [\varphi(x) - \varphi'(x)].
\]

2. Intertwining condition

Let us reduce (1.1) to a set of equations on $\varphi(x)$ and operator coefficients $V$. The following lemma is required.

**Lemma.** Let

\[
D_1 = \sum_{i=0}^I f_i(x) \frac{d^i}{dx^i}, D^{-1}_1 = \sum_{k=1}^K f_k(x) \frac{d^{-k}}{dx^{-k}},
\]

\[
D_2 = \sum_{j=0}^J f_j(x) \frac{d^j}{dx^j}, D^{-1}_2 = \sum_{i=1}^L f_i(x) \frac{d^{-l}}{dx^{-l}},
\]

then

\[
D_1^{-1} + D_2^{-1}s + D_1 + D_2s = 0,
\]

if and only if $D_1 = D_1^{-1} = D_2 = D_2^{-1} = 0$.

**Proof.** Let $D_1^{-1} + D_2^{-1}s + D_1 + D_2s = 0$, in algebra $A^*$, then for $q = \max \{\deg D_1^{-1}, \deg D_2^{-1}\}$ congruence

\[
(D_1^{-1} + D_2^{-1}s + D_1 + D_2s) \frac{d^q}{dx^q} = 0
\]

in algebra $A$ is obtained.

Let us designate via $\widetilde{D}_1 = (D_1^{-1} + D_1) \frac{d^q}{dx^q}$ and $\widetilde{D}_2 = (D_2^{-1} + D_2) \frac{d^q}{dx^q}$. Then [4] we have

\[
\widetilde{D}_1 + \widetilde{D}_2s = 0 \Leftrightarrow \widetilde{D}_1 = \widetilde{D}_2 = 0.
\]

Thus, the coefficients in the operators $\widetilde{D}_1$ and $\widetilde{D}_2$ are zero. Consequently, in operators $D_1, D_1^{-1}, D_2$ and $D_2^{-1}$ the same. Sufficiency is obvious. The proof is complete.

Let us designate via

\[
V = \sum_{i=1}^N P_i(x) \frac{d^{-i}}{dx^{-i}} + \sum_{i=1}^M Q_i(x) \frac{d^{-i}}{dx^{-i}}s + \sum_{i=0}^n p_i(x) \frac{d^i}{dx^i} + \sum_{i=0}^m q_i(x) \frac{d^i}{dx^i}s,
\]

**Assertion.** If the operator of $V$ kind (2.1) satisfies intertwining property (1.2), then $N = M$ and $n = m + 1$. 

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Theorem. We shall further write
\[
\sum_{i=1}^{N} P_i(x) \frac{d^{-i}}{dx^{-i}} + \sum_{i=1}^{M} Q_i(x) \frac{d^{-i}}{dx^{-i}} + \sum_{i=0}^{n} p_i(x) \frac{d^i}{dx^i} + \sum_{i=0}^{m} q_i(x) \frac{d^i}{dx^i} + \sum_{i=1}^{M} Q_i(x) \frac{d^{-i}}{dx^{-i}}
\]
\[
+ \varkappa(x) \sum_{i=1}^{N} (-1)^{i+1} P_i(-x) \frac{d^{-i}}{dx^{-i}} + \varkappa(x) \sum_{i=1}^{M} (-1)^{i+1} Q_i(-x) \frac{d^{-i}}{dx^{-i}}
\]
\[
+ \varkappa(x) \sum_{i=0}^{n} (-1)^{i+1} p_i(-x) \frac{d^i}{dx^i} + \varkappa(x) \sum_{i=0}^{m} q_i(-x) \frac{d^i}{dx^i},
\]
\[
V \frac{d}{dx} = \sum_{i=1}^{N} P_i(x) \frac{d^{-i+1}}{dx^{-i+1}} - \sum_{i=1}^{M} Q_i(x) \frac{d^{-i+1}}{dx^{-i+1}} + \sum_{i=0}^{n} p_i(x) \frac{d^{i+1}}{dx^{i+1}} - \sum_{i=0}^{m} q_i(x) \frac{d^{i+1}}{dx^{i+1}}.
\]

According to the lemma we have
\[
\begin{align*}
\sum_{i=1}^{N} P_i(x) \frac{d^{-i}}{dx^{-i}} + \varkappa(x) \sum_{i=1}^{M} (-1)^{i+1} Q_i(-x) \frac{d^{-i}}{dx^{-i}} &= 0, \\
\sum_{i=1}^{M} Q_i(x) \frac{d^{-i}}{dx^{-i}} + 2 \sum_{i=2}^{M} Q_i(x) \frac{d^{-i}}{dx^{-i}} + \varkappa(x) \sum_{i=1}^{N} (-1)^{i+1} P_i(-x) \frac{d^{-i}}{dx^{-i}} &= 0, \\
\sum_{i=0}^{n} p_i(x) \frac{d^i}{dx^i} + \varkappa(x) \sum_{i=0}^{m} (-1)^{i+1} q_i(-x) \frac{d^i}{dx^i} &= 0, \\
\sum_{i=0}^{m} q_i(x) \frac{d^i}{dx^i} + 2Q_1 + 2 \sum_{i=0}^{m} q_i(x) \frac{d^{i+1}}{dx^{i+1}} + \varkappa(x) \sum_{i=0}^{n} (-1)^{i+1} p_i(-x) \frac{d^i}{dx^i}. 
\end{align*}
\]

Let us look at the second equation in the system (2.2). If \(Q_M = const\), then \(M = N\). This is contrary to the first equation of the system. At \(Q_M \neq const\) we have \(M = N\). From the fourth equation of the system (2.2) follows that \(m + 1 = n\). The proof is complete.

We shall further write
\[
V = \sum_{i=1}^{N} F_i(x) \frac{d^{-i}}{dx^{-i}} + \sum_{i=N+1}^{N+N+1} F_i(x) \frac{d^{-i+N}}{dx^{-i+N}} + \sum_{i=0}^{n} f_i(x) \frac{d^i}{dx^i} + \sum_{i=n+1}^{2n+1} f_i(x) \frac{d^{i-n-1}}{dx^{i-n-1}},
\]

where for convenience \(F(N+1)(x) \equiv 0, f_{2n+1}(x) \equiv 0\).

Theorem. The equation (1.2) for the operator (2.3) is equivalent to the system of the differential-difference equations
\[
\begin{align*}
F_i'(x) + (-1)^{i+1} \varphi(x) F_i(x) &= 0, i = 1, N, \\
F_{i+N}'(x) + 2 F_{i+N+1}(x) + (-1)^{i+1} \varphi(x) F_i(-x) &= 0, i = 1, N, \\
f_i'(x) + (-1)^{i+1} \varphi(x) f_{i+n+1}(x) &= 0, i = 0, n, \\
f_{n+1}'(x) + 2F_{N+1}(x) - \varphi(x)f_0(-x) &= 0, \\
f_{i+n+1}(x) + 2f_{i+n} + (-1)^{i+1} \varphi(x) f_i(-x) &= 0, i = 1, n.
\end{align*}
\]
Proof. As per formula (2.2), taking into account the designations (2.3), we have

\[ \sum_{i=1}^{N} (F_i'(x) + (-1)^{i+1} \varphi(x)F_{i+N}(-x)) \frac{d^{-i}}{dx^{-i}} = 0, \]

\[ \sum_{i=1}^{N-1} (F_{i+N}'(x) + 2F_{i+N+1}(x) + (-1)^{i+1} \varphi(x)F_i(-x)) \frac{d^{-i}}{dx^{-i}} \]

\[ + (F_{2N}' + (-1)^{N+1} \varphi(x)F_N(-x)) \frac{d^{-N}}{dx^{-N}} = 0, \]

\[ \sum_{i=0}^{n-1} (f_i''(x) + (-1)^{i+1} \varphi(x)f_{i+n+1}(-x)) \frac{d^i}{dx^i} + f_n'(x) \frac{d^n}{dx^n} = 0, \]

\[ f_{n+1}'(x) + \varphi(x)f_0(-x) + 2F_{n+1}(x) \]

\[ + \sum_{i=1}^{n} (f_i''(x) + 2f_{i+n} + (-1)^{i+1} \varphi(x)f_i(-x)) \frac{d^i}{dx^i} = 0. \]

Taking into account the lemma, the proof is complete.

3. Important example

Let \( N = 1, n = 1 \), then the system (2.4) shall take the form

\[
\begin{align*}
F_1'(x) + \varphi(x)F_2(-x) &= 0, \\
F_2'(x) + \varphi(x)F_1(-x) &= 0, \\
f_0'(x) - \varphi(x)f_2(-x) &= 0, \\
f_1'(x) &= 0, \\
f_2'(x) + 2F_2 - \varphi(x)f_0(-x) &= 0, \\
2f_2 + \varphi(x)f_1(-x) &= 0.
\end{align*}
\tag{3.1}
\]

Without restriction on generality it is possible to consider that \( f_1(x) \equiv 1 \). Then from the 4th equation of system (3.1) we have \( f_2 = \frac{-\varphi(x)}{2} \). It means that \( f_2'(x) = \frac{-\varphi'(x)}{2} \). Further from the third equation we get \( f_0'(x) = \frac{-\varphi(x)}{2} \in \mathcal{F}_+ \). It means that \( f_0(x) = f_0^-(x) + \text{const} \), where \( f_0^-(x) \in \mathcal{F}_- \). There are two possible cases.

3.1. The case \( f_0(x) \in \mathcal{F}_- \)

From the fifth equation of system (3.1) we get that \( F_2(x) \in \mathcal{F}_+ \). Then \( \int F_1(x)dF_1 = \int F_2(x)dF_2 \), and \( F_2^2(x) = F_1^2(x) + c_1 \).

3.1.1. The case \( c_1 = 0 \) 

If \( c_1 = 0 \), then \( F_1(x) = F_2(x) \), and then, for example, from the second equation of the system (3.1) it follows

\[ \frac{F_1''(x)}{F_2(x)} = -\varphi(x). \]

Taking into account that \( \varphi(x) = (\log |\varphi(x)|)' \), we get

\[ \log |F_2(x)| = -\log |\varphi(x)|. \]
From the fifth equation (3.1), having put $\tilde{c} = 1$, we have

$$\kappa(x)f_0(x) = \frac{\kappa'(x)}{2} - \frac{2}{\varphi(x)}. \quad (3.2)$$

Differentiating the latter, we get

$$\kappa'(x)f_0(x) + \kappa(x)f'_0(x) = \frac{\kappa''(x)\varphi(x)}{2} + \frac{2\varphi'(x)\kappa(x)}{\varphi^2(x)}.$$

Let us multiply the obtained result by $\kappa(x)$:

$$\kappa'(x)\kappa(x)f_0(x) + \kappa^2(x)f'_0(x) = \frac{\kappa''(x)\kappa(x)}{2} + \frac{2\varphi'(x)\kappa(x)}{\varphi^2(x)}.$$

Let us plug here the equation for $\kappa(x)f_0(x)$ from (3.2):

$$\kappa'(x)\left[\frac{\kappa'(x)}{2} - \frac{2}{\varphi(x)}\right] + \kappa^2(x)f'_0(x) = \frac{\kappa''(x)\kappa(x)}{2} + \frac{2\varphi'(x)\kappa(x)}{\varphi^2(x)}.$$

taking into account $\kappa(x) = (\log |\varphi(x)|)'$, we get

$$\varphi'''(x)\varphi'(x)\varphi(x) - \varphi''(x)(\varphi'(x))^2 - (\varphi''(x))^2\varphi(x) + 4\varphi''(x)\varphi(x) = 0. \quad (3.3)$$

**3.1.2. The case $c_1 = 1$**

If $c_1 = 1$, then $F_1(x) = \sqrt{F_2^2(x) + 1}$ and, for example, from the second equation of the system (3.1) we get

$$\frac{F_1'(x)}{\sqrt{F_2^2(x) + 1}} = -\kappa(x).$$

Similar to point 2.1.1, we have

$$\kappa(x)f_0(x) = \frac{\kappa'}{2} - 2\left(-\frac{\varphi(x)}{2} + \frac{1}{2\varphi(x)}\right),$$

$$\kappa'(x)f_0(x) + \kappa(x)f'_0(x) = \frac{\kappa''}{2} + \frac{\varphi'(x)}{\varphi^2(x)},$$

$$\varphi'''(x)\varphi'(x)\varphi(x) - \varphi''(x)(\varphi'(x))^2 - (\varphi''(x))^2\varphi(x) + 2\varphi''(x)\varphi(x) - 2\varphi''(x)\varphi^3(x) + 4(\varphi'(x))^2\varphi^2(x) = 0. \quad (3.4)$$

**3.1.3. The case $c_1 = -1$**

Let then $c_1 = -1$, then

$$F_1(x) = \sqrt{F_2^2(x) - 1},$$

$$\varphi'''(x)\varphi'(x)\varphi(x) - \varphi''(x)(\varphi'(x))^2 - (\varphi''(x))^2\varphi(x) + 2\varphi''(x)\varphi(x) + 2\varphi''(x)\varphi^3(x) - 4(\varphi'(x))^2\varphi^2(x) = 0. \quad (3.5)$$
3.2. The case $f_0(x) = f_0^-(x) + c$
Suppose now that $f_0(x) = f_0^-(x) + c$, where $f_0^-(x) \in \mathcal{F}_-$, $c \in \mathbb{R} \setminus \{0\}$. From the fifth equation of the system (3.1) we get

$$f_2'(x) + 2F_2(x) + \kappa(x)f_0^-(x) - c\kappa(x) = 0.$$  

Let us depict $F_2$ as the sum of even and odd functions

$$F_2(x) = F_2^+(x) + F_2^-(x),$$

then

$$f_2'(x) + 2F_2^+(x) + 2F_2^-(x) + \kappa(x)f_0^-(x) - c\kappa(x) = 0.$$  

(3.6)

Here $f_2^+(x), 2F_2^+(x), \kappa(x)f_0^-(x) \in \mathcal{F}_+$, and $2F_2^-, c\kappa(x) \in \mathcal{F}_-$. Thus,

$$\begin{cases} 2F_2^-(x) = c\kappa(x), \\ f_2'(x) + 2F_2^+(x) + \kappa(x)f_0^-(x) = 0. \end{cases}$$  

(3.7)

Further, depicting $F_1$ as the sum of even and odd functions

$$F_1(x) = F_1^+(x) + F_1^-(x),$$

and using the first equation from (3.7), we shall present the first two equations of the system (3.1) as follows

$$\begin{cases} (F_1^+(x))' + (F_1^-(x))' + \kappa(x)F_2^+(x) - c\kappa^2(x) = 0, \\ (F_2^+(x))' + c\kappa'(x) + \kappa(x)F_1^+(x) - \kappa(x)F_1^-(x) = 0. \end{cases}$$  

(3.8)

From (2.3) we get

$$\begin{cases} (F_1^-(x))' = c\kappa^2(x), \\ \kappa(x)F_1^+(x) = c\kappa^2(x). \end{cases}$$

Where from

$$\kappa^2(x) = \left(\frac{\kappa'(x)}{\kappa(x)}\right)' = \kappa(x),'$$

(3.9)

$$\kappa^2(x) = (\log |\kappa(x)|)'.'$$

(3.10)

is easily integrated [4], and we get

$$\kappa = \pm \frac{1}{x}; \quad \kappa = \pm \frac{1}{\sinh x}; \quad \kappa = \pm \frac{1}{\sinh x}.$$  

Comment. Equation (3.9) can be integrated by reduction of order, replacing $\kappa'(x) = y(\kappa)$, which reduces it to Bernoulli equation.

Thus, as a result we get the following set of nonlinear differential equations

$$\begin{bmatrix} \kappa^2(x) = (\log |\kappa(x)|)'', \\ \varphi'' \varphi' \varphi - (\varphi'')^2 \varphi - \varphi'' (\varphi')^2 + 4\varphi'' \varphi = 0, \\ \varphi'' \varphi' \varphi - (\varphi'')^2 \varphi - \varphi'' (\varphi')^2 + 2\varphi'' \varphi'' + 4(\varphi')^2 \varphi'' = 0, \\ \varphi'' \varphi' \varphi - (\varphi'')^2 \varphi - \varphi'' (\varphi')^2 + 2\varphi'' \varphi'' + 2\varphi'' \varphi'' = 0, \\ \varphi'' \varphi' \varphi - (\varphi'')^2 \varphi - \varphi'' (\varphi')^2 + 4(\varphi')^2 \varphi''^2 = 0, \end{bmatrix}$$

This set actually classifies the operators of the type (1.1): rational, hyperbolic, trigonometrical and their combinations. For example,

$$\kappa = \frac{1}{x}, \quad \kappa = \frac{1}{\sin x}; \quad \kappa = \frac{1}{\sinh x}; \quad \kappa = \frac{\sinh(2x) - 2x}{x\sinh(2x) - 2\sinh^2(x)}.$$  

(3.11)
4. Solution of the equation (3.3)

In section 3.1., we got the equation (3.3) for $\varphi(x)$, provided that $f_0 \in \mathcal{F}_-$. Let us multiply it by $\frac{1}{(\varphi'(x))^2}$, then

$$\frac{\varphi''(x)\varphi'(x)\varphi(x)}{(\varphi'(x))^2} - \frac{\varphi''(x)(\varphi'(x))^2}{(\varphi'(x))^2} - \frac{(\varphi''(x))^2\varphi(x)}{(\varphi'(x))^2} + 4\frac{\varphi''(x)\varphi(x)}{(\varphi'(x))^2} = 0,$$

$$\frac{\varphi''(x)\varphi'(x)\varphi(x)(c_1 + 4x)}{\varphi'(x)\varphi^3(x)} - 2\frac{(\varphi'(x))^2(c_1 + 4x)}{\varphi^3(x)} - \frac{4\varphi'(x)\varphi(x)(c_1 + 4x)}{\varphi'(x)\varphi^3(x)} + 4\frac{\varphi'(x)(c_1 + 4x)x}{\varphi^3(x)} = -c_1 - c_1(x + 4x)\varphi'(x),$$

$$\left(\frac{(c_1 + 4x)\varphi'(x)}{\varphi^2(x)}\right)' - 4\frac{1}{\varphi(x)}\varphi'(x) - 4\left[-\frac{(c_1 + 4x)^2\varphi(x)}{4\varphi^2(x)}\varphi(x) + 4(c_1 + 4x)\varphi^2(x)\right] = 0,$$

$$\left(\frac{(c_1 + 4x)\varphi'(x)}{\varphi^2(x)}\right)' - 4\frac{1}{\varphi(x)}\varphi'(x) - 4\left[\frac{1}{2\varphi^2(x)}\varphi'(x)\right]' = 0,$$

$$\frac{(c_1 + 4x)\varphi'(x)}{\varphi^2(x)} - 4\frac{1}{\varphi(x)}\varphi'(x) - 4\left[\frac{1}{2\varphi^2(x)}\varphi'(x)\right] = c_2,$$

$$\varphi'(x) = \frac{-2c_2\varphi^2(x) - 8\varphi(x) + (c_1 + 4x)^2}{2(c_1 + 4x)^2}. \quad (4.1)$$

$$(4.1)$$ is the Riccati equation. By means of replacement $\varphi(x) = \frac{z'(c_1 + 4x)}{c_2}$ it is reduced to linear

$$z'' = \frac{-8z'}{(c_1 + 4x)} + \frac{c_2}{2}z. \quad (4.2)$$

Solving (4.2) using Kovachich algorithm [7] we get

$$z = c_3 \frac{\sinh(\frac{\sqrt{2}c_2 x}{2})}{c_1 + 4x} + \frac{c_3 \cosh(\frac{\sqrt{2}c_2 x}{2})}{c_1 + 4x}.$$

Coming back to $\varphi(x)$, we shall have

$$\varphi(x) = \frac{(\sqrt{2}c_1c_2c_3 + 4\sqrt{2}c_2c_3 - 8)\sinh(\frac{\sqrt{2}c_2 x}{2}) + (\sqrt{2}c_1c_2 + 4\sqrt{2}c_2 - 8c_3)\cosh(\frac{\sqrt{2}c_2 x}{2})}{2c_2^2 c_3 \cosh(\frac{\sqrt{2}c_2 x}{2}) + \sinh(\frac{\sqrt{2}c_2 x}{2})}.$$

Since we are interested in the odd $\varphi(x)$, we need even or odd $\varphi(x)$: $\varphi(x) = x$, only, or $\varphi(x) = \frac{2x}{\tanh(x)} - 2$.

They correspond to $\varphi(x) = \frac{1}{2}$, or $\varphi(x) = \frac{\sinh(2x) - 2x}{x \sinh(2x) - 2 \sinh(x)}$.

Comment. The explicit solution of the equations (3.4) and (3.5) is not known to us, there is a numerical solution only.
5. Explicit form of the operators $V$

For all $\kappa(x)$, determined by the formulas (3.11), let us calculate the coefficients of the operator $V$, which intertwines $\nabla_{\kappa}$ and $\frac{d}{dx}$.

5.1. The case $\kappa(x) = \frac{1}{x}$

In accordance with the outlined in section 2, from the system (3.1)

$$f_1(x) \equiv 1, f_2(x) = -\frac{\kappa(x)}{2} = -\frac{1}{2x}, f_0'(x) = \kappa(x)f_2(-x) = \frac{1}{2x^2}$$

and

$$V = -\frac{1}{2x} + \frac{d}{dx} - \frac{1}{2x}s.$$  

5.2. The case $\kappa(x) = \frac{1}{\sin x}$

From the system (3.1) we get

$$f_1(x) \equiv 1, f_2(x) = -\frac{\kappa(x)}{2} = -\frac{1}{2\sin x}, f_0'(x) = \kappa(x)f_2(-x) = \frac{1}{2\sin^2 x}$$

and

$$V = -\frac{\cot x}{2} + \frac{d}{dx} - \frac{1}{2\sin x}s.$$  

5.3. The case $\kappa(x) = \frac{1}{\sinh x}$

If $\kappa(x) = \frac{1}{\sinh x}$, then

$$V = -\frac{\coth x}{2} + \frac{d}{dx} - \frac{1}{2\sinh x}s.$$  

5.4. The case $\kappa = \frac{\sinh(2x)-2x}{\sinh(2x)-2\sinh^2(x)}$

If $\kappa = \frac{\sinh(2x)-2x}{x\sinh(2x)-2\sinh^2(x)}$, then

$$V = \frac{\tanh x}{2x-2\tanh x}(d^{-1}_{x} - \coth x - \frac{\sinh 2x - 2x}{2\sinh 2x - 4\sinh^2 x}(s + 1) + \frac{d}{dx}).$$  

6. Eigenfunctions of the operators $\nabla_{\kappa}^2$ and their Connection with Sturm – Liouville problem

From intertwining relations

$$\nabla_{\kappa} V = V \frac{d}{dx}$$

and

$$\nabla_{\kappa}^2 V = V \frac{d^2}{dx^2}$$

it is obvious that eigenfunctions of Dunkl operator and its square can be obtained. In the general case, the operator $\nabla_{\kappa}^2$ shall take the form

$$\nabla_{\kappa}^2 = \frac{d^2}{dx^2} - \kappa^2 - \kappa's.$$  

Thus, its restriction to the space of even $F_+$ or odd $F_-$ functions is the Sturm – Liouville operator

$$\mathcal{L}_+ = \nabla_{\kappa}^2|_{F_+} = \frac{d^2}{dx^2} - [\kappa^2 + \kappa'], \quad \mathcal{L}_- = \nabla_{\kappa}^2|_{F_-} = \frac{d^2}{dx^2} - [\kappa^2 - \kappa'].$$

Based on the results from section 4 we have the following cases.
6.1. Rational type case [8]

\[ \varphi(x) = \frac{1}{x}; \nabla \varphi = \frac{d}{dx} - \frac{1}{x}s; \nabla^2 \varphi = \frac{d^2}{dx^2} + \frac{1}{x^2}(s - 1), \]

\[ V = -\frac{1}{2x} + \frac{d}{dx} - \frac{1}{2x}s. \]

Based on direct calculations get that

\[ V[\sinh x] = \cosh x \in \mathcal{F}_+, \quad V[\cosh x] = \sinh x - \frac{\cosh x}{x} \in \mathcal{F}_-. \]

Thus, eigenfunctions for the following Sturm – Liouville operators (Lagnese – Stellmacher operators) are obtained

\[ L_+ = \frac{d^2}{dx^2}, \quad L_- = \frac{d^2}{dx^2} - \frac{2}{x^2}. \]

6.2. Trigonometrical type case [9]

\[ \varphi(x) = \frac{1}{\sin x}; \nabla \varphi = \frac{d}{dx} - \frac{1}{\sin x}s; \nabla^2 \varphi = \frac{d^2}{dx^2} + \frac{1}{\sin^2(x)}(\cos(x)s - 1), \]

\[ V = -\frac{\cot x}{2} + \frac{d}{dx} - \frac{1}{2\sin x}s. \]

For the Sturm – Liouville operators

\[ L_+ = \frac{d^2}{dx^2} - \frac{1}{2\cos^2(x/2)}, \quad L_- = \frac{d^2}{dx^2} - \frac{1}{2\sin^2(x/2)}. \]

the corresponding eigenfunctions have the following form

\[ V[\sinh(x)] = \cosh(x) + \frac{1}{2}\tan(x/2)\sinh(x), \quad V[\cosh(x)] = \sinh(x) - \frac{\cot(x/2)\cosh(x)}{2}. \]

6.3. Hyperbolic type case [9]

\[ \varphi(x) = \frac{1}{\sinh x}; \nabla \varphi = \frac{d}{dx} - \frac{1}{\sinh x}s; \nabla^2 \varphi = \frac{d^2}{dx^2} + \frac{1}{\sinh^2(x)}(\cosh(x)s - 1), \]

\[ V = \frac{d}{dx} - \frac{1}{2\sinh(x)}(s + \cosh(x)). \]

Eigenfunctions of the operators

\[ L_+ = \frac{d^2}{dx^2} + \frac{1}{2\cosh^2(x/2)}, \quad L_- = \frac{d^2}{dx^2} - \frac{1}{2\sinh^2(x/2)}. \]

have the following form

\[ V[\sinh(x)] = \cosh^2(x/2), \quad V[\cosh(x)] = \frac{\sinh(x) - \coth(x/2)}{2}. \]
6.4. Mixed type case [10]
For
\[ x(x) = \frac{\sinh(2x) - 2x}{x \sinh(2x) - 2 \sinh^2(x)} \]
we have
\[ \nabla_x^2 = \frac{d^2}{dx^2} + \left( \frac{\sinh(2x) - 2x}{x \sinh(2x) - 2 \sinh^2(x)} \right)^2 (s - 1) - \frac{2}{\sinh^2(x)} s, \]
\[ V = \frac{\tanh x}{2x - 2 \tanh x} \left( \frac{d^{-1}}{dx} + 1 \right) - \coth x - \frac{\sinh 2x - 2x}{2 \sinh 2x - 4 \sinh^2 x} (s + 1) + \frac{d}{dx}. \]
Direct calculations show that \( V[\sinh x] = V[\cosh x] = 0. \)

Sturm – Liouville eigenfunctions
\[ L_+ = \frac{d^2}{dx^2} - \frac{2}{\sinh^2(x)}, \quad L_- = \frac{d^2}{dx^2} - 2 \left( \frac{\sinh(2x) - 2x}{x \sinh(2x) - 2 \sinh^2(x)} \right)^2 + \frac{2}{\sinh^2(x)}. \]
have the following form
\[ V[\text{sgn} x \cosh(x)] = -\frac{1}{\sinh|x|}, \quad V[\sinh|x|] = \frac{|x|}{x \cosh(x) - \sinh(x)}. \]

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