Stationary strings near a higher-dimensional rotating black hole

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We study stationary string configurations in a space-time of a higher-dimensional rotating black hole. We demonstrate that the Nambu-Goto equations for a stationary string in the 5D Myers-Perry metric allow a separation of variables. We present these equations in the first-order form and study their properties. We prove that the only stationary string configuration which crosses the infinite red-shift surface and remains regular there is a principal Killing string. A worldsheet of such a string is generated by a principal null geodesic and a timelike at infinity Killing vector field. We obtain principal Killing string solutions in the Myers-Perry metrics with an arbitrary number of dimensions. It is shown that due to the interaction of a string with a rotating black hole there is an angular momentum transfer from the black hole to the string. We calculate the rate of this transfer in a spacetime with an arbitrary number of dimensions. This effect slows down the rotation of the black hole. We discuss possible final stationary configurations of a rotating black hole interacting with a string.

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I. INTRODUCTION

It is well known that the Kerr metric possesses a number what was called by Chandrasekhar [1] ‘miraculous’ properties such as the separability of the geodesic Hamilton-Jacobi equation [2], separability of a scalar field equation [3], separability and decoupling of the massless non-zero-spin field equations [4] and separability of the equilibrium equation for a stationary cosmic string [5,6]. These properties are closely connected with the existence of the second order Killing tensor discovered in the Kerr metric by Carter [2].

A lot of attention has been focused recently on the study the higher dimensional generalizations of the Kerr metric and their properties. This interest is partly connected with brane world scenarios where our physical world is represented by a 4-dimensional brane embedded in the higher dimensional bulk spacetime. A higher dimensional generalization of the metric of a rotating black hole was obtained by Myers and Perry in 1986 [7]. It is remarkable that the five dimensional Myers-Perry (MP) metric possesses ‘miraculous’ properties similar to the Kerr metric. Namely it allows a separation of variables in the geodesic Hamilton-Jacobi equation [8] and the separability of the massless scalar field equation [9]. These properties are also connected with the existence of the second order Killing tensor [8].

In this paper we study stationary equilibrium configuration of a cosmic string in the spacetime of a higher-dimensional rotating black hole. We shall demonstrate that in the 5D spacetime the equation of motion for such a string allows separation of variables. We shall also show that in the five dimensional case there is an analogue of the 4-dimensional ‘string uniqueness theorem’ [10], namely, the only stationary string configuration which crosses the infinite red-shift surface and remains regular there is a principal Killing string. A worldsheet of such a string is generated by principal null geodesics and the timelike at infinity Killing vector field. We obtain the principal Killing string solutions in the Myers-Perry metric with an arbitrary number of dimensions. And finally, we consider the interaction of a stationary string with a higher-dimensional rotating black hole and demonstrate that in a general case there exists an angular momentum transfer from the black hole to the string. As a result of this friction effect, the black hole rotation is slowed down until the final stationary configuration is reached. We discuss possible final stationary states of such systems.

II. A STATIONARY STRING IN A STATIONARY SPACETIME

A general stationary metric in a spacetime \( M \) with \( N \) spatial dimensions can be written in the form \((\mu, \nu = 0 \ldots N)\)

\[
ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu = -F (dt + A_i \, dx^i)^2 + H_{ij} \, dx^i \, dx^j ,
\]

(II.1)

where \( F, A_i, \) and \( h_{ij} \) are functions of spatial coordinates \( x^i \) (\( i, j = 1 \ldots N \)). Denote by \( \xi^\mu \partial_\mu = \partial_t \) the Killing vector. Then
\[ \xi^2 = -F, \quad \xi_i = -F A_i. \]  

We assume that the spacetime is a \( N \)-dimensional foliation of the Killing trajectories and denote \( \mathcal{M} = M/G_1 \) the factor space. Elements of \( \mathcal{M} \) are orbits of \( 1 \)-dimensional group \( G_1 \) generated by \( \xi \). A tensor

\[ H_{\mu\nu} = g_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{F} \]  

is the projector onto the factor space \( M \). It also possesses the property

\[ \mathcal{L}_\xi H = 0, \]  

where \( \mathcal{L}_\xi \) is the Lie derivative in the direction \( \xi \).

Denote by \( H^{ij} \) an \( N \times N \) matrix defined as

\[ H_{ij} H^{jk} = \delta^k_i, \]  

then the contravariant components of the metric \( g \) are

\[ g^{00} = -F^{-1} + F^{-2} H^{ij} \xi_i \xi_j, \]

\[ g^{0i} = F^{-1} H^{ij} \xi_j, \quad g^{ij} = H^{ij}. \]

Any stationary tensor \( T \), \( \mathcal{L}_\xi T = 0 \), can be projected onto \( \mathcal{M} \) by using the projector operator \( H \). Geroch [11] demonstrated that the derivative of a stationary tensor defined as

\[ T_{\nu\cdots}^{\mu\cdots} = H^\mu_\alpha \cdots H^\beta_\gamma H^i_\lambda T_{\beta\cdots\gamma} \]

obeys all axioms of a covariant derivative connected with the metric \( H \).

Suppose that in addition to the Killing vector \( \xi \) the spacetime \( M \) has also either Killing vector \( \eta^\mu \) and/or Killing tensor \( K^{\mu\nu} \),

\[ \eta_{(\mu\nu)} = 0, \quad K_{(\mu\nu;\lambda)} = 0, \]

which obey the conditions

\[ \mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi K = 0. \]

Their projection onto \( \mathcal{M} \) have components \( \eta^i \) and \( K^{ij} \) respectively. It is easy to show (see [6]) that these projections obey the equations

\[ \eta_{(i;j)} = 0, \quad K_{(i;j;k)} = 0, \]

and hence they are a Killing vector and a Killing tensor for the metric \( H_{ij} \), respectively.

We call a 2-dimensional surface \( \Sigma \) stationary if it is tangent to \( \xi \). A stationary surface is formed by a one-dimensional family of the Killing trajectories for the field \( \xi \). We choose coordinates \( \xi^A \) \( (A = 0, 1) \) on \( \Sigma \) so that \( \xi^0 = t \) and denote \( \xi^1 = \sigma \). The embedding of \( \Sigma \) into \( M \) is determined by its projection \( x^i = x^i(\sigma) \) in \( \mathcal{M} \). The induced metric \( G_{AB} \) on \( \Sigma \) is

\[ ds^2 = G_{AB} \, d\xi^A \, d\xi^B = -F \, (dt + A \, d\sigma)^2 + \mathcal{H} \, d\sigma^2, \]

where

\[ A = A_i \frac{dx^i}{d\sigma}, \quad \mathcal{H} = H_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}. \]

A test string worldsheet equation of motion can be obtained as an extremum of the Nambu-Goto action

\[ I [x^\mu (\xi^A)] = - \mu^* \int \sqrt{-\det (G_{AB})}, \]

where \( \mu^* \) is the string tension and \( G_{AB} \) is the induced metric. For a stationary string this action reduces to

\[ I = -\Delta t \, E, \]

where

\[ E = \mu^* \int d\sigma \, \sqrt{h_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}}, \quad h_{ij} = F \, H_{ij}. \]

In other words, a line \( x^i \) representing the stationary string in the projection space \( \mathcal{M} \) is a geodesic in the metric \( h_{ij} \). One can also interpret this result as follows.

\[ dl = \sqrt{H_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma}} \]

is the element of proper length of the string. The proper mass of this element is \( dm = \mu^* \, dl \), while its energy \( dE \) differs from \( dm \) by a red-shift factor \( F^{-1/2} \).

\[ \Delta = (x + a^2)(x + b^2) - r_0^2 \, x. \]

The gravitational radius \( r_0 \) and the rotation parameters, \( a \) and \( b \), are connected with the mass \( M \) of the black hole and its angular momenta as follows

\[ M = \frac{3\pi}{8} r_0^2, \]

\[ \mathcal{H} = M/G_1 \]
\[ J_a = -\frac{2}{3} Ma , \quad J_b = -\frac{2}{3} Mb . \quad \text{(III.5)} \]

(Note that the sign of the rotation parameters in the MP metric is opposite to the one adopted in the Kerr metric.)

Angles \( \phi \) and \( \psi \) take values from the interval \([0, 2\pi]\), while angle \( \theta \) takes values from \([0, \pi/2]\). Note also that instead of the ‘radius’ \( r \) we use the coordinate \( x = r^2 \). This will allow us to simplify calculations and make many of the expressions more compact.

The black hole horizon is located at \( x = x_+ \), where
\[ x_\pm = \frac{1}{2} \left[ B \pm \sqrt{B^2 - 4a^2b^2} \right] , \quad \text{(III.6)} \]
where \( B = r_0^2 - a^2 - b^2 \). The angular velocities \( \Omega_a \) and \( \Omega_b \) and the surface gravity \( \kappa \) are
\[ \Omega_a = \frac{a}{x_+ + a^2} , \quad \Omega_b = \frac{b}{x_+ + b^2} , \quad \kappa = \frac{\partial_\Sigma \Delta}{r_0^2 \sqrt{\Delta}} \bigg|_{x = x_+} . \quad \text{(III.7)} \]

The infinite red-shift surface, which is an external boundary of the ergosphere, is determined by the equation \( \xi^2 = 0 \), or \( r^2 = r_0^2 \).

The metric (III.1) is invariant under the following transformation
\[ a \leftrightarrow b , \quad \theta \leftrightarrow \left( \frac{\pi}{2} - \theta \right) , \quad \phi \leftrightarrow \psi . \quad \text{(III.8)} \]

It possesses 3 Killing vectors, \( \partial_t \), \( \partial_\phi \) and \( \partial_\psi \). For \( a = b \) the metric has 2 additional Killing vectors [12,8]:
\[ \cos \tilde{\theta} - \cot \tilde{\theta} \sin \tilde{\phi} \partial_{\tilde{\phi}} + \frac{\sin \tilde{\phi}}{\sin \theta} \partial_{\tilde{\psi}} , \quad \text{(III.9)} \]
and
\[ -\sin \tilde{\phi} \partial_{\tilde{\theta}} - \cot \tilde{\theta} \cos \tilde{\phi} \partial_{\tilde{\psi}} + \frac{\cos \tilde{\phi}}{\sin \theta} \partial_{\tilde{\psi}} , \quad \text{(III.10)} \]
where \( \tilde{\phi} = \psi - \phi, \tilde{\psi} = \psi + \phi \) and \( \tilde{\theta} = 2\theta \).

Besides the Killing vectors the metric (III.1) has also the Killing tensor \( K^{\mu\nu} \) [8]
\[ K^{\mu\nu} = -P^2 (g^{\mu\nu} + \delta_\mu^\theta \delta_\nu^\theta) + \delta_\mu^\rho \delta_\nu^\rho + \frac{1}{\sin^2 \theta} \delta_\mu^\phi \delta_\nu^\phi + \frac{1}{\cos^2 \theta} \delta_\mu^\psi \delta_\nu^\psi . \quad \text{(III.11)} \]

By representing the metric (III.1) in the form (II.1) one gets
\[ F = -\xi^2 = \frac{C}{\rho^2} , \quad C = \rho^2 - r_0^2 , \quad \text{(III.12)} \]
\[ H_{xx} = \frac{\rho^2}{4\Delta} , \quad H_{\theta\theta} = \rho^2 , \quad H_{\phi\phi} = \frac{ab \sin^2 \theta \cos^2 \psi r_0^2}{\Delta} . \]

\[ H_{\phi\phi} = \left( x + a^2 + \frac{r_0^2 a_2 \sin^2 \theta}{C} \right) \sin^2 \theta , \quad \text{(III.13)} \]
\[ H_{\psi\psi} = \left( x + b^2 + \frac{r_0^2 b^2 \cos^2 \theta}{C} \right) \cos^2 \theta , \]
and obtains the following expression for \( h^{\mu\nu} \)
\[ h^{\phi\phi} = \frac{1}{C} \left[ \frac{1}{\sin^2 \theta} - \frac{a^2 - b^2}{\Delta} \right] , \quad \text{(III.14)} \]
\[ h^{\psi\psi} = \frac{1}{C} \left[ \frac{1}{\cos^2 \theta} + \frac{a^2 - b^2}{\Delta} - \frac{a^2 r_0^2}{\Delta} \right] . \quad \text{(III.15)} \]

We used GRTensor program for the calculations.

To study the geodesics in the metric (III.1) it is convenient to use the Hamilton-Jacobi method (see e.g. [2,13]). The corresponding Hamilton-Jacobi equation reads
\[ \frac{\partial S}{\partial \sigma} + \frac{1}{2} h^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = 0 . \quad \text{(III.15)} \]

For the metric (III.14) this equation allows separation of variables
\[ S = -\frac{1}{2} m^2 \sigma + \Phi \phi + \Psi \psi + S_{\theta} + S_{x} \quad \text{(III.16)} \]
where \( S_{\theta} \) and \( S_{x} \) are functions of \( \theta \) and \( x \) respectively. Substituting this expression into (III.15), one obtains
\[ \left( \frac{\partial S_{\theta}}{\partial \theta} \right)^2 - m^2 P^2 + \frac{1}{\sin^2 \theta} \Phi^2 + \frac{1}{\cos^2 \theta} \Psi^2 = K , \quad \text{(III.17)} \]
and
\[ 4\Delta \left( \frac{\partial S_{x}}{\partial x} \right)^2 + m^2 (r_0^2 - x) - \frac{r_0^2 (x + a^2)(x + b^2)}{\Delta} \Phi^2 \]
\[ - (a^2 - b^2) \left( \frac{\Phi^2}{x + a^2} - \frac{\Psi^2}{x + b^2} \right) = -K , \quad \text{(III.18)} \]
where
\[ \Phi = \frac{a \phi}{x + a^2} + \frac{b \psi}{x + b^2} . \quad \text{(III.19)} \]

Here \( K \) is a separation constant. These equations are similar to the the separated equations for the motion of a particle in 5D MP metric [9], with the following three differences. First, the energy \( E \) has been set to zero, second the mass squared \( m^2 \) has been changed to \(-m^2\) and third, there is an extra term of \( m^2 r_0^2 \) in the \( x \)-equation. The parameter \( m \) depends on the choice of the length
The sign functions \( \sigma \) equations in the following form hand side vanishes. Change of sign occurs when the expression on the right equations are independent of each other. In each equation the parameter \( m \) connected with presence of red-shift factor \( \sqrt{F} \) in (II.16).

We can write equations (III.17) and (III.18) as

\[
\frac{\partial S_\theta}{\partial \theta} = \sigma_\theta \sqrt{\Theta}, \quad \frac{\partial S_x}{\partial x} = \sigma_x \frac{\sqrt{X}}{2\Delta},
\]

(III.20)

with \( \Theta \) and \( X \) given as

\[
\Theta = K + m^2 \rho^2 - \frac{\Phi^2}{\sin^2 \theta} - \frac{\Psi^2}{\cos^2 \theta},
\]

(III.21)

\[
X = \Delta \left[ m^2 (x - r_0^2) - K \right] + r_0^2 (b \Phi + a \Psi)^2
\]

\[
+ (a^2 - b^2) \left[ \Psi^2 (x + b^2) - \Psi^2 (x + a^2) \right].
\]

(III.22)

The sign functions \( \sigma_\theta = \pm \) and \( \sigma_x = \pm \) in the two equations are independent of each other. In each equation the change of sign occurs when the expression on the right hand side vanishes.

We can now write the Hamilton-Jacobi action as

\[
S = \frac{1}{2} m^2 \sigma + \Phi \dot{\phi} + \Psi \dot{\psi} + \sigma_\theta \int^\theta \sqrt{\Theta} d\theta + \sigma_x \int_x \frac{\sqrt{X}}{2\Delta} dx.
\]

(III.23)

By setting the derivatives of \( S \) with respect to \( K, m^2, \Phi, \Psi \) equal to zero, we get a solution for the Hamilton-Jacobi equations in the following form

\[
\int^\theta \frac{d\theta}{\sqrt{\Theta}} = \int_x \frac{dx}{2\sqrt{X}},
\]

(III.24)

\[
\sigma = \int^\theta \frac{\rho^2}{\sqrt{\Theta}} d\theta + \int_x \frac{x - r_0^2}{2\sqrt{X}} dx,
\]

(III.25)

\[
\dot{\phi} = \int^\theta \Phi \frac{d\theta}{\sin^2 \theta \sqrt{\Theta}},
\]

\[
- \int_x \frac{1}{\sqrt{X}} \left[ \frac{a r_0^2 (x + b^2)}{2\Delta} \frac{\epsilon}{\epsilon} + \frac{(a^2 - b^2) \Phi}{2(x + a^2)} \right] dx,
\]

(III.26)

\[
\dot{\psi} = \int^\theta \Psi \frac{d\theta}{\cos^2 \theta \sqrt{\Theta}},
\]

\[
- \int_x \frac{1}{\sqrt{X}} \left[ \frac{b r_0^2 (x + a^2)}{2\Delta} \frac{\epsilon}{\epsilon} - \frac{(a^2 - b^2) \Psi}{2(x + b^2)} \right] dx.
\]

(III.27)

By differentiating these equations with respect to \( \sigma \), one obtains the first order differential equations

\[
C \dot{x} = \sigma_x 2\sqrt{X},
\]

(III.28)

\[
C \dot{\theta} = \sigma_\theta \sqrt{\Theta},
\]

(III.29)

\[
C \dot{\phi} = \frac{\Phi}{\sin^2 \theta} - \frac{a r_0^2 (x + b^2)}{\Delta} \frac{\epsilon}{\epsilon} + \frac{(a^2 - b^2) \Phi}{x + a^2},
\]

(III.30)

\[
C \dot{\psi} = \frac{\Psi}{\cos^2 \theta} - \frac{b r_0^2 (x + a^2)}{\Delta} \frac{\epsilon}{\epsilon} + \frac{(a^2 - b^2) \Psi}{x + b^2}.
\]

(III.31)

These relations are also the integrals of motion and 4 initial conditions for \( \Phi, \Psi, K, r_0 \). The extra term \( r_0 \) at the limit of large \( x \), \( X \) has a leading positive term \( \frac{\epsilon}{\epsilon} \).

IV. TYPES OF STRING CONFIGURATIONS

A. Properties of the radial equation

In a 5D spacetime if one starts with the variation problem (II.16), one obtains 4 second order geodesic equations which require \( 4 \times 2 \) initial data. The integrals of motion \( \Phi, \Psi, K \) and the normalization condition \( h_{i j} x^i x^j = 1 \) allow one to determine the initial data for \( x^i \). Hence the integrals of motion and 4 initial conditions for \( x^i \) uniquely specify a string configuration. Because of the symmetry of the problem, two of these quantities \( \phi(0) \) and \( \psi(0) \) are cyclic. Thus a stationary string in the 5D MP metric is completely determined by \( \Phi, \Psi, K \) and \( x(0) \) and \( \theta(0) \).

We discuss first properties of the radial equation (III.28). For given values of other parameters, \( X \) given by (III.22) considered as a function of \( x \) is a third order polynomial. The allowed configurations must only occur where \( X \geq 0 \), \( X = 0 \) gives radial turning points. In the limit of large \( x \), \( X \) has a leading positive term \( x^3 \), so configurations can extend to infinity. By integrating the equations (III.28)–(III.31) in the region \( x \to \infty \) one obtains

\[
\sigma \sim \sqrt{x} \sim \tau, \quad \theta \sim \theta_0 - \frac{\Theta(\theta_0)}{r},
\]

(IV.1)

\[
\phi \sim \phi_0 - \frac{\Phi}{r \sin^2 \theta_0}, \quad \psi \sim \psi_0 - \frac{\Psi}{r \cos^2 \theta_0}.
\]

(IV.2)
$X$ as a function of $x$ either is a monotonically growing function, or it has one local maximum and one local minimum at $x^{-}$ and $x^{+}$, respectively. By solving the equation $dX/dx = 0$ one finds

$$x^\pm = \frac{1}{3} \left[ 2 - a^2 - b^2 + K \pm \sqrt{B} \right]. \quad \text{(IV.3)}$$

Here

$$B = K^2 + (a^2 + b^2 + 1)K - 3(a^2 - b^2)(\Phi^2 - \Psi^2)$$

$$+(a^2 - b^2)^2 + (1 - a^2)(1 - b^2). \quad \text{(IV.4)}$$

For negative value of $B$ the function $X$ is monotonic. Let us denote

$$K_\pm = \frac{1}{2} \left[ -(1 + a^2 + b^2) \pm \sqrt{C} \right], \quad \text{(IV.5)}$$

$$C = 3 \left\{ 4(a^2 - b^2)(\Phi^2 - \Psi^2) - [(a + b)^2 - 1][-(a - b)^2 - 1] \right\}. \quad \text{(IV.6)}$$

If $C > 0$ then $B$ is positive at $K < K^{-}$ and $K > K^{+}$. If $C < 0$, $B > 0$ for all values of $K$.

The value of $X$ at the 5D black hole horizon (where $\Delta = 0$) is

$$X(x_+) = x_+(\Phi\Omega_a + \Psi\Omega_b)^2. \quad \text{(IV.7)}$$

**B. Properties of the $\theta$ equation**

Let us examine the $\theta$ equation. The string can extend to the subspace $\theta = 0$ only if $\Phi = 0$, and can reach the subspace $\theta = \frac{\pi}{2}$ only if $\Psi = 0$. For $\Psi = 0$ the configuration is in the $\theta = \frac{\pi}{2}$ plane only if $K = \Phi^2 - b^2$. Similarly, for $\Phi = 0$ the configuration is in the $\theta = 0$ plane if $K = \Psi^2 - a^2$.

Consider a special type of configuration when strings are aligned so that $\theta$ remains constant $\theta = \theta_0$. This configuration can occur when

$$\Theta(\theta_0) = \frac{d\Theta}{d\theta}(\theta_0) = 0. \quad \text{(IV.8)}$$

This is equivalent to

$$K + P^2 - \frac{\Phi^2}{\sin^2 \theta_0} - \frac{\Psi^2}{\cos^2 \theta_0} = 0, \quad \text{(IV.9)}$$

$$\frac{\Phi^2}{\sin^4 \theta_0} - \frac{\Psi^2}{\cos^4 \theta_0} - (a^2 - b^2) = 0, \quad \text{(IV.10)}$$

Where for the second equation, we have excluded cases $\theta_0 = 0$ and $\theta_0 = \pi/2$.

**C. A string within a brane**

Higher dimensional black holes are of special interest in the brane world models where a physical world is represented by a $(3+1)$-dimensional brane embedded in a bulk space with large or infinite extra dimensions. In these models the usual matter (bosons, fermions and gauge fields) are localized on the brane. A cosmic string formed from this matter must also be located on the brane. Let us assume that a spacetime has 1 spatial extra dimension and its size is much larger than the gravitational radius of the black hole. Then a stationary cosmic string interacting with such a 5D rotating black hole attached to the brane is described by a special solutions of the equations (III.28)-(III.31). Let us consider this case in more details.

As a result of the black-hole–brane interaction, in the presence of a $(3+1)$-brane a stationary black hole can have only one parameter of the rotation (this follows from the general analysis [14]). We put $b = 0$ and choose $\psi = \text{const}$ as the brane equation. The metric on the brane can be obtained from the 5D MP metric (III.1)-(III.3) by putting $b = 0$ and $\psi = \text{const}$ there. In order to preserve the latter condition one can also put $\Psi = 0$ in the string equations (III.28)-(III.31). As a result one obtains the following equations

$$q^2 \dot{x} = 2\sigma_x \sqrt{\mathcal{X}}, \quad \text{(IV.11)}$$

$$q^2 \dot{\theta} = \sigma_\theta \sqrt{Q}, \quad \text{(IV.12)}$$

$$q^2 \dot{\phi} = \Phi \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{x - 1 + a^2} \right), \quad \text{(IV.13)}$$

where $q^2 = x + a^2 \cos^2 \theta - 1$, and

$$\mathcal{X} = x (x - 1 + a^2)(x - 1 - K) + a^2 \Phi^2, \quad \text{and}$$

$$Q = K + a^2 \cos^2 \theta - \frac{\Phi^2}{\sin^2 \theta}. \quad \text{(IV.14)}$$

For a string lying in the equatorial plane $\theta = \pi/2$ these equations allow further simplification. Since $Q = 0$, $K = \Phi^2$, and

$$\dot{x} = 2\sigma_x V, \quad V = \sqrt{x(x - 1 + a^2 - \Phi^2)}, \quad \text{(IV.15)}$$

$$\dot{\phi} = \frac{\Phi}{x - 1 + a^2}. \quad \text{(IV.16)}$$

From these equations we have

$$\frac{d\phi}{dx} = \frac{\sigma_x}{2} \frac{\Phi}{x - 1 + a^2} \sqrt{\frac{x - 1}{x(x - 1 + a^2 - \Phi^2)}}. \quad \text{(IV.17)}$$
Outside the infinite red-shift surface, where \( x > 1 \), this equation may have a singular point only if \( |\Phi| > a \). This point \( x = 1 - a^2 + \Phi^2 \) is a turning point where the string has a minimal distance to the black hole. For \( |\Phi| = a \) this singular point disappears. For this value of \( \Phi \) the string crosses the ergosphere and enters the 5D horizon.

In the case when \( |\Phi| < a \), the string is not regular at the infinite red-shift surface. To demonstrate this consider the induced metric on the string worldsheet

\[
d\gamma^2 = \left[ \frac{x D^2 - \Phi^2 a^2}{4 x D^2 (D - \Phi^2)} \right] dx^2 - dt^2 + \frac{1}{x} \left[ dt + \frac{a \Phi}{2D} \sqrt{\frac{x - 1}{x(D - \Phi^2)}} \right]^2,
\]

where we have defined \( D = x - 1 + a^2 \) for brevity. The 2D curvature has only one component which for the metric (IV.18) is

\[
R = \frac{2}{x^2(x - 1)^2} \left[ 3(x - 1)^2 + (4x - 3)(a^2 - \Phi^2) \right].
\]

(IV.19)

For \( |\Phi| < a \) the curvature is infinite at the infinite redshift surface, \( x = 1 \). The curvature remains finite at \( x = 1 \) in a special case \( |\Phi| = a \) when it reduces to \( R = 6/x^2 \). To examine the nature of the singularity for \( |\Phi| < a \) let us consider the determinant of the metric (IV.18)

\[
g = -\frac{x - 1}{4x(x^2 - 1 - 4x^2)}.
\]

(IV.20)

We see that for values of \( x > 1 \) the metric has a negative determinant, signaling one positive and one negative eigenvalue, corresponding to one spacelike and one timelike dimension. For \( |\Phi| = a \) it remains negative for \( 0 < x < 1 \). However, for \( |\Phi| < a \) the determinant is positive between \( x = 1 \) and \( x = 1 - a^2 + \Phi^2 \), so that that induced metric has Euclidean signature. The corresponding spacelike surface does not represent any solution for a physical cosmic string.

In the next section we obtain a general solution for a stationary string which enters ergosphere and horizon of 5D rotating black hole and remains regular and timelike there. In the appendix we prove the uniqueness of such a solution.

V. STATIONARY STRINGS ATTACHED TO A ROTATING BLACK HOLE

A. Principal null rays

We discuss now conditions when a stationary string can cross the horizon and enter the black hole. In a four dimensional case a uniqueness theorem was proven [10] according to which a worldsheet of such a string must be generated by the timelike at infinity Killing vector \( \xi = \xi(t) \) and a principal null vector field \( l \). We demonstrate that a similar result is valid in a 5D case.

The principal null vectors in 5D MP metric are defined as a solution of the equation

\[
l_{\pm[a} C_{\beta]gamma} l_{\pm}^a = 0,
\]

(V.1)

where \( C_{\beta]gamma} \) is the Weyl tensor. They are of the form [7,9]

\[
l_{\pm}^a \partial_\mu = \pm 2\sqrt{a} \partial_x,
\]

(IV.19)

\[
+ \frac{(x + a^2)(x + b^2)}{\Delta} \left[ \partial_t - \frac{a}{x + a^2} \partial_\phi - \frac{b}{x + b^2} \partial_\psi \right].
\]

(V.2)

The integral lines of \( l_{\pm}^a \) are geodesics. By analogy with similar congruences in the four-dimensional Kerr geometry, we call the congruences generated by \( l_{\pm}^a \) principal Kerr null congruences.

One can define a convenient basis by accompanying the two null vectors \( l_+ \) and \( l_- \) by the vectors \( m, \bar{m} \) and \( k \) defined as follows

\[
m^\mu \partial_\mu = \frac{1}{\rho \sqrt{2}} \left( \partial_\theta + i \frac{\sin \theta \cos \theta}{\rho} \frac{\beta}{P} \right),
\]

(5.3)

\[
\beta = (b^2 - a^2) \partial_t + \frac{a}{\sin^2 \theta} \partial_\phi - \frac{b}{\cos^2 \theta} \partial_\psi,
\]

(5.4)

\[
k^\mu \partial_\mu = \frac{1}{\sqrt{x} P} \left( ab \partial_t - b \partial_\phi - a \partial_\psi \right).
\]

(5.5)

These vectors obey the following normalization conditions

\[
(m \cdot m) = (\bar{m} \cdot \bar{m}) = 0, \quad (m \cdot \bar{m}) = 1, \quad (l_{\pm} \cdot m) = 0,
\]

(V.6)

\[
(k \cdot k) = 1, \quad (k \cdot m) = (k \cdot \bar{m}) = (k \cdot l_\pm) = 0,
\]

(V.6)

\[
(l_+ \cdot l_) = 2 x P^2 / \Delta, \quad (l_{\pm} \cdot \xi) = -1.
\]

(V.6)

Direct calculations (by using GRTensor) allow one to prove that for this choice of the basis one has

\[
\xi_{\mu \nu} = \frac{\Delta F_x}{\rho^2 \sqrt{x}} l_{\mu} l_{-\nu} - \frac{2 i P (1 - F)}{\rho^2} m_{\mu} \bar{m}_{\nu}.
\]

(5.7)

The relation (V.7) shows that the principal null vectors \( l_{\pm} \) are eigenvectors of \( \xi_{\mu \nu} \)

\[
\xi_{\mu \nu} l_{\pm} = \pm \beta l_{\pm} , \quad \beta = \sqrt{x} F_x = \frac{1}{2} F_x.
\]

(V.8)
B. Principal Killing surfaces

We use now vectors $\xi$ and $l$ to generate a stationary 2D surface. We would like to have a surface which is regular at the future event horizon $H^+$. For this reason we shall use the incoming principal null vector $l_-$ which is linearly independent from $\xi$ at $H^+$ and denote it simply by $l$. The time symmetry implies that $L_{\xi}l = 0$ where $L_{\xi}$ is a Lie derivative with respect to the vector field $\xi$. (For briefness we omit the index $t$.) This condition can be written as

$$[\xi, l] = 0.$$  \hspace{1cm} (V.9)

By the Frobenious theorem the relation (V.9) implies that there exists a 2D surface $\Sigma$ given by the equation $x^\mu = x^\mu(\zeta)$, $(\zeta^A = (v, \lambda))$ such that $x^\mu_\lambda = \xi^\mu$, $x^\mu_\nu = -l^\mu$. We call $\Sigma$ a principal Killing surface. The parameter $v$ coincides with the Killing time, while $\lambda$ is an affine parameter along a principal null geodesics. In these coordinates the metric on $\Sigma$ takes the form

$$d\gamma^2 = G_{AB}d\zeta^Ad\zeta^B = -Fdv^2 + 2\alpha dv\,d\lambda,$$  \hspace{1cm} (V.10)

$$F = -\xi^2, \quad \alpha = -\langle\xi, l\rangle.$$  \hspace{1cm} (V.11)

Let us show that $\alpha$ is constant* on $\Sigma$. Really

$$\frac{d\alpha}{dv} = -\xi^\mu(l^\nu_{\nu})_{;\mu} = \xi_{\nu;\mu}l^\nu + \xi^\mu l^\nu_{\nu;\mu} = 0.$$  \hspace{1cm} (V.12)

The first term in the right hand side vanishes since $\xi_{\mu;\nu}$ is antisymmetric, the second vanishes since the null congruence is geodesic. Similarly

$$\frac{d\alpha}{d\lambda} = -\xi^\mu(l^\nu_{\nu})_{;\mu} = -\xi^\mu(l^\nu_{\nu})_{;\mu} = 0.$$  \hspace{1cm} (V.13)

The latter equality follows from (V.9). By rescaling the affine parameter $\lambda$ one can put the constant $\alpha$ to be equal to 1. We shall use this gauge.

We introduce three vectors, $n^\mu_R$ ($R = 2, 3, 4$) orthogonal to the worldsheet $\Sigma$

$$g_{\mu\nu}n^\mu_Rn^\nu_S = \delta_{RS}, \quad g_{\mu\nu}x^\mu_{;A}n^\nu_R = 0$$  \hspace{1cm} (V.14)

so that they together with $\xi$ and $l$ form a complete set. For this set the following relation is valid

$$g^{\mu\nu} = G^{AB}x^\mu_{;A}x^\nu_{;B} + \delta_{RS}n^\mu_Rn^\nu_S.$$  \hspace{1cm} (V.15)

The concrete form of these normal vectors $n_R$ is not important.

Consider the second fundamental form for the worldsheet $\Sigma$, defined as

$$\Omega_{RAB} = g_{\mu\nu}n^\mu_Rx^\nu_{;A}(x^\nu_{;B})_{;\rho}.$$  \hspace{1cm} (V.16)

For the worldsheet to be minimal, we will need to have a vanishing trace of the second fundamental form

$$\Omega_R = G^{AB}\Omega_{RAB} = (n_R \cdot z),$$  \hspace{1cm} (V.17)

where

$$z^\nu = G^{AB}x^\rho_{;A}(x^\rho_{;B})_{;\rho}.$$  \hspace{1cm} (V.18)

Simple calculations using (V.9) give

$$z^\nu = -2\xi^\nu l^\rho + Fl^\rho l^\nu.$$  \hspace{1cm} (V.19)

Since the integral lines of $l$ are geodesics, the second term on the right hand side vanishes. Using (V.8) we can write (V.19) in the form

$$z^\nu = F_{;\nu}l^\nu.$$  \hspace{1cm} (V.20)

Since $(n_R \cdot l) = 0$ one has $\Omega_R = 0$. This result implies that the principal Killing surface $\Sigma$ is a minimal surface and hence it is a stationary solution of the Nambu-Goto equations. This solution describes a stationary string which crosses the ergosphere and enters the event horizon. It is shown in the appendix that the principal Killing surface is the only stationary string solution which crosses the finite red-shift surface and remains a regular time-like surface there.

C. Explicit form of a solution for a principal Killing string

The expression (V.2) for the principal null vectors implies that

$$p = \frac{(x + a^2)(x + b^2)}{\Delta} \xi - l$$  \hspace{1cm} (V.21)

is a spacelike vector tangent to the string and one has

$$\dot{x}^i = p^i.$$  \hspace{1cm} (V.22)

Thus the string equations (III.28)-(III.31) for this case take the form $\theta = \theta_0 =$const,

$$\dot{x} = \mp 2\sqrt{\Delta}, \quad \dot{\phi} = a(x + b^2)\Delta, \quad \dot{\psi} = b(x + a^2)\Delta.$$  \hspace{1cm} (V.23)

Equation (V.23) shows that $x = \sqrt{\Delta} = \sigma^2$. By comparing these equations with (III.28)-(III.31) one obtains

$$\Phi = a\sin^2 \theta_0, \quad \Psi = b\cos^2 \theta_0,$$  \hspace{1cm} (V.24)

$$K = (a^2 - b^2)(\sin^2 \theta_0 - \cos^2 \theta_0).$$  \hspace{1cm} (V.25)

The remaining two equations of motion
\[ \dot{\phi} = \frac{a(r^2 + b^2)}{(r^2 + a^2)(r^2 + b^2) - r^2}, \quad (V.26) \]
\[ \dot{\psi} = \frac{b(r^2 + a^2)}{(r^2 + a^2)(r^2 + b^2) - r^2}, \quad (V.27) \]
can be integrated and give
\[ \phi = \phi_0 + \frac{a}{2(r^2 + r^2)} \left[ \frac{r_+^2 + b^2}{r_+} \ln \left( \frac{r - r_+}{r + r_+} \right) - \frac{r_+^2 + b^2}{r_-} \ln \left( \frac{r - r_-}{r + r_-} \right) \right], \quad (V.28) \]
\[ \psi = \psi_0 + \frac{b}{2(r^2 + r^2)} \left[ \frac{r_+^2 + a^2}{r_+} \ln \left( \frac{r - r_+}{r + r_+} \right) - \frac{r_+^2 + a^2}{r_-} \ln \left( \frac{r - r_-}{r + r_-} \right) \right], \quad (V.29) \]
with \( \phi_0 \) and \( \psi_0 \) being initial data for the string, and \( r_\pm \) being the horizon locations, defined in (III.6). The last two relations show that when \( a \neq 0 \) and \( b \neq 0 \), the principal Killing string makes infinite number of turns in \( \phi \) and \( \psi \) directions before it reaches the horizon \( r_+ \). This is a pure coordinate effect. In the next section we show that in the ingoing Eddington-Finkelstein coordinates which are regular at the horizon there is no infinite winding.

**VI. THE PRINCIPAL KILLING STRING AS A 2D BLACK HOLE**

To study stationary strings in the 5D black hole exterior we used the MP metric in the Boyer-Lindquist coordinates (III.1). The principal Killing strings cross infinite red-shift surface and enter the 5D black hole horizon. For studying global properties of these solutions instead of Boyer-Lindquist coordinates it is more convenient to use the ingoing Eddington-Finkelstein coordinates which are regular at the future event horizon. The corresponding coordinate transformation is
\[ dv = dt + \frac{(x + a^2)(x + b^2)}{2\Delta x} dx, \quad (VI.1) \]
\[ d\tilde{\phi} = d\phi - \frac{(x + b^2)a}{2\Delta x} dx, \quad (VI.2) \]
\[ d\tilde{\psi} = d\psi - \frac{(x + a^2)b}{2\Delta x} dx. \quad (VI.3) \]
In these coordinates the 5D MP metric is
\[ ds^2 = -dv^2 + (r^2 + a^2) \sin^2 \theta d\tilde{\phi}^2 + (r^2 + b^2) \cos^2 \theta d\tilde{\psi}^2 \]
\[ + \rho^2 d\theta^2 + \frac{1}{\rho^2} \left( dv + a \sin^2 \theta d\tilde{\phi} + b \cos^2 \theta d\tilde{\psi} \right)^2 \]
\[ + 2dr \left( dv + a \sin^2 \theta d\tilde{\phi} + b \cos^2 \theta d\tilde{\psi} \right). \]
It is more convenient to use again the radial coordinate \( r \) instead of \( x = r^2 \). (We still use units in which \( r_0 = 1 \)).

The original Killing vectors \( \partial_t, \partial_\phi \) and \( \partial_\psi \) take the form \( \partial_t, \partial_{\tilde{\phi}}, \) and \( \partial_{\tilde{\psi}} \).

The principal Killing string in these coordinates has a simple form
\[ \theta = \theta_0, \quad \tilde{\phi} = \tilde{\phi}_0, \quad \tilde{\psi} = \tilde{\psi}_0. \quad (VI.5) \]
We use coordinates of \( c^0 = v \) and \( c^1 = r \) as coordinates on \( \Sigma \). The induced metric in these coordinates is
\[ d\gamma^2 = -F dv^2 + 2dr dv, \quad (VI.6) \]
\[ F = 1 - \frac{1}{r^2 + P_0^2}, \quad P_0^2 = a^2 \cos^2 \theta_0 + b^2 \sin^2 \theta_0. \quad (VI.7) \]
This is a metric of a 2D black hole with an event horizon located at
\[ r^2 + P_0^2 = 1. \quad (VI.8) \]
The surface gravity is given by \( \kappa = \frac{1}{2} F_{,r} \), evaluated at the horizon \( F = 0 \). We get
\[ \kappa = \sqrt{1 - P_0^2}. \quad (VI.9) \]
For comparison, we restate the five dimensional surface gravity \( \kappa_5 \) from (III.7) in an explicit form
\[ \kappa_5 = \sqrt{2} \frac{\sqrt{B^2 - 4a^2b^2}}{B + \sqrt{B^2 - 4a^2b^2}}, \quad (VI.10) \]
where \( B = 1 - a^2 - b^2 \). Denote
\[ f(r) = \frac{2r^2 - B}{r}. \quad (VI.11) \]
This function is monotonically increasing. One also has
\[ \kappa_5 = f(r_+), \quad (VI.12) \]
where \( r_+^2 = \frac{1}{2}(B + \sqrt{B^2 - 4a^2b^2}) \). Since \( r_+ \leq B^{1/2} \) one has
\[ \kappa_5 = f(r_+) \leq f(B^{1/2}) = B^{1/2} \leq \kappa_2. \quad (VI.13) \]
This relation shows that in a general case the surface gravity of 2D principal Killing string hole is greater than the surface gravity of the 5D MP black hole. The equality is possible only if
\[ a^2 \sin^2 \theta_0 + b^2 \cos^2 \theta_0 = 0. \quad (VI.14) \]
This occurs either when \( a = b = 0 \) and the black hole is non-rotating, or when one of the rotation parameters, say \( b \), vanishes and the string is in the \( \theta = 0 \) plane.
VII. HIGHER DIMENSIONAL ROTATING BLACK HOLES ATTACHED TO A STRING

A. Higher dimensional MP metric

We demonstrate now that the results concerning principal Killing strings can be generalized to the higher dimensional case, that is when the number \( N \) of spatial dimensions is greater than 4. Corresponding MP metrics [7] have slightly different form for \( N \) even and odd.

For an even number of spatial dimensions

\[
ds^2 = -dt^2 + \sum_i (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\Pi L}{\Pi - mr^2} dr^2 + \frac{mr^2}{\Pi L} (dt + \sum_i a_i \mu_i^2 d\phi_i)^2,
\]

where \( \Pi \) and \( L \) given by

\[
\Pi = \prod (a_i^2 + a_i^2), \quad L = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2},
\]

and \( \mu_i \) obey the relation

\[
\sum_i \mu_i^2 = 1.
\]

For an odd number of spatial dimensions we instead have

\[
ds^2 = -dt^2 + r^2 d\mu^2 + \sum_i (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)
\]

\[
+ \frac{\Pi L}{\Pi - mr^2} dr^2 + \frac{mr^2}{\Pi L} (dt + \sum_i a_i \mu_i^2 d\phi_i)^2.
\]

\[
\mu^2 + \sum_i \mu_i^2 = 1.
\]

Here and later it is assumed that the summation \( \sum_i \) is taken from \( i = 1 \) to \( i = [N/2] \), where \([A]\) means the integer part of \( A \).

The parameter \( m = r_0^{N-2} \), \( r_0 \) being the gravitational radius, is related to the mass \( M \) of the black hole as follows

\[
M = \frac{(N-1)A_{N-1} r_0^{N-2}}{16\pi G_{N+1}},
\]

where

\[
A_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}
\]

is the area of a unit sphere \( S^{N-1} \) and \( G_{N+1} \) is the \( N+1 \)-dimensional gravitational coupling constant which has dimensionality \([\text{length}]^{(N-2)}/[\text{mass}]\).

The angular momenta \( J_i \) of the black hole are defined as

\[
J_i = -\frac{A_{N-1}}{8\pi G_{N+1}} m a_i = -\frac{2}{N-1} M a_i.
\]

(Note that the sign of the rotation parameters in the MP metric is opposite to the one adopted in the Kerr metric.)

The principal null vectors \( l_{\pm} \) are given by

\[
l_{\pm}^\mu \partial_\mu = \Lambda \left( \partial_t - \sum_i \frac{a_i}{r^2 + a_i^2} \partial\phi_i \right) \pm \partial_r.
\]

where

\[
\Lambda = \Pi/(\Pi - mr), \quad \text{for odd } N;
\]

\[
\Lambda = \Pi/(\Pi - mr^2), \quad \text{for even } N.
\]

Let us introduce the ingoing (−) and outgoing (+) Eddington-Finkelstein coordinates \((v_\pm, r, \phi_{\pm i})\)

\[
dv_\pm = dt \mp \Lambda dr,
\]

\[
d\tilde{\phi}_{\pm i} = d\phi_i \pm \frac{\Lambda a_i}{r^2 + a_i^2} dr.
\]

One has

\[
l_{\pm}^\mu \partial_\mu = \pm \partial_r, \quad l_{\pm} \mu dx^\mu = -[dv_\pm + \sum_i \mu_i^2 a_i d\tilde{\phi}_{\pm i}].
\]

For odd \( N \) the MP metric takes the form

\[
ds^2 = -dv_\pm^2 + \sum_i (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)
\]

\[
+ \frac{mr^2}{\Pi L} (l_{\pm} \mu dx^\mu)^2 \pm 2dr(l_{\pm} \mu dx^\mu) + r^2 d\mu^2.
\]

For even \( N \) the metric is similar. The only difference is that the last term is absent and \( mr \) must be changed to \( mr^2 \).

In the MP metrics (VII.1), (VII.4) and (VII.15) the quantities \( \mu \) and \( \mu_i \) are not independent. They obey the restrictions (VII.3) or (VII.5) indicating that these quantities belong to a unit sphere. We denote by \( \theta_a \) \((a = 1, \ldots, [(N-1)/2])\) independent coordinates on the sphere. We shall also use a notation \( \omega_m = (\theta_a, \tilde{\phi}_i, m = (2, \ldots, N)) \) for a total set of ‘angular’ coordinates.

Denote as earlier by \( \xi \) a timelike at infinity Killing vector, \( \xi^\mu \partial_\mu = \partial_t \). Then the principal null vectors are eigen-vectors of \( \xi_{\mu\nu} \),

\[
\xi_{\mu\nu} l_{\pm}^\nu = \pm \frac{1}{2} \partial_\mu F l_{\pm}^\mu,
\]

where \( F = -\xi^2 = -g_{\Theta \Theta} \). This is a generalization of the property (V.8) to the higher dimensional case. To prove this relation we consider an expression...
\[\xi_{\mu;\nu} l^\nu_{\pm} = \xi_{\mu,\nu} l^\nu_{\pm} - \Gamma_{\sigma,\mu\nu} \xi^\sigma l^\nu_{\pm}. \quad \text{(VII.17)}\]

Since \(l^\nu_{\pm} = \pm \delta^\nu_{\pm}\) and \(\xi^\mu = \delta^\mu_{\pm}\), we obtain
\[\xi_{\mu;\nu} l^\nu_{\pm} = \pm g_{\nu,\mu} + \frac{1}{2} (g_{\nu,\mu} + g_{\nu,\mu} - g_{\nu,\mu}) ,\]
\[= \pm \frac{1}{2} g_{\nu,\mu} = \pm \frac{1}{2} F_{\nu, \mu}. \quad \text{(VII.18)}\]

Combining relations (VII.18) and (VII.19) one proves the relation (VII.16).

**B. Higher-dimensional principal Killing strings**

Now we show that a 2D surface \(\Sigma\) spanned by the vectors \(l^\mu_{\pm}\) and \(\xi^\mu\) is a solution to the Nambu-Goto equations of motions, and is thus a minimal surface. We use coordinates \(\xi^0 = v\) and \(\xi^1 = r\) to parametrize \(\Sigma\). The worldsheet equation is
\[X^0 = v, \quad X^1 = r, \quad X^m = \omega_m = \text{const}. \quad \text{(VII.19)}\]

We write the Nambu-Goto equations as follows
\[g_{\mu\nu} \Box X^\nu + G^{AB} \Gamma_{\mu, \alpha\beta} X^\alpha_{A} X^\beta_{B} = 0, \quad \text{(VII.20)}\]
with
\[\Box X^\nu = \frac{1}{\sqrt{-G}} \partial_A (\sqrt{-G} G^{AB} \partial_B X^\nu), \quad \text{(VII.21)}\]
\[G^{AB} \partial_A \partial_B = F \partial^2_r = 2 \partial_r \partial_v, \quad \sqrt{-G} = 1. \quad \text{(VII.22)}\]

The first term in (VII.20) is
\[g_{\mu\nu} \Box X^\nu = g_{\mu\nu} [\partial_r (F \partial_r X^\nu) - 2 \partial_r \partial_v X^\nu] = g_{\mu\nu} \partial_r F, \quad \text{(VII.23)}\]
and the second term reads
\[G^{AB} \Gamma_{\mu, \alpha\beta} X^\alpha_{A} X^\beta_{B} = \Gamma_{\mu, \alpha\beta} [F X^\alpha_{A} X^\beta_{B} + 2 X^\alpha_{A} X^\beta_{B}] \]
\[= \frac{1}{2} F (2 g_{\nu,\mu} - g_{\nu,\mu}) \Rightarrow (g_{\nu,\mu} + g_{\nu,\mu} - g_{\nu,\mu}). \quad \text{(VII.24)}\]

The form of the metric (VII.15) implies that
\[g_{\mu\nu,\nu} = g_{\mu\nu,\nu} = g_{\nu\nu,\mu} = 0. \quad \text{(VII.25)}\]

Thus the left hand side of (VII.20) is
\[g_{\mu\nu} \partial_r F + g_{\nu,\mu} = \pm l^\mu_{\pm} \partial_r F + F_{r, l^\mu_{\pm}} = 0. \quad \text{(VII.26)}\]

This result means that the principal Killing surface (VII.19) is a solution of the Nambu-Goto equations (VII.20). A principal Killing string which is regular at the future event horizon is generated by \(l_-\).

**VIII. INTERACTION OF A HIGHER DIMENSIONAL ROTATING BLACK HOLE WITH A STRING**

Until now we considered a cosmic string as a test object and neglect its action on the black hole. We show now that the interaction between the string and the black hole results in the transfer of the angular momenta from the black hole to the string.

Suppose there exists a distribution of matter outside the rotating black hole. Then the fluxes of energy and angular momenta of this matter through a surface \(r = \text{const}\) are
\[\Delta E = - \int T_{\mu}^\nu \xi^\mu d\sigma^\nu, \quad \Delta J_i = \int T_{\mu}^\nu \xi^\mu d\sigma^\nu. \quad \text{(VIII.1)}\]

Here \(T_{\mu\nu}\) is the stress-energy tensor of the matter, and \(\xi^\mu \partial_{\mu} = \partial_{\xi^i}\) are the Killing vectors connected with the rotational invariance of the MP metric. The expression for \(d\sigma_{\mu}\) can be written as follows
\[d\sigma_{\mu} = r_{,\mu} \sqrt{-g} d\nu d\omega^{N-1}. \quad \text{(VIII.2)}\]
where
\[d\omega^{N-1} = \prod_{m=1}^{N-1} d\omega_m. \quad \text{(VIII.3)}\]

For a stationary configuration the (constant) rate of energy and angular momentum fluxes from the black hole through the \(r = \text{const}\) surface are
\[\dot{E} = \frac{dE}{dv} = - \int d\omega^{N-1} \sqrt{-g} T_{\mu}^\nu \xi^\mu d\sigma^\nu, \quad \text{(VIII.4)}\]
\[\dot{J}_i = \frac{dJ_i}{dv} = \int d\omega^{N-1} \sqrt{-g} T_{\mu}^\nu \xi^\mu d\sigma^\nu. \quad \text{(VIII.5)}\]

If a part of initially infinite string is captured by a black hole its world sheet in the black hole exterior consists of two segments. For a stationary cosmic string each of these segments is a principal Killing string. We calculate the energy and angular momenta transfer for one segment. The stress-energy tensor of the string is defined as follows:
\[\sqrt{-g} G^{\mu\nu} = - \mu^\ast \int d^2 \xi d^2 (X - X(\xi)) \mu^{\mu\nu}, \quad \text{(VIII.6)}\]
\[\mu^{\mu\nu} = \sqrt{-G} G^{AB} X_{A}^{\mu} X_{B}^{\nu}, \quad \text{(VIII.7)}\]
where \(\mu^\ast\) is the tension of the string. For the principal Killing string one has
\[\mu^{\mu\nu} = F^{\mu_1 \nu_1} \xi^\mu \xi^\nu - 2 l^\mu \xi^\nu. \quad \text{(VIII.8)}\]

Since we are considering strings which are regular at the future event horizon, \(l\) is equal to \(l_-\). Using (VII.15) one obtains
Substituting (VIII.6) into (VIII.4) and (VIII.5), taking the integrals and using the relations (VIII.9) one finds
\[ E = 0, \quad \text{(VIII.10)} \]
The angular momentum flux does not depend on \( r \). This is in accordance with the conservation law. The energy flux vanishes. The angular momenta transfer to the string results in the decrease of the corresponding angular momenta of the black hole
\[ J_i^{BH} = -\dot{J}_i. \quad \text{(VIII.12)} \]
Using the expression (VII.8) for the angular momenta of the black hole one can obtain the following equation for the loss of the angular momenta of the black hole
\[ J_i^{BH} = -\mu_i^2(N-1)\frac{\mu_i^*}{M} J_i^{BH}. \quad \text{(VIII.13)} \]
In this equation we took into account that there are 2 string segments attached to the black hole. We choose the second segment to be an inverse image of the first one with respect to the center of the black hole so that for it \( \tilde{\phi} \to \pi + \tilde{\phi} \) (and \( \mu \to -\mu \) for odd \( N \)). This guarantees that under the action of the string the black hole remains at rest as a whole. In 4D case of the Kerr black hole the obtained result coincides with result of [14].

The equation (VIII.13) shows that \( J_i^{BH} = 0 \) either when \( J_i^{BH} = 0 \), that is when the black hole does not rotate in the \( i \)-th bi-plane of rotation, or when \( \mu_i^2 = 0 \). In the 4D case of the Kerr black hole the latter condition implies that the string is directed along the axis of rotation of the black hole.

Equation (VIII.13) shows that the bulk components of the angular momentum \( J_i \) with \( \mu_i \neq 0 \) decrease. In the case when \( \mu^* \) is small (\( \mu^* r_+/M \ll 1 \)) this process is very slow and one can use a quasistationary approximation, that is to consider it as a slow change from one stationary configuration to another one. In this approximation the evolution of the system can be described as a evolution in the space of parameters characterizing quasistationary system black-hole–cosmic-string.

To estimate the characteristic time of the slowing down of the bulk components of the black hole rotation let us consider a case when only one component of the initial angular momentum is non-vanishing and the corresponding \( \mu_i = 1 \). In this case all other \( \mu_i \) vanish. This is a higher dimensional analogue of a cosmic string in the equatorial plane of the 4D Kerr black hole. Since \( \mu^* \) is small and the evolution is adiabatic, the surface area of the black hole
\[ A = r_+^{N-3}(r_+^2 + a^2)A_{N-1}, \quad \text{(VIII.14)} \]
remains constant. Here \( r_+ \) is a position of the event horizon defined by the relation
\[ 16\pi G_{N+1} M = r_+^{N-4}(r_+^2 + a^2)A_{N-1}. \quad \text{(VIII.15)} \]
By using the equations (VIII.13), (VIII.14) and (VII.8) one gets
\[ \dot{r}_+ = \frac{32\pi G_{N+1}\mu^*}{(N-1)A_{N-1}} \frac{a^2}{r_+^{N-5}(r_+^2 + a^2)^2}. \quad \text{(VIII.16)} \]
This equation shows that during the evolution \( r_+ \) does not decrease and it remains constant only when \( a = 0 \). Let us rewrite the relation (VIII.14) in the form
\[ \beta \equiv \frac{\mu_i^2}{r_+} = \frac{A}{A_{N-1}r_+^{N-1}} - 1. \quad \text{(VIII.17)} \]
This relation shows that \( r_+ \) grows until it reaches its final value
\[ r_f = \left( \frac{A}{A_{N-1}} \right)^{1/(N-1)}. \quad \text{(VIII.18)} \]
Using (VIII.18) and (VIII.16) one can obtain the following equation which defines the evolution of the rotation parameter \( \beta \)
\[ \dot{\beta} = -\frac{1}{T} \frac{\beta}{(1 + \beta)^{N-3}}, \quad \text{(VIII.19)} \]
where
\[ T = \frac{M_f}{2(N-1)\mu^*}, \quad \text{(VIII.20)} \]
where \( M_f = A(N-1)/(16\pi G_{N+1} r_f) \) is the final value of mass of the black hole. The parameter \( T \) has dimensionality of time and it determines the characteristic time scale of the process of slowing down the rotation of the black hole.

The equation (VIII.19) can be solved analytically. The solution is
\[ \ln \beta + \frac{\beta^2 F_2(1,1,\frac{N-2}{2};2,2;\beta)}{N-1} = -\frac{t}{T} + C, \quad \text{(VIII.21)} \]
where \( C \) is the constant of integration. The hypergeometric function takes the value 1 when \( \beta = 0 \). In the limit \( t \to \infty \) we have \( \beta \to 0 \) so the logarithm is the leading term in the left hand side. Thus at late time the function \( \beta \) has the following asymptotic form
\[ \beta = \beta_0 \exp \left( -\frac{t}{T} \right). \quad \text{(VIII.22)} \]
IX. DISCUSSIONS

Let us summarize the results obtained in the paper. We considered interaction of a cosmic string with rotating higher dimensional black holes. In 5D case the stationary string equations allow separation of variables. This occurs because of the existence of a sufficient number of integral of motions which, in particular, include the conservation law connected with the Killing tensor. Increasing the number of spacetime dimensions from 5 to 6 does not produce a new Killing vector. Moreover it is probable that the 6D MP spacetime does not have the Killing tensor. Lack of the sufficient number of integrals of motion in six and higher number of spacetime dimensions make it improbable the separation of variables of stationary string equations in such higher dimensional spacetimes.

Among all stationary solutions of string equations there is a special class, principal Killing strings. Their characteristic property is that such solutions describe stationary strings which starting from spatial infinity cross the ergosphere and enter the horizon and remain regular. We demonstrated that these solutions exist in an arbitrary number of dimensions. Internal geometry of these principal Killing strings is a geometry of 2D static black hole with its 2D horizon located at the intersection of the string with infinite red-shift surface. Perturbations propagating along the string cannot escape the region inside the 2D horizon. At the same time, the causal signals propagating in the bulk space can transfer the information about the state of the string inside its 2D horizon up to the point of its intersection with the horizon of the higher-dimensional black hole. A 2D observer would interpret this as extracting the information from the 2D black hole by means of extra dimensions.

An interaction of a rotating black hole with the string results in the reduction of some components of its angular moments. In a general case there exists an angular momentum flux from a black hole to the attached string. The flux of the angular momentum is proportional to the tension of the string $\mu^*$. The characteristic time of the relaxation process during which the black hole reaches its final state is given by relation (VIII.20). In the first order in the string tension $\mu^*$ there is no energy flux. The final stationary black hole may have rotation only in those planes for which $\mu_i = 0$.

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APPENDIX A: UNIQUENESS PROPERTY

We demonstrate now that a principal Killing surface is the only possible worldsheet of a stationary string which crosses the infinite red-shift surface and remain regular.

Consider a stationary surface $\Sigma$. It can be presented as a one parameter family of trajectories of the Killing vector $\xi$. Let $v$ be a Killing time parameter, then $d\xi^A/dv = \xi^A$. We use $v$ as one of the coordinates on $\Sigma$. We use an ambiguity in the choice of the other coordinate, $r$, to put $G_{rr} = 0$. For this choice

$$d\gamma^2 = -F dv^2 + 2\alpha dr dv, \quad \text{(A.1)}$$

$$F = -\xi^2, \quad \alpha = - (\xi \cdot l), \quad l^A \partial_A = - \partial_r. \quad \text{(A.2)}$$

The Killing vector $\xi$ together with a null vector $l$ span $\Sigma$. It is easy to check that $\xi$ is also a Killing vector for the induced metric $G_{AB}$. Using Killing equation $\xi_{(A,B)} = 0$ in the induced metric one has $\partial_r F = \partial_r \alpha = 0$. Using an ambiguity $r \to f(r)$ one can always put $\alpha = 1$. Denote by $n_{\mu}$ a complete set of mutually orthogonal unit vectors orthogonal to $\Sigma$. Then a definition of the second fundamental form (V.16) implies that relations (V.17)-(V.19) are still valid

$$\Omega_R = (n_R \cdot z), \quad \text{(A.3)}$$

$$z^\mu = -2\xi^\mu \cdot l^\rho + F [l^\rho l^\mu]. \quad \text{(A.4)}$$

Since $\xi_{\mu \nu}$ is antisymmetric and $l$ is null one also has

$$(l \cdot z) = 0. \quad \text{(A.5)}$$

The surface $\Sigma$ is minimal if $\Omega_R = 0$. For such a surface

$$\Omega^2 = \Omega_R \Omega^R = (g^{\mu \nu} - G^{AB} x_A^{\mu} x_B^{\nu}) z_\mu z_\nu = 0. \quad \text{(A.6)}$$

Using (A.5) it is easy to check that for our choice of the coordinates

$$G^{AB} x_A^{\mu} x_B^{\nu} z_\mu z_\nu = (l \cdot z) [2(\xi \cdot z) + F (l \cdot z)] = 0. \quad \text{(A.7)}$$

Thus $z$ is a null vector, $g_{\mu \nu} z^\mu z^\nu = 0$. Equation (A.5) implies that

$$z^\mu = q l^\mu. \quad \text{(A.8)}$$

To determine $q$ we multiply this relation by $\xi_\mu$

$$q = l^\rho (\xi^2)^{,\rho} + 2 F \xi_{\nu} l^\nu l^\rho = \frac{dF}{dr}. \quad \text{(A.9)}$$

Using (A.4) and (A.8) we have

$$2 \xi^\mu \cdot l^\rho = F [l^\rho l^\mu] = \frac{dF}{dr} l^\mu. \quad \text{(A.10)}$$

This relation shows that at the infinite red-shift surface, $F = 0$, the null vector $l$ is the eigenvector of $\xi^\mu \cdot l$ with
the eigenvalue $-F_r/2$ and hence it coincides with $l_-$. In
the vicinity of the infinite red-shift surface one has
\[ l = (1 + \lambda)l + \mu m + \bar{\nu} \mu + \nu k, \]  
(A.11)
The term proportional to $l_+$ does not appear since $l \cdot l = 0$. Equations (V.3)–(V.5) implies that
\[ (m \cdot \xi) = \frac{i \sin \theta \cos \theta}{\sqrt{2} \rho P} (a^2 - b^2), \]  
(A.12)
Using these relations and $\xi \cdot l = -1$ we obtain
\[ \lambda = \frac{1}{P} \left[ \frac{i (a^2 - b^2) \sin \theta \cos \theta}{\rho \sqrt{2}} (\bar{\mu} - \mu) - \frac{ab}{\sqrt{x}} \rho \right], \]  
(A.14)
where $P$ is defined by (III.2). Vectors $m$ and $k$ are given
by (V.3) and (V.5), respectively. In our perturbation
analysis we keep only those terms which are of the first
order in $\mu$ and $\nu$, and drop anything that is a multiple of
these two.
If we contract (A.10) with $\bar{\mu}$, and use the relations
\[ \bar{m}_\mu l^\rho \xi^\rho = -\frac{i \bar{\mu}(1 - F)P}{\rho^2}, \]  
(A.15)
\[ \bar{m}_\mu l^\mu = \bar{\mu}, \]  
(A.16)
\[ \bar{m}_\mu l^\rho l^\rho_{\bar{\mu}} = l^\rho \bar{\mu}_{\rho} + \frac{\bar{\mu}}{\rho^2} (2iP - \sqrt{x}), \]  
(A.17)
we arrive at
\[ F l^\rho \bar{\mu}_{\rho} = -F \frac{d\bar{\mu}}{dr} = -\Omega \bar{\mu}, \]  
(A.18)
where
\[ \Omega = \frac{2iP - \sqrt{x}F}{\rho^2} - \frac{dF}{dr}. \]  
(A.19)
So if we define the tortoise coordinate $r^*$ as
\[ \frac{dr}{dr^*} = F, \]  
(A.20)
so that the ergosphere lies at $r^* \to -\infty$, then we can solve for $\bar{\mu}$ as
\[ \bar{\mu} = \bar{\mu}_0 e^{\Omega r^*}, \]  
(A.21)
and since $\text{Re}(\Omega) < 0$ we must have $\bar{\mu}_0 = 0$ to ensure that
the solution is regular at the ergosphere.
If we contract (A.10) with $k_\mu$ and use the following relations
\[ k_\mu l^\rho \xi^\rho_{\bar{\mu}} = 0, \]  
(A.22)
we arrive at
\[ F l^\rho _\nu = -F \frac{d\nu}{dr} = -W \nu, \]  
(A.25)
where
\[ W = -\frac{dF}{dr} - \frac{F}{\sqrt{x}}. \]  
(A.26)
so we can get
\[ \nu = \nu_0 e^{W r^*}. \]  
(A.27)
Again since $\text{Re}(W) < 0$ we must have $\nu_0 = 0$ to have a
solution that is regular at the ergosphere.
We demonstrated that there is no a regular stationary string solution which coincides with $l_-$ at the infinite red-
shift surface but differs from it outside the ergosphere. This completes the proof.

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