Frame hydrodynamics of biaxial nematics from molecular-theory-based tensor models

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Abstract

Starting from a dynamic tensor model about two second-order tensors, we derive the frame hydrodynamics for the biaxial nematic phase using the Hilbert expansion. The coefficients in the frame model are derived from those in the tensor model. The energy dissipation of the tensor model is maintained in the frame model. The model is reduced to the Ericksen–Leslie model if the biaxial bulk energy minimum of the tensor model is reduced to a uniaxial one.

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1 Introduction

Liquid crystals are featured by local orientational order, typically originated from nonuniform orientational distribution of non-spherical rigid molecules. One case that many of us are familiar with is the uniaxial nematic phase formed by rod-like molecules. For the uniaxial nematic phase, the local orientational order can be described by a unit vector $n$. The hydrodynamics of liquid crystals then involves dynamics of the vector $n$, for which the well-known Ericksen–Leslie theory is proposed [8, 15]. The Ericksen–Leslie theory, as well as its variants, has been studied extensively in both analysis [18, 19, 20, 21, 40, 38, 22] and simulation [24, 6, 2, 37, 54]. It has also been coupled with other systems [52, 53]. For a detailed survey on modeling, analysis and computation of liquid crystals, we refer to [39].

Constructed on the assumption of uniaxial local anisotropy, the Ericksen–Leslie theory is opaque to the building blocks of liquid crystals. Although the elastic constants can be related to experimental measurements, several other coefficients in the hydrodynamics are difficult to obtain. This deficiency can be overcome by studying the relation of the Ericksen–Leslie theory to molecular models about the orientation density function [14, 7], or tensor models about a second-order tensor $Q$ [12]. From molecular models or tensor models, one could derive the Ericksen–Leslie theory with its coefficients expressed by those in the molecular models or tensor models. Such derivations are based on the fact that the minimum of the bulk energy must be uniaxial [25, 9]. When the bulk energy dominates, the dynamics can be regarded as constrained in the states such that the bulk energy takes its minimum, so that it reduces to a dynamics of vector field. The whole procedure is done through the Hilbert expansion that has been shown rigorously [41, 42, 17, 16].

Local orientational orders other than the uniaxial type have also been considered, of which the biaxial nematics is discussed more [35, 28, 1, 4]. Its orientational elasticity is written down in various forms that turn out to be equivalent [33, 10, 34]. Biaxial hydrodynamics are also proposed [33, 26, 5, 22, 11] in different forms. Analysis has been carried out for a few simplified models [23]. These works focus on the form of the model, in which many more coefficients are involved. A couple of previous works attempt to relate the elastic constants to molecular parameters [13, 50], while other coefficients in the hydrodynamics are yet to be considered.

In this paper, we derive the hydrodynamics for the biaxial nematic phase from a tensor model. The $Q$-tensor model for rod-like molecules can only possess isotropic and uniaxial bulk energy minima. For this reason, it is necessary to start from a different tensor model that could exhibit other types of bulk energy minima. We consider a dynamic tensor model for bent-core molecules derived from molecular theory [51], which has the biaxial nematic phase as an energy minimum. Its free energy is constructed on molecular architecture by expanding the pairwise molecular interaction kernel [47]. The interaction between the molecule and the fluid is also carefully derived from the molecular architecture. As a result, the form of the dynamic tensor model is determined by molecular symmetry, with all the coefficients calculated from molecular parameters.

Rigorous analyses show that under certain coefficients, the stationary points of the bulk energy can only be isotropic, uniaxial or biaxial [48, 49, 14]. Although further rigorous analysis is still not available, numerical studies indicate that we can indeed find some coefficients such that the biaxial nematic phase is the bulk energy minimum [35, 36, 27, 29, 30, 48, 47, 44]. Therefore, we assume that
it holds and use the Hilbert expansion near this bulk energy minimum. The free energy in the tensor model is rotationally invariant, which is an essential ingredient to be utilized in our derivation. In particular, the rotational invariance of the bulk energy implies that its minimum, if not isotropic, can be freely rotated. The biaxial nematic phase has its own symmetry other than axisymmetry. When axisymmetry does not hold, the orientation of the bulk energy minimum shall generally be described by an orthonormal frame, or an element in $SO(3)$. We would like to call it a ‘frame model’ that gives the elasticity and dynamics of the field of orthonormal frame.

Two key ingredients are needed to be dealt with in the Hilbert expansion. When the tensors are constrained at the biaxial minimum, it actually gives a three-dimensional manifold. We shall constrain the equations of tensors on this manifold to obtain the evolution equation for the orthonormal frame field. The tangent space of the manifold given by the bulk energy minimum gives a zero-eigenvalue subspace of the Hessian of the bulk energy. This subspace is utilized to cancel the non-leading terms in the Hilbert expansion, thus closing the system of the leading order. The free energy about tensors can then be reduced to the orientational elasticity for the biaxial nematic phase, with the elastic constants expressed as the coefficients in the tensor model, which is exactly the results in [50].

Although the free energy can be reduced straightforwardly, we still need to handle several high-order tensors, which call for a closure approximation to express them as functions of the order parameter tensors. Intuitively, these high-order tensors shall be consistent with the symmetry of the biaxial nematic phase, from which the form of high-order tensors can be written down. This intuition can be made rigorously by the closure through minimization of the entropy term. The entropy term can have two choices. One is calculated from the density function of the maximum entropy state, which we call the original entropy. The other is the quasi-entropy, an elementary function about tensors, which maintains essential properties and underlying physics of the original entropy [44]. No matter we choose the original entropy or the quasi-entropy, their fine properties result in the particular form of high-order tensors consistent with the symmetry of the biaxial nematic phase. From these symmetry arguments, we could further arrive at alternative expressions of these high-order tensors that are convenient for us to deduce the coefficients.

Using these properties, we could derive the frame model for the biaxial nematic phase. Its form is actually determined by the symmetry of the biaxial nematic phase, which is consistent with early works [11]. The coefficients, on the other hand, are expressed as functions of the coefficients in the tensor model. Again, since the coefficients in the tensor model are derived from physical parameters, the frame model we obtain is connected to rigid molecules with certain architecture. We shall show that the energy dissipation of the tensor model is maintained in the frame model. Furthermore, we will show that the biaxial hydrodynamics can be reduced to the Ericksen-Leslie theory when the bulk energy has a uniaxial minimum. The corresponding coefficients are also derived, which turn out to be distinct from those derived from the $Q$-tensor hydrodynamics for rod-like molecules.

Below, we begin with introducing some notations for orthonormal frames and tensors in Section 2. The tensor model is briefly described in Section 3. Here, we also claim essential properties on the entropy term, bulk energy minima, and high-order tensors. The Hilbert expansion is carried out in Section 4 from which we derive the biaxial hydrodynamics. The biaxial hydrodynamics can be reduced to Ericksen–Leslie theory if the bulk energy minimum becomes uniaxial, which is shown in Section 5. The relation of our model to the previous ones is discussed in Section 6. Concluding remarks are given in Section 7. Detailed calculations and discussions on high-order tensors are presented in Appendices.

# 2 Preliminary

Let us introduce some notations for orthonormal frames and tensors to be used subsequently. For the rigid molecules forming liquid crystalline states, several essential quantities are defined through the orientational distribution. The orientation of a single rigid molecule is described by an orthonormal, right-handed frame $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ fixed on the molecule. The axes of the frame are typically coincident
with symmetry axes of the molecule. Under a reference frame \((e_1, e_2, e_3)\), the coordinates of the molecular frame can be expressed by

\[
q_{ij} = e_i \cdot m_j, \quad i, j = 1, 2, 3,
\]

which define a \(3 \times 3\) rotation matrix \(q \in SO(3)\). The frame \(q\) can be parametrized by three Euler angles \((\varphi, \psi, \theta)\) as

\[
q = (m_1, m_2, m_3) = \begin{pmatrix}
  m_{11} & m_{12} & m_{13} \\
  m_{12} & m_{22} & m_{23} \\
  m_{13} & m_{23} & m_{33}
\end{pmatrix}
= \begin{pmatrix}
  \cos \varphi & -\sin \varphi \cos \psi & \sin \varphi \sin \psi \\
  \sin \varphi \cos \psi & \cos \varphi \cos \psi - \sin \psi \sin \theta & -\cos \varphi \sin \psi - \sin \varphi \cos \theta \\
  \sin \varphi \sin \psi & \cos \varphi \sin \psi + \sin \psi \sin \theta & -\cos \varphi \cos \psi + \sin \varphi \cos \theta
\end{pmatrix},
\]

where \(\varphi \in [0, \pi]\) and \(\psi, \theta \in [0, 2\pi]\). The corresponding invariant Haar measure on \(SO(3)\) is denoted by

\[
dq = \frac{1}{8\pi^2} \sin \varphi d\varphi d\psi d\theta.
\]

In this paper, we also deal with fields of orthonormal frame. To be distinguished from the molecular frame, we use the notation \(p = (n_1, n_2, n_3)\) for a frame field that is a function of the position \(x\). The notations for \(n_i\) are similar to those for \(m_i\) above.

Next, let us describe notations for tensors. An \(n\)th-order tensor \(U\) in \(\mathbb{R}^3\) can be expressed as a linear combination of tensors generated by the axes of the reference frame \((e_1, e_2, e_3)\), written as

\[
U = U_{i_1 \ldots i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad i_1, \ldots, i_n \in \{1, 2, 3\},
\]

(2.1)

where \(U_{i_1 \ldots i_n}\) are the coordinates of the tensor \(U\). Hereafter, we adopt the Einstein summation convention on repeated indices. For any two \(n\)th-order tensors \(U_1\) and \(U_2\), the dot product \(U_1 \cdot U_2\) is defined as

\[
U_1 \cdot U_2 = (U_1)_{i_1 \ldots i_n} (U_2)_{i_1 \ldots i_n}.
\]

A tensor can be symmetrized by calculating its permutational average,

\[
U_{\text{sym}} = \frac{1}{n!} \sum_\sigma U_{i_{\sigma(1)} \ldots i_{\sigma(n)}} e_{i_1} \otimes \cdots \otimes e_{i_n},
\]

where the summation is taken over all the permutations \(\sigma\) of \(\{1, \ldots, n\}\). If \(U = U_{\text{sym}}\), we say that the tensor \(U\) is symmetric. For an \(n\)th-order symmetric tensor, we define its trace as the contraction of two of its indices, giving an \((n - 2)\)th-order symmetric tensor,

\[
(trU)_{i_1 \ldots i_{n-2}} = U_{i_1 \ldots i_{n-2}kk}.
\]

If a symmetric tensor \(U\) satisfies \(trU = 0\), then \(U\) is called a symmetric traceless tensor. For any symmetric tensor \(U\) of the order \(n\), there exists a unique symmetric traceless tensor \((U)_0\) of the form

\[
(U)_0 = U - (i \otimes W)_{\text{sym}},
\]

(2.2)

where \(W\) is a tensor of the order \((n - 2)\). We call \((U)_0\) the symmetric traceless tensor generated by \(U\).

It could be convenient to express symmetric traceless tensors by polynomials. The basic monomial notation is defined as

\[
m_{1}^{k_1} m_{2}^{k_2} m_{3}^{k_3} t^l = \left( m_{1} \otimes \cdots \otimes m_{1} \otimes m_{2} \otimes \cdots \otimes m_{2} \otimes m_{3} \otimes \cdots \otimes m_{3} \otimes i \otimes \cdots \otimes i \right)_{\text{sym}},
\]

(2.3)
where \( i \) is the second-order identity tensor that can be expressed as

\[
i = m_1^2 + m_2^2 + m_3^2.
\]

This equality holds independent of what frame \((m_1, m_2, m_3)\) is chosen. As we have commented, the above definitions are also suitable for \( n \). When the symbol \( \otimes \) is absent in a product, it means that the resulting tensor is symmetrized. A few examples are given below,

\[
m_1 = m_1,
\]

\[
m_1 m_2 = \frac{1}{2} (m_1 \otimes m_2 + m_2 \otimes m_1),
\]

\[
m_1^2 = m_1 \otimes m_1,
\]

\[
m_1 m_2 m_3 = \frac{1}{6} (m_1 \otimes m_2 \otimes m_3 + m_2 \otimes m_3 \otimes m_1 + m_3 \otimes m_1 \otimes m_2
\]

\[
+ m_1 \otimes m_3 \otimes m_2 + m_2 \otimes m_1 \otimes m_3 + m_3 \otimes m_2 \otimes m_1),
\]

\[
m_1 m_2^2 = \frac{1}{3} (m_1 \otimes m_2 \otimes m_2 + m_2 \otimes m_1 \otimes m_2 + m_2 \otimes m_2 \otimes m_1),
\]

where the permutation of the subscripts on the right-hand side implies that the tensor are symmetric. Using the polynomial notations, the explicit expressions for different bases of symmetric traceless tensors have been identified \[43\]. We shall introduce these explicit expressions when they are encountered afterwards.

The orientational distribution of rigid molecules is denoted by \( \rho(x, q) \). However, it is significant to introduce some simple quantities to classify the local anisotropy given by the density function \( \rho \). Such quantities are defined through the moments of \( m_i \),

\[
\langle m_{i_1} \otimes \cdots \otimes m_{i_n} \rangle = \int_{SO(3)} m_{i_1}(q) \otimes \cdots \otimes m_{i_n}(q) \rho(x, q) dq, \quad i_1, \ldots, i_n = 1, 2, 3. \quad (2.4)
\]

Hereafter, the notation \( \langle \cdot \rangle \) is employed to represent the average of the distribution function \( \rho(x, q) \) on \( SO(3) \). The linearly independent components in these moments are further extracted, for which it is necessary to choose a few symmetric traceless tensors averaged by \( \rho \) \[43\]. These chosen averaged symmetric traceless tensors are the so-called order parameters. For example, the local alignment state of rod-like molecules is described by the second-order symmetric traceless order tensor

\[
\langle m_1^2 - \frac{i}{3} \rangle = \int_{SO(3)} \left( m_1^2 - \frac{i}{3} \right) \rho(x, q) dq.
\]

Two useful quantities that are employed throughout this paper are the Kroneker delta and the Levi-Civita symbol, which are defined respectively by

\[
\delta_{ij} = \begin{cases} 
1, & i = j, \\
0, & i \neq j,
\end{cases}
\]

and

\[
\epsilon^{ijk} = \begin{cases} 
1, & \text{if } i, j \text{ and } k \text{ are unequal and in cyclic order}, \\
-1, & \text{if } i, j \text{ and } k \text{ are unequal and in non-cyclic order}, \\
0, & \text{if any two of } i, j \text{ or } k \text{ are equal}.
\end{cases}
\]

The following two properties are useful:

\[
\epsilon^{ijk} \epsilon^{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp},
\]

\[
\epsilon^{ijk} \epsilon^{ipqr} = \det \begin{pmatrix} 
\delta_{ip} & \delta_{iq} & \delta_{ir} \\
\delta_{jp} & \delta_{jq} & \delta_{jr} \\
\delta_{kp} & \delta_{kq} & \delta_{kr}
\end{pmatrix}.
\]

5
We will frequently encounter derivatives involving orthonormal frames. Let us first define rotational differential operators. For any frame \( p = (n_1, n_2, n_3) \in SO(3) \), its tangent space in \( SO(3) \) is spanned by three matrices, given by \((0, n_3, -n_2), (-n_3, 0, n_1), (n_2, -n_1, 0)\). Thus, we define three differential operators \( \mathcal{L}_j \) by taking the inner products of the above three matrices and \( \partial/\partial p = (\partial/\partial n_1, \partial/\partial n_2, \partial/\partial n_3) \), i.e.,

\[
\begin{align*}
\mathcal{L}_1 &= n_3 \cdot \frac{\partial}{\partial n_2} - n_2 \cdot \frac{\partial}{\partial n_3}, \\
\mathcal{L}_2 &= n_1 \cdot \frac{\partial}{\partial n_3} - n_3 \cdot \frac{\partial}{\partial n_1}, \\
\mathcal{L}_3 &= n_2 \cdot \frac{\partial}{\partial n_1} - n_1 \cdot \frac{\partial}{\partial n_2}.
\end{align*}
\]

(5)

The subscript indicates the differential operator is along the infinitesimal rotation about \( n_j (j = 1, 2, 3) \). This can be verified by acting the differential operators on the axes of the frame, resulting in

\[
\mathcal{L}_j n_k = \epsilon^{jkl} n_l.
\]

(6)

For a frame field \( p(x) \), its orientational elasticity is characterized by an elastic energy about the spatial derivatives of \( p(x) \). Let us express these spatial derivatives under the local frame \( p \). The derivative of \( n_\mu \) along the direction \( n_\lambda \) is given by \( n_\lambda \cdot \nabla n_\mu \). Its \( \nu \)-component in the frame \( p \) can be written as \( n_\lambda n_\nu \partial_\lambda n_\mu \). Using the equality \( n_\mu n_\nu \partial_\lambda n_\mu = \delta_{\mu\nu} \), we obtain the following relation

\[
n_\lambda n_\nu \partial_\lambda n_\mu = -n_\lambda n_\mu \partial_\lambda n_\nu.
\]

(7)

Consequently, the first-order derivatives of the frame \( p \) has nine degrees of freedom:

\[
\begin{align*}
D_{11} &= n_{1i} n_{2j} \partial_i n_{3j}, & D_{12} &= n_{1i} n_{3j} \partial_i n_{1j}, & D_{13} &= n_{1i} n_{1j} \partial_i n_{2j}, \\
D_{21} &= n_{2i} n_{2j} \partial_i n_{3j}, & D_{22} &= n_{2i} n_{3j} \partial_i n_{1j}, & D_{23} &= n_{2i} n_{1j} \partial_i n_{2j}, \\
D_{31} &= n_{3i} n_{2j} \partial_i n_{3j}, & D_{32} &= n_{3i} n_{3j} \partial_i n_{1j}, & D_{33} &= n_{3i} n_{1j} \partial_i n_{2j}.
\end{align*}
\]

(8)

### 3 Tensor model

In tensor models, the local orientational order is described by one or several order parameter tensors. From the structure of nonzero components in the tensors, local anisotropy could be divided into several classes. Each class is recognized as a phase, and phase transitions between them can be described. For examples, the transition between the isotropic and uniaxial nematic phases for rod-like molecules can be described by an energy about a second-order symmetric traceless tensor \( Q \). The dynamic tensor models could either be phenomenological, such as the Beris-Edwards model [3] and the Qian-Sheng model [41] based on the Landau-de Gennes theory, or be derived from the molecular theory [12]. In the vicinity of a bulk energy minimum, the tensors possess the nonzero structure of a certain phase, so that the tensor model is reduced to a model with fewer variables. For the uniaxial nematic phase of rod-like molecules, the models about a field of \( Q \)-tensor, which has five degrees of freedom, could be reduced to models about a field of unit vector, which has two degrees of freedom.

When rigid molecules of more complex architecture are taken into account, such as bent-core and star molecules, the corresponding molecular-theory-based tensor models have also been derived [41, 51]. The most notable feature of this model lies in the fact that its form and coefficients are determined by molecular symmetry and molecular parameters, respectively. Depending on the coefficients, the bulk energy may exhibit isotropic, uniaxial nematic or biaxial nematic phases. The modulated twist-bend nematic phase can also be described together with elastic energy. Since the biaxial nematic phase is included in this tensor model, we choose this model as our starting point.

Compared with the original form in [51], we have made a couple of simplifications that are clarified below.
The model in [51] has three order parameter tensors, one first-order and two second-order. In the biaxial nematic phases, the first-order tensor takes zero. This is also maintained in the leading order of Hilbert expansion. As a result, keeping the first-order tensor makes no difference in our derivation. For this reason, we assume that the first-order tensor is zero to discard all the terms about it.

We ignore the spatial diffusion term. This is also adopted in the derivation from dynamic Q-tensor models to the Ericksen-Leslie model (see [12]).

The tensor model is then about two second-order symmetric traceless tensors, defined as
\[ Q_1 = \langle (m_1^2)_0 \rangle = \langle m_1^2 - \frac{i}{3} \rangle, \quad Q_2 = \langle (m_2^2)_0 \rangle = \langle m_2^2 - \frac{i}{3} \rangle. \]

Denote \( \mathbf{Q} = (Q_1, Q_2)^T \). Let us also define a projection on to symmetric traceless tensors,
\[ (\mathcal{P} R)_{ij} = \frac{1}{2}(R_{ij} + R_{ji}) - \frac{1}{3} R_{kk} \delta_{ij}. \] (3.1)
The projection can also be imposed on an array of second-order tensors:
\[ \mathcal{P}(R_1, \ldots, R_k) = (\mathcal{P} R_1, \ldots, \mathcal{P} R_k). \] (3.1)

3.1 Free energy

Assume that the concentration \( c \) of rigid molecules is constant in space. The free energy contains two parts, the bulk energy and elastic energy,
\[ \frac{\mathcal{F}[\mathbf{Q}, \nabla \mathbf{Q}]}{k_B T} = \int d\mathbf{x} \left( \frac{1}{\varepsilon} F_b(\mathbf{Q}) + F_e(\nabla \mathbf{Q}) \right), \] (3.2)
which is measured by the product of the Boltzmann constant \( k_B \) and the absolute temperature \( T \). The bulk energy density, which can describe transitions between homogeneous phases, consists of an entropy term and pairwise interaction terms,
\[ F_b = c F_{\text{entropy}} + \frac{c^2}{2} \left( c_{02} |Q_1|^2 + c_{03} |Q_2|^2 + 2 c_{04} Q_1 \cdot Q_2 \right). \] (3.3)
The elastic energy density penalizing spatial inhomogeneity contains a few quadratic terms of \( \nabla \mathbf{Q} \),
\[ F_e = \frac{c^2}{2} \left( c_{22} |\nabla Q_1|^2 + c_{23} |\nabla Q_2|^2 + 2 c_{24} \partial_i Q_{1jk} \partial_j Q_{2jk} + c_{28} \partial_i Q_{11k} \partial_j Q_{1jk} + c_{29} \partial_i Q_{21k} \partial_j Q_{2jk} + 2 c_{210} \partial_i Q_{11k} \partial_j Q_{2jk} \right). \] (3.4)

We have introduced a small parameter \( \varepsilon \) in the free energy (3.2). It can be regarded as the squared relative scale \( \tilde{L} \) between the rigid molecule and the domain of observation by a change of variable \( \tilde{x} = x / \tilde{L} \).

The entropy term acts as a stabilizing term that guarantees the lower-boundedness of the bulk energy. There can be different choices, but it is always independent of molecule architecture. Moreover, the entropy term is related to expressing the tensors of higher order by \( Q_1 \) and \( Q_2 \). For this reason, we shall specify the entropy term afterwards.

On the other hand, the coefficients \( c_{ij} \) of the quadratic terms can be calculated as functions of molecular parameters. For instance, if the hardcore molecular interaction is adopted, we are able to compute these coefficients from molecular shape parameters [47]. This is also the case for the dynamic tensor model, which we introduce below.
3.2 Dynamic model

Based on the free energy functional \( \mathcal{F}(Q, \nabla Q) \) and \( \mathcal{G}(Q) \), let us write down the molecular-theory-based dynamic tensor model derived in [51]. We define the variational derivative of (3.2) as

\[
\mu_Q = \frac{1}{ck_B T} \frac{\delta \mathcal{F}(Q, \nabla Q)}{\delta Q} = \frac{1}{ck_B T} \mathcal{P} \left( \frac{1}{\varepsilon} \frac{\partial F_b(Q)}{\partial Q} - \partial_i \left( \frac{\partial F_a(\nabla Q)}{\partial (\partial_i Q)} \right) \right)
\]

\[
def \frac{1}{\varepsilon} \mathcal{J}(Q) + \mathcal{G}(Q),
\]

where \( \mu_Q = (\mu_Q^1, \mu_Q^2)^T \), \( \mathcal{J}(Q) = (J_1(Q), J_2(Q))^T \) and \( \mathcal{G}(Q) = (G_1(Q), G_2(Q))^T \) are calculated as

\[
\mu_Q^1 = \frac{1}{\varepsilon} J_1(Q) + G_1(Q)
\]

\[
= \frac{1}{\varepsilon} \left( \mathcal{P} \frac{\partial F_{\text{entropy}}}{\partial Q^1} + cc_{Q^1} Q^1 + cc_{Q^2} Q^2 \right)
\]

\[
- cc_{Q^2} Q^1 j - cc_{Q^2} Q^2 k - \mathcal{P} (cc_{Q^1} \partial_j \partial_k Q^1 + cc_{Q^2} \partial_j \partial_k Q^2),
\]

\[
\mu_Q^2 = \frac{1}{\varepsilon} J_2(Q) + G_2(Q)
\]

\[
= \frac{1}{\varepsilon} \left( \mathcal{P} \frac{\partial F_{\text{entropy}}}{\partial Q^2} + cc_{Q^1} Q^1 + cc_{Q^2} Q^2 \right)
\]

\[
- cc_{Q^1} Q^1 j - cc_{Q^1} Q^2 k - \mathcal{P} (cc_{Q^1} \partial_j \partial_k Q^1 + cc_{Q^1} \partial_j \partial_k Q^2).
\]

The dynamic tensor model is given by

\[
\frac{\partial Q}{\partial t} + v \cdot \nabla Q = K_Q + \mathcal{W}_Q,
\]

\[
\rho_s \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = - \nabla p + \nabla \cdot \sigma + F^e,
\]

\[
\nabla \cdot v = 0,
\]

where \( \rho_s \) is the density of the fluid and \( v \) the fluid velocity, and \( p \) is the pressure to maintain the incompressibility. Let us denote by \( \kappa_{ij} = \partial_j v_i \) the velocity gradient. The terms \( K_Q = (K_Q^1, K_Q^2) \) and \( \mathcal{W}_Q = (W_Q^1, W_Q^2) \) on the right-hand side of (3.8) characterize the rotational diffusions and rotational convections, respectively. They are given by

\[
-(K_{Q^1})_{ij} = 4\Gamma_2 (\mu_Q^1)_{ij} \langle m_1 m_3 \otimes m_1 m_3 \rangle_{ijkl} + 4\Gamma_3 (\mu_Q^1 - \mu_Q^2)_{ij} \langle m_1 m_2 \otimes m_1 m_2 \rangle_{ijkl},
\]

\[
-(K_{Q^2})_{ij} = 4\Gamma_1 (\mu_Q^2)_{ij} \langle m_2 m_3 \otimes m_2 m_3 \rangle_{ijkl} - 4\Gamma_3 (\mu_Q^1 - \mu_Q^2)_{ij} \langle m_1 m_2 \otimes m_1 m_2 \rangle_{ijkl},
\]

\[
(W_{Q^1})_{ijl} = 2\kappa_{ij} \left( \langle m_1 \otimes m_3 \rangle \langle m_1 m_3 \rangle + \frac{I_{22}}{I_{11} + I_{22}} \langle m_1 \otimes m_2 \rangle \langle m_1 m_2 \rangle \right)
\]

\[
- \frac{I_{11}}{I_{11} + I_{22}} \langle m_2 \otimes m_1 \rangle \langle m_1 m_2 \rangle_{ijkl},
\]

\[
(W_{Q^2})_{ijl} = 2\kappa_{ij} \left( \langle m_2 \otimes m_3 \rangle \langle m_2 m_3 \rangle - \frac{I_{22}}{I_{11} + I_{22}} \langle m_1 \otimes m_2 \rangle \langle m_1 m_2 \rangle \right)
\]

\[
+ \frac{I_{11}}{I_{11} + I_{22}} \langle m_2 \otimes m_1 \rangle \langle m_1 m_2 \rangle_{ijkl},
\]

where \( \Gamma_i = \frac{m_i}{\zeta_i} \) \((i = 1, 2, 3)\) are the diffusion coefficients, \( m_0 \) is the mass of a rigid molecule, \( \zeta \) is the friction constant, and \( I_{ij} (i = 1, 2, 3) \) are diagonal elements of the moment of inertia for a molecule.

In (3.9), the stress tensor \( \sigma \) consists of the viscous stress \( \sigma_v \) and the elastic stress \( \sigma_e \). The viscous stress \( \sigma_v \) includes the contribution of the fluid itself with a viscous coefficient \( \eta \), and the fluid-molecule friction,

\[
\sigma_v = \frac{1}{2} \eta (\kappa + \kappa^T) + \sigma_{vf}.
\]

(3.11)
The second term $\sigma_{ef}$ is determined by the following equation,

$$(\sigma_{ef})_{ij} = c\kappa k_{kl}\left( I_{22}\langle m_1^2 \rangle + I_{11}\langle m_2^4 \rangle + \frac{4I_{11}I_{22}}{I_{11} + I_{22}}\langle m_1 m_2 \rangle m_1 m_2 \right)_{ijkl}.$$ 

The elastic stress $\sigma_e$ can be written as

$$(\sigma_e)_{kl} = 2\kappa k_{kl}\left[ (\mu Q_2)_{ij}\langle m_2 m_3 \otimes (m_2 \otimes m_3) \rangle_{ijkl} + (\mu Q_1)_{ij}\langle m_1 m_2 \otimes (m_1 \otimes m_3) \rangle_{ijkl} \right].$$

The external force $F^e$ is given by

$$F^e = ck_B T \mu Q \cdot \partial_t Q \equiv ck_B T (\mu Q_1 \cdot \partial_t Q_1 + \mu Q_2 \cdot \partial_t Q_2).$$ (3.12)

The system (3.8)-(3.10) obeys the following energy dissipation law (see [51]):

$$\frac{d}{dt}\left( \int dx \frac{q^2}{2} + F(Q, \nabla Q) \right) = \int dx\left[ -c k_B T \left( \Gamma_1 (2(\mu Q_2 \cdot m_3)^2) + \Gamma_2 (2(\mu Q_1 \cdot m_1 m_3)^2) \right) 
+ \Gamma_3 (2(\mu Q_1 - \mu Q_2) \cdot m_1 m_2)^2 - \frac{\eta}{2} \kappa \cdot \frac{\kappa^T}{2} - \frac{\eta}{2} \kappa \cdot \frac{\kappa^T}{2} 
- \frac{c \zeta}{2} \left[ I_{22} (2(\kappa \cdot m_1^2)^2) + I_{11} (2(\kappa \cdot m_2^2)^2) \right] + \frac{I_{11} I_{22}}{I_{11} + I_{22}} \left( (\kappa \cdot m_1 m_2)^2 \right) \right].$$ (3.13)

Note that several fourth-order tensors appear in the dynamic model. In order to close the system, it is necessary to find certain way to express them by $Q_1$ and $Q_2$. The closure approximation can be done by the entropy term, which will be introduced below. Although there might be other ways of closure, one advantage of closure by the entropy term is that it guarantees the non-positiveness of the terms on the right-hand side of (3.13).

### 3.3 Entropy and quasi-entropy

We have mentioned that the entropy term plays a significant role in both free energy and closure approximation. A general approach is to deduce the entropy term by minimizing $\int_{SO(3)} \rho \ln \rho \eta dp$ with the values of the tensors fixed, or finding the maximum entropy state. When the two tensors $Q_1$ and $Q_2$ are involved, the maximum entropy state is given by

$$\rho(q) = \frac{1}{Z} \exp(B_1 \cdot m_1^2 + B_2 \cdot m_2^2),$$ (3.14)

where the normalizing constant $Z$ and two second-order symmetric traceless tensors $B_1$ and $B_2$ are Lagrange multipliers for constraints,

$$Z = \int_{SO(3)} \exp(B_1 \cdot m_1^2 + B_2 \cdot m_2^2) dq,$$

$$Q_i = \frac{1}{Z} \int_{SO(3)} \left( m_i^2 - \frac{1}{3} \right) \exp(B_1 \cdot m_1^2 + B_2 \cdot m_2^2) dq.$$ (3.15)

Taking (3.14) into $\int_{SO(3)} \rho \ln \rho \eta dp$, we obtain

$$F_{\text{entropy}} = B_1 \cdot Q_1 + B_2 \cdot Q_2 - \ln Z.$$ (3.16)
The maximum entropy state (3.14) is unique about $Q_1$ and $Q_2$. Therefore, $F_{\text{entropy}}$ can be viewed as a function about $Q_1$ and $Q_2$. It is observed that $F_{\text{entropy}}$ is invariant under rotations on $Q_1$ and $Q_2$. Generally, we say a tensor $U$ is invariant under rotations, if the coordinates $U_{i_1\ldots i_n}$ in (3.11) are kept but the basis $(e_i)$ is rotated to another frame $(e'_i)$. Specifically, a rotation on $Q_1$ and $Q_2$ is done by choosing certain $t \in SO(3)$ and transforming $Q_i$ into $tQ_it^{-1}$. It is easy to verify that (3.16) is rotationally invariant under this transformation (see Appendix B.1).

The closure approximation can be done with the maximum entropy state, as the high-order tensors can be calculated using the density function (3.14). An equivalent viewpoint is that when $Q_1$ and $Q_2$ are given, the high-order tensors obtained in this way minimize $\int_{SO(3)} \rho \ln \rho \, dp$.

The entropy term defined from maximum entropy state, which we call the original entropy, is given implicitly that involves integrals on $SO(3)$, which could bring difficulties in both analyses and numerical studies. An alternative approach is proposed [44], where the original entropy is substituted with the quasi-entropy. The quasi-entropy is defined by log-determinant covariance matrix, which is an elementary function about the order parameter tensors. To write down the expression of the quasi-entropy, it is necessary to specify the highest tensor order (that shall be even) to be involved. When only second-order tensors are involved, the quasi-entropy for $Q_1$ and $Q_2$ is given by (see discussions in Appendix B.2)

$$F_{\text{entropy}} = \Xi_2(Q_1, Q_2) = \nu \left( -\ln \det \left( Q_1 + \frac{i}{3} \right) - \ln \det \left( Q_2 + \frac{i}{3} \right) - \ln \det \left( \frac{i}{3} - Q_1 - Q_2 \right) \right).$$

Domain: $Q_1 + \frac{i}{3}, Q_2 + \frac{i}{3}, \frac{i}{3} - Q_1 - Q_2$ positive definite. (3.17)

A free parameter $\nu$ is introduced above. It can be estimated as $\nu = 5/9$ from special cases (see Section 6 in [44]), which we adopt in the current work. Moreover, analyses show that the quasi-entropy possesses similar properties with the original entropy. In particular, the results from the quasi-entropy (5.17) are very similar to those from the original entropy (5.10), provided that other terms in the free energy are identical. These results have all been reported in [44].

The quasi-entropy is also suitable for closure approximation. To deduce high-order tensors in the dynamic model, we shall use the log-determinant covariance matrix up to fourth order, denoted by $\Xi_4$ which is provided in (3.5) of Appendix B.2. The fourth-order tensors shall minimize $\Xi_4$ with the given values of $Q_1$ and $Q_2$. Thus, we can see that the closure approximation by the original entropy and the quasi-entropy share the rationale, with the only difference lying in the function to be minimized. In what follows, we shall see that these two approaches of closure approximation lead to high-order tensors of the same form due to the same symmetry arguments that will be shown in Appendix B.

The properties of the quasi-entropy have been discussed previously [44]. Here, let us state those to be utilized in this paper.

**Proposition 3.1.** The two functions $\Xi_2$ (see (3.17)) and $\Xi_4$ (see (3.5)) have the following properties:

- They are invariant under rotations on the tensors.
- They act as barrier functions in the following sense: $\Xi_2$ keeps covariance matrices up to the second order positive definite, while $\Xi_4$ keeps those up to the fourth order positive definite.
- They are strictly convex about the tensors.

As an example, it is straightforward to see the rotational invariance for $\Xi_2$ by taking the rotation $Q_i \rightarrow tQ_it^{-1}$ into (3.17).

The properties stated above are all crucial in our derivation below. The rotational invariance is a foundation for the frame model to be established. The positive-definiteness of covariance matrices is essential for energy dissipation to hold. The strict convexity guarantees that the closure approximation by minimization results in a unique solution.
3.4 Stationary points of bulk energy

There are analyses on the stationary points of the bulk energy $F_b$, but they are far from well-understood. We summarize the main results up to date in the following proposition \[19, 34\]. To simplify the presentation, the conditions on the coefficients are stricter than they could be.

**Proposition 3.2.** Assume that the matrix

$$
\begin{pmatrix}
 c_{02} & c_{04} \\
 c_{04} & c_{03}
\end{pmatrix}
$$

is not negative definite, or is negative but $c_{04}/c_{03} - c_{02} < 2$. No matter the entropy term takes (3.16) or (3.17) (with $\nu = 5/9$), at the stationary points $Q_1$ and $Q_2$ have a shared eigenframe.

When $Q_1$ and $Q_2$ has the same eigenframe, they can be written as

$$Q_i = s_i \left( n_i^2 - \frac{1}{3} \right) + b_i (n_i^2 - n_i^3), \quad i = 1, 2. \tag{3.18}$$

Numerical studies indicate that the global energy minimum could be either uniaxial (where $b_i = 0$) or biaxial (where at least one $b_i \neq 0$). Here, we assume that under certain coefficients $c_{02}, c_{03},$ and $c_{04}$, we have a biaxial global minimum $Q^{(0)} = (Q_1^{(0)}, Q_2^{(0)})$ of the form (3.18).

It shall be noticed that the bulk energy $F_b$ is rotationally invariant, i.e. invariant of $p = (n_1, n_2, n_3)$. This can be observed by combining Proposition 3.2 and the fact that the three $c_{0i}$ terms are rotationally invariant. Thus, a rotation of an energy minimum also results in an energy minimum.

At any energy minimum, we have $\mathcal{J}(Q^{(0)}) = 0$. Let us fix $s_i$ and $b_i$ and let $p = (n_1, n_2, n_3)$ vary, so that $Q^{(0)} = Q^{(0)}(p)$ becomes a function of $p$. Since $Q^{(0)}$ is an energy minimum whatever $p$ is, it implies that $\mathcal{J}(Q^{(0)}(p)) = 0$. We then impose the operators $\mathcal{L}_i$ on it. By the chain rule, we obtain

$$\mathcal{L}_m \mathcal{J}(Q^{(0)})_{ij} = \mathcal{J}'(Q^{(0)})_{ijkl} \mathcal{L}_m Q^{(0)}_{kl} = 0, \quad m = 1, 2, 3. \tag{3.19}$$

This implies that the kernel of the Hessian $\mathcal{J}'(Q^{(0)})$ contains the space spanned by $\mathcal{L}_m Q^{(0)}$.

With the form (3.18), the scalars $s_i$ and $b_i$ shall satisfy

$$\frac{2}{3} s_i + \frac{1}{3} > 0, \quad \frac{1}{3} - \frac{1}{3} s_i \pm b_i > 0, \quad i = 1, 2, 3, \tag{3.20}$$

where we define $s_3 = -s_1 - s_2$ and $b_3 = -b_1 - b_2$. This is exactly the range such that $F_{\text{entropy}}$ is well-defined, no matter it takes the original entropy (3.16) or the quasi-entropy (3.17): for the original entropy, the derivation can be found in [17]; for the quasi-entropy $\Xi_2$, the condition (3.20) is equivalent to the requirement that $Q_1 + i/3, Q_2 + i/3$ and $-Q_1 - Q_2 + i/3$ are positive definite.

3.5 High-order tensors and their symmetry

We have mentioned that the high-order tensors in the dynamic model are determined from closure approximation. However, there are many linear relations between these high-order tensors. It is necessary to specify their linearly independent components, which can be done with the help of symmetric traceless tensors. The use of symmetric traceless tensors turn out to be crucial to figuring out symmetry arguments for these high-order tensors.

For the high-order tensors appearing in the dynamic tensor model, only the tensors below are involved other than $Q_1$ and $Q_2$,

$$\langle m_1 m_2 m_3 \rangle, \langle (m_1^2) \rangle_0, \langle (m_2^2) \rangle_0, \langle (m_1^2 m_2^2) \rangle_0. \tag{3.21}$$

Here, we recall the notation $(U)_0$ in \[29\] for the symmetric traceless tensor generated by $U$. The explicit expressions of these tensors are given in Appendix A.1. Furthermore, in Appendix A.2 we
provide the explicit expressions of the fourth-order tensors in the dynamic model by the above four tensors together with \( Q_1 \) and \( Q_2 \).

Therefore, in closure approximation, our task is to determine the third-order and fourth-order tensors in (3.21). In particular, when \( Q_1 \) and \( Q_2 \) have the form (4.18), the tensors in (3.21) have the form indicated by the following theorem.

**Theorem 3.3.** If \( Q_1 \) and \( Q_2 \) are biaxial of the form (4.18), then the third- and fourth-order symmetric traceless tensors, solved from closure approximation by the original entropy or the quasi-entropy, take the form

\[
\langle m_1 m_2 m_3 \rangle = z n_1 n_2 n_3, \\
\langle m_1^4 \rangle_0 = a_1(n_1^4)_0 + a_2(n_2^4)_0 + a_3(n_3^2 n_2^2)_0, \\
\langle m_2^4 \rangle_0 = \tilde{a}_1(n_1^4)_0 + \tilde{a}_2(n_2^4)_0 + \tilde{a}_3(n_3^2 n_2^2)_0, \\
\langle m_1^2 m_2^2 \rangle_0 = \bar{a}_1(n_1^4)_0 + \bar{a}_2(n_2^4)_0 + \bar{a}_3(n_3^2 n_2^2)_0.
\]  

(3.22)

The scalars \( z, a_i, \tilde{a}_i, \bar{a}_i \) are solved as functions of \( s_i \) and \( b_i \). Furthermore, if \( s_i \) and \( b_i \) satisfy (3.20), these scalars can be uniquely solved.

The proof is left to Appendix B. This result actually determines the form of high-order tensors in the Hilbert expansion, which in turn makes a great difference in finding out the form of the frame hydrodynamics for the biaxial nematic phase.

4 From tensor model to orthonormal frame model

We make the Hilbert expansion (also called the Chapman–Enskog expansion) of solutions to the with respect to the small parameter \( \varepsilon \). The \( O(1) \) system results in the orthonormal frame model for the biaxial nematic phase, with the energy dissipation maintained. The coefficients in the frame model are derived from those in the tensor model. Since the coefficients in the tensor model are derived from physical parameters, we finally build the relation between the frame model and the physical parameters.

For convenience, we denote seven fourth-order tensor moments as follows:

\[
\begin{align*}
R_1 &= \langle (m_1^2 - \frac{1}{3}) \otimes (m_2^2 - \frac{1}{3}) \rangle, \\
R_2 &= \langle (m_1^3 - \frac{1}{3}) \otimes (m_2^2 - \frac{1}{3}) \rangle, \\
R_3 &= 4\langle m_1 m_2 \otimes m_1 m_2 \rangle, \\
R_4 &= 4\langle m_1 m_3 \otimes m_1 m_3 \rangle, \\
R_5 &= 4\langle m_2 m_3 \otimes m_2 m_3 \rangle, \\
\mathcal{V}_{Q_1} &= 2\left( \langle m_1 m_3 \otimes (m_1 \otimes m_3) \rangle + e_1 \langle m_1 m_2 \otimes (m_1 \otimes m_2) \rangle - e_2 \langle m_1 m_2 \otimes (m_1 \otimes m_1) \rangle \right), \\
\mathcal{V}_{Q_2} &= 2\left( \langle m_2 m_3 \otimes (m_2 \otimes m_3) \rangle - e_1 \langle m_1 m_2 \otimes (m_1 \otimes m_2) \rangle + e_2 \langle m_1 m_2 \otimes (m_2 \otimes m_1) \rangle \right),
\end{align*}
\]  

(4.1)

where the coefficients \( e_i (i = 1, 2) \) are expressed by

\[
e_1 = 1 - e_2 = \frac{I_{22}}{I_{11} + I_{22}}.
\]

We frequently deal with contractions between fourth-order tensors and second-order tensors. We could regard a fourth-order tensor as a matrix, and a second-order tensor as a vector, so that the contractions can be formulated as matrix-matrix and matrix-vector multiplications, as we explain below. When a fourth-order tensor is contracted with a second-order tensor, we could write it in short as a matrix-vector product, say

\[
\mathcal{V}_{Q_1}^{ijkl} \kappa^{kl} = (\mathcal{V}_1 \kappa)^{ij}.
\]  

(4.2)

When using this short notation, we always assume that the second last index of the fourth-order tensor is contracted with the first of the second-order tensor, and the last of fourth-order tensor is contracted
with the last of the second-order tensor. By the convention \(4.2\), we could define the transpose of a fourth-order tensor, such as
\[
(V^T_{Q_1})_{ijkl} = (V_{Q_1})_{klij}.
\]

Let us define
\[
\mathcal{M} = \begin{pmatrix}
M_{11} & M_{12} \\
M_{12} & M_{22}
\end{pmatrix} \equiv \begin{pmatrix}
\Gamma_2 R_4 + \Gamma_3 R_3 & -\Gamma_3 R_3 \\
-\Gamma_3 R_3 & \Gamma_1 R_5 + \Gamma_3 R_3
\end{pmatrix},
\]
\[
\mathcal{V} \equiv \begin{pmatrix}
V_{Q_1} \\
V_{Q_2}
\end{pmatrix}, \quad \mathcal{N} \equiv (N_{Q_1}, N_{Q_2}) = (V^T_{Q_1}, V^T_{Q_2}),
\]
\[
\mathcal{P} \equiv c_\zeta (I_{22} R_1 + I_{11} R_2 + e_1 I_{11} R_3).
\]

The system \((3.8)-(3.10)\) can then be rewritten as
\[
\frac{\partial Q}{\partial t} + v \cdot \nabla Q = -\mathcal{M} \mu Q + \mathcal{V} \kappa,
\]
\[
\rho_v \left( \frac{\partial \nu}{\partial t} + v \cdot \nabla \nu \right)_i = -\partial_i p + \eta \Delta v_i + \partial_j (P \kappa)_{ij} + c_k B T \partial_j (N \mu Q)_{ij} + c_k B T \mu Q \cdot \partial_i Q,
\]
\[
\nabla \cdot v = 0,
\]
where \(\mathcal{M} \mu Q\) is carried out by matrix-vector multiplication,
\[
\mathcal{M} \mu Q = \begin{pmatrix}
M_{11} \mu Q_1 + M_{12} \mu Q_2 \\
M_{12} \mu Q_1 + M_{22} \mu Q_2
\end{pmatrix}.
\]

Similarly are the terms involving \(\mathcal{V}\) and \(\mathcal{N}\) interpreted. In the above, we have incorporated some simple calculations for the viscous stress, such as
\[
\langle m_1^i \rangle = (R_1 + Q_1 \otimes i + i \otimes Q_1 + i \otimes i) \kappa
\]
\[
= R_1 \kappa + (Q_1 \cdot \kappa) i,
\]
because the incompressibility can also be written as \(i \cdot \kappa = 0\). Furthermore, the second term in \((4.9)\) can be merged into the pressure \(p\), so that only the term \(R_1 \kappa\) remains in the operator \(\mathcal{P}\).

The fourth-order tensors \(R_i (i = 1, \cdots, 5)\) are positive definite in the sense that for any second-order symmetric traceless tensor \(Y\), we have \(Y \cdot R_i Y \geq 0\) and the equality implies \(Y = 0\). This result comes from the property of the entropy term, which we will show in Appendix B. Consequently, we deduce that for arbitrary second-order symmetric traceless tensors \(Y_1\) and \(Y_2\), it holds
\[
Y_1 \cdot PY_1 = c_\zeta (I_{22} Y_1 \cdot R_1 Y_1 + I_{11} Y_1 \cdot R_2 Y_1 + e_1 I_{11} Y_1 \cdot R_3 Y_1) \geq 0,
\]
\[
(Y_1, Y_2) \mathcal{M} \begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix} = \Gamma_1 Y_2 \cdot R_3 Y_2 + \Gamma_2 Y_1 \cdot R_4 Y_1 + \Gamma_1 (Y_1 - Y_2) \cdot R_3 (Y_1 - Y_2) \geq 0.
\]

The equality in \((4.11)\) leads to \(Y_1 = Y_2 = 0\), so that \(\mathcal{M}(Y_1, Y_2)^T = 0\) implies \(Y_1 = Y_2 = 0\).

### 4.1 The Hilbert expansion

Assume that \((Q(t, x), v(t, x))\) is a solution to the molecular-theory-based \(Q\)-tensor system \((4.0)-(4.8)\). We perform the following Hilbert expansion about \(\varepsilon\):
\[
Q(t, x) = Q^{(0)}(t, x) + \varepsilon Q^{(1)}(t, x) + \varepsilon^2 Q^{(2)}(t, x) + \cdots,
\]
\[
v(t, x) = v^{(0)}(t, x) + \varepsilon v^{(1)}(t, x) + \varepsilon^2 v^{(2)}(t, x) + \cdots.
\]
where \( \mathbf{Q}^{(i)} = (Q_1^{(i)}, Q_2^{(i)})^T \), and \((\mathbf{Q}^{(i)}, \nu^{(i)}) (i = 0, 1, 2, \ldots)\) are independent of the small parameter \( \varepsilon \).

Based on the expansion (4.12)–(4.13), we could write down the expansion of other terms in (4.6)–(4.8), frequently by Taylor expansion. Since we focus on the \( O(1) \) system, we only write down the terms up to \( O(1) \). In \( \mu_Q \), the term \( \mathcal{J}(\mathbf{Q}) = \mathcal{J}(\mathbf{Q}^{(0)}) + \varepsilon \mathcal{J}'(\mathbf{Q}^{(0)}) \mathbf{Q}^{(1)} + O(\varepsilon^2) \)

where \( \mathcal{J}'(\mathbf{Q}^{(0)}) \equiv \mathcal{H}(0) \) is a fourth-order tensor. By (4.14) we can deduce that

\[
\mu_Q = \varepsilon^{-1} \mathcal{J}(\mathbf{Q}) + \mathcal{G}(\mathbf{Q}) = \varepsilon^{-1} \mathcal{J}(\mathbf{Q}^{(0)}) + \mathcal{H}^{(0)} \mathcal{Q}^{(1)} + \mathcal{G}(\mathbf{Q}^{(0)}) + O(\varepsilon).
\]

Since the tensors in (4.11) are solved from closure approximation, they are functions of \( \mathbf{Q} \). Thus, \( \mathcal{M}, \mathcal{V}, \mathcal{N} \) and \( \mathcal{P} \) are functions of \( \mathbf{Q} \). Let us use the notation \( \mathcal{M}^{(0)} \) for the \( \mathcal{M} \) when \( \mathbf{Q} \) takes \( \mathbf{Q}^{(0)} \). Then we have

\[
\mathcal{M} = \mathcal{M}^{(0)} + O(\varepsilon), \quad \mathcal{V} = \mathcal{V}^{(0)} + O(\varepsilon),
\]

\[
\mathcal{N} = \mathcal{N}^{(0)} + O(\varepsilon) = \mathcal{V}^{(0)T} + O(\varepsilon), \quad \mathcal{P} = \mathcal{P}^{(0)} + O(\varepsilon),
\]

where

\[
\mathcal{M}^{(0)} = \begin{pmatrix} M_{11}^{(0)} & M_{12}^{(0)} \\ M_{12}^{(0)} & M_{22}^{(0)} \end{pmatrix}, \quad \mathcal{V}^{(0)} = \begin{pmatrix} V_{Q_1}^{(0)} \\ V_{Q_2}^{(0)} \end{pmatrix}, \quad \mathcal{N}^{(0)} = (\mathcal{N}_{Q_1}, \mathcal{N}_{Q_2}) = (\mathcal{V}_{Q_1}^{(0)}, \mathcal{V}_{Q_2}^{(0)T}),
\]

\[
\mathcal{P}^{(0)} = c\zeta (I_{22} \mathcal{R}_1^{(0)} + I_{11} \mathcal{R}^{(0)}_2 + \varepsilon I_{11} \mathcal{R}_3^{(0)}).
\]

Substituting the above expansion (4.12) and (4.13) into the system (4.6)–(4.8) and collecting the terms with the same order of \( \varepsilon \), we can obtain a series of equations. The \( O(\varepsilon^{-1}) \) system requires that

\[
\mathcal{M}^{(0)} \mathcal{J}(\mathbf{Q}^{(0)}) = 0.
\]

Since \( \mathcal{M}^{(0)} \) is positive definite, the above equation implies that

\[
\mathcal{J}(\mathbf{Q}^{(0)}) = 0.
\]

It means that \( \mathbf{Q}^{(0)} \) is just the critical point of \( F_0(\mathbf{Q}) \). We shall consider the case that \( \mathbf{Q}^{(0)} \) is the biaxial global minimum, which takes the form (3.18).

The terms of the order \( O(1) \) give

\[
\frac{\partial \mathbf{Q}^{(0)}}{\partial t} + \mathbf{v}^{(0)} \cdot \nabla \mathbf{Q}^{(0)} = - \mathcal{M}^{(0)} (\mathcal{H}^{(0)} \mathbf{Q}^{(1)} + \mathcal{G}(\mathbf{Q}^{(0)})) + \mathcal{V}^{(0)T} \mathbf{Q}^{(1)},
\]

\[
\rho \left( \frac{\partial \mathbf{v}^{(0)}}{\partial t} + \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} \right)_i = - \partial_j p^{(0)} + \eta \Delta v_i^{(0)} + \partial_j (\mathcal{P}^{(0)} \kappa_i^{(0)}) + c \kappa_B T \partial_j \left( \mathcal{N}^{(0)} (\mathcal{H}^{(0)} \mathbf{Q}^{(1)} + \mathcal{G}(\mathbf{Q}^{(0)})) \right)_i
\]

\[
+ c \kappa_B T (\mathcal{H}^{(0)} \mathbf{Q}^{(1)} + \mathcal{G}(\mathbf{Q}^{(0)})) \cdot \partial_i \mathbf{Q}^{(0)},
\]

\[

\nabla \cdot \mathbf{v}^{(0)} = 0.
\]

\[
(4.19)
\]

\[
(4.20)
\]

\[
(4.21)
\]
In the $O(1)$ system $\{4.19\} - \{4.21\}$, $Q^{(0)}$ is a function about $p = (n_1, n_2, n_3)$. The high-order tensors with the superscript $(0)$ are functions of $Q^{(0)}$, thus are functions of $p$. Therefore, if we could eliminate $Q^{(1)}$ in the $O(1)$ system, we would arrive at a system about $p$ and $v^{(0)}$. Indeed, this can be done by examining the kernel of $H^{(0)}$.

Our task becomes expressing terms with the superscript $(0)$ in terms of $n_1, n_2, n_3$. It turns out that the form $\{3.18\}$ of $Q^{(0)}$ results in specific form of the following terms.

- The derivatives of $Q^{(0)}$, which are related to the kernel of $H^{(0)}$.
- The variational derivative of the elastic energy, $G(Q^{(0)})$.
- The high-order tensors $M^{(0)}, V^{(0)}, N^{(0)}$ and $P^{(0)}$.

Up to now, all the equations are expressed by the components in the basis generated by the reference frame $(e_1, e_2, e_3)$. In order to facilitate the specific form of the above terms, we shall first rewrite the $O(1)$ system in the basis generated by $p = (n_1, n_2, n_3)$.

### 4.2 Change to the local basis

In what follows, we denote by $A_0$ and $\Omega_0$ the symmetric and skew-symmetric parts of the velocity gradient $\kappa^{(0)}_{ij} = \partial_j v_i^{(0)}$, respectively, i.e.,

$$A_0 = \frac{1}{2}(\kappa^{(0)} + \kappa^{(0) \, T}), \quad \Omega_0 = \frac{1}{2}(\kappa^{(0)} - \kappa^{(0) \, T}).$$

We consider the basis for second-order tensors given by five symmetric traceless tensors,

$$s_1 = n_1^2 - \frac{1}{3} 1, \quad s_2 = n_2^2 - n_3^2, \quad s_3 = n_1 n_2, \quad s_4 = n_1 n_3, \quad s_5 = n_2 n_3,$$

and three asymmetric traceless tensors,

$$a_1 = n_1 \otimes n_2 - n_2 \otimes n_1, \quad a_2 = n_3 \otimes n_1 - n_1 \otimes n_3, \quad a_3 = n_2 \otimes n_3 - n_3 \otimes n_2.$$  \hspace{1cm} (4.22)

Let us look into the fourth-order tensors $M^{(0)}_{11}, M^{(0)}_{12}, M^{(0)}_{22}$. The first two components of $M^{(0)}_{11}$ are symmetric, and the contraction of the first two components gives a zero second-order tensor. So is the contraction of its last two components. Thus, it can be expressed as

$$M^{(0)}_{11} = (M_{11})_{ij} s_i \otimes s_j.$$  \hspace{1cm} (4.24)

Similarly, $M_{12}$ and $M_{22}$ are also defined.

When the last two indices of $M^{(0)}_{11}$ are contracted with a second-order symmetric traceless tensor $Y = y_k s_k$, it gives

$$M^{(0)}_{11} Y = ((M_{11})_{ij} y_k) (s_i \otimes s_j) s_k.$$  \hspace{1cm} (4.25)

By the convention of the a fourth-order tensor times a second-order tensor $[4.2]$, the product $(s_i \otimes s_j) s_k$ gives a second-order tensor $(s_j \cdot s_k) s_i$. Let us define a matrix $\Lambda$ by

$$\Lambda_{ij} = s_i \cdot s_j,$$  \hspace{1cm} (4.26)

which equals

$$\Lambda = \text{diag} \left( \frac{2}{3}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right).$$  \hspace{1cm} (4.27)

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So, $\mathcal{M}^{(0)}_{11}Y$ is written as
\begin{equation}
\mathcal{M}^{(0)}_{11}Y = ((M_{11})_{ij} \lambda_{jk} y_k) s_i = (M_{11} \lambda y)_i s_i.
\end{equation}

In other words, the coordinates of $\mathcal{M}^{(0)}_{11}Y$ under the basis $s_i$ is given by $M_{11}\lambda y$.

For a product involving $\mathcal{M}^{(0)}$, we just combine the coordinates in a single vector. For instance, for the term $\mathcal{M}^{(0)}\mathcal{G}(Q^{(0)})$, let us denote
\begin{equation}
M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \tilde{\lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix},
\end{equation}
where $g_1$ is the vector of the coordinates of $\mathcal{G}_1(Q^{(0)})$, and $g_2$ that of $\mathcal{G}_2(Q^{(0)})$, i.e. $\mathcal{G}_1(Q^{(0)}) = (g_1)_i s_i$, $\mathcal{G}_2(Q^{(0)}) = (g_2)_i s_i$. Then, the term $\mathcal{M}^{(0)}\mathcal{G}(Q^{(0)})$ has the coordinates
\begin{equation}
M\tilde{\lambda}g = \begin{pmatrix} M_{11}\lambda g_1 + M_{12}\lambda g_2 \\ M_{12}\lambda g_1 + M_{22}\lambda g_2 \end{pmatrix}.
\end{equation}

We turn to the tensors $\mathcal{N}^{(0)}_{Q_1}$, $\mathcal{N}^{(0)}_{Q_2}$. For $\mathcal{N}^{(0)}_{Q_1}$, its first two components are no longer symmetric, so that we can express it as
\begin{equation}
\mathcal{N}^{(0)}_{Q_1} = (N^{a}_{i})_{ij} s_i \otimes s_j + (N^{l}_{i})_{ij} a_i \otimes s_j.
\end{equation}
The matrices $N^{a}_{1}$ and $N^{l}_{2}$ are defined in the same way. By $\mathcal{V}^{(0)} = \mathcal{N}^{(0)} T$, we can further write
\begin{equation}
\mathcal{V}^{(0)}_{Q_1} = (N^{a}_{i})_{ij} s_j \otimes s_i + (N^{l}_{i})_{ij} s_j \otimes a_i.
\end{equation}
Denote
\begin{equation}
N = \begin{pmatrix} N^{a}_{1} & N^{l}_{1} \\ N^{a}_{2} & N^{l}_{2} \end{pmatrix}, \quad V = N^T,
\end{equation}
where the matrix $N$ has the size $8 \times 10$, so that $V$ is $10 \times 8$. We have
\begin{equation}
\mathcal{V}^{(0)}_{k^{(0)}} = \begin{pmatrix} (N^{a}_{1})_{ij} s_j \otimes s_i + (N^{l}_{1})_{ij} s_j \otimes a_i \\ (N^{a}_{2})_{ij} s_j \otimes s_i + (N^{l}_{2})_{ij} s_j \otimes a_i \end{pmatrix}_{k^{(0)}}.
\end{equation}
We define an $8 \times 1$ vector $\omega$ by the contraction of $k^{(0)}$ and the vector $u = (s_1, \cdots, s_5, a_1, a_2, a_3)^T$ formed by eight tensors, which is given by
\begin{equation}
\omega = (\omega^T, \omega^T)^T,
\end{equation}
where $\omega_s$ and $\omega_a$ are given by
\begin{equation}
\omega_s = (A_0 \cdot s_1, \cdots, A_0 \cdot s_5)^T, \quad \omega_a = (\Omega_0 \cdot a_1, \Omega_0 \cdot a_2, \Omega_0 \cdot a_3)^T.
\end{equation}
Then, the $10 \times 1$ vector $V\omega$ gives the coordinates of $\mathcal{V}^{(0)}_{k^{(0)}}$.

Similar to the vector $g$, we denote by $\bar{q}$ the coordinates of $\partial_t Q^{(0)}$, by $\bar{q}_i$ the coordinates of $\partial_i Q^{(0)}$, and by $h$ the coordinates of $\mathcal{H}^{(0)} Q^{(1)}$. Then, the coordinates of the material derivative $\dot{Q}^{(0)} = \partial_t Q^{(0)} + \mathcal{V}^{(0)} \cdot \nabla Q^{(0)}$ are given by $q = \bar{q} + v_1^{(0)} \bar{q}_1 + v_2^{(0)} \bar{q}_2 + v_3^{(0)} \bar{q}_3$. Therefore, we could write \[1.19\] in the coordinates,
\begin{equation}
q - V\omega = -M\tilde{\lambda}(h + g).
\end{equation}
For \([4.20]\), the term \(\mathcal{N}^{(0)}(\mathcal{H}^{(0)}Q^{(1)} + \mathcal{G}(Q^{(0)}))\) can be expressed under the basis \(s_i\) together with \(a_i\),

\[
\sigma_1^{(0)} = c_k B T \mathcal{N}^{(0)}(\mathcal{H}^{(0)}Q^{(1)} + \mathcal{G}(Q^{(0)})) \\
= c_k B T (s_1, \ldots, s_5, a_1, a_2, a_3) N \tilde{\Lambda}(h + g). \tag{4.35}
\]

The term \(\mathcal{P}^{(0)}\kappa^{(0)}\) is symmetric traceless, so that it can be written as

\[
\mathcal{P}^{(0)}\kappa^{(0)} = (s_1, \ldots, s_5) P\omega_s. \tag{4.36}
\]

The dot product \((\mathcal{H}^{(0)}(Q^{(1)}) + \mathcal{G}(Q^{(0)})) \cdot \partial_i Q^{(0)}\) is given by \(\tilde{q}^T \tilde{\Lambda}(h + g)\). Thus, \([4.20]\) can be rewritten as

\[
\rho_s \left( \frac{\partial v^{(0)}}{\partial t} + v^{(0)} \cdot \nabla v^{(0)} \right)_i = - \partial_i p^{(0)} + \eta \Delta v_i^{(0)} + \partial_j \left( (s_1, \ldots, s_5) P\omega_s \right)_{ij} \\
+ c_k B T \partial_j \left( (s_1, \ldots, s_5, a_1, a_2, a_3) N \tilde{\Lambda}(h + g) \right)_{ij} \\
+ c_k B T \tilde{q}^T \tilde{\Lambda}(h + g). \tag{4.37}
\]

### 4.3 Expressions of matrices and vectors under the local basis

We begin to write down the matrices and vectors in \([4.34]\) and \([4.37]\). Let us first discuss the derivatives of \(Q^{(0)}\), i.e. \(\tilde{q}\) and \(\tilde{q}_i\). Since \(Q^{(0)}\) is a function of \(p\), any derivative of \(Q^{(0)}\) can be expressed linearly by \(\mathcal{L}_i Q^{(0)}\). For this reason, let us first look into the coordinates of \(\mathcal{L}_i Q^{(0)}\).

Using the relation \([2.6]\) and from \([3.18]\) we obtain

\[
\mathcal{L}_i Q^{(0)} = s_i (\mathcal{L}_1 n_1 \otimes n_1 + n_1 \otimes \mathcal{L}_j n_1) \\
+ b_i (\mathcal{L}_j n_2 \otimes n_2 + n_2 \otimes \mathcal{L}_j n_2 - (\mathcal{L}_j n_3 \otimes n_3 + n_3 \otimes \mathcal{L}_j n_3)), \quad i = 1, 2; \quad j = 1, 2, 3,
\]

which implies that

\[
\mathcal{L}_1 Q^{(0)} = \begin{pmatrix}
4b_1 n_2 n_3 \\
4b_2 n_2 n_3
\end{pmatrix},
\]

\[
\mathcal{L}_2 Q^{(0)} = -\begin{pmatrix}
2(s_1 + b_1) n_1 n_3 \\
2(s_2 + b_2) n_1 n_3
\end{pmatrix},
\]

\[
\mathcal{L}_3 Q^{(0)} = \begin{pmatrix}
2(s_1 - b_1) n_1 n_2 \\
2(s_2 - b_2) n_1 n_2
\end{pmatrix}.
\]

We denote by \(L\) the coordinates of \((\mathcal{L}_3 Q^{(0)}, \mathcal{L}_2 Q^{(0)}, \mathcal{L}_1 Q^{(0)})\), which is a \(10 \times 3\) matrix. The calculation above gives

\[
L = \begin{pmatrix}
0_{2 \times 3} \\
\text{diag}(2(s_1 - b_1), -2(s_1 + b_1), 4b_1) \ & 0_{2 \times 3} \\
\text{diag}(2(s_2 - b_2), -2(s_2 + b_2), 4b_2)
\end{pmatrix}. \tag{4.38}
\]

For any differential operator \(\mathcal{D}\), we have

\[
\mathcal{D} n_1 = (\mathcal{D} n_1 \cdot n_2) n_2 + (\mathcal{D} n_1 \cdot n_3) n_3,
\]

\[
\mathcal{D} n_2 = (\mathcal{D} n_2 \cdot n_1) n_1 + (\mathcal{D} n_2 \cdot n_3) n_3,
\]

\[
\mathcal{D} n_3 = (\mathcal{D} n_3 \cdot n_1) n_1 + (\mathcal{D} n_3 \cdot n_2) n_2,
\]

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and

\[ 0 = D(n_1 \cdot n_2) = Dn_1 \cdot n_2 + Dn_2 \cdot n_1. \]

By the biaxial form (3.18) of \( Q^{(0)} \), we have

\[ DQ_1^{(0)} = (s_1 + b_1)(Dn_1 \otimes n_1 + n_1 \otimes Dn_1) + 2b_1(Dn_2 \otimes n_2 + n_2 \otimes Dn_2) \]
\[ = 2(s_1 + b_1)(Dn_1 \cdot n_2)n_1n_2 + (Dn_1 \cdot n_3)n_1n_3 + 4b_1((Dn_2 \cdot n_1)n_1n_2 + (Dn_2 \cdot n_3)n_2n_3) \]
\[ = 2(s_1 + b_1)(Dn_1 \cdot n_2)s_3 + (Dn_1 \cdot n_3)s_4 + 4b_1((Dn_2 \cdot n_1)s_3 + (Dn_2 \cdot n_3)s_5) \]
\[ = 2(s_1 - b_1)(Dn_1 \cdot n_2)s_3 - 2(s_1 + b_1)(Dn_3 \cdot n_1)s_4 + 4b_1(Dn_2 \cdot n_3)s_5. \]

Similarly is \( DQ_2^{(0)} \) calculated. Choose \( D \) as \( \partial_i \), \( \partial_i \) and the material derivative \( \partial_i + v_i^{(0)} \partial_i \), respectively. Their coordinates are given by

\[
q = L \begin{pmatrix} \partial_i n_1 \cdot n_2 \\ \partial_i n_3 \cdot n_1 \\ \partial_i n_2 \cdot n_3 \end{pmatrix}, \quad \tilde{q}_i = L \begin{pmatrix} \partial_i n_1 \cdot n_2 \\ \partial_i n_3 \cdot n_1 \\ \partial_i n_2 \cdot n_3 \end{pmatrix}, \quad q = L \begin{pmatrix} \tilde{n}_1 \cdot \tilde{n}_2 \\ \tilde{n}_3 \cdot \tilde{n}_1 \\ \tilde{n}_2 \cdot \tilde{n}_3 \end{pmatrix}. \tag{4.39}
\]

Another important thing to be noticed is that (3.19) leads to \( H^{(0)}Q^{(1)} \cdot L_i(Q^{(0)}) = 0 \). Recall that the coordinates of \( H^{(0)}Q^{(1)} \) is \( h \). Thus, when writing this equation by the coordinates, we deduce that

\[ L^T \tilde{h} = 0. \tag{4.40} \]

The calculations of the matrices \( M, V, N \) and \( P \) involve high-order tensors, which are discussed in Appendices. Here, we only present the result. To express these matrices, we introduce six constant \( 5 \times 5 \) matrices \( X_i (i = 1, \cdots, 6) \),

\[
X_1 = \begin{pmatrix} -9 & 0 \\ 0 & -3 \\ -12 & 0 \\ 0 & -12 \\ -12 & -12 \end{pmatrix}, \quad X_2 = \begin{pmatrix} -\frac{3}{2} & 0 \\ 0 & \frac{1}{2} \\ -1 & -1 \\ -1 & 2 \end{pmatrix}, \tag{4.41}
\]
\[
X_3 = \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} \frac{18}{35} & 0 \\ 0 & \frac{1}{35} \\ -\frac{16}{35} & -\frac{16}{35} \\ -\frac{16}{35} & \frac{1}{35} \end{pmatrix}, \tag{4.42}
\]
\[
X_5 = \begin{pmatrix} \frac{27}{140} & -\frac{3}{28} \\ -\frac{3}{28} & \frac{19}{140} \\ -\frac{16}{35} & \frac{4}{35} \\ \frac{4}{35} & -\frac{16}{35} \end{pmatrix}, \quad X_6 = \begin{pmatrix} -\frac{9}{35} & \frac{3}{28} \\ \frac{3}{28} & -\frac{9}{35} \\ \frac{18}{35} & -\frac{2}{35} \\ -\frac{2}{35} & -\frac{2}{35} \end{pmatrix}. \tag{4.43}
\]

They are associated with six tensors given in (C.2). The detailed calculations of (4.41)-(4.43) can be found in Appendices A.3 and C.
By (C.14), (C.16), (C.18) and (C.22), the blocks of the matrix $M$ are expressed as

$$
M_{11} = -\frac{1}{15} (\Gamma_2 + \Gamma_3) X_1 + \frac{4}{7} \left( (\Gamma_2 s_2 - \Gamma_3 (s_1 + s_2)) X_2 + (\Gamma_2 b_2 - \Gamma_3 (b_1 + b_2)) X_3 \right)
- 4 \left( \Gamma_2 (a_1 + \bar{a}_1) X_4 + (\Gamma_2 (a_2 + \bar{a}_2) - \Gamma_3 \bar{a}_2) X_5 \right.
\left. + (\Gamma_2 (a_3 + \bar{a}_3) - \Gamma_3 \bar{a}_3) X_6 \right)
$$

$$
def = \begin{pmatrix}
\alpha_{11} & \alpha_{12} \\
\alpha_{12} & \alpha_{22} \\
\alpha_{33} & \\
\alpha_{44} & \\
\alpha_{55}
\end{pmatrix},
$$

$$
M_{12} = \frac{1}{15} \Gamma_3 X_1 + \frac{4}{7} \Gamma_3 \left( (s_1 + s_2) X_2 + (b_1 + b_2) X_3 \right) - 4 \Gamma_3 \left( \bar{a}_1 X_4 + \bar{a}_2 X_5 + \bar{a}_3 X_6 \right)
$$

$$
def = \begin{pmatrix}
\beta_{11} & \beta_{12} \\
\beta_{12} & \beta_{22} \\
\beta_{33} & \\
\beta_{44} & \\
\beta_{55}
\end{pmatrix},
$$

$$
M_{22} = -\frac{1}{15} (\Gamma_1 + \Gamma_3) X_1 + \frac{4}{7} \left( (\Gamma_1 s_1 - \Gamma_3 (s_1 + s_2)) X_2 + (\Gamma_1 b_1 - \Gamma_3 (b_1 + b_2)) X_3 \right)
- 4 \left( \Gamma_1 (\bar{a}_1 + \bar{a}_1) X_4 + (\Gamma_1 (\bar{a}_2 + \bar{a}_2) - \Gamma_3 \bar{a}_2) X_5 \right.
\left. + (\Gamma_1 (\bar{a}_3 + \bar{a}_3) - \Gamma_3 \bar{a}_3) X_6 \right)
$$

$$
def = \begin{pmatrix}
\gamma_{11} & \gamma_{12} \\
\gamma_{12} & \gamma_{22} \\
\gamma_{33} & \\
\gamma_{44} & \\
\gamma_{55}
\end{pmatrix},
$$

where $a_i$, $\bar{a}_i$, $\bar{a}_i(i = 1, 2, 3)$ are those in (3.22), which we shall keep in mind are functions of $s_i, b_i(i = 1, 2)$.

By (C.32) and (C.33), it follows that

$$
N_1^n = -\frac{1}{15} e_1 X_1 - \frac{2}{7} \left( ((e_1 - e_2) s_1 - 2e_2 s_2) X_2 + ((e_1 - e_2) b_1 - 2e_2 b_2) X_3 \right)
- 2 \left( (a_1 + 2e_2 \bar{a}_1) X_4 + (a_2 + 2e_2 \bar{a}_2) X_5 + (a_3 + 2e_2 \bar{a}_3) X_6 \right)
$$

$$
def = \begin{pmatrix}
\mu_{11} & \mu_{12} \\
\mu_{12} & \mu_{22} \\
\mu_{33} & \\
\mu_{44} & \\
\mu_{55}
\end{pmatrix},
$$

$$
N_2^n = -\frac{1}{15} e_2 X_1 + \frac{2}{7} \left( (2e_1 s_1 - (e_2 - e_1) s_2) X_2 + (2e_1 b_1 - (e_2 - e_1) b_2) X_3 \right)
$$
where $L$ is given by (4.38).

By (C.10), (C.12), (C.14), and (4.17), we obtain

$$P = c \left[ -\frac{1}{45} (I_{22} + I_{11}(1 + 3e_1))X_1 - \frac{4}{21} \left( (I_{22} + 3I_{11}e_1)s_1 + I_{11}(1 + 3e_1)s_2 \right) X_2 
+ (I_{22} + 3I_{11}e_1)b_1 + I_{11}(1 + 3e_1)b_2 \right) X_3
+ (I_{22}a_1 + I_{11}a_1 + 4I_{11}e_1a_1)X_4
+ (I_{22}a_2 + I_{11}a_2 + 4I_{11}e_1a_2)X_5
+ (I_{22}a_3 + I_{11}a_3 + 4I_{11}e_1a_3)X_6 \right]$$

$$\begin{pmatrix}
\vartheta_{11} & \vartheta_{12} \\
\vartheta_{12} & \vartheta_{22}
\end{pmatrix}$$

(4.50)

In the above, we intentionally introduce notations for the components of $M$, $V$ and $P$, to emphasize that these matrices have specific forms. These specific forms are significant in the forthcoming derivations. We have claimed that $M$ and $P$ are positive definite in (4.11) and (4.10). As a result, the corresponding coefficient matrices $M$ and $P$ are also positive definite. We do not consider the expressions of the vectors $h$ and $g$, because the terms involving them will be expressed by variational derivatives of the elastic energy.

### 4.4 Orthonormal frame model

We are now ready to derive the frame hydrodynamics for the biaxial nematic phase from the $O(1)$ system (4.19)-(4.21).

To begin with, we write down the elastic energy for the biaxial nematic phase. In the tensor model, the elastic energy is a functional of $Q$. When $Q$ takes $Q^{(0)}$ that is a function of $p$, the corresponding elastic energy becomes a functional of the frame $p$, which we denote by $F_{B_1}$. Generally, the biaxial elastic energy consists of twelve bulk terms [10, 50], written as

$$\frac{F_{B_1}[p]}{ck_B T} = \int dx \left( \frac{1}{2} \left( K_{1111}D_{11}^2 + K_{2222}D_{22}^2 + K_{3333}D_{33}^2 
+ K_{1212}D_{12}^2 + K_{2121}D_{21}^2 + K_{3232}D_{32}^2 + K_{3232}D_{32}^2 + K_{3131}D_{31}^2 + K_{1313}D_{13}^2 
+ K_{1221}D_{12}D_{21} + K_{2332}D_{23}D_{32} + K_{1331}D_{13}D_{31} \right) \right).$$

(4.51)

We here take no account of three surface terms, such as

$$\partial_i n_{2i} \partial_j n_{2j} - \partial_i n_{2j} \partial_j n_{2i} = 2(D_{33}D_{11} - D_{31}D_{13}).$$

(4.52)
The coefficients in \([4531]\) can be derived from the coefficients in the tensor model \([50]\) as

\[
\begin{align*}
K_{1111} &= J_2, & K_{2222} &= J_1, & K_{3333} &= J_1 + J_2 - J_3, \\
K_{1212} &= K_{3232} = J_1 + J_4, & K_{2121} = K_{3131} &= J_2 + J_5, \\
K_{2323} &= K_{1313} = J_1 + J_3 - J_3 + J_4 + J_5 - J_6, \\
K_{1221} &= -J_6, & K_{2332} &= J_6 - 2J_4, & K_{1331} &= J_6 - 2J_5,
\end{align*}
\tag{4.53}
\]

where

\[
\begin{align*}
J_1 &= 2c(c_{22}(s_1 + b_1)^2 + c_{23}(s_2 + b_2)^2 + 2c_{24}(s_1 + b_1)(s_2 + b_2)), \\
J_2 &= 8c(c_{22}b_1^2 + c_{23}b_2^2 + 2c_{24}b_1b_2), \\
J_3 &= 8c(c_{22}b_1(s_1 + b_1) + c_{23}b_2(s_2 + b_2) + c_{24}[b_1(s_2 + b_2) + b_2(s_1 + b_1)]), \\
J_4 &= c(c_{28}(s_1 + b_1)^2 + c_{29}(s_2 + b_2)^2 + 2c_{21,10}(s_1 + b_1)(s_2 + b_2)), \\
J_5 &= 4c(c_{28}b_1^2 + c_{29}b_2^2 + 2c_{21,10}b_1b_2), \\
J_6 &= 4c(c_{28}b_1(s_1 + b_1) + c_{29}b_2(s_2 + b_2) + c_{21,10}[b_1(s_2 + b_2) + b_2(s_1 + b_1)]).
\end{align*}
\tag{4.54}
\]

Using the chain rule, we deduce that

\[
\mathcal{L} F_{B_i} = \mathcal{L} F_{B_i}(Q^{(0)}(p)) = \frac{\delta F_{B_i}}{\delta Q^{(0)}} \cdot Q^{(0)} = c k_B T G(Q^{(0)}) \cdot Q^{(0)} = c k_B T L^T \tilde{\Lambda} g.
\tag{4.55}
\]

Therefore, it is deduced from \([4.34]\) and \([4.40]\) that

\[
L^T M^{-1}(q - V\omega) + \frac{1}{ck_BT} \mathcal{L} F_{B_i} = 0.
\tag{4.56}
\]

In the above, we notice that \(M\) is positive definite, thus invertible.

To calculate \([4.56]\), we rearrange the rows and columns of the matrices so that they can be divided into blocks appropriately. To this end, we introduce a \(10 \times 10\) permutation matrix

\[
C = (E_1, E_6, E_2, E_7, E_3, E_8, E_4, E_9, E_5, E_{10}),
\tag{4.57}
\]

where \(E_j\) is the \(10 \times 1\) unit vector with the \(j\)-th component equal to one. We have \(CC^T = I_{10}\), which is the \(10 \times 10\) identity matrix. Then we have

\[
C^T MC = \begin{pmatrix} M_0 & M_1 \\ M_1 & M_2 \\ & & M_3 \end{pmatrix},
\tag{4.58}
\]

where the blocks \(M_i (i = 0, 1, 2, 3)\) are given by

\[
M_0 = \begin{pmatrix} \alpha_{11} & \beta_{11} & \alpha_{12} & \beta_{12} \\ \beta_{11} & \gamma_{11} & \beta_{12} & \gamma_{12} \\ \alpha_{12} & \beta_{12} & \alpha_{22} & \beta_{22} \\ \beta_{12} & \gamma_{12} & \beta_{22} & \gamma_{22} \end{pmatrix},
\]

\[
M_1 = \begin{pmatrix} \alpha_{33} & \beta_{33} \\ \beta_{33} & \gamma_{33} \end{pmatrix},
\]

\[
M_2 = \begin{pmatrix} \alpha_{44} & \beta_{44} \\ \beta_{44} & \gamma_{44} \end{pmatrix},
\]

\[
M_3 = \begin{pmatrix} \alpha_{55} & \beta_{55} \\ \beta_{55} & \gamma_{55} \end{pmatrix}.
\]
These blocks are all positive definite because $M$ is. For the matrix $V$, we have
\[
CTV = \begin{pmatrix} V_0 \\ V_1 & 0 & 0 & V_4 & 0 & 0 \\ 0 & V_2 & 0 & 0 & V_5 & 0 \\ 0 & 0 & V_3 & 0 & 0 & V_6 \end{pmatrix},
\]
where the blocks $V_i (i = 0, 1, \cdots, 6)$ are given by
\[
V_0 = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \nu_{11} & \nu_{12} \\ \mu_{12} & \nu_{22} \end{pmatrix}, \quad V_1 = \begin{pmatrix} \mu_{33} \\ \nu_{33} \end{pmatrix}, \quad V_2 = \begin{pmatrix} \mu_{44} \\ \nu_{44} \end{pmatrix}, \quad V_3 = \begin{pmatrix} \mu_{55} \\ \nu_{55} \end{pmatrix}, \quad V_4 = \begin{pmatrix} s_1 - b_1 \\ s_2 - b_2 \end{pmatrix}, \quad V_5 = -\begin{pmatrix} s_1 + b_1 \\ s_2 + b_2 \end{pmatrix}, \quad V_6 = \begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix}.
\]
We also rearrange the indices of the vector $q$ by $C$,
\[
CTq = \begin{pmatrix} 0_{4 \times 1} \\ 2(\hat{n}_1 \cdot n_2)V_4 \\ 2(\hat{n}_3 \cdot n_1)V_5 \\ 2(\hat{n}_2 \cdot n_3)V_6 \end{pmatrix},
\]
where we use $0_{N_1 \times N_2}$ to represent an $N_1 \times N_2$ zero matrix. The matrix $L$ is rearranged as
\[
(C^TL)^T = L^TC = (0_{3 \times 4}, L_1^T),
\]
where
\[
L_1^T = \begin{pmatrix} 2V_4^T \\ 2V_5^T \\ 2V_6^T \end{pmatrix}.
\]
Thus, from (4.58) and (4.61), we have
\[
L^TM^{-1}C = (C^TL)^T(C^TM)^{-1}C = \begin{pmatrix} 2V_4^TM_1^{-1} \\ 2V_5^TM_2^{-1} \\ 2V_6^TM_3^{-1} \end{pmatrix}.
\]
Together with (4.62), we deduce that
\[
L^TM^{-1}q = (L^TC)(C^TM)^{-1}(CTq) = \begin{pmatrix} \chi_3\hat{n}_1 \cdot n_2 \\ \chi_2\hat{n}_3 \cdot n_1 \\ \chi_1\hat{n}_2 \cdot n_3 \end{pmatrix},
\]
where the coefficients $\chi_i (i = 1, 2, 3)$ are given by
\[
\chi_3 = 4V_4^TM_1^{-1}V_4, \quad \chi_2 = 4V_5^TM_2^{-1}V_5, \quad \chi_1 = 4V_6^TM_3^{-1}V_6.
\]
From (4.58), (4.59) and (4.61), we have
\[
L^TM^{-1}V
\]
Using (4.55) and (4.40), we arrive at

\[
\eta
\] where the coefficients

So we have

\[
L^T M^{-1} \Omega = \begin{pmatrix}
2V_4^T M_1^{-1} V_1 \\
2V_5^T M_2^{-1} V_2 \\
2V_6^T M_3^{-1} V_3
\end{pmatrix}
\]

\[
L^T M^{-1} \Omega = \begin{pmatrix}
\eta_3 A_0 \cdot s_3 + \frac{1}{2} \chi_3 \Omega_0 \cdot a_1 \\
\eta_2 A_0 \cdot s_4 + \frac{1}{2} \chi_2 \Omega_0 \cdot a_2 \\
\eta_1 A_0 \cdot s_5 + \frac{1}{2} \chi_1 \Omega_0 \cdot a_3
\end{pmatrix},
\]

(4.65)

where the coefficients \( \eta_i (i = 1, 2, 3) \) are expressed by

\[
\eta_3 = 2V_4^T M_1^{-1} V_1, \quad \eta_2 = 2V_5^T M_2^{-1} V_2, \quad \eta_1 = 2V_6^T M_3^{-1} V_3.
\]

Therefore, using (4.63) and (4.65), the equation (4.66) can be reformulated as

\[
\begin{align*}
\chi_1 n_2 \cdot n_3 &= \frac{1}{2} \chi_1 \Omega_0 \cdot a_3 - \eta_1 A_0 \cdot s_5 + \frac{1}{ck_BT} \mathcal{L}_1 F_{Bi} = 0, \\
\chi_2 n_3 \cdot n_1 &= \frac{1}{2} \chi_2 \Omega_0 \cdot a_2 - \eta_2 A_0 \cdot s_4 + \frac{1}{ck_BT} \mathcal{L}_2 F_{Bi} = 0, \\
\chi_3 n_1 \cdot n_2 &= \frac{1}{2} \chi_3 \Omega_0 \cdot a_1 - \eta_3 A_0 \cdot s_3 + \frac{1}{ck_BT} \mathcal{L}_3 F_{Bi} = 0,
\end{align*}
\]

(4.66-4.68)

where \( \mathcal{L}_i F_{Bi} (i = 1, 2, 3) \) are the variational derivatives along the infinitesimal rotation round \( n_i (i = 1, 2, 3) \), respectively, which are written as

\[
\mathcal{L}_1 F_{Bi} = n_3 \cdot \frac{\delta F_{Bi}}{\delta n_2} - n_2 \cdot \frac{\delta F_{Bi}}{\delta n_3},
\]

\[
\mathcal{L}_2 F_{Bi} = n_1 \cdot \frac{\delta F_{Bi}}{\delta n_3} - n_3 \cdot \frac{\delta F_{Bi}}{\delta n_1},
\]

\[
\mathcal{L}_3 F_{Bi} = n_2 \cdot \frac{\delta F_{Bi}}{\delta n_1} - n_1 \cdot \frac{\delta F_{Bi}}{\delta n_2}.
\]

It remains to derive the equation of the fluid velocity \( v^{(0)} \). From (4.39), we have

\[
\tilde{q}_i^T \tilde{A} (h + g) = \left( \partial_i n_1 \cdot n_2, \partial_i n_3 \cdot n_1, \partial_i n_2 \cdot n_3 \right) L^T \tilde{A} (h + g).
\]

(4.69)

Using (4.55) and (4.40), we arrive at

\[
\tilde{q}_i^T \tilde{A} (h + g) = \frac{1}{ck_BT} \left( \mathcal{L}_3 F_{Bi} \partial_i n_1 \cdot n_2 + \mathcal{L}_2 F_{Bi} \partial_i n_3 \cdot n_1 + \mathcal{L}_1 F_{Bi} \partial_i n_2 \cdot n_3 \right) \overset{\text{def}}{=} \tilde{s}_i.
\]

(4.70)

Taking them into (4.37), we deduce that

\[
\rho_s \left( \frac{\partial v^{(0)}}{\partial t} + v^{(0)} \cdot \nabla v^{(0)} \right)_i = - \partial_i p^{(0)} + \eta \Delta v^{(0)}_i + \partial_j \left( (s_1, \cdots, s_5) P \omega_s \right)_{ij}
\]

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In the above, we recall that the matrix $P$ and

Thus, we have

Noticing $N = V^T$, we obtain

Thus, we have

and

In the above, we recall that the matrix $P$ is given by $\text{Eqn.}$. 

(4.71)

(4.72)

(4.73)
\[
\begin{pmatrix}
V_0^T M_0^{-1} V_0 \\
V_1^T M_1^{-1} V_1 & 0 & 0 & V_4^T M_4^{-1} V_4 & 0 & 0 \\
0 & V_2^T M_2^{-1} V_2 & 0 & 0 & V_2^T M_2^{-1} V_5 & 0 \\
0 & 0 & V_3^T M_3^{-1} V_3 & 0 & 0 & V_3^T M_3^{-1} V_6 \\
V_1^T M_1^{-1} V_1 & 0 & 0 & V_4^T M_4^{-1} V_4 & 0 & 0 \\
0 & V_2^T M_2^{-1} V_2 & 0 & 0 & V_2^T M_2^{-1} V_5 & 0 \\
0 & 0 & V_3^T M_3^{-1} V_3 & 0 & 0 & V_3^T M_3^{-1} V_6
\end{pmatrix}.
\]

(4.74)

It further gives

\[
NM^{-1} V_\omega =
\begin{pmatrix}
V_0^T M_0^{-1} V_0 \omega_0 \\
V_1^T M_1^{-1} V_1 A_0 \cdot s_3 + V_1^T M_1^{-1} V_4 \Omega_0 \cdot a_1 \\
V_2^T M_2^{-1} V_2 A_0 \cdot s_4 + V_2^T M_2^{-1} V_5 \Omega_0 \cdot a_2 \\
V_3^T M_3^{-1} V_3 A_0 \cdot s_5 + V_3^T M_3^{-1} V_6 \Omega_0 \cdot a_3 \\
V_4^T M_4^{-1} V_4 A_0 \cdot s_3 + V_4^T M_4^{-1} V_4 \Omega_0 \cdot a_1 \\
V_5^T M_5^{-1} V_2 A_0 \cdot s_4 + V_5^T M_5^{-1} V_5 \Omega_0 \cdot a_2 \\
V_6^T M_6^{-1} V_3 A_0 \cdot s_5 + V_6^T M_6^{-1} V_6 \Omega_0 \cdot a_3
\end{pmatrix} =
\begin{pmatrix}
V_0^T M_0^{-1} V_0 \omega_0 \\
\beta_3 A_0 \cdot s_3 + \frac{1}{2} \eta_3 \Omega_0 \cdot a_1 \\
\beta_4 A_0 \cdot s_4 + \frac{1}{2} \eta_2 \Omega_0 \cdot a_2 \\
\beta_5 A_0 \cdot s_5 + \frac{1}{2} \eta_1 \Omega_0 \cdot a_3 \\
\frac{1}{2} \eta_3 A_0 \cdot s_3 + \frac{1}{2} \eta_3 \Omega_0 \cdot a_1 \\
\frac{1}{2} \eta_2 A_0 \cdot s_4 + \frac{1}{2} \eta_2 \Omega_0 \cdot a_2 \\
\frac{1}{2} \eta_1 A_0 \cdot s_5 + \frac{1}{2} \eta_1 \Omega_0 \cdot a_3
\end{pmatrix},
\]

(4.75)

where we denote

\[
\omega_0 = (A_0 \cdot s_1, A_0 \cdot s_2)^T,
\]

and the coefficients \( \beta_i (i = 3, 4, 5) \) are given by

\[
\beta_3 = V_1^T M_1^{-1} V_1, \quad \beta_4 = V_2^T M_2^{-1} V_2, \quad \beta_5 = V_3^T M_3^{-1} V_3.
\]

(4.76)

Since \( M_0 \) is positive definite, the \( 2 \times 2 \) matrix \( V_0^T M_0^{-1} V_0 \) is symmetric positive semi-definite. If we write out its components,

\[
V_0^T M_0^{-1} V_0 = \begin{pmatrix}
\beta_1 & \beta_0 \\
\beta_0 & \beta_2
\end{pmatrix},
\]

then \( \beta_i (i = 0, 1, 2) \) satisfy

\[
\beta_i \geq 0, \quad i = 1, 2, \quad \beta_0 \leq \beta_1 \beta_2.
\]

(4.77)

Hence, combining (4.73) with (4.75), we deduce that

\[
NM^{-1} (q - V_\omega) = \begin{pmatrix}
-V_0^T M_0^{-1} V_0 \omega_0 \\
-\beta_3 A_0 \cdot s_3 + \eta_3 (\dot{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) \\
-\beta_4 A_0 \cdot s_4 + \eta_2 (\dot{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) \\
-\beta_5 A_0 \cdot s_5 + \eta_1 (\dot{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3) \\
-\frac{1}{2} \eta_3 A_0 \cdot s_3 + \frac{1}{2} \chi_3 (\dot{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) \\
-\frac{1}{2} \eta_2 A_0 \cdot s_4 + \frac{1}{2} \chi_2 (\dot{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) \\
-\frac{1}{2} \eta_1 A_0 \cdot s_5 + \frac{1}{2} \chi_1 (\dot{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3)
\end{pmatrix},
\]

(4.78)

which further implies

\[
\frac{1}{ck_B T} \sigma^{(0)} = - (s_1, \ldots, s_5, a_1, a_2, a_3) \cdot NM^{-1} (q - V_\omega)
\]

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\[
\begin{align*}
&= \beta_1 (A_0 \cdot s_1)s_1 + \beta_0 (A_0 \cdot s_2)s_1 + \beta_0 (A_0 \cdot s_1)s_2 + \beta_2 (A_0 \cdot s_2)s_2 \\
&+ \beta_3 (A_0 \cdot s_3)s_3 - \eta_3 \left( \hat{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1 \right)s_3 \\
&+ \beta_4 (A_0 \cdot s_4)s_4 - \eta_2 \left( \hat{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2 \right)s_4 \\
&+ \beta_5 (A_0 \cdot s_5)s_5 - \eta_1 \left( \hat{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3 \right)s_5 \\
&+ \frac{1}{2} \eta_3 (A_0 \cdot s_3)a_1 - \frac{1}{2} \chi_3 \left( \hat{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1 \right)a_1 \\
&+ \frac{1}{2} \eta_2 (A_0 \cdot s_4)a_2 - \frac{1}{2} \chi_2 \left( \hat{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2 \right)a_2 \\
&+ \frac{1}{2} \eta_1 (A_0 \cdot s_5)a_3 - \frac{1}{2} \chi_1 \left( \hat{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3 \right)a_3.
\end{align*}
\]

Therefore, from (4.77), the equation of \( v^{(0)} \) reads
\[
\rho_s \left( \frac{\partial v^{(0)}}{\partial t} + v^{(0)} \cdot \nabla v^{(0)} \right)_i = -\partial_i p^{(0)} + \partial_j \left( (\sigma_v^{(0)})_{ij} + (\sigma_e^{(0)})_{ij} \right) + c k_B T \hat{\beta}, \quad \nabla \cdot v^{(0)} = 0.
\]

Here, the viscous stress \( \sigma_v^{(0)} \) is denoted by
\[
\sigma_v^{(0)} = \eta A_0 + (s_1, \cdots, s_5) \rho \omega_s \\
= (s_1, \cdots, s_5) (\eta A^{-1} + \rho) \omega_s,
\]
where we have used the following fact
\[
A_0 = \sum_{i=1}^5 \frac{1}{|s_i|^2} (A_0 \cdot s_i)s_i = (s_1, \cdots, s_5) A^{-1} \omega_s, \quad A^{-1} = \text{diag}(3, 1, 2, 2, 2).
\]

To sum up, the frame hydrodynamics for the biaxial nematic phase is given by (4.66), (4.68), (4.80) and (4.81).

### 4.5 Energy dissipation

Taking the derivative about \( t \) of the biaxial elastic energy (4.51), we deduce that
\[
\frac{dF_{B_i}}{dt} = \int \left( \frac{\delta F_{B_i}}{\delta n_1} \cdot \frac{\partial n_1}{\partial t} + \frac{\delta F_{B_i}}{\delta n_2} \cdot \frac{\partial n_2}{\partial t} + \frac{\delta F_{B_i}}{\delta n_3} \cdot \frac{\partial n_3}{\partial t} \right) dx
\]
\[
= \int \left( \frac{\delta F_{B_i}}{\delta n_1} \cdot (n_2(n_2 \cdot \partial_t n_1) + n_3(n_3 \cdot \partial_t n_1)) + \frac{\delta F_{B_i}}{\delta n_2} \cdot (n_1(n_1 \cdot \partial_t n_2) + n_3(n_3 \cdot \partial_t n_2)) \right) dx
\]
\[
+ \frac{\delta F_{B_i}}{\delta n_3} \cdot (n_1(n_1 \cdot \partial_t n_3) + n_2(n_2 \cdot \partial_t n_3)) \right) dx
\]
\[
= \int \left[ n_{3k} \partial_t n_{2k} \left( n_3 \cdot \frac{\delta F_{B_i}}{\delta n_2} - n_2 \cdot \frac{\delta F_{B_i}}{\delta n_3} \right) + n_{1k} \partial_t n_{3k} \left( n_1 \cdot \frac{\delta F_{B_i}}{\delta n_3} - n_3 \cdot \frac{\delta F_{B_i}}{\delta n_1} \right) \right] dx
\]
\[
+ n_{2k} \partial_t n_{1k} \left( n_2 \cdot \frac{\delta F_{B_i}}{\delta n_1} - n_1 \cdot \frac{\delta F_{B_i}}{\delta n_2} \right) \right) dx
\]
\[
= \int \left( n_{3k} \partial_t n_{2k} \mathcal{L}_1 F_{B_i} + n_{1k} \partial_t n_{3k} \mathcal{L}_2 F_{B_i} + n_{2k} \partial_t n_{1k} \mathcal{L}_3 F_{B_i} \right) dx.
\]

Taking the inner product on the equation (4.80) with \( v^{(0)} \) and using \( \nabla \cdot v^{(0)} = 0 \), we derive that
\[
\frac{\rho_s}{2} \frac{d}{dt} \int |v^{(0)}|^2 dx = -\langle \sigma_v^{(0)} , A_0 \rangle - \langle \sigma_e^{(0)} , \nabla v^{(0)} \rangle + c k_B T \overline{g} \langle \hat{\beta} , v^{(0)} \rangle,
\]

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where
\[
ck_B T(\tilde{3}, \nu^{(0)}) = \int v^{(0)}_1 n_{3k} \partial_k n_{2k} \mathcal{L}_1 F_{B1} + n_{1k} \partial_k n_{3k} \mathcal{L}_2 F_{B1} + n_{2k} \partial_k n_{1k} \mathcal{L}_3 F_{B1}) dx.
\]

Combining (4.83) with (4.84), and using the equations (4.66) – (4.68), we obtain the energy dissipation law,
\[
\frac{d}{dt} \left( \frac{\rho}{2} \int |v^{(0)}|^2 dx + \mathcal{F}_B(p) \right) = -\langle \sigma_v^{(0)}, A_0 \rangle - \langle \sigma_e^{(0)}, \nabla v^{(0)} \rangle + \int \left( (\tilde{n}_2 \cdot n_3) \mathcal{L}_1 F_{B1} + (\tilde{n}_3 \cdot n_1) \mathcal{L}_2 F_{B1} + (\tilde{n}_1 \cdot n_2) \mathcal{L}_3 F_{B1} \right) dx
\]
\[
= -\int \omega_s (\eta A^{-1} + P) \omega_s dx + ck_B T \left( -\beta_1 ||A_0 \cdot s_1||^2_{L^2} - 2\beta_0 \int (A_0 \cdot s_1)(A_0 \cdot s_2) dx - \beta_2 ||A_0 \cdot s_2||^2_{L^2} - \beta_3 ||A_0 \cdot s_3||^2_{L^2} + \eta_3 \int (\tilde{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1)(A_0 \cdot s_3) dx 
\]
\[
= -\beta_4 ||A_0 \cdot s_4||^2_{L^2} + \eta_2 \int (\tilde{n}_3 \cdot n_1)(A_0 \cdot s_4) dx 
\]
\[
-\beta_5 ||A_0 \cdot s_5||^2_{L^2} + \eta_1 \int (\tilde{n}_2 \cdot n_3)(A_0 \cdot s_5) dx 
\]
\[
= -\frac{1}{2} \eta_3 \int (A_0 \cdot s_3) \Omega_0 \cdot a_1 dx + \frac{1}{2} \chi_3 \int (\tilde{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) \Omega_0 \cdot a_1 dx 
\]
\[
-\frac{1}{2} \eta_2 \int (A_0 \cdot s_4) \Omega_0 \cdot a_2 dx + \frac{1}{2} \chi_2 \int (\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) \Omega_0 \cdot a_2 dx 
\]
\[
-\frac{1}{2} \eta_1 \int (A_0 \cdot s_5) \Omega_0 \cdot a_3 dx + \frac{1}{2} \chi_1 \int (\tilde{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3) \Omega_0 \cdot a_3 dx 
\]
\[
+ \int (\tilde{n}_2 \cdot n_3) \left[ -\chi_1 (\tilde{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3) + \eta_1 A_0 \cdot s_5 \right] dx 
\]
\[
+ \int (\tilde{n}_3 \cdot n_1) \left[ -\chi_2 (\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) + \eta_2 A_0 \cdot s_4 \right] dx 
\]
\[
+ \int (\tilde{n}_1 \cdot n_2) \left[ -\chi_3 (\tilde{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) + \eta_3 A_0 \cdot s_3 \right] dx 
\]
\[
= -\int \omega_s (\eta A^{-1} + P) \omega_s dx + ck_B T \left( -\beta_1 ||A_0 \cdot s_1||^2_{L^2} - 2\beta_0 \int (A_0 \cdot s_1)(A_0 \cdot s_2) dx 
\]
\[
-\beta_2 ||A_0 \cdot s_2||^2_{L^2} - \beta_3 ||A_0 \cdot s_3||^2_{L^2} - \beta_4 ||A_0 \cdot s_4||^2_{L^2} - \beta_5 ||A_0 \cdot s_5||^2_{L^2} + 2\eta_3 \int (\tilde{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) \Omega_0 \cdot a_1 dx 
\]
\[
+ 2\eta_2 \int (\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) \Omega_0 \cdot a_2 dx 
\]
\[
+ 2\eta_1 \int (\tilde{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3) \Omega_0 \cdot a_3 dx 
\]
\[
- \chi_1 ||\tilde{n}_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3||^2_{L^2} - \chi_2 ||\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2||^2_{L^2} 
\]
\[
- \chi_3 ||\tilde{n}_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1||^2_{L^2} \right). \quad (4.85)
\]

The dissipation can be recognized by noticing the following facts:
• $\Lambda$ and $P$ are positive definite.
• $\beta_1, \beta_2 \geq 0$ and $\beta_0^2 \leq \beta_1 \beta_2$. This comes from (4.77).
• $\beta_3, \chi_3 \geq 0$ and $\eta_3^2 \leq \beta_3 \chi_3$. To realize this, we use the expressions $\beta_3 = V_1^T M_1^{-1} V_1$, $\chi_3 = 4V_1^T M_1^{-1} V_4$ and $\eta_3 = 2V_1^T M_1^{-1} V_1$ and the fact that $M_1$ is positive definite.

5 Reduction to uniaxial dynamics

In the tensor model, the minimum of the bulk energy (3.3) might be uniaxial in the form

$$Q_i = s_i \left( n_i^2 - \frac{1}{3} \right), \quad i = 1, 2. \quad (5.1)$$

In this case, the local anisotropy is axisymmetric, and the corresponding hydrodynamics is reduced to the Ericksen–Leslie theory, which we derive in the following.

The most important thing is that the form of tensors in Theorem 3.3 will be reduced.

**Theorem 5.1.** Assume that $Q_1$ and $Q_2$ have the uniaxial form (5.1). Then, the high-order symmetric traceless tensors obtained from closure by the original entropy or the quasi-entropy have the following form,

$$\langle m_1 m_2 m_3 \rangle = 0, \quad \langle (m_1^4) \rangle = a_1(n_4^4),$$

$$\langle (m_2^4) \rangle = \bar{a}_1(n_4^4), \quad \langle (m_1 m_2^2 \rangle = \bar{a}_1(n_4^4).$$

The proof is left to Appendix D.

For the elastic energy, we notice that when $b_1 = b_2 = 0$, it holds $J_2 = J_3 = J_5 = J_6 = 0$ in (4.54). According to [192], the elastic energy only depends on $n_1$,

$$F_{Un} = \int \frac{1}{2} \left( \tilde{J}_1 (\partial_i n_{1j})^2 \right) \left( \tilde{J}_4 (\nabla \cdot n_1)^2 + n_{11} n_{1j} \partial_i n_{1k} \partial_j n_{1k} \right) dx. \quad (5.2)$$

Such an $F_{Un}$ can be written in the form of the Oseen-Frank energy (which we omit the derivation here), and the coefficients are given by

$$\tilde{J}_1 = 2k_B T c^2 (c_{22} s_1^2 + c_{23} s_2^2 + 2c_{24} s_1 s_2),$$

$$\tilde{J}_4 = k_B T c^2 (c_{28} s_1^2 + c_{29} s_2^2 + 2c_{2,10} s_1 s_2).$$

From (5.2) we immediately get

$$\frac{\delta F_{Un}}{\delta n_2} = \frac{\delta F_{Un}}{\delta n_3} = 0,$$

which implies that

$$\mathcal{L}_1^{-} F_{Bi} = n_3 \frac{\delta F_{Bi}}{\delta n_2} - n_2 \frac{\delta F_{Bi}}{\delta n_3} = 0, \quad (5.3)$$

$$\mathcal{L}_2^{-} F_{Bi} = n_1 \frac{\delta F_{Bi}}{\delta n_3} - n_3 \frac{\delta F_{Bi}}{\delta n_1} = 0, \quad (5.4)$$

$$\mathcal{L}_3^{-} F_{Bi} = n_2 \frac{\delta F_{Bi}}{\delta n_1} - n_1 \frac{\delta F_{Bi}}{\delta n_2} = 0. \quad (5.5)$$

By $b_1, b_2 = 0$ and Theorem 5.1 in the matrices $M_{ij}$, $N^w_i$ and $P$ (see (4.44)–(4.50)), the coefficients of $X_3$, $X_5$, $X_6$ are all zero. The matrices $X_1$, $X_2$ and $X_4$ are all diagonal matrices with their elements satisfying the following relations:

$$\langle X_i \rangle_{33} = \langle X_i \rangle_{44}, \quad \langle X_i \rangle_{55} = 4 \langle X_i \rangle_{22}, \quad i = 1, 2, 4. \quad (5.6)$$

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Thus, the blocks in (4.38) become

\[
M_0 = \begin{pmatrix}
\alpha_{11} & \beta_{11} & 0 & 0 \\
\beta_{11} & \gamma_{11} & 0 & 0 \\
0 & 0 & \alpha_{22} & \beta_{22} \\
0 & 0 & \beta_{22} & \gamma_{22}
\end{pmatrix} \text{ def } \begin{pmatrix}
M_{01} \\
M_{02}
\end{pmatrix},
\]

\[
M_1 = M_2 = \begin{pmatrix}
\alpha_{33} & \beta_{33} \\
\beta_{33} & \gamma_{33}
\end{pmatrix},
\]

\[M_3 = 4 \begin{pmatrix}
\alpha_{22} & \beta_{22} \\
\beta_{22} & \gamma_{22}
\end{pmatrix},
\]

Similarly, the blocks in (4.59) are reduced to

\[
V_0 = \begin{pmatrix}
\mu_{11} & 0 \\
\nu_{11} & 0 \\
0 & \mu_{22} \\
0 & \nu_{22}
\end{pmatrix} \text{ def } \begin{pmatrix}
V_{01} \\
V_{02}
\end{pmatrix},
\]

\[V_1 = V_2 = \begin{pmatrix}
\mu_{33} \\
\nu_{33}
\end{pmatrix},
\]

\[V_3 = 4 \begin{pmatrix}
\mu_{22} \\
\nu_{22}
\end{pmatrix},
\]

\[V_4 = \begin{pmatrix}
s_1 \\
s_2
\end{pmatrix},
\]

\[V_5 = - \begin{pmatrix}
s_1 \\
s_2
\end{pmatrix} = -V_4, \quad V_6 = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

We know from (5.3) and \(V_6\) is a zero vector that the equation (4.66) disappears. Meanwhile, noting the following relations between coefficients,

\[\chi_3 = \chi_2 = 4V_4^T M_1^{-1} V_4 > 0, \quad \eta_3 = -\eta_2 = 2V_4^T M_1^{-1} V_1,
\]

we could simplify the equations (4.67) and (4.68) as

\[
\begin{aligned}
\chi_2 (\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2) - \eta_2 A_3 \cdot s_4 - \frac{1}{c k_B T} n_3 \cdot \frac{\delta F_{U \delta}}{\delta n_1} &= 0, \\
\chi_2 (\tilde{n}_3 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1) + \eta_2 A_3 \cdot s_3 + \frac{1}{c k_B T} n_2 \cdot \frac{\delta F_{U \delta}}{\delta n_1} &= 0.
\end{aligned}
\]

\[\text{(5.7)}
\]

Denote the rotational derivative of the director and the molecular field as

\[N_1 = \tilde{n}_1 - \Omega_{ij}^{(0)} n_{1j}, \quad h_1 = -\frac{1}{c k_B T} \frac{\delta F_{U \delta}}{\delta \tilde{n}_1}.
\]

Then, based on the facts that

\[
\begin{aligned}
\tilde{n}_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2 &= -\tilde{n}_1 \cdot n_3 + \Omega_{ij}^{(0)} n_{1j} n_{3j} = -N_1 \cdot n_3, \\
A_0 \cdot s_3 &= A_{ij}^{(0)} n_{1i} n_{2j}, \quad A_0 \cdot s_4 = A_{ij}^{(0)} n_{1i} n_{3j},
\end{aligned}
\]

the equations (5.7) can be rewritten as

\[
\begin{aligned}
n_{3j} \left( h_{1j} - \chi_2 N_{1j} - \eta_2 A_{ij}^{(0)} n_{1i} \right) &= 0, \\
n_{2j} \left( h_{1j} - \chi_2 N_{1j} - \eta_2 A_{ij}^{(0)} n_{1i} \right) &= 0,
\end{aligned}
\]

which further implies

\[n_1 \times (h_1 - \chi_2 N_1 - \eta_2 A_0 n_1) = 0.
\]

\[\text{(5.8)}
\]

It remains to reduce the equation (4.80) to the uniaxial case. It follows from the derivations above about the blocks in \(M\) and \(V\) that

\[\beta_0 = 0, \quad \eta_1 = 0, \quad \chi_1 = 0,
\]

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\[\chi_2 = \chi_1, \quad \eta_2 = -\eta_3, \quad \beta_3 = \beta_4, \quad \beta_5 = 4\beta_2,\]

and the coefficients \(\beta_1, \beta_2, \beta_3\) are given by

\[
\beta_1 = V_{01}^T M_{01}^{-1} V_{01}, \quad \beta_2 = V_{02}^T M_{02}^{-1} V_{02}, \quad \beta_3 = V_1^T M_1^{-1} V_1.
\]

To shorten the notations, we define five tensors as follows:

\[
\begin{cases}
G_1 = (A_0 \cdot s_2)s_2 + 4(A_0 \cdot s_5)s_5,
G_2 = (A_0 \cdot s_3)s_3 + (A_0 \cdot s_4)s_4,
G_3 = (A_0 \cdot s_1)a_1 - (A_0 \cdot s_4)a_2,
G_4 = (n_2 \cdot N_1)s_3 + (n_3 \cdot N_1)s_4,
G_5 = (n_2 \cdot N_1)a_1 - (n_3 \cdot N_1)a_2.
\end{cases}
\]

Consequently, the viscosity stress \(\sigma_v^{(0)}\) and the elastic stress \(\sigma_e^{(0)}\) can be reduced to, respectively,

\[
\sigma_v^{(0)} = \left(\frac{3}{2} \eta + \vartheta_{11}\right) (A_0 \cdot s_1)s_1 + \left(\frac{1}{2} \eta + \vartheta_{22}\right) G_1 + \left(2 \eta + \vartheta_{33}\right) G_2,
\]

\[
\frac{1}{\varepsilon k_BT} \sigma_e^{(0)} = \beta_1 (A_0 \cdot s_1)s_1 + \beta_2 G_1 + \beta_3 G_2 - \eta_2 G_4 - \frac{1}{2} \eta_2 G_3 - \frac{1}{2} \chi_2 G_5.
\]

A direct calculation shows that

\[
\begin{aligned}
& (n_2^2 - n_3^2) \otimes (n_2^2 - n_3^2) + 4n_2n_3 \otimes n_2n_3 \\
& = (n_2^2 + n_3^2) \otimes (n_2^2 + n_3^2) - 2n_2^2 \otimes n_2^2 - 2n_3^2 \otimes n_3^2 + n_2 \otimes n_3 \otimes n_2 \otimes n_3 \otimes n_2 \otimes n_3 \\
& = (i - n_1^2) \otimes (i - n_1^2) - (n_2 n_3 - n_3 n_2) (n_2 n_3 - n_3 n_2) \\
& = (i - n_1^2) \otimes (i - n_1^2) - \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \\
& = 2 \delta_{ij} (n_1^2)_{kj} + 2 \delta_{kl} (n_1^2)_{ij} - \delta_{ik} (n_1^2)_{jl} - \delta_{il} (n_1^2)_{jk}.
\end{aligned}
\]

From this, \(G_1\) can be expressed as

\[
G_1 = (A_0 \cdot n_1^2)(n_1^2 - i) + 2A_0 - 2(A_{ik}^{(0)} n_{1k} n_{1j} + n_{1i} n_{1k} A_{kj}^{(0)}).
\]

For the tensor \(G_2\), it follows that

\[
G_2 = \frac{1}{4} A_{ij}^{(0)} (n_{1i} n_{2j} + n_{2i} n_{1j}) (n_{1k} n_{2l} + n_{2k} n_{1l}) + \frac{1}{4} A_{ij}^{(0)} (n_{1i} n_{3j} + n_{3i} n_{1j}) (n_{1k} n_{3l} + n_{3k} n_{1l})
\]

\[
= \frac{1}{4} \left[ A_{ij}^{(0)} (n_{1i} n_{2j} n_{1k} n_{2l} + n_{2i} n_{1j} n_{1k} n_{2l} + n_{1i} n_{2j} n_{2k} n_{1l} + n_{2i} n_{1j} n_{2k} n_{1l})
\right.
\]

\[
+ A_{ij}^{(0)} (n_{1i} n_{3j} n_{1k} n_{3l} + n_{3i} n_{1j} n_{1k} n_{3l} + n_{1i} n_{3j} n_{3k} n_{1l} + n_{3i} n_{1j} n_{3k} n_{1l}))
\]

\[
= \frac{1}{4} \left[ A_{ij}^{(0)} \left( (\delta_{jl} - \delta_{il}) n_{1i} n_{1k} + (\delta_{il} - \delta_{ij}) n_{1j} n_{1k} \right)
\right.
\]

\[
+ A_{ij}^{(0)} \left( (\delta_{jk} - \delta_{il}) n_{1i} n_{1l} + (\delta_{il} - \delta_{ij}) n_{1j} n_{1l} \right)
\]

\[
= \frac{1}{2} (A_{ik}^{(0)} n_{1k} n_{1j} + n_{1i} n_{1k} A_{kj}^{(0)}) - (A_0 \cdot n_1^2) n_1^2.
\]
Using the relation
\[
\left( (n_1 \otimes n_2 - n_2 \otimes n_1) \otimes n_1 n_2 + (n_1 \otimes n_3 - n_3 \otimes n_1) \otimes n_1 n_3 \right)_{ijkl}
= \frac{1}{2} \left[ \left( n_1^2 - \frac{i}{3} \right) \delta_{jl} - \left( n_1^2 - \frac{i}{3} \right) \delta_{il} + \left( n_1^2 - \frac{i}{3} \right) \delta_{jk} - \left( n_1^2 - \frac{i}{3} \right) \delta_{ik} \right],
\]
we obtain
\[
G_4 = A_{kl}^{(0)} \left( (n_1 \otimes n_2 - n_2 \otimes n_1) \otimes n_1 n_2 + (n_1 \otimes n_3 - n_3 \otimes n_1) \otimes n_1 n_3 \right)_{ijkl}
= \frac{1}{2} \left( (n_1^2)_{ki} A_{ki} - \frac{1}{3} A_{ij} - (n_1^2)_{kj} A_{ki} + \frac{1}{3} A_{ij} \right)
+ (n_1^3)_{ij} A_{il} - \frac{1}{3} A_{ij} - (n_1^2)_{ij} A_{il} + \frac{1}{3} A_{ij} \right)
= (n_1^2)_{ki} A_{kj} - (n_1^2)_{kj} A_{ki},
\]
(5.14)
In addition, by virtue of the relation \(i = n_1^2 + n_2^2 + n_3^2\), \(G_4\) and \(G_5\) can be calculated as, respectively,
\[
G_4 = (n_2 \cdot N_1) n_1 n_2 + (n_3 \cdot N_1) n_1 n_3
= \frac{1}{2} n_2 N_{1i} (n_1 n_2 k + n_2 n_1 k) + \frac{1}{2} n_3 N_{1i} (n_1 n_3 k + n_3 n_1 k)
= \frac{1}{2} (\delta_{ik} - n_1 n_1 k) N_{1i} n_1 j + \frac{1}{2} (\delta_{ij} - n_1 n_1 k) N_{1i} n_1 j
= \frac{1}{2} (n_1 \otimes N_1 + N_1 \otimes n_1),
\]
(5.15)
\[
G_5 = (n_2 \cdot N_1) (n_1 \otimes n_2 - n_2 \otimes n_1) + (n_3 \cdot N_1) (n_1 \otimes n_3 - n_3 \otimes n_1)
= n_2 N_{1i} n_1 n_2 k + n_3 N_{1i} n_1 n_3 k - (n_2 N_{1i} n_2 n_1 k + n_3 N_{1i} n_3 n_1 k)
= (\delta_{ik} - n_1 n_1 k) N_{1i} n_1 j - (\delta_{ij} - n_1 n_1 k) N_{1i} n_1 k
= n_1 \otimes N_1 - N_1 \otimes n_1.
\]
(5.16)
Hence, from (5.12) and (5.13), the viscous stress \(\sigma_v^{(0)}\) can be expressed by
\[
\begin{align*}
\sigma_v^{(0)} &= (\vartheta_{11} + \vartheta_{22} - \vartheta_{33})(A_0 \cdot n_1^2) n_1^2 + (\eta + 2\vartheta_{22}) A_0 \\nonumber
&\quad + \left( \frac{1}{2} \vartheta_{33} - 2\vartheta_{22} \right) (A_{ik}^{(0)} n_{1k} n_{1j} + n_{1i} n_{1k} A_{kj}^{(0)}),
\end{align*}
\]
(5.17)
where we have neglected the term \((A_0 \cdot n_1^2)i\), since it can be absorbed into the pressure. Furthermore, from (5.12)-(5.16), the elastic stress \(\sigma_e^{(0)}\) can be expressed by
\[
\begin{align*}
\frac{1}{c k_B T} \sigma_e^{(0)} &= (\beta_1 + \beta_2 - \beta_3)(A_0 \cdot n_1^2) n_1^2 + 2\beta_2 A_0 \\nonumber
&\quad + \left( \frac{1}{2} \beta_3 - 2\beta_2 \right) (A_{ik}^{(0)} n_{1k} n_{1j} + n_{1i} n_{1k} A_{kj}^{(0)}) \nonumber
&\quad - \frac{1}{2} \beta_2 (n_{1i} n_{1k} A_{kj}^{(0)} - A_{ik}^{(0)} n_{1k} n_{1j}) \nonumber
&\quad - \frac{1}{2} \beta_2 (n_1 \otimes N_1 + N_1 \otimes n_1) - \frac{1}{2} \chi_2 (n_1 \otimes N_1 - N_1 \otimes n_1).
\end{align*}
\]
(5.18)
Using \(\frac{\delta F_{U_n}^i}{\delta n_1} = 0\), the body force can be simplified as
\[
ck_B T \hat{\mathbf{F}} = \mathcal{L}_3 F_{B_1} \mathcal{P}_1 n_1 \cdot n_2 + \mathcal{L}_2 F_{B_1} \mathcal{P}_1 n_3 \cdot n_1 + \mathcal{L}_1 F_{B_1} \mathcal{P}_1 n_2 \cdot n_3
= \partial_i n_{1k} n_{3k} \left( n_3 \cdot \frac{\delta F_{U_n}}{\delta n_1} \right) + n_{2k} \partial_i n_{1k} \left( n_2 \cdot \frac{\delta F_{U_n}}{\delta n_1} \right)
\]
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where \( \sigma_{ij} \) satisfy the following relations:

\[
\text{energy dissipation:} \quad \frac{\partial \sigma_{ij}}{\partial t} = \frac{\partial}{\partial n_{ik}} \left( \delta_{ik} \sigma_{ij} \right)
\]

\[
= \partial_{1} n_{ik} \frac{\delta F_{\nu n}}{\delta n_{1a}} \left( \delta_{k\alpha} - n_{1k} n_{1\alpha} \right)
\]

\[
= \partial_{1} n_{ik} \frac{\delta F_{\nu n}}{\delta n_{1a}} = \partial_{j} \sigma^{E}_{ij},
\]  

(5.19)

Therefore, from Proposition 2.2 in [40], it can be seen that the first three terms in (5.22) are negative semi-definite similar to that of Lemma 3.5 in [42].

Therefore, from (5.17), (5.18) and (5.19), the equation of the fluid velocity \( \mathbf{v}^{(0)} \) for the uniaxial case is given by

\[
\rho_{s} \left( \frac{\partial \mathbf{v}^{(0)}}{\partial t} + \mathbf{v}^{(0)} \cdot \nabla \mathbf{v}^{(0)} \right) = -\partial_{1} p^{(0)} + \partial_{j} (\sigma^{L}_{ij} + \sigma^{E}_{ij}).
\]  

(5.20)

Here, the Leslie stress \( \sigma^{L} \) is written as

\[
\sigma^{L} = \sigma_{\nu}^{(0)} + \sigma_{e}^{(0)}
\]

\[
= \alpha_{1} (\mathbf{A}_{0} \cdot \mathbf{n}^{2}) \mathbf{n}^{2} + \alpha_{2} \mathbf{n} \otimes \mathbf{N}_{1} + \alpha_{3} \mathbf{N} \otimes \mathbf{n} + \alpha_{4} \mathbf{A}_{0} + \alpha_{5} n_{1i} n_{1k} A_{ij}^{(0)} + \alpha_{6} A_{ik}^{(0)} n_{1k} n_{1j}.
\]  

(5.21)

where the coefficients \( \alpha_{i} (i = 1, \cdots, 6) \) are given by

\[
\alpha_{1} = \vartheta_{11} + \vartheta_{22} - \vartheta_{33} + ck_{B} T(\beta_{1} + \beta_{2} - \beta_{3}),
\]

\[
\alpha_{2} = -\frac{1}{2} ck_{B} T(\chi_{2} + \eta_{2}), \quad \alpha_{3} = \frac{1}{2} ck_{B} T(\chi_{2} - \eta_{2}),
\]

\[
\alpha_{4} = \eta + 2 \vartheta_{22} + 2 ck_{B} T \beta_{2},
\]

\[
\alpha_{5} = \frac{1}{2} (\vartheta_{33} + ck_{B} T(\beta_{3} - \eta_{2})) - 2(\vartheta_{22} + ck_{B} T \beta_{2}),
\]

\[
\alpha_{6} = \frac{1}{2} (\vartheta_{33} + ck_{B} T(\beta_{3} + \eta_{2})) - 2(\vartheta_{22} + ck_{B} T \beta_{2}),
\]

which satisfy the following relations:

\[
\alpha_{2} + \alpha_{3} = \alpha_{6} - \alpha_{5},
\]

\[
ck_{B} T \chi_{2} = \alpha_{3} - \alpha_{2}, \quad ck_{B} T \eta_{2} = \alpha_{6} - \alpha_{5}.
\]

The equations (5.8) and (5.20) can be called the Ericksen-Leslie system, which also keep the following energy dissipation:

\[
\frac{d}{dt} \left( \frac{\rho_{s}}{2} \int \| \mathbf{v}^{(0)} \|^{2} dx + \mathcal{F}_{\nu n}(\mathbf{n}_{1}) \right)
\]

\[
= -\int \left( \left( \alpha_{1} + ck_{B} T \frac{\eta_{2}}{\chi_{2}} \right) |\mathbf{A}_{0} \cdot \mathbf{n}^{2}|^{2} + \alpha_{4} |\mathbf{A}_{0}|^{2}
\]

\[
+ \left( \alpha_{5} + \alpha_{6} - ck_{B} T \frac{\eta_{2}}{\chi_{2}} \right) |\mathbf{A}_{0} \mathbf{n} \mathbf{l}|^{2} + ck_{B} T \frac{1}{\chi_{2}} |\mathbf{n} \mathbf{l} \times \mathbf{h}_{1}|^{2} \right) dx.
\]  

(5.22)

We denote

\[
\alpha'_{1} = \alpha_{1} + ck_{B} T \frac{\eta_{2}}{\chi_{2}}, \quad \alpha'_{2} = \alpha_{4}, \quad \alpha'_{3} = \alpha_{5} + \alpha_{6} - ck_{B} T \frac{\eta_{2}}{\chi_{2}}.
\]

It can be seen from Proposition 2.2 in [40] that the first three terms in (5.22) are negative semi-definite if and only if

\[
\alpha'_{2} \geq 0, \quad 2 \alpha'_{2} + \alpha'_{3} \geq 0, \quad \frac{3}{2} \alpha'_{2} + \alpha'_{3} + \alpha'_{1} \geq 0.
\]
From our derivation, it holds
\[
\alpha_1' = \eta + 2\vartheta_{22} + 2ck_BT\beta_2, \\
2\alpha_2' + \alpha_3' = 2\eta + \vartheta_{33} + ck_BT \left( \beta_3 - \frac{\eta^2}{\chi_2} \right), \\
\frac{3}{2}\alpha_2' + \alpha_3' + \alpha_4' = \frac{3}{2}\eta + \vartheta_{11} + ck_BT\beta_1.
\]
They are indeed nonnegative, since we have \( \eta > 0, \vartheta_{11}, \vartheta_{22}, \vartheta_{33} > 0 \) from the positive definiteness of \( P \), and \( \beta_1, \beta_2, \beta_3 > 0, \beta_4\chi_2 - \eta^2 > 0 \) from the positive definiteness of \( M \).

6 Comparison with other models

In previous works, the discussion of biaxial hydrodynamics focused on the dissipation function, i.e. (4.55). If the dissipation function is determined, the hydrodynamics can be established by deriving the forces from it and apply the Newton’s law. For this reason, we compare the dissipation function in this work and those in previous works. Although the dissipation function has different expressions previously, they turn out to be equivalent as claimed in (11). Thus, we choose the expression in (11) for comparison.

We use the expressions of \( \Lambda \) and \( P \) in (4.27) and (4.50), respectively. Then, the dissipation function in (4.55) can be rewritten as (to simplify the presentation, we omit \( ck_BT \) below)
\[
- \left( \beta_1 + \vartheta_{11} + \frac{3}{2}\eta \right) \| A_0 \cdot s_1 \|_{L^2}^2 - 2(\beta_0 + \vartheta_{12}) \int (A_0 \cdot s_1)(A_0 \cdot s_2)dx \\
- \left( \beta_3 + \vartheta_{33} + 2\eta \right) \| A_0 \cdot s_3 \|_{L^2}^2 \\
- \left( \beta_4 + \vartheta_{44} + 2\eta \right) \| A_0 \cdot s_4 \|_{L^2}^2 \\
+ 2\vartheta_3 \int \left( n_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1 \right)(A_0 \cdot s_3)dx \\
+ 2\vartheta_2 \int \left( n_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2 \right)(A_0 \cdot s_4)dx \\
+ 2\vartheta_1 \int \left( n_2 \cdot n_3 - \frac{1}{2} \Omega_0 \cdot a_3 \right)(A_0 \cdot s_5)dx \\
- \chi_1 \| n_2 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_3 \|_{L^2}^2 - \chi_2 \| n_3 \cdot n_1 - \frac{1}{2} \Omega_0 \cdot a_2 \|_{L^2}^2 \\
- \chi_3 \| n_1 \cdot n_2 - \frac{1}{2} \Omega_0 \cdot a_1 \|_{L^2}^2.
\]
It has twelve terms that are exactly those given in (11). While the form is identical, we manage to derive the coefficients from the physical parameters, which is not attained previously.

7 Conclusion

Using the Hilbert expansion, we derive a frame hydrodynamics for the biaxial nematic phase from a molecular-theory-based tensor model. Its coefficients are all expressed as those in the tensor model, and the energy dissipation is maintained. The model is further reduced to the Ericksen–Leslie model if the bulk energy minimum becomes uniaxial.

The key ingredient is to recognize the form of the high-order tensors from the properties of the original entropy or the quasi-entropy. This technique is also applicable to other mesoscopic symmetries. It calls for expressions of tensors under other symmetries (14) (15) (16), which we aim to investigate in future works.
A Symmetric traceless tensors

As we have mentioned, any tensor can be decomposed into symmetric traceless tensors. To carry out calculations of high-order tensors, it is necessary to discuss some fundamental ingredients of symmetric traceless tensors.

For a tensor $U$ expressed in the basis generated by $q = (m_1, m_2, m_3)$, let us denote it as a function of $q$, i.e. $U(q)$, to allow $q$ to vary. For example, let us consider a tensor $U(q) = 3m_1 \otimes m_3 - m_3 \otimes m_2$. For another orthonormal frame $q' = (m_1', m_2', m_3')$, we mean $U(q') = 3m_1' \otimes m_3' - m_3' \otimes m_2'$.

A.1 Basis of symmetric traceless tensors

Any symmetric tensor can generate a symmetric traceless tensor in the form (2.2). To write down a basis of symmetric traceless tensors of certain order, we could choose those generated by monomials. Their expressions are derived previously [43]. Below, we list the third-order and fourth-order tensors that we will make use of.

A basis of third-order symmetric traceless tensors can be given by

$$(m_i^3)_0, (m_i^2m_j)_0, (m_im_j^2)_0, (m_i^3)_0,$$

$$(m_i^2m_j)_0, (m_im_j^2)_0, (m_i^2m_j)_0.$$  

Their expressions are given by

$$(m_1m_2m_3)_0 = m_1m_2m_3,$$

$$(m_i^3)_0 = m_i^3 - \frac{3}{5}m_i,$$

$$(m_i^2m_j)_0 = m_i^2m_j - \frac{1}{5}m_i.$$  

The others can be written down by changing the indices. A basis of fourth-order symmetric traceless tensors can be given by

$$(m_i^4)_0, (m_i^3m_j)_0, (m_i^2m_j^2)_0, (m_im_j^3)_0, (m_i^4)_0,$$

$$(m_i^3m_j)_0, (m_i^2m_j^3)_0, (m_i^3m_j)_0, (m_i^2m_j^3)_0.$$  

Their expressions are given by

$$(m_1^4)_0 = m_1^4 - \frac{6}{7}m_1^2 + \frac{3}{35}m_1^2,$$

$$(m_1^3m_2)_0 = m_1^3m_2 - \frac{3}{7}m_1m_2,$$

$$(m_1^2m_2)_0 = m_1^2m_2 - \frac{1}{7}(m_1^3 + m_2^3) + \frac{1}{35}m_1m_2,$$

$$(m_1^2m_2m_3)_0 = m_1^2m_2m_3 - \frac{1}{7}m_2m_3.$$  

An additional note is that for a monomial with the power of $m_3$ not less than two, we could substitute it by $m_i^2 = i - m_i^2 - m_i^2$ to obtain equations such as (cf. the uniqueness of $W$ in [22])

$$(m_i^2)_0 = (i - m_i^2 - m_i^2)_0 = (-m_i^2 - m_i^2)_0, \quad (m_i^4)_0 = ((m_i^2 + m_i^2)^2)_0.$$  

For our discussion afterwards, we introduce the group $\mathcal{D}_2$ that has four elements,

$$i = \text{diag}(1, 1, 1), \quad b_1 = \text{diag}(1, -1, -1), \quad b_2 = \text{diag}(-1, 1, -1), \quad b_3 = \text{diag}(-1, -1, 1).$$  

The tensor $U$ is called invariant of $\mathcal{D}_2$ if $U(qb_i) = U(q)$ (recall the notation at the beginning of Appendix). All the invariant tensors of the order $n$ form a linear subspace of $n$th-order symmetric
traceless tensors, denoted by $A_{\mathbb{R}^2}^\perp$. According to a short discussion in [38], its orthogonal complement $(A_{\mathbb{R}^2}^\perp)^\perp$ consists of all the nth-order symmetric traceless tensors $U$ such that $U(q) + U(qb_1) + U(qb_2) + U(qb_3) = 0$.

It is evident that $qb_i$ transforms two of $m_1$, $m_2$, $m_3$ to their opposites. From the expressions of symmetric traceless tensors written above, we can easily identify the decomposition $A_{\mathbb{R}^2}^\perp$ and $(A_{\mathbb{R}^2}^\perp)^\perp$. For $n = 1, 2, 3, 4$, they are listed below,

\[
\begin{align*}
A_{\mathbb{R}^2,1} &= \{0\}, & (A_{\mathbb{R}^2,1})^\perp &= \text{span}\{m_1, m_2, m_3\}, \\
A_{\mathbb{R}^2,2} &= \text{span}\{(m_1^2)_0, (m_2^2)_0\}, & (A_{\mathbb{R}^2,2})^\perp &= \text{span}\{m_1m_2, m_1m_3, m_2m_3\}, \\
A_{\mathbb{R}^2,3} &= \text{span}\{m_1m_3, m_2m_3\}, \\
(A_{\mathbb{R}^2,3})^\perp &= \text{span}\{(m_1^3)_0, (m_1^2m_2)_0, (m_1^2m_3)_0, (m_2m_3)_0\}, \\
A_{\mathbb{R}^2,4} &= \text{span}\{(m_1^4)_0, (m_1^3m_2)_0, (m_1^3m_3)_0, (m_2^2m_3)_0, (m_2^2m_3)_0\}. & (A.1)
\end{align*}
\]

Let us write down some equalities to be used later. Define

\[
S_1 = m_1^2 - i/3, \quad S_2 = m_2^2 - m_3^2, \quad S_3 = m_1m_2, \quad S_4 = m_1m_3, \quad S_5 = m_2m_3. \tag{A.2}
\]

For third-order tensors, we have

\[
\begin{align*}
m_1m_2m_3 \cdot m_i \otimes S_j &= 0, & \text{if } (i, j) &\neq \{(1, 5), (2, 4), (3, 3)\}, \\
e^{ils}(m_1m_2m_3)_{jks}(S_{\nu} \otimes S_{\nu'})_{ijkl} &= 0, & \text{if } i &\neq j \text{ and } \{\nu, \nu'\} \neq \{1, 2\}. & (A.3)
\end{align*}
\]

If $U \in (A_{\mathbb{R}^2,3})^\perp$, then

\[
U \cdot m_i \otimes S_j = 0, & \text{if } (i, j) = \{(1, 5), (2, 4), (3, 3)\}, \\
e^{ils}U_{jks}(S_{\nu} \otimes S_{\nu'})_{ijkl} &= \epsilon^{ils}U_{jks}(S_1 \otimes S_2)_{ijkl} = \epsilon^{ils}U_{jks}(S_2 \otimes S_1)_{ijkl} = 0. & (A.4)
\]

For fourth-order tensors, if $U \in A_{\mathbb{R}^2,4}$, then

\[
U \cdot S_i \otimes S_j = 0, & \text{if } i &\neq j \text{ and } \{i, j\} \neq \{1, 2\}. \tag{A.5}
\]

If $U \in (A_{\mathbb{R}^2,4})^\perp$, then

\[
U \cdot S_i \otimes S_j = (m_i^2m_2)_0 \cdot S_i \otimes S_2 = 0. & (A.6)
\]

The equalities (A.3)–(A.6) can be recognized straightforwardly by expanding the tensors into several terms of tensor products of $m_i$. The Levi-Civita symbol can be expanded as

\[
e^{ijk} = (m_1 \otimes m_2 \otimes m_3 + m_2 \otimes m_3 \otimes m_1 + m_3 \otimes m_1 \otimes m_2) \\
- m_1 \otimes m_3 \otimes m_2 - m_3 \otimes m_2 \otimes m_1 - m_2 \otimes m_1 \otimes m_3)_{ijk}.
\]

The equations above hold independent of the orthonormal frame we choose. In particular, they are valid if we substitute $m_i$ with $n_i$ and correspondingly $S_i$ with $s_i$ (recall (1.22)).

### A.2 Expressing tensors by symmetric traceless tensors

There are complicated linear relations between high-order tensors. To figure out the linear relations, we shall express them by symmetric traceless tensors that completely give linearly independent components. These linear relations are inherited by averaged high-order tensors. The special forms of averaged high-order tensors are also revealed in this way.

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To express a general tensor $U$ by symmetric traceless tensors, we first decompose it as $U = U_{\text{sym}} + (U - U_{\text{sym}})$. The anti-symmetric part $U - U_{\text{sym}}$ can be written as the sum of several terms of the form

$$U_{ijkl} - U_{ijlk} - U_{iljk} - U_{ijkl} = \epsilon^{ijkl}W_{kl},$$

where $W$ is an $(n-1)$th-order tensor. Carry out this action repeatedly until we express $U$ by symmetric tensors. Then, each symmetric tensor can be expressed by a symmetric traceless tensor and several symmetric tensors of lower order. Also do it repeatedly to finally express $U$ by symmetric traceless tensors.

In what follows, we use this procedure to deal with the tensors

$$m_1m_2 \otimes m_1m_2, \ m_1m_3 \otimes m_1m_3, \ m_2m_3 \otimes m_2m_3.$$ 

From the expressions of symmetric traceless tensors, we can derive that

$$m_1^4 = (m_1^4)_0 + \frac{6}{7}m_1^3i - \frac{3}{35}i^2 = (m_1^4)_0 + \frac{6}{7}(m_1^2 - \frac{1}{3})i + \left(\frac{2}{7} - \frac{3}{35}\right)i^2,$$

$$m_1^2m_2^2 = (m_1^2m_2^2)_0 + \frac{1}{7}(m_1^2 + m_2^2)i - \frac{1}{35}i^2 = (m_1^2m_2^2)_0 + \frac{1}{7}(\frac{1}{3} - m_3^2)i + \left(\frac{2}{21} - \frac{1}{35}\right)i^2,$$

(A.7)

For a second-order tensor $U$, define

$$A(U)_{ijkl} \overset{\text{def}}{=} \delta_{kl}U_{ij} + \delta_{ij}U_{kl} - \frac{3}{4}(\delta_{ik}U_{jl} + \delta_{jl}U_{ik} + \delta_{il}U_{jk} + \delta_{jk}U_{il}),$$

$$B(U)_{ijkl} \overset{\text{def}}{=} U_{kl}\delta_{ij} - U_{kj}\delta_{il} + U_{li}\delta_{jk} - U_{lj}\delta_{ik}.$$

Using the expressions of symmetric traceless tensors and \(A.7\), it follows that

$$\left(\frac{m_1^2 - i}{3}\right)_i \left(\frac{m_1^2 - i}{3}\right)_j = \left((m_1^4)_0 + \frac{6}{7}(m_1^2)_0i + \frac{1}{5}i^2\right)_{ijkl} - \frac{1}{3}\delta_{ij}(m_1^2)_kl - \frac{1}{3}\delta_{kl}(m_1^2)_ij + \frac{1}{9}\delta_{ij}\delta_{kl},$$

$$B(U)_{ijkl} = \frac{1}{21}A((m_1^2)_0)_{ijkl} - \frac{1}{45}\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}).$$

(A.9)

The symmetric tensor $m_1^2m_2^2$ is expressed by

$$(m_1^2m_2^2)_{ijkl} = \frac{1}{6}\left(m_{ij}m_{kl} - m_{il}m_{jk} + m_{il}m_{jk} - m_{ij}m_{kl}ight).$$

(A.10)

Then we obtain from \(A.10\) that

$$(m_1m_2 \otimes m_1m_2)_{ijkl} - (m_1^2m_2^2)_{ijkl}$$
Similar to the calculation of (A.14), we obtain the tensor in the big parenthesis of (A.11) can be calculated as

\[
\frac{1}{12} \left( 2m_1m_1m_2m_2 + 2m_2m_2m_1m_1 - (m_1m_2 + m_2m_1)(m_1m_2 + m_2m_1) \right).
\]

(A.11)

Using the equality

\[ m_1m_2 - m_2m_1 = \epsilon^{ij} m_{3i}, \]

(A.12)

the tensor in the big parenthesis of (A.11) can be calculated as

\[
\begin{align*}
&m_1m_2(m_1m_2m_2 - m_2m_1m_1) + m_1m_2(m_2m_1m_1 - m_1m_2m_2) \\
&+ m_2m_1(m_2m_1m_2 - m_1m_2m_1) \\
&= \frac{1}{3} \left( \delta_{i\ell} \delta_{j\ell} - \delta_{i\ell} \delta_{j\ell} - 2\delta_{i\ell} \delta_{j\ell} - 2\delta_{i\ell} \delta_{j\ell} - 2\delta_{i\ell} \delta_{j\ell} - \delta_{i\ell} \delta_{j\ell} + \delta_{i\ell} \delta_{j\ell} + \delta_{i\ell} \delta_{j\ell} - \delta_{i\ell} \delta_{j\ell} - 2\delta_{i\ell} \delta_{j\ell} - \delta_{i\ell} \delta_{j\ell} \\
&= (\epsilon^{i\ell}s \epsilon^{j\ell}) (m_{3\ell})_{ij}.
\end{align*}
\]

(A.13)

Thus, combining (A.8), (A.11) and (A.13) yields

\[
(m_1m_2 \otimes m_1m_2)_{ijkl}
= (m_1^2m_2^2)_{ij} + \frac{1}{4} A ((m_3^2)_{0})_{ijkl} - \frac{1}{60} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}).
\]

(A.14)

Similar to the calculation of (A.11), we obtain

\[
\begin{align*}
(m_1m_3 \otimes m_1m_3)_{ijkl}
&= ((m_1^2m_3^2)_{0})_{ijkl} + \frac{1}{4} A ((m_3^2)_{0})_{ijkl} - \frac{1}{60} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}),
\end{align*}
\]

(A.15)

\[
\begin{align*}
(m_2m_3 \otimes m_2m_3)_{ijkl}
&= ((m_2^2m_3^2)_{0})_{ijkl} + \frac{1}{4} A ((m_3^2)_{0})_{ijkl} - \frac{1}{60} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}).
\end{align*}
\]

(A.16)

In the same way, we have

\[
\begin{align*}
((m_2^2 - m_3^2) \otimes (m_2^2 - m_3^2))_{ijkl}
&= -\frac{1}{3} \left( 2m_2m_2m_3m_3m_3 + 2m_3m_3m_2m_2m_2 - (m_2m_3 + m_3m_2)(m_2m_3 + m_3m_2) \right).
\end{align*}
\]

(A.17)

Similar to the calculation of (A.13), we obtain

\[
\begin{align*}
&2m_2m_2m_3m_3m_3 + 2m_3m_3m_2m_2m_2 - (m_2m_3 + m_3m_2)(m_2m_3 + m_3m_2)
\end{align*}
\]

(A.18)

\[
\begin{align*}
&= m_2m_2m_2m_3m_3 - m_3m_2m_3m_2 + m_2m_2m_3m_3 - m_3m_2m_3m_2
\end{align*}
\]

(A.19)

\[
\begin{align*}
&= (\epsilon^{i\ell}s \epsilon^{j\ell}) (m_{3\ell})_{ij} + \frac{1}{3} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk})
\end{align*}
\]

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where we have used the fact that

Furthermore, the asymmetric part of (A.20) can be calculated as follows:

Note that

Then, from (A.17) and (A.18), we deduce

The next task is to calculate the fourth-order tensor \((m_1^2 - m_3^2) \otimes (m_2^2 - m_3^2)\). A direct calculation gives

Furthermore, the asymmetric part of (A.20) can be calculated as follows:

Finally, we have used the fact that

\[(e^{jks} \epsilon_{ilt} + e^{ils} \epsilon_{kjt})(m_3^2)_{it} = \frac{1}{3} (\delta_{kl}(m_1^2)_{ij} - \delta_{ij}(m_1^2)_{kl} + \delta_{ij}(m_2^2)_{kl} - \delta_{kl}(m_2^2)_{ij})\]
Then, using (A.10), (A.7)-(A.8), (A.14), (A.20)-(A.21) and the relation
\[ = \epsilon_{ijkl}(m^2_{ij})_{ikl}, \]
and the relation \( m^2_i + m^2_2 + m^2_4 = i \), we obtain
\[
\left( m^2_1 - \frac{1}{3} \right)_{ij} (m^2_2 - m^2_4)_{kl} = \left( m^2_1 - \frac{1}{3} \right)_{ij} (2m^2_2 + m^2_4 - i)_{kl},
\]
where we have also used the following fact
\[
\frac{1}{3} \left( \delta_{kl}(m^2_i),_{ij} - \delta_{ij}(m^2_i),_{kl} + \delta_{ij}(m^2_i),_{kl} - \delta_{kl}(m^2_i),_{ij} \right) - \delta_{kl}(m^2_1),_{ij} - \frac{2}{3} \delta_{ij}(m^2_2),_{kl} - \frac{1}{3} \delta_{ij}(m^2_1),_{kl} - \frac{6}{7}(m^2_3),_{ijkl} + \frac{2}{5}(i^2),_{ijkl} + \frac{6}{7}(m^2_3),_{ijkl} + \frac{1}{5}(i^2),_{ijkl} - \frac{1}{3} \delta_{ij}\delta_{kl}
\]
\[
= -\frac{1}{3} \left( \delta_{kl}(m^2_i),_{ij} + 2 \delta_{ij}(m^2_i),_{kl} + \delta_{kl}(m^2_i),_{ij} + \delta_{ij}(m^2_i),_{kl} \right) + \frac{12}{7}(m^2_3),_{ijkl} + \frac{6}{7}(m^2_3),_{ijkl} - \frac{1}{15}(2 \delta_{ij}\delta_{kl} - 3 \delta_{ik}\delta_{jl} - 3 \delta_{il}\delta_{jk}).
\]
Note that \( m^2_2 - m^2_4 = 2(m^2_2 + m^2_4) \). It follows that
\[
(m^2_2),_{ij} = \frac{1}{4}(m^2_2 - m^2_4 - (m^2_i),_{ij}) \times (m^2_2 - m^2_4 - (m^2_i),_{ij}) = \frac{1}{4}(m^2_2 - m^2_4) \times (m^2_2 - m^2_4) - (m^2_2 - m^2_4) \times (m^2_i),_{ij}.
\]
Thus, combining (A.23) with (A.9), (A.19) and (A.22), we obtain
\[
\begin{aligned}
(m_2^2 - \frac{1}{3}m_0^2)_{ij} (m_2^2 - \frac{1}{3}m_0^2)_{kl}
 &= \frac{1}{4} \left( \left( (m_2^4 - 2m_2^2m_3 + m_3^2)_{ijkl} - \frac{4}{21}A((m_2^2)_0)_{ijkl} - \frac{1}{10}(2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}) 
 + (m_1^4)_{ij} - \frac{4}{21}A((m_1^2)_0)_{ij} - \frac{8}{21}A((m_1^2)_0)_{ijkl} - \frac{16}{21}A((m_2^2)_0)_{ijkl} \right) 
 - 4((m_1^4)_{ijkl} - 2((m_1^4)_{ijkl} - \frac{16}{21}A((m_1^2)_0)_{ijkl} - (m_0^4)_{ijkl} - \frac{1}{2}A((m_0^2)_0)_{ijkl} - (m_1^2m_3)_0 - (m_2^2)_0)_{ijkl} + \epsilon^kjs((m_1^2m_3)_0 - (m_2^2)_0)_{js} + \epsilon^kjs((m_1^2m_3)_0 - (m_2^2)_0)_{js} = 0.
\end{aligned}
\]

where we have employed the cancellation relation
\[
\epsilon^{iks}(m_1^2m_2m_3)_0 + \epsilon^{iks}(m_1^2m_2m_3)_0 = 0.
\]

Next, we calculate the three tensors
\[
m_1 \otimes m_2m_3, \quad m_2 \otimes m_1m_3, \quad m_3 \otimes m_1m_2.
\]

It turns out that
\[
\frac{1}{2}m_{1i}(m_{2j}m_{3k} + m_{3j}m_{2k})
 = (m_1m_2m_3)_{ijk} + \frac{1}{6}((m_1m_2 - m_2m_1)m_3 + (m_1m_3 - m_3m_1)m_2 + 2(m_1m_2 - m_2m_1)m_3 + (m_1m_3 - m_3m_1)m_2) m_{1i}
 = (m_1m_2m_3)_{ijk} + \frac{1}{6}\left( \epsilon^{iks}(m_1^2)_0 - (m_2^2)_0 \right)_{k} + \epsilon^{iks}(m_1^2m_3)_0 - (m_2^2)_0)_{js},
\]
\[
\frac{1}{2}m_{2i}(m_{1j}m_{3k} + m_{3j}m_{1k})
 = (m_1m_2m_3)_{ijk} + \frac{1}{6}\left( \epsilon^{iks}(m_1^2)_0 - (m_2^2)_0 \right)_{k} + \epsilon^{iks}(m_1^2m_3)_0 - (m_2^2)_0)_{js},
\]
\[
\frac{1}{2}m_{3i}(m_{1j}m_{2k} + m_{2j}m_{1k})
 = (m_1m_2m_3)_{ijk} + \frac{1}{6}\left( \epsilon^{iks}(m_1^2)_0 - (m_2^2)_0 \right)_{k} + \epsilon^{iks}(m_1^2m_3)_0 - (m_2^2)_0)_{js}.
\]

The equations (A.25)–(A.27) also hold if we replace \( m_i \) with \( n_i \), which we also need to use later.

To deal with \( Y^{(0)} \) in (4.16), we need to calculate
\[
(m_1 \otimes m_2) \otimes m_1m_2, \quad (m_2 \otimes m_1) \otimes m_1m_2,
(m_1 \otimes m_3) \otimes m_1m_3, \quad (m_2 \otimes m_3) \otimes m_2m_3.
\]

Using the definition of \( R^{(0)}_{i}(i = 3, 4, 5) \), it follows that
\[
2\langle m_{1i}m_{2j}(m_1m_2)_{kl} = \frac{1}{2}R^{(0)}_{3} + \langle (m_{1i}m_{2j} - m_{2i}m_{1j})(m_1m_2)_{kl} >, \quad 2\langle m_{2i}m_{1j}(m_1m_2)_{kl} = \frac{1}{2}R^{(0)}_{3} - \langle (m_{1i}m_{2j} - m_{2i}m_{1j})(m_1m_2)_{kl} >,
\]
2\langle m_{1j}m_{3j}(m_1m_3)_{kl} \rangle = \frac{1}{2} R_4^{(0)} + \langle (m_{1j}m_{3j} - m_{3i}m_{1j})(m_1m_3)_{kl} \rangle,
2\langle m_{2i}m_{3j}(m_2m_3)_{kl} \rangle = \frac{1}{2} R_5^{(0)} + \langle (m_{2i}m_{3j} - m_{3i}m_{2j})(m_2m_3)_{kl} \rangle.

In order to calculate the above tensor moments, it is desirable to utilize the following relation:
\[\epsilon^{ijk} \epsilon^{i'st} = \delta_{js} \delta_{kt} - \delta_{jt} \delta_{ks}.\] (A.28)

Then, using (A.12) and (A.28), we derive that
\[
\langle m_{1i}m_{2j} - m_{2i}m_{1j} \rangle (m_1m_2)_{kl}
= \frac{1}{2} \epsilon^{ij}m_{3s}(m_{1k}m_{2l} + m_{2k}m_{1l})
= \epsilon^{ij} \left( m_{1j}m_{3s} + \frac{1}{6} m_{1k}(m_{2i}m_{3s} - m_{3i}m_{2s}) + \frac{1}{6} m_{2l}(m_{1k}m_{3s} - m_{3k}m_{1s})
+ \frac{1}{6} m_{2k}(m_{1i}m_{3s} - m_{3i}m_{1s}) + \frac{1}{6} m_{1l}(m_{2k}m_{3s} - m_{3k}m_{2s}) \right)
= \epsilon^{ij} \left( m_{1j}m_{3s} + \frac{1}{6} m_{1k}(m_{2i}m_{3s} - m_{3i}m_{2s}) + \frac{1}{6} m_{2l}(m_{1k}m_{3s} - m_{3k}m_{1s})
+ \frac{1}{6} m_{2k}(m_{1i}m_{3s} - m_{3i}m_{1s}) + \frac{1}{6} m_{1l}(m_{2k}m_{3s} - m_{3k}m_{2s}) \right)
= \epsilon^{ij} \left( m_{1j}m_{3s} + \frac{1}{6} \left( \mathcal{B}(m_2^3)_{ijkt} + \mathcal{B}(m_3^3)_{ijkl} \right) \right).
\] (A.29)

In the above, we have encountered several symmetric traceless tensors. They have the following relations.
\[
\langle m_3^2 \rangle_0 = -\langle m_2^2 \rangle_0 - \langle m_1^2 \rangle_0,
\langle m_1^2 m_3^0 \rangle_0 = -\langle m_1^2 \rangle_0 - \langle m_1^2 m_3^0 \rangle_0,
\langle m_2^2 m_3^0 \rangle_0 = -\langle m_2^2 \rangle_0 - \langle m_1^2 m_3^0 \rangle_0.
\]

When averaged tensors are considered, the linear relations obtained above still hold. Therefore, we only need to focus on the following tensors that are the linearly independent:
\[\langle (m_1^2 m_2^3) \rangle_0, \langle (m_1^2 m_3^2) \rangle_0, \langle (m_2^2 m_3^2) \rangle_0, \langle (m_3^2) \rangle_0, i = 1, 2, 3.\]

A.3 Expression involving low-order tensors

When \( Q_i = Q_i^{(0)} \), we will encounter a few terms only involving second-order tensors, which we provide alternative expressions below. They will be useful for matrix manipulations in the main text, and the discussion afterwards.

Let us look into the last tensor in (C.2). Using the relation \( i = n_1^2 + n_2^2 + n_3^2 \), we deduce that
\[
2\delta_{ij} \delta_{kl} - 3 \delta_{ik} \delta_{jl} - 3 \delta_{il} \delta_{jk}
= 2(n_1^2 + n_2^2 + n_3^2)_{ik}(n_1^2 + n_2^2 + n_3^2)_{jl} - 3(n_1^2 + n_2^2 + n_3^2)_{ik}(n_1^2 + n_2^2 + n_3^2)_{jl}
- 3(n_1^2 + n_2^2 + n_3^2)_{il}(n_1^2 + n_2^2 + n_3^2)_{jk}
= 2 \sum_{\alpha \neq \beta} n_\alpha^2 \otimes n_\beta^2 - 4 \sum_{\alpha = 1}^3 n_\alpha^4
- 12(n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_2 n_3 + n_2 n_3 \otimes n_2 n_3),
\]
where we have used the fact that \( \mathbf{n}_1 \mathbf{n}_2 = \frac{1}{4}(\mathbf{n}_1 \otimes \mathbf{n}_2 + \mathbf{n}_2 \otimes \mathbf{n}_1) \). The terms in the second line are expressed linearly by \( s_i \otimes s_j \). We would also like to express the first line in this form. Note that

\[
\mathbf{n}_1 \mathbf{n}_2 - \frac{i}{3} = \frac{1}{3}(2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2).
\] (A.30)

Thus, fitting with a term \( \mathbf{n}^2 - i/3 \) in (A.30) yields

\[
-9s_1 \otimes s_1 = -9\left(\mathbf{n}_1^2 - \frac{i}{3}\right) \otimes \left(\mathbf{n}_2^2 - \frac{i}{3}\right)
= -(2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2)
= -4\mathbf{n}_1^4 + 2(\mathbf{n}_2^2 \otimes \mathbf{n}_3^2 + \mathbf{n}_3^2 \otimes \mathbf{n}_2^2) + (\mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 + \mathbf{n}_3^2).
\] (A.31)

Then the remaining terms are given by

\[
-4\mathbf{n}_1^4 - 4\mathbf{n}_2^4 + 2(\mathbf{n}_2^2 \otimes \mathbf{n}_3^2 + \mathbf{n}_3^2 \otimes \mathbf{n}_2^2) + (\mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 + \mathbf{n}_3^2)
= -3(\mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 - \mathbf{n}_3^2) = -3s_2 \otimes s_2.
\]

Therefore, we arrive at

\[
2\delta_{ij} \delta_{kl} - 3\delta_{ik} \delta_{jl} - 3\delta_{il} \delta_{jk} = -9s_1 \otimes s_1 - 3s_2 \otimes s_2 - 12(s_3 \otimes s_3 + s_4 \otimes s_4 + s_5 \otimes s_5),
\] (A.32)

where the corresponding coordinate \( X_1 \) is given by (1.41).

We note that

\[
s_2 \otimes s_2 = (\mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 - \mathbf{n}_3^2)
= \mathbf{n}_3^2 + \mathbf{n}_1^2 - (\mathbf{n}_2^2 \otimes \mathbf{n}_3^2 + \mathbf{n}_3^2 \otimes \mathbf{n}_2^2).
\] (A.33)

Then \( \mathcal{A}((\mathbf{n}_i^2)_{ij})_{ijkl} \) can be calculated as follows:

\[
\mathcal{A}((\mathbf{n}_i^2)_{ij})_{ijkl} = (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)_{ijkl} \left(\mathbf{n}_1^2 - \frac{i}{3}\right)_{ij} + (\mathbf{n}_2^2 + \mathbf{n}_3^2)_{ijkl} \left(\mathbf{n}_2^2 - \frac{i}{3}\right)_{kl}
- \frac{3}{4}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)_{ik} \left(\mathbf{n}_1^2 - \frac{i}{3}\right)_{jl} - \frac{3}{4}(\mathbf{n}_2^2 + \mathbf{n}_3^2)_{jl} \left(\mathbf{n}_2^2 - \frac{i}{3}\right)_{ik}
- \frac{3}{4}(\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2)_{jl} \left(\mathbf{n}_1^2 - \frac{i}{3}\right)_{jk} - \frac{3}{4}(\mathbf{n}_2^2 + \mathbf{n}_3^2)_{jk} \left(\mathbf{n}_2^2 - \frac{i}{3}\right)_{lj}
= \frac{1}{3} \left((2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2) + (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes (2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2)\right)
- (2\mathbf{n}_1^4 - \mathbf{n}_2^4 - \mathbf{n}_3^4) - (\mathbf{n}_1 \mathbf{n}_2) \otimes (\mathbf{n}_1 \mathbf{n}_2 + \mathbf{n}_1 \mathbf{n}_3 \otimes \mathbf{n}_1 \mathbf{n}_3 - \mathbf{n}_2 \mathbf{n}_3 \otimes \mathbf{n}_2 \mathbf{n}_3),
\]

where we have used the relation \( i = \mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2 \). Using (A.31) and (A.32), we get

\[
\frac{1}{3} \left((2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2) + (\mathbf{n}_1^2 + \mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes (2\mathbf{n}_1^2 - \mathbf{n}_2^2 - \mathbf{n}_3^2)\right)
- (2\mathbf{n}_1^4 - \mathbf{n}_2^4 - \mathbf{n}_3^4)
= \mathbf{n}_3^2 + \mathbf{n}_1^2 + \frac{1}{3} \left(-2\mathbf{n}_1^4 + \mathbf{n}_2^4 + \mathbf{n}_3^4 + (\mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes \mathbf{n}_1^2 - 2(\mathbf{n}_2^2 + \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 + \mathbf{n}_3^2)\right)
= -\frac{3}{2}(\mathbf{n}_1^2 - \frac{i}{3}) \otimes (\mathbf{n}_2^2 - \frac{i}{3}) + \frac{1}{2}(\mathbf{n}_2^2 - \mathbf{n}_3^2) \otimes (\mathbf{n}_2^2 - \mathbf{n}_3^2)
= -\frac{3}{2}s_1 \otimes s_1 + \frac{1}{2}s_2 \otimes s_2.
\]

Consequently, we obtain

\[
\mathcal{A}((\mathbf{n}_i^2)_{ij})_{ijkl} = \frac{3}{2}s_1 \otimes s_1 + \frac{1}{2}s_2 \otimes s_2 - (s_3 \otimes s_3 + s_4 \otimes s_4 - 2s_5 \otimes s_5),
\] (A.34)
where the corresponding coordinate $X_2$ is written by $\mathbf{141}$.

Next we deal with the term $\mathcal{A}(n_2^2 - n_3^2)_{ijkl}$. Note that

$$
\frac{3}{2}(s_1 \otimes s_2 + s_2 \otimes s_1) = \frac{3}{2}\left((n_1^2 - \frac{i}{3}) \otimes (n_2^2 - n_3^2) + (n_2^2 - n_3^2) \otimes (n_1^2 - \frac{i}{3})\right)
= -(n_2^2 - n_3^2) + (n_1^2 \otimes n_2^2 + n_2^2 \otimes n_1^2) - (n_1^2 \otimes n_3^2 + n_3^2 \otimes n_1^2).
$$

Then, $\mathcal{A}(n_2^2 - n_3^2)_{ijkl}$ can be calculated as

$$
\mathcal{A}(n_2^2 - n_3^2)_{ijkl} = (n_1^2 + n_2^2 + n_3^2)_{kl}((n_2^2 - n_3^2)_{ij} + (n_1^2 + n_2^2 + n_3^2)_{ij}(n_2^2 - n_3^2)_{kl}
- \frac{3}{4}(n_1^2 + n_2^2 + n_3^2)_{ik}(n_2^2 - n_3^2)_{jl} - \frac{3}{4}(n_1^2 + n_2^2 + n_3^2)_{jl}(n_2^2 - n_3^2)_{ik}
- \frac{3}{4}(n_1^2 + n_2^2 + n_3^2)_{il}(n_2^2 - n_3^2)_{jk} - \frac{3}{4}(n_1^2 + n_2^2 + n_3^2)_{jk}(n_2^2 - n_3^2)_{il}
= 2(n_2^4 - n_3^4) + (n_1^2 \otimes n_2^2 + n_2^2 \otimes n_1^2) - (n_1^2 \otimes n_3^2 + n_3^2 \otimes n_1^2)
- 3(n_1^4 - n_3^4 + n_1 n_2 \otimes n_1 n_2 - n_1 n_3 \otimes n_1 n_3)
= \frac{3}{2}\left((n_1^2 - \frac{i}{3}) \otimes (n_2^2 - n_3^2) + (n_2^2 - n_3^2) \otimes (n_1^2 - \frac{i}{3})\right)
- 3(n_1 n_2 \otimes n_1 n_2 - n_1 n_3 \otimes n_1 n_3)
= \frac{3}{2}(s_1 \otimes s_2 + s_2 \otimes s_1) - 3(s_1 \otimes s_3 - s_3 \otimes s_1),
$$

(A.35)

where the corresponding coordinate $X_3$ is written by $\mathbf{141}$.

## B Closure approximation: Theorem 3.3

We discuss Theorem 3.3 that recognizes the form of high-order tensors. Theorem 3.3 is stated for the original entropy and the quasi-entropy. So we need to consider them separately.

Theorem 3.3 is actually a special case in previous works: for the original entropy, it is a special case of Theorem 5.2 in $\mathbf{38}$; for quasi-entropy, it is a special case of Theorem 4.8 in $\mathbf{44}$ Nevertheless, both of them were shown for general cases of symmetry and the explicit form (3.22) is not provided. For this reason, we shall explain how those theorems are applied to the current work to obtain Theorem 3.3 and at places show some results.

### B.1 Original entropy

We first discuss the closure by the original entropy. The following result has been shown in Appendix in $\mathbf{47}$.

**Lemma B.1.** If $s_i, b_i$ satisfy (3.20), then there exists a unique density function

$$
\rho(q) = \frac{1}{Z} \exp\left(\sum_{i,j=1,2} \lambda_{ij}(\mathbf{m}_i \cdot \mathbf{n}_j)^2\right),
$$

such that $\langle (m_i^2) \rangle = s_i(n_i^2) + b_i(n_2^2 - n_3^2)$. It minimizes $\int_{SO(3)} \rho \ln \rho \, dq$ when $Q_i$ is fixed.

Recall that $q = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$ and $p = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$. The density function satisfies $\rho(p b_k p^T q) = \rho(q)$ for $k = 1, 2, 3$. This can be seen by noticing that $\mathbf{m}_i \cdot \mathbf{n}_j$ is the $(j, i)$ element of $p^T q$. Thus, when $q$ is replaced by $pb_k p^T q$, the dot product $\mathbf{m}_i \cdot n_j$ becomes the $(j, i)$ element of $p^T (pb_k p^T q) = b_k p^T q = (pb_k^T)^T q$. It suffices to notice the equalities like $pb_1^T = (n_1, -n_2, -n_3)$.

By Theorem 5.2 in $\mathbf{38}$ and the related discussions before the theorem, when an nth-order symmetric traceless tensor is calculated from the density function above, it could be expressed as $W(p)$ for some
$W \in \mathbb{A}^{2,n}$. Using the decomposition written down in (A.1), we arrive at the expression in Theorem 4.3. The positive-definiteness of the averaged tensors $R_i$ in (4.1) is obvious because they are calculated from a positive density function.

**B.2 Quasi-entropy**

To illustrate some ideas, let us start from the second-order quasi-entropy $\Xi_2$. Denote by $r_1$ the vector formed by 1 and $m_i \cdot n_j$, $1 \leq i, j \leq 3$, which is a $10 \times 1$ vector. For a first-order tensor $U$, we define a row vector as

$$\Phi(U)_j = (U \cdot n_j).$$

For a second-order tensor $U$, we define a matrix as

$$\Psi(U)_{ij} = (U \cdot n_i \otimes n_j).$$

The general second-order quasi-entropy, denoted by $\Xi_2$, is defined as the minus log-determinant of the second moment of $r_1$ (hereafter we omit the free parameter $\nu$ introduced in (3.17)),

$$\Xi_2 = -\ln \det (r_1 r_1^T) = -\ln \det \begin{pmatrix}
1 & \Phi_1(m_1) & \Phi_1(m_2) & \Phi_1(m_3) \\
\Phi_1(m_1)^T & \Psi_2(m_1^2) & \Psi_2(m_2^2) & \Psi_2(m_3^2) \\
\Phi_1(m_2)^T & \Psi_2(m_1 \otimes m_2) & \Psi_2(m_2 \otimes m_3) \\
\Phi_1(m_3)^T & \Psi_2(m_3 \otimes m_1) & \Psi_2(m_3 \otimes m_2)
\end{pmatrix}.$$  \hspace{1cm} \text{(B.2)}

In (4.1), the second moment of $r_1$ is replaced by the covariance matrix of the $9 \times 1$ vector formed by the last nine components of $r_1$. It can be seen that these two formulations are equivalent.

Here, we need to emphasize that the notation $\langle \cdot \rangle$ does not assume that they are averaged by certain positive density function, but only implies that the tensors obey linear relations such as what we have obtained in Appendix A.2. For second-order tensors not symmetric, we express them using symmetric traceless tensors, such as $\langle m_1 \otimes m_2 \rangle_{ij} = \langle m_1 m_2 \rangle_{ij} + \epsilon^{ijk} \langle m_3 \rangle_k$. Thus, $\Xi_2$ is a function of symmetric traceless tensors up to second order. If we choose a basis of symmetric traceless tensors, their ‘average’ are independent variables in $\Xi_2$.

Now, for our problem, the tensors specified are $Q_1$ and $Q_2$, which determine $\langle m_1^2 \rangle = Q_1 + i/3$, $\langle m_2^2 \rangle = Q_2 + i/3$, $\langle m_3^2 \rangle = -Q_1 - Q_2 + i/3$. To obtain the quasi-entropy $\Xi_2$ about $Q_1$ and $Q_2$ only, we shall minimize $\Xi_2$ with $Q_1$ and $Q_2$ fixed. At the minimizer many tensors vanish, because we have the following lemma.

**Lemma B.2.** For a symmetric positive-definite matrix $K$, suppose that it is given in blocks as

$$K = \begin{pmatrix}
K_1 & A \\
A^T & K_2
\end{pmatrix}.$$ \hspace{1cm} \text{(B.3)}

Then we have

$$\det K \leq \det K_1 \det K_2.$$ \hspace{1cm} \text{(B.4)}

The equality holds if and only if $A = 0$.

Notice that off-diagonal blocks are functions of $\langle m_i \rangle$ and $\langle m_i m_j \rangle$ for $i \neq j$, which are independent of $Q_1$ and $Q_2$. Using this lemma, we immediately deduce that the minimizer is attained when all the off-diagonal blocks are zero. In this way, we obtain the quasi-entropy $\Xi_2$ in (3.17).

It is worthy noting that for $Q_1$ and $Q_2$, $m_1^2 - i/3$ and $m_2^2 - i/3$ are invariant under $\mathcal{D}_2$. On the other hand, the off-diagonal blocks vanish when averaged over $\mathcal{D}_2$, since in these blocks the times of
The ideas above are also useful when discussing the fourth-order quasi-entropy \( \Xi_4 \). Denote by \( r_2 \) the vector formed by \( 1, m_i \cdot n_j, 1 \leq i, j \leq 3 \) and \( s_i \cdot s_j, 1 \leq i, j \leq 5 \), which has the size \( 35 \times 1 \). The fourth-order quasi-entropy is defined as the minus log-determinant of \( (r_2 r_2^T) \). It is a function of symmetric traceless tensors up to fourth order.

The closure approximation minimizes the quasi-entropy with \( Q_1 \) and \( Q_2 \) fixed. Still, if the times of \( m_1, m_2, m_3 \) are not all odd or not all even, then the tensor vanishes when averaged over \( \mathcal{P}_2 \). Theorem 4.8 in [44] guarantees that when seeking the minimizer with \( Q_1 \) and \( Q_2 \) fixed, these tensors are zero. After setting these tensors as zero in the quasi-entropy, we could get a reduced expression, which we write down below.

For a second-order tensor \( U \), we define a \( 1 \times 5 \) row vector as

\[
\Phi_2(U) = (U \cdot s_j).
\]

For a third-order tensor \( U \), we define a \( 3 \times 5 \) matrix,

\[
\Psi_3(U)_{ij} = (U \cdot n_i \otimes s_j).
\]

For a fourth order tensor \( U \), we define a \( 5 \times 5 \) matrix,

\[
\Psi_4(U)_{ij} = (U \cdot s_i \otimes s_j).
\]

The reduced quasi-entropy is given by

\[
\Xi_4 = -\ln \det \begin{pmatrix}
1 & \Phi_2((m_i^2 - \frac{1}{3})) & \Phi_2((m_j^2 - \frac{1}{3})) \\
\Phi_2((m_2^2 - \frac{1}{3})^T) & \Psi_3((m_2 \otimes m_2 m_3)) \\
\Phi_2((m_3^2 - \frac{1}{3})^T) & \Psi_3((m_3 \otimes m_2 m_3)) & \Phi_2((m_j^2 - \frac{1}{3})) & \Phi_2((m_3^2 - m_2^2)) \\
\Psi_3((m_1 \otimes m_2 m_3)) & \Phi_2((m_1^2 - \frac{1}{3})) & \Phi_2((m_1 m_2 m_3)) & \Phi_2((m_j^2 - \frac{1}{3})) & \Phi_2((m_2^2 - m_3^2)) \\
\Psi_3((m_1 \otimes m_3 m_2)) & \Phi_2((m_1 m_3 m_2)) & \Phi_2((m_1 m_2 m_3)) & \Phi_2((m_2^2 - m_3^2)) & \Phi_2((m_1^2 - \frac{1}{3})) & \Phi_2((m_2^2 - \frac{1}{3})) & \Phi_2((m_3^2 - \frac{1}{3}))
\end{pmatrix}
\]

The first matrix is \( 11 \times 11 \), while the other three are \( 8 \times 8 \). The blocks can be expressed by symmetric traceless tensors as we have calculated in Appendix A.2.

The quasi-entropy \( \Xi_4 \) is defined on the domain such that the four matrices in \( \Xi_4 \) are positive definite. Thus, we conclude that if the high-order tensors are calculated from the constrained minimization of \( \Xi_4 \), the tensors \( R_1 \) in [4.3] are positive definite in the sense of [4.10]. This is because that many of \( R_1, R_3, R_4, R_5 \) are diagonal blocks of \( \Xi_4 \), and for \( R_2 \) we use [A.23].

Now, let us assume that \( Q_1 \) has the biaxial form [B.18]. First, we claim that the domain of quasi-entropy \( \Xi_4 \) is not empty when \( s_i, b_j \) are fixed with the conditions [B.20]. This is because that the high-order tensors calculated from any positive density function must make the covariance matrix positive definite. Such a density function exists because of Lemma B.1.

We are now ready to show Theorem 3.3. By (A.25), (A.27), (A.32), (A.34) and (A.35), the zeroth- and second-order tensors could fill the following entries in the quasi-entropy. In the \( 11 \times 11 \) matrix,
they are labelled as

\[
\begin{pmatrix}
1 & * & * \\
* & * & * \\
* & * & * \\
\end{pmatrix}
\]

In the three \(8 \times 8\) matrices, they are labelled as

\[
\begin{pmatrix}
* & * & * \\
* & * & * \\
* & * & * \\
\end{pmatrix}
\]

The third-order and fourth-order symmetric traceless tensors are expressed by the bases,

\[
\begin{align*}
\langle m_1m_2m_3 \rangle &= n_1n_2n_3 + z'_1(n^3_1)0 + z'_2(n^3_2)0 + z'_3(n^3_1n^3_2)0 + z'_4(n^3_1n^3_3)0 + z'_5(n^3_2n^3_3)0, \\
\langle m^3_1 \rangle &= a_1(n^3_1)0 + a_2(n^3_2)0 + a_3(n^3_3)0 \\
&\quad + a_4(n^3_1n^3_2)0 + a_5(n^3_1n^3_3)0 + a_6(n^3_2n^3_3)0 + a_7(n^3_1n^3_2n^3_3)0 + a_8(n^3_1n^3_3n^3_3)0 + a_9(n^3_2n^3_3n^3_3)0, \\
\langle m^3_2 \rangle &= \bar{a}_1(n^3_1)0 + \bar{a}_2(n^3_2)0 + \bar{a}_3(n^3_3)0 \\
&\quad + \bar{a}_4(n^3_1n^3_2)0 + \bar{a}_5(n^3_1n^3_3)0 + \bar{a}_6(n^3_2n^3_3)0 + \bar{a}_7(n^3_1n^3_2n^3_3)0 + \bar{a}_8(n^3_1n^3_3n^3_3)0 + \bar{a}_9(n^3_2n^3_3n^3_3)0.
\end{align*}
\]

Using (A.3)–(A.9), the terms \(a_i, \bar{a}_i, \bar{a}_i\) for \(i = 1, 2, 3\) and \(z\) contribute only to the starred entries, while the terms \(z'_j\) and \(a_j, \bar{a}_j, \bar{a}_j\) for \(4 \leq j \leq 9\) contribute only to the non-starred entries. Meanwhile, as long as the starred entries form a positive definite matrix, the determinant reaches its unique maximum when the non-starred entries are zero. This can be observed by rearranging the rows and columns of the four matrices in \(\Xi_4\). In the \(11 \times 11\) matrix, we group the indices as \{1, 2, 3, 7, 8\}, \{4, 9\}, \{5, 10\} and \{6, 11\}. In the three \(8 \times 8\) matrices, we group the indices as \{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}. After rearrangement, these matrices become block diagonal. Thus, the determinant must be no less than that the off-diagonal blocks are zero. Therefore, at the minimizer of \(\Xi_4\) we must have \(z'_i = 0\) and \(a_i = \bar{a}_i = \bar{a}_i = 0\), \(i = 4, \ldots, 9\).

\section*{C Explicit expression with biaxial \(Q_i\)}

Next, we calculate the blocks in (B.3) when the tensors take (3.22). They also give the matrices in Section 1.3.

Using (3.22) and the average of (A.14) with respect to the density function, we derive that

\[
\langle m_1m_2 \otimes m_1m_2 \rangle_{ijkl}
\]
Actually, we will see that they only have the following terms:

\[
\frac{1}{60} (2 \delta_{ij} \delta_{kl} - 3 \delta_{ik} \delta_{jl} - 3 \delta_{il} \delta_{jk})
\]

From here, we can see that we shall need to express the six tensors below in the basis of \( s_i \otimes s_j \),

\[
2 \delta_{ij} \delta_{kl} - 3 \delta_{ik} \delta_{jl} - 3 \delta_{il} \delta_{jk}, \quad A((n_1^4)_{ijkl})^\prime, \quad A((n_2^2 - n_3^2)_{ijkl})^\prime, \quad (n_1^4)_0, \quad (n_2^4)_0, \quad (n_1^4 n_2^2)_0.
\]

Actually, we will see that they only have the following terms:

\[
\begin{align*}
&2 \sum_{\alpha \neq \beta} n_\alpha^2 \otimes s_\beta^2 - 4 \sum_{\alpha=1}^3 n_\alpha^4 = -9s_1 \otimes s_1 - 3s_2 \otimes s_2.
\end{align*}
\]

we deduce that

\[
(n_1^4)_0 = n_1^4 - \frac{6}{7} n_1^2 i + \frac{3}{35} i^2
\]

\[
= n_1^4 - \frac{1}{7} \left( n_{11} n_{11} \delta_{kl} + n_{11} n_{12} \delta_{jl} + n_{11} n_{13} \delta_{ik} + n_{11} n_{14} \delta_{il} + n_{12} n_{12} \delta_{kl} + n_{12} n_{13} \delta_{ij} + n_{12} n_{14} \delta_{ij} + n_{13} n_{13} \delta_{ij} \right) + \frac{1}{35} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)
\]

\[
= \frac{1}{7} \left( n_1^4 - ( n_1^2 \otimes n_2^2 + n_2^2 \otimes n_3^2 + n_3^2 \otimes n_4^2 + n_4^2 \otimes n_1^2 ) - 4(n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3 + n_2 n_3 \otimes n_2 n_3) \right)
\]

\[
+ \frac{1}{35} \left( \sum_{\alpha \neq \beta} n_\alpha^2 \otimes n_\beta^2 + 3 \sum_{\alpha=1}^3 n_\alpha^4 + 4(n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3 + n_2 n_3 \otimes n_2 n_3) \right)
\]

\[
= \frac{18}{35} s_1 \otimes s_1 + \frac{1}{35} s_2 \otimes s_2 - \frac{16}{35} (s_3 \otimes s_3 + s_4 \otimes s_4) + \frac{4}{35} s_5 \otimes s_5.
\]

where the corresponding matrix \( X_4 \) is given by \((\text{C.2})\).

Similarly, from \((\text{C.3})\) and \((\text{A.33})\), we derive that

\[
(n_2^4)_0 = n_2^4 - \frac{6}{7} n_2^2 i + \frac{3}{35} i^2
\]

\[
= n_2^4 - \frac{1}{7} \left( n_{21} n_{21} \delta_{kl} + n_{21} n_{22} \delta_{jl} + n_{21} n_{23} \delta_{ik} + n_{21} n_{24} \delta_{il} + n_{22} n_{22} \delta_{kl} + n_{22} n_{23} \delta_{ij} + n_{22} n_{24} \delta_{ij} + n_{23} n_{23} \delta_{ij} \right) + \frac{1}{35} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)
\]

\[
= \frac{1}{7} \left( n_2^4 - ( n_1^2 \otimes n_2^2 + n_2^2 \otimes n_3^2 + n_3^2 \otimes n_4^2 + n_4^2 \otimes n_1^2 ) - 4(n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3 + n_2 n_3 \otimes n_2 n_3) \right)
\]

\[
+ \frac{1}{35} \left( \sum_{\alpha \neq \beta} n_\alpha^2 \otimes n_\beta^2 + 3 \sum_{\alpha=1}^3 n_\alpha^4 + 4(n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3 + n_2 n_3 \otimes n_2 n_3) \right)
\]
\[
\frac{27}{140} s_1 \otimes s_1 + \frac{19}{140} s_2 \otimes s_2 - \frac{3}{28} (s_1 \otimes s_2 + s_2 \otimes s_1) \\
- \frac{16}{35} (s_3 \otimes s_3 + s_5 \otimes s_5) + \frac{4}{35} s_4 \otimes s_4, 
\]

where the associated coefficient matrix \( \Pi \) is given by (4.43).

We may now proceed to deal with the term \((n_1^2 n_2^2)_0\). Analogously, we have

\[
(n_1^2 n_2^2)_0 = n_1^2 n_2^2 - \frac{1}{7} (n_1^2 + n_2^2) i + \frac{1}{35} i^2
\]

\[
= \frac{1}{6} \left( n_1 n_1 n_2 n_2 + n_1 n_2 n_1 n_2 + n_1 n_2 n_1 n_2 + n_2 n_1 n_2 n_1 \right)
- \frac{1}{42} \left( n_1 n_1 n_2 n_2 + n_1 n_2 n_1 n_2 + n_1 n_2 n_1 n_2 + n_2 n_1 n_2 n_1 \right)
- \frac{1}{42} \left( n_1 n_2 n_1 n_2 + n_2 n_1 n_2 n_1 + n_2 n_1 n_2 n_1 + n_2 n_1 n_2 n_1 \right)
- \frac{1}{105} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)
= \frac{1}{6} (n_1^2 \otimes n_2^2 + n_2^2 \otimes n_1^2 + 2 n_1 n_2 \otimes n_1 n_2)
- \frac{1}{42} \left( 6 n_1^2 + \sum_{\alpha \neq \beta} n_2^2 \otimes n_2^2 - (n_2^2 \otimes n_2^2 + n_2^2 \otimes n_2^2) + 4 (n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3) \right)
- \frac{1}{42} \left( 6 n_1^2 + \sum_{\alpha \neq \beta} n_2^2 \otimes n_2^2 - (n_2^2 \otimes n_2^2 + n_2^2 \otimes n_2^2) + 4 (n_1 n_2 \otimes n_1 n_2 + n_2 n_3 \otimes n_2 n_3) \right)
+ \frac{1}{105} \left( \sum_{\alpha \neq \beta} n_2^2 \otimes n_2^2 + 3 \sum_{\alpha = 1} n_2^4 + 4 (n_1 n_2 \otimes n_1 n_2 + n_1 n_3 \otimes n_1 n_3 + n_2 n_3 \otimes n_2 n_3) \right)
= - \frac{9}{35} s_1 \otimes s_1 - \frac{1}{70} s_2 \otimes s_2 + \frac{3}{28} (s_1 \otimes s_2 + s_2 \otimes s_1)
+ \frac{18}{35} s_4 \otimes s_4 - \frac{2}{35} (s_4 \otimes s_4 + s_5 \otimes s_5),
\]

where the corresponding matrix \( X_6 \) is given by (4.43).

A direct calculation leads to

\[
\epsilon^{iit} (n_1 n_2 n_3)_{jkl} + \epsilon^{jkl} (n_1 n_2 n_3)_{iit} = \frac{3}{2} (s_1 \otimes s_2 - s_2 \otimes s_1),
\]

where the associated coefficient matrix \( \Pi \) is given by

\[
\Pi = \begin{pmatrix}
0 & \frac{3}{2} & 0 \\
-\frac{3}{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Based on the above calculations, we immediately give the expressions of \( R^{(0)}_i (i = 1, \cdots, 6) \) under the basis \( s_i \otimes s_j \). Using (3.22) and the averages of (A.9) with respect to the density function, we deduce that

\[
R^{(0)}_i = \left\langle \left( m_i^2 - \frac{1}{3} i \right) \otimes \left( m_i^2 - \frac{1}{3} i \right) \right\rangle_{ijkl}
\]

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\[ = \langle (m_1^4)_{ij} \rangle - \frac{4}{21} A(Q_1^{(0)})_{ij kl} - \frac{1}{45} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}). \]

In the light of Theorem 3.3 and from (A.32), (C.4)-(C.6), we deduce that
\[ R_1^{(0)} = a_1(n_1^4)_{0} + a_2(n_2^4)_{0} + a_3(n_1^2 n_2^2)_{0} - \frac{4}{21} (s_1 A((n_1^2)_{0})_{ij kl} + b_1 A(n_2^2 - n_3^2)_{ij kl} ) \]
\[ - \frac{1}{45} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]
(C.9)

for which the matrix \( R_1 \) is written as
\[ R_1 = -\frac{1}{45} X_1 - \frac{4}{21} (s_1 X_2 + b_1 X_3) + a_1 X_4 + a_2 X_5 + a_3 X_6. \]
(C.10)

Similarly, for \( R_2^{(0)} \), it follows that
\[ R_2^{(0)} = \langle (m_2^4)_{0} - \frac{1}{3} \rangle \otimes (m_2^4 - \frac{1}{3}) \]
\[ = \langle (m_2^4)_{0} \rangle - \frac{4}{21} A(Q_2^{(0)})_{ij kl} - \frac{1}{45} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}) \]
\[ = \tilde{a}_1(n_1^4)_{0} + \tilde{a}_2(n_2^4)_{0} + \tilde{a}_3(n_1^2 n_2^2)_{0} - \frac{4}{21} (s_2 A((n_1^2)_{0})_{ij kl} + b_2 A(n_2^2 - n_3^2)_{ij kl} ) \]
\[ - \frac{1}{45} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]
(C.11)

for which the matrix \( R_2 \) is written as
\[ R_2 = -\frac{1}{45} X_1 - \frac{4}{21} (s_2 X_2 + b_2 X_3) + \tilde{a}_1 X_4 + \tilde{a}_2 X_5 + \tilde{a}_3 X_6. \]
(C.12)

Combining (C.1) with (C.4)-(C.6), the tensor \( R_3^{(0)} \) is expressed by
\[ R_3^{(0)} = 4(\tilde{a}_1(n_1^4)_{0} + \tilde{a}_2(n_2^4)_{0} + \tilde{a}_3(n_1^2 n_2^2)_{0})_{ij kl} - \frac{4}{7} (s_1 + s_2) A((n_1^2)_{0})_{ij kl} \]
\[ + (b_1 + b_2) A(n_2^2 - n_3^2)_{ij kl} ) - \frac{1}{15} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]
(C.13)

for which the matrix \( R_3 \) is denoted by
\[ R_3 = -\frac{1}{15} X_1 - \frac{4}{7} ((s_1 + s_2) X_2 + (b_1 + b_2) X_3) + 4(\tilde{a}_1 X_4 + \tilde{a}_2 X_5 + \tilde{a}_3 X_6). \]
(C.14)

Analogously, the tensor moment \( R_4^{(0)} \) can be expressed by
\[ R_4^{(0)} = -4 (a_1 + \tilde{a}_1)(n_1^4)_{0} + (a_2 + \tilde{a}_2)(n_2^4)_{0} + (a_3 + \tilde{a}_3)(n_1^2 n_2^2)_{0})_{ij kl} \]
\[ + \frac{4}{7} (s_2 A((n_1^2)_{0})_{ij kl} + b_2 A(n_2^2 - n_3^2)_{ij kl} ) - \frac{1}{15} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]
(C.15)

for which the matrix \( R_4 \) is denoted by
\[ R_4 = -\frac{1}{15} X_1 + \frac{4}{7} (s_2 X_2 + b_2 X_3) - 4((a_1 + \tilde{a}_1) X_4 + (a_2 + \tilde{a}_2) X_5 + (a_3 + \tilde{a}_3) X_6). \]
(C.16)

In the same way, we obtain
\[ R_5^{(0)} = -4 (\tilde{a}_1 + \tilde{a}_1)(n_1^4)_{0} + (\tilde{a}_2 + \tilde{a}_2)(n_2^4)_{0} + (\tilde{a}_3 + \tilde{a}_3)(n_1^2 n_2^2)_{0})_{ij kl} \]
\[ + \frac{4}{7} \left( s_1 A((n_1^2)_{0})_{ijkl} + b_1 A(n_2^2 - n_3^2)_{ijkl} \right) - \frac{1}{15} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]  
(C.17)

for which the matrix \( R_5 \) is denoted as

\[ R_5 = -\frac{1}{15} X_1 + \frac{4}{7} (s_1 X_2 + b_1 X_3) - 4((\tilde{a}_1 + \tilde{a}_1) X_4 + (\tilde{a}_2 + \tilde{a}_2) X_5 + (\tilde{a}_3 + \tilde{a}_3) X_6). \]  
(C.18)

By

\[ (m_1^3)_0 = (m_1^4)_0 + (m_2^3)_0 + 2(m_1^2 m_2^2)_0, \quad (m_2^3 m_3^2)_0 = -(m_1^4)_0 - (m_1^2 m_2^2)_0, \]
we derive from (A.19) that

\[ R_6 = \langle (m_1^2 - m_2^2) \otimes (m_2^2 - m_3^2) \rangle \]
\[ = \langle (m_1^4)_0 \rangle + 4\langle (m_2^3)_{0} \rangle + 4\langle (m_1^2 m_2^2)_0 \rangle + \frac{4}{7} A(Q_1^{(0)}) \]
\[ - \frac{1}{15} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}) \]
\[ = (a_1 + 4\tilde{a}_1 + 4\tilde{a}_1)(n_1^4)_0 + (a_2 + 4\tilde{a}_2 + 4\tilde{a}_2)(n_2^4)_0 + (a_3 + 4\tilde{a}_3 + 4\tilde{a}_3)(n_1^2 n_2^2)_0 \]
\[ + \frac{4}{7} (s_1 A((n_2^2)_{0})_{ijkl} + b_1 A(n_2^2 - n_3^2)_{ijkl} \rangle - \frac{1}{15} (2\delta_{ij}\delta_{kl} - 3\delta_{ik}\delta_{jl} - 3\delta_{il}\delta_{jk}), \]  
(C.19)

for which the matrix \( R_6 \) is given by

\[ R_6 = -\frac{1}{15} X_1 + \frac{4}{7} (s_1 X_2 + b_1 X_3) + (a_1 + 4\tilde{a}_1 + 4\tilde{a}_1) X_4 \]
\[ + (a_2 + 4\tilde{a}_2 + 4\tilde{a}_2) X_5 + (a_3 + 4\tilde{a}_3 + 4\tilde{a}_3) X_6. \]  
(C.20)

We turn to the term \( \langle (m_1^2)_{0} \otimes (m_2^2 - m_3^2) \rangle. \) By (A.22), we have

\[ S = \langle (m_1^2)_0 \otimes (m_2^2 - m_3^2) \rangle \]
\[ = \epsilon^{ks} (m_1 m_2 m_3)_{ils} + \epsilon^{ils} (m_1 m_2 m_3)_{kjs} + 2\langle (m_1^2 m_2^2)_0 \rangle + \langle (m_1^4)_0 \rangle \]
\[ + \frac{4}{21} A(Q_1^{(0)}) + \frac{8}{21} A(Q_2^{(0)}) \]
\[ = \epsilon^{ks} (m_1 m_2 m_3)_{ils} + \epsilon^{ils} (m_1 m_2 m_3)_{kjs} \]
\[ + (a_1 + 2\tilde{a}_1)(n_1^4)_0 + (a_2 + 2\tilde{a}_2)(n_2^4)_0 + (a_3 + 2\tilde{a}_3)(n_1^2 n_2^2)_0 \]
\[ + \frac{4}{21} \left( s_1 + 2s_2 \right) A((n_1^2)_0)_{ijkl} + (b_1 + 2b_2) A((n_1^2 m_2^2)_{ijkl}), \]  
(C.21)

where the coefficient matrix \( S \) is given by

\[ S = z\Xi + \frac{4}{21} \left( (s_1 + 2s_2) X_2 + (b_1 + 2b_2) X_3 \right) \]
\[ + (a_1 + 2\tilde{a}_1) X_4 + (a_2 + 2\tilde{a}_2) X_5 + (a_3 + 2\tilde{a}_3) X_6. \]

We are now able to give the matrices \( M \) and \( P. \) By (4.15) and (4.17), the corresponding coordinates \( M_{11}, M_{12} \) and \( M_{22} \) are

\[ M_{11} = \Gamma_2 R_4 + \Gamma_3 R_3, \quad M_{12} = -\Gamma_3 R_3, \quad M_{22} = \Gamma_1 R_5 + \Gamma_3 R_3, \]  
(C.22)

\[ P = c\xi (I_{22} R_1 + I_{11} R_2 + I_{12} \epsilon_1 R_3). \]  
(C.23)

Using the expressions of \( R_i, \) we arrive at (4.44)–(4.46) and (4.50).

The remaining part is to express averages of fourth-order antisymmetric traceless tensors and third-order tensors. From (A.29), we deduce that

\[ \langle (m_1 m_2 - m_2 m_1)_{ijkl} \rangle \]
By virtue of the definition of symmetric tensors and (A.12), it follows that

\[
\begin{align*}
\mathcal{B}(n_1^2)_{ijkl} &= \frac{1}{3} \left( (2n_1^2 - n_2^2 - n_3^2)_{kl} (n_1^2 + n_2^2 + n_3^2)_{ij} - (2n_1^2 - n_2^2 - n_3^2)_{ij} (n_1^2 + n_2^2 + n_3^2)_{kl} \\
&\quad + (2n_1^2 - n_2^2 - n_3^2)_{il} (n_1^2 + n_2^2 + n_3^2)_{jk} - (2n_1^2 - n_2^2 - n_3^2)_{ij} (n_1^2 + n_2^2 + n_3^2)_{lk} \right) \\
&= 2(n_1 \otimes n_2 - n_2 \otimes n_1) \otimes n_1 n_2 + 2(n_1 \otimes n_3 - n_3 \otimes n_1) \otimes n_1 n_3 \\
&\quad + 4(n_2 \otimes n_3 - n_3 \otimes n_2) \otimes n_2 n_3 \\
\mathcal{B}(n_2^2 - n_3^2)_{ijkl} &= (n_2^2 - n_3^2)_{kl} (n_1^2 + n_2^2 + n_3^2)_{ij} - (n_2^2 - n_3^2)_{ij} (n_1^2 + n_2^2 + n_3^2)_{kl} \\
&\quad + (n_2^2 - n_3^2)_{il} (n_1^2 + n_2^2 + n_3^2)_{jk} - (n_2^2 - n_3^2)_{ij} (n_1^2 + n_2^2 + n_3^2)_{lk} \\
&= -2(n_1 \otimes n_2 - n_2 \otimes n_1) \otimes n_1 n_2 + 2(n_1 \otimes n_3 - n_3 \otimes n_1) \otimes n_1 n_3 \\
&\quad + 4(n_2 \otimes n_3 - n_3 \otimes n_2) \otimes n_2 n_3 \\
&= -2(a_1 \otimes s_3 + a_2 \otimes s_4) + 4a_3 \otimes s_5. 
\end{align*}
\]

By virtue of the definition of symmetric tensors and (A.12), it follows that

\[
\varepsilon^{ijkl}(n_1 n_2 n_3)_{kl} = \frac{1}{6} \varepsilon^{ijkl} (n_1 n_2 n_3)_{s} = \frac{1}{3} (a_1 \otimes s_3 + a_2 \otimes s_4 + a_3 \otimes s_5),
\]

(28)
\[ = z \epsilon^{js}(\mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3)_{skl} + \frac{1}{6} \left( (s_1 + 2s_2) B((\mathbf{n}_1^2)_{0})_{ijkl} + (b_1 + 2b_2) B((\mathbf{n}_2^2 - \mathbf{n}_3^2)_{ijkl} \right). \]  

(C.31)

Therefore, taking advantage of the definition of \( \mathcal{N}_{Q_1}^{(0)} \) and combining (C.28) and (C.30) with (C.24)-(C.26) and (C.30), and using \( 1 - e_1 - e_2 = 0 \), we deduce that

\[ \mathcal{N}_{Q_1}^{(0)} = \frac{1}{2} R_4^{(0)} + \frac{1}{2} (e_1 - e_2) R_3^{(0)} - \frac{z}{3} (a_1 \otimes s_3 + a_2 \otimes s_4 + a_3 \otimes s_5) \]
\[ + (e_1 + e_2) \frac{z}{3} (a_1 \otimes s_3 + a_2 \otimes s_4 + a_3 \otimes s_5) \]
\[ + \frac{1}{6} \left( (2s_1 + s_2) B((\mathbf{n}_1^2)_{0})_{ijkl} + (2b_1 + b_2) B((\mathbf{n}_2^2 - \mathbf{n}_3^2)_{ijkl} \right) \]
\[ + (e_1 + e_2) \frac{1}{6} \left( (s_1 - s_2) B((\mathbf{n}_1^2)_{0})_{ijkl} + (b_1 - b_2) B((\mathbf{n}_2^2 - \mathbf{n}_3^2)_{ijkl} \right) \]
\[ = \frac{1}{2} R_4^{(0)} + \frac{1}{2} (e_1 - e_2) R_3^{(0)} + (s_1 - b_1) a_1 \otimes s_3 - (s_1 + b_1) a_2 \otimes s_4 + 2b_1 a_3 \otimes s_5. \]  

(C.32)

Similarly, combining (C.13) and (C.17) with (C.24)-(C.26), (C.29) and (C.31), then we have

\[ \mathcal{N}_{Q_2}^{(0)} = \frac{1}{2} R_5^{(0)} + \frac{1}{2} (e_2 - e_1) R_4^{(0)} + \frac{z}{3} (a_1 \otimes s_3 + a_2 \otimes s_4 + a_3 \otimes s_5) \]
\[ - (e_1 + e_2) \frac{z}{3} (a_1 \otimes s_3 + a_2 \otimes s_4 + a_3 \otimes s_5) \]
\[ + \frac{1}{6} \left( (s_1 + 2s_2) B((\mathbf{n}_1^2)_{0})_{ijkl} + (b_1 + 2b_2) B((\mathbf{n}_2^2 - \mathbf{n}_3^2)_{ijkl} \right) \]
\[ - (e_1 + e_2) \frac{1}{6} \left( (s_1 - s_2) B((\mathbf{n}_1^2)_{0})_{ijkl} + (b_1 - b_2) B((\mathbf{n}_2^2 - \mathbf{n}_3^2)_{ijkl} \right) \]
\[ = \frac{1}{2} R_5^{(0)} + \frac{1}{2} (e_2 - e_1) R_4^{(0)} + (s_2 - b_2) a_1 \otimes s_3 - (s_2 + b_2) a_2 \otimes s_4 + 2b_2 a_3 \otimes s_5. \]  

(C.33)

The equations (1.47)-(1.49) then come from (C.32) and (C.33).

Finally, we deal with the third-order tensors. By a direct calculation, we get

\[ (\mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3)_{ijk} = \frac{1}{6} \left( n_{1i}(n_{2j}n_{3k} + n_{3j}n_{2k}) + n_{2i}(n_{1j}n_{3k} + n_{3j}n_{1k}) \right) \]
\[ + n_{3i}(n_{1j}n_{2k} + n_{2j}n_{1k}) \]
\[ = \frac{1}{3} (\mathbf{n}_1 \otimes s_5 + \mathbf{n}_2 \otimes s_4 + \mathbf{n}_3 \otimes s_3). \]

Meanwhile, we also easily deduce that

\[ \epsilon^{ij}(n_1^2)_{0} \]  
\[ = 2(n_2 \otimes s_4 - n_3 \otimes s_3), \]
\[ \epsilon^{ij}(n_2^2 - n_3^2)_{ks} + \epsilon^{ks}(n_2^2 - n_3^2)_{js} = n_{2k}(n_{3i}n_{1j} - n_{1i}n_{2j}) - n_{3k}(n_{1j}n_{3i} - n_{3i}n_{1j}) \]
\[ + n_{2j}(n_{3i}n_{1k} - n_{1k}n_{3i}) - n_{3j}(n_{1k}n_{2j} - n_{2j}n_{1k}) \]
\[ = 2(n_3 \otimes s_5 - 2n_1 \otimes s_5 + n_2 \otimes s_4). \]

Hence, by using (A.25), we derive from Theorem 3.36 that

\[ (\mathbf{m}_1 \otimes \mathbf{m}_2 \mathbf{m}_3)_{ijk} = (\mathbf{m}_1 \otimes \mathbf{m}_2 \mathbf{m}_3)_{ijk} + \frac{1}{6} \left( \epsilon^{ij}(\mathbf{m}_1^2)_{0} - (\mathbf{m}_2^2)_{0} \right)_{ks} \]
\[ + \epsilon^{kJ}((\mathbf{m}_3^2)_{0}) - (\mathbf{m}_2^2)_{0})_{js} \]
\[ = z(\mathbf{n}_1 \mathbf{n}_2 \mathbf{n}_3)_{ijk} - \frac{1}{6} \left( \epsilon^{ij}((Q_1^{(0)} + 2Q_2^{(0)})_{ks} + \epsilon^{ks}(Q_1^{(0)} + 2Q_2^{(0)})_{ks} \right). \]

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= z(n_1 n_2 n_3)_{ijk} - \frac{1}{6} (s_1 + 2 s_2) \left( \epsilon^{ij s} (n_1^2)_{0 k} + \epsilon^{iks} (n_1^2)_{0 s} \right) \\
- \frac{1}{6} (b_1 + 2 b_2) \left( \epsilon^{ij s} (n_2^2 - n_3^2)_{ks} + \epsilon^{iks} (n_2^2 - n_3^2)_{js} \right) \\
= \frac{1}{3} (z + 2 b_1 + 4 b_2) n_1 \otimes s_5 + \frac{1}{3} (z + s_1 - 2 s_2 - b_1 - 2 b_2) n_2 \otimes s_4 \\
+ \frac{1}{3} (z + s_1 + 2 s_2 - b_1 - 2 b_2) n_3 \otimes s_3, \quad (C.34)
for which the coefficient matrix $T_1$ under the basis $n_i \otimes s_j$ is given by

$$
T_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{3} (z + 2 b_1 + 4 b_2) \\
0 & 0 & 0 & \frac{1}{3} (z - s_1 - 2 s_2) & 0 \\
0 & \frac{1}{3} (z + s_1 + 2 s_2) & 0 & 0 & 0 \\
\end{pmatrix}. \quad (C.35)
$$

Following the same procedure, we obtain

$$
\langle m_2 \otimes m_1 m_3 \rangle_{ijk} = \langle m_1 m_2 m_3 \rangle_{ijk} + \frac{1}{6} \left( \epsilon^{ij s} \left( (n_1^2)_{0 k} - \langle (m_2^2)_{0 s} \right) \\
+ \epsilon^{iks} \left( (m_2^2)_{0 k} - \langle (m_2^2)_{0 s} \right) \\
= z(n_1 n_2 n_3)_{ijk} + \frac{1}{6} (2 s_1 + s_2) \left( \epsilon^{ij s} (n_1^2)_{0 k} + \epsilon^{iks} (n_1^2)_{0 s} \right) \\
+ \frac{1}{6} (2 b_1 + b_2) \left( \epsilon^{ij s} (n_2^2 - n_3^2)_{ks} + \epsilon^{iks} (n_2^2 - n_3^2)_{js} \right) \\
= \frac{1}{3} (z - 4 b_1 - 2 b_2) n_1 \otimes s_5 + \frac{1}{3} (z + 2 s_1 + s_2 + 2 b_1 + b_2) n_2 \otimes s_4 \\
+ \frac{1}{3} (z - s_1 - s_2 + 2 b_1 + b_2) n_3 \otimes s_3, \quad (C.36)
$$

$$
\langle m_3 \otimes m_1 m_2 \rangle_{ijk} = \langle m_1 m_2 m_3 \rangle_{ijk} + \frac{1}{6} \left( \epsilon^{ij s} \left( (m_3^2)_{0 k} - \langle (m_1^2)_{0 s} \right) \\
+ \epsilon^{iks} \left( (m_3^2)_{0 k} - \langle (m_1^2)_{0 s} \right) \\
= z(n_1 n_2 n_3)_{ijk} - \frac{1}{6} (s_1 - s_2) \left( \epsilon^{ij s} (n_1^2)_{0 k} + \epsilon^{iks} (n_1^2)_{0 s} \right) \\
- \frac{1}{6} (b_1 - b_2) \left( \epsilon^{ij s} (n_2^2 - n_3^2)_{ks} + \epsilon^{iks} (n_2^2 - n_3^2)_{js} \right) \\
= \frac{1}{3} (z + 2 b_1 - 2 b_2) n_1 \otimes s_5 + \frac{1}{3} (z - s_1 + s_2 - b_1 + b_2) n_2 \otimes s_4 \\
+ \frac{1}{3} (z + s_1 - s_2 - b_1 + b_2) n_3 \otimes s_3, \quad (C.37)
$$

where the associated coefficient matrices $T_2, T_3$ in (C.36) and (C.37) can be written as

$$
T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{3} (z - 4 b_1 - 2 b_2) \\
0 & 0 & 0 & \frac{1}{3} (z + 2 s_1 + s_2) & 0 \\
0 & \frac{1}{3} (z - 2 s_1 - s_2) & 0 & 0 & 0 \\
\end{pmatrix}, \quad (C.38)
$$
\[
T_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{3}(z + 2b_1 - 2b_2) \\
0 & 0 & 0 & \frac{1}{3}(z - s_1 + s_2) & 0 \\
0 & 0 & \frac{1}{3}(z + s_1 - s_2) & -b_1 + b_2 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\] (C.39)

Define
\[
w_i^T = (s_i, b_i, 0, 0, 0), \quad i = 1, 2,
\]
\[
W_1 = \text{diag}\left(\frac{1}{3}(2s_1 + 1), \frac{1}{3}(1 - s_1) + b_1, \frac{1}{3}(s_1 - 1) - b_1\right),
\]
\[
W_2 = \text{diag}\left(\frac{1}{3}(2s_2 + 1), \frac{1}{3}(1 - s_2) + b_2, \frac{1}{3}(s_2 - 1) - b_2\right),
\]
\[
W_3 = \text{diag}\left(\frac{1}{3}(1 - 2s_1 - 2s_2), \frac{1}{3}(1 + s_1 + s_2) - b_1 - b_2, \frac{1}{3}(1 + s_1 + s_2) + b_1 + b_2\right). \quad (C.40)
\]

Then, the quasi-entropy \(\Xi_4\) can be reduced to
\[
\Xi_{4, Bi} = -\ln \det \begin{pmatrix}
1 & w_1^T & (2w_2 + w_1)^T \\
2w_2 + w_1 & R_1 & S \\
& S^T & R_6
\end{pmatrix}
\]
\[
- \ln \det \begin{pmatrix}
1 & W_1 & T_1 \\
W_1^T & T_1^T & R_3 \\
& & R_6
\end{pmatrix}
\]
\[
- \ln \det \begin{pmatrix}
1 & W_2 & T_2 \\
W_2^T & T_2^T & R_4 \\
& & R_6
\end{pmatrix}
\]
\[
- \ln \det \begin{pmatrix}
1 & W_3 & T_3 \\
W_3^T & T_3^T & R_5 \\
& & R_6
\end{pmatrix}
\]
\[
= -\ln \det \begin{pmatrix}
w_1 & R_1 & S \\
2w_2 + w_1 & S^T & R_6
\end{pmatrix} - \ln \det \begin{pmatrix}
W_1 & T_1 \\
T_1^T & R_3 \\
& & R_6
\end{pmatrix}
\]
\[
- \ln \det \begin{pmatrix}
W_2 & T_2 \\
T_2^T & R_4
\end{pmatrix} - \ln \det \begin{pmatrix}
W_3 & T_3 \\
T_3^T & R_5
\end{pmatrix} - 10 \ln \det \Lambda. \quad (C.41)
\]

The expressions of \(R_i, S, T_i\) can all be found above.

**D** The uniaxial case: Theorem 5.1

Assume that \(Q_i\) are uniaxial, i.e. \(b_i = 0\) so that
\[
Q_i = s_i \left(n_i^2 - \frac{1}{3}\right), \quad i = 1, 2.
\]

By 320, we require that the two scalars \(s_i\) satisfy
\[
-\frac{1}{2} < s_1, s_2, \quad s_1 - s_2 < 1. \quad (D.1)
\]

For the original entropy, the discussion is similar to the biaxial case.
Lemma D.1. If $s_i$ satisfy (D.1), then there exists a unique density function

$$
\rho = \frac{1}{Z} \exp\left( \sum_{i=1,2} \lambda_i (\mathbf{m}_i \cdot \mathbf{n})^2 \right)
$$

such that $\langle (\mathbf{m}_i^2) \rangle_0 = s_i (\mathbf{n}_i^2)$.  

We omit the rest of the derivation since it is the same as the biaxial case. We turn to the quasi-entropy. Here, we need to notice that

$$
2X_6 + X_4 = \begin{pmatrix}
0 & \frac{3}{14} \\
\frac{3}{14} & 0
\end{pmatrix} \overset{\text{def}}{=} \mathbf{X}_{6}', \quad (D.2)
$$

$$
8X_6 + 8X_5 + X_4 = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix} \overset{\text{def}}{=} \mathbf{X}_{5}'. \quad (D.3)
$$

Let us define

$$
a'_1 = a_1 - \frac{1}{2} a_3 + \frac{3}{8} a_2, \quad a'_2 = \frac{1}{8} a_2, \quad a_3 = \frac{1}{2} (a_3 - a_2), \quad a'_3 = \frac{1}{2} (a_3 - a_2),
$$

$$
\hat{a}'_1 = \hat{a}_1 - \frac{1}{2} \hat{a}_3 + \frac{3}{8} \hat{a}_2, \quad \hat{a}'_2 = \frac{1}{8} \hat{a}_2, \quad \hat{a}_3 = \frac{1}{2} (\hat{a}_3 - \hat{a}_2), \quad \hat{a}'_3 = \frac{1}{2} (\hat{a}_3 - \hat{a}_2).
$$

It can be verified that

$$
a_1 X_4 + a_2 X_5 + a_3 X_6 = a'_1 X_4 + a'_2 X_5' + a'_3 X_6'.
$$

In what follows, we show that when $Q_i$ are uniaxial, $\Xi_{4,51}$ reaches its minimum only when $a'_2 = a'_4 = \hat{a}'_2 = a'_3 = \hat{a}'_3 = a'_3 = z = 0$.

Let us discuss each of the log-determinant in (C.41). We could rearrange the indices to arrvive at

$$
-\ln \det \begin{pmatrix}
1 & w_1^T \\
w_1 & R_1 \\
2w_2 + w_1 & S \\
S^T & R_6
\end{pmatrix} = -\ln \det \begin{pmatrix}
1 & (s_1, 2s_1 + s_2) \\
(s_1, 2s_1 + s_2)^T & \mathbf{Y}_1 \\
0 & \frac{4}{7} \Theta_3 + z \Pi_1
\end{pmatrix} \quad (D.4)
$$

where the blocks $\mathbf{Y}_i$, $\Theta_i$ and $\Pi_i$ are given by

$$
\mathbf{Y}_1 = \begin{pmatrix}
\frac{1}{7} + \frac{2}{7} s_1 + \frac{18}{35} a'_1 & -\frac{2}{7} (s_1 + 2s_2) + \frac{18}{35} (a'_1 + 2a'_2) \\
-\frac{2}{7} (s_1 + 2s_2) + \frac{18}{35} (a'_1 + 2a'_2) & \frac{3}{7} - \frac{6}{7} s_1 + \frac{18}{35} (a'_1 + 4a'_2 + 4a'_3)
\end{pmatrix},
$$

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Notice that \( \Psi_i \) does not depend on \( a_i', \bar{a}_i', \bar{a}_i' \) for \( i = 2, 3 \), and \( \Theta_2, \Theta_3 \) only depend on them.

By Lemma B.2 we deduce that

\[
- \ln \det \begin{pmatrix}
1 & (s_1, 2s_1 + s_2) & 0_{1 \times 2} \\
(s_1, 2s_1 + s_2)^T & \Psi_1 & 0_{2 \times 1} \\
0_{2 \times 1} & 0_{1 \times 1} & \Theta_3 - z \Pi_1
\end{pmatrix} \geq - \ln \det \begin{pmatrix}
1 & (s_1, 2s_1 + s_2) \\
(s_1, 2s_1 + s_2)^T & \Psi_1
\end{pmatrix} - \ln \det (\Psi_1 + \Theta_2).
\]  

(D.5)

The equality holds if and only if \( z = 0 \) and \( \Theta_3 = 0 \). In addition, it shall be noticed that \(- \ln \det A\) is strictly convex about \( A \) (see, for example, Lemma 4.5 in [44] for a proof). Therefore, we obtain

\[
- \ln \det (\Psi_2 + \Theta_2) - \ln \det (4\Psi_2 - 4\Theta_2) \geq -2 \ln \det \Psi_2 - 2 \ln 4,
\]

\[
- \ln \det (\Psi_3 + \frac{4}{l} \Theta_3) - \ln \det (\Psi_3 - \frac{4}{l} \Theta_3) \geq -2 \ln \det \Psi_3.
\]  

(D.6)

The equalities hold if and only if \( \Theta_2 = \Theta_3 = 0 \).

Let us look into another log-determinant in (C.41). It follows that

\[
- \ln \det \begin{pmatrix}
W_1 & T_1 \\
T_1^T & R_3
\end{pmatrix} =
- \ln \det \begin{pmatrix}
\xi_1 & \xi_2 \\
\xi_2 & \xi_3 + \frac{1}{3} z \\
\xi_4 & \xi_5 + \bar{a}'_2 \\
\xi_5 + \frac{1}{3} z & \xi_6 + \frac{4}{l} \bar{a}'_2 \\
\xi_6 - \frac{4}{l} \bar{a}'_2 & 4\xi_5 - 4\bar{a}'_5
\end{pmatrix}.
\]

In the above, those \( \xi_i \) are given by

\[
\begin{align*}
\xi_1 &= \frac{1}{3} (2s_2 + 1), \\
\xi_2 &= \frac{1}{3} (1 - s_1), \\
\xi_3 &= \frac{1}{3} (s_1 + 2s_2), \\
\xi_4 &= \frac{3}{5} + \frac{6}{7} (s_1 + s_2) + \frac{72}{35} \bar{a}_1', \\
\xi_5 &= \frac{1}{5} - \frac{2}{7} (s_1 + s_2) + \frac{4}{35} \bar{a}_1'.
\end{align*}
\]

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\[ \xi_6 = \frac{4}{5} + \frac{4}{7} (s_1 + s_2) - \frac{64}{35} a'_1. \]

Since the function \(- \ln x\) is monotonely decreasing and strictly convex, we have the inequality
\[
- \ln \left( \xi_1(4\xi_5 - 4a'_3) - \frac{1}{9} z^2 \right) - \ln \left( \xi_2(\xi_6 - 4\zeta_2) - (\xi_3 - \frac{1}{3} z)^2 \right) \\
- \ln \left( \xi_2(\xi_6 + \frac{4}{7} a'_2) - (\xi_3 + \frac{1}{3} z)^2 \right) - \ln \left( \xi_4(\xi_5 + a'_3) - \left( \frac{3}{14} a_2^2 \right)^2 \right) \\
\geq - \ln \left( \xi_1(4\xi_5 - 4a'_3) \right) - \ln \left( \xi_2(\xi_6 - 4\zeta_2) - \xi_3^2 + \frac{2}{3} \xi_3 z \right) \\
- \ln \left( \xi_2(\xi_6 + \frac{4}{7} a'_2) - \xi_3^2 - \frac{2}{3} \xi_3 z \right) - \ln \left( \xi_4(\xi_5 + a'_3) \right)
= - \ln \xi_1 - \ln \xi_4 - \ln 4 \\
- \ln (\xi_5 - a'_3) - \ln (\xi_5 + a'_3) \\
- \ln \left( \xi_2(\xi_6 - 4\zeta_2) - \xi_3^2 + \frac{2}{3} \xi_3 z \right) - \ln \left( \xi_2(\xi_6 + \frac{4}{7} a'_2) \right)
\geq - \ln \xi_1 - \ln \xi_4 - \ln 4 - 2 \ln 5 - 2 \ln (\xi_2\xi_5 - \xi_3^2).
\]  
(D.7)

The equalities hold if and only if \( \bar{a}_3 = \bar{a}'_3 = z = 0 \).

Similarly, we could deal with the other two log-determinants in (C.41). Summarizing (D.5), (D.6) and (D.7), we conclude that when \( Q_i \) are uniaxial, at the minimizer we must have \( a'_2 = a'_3 = \bar{a}'_2 = \bar{a}'_3 = \bar{a}_3 = z = 0 \).

### E The orientational elasticity

For the readers' convenience, we present the orientational elasticity for the biaxial nematic phases that can be found in [50], where the elastic constants expressed the coefficients in the molecular-theory-based static Q-tensor model. In addition, the variational derivatives with respect to the orthonomal frame \( p = (n_1, n_2, n_3) \) are derived.

We first write down an equivalent formulation of (4.41). Using the following relations
\[
\nabla \cdot n_2 = - D_{31} + D_{13}, \quad n_2 \cdot \nabla \times n_2 = D_{33} + D_{11}, \\
n_3 \cdot \nabla \times n_2 = - D_{23}, \quad n_1 \cdot \nabla \times n_2 = - D_{21}, \\
| n_2 \times \nabla \times n_2 |^2 = (n_1 \cdot \nabla \times n_2)^2 + (n_3 \cdot \nabla \times n_2)^2,
\]
together with (4.51) yields the equivalent expression analogous to the Oseen-Frank energy can be given by
\[
\tilde{F}_{B_i(p)}(\nabla \cdot n_2) = \int dx \frac{1}{2} \left( K_1(\nabla \cdot n_1)^2 + K_2(n_1 \cdot \nabla \times n_1)^2 + K_3(n_1 \cdot \nabla \times n_1)^2 \right) \\
+ K_4(\nabla \cdot n_2)^2 + K_5(n_2 \cdot \nabla \times n_2)^2 + K_6(n_2 \cdot \nabla \times n_2)^2 \\
+ K_7(n_3 \cdot \nabla \times n_3)^2 + K_8(n_3 \cdot \nabla \times n_3)^2 + K_9(n_3 \cdot \nabla \times n_3)^2 \\
+ K_{10}(n_1 \cdot \nabla \times n_3)^2 + K_{11}(n_2 \cdot \nabla \times n_1)^2 + K_{12}(n_3 \cdot \nabla \times n_2)^2, 
\]  
(E.1)

where the elastic coefficients \( K_i (i = 1, \cdots, 12) \) can be expressed by \( K_{ijkl}(i, j, k, l = 1, 2, 3) \) (see [10] for details). In the above, we also neglect the surface terms (4.52).

The next task is to provide the biaxial elastic energy with the form (4.51) derived from the molecular-theory-based static tensor model (4.52), where the elastic coefficients \( K_{ijkl} \) are expressed by molecular parameters. We refer to [50] for more detailed discussion.
Assume that the minimizers of the bulk energy in (3.2) has the following biaxial form:

\[ Q_\alpha = (s_\alpha + b_\alpha) n_1^2 + 2b_\alpha n_2^2 - \left( \frac{1}{3} s_\alpha + b_\alpha \right), \quad \alpha = 1, 2. \]

Then the corresponding derivative terms are calculated as

\[
|\nabla Q_\alpha|^2 = 2(s_\alpha + b_\alpha)^2 (\partial_k n_{1i})^2 + 8b_\alpha^2 (\partial_k n_{2i})^2 + 8b_\alpha (s_\alpha + b_\alpha) n_{1i} n_{2j} \partial_k n_{1j} \partial_k n_{2i},
\]

\[
\partial_i Q_{1jk} \partial_i Q_{2jk} = 2(s_1 + b_1)(s_2 + b_2)(\partial_i n_{1j})^2 + 8b_1 b_2 (\partial_i n_{2j})^2
+ 4[b_1 (s_2 + b_2) + b_2 (s_1 + b_1)] n_{1j} n_{2k} \partial_i n_{1k} \partial_i n_{2j},
\]

\[
\partial_i Q_{\alpha nk} \partial_j Q_{\beta jk} = (s_\alpha + b_\alpha) (s_\beta + b_\beta) (|\nabla \cdot n_1|^2 + n_{1i} n_{1j} \partial_i n_{1k} \partial_j n_{1k})
+ 2[b_\alpha (s_\beta + b_\beta) + b_\beta (s_\alpha + b_\alpha)] (|\nabla \cdot n_1|^2 + n_{1i} n_{2j} \partial_i n_{1k} \partial_j n_{2k})
+ n_{1i} n_{2j} \partial_i n_{1k} \partial_j n_{2k}) + 4b_\alpha b_\beta (|\nabla \cdot n_2|^2 + n_{2i} n_{2j} \partial_i n_{2k} \partial_j n_{2k})).
\]

From which and the elastic energy in (E.2) implies that

\[
\frac{F_{\text{Bi}}(p)}{ck_BT} = \int d^3 \frac{1}{2} \left[ J_1 (\partial_i n_{1j})^2 + J_2 (\partial_i n_{2j})^2 + J_3 n_{1i} n_{2j} \partial_i n_{1j} \partial_i n_{2i}
+ J_4 (|\nabla \cdot n_1|^2 + n_{1i} n_{1j} \partial_i n_{1k} \partial_j n_{1k}) + J_5 (|\nabla \cdot n_2|^2 + n_{2i} n_{2j} \partial_i n_{2k} \partial_j n_{2k})
+ J_6 (|\nabla \cdot n_1|^2 n_{1i} n_{2k} \partial_i n_{1k} + n_{1i} n_{2j} \partial_i n_{1k} \partial_j n_{2k}) \right],
\]

(E.2)

where the coefficients \(J_i(i = 1, \ldots, 6)\) are given by (E.5).

We need to express the derivative terms in (E.2) by the nine invariant \(D_{\lambda k}(\lambda, \delta = 1, 2, 3)\). For example, the following four terms can be respectively expressed as

\[
(\partial_i n_{1j})^2 = \delta_{ij} \delta_{ik} \partial_k n_{1i} \partial_i n_{1j}
= (n_{2i} n_{2i} + n_{3i} n_{3i})(n_{1i} n_{1k} + n_{2i} n_{2k} + n_{3i} n_{3k}) \partial_k n_{1i} \partial_i n_{1j}
= (n_{1i} n_{2j} n_{1k} n_{2i} + n_{2i} n_{2j} n_{2k} n_{2i} + n_{3i} n_{3j} n_{3k} n_{3i} + n_{1i} n_{1j} n_{1k} n_{1i})
+ n_{2i} n_{2j} n_{2k} n_{3i} + n_{3i} n_{3j} n_{3k} n_{3i}) \partial_k n_{1i} \partial_i n_{1j}
= D_{13}^2 + D_{23}^2 + D_{33}^2 + D_{12}^2 + D_{22}^2 + D_{32}^2,
\]

\[
(\partial_i n_{2j})^2 = \delta_{ij} \delta_{ik} \partial_k n_{2i} \partial_i n_{2j}
= (n_{2j} n_{2j} n_{1i} n_{1k} + n_{2k} n_{2k} n_{2j} + n_{3k} n_{3k}) n_{2i} n_{2j} \partial_k n_{1i} \partial_i n_{1j}
= (n_{1i} n_{2j} n_{1i} n_{1k} + n_{2k} n_{2j} n_{2i} n_{2j} + n_{3k} n_{3j} n_{3i} n_{1i}) \partial_k n_{1i} \partial_i n_{1j}
= - (D_{13}^2 + D_{23}^2 + D_{33}^2),
\]

\[
n_{1i} n_{1j} \partial_i n_{1k} \partial_j n_{1k} = \delta_{ik} n_{1i} n_{1j} \partial_i n_{1j} \partial_j n_{1k}
= (n_{2i} n_{2i} + n_{3k} n_{3k}) n_{1i} n_{1j} \partial_i n_{1j} \partial_j n_{1k}
= D_{13}^2 + D_{23}^2.
\]

While the remaining four terms can be similarly expressed as follows:

\[
(\partial_i n_{2j})^2 = D_{13}^2 + D_{23}^2 + D_{33}^2 + D_{11}^2 + D_{21}^2 + D_{31}^2,
\]

\[
(\partial_j n_{2i})^2 = D_{13} - D_{31},
\]

\[
n_{2i} n_{2j} \partial_i n_{2k} \partial_j n_{2k} = D_{21}^2 + D_{23}^2,
\]

\[
n_{1i} n_{1j} \partial_i n_{1k} \partial_j n_{2k} = - D_{12} D_{21}.
\]
Plugging the above eight relations into (E.2), we immediately obtain the biaxial elastic energy (4.51), where the elastic coefficients $K_{ijkl}(i,j,k,l = 1,2,3)$, completely determined by the molecular parameters, are given by (4.53).

Then, we calculate the variational derivative about the frame $p$, and derive the variational derivative along the infinitesimal rotation round $n_i (i = 1,2,3)$. For instance, the variational derivative along the infinitesimal rotation round $n_1$ is given by

$$n_{2\alpha} \frac{\delta}{\delta n_3 \alpha} - n_{3\alpha} \frac{\delta}{\delta n_2 \alpha},$$

where the operator $\frac{\delta}{\delta n_3 \alpha}$ represents the variational derivative about $n_3$ assuming that $n_3$ is an independent vector (ignoring the constraints that $n_3 \cdot n_3 = 1$ and $n_3 \cdot n_1 = n_3 \cdot n_2 = 0$).

Therefore, the variational derivatives of the elastic energy (4.51) with respect to the frame $p$ can be respectively calculated as follows:

\[
\frac{\delta F_{Bi}}{\delta n_{1\alpha}} = K_{1111} D_{11} n_{2\alpha} \partial_\alpha n_{3k} - K_{2222} \partial_\alpha (D_{22} n_{2\alpha} n_{3\alpha}) + K_{3333} D_{33} n_{3k} \partial_\alpha n_{2\alpha} \\
+ K_{1212} (D_{12} n_{3k} \partial_\alpha n_{1\alpha} - \partial_\alpha (D_{12} n_{1\alpha} n_{3\alpha})) + K_{2323} D_{23} n_{2\alpha} \partial_\alpha n_{2\alpha} \\
- K_{3232} \partial_\alpha (D_{32} n_{3k} n_{3\alpha}) + K_{1313} D_{13} n_{1\alpha} (\partial_\alpha n_{2\alpha} + \partial_\alpha n_{2k}) \\
+ \frac{1}{2} K_{1221} (D_{21} n_{3k} \partial_\alpha n_{1\alpha} - \partial_\alpha (D_{21} n_{1\alpha} n_{3\alpha})) \\
+ \frac{1}{2} K_{2332} (D_{32} n_{2\alpha} \partial_\alpha n_{2\alpha} - \partial_\alpha (D_{23} n_{3\alpha} n_{3\alpha})) \\
+ \frac{1}{2} K_{1331} D_{31} n_{1\alpha} (\partial_\alpha n_{2\alpha} + \partial_\alpha n_{2k}) \\
\]  \hspace{1cm} (E.3)

\[
\frac{\delta F_{Bi}}{\delta n_{2\alpha}} = K_{1111} D_{11} n_{1\alpha} \partial_\alpha n_{3k} + K_{2222} D_{22} n_{3k} \partial_\alpha n_{1\alpha} - K_{3333} \partial_\alpha (D_{33} n_{3k} n_{1\alpha}) \\
+ K_{2121} D_{21} n_{2\alpha} (\partial_\alpha n_{3\alpha} + \partial_\alpha n_{3k}) + K_{2323} (D_{23} n_{1\alpha} \partial_\alpha n_{2\alpha} - \partial_\alpha (D_{23} n_{2\alpha} n_{1\alpha})) \\
+ K_{3131} D_{31} n_{3k} \partial_\alpha n_{3\alpha} - K_{3133} \partial_\alpha (D_{31} n_{1\alpha} n_{1\alpha}) \\
+ \frac{1}{2} K_{1221} D_{12} n_{2\alpha} (\partial_\alpha n_{3\alpha} + \partial_\alpha n_{3k}) \\
+ \frac{1}{2} K_{2332} (D_{32} n_{1\alpha} \partial_\alpha n_{2\alpha} - \partial_\alpha (D_{32} n_{2\alpha} n_{1\alpha})) \\
+ \frac{1}{2} K_{1331} (D_{13} n_{3k} \partial_\alpha n_{3\alpha} - \partial_\alpha (D_{31} n_{1\alpha} n_{1\alpha})), \\
\]  \hspace{1cm} (E.4)

\[
\frac{\delta F_{Bi}}{\delta n_{3\alpha}} = - K_{1111} \partial_\alpha (D_{11} n_{1\alpha} n_{2\alpha}) + K_{2222} D_{22} n_{2\alpha} \partial_\alpha n_{1\alpha} + K_{3333} D_{33} n_{3k} \partial_\alpha n_{2\alpha} \\
+ K_{1212} D_{12} n_{1\alpha} \partial_\alpha n_{1\alpha} - K_{2121} \partial_\alpha (D_{21} n_{2\alpha} n_{2\alpha}) + K_{3232} D_{32} n_{2\alpha} (\partial_\alpha n_{1\alpha} + \partial_\alpha n_{1k}) \\
+ K_{3131} (D_{31} n_{2\alpha} \partial_\alpha n_{3k} - \partial_\alpha (D_{31} n_{3k} n_{2\alpha})) \\
+ \frac{1}{2} K_{1221} (D_{21} n_{1\alpha} \partial_\alpha n_{1\alpha} - \partial_\alpha (D_{12} n_{2\alpha} n_{2\alpha})) \\
+ \frac{1}{2} K_{2332} D_{23} n_{3k} (\partial_\alpha n_{1\alpha} + \partial_\alpha n_{1k}) \\
+ \frac{1}{2} K_{1331} (D_{13} n_{2\alpha} \partial_\alpha n_{3k} - \partial_\alpha (D_{13} n_{3k} n_{2\alpha})). \\
\]  \hspace{1cm} (E.5)

Using the variational derivatives (E.3) and (E.5), we have

\[
\frac{n_{2\alpha} \delta F_{Bi}}{\delta n_{3\alpha}} - \frac{n_{3\alpha} \delta F_{Bi}}{\delta n_{2\alpha}} = -K_{1111} n_{2\alpha} \partial_\alpha (D_{11} n_{1\alpha} n_{2\alpha}) - K_{2222} D_{22} (D_{23} + D_{32})
\]
Similarly, we have

\[
\begin{align*}
K_{3333} & \left( D_{33} D_{23} + n_{3 \alpha} \partial_k (D_{33} n_{3 k} n_{1 \alpha}) \right) - K_{1211} D_{12} D_{13} \\
- K_{2121} & \left( D_{21} D_{31} + n_{2 \alpha} \partial_k (D_{21} n_{2 k} n_{2 \alpha}) \right) + K_{3232} D_{32} (D_{22} - D_{33}) \\
- K_{2323} & \left( D_{33} D_{23} - n_{3 \alpha} \partial_k (D_{23} n_{2 k} n_{1 \alpha}) \right) \\
+ K_{3131} & \left( D_{31} D_{21} - n_{2 \alpha} \partial_k (D_{31} n_{3 k} n_{2 \alpha}) \right) + K_{1313} n_{3 \alpha} \partial_k (D_{13} n_{1 k} n_{1 \alpha}) \\
- \frac{1}{2} K_{1211} & \left( D_{21} D_{13} + D_{12} D_{31} + n_{2 \alpha} \partial_k (D_{12} n_{2 k} n_{2 \alpha}) \right) \\
+ \frac{1}{2} K_{2323} & \left( D_{23} (D_{22} - D_{33}) - D_{33} D_{32} + n_{3 \alpha} \partial_k (D_{32} n_{2 k} n_{1 \alpha}) \right) \\
+ \frac{1}{2} K_{1313} & \left( D_{13} D_{21} - n_{2 \alpha} \partial_k (D_{13} n_{3 k} n_{2 \alpha}) + n_{3 \alpha} \partial_k (D_{31} n_{1 k} n_{1 \alpha}) \right). \\
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
n_{3 \alpha} \frac{\delta F_{Bi}}{\delta n_{1 \alpha}} - n_{1 \alpha} \frac{\delta F_{Bi}}{\delta n_{3 \alpha}} &= K_{1111} D_{11} (D_{13} + D_{31}) - K_{2222} \partial_k (n_{2 k} D_{22}) - K_{3333} D_{33} (D_{13} + D_{31}) \\
+ K_{1212} & \left( D_{12} D_{32} - \partial_k (n_{1 k} D_{12}) \right) + (K_{2121} - K_{2323}) D_{23} D_{21} \\
- K_{3232} & \left( D_{12} D_{32} + \partial_k (n_{3 k} D_{32}) \right) + K_{1313} D_{13} (D_{33} - D_{11}) \\
- K_{3131} D_{31} (D_{11} - D_{33}) + \frac{1}{2} K_{1221} & \left( D_{21} D_{32} + D_{12} D_{23} - \partial_k (n_{1 k} D_{21}) \right) \\
- \frac{1}{2} K_{2323} & \left( D_{32} D_{21} + D_{23} D_{12} + \partial_k (n_{3 k} D_{23}) \right) \\
+ \frac{1}{2} K_{1313} & \left( D_{31} D_{13} + D_{33} D_{31} - \partial_k (n_{1 k} D_{31}) \right),
\end{align*}
\]

and

\[
\begin{align*}
n_{1 \alpha} \frac{\delta F_{Bi}}{\delta n_{2 \alpha}} - n_{2 \alpha} \frac{\delta F_{Bi}}{\delta n_{1 \alpha}} &= -K_{1111} D_{11} (D_{12} + D_{21}) + K_{2222} D_{22} (D_{12} + D_{21}) - K_{3333} \partial_k (n_{3 k} D_{33}) \\
+ K_{2121} D_{21} (D_{11} - D_{22}) + K_{2323} & \left( D_{23} D_{13} - \partial_k (n_{2 k} D_{23}) \right) \\
+ (K_{3232} - K_{3131}) D_{31} D_{32} & - K_{1313} D_{13} (D_{23} + \partial_k (n_{1 k} D_{13})) \\
- K_{1212} D_{12} (D_{22} - D_{11}) & + \frac{1}{2} K_{1221} (D_{12} + D_{21}) (D_{11} - D_{22}) \\
+ \frac{1}{2} K_{2323} & \left( D_{32} D_{13} + D_{23} D_{31} - \partial_k (n_{2 k} D_{32}) \right) \\
- \frac{1}{2} K_{1313} & \left( D_{32} D_{13} + D_{23} D_{31} + \partial_k (n_{1 k} D_{31}) \right).
\end{align*}
\]

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