Abstract. We equip the space of convex rational cones with a connected coalgebra structure, which we further generalize to decorated cones by means of a differentiation procedure. Using the convolution product $\ast$ associated with the coproduct on cones we define an interpolator $\mu := I^{(-1)} \ast S$ as the $\ast$ quotient of an exponential discrete sum $S$ and an exponential integral $I$ on cones. A generalization of the algebraic Birkhoff decomposition to linear maps from a connected coalgebra to a space with a linear decomposition then enables us to carry out a Birkhoff-Hopf factorization $S := S^{(-1)} \ast S_+$ on the map $S$ whose “holomorphic part” corresponds to $S_+$. By the uniqueness of the Birkhoff-Hopf factorization we obtain $\mu = S_+$ and $I = S^{(-1)}$ so that this renormalization procedure à la Connes and Kreimer yields a new interpretation of the local Euler-Maclaurin formula on cones of Berline and Vergne. The Taylor coefficients at zero of the interpolating holomorphic function $\mu = S_+$ correspond to renormalized conical zeta values at non-positive integers. When restricted to Chen cones, this yields yet another way to renormalize multiple zeta values at non-positive integers previously investigated by the authors using other approaches.

In the present approach renormalized conical multiple zeta values lie at the cross road of three a priori distinct fields, the geometry on cones with the Euler-Maclaurin formula, number theory with multiple zeta values and renormalization theory with methods borrowed from quantum field theory.

Contents

1. Introduction
2. The coalgebra of cones
   2.1. Convex rational cones
   2.2. A coproduct on cones
   2.3. The coalgebra of decorated open cones
3. Germs of meromorphic functions with linear poles at zero
   3.1. Germs of Meromorphic functions with linear poles
   3.2. A decomposition of germs of meromorphic functions with linear poles
   3.3. A generalized evaluator at zero
4. Algebraic Birkhoff decomposition
5. Conical zeta values and their regularization
   5.1. Regularization
   5.2. Meromorphicity of regularized conical zeta values
6. The Euler-Maclaurin formula on cones and renormalization
   6.1. Euler-Maclaurin formula for closed cones
   6.2. The Euler-Maclaurin formula for open cones
7. Renormalization of conical zeta values and multiple zeta values

Date: May 7, 2014.
2010 Mathematics Subject Classification. 11M32, 11H06, 52C07, 52B20, 65B15, 81T15.
Key words and phrases. convex cones, conical zeta values, multiple zeta values, renormalization, Birkhoff decomposition, Euler-Maclaurin formula, meromorphic functions.
1. Introduction

Our starting point is the one-dimensional Euler-Maclaurin formula for exponential sums which we briefly review in the perspective of a generalization to higher dimensions.

The discrete sum $S(\varepsilon) := \sum_{k=0}^{\infty} e^{\varepsilon k} = \frac{1}{1-e^\varepsilon}$ for positive $\varepsilon$ relates to the integral $I(\varepsilon) := \int_0^{\infty} e^{\varepsilon x} \, dx = \frac{1}{\varepsilon}$ by means of the Euler-Maclaurin formula which gives a Taylor expansion of the interpolator

$$\mu(\varepsilon) := S(\varepsilon) - I(\varepsilon) = -\sum_{n=0}^{\infty} \frac{B_{n+1}}{(n+1)!} \varepsilon^n,$$

which is holomorphic at $\varepsilon = 0$, in terms of the Bernoulli numbers $B_n$. Since the integral $I(\varepsilon)$ corresponds to the pole part of the Laurent expansion of the sum $S(\varepsilon)$ at zero, we have $\mu(\varepsilon) = S_+(\varepsilon) + \frac{\text{Res}_0 S}{\varepsilon}$ where $\text{Res}_0$ stands for the residue at zero. Thus, $\mu(0) = S_+(0) = \frac{1}{2} = B_1$ corresponds to the regularized evaluation of $S$ at zero obtained by what physicists call a minimal subtraction scheme. It relates to the zeta function at zero $\mu(0) = -\zeta(0)$ and can be viewed as the renormalized number of $\sum_{n=0}^{\infty} 1$ of integer points on the one-dimensional closed cone $[0, +\infty)$.

Similarly, the higher Taylor coefficients of the holomorphic function $\mu$ at zero relate to the value of the zeta function at negative integers

$$\mu^{(n)}(0) = -\frac{B_{n+1}}{n+1} = \zeta(-n)$$

and can be viewed as renormalized polynomial sums $\sum_{k=0}^{\infty} k^n$ on integer points on the one-dimensional closed cone $[0, +\infty)$.

These exponential discrete sum and integral on the one-dimensional cone $[0, +\infty)$ generalize to higher dimensional convex rational polyhedral cones together with the various interpretations of the interpolator $\mu$ described above:

- The minimal subtraction scheme used in the one-dimensional case to derive $\mu = S_+$ from $S$ by extracting the pole at $\varepsilon = 0$ (which parameterizes the edge of the cone $[0, +\infty)$) is substituted in higher dimensions by a Birkhoff-Hopf factorization, an algebraic procedure borrowed from the physics literature to extract more complicated poles corresponding to the faces of the cones.
- The Taylor coefficients of $\mu$ which in the one-dimensional case relate to the $\zeta$-values at non-positive integers, relate for higher dimensional cones to conical zeta values at non-positive integers, that generalize the well-known multiple zeta values familiar to number theorists. These correspond here to conical zeta values associated with Chen cones given by $x_k < \cdots < x_1$ in $\mathbb{R}^k$.

This generalization to higher dimensional cones provides a new outlook on the remarkable local Euler-Maclaurin formula on cones derived by N. Berline and M. Vergne [1], which arises here as a straightforward consequence of the Birkhoff-Hopf factorization of discrete exponential sums on the coalgebra of cones. Also, the coefficients of its Taylor expansion at zero, which we relate to renormalized conical zeta values at non-positive integers, can be viewed as renormalized sums of polynomials at integer points of these cones. Thus the renormalization techniques borrowed...
from quantum field theory provide a useful short-cut to the rather involved Euler-Maclaurin formula and highlights its relation to multiple zeta values at non-positive integers. The renormalized multiple zeta values (as conical zeta values, they are associated with Chen cones $x_k < \cdots < x_1$) obtained this way differ from the ones derived by other methods \cite{4}, \cite{14}, all of which nevertheless obey the stuffle relations. Thus conical zeta values seem to offer a good testing ground to approach the difficult concept of an abstract renormalization group which would relate the different renormalized values.

Let us now describe the higher dimensional framework more precisely. To a $k$-dimensional cone $C$ in $V_k := \mathbb{R}^k$ one assigns the discrete sum and discrete integral

$$S(C)(\vec{\varepsilon}) := \sum_{C \subseteq \mathbb{Z}^k} e^{i\langle \vec{\varepsilon}, \vec{x} \rangle}; \quad I(C)(\vec{\varepsilon}) := \int_C e^{i\langle \vec{\varepsilon}, \vec{x} \rangle} d\vec{x}$$

where $\vec{\varepsilon}$ is a vector in the dual space $V^*_k$ such that the duality pairing $\langle \vec{\varepsilon}, \vec{x} \rangle$ is positive on the cone. Both $S(C)$ and $I(C)$ extend to meromorphic functions on $\mathbb{C}^k$ with linear poles corresponding to the equations of the faces of the cones provided the cone is smooth. The Euler-Maclaurin formula on cones derived by Berline and Vergne in \cite{10}, which relates these discrete sum and integral on cones, hints to the existence of an underlying coproduct on the linear space of cones. We indeed prove the existence of a coalgebra structure on cones (Proposition \ref{2.3}), whose coproduct involves the notion of transverse cone introduced in \cite{10}. By means of a differentiation on cones, this coalgebra structure carries out to decorated cones (Theorem \ref{2.12}) which form a differential connected coalgebra in the sense of Definition \ref{2.7}.

Having introducing a linear decomposition (Theorem \ref{3.3}) on the algebra of germs of meromorphic functions in $\mathbb{C}^m$ with linear poles, we then use the coproduct on the linear space of cones to implement a Birkhoff-Hopf factorization $S_+ = S_{+}^{-1} \ast S$ on the map $S$ (Theorem \ref{4.1}). This Birkhoff-Hopf factorization can be carried out either on cones $C$ or on decorated cones $(C, \vec{s})$ for some $\vec{s} = (s_1, \cdots, s_k) \in \mathbb{Z}^k_{\geq 0}$ if $C$ is $k$-dimensional. Differentiating the meromorphic function $\vec{\varepsilon} \mapsto S(C)(\vec{\varepsilon})$, for non-positive integers $s_1, \cdots, s_k$ we get meromorphic functions

$$S(C; \vec{s})(\vec{\varepsilon}) = D_{-e_1}^{-s_1} \cdots D_{-e_k}^{-s_k} S(C)(\vec{\varepsilon})$$

with linear poles at zero. Thanks to the compatibility of the coproduct with the differentiation on decorated cones, we relate the resulting renormalized sums $S_+$ on decorated cones, which are holomorphic at zero, to the ones on ordinary cones via iterated differentiation:

$$S_+(C; \vec{s})(\vec{\varepsilon}) = D_{-e_1}^{-s_1} \cdots D_{-e_k}^{-s_k} S_+(C)(\vec{\varepsilon}). \tag{1}$$

The uniqueness of the Birkhoff-Hopf factorization yields an interpretation of the Euler-Maclaurin formula in terms of a Birkhoff-Hopf factorization (Theorem \ref{6.1}):

$$S = \mu \ast I = S_{+}^{-s(-1)} \ast S_+ \Rightarrow \mu := I^{s(-1)} \ast S = S_+$$

both on ordinary cones and on decorated cones. Here $\ast$ denotes the convolution product associated with the coproduct on cones. In other words, the interpolator $\mu$ between the discrete sum $S$ and the integral $I$ on cones coincides with the "holomorphic" part of $S$ arising in the Birkhoff-Hopf factorization.

The Taylor coefficients in the expansion of $S_+$ at zero are natural candidates for the renormalized conical zeta values at non-positive integers $(s_1, \cdots, s_k)$

$$\zeta_{\mathcal{C}}^{\text{ren}}(s_1, \cdots, s_k) := \left(D_{-e_1}^{-s_1} \cdots D_{-e_k}^{-s_k} S_+(C)\right)(0) = S_+(C; \vec{s})(0),$$
which for Chen cones \( x_k < \cdots < x_1 \) correspond to renormalized multiple zeta values. This yields yet another method (compare with [B, [14]]) to renormalize multiple zeta values at non-positive integers. When viewed as generalized Bernoulli numbers, it comes at no surprise that they are rational.

2. The coalgebra of cones

2.1. Convex rational cones. Let \( k \geq 0 \) be an integer. Let \( V_k := \mathbb{R}^k \) be the standard Euclidean space equipped with the canonical basis \( \{e_1, \ldots, e_k\} \), and let \( \Lambda_k := \oplus_{i=1}^{k} \mathbb{Z}e_i \) be the standard lattice. Using the standard embedding \( \mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1} \) one can equip the vector space \( V := \bigcup_{k=0}^{\infty} V_k = \mathbb{R}^{\infty} \) and the corresponding lattice \( \Lambda := \bigcup_{k=0}^{\infty} \Lambda_k = \mathbb{Z}^{\infty} \) with their natural filtration. Let \( V_k^* \) be the dual space of \( V_k \), then there is a natural extension of elements in \( V_k^* \) to elements in \( V_{k+1}^* \). Let \( V^* = \bigcup_{k=0}^{\infty} V_k^* \), viewed as the the dual space of \( V \).

If \( W \) is a subspace of \( V_k \), the dual space \( (V_k/W)^* \) is canonically identified with

\[
W^\perp := \{ \overline{z} \in V_k^* \mid \langle \overline{z}, w \rangle = 0 \text{ for all } w \in W \},
\]

denotes the dual product. In this paper we identify \( V_k^* \) with \( V_k \) by means of the canonical inner product

\[
\langle \cdot, \cdot \rangle : V_k \otimes V_k \to \mathbb{R}, \quad (a_1, \cdots, a_k) \otimes (b_1, \cdots, b_k) \mapsto \sum_{i=1}^{k} a_ib_i.
\]

Then \( W^\perp \) can be identified with the orthogonal subspace with respect to the canonical inner product, hence the notation. For a subset \( S = \{s_\alpha\}_\alpha \) of \( V_k \), let \( \operatorname{lin} S = \operatorname{lin}(s_\alpha)_\alpha \) denote its linear span and \( S^\perp = \operatorname{lin}^\perp(S) \) the orthogonal subspace.

**Example 2.1.** For \( W = \operatorname{lin}(e_1 + e_2) \subset V_2 = \operatorname{lin}(e_1, e_2) \), we have \( V_2/W \cong W^\perp = \operatorname{lin}(e_1 - e_2) \).

We now collect basic notations and facts (mostly following [B] and [15]) on cones that will be used in this paper. See [B] for a detailed discussion on these facts.

(a) A **rational (polyhedral) cone** in \( V_k \) is the convex set

\[
\langle v_1, \cdots, v_n \rangle := \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n,
\]

where \( v_i \in \Lambda_k, i = 1, \cdots, n \). If \( \operatorname{lin} \langle v_1, \cdots, v_n \rangle = V_k \) we call the cone a full-dimensional cone.

(b) The set \( \{v_1, \cdots, v_n\} \) in the definition of a cone \( C \) is called a **generating set** or a **spanning set** of the cone. Define the dimension of \( C \) by \( \dim C := \dim \operatorname{lin}(C) \).

(c) A cone in \( V_k \) can also be described as the intersection \( \bigcap_i H_{u_i} \) of finitely many half spaces \( H_{u_i} = \{ x \in V_k | u_i(x) \geq 0 \} \) defined by linear functionals \( u_i : V_k \to \mathbb{R} \) (see [15, Theorem 1.3]).

(d) Let \( C_k \) denote the set of rational cones in \( V_k, k \geq 1 \). For \( k = 0 \) we set \( C_0 = \{0\} \) by convention. The natural inclusions \( C_k \to C_{k+1} \) induced by the natural inclusions \( V_k \to V_{k+1}, k \geq 1 \), give rise to the direct limit \( C = \operatorname{lim} C_k \).

(e) A **simplicial cone** or a **simplex cone** is a cone spanned by linearly independent vectors.

(f) A **smooth cone** is a rational cone with a spanning set that is a part of a basis of \( \Lambda_k \subset \mathbb{R}^k \). In this case, the spanning set is unique and is called the **primary set** of the cone.

(g) A cone is called **strongly convex** if it does not contain any linear subspace.
2.2. A coproduct on cones. We consider the linear space \( \mathbb{Q}C \) over \( \mathbb{Q} \) with basis \( C \). To explore its structure, we first borrow the following concept from [1].

**Definition 2.3.** Let \( F \) be a face of a cone \( C \subseteq V_k \). The transverse cone \( t(C, F) \) to \( F \) is the projection \( \pi_{F^\perp}(C) \) of \( C \) on \( V_k/\text{lin}(F) \approx \text{lin}(F)^\perp \subseteq V_k \).

**Remark 2.4.** The definition of \( t(C, F) \) does not depend on the choice of \( k \geq 1 \) such that \( C \subseteq V_k \), so it is well-defined in \( C \).

**Example 2.5.** The transverse cone to the face \( F = \langle e_1 + e_2 \rangle \) in the cone \( C = \langle e_1, e_1 + e_2 \rangle \) is the cone \( t(C, F) = \langle e_1 - e_2 \rangle \). Note that \( t(C, F) \) is not a face of \( C \).

**Lemma 2.6.** Let \( \mathcal{F}(C) \) denote the set of faces of a cone \( C \). The map

\[
\mathcal{F}(C) \rightarrow C \\
F \mapsto t(C, F)
\]

which to a face \( F \) of a cone \( C \) assigns the transverse cone \( t(C, F) \), enjoys the following properties.

(a) **(Transitivity)** \( t(C, F) = t(t(C, F'), t(F, F')) \) if \( F' \) is a face of \( F \).

(b) **(Compatibility with the partial order)** There is a one-to-one correspondence between the set of faces containing \( F \) and the set of faces of \( T(C, F) \).

(c) **(Compatibility with the dimension filtration)** \( \dim(C) = \dim(F) + \dim(t(C, F)) \) for any face \( F \) of \( C \).

**Proof.** (1) By definition, \( t(C, F) \) is the quotient \( (C+\text{lin}(F))/\text{lin}(F) \) and \( \text{lin}(t(C, F)) = \text{lin}(C)/\text{lin}(F) \). Thus we have

\[
t(t(C, F'), t(F', F')) = [t(C, F') + \text{lin}(t(F', F'))]/\text{lin}(t(F', F')) \\
= [(C + \text{lin}(F'))/\text{lin}(F') + \text{lin}(F)/\text{lin}(F')]/[\text{lin}(F)/\text{lin}(F')] \\
= [C + \text{lin}(F)]/\text{lin}(F),
\]

as needed.

(2) Assume that \( F \) is defined by a linear function \( u_F \in V^* \), i.e.,

\[
F = C \cap u_F^\perp = \{v \in C \mid \langle u_F, v \rangle = 0\}.
\]

Let \( G \) be any face of \( C \) containing \( F \) that is defined by \( u_G \in V^* \), then \( u_G|_F = 0 \). But any element \( u \in V^* \) with \( u|_F = 0 \) induces an element \( u \in (\text{lin}(F)^\perp)^* \). So we can view \( u_G \) as an element in \( (\text{lin}(F)^\perp)^* \); it therefore defines a face \( t(G, F) \) of \( t(C, F) \). We can therefore define a map

\[
t(\bullet, F) : \{\text{faces of } C \text{ containing } F\} \rightarrow \{\text{faces of } t(C, F)\}
\]

\[
G \mapsto t(G, F) = t(C, F) \cap u_G^\perp.
\]

To check the bijectivity of \( t(\bullet, F) \), first note that any face of \( T(C, F) \) is defined by some element \( u \in (\text{lin}(F)^\perp)^* \) which can be viewed as an element in \( V^* \) that vanishes on \( \text{lin}(F) \). Hence \( u \) defines a face \( G \) of \( C \) containing \( F \). Thus \( t(\bullet, F) \) is surjective.
Next for two different faces \(G_1, G_2\) containing \(F\) defined by \(u_1, u_2 \in V^*\), there are vectors \(v_1, v_2\) in \(G_1\) and \(G_2\) such that \(\langle u_1, v_2 \rangle > 0\) and \(\langle u_2, v_1 \rangle > 0\). Thus \(T(G_1, F)\) and \(T(G_2, F)\) are different since the image of \(v_1\) is not in \(T(G_2, F)\) and the image of \(v_2\) is not in \(T(G_1, F)\). Hence the map \(T(\bullet, F)\) is one-to-one.

\(^{(1)}\) follows directly from the definition of \(t(C, F)\).

We equip the space \(\mathbb{Q}C\) of cones with the coproduct
\[
\Delta : C \longrightarrow C \otimes C, \quad C \longmapsto \Delta C = \sum_{F \subseteq C} F \otimes t(C, F).
\]
and counit
\[
\varepsilon : C \longrightarrow \mathbb{R}, \quad C \longmapsto \begin{cases} 1, & C = \{0\}, \\ 0, & C \neq \{0\}. \end{cases}
\]

We consider the concept of a connected coalgebra similar to the case of connected bialgebra \(\text{[13]}\). See also \(\text{[6]}, \S\ 2.3\).

**Definition 2.7.** Let \((C, \Delta)\) be a coalgebra over a field \(k\) with counit \(\varepsilon\). It is called

(a) **cograded** if there is a grading \(C = \bigoplus_{n \geq 0} C^{(n)}\) such that
\[
\Delta(C^{(n)}) \subseteq \bigoplus_{p+q=n} C^{(p)} \otimes C^{(q)}, \quad n \geq 0.
\]

Elements in \(C^{(n)}\) are said to have degree \(n\).

(b) **coaugmented** if there is a linear map \(u : k \rightarrow C\), called the **coaugmentation**, such that
\[
\varepsilon \circ u = \text{id}_k.
\]

(c) **connected** if it is cograded, coaugmented and
\[
C^{(0)} = k u(1).
\]

With the coaugmentation \(u, C\) is canonically isomorphic to \(\ker \varepsilon \oplus k u(1)\). Set \(\overline{C} := \ker \varepsilon\) and \(I := u(1)\). The following result is well-known for bialgebras. We give some details for completeness.

**Lemma 2.8.** Let \((C, \Delta)\) be a connected coalgebra. Then
\[
\overline{C} = \bigoplus_{n \geq 1} C^{(n)}.
\]

Further the reduced coproduct
\[
\overline{\Delta} : \overline{C} \rightarrow \overline{C} \otimes \overline{C}, \quad \overline{\Delta}(x) := \Delta(x) - x \otimes I - I \otimes x, \quad x \in C
\]
is well defined and \(\overline{\Delta}^m(C^{(n)}) = 0\) for \(m \geq n \geq 1\) where \(\overline{\Delta}^m, m \geq 2\), is defined by the recursion \(\overline{\Delta}^m = (\text{id} \otimes \overline{\Delta}^{m-1}) \circ \overline{\Delta}\).

The last condition is called the **conilpotency** of \(\Delta\) \([12]\).

**Proof.** For the first statement, we just need to prove \(\varepsilon(x) = 0\) and \(\overline{\Delta}(x) \in \sum_{p+q=n, p,q \geq 0} C^{(p)} \otimes C^{(q)}\) for \(x \in C^{(n)}\) with \(n \geq 1\). We will proceed by induction on \(n\). The case \(n = 1\) follows from the connectedness, which yields
\[
\Delta(x) = y \otimes I + I \otimes z
\]
with \(y, z \in C^{(1)}\). Recall the counitariness of \(\varepsilon\) for \(\Delta\):
\[
(4) \quad \beta_\varepsilon = (\varepsilon \otimes \text{id}) \Delta, \quad \beta_\varepsilon = (\text{id} \otimes \varepsilon) \Delta,
\]
where 
\[ \beta_r : C \to k \otimes C, x \mapsto I \otimes x, \quad \beta_r' : C \to C \otimes k, x \mapsto x \otimes I \]
with 
\[ \beta_r^{-1} : k \otimes C \to C, a \otimes x \mapsto ax, \quad \beta_r'^{-1} : C \otimes k \to C, x \otimes a \mapsto ax. \]

Then we have
\[ x = \beta_r'^{-1}(\text{id} \otimes \varepsilon)(\Delta(x)) = \varepsilon(y)I + \varepsilon(I)z = \varepsilon(y)I + z. \]

Comparing the graded components yields \( \varepsilon(y) = 0 \) and \( x = z \). Similarly, we have \( \varepsilon(z) = 0 \) and \( x = y \). Thus \( \varepsilon(x) = 0 \) and \( \Delta(x) = 0 \) as needed. Assume the claim holds for \( n \leq k \) where \( k \geq 1 \) and consider \( x \in C^{(k+1)} \). Then by the connectedness, we have
\[ \Delta(x) = y \otimes I + I \otimes z + w \]
with \( y, z \in C^{(k+1)} \) and \( w \in \sum_{0+p+q=k+1, p>0, q>0} C^{(p)} \otimes C^{(q)}. \) By the induction hypothesis, we have
\[ (\text{id} \otimes \varepsilon)(w) = 0 = (\varepsilon \otimes \text{id})(w). \] The counitality of \( \varepsilon \) then implies
\[ x = \beta_r'^{-1}(\text{id} \otimes \varepsilon)(\Delta(x)) = \varepsilon(y)I + \varepsilon(I)z = \varepsilon(y)I + z \]
and \( x = \beta_r'^{-1}(\varepsilon \otimes \text{id})(\Delta(x)) = (\varepsilon(I)y + \varepsilon(z))I. \) Thus \( z = x = y \) and \( \varepsilon(x) = 0 \), completing the induction.

We finally prove the conilpotency \( \tilde{\Delta}^m(x) = 0 \) when \( m \geq n \) by induction on the degree \( n \geq 1 \) of \( x \). If \( n = 1 \), then by definition \( \tilde{\Delta}(x) = 0 \) and hence \( \tilde{\Delta}^m(x) = (\text{id} \otimes \tilde{\Delta}^{m-1}) \circ \tilde{\Delta}(x) = 0 \) for \( m \geq 1 \). Assuming that \( \tilde{\Delta}^m(x) = 0 \) for \( x \) of degree \( n \geq 1 \) and \( m \geq n \). Then for \( x \) of degree \( n + 1 \), we have \( \tilde{\Delta}(x) = \sum_{(x)} x' \otimes x'' \) with \( x' \) and \( x'' \) of degree less than or equal to \( n \). Thus by the induction hypothesis, we have
\[ \tilde{\Delta}^m(C) = \sum_{(x)} x' \otimes \tilde{\Delta}^{m-1}(x'') = 0, \quad \text{for all } m \geq n + 1. \]

This completes the induction. \( \square \)

**Proposition 2.9.** \( \langle C, \Delta \rangle \) defines a connected coalgebra.

**Proof.** To prove that \( \mathbb{Q}C \) is a coalgebra, we only need to verify the coassociativity of the coproduct. Let \( C \) be a cone. On one hand we have,
\[ (\text{id} \otimes \Delta)(\Delta(C)) = \sum_{F \subseteq C} (\text{id} \otimes \Delta)(t(C, F) \otimes F) = \sum_{F \subseteq F' \subseteq C} t(C, F) \otimes t(F, F') \otimes F'. \]

On the other hand,
\[ (\Delta \otimes \text{id})(\Delta(C)) = \sum_{F \subseteq C} (\Delta \otimes \text{id})(t(C, F') \otimes F') \]
\[ = \sum_{F \subseteq C} \sum_{H \subseteq C} (t(t(C, F'), H) \otimes H) \otimes F' \]
\[ = \sum_{F' \subseteq F \subseteq C} t(C, F) \otimes t(F, F') \otimes F', \]
where we have used Lemma 2.6 to write \( H = t(F, F') \) and \( t(t(C, F'), t(F, F')) = t(C, F) \).

The counicity in Eq. (13) is clear. Define \( u : \mathbb{Q} \to \mathbb{Q}C \) by \( u(1) := \{0\} \). Then we have \( \mathbb{Q}C = \mathbb{Q}[0] \oplus \ker \varepsilon. \) So \( \mathbb{Q}C \) is augmented. Finally define
\[ C^{(n)} := \{ C \in C \mid \dim C = n \}. \]
Then
\[QC = \bigoplus_{n \geq 0} QC^{(n)},\]
making $QC$ into a connected coalgebra by Lemma 2.10. \(\square\)

2.3. The coalgebra of decorated open cones. For a fixed $k \geq 0$, let
\[DC_k := \{(C; \vec{s}) \mid C \in C_k, \vec{s} \in \mathbb{Z}_\geq 1^\ell, \ell \geq k\},\]
with the convention that $DC_0 := \{(\{0\}; \vec{s}) \mid \vec{s} \in \mathbb{Z}_\geq 1^\ell, \ell \geq 0\}$. Through the natural embeddings $C_k \to C_{k+1}$ and $\mathbb{Z}_\geq 1^\ell \to \mathbb{Z}_\geq 1^{\ell+1}, \ell \geq k$, we obtain a natural embedding $DC_k \to DC_{k+1}$ and hence the direct limit
\[DC := \lim_{\longrightarrow} DC_k = \bigcup_{k \geq 0} DC_k.\]

Define
\[DC^{(n)} := \{(C; \vec{s}) \mid \dim C = n\}, \quad n \geq 0.\]

The $\mathbb{Q}$-space $\mathbb{Q}DC$ carries a family of linear operators:
\[\delta_i : QDC \to QDC, \quad \delta_i(C; \vec{s}) := \begin{cases} (C; \vec{s} + e_i), & \text{Proj}_{\text{lin}(e_i)}(C) \neq 0, \\ 0, & \text{otherwise}. \end{cases}\]

Here $e_i$ is the $i$-th unit vector in $\mathbb{Z}^\ell$. Thus in particular, $\delta_i(\{0\}; \vec{s}) = 0$ for all $\vec{s}$. For $(C, \vec{s}) \in DC$ with $\vec{s} = (s_1, \ldots, s_\ell) \in \mathbb{Z}_\geq 1^\ell$, denote
\[s^*: = (s^*_1, \ldots, s^*_\ell), \quad s^*_i := \begin{cases} s_i, & \text{Proj}_{\text{lin}(e_i)}(C) \neq 0, \\ 0, & \text{otherwise}. \end{cases}\]

Then we immediately obtain

**Lemma 2.10.** For $C \in C_k$ and $\vec{s} \in \mathbb{Z}^\ell$ with $\ell \geq k$, we have
\[(C; \vec{s}) = \delta_1^{s^*_1} \cdots \delta_\ell^{s^*_\ell}(C; \vec{s} - \vec{s}^*) = \delta_{i_1}^{s_{i_1}} \cdots \delta_{i_m}^{s_{i_m}}(C; \vec{s} - \vec{s}^*),\]
where $\{i_1, \ldots, i_m\} = \{i \mid \text{Proj}_{\text{lin}(e_i)}(C) \neq \{0\}\}$.

We define a coproduct $\Delta$ on $\mathbb{Q}DC$ by induction on $n := |\vec{s}^*| \geq 0$. If $n = 0$, then define
\[\Delta(C; \vec{s}) := \sum_{(C)} (C_{(1)}; \vec{s}) \otimes (C_{(2)}; \vec{s}),\]
where $\Delta(C) = \sum_{(C)} C_{(1)} \otimes C_{(2)}$ is the coproduct defined in Eq. (2). In particular, $\Delta(\{0\}; \vec{s}) = (\{0\}; \vec{s}) \otimes (\{0\}; \vec{s})$ for all $\vec{s} \in \mathbb{Z}_\geq 1^\ell, \ell \geq 0$.

Assume that $\Delta(C; \vec{s})$ have been defined for $|\vec{s}^*| = k$ for $k \geq 0$. Then for $(C, \vec{s})$ with $|\vec{s}^*| = k + 1$, there is $1 \leq i \leq \ell$ with $s^*_i \geq 1$. Then we define
\[\Delta \circ \delta_i = D_i \circ \Delta,\]
where $D_i = \delta_i \otimes 1 + 1 \otimes \delta_i$. Explicitly,
\[\Delta(C; \vec{s}) := D_1^{s^*_1} \cdots D_\ell^{s^*_\ell} \Delta(C; \vec{s} - \vec{s}^*).\]

Also expand $\varepsilon$ to
\[\varepsilon : \mathbb{Q}DC \to k, \quad (C; \vec{s}) \mapsto \varepsilon(C).\]
Definition 2.11. A differential cograded coalgebra is a cograded coalgebra \((C, \Delta)\) with \(C = \bigoplus_{n \geq 0} C^{(n)}\) that carries a set of linear maps \(\delta_\sigma : C \to C, \sigma \in \Sigma\), such that
\[
\Delta \circ \delta_\sigma = (1 \otimes \delta_\sigma + \delta_\tau \otimes 1) \circ \Delta, \quad \delta_\sigma(C^{(n)}) \subseteq C^{(n)}, \quad \delta_\sigma \circ \delta_\tau = \delta_\sigma \circ \delta_\tau, \quad \sigma, \tau \in \Sigma.
\]
A differential coaugmented coalgebra is a coaugmented coalgebra \((C, \Delta)\) with coaugmentation \(u : k \to C\) such that \(\delta_\sigma(u(1)) = 0, \sigma \in \Sigma\).

Define a subspace of \(QDC\) by
\[
(12) \quad I := Q \{ (C; \bar{s}) - (C; \bar{t}) \mid C \in C, \bar{s} - \bar{t} = 0 \}.
\]
By convention, \(((0); \bar{s}) - ((0); \bar{t})\) is in \(I\) for all \(\bar{s}\) and \(\bar{t}\). Let \(QDMC\) denote the quotient space \(QDC/I\) whose elements are called decorated cones. Let \(\tilde{\Delta}, \tilde{e}\) and \(\tilde{u}\) denote \(\Delta, e\) and \(u\) modulo \(I\).

Theorem 2.12. (a) The coproduct \(\Delta\) in Eq. (3) and grading in Eq. (5) define a differential cograded coalgebra structure on \(QCD\).
(b) The subspace \(I\) is a differential homogeneous coideal of \(QDC\).
(c) The coproduct \(\Delta\) induces a differential connected coalgebra structure on \(QDMC\).

Proof. (3) We prove the coassociativity by induction on \(|\bar{s}|\) with the case when the length \(|\bar{s}| = s_1 + \cdots + s_k\) of \(s = (s_1, \ldots, s_k)\) vanishes (Proposition 2.3). Provided \((\Delta \otimes 1)\Delta(C; \bar{s} - \bar{t}) = (1 \otimes \Delta)\Delta(C; \bar{s} - \bar{t})\) holds, we have
\[
(\Delta \otimes \text{id})\Delta(C; \bar{s}) = (\Delta \otimes \text{id})D_1\Delta(C; \bar{s} - \bar{t})
\]
\[
= (\delta_1 \otimes \text{id} \otimes \text{id} + \text{id} \otimes \delta_1 \otimes \text{id} + \text{id} \otimes \text{id} \otimes \delta_1)D_2\Delta(C; \bar{s} - \bar{t})
\]
\[
= (\delta_1 \otimes \text{id} \otimes \text{id} + \text{id} \otimes \delta_1 \otimes \text{id} + \text{id} \otimes \text{id} \otimes \delta_1)(\Delta \otimes \text{id})\Delta(C; \bar{s} - \bar{t})
\]
\[
= (\text{id} \otimes \Delta)D_1\Delta(C; \bar{s} - \bar{t})
\]
\[
= (\text{id} \otimes \Delta)\Delta(C; \bar{s} - \bar{t})
\]

The first equation in Eq. (11) is just Eq. (8). The second and third equations follow from the definitions.

(3) By Eq. (12) we get \(D_1 I \subseteq I\). Further
\[
I \cap QDC^{(n)} = Q \{ (C; \bar{s}) - (C; \bar{t}) \mid C \in C^{(n)}, \bar{s} - \bar{t} = 0 \}.
\]
Hence \(I\) is homogeneous. So we just need to show
\[
\Delta((C; \bar{s}) - (C; \bar{t})) \in I \otimes QDC + QDC \otimes I,
\]
for which we use induction on \(|\bar{s}^*| = |\bar{s}^*| \geq 0\). When \(|\bar{s}^*| = 0\), we have
\[
\Delta(((0); \bar{s}) - ((0); \bar{t})) = ((0); \bar{s}) \otimes ((0); \bar{t}) - ((0); \bar{t}) \otimes ((0); \bar{s})
\]
\[
= (((0); \bar{s}) - ((0); \bar{t})) \otimes ((0); \bar{s}) + ((0); \bar{t}) \otimes ((0); \bar{s}) - ((0); \bar{t})((0); \bar{s}),
\]
which is in \(I \otimes QDC + QDC \otimes I\). Then the induction follows from Eq. (8) and the fact that
\[
D_1 (I \otimes QDC + QDC \otimes I) \subseteq I \otimes QDC + QDC \otimes I.
\]

(3) Since \(e(I) = 0\), we have a induced counit \(\tilde{e}\) on \(QDMC\). Since \(I\) is a differential homogeneous coideal, \(QDMC\) is a differential cograded coalgebra.

Define
\[
\tilde{u} : k \to QDMC, 1 \mapsto ((0); 0) \mod I.
\]
Then \(\delta_t(\bar{u}(1)) = 0\) and, by Eq. (14), we have
\[
\mathbb{Q} \mathcal{D} \mathcal{M} \mathcal{C} = k \bar{u}(1) \oplus \ker \bar{e},
\]
since \(\ker \bar{e} = \{(C; \bar{s}) \mod I \mid C \neq \{0\}\}. \) Since \(\mathbb{Q} \mathcal{D} \mathcal{M} \mathcal{C}^{(0)} = \mathbb{Q} \mathcal{D} \mathcal{C}^{(0)}/I \cap \mathbb{Q} \mathcal{D} \mathcal{C}^{(0)} = \mathbb{Q}([0]; 0) \mod I, \mathbb{Q} \mathcal{D} \mathcal{M} \mathcal{C}^{(0)}\) is connected. \(\Box\)

3. Germs of meromorphic functions with linear poles at zero

Germs of meromorphic functions with linear poles at zero arise naturally in various contexts, e.g. in the computation of Feynman integrals or as we shall see below, when evaluating exponential integrals or sums on cones. Theorem 3.3 provides a decomposition of germs of meromorphic functions with linear poles at zero. \[\] We refer to [8,9] for related discussions. The decomposition is used to derive a generalized evaluator in Section 5.3.

3.1. Germs of Meromorphic functions with linear poles.

**Definition 3.1.** A germ of meromorphic functions at 0 in \(C^\infty = \cup C^k\) is the quotient of two holomorphic functions in a neighborhood of 0 inside some \(C^k\). A germ of meromorphic function \(f(\vec{z})\) on \(U \subset C^k\) is said to have linear poles at zero if there exist linearly independent linear forms \(L_1(\vec{z}) = \sum_{i=1}^k a_i z_i, \ldots, L_m(\vec{z}) = \sum_{i=1}^k a_{im} z_i\) and positive integers \(r_1, \ldots, r_k\) such that \(f(\vec{z}) \prod_{i=1}^r L_i(\vec{z})\) is holomorphic at zero. The pole is simple if \(r_i = 1\). We call a meromorphic function with simple linear poles at zero a simple fraction if it is a linear combination of expressions of the type \(\frac{1}{L_1 \cdots L_k}\) with linearly independent linear forms. We call a meromorphic function with simple linear poles at zero a multiple fraction if it is a linear combination of expressions of the type \(\frac{1}{L_1 \cdots L_k^r}\).

The set \(M(C^k)\) of germs of meromorphic functions with linear poles at zero is a linear space over \(C\) and we have standard embedding
\[
M(C^k) \hookrightarrow M(C^{k+1})
\]
and the direct limit
\[
M(C^\infty) := \bigcup_{k=1}^\infty M(C^k).
\]

3.2. A decomposition of germs of meromorphic functions with linear poles. We fix a scalar product \(Q(v, v') = (v, v')\) on \(V := \bigcup_{k=0}^\infty V_k = R^\infty\), and extend it \(C\)-linearly to a non-degenerate bi-linear form on \(C^\infty = V \otimes_R C\). It induces a scalar product on the dual \(V_C = V^* \otimes_R C\) denoted by the same symbol, where \(V^* = \bigcup_{k=0}^\infty V_k^*\).

Let \(M_*(C^k)\) denote the space of germs of holomorphic functions at zero on \(C^k\). Also define the linear subspace
\[
M_*(C^k) = \left\{ \sum_{L_1^{s_1} \cdots L_m^{s_m}} h(L_1, \ldots, \ell_m) \mid h \in M_*(C^k) \right\} \subseteq M(C^k),
\]
where \(s_1, \ldots, s_n\) are positive integers and \(\ell_1, \ldots, \ell_m, L_1, \ldots, L_n\) are linear forms on \(R^k\), with \(L_1, \ldots, L_n\) linearly independent, such that
\[
Q(\ell_i, L_j) = 0, \quad \text{for all } (i, j) \in [m] \times [n].
\]
Here for a positive integer \(p\), denote \([p] = \{1, \ldots, p\}\).

\(^1\)The statement of Theorem 3.3 can be found in some unpublished notes that Michèle Vergne kindly shared with us. We provide a proof here since we could not find one in the literature.
Example 3.2. Let $Q := \langle \cdot, \cdot \rangle$ be the canonical Euclidean inner product on $\mathbb{R}^\infty$. Then the functions \( \frac{(z_1 - z_2)^s}{(z_1 + z_2)^t}, s, t \geq 0 \), lie in $\mathcal{M}_{-}(C^2)$.

We set $\mathcal{M}_{-}(C^\infty) := \cup_{k=0}^\infty \mathcal{M}_{-}(C^k)$ and $\mathcal{M}_{+}(C^\infty) = \cup_{k=0}^\infty \mathcal{M}_{+}(C^k)$, the latter being the algebra of germs of holomorphic functions at zero.

Theorem 3.3. For any positive integer $k$ there is a direct sum decomposition

$$\mathcal{M}(C^k) = \mathcal{M}_{-}(C^k) \oplus \mathcal{M}_{+}(C^k).$$

The projections $\pi_+^k$, $k \geq 1$, onto $\mathcal{M}_{+}(C^k)$ along $\mathcal{M}_{-}(C^k)$ induce a linear map

$$\pi_+ : \mathcal{M}(C^\infty) \to \mathcal{M}_{+}(C^\infty)$$

compatible with the filtration $\{\mathcal{M}(C^k)\}_{k \geq 1}$.

Proof. We first check $\mathcal{M}(C^k) = \mathcal{M}_{-}(C^k) + \mathcal{M}_{+}(C^k)$. For any element $f$ in $\mathcal{M}(C^k)$ we have, in a neighborhood of zero,

$$f = \frac{h}{L_1 \cdots L_k},$$

where $h \in \mathcal{M}_{+}(C^k)$ and $L_1, \cdots, L_k$ are linear functions.

By applying the relation

$$\frac{1}{L_1 \cdots L_n(\sum_{i=1}^n a_i L_i)} = \sum_{i=1}^n \frac{a_i}{L_1 \cdots \hat{L}_i \cdots L_n(\sum_{j=1}^k a_i L_j)^2}$$

repeatedly, we have

$$f = \sum_i \frac{h_i}{L_{j_1}^{s_1} \cdots L_{j_i}^{s_i}},$$

where $h_i \in \mathcal{M}_{+}(C^k)$, $L_{j_1}, \cdots, L_{j_i}$ are linearly independent and $s_1, \cdots, s_i$ are positive integers. So we only need prove the decomposition for

$$f = \frac{h}{L_1^{s_1} \cdots L_{i}^{s_i}}$$

with $h \in \mathcal{M}_{+}(C^k)$ and $L_1, \cdots, L_{i}$ linearly independent is in $\mathcal{M}_{-}(C^k) + \mathcal{M}_{+}(C^k)$.

Extend $\{L_1, \cdots, L_{i}\}$ to a basis $\{L_1, \cdots, L_{i}, \ell_1, \cdots, \ell_{k-i}\}$ of $C^k$ with the additional condition that $Q(L_i, \ell_j) = 0$.

Using a power series expansion of this germ of holomorphic function $h$ in this set of new variables, we reach the conclusion that $\mathcal{M}(C^k) = \mathcal{M}_{-}(C^k) + \mathcal{M}_{+}(C^k)$.

We next show that $\mathcal{M}_{+}(C^k) \cap \mathcal{M}_{-}(C^k) = \{0\}$. Assume $f \in \mathcal{M}_{+}(C^k) \cap \mathcal{M}_{-}(C^k)$. We show that $f = 0$ by induction on the number $d$ of linear independent linear forms in the denominator of $f$.

If $d = 1$, and $f \neq 0$, let $L$ be the only linear form in the denominators of $f$, then

$$f = \sum_{i=1}^N g_i(\ell_1, \cdots, \ell_{k-1})L^{-i},$$

with $g_N \neq 0$, where $g_i$ is holomorphic and $\{\ell_1, \cdots, \ell_{k-1}\}$ is a basis of $L^\perp$. Since $g_N \neq 0$, we can find $(\ell_0^0, \cdots, \ell_0^0)$, such that $g_N(\ell_0^0, \cdots, \ell_0^0) \neq 0$. By plugging in these values, we have a nonzero Laurent series in $L$ which is holomorphic, a contradiction.
Assume the claim holds for \( d \leq n, n \geq 1 \). Let \( 0 \neq f \in \mathcal{M}_-(\mathbb{C}^k) \cap \mathcal{M}_+(\mathbb{C}^l) \) be a meromorphic function with \( n+1 \) linearly independent linear forms \( L_1, \ldots, L_{n+1} \) in the denominator. Expressing \( f \) in powers of \( L_{n+1} \), we have
\[
f = \sum_{i=0}^{N} g_i L^{-i}_{n+1}.
\]
If \( g_i = 0 \) for \( i = 1, \ldots, N \), then \( f = g_0 \) and we reach the conclusion. Otherwise, assume that \( g_N \neq 0 \). Let \( h = h(L_1, \ldots, L_n) \) be the product of all the denominators in \( g_N \), which is a monomial in the independent variables \( L_i \); the function \( hf \) is holomorphic since \( f \) is by our assumption. We have
\[
L_{n+1}^N hf = g_n h + \sum_{i=0}^{N-1} g_i h L^{-i}_{n+1} = g_n h (L_1', \ldots, L_n') L_{n+1}^{-N} + \text{higher power terms in } L_{n+1},
\]
where \( L_i' \) is the projection of \( L_i \) on \( L_{n+1}' := L_i - \frac{\partial(L_{n+1})}{\partial(L_{n+1})} L_{n+1} \). Note that, by our assumption on \( \{L_i\}_{1 \leq i \leq n+1} \), \( L_i' \neq 0 \) for \( i = 1, \ldots, n \).

Using the same argument as in the case of \( d = 1 \), we know that \( g_n h (L_1', \ldots, L_n') = 0 \). But \( h(L_1', \ldots, L_n') \) is a monomial, implying \( g_N = 0 \) which is a contradiction. \( \square \)

### 3.3. A generalized evaluator at zero.

Combining the projections \( \pi_k : \mathcal{M}(\mathbb{C}^\infty) \to \mathcal{M}_+(\mathbb{C}^\infty) \) introduced in Theorem 3.3 with the evaluation at zero on holomorphic functions \( \text{ev}_0 : \mathcal{M}_+(\mathbb{C}^\infty) \to \mathbb{C} \), gives rise to a linear map
\[
\mathcal{E}_0 := \text{ev}_0 \circ \pi_+: \mathcal{M}(\mathbb{C}^\infty) \to \mathbb{C}.
\]

**Proposition 3.4.** Let \( \langle \cdot, \cdot \rangle \) be the canonical inner product on \( \mathbb{C}^\infty \). The map \( \mathcal{E}_0 \) defines a generalized evaluator at zero in the sense of 7, namely

(a) It extends the evaluation at zero on holomorphic functions:
\[
\mathcal{E}_0(f) = \text{ev}_0(f), \quad \text{for all } f \in \mathcal{M}_+(\mathbb{C}^\infty).
\]

(b) It is multiplicative on tensor products:
\[
\mathcal{E}_0(f \otimes g) = \mathcal{E}_0(f) \mathcal{E}_0(g), \quad \text{for all } f \in \mathcal{M}(\mathbb{C}^k), \quad g \in \mathcal{M}(\mathbb{C}^l).
\]

It moreover extends the minimal subtraction scheme\(^2\) defined as follows on a meromorphic germ of functions \( f(z) \) in one complex variable with a pole of order at most \( k \) at zero:
\[
\text{ev}_0^{\text{reg}}(f) := \text{ev}_0 \left( f - \sum_{j=1}^{k} \frac{1}{z_j} \text{Res}_{z=0}^j f(z) \right),
\]
where \( \text{Res}_{z=0}^j f(z) \) is the \( j \)-th residue of the pole of order \( j \). Namely we have:
\[
\mathcal{E}_0(f) = \text{ev}_0^{\text{reg}}(f).
\]

**Proof.**

(a) The first property is trivially satisfied since \( f \in \mathcal{M}_+(\mathbb{C}^\infty) \) implies that \( \pi_+ f = f \).

(b) Given \( f \in \mathcal{M}(\mathbb{C}^k), \quad g \in \mathcal{M}(\mathbb{C}^l) \) the tensor product \( f \otimes g \) lies in \( \mathcal{M}(\mathbb{C}^{k+l}) \). The canonical inner product \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^\infty \) induces the canonical product \( \langle \cdot, \cdot \rangle_k \) on each \( \mathbb{C}^k \) and we have the following factorization property
\[
\langle f \otimes g, f' \otimes g' \rangle_{k+l} = \langle f, f' \rangle_k \langle g, g' \rangle_l \quad \text{for all } f, f' \in \mathcal{M}(\mathbb{C}^k), \quad g, g' \in \mathcal{M}(\mathbb{C}^l).
\]

\(^2\)This terminology is borrowed from physics, see e.g. [3] for a precise mathematical formulation.
It follows that
\[ \pi^k_+(f \otimes g) = \pi^k_+(f) \otimes \pi^l_+(g) \quad \text{for all } f \in \mathcal{M}(\mathbb{C}^k), \quad g \in \mathcal{M}(\mathbb{C}^l). \]

Implementing the evaluator at zero then yields
\[ \mathcal{E}_0(f \otimes g) = \mathcal{E}_0(f) \mathcal{E}_0(g) \quad \text{for all } f \in \mathcal{M}(\mathbb{C}^k), \quad g \in \mathcal{M}(\mathbb{C}^l). \]

(c) To prove the last property we write a function \( f \in \mathcal{M}(\mathbb{C}) \) as \( f(z) = \frac{h(z)}{z} \) for some non-negative integer \( k \) and a holomorphic germ \( h \) at zero. A Taylor expansion at zero yields \( h(z) = \sum_{i=0}^k a_i z^i + o(z^k) \) and hence \( f(z) = \sum_{i=0}^k \frac{a_i}{z^{i+1}} + o(1) \). It follows that \( \pi_-(f) = \sum_{i=0}^{k-1} \frac{a_i}{z^{i+1}} \) and
\[
\text{ev}_0 \circ \pi_+(f) = a_k = \text{ev}_0 \left( f - \sum_{j=1}^k \frac{1}{z_j^j} \text{Res}_{z_j=0} f(z) \right).
\]

**Remark 3.5.** This generalized evaluator differs from the generalized evaluators introduced in [7] defined on \( \mathcal{M}(\mathbb{C}^k) \) as
\[
\overline{\text{ev}}_{0,\text{ren}} := \text{ev}_{\text{reg},z_1=0} \circ \cdots \circ \text{ev}_{\text{reg},z_k=0} \quad \text{and} \quad \overline{\text{ev}}_{0,\text{ren}}^\prime := \text{ev}_{\text{reg},z_1=0} \circ \cdots \circ \text{ev}_{\text{reg},z_k=0},
\]
where \( \text{ev}_{\text{reg}} \) stands for the regularized evaluator at zero in the variable \( z_j \).

Let \( f(z_1, z_2) = \frac{z_1}{z_1 + z_2} \), then
\[
\overline{\text{ev}}_{0,\text{ren}}(f) = 1, \quad \overline{\text{ev}}_{0,\text{ren}}^\prime(f) = 0; \quad \mathcal{E}_0(f) = \text{ev}_0 \circ \pi_+ \left( \frac{1}{2} + \frac{1}{2 z_1 + z_2} \right) = \frac{1}{2}.
\]

### 4. Algebraic Birkhoff decomposition

We give the following generalization of the Algebraic Birkhoff Decomposition of Connes-Kreimer [9] and its differential variation [10] for conilpotent coalgebras without the need for a Hopf algebra and a Rota-Baxter algebra.

**Theorem 4.1.** Let \( \mathbb{C} = \bigoplus_{n \geq 0} \mathbf{k} \mathbb{C}^{(n)} \) be a differential connected coalgebra with derivations \( d_{\sigma}, \sigma \in \Sigma \). Let \( A \) be a unitary differential algebra with derivations \( \partial_{\sigma}, \sigma \in \Sigma \) and with a linear decomposition \( A = A_+ \oplus A_- \) such that \( 1 \in A_+ \) or \( 1 \in A_- \), either \( A_- \) or \( A_+ \) is a subalgebra of \( A \) and \( \partial_{\sigma}(A_\pm) \subseteq A_\pm \). Let \( P \) denote the projection from \( A \) to \( A_- \) along \( A_+ \). Let \( \varphi : \mathbb{C} \to A \) be a linear map such that \( \varphi(1) = 1_A \) and \( \partial_{\sigma} \varphi = \varphi d_{\sigma}, \sigma \in \Sigma \). Define \( \varphi_{\pm} : \mathbb{C} \to A \) by \( \varphi_{\pm}(1) = 1_A \) and by the following recursions on \( \ker \varepsilon \):

\[
\varphi_-(x) = -P(\varphi(x) + \sum_{(x)} \varphi_-(x') \varphi(x'')) \subseteq \text{Im} P = A_-,
\]
(15)  
\[
\varphi_+(x) = (\text{id}_A - P)(\varphi(x) + \sum_{(x)} \varphi_-(x') \varphi(x'')) \subseteq \text{Im}(\text{id}_A - P) = A_+.
\]
(16)

(a) We have \( \varphi_{\pm} : \mathbb{C} \to \mathbf{k} 1_A + A_{\pm}, \varphi_{\pm}(1) = 1_A, \varphi_{\pm}(\ker \varepsilon) \subseteq A_{\pm}, \partial_{\sigma} \varphi_{\pm} = \varphi_{\pm} d_{\sigma}, \sigma \in \Sigma, \) and
\[
\varphi = \varphi_{\pm}^{(1)} \ast \varphi_+.
\]
(17)

(b) \( \varphi_{\pm} \) are the unique maps from \( \mathbb{C} \) to \( A \) that satisfy the conditions in Item (7).
When the derivations $d_{\sigma}$ and $\partial_{\sigma}, \sigma \in \Sigma$, are taken to be the zero maps, we obtain a generalization of the Algebraic Birkhoff Decomposition of Connes and Kreimer [3] which does not involve the differential structure.

**Proof.** We use Sweedler’s notation. It is clear that the maps $\varphi_{\pm}$ have images in $k1 + A_{\pm}$ and $\varphi_{\pm}(1) = 1_{A}$. Furthermore

$$\varphi_{\pm}(x) = (id_{A} - P) \left( \varphi(x) + \sum_{(x)} \varphi_{-}(x')\varphi(x'') \right)$$

$$= \varphi(x) + \varphi_{-}(x) + \sum_{(x)} \varphi_{-}(x')\varphi(x'')$$

$$= (\varphi_{-} \ast \varphi)(x).$$

Thus to prove Eq. (17) we only need to show that $\varphi^{(-1)}$ exists for which we derive $\varphi^{(-1)}(x)$ by induction on the grading degree of $x$. When the degree of $x$ is zero, we have $x = c1$ for $c \in k$ and $\varphi^{(-1)}(c1) = c1$. Assume that $\varphi^{(-1)}(x)$ has been defined for $x$ with degree $\leq n$. Then for $x$ with degree $n+1$, from

$$0 = (ue)(x) = (\varphi^{(-1)} \ast \varphi)(x) = \varphi^{(-1)}(x) + \sum_{(x)} \varphi^{(-1)}(x')\varphi(x'') + \varphi(x),$$

we obtain

$$\varphi^{(-1)}(x) = -\varphi(x) - \sum_{(x)} \varphi^{(-1)}(x')\varphi(x'').$$

It remains to verify $\partial_{\sigma}\varphi_{\pm} = \varphi_{\pm}d_{\sigma}, \sigma \in \Sigma$. The proof follows from the same inductive argument as in [14].

(3) Let

$$\varphi = \varphi_{-}^{(-1)} \ast \varphi_{+} = \psi_{-}^{(-1)} \ast \psi_{+},$$

with $\varphi_{\pm}, \psi_{\pm} : C \to k1_{A_{\pm}} + A_{\pm}$, $\varphi_{\pm}(1) = \psi_{\pm}(1) = 1_{A}$ and $\psi_{\pm}(\ker \epsilon) \subseteq A_{\pm}$. Let $C = \oplus_{k \geq 0} C^{(k)}$ be the grading. We use induction on $k \geq 1$ to prove that $\varphi_{\pm} = \psi_{\pm}$ on $C^{(k)}$.

Assume that $A_{-}$ is a subalgebra. We first use induction on $k \geq 1$ to show that $\varphi^{(-1)}(C^{(k)})$ is contained in $k1 + A_{-}$ if $\varphi : C \to k1 + A_{-}$. It is obvious for $k = 0$. Assume that $\varphi^{(-1)}(C^{(k)}) \subseteq k1 + A_{-}$. Then for any $x \in C^{(k+1)}$, from $\varphi^{(-1)} \ast \varphi = e = u \circ \epsilon$, we obtain

$$0 = e(x) = (\varphi^{(-1)} \ast \varphi)(x)$$

$$= \varphi^{(-1)}(x)\varphi(1) + \varphi^{(-1)}(1)\varphi(x) + \sum_{(x)} \varphi^{(-1)}(x')\varphi(x'')$$

$$= \varphi^{(-1)} + \varphi(x) + \sum_{(x)} \varphi^{(-1)}(x')\varphi(x'').$$

From the induction hypothesis, we then conclude

$$\varphi^{(-1)}(x) \in k1 + A_{-}.$$

Now we prove the uniqueness by showing that for two maps $\varphi$ and $\psi$ satisfying the requirements we have $\varphi_{\pm}(x) = \psi_{\pm}(x)$ for $x \in C^{(k)}$ by induction on $k \geq 0$. This holds for $k = 0$. Assuming that
the equation holds for $C^{(k)}$, then for $x \in C^{(k+1)} \subseteq \ker(\epsilon)$, we have
\[
\varphi^{(1)}_{-}(x)\varphi_{+}(1) + \varphi^{(1)}_{-}(1)\varphi_{+}(x) + \sum_{(x)} \varphi^{(1)}_{-}(x')\varphi_{+}(x''),
\]
\[
= \psi^{(1)}_{-}(x)\psi_{+}(1) + \psi^{(1)}_{-}(1)\psi_{+}(x) + \sum_{(x)} \psi^{(1)}_{-}(x')\psi_{+}(x''),
\]
which means
\[
\varphi^{(1)}_{-}(x) + \varphi_{+}(x) = \psi^{(1)}_{-}(x) + \psi_{+}(x).
\]
So by the direct sum $A = A_{+} \oplus A_{-}$, we have
\[
\varphi^{(1)}_{-}(x) = \psi^{(1)}_{-}(x), \quad \varphi_{+}(x) = \psi_{+}(x).
\]
Suppose that $A_{+}$ is a subalgebra. Then by the uniqueness of factorization of $\varphi^{(1)}$ with respect to the projection to $A_{+}$ along $A_{-}$, we reach the conclusion. \hfill \Box

5. Conical zeta values and their regularization

5.1. Regularization. We recall some background on conical zeta values from [3].

**Definition 5.1.** For a (closed) cone $C$ in the first orthant, let $C^{o}$ denote the open cone of the interior of $C$. Let $C$ be a smooth cone in $\mathbb{R}_{>0}^{k}$. Define the **open conical zeta function** of $C$ by
\[
\zeta^{o}(C; \vec{s}) = \sum_{(n_{1}, \ldots, n_{k}) \in C^{o} \cap \mathbb{Z}^{k}} \frac{1}{n_{1}^{s_{1}} \cdots n_{k}^{s_{k}}}, \vec{s} \in \mathbb{C}^{k},
\]
if the sum converges. Here we have used the convention that $0^{i} = 1$ for any $s$.

With this convention, the definition of $\zeta^{o}(C; \vec{s})$ does not depend on the integer $k$ such that $C \subseteq \mathbb{R}_{>0}^{k}$ and $\vec{s} \in \mathbb{C}^{k}$. Thus we can use $\zeta^{o}(C; \vec{s})$ without referring to $k$. When $s_{1}, \ldots, s_{k}$ are taken to be integers, we call $\zeta^{o}(C; \vec{s})$ a **open conical zeta value (CZV)**. The adjective ”open” is used to distinguish it from the closed conical zeta values to be introduced below.

We know that $\zeta^{o}(C; \vec{s})$ is convergent for $\vec{s} \in \mathbb{Z}^{n}$ with $s_{1} \geq 2$.

A **Chen cone** of dimension $k$ is a cone $C_{k,\sigma}$ spanned by the vectors $\{e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)}, \ldots, e_{\sigma(1)} + \cdots + e_{\sigma(k)}\}$ where $\{e_{1}, \ldots, e_{n}\}$ is the standard basis of $\mathbb{Z}^{n}$ and $\sigma \in S_{n}, S_{n}$ is the symmetric group on $\{1, \ldots, n\}$. Let $C_{k}$ denote the standard (open or closed) Chen cone spanned by $\{e_{1}, e_{1} + e_{2}, \ldots, e_{1} + \cdots + e_{k}\}$.

For any open Chen cone $C_{k,\sigma}$, $k \geq 1, \sigma \in S_{n},$ we have
\[
\zeta^{o}(C_{k,\sigma}; s_{1}, \ldots, s_{n}) = \zeta(s_{\sigma(1)}, \ldots, s_{\sigma(k)}),
\]
where the right hand side is the multiple zeta value (MZV).

Recall a variation of a MZV, called a **multiple zeta-star value (MZSV)**, defined by
\[
\zeta^{*}(s_{1}, \ldots, s_{k}) := \sum_{n_{1} \geq n_{2} \geq \cdots \geq n_{k} \geq 1} \frac{1}{n_{1}^{s_{1}} \cdots n_{k}^{s_{k}}},
\]
for integers $s_{1} \geq 2, s_{i}, 1 \leq i \leq k$. When $k = 2$, it is this variation that was first studied by Goldbacher and Euler [11].

In the conical context, we similarly define a **closed conical zeta value** or **conical zeta-star value** for a cone $C$ to be
5.2. Meromorphicity of regularized conical zeta values. Let $\mathcal{C} = \langle v_1, \ldots, v_k \rangle$ be a $k$-dimensional smooth cone in $\mathbb{R}^n$ and $\vec{\varepsilon}$ be in $\vec{C}^- = \{ \varepsilon | \langle \vec{x}, \varepsilon \rangle < 0, \text{ for all } \vec{x} \in C \}$, we define the regularized zeta values by

$$\zeta^o(C; \vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^k} \frac{e^{\langle \vec{n}, \vec{s} \rangle}}{\vec{n}!} = \sum_{(n_1, \ldots, n_k) \in \mathbb{Z}^k} \frac{e^{n_1 \varepsilon_{11} \cdots e_n \varepsilon_{kk}}}{n_1! \cdots n_k!}.$$ 

The regularization of a close cone zeta value is defined similarly.

We in particular denote

$$S^o(C)(\vec{\varepsilon}) := \zeta(C; \vec{0})(\vec{\varepsilon}) = \sum_{\vec{n} \in \mathbb{Z}^k} e^{\langle \vec{n}, \vec{\varepsilon} \rangle},$$

for an open cone $C$ and

$$S^c(C)(\vec{\varepsilon}) := \zeta(C; \vec{0})(\vec{\varepsilon}) = \sum_{\vec{n} \in \mathbb{Z}^k} e^{\langle \vec{n}, \vec{\varepsilon} \rangle},$$

for a closed cone $C$. When the openness of $C$ is clear from the context, we also omit the superscripts. $S^o$ and $S^c$ define functions on $\mathbb{Q}^C$.

5.2. Meromorphicity of regularized conical zeta values. Let $\mathcal{C} = \langle v_1, \ldots, v_k \rangle$ be a $k$-dimensional smooth cone in $\mathbb{R}^n$ and $\vec{\varepsilon}$ be in $\vec{C}^-$. Since an element $\vec{x}$ in $C$ can be written in a unique way as $\sum_{j=1}^k n_j v_j$, we have

$$S^o(C)(\vec{\varepsilon}) := \prod_{j=1}^k \sum_{n_j \in \mathbb{Z}_{\geq 0}} e^{n_j \langle v_j, \varepsilon \rangle} = \prod_{j=1}^k \frac{e^{\langle \vec{\varepsilon}, v_j \rangle}}{1 - e^{\langle \vec{\varepsilon}, v_j \rangle}} = \prod_{j=1}^k \frac{e^{L_j(\vec{\varepsilon})}}{1 - e^{L_j(\vec{\varepsilon})}},$$

and

$$S^c(C)(\vec{\varepsilon}) := \prod_{j=1}^k \sum_{n_j \in \mathbb{Z}_{\geq 0}} e^{n_j \langle v_j, \varepsilon \rangle} = \prod_{j=1}^k \frac{1}{1 - e^{\langle \vec{\varepsilon}, v_j \rangle}} = \prod_{j=1}^k \frac{1}{1 - e^{L_j(\vec{\varepsilon})}},$$

where as before, we have set $\vec{L}(\vec{\varepsilon}) = (L_1(\vec{\varepsilon}), \ldots, L_k(\vec{\varepsilon}))$ with $L_j(\vec{\varepsilon}) = \sum_{i=1}^n a_{ij} \varepsilon_i$. It is holomorphic on $\vec{C}^-$ and extends to a meromorphic map on $\mathbb{C}^n$ with simple linear poles at zero $L_1(\vec{\varepsilon}) = 0, \ldots, L_k(\vec{\varepsilon}) = 0$. By subdivision, this definition extends to any cone. (see also [I]).

**Proposition 5.2.** $S^o$ and $S^c$ define linear maps from $\mathbb{Q}^C$ to $\mathcal{M}(\mathbb{C}^n)$.

Let $\vec{r}$ be a vector with nonpositive components. Differentiating with respect to $\vec{\varepsilon}$ the meromorphic functions defined from Eqs. (25) and (26) yields, on a smooth cone $C$ in $\mathbb{R}^n$, the expression

$$D_{\vec{\varepsilon}}^\vec{r} S^o(C)(\vec{\varepsilon}) = \sum_{\vec{n} \in \mathbb{Z}^k} \vec{n}^\vec{r} e^{\langle \vec{n}, \varepsilon \rangle},$$
and
\[ D_\vec{r}^{-\vec{r}} S^c(C)(\vec{\varepsilon}) = \sum_{C \cap \mathbb{Q}^2} \vec{n}^{-\vec{r}} e^{(\vec{n},\vec{r})}, \]

These maps can be extended by subdivision.

**Proposition 5.3.** \( S^o(C; \vec{r})(\vec{\varepsilon}) = D_\vec{r}^{-\vec{r}} S^o(C)(\vec{\varepsilon}) \) and \( S^c(C; \vec{r})(\vec{\varepsilon}) = D_\vec{r}^{-\vec{r}} S^c(C)(\vec{\varepsilon}) \) define linear maps
\[ S^o : \mathbb{QDMC} \to M(\mathbb{C}^\infty), \]
and
\[ S^c : \mathbb{QDMC} \to M(\mathbb{C}^\infty). \]

6. The Euler-Maclaurin formula on cones and renormalization

We implement the algebraic Birkhoff decomposition on regularized closed conical zeta values to give an interpretation of the Euler-Maclaurin formula \([1, 5]\) for lattice points in (closed) cones. This approach also provides an Euler-Maclaurin formula for open cones. With the notation of Theorems 3.3 and 4.1, we set
\[ A = M(\mathbb{C}^\infty), \quad A_+ = M_+(\mathbb{C}^\infty), \quad P = 1 - \pi_+. \]

6.1. Euler-Maclaurin formula for closed cones. We first recall the Euler-Maclaurin formula for closed cones derived in \([1]\), see also \([5]\). There are unique maps \( I : \mathbb{Q}C \to M(\mathbb{C}^\infty) \) and \( \mu : \mathbb{Q}C \to M(\mathbb{C}^\infty) \) such that for any closed cone \( C \) we have
\[ S(C) = \sum_{F \subseteq C} I(F) \mu(t(C, F)), \]
where \( S(C) : \vec{z} \mapsto S^c(C)(\vec{z}) \) is the regularized sum defined in Eq. (24), \( \Delta(C) = \sum_{F \subseteq C} F \otimes t(C, F) \) is the coproduct of cones defined in Eq. (3). \( I \) takes values in \( 1 + M_+ \), and \( \mu \) takes values in \( M_+ \). \( I(\emptyset) = 1_A \) and \( \mu(\emptyset) = 1_A \).

**Theorem 6.1.** The Euler-Maclaurin formula in Eq. (27) coincides with the Algebraic Birkhoff Decomposition for \( S^c \):
\[ (S^c)^{(a)}_+ * (S^c)_+. \]

More precisely, we have
\[ I = (S^c)^{(a)}_+, \quad \mu = (S^c)_+. \]

**Proof.** From the linear map
\[ S^c : \mathbb{QDMC} \to M(\mathbb{C}^\infty), \]
and the decomposition
\[ M(\mathbb{C}^\infty) = M_-(\mathbb{C}^\infty) \oplus M_+(\mathbb{C}^\infty), \]
where \( M_+(\mathbb{C}^\infty) \) of homomorphic functions is a subalgebra, the corresponding Algebraic Birkhoff Decomposition in Theorem 4.1 is \( (S^c) = (S^c)^{(a)}_+ * (S^c)_+ \) with \( (S^c)^{(a)}_+ : \mathbb{QDMC} \to M_- \) and \( (S^c)_+ : \mathbb{QDMC} \to M_+ \). By the uniqueness of the decomposition proved in Theorem 4.1, we conclude
\[ I = (S^c)^{(a)}_+, \quad \mu = (S^c)_+. \]

□
**Corollary 6.2.** For any closed cone $C$, we have
\[ \mu(C) = S^\circ(C) = \pi_+(S^\circ(C)) \]
where $\pi_+$ is the projection defined in Theorem 3.2.

**Proof.** Note that the linear forms in $F$ and $t(C,F)$ are perpendicular. So by Eq. (16) and our choice of the decomposition of $M$, the sum
\[ \sum_{F \prec C} (S^\circ)^{(-1)}(F)S^\circ(t(C,F)) \]
is in $M_-(\mathbb{C}^\infty)$. Hence by Eq. (16),
\[ \mu(C) = S^\circ(C) = (id - P)(S^\circ(C)). \]

6.2. **The Euler-Maclaurin formula for open cones.** We next establish an open variation of the Euler-Maclaurin formula by the Algebraic Birkhoff Decomposition in Theorem 4.1.

**Lemma 6.3.** There exists a function $\nu : \mathbb{Q}C \to M_+(\mathbb{C}^\infty)$ with $\nu([0]) = 1_A$, such that $S^\circ(C) = (I * \nu)(C) = \sum_{F \subseteq C} I(F)\nu(t(C,F)).$

**Proof.** We prove this lemma for $C \in F_k(C)$ by induction on $k \geq 0$. For $k = 0$, we have $C = \{0\}$ and
\[ S^\circ([0]) = I([0])\nu([0]). \]
Assume the lemma holds for cones of dimension $k$. For a cone $C$ of dimension $k + 1$, by Eq. (21), for $\tilde{s} = \tilde{0}$ we have
\[
S^\circ(C) = S^\circ(C) - \sum_{F \subseteq C} S^\circ(F)
\]
\[ = \sum_{G \subseteq C} I(G)\mu(t(C,G)) - \sum_{F \subseteq C} \sum_{G \subseteq F} I(G)\nu(t(F,G))
\]
\[ = I(C)\nu([0]) + \sum_{G \subseteq C} I(G)(\mu(t(C,G)) - \sum_{G \subseteq F \subseteq C} \nu(t(F,G))).
\]
So by the induction hypothesis, $\nu(C)$ is defined by
\[ \nu(C) = \mu(C) - \sum_{F \subseteq C} \nu(F). \]

**Theorem 6.4.** Let $C$ be a (closed) smooth cone in the first orthant. Then
\[ \nu(C) = S^\circ_+(C) = \pi_+(S^\circ(C)) \]
and
\[ S^\circ(C) = \sum_{F \subseteq C} I(F)\pi_+(S^\circ(t(C,F))). \]
Proof. Applying the Algebraic Birkhoff Decomposition to the linear map $S^o : \mathbb{Q}C \rightarrow \mathcal{M}(\mathbb{C}^o)$, we obtain

$$S^o = (S^o)^{(1)} * S^o.$$  

Thus we just need to prove

$$S^o_+(C) = (\text{id} - P)(S^o(C)), \quad (S^o_+)^{(1)}(C) = I(C).$$  

By Eq. (16), we have

$$S^o_+(C) = (\text{id} - P)\left(S^o(C) + \sum \right) S^o(F)S^o(t(C, F))\).$$  

By the construction of the transverse cone $t(C, F)$, the linear forms in $S^o(t(C, F))$ are orthogonal to those in $S^o_+(F)$. Therefore

$$\sum \left[S^o(F)S^o(t(C, F) \in \mathcal{M}_-(\mathbb{C}^o),$$

and we have

$$S^o_+(C) = (\text{id} - P)(S^o(C)).$$

Now compare with Lemma 5.2, we know the second equation in Eq. (29), and

$$v(C) = S^o_+(C).$$

\[ \square \]

7. Renormalization of conical zeta values and multiple zeta values

In this section, we relate the renormalized conical zeta values to the Euler-Maclaurin formula in Section 7.1 and consider the case of two variables in Section 7.2.

7.1. Renormalization of conical zeta values. The connected cograded coalgebra structure on $\mathcal{QDMC}$ together with the decomposition of $\mathcal{M}(\mathbb{C}^o)$ in Theorem 3.3 allows us to apply the Birkhoff decomposition in Theorem 3.1 to the linear map $\varphi : (= S^o$ or $S^c) : \mathcal{QDMC} \rightarrow \mathcal{M}(\mathbb{C}^o)$ and give renormalized maps

$$\varphi_+ : \mathcal{QDMC} \rightarrow \mathcal{M}_+(\mathbb{C}^o).$$

**Theorem 7.1.** For a decorated open cone $(C; \vec{s})$, where $\vec{s}$ has nonpositive components, we have

$$S^o_+(C; \vec{s}) = \pi_+(S^o(C; \vec{s})) = D^{-\vec{s}}v(C) = D^{-\vec{s}}\pi_+(S^o(C)),$$

$$S^c_+(C; \vec{s}) = \pi_+(S^c(C; \vec{s})) = D^{-\vec{s}}\mu(C) = D^{-\vec{s}}\pi_+(S^c(C)),$$

where $\pi_+ = \text{id} - P$.

**Proof.** We first observe that $S^o_+(C; \vec{s}) = (\text{id} - P)(S^o(C; \vec{s}))$ and $S^c_+(C; \vec{s}) = (\text{id} - P)(S^c(C; \vec{s}))$ as a result of the linear forms for $F$ and $t(C, F)$ being perpendicular. Then, the fact that $(\mathcal{QDMC}, \Delta, \delta_i)$ defines a conilpotent differential coalgebra for any $i \in \mathbb{N}$ combined with the fact that derivations preserve the decomposition of $\mathcal{M}(\mathbb{C}^o)$ gives the remaining identities. \[ \square \]

Let us now evaluate at $\vec{s} = 0$.

$$\text{ev}_0 \circ \varphi_+ : \mathcal{QDMC} \rightarrow \mathbb{C}, (C; \vec{s}) \mapsto \varphi_+(C; \vec{s})(0), \quad (C; \vec{s}) \in \mathcal{DMC}.\]
**Definition 7.2.** The value 
\[ \zeta^o(C; \vec{s}) := \text{ev}_0 \circ S^o_\vec{s}(C; \vec{s})(0) \]
is called the **renormalized open conical zeta value** of \((C; \vec{s})\), and the value 
\[ \zeta^c(C; \vec{s}) := \text{ev}_0 \circ S^c_\vec{s}(C; \vec{s})(0) \]
is called the **renormalized closed conical zeta value** of \((C; \vec{s})\).

**Corollary 7.3.** Let \(\langle \cdot, \cdot \rangle\) be the canonical inner product on \(\mathbb{C}^\infty\). For a decorated cone \((C; \vec{s})\),
\[ \zeta^o(C; \vec{s}) = D^{-\vec{s}}S^o(C)(0) = \mathcal{E}_0 \left( D^{-\vec{s}}S^o(C) \right), \quad \zeta^c(C; \vec{s}) = D^{-\vec{s}}S^c(C)(0) = \mathcal{E}_0 \left( D^{-\vec{s}}S^c(C) \right), \]
so that multiple zeta values are derived from the differentiated exponential sums \(D^{-\vec{s}}S^o(C)\) and \(D^{-\vec{s}}S^c(C)\) by implementing the renormalized evaluator \(\mathcal{E}_0\) introduced in \(\mathcal{E}_4\).

Consequently, renormalized multiple zeta values can be equivalently derived
- from the differentiated exponential sums \(D^{-\vec{s}}S^o(C)\) and \(D^{-\vec{s}}S^c(C)\) by implementing the renormalized evaluator \(\mathcal{E}_0\) introduced in \(\mathcal{E}_4\), or
- from a Birkhoff-Hopf factorization of the discrete exponential sums \(S^o(C)\) and \(S^c(C)\) followed by differentiation and an ordinary evaluation at zero of the "holomorphic part".

The coincidence of the two methods reflects an analogy between the coproduct on cones involving the transverse cone of a face, which is the cone projected orthogonally to the face and the decomposition of the meromorphic functions with linear poles which involves projecting linear forms perpendicularly to the poles. We hope to investigate this analogy in future work.

### 7.2. Tables of double zeta values.

The tables in page 21 for double zeta values are all derived via a Birkhoff-Hopf factorization and satisfy the stuffle relations:

- Table [\(\mathcal{E}_5\)] was derived in \(\mathcal{E}_9\) using a deconcatenation product (on words) and a heat-type regularization.
- Table \(\mathcal{E}_6\) was derived in \(\mathcal{E}_14\) using a deconcatenation product (on symbols) and a Riesz (called "modified dimensional" by physicists) regularization.
- Table \(\mathcal{E}_7\) is derived from the coproduct on open Chen cones introduced in this paper and uses a heat-type regularization.

These three tables coincide on the diagonal \(a = b\) and when \(a+b\) is odd, in which case \(\zeta(-a, -b) = -1/2\zeta(-a-b)\) (see Remark 8 on page 44 of \(\mathcal{E}_2\)).

The fact that these tables nevertheless differ raises a natural question; how do they relate, namely is there a renormalization group relating them? The methods obtained to derive the first two tables differ by the regularization method, a heat-kernel type one \(x^{-\vec{s}} \mapsto x^{-\vec{s}}e^{\vec{s}x}\) in the first case, a Riesz type one \(x^{\vec{s}} \mapsto x^{\vec{s}}e^{\vec{s}x}\) in the second case. The coproduct implemented in the Birkhoff-Hopf factorization is the same deconcatenation coproduct. In one variable, the two regularization methods are related by a Mellin transform \(x^{\vec{s}^{-\vec{s}x}} = \frac{\chi^i}{\pi(i)} \int_0^\infty \varepsilon^{\vec{s}-1}e^{-\varepsilon x}d\varepsilon\). It is unclear how this generalizes to several variables constrained by conical conditions.

The methods obtained to derive the first and the third tables differ by the choice of the coproduct and the projector operator used in the Birkhoff-Hopf factorization, since the regularization method is essentially the same (up to a small technical variation in \(\mathcal{E}_5\)) heat-kernel one in both cases.
Table 1. Values of renormalized double zeta values from $[3]$

| $\zeta(-s_1, -s_2)$ | $s_1 = 1$  | $s_1 = 2$  | $s_1 = 3$  | $s_1 = 4$  | $s_1 = 5$  | $s_1 = 6$  |
|---------------------|------------|------------|------------|------------|------------|------------|
| $s_2 = 1$           | $\frac{1}{2}$ | $-1$    | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 2$           | $-\frac{1}{2}$ | $0$     | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $s_2 = 3$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 4$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 5$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 6$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table 2. Values of renormalized double zeta values from $[4]$

| $\zeta(-s_1, -s_2)$ | $s_1 = 1$  | $s_1 = 2$  | $s_1 = 3$  | $s_1 = 4$  | $s_1 = 5$  | $s_1 = 6$  |
|---------------------|------------|------------|------------|------------|------------|------------|
| $s_2 = 1$           | $\frac{-1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $s_2 = 2$           | $-\frac{1}{2}$ | $0$     | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $s_2 = 3$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 4$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 5$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 6$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table 3. Values of renormalized conical double zeta values

| $\zeta(-s_1, -s_2)$ | $s_1 = 1$  | $s_1 = 2$  | $s_1 = 3$  | $s_1 = 4$  | $s_1 = 5$  | $s_1 = 6$  |
|---------------------|------------|------------|------------|------------|------------|------------|
| $s_2 = 1$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 2$           | $-\frac{1}{2}$ | $0$     | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $s_2 = 3$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 4$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 5$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 6$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table 4. Values of conical double star zeta values

| $\zeta(-s_1, -s_2)$ | $s_1 = 1$  | $s_1 = 2$  | $s_1 = 3$  | $s_1 = 4$  | $s_1 = 5$  | $s_1 = 6$  |
|---------------------|------------|------------|------------|------------|------------|------------|
| $s_2 = 1$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 2$           | $-\frac{1}{2}$ | $0$     | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |
| $s_2 = 3$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 4$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 5$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $s_2 = 6$           | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

Table 4 for star-double zeta values is derived from the coproduct on closed Chen cones introduced in this paper and uses a heat-type regularization. It coincides with a table of values derived by M. Vergne via a Taylor expansion from her results obtained with N. Berline in $[1],[2]$. 

---

3 The second author would like to thank M. Vergne for communicating this table to her in 2007 following discussions that served as a source of inspiration for the present paper.
Such a situation with a variety of renormalization methods which all lead to renormalized values obeying the same requirements (here multiple zeta values obeying stuffle relations) to our knowledge, does not occur in physics where the methods usually differ only by a choice of regularization, the coproduct being the Connes-Kreimer coproduct. This new situation calls for further investigations as conical zeta values seem to offer a good testing ground to approach the difficult concept of an abstract renormalization group which would related different renormalized values.

Acknowledgements: L. Guo acknowledges support from NSF grant DMS 1001855. B. Zhang thanks support from NSFC grant 11071176 and 11221101.

References

[1] N. Berline, M. Vergne, Local Euler-Maclaurin formula for polytopes, Mosc. Math. J., 7 (2007) 355-386.
[2] M. Brion, M. Vergne, Arrangement of hyperplanes I Rational functions and Jeffrey-Kirwan residue, Annales scientifiques de l’École Normale Supérieure (1999), Volume: 32, Issue: 5, page 715-741.
[3] A. Connes, D. Kreimer, Hopf algebras, Renormalization and Noncommutative Geometry, Comm. Math. Phys. 199 (1988) 203-242.
[4] W. Fulton: Introduction to toric varieties, Princeton University Press, 1993.
[5] S. Garoufalidis, J. Pommersheim, Sum-integral interpolators and the Euler-Maclaurin formula for polytopes, Trans. Amer. Math. Soc. 364 (2012), 2933-2958.
[6] L. Guo, An Introduction to Rota-Baxter Algebra, International Press, 2012.
[7] L. Guo, S. Paycha and B. Zhang, Renormalization by Birkhoff factorization and by generalized evaluators; a study case, in “Noncommutative Geometry, Arithmetic and Related Topics”, Ed. A. Connes, K. Consani John Hopkins University Press (2011) 183-211.
[8] L. Guo, S. Paycha and B. Zhang, Conical zeta values and their double subdivision relations, arXiv:1301.3370.
[9] L. Guo and B. Zhang, Renormalization of multiple zeta values J. Algebra 319 (2008) 3770-3809.
[10] L. Guo and B. Zhang, Differential Birkhoff decomposition and renormalization of multiple zeta values, J. Number Theory 128 (2008), 2318-2339.
[11] A.P. Juskevic and E. Winter, eds., Leonhard Euler und Christian Goldbach: Briefwechsel 1729–1764, Akademie-Verlag, Berlin, 1965.
[12] J.-L. Loday and B. Vallette, Algebraic operads, Grundlehren Math. Wiss. 346, Springer, Heidelberg, 2012.
[13] D. Manchon, Hopf algebras, from basics to applications to renormalization, Comptes-rendus des Rencontres mathématiques de Glanon 2001 (2003); Hopf algebras in renormalization, Handbook of algebra, Vol. 5 (M. Hazewinkel ed.) (2008).
[14] D. Manchon and S. Paycha, Nested sums of symbols and renormalised multiple zeta values, Int. Math. Res. Papers 2010 issue 24, 4628-4697 (2010).
[15] G. Ziegler, Lectures on polytopes, Graduate Texts in Mathematics, Springer Verlag, 2nd edition 1994.