Study of the phase diagram of dense two-color QCD within lattice simulation

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In this paper we carry out a low-temperature scan of the phase diagram of dense two-color QCD with \( N_f = 2 \) quarks. The study is conducted using lattice simulation with rooted staggered quarks. At small chemical potential we observe the hadronic phase, where the theory is in a confining state, chiral symmetry is broken, the baryon density is zero and there is no diquark condensate. At the critical point \( \mu = m_\pi/2 \) we observe the expected second order transition to Bose-Einstein condensation of scalar diquarks. In this phase the system is still in confinement in conjunction with non-zero baryon density, but the chiral symmetry is restored in the chiral limit. We have also found that in the first two phases the system is well described by chiral perturbation theory. For larger values of the chemical potential the system turns into another phase, where the relevant degrees of freedom are fermions residing inside the Fermi sphere, and the diquark condensation takes place on the Fermi surface. In this phase the system is still in confinement, chiral symmetry is restored and the system is very similar to the quarkyonic state predicted by SU(\( N_c \)) theory at large \( N_c \).

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I. INTRODUCTION.

The phase diagram of QCD is of high importance for several fields of observational physics like cosmology and astrophysics. One field of experimental physics, located at the frontier of nuclear physics and high energy physics, is the study of hadronic matter created by relativistic heavy ion collisions. Such experiments are addressing the structure of the phase diagram, although the understanding and modeling of an actual collision requires much more than the knowledge of the equilibrium phase diagram. However, equilibrium observables like the equation of state and transport coefficients are highly needed to be used in hydrodynamical approaches which serve to probe various scenarios.

The region of high temperature and vanishing baryonic density of QCD phase diagram is well explored at LHC and RHIC. The theoretical study of this part of the phase diagram can be carried out with lattice gauge theory (LGT), based on the first principles of QCD. Today this approach has reached a high level of maturity and the results obtained within LGT for small \( \mu/T \) are in good agreement with experiments [1,2,3].

In the 2010-th years, a “low-energy frontier” of heavy ion physics has opened (with the beam energy scan program at RHIC) focussing at the region of high baryonic density and lower temperatures. The new experimental facilities presently under construction, FAIR and NICA, hosting the future experiments CBM, BM&N, and MPD, respectively, will be suitable for this region of the phase diagram. This situation is urging theorists to study QCD with large chemical potential.

Unfortunately, lattice simulation of QCD cannot be applied today to arbitrary chemical potential because of the sign problem [3]. The origin of the sign problem is that the fermion determinant becomes complex-valued, and direct simulation by importance sampling of gauge field configurations is not possible. In the absence of straightforward results from LGT one applies different approaches to study the \((T, \mu)\) phase diagram: for instance, mean field approaches [4], the method of Dyson-Schwinger equations [5], the large-\( N_c \) approach [6, 7], perturbative QCD coupled to HRG models [8], exploring the phase diagram of QCD with isospin chemical potential [9,10,11] and others. Although the results obtained within these approaches are very interesting, they may still be rather schematic and require confirmation.

An alternative to lattice simulation of SU(3) QCD with \( \mu \neq 0 \) is the simulation of SU(2) QCD (also called QC2D). Introduction of a chemical potential to the latter theory does not lead to a sign problem, so one can apply the standard lattice approach to study this theory. Although a two-color world differs from the three-color world, lattice study of QC2D with chemical potential can provide us with important information about the properties of QCD with non-zero baryon density. In particular we believe that some physical properties of the regions of the phase diagram where relevant degrees of

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freedom are quarks and gluons are similar for the SU(2) and SU(3) theories. As an example one could mention equation of state, some properties of gluon propagator (for instance, Debye screening), generation of the fermion mass gap, etc. In addition one can use SU(2) QCD to study how non-zero density influences different observables and phenomena. We would like also to note that the QC2D phase diagram has a rich structure and it is interesting to study by its own.

The properties of QC2D were studied theoretically within the following approaches: chiral perturbation theory (ChPT), the NJL model, functional renormalization group (fRG), random matrix theory (RMT). Principally, these studies have revealed the following phase structure of low temperature QC2D with three subsequent phases: (1) $0 < \mu < \mu^c$ (hadronic phase), (2) $\mu^c < \mu < \mu^d$ ("baryon onset" with a superfluid condensate due to Bose-Einstein condensation [BEC]) and (3) $\mu^d < \mu$ (the phase with diquark condensation due to the Bardeen-Cooper-Schrieffer mechanism [BCS]).

The first lattice study of QC2D with chemical potential and Wilson fermions was performed by A. Nakamura in 1981. Further lattice investigation of dense two-color QCD was continued by J. Kogut and collaborators using staggered quarks. The staggered Dirac operator without rooting describes $N_f = 4$ flavors. Making the whole fermion matrix hermitian positive definite by doubling the number of flavors has lead to the eight-flavor theory investigated in the pioneering paper. Following this work, introduction of the rooting trick for the staggered fermion determinant has allowed to investigate the case of $N_f = 4$ flavors in more detail by factors.

In this paper we are going to study the QC2D phase diagram with $N_f = 2$ flavors going back to the lattice simulation of staggered fermions using the rooting procedure. The advantage of the staggered fermion formulation is the approximate residual chiral symmetry of the Dirac operator. Therefore we have chosen this formulation to complement the Swansea studies by an alternative study of the two flavors case. In our first paper we have calculated the Polyakov loop and the chiral condensate as functions of temperature for different values of the chemical potential $\mu$. In the present paper we are going to carry out a $\mu$ scan at low temperature of the QC2D phase diagram.

The paper is organized as follows. In sect. II we specify details of the lattice set-up to be used: action, the way of the diquark source introduction, details of the simulation. In sect. III we present the numerical results of this study. The last section is devoted to the discussion of the results and to some conclusions to be drawn.

II. THE LATTICE SET-UP

A. Partition function

In our simulations we used the Wilson action for the SU(2) gauge fields

$$S_G = \beta \sum_x \sum_{\mu<\nu=1} \left(1 - \frac{1}{2} Tr U_{x,\mu\nu}\right).$$

(1)

For the fermionic degrees of freedom we used staggered fermions with an action of the form

$$S_F = \sum_{x,y} \psi_x^\dagger M(\mu, m)_{x,y} \psi_y + \frac{\lambda}{2} \sum_x \left(\psi_x^\dagger T_2 \psi_x + \overline{\psi}_x T_2 \overline{\psi}_x\right),$$

(2)

$$M_{xy} = m \delta_{xy} + \frac{4}{\beta} \sum_{\mu=1}^4 \eta_\mu(x) \left[U_{x,\mu} \delta_{x+\mu, y} e^{\mu a \delta_{\mu,4}} - U_{x-\mu, y} \delta_{x-\mu, y} e^{-\mu a \delta_{\mu,4}}\right].$$

(3)

where $\overline{\psi}, \psi$ are staggered fermion fields, $a$ is the lattice spacing, $m$ is the bare quark mass, and $\eta_\mu(x)$ are the standard staggered phase factors: $\eta_1(x) = 1$, $\eta_2(x) = (-1)^{x_1+\ldots+x_n}$, $\mu = 2, 3, 4$. The chemical potential $\mu$ is introduced into equation (3) through the multiplication of the links along and opposite to the temporal direction by factors $e^{\pm\mu a}$ respectively. This way of introducing the chemical potential makes it possible to avoid additional divergences and to reproduce well known continuum results.

In addition to the standard staggered fermion action we add a diquark source term to equation (2). The diquark source term explicitly violates $U_V(1)$ and allows to observe diquark condensation even on finite lattices, because this term effectively chooses one vacuum from the family of $U_V(1)$-symmetric vacuums. The results presented in this paper are obtained as follows: we carry out simulations at small but nonzero parameter $\lambda \ll ma$, and then extrapolate obtained data to $\lambda \to 0$. Notice that similar to the diquark source term an additional pion term was introduced to the fermion action during the studies of QCQD phase diagram with isospin chemical potential.

Integrating out the fermion fields the partition function for the theory with the action $S = S_G + S_F$ can be written in the form

$$Z = \int DU e^{-S_G} \cdot Pf \left(\begin{array}{cc} \lambda \tau_2 & M \\ -M^T & \lambda \tau_2 \end{array}\right)$$

$$= \int DU e^{-S_G} \cdot \left(\text{det}(M^T M + \lambda^2)\right)^{\frac{1}{4}},$$

(4)

which corresponds to $N_f = 4$ dynamical fermions in the continuum limit. Note that the pfaffian $Pf$ is strictly positive, such that one can use Hybrid Monte-Carlo methods to study this system. The lattice study of the...
theory with partition function \([4]\) was carried out in papers \([33,35]\). In the present paper we are going to study the theory with the partition function

\[
Z = \int D\phi e^{-S_{\phi}} \cdot (\text{det}(M^{\dagger}M + \lambda^2))^\frac{1}{2}, \tag{5}
\]

which corresponds to \(N_f = 2\) dynamical fermions in the continuum limit. Notice that the diquark source term lifts the lowest eigenvalues of the matrix in determinant \([5]\) and thus lowers the cost of numerical simulations.

It is known that the symmetries of the staggered fermion action are different from those of two-color QCD with fundamental quarks \([32]\). In particular, the symmetry breaking pattern of QC\(_2\)D with fundamental quarks is SU(2\(N_f\)) \(\rightarrow\) Sp(2\(N_f\)), whereas for staggered quarks it is SU(2\(N_f\)) \(\rightarrow\) O(2\(N_f\)). However, it is easy to show that the diquark source term in the continuum limit can be written as

\[
\frac{\lambda}{2} \sum_x \left( \bar{\psi}_x \gamma^2 \tau_2 \psi_x - \bar{\psi}_x \tau_2 \psi_x \right) \bigg|_{\alpha \to 0} = \frac{\lambda}{2} \int d^4x \left( \bar{q}_T C\gamma_5 \tau_2 q_j + \bar{q}_i C\gamma_5 \tau_2 \bar{q}_j \right) \times \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}_{ij}.
\]

So in the naive continuum limit for the diquark source term we have two copies of fundamental fermions. Thus, one can expect that the partition function \([5]\) after rooting procedure corresponds to QC\(_2\)D with \(N_f = 2\) fundamental fermions. Moreover, for sufficiently small lattice spacing \(a\) the \(\beta\)-function of the theory \([5]\) corresponds to the \(\beta\)-function of QC\(_2\)D with two fundamental flavors (see below). For these reasons we believe, that the partition function \([5]\) in the continuum limit describes QC\(_2\)D with \(N_f = 2\) fundamental fermions.

**B. Observables**

In our simulations we measured the following observables:

- The Polyakov loop:
  \[
  \langle L \rangle = \frac{1}{N^3_s} \sum_{x_1,x_2,x_3=0}^{N-1} \frac{1}{2} \left\{ \text{Tr} \left[ \prod_{x_4=0}^{N-1} U_{x,4} \right] \right\}; \tag{6}
  \]

- The time-like Wilson loop around a rectangular contour \(C = R \times T\):
  \[
  W(R,T) = \left\{ \text{Tr} \left[ \prod_C U_{x,\mu} \right] \right\}; \tag{7}
  \]

- The chiral condensate:
  \[
  a^3 \langle \bar{q}q \rangle = a^3 \langle \bar{q}_{i\alpha} q_{i\alpha} \rangle = -\frac{1}{N^2_s N_{\tau}} \frac{\partial (\ln Z)}{\partial (ma)}; \tag{8}
  \]

- The baryon density:
  \[
  a^3 n_B = a^3 \frac{1}{2} \langle \bar{q}_{i\alpha} \gamma_0 q_{i\alpha} \rangle = \frac{1}{2} \frac{N^2_s N_{\tau}}{\partial (ma)} \frac{\partial (\ln Z)}{\partial (ma)}; \tag{9}
  \]

- The diquark condensate:
  \[
  a^3 \langle \bar{q}q \rangle = -\frac{1}{N^2_s N_{\tau}} \frac{\partial (\ln Z)}{\partial \lambda} = a^3 \langle \bar{q}_{i\alpha} \gamma_5 (\tau_2)_{ij} (\sigma_2)_{\alpha\beta} q_{j\beta} \rangle. \tag{10}
  \]

In formulae \((8) - (10)\) the fields \(\bar{q}, q\) are quark fields in the continuum theory, \(\bar{C}\) is the matrix of charge conjugation, \(\tau_2\) and \(\sigma_2\) are flavor and colour Pauli matrices, respectively. The quark fields have Dirac (not shown for the sake of brevity), colour \((\alpha, \beta)\) and flavor indices \((i, j)\). Summation over repeated indices is tacitly understood. Notice that in addition to the quark contribution there is similar antiquark contribution to equation \((10)\), which is not shown. This is because we work with positive chemical potential and in this region antiquark contribution to the equation \((10)\) is exponentially suppressed. In numerical calculations of the diquark condensate we have taken into account both quark and antiquark contributions.

The Polyakov and Wilson loops are meant to be sensitive to an eventual confinement/deconfinement phase transition. The chiral condensate is sensitive to breaking/restoration of the chiral symmetry. The diquark condensate is an order parameter for the transition to a phase, where scalar diquarks are condensed.

**C. Details of the simulation**

To study the phase diagram of QC\(_2\)D with \(N_f = 2\) flavors we used a \(16^3 \times 32\) lattice, simulating with \(\beta = 2.15\) and \(ma = 0.005\), what corresponds to a fixed temperature \(T \approx 55\) MeV, lattice spacing \(a = 0.112(1)\) fm, pion mass \(M_\pi = 378(4)\) MeV and \(m_\pi L_\pi \approx 3.4\) (see the section \([D]\)). The simulation was carried out for a set of values of the chemical potential \(\mu\) spanning the region \(\mu \in [0; 1759]\) MeV (\(\mu a \in [0.0; 1.0]\)). For each value of \(\mu\) in the region \(\mu \in [0; 1055]\) MeV (\(\mu a \in [0.0; 0.6]\)) we carried out the simulation at three values of the diquark source \(\lambda = 0.001, 0.00075\) and 0.0005. The measured data have been then extrapolated to \(\lambda = 0\). In the vicinity of the phase transition from the hadronic phase to the BEC phase \(\mu = 176, 211, 246\) MeV (\(\mu a = 0.1, 0.12, 0.14\) respectively) we carried out simulations at five values of the diquark source: \(\lambda = 0.001, 0.000875, 0.00075, 0.000625\) and 0.0005. Simulations with higher \(\mu\) are more computationally demanding, thus for \(\mu > 1055\) MeV (\(\mu a > 0.6\)) only one value of the diquark source, \(\lambda = 0.0005\), was used.

In the simulations we used the RHMC algorithm \([44,45]\). The fourth root in the action evaluation was approximated with the accuracy \(\sim O(10^{-15})\). For each pair of \(\mu\) and \(\lambda\) we generated 1000 — 1500 MD trajectories after thermalization and performed measurements.
of the Polyakov loop \( \langle 0 \rangle \) at each trajectory and of the fermionic observables \( \langle S \rangle \) at each 10th trajectory. We employed the stochastic estimation technique with Gaussian random sources to calculate fermionic traces and used 100 — 250 Gaussian random vectors per trace.

It is worth to mention, that we carried out a check of our simulation program through the comparison of our results with the QC\( \beta \)D results existing in the literature. In particular, we compared with the results of simulation of staggered fermions without rooting and chemical potential \( \mu = 0 \) \[47\], of staggered \( N_f = 2 \) flavors and with \( \mu = 0 \) \[47\], and of staggered \( N_f = 4 \) flavors with non-zero chemical potential and non-zero diquark source \[32\]. For all these cases we found good agreement.

D. Scale setting and pion mass

First we performed additional measurements at zero values of the baryon chemical potential \( \mu \) in order to calculate the \( \beta \)-function and the pion mass, because the behaviour of the \( \beta \)-function provides a natural check for the correct continuum limit. In these simulations we used a lattice with the size \( 16^3 \times 32 \) as well. To fix the physical scale, we extracted the heavy quark potential from smeared Wilson loops \( 1 \) HYP smearing \[18\] step for temporal links was employed followed by \( 20 \) APE smearing \[29\] steps for spatial links, the details are described in \[50\]). From this potential we extracted the Sommer scale parameter \( r_0 \). Assuming, that it is equal to the Sommer scale parameter in real QCD, \( r_0 = 0.468(4) \) fm in physical units \[51\], we determined the lattice spacing.

To carry out the scale setting we fixed the quark mass \( ma = 0.005 \), the diquark source \( \lambda = 0.0005 \) and varied \( \beta \in [2.1; 2.25] \). 4000 MD trajectories were generated for each value of \( \beta \), measurements were performed at every 10th trajectory. The results of the simulation are presented in Tab.\[I\] and in Fig.\[I\].

We found that for all considered values of \( \beta \) the dependence of the lattice spacing \( a \) on the inverse coupling \( \beta = 4/g^2 \).

We performed additional measurements at zero values of the baryon chemical potential \( \mu \) in order to calculate the \( \beta \)-behaviour of the lattice with the size \( 16 \times 0 \) \[47\], and of staggered \( N_f = 2 \) flavors and with \( \mu = 0 \) \[47\], and of staggered \( N_f = 4 \) flavors with non-zero chemical potential and non-zero diquark source \[32\]. For all these cases we found good agreement.

![FIG. 1. (Color online) The dependence of the lattice spacing on the inverse coupling \( \beta = 4/g^2 \).](image)

| \( \beta \) | \( a, \text{fm} \) | \( M_\pi, \text{MeV} \) |
|---|---|---|
| 2.1 | 0.129(1) | 329(3) |
| 2.15 | 0.112(1) | 377(4) |
| 2.2 | 0.095(2) | 493(8) |
| 2.25 | 0.082(1) | 561(9) |

**TABLE I.** The lattice spacing \( a \) and the pion mass \( m_\pi \) for various values of the inverse coupling \( \beta \) for the bare quark mass \( ma = 0.005 \) and \( \lambda = 0.0005 \).

In this section we are going to study the diquark condensate. It was noted above, that in the region \( \mu \in [0.0; 1055] \) MeV \( (\mu a \in [0.0; 0.6]) \) the condensate is calculated for three values of the diquark source: \( \lambda = 0.0005\), \( 0.00075 \), and \( 0.001 \). To extrapolate our results to \( \lambda \to 0 \) we used a linear fit \[52\] of the data for all values of the chemical potential under investigation. The linear fit turned out to be good \( (\chi^2/\text{dof} \sim 1) \) in the region \( \mu \leq 141 \) MeV \( (\mu a \leq 0.08) \) and \( \mu \geq 263 \) MeV \( (\mu a \geq 0.15) \). For the values \( \mu a = 176 \) MeV, \( 211 \) MeV and \( 246 \) MeV \( (\mu a = 0.1, 0.12, 0.14, \text{respectively}) \) a linear fit does not describe the data well. We believe, that this fact can be explained by the closeness of these \( \mu \) values to the critical chemical potential \( \mu^\ast \), where the system undergoes
the phase transition from the hadronic phase to the phase
with \( \langle qq \rangle \neq 0 \).

In Fig. 2 we plot the diquark condensate \( \langle qq \rangle \) (obtained
by linear extrapolation to \( \lambda = 0 \) ) as a function of \( \mu \) in the
region \( \mu \in [0.0; 440] \text{ MeV} \) (\( \mu \alpha \in [0.0; 0.25] \)). It may
be seen, that for \( \mu \leq 141 \text{ MeV} \) (\( \mu \alpha \leq 0.08 \) ) the diquark
condensate \( \langle qq \rangle \) is compatible with zero. However, for
\( \mu \geq 176 \text{ MeV} \) (\( \mu \alpha \geq 0.1 \) ) the diquark condensate starts
deviate from zero. If we are sufficiently far from the
position of the phase transition, one can try to use ChPT
to describe the data \[14\]. In particular, ChPT predicts
that the transition from the hadronic phase to the
phase with \( \langle qq \rangle \neq 0 \) takes place at \( \mu = m_\pi / 2 \), and the
behaviour of the diquark condensate above the transition
would be given by the formula:

\[
\langle qq \rangle = \langle qq \rangle_0 \sqrt{1 - \left( \frac{\mu}{\mu_0} \right)^4},
\]

where \( \langle qq \rangle_0 \) is the chiral condensate at zero chemical
potential. If one uses formula \[12\] to fit our data in the
region \( \mu \in [263; 352] \text{ MeV} \) (\( \mu \alpha \in [0.15; 0.20] \)), one gets
\( \mu = 215(10) \text{ MeV} \) (\( \alpha \mu = 0.12(6) \)) with \( \chi^2/dof = 2.5 \).
We plot the function \[12\] in Fig. 2.

One can try to fit the data by another function. To
build it we recall that in the ChPT the diquark condensate
can be determined from the equation \( \langle qq \rangle = \sqrt{\langle qq \rangle_0^2 - \langle \bar{q}q \rangle^2} \). In the ChPT for the \( \mu > \mu_c \) the
chiral condensate \( \langle \bar{q}q \rangle \) drops with the chemical potential
as \( \sim 1/\mu^2 \) and thus one gets \[12\]. However, our data
show (see below), that the chiral condensate drops slower:
\( \langle qq \rangle \sim 1/\mu^\alpha \) with \( \alpha = 0.78(2) \). Thus it is reasonable
to fit the data by the formula

\[
\langle qq \rangle = \langle qq \rangle_0 \sqrt{1 - \left( \frac{\mu}{\mu_0} \right)^{2\alpha}},
\]

with the power \( \alpha \) mentioned above. The fit of the data
by formula \[13\] in the region \( \mu \in [263; 352] \text{ MeV} \) (\( \mu \in [0.15; 0.20] \)) gives \( \mu = 193(10) \text{ MeV} \) (\( \alpha \mu = 0.110(6) \))
with \( \chi^2/dof = 1.4 \).

From these examples one sees, that the position of
the critical point determined from the fitting procedure
strongly depends on the fitting function. Nevertheless,
one can state, that the results for \( \mu_c \) are in reasonable
agreement with ChPT.

It is interesting to study the limit \( \lambda \to 0 \) of our data
in the vicinity of the phase transition at \( \mu = 176, 210 \)
and 246 MeV. For these values of the chemical potential
the diquark condensate was measured at five points
\( \lambda = 0.0005, 0.000625, 0.00075, 0.000825 \) and 0.001.
From ChPT we know, that at the critical chemical potential
\( \mu = \mu_c \) the behaviour of the diquark condensate should be
like \( \langle qq \rangle \sim \lambda^{1/3} \). Thus it is reasonable to fit the data
in the vicinity of the phase transition by the function
\( \langle qq \rangle = A + B \lambda^{1/3} \). The results of the fit are shown in
Fig. 3. For all three values of the chemical potential the
fit is good: \( \chi^2/dof \sim 1 \). We found, that for the smallest
chemical potential value \( \mu = 176 \text{ MeV} \) (\( \mu \alpha = 0.1 \) )
the extrapolated value of the diquark condensate is negative:
\( \langle qq \rangle \big|_{\lambda \to 0} = -0.012(2) \). Negative value of the condensate
can be attributed to the fact that the value \( \mu = 176 \text{ MeV} \)
is rather far from the critical point. For the next value
\( \mu = 211 \text{ MeV} \) (\( \mu \alpha = 0.12 \) ) the condensate is compatible
to zero: \( \langle qq \rangle \big|_{\lambda \to 0} = -0.0021(12) \). Finally, for the largest
value \( \mu = 246 \text{ MeV} \) (\( \mu \alpha = 0.14 \) ) the condensate is greater
than zero: \( \langle qq \rangle \big|_{\lambda \to 0} = 0.0058(14) \). These results indicate
that \( \mu = 211 \text{ MeV} \) (\( \mu \alpha \approx 0.12 \) ) is closer to the critical
point than \( \mu = 176 \text{ MeV} \) (\( \mu \alpha = 0.1 \) ) and \( \mu = 246 \text{ MeV} \)
(\( \mu \alpha = 0.14 \) ), which agrees within the uncertainty with the
value of the critical point obtained above.

To summarize: in the region \( \mu < \mu_c \) the system is in
the hadronic phase with zero diquark condensate. In
the region \( \mu > \mu_c \) the system is in the BEC phase with
nonzero diquark condensate. In the region \( \mu \in [0.0;352] \) MeV (\( \mu a \in [0.0;0.20] \)) our results for the diquark condensate are in good agreement with ChPT predictions. From Fig. 2 one sees, that in the region \( \mu > 352 \) MeV (\( \mu a > 0.2 \)) the data start to deviate from ChPT description.

Let us consider the region of larger chemical potential \( \mu > 352 \) MeV (\( \mu a > 0.2 \)). To understand what happens in this region, we plot in Fig. 4 the linearly extrapolated diquark condensate, divided by \( T \mu^2 \), as a function of \( \mu \). As visible from this plot, in the region \( \mu > 352 \) MeV (\( \mu a \in [0.3;0.6] \)) there is a plateau, i.e. the value of the diquark condensate is proportional to the surface of a sphere with the radius \( \mu \): \( \langle qq \rangle \sim \mu^2 \). This is a characteristic property of the BCS theory, where the condensate appears on the Fermi surface and where it is proportional to the density of states on this surface. Thus we conclude, that for \( \mu > 528 \) MeV (\( \mu a > 0.3 \)) the system reveals properties of the BCS phase, and that the transition from the BEC to the BCS phase is smooth.

It is worth to note, that in [35] \( N_f = 4 \) theory was simulated on the \( 16^4 \) lattice at \( \beta = 1.85 \) with \( ma = 0.05 \) and the BCS phase has not been observed. According to [40], the lattice spacing for this set of parameters is larger than lattice spacing in our simulations. In our study of the critical chemical potential is \( \mu^* a \simeq 0.12 \), whereas in [35] it was found that \( \mu^* a \simeq 0.29 \). From the relation \( \mu^* = m_\pi/2 \) one might conclude that in [35] the pion is more than two times heavier than in our simulations. This remarkable physical difference may be the reason why in the previous studies with \( N_f = 4 \) the BCS phase has not been realized.

In the region \( \mu > 1055 \) MeV (\( \mu a > 0.6 \)) the simulations become very computationally demanding. At the same time in this region the value of the diquark condensate becomes less sensitive to the value of the source \( \lambda \), compared to the BEC phase. We believe that this might be related to the fact that the larger the \( \mu \) the larger the fermion mass gap, which plays a role of the regulator of the fermion determinant. For this reason for \( \mu > 1055 \) MeV (\( \mu a > 0.6 \)) we used \( \langle qq \rangle |_{\lambda=0.0005} \) as the estimate of the value of the condensate at \( \lambda = 0 \). In Fig. 6 we plot the diquark condensate \( \langle qq \rangle \) as a function of \( \mu \) throughout the whole region under study. In the region \( \mu > 1055 \) MeV (\( \mu a > 0.6 \)) the condensate starts to deviate from the BCS behaviour, and after \( \mu > 1410 \) MeV (\( \mu a > 0.8 \)) the condensate decreases. Such a descent of the diquark condensate \( \langle qq \rangle \) in the region \( \mu a \sim 1 \) has already been observed in refs. [34, 35]. This behaviour might be connected with a saturation effect, and therefore can be considered as a lattice artifact.
In Fig. 7 we plot the combination \( \sqrt{(q\bar{q})^2 + \langle q\bar{q} \rangle^2} / T^3 \) of diquark and chiral condensates as a function of \( \mu \). The chemical potential is expressed in physical units (lower scale) and in lattice units (upper scale).

**B. The chiral condensate**

Next let us consider the chiral condensate \( \langle q\bar{q} \rangle \). In Fig. 8 we plot the chiral condensate as a function of \( \mu \) in the region \( \mu \in [0; 1440] \) MeV (\( \mu a \in [0.0; 0.25] \)) for the following three values of the diquark source: \( \lambda = 0.001 \), \( 0.00075 \) and \( 0.0005 \). From the Fig. 8 it is obvious, that for the values \( \lambda \) the chiral condensate does not depend on the chemical potential. In the region \( \mu > 176 \) MeV, where the system is in the vicinity of the transition to the BEC phase, the chiral condensate starts to decrease. These properties are in agreement with ChPT predictions (see Figures 4 and 5 in paper [16]). An interesting prediction of ChPT is that in the whole region, where ChPT is applicable, a relation between the chiral condensate and the diquark condensate holds: \( (q\bar{q})^2 + \langle q\bar{q} \rangle^2 = \text{const} \). Note that this “circle relation” is valid only in the leading order approximation, and it is violated by the next-to-leading order corrections [17]. Our lattice results allow us to address the question how well this relation is satisfied. In Fig. 7 we plot the combination \( \sqrt{(q\bar{q})^2 + \langle q\bar{q} \rangle^2} / T^3 \) as a function of \( \mu \). From this plot one sees, that the deviation from the “circle relation” is well satisfied up to \( \mu < 263 \) MeV (\( \mu a < 0.15 \)). For bigger \( \mu \) values one can see the deviation from the “circle relation” for all values of the diquark source \( \lambda \) under consideration. Note also that for the values \( \lambda = 0.00075 \) and \( 0.0005 \), which move the system closer to the phase transition, the deviation from the “circle relation” is clearly seen in the transition region \( \mu \in [176; 246] \) MeV (\( \mu a \in [0.1; 0.14] \)). The smaller the source \( \lambda \) is chosen, the larger is the deviation. We believe, that the deviation of our results from the relation \( \sqrt{(q\bar{q})^2 + \langle q\bar{q} \rangle^2} / T^3 = \text{const} \) in the region \( \mu \in [176; 246] \) MeV can be explained by the closeness of the system to the transition point, where a mean field study of ChPT is not applicable.

Now let us consider the chiral condensate throughout the full region \( \mu \in [0; 1759] \) MeV (\( \mu a \in [0.0; 1.0] \)). In Fig. 8 we plot the chiral condensate calculated for the smallest diquark source \( \lambda = 0.0005 \) as a function of \( \mu \). It was noted above that the chiral condensate is practically insensitive to the values of \( \lambda \), thus the value of the chiral condensate at \( \lambda = 0.0005 \) can be taken as the value at \( \lambda = 0 \). According to ChPT, at \( \mu > \mu^c \) the chiral condensate drops as

\[
\langle q\bar{q} \rangle = \langle q\bar{q} \rangle_0 \left( \frac{\mu^c}{\mu} \right)^\alpha,
\]

where \( \langle q\bar{q} \rangle_0 \) denotes the chiral condensate at zero chemical potential. To check this prediction in the region \( \mu \in [263; 352] \) MeV (\( \mu a \in [0.15; 0.20] \)) we fit our data by a power law \( \langle q\bar{q} \rangle = A / \mu^\alpha \). This ansatz fits our data well \((\chi^2/\text{dof} = 0.3) \) with the exponent \( \alpha = 0.78(2) \). It is interesting to note, that this fit gives a satisfactory description of the data up to \( \mu \sim 1055 \) MeV (\( \mu a \sim 0.6 \)). Thus, one sees that the chiral condensate drops slower with increasing chemical potential than ChPT predicts. Similar slower decrease of the form \( \langle q\bar{q} \rangle \sim 1 / \mu \) was observed in [38] on the gauge ensembles generated with \( N_f = 2 \) Wilson quarks.

Good agreement with the LO ChPT prediction for the chiral condensate dependence on the chemical potential was found in [54] for \( N_f = 1 \) adjoint flavor, and in [34] for \( N_f = 4 \) fundamental flavors, the latter study being carried out at \( \beta = 1.5 \). On the other hand, in [33] another lattice study at \( \beta = 1.85 \) has been performed, where the chiral condensate was found to decrease as \( \langle q\bar{q} \rangle \sim 1 / \mu^\alpha \) with \( \alpha = 1 \cdots 1.3 \) depending on the \( \lambda \) value (see
Table 3 of the Ref. [32]). The same dependence of the $\langle \bar{q}q \rangle$ on the baryon chemical potential was also observed in [32] for $N_f = 8$ fundamental flavors at $\beta = 1.3$. We conjecture that the behaviour of the chiral condensate is rather sensitive on the coupling regime of the theory. If $\beta$ is small enough and the system is in the strong coupling regime the leading order of ChPT is sufficient, and higher order effects are weak.

Finally, it is interesting to study the chiral symmetry breaking in the chiral limit for different regions of the chemical potential. In Fig. 9 we plot the chiral condensate for different values of the chemical potential as function of the quark mass. As an example we took a few values of the chemical potential in the hadronic phase: $\mu = 0, 70$ and 141 MeV ($\mu a = 0.0, 0.04, 0.08$, respectively), in the BEC phase: $\mu = 246, 281, 352$ MeV ($\mu a = 0.14, 0.16, 0.20$, respectively), and in the BCS phase: $\mu = 615$ MeV ($\mu a = 0.35$). At these fixed values of the chemical potential we linearly extrapolate our data to $ma = 0$. It is seen from Fig. 9 that chiral symmetry breaking exists in the chiral limit within the hadronic phase (values $\mu = 0, 70$ and 141 MeV), whereas there is no chiral symmetry breaking in the chiral limit in the BCS phase ($\mu = 615$ MeV). We also found, that the chiral limit of the chiral condensate at the points $\mu = 246, 281, 352$ MeV (in the BEC phase) are vanishing, but it is difficult to claim, that there is no chiral symmetry breaking in the whole BEC phase: when we take the chiral limit, we change the pion mass and thus shift the critical point closer $\mu^c$ to zero. This effect is not important for the values of the chemical potential sufficiently far from the phase transition, but it might be important close to the phase transition. Note, that the absence of chiral symmetry breaking in the chiral limit within the BEC phase agrees with ChPT predictions.

C. The baryon density

In this section we are going to consider the baryon number density $n_B$. It clear, that for all $\mu < 176$ MeV ($\mu a < 0.1$) the baryon density is vanishing within the uncertainty of the calculation. In the vicinity of the phase transition ($\mu \geq 176$ MeV) the baryon density starts to deviate from zero, and for larger values of the chemical potential it rises with increasing $\mu$. ChPT predicts, that the dependence of the baryon density on the chemical potential above $\mu^c$ is given by a formula $n_B \sim \mu - \mu^c/\mu^3$. In the region $\mu \in [263; 352]$ MeV ($\mu a \in [0.15; 0.20]$) we fit our data by this formula in order to extract the critical chemical potential $\mu^c$. The fit is of good quality, $\chi^2/dof = 1.2$, and the result is $\mu^c = 207(7)$ MeV ($a\mu^c = 0.118(4)$). This value is in agreement with our previous results for $\mu^c$, obtained from the $\langle qq \rangle$ fits. From Fig. 10 it is visible, that for bigger chemical potential, $\mu > 352$ MeV ($\mu a > 0.2$), our data deviate from the ChPT prediction.

Next, let us consider the baryon density at even larger values of the chemical potential. In Fig. 10 we plot the ratio $n_B/n_0$ as a function of $\mu$, where for the square points the reference density $n_0$ is the baryon density for free continuum fermions at $T = 0$, $n_0 = (2\mu^3)/(3\pi^2)$, and for the circle points $n_0$ is the baryon number density for free lattice fermions. It can be seen, that in the region $\mu \in [528; 1055]$ MeV ($\mu a \in [0.3; 0.6]$) these ratios are slowly varying functions of the chemical potential, tak-
FIG. 11. (Color online) The ratio \( n_B/n_0 \) as a function of the chemical potential \( \mu \). For the square points, the reference density \( n_0 \) denotes the baryon density for free continuum fermions, \( n_0 = (2\mu^3)/(3\pi^2) \), whereas for the circle points the reference density \( n_0 \) denotes the baryon density for free lattice fermions. The chemical potential is expressed in physical units (lower scale) and in lattice units (upper scale).

FIG. 12. (Color online) The time-like Wilson loops for the contours 8 × 8 and 10 × 10 as a functions of the chemical potential \( \mu \).

D. The gluon observables

In this section we study the gluon observables Polyakov loop \( \langle \Omega \rangle \) and Wilson loops \( W \). Similarly to the chiral condensate the gluon observables are not sensitive to the value of the \( \lambda \), thus we take these observables calculated at the smallest value \( \lambda = 0.00005 \) as their values at the \( \lambda = 0 \).

We measured the average of the Polyakov loop as a function of the chemical potential. The result of this measurement is that for all values of the chemical potential studied in this paper the average Polyakov loop is vanishing within the uncertainty of the calculation.

Furthermore, in order to investigate the confinement properties of the system, we have calculated time-like Wilson loops \( W \) for the quadratic contours of the size 8 × 8 and 10 × 10 (for larger Wilson loops we obtained results compatible with zero) as functions of the chemical potential. The same smearing strategy, as discussed in the section II.D, was employed for these Wilson loops measurements. The results are shown in Fig. 12. One learns from this plot, that for \( \mu > 352 \) MeV (\( \mu > 0.2 \)) the Wilson loops decrease with the growth of the chemical potential. At small \( \mu \), for \( \mu \in [0; 263] \) MeV (\( \mu \in [0.0; 0.15] \)), a plateau for both Wilson loops may be noticed. From these results one can conclude, that the system is in a confined phase for all values of the chemical potential under consideration. The possible explanation for this behaviour may be the absence of the Debye screening in two-color QCD at zero temperature \[18, 55\].

IV. DISCUSSION AND CONCLUSION

In conclusion, in this paper we have carried out a low-temperature scan of the phase diagram of dense two-color QCD with \( N_f = 2 \) quarks. The study has been conducted using lattice simulations with rooted staggered quarks.

Our results can be summarized as follows. At small chemical potential \( \mu < \mu^c = m_\pi/2 \sim 200 \) MeV we observe a hadronic phase. In this phase QC2D matter is in confinement, chiral symmetry is broken, the diquark condensate \( \langle \Omega \rangle \) vanishes and the baryon number density is also zero. Relevant degrees of freedom in this phase are Goldstone bosons.

In the region \( \mu^c < \mu < \mu^d \sim 352 \) MeV we observe the BEC phase. Characteristic feature of this phase is Bose-Einstein condensation of scalar diquarks. The order parameter for the transition to the BEC phase is the diquark condensate, which develops a non-zero value in the region \( \mu > \mu^c \). Within the uncertainty of the calculation \( \mu^c = m_\pi/2 \), where \( m_\pi \) is the pion mass at zero chemical potential. In this phase, QC2D matter has also confining properties, whereas the baryon density is non-zero. Based on our detailed results for the onset of...
the diquark condensate we believe, that the transition from the hadronic to the BEC phase should be of the second order. Relevant degrees of freedom in the BEC phase are Goldstone bosons as well.

We have also found, that the chiral limits of the chiral condensate at the points \( \mu = 246, 281, 352 \text{ MeV} \) in the BEC phase are vanishing. Nevertheless, it is difficult to claim, that there is no chiral symmetry breaking in the whole BEC phase, since when we take the chiral limit – we change the pion mass and thus shift the critical point \( \mu_c \). This effect is not important for the values of the chemical potential sufficiently far from the phase transition, but it might be essential close to the phase transition.

It is important to notice, that for all values of the chemical potential \( \mu < \mu^d \), our results are in good agreement with the predictions of ChPT. An exception is the chiral condensate, which drops with increasing chemical potential slower than ChPT predicts in leading order. This behaviour of the chiral condensate might be explained by higher radiative corrections.

In the region \( \mu > \mu^d \), our data start to deviate from ChPT predictions. The physical origin of this deviation can be understood as follows. At \( \mu = \mu^d \), the baryon number density is \( n_B \sim 1 \text{ fm}^{-3} \) (see Fig. 10). In SU(3) theory, a baryon density \( n_B \sim 1 \text{ fm}^{-3} \) is of the order, when a gas of baryons can not be considered anymore as dilute. The interactions of baryons at a density \( n_B > 1 \text{ fm}^{-3} \) play an important role and can not be taken into account as a perturbation, as it is done within ChPT. On the contrary, for a density \( n_B < 1 \text{ fm}^{-3} \) a gas of baryons can be considered as dilute and ChPT is applicable. From this consideration one may conclude, that QC\(_2\)D in the region \( \mu^c < \mu < \mu^d \) is an analog of the dilute baryon gas of SU(3) QCD. It is remarkable, that the density at which one expects the transition from dilute gas to dense baryon matter in the SU(3) QCD is very close to that in the QC\(_2\)D.

If we further increase the chemical potential, starting from \( \mu \sim 500 - 600 \text{ MeV} \), one can observe that the diquark condensate scales as \( \langle qq \rangle \propto \mu^2 \) and the baryon density scales as \( n_B \propto \mu^3 \). Physically, this implies that the relevant degrees of freedom are quarks, which are mostly living inside the Fermi sphere with a condensate of Cooper pairs on the Fermi surface. These properties are clear hints in favor of the **BEC phase**. In this phase the chiral symmetry is restored in the chiral limit. Our measurements of the time-like Wilson loops imply, that the system still keeps the confinement property in this phase. In addition our data confirm, that the transition from the BEC to the BCS phase is smooth, if there is a phase transition at all.

The BCS phase extends up to \( \mu \sim 1000 - 1100 \text{ MeV} \). In the region \( \mu \in \{1100; 1140\} \text{ MeV} \) the ratio \( \langle qq \rangle/\mu^2 \) drops, the baryon density scaling is \( n_B \sim \mu^3 \), the chiral condensate is very small, and the system is still retaining the confinement property. It is not quite clear, what happens in this region, but most likely we are facing with lattice artifacts, related with the fact, that \( \mu a \) is close to 1. In the region \( \mu > 1410 \text{ MeV} \) (\( \mu a > 0.8 \)) the diquark condensate begins to drop, and \( n_B \) is close to saturation.

The results obtained in this paper are in reasonable agreement with the results of Refs. [32, 33]. In these papers the authors studied the phase diagram of QC\(_2\)D with \( N_f = 4 \) flavors of staggered fermions. What concerns a low temperature scan of the phase diagram, these authors observed the succession of a hadronic phase and the BEC phase, with their properties well described by ChPT, but they didn’t find a BCS phase.

In Refs. [32, 33] the QC\(_2\)D phase diagram with \( N_f = 2 \) flavors was studied through lattice simulation with Wilson fermions. In a low temperature scan of the phase diagram the authors observed a hadronic phase, followed by the BCS phase with deconfinement. Probably, the BEC phase has been missed in their simulations due to the violation of chiral symmetry by Wilson fermions. In addition these authors observed the transition to the deconfinement phase at \( \mu \sim 800 \text{ MeV} \) for a temperature \( T = 47 \text{ MeV} \) [38]. In our study we don’t observe the transition to the deconfinement phase up to the chemical potential \( \mu \sim 1410 \text{ MeV} \). In order to understand the origin of the disagreement between our results and the results of the other groups one should carry out more numerical simulations with different set of lattice parameters, but with the same \( N_f \) and at the same physical point.

It is interesting to mention the results of Ref. [3], where the phase diagram of SU\(_N\text{c} \) QCD was studied in the limit \( N_c \to \infty \). The authors of this paper predicted the following phases: firstly, a hadronic phase is observed at sufficiently small chemical potential; when the chemical potential reaches \( \mu = m_N/N_c \), the baryonic density ceases to vanish, and there starts a phase of a dilute nuclear gas, which is similar to the BEC phase of the QC\(_2\)D theory. Further enhancing the chemical potential, this study has ended with the so-called “quarkyonic phase”. In this phase there is a Fermi sphere of quarks, at the surface of which baryons are living. The system is in confinement, but chiral symmetry is restored. The described “quarkyonic phase” at large \( N_c \) may be similar to the BCS phase of the QC\(_2\)D theory. Using this physical picture it is not difficult to estimate the value of the chemical potential, where the quarkyonic phase becomes manifest. To do this we note, that the thickness of the surface layer, where strong interactions are important, is \( \sim \Lambda_{QCD} \). Then the “quarkyonic phase” becomes manifest, when the volume inside the Fermi sphere \( \sim 4/3 \pi \mu^3 \) becomes larger than the volume of the surface layer, modified by strong interactions which is \( \sim 4\mu^2\Lambda_{QCD} \). Thus we get \( \mu > 3\Lambda_{QCD} \).

If we take \( \Lambda \sim 200 \text{ MeV} \), the “quarkyonic phase” starts at \( \mu > 600 \text{ MeV} \), what is in good agreement with the result of our present paper. One can also expect, that the value of the chemical potential, where the “quarkyonic phase” starts in SU(3) theory is close to that in QC\(_2\)D, \( \mu \sim 600 \text{ MeV} \), since the \( \Lambda_{QCD} \) values in both theories are close to each other.
Finally, we summarize that in this paper we have carried out a low temperature scan of the phase diagram for the QC2D theory with two flavors of quarks. We have shown that the phase structure of this theory has a lot of similarities with SU(Nc) theory at large Nc. Since the predictions of the SU(Nc) theory at large Nc start to work already at Nc = 2, one can use QC2D to make quantitative estimates for SU(3) QCD with chemical potential, which is directly inaccessible due to the sign problem.

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