Quasiconvex risk measures with markets volatility

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Abstract Since the quasiconvex risk measures is a bigger class than the well known convex risk measures, the study of quasiconvex risk measures makes sense especially in the financial markets with volatility. In this paper, we will study the quasiconvex risk measures defined on a special space $L^{p(\cdot)}$ where the variable exponent $p(\cdot)$ is no longer a given real number like the space $L^p$, but a random variable, which reflects the possible volatility of the financial markets. The dual representation for this quasiconvex risk measures will also provided.

Keywords quasiconvex; risk measure; variable exponent; dual representation;

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1 Introduction

In their seminal paper, Artzner et al. (1997, 1999) firstly introduced the class of coherent risk measures, by proposing four basic properties to be satisfied by every sound financial risk measure. Further, Föllmer and Schied (2002), and independently, Frittelli and Rosazza Gianin (2002) introduced the broader class, named convex risk measures, by dropping one of the coherency axioms.

Recent years, risk measures on general linear spaces have attracted many attentions. Kountzakis (2011) studied the coherent and convex risk measures on ordered non-reflexive Banach spaces. Konstantinides and Kountzakis (2011) studied the coherent and convex risk measures on partially ordered normed linear spaces. The coherent risk measures defined on ordered Banach space were studied by Kountzakis and Polyrakis (2013), while convex risk measures defined on appropriate wedges of a space of financial positions were studied by Konstantinides and Kountzakis (2014). In all the above-mentioned works on risk measures, an axiom of convexity is employed. However, as pointed out by Cerreia-Vioglio et al. (2011), once the cash additivity (translation invariance) is replaced with the economically sounder assumption of cash sub-additivity, the sounder mathematical translation of the ‘diversification’ should be the so-called quasiconvexity. They also claimed that when a decision problem under uncertainty is regarded as a game against nature, the quasiconvex function can be interpreted as nature’s cost function. These observations motivated us to study the quasiconvex risk measures on a more general space.

In most frameworks of quasiconvex risk measures, the spaces of financial positions are

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described by the linear space of bounded random variables, which can be regard as the subspace of $L^p$ with $p \in [1, \infty)$. However, the financial markets are becoming much more complicated that the usual risk measures may not deal with the risk management availably. This arise the awareness of the urgent need for designing more appropriate risk measures under a financial systems with greater uncertainty and volatility. The current volatility of risk is reflected in the potentially conflicting views on the relationship between the structure of the financial network and the extent of financial contagion. In other words, it is the volatility of the financial markets. Taking this into consideration, we would like to emphasize that our study of quasiconvex risk measures will not focus on the common space of financial positions, but on a special space: the variable exponent Bochner-Lebesgue space, which is denoted by $L^{p(\cdot)}$. Under this space, the order $p(\cdot)$ is no longer a fixed positive number like $L^p$, but a measurable function. More concretely, the variable exponent $p(\cdot)$ reflects the uncertainty and volatility of the financial markets.

The variable exponent Lebesgue spaces appeared firstly in Orlicz (1931). For more studies on variable exponent Lebesgue spaces, see Harjulehto et al. (2010), Kempka (2010), Diening et al. (2009), Hästö (2009), Kempka (2009), Xu (2009), Xu (2008), Almeida et al. (2008), Kováčik and Rákosník (1991), Musielak (1983), Nakano (1950) and the references therein.

In the present paper, we will introduce a new class of quasiconvex risk measures, which was defined on a special space of financial positions, the variable exponent Bochner-Lebesgue space. Dual representation of this class of quasiconvex risk measures is provided.

The rest of the paper is organized as follows. In Section 2, we will briefly review the definition and the main properties of variable exponent Bochner-Lebesgue spaces. Section 3 is devoted to the definition of quasiconvex risk measures on the variable exponent Bochner-Lebesgue spaces. Finally, in Section 4, we will study the dual representation of quasiconvex risk measures.

2 Preliminaries

In this section, we will briefly introduce the definition and the main properties of variable exponent Bochner-Lebesgue spaces and the preliminaries.

From now on, let $(\Omega, \mathcal{F}, \mu)$ be a $\sigma$-finite complete measurable space, $E$ be a given reflexive Banach space with zero element $\theta$ and dual space $E^*$. Throughout this paper, we always assume $E^*$ is partially ordered by a given cone $K$, $E$ is partially ordered by $K_0$ where $K_0 := \{f \in E : \langle X, f \rangle \geq 0 \text{ for any } X \in K\}$ is the positive dual cone of $K$.

**Remark 2.1.** The partial order relation $\geq_{K_0}$ is defined as follows, for any $X, Y \in E$,

$$X \geq_{K_0} Y \iff X - Y \in K_0.$$  

**Remark 2.2.** The cone $K$ is consisted of the ‘admissible’ price functionals. On the other hand, the cone $K$ is also introduced to play the role of the solvency set of financial positions which denotes the way that a set of investors jointly interprets the common notion of the cost of financial positions.
We suppose that the numeraire asset $z$ is some interior point of $K_0$. The asset $z$ is actually either a ‘reference cash stream’ according to Stoica (2006), or a ‘relatively secure cash stream’ according to Jaschke and Küchler (2001).

The Banach space valued Bochner-Lebesgue spaces with variable exponent were first introduced by Cheng and Xu (2013). Now, we will recall the definition and the related properties of this special space. We denote by $S(\Omega, \mu)$ the set of all $\mathcal{F}$-measurable functions $p(\cdot) : \Omega \to [1, \infty]$, which are called variable exponent on $\Omega$. For a function $p(\cdot) \in S(\Omega, \mu)$, we define $p'(\cdot) \in S(\Omega, \mu)$ by $1/p(y) + 1/p'(y) = 1$.

The following definitions come from Cheng and Xu (2013).

**Definition 2.1.** A function $f : \Omega \to E$ is strongly $\mathcal{F}$-measurable if there exists a sequence $\{f_n\}_{n \geq 1}$ of $\mu$-simple functions converging to $f \mu$-almost everywhere.

**Definition 2.2.** The Bochner-Lebesgue space with variable exponent, which is denoted by $L^{p(\cdot)}(\Omega, E)$, is the collection of all strongly $\mathcal{F}$-measurable functions $f : \Omega \to E$ endowed with the norm

$$\|f\|_{L^{p(\cdot)}(\Omega, E)} := \inf\{\lambda > 0, \rho_{p(\cdot)}(f/\lambda) \leq 1\}$$

where

$$\rho_{p(\cdot)}(f) := \int_{\Omega} \|f(y)\|^{p(y)} d\mu(y) \quad \text{and} \quad p(\cdot) \in S(\Omega, \mu).$$

**Remark 2.3.** The variable exponent Bochner-Lebesgue space $L^{p(\cdot)}(\Omega, E)$ was consisted of vector-valued measurable functions, which take values in a given Banach space $E$. Moreover, if $E$ is reflexive Banach space, then $E^*$ is also reflexive. By Diestel and J.Uhl (1977), $E^*$ has the Randon-Nikodym property. Under this condition, $L^{p(\cdot)}(\Omega, E)$ is a reflexive Banach space itself. See Cheng and Xu (2013).

**Remark 2.4.** If $E$ is a reflexive Banach space, then the dual of $L^{p(\cdot)}(\Omega, E)$ is characterized by the mapping $g \mapsto V_g$, $L^{p'(\cdot)}(\Omega, E^*) \to (L^{p(\cdot)}(\Omega, E))^*$ as follows

$$\langle V_g, f \rangle = \int_{\Omega} \langle g, f \rangle d\mu, \quad \text{for any} \quad f \in L^{p(\cdot)}(\Omega, E).$$

See Cheng and Xu (2013).

From now on, we denote by $L^{p(\cdot)} := L^{p(\cdot)}(\Omega, E)$ the variable exponent Bochner-Lebesgue space in the absence of ambiguity. We also denote $L^{p(\cdot)}(K_0) := \{f \in L^{p(\cdot)} : f : \Omega \to K_0\}$ and $L^{p'(\cdot)}(K) := \{g \in L^{p'(\cdot)} : g : \Omega \to K\}$.

**Remark 2.5.** Since the variable exponent $p(\cdot)$ is strongly related to the uncertainty of financial markets, we will use the variable exponent Bochner-Lebesgue space to describe the space of financial positions. This is based on two considerations. From the perspective of the markets, one is hard to evaluate a deterministic order $p$ due to the possible volatility of the markets. On the other hand, from the Bayesian statistical point, the order $p$ can be considered as a kind of parameter, and hence the $p$ should be assumed to be a random variable.
3 Quasiconvex risk measures on $\mathbb{L}_{p(\cdot)}$

When a decision problem under uncertainty is regarded as a game against nature, the quasiconvex function can be interpreted as nature’s cost function. Thus, the study of quasiconvex risk measures have its own financial sense.

In this section, the theory of quasiconvex risk measures will be extended to the case where the space of financial positions is a Bochner-Lebesgue space with variable exponent. Other papers about quasiconvex risk measures, see Drapeau et al. (2015), Mastrogiacomo and Gianin (2015), De-Jian and Long (2015), Drapeau and Kupper (2013), Cerreia-Vioglio et al. (2011b), Cerreia-Vioglio et al. (2011), Drapeau et al. (2011), Frittelli and Maggis (2011) and references therein. The main target of this section is to study the properties of quasiconvex risk measures defined on variable exponent Bochner-Lebesgue spaces.

Remark 3.1. By the definition of $\mathbb{L}_{p(\cdot)}$, each $f \in \mathbb{L}_{p(\cdot)}$ is a $E$-valued measurable function and $E$ is partially ordered by $K_0$. Hence, in the absence of ambiguity, we also regard that the $\mathbb{L}_{p(\cdot)}$ is also partially ordered by $K_0$.

Now, the definition of quasiconvex risk measures on $\mathbb{L}_{p(\cdot)}$ will be introduced by axiomatic approach.

Definition 3.1. Let $E$ be a Banach space ordered by the partial ordering relation induced by a cone $K_0$ and $\mathbb{L}_{p(\cdot)}$ is a variable exponent Bochner-Lebesgue space. A mapping $\varrho : \mathbb{L}_{p(\cdot)} \to [-\infty, +\infty]$ called quasiconvex risk measure if it satisfies

A1 Monotonicity: for any $f_1, f_2 \in \mathbb{L}_{p(\cdot)}$, $f_1 \leq_{K_0} f_2 \Rightarrow \varrho(f_1) \leq \varrho(f_2)$;

A2 Quasiconvexity: for any $f_1, f_2 \in \mathbb{L}_{p(\cdot)}$ and $\lambda \in [0, 1]$, $\varrho(\lambda f_1 + (1-\lambda)f_2) \leq \max\{\varrho(f_1), \varrho(f_2)\}$.

Remark 3.2. Note that, the quasiconvex risk measures need not satisfy the property of translation invariance, which is a key axiom for convex risk measures. Which makes the quasiconvex risk measures to be a special class of risk measures. On the other hand, the quasiconvexity also make the quasiconvex risk measures distinguish from the convex risk measures.

In order to study the dual representation of quasiconvex risk measures, we need to introduce the concept of risk functions.

Definition 3.2. Let $R(\mathbb{L}_{p(\cdot)} \times (\mathbb{L}_{p(\cdot)})^*)$ denotes the class of risk functions $R : \mathbb{L}_{p(\cdot)} \times (\mathbb{L}_{p(\cdot)})^* \to [-\infty, +\infty]$ that satisfy the following requirements:

B1 Monotonicity: for any $f_1, f_2 \in \mathbb{L}_{p(\cdot)}$ and $g \in (\mathbb{L}_{p(\cdot)})^*$, $f_1 \leq_{K_0} f_2 \Rightarrow R(f_1, g) \leq R(f_2, g)$;

B2 Quasiconvexity: for any $f_1, f_2 \in \mathbb{L}_{p(\cdot)}$, $g \in (\mathbb{L}_{p(\cdot)})^*$ and $\lambda \in (0, 1)$, $R(\lambda f_1 + (1-\lambda)f_2, g) \leq \max\{R(f_1, g), R(f_2, g)\}$;

B3 Lower semicontinuity: $R$ is lower semicontinuous in the first component.
Now, the acceptance sets of quasiconvex risk measures should be defined.

**Definition 3.3.** Given a quasiconvex risk measure $\varrho$, the acceptance set at level $\nu \in \mathbb{R}$ is denoted by $A_\nu$ as follows

$$A_\nu := \{ f \in L^p(\cdot) : \varrho(f) \leq \nu \}. \quad (3.1)$$

**Remark 3.3.** Given a quasiconvex risk measure $\varrho$, it is easy to check that $A_\nu$ is a closed convex set and have the monotonicity, i.e. $\nu_1 \leq \nu_2$ implies $A_{\nu_1} \subseteq A_{\nu_2}$. In fact, by $A2$, for any $f_1, f_2 \in A_\nu$ and $\lambda \in [0,1],

$$\varrho(\lambda f_1 + (1-\lambda)f_2) \leq \max\{\varrho(f_1), \varrho(f_2)\}. $$

Since $f_1, f_2 \in A_\nu$, we have $\varrho(f_1) \leq \nu$ and $\varrho(f_2) \leq \nu$, which implies

$$\max\{\varrho(f_1), \varrho(f_2)\} \leq \nu.$$

Hence,

$$\varrho(\lambda f_1 + (1-\lambda)f_2) \leq \nu.$$

By (3.1), we know that $\lambda f_1 + (1-\lambda)f_2 \in A_\nu$, which means $A_\nu$ is a convex set. It is also easy to show that $A_\nu$ is a closed set and have the monotonicity.

**Lemma 3.1.** Let $A_\nu$ defined as Definition 3.3. Then, we have

$$f \in A_\nu \quad \text{if and only if} \quad \langle g, f \rangle \leq \sup_{X \in A_\nu} \langle g, X \rangle \quad (3.2)$$

for all $g \in Q_{p(\cdot)}$ where

$$Q_{p(\cdot)} := \left\{ g \in (L^{p(\cdot)})^* : \frac{dg}{d\mu} \in L^{p(\cdot)}(K) \right\}. $$

**Proof.** We first show the ‘only if’ part. If $A_\nu = \emptyset$, the implication is obvious. If $A_\nu \neq \emptyset$, the following implication

$$f \in A_\nu \quad \text{implies} \quad \langle g, f \rangle \leq \sup_{X \in A_\nu} \langle g, X \rangle \quad \text{for all} \quad g \in Q_{p(\cdot)}$$

is also straightforward. Next, we show the ‘if’ part. From Remark 3.3, $A_\nu$ is a closed convex set. Thus, by Hahn-Banach theorem, for any $f \in L^{p(\cdot)} \setminus A_\nu$, there exits a $\widehat{g} \in (L^{p(\cdot)})^*$, such that

$$\langle \widehat{g}, f \rangle > \sup_{X \in A_\nu} \langle \widehat{g}, X \rangle.$$

Now, we only need to show $\widehat{g} \in Q_{p(\cdot)}$. In fact, by Remark ??, we have

$$\langle \widehat{g}, X \rangle = \int_{\Omega} \langle \widehat{h}, X \rangle d\mu$$
where \( \hat{h} = d\hat{g}/d\mu \in L^{p(\cdot)}(\Omega, E^*) \). Then, with the monotonicity of \( \varrho \), it is easy to check \( A_{\nu} = A_{\nu} - K_0 \). Hence
\[
\langle \hat{g}, f \rangle > \langle \hat{g}, X - k \rangle = \int_{\Omega} \langle \hat{h}, X - k \rangle d\mu \\
= \int_{\Omega} \langle \hat{h}, X \rangle d\mu - \int_{\Omega} \langle \hat{h}, k \rangle d\mu \\
= \langle \hat{g}, X \rangle - \int_{\Omega} \langle \hat{h}, k \rangle d\mu
\]
for all \( k \in K_0 \) and \( X \in A_{\nu} \). Thus, \( \int_{\Omega} \langle \hat{h}, k \rangle d\mu \geq 0 \) for all \( k \in K_0 \), which implies \( \hat{h} \in L^{p(\cdot)}(\Omega, K) \). By the definition of \( Q_{p(\cdot)} \), we have \( \hat{g} \in Q_{p(\cdot)} \).

4 Dual representation

In this section, we will study the dual representation of quasiconvex risk measures defined on variable exponent Bochner-Lebesgue spaces, which is also the main result of this paper.

**Theorem 4.1.** A mapping \( \varrho : L^{p(\cdot)} \to [-\infty, +\infty] \) is a lower semicontinuous quasiconvex risk measure if and only if for any \( f \in L^{p(\cdot)} \),
\[
\varrho(f) = \sup_{g \in Q_{p(\cdot)}} R(f, g) \tag{4.1}
\]
where \( R \in \mathfrak{R}(L^{p(\cdot)} \times Q_{p(\cdot)}) \) is expressed as
\[
R(f, g) := \inf_{\nu \in \mathfrak{R}} \left\{ \nu : \langle g, f \rangle \leq \sup_{X \in A_{\nu}} \langle g, X \rangle \right\} \tag{4.2}
\]
and
\[
Q_{p(\cdot)} := \left\{ g \in (L^{p(\cdot)})^* : \frac{dg}{d\mu} \in L^{p'(\cdot)}(K) \right\}. \tag{4.3}
\]

**Proof.** We first show the ‘only if’ part. Suppose \( \varrho \) is a lower semicontinuous quasiconvex risk measure, we claim that \( \varrho \) can be expressed as
\[
\varrho(f) = \inf \{ \nu \in \mathbb{R} : f \in A_{\nu} \}, \quad f \in L^{p(\cdot)}. \tag{4.4}
\]
In fact, define \( \varrho_A(f) := \inf \{ \nu \in \mathbb{R} : f \in A_{\nu} \} \), it is easy to check that \( \varrho_A \) is a lower semicontinuous quasiconvex risk measure. Thus, we only need to show \( \varrho_A(f) = \varrho(f) \) for any \( f \in L^{p(\cdot)} \). If \( f \in L^{p(\cdot)} \) is such that \( \varrho(f) = +\infty \), we have \( \varrho_A(f) = \varrho(f) = +\infty \). The same argumentation holds for those \( f \in L^{p(\cdot)} \) satisfying \( \varrho(f) = -\infty \). If \( \varrho(f) \in \mathbb{R} \), we have \( f \in A_{\varrho(f)} \), which implies \( \varrho_A(f) \leq \varrho(f) \). On the other hand, we have \( f \notin A_{\varrho(f)} \) for any \( r < \varrho(f) \). Thus, \( r < \varrho_A(f) \), which implies \( \varrho(f) \leq \varrho_A(f) \). Hence, for any \( f \in L^{p(\cdot)} \)
\[
\varrho_A(f) = \varrho(f) = \inf \{ \nu \in \mathbb{R} : f \in A_{\nu} \}.
\]
By Lemma 3.1, we have
\[ f \in A \nu \iff \langle g, f \rangle \leq \sup_{X \in A \nu} \langle g, X \rangle \] (4.5)
for all \( g \in Q_{p(\cdot)} \). Then, from (4.4) and (4.5), we have
\[ \varrho(f) = \inf \{ \nu \in \mathbb{R} : \langle g, f \rangle \leq \sup_{X \in A \nu} \langle g, X \rangle \text{ for all } g \in Q_{p(\cdot)} \}. \] (4.6)

Our goal is to show that
\[ \varrho(f) = \sup_{g \in Q_{p(\cdot)}} \inf_{\nu \in \mathbb{R}} \{ \nu : \langle g, f \rangle \leq \sup_{X \in A \nu} \langle g, X \rangle \} = \sup_{g \in Q_{p(\cdot)}} R(f, g). \] (4.7)

To this end, by (4.6), we know that
\[ \varrho(f) \geq \sup_{g \in Q_{p(\cdot)}} \inf_{\nu \in \mathbb{R}} \{ \nu : \langle g, f \rangle \leq \sup_{X \in A \nu} \langle g, X \rangle \}. \]

Next, we will show the reverse inequality. Suppose \( \varrho(f) > -\infty \), otherwise (4.7) is trivial. Now, we fix \( m < \varrho(f) \) and define
\[ \mathcal{B} := \{ X \in L^p(\cdot) : \varrho(X) \leq m \}. \]

By Remark 3.3, we know that \( \mathcal{B} \) is a closed convex set and have the monotonicity. Since \( f \not\in \mathcal{B} \), by the Hahn-Banach theorem, there exits a \( \hat{g} \in (L^p(\cdot))^* \), such that
\[ \langle \hat{g}, f \rangle > \sup_{X \in \mathcal{B}} \langle \hat{g}, X \rangle. \] (4.8)

We claim that \( \hat{g} \in Q_{p(\cdot)} \). In fact, by Remark ??, we have
\[ \langle \hat{g}, X \rangle = \int_{\Omega} \langle \hat{h}, X \rangle d\mu \]
where \( \hat{h} = d\hat{g}/d\mu \in L^p(\cdot)(\Omega, E^*) \). Then, with the monotonicity of \( \varrho \), it is easy to check \( \mathcal{B} = \mathcal{B} - K_0 \). Hence, by (4.8)
\[ \langle \hat{g}, f \rangle > \langle \hat{g}, X - k \rangle = \int_{\Omega} \langle \hat{h}, X - k \rangle d\mu \]
\[ = \int_{\Omega} \langle \hat{h}, X \rangle d\mu - \int_{\Omega} \langle \hat{h}, k \rangle d\mu \]
\[ = \langle \hat{g}, X \rangle - \int_{\Omega} \langle \hat{h}, k \rangle d\mu \]
for all \( k \in K_0 \) and \( X \in \mathcal{B} \). Thus, \( \int_{\Omega} \langle \hat{h}, k \rangle d\mu \geq 0 \) for all \( k \in K_0 \), which implies \( \hat{h} \in L^p(\cdot)(\Omega, K) \). By the definition of \( Q_{p(\cdot)} \), we have \( \hat{g} \in Q_{p(\cdot)} \).

For all \( \nu \leq m \), we have \( A_{\nu} \subseteq \mathcal{B} \). Then
\[ \sup_{X \in A_{\nu}} \int_{\Omega} \langle \hat{h}, X \rangle d\mu \leq \sup_{X \in \mathcal{B}} \int_{\Omega} \langle \hat{h}, X \rangle d\mu. \] (4.9)
Thus, by (4.8) and (4.9)
\[ \langle \hat{g}, f \rangle - \sup_{X \in \mathcal{A}_\nu} \langle \hat{g}, X \rangle \geq \langle \hat{g}, f \rangle - \sup_{X \in \mathcal{B}} \langle \hat{g}, X \rangle > 0. \tag{4.10} \]

Since for each \( \nu \leq m \), we can imply (4.10) and by the fact that \( \nu \mapsto \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \) is nondecreasing, we have
\[ m \leq \sup_{g \in Q_{p(\cdot)}} \inf_{\nu \in \mathbb{R}} \{ \nu : \langle g, f \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \}. \tag{4.11} \]

This relation holds for each \( m < \varrho(f) \). Hence
\[ \varrho(f) \leq \sup_{g \in Q_{p(\cdot)}} \inf_{\nu \in \mathbb{R}} \{ \nu : \langle g, f \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \}. \]

Then,
\[ \varrho(f) = \sup_{g \in Q_{p(\cdot)}} R(f, g). \tag{4.12} \]

Now, we only need to show \( R \in \mathcal{R}(L^{p(\cdot)} \times Q_{p(\cdot)}) \). First, by the monotonicity and lower semicontinuity of \( \varrho \) with (4.12), it is easy to check that \( R \) satisfies \( \textbf{B1} \) and \( \textbf{B3} \). Next, we will show that \( R \) satisfies \( \textbf{B2} \). By Remark \( \textbf{??} \), we have
\[ \langle g, X \rangle = \int_{\Omega} \langle h, X \rangle d\mu \]

where \( h = dg/d\mu \in L^{p(\cdot)}(\Omega, E^*) \). For any \( f_1, f_2 \in L^{p(\cdot)}, \alpha \in (0, 1) \) and \( g \in Q_{p(\cdot)} \),
\[ R(\alpha f_1 + (1 - \alpha)f_2, g) = \inf_{\nu \in \mathbb{R}} \left\{ \nu : \langle g, \alpha f_1 + (1 - \alpha)f_2 \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\} \]
\[ = \inf_{\nu \in \mathbb{R}} \left\{ \nu : \int_{\Omega} \langle h, \alpha f_1 + (1 - \alpha)f_2 \rangle d\mu \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\} \]
\[ = \inf_{\nu \in \mathbb{R}} \left\{ \nu : \alpha \int_{\Omega} \langle h, f_1 \rangle d\mu + (1 - \alpha) \int_{\Omega} \langle h, f_2 \rangle d\mu \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\} \]
\[ = \inf_{\nu \in \mathbb{R}} \left\{ \nu : \alpha \langle g, f_1 \rangle + (1 - \alpha) \langle g, f_2 \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\}. \]

Without loss of generality, let \( \langle g, f_1 \rangle \geq \langle g, f_2 \rangle \). Then
\[ R(\alpha f_1 + (1 - \alpha)f_2, g) \leq \inf_{\nu \in \mathbb{R}} \left\{ \nu : \langle g, f_1 \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\} \]
\[ = R(f_1, g) \]
\[ \leq \max\{R(f_1, g), R(f_2, g)\}, \]
which means \( R \) satisfies \( \textbf{B2} \). Therefore, \( R \in \mathcal{R}(L^{p(\cdot)} \times Q_{p(\cdot)}) \).

Now, we will show the ‘if’ part. Suppose that \( \varrho(f) = \sup_{g \in Q_{p(\cdot)}} R(f, g) \) for a risk function \( R \in \mathcal{R}(L^{p(\cdot)} \times Q_{p(\cdot)}) \) where \( R(f, g) = \inf_{\nu \in \mathbb{R}} \left\{ \nu : \langle g, f \rangle \leq \sup_{X \in \mathcal{A}_\nu} \langle g, X \rangle \right\} \). The properties of monotonicity and lower semicontinuity of \( \varrho \) are the direct consequences of \( \textbf{B1} \) and \( \textbf{B3} \).
Now, we only need to show that \( \rho \) satisfies A2. Since \( R \) satisfies B2, for any \( \lambda \in (0,1) \) and \( f_1, f_2 \in L^{p(\cdot)} \),
\[
R(\lambda f_1 + (1 - \lambda) f_2, g) \leq \max\{R(f_1, g), R(f_2, g)\}.
\]
(4.13)

Thus, it follows that
\[
\rho(\lambda f_1 + (1 - \lambda) f_2) = \sup_{g \in Q^{p(\cdot)}} R(\lambda f_1 + (1 - \lambda) f_2, g)
\leq \sup_{g \in Q^{p(\cdot)}} \max\{R(f_1, g), R(f_2, g)\}
\leq \max\{\rho(f_1), \rho(f_2)\}.
\]

Hence, \( \rho \) is a lower semicontinuous quasiconvex risk measure. \( \square \)

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