Hirota’s method for a spin model with self-consistent potential

G. Nugmanova, Zh. Sagidullayeva, R. Myrzakulov
L.N. Gumilyov Eurasian National University,
Satpayev Str., 2, Astana, Kazakhstan
E-mail: nugmanovagn@gmail.com

Abstract. In this work an integrable generalization of the Landau-Lifshitz equation with self-consistent vector potential is studied. It was established that self-consistence of spin vector and potential happen by the relation between solutions the potential and linear system, the compatibility condition of which corresponds to the equation considered by us. By generalizing Hirota method, it’s exact solutions, defining self-consistent motion of the potential and soliton, are constructed.

1. Introduction
Solutions of integrable nonlinear differential equations which is a solitary wave and have elastic properties of interaction with the same another solution have a variety of applications in many areas of the natural sciences. Analytical studies of processes of interaction of the solitary waves is one of the main tasks of the theory of solitons. The development of nonlinear theory of magnetism, in turn, has put the problem of constructing integrable generalization of the Landau-Lifshitz equation with self-consistent potential.

One of such generalizations with self-consistent scalar potential was proposed in [1]. Various algebraic-geometric aspects of such models were studied in [2-5]. The generalized Landau-Lifshitz equation with self-consistent vector potential was obtained in [6], and also their connection with the movement of curves and surfaces are established.

2. Generalized Landau-Lifshitz equation
The Generalized Landau-Lifshitz equation (GLL) equation with self-consistent potential reads as

\[
i S_t + \frac{1}{2}[S, S_{xx}] + \frac{1}{a}[S, W] = 0, \tag{2.1}
\]

\[
i W_x + a[S, W] = 0, \tag{2.2}
\]

where \( a = \text{const} \), \( S = \sum_{j=1}^{3} S_j(x, y, t)\sigma_j \) is a matrix analogue of the spin vector, \( W \) - potential with the matrix form \( W = \sum_{j=1}^{3} W_j(x, y, t)\sigma_j \), and

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
are Pauli matrixes.

The GLL equation with self-consistent potential is integrable by the IST. Its Lax representation can be written in the form

\[ \Phi_x = U \Phi, \]
\[ \Phi_t = V \Phi, \]

where the matrix operators \( U \) and \( V \) have the form

\[ U = -i \lambda S, \]
\[ V = \lambda^2 V_2 + \lambda V_1 + \left( \frac{i}{\lambda + a} - \frac{i}{a} \right) W. \]

Here

\[ V_2 = -2i S, \quad V_1 = SS_x. \]

3. Bilinearization

In this section we construct a bilinear form of equations (2.1) - (2.2). Now for our convenience we write systems (2.1) - (2.2) in the component form

\[ iS_t^+ + S^+ S_{3xx} - S_3 S_{xx}^+ + \frac{2}{a} (S^+ W_3 - S_3 W^+) = 0, \]
\[ iS_t^- - S_{3xx} S^- + S_3 S_{xx}^- - \frac{2}{a} (S^- W_3 - S_3 W^-) = 0, \]
\[ 2iS_{3t} + S_{xx}^+ S^- - S^+ S_{xx}^- + \frac{2}{a} (S^- W^+ - S^+ W^-) = 0, \]
\[ iW_{xx}^+ - 2a (S_3 W^+ - S^+ W_3) = 0, \]
\[ iW_{xx}^- - 2a (S^- W_3 - S_3 W^-) = 0, \]
\[ iW_{3xx} - a (S^- W^- - S^+ W^+) = 0. \]

Self-consistent potential \( W \) is given as

\[ W_3 = |\varphi_1|^2 - |\varphi_2|^2, \quad W^+ = 2\varphi_1^* \varphi_2, \quad W^- = 2\varphi_1 \varphi_2^*, \]

where \( \varphi_1 \) and \( \varphi_2 \) are solutions of the matrix systems (2.3)-(2.4). With considering (3.7) we rewrite equations (3.1)-(3.6) as

\[ iS_t^+ + S^+ S_{3xx} - S_3 S_{xx}^+ + \frac{2}{a} (S^+ |\varphi_1|^2 - |\varphi_2|^2) - 2S_3 \varphi_1^* \varphi_2 = 0, \]
\[ iS_t^- - S_{3xx} S^- + S_3 S_{xx}^- - \frac{2}{a} (S^- |\varphi_1|^2 - |\varphi_2|^2) - 2S_3 \varphi_1 \varphi_2^* = 0, \]
\[ 2iS_{3t} + S_{xx}^+ S^- - S^+ S_{xx}^- + \frac{4}{a} (S^- \varphi_1^* \varphi_2 - S^+ \varphi_1 \varphi_2^*) = 0, \]
\[ \varphi_{1x} - ia (S_3 \varphi_1 + S^- \varphi_2) = 0, \]
\[ \varphi_{2x} - ia (S^+ \varphi_1 - S_3 \varphi_2) = 0. \]

Let us consider the following stereographic transformation:

\[ S^+ = \frac{2\omega}{1 + |\omega|^2}, \quad S^- = \frac{2\omega^*}{1 + |\omega|^2}, \quad S_3 = \frac{1 - |\omega|^2}{1 + |\omega|^2}. \]
Hence we obtain
\[ \omega = \frac{S^+}{1 + S^3}. \] (3.14)

So we have
\[ i\omega_t - \omega_{xx} + \frac{2\omega^*\omega_x^2}{1 + |\omega|^2} + \frac{2}{a}(\omega\varphi_2^* + \varphi_1^*)(\omega\varphi_1 - \varphi_2) = 0, \] (3.15)
\[ \varphi_{1x} - \frac{ia}{1 + |\omega|^2} (1 - |\omega|^2)\varphi_1 + 2\omega^*\varphi_2 = 0, \] (3.16)
\[ \varphi_{2x} - \frac{ia}{1 + |\omega|^2} (2\omega\varphi_1 - (1 - |\omega|^2)\varphi_2) = 0. \] (3.17)

Functions \( \varphi_1 \) and \( \varphi_2 \) can be given by
\[ \varphi_1 = \frac{p}{f}e^{iax}, \quad \varphi_2 = \frac{q}{f}e^{iax}, \] (3.18)
and \( \omega = \frac{g}{f} \), where \( f \) is a real function, \( g, p, q \) are complex functions. In view of this further transforming equations (3.15) - (3.17) we obtain bilinear form for equations (3.8) - (3.12)
\[ [iD_t + D_x^2](g \cdot f) - \frac{2}{a}p^*q = 0, \] (3.19)
\[ D_x^2(f \cdot f) - \frac{2}{a}q^*q = 0, \] (3.20)
\[ D_x(p \cdot f) - 2ia g^* q = 0, \] (3.21)
\[ D_x(q \cdot f) + 2iaqf = 0, \] (3.22)
where Hirota’s operators defined by
\[ D_x^nf(x, t) \cdot g(x, t) = (\partial_x - \partial_{x'})^n(\partial_t - \partial_{t'})^nf(x, t) \cdot g(x', t') \bigg|_{x=x', t=t'}. \] (3.23)

4. Solutions
Using above obtained the bilinear forms, now we can construct soliton solutions for spin systems. For this, we expand the functions \( g, f, p \) and \( q \) in formal series on an arbitrary constant \( \varepsilon \):
\[ g(x, t) = \varepsilon g_1(x, t) + \varepsilon^3 g_3(x, t) + \cdots, \] (4.1)
\[ f(x, t) = 1 + \varepsilon^2 f_2(x, t) + \varepsilon^4 f_4(x, t) + \cdots, \] (4.2)
\[ p(x, t) = 1 + \varepsilon^2 p_2(x, t) + \varepsilon^4 p_4(x, t) + \cdots, \] (4.3)
\[ q(x, t) = \varepsilon q_1(x, t) + \varepsilon^3 q_3(x, t) + \cdots. \] (4.4)

In order to obtain one-soliton solution for equations (3.19) - (3.22), we choose \( g = \varepsilon g_1, \)
\( p = 1 + \varepsilon^2 p_2, \quad q = \varepsilon q_1, \quad f = 1 + \varepsilon^2 f_2. \)
Substituting expressions into bilinear forms (3.19) - (3.22) we have

\[ ig_{tt} + g_{1xx} - \frac{2a}{g} q_1 = 0, \]  \hspace{1cm} (4.5)  

\[ ig_{tt} f_2 - ig_1 f_{2x} + g_{1xx} f_2 - 2g_1 f_{2x} + g_1 f_{2xx} - \frac{2}{a} f_2 q_1 = 0, \]  \hspace{1cm} (4.6)  

\[ f_{2xx} - \frac{1}{a} q_1 g_1 = 0, \]  \hspace{1cm} (4.7)  

\[ f_2 f_{2xx} - f_{2x} f_{2x} = 0, \]  \hspace{1cm} (4.8)  

\[ p_{2x} - f_{2x} - 2iag_1 q_1 = 0, \]  \hspace{1cm} (4.9)  

\[ p_{2x} f_2 - p_2 f_{2x} = 0, \]  \hspace{1cm} (4.10)  

\[ q_{1x} + 2ia q_1 = 0, \]  \hspace{1cm} (4.11)  

\[ q_1 f_2 - q_1 f_{2x} + 2ia q_1 f_2 = 0. \]  \hspace{1cm} (4.12)  

If we choose function \( g_1 \) as \( g_1 = e^{i\theta} \), with \( \theta = k_1 x + l_1 t + m_1 \), where \( k_1, l_1 \) and \( m_1 \) are complex constants. Then we can construct solutions for \( f, q_1, p_2 \) in forms:

\[ q_1 = - \frac{a}{2} (l_1 + k_1^2) e^{i\theta}, \]  \hspace{1cm} (4.13)  

\[ f_2 = - \frac{a (l_1 + k_1^2)(l_1^* + k_1^{*2})}{(k_1 - k_1^*)^2} e^{i(\theta - \theta^*)}, \]  \hspace{1cm} (4.14)  

\[ p_2 = - \frac{a (l_1 + k_1^2)}{(k_1 - k_1^*)} \left( \frac{a (l_1^* + k_1^{*2})}{4(k_1 - k_1^*)^2} + a \right) e^{i(\theta - \theta^*)}. \]  \hspace{1cm} (4.15)  

Equation (3.13) based on components of matrix \( S \) can be given as

\[ S^+ = \frac{2fg}{f^2 + |g|^2}, \quad S^- = \frac{2fg^*}{f^2 + |g|^2}, \quad S_3 = \frac{f^2 - |g|^2}{f^2 + |g|^2}. \]  \hspace{1cm} (4.16)  

Finally, we can find solutions for equations (2.1)-(2.2)

\[ S^+ = \frac{2 \left( 1 - \frac{a(k_1^2 + l_1^2)(k_1^2 + l_1^*) e^{i(\theta - \theta^*)}}{4(k_1 - k_1^*)^2} \right) e^{i\theta}}{\left( 1 - \frac{a(k_1^2 + l_1^2)(k_1^2 + l_1^*) e^{i(\theta - \theta^*)}}{4(k_1 - k_1^*)^2} \right)^2 + e^{i(\theta - \theta^*)}}, \]  \hspace{1cm} (4.17)  

\[ S_3 = \left( 1 - \frac{a(k_1^2 + l_1^2)(k_1^2 + l_1^*) e^{i(\theta - \theta^*)}}{4(k_1 - k_1^*)^2} \right)^2 - e^{i(\theta - \theta^*)}. \]  \hspace{1cm} (4.18)  

From equation (3.7) for the potential we get

\[ W^+ = - \frac{a \left( 1 - \frac{a(l_1 + k_1^2)}{(k_1 - k_1^*)} \left( \frac{(l_1^* + k_1^{*2})}{4(k_1 - k_1^*)^2} + a \right) e^{i(\theta - \theta^*)} \right)(l_1 + k_1^2) e^{i\theta}}{\left( 1 - \frac{a(k_1^2 + l_1^2)(k_1^2 + l_1^*) e^{i(\theta - \theta^*)}}{4(k_1 - k_1^*)^2} \right)^2}, \]  \hspace{1cm} (4.19)  

\[ \text{doi:10.1088/1742-6596/804/1/012035} \]
\[
W_3 = \left(1 - \frac{a(l_1 + k_1^2)}{(k_1 - k_1^2)} \left(\frac{a(l_1^* + k_1^2)}{4(k_1 - k_1^2)} + a\right) e^{i(\theta - \theta^*)}\right) \left(1 - \frac{a(l_1^2 + l_1)}{(k_1 - k_1^2)} \left(\frac{a(l_1 + k_1^2)}{4(k_1 - k_1^2)} + a\right) e^{i(\theta - \theta^*)}\right) \left(1 - \frac{a(l_1^2 + l_1)}{(k_1 - k_1^2)} \left(\frac{a(l_1 + k_1^2)}{4(k_1 - k_1^2)} + a\right) e^{i(\theta - \theta^*)}\right) + \frac{a^2(l_1 + k_1^2)(l_1^* + k_1^2)}{4(1 - \frac{a(l_1^2 + l_1)}{(k_1 - k_1^2)} \left(\frac{a(l_1 + k_1^2)}{4(k_1 - k_1^2)} + a\right) e^{i(\theta - \theta^*)})^2}. \tag{4.20}
\]

Graphics for obtained solutions (4.18)-(4.20), constant parameters: \( k_1 = 25 - i, l_1 = 10 - i, \) \( m_1 = 1. \) We can clearly see self-consistent behavior of spin matrix \( S \) and potential \( W, \) as shown on figures (1)- (3) in the time interval from 0 to 10.

5. Conclusions
The present work is a continuation of our previous researches in the area of integrable spin systems, specifically, generalized Landau-Lifshitz equation with self-consistent scalar potential. The object of the current research was the generalization of the Landau-Lifshitz equation with self-consistent source, the role of which is played by three-component vector with variable length. The components of this vector are the functions of two variables. For this reason the generalized Landau-Lifshitz equation with the self-consistent source considered by us is not limited by conclusions and laws formulated for spin model with the scalar potential. The main result of this work is finding the relation defining self-consistent motion of the vector and soliton wave.

6. References
[1] Myrzakulov R, Vijayalakshmi S, Nugmanova G and Lakshmanan M 1999 Phys. Lett. A 233 391
[2] Myrzakulov R, Nugmanova G and Danilybaeva A 1999 Theoretical and Mathematical Physics 118 441
[3] Myrzakulov R, Nugmanova G, Syzdykova R 1998 J. Phys. A: Math. & Theor. 31 147
[4] Zhang Zh-H, Deng M, Zhao W-Zh, Ke W 2006 arXiv:nlin/0603069v1 [nlin.SI]
[5] Chen Ch, Zhou Zi-X 2009 Chin. Phys. Lett. 26 080504
[6] Myrzakulov R, Mamyrbekova G, Nugmanova G, Lakshmanan M 2015 Symmetry 7 1352