Existence, multiplicity and classification results for solutions to $k$-Hessian equations with general weights

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Abstract

The present paper is concerned with negative classical solutions to a $k$-Hessian equation involving a nonlinearity with a general weight

\begin{equation}
\begin{aligned}
S_k(D^2 u) = \lambda \rho(|x|)(1-u)^q & \quad \text{in } B, \\
 u = 0 & \quad \text{on } \partial B.
\end{aligned}
\end{equation}

(P)

Here, $B$ denotes the unit ball in $\mathbb{R}^n$, $n > 2k$, $\lambda$ is a positive parameter and $q > k$ with $k \in \mathbb{N}$. The function $\rho'(r)/\rho(r)$ satisfies very general conditions in the radial direction $r = |x|$. We show the existence, nonexistence, and multiplicity of solutions to Problem (P). The main technique used for the proofs is a phase-plane analysis related to a non-autonomous dynamical system associated to the equation in (P). Further, using the aforementioned non-autonomous system, we give a comprehensive characterization of $P_2^-, P_3^+, P_4^-$-solutions to the related problem

\begin{equation}
\begin{aligned}
S_k(D^2 w) = \rho(|x|)(-w)^q, \\
w < 0,
\end{aligned}
\end{equation}

(\hat{P})

given on the entire space $\mathbb{R}^n$. In particular, we describe new classes of solutions: fast decay $P_3^+$-solutions and $P_4^+$-solutions.

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1 Introduction

This paper aims to address the question of the existence, nonexistence and multiplicity of radially symmetric bounded solutions to Problem (P) involving a $k$-Hessian. Furthermore, it aims to characterize $P_2^-, P_3^+, P_4^+$-solutions to Problem (\hat{P}) (see the definition in Section 4) and to find their asymptotic behavior in a neighborhood of $+\infty$.

The motivation for studying problems like (P) and (\hat{P}) is twofold. First of all, in the Laplacian case, Problem (\hat{P}) appears in connection with the stationary case of the Vlasov-Poisson system describing stellar dynamics. More specifically, one needs to construct a stationary spherically symmetric stellar dynamics model $(\Phi, \rho, U)$, where $\Phi$ is the first integral of the stationary Vlasov-Poisson system, $\rho$ is a local density, and $U$ is the Newtonian potential, which turns out to be a positive solution to the radial version of the equation $\Delta U + h_\varphi(|x|, U) = 0$ (see [2]). Above, the function $\varphi$ is to be fixed a priori, and the model $(\Phi, \rho, U)$ is constructed via this function $\varphi$, while different choices of $\varphi$ lead to different stellar dynamic models (see [2]). In [3], the authors considered only two particular types of stellar dynamics models determined by two particular choices of the function $\varphi$. This led to the Matukuma and Emden-Fowler equations, i.e., to two particular types of our weight function $\rho$. Thus, even in the Laplacian case, Problem (\hat{P}) appears to be an interesting object to study as long as general weight functions $\rho$ are concerned. The second goal, which is related to the presence of the $k$-Hessian operator on the left-hand sides of (P) and (\hat{P}), is motivated by the extension of the results of [17] and [21] to a more general class of weights.
Furthermore, with respect to the work [3], apart from more general weights, our results on extending, particularly, the results obtained in [3, 17, 20, 21] by considering a large class of weights characterized in the present work.

Our proofs do not involve any energy function obtained via Pohozaev-type identities. Following the strategy developed in [21] and using ideas from [16], recently, in [17], it has been investigated the existence and multiplicity of radially symmetric bounded solutions for problem (P) with a particular weight of Matukuma-type given by \( \rho(|x|) = |x|^{\mu-2}(1+|x|^2)^{-\frac{\delta}{2}} \), where \( \mu \geq 2 \) is an additional parameter and \( q > k \). Furthermore, in [3], the authors studied \( P_2^- \) and \( P_3^+ \)-solutions for a particular case of (P), where the equation was in the dimension three, containing the Laplacian, and the source was of Matukuma-type.

In [17], where the weight \( \rho \) is not a pure power as in [21], the problem was reduced to a two-dimensional non-autonomous Lotka-Volterra system, which was considered as an asymptotically autonomous system in the sense of H. Thieme [22]. In studying bounded (regular) solutions, we note that a structural property of the equations considered in [20, 21], as the scaling invariance, played an important role. The similar property in [17] is an asymptotic scaling invariance of the equation. On the other hand, to obtain multiplicity, a key point is to construct a singular solution and study the intersection number between regular and singular solutions, which is a common strategy for studying these relationships.

We mention that a dynamical system approach to a class of radial weighted fully nonlinear equations involving Pucci extremal operators was recently applied in [13]. The resulting dynamics are induced by an autonomous quadratic system, obtained after a suitable transformation, similar to our change of variables. Nevertheless, their approach is quite simple, and the presence of the weight \(|x|^\sigma\) in the equation does not produce additional difficulties. However, in our case, analyzing the flow generated by the non-autonomous quadratic dynamical system is more delicate. Furthermore, our proofs do not involve any energy function obtained via Pohozaev-type identities.

An important ingredient of the current study is identifying the function \( R(r) = r\rho'(r)/\rho(r) \) and understanding that the existence of the finite limit \( l_0 = \lim_{r \to 0} R(r) \) is responsible for the multiplicity of solutions to Problem (P). We remark that, even in light of [17] and [21], the aforementioned observation is not obvious; importantly, it led to important generalizations and further developments of the results of [17] and [21]. As such, the existence and the value of the finite limit \( l_\infty = \lim_{r \to +\infty} R(r) \) is responsible for the behavior of solutions to Problem (P) in a neighborhood of \( +\infty \), as well for their classification as a \( P_2^- \), \( P_3^- \), or \( P_4^- \)-solutions. We reinforce that the previous results on the existence and multiplicity of solutions to Problem (P) [17, 21] were obtained only for weight functions of particular types, e.g., the Matukuma weight, so the important advance of this work is describing the most general class of weights \( \rho(\cdot) \) to which the results of [17] and [21] can be extended. On the other hand, identifying the function \( R(r) \) and specifying its behavior as \( r \to +\infty \) allows to use the associated Lotka-Volterra system for studying the behavior of solutions to the related Problem (P) at a neighborhood of \( +\infty \) and for classifying respective solutions as \( P_2, P_3^+, \) or \( P_4^+ \). This classification, in particular, is valid for the Matukuma weight considered in [17] and for the power weight \(|x|^\sigma\), considered in [21], so it complements the latter results.

In characterizing \( P_3^+ \)- and \( P_4^+ \)-solutions to Problem (P) an important role is played by a certain parameter \( \delta \), which, in the case of \( P_3^+ \)- and \( P_4^+ \)-solutions, determines the rate of decay. The particular case considered in [3] corresponds to the situation \( \delta = 0 \) and leads to \( P_3^+ \)-solutions of slow log-like decay. To the authors’ knowledge, \( P_4^+ \)-solutions of fast algebraic decay, corresponding to the case \( \delta > 0 \), and \( P_4^+ \)-solutions, corresponding to the case \( \delta < 0 \), were first introduced and characterized in the present work.

Thus, in this paper, we proceed further to study weighted problems for Hessian equations extending, particularly, the results obtained in [3, 17, 20, 21] by considering a large class of weights. Furthermore, with respect to the work [3], apart from more general weights, our results on \( P_{2^-}, P_3^-, \) and \( P_4^+ \)-solutions are valid for space dimensions \( n \geq 3 \) and numbers \( k < \frac{n}{2} \), while in [3], \( n = 3 \) and \( k = 1 \).

The change of variables introduced in [21] (see Subsection 3.2.1 and Appendix for details) reduces
the equation in (P) to the non-autonomous Lotka-Volterra system:

\[
\begin{align*}
\frac{dx}{dt} &= x(\nu(t) - x - qy), \\
\frac{dy}{dt} &= y(-\frac{n-2k}{k} + \frac{x}{x} + y),
\end{align*}
\tag{TS_{q,v}}
\]

where

\[
\nu(t) = n + R(r) \quad \text{and} \quad t = \ln(r).
\]

The cases \(\rho(r) \equiv 1\) and \(\rho(r) = r^\sigma \quad (\sigma > 0)\) were recently studied in [20] and [21], respectively. Note that both cases lead to an autonomous Lotka-Volterra system. Hence, it is interesting to examine what happens with the solutions when the function \(R(r)\) is not constant. Note that this function appears naturally in the system \((TS_{q,v})\) as the definition of \(\nu(t)\) shows. On the other hand, an important exponent appearing in the main results of [20] and [21] is

\[
q^*(k, \sigma) := \frac{(n+2)k+\sigma(k+1)}{n-2k}, \quad \sigma \geq 0.
\tag{1}
\]

If the function \(R\) is not constant, then the “critical exponent” \(\frac{(n+2)k+R(r)(k+1)}{n-2k}\) will vary with \(r\) and the structure of the solution set will be more complex. A nonlinear model where this situation occurs was studied in [17]. We point out that the results obtained here cannot be derived using the standard theory by the classical Emden-Fowler transformation.

In the first part of the paper, dedicated to Problem (P), we assume that the weight function \(\rho\) satisfies the conditions (\(\rho.1\))–(\(\rho.3\)):

(\(\rho.1\)) \(\rho \in C^2(0, \infty) \cap C[0, \infty)\) with \(\rho(r) > 0\) for \(r > 0\).

(\(\rho.2\)) For the function \(R(r) = r^{\nu(r)/\rho(r)}\) it holds that the limit \(l_0 = \lim_{r \to 0} R(r)\) exists and one of the conditions, (1) or (2), is fulfilled

- (1) \(l_0 > R(r)\) for all \(r > 0\) and \(q \geq q^*(k, l_0)\),
- (2) \(l_0 \geq R(r)\) for all \(r > 0\) and \(q > q^*(k, l_0)\).

where \(q^*(k, l_0)\) is defined by (1).

(\(\rho.3\)) The function \(K(r) := r^{-l_0} \rho(r)\) satisfies the condition \(0 < K(r) < L\) for all \(r > 0\) and for some constant \(L > 0\).

In Remark 2.3, we will show the existence of the limit \(\lim_{r \to 0} K(r) > 0\), denoted by \(K(0)\).

When the limits \(\nu_\pm := \lim_{r \to \pm \infty} (n + R(r))\) exist, we may consider the system \((TS_{q,v})\) as an asymptotically autonomous system in the sense of Thieme [22]. We will denote these autonomous systems by \((LV_{q,v},_+)\) and \((LV_{q,v},_-)\), respectively. Thus, we can describe the flow of \((TS_{q,v})\) from this autonomous systems. To do so, we follow the same approach as in [17], i.e., we use dynamical-systems tools, the intersection number between a regular and a singular solution and the method of super and subsolutions. We mention that system \((LV_{q,v},-)\) coincides with the autonomous Lotka-Volterra system used for studying Problem (P) with \(\rho(|x|) = |x|^{l_0}\). In this case, the multiplicity of solutions is related to the Tso and Joseph-Lundgren type exponents (see [21]). On the other hand, two stationary points of \((LV_{q,v},-)\), denoted by \(P_4(\hat{x}, \hat{y})\) and \(P_3(n + l_0, 0)\), are relevant to obtaining a singular solution and a bounded solution for the radial version of (P), denoted by \((P_\lambda)\). Namely, we prove that the trajectories of \((TS_{q,v})\) that start at the stationary point \(P_3(n + l_0, 0)\) are characterized by the existence of bounded solutions to Problem \((P_\lambda)\) (see Proposition 3.5). In turn, the trajectories of \((TS_{q,v})\) that start at the stationary point \(P_4(\hat{x}, \hat{y})\) yield the existence of a singular solution to \((P_\lambda)\) for some \(\lambda > 0\) (see Proposition 3.4).

In the second part of the paper, dealing with \(P_3-, P_3^+, P_4-,\ and \(P_4^+\)-solutions to Problem \((P)\), we use the above assumptions (\(\rho.1\)), (\(\rho.2\)) and the assumptions (\(\rho.4\)), (\(\rho.5\)) below. Also, to characterize \(P_3^+\) and \(P_4^+\)-solutions to Problem \((P)\), we introduce the parameter

\[
\delta = -\frac{2k + l_\infty}{k},
\tag{2}
\]
whose sign determines whether a solution is $P^+_3$ or $P^+_4$; moreover, $\delta$ characterizes the rate of decay. Assumptions (p.4) and (p.5) read as follows:

(p.4) The limit $l_\infty = \lim_{r \to \infty} R(r)$ exists and one of the conditions, (1) or (2), is fulfilled

1. $l_\infty < l_0$ and $q > q^*(k, l_0)$,
2. $l_\infty \leq l_0$ and $q > q^*(k, l_0)$.

(p.5) There exists $\vartheta > 0$ such that $R(r) - l_\infty = O(r^{-\vartheta})$ as $r \to +\infty$.

In order to obtain a sharper asymptotic representation for the component $x(t)$ of the orbit of $(TS_{q, \nu})$ associated to a $P^+_3$-solution, we use the additional assumption (p.6) which reads as follows:

(p.6) One of the conditions, (1) or (2), is fulfilled:

1. $\lim_{r \to +\infty} r^k R(r) - l_\infty \in \mathbb{R}$ and $n + l_\infty > \delta$;
2. $\lim_{r \to +\infty} r^k |R(r) - l_\infty| = +\infty$; the limit $\nu = -\lim_{r \to +\infty} \frac{\ln |R(r) - l_\infty|}{\ln r}$ exists and $n + l_\infty > \nu$.

Examples of weights satisfying the above conditions are $\rho(|x|) = a|x|^l / (a^l + |x|^\sigma)$ with $0 < \sigma \leq l$ and $a, \tilde{a} > 0$, $\rho(|x|) = c|x|^\sigma$, $\sigma > 0$, and $\rho(x) = c$, where $c > 0$ is a constant. For more examples, see Remark 2.4.

When Problem $(\hat{P})$ is concerned, the autonomous system $(LVS_{q, \nu})$ plays an important role. More specifically, $(LVS_{q, \nu})$ along with Thieme’s theorem [22] allows to conclude that the $\omega$-limit set for solutions to Problem $(\hat{P})$ may consist either of point $P_2(0, n - 2k)$ or point $P^+_3(n + l_\infty, 0)$ or point $P^+_4(\frac{1}{\nu}(n - 2k) - k(n + l_\infty), \frac{2k + l_\infty}{\nu})$. These three options correspond to $P^+_2$, $P^+_3$, or $P^+_4$-solutions, respectively. Moreover, we show that the behavior of solutions as $r \to +\infty$ is determined by the intersection of the associated trajectory $\varphi(t)$ of the Lotka-Volterra system $(TS_{q, \nu})$ with the area

$$G = \{(x, y) \in \mathbb{R}_+^2 : (k + 1)x + k(y + 1) \leq (n - 2k)(q + 1)\}$$

and by the parameter $\delta$, introduced above. More specifically, our characterization can be summarized in the following table:

| $\delta > 0$, $\exists \tau_0$: $\varphi(t_0) \in G_-$ | $\delta > 0$, $\exists \tau_0$: $\varphi(t_0) \in G_-$ | $\delta < 0$, $\exists \tau_0$: $\varphi(t_0) \in G_-$ | $\delta \in \mathbb{R}$, $\varphi(t) \in G_+$ $\forall t$ |
|---|---|---|---|
| $P^+_3$-solution of fast decay | $P^+_3$-solution of slow decay | $P^+_4$-solution | $P_2$-solution |

Furthermore, we establish that if the solution is $P^+_3$, then its rate of decay is of order $r^{-\delta}$ if $\delta > 0$ (fast decay) and of order $(\ln r)^{-\frac{\delta}{\nu}}$ if $\delta = 0$ (slow decay); if the solution is $P^+_4$, then its rate of decay is of order $r^{\frac{\delta}{\nu}}$ ($\delta < 0$). We reinforce that in the situation considered in [3], one has $l_\infty = -2$ and $k = 1$, which corresponds to the case $\delta = 0$, so the work [3] only deals with $P_2$-solutions and slow decay $P^+_3$-solutions. The situation of [3] is reflected in the second and the fourth columns of the above table. The cases displayed in the first and the third columns are new and studied in the present work for the first time.

We indicate the contents of the individual sections. In Section 2, we briefly introduce the $k$-Hessian operator and provide necessary definitions. Moreover, the main results of this work are announced in this section. The detailed description of the main results is splitted into Sections 3 and 4. Section 3 is dedicated to the existence, non-existence, and multiplicity of solutions to Problem $(P)$. We would like to emphasize that compared to [17], most of the proofs of Section 3 contain significant changes due to the presence of the general weight $\rho$, the number $l_0$, and the asymptotic representation $\rho \sim K(0)r^{l_0}$ as $r \to 0$. The contents of the subsections of Section 3 is as follows. In Subsection 3.1, we give an existence and a non-existence result of classical solutions to Problem $(P)$ (Theorem 2.1). In Subsection 3.2, we construct a singular solution and obtain its associated parameter $\lambda$. In Subsection 3.3, we study the number of intersections points between a
regular solution and a singular solution. Further, using scaling arguments, we study the convergence of regular solutions to singular solutions and prove Theorem 2.2 on the multiplicity of solutions to Problem (P). Finally, Section 4 is dedicated to a characterization of $P_2^-$, $P_3^+$, and $P_4^+$-solutions to Problem (P). We remark that most of the results of Section 4 do not have analogs in the existing literature.

## 2 Preliminaries and main results

First, we review fundamental concepts and properties for the $k$-Hessian operators. For $k \in \{1, \ldots, n\}$, let $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ denote the $k$-th elementary symmetric function

$$\sigma_k (\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

and let $\gamma_k$ denote the set $\gamma_k = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \sigma_1 (\lambda) \geq 0, \ldots, \sigma_k (\lambda) \geq 0 \}$. For a twice differentiable function $u$ defined on a smooth domain $\Omega \subset \mathbb{R}^n$, the $k$-Hessian operator is defined by $S_k (D^2 u) = \sigma_k (\lambda (D^2 u))$, where $\lambda (D^2 u)$ are the eigenvalues of $D^2 u$. Equivalently, $S_k (D^2 u)$ is the sum of the $k$-th principal minors of the Hessian matrix. See e.g. [27, 28]. Two relevant examples in this family of operators are the Laplace operator $S_1 (D^2 u) = \Delta u$ and the Monge-Ampère operator $S_n (D^2 u) = \det (D^2 u)$. They are fully nonlinear when $k \geq 2$, but not elliptic in the whole space $C^2 (\Omega)$. In order to overcome this inconvenient L. Caffarelli, L. Nirenberg and G. Spruck [4] consider the class of functions

$$\Phi^k (\Omega) = \{ u \in C^2 (\Omega) \cap C (\overline{\Omega}) : \lambda (D^2 u) \in \gamma_k, i = 1, \ldots, k \}.$$

The functions in $\Phi^k (\Omega)$ are called admissible or $k$-convex functions. Further, $S_k (D^2 u)$ turns to be elliptic in the class of $k$-convex functions. Denote by $\Phi^k_0 (\Omega)$ the set of functions in $\Phi^k (\Omega)$ that vanish on the boundary $\partial \Omega$. An interesting property of the set $\Phi^k_0 (\Omega)$ is that their elements are negative in $\Omega$. Further, the $k$-Hessian operators have a divergence structure (see, for instance, [28]). The study of $k$-Hessian equations has many applications in geometry, optimization theory and other related fields. See [27]. These operators have been studied extensively, starting with the seminal work [4]. See, e.g., [10, 11, 12, 25, 26]. Recently, this class of operators has attracted renewed interest. See e.g. [7, 9, 17, 19, 29, 30, 31].

Let $\Omega = B$ be the unit ball in $\mathbb{R}^n$. It is well known that the $k$-Hessian operator, when acting on radially symmetric $C^2$-functions, can be written as

$$S_k (D^2 u) = c_{n,k} r^{1-n} \left(r^{n-k}(u')^k\right)' = kc_{n,k} r^{1-k}(u')^{k-1} \left(u'' + \frac{n-k}{k} u' - \frac{n}{r} \right),$$

where $r = |x| > 0$ and $c_{n,k}$ is defined by $c_{n,k} = C_n / n$. Here $u'$ denote the radial derivative of the radial function $u$.

Consider the problem

$$\begin{cases}
    c_{n,k} r^{1-n} \left(r^{n-k}(u')^k\right)' = \lambda \rho (r)(1-u)^{q}, & 0 < r < 1, \\
    u(r) < 0, & 0 \leq r < 1, \\
    u(1) = 0.
\end{cases}$$

(\lambda)

As in [17], we consider the space of functions $\Phi^k_0$ defined on $I = (0,1)$ for Problem (P$_\lambda$): $\Phi^k_0 = \{ u \in C^2 (I) \cap C^1 (\overline{I}) : (r^{n-1}(u')^k)' \geq 0 \text{ in } I, i = 1, \ldots, k, u'(0) = u(1) = 0 \}$.

**Remark 2.1.** Note that, if $u \in \Phi^k_0$ is a solution of (P$_\lambda$) then, in particular, $S_1 (D^2 u) = r^{1-n}(r^{n-1}u')' \geq 0$. Thus $G(r) = r^{n-1}u'$ is nondecreasing, since $G(0) = 0$, we deduce that $G \geq 0$ and hence $u$ is nondecreasing. As a consequence, $u' \geq 0$ and $u < 0$ on $[0,1)$. Further, using the integral form of equation in (P$_\lambda$) we see that $u$ is strictly increasing, and therefore, $u'(r) > 0$ for $0 < r \leq 1$. 

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Thus, negative radial solutions to Problem \((P)\) are those that solve \((P_\lambda)\) and vice versa.

**Definition 2.1.** We say that a function \(u\) is:

(i) a classical solution to \((P_\lambda)\) if \(u \in \Phi^k_0\) and the equation in \((P_\lambda)\) holds;

(ii) an integral solution to \((P_\lambda)\) if \(u\) is absolutely continuous on \((0, 1]\), \(\int_0^1 s^{n-1} \rho(s)(1-u(s))^q ds < \infty\), \(u(1) = 0\), and the equality

\[
c_n,k r^{n-k}(u'(r))^k = \lambda \int_0^r s^{n-1} \rho(s)(1-u(s))^q ds
\]

holds for all \(r \in I\).

**Remark 2.2.** Note that a classical solution is always an integral solution since \(u'(0) = 0\). In this case, the equation in \((P_\lambda)\) is equivalent to (4). Furthermore, since for all \(0 < r_0 < r < 1\),

\[
c_n,k r^{n-k}(u'(r))^k - r_0^{n-k}(u'(r_0))^k = \lambda \int_{r_0}^r s^{n-1} \rho(s)(1-u(s))^q ds,
\]

a solution to \((P_\lambda)\) is an integral solution if and only if \(\lim_{r \to 0} (r^{n-k}(u'(r))^k) = 0\).

Now we give the definition of super and subsolutions for \((P)\). For the method of super and subsolutions, see [28, Theorem 3.3].

**Definition 2.2.** A function \(u \in \Phi^k(B) := \{u \in C^2(B) \cap C(\overline{B}) : S_k(D^2u) \geq 0\} \) in \(B\), \(i = 1, \ldots, k\) is called a subsolution (resp. supersolution) to Problem \((P)\) if

\[
\begin{align*}
S_k(D^2u) \geq (resp. \leq) \lambda \rho(s)(1-u)^q & \quad \text{in } B, \\
u \leq (resp. \geq) 0 & \quad \text{on } \partial B.
\end{align*}
\]

In what follows, we will need the notion of maximal solutions.

**Definition 2.3.** We say that a function \(v\) is a maximal solution to \((P)\) if \(v\) is a solution to \((P)\) and, for each subsolution \(u\) to \((P)\), we have \(u \leq v\).

**Definition 2.4.** We say that \(u\) is a maximal solution to \((P_\lambda)\) if it is a solution to \((P_\lambda)\) and \(v(x) = u(|x|)\) is a maximal solution to \((P)\) in the sense of Definition 2.3.

We shall prove the following.

**Theorem 2.1.** Assume \(n > 2k\), \(q > k\), \(l_0 > 2k\) and the conditions \((\rho.1)\)–\((\rho.3)\) hold. Then, there exists \(\lambda^* > 0\) such that for each \(\lambda \in (0, \lambda^*)\), Problem \((P_\lambda)\) admits a maximal bounded solution. Moreover, there is at least one integral solution for \(\lambda = \lambda^*\), possibly unbounded, and no classical solutions for all \(\lambda > \lambda^*\). Furthermore, we have the following lower bound for \(\lambda^*\):

\[
\lambda^* \geq C^{-1} \left( \frac{n}{k} \left( \frac{q-k}{q} \right)^q \left( \frac{2k}{q-k} \right)^k \right),
\]

where \(C = C(\rho) := \max_{r \in [0,1]} \rho(r) > 0\).

We mention that the above result gives a natural extension, valid for our general class of weights, of the corresponding results in [17].

To establish our next result on multiplicity of solutions, we need to introduce a second relevant exponent recall that the first exponent \(q^*(k, \sigma)\) is defined by (1), namely

\[
q_{JL}(k, \sigma) := \begin{cases} k(k+1)n-k^2(2-\sigma)+2k+2\sigma-2\sqrt{k(3k+\sigma)[(k+1)n-k(2-\sigma)]}, & n > 2k + 8 + \frac{4\sigma}{k}, \\
2k < n \leq 2k + 8 + \frac{4\sigma}{k}. & \infty,
\end{cases}
\]

This is the Joseph-Lundgren-type exponent. The exponent \(q_{JL}(k, \sigma)\) was found in [21] in the study of the multiplicity of radial bounded solutions to \((P)\) with \(\rho(|x|) = |x|^\sigma\). We remark that for \(k > 1\), \(q_{JL}(k, 0)\) is a critical exponent for the existence of intersection points of any two positive radial solutions to \((P)\) with \(\rho \equiv 1\) (see [16, 18]). It is also critical for the existence of non-trivial stable solutions to the \(k\)-Hessian equation \(S_k(D^2V) = (-V)^p\) on \(\mathbb{R}^n\) (see [29]).
We are now in a position to state our result on multiplicity of solutions.

**Theorem 2.2.** Let \( n > 2k \) and \( q > k \). Assume that conditions (p.1)-(p.3) hold and, moreover, \( \rho(r) \) has at most polynomial growth as \( r \to \infty \). Further assume that \( q^*(k, l_0) < q < q_{JL}(k, l_0) \). Then, there exists \( \lambda \in (0, \lambda^*) \) such that for each \( N \in \mathbb{N} \), one can find an \( \varepsilon > 0 \) such that for \( |\lambda - \lambda^*| < \varepsilon \), (P\(_3\)) has at least \( N \) solutions. In particular, if \( \lambda = \lambda^* \), problem (P\(_3\)) has infinitely many solutions.

Note that the exponents \( q^*(k, l_0) \) and \( q_{JL}(k, l_0) \) retain their role in Problem (P\(_3\)) independently of the weight functions considered. On the other hand, the parameter \( \lambda \) and its associate unbounded (singular) solution are explicit in the case of a weight of the form \( |x|^7 \) (See [21, Theorem 3.1 (I)]).

In the more general setting under consideration, the parameter \( \lambda \) in Theorem 2.2 can be written in terms of an orbit (from which we construct a singular solution of (P\(_3\))) and the values of \( \rho \) on the boundary of the unit ball (see Proposition 3.4 below). See Sections 3.2 and 3.3 for more comments on this topic.

We finally state our results on the second problem of our interest, Problem (P\(_4\)). These results characterize \( P_{2^-}, P_{3^+}, \) and \( P_{4^+} \)-solutions. First of all, we remark that as a corollary of Thieme’s theorem [22], in Subsection 4.2, we prove that the \( \omega \)-limit set of the Lotka-Volterra system \((TS_{q,\nu})\) can be one of the points: \( P_2(0, \frac{2\pi n}{k}) \), \( P_3^*(n + l_\infty, 0) \), or \( P_4^*(\frac{2(n - 2k)}{n - 2k}, \frac{2k + l_\infty}{n - 2k}, 2k + l_\infty) \). Roughly speaking, \( P_{2^-}, P_{3^+}, \) and \( P_{4^+} \)-solutions are defined by the \( \omega \)-limit sets of the associated orbits of the non-autonomous Lotka-Volterra system \((TS_{q,\nu})\).

**Theorem 2.3 (On \( P_2 \)-solutions).** Let (p.1), (p.2), (p.4), and (p.5) hold. Then, the following conditions are equivalent:

(i) \( w \) is a \( P_2 \)-solution.

(ii) If \( \varphi(t) = (x(t), y(t)) \) is the associated orbit of the non-autonomous Lotka-Volterra system \((TS_{q,\nu})\), then there exist constants \( c_1 > 0 \) and \( c_2 = c_1(\frac{k}{n - 2k} - k) \) such that

\[
x(t) = c_1 e^{-\gamma t}(1 + o(1)), \quad y(t) = \frac{n - 2k}{k} + c_2 e^{-\gamma t}(1 + o(1)) \quad (t \to +\infty),
\]

where \( \gamma = \frac{k}{k}(n - 2k) - (n + l_\infty) > 0 \).

(iii) \( w(r) = -c_3 r^{-\frac{n - 2k}{k}}(1 + o(1)), \quad w'(r) = c_4 r^{-\frac{n - k}{k}}(1 + o(1)) \quad (r \to +\infty), \)

where \( c_3 = (c_{n,k} c_1 c_{\rho}^{-1})^\frac{1}{n - 2k} \), \( c_4 = (c_{n,k} c_1 c_{\rho}^{-1})^\frac{1}{n} \).

(iv) \( \varphi(t) \in G_+ \) for all \( t \in \mathbb{R} \).

Moreover, in a neighborhood of \( P_2 \), \( y \) can be represented as a function of \( x \), which we denote by \( \tilde{y}(x) \), and

\[
\lim_{x \to 0} \tilde{y}'(x) = \tilde{y}'(0) = -\frac{n - 2k}{k^2 \gamma + k(n - 2k)}.
\]

In the above item (iii), the constant \( c_{\rho} = \lim_{r \to \infty} \rho(r) r^{-l_\infty} \) (see Lemma 4.6).

Finally, in Theorems 2.4 and 2.5 below, \( W_- \) is defined similar to \( G_- \) (see (3)), but via a different linear function. Remark that all the above-mentioned subsets are rigorously defined in Subsection 3.2.3. Also define \( \zeta(t) = \nu(t) - n - l_\infty \).

**Theorem 2.4 (On \( P_{3^+} \)-solutions of algebraic fast decay).** Assume (p.1), (p.2), (p.4), (p.5) and let \( l_\infty < -2k \) (\( \delta > 0 \)). Then, the following conditions are equivalent:

(i) \( w \) is a \( P_{3^+} \)-solution.

(ii) If \( \varphi(t) = (x(t), y(t)) \) is the associated orbit of the non-autonomous Lotka-Volterra system \((TS_{q,\nu})\), then there exists a constant \( e > 0 \) such that

\[
x(t) = \nu_+(1 + o(1)), \quad y(t) = c e^{-\delta t}(1 + o(1)) \quad (t \to +\infty).
\]
Theorem 2.6. Assume (\(\rho.1\)), (\(\rho.2\)), (\(\rho.4\)), (\(\rho.5\)) and let \(l_\infty > -2k\) (\(\delta < 0\)). Then, the following conditions are equivalent:

(i) \(w(r)\) is a \(P_+\)-solution.

(ii) \(\varphi(t) = (x(t), y(t))\) is the associated orbit of \((T_{q,w})\), then

\[
\begin{align*}
x(t) &= n - 2k - \frac{qk}{q - k} \frac{1}{t} (1 + o(1)), \\
y(t) &= \frac{k}{q - k} \frac{1}{t} (1 + o(1)) \quad (t \to +\infty),
\end{align*}
\]

(iii) It holds that

\[
\begin{align*}
w(r) &= -c_3 (\ln r)^{1 - \frac{k}{q - k}} (1 + o(1)), \\
w'(r) &= c_4 r^{1 - \frac{1}{q} - \frac{k}{q - k}} (1 + o(1)),
\end{align*}
\]

where \(c_3 = (c_{\rho}^{-1} c_{n,k} \left(\frac{k}{q - k}\right))^{1/q} (n - 2k)\), \(c_4 = (c_{\rho}^{-1} c_{n,k} \left(\frac{k}{q - k}\right))^\frac{q}{q - k} (n - 2k)\).

(iv) There exists \(t_0 \in \mathbb{R}\) such that \(\varphi(t_0) \in G_+\).

(v) There exists \(t_0 \in \mathbb{R}\) such that \(\varphi(t_0) \in W_+\).

Moreover, in a neighborhood of \(P_+\), \(y\) can be represented as a function of \(x\), which we still denote by \(\hat{y}(x)\), and

\[
\lim_{x \to n-2k} \hat{y}'(x) = \hat{y}'(n - 2k) = -\frac{1}{q}.
\]
(ii) It holds that as \( r \to +\infty \),
\[
w(r) = c_3 r^{\frac{4k}{n}} (1 + o(1)), \quad w'(r) = -c_4 r^{\frac{2k}{n}-1}(1 + o(1)),
\]
where \( c_3 = (c_{n,k}c_{\rho}^{-1} x \tilde{y}^k)^{\frac{1}{n-k}} \), \( c_4 = (c_{n,k}c_{\rho}^{-1} x \tilde{y}^q)^{\frac{1}{n-k}} \), \( P_4^+(\tilde{x}, \tilde{y}) = \left( \frac{q(n-2k)-(k n + l_{\infty})}{q-k}, \frac{2k+l_{\infty}}{q-k} \right) \).

(iii) There exists \( t_0 \in \mathbb{R} \) such that \( \varphi(t_0) \in G_\omega \).

If, additionally,
\[
l_\infty < \frac{q(n-2k) - k(n+2 \mu_2)}{k + \mu_2} \quad \text{or} \quad l_\infty > \frac{q(n-2k) - k(n+2 \mu_1)}{k + \mu_1}
\]
(12)
where \( \mu_{1,2} = \frac{2q}{k} - 1 \mp 2 \sqrt{(\frac{q}{k})^2 - \frac{2}{k}} (\mu_1 < \mu_2) \), then items (i), (ii), (iii) are equivalent to the following:

(iv) There exist constants \( c_1, c_2 \in \mathbb{R} \) such that
\[
x(t) = \tilde{x} + e^{\lambda_1 t}(c_1 + o(1)), \quad y(t) = \tilde{y} + e^{\lambda_1 t}(c_2 + o(1)) \quad (t \to +\infty),
\]
where \( \lambda_1 < 0 \) is the maximal eigenvalue of the linearized system \((LVS_q, \nu, \nu)\) at the stationary point \( P_4^+(\tilde{x}, \tilde{y}) = \left( \frac{q(n-2k)-(k n + l_{\infty})}{q-k}, \frac{2k+l_{\infty}}{q-k} \right) \).

Remark 2.3. Note that under \((\rho,3)\), the positive limit \( K(0) = \lim_{r \to 0} r^{-l_0} \rho(r) \) exists. Indeed,
\[
(ln K(r))' = -\frac{l_0}{r} + \frac{\rho'(r)}{\rho(r)} = \frac{R(r) - l_0}{r}.
\]
(13)
Integrating from \( \frac{1}{n} \) to \( r \), we obtain that
\[
0 \leq \int_0^r \left[ \frac{l_0 - R(s)}{s} \right] ds = \ln K\left( \frac{1}{n} \right) - \ln K(r) \leq \ln L - \ln K(r).
\]

By the monotone convergence theorem, the finite limit
\[
\lim_{n \to +\infty} \int_0^r \frac{l_0 - R(t)}{t} dt = \int_0^r \frac{l_0 - R(t)}{t} dt.
\]
exists. Therefore, the finite limit \( \lim_{n \to +\infty} \ln K\left( \frac{1}{n} \right) = K(0) \) also exists. This implies the existence of a positive limit \( \lim_{n \to +\infty} K\left( \frac{1}{n} \right) = K(0) \).

Remark 2.4. Formula (13) along with the argument in Remark 2.3 imply that a general construction of weights \( \rho(r) \), satisfying \((\rho,1)-(\rho,3)\), can be done by the formula
\[
\rho(r) = r^{l_0} K(r) = K(0)r^{l_0} \exp \left\{ \int_0^r \frac{R(s) - l_0}{s} ds \right\},
\]
(14)
where \( R(\cdot) \) is to be chosen as in assumption \((\rho,2)\).

3 Existence and multiplicity results for Problem \((P)\)

3.1 Existence and non-existence of solutions of Problem \((P_\lambda)\)

In the following, we prove an existence result for classical solutions of Problem \((P_\lambda)\).

Lemma 3.1. Let \( n > 2k, q > k, \) and \( l_0 > 2k \). Assume that conditions \((\rho,1)-(\rho,3)\) hold. Further assume there exists a negative subsolution \( w \in C^2([0,1]) \) to Problem \((P_\lambda)\). Then, there exists a classical maximal bounded solution to Problem \((P_\lambda)\). Furthermore, the classical maximal bounded solutions form a decreasing sequence as \( \lambda \) increases.
Proof. Let $M_0 = \sup_y |w|$ and let $\zeta(y)$ be a smooth cutting function with the support $(-M_0 - \frac{1}{2}, \frac{1}{2})$ such that $\zeta(y) = 1$ for $y \in [-M_0, 0]$. $\zeta(\cdot)$ could be a mollified indicator function for $1_{[-M_0 - \frac{1}{2}, \frac{1}{2}]}(\cdot)$. Then, $w$ is also a subsolution to the problem obtained from $(P_\lambda)$ by replacing the right-hand side of the equation in $(P_\lambda)$ with $f(x, u) = \lambda \rho(|x|)(1 - u)^q x^k(u)$. We will use the notation $(P_\lambda)$ for the above-described problem. In what follows, we will make use of Theorem 3.3 in [28]. Introducing $\tilde{f}(x, u)$ as above, we are able to verify the following conditions required in this theorem:

$$D_x^2(\tilde{f}) \in \mathcal{C}^1(B \times \mathbb{R}), \quad D_u^2(\tilde{f}) \geq -C_0 > -\infty.$$ 

Indeed, for $r = |x|$, we have $D_{x_i}^2 \rho(r) \frac{1}{r} = (D_r^2 \rho(r)) \frac{1}{r} = \frac{D_r \rho(r)}{r}$ and $D_u^2 \rho(r) = (D_r^2 \rho(r)) \frac{1}{r}$. Thus, $D_r^2 \rho(r) \frac{1}{r} \sim \frac{K(r)}{r^2} (r \to 0)$ for some constants $K(r)$. Next, a straightforward computation shows that $\frac{D_u^2 \rho(r)}{\rho(r)} = \frac{c_1 \rho(r)}{r} + \frac{c_2 \rho(r)}{r}$ for some constants $c_1$ and $c_2$. Note that under $\rho(1)$, $\rho(|x|)$ is twice continuously differentiable at every point $x \neq 0$. Thus, to ensure that $\rho(|x|) \in C^2$, it suffices to notice that $\frac{\rho(r)}{r} \sim K(0) r^2$ $(r \to 0)$ in this case. This implies that $\frac{\rho(r)}{r} \sim \frac{c_1}{r}$ $(r \to 0)$ for some constant $c$. Next, a straightforward computation shows that $\frac{D_u^2 \rho(r)}{\rho(r)} = \frac{c_1 \rho(r)}{r} + \frac{c_2 \rho(r)}{r}$ for some constants $c_1$ and $c_2$. Note that under $\rho(1)$, $\rho(|x|)$ is twice continuously differentiable at every point $x \neq 0$. Thus, to ensure that $\rho(|x|) \in C^2$, it suffices to notice that $\frac{\rho(r)}{r} \sim K(0) r^2$ $(r \to 0)$. Since $u$ is a sub-solution of Problem $(P_\lambda)$, we are able to verify the following conditions required in this theorem:

$$\left\{ \begin{array}{l} S_k(D^2 u_i) = \lambda \rho(|x|)(1 - u_{i-1})^q \quad \text{in } B, \\ u_i = 0 \quad \text{on } \partial B. \end{array} \right. \quad (15)$$

where $u_0 = 0$. By Remark 2.1, $u_i < 0$ in $B$. Let us show that $u_i, i = 1, 2, \ldots$, form a decreasing sequence. By the comparison principle (see, e.g., [23]), $u_1 \geq u_2$. Suppose, as the induction hypothesis, that $u_{i-1} \geq u_i$. Then, $S_k(D^2 u_{i+1}) \geq \lambda \rho(|x|)(1 - u_{i-1})^q$. By the same principle, $u_i \geq u_{i+1}$. Furthermore, we note that $u_i \geq u$ for all $i$. Indeed, the inequality $u_1 \geq u$ follows from the comparison principle. Suppose, as the induction hypothesis, that $u_{i-1} \geq u$; then, $S_k(D^2 u_i) \leq \lambda \rho(|x|)(1 - u_i)^q$. Therefore, $u_i \geq u$ by comparison. Thus, there exists a function, we denote it by $u_{\max}$, which is the pointwise limit of $u_i$ as $i \to +\infty$. Note that $u_{\max}$ is a radial function. Moreover, it is a solution to $(P_\lambda)$. Indeed, Problem (15), can be reduced to an integral form. Passing to the limit, we obtain that $u_{\max}$ satisfies the integral equation equivalent to $(P_\lambda)$.

It remains to show that $u_{\max}$ is the maximal solution to $(P)$. Let $v$ be a sub-solution to $(P)$. Since the identical zero is a supersolution to $(P)$, then, by [28, Theorem 3.3], $v \leq u$. Then, $v$ is a subsolution to (15) for $i = 1$; therefore, $u_1 \geq v$. By the same argument as above (i.e., involving $v$ instead of $u$), we obtain that $u_i \geq v$ for all $i$. This implies that $u_{\max} \geq v$. Now let $\lambda_1 < \lambda_2$ and $u_{\lambda_1}, u_{\lambda_2}$ be maximal solutions of $(P_\lambda)$ (i = 1, 2), respectively. Since $u_{\lambda_2}$ is a subsolution of $(P_{\lambda_1})$, we have $u_{\lambda_2} \leq u_{\lambda_1}$ by the maximality of $u_{\lambda_1}$. \hfill \Box

3.1.1 Proof of Theorem 2.1

Proof. Let $B_2$ be the ball of radius 2 centered at zero and the constant $C > 0$ be as defined in the statement of the theorem. It is straightforward to verify that $\phi(x) = \frac{1}{2}(\frac{C}{u_{\max}})^{\frac{1}{2}}(|x|^2 - 4)$ is a radial classical solution to

$$\left\{ \begin{array}{l} S_k(D^2 \phi) = C \quad \text{in } B_2, \\ \phi = 0 \quad \text{on } \partial B_2. \end{array} \right.$$
Let $\gamma < 0$ be a constant such that $\phi < \gamma < 0$ on $\partial B$. Set $M = 2\left(\frac{C}{\pi n} - \frac{C}{\pi n} \right)^{\frac{1}{n}} = \max_{x \in \overline{B}} |\phi(x)|$ and take $\lambda < (1 + M)^{-q}$. Then, since $0 \leq 1 - \phi \leq 1 + M$ and $\rho(|x|) \leq C$, for $x \in B$, it holds that

$$S_k(D^2\phi) = C \geq \lambda \rho(|x|)(1 - \phi)^q.$$  

Since $\phi$ is a subsolution to Problem $(P_\lambda)$ and the identical zero is a supersolution, by Lemma 3.1, for every $\lambda \in (0, (1 + M)^{-q})$, there exists a solution $u_\lambda \in C^2([0, 1])$ to $(P_\lambda)$ such that $\phi \leq u_\lambda \leq 0$. Define

$$\lambda^* = \sup\{\lambda > 0 : \text{ there exists a solution } u_\lambda \in C^2([0, 1]) \text{ to } (P_\lambda)\}. \quad (16)$$

It is obvious that $\lambda^* > 0$. To show that $\lambda^*$ is finite, we consider the Newton inequality

$$\Delta u \geq C(n, k)[S_k(D^2u)]^{\frac{1}{n}}, \quad (17)$$

where $C(n, k) > 0$ is a constant and $u \in \Phi^k(B)$, see e.g. [28, Proposition 2.2, part (4)]. We also need to consider the weighted eigenvalue problem

$$\begin{cases}
-\Delta u = \lambda w(x)u & \text{in } B,
 u = 0 & \text{on } \partial B,
\end{cases} \quad (E_w)$$

where $w(x) := \rho(|x|)^{\frac{1}{n}} \in L^\infty(B)$. Let $\lambda_{1,w} > 0$ be the first eigenvalue and $\phi_{1,w} > 0$ their associated eigenfunction for problem $(E_w)$, see e.g., [1, Theorem 0.6]. It is easy to see that there exists a constant $L > 0$ such that for every $u < 0$, we have $(1 - u)^{\frac{1}{n}} \geq \frac{L|u|}{C(n,k)}$.

Let $\lambda \in (0, \lambda^*)$ and let $u_\lambda \in C^2([0, 1])$ be a solution to $(P_\lambda)$. Remark that $u_\lambda \leq 0$. Then, by (17) and the integration-by-parts formula,

$$\lambda_{1,w} \int_B (-u_\lambda)w(x)\phi_{1,w}dx = \int_B (-u_\lambda)\Delta\phi_{1,w}dx \geq L\lambda^{\frac{1}{n}} \int_B (-u_\lambda)w(x)\phi_{1,w}dx,$$

which implies that $\lambda > 0$ is bounded from above, and hence, $\lambda^*$ is finite.

Now, let $\lambda \in (0, \lambda^*)$. Take $\delta \in (0, \lambda^* - \lambda)$ and note that $u_{\lambda+\delta}$ is a subsolution for $(P_\lambda)$. By Lemma 3.1, $u_\lambda \in C^2([0, 1])$ is a maximal bounded solution to $(P_\lambda)$. By the same lemma, $u_\lambda$ decreases as $\lambda$ increases. Define $u_{\lambda^*} = \lim_{\lambda \to \lambda^*} u_\lambda$. Rewriting $(P_\lambda)$ in the integral form, we obtain

$$-u_\lambda(r) = K_\lambda \int_1^r \left(\int_0^s \rho(s)(1 - u_\lambda(s))^{q} ds\right)^{\frac{1}{n}} ds, \quad (18)$$

where $K_\lambda = c_{n,w}^{\frac{1}{n}} \lambda^{\frac{1}{n}}$. Let us show that in the above identity, we can pass to the limit as $\lambda \to \lambda^*$ on any compact interval $[\epsilon, 1] \subset (0, 1)$.

To prove this, we show first the boundedness of the family $u_\lambda$, uniformly in $\lambda \in (0, \lambda^*)$, on $[\epsilon, 1]$. It is known that the change of variable $u(r) = -u_\lambda(r\lambda^{-\frac{1}{n}})$ (see, e.g., [8]) reduces Problem $(P_\lambda)$ to

$$\begin{cases}
-S_k(D^2u) = \rho_1(r)(1 + u)^q,
 u(0) = a,
\end{cases} \quad (19)$$

where $a = -u_\lambda(0)$ and $\rho_1(r) = \rho(r\lambda^{-\frac{1}{n}})$. Let $u(r, a)$ be the $C^2([0, 1])$-solution to (19) (satisfying $u'(0) = 0$) and let $D \geq 0$. Define $R(D, a) = u^{-1}(\cdot, a)$. Let us show that $R(D, a)$ is well-defined for $a > D$, that is, a zero of $u(r, a) = D$ exists. Suppose this is not the case. Since $u(r, a)$ is decreasing, then $u(r, a) > D$ for all $r \geq 0$. From (19), we obtain that $u(r, a) \leq a - \frac{1}{2}(1 + D)^{-\frac{1}{n}} r^2 \to -\infty$, as $r \to +\infty$. This is a contradiction. Hence, $R(D, a)$ is well-defined. Let us show that the function $[D, +\infty) \to \mathbb{R}$, $a \mapsto R(D, a)$ is bounded from above. Rescale the equation in (19), by setting $\mu = R(D, a)^{2k}$, as follows:

$$-S_k(D^2u(\mu^{\frac{1}{n}} r)) = \mu \rho_1(\mu^{\frac{1}{n}} r)(1 + u(\mu^{\frac{1}{n}} r))^q.$$
Rewriting it in the integral form, we obtain

\[
\frac{\lambda}{\Pi} r = D + R(D, a)^2 \int_0^1 t^{-\frac{k}{n}} \left( \int_0^t s^{n-1} \rho_1(\mu \frac{\lambda}{\Pi} r) \left( 1 + u(\mu \frac{\lambda}{\Pi} r) \right)^q ds \right)^{\frac{1}{q}} dt
\]
\[
\geq D + R(D, a)^2 \int_0^1 t^{-\frac{k}{n}} dt \left( \int_0^1 s^{n-1} \rho_1(\mu \frac{\lambda}{\Pi} r) \left( 1 + u(\mu \frac{\lambda}{\Pi} r) \right)^q ds \right)^{\frac{1}{q}}.
\]

Next, by (ρ.3) and Remark 2.3, \(\rho_1(\mu \frac{\lambda}{\Pi} r) = \rho(\frac{\lambda}{\Pi} \mu \frac{\lambda}{\Pi} r) \geq L_1 \mu \frac{\lambda}{\Pi} s^l \) for some constant \(L_1\). Hence,

\[
\sup_{u \geq D} \{ (u - D) (1 + u)^{-\frac{q}{r}} \} \geq (u(\mu \frac{\lambda}{\Pi} r) - D)(1 + u(\mu \frac{\lambda}{\Pi} r))^{\frac{1}{q}} \geq L_2 R(D, a)^{2 + l \rho} (\frac{\lambda}{\Pi} \mu \frac{\lambda}{\Pi} r - \frac{n + \alpha}{\lambda}),
\]

where \(L_2 > 0\) is a constant. Evaluating the right-hand side at \(\tau = \frac{1}{2}\), we obtain an upper bound \(\tilde{R}(D)\) for \(R(D, a)\) whenever \(a \geq D\). It is straightforward to verify that the supremum on the left-hand side is \((1 + D)^{1 - \frac{q}{r}}\) up to a multiplicative constant. Therefore, \(\lim_{\rho \rightarrow +\infty} \tilde{R}(D) = 0\).

Note that \(u(\lambda \frac{\lambda}{\Pi}, a) = 0\) for all \(a > 0\). Take \(\varepsilon > 0\). Let \(D > 0\) be such that \(\tilde{R}(D) < \lambda \frac{\lambda}{\Pi} \varepsilon\).

Then, on \([\lambda \frac{\lambda}{\Pi}, a] \setminus [\lambda \frac{\lambda}{\Pi}, \varepsilon]\), \(u(\cdot, a) < D\) for \(a > D\). If \(a \leq D\), then clearly, \(u(\cdot, a) \leq a \leq D\). The family \(u(\lambda \frac{\lambda}{\Pi}, a)\), parametrized by \(a\), is therefore uniformly bounded on \([\lambda \frac{\lambda}{\Pi}, \varepsilon]\) for each fixed \(\lambda \in (0, \lambda^*)\). Consequently, the family \(u_{\lambda^*}\), parametrized by \(\lambda \in (0, \lambda^*)\), is uniformly bounded on \([\varepsilon, 1]\). Therefore, for \(r \in [\varepsilon, 1]\) and for \(N \in \mathbb{N}\),

\[
\lim_{\lambda \rightarrow \lambda^*} K_{\lambda} \int_r^1 \tau^{-\frac{k}{n}} \left( \int_0^\tau s^{n-1} \rho(s) (1 - u_{\lambda^*}(s))^q ds \right)^{\frac{1}{q}} d\tau
\]
\[
= K_{\lambda^*} \int_r^1 \tau^{-\frac{k}{n}} \left( \int_0^\tau s^{n-1} \rho(s) (1 - u_{\lambda^*}(s))^q ds \right)^{\frac{1}{q}} d\tau \leq \lim_{\lambda \rightarrow \lambda^*} -u_{\lambda^*}(r) = -u_{\lambda^*}(r) \leq D.
\]

Indeed, the passage to the limit is possible by the monotone convergence theorem and the uniform boundedness in \(\lambda \) of \(-u_{\lambda}\) on \([\frac{1}{N}, 1]\). Now let \(N \rightarrow \infty\) and apply the monotone convergence theorem once again. The limit function \(\tau^{-\frac{k}{n}} \left( \int_0^\tau s^{n-1} \rho(s) (1 - u_{\lambda^*}(s))^q ds \right)^{\frac{1}{q}}\) is then integrable in \(\tau\) on \([r, 1]\). Therefore, \(s^{n-1} \rho(s) (1 - u_{\lambda^*}(s))^q\) is integrable on \([0, \tau]\). This allows to pass to the limit in (18) as \(\lambda \rightarrow \lambda^*\) by the dominated convergence theorem. In addition, (18) implies that \(u_{\lambda^*}\) is an integral solution to \((P_{\lambda^*})\).

It remains to obtain a uniform lower bound for \(\lambda^*\). For this, consider the function \(v(r) = \frac{k}{q-k} (r^2 - 1)\) and note that \(v(|x|) < 0\) on \(B\) and

\[
S_k(D^2 v) = n c_n(2k)^k (q-k)^{-k} \geq \left( \frac{n}{k} \right) (2k)^k \frac{(q-k)^{-k}}{q^q} (1 - v)^q
\]
\[
\geq C^{-1} \left( \frac{n}{k} \right) (2k)^k \frac{(q-k)^{-k}}{q^q} \rho(|x|)(1 - v)^q.
\]

Hence, \(v(r)\) is a \(C^2([0,1])\)-subsolution to \((P_\lambda)\) for all \(\lambda \leq C^{-1} \left( \frac{n}{k} \right) (2k)^k \frac{(q-k)^{-k}}{q^q}\). By Lemma 3.1, there exists a maximal bounded classical solution to \((P_\lambda)\). This implies (5) and concludes the proof.

\[\square\]

### 3.2 Singular solution to \((P_\lambda)\)

The main part of this subsection consists of the construction of a singular solution, which is deduced directly from the dynamical system approach. For the reader convenience, we reintroduce some elements of this approach exposed in subsections 4.1, 4.2, 4.3 from [17], such as the Lotka-Volterra system, the stationary points of \((LV_{S,q,v_\cdot})\), and some characteristics of the flow of \((TS_{q,v_\cdot})\).
3.2.1 The new variables and the Lotka-Volterra system

Rewrite Problem \((P_\lambda)\) as follows:

\[
\begin{cases}
(r^{n-k}(w')^k)' = c_{n,k}^{-1} r^{n-1} f(r,u), & 0 < r < 1, \\
u(r) < 0, & 0 \leq r < 1, \\
u(1) = 0,
\end{cases}
\] (20)

where \(f(r,u) := \lambda r \rho(r)(1-u)^q\). Let \(u\) be a solution of (20) and set \(w = u - 1\). Then \(w\) solves the equation

\[
(r^{n-k}(w')^k)' = r^{n-1} c_{n,k}^{-1} f(r,w+1), \quad (r > 0).
\] (21)

The following variables transform this equation into a Lotka-Volterra system:

\[
x(t) = v, \quad y(t) = \frac{w'}{w}, \quad t = \ln(r),
\] (22)

whenever \(r > 0\) is such that \(w(r) \neq 0\) and \(w'(r) \neq 0\). We consider the phase plane \((x,y) \in \mathbb{R}^2\). Since we are studying negative solutions, the points \((x(t),y(t))\) belong to the first quadrant when \(w' > 0\) (see Remark 2.1). We also remark that for \(f(r,w+1) = \lambda \rho(r)(-w)^q\), this change of variables becomes optimal for equation (21) since, depending on the weight \(\rho\), it leads to either an autonomous or a non-autonomous Lotka-Volterra system. In fact, the variables in (22) transform equation (21) into the following quadratic non-autonomous system of differential equations of Lotka-Volterra-type

\[
\begin{align*}
\frac{dx}{dt} &= x \left( \nu(t) - x - qy \right), \\
\frac{dy}{dt} &= y \left( -\frac{n-2k}{k} + \frac{x}{k} + y \right)
\end{align*}
\]

(23)

previously denoted by \((TS_{q',\nu})\), where \(\nu(t) = n + r \frac{\rho'(r)}{\rho(r)}\) and \(t = \ln(r)\). In the Appendix, we provide a detailed derivation of the Lotka-Volterra system \((TS_{q',\nu})\). Note that we can recover the function \(w\) by the formula

\[
w(r) = - (c_{n,k}^{-1} \lambda r^{2k} \rho(r))^{1/2} (x(t)y(t)^k)^{1/2}, \quad \text{where} \quad r = e^t.
\] (23)

More specifically, if \((x(t),y(t))\) is a solution to \((TS_{q',\nu})\), it is straightforward to verify that \(w(r)\), given by (23), solves (21).

Since the limit \(\lim_{t \to -\infty} \nu(t) = \lim_{t \to -\infty} \left( n + \frac{\rho'(e^t)}{\rho(e^t)} \right) =: \nu_-\) exist, we may consider the system \((TS_{q',\nu})\) as an asymptotically autonomous system in the sense of H. Thieme (see \([22]\)). Thus we can describe the flow of \((TS_{q',\nu})\) from the limit autonomous system \((LVS_{q,\nu_-})\).

Remark 3.1. In \((TS_{q',\nu})\), TS stands for “Thieme system”.

3.2.2 Stationary points and local analysis

Since in the first part of the paper we study radially symmetric bounded solutions, we only focus on the autonomous system \((LVS_{q,\nu_-})\). In this case the decomposition \(\nu(t) = n + t_0 - \kappa(t)\) is useful when \(t \to -\infty\), where the function \(\kappa\) is such that \(\lim_{t \to -\infty} |\kappa(t)| = 0\).

Let \(P = (a,b)\) be a stationary point of \((LVS_{q,\nu_-})\). Introducing the coordinates \(\bar{x} := x - a\), \(\bar{y} := y - b\) and using the above decomposition we can write \((TS_{q',\nu})\) as a time-dependent perturbation of \((LVS_{q,\nu_-})\):

\[
\begin{pmatrix}
\frac{d\bar{x}}{dt} \\
\frac{d\bar{y}}{dt}
\end{pmatrix} = A \begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} + \begin{pmatrix}
-\bar{x}^2 - q\bar{x}\bar{y} \\
\frac{\bar{x}\bar{y}}{\kappa} + \bar{y}^2
\end{pmatrix} + \begin{pmatrix}
-(\bar{x} + a)\kappa(t) \\
0
\end{pmatrix}
\] (24)
with

\[ A := \begin{pmatrix} n + l_0 - 2a - qb & -qa \\ \frac{b}{k} & \frac{a}{k} + 2b - \frac{n-2k}{k} \end{pmatrix} \]

The stationary points of \((LVS_{q,ν})\) are \(P_1(0,0), P_2(0, \frac{n-2k}{k}), P_3(n + l_0, 0)\), and

\[ P_4 \left( \frac{q(n-2k) - k(n+l_0)}{q-k}, \frac{2k+l_0}{q-k} \right) =: P_4(χ, ˆy). \quad (25) \]

Note that, under the assumptions \(2k < n, k < q\) and \(-2k < l_0\), the first three critical points belong to \(\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}\) and they are saddle points. The fourth critical point \((χ, ˆy)\) belongs to the interior of \(\mathbb{R}^2_+\) if, and only if, \(q > k(n + l_0)/(n - 2k)\). Further, \((χ, ˆy)\) is a stable node for \(q > q^*(k, l_0) = \frac{(n+2k+l_0(k+1))}{n-2k}\). It is not difficult to see that the (bounded) orbit \((x(t), y(t))\) of \((LVS_{q,ν})\) starts at \(P_3(n + l_0, 0)\) (see [21]).

### 3.2.3 The flow lines of \((TS_{q,ν})\)

We define

\[ S(t, x, y) := ν(t) - x - qy, \quad t ∈ \mathbb{R}, \]

\[ W(x, y) := -\frac{n - 2k}{k} + \frac{x}{k} + y \]

and write \((TS_{q,ν})\) in the form \(x' = xS(t, x, y), y' = yW(x, y)\). For a function \(F : \mathbb{R}^+ × \mathbb{R}^+ \rightarrow \mathbb{R}\), we define

\[ F_0 := F^{-1}(0), \quad F_+ := F^{-1}[\mathbb{R}^+), \quad F_- := F^{-1}(\mathbb{R}^-). \]

Observe that \(W_0\) is the fixed straight line \(y = \frac{n-2k}{k} - \frac{x}{k}\). Furthermore, we define

\[ G(x, y) := x + \frac{(n-2k)(q+1)}{k+1} \left( \frac{k}{n-2k} y - 1 \right). \]

Note that \(G_0\) is the straight line \(y = -\frac{k+1}{k+1} x + \frac{n-2k}{k}\) with intercepts \(\left( \frac{(n-2k)(q+1)}{k+1}, 0 \right)\) and \((0, \frac{n-2k}{k})\).

**Lemma 3.2.** Assume \((ρ, 2)\). Let \(φ\) be a solution of \((TS_{q,ν})\) on \((T_0, T)\).

i) If \(φ(t_0) ∈ G_- ∪ G_0\) for \(t_0 ≥ -∞\), then \(φ(t) ∈ G_-\) for \(t > t_0\).

ii) If \(φ(t_0) ∈ G_+ ∪ G_0\) for \(t_0 > -∞\), then \(φ(t) ∈ G_+\) for \(-∞ ≤ t < t_0\).

**Proof.** For the proof see Lemma 4.1 in [17] (with the obvious changes). \(\square\)

### 3.2.4 Construction of a singular solution

To construct a singular solution of \((P_λ)\), we need the following technical lemma. For the proof, we redirect the reader to [17, Lemma 4.3], where one needs to substitute \(μ\) with \(l_0 + 2\) and \(\frac{μe^{2t}}{1+e^{2t}}\) with \(κ(t) = R(e^t) - l_0\).

**Lemma 3.3.** Assume conditions \((ρ, 1)–(ρ, 3)\) hold. Then, there exists a \(t_0 ∈ \mathbb{R}\) such that \((TS_{q,ν})\) admits a trajectory \((x(t), y(t))\) with the property \(x(t), y(t) ∈ C^1(-∞, t_0)\) and

\[ (x(t), y(t)) → (χ, ˆy) \text{ as } t → -∞, \]

where \((χ, ˆy)\) is defined by \((25)\).
Now we are in position to construct a singular solution of \((P_\lambda)\). For this, we use the trajectory \((x(t), y(t))\) obtained in Lemma 3.3. As we will see, the value \(\lambda \bar{r}\) associated with the singular solution will depend on both the trajectory and the weight function \(\rho\).

**Proposition 3.4.** Assume that conditions (\(\rho.1\))-(\(\rho.3\)) hold. Let \((x(t), y(t))\) be the trajectory obtained in Lemma 3.3. Then, for

\[
\lambda = \lambda := \frac{c_{n,k} y(0)^k}{\rho(1)},
\]

Problem \((P_\lambda)\) possesses a singular solution \(u\) with the following asymptotic representation as \(r \to 0:\)

\[
\hat{u}(r) = -\left(\frac{c_{n,k} x y^k}{\lambda K(0)}\right)^{\frac{1}{q+r}} r^{-\frac{2k+l_0}{q+r}} (1 + o(1)).
\]

**Proof.** Since \((x(t), y(t)) \to (\hat{x}, \hat{y})\) as \(t \to -\infty\) and \(\rho(r) \sim K(0)r^{l_0}\) at \(r = 0\), formula (23) implies

\[
u(r) := 1 + w(r) = 1 - \left(\frac{e^{-1}}{c_{n,k} \lambda r^k \rho(r)}\right)^{\frac{1}{q+r}} (x(t)y(t)k)^{\frac{1}{q+r}} \to -\infty \quad \text{as} \quad r \to 0.
\]

Then, \(u(r)\) is a singular solution to the equation in \((P_\lambda)\). Derivating the above identity in \(r\), we obtain

\[
\left(\frac{c_{n,k} \lambda}{r^{q+r}} u'(r)\right) = \left(r^{2k} \rho(r)\right)^{\frac{1}{q+r}} r^{-1} \frac{1}{q-k} (x(t)y(t)k)^{\frac{1}{q+r}} \left\{2k + \frac{q'}{q-r^k \rho(r)} - \left(\frac{x'(t)}{x(t)} + k \frac{y'(t)}{y(t)}\right)\right\}
\]

\[
= r^{-\frac{q+k}{q-k}} \rho(r)^{\frac{1}{q+r}} \frac{1}{q-k} (x(t)y(t)k)^{\frac{1}{q+r}} (q-k)y(t).
\]

Therefore, since \(\rho(r) \sim K(0)r^{l_0}\) at \(r = 0\), there exists a constant \(C > 0\) such that as \(r \to 0\),

\[
|r^{n-k}(w'(r))^k| \leq C r^{\frac{q(n-2k)-k(n+l_0)}{q-k}} \to 0.
\]

Since \((TS_{q,\nu})\) has trajectories on the \(x\)-axis and \(y\)-axis, by the uniqueness of a solution for \((TS_{q,\nu})\), the trajectory \((x(t), y(t))\) is not on the \(x\)-axis nor on the \(y\)-axis. Hence \(x(t) > 0\) and \(y(t) > 0\) as long as the solution \((x(t), y(t))\) exists. By (23),

\[
w(r) < 0.
\]

The equation in \((P_\lambda)\) can be rewritten as

\[
r^{n-k}(w'(r))^k - s^{n-k}(w'(s))^k = \int_s^r t^{n-1} c_{n,k} \lambda \rho(t)(-w(t))^q d\tau.
\]

By (29), the integral on the right-hand side converges at 0. Therefore,

\[
r^{n-k}(w'(r))^k = \int_0^r t^{n-1} c_{n,k} \lambda \rho(t)(-w(t))^q d\tau.
\]

Next we define

\[
\bar{\sigma} := \sup\{\sigma > 0 : \text{the solution } w(r) \text{ of (21) exists on } (0, \sigma)\}.
\]

We show by contradiction that \(\bar{\sigma} = \infty\). Suppose \(\bar{\sigma} < \infty\). By (30) and (31), \(w'(r) > 0\) on \((0, \bar{\sigma})\).

Take \(r_0 \in (0, \bar{\sigma})\). Integrating equation (21) from \(r_0\) to \(\bar{\sigma}\), we obtain

\[
r^{n-k}(w'(r))^k = r_0^{n-k}(w'(r_0))^k + \int_{r_0}^r t^{n-1} c_{n,k} \lambda \rho(t)(-w(t))^q d\tau > r_0^{n-k}(w'(r_0))^k.
\]
Therefore, \( w'(r) > \left(\frac{n}{\alpha}\right)^{\frac{n-k}{r}} w'(r_0) > 0 \). Integrating the latter inequality gives
\[
0 > w(r) > w(r_0) + \int_{r_0}^{r} \left(\frac{r_0}{r}\right)^{\frac{n-k}{r}} w'(r_0) \, dr > -\infty.
\]
Hence, the finite limit \( \lim_{r \to \infty} w(r) \) exists. By (31), the finite limit \( \lim_{r \to \infty} w'(r) \) also exists. Since \( w(\bar{r}) < 0 \), \( w'(\bar{r}) \neq 0 \), and \( w \) satisfies (21), \( w(r) \) can be locally defined as the solution of (21) in a right neighborhood of \( r = \bar{r} \). This contradicts the definition of \( \bar{r} \), and therefore \( \bar{r} = \infty \).

Since \( w(r) \) is defined on \((0, \infty)\), (22) yields that \((x(t), y(t))\) is defined for all \( t \in \mathbb{R} \). Evaluating (28) as \( r = 1 \), we obtain that \( \hat{\lambda} \) is indeed defined by (26).

Finally, we define \( \hat{u} \) by (28) with \( \lambda = \hat{\lambda} \) obtaining \((\hat{\lambda}, \hat{u}(r))\) as a desired singular solution. \( \square \)

### 3.3 Intersection number and proof of Theorem 2.2

Here we are interested in studying the intersection number between a regular and a singular solution of suitable equations. This intersection number is the most important ingredient in the proof of Theorem 2.2 on the multiplicity of solutions to Problem \((P_3)\). We start by showing the existence of a regular solution to the initial value problem
\[
\begin{align*}
\left(\begin{array}{l}
(r^{n-k}(w'(r))^k)' = \lambda c_{n,k}^{-1} r^{n-1} \rho(r)(-w(r))^q, \\
w(0) = w_0 \in (-\infty, 0), \ w'(0) = 0
\end{array}\right)
\end{align*}
\]
and its characterization via a proper trajectory of system \((TS_{q,v})\). The existence result for problem (32) will be used in the proof of Theorem 2.2.

**Proposition 3.5.** Assume that conditions (\(\rho.1\))–(\(\rho.3\)) hold. Further assume that \( \rho(r) \) has at most polynomial growth as \( r \to \infty \). Then there exists a unique global solution \( w \) of (32) in the regularity class \( C^2(0, \infty) \cap C^1[0, \infty) \). Furthermore, the function \( w \) defined by (33) is the unique solution of (32) if, and only if, the trajectory \((x(t), y(t))\) of system \((TS_{q,v})\) given by (22) starts at the point \( P_3(n + l_0, 0) \).

**Proof.** To apply Theorem 4.1 from [6], we define \( B(r) = \int_0^r s \left(\frac{\alpha - \beta(q+1)}{\beta + 1}\right) (a(s)s^q)' ds, \) with
\[
\alpha = n - k; \quad \beta = k; \quad \gamma = n - 1; \quad a(s) = \lambda c_{n,k}^{-1} \rho(s), \quad \theta = \frac{\gamma + 1}{\beta + 1} - \frac{(\alpha - \beta)(q+1)}{\beta + 1}
\]
and note that the condition \( B(r) \leq 0 \) on \((0, \infty)\) follows from (\(\rho.2\)). Indeed, we have
\[
\theta + R(s) < n + l_0 - \frac{(n - 2k)(q+1)}{k+1} < 0,
\]
where the last inequality follows from condition (\(\rho.2\)).

Then the global existence of (32) follows from [6, Theorem 4.1]. The uniqueness follows from a contraction mapping argument, as in [20].

Next, let \( w(r) \) be the unique solution of (32). By (22) and (32), the function \( y = y(t) \) satisfies
\[
\lim_{t \to -\infty} y(t) = \lim_{r \to 0} r \frac{w'(r)}{w(r)} = 0
\]
and for \( x = x(t), \) we have
\[
\lim_{t \to -\infty} x(t) = \lim_{r \to 0} \lambda c_{n,k}^{-1} (-w(r))^q \frac{r^k \rho(r)}{w'(r))^k}.
\]
Now, by the equation in (32) and L’Hôpital’s rule, we have
\[
\lim_{r \to 0} \frac{r^k \rho(r)}{(w'(r))^k} = \lim_{r \to 0} \frac{r^n \rho(r)}{r^{n-k}(w'(r))^k} = \lim_{r \to 0} \frac{r^n \rho'(r) + nr^{n-1} \rho(r)}{\lambda c_{n,k}^{-1} r^{n-1} \rho(r)(-w(r))^q}
\]
and
\[
\lim_{r \to 0} \frac{R(r) + n}{\lambda c_{n,k}^{-1} (-w(r))^q} = \lim_{r \to 0} \frac{\nu(ln r)}{\lambda c_{n,k}^{-1} (-w(r))^q} = \frac{n + l_0}{\lambda c_{n,k}^{-1} (-w_0)^q}.
\]
Thus, \( \lim_{t \to -\infty} x(t) = n + l_0 \). From here and (33) we conclude that
\[
\lim_{t \to -\infty} (x(t), y(t)) = (n + l_0, 0) = P_3(n + l_0, 0).
\] (34)

Conversely, suppose that \((x(t), y(t)) \to P_3(n + l_0, 0)\) as \(t \to -\infty\). Rewriting the equation for \(y\) in \((TS_{q,v})\), we have
\[
y' = y \left( -\frac{n - 2k}{k} + \frac{x}{k} + y^2 \right).
\]

Let \(z = y^{-1}\). Then, the above equation reduces to
\[
z' = z \left( \frac{n - 2k}{k} - \frac{x}{k} - 1 \right).
\]

By simple computations, we obtain
\[
z(t) = z(t_0)e^{\gamma(t_0)} - e^{\gamma(t)} \int_{t_0}^t e^{-\gamma(s)} ds
\]
for some \(t_0 \in (T_0, T)\) and \(\gamma(t) = \int_{t_0}^t \left( \frac{n - 2k}{k} - \frac{x}{k} \right) ds\). Since \(x(t) \to n + l_0\) as \(t \to -\infty\), we see that \(\gamma(t) = -\frac{\lambda}{k} + 2k t + o(t)\), which also implies that \(z(t) = O \left( e^{-\frac{\lambda}{k} + 2k t} \right)\) as \(t \to -\infty\); in other words, we have \(y(t) = O \left( e^{\frac{\lambda}{k} - 2k t} \right)\). By the inverse transformation (23) and the fact that \(\rho(r) \sim K(0)r^{l_0}\) at \(r = 0\), we obtain
\[
(-w(r))^{q-k} = \frac{c_{n,k}}{\lambda} \frac{x(ln r)(y(ln r))^k}{r^{2k} \rho(r)} = \frac{c_{n,k}}{\lambda} x(ln r) \left( C r^{\frac{\lambda}{k} + 2k} + o \left( r^{\frac{\lambda}{k} + 2k} \right) \right)^k
\]
\[
\to \frac{c_{n,k}}{\lambda} (n + l_0)^k \sim: (-w_0)^{q-k} \ (r \to 0),
\]
where \(C\) and \(\tilde{C}\) are positive constants independent of \(t\).

On the other hand, differentiating the function in (23) with respect to \(r\), we obtain
\[
w'(r) = \left( c_{n,k}^{-1} \lambda r^{2k} \rho(r) \right)^{-\frac{1}{2}} r^{-1} \frac{1}{q - k} (x(t)y(t))^k \left\{ 2k + \frac{\rho'(r)}{\rho(r)} - \left( x'(t) + k \frac{y'(t)}{y(t)} \right) \right\}
\]
\[
= -\frac{1}{q - k} \frac{w(r)}{r} \left\{ 2k + R(r) - \left( n + R(r) - x - qy + k \left( -\frac{n - 2k}{k} + \frac{x}{k} + y \right) \right) \right\}
\]
\[
= -w(r) \frac{y(ln r)}{r} \to 0 \ (r \to 0)
\] (35)
since \(y(ln r) = O(r^{\frac{\lambda}{k} + 2})\). Hence, the function \(w\), defined by (23), is the unique solution of Problem (32) by the first statement of this proposition. This ends the proof. \(\square\)

**Remark 3.2.** The condition on the polynomial growth of \(\rho\) is required to satisfy the condition \(r^n |\rho'(r)| \leq Cr^\mu\) (fulfilled for some constants \(C > 0\) and \(\mu > -1\)) in order to apply Theorem 4.1 from [6]. Remark that by \((p,2), \ |\rho'(r)| \leq l_0 \rho(r)r^{-1}\) and, by \((p,1), \ \rho(r)\) is bounded on any interval \([0,M], \ M > 0\). Therefore, the above condition, required by Theorem 4.1 from [6], is fulfilled if \(\rho(r)\) is of at most polynomial growth at \(+\infty\).

Let \(\hat{\lambda}\) be as in Proposition 3.4 and consider the problem
\[
\begin{align*}
(r^{n-k}(U' r^k))' &= c_{n,k}^{-1} \hat{\lambda} K(0) r^{n+l_0-1}(-U(r))^g, \quad r > 0, \\
U(0) &= -1, \quad U'(0) = 0.
\end{align*}
\] (36)
Further let
\[ \tilde{U}(r) := - \left( \frac{c_{n,k} \hat{x} \hat{y}^k}{\tilde{\lambda} K(0)} \right)^{\frac{1}{\gamma}} r^{\frac{2k+l_0}{\gamma}} . \]

Note that \( \tilde{U}(r) \) is a singular solution of the first equation in (36), which is an Emden-Fowler-type equation corresponding to the system \((LV S_{q,p})\) with \( \rho(r) = K(0)r^{l_0} \).

In this case, substituting the constant trajectory \((x(t), y(t)) = (\hat{x}, \hat{y})\), the number \( \lambda = \tilde{\lambda} \), and \( \rho(r) = K(0)r^{l_0} \) into (23), we obtain the function \( \tilde{U}(r) \). Indeed, it suffices to compare the value \( \tilde{\lambda} \) with the one defined in [21, Theorem 3.1 (I)], as well as the corresponding singular solutions.

The next result say that there is infinitely many intersections between the singular solution \( \tilde{U}(r) \) and the regular solution to (36).

**Proposition 3.6.** Let \( q^*(k,l_0) < q < q_{JL}(k,l_0) \) and let \( U(r) \) be the unique solution to (36). Then
\[ Z_{[0,\infty)}[\tilde{U} (\cdot) - U (\cdot)] = \infty, \]
where \( Z_I[\varphi] \) denotes the number of the zeros of the function \( \varphi \) in the interval \( I \subset \mathbb{R} \).

**Proof.** By the local analysis at the point \((\hat{x}, \hat{y})\) discussed in [21, Section 6], we see that this point is a stable spiral for \( q^*(k,l_0) < q < q_{JL}(k,l_0) \). Therefore, there exists a strictly increasing sequence \( \{t_n\}_{n=1}^\infty \) such that for all \( n \in \mathbb{N} \), \( y(t_n) = \hat{y} \) and \( x(t_2) < x(t_4) < \cdots < x(t_{2n}) < \cdots < \hat{x} < \cdots < x(t_{2n-1}) < \cdots < x(t_3) < x(t_1) \). Let \( r_n := e^{\gamma n} \). By (23) with \( \rho(r) = K(0)r^{l_0} \) and \( \lambda = \tilde{\lambda} \), we have
\[ \frac{\tilde{U}(r_n)}{U(r_n)} = \left( \frac{\hat{x}}{x(t_n)} \right)^{\frac{1}{\gamma}} \begin{cases} < 1, & \text{if } n \text{ is odd}, \\ > 1, & \text{if } n \text{ is even}. \end{cases} \]

Therefore \( Z_{[0,\infty)}[\tilde{U} (\cdot) - U (\cdot)] = \infty \).

**Lemma 3.7.** Let \( \tilde{u}(r) \) be the singular solution constructed in Proposition 3.4. Further let \( \tilde{w}(r) = \tilde{u}(r) - 1 \) and \((F_a \tilde{w})(r) = \frac{1}{a} \tilde{w}(\frac{r}{a})\) for \( r > 0 \) and \( a > 0 \), where \( \gamma := (q-k)/(2k+l_0) \).

Then, as \( a \to \infty \),
\[ (F_a \tilde{w})(r) \to \tilde{U}(r) \text{ in } C_{loc}(0, \infty). \]

**Proof.** By Proposition 3.4,
\[ \tilde{w}(r) = - \left( \frac{c_{n,k} \hat{x} \hat{y}^k}{\tilde{\lambda} K(0)} \right)^{\frac{1}{\gamma}} r^{-\frac{1}{\gamma}} (1 + \epsilon(r)), \]
where \( \epsilon(r) \) satisfies \( \limsup_{r \to 0} \epsilon(r) = 0 \). Hence,
\[ \frac{1}{a} \tilde{w} \left( \frac{r}{a^\gamma} \right) = - \left( \frac{c_{n,k} \hat{x} \hat{y}^k}{\tilde{\lambda} K(0)} \right)^{\frac{1}{\gamma}} r^{-\frac{1}{\gamma}} \left( 1 + \epsilon \left( \frac{r}{a^\gamma} \right) \right) \to \tilde{U}(r) \text{ in } C_{loc}(0, \infty) \text{ as } a \to \infty. \]

This concludes the proof.

**Lemma 3.8.** Under the assumptions of Proposition 3.5, we let \( w(r,a) \) be the unique solution to Problem (32) with \( \lambda = \tilde{\lambda} \) and \( w_0 = -a \) and let \((F_a w)(r,a)\) be as in Lemma 3.7. Then, as \( a \to \infty \),
\[ (F_a w)(r,a) \to U(r) \text{ in } C_{loc}[0, \infty), \]
where \( U(r) \) is the unique solution to problem (36).

**Remark 3.3.** In Proposition 3.6 and Lemma 3.8, the existence of unique solutions is known due to Proposition 3.5.
Proof of Lemma 3.8. Note that the function \( \tilde{w}(r,a) := (F_a w) (r,a) \) is a solution to

\[
\begin{cases}
(r^{n-k}(\tilde{w})')' = r^{n-k-1}c_{n,k} \tilde{\gamma} r^l \Phi (\frac{r}{a}) (-\tilde{w})^q, & r > 0, \\
\tilde{w}(0,a) = -1, & \tilde{w}_r(0,a) = 0.
\end{cases}
\]

Since \(-a \leq w(r,a) \leq a\) for \( r \geq 0 \), we see that \(-1 \leq \tilde{w}(r,a) \leq 0 \) for \( r \geq 0 \). Integrating the above equation over \([0,r] \) and recalling that \( 0 < K(r) < L \) for \( r > 0 \) by (p.3), we obtain

\[
r^{n-k}(\tilde{w}(r)) = \int_0^r c_{n,k} \tilde{\gamma} s^{n+l_0-1} K \left( \frac{s}{a} \right) (-\tilde{w})^q ds.
\]

Then,

\[
|\tilde{w}_r(r,a)| \leq \left( r^{-n+k} \int_0^r c_{n,k} \tilde{\gamma} s^{n+l_0-1} ds \right)^{1/k} \leq \left( \frac{c_{n,k} \tilde{\gamma} L}{n+l_0} \right)^{1/k} \frac{r^{l+l_0}}{r^{l+k}}.
\]

Let \( I \subset [0,\infty) \) be an arbitrary compact interval containing 0. Note that the family \( \tilde{w}(r,a) \) is uniformly bounded and equicontinuous on \( I \). By the Ascoli-Arzelà theorem, there exists a sequence \( a_m \to +\infty \) (as \( m \to \infty \)) and a function \( \tilde{w}^*(r) \in C(I) \) such that as \( m \to \infty \), \( \tilde{w}(r,a_m) \to \tilde{w}^*(r) \) in \( C(I) \). By (39),

\[
\tilde{w}(r,a_m) = -1 + \int_0^r \left( t^{-n+k} \int_0^t c_{n,k} \tilde{\gamma} s^{n+l_0-1} K \left( \frac{s}{a_m} \right) (-\tilde{w}(s,a_m))^q ds \right)^{1/k} dt.
\]

Passing to the limit in this equation as \( m \to \infty \) and noticing that \( K(\frac{s}{a_m}) \to K(0) \), we obtain

\[
\tilde{w}^*(r) = -1 + \int_0^r \left( t^{-n+k} \int_0^t c_{n,k} \tilde{\gamma} s^{n+l_0-1} K(0) (-\tilde{w}^*(s))^q ds \right)^{1/k} dt \text{ for } r \in I.
\]

Therefore, \( \tilde{w}^*(r) \in C^2(\tilde{I}) \cap C^1(I) \) (here \( \tilde{I} \) denotes the interior of \( I \) and \( \tilde{w}^*(r) \) is the solution to (36). By uniqueness, \( \tilde{w}^*(r) = U(r) \) for \( r \geq 0 \). Convergence (38) holds by uniqueness of the limit point for the family \( F_a w \).

Lemma 3.9. Under the assumptions of Proposition 3.6,

\[
Z_{[0,1]}[\tilde{w}(\cdot) - w(\cdot, a)] \to \infty \text{ as } a \to \infty.
\]

Proof. Let \( U \) and \( \tilde{U} \) be the regular and singular solutions to the equation in (36) determined above. Define \( V = U - \tilde{U} \). Then, \( V \) satisfies the ODE

\[
k r^{n-k}(\tilde{U})^{k-1} V'' + (k r^{n-k} U' \Phi_1 + (n - k) r^{n-k-1} \Phi_2) V' - r^{n+l_0-1} c_{n,k} \tilde{\gamma} K(0) \Phi_3 V = 0,
\]

where \( \Phi_1, \Phi_2, \) and \( \Phi_3 \) are continuous functions in \( r \). Since \( \tilde{U}'' \neq 0 \), the ODE (41) is of second order. We claim that

\[
V'(r_0) \neq 0 \text{ whenever } V(r_0) = 0.
\]

The proof of this claim is exposed in the beginning of the proof of Theorem 4.3 in [8]. In this proof, one has to take \( f(x) = q \ln(-x), u = -U \) and \( u^* = -\tilde{U} \).

By the property (42), each zero of \( V \) is simple, and furthermore, the set of zeros of \( V \) has no accumulation points. Therefore, each zero is isolated. By Proposition 3.6, for every \( N \in \mathbb{N} \), sufficiently large, there exists a number \( M > 0 \) such that \( Z_{[0,M]}(\tilde{U} - U) \geq N + 1 \). By Lemmas 3.7 and 3.8, as \( a \to \infty \),

\[
F_a \tilde{w} \to \tilde{U} \text{ and } F_a w \to U \text{ in } C_{loc}(0,\infty),
\]

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Therefore, in a neighborhood of each zero of $\tilde{U} - U$, there exists at least one zero of $F_a \tilde{w} - F_a w$ when $a$ is sufficiently large. Hence,

$$Z_{[0,M]}(F_a \tilde{w} - (F_a w)(\cdot, a)) \geq N.$$ 

Since $Z_{[0,M]}(F_a \tilde{w} - (F_a w)(\cdot, a)) = Z_{[0,a^{-\gamma}M]}(\tilde{w} - w(\cdot, a))$, where $\gamma = (q - k)/(2k + l_0)$, we obtain

$$Z_{[0,a^{-\gamma}M]}[\tilde{w} - w(\cdot, a)] \geq N.$$ 

When $a > 0$ is large, $[0,a^{-\gamma}M] \subset [0,1]$, and hence,

$$Z_{[0,1]}[\tilde{w} - w(\cdot, a)] \geq Z_{[0,a^{-\gamma}M]}[\tilde{w} - w(\cdot, a)] \geq N.$$ 

The lemma is proved.

\textbf{Proof of Theorem 2.2.} Let $\lambda$ be defined by (26) and let $w(r, a)$ be the solution to (32) with $\lambda = \lambda$ and $w_0 = -a$, whose existence and uniqueness is known due to Proposition 3.5. It is straightforward to verify that $\tilde{w}(r, a) := (\lambda/\lambda)^{1/(q-k)}w(r, a)$ is a solution to

$$w(t, a) = - \left( \frac{\lambda}{\lambda} \right)^{1/(q-k)} a, \quad \tilde{w}(0, a) = 0.$$ 

The reminder of the proof follows the lines of the proof of Theorem 2.2 in [17], so we omit it.

\section{\textbf{4} $P_2$, $P_3^+$, and $P_4^+$-solutions to Problem $\hat{P}$}

In this section, we characterize $P_2$, $P_3^+$, and $P_4^+$-solutions to Problem $\hat{P}$. Since we are only interested in the behavior of these solutions at a neighborhood of $+\infty$, we will define these solutions on an interval $(M, +\infty)$ for some $M > 0$ large enough.

\textbf{Definition 4.1.} A $C^2(M, +\infty)$-solution $w(r)$ to Problem $\hat{P}$ is called a $P_2$, $P_3^+$, or $P_4^+$-solution if the associated orbit $(x(t), y(t))$, $t = \ln(r)$, of the non-autonomous Lotka-Volterra system (TS)$_{q,\nu}$ tends to $(0, n(2k)/k, n + l_\infty, 0)$, respectively, as $t \to +\infty$.

The main tool for characterizing these solutions is the associated non-autonomous Lotka-Volterra system (TS)$_{q,\nu}$, which it will be convenient to rewrite with the help of the function $\zeta(t) = \nu(t) - n - l_\infty$ in one of the following ways:

$$\begin{align*}
\begin{cases}
\dot{x} = x(\nu_+ + \zeta(t) - x - qy), \\
\dot{y} = y(-\frac{n-2k}{k} + \frac{\nu_+}{k} + y).
\end{cases}
\end{align*}$$

Note that $\lim_{t \to +\infty} \zeta(t) = 0$.

As we mentioned in the Introduction, an important role will be played by the parameter $\delta$, given by (2). More specifically, we will show that $P_2$-solutions to $\hat{P}$ can exist for all values of $\delta$; $P_3^+$-solutions can exist only if $\delta \geq 0$, and $P_4^+$-solutions can exist only if $\delta < 0$.

\subsection{Classification of the stationary points of (LV$S_{q,\nu}$+$+$)}

In this subsection, we assume that (p.4) is in force. Consider the autonomous system

$$\begin{align*}
\begin{cases}
\dot{x} = x(n + l_\infty - x - qy), \\
\dot{y} = y(-\frac{n-2k}{k} + \frac{\nu_+}{k} + y).
\end{cases} \quad (LVS_{q,\nu})
\end{align*}$$

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whose right-hand side is the limit of \((TS_{q,\nu})\) as \(t \to +\infty\). Let \(P = (a, b)\) be a stationary point of \((LVS_{q,\nu})\). Introduce the coordinates \(\bar{x} = x - a\) and \(\bar{y} = y - b\). Then, one can rewrite \((LVS_{q,\nu})\) as follows:

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{pmatrix} = A \begin{pmatrix}
\bar{x} \\
\bar{y}
\end{pmatrix} + \begin{pmatrix}
-\bar{x}^2 - q\bar{x}\bar{y} \\
\bar{x}^2 + \bar{y}^2
\end{pmatrix}
\]

with

\[
A := \begin{pmatrix}
n + l_\infty - 2a - qb & -qa \\
\frac{b}{k} & \frac{a}{k} + 2b - \frac{n - 2k}{k}
\end{pmatrix}.
\]

The critical points of \((LVS_{q,\nu})\) are \(P_1(0,0), P_2(0, \frac{n - 2k}{k}), P_3^+(n + l_\infty, 0), \) and \(P_4^+ \) as follows:

\[
P_4^+ \left( \frac{q(n - 2k) - k(n + l_\infty)}{q - k}, \frac{2k + l_\infty}{q - k} \right) = P_4^+(\bar{x}, \bar{y}).
\]

Below we classify \(P_1, P_2, P_3^+\) and \(P_4^+\). First, we note that the inequality \(q \geq q^* (k, l_0)\) is equivalent to \(\frac{n + l_0}{n - 2k} \leq \frac{n + 1}{k + 1}\) and \(q > q^*(k, l_0)\) is equivalent to \(\frac{n + l_0}{n - 2k} < \frac{q + 1}{k + 1}\). By \((\rho.4)\),

\[
\frac{n + l_\infty}{n - 2k} < \frac{q + 1}{k + 1} < \frac{q}{k}.
\]

**Case** \(n + l_\infty > 0\). It is straightforward to verify that \(P_1\) and \(P_2\) are saddle points. We will classify \(P_3^+\) depending on the parameter \(\delta\) defined by \((2)\). If \(\delta > 0\), then \(P_3^+\) is a node, if \(\delta < 0\), then \(P_3^+\) is a saddle. Furthermore, if \(\delta = 0\), then \(P_3^+ = P_3^+ = (n - 2k, 0)\). In the latter case, the eigenvalues of the matrix \(A\) with \((a, b) = (n - 2k, 0)\) are \(\lambda_1 = 0, \lambda_2 = -(n - 2k)\). According to the results of [5] (Section 5, item (B-3) on p. 811), the point \(P_3^+ = P_4^+ = (n - 2k, 0)\) classifies as a saddle-node.

Consider now \(P_4^+(\bar{x}, \bar{y})\), where \((\bar{x}, \bar{y})\) is defined by \((44)\). A straightforward computation shows that at \(P_4^+\)

\[
A = \begin{pmatrix}
-\bar{x} & -q\bar{x} \\
\frac{\bar{y}}{k} & \bar{y}
\end{pmatrix}.
\]

If \(\delta > 0\) \((l_\infty < 2k)\), then \(\bar{x} > 0\) \((by \,(45))\) and \(\bar{y} < 0\), so it is straightforward to compute that for the eigenvalues of \(A\) holds that \(\lambda_1 > 0\) and \(\lambda_2 < 0\). Therefore, \(P_4^+\) is a saddle. If \(\delta = 0\), then \(P_4^+ = P_3^+\); this case was analyzed above. Finally, if \(\delta < 0\), then \(\bar{x} > 0\) and \(\bar{y} > 0\). Further, by \((45)\), \(\bar{x} - \bar{y} > 0\). The characteristic equation takes the form \(\lambda^2 + (\bar{x} - \bar{y})\lambda + \left(\frac{q}{k} - 1\right)\bar{x}\bar{y} = 0\) with the discriminant

\[
D = (\bar{x} - \bar{y})^2 - 4\left(\frac{q}{k} - 1\right)\bar{x}\bar{y} = \bar{y}^2\left(\mu^2 + 2\mu \left(1 - \frac{2q}{k}\right) + 1\right), \quad \mu = \frac{\bar{x}}{\bar{y}}.
\]

Note that the quadratic polynomial \((with\ respect\ to\ \mu)\) on the right-hand side can take negative, zero or positive values. Indeed, it has two positive roots. Therefore, \(P_4^+\) can be either a focus or a node. Let us find the condition on \(l_\infty\) implying that \(P_4^+\) is a node. The roots of the polynomial are \(\mu_{1,2} = \frac{2q}{k} - 1 \mp 2\sqrt{(\frac{q}{k})^2 - \frac{q}{k}}\) \((\mu_1 < \mu_2)\). Then, the condition of the positivity of \(D\) is \(\frac{\bar{x}}{\bar{y}} < \mu_2\) or \(\frac{\bar{x}}{\bar{y}} < \mu_1\). Equivalently, it can be written as follows:

\[
l_\infty < \frac{q(n - 2k) - k(n + 2\mu_2)}{k + \mu_2} \quad \text{or} \quad l_\infty > \frac{q(n - 2k) - k(n + 2\mu_1)}{k + \mu_1}.
\]

Under this condition, we have that \(\lambda_1 < 0, \lambda_2 < 0,\) and \(\lambda_1 \neq \lambda_2,\) so \(P_4^+\) is a stable node. Note that if \(D = 0,\) then \(P_4^+\) is a stable degenerate node.

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Remark 4.1. The aforementioned condition on \( l_\infty \) is equivalent to \( q > q_{IL}(k, l_\infty) \). See Section 6 in [21], p. 702.

Case \( n + l_\infty < 0 \). \( P_1 \) is now a node and \( P_2 \) remains a saddle. Note that if \( n + l_\infty < 0 \), then \( \delta = \frac{-2k + l_\infty}{k} > \frac{n - 2k}{k} > 0 \). Therefore, \( P_3^{+} \) is a saddle. Furthermore, the analysis of the previous case shows that \( P_4^{+} \) is a saddle.

Case \( n + l_\infty = 0 \). In this case, \( P_2 \) and \( P_4^{+} \) remain saddle points. Note that \( P_3^{+} = P_1 \). According to the classification in [5], Section 5, item (B-4-b-1-j) on p. 812, \( P_3^{+} = P_1 \) is a saddle-node.

Remark that the classification of \( P_1 \), \( P_3^{+} \) and \( P_4^{+} \) depends on how the number \( l_\infty \) is positioned with respect to \(-n\) and \(-2k\). On the other hand, we will only be interested in stationary points located in the first quadrant since the orbits of \((T_{q,v})\) are located in this quadrant due to the transformation \((22)\). For the reader’s convenience, we summarize the results of this subsection in the following table:

| \( l_\infty < -n \) | \( l_\infty = -n \) | \(-n < l_\infty < -2k \) | \( l_\infty = -2k \) | \( l_\infty > -2k \) |
|-----------------|-----------------|---------------------|-----------------|-----------------|
| \( P_2 \) is a saddle, \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{x \geq 0\} \), \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{y \geq 0\} \), \( P_4^{+} \) is a saddle, \( P_3^{+} \) is a saddle-node | \( P_2 \) is a saddle, \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{x \geq 0\} \), \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{y \geq 0\} \), \( P_4^{+} \) is a saddle, \( P_3^{+} \) is a saddle-node | \( P_1, P_2 \) are saddles, \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{x \geq 0\} \), \( P_4^{+} \) is a saddle-node | \( P_1, P_2 \) are saddles, \( P_3^{+} \) is a saddle but \( P_4^{+} \notin \{x \geq 0\} \), \( P_4^{+} \) is a saddle-node | \( P_1, P_2, P_3^{+} \) are saddles, \( P_4^{+} \) is either a node or a focus |

Table 1: Classification of the stationary points of \((LV_{q,v})\) depending on \( l_\infty \).

4.2 Useful results

The following lemma is a slight reformulation of Lemma 5.5 from [3].

Lemma 4.1. Consider the differential equation \( \dot{z} = r(t)z + b(t) \) on \([t_0, +\infty)\). Assume \( \lim_{t \to +\infty} r(t) = r_0 < 0 \) and \( \lim_{t \to +\infty} b(t) = b_0, b_0 \in \mathbb{R} \). Then,

\[
\lim_{t \to +\infty} z(t) = -\frac{b_0}{r_0}.
\]

Proof. The result follows immediately from Lemma 5.5 if we rewrite the ODE with respect to \( \dot{z}(t) = z(-t) \) and introduce the new coefficients \( \tilde{r}(t) = -r(-t) \) and \( \tilde{b}(t) = -b(-t) \).

Proposition 4.2. Suppose \( l_\infty > -2k \) (equivalently, \( \delta < 0 \)). Then a solution to Problem \((\tilde{P})\) of class \( C^2(M, +\infty) \) cannot be a \( P_3^{+} \)-solution.

Proof. Suppose \((x(t), y(t))\) is the orbit of \((T_{q,v})\) associated to a \( P_3^{+} \)-solution. Computing the derivative of \( \frac{1}{y} \) and using \((T_{q,v})\), we obtain

\[
\left( \frac{1}{y} \right)' = -\frac{1}{y} \frac{\dot{y}}{y} = \frac{1}{y} \left( \delta + \frac{\nu_+ - x}{k} \right) - 1.
\]

(47)

Since \( x \to \nu_+ \) as \( t \to +\infty \), by Lemma 4.1, \( \lim_{t \to +\infty} \frac{1}{y(t)} = \frac{1}{\delta} < 0 \). This contradicts to the fact that \( \lim_{t \to +\infty} y(t) = 0 \).

Lemma 4.3. Let \((p.1), \( (p.2), \( (p.4), \( (p.5)\) hold. Further let \((x, y)\) be either an orbit of \((T_{q,v})\) or an orbit of \((LV_{q,v})\) tending to \( P_2 \) as \( t \to +\infty \) in such a way that \( x(t) > 0 \) in a neighborhood of \(+\infty\). Then, in a neighborhood of \( P_2 \), \( y \) can be represented as a function of \( x \), which we denote by \( \hat{y}(x) \). If, moreover, \( \hat{y}'(0) \) exists, then, in a neighborhood of \( P_2 \), the graph of \( \hat{y}(x) \) lies in \( G_+ \) and

\[
\hat{y}'(0) = -\frac{n - 2k}{k^2 \gamma + k(n - 2k)}.
\]
Proof. Note that by (45), \( \lim_{t \to +\infty} (\nu_+ + \zeta(t) - x - qy) = n + l_\infty - \frac{q}{k} (n - 2k) < 0 \). Therefore, in a neighborhood of \(+\infty, \dot{x} > 0\) and one can express \( t \) as a function of \( x \). In this neighborhood of \(+\infty\), we introduce the function

\[
\hat{y}(x) = y(t(x)),
\]

where \( t(x) \) is the inverse function to \( x(t) \). Next, since \( \hat{y}'(0) \) exists, from \((TS_{q,v})\) we obtain

\[
\hat{y}'(0) = \lim_{x \to 0^+} \hat{y}'(x) = \lim_{t \to +\infty} \frac{\hat{y}(t)}{\dot{x}(t)} = \frac{n - 2k}{k} \left( \frac{1}{k} + \lim_{t \to 0^+} \frac{\hat{y} - \frac{n + l_\infty}{k}}{x} \right) = -\frac{(n - 2k)(1 + k\hat{y}'(0))}{k^2 \gamma},
\]

where \( \gamma = \frac{q}{k} (n - 2k) - (n + l_\infty) > 0 \) by (45). From the above equation, we obtain the expression for \( \hat{y}'(0) \). Using this expression, we compare \( \hat{y}'(0) \) with the slope of \( G_0 \) which equals \(-\frac{k + 1}{k(q + 1)}\).

Observe that \( \frac{1}{|y'(0)|} > \frac{k(q + 1)}{k + 1} \). Indeed, the latter inequality is equivalent to \( \frac{n + l_\infty}{n - 2k} < \frac{q + 1}{k + 1} \), which is the same as (45). This implies that in a neighborhood of \( P_1 \), the graph of \( \hat{y}(x) \) lies in \( G_0 \). \( \square \)

The following theorem by Thieme will be useful. Although the theorem holds for a large class of asymptotically autonomous 2-dimensional systems (see Theorem 1.5 in [22]), we will formulate it only for systems \((TS_{q,v})\) and \((LVS_{q,v})\). We take into account that we have at most four equilibrium points, so the assumptions of Thieme’s theorem are fulfilled.

Theorem 4.4. For the \( \omega \)-limit set \( \omega \) of \((TS_{q,v})\), the following trichotomy holds:

(a) \( \omega \) consists of an equilibrium point of \((LVS_{q,v})\).

(b) \( \omega \) is the union of periodic orbits of \((LVS_{q,v})\) and possibly centers of \((LVS_{q,v})\) surrounded by periodic orbits lying in \( \omega \).

(c) \( \omega \) contains equilibria of \((LVS_{q,v})\) that are cyclically chained to each other by orbits of \((LVS_{q,v})\).

The following corollary of Theorem 4.4 will be useful.

Corollary 4.5. Assume \((p.1), (p.2), (p.4), (p.5)\). Then, for orbits of \((TS_{q,v})\) associated to solutions of \((\hat{P})\), only option (a) of Theorem 4.4 can be realized. Furthermore, if \( l_\infty < -2k \), the \( \omega \)-limit set of \((TS_{q,v})\) is either \( P_2 \) or \( P_3^+ \); if \( l_\infty > -2k \), the \( \omega \)-limit set of \((TS_{q,v})\) is either \( P_2 \) or \( P_4^+ \).

Proof. Note that any solution to Problem \((\hat{P})\) gives rise to the associated orbit of \((TS_{q,v})\) totally located in the first quadrant. Therefore, the \( \omega \)-limit set of such an orbit is contained in the domain \( \{ x \geq 0 \} \cap \{ y \geq 0 \} \). We now analyze the \( \omega \)-limit set of \((TS_{q,v})\) by using Theorem 4.4.

First, we note that according to Theorem 2.1 in [5], \((TS_{q,v})\) does not have non-trivial closed orbits in the first quadrant. This excludes option (b) of Theorem 4.4. Theorem 2.1 also implies that in the hypothesis that option (c) is realized, the \( \omega \)-limit set of \((TS_{q,v})\) cannot have homoclinic orbits connecting saddles of \((TS_{q,v})\) to themselves.

First, consider the case \( l_\infty \leq -2k \) or, equivalently, \( \delta \geq 0 \). Let us show that this condition fully excludes option (c). For this, we analyze the behavior of heteroclinic orbits of \((LVS_{q,v})\) that can possibly start or end at the saddle points of \((LVS_{q,v})\). Since \( l_\infty \leq -2k \), we have to separately analyze the situation described in each of the first four columns of Table 1. Note that we can immediately disregard the first two columns since there is only one saddle point \( P_2 \) in the domain \( \{ x \geq 0 \} \cap \{ y \geq 0 \} \). Thus, without loss of generality, we assume that \( n + l_\infty > 0 \). Note that we have to consider only the saddle points \( P_1 \) and \( P_2 \). Indeed, if \( l_\infty < -2k \), \( P_2^+ \) is also a saddle, but \( P_3^+ \notin \{ y \geq 0 \} \); so orbits starting or ending at \( P_3^+ \) are excluded. If \( l_\infty = -2k \), then \( P_3^+ = P_4^+ \) is a saddle-node. Thus, we analyze heteroclinic orbits that can possibly connect \( P_1 \) and \( P_2 \).

Below, \( (x(t), y(t)) \) denotes such an orbit. Suppose first that \( x(t) \) and \( y(t) \) are not identically equal zero at a neighborhood of \( P_1 \). Then, at \( P_1 \) we have \( (ln x)' = n + l_\infty > 0 \) and \( (ln y)' = -\frac{n - 2k}{k} < 0 \). This implies that as \( t \to +\infty \), \( x \) increases and \( y \) decreases. Therefore, any heteroclinic orbit can
enter $P_1$ as $t \to +\infty$ only from quadrant ii. In a similar manner, any heteroclinic orbit can enter $P_1$ as $t \to -\infty$ only from quadrant iv. Thus, the aforementioned case is excluded since the respective orbits cannot attract trajectories of $(TS_{q,v})$ corresponding to solutions of $(P)$. Next, if $x(t)$ equals zero in a neighborhood of $P_1$, then $y(t)$, in a neighborhood of $P_1$, solves the ODE $\dot{y} = y^2 - \frac{n-2k}{k}y$ whose solution is

$$y(t) = \frac{n - 2k}{k} \left(1 + Ce^{-\frac{n-2k}{k}t}\right)^{-1}, \tag{48}$$

where $C \neq 0$ is a constant. If $C = 0$, then $(x(t), y(t)) = P_2$ is a trivial orbit and shall be considered within the analysis of case (a) of Theorem 4.4. Note that, by uniqueness, $(0, y(t))$, where $y(t)$ is given by (48), is the only solution to $(LVS_{q,v})$ with the property that $x = 0$ in a neighborhood of $P_1$. It represents a heteroclinic orbit that starts at $P_2$ at $t = -\infty$ and ends at $P_1$ at $t = +\infty$. Furthermore, if $y(t) = 0$ in a neighborhood of $P_1$, then $x(t)$, in a neighborhood of $P_1$, solves the ODE $\dot{x} = x(n + l_\infty - x)$ whose solution is

$$x(t) = \frac{n + l_\infty}{Ce^{-(n + l_\infty)n + 1}}, \tag{49}$$

where $C \neq 0$ is a constant. Remark that if $C = 0$, then $(x(t), y(t)) = P^+_3$ is a trivial orbit and shall be considered within the analysis of case (a) of Theorem 4.4. Also remark that, by uniqueness, $(x(t), 0)$, where $x(t)$ is given by (49), is the only solution to $(LVS_{q,v})$ with the property that $y = 0$ in a neighborhood of $P_1$. However, the above-described orbit connects $P_1$ with $P^+_3$, not with $P_2$.

Therefore, the segment $[0, \frac{n-2k}{k}]$ is the only heteroclinic orbit connecting $P_1$ and $P_2$. Let us prove that this orbit cannot belong to the $\omega$-limit set of any trajectory of $(TS_{q,v})$. Arguing by contradiction, we assume that $(\tilde{x}(t), \tilde{y}(t))$ is such a trajectory. Fix $\varepsilon \in (0, n - 2k)$, sufficiently small, and find $\tau_1 > 0$ such that $\tilde{x}(t) < \varepsilon$ for $t > \tau_1$. Fix $y_0 \in (0, \frac{n-2k-\varepsilon}{k})$. There exists a sequence $t_n \to +\infty$ such that $\tilde{y}(t_n) \to y_0$. Find $\tau_2 > \tau_1$ such that $\tilde{y}(\tau_2) < \frac{n-2k-\varepsilon}{k}$. It holds that $\tilde{y}'(\tau_2) < 0$, and therefore, $\tilde{y}' < 0$ on $(\tau_2, +\infty)$. Since $(0, 0)$ cannot be a limit point of any trajectory of (43), $y_1 = \lim_{t \to +\infty} \tilde{y}(t) > 0$. Hence, the interval $[0, y_1)$ cannot belong to the $\omega$-limit set of $(\tilde{x}(t), \tilde{y}(t))$ which is a contradiction.

Thus, we proved that option (c) of Theorem 4.4 is also excluded. The only remaining option is (a). Now we have to analyze trajectories of $(TS_{q,v})$ possibly ending at the stationary points $P_i$, $i = 1, 2, 3, 4$. The same argument as we used in the analysis of heteroclinic orbits of $(LVS_{q,v})$ shows that orbits of $(TS_{q,v})$ cannot end at $P_1$ or $P^+_3$, while $P^+_4$ is considered only when $l_\infty < -2k$.

Thus, the $\omega$-limit set of $(TS_{q,v})$ can only consist of $P_2$ or $P^+_3$.

Suppose now $l_\infty > -2k$ or, equivalently, $\delta < 0$. Suppose item (c) of Theorem 4.4 takes place. This situation is reflected in the fifth column of Table 1. The $\omega$-limit set consists then of three saddle points $P_1$, $P_2$, and $P^+_3$ and heteroclinic orbits connecting them. In particular, by the above argument, the orbit $(0, y(t))$, where $y(t)$ is given by (48), goes from $P_2$ to $P_1$; and the orbit $(x(t), 0)$, where $x(t)$ is given by (49), goes from $P_2$ to $P^+_3$. Since $[P_2, P_1]$ and $[P_1, P^+_3]$ attract trajectories $\varphi(t)$ of $(TS_{q,v})$, by Lemma 3.2, it holds that $\varphi(t) \in G_+ \cap \{x \geq 0\} \cap \{y \geq 0\}$ for all $t > t_0$ for some $t_0 \in \mathbb{R}$. Therefore, any heteroclinic orbit, name it $\psi(t)$, starting at $P^+_3$ and ending at $P_2$, should be located in the domain $(G_- \cup G_0) \cap \{x \geq 0\} \cap \{y \geq 0\}$.

However, Lemma 4.3 implies that in a neighborhood of $+\infty$, $\psi(t)$ enters $P_2$ from $G_+$. Therefore, $\psi(t)$ cannot belong to the $\omega$-limit set of $(TS_{q,v})$. The conditions of Lemma 4.3 are fulfilled since in a neighborhood of $P_2$, $\psi(t)$ is represented as a graph of function $\tilde{y}(x)$ and $\tilde{y}'(0)$ exists by item (ii) of Theorem 2.3, applied to system $(LVS_{q,v})$. Indeed, the asymptotic representations obtained in item (ii) of Theorem 2.3 remain valid for $\psi(t)$, since the proof also works in the situation when $\zeta(t) = 0$. Thus, we obtained a contradiction to option (c). The only remaining option is (a). Since the coordinates of $P^+_4$ are positive and $P^+_4 \in G_-$, it can belong to the $\omega$-limit set of $(TS_{q,v})$. To see that $P^+_4 \in G_-$, we substitute $y = \frac{2k+1}{k+1} + \frac{n}{k+1} (n - 2k)$. By (45), we obtain

$$-\frac{k(q+1)}{k+1} \frac{2k + l_\infty}{q - k} + \frac{q + 1}{k+1} (n - 2k) = \frac{q(n - 2k) - k(n + l_\infty)}{q - k} > \frac{q(n - 2k) - k(n + l_\infty)}{q - k}.$$
This implies that $P^+_4 \in G_-$. Further, by Lemma 4.2, $P^+_3$ cannot belong to the $\omega$-limit set. Therefore, if $l_\infty > -2k$, the $\omega$-limit set of $(TS_{q,\rho})$ consists either of $P_2$ or $P^+_4$.

Remark 4.2. Remark that, under $(p.1)$, $(p.2)$, $(p.4)$, $(p.5)$, by the existence result obtained in Proposition 3.5 global solutions to Problem (32) exist. Furthermore, by Corollary 4.5, these solutions classify as $P_2$, $P^+_3$, or $P^+_4$.

Lemma 4.6. Assume $(p.5)$. Then, there exists a constant $c_\rho > 0$ such that as $r \to +\infty$,
\[ \rho(r) = c_\rho e^{r^\delta} (1 + o(1)). \]

Proof. By the definition of $R(r)$,
\[ \ln \rho(r) = \ln \rho_0 + \int_{\rho_0}^r \frac{R(s) - l_\infty}{s} ds + l_\infty \int_{\rho_0}^r \frac{ds}{s}. \]
By $(p.5)$, the integral containing $R(s)$ converges as $r \to +\infty$, which implies the statement.

In what follows, it will be rather convenient to formulate a version of assumption $(p.6)$ in terms of the function $\zeta(t) = R(e^t) - l_\infty$.

$(p.6)$ One of the assumptions, $(1)$ or $(2)$, is fulfilled:
\begin{enumerate}
  \item $\lim_{t \to +\infty} e^{\delta t} \zeta(t) \in [0, +\infty)$ and $\nu_+ > \delta$;
  \item $\lim_{t \to +\infty} e^{\delta t} \zeta(t) = +\infty$, the limit $\nu = \lim_{t \to +\infty} \left( -\frac{\zeta(t)}{\zeta'(t)} \right)$ exists, and $\nu_+ > \nu$.
\end{enumerate}

First of all note that by L'Hôpital's rule and since $r = e^t$,
\[ \lim_{r \to +\infty} \frac{\ln |R(r) - l_\infty|}{\ln r} = \lim_{t \to +\infty} \frac{\ln |\zeta(t)|}{t} = \lim_{t \to +\infty} \frac{\zeta'(t)}{\zeta(t)}. \] (50)

For the purpose of examples of weights given in Subsection 4.6, we formulate an alternative assumption $(p.6')$ which is stronger than $(p.6)$, but it is easier to be verified.

$(p.6')$ Define $\psi(r) = r^\theta |R(r) - l_\infty|$. We assume (a) $\liminf_{r \to +\infty} \psi(r) > 0$; (b) $\nu_+ > \min \{\delta, \theta\}$; and (c) $\lim_{r \to +\infty} \psi(r) > 0$ exists in the case $\delta = \theta$.

Lemma 4.7. $(p.6')$ implies $(p.6)$.

Proof. Note that options $(1)$ and $(2)$ in the assumption $(p.6)$ correspond to the cases $\delta < \theta$ and $\delta > \theta$, respectively. Further we note that for all values of $\delta$ and $\theta$, the limit $\lim_{r \to +\infty} r^\delta |R(r) - l_\infty| = \ell$ exists, finite or infinite. More specifically, if $\delta < \theta$, then $\ell = 0$; if $\delta = \theta$, then $\ell \in (0, +\infty)$; if $\delta > \theta$, then $\ell = +\infty$. In the latter case, by the definition of $\psi(r)$, L'Hôpital's rule, and (50),
\[ \theta = -\frac{\ln r^{-\theta}}{\ln r} - \lim_{r \to +\infty} \frac{\ln \psi(r)}{\ln r} = -\lim_{r \to +\infty} \frac{\ln |R(r) - l_\infty|}{\ln r} = -\lim_{t \to +\infty} \frac{\zeta'(t)}{\zeta(t)} = \nu. \]

This proves $(p.6)$.

Corollary 4.8. Let $\liminf_{r \to +\infty} \psi(r) > 0$ and $\delta \neq \theta$. Then $(p.6)$ is equivalent to $(p.6')$.

$(p.6')$ $\nu_+ > \min \{\delta, \theta\}$.

Corollary 4.9. Let $\lim_{r \to +\infty} \psi(r) > 0$ exist and $\delta = \theta$. Then, $(p.6)$ is equivalent to $(p.6')$, where the latter is reduced to the inequality $\delta < \frac{n-2k}{k+1}$.

4.3 $P_2$-solutions

From Remark 4.2, we know that if $q \geq q^*(k,\ell_0)$, then solutions to Problem $(\hat{P})$ exist and can classify as $P_2$, $P^+_3$, or $P^+_4$. In this subsection, we characterize $P_2$-solutions.
4.3.1 Proof of Theorem 2.3

We are ready to prove Theorem 2.3. The proof uses some ideas from the related result in [3] (Theorem 7.2).

Remark 4.3. Note that the number \( \gamma = \frac{2}{k}(n-2k) - (n + l_\infty) \), defined in the statement of Theorem 2.3, is positive by (45) and (p.2).

Proof of Theorem 2.3. Step 1. (i) \( r. \)

2.3. is positive by (45) and (4.3).

Remark (Theorem 7.2).

We are ready to prove Theorem 2.3. The proof uses some ideas from the related result in [3].

4.3.1 Proof of Theorem 2.3

where \( t > \tau \) and \( \bar{\psi} \).

Note that \( I \)

decaying estimate for \(|\psi(t)|\) by using Lemma 7.1 from [3]. First, we find \( \sigma > 0 \) such that

\[
\beta := C\sigma\left(\frac{1}{|\lambda_1|} + \frac{1}{|\lambda_2|}\right) < 1.
\]
Note that
\[ |\vartheta(\psi(t))| \leq (|x(t)| + q|\theta(t)|)|\psi(t)| \quad \text{and} \quad |\eta(t, \psi(t))| \leq |\zeta(t)||\psi(t)|; \quad (57) \]
so the number \( \tau \) in (55) can be chosen in such a way that \( |\vartheta(\psi(t))| + |\eta(t, \psi(t))| < \sigma|\psi(t)| \) for all \( t \geq \tau \). Therefore, by (53),
\[ P_i(\vartheta(\psi(s)) + \eta(s, \psi(s))) \leq C\sigma|\psi(t)|, \quad i = 1, 2. \quad (58) \]
Estimate (58) and equation (55) imply that there exists a constant \( K_1 > 0 \) such that
\[ |\psi(t)| \leq K_1 e^{\lambda_1 t} + C\sigma \int_{\tau}^{t} e^{\lambda_1 (t-s)}|\psi(s)|ds + C\sigma \int_{0}^{+\infty} e^{-\lambda_2 s}|\psi(s + t)|ds. \]
By Lemma 7.1 from [3], there exist a constant \( K_2 > 0 \) such that
\[ |\psi(t)| \leq K_2 e^{(\frac{C\sigma}{1-\rho})t}. \quad (59) \]
Furthermore, (57) imply that there exists a constant \( K_3 > 0 \) such that
\[ |\vartheta(\psi(t))| \leq (1 + q)|\psi(t)|^2 \quad \text{and} \quad |\eta(t, \psi(t))| \leq K_3 e^{-\theta t}|\psi(t)|. \quad (60) \]
By (59) and (60), we have the following estimates for \( I(t) \) and \( J(t) \) in (55):
\[ |I(t)| \leq K_4 \int_{\tau}^{t} (e^{(\frac{C\sigma}{1-\rho})s + \lambda_1 s}) ds, \]
\[ |J(t)| \leq K_5 e^{(\lambda_2 - \lambda_1) t} \int_{\tau}^{+\infty} (e^{(\frac{C\sigma}{1-\rho})s + 2\lambda_1 - \lambda_2 s}) ds, \]
where \( K_4, K_5 > 0 \) are constants. In (56), we can take \( \sigma > 0 \) even smaller so that \( \frac{2C\sigma}{1-\rho} + \lambda_1 < 0 \) and \( \frac{C\sigma}{1-\rho} - \theta < 0 \). With this choice of \( \sigma \), \( \lim_{t \to +\infty} I(t) \) is finite and \( \lim_{t \to +\infty} J(t) = 0 \). Therefore, \( \lim_{t \to +\infty} e^{-\gamma t}|\psi(t)| \) exists. We denote it by \( \left( \frac{c_1}{c_2} \right) \). Note that \( c_1 = \lim_{t \to +\infty} e^{-\gamma t}|x(t)| \geq 0 \). In Step 3, we will prove that \( c_1 > 0 \); the expression for \( c_2 \) via \( c_1 \) is so far unknown and will be obtained at the end of the proof.

**Step 2**. (ii) \( \rightarrow \) (i). This implication is straightforward.

**Step 3**. (ii) \( \rightarrow \) (iii). We compute \( w(r) \) using formula (23), by finding \( x(y)^k \) and substituting \( e^t \) by \( r \). Noticing that \( y = \frac{n-2k}{k} (1 + o(1)) \), we obtain the expression for \( w(r) \) from (iii). Furthermore, by (22), \( w'(r) = -\frac{w(r)y(t)}{r} = -\frac{w n-2k}{k} (1 + o(1)) \) which implies the expression for \( w'(r) \) from item (iii).

Let us show now that the constant \( c_1 \) from Step 1 is strictly positive. From the expression for \( w' \) in item (iii), it follows that \( r^{n-k}w'(r)^k = c_3^k(1 + o(1)) \). Furthermore, integrating (21) gives
\[ r^{n-k}w'(r)^k = r_0^{n-k}w'(r_0)^k + \int_{r_0}^{r} s^{n-1} \eta_{n,k} \rho(s)(-w(s))^9 ds \geq r_0^{n-k}w'(r_0)^k > 0, \quad \text{where} \quad r_0 > 0. \]
Passing to the limit as \( r \to +\infty \), we obtain that \( c_4^k > 0 \). But \( c_4 \sim c_1^{\frac{1}{n-k}} \), which implies that \( c_1 > 0 \).

**Step 4**. (iii) \( \rightarrow \) (i). Since \( w'(r) = -\frac{w(r)y(t)}{r} \), one obtains \( y(t) = \frac{n-2k}{k} (1 + o(1)) \). By (23),
\[ x(t) = c_n^{-1} t^{2k} \rho(r)(-w)^9 y(t)^{-k} = c_1 r^{(1 + o(1))}, \quad \text{with} \quad r = e^t. \]
The above representations for \( x(t) \) and \( y(t) \) imply (i).

**Step 5**. (i) \( \rightarrow \) (iv). It follows from Lemma 3.2 that \( \varphi(t) \in G_+ \) for all \( t \in \mathbb{R} \).

**Step 6**. (iv) \( \rightarrow \) (i). Note that \( P_4^+ \) and \( P_4^+ \) are always in \( G_- \). Indeed, we already proved in Corollary 4.5 that \( P_4^+ \in G_- \). Further, we note that \( G_0 \) intersects the x-axis at \( (0, \frac{t}{k+1}(n-k)) \) and
\[ n + l_{\infty} < \frac{q+1}{k+1}(n - 2k) \] by (45). Now it follows from Corollary 4.5 that only point \( P_2 \) can belong to the \( \omega \)-limit set of the orbit associated to a solution of \( \hat{P} \).

It remains to show (7). By Lemma 4.3, in a neighborhood of \( P_2 \), \( y \) is a function of \( x \) (denoted by \( \hat{y}(x) \)). From (6), we observe that

\[
\lim_{x \to 0^+} \frac{\hat{y}(x) - \frac{n-2k}{k}}{x} = \lim_{t \to +\infty} \frac{y(t) - \frac{n-2k}{k}}{x(t)} = \frac{c_2}{c_1}.
\]

Formula (7) holds now by Lemma 4.3. Furthermore, since \( \hat{y}'(0) = \frac{c_2}{c_1} \), we find that \( c_2 = c_1 \left( \frac{k^2 + n - 2k}{n - 2k} \right) \).

At the same time, finding \( c_2 \) concludes the proof of Step 1. \( \square \)

### 4.3.2 Examples of \( P_2 \)-solutions

Here we give examples of radial \( P_2 \)-solutions to Problem \( \hat{P} \) for some explicitly given weight functions \( \rho(|x|) \). Our examples also contain explicit expressions for the solutions. Remark that in both examples, the asymptotic representations at a neighborhood of \( +\infty \), obtained in Theorem 2.3, agree with the explicit formulas for the respective solutions.

**Example 1.** Let \( q > k \) and \( n > 2k \). Define

\[ u(x) = w(r) = -(1 + r^2)^{-\theta}, \quad r = |x|, \]

where \( \theta = \frac{n-2k}{2k} \). An easy calculation shows that

\[ S_k(D^2w) = c_{n,k} \frac{(2\theta)^k(n-k)}{1 + r^2} (1 + r^2)^{-k(\theta+1)}. \]

Now put \( k(\theta + 1) = \theta q \), so that \( \theta = \frac{k}{q-1} \). Therefore,

\[ w(r) = -(1 + r^2)^{-\frac{n-2k}{2k}} \]

is a \( P_2 \)-solution of \( S_k(D^2w) = \rho(|x|)(-w)^q, \ x \in \mathbb{R}^n, \) with \( \rho(|x|) = c_{n,k} \frac{(n-2k)^k(n-k)}{1 + |x|^2} \). In this case, \( R(r) = -\frac{2r^2}{1 + r^2} \). Hence, \( l_0 = 0 \) and \( l_{\infty} = -2 \).

**Example 2.** Let \( c > 0 \) any constant and \( \sigma > 0 \). Consider the family of “Bliss functions”

\[ w_c(|x|) = -\frac{K}{(c + |x|)^{\frac{2k+2}{k}}} \]

where 

\[ K = \left[ c_{n,k}(n + \sigma) \left( \frac{n-2k}{k} \right)^k \right]^{\frac{n-2k}{n+\sigma(k+1)}}. \]

Then \( w_c \) is a \( P_2 \)-solution of

\[ S_k(D^2w) = |x|^q (-w)^q, \ x \in \mathbb{R}^n, \]

where \( q = q^*(k, \sigma) = \frac{(n+2)k+\sigma(k+1)}{n-2k} \). In this case, we have \( l_0 = l_{\infty} = \sigma \).

### 4.4 \( P_3^+ \)-solutions

In this section, we characterize \( P_3^+ \)-solutions.

#### 4.4.1 \( P_3^+ \)-solutions of algebraic fast decay: Proof of Theorem 2.4

Here we consider the case \( l_{\infty} < -2k \) or \( \delta > 0 \), the necessary condition when fast decay \( P_3^+ \)-solutions may exist.
Proof of Theorem 2.4. Step 1. (i) → (ii): The expression \( x(t) = \nu_+(1 + o(1)) \) follows immediately from the definition of a \( P^+_\delta \)-solution. Next, define

\[
z(t) = \frac{\nu_+ - x}{y}.
\]

One can rewrite the ODE (47) as follows:

\[
\left( \frac{1}{y} \right)' - \delta \frac{1}{y} - \frac{z}{k} + 1 = 0,
\]

whose solution is

\[
y(t) = \frac{e^{-\delta t}}{e^{-\delta t} y(t_0)^{-1} + \int_{t_0}^t e^{-\delta s} \left( \frac{z(s)}{k} - 1 \right) ds}, \quad t_0 \in \mathbb{R}. \tag{61}
\]

Let us determine the behavior of the term \( e^{-\delta t} z(t) \). Differentiating \( z(t) \) and taking into account that \( \nu_+ - x = zy \), we obtain

\[
\dot{z} = -\frac{x}{y} - z \frac{\dot{y}}{y} = -\frac{x}{y} (yz + \zeta(t) - qy) - z \left( -\delta + \frac{x - \nu_+}{k} + y \right)
\]

\[
= \left( \delta + \frac{\nu_+ - x}{k} - y - x \right) z + qx - \frac{x}{y} \zeta(t). \tag{62}
\]

Next, for a differentiable function \( \varphi \), we have

\[
(z \varphi)' = z' \varphi + \varphi' z = \left( \delta + \frac{\nu_+ - x}{k} - y - x + \frac{\varphi'}{\varphi} \right) z \varphi + qx \varphi - \frac{x}{y} \zeta \varphi. \tag{63}
\]

Taking \( \varphi(t) = e^{-(\delta - \varepsilon) t} \), we obtain

\[
(z e^{-(\delta - \varepsilon) t})' = \left( \delta + \frac{\nu_+ - x}{k} - y - x + \varepsilon \right) z e^{-(\delta - \varepsilon) t} + qx e^{-(\delta - \varepsilon) t} + \frac{x e^{-\delta t}}{y} e^{\varepsilon t} \zeta(t).
\]

Choose \( \varepsilon \in (0, \min\{ \frac{\pi}{2}, \nu_+, \delta \}) \). With this choice, \( e^{2\varepsilon t} \zeta(t) \to 0 \) as \( t \to +\infty \) and \( \lim_{t \to +\infty} (\nu_+ - x - y - x + \varepsilon) = -\nu_+ + \varepsilon < 0 \). We need to analyze the factor \( \frac{e^{-\delta t}}{y} \) appearing in the last term. Equation (47) implies

\[
\left( \frac{e^{-(\delta + \varepsilon) t}}{y} \right)' = \frac{e^{-(\delta + \varepsilon) t}}{y} \left( \frac{\nu_+ - x}{k} - \varepsilon \right) - e^{-(\delta + \varepsilon) t}.
\]

By Lemma 4.1, \( \lim_{t \to +\infty} \frac{e^{-(\delta + \varepsilon) t}}{y} = 0 \). Therefore, for all \( \varepsilon > 0 \),

\[
\frac{e^{-\delta t}}{y} = e^{\varepsilon t} o(1). \tag{64}
\]

By Lemma 4.1, \( \lim_{t \to +\infty} z e^{-(\delta - \varepsilon) t} = 0 \). This implies that as \( t \to +\infty \),

\[
|z| e^{-\delta t} = e^{-\varepsilon t} o(1).
\]

Therefore, the integral in the denominator of (61) converges. There are two choices:

\[
e^{-\delta t_0} y(t_0)^{-1} + \int_{t_0}^{+\infty} e^{-\delta s} \left( \frac{z(s)}{k} - 1 \right) ds \neq 0 \quad \text{for some } t_0 \in \mathbb{R} \tag{a}
\]

\[
e^{-\delta t} y(t)^{-1} + \int_t^{+\infty} e^{-\delta s} \left( \frac{z(s)}{k} - 1 \right) ds = 0, \quad \text{for all } t \in \mathbb{R}. \tag{b}
\]
If \( (a) \) holds, we obtained (8). Remark that the constant \( c \) is positive by (22). In the case \( (b) \),
\[
\left| \frac{1}{ye^{st}} \right| \leq \frac{1}{k} \int_t^{+\infty} |z|e^{-\delta s} ds + \int_t^{+\infty} e^{-\delta s} ds \leq K e^{-\epsilon t}
\]
for some constant \( K > 0 \), depending on \( \epsilon \). On the other hand, (43) implies that for any \( \xi > 0 \),
\[
(ye^{(s-\xi)t})' = \left( \frac{x - \nu_+}{\kappa} + y - \xi \right) ye^{(s-\xi)t}.
\]
By Lemma 4.1, \( \lim_{t \to +\infty} ye^{(s-\xi)t} = 0 \). Therefore, for all \( \xi > 0 \),
\[
y e^{\delta t} = e^{\xi t} o(1).
\]
Inequality (65) can be written as \( ye^{\delta t} o(1) = e^{\xi t} \). Substituting there \( ye^{\delta t} \) from the previous identity, we obtain
\[
e^{(\xi-\xi) t} = o(1) \quad \text{for all } \varepsilon, \xi > 0 \text{ sufficiently small.}
\]
Clearly, the above identity cannot hold for all (even sufficiently small) \( \varepsilon, \xi > 0 \). This implies that the case \( (b) \) cannot be realized. Thus, we obtained \( (ii) \).

**Step 2.** (ii) \( \to \) (i). This implication is straightforward.

**Step 3.** (ii) \( \to \) (iii). We apply formula (23). First, we note that \( yk x = \nu_+ e^k e^{-\delta t}(1 + o(1)) \). Substituting this expression into (23), by Lemma 4.6, we obtain
\[
w(r) = -c_1 (\rho(r) e^{2k + \delta k})^{-\frac{1}{q-\nu}} (1 + o(1)) = -c_1 (1 + o(1)),
\]
where \( c_1 = \left( \frac{\nu_+ - c_0, k e^k}{\kappa} \right)^{\frac{1}{q-\nu}} \). Above, we used the identity \( 2k + \delta k = -l_\infty \) following from the definition of \( \delta \). Next, by (22), \( w'(r) = -w(r) y (\ln r) \nu^{-1} \) which immediately implies the second expression in item (iii). Integrating the expression for \( w' \) from \( r \) to \( +\infty \) and taking into account that \( w(+\infty) = -c_1 \), we obtain the expression for \( w(r) \) in (iii).

**Step 4.** (iii) \( \to \) (i). It is straightforward to obtain (8) by the second formula in (22) and the expressions for \( w \) and \( w' \) in (iii). Indeed, \( y(t) = r \frac{w'}{w} = \frac{2e}{\kappa} e^{-\delta t}(1 + o(1)) \). Furthermore, by the first formula in (22),
\[
x(t) = e^{-\frac{1}{k} \nu_+ p(r)} \left( \frac{-w}{w'} \right)^k (-w)^{q-k} = \nu_+ (1 + o(1)).
\]
Thus, we proved that \( (x(t), y(t)) \) tends to \( P_{3}^+ \) as \( t \to +\infty \).

**Step 5.** (i) \( \to \) (v). Note that \( \lim_{t \to +\infty} W(x(t), y(t)) = -\frac{x - \nu_+}{\kappa} + \frac{\nu_+ - 1}{\nu_+ - \delta} < 0 \) which immediately implies (v).

**Step 6.** (i) \( \to \) (iv). Note that \( W(x, y) < 0 \) is equivalent to \( \frac{k}{\nu_+ - 2\xi} y - 1 < -\frac{x}{\nu_+ - 2\xi} \). The latter implies that \( G(x, y) < 0 \). Therefore, \( W_- \subset G_- \).

**Step 7.** (iv) \( \to \) (i). By Corollary 4.5, the \( \omega \)-limit set of a trajectory of (43) is either \( P_2 \) or \( P_{3}^+ \). Suppose it is \( P_2 \). Then, item (iv) of Theorem 2.3 contradicts to item (iv) of the current theorem. Therefore, \( \varphi(t) \) is a \( P_3 \)-solution.

**Step 8.** (v) \( \to \) (i). This follows from the inclusion \( W_\infty \subset G_- \) shown above.

Now we let (p.6) be in force and let us obtain (9).

**Step 9: Asymptotic representation for x under (p.6)-(1).** Recall that \( \nu = \lim_{t \to +\infty} e^{\delta t} \zeta(t) \). Since \( \nu_+ > \delta \), from (62), by Lemma 4.1, we obtain that \( \lim_{t \to +\infty} z(t) = \frac{(q - \nu_+)(\nu_+ - \delta)}{\nu_+ - \delta} \). Therefore,
\[
\nu_+ - x = yz = \frac{(q - \nu_+)(\nu_+ - \delta)}{\nu_+ - \delta} e^{-\delta t}(1 + o(1)),
\]
which is the same as the first expression in (9).
Step 10: Asymptotic representation for x under (ρ.6)-(2). Applying (63) with \( \varphi(t) = \frac{1}{e^{r\xi(t)}} \) and introducing \( \tilde{z} = \frac{y}{e^{r\xi}} \), we obtain

\[
z' = \left( \frac{\nu_+ - x}{k} - y - x - \frac{\zeta'}{\zeta} \right) \tilde{z} + \frac{q_x}{e^{r\xi}} \tilde{z} + \frac{x(1 + o(1))}{c}.
\]

By Lemma 4.1, \( \lim_{t \to +\infty} \tilde{z} = -\frac{\nu_+}{c(\nu_+ - \nu)} \). This implies that

\[
z(t) = -e^{\delta t \xi(t)} \frac{\nu_+}{c(\nu_+ - \nu)} (1 + o(1)),
\]

\[
\nu_+ - x = zy = -\frac{\nu_+}{\nu_+ - \nu} \xi(t)(1 + o(1)).
\]

Hence, the second formula in (9) holds.

Step 11: Obtaining (10). First, we note that under (ρ.6)-(1), \( x \) can be represented as a function of \( y \) which we denote by \( \hat{x}(y) \). Indeed, it follows from (43) that in a neighborhood of \( +\infty \), \( \hat{x} \) is a function of \( y \). Let us show now that under (ρ.6)-(2), in a neighborhood of \( +\infty \), \( y \) can be represented as a function of \( x \). Indeed, \( e^{-\delta t} = o(\xi(t)) \) and \( \hat{x} = x( -\frac{\nu_+}{\nu_+ - \nu} \xi(t) + o(\xi(t))) \). Therefore, in a neighborhood of \( +\infty \), \( \hat{x} < 0 \) and we can express \( t \) as a function of \( x \). Finally, we compute \( \hat{x}'(0) \) as \( \lim_{y \to 0+} \frac{\hat{x}(y) - \nu_+}{y} = \lim_{t \to +\infty} \frac{x(t) - \nu_+}{\nu_+} \) and \( \hat{y}'(\nu_+) \) as \( \lim_{x \to \nu_+} \frac{\hat{y}(x)}{x - \nu_+} = \lim_{t \to +\infty} \frac{\nu(t)}{x(t) - \nu_+} \) by (8) and (9). These computations imply (10). \( \square \)

4.4.2 \( P_3^+ \)-solutions of slow log-like decay: Proof of Theorem 2.5

Recall that the necessary condition for slow-decay \( P_3^+ \)-solutions to exist is \( l_\infty = -2k \) (or \( \delta = 0 \)).

Proof of Theorem 2.5. Step 1. (i) \( \to \) (ii). Formula (61) with \( \delta = 0 \) implies

\[
y = \frac{1}{t \varphi(y(t))} + \frac{1}{t} \int_{t_0}^{t} \left( \frac{2(s)}{k} - 1 \right) ds.
\]

(67)

Let us derive an asymptotic representation for \( z(t) \) by formula (62). By (64), for all \( \varepsilon > 0 \),

\[
\frac{1}{y} = e^{\varepsilon\delta}(1).
\]

Note that the condition \( k < \frac{\alpha}{2} \) is equivalent to the condition \( \nu_+ > 0 \). From (62) and Lemma 4.1, it follows that

\[
z(t) = q(1 + o(1)).
\]

Therefore, in (67), the integral diverges. By L’Hopital’s rule,

\[
y(t) = \frac{1}{t} \frac{k}{q - k}(1 + o(1)).
\]

Finally, \( \nu_+ - x = n - 2k - x = zy = \frac{1}{t} \frac{k}{q - k}(1 + o(1)). \)

Step 2. (ii) \( \to \) (i). This implication is straightforward.

Step 3. (ii) \( \to \) (iii). Note that \( x(1 + o(1)) = (n - 2k)(1 + o(1)) \) and \( y^k x = \frac{1}{k} \left( \frac{k}{q - k} \right)^k (n - 2k)(1 + o(1)) \). This immediately implies the expression for \( w(r) \) in (iii). The second expression in (iii) is implied by the formula \( w'(r) = -w(r)y(\ln r)\)\(^{-1} \).

Step 4. (iii) \( \to \) (i). It is straightforward to obtain the expression for \( y \), given in item (ii), by the second formula in (22) and the expressions for \( w \) and \( w' \) in (iii). Furthermore, by (66), \( x(t) = \nu_+(1 + o(1)) \). This implies that \( (x(t), y(t)) \to P_3^+ \) as \( t \to +\infty \).

Step 5. (ii) \( \to \) (v). Note that \( W_- = \left\{ y + \frac{x}{k} < \frac{n - 2k}{k} \right\} \). Representation (ii) implies

\[
y + \frac{x}{k} = \frac{n - 2k}{k} - \frac{q}{q - k} \frac{1}{t}(1 + o(1)) + \frac{k}{q - k} \frac{1}{t}(1 + o(1)) = \frac{n - 2k}{k} - \frac{1}{t}(1 + o(1))
\]

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which immediately implies (v).

Step 6. (v) → (iv). This implication is straightforward since $W_- \subset G_-.$

Step 7. (iv) → (i). The argument of Step 7 of the proof of Theorem 2.4 works here without changes.

It remains to prove (11). The second equation in $(TS_{q, n})$, along with the representations for $x(t)$ and $y(t)$ from item (ii), implies that $\dot{y} = -y(\frac{1}{2} + o(\frac{1}{x}))$. Therefore, $\dot{y} < 0$ at some neighborhood of $+\infty$ and we can express $t$ as a function of $y$. This implies that $x$ is a function of $y$, which we denote by $\tilde{x}(y)$. By using the representations for $x(t)$ and $y(t)$, we obtain

$$\tilde{x}'(0) = \lim_{y \to 0} \frac{\tilde{x}(y) - (n - 2k)}{y} \frac{x(t) - (n - 2k)}{y(t)} = -q.$$  

From here and item (ii), we obtain the existence of the limit $\lim_{y \to 0} \tilde{x}'(y) = \lim_{t \to +\infty} \frac{\tilde{x}(0)}{y} = \tilde{x}'(0).$ By the continuity of $\tilde{x}'(y)$ in 0, we obtain that $\tilde{x}(y)$ is decreasing in a neighborhood of 0; therefore, $y$ is a function of $x$ in a neighborhood of $n - 2k$. As before, we denote this function by $\hat{y}(x)$. Formula (11) follows now from the above expression for $\tilde{x}'(0).$ □

4.5 $P^+_4$-solutions

It was established in Corollary 4.5 that $P^+_4$-solutions can exist only if $l_\infty > -2k$ or, equivalently, $\delta < 0$. Below, we prove Theorem 2.6 characterizing $P^+_4$-solutions.

4.5.1 Proof of Theorem 2.6

Proof. Step 1: (i) → (ii). Since $x(t) = \tilde{x}(1 + o(1))$ and $y(t) = \hat{y}(1 + o(1))$, item (ii) follows from (22), (23), and Lemma 4.6.

Step 2: (ii) → (i). From (22) and (23), it follows that $x(t) = \tilde{x}(1 + o(1))$ and $y(t) = \hat{y}(1 + o(1))$, which implies (i).

Step 3: (i) → (iii). As it was shown in Corollary 4.5, $P^+_4 \in G_-$ and is located in the first quadrant (since $l_\infty > -2k$). This proves (iii).

Step 4: (iii) → (i). By Corollary 4.5, $w(r)$ can be either $P_2$ or $P^+_4$-solution. By Theorem 2.3, if $w(r)$ is a $P_2$-solution, then $\varphi(t) \in G_+$ for all $t$. Therefore, $w(r)$ is a $P^+_4$-solution.

Step 5: (i) → (iv) under condition (12). By the results of Subsection 4.1, condition (12) implies that the roots of the characteristic equation of the linearized system $(LV_{S_{q, n}})$, denoted by $\lambda_1$ and $\lambda_2$, are real, negative, and different, so $P^+_4$ is a stable node. Let $\lambda_1 > \lambda_2$. In a neighborhood of $P^+_4$, one has the representation of $(TS_{q, n})$ as a time-dependent perturbation of $(LV_{S_{q, n}})$ given by (51) with $\tilde{x} = x - \hat{x}, \tilde{y} = y - \hat{y}$, where $(\tilde{x}, \tilde{y})$ is given by (44) and the matrix $A$ given by (46). As in the case of $P_2$-solutions, we rewrite (51) with respect to $\psi(t) = (\tilde{x}(t), \tilde{y}(t))$ as follows:

$$\psi'(t) = A\psi(t) + \vartheta(\psi(t)) + \eta(t, \psi(t)),$$

where $\vartheta(\psi(t))$ and $\eta(t, \psi(t))$ denote the two last terms (respectively) on the right-hand side.

Let $v_1$ and $v_2$ be the unit eigenvectors corresponding to $\lambda_1$ and $\lambda_2$, respectively. Since the roots are real and different, $v_1$ and $v_2$ are non-colinear. Therefore, each vector $z \in \mathbb{R}^2$ is a linear combination $z = z_1v_1 + z_2v_2$. Define the projection operators $P_1z = z_1v_1$ and $P_2z = z_2v_2$. By the continuity of $P_1$ and $P_2$, there exists a constant $C > 0$ such that

$$|P_iz| \leq C|z|, \quad i = 1, 2.$$  

In the same manner as in the proof of Theorem 2.3, one can write equation (54). Setting $\tau = \tilde{\tau}$, we
$$e^{-\lambda_1 t}\psi(t) = (e^{-\lambda_1 \tau} P_1 + e^{-\lambda_2 \tau} P_2)\psi(\tau) + e^{(\lambda_2 - \lambda_1) t} P_2 \psi(\tau) + \int_{\tau}^{t} e^{-\lambda_1 s} P_1 (\vartheta(\psi(s)) + \eta(s, \psi(s)))\,ds$$

$$+ e^{-\lambda_1 t} \int_{\tau}^{t} e^{\lambda_2 (t-s)} P_2 (\vartheta(\psi(s)) + \eta(s, \psi(s)))\,ds = (e^{-\lambda_1 \tau} P_1 + e^{-\lambda_2 \tau} P_2)\psi(\tau) + e^{(\lambda_2 - \lambda_1) t} P_2 \psi(\tau) + I(t) + J(t).$$

(68)

Note that $$\lim_{t \to +\infty} e^{(\lambda_2 - \lambda_1) t} P_2 \psi(\tau) = 0$$. Let us show that $$I(t)$$ and $$J(t)$$ converge to finite limits. To do this, we find an exponentially decaying estimate for $$|\psi(t)|$$ by using Lemma 7.1 from [3]. First, we find $$\sigma > 0$$ such that

$$\beta := \frac{C\sigma}{|\lambda_1|} < 1.$$

(69)

Estimate (58) can be obtained likewise. This estimate and equation (68) imply that there exist a constant $$K_1 > 0$$ such that

$$|\psi(t)| \leq K_1 e^{\lambda_1 t} + C \sigma \int_{\tau}^{t} e^{\lambda_1 (t-s)}|\psi(s)|ds.$$

By Lemma 7.1 from [3], there exist a constant $$K_2 > 0$$ such that $$|\psi(t)| \leq K_2 e^{(\frac{C\sigma}{|\lambda_1|} + \lambda_1) t}$$. Furthermore, estimates (60) can be obtained likewise: namely, $$|\vartheta(\psi(t))| \leq (1 + q)|\psi(t)|^2$$ and $$|\eta(t, \psi(t))| \leq K_3 e^{-\theta t} |\psi(t)|$$ for some constant $$K_3 > 0$$. Therefore,

$$|I(t)| \leq K_4 \int_{\tau}^{t} (e^{(\frac{2C\sigma}{|\lambda_1|} + \lambda_1) s} + e^{(\frac{C\sigma}{|\lambda_1|} - \theta) s})\,ds,$$

$$|J(t)| \leq K_5 e^{(\lambda_2 - \lambda_1) t} \int_{\tau}^{t} (e^{(\frac{2C\sigma}{|\lambda_1|} + 2\lambda_1 - \lambda_2) s} + e^{(\frac{C\sigma}{|\lambda_1|} - \theta + \lambda_1 - \lambda_2) s})\,ds.$$

(70)

where $$K_4, K_5 > 0$$ are constants. In (69), we can take $$\sigma > 0$$ even smaller so that $$\frac{2C\sigma}{|\lambda_1|} + \lambda_1 < 0$$ and $$\frac{C\sigma}{|\lambda_1|} - \theta < 0$$. With this choice of $$\sigma$$, the integral $$I(+\infty)$$ converges, so $$\lim_{t \to +\infty} I(t)$$ is finite. Furthermore, integrating in the right-hand side of (70) and taking into account the factor $$e^{(\lambda_2 - \lambda_1) t}$$, we conclude that $$\lim_{t \to -\infty} J(t) = 0$$. Therefore, $$\lim_{t \to +\infty} e^{-\lambda_1 t} \psi(t)$$ exists. We denote it by $$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$.

Step 6: (iv) $$\to$$ (i) under condition (12). This implication is straightforward. 

4.6 Weights satisfying the entire set of assumptions (ρ.1)−(ρ.6)

Example 1 below describes a construction of the weight $$\rho$$ based on the choice of the parameters $$\delta$$ and $$\vartheta$$ and the function $$\psi$$ with values in a closed subinterval of $$(0, +\infty)$$.

4.6.1 Example 1: General construction of the weight function $$\rho(r)$$.

Here we describe how to choose the function $$R(r)$$, and then give a formula explicitly defining $$\rho(r)$$ via $$R(r)$$, so that assumptions (ρ.1)−(ρ.3) and (ρ.4)−(ρ.6) are satisfied.

Let $$R(r)$$ be a continuous function on $$(0, +\infty)$$ such that $$l_0 = \lim_{r \to 0} R(r)$$ exists and $$l_0 \geq R(r)$$ for all $$r > 0$$. We specify properties of $$R(r)$$ at neighborhoods of $$+\infty$$ and 0. We note that these additional properties in a neighborhood of 0 are needed only if we want to satisfy (ρ.3); in the current section on $$P_{2r}^-$$, $$P_{3r}^+$$, and $$P_{4r}^+$$-solutions, we do not use (ρ.3). Outside of neighborhoods of 0 and $$+\infty$$, the behavior of $$R(r)$$ can be arbitrary.

First of all, choose parameters $$\delta \in \mathbb{R}$$ and $$\vartheta > 0$$ satisfying

$$\delta < \min \left\{ \frac{n - 2k}{k + 1}, \vartheta \right\}$$ or $$\vartheta < \min \{\delta, n - 2k - \delta k\}$$ or $$\delta = \vartheta < \frac{n - 2k}{k + 1}.$$
The latter is equivalent to the condition \( \nu_0 > \min\{\delta, \theta\} \) from (\( \rho, \delta' \)). Define \( l_\infty = -\delta k - 2k \) and choose \( l_0 \geq 0 \) in such a way that \( q > q^*(k, l_0) \).

Construction of \( R(r) \) at a neighborhood of +\( \infty \). Choose \( \psi(r) \) (in a neighborhood of +\( \infty \)) taking values in a closed subinterval of (0, +\( \infty \)) Define \( R(r) = r^{-\theta} \psi(r) + l_\infty \). If \( \theta = \delta \), we also assume the existence of a finite limit \( \lim_{r \to +\infty} \psi(r) \).

Construction of \( R(r) \) at a neighborhood of 0. This step is necessary only if we need to satisfy assumption (\( \rho, 3 \)); otherwise, we do not need to specify the behavior of \( R(r) \) in a neighborhood of 0. Choose \( R(r) \) in such a way that

\[
\text{the integral } \int_0^r \frac{R(s) - l_0}{s} \, ds \text{ converges.}
\]

Remark that the convergence of the above integral implies that \( \lim_{r \to 0} R(r) = l_0 \). For the function \( K(r) = r^{-\theta} \rho(r) \) we have

\[
\ln K(r) = \ln K(\epsilon) + \int_\epsilon^r \frac{R(t) - l_0}{t} \, dt.
\]

By the above choice of \( R(\cdot) \), \( \lim_{r \to 0} K(\epsilon) > 0 \) exists; it will be referred to below as \( K(0) \).

Formulas for \( \rho(r) \). There are several ways to compute \( \rho(r) \) if we know \( R(r) \). Indeed, the definition of \( R(\cdot) \) is equivalent to \( (\ln \rho(r))' = \frac{R(\cdot)}{r} \). If we need to satisfy (\( \rho, 3 \)), then \( \rho(r) \) can be computed by formula \( (14) \). If we do not need to satisfy (\( \rho, 3 \)), then we use Lemma 4.6.

Introduce the function \( K(\cdot) = \rho(\cdot) r^{-\theta} \). One easily verifies that \( (\ln K(\cdot))' = \frac{R(r) - l_\infty}{r} \). By (\( \rho, 5 \)), \( \int_r^{+\infty} \frac{R(s) - l_\infty}{s} \, ds \to 0 \) as \( r \to +\infty \). Integrating the aforementioned identity from \( r \) to +\( \infty \) and noticing that, by Lemma 4.6, \( K(\cdot) \) is strictly increasing, we obtain the formula for computing \( \rho(r) \):

\[
\rho(r) = c_r r^{\frac{\sigma}{\alpha}} \exp \left\{ \int_r^{+\infty} \frac{l_\infty - R(s)}{s} \, ds \right\}.
\]

Let us verify the assumptions (\( \rho, 1 \)–(\( \rho, 3 \)) and (\( \rho, 4 \)–(\( \rho, 6 \)). First, we note that (\( \rho, 1 \), (\( \rho, 2 \) and (\( \rho, 4 \), (\( \rho, 5 \) are clearly satisfied. Assumption (\( \rho, 6 \)) is fulfilled by Corollaries 4.8 and 4.9. Finally, if we are interested in weights satisfying (\( \rho, 3 \)), we notice that by \( (14) \) and since \( R(r) \leq l_0 \) for all \( r > 0 \), \( K(r) \leq K(0) \).

4.6.2 Example 2: Concrete examples of \( \rho(r) \).

Define \( \rho(r) = \frac{\alpha r^\beta}{\alpha + \gamma} \), where \( \alpha, \bar{\alpha}, \beta, \gamma \) are positive constants. In this case, \( R(r) = \beta - \frac{\gamma r^\gamma}{\alpha + \gamma} \), so that \( l_0 = \beta \) and \( l_\infty = \beta - \gamma \). Furthermore, \( R(r) - l_\infty = \frac{\gamma r^\gamma}{\alpha + \gamma} \), and hence, \( \theta = \gamma \) and \( \psi(r) = \frac{\gamma}{\alpha + \gamma} \). By Corollary 4.8, we have to choose \( \beta \) and \( \gamma \) in agreement with the inequality \( n + \beta - \gamma > \min\{\delta, \gamma\} = \delta \) since \( \delta = \frac{\gamma - \bar{\alpha}}{\alpha} - 2 \). Therefore, \( \beta \) and \( \gamma \) have to satisfy \( \beta - \gamma > - \frac{k(n+2)}{k+1} \).

Another example is \( \rho(r) = r^\sigma \), where \( \sigma > 0 \) and such that \( q > q^*(k, \sigma) \). In this case \( R(r) = \sigma \), so (\( \rho, 1 \)–(\( \rho, 6 \) are obviously satisfied.

Appendix

Derivation of the Lotka-Volterra system

We transform the equation \( c_{n,k} r^{1-n}(r^{n-k}(w_r)^k)_r = f(r, w + 1) \), where \( f(r, w + 1) := \lambda \rho(r)(-w)^q \), into a Lotka-Volterra-type system. For this, define \( \Phi(r) := r^{n-k}(w_r)^k \), \( g(r, w) := c_{n,k} r^{n-1} f(r, w + 1) \), and introduce the variables

\[
x(t) = r^k c_{n,k} f(r, w + 1) \frac{w_r}{(w_r)^k}, \quad y(t) = r \frac{w_r}{-w}, \quad t = \ln(r).
\]
The change of variable (71) was first introduced by Milne [14, 15] for the case \( k = 1 \) and \( \rho \equiv 1 \). Derivating \( x(t) \) in (71), we obtain

\[
x' = \frac{dx}{dt} = r \left( \frac{rg}{\Phi} \right)_r = \frac{r}{\Phi^2} \left[ \Phi (r(g_r + g_w w_r) + g) - rg \Phi_r \right]
\]

\[
= \frac{rg_w w_r}{\Phi} + \frac{r^2 g_r + rg}{\Phi} - \left( \frac{rg}{\Phi} \right)^2
\]

\[
= -u g_w r w_r \Phi - w + \left( \frac{g_r}{g} + 1 \right) \frac{rg}{\Phi} - \left( \frac{rg}{\Phi} \right)^2
\]

\[
= -w \frac{g_w y}{y} + \left( \frac{g_r}{g} + 1 \right) x - x^2
\]

\[
= x \left( \frac{g_r}{g} + 1 \right) - x - w \frac{g_w y}{y} = x(\nu(t) - x - gy)
\]

where the last equality follows from the definition of \( g \). Next, we note that \( c_n^{-1} r^{n-1} f(r, w+1) = \Phi_r \). Furthermore,

\[
\Phi_r = r^{n-2k}(k(-wy)^{k-1}(-wy e^{-t} - w_r y)) + (n-2k) r^{n-2k-1}(-wy)^{k-1}
\]

\[
= r^{n-2k}\left( -k \frac{y'}{y}(-wy)^{k-1} - kw_r y(-wy)^{k-1} \right) + (n-2k) r^{n-2k-1}(-wy)^{k-1}
\]

\[
= r^{n-2k}\left( \frac{k y'}{y}(-wy)^{k-1} - ky \left( -\frac{wy}{r} \right)(-wy)^{k-1} \right) + (n-2k) r^{n-2k-1}(-wy)^{k-1}
\]

\[
= r^{n-2k-1}(-wy)^{k-1}\left( \frac{y'}{y} - ky + n - 2k \right)
\]

\[
= r^{n-2k-1}(rw_r)^k\left( \frac{y'}{y} - ky + n - 2k \right).
\]

The definition of \( x(t) \) by (71) and the last equality imply that \( k \frac{y'}{y} - ky + n - 2k = x \), which is equivalent to \( y' = y \left( -\frac{n-2k}{k} + \frac{t}{k} + y \right) \).

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**References**

[1] A. Ambrosetti and G. Prodi. *A primer of nonlinear analysis*, volume 34 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.

[2] J. Batt, W. Faltenbacher, E. Horst, Stationary spherically symmetric models in stellar dynamics, Arch. Rational Mech. Anal. 93: 159–183, 1986.

[3] J. Batt, Y. Li, The positive solutions of the matukuma equation and the problem of finite radius and finite mass, Arch. Rational Mech. Anal. 198: 613–675, 2010.

[4] L. Caffarelli, L. Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian. *Acta Math.*, 155(3-4):261–301, 1985.
[5] F. Cao, J. Jiang. The Classification on the Global Phase Portraits of the Two-dimensional Lotka-Volterra System. J. Dyn. Diff. Equat. 20: 797-830. 2008

[6] Ph. Clément, R. Manásevich, and E. Mitidieri. Some existence and non-existence results for a homogeneous quasilinear problem. Asymptot. Anal., 17(1):13–29, 1998.

[7] G. Dai. Bifurcation and admissible solutions for the Hessian equation. J. Funct. Anal., 273(10):3200–3240, 2017.

[8] J.M. do Ó, E. Shamarova, E. da Silva, Singular solutions to $k$-Hessian equations with fast-growing nonlinearities, Nonlinear Anal. 222, Paper no 113000, 2022.

[9] J.F. de Oliveira, J.M. do Ó, and P. Ubilla. Existence for a $k$-Hessian equation involving supercritical growth. J. Differential Equations, 267(2):1001–1024, 2019.

[10] J. Jacobsen. Global bifurcation problems associated with $k$-Hessian operators. Topol. Methods Nonlinear Anal., 14:81–130, 1999.

[11] J. Jacobsen. A Liouville-Gelfand equation for $k$-Hessian operators. Rocky Mountain J. Math., 34(2):665–683, 2004.

[12] J. Jacobsen and K. Schmitt. The Liouville-Bratu-Gelfand problem for radial operators. J. Differential Equations, 184(1):283–298, 2002.

[13] L. Maia, G. Norber, and F. Pacella, A dynamical system approach to a class of radial weighted fully nonlinear equations. Commun. Partial Differ. Equ., 46(4): 573–610, 2021.

[14] E.A. Milne, The analysis of stellar structure. Mon. Not. R. Astron. Soc., 91: 4–55, 1930.

[15] E.A. Milne, The analysis of stellar structure. II. Mon. Not. R. Astron. Soc., 92: 610–643, 1932.

[16] Y. Miyamoto. Intersection properties of radial solutions and global bifurcation diagrams for supercritical quasilinear elliptic equations. NoDEA Nonlinear Differential Equations Appl., 23(2):Art. 16, 24, 2016.

[17] Y. Miyamoto, J. Sánchez, and V. Vergara. Multiplicity of bounded solutions to the $k$-Hessian equation with a Matukuma-type source. Rev. Mat. Iberoam., 35(5):1559–1582, 2019.

[18] Y. Miyamoto and K. Takahashi. Generalized Joseph-Lundgren exponent and intersection properties for supercritical quasilinear elliptic equations. Arch. Math. (Basel), 108(1):71–83, 2017.

[19] M.A. Navarro and J. Sánchez. Sharp estimates of semistable radial solutions of $k$-Hessian equations. Proc. Roy. Soc. Edinburgh Sect. A, 150(4):2083–2115, 2020.

[20] J. Sánchez and V. Vergara. Bounded solutions of a $k$-Hessian equation in a ball. J. Differential Equations, 261(1):797–820, 2016.

[21] J. Sánchez and V. Vergara. Bounded solutions of a $k$-Hessian equation involving a weighted nonlinear source. J. Differential Equations, 263(1):687–708, 2017.

[22] H.R. Thieme. Asymptotically autonomous differential equations in the plane. Rocky Mountain J. Math., 24(1):351–380, 1994. 20th Midwest ODE Meeting (Iowa City, IA, 1991).

[23] N.S. Trudinger and X.-J. Wang. Hessian measures. I. Dedicated to Olga Ladyzhenskaya. Topol. Methods Nonlinear Anal. 10(2): 225–239, 1997

[24] N.S. Trudinger and X.-J. Wang. Hessian measures. II. Ann. of Math. (2), 150(2):579–604, 1999.
[25] K. Tso. On symmetrization and Hessian equations. *J. Analyse Math.*, 52:94–106, 1989.

[26] K. Tso. Remarks on critical exponents for Hessian operators. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(2):113–122, 1990.

[27] X.-J. Wang. The $k$-Hessian equation. In *Geometric analysis and PDEs. Lectures from the C.I.M.E. Summer School held in Cetraro, June 11-16, 2007* (eds. A. Ambrosetti, S.-Y. A. Chang and A. Malchiodi). Volume 1977 of Lecture Notes in Mathematics, pp. 177-252 (Dordrecht: Springer; Florence: Fondazione C.I.M.E., 2009).

[28] X.-J. Wang. A class of fully nonlinear elliptic equations and related functionals. *Indiana Univ. Math. J.*, 43(1):25–54, 1994.

[29] Y. Wang and Y. Lei. On critical exponents of a $k$-Hessian equation in the whole space. *Proc. Roy. Soc. Edinburgh Sect. A*, 149(6):1555–1575, 2019.

[30] W. Wei. Uniqueness theorems for negative radial solutions of $k$-Hessian equations in a ball. *J. Differential Equations*, 261(6):3756–3771, 2016.

[31] W. Wei. Existence and multiplicity for negative solutions of $k$-Hessian equations. *J. Differential Equations*, 263(1):615–640, 2017.

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