CHAOS AND PERIODICITY ON STAR GRAPHS

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Abstract. For a continuous self-map of a star graph to be Li-Yorke chaotic and to have full periodicity, we prove some new sufficient conditions on the orbit of the center.

1. Introduction and main results

By the \( n \)-od, we mean a topological space \( X_n \) that is homeomorphic to the star graph of order \( n \), also known as the \( n \)-star \( S_n \). The triod is \( X_3 \), which is also known as the simple dendrite or as \( Y \). The center of \( X_n \) is its vertex of order \( n \), which we denote by \( o \). A proper branch of \( X_n \) is a connected component of \( X_n \setminus \{o\} \); fix an enumeration \( \beta_1, \ldots, \beta_n \) of these proper branches. A branch of \( X_n \) is the closure of a proper branch.

The original motivation for our results was to find a new generalization to the triod of Li and Yorke’s “Period three implies chaos” for the interval, and to avoid the uninteresting case of maps \( f: X_3 \to X_3 \) of the form \( \iota \circ g \circ r \) where \( r \) is a retraction of \( X_3 \) to \([0, 1]\), \( \iota \) is its unique right inverse, and \( g: [0, 1] \to [0, 1] \). As a special case of Corollary 1 below, we meet this goal: if \( f: X_3 \to X_3 \) and the orbit of \( o \) intersects each proper branch exactly once, then \( f \) is Li-Yorke chaotic and has full periodicity. (We assume all maps are continuous.)

Theorem 1. If \( f: X_n \to X_n \) and \( f^3(o) \) is not on the same branch as \( f(o) \), then \( f \) has points of all periods.

Theorem 2. If \( f: X_n \to X_n \) and \( f^3(o) \) is not on the same branch as \( f(o) \), then \( f \) scrambles an uncountable set.

Here \( S \subseteq X_n \) is scrambled \([3]\) by \( f: X_n \to X_n \) if, for all distinct \( p, q \in S \),

\[
\lim_{i \to \infty} \inf d(f^i(p), f^i(q)) = 0 < \lim_{i \to \infty} \sup d(f^i(p), f^i(q))
\]

where \( d \) is a metric compatible with the topology of \( X_n \). Because \( X_n \) is compact, whether \( S \) is scrambled or not does not depend on \( d \): the identity map from \( (X_n, d_1) \) to \( (X_n, d_2) \) is uniformly continuous for all pairs \( (d_1, d_2) \) of compatible metrics.

\( f: X_n \to X_n \) is called Li-Yorke chaotic if it scrambles an uncountable set.

Theorems 1 and 2 are proved in section 3. The proof of Theorem 2 mainly uses ideas from Li and Yorke’s scrambled set construction \([3]\). The proof of Theorem 1 leans more heavily on techniques involving “basic intervals” similar to Baldwin’s \([2]\).

Corollary 1. If \( n \geq 2 \), \( f: X_n \to X_n \), and the orbit of \( o \) has size \( n+1 \) and intersects every proper branch, then \( f \) is Li-Yorke chaotic and has full periodicity.

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For comparison, Alsedà and Moreno [1] proved that, for an arbitrary \( f: X_3 \to X_3 \), if the periodicity of \( f \) does not contain \( \{2, 3, 4, 5, 7\} \), then \( f \) may not have full periodicity. (By periodicity of \( f \), we mean the set of all \( f \)-periods of points in \( X_3 \).) If \( n = 3 \) in Corollary 1 then period 4 for an “interesting” orbit of the center implies full periodicity. In section 2, we compare Corollary 1 to Baldwin’s characterizations of periodicity sets of self maps of \( X_n \).

In section 2 we show that “\( n + 2 \)” can replace “\( n + 1 \)” in Corollary 1 at the cost of assuming \( n \geq 3 \) and weakening “full periodicity” to “all periods except 3.” We show by example that period 3 can indeed be avoided. We also give an example showing that all odd periods \( \geq 3 \) can be avoided if “\( n + 3 \)” replaces “\( n + 1 \).”

2. Relation to Baldwin’s characterization

Baldwin [2] defines, given a topological space \( X \), a preorder (\( i.e., \) transitive and reflexive relation) \( \leq_X \) of \( \mathbb{N} \) by \( p \leq_X q \) iff every \( f: X \to X \) with a point of period \( q \) also has a point of period \( p \). When \( X \) is the \( n \)-od, this preorder is also a partial order (\( i.e., \) antisymmetric) and is characterized in [2] by \( \leq_{X_n} = \bigcap_{b \leq n, \leq_i} \), where each \( \leq_i \) is a partial ordering defined below. Baldwin actually proves something stronger, that if \( f: X_n \to X_n \), then the set of \( f \)-periods is a finite union of sets each a \( \leq_i \)-initial segment for some \( t \leq n \).

First, \( \leq_1 \) is the Šarkovskii linear ordering defined by \( 2^i(2a + 1) \leq_1 2^i(2b + 1) \) iff

\[
\begin{align*}
&\bullet a = 0 = b \text{ and } i \leq j, \\
&\bullet a = 0 < b, \\
&\bullet 0 < a, b \text{ and } i > j, \text{ or} \\
&\bullet 0 < b < a \text{ and } i = j,
\end{align*}
\]

for all \( a, b, i, j \geq 0 \). ((\( \mathbb{N}, \leq_1 \)) has order type \( \omega + (\omega^*)^2 \).) Second, given \( n > 1 \) and \( m, k \geq 1 \):

\[
m \leq_{n} k \iff \begin{cases} 
\text{case } k = 1 : & m = 1 \\
\text{case } n \mid k : & m = 1 \text{ or } n \mid m \text{ and } m/n \leq_{1} k/n \\
\text{case } n \nmid k \neq 1 : & m \in \{1, k\} \cup \{ik + jn : i \geq 0 \text{ and } j \geq 1\}
\end{cases}
\]

((\( \mathbb{N}, \leq_{n} \)) is a disjoint union of \( n \) chains, one chain of type \( \omega + (\omega^*)^2 \) below \( n - 1 \) chains of type \( \omega^* \)).

Baldwin proves a result related to Corollary 1. To state it, we must first give his classification of the finite orbits of a given \( f: X_n \to X_n \) into types. If \( o \) is in a finite orbit \( O \) then \( O \) has type 1. (Thus, any \( f \) satisfying the hypotheses of Theorem 1 has an orbit of type 1.) On the other hand, if \( o \) is not in \( O \), then \( O \) has type \( p \) for each period \( p \) of the partial map \( f_O: [n] \to [n] \) where \( f_O(i) = j \) if \( O \cap \beta_i \) is nonempty and \( f \) maps to \( \beta_j \) the point in \( O \cap \beta_i \) closest to \( o \). Baldwin proved that if \( f \) has an orbit of size \( k \) that has type \( p \), then, for each \( m \leq_{p} k \), \( f \) has a point of period \( m \). Since, for example, \( x \leq_1 4 \iff x \in \{1, 2, 4\} \), the full periodicity of case \( n = 3 \) of Corollary 1 is not a corollary of Baldwin’s type-based analysis.

3. Proofs of Theorem 1 and 2

**Definition 1.** Given \( x, y \in X_n \), let the closed interval \([x, y]\) denote the unique arc with endpoints \( x \) and \( y \). Define open and half-open intervals as closed intervals with appropriate points removed. Given arcs \( I, J \) of \( X_n \) and \( g: X_n \to X_n \), we say that \( I \) \( g \)-covers \( J \) and write \( I \succ_g J \) if \( g(I) \supset J \).
The next two propositions are fundamental properties of star graphs that we will use without comment.

**Proposition 1.** If \(a, b \in X_n\) and \(g: X_n \to X_n\), then \([a, b] \supseteq [g(a), g(b)]\).

**Proof.** \(g([a, b])\) is connected and \([g(a), g(b)]\) is the smallest connected superset of \([g(a), g(b)]\). \(\square\)

**Definition 2.** Given an arc \(I \subset X_n\), a compatible ordering of \(I\) is a linear ordering of \(I\) such that the order topology on \(I\) equals the subspace topology inherited from \(X_n\).

Order each branch \(\overline{g_i}\) of \(X_n\) by the unique compatible ordering \(\leq_i\) such that \(o = \min(\overline{g_i})\). We will omit the subscript of \(\leq_i\) when safe to do so.

**Proposition 2.** If \(a, b, c \in X_n\) and \(o \not\in (a, b)\), then \(x \in [y, z]\) for some permutation \(x, y, z\) of \(a, b, c\).

**Proof.** The points \(a\) and \(b\) must be on the same branch, and if \(c\) is also on that branch, then the proposition is clearly true. If \(c\) is not in the same branch as \(a\) and \(b\), then, letting \([x \leq y] = \{a, b\}\), we have \([c, y] \supseteq [o, y] \supseteq [x, y]\). \(\square\)

**Definition 3.** Given \(g: X_n \to X_n\), by a \(g\)-cascade we mean a finite or infinite sequence of arcs \(I_0, I_1, I_2, \ldots\) such that for all \(i \geq 1\) we have \(I_{i-1} \supseteq g I_i\) and \(o \not\in I_i\) where \(Y^o\) denotes the interior of \(Y\). By a \(g\)-loop we mean a \(g\)-cascade \(I_0, \ldots, I_m\) such that \(I_m \supset I_0\).

**Lemma 1.** If \(I_0, I_1, I_2, \ldots\) is a \(g\)-cascade, then there is a descending chain of arcs \(I_0 = Q_0 \supset Q_1 \supset Q_2 \supset \cdots\) such that \(g^i(Q_i) = I_i\) for all \(i\).

**Proof.** Construct \(Q_0, Q_1, \ldots, Q_m, \ldots\) by recursion on \(m\). Given \(Q_{m-1}\), let \(h = g^m\) and observe that \(h(Q_{m-1}) = g(I_{m-1}) \supset I_m\). Choose \(Q_m = [a, b]\) minimal among the subarcs of \(Q_{m-1}\) that \(h\)-cover \(I_m\). Then \(I_m = [h(a), h(b)]\) because if \(I_m = [h(c), h(d)]\) then \([c, d]\) is not a proper subinterval of \([a, b]\). Moreover, if \(z \in (a, b)\) and \(h(z) \not\in I_m\), then, since \(o \not\in I_m\), there is a permutation \(x, y, z\) of \(a, b, s\) such that \(h(x) \in [h(y), h(z)]\), which implies there is \(w \in [y, z]\) such that \(I_m = [h(y), h(w)]\) in contradiction with the minimality of \(Q_m\). Thus, \(h(Q_m) = I_m\). \(\square\)

**Lemma 2.** If \(I_0, \ldots, I_m\) is a \(g\)-loop then for some \(x \in I_0\) we have \(g^m(x) = x\) and \(g^i(x) \in I_i\) for all \(i\).

**Proof.** Let \(Q_0, \ldots, Q_m\) be as in Lemma 1 Then \(g^m(Q_m) = I_m \supset I_0 \supset Q_m\). Since \(I_m\) and \(Q_m\) are arcs, we may assume that \(I_m = [0, 1]\) and \(Q_m = [a, b] \subset [0, 1]\). Applying the Intermediate Value Theorem, \(g^m\) has a fixed point \(x\) in \(Q_m\). Finally, \(g^i(x) \in g^i(Q_m) \subset g^i(Q_i) = I_i\). \(\square\)

We prove Lemma 3 below using the well-known (see [2] for citations) technique of analyzing the restriction of \(\supseteq g\) to pairs of minimal elements of the set of intervals with endpoints in a fixed \(g\)-orbit.

**Definition 4.** Given \(g: X_n \to X_n\), a \(g\)-basic interval is a minimal element of the set of closed intervals of the form \([a, b]\) where \(a\) and \(b\) are distinct elements of the \(g\)-orbit of \(o\).

In [2], Baldwin defines “basic intervals” as above but assumes \(g(o) = o\) and replaces the orbit of \(o\) with the union of \(\{o\}\) and another fixed finite orbit.
Lemma 3. If \( g: X_n \to X_n \), \( n \geq 2 \), and \( B_{-1}, B_0, B_1, B_2, \ldots, B_m \) is a \( g \)-cascade of \( g \)-basic intervals such that \( B_{-1} = B_0 = B_m \) and \( B_i \neq B_j \) for all \( i < j \) \( \subset \) \( \{0, \ldots, m-1\} \), then, for all \( p \geq m, g \) has a point of period \( p \).

Proof. Fix \( p \geq m \) such that \( p \) is not the period of \( o \). Every sequence of the form \( B_0, B_0, B_0, \ldots, B_0 \), \( B_1, B_2, B_3, \ldots, B_m \) is a \( g \)-loop. Therefore, by Lemma 2 there exists \( x \in B_0 \) such that \( g^p(x) = x \), \( g^i(x) \in B_0 \) for all \( i \in [0, p-m] \), and \( g^i(x) \in B_i+m-p \) for all \( i \in [p-m, p] \). Let \( q \) be the period of \( x \).

Seeking a contradiction, suppose that \( q < p \). If \( m-1 \leq q < p \), then \( g^{p-1}(x) \) is in the orbit of \( o \) because \( B_{m-1} \ni g^{p-1}(x) = g^{p-q-1}(x) \in B_0 \); if \( q < m \), then \( g^{p-q}(x) \) is in the orbit of \( o \) because \( B_{m-g} \ni g^{p-q}(x) = g^p(x) \in B_0 \). Therefore, \( x \) and \( o \) have the same orbit. Since \( B_0 \neq B_1 \), the orbit of \( o \) must have at least 3 points. Therefore, \( x, g(x), \) and \( g^2(x) \) are 3 distinct points in the orbit of \( o \) and so cannot all be endpoints of \( B_0 \). Therefore, \( p-m \leq 1 \) and, hence, \( q \leq m \).

For each basic interval \( I, \max(I) \) is well-defined and not \( o \). Moreover, \( \max(I) \neq \max(J) \) for all distinct basic intervals \( I \) and \( J \). Therefore, \( m \leq q-1 \), in contradiction with \( q \leq m \). \( \square \)

Lemma 4. Suppose that \( g: X_n \to X_n \), \( u, v \in X_n \), \( \leq \) is a compatible ordering of \( [g(u), g(v)] \) such that \( g(v) < u < v \leq g(u) \), and \( B_0, \ldots, B_p \) is a \( g \)-loop such that \( B_0 = [u, v], B_1 \subset [g(v), u], \) and \( B_{p-1} \) is disjoint from \( (u, v) \). Then \( g \) is Li-Yorke chaotic.

Proof. Inductively construct an infinite sequence \( x_0, x_1, x_2, \ldots \) as follows. Let \( x_0 = v \) and choose \( x_1 \in [u, v) \) such that \( g(x_1) = x_0 \). Observe that \( x_1 \in (g(x_0), g(x_1)) \).

Inductively assume we have \( m > 0, g(x_m) = x_{m-1} \), and \( x_m \in (g(x_{m-1}), g(x_m)) \).

Choose \( x_{m+1} \in (x_{m-1}, x_m) \) such that \( g(x_{m+1}) = x_m \). Now we have \( x_{m+1} \in (x_{m-1}, x_m) = (g(x_m), g(x_{m+1})) \); hence, the inductive hypotheses have been preserved. This completes the construction of \( \bar{x} \). Next, observe that \( x_1 < x_0 \) and \( x_{m+1} \in (x_{m-1}, x_m) \) for all \( m \geq 1 \), so \( x_1 < x_3 < x_5 \leq \cdots < x_2 < x_0 \).

Let \( a = \lim x_{2i+1} \) and \( b = \lim x_{2i} \); observe that \( g(a) = b = g(b) \). Let \( A_{2i+1} = [x_{2i+1}, a] \) and \( A_{2i} = [b, x_{2i}] \). Since \( g(a) = b < x_{2i} = g(x_{2i+1}) \) for all \( i \geq 0 \) and \( g(x_{2i}) = x_{2i-1} < x_{2i+1} < a = g(b) \) for all \( i \geq 1 \), we have \( A_{j+1} \ni g A_j \) for all \( j \geq 0 \).

We may assume \( p \) is even, for we may replace \( B_0, \ldots, B_p \) with \( B_0, \ldots, B_p, B_0, \ldots, B_p \) without loss. For each real \( r \in [0, 1] \), choose \( E_r \subset \mathbb{N} \) with asymptotic density \( r \) and define an infinite sequence \( I_r(0), I_r(1), I_r(2), \ldots \) as the concatenation of the infinite sequence of finite sequences \( \bar{C}_k, \bar{D}_k, \ldots \) where

\[
\bar{C}_k = A_{2k}, A_{2k-1}, A_{2k-2}, \ldots, A_0
\]

\[
\bar{D}_k = \begin{cases} A_{p-1}, A_{p-2}, A_{p-3}, \ldots, A_1 & \text{if } k \in E_r \\ B_1, B_2, B_3, \ldots, B_{p-1} & \text{if } k \notin E_r. \end{cases}
\]

This sequence is a \( g \)-cascade because:

- \( A_{j+1} \ni g A_j \) for all \( j \geq 0 \).
- \( B_j \ni g B_{j+1} \) for all \( j < p \).
- \( A_0 = [b, v] \ni g [g(v), a] \ni B_1 \cup A_{p-1} \).
- \( A_1 \ni g A_0 \ni A_{2k} \).
- \( B_{p-1} \ni g [u, v] \ni A_{2k} \).
Applying Lemma 1 (and compactness), choose \( y_r \in I_r(0) \) such that \( g^i(y_r) \in I_r(i) \) for all \( i \geq 0 \).

Define a compatible metric \( d \) on \( X_n \) by requiring each branch to be isometric to \([0, 1]\) and requiring \( d(x, y) = d(x, o) + d(o, y) \) if \( x \) and \( y \) are on different branches. Since \( o \notin (u, v) \), we have \( d(x, y) \geq \min_{w \in \{u, v\}} d(x, w) \) for all \( x \in (u, v) \) and \( y \notin (u, v) \). Let \( \delta = \min\{d(u, b), d(b, v)\} \). Choose \( \varepsilon > 0 \) such that \( d(x, y) < \varepsilon \) implies \( d(g(x), g(y)) < \delta/2 \) for all \( x \in X_n \) and \( y \in \{u, v\} \).

Claim. Given \( 0 \leq r < s \leq 1 \), \( d(g^r(y_r), g^s(y_s)) \geq \varepsilon \) infinitely often.

Proof. Let \( H = \{i : I_r(i) = B_{k-1} \text{ and } I_s(i) = A_1\} \), which is infinite. For each \( i \in H \), we have \( g^{i+1}(y_r) \in A_{2k} \) where \( k \) is such that \( i + 1 \) is the sum of the lengths of \( \tilde{C}_1, \tilde{D}_1, \ldots, \tilde{C}_{k-1}, \tilde{D}_{k-1} \). Hence, for all sufficiently large \( i \in H \), we have

\[
d(b, g^{i+1}(y_r)) \leq \delta/2 \Rightarrow \forall u \in \{u, v\} \ d(g^{i+1}(y_r), u) \geq \delta/2
\]

\[
\Rightarrow \forall z \in \{g(v), g(u)\} \ d(g^{i+1}(y_r), z) \geq \delta/2
\]

\[
\Rightarrow \forall w \in \{v, u\} \ d(g^i(y_s), w) \geq \varepsilon
\]

\[
\Rightarrow d(g^i(y_s), g^i(y_r)) \geq \varepsilon.
\]

Finally, since \( \text{diam}(A_k) \to 0 \) as \( k \to \infty \),

\[
\lim_{i \to \infty} \inf d(g^i(y_r), g^i(y_s)) = 0
\]

for all \( r, s \in [0, 1] \).

Proof of Theorems 2 and 3. Let \( n \geq 2 \), \( f : X_n \to X_n \), \( f(o) \in \beta_1 \), and \( f^3(o) \notin \beta_1 \). There are three cases:

1. \( f^2(o) \notin \beta_1 \): let \( u = o, v = f(o) \), and \( B = [f^2(o), o] \).
2. \( o < f(o) < f^2(o) \): let \( u = f(o), v = f^2(o) \), and \( B = [o, f(o)] \).
3. \( o < f^2(o) < f(o) \): let \( u = f^2(o), v = o \), and \( B = [f(o), f^2(o)] \).

In all three cases, let \( A = [u, v] \) and verify that \( A \) is a basic interval, that \( A \supseteq f A \supseteq B \supseteq f A \), that \( B \subset [f(v), u] \), and that \( [f(u), f(v)] \) has a compatible ordering such that \( f(v) < u < v \leq f(u) \). By Lemmas 3 and 4, \( f \) has points of all periods \( \geq 2 \) and is Li-Yorke chaotic. Since \( X_n \) is a dendroid, \( f \) also has a fixed point.

4. Orbits of \( o \) of size \( \geq n + 2 \)

Example 1. There exists \( f : X_3 \to X_3 \) such that \( o \) has period 5 and intersects every proper branch, but \( f \) lacks period 3.

Proof. Let \( x_2 = \max(\beta_2) \), \( x_4 = \max(\beta_3) \), and \( o = x_0 < x_1 < x_2 = \max(\beta_1) \). (See the diagram below.) Declare \( f(x_i) = x_j \) where \( j = i + 1 \mod 5 \). For convenience, we will write simply \( i \) for \( x_i \).

\[
\begin{array}{cccc}
4 & \text{0} & \text{1} & \text{3} \\
\text{2} &
\end{array}
\]

Then, for each minimal arc of the form \([i, j]\), extend \( f \) to include a homeomorphism from \([i, j]\) to \([f(i), f(j)]\). To show that \( f \) does not have period 3, we again use the method of analyzing the digraph \( G \) consisting of the restriction of \( \supseteq f \) to pairs
of \(f\)-basic intervals. \(G\) is easily computed (see the diagram below), and its only 3-cycle is

\[
[0, 1] \supset_f [0, 1] \supset_f [0, 1] \supset_f [0, 1].
\]

Seeking a contradiction, suppose \(y \in X_3\) has period 3. Since the orbit of \(y\) cannot intersect that of \(o\), there exist \(I_0, I_1, I_2, I_3\) in \(G\) such that \(f^i(y) \in I_i^p\) for all \(i \leq 3\). Moreover, \(I_0, I_1, I_2, I_3\) must be an \(f\)-loop. Therefore, \(0 < f^i(y) < 1\) for all \(i\). But \(f\) is order-reversing on \(D = [0, 1] \cap f^{-1}[0, 1]\), so there are no orbits of size 3 in \([0, 1]\).

**Theorem 3.** If \(n \geq 3\), \(f: X_n \rightarrow X_n\), and the orbit of \(o\) has size \(n+2\) and intersects every proper branch, then \(f\) is Li-Yorke chaotic and has all periods except possibly \(3\).

**Proof.** We may assume \(f(o) \in \beta_1\). By Theorem\(^4\) we may assume also \(f^3(o) \in \beta_1\). Therefore, the orbit of \(o\) intersects \(\beta_1\) at exactly \(f(o)\) and \(f^3(o)\) and intersects each other proper branch at exactly one point. In particular, \(f^3(o) \notin \beta_1\) and we may assume that \(f^2(o) \in \beta_2\) and \(f^4(o) \in \beta_3\). There are two cases:

1. \(o < f^3(o) < f(o)\): let \(u = o, v = f^3(o)\) and \(B_1 = [o, f^4(o)]\).
2. \(o < f(o) < f^3(o)\): let \(u = o, v = f(o)\), \(B_1 = [o, f^2(o)]\), \(B_2 = [f(o), f^3(o)]\), and \(B_3 = [o, f^4(o)]\).

In both cases, let \(A = [u, v]\). In Case\(^1\) \(A \supset_f A \supset_f B_1 \supset_f A\). In Case\(^2\) \(A \supset_f A \supset_f B_1 \supset_f B_2 \supset_f B_3 \supset_f A\). Therefore, by Lemma\(^5\) \(f\) has points of all periods \(\geq 2\) in Case\(^1\) and points of all periods \(\geq 4\) in Case\(^2\). Since \(X_n\) is a dendroid, \(f\) also has a fixed point. Moreover, in Case\(^2\) \(B_1 \supset_f B_2 \supset_f B_3\), which, by Lemma\(^2\) implies \(x \in B_1\) such that \(f(x) \in B_2\) and \(f^2(x) = x\). Since \(B_1\) and \(B_2\) are disjoint, any such \(x\) has period 2.

In both Case 1 and Case 2, \(B_1 = [f(v), u]\) and \([f(u), f(v)]\) has a compatible ordering such that \(f(v) < u < v \leq f(u)\). By Lemma\(^4\) \(f\) is Li-Yorke chaotic. \(\square\)

**Example 2.** There exists Li-Yorke chaotic \(f: X_3 \rightarrow X_3\) such that \(o\) has period 6 and intersects every proper branch but the periodicity of \(f\) is \(\{1\} \cup 2\mathbb{N}\).

**Proof.** Let \(x_2 = \max(\beta_2)\), \(x_4 = \max(\beta_3)\), and \(o = x_0 < x_1 < x_3 < x_5 = \max(\beta_1)\). (See the diagram below.) Declare \(f(x_i) = x_j\) where \(j = i + 1 \mod 5\). For convenience, we will write simply \(i\) for \(x_i\).

\[
\begin{array}{c c c c c}
4 & 0 & 1 & 3 & 5 \\
\downarrow & & & & \\
2 & & & & \\
\end{array}
\]

Then, for each minimal arc of the form \([i, j]\), extend \(f\) to include a homeomorphism from \([i, j]\) to \([f(i), f(j)]\). Like in our previous example, to show that a given \(y \in X_3\) does not have a given odd period \(p \geq 3\), we analyze the digraph \(G\):

\[
\begin{array}{c c c c c c c c c}
[0, 1] & \longrightarrow & [0, 2] & \longrightarrow & [1, 3] & \longrightarrow & [0, 4] & \longrightarrow & [3, 5] \\
\end{array}
\]

Since the orbit of \(y\) cannot intersect that of \(o\), there is an \(f\)-loop \(I_0, \ldots, I_p\) of elements of \(G\) such that \(f^i(y) \in I_i^p\) for all \(i \leq p\). But all odd cycles of \(G\) are of
the form $[0, 1], \ldots, [0, 1]$, so $0 < f^i(y) < 1$ for all $i$. But $f$ is order-reversing on $D = [0, 1] \cap f^{-1}[0, 1]$, so there are no odd orbits in $[0, 1]$ except for fixed points.

It now suffices to show that $g = f^2$ is Li-Yorke chaotic and has full periodicity. By Lemma 3 and 4 this is indeed the case: letting $u = 0, v = 2, A = [u, v]$, and $B = [4, 0]$, we have $A \supset g A \supset g B \supset g A, B = [g(v), u]$ and $g(u) = v$. □

5. Open problems

We should not be surprised that $\leq X_m$ is weaker than $\leq X_n$ for $m \leq n$ because, choosing a retraction $r: X_n \rightarrow X_m$ and letting $\iota: X_m \rightarrow X_n$ be its unique right inverse, we have, for all $g: X_m \rightarrow X_m$ and $p \geq 0$, that $\iota \circ g \circ r: X_n \rightarrow X_n$ and $(\iota \circ g \circ r)^p = \iota \circ g^p \circ r$. On the other hand, it is natural to wonder if other interesting weakenings $\leq F$ of $\leq X_n$ can be found by restricting to various sets $F$ of maps $f: X_n \rightarrow X_n$ not of the form $\iota \circ g \circ r$ above. (To be precise, $p \leq F q$ means that every $f \in F$ with a point of period $q$ also has a point of period $p$.)

An obvious candidate for $F$ is the set $T_n$ of $f: X_n \rightarrow X_n$ with an orbit intersecting every proper branch.

Problem 1. Characterize $\leq T_n$.

Problem 2. If $f \in T_n$ is witnessed by the orbit of $o$ intersecting every proper branch, then what does the period of $o$ imply about the set of all periods of $f$?

Theorem 2 can be interpreted as a modest partial solution to these problems. Moreover, conjectured answers to the second problem can be tested computationally if we limit the size of the orbit of $f$ to, say, at most 10. Then an exhaustive computer search for when the conditions of Lemma 3 are satisfied by an interand of $f$ becomes quite feasible. It would also then be feasible to automate a search for absent digraph cycle lengths like in the examples of section 4. In fact, the $f$ of Example 2 is in one of only 24 classes of $f \in T_3$ where a 6-point orbit of $o$ hits every branch yet Theorem 1 does not apply. Manual analysis of 24 digraphs shows that the periodicity is always cofinite or $\{1\} \cup 2\mathbb{N}$.

We are also interested in proving Li-Yorke chaos from larger orbits of $o$.

Problem 3. If $f \in T_n$ is witnessed by the orbit of $o$ intersecting every proper branch, and the orbit of $o$ has cardinality in $[n + 3, \infty)$, then is $f$ Li-Yorke chaotic?

For small orbit sizes, we can exhaustively search for small iterands $g$ of $f$ and $g$-loops $B_0, \ldots, B_p$ that satisfy the hypotheses of Lemma 4 where the endpoints of $B_0, \ldots, B_p$ come from the orbit of $f$. For orbits of size 6 in $X_3$ that hit every branch, there are only 24 cases not covered by Theorem 2. Manual analysis reveals that Lemma 4 applies to $f$ or to $f^2$ in every case.

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