TVD FIELDS AND ISENTROPIC GAS FLOW

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ABSTRACT. Little is known about global existence of large-variation solutions to Cauchy problems for systems of conservation laws in one space dimension. Besides results for $L^\infty$ data via compensated compactness, the existence of global BV solutions for arbitrary BV data remains an outstanding open problem. In particular, it is not known if isentropic gas dynamics admits an a priori variation bound which applies to all BV data.

In a few cases such results are available: scalar equations, Temple class systems, $2 \times 2$-systems satisfying Bakhvalov’s condition, and, in particular, isothermal gas dynamics. In each of these cases the equations admit a TVD (Total Variation Diminishing) field: a scalar function defined on state space whose spatial variation along entropic solutions does not increase in time.

In this paper we consider strictly hyperbolic $2 \times 2$-systems and derive a representation result for scalar fields that are TVD across all pairwise wave interactions, when the latter are resolved as in the Glimm scheme. We then use this to show that isentropic gas dynamics with a $\gamma$-law pressure function does not admit any nontrivial TVD field of this type.

CONTENTS

1. Introduction 2
2. TVD fields for strictly hyperbolic $2 \times 2$ - systems 3
  2.1. Systems and assumptions 3
  2.2. Exact and Glimm-type TVD fields 4
  2.3. Representation of Glimm-type TVD fields 4
3. Isentropic gas dynamics 8
  3.1. Eigen-structure and wave curves 8
  3.2. Riemann problems and vacuum criterion 9
  3.3. Change in scalar functions of the Riemann invariants across shocks 10
4. Pairwise interactions in isentropic flow 10
  4.1. Group I: Head-on interactions 11
  4.2. Group II: Overtaking interactions 12
5. Non-existence of Glimm-type TVD fields for isentropic flow 12
  5.1. Step 1: Monotonicity of $\theta$ and $\psi$ 13
  5.2. Step 2: Completing the proof of Theorem 5.2 15
6. Final remarks 19
References 20

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1. Introduction

Consider the one-dimensional Cauchy problem for the $p$-system in Lagrangian coordinates $(t, X)$:

\begin{align*}
\tau_t - u X &= 0 \quad (1.1) \\
u_t + p(\tau)_X &= 0. \quad (1.2)
\end{align*}

We are primarily interested in the cases

\[ p(\tau) = \tau - \gamma, \quad \gamma = \text{const.} \geq 1. \quad (1.3) \]

In gas dynamics $\tau$ denotes specific volume, $u$ is the particle velocity, $p$ is pressure, and $\gamma$ is the adiabatic exponent. The initial data $\tau_0$ and $u_0$ are prescribed, and the state space is the open, right half-plane $\mathbb{R}^+ \times \mathbb{R} = \{(\tau, u) | \tau > 0\}$.

The case $\gamma = 1$ models isothermal gas flow, while $\gamma > 1$ models isentropic flow of an ideal, polytropic gas. The former case was resolved by Nishida [12] who established global-in-time existence of weak, entropic solutions for any BV-data. The proof employs the Glimm scheme [8] and provides a uniform variation bound for the solution by exploiting translation invariance of the shock curves when $\gamma = 1$. Bakhvalov [1] generalized Nishida’s analysis by formulating conditions on wave interactions in systems of the form (1.1)-(1.2) which guarantee uniform BV-bounds. These conditions do not hold for (1.1)-(1.3).

Several large variation results have been established for isentropic flow, see [5,13]. These require that $(\gamma - 1) \times \text{Var}(\tau_0, u_0)$ is sufficiently small. Results of the latter type have been established for the full Euler system as well, see [10,11,14]. Also, Glimm & Lax [9] established global existence and decay for a class of systems (including isentropic gas dynamics) requiring only that the oscillation of the initial data be small. Finally, the method of compensated compactness has been used to establish global weak solutions for arbitrary $L^\infty$ data, [4,6,7].

On the other hand it remains an open problem to establish, or rule out, uniform variation bounds for solutions (1.1)-(1.3), when $\gamma > 1$, for general BV data. By this we mean bounds of the form

\[ \text{Var} U(t, \cdot) \leq C_0 \quad \text{for all times } t \geq 0, \]

where $C_0$ depends only on the initial data and $U = (\tau, u)$.

For near-equilibrium solutions, Glimm’s theorem [8] establishes such an estimate for systems of conservation laws, provided the initial data have sufficiently small total variation. The proof relies on the so-called Glimm functional, a non-local quantity which is made to decrease in time through a careful balance of increase in variation against decrease in potential for future interaction.

In contrast to this let us consider three cases where existence results for large-variation data are available without the use of a Glimm functional:

1. Scalar equations: $v_t + f(v)_X = 0$, $v(t, x) \in \mathbb{R}$;
2. Temple class systems: i.e. systems equipped with a full set of Riemann coordinates and such that shock curves and rarefaction curves coincide (Temple [15] introduced these systems and characterized all possible $2 \times 2$-systems of this type; electrophoresis [3] provides an example);
3. Isothermal gas dynamics ($\gamma = 1$), and more generally systems satisfying Bakhvalov’s conditions (see [1,5,11,12]).

For a scalar equation it is well-known that the total variation of the solution is non-increasing in time [3]. Temple class systems enjoy the same property provided the variation of the solution is measured correctly, namely in terms of changes in the Riemann coordinates. For isothermal gas-dynamics, which is not of Temple class, Nishida [12] defined the functional that records variation of one Riemann coordinate $r$ across backward shocks, plus variation of another Riemann coordinate $s$ across forward shocks, i.e.

\[ N(t) := \frac{\text{Var} r(t)}{S} + \frac{\text{Var} s(t)}{S}. \quad (1.4) \]

1Throughout the paper “Var” denotes total variation with respect to the spatial variable $X$. \]
It turns out that, for the specific system of isothermal gas dynamics, and more generally for systems satisfying Bakhvalov’s conditions \[1\], the property that \(N(t)\) is non-increasing is equivalent to the statement that the functional
\[
L(t) := \text{Var}(s(t) - r(t))
\]
is non-increasing; see Theorem 2.3 in \[11\].

By exploiting the existence of scalar fields whose variation is non-increasing along solutions, global existence for general BV data has been established for each of the cases (1)-(3). (In the case of (3) it is also important that the Riemann invariants vary in opposite directions across rarefactions and shocks.) It is therefore natural to ask if such TVD fields exist also for isentropic flow. The main results of the present work are, roughly stated, the following.

I. Representation result: any TVD field \(\varphi\) for a strictly hyperbolic \(2 \times 2\)-system is of the form \(\theta(s) - \psi(r)\), where \(r\) and \(s\) are Riemann invariants; see Theorem 2.4.

II. Non-existence result: the system \([1.1]-[1.3]\) for isentropic gas dynamics, with \(\gamma > 1\), does not admit any non-trivial TVD field; see Theorem 5.2.

In what follows we provide a precise definition of TVD fields for general systems. We then establish part I above under some mild regularity conditions on the TVD field. Finally we consider isentropic gas dynamics and prove part II. Both results are based on a study of pairwise wave interactions and apply to “Glimm-type” TVD fields; see Section 2.2.

2. TVD FIELDS FOR STRICTLY HYPERBOLIC \(2 \times 2\) - SYSTEMS

2.1. Systems and assumptions. We consider a strictly hyperbolic system of two equations for the unknown \(U(t, X) \in \mathcal{U}^{\text{open}} \subset \mathbb{R}^{2 \times 1}\):
\[
U_t + f(U)_X = 0, \quad t \geq 0, \quad X \in \mathbb{R}.
\]
Let the diagonalization of \(Df\) be
\[
Df(U) = \begin{bmatrix}
\tilde{R}(U) & \tilde{R}(U) \\
\tilde{\lambda}(U) & \lambda(U)
\end{bmatrix}
\begin{bmatrix}
0 & \tilde{L}(U) \\
\lambda(U) & \tilde{L}(U)
\end{bmatrix},
\]
where \(\tilde{\lambda}(U) < \lambda(U)\) denote the slow and fast characteristic speeds. The corresponding right eigenvectors \(\tilde{R}, \tilde{R} \in \mathbb{R}^{2 \times 1}\) and left eigenvectors \(\tilde{L}, \tilde{L} \in \mathbb{R}^{1 \times 2}\) satisfy
\[
\tilde{R} \cdot \tilde{L} \equiv 0, \quad \tilde{R} \cdot \tilde{L} \equiv 0.
\]
For the following terminology we refer to \[3\]. We assume that each characteristic field of \([2.1]\) is either genuinely non-linear or linearly degenerate throughout \(\mathcal{U}\). We will also assume that any Riemann problem \((U_l, U_r)\), with \(U_l, U_r \in \mathcal{U}\), has a unique, self-similar solution consisting of one slow elementary wave (i.e., shock, contact, or centered rarefaction) and one fast elementary wave connecting \(U_l\) to \(U_r\) through a middle state. The eigenvectors are normalized according to
\[
\nabla \tilde{\lambda} \cdot \tilde{R} \geq 0, \quad \nabla \tilde{\lambda} \cdot \tilde{R} \geq 0, \quad \tilde{R} \cdot \tilde{L} \equiv 1, \quad \tilde{R} \cdot \tilde{L} \equiv 1.
\]
We denote the corresponding wave curves through a base point (left state) \(\tilde{U}\) by \(\tilde{W}(\varepsilon; \tilde{U})\) and \(\tilde{W}(\varepsilon; \tilde{U}), \varepsilon \in \mathbb{R}\). The parameter value \(\varepsilon\) is also referred to as the strength of the waves \((\tilde{U}, \tilde{W}(\varepsilon; \tilde{U}))\) and \((\tilde{U}, \tilde{W}(\varepsilon; \tilde{U}))\). In this section we assume that the wave curves are parametrized such that:
\begin{itemize}
  \item \(\tilde{W}(0; \tilde{U}) = \tilde{U}\),
  \item \(\varepsilon > 0\) gives the states \(\tilde{W}(\varepsilon; \tilde{U})\) on the right of backward rarefaction waves with left state \(\tilde{U}\),
  \item \(\varepsilon < 0\) gives the states \(\tilde{W}(\varepsilon; \tilde{U})\) on the right of backward shock waves with left state \(\tilde{U}\).
\end{itemize}

\[\text{2.} \quad \text{I.e. “Total Variation Diminishing;” this allows for fields whose variation is just non-increasing in time.}\]
Similarly for $\tilde{W}(\varepsilon; \tilde{U})$. For now we do not specify further the parametrizations: $\tilde{R}$ and $\tilde{\overline{R}}$ denote any fixed versions of the eigen-fields (subject to (2.3)). We shall later choose these to commute.

For any fixed choice of the eigen-fields $\tilde{R}$, $\tilde{\overline{R}}$ we introduce Riemann coordinates $r$ and $s$ satisfying

\begin{align}
\nabla r \cdot \tilde{R} &= 1, \quad \nabla r \cdot \tilde{\overline{R}} = 0, \\
\nabla s \cdot \tilde{R} &= 0, \quad \nabla s \cdot \tilde{\overline{R}} = 1.
\end{align}

(See (3.2) for the case of the $p$-system (1.1)-(1.3).) Finally, the map $(r, s) \mapsto \tilde{R}(r, s)$ is assumed to be a diffeomorphism.

### 2.2. Exact and Glimm-type TVD fields.

For clarification we introduce two types of TVD fields, exact and Glimm-type. However, both the representation result (Theorem 2.4) and the non-existence result for the $p$-system (Theorem 5.2) will be formulated for Glimm-type TVD fields only.

First, $U(t, X)$ is an entropic weak solution to (2.1) provided it is a weak (distributional) solution of (2.1), and satisfies

\[
\eta(U)_t + q(U)_X \leq 0 \quad \text{in} \ D',
\]

whenever $\eta, q : U \to \mathbb{R}$ are such that $\nabla_U q = \nabla_U \eta D_U f$ and $\eta$ is convex. (We refer to [3] for details.) Given a scalar field $\varphi : U \to \mathbb{R}$ and a weak solution $U(t, X)$ of (2.1) such that $U(t) \equiv U(t, \cdot) \in \text{BV}(\mathbb{R})$, the variation of the function $X \mapsto \varphi(U(t, X))$ is denoted $\text{Var} \varphi(U(t))$, or just $\text{Var} \varphi$.

**Definition 2.1.** A scalar field $\varphi : U \to \mathbb{R}$ is an exact TVD field for the system (2.1) provided it satisfies the following holds: whenever $U(t, X)$ is an entropic weak solution of (2.1) with $U(t) \in \text{BV}(\mathbb{R})$ for $t > 0$, the map $t \mapsto \text{Var} \varphi(U(t))$ is non-increasing on $\mathbb{R}_+$.

Next, we recall how pairwise wave interactions are resolved in the Glimm scheme [8]. Consider two approaching elementary waves (shocks, contact, or centered rarefactions) with left, middle, and right states $U_l$, $U_m$, and $U_r$, respectively. The interaction of these waves is resolved by solving the interaction Riemann problem $(U_l, U_r)$, whose solution we denote by $U(t, x)$ with $t > t^*$, $t^*$ being the time of interaction.

**Definition 2.2.** A scalar field $\varphi : U \to \mathbb{R}$ is a Glimm-type TVD field for the system (2.1) provided the following holds: whenever $U_l$, $U_m$, and $U_r$ are left, middle, and right states, respectively, in a pairwise interaction as described above, the spatial variation of $\varphi(U(t, X))$ for $t > t^*$ is less than or equal to its spatial variation across the incoming waves $(U_l, U_m)$ and $(U_m, U_r)$ combined.

We next derive a necessary condition on the functional form of any Glimm-type TVD field for (2.1).

### 2.3. Representation of Glimm-type TVD fields.

We shall obtain the general form of a Glimm-type TVD field in terms of Riemann coordinates by considering weak, head-on interactions. To analyze these we use the Glimm interaction estimate ([3], Section 9.9) together with the freedom of choosing commuting versions of the eigenvector fields $\tilde{R}$ and $\tilde{\overline{R}}$.

Changing notation slightly we let the extreme left state in the interactions be $\tilde{U}$. In the following computation all quantities are evaluated at $\tilde{U}$ unless indicated otherwise. The far right state is denoted $U$, while the incoming middle state and the outgoing middle state are denoted $U_0$ and $\tilde{U}$, respectively. Consider the head-on interaction on the left in Figure 1. Let $\alpha, \alpha', \beta$, and $\beta'$ denote...
the parameter values corresponding to incoming and outgoing waves in the slow and fast families, respectively. The total variation of \( \varphi(U) \) before and after interaction are, respectively,

\[
\text{Var} \varphi^- \equiv |\varphi(U) - \varphi(U_0)| + |\varphi(U) - \varphi(\hat{U})|,
\]

and

\[
\text{Var} \varphi^+ \equiv |\varphi(U) - \varphi(\hat{U})| + |\varphi(\hat{U}) - \varphi(\hat{U})|.
\]

To evaluate these for weak interactions we use Taylor expansion and Glimm’s interaction estimate \( [3,8] \). Consider the variation of \( \varphi(U) \) before and after interaction. The middle state \( U_0 \) is given by

\[
U_0 = \hat{W}(\beta; \hat{U}) = \hat{U} + \beta \hat{R} + \frac{1}{2} \beta^2 (D \hat{R}) \hat{R} + O^3(\alpha),
\]

while the right state \( U \) is given by

\[
U = \hat{W}(\alpha; U_0) = U_0 + \alpha \hat{R} |_{U_0} + \frac{1}{2} \alpha^2 (D \hat{R}) \hat{R} |_{U_0} + O(\alpha^3)
\]

where we have used \( [2.8] \). We use \( O^3(\alpha, \beta) \) to indicate third order terms such as \( \alpha^3, \alpha^2 \beta \) etc., and recall that all quantities are evaluated at \( \hat{U} \) unless otherwise indicated.

Taylor expanding \( \varphi \) about \( U_0 \), and using \( [2.8] \) and \( [2.9] \), we obtain

\[
\varphi(U) - \varphi(U_0) = \alpha \nabla \varphi \hat{R} + \left[ \frac{\alpha^2}{2} \nabla \varphi (D \hat{R}) \hat{R} + \frac{\alpha^2}{2} \hat{R}^T (D^2 \varphi) \hat{R} + \alpha \beta \nabla \varphi (D \hat{R}) \hat{R} + \alpha \beta \hat{R}^T (D^2 \varphi) \hat{R} \right] + O^3(\alpha, \beta).
\]

(Note: gradients are row vectors, while \( \hat{R} \) and \( \hat{R} \) are column vectors. A ‘\( T \)’ superscript denotes transpose, and \( D^2 \varphi \) denotes the Hessian of \( \varphi \). Here and below \( \nabla \) and \( D \) are with respect to \( U \) unless stated otherwise.) Similarly we have

\[
\varphi(U_0) - \varphi(\hat{U}) = \beta \nabla \varphi \hat{R} + \frac{\beta^2}{2} \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + O^3(\alpha, \beta).
\]

Thus

\[
\text{Var} \varphi^- = \left| \alpha \nabla \varphi \hat{R} + \frac{\alpha^2}{2} \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + \alpha \beta \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + O^3(\alpha, \beta) \right|
\]

and

\[
\text{Var} \varphi^+ = \left| \beta \nabla \varphi \hat{R} + \frac{\beta^2}{2} \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + O^3(\alpha, \beta) \right|
\]

Next, for the states after interaction we first use similar Taylor expansions with respect to the outgoing strengths \( \alpha' \) and \( \beta' \), to obtain a corresponding expression for \( \text{Var} \varphi^+ \). We shall then recall the Glimm interaction estimate which provides \( \alpha' \) and \( \beta' \) in terms of \( \alpha \) and \( \beta \) to leading orders. Combining these yields an expression for \( \text{Var} \varphi^+ \) in terms of \( \alpha \) and \( \beta \) which we can compare to \( \text{Var} \varphi^- \) as given by \( [2.12] \). As above we have

\[
\hat{U} = \hat{W}(\alpha'; \hat{U}) = \hat{U} + \alpha' \hat{R} + \frac{1}{2} \alpha'^2 (D \hat{R}) \hat{R} + O^3(\alpha'),
\]

and

\[
U = \hat{U} + \beta' \hat{R} + \alpha' \beta' (D \hat{R}) \hat{R} + \frac{1}{2} \beta'^2 (D \hat{R}) \hat{R} + O^3(\alpha', \beta').
\]

It follows that

\[
\varphi(U) - \varphi(\hat{U}) = \beta' \nabla \varphi \hat{R} + \alpha' \beta' \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + \frac{\beta'^2}{2} \left[ \nabla \varphi (D \hat{R}) \hat{R} + \hat{R}^T (D^2 \varphi) \hat{R} \right] + O^3(\alpha', \beta'),
\]
Hence,

\[ \varphi(\bar{U}) - \varphi(\bar{U}) = a'\nabla\varphi\bar{R} + \frac{a'^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha', \beta'). \]

Hence,

\[ \text{Var } \varphi^+ = \left| \beta'\nabla\varphi\bar{R} + \alpha'\beta' \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + \frac{\beta'^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha', \beta') \right| + \left| \alpha'\nabla\varphi\bar{R} + \frac{a'^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha', \beta') \right|. \] (2.13)

We now invoke Glimm’s interaction estimate (3, Section 9.9) according to which

\[ \alpha' = \alpha + \alpha\beta L[\bar{R}, \bar{R}] + O^3(\alpha, \beta) \quad \text{and} \quad \beta' = \beta + \alpha\beta L[\bar{R}, \bar{R}] + O^3(\alpha, \beta), \]

where \([X, Y] = (DY)X - (DX)Y\) denotes the commutator. We then scale \(\bar{R}, \bar{R}\) such that their commutator vanishes. (This is always possible for planar vector fields.) With these versions of \(\bar{R}\) and \(\bar{R}\) fixed we obtain that

\[ \alpha' = \alpha + O^3(\alpha, \beta) \quad \text{and} \quad \beta' = \beta + O^3(\alpha, \beta), \]

and substitution into (2.13) yields

\[ \Delta \text{Var } \varphi = \text{Var } \varphi^+ - \text{Var } \varphi^- = \left| \beta\nabla\varphi\bar{R} + \alpha\beta \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + \frac{\beta^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha, \beta) \right| + \left| \alpha\nabla\varphi\bar{R} + \frac{a^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha, \beta) \right| - \left| \alpha\nabla\varphi\bar{R} + \frac{a^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha, \beta) \right| - \left| \beta\nabla\varphi\bar{R} + \frac{\beta^2}{2} \left[ \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \right] + O^3(\alpha, \beta) \right|. \] (2.14)

We now use that \([\bar{R}, \bar{R}] = 0\) and \(\bar{R}^T(D^2\varphi)\bar{R} \equiv \bar{R}^T(D^2\varphi)\bar{R}\), and set

\[ A := \nabla\varphi\bar{R} \]
\[ B := \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \]
\[ C := \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} \]
\[ D := \nabla\varphi\bar{R} \]
\[ E := \nabla\varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R}, \]

such that (2.14) reads

\[ \Delta \text{Var } \varphi = \left| \beta A + \alpha \beta B + \frac{\beta^2}{2} C + O^3(\alpha, \beta) \right| + \left| \alpha D + \frac{a^2}{2} E + O^3(\alpha, \beta) \right| - \left| \alpha D + \frac{a^2}{2} E + O^3(\alpha, \beta) \right| - \left| \beta A + \frac{\beta^2}{2} C + O^3(\alpha, \beta) \right| =: |\Delta_1| + |\Delta_2| - |\Delta_3| - |\Delta_4|. \] (2.15)

Recall that \(A, B, C, D, E\) are all evaluated at \(\bar{U}\). We consider the situation when both \(A = \nabla\varphi\cdot\bar{R}\) and \(D = \nabla\varphi\cdot\bar{R}\) are non-vanishing on some open set \(U' \subset U\). With \(\bar{U}\) fixed in \(U'\) we then consider weak interactions. Specifically, we choose \(|\alpha|, |\beta| \neq 0\) and so small that

\[ \text{sgn } \Delta_1 = \text{sgn } (\beta A) = \text{sgn } \Delta_4, \]
and
\[ \text{sgn} \Delta_2 = \text{sgn}(\alpha D) = \text{sgn} \Delta_3. \]
As we assume \( A \) and \( D \) are non-vanishing there are four possibilities:
(i) \( \beta A, \alpha D > 0 \)
(ii) \( \beta A, \alpha D < 0 \)
(iii) \( \beta A < 0 < \alpha D \)
(iv) \( \alpha D < 0 < \beta A. \)

For these we get, respectively,
(i) \( \Delta \text{Var} \varphi = \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 = O^3(\alpha, \beta) \)
(ii) \( \Delta \text{Var} \varphi = -\Delta_1 - \Delta_2 + \Delta_3 + \Delta_4 = O^3(\alpha, \beta) \)
(iii) \( \Delta \text{Var} \varphi = -\Delta_1 + \Delta_2 - \Delta_3 + \Delta_4 = -2\alpha \beta B + O^3(\alpha, \beta) \)
(iv) \( \Delta \text{Var} \varphi = \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = +2\alpha \beta B + O^3(\alpha, \beta). \)

Now, assume for contradiction that \( B \neq 0 \). Whatever the signs of \( A \) and \( D \) are, we consider a head-on interaction with left state \( \bar{U} \), and with \( \alpha \)-value \( \alpha_1 \neq 0 \) and \( \beta \)-value \( \beta_1 \neq 0 \) chosen so that Case (iii) holds. By reducing further (if necessary) \( |\alpha_1| \) and \( |\beta_1| \), we obtain from (iii)' that
\[ -2\alpha_1 \beta_1 B < 0. \tag{2.16} \]
We then consider the head-on interaction with the same left state \( \bar{U} \), but now with \( \alpha \)- and \( \beta \)-values \( \alpha_2 := -\alpha_1 \) and \( \beta_2 := -\beta_1 \). This latter interaction then belongs to Case (iv), and (iv)' gives
\[ 2\alpha_2 \beta_2 B = 2\alpha_1 \beta_1 B < 0. \]
This contradicts (2.16) and shows that we must have \( B = 0 \). Thus, in order that the total \( X \)-variation of \( \varphi(U(t, X)) \) be non-increasing across all head-on Glimm interactions, it is necessary that
\[ B = \nabla \varphi(D\bar{R})\bar{R} + \bar{R}^T(D^2\varphi)\bar{R} = 0, \tag{2.17} \]
which is equivalent to
\[ \nabla(\nabla \varphi \cdot \bar{R}) \cdot \bar{R} = 0. \tag{2.18} \]

**Remark 2.3.** Performing the same type of analysis for weak, overtaking interactions shows that \( \Delta \text{Var} \varphi = O^3(\alpha, \beta) \) in all such interactions. Thus, no similar constraint follows from non-increase of variation across weak, overtaking interactions.

For the versions of \( \bar{R} \) and \( \bar{R} \) fixed above we let \( r \) and \( s \) be Riemann coordinates satisfying (2.4) and (2.5). By changing to these coordinates and considering \( \varphi \) as a function of \( r \) and \( s \), the necessary condition (2.17) takes the simple form
\[ \partial^2_{rs} \varphi = 0. \tag{2.19} \]
This shows that \( \varphi \) must admit a representation of the form
\[ \varphi(r, s) = \theta(s) - \psi(r) \tag{2.19} \]
in any convex subset of \( V := \{ (r, s) \mid U(r, s) \in U' \} \). (The minus sign is chosen for convenience in formulating later results.) This conclusion was reached under the assumption that \( \nabla_U \varphi \cdot \bar{R} \) and \( \nabla_U \varphi \cdot \bar{R} \) are both non-vanishing in \( U' \). We note that if one of these vanishes identically in \( U' \), then \( \varphi \) is again of the form (2.19) in \( U' \) (with \( \theta \) or \( \psi \) vanishing). We summarize our findings:

**Theorem 2.4.** Given a \( 2 \times 2 \)-system (2.1) satisfying the standard structural assumptions in Section 2.1. Let its eigen-structure be given by (2.2), with \( \bar{R} \) and \( \bar{R} \) scaled to commute (\( [\bar{R}, \bar{R}] \equiv 0 \)) in \( U \), and let \( r, s \) denote the corresponding Riemann coordinates satisfying (2.4)-(2.5).

Assume that \( \varphi \in C^2(U) \) is a Glimm-type TVD field for (2.1), and consider \( \varphi \) as a function of \( r \) and \( s \). Define the sets
\[ V := \{ (r, s) \in \mathbb{R}^2 \mid U(r, s) \in U \} \]
and
\[ V' := \{ (r, s) \in V \mid \partial_r \varphi(r, s) \neq 0 \text{ and } \partial_s \varphi(r, s) \neq 0 \}. \]
Then \( \varphi \) admits a representation of the form \( \varphi(r,s) = \theta(s) - \psi(r) \) on any open, convex subset of \( \mathcal{V} \). Furthermore, if \( \mathcal{V} \) is convex and \( \mathcal{V}' \) is dense in \( \mathcal{V} \), then there exist functions \( \theta(s) \), \( \psi(r) \) such that \( \varphi \) has this representation throughout \( \mathcal{V} \).

**Proof.** It only remains to argue for the last part. So assume \( \mathcal{V}' \) is dense in \( \mathcal{V} \) and that the latter is convex. Given any point \(( \bar{r}, \bar{s} )\) in the open set \( \mathcal{V}' \), we choose a ball \( B \subset \mathcal{V}' \) about \(( \bar{r}, \bar{s} )\). According to the analysis above \( \partial^2_r \varphi \) vanishes identically on \( B \), and in particular \( \partial^2_r \varphi(\bar{r}, \bar{s}) = 0 \). As \( \partial^2_r \varphi \) is continuous on \( \mathcal{V} \) we conclude that \( \partial^2_r \varphi \equiv 0 \) throughout \( \mathcal{V} \). Finally, by convexity of \( \mathcal{V} \) it follows that there is a single pair of functions \( \theta \) and \( \psi \) such that \( \varphi(r,s) = \theta(s) - \psi(r) \) throughout \( \mathcal{V}' \). \( \square \)

**Remark 2.5.** Note that the decay of the Liu functional \( L(t) \) in \((1.3)\) shows that \( \varphi(r,s) = s - r \) is a Glimm-type TVD field for any \( 2 \times 2 \)-systems satisfying Bakhvalov’s conditions.

### 3. Isentropic Gas Dynamics

In the remainder of this paper we consider the specific case of isentropic gas dynamics with a standard \( \gamma \)-law pressure function \[ (1.1), (1.3). \]

#### 3.1. Eigen-structure and wave curves.

The characteristic speeds are
\[
\lambda = -\sqrt{\gamma} \tau^{-\alpha - 1} \quad \text{and} \quad \bar{\lambda} = \sqrt{\gamma} \tau^{-\alpha - 1}, \tag{3.1}
\]
where
\[
\alpha := \frac{\gamma - 1}{2}.
\]
As corresponding right eigenvectors of the Jacobian of the flux \((-u, p(\tau))^T\) we choose
\[
\vec{R} = \frac{1}{\gamma} \begin{bmatrix} \frac{1}{\sqrt{\gamma}} \tau^{\alpha + 1} \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{R} = \frac{1}{\gamma} \begin{bmatrix} \frac{1}{\sqrt{\gamma}} \tau^{\alpha + 1} \\ 1 \end{bmatrix}.
\]
These satisfy \( \nabla_{(\tau,u)} \vec{\lambda} \cdot \vec{R} > 0 \), \( \nabla_{(\tau,u)} \bar{\lambda} \cdot \vec{R} > 0 \), and \([\vec{R}, \vec{R}] \equiv 0\). As Riemann invariants we use
\[
 r := u - \kappa \tau^{-\alpha} \quad \text{and} \quad s := u + \kappa \tau^{-\alpha}, \quad \kappa := \frac{\sqrt{\gamma}}{\alpha}, \tag{3.2}
\]
such that
\[
\nabla_{(\tau,u)} r \cdot \vec{R} \equiv 1, \quad \nabla_{(\tau,u)} s \cdot \vec{R} \equiv 0, \quad \nabla_{(\tau,u)} s \cdot \vec{R} \equiv 0, \quad \nabla_{(\tau,u)} s \cdot \vec{R} \equiv 1.
\]
We parametrize the wave curves using \( \xi \)-ratios where
\[
\xi := \rho^\alpha = \tau^{-\alpha}; \tag{3.3}
\]
see Remark \[4.1\] The Riemann invariants are then
\[
r = u - \kappa \xi, \quad s = u + \kappa \xi. \tag{3.4}
\]
We next give the parametrizations in the \((\xi, u)\)-plane of the backward and forward wave curves of the “first type.” That is, given a base point \((\bar{\xi}, \bar{u})\), we consider the curves of points \((\xi, u)\) such that the Riemann problem with left state \((\bar{\xi}, \bar{u})\) and right state \((\xi, u)\) yields a single backward or forward wave (rarefaction or entropic shock). Letting \( b \) and \( f \) denote the \( \xi \)-ratios \( \xi_{\text{right}}/\xi_{\text{left}} \) across backward and forward waves, respectively, the parametrizations are given by:

**Backward waves:**
\[
\vec{V}(b; \bar{\xi}, \bar{u}) = \left( \begin{array}{c} \frac{b \xi}{\bar{u} - \bar{\phi}(b) \xi} \\ \bar{u} - \bar{\phi}(b) \xi \end{array} \right) \quad \left\{ \begin{array}{l} \vec{R} : \quad 0 < b < 1 \\ \vec{S} : \quad b > 1 \end{array} \right. \tag{3.5}
\]

**Forward waves:**
\[
\vec{V}(f; \bar{\xi}, \bar{u}) = \left( \begin{array}{c} \frac{f \xi}{\bar{u} + \bar{\phi}(f) \xi} \\ \bar{u} + \bar{\phi}(f) \xi \end{array} \right) \quad \left\{ \begin{array}{l} \vec{R} : \quad f > 1 \\ \vec{S} : \quad 0 < f < 1, \end{array} \right. \tag{3.6}
\]

\(3\)Some restriction on the domain is required for the existence of one pair \( \theta, \psi \) such that \[2.19\] holds in all of \( \Omega \). E.g., on \( \Omega := (-2,2)^2 \setminus [-1,1]^2 \) let \( \varphi(r,s) = \eta(s) \chi((-1,1) \times (-1,1)) \), where \( \eta \neq 0 \) is smooth and with support in \((-1,1)\). Then \( \partial^2_r \varphi \equiv 0 \) on \( \Omega \), but \( \varphi \) does not have one representation of the form \[2.19\] which is valid throughout \( \Omega \).
Figure 2. The auxiliary $\phi$-functions (backward=solid, forward=dashed)

where the auxiliary functions $\overset{\leftarrow}{\phi}$ and $\overset{\rightarrow}{\phi}$ are given by

$$
\overset{\leftarrow}{\phi}(x) = \begin{cases} 
\kappa(x-1) & 0 < x < 1 \\
\sqrt{(1-x^{-\frac{1}{\alpha}})(x^{\frac{2}{\alpha}}-1)} & x > 1
\end{cases}
$$

$$
\overset{\rightarrow}{\phi}(x) = \begin{cases} 
-x\overset{\leftarrow}{\phi}(\frac{1}{x}) & 0 < x < 1 \\
\kappa(x-1) & x > 1.
\end{cases}
$$

We refer to the $\xi$-ratios $b$ and $f$ as wave strengths. Note that the auxiliary functions $\overset{\leftarrow}{\phi}$ and $\overset{\rightarrow}{\phi}$ are both strictly increasing and satisfy

$$
\overset{\rightarrow}{\phi}(x) = -x\overset{\leftarrow}{\phi}(\frac{1}{x}).
$$

A calculation shows that the map $x \mapsto \sqrt{(1-x^{-\frac{1}{\alpha}})(x^{\frac{2}{\alpha}}-1)}$ is convex up for $x > 1$. It follows that $\overset{\leftarrow}{\phi}$ is convex up, while $\overset{\rightarrow}{\phi}$ is convex down. Furthermore, it is immediate to verify that

$$
\lim_{x \to +\infty} \overset{\rightarrow}{\phi}(x) = +\infty \quad \text{and} \quad \lim_{x \to 0^+} \overset{\rightarrow}{\phi}(x) = -\infty.
$$

For later reference we also record the fact that

$$
\lim_{x \to +\infty} \frac{\overset{\rightarrow}{\phi}(x) + \kappa(x+1)}{\overset{\rightarrow}{\phi}(x) + \kappa} \to +\infty \quad \text{as} \quad x \to +\infty.
$$

The wave curves in the $(r,s)$-plane are illustrated in Figure 3. Note that the no-vacuum region $\{ \rho > 0 \}$ corresponds to the half-plane $H := \{ s > r \}$ in $(r,s)$-coordinates.

3.2. Riemann problems and vacuum criterion. Consider the Riemann problem with left state $(\bar{\xi}, \bar{u})$ and right state $(\xi, u)$, and let $b$ and $f$ denote the $\xi$-ratios of the resulting backward and forward waves, respectively. A short calculation shows that $b$ is given as the root of

$$
\xi\overset{\rightarrow}{\phi}(b) + \xi\overset{\leftarrow}{\phi}\left(\frac{b\xi}{\kappa}\right) = \bar{u} - u,
$$

(3.10)
and \( f = \frac{\xi}{\bar{\xi}} \).

As the left-hand side of (3.10) is increasing with respect to \( b \), it follows that the Riemann problem \( ((\bar{\xi}, \bar{u}), (\xi, u)) \) has a unique solution without vacuum provided \( u - \bar{u} < \kappa (\xi + \bar{\xi}) \) (no vacuum). (3.11)

3.3. Change in scalar functions of the Riemann invariants across shocks. For later reference we record how scalar fields \( h(r) \) and \( k(s) \) change across a shock wave. Let \((\bar{u}, \bar{\xi})\) denote the state on the left of a backward shock wave with strength (i.e., \( \xi \)-ratio) \( x \). The state on the right of the shock is then \((\tilde{\xi}, \tilde{u})\), where

\[
\tilde{\xi} = x\bar{\xi}, \quad \tilde{u} = \bar{u} - \tilde{\phi}(x)\bar{\xi},
\]

and the corresponding values of the Riemann invariants are

\[
\bar{r} = \bar{u} - \kappa \bar{\xi}, \quad \bar{s} = \bar{u} + \kappa \bar{\xi} \quad \text{(left state)},
\]

and

\[
\tilde{r} = \tilde{u} - \tilde{\phi}(x)\tilde{\xi} - \kappa x\bar{\xi}, \quad \tilde{s} = \tilde{u} - \tilde{\phi}(x)\bar{\xi} + \kappa x\tilde{\xi} \quad \text{(right state)}.
\]

We then have

\[
\Delta_x h(r) := h(\bar{r}) - h(\bar{\xi}) = -\xi \int_1^x h'(\bar{u} - \tilde{\phi}(\sigma)\bar{\xi} - \kappa \sigma \bar{\xi})(\tilde{\phi}'(\sigma) + \kappa) \, d\sigma, \quad (3.12)
\]

and

\[
\Delta_x k(s) := k(\tilde{s}) - k(\bar{s}) = -\xi \int_1^x k'(\bar{u} - \tilde{\phi}(\sigma)\bar{\xi} + \kappa \sigma \tilde{\xi})(\tilde{\phi}'(\sigma) - \kappa) \, d\sigma. \quad (3.13)
\]

Similar identities hold for changes across forward waves.

4. Pairwise interactions in isentropic flow

There are six essentially distinct types of pairwise wave interactions:

\[
\begin{align*}
\text{Ia:} & \quad \bar{R}\bar{R} & \quad \text{head-on interactions} \quad (4.1) \\
\text{Ib:} & \quad \bar{R}\bar{S} \\
\text{Ic:} & \quad \bar{S}\bar{S} \\
\text{IIa:} & \quad \bar{S}\bar{S} \\
\text{IIb:} & \quad \bar{S}\tilde{R} \\
\text{IIc:} & \quad \bar{R}\tilde{S} & \quad \text{overtaking interactions}.
\end{align*}
\]
The head-on interaction

\[ \text{Ib'}: \: \vec{S}\vec{R}, \]

and the overtaking interactions

\[ \begin{align*}
\text{IIa'}: & \: \vec{SS} \\
\text{IIb'}: & \: \vec{R}\vec{S} \\
\text{IIc'}: & \: \vec{S}\vec{R},
\end{align*} \tag{4.3} \]

are qualitatively the same as those in Ib and IIa-IIc, respectively. Finally, for cases IIb, IIc, IIb’, and IIc’, there are two possible outcomes depending on the relative strengths of the incoming waves.

The analysis of interaction Riemann problems for the \( p \)-system has been carried out, see [2]. In cases Ic, IIa, and IIa’, where both incoming waves are shocks, this gives the exact, weak entropy solution of the wave interaction. When one of the incoming waves is a rarefaction, the actual solution involves penetration of a rarefaction wave, to which the interaction Riemann problem provides an approximate solution. For completeness we include a brief description of the results.

4.1. Group I: Head-on interactions. Let the state on the far left be \((\bar{\xi}, \bar{u})\) and let the \( \xi \)-ratios of the incoming (outgoing) backward and forward waves be \( b \) and \( f \) (\( B \) and \( F \)), respectively (see Figure 4, left diagram). Traversing the waves before and after interaction shows that \( B \) and \( F \) satisfy

\[ BF = bf \]

\[ \vec{\phi}(B) - B \vec{\phi}(F) = f \vec{\phi}(b) - \vec{\phi}(f). \tag{4.4} \]

\[ \vec{\phi}(B) - \vec{\phi}(b) = 0. \tag{4.5} \]

Remark 4.1. Parametrizing the wave curves in terms of \( \xi \)-ratios as in (3.5) and (3.6) implies that \( B \) and \( F \) are determined independently of the left state \((\bar{\xi}, \bar{u})\) in the interaction. The same is true for overtaking interactions. This is advantageous when we later want to search for interactions where new variation is created; see Section 5.2.

Applying (3.7) we get that the \( \xi \)-ratio \( B = B(b, f) \) solves the equation

\[ \mathcal{F}(B; b, f) := \vec{\phi}(B) + bf \vec{\phi} \left( \frac{B}{bf} \right) + \vec{\phi}(f) - f \vec{\phi}(b) = 0. \tag{4.6} \]

The condition for having no vacuum is that \( B > 0 \), or

\[ \vec{\phi}(b) + \kappa b > \frac{1}{f} \left( \vec{\phi}(f) - \kappa \right) \quad \text{(no vacuum in head-on).} \]

A direct calculation shows that there is no vacuum in Ib (\( \vec{R}\vec{S} \)) and Ic (\( \vec{S}\vec{S} \)) interactions, while there is no vacuum in Ia (\( \vec{R}\vec{R} \)) interaction if and only if the incoming ratios satisfy \( b + \frac{1}{f} > 1 \).

By analyzing the interactions in the \((r, s)\)-plane of Riemann invariants, we obtain that: Ia interactions always yield \( \vec{R}\vec{R} \); Ib interactions always yield \( \vec{S}\vec{R} \) (and Ib’ interactions always yield \( \vec{R}\vec{S} \)); Ic interactions always yield \( \vec{S}\vec{S} \).
4.2. Group II: Overtaking interactions. Let the state on the far left be \((\xi, \bar{u})\) and let the \(\xi\)-ratios of the incoming backward waves be \(x\) (leftmost) and \(y\) (rightmost). As above let \(B\) and \(F\) denote the \(\xi\)-ratios of the outgoing backward and forward waves, respectively. Traversing the waves before and after interaction yields

\[
BF = xy \\
\tilde{\phi}(B) - B\tilde{\phi}(F) = \tilde{\phi}(x) + x\tilde{\phi}(y).
\]

Using (4.7) we get that the \(\xi\)-ratio \(B\) solves the equation

\[
G(B; x, y) := \tilde{\phi}(B) + xy\tilde{\phi} \left( \frac{B}{xy} \right) - \tilde{\phi}(x) - x\tilde{\phi}(y) = 0.
\]

The condition for having no vacuum is that \(B > 0\), or

\[
\kappa(1 + xy) + \tilde{\phi}(x) + x\tilde{\phi}(y) > 0 \quad \text{(no vacuum in overtaking)}.
\]

A direct calculation shows that there is never vacuum formation in any of the overtaking interactions. Furthermore, by analyzing the interactions in the \((r, s)\)-plane of Riemann invariants, we obtain that: Ia \((\hat{S}\hat{S})\) interactions always yield \(\hat{S}\hat{R}\); Ib \((\hat{S}\hat{R})\) interactions yield either \(\hat{S}\hat{S}\) (when the incoming shock is strong relative to the incoming rarefaction), or \(\hat{R}\hat{S}\) (when the incoming shock is weak relative to the incoming rarefaction); Ic \((\hat{R}\hat{S})\) interactions yield either \(\hat{S}\hat{S}\) (when the incoming shock is strong relative to the incoming rarefaction), or \(\hat{R}\hat{S}\) (when the incoming shock is weak relative to the incoming rarefaction). Similar statements apply to Ia’, Ib’, and Ic’ interactions.

5. Non-existence of Glimm-type TVD fields for isentropic flow

The representation of Glimm-type TVD fields in Theorem 2.4 was obtained by analyzing only head-on interactions. To show that no such field exists for the \(p\)-system we will consider specific overtaking interactions. We work in the Riemann coordinates \(r, s\), given in (3.2), which vary over

\[
\mathcal{H} = \{ (r, s) \mid s > r \}.
\]

We restrict attention to non-degenerate fields:

**Definition 5.1.** A scalar field \(\varphi : \mathcal{H} \to \mathbb{R}\) is called non-degenerate provided the set

\[
\mathcal{H}_\varphi := \{ (r, s) \in \mathcal{H} \mid \partial_r \varphi(r, s) \neq 0 \text{ or } \partial_s \varphi(r, s) \neq 0 \} \quad \text{is dense in } \mathcal{H}.
\]

Our main result is the following:

**Theorem 5.2.** There is no non-degenerate \(C^2\)-smooth TVD field of Glimm type for (1.1)-(1.3), with \(p > 1\), which is defined on all of \(\mathcal{H}\).

Towards the proof of Theorem 5.2 we shall first show that the non-degeneracy condition (5.1), together with the TVD property, implies strict non-degeneracy (cf. Theorem 2.4):

**Definition 5.3.** A scalar field \(\varphi : \mathcal{H} \to \mathbb{R}\) is called strictly non-degenerate provided the set

\[
\{ (r, s) \in \mathcal{H} \mid \partial_r \varphi(r, s) \neq 0 \text{ and } \partial_s \varphi(r, s) \neq 0 \} \quad \text{is dense in } \mathcal{H}.
\]

**Lemma 5.4.** Any \(C^2\)-smooth, non-degenerate Glimm-type TVD field for (1.1)-(1.3) is necessarily strictly non-degenerate.

**Proof.** Assume for contradiction that \(\varphi\) is non-degenerate but not strictly non-degenerate. By definition there then exists an open set \(\mathcal{A} \subset \mathcal{H}\) such that for any point \((r, s) \in \mathcal{A}\), either \(\partial_r \varphi(r, s) = 0\) or \(\partial_s \varphi(r, s) = 0\). As \(\varphi\) is non-degenerate, \(\mathcal{A}\) must meet the open set \(\mathcal{H}_\varphi\). Thus, there exists \((r^*, s^*) \in \mathcal{A} \cap \mathcal{H}_\varphi\), with \(\partial_r \varphi(r^*, s^*) \neq 0\), say. By smoothness of \(\varphi\) it follows that there is an open rectangle \(R = (r_1, r_2) \times (s_1, s_2) \subset \mathcal{A} \cap \mathcal{H}_\varphi\) where \(\partial_r \varphi(r, s) \neq 0\), and at the same time \(\partial_s \varphi(r, s) = 0\) (since \(R \subset \mathcal{A}\)). Thus, \(\varphi(r, s) = \theta(s)\) for \((r, s) \in (r_1, r_2) \times (s_1, s_2)\), and \(\theta'(s) \neq 0\) for \(s \in (s_1, s_2)\).
Now choose \((\bar{r}, \bar{s}) \in R\) and consider \(\tilde{SS}\)-interactions with left state \((\bar{r}, \bar{s})\). Let the incoming waves be so weak that all \((r, s)\)-states in the solution belongs to \(R\), and also such that \(\text{sgn} \theta'(s) = \text{sgn} \theta'({\bar{s}})\) for all \(s\)-values in the solution; this is possible by smoothness of \(\theta\). The outgoing wave-pattern is \(\tilde{S}\tilde{R}\).

By considering the wave-curves in the \((r, s)\)-plane it follows that \(x \mapsto \theta(s(t, x))\) is strictly monotone at times \(t\) before the interaction, whereas it is non-monotone after interaction. We conclude that \(\varphi\) cannot be a Glimm-type TVD field. If instead \(\partial_r \varphi(r^*, s^*) \neq 0\), then a similar argument with weak \(\tilde{S}\tilde{S}\)-interactions yields the same conclusion.

As a consequence of Lemma 5.4 to establish Theorem 5.2 we only need to prove the non-existence of strictly non-degenerate, Glimm-type TVD fields. To do so we assume for contradiction that \(\psi\) admits one representation of the form

\[
\varphi(r, s) = \theta(s) - \psi(r) \quad \text{on all of } \mathcal{H}.
\]  

(5.3)

To reach a contradiction we proceed in two steps:

**Step 1:** By considering the change in \(\text{Var} \ \varphi\) across \(\tilde{S}\tilde{S}\) and \(\tilde{S}\tilde{S}\)-interactions, we obtain that \(\theta\) and \(\psi\) must have the same, strict monotonicity. (See Proposition 5.6.)

**Step 2:** Then, assuming without loss of generality that \(\theta\) and \(\psi\) are both strictly increasing, there are three possibilities: \(\exists a \in \mathbb{R}\) where \(\theta'(a) > \psi'(a)\), vice versa, or \(\theta\) and \(\psi\) differ by a constant. For each case we analyze carefully chosen interactions and obtain that \(\text{Var} \varphi\) increases across these, contradicting the assumed TVD property.

**5.1. Step 1: Monotonicity of \(\theta\) and \(\psi\).** In what follows we use the following notation:

- \(\Delta \ \text{Var} \ \varphi\) denotes the change in spatial variation of \(\varphi\) across a pairwise wave interaction; see (2.6), (2.7), and (2.14).
- For any function \(h \equiv h(s, r)\) and any \(\tilde{S}, \tilde{S}, \tilde{R}, \text{ or } \tilde{R}\) wave with associated \(\xi\)-ratio \(q\) (see (3.5)-(3.6)), we set \(\Delta_q h := h_{\text{right}} - h_{\text{left}}\), where \(h_{\text{right}}\) and \(h_{\text{left}}\) denote the values of \(h\) on the right and on the left of the wave, respectively.

**Lemma 5.5.** We have:

(i) \(\Delta_q s < 0\) and \(\Delta_q r < 0\) for \(\tilde{S}\)-waves and \(\tilde{S}\)-waves;

(ii) \(\Delta_q s > 0\) and \(\Delta_q r = 0\) for \(\tilde{R}\)-waves;

(iii) \(\Delta_q r > 0\) and \(\Delta_q s = 0\) for \(\tilde{R}\)-waves;

**Proof.** Immediate from the form of the wave curves in the Riemann coordinate plane (Figure 3).

The following result narrows down the possible structure of strictly non-degenerate TVD fields in terms of the monotonicity of its two parts.

**Proposition 5.6.** Let \(\varphi\) be a strictly non-degenerate \(C^2\)-smooth TVD field of Glimm type for \(\{1.1\}\)-\(\{1.3\}\), and thus admitting a representation of the form \(\varphi(r, s) = \theta(s) - \psi(r)\) on all of \(\mathcal{H}\). Then \(\theta(s)\) and \(\psi(r)\) are either both strictly increasing functions or both are strictly decreasing functions.

The proof of Proposition 5.6 will follow from Lemma 5.7 - Lemma 5.10.

**Lemma 5.7.** With the notation and assumptions of Proposition 5.6 we have that neither \(\theta'\) nor \(\psi'\) can vanish identically on any open interval.

**Proof.** Suppose for contradiction that \(\psi' \equiv 0\) on an \(r\)-interval \((r_1, r_2)\). By strict non-degeneracy (5.1), there exist an \(s \in (r_1, r_2)\) with \(\theta'(s) \neq 0\). We can now argue as in the proof of Lemma 5.4 to see that the variation of \(\varphi\) along the solution of weak \(\tilde{S}\tilde{S}\)-interactions with left state \((\bar{r}, \bar{s})\), must necessarily increase. Thus, on any open \(r\)-interval, \(\psi' \neq 0\). A similar argument shows that \(\theta' \neq 0\) on any open \(s\)-interval.
Lemma 5.8. With the notation and assumptions of Proposition 5.6 we have that
\[ \psi'(r)\theta'(s) \geq 0 \quad \text{whenever } (r, s) \in \mathcal{H}. \] (5.4)

Proof. If not there exist a point \((\bar{r}, \bar{s}) \in \mathcal{H}\) such that either
\[ \psi'(\bar{r}) < 0 \quad \text{and} \quad \theta'(\bar{s}) > 0, \] (5.5)
or
\[ \psi'(\bar{r}) > 0 \quad \text{and} \quad \theta'(\bar{s}) < 0. \] (5.6)

Suppose (5.5) holds. Then, by continuity, there are intervals \((r_1, r_2) \ni \bar{r}\) and \((s_1, s_2) \ni \bar{s}\) with
\[ \psi'(r) < 0 \quad \text{on } (r_1, r_2) \quad \text{and} \quad \theta'(s) > 0 \quad \text{on } (s_1, s_2). \]

If necessary we reduce the lengths of these intervals so as to have \((r_1, r_2) \times (s_1, s_2) \subset \mathcal{H}\). We then let \((\bar{r}, \bar{s})\) be the left state in an \(SS\)-interaction and choose the strengths of the two incoming shocks sufficiently weak to guarantee that all \((r, s)\)-states in the solution of the interaction belong to \((r_1, r_2) \times (s_1, s_2)\). Let \(x, y, B\) and \(F\) denote the \(\xi\)-ratios (see (3.3)) across the incoming left shock, incoming right shock, outgoing backward shock, and outgoing forward rarefaction, respectively. See the right diagram in Figure 4. By parts (i) and (ii) of Lemma 5.5 we have

\[ \Delta \text{Var} \varphi = |\Delta_B \varphi| + |\Delta_F \varphi| - |\Delta_x \varphi| - |\Delta_y \varphi| \]
\[ = |\Delta_B \theta(s) - \Delta_B \psi(r)| + |\Delta_F \theta(s) - \Delta_F \psi(r)| - |\Delta_x \theta(s) - \Delta_x \psi(r)| - |\Delta_y \theta(s) - \Delta_y \psi(r)| \]
\[ = -\Delta_B \theta + \Delta_B \psi + \Delta_F \theta - \Delta_F \psi + \Delta_x \theta - \Delta_x \psi + \Delta_y \theta - \Delta_y \psi \]
\[ = 2\Delta_F \theta > 0, \] (5.7)

where we have used that
\[ \Delta_B \theta(s) + \Delta_F \theta(s) = \Delta_x \theta(s) + \Delta_y \theta(s) \quad \text{and} \quad \Delta_B \psi(r) = \Delta_x \psi(r) + \Delta_y \psi(r). \] (5.8)

This contradicts the assumed TVD property of \(\varphi\). A similar argument (again for an \(SS\)-interaction) shows that (5.6) contradicts the assumed TVD property. \(\square\)

Lemma 5.9. With the notation and assumptions of Proposition 5.6 we have that \(\theta(s)\) and \(\psi(r)\) cannot change their monotonicity:
\[ \forall r_1, r_2 \in \mathbb{R}: \quad \psi'(r_1)\psi'(r_2) \geq 0, \] (5.9)

and
\[ \forall s_1, s_2 \in \mathbb{R}: \quad \theta'(s_1)\theta'(s_2) \geq 0. \] (5.10)

Proof. For contradiction assume that (5.9) does not hold: there are \(r_1, r_2 \in \mathbb{R}\) with \(r_1 < r_2\) and such that \(\psi'(r_1)\psi'(r_2) < 0\). Then, either
\[ \psi'(r_1) < 0 < \psi'(r_2), \] (5.11)
or
\[ \psi'(r_2) < 0 < \psi'(r_1). \] (5.12)

Since \(\theta'\) cannot vanish identically on \((r_2, \infty)\) (by Lemma 5.7), there exists \(\hat{s} \in (r_2, \infty)\) with \(\theta'(\hat{s}) \neq 0\). We then have:
- if \(\theta'(\hat{s}) > 0\) and (5.11) holds, or if \(\theta'(\hat{s}) < 0\) and (5.12) holds, then the point \((r_1, \hat{s}) \in \mathcal{H}\) yields a contradiction with Lemma 5.8.
- if \(\theta'(\hat{s}) < 0\) and (5.11) holds, or if \(\theta'(\hat{s}) > 0\) and (5.12) holds, then the point \((r_2, \hat{s}) \in \mathcal{H}\) yields a contradiction with Lemma 5.8.

This shows that (5.9) must hold. An analogous argument shows that (5.10) holds. \(\square\)

Lemma 5.10. With the notation and assumptions of Proposition 5.6 we have that \(\psi(r)\) and \(\theta(s)\) are both strictly monotonic.
Proof. By Lemma 5.9, \( \psi \) is a monotone function. Lemma 5.7 then gives that \( \psi \) cannot take the same value twice and it is thus strictly monotonic. The same argument applies to \( \theta \). \( \square \)

**Proof of Proposition 5.6** Without loss of generality assume that \( \psi(r) \) is strictly increasing. If \( \theta(s) \) is not strictly increasing then it is strictly decreasing, by Lemma 5.10. Thus there exists an open \( s \)-interval \( I \) such that \( \theta'(s) < 0 \) for \( s \in I \). Since \( \psi(r) \) is strictly increasing there is an \( \bar{r} \in I \) with \( \psi'(ar{r}) > 0 \). But then \( \psi'(ar{r})\theta'(s) < 0 \) for \( s \in I, s > \bar{r} \), contradicting Lemma 5.8. A similar argument applies if \( \psi(r) \) is strictly decreasing. This concludes the proof of Proposition 5.6. \( \square \)

5.2. **Step 2: Completing the proof of Theorem 5.2** As already indicated we shall assume, for contradiction, that \( \varphi : \mathcal{H} \to \mathbb{R} \) is a strictly non-degenerate \( C^2 \)-smooth TVD field of Glimm type. Theorem 2.4 and Proposition 5.6 provide functions \( \psi(r) \) and \( \theta(s) \) with the same strict monotonicity, and such that the representation (5.3) holds throughout \( \mathcal{H} \).

Since \( \text{Var} \varphi = \text{Var}(\varphi - \varphi) \) we need only consider the case where both \( \psi(r) \) and \( \theta(s) \) are strictly increasing. There are then three mutually exclusive possibilities:

- **Case 1:** \( \exists a \in \mathbb{R} \) such that \( \theta'(a) > \psi'(a) > 0 \), or
- **Case 2:** \( \exists a \in \mathbb{R} \) such that \( \psi'(a) > \theta'(a) > 0 \), or
- **Case 3:** \( \psi = \theta + C \) for some constant \( C \).

In each case we shall show that there is an interaction for which \( \Delta \text{Var} \varphi > 0 \), contradicting the assumed TVD property of \( \varphi \). This will then complete the proof of Theorem 5.2. The interactions we consider are \( \overline{S\bar{S}} \), \( \overline{S\bar{S}} \), and \( \overline{R\bar{S}} \) for each of Cases 1, 2, and 3, respectively.

**Case 1.** By continuity there is an open interval \( J \) and constants \( M > \delta > 0 \) such that

\[
\theta'(s) > M > \psi'(r) + \delta > \delta \quad \text{whenever} \quad r, s \in J.
\]

(5.13)

We consider an \( \overline{S\bar{S}} \)-interaction in which we first choose the \( \xi \)-ratios \( x \) and \( y \) across the incoming left and right shocks, respectively (see Figure 4, right diagram). The \( \xi \)-ratios across the outgoing backward shock and the outgoing forward rarefaction are denoted \( B \) and \( F \), respectively. These are uniquely determined by \( x \) and \( y \) alone, according to (4.7)-(4.8), and the results in Section 4 show that \( x, y, F, B > 1 \). The far left state \( (\bar{u}, \bar{\xi}) \) will then be chosen appropriately; see Remark 4.1.

By (3.5) the \( (u, \xi) \)-state between the two incoming waves is given by

\[
\bar{\xi} = x \xi \quad \text{and} \quad \bar{u} = \bar{u} - \bar{\phi}(x)\xi,
\]

(5.14)

and the \( (u, \xi) \)-state on the far right is given by

\[
\xi = y \bar{\xi} \quad \text{and} \quad u = \bar{u} - \bar{\phi}(y)\bar{\xi}.
\]

(5.15)

We proceed to calculate the changes in \( \varphi(r, s) \) across the incoming waves \( x \) and \( y \). According to (3.12) and (3.13) we have

\[
\Delta_x \varphi = \Delta_x \theta - \Delta_x \psi = -\bar{\xi} \int_1^x \left[ \theta'(\bar{u} - \bar{\phi}(\sigma)\xi + \kappa \sigma \xi)(\bar{\phi}'(\sigma) - \kappa) - \psi'(\bar{u} - \bar{\phi}(\sigma)\xi - \kappa \sigma \xi)(\bar{\phi}'(\sigma) + \kappa) \right] d\sigma,
\]

(5.16)

and

\[
\Delta_y \varphi = \Delta_y \theta - \Delta_y \psi = -\bar{\xi} \int_1^y \left[ \theta'(\bar{u} - \bar{\phi}(\sigma)\xi + \kappa \sigma \xi)(\bar{\phi}'(\sigma) - \kappa) - \psi'(\bar{u} - \bar{\phi}(\sigma)\xi - \kappa \sigma \xi)(\bar{\phi}'(\sigma) + \kappa) \right] d\sigma.
\]

(5.17)

We fix \( x, y > 1 \) such that

\[
\bar{\phi}(x) > \kappa \left( \frac{2M}{x} - 1 \right) (x - 1) \quad \text{and} \quad \bar{\phi}(y) > \kappa \left( \frac{2M}{y} - 1 \right) (y - 1),
\]

(5.18)

which is possible according to (3.8) since \( M > \delta \). Next, fix any \( \bar{u} \in J \) and \( \bar{\xi} > 0 \) so small that

\[
\bar{u} - \bar{\phi}(\sigma)\xi \pm \kappa \sigma \xi \in J \quad \text{for all} \ \sigma \in [1, x],
\]

(5.19)
and
\[ \hat{u} - \tilde{\phi}(\sigma)\xi \pm \kappa\sigma\xi = \hat{u} - \left( \tilde{\phi}(x) + \tilde{\phi}(\sigma)x \mp \kappa\sigma x \right) \xi \in J \quad \text{for all } \sigma \in [1, y], \tag{5.20} \]

This guarantees that the arguments of \( \theta' \) and \( \psi' \) in (5.10) and (5.17) are evaluated at points in \( J \). Finally, the state \((\hat{\xi}, \hat{u})\) between the two outgoing waves is given by (3.5) as
\[ \hat{\xi} = B\hat{\xi} \quad \text{and} \quad \hat{u} = \hat{u} - \tilde{\phi}(B)\hat{\xi}, \]
where \( B \) is determined entirely by \( x \) and \( y \) (see Remark 4.1). The corresponding \( r \) and \( s \) values are \( \hat{u} - \tilde{\phi}(B)\hat{\xi} \pm \kappa\hat{\xi} \).

We now combine this with (5.13) to determine the signs of \( \Delta_x \varphi \) and \( \Delta_y \varphi \). Applying (5.13) and (5.19)-(5.20) in (5.16) and (5.17), and using that \( \tilde{\phi}'(\sigma) > \kappa \) for \( \sigma > 1 \), we get
\[ \Delta_x \varphi < \hat{\xi} \int_1^\theta \kappa(2M - \delta) - \tilde{\phi}'(\sigma) d\sigma = \hat{\xi} \left[ \kappa(2M - \delta)(x - 1) - \tilde{\phi}(x) \right], \]
and
\[ \Delta_y \varphi < \hat{\xi} \int_1^\theta \kappa(2M - \delta) - \tilde{\phi}'(\sigma) d\sigma = \hat{\xi} \left[ \kappa(2M - \delta)(y - 1) - \tilde{\phi}(y) \right]. \]

By (5.18) we therefore have
\[ \Delta_x \varphi, \Delta_y \varphi < 0. \]

Recalling Lemma 5.5 and using that \( \theta \) and \( \psi \) are both strictly increasing, we conclude that
\[ \Delta \text{Var } \varphi = |\Delta_B \varphi| + |\Delta_F \varphi| - |\Delta_x \varphi| - |\Delta_y \varphi| \]
\[ = |\Delta_B \theta - \Delta_B \psi| + |\Delta_F \theta - \Delta_F \psi| + \Delta_x \varphi + \Delta_y \varphi \]
\[ = |\Delta_B \theta - \Delta_B \psi| + \Delta_F \theta + \Delta_x \theta - \Delta_x \psi + \Delta_y \theta - \Delta_y \psi \]
\[ = |\Delta_B \theta - \Delta_B \psi| + (\Delta_B \theta - \Delta_B \psi) + 2\Delta_F \theta \]
\[ \geq 2\Delta_F \theta > 0, \]

where we have used (5.8). We have thus provided an \( S \bar{S} \)-interaction in which the total variation of \( \varphi \) increases. That is, Case 1 cannot apply if \( \varphi \) is a strictly non-degenerate TVD field of Glimm type.

**Case 2:** The argument is similar to that of Case 1. For completeness we include the details. By continuity there is an open interval \( J \) and constants \( M > \delta > 0 \) such that
\[ \psi'(r) > M > \theta'(s) + \delta \quad \text{whenever } r, s \in J. \tag{5.21} \]

We then consider an \( S \bar{S} \)-interaction for which we first choose the \( \xi \)-ratios \( x \) and \( y \) across the incoming left and right shocks, respectively. The \( \xi \)-ratios across the outgoing backward rarefaction and the outgoing forward shock are denoted \( B \) and \( F \), respectively. As before these are uniquely determined by \( x \) and \( y \). By the results in Section 4, we have \( 0 < x, y, F, B < 1 \). The far left state \((\hat{u}, \hat{\xi})\) will be chosen appropriately; see Remark 4.1.

By (3.6) the \((u, \xi)\)-state between the two incoming waves is given by
\[ \hat{\xi} = x\xi \quad \text{and} \quad \hat{u} = \hat{u} + \tilde{\phi}(x)\xi, \tag{5.22} \]
and the \((u, \xi)\)-state on the far right is given by
\[ \xi = y\xi \quad \text{and} \quad u = \hat{u} + \tilde{\phi}(y)\xi. \tag{5.23} \]

We proceed to calculate the changes in \( \varphi(r, s) \) across the incoming waves \( x \) and \( y \). By using expressions similar to those in (3.12) and (3.13), but now for changes across forward waves, we have
\[ \Delta_x \varphi = \Delta_x \theta - \Delta_x \psi \]
\[ = \tilde{\xi} \int_x^1 \left[ - \psi'(u + \tilde{\phi}(\sigma)\xi + \kappa\sigma \xi)(\tilde{\phi}'(\sigma) + \kappa) + \psi'(u + \tilde{\phi}(\sigma)\xi - \kappa\sigma \xi)(\tilde{\phi}'(\sigma) - \kappa) \right] d\sigma, \tag{5.24} \]
Finally, the state \((\hat{\xi}, \hat{u})\) in (5.24) and (5.25), and using that
\[
\Delta \varphi = \Delta_x \theta - \Delta_x \psi
\]
\[
= \hat{\xi} \int_y^1 \left[ -\theta'(\hat{u} + \bar{\phi}(\sigma)\hat{\xi} + \kappa \sigma \hat{\xi})(\bar{\phi}'(\sigma) + \kappa) + \psi'(\hat{u} + \bar{\phi}(\sigma)\hat{\xi} - \kappa t \hat{\xi})(\bar{\phi}'(\sigma) - \kappa) \right] d\sigma. \tag{5.25}
\]
Using (3.8) and \(M > \delta\), we fix \(x, y \in (0, 1)\) such that
\[
\bar{\phi}(x) < \kappa (1 - \frac{2M}{\delta}) (1 - x) \quad \text{and} \quad \bar{\phi}(y) < \kappa (1 - \frac{2M}{\delta}) (1 - y). \tag{5.26}
\]
Next, fix any \(\bar{u} \in J\) and choose \(\hat{\xi} > 0\) so small that
\[
\bar{u} + \bar{\phi}(\sigma)\hat{\xi} \pm \kappa \sigma \hat{\xi} \in J \quad \text{for all} \ \sigma \in [x, 1], \tag{5.27}
\]
and
\[
\bar{u} + \bar{\phi}(\sigma)\hat{\xi} \pm \kappa \sigma \hat{\xi} = \bar{u} + \left(\bar{\phi}(x) + \bar{\phi}(\sigma)x \pm \kappa \sigma x\right) \hat{\xi} \in J \quad \text{for all} \ \sigma \in [y, 1]. \tag{5.28}
\]
This guarantees that the arguments of \(\theta'\) and \(\psi'\) in (5.21) and (5.25) are evaluated at points in \(J\).
Finally, the state \((\hat{\xi}, \hat{u})\) between the two outgoing waves is given by (3.5) as
\[
\hat{\xi} = B\hat{\xi} \quad \text{and} \quad \hat{u} = \bar{u} - \bar{\phi}(B)\hat{\xi},
\]
where \(B\) is determined entirely by \(x\) and \(y\) (see Remark 4.1). The corresponding \(r\)- and \(s\)-values are \(\bar{u} - \bar{\phi}(B)\hat{\xi} \pm \kappa \hat{\xi}\).

We now combine this with (5.24) to determine the signs of \(\Delta_x \varphi\) and \(\Delta_y \varphi\). Applying (5.21) and (5.27) in (5.24) and (5.25), and using that \(\bar{\phi}'(\sigma) > \kappa\) for \(0 < \sigma < 1\), we get
\[
\Delta_x \varphi > \hat{\xi} \int_x^1 \kappa(\delta - 2M) + \delta \bar{\phi}'(\sigma) d\sigma = \hat{\xi} \left[\kappa(\delta - 2M)(1 - x) - \delta \bar{\phi}(x)\right],
\]
and
\[
\Delta_y \varphi > \hat{\xi} \int_y^1 \kappa(\delta - 2M) + \delta \bar{\phi}'(\sigma) d\sigma = \hat{\xi} \left[\kappa(\delta - 2M)(1 - y) - \delta \bar{\phi}(y)\right].
\]
By (5.26) we therefore have
\[
\Delta_x \varphi, \Delta_y \varphi > 0.
\]
Recalling Lemma 5.5 and using that \(\theta\) and \(\psi\) are both strictly increasing, we conclude that
\[
\Delta \mathrm{Var}(\varphi) = |\Delta_B \varphi| + |\Delta_F \varphi| - |\Delta_x \varphi| - |\Delta_y \varphi|
\]
\[
= |\Delta_B \bar{\theta} - \Delta_B \bar{\psi}| + |\Delta_F \bar{\theta} - \Delta_F \bar{\psi}| - \Delta_x \varphi - \Delta_y \varphi
\]
\[
= \Delta_B \bar{\psi} + |\Delta_F \bar{\theta} - \Delta_F \bar{\psi}| - \Delta_x \psi - \Delta_y \psi + \Delta_x \theta + \Delta_y \theta + \Delta_x \psi + \Delta_y \psi
\]
\[
= 2\Delta_B \bar{\psi} + |\Delta_F \bar{\theta} - \Delta_F \bar{\psi}| - (\Delta_F \bar{\theta} - \Delta_F \bar{\psi}) > 0
\]
\[
\geq 2\Delta_B \bar{\psi} > 0,
\]
where we have used
\[
\Delta_x \theta(s) + \Delta_y \theta(s) = \Delta_F \theta(s) \quad \text{and} \quad \Delta_x \psi(r) + \Delta_y \psi(r) = \Delta_F \psi(r) + \Delta_B \psi(r).
\]
This provides an interaction in which the total variation of \(\varphi\) increases, such that Case 2 cannot apply if \(\varphi\) is a strictly non-degenerate TVD field of Glimm type.

The only remaining possibility for a strictly non-degenerate TVD field of Glimm type \(\varphi(r, s) = \theta(s) - \psi(r)\), is that Case 3 holds: \(\theta(s)\) and \(\psi(r)\) differ by a constant. Note that the Liu functional \(L(t)\), defined in (1.5) for \(\gamma = 1\), falls into this case with \(\theta \equiv \psi \equiv \text{id}\). However, as we show next, for \(\gamma > 1\) there are interactions in which the total variation of such \(\varphi\) must necessarily increase.
we apply (5.31) in (5.30) to obtain

$$x_\xi$$

strictly increasing, there are positive constants $B$, $N_U$, $N_L$, and $N_U$ such that

$$|\theta''(s)| < M_U, \quad N_U > \theta'(s) > N_L > 0, \quad \forall s \in I. \quad (5.29)$$

As before we denote the far left state in the interaction by $(\bar{u}, \bar{\varepsilon})$, while $x < 1$ and $y > 1$ denote the $\xi$-ratios across the incoming rarefaction and shock waves, respectively. Again, $B$ and $F$ denote the $\xi$-ratios across the outgoing backward and forward waves. We recall that these are functions of $x$ and $y$ alone. In particular, $B = B(x, y)$ is given as the solution of (4.9), and we proceed to Taylor expand it about $x = 1$. Differentiating (4.9) with respect to $x$, evaluating at $x = 1$, solving for $\partial_x B(1, y)$, and using that $B(1, y) = y$, we obtain the expansion

$$B(x, y) = y + \left( \frac{\phi(y) + \kappa(y + 1)}{\phi(y) + \kappa} \right) (x - 1) + O_y ((x - 1)^2)$$

$$= xy + \left( y - \frac{\phi(y) + \kappa(y + 1)}{\phi(y) + \kappa} \right) (1 - x) + O_y ((x - 1)^2), \quad (5.30)$$

where $O_y$ indicates that the last term depends also on $y$. We now fix any $\varepsilon > 0$ and then $y > 1$ so large that

$$y - \left( \frac{\phi(y) + \kappa(y + 1)}{\phi(y) + \kappa} \right) > \frac{N_U + 2\varepsilon}{N_L}, \quad (5.31)$$

where we use the fact recorded earlier in (3.9). We then fix $x < 1$ sufficiently close to 1 to guarantee that $B = B(x, y) > 1$, such that the outcome of the $\bar{R}\bar{S}$-interaction is $\bar{S}\bar{S}$. If necessary we further increase $x$ towards 1 so as to guarantee that $\varepsilon(1 - x)/N_L + O_y ((x - 1)^2) > 0$. With these choices we apply (5.31) in (5.30) to obtain

$$B > xy + \frac{N_U + 2\varepsilon}{N_L} (1 - x) + O_y ((x - 1)^2) > xy + \frac{N_U + \varepsilon}{N_L} (1 - x). \quad (5.32)$$

At this point the strengths $x < 1$, $y > 1$, $B = B(x, y) > 1$, and $F(x, y) < 1$, are all fixed. We finally rewrite (5.32), using (4.7), as

$$N_L B(1 - F) > (N_U + \varepsilon)(1 - x). \quad (5.33)$$

We proceed to compute the changes in $\varphi(x, s) = \theta(s) - \theta(r) + Const.$ across each of the waves in the interaction. Applying the identities (3.12) and (3.13) for changes across backward waves we obtain the following expressions

$$\Delta_x \varphi = \Delta_x \varphi(\xi) = \int_{x}^{1} \left( \frac{\phi'(\sigma) + \kappa}{\phi'(\sigma) + \kappa} \right) \theta'(\bar{u} - \phi(\sigma)\xi - \kappa\sigma\xi) d\sigma, \quad (5.34)$$

$$\Delta_y \varphi = \Delta_y \varphi(\sigma) - \Delta_y \varphi(r)$$

$$= -\xi \int_{1}^{y} \left( \phi'(\sigma) - \kappa \right) \theta'(\bar{u} - \phi(\sigma)\xi + \kappa\sigma\xi) - \theta'(\bar{u} - \phi(\sigma)\xi - \kappa\sigma\xi)(\phi'(\sigma) + \kappa) d\sigma$$

$$= -\xi \int_{1}^{y} \left( \phi'(\sigma) - \kappa \right) \left[ \theta'(\bar{u} - \phi(\sigma)\xi - \kappa\sigma\xi) d\sigma - 2\kappa \int_{1}^{y} \theta'(\bar{u} - \phi(\sigma)\xi - \kappa\sigma\xi) d\sigma \right], \quad (5.35)$$
where \( \tilde{u} = \bar{u} - \tilde{\phi}(x) \tilde{\xi}, \) \( \tilde{\xi} = x \tilde{\xi} \) are the \( u- \) and \( \xi- \)values of the state between the two incoming waves. Similarly,

\[
\Delta_B \varphi = \Delta_B \theta(s) - \Delta_B \theta(r) \\
= -\tilde{\xi} \left[ \int_1^B \left( \phi'(\sigma) - \kappa \right) \int_{\tilde{u} - \tilde{\phi}(\sigma) \tilde{\xi} - \kappa \sigma \tilde{\xi}}^\theta(\mu) d\sigma - 2\kappa \int_1^B \theta'(\tilde{u} - \tilde{\phi}(\sigma) \tilde{\xi} - \kappa \sigma \tilde{\xi}) d\sigma \right], \tag{5.36}
\]

The corresponding identities for forward waves yield

\[
\Delta_F \varphi = \Delta_F \theta(s) - \Delta_F \theta(r) \\
= -\tilde{\xi} \left[ \int_F^1 \left( \phi'(\sigma) + \kappa \right) \int_{\tilde{u} - \tilde{\phi}(\sigma) \tilde{\xi} + \kappa \sigma \tilde{\xi}}^\theta(\mu) d\sigma + 2\kappa \int_F^1 \theta'(\tilde{u} - \tilde{\phi}(\sigma) \tilde{\xi} + \kappa \sigma \tilde{\xi}) d\sigma \right], \tag{5.37}
\]

where

\( \hat{u} = \bar{u} - \tilde{\phi}(B) \tilde{\xi}, \) \( \hat{\xi} = B \tilde{\xi} \)

are the \( u- \) and \( \xi- \)values of the state between the two outgoing waves.

It follows from the expressions for \( \hat{\xi}, \hat{u}, \xi, u, \) that for \( \hat{u} \in I \) and \( \xi \) sufficiently small, all arguments of \( \theta' \) and \( \theta'' \) occurring in \( \Delta_F \varphi \) and \( \Delta_B \varphi \), and indeed all \( r- \) and \( s- \)values of all states involved in the interaction, belong to \( I \). Next, it is clear from \( \Delta_F \varphi < 0 \) (5.38)

Furthermore, it follows from \( \Delta_B \varphi > 0 \) (5.39)

provided \( O(1) \xi + N_L B(1 - F) > N_U (1 - x) \), where the \( O(1) \) term depends only on \( B, F, M_U, \) and \( \kappa \). According to \( \xi > 0 \) (5.33) this latter condition is satisfied for \( \tilde{\xi} \) sufficiently small. Hence, by \( \Delta_F \varphi < \Delta_B \varphi \) (5.40) and Lemma 5.5 we obtain that whenever \( \xi \) is sufficiently small, then

\[
\Delta \text{Var } \varphi = |\Delta_B \varphi| + |\Delta_F \varphi| - |\Delta_B \varphi| - |\Delta_F \varphi| \\
= |\Delta_B \varphi - \Delta_F \varphi| + |\Delta_B \varphi - \Delta_F \varphi| \\
= \left\{ |\Delta_B \theta(s) - \Delta_B \theta(r)| - |\Delta_F \theta(s) - \Delta_F \theta(r)| \right\} + \left\{ -\Delta_x \theta(r) - [\Delta_y \theta(s) - \Delta_y \theta(r)] \right\} \\
= -2\Delta_F \theta(s) - 2\Delta_x \theta(r) + 2\Delta_F \theta(r) \\
= -2\Delta_F \varphi + 2\Delta_B \varphi > 0, \tag{5.41}
\]

where we have used the identities

\[
\Delta_B \theta(s) = \Delta_y \theta(s) - \Delta_F \theta(s), \quad \Delta_y \theta(r) = -\Delta_x \theta(r) + \Delta_B \theta(r) + \Delta_F \theta(r).
\]

**Proof of Theorem 5.2** Taken together, Case 1, Case 2, and Case 3 show that the system (1.1)–(1.3) for isentropic flow of an ideal, polytropic gas with \( \gamma > 1 \), does not admit any strictly non-degenerate Glimm-type TVD field \( \varphi \).

6. Final remarks

The non-existence of TVD fields for the \( p \)-system is similar in spirit to Temple’s result [16] that no metric on state space yields \( L^1 \)-contraction for systems.\(^4\) We note that our analysis does not rule out decay for other types of Nishida-like functionals such as

\[
\mathfrak{M}(t) := \varphi_1(r(t)) + \varphi_2(s(t)),
\]

\(^4\)Temple’s result applies to general \( 2 \times 2 \)-systems and was proved by analyzing solutions without interactions.
for suitably chosen scalar fields $\varphi_1, \varphi_2$, say. Furthermore, our results do not rule out the existence of “local” TVD fields, i.e. scalar fields $\varphi$ whose variation decays along all solutions with values in a fixed compact subset of the no-vacuum set $\mathcal{H}$. However, it remains an open problem to prove or disprove that solutions of $(1.1)-(1.3)$ are bounded away from vacuum unless a vacuum appears immediately at time $0^+$. In connection to this we note that our proofs above utilize sufficiently strong interactions, sufficiently close to vacuum.

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