Singular perturbations and first order PDE on manifolds

David Holcman \(^*\) and Ivan Kupka \(^\dagger\)

October 1999

Abstract

In this note we present some results concerning the concentration of sequences of first eigenfunctions on the limit sets of a Morse-Smale dynamical system on a compact Riemannian manifold. More precisely a renormalized sequence of eigenfunctions converges to a measure \(\mu\) concentrated on the hyperbolic sets of the field. The set of all possible measure turns out to be a sum of a finite Dirac distributions localized at the critical point of the field and absolutely continuous measure with respect to the Lebesgue measure on each limit cycles: the coefficients which appear in the limit measure can be characterized using the concentration theory.

In the second part, certain aspects of some first order PDE on manifolds are studied. We study the limit of a sequence solutions of a second order PDE, when a parameter of viscosity tends to zero. Under some explicit assumptions on some vector fields, bounded and differentiable solutions are obtained. We exhibit the role played by the limit sets of the dynamical systems and provide in some cases an explicit representation formula.

1 Introduction

\((V_n, g)\) denotes a compact riemannian manifold of dimension \(n \geq 2\), with no boundary and \(\Delta_g = -\nabla_i \nabla^i\) is the Laplace-Beltrami operator. This note concerns the study of the operator \(L_\epsilon = \epsilon \Delta + \sum_{i=1}^{n} b_i \partial_i + c\) acting on smooth functions, when the parameter \(\epsilon\) converges to zero. \(b\) is a regular vector field and \(c\) is a positive function.

In the first part, we are interested to study the behavior of the first eigenfunction sequence \(u_\epsilon\), associated to the the smallest eigenvalues \(\lambda_\epsilon > 0\) of the operator \(L_\epsilon(u_\epsilon) = \lambda_\epsilon u_\epsilon\), when the parameter \(\epsilon\) converges to zero.

Some local results about these sequences are known on bounded domain of \(\mathbb{R}^n\) (see Friedmann \(\text{[1]}\) \(\text{[2]}\) \(\text{[3]}\), Friedlin Ventcell’ \(\text{[4]}\) ): when \(b\) has only one attracting point, \(u_\epsilon\) converges uniformly on every compact set to a constant as \(\epsilon\) converges to 0, but when the point is repulsive the sequence converges in the distribution sense to a Dirac distribution centered at this point.

The behavior of the first eigenvalue \(\lambda_\epsilon\) as \(\epsilon\) goes to zero is well known for a large class of dynamical system: \(\lambda_\epsilon\) converges to some quantity called the topological pressure \(P\). This number is characterized by a variational problem on the set of probability measures: when the field \(b\) has a finite number of hyperbolic invariant sets \(K\), then the topological pressure is

\[
P = \sup(h_\mu + \int_{V_n} (c - \frac{d \det D\phi_t}{dt}) d\mu | \text{supp } \mu \subset K \text{ and } \mu \phi_t \text{ invariant})
\]  

\(^*\text{Scuola Normale Superiore di Pisa, 7 Piazza dei Cavalieri Italy}\)
\(^\dagger\text{Weizmann Institute of Science, Rehovot 76100, Israel}\)
\(^\ddagger\text{Université Paris VI, Tour 46, 5 etage, 4 Place Jussieu 75005 Paris, France}\)
The set of measures considered here is all the measures with support in $K$ and invariant by the flow. $h_\mu$ is metric entropy (see (3)), $\phi_t$ is the flow induced by the vector field $b$ and $D\phi_t^*\mu$ is the differential of $\phi$ restricted to the unstable bundle of $K$.

The topological pressure $P$ is attained at a measure, called the equilibrium state (see (6)). When the recurrence set of the field $b$ consists of $p$ hyperbolic sets $K_i$, $i = 1..p$ the measure $\mu$ is concentrated on the union of the $K_i$'s. More precisely, $\mu = \sum_{i=1}^p p_i \mu_{K_i}$, where $\mu_{K_i}$ is the equilibrium state associate to $K_i$, $p_i \geq 0$ and $\sum_{i=1}^p p_i = 1$ (see theorem 3.4 in (6)).

Unfortunately this fruitful approach is not adapted to the study of the first eigenfunction problem since the equilibrium measures are not invariant by the flow, in general. However we shall see that the limit measures have their supports on the recurrent sets of the field $b$ although they are not invariant by the flow.

In the second part, we gives some results about the behavior of the solutions of the equation $L_\epsilon u_\epsilon = f$ when $\epsilon$ goes to zero, for a given smooth positive function $f$. Our main interest is to find bounded $C^1$-solutions and to understand the interaction between the geometry of the characteristic curves (trajectories) of the field $b$ and the behaviour of the solutions of $L_\epsilon u_\epsilon = f$ when $\epsilon$ goes to zero. The fascinating point here is that when $\epsilon$ goes to zero the elliptic equation $L_\epsilon u_\epsilon = f$ tends to a hyperbolic one. In this last case the singularities of the solutions propagate along the characteristic whereas, in the elliptic case there is no propagation of singularities.

The fields considered here are Morse-Smale (see (6)), that is 1-) the recurrent set consists of a finite number of hyperbolic stationary points and periodic orbits 2-) the stable and unstable manifolds of the recurrent orbits are pairwise transversal (for all $p,q$, the unstable manifold $W^u(p)$ of $p$ and the stable manifold $W^s(q)$ of $q$ intersect transversally : $T_n(W^u(p)) \oplus T_n(W^s(q)) = T_n(V_n)$). A more general class of vector fields will be considered elsewhere.

**Theorem 1** Suppose that the first eigenvalue of the operator $\Delta_g + a$ is positive and $a$ is a positive function with a finite number of minimum points which are not degenerate (in the sense of a Morse). Consider the first eigenvalue problem (which has the following variational formulation)

$$\lambda_\epsilon = \inf_{u \in H_1(V_n)-\{0\}} \frac{\epsilon \int_{V_n} |\nabla u|^2 + au^2}{\int_{V_n} u^2}$$

(2)

Then, when $\epsilon$ converges to zero the sequence $\lambda_\epsilon$ converges to the minimum of the function $a$ and the set of limits for the weak topology, when $\epsilon$ goes to zero, of the family of measures $\frac{\epsilon \int_{V_n} u^2 dV_n}{\int_{V_n} u^2 dV_n}$ defined by the positive solutions $u_\epsilon$ of the PDE,

$$\epsilon \Delta_g u_\epsilon + au_\epsilon = \lambda_\epsilon u_\epsilon \text{ on } V_n$$

(3)

is contained in the simplex $M = \{\nu = \sum_{i=1}^n c_i^2 \delta_{P_i}, \sum_{i=1}^n c_i = 1\}$ of all probability measures with support in the finite set $\{P_i \mid i = 1..n\}$ where $\delta_P$ denotes the Dirac measure at the point $P$.

Remark: $u_\epsilon$ is uniquely defined up to a multiplicative constant by the Krein-Rutman theorem.

Consider now the case when $b = \nabla \phi$ and the function $c$ is chosen so that the eigenvalue $\lambda_\epsilon$ of the operator $L_\epsilon = \epsilon \Delta + \sum_{i=1}^n b_i \partial_i + c$ is positive on the manifold. To study the family $u_\epsilon$ of solutions of the PDE

$$\epsilon \Delta u_\epsilon + \sum_{i=1}^n b_i \partial_i u_\epsilon + cu_\epsilon = \lambda_\epsilon u_\epsilon \text{ on } V_n,$$

(4)
we apply the transformation $b = \nabla \phi = -2\epsilon \nabla \psi$, and consider the new variable $v_\epsilon = u_\epsilon \psi_\epsilon$. Equation (4) is transformed into the following PDE where the vector field disappear:

$$
\epsilon^2 \Delta v_\epsilon + a_\epsilon v_\epsilon = \epsilon \lambda_\epsilon v_\epsilon \text{ on } V_n
$$

where $a_\epsilon = c\epsilon + |\nabla \phi|^2/4 + \epsilon \Delta \phi/2$.

**Proposition 1** Suppose that the following condition is satisfied : at each minimum points $P_i$ of the function $c$, $c(P_i) + \Delta \phi(P_i)/2 \geq 0$. Let $v_\epsilon$ be a minimizer of the following variational problem

$$
\epsilon \lambda_\epsilon = \inf_{u \in H_1(V_n) - \{0\}} \frac{\epsilon^2 \int_{V_n} |\nabla u|^2 + a_\epsilon u^2}{\int_{V_n} u^2}
$$

then

- $\lim_{\epsilon \to 0} \lambda_\epsilon = \inf_{V_n} (\nabla \phi)^2 = 0$
- The set of limits for the weak topology of the measures $v_\epsilon^2 dV_g$ is localized at the points where the function $\lim_{\epsilon \to 0} \min a_\epsilon$ is zero. These measures have support on the critical points of the vector field $b$.

Moreover $\sup_{V_n} v_\epsilon$ converges to $+\infty$

Going back to the sequence of eigenfunctions, we get that for some subsequence of $\epsilon$ tending to zero

$$
\lim_{\epsilon \to 0} \frac{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2 a_\epsilon}{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2} = \min(\nabla \phi)^2 = 0
$$

and for any function $\psi \in C^\infty(V_n)$ we have for some subsequence of $\epsilon$ tending to zero

$$
\frac{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2 \psi}{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2} \to \sum_{i=1}^n c_i^2 \psi(P_i)
$$

where $c_i^2 = \lim_{\epsilon \to 0} \frac{\int_{B_{\epsilon\delta_i}(\delta)} e^{-\phi/\epsilon} u_\epsilon^2}{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2}$ (the limit is independant of $\delta$). The set of weak limits of the family of measure $\frac{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2}{\int_{V_n} e^{-\phi/\epsilon} u_\epsilon^2} dV_g$ is given by $\sum_{i=1}^n c_i^2 \delta_{P_i}$ where $P_i$ are the critical points of the function $\phi$, the zero of the vector field $b = \nabla \phi$.

Next we will consider vector fields of the following form $b = -\nabla L + \Omega$ where $\Omega$ is fixed but not of gradient type and we will construct $L$ to be a type of Lyapunov function of $\Omega$. Using this vector field, we study the behavior of the sequence of the first eigenfunction for the operator $L_\epsilon$ and we obtain the following theorem:

**Theorem 2** On a compact Riemannian manifold $V_n$, consider a Morse-Smale vector field $\Omega$ not a gradient whose recurrent set consists of the stationary points $P_1, \ldots, P_M$ and of the periodic orbits $\Gamma_1, \ldots, \Gamma_N$. $L$ denotes a special Lyapunov function associated with $\Omega$ defined in the proof of Theorem 2.

For $\epsilon > 0$ let $\lambda_\epsilon$, $u_\epsilon$ denotes respectively the first eigenvalue and an associated eigenfunction of the operator

$$
\epsilon \Delta_g + b \nabla + c \text{ on } V_n
$$

(9)
Then the set of all weak limits as $\epsilon$ goes to zero, of the family of normalized measures

$$
\frac{e^{-L/\epsilon} u^2 dV}{\int_{V_n} e^{-L/\epsilon} u^2 dV} \tag{10}
$$

is contained in the set $M = \{ \nu \mid \nu \text{ Borel measure, support } \nu \subset \bigcup_{j=1}^N \Gamma_j \cup \{ P_i \mid 1 \leq i \leq M \} \}$ and $\nu = \sum_{j=1}^N f_j^2(l_j) dl_j + \sum_{i=1}^M c_i^2 \delta_{P_i}$ where $dl_j$ denotes the arc length on $\Gamma_j$, for the metric $g$, $\delta_{P_i}$ the Dirac measure with support $P_i$, $f_j : \Gamma_j \to \mathbb{R}$ are continuous functions $1 \leq j \leq N$ and the $c_i$ are constants.

To prove the theorem, we will need 3 lemma

**Lemma 1** There exists a Lyapunov function $L$ for the field $\Omega$, such that $L$ is twice differentiable in the neighborhood of the recurrent set of $\Omega$ and reaches its minimum on the union of the neighborhoods on the recurrent sets only. Outside these neighborhoods, $L$ can be arbitrary such that $\Psi(L) = \frac{1}{4}(\| \nabla L \|^2 + 2(\nabla L, \Omega)) \geq 0$

The same type of Lyapunov function $L$ was constructed by Kamin [7] [8] in the case of an attractive point.

**Lemma 2** Under the assumptions of Theorem 2 on the vector field $\Omega$,

$$
\lim_{\epsilon \to 0} \epsilon \lambda_\epsilon = 0 = \min_{V_n} \Psi \tag{11}
$$

where $\Psi = \frac{(\nabla L)^2}{4} + \frac{(\Omega, \nabla L)}{2}$.

**Lemma 3** Under the assumptions of Theorem 2 on the vector field $\Omega$, all weak limits of measures $\nu^2 dV$ as $\epsilon$ goes to zero are concentrated on the minimum set of the function $\Psi$.

The measure $\mu$ is absolutely continuous with respect to the measure induced by the length along the periodic orbit. This is true on each pairwise orbit of the vector field. We have

$$
\frac{d\mu}{dl} = f^2(l) = \lim_{\epsilon \to 0} \int_{H_l} u^2 d\Sigma_g \tag{12}
$$

$H_l$ denotes any hypersurface cutting the orbit at the point of abscisse $l$ transversally.

Remark: The limit is independant of the choice of the hypersurface.

## 2 First order PDE on manifolds

We study the limit of the solution $u_\epsilon$ of $L_\epsilon(u_\epsilon) = f$, where $f$ is a given smooth function on the manifold, when $\epsilon$ tends to zero. The limit of the sequence when $\epsilon$ goes to zero, solves some first order PDE. For some previous works see [13], [19] and [11]. $c$ and $f$ will denote two given positive smooth functions and $b$ a vector field. We suppose that $c_0 = \inf_{V_n} c > 0$ and $b_0 = \sup_{X \in T V_n} 1/2(\nabla_i b^k + \nabla_k b^i)(X_i, X_k)$ a finite number. $S$ denotes the set of separatrices associates to the dynamical system. We prove the following theorem:

**Theorem 3** On a compact Riemannian manifold, consider a Morse-Smale vector field $b$ and let $c$ be a positive function satisfying $c(x) \geq c_0 > 0$ and $c_0 - b_0 > 0$. $f$ is a differentiable
function. Under these assumptions, there exists a solution \( u \in C^0(V_n) \) such that \( |\nabla u| \in L_\infty(V_n) \) of the first order PDE:

\[
< b, \nabla u > + cu = f \text{ on } V_n
\]  

Moreover if the limit set of the vector field \( b(x) \) consist of a finite number of points \( P_1, \ldots, P_p \), and \( c = c_0 \) is constant larger than the eigenvalues of \( Db \), then \( u \) is in \( C^1(V_n - S) \) and \( u \) is unique and is completely determined by the values \( u(P_i) = \frac{f(P_i)}{c(P_i)} \). When \( b \) possess some limit cycles, the solution is not unique and has no limit near the limit cycles.

We adapt the previous theorem to the nonlinear first order PDE on a compact manifold.

\[
< b(u, x), \nabla u > + cu(x)u = f
\]  

where \( b(\lambda, x) \) is a regular vector field, \( \lambda \) is parameter and \( c(\lambda, x) \) and \( f \) are two given functions. The purpose of this part is to find some conditions on \( f, c \) and \( b \) to insure regular solutions, since there exists examples where shocks occur. To prove the existence of solutions for (13), we use an elliptic regularization and proceed by successive approximations, (see Jausselin et al. [12]).

For the last theorem we need the following notations:

\[
 b_0 = \frac{1}{2} \sup_{\|X\| = 1, X \in V_n, \lambda \in \mathbb{R}} (\nabla, b(x, \lambda) + \nabla b(x, \lambda))(X, X)
\]

and \( \gamma = \sup_{\lambda \in \mathbb{R}, x \in V_n} |\partial \lambda b(\lambda, x)| \), \( a_0 = \inf_{\lambda, X} c(\lambda, x) - b_0 \), \( A = \sup_{V_n} |\nabla f| + \sup_{V_n} (f/c) \times \sup_{V_n} |\partial c| \), \( \beta = \sup_{V_n} |c'| \times \sup_{V_n} \frac{L}{c} \), where we use the notation \( c' = \frac{dc}{dx} \).

We make the following assumptions for the rest of this section:

1. \( a_0 > \beta \) that is \( \inf_{\lambda, x} c(\lambda, x) - b_0 > \sup_{V_n} |c'| \sup_{V_n} \frac{L}{c} \)
2. \( (\inf_{\lambda, x} c(\lambda, x) - b_0)^2 + (\sup_{V_n} |c'| \sup_{V_n} \frac{L}{c})^2 \geq 2(\inf_{\lambda, x} c(\lambda, x) - b_0)(\sup_{V_n} |c'| \sup_{V_n} \frac{L}{c} \sup_{V_n} |\nabla f|) \)
3. If \( \Lambda = \inf_{\lambda, x} c(\lambda, u) + uc'(\lambda, u) - b_0 \) and \( \Lambda^2 - 4\Lambda \gamma \geq 0 \)

These assumptions mean that the minimum of the function \( c \) must be large enough with respect to the vector field \( b \) : it is a hyperbolicity condition.

**Theorem 4** On a compact manifold, consider the parametrized smooth vector field \( b(\lambda, x) \), and let \( c(\lambda, x) \) satisfy \( c(\lambda, x) \geq c_0 > 0 \) and \( c_0 \) large so that \( c_0(c_0 - b_0) > \sup_{V_n} (f/c') \) and the conditions above satisfied. Then there exists a solution \( u \in W^{\infty, -1}(V_n) \) such that the first order PDE:

\[
< b(u, x), \nabla u > + cu(x)u = f \text{ on } V_n
\]

Moreover if the limit set of the vector field \( b(u, x) \) is a union of a finite number of points \( P_1, \ldots, P_p \), \( u \) is in \( C^1(V_n - S) \), where \( S \) is the set of separatrices of the field \( b(u, x) \). If the number of solution of \( c(u(P_i), P_i)u(P_i) = f(P_i) \) is finite and equal to \( k \), the number of solutions \( u \) is \( k^p \).

**References**

[1] M.D. Donsker S. Varadham On a variational formula for the principal eigenvalue for the operators with maximum principle, Proc. Nat. Acad. Sci. USA, 72, 1975, p780-783.
[2] M. Friedlin–A. Wentzell–Random Pertubations of Dynamical Systems, Springer-Verlag, 1984

[3] A. Devinatz–A. Friedman–The Asymptotic behavior of the solution of a singular perturbed Dirichlet problem Indiana Univ. Math. J. 27, No. 3, 1978 p527-537.

[4] A. Friedman–The Asymptotic behavior of the First Real Eigenvalue of Second Order Elliptic Operators with a Small Parameter in the Highest Derivative, Indiana Univ. Math. J. 22, No. 10, 1973 p1005-1015.

[5] A. Devinatz– R. Ellis–A. Friedman–The Asymptotic behavior of the First Real Eigenvalue of Second Order Elliptic Operators with a Small Parameter in the Highest Derivative, II Indiana Univ. Math. J. 23, No. 11, 1974 p991-1011.

[6] A. Devinatz– A. Friedman– Asymptotic Behavior of the Principal Eigenfunction for a Singularly Perturbed Dirichlet Problem Indiana Univ. Math. J. 27, No. 1, 1978 p143-157.

[7] S. Kamin–Exponential descent of solutions of elliptic singular pertubation problems, Comm P.D.E, 9(2) 1984 p197-213.

[8] S. Kamin– Singular pertubation problems and Hamilton-Jacobi Integral Equations and Operator Theory, Vol. 9, 1986 p95-105.

[9] Y. Kifer– Principal eigenvalues, Topological pressure, and stochastic stability of equilibrium states, Israel Journal of Math. Vol. 70, No.1 1990

[10] Y. Kifer– Random Perturbations of Dynamical Systems, Birkhäuser, 1988

[11] J. Moser– A rapidly convergent iteration method and non-linear partial differential equations -I Ann. Scuola Norm. Sup. Pisa,8, 1965 p 290- 313.

[12] H. Jausselin–H. Kreiss– J. Moser– On the forced Burgers equation with periodic Boundary conditions Proceeding of the A.M.S. ,8, 1999 p 290- 313.

[13] C. Robinson– Dynamical Systems, CRC press 1995.

[14] S. Smale– Differential Dynamical Systems, Bull. of A.M.S, 1967 p747-830.

[15] R. Sacker– A new approach to the pertubation theory of invariant surfaces Comm. on Pure and Appl. math. VolXVIII, 1965 p.717-732 .

[16] R. Sacker– A pertubation theorem for invariant Riemannian manifolds, Symp. on Diff Eq. Puerto Rico 1965 p43-54.