ZERO-ENTROPY DYNAMICAL SYSTEMS WITH GLUING ORBIT PROPERTY

PENG SUN
China Economics and Management Academy
Central University of Finance and Economics
Beijing 100081, China

Abstract. We show that a dynamical system with gluing orbit property and zero topological entropy is equicontinuous, hence it is topologically conjugate to a minimal rotation. Moreover, we show that a dynamical system with gluing orbit property has zero topological entropy if and only if it is minimal. These results also lead to some interesting corollaries.

1. Introduction

Gluing orbit property is a much weaker form of specification property. It is introduced in [16], [5] and [3] to study various properties of non-uniformly hyperbolic systems. In [2] it is illustrated that a system with gluing orbit property, which shall be called a GO system in this article for short, may have zero topological entropy, which differs from the case for specification property. However, we perceive that such a system should be quite simple and special. In [15], we have shown that a zero-entropy GO system must be minimal and every equicontinuous minimal system is GO. In this article, we give a complete characterization of zero-entropy GO systems.

Theorem 1.1. A topological dynamical system is a GO system of zero topological entropy if and only if it is minimal and equicontinuous, i.e. it is topologically conjugate to a minimal rotation on a compact abelian group.

Based on the known results from [15], to prove Theorem 1.1 we just need to show that a zero-entropy GO system is equicontinuous. This is made possible by taking advantage of a sequence of notions in topological dynamics, among which the most important one is uniform rigidity. We are actually able to prove that the following list of conditions are equivalent to each other.

Theorem 1.2. Let \((X, f)\) be a GO system. The followings are equivalent:

1. \((X, f)\) has zero topological entropy.
2. \((X, f)\) is minimal.
3. \((X, f)\) is equicontinuous (uniformly almost periodic).
4. \((X, f)\) is uniformly rigid.

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2 PENG SUN

(5) \((X, f)\) is uniquely ergodic.

It is clear that \((3) \implies (1)\). We shall prove \((1) \implies (4)\) in Proposition 3.6. By [17, Theorem 3.1.3], equicontinuity is equivalent to uniform almost periodicity. Finally, we prove \((4) \implies (3)\) in Section 4. Moreover, \((2)(3) \implies (5)\) and we have shown in [15] that \((5) \implies (2)\).

The following is a simple corollary of the above theorems obtained from uniform rigidity (cf. Proposition 2.18). It is also a bit unexpected that there is no zero-entropy GO system that is either non-invertible or symbolic.

**Corollary 1.3.** Let \((X, f)\) be a GO system. Then it must have positive topological entropy if either of the followings holds.

1. \(f\) is non-invertible.
2. \((X, f)\) is expansive.

To show that minimality implies zero entropy for GO systems (Proposition 4.9), we introduce a symbolic system obtained from gluing orbit property and construct compact invariant subsets of small entropies. Along with a result in [4] this leads to a byproduct, which gives a partial result related to a conjecture of Katok on intermediate metric entropies. Analogous results can be found in [14, 11, 7].

**Theorem 1.4.** Let \((X, f)\) be an expansive GO system. Then the set 
\[
\mathcal{E}(f) := \{ h_\mu(f) : \mu \text{ is an ergodic measure for } (X, f) \}
\]
is dense in the interval \([0, h(f)]\).

It is shown in [1] that topologically, most dynamical systems are GO. However, as a corollary of Theorem 1.2 (Proposition 4.9), we note an example that is somewhat contrary to [1, Corollary 2]. In [8], Herman considered a family of \(C^\infty\) diffeomorphisms \(\{F_\alpha : \alpha \in \mathbb{T}^1\}\) on \(X = \mathbb{T}^1 \times \text{SL}(2, \mathbb{R})/\Gamma\), where \(\Gamma\) is a cocompact discrete subgroup. For each \(\alpha \in \mathbb{T}^1\), \(F_\alpha(\theta, g\Gamma) = (R_\alpha(\theta), A_\theta(g\Gamma))\) is a skew product, where \(R_\alpha(\theta) = \theta + \alpha\) is the rotation on \(\mathbb{T}^1\),
\[
A_\theta(g\Gamma) = \begin{pmatrix}
\cos 2\pi \theta & -\sin 2\pi \theta \\
\sin 2\pi \theta & \cos 2\pi \theta \\
\end{pmatrix}
\begin{pmatrix}
\lambda & 0 \\
0 & 1/\lambda \\
\end{pmatrix}
\]
g\(\Gamma\) for each \(g\\Gamma \in \text{SL}(2, \mathbb{R})/\Gamma\)
and \(\lambda > 1\) is a fixed number.

**Theorem 1.5** ([8]). \(h(F_\alpha) > 0\) for every \(\alpha \in \mathbb{T}^1\). There is dense \(G_5\) subset \(W \subset \mathbb{T}^1\) such that \(F_\alpha\) is minimal for every \(\alpha \in W\).

By Proposition 4.9, \(F_\alpha\) must not be GO for every \(\alpha \in W\). Actually, we doubt if there is any element \(F_\alpha\) that is GO.

**Corollary 1.6.** There is a one-parameter family \(\{F_\alpha : \alpha \in \mathbb{T}^1\}\) of \(C^\infty\) diffeomorphism and a residual subset \(W \subset \mathbb{T}^1\) such that \(F_\alpha\) is not GO for every \(\alpha \in W\).

2. Preliminaries

Let \((X, d)\) be a compact metric space. Throughout this article, we assume that \(X\) is infinite to avoid trivial exceptions. Let \(f : X \to X\) be a continuous map. Then \((X, f)\) is conventionally called a topological dynamical system or just a system. We shall denote by \(\mathbb{Z}^+\) the set of all positive integers and by \(\mathbb{N}\) the set of all nonnegative integers, i.e. \(\mathbb{N} = \mathbb{Z}^+ \cup \{0\}\). For \(M \in \mathbb{Z}^+\), we denote by
\[
\Sigma_M := \{1, 2, \cdots, M\}^\mathbb{Z}^+.
\]
the space of sequences in \( \{1, 2, \cdots, M\} \).

2.1. Gluing orbit property.

**Definition 2.1.** We call a sequence \( C = \{(x_j, m_j)\}_{j \in \mathbb{Z}^+} \) of ordered pairs in \( X \times \mathbb{Z}^+ \) an orbit sequence. A gap for an orbit sequence is a sequence \( G = \{t_j\}_{j \in \mathbb{Z}^+} \) of positive integers. For \( \epsilon > 0 \), we say that \((C, G)\) can be \( \epsilon \)-shadowed by \( z \in X \) if for every \( j \in \mathbb{Z}^+ \),

\[
d(f^{s_j+t_l}(z), f^l(x_j)) \leq \epsilon \quad \text{for every } l = 0, 1, \cdots, m_j - 1, \tag{1}
\]

where

\[
s_1 = 0 \quad \text{and} \quad s_j = \sum_{i=1}^{j-1} (m_i + t_i - 1) \quad \text{for } j \geq 2.
\]

**Definition 2.2.** \((X, f)\) is said to have gluing orbit property, or called a GO system, if for every \( \epsilon > 0 \) there is \( M = M(\epsilon) > 0 \) such that for any orbit sequence \( C \), there is a gap \( G \in \Sigma_M \) such that \((C, G)\) can be \( \epsilon \)-shadowed.

**Remark.** By [15, Lemma 2.10], Definition 2.2 is equivalent to the original definitions of gluing orbit property (cf. [15, Definition 2.8] and [3, Definition 2.1]). A difference to note is that we allow equalities in (1) for our convenience.

2.2. Topological entropy.

**Definition 2.3.** Let \( K \) be a subset of \( X \). For \( n \in \mathbb{Z}^+ \) and \( \epsilon > 0 \), a subset \( E \subset K \) is called an \((n, \epsilon)\)-separated set in \( K \) if for any distinct points \( x, y \in E \), there is \( k \in \{0, \cdots, n-1\} \) such that

\[
d(f^k(x), f^k(y)) > \epsilon.
\]

Denote by \( s(K, n, \epsilon) \) the maximal cardinality of \((n, \epsilon)\)-separated subsets of \( K \). Then the topological entropy of \( f \) on \( K \) is

\[
h(K, f) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\ln s(K, n, \epsilon)}{n}.
\]

In particular, \( h(f) := h(X, f) \) is the topological entropy of the system \((X, f)\).

**Proposition 2.4.** If \( K \) is a compact subset of \( X \) such that \( f(K) \subset K \). Then

\[
h(K, f) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{\ln s(K, n, \epsilon)}{n}.
\]

2.3. Almost periodicity and Equicontinuity.

**Definition 2.5.** A set \( A \subset \mathbb{N} \) is called syndetic if there is \( L > 0 \) such that

\[
A \cap [k, k + L - 1] \neq \emptyset \quad \text{for any } k \in \mathbb{N}.
\]

**Definition 2.6.** For \( x \in X \) and \( \epsilon > 0 \), denote \( \mathcal{R}(x, \epsilon) := \{n \in \mathbb{Z}^+ : d(x, f^n(x)) \leq \epsilon\} \).

A point \( x \in X \) is called almost periodic in \((X, f)\) if \( \mathcal{R}(x, \epsilon) \) is syndetic for every \( \epsilon > 0 \).
Definition 2.7. For $\varepsilon > 0$, denote
\[ R(\varepsilon) := \bigcap_{x \in X} R(x, \varepsilon) = \{ n \in \mathbb{Z}^+ : d(x, f^n(x)) \leq \varepsilon \text{ for every } x \in X \}. \]
The system $(X, f)$ is called \textit{uniformly almost periodic} if $R(\varepsilon)$ is syndetic for every $\varepsilon > 0$.

Definition 2.8. $(X, f)$ is called \textit{equicontinuous} if the family $\{ f^n \}_{n=1}^{\infty}$ is equicontinuous, i.e. for every $\varepsilon > 0$, there is $\delta > 0$ such that for any $x, y \in X$ with $d(x, y) < \delta$, we have
\[ d(f^n(x), f^n(y)) < \varepsilon \text{ for every } n \geq 0. \]

Proposition 2.9 ([17, Theorem 3.1.3]). $(X, f)$ is equicontinuous if and only if it is uniformly almost periodic.

Proposition 2.10. Every equicontinuous system is has zero topological entropy.

2.4. Transitivity, minimality and weak mixing.

Definition 2.11. Denote by
\[ O(x) := \{ f^n(x) : n \in \mathbb{N} \} \]
the (forward) orbit of $x \in X$ under $f$. Denote by
\[ \text{Tran}(X, f) := \{ x \in X : O(x) = X \} \]
the set of all points whose orbits are dense. Then
\begin{enumerate}
  \item We say that $(X, f)$ is \textit{topologically transitive} if $\text{Tran}(X, f)$ is dense in $X$.
  \item We say that $(X, f)$ is \textit{minimal} if $\text{Tran}(X, f) = X$.
\end{enumerate}
The following facts are clear.

Proposition 2.12. Let $(X, f)$ be a topological dynamical system.
\begin{enumerate}
  \item If $(X, f)$ is GO, then $(X, f)$ is topologically transitive.
  \item $x \in X$ is almost periodic if and only if $(\overline{O(x)}, f|_{\overline{O(x)}})$ is minimal.
\end{enumerate}

A system that is both equicontinuous and minimal is very special and simple. Such a system can be characterized in various ways. The following proposition provides an incomplete summary of them.

Proposition 2.13. Let $(X, f)$ a topological dynamical system. The followings are equivalent:
\begin{enumerate}
  \item $(X, f)$ is equicontinuous and minimal.
  \item $(X, f)$ is equicontinuous and topologically transitive.
  \item $(X, f)$ is minimal and uniformly almost periodic (cf. Proposition 2.9).
  \item $(X, f)$ is minimal and it has (topologically) discrete spectrum.
  \item $(X, f)$ is topologically conjugate to a minimal isometry.
  \item $(X, f)$ is topologically conjugate to a minimal rotation on a compact abelian group.
\end{enumerate}

Definition 2.14. We say that $(X, f)$ is \textit{weakly mixing} if $(X \times X, f \times f)$ is topologically transitive.

Proposition 2.15. Every equicontinuous system is not weakly mixing.
2.5. **Uniform Rigidity.**

The notion of uniform rigidity plays a key role in this article.

**Definition 2.16.** $(X, f)$ is called **uniformly rigid** if there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$f^{n_k} \to \text{Id} \text{ uniformly as } k \to \infty.$$  

Uniform rigidity is closely related to recurrence and almost periodicity.

**Proposition 2.17** (cf. [17, Lemma 9.2.5]). If $(X, f)$ is uniformly rigid, then for every $\varepsilon > 0$, $R(\varepsilon)$ is infinite. In fact, $R(\varepsilon)$ is an IP-set in $\mathbb{Z}^+$.  

The following proposition is needed for Corollary 1.3.

**Proposition 2.18.** Let $(X, f)$ be a uniformly rigid system. Then the followings hold:

1. $f$ is a homeomorphism (cf. [17, Lemma 3.2.11]).
2. $h(f) = 0$ (cf. [6, Proposition 6.3]).

### 3. Proof of Equicontinuity

3.1. **A sufficient condition for uniform rigidity.**

Our key to take advantage of uniform rigidity for GO systems is the point whose orbit stays close to a shift of itself.

**Lemma 3.1.** Suppose that there are $\gamma > 0$, $p \in \text{Tran}(X, f)$ and $m \in \mathbb{Z}^+$ such that

$$d(f^n(p), f^n(f^m(p))) \leq \gamma \text{ for every } n \in \mathbb{N}.$$  

Then for every $x \in X$,

$$d(f^n(x), f^n(f^m(x))) \leq \gamma \text{ for every } n \in \mathbb{N}.$$  

**Proof.** Take any $x \in X$. There is a sequence $\{n_k\}_{k=1}^{\infty}$ in $\mathbb{N}$ such that

$$\lim_{k \to \infty} f^{n_k}(p) = x.$$  

Then for every $n \in \mathbb{N}$,

$$d(f^n(x), f^n(f^m(x))) = \lim_{k \to \infty} d(f^n(f^{n_k}(p)), f^n(f^{n_k}(p))) \leq \gamma.$$  

□

**Corollary 3.2.** If $(X, f)$ is not uniformly rigid, then there is $\gamma > 0$ such that for every $p \in \text{Tran}(X, f)$ and every $m \in \mathbb{Z}^+$, there is $\tau = \tau(p, m) \in \mathbb{N}$ such that

$$d(f^\tau(p), f^\tau(f^m(p))) > \gamma.$$  

3.2. **Zero entropy implies uniform rigidity.**

Based on Corollary 3.2, we can show that a GO system of zero topological entropy must be uniformly rigid.

**Proposition 3.3.** Let $(X, f)$ be a GO system that is not uniformly rigid. Then $h(f) > 0$. 

Let \((X, f)\) be a GO system that is not uniformly rigid. By Corollary 3.2, there are \(p \in X\), \(\gamma > 0\) and \(0 < \varepsilon < \frac{1}{2} \gamma\) such that for \(M = M(\varepsilon)\) and for each \(k = 1, 2, \cdots, 2M - 1\), there is \(\tau_k \in \mathbb{N}\) such that
\[
d(f^{\tau_k}(p), f^{\tau_k}(f^k(p))) > \gamma.
\]
Let
\[
T := 2M + \max\{\tau_k : k = 1, \cdots, 2M - 1\}.
\]
Denote
\[
m_1 := T + M \text{ and } m_2 := T.
\]
For each \(\xi = \{\xi(k)\}_{k=1}^{\infty} \in \Sigma_2 := \{1, 2\}^{\mathbb{Z}^+}\), denote
\[
\mathcal{G}_{\xi} := \{ (p, m_{\xi(k)} + 1) \}_{k=1}^{\infty}.
\]
Then there are \(z_\xi \in X\) and
\[
\mathcal{G}_{\xi'} := \{ t_k(\xi) \}_{k=1}^{\infty} \in \Sigma_M
\]
such that \((\mathcal{G}_{\xi}, \mathcal{G}_{\xi'})\) is \(\varepsilon\)-shadowed by \(z_\xi\).

**Lemma 3.4.** If there is \(n \in \{1, \cdots, N\}\) such that \(\xi(n) \neq \xi'(n)\), then \(z_\xi\) and \(z_{\xi'}\) are \(((N + 1)(T + 2M), \varepsilon)\)-separated.

**Proof.** We may assume that \(\xi(n) = 1\), \(\xi'(n) = 2\) and
\[
\xi(k) = \xi'(k) \text{ for each } k = 1, 2, \cdots, n - 1.
\]
Our discussion splits into two cases:

**Case 1.** Assume that
\[
t_k(\xi) = t_k(\xi') \text{ for each } k = 1, 2, \cdots, n - 1.
\]
Denote
\[
s := \sum_{k=1}^{n-1} (m_{\xi(k)} + t_k(\xi)) = \sum_{k=1}^{n-1} (m_{\xi'(k)} + t_k(\xi')) \leq (n - 1)(T + 2M).
\]
Consider the shadowing properties of \(z_\xi\) and \(z_{\xi'}\) for the \((n + 1)\)-th orbit segment. We have
\[
d(f^{s+T+M+t_n(\xi)+1}(z_\xi), f^l(p)) \leq \varepsilon, \text{ for each } l = 0, 1, \cdots, T;
\]
\[
d(f^{s+T+t_n(\xi')+1}(z_{\xi'}), f^l(p)) \leq \varepsilon, \text{ for each } l = 0, 1, \cdots, T.
\]
Note that \(1 \leq M + t_n(\xi) - t_n(\xi') \leq 2M - 1\). This implies that there is
\[
\tau_* = \tau_{M+t_n(\xi)-t_n(\xi')} \leq T - 2M
\]
such that
\[
d(f^{\tau_*}(p), f^{\tau_*}(f^{M+t_n(\xi)-t_n(\xi')} (p))) > \gamma.
\]
Then
\[
d(f^{s+T+M+t_n(\xi)+\tau_*}(z_\xi), f^{s+T+M+t_n(\xi)+\tau_*}(z_{\xi'})) \geq d(f^{\tau_*}(p), f^{M+t_n(\xi)-t_n(\xi')} + \tau_*(p))
\]
\[
- d(f^{s+T+M+t_n(\xi)+\tau_*}(z_\xi), f^{\tau_*}(p))
\]
\[
- d(f^{s+T+M+t_n(\xi)+\tau_*}(z_{\xi'}), f^{M+t_n(\xi)-t_n(\xi')} + \tau_*(p))
\]
\[
> \gamma - 2\varepsilon > \varepsilon.
\]
Moreover, we have
\[ s + T + M + t_n(\xi) + \tau_* < n(T + 2M) + T \leq (N + 1)(T + 2M) \]

**Case 2.** Assume that
\[ K := \min\{k \in \mathbb{Z}^+ : t_k(\xi) \neq t_k(\xi')\} \leq n - 1. \]
We may assume that \( t_K(\xi) > t_K(\xi') \). Note that
\[ 1 \leq t_K(\xi) - t_K(\xi') \leq M - 1. \]
This implies that there is
\[ \tau_* = \tau_{t_K(\xi) - t_K(\xi')} \leq T - 2M \]
such that
\[ d(f^{\tau_*}(p), f^{t_{t_{K}(\xi)} - t_{t_{K}(\xi')}}(p)) > \gamma. \]

Denote
\[ s := \sum_{k=1}^{K} (m_{\xi(k)} + t_k(\xi)) = \sum_{k=1}^{K} (m_{\xi'(k)} + t_k(\xi')) + t_K(\xi) - t_K(\xi'). \]

Shadowing properties yield
\[ d(f^{s+l}(z_\xi), f^l(p)) \leq \varepsilon, \text{ for each } l = 0, 1, \cdots, T; \]
\[ d(f^{s-(t_{t_K(\xi)} - t_{t_K(\xi')}}+l)(z_\xi'), f^l(p)) \leq \varepsilon, \text{ for each } l = 0, 1, \cdots, T. \]

Then
\[
\begin{align*}
& \quad \quad d(f^{s+\tau_*}(z_\xi), f^{s+\tau_*}(z_{\xi'})) \\
& \quad \quad \quad \geq d(f^{\tau_*}(p), f^{t_{t_{K}(\xi)} - t_{t_{K}(\xi')}}\tau_*(p)) \\
& \quad \quad \quad \quad - d(f^{s+\tau_*}(z_\xi), f^{\tau_*}(p)) \\
& \quad \quad \quad \quad - d(f^{s+\tau_*}(z_{\xi'}), f^{t_{t_{K}(\xi)} - t_{t_{K}(\xi')}}\tau_*(p)) \\
& \quad \quad \quad \quad > \gamma - 2\varepsilon > \varepsilon.
\end{align*}
\]
Clearly,
\[ s + \tau_* < K(T + 2M) + T \leq (N + 1)(T + 2M). \]

By Lemma 3.4, for each \( N \), there is an \(((N + 1)(\tau + 2M), \varepsilon)\)-separated set whose cardinality is \( 2^N \). This yields that
\[ h(f) \geq \limsup_{N \to \infty} \frac{\ln 2^N}{(N + 1)(\tau + 2M)} = \frac{\ln 2}{\tau + 2M} > 0. \]

### 3.3. Uniform rigidity implies uniform almost periodicity.

We remark that there are minimal and uniformly rigid systems that are weakly mixing (cf. [6]). To complete the Proof of Theorem 1.1, we need to show that a uniform rigid GO system is uniformly almost periodic.

**Lemma 3.5.** Let \((X, f)\) be a uniformly rigid GO system. Then for every \( \varepsilon > 0 \) and any \( x, y \in X \), there is \( m \leq M := M(\frac{\tau}{\varepsilon}) \) such that
\[ d(f^n(x), f^m(f^m(y))) \leq \varepsilon \text{ for every } n \in \mathbb{N}. \]
Proof. As $R(\frac{\varepsilon}{3})$ is infinite, we can find a sequence $\{n_k\}_{k=1}^{\infty}$ in $R(\frac{\varepsilon}{3})$ such that

$$n_k \to \infty \text{ and } n_k > M \text{ for every } k \in \mathbb{Z}^+.$$ 

For each $k$, consider

$$C_k = \{(x, n_k - M + 1), (y, n_k)\}.$$ 

There are $z_{n_k} \in X$ and $t_{n_k} \in \{1, 2, \cdots, M\}$ such that $(C_k, \{t_{n_k}\})$ is $\frac{\varepsilon}{3}$-shadowed by $z_k$. We can find a subsequence $\{n_l\}_{l=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ and $t \in \{1, 2, \cdots, M\}$ such that

$$t_{n_l} = t \text{ for every } l \in \mathbb{Z}^+.$$ 

Then

$$M - t \in \{0, 1, \cdots, M - 1\}.$$ 

For every $l \in \mathbb{Z}^+$, the fact of shadowing provides that

$$d(f^j(z_{n_l}), f^j(x)) \leq \frac{\varepsilon}{3} \text{ for every } j = 0, 1, \cdots, n_l - M;$$

$$d(f^{n_l-M+t+j}(z_{n_l}), f^j(y)) \leq \frac{\varepsilon}{3} \text{ for every } j = 0, 1, \cdots, n_l.$$ 

Then for every $l \in \mathbb{Z}^+$ and every $n \leq n_l - M$, we have

$$d(f^n(x), f^n(f^{M-t}(y))) \leq d(f^n(x), f^n(z_{n_l})) + d(f^n(z_{n_l}), f^{n_l}(z_{n_l})) + d(f^{n_l}(z_{n_l}), f^n(f^{M-t}(y)))$$

$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$ 

As $n_l \to \infty$, we have

$$d(f^n(x), f^n(f^{M-t}(y))) \leq \varepsilon \text{ for every } n \in \mathbb{N}. \quad \Box$$

**Proposition 3.6.** Let $(X, f)$ be a uniformly rigid GO system. Then $(X, f)$ is uniformly almost periodic.

**Proof.** Take $x \in \text{Tran}(X, f)$. For every $\varepsilon > 0$, let $M = M(\frac{\varepsilon}{3})$. For every $k \in \mathbb{N}$, apply Lemma 3.5 for $x$ and $f^k(x)$. There is $m < M$ such that

$$d(f^n(x), f^n(f^m(f^k(x)))) \leq \varepsilon \text{ for every } n \in \mathbb{N}.$$ 

By Lemma 3.1, this implies that

$$d(f^n(y), f^n(f^{m+k}(y))) \leq \varepsilon \text{ for every } n \in \mathbb{N} \text{ and for every } y \in X.$$ 

Hence $k + m \in R(\varepsilon)$ and $R(\varepsilon)$ is syndetic. Then $(X, f)$ is uniformly almost periodic. \quad \Box

### 4. Invariant Sets of Small Entropy

#### 4.1. The induced shift map.

Let $(X, f)$ is a GO system of positive topological entropy $h(f) > 0$. Then by Proposition 2.18, $(X, f)$ is topologically transitive and not uniformly rigid. By Corollary 3.2, there is $\gamma = \gamma(f) > 0$ such that for every $p \in \text{Tran}(X, f)$ and for each $k \in \mathbb{Z}^+$ there is there is $\tau_k = \tau_k(p) \in \mathbb{Z}$ such that

$$d(f^{\tau_k}(p), f^{\tau_k}(f^k(p))) > \gamma. \quad (2)$$
We fix $p \in \text{Tran}(X, f)$. Let $0 < \varepsilon < \frac{1}{2} \gamma$, $M = M(\varepsilon)$ and
\[ \tau > M + \max\{\tau_k : k = 1, \ldots, M - 1\}. \]
We fix
\[ C = C(p, \tau) := \{(p, \tau + 1)\}^{\mathbb{Z}^+}. \]
For each $G \in \Sigma_M$, denote
\[ Y_G := \{x \in X : (C, G) \text{ is } \varepsilon\text{-shadowed by } x\}. \]
Denote
\[ \Sigma = \Sigma(\tau, \varepsilon) := \{G \in \Sigma_M : Y_G \neq \emptyset\} \subset \Sigma_M. \]
Let
\[ Y = Y(\tau, \varepsilon) := \bigcup_{G \in \Sigma} Y_G = \bigcup_{G \in \Sigma} Y_G. \]

**Lemma 4.1.** Assume that $G = \{t_k\}_{k=1}^\infty$ and $\mathcal{G}' = \{t_k'\}_{k=1}^\infty$ in $\Sigma_M$ such that $(C, G)$ is shadowed by $x$, $(C, G')$ is shadowed by $y$ and
\[ t_k = t_k' \text{ for } k = 1, \ldots, n - 1 \text{ and } t_n \neq t_n'. \]
Then $x, y$ are $((n + 1)(\tau + M), \varepsilon)$-separated.

**Proof.** We may assume that $t_n < t_n'$. Note that
\[ 1 \leq t_n' - t_n \leq M - 1. \]
This implies that there is
\[ \tau_s = \tau_{t_n' - t_n} \leq \tau - M \]
such that
\[ d(f^{\tau_s}(p), f^{\tau_s}(f^{t_n'-t_n}(p))) > \gamma. \]
However, for
\[ s := \sum_{k=1}^n (\tau + t_k) \text{ and } s' := \sum_{k=1}^n (\tau + t_k') = s + (t_n' - t_n), \]
shadowing properties yield
\[
\begin{align*}
&d(f^{s'+\tau_s}(x), f^{s'+\tau_s}(y)) \\
&\geq d(f^{\tau_s}(p), f^{\tau_s}(f^{t_n'-t_n}(p))) - d(f^{s'+\tau_s}(x), f^{\tau_s}(p)) \\
&\quad - d(f^{s'+t_n'-t_n+\tau_s}(y), f^{t_n'-t_n+\tau_s}(p)) \\
&> \gamma - 2\varepsilon > \varepsilon.
\end{align*}
\]
This implies that $x, y$ are $((n + 1)(\tau + M), \varepsilon)$-separated, as
\[ s' + \tau_s \leq n(\tau + M) + \tau_s < (n + 1)(\tau + M). \]
\[ \square \]

As a simple corollary of Proposition 4.1, we have:

**Corollary 4.2.** For every $x \in Y$ there is a unique $G = G(x)$ such that $(C, G)$ is $\varepsilon$-shadowed by $x$. Hence $\Sigma = \{G(x) : x \in Y\}$.

Note that $\Sigma_M$ is a symbolic space on which there are a shift map $\sigma$ and a product topology that can be induced by the metric
\[ \rho(\{t_k\}_{k=1}^\infty, \{t_k'\}_{k=1}^\infty) = 2^{-\min\{k \in \mathbb{Z}^+: t_k \neq t_k'\}}. \]
Corollary 4.3. The map $G : Y \to \Sigma_M$ is uniformly continuous.

Proof. Let $n \in \mathbb{Z}^+$. As $f$ is continuous and $X$ is compact, there is $\delta > 0$ such that for each $k = 0, 1, \cdots, (n+1)(\tau + M)$, we have
\[
d(f^k(x), f^k(y)) < \varepsilon \quad \text{whenever } x, y \in Y \text{ and } d(x, y) < \delta.
\]
This implies that
\[
\rho(G(x), G(y)) < 2^{-n} \quad \text{whenever } x, y \in Y \text{ and } d(x, y) < \delta.
\]
Hence $\mathcal{G}$ is continuous. \hfill $\square$

Denote by $\sigma$ the shift map on $\Sigma_M$. It is clear that for every $x \in Y$ and $G(x) = \{t_k\}_{k=1}^\infty$, we have $\sigma(G(x)) = G(f^{\tau t_k}(x))$, i.e. $(\mathcal{E}, \sigma(G(x)))$ can be $\varepsilon$-shadowed by $f^{\tau t_k}(x)$. This implies that $\sigma(\Sigma) \subset \Sigma$. The following lemma shows that both $Y$ and $\Sigma$ are compact.

Lemma 4.4. Let $\{x_n\}$ be a sequence in $Y$ such that $x_n \to x$ in $X$. Then there is $\mathcal{G} \in \Sigma_M$ such that $G(x_n) \to \mathcal{G}$ and $(\mathcal{E}, \mathcal{G})$ can be $\varepsilon$-shadowed by $x$.

Proof. As $\{x_n\}$ is a Cauchy sequence and $G$ is uniformly continuous, $\{G(x_n)\}$ is a Cauchy sequence in $\Sigma_M$. By compactness of $\Sigma_M$, there is $\mathcal{G} = \{t_k\}_{k=1}^\infty \in \Sigma_M$ such that $G(x) \to \mathcal{G}$.

For each $n \in \mathbb{Z}^+$, denote
\[
G(x_n) = \{t_k(n)\}_{k=1}^\infty \text{ and } s_k(n) = \sum_{i=1}^{k-1}(\tau + t_i(n)) \text{ for each } k.
\]
For each $k \in \mathbb{Z}^+$, there is $N$ such that for every $n > N$,
\[
t_j(n) = t_j \text{ for each } j = 1, 2, \cdots, k \text{ and hence } s_k(n) = s_k := \sum_{i=1}^{k-1}(\tau + t_i).
\]
Then for every $k \in \mathbb{Z}^+$ and every $l = 0, 1, \cdots, \tau$, we have
\[
d(f^{n \tau + l}(x), f^l(p)) = \lim_{n \to \infty} d(f^{n \tau + l}(x_n), f^l(p))
= \lim_{n \to \infty} d(f^{n \tau + l}(x_n), f^l(p)) \text{ (for } n > N)
\leq \varepsilon.
\]
i.e. $(\mathcal{E}, \mathcal{G})$ can be $\varepsilon$-shadowed by $x$. \hfill $\square$

Corollary 4.5. $Y$ is a compact subset of $X$. $\Sigma$ is a compact subset of $\Sigma_M$ and is invariant of the shift map $\sigma$.

4.2. Invariant sets and entropy estimates.

A set of the form
\[
B_n(x, \varepsilon) = \{y \in X : d(f^{k}(y), f^{k}(x)) < \varepsilon, k = 0, 1, \cdots, n-1\}
\]
is called an $(n, \varepsilon)$-ball in $(X, f)$. A subset $E$ of $X$ is called an $(n, \varepsilon)$-spanning set if
\[
X = \bigcup_{x \in E} B_n(x, \varepsilon).
\]
For $0 < \varepsilon < \frac{\lambda}{4}$, let $Y = Y(\tau, \varepsilon)$ and $E_\varepsilon$ be a fixed $(M(\varepsilon) - 1, \varepsilon)$-spanning subset of $X$ with smallest cardinality. Denote by
\[
C_{w_1 \cdots w_n} = \{\{t_k\}_{k=1}^\infty \in \Sigma : t_j = w_j \text{ for each } j = 1, \cdots, n\}
\]
a cylinder of rank \( n \) in \( \Sigma \). For each cylinder \( C \), denote
\[
Y_C = \bigcup_{g \in g} Y_g.
\]

Lemma 4.6. For every \( k \in \{0, \cdots, \tau + M\} \), every \( n \in \mathbb{Z}^+ \) and every cylinder \( C = C_{t_1 \cdots t_{n+2}} \) in \( \Sigma \) of rank \( n + 2 \), there are at most \(|E_\varepsilon|^{n+2}\) points in \( f^k(Y_C) \) that are \((n\tau, 2\varepsilon)\)-separated.

Proof. Denote
\[
s_1 = 0 \text{ and } s_k = \sum_{i=1}^{k-1} (\tau + t_i) \text{ for } k \geq 2.
\]
Assume that \( x_1, x_2 \in f^k(Y_C) \) are \((n\tau, 2\varepsilon)\)-separated. There are \( y_1, y_2 \in Y_C \) such that
\[
f^k(y_1) = x_1 \text{ and } f^k(y_2) = x_2.
\]
Then \( y_1, y_2 \) are \((n\tau + k, 2\varepsilon)\)-separated and hence \(((n+2)\tau, 2\varepsilon)\)-separated since
\[
k \leq \tau + M < 2\tau \text{ when } \tau > M.
\]
But in this time period \( y_1 \) and \( y_2 \) can only be separated when their orbits are not shadowing \( C \). So there must be \( k \leq n + 2 \) and \( 1 \leq t \leq t_k - 1 \) such that
\[
d(f^{s_k+\tau+t}(y_1), f^{s_k+\tau+t}(y_2)) > 2\varepsilon.
\]
This implies that \( f^{s_k+\tau+1}(y_1) \) and \( f^{s_k+\tau+1}(y_2) \) are \((M-1, 2\varepsilon)\)-separated. They must belong to distinct \((M-1, \varepsilon)\)-balls centered at distinct points in \( E_\varepsilon \). Then the result follows.

Corollary 4.7. For every \( k \in \{0, \cdots, \tau + M\} \) and every \( n \in \mathbb{Z}^+ \), we have
\[
s(f^k(Y), n\tau, 2\varepsilon) \leq M^{n+2}|E_\varepsilon|^{n+2}
\]

Proposition 4.8. For \( Y = Y(\tau, \varepsilon) \), let
\[
\Lambda = \Lambda(\tau, \varepsilon) := \bigcup_{k=0}^{\tau+M-1} f^k(Y).
\]
Then \( \Lambda \) is a compact invariant set in \((X, f)\) and for each \( n \in \mathbb{Z}^+ \), we have
\[
s(\Lambda, n\tau, 2\varepsilon) \leq (\tau + M)M^{n+2}|E_\varepsilon|^{n+2}
\]

Proof. \( \Lambda \) is compact since \( Y \) is compact.
For every \( x \in \Lambda \), there is \( y \in Y \) and \( r \in \{0, \cdots, \tau + M - 1\} \) such that \( f^r(y) = x \).
If \( r < \tau + M - 1 \), then \( f(x) = f^{r+1}(y) \in \Lambda \). Assume that \( \mathcal{G}(y) = \{t_k\}_{k=1}^\infty \). Note that \( 1 \leq t_1 \leq M \) and
\[
f^{\tau+t_1}(y) \in Y_\varepsilon(\mathcal{G}) \subset Y.
\]
Hence if \( r = \tau + M - 1 \), then
\[
f(x) = f^{r+1}(y) = f^{\tau+M-(\tau+t_1)}(f^{\tau+t_1}(y)) \in f^{M-t_1}(Y) \subset \Lambda.
\]
So we can conclude that \( f(\Lambda) \subset \Lambda \).
By Corollary 4.7, we have for each \( n \in \mathbb{Z}^+ \),
\[
s(\Lambda, n\tau, 2\varepsilon) \leq \sum_{k=0}^{\tau+M-1} s(f^k(Y), n\tau, 2\varepsilon) \leq (\tau + M)M^{n+2}|E_\varepsilon|^{n+2}.
\]
\[\square\]
4.3. Minimality implies zero entropy.

Proposition 4.9. Let \((X, f)\) be a GO system with positive topological entropy. Then \((X, f)\) is not minimal.

Proof. Take \(\beta \in (0, h(f))\). There is \(\varepsilon_0 \in (0, \frac{2}{3}\gamma)\), where \(\gamma\) is as in (2), such that

\[
\liminf_{n \to \infty} \frac{1}{n} \ln s(X, n, \varepsilon_0) > \beta.
\]  

(3)

For \(M = M(\varepsilon_0)\), we can find a large \(\tau\) and \(N_1\) such that

\[(\tau + M)M^{(n+2)}E_{\varepsilon_0}^{(n+2)} < e^{n\tau\beta}\] for every \(n > N_1\).

By (3), there is \(N_2\) such that

\[s(X, n\tau, \varepsilon_0) > e^{n\tau\beta}\] for every \(n > N_2\).

However, for \(\Lambda = \Lambda(\tau, \varepsilon_0)\), by Proposition 4.8 we have

\[s(\Lambda, n\tau, \varepsilon_0) \leq (\tau + M)M^{(n+2)}E_{\varepsilon_0}^{(n+2)} < e^{n\tau\beta}\] for \(n > \max\{N_1, N_2\}\).

This implies that \(\Lambda(\tau, \varepsilon_0)\) is a proper compact invariant subset of \(X\) and hence \((X, f)\) is not minimal. \(\square\)

4.4. Denseness of metric entropy.

Denote \(\mathcal{I}(f) := [0, h(f))\) and

\[
\mathcal{E}(f) := \{h_\mu(f) : \mu \text{ is an ergodic measure for } (X, f)\}.
\]

Katok [9] showed that \(\mathcal{E}(f) \supset \mathcal{I}(f)\) for any \(C^{1+\alpha}\) diffeomorphism on any surface and conjectured that this holds in any dimension. The author has obtained a sequence of partial results on this conjecture [12, 13, 14]. In [11] Quas and Soo showed that the conjecture holds when \(f\) satisfies almost weak specification (also called almost specification by some other authors, e.g. [10]), asymptotically entropy expansivity and small boundary property. Taking advantage of the result of Quas and Soo, the author and collaborators [7] were able to show that the conjecture holds for certain homogeneous dynamical systems. We also showed in [7] that for every \(f\) satisfying almost weak specification and asymptotically entropy expansivity (without small boundary property), we have \(\mathcal{E}(f) \supset \mathcal{I}(f)\), i.e. the metric entropies of ergodic measures are dense in \(\mathcal{I}(f)\).

Recently, Constantine, Lafont and Thompson [4] showed that every expansive GO system is entropy dense, i.e. for every invariant measure \(\nu\), every neighborhood \(\mathcal{N}\) of \(\nu\) and every \(\varepsilon > 0\), there is an ergodic measure \(\mu \in \mathcal{N}\) such that

\[|h_\mu(f) - h_\nu(f)| < \varepsilon.\]

However, entropy denseness itself, even along with expansivity, does not rule out absence of invariant measures of small entropy. This can be fulfilled in Proposition 4.11. Since every compact invariant subset supports ergodic measures, the following result is verified.

Theorem 4.10. Let \((X, f)\) be an expansive GO system. Then \(\overline{\mathcal{E}(f)} \supset \mathcal{I}(f)\).

Proposition 4.11. Let \((X, f)\) be GO system that is entropy expansive, then for every \(\beta > 0\), there is a compact invariant subset \(\Lambda_\beta\) of \(X\) such that

\[h(\Lambda_\beta, f) \leq \beta.\]
Proof. The statement is trivial if \( h(f) = 0 \). Now we assume that \( h(f) > 0 \) and apply the results we have obtained in this section.

By entropy expansiveness, there is \( \varepsilon_0 > 0 \) such that

\[
h(K, f) = \liminf_{n \to \infty} \frac{\ln s(K, n, \varepsilon_0)}{n}
\]

for every compact invariant subset \( K \) of \( X \). For given \( \beta > 0 \) and \( M = M(\frac{\varepsilon_0}{2}) \), we can find a large \( \tau \) and \( N \) such that

\[
(\tau + M)M^{(n+2)}|E_{\frac{\varepsilon_0}{2}}|^{(n+2)} < e^{n\tau\beta}
\]

for every \( n > N \).

Then \( \Lambda_\beta := \Lambda(\tau, \varepsilon_0) \) is a compact invariant subset of \( X \). By Proposition 4.8 we have

\[
s(\Lambda_\beta, n\tau, \varepsilon_0) \leq (\tau + M)M^{(n+2)}|E_{\frac{\varepsilon_0}{2}}|^{(n+2)} < e^{n\tau\beta}
\]

for \( n > N \).

Hence

\[
h(\Lambda_\beta, f) = h(\Lambda_\beta, f, \varepsilon_0) = \liminf_{n \to \infty} \frac{1}{n} \ln s(\Lambda_\beta, n, \varepsilon_0) \leq \liminf_{n \to \infty} \frac{1}{n\tau} \ln s(\Lambda_\beta, n\tau, \varepsilon_0) \leq \beta.
\]

\[\Box\]

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\textit{E-mail address: sunpeng@cufe.edu.cn}