GLEASON-TYPE POLYNOMIALS FOR RATIONAL MAPS WITH NONTRIVIAL AUTOMORPHISMS

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Abstract. In this paper, we consider a family of degree $d \geq 2$ rational maps with an automorphism group containing the cyclic group of order $d$. We construct a polynomial whose roots are the values of parametrizing variable of the family which make the corresponding map be post-critically finite with a certain dynamical portrait. Then we prove that the polynomial is irreducible in certain cases.

1. Introduction and main results

One of main topics in Arithmetic Dynamics is the study of post-critically finite rational maps. Specifically, if we have a family of rational maps parametrized by a variable (or variables,) it is natural to ask which values of the variable(s) make the map be post-critically finite with a certain dynamical portrait. Most studies were about the family of polynomial maps $z \mapsto z^d + c$ [2], and some papers studied certain families of rational maps of degree 2 [1] [4]. There is not much research about rational maps of higher degree, since in most cases there are lots of critical points whose behaviors are different. However, if we consider a family of rational maps which have certain nontrivial automorphisms, those automorphisms may let dynamical behavior of critical points be similar. In this manner, we study the values of variable(s) which make such maps post-critically finite with a certain dynamical portrait.

Specifically, in this paper, we consider a single-parameter family $\{\phi_a = az/(z^d + d - 1)\}$ of degree $d$ rational maps with an automorphism group containing the cyclic group of order $d$, and construct a polynomial $G_m \in \mathbb{Z}[a]$, which is called the $m$-th Gleason polynomial, whose roots make $\phi_a$ be post-critically finite with preperiod $m$ and period 1. Then we can prove the following.

Theorem 1. Suppose that $d \geq 3$ is prime. Then the $m$-th Gleason polynomial $G_m$ is irreducible over $\mathbb{Q}$ for all $m \leq 3$.

We also have the following conjecture, which is supported by some calculation data.

Conjecture 2. Suppose that $d \geq 2$ is a positive integer. Then the $m$-th Gleason polynomial $G_m$ is irreducible over $\mathbb{Q}$ for all $m$.

2. Construction of Gleason polynomial

The family of degree $d$ rational maps $\phi : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})$, up to $PGL_2$-conjugation, forms a moduli space $\mathcal{M}_d = \text{Rat}_d/SL_2$. In $\mathcal{M}_d$, a subfamily of rational maps with an automorphism group containing a cyclic group $C_d$ of order $d$ has dimension 1. Explicitly, any such rational map is $PGL_2$-conjugate to a map

$$\phi_a([x:y]) = [axy^{d-1}, x^d + (d-1)y^d]$$
where $a \neq 0$. $\phi_a$ has $(d+1)$ fixed points

$$[0, 1], \ \left[\sqrt[d]{a - d + 1}, 1\right], \ \left[\zeta_d \sqrt[d]{a - d + 1}, \ 1\right], \ \cdots, \ \left[\zeta_d^{d-1} \sqrt[d]{a - d + 1}, 1\right]$$

and $d+1$ critical points

$$z = [1, 0], \ [1, 1], \ \zeta_d, 1, \ \cdots, \ \zeta_d^{d-1}, 1$$

where $\zeta_d$ is a primitive $d$-th root of unity. The trivial critical point $[1, 0]$ is always preperiodic, since its forward orbit is $[1, 0] \rightarrow [0, 1] \rightarrow [0, 1] \rightarrow \cdots$. Also, since $[x, y] \mapsto [\zeta_d x, y]$ generates $C_d$ in the automorphism group, all other critical points have the same dynamical aspects.

In this manner, we can construct the Gleason-type polynomial associated to this family of maps as follows: we define the dynatomic polynomial

$$\Phi_n(x, y) = \prod_{d|n}(Q_n x - P_n y)^{\mu(n/d)}$$

where $[P_n(x, y), Q_n(x, y)] = \phi_a^{(n)}([x, y])$ so that

$$P_{n+1} = a P_n Q_n^{d-1}, \ Q_{n+1} = P_n^d + (d-1)Q_n^d$$

and $\mu$ is the Mobius function. Next, we define the generalized dynatomic polynomial

$$\Phi_{m,n}(x, y) = \begin{cases} 
\Phi_n(P_m, Q_m)/\Phi_n(P_{m-1}, Q_{m-1}) & \text{if } m > 0, \\
\Phi_n(x, y) & \text{if } m = 0.
\end{cases}$$

Finally, we define the pre-Gleason polynomial as

$$PG_{m,n} = \Phi_{m,n}(1, 1).$$

For example,

$$\Phi_1(x, y) = (x^d + (d-1)y^d)x - (axy^{d-1})y = x^{d+1} - (a - d + 1)xy^d = x(x^d - (a - d + 1)y^d)$$

so

$$PG_{0,1} = \Phi_1(1, 1) = 1 - (a - d + 1) = -(a - d).$$

We are currently interested in the nontrivial prefixed cases; where $m > 0$ and $n = 1$. In these cases, we can define

$$PG_m := PG_{m,1} = \Phi_{m,1}(1, 1) = \frac{Q_{m+1}(1,1)P_{m}(1,1) - P_{m+1}(1,1)Q_{m}(1,1)}{Q_{m}(1,1)P_{m-1}(1,1) - P_{m}(1,1)Q_{m-1}(1,1)} = \frac{q_{m+1}p_{m} - p_{m+1}q_{m}}{q_{m}p_{m-1} - p_{m}q_{m-1}}$$

where

$$p_n = P_n(1,1), \ q_n = Q_n(1,1) \in \mathbb{Z}[a].$$

Note that $p_0 = q_0 = 1$. We now prove the following proposition:

**Proposition 3.** Suppose that $m \geq 1$. For given $d \geq 2$, we have

$$PG_m = a(a - d)q_{m-1}^{d-1}G_m$$

where

$$G_m = \frac{1}{a - d} \left[ a^d q_{m-1}^{d(d-1)} - (a - d + 1) \frac{q_m^d - (aq_m^{d})}{q_m - aq_m^{d-1}} \right] \in \mathbb{Z}[a].$$

We call $G_m$ to be the $m$-th Gleason polynomial. Here we state our main results again.
Theorem 1. Suppose that $d \geq 3$ is prime. Then the $m$-th Gleason polynomial $G_m$ is irreducible over $\mathbb{Q}$ for all $m \leq 3$.

Conjecture 2. Suppose that $d \geq 2$ is a positive integer. Then the $m$-th Gleason polynomial $G_m$ is irreducible over $\mathbb{Q}$ for all $m$.

Calculation shows that this conjecture is true for all $d \leq 12$ and $m \leq 4$.

Proof of Proposition 3. Similarly as $P_n$ and $Q_n$ in (1), we have

$$p_{n+1} = ap_0q_{n+1}^{d-1}, \quad q_{n+1} = p_n^d + (d-1)q_n^d. \tag{2}$$

Therefore,

$$PG_m = \frac{q_{m+1}p_m - p_{m+1}q_m}{q_mp_{m-1} - p_mq_{m-1}}$$

$$= \frac{(p_m^d + (d-1)q_m^d)p_m - ap_mq_m}{(p_m^d + (d-1)q_m^d)p_m - ap_mq_m^d}$$

$$= \frac{p_m(p_m^d - (a - d + 1)q_m^d)}{p_m(p_m^d - (a - d + 1)q_m^d)}$$

$$= \frac{aq_m^{d-1}(p_m - (a - d + 1)q_m^d)}{p_m^{d-1} - (a - d + 1)q_m^{d-1}}$$

For convenience, here we use substitutions $X = p_m^d$ and $Y = q_m^d$. Then

$$PG_m = \frac{a^dX^{d-1} - (a - d + 1)(X + (d-1)Y)d}{X - (a - d + 1)Y}$$

$$= \frac{a^dX^{d-1} - a^d(a - d + 1)Yd - (a - d + 1)((X + (d-1)Y)d - a^dYd)}{X - (a - d + 1)Y}$$

$$= a^dY^{d-1} - (a - d + 1)(X + (d-1)Y)d - (aY)^d$$

$$= (X + (d-1)Y)d - (aY)^d$$

Substituting back and using $X + (d-1)Y = p_m^d + (d-1)q_m^d = q_m$, we get

$$PG_m = aq_m^{d-1} \left[ a^d q_m^{d-1} - (a - d + 1) \frac{q_m^d - (aq_m^{d-1})^d}{q_m - aq_m^{d-1}} \right].$$

Therefore, it suffices to prove that

$$(a - d) \left| \frac{PG_m}{aq_m^{d-1}} \right. \tag{3}$$
where
\[
\frac{PG_m}{a^d q_{m-1}^d} = a^d q^d_{m-1} - (a - d + 1) \frac{q_m^d - (aq_{m-1}^d)^d}{q_m - aq_{m-1}^d}.
\]

However, if \(a = d\) we can prove that
\[
p_n = q_n = dq_{n-1}^d
\]
for all \(n\) by induction from (2), so
\[
\frac{PG_m}{a^d q_{m-1}^d} \bigg|_{a=d} = a^d q^d_{m-1} - \sum_{i=0}^{d-1} q_m^i (dq_{m-1})^{d-1-i} = a^d q^d_{m-1} - d(q_{m-1}^d)^{d-1} = 0,
\]
which implies (3).

For example, here are the first few Gleason polynomials for \(d = 3\).

\[
G_1 = -a - 6
\]
\[
G_2 = -a^6 - a^5 - 30a^4 - 144a^3 - 216a^2 - 648a - 1944
\]
\[
G_3 = 8a^{17} - 705a^{16} - 3573a^{15} - 2295a^{14} - 142983a^{13} - 1142586a^{12} - 1070172a^{11} - 9587808a^{10} - 41518008a^9 - 48341448a^8 - 259815600a^7 - 1009029312a^6 - 1088391168a^5 - 3265173504a^4 - 9795520512a^3 - 7346640384a^2 - 22039921152a - 66119763456
\]

### 3. Newton Polygons

In the proof of Theorem 1, we will use the theory of Newton polygons. For a prime \(p\) and a polynomial \(f(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{C}_p[z]\) where \(a_n \neq 0\), the Newton polygon of \(f\) is defined as the convex hull of the set of points
\[
\{(i, \text{ord}_p(a_i)) : i = 0, 1, 2, \ldots, n\}
\]
defined on \(0 \leq x \leq n\). Here \(\text{ord}_p\) is the \(p\)-adic valuation in \(\mathbb{C}_p\). For convenience, we introduce a notation
\[
v_{i,p}(f) := \text{ord}_p(a_i),
\]
which will be used throughout this paper. When \(p\) is obvious, we often write just \(v_i(f) := v_{i,p}(f)\). By definition, the Newton polygon is composed of line segments of increasing slopes. (If \(a_0 = 0\), the first line segment may have slope \(-\infty\).) The points where the slope changes, along with two endpoints \((0, \text{ord}_p(a_0))\) and \((n, \text{ord}_p(a_n))\), are called vertices of the Newton polygon. Note that any given set of vertices determines at most one Newton polygon. The slopes of the Newton polygon gives a nice explanation about roots of the associated polynomial.

**Proposition 4.** Consider the Newton polygon of \(f\) as above. If a line segment in the Newton polygon has slope \(-m\) and horizontal length \(\ell\), then exactly \(\ell\) roots, counted with multiplicity, of \(f\) in \(\mathbb{C}_p\) have \(p\)-adic valuation \(m\).

**Proof.** See [3, IV.4, Lemma 4]. The proof assumes that the constant term is 1, but the same method can be applied to the case where the constant term is nonzero. Furthermore, using the convention \(\text{ord}_p(0) = \infty\) we can extend the proof to the general case. \(\square\)
In other words, if we make two multisets, one with roots of \(f\) and another with slopes of line segments in the Newton polygon of \(f\), then these two multisets have a natural one-to-one correspondence. With this idea, we can find one important property of Newton polygons.

**Proposition 5.** Let \(f_i(z) = \sum_j a_{i,j} z^j \in \mathbb{C}_p[z]\) be a finite number of polynomials. Then the Newton polygon of \(\prod f_i\) is the ‘rearranged concatenation’ of Newton polygons of \(f_i\)’s, constructed by the following method: we gather all line segments of all Newton polygons, arrange them by slope in increasing order, and attach them from the starting point \((0, \text{ord}_p(\prod a_{i,0}))\).

There are some applications of this theory to polynomials with rational coefficients. Here we define the \(p\)-Newton polygon of \(f(z) \in \mathbb{Q}[z]\), denoted by \(N_p(f)\), as the Newton polygon of \(f(z)\) with the embedding \(\mathbb{Q} \hookrightarrow \mathbb{C}_p\). It is actually equivalent to the convex hull of the set of points as above, using the \(p\)-adic valuation in \(\mathbb{Q}\).

**Corollary 6.** Suppose that \(f(z) \in \mathbb{Q}[z]\) has nonzero constant term. If the \(p\)-Newton polygon of \(f(z)\) contains exactly \(k\) lattice points, i.e., \(N_p(f) \cap \mathbb{Z}^2\) contains exactly \(k\) points, then there are at most \(k - 1\) factors of \(f\) over \(\mathbb{Q}\).

**Proof.** Suppose that \(f(z)\) can be represented as a product of \(k\) factors in \(\mathbb{Q}[z]\). We can consider this factorization in \(\mathbb{C}_p[z]\). Then by Proposition 5 the \(p\)-Newton polygon of \(f\) should be the rearranged concatenation of \(p\)-Newton polygons of those factors. However, since two endpoints of the \(p\)-Newton polygon of any polynomial in \(\mathbb{Q}[z]\) are always lattice points, so the rearranged concatenation has at least \(k + 1\) lattice points, contradiction. \(\square\)

**Corollary 7** (Eisenstein’s criterion). If the \(p\)-Newton polygon of \(f(z) \in \mathbb{Q}[z]\) is composed of only one line segment, with no lattice point except for the vertices, then \(f(z)\) is irreducible over \(\mathbb{Q}\).

### 4. Irreducibility of \(G_1\)

In this section, we prove Theorem 1 for \(m = 1\). From now on, unless it is specified otherwise, \(d\) is an odd prime. First we note that a direct application of Corollary 6 does not work; for example, from Proposition 3 we have

\[
G_1 = \frac{1}{a - d} \left( d^d - \frac{a^d - d^d}{a - d} \right),
\]

and it turns out that the \(d\)-Newton polygon of \(G_1\) is composed of a single line segment between \((0, d - 2)\) and \((d - 2, 0)\), which contains many lattice points.

To apply the theory of Newton polygons, we first do a change of variable

\[a \mapsto (b + 1)d.\]

That is, now we consider the family of maps

\[\varphi_b([x, y]) = [(b + 1)dx y^{d-1}, x^d + (d - 1)y^d]\]

where \(b \neq -1\). Let

\[\varphi_b^{(n)}([x, y]) = [R_n(x, y), S_n(x, y)]\]

and

\[r_n = R_n(1, 1), \quad s_n = S_n(1, 1) \in \mathbb{Z}[b].\]

Then similarly as above we can construct the pre-Gleason polynomial \(\mathcal{P}_m\) and the Gleason polynomial \(\mathcal{G}_m\) in \(\mathbb{Z}[b]\), satisfying the following equations which directly come from Proposition 6.
Proposition 8. \( P_G_m \) and \( G_m \) satisfy the following equations.

\[
P_G_m = (b + 1)bd^2 s_{m-1}^{d-1} G_m,
\]

\[
G_m = \frac{1}{bd} \left[ (b + 1)^d d^d s_{m-1}^{d(d-1)} - (bd + 1) \frac{s_m^d - ((b + 1)ds_{m-1}^d)^d}{s_m - (b + 1)ds_{m-1}^d} \right] \in \mathbb{Z}[b].
\]

Note that the irreducibility of \( G_m \) over \( \mathbb{Q} \) is the same as that of \( G_m \), since the change of variable is linear.

Proposition 9. \( G_1 \) is irreducible over \( \mathbb{Q} \), and thus so is the first Gleason polynomial \( G_1 \).

Proof. From (4), we have

\[
G_1 = \frac{1}{bd} \left( d^d - \frac{(b + 1)^d d^d - d^d}{bd} \right)
\]

\[
= -\frac{1}{bd} \left( \sum_{i=0}^{d-1} \left( \frac{d}{i+1} \right) b^i d^{d-1} - d^d \right)
\]

\[
= -\sum_{i=0}^{d-2} \left( \frac{d}{i+2} \right) b^i d^{d-2}.
\]

Since

\[
\text{ord}_d \left( \binom{d}{i+2} \right) = \begin{cases} 
1 & \text{if } i = 0, \ldots, d - 3, \\
0 & \text{if } i = d - 2,
\end{cases}
\]

we have

\[
v_i(G_1) = \begin{cases} 
-d - 1 & \text{if } i = 0, \ldots, d - 3, \\
d - 2 & \text{if } i = d - 2.
\end{cases}
\]

It follows that the \( d \)-Newton polygon of \( G_1 \) is composed of a single line segment between \((0, d - 1)\) and \((d - 2, d - 2)\) which has no other lattice point except for the vertices. Therefore, by Eisenstein’s criterion \( G_1 \) is irreducible over \( \mathbb{Q} \). \( \square \)

5. Irreducibility of \( G_2 \)

In this section, we prove Theorem 1 for \( m = 2 \). From Proposition 8 we have

\[
bd G_2 = (b + 1)^d d^d s_1^{d(d-1)} - (bd + 1) \frac{s_2^d - ((b + 1)ds_1^d)^d}{s_2 - (b + 1)ds_1^d}. \quad (5)
\]

If we let

\[
\sigma = (b + 1)ds_1^d, \quad \tau = s_2 - \sigma = s_2 - (b + 1)ds_1^d,
\]

then

\[
(b + 1)^d d^d s_1^{d(d-1)} = (b + 1)d\sigma^{d-1}
\]

and

\[
\frac{s_2^d - ((b + 1)ds_1^d)^d}{s_2 - (b + 1)ds_1^d} = \frac{(\sigma + \tau)^d - \sigma^d}{\tau} = \sum_{k=0}^{d-1} \binom{d}{k} \sigma^k \tau^{d-1-k}.
\]
Therefore,

\[ bdG_2 = (b + 1)d\sigma^{d-1} - (bd + 1) \sum_{k=0}^{d-1} \binom{d}{k} \sigma^k \tau^{d-1-k} \]

\[ = -bd(d - 1)\sigma^{d-1} - (bd + 1) \sum_{k=0}^{d-2} \binom{d}{k} \sigma^k \tau^{d-1-k}. \]

(6)

**Proposition 10.** \( G_2 \) is irreducible over \( \mathbb{Q} \), and thus so is the second Gleason polynomial \( G_2 \).

**Proof.** We first investigate the \( d \)-Newton polygons of \( \sigma \) and \( \tau \). Since

\[ s_1 = d, \quad s_2 = ((b + 1)d)^d + (d - 1)d = ((b + 1)^d + d - 1)d, \]

we have

\[ \sigma = (b + 1)d^{d+1}, \quad \tau = d^d \sum_{i=2}^{d} \binom{d}{i} b^i. \] (7)

Therefore the \( d \)-Newton polygon of \( \sigma \) is defined by two vertices \((0, d+1)\) and \((1, d+1)\), while the \( d \)-Newton polygon of \( \tau \) is defined by three vertices \((0, \infty), (2, d+1), \) and \((d, d)\).

Now we investigate the \( d \)-Newton polygon of \( bdG_2 \) with (6) and the \( d \)-Newton polygons of \( \sigma \) and \( \tau \). Explicitly, we claim the following:

(i) \( bdG_2 \) is divisible by \( b \), and \( v_1(bdG_2) = d^2 \).

(ii) The degree of \( bdG_2 \) is \( d^2 - d + 1 \), and

\[ v_{d^2 - d}(bdG_2) = d^2 - d, \quad v_{d^2 - d + 1}(bdG_2) = d^2 - d + 1. \]

(iii) Let \( \ell \) be the line in the \( xy \)-plane passing through the two points \((1, d^2)\) and \((d^2 - d, d^2 - d)\).

Then \((i, v_1(bdG_2)) \) is on or above the line \( \ell \) for all \( i = 1, 2, \ldots, d^2 - d \).

For (i), we observe that \( \tau \) is divisible by \( b^2 \). Therefore,

\[ bdG_2 \equiv -bd(d - 1)\sigma^{d-1} = -bd(d - 1)(b + 1)d^{d-1}d^{d-1} \equiv -bd^2(d - 1) \quad (\text{mod } b^2). \]

This proves (i).

For (ii), we observe that \( \deg(\sigma) = 1 \) while \( \deg(\tau) = d \). Therefore,

\[ \deg(bd(d - 1)\sigma^{d-1}) = d \]

and

\[ \deg \left( (bd + 1) \binom{d}{k} \sigma^k \tau^{d-1-k} \right) = 1 + k + d(d - 1 - k) = d^2 - d + 1 - (d - 1)k \]

for each \( k = 0, \ldots, d - 2 \). With (6), this implies that \( \deg(bdG_2) = d^2 - d + 1 \), and moreover terms with degree \( d^2 - d + 1 \) and \( d^2 - d \) come from the \( k = 0 \) case only. It gives

\[ -(bd + 1)\tau^{d-1} = -(bd + 1)d^{d-2-d}(b^d + db^{d-1} + \ldots)^d \]

\[ = -(bd + 1)d^{d-2-d}(b^{d-2} + (d^2 - d)b^{d-2-d-1} + \ldots) \]

\[ = -d^{d-2-d}(db^{d-2-d+1} + (d^3 - d^2 + 1)b^{d-2-d} + \ldots), \]

and (ii) directly follows.
To prove (iii), we investigate the $d$-Newton polygon of each term in $[6]$. First, the $d$-Newton polygon of $bd(d-1)\sigma^{d-1}$ is defined by three vertices $(0, \infty)$, $(1, d^2)$, and $(d-1, d^2)$, and all vertices are above $\ell$. On the other hand, if $k \neq 0$, then the $d$-Newton polygon of

$$(bd + 1)\binom{d}{k} \sigma^k \tau^{d-1-k}$$

is defined by five vertices

$$(0, \infty), \ (2(d - 1 - k), d^2), \ (d(d - 1 - k), d^2 - (d - 1 - k)),$$

$$(d(d - 1 - k) + k, d^2 - (d - 1 - k)), \ (d(d - 1 - k) + k + 1, d^2 - (d - 1 - k) + 1).$$

Note that

$$\text{ord}_d \left( \binom{d}{k} \right) = 1$$

for $k = 1, \cdots, d - 2$, which increases the $d$-adic valuation by 1. All of these vertices are on or above $\ell$. Finally, even if $k = 0$, the $d$-Newton polygon of

$$(bd + 1)\tau^{d-1}$$

is defined by four vertices

$$(0, \infty), \ (2(d - 1), d^2 - 1), \ (d^2 - d, d^2 - d), \ (d^2 - d + 1, d^2 - d + 1),$$

and all of these vertices are on or above $\ell$. Therefore, since the $d$-adic valuation satisfies the non-Archimedean triangle inequality, the $d$-Newton polygon of $bdG_2$ is on or above $\ell$ as well. This proves (iii).

Now (i), (ii), and (iii) says that the $d$-Newton polygon of $bdG_2$ is defined by four vertices

$$(0, \infty), \ (1, d^2), \ (d^2 - d, d^2 - d), \ (d^2 - d + 1, d^2 - d + 1),$$

or equivalently the $d$-Newton polygon of $G_2$ is defined by three vertices $(0, d^2 - 1)$, $(d^2 - d - 1, d^2 - d - 1)$, and $(d^2 - d, d^2 - d)$.

Since the line segment between $(0, d^2 - 1)$ and $(d^2 - d - 1, d^2 - d - 1)$ contains no lattice point except for the endpoints, Corollary [6] says that $G_2$ has at most two factors over $\mathbb{Q}$, so also over $\mathbb{Z}$. Moreover, in the proof of Corollary [6] if $G_2$ has two factors then one should be associated to the line segment between $(d^2 - d - 1, d^2 - d - 1)$ and $(d^2 - d, d^2 - d)$. This factor should be linear, whose $d$-Newton polygon is composed of a single line segment of slope 1. Since we calculated above that

$$bdG_2 = -d^{2d}(d - 1)b - \cdots - d^{2d-d+1}b^{d^2-d+1}$$

so

$$G_2 = -d^{d^2-1}(d - 1) - \cdots - d^{d^2-d}b^{d^2-d} = -d^{d^2-d-1}(-d^d(d - 1) - \cdots - db^{d^2-d}).$$

This means that the only possible factors are

(a divisor of $(d - 1)) \pm db.$

We claim any such linear polynomial cannot be a factor of $G_2$, so $G_2$ is indeed irreducible over $\mathbb{Q}$. It is equivalent to show that any $b = e/d$, where $e$ is a (positive or negative) divisor of $d - 1$, cannot be a root of $G_2$. For the sake of contradiction, suppose that there is a root $b = e/d$ such that $G_2(b) = 0$. For such $b$, ([6]) says that

$$bd(d - 1)\sigma^{d-1} = -(bd + 1) \sum_{k=0}^{d-2} \binom{d}{k} \sigma^k \tau^{d-1-k}$$
so
\[ e(d - 1)\sigma^{d-1} = -(e + 1) \sum_{k=0}^{d-2} \binom{d}{k} \sigma^k \tau^{d-1-k}. \] (8)

From (7), we have
\[ \text{ord}_d(\sigma) = \text{ord}_d((d + e)d^2) = d, \]
so the left hand side of (8) has a \(d\)-adic valuation of \(d(d - 1)\).

On the other hand, also from (7) we have
\[ \text{ord}_d(\tau) = \text{ord}_d(d^d (b^d + db^{d-1} + \cdots + \left(\frac{d}{2}\right)b^2)) = \text{ord}_d \left( e^d + d^2e^{d-1} + \cdots + d^3e^{d-2} \right) = 0. \]
Therefore, if \(0 < k < d\) then
\[ \text{ord}_d \left( \binom{d}{k} \sigma^k \tau^{d-1-k} \right) = 1 + dk, \]
while
\[ \text{ord}_d \left( \binom{d}{0} \sigma^0 \tau^{d-1} \right) = \text{ord}_d(\tau^{d-1}) = 0. \]
This implies
\[ \text{ord}_d \left( \sum_{k=0}^{d-2} \binom{d}{k} \sigma^k \tau^{d-1-k} \right) = 0. \]
Since \(e\) is a divisor of \(d - 1\), \(\text{ord}_d(e + 1) \leq 1\) unless \(e = -1\), which makes the \(d\)-adic valuation of the right hand side of (8) be either \(\infty\) or at most 1. Therefore (8) cannot be true, contradiction. This ends the proof. \(\square\)

6. Irreducibility of \(G_3\)

Now we prove that \(G_3\) is irreducible over \(\mathbb{Q}\), which directly implies the irreducibility of \(G_3\) as well. That will finish the proof of Theorem 1. As in the previous section, Proposition 8 gives
\[ G_3 = \frac{1}{bd} \left[ (b + 1)d^d s_2^{d(d-1)} - (bd + 1) \frac{s_2^d - ((b + 1)ds_2^d)^d}{s_3 - (b + 1)ds_2^d} \right]. \] (9)

Similarly, we let
\[ \sigma = (b + 1)ds_2^d, \quad \tau = s_3 - \sigma = s_3 - (b + 1)ds_2^d. \]
Then as in (5), we have
\[ bdG_3 = -bd(d - 1)\sigma^{d-1} - (bd + 1) \sum_{k=0}^{d-2} \binom{d}{k} \sigma^k \tau^{d-1-k}. \] (10)

Before we prove the irreducibility of \(G_3\), we investigate the \(d\)-Newton polygons of \(\sigma\) and \(\tau\). First,
\[ \sigma = (b + 1)ds_2^d = (b + 1)dF, \]
where
\[ F = s_2^d = ((b + 1)^d + d - 1)^d d^2. \]
It follows that the \(d\)-Newton polygon of \(\sigma\) is defined by three vertices \((0, d^2 + d + 1), (d^2, d^2 + 1),\) and \((d^2 + 1, d^2 + 1)\).
\( \tau \) is more complicated. We can first observe that

\[
r_2 = (b+1)^2d^{d+1},
\]

so

\[
\tau = s_3 - \sigma = r_2^d + (d-1)s_2^d - (b+1)ds_2^d = (b+1)^{2d}d^{2d+d} - (bd+1)F. \tag{11}
\]

**Proposition 11.** The \( d \)-Newton polygon of \( \tau \) is defined by five vertices

\( (0, \infty), \ (3, d^2+d+1), \ (d+1, d^2+d), \ (d^2, d^2), \ (d^2+1, d^2+1) \).

**Proof.** We first directly calculate the terms in \( \tau \) with degree at most 3 to show that

\[
\tau = -\frac{(d-1)^2}{2}d^{2d+d+1} + \text{(higher terms)},
\]

which gives the first two vertices. Also, since \( \tau \) is equal to \( -(bd+1)F \) for the terms with degree at least \( 2d+1 \), the last two vertices follow from the \( d \)-Newton polygon of \( (bd+1)F \).

For the remaining vertex \( (d+1, d^2+d) \), we introduce the following lemma.

**Lemma A.** For any \( k \geq 1 \),

\[
v_i(f^k) \geq \min_{0 \leq \alpha_1 \leq \cdots \leq \alpha_k, \sum \alpha_j = i} \left( \frac{k!}{N(\alpha_1, \cdots, \alpha_k)} + \sum v_{\alpha_j}(f) \right), \tag{12}
\]

where \( N(\alpha_1, \cdots, \alpha_k) \) is the number of permutations \( (l_1, \cdots, l_k) \) of \( (1, \cdots, k) \) such that \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_k \).

**Proof of lemma.** In the expansion of \( f^k \) for \( f(z) = a_nz^n + \cdots + a_0 \) where \( a_n \neq 0 \), the coefficient of \( z^i \) is the sum of terms of the form

\[
a_{\alpha_1 \cdots a_{\alpha_k}},
\]

where \( \alpha_1 + \cdots + \alpha_k = i \).

For a fixed \( k \)-tuple \( (\alpha_1, \cdots, \alpha_k) \) such that \( \alpha_1 \leq \cdots \leq \alpha_k \) and \( \alpha_1 + \cdots + \alpha_k = i \), there are exactly

\[
\frac{k!}{N(\alpha_1, \cdots, \alpha_k)}
\]

terms which appears in the coefficient of \( z^i \) in the expansion of \( f^k \) which are equal to \( a_{\alpha_1 \cdots a_{\alpha_k}} \).

Therefore the coefficient of \( z^i \) is

\[
\sum_{0 \leq \alpha_1 \leq \cdots \leq \alpha_k, \sum \alpha_j = i} \frac{k!}{N(\alpha_1, \cdots, \alpha_k)} a_{\alpha_1 \cdots a_{\alpha_k}}.
\]

Now the properties of \( d \)-valuation give \((12)\). \( \diamond \)

From Lemma A it turns out that if \( k = d \) is prime and \( d \mid i \) then

\[
v_i(f^d) \geq 1 + \min_{0 \leq \alpha_1 \leq \cdots \leq \alpha_d, \sum \alpha_j = i} \left( \sum v_{\alpha_j}(f) \right).
\]

If \( d \mid i \), then letting \( i = de \), due to the possibility of \( \alpha_1 = \cdots = \alpha_k \) we have

\[
v_{de}(f^d) \geq \min \left[ 1 + \min_{0 \leq \alpha_1 \leq \cdots \leq \alpha_d, \sum \alpha_j = de} \left( \sum v_{\alpha_j}(f) \right), dv_e(f) \right].
\]

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Applying this to $f = (b + 1)^d + d - 1$, where $v_i(f) = 0$ for $i = d$ and $v_i(f) = 1$ for all $i < d$, we can conclude that

$$v_i(f^d) \begin{cases} 
  \geq d + 1 - \left\lfloor \frac{i}{d} \right\rfloor & \text{if } d \nmid i, \\
  \geq \min(d + 1 - e, d) & \text{if } i = de, e < d, \\
  = 0 & \text{if } i = d^2.
\end{cases}$$

Note that

$$d \geq d + 1 - e = d + 1 - \left\lfloor \frac{i}{d} \right\rfloor$$

for all $i = de$ where $0 < e < d$. Moreover, since the constant term of $f$ is $d$,

$$v_0(f^d) = \ord_d(d^d) = d.$$

Therefore, we can simplify the above inequality to

$$v_i(f^d) \begin{cases} 
  = 0 & \text{if } i = d^2, \\
  = d & \text{if } i = 0, \\
  \geq d + 1 - \left\lfloor \frac{i}{d} \right\rfloor & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (13)

Now we claim the following:

(i) $v_i(\tau) \geq d^2 + d + 1$ for $i = 4, \cdots, d$

(ii) $v_{d+1}(\tau) = d^2 + d$.

(iii) Let $\ell$ be the line in the $xy$-plane passing through the two points $(d + 1, d^2 + d)$ and $(d^2, d^2)$.

Then $(i, v_i(\tau))$ is on or above the line $\ell$ for all $i = d + 2, \cdots, d^2 - 1$.

For (i), when $i = 4, \cdots, d - 1$ we have

$$v_i(f^d) \geq d + 1,$$

so

$$v_i((bd + 1)^d) \geq \min(v_{i-1}(F) + 1, v_i(F)) = \min(v_{i-1}(f^d) + d^2 + 1, v_i(f^d) + d^2) \geq d^2 + d + 1.$$  \hspace{1cm} (11)

On the other hand,

$$v_i((b + 1)^{2d}d^{d^2+d}) = \ord_d \left( \binom{2d}{i} \right) + (d^2 + d) = d^2 + d + 1$$

so from (11) we have

$$v_i(\tau) \geq d^2 + d + 1.$$  \hspace{1cm} (13)

When $i = d$, the coefficient of $b^d$ in (11) is

$$d^{d^2+d} \binom{2d}{d} - (dc_{d-1}(F) + c_d(F)) = d^{d^2+d} \binom{2d}{d} - d^{d^2+1}c_{d-1}(f^d) - d^{d^2}c_d(f^d),$$  \hspace{1cm} (14)

where $c_j(f^d)$ is the coefficient of $b^j$ in $f^d$. We have shown above that

$$\ord_d(c_{d-1}(f^d)) = v_{d-1}(f^d) \geq d + 1,$$

so the second term has $d$-adic valuation at least $d^2 + d + 2$. For $c_d(f^d)$, considering (12) in Lemma A the least possible $d$-valuation $d$ appears only when $\alpha_1 = \cdots = \alpha_d = 1$ or $\alpha_1 = \cdots = \alpha_{d-1} = 0$.
and $\alpha_d = d$. In other words,
\[
c_d(f^d) = \sum_{\alpha_1 = \cdots = \alpha_d = 1}^{d} + \sum_{\alpha_1 = \cdots = \alpha_{d-1} = 0, \; \alpha_d = d} d \cdot d^{d-1} + \text{(divisible by } d^{d+1})
\]
\[
= 2d^d + \text{(divisible by } d^{d+1}).
\]
Therefore, (14) becomes
\[
d^{d^2+d} \left( \frac{2d^d}{d} \right) - 2 + \text{(divisible by } d^{d+1}),
\]
and it is actually divisible by $d^{d^2+d+1}$ since
\[
(x + y)^{2d} = ((x + y)^d)^2 \equiv (x^d + y^d)^2 = x^{2d} + 2x^dy^d + y^{2d} \pmod{d}
\]
so
\[
\left( \frac{2d^d}{d} \right) \equiv 2 \pmod{d}.
\]
This proves that $v_d(\tau) \geq d^2 + d + 1$ as well.

For (ii), the coefficient of $b^{d+1}$ in (11) is
\[
d^{d^2+d} \left( \frac{2d^d}{d+1} - dc_d(F) + c_{d+1}(F) \right) = d^{d^2+d} \left( \frac{2d^d}{d+1} \right) - d^{d^2+1}c_d(f^d) - d^2c_{d+1}(f^d),
\]
and it turns out that the first two terms have the $d$-adic valuation $d^2 + d + 1$. On the other hand, in $c_{d+1}(f^d)$ the least possible $d$-valuation $d$ appears only when $\alpha_1 = \cdots = \alpha_{d-2} = 0$, $\alpha_{d-1} = 1$, and $\alpha_d = d$ in (12), so
\[
c_{d+1}(f^d) = d(d - 1)d^{d-1} + \text{(divisible by } d^{d+1}) = (d - 1)d^d + \text{(divisible by } d^{d+1}).
\]
This implies that (15) has the $d$-adic valuation exactly $d$, which implies (ii).

For (iii), we observe that
\[
v_i((b + 1)^2d^2d^2 + d) \geq d^2 + d
\]
so it suffices to prove the same statement for $(bd + 1)F$ instead of $\tau$. However, using (13) we have
\[
v_i((bd + 1)F) \geq \min(v_{i-1}(F) + 1, v_i(F)) = \min(v_{i-1}(f^d) + d^2 + 1, v_i(f^d) + d^2) \geq d^2 + d + 1 - \left\lfloor \frac{i}{d} \right\rfloor
\]
for all $i = d + 2, \cdots , d^2 - 1$, and for such $i$, $(i, d^2 + d + 1 - \left\lfloor \frac{i}{d} \right\rfloor)$ is on or above $\ell$. This proves (iii).

Now (i), (ii), and (iii) says $(d + 1, d^2 + d)$ is the only other vertex in the $d$-Newton polygon of $\tau$.

Now we are ready to prove our main result.

**Proof of Theorem** As in Proposition we investigate the $d$-Newton polygon of $bdG_3$ with (10) and the $d$-Newton polygons of $\sigma$ and $\tau$. Explicitly, we claim the followings:

(i) $bdG_3$ is divisible by $b$, and $v_1(bdG_3) = d^3$.

(ii) The degree of $bdG_3$ is $d^3 - d^2$, and
\[
v_{d^3 - d^2}(bdG_3) = d^3 - d^2.
\]

(iii) Let $\ell$ be the line in the $xy$-plane passing through the two points $(1, d^3)$ and $(d^3 - d^2, d^3 - d^2)$. Then $(i, v_i(bdG_3))$ is on or above the line $\ell$ for all $i = 1, 2, \cdots , d^3 - d^2$. 

For (i), we observe that \( \tau \) is divisible by \( b^2 \). Therefore,

\[
bdG_3 \equiv -bd(d - 1)\sigma^{d-1} = -bd(d - 1)((b + 1)dF)^{d-1} = -bd(d - 1)((b + 1)((b + 1)^d + d - 1)^d(d^{d+1})^{d-1} = bdG_3(d - 1) \mod b^2.
\]

This proves (i).

For (ii), we observe that \( \deg(\sigma) = \deg(\tau) = d^2 + 1 \), so

\[
\deg(\sigma^k \tau^{d-1-k}) = (d^2 + 1)(d - 1) = d^3 - d^2 + d - 1
\]

for all \( k \). However, in the expression (11) of \( \tau \), \((b + 1)^{2d}d^{d^2+d}\) has degree \( 2d \), so any term in \( \sigma^k \tau^{d-1-k} \) which comes from multiplying \((b + 1)^{2d}d^{d^2+d}\) has degree at most

\[
(d^3 - d^2 + d - 1) - (d^2 - 2d + 1) = d^3 - 2d^2 + 3d - 2 < d^3 - d^2.
\]

In other words, \((b + 1)^{2d}d^{d^2+d}\) cannot affect the terms in (10) with degree at least \( d^3 - d^2 \), or equivalently those terms are the same when we replace \( \tau \) by \(-(bd + 1)F\) in (10). However, then (10) becomes

\[
-bd(d - 1)\sigma^{d-1} - (bd + 1) - (bd + 1) \sum_{k=0}^{d-2} \left( \begin{array}{c} d \\ k \end{array} \right) \sigma^k (-bd + 1)F^{d-1-k}
\]

\[
= -bd(d - 1)((b + 1)dF)^{d-1} - (bd + 1) \sum_{k=0}^{d-2} \left( \begin{array}{c} d \\ k \end{array} \right) ((b + 1)dF)^k (-bd + 1)F^{d-1-k}
\]

\[
= F^{d-1} \left[ -bd(d - 1)((b + 1)d)^{d-1} - (bd + 1) \sum_{k=0}^{d-2} \left( \begin{array}{c} d \\ k \end{array} \right) ((b + 1)d)^k (-bd + 1)d^{d-1-k} \right]
\]

\[
= F^{d-1} \left[ -bd(d - 1)((b + 1)d)^{d-1} + \sum_{k=0}^{d-2} \left( \begin{array}{c} d \\ k \end{array} \right) ((b + 1)d)^k (-bd + 1)d^{d-1-k} \right]
\]

\[
= F^{d-1} \left[ (b + 1)d - (bd + 1)d \right] = F^{d-1}(bd - (bd + 1))^d = F^{d-1}(d - 1)^d = (bd - d)^{d-1}d^{d^2-1}(bdG_3) = d^3 - d^2.
\]

This proves (ii).

Finally, for (iii), we investigate the \( d \)-Newton polygon of each term in (10). First, the \( d \)-Newton polygon of \( bd(d - 1)\sigma^{d-1} \) is defined by four vertices

\[
(0, \infty), \quad (1, d^3), \quad (d^3 - d^2 + 1, d^3 - d^2 + d), \quad (d^3 - d^2 + d, d^3 - d^2 + d),
\]
and all of these vertices are on or above \( \ell \). On the other hand, if \( k \neq 0 \), then the \( d \)-Newton polygon of 

\[
(bd + 1) \left( \frac{d}{k} \right) \sigma^k \tau^{d-1-k}
\]

is defined by seven vertices 

\[
(0, \infty), \quad (3(d - 1 - k), d^3), \quad ((d + 1)(d - 1 - k), d^3 - (d - 1 - k)), \\
(d^2(d - 1 - k), d^3 - (d + 1)(d - 1 - k)), \quad (d^2(d - 1), d^3 - (d + 1)(d - 1 - k) - dk), \\
(d^2(d - 1) + k, d^3 - (d + 1)(d - 1 - k) - dk), \quad (d^2(d - 1) + d, d^3 - (d + 1)(d - 1 - k) - dk + d - k), \\
\]

(note that \( \binom{d}{k} \) gives the additional \( d \)-valuation of 1) and all of these vertices are on or above \( \ell \). Even if \( k = 0 \), the \( d \)-Newton polygon of \((bd + 1)\tau^{d-1}\) is defined by five vertices 

\[
(3(d - 1), d^3 - 1), \quad ((d + 1)(d - 1), d^3 - d), \quad (d^2(d - 1), d^3 - d^2), \quad (d^2(d - 1) + d, d^3 - d^2 + d),
\]

and all of these vertices are on or above \( \ell \). Then the non-Archimedean triangle inequality of the \( d \)-adic valuation proves (iii).

Now (i), (ii), and (iii) say that the \( d \)-Newton polygon of \( bdG_3 \) is defined by three vertices \((0, \infty), (1, d^3), \) and \((d^3 - d^2, d^3 - d^2)\), or equivalently the \( d \)-Newton polygon of \( G_3 \) is defined by two vertices \((0, d^3 - 1)\) and \((d^3 - d^2 - 1, d^3 - d^2 - 1)\). Since the line segment between \((0, d^3 - 1)\) and \((d^3 - d^2 - 1, d^3 - d^2 - 1)\) contains no lattice point except for the endpoints, Corollary [7] says that \( G_3 \) is irreducible over \( \mathbb{Q} \). 

\[\square\]

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