STABILITY OF TRAVELING WAVES FOR NONLOCAL TIME-DELAYED REACTION-DIFFUSION EQUATIONS

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Abstract. This paper is concerned with the stability of noncritical/critical traveling waves for nonlocal time-delayed reaction-diffusion equation. When the birth rate function is non-monotone, the solution of the delayed equation is proved to converge time-exponentially to some (monotone or non-monotone) traveling wave profile with wave speed \( c > c^* \), where \( c^* > 0 \) is the minimum wave speed, when the initial data is a small perturbation around the wave. However, for the critical traveling waves \( (c = c^*) \), the time-asymptotical stability is only obtained, and the decay rate is not gotten due to some technical restrictions. The proof approach is based on the combination of the anti-weighted method and the nonlinear Halanay inequality but with some new development.

1. Introduction. The object of this paper is to consider a nonlocal time-delayed reaction-diffusion equation

\[
\frac{\partial u(t,x)}{\partial t} - D \frac{\partial^2 u(t,x)}{\partial x^2} + d(u) = \int_{-\infty}^{\infty} f_\alpha(y) b(u(t-r,x-y))dy, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},
\]

with the following initial data

\[
u(s,x) = u_0(s,x), \quad (s,x) \in (-r,0) \times \mathbb{R}.
\]

Here \( u(t,x) \) denotes the total mature population of the species at time \( t \) and position \( x \), \( D > 0 \) is the spatial diffusion rate for the mature population, \( r > 0 \) is the maturation delay, the time required for a newborn to become matured. Here \( \alpha > 0 \) is the total amount of the immature species, and \( f_\alpha(y) \) is the heat kernel in the form of

\[
f_\alpha(y) = \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{y^2}{4\alpha}} \quad \text{with} \quad \int_{-\infty}^{\infty} f_\alpha(y)dy = 1.
\]

The nonlinear functions \( d(u) \) and \( b(u) \) denote the death and birth rates of the mature population respectively, and satisfy the following hypotheses:

\( (H_1) \) There are two constant equilibria saying \( u_\pm: u_- = 0 \) is unstable and \( u_+ > 0 \) is stable. That is \( d(u_{\pm}) = b(u_{\pm}) \), and \( d'(0) < b'(0), \quad d'(u_+) > b'(u_+) \);

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(H2) \( b(u) \in C^2[0, \infty), b(u) \geq 0, \text{ and } |b'(u)| \leq b'(0) \text{ for } u \in [0, \infty); \)
(H3) \( d(u) \in C^2[0, \infty), d(u) \geq 0, d'(u) \geq d'(0) > 0, \text{ and } d''(u) \geq 0 \text{ for } u \in [0, \infty). \)

As far as we know, the non-monotonicity of \( b(u) \) causes the delayed reaction-diffusion equation (1) to be non-monotone and loss its comparison principle. The corresponding traveling waves solutions may also be oscillating when the time delay is big or the wave speed is big.

The equation (1) satisfying the above hypotheses includes a lot of time-delayed equations with diffusion coming from the population dynamics in ecology. Two typical examples of the equation (1) satisfying (H1) – (H3) are:

- Nicholson’s blowflies model [8],[9],[13],[15],[18]-[27]: Taking the death and birth rate functions as follows:
  \[
  d(u) = \delta u, \text{ and } b(u) = pue^{-\alpha u},
  \]
  where \( \delta > 0, p > 0 \text{ and } a > 0. \) There are two constant equilibria \( u_- = 0 \text{ and } u_+ = \frac{1}{a} \ln \frac{p}{\delta} > 0. \) Obviously, \( |b'(u)| = p|1 - au e^{-\alpha u} < b'(0) \text{ for } u \in [0, u_+]. \)

- Mackey-Glass model [20, 23]: The death and birth rate functions are taken
  \[
  d(u) = \delta u, \text{ and } b(u) = \frac{pu}{1 + au^q},
  \]
  where \( \delta > 0, p > 0 \text{ and } a > 0. \) Two equilibria are \( u_- = 0, \text{ and } u_+ = \left( \frac{p}{\delta a} \right)^{\frac{1}{q}}. \)
  It is easily checked that \( |b'(u)| \leq b'(0) \text{ for } u \in [0, u_+]. \)

Furthermore, if we assume that the immature species is almost non-mobile, i.e., \( \alpha \to 0, \) using the property of the heat kernel, we can obtain
  \[
  \lim_{\alpha \to 0} \int_{-\infty}^{\infty} f_\alpha(y)b(u(t - r, x - y))dy = b(u(t - r, x)).
  \]
In this case, the nonlocal equation (1) is reduced to a local time-delayed reaction-diffusion equation for \( \alpha = 0: \)
  \[
  \frac{\partial u(t, x)}{\partial t} - D \frac{\partial^2 u(t, x)}{\partial x^2} + d(u(t, x)) = b(u(t - r, x - y)).
  \]

A traveling wave of (1) connecting the constant states \( u_\pm \) is a special solution of the form of \( \phi(x + ct), \) where \( c > 0 \) is the wave speed. Thus, it satisfies
  \[
  c\phi'(\xi) - D\phi''(\xi) + d(\phi(\xi)) = \int_{-\infty}^{\infty} f_\alpha(y)b(\phi(\xi - cr - y))dy, \phi(\pm \infty) = u_\pm, \xi := x + ct.
  \]
In this paper, we will show that these traveling waves, including non-monotone noncritical/critical waves, are asymptotically stable as \( t \to \infty. \)

The non-local reaction-diffusion equations with time delay are the important and interesting models from both physics and biology. For the non-critical traveling wave with wave speed \( c > c_*, \) Mei et al. [19] obtained the exponential stability by using combination of the weighted energy method and the comparison principle for the monotone equation. Later on, the global stability of critical wavefronts with optimal convergent rates was proved by Mei-Ou-Zhao [20] by using the Fourier transform and Green’s function method plus energy estimate. For the local equation, by using the \( L^2 \)-weighted energy and Hanalay’s inequality, Lin-Lin-Lin-Mei [14] first obtained the exponential stability for the non-critical oscillating waves. Then, the stability of critical traveling waves was obtained by Chern-Mei-Yang-Zhang [2] by using the anti-weighted technique and energy estimates. For nonlocal diffusion problem with time delay, Huang-Mei-Wang [10] got the global stability and the optimal rates for the planar wavefronts by Fourier transform and the weighted
energy method. Recently, using the anti-weighted method, the stability of oscillating traveling wave for time-delayed nonlocal dispersion equations was obtained by Huang-Mei-Zhang-Zhang [11]. On the other hand, for the initial boundary values problem, Jiang-Zhang [12] obtain the global stability for the monotone equation by using the weighted energy method and the squeeze theorem, and the local stability for the non-monotone equation by using the weighted energy method for the noncritical wave.

For the precious works, Mei-Lin-Lin-So [19], Huang-Mei-Wang [10] and Mei-Ou-Zhao [20] showed the stability for traveling waves but it sufficiently depends on the advantage of the monotonicity of both the equation and the traveling waves. Notice that, in this paper, the equation loses monotonicity and the traveling wave may be oscillating when the time-delay \( r \) is sufficiently big. So, the above approaches, including Fourier transform method, the monotone technique and \( L^1 \)-weighted energy method seem to fail in getting the stability of the traveling waves for (1). On the other hand, since the nonlocal birth term exists, the regular \( L^2 \)-weighted energy method [14] can not be applied to deal with the stability of these slower waves.

In this paper, the equation (1) is nonlocal and non-monotone, and the traveling waves may be oscillating when the delay \( r \) is large. Inspired by the study on the \( p \)-system with viscosity by Matsumura-Mei [17] and the study on the stability of the critical traveling waves for time-delayed reaction-diffusion equations with local non-monotone nonlinear term [2], where they give a suitable transform function (i.e. the anti-weight) to change the equation to a new equation, we realize that we can overcome the difficulty caused by the integral terms in the \( L^2 \) weighted energy estimates. Similar to [14], we work out the trouble caused by these oscillations by the nonlinear Halanay’s inequality.

The rest of the paper is organized as follows. In Section 2, we give our main stability results after introducing some necessary notations. When \( c > c_* \), the solution will be expected to exponentially converge to its corresponding traveling wave if the initial perturbation around the wave is small enough in a certain weighted Sobolev space. However, when \( c = c_* \), we can only obtain the time asymptotic stability for the critical wave. In Section 3, we reformulate the original equation to the perturbed equation around the given traveling wave and give the corresponding stability theorem for the new equation. In Section 4, we use the anti-weighted technique to establish the desired a priori estimates and use the nonlinear Halanary inequality to treat the case when the traveling wave near \( u_+ \). This plays a crucial role in the proof of stability.

2. Preliminaries and main results. At first, we state some notations used throughout this paper. \( C \) denotes a generic constant, while \( C_i > 0 \) \((i = 0, 1, 2, \ldots)\) represents a specific constant. Let \( L^2(R) \) be the space of the square integrable functions defined on \( R \), and \( H^k(R)(k \geq 0) \) is the Sobolev space of the \( L^2(R) \)-functions \( f(x) \) defined on the \( R \) whose derivatives \( \frac{d^i f}{dx^i}(i = 1, \ldots, k) \) also belong to \( L^2(R) \). Let \( T \) be a positive number and \( B \) a Banach space. We denote by \( C([0,T];B) \) to be the space of the \( B \)-valued \( L^2 \)-function on \([0,T]\).

2.1. Traveling wave. The traveling wave of (1) connecting \( u_- \) and \( u_+ \) is a special solution to (1) of the form \( \phi(x + ct) \geq 0 \). Plugging \( \phi(x + ct) \geq 0 \) into (1), we get

\[
\begin{align*}
\text{or}'(\xi) - D\phi''(\xi) + d(\phi(\xi)) &= \int_{-\infty}^{\infty} f_\alpha(y)b(\phi(\xi - cy - y)dy, \\
\phi(\pm \infty) &= u_\pm,
\end{align*}
\]
where \( \xi = x + ct', \quad t' = \frac{t}{\pi t}, \) and \( c > 0 \) is the wave speed. The traveling wave can be monotone or oscillatory. The existence and uniqueness of the monotone or oscillatory traveling waves of (1) have been studied extensively [1],[3]-[7],[16],[27]-[30]. We briefly describe the results we need below.

Since \( \phi(\xi) \to u_- = 0 \) as \( \xi \to -\infty \), we expect that \( \phi(\xi) \) is close to a function \( u(\xi) \) which satisfies the linearized equation of (1) around \( u_- \) for \( \xi \to -\infty \):

\[
\begin{cases}
  c\phi'(\xi) - D\phi''(\xi) + d'(0)\phi(\xi) = b'(0) \int_{-\infty}^{\infty} f_\alpha(y)e^{-\lambda(y+cr)}dy, \\
  \phi(\pm\infty) = u_\pm.
\end{cases}
\]

By plugging \( \phi(\xi) = e^{\lambda \xi} \) into (4), we get the following characteristic equation for \( \lambda > 0 \):

\[
c\lambda - DL^2 + d'(0) = b'(0)e^{\alpha\lambda^2 - cr\lambda}.
\]

Denote

\[
F_c(\lambda) := c\lambda - DL^2 + d'(0) \quad \text{and} \quad G_c(\lambda) := b'(0)e^{\alpha\lambda^2 - cr\lambda},
\]

there exists a unique \( c_* = c_*(r) > 0 \) at which the two graphs of \( F_c \) and \( G_c \) are tangent at \( \lambda_* \) [19]. This means that \( (c_*, \lambda_*) \) are determined by

\[
F_{c_*}(\lambda_*) = G_{c_*}(\lambda_*) \quad \text{and} \quad F'_{c_*}(\lambda_*) = G'_{c_*}(\lambda_*),
\]

namely

\[
\begin{cases}
  c_*\lambda_* - DL_*^2 + d'(0) = b'(0)e^{\alpha\lambda_*^2 - cr\lambda_*}, \\
  c_* - 2DL_* = b'(0)(2\alpha\lambda_* - c_*r)e^{\alpha\lambda_*^2 - \lambda_* r}.
\end{cases}
\]

- When \( c < c_* \), there will be no traveling wave for equation (1);
- When \( c \geq c_* \), if the traveling wave exists, it should satisfy

\[
\begin{cases}
  \phi(\xi) = O(1)e^{\lambda_1 \xi} \to 0, \quad \text{as} \quad \xi \to -\infty, \quad \text{for} \quad c > c_*, \\
  \phi(\xi) = O(1)e^{\lambda_2 \xi} \to 0, \quad \text{as} \quad \xi \to -\infty, \quad \text{for} \quad c = c_*.
\end{cases}
\]

where \( c > c_*, \lambda_1 = \lambda_1(c) > 0, \lambda_2 = \lambda_2(c) > 0, \) are two roots of the equation (5) satisfying

\[
c\lambda - DL^2 + d'(0) > b'(0)e^{\alpha\lambda^2 - cr\lambda} \quad \text{for} \quad \lambda_1 < \lambda < \lambda_2.
\]

Similarly, we can get the asymptotic behavior of the traveling wave as \( \xi \to \infty \).

\[
|u_+ - \phi(\xi)| = \begin{cases}
  O(1)e^{-\lambda_+ \xi}, & \text{as} \quad \xi \to \infty, \quad \text{for} \quad c > c_*, \\
  O(1)e^{-\lambda_+ \xi}, & \text{as} \quad \xi \to \infty, \quad \text{for} \quad c = c_*.
\end{cases}
\]

for some positive constants \( \lambda_+ \) and \( \lambda_+^* \) determined by

\[
\begin{cases}
  -c\lambda_+ - DL_+^2 + d'(u_+) = b'(u_+)e^{\lambda_+ cr}, \\
  -c\lambda_+^* - DL_+^* + d'(u_+) = b'(u_+)e^{\lambda_+^* cr}.
\end{cases}
\]

For the exciting traveling waves, from [7], [14], [28] and [29], we know that the traveling waves may be non-monotone and oscillatory around \( u_+ \), when the time-delay \( r \) is suitably big. Furthermore, when \( d'(u_+) < |b'(u_+)| \), there exists a positive constant \( \tau \), such that, if the time-delay \( r \) satisfies \( r > \tau \), there will be no traveling waves [3]. As far as we know, when traveling waves lose monotonicity, the perturbation equation around the traveling wave will not be monotone. Furthermore, the monotone technique can be no longer applied for the stability of the traveling waves.
2.2. Main results. As follows, we give the main results of this paper. Let \( \phi(x + ct) \) be a given traveling wave with \( c \geq c_* \), even if the traveling wave is monotone or slowly oscillatory around \( u_+ \). Here \( c_* = c_*(r) \) and \( \lambda_* = \lambda_*(r) \) satisfy (6). We define two weight functions by

\[
\begin{align*}
  w_1(\xi) &= e^{-2\lambda \xi}, \quad \lambda \in (\lambda_1, \lambda_2) \\
  w_2(\xi) &= e^{-2\lambda_* \xi},
\end{align*}
\]

where \( \lambda > 0 \), \( \lambda_1 \) and \( \lambda_2 \), stated in (8), are the eigenvalues of the characteristic equation (5), and \( \lambda_* \) stated in (6).

Now we state the asymptotic behavior of non-critical and critical traveling waves.

**Theorem 2.1** (Stability of non-critical traveling waves). Suppose that the birth rate \( b(u) \) and the death rate \( d(u) \) satisfy \((H_1)-(H_3)\), and assume either \( d'(u_+)^{\frac{1}{2}} \geq |b'(u_+)| \) with any time-delay \( r > 0 \), or \( d'(u_+)^{\frac{1}{2}} < |b'(u_+)| \) but with a small time-delay \( 0 < r < \hat{r} \) where

\[
\hat{r} = \pi - \arctan(\sqrt{|b'(u_+)|^{2} - |d'(u_+)|^{2}}/d'(u_+)).
\]

For given traveling wave \( \phi(x+ct) \) with \( c > c_* \) to (1), no matter the wave is monotone or oscillatory. Suppose that

\[
v_0(s, x) := u_0(s, x) - \phi(x + cs) \in C([-r, 0]; C(R)) \cap C_{unif}[-r, 0],
\]

and

\[
\sqrt{w_1}v_0(s, x) \in C([-r, 0]; H^1(R)) \cap L^2([-r, 0]; H^1(R)).
\]

When the initial perturbation is small:

\[
\max_{-r \leq s \leq 0} \|v_0(s)\|^2_{C(R)} + \int_{-r}^0 \|\sqrt{w_1}v_0(s)\|^2_{H^1} \, ds + \|\sqrt{w_1}v_0(0)\|^2_{H^1} \leq \delta^2_1,
\]

for some positive number \( \delta_1 \), then the solution \( u(t, x) \) of (1)-(2) is unique, and exists globally in time, and satisfies

\[
\begin{align*}
  u(t, x) - \phi(x + ct) &\in C_{unif}[-r, \infty), \\
  \sqrt{w_1}[u(t, x) - \phi(x + ct)] &\in C([-r, \infty); H^1(R)), \\
  \partial_2(\sqrt{w_1}[u(t, x) - \phi(x + ct)]) &\in L^2([-r, \infty); H^1(R)),
\end{align*}
\]

and

\[
\sup_{x \in R} |u(t, x) - \phi(x + ct)| \leq Ce^{-\mu t}.
\]

for some constant \( \mu > 0 \), where \( C_{unif}[-r, T] \) is defined by, for \( 0 < T \leq \infty \),

\[
C_{unif}[-r, T] := \{u(t, x) \in C([-r, T] \times R) \text{ such that} \}
\]

\[
\lim_{x \to -\infty} u(t, x) \text{ uniformly exists in } t \in [-r, T], \\
\lim_{x \to -\infty} u_x(t, x) = \lim_{x \to -\infty} u_{xx}(t, x) = 0
\]

uniformly with respect to \( t \in [-r, T] \).

**Theorem 2.2** (Stability of critical traveling waves). Under the assumptions of Theorem 2.1, for the critical traveling wave \( \phi(x + c_* t) \), suppose that the initial perturbation

\[
u_0(s, x) - \phi(x + c_* s) \in C([-r, 0]; C(R)) \cap C_{unif}[-r, 0],
\]

and
\[
\sqrt{w_2}|u_0(s, x) - \phi(x + cs)| \in C([-r, 0]; H^1(R)) \cap L^2([-r, 0]; H^2(R)),
\]
and is small enough, namely, there exists a constant \(\delta_2 > 0\), such that
\[
\max_{-r \leq s \leq 0} \{\|u_0 - \phi\|^2_{C(R)} + \|\sqrt{w_2}(u_0 - \phi)\|^2_{H^1} \} + \int_{-r}^0 \|\sqrt{w_2}u_0 - \phi\|^2_{H^2} ds \leq \delta_2^2.
\]
Then the solution \(u(t, x)\) of (1)-(2) is unique, and exists globally in time, and satisfies
\[
u(t, x) - \phi(x + cs_t) \in C_{unif}[-r, \infty),
\]
\[
\sqrt{w_2}[u(t, x) - \phi(x + cs_t)] \in C([-r, \infty); H^1(R)),
\]
\[
\partial_x(\sqrt{w_2}[u(t, x) - \phi(x + cs_t)]) \in L^2([-r, \infty); H^1(R)),
\]
and
\[
\lim_{t \to \infty} \sup_{x \in R} |u(t, x) - \phi(x + cs_t)| = 0.
\]

**Remark 1.** In Theorem 2.2, the critical traveling waves \(\phi(x + cs_t)\), no matter they are monotone or oscillatory, are proved to be time-asymptotically stable when the initial perturbations are small enough. However, due to some technical restrictions, the convergence rate can not be obtained in this paper.

3. **Reformulation of the problem.** This section is devoted to the proof of the Theorem 2.1 and Theorem 2.2 for the stability of those non-critical and critical traveling waves of (1).

Let \(\phi(x + ct) = \phi(\xi)\) be a given traveling wave with the speed \(c > c_*\), and \(u(t, x)\) be the solution of (1) with a small initial perturbation around the wave \(\phi(x + cs)\) for \(s \in [-r, 0]\). Denote
\[
v(t, \xi) := u(t, x) - \phi(\xi),
\]
\[
v_0(s, \xi) := u_0(s, x) - \phi(x + cs),
\]
Then, from (1)-(2), \(v(t, \xi)\) satisfies
\[
\begin{cases}
\rho(t) \partial_v + c \partial_v - D \partial^2_v = -Q_1(v) + \int_{R} f_0(y)Q_2(v(t - r, \xi - cr - y)) dy + \partial^d(\phi)v = 0, \quad t \in R_+, \; \xi \in R; \\
v(s, \xi) = v_0(s, \xi), \quad s \in [-r, 0], \; \xi \in R,
\end{cases}
\]
where
\[
Q_1(v(t, \xi)) := d(v(t, \xi) + \phi(\xi)) - d(\phi(\xi))v(t, \xi),
\]
\[
Q_2(v(t - r, \xi)) := b(v(t - r, \xi) + \phi(\xi - cr)) - b(\phi(\xi - cr)) - b'\phi(\xi - cr)v(t - r, \xi).
\]
Let \(0 \leq T \leq \infty\), we define the solution space for (20) as follows
\[
X(-r, T) = \{v|v(t, x) \in C([-r, T]; C_{unif}[-r, T]), \sqrt{w_1}v \in C([-r, T]; H^1(R)), \sqrt{w_1}v \in L^2([-r, T]; H^1(R))\},
\]
equipped with the norm
\[
N^2(T) = \sup_{t \in [-r, T]} \left(\|v(t)\|^2_{C} + \|\sqrt{w_1}v(t)\|^2_{H^1}\right).
\]
For the critical wave, similarly as the non-critical wave, we denote
\[ V(t, \xi) := u(t, x) - \phi(\xi), \]
where \( \xi = x + c_\ast t \). Then \( V(t, x) \) satisfies
\[
\begin{aligned}
&\frac{\partial V}{\partial t} + c_\ast \frac{\partial V}{\partial \xi} - D \frac{\partial^2 V}{\partial \xi^2} + d'(\phi)V \\
&- \int_R f_\alpha(y) b'(\phi(\xi - c_\ast r - y))V(t - r, \xi - c_\ast r - y)dy \\
&= -Q_1(V) + \int_R f_\alpha(y)Q_2(V(t - r, \xi - c_\ast r - y))dy, \quad t \in \mathbb{R}_+, \quad \xi \in \mathbb{R}; \\
V(s, \xi) &= V_0(s, \xi), \quad s \in [-r, 0], \quad \xi \in \mathbb{R}.
\end{aligned}
\]
(26)

The solution space for (26) is defined as follows. Let \( 0 \leq V \leq 2.2 \), suppose that
\[
\begin{align*}
&0 \leq V \leq 2.2, \\
&\text{sup} \{ \|V(t)\|_0 + \|\sqrt{w_2 V(t)}\|_{H^1}^2 \} \\
&\quad + \int_0^t \|\sqrt{w_2 V(s)}\|_{L^2}^2 ds = \int_0^t \|\partial_\xi(\sqrt{w_2 V(s)})\|_{L^2}^2 ds.
\end{align*}
\]
(27)

The solution \( V(t, x) \) of (20) and (26) exists uniquely and globally in \( X(\mathbb{R}, 0) \) and \( N(0) \leq \delta_1 \). Then the solution \( v(t, x) \) of (20) exists uniquely and globally in \( X(\mathbb{R}, \infty) \) and satisfies
\[
\|v(t, \xi)\|_{C(\mathbb{R})} \leq Ce^{-\mu t}.
\]
(29)

Theorem 3.2 (Stability of the critical waves). Under the conditions of Theorem 2.2, suppose that \( V_0(s, \xi) \in Y(\mathbb{R}, 0) \) and \( M(t) \leq \delta_2 \). Then the solution \( V(t, x) \) of (26) exists uniquely and globally in \( Y(\mathbb{R}, \infty) \) and satisfies
\[
\lim_{t \to \infty} \sup_{\xi \in \mathbb{R}} |V(t, \xi)| = 0.
\]
(30)

Notice that, Theorem 2.1 is equivalent to Theorem 3.1, and Theorem 2.2 is equivalent to Theorem 3.2.

4. Proof of the main results. In this section, we are going to show the main results. The adopted method is the so-called transformed energy method (i.e. anti-weighted method) combining with the nonlinear Halanay’s inequality.

4.1. Non-critical traveling wave. At first, we give the following transformation. Let \( \tilde{v} = \sqrt{w_1}v \). Substituting \( v = w_1^{-\frac{1}{2}} \tilde{v} \) to (20), then we derive the following equation for \( \tilde{v}(t, \xi) \)
\[
\begin{aligned}
&\frac{\partial \tilde{v}}{\partial t} + (c - 2DA) \frac{\partial \tilde{v}}{\partial \xi} - D \frac{\partial^2 \tilde{v}}{\partial \xi^2} + [c\lambda - D\lambda^2 + d'(\phi)]\tilde{v} \\
&- e^{-\lambda r} \int_R f_\alpha(y)e^{-\lambda r} b'(\phi(\xi - cr - y))\tilde{v}(t - r, \xi - cr - y)dy \\
&= w_1^{-\frac{1}{2}}[-Q_1(v) + \int_R f_\alpha(y)Q_2(v(t - r, \xi - cr - y))dy], \quad t \in \mathbb{R}_+, \quad \xi \in \mathbb{R}; \\
\tilde{v}(s, \xi) &= \sqrt{w_1}(v(s, \xi)) =: \tilde{v}_0(s, \xi), \quad s \in [-r, 0], \quad \xi \in \mathbb{R}.
\end{aligned}
\]
(31)
Next, we have the \textit{a priori} estimates by several lemmas.

\textbf{Lemma 4.1.} It holds that
\begin{equation}
\|\tilde{v}(t)\|^2_{L^2} + \int_0^t e^{-2\mu(t-s)} \int_R [B_\mu(\xi) - CN(T)]\tilde{v}(s)^2 d\xi ds \\
\leq Ce^{-2\mu t} \left( \|\tilde{v}_0(0)\|^2_{L^2} + \int_{-r}^0 \|\tilde{v}_0(s)\|^2_{L^2} ds \right)
\end{equation}
for \( t \in [0,T] \), where
\[
B_\mu(\xi) = c\lambda - D\lambda^2 + d'(\phi(\xi)) - \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} b'(0) - \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} e^{2\mu\lambda} b'(0) - \mu.
\]

\textit{Proof.} Multiplying (31) by \( e^{2\mu t}\tilde{v} \), where \( \mu > 0 \) will be selected later, we get
\begin{equation}
\begin{align*}
\frac{1}{2} e^{2\mu t} \tilde{v}^2(t) - \mu e^{2\mu t}\tilde{v}^2 + \frac{1}{2} (c - 2D\lambda)e^{2\mu t}\tilde{v}^2 + (De^{2\mu t}\tilde{v}\tilde{v})' &\leq -2\lambda cr \int_0^t \int_{-\infty}^{\infty} f_\alpha(y)e^{-\lambda y} b'(\phi(\xi - cr - y))\tilde{v}(s - r, \xi - cr - y) dy d\xi ds \\
&= w_1^2 e^{2\mu t} \tilde{v}[-Q_1(v) + \int_{-\infty}^\infty f_\alpha(y)Q_2(v(t - r, \xi - cr - y)) dy] dy d\xi ds.
\end{align*}
\end{equation}

Integrating (33) over \( R \times [0,t] \) with respect to \( \xi \) and \( t \), using the property of Sobolev space \( H^2(R) \), we further have
\begin{equation}
\begin{align*}
\frac{1}{2} e^{2\mu t} \|\tilde{v}(t)\|^2_{L^2} - \frac{1}{2} e^{2\mu t} \|\tilde{v}(0)\|^2_{L^2} &+ \int_0^t \int_{-\infty}^{\infty} De^{2\mu t}\tilde{v}\tilde{v} d\xi ds \\
&+ \int_0^t \int_{-\infty}^{\infty} (c\lambda - D\lambda^2 + d'(\phi(\xi)) - \mu)e^{2\mu t}\tilde{v}^2 d\xi ds \\
&- e^{-\lambda cr} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu s} f_\alpha(y)e^{-\lambda y} b'(\phi(\xi - cr - y))\tilde{v}(s - r, \xi - cr - y) dy d\xi ds \\
&= \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} w_1^2 \tilde{v}[-Q_1(v) + \int_{-\infty}^\infty f_\alpha(y)Q_2(v(s - r, \xi - cr - y)) dy] dy d\xi ds.
\end{align*}
\end{equation}

Again, by using Cauchy-Schwarz inequality and by change of variables, we have
\begin{equation}
\begin{align*}
&= \frac{1}{2} e^{-\lambda cr} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu s} f_\alpha(y)e^{-\lambda y} b'(\phi(\xi - cr - y))\tilde{v}(s - r, \xi - cr - y) dy d\xi ds \\
&\leq \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} b'(0) \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu s} \|\tilde{v}(s)\|^2_{L^2} ds + \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} b'(0) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{v}^2(s, \xi) d\xi ds \\
&\leq \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} b'(0) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \|\tilde{v}(s)\|^2_{L^2} ds + \frac{1}{2} \epsilon^{\alpha\lambda^2 - cr\lambda} b'(0) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{v}^2(s, \xi) d\xi ds.
\end{align*}
\end{equation}
Substituting (35) into (34), we get
\[
\frac{1}{2}e^{2\mu t}\|\tilde{v}(t)\|_{L^2}^2 + D \int_0^t \|\tilde{v}(s)\|_{L^2}^2 ds + \int_0^t \int_{-\infty}^{\infty} B_\mu(\xi) e^{2\mu s} \tilde{v}(s, \xi) d\xi ds \\
\leq \frac{1}{2}\|\tilde{v}_0(0)\|_{L^2}^2 + \frac{1}{2}e^{\alpha \lambda^2 - cr\lambda + 2\mu r}B'(0) \int_{-r}^{0} e^{2\mu s}\|\tilde{v}_0(s)\|_{L^2}^2 ds \\
+ \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{w}_1^\frac{1}{2} \tilde{v}[-Q_1(v) + \int_{-\infty}^{\infty} f_\alpha(y)Q_2(v(s-r, \xi - c\xi - y)) dy d\xi ds.
\]

(36)

Next, we estimate the nonlinear terms.
\[
\int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{w}_1^\frac{1}{2} \tilde{v}[-Q_1(v)] d\xi ds \leq C \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{w}_1^\frac{1}{2} \tilde{v}^2 d\xi ds \\
\leq CN(T) \int_0^t e^{2\mu s} \|\tilde{v}\|_{L^2}^2 ds,
\]

(37)

\[
\int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{w}_1^\frac{1}{2} \tilde{v} \int_{-\infty}^{\infty} f_\alpha(y)Q_2(v(s-r, \xi - c\xi - y)) dy d\xi ds \\
\leq C \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \tilde{w}_1^\frac{1}{2} \int_{-\infty}^{\infty} f_\alpha(y) v^2(s-r, \xi - c\xi - y) dy d\xi ds \\
\leq CN(T) \int_0^t \int_{-\infty}^{\infty} e^{2\mu s} \int_{-\infty}^{\infty} \tilde{w}_1^\frac{1}{2} (\xi) f_\alpha(y) \|\tilde{v}(s-r, \xi - c\xi - y)\| dy d\xi ds \\
= Ce^{-\lambda r} N(T) \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\mu s} \|\tilde{v}(s)\|_{L^2}^2 ds + CN(T) e^{\alpha \lambda^2 - cr\lambda} \int_{-r}^{0} e^{2\mu s} \|\tilde{v}(s)\|_{L^2}^2 ds \\
\leq CN(T) e^{\alpha \lambda^2 - cr\lambda} \int_{-r}^{0} \int_{-\infty}^{\infty} e^{2\mu s} \|\tilde{v}(s)\|_{L^2}^2 ds + C e^{\alpha \lambda^2 - cr\lambda} \int_{-r}^{0} e^{2\mu s} \|\tilde{v}_0(s)\|_{L^2}^2 ds.
\]

(38)

From (36) to (38), we obtain (32).

Lemma 4.2. It holds that

\[
B_\mu(\xi) \geq C_0
\]

for \(0 < \mu < \mu_0\), where \(C_0\) is some constant, and \(\mu_0\) is the root of the following equation

\[
c\lambda - D\lambda^2 + d'(0) - e^{\alpha \lambda^2 - cr\lambda} b'(0) - \mu - \frac{1}{2} e^{\alpha \lambda^2 - cr\lambda} b'(0)(e^{2\mu r} - 1) = 0
\]

(40)

Proof. By using Condition (H2), we get \(d'(\phi) \geq d'(0),\) and \(|b'(\phi(c\xi - cr))| \leq b'(0)\).

Combining with (H1), we have

\[
B_\mu(\xi) = c\lambda - D\lambda^2 + d'(\phi(\xi)) - \frac{1}{2} e^{\alpha \lambda^2 - cr\lambda} b'(0) - \frac{1}{2} e^{\alpha \lambda^2 - cr\lambda} e^{2\mu r} b'(0) - \mu \\
\geq c\lambda - D\lambda^2 + d'(0) - e^{\alpha \lambda^2 - cr\lambda} b'(0) - \mu - \frac{1}{2} e^{\alpha \lambda^2 - cr\lambda} b'(0)(e^{2\mu r} - 1)
\]

(41)

Then, there exists \(\mu_0 > 0\), such that if \(0 < \mu < \mu_0\), \(B_\mu(\xi) \geq C_0 > 0\).

Combining Lemma 4.1 and Lemma 4.2, we get the following estimate.
Lemma 4.3. Let \( v(t, \xi) \in X(-r, T) \), then there exists a constant \( \mu > 0 \), such that it holds that

\[
\| \sqrt{w_1} v(t) \|_{L^2}^2 + \int_0^t e^{-2\mu(t-s)} \| \sqrt{w_1} v(s) \|_{L^2}^2 ds \\
\leq C e^{-2\mu t} (\| \sqrt{w_1} v_0(0) \|_{L^2}^2 + \int_{-r}^0 \| \sqrt{w_1} v_0(s) \|_{L^2}^2 ds)
\]

provided \( N(T) \ll 1 \).

Next we derive the estimates for the higher order derivatives of the solution.

Lemma 4.4. It holds that

\[
\| \partial_t \overset{\sim}{v}(t) \|_{H^1}^2 + \int_0^t e^{-2\mu(t-s)} \| \partial_t \overset{\sim}{v}(s) \|_{H^1}^2 ds \\
\leq C e^{-2\mu t} (\| \overset{\sim}{v}_0(0) \|_{H^1}^2 + \int_{-r}^0 e^{2\mu s} \| \overset{\sim}{v}_0(s) \|_{H^1}^2 ds)
\]

Proof. Differentiating (31) with respect to \( \xi \) and multiplying it by \( \overset{\sim}{v}_\xi \), then integrating the resultant equation with respect to \( \xi \) and \( t \) over \( R \times [0, t] \), we can similarly prove (43) provided \( N(T) \ll 1 \). The detail is omitted.

Thus, combining (42) and (43), we will establish the following energy estimate.

Lemma 4.5. It holds that

\[
\| \partial_t \overset{\sim}{v}_\xi(t) \|_{H^1}^2 + \int_0^t e^{-2\mu(t-s)} \| \partial_t \overset{\sim}{v}_\xi(t) \|_{H^1}^2 ds \\
\leq C e^{-2\mu t} (\| \overset{\sim}{v}_\xi(0) \|_{H^1}^2 + \int_{-r}^0 e^{2\mu s} \| \overset{\sim}{v}_\xi(s) \|_{H^1}^2 ds)
\]

By using the Sobolev inequality, we can get the following lemma.

Lemma 4.6. Let \( \xi_1 \) be a big enough positive constant, then it holds that

\[
\| v(t, \xi) \|_C \leq C \delta_0 e^{-\mu t},
\]

provided \( \xi \leq \xi_1 \).

Proof. By using the Sobolev imbedding theorem \( H^1(R) \hookrightarrow C(R) \) and Lemma 4.5, we have

\[
\| \sqrt{w_1} v \|_C = \| \overset{\sim}{v} \|_C \leq C \| \overset{\sim}{v} \|_{H^1} \leq C \delta_0 e^{-\mu t}
\]

Thus, we can choose \( \xi_1 \gg 1 \), when \( \xi \leq \xi_1 \), we get

\[
e^{-\lambda \xi_1} \sup_{\xi \in R} |v(t, \xi)| \leq \| \sqrt{w_1} v \|_C \leq C \delta_0 e^{-\mu t},
\]

\[
e^{-\lambda \xi_1} \sup_{\xi \in (-\infty, \xi_1)} |v(t, \xi)| \leq \sup_{\xi \in (-\infty, \xi_1)} |\overset{\sim}{v}(t, \xi)| \leq \| \overset{\sim}{v} \|_C \leq C \delta_0 e^{-\mu t}.
\]

So, we obtain (45).
uniformly for \( t \in [0, T] \). Denote \( z(t) = v(t, \infty) \) and \( z_0(s) = v(s, \infty) \) for \( s \in [-r, 0] \).

Let us take the limit to (20) as \( \xi \to \infty \), then

\[
\begin{aligned}
\left\{ \begin{array}{l}
z'(t) + d'(u_+)z(t) - b'(u_+)z(t - r) = -Q_1(z(t)) + Q_2(z(t - r)), \\
z(s) = z_0(s), s \in [-r, 0].
\end{array} \right.
\tag{46}
\end{aligned}
\]

As shown in [14], we can have the following exponential decay for the solution \( z(t) \) of the time-delayed ODE (46).

**Lemma 4.7.** Under the conditions of Theorem 2.1, it holds that

\[ |v(t, \infty)| = |z(t)| \leq C\delta_0 e^{-\mu t} \]

for some constant \( \mu > 0 \), provided \( |z_0| \ll 1 \).

From Lemma 4.7, there exists \( \xi_0 \), such that when \( \xi \geq \xi_0 \), \( |v(t, \xi)| \leq C\delta_0 e^{-\mu t} \). Choose \( \delta_1 = \xi_0 \) in Lemma 4.6, combining Lemma 4.6 and Lemma 4.7, we can get the following lemma.

**Lemma 4.8.** It holds that

\[ \|v(t, \xi)\|_C \leq Ce^{-\mu t} \]

provided \( N(T) \ll 1 \).

4.2. **Critical traveling wave.** For the critical waves, since the inequality (39) is no longer satisfied, the decay rate can not be obtained by the same method for the non-critical waves. Inspired by the reference [2], we can only get the time-asymptotic result by the anti-weighted energy method.

Taking the following transformation (or say, anti-weight)

\[ \hat{V}(t, \xi) = \sqrt{w_2(\xi)}V(t, \xi), \]

we get the following equations for \( \hat{V}(t, \xi) \)

\[
\begin{aligned}
\begin{aligned}
\frac{\partial \hat{V}}{\partial t} + (c_s - 2D\lambda_s\lambda) \frac{\partial \hat{V}}{\partial \xi} - D \frac{\partial^2 \hat{V}}{\partial \xi^2} + [c_s\lambda_s - D\lambda_s^2 + d'(\phi)] \hat{V} \\
- e^{-\lambda_s c_s r} \int_R f_\alpha(y)e^{-\lambda_s y}b'(\phi(\xi - c_s r - y))\hat{V}(t - r, \xi - c_s r - y)dy \\
= w_2^2 [-Q_1(V) + \int_R f_\alpha(y)Q_2(V(t - r, \xi - c_s r - y))dy], \quad t \in R_+, \quad \xi \in R;
\end{aligned}
\end{aligned}
\tag{47}
\]

Now we are going to establish the estimates of the solution \( V \in Y(-r, T) \) by several lemmas.

**Lemma 4.9.** It holds that

\[ \|\hat{V}(0)\|_{L^2}^2 + \int_0^t \|\hat{V}_\xi(s)\|_{L^2}^2 ds + \int_0^t \int_{-\infty}^\infty \phi \hat{V}_\xi^2 d\xi ds \leq C(\|\hat{V}(0)\|_{L^2}^2 + \int_0^t \int_{-\infty}^\infty \phi \hat{V}_\xi^2 d\xi ds) \]

provided \( N(T) \ll 1 \).

In order to prove Lemma 4.9, we need the following lemmas.
Lemma 4.10. It holds that
\[
\frac{1}{2} \| \dot{V}(t) \|^2_{L^2} + D \int_0^t \| \dot{V}(s) \|^2_{L^2} ds + \int_0^t \int_{-\infty}^{\infty} A(\xi) \dot{V}^2 d\xi ds \\
\leq \frac{1}{2} \| \dot{V}(0) \|^2_{L^2} + \frac{1}{2} e^{\alpha \lambda^2 - c.r. \lambda} \int_{-r}^0 \| \dot{V}_0(s) \|^2_{L^2} ds \\
+ \int_0^t \int_{-\infty}^{\infty} \dot{V} W_2^2[-Q_1(V) + \int_{-\infty}^{\infty} Q_2(V) dy] d\xi ds
\]
(49)
where \( A(\xi) = c_s \lambda_s - D \lambda^2_s + d'(\phi) - \frac{1}{2} e^{\alpha \lambda^2_s - c_r r \lambda_s} (b'(0) + |b'(\phi)|) \).

Proof. Multiplying Eq. (47) by \( \dot{V} \) and integrating it with respect to \( \xi \) and \( t \) over \( R \times [0, t] \), we have
\[
\frac{1}{2} \left[ \frac{d}{dt} \int_R \| \dot{V}(\xi, t) \|^2_{L^2} d\xi \right] - \int_{-\infty}^{\infty} f_{\alpha}(y) e^{-\lambda_s y} b'(\phi(\xi - c_s r - y)) \dot{V} \dot{V}(s - r, \xi - c_s r - y) dy d\xi ds \\
= \int_0^t \int_{-\infty}^{\infty} \frac{d}{dt} \left[ \int_{-\infty}^{\infty} e^{-\lambda_s y} b'(\phi - c_s r - y) \dot{V} \dot{V}(s - r, \xi - c_s r - y) dy d\xi ds \right]
\]
(50)
By using Cauchy-Schwarz inequality and by change of variables, we get
\[
\int_0^t \int_{-\infty}^{\infty} e^{-\lambda_s y} f_{\alpha}(y) e^{-\lambda_s y} b'(\phi - c_s r - y) \dot{V} \dot{V}(s - r, \xi - c_s r - y) dy d\xi ds \\
\leq \frac{1}{2} e^{\alpha \lambda^2_s - c_r r \lambda_s} b'(0) \int_0^t \| \dot{V}(s) \|^2_{L^2} ds + \frac{1}{2} e^{\alpha \lambda^2_s - c_r r \lambda_s} \int_0^t \int_{-\infty}^{\infty} |b'(\phi(\xi))| \dot{V}^2(s) d\xi ds \\
+ \frac{1}{2} e^{\alpha \lambda^2_s - c_r r \lambda_s} b'(0) \int_{-r}^0 \int_{-\infty}^{\infty} \dot{V}_0^2(s) d\xi ds
\]
(51)
Substituting (51) into (50), we get (49).

Now we are going to give the estimate of \( A(\xi) \).

Lemma 4.11. It holds that
\[
A(\xi) \geq C \phi(\xi).
\]
(52)

Proof. From \( \lim_{\xi \to -\infty} \phi(\xi) = 0 \), then for any given \( \varepsilon > 0 \), there exists \( \xi_2 < 0 \), \( |\xi_2| \geq 1 \), when \( \xi < \xi_2 \), such that
\[
|b'(0) - |b'(\phi(\xi))|| < \varepsilon.
\]
So, we can get
\[
\lim_{\xi \to -\infty} \frac{b'(0) - |b'(\phi(\xi))|}{\phi(\xi)} = \lim_{\xi \to -\infty} \frac{b'(0) - b'(\phi(\xi))}{\phi(\xi)} = \lim_{\xi \to -\infty} b''(\phi(\xi)) = b''(0) > 0,
\]
(53)
where \( \phi \) lies in between 0 and \( \phi \). From (6) and (53), we can obtain when \( \xi \in (-\infty, \xi_2) \),
\[
A(\xi) \geq C \phi(\xi).
\]
(54)
When $\xi \in [\xi_2, \infty)$, by using the boundedness of the wave $\phi(\xi)$, we have $0 < m \leq \phi(\xi) \leq M$, where $m$ and $M$ are two constants. Then

$$A(\xi) = \frac{1}{\phi(\xi)} \left[ \frac{\phi(\xi)'}{\phi(\xi)} + \frac{d'(\phi)}{\phi(\xi)} \right] \geq \frac{1}{2} C^{\alpha^2 - c_\ast r_\ast} \frac{b'(0) - \max_{r \in [m, M]} |b'(r)|}{M} + d''(\tilde{\phi}_2)$$

where $\tilde{\phi}_2$ lies in between 0 and $\phi$. Combining (54) and (55), we have $\phi(t)$.

Next, we will establish the estimates of the nonlinear terms.

**Lemma 4.12.** It holds that

$$\left| \int_0^t \int_{-\infty}^{\infty} \tilde{V}_w \frac{1}{2} [-Q_1(V(s, \xi))] d\xi ds \right| \leq CM(T) \int_0^t \int_{-\infty}^{\infty} \phi \tilde{V} \phi d\xi ds$$

(56)

$$\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^2 \tilde{V} f_0(y) Q_2(V(s - r, \xi - c_\ast r - y)) dy d\xi ds$$

(57)

$$\leq CM(T)C^{\alpha^2 - c_\ast r_\ast} \left[ \int_0^t \int_{-\infty}^{\infty} \phi \tilde{V}^2(s, \xi) d\xi ds + \int_{-r}^{0} \int_{-\infty}^{\infty} w_2 \phi \tilde{V}_0^2(s, \xi) d\xi ds \right]$$

**Proof.** From $\phi(\xi) = O(1)|\xi|^{-1-\lambda},$ as $\xi \to -\infty$, there exists $\xi_3 < 0, |\xi_3| \geq 1$, such that

$$\frac{1}{\phi(\xi)} \leq \begin{cases} 
Cw^2_2, & \text{for } \xi \in (-\infty, \xi_3), \\
C, & \text{for } \xi \in [\xi_3, \infty). 
\end{cases}$$

(58)

Therefore,

$$\left| \frac{V(t, \xi)}{\phi(\xi)} \right| \leq \sup_{\xi \in (-\infty, \xi_3)} C \sqrt{w_2} |\tilde{V}(t, \xi)| + \sup_{\xi \in [\xi_3, \infty)} \sup_{\xi \in R} |\tilde{V}(t, \xi)|$$

$$\leq \sup_{\xi \in R} C \sqrt{w_2} |\tilde{V}(t, \xi)| + \sup_{\xi \in R} |\tilde{V}(t, \xi)|$$

$$\leq C \|\sqrt{w_2} \tilde{V}(t, \xi)\|_{H^1} + C \|\tilde{V}(t, \xi)\|_{C(R)}$$

$$\leq CM(T).$$

(59)

Thus,

$$\left| \int_0^t \int_{-\infty}^{\infty} \tilde{V}_w \frac{1}{2} [-Q_1(V(s, \xi))] d\xi ds \right|$$

$$\leq C \left| \int_0^t \int_{-\infty}^{\infty} \tilde{V}_w \tilde{V}_0^2(s, \xi) d\xi ds \right|$$

$$= C \left| \int_0^t \int_{-\infty}^{\infty} \tilde{V}_w \tilde{V}_0^2(s, \xi) d\xi ds \right|$$

$$= C \int_0^t \int_{-\infty}^{\infty} \frac{|V(s, \xi)|}{\phi(\xi)} \phi \tilde{V}_0^2(s, \xi) d\xi ds$$

$$\leq CM(T) \int_0^t \int_{-\infty}^{\infty} \phi \tilde{V}_0^2(s, \xi) d\xi ds. \tag{60}$$

Similarly as the proof of (56), we can get (57).
From Lemma 4.10 to Lemma 4.12, we can easily show Lemma 4.9. Next, we establish the estimates of the higher order derivatives of the solution.

**Lemma 4.13.** It holds that

\[
\|\hat{V}_\xi(t)\|_{L^2}^2 + \int_0^t \|\hat{V}_\xi(s)\|_{L^2}^2 ds + \int_0^t \int_{-\infty}^{\infty} \phi_\xi^2 d\xi ds \\
\leq C\|\hat{V}_0(0)\|_{H^1}^2 + \int_{-\infty}^{\infty} \phi(\hat{V}_0^2 + \hat{V}_\xi^2) d\xi ds.
\]

(61)

**Proof.** Differentiating (47) with respect to \(\xi\), we get

\[
\hat{V}_\xi - D\hat{V}_\xi\xi + (c_\ast - 2D\lambda_\ast)\hat{V}_\xi + [c_\lambda - D\lambda_\ast^2 + d'(\phi)]\hat{V}_\xi
\]

\[
- \int_{-\infty}^{\infty} e^{-\lambda_\ast c_\ast r} e^{-\lambda_\ast y} f_\alpha(y)b'\phi(\xi - c_\ast r - y))\hat{V}_\xi(t - r; \xi - c_\ast r - y)dy + d''(\phi)\phi'\hat{V}_\xi
\]

\[
- \int_{-\infty}^{\infty} e^{-\lambda_\ast c_\ast r} e^{-\lambda_\ast y} f_\alpha(y)b''(\phi(\xi - c_\ast r - y))\phi'(\xi - c_\ast r - y)\hat{V}(t - r, \xi - c_\ast r - y)dy
\]

\[
= (w_2^3)[-Q_1(V(t, \xi)) + \int_{-\infty}^{\infty} f_\alpha(y)Q_2(V(t - r, \xi - c_\ast r - y))] \\
+ w_2^3[-Q_1(V(t, \xi)) + \int_{-\infty}^{\infty} f_\alpha(y)Q_2(V(t - r, \xi - c_\ast r - y))]\xi
\]

(62)

Multiplying (62) by \(\hat{V}_\xi\) and integrating it with respect to \(\xi\) and \(t\) over \(R \times [0, t]\), as showed in Lemma 4.10, we have

\[
\|\hat{V}_\xi(t)\|_{L^2}^2 + \int_0^t \|\hat{V}_\xi(s)\|_{L^2}^2 ds + \int_0^t \int_{-\infty}^{\infty} \phi_\xi^2 d\xi ds \\
\leq C\|\hat{V}_0(0)\|_{H^1}^2 + \int_{-\infty}^{\infty} \phi(\hat{V}_0^2 + \hat{V}_\xi^2) d\xi ds - \int_0^t \int_{-\infty}^{\infty} d''(\phi)\phi'\hat{V}_\xi d\xi ds
\]

\[
+ \int_0^t \int_{-\infty}^{\infty} e^{-\lambda_\ast c_\ast r} e^{-\lambda_\ast y} f_\alpha(y)b''(\phi(\xi - c_\ast r - y))\hat{V}_\xi\hat{V}(s - r, \xi - c_\ast r - y)dy d\xi ds
\]

\[
+ \int_0^t \int_{-\infty}^{\infty} w_2^3 \hat{V}_\xi[-Q_1(V(t, \xi)) + \int_{-\infty}^{\infty} f_\alpha(y)Q_2(V(t - r, \xi - c_\ast r - y))]\xi d\xi ds
\]

\[
= : C\|\hat{V}_0(0)\|_{H^1}^2 + \int_{-\infty}^{\infty} \phi(\hat{V}_0^2 + \hat{V}_\xi^2) d\xi ds - I_1 + I_2 + I_3.
\]

(63)

Next, we give the estimates of \(I_1\), \(I_2\) and \(I_3\).

\[
I_1 = \int_0^t \int_{-\infty}^{\infty} d''(\phi)\phi'\hat{V}_\xi d\xi ds \\
\leq \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \|d''(\phi)\phi'\|\hat{V}_\xi^2 d\xi ds + \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \|d''(\phi)\phi'\|\hat{V}_\xi^2 d\xi ds
\]

\[
\leq C \int_0^t \int_{-\infty}^{\infty} \hat{V}_\xi^2 d\xi ds
\]

(64)
\[ I_2 = \int_0^t \int_0^\infty \int_{-\infty}^\infty e^{-\lambda \tau - \lambda s} f_o(y) b''(\phi(x - c \tau - y)) \tilde{V}_\xi \hat{V}(s - r, \xi - c \tau - y) dy d\xi ds \]
\[ \leq \frac{1}{2} \int_0^t \int_0^\infty \int_{-\infty}^\infty e^{-\lambda \tau - \lambda s} f_o(y) b''(\phi(x - c \tau - y)) \phi'(\xi - c \tau - y) \hat{V}_\xi^2 dy d\xi ds \]
\[ + \frac{1}{2} \int_0^t \int_0^\infty \int_{-\infty}^\infty e^{-\lambda \tau - \lambda s} f_o(y) b''(\phi(x - c \tau - y)) \phi'(\xi - c \tau - y) \hat{V}_\xi^2 dy d\xi ds \]
\[ \leq C \int_0^t \| \tilde{V}_\xi(s) \|^2_{L^2} ds + C \int_{-r}^t \int_{-\infty}^\infty \phi(\xi) \hat{V}^2(s, \xi) d\xi ds \]
\[ \leq C \int_0^t \| \tilde{V}_\xi(s) \|^2_{L^2} ds + \int_0^t \int_{-\infty}^\infty \phi(\xi) \hat{V}^2(s, \xi) d\xi ds + \int_0^t \int_{-\infty}^\infty \phi(\xi) \hat{V}^2(s, \xi) d\xi ds \]
(65)

Rewriting \( Q_1(V) \), and we get
\[ [Q_1(V)]_\xi = [d(V) - d(\phi) - d'(\phi)]_\xi \]
\[ = d'(\phi + V) (\phi' + \tilde{V}_\xi) - d'(\phi) \phi' - d'(\phi) V_\xi - d''(\phi) \phi' V \]
\[ = [d'(\phi + V) - d'(\phi)] \phi' + [d'(\phi + V) - d'(\phi)] V_\xi - d''(\phi) \phi' V \]
\[ = d''(\tilde{\phi}_2) \phi' V + d''(\tilde{\phi}_2) V_\xi - d''(\phi) \phi' V \]
(66)

\[ \int_0^t \int_{-\infty}^\infty \tilde{V}_\xi w_2^\frac{3}{2} [Q_1(V)]_\xi d\xi ds \]
\[ = \int_0^t \int_{-\infty}^\infty d''(\tilde{\phi}_2) \phi' w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \]
\[ + \int_0^t \int_{-\infty}^\infty d''(\tilde{\phi}_2) w_2^\frac{3}{2} \tilde{V}_\xi V_\xi d\xi ds - \int_0^t \int_{-\infty}^\infty d'(\phi) \phi' w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \]
\[ =: I_{31} + I_{32} - I_{33} \]
(67)

\[ |I_{31}| = \left| \int_0^t \int_{-\infty}^\infty d''(\tilde{\phi}_2) \phi' w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \right| \]
\[ \leq C \int_0^t \int_{-\infty}^\infty \phi w_2 V^2 d\xi ds + C \int_0^t \int_{-\infty}^\infty \phi w_2 \tilde{V}_\xi d\xi ds \]
(68)

\[ |I_{32}| = \left| \int_0^t \int_{-\infty}^\infty d''(\tilde{\phi}_2) \phi' w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \right| \]
\[ \leq CM(T) \int_0^t \int_{-\infty}^\infty w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \]
\[ = CM(T) \int_0^t \int_{-\infty}^\infty (\tilde{V}_\xi + \lambda \tilde{V}) \tilde{V}_\xi d\xi ds \]
\[ = CM(T) \int_0^t \| \tilde{V}_\xi(s) \|^2_{L^2} ds \]
(69)

\[ |I_{33}| = \left| \int_0^t \int_{-\infty}^\infty d''(\phi) \phi' w_2^\frac{3}{2} \tilde{V}_\xi d\xi ds \right| \]
\[ \leq C \int_0^t \int_{-\infty}^\infty \phi w_2 V^2 d\xi ds + C \int_0^t \int_{-\infty}^\infty \tilde{V}_\xi^2 d\xi ds \]
(70)
Similarly, rewriting $Q_2(V(t-r,\xi-c_*r-y))$, we have

$$[Q_2(V(t-r,\xi-c_*r-y))]_\xi$$

$$= b''(\tilde{\phi}_3)\phi'(\xi-c_*r-y)V(t-r,\xi-c_*r-y)$$

$$+ b''(\tilde{\phi}_4)V(t-r,\xi-c_*r-y)V_\xi(t-r,\xi-c_*r-y)$$

$$- b''(\phi(t-r,\xi-c_*r-y))\phi'(t-r,\xi-c_*r-y)V(t-r,\xi-c_*r-y).$$

Then, we have

$$\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\alpha(y)w_2^{1/2} \tilde{V}_\xi Q_2(V(s-r,\xi-c_*r-y))d\xi ds dy$$

$$= \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b''(\tilde{\phi}_3)f_\alpha(y)w_2^{1/2} \phi'(\xi-c_*r-y)\tilde{V}_\xi V(s-r,\xi-c_*r-y)d\xi ds dy$$

$$+ \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b''(\tilde{\phi}_4)f_\alpha(y)w_2^{1/2} V(s-r,\xi-c_*r-y)\tilde{V}_\xi V_\xi(s-r,\xi-c_*r-y)d\xi ds dy$$

$$- \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b''(\phi(t-c_*r-y))f_\alpha(y)w_2^{1/2} \phi'(\xi-c_*r-y)\tilde{V}_\xi V(s-r,\xi-c_*r-y)d\xi ds dy$$

$$=: I_{41} + I_{42} - I_{43}$$

Next, by using the property of weighted function $w_2$ and the traveling wave, we estimate $I_{41}$, $I_{42}$ and $I_{43}$.

$$|I_{41}|$$

$$= |\int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b''(\tilde{\phi}_3)f_\alpha(y)w_2^{1/2} \phi'(\xi-c_*r-y)\tilde{V}_\xi V(s-r,\xi-c_*r-y)d\xi ds dy|$$

$$\leq \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |b''(\tilde{\phi}_3)f_\alpha(y)\phi'(\xi-c_*r-y)|w_2V^2(s-r,\xi-c_*r-y)d\xi ds dy$$

$$+ \frac{1}{2} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |b''(\tilde{\phi}_3)f_\alpha(y)\phi'(\xi-c_*r-y)|\tilde{V}_\xi^2 d\xi ds dy$$

$$\leq C \int_0^t \int_{-\infty}^{\infty} f_\alpha(y) \left\| \frac{w_2(\xi)}{w_2(\xi-c_*r-y)} \phi(\xi-c_*r-y)w_2(\xi-c_*r-y) \right\|^2_{L^2} ds dy$$

$$\cdot V^2(s-r,\xi-c_*r-y)d\xi ds + C \int_0^t \left\| \tilde{V}_\xi(s) \right\|^2_{L^2} ds dy$$

$$\leq C \int_0^t \int_{-\infty}^{\infty} \phi(\xi)\tilde{V}^2(s,\xi)d\xi ds + \int_0^t \int_{-\infty}^{\infty} \phi(\xi)\tilde{V}_\xi^2(s,\xi)d\xi ds + C \int_0^t \left\| \tilde{V}_\xi(s) \right\|^2_{L^2} ds dy$$

$$\leq |I_{42}|$$

$$\leq \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\alpha(y)e^{-\lambda_*y-c_*r}w_2$$

$$\cdot V(s-r,\xi-c_*r-y)\tilde{V}_\xi(s-r,\xi-c_*r-y)\tilde{V}_\xi d\xi ds dy$$

$$= C \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\alpha(y)e^{-\lambda_*y-c_*r} V(s-r,\xi-c_*r-y)\tilde{V}_\xi(s-r,\xi-c_*r-y)$$

$$+ \lambda_* \tilde{V}(s-r,\xi-c_*r-y)\tilde{V}_\xi d\xi ds dy$$
\[ \leq CM(T) \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\alpha(y) e^{-\lambda_* y} |\tilde{V}_\xi\tilde{V}(s-r, \xi-c_*r-y)| dyd\xi ds \]
\[ + \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\alpha(y) e^{-\lambda_* y} |\tilde{V}_\xi\tilde{V}(s-r, \xi-c_*r-y)| dyd\xi ds \]
\[ \leq CM(T) \int_0^t \|\tilde{V}_\xi(s)\|_{L^2}^2 ds + CM(T) \int_0^t \int_{-\infty}^{\infty} V_{0\xi}^2(s, \xi) d\xi ds \]
\[ + CM(T) \int_0^t \int_{-\infty}^{\infty} \phi V^2(s, \xi) d\xi ds \]  \hspace{1cm} (74)

From (63) to (74), we can show Lemma 4.13.

**Lemma 4.14.** It holds that
\[ \|V(t)\|_C \leq C \sqrt{M(\infty) + 1} M(0), \quad t \in [0, \infty), \]  \hspace{1cm} (75)
provided \( M(\infty) \leq \delta_4. \)

By the former estimate, when \( M(0) \leq \delta_0, \) we have
\[ \|V(t)\|_C^2 + \|\tilde{V}(t)\|_{H^1}^2 + \int_0^t \int_{-\infty}^{\infty} \phi (\tilde{V}^2 + \tilde{V}_\xi^2) d\xi ds \]
\[ + \int_0^t \|\tilde{V}_\xi(s)\|_{H^1}^2 ds + \int_0^t \frac{d}{ds} \|\tilde{V}_\xi(s)\|_{L^2}^2 ds \]  \hspace{1cm} (66)

Denote
\[ g(t) := \|\tilde{V}(t)\|_{L^2}^2. \]

From (76), we know that
\[ 0 \leq g(t) \leq CM^2(0), \quad \int_0^\infty g(t) dt \leq CM^2(0), \quad \int_0^\infty |g'(t)| dt \leq CM^2(0). \]

This implies
\[ \lim_{t \to \infty} g(t) = 0, \quad \text{i.e.,} \quad \lim_{t \to \infty} \|\tilde{V}_\xi(t)\|_{L^2}^2 = 0. \]  \hspace{1cm} (77)

By using Sobolev imbedding theorem \( H^1(R) \hookrightarrow C(R), \)
\[ \|\tilde{V}(t)\|_C \leq \sqrt{2} \|\tilde{V}(t)\|_{L^2}^\frac{1}{2} \|\tilde{V}_\xi(t)\|_{L^2}^\frac{3}{2}, \]
the boundedness of \( \|\tilde{V}(t)\|_{L^2} \leq CM(0) \) and the convergence of (77), then we prove
\[ \lim_{t \to \infty} \sup_{\xi \in R} |\sqrt{w_2(\xi)} V(t, \xi)| = \lim_{t \to \infty} \|\tilde{V}(t)\|_C = 0. \]  \hspace{1cm} (78)

Similar to the \( c > c_* \) case, using the nonlinear Halanay inequality and (78), we can get the convergence
\[ \lim_{t \to \infty} \sup_{\xi \in R} |V(t, \xi)| = 0. \]

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