A realistic interpretation of the density matrix
I: Basic concepts

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1 Introduction

The debate about the meaning of the wave function in quantum mechanics is still going on, seventy years after the birth of the Copenhagen interpretation. Despite the enormous successes achieved by quantum mechanics and quantum field theory, we are not yet able to answer the following simple question: should we consider the wave function as a real objective field, or merely as a useful tool for computing probability amplitudes?

There are (at least) two obstacles to a realistic interpretation of the wave function:

1. For a N-body system, the wave function is defined in a 3N-dimensional configuration space, while we would expect a “real” field to be defined in ordinary 3-dimensional space. As a consequence, entanglement between states of distant systems is allowed, and this gives rise to the counter-intuitive behaviour known as “quantum non-locality”.

2. The wave function spreads out with time. This property is strictly related to the Heisenberg uncertainty principle: if a particle is well localized in position space at a certain time, then it will be poorly localized in momentum space, and therefore the dispersion of the wave packet will grow with time. Our experience with macroscopic objects tells us that they do not spread out with time, but in this case the quantum mechanical explanation may be that for a macroscopic object the time needed to obtain an observable dispersion of its wave packet is greater than the age of the universe. However, microscopic objects too are always detected in small space regions and it is hard to believe that immediately before their detection they were spread out at long distances from the detection point.

In this paper we conjecture that maybe the two above-mentioned problems are a consequence of the particular choice of representation for the wave function: with a suitable change of representation, it is possible to find solutions to the wave equations which do not spread out with time. We’ll see that this result can be obtained at the cost of introducing a second space-time coordinate $x_D$; we will consider this new coordinate as an “auxiliary” coordinate, in addition to the
“physical” coordinate $x_S$. Roughly speaking, the wave function will be spread out in the $x_D$ direction, while being perfectly localized in the “physical” $x_S$ direction.

We will first examine the non-relativistic Schrödinger equation (Section 2) and then the Dirac equation (Section 3), both for free fields and for interacting fields (we will focus on the electromagnetic interaction); finally (Section 4) we will draw some conclusions and propose some possible interpretations of the results obtained in Sections 2 and 3.

2 The non-relativistic Schrödinger equation

Let’s consider a quantum system describing a non-relativistic particle moving in one dimension under the effect of an external potential $V(x)$. If we choose the basis formed by the eigenvectors $|x⟩$ of the position operator $Q$, then the representation of an arbitrary state is given by the wave function $ψ(x) = ⟨x|ψ⟩$ and the Schrödinger equation is

$$i\hbar \frac{∂ψ}{∂t} = Hψ = -\frac{\hbar^2}{2m} \frac{∂^2ψ}{∂x^2} + V(x)ψ$$  \hspace{1cm} (1)

The time evolution of the system is a unitary transformation, and can be written shortly as $ψ(x,t) = e^{-i\frac{H}{\hbar}t}ψ(x,0)$. Besides, the position operator $Q$ and the momentum operator $P$ are hermitian operators defined by

$$Qψ = xψ \hspace{1cm} Pψ = -i\hbar \frac{∂ψ}{∂x}$$  \hspace{1cm} (2)

and they too generate unitary transformations of the form $e^{iαQ}$ and $e^{iαP}$; the transformations generated by the momentum operator are simply the space displacements, i.e. $e^{i\Delta x P/\hbar}ψ(x) = ψ(x + \Delta x)$.

We now define a new field $ϕ(x,y)$, which depends on the wave function $ψ(x)$ through the following relation:

$$ϕ(x,y) = ψ(x)ψ^*(y)$$  \hspace{1cm} (3)

If we apply an arbitrary unitary transformation to the wave function $ψ(x)$, the field $ϕ(x,y)$ will undergo a corresponding transformation. Through this correspondence, we can obtain the form of the generators in a new representation of the group of unitary transformations. In the case of the hamiltonian operator $H$, which generates the time evolution of the system, we easily obtain:

$$i\hbar \frac{∂ϕ}{∂t} = Hϕ = -\frac{\hbar^2}{2m} \left( \frac{∂^2ϕ}{∂x^2} - \frac{∂^2ϕ}{∂y^2} \right) + [V(x) - V(y)]ϕ$$  \hspace{1cm} (4)

Equation (4) is equal to the Schrödinger equation for density matrices, which defines the time evolution for mixed states, and therefore we will sometimes refer to the new representation as to the “density matrix representation”; in this paper, however, we will try to interpret the field $ϕ$ as a real objective field, rather than
just a tool for computing probabilities. As for the generators $Q$ and $P$, in the new representation we obtain the expressions

$$Q \varphi = (x - y) \varphi$$

$$P \varphi = -i\hbar \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y} \right)$$

The definitions (4) and (5) can be used for an arbitrary complex field $\varphi(x, y)$, not only for the fields expressed by the relation (3). However, we will consider only fields for which

$$\varphi(y, x) = \varphi^*(x, y)$$

This condition is needed to obtain real values for the observable quantities, as shown by expressions (13)-(15) below. From (6) it follows that we could use a real representation for the field $\varphi$, simply by defining the real field

$$\varphi_R = \text{Re} \{ \varphi \} + \text{Im} \{ \varphi \}$$

Indeed, $\text{Re} \{ \varphi \}$ is symmetric and $\text{Im} \{ \varphi \}$ is antisymmetric with respect to the permutation $x \leftrightarrow y$, and therefore they can be unambiguously obtained from the real field $\varphi_R$. However, in the rest of this section we will use the complex representation because it produces simpler expressions.

Now we want to define in the density matrix representation the observable quantities, which are the position, the momentum and the energy. In the original representation the natural definitions are the following:

$$Q = \int \psi^* (x) x \psi (x) \, dx$$

$$P = \int \psi^* (x) \left( -i\hbar \frac{d}{dx} \right) \psi (x) \, dx$$

$$E = \int \psi^* (x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi (x) \, dx$$

In the probabilistic interpretation of (1), the above definitions are simply the mean values of the corresponding quantum observables. However, if we consider (1) as a classical field equation and set $V(x) = 0$, then it is easily seen that (9) is the conserved quantity associated to the space invariance of the system, (10) is the conserved quantity associated to the time invariance of the system, while (8) is the center of mass, since (8) and (9) satisfy the following classical relation

$$Q(t) = P \frac{t}{m} + Q(0)$$

Besides, (11) is exactly the hamiltonian functional needed for applying to (1) the canonical hamiltonian formalism. Thus the definitions (8), (9) and (10) for position, momentum and energy are reasonable also within a classical interpretation of the Schrödinger equation (1).
For extending the above definitions to our new representation it is convenient to introduce a new pair of space coordinates
\[ x_S = \frac{1}{2} (x + y) \quad x_D = y - x \] (12)

In the rest of this section we will write loosely \( \varphi(x,y) \) or \( \varphi(x_S,x_D) \), meaning the same field expressed in two different coordinate systems. By means of the relation (3) we obtain the following expressions in the new representation:

\[ Q = \int x_S \varphi(x_S,x_D) \bigg|_{x_D=0} dx_S \] (13)

\[ P = i\hbar \int \frac{\partial \varphi(x_S,x_D)}{\partial x_D} \bigg|_{x_D=0} dx_S \] (14)

\[ E = \int \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \varphi(x_S,x_D)}{\partial x^2_D} + V(x_S) \varphi(x_S,x_D) \right]_{x_D=0} dx_S \] (15)

It can be shown that \( P \) and \( E \) are still conserved in the case \( V(x) = 0 \), while in the same case the classical relation (11) is still satisfied.

Thus the physical quantities depend only on the value of the field \( \varphi \) (and its space derivatives) in the region where \( x_D = 0 \), or \( x = y \). Therefore we are induced to consider \( x_S \) as the “physical” space coordinate, while \( x_D \) will be considered as an “auxiliary” coordinate, which has physical effects only around the point \( x_D = 0 \). Besides it can be shown that the momentum operator \( P \), as defined in (5), satisfies the following relation:

\[ e^{i\Delta x P/\hbar} \varphi(x_S,x_D) = \varphi(x_S + \Delta x, x_D) \] (16)

Since we already know that the momentum operator generates the spatial displacements, expression (16) adds more evidence to our belief that \( x_S \) is the “physical” position coordinate.

Now we want to express in the density matrix representation the unitary transformation which allows to switch from the position space to the momentum space. We know that this transformation for the original Schrödinger equation is given by the Fourier transform

\[ \Psi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x) e^{-ikx} dx \] (17)

We will again use the fundamental relation (3), exploiting the property that it commutes with all unitary transformations, and then extend the result to an arbitrary field \( \varphi \). The final expression is

\[ \Phi(k_x,k_y) = \frac{1}{2\pi} \int \int \varphi(x,y) e^{-ik_xx+ik_yy} dx dy \] (18)

From (18) we easily find that in momentum space the operator \( P \) is given by

\[ P \Phi = (p_x - p_y) \Phi \] (19)
where we set \( p_x = \hbar k_x \) and \( p_y = \hbar k_y \); we note that expression (19) has exactly the same form as the definition (5) for the position operator \( Q \). Besides, the quantity \( P \) defined by (14) is expressed in momentum space by

\[
P = \int p_S \Phi(k_S, k_D) \bigg|_{k_D=0} dk_S
\]

where we have defined

\[
k_S = \frac{1}{2} (k_x + k_y) \quad k_D = k_y - k_x
\]

while obviously \( p_S = \hbar k_S \). Again we note that expression (20) has exactly the same form as the definition (13) of the quantity \( Q \) in position space. This formal equivalence between the operators \( P \) and \( Q \), and between the observable quantities \( P \) and \( Q \), extends to our new representation the symmetry between position space and momentum space which is well known in non-relativistic quantum mechanics.

In the free particle case, where \( V(x) = 0 \), the time evolution of the system in momentum space is simply given by

\[
i\hbar \frac{\partial \Phi}{\partial t} = \left( \frac{p_x^2}{2m} - \frac{p_y^2}{2m} \right) \Phi
\]

while the energy \( E \) becomes

\[
E = \int \frac{p_S^2}{2m} \Phi(k_S, k_D) \bigg|_{k_D=0} dk_S
\]

From (21) and (23) we are again induced to consider \( k_S \) as the “physical” momentum coordinate, while \( k_D \) will be considered as an “auxiliary” coordinate.

Now we will answer a very interesting question: what happens to the Heisenberg uncertainty principle in the density matrix representation? In the original representation the Heisenberg principle follows from the fact that the operators \( Q \) and \( P \) do not commute:

\[
[ Q, P ] = Q P - P Q = i\hbar
\]

On the contrary, it can be easily shown that in our new representation the operators \( P \) and \( Q \), as defined by (5), do indeed commute. So it seems that the Heisenberg principle is not true in this representation, and indeed we will now show that there exist fields \( \varphi \) which are perfectly localized both in position space and in momentum space: let’s consider the field

\[
\varphi_0(x_S, x_D) = \delta(x_S - x_0) e^{-ik_0 x_D}
\]

Having identified \( x_S \) as the “physical” position, we can say that \( \varphi_0 \) is perfectly localized at the position \( x_0 \), since we have \( \varphi_0 = 0 \) for \( x_S \neq x_0 \). Applying the transformation (18) to \( \varphi_0 \) we obtain in momentum space

\[
\Phi_0(k_S, k_D) = \delta(k_S - k_0) e^{ik_D x_0}
\]
and again, having identified $k_S$ as the “physical” momentum coordinate, we can say that $\Phi_0$ is perfectly localized in momentum space, with momentum $p_0 = \hbar k_0$.

Besides, applying to $\phi_0$ the expression (13), we indeed obtain $Q = x_0$, while applying to $\varphi_0$ the expression (14) we obtain $P = \hbar k_0 = p_0$ as expected; the same value for $P$ is obtained in momentum space, i.e. applying the expression (20) to $\Phi_0$.

In the case of a free particle, where $V(x) = 0$, we easily find that the energy, obtained from (15) or from (23), has the correct classical value, i.e. $E = \hbar^2 k_0^2 / 2m = p_0^2 / 2m$. But there is a far more interesting result: by setting

$$x_0 = v_0 t, \quad k_0 = \frac{p_0}{\hbar} = \frac{mv_0}{\hbar}$$

which are the classical position and momentum for a particle moving with constant speed $v_0$, we find that (25) is a solution of the motion equations (3). Thus (25) and (27) define a field $\varphi_0(x, y, z, t)$ which can be interpreted as a non-relativistic point-like particle moving at constant speed $v_0$ with position and momentum perfectly known at every time $t$. This result, which seemed incompatible with quantum mechanics, has been obtained simply by carrying the Schrödinger equation (1) to the density matrix representation.

Now we turn to the general case of a potential $V(x) \neq 0$. We are interested in finding the stationary states, i.e. those states which are sinusoidal functions of the time $t$. In the original representation, the stationary states $\psi_{\alpha}$ were the eigenfunctions of the Hamiltonian

$$i\hbar \frac{\partial \psi_{\alpha}}{\partial t} = H \psi_{\alpha} = E_{\alpha} \psi_{\alpha}$$

while the set of all eigenvalues $E_{\alpha}$ was the energy spectrum of the system. The same is true in our new representation, but now the Hamiltonian $H$ is expressed by equation (4). It is not difficult to show that in our case the eigenfunctions $\varphi_{\alpha\beta}$ and the eigenvalues $E_{\alpha\beta}$ are simply given by

$$\varphi_{\alpha\beta}(x, y) = \psi_{\alpha}(x) \psi^*_\beta(y) \quad E_{\alpha\beta} = E_{\alpha} - E_{\beta}$$

Thus the natural frequencies in our representation correspond to the difference between two energy levels of the original quantum system. But we know very well that the individual energy levels are not really observable; the only observable quantities are exactly the “jumps” between different energy levels expressed by (29). Therefore we can say that we have reproduced the same energy spectrum as the original quantum system; besides, we have eliminated the ambiguity about the energy zero-point which is typical of all quantum systems.

In conclusion, simply by changing the representation of the Schrödinger equation we have succeeded in bringing together two concepts which are widely held as incompatible: on one hand we describe point-like particles, with position and momentum perfectly known at every time $t$, on the other hand we allow the existence of discrete energy spectra. Besides, in the case $V(x) = 0$ we can add together an arbitrary number of solutions of the form (23), thus obtaining a system of non-interacting particles; the field describing this system depends always on the same
two variables $x_S$ and $x_D$, so the dimension of the space on which the field is defined does not change with the number of the particles. Finally, the transition to our new representation involves the definition of an “auxiliary” position coordinate $x_D$, in addition to the “physical” coordinate $x_S$. The existence of extra space-time dimensions is obviously not an original idea: it was first proposed by Kaluza and then developed by Klein [2]. In recent years, I find very interesting the attempt made by Hasselmann [3] to derive all elementary particles properties from Einstein’s gravitational field equations in a higher-dimensional matter-free space. Our model has in common with the cited works the periodicity of the solutions with respect to the extra space dimension, as shown by (25).

3 The Dirac equation and the electromagnetic interaction

We now try to extend the results obtained in Section 2 to the case of a relativistic wave equation. We choose the Dirac equation, rather than the Klein-Gordon equation, for the simple reason that all known elementary particles are half-spin fermions, while no elementary spinless particle is known to exist. We start from the non-covariant form of the Dirac equation, which separates the time derivative from the space derivatives:

\[ i \frac{\partial \psi}{\partial t} = \left( -i \alpha_k \partial_k + m \beta \right) \psi = H_0 \psi \]  

(30)

where the implicit sum is over $k = 1, 2, 3$.

The operator $H_0$ defined by equation (30) generates a unitary transformation on the spinor field $\psi(x)$, where now $\psi$ depends upon a three-dimensional space coordinate $x$; we then follow the same method already seen in the previous section and define a new representation of the group of unitary transformations by means of the fundamental relation:

\[ \varphi(x, y) = \psi(x) \psi^\dagger(y) \]  

(31)

The time evolution in the new representation is then given by

\[ i \frac{\partial \varphi}{\partial t} = H_0 \varphi = -i (\alpha_k \partial_x \varphi + \partial_y \varphi \alpha_k) + m (\beta \varphi - \varphi \beta) \]  

(32)

We now extend equation (32) to an arbitrary $4 \times 4$ complex matrix $\varphi(x, y)$; however, as in the previous section, to obtain real values for the observable quantities the anti-hermitian part of the matrix field $\varphi$ must vanish, and therefore we will only consider fields $\varphi(x, y)$ satisfying the relation

\[ \varphi(y, x) = \varphi^\dagger(x, y) \]  

(33)

Again we could use a real representation by defining $\varphi_R = Re \{ \varphi \} + Im \{ \varphi \}$, but we prefer the complex representation because it produces simpler expressions.

We are now able to find solutions to the motion equations (32) which are perfectly localized in physical space and do not depend on time; these solutions
are thus good candidates to represent point-like particles at rest. To obtain this result, we first define a new pair of space coordinates
\[ x_S = \frac{1}{2} (x + y) \quad x_D = y - x \] (34)
and then choose a particular representation for the matrices \( \beta \) and \( \alpha_k \)
\[ \beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \alpha_k = \begin{bmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{bmatrix} \] (35)
where \( \sigma_k \) are the well known Pauli matrices, while \( I \) is the \( 2 \times 2 \) identity matrix. Our solutions are then given by:
\[ \varphi_A = \frac{1}{4m} \begin{bmatrix} 4m \delta & 0 & i \partial_3 \delta & i \partial_1 \delta + \partial_2 \delta \\ 0 & 0 & 0 & 0 \\ -i \partial_3 \delta & 0 & 0 & 0 \\ \partial_2 \delta - i \partial_1 \delta & 0 & 0 & 0 \end{bmatrix} \] (36)
\[ \varphi_B = \frac{1}{4m} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 4m \delta & i \partial_1 \delta - \partial_2 \delta & -i \partial_3 \delta \\ 0 & -i \partial_1 \delta - \partial_2 \delta & 0 & 0 \\ 0 & i \partial_2 \delta & 0 & 0 \end{bmatrix} \] (37)
\[ \varphi_C = \frac{1}{4m} \begin{bmatrix} 0 & 0 & -i \partial_3 \delta & 0 \\ 0 & 0 & \partial_2 \delta - i \partial_1 \delta & 0 \\ i \partial_3 \delta & i \partial_1 \delta + \partial_2 \delta & -4m \delta & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] (38)
\[ \varphi_D = \frac{1}{4m} \begin{bmatrix} 0 & 0 & 0 & -i \partial_1 \delta - \partial_2 \delta \\ 0 & 0 & 0 & i \partial_3 \delta \\ i \partial_1 \delta - \partial_2 \delta & -i \partial_3 \delta & 0 & -4m \delta \\ i \partial_3 \delta - \partial_2 \delta & -i \partial_1 \delta & 0 & 0 \end{bmatrix} \] (39)
where we wrote simply \( \delta \) for \( \delta (x_S) \). If we identify, as in the previous section, the physical position with the coordinate \( x_S \), then the solutions (36)-(39) are perfectly localized in the physical position \( x_S = 0 \). At this point we would like to find solutions corresponding to particles moving at constant speed; these solutions will be easily obtained from (36)-(39) by means of Lorentz transformations, but first we must investigate the Lorentz covariance of our new representation.

There is only one way to recover a formal Lorentz covariance in our new representation: since special relativity treats position and time as components of the same vector \( x^\mu \), and since we have duplicated the position coordinate \( x \), we inevitably must duplicate also the time coordinate \( t \). Thus we are led to introduce two time coordinates \( t_x, t_y \) and a second pair \( t_S, t_D \) obtained through a relation analogous to (34). In expressions (31) and (33) then the three-vectors \( x \) and \( y \) must be replaced by the four-vectors \( x \equiv (t_x, x) \) and \( y \equiv (t_y, y) \), while the motion equations (32) are extended to the following covariant form
\[ i \gamma^\mu \frac{\partial}{\partial x^\mu} \varphi(x, y) - m \varphi(x, y) = 0 \] (40)
which is formally identical to the covariant Dirac equation in the usual representation. Equation (40) can be written in the equivalent form

\[ i \frac{\partial}{\partial y} \gamma^0 \gamma^\mu \varphi(x, y) + m \varphi(x, y) \gamma^0 = 0 \] (41)

which is obtained by taking the adjoint of (40) and by exchanging \( x \) with \( y \). If we separate the time derivative from the space derivatives, (40) and (41) become

\[ i \partial_t \varphi = -i \alpha_k \partial_x \varphi + m \beta \varphi \] (42)

\[ i \partial_y \varphi = -i \partial_y \varphi \alpha_k - m \beta \varphi \] (43)

where we applied the usual definitions \( \beta = \gamma^0 \) and \( \alpha_k = \gamma^0 \gamma^k \). By adding together (42) and (43), we obtain the derivative with respect to \( t = (t_x + t_y) / 2 \), which is equal to the motion equations (32) in the case \( t_x = t_y = t \); by subtracting (42) from (43) and dividing by two, we obtain the derivative with respect to \( t_D = t_y - t_x \).

As for the Lorentz transformations, it is well known that an infinitesimal Lorentz transformation acts on a four-vector \( x^\mu \) and on a spinor \( \psi \) in the following way:

\[ x'^\mu = x^\mu + \epsilon_{\mu\nu} x^\nu \] (44)

\[ \psi' = \psi + \frac{1}{4} \epsilon_{\mu\nu} \gamma^\mu \gamma^\nu \psi \] (45)

where \( \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = 0 \). By means of the fundamental relation (31), we extend (45) to our new representation, obtaining:

\[ \varphi' \gamma^0 = \varphi \gamma^0 + \frac{1}{4} \epsilon_{\mu\nu} \left( \gamma^\mu \gamma^\nu \varphi \gamma^0 + \varphi \gamma^0 \gamma^\nu \gamma^\mu \right) \] (46)

We then separate the boosts from the spatial rotations rewriting (46) as follows

\[ \varphi' = \varphi + i \epsilon_R \cdot R \varphi + i \epsilon_B \cdot B \varphi \] (47)

where the vectors \( \epsilon_R \) and \( \epsilon_B \) are

\[ \epsilon_R = \begin{bmatrix} \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{bmatrix} \quad \epsilon_B = \begin{bmatrix} \epsilon_{01} \\ \epsilon_{02} \\ \epsilon_{03} \end{bmatrix} \] (48)

and the components of the two vector operators \( R \) and \( B \) are expressed by

\[ R_k \varphi = -\frac{1}{2} \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \varphi + \frac{1}{2} \varphi \begin{bmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{bmatrix} \] (49)

\[ B_k \varphi = -\frac{i}{2} \alpha_k \varphi - \frac{i}{2} \varphi \alpha_k \] (50)
with $\alpha_k$ given by (35).

Now we are able to find solutions of the motion equations representing particles moving with arbitrary speed $v = v_n$. Let’s call $\varphi_0$ one of the four solutions (36)-(39) representing particles at rest. The first step is to solve equations (42) and (43) with respect to $t_D$, obtaining a complete solution $\varphi_0(x_S, t_D)$ independent from $x_D$ and $t_S$; the solutions (36)-(39) are recovered in the case $t_D = 0$. Then we apply a boost with speed $v$ along the unit vector $n$, which corresponds to the following transformations:

$$\varphi' = e^{i n \cdot B} \varphi$$

$$t'_S = t_S \cosh \xi + (x_S \cdot n) \sinh \xi$$

$$x'_S = x_S + (x_S \cdot n) (\cosh \xi - 1) + n t_S \sinh \xi$$

where the parameter $\xi$ is defined by $\tanh \xi = v$; obviously $t_D$ and $x_D$ transform like $t_S$ and $x_S$. Thus we obtain the following solution:

$$\varphi'_{v}(t'_S, x'_S, t'_D, x'_D) = e^{i n \cdot B} \varphi_0(x_S, t_D)$$

It is clear that in the “physical” case $t'_D = x'_D = 0$ the solution vanishes outside the region defined by $x'_S = v t'_S$; therefore it represents a particle moving with speed $v$ perfectly localized in the position $x'_S = v t'_S$.

Now we will express in our new representation the physical quantities associated to the Dirac particles, which are energy, momentum, charge and spin. Let’s start from the charge $Q$, whose space density in the original representation was the time component of the four-current

$$J^\mu (x) = \psi^\dagger (x) \gamma^0 \gamma^\mu \psi (x)$$

By means of the fundamental relation (31) we obtain in the new representation the following expression

$$J^\mu (x) = \text{Tr} \left\{ \varphi (x_S, x_D) \gamma^0 \gamma^\mu \right\} \bigg|_{x_S = x, x_D = 0}$$

The time conservation of the charge $Q$ is a consequence of the continuity equation $\partial_\mu J^\mu = 0$, which is true in both representations. The explicit form of the charge $Q$ in the new representation is simply given by

$$Q = \int d^3x \sum_{k=1}^4 \varphi_{kk} (x_S, x_D) \bigg|_{x_S = x, x_D = 0}$$

In the same way, starting from the energy-momentum tensor in the original representation, we easily obtain the following expression for the same tensor in our new representation:

$$T^\mu_\nu (x) = \text{Tr} \left\{ -i \frac{\partial}{\partial x^D_\nu} \varphi (x_S, x_D) \gamma^0 \gamma^\mu \right\} \bigg|_{x_S = x, x_D = 0}$$
In both representations the conservation of energy and momentum is expressed by the continuity equation $\partial_{\mu} T^{\mu\nu} = 0$, while the definitions of energy and momentum in the new representation are

$$E = \int \mathcal{E}(x) \, d^3x \quad \quad P_i = \int \mathcal{P}_i(x) \, d^3x$$

with the spatial densities $\mathcal{E}$ and $\mathcal{P}_i$ given by

$$\mathcal{E} = T^{00} = \text{Tr} \left\{ i \alpha_k \frac{\partial \varphi}{\partial x_{D,k}} + m \beta \varphi \right\}_{x_S=x, x_D=0}$$

$$\mathcal{P}_i = T^{0i} = \sum_{k=1}^{4} i \frac{\partial \varphi_{kk}}{\partial x_{D,i}} \bigg|_{x_S=x, x_D=0}$$

Finally, from the angular momentum tensor in the original representation, we obtain the following expression for the same tensor in the new representation:

$$M^{\mu\lambda\nu}(x) = x^\lambda T^{\mu\nu}(x) - x^\nu T^{\mu\lambda}(x)$$

$$+ \text{Tr} \left\{ \frac{i}{4} \varphi(x_S, x_D) \gamma^0 \gamma^\mu \left( \gamma^\lambda \gamma^\nu - \gamma^\nu \gamma^\lambda \right) \right\}_{x_S=x, x_D=0}$$

In both representations the angular momentum conservation is expressed by the continuity equation $\partial_{\mu} M^{\mu\lambda\nu} = 0$. Besides the angular momentum spatial densities are given by:

$$M^{0ij}(x) = \epsilon_{ijk} \left( \mathcal{L}_k(x) + \mathcal{S}_k(x) \right) \quad i, j, k = 1, 2, 3$$

where $\epsilon_{ijk}$ is the rank three antisymmetric tensor; $\mathcal{L}$ is the orbital angular momentum density, while $\mathcal{S}$ is the spin density. The spin components are then given by:

$$S_i = \int S_i(x) \, d^3x = \int d^3x \text{ Tr} \left\{ \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \varphi(x_S, x_D) \right\}_{x_S=x, x_D=0}$$

From expressions (56), (58) and (62), it is clear that all observable quantities depend on the value of the field $\varphi$ (and its space-time derivatives) in the region where $x_D = 0$, or $x = y$. Once again we are induced to consider $x_S$ as the “physical” space-time coordinate, while $x_D$ is an “auxiliary” coordinate, which has observable effects only around the point $x_D = 0$.

We are now able to associate to the solutions defined in (36)-(39) their physical properties. From the above definitions, we see that $\varphi_A$, $\varphi_B$, $\varphi_C$ and $\varphi_D$ have energy $E = m$, momentum $P = 0$ and spin components $S_1 = S_2 = 0$. As for the charge $Q$ and the third spin component $S_3$, we have the following situation: $\varphi_A$ has $Q = +1$ and $S_3 = +1/2$, thus representing a spin up positron; $\varphi_B$ has $Q = +1$ and $S_3 = -1/2$, thus representing a spin down positron; $\varphi_C$ has $Q = -1$ and $S_3 = -1/2$, thus representing a spin down electron; finally, $\varphi_D$ has $Q = -1$ and $S_3 = +1/2$, thus representing a spin up electron. Besides, if we apply a rotation or
a boost to these solutions, we are certain that the physical quantities associated
to the new solutions will have the usual relativistic values; this is true because we
have expressed all the observable quantities in a covariant form.

Until now, we have shown that our new representation admits solutions which
can be interpreted as relativistic point-like particles moving at constant speed in
an arbitrary direction, with the correct values for energy and momentum. In their
rest frame, the particles have spin \( S = 1/2 \), while the spin orientation is arbitrary;
as for the charge \( Q \), we have solutions with \( Q = +1 \), to be interpreted as positrons,
and solutions with \( Q = -1 \), to be interpreted as electrons. By adding together
an arbitrary number of such solutions, we obtain a system of non-interacting
particles. Although attractive, this description is far from complete: indeed, there
are a lot of solutions for which such particle interpretation is not possible and our
model does not explain why the physical solutions should be acceptable while the
non-physical solutions should be discarded.

At this point, we must put forward the following consideration: real particles
interact with each other and we can observe them only because they interact
with our measuring devices; therefore, all mathematical models describing non-
interacting particles are equally useless for explaining physical phenomena. To
complete our model, then, we need an interaction between particles; we will choose
the electromagnetic interaction, which has provided the first striking application
of quantum field theory, as well as the first example of a modern gauge theory.

To introduce the electromagnetic interaction in our model, we return to the
original representation, where we make the usual substitution

\[
\frac{\partial}{\partial x^\mu} \rightarrow \frac{\partial}{\partial x^\mu} - ieA_\mu
\]  

(65)

If we insert (65) in the non-covariant Dirac equation (30), and then switch to our
new representation by means of (31), we obtain the following time evolution:

\[
i \frac{\partial \varphi}{\partial t} = H_0 \varphi + eH_1 \varphi
\]  

(66)

where \( H_0 \) is still defined as in (32), while \( H_1 \) is given by

\[
H_1 \varphi = A^0(y) \varphi - A^0(x) \varphi + A^k(x) \alpha_k \varphi - A^k(y) \varphi \alpha_k
\]  

(67)

where, as usual, the implicit sum is over \( k = 1, 2, 3 \).

As for the covariant equations (40) and (41), in the presence of an external
electromagnetic field they become:

\[
i \gamma^\mu \left( \frac{\partial}{\partial x^\mu} - ieA_\mu(x) \right) \varphi - m \varphi = 0
\]  

(68)

\[
i \left( \frac{\partial}{\partial y^\mu} + ieA_\mu(y) \right) \varphi \gamma^0 \gamma^\mu + m \varphi \gamma^0 = 0
\]  

(69)

As in the free field case, (69) can be obtained by taking the adjoint of (68) and
by exchanging \( x \) with \( y \). The non-covariant time evolution (66) can be recovered
from (68) and (69) simply by setting \( t_x = t_y = t \).
Let’s spend a few words about charge conjugation and gauge invariance. In the original representation, if \( \psi \) is a solution of the Dirac equation interacting with an electromagnetic field

\[
    i \gamma^\mu (\partial_\mu - i e A_\mu) \psi - m \psi = 0 \tag{70}
\]

then the field

\[
    \psi'(x) = \gamma^2 \psi^*(x) \tag{71}
\]

is a solution of the equation obtained from (70) by changing the sign of the interaction parameter \( e \):

\[
    i \gamma^\mu (\partial_\mu + i e A_\mu) \psi' - m \psi' = 0 \tag{72}
\]

The transformation (71) is called charge conjugation and depends on the particular representation \( (35) \) we have chosen for the Dirac matrices. It is easily seen that \( \psi \) and \( \psi' \) have the same charge \( Q = \int \psi^\dagger(x) \psi(x) d^3x \); we would expect the charge conjugation to change the sign of the charge, as it happens for instance in the case of the Klein-Gordon equation, but this is of course impossible for the Dirac equation, since the charge is always positive in the original representation. If we now switch to our new representation, we find that the transformation (71) becomes

\[
    \varphi'(x,y) = -\gamma^2 \varphi^T(y,x) \gamma^2 \tag{73}
\]

Again we find that \( \varphi' \) satisfies the motion equations obtained by changing the sign of the interaction parameter \( e \), and again \( \varphi \) and \( \varphi' \) have the same charge, i.e. \( Q' = Q \). However, if we now change the sign of (73), we still have a solution of the same motion equations, but now we have \( Q' = -Q \). Therefore in our new representation the correct definition for the charge conjugation is

\[
    \varphi'(x,y) = +\gamma^2 \varphi^T(y,x) \gamma^2 \tag{74}
\]

It can be shown that (74) changes the sign of the whole four-current \( J^\mu \), so that the field \( \varphi' \) satisfies the conjugate Maxwell equations for the electromagnetic field; this property was obviously not true in the original representation. Besides, the transformation (74) leaves unchanged all other observable quantities, namely energy, momentum and spin, and this is exactly what we expect from a well defined charge conjugation transformation; for instance, if we consider the four solutions \( (36)-(39) \), we easily see that \( \varphi_D (\varphi_C) \) is obtained by charge conjugation from \( \varphi_A (\varphi_B) \) and vice versa.

Turning now to the gauge invariance, we remember that in the original representation a gauge transformation is given by a local phase rotation on the field \( \psi \)

\[
    \psi'(x) = e^{i \theta(x)} \psi(x) \tag{75}
\]

together with the transformation

\[
    A'_\mu(x) = A_\mu(x) + \frac{1}{e} \frac{\partial}{\partial x^\mu} \theta(x) \tag{76}
\]
for the electromagnetic field. Gauge invariance means that if $\psi$ and $A^\mu$ are solutions of the motion equations, the same is true for $\psi'$ and $A'^\mu$. In our new representation (75) becomes

$$\varphi'(x, y) = e^{i [\theta(x) - \theta(y)]} \varphi(x, y)$$  (77)

and it is easily seen that the motion equations (68), or (69), are invariant with respect to the transformations (76) and (77), i.e. they are gauge invariant.

Until now we have considered the electromagnetic field as an external source, rather than a dynamical component of our model. If we want to include the field $A^\mu$ in our model, we have to define its time evolution in interaction with the “material” field $\varphi$. The dynamical behaviour of the electromagnetic field will be described by the classical Maxwell equations:

$$\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \partial^\mu F_{\mu\nu} = J_\nu$$  (78)

We simply have to define the current $J_\nu$ as a function of the field $\varphi$. To obtain the electric current density from (56) we multiply by the unit electric charge $e$, obtaining:

$$J^\mu (x) = e \text{Tr} \left\{ \varphi(x_S, x_D) \gamma^0 \gamma^\mu \right\} \bigg|_{x_S=x, x_D=0}$$  (79)

The Maxwell equations (78) are again gauge invariant: the left-hand side depends only on the gauge invariant tensor $F_{\mu\nu}$, while the right-hand side depends only on the values of $\varphi$ for $x_D = 0$, or $x = y$, which are not affected by the transformation (77).

In conclusion, we have built a model which describes the “material” Dirac field $\varphi$ together with the electromagnetic field $A^\mu$. The model is defined by the time evolution of the field $\varphi$, (68) or (69), and by the Maxwell equations (78) where the current is given by (79). The field $\varphi$ depends on two space-time coordinates $x_S$ and $x_D$; however, we point out that the duplication of the time coordinate is needed only for formal requirements: the motion equations (68) or (69) can be solved with respect to the “physical” time ($t_S = t$ and $t_D = 0$) without even mentioning a second time coordinate. On the contrary, the electromagnetic field $A^\mu$ depends on a single space-time coordinate $x$ and its time evolution is given by the classical Maxwell equations; it is interesting here to note that some authors (see Marshall and Santos [4]) reject the quantum concept of an electromagnetic field made up by point-like massless “photons”, and believe that all experimental data involving light fields can be explained by means of the unquantized Maxwell equations.

4 Discussion

Let me explain the basic ideas (or maybe prejudices) on which the present paper is built. The main idea is that quantum mechanics cannot be a fundamental theory: I cannot force myself to believe that probabilities are an objective feature of the physical world, and therefore I maintain the old-fashioned belief that the
fundamental laws of nature must be deterministic and quantum mechanics must be just a statistical formulation of these fundamental laws. The obvious example for clarifying this concept is the classical motion of a free non-relativistic particle, with position $x$ and momentum $p$: the “wave equation”

$$\frac{\partial f}{\partial t} = -\frac{p}{m} \frac{\partial f}{\partial x} \quad (80)$$

for the joint probability distribution $f(x,p)$ is just a statistical formulation of the deterministic state equations

$$\dot{x} = \frac{p}{m} \quad \dot{p} = 0 \quad (81)$$

We are not worried about the “collapse of the wave function” at the time when the position $x$ is measured, because we do not interpret $f(x,p)$ as a real objective field; in any case, such interpretation would be strongly hampered by the fact that $f(x,p)$ is defined in configuration space, whose dimension depends on the number of particles described.

Now, what mathematical form should have this deterministic law which will finally replace quantum mechanics and quantum field theory? I am convinced that it should be a classical (unquantized) field equation, satisfying the principle of relativistic covariance. However, since until now all efforts at finding this unified field equation have failed, we probably need to extend the classical concept of field, to gain some room for maneuver. In my opinion, the most natural extension is to allow for extra space-time dimensions, following the idea which was originally proposed by Kaluza and Klein [2], and has been recently revived by Hasselmann [3]. Another possible choice could be to abandon the concept of a continuous space-time, and suppose that the ultimate structure of space-time is discrete (see for instance Budnik [5]); however, besides having a prejudice against “objective probabilities”, I also have a prejudice in favour of Newton’s “Natura non facit saltus”, basing this prejudice mainly on the wish to maintain the beautiful space-time symmetries and conservation laws which are strictly connected to the continuous structure of space-time.

Thus, we are looking for a relativistic field equation (or a system of equations) defined on a space-time with more than four dimensions. But what should be our starting point for this long and perilous search? The authors cited above stated immediately their ambitious goal: they wanted to unify gravity with all other known interactions (in the Kaluza-Klein case it was just the electromagnetic interaction, while Hasselmann’s “metron” model includes all Standard Model’s interactions). Even if this is indeed the final goal, I propose here a more cautious approach: instead of trying to guess from the beginning the final form of our field equations, let’s start from the field equations on which quantum mechanics and quantum field theory are based (the non-relativistic Schrödinger equation and the relativistic Dirac equation) and let’s try to extend them in such a way that their solutions may now be interpreted as real objective fields.

In the introduction to this paper (Section [2]) I listed the two main obstacles to a realistic interpretation of the wave function in non-relativistic quantum mechanics:
(1) the wave function spreads out with time and (2) it is defined on a variable dimension configuration space. Well, in Section 2 we saw that if we choose the Schrödinger equation for density matrices, instead of the usual equation for pure states, it is possible to find solutions which are perfectly localized both in position space and in momentum space. In the free field case, these solutions behave exactly as non-relativistic point-like particles moving at constant speed, with the correct values for the observable quantities (position, energy and momentum). By adding together an arbitrary number of such solutions we obtain a field describing a system of non-interacting particles; this field depends always on the same two space coordinates, the “physical” coordinate $x_S$ and the “auxiliary” coordinate $x_D$. Both problems (1) and (2) are solved, and I believe that this is a first (small) step in the right direction.

It is interesting here to note that some authors already consider the Schrödinger equation for density matrices more fundamental than the one for pure states. For instance Olavo [6] explicitly names the equation for density matrices “First Schrödinger’s Equation”, while the equation for pure states, named “Second Schrödinger’s Equation”, is obtained from the first in the less general case where the density matrix is a product of pure states. Olavo also defines the position operator and the momentum operator for density matrices and shows that they commute, similarly to what we did in Section 2; from this property Olavo infers the “negation of ontological origin of Heisenberg’s uncertainty relations” and I do indeed agree with him. However, Olavo’s purpose is to demonstrate that quantum mechanics can be derived from classical mechanics, and therefore he is not interested in giving a realistic interpretation of the density matrix.

A second example can be found in Mermin’s “Ithaca interpretation” [7]: in this paper the author lists six desiderata for a satisfactory interpretation of quantum mechanics, and then writes “If you take Desideratum (5) seriously (i.e. generalized Einstein locality), then there can be no more objective reality to the possible different realizations of a density matrix, than there is to the different possible ways of expanding a pure state in terms of different complete orthonormal sets ... In the case of an individual system the density matrix must be a fundamental and irreducible objective property, whether or not it is a pure state”. This last statement sounds quite similar to the idea that I am suggesting in the present paper, i.e. the density matrix as a real objective field. However, this resemblance disappears when we read Desideratum (6), which states that probabilities are objective intrinsic properties of individual physical systems; unlike Mermin, I still believe that probability is “just a way of dealing systematically with our own ignorance”, and not a fundamental feature of the physical world.

Let’s turn now to the relativistic Dirac equation. We immediately see a new obstacle to a realistic interpretation of its solutions: (3) the charge density $\psi^*\psi$ is always positive, and therefore the classical Dirac field does not allow us to describe particles with both positive and negative charge, i.e. positrons and electrons. The second quantization procedure solves point (3) by means of the anti-commuting properties of the creation-annihilation operators, but the points (1) and (2) listed above remain unsolved; furthermore, the final wave function (or state vector) is defined in a Hilbert space whose dimension is dramatically increased. On the
contrary, in Section 3 we saw that, by extending to the Dirac equation the same representation already used for the non-relativistic Schrödinger equation, it is possible to find solutions which behave as relativistic point-like particles moving at constant speed, with the correct values for all observable quantities. By adding together an arbitrary number of such solutions we can describe a system of non-interacting particles, and the resulting field depends always on the same six space coordinates, three “physical” coordinates $x_S$ and three “auxiliary” coordinates $x_D$. By introducing also two time coordinates (a “physical” time coordinate $t_S$ and an “auxiliary” time coordinate $t_D$) we can recover the formal Lorentz covariance of the Dirac equation. Finally, in the new representation the charge can have both signs, and more generally it is possible to define a charge conjugation transformation which changes the sign of the charge and leaves unchanged all other physical quantities. Therefore, points (1), (2) and (3) are all solved, and I think that this is a second step in the right direction.

Thus, for both the non-relativistic Schrödinger equation and the relativistic Dirac equation we were able to find solutions which can be interpreted as non-interacting point-like particles. At this point, however, we must face a big problem: these solutions are only a limited subset of all possible solutions, while for most solutions this particle interpretation is not possible; until now our model does not explain why the “particle” solutions should be accepted while the other solutions should be discarded. I will discuss this problem only for the Dirac equation, neglecting the non-relativistic case for the following reasons: first, not all potentials $V(x)$ have physical meaning; second, since non-relativistic quantum mechanics is just an approximation for relativistic quantum field theory, if we find an acceptable model for quantum field theory then the corresponding model for quantum mechanics will follow in the non-relativistic limit.

In my opinion, there is only one way to single out the “particle” solutions of the Dirac equation from all other solutions: we must add a non-linear interaction to the model. The interaction must be non-linear for obvious reasons: for instance, the field obtained by changing the sign to a positive energy solution cannot be itself a solution, because it would have negative energy (negative mass in its rest frame) and therefore it would be non-physical. Thus, in the second part of Section 3 we inserted in our model the electromagnetic interaction, by means of the usual “minimal coupling” prescription. We also included in our model the dynamical evolution of the electromagnetic field, in the form of the classical Maxwell equations; the four-current in the Maxwell equations depends on the Dirac field through equation (79). Our final model provides a complete description of the mutual interactions between the “material” Dirac field and the “gauge” field, i.e. the classical electromagnetic field.

Now the question is: does this final model really solve the problem of eliminating the non-physical solutions? This will be true if the only stable solutions for the “material” field $\varphi$ are the vacuum (i.e. the solution $\varphi = 0$) and the single particle solutions, which must be strongly localized in physical space and must provide the correct values for all observable quantities; we can relax this last requirement by accepting some form of renormalization, i.e. the physical mass and charge $m_{ph}$ and $e_{ph}$ could be different from the “bare” quantities $m$ and $e$ which
appear in the model equations. Unfortunately, the task of finding the solutions to our model (either by symbolic calculations or by computer simulations) seems to me extraordinarily complex, so our question will find no answer in the present paper. However, let’s suppose that the answer may be affirmative; if this were true, then we could attempt to interpret our model as a deterministic alternative to quantum electrodynamics. To reach this ambitious goal, we would have to satisfy (at least) two further requirements: first, in scattering processes our model should be able to reproduce the well known phenomenology of QED, with the same probability amplitudes; second, by adding an external source to our model (for instance the charge of the hydrogen atom nucleus), the stable solutions should reproduce the same energy spectra which can be obtained from standard QED.

On the contrary, if the answer to our question were a plain no, this would not necessarily entail that all our previous efforts have been useless; we could try to make some changes in our model, along one of the following lines:

- Maybe the electromagnetic interaction alone is not enough to single out the “particle” solutions from all other solutions: maybe it is necessary to insert in our model the other known interactions, or even some new (yet unknown) interaction, which would be responsible for the strong localization of the electron and positron fields. My guess that the electromagnetic interaction may suffice is based only on the fact that QED can exist as an independent theory even if we know that it is really a part of a more general theory, i.e. the Weinberg-Salam model for electroweak interactions.

- In this paper we suppose that the “auxiliary” space-time, indexed by the coordinate $x_D$, is infinite and flat. It could as well be finite and strongly curved, similarly to the fifth dimension in the Klein version of the original Kaluza theory. If the curvature radius were really small compared to macroscopic space-time distances, this could explain why we do not observe the extra space-time dimensions.

- Finally, we could look for a way to extend the electromagnetic field to the extra space-time dimensions, obtaining a new field $A^\mu(x_S, x_D)$ which would be equal to the classical electromagnetic field in the case $x_D = 0$. The resulting model should have a better symmetry between the material field and the gauge field.

Of course, I have no absolute certainty that the method presented in this paper will finally produce acceptable explanations of physical phenomena: I just tried to find a new approach to deal with a very old problem. However, it can be shown that if we apply this method to simple non-physical systems (two- and three-dimensional Hilbert spaces) we indeed obtain deterministic models equivalent to the corresponding quantum systems; in this case, the density matrix representation becomes simply the adjoint representation of the SU(2) and SU(3) groups.

In conclusion, I will point out some features of our model which could be related to the debate about quantum non-locality; let’s briefly summarize the main results of this long lasting debate. In 1964 Bell showed that no local hidden
variable (LHV) model can reproduce the same probability correlations obtained from quantum mechanics; he did so by constructing an explicit relation (the first famous Bell inequality) which must be satisfied by every LHV model and which is violated by some quantum states. In the early 80’s, experiments were performed [9, 10] which seemed to indicate that Bell inequalities are indeed violated by nature. If this were true, then we would be forced to choose between determinism and locality; for instance, Mermin [7] chooses to abandon determinism and to retain locality. However, the debate about the experimental violation of Bell inequalities is still open: it seems that this violation can be inferred from the existing experimental data only by introducing some supplementary assumptions (no enhancement, faithful sampling, . . . ) and therefore many authors [11, 12] believe that until now no incontrovertible violation has been really observed. Besides, Marshall and Santos [4, 13] are convinced that the corpuscular interpretation of light plays an important role in the above-mentioned assumptions, and were able to build a local “wave” theory of light, derived from Maxwell’s classical equations, whose predictions are in good agreement with the experimental data [14].

The model presented in Section 3 of the present paper is expressed in a relativistic covariant form and therefore it must satisfy the principle of relativistic causality (space-like separated events cannot influence each other and no interaction can propagate at superluminal speed). However, the following features seem to indicate some form of non-local behaviour:

- Expression (33) establishes an instant relation between the fields \( \varphi(x, y) \) and \( \varphi(y, x) \). However, this non-local relation is confined to extra-space, while in the physical case \( x = y \) we obtain a local hermiticity property for the matrix field \( \varphi(x, x) \).

- The solutions (36)-(39) representing particles at rest are perfectly localized in the physical position \( x_S = 0 \) and do not depend on the auxiliary coordinate \( x_D \). However, it can be shown that the solutions representing moving particles, while being perfectly localized at \( x_S = vt \) when \( x_D = 0 \), spread out along the \( x_S \) axis when \( x_D \neq 0 \). Indeed, if \( n \) is the direction of motion, then the corresponding field is different from zero inside the region defined by \( |x_S \cdot n - vt| \leq v |x_D \cdot n|/2 \). Therefore, if we have two such solutions localized at two distant physical points, their fields will anyway overlap in some extra-space regions. This does not mean that the two fields interact, since in the free field case the motion equations are linear, but in any case this is again a form of non-locality in extra-space.

- Finally, equation (67) clearly shows that the electromagnetic field \( A^\mu(z) \) acts instantly on the field \( \varphi(x, y) \) at all those points having \( x = z \) or \( y = z \). Therefore an interaction can propagate between two physical points \( A = (z_A, z_A) \) and \( B = (z_B, z_B) \) through the following mechanism: the field \( \varphi(z_A, z_A) \) acts instantly on the electromagnetic field \( A^\mu(z_A) \) through the current (79); the field \( A^\mu(z_A) \) acts, again instantly, on a extra-space point \( C = (z_A, z_B) \); finally, the interaction propagates from \( C \) in the \( x \) direction until it reaches the point \( B = (z_B, z_B) \), this time moving at the speed of light. The time
elapsed is exactly the time needed for a light signal to travel from $A$ to $B$, but the interaction has been propagated entirely through extra-space: at intermediate times, the effect of the interaction is not observable at any physical point between $A$ and $B$. This seems to be a non-local behaviour with physical effects, and it would be interesting to investigate if this kind of non-locality is in some way related to the non-local features emerging from quantum mechanics.
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