On cluster $C^*$-algebras

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Abstract

We introduce a $C^*$-algebra $\mathbb{A}(x, Q)$ attached to the cluster $x$ and a quiver $Q$. If $Q_T$ is the quiver coming from a triangulation $T$ of the Riemann surface $S$ with a finite number of cusps, we prove that the primitive spectrum of $\mathbb{A}(x, Q_T)$ times $\mathbb{R}$ is homeomorphic to a generic subset of the Teichmüller space of surface $S$. We conclude with an analog of the Tomita-Takesaki theory and the Connes invariant $T(\mathcal{M})$ for the algebra $\mathbb{A}(x, Q_T)$.

Key words and phrases: cluster algebras, Riemann surfaces, $C^*$-algebras

MSC: 13F60 (cluster algebras); 14H55 (Riemann surfaces); 46L85 (noncommutative topology)

1 Introduction

Cluster algebras of rank $m$ are a class of commutative rings introduced by [Fomin & Zelevinsky 2002] [10]. Among these algebras one finds coordinate rings of important algebraic varieties, like the Grassmannians and Schubert varieties; cluster algebras appear in the Teichmüller theory [Fomin, Shapiro & Thurston 2008] [9]. Unlike the coordinate rings, the set of generators $x_i$ of cluster algebra is usually infinite and defined by induction from a cluster $x = (x_1, \ldots, x_m)$ and a quiver $Q$, see [Williams 2014] [22] for an excellent survey; the cluster algebra is denoted by $\mathbb{A}(x, Q)$. Notice that the $\mathbb{A}(x, Q)$ has an additive structure of countable (unperforated) abelian group with an order satisfying the Riesz interpolation property; see Remark 8. In other words, the cluster algebra $\mathbb{A}(x, Q)$ is a dimension group by the Effros-Handelman-Shen Theorem [Effros 1981, Theorem 3.1] [7].
The subject of our note is an operator algebra $\mathcal{A}(x, Q)$, such that $K_0(\mathcal{A}(x, Q)) \cong \mathcal{A}(x, Q)$; here $K_0(\mathcal{A}(x, Q))$ is the dimension group of $\mathcal{A}(x, Q)$ and $\cong$ is an isomorphism of the ordered abelian groups [Blackadar 1986, Chapter 7] [2]. The $\mathcal{A}(x, Q)$ is an Approximately Finite $C^*$-algebra (AF-algebra) given by a Bratteli diagram derived explicitly from the pair $(x, Q)$. The AF-algebras were introduced and studied by [Bratteli 1972] [4]; we refer to $\mathcal{A}(x, Q)$ as a cluster $C^*$-algebra.

An exact definition of $\mathcal{A}(x, Q)$ can be found in Section 2.4; to give an idea, recall that the pair $(x, Q)$ is called a seed and the cluster algebra $\mathcal{A}(x, Q)$ is generated by seeds obtained via mutation of $(x, Q)$ (and its mutants) in all directions $k$, where $1 \leq k \leq m$ [Williams 2014, p.5] [22]. The mutation process can be described by an oriented regular tree $\mathcal{T}_m$; the vertices of $\mathcal{T}_m$ correspond to the seeds and the outgoing edges to the mutations in directions $k$. The quotient $\mathcal{B}(x, Q)$ of $\mathcal{T}_m$ by a relation identifying equivalent seeds at the same level of $\mathcal{T}_m$ is a graph with cycles. (For a quick example of such a graph, see Figure 3.) The cluster $C^*$-algebra $\mathcal{A}(x, Q)$ is an AF-algebra given by the $\mathcal{B}(x, Q)$ regarded as a Bratteli diagram [Bratteli 1972] [4].

Let $S_{g,n}$ be a Riemann surface of genus $g \geq 0$ with $n \geq 1$ cusps and such that $2g - 2 + n > 0$; denote by $T_{g,n} \cong \mathbb{R}^{6g - 6 + 2n}$ the (decorated) Teichmüller space of $S_{g,n}$, i.e. a collection of all Riemann surfaces of genus $g$ with $n$ cusps endowed with the natural topology [Penner 1987] [19]. In what follows, we focus on the algebras $\mathcal{A}(x, Q_{g,n})$ with quivers $Q_{g,n}$ coming from an ideal triangulation of $S_{g,n}$; the corresponding cluster algebra $\mathcal{A}(x, Q_{g,n})$ of rank $m = 6g - 6 + 3n$ is related to the Penner coordinates in $T_{g,n}$ [Fomin, Shapiro & Thurston 2008] [9].

![Figure 1: The Markov quiver $Q_{1,1}$.](image)

**Example 1** Let $S_{1,1}$ be a once-punctured torus. The ideal triangulation of
$S_{1,1}$ defines the Markov quiver $Q_{1,1}$ shown in Figure 1, see [Fomin, Shapiro & Thurston 2008, Example 4.6] [9]. The corresponding cluster $C^*$-algebra $A(x, Q_{1,1})$ of rank 3 can be written as:

$$A(x, Q_{1,1}) \cong \mathfrak{m}/I_0,$$

where $I_0$ is a primitive ideal of an $AF$-algebra $\mathfrak{m}$. The unital $AF$-algebra $\mathfrak{m}$ was originally defined by [Mundici 1988, Section 3] [11]; the genuine notation for such an algebra was $\mathfrak{m}_1$, because $K_0(\mathfrak{m}_1) = (M_1, 1) :=$ free one-generator unital $\ell$-group, i.e. a finitely piecewise affine linear continuous real-valued functions on $[0, 1]$ with integer coefficients. The $\mathfrak{m}_1$ was subsequently rediscovered after two decades by [Boca 2008] [3] and denoted by $\mathfrak{m}$. The remarkable properties of $\mathfrak{m}_1$ include the following features. Every primitive ideal of $\mathfrak{m}_1$ is essential [Mundici 2011, Theorem 4.2] [13]. The $\mathfrak{m}_1$ is equipped with a faithful invariant tracial state [Mundici 2009, Theorem 3.1] [12]. The center of $\mathfrak{m}_1$ coincides with the $C^*$-algebra $C[0, 1]$ of continuous complex valued functions on $[0, 1]$ [Boca 2008, p. 976] [3]. There is an affine weak $*$-homeomorphism of the state space of $C[0, 1]$ onto the space of tracial states on $\mathfrak{m}_1$ [Mundici 2011, Theorem 4.5] [13]. Any state of $C[0, 1]$ has precisely one tracial extension to $\mathfrak{m}_1$ [Eckhardt 2011, Theorem 2.5] [6]. The automorphism group of $\mathfrak{m}_1$ has precisely two connected components [Mundici 2011, Theorem 4.3] [13]. The Gauss map – a Bernoulli shift for continued fractions – is generalized in [Eckhardt 2011] [6] to the noncommutative framework of $\mathfrak{m}_1$. In the light of the original definition of $\mathfrak{m}_1$ and the fact that the $K_0$-functor preserves exact sequences (see, e.g. [Effros 1981, Theorem 3.1] [7]), the primitive spectrum of $\mathfrak{m}_1$ and its hull-kernel topology is widely known to the lattice-ordered group theorists and the MV-algebraists long ago before the laborious analysis in [Boca 2008] [3], where $\mathfrak{m}_1$ is defined in terms of the Bratteli diagram. We refer the reader to the final part of a paper by [Panti 1999] [18] for a general result encompassing the characterization of the prime spectrum of $(M_1, 1) \cong Prim \mathfrak{m}_1$. Moreover, the $AF$-algebras $A_\theta$ introduced by [Effros & Shen 1980] [8] are precisely the infinite-dimensional simple quotients of $\mathfrak{m}_1$; this fact was first proved by [Mundici 1988, Theorem 3.1(i)] [11] and rediscovered independently by [Boca 2008] [3]. Summing up the above, the primitive ideals $I_\theta \subset \mathfrak{m}$ are indexed by numbers $\theta \in \mathbb{R}$; if $\theta$ is irrational, the quotient $\mathfrak{m}/I_\theta \cong A_\theta$, where $A_\theta$ is the Effros-Shen algebra. In view of

\footnote{Such a quiver is related to solutions in the integer numbers of the equation $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$ considered by A. A. Markov; hence the name.}
the algebra \( \mathfrak{M} \) is a non-commutative coordinate ring of the Teichmüller space \( T_{1,1} \). Moreover, there exists an analog of the Tomita-Takesaki theory of modular automorphisms \( \{ \sigma_t \mid t \in \mathbb{R} \} \) for algebra \( \mathfrak{M} \), see Section 4; such automorphisms correspond to the Teichmüller geodesic flow on \( T_{1,1} \) [Veech 1986] [21]. The \( \sigma_t(I_\theta) \) is an ideal of \( \mathfrak{M} \) for all \( t \in \mathbb{R} \), where \( \sigma_0(I_\theta) = I_\theta \). The quotient algebra \( \mathfrak{M}/\sigma_t(I_\theta) \) can be viewed as a non-commutative coordinate ring of the Riemann surface \( S_{1,1} \); in particular, the pairs \( (\theta, t) \) are coordinates in the space \( T_{1,1} \cong \mathbb{R}^2 \). We refer the reader to [14] for a construction of the corresponding functor.

Motivated by Example 1, denote by \( \mathbb{A}(x, Q_{g,n}) \) the cluster C*-algebra corresponding to a quiver \( Q_{g,n} \); let \( \sigma_t : \mathbb{A}(x, Q_{g,n}) \to \mathbb{A}(x, Q_{g,n}) \) be the Tomita-Takesaki flow on \( \mathbb{A}(x, Q_{g,n}) \), see Section 4 for the details. Denote by \( \text{Prim } \mathbb{A}(x, Q_{g,n}) \) the set of all primitive ideals of \( \mathbb{A}(x, Q_{g,n}) \) endowed with the Jacobson topology and let \( I_\theta \in \text{Prim } \mathbb{A}(x, Q_{g,n}) \) for a generic value of index \( \theta \in \mathbb{R}^{6g-7+2n} \). Our main result can be stated as follows.

**Theorem 1** There exists a homeomorphism

\[
h : \text{Prim } \mathbb{A}(x, Q_{g,n}) \times \mathbb{R} \to \{ U \subseteq T_{g,n} \mid U \text{ is generic} \}
\]

given by the formula \( \sigma_t(I_\theta) \mapsto S_{g,n} \); the set \( U = T_{g,n} \) if and only if \( g = n = 1 \). The \( \sigma_t(I_\theta) \) is an ideal of \( \mathbb{A}(x, Q_{g,n}) \) for all \( t \in \mathbb{R} \) and the quotient algebra \( \mathbb{A}(x, Q_{g,n})/\sigma_t(I_\theta) \) is a non-commutative coordinate ring of the Riemann surface \( S_{g,n} \).

**Remark 1** Theorem 1 is valid for \( n \geq 1 \), i.e. the Riemann surfaces with at least one cusp. This cannot be improved, since the cluster structure of algebra \( \mathbb{A}(x, Q_{g,n}) \) comes from the Ptolemy relations satisfied by the Penner coordinates; so far such coordinates are available only for the Riemann surfaces with cusps [Penner 1987] [19]. It is likely, that the case \( n = 0 \) has also a cluster structure; we refer the reader to [15], where a functor from the Riemann surfaces \( S_{g,0} \) to the AF-algebras \( \mathbb{A}(x, Q_{g,0})/\sigma_t(I_\theta) \) was constructed.

**Remark 2** The braid group \( B_{2g+n} \) with \( n \in \{1, 2\} \) admits a faithful representation by projections in the algebra \( \mathbb{A}(x, Q_{g,n}) \); such a construction is based on the Birman-Hilden Theorem for the braid groups. This observation and the well-known Laurent phenomenon in the cluster algebra \( K_0(\mathbb{A}(x, Q_{g,n})) \) allow to generalize the Jones and HOMFLY invariants of knots and links to an arbitrary number of variables, see [17] for the details.
The article is organized as follows. We introduce preliminary facts and notation in Section 2. Theorem 1 is proved in Section 3. An analog of the Tomita-Takesaki theory of modular automorphisms and the Connes invariant $T(A(x, Q_{g,n}))$ of the cluster $C^*$-algebra $A(x, Q_{g,n})$ is constructed.

2 Notation

In this section we introduce notation and briefly review some preliminary facts. The reader is encouraged to consult [Bratteli 1972] [4], [Fomin, Shapiro & Thurston 2008], [Fomin & Zelevinsky 2002] [10], [Penner 1987] [19] and [Williams 2014] [22] for the details.

2.1 Cluster algebras

A cluster algebra $A$ of rank $m$ is a subring of the field $Q(x_1, \ldots, x_m)$ of rational functions in $n$ variables. Such an algebra is defined by a pair $(x, B)$, where $x = (x_1, \ldots, x_m)$ is a cluster of variables and $B = (b_{ij})$ is a skew-symmetric integer matrix; the new cluster $x'$ is obtained from $x$ by an excision of the variable $x_k$ and replacing it by a new variable $x'_k$ subject to an exchange relation:

$$x_k x'_k = \prod_{i=1}^{m} x_i^{\max(b_{ik}, 0)} + \prod_{i=1}^{m} x_i^{\max(-b_{ik}, 0)}.$$  (3)

Since the entries of matrix $B$ are exponents of the monomials in cluster variables, one gets a new pair $(x', B')$, where $B' = (b'_{ij})$ is a skew-symmetric with:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + |b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases}$$  (4)

For brevity, the pair $(x, B)$ is called a seed and the seed $(x', B') := (x', \mu_k(B))$ is obtained from $(x, B)$ by a mutation $\mu_k$ in the direction $k$, where $1 \leq k \leq m$; the $\mu_k$ is an involution, i.e. $\mu_k^2 = Id$. The matrix $B$ is called mutation finite if only finitely many new matrices can be produced from $B$ by repeated matrix mutations. The cluster algebra $A(x, B)$ can be defined as the subring of $Q(x_1, \ldots, x_m)$ generated by the union of all cluster variables obtained from the initial seed $(x, B)$ by mutations of $(x, B)$ (and its iterations) in all possible directions. We shall write $T_m$ to denote an oriented tree whose vertices are seeds $(x', B')$ and $m$ outgoing arrows in each vertex correspond to mutations $\mu_k$ of the seed $(x', B')$. The Laurent phenomenon proved by [Fomin &
Zelevinsky 2002 \[10\] says that \( \mathcal{A}(x, B) \subset \mathbb{Z}[x^\pm 1] \), where \( \mathbb{Z}[x^\pm 1] \) is the ring of the Laurent polynomials in variables \( x = (x_1, \ldots, x_n) \); in other words, each generator \( x_i \) of algebra \( \mathcal{A}(x, B) \) can be written as a Laurent polynomial in \( n \) variables with the integer coefficients.

**Remark 3** The Laurent phenomenon turns the additive structure of cluster algebra \( \mathcal{A}(x, B) \) into a totally ordered abelian group satisfying the Riesz interpolation property, i.e. a dimension group [Effros 1981, Theorem 3.1] \[7\]; the abelian group with order comes from the semigroup of the Laurent polynomials with positive coefficients, see \[16\] for the details. A background on the partially and totally ordered, unperforated abelian groups with the Riesz interpolation property can be found in [Effros 1981] \[7\].

To deal with mutation formulas (3) and (4) in geometric terms, recall that a **quiver** \( Q \) is an oriented graph given by the set of vertices \( Q_0 \) and the set of arrows \( Q_1 \); an example of quiver is given in Figure 1. Let \( k \) be a vertex of \( Q \); the mutated at vertex \( k \) quiver \( \mu_k(Q) \) has the same set of vertices as \( Q \) but the set of arrows is obtained by the following procedure: (i) for each sub-quiver \( i \rightarrow k \rightarrow j \) one adds a new arrow \( i \rightarrow j \); (ii) one reverses all arrows with source or target \( k \); (iii) one removes the arrows in a maximal set of pairwise disjoint 2-cycles. The reader can verify, that if one encodes a quiver \( Q \) with \( n \) vertices by a skew-symmetric matrix \( B(Q) = (b_{ij}) \) with \( b_{ij} \) equal to to the number of arrows from vertex \( i \) to vertex \( j \), then mutation \( \mu_k \) of seed \( (x, B) \) coincides with such of the corresponding quiver \( Q \). Thus the cluster algebra \( \mathcal{A}(x, B) \) is defined by a quiver \( Q \); we shall denote such an algebra by \( \mathcal{A}(x, Q) \).

### 2.2 Cluster algebras from Riemann surfaces

Let \( g \) and \( n \) be integers, such that \( g \geq 0, \ n \geq 1 \) and \( 2g - 2 + n > 0 \). Denote by \( S_{g,n} \) a Riemann surface of genus \( g \) with the \( n \) cusp points. It is known that the fundamental domain of \( S_{g,n} \) can be triangulated by \( 6g - 6 + 3n \) geodesic arcs \( \gamma \), such that the footpoints of each arc at the absolute of Lobachevsky plane \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \) coincide with a (pre-image of) cusp of the \( S_{g,n} \). If \( l(\gamma) \) is the hyperbolic length of \( \gamma \) measured (with a sign) between two horocycles around the footpoints of \( \gamma \), then we set \( \lambda(\gamma) = e^{\frac{2}{3}l(\gamma)} \); the \( \lambda(\gamma) \) are known to satisfy the **Ptolemy relation**:

\[
\lambda(\gamma_1)\lambda(\gamma_2) + \lambda(\gamma_3)\lambda(\gamma_4) = \lambda(\gamma_5)\lambda(\gamma_6),
\]

(5)
where $\gamma_1, \ldots, \gamma_4$ are pairwise opposite sides and $\gamma_5, \gamma_6$ are the diagonals of a geodesic quadrilateral in $\mathbb{H}$.

Denote by $T_{g,n}$ the decorated Teichmüller space of $S_{g,n}$, i.e. the set of all complex surfaces of genus $g$ with $n$ cusps endowed with the natural topology; it is known that $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$.

**Theorem 2** ([Penner 1987] [19]) The map $\lambda$ on the set of $6g - 6 + 3n$ geodesic arcs $\gamma_i$ defining a triangulation of $S_{g,n}$ is a homeomorphism with the image $T_{g,n}$.

**Remark 4** Notice that among $6g - 6 + 3n$ real numbers $\lambda(\gamma_i)$ there are only $6g - 6 + 2n$ independent, since such numbers must satisfy $n$ Ptolemy relations (5).

Let $T$ be a triangulation of surface $S_{g,n}$ by $6g - 6 + 3n$ geodesic arcs $\gamma_i$; consider a skew-symmetric matrix $B_T = (b_{ij})$, where $b_{ij}$ is equal to the number of triangles in $T$ with sides $\gamma_i$ and $\gamma_j$ in clockwise order minus the number of triangles in $T$ with sides $\gamma_i$ and $\gamma_j$ in the counter-clockwise order. It is known that matrix $B_T$ is always mutation finite. The cluster algebra $A(x, B_T)$ of rank $6g - 6 + 3n$ is called associated to triangulation $T$.

**Example 2** Let $S_{1,1}$ be a once-punctured torus of Example 1. The triangulation $T$ of the fundamental domain $\mathbb{R}^2/\mathbb{Z}^2$ of $S_{1,1}$ is sketched in Figure 2 in the charts $\mathbb{R}^2$ and $\mathbb{H}$, respectively. It is easy to see that in this case $x = (x_1, x_2, x_3)$ with $x_1 = \gamma_{23}, x_2 = \gamma_{34}$ and $x_3 = \gamma_{24}$, where $\gamma_{ij}$ denotes a geodesic arc with the footpoints $i$ and $j$. The Ptolemy relation (5) reduces to $\lambda^2(\gamma_{23}) + \lambda^2(\gamma_{34}) = \lambda^2(\gamma_{24})$; thus $T_{1,1} \cong \mathbb{R}^2$. The reader is encouraged to verify, that matrix $B_T$ has the form:

$$B_T = \begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$ (6)

**Theorem 3** ([Fomin, Shapiro & Thurston 2008] [9]) The cluster algebra $A(x, B_T)$ does not depend on triangulation $T$, but only on the surface $S_{g,n}$; namely, replacement of the geodesic arc $\gamma_k$ by a new geodesic arc $\gamma'_k$ (a flip of $\gamma_k$) corresponds to a mutation $\mu_k$ of the seed $(x, B_T)$.

**Remark 5** In view of Theorems 2 and 3 the $A(x, B_T)$ corresponds to an algebra of functions on the Teichmüller space $T_{g,n}$; such an algebra is an analog of the coordinate ring of $T_{g,n}$. 

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**Note:** The text is a continuation of the previous page, but the page number is not visible in the image provided. The content is a continuation of the mathematical exposition on Teichmüller spaces, cluster algebras, and their applications. The equations and relations are presented in a manner consistent with the previous context, maintaining the integrity of the mathematical discussion. This section likely continues the exploration of specific theorems, examples, and remarks related to the study of geometric and algebraic structures in the context of Teichmüller spaces. The reader is encouraged to follow the logical progression of ideas and proofs, which are typically presented in a coherent and methodical manner in mathematical literature.
2.3 \( C^* \)-algebras

A \( C^* \)-algebra is an algebra \( A \) over \( \mathbb{C} \) with a norm \( a \mapsto ||a|| \) and an involution \( a \mapsto a^* \) such that it is complete with respect to the norm and \( ||ab|| \leq ||a|| \cdot ||b|| \) and \( ||a^*a|| = ||a^2|| \) for all \( a, b \in A \). Any commutative \( C^* \)-algebra is isomorphic to the algebra \( C_0(X) \) of continuous complex-valued functions on some locally compact Hausdorff space \( X \); otherwise, \( A \) represents a noncommutative topological space.

An \( AF \)-algebra (Approximately Finite \( C^* \)-algebra) is defined to be the norm closure of an ascending sequence of finite dimensional \( C^* \)-algebras \( M_n \), where \( M_n \) is the \( C^* \)-algebra of the \( n \times n \) matrices with entries in \( \mathbb{C} \). Here the index \( n = (n_1, \ldots, n_k) \) represents the semi-simple matrix algebra \( M_n = M_{n_1} \oplus \cdots \oplus M_{n_k} \). The ascending sequence mentioned above can be written as

\[
M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \ldots,
\]

where \( M_i \) are the finite dimensional \( C^* \)-algebras and \( \varphi_i \) the homomorphisms between such algebras. The homomorphisms \( \varphi_i \) can be arranged into a graph as follows. Let \( M_i = M_{i_1} \oplus \cdots \oplus M_{i_k} \) and \( M_{i'} = M'_{i_1} \oplus \cdots \oplus M'_{i_k} \) be the semi-simple \( C^* \)-algebras and \( \varphi_i : M_i \to M_{i'} \) the homomorphism. One has two sets of vertices \( V_{i_1}, \ldots, V_{i_k} \) and \( V'_{i_1}, \ldots, V'_{i_k} \) joined by \( b_{rs} \) edges whenever the summand \( M_{i_r} \) contains \( b_{rs} \) copies of the summand \( M'_{i_s} \) under the embedding \( \varphi_i \). As \( i \) varies, one obtains an infinite graph called the Bratteli diagram of the \( AF \)-algebra. The matrix \( B = (b_{rs}) \) is known as a partial multiplicity matrix; an infinite sequence of \( B_i \) defines a unique \( AF \)-algebra.

Let \( \theta \in \mathbb{R}^{n-1} \); recall that by the Jacobi-Perron continued fraction of vector...
one understands the limit:

\[
\begin{pmatrix}
1 \\
\theta_1 \\
\vdots \\
\theta_{n-1}
\end{pmatrix} = \lim_{k \to \infty} \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_{1}^{(1)} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(1)}
\end{pmatrix} \cdots \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & b_{1}^{(k)} \\
\vdots & \vdots & & \vdots & \vdots \\
0 & 0 & \ldots & 1 & b_{n-1}^{(k)}
\end{pmatrix} \begin{pmatrix} 0 \\
0 \\
0 \\
\end{pmatrix},
\]

where \(b_{j}^{(j)} \in \mathbb{N} \cup \{0\}\), see e.g. [Bernstein 1971] \[1\]; the limit converges for a generic subset of vectors \(\theta \in \mathbb{R}^{n-1}\). Notice that \(n = 2\) corresponds to (a matrix form of) the regular continued fraction of \(\theta\); such a fraction is always convergent. Moreover, the Jacobi-Perron fraction is finite if and only if vector \(\theta = (\theta_{i})\), where \(\theta_{i}\) are rational. The \(AF\)-algebra \(A_{\theta}\) associated to the vector \((1, \theta)\) is defined by the Bratteli diagram with the partial multiplicity matrices equal to \(B_{k}\) in the Jacobi-Perron fraction of \((1, \theta)\); in particular, if \(n = 2\) the \(A_{\theta}\) coincides with the Effros-Shen algebra [Effros & Shen 1980] \[8\].

### 2.4 Cluster \(C^*\)-algebras

Notice that the mutation tree \(\overrightarrow{T}_{m}\) of a cluster algebra \(A(\mathbf{x}, B)\) has a grading by levels, i.e. a distance from the root of \(\overrightarrow{T}_{m}\). We shall say that a pair of clusters \(\mathbf{x}\) and \(\mathbf{x}'\) are \(\ell\)-equivalent, if:

(i) \(\mathbf{x}\) and \(\mathbf{x}'\) lie at the same level;

(ii) \(\mathbf{x}\) and \(\mathbf{x}'\) coincide modulo a cyclic permutation of variables \(x_{i}\);

(iii) \(B = B'\).

It is not hard to see that \(\ell\) is an equivalence relation on the set of vertices of graph \(\overrightarrow{T}_{m}\).

**Definition 1** By a cluster \(C^*\)-algebra \(\mathfrak{A}(\mathbf{x}, B)\) one understands an \(AF\)-algebra given by the Bratteli diagram \(\mathfrak{B}(\mathbf{x}, B)\) of the form:

\[
\mathfrak{B}(\mathbf{x}, B) := \overrightarrow{T}_{m} \mod \ell.
\]

The rank of \(\mathfrak{A}(\mathbf{x}, B)\) is equal to such of cluster algebra \(A(\mathbf{x}, B)\).

**Example 3** If \(B_T\) is matrix \[6\] of Example \[2\] then \(\mathfrak{B}(\mathbf{x}, B_T)\) is shown Figure 3. (We refer the reader to Section 4 for a proof.) Notice that the graph \(\mathfrak{B}(\mathbf{x}, B_T)\) is a part of of the Bratteli diagram of the Mundici algebra \(\mathfrak{M}\), compare [Mundici 2011, Figure 1] \[13\].

**Remark 6** It is not hard to see that \(\mathfrak{B}(\mathbf{x}, B)\) is no longer a tree and \(\mathfrak{B}(\mathbf{x}, B)\) is a finite graph if and only if \(A(\mathbf{x}, B)\) is a finite cluster algebra.
3 Proof

Let \( m = 3(2g - 2 + n) \) be the rank of cluster \( C^* \)-algebra \( \mathcal{A}(x, Q_{g,n}) \). For the sake of clarity, we shall consider the case \( m = 3 \) and the general case \( m \in \{3, 6, 9, \ldots\} \) separately.

(i) Let \( \mathcal{A}(x, B_T) \) be the cluster \( C^* \)-algebra of rank 3. In this case \( 2g - 2 + n = 1 \) and either \( g = 0 \) and \( n = 3 \) or else \( g = n = 1 \). Since \( T_{0,3} \cong \{pt\} \) is trivial, we are left with \( g = n = 1 \), i.e. the once-punctured torus \( S_{1,1} \).

Repeating the argument of Example 2, we get the seed \( (x, B_T) \), where \( x = (x_1, x_2, x_3) \) and the skew-symmetric matrix \( B_T \) is given by formula (6).

Let us verify that matrix \( B_T \) is mutation finite; indeed, for each \( k \in \{1, 2, 3\} \) the matrix mutation formula (4) gives us \( \mu_k(B_T) = -B_T \).

Therefore, the exchange relations (3) do not vary; it is verified directly that such relations have the form:

\[
\begin{align*}
    x_1x_1' &= x_2^2 + x_3^2, \\
    x_2x_2' &= x_1^2 + x_3^2, \\
    x_3x_3' &= x_1^2 + x_2^2.
\end{align*}
\]  

(9)

Consider a mutation tree \( \overrightarrow{T}_3 \) shown in Figure 4; the vertices of \( \overrightarrow{T}_3 \) correspond to the mutations of cluster \( x = (x_1, x_2, x_3) \) following the exchange rules (9).

The reader is encouraged to verify that modulo a cyclic permutation of variables \( x_1' = x_2, x_2' = x_3, x_3' = x_1 \) and \( x_1' = x_3, x_2' = x_1, x_3' = x_2 \) one obtains (respectively) the following equivalences of clusters:

\[
\begin{align*}
    \mu_{13}(x) &= \mu_{21}(x), \\
    \mu_{23}(x) &= \mu_{31}(x).
\end{align*}
\]  

(10)
where $\mu_{ij}(x) := \mu_j(\mu_i(x))$; there are no other cluster equivalences for the vertices of the same level of graph $\overrightarrow{T}_3$.

To determine the graph $\mathfrak{B}(x, B_T)$ one needs to take the quotient of $\overrightarrow{T}_3$ by the $\ell$-equivalence relations (10); since the pattern repeats for each level of $\overrightarrow{T}_3$, one gets the $\mathfrak{B}(x, B_T)$ shown in Figure 3. The cluster $C^*$-algebra $A(x, B_T)$ is an $AF$-algebra with the Bratteli diagram $\mathfrak{B}(x, B_T)$.

Notice that the Bratteli diagram $\mathfrak{B}(x, B_T)$ of our $AF$-algebra $A(x, B_T)$ and such of the Mundici algebra $\mathfrak{m}$ are distinct, compare [Mundici 2011, Figure 1] [13]; yet there is an obvious inclusion of one diagram into another. Namely, if one erases a “camel’s back” (i.e. the two extreme sides of the diagram) in the Bratteli diagram of $\mathfrak{m}$, then one gets exactly the diagram in Figure 3. Formally, if $G$ is the Bratteli diagram of the Mundici algebra $\mathfrak{m}$, the complement $G - \mathfrak{B}(x, B_T)$ is a hereditary Bratteli diagram which gives rise to an ideal $I_0 \subset \mathfrak{m}$, such that:

$$A(x, B_T) \cong \mathfrak{m}/I_0,$$

see [Bratteli 1972, Lemma 3.2] [4]; the $I_0$ is a primitive ideal *ibid.*, Theorem
3.8. (It is interesting to calculate the group $K_0(I_0)$ in the context of the work of [Panti 1999] [18].)

On the other hand, the space $Prim \mathfrak{m}$ (and hence $Prim \mathcal{A}(x, B_T)$) is well understood, see e.g. [Panti 1999] [18] or [Boca 2008, Proposition 7] [3]. Namely,

$$Prim (\mathfrak{m}/I_0) = \{I_\theta \mid \theta \in \mathbb{R}\},$$

where $I_\theta \subset \mathfrak{m}$ is such that $\mathfrak{m}/I_\theta \cong A_\theta$ is the Effros-Shen algebra [Effros & Shen 1980] [8] if $\theta$ is an irrational number or $\mathfrak{m}/I_\theta \cong M_q$ is finite-dimensional matrix $C^*$-algebra (and an extension of such by the $C^*$-algebra of compact operators) if $\theta = \frac{p}{q}$ is a rational number. (Note that the third series of primitive ideals of [Boca 2008, Proposition 7] [3] correspond to the ideal $I_0$.) Moreover, given the Jacobson topology on $Prim \mathfrak{m}$, there exists a homeomorphism

$$h : Prim (\mathfrak{m}/I_0) \rightarrow \mathbb{R}$$

defined by the formula $I_\theta \mapsto \theta$, see [Boca 2008, Corollary 12] [3].

Let $\sigma_t : \mathfrak{m}/I_0 \rightarrow \mathfrak{m}/I_0$ be the Tomita-Takesaki flow, i.e. a one-parameter automorphism group of $\mathfrak{m}/I_0$, see Section 4. Because $I_\theta \subset \mathfrak{m}/I_0$, the image $\sigma^t(I_\theta)$ of $I_\theta$ is correctly defined for all $t \in \mathbb{R}$; the $\sigma_t(I_\theta)$ is an ideal of $\mathfrak{m}/I_0$ but not necessarily primitive. Since $\sigma_t$ is nothing but (an algebraic form of) the Teichmüller geodesic flow on $T_{1,1}$ [Veech 1986] [21], one concludes that that the family of ideals

$$\{\sigma_t(I_\theta) \subset \mathfrak{m}/I_0 \mid t \in \mathbb{R}, \theta \in \mathbb{R}\}$$

(14)
can be taken for a coordinate system in the space $T_{1,1} \cong \mathbb{R}^2$. In view of (13) and $\mathfrak{m}/I_0 \cong \mathcal{A}(x, Q_{1,1})$, one gets the required homeomorphism

$$h : Prim \mathcal{A}(x, Q_{1,1}) \times \mathbb{R} \rightarrow T_{1,1},$$

(15)
such that the quotient algebra $\mathcal{A}(x, Q_{1,1})/\sigma_t(I_\theta)$ is a non-commutative coordinate ring of the Riemann surface $S_{1,1}$.

**Remark 7** The family of algebras $\{\mathcal{A}(x, Q_{1,1})/\sigma_t(I_\theta) \mid \theta = Const, t \in \mathbb{R}\}$ are in general pairwise non-isomorphic. (For otherwise all ideals $\{\sigma_t(I_\theta) \mid t \in \mathbb{R}\}$ were primitive.) Yet their Grothendieck semi-groups $K_0^+$ are, see [Effros & Shen 1980] [8]; the action of $\sigma_t$ is given by the formula (see Section 4):

$$K_0^+(\mathcal{A}(x, Q_{1,1})/\sigma_t(I_\theta)) \cong e^t(\mathbb{Z} + \mathbb{Z} \theta).$$

(16)
(ii) The general case \( m = 3k = 3(2g - 2 + n) \) is treated likewise. Notice that if \( d = 6g - 6 + 2n \) is dimension of the space \( T_{g,n} \), then we have \( m - d = n \); in particular, rank \( m \) of the cluster \( C^* \)-algebra \( \mathcal{A}(x, Q_{g,n}) \) determines completely the pair \((g, n)\) provided \( d \) is a fixed constant. (If \( d \) is not fixed, there is only a finite number of different pairs \((g, n)\) for given rank \( m \).)

Let \((x, B_T)\) be the seed given by the cluster \( x = (x_1, \ldots, x_{3k}) \) and the skew-symmetric matrix \( B_T \). Since matrix \( B_T \) comes from a triangulation of the Riemann surface \( S_{g,n} \), \( B_T \) is mutation finite, see [Williams 2014, p.18] [22]; the exchange relations (3) take the form:

\[
\begin{align*}
    x_1 x'_1 &= x_2^2 + x_3^2 + \ldots + x_{3k}^2, \\
    x_2 x'_2 &= x_1^2 + x_3^2 + \ldots + x_{3k}^2, \\
    \vdots \\
    x_{3k} x'_{3k} &= x_1^2 + x_2^2 + \ldots + x_{3k-1}^2.
\end{align*}
\]

(17)

One can construct the mutation tree \( \mathcal{T}_{3k} \) using relations (17); the reader is encouraged to verify, that the \( \mathcal{T}_{3k} \) is similar to the one shown in Figure 4, except for the number of the outgoing edges at each vertex is equal to \( 3k \).

A tedious but straightforward calculation shows that the only equivalent clusters at the same level of \( \mathcal{T}_{3k} \) are the ones at the extremities of tuples \((x'_1, \ldots, x'_{3k})\); in other words, one gets the following system of equivalences of clusters:

\[
\begin{align*}
    \mu_{1,3k}(x) &= \mu_{21}(x), \\
    \mu_{2,3k}(x) &= \mu_{31}(x), \\
    \vdots \\
    \mu_{3k-1,3k}(x) &= \mu_{3k,1}(x),
\end{align*}
\]

(18)

where \( \mu_{ij}(x) := \mu_j(\mu_i(x)) \).

The graph \( \mathfrak{B}(x, B_T) \) is the quotient of \( \mathcal{T}_{3k} \) by the \( \ell \)-equivalence relations (18); for \( k = 2 \) such a graph is sketched in Figure 5. The \( \mathcal{A}(x, Q_{g,n}) \) is an \( AF \)-algebra given by the Bratteli diagram \( \mathfrak{B}(x, B_T) \).

**Lemma 1** The set

\[
\text{Prim } \mathcal{A}(x, Q_{g,n}) = \{ I_\theta \mid \theta \in \mathbb{R}^{6g-7+2n} \text{ is generic} \},
\]

(19)

where \( \mathcal{A}(x, Q_{g,n}) / I_\theta \) is an \( AF \)-algebra \( \mathcal{A}_\theta \) associated to the convergent Jacobi-Perron continued fraction of vector \((1, \theta)\), see Section 2.3.
Figure 5: The Bratteli diagram of a cluster $C^*$-algebra of rank 6.

Proof. We adapt the argument of [Boca 2008, case $k = 1$] to the case $k \geq 1$. Let $d = 6g - 6 + 2n$ be dimension of the space $T_{g,n}$. Roughly speaking, the Bratteli diagram $\mathfrak{B}(x, B_T)$ of algebra $A(x, Q_{g,n})$ can be cut in two disjoint pieces $G_\theta$ and $\mathfrak{B}(x, B_T) - G_\theta$, as it is shown by [Boca 2008, Figure 7]. The $G_\theta$ is a (finite or infinite) vertical strip of constant “width” $d$, where $d$ is equal to the number of vertices cut from each level of $\mathfrak{B}(x, B_T)$. The reader is encouraged to verify, that $G_\theta$ is exactly the Bratteli diagram of the AF-algebra $A_\theta$ associated to the convergent Jacobi-Perron continued fraction of a generic vector $(1, \theta)$, see Section 2.3.

On the other hand, the complement $\mathfrak{B}(x, B_T) - G_\theta$ is a hereditary Bratteli diagram, which defines an ideal $I_\theta$ of algebra $A(x, Q_{g,n})$, such that:

$$A(x, Q_{g,n})/I_\theta = A_\theta,$$

see [Bratteli 1972, Lemma 3.2]. Moreover, $I_\theta$ is a primitive ideal [Bratteli 1972, Theorem 3.8]. (An extra care is required if $\theta = (\theta_i)$ is a rational vector; the complete argument can be found in [Boca 2008, pp. 980-985].) Lemma 1 follows.

Lemma 2 The sequence of primitive ideals $I_{\theta_n}$ converges to $I_\theta$ in the Jacobson topology in $\text{Prim } A(x, Q_{g,n})$ if and only if the sequence $\theta_n$ converges to $\theta$ in the Euclidean space $\mathbb{R}^{6g-6+2n}$.

Proof. The proof is a straightforward adaption of the argument in [Boca 2008, pp. 986-988]; we leave it as an exercise to the reader.

Let $\sigma_t : A(x, Q_{g,n}) \to A(x, Q_{g,n})$ be the Tomita-Takesaki flow, i.e. the group \{\sigma_t | t \in \mathbb{R}\} of modular automorphisms of algebra $A(x, Q_{g,n})$, see Section 4. Because $I_\theta \subset A(x, Q_{g,n})$, the image $\sigma^t(I_\theta)$ of $I_\theta$ is correctly defined for all $t \in \mathbb{R}$; the $\sigma_t(I_\theta)$ is an ideal of $A(x, Q_{g,n})$ but not necessarily a primitive
ideal. Since $\sigma_t$ is an algebraic form of the Teichmüller geodesic flow on the space $T_{g,n}$ [Veech 1986] [21], one concludes that that the family of ideals:

$$\{\sigma_t(I_\theta) \subset \mathbb{A}(x, Q_{g,n}) \mid t \in \mathbb{R}, \theta \in \mathbb{R}^{6g-7+2n}\}$$

(21)
can be taken for a coordinate system in the space $T_{g,n} \cong \mathbb{R}^{6g-6+2n}$. In view of Lemmas [1] and [2] one gets the required homeomorphism

$$h : \text{Prim} \mathbb{A}(x, Q_{g,n}) \times \mathbb{R} \to \{U \subseteq T_{g,n} \mid U \text{ is generic}\},$$

(22)
such that the quotient algebra $A_\theta = \mathbb{A}(x, Q_{g,n})/\sigma_t(I_\theta)$ is a non-commutative coordinate ring of the Riemann surface $S_{g,n}$.

Theorem [1] is proved.

4 An analog of modular flow on $\mathbb{A}(x, Q_{g,n})$

A. Modular automorphisms $\{\sigma_t \mid t \in \mathbb{R}\}$. Recall that the Ptolemy relations (5) for the Penner coordinates $\{\lambda(\gamma_i)\}$ in the space $T_{g,n}$ are homogeneous; in particular, the system $\{t\lambda(\gamma_i) \mid t \in \mathbb{R}\}$ of such coordinates will also satisfy the Ptolemy relations. On the other hand, for the cluster $C^*$-algebra $\mathbb{A}(x, Q_{g,n})$ the variables $x_i = \lambda(\gamma_i)$ and one gets an obvious isomorphism $\mathbb{A}(x, Q_{g,n}) \cong \mathbb{A}(tx, Q_{g,n})$ for all $t \in \mathbb{R}$. Since $\mathbb{A}(tx, Q_{g,n}) \subseteq \mathbb{A}(x, Q_{g,n})$, one obtains a one-parameter group of automorphisms:

$$\sigma_t : \mathbb{A}(x, Q_{g,n}) \longrightarrow \mathbb{A}(x, Q_{g,n}).$$

(23)

By analogy with [Connes 1978] [5], we shall call $\sigma_t$ a \textit{Tomita-Takesaki flow} on the cluster $C^*$-algebra $\mathbb{A}(x, Q_{g,n})$. The reader is encouraged to verify, that $\sigma_t$ is an algebraic form of the \textit{geodesic flow} $T^t$ on the Teichmüller space $T_{g,n}$, see [Veech 1986] [21] for an introduction. Roughly speaking, such a flow comes from the one-parameter group of matrices

$$\begin{pmatrix}
e^t & 0 \\
0 & e^{-t}
\end{pmatrix}$$

(24)
acting on the space of holomorphic quadratic differentials on the Riemann surface $S_{g,n}$; the latter is known to be isomorphic to the Teichmüller space $T_{g,n}$. 

15
B. Connes invariant $T(\mathcal{A}(x, Q_{g,n}))$. Recall that an analogy of the Connes invariant $T(\mathcal{M})$ for a $C^*$-algebra $\mathcal{M}$ endowed with a modular automorphism group $\sigma_t$ is the set $T(\mathcal{M}) := \{ t \in \mathbb{R} \mid \sigma_t \text{ is inner} \}$ [Connes 1978] [5]. The group of inner automorphisms of the space $T_{g,n}$ and algebra $\mathcal{A}(x, Q_{g,n})$ is isomorphic to the mapping class group $\text{Mod } S_{g,n}$ of surface $S_{g,n}$. The automorphism $\phi \in \text{Mod } S_{g,n}$ is called pseudo-Anosov, if $\phi(\mathcal{F}_\mu) = \lambda_\phi \mathcal{F}_\mu$, where $\mathcal{F}_\mu$ is an invariant measured foliation and $\lambda_\phi > 1$ is a constant called dilatation of $\phi$; the $\lambda_\phi$ is always an algebraic number of the maximal degree $6g - 6 + 2n$ [Thurston 1988] [20]. It is known, that if $\phi \in \text{Mod } S_{g,n}$ is pseudo-Anosov then there exists a trajectory $\mathcal{O}$ of the geodesic flow $T^t$ and a point $S_{g,n} \in T_{g,n}$, such that the points $S_{g,n}$ and $\phi(S_{g,n})$ belong to $\mathcal{O}$ [Veech 1986] [21]; the $\mathcal{O}$ is called an axis of the pseudo-Anosov automorphism $\phi$. The axis can be used to calculate the Connes invariant $T(\mathcal{A}(x, Q_{g,n}))$ of the cluster $C^*$-algebra $\mathcal{A}(x, Q_{g,n})$; indeed, in view of formula (24) one must solve the following system of equations:

$$\begin{align*}
\sigma_t(x) &= e^t x \\
\phi(x) &= \lambda_\phi x,
\end{align*}$$

for a point $x \in \mathcal{O}$. Thus $\sigma_t(x)$ coincides with the inner automorphism $\phi(x)$ if and only if $t = \log \lambda_\phi$. Taking all pseudo-Anosov automorphisms $\phi \in \text{Mod } S_{g,n}$, one gets a formula for the Connes invariant:

$$T(\mathcal{A}(x, Q_{g,n})) = \{ \log \lambda_\phi \mid \phi \in \text{Mod } S_{g,n} \text{ is pseudo-Anosov} \}. \quad (26)$$

**Remark 8** The Connes invariant (26) says that the family of cluster $C^*$-algebras $\mathcal{A}(x, Q_{g,n})$ is an analog of the type $\text{III}_\lambda$ factors of von Neumann algebras, see [Connes 1978] [5].

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References

[1] L. Bernstein, The Jacobi-Perron Algorithm, its Theory and Applications, Lect. Notes in Math. 207, Springer 1971.

[2] B. Blackadar, $K$-Theory for Operator Algebras, MSRI Publications, Springer, 1986.

[3] F. P. Boca, An AF-algebra associated with the Farey tessellation, Canad. J. Math. 60 (2008), 975-1000.

[4] O. Bratteli, Inductive limits of finite dimensional $C^*$-algebras, Trans. Amer. Math. Soc. 171 (1972), 195-234.

[5] A. Connes, Von Neumann algebras, Proceedings of the ICM, Helsinki, 1978, pp. 97-109.

[6] C. Eckhardt, A noncommutative Gauss map, Math. Scand. 108 (2011), 233-250.

[7] E. G. Effros, Dimensions and $C^*$-Algebras, in: Conf. Board of the Math. Sciences, Regional conference series in Math., No.46, AMS, 1981.

[8] E. G. Effros and C. L. Shen, Approximately finite $C^*$-algebras and continued fractions, Indiana Univ. Math. J. 29 (1980), 191-204.

[9] S. Fomin, M. Shapiro and D. Thurston, Cluster algebras and triangulated surfaces, I. Cluster complexes, Acta Math. 201 (2008), 83-146.

[10] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, J. Amer. Math. Soc. 15 (2002), 497-529.

[11] D. Mundici, Farey stellar subdivisions, ultrasimplicial groups, and $K_0$ of AF $C^*$-algebras, Adv. in Math. 68 (1988), 23-39.

[12] D. Mundici, Recognizing the Farey-Stern-Brocot AF-algebra, Dedicated to the memory of Renato Caccioppoli, Rend. Lincei Mat. Appl. 20 (2009), 327-338.

[13] D. Mundici, Revisiting the Farey AF-algebra, Milan J. Math. 79 (2011), 643-656.
[14] I. Nikolaev, On a Teichmüller functor between the categories of complex tori and the Effros-Shen algebras, New York J. Math. 15 (2009), 125-132.

[15] I. Nikolaev, Riemann surfaces and AF-algebras, Ann. Funct. Anal. 7 (2016), 371-380.

[16] I. Nikolaev, K-theory of cluster $C^*$-algebras, arXiv:1512.00276

[17] I. Nikolaev, Cluster $C^*$-algebras and knot polynomials, arXiv:1603.01180

[18] G. Panti, Prime ideals in free $\ell$-groups and free vector lattices, J. of Algebra 219 (1999), 173-200.

[19] R. C. Penner, The decorated Teichmüller space of punctured surfaces, Comm. Math. Phys. 113 (1987), 299-339.

[20] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. 19 (1988), 417-431.

[21] W. A. Veech, The Teichmüller geodesic flow, Annals of Math. 124 (1986), 441-530.

[22] L. K. Williams, Cluster algebras: An introduction, Bull. Amer. Math. Soc. 51 (2014), 1-26.

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