Fragile topology of triangular kagome lattices

Yun-Feng Chen and Dao-Xin Yao*
State Key Laboratory of Optoelectronic Materials and Technologies,
School of Physics, Sun Yat-Sen University, Guangzhou 510275, China
(Dated: February 25, 2022)

In this work, we examine the fragile topological phases existing in a two-dimensional (2D) triangular kagome lattice (TKL) by the theory of “topological quantum chemistry”. We find that TKLs have more frustrated structures in comparison with kagome lattices and therefore can host very rich topological physics. We show the non-trivial windings of Berry phases along the Wilson loops which are protected by $C_2$T, and the connecting patterns in the Hofstadter spectrum as evidences, to diagnose the fragile topology. We also examine the bulk-boundary correspondence of these fragile phases in the TKL by showing the second-order topological states, i.e., mid-gap corner states of 2D models, which are hosted by obstructed atomic limits. After introducing magnetic fields, the second-order topological states can be eliminated and replaced by first-order states, i.e., chiral edge states, resulting in topological phase transitions from second-order topological phases to first-order topological phases which is perhaps usable in quantum computing and quantum simulation.

I. INTRODUCTION

The prosperity of studies for topological insulators has been through decades and undoubtedly deepened our understanding of condensed matter physics. Since the discovery of integer quantum Hall effect, robust bulk-boundary correspondence has become a hallmark of non-trivial topology. Later, the discovery of high-order topological insulators (HOTIs) enrich the physics with robust high-order bulk-boundary correspondence, e.g., bulk-hinge and bulk-corner correspondence [1–6]. Namely, for an $N$-dimensional crystalline topological insulator hosting $n$-dimensional gapless states residing in bulk gap, its topology is of $(N-n)$th order.

Recently, however, this seemly complete scheme is challenged by the fragile topological phases which are expected to appear in twisted bilayer graphene [7–14], for they being non-trivial yet not exhibiting robust gapless modes on boundaries under normal open-boundary conditions [15–22]. Under specific twisted boundary conditions, however, fragile phases have been identified theoretically and experimentally to host non-trivial edge modes [23–27]. While various topological invariants (e.g., Chern number, Kane-Mele invariant, etc.) have been proposed to diagnose stable topological phases, they fail to capture the fragile topology due to its removable Wannier obstruction, i.e., the fragility of being trivialized by addition of exponentially localized symmetric Wannier functions (known as atomic limits), which is outside the K-theory framework. Therefore, a systematic method, namely the “topological quantum chemistry” (TQC) [15–20], has been developed to characterize the fragile topology. Its central idea is to construct homomorphic mapping between real-space occupying configurations of atomic orbitals, and band representations (BRs) consisting of irreducible representations (irreps) of crystalline symmetries preserved at points in Brillouin zone (BZ). For instance, localized orbitals at maximal Wyckoff positions (high-symmetry points of certain space group in the unit cell), i.e., atomic-limit insulators, can induce elementary band representations (EBRs) in momentum space. Naturally, integer sum of EBRs maps to a trivial set of electronic bands while the difference or, equivalently, the partition of EBRs indicates the existence of the fragile topology [15–20]. In most cases, split EBR can be divided into two separated-by-gap subspaces, i.e., an occupied fragile subspace, and a complementary unoccupied subspace which is trivial yet still wannierized-obstructed for being induced by orbitals formally localized at non-existent sites in the unit cell. According to [28–32], electronic bands with this practically obstruction, named obstructed atomic bands (OABs), are able to host symmetry-protected fractional corner states. Because the obstruction can cause mismatch of localization of occupied charge centers and ionic sites in real space, and eventually give rise to spare charges distributed to corners of finite-size lattice.

Hofstadter butterfly is a fractal structure firstly discovered by Hofstadter in 1970s [33]. The energy spectrum containing this structure, namely the Hofstadter spectrum describes electrons moving in 2D lattice under perpendicular magnetic field. Interestingly, when subjected to a magnetic field, fragile topological bands (FTBs) will have different patterns in the Hofstadter spectrum in comparison with stable topological bands. More concretely, It is generally believed that energy range of an electronic band at zero-field limit offers upper and lower boundaries of its landau levels (LLs), which means that no connection of LLs, or equivalently no connection of Hofstadter butterflies shall happen if each set of them originates from different bands that are disconnected from each other at zero field. However, such connection has been witnessed in the Hofstadter spectrum containing FTBs [10–14, 32, 34–36]. For this reason, the Hofstadter butterflies of these structures have been widely analyzed to study the physics behind the fragile topol-
FIG. 1: (a) Diagram of the TKL structure and its first BZ. $\mathbf{a}_1$, $\mathbf{a}_2$ are basis vectors of the primitive unit cell, highlighted by black dashed lines, and $\mathbf{b}_1$, $\mathbf{b}_2$ are basis vectors of BZ, highlighted by blue solid lines. Orange and black bonds correspond respectively to hoppings with energy $t$ and $t'$. Anti-clockwise dashed green arrows mark the Haldane-model-like hoppings inside $\beta$ sites, with energy $t_{soc}$. NNN hoppings with energy $t_{NNN}$ are marked by yellow dashed bonds, and similar hoppings with energy $t'_{NNN}$ are marked by deep blue solid bonds. Wyckoff positions 1a and 3c are indicated in this diagram. (b) Electronic band structure of $H_{TB}$ along $\Gamma-K-M$, with $t = 1$, $t' = 0.2t$. Top, middle and bottom band group and their corresponding EBRs are marked by green, blue and red, respectively. (c)∼(d) The Wilson-loop spectrum of up and down sets of FTBs, respectively. The Wilson loop along $\mathbf{b}_2$ is indicated by yellow dashed line, and integral loop along $\mathbf{b}_1$ is indicated by black dashed line in BZ. Red and blue spectral lines belong to different FTB inside each set. Both of them winds from $-\pi$ to $\pi$.

Ref. [37, 38] have offered two theoretical methods, namely the split-graph and line-graph methods to rebuild bipartite lattices into new lattices with created or increased frustration to host the fragile topology. We find that the split-graph method can be used to construct an existent frustrated lattice from kagome lattice, which is referred as triangular kagome lattice (TKL) and shown in Fig.1(a). Extra small triangles placed inside the big triangles of kagome lattice make the TKL more frustrated and therefore can cause more complicated obstruction of Wannier bands, which is an advantage for hosting the fragile topology. Several magnetic structures based on this lattice have been studied [39–44], and materials with this kind of lattice structure have already been synthesized in experiments [45–47]. In this paper, we mainly focus on studying the topology which closely correlates to the crystalline symmetries of the TKL. Last but not least, a topological phase transition from second-order topological phase (SOTP) to first-order topological phase (FOTP) realized in this lattice will be mentioned later in Sec.II C and Sec.III B.

The rest of this paper is organized as follows: In Sec.II, we use the theory of TQC to offer a symmetry analysis of our model and diagnose the topology of its electronic bands. The Wilson-loop spectrum of FTBs and corner states hosted by OAB are also discussed in this section. In Sec.III, we show the Hofstadter spectrum of the TKL
and explain how the transition from SOTP to FOTP can be realized under magnetic field. We conclude in Sec.IV.

II. FRAGILE TOPOLOGY OF THE TKL

A. Symmetry analysis based on TQC

Firstly, we consider a spinless nearest-neighbor (NN) tight-binding (TB) model of the TKL, defined as:

\[
H^\text{TB}_{\text{NN}} = \sum_{\langle i,j \rangle} \left[ t \sum_{\alpha,\beta} C_{i,\alpha}^\dagger C_{j,\alpha} + t' \sum_{\beta',\beta} C_{i,\beta'}^\dagger C_{j,\beta} \right] + \text{H.c.},
\]

where \(\langle \rangle\) denotes that only the hoppings between nearest sites are considered in this summation, and \(\alpha(\alpha')\), \(\beta(\beta')\) are sublattice indices of the TKL, with \(\alpha(\alpha')=1,2,3\) assigned to sites of big triangles and \(\beta(\beta')=4,5,6,7,8,9\) assigned to sites of small triangles. Hopping bonds connecting \(\alpha\) and \(\beta\) sites contribute energy \(t\), and bonds between \(\beta\) sites contribute energy \(t'\) (see Fig.1(a)). Space group of this model is P6mm (No.183) and point groups of three high-symmetry momentum in BZ, labeled by \(\Gamma=(0,0), K=(\frac{\pi}{3},\frac{2\pi}{3}), M=(\frac{\pi}{3},0)\), are respectively \(C_6v\), \(C_{3v}\) and \(C_{2v}\). We assume that \(t=1\) and the ratio between \(t\) and \(t'\) is \(t'/t=0.2\). Resulting band structure is displayed in Fig.1(b), where nine bands are divided into top, middle and bottom groups respectively. According to TQC, all of these band groups are trivial because they can be classified into two kinds of EBR: \(A_1 \uparrow G(3c)\) and \(B_2 \uparrow G(3c)\) (see Table.1), which are generated by orbitals \((q_x, q_y)\) respectively localized at 3c wyckoff positions (see Fig.1(a)) where actual sites also lie. After introducing next-nearest-neighbor (NNN) hopping terms (see Fig.1(a)), the TB model becomes:

\[
H^\text{TB}_{\text{NNN}} = H_{\text{NN}} + H_{\text{NNN}},
\]

\[
H_{\text{NNN}} = \sum_{\langle\langle i,j\rangle\rangle} \left[ t_{\text{NNN}} \sum_{\alpha,\alpha'} C_{i,\alpha}^\dagger C_{j,\alpha'} + t'_{\text{NNN}} \sum_{\beta',\beta} C_{i,\beta'}^\dagger C_{j,\beta} \right] + \text{H.c.},
\]

where \(\langle\langle\rangle\rangle\) denotes that hoppings between next-nearest sites are considered in this summation. We assume that NNN hoppings between \(\alpha\) sites contribute energy \(t_{\text{NNN}}\), and similar hoppings between \(\beta\) sites contribute \(t'_{\text{NNN}}\). By setting the ratios of \(t_{\text{NNN}} = 0.6, t'_{\text{NNN}} = 0.4\), we obtain the energy spectrum shown in Fig.2(a), where top and bottom groups are both separated into an OAB and two crossing FTBs. Even though the band structure seems completely changed after adding NNN hoppings, all degeneracy protected by crystalline symmetries at high-symmetry points are still preserved because the space-group symmetry is unchanged. In fact, these NNN hoppings can not change the geometry of the unit cell of the TKL because they are symmetric about the center of all symmetrical operations, i.e., the 1a position (see Fig.1(a)) which has the maximal point-group symmetry of P6mm. Besides, they have no effect on time-reversal symmetry \(\mathcal{T}\) due to their real hopping energies. Therefore, we can use irreps from P6mm to diagnose the topology of \(H^\text{TB}_{\text{NNN}}\). The two sets of FTBs, named up and down sets respectively, can be written as a same formal subtraction of EBRs (see Table.1): \(A_1 \uparrow G(3c) \oplus A_2 \uparrow G(1a)\). It indicates that they have similar topological property, which can be diagnosed by the Wilson-loop spectrum introduced in Sec.IIB. \(A_1 \uparrow G(1a)\) is practically obstructed since no actual site lies at 1a position of the unit cell. Later in Sec.IIC, we will demonstrate how to lift the degeneracy at high-symmetry points by introducing hoppings with chiral direction.

B. Wilson loop spectrum

Even though the topology of FTBs can not be diagnosed by stable indices such as Chern number, their Wilson-loop spectrum still can display non-trivial gauge-invariant windings which are irrelevant to any specific choice of the loop. To obtain such spectrum of the two sets of FTBs, first of all, we need to define the Wilson loop operator \(\mathbf{w}_k\) which is constructed by wave functions \(\psi^1\) and \(\psi^2\) of each set of FTBs [16, 20, 22]:

\[
\mathbf{w}_k = [\psi^1_k, \psi^2_k],
\]

where \(k = k_1 \cdot \mathbf{b}_1 + k_2 \cdot \mathbf{b}_2\), with \(k_1, k_2 \in [0,2\pi]\). Next, we discretize the BZ by setting periodic conditions on the boundary of an \(N \times N\) lattice. Keeping \(k_2\) fixed, we multiply all discretized Wilson operators along an integral loop in \(\mathbf{b}_1\) direction [16, 20, 22]:

\[
\text{W}(k_2) = \prod_{n=0}^{N-1} \mathbf{w}_{\frac{n \cdot 2\pi}{N}} \cdot \mathbf{w}_{\frac{(n+1) \cdot 2\pi}{N} - k_2}.
\]

Finally, we collect all eigenvalues of \(-i\log \text{W}(k_2)\) to draw the Wilson spectrum along the Wilson loop in \(\mathbf{b}_2\) direction. All gauge-invariant topological information of the model are contained in this spectrum, since all parallel integral loops starting from points of the Wilson loop weave a network which can cover the whole BZ. The maximal change of eigenvalues (divided by \(2\pi\)) counts the winding number of Berry phase, which acts as a Z classification to diagnose the fragile topology. In fact, this Z classification is formally introduced by combined symmetry \(C_{2z}\mathcal{T}\) which can protect the fragile topology of two crossing FTBs [16, 20, 22]. \(C_{2z}\mathcal{T}\) keeps every momentum \(k\) inside BZ fixed, considering that \(C_{2z}\) rotates \(k\) to \(-k\) around the out-of-plane \(z\) axis, and time-reversal symmetry \(\mathcal{T}\) reverses \(-k\) back to \(k\). Therefore, \(C_{2z}\mathcal{T}\) is preserved along \(\mathbf{b}_2\) and ensures the continuity of eigenvalues arranging along the Wilson loop.

In 2D spinless model, \(C_{2z}\) can be treated as parity
FIG. 2: (a) Diagram of electronic band structure of $H^T_{N NN}$ with $t_{NNN} = 0.6 t$, $t'_{NNN} = 0.4 t'$. (b) The energy spectrum of $H^T_{N NN}$ for a cylindrical geometry, with $a_1$ opened and $a_2$ remaining periodic. (c) In this diagram, we place the energy spectrum with $a_1$, $a_2$ both opened on the left. Its energy range is limited inside the black dashed frame of (a). We mark the corner states exhibited in this spectrum by red dots and show the weight of their wave functions on a finite $30 \times 30$ TKL in the right picture, with color bars indexing the weight (Dispersion of the weight from corners into bulk due to finite-size effect). The insert placed at the bottom left displays a zoom-in spectrum of (b) which shares the same energy range with the black dashed frame of (a). (d) Similar diagrams of $H^T_{SOC}$, with $t_{soc} = 0.2 t$. In (b), no chiral edge state occurs, while in (e), edge states connecting up and down bulk levels appear in gap. We mark the chiral edge states in the first and second gaps (counting from bottom to top) with red color and display them in the insert of (f).

symmetry $P$. As we know, the change of parity at high-symmetry points symbolizes non-trivial Berry phase along the loop going through these points [51]. For this reason, the sign change of $C_{2z}$ eigenvalues implies non-trivial winding of the Wilson loop. From $\Gamma$ to $M$ along $b_2$, two sets of FTBs experience sign change of their $C_{2z}$ eigenvalues (see Table.II). Therefore, in Fig.1(c)–1(d), we can see that both spectral lines finish a complete winding by passing from $-\pi$ to $\pi$ ($or \pi$ to $-\pi$) once, displaying their non-trivial topology protected by $C_{2z} T$.

C. Corner states hosted by OAB

Under open-boundary conditions, we find fractional corner states existing in the gap above the up set of FTBs (Fig.2(c)). As explained in Sec.I, they are hosted by OAB from top set due to the mismatch of localization of ionic sites and charge centers. When we replace the NNN terms with Haldane-model-like spin-orbit-coupling (SOC) terms (Fig.1(a)) to break $T$, similar OAB also appears and hosts corner states likewise in the gap above
TABLE I: For each band group from $H_{NN}$ and each set of FTBs from $H_{NN}$, we tabulate their irreducible representations of high-symmetry points. For better illustration, we also tabulate three kinds of EBRs from P6mm to determine the BRs of these groups. In this table, number in bracket behind an irrep marks its dimension. For example, the BR of top group at $\Gamma$ is written as a direct sum of one-dimensional irrep $\Gamma_1$ due to the two-fold degeneracy at $\Gamma$. Notation used in this paper is borrowed from the Bilbao Crystallographic Server [48–50].

| BRs   | $\Gamma$ | $K$         | $M$                     |
|-------|-----------|-------------|-------------------------|
| top group | $\Gamma_1(1) \oplus \Gamma_5(2)$ | $K_1(1) \oplus K_3(2)$ | $M_1(1) \oplus M_3(1) \oplus M_4(1)$ |
| middle group | $\Gamma_4(1) \oplus \Gamma_6(2)$ | $K_2(1) \oplus K_3(2)$ | $M_1(1) \oplus M_2(1) \oplus M_4(1)$ |
| bottom group | $\Gamma_1(1) \oplus \Gamma_5(2)$ | $K_1(1) \oplus K_3(2)$ | $M_1(1) \oplus M_5(1) \oplus M_4(1)$ |

EBRs

| $A_1 \uparrow G(3c)$ | $\Gamma_1(1) \oplus \Gamma_5(2)$ | $K_1(1) \oplus K_3(2)$ | $M_1(1) \oplus M_3(1) \oplus M_4(1)$ |
| $B_2 \uparrow G(3c)$ | $\Gamma_4(1) \oplus \Gamma_6(2)$ | $K_2(1) \oplus K_3(2)$ | $M_1(1) \oplus M_2(1) \oplus M_4(1)$ |
| $A_1 \uparrow G(1a)$ | $\Gamma_1(1)$ | $K_1(1)$ | $M_1(1)$ |

FTBs

| FTBs   | $\Gamma$ | $M$         |
|--------|----------|-------------|
| up set | $\Gamma_5(2)$ | $K_3(2)$ | $M_3(1) \oplus M_4(1)$ |
| down set | $\Gamma_5(2)$ | $K_3(2)$ | $M_3(1) \oplus M_4(1)$ |

TABLE II: For the two sets of FTBs, we tabulate their $C_2$-eigenvalues at $\Gamma$ and $M$ where $C_2$ symmetry is preserved.

| FTBs   | $\Gamma$ | $M$         |
|--------|----------|-------------|
| up set | 9        | $-2$        |
| down set | 2        | $-2$        |

bottom group. The TB Hamiltonian then becomes:

$$H_{TB} = H_{NN} + H_{SOC},$$

$$H_{SOC} = \sum_{\beta', \beta} \exp(i \phi (r_{\beta'} - r_{\beta})) t_{soc} C_{\beta'}^\dagger C_{\beta} + H.c.,$$

where the phase factor in front of $t_{soc}$ contributes $i$ when vector $(r_{\beta'} - r_{\beta})$ aligns to anti-clockwise direction (as pointed by arrows in Fig.1(a)), and $-i$ when the vector aligns to clockwise direction. These directional phase factors endow the SOC hoppings with in-plane chirality (according to right-hand rule) that is actually created by magnetic flux which threads through the triangles (see Fig.1(a)) with total amount constrained to zero. As a result, $\Gamma$ is broken due to the imaginary phase factors in front of hopping energies, and all in-plane mirror symmetries of P6mm are also broken since in-plane chirality can not preserve under in-plane mirror reflection. Therefore, by introducing SOC, we reduce the space-group symmetry from P6mm (No.183) to P6 (No.168) and then lift all degeneracy at $\Gamma$ and $K$ (Fig.2(d)). Now the bottom group splits into an OAB in the middle and two bands under and above it. While the OAB is represented by $E_1 \uparrow G(1a)$, the two bands can not be represented by any one-dimensional EBR or formal subtraction of EBRs, of P6. Therefore, they have stable topology which usually accompanies with first-order chiral modes. Under open-boundary conditions, we find such modes in two gaps between the two non-trivial bands and the OAB (Fig.2(e)–2(f)). Since the whole group including OAB (whose Chern number is zero) is trivial, Chern numbers of the two bands have to cancel with each other when the whole group is occupied, which consequently causes the annihilation of those chiral edge modes. Finally, fractional corner states hosted by OAB are able to exhibit themselves in the gap above the bottom band group (Fig.2(f)).

If we change the filling factor $\nu$ from $\frac{1}{3}$ to $\frac{2}{3}$, the topological phase will transit from second-order to first-order, without breaking any crystalline symmetry. However, it is experimentally difficult to realize this topological transition in TKLs by adding Haldane-like hoppings, since it requires that the flux constrained inside the small triangles has to offset the flux outside to keep the total flux through the whole unit cell at zero. Therefore, we find an alternative way for realizing the transition under magnetic field, which will be discussed latter in Sec.III.

III. BEHAVIORS OF FTBS UNDER MAGNETIC FIELD

A. Hofstadter butterflies of FTBs

We choose the Landau gauge: $A_{a_2} = 0$, $A_{a_1} = A_2 = 0$ to introduce magnetic field, where $A$ is the magnetic vector potential, $\phi$ is the magnetic flux per unit cell, and $a_1$, $a_2$ are real-space basis vectors of the unit cell of the TKL. Under this gauge, translational invariance is preserved along $a_2$ while broken along $a_1$. However, commensurate
\( \phi = \frac{p}{q} \) with coprime \( p, q \) can rebuild translation symmetry along \( a_2 \), with the cost of having \( q \)-times-larger magnetic unit cells. Correspondingly, area of first BZ shrinks to \( \frac{1}{q} \), with \( k_1 \in [0, 2\pi] \) and \( k_2 \in [0, \frac{2\pi}{q}] \). In periodic crystal field, via Peierls substitution, the correction of momentum induced by magnetic field, i.e., \( q' = q - eA \), is substituted by an extra phase factor \( \exp(i\theta(r - r')) \) in front of hopping energy, where the argument \( \theta(r - r') \) can be written as the integral of \( A \) along a Peierls path:

\[
\theta(r - r') = \int_{C_{r' \rightarrow r}} A(r) \cdot dr. \tag{6}
\]

In this paper, we choose the integral paths to be straight lines that connect nearest or next-nearest sites. Then, we can give the Hofstadter Hamiltonian of the TKL via Peierls substitution:

\[
H_{\text{hof}} = \sum_{i,j,\alpha,\beta} \exp(i\theta(r_{i,\alpha} - r_{j,\beta})) t_{i,j,\alpha,\beta} \cdot C_{i,\alpha}^\dagger C_{j,\beta} + H.c.. \tag{7}
\]

We define \( H_{\text{hof}}^{\text{NN}} \) when only NN hoppings are considered, and \( H_{\text{hof}}^{\text{NNN}} \) when NNN hoppings are also included. Finally, we diagonalize this Hamiltonian in the enlarged magnetic unit cell and collect all eigenvalues depending on \( \phi \) to draw the Hofstadter spectrum.

From Fig.3(a), we can see that LLs originating from each band group are separated by gaps, thus no connection of Hofstadter butterflies appears. After introducing NNN hoppings, LLs of FTBs (both the up and down sets) connect with LLs of OABs above them, displaying the expected connecting pattern in Fig.3(b). In Fig.3(a) and Fig.3(b), LLs of middle group stay compact all along, since no FTB is separated from this band group.

Generally speaking, magnetic field can break \( C_{2z}T \)
symmetry which protects the fragile topology by breaking $T$. Therefore, after introducing magnetic field, degenerate FTBs disconnect each other and turn into several Chern bands. Consequently, the energy expansion of this set of bands becomes wider than before. Besides, Chern bands themselves also have anomalous expansion outside its zero-field spreading induced by magnetic filed [34, 35]. Combining these expanding effects together, we can qualitatively explain the mechanism of the unusual connection of Hofstadter butterflies in Fig.3(b).

B. Topologcial phase transition from SOTP to FOTP driven by magnetic field

We find that by introducing and strengthening the magnetic field, LLs split by crystal field of the TKL can be shifted across gaps. It inspires us to realize the transition from SOTP to FOTP in the TKL under the action of magnetic field. Firstly, we search for a gap where isolated corner states exist under open-boundary conditions, and assume that all levels under it are filled. After the shifting of LLs, if first-order non-trivial levels have been moved from unoccupied subspace to occupied subspace across the gap, the occupied subspace will own non-zero Chern number and contribute chiral edge states to replace corner states in the gap, which symbolizes the completion of the transition. Based on the discussion in Sec.IIC, we choose the gap above the up set of FTBs as our candidate, where corner states already exist before the presence of magnetic fields. While the magnetic field is merely introduced with $\phi=\frac{2\pi}{q}$, we find under open-boundary conditions that corner states still exist in this gap. As $\phi$ increasing to $2\pi$, Chern number of the occupied bands under the gap becomes 1 from zero, and then chiral edge states arise to replace corner states (see Fig.3(c)). Generally speaking, it is still experimentally impossible to create such strong magnetic field which can squeeze a magnetic flux quantum into one unit cell. However, inspired by the enormous unit cell of twisted bilayer graphene (TBG) [52, 53], we consider that it could be experimentally accessible to accumulate enough amount of magnetic flux inside the unit cell of a twisted-bilayer TKL to realize the transition.

IV. CONCLUSION AND DISCUSSION

Considering the recent popularity of kagome metals [54–56] and insulators with nontrivial topology, we believe that it is of great interest to explore the topological physics behind the more frustrated kagome-like lattices, which can simultaneously enhance Wannier obstructions and the electronic correlations. Our work is a very beginning study in this direction. In Sec.II and Sec.III, we have diagnosed the fragile topology of the TKL by looking into the Wilson-loop spectrum and Hofstadter spectrum. Complete windings of Berry phases of FTBs ensure their non-trivial topology in spite of the trivial stable indices. During calculation, we realize that magnetic fields can release implicit details of fragile topological phases to introduce fine structure into energy spectrum. Thanks to this fine structure, we find the topological phase transition from SOTP to FOTP which is potentially usable in quantum computing and quantum simulation. In further study, we are going to explore possible applications of this transition, and more applications of the fragile topology under magnetic fields.

V. ACKNOWLEDGMENTS

We thank Zhong-Bo Yan for enlightening advices, and Jun Li, Jia-Zheng Ma, Zhi-Hui Luo, Shan-Bo Zhou, Biao Lv, Xun-Wu Hu, Ze-Nan Liu for helpful discussions. This project is supported by NKRDPC-2017YFA0206203, NKRDPC-2018YFA0306001, NSFC-11974432, NSFC-92165204, GBABRF-2019A15151337, Shenzhen Institute for Quantum Science and Engineering (Grant No. SIE202102), and Leading Talent Program of Guangdong Special Projects (201626003).

[1] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. Parkin, B. A. Bernevig, and T. Neupert, Higher-order topological insulators, Science advances 4, eaat0346 (2018).
[2] F. Schindler, Z. Wang, M. G. Vergniory, A. M. Cook, A. Murani, S. Sengupta, A. Y. Kasumov, R. Debloch, S. Jeon, I. Drozdov, et al., Higher-order topology in bismuth, Nature physics 14, 918 (2018).
[3] Y. Xu, Z. Song, Z. Wang, H. Weng, and X. Dai, Higher-order topology of the axion insulator euin2 as 2 as 2, Physical review letters 122, 256402 (2019).
[4] F. K. Kunst, G. van Miert, and E. J. Bergholtz, Lattice models with exactly solvable topological hinge and corner states, Physical Review B 97, 241405(R) (2018).
[5] H. Xue, Y. Yang, G. Liu, F. Gao, Y. Chong, and B. Zhang, Realization of an acoustic third-order topological insulator, Physical review letters 122, 244301 (2019).
[6] M. Rodriguez-Vega, A. Kumar, and B. Seradjeh, Higher-order floquet topological phases with corner and bulk bound states, Physical Review B 100, 085138 (2019).
[7] L. Zou, H. C. Po, A. Vishwanath, and T. Senthil, Band structure of twisted bilayer graphene: Emergent symmetries, commensurate approximants, and wannier obstructions, Physical Review B 98, 085435 (2018).
[8] J. Ahn, S. Park, and B.-J. Yang, Failure of nielsen-ninomiya theorem and fragile topology in two-dimensional systems with space-time inversion symmetry: Application to twisted bilayer graphene at magic
angle, Physical Review X 9, 021013 (2019).

[9] Z.-D. Song, B. Lian, N. Regnault, and B. A. Bernevig, Twisted bilayer graphene. ii. stable symmetry anomaly, Physical Review B 103, 205412 (2021).

[10] B. Lian, F. Xie, and B. A. Bernevig, Landau level of fragile topology, Physical Review B 102, 041402(R) (2020).

[11] Q. Wu, J. Liu, Y. Guan, and O. V. Yazyev, Landau levels as a probe for band topology in graphene moiré superlattices, Physical Review Letters 126, 056401 (2021).

[12] Y.-Z. Chou, F. Wu, and S. D. Warma, Hofstadter butterfly and floquet topological insulators in minimally twisted bilayer graphene, Physical Review Research 2, 033271 (2020).

[13] X. Lu, B. Lian, G. Chaudhary, B. A. Piot, G. Romagnoli, K. Watanabe, T. Taniguchi, M. Poggio, A. H. MacDonald, B. A. Bernevig, et al., Multiple flat bands and topological hofstadter butterfly in twisted bilayer graphene close to the second magic angle, Proceedings of the National Academy of Sciences 118 (2021).

[14] X. Lu, B. Lian, G. Chaudhary, B. A. Piot, G. Romagnoli, K. Watanabe, T. Taniguchi, M. Poggio, A. H. MacDonald, B. A. Bernevig, et al., Fingerprints of fragile topology in the hofstadter spectrum of twisted bilayer graphene close to the second magic angle, arXiv e-prints, arXiv (2020).

[15] B. Bradlyn, L. Elcoro, J. Cano, M. Vergniory, Z. Wang, C. Felser, M. I. Aroyo, and B. A. Bernevig, Topological quantum chemistry, Nature 547, 298 (2017).

[16] B. Bradlyn, Z. Wang, J. Cano, and B. A. Bernevig, Disconnected elementary band representations, fragile topology, and wilson loops as topological indices: An example on the triangular lattice, Physical Review B 99, 045140 (2019).

[17] J. Cano, B. Bradlyn, Z. Wang, L. Elcoro, M. Vergniory, C. Felser, M. I. Aroyo, and B. A. Bernevig, Topology of disconnected elementary band representations, Physical review letters 120, 266401 (2018).

[18] J. Cano, B. Bradlyn, Z. Wang, L. Elcoro, M. Vergniory, C. Felser, M. I. Aroyo, and B. A. Bernevig, Building blocks of topological quantum chemistry: Elementary band representations, Physical Review B 97, 035139 (2018).

[19] H. C. Po, H. Watanabe, and A. Vishwanath, Fragile topology and wannier obstructions, Physical review letters 121, 126402 (2018).

[20] A. Bouhon, A. M. Black-Schaffer, and R.-J. Slager, Wilson loop approach to fragile topology of split elementary band representations and topological crystalline insulators with time-reversal symmetry, Physical Review B 100, 195135 (2019).

[21] A. Bouhon, T. Bzdúšek, and R.-J. Slager, Geometric approach to fragile topology beyond symmetry indicators, Physical Review B 102, 115135 (2020).

[22] V. Peri, Z.-D. Song, B. A. Bernevig, and S. D. Huber, Fragile topology and flat-band superconductivity in the strong-coupling regime, Physical review letters 126, 027002 (2021).

[23] Z.-D. Song, L. Elcoro, and B. A. Bernevig, Twisted bulk-boundary correspondence of fragile topology, Science 367, 794 (2020).

[24] M. B. de Paz, M. G. Vergniory, D. Bercioux, A. García-Etxarri, and B. Bradlyn, Engineering fragile topology in photonic crystals: Topological quantum chemistry of light, Physical Review Research 1, 032005(R) (2019).

[25] R.-X. Zhang and Z.-C. Yang, Tunable fragile topology in floquet systems, Physical Review B 103, L121115 (2021).

[26] V. Peri, Z.-D. Song, M. Serra-Garcia, P. Engeler, R. Queiroz, X. Huang, W. Deng, Z. Liu, B. A. Bernevig, and S. D. Huber, Experimental characterization of fragile topology in an acoustic metamaterial, Science 367, 797 (2020).

[27] J. Yu, Y. Ge, and S. D. Sarma, Dynamical fragile topology in floquet crystals, Physical Review B 104, L180303 (2021).

[28] A. Luo, Z. Song, and G. Xu, Fragile topologically flat band in the checkerboard antiferromagnetic monolayer fese, arXiv preprint arXiv:2107.05433 (2021).

[29] C. Shang, X. Zang, W. Gao, U. Schwingenschlög, and A. Manchon, Second-order topological insulator and fragile topology in topological circuitry simulation, arXiv preprint arXiv:2009.09167 (2020).

[30] Z. Wang, B. J. Wieder, J. Li, B. Yan, and B. A. Bernevig, Higher-order topology, monopole nodal lines, and the origin of large fermi arcs in transition metal dichalcogenides, Phys. Rev. Lett. 123, 186401 (2019).

[31] G. F. Lange, A. Bouhon, and R.-J. Slager, Subdimensional topologies, indicators, and higher order boundary effects, Physical Review B 103, 195145 (2021).

[32] J. Herzog-Arbeitman, Z.-D. Song, N. Regnault, and B. A. Bernevig, Hofstadter topology: Noncrystalline topological materials at high flux, Physical review letters 125, 236804 (2020).

[33] D. R. Hofstadter, Energy levels and wave functions of bloch electrons in rational and irrational magnetic fields, Physical review B 14, 2239 (1976).

[34] J.-W. Rhim, K. Kim, and B.-J. Yang, Quantum distance and anomalous landau levels of flat bands, Nature 584, 59 (2020).

[35] Y. Hwang, J.-W. Rhim, and B.-J. Yang, Geometric characterization of anomalous landau levels of isolated flat bands, Nature communications 12, 1 (2021).

[36] Y. Guan, A. Bouhon, and O. V. Yazyev, Landau levels of the euler class topology, arXiv preprint arXiv:2108.10353 (2021).

[37] C. S. Chiu, D.-S. Ma, Z.-D. Song, B. A. Bernevig, and A. A. Houck, Fragile topology in line-graph lattices with two, three, or four gapped flat bands, Physical Review Research 2, 043410 (2020).

[38] D.-S. Ma, Y. Xu, C. S. Chiu, N. Regnault, A. A. Houck, Z. Song, and B. A. Bernevig, Spin-orbit-induced topological flat bands in line and split graphs of bipartite lattices, Physical review letters 125, 266403 (2020).

[39] L. Wang and D.-X. Yao, Coexistence of spin-1 fermion and dirac fermion on the triangular kagome lattice, Physical Review B 98, 161403(R) (2018).

[40] D.-X. Yao, Y. L. Loh, E. W. Carlson, and M. Ma, X x z and ising spins on the triangular kagome lattice, Physical Review B 78, 024428 (2008).

[41] Y.-H. Chen, H.-S. Tao, D.-X. Yao, and W.-M. Liu, Kondo metal and ferrimagnetic insulator on the triangular kagome lattice, Physical review letters 108, 246402 (2012).

[42] Y. L. Loh, D.-X. Yao, and E. W. Carlson, Dimers on the triangular kagome lattice, Physical Review B 78, 224410 (2008).

[43] Y. L. Loh, D.-X. Yao, and E. W. Carlson, Thermodynamics of ising spins on the triangular kagome lattice: Exact analytical method and monte carlo simulations, Physical
[44] S. Kim, W. H. Han, I.-H. Lee, and K. Chang, Boron triangular kagome lattice with half-metallic ferromagnetism, Scientific reports 7, 1 (2017).

[45] M. Gonzalez, F. Cervantes-lee, and L. W. ter Haar, Structural and magnetic properties of the topologically novel 2-d material cu9f2cpa) 6: A triangulated kagome-like hexagonal network of cu (ii) trimers interconnected by cu (ii) monomers, Molecular Crystals and Liquid Crystals Science and Technology. Section A. Molecular Crystals and Liquid Crystals 233, 317 (1993).

[46] S. Maruti and L. W. ter Haar, Magnetic properties of the two-dimensional “triangles-in-triangles”kagomé lattice cu9 x 2 (cpa) 6 (x= f, cl, br), Journal of Applied Physics 75, 5949 (1994).

[47] M. Mekata, M. Abdulla, T. Asano, H. Kikuchi, T. Goto, T. Morishita, and H. Hori, Magnetic ordering in triangulated kagomé lattice compound, cu9cl2 (cpa) 6· nh2o, Journal of magnetism and magnetic materials 177, 731 (1998).

[48] M. I. Aroyo, J. Perez-Mato, D. Orobenchoa, E. Tasci, G. de la Flor, and A. Kirov, Crystallography online: Bilbao crystallographic server, Bulg. Chem. Commun 43, 183 (2011).

[49] M. I. Aroyo, J. M. Perez-Mato, C. Capillas, E. Kroumova, S. Ivantchev, G. Madariaga, A. Kirov, and H. Wondratschek, Bilbao crystallographic server: I. databases and crystallographic computing programs, Zeitschrift für Kristallographie-Crystalline Materials 221, 15 (2006).

[50] M. I. Aroyo, A. Kirov, C. Capillas, J. Perez-Mato, and H. Wondratschek, Bilbao crystallographic server: ii. representations of crystallographic point groups and space groups, Acta Crystallographica Section A: Foundations of Crystallography 62, 115 (2006).

[51] D. Vanderbilt, Berry Phases in Electronic Structure Theory: Electric Polarization, Orbital Magnetization and Topological Insulators (Cambridge University Press, 2018).

[52] J.-P. Liu et al., Topological properties and orbital magnetism in twisted graphene systems, Acta Physica Sinica 69, 20200506.

[53] A. Nimbalkar and H. Kim, Opportunities and challenges in twisted bilayer graphene: a review, Nano-Micro Letters 12, 1 (2020).

[54] F. Yu, T. Wu, Z. Wang, B. Lei, W. Zhuo, J. Ying, and X. Chen, Concentration of anomalous hall effect and charge density wave in a superconducting topological kagome metal, arXiv preprint arXiv:2102.10987 (2021).

[55] Z. Zhang, Z. Chen, Y. Zhou, Y. Yuan, S. Wang, J. Wang, H. Yang, C. An, L. Zhang, X. Zhu, et al., Pressure-induced reemergence of superconductivity in the topological kagome metal cs v 3 sb 5, Physical Review B 103, 224513 (2021).

[56] B. R. Ortiz, S. M. Teicher, Y. Hu, J. L. Zuo, P. M. Sarte, E. C. Schueller, A. M. Abeykoon, M. J. Krogstad, S. Rosenkranz, R. Osborn, et al., Cs v 3 sb 5: A z 2 topological kagome metal with a superconducting ground state, Physical Review Letters 125, 247002 (2020).