Quantum corrections for pion correlations involving resonance decays

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A method is presented to include quantum corrections into the calculation of two-pion correlations for the case where particles originate from resonance decays. The technique uses classical information regarding the space-time points at which resonances are created. By evaluating a simple thermal model, the method is compared to semiclassical techniques that assume exponential decaying resonances moving along classical trajectories. Significant improvements are noted when the resonance widths are broad as compared to the temperature.

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I. INTRODUCTION

Analyses of two-boson correlations have provided intangible information regarding the space-time development of hadronic reactions [1,2]. Pions, kaons, and photons have all been exploited for their bosonic nature which results in a positive correlation at small relative momentum. Numerous other correlations, involving nucleons [3] or light nuclei [4] that are correlated due to the strong or Coulomb interaction as well as identical-particle statistics, have also been analyzed and have given further information regarding collision dynamics. Source sizes and time scales have been extracted from collisions covering a wide assortment of reactions, from heavy-ion collisions at a few MeV per nucleon, where time scales of thousands of fm/$c$ have been determined, to $e^+e^-$ → jets, where lifetimes of a fraction of a fm/$c$ have been observed [5].

The comparison of theoretically predicted correlation functions with experimental results provides an important test of the dynamical properties of reaction models. Most models provide semiclassical information about the source function $S(p,x)$, the probability of emitting a particle of momentum $p$ from the space-time point $x$. By convoluting the source functions for particles of momenta $p_1$ and $p_2$ with the squared relative wave function $|\phi_q(x_1 - x_2)|^2$, one is able to predict the correlation function $C(p_1,p_2)$. Source functions are usually obtained from semiclassical simulations, where the source points are associated with the last point of interaction [10]. Particles from resonances are usually assumed to be emitted according to an exponential decay law, with the characteristic time usually chosen to be independent of the energy of the resonance. Quantum considerations have been explored by Lednicky and Progulova [11] and by Bertsch, Danielewicz and Herrmann [12].

In this study, we pursue two goals. First, we wish to quantify the importance of quantum treatments by comparing to semiclassical forms for a simple thermal model. Although the formalism we present is not much different from that presented previously in the literature [11,12], the differences with semiclassical treatments have not been quantitatively documented. We find that quantum corrections become important when kinematics constrain the resonances to be off shell. Second, we propose an alteration to the methods for extracting correlations from classical simulations to better account for quantum effects. We show that a simple modification of the semiclassical treatment can account for the quantum corrections by incorporating information regarding the off-shell energy of the decaying resonance. In this study we neglect any interaction between the particles aside from the constraints imposed by symmetrization.

In this paper, we will first briefly review correlation functions for direct sources, which will provide the foundation for correlations from resonant sources in the following section. Section [V] contains a calculation for a simple Breit-Wigner resonance to demonstrate the importance of quantum treatments. Modified correlation weights are presented in Sec. [V] as a means to better calculate correlations from classical simulations when resonant decays are involved. Finally, we will compare the various methods mentioned in this paper by considering a simple thermal model in Sec. [VI].

II. REVIEW: CORRELATIONS FROM DIRECT SOURCES

The two-particle correlation function is usually defined as
Before describing the two-particle probability, we first derive an expression for the one-particle probability, which also introduces the definition of the source function. One-particle probabilities can be determined by matrix elements \( T_f(x) \) where \( f \) describes the remainder of the system, and \( x \) is the point at which the pion has the final interaction with the system. Without loss of generality, one can write

\[
2E_p \frac{dN^{(1)}}{dp} = \sum_f \left| \int d^4x \ T_f^*(x) e^{ip \cdot x} \right|^2.
\]

The definition of the source function for pions is

\[
S(p, x) = \sum_f \int d^4 \delta x \ T_f^*(x + \delta x/2) T_f(x - \delta x/2) e^{-ip \cdot \delta x},
\]

which leads to the simple relation

\[
2E_p \frac{dN^{(1)}}{dp} = \int d^4x \ S(p, x) \bigg|_{p_0 = E_p}.
\]

The source function can be interpreted as the probability per unit space-time for creating a pion of momentum \( p \). Since source functions can be extracted from semiclassical simulations or thermal models, it has proven useful to also express two-particle probabilities in terms of source functions. The two-particle probability requires a two-particle matrix element \( T_f^{(2)}(x_a, x_b) \). Assuming independent, or uncorrelated, emission means that the two-particle element factorizes, \[12\].

\[
T_f^{(2)}(x_a, x_b) \rightarrow T_{f_a}(x_a) T_{f_b}(x_b)
\]

Here, \( a \) and \( b \) label independent sources. The two-particle probability then becomes

\[
(2E_1)(2E_2) \frac{dN^{(2)}}{dp_1 dp_2} = \sum_{a,b,f_a,f_b} \left| \int d^4x_a d^4x_b \ T_{f_a}(x_a) T_{f_b}(x_b) \frac{1}{\sqrt{2}} \bigg\{ \exp[ip_1 \cdot x_a + ip_2 \cdot x_b] + \exp[ip_2 \cdot x_a + ip_1 \cdot x_b] \bigg\} \right|^2
\]

\[
= \sum_{a,b} \left\{ \int d^4x_a d^4x_b \ S_a(p_1, x_a) S_b(p_2, x_b)
\right.
\]

\[
+ \int d^4x_a d^4x_b \ S_a(\bar{p}, x_a) S_b(\bar{p}, x_b) \exp[ip_2 - p_1 \cdot (x_a - x_b)] \bigg\}
\]

\[
= \sum_{a,b} \left\{ I_a(p_1, p_1) I_b(p_2, p_2) + I_a(p_1, p_2) I_b(p_2, p_1) \bigg\},
\]

where \( \bar{p} = (p_1 + p_2)/2 \), and the functions \( I_{a,b} \) are defined as

\[
I_{a,b}(p_1, p_2) = \int d^4x \ S_{a,b} \left( \frac{p_1 + p_2}{2}, x \right) e^{ip_2 - p_1 \cdot x}.
\]

The above allows one to write the correlation function as

\[
C(p_1, p_2) = 1 + \frac{\sum_{a,b} I_a(p_1, p_2) I_b(p_2, p_1)}{\sum_{a,b} I_a(p_1, p_1) I_b(p_2, p_2)}.
\]

This formulation is similar to that originally used by Shuryak \[13\]. Simulations typically provide a sampling of the on-shell source function. The application of Eq. \[8\] to simulations is made difficult because the source functions are evaluated at \( \bar{p}_0 \neq E_{(p_1 + p_2)/2} \) meaning that they require off-shell information. The above formalism can be related to simulations through the smoothness approximation \[14\] \& \[15\].
\[ S_a \left( \frac{p_1 + p_2}{2}, x_0 \right) S_b \left( \frac{p_1 + p_2}{2}, x_0 \right) \rightarrow S_a (p_1, x_0) S_b (p_2, x_0). \]  

(11)

For thermal sources, one can justify a form for the off-shell behavior of \( S \) and the smoothness approximation can be averted. A truly quantum theory would provide the \( T \) matrices that would also allow off-shell evaluation of \( S \), and in fact, formalisms have been developed where classical simulations are augmented by converting the point particles into wave packets [10]. This also allows one to forego the smoothness approximation, but at the price of inserting an ansatz for the quantum behavior that has some peculiarities. We will sidestep the issue of the smoothness approximation in this study as we wish to focus on quantum aspects associated with the propagation of off-shell particles.

When calculating correlation functions from simulations, particles from resonances are usually included in the source function by using the space-time points from which the resonances decay. The lifetime of the resonance affects the correlation function through the exponential decay that is simulated in the transport model. As will be demonstrated later in the paper, the exponential decay law is modified if the dynamics emit resonances with a particular mass or range of masses. In this case, the form of the source function becomes nonexponential as will be explained in the next two sections.

### III. CORRELATIONS FROM RESONANT SOURCES

In this section we present a formalism for calculating two-particle correlation functions from resonance decays given that one or both of the pions might originate from a resonance. The result will depend on the source function that represents the creation of the resonance rather than the source function that represents the points at which the final-state pions are created. The evolution and decay of the resonance will be accounted for through the quantum propagator of the resonance. The space-time point at which the resonance decayed to produce the final-state pions will be treated as an intermediate quantum step in the evolutionary path between the initial creation of the resonance and the asymptotic momentum states of the decay products. The expressions derived here thus incorporate an integration over all points at which the resonance might have decayed relative to the points at which the resonance is created.

We will consider three examples, a hypothetical scalar \( A \) that decays into two pions, \( \rho \), which is a vector resonance also decaying into two pions, and \( \omega \), which is also a vector resonance, but decays into three pions. The matrix element for creating the pion and a second particle with momentum \( k \) through a scalar resonance \( A \) is

\[
T_{\pi,f_A,k}(x) = g \int d^4x_A \tilde{G}_A(x - x_A)T_{A,f_A}(x_A)e^{ik \cdot x},
\]

(12)

where \( \tilde{G}_A \) is the Fourier transform of the propagator for the resonance and \( f_A \) refers to the state of the remainder of the source. Here \( T_A \) is effectively the \( T \)-matrix element for emission of the pion, while \( T_A \) is the \( T \)-matrix element that would describe emission of the resonance if the resonance were stable.

Following the same method as in the previous section, one can use \( T_\pi \) to create the pion source function using Eq. (3) to express \( T_\pi^2(x)T_A(x') \) in terms of the source function of the resonance. The resulting expression for the pion source function can be used to generate \( I_A \), as defined in Eq. (3), which is all that is needed to calculate correlation functions.

\[
I_A(p_1, p_2) = g^2 \int \frac{d^3k}{2E_k} \int d^4x_A \exp[i(p_2 - p_1) \cdot x_A]S_A \left( \frac{p_1 + p_2}{2} + k, x_A \right) G_A(p_1 + k)G_A^*(p_2 + k)
\]

(13)

\[
G_A(p) = \frac{i}{p^2 - M_A^2 + i\Pi_A(p^2)}
\]

(14)

\[
\Pi_A(m^2) = M_A \Gamma_A \frac{q}{q_R} \frac{M_A}{m}
\]

(15)

Here, \( m^2 = p_A^2 \), \( M_A \) and \( \Gamma_A \) are the mass and width of the resonance, respectively. The relative momentum of the outgoing pions in the frame of the resonance is \( q^2 = m^2/4 - m_A^2 \) and \( q_R \) is the same quantity for an on-shell resonance. We emphasize that the interference term can be calculated in Eq. (3) without reference to the direct source of the pions, as the source function of the resonance becomes the required input.

In order to understand the role of the propagators, we reconsider the case of emitting a pion pair, with momenta \( p \) and \( k \) through a single, scalar resonance with momentum \( p_A = p + k \),

\[
2E_p \frac{dN}{d^3p} = I_A(p, p)
\]

(16)
\[ dN = \frac{d^3p}{2E_k} \frac{d^3k}{2E_k} \frac{d^4x}{4\pi^2} \frac{g^2}{|(p + k)^2 - M_A^2 + i\Pi_A((p + k)^2)|^2} \]  
\[ = \frac{d^3p_A}{2E_A} \frac{d^4x}{4\pi^2} S_A(p_A, x) \frac{1}{m^2 - M_A^2 - i\Pi_A(p_A^2)}. \]

The spectral function describes the density of states of resonances of invariant masses \( m \),

\[ \frac{1}{2\pi} \Im \frac{1}{m^2 - M_A^2 - i\Pi_A(p_A^2)} = \frac{dn}{dm^2}. \]

Thus, one can see that the source function does not provide any information regarding the mass or width of the resonance. However, combined with the spectral function, which derives from the product of propagators, it provides the probability of creating the resonance at space time point \( x \) with momentum \( p \) and with invariant mass \( m \). For the direct case considered in the last section, the source function was always evaluated on shell, i.e., the spectral function was effectively a delta function.

One can also perform the same calculation for vector resonances such as the \( \rho \) meson. In that case, where the coupling of the pions to the vector meson is

\[ \mathcal{L}_{\text{int}} = i\lambda (\pi\partial_\mu\pi) \rho^\mu, \]
the expression for \( I_\rho \) is similar to Eq. (13) with the source function and propagator accounting for the vector nature of the \( \rho \),

\[ I_\rho(p_1, p_2) = \lambda^2 \int \frac{d^3k}{2E_k} \int d^4x_\rho \exp[i(p_2 - p_1) \cdot x_\rho](p_1 - k)_{\alpha}G_{\rho\beta}^\alpha(p_1 + k)S_{\beta\gamma}^\rho \left( \frac{p_1 + p_2}{2} + k, x_\rho \right) G_{\rho\delta}^\gamma(p_2 + k)(p_2 - k)_{\delta}. \]

The propagator for the vector resonance is

\[ G_{\rho}^{\alpha\beta}(p) = i \frac{-g^{\alpha\beta} + p^\alpha p^\beta/p^2}{p^2 - M_A^2 + i\Pi_\rho(p^2)}. \]

For the vector case the self-energy scales differently as a function of the resonance mass than in the scalar case,

\[ \Pi_\rho(m^2) = M_\rho\Gamma_\rho \frac{q_\rho^3 M_\rho}{q_\rho^3 M_\rho}, \]

where the same notation as in Eq. (19) was used.

As a third example, we consider the propagation of an \( \omega \) meson, which is also a vector resonance, but decays into three pions through

\[ \mathcal{L}_{\text{int}} = i\kappa\epsilon_{\mu\nu\xi\omega} \partial^\nu \pi^\mu + \partial^\xi \pi^0 \partial^\omega \pi^-. \]

In this case the expression for \( I_\omega \) becomes even more complicated than the \( \rho \) example.

\[ I_\omega(p_1, p_2) = \kappa^2 \int \frac{d^3k}{2E_k} \int d^4x_\omega \exp[i(p_2 - p_1) \cdot x_\omega] \times \epsilon_{\alpha\mu\nu\xi} \hat{P}_\omega^\mu \hat{P}_\omega^\nu \epsilon_\xi G_{\omega}^{\alpha\beta}(p_1 + k + l)S_{\beta\gamma}^\omega \left( \frac{p_1 + p_2}{2} + k + l, x_\omega \right) G_{\omega\delta}^\gamma(p_2 + k + l)\epsilon_{\delta\mu\nu\xi}. \]

The expression for the self-energy is also somewhat more complicated.

\[ \Pi_\omega(m^2) = B \int \frac{d^3k}{2E_k} \frac{d^4l}{2E_l} \delta \left( (m - E_k - E_l)^2 - E_{k+l-1}^2 - 2kl \cos \theta \right) m^2 |\mathbf{k} \times \mathbf{l}|^2. \]
IV. LIMIT OF A NARROW RESONANCE

For this discussion we shall, for simplicity, consider the source function for a narrow Breit-Wigner resonance $A$. Then, according to Eqs. (2) and (12), the probability of emitting a pion pair with momenta $k$ and $p$ is

$$
(2E_k)(2E_p) \frac{dN}{d^3 p \; d^3 k} = \sum_f g \int d^4 x d^4 x_A \; G_A(x-x_A) T_{A,f,k}(x_A)e^{i(p+k) \cdot x} \right|^2
$$

$$
= g^2 \int d^4 x d^4 x \; S_A(p+k,x_A)K(p+k,x-x_A).
$$

Here, $K$ represents the probability of a resonance carrying momentum $p+k$ propagating from $x_A$ to $x$. It may be expressed in terms of the propagators,

$$
K(p,x) = \int \frac{d^4 \delta q}{(2\pi)^4} e^{i\delta q \cdot x} G_A^*(p+\delta q/2)G_A(p-\delta q/2).
$$

In general $K$ is a complicated function. In order to illustrate the quantum nature, we consider the limit of a narrow resonance that allows the propagator to be expressed in a simplified form,

$$
G_A(p) = \frac{1}{2M_A p_0 - E_p + i\Gamma/2}.
$$

Furthermore, we consider the case where the particles are emitted with equal and opposite momentum. Integrating over spatial coordinates gives the probability of the resonance propagating for a time $t$ with off-shellness $\Delta E$,

$$
\mathcal{R}(\Delta E, t) = \frac{2\Gamma M_A^2}{\pi} \int d^3 x \; K(p,x) = \frac{\Gamma}{\pi} \Theta(t) e^{-\Gamma t} \frac{\sin(2\Delta E t)}{\Delta E}.
$$

where $\Delta E = E_k + E_p - M_A$ is the off-shellness. If one integrates over the off-shellness, the expected exponential behavior is obtained.

$$
\int d\Delta E \; \mathcal{R}(\Delta E, t) = \Gamma e^{-\Gamma t},
$$

whereas integrating over $t$ describes the preference for emitting the particle with energy close to on-shell.

$$
\int dt \; \mathcal{R}(\Delta E, t) = \frac{1}{\pi} \frac{\Gamma/2}{\Delta E^2 + (\Gamma/2)^2}.
$$

The oscillating term $\sin(2\Delta E t)$, which is responsible for preferentially emitting resonances with small $\Delta E$, also governs the distribution of emission times. Classical simulations, which are typically based on Monte Carlo algorithms, cannot easily incorporate regions with negative probabilities as suggested by this form. The mean propagation time in a transport simulation could be altered to match the mean time of the quantum propagator,

$$
\langle t \rangle = \frac{\int dt \; \mathcal{R}(\Delta E, t)}{\int dt \; \mathcal{R}(\Delta E, t)} = \frac{\Gamma/2}{\Delta E^2 + (\Gamma/2)^2}.
$$

However, the second moment for the time would not correspond to that expected for an exponential decay with the same average time.

$$
\langle t^2 \rangle = \frac{1}{8} \frac{3\Gamma^2 - 4\Delta E^2}{(\Delta E^2 + \Gamma^2/4)^2} \neq 2\langle t \rangle^2.
$$

Not only is this form inconsistent with exponential decay, $\langle t^2 \rangle$ can even become negative for resonances far off shell. This illustrates, on a formal level, the need for performing the quantum corrections described in this paper. Quantitative comparisons follow in the next section.
Simulations of heavy-ion collisions usually provide the creation points of pions along with their outgoing momenta. Neglecting other interactions besides symmetrization, correlation weights for pions of momenta $p_1$ and $p_2$ originating from space-time points $x_a$ and $x_b$ are usually determined by calculating the average symmetrization weight of all pairs satisfying the imposed binning or acceptance condition.

\[
C(p_1, p_2) = 1 + \frac{\sum_{a,b} \int d^4x_a d^4x_b S_a(p_1, x_a) S_b(p_2, x_b) w_a(p_1, p_2) w_b(p_2, p_1)}{\sum_{a,b} \int d^4x_a d^4x_b S_a(p_1, x_a) S_b(p_2, x_b) w_a(p_1, p_2) w_b(p_2, p_1)}
\]

\[\equiv 1 + \langle w_a(p_1, p_2) w_b(p_2, p_1) \rangle.\]  

(36)

(37)

Inspection of Eqs. (39), (40), and (41) reveals the weight for direct sources,

\[w_d(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_d] \frac{S_A(\frac{p_1 + p_2}{2}, x_d)}{S(p_1, x_d)},\]  

(38)

\[w_d^{(sc)}(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_d],\]  

(39)

where the semiclassical form assumes the smoothness approximation.

If the particle originates from the decay of a scalar resonance $A$, the weight takes on a different form as can be seen from inspecting Eq. (43).

\[w_A(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_A] \frac{S_A(\frac{p_1 + p_2}{2}, k, x_A) (p_1 + k)^2 - M_A^2 - i\Pi_A((p_1 + k)^2)}{S_A(p_1 + k, x_A) (p_2 + k)^2 - M_A^2 - i\Pi_A((p_2 + k)^2)} \cdot \]  

(40)

Here the resonance was created at $x_A$ and decayed into pions of momenta $p_1$ and $k$. The space-time coordinate of the decay does not enter as weight decay points have been considered. If the decay is of a vector resonance such as a $\rho$ meson, the weights are slightly different,

\[w_\rho(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_\rho] \frac{S_\rho(p_1, p_2, k) (p_1 + k)^2 - M_\rho^2 - i\Pi_\rho((p_1 + k)^2)}{S_\rho(p_1, p_2, k) (p_2 + k)^2 - M_\rho^2 - i\Pi_\rho((p_2 + k)^2)} \cdot \]  

(41)

\[S_\rho(p_1, p_2, k, x_\rho) = (p_1 - k)'^\alpha S^\rho_{\alpha\beta}(\frac{p_1 + p_2}{2} + k, x_\rho) (p_2 - k)^\beta.\]  

(42)

In the derivation of $S_\rho$, we have assumed the two pions involved in the resonance decay to have equal mass. Any sort of resonance can be included in this manner, including resonances that decay into three or more bodies. One such example is $\omega$, which decays into three pions, one of each species. Labeling the momenta of the two pions, whose symmetrization we ignore, as $k$ and $l$, the weights turn out to be

\[w_\omega(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_\omega] \frac{S_\omega(p_1, p_2, k, l) (p_1 + k + l)^2 - M_\omega^2 - i\Pi_\omega((p_1 + k + l)^2)}{S_\omega(p_1, p_2, k, l) (p_2 + k + l)^2 - M_\omega^2 - i\Pi_\omega((p_2 + k + l)^2)} \cdot \]  

(43)

\[S_\omega(p_1, p_2, k, l, x_\omega) = e^\alpha_{\mu\nu\xi} P_\mu^k k'\nu'k'^\xi S^\omega_{\alpha\beta}(\frac{p_1 + p_2}{2} + k + l, x_\omega) \epsilon^{\beta}_{\mu'\nu'\xi'} P_\mu^\nu' k'\nu'k'^\xi'.\]  

(44)

Thus, weights can be used to calculate correlation functions for the decay of any resonance in a rather straightforward manner. The formalism coherently accounts for all points at which the resonance may have decayed, but requires information regarding the points at which the resonances were created as well as information about the accompanying particles in the decay. The only difficulty comes in assigning the ratios of the source functions, i.e., the smoothness problem.

For a thermal source, $S_A(p, x) \sim p_0 e^{-p_0/T}$ in the scalar case and $S^\rho_{\alpha\beta}(p, x) \sim p_0 e^{-p_0/T} (-g_{\alpha\beta} + p_\alpha p_\beta/p^2)$ in the vector case. In the thermal cases the Boltzmann factor cancels out of the ratios used to calculate weights. Thus for the thermal example, the weights become a product of four factors, a phase arising from the points at which the resonance $R$ ($= A, \rho, \omega$) is created, a ratio of energies, a spin factor, and a ratio of propagator denominators.

\[w^{(th)}_R(p_1, p_2) = \exp[i(p_1 - p_2) \cdot x_R] \frac{(p_1 + p_2)}{(p_1 + k') \cdot n} \chi_R(p_1, p_2) (p_1 + k')^2 - M_R^2 - i\Pi_R((p_1 + k')^2) \cdot \]  

(45)
where \( n \) refers to the frame of the thermal source and \( k' \) is equal to either \( k \) for a two-body decay, or to \( k + l \) for a three-body decay.

The spin factors \( \chi_R \) depend on the sort of resonance being considered.

\[
\chi_A(p_1, p_2) = 1
\]
\[
\chi_\rho(p_1, p_2) = (p_1 - k)^\alpha \left(-g_{\alpha\beta} + \vec{p}_\alpha \vec{p}_\beta / p^2\right)(p_2 - k)^\beta, \tag{47}
\]
\[
\chi_\omega(p_1, p_2) = e^{i\mu \nu \xi} p_1^\mu k^\nu k^\xi \left(-g_{\alpha\beta} + \vec{p}_\alpha \vec{p}_\beta / p^2\right) e^{\beta \nu \nu \xi^*} p_2^\mu k^\nu k^\xi, \tag{48}
\]
where \( \vec{p} = (p_1 + p_2)/2 + k' \).

However, one cannot easily apply this technique to nonthermal sources because the off-shell behavior of the source function is not always known. This problem also confronts calculations with direct sources, and forces one to either invoke the smoothness approximation, or assume some form for the off-shell behavior of the source function.

In analogy to the smoothness approximation one might use the thermal weight \( w_R^{(th)} \) assuming a reference frame \( n^\mu \) or simply neglect the ratio of energies in Eq. (13). In fact, the ratio \( \chi_R(p_1, p_2)/\chi_R(p_1, p_1) \) can also be neglected to a reasonable approximation as the structure of the correlation function derives largely from the last factor in Eq. (13), the ratio of propagator denominators.

**VI. COMPARISON WITH SEMICLASSICAL MODELS**

In this section we compare calculations of the quantum type described in Sec. III with semiclassical calculations where the resonance is assumed to propagate classically and decay according to an exponential form \( \exp(-t/\tau) \). For the purposes of this comparison we choose to model two simplified systems, one of decaying \( \rho \) resonances and a second of decaying \( \omega \)’s. For each case resonances are produced and decayed with a Monte Carlo procedure according to a thermal distribution characterized by a temperature of 150 MeV. The mass of \( \rho \) is 770 MeV and the width is chosen to be 150 MeV, while the mass of \( \omega \) is 783 MeV and the width is 8.4 MeV.

To model the uncorrelated emission of pion pairs, we thermally create particles of momentum \( k \) (and \( l \) for the \( \omega \)) by Monte Carlo, then add to the particles the weight

\[
W_i = \frac{E_k \{+E_l\} + E_i}{E_k \{E_l\} E_i} \frac{\chi_R(p_1, p_i)}{|(p_i + k')^2 - M_R^2 + i\Pi[(p_i + k')^2]|}, \tag{49}
\]

where \( i = 1, 2 \) and the braces indicate terms that appear for \( \omega \) only. This weight accounts for the spectral function of the resonances as described in Eq. (19) and for the spin factors \( \chi_R \) as in Eqs. (13), (18). We note that the weights \( W_{1,2} \) are merely used to generate resonances and their products, and are not related to the correlation weights. If these weights were included through a keep/reject prescription, they would not need to appear in any of the following expressions.

Once the pions are statistically generated, one simply calculates the average weights described previously to generate the correlation functions.

\[
C(p_1, p_2) = 1 + \Re \left( \sum_a W_{1a} w_R^{(th),a}(p_1, p_2) \sum_b W_{2b} w_R^{(th),b}(p_2, p_1) \right) / \left( \sum_a W_{1a}^2 \sum_b W_{2b}^2 \right). \tag{50}
\]

For our comparison \( w_R \) is either the weight for \( \rho, w_\rho \), or the one for \( \omega, w_\omega \). Note that \( a \) and \( b \) label individual resonances.

In the semiclassical descriptions, the weights are determined by calculating the expectation value

\[
w_R^{(sc)}(p_1, p_2, x_R) = \langle e^{i(p_1 - p_2) \cdot x_R} \rangle
\]
\[
= \exp[i(p_1 - p_2) \cdot x_R] \int d^4(x - x_R) \delta^3[x - x_R - \nu_R(t - t_R)] \exp[i(p_1 - p_2) \cdot (x - x_R)]
\]
\[
\times \frac{1}{\gamma_R \tau} \exp[-(t - t_R)/\gamma_R \tau] \Theta(t - t_R)
\]
\[
= \exp[i(p_1 - p_2) \cdot x_R] \frac{m_R/\gamma_R \tau}{m_R/\gamma_R \tau + i\nu_R \cdot (p_1 - p_2)}. \tag{53}
\]
where \( \gamma_R \) is the Lorentz factor due to the motion of the resonance. Here, \( w^{(sc)} \) assumes an exponential form for the pion emission, which is characterized by a lifetime \( \tau \). The same form for exponential decays was developed by Padula and Gyulassy [17].

Based on one’s perspective, one might choose any of several prescriptions for the energy dependence of the lifetime \( \tau(m) \). We investigate three possibilities: (1) The lifetime is chosen such that \( m/\tau = \Pi(m^2) \). This choice would be motivated by the form of the propagator. (2) The lifetime is chosen to correspond to the average emission time as described in Sec. [V], except that the relativistic generalization of Eq. (24) is used \( \tau = 2m \Im(m^2 - M_R^2 - \Pi)^{-1} \). (3) A fixed lifetime \( 1/T \) is used.

If the resonance distribution function in coordinate space is independent from that in momentum space, the interference term in the correlation function factorizes into a term stemming from the space-time extent of the resonance source itself and one that arises from the decay process.

\[
C(p_1, p_2) - 1 = \Re \{ \langle \exp[i(p_1 - p_2) \cdot (x_A - x_B)] \rangle (C'(p_1, p_2) - 1) \}
\]

Here, \( x_A \) and \( x_B \) refer to points at which the resonances are created, and \( \langle \exp[i(p_1 - p_2) \cdot (x_A - x_B)] \rangle \) represents the weighted average using the product of the source functions as the weight.

\[
\langle \exp[i(p_1 - p_2) \cdot (x_A - x_B)] \rangle = |J_R(p_1 - p_2)|^2
\]

\[
J_R(p_1 - p_2) = \frac{\sum_A |d^4x A| S_A((p_1 + p_2)/2 + k_A, x_A) \exp[i(p_1 - p_2) \cdot x_A]}{\sum_A |d^4x A| S_A((p_1 + p_2)/2 + k_A, x_A)}
\]

The reduced correlation function, \( C'(p_1, p_2) - 1 \), is similar to the average of the weights \( \langle w_a w_b \rangle \) from Sec. [V] only with the factors of \( \exp[i(p_1 - p_2) \cdot x_R] \) removed from the weights in Eq. (52). Since \( C' \) contains all the relevant information about the decay, we will focus on the reduced correlation function for our comparisons.

Figures 3 and 4 display the reduced correlation function for the cases where \( (p_1 + p_2)/2 = 200 \text{ MeV}/c \) and 800 MeV/c, respectively. The upper panel of each figure displays results for the case where \( p_1 \) is parallel to \( p_2 \) while the lower panel displays the results for the case where the two momenta are perpendicular. All three semiclassical results exhibit significant deviations from the quantum calculations.

Figures 5 and 6 display the same information but for a thermal source of \( \omega \) mesons. In this case, the semiclassical treatments are significantly more accurate. This was expected as the width of the \( \omega \) is much less than the temperature, which allows the spectral function of the \( \omega \) to be sampled evenly. As described in Sec. [V], the distribution of decay times become exponential when evenly averaged over all masses.

The overall width of the correlation functions in the upper panels of Figs. 3 and 4 can be understood by noting that the correlation’s width should be determined by the condition \( \Delta E \tau = 1 \), where \( \Delta E \approx v_{RQ} \). The width of the correlation function for the lower panels, where \( p_1 \perp p_2 \), is more complicated since it is more sensitive to the spatial movement of the resonance while it decays. The correlation functions for the \( \rho \) in Figs. 1 and 2 are even more complicated since they extend to large relative momenta where \( \Delta E \) approaches \( q \). Given the complicated kinematics involved, it is not surprising that the result is sensitive to the exact form for the semiclassical treatment.

In the case of the \( \rho \), none of the three semiclassical prescriptions for \( \tau(m) \) provides a consistently good approximation to the quantum result for all scenarios shown in Figs. 1 and 2. They all exhibit significant deviations from the quantum result for high relative momentum of the pion pair, above 200 MeV/c. In the case of the \( \omega \), where only small relative momenta, \( q < 200 \text{ MeV}/c \), are relevant, prescription (1) seems to best reproduce the quantum result, as can be seen in Figs. 3 and 4.

It should be emphasized that the correlation from the non-zero extent of the resonance source function has been factored out in this calculation. The deviations of the semiclassical results occur for relative momenta of a hundred MeV/c or more. In a heavy-ion collision the factor \( \langle \exp[i(p_1 - p_2) \cdot (x_A - x_B)] \rangle \) in Eq. (53) would tend to zero for relative momenta much greater than 50 MeV/c due to the large spatial sizes of the emitting regions. Thus, the form of \( C' \) becomes irrelevant for higher relative momenta unless the source sizes are small, e.g., pp collisions.

VII. CONCLUSION

Our findings imply that the semiclassical treatments work quite well for the larger sources considered with heavy-ion collisions. However, for smaller sources, especially when the resonance widths are comparable to the temperature and resonances are produced far off-shell, the semiclassical treatments significantly deviate from the quantum result. This can be linked to the failure of the usual exponential decay law when off-shell resonances are involved as was shown in Sec. [V].
In Sec. V, we proposed a method to correctly calculate correlation functions from semiclassical models, exploiting the source function of the resonance and its creation point in space-time as opposed to the creation points of the final-state pions. This method could be easily applied to generate correlation functions from the event histories of simulations. When using direct pions, the creation points of the final-state pions provide all the necessary information for creating correlation functions. By considering the creation points of the resonances that decayed into the final-state pions, one is able to coherently account for all space-time points at which the resonance might have decayed by modifying the prescription for generating correlation weights as described in Sec. V.

It should be emphasized that the quantum formalism discussed here becomes important only for sources that themselves are quantum-mechanical in nature. If the source is large and the product of the momentum and spatial uncertainties are large, $\Delta p \Delta x \gg \hbar$, the behavior of the correlation function is dominated by the term $e^{i q (x_A - x_B)}$, which is determined by the points at which the resonance is created. Hence $C'$ plays little role in the correlation function of the $\rho$ resonance described in Fig. 1 when the overall source size is many Fermi as is the case for heavy-ion collisions. Quantum considerations in resonant decays could play an important role when considering the decay of small sources that push the limits of the uncertainty principle. However, such sources are also accompanied by questions regarding the quantum nature of the source functions responsible for initial creation of the resonances, i.e., the smoothness approximation might not be justified. For such problems, unless one knows the off-shell behavior of the source functions as is the case for a thermal model, the quantum treatments presented here address only half the problem.

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FIG. 1. Reduced correlation function for a source of $\rho$ mesons at a temperature of 150 MeV. The average momenta of the two pions is fixed at 200 MeV/c. By factoring out the space-time dependence of the $\rho$, the manifestation of the $\rho$ lifetime is singled out. The exact quantum treatment is shown to differ from the three semiclassical treatments that are described in the text. The failure of the semiclassical descriptions owes itself to the fact that the $\rho$ width is sufficiently large for the thermal source to effectively emphasize a specific region of off-shellness.
FIG. 2. Same as Fig. 1, except that $\rho$'s have average momentum $P = 800$ MeV/c.
FIG. 3. Same as Fig. 1, but using a thermal source of $\omega$'s at a temperature of 150 MeV. In this case, since the width of the $\omega$ is much less than the temperature, semiclassical treatments work remarkably well.
FIG. 4. Same as Fig. 3 except that $\omega$'s have average momentum $P = 800$ MeV/c.