SO AND USp (HYPER)KÄHLER QUOTIENTS AND LUMPS

SVEN BJARKE GUDNASON

Department of Physics, University of Pisa,
INFN, Sezione di Pisa,
Largo Pontecorvo, 3, Ed. C, 56127 Pisa, Italy
E-mail: gudnason(at)df.unipi.it

The properties of supersymmetric gauge theories in the Higgs phase at low energies can appropriately be studied by means of a non-linear $\sigma$ model, which has the target space being Kähler for $\mathcal{N}=1$ supersymmetric models and hyperKähler for $\mathcal{N}=2$ models. By construction of the Kähler and hyperKähler quotients for the gauge theories with SO and USp gauge symmetries, we obtain the explicit metrics on their respective manifolds. Furthermore, we study the lumps in the non-linear $\sigma$ models and their effective description, using the Kähler quotients.

Keywords: Low energy effective theories, non-linear $\sigma$ model lumps, (hyper)Kähler quotients

1. Introduction

The target space of the $\mathcal{N}=2$ supersymmetric non-linear sigma model (NL$\sigma$M), with eight supercharges, must be hyperKähler. By using this fact, the notion of the hyperKähler quotient was first found in physics and was later formulated mathematically. A $U(1)$ hyperKähler quotient recovers the Calabi metric on the cotangent bundle over the projective space, $T^*\mathbb{C}P^{N-1}$, while its $U(N)$ generalization leads to the cotangent bundle over the complex Grassmann manifold, $T^*G_{M,N}$. The hyperKähler quotient offers a powerful tool to construct hyperKähler manifolds such as instanton moduli spaces, gravitational instantons and monopole moduli spaces.

The Higgs branch of $\mathcal{N}=2$ supersymmetric QCD is hyperKähler. The low energy effective theory on the Higgs branch is described by an $\mathcal{N}=2$ NL$\sigma$M on the hyperKähler manifold. In the cases of an $SU(N)$ or a $U(N)$ gauge theory with hypermultiplets charged commonly under $U(1)$, the explicit metrics on the Higgs branch and their Kähler potentials are
known explicitly. The latter is nothing but the Lindström-Roček metric.\cite{4,12} A $U(1)\times U(1)$ gauge theory for instance gives the space: $T^*F_n$ with $F_n$ being the Hirzebruch surface.\cite{13}

This contribution has two main concerns, the first is the construction of the metric and Kähler potential on the Higgs branch of $\mathcal{N}=2$ supersymmetric gauge theories with $SO(2N)$, $USp(2N)$, $U(1)\times SO(2N)$, $U(1)\times USp(2N)$ gauge groups.\cite{1}

The second motive is concerned with $\sigma$ model lumps, or $\sigma$ model instantons. A lump solution was first found in the $O(3)$ sigma model, or the $\mathbb{C}P^1$ model.\cite{14} It was then generalized to the $\mathbb{C}P^n$ model and the Grassmann model. Gauge theories coupled to several Higgs fields often admit semi-local vortex-strings.\cite{15} In the strong gauge coupling limit, the gauge theories reduce to NL$\sigma$Ms and in this limit, semi-local strings reduce to lump-strings. In the gauge theories at finite couplings, the large distance behavior of semi-local strings is well approximated by lump solutions. It was demonstrated in Ref. 16 that non-Abelian semi-local strings\cite{17,18} in a $U(N)$ gauge theory reduce to the Grassmann lumps at large distance.

This work has been done in collaboration with M. Eto, T. Fujimori, K. Konishi, T. Nagashima, M. Nitta, K. Ohashi and W. Vinci. Many details are left in the Ref. 1.

2. Obtaining the Low Energy Effective Theory

Obtaining the low-energy effective theory of supersymmetric gauge theories on the Higgs branch has been well studied in the literature.

To obtain the target space, we can do one of the following\cite{19}

(i) Fix the gauge to the Wess-Zumino gauge and find the potential zeroes $(D = 0, \ F = 0)$ and then mod out the remaining gauge group.

(ii) Take the infinite gauge coupling limit immediately and then mod out the full complexified gauge group.

(iii) Construct all gauge invariants and find all relations between them. This set constitutes the target space.

In the next Section we will construct the metrics following the method (ii) and rewrite them according to method (iii) for the gauge theories with $SO(N_c)$ and $USp(2M_c)$ gauge groups. Similarly, the result can be transformed onto the form of method (i).
3. The $SO(N_c)$ and $USp(2M_c)$ Kähler Quotients

The Kähler potential for an $SO(N_c)$ or a $USp(2M_c)$ gauge theory is given by

$$K_{SO,USp} = \text{Tr} \left[ Q Q^\dagger e^{-V'} \right],$$

where $V'$ takes a value in the $so(N_c)$ or $usp(2M_c)$ algebra and hence satisfies

$$V'^T J + JV' = 0 \leftrightarrow e^{-V'^T} J e^{-V'} = J.$$  \hspace{1cm} (2)

Note that, this condition implies that $\det(e^{-V'}) = 1$. Here the matrix $J$ is the invariant tensor of the $SO$ or $USp$ group, $g^T J g = J$ with $g \in SO(N_c), USp(2M_c)$, satisfying

$$J^T = \epsilon J, \quad J^\dagger J = 1_N, \quad \epsilon = \begin{cases} +1 & \text{for } SO(N_c), \\ -1 & \text{for } USp(N_c = 2M_c). \end{cases}$$

Conversely, a matrix $J$ satisfying the above equations defines a representation of the $SO$ and $USp$ groups. We will (mainly) use the convention (in the even case)

$$J = \begin{pmatrix} 0_{M_c} & 1_{M_c} \\ \epsilon 1_{M_c} & 0_{M_c} \end{pmatrix}.$$  \hspace{1cm} (3)

First we will discuss the breaking pattern of the gauge symmetry and the flat directions of the vacuum. For this we will consider both the gauge and the global symmetries. The vacuum expectation value of $Q_{wz}^{SO}$ in the case of $SO(N_c)$ can be put on diagonal form after fixing both the local and the global symmetry

$$Q_{wz}^{SO} = \left( A_{N_c \times N_c}, 0_{N_c \times (N_f - N_c)} \right),$$

with $A_{N_c \times N_c} = \text{diag}(a_1, a_2, \cdots, a_{N_c})$, $J = 1_{N_c}$,

where all the parameters $a_i$ are taken to be real and positive, which indeed parametrize flat directions of the Higgs branch. In generic points on the vacuum manifold with non-degenerate $a_i$, there is no global symmetry in the vacuum and the flavor symmetry is $U(N_f)$ apart from $U(N_f - N_c)$ which freely acts on the vacuum configuration. At a generic point, the vacuum manifold can be written as

$$M_{SO(N_c)}^{\text{gen}.K} \simeq \mathbb{R}_{\geq 0}^{N_c} \times \frac{U(N_f)}{U(N_f - N_c)}.$$  \hspace{1cm} (3)

\footnote{In the $SO(N_c)$ cases, we remove an integral region with $\det e^{-V} = -1$ in the functional integral of $V$.}
The flat directions $\mathbb{R}_{\geq 0}^{N_c}$ have no origin due to symmetry breaking, and their coordinates $\{a_i\}$ are quasi Nambu-Goldstone (NG) modes. When two parameters coincide, $a_i = a_j, (i \neq j)$, a color-flavor locking symmetry $SO(2)$ emerges. In such degenerate points on the manifold, the above quotient space attached to $\mathbb{R}_{\geq 0}^{N_c}$ shrinks to a space with a smaller dimension. The existence of quasi-NG modes are strongly related to the emergence of an unbroken phase (classically). When any two of the $a_i$'s vanish, an $SO(2)$ subgroup of the gauge symmetry remains unbroken. Thus, in the Higgs phase with completely broken gauge symmetry, the rank of $Q_{wx}$ has to be greater than $N_c - 2$. We will only consider this latter case here, namely the models with $N_f \geq N_c - 1$.

For the $USp(2M_c)$ case, it is known that the flat directions are parametrized by

$$Q_{wx}^{USp} = (A_{M_c \times M_c}, 0_{M_c \times (M_f - M_c)}) \otimes 1_2,$$

where the number of flavors is even $N_f = 2M_f$. Even in generic points with non-degenerate $\{a_i\}$, the color-flavor symmetry $USp(2)^{M_c} \simeq SU(2)^{M_c}$ remains unbroken in the vacuum. Therefore, the vacuum manifold can, at generic points, be written as

$$M_{USp(2M_c)}^{\text{generic},K} \simeq \mathbb{R}_{\geq 0}^{M_c} \times \frac{U(N_f)}{SU(2)^{M_c} \times U(N_f - 2M_c)},$$

except for sub-manifolds where the quotient space shrinks. In this case the completely broken gauge symmetry needs $M_f \geq M_c$.

The $D$-flatness conditions in the Wess-Zumino gauge (i.e. method (i)) are

$$D^A = \text{Tr}_f \left( Q_{wx}^\dagger T^A Q_{wx} \right) = 0,$$

with $T_A$ being the generators in the Lie algebra $\mathfrak{so}$ or $\mathfrak{usp}$. However, these conditions are rather difficult to solve. Without taking the Wess-Zumino gauge, we can eliminate the superfield $V'$ directly within the superfield formalism by using the following trick; let us consider $V'$ taking a value in a larger algebra, namely $\mathfrak{u}(N_c)$ and then introduce an $N_c \times N_c$ matrix of Lagrange multipliers $\lambda$ to restrict $V'$ to take a value in the $\mathfrak{so}(N_c)$ or the $\mathfrak{usp}(N_c)$ algebras.

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*b*To our knowledge the $D$-flatness conditions have not been solved in the case of an $SO$ or a $USp$, $\mathcal{N} = 1$ supersymmetric gauge theory.

*c*Hermiticity of $\lambda$ is defined such that $\lambda e^{-V'^T} J$ is a vector superfield, that is, $\lambda^\dagger = e^{V'^T} J \lambda e^{-V'^T} J$. 
usp(N_c = 2M_c) sub-algebra which leaves us with

\[ K_{SO,USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} + \lambda \left( e^{-V'T} J e^{-V'} - J \right) \right], \]  

(5)

where \( Q \) are \( N_f \) chiral superfields as earlier and \( V' \) is a vector superfield of \( U(N_c) \). The added term breaks the complexified gauge transformations down to \( SO(N_c) \), \( USp(2M_c) \) and the equations of motion for \( \lambda \) gives the constraint (2) which reduces the Kähler potential (5) back to Eq. (1). The equation of motion for \( V' \) takes the form

\[ QQ^\dagger e^{-V'} + \left( \lambda + \epsilon \lambda^T \right) J = 0, \]

where we have used (2). \( \lambda \) can be eliminated by combining the equations of motion with their transposed ones. The resultant equation contains the square of the manifest Hermitian and positive definite matrix \( X \) that traces to the Kähler potential (1) that we set out to find:

\[ X^2 = \left( Q^T J \sqrt{QQ^\dagger} \right)^\dagger \left( Q^T J \sqrt{QQ^\dagger} \right), \quad X \equiv \sqrt{QQ^\dagger} e^{-V'} \sqrt{QQ^\dagger}. \]

We can uniquely obtain \( V' \) from \( X \) if and only if \( \text{rank} M > N_c - 2 \), where \( M \) are the holomorphic invariants for the \( SO,USp \) gauge theories, that is, the vacuum is in the full Higgs phase. See Appendix B in Ref. 1 for a uniqueness proof, in the case of \( \text{rank} M = N_c - 1 \). It is possible to switch to \( Q_{xz} \) from \( Q \) by the complexified gauge transformation \( Q_{xz} = u'^{-1} Q \) with \( uu'^T = e^{V'} \). Without using an explicit solution for \( V' \), we obtain the Kähler potential of the NLσM (according to method (ii))

\[ K_{SO,USp} = \text{Tr} X = \text{Tr} \sqrt{Q^T J \sqrt{QQ^\dagger} \left( Q^T J \sqrt{QQ^\dagger} \right)^\dagger}. \]  

(6)

Now we can naturally switch to an expression according to method (iii) for this NLσM. With the help of \( \text{Tr}_\epsilon \sqrt{AA^\dagger} = \text{Tr}_\epsilon \sqrt{A^T A} \), one can rewrite the Kähler potential (6) as follows

\[ K_{SO,USp} = \text{Tr} X = \text{Tr} \sqrt{MM^\dagger}, \quad M^T = \epsilon M, \]  

(7)

where \( M \) is nothing but the holomorphic invariants of the gauge symmetry

\[ M \equiv Q^T J Q, \quad B^{(A)} \equiv \det Q^{(A)}. \]

The first one is the “mesonic” invariant while the second is the “baryonic” one which appears for \( N_f \geq N_c \) and \( \langle A \rangle \equiv \langle A_1, \ldots, A_{N_c} \rangle \) and \( Q^{(A)} \) is the \( N_c \)-by-\( N_c \) minor matrix, i.e. \( (Q^{(A)})_{i,j} \equiv Q_{i,A_j} \) with \( A_j \in [1, N_f] \). The two kinds
of invariants should be subject to constraints in order to correctly express the NL$\sigma$M. There are relations between the mesons and the baryons:

\[
SO(N_c) : \det(J) B^{(A)} B^{(B)} = \det M^{(A)(B)},
\]

\[
USp(2M_c) : Pf(J) B^{(A)} = Pf M^{(A)(A)}.
\]

The Plücker relation among the baryonic invariants $B^{(A)}$ is derived from the above relation. Actually, from the invariants $M$ and $B^{(A)}$ with the constraints we can reconstruct $Q$ modulo the complexified gauge symmetry as follows. By using an algorithm similar to the Cholesky decomposition of an Hermitian matrix, we show in Ref. 1 that

An arbitrary $n \times n$ (anti-)symmetric complex matrix $X$ can always be decomposed as $X = p^T J p$ with $p$ being a rank$(X) \times n$ matrix.

In the $USp$ case, with a decomposition of the meson $M$, we can completely reconstruct $Q$ modulo $USp(2M_c)$ transformations. This fact corresponds to that there are no independent baryons $B^{(A)}$ in a $USp(2M_c)$ theory and only the meson fields describe the full Higgs phase

\[
\mathcal{M}_{USp} = \{ M \mid M \in N_f \times N_f \text{ matrix}, \quad M^T = -M, \quad \text{rank} \ M = 2M_c \}.
\]

On the contrary, in the $SO(N_c)$ case, a decomposition of $M$ gives $Q$ modulo $O(N_c)^C$ and one finds two candidates for $Q$ since $\mathbb{Z}_2 \simeq O^C/SO^C$ which is fixed by the sign of the baryons.\(^d\) Therefore we have to take the degrees of freedom of the baryons into account to consider the full Higgs phase

\[
\mathcal{M}_{SO} = \{ M, B^{(A)} \mid M : \text{symmetric} \ N_f \times N_f, \quad \text{Relation } (8), \quad N_c - 1 \leq \text{rank} \ M \leq N_c \}.
\]

For large $N_c$, it is a hard task to obtain an explicit metric from the formula (7), since we need to calculate the eigenvalues of $MM^T$. In Ref. 1 we calculate explicitly an expansion of the Kähler potential around its vacuum value from which we are able to obtain the metric and curvature.

4. The $U(1) \times SO(N_c)$ and $U(1) \times USp(2M_c)$ Kähler Quotients

Next, we would like to consider a Kähler quotient by gauging an overall $U(1)$ phase in addition to the $SO(N_c)$ or $USp(2M_c)$ gauge symmetry. We

\(^d\)In the case of rank $M = N_c - 1$, $g \in \mathbb{Z}_2$ acts trivially on $Q$ as $gQ = Q$, although all the baryons vanish.
turn on the FI $D$-term associated with the additional $U(1)$ gauge group. The Kähler potential can be written as

$$K_{U(1) \times (SO, USp)} = \text{Tr} \left[ QQ^\dagger e^{-V'} e^{-V_c} + \lambda \left( e^{-V'^T} J e^{-V'} - J \right) \right] + \xi V_c ,$$

where $V_c$ is the vector multiplet of the additional $U(1)$ gauge field. We have already solved the $SO(N_c)$ and $USp(2M_c)$ part in the previous section, so the Kähler potential can be rewritten as

$$K_{U(1) \times (SO, USp)} = \text{Tr} \left[ \sqrt{MM^\dagger} \right] e^{-V_c} + \xi V_c .$$

The equation of motion for $V_c$ can be solved by $V_c = log \left[ \text{Tr} \left( \sqrt{MM^\dagger} \right) / \xi \right]$. Plugging this into the Kähler potential, we obtain

$$K_{U(1) \times (SO, USp)} = \xi \log \left[ \text{Tr} \left( \sqrt{MM^\dagger} \right) \right] , \quad M \equiv Q^T J Q . \quad (10)$$

5. The $SO(N_c)$, $USp(2M_c)$ HyperKähler Quotients

Our next task is lifting up the $SO(N_c)$ and $USp(N_c = 2M_c)$ Kähler quotients to hyperKähler quotients. In order to construct the $SO(N_c), USp(2M_c)$ hyperKähler quotients we need to consider $\mathcal{N} = 2$ hypermultiplets. Hence, we consider an $\mathcal{N} = 2$ extension of the $\mathcal{N} = 1$ Kähler potential (5), together with the superpotential

$$\tilde{K}_{SO, USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} + \tilde{Q}^\dagger \tilde{Q} e^{-V'} + \lambda \left( e^{-V'^T} J e^{-V'} - J \right) \right] , \quad (11)$$

$$W = \text{Tr} \left[ Q \tilde{Q} \Sigma' + \chi \left( \Sigma'^T J + J \Sigma' \right) \right] , \quad (12)$$

where $(V', \Sigma')$ denote the $SO(N_c)$ and $USp(2M_c)$ vector multiplets, $(Q, \tilde{Q})$ are $N_f$ hypermultiplets in the fundamental representation of $SO(N_c)$ or $USp(2M_c)$, and $(\lambda, \chi)$ are the Lagrange multipliers which are $N_c \times N_c$ matrix valued superfields. We can rewrite the Kähler potential (11) as follows

$$\tilde{K}_{SO, USp} = \text{Tr} \left[ QQ^\dagger e^{-V'} \right] , \quad \text{with} \quad Q \equiv \left( Q, J \tilde{Q}^T \right) ,$$

where we have used that $e^{V'^T} = J^T e^{-V'} J$. This trick relates the superfields in the anti-fundamental representation with those of the fundamental representation via the algebra. This Kähler potential is nothing but the $\mathcal{N} = 1$ Kähler potential of $SO(N_c)$ and $USp(2M_c)$ with $Q$ being a set of $2N_f$ chiral superfields. We can straightforwardly borrow the result of Sec. 3 and hence the Kähler potential reads

$$\tilde{K}_{SO, USp} = \text{Tr} \left[ \sqrt{M M^\dagger} \right] , \quad \mathcal{M} \equiv Q^T J Q . \quad (13)$$
The constraint coming from the superpotential (12) is
\[ Q \tilde{J} Q^T = 0, \quad \text{with} \quad \tilde{J} \equiv \begin{pmatrix} 0_{N_f} & 1_{N_f} \\ -\epsilon 1_{N_f} & 0_{N_f} \end{pmatrix}. \]
Therefore, we again find the constraints for the meson field \( \mathcal{M} \)
\[ \mathcal{M}^T = \epsilon \mathcal{M}, \quad \mathcal{M} \tilde{J} \mathcal{M} = 0, \quad N_c - 2 < \text{rank} \mathcal{M} \leq N_c. \]
As is well-known, the \( SO(N_c) \) case has a \( USp(2N_f) \) flavor symmetry while
the \( USp(2M_c) \) case has an \( SO(2N_f) \) flavor symmetry.
Like in the case of the Kähler manifolds, the vacuum manifolds in the
hyperKähler case can be written down for a generic point, which for the
\( SO(N_c) \) case contains the space of Eq. (3)
\[ \mathcal{M}_{\text{generic,HK}}^{SO(N_c)} \cong \mathbb{R}^{N_c} \times \frac{USp(2N_f)}{USp(2N_f - 2N_c)} \supset \mathcal{M}_{\text{Kähler submfd.}}^{SO(N_c)}. \]
Similarly, in a generic point on the vacuum manifold of the \( USp(2M_c) \)
hyperKähler case, we can write\n\[ \mathcal{M}_{\text{generic,HK}}^{USp(2M_c)} \cong \mathbb{R}^{M_c} \times \frac{SO(2N_f)}{SO(2N_f - 4M_c) \times SU(2)} \supset \mathcal{M}_{\text{Kähler submfd.}}^{USp(2M_c)}, \]
where Eq. (4) is a special Lagrangian sub-manifold of the hyperKähler
manifold and analogously for the \( SO \) case.
Let us make a comment on the hyperKähler quotient of the \( U(1) \times \)
\( SO(N_c) \) and \( U(1) \times USp(2M_c) \) theories. We succeeded in constructing the
hyperKähler quotient of \( SO(N_c) \) and \( USp(2M_c) \) thanks to the fact that
\( JQ^T \) is in the anti-fundamental representation, which is the same repre-
sentation as \( Q \). Although, we want to make use of the same strategy for
\( U(1) \times SO(N_c) \) and \( U(1) \times USp(2M_c) \) as before, \( JQ^T \) still has charge \(-1\)
with respect to the the \( U(1) \) gauge symmetry while \( Q \) has \( U(1) \) charge
\(+1\). Therefore, it is not an easy task to construct the \( U(1) \times SO(N_c) \) and
\( U(1) \times USp(2M_c) \) hyperKähler quotients.

6. 1/2 BPS states: NLσM Lumps
In this section, we will study the \( \sigma \) model lumps which are 1/2 BPS states.
Lumps are stringy topological defects in \( d = 1 + 3 \) dimensional spacetime
and are supported by the non-trivial homotopy group \( \pi_2(\mathcal{M}) \) associated
with a holomorphic mapping from the 2 dimensional spatial plane \( z = x_1 + ix_2 \) to the target space of the NLσMs.
Let us concentrate on $U(1) \times G'$ Kähler quotient models as NL$\sigma$Ms. In these cases, (inhomogeneous) complex coordinates on the Kähler manifold $\{\phi^\alpha\}$, which are the lowest scalar components of the chiral superfields, are given by some set of holomorphic $G'$ invariants: $I^i$ modulo $U(1)^C$, $\phi^\alpha \in \{I^i\} \backslash U(1)^C$. Lump solutions can be obtained by just imposing $\phi^\alpha$ to be a holomorphic function with respect to $z$

$$\phi^\alpha(t, z, \bar{z}, x^3) \to \phi^\alpha(z; \varphi^i) ,$$

where $\varphi^i$ denote complex constants. The mass of the lumps can be obtained by plugging the solution back into the Lagrangian

$$E = 2 \int C K_{a\bar{b}}(\phi, \bar{\phi}) \left. \frac{\partial \phi^a}{\partial \phi^a} \frac{\partial \bar{\phi}^\bar{\beta}}{\partial \bar{\phi}^\bar{\beta}} \right|_{\phi \to \phi(z)} .$$

We would like to stress that all the parameters $\varphi^i$ are nothing but the moduli parameters of the 1/2 BPS lumps.

We assume that the boundary i.e. $z = \infty$ is mapped to a point $\phi^\alpha = \phi^\alpha_{\text{vev}}$ on the vacuum manifold in a lump solution. Since the functions $\phi^\alpha(z)$ should be single valued, $\phi^\alpha(z)$ can be expressed with a finite number of poles as

$$\phi^\alpha(z) = \phi^\alpha_{\text{vev}} + \sum_{i=1}^{k} \frac{\phi^\alpha_i}{z - z_i} + O(z^{-2}) .$$

Strictly speaking, we have to change patches of the manifold at the poles in order to describe the solutions correctly. To pick up such global information of lumps thoroughly, it is convenient to use the holomorphic invariants $I^i$ as homogeneous coordinates. By fixing some components of $I^i$ to be constants, we can construct the invariants $I^i$ in terms of $\phi^\alpha(z)$ and find that $I^i$ also be holomorphic functions $I^i(z) = I^i_{\text{vev}} + O(z^{-1})$. We can redefine the functions $I^i(z)$ by using $U(1)^C$ transformations $I^i(z) \simeq I^i(z) = (z^\nu)^{n_i} I^i(z)$, such that all the invariants $I^i(z)$ are polynomials

$$I^i(z) = I^i_{\text{vev}} z^{n_i \nu} + O(z^{n_i \nu - 1}) ,$$

where $n_i$ is the $U(1)$ charge of the holomorphic invariant $I^i$ and $n_i \nu \in \mathbb{Z}_{>0}$. These polynomials are basic tools to study lump solutions and their moduli, and $\phi^\alpha(z)$ can be written as ratios of these polynomials, which are known as rational maps in the Abelian case. Here we assume that the invariants $I^i(z)$ have no common zeroes, in order to fix $U(1)^C$, $I^i(z) \simeq (z - a) I^i(z)$.

If a common zero accidentally emerges by varying the moduli parameters, the behavior of lumps cannot be defined from the viewpoint of the NL$\sigma$M, since a common zero corresponds to the Coulomb phase for $U(1)$ in the
original gauge theory. This can also be understood as the emergence of a local vortex (see Ref. 1 for details).

Using the so-called moduli matrix, which describes different BPS solitons in supersymmetric gauge theories, we can indeed identify the lowest component \( \phi^\alpha \) with the moduli matrix. The key observation is that the gauge symmetry \( G \) in the supersymmetric theory is naturally complexified: \( G^C \). Hence, the moduli matrix naturally appears in the superfield formulation, while if we fix \( G^C \) in the Wess-Zumino gauge, the scalar field \( Q_{wz} \) appears as the usual bosonic component in the Lagrangian.

### 7. Effective Action of Lumps

Now we have a great advantage, thanks to the above superfield formulation of the NL\( \sigma \)Ms. A supersymmetric low energy effective theory of the 1/2 BPS lumps is immediately obtained merely by plugging the 1/2 BPS solution (14) into the Kähler potential which we have obtained in the previous section after promoting the moduli parameters \( \varphi^i \) to fields on the lump world-volume

\[
\phi^\alpha(t, z, \bar{z}, x^3) \rightarrow \phi^\alpha(z; \varphi^i(t, x^3)).
\]

The resulting (effective) expression for the Kähler potential is

\[
K_{\text{lump}} = \int_{\mathbb{C}} K \left( \phi(z, \varphi^i(t, x^3), \varphi^i(z; t, x^3) \right).
\]

### 8. Lumps in \( U(1) \times SO(N_c) \) Kähler Quotients

Let us start with a very simple example of a theory with the gauge group \( U(1) \times SO(2) \) and only two flavors \( N_f = 2 \). The target space is \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). Lump solutions are classified by a pair of integers \( (k_+, k_-) \) and

\[
\pi_2 \left( \mathcal{M}_{N_f=2}^{U(1) \times SO(2)} \right) = \mathbb{Z}_+ \otimes \mathbb{Z}_-,
\]

where \( \mathbb{Z}_\pm \) denote integers. A solution with \( (k_+, k_-) \) lumps is given by

\[
Q(z) = \begin{pmatrix}
Q_{+1}(z) & Q_{+2}(z) \\
Q_{-1}(z) & Q_{-2}(z)
\end{pmatrix} = \begin{pmatrix}
\bar{Q}_+ \\
\bar{Q}_-
\end{pmatrix},
\]

where \( Q_{\pm i}(z) \) are holomorphic functions of \( z \) of degree \( k_\pm \), respectively. The energy reads

\[
E = 2 \int_{\mathbb{C}} \partial \bar{\partial} K_{U(1) \times SO(2)} = \pi \xi (k_+ + k_-) \equiv \pi \xi k,
\]

\[
K_{U(1) \times SO(2)} = \frac{\xi}{2} \log |\bar{Q}_+|^2 + \frac{\xi}{2} \log |\bar{Q}_-|^2.
\]
Interestingly, the tension of the minimal lump \((k_+, k_-) = (1, 0), (0, 1)\) is half of \(2\pi \xi\) which is that of the minimal lump in the usual \(CP^1\) model.

We would now like to consider lump configurations in slightly more complicated models by considering general \(SO(2M_c)\) Kähler quotients, where we set \(M_c \geq 2\), \(N_f = 2M_c\) and \(M_{	ext{cov}} = J\). However, we should take into account the following constraint on the holomorphic invariants of the \(SO(2M_c)\) group for \(k\) lump configurations

\[
M_{SO(2M_c)} = Q^T(z)JQ(z) = Jz^k + O(z^{k-1}).
\]  

(15)

As an example for \(k = 1\), we take

\[
Q_{k=1} = \begin{pmatrix} z1_{M_c} - A & C \\ 0 & 1_{M_c} \end{pmatrix}, \quad \begin{cases} A = \text{diag}(z_1, z_2, \cdots, z_{M_c}) \\ C = \text{diag}(c_1, c_2, \cdots, c_{M_c}) \end{cases}.
\]

This choice of diagonal matrices allows us to treat the invariants as if they were simply invariants of \(M\) different \(SO(2)\) subgroups. Note that non-zero parameters \(c_i\) keep the rank \(M \geq 2M_c\) even at \(z = z_i\). Thus, the Kähler potential in Eq. (10) becomes

\[
K = \xi \log \left( 2 \sum_{i=1}^{M} \sqrt{|z - z_i|^2 + |c_i|^2} \right).
\]  

(16)

The corresponding energy density is obtained by calculating \(E = 2\partial \bar{\partial}K\) with this potential. If we take some \(c_i\) to vanish, the energy density becomes singular at \(z = z_i\)

\[
E = 2\xi \partial \bar{\partial} \log \left[ \sqrt{|z - z_i|^2 + \cdots} \right] \sim \text{const.} \times \frac{1}{|z - z_i|} + O \left( (z - z_i)^0 \right),
\]

This is due to the occurrence of a curvature singularity in the manifold when \(\text{rank} M = 2M_c - 2\). The trace part of \(C\) determines the overall size of the configuration and the trace part of \(A\) the center of mass, where only the latter turns out to be a normalizable mode.\(^1\)

9. Conclusion

We have studied the NL\(\sigma\)M lumps in gauge theories with \(SO(N_c)\), \(USp(2M_c)\), \(U(1) \times SO(N_c)\) and \(U(1) \times USp(2M_c)\) gauge groups and obtained Kähler metrics and for the two first cases also the hyperKähler metrics. Furthermore, we have constructed NL\(\sigma\)M lumps in these models.

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References

1. M. Eto, T. Fujimori, S. B. Gudnason, M. Nitta and K. Ohashi, arXiv:0809.2014 [hep-th].
2. L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. **80**, 443 (1981).
3. T. L. Curtright and D. Z. Freedman, Phys. Lett. B **90**, 71 (1980) [Erratum-ibid. B **91**, 487 (1980)]; L. Alvarez-Gaume and D. Z. Freedman, Phys. Lett. B **94**, 171 (1980); M. Rocek and P. K. Townsend, Phys. Lett. B **96**, 72 (1980).
4. U. Lindström and M. Roček, Nucl. Phys. B **222**, 285 (1983).
5. N. J. Hitchin, A. Karlhede, U. Lindström and M. Roček, Commun. Math. Phys. **108**, 535 (1987).
6. M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, Phys. Lett. A **65**, 185 (1978).
7. P. B. Kronheimer, J. Diff. Geom. **29**, 665 (1989).
8. G. W. Gibbons, R. Goto and P. Rychenkova, Commun. Math. Phys. **186**, 585 (1997) [arXiv:hep-th/9608085].
9. P. C. Argyres, M. Ronen Plesser and A. D. Shapere, Nucl. Phys. B **483**, 172 (1997) [arXiv:hep-th/960129].
10. N. Seiberg and E. Witten, Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].
11. P. C. Argyres, M. R. Plesser and N. Seiberg, Nucl. Phys. B **471**, 159 (1996) [arXiv:hep-th/9603042].
12. I. Antoniadis and B. Pioline, Int. J. Mod. Phys. A **12**, 4907 (1997) [arXiv:hep-th/9607058].
13. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, Phys. Rev. D **71**, 105009 (2005) [arXiv:hep-th/0503033].
14. A. M. Polyakov and A. A. Belavin, JETP Lett. **22**, 245 (1975) [Pisma Zh. Eksp. Teor. Fiz. **22**, 503 (1975)].
15. T. Vachaspati and A. Achucarro, Phys. Rev. D **44**, 3067 (1991); A. Achucarro and T. Vachaspati, Phys. Rept. **327**, 347 (2000) [arXiv:hep-ph/9904229].
16. M. Eto et al., Phys. Rev. D **76**, 105002 (2007) [arXiv:0704.2218 [hep-th]].
17. M. Shifman and A. Yung, Phys. Rev. D **73**, 125012 (2006) [arXiv:hep-th/0603134].
18. A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].
19. M. A. Luty and W. Taylor, Phys. Rev. D **53**, 3399 (1996) [arXiv:hep-th/9506098].
20. M. Bando, T. Kuramoto, T. Maskawa and S. Uehara, Phys. Lett. B **138**, 94 (1984).
21. K. A. Intriligator and N. Seiberg, Nucl. Phys. B **444**, 125 (1995) [arXiv:hep-th/9503179].
22. K. A. Intriligator and P. Pouliot, Phys. Lett. B **353**, 471 (1995) [arXiv:hep-th/9505006].
23. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170].
24. M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi and W. Vinci, Phys. Lett. B **669**, 98 (2008) [arXiv:0802.1020 [hep-th]].