EXISTENCE AND STABILITY OF A TWO-PARAMETER FAMILY OF SOLITARY WAVES FOR A 2-COUPLED NONLINEAR SCHRÖDINGER SYSTEM

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Abstract. In this paper, the existence and stability results for a two-parameter family of vector solitary-wave solutions (i.e. both components are nonzero) of the nonlinear Schrödinger system
\[
\begin{align*}
    iu_t + u_{xx} + (a|u|^2 + b|v|^2)u &= 0, \\
v_t + v_{xx} + (b|u|^2 + c|v|^2)v &= 0,
\end{align*}
\]
where \(u, v\) are complex-valued functions of \((x, t) \in \mathbb{R}^2\), and \(a, b, c \in \mathbb{R}\) are established. The results extend our earlier ones as well as those of Ohta, Cipolatti and Zumpichiatti and de Figueiredo and Lopes. As opposed to other methods used before to establish existence and stability where the two constraints of the minimization problems are related to each other, our approach here characterizes solitary-wave solutions as minimizers of an energy functional subject to two independent constraints. The set of minimizers is shown to be stable; and depending on the interplay between the parameters \(a, b\) and \(c\), further information about the structures of this set are given.

1. Introduction. The nonlinear Schrödinger (NLS) equation
\[
iu_t + u_{xx} \pm |u|^2u = 0,
\]
where \(u\) is a complex-valued function of \((x, t) \in \mathbb{R}^2\) arises in several applications. It has been derived in such diverse fields as deep water waves [30], plasma physics [31], nonlinear optical fibers [15, 16], magneto-static spin waves [32], to name a few.

The coupled nonlinear Schrödinger (CNLS) system
\[
\begin{align*}
    iu_t + u_{xx} + (a|u|^2 + b|v|^2)u &= 0, \\
v_t + v_{xx} + (b|u|^2 + c|v|^2)v &= 0,
\end{align*}
\]
where \(u, v\) are complex-valued functions of \((x, t) \in \mathbb{R}^2\), and \(a, b, c \in \mathbb{R}\), arises physically under conditions similar to those described by (1.1) when there are two wave-trains moving with nearly the same group velocities [26, 29]. The CNLS system

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also models physical systems in which the field has more than one components; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. The CNLS system also arises in the Hartree-Fock theory for a double condensate. Readers are referred to the works [7, 15, 16, 30, 31] for the derivation as well as applications of this system.

The system (1.2) has the following conserved quantities

\[
E(u, v) = \int_{-\infty}^{\infty} \left[ |u_x(x, t)|^2 + |v_x(x, t)|^2 - \frac{a}{2} |u(x, t)|^4 - \frac{c}{2} |v(x, t)|^4 - b |u(x, t)|^2 |v(x, t)|^2 \right] dx,
\]

(1.3)

\[
Q(u) = \int_{-\infty}^{\infty} |u(x, t)|^2 dx,
\]

(1.4)

\[
Q(v) = \int_{-\infty}^{\infty} |v(x, t)|^2 dx.
\]

(1.5)

It is our aim in this manuscript to prove existence and stability results for solitary-wave solutions of (1.2). Such solutions are of the form

\[
\begin{align*}
  u(x, t) &= e^{i(\omega_1 - \sigma^2)t + i\sigma x} \phi(x - 2\sigma t), \\
  v(x, t) &= e^{i(\omega_2 - \sigma^2)t + i\sigma x} \psi(x - 2\sigma t),
\end{align*}
\]

(1.6)

where \(\omega_1, \omega_2, \sigma \in \mathbb{R}\), and \(\phi, \psi : \mathbb{R} \to \mathbb{R}\) are functions that vanish at infinity in the sense that \(\phi, \psi \in H^1_{\mathbb{C}}\). (Here \(H^1_{\mathbb{C}}\) is the usual Sobolev space consisting of complex-valued, measurable functions such that both \(f\) and \(f_x\) are in \(L^2\).) When \(\sigma = 0\), solutions to (1.6) are usually referred to as standing-wave solutions. Notice that if \((u(x, t), v(x, t))\) as defined in (1.6) is a solution of (1.2), then \((\phi, \psi)\) solves the system of ordinary differential equations

\[
\begin{cases}
-\phi'' + \omega_1 \phi = a|\phi|^2 \phi + b|\psi|^2 \phi, \\
-\psi'' + \omega_2 \psi = c|\psi|^2 \psi + b|\phi|^2 \psi.
\end{cases}
\]

(1.7)

In the last several years there have been intensive works in studying the existence of standing waves for nonlinear Schrödinger systems of the form studied in this paper, for example, see [2, 3, 4, 5, 6, 12, 13, 19, 22, 27] and references therein. Few papers address the issue of stability of solitary-wave solutions to the CNLS systems [11, 23, 24, 25, 28]. The stability results obtained from those papers are all for one-parameter family of solitary waves. For example, in [11], the authors considered the variational problem of minimizing \(E\) with one constraint being the sum of the \(L^2\)-norms of the two components, while in [23, 24, 25] the constraints were not independently chosen. In this paper we establish the existence and stability of a two-parameter family of solitary waves of (1.2).

**Notation.** For \(1 \leq p \leq \infty\), we denote by \(L^p = L^p(\mathbb{R})\) the space of all measurable functions \(f\) on \(\mathbb{R}\) for which the norm \(|f|_p = (\int_{-\infty}^{\infty}|f|^p dx)^{1/p}\) is finite for \(1 \leq p < \infty\) and \(|f|_\infty\) is the essential supremum of \(|f|\) on \(\mathbb{R}\). Whether we intend the functions in \(L^p\) to be real-valued or complex-valued will be clear from the context. \(H^1_{\mathbb{C}}(\mathbb{R})\),
as mentioned above, is the usual Sobolev space consisting of complex-valued, measurable functions on $\mathbb{R}$ such that both $f$ and $f'$ are in $L^2$, furnished with the norm

$$
\|f\|_1^2 = \int_{-\infty}^{\infty} (|f|^2 + |f'|^2) dx.
$$

We define the space $X$ to be the Cartesian product $H^1_x(\mathbb{R}) \times H^1_x(\mathbb{R})$, equipped with the product norm

$$
\|(f, g)\|_X^2 = \|f\|_1^2 + \|g\|_1^2.
$$

If $T > 0$ and $Y$ is any Banach space, we denote by $\mathcal{C}([0, T], Y)$ the Banach space of continuous maps $f : [0, T] \to Y$, with norms given by $\|f\|_{\mathcal{C}([0, T], Y)} = \sup_{t \in [0, T]} \|f(t)\|_Y$.

For fixed $s > 0$ and $t > 0$, define the real number $I = I(s, t)$ as follows:

$$
I(s, t) = \inf \{E(f, g) : (f, g) \in X, \|f\|_2^2 = s, \|g\|_2^2 = t\}. \tag{1.8}
$$

The set of minimizers for $I(f, g)$ is

$$
S_{s, t} = \{(f, g) \in X : E(f, g) = I(f, g), \|f\|_2^2 = s, \|g\|_2^2 = t\}.
$$

For the single equation (1.1), stability of solitary waves is a direct consequence of the minimization problem of the energy functional subject to the one constraint of the $L^2$-norm being kept constant. One crucial point in preventing dichotomy of minimizing sequences is establishing the strict sub-additivity of $I$, as is well-known from Lions’ pioneer work [20, 21]. The strict sub-additivity of $I$ seems to be much more challenging for the two-parameter variational problem posed in (1.8).

Following the same approach used by Albert et al. [1] which in turn relied on an argument due to [8, 14], we utilize the fact that the $H^1$-norms of some functions are strictly decreasing when the mass of the functions are symmetrically rearranged. The set of minimizers $S_{s, t}$ is shown to be stable; and depending on the interplay between the parameters $a, b$, and $c$, further information about the structures of this set such as the minimizer $(\phi, \psi)$, the Lagrange multipliers $\omega_1, \omega_2$ are given. The set $S_{s, t}$ form a true two-parameter family in the sense that if $(s_1, t_1) \neq (s_2, t_2)$, then the two sets $S_{s_1, t_1}$ and $S_{s_2, t_2}$ are disjoint. To the best of our knowledge, such existence and stability results are the first for the system (1.2). (See also the Remark following Theorem 2.1 below.) After this paper was submitted we learnt the interesting work [17] in which a new type of solutions called multi-speeds solitary waves were constructed with each component behaving as a solitary wave to a scalar equation and the two components travelling in relatively large different speeds. The stability of these solutions are still unknown.

Naturally, prior to a discussion of stability should be a theory for the initial-valued problem itself. It has been proved (see, for example, [9, 10]) that for all $(u(x, 0), v(x, 0)) \in X$, exists unique $(u(x, t), v(x, t))$ of (1.2) in $\mathcal{C}(\mathbb{R}; X)$ emanating from $(u(x, 0), v(x, 0))$, and $(u(x, t), v(x, t))$ satisfies

$$
Q(u(x, t)) = Q(u(x, 0)), \quad Q(v(x, t)) = Q(v(x, 0)),
$$

$$
E(u(x, t), v(x, t)) = E(u(x, 0), v(x, 0)).
$$

This manuscript is organized as follows. In Section 2, the main contributions of this manuscript are presented and discussed. The proofs of Theorems 2.1 and 2.2 are accomplished through several Lemmas and Propositions in Section 3. Lemmas 3.5 and 3.6 are crucial in establishing the proof of the relative compactness of minimizing sequences for the variational problem which defines the solitary wave.
solutions of (1.2). An immediate consequence of this fact is that the set of minimizers $S_{s,t}$ is stable. And depending on the interplay between the parameters $a, b$ and $c$, further information about the structures of this set such as the minimizer $(\phi, \psi)$, the Lagrange multipliers $\omega_1, \omega_2$ are given.

2. Statement of results. Our existence and stability results are as follows.

Theorem 2.1. Suppose $a, b, c > 0$. Then the following statements are true for all $s > 0$ and all $t > 0$.

1. Every minimizing sequence $\{(f_n, g_n)\} \in X$ for $I(s, t)$ is relatively compact in $X$ up to translations. In particular, the set $S_{s,t}$ is non-empty.

2. Each function $(\phi, \psi) \in S_{s,t}$ is a solution of (1.7) for some $\omega_1 > 0$ and $\omega_2 > 0$, and therefore when substituted into (1.6) yields a (standing-wave) solitary-wave solution of (1.2). Moreover, if $0 < a < b < c$, then $0 < \omega_1 < \omega_2$; and if $0 < c < b < a$, then $0 < \omega_2 < \omega_1$.

3. For every $(\phi, \psi) \in S_{s,t}$, there exist numbers $\theta_1, \theta_2 \in \mathbb{R}$ and functions $\hat{\phi}(x) > 0$, $\hat{\psi}(x) > 0$ for all $x \in \mathbb{R}$ such that $\phi(x) = e^{i\theta_1} \hat{\phi}(x)$, and $\psi(x) = e^{i\theta_2} \hat{\psi}(x)$. Moreover, the functions $\phi$ and $\psi$ are infinitely differentiable.

4. For every $\epsilon > 0$ given, there exists $\delta > 0$ such that if

$$\inf_{(\phi, \psi) \in S_{s,t}} \| (u_0, v_0) - (\phi, \psi) \|_X < \delta,$$

then the solution $(u(x,t), v(x,t))$ of (1.2) with $(u(x,0), v(x,0)) = (u_0, v_0)$ satisfies

$$\inf_{(\phi, \psi) \in S_{s,t}} \| (u(\cdot, t), v(\cdot, t)) - (\phi, \psi) \|_X < \epsilon,$$

for all $t \in \mathbb{R}$.

Remark.

1. The set of minimizers $S_{s,t}$ form a true two-parameter family in the sense that if $(s_1, t_1) \neq (s_2, t_2)$, then the two sets $S_{s_1,t_1}$ and $S_{s_2,t_2}$ are disjoint.  

2. Statements 2) and 3) say that when $a, b, c > 0$ with $b$ between $a$ and $c$, (1.7) still has vector solutions, and the set of minimizers $S_{s,t}$ is stable and consists of $\{(\phi, \psi)\}$ with $\hat{\phi}(x) > 0$, $\hat{\psi}(x) > 0$ for all $x \in \mathbb{R}$ with $\omega_1 \neq \omega_2$. In fact, by the non-existence result in [5] for (1.7), we must have $0 < \omega_1 < \omega_2$ if $0 < a < b < c$ and $0 < \omega_2 < \omega_1$ if $0 < c < b < a$. Our existence result also supplements those of [13] in which existence results were given for a range of the coupling constant $b > 0$ depending on $a, c > 0$, in terms of fixed $\omega_1$ and $\omega_2$. Moreover, our stability result established here for $a, b, c > 0$ with $b$ between $a$ and $c$ is new as this case has never been considered before.

3. When the two constraints are not independent, such as when $s = \frac{b - c}{b - a}$ with either

(A1) $0 < b < \min\{a, c\}$;

or

(A2) $b > 0$ with $b > \max\{a, c\}$ and $b^2 > ac$,

it was proved in [23] that the set of minimizers consists of, up to translations, vector solutions with each component being multiple of the hyperbolic function $\sech$. (Notice that in the condition (A2), the numbers $a, c$ are allowed to be negative as well.)
Regarding item 3) in the above Remark, we will show next that this is exactly the case for our set of minimizers $S_{s,t}$ when $s, t$ are so restricted. Hence, the results here include those in [23].

**Theorem 2.2.** Suppose $a, b, c \in \mathbb{R}$ such that either (A1) or (A2) holds. For any fixed $\omega > 0$, let $s = 4\sqrt{\omega}\frac{b-c}{b^2-ac}$ and $t = 4\sqrt{\omega}\frac{b-a}{b^2-ac}$. Then the following statements are true regarding the variational problem: $I(s,t) = \inf\{E(f,g) : f, g \in H^1_x(\mathbb{R}), \|f\|_2^2 = s, \|g\|_2^2 = t\}$.

1. Every minimizing sequence $\{(f_n,g_n)\} \subset X$ for $I(s,t)$ is relatively compact in $X$ up to translations. In particular, the set $S_{s,t}$ is non-empty.

2. Each function $(\phi, \psi) \in S_{s,t}$ is a solution of (1.7) for $\omega_1 = \omega_2 = \omega > 0$, and therefore when substituted into (1.6) yields a (standing-wave) solitary-wave solution of (1.2). Moreover,

$$S_{s,t} = \{(e^{i\theta_1}\sqrt{b-c \over b^2-ac}\Phi(x), e^{i\theta_2}\sqrt{b-a \over b^2-ac}\Phi(x))\},$$

where $\Phi(x) = \sqrt{2\omega}\text{sech}(\sqrt{\omega}x)$.

3. For every $\epsilon > 0$ given, there exists $\delta > 0$ such that if $(u_0, v_0) \in X$ satisfies

$$\inf_{\theta_1, \theta_2, y \in \mathbb{R}} \left\| u_0 - \sqrt{b-c \over b^2-ac} e^{i\theta_1} \tau_y \Phi \right\|_{H^1_x(\mathbb{R})} + \left\| v_0 - \sqrt{b-a \over b^2-ac} e^{i\theta_2} \tau_y \Phi \right\|_{H^1_x(\mathbb{R})} < \delta,$$

then the solution $(u(x,t), v(x,t))$ with $(u(\cdot,0), v(\cdot,0)) = (u_0,v_0)$ satisfies

$$\sup_{t \in \mathbb{R}} \inf_{\theta_1, \theta_2, y \in \mathbb{R}} \left\| u(\cdot,t) - \sqrt{b-c \over b^2-ac} e^{i\theta_1} \tau_y \Phi \right\|_{H^1_x(\mathbb{R})} + \left\| v(\cdot,t) - \sqrt{b-a \over b^2-ac} e^{i\theta_2} \tau_y \Phi \right\|_{H^1_x(\mathbb{R})} < \epsilon,$$

where $\tau_y f(x) = f(x-y)$, for all $t \in \mathbb{R}$.

3. **Variational problem.** In preparation for the proof of Theorem 2.1, several Lemmas and Propositions are established first. Recall that $a, b, c > 0$ in Theorem 2.1. Our first Lemma states that the infimum must be finite and negative and that minimizing sequences are bounded uniformly.

**Lemma 3.1.** Every minimizing sequence for $I(s,t)$ is bounded in $X$ and

$$-\infty < I(s,t) < 0.$$

**Proof.** Let $(f_n,g_n) \in X$ be a minimizing sequence. Using Gagliardo-Nirenberg inequality, the following estimates are clear:

$$a) \int_{-\infty}^{\infty} |f_n|^4 dx \leq C|f_n|_{4}^2 |f_n|_{2}^2 \leq C|f_n|_{2}^4;$$

$$b) \int_{-\infty}^{\infty} |f_n|^2 |g_n|^2 dx \leq \frac{1}{2} (|f_n|_{4}^4 + |g_n|_{4}^4) \leq C(|f_n|_{2}^4 + |g_n|_{2}^4),$$

where $C$ denotes various constants whose precise values are not of importance. Rewrite

$$\|(f_n, g_n)\|_{X}^2$$
Thus, let
\[ E(f_n, g_n) = (s + t) + \int_{-\infty}^{\infty} \left( \frac{a}{2} |f_n|^4 + \frac{c}{2} |g_n|^4 + b|f_n|^2|g_n|^2 \right) dx \]
\[ \leq C(\|f_n\|_1 + \|g_n\|_1) \]
where an application of (3.1) is used to estimate the integral. As the norm of the minimizing sequence \((f_n, g_n)\) is bounded by itself but with a smaller power, it follows that the minimizing sequence must be bounded uniformly in \(X\). A finite lower bound is now immediate using again (3.1) and the fact that \((f_n, g_n)\) is bounded.

To see that \(I(s, t) < 0\), let \((f, g) \in X\) such that \(|f|^2 = s\) and \(|g|^2 = t\). For each \(r > 0\), set
\[
\begin{cases}
  f_r(x) = \sqrt{r} f(rx), \\
  g_r(x) = \sqrt{r} g(rx).
\end{cases}
\]
Then \(|f_r|^2 = s\) and \(|g_r|^2 = t\) and
\[
E(f_r, g_r) = \int_{-\infty}^{\infty} \left( |f_r'|^2 + |g_r'|^2 - \frac{a}{2} |f_r|^4 - \frac{c}{4} |g_r|^4 - b|f_r|^2|g_r|^2 \right) dx 
\leq r^2 \int_{-\infty}^{\infty} (|f'|^2 + |g'|^2) dx - \min\{a, b, c\} r \int_{-\infty}^{\infty} (|f|^4 + |g|^4 + |f|^2|g|^2) dx.
\]
Thus, \(E(f_r, g_r) < 0\) for sufficiently small \(r\).

**Lemma 3.2.** Let \((f_n, g_n) \in X\) be a minimizing sequence for \(I(s, t)\). Then for all sufficiently large \(n\),
\begin{enumerate}[i)]
\item if \(s > 0\) and \(t \geq 0\), then there exists \(\delta_1 > 0\) such that \(|f_n|^2 \geq \delta_1\);
\item if \(s \geq 0\) and \(t > 0\), then there exists \(\delta_2 > 0\) such that \(|g_n|^2 \geq \delta_2\);
\item if \(s > 0\) and \(t > 0\), then there exists \(\delta_3 > 0\) such that
\[
\int_{-\infty}^{\infty} \left( |f_n'|^2 - \frac{a}{2} |f_n|^4 - b|f_n|^2|g_n|^2 \right) dx \leq -\delta_3 < 0,
\]
\[
\int_{-\infty}^{\infty} \left( |g_n'|^2 - \frac{c}{2} |g_n|^4 - b|f_n|^2|g_n|^2 \right) dx \leq -\delta_3 < 0.
\]
\end{enumerate}

**Proof.** Suppose to the contrary that i) is false, then by passing to a subsequence if necessary, we may assume there exists a minimizing sequence such that \(\lim_{n \to \infty} |f_n'|^2 = 0\). By Gagliardo-Nirenberg inequality,
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} |f_n|^4 dx = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{-\infty}^{\infty} |f_n|^2|g_n|^2 dx = 0.
\]
Thus,
\[
I(s, t) = \lim_{n \to \infty} E(f_n, g_n) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( |g_n'|^2 - \frac{c}{2} |g_n|^4 \right) dx. \tag{3.2}
\]
Now, pick any \(\psi \in H_x^1(\mathbb{R})\) such that \(|\psi|^2 = s\) and let \(\psi_r(x) = \sqrt{r} \psi(rx)\). Hence, for all \(n \in \mathbb{N}\),
\[
I(s, t) \leq E(\psi_r, g_n).
\]
On the other hand, if we define
\[
\eta = r^2 \int_{-\infty}^{\infty} |\psi'|^2 dx - \frac{a}{2} r \int_{-\infty}^{\infty} |\psi|^4 dx, \tag{3.3}
\]
then \( \eta < 0 \) for sufficiently small \( r > 0 \). Consequently, for all \( n \in \mathbb{N} \),
\[
I(s, t) \leq E(\psi_r, g_n) \leq \int_{-\infty}^{\infty} \left( |g_n'|^2 - \frac{c}{2} |g_n|^4 \right) dx + \eta.
\]
But then
\[
I(s, t) \leq \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( |g_n'|^2 - \frac{c}{2} |g_n|^4 \right) dx + \eta,
\]
a contradiction to (3.2) and (3.3). The case ii) can be proved similarly.

To see iii), suppose the statement is false. By passing to a subsequence if necessary, we may assume that there exists a minimizing sequence \((f_n, g_n)\) for which
\[
\liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |f'_n|^2 - \frac{a}{2} |f_n|^4 - b|f_n|^2 |g_n|^2 \right) dx \geq 0.
\]
Hence,
\[
I(s, t) = \lim_{n \to \infty} E(f_n, g_n) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |g_n'|^2 - \frac{c}{2} |g_n|^4 \right) dx = \frac{1}{3} \left( \frac{c}{4} \right)^2 t^3,
\]
where the minimum is attained at \( \phi_t(x) = \sqrt{2c} \frac{t}{4} \text{sech} \left( \frac{ct}{4} x \right) \). On the other hand, take any \( f \in H^1_{\mathbb{R}} \) with \( \|f\| = s \) such that
\[
\int_{-\infty}^{\infty} \left( |f'|^2 - \frac{a}{2} |f|^4 - b|f|^2 |\phi_t|^2 \right) dx < 0,
\]
we have
\[
I(s, t) \leq E(f, \phi_t) = \int_{-\infty}^{\infty} \left( |f'|^2 - \frac{a}{2} |f|^4 - b|f|^2 |\phi_t|^2 \right) dx - \frac{1}{3} \left( \frac{c}{4} \right)^2 t^3 < \frac{1}{3} \left( \frac{c}{4} \right)^2 t^3,
\]
which contradicts (3.4). Similar argument can be used to prove the other case.

Following approach used in [1], we will show in the next two Lemmas that the value of \( E(f, g) \) decreases when \( f \) and \( g \) are replaced by \( |f| \) and \( |g| \), and when \( |f| \) and \( |g| \) are symmetrically rearranged. It is straightforward to see the next Lemma, using the fact that
\[
\int_{-\infty}^{\infty} |f|_x^2 dx \leq \int_{-\infty}^{\infty} |f_x|^2 dx.
\]

**Lemma 3.3.** For all \((f, g) \in X\), one has
\[
E(|f|, |g|) \leq E(f, g).
\]

We recall here (see also [1, 18]) the definition of symmetric decreasing rearrangement of a function. Let \( w : \mathbb{R} \to [0, \infty) \) be a non-negative function. If \( \{ x : w(x) > y \} \) has finite measure \( m(w, y) \) for all \( y > 0 \), then the symmetric decreasing rearrangement \( w^* \) of \( w \) is defined by
\[
w^*(x) = \inf \{ y \in (0, \infty) : \frac{1}{2} m(w, y) \leq x \}.
\]
Notice that if \((f, g) \in X\), then \(|f|, |g| \in H^1\), and thus \(|f|^* \) and \(|g|^* \) are well-defined.

**Lemma 3.4.** For all \((f, g) \in X\), it must be true that
\[
E(|f|^*, |g|^*) \leq E(f, g).
\]
Proof. Using the following important facts (for the proofs of those, see [18]):

\begin{align*}
a) & \int_{-\infty}^{\infty} (|f|^p) dx = \int_{-\infty}^{\infty} |f|^p dx. \\
b) & \int_{-\infty}^{\infty} (|f|^p)(|g|^p) dx \geq \int_{-\infty}^{\infty} |f|^2 |g|^p dx. \\
c) & \int_{-\infty}^{\infty} |(|f^*|^p)_x|^2 dx \leq \int_{-\infty}^{\infty} ||f_x|^p|^2 dx.
\end{align*}

and \( b > 0 \), the Lemma is clear.

The next Lemma is crucial in obtaining the strict sub-additivity of the function \( I(s,t) \) needed in ruling out dichotomy of minimizing sequences. We refer readers to [1] for the proof of this.

Lemma 3.5. Suppose \( u \) and \( v \) are non-negative, even, \( C^\infty \)-functions with compact support in \( \mathbb{R} \), which are non-increasing on \( \{ x : x \geq 0 \} \). Let \( x_1 \) and \( x_2 \) be numbers such that \( u(x + x_1) \) and \( v(x + x_2) \) have disjoint supports, and define

\[ w(x) = u(x + x_1) + v(x + x_2). \]

Let \( w^* : \mathbb{R} \to \mathbb{R} \) be the symmetric decreasing rearrangement of \( w \). Then the (distributional) derivative \( (w^*)' \) is in \( L^2 \) and satisfies

\[ |(w^*)'|_2^2 \leq |w'|_2^2 - \frac{3}{4} \min\{|u'|_2^2, |v'|_2^2\}. \tag{3.6} \]

Lemma 3.6. Let \( s_1, s_2, t_1, t_2 \geq 0 \) be given, and suppose that \( s_1 + s_2 > 0, t_1 + t_2 > 0, \)
\( s_1 + t_1 > 0, \) and \( s_2 + t_2 > 0. \) Then

\[ I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2). \]

Proof. Following closely the argument used in [1], we claim that for \( i = 1, 2, \) one can choose minimizing sequences \( (f_n^{(i)}, g_n^{(i)}) \) for \( I(s_i, t_i) \) such that for all \( n \in \mathbb{N}, f_n^{(i)} \)
and \( g_n^{(i)} : 
\begin{enumerate}
\item[i)] are real-valued and non-negative on \( \mathbb{R}; \)
\item[ii)] belong to \( H^1 \) and have compact support;
\item[iii)] are even functions;
\item[iv)] are non-increasing functions of \( x \), for all \( x \geq 0; \)
\item[v)] are \( C^\infty \)-functions; and
\item[vi)] \( |f_n^{(i)}|^2 = s_i \) and \( |g_n^{(i)}|^2 = t_i. \)
\end{enumerate}

To see this, we can take, without loss of generality, \( i = 1 \) as the case \( i = 2 \) is exactly the same. Moreover, we can assume that \( s_1 > 0 \) and \( t_1 > 0, \) as otherwise just simply take \( f_n^{(1)} = 0 = g_n^{(1)}. \) Now, let \( (w_n^{(1)}, z_n^{(1)}) \) be any minimizing sequence for \( I(s_1, t_1). \) Since functions with compact support are dense in \( H^1, \) and \( E : X \to \mathbb{R} \)
is continuous, we can approximate \( (w_n^{(1)}, z_n^{(1)}) \) by functions \( (w_n^{(2)}, z_n^{(2)}) \) which have compact support and which still form a minimizing sequence for \( I(s_1, t_1). \) Then by Lemma 3.4,

\[ (w_n^{(3)}, z_n^{(3)}) = (|w^{(2)}|^*, |z^{(2)}|^*) \]
is still a minimizing sequence for \( I(s_1, t_1), \) and for each \( n \in \mathbb{N}, w_n^{(3)}, z_n^{(3)} \) satisfy (i)-(iv).
Notice next that if \( \psi \) is any non-negative, even, \( C^\infty \), decreasing function for \( x \geq 0 \) with compact support, then the convolution of \( \psi \) with any function \( f \) satisfying properties (i)-(iv)

\[
f \ast \psi(x) = \int_{-\infty}^{\infty} f(x - y)\psi(y)dy
\]

also satisfies (i)-(iv). Using “approximation to the identity”

\[
\psi_{\epsilon}(x) = \frac{1}{\epsilon^2} \psi\left(\frac{x}{\epsilon}\right), \quad \text{for } \epsilon > 0 \quad \text{with } \int_{-\infty}^{\infty} \psi(x)dx = 1,
\]

then \( f \ast \psi_{\epsilon} \longrightarrow f \) as \( \epsilon \to 0 \). Thus, by choosing \( \psi(x) \) satisfying \( \int_{-\infty}^{\infty} \psi(x)dx = 1 \) to be any non-negative, even, \( C^\infty \), decreasing function for \( x \geq 0 \) with compact support, and defining

\[
(w_n^{(4)}, z_n^{(4)}) = (w_n^{(3)} \ast \psi_{\epsilon n}, z_n^{(3)} \ast \psi_{\epsilon n})
\]

with \( \epsilon_n \) appropriately small for \( n \) large, then \( (w_n^{(4)}, z_n^{(4)}) \) satisfies all (i)-(v).

Set

\[
f_n^{(1)} = \sqrt{s_{n^2}w_n^{(4)}} \quad \text{and} \quad g_n^{(1)} = \frac{\sqrt{t_{n^2}z_n^{(4)}}}{|z_n^{(4)}|_2},
\]

(which is possible as \( |w_n^{(4)}|_2 > 0 \) and \( |z_n^{(4)}|_2 > 0 \) for \( n \) large), then \( (f_n^{(1)}, g_n^{(1)}) \) satisfies all (i)-(vi). For each \( n \), choose a number \( x_n \) such that \( f_n^{(1)}(x) \) and \( \tilde{f}_n^{(2)}(x) \equiv f_n^{(2)}(x + x_n) \) have disjoint support, and \( g_n^{(1)}(x) \) and \( \tilde{g}_n^{(2)}(x) \equiv g_n^{(2)}(x + x_n) \) have disjoint support. Define:

\[
f_n = (f_n^{(1)} + \tilde{f}_n^{(2)})^*; \quad g_n = (g_n^{(1)} + \tilde{g}_n^{(2)})^*.
\]

Then \( |f_n|^2 = s_1 + s_2 \); and \( |g_n|^2 = t_1 + t_2 \); so

\[
I(s_1, s_2, t_1 + t_2) \leq E(f_n, g_n). \tag{3.7}
\]

On the other hand, Lemma 3.5 guarantees that

\[
\int_{-\infty}^{\infty} (|f'_n|^2 + |g'_n|^2)dx
\]

\[
\leq \int_{-\infty}^{\infty} \left( \left( |f_n^{(1)} + \tilde{f}_n^{(2)}|_x \right|^2 + \left( |g_n^{(1)} + \tilde{g}_n^{(2)}|_x \right|^2 \right)dx - K_n \tag{3.8}
\]

\[
= \int_{-\infty}^{\infty} \left( |(f_n^{(1)})_x|^2 + |(\tilde{f}_n^{(2)})_x|^2 + |(g_n^{(1)})_x|^2 + |(\tilde{g}_n^{(2)})_x|^2 \right)dx - K_n
\]

where

\[
K_n = \frac{3}{4} \left( \min\{||f_n^{(1)}|_x|^2, ||f_n^{(2)}|_x|^2\} + \min\{||g_n^{(1)}|_x|^2, ||g_n^{(2)}|_x|^2\} \right). \tag{3.9}
\]

Moreover, from properties of rearrangement, we have

\[
\int_{-\infty}^{\infty} |f_n|^4dx = \int_{-\infty}^{\infty} |f_n^{(1)}|^4dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^4dx;
\]

\[
\int_{-\infty}^{\infty} |g_n|^4dx = \int_{-\infty}^{\infty} |g_n^{(1)}|^4dx + \int_{-\infty}^{\infty} |g_n^{(2)}|^4dx;
\]

\[
\int_{-\infty}^{\infty} |f_n|^2|g_n|^2dx \geq \int_{-\infty}^{\infty} |f_n^{(1)}|^2|g_n^{(1)}|^2dx + \int_{-\infty}^{\infty} |f_n^{(2)}|^2|g_n^{(2)}|^2dx.
\]
Thus, (3.8)-(3.9) give, for all \( n \)
\[
I(s_1 + t_1, s_2 + t_2) \leq E(f_n, g_n) \leq E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) - K_n.
\]
Hence
\[
I(s_1 + t_1, s_2 + t_2) \leq I(s_1, t_1) + I(s_2, t_2) - \liminf_{n \to \infty} K_n. \tag{3.10}
\]
Since \( t_1 + t_2 > 0 \), either both are positive or one is zero while the other is positive. Thus, we have three cases to consider:

i) \( t_1 > 0 \) and \( t_2 > 0 \);

ii) \( t_1 = 0 \), \( t_2 > 0 \), and \( s_2 > 0 \);

iii) \( t_1 = 0 \), \( t_2 > 0 \), and \( s_2 = 0 \).

**Case 1.** When \( t_1 > 0 \) and \( t_2 > 0 \), Lemma 3.2 guarantees that there exist \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that for all \( n \) large,
\[
|g_n^{(1)}(x)|_2 \geq \delta_1 \quad \text{and} \quad |g_n^{(2)}(x)|_2 \geq \delta_2.
\]
Let \( \delta = \min\{\delta_1, \delta_2\} \); then (3.9) implies that \( K_n \geq 3\delta/4 \), for all \( n \) large. From (3.10), we have
\[
I(s_1 + t_1, s_2 + t_2) \leq I(s_1, t_1) + I(s_2, t_2) - \frac{3\delta}{4} < I(s_1, t_1) + I(s_2, t_2).
\]

**Case 2.** \( t_1 = 0 \), \( t_2 > 0 \), \( s_2 > 0 \) and since \( s_1 + t_1 > 0 \), \( s_1 > 0 \) too. By Lemma 3.2, there exist \( \delta_3 > 0 \) and \( \delta_4 > 0 \) such that for all \( n \) large,
\[
|f_n^{(1)}(x)|_2 \geq \delta_3 \quad \text{and} \quad |f_n^{(2)}(x)|_2 \geq \delta_4.
\]
Let \( \delta = \min\{\delta_3, \delta_4\} \); then again (3.9) implies that \( K_n \geq 3\delta/4 \), for all \( n \) large and
\[
I(s_1 + t_1, s_2 + t_2) \leq I(s_1, t_1) + I(s_2, t_2) - \frac{3\delta}{4} < I(s_1, t_1) + I(s_2, t_2).
\]

**Case 3.** \( s_1 > 0 \) and \( t_2 > 0 \), and we have to prove that
\[
I(s_1, t_2) < I(s_1, 0) + I(0, t_2).
\]
In this case, it is well-known that

1) \( I(s_1, 0) \)
\[
= \inf \left\{ \int_{-\infty}^{\infty} (|f'|^2 - \frac{a}{2} |f|^4) dx : f \in H^1_\infty(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} |f|^2 dx = s_1 > 0 \right\}
\]
\[
= -\frac{1}{3}\left(\frac{a}{4}\right)^2 s_1^3,
\]
where the minimum is achieved at \( \phi_{s_1}(x) = \sqrt{2a}s_1 \sech\left(\frac{as_1}{4}x\right) \).

2) \( I(0, t_2) \)
\[
= \inf \left\{ \int_{-\infty}^{\infty} (|f'|^2 - \frac{c}{2} |f|^4) dx : f \in H^1_\infty(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} |f|^2 dx = t_2 > 0 \right\}
\]
\[
= -\frac{1}{3}\left(\frac{c}{4}\right)^2 t_2^3,
\]
where the minimum is achieved at \( \phi_{t_2}(x) = \sqrt{2c}t_2 \sech\left(\frac{ct_2}{4}x\right) \).
Thus, we have
\[ I(s_1, t_2) \leq E(\phi_{s_1}, \phi_{t_2}) \]
\[ = \int_{-\infty}^{\infty} (|\phi'_{s_1}|^2 - \frac{a}{2} |\phi_{s_1}|^4) \, dx + \int_{-\infty}^{\infty} (|\phi'_{t_2}|^2 - \frac{c}{2} |\phi_{t_2}|^4) \, dx \]
\[ - b \int_{-\infty}^{\infty} |\phi_{s_1}|^2 |\phi_{t_2}|^2 \, dx \]
\[ = I(s_1, 0) + I(0, t_2) - b \int_{-\infty}^{\infty} |\phi_{s_1}|^2 |\phi_{t_2}|^2 \, dx < I(s_1, 0) + I(0, t_2). \]

The Lemma is hence proved. \( \Box \)

Let \( \{(f_n, g_n)\} \in X \) be any minimizing sequence for \( E \) and consider a sequence of nondecreasing functions \( M_n : [0, \infty) \to [0, s + t] \) as follows
\[ M_n(r) = \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} (|f_n(x)|^2 + |g_n(x)|^2) \, dx. \]
As \( M_n(r) \) is a uniformly bounded sequence of nondecreasing functions in \( r \), one can show that it has a subsequence, which is still denoted as \( M_n \), that converges point-wisely to a nondecreasing limit function \( M(r) : [0, \infty) \to [0, s + t] \). Let
\[ \gamma = \lim_{r \to \infty} M(r) \equiv \lim_{r \to \infty} \lim_{n \to \infty} M_n(r) = \lim_{r \to \infty} \limsup_{n \to \infty} \int_{y-r}^{y+r} (|f_n(x)|^2 + |g_n(x)|^2) \, dx. \]
Then \( 0 \leq \gamma \leq s + t \).

The following Lemma is well-known. (See, for example, [1, 18].)

**Lemma 3.7.** Suppose \( w_n \) is a sequence of functions which is bounded in \( H^1 \) and which satisfies for some \( R > 0 \),
\[ \lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} w_n^2 \, dx = 0. \]
(3.11)
Then \( \forall r > 2, \lim_{n \to \infty} |w_n|_r = 0 \).

The next Lemma says that vanishing of minimizing sequences cannot occur.

**Lemma 3.8.** For any minimizing sequence \( (f_n, g_n) \in X \), \( \gamma > 0 \).

**Proof.** Suppose to the contrary that \( \gamma = 0 \). Then (3.11) holds for both \( w_n = |f_n| \) and \( w_n = |g_n| \). Thus, Lemma 3.7 says that for all \( r > 2 \), \( f_n \) and \( g_n \to 0 \) in \( L^r \)-norm. In particular, \( \int_{-\infty}^{\infty} |f_n|^4 \, dx \to 0 \), \( \int_{-\infty}^{\infty} |g_n|^4 \, dx \to 0 \) and \( \int_{-\infty}^{\infty} |f_n|^2 |g_n|^2 \, dx \leq |f_n|^2 |g_n|^2 \to 0 \). Hence
\[ I(s, t) = \lim_{n \to \infty} E(f_n, g_n) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} (|f_n'|^2 + |g_n'|^2) \, dx \geq 0, \]
a contradiction. \( \Box \)

**Lemma 3.9.** There exist numbers \( s_1 \in [0, s] \) and \( t_1 \in [0, t] \) such that \( \gamma = s_1 + t_1 \), and
\[ I(s_1, t_1) + I(s - s_1, t - t_1) \leq I(s, t). \]
Proof. Let $\rho, \sigma \in C^\infty(\mathbb{R})$ such that $\rho^2 + \sigma^2 = 1$ and $\rho \equiv 1$ on $[-1, 1]$ and has support in $[-2, 2]$. Set, for $\omega > 0$,

$$\rho_\omega(x) = \rho(\frac{x}{\omega}) \quad \text{and} \quad \sigma_\omega(x) = \sigma(\frac{x}{\omega}).$$

We claim now that for $\epsilon > 0$ given, $\exists \omega > 0$ and a sequence $y_n$ such that, after passing to a subsequence, the functions

$$(f_n^{(1)}(x), g_n^{(1)}(x)) = \rho_\omega(x - y_n)(f_n(x), g_n(x));$$

$$(f_n^{(2)}(x), g_n^{(2)}(x)) = \sigma_\omega(x - y_n)(f_n(x), g_n(x));$$

satisfy

$$|f_n^{(1)}|_{L^2}^2 \to s_1; \quad |f_n^{(2)}|_{L^2}^2 \to s - s_1;$$

$$|g_n^{(1)}|_{L^2}^2 \to t_1; \quad |g_n^{(2)}|_{L^2}^2 \to t - t_1;$$

where $|(s_1 + t_1) - \gamma| < \epsilon$, and for all $n$

$$E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) \leq E(f_n, g_n) + C\epsilon. \quad (3.12)$$

To see (3.12), notice that

$$E(f_n^{(1)}, g_n^{(1)})$$

$$= \int_{-\infty}^{\infty} \rho_\omega^2 \left( |f_n'|^2 + |g_n'|^2 - \frac{a}{2} |f_n|^4 - \frac{c}{2} |g_n|^4 - b|f_n|^2|g_n|^2 \right) \, dx$$

$$+ \int_{-\infty}^{\infty} \left( (\rho_\omega')^2 (|f_n|^2 + |g_n|^2) + 2\rho_\omega \rho_\omega' (\Re(f_n)(\bar{f}_n) + \Re(g_n)(\bar{g}_n)) \right) \, dx$$

$$+ \int_{-\infty}^{\infty} \left( \rho_\omega^2 - \rho_\omega' \right) \left( \frac{a}{2} |f_n|^4 + \frac{c}{2} |g_n|^4 + b|f_n|^2|g_n|^2 \right) \, dx$$

$$= \int \rho_\omega^2 \left( |f_n'|^2 + |g_n'|^2 - \frac{a}{2} |f_n|^4 - \frac{c}{2} |g_n|^4 - b|f_n|^2|g_n|^2 \right) \, dx + C\epsilon,$$

where $\Re(f_n)(\bar{f}_n)$ denotes the real part of $(f_n)(\bar{f}_n)$, because of the following:

i) $|\rho_\omega'|_\infty \leq \frac{2}{\omega} |\rho'|_\infty \leq \frac{2}{\omega} \leq C\epsilon$, by taking $\omega$ sufficiently large.

ii)

$$| \int_{-\infty}^{\infty} (\rho_\omega^2 - \rho_\omega') \left( \frac{a}{2} |f_n|^4 + \frac{c}{2} |g_n|^4 + b|f_n|^2|g_n|^2 \right) \, dx |$$

$$\leq C \int_{\omega \leq |x - y_n| \leq 2\omega} (|f_n|^2 + |g_n|^2) \, dx$$

$$\leq C\epsilon,$$

since for each $n \geq N$, we can find $y_n$ such that

$$\int_{y_n - \omega}^{y_n + \omega} (|f_n|^2 + |g_n|^2) \, dx > s_1 + t_1 - \epsilon,$$

$$\int_{y_n - 2\omega}^{y_n + 2\omega} (|f_n|^2 + |g_n|^2) \, dx < s_1 + t_1 + \epsilon.$$

Similarly, we have

$$E(f_n^{(2)}, g_n^{(2)}) = \int \sigma_\omega^2 \left( |f_n'|^2 + |g_n'|^2 - \frac{a}{2} |f_n|^4 - \frac{c}{2} |g_n|^4 - b|f_n|^2|g_n|^2 \right) \, dx + C\epsilon.$$
Consequently,
\[ E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) \leq E(f_n, g_n) + C \epsilon \]
because \( s_n^2 + \sigma_n^2 = 1 \). Hence \((3.12)\) follows.

Now, if \( s_1, t_1, s - s_1 \) and \( t - t_1 \) are all positive, then by re-scaling \( f_n^{(i)} \) and \( g_n^{(i)} \) for \( i = 1, 2 \) so that
\[
|f_n^{(i)}|^2 = s_i, \quad |g_n^{(i)}|^2 = t_i,
\]
that is, let
\[
\alpha_n = \frac{\sqrt{s_1}}{|f_n^{(1)}|^2}, \quad \beta_n = \frac{\sqrt{t_1}}{|g_n^{(1)}|^2},
\]
\[
\gamma_n = \frac{\sqrt{s - s_1}}{|f_n^{(2)}|^2}, \quad \theta_n = \frac{\sqrt{t - t_1}}{|g_n^{(2)}|^2},
\]
which gives
\[
|\alpha_n f_n^{(1)}|^2 = s_1, \quad |\beta_n g_n^{(1)}|^2 = t_1,
\]
\[
|\gamma_n f_n^{(2)}|^2 = s - s_1, \quad |\theta_n g_n^{(2)}|^2 = t - t_1.
\]
As all the scaling factors tend to 1 as \( n \to \infty \),
\[
\liminf_{n \to \infty} E(f_n^{(1)}, g_n^{(1)}) + E(f_n^{(2)}, g_n^{(2)}) \geq I(s_1, t_1) + I(s - s_1, t - t_1).
\]
If \( s_1 = 0 \) and \( t_1 > 0 \), then
\[
\lim_{n \to \infty} E(f_n^{(1)}, g_n^{(1)}) = \lim_{n \to \infty} \int_{-\infty}^{\infty} \left( |(f_n^{(1)})_x|^2 + |(g_n^{(1)})_x|^2 - \frac{c}{2}|g_n^{(1)}|^4 \right) dx
\]
\[
\geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |(g_n^{(1)})_x|^2 - \frac{c}{2}|g_n^{(1)}|^4 \right) dx \geq I(0, t_1).
\]
Similar estimates hold if \( t_1, s - s_1 \) or \( t - t_1 \) are zero. Thus, in all the cases we have the limit inferior as \( n \to \infty \) of the left hand side of \((3.12)\) \( \geq I(s_1, t_1) + I(s - s_1, t - t_1) \).
Consequently,
\[
I(s_1, t_1) + I(s - s_1, t - t_1) \leq I(s, t) + C \epsilon,
\]
which implies that
\[
I(s_1, t_1) + I(s - s_1, t - t_1) \leq I(s, t),
\]
as \( \epsilon > 0 \) is arbitrary. \( \square \)

The following Lemma rules out the possibility of dichotomy of minimizing sequences.

**Lemma 3.10.** For any minimizing sequence \((f_n, g_n)\), we have
\[
\gamma = s + t.
\]

**Proof.** Suppose not, then \( 0 < \gamma < s + t \). Let \( s_1, t_1 \) be defined as in Lemma 3.9, and let \( s_2 = s - s_1, t_2 = t - t_1 \). Then \( s_2 + t_2 = (s + t) - \gamma > 0 \), and \( s_1 + t_1 = \gamma > 0 \).
Since \( s_1 + s_2 = s > 0 \), and \( t_1 + t_2 = t > 0 \), Lemma 3.9 implies then that
\[
I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2).
\]
But this gives a contradiction to Lemma 3.9. \( \square \)

With all the above calculations at hand, we now proceed to prove Theorem 2.1.
Proof. (of Theorem 2.1) As the vanishing and dichotomy of minimizing sequences both have been ruled out, the Concentration Compactness Lemma [20, 21] asserts that minimizing sequences must be compact up-to translations. Hence statement 1) follows directly. Moreover, since the functionals $E$ and $Q$ are all invariant under translations, an immediate consequence of the above result is that the set of minimizers $S_{s,t}$ is stable. (See, for example, [23].) Thus, statement 4) is also clear.

To see the validity of statements 2) and 3), notice that the Lagrange multiplier principle guarantees that there are real numbers $\omega_1, \omega_2$ such that

$$E'(\phi, \psi) + \omega_1 Q'(\phi) + \omega_2 Q'(\psi) = 0,$$

where the prime denotes the Fréchet derivative. Thus, the following equations

$$\begin{align*}
\int -\phi'' + \omega_1 \phi &= a|\phi|^2 \phi + b|\psi|^2 \phi, \\
\int -\psi'' + \omega_2 \psi &= c|\psi|^2 \psi + b|\phi|^2 \psi,
\end{align*}$$

(3.13)

hold at least in the sense of distributions. A straightforward bootstrapping argument reveals that indeed (3.13) holds true in the classical sense as well.

Multiplying the equations (3.13) by $(\tilde{\phi}, \tilde{\psi})$ and integrating over $\mathbb{R}$, we obtain

$$\begin{align*}
\int_{-\infty}^{\infty} (|\phi'|^2 - a|\phi|^4 - b|\phi|^2|\psi|^2) dx &= -\omega_1 \int_{-\infty}^{\infty} |\phi|^2 dx = -\omega_1 s, \\
\int_{-\infty}^{\infty} (|\psi'|^2 - c|\psi|^4 - b|\phi|^2|\psi|^2) dx &= -\omega_2 \int_{-\infty}^{\infty} |\psi|^2 dx = -\omega_2 t.
\end{align*}$$

(3.14)

But part iii) in Lemma 3.2 implies that

$$\begin{align*}
\int_{-\infty}^{\infty} (|\phi'|^2 - a|\phi|^4 - b|\phi|^2|\psi|^2) dx \\
= \int_{-\infty}^{\infty} (|\phi'|^2 - \frac{a}{2}|\phi|^4 - b|\phi|^2|\psi|^2) dx - \int_{-\infty}^{\infty} \frac{a}{2} |\phi|^4 dx < 0,
\end{align*}$$

(3.15)

$$\int_{-\infty}^{\infty} (|\psi'|^2 - c|\psi|^4 - b|\phi|^2|\psi|^2) dx \\
= \int_{-\infty}^{\infty} (|\psi'|^2 - \frac{c}{2}|\psi|^4 - b|\phi|^2|\psi|^2) dx - \int_{-\infty}^{\infty} \frac{c}{2} |\psi|^4 dx < 0,$$

since $a, c > 0$. Consequently, (3.14) and (3.15) assert that $\omega_1, \omega_2 > 0$.

To prove the rest of statement 2) in Theorem 2.1, assume for the moment the validity of statement 3). That is, for every $(\phi, \psi) \in S_{s,t}$, there exist numbers $\theta_1, \theta_2 \in \mathbb{R}$ and functions $\tilde{\phi}(x) > 0$, $\tilde{\psi}(x) > 0$ for all $x \in \mathbb{R}$ such that $\phi(x) = e^{i\theta_1} \tilde{\phi}(x)$, and $\psi(x) = e^{i\theta_2} \tilde{\psi}(x)$. Then, (3.13) also holds with $\phi$ and $\psi$ being replaced by $\tilde{\phi}$ and $\tilde{\psi}$. Suppose to the contrary that $0 < a < b < c$ but $\omega_1 \geq \omega_2$. After multiplying the first equation by $\tilde{\psi}$ and the second by $\tilde{\phi}$ and subtracting them, we have

$$\int_{-\infty}^{\infty} \tilde{\phi} \tilde{\psi} (a \tilde{\phi}^2 - c \tilde{\psi}^2 + b \tilde{\psi}^2 - b \tilde{\phi}^2) dx = \int_{-\infty}^{\infty} (\omega_1 - \omega_2) \tilde{\phi} \tilde{\psi} dx.$$ 

(3.16)

Since $\tilde{\phi}(x) > 0$, $\tilde{\psi} > 0$ for all $x \in \mathbb{R}$, and $\omega_1 \geq \omega_2$, (3.16) implies that $(a-b) \tilde{\phi}^2 + (b-c) \tilde{\psi}^2 \geq 0$, which is impossible. Similar argument shows that when $0 < c < b < a$, then $\omega_1 > \omega_2$.

It is left to prove statement 3). Rewrite the complex-valued functions $\phi$ and $\psi$ as

$$\phi(x) = e^{i\theta_1(x)} \tilde{\phi}(x), \quad \psi(x) = e^{i\theta_2(x)} \tilde{\psi}(x)$$
where \( \tilde{\phi}(x) = |\phi(x)| \geq 0 \), \( \tilde{\psi}(x) = |\psi(x)| \geq 0 \), and \( \theta_1, \theta_2 : \mathbb{R} \rightarrow \mathbb{R} \). By Lemma 3.3, \((\tilde{\phi}, \tilde{\psi})\) is also a minimizer of the variational problem, hence (3.13) is satisfied by \((\phi, \psi)\). (The Lagrange multipliers stay the same as they are determined by (3.14), which are unchanged when \((\phi, \psi)\) is replaced by \((\tilde{\phi}, \tilde{\psi})\).) Computing the second derivative yields

\[
\phi'' = e^{i\theta_1(x)} \left( \phi'' - (\theta'_1(x))^2 \phi + 2i\theta'_1(x)\phi' + i\theta''_1(x)\phi \right) = e^{i\theta_1(x)} \left( \omega_1 \phi - a|\phi|^2 \phi - b|\psi|^2 \phi - (\theta'_1(x))^2 \phi + 2i\theta'_1(x)\phi' + i\theta''_1(x)\phi \right). \quad (3.17)
\]

On the other hand, the first equation in (3.13) gives

\[
\phi'' = \omega_1 \phi - a|\phi|^2 \phi - b|\psi|^2 \phi = \omega_1 e^{i\theta_1(x)} \tilde{\phi}(x) - a|\phi|^2 e^{i\theta_1(x)} \tilde{\phi} - b|\psi|^2 e^{i\theta_1(x)} \tilde{\phi}. \quad (3.18)
\]

Comparing (3.17) and (3.18), we arrive at the following

\[
(\theta'_1(x))^2 \phi(x) - 2i\theta'_1(x)\phi'(x) - i\theta''_1(x)\phi(x) = 0,
\]

holding for all \( x \in \mathbb{R} \). Equating the real and imaginary parts of the latter, we conclude that \( \theta'_1(x) = 0 \). Hence, \( \theta_1(x) = \text{constant} = \theta_1 \). Similarly, \( \theta_2(x) = \text{constant} = \theta_2 \).

Straightforward exercise (using Fourier Analysis, for example), we can rewrite the Lagrange equations associated with \((\phi, \psi)\) as

\[
\tilde{\phi} = K_{\omega_1} \ast (a|\phi|^2 \phi + b|\psi|^2 \phi), \quad \tilde{\psi} = K_{\omega_2} \ast (a|\tilde{\psi}|^2 \tilde{\psi} + b|\tilde{\phi}|^2 \tilde{\psi}),
\]

where

\[
K_{\omega}(x) = \frac{1}{2\sqrt{\omega}} e^{-\sqrt{\omega}|x|}.
\]

Since the convolution of \( K_{\omega} \) with a function that is everywhere non-negative and not identically zero gives an everywhere positive function, it follows that \( \phi(x) > 0 \) and \( \psi(x) > 0 \), \( \forall x \in \mathbb{R} \). Thus, Theorem 2.1 is proved. \( \square \)

**Proof.** (of Theorem 2.2) It is clear that in order to prove Theorem 2.2, we only need to justify the validity of Lemmas 3.1, 3.2 and 3.6 in the presence of the condition \( a, c < 0 \) and \( b > 0 \) such that \( b^2 > ac \). Recall that, for any fixed \( \omega > 0 \), we now take \( s = 4\sqrt{\omega} \frac{b-c}{b^2-ac} \) and \( t = 4\sqrt{\omega} \frac{b-a}{b^2-ac} \). We make the following claims.

**Claim 1.** Let \( a, c < 0 \) and \( b > 0 \) such that \( b^2 > ac \). Then Lemma 3.1 still holds.

**Proof.** (of Claim 1). This has been proved in Lemma 3.1 of [23] under either condition (A1) or (A2). \( \square \)

**Claim 2.** Let \( a, c < 0 \) and \( b > 0 \) such that \( b^2 > ac \). Then Lemma 3.2 is still valid.

**Proof.** (of Claim 2). Suppose to the contrary that i) is false, then by passing to a subsequence if necessary, we may assume there exists a minimizing sequence such that \( \lim_{n \rightarrow \infty} |f'_n|_2 = 0 \). By Gagliardo-Nirenberg inequality,

\[
\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_n|^4 dx = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f_n|^2 |g_n|^2 dx = 0.
\]

Thus,

\[
I(s, t) = \lim_{n \rightarrow \infty} E(f_n, g_n) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \left( |g'_n|^2 - \frac{c}{2} |g_n|^4 \right) dx \geq 0, \quad (3.19)
\]
since $c < 0$, which is a contradiction to Lemma 3.1. The case ii) can be proved similarly.

To see iii), suppose the statement is false. By passing to a subsequence if necessary, we may assume that there exists a minimizing sequence $(f_n, g_n)$ for which

$$\liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |f'_n|^2 - \frac{a}{2} |f_n|^4 - b |f_n|^2 |g_n|^2 \right) dx \geq 0.$$  

Hence,

$$I(s, t) = \lim_{n \to \infty} E(f_n, g_n) \geq \liminf_{n \to \infty} \int_{-\infty}^{\infty} \left( |g'_n|^2 - \frac{c}{2} |g_n|^4 \right) dx \geq 0,$$

which again contradicts Lemma 3.1. Similar argument can be used to prove the other case.

**Claim 3.** Let $a, c < 0$ and $b > 0$ such that $b^2 > ac$. Then Lemma 3.6 is still valid.

**Proof (of Claim 3).** One only needs to see that in Case 3 when $s_1 > 0$ and $t_2 > 0$,

$$I(s_1, 0) = \inf \left\{ \int_{-\infty}^{\infty} (|f'|^2 - \frac{a}{2} |f|^4) dx : f \in H^1_\mathbb{R}, \int_{-\infty}^{\infty} |f|^2 dx = s_1 > 0 \right\} \geq 0$$

and

$$I(0, t_2) = \inf \left\{ \int_{-\infty}^{\infty} (|f'|^2 - \frac{c}{2} |f|^4) dx : f \in H^1_\mathbb{R}, \int_{-\infty}^{\infty} |f|^2 dx = t_2 > 0 \right\} \geq 0.$$

Thus, we have $I(s_1, t_2) < I(s_1, 0) + I(0, t_2)$. □

The proof of Theorem 2.2 hence is now complete. □

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