An Optimization Approach to Jacobian Conjecture

Jiang Liu
Chongqing Institute of Green and Intelligent Technology
Chinese Academy of Sciences
✉: liujiang@cigit.ac.cn

Abstract

Let \( n \geq 2 \) and \( \mathbb{K} \) be a number field of characteristic 0. Jacobian Conjecture asserts for a polynomial map \( \mathcal{P} \) from \( \mathbb{K}^n \) to itself, if the determinant of its Jacobian matrix is a nonzero constant in \( \mathbb{K} \) then the inverse \( \mathcal{P}^{-1} \) exists and is also a polynomial map. This conjecture was firstly proposed by Keller in 1939 for \( \mathbb{K}^n = \mathbb{C}^2 \) and put in Smale’s 1998 list of Mathematical Problems for the Next Century. This study is going to present a proof for the conjecture. Our proof is based on Drużkowski Map and Hadamard’s Diffeomorphism Theorem, and additionally uses some optimization idea.

Index Terms

D-map, Jacobian Conjecture, Polynomial Automorphism, Proper Map

I. INTRODUCTION

Let \( \mathbb{K} \) denote a number field, on which \( \mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables \( X = (x_1, \ldots, x_n) \). Each polynomial vector \( \mathcal{P}(X) := (\mathcal{P}_1(X), \ldots, \mathcal{P}_n(X)) \in \mathbb{K}[X]^n \) defines a map from \( \mathbb{K}^n \) to \( \mathbb{K}^n \). The \( n \times n \) Jacobian matrix \( \mathbf{J}_\mathcal{P}(X) \) (\( \mathbf{J}_\mathcal{P} \) for short) of \( \mathcal{P}(X) \) consists of the partial derivatives of \( \mathcal{P}_i \) with respect to \( x_j \), that is, \( \mathbf{J}_\mathcal{P} = [\frac{\partial \mathcal{P}_i}{\partial x_j}, i, j = 1, \ldots, n] \). Then the determinant \( \det(\mathbf{J}_\mathcal{P}) \) of \( \mathbf{J}_\mathcal{P} \) is a polynomial function of \( X \). Jacobian Condition is that \( \det(\mathbf{J}_\mathcal{P}) \) is a nonzero constant in \( \mathbb{K} \). A \( \mathcal{P} \in \mathbb{K}[X]^n \) is called Keller map if it satisfies Jacobian Condition.

This work was partially supported by NSF of China (No. 61672488), CAS Youth Innovation Promotion Association (No. 2015315).
A polynomial map \( P \in \mathbb{K}[X]^n \) is an automorphism of \( \mathbb{K}^n \) if the inverse \( P^{-1} \) exists and is also a polynomial map. Due to Osgood’s Theorem, the Jacobian Condition is necessary for \( P \) being an automorphism. Keller [2] proposed the Jacobian Conjecture (\( JC \)) for \( \mathbb{K}^n = \mathbb{C}^2 \) in 1939, where \( \mathbb{C} \) is the complexity field. We refer the reader to [1], [3]–[6] for a nice survey paper containing some history and updated progresses of the Jacobian Conjecture. Abhyankar [7] gave its modern style as follows.

**Jacobian Conjecture** — \( JC(\mathbb{K}, n) \): If \( P \in \mathbb{K}[X]^n \) is a Keller map where \( \mathbb{K} \) is an arbitrary field of characteristic 0 and the integer \( n \geq 2 \), then \( P \) is an automorphism.

\( JC(\mathbb{K}, 1) \) is trivially true. Besides, there are simple counterexamples [3], [8] for \( JC(\mathbb{K}, n) \) when the number field \( \mathbb{K} \) has characteristic > 0. So the characteristic 0 condition is necessary. These two assumptions are always supposed to be true in the rest of this article unless a related statement is specified particularly. Let \( JC(\mathbb{K}) \) stand for \( JC(\mathbb{K}, n) \) being true for all positive integers \( n \). Due to Lefschetz Principle, it suffices to deal only with \( JC(\mathbb{C}) \). For polynomial maps in \( \mathbb{C}[X]^n \), it is well known that

**Theorem 1.1** ([3], [9]–[11]): If a polynomial map \( P \in \mathbb{C}[X]^n \) is injective then it must be an automorphism.

Thus, it suffices to show that each Keller map in \( \mathbb{C}[X]^n \) is injective. Furthermore, Yag\’\'zov [10] and independently Bass et al. [3] showed that it is sufficient to consider Keller maps of form \( P(X) := X + \mathcal{H}(X) \in \mathbb{C}[X]^n \) which is called Yag\’\'zov map, where \( \mathcal{H}(X) \) is a homogeneous polynomial vectors of degree 3. Based on this result, Dr\’\'zkowski [12] further showed that it suffices to consider Keller maps of so-called cubic linear form \( P(X) := X + (AX)^{\times 3} \in \mathbb{C}[X]^n \), nowadays, called Dr\’\'zkowski map (D-map for short), wherein \( (AX)^{\times 3} \) be the vector whose \( i \)-th element is \( (A^T_i)^X \) and \( A^T_i \) is the \( i \)-th row vector of \( A \). Interestingly, it has been pointed out in [1], [13] that for each D-map \( P(X) \in \mathbb{C}[X]^n \) there is a Yag\’\'zov map \( P^*(Y) \in \mathbb{R}[Y]^{2n} \) such that \( P(X) \) is injective iff \( P^*(Y) \) is injective. We applied Dr\’\'zkowski’s reduction in [12] again to get that there is some D-map \( P@^*(Z) \in \mathbb{R}[Z]^m \) with \( m \geq n \) such that \( P^*(Y) \) is injective iff \( P@^*(Z) \) is injective. In consequence,

**Proposition 1.2**: If each D-map in \( \mathbb{R}[X]^n \) is injective for all \( n \), then each Keller map \( P \in \mathbb{C}[X]^n \) is injective and so an automorphism.

In the study, we will use Hadamard’s Diffeomorphism Theorem to show the injectivity of D-maps in \( \mathbb{R}[X]^n \). By Jacobian Condition, each Keller map is a local homeomorphism. Even for analytic maps, Hadamard’s Diffeomorphism Theorem presents a necessary and sufficient
condition for local homeomorphism being diffeomorphism in terms of the proper map. A
differentiable map $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is an *diffeomorphism* if it is bijective and its inverse is
also differentiable. A continuous map $H : \mathbb{R}^n \mapsto \mathbb{R}^n$ is *proper* if $H^{-1}(C)$ is compact for each
compact set $C \subset \mathbb{R}^n$. Equivalently, a map $H$ is proper iff it maps each unbounded set $\mathcal{B}$ into
an unbounded set $H(\mathcal{B})$. Based on these notions, in (\cite{14}, \cite{15}) it was showed that

**Hadamard Theorem 1.3:** For an analytic map $H : \mathbb{R}^n \mapsto \mathbb{R}^n$, $H$ is a diffeomorphism iff
$\det JH(X) \neq 0$ for all $X \in \mathbb{R}^n$ and $H$ is proper.

Therefore, it suffices to show the properness of the D-maps in $\mathbb{R}[X]^n$ for their injectivity. To
this end, We are going to show the following crucial result in the next section.

**Proper Map Theorem 1.4:** Each D-map in $\mathbb{R}[X]^n$ is proper and so injective.

Now by Proposition \ref{prop1.2} we can conclude

**Polynomial Automorphism Theorem 1.5:** For every $n \geq 2$ and any field $\mathbb{K}$ of characteristic 0,
each Keller map $P \in \mathbb{K}[X]^n$ is an automorphism.

In the rest of this article, we assume $n \geq 2$ for the dimension of $\mathbb{R}^n$ unless it is specified
in particular. For clarity, let capital letters denote vectors, blackboard bold letters denote sets
of vectors (points), fraktur letters denote vector functions, little letters denote integers, Greek
letters denote real numbers and real functions.

## II. Proof of Proper Map Theorem

Theorem \ref{thm1.4} will be proved by contradiction. To this end, it needs to study the properties of
D-map $D(X) := X + (AX)^{\Delta^3}$ based on the matrix $A$. Let’s start with some basic notions. A set
$\mathcal{B} \subset \mathbb{R}^n$ is called *unbounded* if for any $\gamma \in \mathbb{R}$ there is an element $X \in \mathcal{B}$ such that $||X|| > \gamma$,
where $|| \cdot ||$ is the Euclidean norm, that is, $||X|| := \sqrt{\sum_{1 \leq i \leq n} X_i^2}$.

Given a matrix $A \in \mathbb{R}^{n \times n}$, let $(AX)^{\Delta^2}$ denote the diagonal matrix whose diagonals are
$(A^2_{11}X^2, \ldots, A^2_{nn}X^2)$. For a D-map $D(X) := X + (AX)^{\Delta^3}$, its Jacobian matrix is
$J_D = I + 3(AX)^{\Delta^2}A$ where $I$ is the $n \times n$ identity matrix. In \cite{12}, it was showed that

**Theorem 2.1:** $D(X) := X + (AX)^{\Delta^3}$ is a D-map iff $(AX)^{\Delta^2}A$ is a nilpotent matrix for each
$X$. And each D-map has only one zero root: vector $0$.

From this theorem, the following proposition is evident.

**Proposition 2.2:** If $D(X) := X + (AX)^{\Delta^3}$ is a D-map, then for all $\lambda \in \mathbb{R}$, the map $D_\lambda(X) :=
X + \lambda(AX)^{\Delta^3}$ is also a D-map, and for any nonzero $X$, $D_\lambda(X) \neq 0$ or equivalently $||D_\lambda(X)|| > 0$.

Immediately, we have

March 12, 2020 DRAFT
Corollary 2.3: For any D-map $\mathcal{D}(X) := X + (AX)^3$, no $X$ satisfies $X^T(AX)^3 = \pm ||(AX)^3|| \neq 0$. This means $\frac{X^T(AX)^3}{||(AX)^3||} \neq \pm 1$ for all $X$.

Proof: We prove this result by contradiction. Given a D-map $\mathcal{D}(X)$, suppose that there is some $X$ such that $X^T(AX)^3 = \delta ||(AX)^3|| \neq 0$, where $\delta = -1$ if $X^T(AX)^3 < 0$ and $\delta = 1$ if $X^T(AX)^3 > 0$.

By Proposition 2.2, $\mathcal{D}_\delta(X) := X - \delta(AX)^3$ is also a Keller map. Let’s consider the following equation

$$||\mathcal{D}_\delta(\sqrt{\lambda X})||^2 = \lambda - 2\lambda^2 \delta X^T(AX)^3 + \lambda^3 ||(AX)^3||^2 = 0$$

It has a zero root $\lambda = \frac{\delta X^T(AX)^3}{||X^T(AX)^3||^2} > 0$. That is, $\mathcal{D}(X)$ has a nonzero root, which contradicts with Proposition 2.2. Therefore, such $X$ does not exist.

Given a matrix $A \in \mathbb{R}^{n \times n}$, let $\mathfrak{A}^\perp := \{X \in \mathbb{R}^n \mid AX = 0\}$ denote the linear space in which each element is a solution of $AX = 0$, and $\mathfrak{A}^{r Sp}$ stand for the linear space spanned by the row vectors of $A$. Regarding the elements of $\mathfrak{A}^{r Sp}$ as column vectors, then $\mathfrak{A}^{r Sp} = \{Y \mid Y = A^T Z, Z \in \mathbb{R}^n\}$. Furthermore, for any $X \in \mathfrak{A}^\perp$ and $Y = A^T Z \in \mathfrak{A}^{r Sp}$, we have $Y^T X = Z^T AX = 0$, that is, $Y \perp X$. In consequence,

Proposition 2.4: The spaces $\mathfrak{A}^\perp$ and $\mathfrak{A}^{r Sp}$ have the following properties:

1) $\mathfrak{A}^\perp$ and $\mathfrak{A}^{r Sp}$ are orthogonal to each other.

2) $\mathfrak{A}^\perp \cap \mathfrak{A}^{r Sp} = 0$, and so $||AY|| \neq 0$ for any nonzero $Y \in \mathfrak{A}^{r Sp}$.

3) Both $\mathfrak{A}^\perp$ and $\mathfrak{A}^{r Sp}$ are closed sets.

4) For any nonzero $Z \in \mathbb{R}^n$, it can be uniquely decomposed as $Z = X + Y$ such that $X \perp Y$, $X \in \mathfrak{A}^\perp$, and $Y \in \mathfrak{A}^{r Sp}$.

Let $\mathcal{U} := \{X \in \mathbb{R}^n \mid ||X|| = 1\}$, which is a compact set in $\mathbb{R}^n$. Then

Proposition 2.5: Both $\mathcal{U}(\mathfrak{A}^\perp) := \mathcal{U} \cap \mathfrak{A}^\perp$ and $\mathcal{U}(\mathfrak{A}^{r Sp}) := \mathcal{U} \cap \mathfrak{A}^{r Sp}$ are closed and so compact. Moreover, $||AY|| \neq 0$ for any $Y \in \mathcal{U}(\mathfrak{A}^{r Sp})$, and furthermore $\inf_{Y \in \mathcal{U}(\mathfrak{A}^{r Sp})} ||(AY)^3|| = \epsilon > 0$ and $\sup_{Y \in \mathcal{U}(\mathfrak{A}^{r Sp})} ||(AY)^3|| = \psi < \infty$ for some positive real numbers $\epsilon$ and $\psi$.

Fix an $X \in \mathcal{U}$, we consider $\mathcal{D}(\lambda, X) := \mathcal{D}(\sqrt[3]{\lambda X}) = \sqrt[3]{\lambda} X + (\sqrt{\lambda})^3 (AX)^3$ for $\lambda > 0$. We study the change of $||\mathcal{D}(\lambda, X)||$ with respect to $\lambda > 0$. First,

$$||\mathcal{D}(\lambda, X)||^2 = \lambda + 2\lambda^2 X^T(AX)^3 + \lambda^3 ||(AX)^3||^2$$

(2)
As a result, $\|D(\lambda, X)\| \to \infty$ as $\lambda \to \infty$ no matter what is $AX$. Suppose $\|(AX)^{*3}\| \neq 0$, then its derivative with respect to $\lambda$ is

$$1 + 4\lambda X^T(AX)^{*3} + 3\lambda^2\|(AX)^{*3}\|^2$$  \hspace{1cm} (3)$$

When $4(AX)^{*3})^2 - 3\|(AX)^{*3}\|^2 \geq 0$, (3) has zero roots:

$$\lambda(X) := -\frac{2X^T(AX)^{*3} \pm \sqrt{4(AX)^{*3})^2 - 3\|(AX)^{*3}\|^2}}{3\|(AX)^{*3}\|^2}$$  \hspace{1cm} (4)$$

Let $D := \{X \mid 4(AX)^{*3})^2 - 3\|(AX)^{*3}\|^2 \geq 0 \& X^T(AX)^{*3} \leq 0\} \cap U$. Suppose $X \in D$, then the smaller solution of (3) is

$$\lambda_1(X) := -\frac{2X^T(AX)^{*3} - \sqrt{4(AX)^{*3})^2 - 3\|(AX)^{*3}\|^2}}{3\|(AX)^{*3}\|^2}$$  \hspace{1cm} (5)$$

and the bigger solution of (3):

$$\lambda_2(X) := -\frac{2X^T(AX)^{*3} + \sqrt{4(AX)^{*3})^2 - 3\|(AX)^{*3}\|^2}}{3\|(AX)^{*3}\|^2} \geq \lambda_1(X)$$  \hspace{1cm} (6)$$

Note that, for any $X \in D$ it must be $\lambda_2(X) \geq 0$. For any fixed $X \in D$, if $\lambda_2(X) > \lambda_1(X)$ then $\lambda_2(X)$ is a local minimal point of $\|D(\lambda, X)\|$ with respect to $\lambda > 0$, since

(a) $\lambda_1(X) < \lambda < \lambda_2(X)$ implies $1 + 4\lambda X^T(AX)^{*3} + 3\lambda^2\|(AX)^{*3}\|^2 < 0$ and

(b) $\lambda > \lambda_2(X)$ implies $1 + 4\lambda X^T(AX)^{*3} + 3\lambda^2\|(AX)^{*3}\|^2 > 0$.

On the other side, if $\lambda_2(X) = \lambda_1(X)$ then $1 + 4\lambda X^T(AX)^{*3} + 3\lambda^2\|(AX)^{*3}\|^2 \geq 0$ for all $\lambda$. As a result, $\|D(\lambda, X)\|^2$ is an increasing function with respect to $\lambda$. In any case, we have

**Proposition 2.6:** For any $X \in D$ such that $\|(AX)^{*3}\| \neq 0$, the following equations hold

$$\inf_{\lambda > \lambda_1(X)} \|D(\lambda, X)\|^2 = \|D(\lambda_2(X), X)\|^2$$  \hspace{1cm} (7)$$

$$= \frac{16\theta^3_X - 18\theta_X + 6\sqrt{4\theta^2_X - 3} - 8\theta^2_X \sqrt{4\theta^2_X - 3}}{27\|(AX)^{*3}\|}$$  \hspace{1cm} (8)$$

where $\theta_X := \frac{X^T(AX)^{*3}}{\|(AX)^{*3}\|}$, $X \in D$.

Let $\phi(\tau) := 16\tau^3 - 18\tau + 6\sqrt{4\tau^2 - 3} - 8\tau^2 \sqrt{4\tau^2 - 3}$, then we have

**Proposition 2.7:** For $\tau \in [-1, -\frac{\sqrt{3}}{2}]$, $\phi(\tau) \geq 0$ and is a strictly increasing function, herein the equality holds only when $\tau = -1$.

Let’s consider a subset $\mathcal{R}$ of $\mathcal{U}(\mathbb{R}^d_x) \times \mathcal{U}(\mathbb{R}^d_y)$ defined by

$$\mathcal{R} := \{(V, W) \in \mathcal{U}(\mathbb{R}^d_x) \times \mathcal{U}(\mathbb{R}^d_y) \mid W^T(AW)^{*3} \leq 0 \& \frac{V^T(AW)^{*3}}{\|(AW)^{*3}\|} \leq -\frac{2\sqrt{2}}{3}\}$$  \hspace{1cm} (9)$$
By the definition, using Proposition 2.4 (2) can show that \( \mathcal{K} \) is closed. By Proposition 2.5 both \( \mathcal{U}(\mathfrak{A}^\perp) \) and \( \mathcal{U}(\mathfrak{A}^{rSp}) \) are bounded. Therefore,

**Proposition 2.8:** \( \mathcal{K} \) is a compact set of \( \mathbb{R}^{2n} \).

Note that \( \mathcal{K} \) has a clear topological structure. Because \( \mathcal{U}(\mathfrak{A}^\perp) \) and \( \mathcal{U}(\mathfrak{A}^{rSp}) \) are defined by algebraic inequalities, so they are semi-algebraic sets [16]. Because \( ||(AW)^{*3}|| > 0 \) for any \( W \in \mathcal{U}(\mathfrak{A}^{rSp}) \), \( \mathcal{K} \) is actually defined by several algebraic inequalities with some existential quantifiers. In fact, \( \mathcal{K} \) is the projection of a semi-algebraic set, which is a union of finitely many disjoint connected sets by Tarski’s Theorem.

Given a \( (V, W) \in \mathcal{K} \), let \( X_\alpha := \alpha V + \sqrt{1 - \alpha^2}W \) for \( \alpha \in [0, 1] \) and define an arc \( \mathfrak{A}(V, W; \alpha) \) by \( \mathfrak{A}(V, W; \alpha) := \{ X_\alpha \mid \alpha \in [0, 1] \} \) which is a curve on the unit sphere. For \( X_\alpha \in \mathfrak{A}(V, W; \alpha) \) with \( (V, W) \in \mathcal{K} \), we set \( \alpha^\# := \frac{3\sqrt{3}}{4\sqrt{2}} < 1 \) then \( \theta_{X_\alpha} := \frac{X_\alpha^T (AX_\alpha)^3}{\|\|D AX_\alpha\|\|} \leq \frac{\alpha V^T (AW)^{*3}}{|||D (AW)^{*3}|||} \leq -\frac{\sqrt{3}}{2} \) for all \( \alpha \in [\alpha^\#, 1] \neq \emptyset \) by (2). That is, \( \theta_{X_\alpha} \leq -\frac{\sqrt{3}}{2} \) for all adequate large \( \alpha \in [0, 1] \). Based on this observation, we obtain the tendency of \( \inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)||^2 \) when \( \alpha \) tends to 1 as follows.

**Lemma 2.9:** For any fixed \( (V, W) \in \mathcal{K} \), \( \lim_{\alpha \to 1} \inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)|| = \infty \).

**Proof:** As \( (V, W) \in \mathcal{K} \), for \( \alpha \in [\alpha^\#, 1] \) it has \( X_\alpha \in \mathcal{D} \) and \( ||AX_\alpha||^3 \neq 0 \). So by Proposition 2.6 for \( \alpha \in [\alpha^\#, 1] \) we have

\[
\inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)||^2 = \frac{16\theta^3_{X_\alpha} - 18\theta_{X_\alpha} + 6\sqrt{4\theta^2_{X_\alpha} - 3} - 3\theta^2_{X_\alpha} \sqrt{4\theta^2_{X_\alpha} - 3}}{27 (\sqrt{1 - \alpha^2})^3 ||(AW)^{*3}||} (10)
\]

\[
= \frac{\phi(\theta_{X_\alpha})}{27 (\sqrt{1 - \alpha^2})^3 ||(AW)^{*3}||} (11)
\]

where \( \theta_{X_\alpha} = v\alpha + \omega\sqrt{1 - \alpha^2} \) with \( v := \frac{V^T (AW)^{*3}}{|||D (AW)^{*3}|||} \leq -\frac{\sqrt{3}}{3} \) and \( \omega := \frac{W^T (AW)^{*3}}{|||D (AW)^{*3}|||} \leq 0 \), since \( (V, W) \in \mathcal{K} \). By Proposition 2.4 (2), \( \inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)||^2 \) is always well defined and \( > 0 \) for any \( \alpha \in [\alpha^\#, 1] \), since \( (V, W) \in \mathcal{K} \). For a fixed \( (V, W) \in \mathcal{K} \), \( \inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)||^2 \) is a function of \( \alpha \), precisely, a rational function of \( \alpha \) and \( \sqrt{1 - \alpha^2} \), in the interval \( [\alpha^\#, 1] \). So we can compute the limit by cases as follows.

**Case I:** \( v \in (-1, -\frac{2\sqrt{3}}{3}] \). By the method in [17], we have

\[
\lim_{\alpha \to 1} \inf_{\lambda > \lambda_1(X_\alpha)} ||D(\lambda, X_\alpha)||^2 = \lim_{\alpha \to 1} \frac{\phi(v\alpha + \omega\sqrt{1 - \alpha^2})}{27 (\sqrt{1 - \alpha^2})^3 ||(AW)^{*3}||} = \text{sign}(-4v^2\sqrt{4v^2 - 3} + 8v^3 + 3\sqrt{4v^2 - 3} - 9v) \cdot \infty (12)
\]

\[
= \infty (13)
\]
wherein $v \leq -\frac{2\sqrt{2}}{3}$ and $\text{sign}$ is the sign function. Thus $4v^2 > 3$. Note that, $-4v^2\sqrt{4v^2 - 3} + 8v^3 + 3\sqrt{4v^2 - 3} - 9v > 0$ for any $v \in (-1, -\frac{\sqrt{3}}{2})$ and so for all $v \in (-1, -\frac{2\sqrt{2}}{3})$.

Case II: $v = -1$, which implies $V = \ell(AW)^{p}$ for some $\ell < 0$. As a result, it must be $\omega = 0$ since $V \in \mathfrak{A}_{\lambda}$ and $W \in \mathfrak{A}^{\lambda}_{\mu}$. So $\theta_{\lambda} = -\alpha$ and then

$$\lim_{\alpha \to 1} \inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})||^2 = \lim_{\alpha \to 1} \frac{\phi(-\alpha)}{27 (\sqrt{1 - \alpha^2})^3 ||(AW)^{p}||} = \infty$$

(14)

In summary, $\lim_{\alpha \to 1} \inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| = \infty$ for all $(V, W) \in \mathfrak{A}$.

By Lemma 2.9, for any $(V, W) \in \mathfrak{A}$ and $\mu > 0$ there is some $\alpha \in [0, 1)$ such that

$$\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| \geq \mu$$

In fact, we have a more strong result as follows.

Lemma 2.10: For any $(V, W) \in \mathfrak{A}$ and $\mu > 0$, there is an $\alpha(V, W, \mu) < 1$ such that $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| \geq \mu$ for all $\alpha \in [\alpha(V, W, \mu), 1)$.

Proof: For a fixed $(V, W) \in \mathfrak{A}$, $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})||^2$ is a function of $\alpha \in [\alpha^{\#}, 1)$ by (11), precisely, is a rational function of $\alpha$ and $\sqrt{1 - \alpha^2}$. As a result, its derivative with respect to $\alpha$ has at most finitely many zero in $[\alpha^{\#}, 1)$. Therefore, its monotonicity can change at most finitely many times. Because

$$\lim_{\alpha \to 1} \inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| = \infty$$

there must exist $\alpha(V, W) < 1$, which is the greatest zero root of the derivative, such that $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})||$ is increasing in the interval $[\alpha(V, W), 1)$. Given a positive number $\mu$, let $\alpha(V, W, \mu)$ be the least number $\alpha$ such that $\alpha \geq \alpha(V, W)$ and $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| \geq \mu$. Such $\alpha(V, W, \mu)$ must exist and $< 1$ by Lemma 2.9. So $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})||$ is increasing with respect to $\alpha \in [\alpha(V, W, \mu), 1)$. Therefore $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| \geq \mu$ for all $\alpha \in [\alpha(V, W, \mu), 1)$.

Given a $\mu > 0$, let $\alpha(\mathfrak{A}, \mu) = \max_{(V, W) \in \mathfrak{A}} \alpha(V, W, \mu)$, then we have

Lemma 2.11: $\alpha(\mathfrak{A}, \mu) < 1$ for any $\mu > 0$. Furthermore, for any $\alpha \in [\alpha(\mathfrak{A}, \mu), 1)$ and $(V, W) \in \mathfrak{A}$, $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})|| \geq \mu$.

Proof: In the proof of Lemma 2.10 it has shown that $\alpha(V, W, \mu) < 1$ for each $(V, W) \in \mathfrak{A}$ and $\mu > 0$. By (11), $\inf_{\lambda > \lambda_{1}(X_{\alpha})} ||D(\lambda, X_{\alpha})||$ is a continuous function with respect to $V, W, \alpha$. So $\alpha(V, W)$ in the proof of Lemma 2.10 is a continuous function with respect to $V$ and $W$. Thus,
for any fixed $\mu$, $\alpha(V,W,\mu)$ is a continuous function with respect to $(V,W)$. Note that $\mathcal{R}$ is a compact set by Proposition 2.8. Therefore $\alpha(\mathcal{R},\mu) = \alpha(V,\mu) < 1$ for some $(V,\mu) \in \mathcal{R}$.

For any $\alpha \in [\alpha(\mathcal{R},\mu), 1)$ and $(V,W) \in \mathcal{R}$, we have $\alpha \geq \alpha(\mathcal{R},\mu) \geq \alpha(V,W,\mu)$. That is, $\alpha \in [\alpha(V,W,\mu), 1)$. By Lemma 2.10 \[ \inf_{\lambda > \lambda_t(X,\alpha)} ||D(\lambda,X,\alpha)|| \geq \mu. \]  

Given a $\mu > 0$ and $Y \in \mathcal{U}(\mathcal{A}^\perp)$, we take an open neighbourhood $\mathcal{N}_\mu(Y)$ of $(\mu + 1)Y$ such that $||D(X)|| > \mu$ for all $X \in \mathcal{N}_\mu(Y)$. This can be done, because $||D((\mu + 1)Y)|| = ||(\mu + 1)Y|| = \mu + 1$ and $||D(X)||$ is a continuous function about $X$. Given a point $P$ and $\theta \in [0,1)$, let $\mathcal{C}(P,\alpha) = \{X \mid \frac{X^TP}{||X|| ||P||} > \theta\}$. Then $\mathcal{C}(P,\alpha)$ is an open set, and contains $P$ since $\frac{P^TP}{||P||^2} = 1 > \theta$. In the view of geometry, $\mathcal{C}(P,\alpha)$ is an open cone with vertex 0 such that the angle $\angle(P,X)$ between $X$ and $P$ is no more than $\arccos \theta$ for any $X \in \mathcal{C}(P,\alpha)$. So $\mathcal{C}(P,\alpha)$ is an unbounded open neighbourhood of $P$. Note that, the cone $\mathcal{C}(P,\alpha)$ is symmetry in the sense that if $X = \lambda P + Y$ is in $\mathcal{C}(P,\alpha)$ for some $Y \perp P$ and real number $\lambda$ then $X^* := \lambda P - Y$ is also in $\mathcal{C}(P,\alpha)$. Another important property of $\mathcal{C}(P,\alpha)$ is the following invariant.

**Proposition 2.12:** If $X \in \mathcal{C}(P,\alpha)$ then $\lambda X \in \mathcal{C}(P,\alpha)$ for all $\lambda > 0$. In particular, $\frac{X}{||X||} \in \mathcal{C}(P,\alpha)$.

Now, we consider $\mathcal{G}_\mu(Y) := \mathcal{N}_\mu(Y) \cap \mathcal{C}(Y,\alpha(\mathcal{R},\mu)) \ni Y$ for $\mu > 0$ and $Y \in \mathcal{U}(\mathcal{A}^\perp)$. Note that $\mathcal{G}_\mu(Y)$ is an open set because $\alpha(\mathcal{R},\mu) < 1$. Take an $X \in \mathcal{G}_\mu(Y)$, we decompose it as $X = \zeta(\mu + 1)Y + \beta Z$ for some $Z \perp Y$ with $Z \in \mathcal{U}$ and real numbers $\zeta, \beta \geq 0$. We further decompose $Z$ as $Z = \rho E + \xi H$ such that $E \in \mathcal{U}(\mathcal{A}^\perp)$ and $H \in \mathcal{U}(\mathcal{A}^{rSp})$ for some real numbers $\rho, \xi \geq 0$. Then

$$X = \zeta(\mu + 1)Y + \beta Z$$

$$= \zeta(\mu + 1)Y + \beta \rho E + \beta \xi H$$

As a result, $Y \perp E$, $Y \perp H$ and so $\zeta(\mu + 1)Y + \beta \rho E \in \mathcal{A}^\perp$. Now, we consider the decomposition \[ \frac{X}{||X||} = \alpha V + \sqrt{1 - \alpha^2} W \] such that $V \in \mathcal{U}(\mathcal{A}^\perp)$ and $W \in \mathcal{U}(\mathcal{A}^{rSp})$, with $\alpha \in [0,1]$. Note that, such decomposition exists and is unique. As a result, we have \[ \alpha V = \frac{\zeta(\mu + 1)Y + \beta \rho E}{||X||}, \quad W = \frac{\rho E}{||W||}. \]
and
\[\alpha = \frac{\sqrt{\zeta^2(\mu+1)^2||Y||^2 + \beta^2\rho^2||E||^2}}{||X||}\]  \hspace{1cm} (17)
\[\geq \frac{\zeta(\mu+1)||Y||}{||X||} \hspace{1cm} (18)
\[= \frac{(\mu+1)Y^TX}{||(\mu+1)||^2 \cdot ||X||} \hspace{1cm} (19)
\[\geq \alpha(\mathfrak{R}, \mu) \hspace{1cm} (20)\]

where equality (19) holds by (15) and inequality (20) holds by \(X \in \mathfrak{G}_\mu(Y)\). By inequalities (18)-(20), if \((V, W) \in \mathfrak{R}\) then the corresponding \(\alpha\) with the decomposition of \(\frac{X}{||X||}\) satisfies \(\alpha \geq \alpha(\mathfrak{R}, \mu)\) and so \(\inf_{\lambda > \lambda_1(X)} \|\mathcal{D}(\lambda, \frac{X}{||X||})\| \geq \mu\) by Lemma 2.11. Note that, \(X \in \mathfrak{H}_\mu(Y)\) means \(\|\mathcal{D}(X)\| \geq \mu\). Furthermore,

**Lemma 2.13:** For a given \(X \in \mathfrak{G}_\mu(Y)\), if \(\frac{X}{||X||} = \alpha V + \sqrt{1-\alpha^2} W\) for some \(\alpha \in [0, 1), V \in \mathfrak{H}(\mathfrak{R})\) and \(W \in \mathfrak{H}(\mathfrak{R}^{Sp})\) such that \((V, W) \in \mathfrak{R}\), then \(\|\mathcal{D}(\ell X)\| \geq \mu\) for all \(\ell \geq 1\).

**Proof:** This result will be proved by cases. If \(\|X\| \geq \lambda_2(\frac{X}{||X||})\), then \(\|\mathcal{D}(\ell \frac{X}{||X||})\|\) is increasing for \(\ell \in [\lambda_2(\frac{X}{||X||}), \infty)\) and so for \(\ell \in [\|X\|, \infty)\). Thus, \(\|\mathcal{D}(\ell X)\| \geq \|\mathcal{D}(X)\|\) for all \(\ell \geq 1\) by the monotonicity. As \(X \in \mathfrak{G}_\mu(Y)\), \(\|\mathcal{D}(X)\| \geq \mu\). Therefore, \(\|\mathcal{D}(\ell X)\| \geq \mu\) for all \(\ell \geq 1\).

If \(\|X\| < \lambda_2(\frac{X}{||X||})\), there are two subcases.

**Subcase I:** \(\|X\| < \lambda_1(\frac{X}{||X||})\). Now \(\|\mathcal{D}(\ell X)\|\) is increasing for \(\ell \in [1, \lambda_1(\frac{X}{||X||})]\). So \(\|\mathcal{D}(\ell X)\| \geq \mathcal{D}(X)\| \geq \mu\) for \(\ell \in [1, \lambda_1(\frac{X}{||X||})]\). For \(\ell \in [\lambda_1(\frac{X}{||X||}), \infty)\), \(\|\ell X\| \geq \lambda_1(\frac{X}{||X||})\) and so \(\|\mathcal{D}(\ell X)\| \geq \inf_{\lambda > \lambda_1(\frac{X}{||X||})} \|\mathcal{D}(\lambda, \frac{X}{||X||})\| \geq \mu\) by \(X \in \mathfrak{G}_\mu(Y)\), \((V, W) \in \mathfrak{R}\) and Lemma 2.11 using \(\alpha \geq \alpha(\mathfrak{R}, \mu)\).

In summary, \(\|\mathcal{D}(\ell X)\| \geq \mu\) for all \(\ell \geq 1\) in this subcase.

**Subcase II:** \(\|X\| \geq \lambda_1(\frac{X}{||X||})\). In this case, \(\|\mathcal{D}(\ell X)\| \geq \inf_{\lambda > \lambda_1(\frac{X}{||X||})} \|\mathcal{D}(\lambda, \frac{X}{||X||})\| \geq \mu\) for all \(\ell \in [1, \infty)\), by \(X \in \mathfrak{G}_\mu(Y)\), \((V, W) \in \mathfrak{R}\) and Lemma 2.11 using \(\alpha \geq \alpha(\mathfrak{R}, \mu)\).

Now, we are ready to prove Theorem 1.4. By Hadamard Theorem 1.3, if a D-map \(\mathcal{D}(X) := X + (AX)^{*3}\) is not an automorphism, then there must be an unbounded sequence \(\{Y_i \in \mathbb{R}^n | i = 1, 2, \ldots , \infty\}\) \(((\{Y_i\})_i\) for simplicity, and such notation applied for other similar set in the article\) such that \(\{\mathcal{D}(Y_i)\}_i\) is a bounded sequence. Without loss of generality, we suppose that \(||Y_{i+1}|| > ||Y_i||\) for all \(i\). By the boundness of \(\{\mathcal{D}(Y_i)\}_i\), there is some real number \(\sigma < \infty\) such that
\[Y_i^T Y_i + 2Y_i^T (AY_i)^{*3} + ||(AY_i)^{*3}||^2 \leq \sigma^2 \hspace{1cm} (21)\]
for all $i$. Because $\{Y_i\}_i$ is unbounded and $D(X)$ has a unique zero root, it must be $Y_i^T(AY_i)^{3} < 0$ for almost all $i$. Without loss of generality, we suppose $Y_i^T(AY_i)^{3} < 0$ for all $i$. As a result, $(AY_i)^{3} \neq 0$ and so $||(AY_i)^{3}|| > 0$ for all $i$. Then by optimization theory [13], we have

$$ \| (AY_i)^{3} \|^2 \gamma^2 + 2Y_i^T(AY_i)^{3} \gamma + Y_i^T Y_i \geq Y_i^T Y_i - \frac{(Y_i^T(AY_i)^{3})^T}{\| (AY_i)^{3} \|^2} $$

for any $\gamma \in \mathbb{R}$ and all $i$. When $\gamma = 1$, we get

$$ Y_i^T Y_i - \frac{(Y_i^T(AY_i)^{3})^T}{\| (AY_i)^{3} \|^2} \leq Y_i^T Y_i + 2Y_i^T(AY_i)^{3} + \| (AY_i)^{3} \|^2 \leq \sigma^2 $$

Therefore,

$$ \lim_{i \to \infty} \left( 1 - \frac{(Y_i^T(AY_i)^{3})^T}{\| (AY_i)^{3} \|^2} \right) \leq \lim_{n \to \infty} \frac{\sigma^2}{Y_i^T Y_i} = 0 $$

since $\{Y_i\}_i$ is unbounded. Let $X_i = \frac{Y_i}{\|Y_i\|}$, then $X_i \in \mathcal{U}$, $(AX_i)^{3} \neq 0$ by the assumption $Y_i^T (AY_i)^{3} < 0$ for all $i$, and then

**Lemma 2.14:**

$$ \lim_{i \to \infty} \frac{X_i^T (AX_i)^{3}}{\| (AX_i)^{3} \|} = \lim_{i \to \infty} \frac{Y_i^T (AY_i)^{3}}{\| (AY_i)^{3} \|} = -1 $$

For each $i$, we decompose $X_i$ as $X_i = \alpha_i V_i + \sqrt{1 - \alpha_i^2} W_i$ such that $V_i \in \mathcal{U}(\mathfrak{A}^\perp)$, $W_i \in \mathcal{U}(\mathfrak{A}^{r_S p})$, and $\alpha_i \in [0, 1]$. By assumption $Y_i^T (AY_i)^{3} < 0$ for all $i$, we have

**Corollary 2.15:** $\alpha_i < 1$ for all $i$.

Because $V_i, W_i \in \mathcal{U}$ for all $i$, we can choose a subsequence $\{X_{ik}\}_k$ of $\{X_i\}_i$ such that $\lim_{k \to \infty} V_{ik} = V_\infty \in \mathbb{R}^n$ and $\lim_{k \to \infty} W_{ik} = W_\infty \in \mathbb{R}^n$. As $\mathcal{U}(\mathfrak{A}^\perp)$ and $\mathcal{U}(\mathfrak{A}^{r_S p})$ are closed, it must be $V_\infty \in \mathcal{U}(\mathfrak{A}^\perp)$ and $W_\infty \in \mathcal{U}(\mathfrak{A}^{r_S p})$. Furthermore, we can choose a subsequence $\{X_{ikj}\}_j$ of $\{X_{ik}\}_k$ such that $\lim_{j \to \infty} \alpha_{ikj} = \alpha_\infty \in [0, 1]$. Without loss of generality, we assume $\{X_i\}_i$ has such properties of $\{X_{ikj}\}_j$, that is,

(i) $\lim_{i \to \infty} \alpha_i = \alpha_\infty \in [0, 1]$.

(ii) $\lim_{i \to \infty} V_i = V_\infty \in \mathcal{U}(\mathfrak{A}^\perp)$ and $\lim_{i \to \infty} W_i = W_\infty \in \mathcal{U}(\mathfrak{A}^{r_S p})$

Then

$$ \lim_{i \to \infty} \frac{X_i^T (AX_i)^{3}}{\| (AX_i)^{3} \|} = \lim_{i \to \infty} \frac{(\alpha_i V_i + \sqrt{1 - \alpha_i^2} W_i)^T (\sqrt{1 - \alpha_i^2} AW_i)^{3}}{\| (\sqrt{1 - \alpha_i^2} AW_i)^{3} \|} $$

$$ = \frac{(\alpha_\infty V_\infty + \sqrt{1 - \alpha_\infty^2} W_\infty)^T (AW_\infty)^{3}}{\| (AW_\infty)^{3} \|} $$

$$ = -1 $$

Now, let $X_\infty = \alpha_\infty V_\infty + \sqrt{1 - \alpha_\infty^2} W_\infty$. There are two cases.
Case (i): $\alpha_\infty < 1$. In this case $X_\infty \notin \mathcal{H}$. Thus $\frac{X_i^T (A X_\infty)^{s_3}}{|| (A X_\infty)^{s_3} ||} = -1$ by (27) and (28). This contradicts with Corollary 2.3.

Case (ii): $\alpha_\infty = 1$. Because $\lim_{i \to \infty} \frac{X_i^T (A X_\infty)^{s_3}}{|| (A X_\infty)^{s_3} ||} = -1$, there must exist an integer $i^\#$ such that $\frac{V_i^T (A W_i)^{s_3}}{|| (A W_i)^{s_3} ||} < -\frac{2\sqrt{2}}{3}$ for all $i \geq i^\#$. Without the loss of generality, we suppose that $\frac{V_i^T (A W_i)^{s_3}}{|| (A W_i)^{s_3} ||} < -\frac{2\sqrt{2}}{3}$ for all $i$. There are two subcases.

Subcase (ii.a): There are infinitely many $i$ such that $\frac{W_i^T (A W_i)^{s_3}}{|| (A W_i)^{s_3} ||} \geq 0$, say $i_s$ for $s = 1, 2, ..., \infty$. By definition, $Y_{i_s} = || Y_{i_s} || X_{i_s}$, and so $Y_{i_s} = \rho_{i_s} V_{i_s} + \tau_{i_s} W_{i_s}$ where $\rho_{i_s} = || Y_{i_s} || \alpha_{i_s}$ and $\tau_{i_s} = || Y_{i_s} || \sqrt{1 - \alpha_{i_s}^2}$. As a result,

$$|| D(Y_{i_s}) ||^2 = \rho_{i_s}^2 + \tau_{i_s}^2 + \tau_{i_s}^6 ((A W_{i_s})^{s_3})^2 + 2 \rho_{i_s} \tau_{i_s}^3 V_{i_s}^T (A W_{i_s})^{s_3} + 2 \tau_{i_s}^4 W_{i_s}^T (A W_{i_s})^{s_3} \geq \rho_{i_s}^2 + \tau_{i_s}^2 + \tau_{i_s}^6 ((A W_{i_s})^{s_3})^2 + 2 \rho_{i_s} \tau_{i_s}^3 V_{i_s}^T (A W_{i_s})^{s_3} \geq \rho_{i_s}^2 + \tau_{i_s}^2 + \tau_{i_s}^6 ((A W_{i_s})^{s_3})^2 - 2 \rho_{i_s} \tau_{i_s}^3 || (A W_{i_s})^{s_3} ||$$

(29)

(30)

(31)

Wherein, inequality (29) holds since $\frac{W_i^T (A W_i)^{s_3}}{|| (A W_i)^{s_3} ||} \geq 0$ by assumption, and inequality (31) holds because $\frac{X_i^T Y}{|| X || || Y ||} \geq -1$ for any vector $X$ and $Y$ by its geometric meaning. By the unboundedness assumption on $\{Y_i\}$, $\rho_{i_s}^2 + \tau_{i_s}^2 \to \infty$ as $s \to \infty$. If $\{\tau_{i_s}\}_s$ is unbounded, then $|| D(Y_{i_s}) ||$ must be unbounded by (31). If $\{\tau_{i_s}\}_s$ is bounded, then $\{\rho_{i_s}\}_s$ must be unbounded. Thus $\{|| (A W_{i_s})^{s_3} ||\}_s$ must be unbounded since $\{|| (A W_{i_s})^{s_3} ||\}_s$ is bounded by Proposition 2.4. Therefore $|| D(Y_{i_s}) ||$ is unbounded again by (31). In consequence, $\{D(Y_i)\}_i$ is unbounded.

Subcase (ii.b): There are only finitely many $i$ such that $\frac{W_i^T (A W_i)^{s_3}}{|| (A W_i)^{s_3} ||} \geq 0$, i.e., $W_i^T (A W_i)^{s_3} \geq 0$. Without the loss of generality, we suppose $W_i^T (A W_i)^{s_3} < 0$ for all $i$. Then $(V_i, W_i) \in \mathcal{H}$ for all $i$. In this subcase, we can still show that $\{D(Y_i)\}_i$ must be unbounded.

**Lemma 2.16**: $\{D(Y_i)\}_i$ must be unbounded.

**Proof**: For any $\mu > 0$, set $Y := X_\infty$ and for which let’s take a neighbourhood $\mathcal{G}_\mu(Y)$ as before. Because $\lim_{i \to \infty} \frac{Y_i^T Y}{|| Y_i ||} = \lim_{i \to \infty} X_i^T Y = \lim_{i \to \infty} X_i^T X_\infty = 1$, there must exist an integer $i_\mu$ such that $Y_i \in \text{Con}(Y, \alpha(\mathcal{H}, \mu))$ for all $i \geq i_\mu$. For each $i \geq i_\mu$, by Proposition 2.12 we have $(\mu + 1) \frac{Y_i}{|| Y_i ||} \in \text{Con}(Y, \alpha(\mathcal{H}, \mu))$. Thus $\lim_{i \to \infty} \frac{Y_i}{|| Y_i ||} = Y$ implies $\lim_{i \to \infty} (\mu + 1) \frac{Y_i}{|| Y_i ||} = (\mu + 1) Y$, therefore for the neighbourhood $\mathcal{H}_\mu(Y)$ there is some integer $i_\mu^\#$ such that $(\mu + 1) \frac{Y_{i_\mu^\#}}{|| Y_{i_\mu^\#} \in \mathcal{H}_\mu(Y) \text{ for all } i \geq i_\mu^\#$. So, $(\mu + 1) \frac{Y_{i_\mu^\#}}{|| Y_{i_\mu^\#} \in \mathcal{H}_\mu(Y) \text{ for all } i \geq i_\mu^\#. As } \{Y_i\}_i \text{ is unbounded, there must exist some } i_\mu^* > i_\mu^\# \text{ such that } || Y_{i_\mu^*} || \geq \mu + 1$. Then by Lemma 2.18 and Corollary 2.15 $|| D(Y_{i_\mu^*}) || \geq \mu$. By the arbitrariness of $\mu$, $\{D(Y_i)\}_i$ must be unbounded.
Lemma 2.16 contradicts with the assumption again. So, for a given D-map \( D(X) \), there must have no unbounded sequence \( \{ Y_i \} \) such that \( \{ D(Y_i) \} \) is bounded. Therefore, each D-map must be a proper map.

This completes the proof of Theorem 1.4.

III. DISCUSSION AND RELATED WORK

Recall the history of \( J\mathcal{C} \), there are many excellent works. First of all, Keller proposed \( J\mathcal{C} \) for \( \mathbb{K}^n = \mathbb{C}^2 \) in 1939 [2]. And Abhyankar gave its modern style in his lectures [7], [8]. Let \( (J\mathcal{C})^k \) denote \( J\mathcal{C}(\mathbb{K}, n) \) in which the polynomial degrees are not greater than \( k \). A remarkable progress is Wang’s result [19] that \( (J\mathcal{C})^2 \) is true for arbitrary field of characteristic \( \neq 2 \). This led to the studies on the reduction of \( J\mathcal{C} \) to specific \( (J\mathcal{C})^k \) for some small integer \( k \). In the line of this, Yagzhev [10] and independently Bass et al. [3] showed that \( (J\mathcal{C})^3 \) implies \( J\mathcal{C} \). Furthermore, Drużkowski [12] showed that it suffices to show D-maps being automorphism for \( J\mathcal{C} \). Anyway, it cannot make a reduction of \( J\mathcal{C} \) to \( (J\mathcal{C})^2 \) on the field \( \mathbb{C} \). That is, we must solve \( (J\mathcal{C})^3 \) to the end of \( J\mathcal{C} \). Anyway, many studies [21], [22] showed that it suffices to prove specific structure D-maps for \( J\mathcal{C} \). Furthermore, \( J\mathcal{C} \) also appeared to be connected to questions in noncommutative algebra, for example, \( J\mathcal{C} \) is equivalent to the Dixmier Conjecture which asserts that each endomorphism of the Weyl algebra is surjective (hence an automorphism) [23]–[25]. \( J\mathcal{C} \) is proved equivalent to various conjectures, such as, Kernel Conjecture [5], Hessian Conjecture [22], [26], Eulerian Conjecture [27], etc. There are many partial results of \( J\mathcal{C} \) on special categories of polynomials, for instance, the “non-negative coefficients” D-map [28], D-maps in low dimension space [29], [30], a special class of D-maps in dimension 9 [31], tame automorphisms [32]–[34], etc, refer to surveys [1], [3]–[6] for more related results. There were some studies about \( J\mathcal{C} \) for fixed number of variables, even for 2-variables, such as, \( J\mathcal{C} \) [35] for 2-variables \( \leq 100 \)-degree, sufficient conditions via polynomial flows in [36] for \( J\mathcal{C}(\mathbb{R}, 2) \), a Hamiltonian flows approach in [37] for \( J\mathcal{C}(\mathbb{C}, 2) \).

\( J\mathcal{C} \) must depend on Jacobian Condition, polynomial type, and the number field of characteristic 0. For the Keller maps over number fields of characteristic 0, injectivity always implies subjectivity [3], [9], [10], [38], [39]. But, this property completely fails for the nonpolynomial maps, already for \( n = 2 \). There is a counterexample in [3]: \( F_1(X) = e^{x_1}, F_2 = x_2e^{-x_1} \) whose Jacobian is 1, but \( F(\mathbb{C}^2) \) excludes exactly the axis \( x_1 = 0 \). That is, injectivity does not mean surjectivity for a generic analytic map. Even for rational maps, there are counterexamples in
As to the zero characteristic condition, there are counterexamples in [3], [8] for \( J_C \) of characteristic \( > 0 \). The Jacobian Condition also cannot be relaxed. A generalization of \( J_C \) is the real Jacobian problem [40] (also called strong Jacobian Conjecture in [41]), that is, whether a polynomial mapping \( F : \mathbb{R}^2 \mapsto \mathbb{R}^2 \) with a nonvanishing Jacobian determinant is an automorphism. The strong Jacobian Conjecture has a negative answer [41]. Pinchuk presented a beautiful example of a non-injective polynomial mapping \( F(x_1, x_2) \) of \( \mathbb{R}^2 \) into itself, of degree \( (x_1, x_2) = (10, 40) \), whose Jacobian determinant is everywhere positive on \( \mathbb{R}^2 \). Therefore, Polynomial Automorphism Theorem (PAT for short) is the best in all of what we can get.

From PAT, it immediately gets that the Roll Theorem is true for polynomial functions over any algebraically closed field. Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero, \( \mathcal{P} \in \mathbb{K}[X]^n \). Then the determinant \( det(J_{\mathcal{P}}(X)) \) is a polynomial and must have an \( X_0 \in \mathbb{K}^n \) such that \( det(J_{\mathcal{P}}(X_0)) = 0 \) if \( det(J_{\mathcal{P}}(X)) \) is not a constant. By Theorem 1.5 using contradiction argument we can obtain

**High Dimension Roll Theorem 3.1:** If there are \( X \neq Y \in \mathbb{K}^n \) such that \( \mathcal{P}(X) = \mathcal{P}(Y) \) then there is some \( X_0 \) such that \( det(J_{\mathcal{P}}(X_0)) = 0 \).

In this theorem, the requirement “algebraically closed” is necessary. Otherwise, the Pinchuk’s counterexample will be a counterexample over \( \mathbb{R}^2 \).

Another immediate result of PAT is that the inverse flows are actually polynomials in \( X \) and \( t \). So, high order Lie derivatives vanish at some stages. That is, the Lie derivatives are locally nilpotent or finite [42], [43]. Therefore, it actually gives a termination criterion for computing inverse polynomial through Lie derivatives [44].

When a polynomial map is an automorphism, there are several different approaches to the inversion formulas. An early one is the Abhyankar-Gurjar inversion formula [8]. In [3] Bass et al. presented a formal expansion for the inverse. Nousiainen and Sweedler [44] provided an inversion formula through Lie derivatives. For specific polynomials, Wright [45] and respectively Zhao [46], [47] gave advanced inversion formulas. Anyway, the degree of inverse polynomial is bounded by \( deg(F^{-1}) \leq deg(F)^{n-1} \) [3], [48], [49].

Besides Jacobian Condition, van den Essen [50] using Gröbner base gave an algebraic criterion for the invertibility of polynomial maps.

**Essen Theorem 3.2** ([50]): Any map \( F = (F_1(X), \ldots, F_n(X)) \in \mathbb{K}[X] \) on arbitrary field \( \mathbb{K} \) is an automorphism iff there are polynomials \( G_1(Y), \ldots, G_n(Y) \in \mathbb{K}[Y] \) such that \( Y_1 - F_1(X), \ldots, Y_n - F_n(X) \) and \( X_1 - G_1(Y), \ldots, X_n - G_n(Y) \) generate the same polynomial idea
in $\mathbb{K}[X, Y]$.

This criterion is not limited to the characteristic zero cases but holds in all characteristics. At the same time Theorem 3.2 also provides an algorithm to decide if a polynomial map has an inverse and compute the inverse if it exists. The theory of Gröbner bases for polynomial ideals [51] is the foundation of the Essen Theorem. In contrast with this, PAT is an analytic criterion for the global invertibility of polynomial maps. In particular, PAT can be efficiently implemented for sparse polynomial maps, by testing the Jacobian Condition through random inputs.

**IV. CONCLUDING REMARKS**

In this study, we gave an affirmative answer to Jacobian Conjecture. Based on D-map, our proof used (1) algebraic property like nilpotency property in Case (i), and (2) geometric properties like an open cone, arc, and cosines between vectors as showed in Subcase (ii.a). To study the direction changing tendency of the unbounded sequence $\{Y_i\}$, we used an optimization method to obtain the key limit equation (25). To obtain the unboundedness of the images, we used an optimization idea again in Subcase (ii.b). Jacobian Conjecture is an algebraic geometry problem. It is no surprise to use algebraic and geometric methods in the proof, but the optimization method is really an extra auxiliary. So, this proof is said as an optimization-based method. Our proof takes much advantage of D-maps’ nice algebraic and geometric properties. Although Yagzhev map is a little extension of D-map, it has no such good properties. At least, so far our proof does not work on Yagzhev map for which we cannot construct something like $\mathfrak{A}$. By the way, in the definition of $\mathfrak{A}$, $-\frac{2\sqrt{2}}{3}$ can be replaced by any real number $\eta$ in the interval $(-1, -\sqrt{\frac{3}{2}})$. Such numbers still work for the proof. We would like to stress that such $\eta$ can not be $-1$ or $-\frac{2\sqrt{3}}{3}$ for the sake of getting Lemma 2.11. From our proof, it can be seen if a cubic linear form $\mathcal{C}(X) := X + (AX)^3$ has only finitely many zero roots then nonproperness of $\mathcal{C}(X)$ will imply Lemma 2.14. For D-map, the Jacobian Condition implies a unique zero root. But for a general cubic linear form $\mathcal{C}(X)$, the finiteness of zero roots is not necessarily true. In Case (i), Jacobian Condition is used again. From the proof, we can clearly see how and why Jacobian Condition makes D-map being injective. However, we should note that an injective polynomial map in $\mathbb{R}[X]^n$ is not necessarily an automorphism. The automorphism property of D-maps in $\mathbb{R}[X]^n$ is not a direct result of their injectivity but derived from a series of deductions involved with Lefschetz Principle, Drużkowski’s reduction, automorphism property of D-maps in $\mathbb{C}[X]^n$, etc.
Given a polynomial map $\mathcal{P}(X) := (\mathcal{P}_1(X), \ldots, \mathcal{P}_n(X)) \in \mathbb{K}[X]^n$, $\mathcal{P}(X)$ is an automorphism from $\mathbb{K}^n$ onto $\mathbb{K}^n$ iff the induced endomorphism $\mathcal{R}_\mathcal{P} : \mathbb{K}[X]^n \rightarrow \mathbb{K}[X]^n$ by $\mathcal{R}_\mathcal{P}(X_i) = \mathcal{P}_i(X)$ for $i = 1, \ldots, n$ is an automorphism of the ring $\mathbb{K}[X]^n$. If we consider the derivatives as algebraic operators, then the Jacobian Conjecture is purely an algebraic problem. So a purely algebraic proof is really an interesting thing. It is already known that an analytic map satisfying Jacobian Condition is not necessarily a diffeomorphism. In fact, even for the rational maps, the Jacobian Condition is not a sufficient condition for this type of map being diffeomorphisms. For analytic maps over Euclidean space, Hadamard’s Diffeomorphism Theorem has provided a nice criterion for diffeomorphism. However, this is not a computable approach like Jacobian Condition. To the best of our knowledge, so far there is no computable method to directly check the properness of a map. In practice, we may need to computably determine whether a given concrete map is a diffeomorphism. In such context, Hadamard’s Diffeomorphism Theorem helps a little and the Jacobian Condition is not a correct criterion for nonpolynomial maps. In the physical world, we are usually concerned with elementary maps which are composed of elementary expressions like $e^X, X^p, \sin X, \cos X$, etc. So for analytic maps, it is natural to ask whether there is some computable diffeomorphism criterion. Another basic question about polynomial automorphisms is how many of them, or what is the ratio of polynomial automorphisms to all polynomials, or what is their distribution. By Weierstrass Approximation Theorem, each continuous real function on some closed interval can be uniformly approximated by polynomials. In comparison with this, it is an interesting problem whether the automorphism polynomials are dense in the set of all analytic diffeomorphisms. The study of the injectivity has given rise to several surprising results and interesting relations in various directions and from different perspectives. In this study, we proved Jacobian Conjecture by showing the injectivity of D-maps. In fact, the injectivity itself has received attention from not only the mathematical field but also by the economic field [52], physical field [53], and chemical field [54]. This study only verifies the injectivity of a special class of polynomial maps in $\mathbb{R}[X]^n$. The proof heavily depends on the Jacobian Condition and the form of D-map. It has been proved that the Samuelson Conjecture in [52] is true for any polynomial map [55] and arbitrary rational map [56], but fails for generic analytic maps [57]. For a general differentiable map $F$ on $\mathbb{R}^n$, Chamberland Conjecture [58] asserts if all the eigenvalues of $J_F(X)$ are bounded away from zero then $F$ must be injective. This conjecture is still open.
REFERENCES

[1] A. van den Essen, “Polynomial automorphisms and the Jacobian conjecture,” in Progress in Mathematics Vol 190. Boston; Berlin: Birkhäuser, 2000.
[2] O. Keller, “Ganze Cremona-transformationen,” Monatshefte für Mathematik, vol. 47, no. 1, pp. 299–306, 1939.
[3] H. Bass, E. H. Connell, and D. Wright, “The Jacobian conjecture: Reduction of degree and formal expansion of the inverse,” Bulletin of the American Mathematical Society, vol. 7, no. 2, pp. 287–330, 1982.
[4] L. M. Drużkowski, “Partial results and equivalent formulations of the Jacobian conjecture,” Rendiconti Del Seminario Matematico, no. 4, pp. 275–282, 1997.
[5] A. van den Essen, “Polynomial automorphisms, and the Jacobian conjecture,” The Mathematical Gazette, vol. 85, no. 504, p. 572, 2001.
[6] D. Wright, “The Jacobian conjecture: ideal membership questions and recent advances,” pp. 261–276, 2005.
[7] S. S. Abhyankar, “Lectures on expansion techniques in algebraic geometry. with notes by balwant singh,” Tata Institute of Fundamental Research Bombay, 1977.
[8] ——, “Lectures in algebraic geometry,” Notes by Chris Christensen, Purdue University, 1974.
[9] T. Winiarski, “Inverse of polynomial automorphisms of $\mathbb{C}^n$,” Journal of the Institute of Image Information & Television Engineers, vol. 27, no. 1, pp. 9–11, 1979.
[10] A. V. Yagzhev, “Keller’s problem,” Siberian Mathematical Journal, vol. 21, no. 5, pp. 747–754, 1980.
[11] W. Rudin, “Injective polynomial maps are automorphisms,” American Mathematical Monthly, vol. 102, no. 6, pp. 540–543, 1995.
[12] L. M. Drużkowski, “An effective approach to Keller’s Jacobian conjecture,” Mathematische Annalen, vol. 264, no. 3, pp. 303–313, 1983.
[13] ——, “The Jacobian conjecture: survey of some results,” Banach Center Publications, vol. 31, no. 1, pp. 163–171, 1995.
[14] J. Hadamard, “Sur les transformations ponctuelles,” Bull.soc.math.france, pp. 71–84, 1906.
[15] W. B. Gordon, “On the diffeomorphisms of Euclidean space,” American Mathematical Monthly, vol. 79, no. 7, pp. 755–759, 1972.
[16] B. Xia and L. Yang, Automated Inequality Proving and Discovering. WORLD SCIENTIFIC, 2016.
[17] K. O. Geddes and G. H. Gonnet, “A new algorithm for computing symbolic limits using hierarchical series,” in Symbolic and Algebraic Computation, International Symposium ISSAC’88, July 4-8, 1988.
[18] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge University Press, 2004.
[19] S. S. Wang, “A Jacobian criterion for separability,” Journal of Algebra, vol. 65, no. 2, pp. 453–494, 1980.
[20] L. A. Campbell, “A condition for a polynomial map to be invertible,” Mathematische Annalen, vol. 205, no. 3, pp. 243–248, 1973.
[21] L. M. Drużkowski, “The Jacobian conjecture: symmetric reduction and solution in the symmetric cubic linear case,” Annales Polonici Mathematici, vol. 87, no. 87, pp. 83–92, 2005.
[22] A. van den Essen, “A reduction of the Jacobian conjecture to the symmetric case,” Proceedings of the American Mathematical Society, vol. 133, no. 8, pp. 2201–2205, 2005.
[23] Y. Tsuchimoto, “Endomorphisms of Weyl algebra and p-curvatures,” Osaka Journal of Mathematics, vol. 42, no. 2, pp. 435–452, 2005.
[24] A. Belovkanel and M. Kontsevich, “The Jacobian conjecture is stably equivalent to the Dixmier conjecture,” Moscow Mathematical Journal, vol. 7, no. 2, pp. 209–218, 2007.
[25] P. K. Adjamagbo and A. V. D. Essen, “A proof of the equivalence of the Dixmier, Jacobian and Poisson conjectures,” Acta Mathematica Vietnamica, vol. 32, no. 2, pp. 205–14, 2007.

[26] G. Meng, “Legendre transform, Hessian conjecture and tree formula,” Applied Mathematics Letters, vol. 19, no. 6, pp. 503–510, 2006.

[27] K. Adjamagbo and A. van den Essen, “Eulerian systems of partial differential equations and the Jacobian conjecture,” Journal of Pure and Applied Algebra, vol. 74, no. 1, pp. 1–15, 1991.

[28] L. M. Drużkowski, “The Jacobian conjecture in case of “non-negative coefficients”,” Annales Polonici Mathematici, vol. 66, no. 1, pp. 67–75, 1997.

[29] C. C. Cheng, “Power linear Keller maps of dimension four,” Journal of Pure and Applied Algebra, vol. 169, pp. 153–158, 2002.

[30] M. D. Bondt and A. van den Essen, “The Jacobian conjecture: Linear triangularization for homogeneous polynomial maps in dimension three,” Journal of Algebra, vol. 294, no. 1, pp. 294–306, 2005.

[31] D. Yan, “A note on the Jacobian conjecture,” Linear Algebra and its Applications, vol. 435, no. 9, pp. 2110–2113, 2011.

[32] M. K. Smith, “Stably tame automorphisms,” Journal of Pure and Applied Algebra, vol. 58, no. 2, pp. 209–212, 1989.

[33] V. Drensky, A. van den Essen, and D. Stefanov, “New stably tame automorphisms of polynomial algebras,” Journal of Algebra, vol. 226, no. 1, pp. 629–638, 2000.

[34] I. Shestakov and U. Umirbaev, “The tame and the wild automorphisms of polynomial rings in three variables,” Journal of the American Mathematical Society, vol. 17, no. 1, pp. 197–227, 2004.

[35] T. T. Moh, “On the Jacobian conjecture and the configurations of roots,” Journal Für Die Reine Und Angewandte Mathematik, vol. 1983, no. 340, pp. 140–213, 1983.

[36] H. Bass and G. H. Meisters, “Polynomial flows in the plane,” Advances in Mathematics, vol. 55, no. 2, pp. 173–208, 1985.

[37] L. A. Campbell, “Jacobian pairs and Hamiltonian flows,” Journal of Pure and Applied Algebra, vol. 115, no. 1, pp. 15–26, 1997.

[38] A. Bailynicki-Birula and M. Rosenlicht, “Injective morphisms of real algebraic varieties,” Proceedings of the American Mathematical Society, vol. 13, no. 2, pp. 200–203, 1962.

[39] K. Kurdyka and K. Rusek, “Surjectivity of certain injective semialgebraic transformations of $\mathbb{R}^n$,” Mathematische Zeitschrift, vol. 200, no. 1, pp. 141–148, 1988.

[40] J. D. Randall, “The real Jacobian problem,” 1983, pp. 411–414.

[41] S. Pinchuk, “A counterexample to the strong real Jacobian conjecture,” Mathematische Zeitschrift, vol. 217, no. 1, pp. 1–4, 1994.

[42] A. van den Essen, “Locally finite and locally nilpotent derivations with applications to polynomial flows and polynomial morphisms,” Proceedings of the American Mathematical Society, vol. 116, no. 3, pp. 861–871, 1992.

[43] ——, “Locally finite and locally nilpotent derivations with applications to polynomial flows, morphisms and $\mathcal{U}_n$-actions. II.” Proceedings of the American Mathematical Society, vol. 121, no. 3, pp. 667–678, 1994.

[44] P. Nousiainen and M. Sweedler, “Automorphisms of polynomial and power series rings,” Journal of Pure and Applied Algebra, vol. 29, no. 1, pp. 93–97, 1983.

[45] D. Wright, “Formal inverse expansion and the Jacobian conjecture,” Journal of Pure and Applied Algebra, vol. 48, pp. 199–219, 1987.

[46] W. Zhao, “Recurrent inversion formulas,” in Some Properties and Open Problems of Hessian Nilpotent polynomials, In preparation. Department of Mathematics, Illinois State University, Normal, IL 61790-4520. E-mail: wzhaoo@ilstu.edu.

[47] ——, “Inversion problem, Legendre transform and inviscid Burgers’ equations,” Journal of Pure and Applied Algebra, vol. 199, no. 1, pp. 299–317, 2005.
[48] K. Rusek and T. Winiarski, “Polynomial automorphisms of $\mathbb{C}$,” vol. 661, 1984.

[49] J. Furter, “On the degree of the inverse of an automorphism of the affine space,” Journal of Pure and Applied Algebra, vol. 130, no. 3, pp. 277–292, 1998.

[50] A. van den Essen, “A criterion to decide if a polynomial map is invertible and to compute the inverse,” Communications in Algebra, vol. 18, no. 10, pp. 3183–3186, 1990.

[51] D. A. Cox, J. Little, and D. O’Shea, Ideals, Varieties, and Algorithms. Springer, 1997.

[52] P. A. Samuelson, “Prices of factors and good in general equilibrium,” Review of Economic Studies, vol. 21, no. 1, pp. 1–20, 1953.

[53] A. Abdesselam, “The Jacobian conjecture as a problem of perturbative quantum field theory,” Annales Henri Poincaré, vol. 4, no. 2, pp. 199–215, 2003.

[54] S. Muller, J. Hofbauer, and G. Regensburger, “On the bijectivity of families of exponential/generalized polynomial maps,” SIAM Journal on Applied Algebra and Geometry, vol. 3, no. 3, pp. 412–438, 2019.

[55] A. V. Den Essen and T. Parthasarathy, “Polynomial maps and a conjecture of Samuelson,” Linear Algebra and its Applications, vol. 177, pp. 191–195, 1992.

[56] L. A. Campbell, “Rational Samuelson maps are univalent,” Journal of Pure and Applied Algebra, vol. 92, no. 3, pp. 227–240, 1994.

[57] D. Gale and H. Nikaido, “The Jacobian matrix and global univalence of mapping,” Mathematische Annalen, vol. 159, no. 2, pp. 81–93, 1965.

[58] M. Chamberland and G. Meisters, “A mountain pass to the Jacobian conjecture,” Canadian Mathematical Bulletin, vol. 41, no. 4, pp. 442–451, 1998.