Buckling patterns of complete spherical shells filled with an elastic medium under external pressure

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Abstract

The critical buckling characteristics of hydrostatically pressurized complete spherical shells filled with an elastic medium are demonstrated. A model based on small deflection thin shell theory, the equations of which are solved in conjunction with variational principles, is presented. In the exact formulation, axisymmetric and inextensional assumptions are not used initially and the elastic medium is modelled as a Winkler foundation, i.e. using uncoupled radial springs with a constant foundation modulus that is independent of wave numbers of shell buckling modes. Simplified approximations based on a Rayleigh–Ritz approach are also introduced for critical buckling pressure and mode number with a considerable degree of accuracy. Characteristic modal shapes are demonstrated for a wide range of material and geometric parameters. A phase diagram is established to obtain the requisite thickness to radius, and stiffness ratios for a desired mode profile. The present exact formulation can be readily extended to apply to more general cases of non-axisymmetric buckling problems.

Keywords: Hydrostatically pressurized buckling, complete spherical shell, Winkler foundation

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1. Introduction

The analytical study of the structural behaviour of spherical shells is of great importance in the fields of not only civil, mechanical and aeronautical engineering but also nanoscience and biomechanics; Notable examples include pressure vessels, spherical honeycombs [1], carbon onions [2] and spherical viruses [3] and so on. In the engineering research fields, some pioneering theoretical works on the elastic instability issues of empty complete spherical shells were conducted in 1960s by Thompson [4], Hutchinson [5] and Koiter [6].

In recent years, analytical works which considered an interaction between a spherical shell and an internal elastic medium have also been conducted. The interaction effect leads to novel buckling patterns which depend on the stiffness and thickness to radius ratios. Recently for example, Yin et al. [7] successfully demonstrated the stress-driven buckling patterns in spheroidal core/shell structures, where the core implies an elastic medium, by using the finite element approach. Interestingly, the authors insisted in that work [7] that shapes of many natural fruits and vegetables can be reproduced by anisotropic stress-driven buckles on the spheroidal core/shell system. Moreover, the morphology of a pressurized spherical shell can give clues to the structure–property relationship of spherically-shaped nanostructures such as carbon onions [8, 9], core/shell semiconductor nanoparticles [10], and the “nano-matryoshka” [11]. The last system refers to a concentric multilayered structure comprising various metallic and dielectric materials, showing great tunability in the optical response; it is thus of interest to explore how pressure-induced deformation affects the electromagnetic properties of the nanoshells and their resonant frequencies.

Purely from a structural mechanics perspective, Timoshenko [12], and Flügge [13] introduced the formulation for buckling of a complete spherical shell in their respective books that are now regarded as classics in the field. In these books, the exact approaches used to solve the hydrostatically pressurized buckling of a complete spherical shell without an internal elastic medium were introduced. However, both of these formulations were based on the axisymmetric assumption and formulations for more general cases including non-axisymmetric deformations were not included. More recently, Fok and Allwrite [14] analysed the elastic axisymmetric buckling behaviour of a complete spherical shell embedded in an elastic material and loaded by a far-field hydrostatic pressure. In that study, the energy method in conjunction with a Rayleigh–Ritz trial function was used for simplicity but the validation of the obtained results was not discussed in detail with buckling deformation modes being omitted. Hence, to our knowledge, no general
non-axisymmetric formulation, in conjunction with exact methods of solution, has been developed for the buckling behaviour of a complete spherical shell with an internal elastic medium thus far.

Moreover, much attention has been given to the structural morphology of the core/shell structure. Some authors have recently demonstrated the cross-sectional morphology of carbon nanotubes embedded in an elastic medium [15, 16, 17]. These results clearly show that interactions between shells and cores lead to some novel wavy-shaped buckling deformation patterns.

From the background described above, the buckling properties of hydrostatically pressurized complete spherical shells filled with an elastic medium is demonstrated currently. The elastic medium is modelled as a Winkler foundation, i.e. with uncoupled radial springs and a constant foundation modulus. An exact approach for solving the developed equations based on the formulation without using the axisymmetric and inextentional assumptions is presented. This approach therefore avoids any discussion about the validity of the solution and allows the model to be extended to cover more generic non-axisymmetric cases with relative ease. The analytical results are presented using a phase diagram and illustrative buckling modes. In addition to this, simplified approximations based on a Rayleigh–Ritz approach are also introduced for the critical buckling pressure and the corresponding mode number. Comparative studies between the exact and simplified approaches are conducted to validate the approximation and it is shown that the approximate formulations enables sufficiently accurate values to be obtained.

2. Complete spherical shell model

The critical pitchfork bifurcation phenomenon of the hydrostatically pressurized complete spherical shell is investigated, as shown in Fig.1. The spherical shell is constructed from an homogeneous and isotropic linear elastic material with Young’s modulus $E$ and Poisson’s ratio $\nu$, which is filled with an elastic material that is modelled as a Winkler foundation, i.e. with uncoupled springs in the radial direction and a constant foundation modulus $k_f$. For a complete spherical shell with radius $a$ and thickness $h$, spherical angular coordinates in the latitude and meridian directions $\theta$ and $\phi$ are used. Displacement functions: $u$, $v$ and $w$ are in the $\theta$, $\phi$ and the (outward) radial directions, respectively.
Figure 1: Hydrostatically pressurized spherical shell filled with an elastic medium.

3. Formulation

3.1. Exact approach

3.1.1. Energy formulation

The following analysis is based on classical small deformation theory of thin shells \[18, 19\]. The total potential energy \( V \) is expressed by the sum of the strain energy and work done by external load as:

\[
V = U_M + U_B + U_F + \Omega,
\]

in which \( U_M \) is the membrane (in-plane) strain energy term, where:

\[
U_M = U_M^\phi + U_M^\theta + U_M^{\phi\theta},
\]

\[
U_M^\phi = \frac{1}{2} \int_\phi \int_\theta N_\phi \varepsilon_\phi a^2 \sin \phi \ d\phi d\theta;
\]

\[
U_M^\theta = \frac{1}{2} \int_\phi \int_\theta N_\theta \varepsilon_\theta a^2 \sin \phi \ d\phi d\theta;
\]

\[
U_M^{\phi\theta} = \int_\phi \int_\theta N_{\phi\theta} \varepsilon_{\phi\theta} a^2 \sin \phi \ d\phi d\theta,
\]
$U_B$ is the bending (out-of-plane) component, where:

$$U_B = U_{B\phi} + U_{B\theta} + U_{B\phi\theta},$$

$$U_{B\phi} = \frac{1}{2} \int_{\phi} \int_{\theta} M_{\phi}\chi_{\phi} a^2 \sin \phi \, d\phi \, d\theta,$$

$$U_{B\theta} = \frac{1}{2} \int_{\phi} \int_{\theta} M_{\theta}\chi_{\theta} a^2 \sin \phi \, d\phi \, d\theta,$$

$$U_{B\phi\theta} = \int_{\phi} \int_{\theta} M_{\phi\theta}\chi_{\phi\theta} a^2 \sin \phi \, d\phi \, d\theta, \quad (3)$$

$U_F$ is the strain energy term due to a Winkler foundation, thus:

$$U_F = \frac{1}{2} \int_{\phi} \int_{\theta} a^2 k_f w^2 \sin \phi \, d\phi \, d\theta, \quad (4)$$

and $\Omega$ is the potential energy of the applied pressure:

$$\Omega = \int_{\phi} \int_{\theta} a^2 pw \sin \phi \, d\phi \, d\theta. \quad (5)$$

Based on the assumptions of thin shell theory [18], the strain–displacement relations can be expressed as:

$$\varepsilon_{\phi} = \frac{1}{a} (v_{\phi} + w),$$

$$\varepsilon_{\theta} = \frac{1}{a} \left( \cot \phi v + \frac{u_{\theta}}{\sin \phi} + w \right),$$

$$\varepsilon_{\phi\theta} = \frac{1}{a} \left( \frac{v_{\theta}}{\sin \phi} - \cot \phi u + u_{\phi} \right),$$

$$\beta_{\phi} = \frac{1}{a} (-w_{\phi} + v),$$

$$\beta_{\theta} = \frac{1}{a} \left( -\frac{w_{\theta}}{\sin \phi} + u \right),$$

$$\chi_{\phi} = \frac{1}{a^2} (-w_{\phi\phi} + v_{\phi}),$$

$$\chi_{\theta} = \frac{1}{a^2} \left[ -\frac{w_{\theta\theta}}{\sin^2 \phi} + u_{\theta} + \cot \phi (v - w_{\phi}) \right],$$

$$\chi_{\phi\theta} = \frac{\sin \phi u_{\phi} + v_{\phi} - 2w_{\phi\phi} + 2\cot \phi w_{\phi\theta} - \cos \phi u}{2a^2 \sin \phi}, \quad (6)$$
where “$\cdot_x$” denotes differentiation with respect to $x$, and the constitutive relations are as follows:

\[
N_\phi = C(\varepsilon_\phi + \nu \varepsilon_\theta), \quad N_\theta = C(\varepsilon_\theta + \nu \varepsilon_\phi),
\]
\[
N_{\phi\theta} = C \frac{1 - \nu}{2} \varepsilon_{\phi\theta}, \quad M_\phi = D(\chi_\phi + \nu \chi_\theta),
\]
\[
M_\theta = D(\chi_\theta + \nu \chi_\phi), \quad M_{\phi\theta} = D \frac{1 - \nu}{2} \chi_{\phi\theta}, \quad (7)
\]

where $C = Eh/(1 - \nu^2)$ is the membrane stiffness and $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the spherical shell, respectively. Substituting the expressions in Eqs.(6) and (7) into Eqs.(2)–(4) gives the energy expressions in terms of the displacements $u$, $v$ and $w$.

3.1.2. Fundamental state

When uniform hydrostatic pressure $p$ is acting on the spherical shell, only the inward radial static displacement $w_0$ can occur in the pre-buckling state. Hence, the stress-strain relationship of the spherical shell is assumed to be linear up to the point of instability. In the fundamental pre-buckling state, the potential energies $V^{(0)}$ are given by:

\[
V^{(0)} = U_M^{(0)} + U_B^{(0)} + U_F^{(0)} + \Omega^{(0)} = \int_\phi \int_\theta F^{(0)} \, d\phi \, d\theta \quad (8)
\]

where

\[
U_M^{(0)} = C(1 + \nu) \int_\phi \int_\theta w_0^2 \, d\phi \, d\theta, \quad U_B^{(0)} = 0,
\]
\[
U_F^{(0)} = \frac{1}{2} \int_\phi \int_\theta a^2 k_f w^2 \sin \phi \, d\phi \, d\theta,
\]
\[
\Omega^{(0)} = \int_\phi \int_\theta a^2 p w_0 \sin \phi \, d\phi \, d\theta. \quad (9)
\]

For an equilibrium state, the first variation of the total potential energy $V$ must equal zero. This condition gives the following static displacement under uniform hydrostatic pressure:

\[
w_0 = -\frac{a^2(1 - \nu)}{2Eh + a^2 k_f (1 - \nu)} p. \quad (10)
\]
3.1.3. Critical buckling analysis

To obtain an expression for the second variation of the total potential energy, the following infinitesimally small increments are defined:

\[ u = u_1, \quad v = v_1, \quad w = w_0 + w_1, \quad (11) \]

which correspond to buckling displacement modes. The expression for the potential energy due to the external force is a linear functional of the displacement components and makes no contribution to the second variation expression, which is \[ \delta^2 \Omega = 0 \] \[18\]. Consequently, the second variation of the total potential energy becomes

\[ \delta^2 U = \delta^2 U_M + \delta^2 U_B + \delta^2 U_F = \int \int F \ d\phi \ d\theta, \quad (12) \]

where

\[ \delta^2 U_M = \delta^2 U_{\text{stretch}} + \delta^2 U_{\text{shear}}, \]

\[ \delta^2 U_{\text{stretch}} = \frac{C w_0 (1 + \nu)}{a} \int \int [(v_1 + w_1, \phi)^2 + (w_1, \theta \csc \phi)^2] \sin \phi \ d\phi \ d\theta \]

\[ + C \int \int [(v_1 + w_1, \phi)^2 + (u_1 - w_1, \theta \csc \phi)^2] \ d\phi \ d\theta, \]

\[ \delta^2 U_{\text{shear}} = \frac{C (1 - \nu)}{2} \int \int (\cot \phi u_1 + \csc \phi v_1, \theta + u_1, \phi)^2 \sin \phi \ d\phi \ d\theta, \quad (13) \]

\[ \delta^2 U_B = \delta^2 U_{\text{bend}} + \delta^2 U_{\text{twist}}, \]

\[ \delta^2 U_{\text{bend}} = \frac{D}{a^2} \int \int ((\cot \phi u_1 + \csc \phi v_1, \theta - \csc^2 \phi w_1, \theta \theta - \cot \phi w_1, \phi)^2 \]

\[ (v_1, \phi + w_1, \phi \phi)[v_1, \phi + w_1, \phi \phi + 2 \nu (\cot \phi v_1 + \csc \phi u_1, \theta - \csc^2 \phi w_1, \theta \theta \]

\[ - \cot \phi w_1, \phi)] \sin \phi \ d\phi \ d\theta, \]

\[ \delta^2 U_{\text{twist}} = \frac{D (1 - \nu)}{2 a^2} \int \int [2 (1 + \nu) w_1 (v_1 \cot \phi + u_1, \theta + u_1, \phi) \]

\[ + u_1, \phi (2 \nu v_1 \cot \phi + 2 \nu u_1, \theta + v_1, \phi) \]

\[ + (v_1 \cot \phi + u_1, \theta \csc \phi)^2] \sin \phi \ d\phi \ d\theta, \quad (14) \]

\[ \delta^2 U_F = \int \int a^2 k_F w_1^2 \sin \phi d\phi d\theta. \quad (15) \]
According to the Trefftz criterion, the buckling equations can be obtained by introducing $F$ in Eq. (12) into the Euler–Lagrange equations with the calculus of variations. The Euler–Lagrange equations in this case are as follows [18]:

\[
\frac{\partial F}{\partial u_1} - \frac{\partial}{\partial \phi} \left( \frac{\partial F}{\partial u_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial F}{\partial u_{1,\theta}} \right) = 0,
\]

\[
\frac{\partial F}{\partial v_1} - \frac{\partial}{\partial \phi} \left( \frac{\partial F}{\partial v_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial F}{\partial v_{1,\theta}} \right) = 0,
\]

\[
\frac{\partial F}{\partial w_1} - \frac{\partial}{\partial \phi} \left( \frac{\partial F}{\partial w_{1,\phi}} \right) - \frac{\partial}{\partial \theta} \left( \frac{\partial F}{\partial w_{1,\theta}} \right) + \frac{\partial^2}{\partial \phi^2} \left( \frac{\partial^2 F}{\partial w_{1,\phi\phi}} \right) + \frac{\partial^2}{\partial \phi \partial \theta} \left( \frac{\partial^2 F}{\partial w_{1,\phi \theta}} \right) + \frac{\partial^2}{\partial \theta^2} \left( \frac{\partial^2 F}{\partial w_{1,\theta \theta}} \right) = 0. \tag{16}
\]

Substitution of a solution of the form

\[
\begin{align*}
  u_1 &= \sum \bar{u}(\phi) \sin n \theta, \\
  v_1 &= \sum \bar{v}(\phi) \cos n \theta, \\
  w_1 &= \sum \bar{w}(\phi) \cos n \theta,
\end{align*}
\]

where $n$ is a positive integer, into Eqs. (12)-(15) and some rearrangement gives the following ordinary differential equations for an arbitrary $n$

\[
\begin{align*}
  a_u^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_u^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_u^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} - \frac{w_0}{a} (1 + \nu) (\bar{u} \sin \phi + n \bar{v}) &= 0, \tag{18} \\
  a_v^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_v^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_v^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} - \frac{w_0}{a} (1 + \nu) \sin \phi (\bar{v} - \frac{\partial \bar{w}}{\partial \phi}) &= 0, \tag{19} \\
  a_w^{(i)} \frac{\partial^i \bar{u}}{\partial \phi^i} + b_w^{(i)} \frac{\partial^i \bar{v}}{\partial \phi^i} + c_w^{(i)} \frac{\partial^i \bar{w}}{\partial \phi^i} + \frac{a^2 k_f}{C} \bar{w} \sin \phi \\
  + \frac{w_0}{a} (1 + \nu) \sin \phi \left( \frac{\partial \bar{v}}{\partial \phi} + \bar{v} \cot \phi + \frac{n \bar{u}}{\sin \phi} - \frac{\partial^2 \bar{w}}{\partial \phi^2} - \frac{\partial \bar{w}}{\partial \phi} \cot \phi + \frac{n^2}{\sin^2 \phi} \bar{w} \right) &= 0, \tag{20}
\end{align*}
\]

in which the implied summation over $i$ cover the range of non-zero coefficients, which are described in the Appendix. These governing differential equations (18)-(20) are difficult to solve exactly by only assuming periodic functions in $\bar{u}, \bar{v}$.
Differentiation of Eq. (26) with respect to \( \phi \) leads to the following:

\[
H(\ldots) = (\ldots)^{\prime\prime} + (\ldots)^{\prime} \cot \phi + (\ldots) \left( 2 - \frac{n^2}{\sin^2 \phi} \right),
\]

(21)

where primes denotes differentiation with respect to \( \phi \). From the definition of \( H \) in Eq. (21), the following relations are readily obtained:

\[
HH(\ldots) = (\ldots)^{\prime\prime\prime} + 2(\ldots)^{\prime\prime} \cot \phi + (\ldots)^{\prime\prime} \left( 3 - \frac{1 + 2n^2}{\sin^2 \phi} \right) + (\ldots)^{\prime} \cot \phi \left( 4 + \frac{1 + 2n^2}{\sin^2 \phi} \right) + (\ldots) \left( 4 - \frac{2n^2}{\sin^2 \phi} - \frac{n^2(4 - n^2)}{\sin^4 \phi} \right),
\]

(22)

\[
H'(\ldots) = (\ldots)^{\prime\prime\prime} + (\ldots)^{\prime\prime} \cot \phi + (\ldots)^{\prime} \cot \phi \left( 2 - \frac{1 + n^2}{\sin^2 \phi} \right) + 2n^2(\ldots) \frac{\cos \phi}{\sin^3 \phi}.
\]

(23)

In addition, a new variable \( \tilde{u} \) is defined with regard to \( \bar{u} \) by

\[
\tilde{u} = \bar{u}'.
\]

(24)

By making use of the relations of Eqs. (21)–(23) and the new definition of Eq. (24), Eqs. (18) and (19) can be rewritten respectively as:

\[
(1 + k) \sin \phi \frac{\partial}{\partial \phi} \left( \frac{1 - \nu}{2} H(\bar{u}) - \frac{1 + \nu}{2} \frac{\bar{\nu} \sin \phi + n\bar{\nu}}{\sin^2 \phi} \right) - 2(1 + k)n \frac{\bar{\nu} \sin \phi + n\bar{\nu}}{\sin^2 \phi} \cos \phi - (1 + k)(1 + \nu) nw + knH(w) - \frac{\omega_0(1 + \nu)}{\alpha} (\bar{\nu}' + n\bar{\nu}) = 0,
\]

(25)

\[
(1 + k) \left[ \frac{1 + \nu}{2} (\bar{\nu} \sin \phi + n\bar{\nu})' + \frac{1 - \nu}{2} H(\bar{\nu} \sin \phi) - \frac{3 - \nu}{2} \cot \phi (\bar{\nu} \sin \phi + n\bar{\nu})' + (1 + \nu) w' \sin \phi \right] - k \sin \phi H'(w) - \frac{\omega_0(1 + \nu)}{\alpha} \sin \phi (\bar{\nu}' - \bar{\nu}) = 0.
\]

(26)

Differentiation of Eq. (26) with respect to \( \phi \) and then multiplying it by \( \sin \phi / n \) leads to the following:

\[
(1 + k) \frac{\sin \phi}{n} \left\{ \frac{1 + \nu}{2} (\bar{\nu} \sin \phi + n\bar{\nu})^{\prime\prime} + \frac{1 - \nu}{2} H'(\bar{\nu} \sin \phi) + \frac{3 - \nu}{2} \left[ \frac{1}{\sin^2 \phi} (\bar{\nu} \sin \phi + n\bar{\nu})' - \cot \phi (\bar{\nu} \sin \phi + n\bar{\nu})^{\prime\prime} \right] \right\} - k \frac{\sin \phi}{n} [\cos \phi H'(w) + \sin \phi H''(w)] = 0.
\]

(27)
By adding Eq. (27) to Eq. (25), some rearrangement gives:

\[
(1 + k) \left[ H(\Psi) - (1 + \nu)(\Psi - H(w) + 2w) \right] \\
- k \left[ HH(w) - 2H(w) \right] - \frac{w_0}{a} (1 + \nu)(\Psi - H(w) + 2w) = 0.
\]

(28)

in which:

\[
\Psi = \frac{(\bar{v} \sin \phi + n\hat{u})'}{\sin \phi}.
\]

(29)

Similarly, Eq. (20) can be written using the operator \( H \) and the new variable \( \Psi \) defined by Eq. (29) as follows:

\[
(1 + k)(1 + \nu)(\Psi + 2w) + k[HH(w) - (3 + \nu)H(w) - H(\Psi)] \\
+ \frac{a^2 k_F}{C} w + (1 + \nu) \frac{w_0}{a} (\Psi - H(w) + 2w) = 0.
\]

(30)

Equations (28) and (30) are the governing equations to be solved. Now, the solutions of \( \Psi \) and \( w \) of Eq. (28) and (30) is assumed to be in the form

\[
\Psi = \sum_{m=0}^{\infty} A_m P_m(\cos \phi), \quad w = \sum_{m=0}^{\infty} B_m P_m(\cos \phi),
\]

(31)

where \( A_m \) and \( B_m \) are the constant deformation amplitudes, and the spherical harmonic \( P_m(\cos \phi) \) is a series of Legendre functions of degree \( n \) which satisfies the following Legendre differential equation,

\[
P_m'' + \frac{m}{\cos \phi} P_m' + m(m + 1)P_m = 0.
\]

(32)

By using the operator \( H \) defined in Eq. (27), the following relation can be obtained:

\[
H(P_m) = -\lambda_m P_m,
\]

(33)

where

\[
\lambda_m = m(m + 1) - 2,
\]

(34)

and

\[
HH(P_m) = -\lambda_m^2 P_m.
\]

(35)
Substituting the series of Eq. (31) into Eqs. (28) and (30) and using Eqs. (33) and (35), we obtain:

\[
\sum_{m=0}^{\infty} (c_m^{(11)} A_m + c_m^{(12)} B_m) P_m (\cos \phi) = 0,
\]

\[
\sum_{m=0}^{\infty} (c_m^{(21)} A_m + c_m^{(22)} B_m) P_m (\cos \phi) = 0,
\]

in which

\[
c_m^{(11)} = (1 + k)(\lambda_m + 1 + \nu) + \frac{w_0}{a} (1 + \nu),
\]

\[
c_m^{(12)} = (1 + k)(1 + \nu)(\lambda_m + 2) + k(\lambda^2_m + 2k\lambda_m) + \frac{w_0}{a} (1 + \nu)(1 + \lambda_m),
\]

\[
c_m^{(21)} = (1 + k)(1 + \nu) + k\lambda_m + \frac{w_0}{a} (1 + \nu),
\]

\[
c_m^{(22)} = 2(1 + k)(1 + \nu) + k\lambda_m(\lambda_m + 3 + \nu) + a^2 k_F C + \frac{w_0}{a} (1 + \nu)(\lambda_m + 2).
\]

Hence, homogeneous linear equations for \(A_m\) and \(B_m\) are obtained for all \(m\) as follows:

\[
\begin{bmatrix}
  c_m^{(11)} & c_m^{(12)} \\
  c_m^{(21)} & c_m^{(22)}
\end{bmatrix}
\begin{bmatrix}
  A_m \\
  B_m
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
\]

This is a standard eigenvalue problem for which nontrivial values are obtained when the coefficient matrix in Eq. (38) becomes singular. The corresponding pressure \(p\) is thus calculated from the determinantal equations:

\[
\det \begin{bmatrix}
  c_m^{(11)} & c_m^{(12)} \\
  c_m^{(21)} & c_m^{(22)}
\end{bmatrix} = 0.
\]

By solving Eq. (39), the pressure \(p\) with the corresponding mode \(m\) for several combinations of geometric and material constants can be obtained as an eigenvalue. For the particular values of the constants, the minimum value of the eigenvalues is the critical buckling pressure \(p_{cr}\). It should be noted that the above equation is independent of the mode number \(n\). This shows that only axisymmetric modes can occur in this problem despite the inclusion of the Winkler foundation term.
3.2. Simplified approach

Next, we consider the simplified formulation by the Rayleigh–Ritz approach. Here an inextensional and axisymmetric buckling deformation of the shell are assumed with shear and torsional strains being neglected for simplicity. Thus, $u$, $\epsilon_{\phi\theta}$, and $\chi_{\phi\theta}$ are assumed to be zero.

The buckling deformation forms are given as follows:

$$v_1 = \sqrt{1 - c^2} \sum_{m=2}^{\infty} \tilde{v}_m \frac{dP_m(c)}{dc},$$  \hspace{1cm} (40)

$$w_1 = \sum_{m=2}^{\infty} \tilde{w}_m P_m(c),$$ \hspace{1cm} (41)

where $c = \cos \phi$.

The first variation $\delta U_M$ of the membrane (in-plane) strain energies is given by

$$\delta U_M = 2\pi C (1 + \nu) \int_\phi w_0 \left(v'_1 + v_1 \cot \phi + 2w_1\right) \sin \phi d\phi. \hspace{1cm} (42)$$

The integrand in Eq. (42),

$$v'_1 + v_1 \cot \phi + 2w_1 = 0,$$ \hspace{1cm} (43)

gives the inextensional buckling condition and when the assumed deformation functions in Eq. (40) and (41) satisfy the condition, the following relation can be obtained

$$\tilde{w}_m = \frac{m(m+1)}{2} \tilde{v}_m.$$ \hspace{1cm} (44)

The second variation of the strain energies are

$$\delta^2 V = \delta^2 U_M + \delta^2 U_B + \delta^2 U_F = 2\pi \int_\phi F d\phi,$$ \hspace{1cm} (45)

in which the each term can be expressed by the strain–displacement relations and Eqs. (40) and (41) as:

$$\delta^2 U_M = \pi C \sum_{m=2}^{\infty} \left[ \frac{2(m-1)m(m+1)(m+2)(1-\nu)}{2m+1} \right. \left. + \frac{w_0}{a} \frac{(m-1)^2m(m+1)(m+2)^2(1+\nu)}{2m+1} \right] \tilde{v}_m^2,$$ \hspace{1cm} (46)
\[ \delta^2 U_B = \frac{\pi D}{a^2} \sum_{m=2}^{\infty} \frac{(m-1)^2m(m+1)(m+2)^2[m(m+1) - 1 + \nu]}{2m+1} v_m^2, \]  
(47)

\[ \delta^2 U_F = \pi a^2 k_F \sum_{m=2}^{\infty} \frac{m^2(m+1)^2}{2m+1} v_m^2. \]  
(48)

The stability criterion
\[ \frac{\partial (\delta^2 V)}{\partial \tilde{v}_m} = 0, \]  
(49)

gives the following critical pressure
\[ p_{cr} = \frac{2Eh + ka^2(1 - \nu)}{aEh(m-1)^2(m+2)^2} \left\{ \frac{2Eh + (m-1)(m+2)}{1 + \nu} + \frac{D(m-1)^2(m+2)^2[m(m+1) - 1 + \nu]}{a^2} + a^2 k_f m(m+1) \right\}. \]  
(50)

Equation (50) has a minimum value when
\[ \frac{\partial p_{cr}}{\partial m} = 0. \]  
(51)

In such spherical shell buckling, \( m \) is usually a large integer. This allows us to approximate \((m-1)^2(m+2)^2 \approx m^4, m(m+1) \approx m^2, \) and so on. By applying the approximations to Eq. (50) and substituting the resulting equation into (51), the mode number \( \tilde{m}_{cr} \) associated with the lowest critical pressure and the corresponding critical pressure \( \tilde{p}_{cr} \) can be obtained as follows:
\[ \tilde{m}_{cr} = \sqrt{\frac{a^2[2Eh + a^2 k(1 + \nu)]}{D(1 + \nu)}}, \]  
(52)

\[ \tilde{p}_{cr} = \frac{2Eh + ka^2(1 - \nu)}{aEh\tilde{m}_{cr}^2} \left[ \frac{2Eh + ka^2(1 + \nu)}{1 + \nu} + \frac{D\tilde{m}_{cr}^2(\tilde{m}_{cr}^2 - 1 + \nu)}{a^2} \right]. \]  
(53)

4. Analytical results and discussion

4.1. Comparison of buckling pressures between exact and simplified approaches

Figure 2 shows the variation of the nondimensionalized critical buckling pressure \( \tilde{p}_{cr}/\tilde{p}_0 \), in which \( \tilde{p}_0 \) is the critical pressure for the empty complete spherical
Figure 2: Comparison of $p_{cr}$ ($a/h = 100$).

Figure 3: Relative error of $p_{cr}$ ($a/h = 100$).
Figure 4: Comparison of $m_{cr} (a/h = 100)$.

Figure 5: Relative error of $m_{cr} (a/h = 100)$.
shell, which is given by:

\[
p_0 = \frac{2E}{\sqrt{3(1 - \nu^2)}} \left( \frac{h}{a} \right)^2.
\]  

(54)

The solid lines are the exact values obtained from Eq.(39) and the dotted lines correspond to Eq.(53) from the Rayleigh–Ritz analysis. As can be easily seen in this figure, the value of \( p_{cr}/p_0 \) increases with increasing stiffness ratio \( ak_f/E \) and the exact and simplified \( p_{cr} \) curve agree with each other. Fig. 3 gives the relative error between exact and simplified values, which is defined by

\[
\frac{p_{cr(\text{simplified})} - p_{cr(\text{exact})}}{p_{cr(\text{exact})}}.
\]  

(55)

The relative error converges with the increase of \( ak_f/E \).

The comparison of the critical buckling mode number \( m_{cr} \) obtained from the exact and simplified approaches is shown in Fig. 4. Larger values of \( ak_f/E \) cause higher buckling modes, as in the case of beams on an elastic Winkler foundation [20]. The relative error

\[
\frac{m_{cr(\text{simplified})} - m_{cr(\text{exact})}}{m_{cr(\text{exact})}}.
\]  

(56)

is given in Fig. 5. We can also see the convergence behaviour with the increase of \( ak_f/E \). These comparisons in Figs. 3 and 5 confirm the validity of the proposed simplified formulations for \( p_{cr} \) and \( m_{cr} \) especially for higher stiffness ratios.

4.2. Eigenmodes

Figure 6 shows a phase diagram of the buckling mode number \( m \). This figure gives significant information on the various axisymmetric buckling modes depending on the values of \( ak_f/E \) and \( a/h \). It is observed from this figure that larger \( ak_f/E \) and \( a/h \) values favour buckling modes with higher mode numbers. We can also find that when the spherical shells are relatively thin with larger \( a/h \) values, the effect of the foundation modulus becomes more significant when compared with the case for the relatively thick shells with smaller \( a/h \) values. It should be noted that The obtained mode number \( m=18 \) for the condition of \( a/h = 100 \) and \( ak_f/E = 0.1 \) is same as that found by Koiter [6].

Figure 7 illustrates the buckling eigenmodes for the empty complete spherical shell of \( a/h = 50 \). The characteristic wavy-shaped axisymmetric buckling deformation with the mode number \( m = 12 \) can be found in this case.
Figure 6: Phase diagram of the buckling mode number $m$. Various axisymmetric buckling modes can be found depending on the values of $ak_f/E$ and $a/h$.

On the other hand, Figs. 8 and 9 show the eigenmodes for the “filled” spherical shell ($ak_f/E = 0.1$) with $a/h=50$ and $100$, respectively; the axisymmetric buckling deformation of the mode numbers $m = 20$ and $33$ are seen in these figures. As can be understood in the developed formulation, non-axisymmetric modes cannot occur under the condition of constant foundation modulus. However, the cross-sectional shape for Fig. 9 is moderately different from that for Figs. 7 and 8.

The results shown in Figs. 7-9 suggest that the eigenmodes change corresponding to the mode number $m$. When $m$ is even, the corresponding deformation of the spherical shell is symmetric about the equator, as shown in Figs. 7 and 8. On the other hand, when $m$ is odd, the deformation is “anti-symmetric” about the equator, that is, an inward deformation at the north pole is accompanied by an outward deformation at the south pole. This is due to the property of the geometry of the sphere and the Legendre functions, not whether the shell is empty or otherwise.
Figure 7: Buckling modes of a hydrostatical pressurized "empty" complete spherical shell with $a/h = 50$ and $ak_f/E = 0$. (Left) 3-D view. (Right) Cross-sectional view through a great circle. The buckling mode number $m = 12$ is found.

Figure 8: Buckling modes of a hydrostatical pressurized "filled" complete spherical shell with $a/h = 50$ and $ak_f/E = 0.1$. (Left) 3-D view. (Right) Cross-sectional view through a great circle. The buckling mode number $m = 20$ is found.

Figure 9: Buckling modes of a hydrostatical pressurized "filled" complete spherical shell with $a/h = 100$ and $ak_f/E = 0.1$. (Left) 3-D view. (Right) Cross-sectional view through a great circle. The buckling mode number $m = 33$ is found.
5. Conclusions

The characteristic buckling modes in hydrostatically pressurized complete spherical shells filled with an elastic medium have been demonstrated. A theoretical formulation based on small-displacement thin shell theory has derived governing equations of equilibrium that have been solved using an exact methodology. In addition, the simplified formulations for estimating the critical pressure and the mode number are proposed with a Rayleigh–Ritz approach. The formulations obtained from the Rayleigh–Ritz methodology is shown to give sufficiently accurate results compared with the exact values.

From the formulation developed currently only axisymmetric eigenmodes can occur in spite of adding the Winkler foundation term; critical modes that are symmetric or anti-symmetric about the equator may be determined depending on the combination of the foundation stiffness and the radius to wall-thickness ratios. A phase diagram has been established to obtain the requisite values of $a/h$ and $ak_f/E$ for observing a corresponding buckling mode.

This work is a fundamental investigation that has been developed with future studies in mind. These will address the post-buckling behaviour and non-axisymmetric modes which may occur when a foundation modulus is not a constant value and a function of the wave number of the shell deformation. The Rayleigh–Ritz approach developed currently is more likely to be employed for the nonlinear studies given that it allows for a simpler solution methodology; it has been shown in the present work that it provides an excellent approximation to the linear case. This will also provide an insight into the electrical properties of carbon onions, i.e., quasi-spherical nanoparticles consisting of concentric graphitic shells. They possess a high specific surface area ($\sim 500 \text{ m}^2 \text{ per gram}$) with no porosity and high field-emission stability owing to the endurance of carbon-carbon atomic bonds. These two features are appealing in the design of electrochemical capacitors and point electron sources, in which the radial buckling of outer layers may lead positive effects in operation. It was also suggested in that the electronic structure of carbon onions is sensitive to the radii of concentric layers and their separations. Hence, their asymmetric buckling under pressure should cause a significant change in the electronic states, where the mechanical deformation of the underlying carbon structure induces a effective electrostatic potential that exerts on the mobile electrons on the layers. The present findings will be helpful for addressing the deformation-driven carbon-onion behaviours from both academic and practical viewpoints.
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Appendix A. The coefficients of the governing equations

The coefficients of the governing equations (18)–(20) are written as follows:
\[ a_u^{(0)} = (1 + k) \left[ \frac{1 - \nu}{2} (1 - \cot^2 \phi) \sin \phi - \frac{n^2}{\sin \phi} \right], \]
\[ a_u^{(1)} = (1 + k) \frac{1 - \nu}{2} \cos \phi, \quad a_u^{(2)} = (1 + k) \frac{1 - \nu}{2} \sin \phi, \quad a_u^{(3)} = a_u^{(4)} = 0, \]
\[ b_u^{(0)} = (1 + k) \frac{v - 3}{2} n \cot \phi, \quad b_u^{(1)} = -(1 + k) \frac{1 + v}{2} n, \quad b_u^{(2)} = b_u^{(3)} = b_u^{(4)} = 0, \]
\[ c_u^{(0)} = kn \left( 2 - \frac{n^2}{\sin^2 \phi} \right), \quad c_u^{(1)} = kn \cot \phi, \quad c_u^{(2)} = kn, \quad c_u^{(3)} = c_u^{(4)} = 0, \]
\[ a_v^{(0)} = (1 + k) \frac{v - 3}{2} n \cot \phi, \quad a_v^{(1)} = (1 + k) \frac{1 + v}{2} n, \quad a_v^{(2)} = a_v^{(3)} = a_v^{(4)} = 0, \]
\[ b_v^{(0)} = -\frac{1 + k}{\sin \phi} \left( \cos^2 \phi + \nu \sin^2 \phi + \frac{1 - \nu}{2} n^2 \right), \]
\[ b_v^{(1)} = (1 + k) \cos \phi, \quad b_v^{(2)} = (1 + k) \sin \phi, \quad b_v^{(3)} = b_v^{(4)} = 0, \]
\[ c_v^{(0)} = -2kn^2 \frac{\cot \phi}{\sin \phi}, \quad c_v^{(1)} = -k \left( 2 - \frac{1 + n^2}{\sin^2 \phi} \right) \sin \phi, \]
\[ c_v^{(2)} = -k \cos \phi, \quad c_v^{(3)} = -k \sin \phi, \quad c_v^{(4)} = 0, \]
\[ a_w^{(0)} = (1 + k)(1 + v)n - kn \left( 2 + \frac{1 - n^2}{\sin^2 \phi} \right), \]
\[ a_w^{(1)} = kn \cot \phi, \quad a_w^{(2)} = -kn, \quad a_w^{(3)} = a_w^{(4)} = 0, \]
\[ b_w^{(0)} = (1 + k)(1 + v) \cos \phi - k \cos \phi \left( 2 + \frac{1 - n^2}{\sin^2 \phi} \right), \]
\[ b_w^{(1)} = (1 + k)(1 + v) \sin \phi + \frac{k(n^2 + \cos^2 \phi)}{\sin \phi}, \]
\[ b_w^{(2)} = -2k \cos \phi, \quad b_w^{(3)} = -k \sin \phi, \quad b_w^{(4)} = 0, \]
\[ c_w^{(0)} = -k \left[ 2(1 + v) \sin \phi + n^2 \frac{3 - v + 4 \cot^2 \phi}{\sin \phi} - \frac{n^4}{\sin^3 \phi} \right], \]
\[ c_w^{(1)} = \left( 1 - v + \frac{1 + 2n^2}{\sin^2 \phi} \right) \cos \phi, \quad c_w^{(2)} = -\frac{1 + 2n^2 + v \sin^2 \phi}{\sin \phi}, \]
\[ c_w^{(3)} = 2k \cos \phi, \quad c_w^{(4)} = k \sin \phi. \quad (A.1) \]
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