Interaction of Double Sine-Gordon Solitons with External Potentials: an Analytical Model

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Interaction of double sine-Gordon solitons with a space dependent potential well as well as a potential well is investigated by employing an analytical model based on the collective coordinate approach. The potential is added to the model through a suitable nontrivial metric for the background spacetime. The model is able to predict most of the features of the soliton-potential interaction. It is shown that a soliton can pass through a potential barrier if its velocity is larger than a critical velocity which is a function of the initial soliton conditions and also the characters of the potential. It is interesting that the solitons of the double sine-Gordon model can be trapped by a potential barrier and oscillate there. This situation is very important in applied physics. The soliton-well system is investigated by using the presented model. Analytical results are also compared with the results of the direct numerical solutions.

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The sine-Gordon equation has attracted a great deal of interest from physicists. This equation is a non-linear partial differential equation appearing naturally in different physical systems such as atomic physics,[1] electromagnetism,[2] superconductivity,[3] field theory,[4] biophysics,[5–7] and statistical mechanics.[8] It also has plenty of applications in condensed matter systems[9,10] and nonlinear optics.[11] Moreover, the solitons and kinks of the SU(5) generalized sine-Gordon model (SGS) are shown to describe the baryonic spectrum of two-dimensional quantum chromodynamics (QCD2).[12] As a natural development of the studies on integrable quantum field theories, there has been recently an increasing interest in studying the properties of such non-integrable quantum field theories in (1+1) dimensions (like the double sine-Gordon (DSG) model) for both theoretical reasons and their applications. The Lagrangian of a realistic physical system often gives a more complicated equation of motion than the sine-Gordon equation. For example, a quantum spin chain is mapped into a Lagrangian with several potential terms.[6] Systems with nonlinear optical properties also give rise to more complicated wave equations.[7] Thus a more enhanced model is desirable. This leads to the DSG equation.

The DSG model has been applied to model a variety of systems in condensed matter, quantum optics, and particle physics.[13] Condensed-matter applications include the spin dynamics of superfluid $^3$He,[14] magnetic chains,[15] commensurate-incommensurate phase transitions,[16] surface structural reconstructions,[17] domain walls[18,19] and fluxon dynamics in the Josephson junction.[20] In quantum field theory and quantum optics, DSG applications include quark confinement[21] and self-induced transparency.[22]

In the ideal DSG equation, the parameters of the model are spacetime independent fixed parameters. However, it is clear that in a realistic system, such parameters are the function of space or time, for example, considering the long Josephson junction. For a sufficiently wide class of Josephson junctions, the superconducting Josephson current (phase difference of superconductor’s wave functions) can be represented as a sine series. Using only the first two terms of this expansion one can show[20,21] that the distribution of the magnetic flux along the $z$-axis of the junction in the static regime satisfies the DSG equation whose parameters are dependent on the preparation technology of junctions which naturally cannot be fixed along the junction. This means that the parameters of the model become space dependent because of medium disorders. In this situation the localized solution encounters some kinds of space dependent potentials which greatly affect the behavior of the soliton.

There has been increasing interest in the scattering of solitons from defects or impurities, which generally come from medium properties. As mentioned above, the motivations come from both theoretical and applied aspects of physics. The effects of medium disorders and impurities can be added to the equation of motion as perturbative terms.[24] These effects can also be generated by making some parameters of the equation of motion. The parameters are a function of space or time.[25,26] There still exists another interesting method which is mainly suitable for working with topological solitons.[27–29] In this method, one can add such effects to the Lagrangian of the system by introducing a suitable nontrivial metric for the background spacetime without losing the topological boundary conditions.

Numerical simulation is the main tool which is used for the investigation of the soliton behavior in a defective medium. Although one could always rely on numerical methods to shed some light on their properties, it would obviously be useful to develop some theoretical tools to control them analytically. Motivated by this situation an analytical model is presented to investigate the interaction of solitons of the DSG model with defects using a collective coordinate ap-
The general form of the Lagrangian density for a real scalar field \( \phi \) is

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - U(\phi).
\]  

(1)

The DSG potential which contains a constant and a harmonic term in addition to the self-interaction potential of the ordinary sine-Gordon equation is considered as follows:

\[
U(\phi) = 1 - \cos(\phi) + A(1 - \cos(2\phi)),
\]  

(2)

where \( A \) is a constant. This potential has absolute minima at \( \phi = 2n\pi \) as the true vacua, and the metastable, local minima at \( \phi = (2n + 1)\pi \) as the false vacua. The harmonic term in this potential can result from the Fourier expansion of an arbitrary, periodic potential \( V(\phi) = V(\phi + 2n\pi) \). One does not expect the system to remain integrable by adding these extra terms. The potential reduces to the ordinary SG potential in the limit \( A \to 0 \). We focus on a model with \( A = 1 \). One solution for the DSG equation can be written as:

\[
\phi = k\pi - 2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sinh\left(\sqrt{\frac{3}{5}}(x_0 - \dot{X}t)\right)\right),
\]  

(3)

where \( x_0 \) and \( \dot{X} \) are the soliton initial position and its velocity, respectively.

As mentioned above, one can add effects of medium disorders to the Lagrangian of the system by introducing a suitable nontrivial metric for the background space time. In other words, the metric carries the information of the medium. The suitable metric in the presence of a weak potential \( V(x) \) is

\[
g_{\mu\nu}(x) = \begin{pmatrix}
1 + V(x) & 0 \\
0 & -1
\end{pmatrix}= (\dot{X} - 2)50\sqrt{\frac{5}{\pi}}\epsilon
\cdot \cosh(\sqrt{5}(x - X))
\cdot \left(3 - \sinh^{2}\left(\sqrt{5}(x - X)\right)\right)\left(5 + \sinh^{2}\left(\sqrt{5}(x - X)\right)\right)^{4},
\]  

(9)

where \( M_0 = \int_{-\infty}^{\infty} dx 100\cosh(\sqrt{5}(x - X)) = \ln(\frac{\pi + 2\sqrt{5}}{\pi - 2\sqrt{5}}) + 4\sqrt{5}\).

The above equation shows that the peak of the soliton energy moves under the influence of a complicated force which is a function of the soliton position, its velocity and also characters of the external potential, \( V(x) \). Equation (10) clearly shows that the soliton mass is a space dependent function which is an interesting non-classical behavior. The soliton energy in the presence of the potential \( V(x) = \epsilon \delta(x) \) is calculated by

\[
E = \frac{1}{2} \dot{X}^{2}\left(M_0 + 50\epsilon \cosh^{2}\left(\sqrt{5}X\right)\right)
+ 50\epsilon \cosh^{2}\left(\sqrt{5}X\right)
\cdot \left(5 + \sinh^{2}\left(\sqrt{5}X\right)\right) + M_0.
\]  

(11)
It is the energy of a particle with a space dependent mass

\[ M(X) = M_0 - 50 \varepsilon \frac{\cosh^2(\sqrt{5}X)}{(5 + \sinh^2(\sqrt{5}X))^2}, \tag{12} \]

and velocity \( \dot{X} \) which is moved under the influence of the external effective potential. By calculating \( X^2 \) from Eq. (10) and inserting it into the soliton energy Eq. (12), one can show that the soliton total energy is a function of both the soliton initial position \( X_0 \) and the initial velocity \( \dot{X}_0 \). Therefore, the energy of a soliton remains conserved during the interaction. Figure 1 (a) presents the energy of a static soliton as a function of its position in the potential \( V(x) = +0.35 \delta(x) \). Due to the extended nature of the soliton, the effective potential is not an exact delta function.

![Figure 1](image)

**Figure 1.** (a) The soliton potential energy as a function of collective variable \( X \), for \( \varepsilon = +0.35 \). (b) The potential energy of soliton as a function of the soliton position with \( \varepsilon = -0.35 \).

Figure 1 (a) also shows that the energy has two absolute maxima (\( E_{\text{max}} \)) in \( X_m = \pm \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}) \) and a local minimum (\( E_{\text{min}} \)) in \( X = 0 \). This configuration creates very interesting features for the soliton during the interaction with the potential. There are three different trajectories for a soliton according to its initial conditions and the potential character.

Consider a soliton which goes toward the potential barrier (\( \varepsilon > 0 \)) from an initial position \( |X_0| > \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}) \) with an initial velocity \( \dot{X}_0 \). The soliton will reflect back if its total energy is less than the potential maximum (\( E_{\text{max}} \)). In this case the soliton initial velocity \( \dot{X}_0 \) is lower than the critical velocity

\[ v_c = \sqrt{\frac{\varepsilon}{M_0 - 50 \varepsilon \frac{\cosh^2(\sqrt{5}X_0)}{(5 + \sinh^2(\sqrt{5}X_0))^2} \cdot \frac{5|3 - \sinh^2(\sqrt{5}X_0)|}{2(5 + \sinh^2(\sqrt{5}X_0))}}}. \tag{13} \]

The soliton passes through the barrier if its initial velocity is larger than the critical velocity. In this situation the soliton energy will be larger than \( E_{\text{max}} \). Figure 1 (b) demonstrates these two soliton trajectories during the interaction with the potential barrier of \( \varepsilon = 0.2 \) plotted by solving Eq. (10) numerically. The dashed line presents the trajectory of a soliton with an initial velocity lower than the critical velocity. The soliton climbs the barrier while it cannot pass the barrier and reflects back. The solid line shows that a soliton with a velocity larger than \( v_c \) passes through the barrier. Note the small fluctuations in the soliton trajectory on top of the barrier. This part of the trajectory contains different physics which are discussed with more details in the following.

**Figure 2.** Soliton trajectories as a function of time with different initial velocities. For the dashed line, the initial velocity is taken to be lower than the critical velocity while the solid line shows the trajectory of a soliton with the initial velocity larger than the critical velocity. The initial position is \( X_0 = -3 \).

Now consider a soliton which is initially located somewhere in the valley between two peaks of the potential, i.e., \( |X_0| < \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3}) \) with an energy less than the potential maximum, \( E < E_{\text{max}} \). The soliton oscillates around the potential minimum which is located at the origin. This situation is unique for a soliton-barrier interaction. It is very interesting and important behavior. A soliton in this situation can be trapped by a potential barrier which has not been observed.

For small amplitude oscillation (sufficiently small \( X_0 \)) one can use a Taylor expansion for the mass term and the potential energy around \( X = 0 \) to find the oscillation frequency as

\[ M(X) = M_0 - 2\varepsilon + O(X^2), \tag{14} \]

\[ U(X) = 2\varepsilon + 6\varepsilon X^2 + O(X^3). \tag{15} \]

With this approximation the angular frequency of the soliton oscillation becomes

\[ \omega_0 = \sqrt{\frac{12\varepsilon}{M_0 - 2\varepsilon}}. \tag{16} \]

A potential well can be created with a negative \( \varepsilon \). \( \varepsilon < 0 \). Therefore, the soliton equation of motion becomes

\[ \ddot{X} \left( M_0 + 50 \varepsilon \frac{\cosh^2(\sqrt{5}X)}{(5 + \sinh^2(\sqrt{5}X))^2} \right) \]

\[ = - (X^2 - 2)50\sqrt{\varepsilon} \cdot \cosh(\sqrt{5}X) \sinh(\sqrt{5}X)(3 - \sinh^2(\sqrt{5}X)) (5 + \sinh^2(\sqrt{5}(x - X)))^4 \]

and its energy is

\[ E = \frac{1}{2} \dot{X}^2 \left( M_0 + 50 \varepsilon \frac{\cosh^2(\sqrt{5}X)}{(5 + \sinh^2(\sqrt{5}X))^2} \right) \]

\[ - 50 \varepsilon \frac{\cosh^2(\sqrt{5}X)}{(5 + \sinh^2(\sqrt{5}(X)))^2} + M_0, \tag{17} \]

which shows that the soliton moves under the influence of an attractive potential. Figure 2 (a) shows the potential well with \( \varepsilon = -0.35 \). Assume that a soliton moves toward the center of the potential well from an
initial position $X_0$ with an initial velocity $\dot{X}_0$. The soliton can escape to infinity if its initial velocity is larger than the escape velocity $v_{\text{escape}} = \frac{\varepsilon}{M_0} \cdot \frac{10 \cosh(\sqrt{5}X_0)}{\sqrt{\frac{5\varepsilon}{M_0}} \cosh^2(\sqrt{5}X_0) + (5 + \sinh^2(\sqrt{5}X_0))^2}$. \hfill (19)

However, the soliton will be captured by the well and oscillate there if its initial speed is lower than the escape velocity $v_{\text{escape}}$. The maximum distance between the soliton and the center of the potential is calculated by using Eqs. (17) and (18) as

$$100\varepsilon \frac{\cosh^2(\sqrt{5}X_{\text{max}})}{(5 + \sinh^2(\sqrt{5}X_{\text{max}}))^2} = (2 - \dot{X}_0^2)(M_0 + 50\varepsilon) \frac{\cosh^2(\sqrt{5}X_0)}{(5 + \sinh^2(\sqrt{5}X_0))^2} - 2M_0. \hfill (20)$$

The captured soliton has two different oscillation modes according to its total energy. If its energy is larger than $M_0 - 2\varepsilon$, it will oscillate around the center of the well, $X = 0$. The angular frequency can be calculated by using the Taylor series expansions of $M(X)$ and $U(X)$ around $X = 0$ as

$$M(X) = M_0 + 2\varepsilon + O(X^2), \hfill (21)$$

$$U(X) = 2\varepsilon + \frac{12\varepsilon}{2} X^2 + O(X^3). \hfill (22)$$

Therefore, we have

$$\omega_w = \sqrt{\frac{12\varepsilon}{M_0 + 2\varepsilon}}. \hfill (23)$$

A trapped soliton with an energy less than $M_0 - 2\varepsilon$ oscillates around one of the two degenerate minima of the potential, $X_{\text{min}} = \pm \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{3})$. The angular frequency for this oscillation mode is calculated as follows:

$$M(X) = M(X = X_1) + (X - X_1) \frac{\partial M}{\partial X} \bigg|_{X = X_1}$$
$$+ O((X - X_1)^2)$$
$$= M_0 + \frac{25}{8} \varepsilon + O((X - X_1)^2), \hfill (24)$$

and

$$U(X) = \frac{25}{2} \varepsilon + \frac{375\varepsilon}{32} (X - X_1)^2 + O((X - X_1)^3), \hfill (25)$$

and therefore

$$\omega_1 = \sqrt{\frac{375\varepsilon}{16M_0 + 50\varepsilon}}. \hfill (26)$$

The dynamics of soliton-potential interaction can be studied by using the above results theoretically. Equation 7 has also been solved numerically by using the fourth-order Runge-Kutta method for time derivatives. Space derivatives were expanded by using the finite difference method. The Hamiltonian density has been calculated by using the finite difference method in each time step. The delta function has been simulated by using the Gaussian function $V(x) = \sqrt{\frac{\varepsilon}{2\pi}} e^{-ax^2}$. The stability of numerical procedures and the validity of the results have been checked by carrying out the calculations with different values for the grid space lengths (0.01, 0.05 and 0.001). Time steps were chosen to be less than $\frac{1}{4}$ of the grid space steps due to the numerical stability considerations.

It is clear that the results of numerical simulations are different from the analytical results due to some used approximations. However, one can fit the numerical outcome on the analytical equations using an effective value for the potential strength $\varepsilon$. Thus the numerical results for the critical velocity of a soliton to pass over the potential barrier has been fitted on the derived analytical Eq. (13). The critical velocity can be found numerically by sending a soliton with different initial speeds and observing the final situation after the interaction (falling back or getting over the potential). The critical velocity for a soliton which goes toward the potential from infinity is

$$v_c = \frac{5}{2} \sqrt{\frac{\varepsilon}{M_0}}. \hfill (27)$$

The soliton initial position $X_0 = -8$ was chosen in the numerical simulations and the critical velocity has been found for different values of the potential strength. An effective strength can be found by fitting the simulation results on the function $\frac{5}{2} \sqrt{\frac{\varepsilon_{\text{eff}}}{M_0}}$, where $\varepsilon_{\text{eff}} = p_1 + p_2 \varepsilon$. Figure 2(b) presents the critical velocity (27) and corresponding numerical results. The effective potential is fitted on the analytic model by $p_1 = 0.03348 \pm 0.00106$ and $p_2 = 0.49791 \pm 0.00357$ with standard deviation of $4.185 \times 10^{-7}$, which means $\varepsilon_{\text{eff}} \approx \frac{5}{2}$. As mentioned above, the effective potential in the curved spacetime is two times the corresponding potential in the flat spacetime. This clearly shows that the theoretical model describes the real situation with a very good approximation.

Equation (13) shows that the soliton critical velocity is a function of its initial position. It is expected that one can successfully fit the numerical results on this equation using the calculated $\varepsilon_{\text{eff}}$. Figure 3(a) presents the critical velocity as a function of the soliton initial position. The dashed line is plotted by using Eq. (13) and the solid line shows the numerical results. This figure demonstrates a good agreement between the analytical and numerical results.

Unfortunately such an agreement between the numerical and analytical results has not been seen in some of the other predictions. Figure 3(b) demonstrates the period of oscillation in a soliton-barrier system. The dashed line is plotted by using Eq. (13) while the solid line shows the numerical results with the corresponding $\varepsilon_{\text{eff}}$. Both the curves show that the oscillation period decreases with increasing $\varepsilon$. However, the predicted period from the analytical model is very different from the numerical results. The situ-
tion is better for small oscillations in the soliton-well system as shown in Fig. 4. The results of numerical simulations are shown with the solid line, while the dashed line is plotted by using Eq. (26).

Fig. 3. (a) Critical velocity as a function of $\varepsilon$. The solid line presents the numerical results while the dashed line is the fitted line on the numerical outcome. (b) Critical velocity as a function of soliton initial position, $X_0$, with $\varepsilon = -0.3$. The solid line shows the numerical results, and the dashed line is corresponding to the analytical prediction.

Fig. 4. (a) The soliton oscillation period (captured in the center of the barrier) as a function of $\varepsilon$. The solid line shows the numerical results and the dashed line is plotted by using Eq. (13). (b) Period of oscillation around one of the two degenerate minima of the potential well as a function of $\varepsilon$. The dashed line presents the analytical prediction, and the solid line shows the numerical results. Interaction of DSG solitons with external defect has also been studied in Ref. [39] by using the collective coordinate approach calculated with different methods for adding the potential to the equation of motion. Comparing the predictions with related numerical simulations has not been carried out in this study. The general behavior of the soliton predicted in Ref. [39] is the same as that presented here. This means that the two models describe the general features of the soliton-potential interaction correctly. However, there are some important differences between the two models in the details of interactions. This is due to the different natures of the ways for adding the potential. In our model, solitons have dynamical space-dependent mass while the soliton mass in the model [39] is constant. Our presented model saves the symmetries and also topological properties of the theory in the presence of the external potential, while the model in Ref. [39] is not able to carry all of the field properties correctly. Note that the solitonic solution in the sine-Gordon model is essentially established due to the topological boundary conditions.

In summary, an analytical model for soliton-potential interaction has been presented in the double sine-Gordon field theory. Most of the soliton characters during the interaction with potential walls and also potential wells have been derived theoretically. The critical velocity of the soliton to pass over the potential barrier is derived as a function of soliton initial conditions and potential characters. The model predicts that the soliton of a double sine-Gordon model can be trapped by a potential barrier, which is a very interesting situation. Outcomes of this behavior in applied physics are very important. The period of small amplitude oscillations in this situation has been calculated theoretically. In a potential well, a soliton needs a velocity larger than an escape velocity (or a minimum energy) to be able to go to infinity. The escape velocity has been calculated by using the presented model. Two different modes of soliton oscillation in the potential well have also been calculated.

The analytical results have been compared with the results of the direct numerical simulation of the soliton-potential interaction. In most of the cases, analytical and numerical results are in agreement with each other, while there are meaningful differences between the oscillation periods which are derived from the analytical model and that calculated by numerical simulations. These differences need more attention. It is possible that the differences come from the interaction between the soliton internal modes with the potential. It is expected that one can resolve this point with an improved model containing better collected coordinate systems. This approach can also be used to create suitable analytical models in other field theories.

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