ON A FREE BOUNDARY PROBLEM FOR A NONLOCAL REACTION-DIFFUSION MODEL

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Abstract. This paper is concerned with the spreading or vanishing dichotomy of a species which is characterized by a reaction-diffusion Volterra model with nonlocal spatial convolution and double free boundaries. Compared with classical reaction-diffusion equations, the main difficulty here is the lack of a comparison principle in nonlocal reaction-diffusion equations. By establishing some suitable comparison principles over some different parabolic regions, we get the sufficient conditions that ensure the species spreading or vanishing, as well as the estimates of the spreading speed if species spreading happens. Particularly, we establish the global attractivity of the unique positive equilibrium by a method of successive improvement of lower and upper solutions.

1. Introduction. It is quite well understood now that the mathematical modelling of the dynamical processes in physical, biological and applied sciences requires differential equations, equations with delay, integro-differential equations, and other kinds of functional equations. For example, Volterra [28] introduced equations of the form

\[
\frac{du}{dt} = au - bu^2 - ru \int_{t_0}^t \phi(t-s)u(s)ds, \quad t > 0
\]

(1)

to describe the evolution of a single population, where \( t_0 = 0 \) or \(-\infty \), \( u \) is the population size, \( a - bu \) denotes the intrinsic rate and \( \int_{t_0}^t \phi(t-s)u(s)ds \) is a temporal convolution representing the memory rate and containing the effect of the past history on the actual population development. The initial function for (1) satisfies

\[
u(0) = u_0 \quad \text{if} \quad t_0 = 0 \quad (\text{resp.} \quad u(t) = q(t) \quad \text{for} \quad t < 0 \quad \text{if} \quad t_0 = -\infty).
\]

In (1), the logistic term which expresses the crowding effect, is separated into two parts: a non-delay term \( bu \) and a hereditary term \( \int_{t_0}^t \phi(t-s)u(s)ds \). During the last 60–80 years, Volterra equations have emerged vigorously in applied fields such as automatic control theory, network theory, and the dynamics of nuclear reactors.

Concerned with the asymptotic behavior of solutions of (1), Miller [21] studied (1) and obtained that for any positive \( u_0 \) (resp. for any positive, continuous and...
bounded function \( q(t) \), problem (1) admits a unique positive solution \( u \) defined for all \( t > 0 \) and \( \lim_{t \to +\infty} u(t) = a \left( b + r \int_0^\infty \phi(s)ds \right)^{-1} \) by assuming

\[
a, b > 0, \quad \phi \in C(0, \infty) \cap L^1(0, \infty), \quad \phi(x) = \phi(-x), \quad \phi \neq 0 \quad \text{and} \quad b > r \int_0^\infty |\phi(s)|ds.
\]

Redlinger [23] extended Miller’s results to the situation that a diffusion term is added to (1) and supposed that the population has no interaction with the exterior, that is

\[
\begin{cases}
  u_t = \Delta u + au - bu^2 - ru \int_0^t \phi(t-s)u(s)ds, & \text{in } (0, \infty) \times \Omega, \\
  u(0, x) = u_0(x), & \text{for } x \in \bar{\Omega}
\end{cases}
\]

subject to the homogeneous Neumann boundary condition, where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( C^2 \) boundary and \( u_0 \in C^1(\bar{\Omega}) \). Redlinger obtained the convergence result \( \lim_{t \to +\infty} u(t, x) = a \left( b + r \int_0^\infty \phi(s)ds \right)^{-1} \) uniformly for \( x \in \bar{\Omega} \) in the case that \( b > r \) by a recursively defined sequence of lower and upper solutions, and also emphasized that those conclusions still remained true if \( \Delta \) is replaced by the linear, uniformly elliptic operator

\[
(\mathcal{L}u)(t,x) = \sum_{i,j=1}^{n} a_{ij}(t, x)u_{x_i x_j} + \sum_{i=1}^{n} a_i(t, x)u_{x_i}
\]

with coefficient functions \( a_{ij}, a_i \) that are uniformly Hölder continuous in \( (0, \infty) \times \bar{\Omega} \) and satisfy \( a_{ij} = a_{ji}, \; a_i(0, \cdot) \in C^1(\bar{\Omega}) \) for \( i, j = 1, 2, \cdots, n \). The main theorem stated in [23] has also been proved by Schiaffino [25] if \( \phi \) is nonnegative, decreasing and \( \min_{x \in \Omega} u_0(x) > 0 \), and by Yamada [33] if \( \phi \in C^1(0, \infty) \) is nonnegative and satisfies \( t \phi \in L^1(0, \infty) \). If \( u_0 \) is close to \( a \left( b + r \int_0^\infty \phi(s)ds \right)^{-1} \) and \( n < 3 \), Tesei [27] established the same convergence results as in [23]. Moreover, please see Schiaffino and Tesei [26] for Dirichlet boundary condition involved in problem (2). We also refer to Deng et al. [6] for a more general reaction-diffusion model on \( \mathbb{R}^N \) with a nonlocal spatial convolution \( (\phi * u) = \int_{\mathbb{R}^N} \phi(x-y)u(t,y)dy \). Integral equations with diffusion are studied by many authors, see Cordoneanu [5] for complete references.

The aforementioned models and results showed that either in a bounded domain, or through the whole \( \mathbb{R}^N \), the associated solutions are always positive, and eventually, converge to the corresponding positive equilibrium for all \( x \) once \( t \) is positive, regardless of the initial data and supporting area. In population biology, this means that the population may spread very quickly to the whole environment even though there is a small amount of individuals in the very early stage of its introduction, which does not match the fact that any population always spreads gradually.

Recently, free boundary problems have been studied intensively in many fields to describe a precise gradual spreading process, together with a changing underlying area. In particular, the well-known Stefan condition has been used to describe the spreading front in many applied problems. It was used to describe the melting of ice in contact with water [24], the wound healing [3] and the tumor growth [4]. In order to get a more precise prediction of the location of the spreading front of an invading species, Du et al. [9] firstly studied the spreading-vanishing dichotomy of some invasion species which is described by a diffusive logistic model in the homogenous environment of a one dimensional space. Since then, more results for reaction-diffusion equations with more general free boundaries have been obtained.
in [7, 11, 8, 32] for single species model, [20, 10, 16, 29, 31] for Lotka-Volterra systems, and [17, 15] fo"{e}r epidemic models. More relevant theoretical advances can be found in [13, 22, 30]. Especially, please refer to [34] for a double fronts free boundary problem with a nonlocal nonlinear reaction term.

Due to the results above, this paper will focus on the dynamics of positive solutions \((u, g, h)\) of the following free boundary problem in a one dimensional space

\[
\begin{align*}
    u_t &= du_{xx} + u(a - bu - r* u), \quad t > 0, \quad g(t) < x < h(t), \\
    u(t, g(t)) &= u(t, h(t)) = 0, \quad t > 0, \\
    h'(t) &= -\mu u_x(t, h(t)), \quad h(0) = h_0, \quad t > 0, \\
    g'(t) &= -\mu u_x(t, g(t)), \quad g(0) = -h_0, \quad t > 0, \\
    u(0, x) &= u_0(x), \quad -h_0 \leq x \leq h_0,
\end{align*}
\]

where \(d, a, b, r\) are positive constants, \(x = g(t)\) and \(x = h(t)\) are moving boundaries that will be determined together with \(u\); the parameter \(\mu > 0\) can be understood as the expanding ability, i.e., the larger \(\mu\) is, then the easier the population can move to a new area; \(bu\) measures the competition for local space while \((\phi * u)(t, x) = \int_{\mathbb{R}} \phi(x - y)u(t, y)dy\) denotes the competition for resources in the neighborhood of an individual. The initial function \(u_0\) satisfies

\[
u_0(x) \in C^2([-h_0, h_0]), \quad u_0(-h_0) = u_0(h_0) = 0 \quad \text{and} \quad u_0(x) > 0 \quad \text{for} \quad x \in (-h_0, h_0)
\]

and the convolution kernel function \(\phi\) satisfies

\[
\phi \in C(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \phi \geq 0, \quad \phi(x) = \phi(-x) \quad \text{and} \quad \int_{\mathbb{R}} \phi(x)dx = 1. \tag{K}
\]

Problem (3)-(4) indicates that the individuals occupy the initial region \([-h_0, h_0]\) at the beginning and spread into the environment from two ends of the initial region. The one-phase Stefan conditions \(g'(t) = -\mu u_x(t, g(t))\) and \(h'(t) = -\mu u_x(t, h(t))\) show that the speed of the spreading frontiers are proportional to the population gradient at the front.

A kernel of the form \(\phi_{\sigma}(x) = \frac{1}{\sigma} \phi(\frac{x}{\sigma})\) can be used to quantify the nonlocal interaction and if \(\sigma \rightarrow 0\), then \(\phi_{\sigma} * u \rightarrow u\), and problem (3) reduces to the classical logistic model described in Du et al. [9]. We are mainly interested in comparing the behavior of our nonlocal model (3) with the corresponding local one (i.e., \(\phi(x) = \delta(x)\)). Caused by the nonlocal nature of the nonlinear term, the lack of an order-preserving property is the main difficulty in our analysis. Due to this, many key techniques including the general comparison principle for handling similar problems collapse here, while many of our results such as the long time behaviors of solutions and the estimates of the spreading speed rely on comparison principles. We introduce some comparison principles over suitable parabolic regions to overcome those difficulties, and get the expected attractivity of the positive steady state by the comparison principle (Lemma 3.1) and the method of successive improvement of lower and upper solutions.

The organization of this paper is as follows. In Section 2, we prove the general existence and uniqueness result, which implies in particular that problem (3)-(4) has a unique positive solution that is defined for all \(t > 0\), the method is inspired by [9, 7, 11]. In Section 3, we establish the spreading-vanishing dichotomy, that is, the asymptotic behavior of solutions \((u, g, h)\). Section 4 is concerned with the estimate of the spreading speed.
2. Global positive solutions of (3)-(4). We firstly prove the following local existence and uniqueness result by the contraction mapping theorem and then use a priori estimates to show that the solution is defined for all $t > 0$. The proof can be done by modifying the arguments of Du et al. [9]. For readers’ convenience, we give a detailed proof here.

**Theorem 2.1.** Assume that $(K)$ holds. Then for any given $u_0$ satisfying (4), there is a $T > 0$ such that problem (3) admits a unique positive solution

\[(u, g, h) \in C^{1+\frac{1}{1+\alpha}}(\Omega(T)) \times C^{1+\frac{2}{1+\alpha}}([0, T]) \times C^{1+\frac{2}{1+\alpha}}([0, T])\]

satisfying

\[\|u\|_{C^{1+\frac{1}{1+\alpha}}(\Omega(T))} + \|g\|_{C^{1+\frac{2}{1+\alpha}}([0, T])} + \|h\|_{C^{1+\frac{2}{1+\alpha}}([0, T])} \leq C,\]

where $\alpha \in (0, 1)$ and $\Omega(T) = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], x \in [g(t), h(t)]\}, C$ and $T$ depend on $h_0, \alpha$ and $\|u_0\|_{C^2([-h_0, h_0])}$.

**Proof.** As in [34], we first straighten the free boundaries by the transformation

\[(t, x) \rightarrow (t, s) \text{ with } s = \frac{2h_0x}{h(t) - g(t)} - \frac{h_0(h(t) + g(t))}{h(t) - g(t)},\]

which changes $x = h(t)$ and $s = g(t)$ to fixed lines $s = h_0$ and $s = -h_0$ respectively, and there holds

\[\begin{align*}
\frac{\partial s}{\partial x} &= \frac{2h_0}{h(t) - g(t)} := \sqrt{B(g(t), h(t))}, \\
\frac{\partial^2 s}{\partial x^2} &= 0, \\
\frac{\partial s}{\partial t} &= -s \frac{h'(t) - g'(t)}{h(t) - g(t)} - h_0 \frac{h'(t) + g'(t)}{h(t) - g(t)} := -A(g(t), h(t), s).
\end{align*}\]

If we set $A = A(g(t), h(t), s), B = B(g(t), h(t))$ and $u(t, x) = u(t, \frac{h(t) - g(t)}{2h_0}s + \frac{h(t) + g(t)}{2}) = v(t, s)$, then

\[\begin{align*}
u_t &= v_s \frac{\partial s}{\partial t} + v = v_t - Av_s, \\
u_s &= v_s \frac{\partial s}{\partial x} = \sqrt{B}v_s, \\
u_{ss} &= v_s \left(\frac{\partial s}{\partial x}\right)^2 + v_r \frac{\partial^2 s}{\partial x^2} = Bv_{ss},
\end{align*}\]

and the free boundary problem (3) becomes the following

\[\begin{align*}
v_t - Av_s - dBv_{ss} &= v \left[a - bv - r \int_\mathbb{R} \phi(y)v(t, s - \varphi(t)y) \, dy\right], & -h_0 < s < h_0, \\
v(t, s) &= 0, & s \not\in (-h_0, h_0), \\
h'(t) &= -\mu \sqrt{B}v_s(t, h_0), & h(0) = h_0, & -h_0 < s < h_0, \\
g'(t) &= -\mu \sqrt{B}v_s(t, -h_0), & g(0) = -h_0, & -h_0 < s < h_0, \\
v(0, s) &= u_0(s), & -h_0 \leq s \leq h_0
\end{align*}\]

for $0 < t < T$, where $g(t) = 2h_0/(h(t) - g(t))$. Denote $h_1 = -\mu v_0'(h_0), g_1 = -\mu v_0'(-h_0)$, and for $0 < T < \frac{h_0}{8(1+|h_1|+|g_1|)}$, define

\[H_T = \{h \in C^1([0, T]), h(0) = h_0, h'(0) = h_1, \|h' - h_1\|_{C([0, T])} \leq 1\},\]

\[G_T = \{g \in C^1([0, T]), g(0) = -h_0, g'(0) = g_1, \|g' - g_1\|_{C([0, T])} \leq 1\}.\]
Clearly, $H_T$ and $G_T$ are bounded and closed convex sets of $C^1([0, T])$. For any $h(t) \in H_T$ and $g(t) \in G_T$,
\[
|h(t) - h(0)| \leq \left| \frac{h(t) - h_0}{t} \right| T \leq T \|h'\|_{L^\infty} < \frac{h_0}{8(1 + |h_1| + |g_1|)} \leq \frac{h_0}{8},
\]
\[
|g(t) - g(0)| \leq \left| \frac{g(t) + h_0}{t} \right| T \leq T \|g'\|_{L^\infty} < \frac{h_0}{8(1 + |h_1| + |g_1|)} \leq \frac{h_0}{8},
\]
which implies that $\frac{7}{8} h_0 \leq h(t) - g(t) \leq \frac{9}{8} h_0$ for all $t \in [0, T]$. Then the transformation from $(t, x)$ to $(t, s)$ is well defined. Define
\[
V_T = \left\{ v \in C(\Delta_T) \mid v(y, 0) = v_0(y), \|v - v_0\|_{L^\infty(\Delta_T)} = \|v - v_0\|_{C(\Delta_T)} \leq 1 \right\}
\]
with $\Delta_T = [0, T] \times [-h_0, h_0]$. It is easily seen that $D := V_T \times G_T \times H_T$ is a complete metric space with the measure
\[
d((v_1, g_1, h_1), (v_2, g_2, h_2)) = \|v_1 - v_2\|_{C(\Delta_T)} + \|g_1' - g_2'\|_{C([0, T])} + \|h_1' - h_2'\|_{C([0, T])}.
\]

Furthermore, for any $h_1, h_2 \in H_T$ and $g_1, g_2 \in G_T$, $h_1(0) = h_2(0) = h_0$, $g_1(0) = g_2(0) = -h_0$, and
\[
|h_1 - h_2|_{C([0, T])} \leq T|h_1' - h_2'|_{C([0, T])}, \|g_1 - g_2\|_{C([0, T])} \leq T|g_1' - g_2'|_{C([0, T])}. \tag{7}
\]
Moreover, for any $v \in V_T$ and $p > 1$, there holds
\[
\left\| v \left[ a - bv - r \int_R \phi(y)v(t, s - g(t)y) dy \right] \right\|_{L^p} \leq a\|v\|_{L^p} + b\|v^2\|_{L^p} + r\|v\|_{L^\infty}^2 \cdot \|\phi\|_{L^1} < \infty, \tag{8}
\]
where $\tilde{c}_1$, $\tilde{c}_2$ and $\tilde{c}_3$ are some positive constants. Applying standard $L^p$ theory and then the Sobolev embedding theorem [19], we can find that for any $v \in V_T$, the following initial boundary value problem
\[
\begin{aligned}
\tilde{v}_t - A\tilde{v}_s - dB\tilde{v}_{ss} &= v \left[ a - bv - r \int_R \phi(y)v(t, s - g(t)y) dy \right], \quad -h_0 < s < h_0, \\
\tilde{v}(t, 0) &= 0, \\
\tilde{v}(0, s) &= u_0(s),
\end{aligned}
\tag{9}
\]
for $0 < t < T$ admits a unique solution $\tilde{v} \in W^{1,2}_p(\Delta_T) \hookrightarrow C^{1+\alpha, 1+\alpha}(\Delta_T)$ with $p > \frac{3}{1-\alpha}$, and
\[
\left\| \tilde{v} \right\|_{W^{1,2}_p(\Delta_T)}, \left\| \tilde{v} \right\|_{C^{1+\alpha, 1+\alpha}(\Delta_T)} \leq C_1 = C_1(h_0, \alpha, \|u_0\|_{C^2([-h_0, h_0])}). \tag{10}
\]

Let $(\tilde{h}, \tilde{g})$ be the unique solution of
\[
\begin{aligned}
h_t(t) &= -\mu \sqrt{B}\tilde{v}_s(t, h_0), \quad h(0) = h_0, \quad 0 \leq t \leq T, \\
g_t(t) &= -\mu \sqrt{B}\tilde{v}_s(t, -h_0), \quad g(0) = -h_0, \quad 0 \leq t \leq T.
\end{aligned}
\tag{11}
\]

Then $\tilde{h}', \tilde{g}' \in C^2([0, T])$ and $\left\| \tilde{h}' \right\|_{C^2([0, T])}, \left\| \tilde{g}' \right\|_{C^2([0, T])} \leq \mu C_1 := C_2. \tag{12}$

In what follows, we define a map $F : D \rightarrow C(\Delta_T) \times C^1([0, T]) \times C^1([0, T])$ by $F(v, g, h) = (\tilde{v}, \tilde{g}, \tilde{h})$. 

It is clear that \((v, g, h) \in \mathcal{D}\) is a fixed point of \(\mathcal{F}\) if and only if it solves (6). According to (10) and (11), we see that

\[
\|\hat{h}' - h_1\|_{C([0, T])} \leq \|\hat{h}'\|_{C^\alpha([0, T])} T^{\frac{\alpha}{2}} \leq C_2 T^{\frac{\alpha}{2}},
\]

\[
\|\hat{g}' - g_1\|_{C([0, T])} \leq \|\hat{g}'\|_{C^\alpha([0, T])} T^{\frac{\alpha}{2}} \leq C_2 T^{\frac{\alpha}{2}},
\]

and

\[
\|\hat{v} - v_0\|_{C(\Delta_T)} \leq \|\hat{v} - v_0\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} T^{\frac{\alpha}{2}} \leq C_1 T^{\frac{\alpha}{2}}.
\]

Hence, \(\mathcal{F}\) maps \(\mathcal{D}\) into itself provided that \(T < \min \left\{ \frac{\alpha}{C_2}, \frac{\alpha}{C_1} \right\} \).

Now, we are in the position to prove that \(\mathcal{F}\) is a contraction mapping on \(\mathcal{D}\), and hence it admits a unique fixed point in \(\mathcal{D}\). For \(0 < T \ll 1\), let \((v_i, g_i, h_i) \in \mathcal{D}, \ i = 1, 2,\) and denote \(\mathcal{F}(v_i, g_i, h_i) = (\hat{v}_i, \hat{g}_i, \hat{h}_i)\), then it follows from (10) and (12) that

\[
\|\hat{v}_1\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} \leq C_1, \quad \|\hat{g}_1\|_{C^\alpha([0, T])} \leq C_2 \quad \text{and} \quad \|\hat{h}_1\|_{C^\alpha([0, T])} \leq C_2.
\]

Setting \(\hat{V} = \hat{v}_1 - \hat{v}_2\), we then find that for \(0 < t < T\) and \(-h_0 < s < h_0\), \(\hat{V}(t, s)\) satisfies

\[
\hat{V}_t - A_2 \hat{V}_s - d B_2 \hat{V}_{ss} = (A_1 - A_2) \hat{v}_{1,s} + d (B_1 - B_2) \hat{v}_{1,ss}
\]

\[
+ v_1 \left[ a - bv_1 - r \int_{R^3} \phi(y) v_1(t, s - g_1(t)y) dy \right]
\]

\[
- v_2 \left[ a - bv_2 - r \int_{R^3} \phi(y) v_2(t, s - g_2(t)y) dy \right]
\]

and

\[
\hat{V}(t, -h_0) = V(t, h_0) = 0 \quad \text{and} \quad \hat{V}(0, s) = 0 \quad \text{in} \ [-h_0, h_0],
\]

in which \(g_1(t) = \frac{2h_0}{h_1(t) - g_1(t)}, \ g_2(t) = \frac{2h_0}{h_2(t) - g_2(t)}, \ A_1 = A(g_1(t), h_1(t), s), \ A_2 = A(g_2(t), h_2(t), s), \ B_1 = B(g_1(t), h_1(t), s) \) and \(B_2 = B(g_2(t), h_2(t), s)\). As in (8), we have

\[
\left\| v_1 \left[ a - bv_1 - r \int_{R^3} \phi(y) v_1(t, s - g_1(t)y) dy \right] \right\|_{L^p(\Delta_T)} < \infty,
\]

\[
\left\| v_2 \left[ a - bv_2 - r \int_{R^3} \phi(y) v_2(t, s - g_2(t)y) dy \right] \right\|_{L^p(\Delta_T)} < \infty.
\]

Using \(L^p\) estimates for parabolic equations and Sobolev embedding theorem again, we have

\[
\left\| \hat{V} \right\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} \leq C_3 \left( \|v_1 - v_2\|_{C(\Delta_T)} + \|g_1 - g_2\|_{C^\alpha([0, T])} + \|h_1 - h_2\|_{C^\alpha([0, T])} \right),
\]

where \(C_3\) is a constant which depends on \(C_1, C_2\) and \(C^*\), where \(C^*\) denotes the uniform bound of the coefficients \(A_1, A_2, B_1\) and \(B_2\). Moreover, \(C_3\) depends on the elliptic constant \(d(\frac{\alpha}{2})^2\). From (11), we find that

\[
\|\hat{h}_1' - \hat{h}_2'\|_{C^\alpha([0, T])}, \quad \|\hat{g}_1' - \hat{g}_2'\|_{C^\alpha([0, T])} \leq \mu C_4 \|\hat{v}_{1,s} - \hat{v}_{2,s}\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)}
\]

Combining (7), (13) and (14), we obtain that

\[
\left\| \hat{v}_1 - \hat{v}_2 \right\|_{C^{\frac{\alpha}{2}, \alpha}(\Delta_T)} + \left\| \hat{g}_1' - \hat{g}_2' \right\|_{C^\alpha([0, T])} + \left\| \hat{h}_1' - \hat{h}_2' \right\|_{C^\alpha([0, T])}
\]

\[
\leq C_5 \left( \|v_1 - v_2\|_{C(\Delta_T)} + \|g_1' - g_2'\|_{C([0, T])} + \|h_1' - h_2'\|_{C([0, T])} \right),
\]
where $C_5 = C_5(\mu, C_3)$. Furthermore, there is
\[
\|\tilde{v}_1 - \tilde{v}_2\|_{C(\Delta_T)} + \|g'_1 - g'_2\|_{C([0,T])} + \|h'_1 - h'_2\|_{C([0,T])} \leq T^{1+\alpha} \|\tilde{v}_1 - \tilde{v}_2\|_{C^{1+\alpha}(\Delta_T)} + T^{\frac{\alpha}{2}} \|g'_1 - g'_2\|_{C^{\frac{\alpha}{2}}([0,T])} + T^{\frac{\alpha}{2}} \|h'_1 - h'_2\|_{C^{\frac{\alpha}{2}}([0,T])}
\]
\[
\leq T^{\frac{\alpha}{2}} C_5 \left( \|v_1 - v_2\|_{C(\Delta_T)} + \|g'_1 - g'_2\|_{C([0,T])} + \|h'_1 - h'_2\|_{C([0,T])} \right)
\]
\[
\leq \frac{1}{2} \left( \|v_1 - v_2\|_{C(\Delta_T)} + \|g'_1 - g'_2\|_{C([0,T])} + \|h'_1 - h'_2\|_{C([0,T])} \right)
\]
provided that
\[
T < \min \left\{ 1, \frac{h_0}{8(1 + |h_1| + |g_1|)}, C_2 \frac{\alpha}{a}, C_1 \frac{1}{\alpha}, (2C_5)^{-\frac{\alpha}{2}} \right\}.
\]
It follows that $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$ for our choice of $T$. Now, $\mathcal{F}$ admits a unique fixed point $(v, g, h)$ in $\mathcal{D}$. Moreover, by the Schauder’s estimate, we get additional regularity for $(u, g, h)$ as a solution of (6), namely, $h(t)$, $g(t) \in C^{1+\frac{\alpha}{2}}([0,T])$ and $v \in C^{1+\frac{\alpha}{2}+\alpha}(\Delta_T)$. In other words, $(v, g, h)$ is the unique classical solution of problem (6). This completes the proof. \[\Box\]

To show that the local solution obtained in Theorem 2.1 can be extended to all $t > 0$, we need the following estimate.

**Theorem 2.2.** Assume that (K) holds and let $(u, g, h)$ be the solution of problem (3) for $t \in (0, T_0)$ with some $T_0 \in (0, +\infty)$. Then, we have
\[
0 < u(t, x) \leq M_1, \quad 0 < h'(t), -g'(t) \leq M_2 \quad \text{for} \quad t \in (0, T_0) \quad \text{and} \quad x \in (g(t), h(t)),
\]
in which $M_1$ and $M_2$ are positive constants independent of $T_0$.

**Proof.** For any $(t, x) \in (0, T_0) \times [g(t), h(t)]$, it follows from the comparison principle that $u(t, x) \leq \bar{u}(t)$, where
\[
\bar{u}(t) = \frac{a}{b} e^{\alpha t} \left( e^{\alpha t} - 1 + \frac{a}{b} \|u_0\|_{L^\infty} \right)^{-1},
\]
which is a solution of
\[
\frac{d\bar{u}(t)}{dt} = \bar{u}(a - b\bar{u}) \quad \text{for} \quad t > 0 \quad \text{and} \quad \bar{u}(0) = \|u_0\|_{L^\infty}.
\]
Therefore, for all $(t, x) \in (0, T_0) \times [g(t), h(t)]$, there holds
\[
u(t, x) \leq \sup_{t \geq 0} \bar{u}(t) = \max \left\{ \frac{a}{b}, \|u_0\|_{L^\infty} \right\} := M_1.
\]

The strong maximum principle and the Hopf lemma imply that
\[
u(t, x) > 0, \quad u_x(t, h(t)), -u_x(t, g(t)) < 0 \quad \text{for} \quad t \in (0, T_0) \quad \text{and} \quad x \in (g(t), h(t)).
\]
Thus, we have $h'(t), -g'(t) > 0$ for $t \in (0, T_0)$ by the Stefan conditions.

It remains to show that $h'(t), -g'(t) \leq M_2$ for all $t \in (0, T_0)$ with some constant $M_2 > 0$ independent of $T_0$. As in Du et al. [9], we define
\[
\Omega^h_M := \{(t, x) \in \mathbb{R}^2 : 0 < t < T_0, \quad h(t) - M^{-1} < x < h(t)\},
\]
\[
\bar{u}(t, x) = M_1 \left[ 2M(h(t) - x) - M^2(h(t) - x)^2 \right],
\]
in which $M > 0$ is a constant that will be chosen later. For any $(t, x) \in \Omega^h_M$, we have
\[
u_t = 2M M_1 [1 - M(h(t) - x)] h'(t) \geq 0, \quad -\Delta \bar{u} = 2M^2 M_1,
\]
\[
u(a - b) \leq a M_1 \quad \text{and} \quad \nu_t - d\Delta \bar{u} \geq 2dM^2 M_1 \geq a M_1 \quad \text{if} \quad M \geq \sqrt{\frac{a}{2d}}.
\]
Furthermore, there is \( \bar{u}(t, h(t) - M^{-1}) = M_1 \geq u(t, h(t) - M^{-1}) \) and \( \bar{u}(t, h(t)) = u(t, h(t)) = 0 \). On the other hand, for any \( x \in [h_0 - M^{-1}, h_0] \),
\[
\bar{u}(0, x) = M_1 \left[ 2M(h_0 - x) - M^2(h_0 - x)^2 \right] \geq MM_1(h_0 - x),
\]
\[
u_0(x) = -\int_{x}^{h_0} \bar{u}'(s)\,ds \leq (h_0 - x)\|\nu_0\|_{C([-h_0, h_0])}
\]
and \( \bar{u}(x, 0) \geq u_0(x) \) in \([h_0 - M^{-1}, h_0]\) if \( MM_1 \geq \|u_0\|_{C([-h_0, h_0])} \). Hence, if we choose
\[
M = \max \left\{ \frac{\|u_0\|_{C([-h_0, h_0])}}{M_1}, \sqrt{\frac{a}{2d}}, \frac{1}{2h_0} \right\},
\]
then by the maximum principle there holds \( \bar{u}(t, x) \geq u(t, x) \) for \((t, x) \in \Omega_{M}^{\epsilon} \). Therefore, \( u_x(t, h(t)) \geq \bar{u}_x(t, h(t)) = -2MM_1 \). And hence,
\[
h'(t) = -\mu u_x(t, h(t)) \leq -\mu \bar{u}_x(t, h(t)) = 2\mu MM_1 := M_2.
\]

In the region \( \Omega_{M_2}^{\epsilon} := \{(t, x) \in \mathbb{R}^2 : 0 < t < T_0, \; g(t) < x < g(t) + M^{-1}\} \), we compare \( \bar{u}(t, x) \) and \( u(t, x) \) as above and get \(-g'(t) \leq M_2 \). This completes the proof. \( \square \)

**Theorem 2.3.** Assume that \((K)\) holds, then the solution of problem (3) exists and is unique for all \( t \in (0, \infty) \).

*Proof.* It follows from the uniqueness of the solutions of (3) that there is some \( T_{\text{max}} > 0 \) such that \([0, T_{\text{max}})\) is the maximal time interval where the solution exists. It remains to show \( T_{\text{max}} = \infty \). We get the conclusion by deriving a contradiction. Suppose that \( T_{\text{max}} < \infty \), then as in Theorem 2.2, there exist positive constants \( M_1 \) and \( M_2 \) independent of \( T_{\text{max}} \) such that
\[
0 < u(t, x) \leq M_1 \quad \text{for} \quad t \in [0, T_{\text{max}}) \quad \text{and} \quad x \in [g(t), h(t)],
\]
\[
0 < h'(t), \quad -g'(t) \leq M_2 \quad \text{for} \quad t \in [0, T_{\text{max}}).
\]

According to the standard \( L^p \) estimates, the Sobolev embedding theorem and the Hölder estimates for parabolic equations, for some fixed \( \gamma^* \in (0, T_{\text{max}}) \), we can find a positive constant \( M_3 \) depending on \( \gamma^* \), \( M_1 \) and \( M_2 \) such that \( \|u(t, \cdot)\|_{C^{2\gamma^*}([g(t), h(t)])} \leq M_3 \) for all \( t \in [\gamma^*, T_{\text{max}}) \). Then, there is a \( \tau > 0 \) depending on \( M_i \) \((i = 1, 2, 3)\) such that the solution of problem (3) with initial time \( T_{\text{max}} - \frac{\tau}{2} \) can be extended uniquely to \( T_{\text{max}} - \frac{\tau}{2} + \tau \), which contradicts to the definition of \( T_{\text{max}} \). Thus, our result follows. \( \square \)

Theorem 2.2 establishes the monotonicity of the free boundaries \( g(t) \) and \( h(t) \). The following result follows as a similar result in [11, 34], which implies that \( g(t) \) and \( h(t) \) are both finite or infinite at the same time. We omit the proof for brevity.

**Theorem 2.4.** Suppose that \((K)\) holds and let \((u, g, h)\) be the solution of problem (3), then \(-2h_0 < g(t) + h(t) < 2h_0 \) for all \( t > 0 \).

As a concluding remark, we discuss the relations and differences among our free boundary problem (3), the corresponding Cauchy problem and the initial-boundary value problem with fixed boundary.

Recalling the initial-boundary value problem (2) stated in section 1, Redlinger [23] (see also [25]) has showed that the positive uniform equilibrium \( \frac{a}{p+q} \) is globally asymptotically stable in the case that \( \phi \ast u \) is a temporal convolution and \( b > r \).
Moreover, it is concluded in Deng et al. [6] that the positive solution \( u(t, x) \) of the following Cauchy problem

\[
\begin{cases}
  u_t = \Delta u + au - bu^2 - ru \int_{\mathbb{R}^N} \phi(x-y)u(y, t)dy, & \text{in } (0, \infty) \times \mathbb{R}^N, \\
  u(0, x) = u_0(x), & \text{on } \mathbb{R}^N
\end{cases}
\]  

(15)

converges to \( \frac{a}{b+r} \) uniformly in \( x \) provided that \( u_0(x) > 0 \) and \( b > r \). Indeed, once \( b > r \), any nontrivial initial population \( u_0(x) \) will spread successfully regardless of its initial size, supporting area and expanding ability. For our free boundary problem (3), we show that

(i): if \( h_0 \geq h^* \), then spreading happens, that is, \( h_\infty = -g_\infty = \infty \);
(ii): if \( h_0 < h^* \), then we can find a \( \mu_* > 0 \) such that \( h_\infty = -g_\infty = \infty \) if \( \mu \geq \mu_* \);
(iii): if \( h_0 < h^* \), then there exists \( \mu^* = \mu^*(u_0(x)) > 0 \) such that \( h_\infty - g_\infty < \infty \) if \( \mu \leq \mu^* \).

Our results show that for the initial supporting area \( h_0 \), there exists a sharp criteria \( h^* \), for \( h_0 \geq h^* \), the new species successful spreading in the long run. While for the case that \( h_0 < h^* \), whether spreading happens or not depending on the expanding ability \( \mu \) even though the initial function \( u_0 \) is nontrivial.

Moreover, if we chose \( \phi(x) = \delta(x) \), then problem (3) becomes the classical logistic model with double free boundary, which has been studied by Du et al. [9], and our results correspond to those in [9].

3. Long time behaviors of \( (u, g, h) \). In this section, we mainly analyze the asymptotic behavior of positive solution \( (u, g, h) \) of problem (3) obtained in section 2, and our conclusions here are mainly based on the comparison principles that are set over some suitable parabolic regions. In what follows, we discuss the comparison principle for problem (3), please refer to Deng et al. [6] for similar results while in \( \mathbb{R}^N \). The proof of Lemma 3.1 is motivated by Lemma 2.6 in [10].

**Lemma 3.1.** (Comparison Principle) Assume that \( T \in (0, \infty) \) and denote \( D_T^+ = \{(t, x) : t \in [0, T], x \in (\gamma(t), h(t)) \} \), \( D_T^- = \{(t, x) : t \in [0, T], x \in (g(t), h(t)) \} \). Let \( \gamma, \ h \in C^1([0, T]) \) and \( u \in C^{1,2}(D_T^+) \cap C(\overline{D_T^+}) \) and \( \overline{\gamma} \in C^1([0, T]) \) and \( \overline{h} \in C^1([0, T]) \) and \( \overline{u} \in C^{1,2}(D_T^-) \cap C(\overline{D_T^-}) \) satisfy

\[
\begin{cases}
  \overline{u}_t - d\Delta \overline{u} \geq \overline{\gamma}(a - b\overline{u} - r\overline{\phi} \ast \overline{u}), & 0 < t \leq T, \ \overline{\gamma}(t) \times \overline{h}(t), \\
  \overline{u}_t - d\Delta \overline{u} \leq \overline{u}(a - b\overline{u} - r\overline{\phi} \ast \overline{u}), & 0 < t \leq T, \ \overline{\gamma}(t) \times \overline{h}(t), \\
  \overline{\gamma}(t) = \overline{\gamma}(t, \overline{h}(t)) = 0, & 0 < t \leq T, \\
  \overline{u}(t, \overline{h}(t)) = \overline{u}(t, \overline{h}(t)) = 0, & 0 < t \leq T, \\
  \overline{\gamma}(t) \\overline{\gamma}(t) - \mu \overline{\gamma}(t, \overline{h}(t)), \ \overline{h}(t) \leq -\mu \overline{u}(t, \overline{h}(t)), & 0 < t \leq T, \\
  \overline{g}(t) \leq -\mu \overline{u}(t, \overline{g}(t)), \ \overline{g}(t) \geq -\mu \overline{u}(t, \overline{g}(t)), & 0 < t \leq T, \\
  \overline{\gamma}(0, x) \geq \overline{u}_0(x) \geq \overline{\gamma}(0, x), & x \in [-h_0, h_0] \\
  \end{cases}
\]

(16)

with \( \overline{h}(0) \leq h_0 \leq \overline{h}_0 \) and \( g(0) \geq -h_0 \geq \overline{g}(0) \) for all \( 0 < t \leq T \). Then the unique solution \( (u, g, h) \) of problem (3) satisfies \( \overline{h}(t) \leq h(t) \leq \overline{h}(t), \ g(t) \geq g(t) \geq \overline{g}(t) \) for \( t \in (0, T) \) and \( \overline{u}(t, x) \geq \overline{u}(t, x) \geq \overline{u}(t, x) \) for \( (t, x) \in (0, T] \times \mathbb{R} \).
Proof. Let $M^* > 0$ be the upper bound of $u, \bar{u}$ and $\underline{u}$ in $[0, T] \times (-\infty, +\infty)$, denote $v = M^* - u$ and $\overline{v} = M^* - \underline{u}$, then $(\overline{v}, \underline{v})$ satisfies

$$
\begin{align*}
\bar{v}_t - d\Delta \bar{v} &\geq \overline{\Pi} [a - b \bar{u} - r \phi * (M^* - \overline{\Pi})], & 0 < t \leq T, & x \in (\gamma(t), \overline{h}(t)), \\
\overline{v}_t - d\Delta \overline{v} &\geq -(M^* - \overline{\Pi}) [a - b(M^* - \overline{\Pi}) - r(\phi * \overline{\Pi})], & 0 < t \leq T, & x \in (\gamma(t), \overline{h}(t)), \\
\overline{v}(t, \gamma(t)) &= \overline{v}(t, \overline{h}(t)) = 0, & 0 < t \leq T, \\
\overline{v}(t, \gamma(t)) &= \overline{v}(t, \overline{h}(t)) = M^*, & 0 < t \leq T, \\
\overline{h}'(t) &\geq -\mu \overline{\Pi}_x(t, \overline{h}(t)), \quad \overline{h}'(t) = \mu \overline{\Pi}_x(t, \overline{h}(t)), & 0 < t \leq T \\
\overline{v}'(t) &\leq -\mu \overline{\Pi}_x(t, \gamma(t)), \quad \overline{v}'(t) \geq \mu \overline{\Pi}_x(t, \gamma(t)), & 0 < t \leq T.
\end{align*}
$$

(17)

with $\overline{h}(0) \leq h_0 \leq \overline{h}_0$, $g(0) \geq -h_0 \geq \overline{\Pi}(0)$ for all $0 < t \leq T$, and $\overline{\Pi}(0, x) \geq u_0(x) \geq M^* - \overline{\Pi}(0, x)$ for $x \in [-h_0, h_0]$.

Firstly, we assume $\overline{h}(0) < h_0 < \overline{h}(0)$ and $g(0) > -h_0 > \overline{\Pi}(0)$, and claim that $h(t) < \overline{h}(t)$ and $g(t) > \overline{\Pi}(t)$ (resp. $h(t) > \overline{h}(t)$ and $g(t) < \overline{\Pi}(t)$) for all $t \in (0, T]$. This is clear for small $t > 0$. If our claim does not hold, then we can find a first $t^* \in (0, T]$ such that $h(t) < \overline{h}(t)$, $g(t) > \overline{\Pi}(t)$ for all $t \in (0, t^*)$, and either $h(t^*) = \overline{h}(t^*)$ or $g(t^*) = \overline{\Pi}(t^*)$ holds (resp. $h(t^*) > \overline{h}(t^*)$ and $g(t^*) < \overline{\Pi}(t^*)$).

We first suppose that $h(t^*) = \overline{h}(t^*)$ (resp. $h(t^*) > \overline{h}(t^*)$) holds. It follows that $\mu^2(t^*)$ (resp. $\mu^2(t^*)$) holds. Letting $U = (\overline{\Pi} - \overline{v})e^{-Kt}$ and $V = (\bar{u} - \overline{v})e^{-Kt}$, then we get

$$
\begin{align*}
U_t - d\Delta U + (K - c_1(t, x))U &\geq r\overline{\Pi}(\phi * V), & 0 < t \leq t^*, & x \in (\gamma(t), h(t)), \\
V_t - d\Delta V + (K - c_2(t, x))V &\geq ru(\phi * U), & 0 < t \leq t^*, & x \in (\gamma(t), h(t)), \\
U(t, g(t)) &\geq 0, \quad U(t, h(t)) \geq 0, & 0 < t \leq t^*, \\
V(t, g(t)) = V(t, h(t)) = 0, & 0 < t \leq t^*, \\
U(0, x) &\geq 0, \quad V(0, x) \geq 0, & x \in \mathbb{R},
\end{align*}
$$

where $c_1(t, x) = a - b(\overline{\Pi} + u) - r(\phi * u)$, $c_2(t, x) = a + b(\overline{\Pi} + v) - 2bM^* - r(\phi * \overline{\Pi})$ and $K > 0$ is sufficiently large such that

$$
K \geq 1 + \max \left\{ \max_{(t, x) \in [0, t^*] \times \mathbb{R}} \frac{c_1(t, x)}{x^2}, \max_{(t, x) \in [0, t^*] \times \mathbb{R}} \frac{c_2(t, x)}{x^2} \right\}.
$$

We now show that $U \geq 0$ and $V \geq 0$ in $[0, t^*] \times \mathbb{R}$. Denoting $D_t^* = [0, t^*] \times \mathbb{R}$, $\tau = \min \{\min_{D_t^*} U, \min_{D_t^*} V \}$ and suppose to the contrary that $\tau < 0$. Then there exists $(t_1, x_1) \in D_t^*$ such that $U(t_1, x_1) = \tau < 0$, or there exists $(t_2, x_2) \in D_t^*$ such that $V(t_2, x_2) = \tau < 0$. In order to deduce some contradictions, we introduce the following two auxiliary functions (see Deng et al. [6])

$$
\begin{align*}
w_1 &= \frac{U}{1 + \gamma t + x^2} \quad \text{and} \quad w_2 = \frac{V}{1 + \gamma t + x^2},
\end{align*}
$$

where $\gamma > 0$ is a constant to be determined. It is obvious that $w_1, w_2 \to 0$ as $|x| \to \infty$ and

$$
\begin{align*}
\theta(t, x) [w_1]_t - d\Delta w_1 + (K - c_1)w_1 - 4dx(w_1)_x + (\gamma - 2d)w_1 &\geq r\overline{\Pi}(\phi * V), \\
\theta(t, x) [w_2]_t - d\Delta w_2 + (K - c_2)w_2 - 4dx(w_2)_x + (\gamma - 2d)w_2 &\geq ru(\phi * U), \\
w_1(0, x) &\geq 0, \quad w_2(0, x) \geq 0
\end{align*}
$$

(19)
with $\theta(t, x) = 1 + \gamma t + x^2$, $c_1 = c_1(t, x)$ and $c_2 = c_2(t, x)$ for all $(t, x) \in [0, t^*] \times \mathbb{R}$.

Letting $T^* \leq t^*$ and assume that there exists $\{t_1, x_1\} \in \mathcal{D}_{T^*}$ such that $U(t_1, x_1) = \tau < 0$, then the negative minimum of $w_1$ denoted by $w_{1, \text{min}}$ satisfies $w_{1, \text{min}} = \min_{(t, x) \in \mathcal{D}_{T^*}} \min_{x \in \mathcal{D}_{t^*}} \frac{U(t, x)}{1 + c_1(t, x) + c_2(t, x)} \leq \frac{-\gamma t}{x^2} < 0$. In addition, if we assume that $w_1$ attains its minimum at some point $(\tilde{t}, \tilde{x}) \in \mathcal{D}_{T^*}$, then $w_1|_{(\tilde{t}, \tilde{x})} \leq 0$, $\Delta w_1|_{(\tilde{t}, \tilde{x})} \geq 0$ and $(w_1)'|_{(\tilde{t}, \tilde{x})} = 0$. Therefore, by the first inequality of (19), there holds

$$(\gamma - 2d)w_{1, \text{min}} \geq r \pi(\tilde{t}, \tilde{x})(\phi \star V)|_{(\tilde{t}, \tilde{x})}.$$ 

Due to the inequality satisfied by $\pi$ in (16), it follows from the maximum principle over $\{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, g(t) \leq x \leq h(t)\}$ that $\pi \geq 0$. Hence, there is $(\gamma - 2d)w_{1, \text{min}} \geq r \pi(\tilde{t}, \tilde{x})\tau$, which deduces that

$$\gamma - 2d \leq r \pi(\tilde{t}, \tilde{x})\tau w_{1, \text{min}} \leq r M^*\tau w_{1, \text{min}} \leq r M^*(1 + \gamma T^* + x_1^2).$$

Then, if we choose $T^* = \min \{t^*, \frac{1}{r M^*}\}$ and $\gamma$ sufficiently large such that

$$(1 - r M^* T^*)\gamma > r M^*(1 + x_1^2) + 2d,$$

a contradiction occurs. If there exists $(t_2, x_2) \in \mathcal{D}_{T^*}$ such that $V(t_2, x_2) = \tau < 0$, we obtain a similar contradiction.

For the case that $g(t^*) = \tilde{g}(t^*)$ (resp. $g(t^*) = g(t^*)$), we can deduce a similar contradiction by the same arguments as above. Therefore, $U \geq 0$ and $V \geq 0$ in $\mathcal{D}_{T^*}$, which further implies that $\pi \geq u$ and $\pi \geq v$ in $\mathcal{D}_{T^*}$.

We now compare $\pi$, $u$ and $v$ over $\Omega_{T^*} = \{(t, x) : 0 < t \leq t^*, g(t) < x < h(t)\}$. Let $W_1 = \pi - u$ and $W_2 = u - v$, then $W_1$ and $W_2$ satisfy following

$$\begin{cases} (W_1)_t - d\Delta W_1 \geq c_1(t, x)W_1 + r \pi(\phi \ast W_2) & \text{in } \Omega_{T^*}, \\ (W_2)_t - d\Delta W_2 \geq c_2(t, x)W_2 + ru(\phi \ast W_1) & \text{in } \Omega_{T^*}, \\ W_1(0, x) \geq 0, W_2(0, x) \geq 0 & \text{in } [-h_0, h_0], \end{cases}$$

Where $c_1(t, x)$ and $c_2(t, x)$ are the same as in (18). The strong maximum principle yields that $W_1 > 0$ and $W_2 > 0$ in $\Omega_{T^*}$. Further, we find that $W_1(t^*, h(t^*)) = W_1(t^*, g(t^*)) = W_2(t^*, h(t^*)) = W_2(t^*, g(t^*)) = 0$. Then, it follows from the Hopf boundary lemma that $(W_1)_x(t^*, h(t^*)) < 0$, $(W_1)_x(t^*, g(t^*)) > 0$, $\frac{(W_2)_x(t^*, h(t^*))}{x_1^2} < 0$ and $(W_2)_x(t^*, g(t^*)) > 0$, which indicate that $h'(t^*) < \tilde{h}'(t^*)$ and $g'(t^*) > \tilde{g}'(t^*)$ (resp. $h'(t^*) > \tilde{h}'(t^*)$ and $g'(t^*) < \tilde{g}'(t^*)$), contradictions occur. Therefore, $h(t) < \tilde{h}(t)$ and $g(t) > \tilde{g}(t)$ (resp. $h(t) > \tilde{h}(t)$ and $g(t) < \tilde{g}(t)$) for all $t \in (0, T)$. We can apply the usual comparison principle over $\Omega_T$ to obtain that $\pi > u$ and $v > u$ in $\Omega_T$.

For the case that $h(0) = h_0$ (resp. $h(0) = \tilde{h}(0)$ and $g(0) = -h_0$ (resp. $g(0) = -\tilde{h}(0)$), choose $\epsilon > 0$ is small, set $h_0' = h_0(1 - \epsilon)$ (resp. $h_0' = \tilde{h}_0(1 + \epsilon)$, $-h_0' = -h_0(1 + \epsilon)$ (resp. $-h_0' = -\tilde{h}_0(1 - \epsilon)$), and let $(u_\epsilon, g_\epsilon, h_\epsilon)$ denote the unique solution of (3) with $h_0$ and $-h_0$ replaced by $h_0'$ and $-h_0'$, respectively. Since the unique solution of (3) depends continuously on the parameters in (3), as $\epsilon \to 0$, $(u_\epsilon, g_\epsilon, h_\epsilon)$ converges to $(u, g, h)$, which is the unique solution of (3). Then the desired results are obtained by letting $\epsilon \to 0$. \hfill \square

The following comparison principle is an immediately result of Lemma 3.1.
Lemma 3.2. Assume that $T \in (0, \infty)$. Let $\overline{\pi} \in C^{1,2}(\overline{D}_T) \cap C(\overline{D}_T)$ and $\pi \in C^{1,2}(D_T) \cap C(\overline{D}_T)$ with $D_T = \{(t, x) \in \mathbb{R}^2 : t \in [0, T], x \in (g(t), h(t))\}$ satisfy

$$
\begin{cases}
\pi_t - d\Delta \pi \geq \pi(a - b\pi - r\phi * \pi), & 0 < t \leq T, \quad g(t) < x < h(t), \\
\pi_t - d\Delta \pi \leq \pi(a - b\pi - r\phi * \pi), & 0 < t \leq T, \quad g(t) < x < h(t), \\
\pi(t, g(t)) \geq 0, \quad \pi(t, h(t)) \geq 0, & 0 < t \leq T, \\
\pi(0, x) \geq u_0(x) \geq \pi(0, x), & x \in [-h_0, h_0].
\end{cases}
$$

(21)

Then the unique solution $(u, g, h)$ of problem (3) satisfies $\pi(t, x) \geq u(t, x) \geq \pi(t, x)$ for $(t, x) \in (0, T) \times [g(t), h(t)]$.

Below are two necessary lemmas that will be used later.

Lemma 3.3. (\cite{12, Lemma 2.2}) Let $\Omega$ be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary. Suppose that $\alpha$ and $\beta$ are smooth positive functions on $\Omega$, and let $\lambda_1$ denote the first eigenvalue of $-\Delta u = \lambda \alpha(x) u$ on $\Omega$ under Dirichlet boundary conditions on $\partial \Omega$. Then the problem

$$
-Lu = \lambda u[\alpha(x) - \beta(x) u^{p-1}] \quad \text{and} \quad u|_{\partial \Omega} = 0 \quad (p > 1)
$$

(22)

with $Lu = \sum_{ij} (a_{ij}(x) u_x)_x$ smooth in $\mathbb{R}^N$, $a_{ij} = a_{ji}$ and

$$
\sigma_1 |\xi|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq \sigma_2 |\xi|^2
$$

for some positive constants $\sigma_1$, $\sigma_2$ and all $\xi \in \mathbb{R}^N$ has a unique positive solution for every $\lambda > \lambda_1$, and the unique positive solution $u_\lambda$ satisfies $u_\lambda(x) \rightarrow [\alpha(x)/\beta(x)]^{1/(p-1)}$ uniformly on any compact subset of $\Omega$ as $\lambda \rightarrow \infty$.

Lemma 3.4. (\cite{2, Corollary 3.4}) For any given $l > 0$ and $\Omega_0 \subset \mathbb{R}$ with $|\Omega_0| = 1$, the positive solutions of following reaction-diffusion model

$$
\begin{cases}
\begin{aligned}
u_t &= d\Delta \nu + g(\nu), & \text{in} \quad \Omega \times (0, \infty) = \Omega_0 \times (0, \infty), \\
u &= 0, & \text{on} \quad \partial \Omega \times (0, \infty)
\end{aligned}
\end{cases}
$$

(23)

with $g(\nu)$ is Lipschitz and decreasing in $u$, $g(0) > 0$, $g(u) < 0$ for $u > K > 0$ (i) approach 0 in $C(\Omega)$ as $t \rightarrow \infty$ if $l \leq \frac{d}{2} \sqrt{\frac{d}{g(0)}}$; (ii) approach $u^*$ in $C(\Omega)$ as $t \rightarrow \infty$ if $l > \frac{d}{2} \sqrt{\frac{d}{g(0)}}$, where $u^*(x)$ depending on $l$ is the unique positive stationary solution of (23).

Theorem 3.5. Let $(u, g, h)$ be the solution of problem (3). If $h_\infty - g_\infty < \infty$, then $h'(t), -g'(t) \rightarrow 0$ and $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. As in the proof of Theorem 2.1, our free boundary problem can be transformed into a fixed boundary one and we see that $v(t, s) := u(t, x)$ has a $C^{1,\alpha}$ bound for $s \in [-h_0, h_0]$. Hence, by the Stefan conditions in (3) and the same conclusion as Theorem 2.1 in [32], we see that there exists a positive constant $c^* = c^*(\mu, h_\infty, g_\infty, M_2)$ (M_2 is defined in Theorem 2.2) such that

$$
\|g\|_{C^{\frac{\alpha}{2}}([n+1, n+3])} + \|h\|_{C^{\frac{\alpha}{2}}([n+1, n+3])} \leq c^* \quad \text{for any integer} \quad n \geq 0.
$$

(24)

Since intervals $[n + 1, n + 3]$ overlap and constant $c^*$ is independent of $n$, it follows that

$$
\|g\|_{C^{\frac{\alpha}{2}}([1, \infty))} + \|h\|_{C^{\frac{\alpha}{2}}([1, \infty))} \leq c^* \quad \text{and} \quad \|v(t, \cdot)\|_{C^{\alpha}([g(t), h(t)])} \leq c^* \quad \text{for} \quad t \geq 1.
$$

(25)
Then we arrive at \( g'(t) \to 0, h'(t) \to 0 \) as \( t \to \infty \) since \( g(t) \) and \( h(t) \) are bounded.

Next, we prove that \( \lim_{t \to +\infty} \| u(t, \cdot) \|_{C([g(t), h(t)])} = 0 \). Assume for the contrary that \( \limsup_{t \to +\infty} \| u(t, \cdot) \|_{C([g(t), h(t)])} = \varepsilon > 0 \). Then there exists a sequence \( (t_k, x_k) \) in \((0, +\infty) \times (g(t), h(t))\) such that \( u(t_k, x_k) \geq \frac{\varepsilon}{2} \) for all \( k \in \mathbb{N} \) and \( t_k \to \infty \) as \( k \to \infty \). Since \(-\infty < g_\infty < g(t) < x_k < h(t) < h_\infty < \infty\), we then have that a subsequence of \( \{x_k\} \) converges to \( x_0 \in (g_\infty, h_\infty) \). Without loss of generality, we assume \( x_k \to x_0 \) as \( k \to \infty \). Define

\[
u_k(t, x) = u(t_k + t, x) \quad \text{for} \quad (t, x) \in (-t_k, \infty) \times (g(t_k + t), h(t_k + t)).
\]

It follows from the parabolic regularity theory that \( \{u_k\} \) has a subsequence \( \{u_{k_i}\} \) such that \( u_{k_i} \to \tilde{u} \) as \( i \to \infty \), where \( \tilde{u} \) satisfies

\[
\begin{align*}
\begin{cases}
\tilde{u}_t &= d\Delta \tilde{u} + \tilde{u}(a - b \tilde{u} - r\phi * \tilde{u}), \quad (t, x) \in \mathbb{R} \times (g_\infty, h_\infty), \\
\tilde{u}(t, g_\infty) &= \tilde{u}(t, h_\infty) = 0, \quad t \in \mathbb{R}.
\end{cases}
\end{align*}
\]

Note that \( \tilde{u}(0, x_0) \geq \frac{\varepsilon}{2} \), therefore \( \tilde{u} > 0 \) in \( \mathbb{R} \times (g_\infty, h_\infty) \). It is clear that \( \| a - b \tilde{u} - r\phi * \tilde{u} \|_{L_\infty} := M < \infty \), then by using the Hopf lemma to \( \tilde{u}_t - d\Delta \tilde{u} - M \tilde{u} \) at \((0, h_\infty)\) and \((0, g_\infty)\) yields that \( \tilde{u}_x(0, h_\infty) \leq -\sigma_0 < 0 \) and \( \tilde{u}_x(0, g_\infty) \geq \sigma_0 > 0 \) respectively for some \( \sigma_0 > 0 \).

Since \( h'(t) \to 0, g'(t) \to 0 \) as \( t \to \infty \), then by Stefan conditions we have

\[
\begin{align*}
u_x(t_k, h(t_k)) &= 0 \quad \text{and} \quad \nu_x(t_k, g(t_k)) \to 0 \quad \text{as} \quad t_k \to \infty.
\end{align*}
\]

However, from \( \| u(t, \cdot) \|_{C([g(t), h(t)])} \leq c^* \) we have \( \nu_x(t_k, h(t_k)) = (u_{k_i})_x(0, h(t_k)) \to \tilde{u}_x(0, h_\infty) \) and \( \nu_x(t_k, g(t_k)) = (u_{k_i})_x(0, g(t_k)) \to \tilde{u}_x(0, g_\infty) \) as \( k \to \infty \), which contradicts to \( \tilde{u}_x(0, h_\infty) < 0 \) and \( \tilde{u}_x(0, g_\infty) > 0 \). And hence if \( h_\infty - g_\infty < \infty \), there holds \( \lim_{t \to +\infty} \| u(t, \cdot) \|_{C([g(t), h(t)])} = 0 \).

\[\square\]

**Theorem 3.6.** Assume that \( b > r \). Let \((u, g, h)\) be the solution of problem (3) with \( h_\infty - g_\infty \geq 2h^* := \pi \sqrt{\frac{d}{a(1 - \frac{r}{b})}} \), then \( h_\infty = -g_\infty = \infty \).

**Proof.** For contradiction, we assume that \( 2h^* \leq h_\infty - g_\infty < \infty \), then there exists some \( T_1 > 0 \) such that \( l_1 = \frac{h(T_1) - g(T_1)}{2} > h^* \). Let \( \lambda_1^{l_1} \) denote the first eigenvalue of

\[
d\omega'' + \lambda\omega = 0 \quad \text{in} \quad (-l_1, l_1) \quad \text{and} \quad \omega(\pm l_1) = 0.
\]

It follows that \( \lambda_1^{l_1} < a(1 - \frac{\pi}{b}) \). Moreover, for all small \( \eta > 0 \), problem

\[
d\omega'' + dr\omega' + \lambda\omega = 0 \quad \text{in} \quad (-l_1, l_1) \quad \text{and} \quad \omega(\pm l_1) = 0,
\]

admits a solution pair \((\lambda_1^{l_1, \eta}, \omega_1) = \left( \frac{dr^2}{4r^2} + \frac{d}{4r^2}, \frac{d}{4r^2} \cos(\frac{\pi x}{2l_1}) \right)\) with \( \lambda_1^{l_1, \eta} \leq a(1 - \frac{\pi}{b}) \).

For such a fixed \( \eta \) and some \( T_2 > T_1 \), choosing

\[
l := \frac{h(T_2) - g(T_2)}{2} \geq \frac{\pi}{2} \sqrt{\frac{d}{a - r(\frac{a}{b} + \eta)}} > l_1
\]

such that \( h'(t), -g'(t) \leq \eta \) for all \( t \geq T_2 \). It is obvious that problem (28) with \( l_1 \) replaced by \( l \) has a solution pair \((\lambda_1^{l_1, \eta}, \omega_1) = \left( \frac{4r^2}{4r^2} + \frac{d}{4r^2}, \frac{d}{4r^2} \cos(\frac{\pi x}{2l_1}) \right)\).

Moreover, it follows from the proof of Theorem 2.2 that \( \limsup_{t \to +\infty} u(t, x) \leq \frac{a}{b} \). Then there exists \( T_0 > 0 \) is large such that \( u(t, x) \leq \frac{a}{b} + \eta \) for all \((t, x) \in [T_0, +\infty) \times [g(t), h(t)]\).
Define \( \Omega_T = \{(t,x) \in \mathbb{R}^2 : t \geq T, \ 0 \leq x \leq h(t) \} \) with \( T \gg T_0 \) and 
\[
\bar{u}(t,x) = \frac{a}{b} + \eta \text{ for all } (t,x) \in \Omega_T,
\]
\[
w(t,x) = \delta \omega_1 \left( \frac{lx}{h(t)} \right) \text{ with } \delta > 0 \text{ is to be determined}
\]
For all \( (t,x) \in \Omega_T \), direct calculation shows that \( \bar{u}_t = 0 > d \Delta \bar{u} + \bar{u}(a-b\bar{u} - r\phi * \bar{u}) \), \( \bar{u}(t,h(t)) > 0 = u(t,h(t)) \) and 
\[
\bar{u}_t - d \Delta \bar{u} - \bar{u}(a-b\bar{u} - r\phi * \bar{u}) \nonumber \\
= -\delta \frac{lx}{h^2} \omega'_1 - \delta \frac{d^2}{h^2} \omega''_1 - \delta \omega_1 \left[ a - b\delta \omega_1 - r \left( \frac{a}{b} + \eta \right) \right] \\
= \frac{\nu^2}{h^2} \left[ -\frac{x}{l} \omega'_1 - d \omega''_1 \right] - \delta \omega_1 \left[ a - b\delta \omega_1 - r \left( \frac{a}{b} + \eta \right) \right].
\]
Since \( h'(t) \to 0 \) as \( t \to +\infty \), then for all \( t \geq T \), there is \( h'(t) \leq \frac{1}{h(T)} \) and hence for \( (t,x) \in [T, +\infty) \times [0, h(t)] \), we have 
\[
xh'(t) < \frac{h(t)}{T} \cdot \frac{l}{h(T)} \eta < \eta.
\]
From \( \frac{l}{h(T)} \leq 1 \) and \( \omega'_1 \leq 0 \) in \([0, l]\), we have 
\[
u_t - d \Delta \nu = \nu(a-b\nu - r\phi * \bar{u}) \nonumber \\
\leq -\delta \nu \omega'_1 + \delta \left( d \nu \omega'_1 + \lambda'_{1,\eta} \omega_1 \right) - \delta \omega_1 \left[ a - b\delta \omega_1 - r \left( \frac{a}{b} + \eta \right) \right].
\]
We are aim at obtaining that 
\[
u_t - d \Delta \nu - \nu(a-b\nu - r\phi * \bar{u}) \leq 0. \tag{29}
\]
Inequality (29) holds provided that \( \eta, \delta \) are small enough and \( \lambda'_{1,\eta} - a + r \left( \frac{a}{b} + \eta \right) \leq 0 \), which is true by \( b > r \) and the formulas of \( \lambda'_{1,\eta} \) and \( l \).
On the other hand, by Lemma 3.4 and the choice of \( l \), we can find some \( w > 0 \) such that 
\[
dw'' + w \left[ a - r \left( \frac{a}{b} + \eta \right) - bw \right] = 0 \text{ in } (-l,l) \text{ and } w(\pm l) = 0.
\]
Choosing \( \rho > 0 \) small and \( T_3 > T_0 \) large, we may get \( u(t,x) \geq \rho w \) for all \( t \geq T_3 \) and \( x \in [-l,l] \). Thus, we can obtain that \( u(t,0) \geq \Theta := \rho w(0) \) for all \( t \geq T_3 \). In addition, we have \( u(t,h(t)) = \nu(t,h(t)) = 0 \).
Now we choose \( 0 < \delta < 1 \) and \( T > \max \{ T_3, T_2 \} \) such that \( u(t,0) \geq \Theta \geq \nu(t,0) \) for all \( t \geq T \), and \( u(T,x) \geq \nu(T,x) \) for all \( x \in [0, h(T)] \).
It follows from Lemma 3.1 that \( u(t,x) \leq \nu(t,x) \) over \( \Omega_T \). In addition, \( u(t,h(t)) = \nu(t,h(t)) \) yields that 
\[
u_x(t,h(t)) \geq u_x(t,h(t)) \text{ for all } t > T.
\]
Then, by taking \( t \to +\infty \), we arrive at \(-\frac{1}{h}h'(\infty) = 0 \leq \frac{\nu}{h} \omega'_1(l) < 0 \), a contradiction. Hence, we conclude that \( h_\infty = -g_\infty = \infty \text{ if } h_\infty - g_\infty > 2h^* \).

It follows from Theorems 3.6 and 3.5 that if \( h_\infty - g_\infty < \infty \), then \( h_\infty - g_\infty < 2h^* \) and \( \lim_{t \to +\infty} \| u(t,\cdot) \|_{C([g(t),h(t)])} = 0 \), that is vanishing happens. Following theorem establishes the spreading or persistence of the species in the case that \( h_\infty = -g_\infty = \infty \).

**Theorem 3.7.** Let \((u,g,h)\) be the solution of problem (3) with \( h_\infty = -g_\infty = \infty \). Then \( \lim_{t \to +\infty} u(t,x) = \frac{u}{e^{\theta t}} \) uniformly in any bounded subset of \( \mathbb{R} \).
Proof. It follows from Theorem 2.2 that there exists a large $T_0$ such that $u(t, x) \leq \frac{q}{2} + \bar{\epsilon}_1$ for all $t \geq T_0$ and $x \in [g(t), h(t)]$, where $\bar{\epsilon}_1 > 0$ is small. We denote $p_1 = \frac{q}{2}$ for brevity. Fix $l_{p_1}$ such that $l_{p_1} > \max \{ h_0, \frac{d}{a-r(p_1+\bar{\epsilon}_1)} \}$, then we can find some $t_{l_{p_1}}$ such that $l_{p_1} = \frac{h(t_{l_{p_1}}) - g(t_{l_{p_1}})}{2}$, since $h_\infty = -g_\infty = \infty$.

Letting $T_{l_{p_1}} = \max(T_0, t_{l_{p_1}})$, by the comparison principle, we get $p_1 + \bar{\epsilon}_1 \geq u(t, x) \geq u^{l_{p_1}}(t, x)$ in $[T_{l_{p_1}}, \infty) \times [-l_{p_1}, l_{p_1}]$, where $u^{l_{p_1}}$ is the positive solution of

$$
\begin{cases}
(u^{l_{p_1}})_t = d(u^{l_{p_1}})_{xx} + u^{l_{p_1}} \left[ a - r (p_1 + \bar{\epsilon}_1) - bu^{l_{p_1}} \right], & t > T_{l_{p_1}}, -l_{p_1} < x < l_{p_1}, \\
(u^{l_{p_1}})(t, -l_{p_1}) = u^{l_{p_1}}(t, l_{p_1}) = 0, & t > T_{l_{p_1}}, \\
(u^{l_{p_1}})(T_{l_{p_1}}, x) = u(T_{l_{p_1}}, x), & -l_{p_1} \leq x \leq l_{p_1}.
\end{cases}
$$

Note that we have used $p_1 + \bar{\epsilon}_1$ and $u^{l_{p_1}}(t, x)$ in $[T_{l_{p_1}}, \infty) \times [-l_{p_1}, l_{p_1}]$ as a pair of upper and lower solutions to (3). Since $l_{p_1} > \frac{q}{2} \sqrt{\frac{d}{a-r(p_1+\bar{\epsilon}_1)}}$, it follows from Lemma 3.4 that $u^{l_{p_1}}(t, x) \rightarrow u^{l_{q_1}}(x)$ in $C([-l_{p_1}, l_{p_1}])$ as $t \rightarrow +\infty$ uniformly in $x \in [-l_{p_1}, l_{p_1}]$, where $u^{l_{q_1}}$ is the unique positive solution of

$$
\begin{cases}
du_{xx} + u \left[ a - r (q_1 + \epsilon_1) - bu \right] = 0, & -l_{q_1} < x < l_{q_1}, \\
u(-l_{q_1}) = u(l_{q_1}) = 0
\end{cases}
$$

and $u^{l_{q_1}}(x) \rightarrow p_1 - \frac{q_1}{2} (p_1 + \epsilon_1)$ as $l_{p_1} \rightarrow \infty$ by Lemma 3.3. Thus, $\liminf_{t \rightarrow +\infty} u(t, x) \geq p_1 - \frac{q_1}{2} (p_1 + \epsilon_1)$. Letting $\epsilon_1 \rightarrow 0^+$, it follows that $\liminf_{t \rightarrow +\infty} u(t, x) \geq p_1 - \frac{q_1}{2} p_1$.

Denoting $q_1 = p_1 - \frac{q_1}{2} p_1$, it is easy to see that $q_1 < \frac{\sqrt{d}}{a-r} < p_1$. Again, we can find a large $T_{q_1}$ and a small $\xi_1 > 0$ such that $u(t, x) \geq q_1 - \xi_1$ for all $t \geq T_{q_1}$ and $x \in [-l_{q_1}, l_{q_1}]$, where $l_{q_1}$ is the one such that $l_{q_1} \geq \frac{\sqrt{d}}{a-r(q_1-\xi_1)}$. In addition, the comparison principle yields that $q_1 - \xi_1 \leq u(t, x) \leq \bar{\pi}^{q_1}(t, x)$ in $[T_{q_1}, \infty) \times [-l_{q_1}, l_{q_1}]$, in which $\bar{\pi}^{q_1} > 0$ solves

$$
\begin{cases}
(\bar{\pi}^{q_1})_t = d(\bar{\pi}^{q_1})_{xx} + \bar{\pi}^{q_1} \left[ a - r (q_1 - \xi_1) - b\bar{\pi}^{q_1} \right], & t > T_{q_1}, -l_{q_1} < x < l_{q_1}, \\
\bar{\pi}^{q_1}(t, -l_{q_1}) = \bar{\pi}^{q_1}(t, l_{q_1}) = 0, & t > T_{q_1}, \\
\bar{\pi}^{q_1}(T_{q_1}, x) = u(T_{q_1}, x), & l_{q_1} \leq x \leq l_{q_1}
\end{cases}
$$

and $\bar{\pi}^{q_1}(t, x) \rightarrow u^{l_{q_1}}(x)$ in $C([-l_{q_1}, l_{q_1}])$ as $t \rightarrow +\infty$ uniformly in $x \in [-l_{q_1}, l_{q_1}]$ by Lemma 3.4. It is clear that $u^{l_{q_1}}(x)$ is the unique positive solution to

$$
\begin{cases}
du_{xx} + u \left[ a - r (q_1 - \xi_1) - bu \right] = 0, & -l_{q_1} < x < l_{q_1}, \\
u(-l_{q_1}) = u(l_{q_1}) = 0
\end{cases}
$$

and $u^{l_{q_1}}(x) \rightarrow p_1 - \frac{q_1}{2} (q_1 - \xi_1)$ as $l_{q_1} \rightarrow \infty$. Here, $q_1 - \xi_1$ and $\bar{\pi}^{q_1}(t, x)$ in $[T_{q_1}, \infty) \times [-l_{q_1}, l_{q_1}]$ are a pair of lower and upper solutions to (3). And hence, by letting $\xi_1 \rightarrow 0^+$ immediately deduce that $\limsup_{t \rightarrow +\infty} u(t, x) \leq p_1 - \frac{q_1}{2} q_1$.

By putting $p_2 = p_1 - \frac{q_2}{2} q_1$, we then find that $\frac{q_1}{2^2} < p_2 < p_1$. Continuing above procedure, we can similarly get that $\liminf_{t \rightarrow +\infty} u(t, x) \geq p_1 - \frac{q_1}{2} p_2$. Letting $q_2 = p_1 - \frac{q_2}{2} p_2$, then $0 < q_1 < q_2 < \frac{a}{b_\infty}$.

Note that above procedure leads to a larger lower bound and a smaller upper bound for $u(t, x)$, respectively. This approach is a similarly use of the method of
successive improvement of lower and upper solutions. To arrive at our conclusion, define sequences \( \{q_n\}_{n=2}^{\infty} \) and \( \{p_n\}_{n=2}^{\infty} \) by following
\[
q_n = p_1 - \frac{r}{b} p_n \quad \text{and} \quad p_n = p_1 - \frac{r}{b} q_{n-1}.
\]
It follows from the comparison principle that \( q_n - \xi_n \leq u(t,x) \leq \bar{\pi} q_n \) in \([T_{q_n}, \infty) \times [-q_n, l_{q_n}]\), where \( \bar{\pi} q_n > 0 \) satisfying following
\[
\begin{cases}
(\bar{\pi} q_n)_t = d(\bar{\pi} q_n)_{xx} + \bar{\pi} q_n [a - r (q_n - \xi_n) - b \bar{\pi} q_n], & t > T_{q_n}, \quad -q_n < x < q_n, \\
\bar{\pi} q_n (t, -q_n) = \bar{\pi} q_n (t, q_n) = 0, & t > T_{q_n}, \\
\bar{\pi} q_n (T_{q_n}, x) = u(T_{q_n}, x), & -q_n \leq x \leq q_n
\end{cases}
\]
and \( \bar{\pi} q_n (t,x) \to \bar{\pi} q_n (x) \in \mathcal{C}([-q_n, l_{q_n}]) \) as \( t \to +\infty \) uniformly in \( x \in [-q_n, l_{q_n}] \).
Also, \( \bar{\pi} q_n (x) \) is the unique positive solution to
\[
\begin{cases}
du_{xx} + u [a - r (q_n - \xi_n) - b u] = 0, & -q_n < x < q_n, \\
u(-q_n) = u(q_n) = 0
\end{cases}
\]
and \( \bar{\pi} q_n (x) \to p_1 - \frac{r}{b} (q_n - \xi_n) \) as \( q_n \to \infty \).
It is obvious that \( 0 < q_1 < q_2 < \cdots < q_n < \frac{a}{b + r} < p_n < \cdots < p_2 < p_1 \), then there exist \( q^* > 0 \) and \( p^* > 0 \) such that \( q_n \to q^* \) and \( p_n \to p^* \) as \( n \to \infty \), respectively. If we can prove that \( p^* = q^* = \frac{a}{b + r} \), then by letting \( n \to \infty \) in \( q_n \leq \liminf_{t \to -\infty} u(t,x) \leq \limsup_{t \to -\infty} u(t,x) \leq p_n \) directly leads to \( \lim_{t \to -\infty} u(t,x) = \frac{a}{b + r} \).
In deed, \( q^* \) and \( p^* \) satisfying following equations
\[
q^* = p_1 - \frac{r}{b} p^* \quad \text{and} \quad p^* = p_1 - \frac{r}{b} q^*.
\]
which indicates that \( p^* = q^* = \frac{a}{b + r} \). Then \( \lim_{t \to -\infty} u(t,x) = \frac{a}{b + r} \) follows. \( \square \)

Now we can deduce the following spreading-vanishing dichotomy.

**Theorem 3.8.** Let \( (u,g,h) \) be the solution of problem (3). Then the following alternative holds: Either

(i): spreading: \( h_\infty = -g_\infty = \infty \) and \( \lim_{t \to +\infty} u(t,x) = \frac{a}{b + r} \) uniformly in any bounded subset of \( \mathbb{R} \); or

(ii): vanishing: \( h_\infty = -g_\infty < 2h^* \) and \( \lim_{t \to +\infty} \|u(t,x)\|_{\mathcal{C}[\{g(t), h(t)\}]} = 0 \).

Theorem 3.6 and the strictly monotonicity of \( h'(t) \) and \( g'(t) \) show that \( h_\infty = -g_\infty = \infty \) if \( h_0 \geq h^* \). For \( h_0 < h^* \), we give two lemmas to illustrate the dependence of the spreading or vanishing of the species on the expanding ability \( \mu \).

**Lemma 3.9.** Assume that \( h_0 < h^* \), then there exists \( \mu_* > 0 \) depending on \( u_0(x) \) such that spreading happens, that is \( h_\infty = -g_\infty = \infty \) if \( \mu > \mu_* \).

**Proof.** We argue indirectly. To obtain the formula of \( \mu_* \), supposing on the contrary that vanishing happens for all \( \mu > 0 \) if \( h_0 \geq h^* \). Then the positive solution \( (u(t,x), h(t), g(t)) \) of (3) satisfies \( h_\infty = -g_\infty < \infty \) for all \( \mu > 0 \).

It follows from Theorem 3.5 that \( \lim_{t \to +\infty} \|u(t,\cdot)\|_{\mathcal{C}[\{g(t), h(t)\}]} = 0 \). In addition, there is \( h_\infty - g_\infty \leq 2h^* \) by Theorem 3.6. And hence, we can find some \( \bar{t} > 0 \) such that \( u(t,x) \leq \frac{a}{b + r} \) for all \( t \geq \bar{t} \) and \( x \in \{g(t), h(t)\} \), here \( \bar{t} \) is independent of \( \mu \).
The idea below mainly based on the proof of Lemma 3.7 in Du et al. [9]. For all \( t \geq \tilde{t} \) and \( x \in [g(t), h(t)] \), direct calculation yields that
\[
\frac{d}{dt} \int_{g(t)}^{h(t)} u(t, x) \, dx = \int_{g(t)}^{h(t)} u_t(t, x) \, dx + u(t, h(t))h'(t) - u(t, g(t))g'(t)
\]
\[
= \int_{g(t)}^{h(t)} d\Delta u \, dx + \int_{g(t)}^{h(t)} u(a - bu - r\phi \ast u) \, dx
\]
\[
\geq -\frac{d}{\mu} h'(t) + \frac{d}{\mu} g'(t).
\]

Integrating from \( \tilde{t} \) to \( t \) deduces that
\[
\int_{g(t)}^{h(t)} u(t, x) \, dx - \int_{g(\tilde{t})}^{h(\tilde{t})} u(\tilde{t}, x) \, dx \geq \frac{d}{\mu} \left( h(t) - h(\tilde{t}) \right) + \frac{d}{\mu} \left( g(t) - g(\tilde{t}) \right).
\]

Then by letting \( t \to \infty \) immediately leads to
\[
- \int_{g(\tilde{t})}^{h(\tilde{t})} u(\tilde{t}, x) \, dx \geq \frac{d}{\mu} \left( h_\infty - g_\infty \right) + \frac{d}{\mu} \left( h(\tilde{t}) - g(\tilde{t}) \right).
\]

Since \( h_\infty - g_\infty \leq 2h^* \) and \( h(\tilde{t}) - g(\tilde{t}) > 2h_0 \), then the above inequality implies that \( \mu < 2d(h^* - h_0) \left( \int_{-h_0}^{h_0} u(\tilde{t}, x) \, dx \right)^{-1} \). Note that \( u(\tilde{t}, x) \) here depends on \( \mu \).

We are now in a position to show the expected \( u_* \). For our choice of \( \tilde{t} \), it follows that \( u(t, x) \leq \frac{a}{b+r} + \tilde{e} \) for all \( t \geq \tilde{t} \), where \( \tilde{e} \) is chosen as \( 0 < \tilde{e} \leq a \left( \frac{1}{r} - \frac{1}{b+r} \right) \). And hence, \( (u, h, g) \) satisfies
\[
\begin{cases}
  u_t - d\Delta u \geq u \left( a - r \left( \frac{a}{b + r} + \tilde{e} \right) - bu \right), & t > \tilde{t}, \ g(t) < x < h(t), \\
  u(t, g(t)) = u(t, h(t)) = 0, & t > \tilde{t}, \\
  h'(t) = -\mu u_x(t, h(t)), & h(\tilde{t}) > h_0, \quad t > \tilde{t}, \\
  g'(t) = -\mu u_x(t, g(t)), & g(\tilde{t}) < -h_0, \quad t > \tilde{t}, \\
  u(\tilde{t}, x) > 0, & g(\tilde{t}) \leq x \leq h(\tilde{t}).
\end{cases}
\]

It follows from the comparison principle that \( u(t, x) \geq w(t, x) \) in \( [\tilde{t}, \infty) \times [-h_0, h_0] \). Particularly, there is \( u(\tilde{t}, x) \geq w(\tilde{t}, x) \) in \( [-h_0, h_0] \), where \( w(t, x) \) is the unique positive solution to the following initial-boundary value problem
\[
\begin{cases}
  w_t - d\Delta w = w \left( a - r \left( \frac{a}{b} + \tilde{e} \right) - bw \right), & t > 0, \ -h_0 < x < h_0, \\
  w(t, -h_0) = w(t, h_0) = 0, & t > 0, \\
  w(0, x) = u_0(x), & -h_0 \leq x \leq h_0.
\end{cases}
\]

It is obvious that \( w(t, x) \) is independent of \( \mu \), as well as \( w(\tilde{t}, x) \), which in turn indicates that
\[
\mu \leq 2d(h^* - h_0) \left( \int_{-h_0}^{h_0} w(\tilde{t}, x) \, dx \right)^{-1} := \mu_*.
\]

As a result, inversely, if \( \mu > \mu_* \), then \( h_\infty = -g_\infty = \infty \). Then the conclusion follows.
Following, we will give a positive number $\mu^*$ such that $h_\infty - g_\infty < \infty$ (i.e. $u \to 0$) if $h_0 < h^*$ and $\mu \leq \mu^*$. In order to illustrate this, we will use the following comparison principle for the free boundary problem in a scalar equation, which is an extension of Lemma 5.1 in Guo et al. [16], see also Lemma 3.5 in Du et al. [9], we omit the proof here for brevity.

**Lemma 3.10.** Assume that $(\bar{u}, \bar{g}, \bar{h}) \in C^{1,2}(D^*) \times C^1([0, +\infty)) \times C^1([0, +\infty))$ with $D^* = \{(t, x) \in \mathbb{R}^2: t > 0, \bar{g}(t) \leq x \leq \bar{h}(t)\}$ satisfying

\[
\begin{cases}
\bar{u}_t \geq d\Delta \bar{u} + \bar{u}(a - \bar{b}u), & t > 0, \bar{g}(t) < x < \bar{h}(t), \\
\bar{u}(t, \bar{g}(t)) = \bar{u}(t, \bar{h}(t)) = 0, & t > 0, \\
\bar{h}(t) \geq -\mu \bar{u}_x(t, \bar{h}(t)), \quad \bar{g}(t) \leq -\mu \bar{u}_x(t, \bar{g}(t)), & t > 0, \bar{g}(t) < x < \bar{h}(t)
\end{cases}
\]

with $h(0), \bar{g}(0) \leq h_0$ and $\bar{u}(0, x) \geq u_0(x)$ for all $x \in [-h_0, h_0]$. Then the unique positive solution $(u, g, h)$ of (3) satisfies $u \leq \bar{u}$, $g \geq \bar{g}$ and $h \leq \bar{h}$ for all $t > 0$ and $x \in [g(t), h(t)]$.

**Lemma 3.11.** If $h_0 < h^*$. Then there exists $\mu^* > 0$ depending on $u_0(x)$ such that $h_\infty - g_\infty < \infty$ if $\mu \leq \mu^*$.

**Proof.** We are going to construct a suitable upper solution to (3) and then apply Lemma 3.10 to prove this. As the auxiliary functions constructed in Du et al. [9], we define

\[
\begin{align*}
\sigma(t) &= h_0(1 + \nu - \frac{\nu}{2}e^{-ct}), \\
\vartheta(t) &= -\sigma(t) \quad \text{for} \quad t \geq 0, \\
V(y) &= \cos\left(\frac{\pi}{2}y\right) \quad \text{for} \quad -1 \leq y \leq 1, \\
\bar{u}(t, x) &= \hat{M}e^{-ct}V\left(\frac{x}{\sigma(t)}\right) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \vartheta(t) \leq x \leq \sigma(t),
\end{align*}
\]

where $\nu$, $\varsigma$ and $\hat{M}$ are positive constants to be determined later. Direct calculation shows that $\bar{u}(t, \sigma(t)) = \bar{u}(t, -\sigma(t)) = 0$, and

\[
\begin{align*}
\bar{u}_t - d\Delta \bar{u} - \bar{u}(a - \bar{b}u) &= \hat{M}e^{-ct} \left[-\varsigma V - \frac{x\sigma'}{\sigma^2}V' - \frac{dV''}{\sigma^2} - V(a - \bar{b}V)e^{-ct}V\right] \\
&\geq \hat{M}e^{-ct}V \left[-\varsigma + \frac{d\pi^2}{4}\left(\frac{1}{(1 + \nu)^2h_0^2} - a\right)\right].
\end{align*}
\]

Since $h_0 < h^* = \frac{\pi}{2}\sqrt{\frac{d}{\sigma'(1/\sigma^2)}}$, then there holds $a(1 - \frac{\varsigma}{\hat{M}}) < \frac{d\pi^2}{4}\pi^2h_0^2$ and it is not hard to find some $\nu > 0$ such that

\[
\frac{d\pi^2}{4}\left(\frac{1}{(1 + \nu)^2h_0^2} - a\right) = \frac{1}{2}\left[\frac{d\pi^2}{4}\frac{1}{h_0^2} - a(1 - \frac{\varsigma}{\hat{M}})\right] > 0.
\]

Then it is noticed that $\bar{u}_t - d\Delta \bar{u} - \bar{u}(a - \bar{b}u) \geq 0$ if $\varsigma = \frac{1}{2}\left[\frac{d\pi^2}{4}\frac{1}{h_0^2} - a(1 - \frac{\varsigma}{\hat{M}})\right]$. In addition, if we choose $\hat{M} \geq \|u_0\|_{L^\infty}(\cos\frac{\pi}{2 + \nu})^{-1}$, then there is

\[
\bar{u}(0, x) = \hat{M}V\left(\frac{x}{\sigma(0)}\right) = \hat{M} \cos\frac{\pi x}{(2 + \nu)h_0} \geq \|u_0\|_{L^\infty} \geq u_0(x).
\]
On the other hand, for $\mu \leq \mu^* := \frac{2\sqrt{\Delta}}{\pi M}$, we have

$$
s'(t) = \frac{\nu}{2} h_0 e^{-\nu t} \geq -\mu u_x(t, \sigma(t)) = \frac{\pi}{2} \mu \bar{M} e^{-\nu t} \frac{1}{h_0(1 + \nu - \frac{\nu}{2} e^{-\nu t})}.
$$

Hence, for $\nu$, $\zeta$, $\bar{M}$ and $\mu^*$ defined above, we have

$$
\begin{aligned}
\bar{u}_t &\geq d\Delta \bar{u} + \bar{u}(a - b \bar{u}), \\
\bar{u}(t, \vartheta(t)) = &\bar{u}(t, \sigma(t)) = 0, \\
\sigma'(t) &\geq -\mu \bar{u}_x(t, \sigma(t)), \quad \vartheta'(t) \leq -\mu \bar{u}_x(t, \vartheta(t)), \\
\bar{u}(0, x) &\geq u_0(x), \quad -h_0(1 + \frac{\nu}{2}) \leq x \leq h_0(1 + \frac{\nu}{2}).
\end{aligned}
$$

It follows from Lemma 3.10 that $h(t) \leq \sigma(t)$ and $g(t) \geq \vartheta(t)$ for all $t \geq 0$. Taking $t \to +\infty$ yields $h_\infty \leq g_\infty \leq h_0(1 + \nu) < \infty$. This completes the proof.

Similar to Theorem 5.11 of [9], we get the following sharp criteria $\tilde{\mu}$ that governing the spreading-vanishing dichotomy for the case that $h_0 < h^*$.

**Theorem 3.12.** If $h_0 < h^*$. Then there exists $\tilde{\mu} > 0$ depending on $u_0(x)$ such that $h_\infty - g_\infty < \infty$ if $\mu \leq \tilde{\mu}$, and $h_\infty = -g_\infty = \infty$ if $\mu > \tilde{\mu}$.

The results stated in Theorem 3.6, Lemmas 3.9 and 3.11 show that for the initial supporting area $h_0$, there exists a sharp criteria $h^*$, for $h_0 \geq h^*$, the new species spread successfully in the long run. While for the case that $h_0 < h^*$, whether spreading happens or not depending on the expanding ability $\mu$ even though the initial function $u_0$ is nontrivial, and the sharp criteria $\tilde{\mu}$ strongly depends on $u_0(x)$.

4. Spreading speed. This section is concerned with the estimate for the spreading speed, which indicates that the asymptotic spreading speed (if exists) for problem (3) cannot be faster than the minimal speed of traveling wavefront solutions to

$$
u_t = d\Delta u + u(a - b u) \quad \text{for} \quad (t, x) \in \mathbb{R}^2. \quad (33)
$$

Recalling from the pioneering works of Fisher [14] and Kolmogorov, Petrovski and Piskunov [18] that for any $|c| \geq c^* := 2\sqrt{ad}$, problem (33) has a traveling wavefront solution of the form $u = U(x - ct)$ satisfying

$$
U(-\infty) = \frac{a}{b}, \quad U(+\infty) = 0 \quad \text{and} \quad U'(x) < 0 \quad \text{for} \quad x \in \mathbb{R},
$$

while no such solution exists for $|c| < c^*$. The number $c^*$ is called the minimal speed of the traveling wavefront.

We shall use Lemma 3.10 to prove $h(t), -g(t) < \sigma(t) := \sigma_0 + c^* t$ for all $t > 0$ with $\sigma_0$ is to be determined, which further shows that $\lim \sup_{t \to +\infty} \frac{h(t)}{t} \leq c^*$ and $\lim \sup_{t \to +\infty} -\frac{g(t)}{t} \leq c^*$.

**Theorem 4.1.** Let $(u, g, h)$ be the solution of (3) with $h_\infty = -g_\infty = +\infty$, then we can obtain that $\lim \sup_{t \to +\infty} \frac{h(t)}{t}, \lim \sup_{t \to +\infty} -\frac{g(t)}{t} \leq c^* = 2\sqrt{ad}$.

**Proof.** Let $U(\xi)$ with $\xi = x - c^* t$ and $U(0) = \frac{a}{b}$ be the solution of

$$
\begin{aligned}
&c^* U'(\xi) + dU''(\xi) + U(\xi)(a - bU(\xi)) = 0, \quad \xi \in \mathbb{R}, \\
&U(-\infty) = \frac{a}{b}, \quad U(+\infty) = 0, \quad U'(\xi) < 0, \quad \xi \in \mathbb{R}.
\end{aligned}
$$


Now, we use $U(\xi)$ to construct an upper solution that satisfies (32). Define
\[
\sigma(t) = \sigma_0 + c^*t, \quad \vartheta(t) = -\sigma(t), \quad t > 0,
\]
\[
\bar{u}(t, x) = \begin{cases} 
  kU(x - c^*t) - kU(\sigma_0), & t > 0, \quad x \in [0, \sigma(t)], \\
  kU(-x - c^*t) - kU(\sigma_0), & t > 0, \quad x \in [\vartheta(t), 0],
\end{cases}
\]
where $k > 1$ is a constant such that $\|u_0\|_{L^\infty} < kU(\xi)$ for all $\xi \in [-h_0, h_0]$, and $\sigma_0 > h_0$ depending on $k, \mu, a, b$ such that
\[
c^* > -k\mu U'(\sigma_0), \quad 0 < U(\sigma_0) \leq \left(1 - \frac{1}{k}\right) \frac{\alpha}{b} \text{ and } U(\sigma_0) \leq \min_{\xi \in [-h_0, h_0]} \left\{ U(\xi) - \frac{u_0}{k} \right\}.
\]
Direct calculation shows that
\[
\bar{u}_t - d\Delta \bar{u} - \bar{u}(a - b\bar{u}) = -c^*kU'' - dkU''' - k(U - U(\sigma_0))[a - bk(U - U(\sigma_0))]
\]
\[
= k[-c^*U' - dt'' - (a - bU) + bk(1-k)U^2] - 2bk^2U\sigma_0 + akU(\sigma_0) + bk^2U^2
\]
\[
= k[b(k-1)U^2 - 2bkU(\sigma_0) + aU(\sigma_0) + bkU^2]
\]
\[
= k \left[b(k-1) \left( U - k \frac{U(\sigma_0)}{k-1} \right)^2 + U(\sigma_0) \left( a - \frac{bk}{k-1} U(\sigma_0) \right) \right].
\]
Below, we compare $(\bar{u}(t, x), \sigma(t), \vartheta(t))$ and $(u(t, x), g(t), h(t))$ over $[0, +\infty) \times [0, \sigma(t)]$ and $[0, +\infty) \times [\vartheta(t), 0]$, respectively. For $(t, x) \in [0, +\infty) \times [0, \sigma(t)]$, we have
\[
\bar{u}_t - d\Delta \bar{u} - \bar{u}(a - b\bar{u}) \geq 0, \quad \sigma'(t) = c^* > -k\mu U'(\sigma_0) = -\mu \bar{u}_x(t, \sigma(t)),
\]
\[
\bar{u}(t, \sigma(t)) = 0 \quad \text{and} \quad \bar{u}(0, x) = k[U(x) - U(\sigma_0)] \geq u_0(x).
\]
It follows from Lemma 3.10 that $h(t) \leq \sigma(t)$ and then $\limsup_{t \to +\infty} \frac{h(t)}{t} \leq c^*$.

For $(t, x) \in [0, +\infty) \times [\vartheta(t), 0]$, we have
\[
\bar{u}_t - d\Delta \bar{u} - \bar{u}(a - b\bar{u}) \geq 0, \quad \vartheta'(t) = -c^* < k\mu U'(\sigma_0) = -\mu \bar{u}_x(t, \vartheta(t)),
\]
\[
\bar{u}(t, \vartheta(t)) = 0 \quad \text{and} \quad \bar{u}(0, x) = k[U(-x) - U(\sigma_0)] \geq u_0(x).
\]
It follows from Lemma 3.10 that $g(t) \geq \vartheta(t)$ and then $\limsup_{t \to +\infty} \frac{g(t)}{t} \geq -c^*$. This completes the proof.

Now we are in a position to establish the lower bound of the spreading speed when spreading happens. To this end, we firstly summarize a result in [1] as follows.

**Lemma 4.2.** For any given constants $a > 0$, $b > 0$, $d > 0$ and $k \in [0, 2\sqrt{ad}]$, the problem
\[
- dU'' + kU' = aU - bU^2 \quad \text{in } [0, +\infty), \quad U(0) = 0 \quad (34)
\]
admits a unique positive solution $U_k = U_{a,b,d,k}$ and it satisfies $U(x) \to \frac{a}{b}$ as $x \to \infty$. Moreover, $U_k'(x) > 0$ for $x \geq 0$, $U'_k(0) > U''_k(0)$, $U_k(x) > U_k'(x)$ for $x > 0$ and $k_1 < k_2$, and for each $\mu > 0$, there exists a unique $k_0 = k_0(\mu, a, b, d) \in (0, 2\sqrt{ad})$ such that $\mu U'_{k_0}(0) = k_0$.

**Theorem 4.3.** Let $(u, g, h)$ be the solution of (3) with $h(\infty) = -g(\infty) = +\infty$, then we have $\liminf_{t \to +\infty} \frac{h(t)}{t}$, $\liminf_{t \to +\infty} -\frac{g(t)}{t} \geq k_0 = k_0(\mu, a(1 - \frac{r}{k_0^2}), b, 1)$. 

\]
Proof. Since \( h_\infty = -g_\infty = +\infty \), it follows from Theorem 3.7 that \( \lim_{t \to +\infty} u(t, x) = \frac{a}{b + r} \) uniformly in any bounded subset of \( \mathbb{R} \). For any small \( \varsigma_1, \varsigma_2 > 0 \) and some given \( l > 0 \), we consider the following problem

\[
\begin{align*}
-dV'' + (1 - \varsigma_2)k_0 V' &= V \left[ a \left( 1 - \frac{r}{b + r} - \frac{b}{a} \varsigma_1 \right) - bV \right] \text{ in } (0, l), \\
V(0) &= V(l) = 0.
\end{align*}
\]

As in lemma 4.2, we see that for all large \( l > 0 \), problem (35) admits a unique positive solution \( V_l \) satisfying \( V_l(x) < U_{(1-\varsigma_2)k_0}(x) \) for \( 0 < x < l \), where \( U_{(1-\varsigma_2)k_0}(x) \) satisfies (34) with \( a \) replaced by \( a \left( 1 - \frac{r}{b + r} - \frac{b}{a} \varsigma_1 \right) \).

In addition, \( U'_{(1-\varsigma_2)k_0}(0) > U'_{k_0}(0) \) by Lemma 4.2, and \( V_l(x) \to U_{(1-\varsigma_2)k_0}(x) \) as \( l \to \infty \) in \( C^1_{loc}\left([0, \infty)\right) \) norm. Then, we can find \( l_0 > 0 \) large enough such that \( V'_{l_0}(0) > U'_{k_0}(0) = \frac{a}{b} \) and \( V_l(0) = V_{l_0}(l_0) = 0 \). Since \( h_\infty = \infty \), we also can find some large \( T \) such that \( h(T) > l_0 \).

Inspired by [9], we introduce

\[
\begin{align*}
\eta(t) &= (1 - \varsigma_2)k_0(t - T) + l_0, \quad t \geq T, \\
\bar{u}(t, x) &= V_{l_0}(\eta(t) - x), \quad t \geq T, \quad \eta(t) - l_0 \leq x \leq \eta(t).
\end{align*}
\]

Direct calculations show that

\[
\begin{align*}
\bar{u}(t, \eta(t) - l_0) &= V_{l_0}(l_0) = 0, \\
\bar{u}(t, \eta(t)) &= V_l(0) = 0, \\
\eta'(t) &= (1 - \varsigma_2)k_0 - \mu_\infty \bar{u}(t, \eta(t)) = (1 - \varsigma_2)V_{l_0}(0) > (1 - \varsigma_2)k_0, \\
\eta(t) - l_0 &= (1 - \varsigma_2)k_0(t - T) \geq 0 > g(t), \\
\eta(T) &= l_0, \quad \bar{u}(T, x) = V_{l_0}(\eta(T) - x) = V_{l_0}(l_0 - x) \quad \text{ for } x \in [0, l_0].
\end{align*}
\]

Moreover, for \( t \geq T \) and \( \eta(t) - l_0 \leq x \leq \eta(t) \), there is

\[
\bar{u}_t - d\bar{u}_{xx} = (1 - \varsigma_2)k_0 V''_{l_0} - V''_{l_0} = \bar{u} \left[ a \left( 1 - \frac{r}{b + r} - \frac{b}{a} \varsigma_1 \right) - b\bar{u} \right].
\]

We now show that \( \bar{u}(T, \cdot) \leq u(T, \cdot) \) in \([0, l_0]\). Indeed, we have \( \lim_{t \to +\infty} u(t, x) = \frac{a}{b + r} \) uniformly in any bounded subset of \( \mathbb{R} \) by Theorem 3.7, that is, for our choice of \( T \), there holds \( u(T, \cdot) \geq \frac{a}{b + r} - \varsigma_1 \) in \([0, l_0]\). On the other hand, we have \( \bar{u}(T, x) = V_{l_0}(l_0 - x) \) and \( V_{l_0}(x) < \frac{a}{b} \left( 1 - \frac{r}{b + r} \right) - \varsigma_1 \) in \([0, l_0]\). Hence, we acquire that

\[
\bar{u}(T, \cdot) < \frac{a}{b} \left( 1 - \frac{r}{b + r} \right) - \varsigma_1 = \frac{a}{b + r} - \varsigma_1 \leq u(T, \cdot) \quad \text{in } [0, l_0].
\]

Then by Lemma 3.1, we have

\[
\eta(t) \leq h(t), \quad \eta(t) - l_0 \geq g(t) \quad \text{and} \quad \bar{u} \leq u \quad \text{for } t \geq T \quad \text{and} \quad \eta(t) - l_0 \leq x \leq \eta(t).
\]

Hence, there is

\[
\begin{align*}
\liminf_{t \to +\infty} \frac{h(t)}{t} &\geq \liminf_{t \to +\infty} \frac{\eta(t)}{t} = \liminf_{t \to +\infty} \frac{(1 - \varsigma_2)k_0(t - T) + l_0}{t} = (1 - \varsigma_2)k_0 \\
(\text{resp. } \liminf_{t \to +\infty} \frac{g(t)}{t} &\leq \liminf_{t \to +\infty} \frac{\eta(t) - l_0}{t} = \liminf_{t \to +\infty} \frac{(1 - \varsigma_2)k_0(t - T)}{t} = (1 - \varsigma_2)k_0).
\end{align*}
\]

By taking \( \varsigma_2 \to 0 \) and using the continuous dependence on parameters of \( k_0 \), we obtain that

\[
\liminf_{t \to +\infty} \frac{h(t)}{t} \left( \text{resp. } \liminf_{t \to +\infty} \frac{g(t)}{t} \right) \geq k_0 = k_0 \left( \mu, a \left( 1 - \frac{r}{b + r} \right), b, 1 \right).
\]
This completes our proof.

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REFERENCES

[1] G. Bunting, Y. Du and K. Krakowski, Spreading speed revisited: Analysis of a free boundary model, *Netw. Heterog. Media*, 7 (2012), 583–603.
[2] R. Cantrell and C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, Wiley Series in Mathematical and Computational Biology, 2003.
[3] X. Chen and A. Friedman, A free boundary problem arising in a model of wound healing, *SIAM J. Math. Anal.*, 32 (2000), 778–800.
[4] X. Chen and A. Friedman, A free boundary problem for an elliptic-hyperbolic system: An application to tumor growth, *SIAM J. Math. Anal.*, 35 (2003), 974–986.
[5] C. Corduneanu, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, London, 1973.
[6] K. Deng and Y. Wu, Global stability for a nonlocal reaction-diffusion population model, *Nonlinear Anal. Real World Appl.*, 25 (2015), 127–136.
[7] Y. Du and Z. Guo, Spreading-Vanishing dichotomy in a diffusive logistic model with a free boundary II, *J. Differential Equations*, 250 (2011), 4336–4366.
[8] Y. Du and X. Liang, Pulsating semi-waves in periodic media and spreading speed determined by a free boundary model, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32 (2015), 279–305.
[9] Y. Du and Z. Lin, Spreading-Vanishing dichotomy in the diffusive logistic model with a free boundary, *SIAM J. Math. Anal.*, 42 (2010), 377–405.
[10] Y. Du and Z. Lin, The diffusive competition model with a free boundary: Invasion of a superior or inferior competitor, *Discrete Contin. Dyn. Syst. Ser. B*, 19 (2014), 3105–3132.
[11] Y. Du and B. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, *J. Eur. Math. Soc.*, 17 (2015), 2673–2724.
[12] Y. Du and L. Ma, Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions, *J. London Math. Soc.*, 64 (2001), 107–124.
[13] Y. Du, H. Matsuzawa and M. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, *SIAM J. Math. Anal.*, 46 (2014), 375–396.
[14] R. A. Fisher, The wave of advance of advantageous, *Ann. Eugenic.*, 7 (1937), 355–369.
[15] J. Ge, K. Kim, Z. Lin and H. Zhu, A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, *J. Differential Equations*, 259 (2015), 5486–5509.
[16] J. S. Guo and C. H. Wu, On a free boundary problem for a two-species weak competition system, *J. Dynam. Differential Equations*, 24 (2012), 873–895.
[17] H. Huang and M. Wang, The reaction-diffusion system for an SIR epidemic model with a free boundary, *Discrete Contin. Dyn. Syst. Ser. B*, 20 (2015), 2039–2050.
[18] A. N. Kolmogorov, I. G. Petrovski and N. S. Piskunov, Étude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. État. Moscou Sér. Intern. A* 1 (1937) 1–26; English transl. in: P. Pelcé (Ed.), *Dynamics of Curved Fronts*, Academic Press, 1988, 105–130.
[19] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural’ceva, *Linear and Quasilinear Equations of Parabolic Type*, Academic Press, New York, London, 1968.
[20] Z. Lin, A free boundary problem for a predator-prey model, *Nonlinearity*, 20 (2007), 1883–1892.
[21] R. Miller, On Volterra’s population equation, *SIAM J. Appl. Math.*, 14 (1966), 446–452.
[22] R. Peng and X. Q. Zhao, The diffusive logistic model with a free boundary and seasonal succession, *Discrete Contin. Dyn. Syst.*, 33 (2013), 2007–2031.
[23] R. Redlinger, On Volterra’s population equation with diffusion, *SIAM J. Math. Anal.*, 16 (1985), 135–142.
[24] L. I. Rubinstein, *The Stefan Problem*, American Mathematical Society, Providence, RI, 1971.
[25] A. Schiaffino, On a diffusion Volterra equation, *Nonlinear Anal.*, 3 (1979), 595–600.
[26] A. Schiaffino and A. Tesei, Monotone methods and attractivity results for Volterra integro-partial differential equations, *Proc. Roy. Soc. Edinburgh Sect. A*, 89 (1981), 135–142.
[27] A. Tesei, Stability properties for partial Volterra integro-differential equations, *Ann. Mat. Pura Appl.*, 126 (1980), 103–115.
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[28] V. Volterra, *Lecons sur la Théorie Mathématique de la Lutte Pour la vie*, Reprint of the 1931 original. Les Grands Classiques Gauthier-Villars. Éditions Jacques Gabay, Sceaux, 1990.

[29] J. Wang and L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment, *J. Math. Anal. Appl.*, 423 (2015), 377–398.

[30] M. Wang, The diffusive logistic equation with a free boundary and sign-changing coefficient, *J. Differential Equations*, 258 (2015), 1252–1266.

[31] M. Wang and J. Zhao, Free boundary problem for a Lotka-Volterra competition system, *J. Dynam. Differential Equations*, 26 (2014), 655–672.

[32] M. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, *J. Funct. Anal.*, 270 (2016), 483–508.

[33] Y. Yamada, On a certain class of semilinear Volterra diffusion equations, *J. Math. Anal. Appl.*, 88 (1982), 433–451.

[34] P. Zhou and Z. Lin, Global existence and blowup of a nonlocal problem in space with free boundary, *J. Funct. Anal.*, 262 (2012), 3409–3429.

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