Detection of genuine tripartite entanglement in quantum network scenario

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Abstract
Experimental demonstration of entanglement needs to have a precise control of experimentalist over the system on which the measurements are performed as prescribed by an appropriate entanglement witness. To avoid such trust problem, recently, device-independent entanglement witnesses (DIEWs) for genuine tripartite entanglement have been proposed where witnesses are capable of testing genuine entanglement without precise description of Hilbert space dimension and measured operators i.e. apparatus are treated as black boxes. Here, we design a protocol for enhancing the possibility of identifying genuine tripartite entanglement in a device independent manner. We consider three mixed tripartite quantum states none of whose genuine entanglement can be detected by applying certain DIEWs, but their genuine tripartite entanglement can be detected by applying the same when distributed in some suitable entanglement swapping network.

Keywords Genuine tripartite entanglement · Entanglement swapping
1 Introduction

Entanglement is one of the most intriguing and most fundamentally non-classical phenomena in quantum physics. A bipartite quantum state without entanglement is called separable. A multipartite quantum state that is not separable with respect to any bipartition is said to be genuinely multipartite entangled [1]. This type of entanglement is important not only for research concerning the foundations of quantum theory but also in quantum information protocols and quantum tasks such as extreme spin squeezing [2], high sensitivity in some general metrology tasks [3], quantum computing using cluster states [4], measurement-based quantum computation [5] and multiparty quantum network [6–9]. Several experimental proposals have been forwarded to create genuine entanglement in a laboratory. [10–12]. However, detection of this kind of resource in an experiment turns out to be quite difficult. Experimental demonstration of genuine multipartite entanglement is generally performed with one of the two following techniques: tomography of the full quantum state [13,14], or evaluation of an entanglement witness [1]. But both of these techniques face some common drawbacks viz. requirement of precise control (by the experimentalist) over the system subjected to measurements and sensitivity of these techniques to systematic errors [15].

However, there exists a way to get around this difficulty by looking for a violation of a Bell inequality [16]. Till date, various types of Bell inequalities [17] have been designed for the purpose of detection of nonlocality of correlations where any precise control of the device by the experimentalist is not needed. However, some specific type of Bell inequalities have also been proposed to detect entanglement, more specifically genuine multipartite entanglement (GME) certified from statistical data only. To be precise, if the value of a Bell expression in multipartite scenario exceeds the value obtained due to measurements on biseparable quantum states, then the presence of genuine entanglement can be guaranteed. This technique to detect genuine multipartite entanglement in device-independent manner was first introduced in [18–21] followed by an extensive formalization by Bancal et al. [22]. In particular, they introduced the term Device-Independent Entanglement Witness (DIEW) of genuine multipartite entanglement. Later, Pal [23] and Liang et al. [24] developed other DIEWs for detecting genuine multipartite entanglement. Throughout the paper, we refer to the procedure of detecting genuine entanglement as device-independent entanglement detection (DIED) and the entanglement detected in device-independent way as device-independent entanglement (DIE).

Apart from entanglement, another remarkable feature of quantum systems is the fact that systems that have never interacted can become entangled. This process is known as entanglement swapping [25,26], in which two independent pairs of entangled particles are first created and then one particle from each pair is jointly measured. As a result, the other particles become entangled. In the present paper, we address the following questions:

**Question.** Consider some tripartite states whose genuine entanglement cannot be detected by applying some standard DIEWs [18,20,22,24], now is it possible to find some suitable entanglement swapping process, after which the genuine entanglement of swapped state can be detected by those DIEWs? We answer this question affirmatively by framing a protocol based on entanglement swapping procedure. It involves
three different tripartite quantum states. DIE of these states can not detected by the DIEWs [18,20,22,24]. When used in the multiple entanglement swapping protocol (MESP), a quantum state is probabilistically generated whose DIE is detected by those DIEWs. We show that MESP can enhance the region of detectability of DIE for certain class of tripartite quantum states. In this context, another important question is whether one can enhance detection of genuine entanglement in a semi-device-independent way (corresponding to phenomenon of quantum steering). We also answer this question affirmatively.

Interestingly, our MESP can also be used as an entanglement purification protocol. The first work in this direction was given in [27] where Bennett et al. introduced an entanglement purification protocol for mixed states. Since then, many theoretical and experimental works have been done in the field of entanglement purification [28–44]. In particular, entanglement purification was performed via entanglement swapping for both pure and mixed bipartite states in [36,42] and only for pure tripartite states in [43]. If we can extend the entanglement swapping procedure to the purification of mixed tripartite states cases, then Bell basis measurements can be used as a replacement of the original controlled-NOT (CNOT) operations in the tripartite entanglement purification process [44]. In general, for unknown mixed tripartite states, whether entanglement swapping can be used to purify the entanglement remains unknown. Here, we have shown that genuine entanglement of some mixed tripartite states can be purified by using our MESP.

The rest of this paper has been organized as follows: Sect 2 deals with some mathematical preliminaries and a brief overview of some standard DIEWs. In Sect. 3, we design the MESP followed by detailed discussion on enhancement of DIED by using the MESP in Sect. 4. In Sect. 5, we have used this MESP to enhance genuine entanglement in a semi-device-independent way. Section 6 shows that our MESP can be used to purify genuine entanglement of tripartite states. Finally, we conclude with a brief discussion regarding the importance of this work and possible further extensions in Sect. 7.

## 2 Background

### 2.1 Notion of DIEWs

Violation of Bell inequality by quantum mechanical systems always indicates the presence of entanglement. Thus, a Bell inequality can be considered as a suitable candidate for detecting the presence of entanglement in a device-independent way unlike the standard procedures like state tomography or use of entanglement witnesses where experimentalist needs to trust the experimental apparatus. This is because detection of entanglement using Bell inequality solely depends on the statistical data. To characterize genuine entanglement in a device-independent way for tripartite scenario, one of $m$ possible measurements can be performed for each of the three subsystems, yielding one of two possible outcomes. The measurement settings are denoted by $x, y, z \in \{0, 1, 2, \ldots, m - 1\}$ and their outputs by $a, b, c \in \{-1, 1\}$ for Alice, Bob and Charlie, respectively. The experiment is thus characterized by the joint probability distribution
The correlations $P(abc|xyz)$ can be categorized as biseparable if they can be reproduced through the measurements on a tripartite biseparable state $\rho_{bi}$ where

$$
\rho_{bi} = \sum_{\lambda} p_{\lambda} \rho_{\lambda}^A \otimes \rho_{\lambda}^{BC} + \sum_{\mu} p_{\mu} \rho_{\mu}^B \otimes \rho_{\mu}^{AC} + \sum_{\nu} p_{\nu} \rho_{\nu}^C \otimes \rho_{\nu}^{AB}.
$$

(1)

Here $0 \leq p_{\lambda}, p_{\mu}, p_{\nu} \leq 1$ and $\sum_{\lambda} p_{\lambda} + \sum_{\mu} p_{\mu} + \sum_{\nu} p_{\nu} = 1$. To be precise, if there exists a state of the form given by Eq. (1) in some Hilbert space $\mathcal{H}$ and some suitable local measurement operators $M_{a|x}, M_{b|y}$ and $M_{c|z}$ (without loss of generality these operators can be considered to be projection operators satisfying the restriction $M_{a|x} M_{a|x}' = \delta_{a,a'}$, $M_{a|x}$ and $\sum_{a} M_{a|x} = I$) such that:

$$
\rho_{bi} = tr[M_{a|x} \otimes M_{b|y} \otimes M_{c|z} \rho_{bi}]
$$

(2)

If the correlations are not biseparable, then the state used is surely a genuine tripartite entangled state. Such a conclusion can be drawn independent of the corresponding Hilbert space dimension. Equivalently, biseparable quantum correlations can also be decomposed as,

$$
P(abc|xyz) = \sum_{k} P_{Q}^k (ab|xy) P_{Q}^k (c|z) + \sum_{k} P_{Q}^k (ac|xz) P_{Q}^k (b|y) + \sum_{k} P_{Q}^k (bc|yz) P_{Q}^k (a|x)
$$

(3)

where $P_{Q}^k (ab|xy)$ and $P_{Q}^k (c|z)$ denote arbitrary two party and one party quantum correlations, respectively. So they are of the form: $P_{Q}^k (ab|xy) = tr[M_{a|x}^k \otimes M_{b|y}^k \rho_{AB}^k]$ and $P_{Q}^k (c|z) = tr[M_{c|z}^k \rho_{C}^k]$ for some unnormalized quantum states $\rho_{AB}^k$, $\rho_{C}^k$ and measurement operators $M_{a|x}^k, M_{b|y}^k, M_{c|z}^k$.

Let $Q_3$ denotes the set of tripartite quantum correlations and $Q_{2|1}$ denotes the set of biseparable quantum correlations. Clearly, $Q_{2|1} \subseteq Q_3$. The set $Q_{2|1}$ being convex can be characterized by linear inequalities. DIEWs of genuine tripartite entanglement correspond to those inequalities (Bell inequalities) that separate the sets of $Q_3$ and $Q_{2|1}$. Now, as $Q_{2|1}$ has infinite number of extremal points, so there exist many such DIEWs separating genuine entanglement from biseparable entanglement. In recent times, many such DIEWs are designed for detecting genuine tripartite entanglement in a device-independent way [18–24]. As already mentioned in the introduction, Bancal [22] was the first to formalize the concept of device-independent detection of entanglement introducing the term DIEW for detecting genuine multipartite entanglement. In this context, one can consider the DIEW provided by the Mermin polynomial [45] as the most simple example for detecting genuine tripartite entanglement [18].

In [20] Uffink, designed another nonlinear Bell-type inequality which has been extensively used for this purpose.

In recent times, Bancal et al. gave more efficient 3-settings Bell inequality which can be used as a DIEW to detect genuine tripartite entanglement [22].
More than 3 setting DIEWs are also provided in [23]. However, in our present topic of discussion, we restrict our search for not more than 3-settings Bell inequalities due to obvious computational complexity. More recently, another 2 setting DIEW was designed by Liang et al. [24] (see Appendix 1).

As detection of genuine nonlocality by any Bell inequality implies genuine entanglement [46,47], so it is a DIEW for detecting genuine entanglement. However, the converse is not necessarily true.

### 2.2 Genuine multipartite concurrence (CGM)

In order to facilitate the discussion of our results, we briefly describe the genuine multipartite concurrence (CGM) which is a measure of genuine tripartite entanglement. For pure $n$-partite states $|\psi\rangle$, this measure has been defined as [48]:

$$C_{GM}(|\psi\rangle) := \min_j \sqrt{2(1 - \Pi_j(|\psi\rangle))},$$

where $\Pi_j(|\psi\rangle)$ is the purity of the $j$th bipartition of $|\psi\rangle$. In [49], they derived the expression of $C_{GM}$ for $X$ states. For tripartite $X$ states,

$$C_{GM} = 2 \max_i \{0, |z_i| - w_i\}$$

with $w_i = \sum_{j \neq i} \sqrt{a_j b_j}$ where $a_j, b_j$ and $z_j$ ($j = 1, 2, 3, 4$) are the elements of the density matrix of tripartite $X$ state:

$$
\begin{bmatrix}
    a_1 & 0 & 0 & 0 & 0 & 0 & z_1 \\
    0 & a_2 & 0 & 0 & 0 & 0 & z_2 \\
    0 & 0 & a_3 & 0 & 0 & z_3 & 0 \\
    0 & 0 & 0 & a_4 & z_4 & 0 & 0 \\
    0 & 0 & 0 & z_4^* & b_4 & 0 & 0 \\
    0 & 0 & z_3^* & 0 & b_3 & 0 & 0 \\
    z_2^* & 0 & 0 & 0 & 0 & b_2 & 0 \\
    z_1^* & 0 & 0 & 0 & 0 & 0 & b_1
\end{bmatrix}
$$

After discussing about DIEW and genuine multipartite concurrence, we are now in a position to use them for our purpose of detecting genuine entanglement in an entanglement swapping protocol. This in turn helps to enhance the chance of genuine tripartite entanglement being detected in a device-independent way. But before that, we illustrate our MESP.

### 3 Multiple entanglement swapping procedure

Consider the MESP given in Fig. 1. It is a network of six space-like separated observers. Three tripartite quantum states $\rho_i$ ($i = 1, 2, 3$) are used in the network. State $\rho_1$ is shared among the parties $A_i$ ($i = 1, 2, 3$) such that $j$th particle ($\rho_{1j}^i$) of $\rho_1$ is with party $A_j$ ($j = 1, 2, 3$), respectively. State $\rho_2$ is shared among $A_2$, $A_3$ and $A_4$ with the specification that $j$th qubit ($\rho_{2j}^i$) is sent to party $A_{j+1}$ ($j = 1, 2, 3$). The remaining
state $\rho_3$ is shared among $A_4$, $A_5$ and $A_6$ such that party $A_{j+3}$ holds $j^{th}$ ($j = 1, 2, 3$) particle of $\rho_3(\rho_3^j)$. So each of the three parties $A_2$, $A_3$ and $A_4$ holds two particles: $A_2$ holds $\rho_1^2$ and $\rho_2^1$; $A_3$ holds $\rho_1^3$ and $\rho_2^2$; $A_4$ holds $\rho_3^2$ and $\rho_1^3$. Now in the preparation stage, each of the three parties $A_i$ ($i = 2, 3, 4$) performs Bell basis measurements on two particles that each of them holds: $A_2$ performs Bell basis measurement on $2^{nd}$ particle of $\rho_1^2$ and $1^{st}$ particle of $\rho_2^1$; $A_3$ performs Bell basis measurement on $3^{rd}$ particle of $\rho_1^3$ and $2^{nd}$ particle of $\rho_2^2$; $A_4$ performs Bell basis measurement on $3^{rd}$ particle of $\rho_2^3$ and $1^{st}$ particle of $\rho_3^1$. After all the three parties have performed Bell basis measurement on their respective particles, they communicate the results among themselves, as a result of which the final state $\rho_4$ shared between $A_1$, $A_5$ and $A_6$ is generated from the composite state $\rho_1 \otimes \rho_2 \otimes \rho_3$ shared between $A_i$ ($i = 1, 2, \ldots, 6$). Clearly $\rho_4$ varies with the output of the Bell measurements. The final state $\rho_4$ is obtained from the initial states $\rho_i$ ($i = 1, 2, 3$) by means of post-selecting on particular results of local measurements, in particular Bell basis measurements performed on these states ($\rho_i$ ($i = 1, 2, 3$)). For example, let us consider the case when $A_i$ ($i = 2, 3, 4$) obtain outputs corresponding to $|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$. If the output of all measurements correspond to $|\psi^\pm\rangle$ the resultant state $\rho_4^\pm$ given by:

$$\rho_4^\pm = \frac{\langle \psi^\pm | A_2 \otimes \langle \psi^\pm | A_3 \otimes \langle \psi^\pm | A_4}{\rho_1 \otimes \rho_2 \otimes \rho_3} | \psi^\pm \rangle^2 \otimes | \psi^\pm \rangle^3 \otimes | \psi^\pm \rangle^4.$$ (5)

So preparation stage of this protocol can be considered as a particular instance of Stochastic Local Operation and Classical Communication (SLOCC). After $\rho_4^\pm$ is generated, each of the three parties $A_1$, $A_5$ and $A_6$ performs projective measurement on the state $\rho_4^\pm$ in the measurement stage. Now, if the correlations generated from $\rho_4^\pm$ exhibit violation of any DI EW under the context that the initial states $\rho_i$ ($i = 1, 2, 3$) fail to reveal the same, then that guarantees enhancement of DIED via our protocol.
4 Enhancement of device-independent entanglement detection possibility

In this section, we deal with the procedure of enhancing DIED of tripartite quantum states by using the MESP described in Fig. 1. For this, we provide an explicit example. Initially, we consider three tripartite quantum states $\rho_i (i = 1, 2, 3)$ with some restricted range of state parameters, for each of which none of the DIEWs proposed in the literature [22,24,45] can detect genuine entanglement. These states, after being used in the MESP, generate a state $\rho_4$ whose genuine entanglement can be detected in a device-independent manner.

Let the three initial states be given by:

$$\rho_1 = p|\psi_0\rangle\langle\psi_0| + (1 - p)|001\rangle\langle001|$$  \hspace{1cm} (6)

with $|\psi_0\rangle = \cos \theta |000\rangle + \sin \theta |111\rangle$, $0 \leq \theta \leq \frac{\pi}{4}$ and $0 \leq p \leq 1$;

$$\rho_2 = p_1|\psi_\frac{\pi}{4}\rangle\langle\psi_\frac{\pi}{4}| + (1 - p_1)|010\rangle\langle010|$$  \hspace{1cm} (7)

with $|\psi_\frac{\pi}{4}\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$ and $0 \leq p_1 \leq 1$;

$$\rho_3 = p|\psi_{\frac{\pi}{o}}\rangle\langle\psi_{\frac{\pi}{o}}| + (1 - p)|100\rangle\langle100|$$  \hspace{1cm} (8)

with $|\psi_{\frac{\pi}{o}}\rangle = \sin \theta |000\rangle + \cos \theta |111\rangle$. Now, each of the three parties $A_2$, $A_3$ and $A_4$ performs Bell basis measurement on their respective particle. As already stated before, the output state depends on the outputs of the Bell measurements performed. For instance, when $|\psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$ is obtained as the output of odd number of parties, a resultant state $\rho_4^\pm$ is obtained which after correcting phase term is given by:

$$\rho_4^\pm = p_f^\pm|\psi_m^\pm\rangle\langle\psi_m^\pm| + (1 - p_f^\pm)|100\rangle\langle100|$$  \hspace{1cm} (9)

where $|\psi_m^\pm\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}$ and $p_f = \frac{2p\cos^2 \theta}{1 + p\cos 2\theta}$. Clearly, $\rho_4^\pm$ is independent of $p_1$, but the probability of obtaining $\rho_4^\pm$ directly depends on it. To detect DIE of each of the states $\rho_i (i = 1, 2, 3, 4)$, we obtain the condition for which they violate each of the DIEWs (see Appendix 1) given in [18,20,22,24]. Among all, the 3-settings Bell inequality

$$\frac{\sqrt{3}}{2} ((A_0B_0C_0) - (A_2B_0C_0) - (A_1B_1C_0) - (A_2B_1C_0)$$
$$- (A_0B_2C_0) - (A_1B_2C_0) - (A_1B_0C_1) - (A_2B_0C_1)$$
$$- (A_0B_1C_1) - (A_1B_1C_1) - (A_0B_2C_1)$$
$$+ (A_2B_2C_1) - (A_0B_0C_2) - (A_1B_0C_2) - (A_0B_1C_2)$$
$$+ (A_2B_1C_2) + (A_1B_2C_2) - (A_2B_2C_2)) \leq 9$$  \hspace{1cm} (10)
given by Bancal et al. [22] is the most efficient DIEW for each of \( \rho_i (i = 1, 2, 3, 4) \) (see Table 1). Here, \( \langle A_\alpha B_\beta C_\gamma \rangle \) designate the expected value of the product of three ±1 observables \( A_\alpha, B_\beta, \) and \( C_\gamma \).

Now, the initial states \( \rho_i (i = 1, 2, 3) \) do not violate Eq. (10) if and only if (see Appendix 1)

\[
p_1 \leq \frac{2}{3}
\]

and

\[
p \leq \frac{2}{3 \sin 2\theta}.
\]

The condition of violation of Eq. (10) for the final states \( \rho_4^\pm \) is given by (see Appendix 1):

\[
p > \frac{1}{\cos^2 \theta + 1}.
\]

There exists a range of the state parameters where the initial states \( \rho_i (i = 1, 2, 3) \) do not violate Bancal’s 3-settings Bell inequality, but after distributing them in the MESP, final states \( \rho_4^\pm \) (Eq. 9) violates it. The range of state parameters in which detection of DIE is enhanced by this entanglement swapping protocol (see Fig. 2) is given by:

\[
p_1 \leq \frac{2}{3} \quad \text{and} \quad \frac{1}{\cos^2 \theta + 1} < p \leq \frac{2}{3 \sin 2\theta}.
\]

The restrictions imposed on the state parameters (Fig. 2) indicate that DIED is enhanced at the end of the swapping procedure.

In this context, it is interesting to note that the probability of success of this protocol \( p^{\text{succ}} \) is given by

\[
p^{\text{succ}} = \frac{1}{2} p p_1 [1 + p \cos 2\theta] \sin^2 \theta.
\]

Table 1 The condition of violation of each of the DIEWs given in [20,22,24,45] for each of the states \( \rho_i (i = 1, 2, 3) \) are enlisted here.

| DIEW                | Violation by \( \rho_1 \) | Violation by \( \rho_2 \) | Violation by \( \rho_3 \) | Enhanced range          |
|---------------------|--------------------------|--------------------------|--------------------------|-------------------------|
| Mermin [45]         | \( p > \frac{1}{\sqrt{2} \sin 2\theta} \) | \( p_1 > \frac{1}{\sqrt{2}} \) | \( p > \frac{1}{\sqrt{2} \sin 2\theta} \) | \( \frac{1}{(2\sqrt{2} - 2) \cos^2 \theta + 1} < p \leq \frac{1}{\sqrt{2} \sin 2\theta} \) |
| Uffink [20]         | \( p > \frac{1}{\sqrt{2} \sin 2\theta} \) | \( p_1 > \frac{1}{\sqrt{2}} \) | \( p > \frac{1}{\sqrt{2} \sin 2\theta} \) | \( \frac{1}{(2\sqrt{2} - 2) \cos^2 \theta + 1} < p \leq \frac{1}{\sqrt{2} \sin 2\theta} \) |
| Bancal et al. [22]  | \( p > \frac{2}{3 \sin 2\theta} \) | \( p_1 > \frac{2}{3} \) | \( p > \frac{2}{3 \sin 2\theta} \) | \( \frac{1}{\cos^2 \theta + 1} < p \leq \frac{2}{3 \sin 2\theta} \) |
| Liang et al. [24]   | \( p > \frac{3\sqrt{3}}{5 \sin 2\theta} \) | \( p_1 > \frac{3\sqrt{3}}{5} \) | \( p > \frac{3\sqrt{3}}{5 \sin 2\theta} \) | \( \frac{1}{(2\sqrt{2} - 2) \cos^2 \theta + 1} < p \leq \frac{3\sqrt{3}}{5 \sin 2\theta} \) |

These conditions restrict the state parameters of the corresponding states such that these in turn give the enhanced region for detection of DIE in the swapping procedure. Moreover, comparison of these restrictions (for each of these states) in turn clearly justifies our claim that the DIEW given by Bancal et al. [22] emerges to be efficient tool for the detection of genuine entanglement in a device-independent way.
Fig. 2 (color online) The shaded region denotes the range of parameters of states \( \rho_1 \) and \( \rho_3 \) (\( p_1 \leq \frac{2}{3} \)) for which the 3-settings Bell inequality (Eq. 10) is violated only after distributing them in the multiple entanglement swapping network as described in Sect. 3, i.e. neither of the three initial states violate Eq. (10), whereas the final state violates it. Hence, this region gives the range where DIED is enhanced.

In recent times, there has been experimental implementation of DIED [50] and also experimental demonstration of entanglement swapping [51,52]. So our procedure of enhancing DIED method can also be demonstrated experimentally within the scope of current technology. The fact that this explicit example shows enhancement of DIED indicates that for any genuinely entangled tripartite state \( (\rho, \text{say}) \), it may be possible to design a suitable swapping protocol (like MESP) via which DIE of the final state resulting from the protocol using many copies of the initial state \( (\rho) \), can be detected even when the same cannot be detected for \( \rho \) itself.

5 Enhancement of semi device-independent entanglement detection possibility

In a tripartite scenario, if one of the three devices is trusted (let us say the third one), Cavalcanti et al. have provided an inequality in [53] which detects genuine entanglement in semi-device-independent way. The inequality looks like:

\[
1 - 0.1831(\langle A_3 B_3 \rangle + \langle A_3 Z \rangle + \langle B_3 Z \rangle) - 0.2582(\langle A_1 B_1 X \rangle - \langle A_1 B_2 Y \rangle - \langle A_2 B_1 Y \rangle - \langle A_2 B_2 X \rangle) \geq 0
\] (14)

where \( X, Y, \) and \( Z \) represent the pauli operators. Following the same procedure as that for device-independent case, it is observed that the initial states \( \rho_i (i = 1, 2, 3) \) do not violate Cavalcanti et al. inequality [53] if and only if

\[
P \geq \frac{1.1831}{0.7324 + 1.0328 \sin[2\theta]},
\]

and

\[
p_1 \geq 0.670236.
\] (15)
Fig. 3 Shaded region gives the restrictions imposed on the state parameters for which enhancement of semi-device-independent entanglement is observed via the MESP under the restriction of $p_1 \geq 0.670236$ over the state parameter $p_1$ of $\rho_2$

Now, the condition of violation of the inequality (Eq. 14) for the final state $\rho_4$ is given by:

$$p > \frac{2p \cos^2[\theta]}{1 + p \cos[2\theta]}.$$  \hspace{1cm} (16)

Thus, there exists a range of the state parameters ($p, p_1$) where the initial states $\rho_i (i = 1, 2, 3)$ do not violate the inequality Eq. (14), but after distributing them in the MESP and executing the protocol, final state $\rho_4$ violates it. The range of state parameters in which enhancement of detection of semi-device-independent entanglement is observed by our protocol is given by: $p_1 \leq 0.670236$ and

$$\frac{2p \cos^2[\theta]}{1 + p \cos[2\theta]} < p < \frac{1.1831}{0.7324 + 1.0328 \sin[2\theta]}$$  \hspace{1cm} (17)

which indicates a clear advantage as shown in Fig. 3.

6 Entanglement purification

Till now, we have discussed about enhancing DIED ability of tripartite quantum states by using the MESP. In this section, we proceed to see that this MESP can also be effective for purification of genuinely tripartite mixed entangled states. Entanglement purification provides a way to extract a small number of entangled pairs with a relatively high degree of entanglement from a large number of less entangled pairs using only local operations and classical communication (LOCC). For the purpose of realizing a purification process, the amount of genuine entanglement of the final state $\rho_4$ (Eq. 9) after entanglement swapping should be larger than that of the initial states $\rho_i (i=1, 2, 3)$ (Eqs. 6, 7, 8). For the purpose of comparing the amount of genuine entanglement ($C_{GM}$) of the states, we consider $p_1 = p$ for $\rho_2$. Since both the initial $(\rho_i, i = 1, 2, 3)$ and final $(\rho_4)$ states belong to the class of tripartite X state, their
amount of genuine entanglement can be measured by Eq. (4):

\[ C_{GM}^{\rho_2} = p, \]

and

\[ C_{GM}^{\rho_1} = C_{GM}^{\rho_3} = p \sin 2\theta \]  \hspace{1cm} (18)

whereas that for \( \rho_4 \) is given by

\[ C_{GM}^{\rho_4} = \frac{2p \cos^2 \theta}{1 + p \cos 2\theta}. \]  \hspace{1cm} (19)

The amount of genuine entanglement of the final state can be increased as long as the following relation holds: \( C_{GM}^{\rho_4} > C_{GM}^{\rho_i} \) \((i = 1, 2, 3)\). It is straightforward to verify that \( C_{GM}^{\rho_4} > C_{GM}^{\rho_i} \) \((i = 1, 2, 3)\) corresponds to the inequalities \( p \cos 2\theta(1 - p \sin 2\theta) + p(1 - \sin 2\theta) > 0 \) and \( p \cos 2\theta(1 - p) > 0 \). Both of these inequalities hold when \( \theta < \frac{\pi}{4} \) and \( p < 1 \). Hence, genuine entanglement of the final state \( \rho_4 \) is larger than that of the initial states \( \rho_i \) \((i = 1, 2, 3)\) for any nonzero value of the state parameters except \( p = 1 \) and \( \theta = \frac{\pi}{4} \). So, if \( 0 < \theta < \frac{\pi}{4} \) and \( 0 < p < 1 \), genuine entanglement can always be purified by our MESP. Here, the final state \( \rho_4 \) (Eq. 9) is obtained when each of the three parties \( A_2, A_3, A_4 \) takes \( |\psi^\pm\rangle (\frac{|10\rangle \pm |01\rangle}{\sqrt{2}}) \) as their output of their respective Bell basis measurement and the probability of obtaining \( |\psi^\pm\rangle \) in each Bell basis measurement is \( \frac{1}{16} p_1^2 \sin^2 \theta_1(1 + p_1 \cos 2\theta_1) \). Of course, in case of other possible outcomes of the three Bell basis measurements i.e., at least one of the parties \( A_i \) \((i = 2, 3, 4)\) obtains \( |\phi^\pm\rangle (\frac{|00\rangle \pm |11\rangle}{\sqrt{2}}) \) instead of \( |\psi^\pm\rangle \) as a result of their measurement, amount of genuine entanglement content of final states resulting from the MESP cannot be larger than that of the initial states in the range \( 0 < \theta < \frac{\pi}{4} \) and \( 0 < p < 1 \). For this reason, we can consider this MESP suitable for purification of genuine entanglement only probabilistically. Again our MESP can be used to generate GHZ state \( (\frac{|000\rangle + |111\rangle}{\sqrt{2}}) \).

For that, we require to fix the state parameter of the third initial state \( \rho_3 \) to be \( p = 1 \) whereas no such restriction over the state parameters of the other two initial states \( \rho_1 \) and \( \rho_2 \) is to be considered. So this protocol can be used for enhancement of the amount of genuine entanglement maximally using some non-maximally genuinely entangled tripartite states. In recent times, there has been experimental demonstration of entanglement purification by using entanglement swapping protocol [54], so our procedure of purification of genuinely tripartite mixed entangled states can also be demonstrated experimentally.

### 7 Conclusion

In a nutshell, our present topic of discussion may be considered as a contribution in the field of device-independent entanglement detection which minimizes the requirement of precise control over measurement devices by an experimentalist in an experimental detection of entanglement. More precisely, in our work, we have shown that it is possible to enhance device-independent detection of genuine tripartite entanglement in
some suitable measurement context. For our purpose, we have considered four DIEWs given by Mermin [45], Uffink [20], Bancal [22] and Liang et al. [24], out of which the DIEW given by Bancal et al. [22] emerges to be the most efficient. We have designed a state preparation protocol (prior to receiving final measurements), particularly an entanglement swapping procedure involving six distant observers via which genuine tripartite entanglement of the resultant (swapped) state, generated by using three initial tripartite states (whose entanglement cannot be detected by the standard DIEWs), can be detected by these standard DIEWs (used for testing entanglement of the initial states) after performing the state preparation.

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Appendix A: Condition for violation of DIEWs

We are now going to enlist the DIEWs which are used as tools for DIED in main text. Mermin DIEW [18]:

\[ M = |\langle A_1 B_0 C_0 \rangle + \langle A_0 B_1 C_0 \rangle + \langle A_0 B_0 C_1 \rangle - \langle A_1 B_1 C_1 \rangle| \leq 2\sqrt{2}. \]  

(A1)

Uffink DIEW [20]:

\[ \langle A_1 B_0 C_0 + A_0 B_1 C_0 + A_0 B_0 C_1 - A_1 B_1 C_1 \rangle^2 + \langle A_1 B_1 C_0 + A_0 B_1 C_1 + A_0 B_0 C_1 - A_0 B_0 C_0 \rangle^2 \leq 8. \]  

(A2)

Bancal et al. DIEW [22]: already discussed in Eq. (10).

Liang et al. DIEW [24]:

\[ \frac{1}{4}(\langle A_0 B_0 C_0 \rangle + \langle A_0 B_1 C_0 \rangle + \langle A_0 B_0 C_1 \rangle + \langle A_0 B_1 C_1 \rangle + \langle A_1 B_0 C_0 \rangle + \langle A_1 B_1 C_0 \rangle + \langle A_1 B_0 C_1 \rangle + \langle A_1 B_1 C_1 \rangle - 3\langle A_1 B_1 C_1 \rangle) \leq \sqrt{2}. \]  

(A3)

Now, we present the detailed proofs of the results stated in the main text. To obtain the condition of violation of each of the DIEWs (Eqs. A1, A2, 10, A3) in terms of state parameters for each of the initial states \( \rho_i (i = 1, 2, 3) \) and final state \( \rho_4 \), we apply the same method as used in [55]. First, we find the condition of violation (in terms of state parameters) of the DIEW given in Eq. (A1) for the initial state \( \rho_1 \).

We consider the following measurements: \( A_0 = \vec{x} \cdot \sigma_1 \) or \( A_1 = \vec{x} \cdot \sigma_1 \) on 1st qubit, \( B_0 = \vec{y} \cdot \sigma_2 \) or \( B_1 = \vec{y} \cdot \sigma_2 \) on 2nd qubit, and \( C_0 = \vec{z} \cdot \sigma_3 \) or \( C_1 = \vec{z} \cdot \sigma_3 \) on 3rd qubit, where \( \vec{x}, \vec{y}, \vec{y}, \vec{z} \) and \( \vec{z}, \vec{z} \) are unit vectors and \( \sigma_i \) are the spin projection operators that can be written in terms of the Pauli matrices. Representing the unit vectors in spherical coordinates, we have, \( \vec{x} = (\sin \theta_0 \cos \phi_0, \sin \theta_0 \sin \phi_0, \cos \theta_0) \), \( \vec{y} = (\sin \alpha_0 \cos \beta_0, \sin \alpha_0 \sin \beta_0, \cos \alpha_0) \) and \( \vec{z} = (\sin \zeta_0 \cos \eta_0, \sin \zeta_0 \sin \eta_0, \cos \zeta_0) \).
\[
\cos \zeta_0 \text{ and similarly, we define } \vec{x}, \vec{y}, \text{ and } \vec{z} \text{ by replacing } 0 \text{ in the indices by } 1. \text{ Then, the value of the operator } M \text{ (Eq. A1) with respect to the state } \rho_1 \text{ (Eq. 6) gives:}
\]

\[
\begin{align*}
M(\rho_1) &= - \cos ab_1(-1 + p + p \cos 2\theta)(\cos \zeta_0 \cos \theta a_1 + \cos \zeta_1 \cos \theta a_0) \\
&\quad - \sin ab_1(p \sin 2\theta)(\cos \beta b_1 + \eta c_1 + \phi a_1) \sin \zeta_1 \sin \theta a_0 \\
&\quad + \cos ab_0(-1 + p + p \cos 2\theta)(\cos \zeta_0 \cos \theta a_0 - \cos \zeta_1 \cos \theta a_1) \\
&\quad + \sin ab_0(p \sin 2\theta)(\cos \beta b_0 + \eta c_0 + \phi a_0) \sin \zeta_0 \sin \theta a_0 \\
&\quad - \cos(\beta b_0 + \eta c_1 + \phi a_1) \sin \zeta_1 \sin \theta a_1). \quad (A4)
\end{align*}
\]

Hence, in order to get maximum value of \( S(\rho_1) \), we have to perform maximization over 12 measurement angles. Now if we maximize the last equation with respect to \( \alpha b_0 \) and \( \alpha b_1 \), we have

\[
M(\rho_1) \leq \sqrt{((X)(\cos \zeta_0 \cos \theta a_1 + \cos \zeta_1 \cos \theta a_0))^2 + (Y)^2 (A_{110} \sin \zeta_1 \sin \theta a_0 + A_{101} \sin \zeta_0 \sin \theta a_1)^2} \\
+ \sqrt{((X)(\cos \zeta_0 \cos \theta a_0 - \cos \zeta_1 \cos \theta a_1))^2 + (Y)^2 (A_{000} \sin \zeta_0 \sin \theta a_0 - A_{011} \sin \zeta_1 \sin \theta a_1)^2} \quad (A5)
\]

Where \( X = -1 + p + p \cos 2\theta, Y = p \sin 2\theta, \) and \( A_{ijk} = \cos(\beta b_i + \eta c_j + \phi a_k)(i, j, k \in \{0, 1\}) \). The last inequality is obtained by using the inequality \( x \cos \theta + y \sin \theta \leq \sqrt{x^2 + y^2} \). It is clear from the symmetry of the measurement angles \( \theta a_0, \zeta_0 \) and \( \theta a_1, \zeta_1 \) that the right-hand side of Eq. (A5) gives maximum value when \( \theta a_0 = \zeta c_0 \) and \( \theta a_1 = \zeta c_1 \). Hence, Eq. (A5) takes the form:

\[
M(\rho_1) \leq \sqrt{((X)(2 \cos \theta a_0 \cos \theta a_1))^2 + (Y \sin \theta a_0 \sin \theta a_1)^2 (A_{110} + A_{101})^2} \\
+ \sqrt{((X)(\cos^2 \theta a_0 - \cos^2 \theta a_1))^2 + (Y)^2 (A_{000} \sin^2 \theta a_0 - A_{011} \sin^2 \theta a_1)^2} \quad (A6)
\]

Again, we maximize it with respect to \( \theta a_1 \). Critical point 0 or \( \frac{\pi}{2} \) gives the maximum value depending on values of the state parameters. For the critical point 0, Eq. (A6) becomes

\[
M(\rho_1) \leq \sqrt{2X^2 \cos \theta a_0)^2 + \sqrt{\sin^4 \theta a_0 (X^2 + Y^2)^2}} \quad (A7)
\]

where we have chosen \( A_{000}^2 = 1 \). Maximizing over \( \theta a_0 \), we get

\[
M(\rho_1) \leq \frac{2X^2 + Y^2}{\sqrt{X^2 + Y^2}} \quad (A8)
\]

the maximum being obtained for \( \cos \theta a_0 = \frac{|X|}{\sqrt{X^2 + Y^2}} \). For the other critical point \( \frac{\pi}{2} \), Eq. (A6) takes the form:

\[
M(\rho_1) \leq \sqrt{(Y \sin \theta a_0)^2 (A_{110} + A_{101})^2} \\
+ \sqrt{X^2 \cos^4 \theta a_0 + Y^2 (A_{000} \sin^2 \theta a_0 - A_{011})^2}
\]
\[ \leq \sqrt{4(Y \sin \theta a_0)^2 + X^2 \cos^4 \theta a_0 + Y^2 (\sin^2 \theta a_0 + 1)^2} \leq 4|Y| \quad (A9) \]

The second inequality in Eq. (A9) is obtained from the first by setting \( A_{110} = 1, A_{101} = 1, A_{000} = 1 \) and \( A_{011} = -1 \). The final inequality is achieved when \( \theta a_0 = \frac{\pi}{2} \).

Two sets of measurement angles which realize the two values \( \frac{2X^2 + Y^2}{\sqrt{X^2 + Y^2}} \) (Eq. A8) and \( 4|Y| \) (Eq. A9), are \( \theta a_0 = \alpha b_0 = \xi c_0 = \cos^{-1}(-\frac{|X|}{\sqrt{X^2 + Y^2}}), \theta a_1 = \alpha b_1 = \xi c_1 = 0, \beta b_i = \eta c_i = \phi a_i = 0 \) (i = 0, 1) and \( \theta a_i = \alpha b_i = \xi c_i = \frac{\pi}{2} \) (i = 0, 1), \( \beta b_0 = \eta c_0 = \phi a_0 = 0, \beta b_1 = -\eta c_1 = -\phi a_1 = \frac{\pi}{2} \), respectively. Hence, from Eq. (A8) and Eq. (A9), we have

\[ M(\rho_1) \leq \max[\frac{2X^2 + Y^2}{\sqrt{X^2 + Y^2}}, 4|Y|]. \quad (A10) \]

Clearly, \( \frac{2X^2 + Y^2}{\sqrt{X^2 + Y^2}} \leq 2 < 2\sqrt{2} \) for any value of \( p \in [0, 1] \) and \( 0 \leq \theta \leq \frac{\pi}{4} \). So the initial state \( \rho_1 \) violates the DIEW based on Mermin expression (Eq. A1) if

\[ 4|Y| = 4|p| \sin 2\theta > 2\sqrt{2}. \quad (A11) \]

The last inequality is considered as the condition of violation of the DIEW based on Mermin expression for the initial state \( \rho_1 \). We have applied the same method over other states \( \rho_i \) (i = 2, 3, 4) to find the condition of violation of the DIEW based on Mermin expression. For other DIEWS (Eqs. A2, 10, A3), we have made analysis in similar manner so as to obtain the condition of violation for each of states \( \rho_i \). All the conditions are summarized in Table 1. However, among the four DIEWS given by Mermin (Eq. A1), Uffink (Eq. A2), Bancal et al. (Eq. 10) and Liang et al. (Eq. A3), the one given by Bancal et al. turns out to be the most efficient for this purpose. The DIEW based on Bancal et al. polynomial (Eq. 10) can thus detect genuine tripartite entanglement in a device-independent way in \( \rho_1 \) for \( p > \frac{2}{3 \sin 2\theta} \) (see Table 1). As \( \frac{2}{3 \sin 2\theta} < \frac{1}{\sqrt{2 \sin 2\theta}} < \frac{3\sqrt{2}}{5 \sin 2\theta} \), so the DIEW based on Bancal et al. polynomial (Eq. 10) is the most efficient DIEW for the state \( \rho_1 \) to detect genuine tripartite entanglement among all the standard DIEWS considered in Eqs. (A1), (A2), (10), (A3).

Similarly by comparing the range of violation of \( \rho_1 \) (for the state \( \rho_2 \)) and \( p \) (for the state \( \rho_3, \rho_4 \)), one can check that Bancal et al. Bell inequality is the best DIEW for the other states \( \rho_i \) (i = 2, 3, 4) to detect genuine tripartite entanglement compared to other standard DIEWS.

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