Nonlinear singular perturbations of the fractional Schrödinger equation in dimension one

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Abstract
The paper discusses nonlinear singular delta-type perturbations of the fractional Schrödinger equation \( \mathbb{i} \partial_t \psi = (-\Delta)^s \psi \), with \( s \in (\frac{1}{2}, 1] \), in dimension one. In particular, we investigate local and global well-posedness (in a strong sense), conservation laws and the existence of blow-up solutions and standing waves.

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1. Introduction

The appearance of the term ‘fractional Schrödinger equation’ (FSE) traces back to the pioneering papers by Laskin [35, 36]. In those papers, this equation, i.e.

\[ \mathbb{i} \partial_t \psi = (-\Delta)^s \psi, \]  

(1)

follows when the Feynman path integral is extended from Brownian-like to Lévy-like quantum mechanical paths. On the other hand, in potential theory, the fractional Laplacian was also introduced as the infinitesimal generator of some Lévy processes [7, 34].

In the most recent physics literature, the fractional nonlinear Schrödinger equation (FNSE) has been exploited in many different frameworks.
The FNSE arises, for instance, in the investigation of the quantum effects in Bose–Einstein condensation [46]. The condensation is usually described by the Gross–Pitaevskii equation when a boson system is ideal, i.e. at low temperatures with weak interactions. If the medium is inhomogeneous with long-range interactions, it has been proven that the dynamics can be described by a fractional Gross–Pitaevskii equation. This phenomenon is due to a general feature of fractional equations, which are able to take decoherence and turbulence effects into account. In fact, these equations can also be used to investigate decoherence effects in several physical models [31].

However, physical models based on the FNSE as an effective equation are not restricted to the quantum domain. The study of the long-time behavior of the solutions of the water waves equation in $\mathbb{R}^2$ relies, for instance, on the FNSE (see [30] and references therein). Furthermore, it has been recently used in optics [38], in solid state physics [44] and biological systems [22, 39].

From a mathematical point of view, the FNSE has been widely studied over the last few years. We mention, for instance, [8, 28, 32] for the standard FNSE and [17–19, 29] for the variant (relevant for physical applications) provided by Hartree-type nonlinearities.

On the other hand, the first discussion of an FSE in the presence of delta potentials is due to [37]: the main result contained in this paper is the analysis of an anomalous spreading of the solutions of the Cauchy problem characterized by a power-law behavior and related to the order of the fractional operator.

More recently, [40, 41, 45] addressed the problem of singular perturbations of the fractional Laplacian. In particular, they show that rank-one linear singular perturbations of the $d$-dimensional fractional Laplacian can be obtained as the norm-resolvent limit of fractional Schrödinger operators with shrinking potentials.

In this paper, we discuss nonlinear singular delta-type perturbations of (1) in dimension one, with $s \in (\frac{1}{2}, 1]$. The main purpose is to investigate the well-posedness and dynamical features of the associated Cauchy problem.

The major characteristic of these kinds of equations is that, albeit nonlinear, they fall under the so-called solvable models: the investigation of the time evolution can be reduced to that of an ODE-type equation (see, for example, [6]). For the ordinary Laplacian in $\mathbb{R}$, the case of a concentrated nonlinearity has been studied in [5, 11] (see also [15]) and then extended to $\mathbb{R}^2$ and $\mathbb{R}^3$ in [2, 13, 14, 16] and [3, 4] (respectively), and to the 1-dimensional Dirac equation in [10].

The paper is organized as follows:

- in section 2 we provide a brief review on the linear FSE in the free case and in the case of a singular delta-type perturbation, and then we present the main results on nonlinear perturbations;
- in section 3 we present the proofs of the main results of the paper (i.e. theorems 2.1 and 2.2);
- in the appendix, we present the proof of some further results on linear perturbations (i.e. proposition 2.1).

2. Setting and main results

Before stating the main results of the paper, it is necessary to fix some notation and recall some well known facts about the linear FSE both in the free case and in the case of delta interactions.
2.1. The free case

Preliminarily, we recall that, for every $s \in (0, 1)$, the fractional Laplace operator on (an open set) $\Omega \subset \mathbb{R}$ is defined by

$$(-\Delta)^s_{\Omega} u(x) := c(s) P.V. \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{1+2s}} \, dy$$

where $c(\cdot)$ is a specific normalization constant such that

$$(-\Delta)^s u(k) := |k|^{2s} \hat{u}(k),$$

with $(-\Delta)^s = (-\Delta)^s_{\Omega}$ and $\hat{\cdot}$ representing the unitary Fourier transform on $\mathbb{R}$ (see, for instance, [23] for an explicit expression). We also recall the definition of the fractional Sobolev space $H^\mu(\Omega)$, with $\mu \in (0, 1)$, i.e.

$$H^\mu(\Omega) := \left\{ u \in L^2(\Omega) : u \in \dot{H}^\mu(\Omega) \right\} = \left\{ u \in L^2(\Omega) : [u]_{\dot{H}^\mu(\Omega)} < \infty \right\}$$

(3)

where $[u]_{\dot{H}^\mu(\Omega)}$ denotes the usual Gagliardo semi-norm, i.e.

$$[u]_{\dot{H}^\mu(\Omega)}^2 := \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{1+2\mu}} \, dx \, dy.$$

When $\Omega = \mathbb{R}$, it is (norm-)equivalent to

$$H^\mu(\mathbb{R}) = \left\{ u \in \mathcal{S}'(\mathbb{R}) : \int_\mathbb{R} (1 + |k|^{2s}) |\hat{u}(k)|^2 \, dk < \infty \right\},$$

(4)

where $\mathcal{S}'(\mathbb{R})$ denotes the space of the tempered distributions and

$$[u]_{\dot{H}^\mu(\mathbb{R})}^2 := \|(-\Delta)^{\mu/2} u\|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} |k|^{2\mu} |\hat{u}(k)|^2 \, dk$$

(5)

(see, again, [23]). Moreover, (4) holds for $\mu > 1$ as well, whereas (3) has to be modified in

$$H^\mu(\Omega) := \left\{ u \in H^{[\mu]}(\Omega) : \frac{d^{[\mu]}}{dx^{[\mu]}} u \in \dot{H}^{\mu - [\mu]}(\Omega) \right\}$$

(with $[\mu]$ the integer part of $\mu$).

On the other hand, it is well known that, for every $s \in (0, 1]$, the operator $\mathcal{H}^s_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by

$$\mathcal{D}(\mathcal{H}^s_0) := H^{2s}(\mathbb{R}) \quad \text{and} \quad \mathcal{H}^s_0 \psi := (-\Delta)^s \psi, \quad \forall \psi \in \mathcal{D}(\mathcal{H}^s_0),$$

is self-adjoint, and hence by Stone’s theorem the Cauchy problem

$$\begin{cases}
\frac{\partial}{\partial t} \psi(x) = \mathcal{H}^s_0 \psi \\
\psi(0, \cdot) = \psi_0(\cdot) \in H^{2s}(\mathbb{R})
\end{cases}$$

has a unique solution

$$\psi(t, x) := (\mathcal{U}_s(t) \psi_0)(x) \in C^0([0, T]; H^{2s}(\mathbb{R})) \cap C^1([0, T]; L^2(\mathbb{R})), \quad \forall T > 0,$$

with $\mathcal{U}_s(t)$ the convolution unitary operator defined by the integral kernel

$$\mathcal{U}_s(t, x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{iks} e^{-i|k|^2 t} \, dk.$$
nearly

\[ U_s(t, \cdot)(k) = \frac{e^{-|k|^2t}}{\sqrt{2\pi}}. \]

In addition, we recall that the quadratic form associated with \( \mathcal{H}_0^s \) is

\[ \mathcal{D}(\mathcal{F}_0^s) := H^s(\mathbb{R}) \quad \text{and} \quad \mathcal{F}_0^s(\psi) := [\psi]^2_{H^s(\mathbb{R})}, \quad \forall \psi \in \mathcal{D}(\mathcal{F}_0^s). \]

Finally, we point out that throughout the paper, we denote by \( (\cdot, \cdot) \) and \( \| \cdot \| \) the standard scalar product and norm of \( L^2(\mathbb{R}) \).

### 2.2. Linear delta perturbations

The problem of linear delta-type singular perturbations of the fractional Laplacian has been widely discussed in [41]. Here, we limit ourselves to mentioning some known facts and notation, which are useful in the following.

For every \( \lambda > 0 \) and \( s \in (0, 1] \), we denote by \( G^\lambda_s \) the Green’s function of \((-\Delta)^s + \lambda\), namely the unique solution in \( L^2(\mathbb{R}) \) of

\[ ((-\Delta)^s + \lambda) G^\lambda_s = \delta, \quad (6) \]

nearly

\[ \hat{G}^\lambda_s(k) = \frac{1}{\sqrt{2\pi (|k|^{2s} + \lambda)}}. \quad (7) \]

Note that \( G^\lambda_s \in L^2(\mathbb{R}) \) for every \( s > 1/4 \) and \( \hat{G}^\lambda_s \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R}) \) for every \( s > \frac{1}{2} \).

In addition, for all fixed \( s \in (\frac{1}{2}, 1] \) and \( \alpha \in \mathbb{R} \), we define the operator \( \mathcal{H}_\alpha^s : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \)

\[ \mathcal{D}(\mathcal{H}_\alpha^s) := \{ \psi \in L^2(\mathbb{R}) : \psi = \phi_\lambda - \alpha \hat{G}^\lambda_s(\cdot) \psi(0), \phi_\lambda \in H^2(\mathbb{R}), \lambda > 0 \}, \quad (8) \]

\[ (\mathcal{H}_\alpha^s + \lambda) \psi := ((-\Delta)^s + \lambda) \phi_\lambda, \quad \forall \psi \in \mathcal{D}(\mathcal{H}_\alpha^s). \quad (9) \]

It is self-adjoint and hence

\[ \begin{aligned}
\frac{\partial \psi}{\partial t} &= \mathcal{H}_\alpha^s \psi \\
\psi(0, \cdot) &= \psi_0(\cdot) \in \mathcal{D}(\mathcal{H}_\alpha^s)
\end{aligned} \quad (10) \]

has a unique solution

\[ \psi \in C^0([0, T]; \mathcal{D}(\mathcal{H}_\alpha^s)) \cap C^1([0, T]; L^2(\mathbb{R})), \]

where \( \mathcal{D}(\mathcal{H}_\alpha^s) \) is endowed with the graph norm. Notice that the definition of the operator domain (8) does not depend on \( \lambda \), provided that \( \lambda > 0 \), which then plays the role of a mere regularizing parameter.

It is also worth mentioning that [41] provides an approximation result for the operator \( \mathcal{H}_\alpha^s \), which actually explains why we consider this operator to be a singular delta-type perturbation of the fractional Laplacian. In particular, the paper shows that (a wide class of) compactly supported potentials \( V \) exist, with \( \alpha = \int_\mathbb{R} V(x) \, dx \), such that the sequence of self-adjoint operators

\[ \mathcal{D}(\mathcal{H}_\alpha^s) := H^2(\mathbb{R}) \quad \text{and} \quad \mathcal{H}_\alpha^s \psi := \left( (-\Delta)^s + \frac{1}{\epsilon^2} V \left( \frac{\cdot}{\epsilon} \right) \right) \psi \]
converges in the norm resolvent sense to $H_\alpha^s$. Therefore, $H_\alpha^s$ defined by (8) and (9) provides a rigorous version of the informal expression $(-\Delta)^\gamma + \delta$.

**Remark 2.1.** It is clear that the case $\alpha = 0$ represents the unperturbed case.

Finally, we provide here a different representation of the operator $H_\alpha^s$. Let us introduce the fractional differential operator $D^\mu u(x) := -\frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} |k|^\mu \text{sgn}(k) \hat{u}(k) \, dk$, $\mu \in (0, 1]$, that is

$$\hat{D^\mu u}(k) = |k|^{\mu} \text{sgn}(k) \hat{u}(k).$$

Note that this operator is bounded $H^\mu + \delta(\mathbb{R}) \to H^\delta(\mathbb{R})$, for all $\delta \geq 0$.

**Remark 2.2.** Note that when $\mu = 1$, this results in $D^1 \equiv \frac{d}{dx}$ (in a distributional sense).

**Proposition 2.1.** Let $s \in (\frac{1}{2}, 1]$ and $\alpha \in \mathbb{R}$. Then, the following representation of the operator $H_\alpha^s$ is equivalent to (8) and (9):

$$D(\mathcal{H}_\alpha^s) = \{ \psi \in H^1(\mathbb{R}) : D^{2s-1}\psi \in H^1(\mathbb{R}\setminus\{0\}), [D^{2s-1}\psi](0) = \alpha\psi(0) \} \tag{12}$$

$$\mathcal{H}_\alpha^s \psi = (-\Delta)^{s/2}\psi, \quad \forall x \neq 0, \tag{13}$$

where $[D^{2s-1}\psi](0) := D^{2s-1}\psi(0^+) - D^{2s-1}\psi(0^-)$. Moreover, the quadratic form associated with $\mathcal{H}_\alpha^s$ is defined as

$$\mathcal{F}_\alpha^s(\psi) := (\psi, H_\alpha^s \psi) = \|(-\Delta)^{s/2}\psi\|^2 + \alpha|\psi(0)|^2 = \|\psi\|^2_{H^s(\mathbb{R})} + \alpha|\psi(0)|^2 \tag{14}$$

with domain $\mathcal{D}(\mathcal{F}_\alpha^s) := H^1(\mathbb{R})$.

For the proof, see the appendix. Here, we limit ourselves to observing that the previous proposition also allows us to write delta perturbations as ‘jump conditions’ in the fractional case. However, in the integer case, this fact arises as a natural consequence of the definition of the action and the domain of the operator; in this case, it is necessary to detect (through a careful analysis of the properties of the Green’s function — see the appendix) the proper fractional differential operator whose jump encodes the delta interaction.

### 2.3. Main results: nonlinear delta perturbations

Let us introduce the nonlinear analogue of (10) by posing

$$\alpha = \alpha(\psi) = \beta|\psi(0)|^{2\sigma}, \quad \sigma > 0, \quad \beta \in \mathbb{R},$$

in (8); that is
\[ \mathcal{D}(\mathcal{H}^n) := \{ \psi \in H^1(\mathbb{R}) : \psi = \phi \lambda - \beta |\psi(0)|^{2\sigma} \psi(0) \mathcal{G}^\lambda, \, \phi \in H^2(\mathbb{R}), \, \lambda > 0 \}, \]
\[ (\mathcal{H}^n + \lambda)\psi := ((-\Delta)^{1/2} + \lambda)\phi, \quad \forall \psi \in \mathcal{D}(\mathcal{H}^n). \]  

On the other hand, in view of proposition 2.1, \( \mathcal{H}^n \) can be equivalently represented as
\[ \mathcal{D}(\mathcal{H}^n) := \{ \psi \in H^1(\mathbb{R}) : D^{2\sigma} \psi \in H^1(\mathbb{R} \setminus \{0\}), \, |D^{2\sigma} \psi|(0) = \beta |\psi(0)|^{2\sigma} \psi(0) \} , \]
\[ H^n_\psi = (-\Delta)^{1/2} \psi, \quad \forall x \neq 0. \]

A solution of the nonlinear analogue of (10) is a function \( \psi \), such that \( \psi(t, \cdot) \in \mathcal{D}(\mathcal{H}^n) \), for all \( t > 0 \), and
\[ \begin{cases} \frac{\partial \psi}{\partial t} = H^n_\psi \psi \\ \psi(0, \cdot) = \psi_0(\cdot) \in \mathcal{D}(\mathcal{H}^n). \end{cases} \]  

In fact, we discuss the integral form of the problem, provided by the Duhamel formula, which reads
\[ \psi(t, x) = (\mathcal{U}_t(\psi_0))(x) + \beta \int_0^t \mathcal{U}_t(t - \tau, x)|\psi(\tau, 0)|^{2\sigma} \psi(\tau, 0) \, d\tau. \]  

Furthermore, one can set
\[ q(t) := \psi(t, 0), \]  
which is usually referred to as charge and, from (17), has to satisfy
\[ q(t) + \sigma a(s) \beta \int_0^t \frac{|q(\tau)|^{2\sigma} q(\tau)}{(t - \tau)^{1/2}} \, d\tau = f(t) \]  
(i.e. a singular nonlinear Volterra integral equation of the second kind), where
\[ f(t) := (\mathcal{U}_t(\psi_0))(0) \]  
and
\[ a(s) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i|\rho|^{2\sigma}} \, d\rho = \mathcal{U}_s(1, 0) \in \mathbb{C} \]  
(note that \( \mathcal{U}_s(1, 0) = a(s)/i^{2\sigma} \)). As a consequence, if (19) uniquely determines \( q \), then \( \psi \) is defined by
\[ \psi(t, x) := (\mathcal{U}_t(\psi_0))(x) - \beta \int_0^t \mathcal{U}_t(t - \tau, x)|q(\tau)|^{2\sigma} q(\tau) \, d\tau. \]

Thus, ‘solving the nonlinear analogue of (10)’ means searching for a solution of (19) such that (21) satisfies (16).

**Theorem 2.1 (Well posedness).** Let \( s \in (\frac{1}{2}, 1] \) and \( \psi_0 \in \mathcal{D}(\mathcal{H}^n) \). Then:

(i) **Local well-posedness.** \( T > 0 \) exists such that the function \( \psi \) defined by (19)–(21) is the unique solution of (16) (where the former is meant as an equality in \( L^2(\mathbb{R}) \)). In addition,
\[ \psi \in C^0([0, T]; \mathcal{D}(\mathcal{H}^n)) \cap C^1([0, T]; L^2(\mathbb{R})). \]

(ii) **Conservation laws.** The mass and energy, namely
\[ M(t) = M(\psi(t, \cdot)) := \|\psi(t, \cdot)\| \]
\[ E(t) = E(\psi(t, \cdot)) := [\psi(t, \cdot)]^2_{\text{IP}(\mathbb{R})} + \frac{\beta}{\sigma + 1}|\psi(t, 0)|^{2\sigma + 2}, \]

are preserved quantities along the flow.

(iii) Global well-posedness. If one of the following conditions is satisfied:
- \( \beta \geq 0 \),
- \( \beta < 0 \) and \( \sigma < \sigma_c(s) := 2s - 1 \),

then the solution is global in time. In addition, when \( \beta < 0 \) and \( \sigma = \sigma_c(s) \), there exists \( C(s, \beta) > 0 \) such that, if \( \|\psi_0\| < C(s, \beta) \), then the solution is global in time as well.

(iv) Blow-up solutions. If \( \beta < 0 \), \( \sigma \geq \sigma_c(s) \) and \( \psi_0 \) have the regular part \( \phi_{\lambda,0} \) in \( S(\mathbb{R}) \) and satisfy
\[ E(\psi_0) < 0, \]

then there exists \( T^* \in [0, \infty) \) such that
\[ \limsup_{t \to T^*} |q(t)| = +\infty, \]

namely the solution blows-up in a finite time.

Remark 2.3. Recall that the case \( \beta > 0 \) is usually called a defocusing or repulsive case, whereas case \( \beta < 0 \) is usually called a focusing or attractive case, in analogy with the standard NLS. In addition, when \( \beta < 0 \) and \( \sigma = \sigma_c(s) \) is said to be sub-critical, case \( \sigma = \sigma_c(s) \) is said to be critical and case \( \sigma > \sigma_c(s) \) is said to be super-critical. According to this, \( \sigma_c(s) \) is called a critical exponent.

Remark 2.4. Notice that if \( \|\psi_0\| < C(s, \beta) \), then \( E(0) \geq 0 \) (see \( 48 \)) and thus (consistently) the blow-up condition is not fulfilled.

Remark 2.5. Throughout the paper, we consider the case of an interaction based at \( x = 0 \). However, this is not restrictive since all the results are valid as well as for any other choice. In addition, one could also consider the case of \( N \) distinct singular \( \delta \)-type perturbations, as well as more general nonlinear dependence for \( \alpha(\psi) \). Anyway, this would produce only computational issues, and hence, for the sake of simplicity, we limit ourselves to the case of a single perturbation with a power-type nonlinearity.

Two comments on item (iv) of the previous theorem are in order. First, as highlighted in remark 3.5, the extra assumption on the smoothness of the regular part of the initial datum is just a technical point required in the proof of proposition 3.4. In addition, blow-up is a phenomenon linked to the magnitude of the initial singular part rather than to the smoothness of the regular part. Hence, a detailed discussion of the minimal regularity assumptions is not that relevant.

On the other hand, it is clear that in the super-critical focusing case, it is possible to choose an initial datum with \( E(0) < 0 \). For instance, consider a generic function \( u \in D(\mathcal{H}^s) \) with a regular part in the class of Schwartz functions and such that \( u(0) \neq 0 \). Then, for every \( \nu > 0 \), define
\[ u_\nu(x) = u(\nu x). \]
Recalling that \( \hat{u}(\nu \cdot (k)) = \frac{1}{\nu} \hat{u}(\frac{k}{\nu}) \), a simple scaling argument shows that (in the focusing case)

\[
E(u_\nu) = \nu^{2s-1} |u_\nu|^2 - \frac{|\beta|}{\sigma+1} |u(0)|^{2\sigma+2}
\]

and hence, if \( \nu \) is sufficiently small, then \( E(u_\nu) < 0 \).

Before stating the second result, we recall that a standing wave is a function \( \psi_\omega(t,x) = e^{\nu t} u_\omega(x), \) with \( \omega \in \mathbb{R} \) and \( 0 \neq u_\omega \in \mathcal{D}(\mathcal{H}_s^0) \), that satisfies (16), namely such that

\[
\mathcal{H}_s^0 u_\omega + \omega u_\omega = 0.
\]

**Theorem 2.2 (Standing waves).** Let \( s \in (\frac{1}{2},1] \). Every positive standing wave of (16) is given by

\[
u^{s} \left( -\sigma \left( \frac{\pi}{2s} \right) \right)^{\frac{\sigma+1}{\sigma}} \omega^{\frac{2s-1}{\sigma} + 1} \mathcal{G}_s^\omega(x),
\]

with \( \omega > 0 \) and \( \beta < 0 \). Any other standing wave with frequency \( \omega \) differs from \( u_\omega \) in (24) by a constant phase factor.

In addition:

(i) if \( \sigma < \sigma_c(s) \), then \( E(u_\omega) < 0 \) for all \( \omega > 0 \), and \( \inf_{\omega \in \mathbb{R}^+} E(u_\omega) = -\infty \)

(ii) if \( \sigma > \sigma_c(s) \), then \( E(u_\omega) > 0 \) for all \( \omega > 0 \), and \( \inf_{\omega \in \mathbb{R}^+} E(u_\omega) = 0 \)

(iii) if \( \sigma = \sigma_c(s) \), then \( E(u_\omega) = 0 \).

Theorem 2.2 has some important consequences. First, it states that no standing wave may exist in the defocusing case. On the other hand, one can immediately see that in the supercritical focusing case, the infimum level of the energies of standing waves is exactly the threshold under which there is blow-up.

The assumption \( s \in (\frac{1}{2},1] \) in theorems 2.1 and 2.2 is explained as follows. First, one can observe that linear \( \delta \)-type perturbations of the fractional Laplacian exist only if \( s > \frac{1}{2} \), since otherwise the Green’s function of \(-\Delta\)^s + \lambda is not square integrable (for details see [40, 41]). On the other hand, as we stressed in section 2.2, when \( s \in (\frac{1}{2}, 1) \), the Green’s function is also bounded and continuous, and this means that the strategy for the proofs of the main results (up to several technical modifications) retraces the one in [5], since the problem still displays the specific features of the so-called models in codimension one. In contrast, when \( s \in (\frac{1}{2}, \frac{1}{2}) \), the issue presents the structure of higher codimensional models, such as nonlinear \( \delta \)-perturbations of the Laplacian in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) (in particular, \( s \in (\frac{1}{2}, \frac{1}{2}) \) retracts delta in 3d, while \( s = \frac{1}{2} \) retracts delta in 2d). In these cases, the strategy for the proof of global and local well-posedness (and blow-up) is considerably different (for instance, it is no more true that \( q(t) = \psi(t,0) \)) and, mainly in the 2d case [2, 14], requires a more refined analysis of the kernel of the associated charge equation, where it is not clear how to adapt to the fractional case at the moment. Hence, we think that in a first work on nonlinear perturbations of the fractional Laplacian, it is worth starting from one-codimensional problems, leaving higher-codimensional ones to forthcoming papers.

However, also in this one-codimensional case, although the strategy is analogous, some relevant and interesting differences can be detected between this work and [5]. First, here we present a finer discussion of the regularizing properties of the integral kernel \( t^{-\frac{s}{2}} \). In particular, such a discussion allows us to prove a far better regularity for the solution of the charge equation, which is subsequently exploited to show that the wave function defined in (21) is a solution of (16) in a strong sense. In view of this, the paper contains a stronger result (which
holds for $s = 1$ too) with respect to [5], where only the integral formulation and distributional solutions were treated (recall that in [5], the main achievement was a feasible strategy for the investigation of nonlinear delta perturbations).

On the other hand, concerning the blow-up analysis, here it is necessary to introduce a fractional moment of inertia, as in the standard fractional NLSE. The study of such a fractional moment of inertia in the context of nonlinear deltas requires several technical modifications in the proofs (which follow, again, the same strategy of the integer problem), which also make the detection of the fractional critical power possible.

Finally, in addition to [5], the paper contains a complete classification of the standing waves of (16) and shows the connection between their energy level and the threshold for the blow-up.

**Remark 2.6.** It is worth recalling that the different strategy used here with respect to the case of the standard NLSE is due to the singular structure of the nonlinearity and to the consequent singular structure of the functions in the ‘operator’ domain. In particular, the Schrödinger equation with a nonlinear delta is not a semilinear equation (as in the standard NLSE), since the nonlinearity is encoded in the domain of the ‘operator’ and not in its action. However, the fact that the problem is the nonlinear version of a so-called solvable model allows us to trace back to a one-dimensional integral problem (the charge equation), whose discussion (although completely different) then plays the role of Strichartz estimates in the proof of local and global well-posedness for standard NLSE.

**Remark 2.7.** Notice that even if the fractional Laplacian can exhibit very different behavior with respect to the integer one in the stationary case (see, for example, [21]), this is not the case in our evolution problem. For instance, $U_s$ shares some dispersive properties with the free propagator $e^{it\Delta}$, namely a $L^1 \to L^\infty$ estimate with a different exponent. Moreover, if we look at equation (19), whose analysis is at the heart of this paper, for $s = 1$ it is still a Volterra integral equation with a different Abel kernel with similar qualitative behavior.

### 3. Proof of the main results

This section is devoted to the proof of theorems 2.1 and 2.2. The first four subsections discuss the four items of theorem 2.1, while the last one completely focuses on theorem 2.2.

#### 3.1. Local well-posedness

In order to prove the local well-posedness of (16), a detailed analysis of the charge equation (19) is required. However, we need some preliminary results concerning fractional Sobolev spaces in $d = 1$ and the regularizing properties of the $\frac{1}{2}$-Abel integral kernel $t^{-\frac{1}{2}}$.

**Lemma 3.1.** Let $s \in (\frac{1}{2}, 1]$ and $\ell \in H^\mu(\mathbb{R})$, with $\mu \geq 0$ and $\text{supp}\{\ell\} \subset [0, r)$, for some $r \in \mathbb{R}^+$. Then, the function

$$
L(t) := \int_0^t \frac{\ell(\tau)}{(t - \tau)^{\frac{1}{2}}} \, d\tau, \quad t \in \mathbb{R},
$$

(25)

belongs to $L^2_{\text{loc}}(\mathbb{R}) \cap \dot{H}^{\mu+1-\frac{1}{2}}(\mathbb{R})$. 

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Proof. Clearly, 
\[ \mathcal{L}(t) = (h \ast \ell)(t), \quad \forall t \in \mathbb{R}, \quad \text{where} \quad h(t) := \frac{H(t)}{t^{s}} \]  \tag{26}
and \( H \) is the Heaviside function. In addition, for every \( t \in [0, r] \), \( \mathcal{L}(t) = \mathcal{L}_{r}(t) \), where 
\[ \mathcal{L}_{r}(t) := (h_{r} \ast \ell)(t), \quad \forall t \in \mathbb{R}, \quad \text{with} \quad h_{r}(t) := \frac{H(t) - H(t - r)}{t^{s}}. \]
As a consequence, simply recalling basic properties of the convolution product 
\[ \| \mathcal{L} \|_{L^{2}[0, r]} \leq \| \mathcal{L}_{r} \|_{L^{2}[\mathbb{R}]} \leq \| h_{r} \|_{L^{r}[\mathbb{R}]} \| \ell \|_{L^{r}[\mathbb{R}]} = r^{1 - \frac{1}{s}} \| \ell \|_{L^{r}(0, r)} < \infty. \]
On the other hand, one can easily see that \( \| \mathcal{L} \|_{L^{2}[a, b]} \) is finite, for every \( a, b \in \mathbb{R} \), since \( \mathcal{L}(t) = 0 \), for all \( t \leq 0 \), and since, if \( b > r \), then \( \| \mathcal{L} \|_{L^{2}(0, b)} \) can be estimated arguing as before (simply replacing \( r \) with \( b \)).
Therefore, it is left to prove that \( \mathcal{L} \in \dot{H}^{\mu + 1 - \frac{1}{s}}(\mathbb{R}) \). From (26) and [27, equations (3.761.4) and (3.761.9)] we see that 
\[ \hat{\mathcal{L}}(k) = \sqrt{2\pi} \hat{h}(k)\hat{\ell}(k) \]
where 
\[ \hat{h}(k) = \frac{d(s)}{k^{1 - s}}, \quad d(s) := \frac{e^{-\frac{i\pi}{2}(1 - \frac{s}{2})}}{\sqrt{2\pi}} \Gamma\left(1 - \frac{1}{2s}\right). \]
Hence 
\[ \int_{\mathbb{R}} |k|^{\mu + 1 - \frac{1}{s}} |\hat{\mathcal{L}}(k)|^{2} \, dk \leq C_{r} \int_{\mathbb{R}} |k|^{\mu} |\hat{\ell}(k)|^{2} \, dk < \infty, \]
which concludes the proof. \( \square \)

Lemma 3.2. Let \( s \in \left(\frac{1}{2}, 1\right) \) and \( \ell \in H^{\mu}(0, r) \), with \( \mu \geq 0 \) and \( r \in \mathbb{R}^{+} \). Then:
(i) if \( \mu \in \left[0, \frac{1}{2}\right) \), then \( \mathcal{L} \in H^{\mu + 1 - \frac{1}{s}}(0, r) \);
(ii) if \( \mu \in \left(\frac{1}{2}, \frac{1}{s}\right) \) and \( \ell(0) = 0 \), then \( \mathcal{L} \in H^{\mu + 1 - \frac{1}{s}}(0, r) \);
(with \( \mathcal{L} \) defined by (25)). In particular, 
\[ \| \mathcal{L} \|_{H^{\mu + 1 - \frac{1}{s}}(0, r)} \leq C_{r} \| \ell \|_{H^{\mu}(0, r)}. \]

Remark 3.1. Note that in lemma 3.2, the regularity gain is provided by the \( \frac{1}{2s} \)-Abel kernel 
\( 1 - \frac{1}{s} \in \left(\frac{1}{2}, 1\right) \), for all \( s \in \left(\frac{1}{2}, 1\right) \), and is independent of \( \mu \). These two features are at the core of the bootstrap argument used in the proof of proposition 3.2.
Proof of lemma 3.2. Let

\[ \tilde{\ell}(t) := \begin{cases} \ell(t), & \text{if } t \in [0, r], \\ \ell(2r - t), & \text{if } t \in (r, 2r], \\ 0, & \text{if } t \in \mathbb{R} \setminus [0, 2r]. \end{cases} \]

Easy computations yield

\[ \|\tilde{\ell}\|_{H^\mu([0, 2r])} \leq 2\|\ell\|_{H^\mu([0, r])}. \]

and, on the other hand, [12, lemma 2.1] implies

\[ \|\tilde{\ell}\|_{H^\mu(\mathbb{R})} \leq C\|\ell\|_{H^\mu([0, 2r])}. \]

As a consequence, \( \tilde{\ell} \) satisfies the assumptions of lemma 3.1 (with support in \([0, 2r]\), instead of \([0, r]\)), so that

\[ \tilde{L}(t) := \int_0^t \tilde{\ell}(\tau) \, d\tau \in L^2_{\text{loc}}(\mathbb{R}) \cap H^{\mu+1 - \frac{s}{2}}(\mathbb{R}). \]

Finally, observing that \( \tilde{L}(t) = L(t) \), for all \( t \in [0, r] \), concludes the proof. \( \square \)

Remark 3.2. In lemma 3.2, \( \mu = \frac{1}{2} \) is excluded, since, in this case, [12, lemma 2.1] is not valid due to the failure of the Hardy inequality (see also [33]).

Lemma 3.3. Let \( \ell \in H^\mu(0, r) \cap C^0[0, r] \), with \( r \in \mathbb{R}^+ \) and \( \mu \in [0, 1] \). Then, for every \( \sigma \geq 0 \), \(|\ell|^{2\sigma} \ell \in H^\mu(0, r)\).

Proof. The cases \( \mu = 0, 1 \) are trivial (recalling that \( H^0 = L^2 \)). Then, consider the case \( \mu \in (0, 1) \). Clearly, one sees that

\[ |\ell(t)|^{2\sigma} \ell(t) - |\ell(\tau)|^{2\sigma} \ell(\tau) | \leq C\|\ell\|_{C^0[0, r]}\|\ell(t) - \ell(\tau)\|, \quad \forall t, \tau \in [0, r], \]

which immediately concludes the proof. \( \square \)

Now, we can focus on the charge equation. The first step is proving that it admits a unique continuous solution on a sufficiently small interval.

Proposition 3.1. Let \( \sigma \in \left( \frac{1}{2}, 1 \right) \) and \( \psi_0 \in D(H^\sigma_n) \). Then, there exists \( T \in \mathbb{R}^+ \) such that (19) has a unique solution \( q \in C^0[0, T] \).

Proof. Define, preliminarily,

\[ h(t, \tau, q) := u_0(s) \beta \frac{|q|^{2\sigma} q}{(t - \tau)^{\frac{s}{2}}}. \]

By [42, corollary 2.7], in order to reach a conclusion, it is sufficient to prove that:

(i) \( f \) is continuous on \([0, \infty)\);

(ii) for every \( t \in \mathbb{R}^+ \) and every bounded set \( B \subset \mathbb{C} \), there exists a measurable function \( m(t, \tau) \) such that
\[ |h(t, \tau, q)| \leq m(t, \tau) \quad \forall 0 \leq \tau \leq t \leq \bar{t}, \quad \forall q \in B, \]
\[
\sup_{t \in [0, \bar{t}]} \int_0^t m(t, \tau) \, d\tau < \infty \quad \text{and} \quad \int_0^t m(t, \tau) \, d\tau \rightarrow 0; \]

(iii) for every compact interval \( I \subset [0, \infty) \), every continuous function \( \varphi : I \rightarrow \mathbb{C} \) and every \( t_0 \in \mathbb{R}^+ \)
\[
\lim_{t \rightarrow t_0} \int_I \left( h(t, \tau, \varphi(\tau)) - h(t_0, \tau, \varphi(\tau)) \right) \, d\tau = 0; \]

(iv) for every \( \tilde{t} \in \mathbb{R}^+ \) and every bounded set \( B \subset \mathbb{C} \), a measurable function \( n(t, \tau) \) exists such that
\[
|h(t, \tau, q_1) - h(t, \tau, q_2)| \leq n(t, \tau)|q_1 - q_2| \quad \forall 0 \leq \tau \leq t \leq \tilde{t}, \quad \forall q_1, q_2 \in B, \]
\[
n(t, \cdot) \in L^1(0, t), \quad \forall t \in [0, \tilde{t}], \quad \text{and} \quad \int_{t-\varepsilon}^{t+\varepsilon} n(t + \varepsilon, \tau) \, d\tau \rightarrow 0. \]

However, (ii)–(iv) can be easily proved, setting \( m, n \) equal to the \( \frac{1}{\pi^2} \)-Abel kernel, up to some suitable multiplicative constants, and exploiting its integrability properties (see, for instance, [3, 5, 14]).

Hence, it only remains to show (i). Preliminarily, note that, as \( \psi_0 \in \mathcal{D}(\mathcal{H}_s^0) \) (and recalling that \( \psi_0(0) = q(0) \)),
\[
f(t) = (\mathcal{U}(t)\phi_{\lambda,0})(0) - \beta|q(0)|^2 q(0)(\mathcal{U}(t)G_{\lambda}^1)(0) =: f_1(t) + f_2(t). \quad (27)\]

Let us discuss the two terms separately. First, simply using Cauchy–Schwarz inequality, we find that for all \( T \in \mathbb{R}^+ \)
\[
\|f_1\|_{L^2(0,T)}^2 \leq C_1 \int_0^T \left( \int_{\mathbb{R}} \left| \hat{\phi}_{\lambda,0}(k) \right|^2 \, dk \right) \, dt \leq C_1 \int_0^T \left( \int_{\mathbb{R}} \left( 1 + |k|^2 \right)^2 \left| \hat{\phi}_{\lambda,0}(k) \right|^2 \, dk \right) \left( \int_{\mathbb{R}} \frac{1}{\left| k \right|^2 + 1} \, dk \right) \, dt \leq C_1 T \|\phi_{\lambda,0}\|_{\mathcal{H}_s^0}^2 < \infty. \quad (28)\]

On the other hand, denoting by \( \hat{\phi}_p \) the even part of \( \hat{\phi}_{\lambda,0} \), and setting \( \omega = k^{2s} \), with some computations, one obtains
\[
f_1(t) = \frac{1}{\pi} \int_0^\infty e^{-ik^2 \tilde{\tau}} \hat{\phi}_p(k) \, dk = \frac{1}{2\pi s} \int_{\mathbb{R}} e^{-i\omega \tilde{\tau}} H(\omega) \left[ \frac{1}{\sqrt{2\pi}} \hat{\phi}_p(|\omega|) \right] \frac{d\omega}{|\omega|} = \frac{\hat{G}(t)}{s\sqrt{2\pi}} = \frac{\hat{G}(-t)}{s\sqrt{2\pi}} \quad (29)\]

(where again \( H \) denotes the Heaviside function), that is
\[
\tilde{f}_1(\omega) = \frac{G(-\omega)}{s\sqrt{2\pi}}. \]

As a consequence (using the same change of variable as before)

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\[
[f_t]_H^{1-rac{1}{4}} := \frac{1}{2} |\phi_{f}(\omega)|^2 \leq C_s \int_0^\infty \frac{\omega^{\frac{1}{2}} |\tilde{\phi}(\omega)|^2}{\omega^2} \, d\omega
\]
\[
\leq C_s \int_0^\infty \omega^k |\tilde{\phi}(\omega)|^2 \, d\omega \leq C_s \int_0^\infty |\tilde{\phi}(\omega)|^2 \, d\omega < \infty, 
\]
since \( \phi_{\lambda,0} \in H^{2\lambda}(\mathbb{R}) \) by assumption, so that, combining with (28) results in \( f_t \in H^{1-rac{1}{4}}(0, T) \) for all \( T \in \mathbb{R}^+ \).

Let us now consider \( f_2 \). In particular, we focus on
\[
f_2(t) := -\frac{f_2(t)}{|q(0)|^{2s}q(0)}. 
\]
As \( G_s^\lambda \in H^s(\mathbb{R}) \), arguing as in (28), one sees that \( f_2 \in L^2_{\text{loc}}([0, \infty)) \). On the other hand, arguing as in (29) results in
\[
f_2(t) = C_s \int_\mathbb{R} e^{-i\omega t} H(\omega) \frac{\omega^{\frac{1}{2}}}{|\omega|^2 + \lambda} \, d\omega = C_s \hat{f}(-t), 
\]
so that
\[
[f_2]^2_{\text{loc}}(\mathbb{R}) = C_s \int_0^\infty \frac{\omega(\mu - 1) + \frac{1}{2}}{(\omega + \lambda)^2} \, d\omega < \infty, \quad \forall \mu \in [0, \frac{3}{2} - \frac{1}{2s}].
\]
Hence, as \( \frac{3}{2} - \frac{1}{2s} \in \left(\frac{1}{2}, 1\right) \) for \( s \in \left(\frac{1}{2}, 1\right) \), this implies that \( f_2 \) is continuous on \([0, \infty)\), which concludes the proof.

**Remark 3.3.** Equation (32) also shows that \( f_2 \in L^2(\mathbb{R}) \), not only in \( L^2_{\text{loc}}([0, \infty)) \).

The last step of the discussion of the charge equation is proving a suitable Sobolev regularity for the solution \( q \). To this aim, it is also convenient to define the maximal existence interval \([0, T^*)\) for (19), i.e.
\[
T^* := \sup\{T > 0 : \text{there exists a unique solution } q \in C^0[0, T] \text{ of (19)}\}. 
\]

**Proposition 3.2.** Let \( s \in \left(\frac{1}{2}, 1\right) \) and \( \psi_0 \in D(H^s) \). Then, the solution \( q \) of (19), provided by proposition 3.1, belongs to \( H^{1-rac{1}{4}} \) for every \( T \in (0, T^*) \).

**Remark 3.4.** Observe that, as \( s \in \left(\frac{1}{2}, 1\right) \), then \( \frac{3}{2} - \frac{1}{2s} \in (1, \frac{3}{2}) \), so that (in particular) the solution of the charge equation is absolutely continuous on all closed and bounded subintervals of \([0, T^*)\). This justifies any integration of the derivative of the charge present throughout the paper.

**Proof of proposition 3.2.** Fix an arbitrary \( T \in (0, T^*) \). The proof can be divided into two parts.
Part(i): regularity of the forcing term. The first point is to show that \( f \in H^{\frac{1}{2} - \frac{1}{s}}(0, T) \). However, as we already proved in (28)–(30), \( f_1(t) := (\mathcal{U}_t(t) \phi_3(t)) \) belongs to \( H^{\frac{1}{2} - \frac{1}{s}}(0, T) \) according to the decomposition of \( f \) pointed out in (27), so we focus on \( f_2 \).

In (33), we proved that \( f_3 \), and hence \( f_2 \) (see (31)), is in \( H^\mu(0, T) \) for all \( \mu \in [0, \frac{3}{2} - \frac{1}{s}] \), which is not sufficient to our purposes. However, by suitably manipulating \( f_3 \), one can find an equivalent formulation of (19), which presents a forcing term with the proper regularity.

First, letting

\[
\beta_1(t, \phi_3) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-|\rho|^{2s}} - 1}{|\rho|^{2s}} \, d\rho,
\]

we can define

\[
\tilde{f}_3(t) := f_3(t) - f_3(0) - t^{\frac{3}{2} - \frac{1}{s}} \beta_1(s).
\]

Using the change of variable \( \omega = k^{2s} \) and the fundamental theorem of calculus, one finds that

\[
\tilde{f}_3(t) = -\frac{\lambda}{\pi} \int_{0}^{\infty} \frac{e^{-ik^2\tau} - 1}{k^{2s}(k^{2s} + \lambda)} \, dk = -\frac{\lambda}{\pi} \int_{0}^{\infty} \frac{(e^{-i\omega s} - 1)\omega^{\frac{s}{2}}}{\omega^2(\omega + \lambda)} \, d\omega = \frac{\lambda}{\pi} \int_{0}^{\infty} \frac{\omega^{\frac{s}{2}}}{\omega^2(\omega + \lambda)} \int_{0}^{\infty} e^{-i\omega \mu} \, d\omega \, d\omega.
\]

As a consequence

\[
\tilde{f}_3(t) = \frac{\lambda}{\pi} \int_{0}^{\infty} e^{-i\omega \mu} H(\omega) \frac{|\omega|^{\frac{s}{2}}}{|\omega|(|\omega| + \lambda)} \, d\omega
\]

and hence, arguing as in (32) and (33) (in view of remark 3.3), one has that \( \tilde{f}_3 \in H^\nu(\mathbb{R}) \) for all \( \nu \in [0, \frac{3}{2} - \frac{1}{s}] \). Then, \( \tilde{f}_3 \in H^\nu(\mathbb{R}) \) for all \( \mu \in [0, \frac{3}{2} - \frac{1}{s}] \), which clearly implies \( \tilde{f}_3 \in H^{\frac{1}{2} - \frac{1}{s}}(\mathbb{R}) \).

Summing up, (observing that \( f_3(0) = G_3(0) \))

\[
f_2(t) = -\beta(q(0))^{2s} q(0) G_3(0) - \beta|q(0)|^{2s} q(0)b(s) t^{\frac{3}{2} - \frac{1}{s}} - \beta|q(0)|^{2s} q(0) \tilde{f}_3(t).
\]

In addition, we find that

\[
t^{\frac{3}{2} - \frac{1}{s}} = \frac{2s}{2s} \int_{0}^{t} \frac{1}{(t - \tau)^{\frac{s}{2}}} \, d\tau
\]

and that (recalling (20))

\[
b(s) = \frac{-1}{\pi} \int_{0}^{\infty} \int_{0}^{1} e^{-ik^2\tau} \, dk = \frac{-1}{\pi} \int_{0}^{1} \frac{e^{\frac{1}{2} - \frac{s}{2}} \int_{0}^{\infty} e^{-i\omega \mu} \, d\omega \, d\omega}{2s - 1}.
\]
Consequently, one can suitably rearrange the terms in (19) in order to obtain
\[q(t) = \tilde{f}(t) - w(a(s)β) \int_0^t |q(\tau)|^{2\sigma} q(\tau) - |q(0)|^{2\sigma} q(0) \, d\tau,\]
where now
\[
\tilde{f}(t) = f_1(t) - \beta|q(0)|^{2\sigma} q(0)q^\lambda(0) - \beta|q(0)|^{2\sigma} q(0)f_2(t)
\]
belongs to \(H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T)\).

Part (ii): bootstrap argument. Now, we can apply a bootstrap argument on (35) in order to obtain that \(q \in H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T)\).

We know that the unique solution of \(q\) in (35) belongs to \(L^2(0, T) \cap C^0[0, T]\). Hence, lemma 3.3 results in \(|q(\cdot)|^{2\sigma} q(\cdot) - |q(0)|^{2\sigma} q(0) \in L^2(0, T) \cap C^0[0, T]\) and (consequently), from lemma 3.2 (item (i)), \(q \in H^{1+\frac{1}{\sigma}} (0, T) \cap C^0[0, T]\), so that \(q \in H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T) \cap C^0[0, T]\)

Repeating the same argument, one can easily prove, with an iterative process, that \(q \in H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T) \cap C^0[0, T]\), which concludes the proof.

However, some provisos are required:

(1) one uses item (i) of lemma 3.2 until the starting index of the iteration is smaller than \(\frac{1}{\sigma}\) and item (ii) when the starting index becomes greater than \(\frac{1}{\sigma}\);
(2) if at some iteration one runs into \(H^{\frac{3}{2}} (0, T)\), which is not covered by lemma 3.2, then it is sufficient to observe that \(q \in H^{\mu}(0, T)\), with \(\mu = \frac{1}{2} - \varepsilon\) and (for instance) \(\varepsilon < \frac{1}{2} - \frac{1}{2\sigma}\), so that one can use lemma 3.2 to leap over \(\frac{1}{\sigma}\) (since \(\mu + 1 - \frac{1}{2\sigma} > \frac{1}{2}\)) and move on;

(3) if at some iteration one finds \(q \in H^\mu(0, T)\) with \(\mu \in (1, \frac{3}{2} - \frac{1}{2\sigma})\), which is not covered by lemma 3.3, then it is sufficient to observe that \(q \in H^{\frac{1}{2} + \frac{1}{\sigma}} (0, T)\), and since \(\frac{1}{2} + \frac{1}{2\sigma} < 1\) one can again use lemmas 3.3 and 3.2 to obtain that \(q \in H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T)\).

Note also that the iterative process must end in a finite number of steps because the regularity gain at each step is always the same (as highlighted in remark 3.1).

We can now prove the first point of theorem 2.1. It is worth pointing out that the following proof holds for every \(T \in (0, T^*)\).

Proof of theorem 2.1: item (i). From propositions 3.1 and 3.2, there is \(T \in (0, T^*)\) for which there exists a unique solution of (19) belonging to \(H^{\frac{3}{2} - \frac{1}{\sigma}} (0, T)\). As a consequence, the function
\[
r(t) := \beta|q(t)|^{2\sigma} q(t)
\]
belongs to \(H^1(0, T)\) (by lemma 3.3, observing that \(\frac{1}{2} - \frac{1}{2\sigma} > 1\)). Then, we can split the proof into three parts.

Part (1): regularity of \(\psi\). Using the Fourier transform in (21) and recalling definition (15), we have
\[
\hat{\psi}(t, k) = e^{-i |k|^{2r}} \hat{\phi}_\lambda(0) - \frac{r(0) e^{-i |k|^{2r}}}{\sqrt{2\pi (|k|^{2r} + \lambda)}} - \frac{t}{\sqrt{2\pi}} \int_0^t e^{-i |k|^{2r} (t-\tau)} r(\tau) \, d\tau.
\] (37)

In addition, an integration by parts yields
\[
\hat{\psi}(t, k) = e^{-i |k|^{2r}} \left\{ \hat{\phi}_\lambda(0) + \frac{1}{\sqrt{2\pi (|k|^{2r} + \lambda)}} \int_0^t e^{i |k|^{2r} \tau} (\hat{r}(\tau) - i \lambda r(\tau)) \, d\tau \right\} - \frac{r(t)}{\sqrt{2\pi (|k|^{2r} + \lambda)}},
\] (38)

so that
\[
\psi(t, x) = \phi_\lambda(t, x) - r(t) G^\lambda_t(x).
\]

Hence, in order to prove that \( \psi(t, \cdot) \in \mathcal{D}(\mathcal{H}_r^n) \), for all \( t > 0 \) it suffices to prove that \( \hat{\phi}_\lambda(t, \cdot) \in L^2(\mathbb{R}, |k|^{4r} \, dk) \) (since it is clearly in \( L^2(\mathbb{R}) \)). First, we can easily see that \( e^{-i |k|^{2r}} \hat{\phi}_\lambda(0) \in L^2(\mathbb{R}, |k|^{4r} \, dk) \) by the properties of the free propagator. Concerning the remaining part, as \( r \) is trivially more regular than \( \hat{r} \), it is sufficient to show that
\[
\frac{1}{|k|^{2r} + \lambda} \int_0^t e^{i |k|^{2r} \omega \tau} \hat{r}(\tau) \, d\tau \in L^2(\mathbb{R}, |k|^{4r} \, dk),
\]

namely, that
\[
g(t, k) := \int_0^t e^{i |k|^{2r} \omega \tau} \hat{r}(\tau) \, d\tau \in L^2(\mathbb{R})
\]

(as functions of \( k \)). Setting \( \rho_t(\tau) = \hat{r}(\tau) \chi_{|\omega|}(\tau) \) and observing that \( \hat{\rho}_t \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) (since \( \rho_t \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \)), one obtains
\[
\int_{\mathbb{R}} |g(t, k)|^2 \, dk = 2 \int_0^\infty \left| \int_0^t e^{ik|\omega|\tau} \hat{r}(\tau) \, d\tau \right|^2 \, dk = 2 \int_0^\infty \frac{1}{\omega^{1-2r}} \left| \int_0^t e^{i\omega\tau} \hat{r}(\tau) \, d\tau \right|^2 \, d\omega
\]
\[
= 2 \int_0^\infty \frac{\hat{\rho}_t(-\omega)^2}{\omega^{1-2r}} \, d\omega \leq C(1 + \sqrt{t})(1 + \|\hat{r}\|_{L^2})
\]

(as \( 1 - \frac{1}{2r} \in (0, \frac{1}{2}) \)), which proves the claim. Furthermore, one can prove with analogous computations that
\[
\phi_\lambda \in C([0, T]; H^{2r}(\mathbb{R}))
\] (39)

(for details, see [12]). However, if we endow the domain \( \mathcal{D}(\mathcal{H}_r^n) \) with the graph norm \( \| \cdot \|_{\mathcal{D}(\mathcal{H}_r^n)} \), then
\[
\| \psi \|_{\mathcal{D}(\mathcal{H}_r^n)} := \| \psi \| + \| \mathcal{H}_r^n \psi \| \leq C \left( \| \psi \| + \| (\mathcal{H}_0^n + \lambda) \phi_\lambda \| + |r| \right) \leq C \left( \| \phi_\lambda \|_{H^r(\mathbb{R})} + |q|^{2r+1} \right)
\]
and thus, combining with (39),
\[
\psi \in C([0, T]; \mathcal{D}(\mathcal{H}_r^n)).
\]
Part (2): regularity of $\frac{\partial \hat{\psi}}{\partial t}$. Let us compute, then, the time derivative of $\hat{\psi}$. We have that
\[
\frac{\partial \hat{\psi}}{\partial t}(t,k) = \frac{\partial \hat{\phi}_\lambda}{\partial t}(t,k) - \dot{r}(t)\hat{G}_\lambda^s.
\]
In addition,
\[
\frac{\partial \hat{\phi}_\lambda}{\partial t}(t,k) = -i|k|^{2\nu}\hat{\phi}_\lambda(t,k) + \frac{\dot{r}(t) - i\lambda r(t)}{\sqrt{2\pi(|k|^{2\nu} + \lambda)}}.
\]
so that
\[
\frac{\partial \hat{\psi}}{\partial t}(t,k) = -i|k|^{2\nu}\hat{\phi}_\lambda(t,k) - \frac{i\lambda r(t)}{\sqrt{2\pi(|k|^{2\nu} + \lambda)}}.
\]
(40)
As a consequence, arguing as in part (1), one can easily see that
\[
\hat{\psi} \in C^1([0,T]; L^2(\mathbb{R})),
\]
which then proves that $\psi \in C^1([0,T]; L^2(\mathbb{R}))$.

Part (3): solution of (16). Now, since the initial condition is clearly satisfied by construction, we have to prove that
\[
{i\frac{\partial \hat{\psi}}{\partial t} = \mathcal{H}_0^n \psi \quad \text{in} \quad L^2(\mathbb{R}), \quad \forall t \in [0,T].}
\]
(41)
By (40),
\[
{i\frac{\partial \hat{\psi}}{\partial t}(t,k) = |k|^{2\nu}\hat{\phi}_\lambda(t,k) + \lambda r(t)\hat{G}_\lambda^s(k).}
\]
On the other hand,
\[
\mathcal{H}_0^n \psi(t,k) = (\mathcal{H}_0^n + \lambda)\phi_\lambda(t,k) - \lambda \hat{\psi}(t,k) = |k|^{2\nu}\phi_\lambda(t,k) + \lambda(\phi_\lambda(t,k) - \hat{\psi}(t,k)),
\]
which concludes the proof. \(\square\)

3.2. Conservation laws

This section is devoted to the proofs of the conservation laws associated with (16). In particular, for the mass conservation, we will prove that
\[
\frac{dM^2}{dt}(t) = 0, \quad \forall t \in [0,T'),
\]
whereas we will show the energy conservation by a direct inspection of the equality
\[
E(t) = E(0), \quad \forall t \in [0,T').
\]
Proof of theorem 2.1: item (ii). The proof can be divided into two parts.

Part (1): mass conservation. First, we observe that
\[
\frac{dM^2}{dt}(t) = 2 \text{Re} \left\{ \int_R \overline{\psi(t,x)} \frac{\partial \psi}{\partial t}(t,x) \, dx \right\},
\]
so we have to prove that \( A \) is purely imaginary.

First, from \((41)\)
\[
\frac{\partial \psi}{\partial t}(t,x) = -i H'_1 \phi_\lambda(t,x) + i r(t) G_\lambda(x)
\]
(with \( r \) defined by \((36)\)). As a consequence
\[
A = i \left( (\phi_\lambda(t,\cdot), H'_1 \phi_\lambda(t,\cdot)) - \lambda |r(t)|^2 \| G_\lambda \|^2 \right) + i \left( \overline{r(t)}(G_\lambda^\lambda, \phi_\lambda(t,\cdot)) + r(t)(\phi_\lambda(t,\cdot), G_\lambda^\lambda) \right),
\]
where we easily see that, since \( H'_1 \) is self-adjoint, \( A_1 \) is real-valued. Now, a computation shows that
\[
\lambda A_2 = i \left( \overline{r(t)}(G_\lambda^\lambda, (H'_1 + \lambda) \phi_\lambda(t,\cdot)) - 2 \lambda \text{Re} \left\{ \overline{r(t)}(G_\lambda^\lambda, \phi_\lambda(t,\cdot)) \right\} \right).
\]

Finally, as \( B_2 \) is clearly real-valued, it remains to prove that \( B_1 \) is real-valued too. Exploiting the definition of the Green’s function \((6)\) (which is also a real-valued function), the decomposition of the domain \( D(H'_1) \) and \((36)\), we have
\[
B_1 = \overline{r(t)}((H'_1 + \lambda) G_\lambda^\lambda, \phi_\lambda(t,\cdot)) = \overline{r(t)} \phi_\lambda(t,0) = |q(t)|^{2\sigma+2} + |r(t)|^2 G_\lambda^\lambda(0),
\]
that is in fact real-valued, which concludes the proof.

Part (2): energy conservation. From \((5)\) and \((22)\), the energy at time \( t \) reads
\[
E(t) = \int_R |k|^2 |\tilde{\psi}(t,k)|^2 \, dk + \frac{\beta}{\sigma + 1} |q(t)|^{2\sigma+2}.
\]

Using the representation of \( \tilde{\psi}(t,k) \) given by \((37)\), the kinetic part of the energy turns out to be
\[
\int_R |k|^2 |\tilde{\psi}(t,k)|^2 \, dk = \int_R |k|^2 |\tilde{\psi}_0(t,k)|^2 - \frac{i}{\sqrt{2\pi}} \int_0^t \int_R |k|^2 \overline{\tilde{\psi}_0(k)} \left( e^{i |k|^2 \tau} r(\tau) \right) d\tau \, dk
\]
\[
= \int_R |k|^2 |\tilde{\psi}_0(t,k)|^2 \, dk + 2 \text{Re} \left\{ \frac{i}{\sqrt{2\pi}} \int_R |k|^2 \overline{\tilde{\psi}_0(k)} \left( \int_0^t e^{-i |k|^2 \tau} r(\tau) \, d\tau \right) \, dk \right\}
\]
\[
+ \frac{i}{2\pi} \int_R |k|^2 \int_0^t e^{-i |k|^2 (\tau_2 - \tau_1)} r(\tau_1) \overline{r(\tau_2)} \, d\tau_2 \, d\tau_1 \, dk.
\]

(42)
An integration by parts yields
\[ E_1 = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{\psi}_0(k) \left( e^{-i|k|^2 r} r(t) - r(0) - \int_0^t e^{-i|k|^2 \tau} \widehat{r(\tau)} \, d\tau \right) \, dk \]
\[ = -\frac{r(t)}{\sqrt{2\pi}} (\mathcal{U}_t(\psi_0)(0) + r(0) (\mathcal{U}_t(0) \psi_0)(0)) + \int_0^t \frac{r(\tau)}{\sqrt{2\pi}} (\mathcal{U}_t(\tau) \psi_0)(0) \, d\tau, \]
and recalling that \((\mathcal{U}_t(0) \psi_0)(0) = \psi_0(0) = q(0),
\[ E_1 = -\frac{r(t)}{\sqrt{2\pi}} (\mathcal{U}_t(\psi_0)(0) + \beta |q(0)|^{2\sigma + 2} + \int_0^t \frac{r(\tau)}{\sqrt{2\pi}} (\mathcal{U}_t(\tau) \psi_0)(0) \, d\tau. \] (43)

On the other hand, observing that
\[ \int_0^t \int_0^s e^{i|k|^2 (t_1 - t_2)} r(t_1) \widehat{r(t_2)} \, dt_1 \, dt_2 = 2 \text{Re} \left\{ \int_0^t \int_0^s e^{i|k|^2 (t_1 - t_2)} r(t_1) \widehat{r(t_2)} \, dt_1 \, dt_2 \right\}, \]
one has
\[ E_2 = \frac{1}{\pi} \text{Re} \left\{ \int_{\mathbb{R}} |k|^2 \int_0^t e^{i|k|^2 t_1} r(t_1) \int_0^t e^{-i|k|^2 t_2} \widehat{r(t_2)} \, dt_1 \, dt_2 \, dk \right\}. \]

Now, another integration by parts shows that
\[ \int_0^t e^{i|k|^2 t} e^{-i|k|^2 \tau} r(t_1) \int_0^t \frac{e^{i|k|^2 \tau} \widehat{r(t_2)}}{\sqrt{2\pi}} \, dt_2 \, dt_1 \]
\[ = e^{i|k|^2 t} r(t) \int_0^t e^{-i|k|^2 \tau} \frac{\widehat{r(t)}}{\sqrt{2\pi}} \, d\tau - \int_0^t e^{i|k|^2 \tau} \frac{\widehat{r(t)}}{\sqrt{2\pi}} \, d\tau - \int_0^t \frac{r(\tau)}{\sqrt{2\pi}} \, d\tau, \]
so that (with some computations)
\[ E_2 = 2 \text{Re} \left\{ \frac{r(t)}{\sqrt{2\pi}} \int_0^t \mathcal{U}_t(t - \tau, 0) r(\tau) \, d\tau + i \int_0^t \frac{\dot{r}(\tau)}{\sqrt{2\pi}} \int_0^t \mathcal{U}_t(t_2 - \tau_1, 0) \frac{\dot{\widehat{r}(t_2)}}{\sqrt{2\pi}} \, dt_2 \, dt_1 \right\}. \] (44)

Plugging (43) and (44) in (42), we get
\[ \int_{\mathbb{R}} |k|^2 |\hat{\psi}(t, k)|^2 \, dk = \int_{\mathbb{R}} |k|^2 |\hat{\psi}(0, k)|^2 \, dk + 2\beta |q(0)|^{2\sigma + 2} - 2 \text{Re} \left\{ \frac{r(t)}{\sqrt{2\pi}} (\mathcal{U}_t(\psi_0)(0) \, d\tau \right\}
\[ + 2 \text{Re} \left\{ \int_0^t \frac{r(\tau)}{\sqrt{2\pi}} (\mathcal{U}_t(\tau) \psi_0)(0) \, d\tau \right\} + 2 \text{Re} \left\{ \frac{r(t)}{\sqrt{2\pi}} \int_0^t \mathcal{U}_t(t - \tau, 0) r(\tau) \, d\tau \right\}
\[ + 2 \text{Re} \left\{ i \int_0^t \frac{\dot{r}(\tau)}{\sqrt{2\pi}} \int_0^t \mathcal{U}_t(t_2 - \tau_1, 0) \frac{\dot{\widehat{r}(t_2)}}{\sqrt{2\pi}} \, dt_2 \, dt_1 \right\}. \]
Consequently, (18) and (21) results in
\[ \frac{r(t)}{\sqrt{2\pi}} \left( i \int_0^t \mathcal{U}_t(t - \tau, 0) r(\tau) - (\mathcal{U}_t(t) \psi_0)(0) \right) = -\frac{r(t)}{\sqrt{2\pi}} q(t) = -\beta |q(t)|^{2\sigma + 2} \]
and
\[ \int_0^t r(\tau) (\mathcal{U}_t(\tau) \psi_0)(0) \, d\tau = \int_0^t r(\tau) q(\tau) \, d\tau + \left( -i \int_0^t \frac{\dot{r}(\tau)}{\sqrt{2\pi}} \int_0^t \mathcal{U}_t(\eta - \tau, 0) r(\eta) \, d\eta \, d\tau \right). \]
and thus (with a further integration by parts)

\[
\begin{align*}
\int_{\mathbb{R}} |k|^{2s} |\widehat{\psi}(t,k)|^2 \, dk &= \int_{\mathbb{R}} |k|^{2s} |\widehat{\psi}_0(k)|^2 \, dk + 2\beta (|q(0)|^{2\sigma+2} - |q(t)|^{2\sigma+2}) \\
&\quad + 2\text{Re} \left\{ \int_0^t \overline{r(\tau)} q(\tau) \, d\tau \right\} = -2 \int_0^t \text{Re} \left\{ \overline{r(\tau)} q(\tau) \right\} \, d\tau.
\end{align*}
\]

Finally, as

\[
\frac{\beta}{\sigma + 1} \frac{d}{dt} |q(t)|^{2\sigma+2} = 2\beta |q(t)|^{2\sigma} \text{Re} \left\{ \overline{q(t)} \dot{q}(t) \right\} = 2\text{Re} \left\{ \overline{r(t)} \dot{r}(t) \right\},
\]

then

\[
\int_{\mathbb{R}} |k|^{2s} |\widehat{\psi}(t,k)|^2 \, dk = \int_{\mathbb{R}} |k|^{2s} |\widehat{\psi}_0(k)|^2 \, dk - \frac{\beta}{\sigma + 1} |q(t)|^{2\sigma+2} + \frac{\beta}{\sigma + 1} |q(0)|^{2\sigma+2},
\]

which (recalling (18)) concludes the proof.

3.3. Global well-posedness

Exploiting conservation laws, and in particular energy conservation, it is possible to prove that the solution of (16) provided by (21)–(19) is global in time, namely

\[ T^* = +\infty \]

(with \( T^* \) defined by (34)), in the defocusing and in the sub-critical/critical focusing cases.

In the focusing case, the main ingredient is a fractional version of the Gagliardo–Nirenberg inequality, namely

\[ \|f\|_{L^\infty(\mathbb{R})} \leq C_s \|f\|^{1-\frac{1}{2s}} \|f\|_{H^s(\mathbb{R})}^{\frac{1}{2}} \]

(45)

(for the proof see, for instance, [9, 25, 43]).

**Proof of theorem 2.1: item (iii).** Consider, first, the defocusing case, i.e. \( \beta > 0 \). From energy conservation, the result is that:

\[ \limsup_{t \to T^*} |q(t)| = C < \infty \]  

(46)

and, by [42, theorem 2.3], this entails that \( T^* = +\infty \).

On the other hand, in the sub-critical focusing case, i.e. \( \beta < 0 \) and \( \sigma < \sigma_c(s) \), from (45) and (18),

\[
E(0) = E(t) \geq \|\psi(t, \cdot)\|_{H^s(\mathbb{R})}^2 + \frac{\beta}{\sigma + 1} C_{s}^{2\sigma+2} \|\psi_0\|^{\frac{(2\sigma+2)(2\sigma+2)}{\sigma}} \|\psi(t, \cdot)\|_{H^s(\mathbb{R})}^{2\sigma+2}.
\]

(47)

Since \( \frac{2\sigma+2}{\sigma} < 2 \) whenever \( \sigma < \sigma_c(s) \), one obtains again (46) and therefore the claim follows the argument as before.

Finally, if \( \sigma = \sigma_c(s) \), then (47) reads...
\[
E(0) = E(t) \geq \left| \psi(t, \cdot) \right|^2_{H^s(\mathbb{R})} \left( 1 - \frac{\beta |C^s_\lambda|}{2s} \|\psi_0\|^2(2s-1) \right)
\]

and hence (46) is satisfied whenever the quantity in brackets is bigger than 0, namely, whenever
\[
\|\psi_0\| < \left( \frac{2s}{|\beta| C^s_\lambda} \right)^{\frac{1}{2s-1}} =: C(s, \beta),
\]
which concludes the proof. \(\square\)

### 3.4. Blow-up solutions

In order to prove the rise of blow-up solutions, we use the classical Glassey method (see, for example, [26]) based on the definition of a moment of inertia and on the proof of the so-called virial identity.

Due to the different scaling properties of the fractional Laplacian, when \(s < 1\), it is necessary to slightly modify the standard definition of the moment of inertia. In particular, we set
\[
I(t) = I(\psi(t, \cdot)) := \|(-\Delta)^{\frac{\beta}{2s}} x\psi(t, \cdot)\|^2 = \|k|^{1-s} \partial_k \hat{\psi}(t, \cdot)\|^2
\]
where \(\psi\) (henceforth) is the solution of (16) provided by (19) and (21) (see, for instance, [8] and the references therein). We notice that the fractional momentum operator defined in the appendix in [20] bears some similarities to (49). It is unclear at the moment if there is a deeper connection between the two objects.

The first point is to prove that \(I\) is well defined on the maximal existence time of \(\hat{\psi}\), i.e. \([0,T]\) (with \(T\) defined by (34)).

**Lemma 3.4.** Let \(s \in \left( \frac{1}{2}, 1 \right]\), \(\psi_0 \in \mathcal{D}(\mathcal{H}^s_0)\) and \(I(\psi_0) < \infty\). Then, for every \(T < T^*\),
\[
I(t) \leq C_T < \infty, \quad \forall t \in [0,T].
\]

**Proof.** Let \(T < T^*\). Recalling that \(\hat{\psi}_0(k) = \hat{\phi}_{\lambda,0} - r(0)/\sqrt{2\pi(|k|^{2s} + \lambda)}\) and differentiating (37) in \(k\),
\[
\|k|^{1-s} \partial_k \hat{\psi}(t, k) = -t2sw|k|^s \text{sgn}(k) \hat{\psi}(t, k) + \frac{2sw|k|^s \text{sgn}(k)}{\sqrt{2\pi}} \int_0^t e^{-i|k|^2(\tau-r)} r(\tau) d\tau =: A_1(t, k)
\]
and differentiating (37) in \(t\),
\[
\|k|^{1-s} \partial_t \hat{\psi}_0(k) = -Aw|k|^s \text{sgn}(k) \hat{\psi}(t, k) + \frac{2sw|k|^s \text{sgn}(k)}{\sqrt{2\pi}} \int_0^t e^{-i|k|^2(\tau-r)} r(\tau) d\tau =: A_2(t, k).
\]

Now, as \(\psi \in C^0([0,T]; \mathcal{D}(\mathcal{H}^s_0)), \psi \in C^0([0,T]; H^s(\mathbb{R}))\) as well, so that
\[
\|A_1(t, \cdot)\|^2 \leq CT \|\psi\|_{C^0([0,T]; H^s(\mathbb{R}))}^2 < \infty.
\]
On the other hand
\[ \| A_3(t, \cdot) \|^2 = I(0) < \infty, \]
by assumption. It is, then, left to discuss \( A_3 \). An integration by parts shows that
\[ A_2(t, k) = \frac{\iota 2s |k|^s \text{sgn}(k)}{\sqrt{2\pi} |k|^{2s} + \lambda} \left( -r_1(t) + \int_0^t e^{-i|k|^2(t-\tau)} (r_1(\tau) - i\lambda r_1(\tau)) \, d\tau \right) \]
with \( r_1(\tau) := \tau r(\tau) \) (which is clearly at least as regular as \( r(\tau) \)). Consequently, arguing as in the proof of theorem 2.1 (in particular, item (i), part 1)), one immediately sees that (50) is satisfied. \( \Box \)

As a second point, we have to prove the fractional virial identity.

**Proposition 3.3.** Let \( s \in (\frac{1}{2}, 1], \psi_0 \in D(D^s_0) \) and \( I(\psi_0) < \infty \). Then, \( I(\cdot) \in C^1[0,T^*) \) and
\[ \dot{I}(t) = 4s \text{Im} \left\{ \int_{\mathbb{R}} k \hat{\psi}(t,k) \frac{\hat{\psi}}{\partial k}(t,k) \, dk \right\}, \quad \forall t \in [0,T^*). \]  
(51)

**Proof.** We start by computing the derivative of the integrand of \( I(t) \) (given by (49)), namely \( A(t,k) := |k|^{2s-2} \partial_k \hat{\psi}(t,k) \). First we note that
\[ \frac{\partial A}{\partial t}(t,k) = 2|k|^{2s-2} \text{Re} \left\{ \frac{\partial \hat{\psi}}{\partial t}(t,k) \frac{\partial^2 \hat{\psi}}{\partial k^2}(t,k) \right\}. \]  
(52)

Now, recalling that
\[ \frac{\partial \hat{\psi}}{\partial k}(t,k) = -\iota 2s |k|^{2s-1} \text{sgn}(k) \hat{\psi}(t,k) \]
\[ \quad + \frac{2s |k|^{2s-1} \text{sgn}(k)}{\sqrt{2\pi}} \int_0^t e^{-i|k|^2(t-\tau)} \tau r(\tau) \, d\tau + e^{-i|k|^2 t} \hat{\psi}_0 \frac{\partial \hat{\psi}_0}{\partial k}(k) \]  
(53)
and, differentiating in \( t \), with some computations, results in
\[ \frac{\partial}{\partial t} \frac{\partial \hat{\psi}}{\partial k}(t,k) = -\iota |k|^{2s-1} \text{sgn}(k) \hat{\psi}(t,k) \]
\[ \quad -\iota |k|^2 \left( e^{-i|k|^2 t} \frac{\partial \hat{\psi}}{\partial k}(k) + \frac{2s |k|^{2s-1} \text{sgn}(k)}{\sqrt{2\pi}} \int_0^t e^{-i|k|^2(t-\tau)} \tau r(\tau) \, d\tau \right) \]
\[ \quad + \frac{2s |k|^{2s-1} \text{sgn}(k) t r(t)}{\sqrt{2\pi}} - \iota 2s |k|^{2s-1} \text{sgn}(k) \frac{\partial \hat{\psi}}{\partial t}(t,k). \]

Then, since by (40)
\[ \frac{\partial \hat{\psi}}{\partial t}(t,k) = -\iota |k|^{2s} \hat{\psi}(t,k) - \frac{t r(t)}{\sqrt{2\pi}}, \]
we find
\[ \frac{\partial}{\partial t} \hat{\psi}(t,k) = -i|k|^{2s-1} \text{sgn}(k) \hat{\psi}(t,k) - i2\pi \left( -i2\pi |k|^{2s-1} \text{sgn}(k) \hat{\psi}(t,k) + \frac{2i|k|^{2s-1} \text{sgn}(k)}{\sqrt{2\pi}} \int_0^\infty e^{-i\|k\|(|\tau| - \tau)} d\tau + e^{-i\|k\|\tau} \right) \]

and, hence, using again (53),

\[ \frac{\partial^2 \hat{\psi}}{\partial t \partial k}(t,k) = -i2\pi |k|^{2s-1} \text{sgn}(k) \hat{\psi}(t,k) - i|k|^{2s} \frac{\partial \hat{\psi}}{\partial k}(t,k). \]

Thus, plugging into (52) results in

\[ \frac{\partial A}{\partial t}(t,k) = 4\pi \text{Im} \left\{ k \frac{\partial \hat{\psi}}{\partial k}(t,k) \right\}. \] (54)

On the other hand, fix an arbitrary \( T < T' \). An integration by parts in (53) shows that

\[ k \frac{\partial \hat{\psi}}{\partial k}(t,k) = -i2\pi |k|^{2s} \text{sgn}(k) \hat{\psi}(t,k) \]

\[ + i2\pi |k|^{2s} \text{sgn}(k) \int_0^\infty e^{-i\|k\|(|\tau| - \tau)} (\hat{r}_1(\tau) - i\lambda r_1(\tau)) d\tau + e^{-i\|k\|\tau} \frac{\partial \hat{\psi}}{\partial k}(t,k), \]

(whence again \( r_1(\cdot) := \tau r(\cdot) \in H^1(0,T) \)). Hence

\[ \left| k \frac{\partial \hat{\psi}}{\partial k}(t,k) \right| \leq C_T \left( |k|^{2s} |\phi_\lambda(t,k)| + 1 + \left| k \frac{\partial \hat{\psi}_0}{\partial k}(k) \right| \right), \quad \forall t \in [0,T]. \]

In addition, since by (38)

\[ |\hat{\psi}(t,k)| \leq C_T \left( |\phi_\lambda(t,k)| + \frac{1}{|k|^{2s} + \lambda} \right), \quad \forall t \in [0,T], \]

results in

\[ \left| \frac{\partial A}{\partial t}(t,k) \right| \leq C_T \left( |k|^{2s} |\phi_\lambda(t,k)|^2 + |\hat{\phi}_\lambda(t,k)|^2 + \frac{1}{|k|^{2s} + \lambda} + \right. \]

\[ + \left| \hat{\phi}_\lambda(t,k) \right| \left| k \frac{\partial \hat{\psi}_0}{\partial k}(k) \right| + \left| \frac{|k|}{|k|^{2s} + \lambda} \right| \left| \frac{\partial \hat{\psi}_0}{\partial k}(k) \right| \right), \quad \forall t \in [0,T]. \]

Now,

\[ |\hat{\phi}_\lambda(t,k)| \leq C_T \left( |\hat{\phi}_{\lambda,0}(k)| + \frac{1}{|k|^{2s} + \lambda} \right), \quad \forall t \in [0,T], \]

results in

\[ \left| \frac{\partial A}{\partial t}(t,k) \right| \leq C_T \left( |k|^{2s} |\hat{\phi}_{\lambda,0}(k)|^2 + |\hat{\phi}_{\lambda,0}(k)| + \frac{1}{|k|^{2s} + \lambda} + \right. \]

\[ + \left| \hat{\phi}_{\lambda,0}(k) \right| \left| k \frac{\partial \hat{\psi}_0}{\partial k}(k) \right| + \left| \frac{|k|}{|k|^{2s} + \lambda} \right| \left| \frac{\partial \hat{\psi}_0}{\partial k}(k) \right| \right), \quad \forall t \in [0,T], \]
which is clearly integrable by the regularity of \( \phi_{\lambda,0} \) and since \( I(\psi_0) < \infty \). Hence, by (54) and dominated convergence, one obtains (51) (and the continuity of \( \dot{I} \) on \([0, T^*)\)).

As a third point, we can compute the second derivative of the moment of inertia.

**Proposition 3.4.** Let \( s \in (\frac{1}{2}, 1] \) and \( \psi_0 \in \mathcal{D}(H^s) \) with \( \phi_{\lambda,0} \in \mathcal{S}(\mathbb{R}) \). Then, \( I(\cdot) \in C^2(0, T^*) \) and

\[
\ddot{I}(t) = 8s^2E(0) + \frac{4s^2\beta(\sigma - \sigma_c(s))}{\sigma + 1}|q(t)|^{2\sigma+2}, \quad \forall t \in [0, T^*).
\]

**(55)**

**Proof.** Fix an arbitrary \( T < T^* \) and focus on the interval \([0, T]\). Preliminarily, we observe that the assumptions entail that \( I(\psi_0) < \infty \) (so that lemma 3.4 and proposition 3.3 are valid). On the other hand, setting

\[
B(t,k) := \text{Im} \left\{ k\hat{\psi}(t,k) \frac{\partial \hat{\psi}}{\partial k}(t,k) \right\},
\]

one sees that

\[
\frac{\partial B}{\partial t}(t,k) = 2s|k|^2|\hat{\psi}(t,k)|^2 - \frac{1}{\sqrt{2\pi}} \text{Re} \left\{ r(t)k\frac{\partial \hat{\psi}}{\partial k}(t,k) \right\}.
\]

**(56)**

Arguing as in the proof of proposition 3.3, for every fixed \( R > 0 \),

\[
\frac{\partial B}{\partial t}(t,k) \leq g_{R,T}(k) \in L^1(-R,R), \quad \forall t \in [0, T],
\]

so that, if we set

\[
\dot{I}_R(t) := 4s \int_{-R}^{R} B(t,k) \, dk,
\]

then by dominated convergence

\[
\ddot{I}_R(t) = 4s \int_{-R}^{R} \frac{\partial B}{\partial t}(t,k) \, dk,
\]

which is, in addition, a continuous function in \([0, T]\). As a consequence, if one can prove that \( \dot{I}_R \) converges pointwise a.e. in \([0, T]\) and that \( \dot{I}_R(t) \leq f_T(t) \), for a.e. \( t \in [0, T] \), with \( f_T(t) \in L^1(0, T) \), then (since clearly \( \dot{I}_R(t) \to \dot{I}(t) \) pointwise) by dominated convergence, the results are that

\[
\dot{I}(t) = \dot{I}(0) + \lim_{R \to \infty} \int_{0}^{t} \dot{I}_R(\tau) \, d\tau = \dot{I}(0) + \int_{0}^{t} \lim_{R \to \infty} \dot{I}_R(\tau) \, d\tau,
\]

**(57)**

hence

\[
\lim_{R \to \infty} \dot{I}_R(\tau) = \dot{I}(\tau).
\]

**(58)**
\[ \bar{I}(\tau) = 8s^2 \int_{-R}^{R} k^2 |\hat{\psi}(\tau, k)|^2 \, dk - \frac{4s}{\sqrt{2\pi}} \text{Re} \left\{ \tau(\tau) \int_{-R}^{R} k \frac{\partial \hat{\psi}}{\partial k} (\tau, k) \, dk \right\}. \]

First we see that \( \Phi(\tau, R) \to 8s^2 |\hat{\psi}(\tau, \cdot)|^2_{H^2(\mathbb{R})} \) and that \( |\Phi(\tau, R)| \leq C \|\psi\|_{C^0([0,T];H^2(\mathbb{R}))} \), for every \( \tau \in [0,T] \). Furthermore, an integration by parts yields
\[
\int_{-R}^{R} k \frac{\partial \hat{\psi}}{\partial k} (\tau, k) \, dk = R(\hat{\psi}(\tau, R) + \hat{\psi}(\tau, -R)) - \int_{-R}^{R} \hat{\psi}(t, k) \, dk.
\]

Now,
\[ A_1(\tau, R) \to 0, \quad \text{as} \quad R \to \infty, \quad \forall \tau \in [0,T], \]
and
\[ |A_1(\tau, R)| \leq C_T, \quad \forall R > 0, \quad \forall \tau \in [0,T], \]
from (38) and the assumptions on \( \phi_{\lambda,0} \). On the other hand,\[ A_2(\tau, R) \to \sqrt{2\pi} q(\tau), \quad \text{as} \quad R \to \infty, \quad \forall \tau \in [0,T]\]
and
\[ |A_2(\tau, R)| \leq C \|\psi\|_{C^0([0,T];H^2(\mathbb{R}))}, \quad \forall R > 0, \quad \forall \tau \in [0,T]. \]

Thus, we can pass to the limit in (57) by dominated convergence, and from (58) we obtain
\[ \bar{I}(\tau) = 8s^2 |\hat{\psi}(t, \cdot)|^2_{H^2(\mathbb{R})} + 4s^2 \beta (q(\tau))^{2\beta+2}. \]

Finally, this immediately implies that \( \bar{I} \) is continuous on \([0,T]\) and, with some easy computations, that (55) is satisfied.

\[ \square \]

**Remark 3.5.** In the proof of proposition 3.4, the only point where the assumption \( \phi_{\lambda,0} \in \mathcal{S}(\mathbb{R}) \) is required is in the discussion of \( A_1(\tau, R) \). It is then clear that it is not the minimal one. We refer to section 2.3 for the reason behind such a choice.

Finally, we can show the proof of item (iv) of theorem 2.1.

**Proof of theorem 2.1: item (iv).** Let \( \beta < 0 \). Hence (55) reads
\[ \bar{I}(t) = 8s^2 E(0) - \frac{4s |\beta| (\sigma - \sigma(t))}{\sigma + 1} |q(t)|^{2\sigma+2}, \quad \forall t \in [0,T^*). \]
If \( \sigma \geq \sigma(t) \) and \( E(0) < 0 \), then \( \bar{I} \) is uniformly concave in \([0,T^*)\), namely
\[ \bar{I}(t) \leq C < 0, \quad \forall t \in [0,T^*). \]
As a consequence
\[ I(t) \leq I(0) + \dot{I}(0)t + \frac{Ct^2}{2}, \quad \forall t \in [0, T^*]. \]

Assume, therefore, by contradiction that \( T^* = +\infty \). Then \( \lim_{t \to T^*} I(t) = -\infty \) but this is prevented by the fact that \( I(t) \geq 0 \), for all \( t \in [0, T^*] \).

3.5. Stationary states

The last part of the paper is devoted to the proof of theorem 2.2. Preliminarily, we recall some computations that descend from some easy changes of variables and [27, equation (3.194.3)].

Let \( \omega > 0 \) and \( s \in \left( \frac{1}{2}, 1 \right] \). Denoting by \( B(\cdot, \cdot) \) the Beta function [24]
\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad \text{Re} \, x > 0, \, \text{Re} \, y > 0,
\]
and by \( \Gamma(\cdot) \) the Euler Gamma function [24]
\[
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt, \quad \text{Re} \, z > 0,
\]
results in
\[
\frac{1}{\pi} \int_0^\infty \frac{1}{k^{2s} + \omega} \, dk = \frac{1}{2\pi s\omega} \int_0^\infty \frac{\eta^{\frac{1}{2s}-1}}{1 + \frac{\eta}{\omega}} \, d\eta = \frac{\omega^{\frac{1}{2s}} B\left(\frac{1}{2s}, 1 - \frac{1}{2s}\right)}{2\pi s} = \frac{\omega^{\frac{1}{2s}} - 1}{2\pi s > 0}
\]
and
\[
\frac{1}{\pi} \int_0^\infty \frac{1}{(k^{2s} + \omega)^2} \, dk = \frac{1}{2\pi s\omega^2} \int_0^\infty \frac{\eta^{\frac{1}{2s}-1}}{(1 + \frac{\eta}{\omega})^2} \, d\eta = \frac{\omega^{\frac{1}{2s}} B\left(\frac{1}{2s}, 2 - \frac{1}{2s}\right)}{4\pi s^2} = \frac{(2s - 1)\omega^{\frac{1}{2s}} - 2}{4\pi s^2} > 0.
\]

**Proof of theorem 2.2.** We divide the proof into three parts.

*Part (1):* \( \omega > 0 \). Assume that \( u^\omega \) is a standing wave. As it belongs to \( \mathcal{D}(H_n^s) \),
\[
u^\omega = \phi^\omega_\lambda - \beta \nu^\omega(0) |\nu^\omega(0)|^{\lambda-1} \nu^\omega_\lambda, \quad \forall \lambda > 0,
\]
and, since it must satisfy (23), it results in
\[
(-\Delta)^s + \lambda \phi^\omega_\lambda = (\lambda - \omega) u^\omega,
\]
hence
\[ ((-\Delta)^s + \omega)\phi_\lambda^\omega = -(\lambda - \omega)\psi_\lambda G^\lambda_\omega. \]

Now, by means of the Fourier transform, the previous equality reads
\[ \widehat{\phi_\lambda^\omega} = \frac{r(\omega - \lambda)}{\sqrt{2\pi(|k|^{2s} + \lambda)(|k|^{2s} + \omega)}} \]
and, since \( r(\omega - \lambda) \neq 0 \) and \( \phi_\lambda^\omega \in H^{2s}(\mathbb{R}) \) for all \( \lambda > 0 \), it results in \( \omega > 0 \).

**Part (2): proof of (24).** As \( \omega > 0 \), let us choose \( \omega = \lambda \), so that (61) reads
\[ ((-\Delta)^s + \omega)\phi_\omega^\omega = 0 \iff \phi_\omega^\omega \equiv 0. \]

As a consequence, \( \psi_\omega \) has to satisfy
\[ \psi_\omega(x) = -\beta \psi_\omega(0)|\psi_\omega(0)|^{2s} G^\omega_\omega(x) \] (62)
and thus
\[ \psi_\omega(0) = -\beta \psi_\omega(0)|\psi_\omega(0)|^{2s} G^\omega_\omega(0), \]
or, equivalently (since \( \psi_\omega(0) \neq 0 \)),
\[ 1 = -\beta |\psi_\omega(0)|^{2s} G^\omega_\omega(0). \] (63)

Since \( G^\omega_\omega(0) > 0 \), clearly if \( \beta > 0 \), then (63) cannot be satisfied. Therefore, no standing wave can exist in the defocusing case.

Let us consider the focusing case (where (63) can be fulfilled). It is clear that, up to the multiplication of a constant phase factor, \( \psi_\omega(0) > 0 \) and that
\[ \psi_\omega(0) = \left( \frac{1}{|\beta| G^\omega_\omega(0)} \right)^{\frac{1}{2s}}. \] (64)

Now, recalling that
\[ G^\omega_\omega(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|k|^{2s} + \omega} \, dk = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{k^{2s} + \omega} \, dk \] (65)
and combining with (59) and (62), (24) follows.

**Part (3): proof of (i)–(iii).** The energy of a standing wave (in the focusing case) is given by
\[ E(\psi^\omega) = \frac{|\omega|^{2s}}{\mathcal{H}^s(\mathbb{R})} \left[ \frac{|\psi_\omega(0)|^{2s+2}}{\mathcal{P}(\psi^\omega)} \right]. \]

Combining (64), (65) and (59), with some computations, one sees that
\[ \mathcal{P}(\psi^\omega) = \frac{(2\sin(\frac{\omega}{2}) \frac{1}{2s+1} \omega^{\frac{\sigma+1}{2s}})}{|\beta|^{\frac{1}{2s+1}}(\sigma + 1)}. \]

On the other hand, combining (62), (60), (65) and (64)
\[ C(u^\omega) = \frac{\beta^2 |u^\omega(0)|^{4\sigma+2}}{2\pi} \int_\mathbb{R} \frac{|k|^{2s}}{(|k|^{2s} + \omega)^2} \, dk \]
\[ = |\beta|^2 |u^\omega(0)|^{4\sigma+2} \left( G_x^\omega(0) - \frac{\omega}{\pi} \int_\mathbb{R} \frac{1}{(|k|^{2s} + \omega)^2} \, dk \right) \]
\[ = |\beta|^2 |u^\omega(0)|^{4\sigma+2} \left( G_x^\omega(0) - \frac{(2s-1)\omega^{\frac{1}{2s}} - 1}{4s^2\sin\left(\frac{\pi}{2s}\right)} \right) \]
\[ = |\beta|^2 |u^\omega(0)|^{4\sigma+2} \left( \frac{(2s)^{\frac{1}{2s}}}{4s^2\sin\left(\frac{\pi}{2s}\right)} \right). \]

Summing up,
\[ E(u^\omega) = \frac{(2s)^{\frac{1}{2s}}(\sin\left(\frac{\pi}{2s}\right))^{1+\frac{1}{2s}}\omega^{\frac{(\sigma+1)(2s-1)}{2s}}}{|\beta|^{\frac{1}{2s}}} \left( 1 - \frac{2s}{\sigma+1} \right), \]
which clearly proves (i)–(iii).

**Appendix. Proof of proposition 2.1**

In order to prove proposition 2.1, some further information on the regularity properties of the Green’s function is required.

**Lemma A.1.** Let \( s \in (\frac{1}{2}, 1) \) and \( \lambda > 0 \). Then
\[ D^{2s-1} G_\lambda^x \in H^1(\mathbb{R}\setminus\{0\}), \]
\[ [D^{2s-1} G_\lambda^x](0) = -1, \]
\[ G_\lambda^x(0) = \lambda\|G_\lambda^x\|^2 + \|(-\Delta)^{\frac{s}{2}} G_\lambda^x\|^2. \]

**Proof.** We divide the proof into three parts.

**Part (i): proof of (A.1).** Combining (7) and (11) yields
\[ D^{2s-1} G_\lambda^x(x) = \frac{1}{2\pi} \int_\mathbb{R} e^{ikx} \frac{|k|^{2s-1} \text{sgn}(k)}{|k|^{2s} + \lambda} \, dk, \]
which clearly belongs to \( L^2(\mathbb{R}) \). Hence, one can easily check (using the Fourier transform) that
\[ \frac{d}{dx} D^{2s-1} G_\lambda^x = \frac{1}{\sqrt{2\pi}} (\lambda G_\lambda^x - \delta) \quad \text{in} \quad D'(\mathbb{R}), \]
so that
\[ (D^{2s-1} G_\lambda^x, \varphi') = -\lambda(G_\lambda^x, \varphi) \quad \forall \varphi \in C_0^\infty(\mathbb{R}\setminus\{0\}), \]
which then proves (A.1).

**Part (ii): proof of (A.2).** Let us compute, then,
\[ D^{2s-1} \mathcal{G}_s^\lambda (0^+) := \lim_{\epsilon \downarrow 0} \frac{i}{2\pi} \int_{\mathbb{R}} e^{ikx} \frac{|k|^{2s-1} \text{sgn}(k)}{|k|^{2s} + \lambda} \, dk. \]

First, for every \( x > 0 \), setting \( p := xk \) and observing that
\[
\int_{|p| \leq 1} \frac{|p|^{2s} \text{sgn}(p)}{|p| \left( |p|^{2s} + \lambda x^{2s} \right)} \, dp = 0,
\]
one obtains
\[
D^{2s-1} \mathcal{G}_s^\lambda (x) = \frac{i}{2\pi} \int_{|p| \leq 1} \frac{e^{ipx}}{|p|^{2s} + \lambda x^{2s}} \, dp + \frac{i}{2\pi} \int_{|p| > 1} \frac{e^{ipx} |p|^{2s}}{|p| \left( |p|^{2s} + \lambda x^{2s} \right)} \, dp
\]
\[= \frac{i}{2\pi} \int_{|p| \leq 1} \frac{(e^{ipx} - 1) |p|^{2s}}{|p|^{2s} + \lambda x^{2s}} \, dp + \frac{i}{2\pi} \int_{|p| > 1} \frac{e^{ipx}}{|p| \left( |p|^{2s} + \lambda x^{2s} \right)} \, dp - \frac{i\lambda}{2\pi} \int_{|p| > 1} \frac{|p|^{2s}}{|p| \left( |p|^{2s} + \lambda x^{2s} \right)} \, dp. \]

Now, clearly \( I_2 \) is independent of \( x \) and \( s \) and is finite as a Fresnel integral (for details, see [1, equations (5.2.1) and (5.2.2)] and [27, equations (3.722.1) and (3.722.3)]). Furthermore,
\[
\left| \frac{(e^{ipx} - 1) |p|^{2s}}{|p|^{2s} + \lambda x^{2s}} \right| \leq C, \quad \forall p \in [-1, 1],
\]
and
\[
\left| \frac{e^{ipx}}{|p|^{2s+2} + \lambda x^{2s}} \right| \leq \frac{C}{|p|^{2s+1}} \in L^1(\mathbb{R}\setminus[-1, 1]), \quad \forall p \in \mathbb{R}\setminus[-1, 1],
\]
so that
\[
I_1 \to \frac{i}{2\pi} \int_{|p| \leq 1} \frac{e^{ipx} - 1}{p} \, dp, \quad I_3 \to 0, \quad \text{as} \quad x \downarrow 0,
\]
which are independent of \( s \) as well. Consequently,
\[
D^{2s-1} \mathcal{G}_s^\lambda (0^+) = \lim_{x \downarrow 0} D^{2s-1} \mathcal{G}_s^\lambda (x) \bigg|_{x=1} = \lim_{x \downarrow 0} \frac{d}{dx} \mathcal{G}_s^\lambda (x) = -\frac{1}{2}.
\]

In the very same way, one can prove that \( D^{2s-1} \mathcal{G}_s^\lambda (0^-) = \frac{1}{2} \), and thus (A.2) follows immediately.

**Part (iii): proof of (A.3).** First we note that from (2) and (7), \((-\Delta)^{s/2} \mathcal{G}_s^\lambda \in L^2(\mathbb{R})\). In addition, by definition one finds
\[
\mathcal{G}_s^\lambda (0) = (\mathcal{G}_s^\lambda, (\mathcal{H}_s^0 + \lambda) \mathcal{G}_s^\lambda) = \lambda \| \mathcal{G}_s^\lambda \|^2 + (\mathcal{G}_s^\lambda, \mathcal{H}_s^0 \mathcal{G}_s^\lambda) = \lambda \| \mathcal{G}_s^\lambda \|^2 + (-\Delta)^{s/2} \mathcal{G}_s^\lambda, (-\Delta)^{s/2} \mathcal{G}_s^\lambda),
\]
which then concludes the proof. \( \square \)

**Proof of proposition 2.1.** The proof can be divided into two parts.

**Part (i): proof of (12) and (13).** First we focus on the inclusion
\[
D(\mathcal{H}_s^0) \subset \{ \psi \in H^1(\mathbb{R}) : D^{2s-1} \psi \in H^1(\mathbb{R}\setminus\{0\}), [D^{2s-1} \psi](0) = \alpha \psi(0) \}.
\]
If $\psi \in \mathcal{D}(\mathcal{H}_s^H)$, then
\[
\psi(x) = \phi_\lambda(x) - \alpha \psi(0) G_\lambda^s(x), \quad \phi_\lambda \in H^2(\mathbb{R}), \quad \lambda > 0.
\]

As a consequence, since $G_\lambda^s \in H^1(\mathbb{R})$, one immediately finds that $\psi \in H^1(\mathbb{R})$. On the other hand, as $D^{2s-1} \phi_\lambda \in H^1(\mathbb{R})$, recalling (A.1) and (A.2), one obtains that $D^{2s-1} \psi \in H^1(\mathbb{R}\setminus\{0\})$ and that
\[
[D^{2s-1} \psi](0) = -\alpha \psi(0) [D^{2s-1} G_\lambda^s](0) = \alpha \psi(0),
\]
thus proving (A.4).

On the other hand, in order to prove
\[
\mathcal{D}(\mathcal{H}_s^H) \supset \{ \psi \in H^1(\mathbb{R}) : D^{2s-1} \psi \in H^1(\mathbb{R}\setminus\{0\}), \ [D^{2s-1} \psi](0) = \alpha \psi(0) \}
\]  
(A.6)

it is sufficient to show that, if $\psi$ belongs to the rhs of (A.6), then
\[
\phi_\lambda := \psi + \alpha \psi(0) G_\lambda^s \in H^2(\mathbb{R}).
\]

Preliminarily, we note that $\phi_\lambda \in H^1(\mathbb{R})$ and that $D^{2s-1} \phi_\lambda \in H^1(\mathbb{R}\setminus\{0\})$. However, (A.5) immediately entails that $[D^{2s-1} \phi_\lambda](0) = 0$ and hence $D^{2s-1} \phi_\lambda \in H^1(\mathbb{R})$, which completes the proof.

Finally, one easily sees that for $x \neq 0$, $\mathcal{H}_s^0 G_\lambda^s = -\lambda G_\lambda^s$, and thus
\[
\mathcal{H}_s^0 \psi = (\mathcal{H}_s^0 + \lambda) \phi_\lambda - \lambda \psi = \mathcal{H}_s^0 \phi_\lambda + \alpha \lambda \psi(0) G_\lambda^s = \mathcal{H}_s^0 \psi,
\]
which proves (13).

**Part (ii): proof of (14).** From (6) and (8), with some (easy) computations one has
\[
(\psi, \mathcal{H}_s^0 \psi) = (\phi_\lambda, \mathcal{H}_s^0 \phi_\lambda) + 2\alpha \lambda \text{Re} \left\{ \overline{\psi(0)} (G_\lambda^s, \phi_\lambda) \right\} - \alpha \phi_\lambda \overline{\psi(0)} - \alpha^2 \lambda |\psi(0)|^2 \|G_\lambda^s\|^2
\]
for all $\psi \in \mathcal{D}(\mathcal{H}_s^H)$. On the other hand, we first observe that $\|(-\Delta)^{1/2} \psi\|^2 < \infty$ as $\psi \in H^1(\mathbb{R})$, and then, arguing in an analogous way results in
\[
\|(-\Delta)^{1/2} \psi\|^2 + \alpha |\psi(0)|^2 = (\phi_\lambda, \mathcal{H}_s^0 \phi_\lambda) + 2\alpha \lambda \text{Re} \left\{ \overline{\psi(0)} (G_\lambda^s, \phi_\lambda) \right\} - 2\alpha \text{Re} \left\{ \phi_\lambda(0) \overline{\psi(0)} \right\} + \alpha^2 |\psi(0)|^2 \|(-\Delta)^{1/2} G_\lambda^s\|^2 + \alpha |\psi(0)|^2.
\]

As a consequence, recalling that the boundary conditions imply
\[
\phi_\lambda(0) = (1 + \alpha G_\lambda^s(0)) \psi(0),
\]
one sees that (14) is satisfied if and only if
\[
G_\lambda^s(0) |\psi(0)|^2 - \lambda \|G_\lambda^s\|^2 |\psi(0)|^2 = \|(-\Delta)^{1/2} G_\lambda^s\|^2 |\psi(0)|^2 \quad \forall \psi \in \mathcal{D}(\mathcal{H}_s^H).
\]

However, since this is clearly true by means of (A.3), one obtains that (14) holds for all $\psi \in \mathcal{D}(\mathcal{H}_s^H)$. Finally, one can easily check that the set of the functions in $L^2(\mathbb{R})$ such that $\mathcal{F}_s^H(\psi) < \infty$ is $H^1(\mathbb{R})$, thus concluding the proof.
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