Research Article

Exact Solutions to the Nonlinear Schrödinger Equation with Time-Dependent Coefficients

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1. Introduction

It is well known that many physical phenomena can be described by a nonlinear Schrödinger equation (NLSE), which is found in many diverse fields such as plasma physics [1], fluid dynamics [2], nonlinear optics [3], quantum mechanics [4], hydrodynamics [5], and biology. Thus, finding the exact solutions to the NLSE has an important theoretical and practical significance in understanding the physical phenomena described by the NLSE.

Recently, some useful and powerful methods have been proposed to explore its exact solutions. For example, these methods include the homogeneous balance method [6], the tanh function expansion method and its extension [7], the sine-cosine methods [7], the exp-function method [7], the multiple exp-function method [8], the first integral method [9], the Jacobi elliptic function expansion method [10], the sub-ODE method [11], the \((G'/G)\)-expansion method [12], the modified simple equation method [13], the extended auxiliary equation method [14], the exp \((-\phi(\xi))\)-expansion method [15], and the trial function method and its generalization [16].

It should be recognized that most of the methods mentioned above are related to constant coefficient models. Undoubtedly, it becomes more difficult than those constant coefficient counterparts when we study the NLSE with time-dependent coefficients. Up to now, considerable attention has been paid to the varying coefficient NLSE, and many authors put forward different approaches [16–22]. Among them, Liu [21] proposed a trial function method to deal with both real and complex equations with varying coefficients. In this paper, we are going to apply Liu’s method to find the exact solutions of the following cubic-quintic NLSE with time-dependent coefficients:

\[
\begin{align*}
&i\frac{\partial q}{\partial t} + f(t)q_{xx} + g(t)(|q|^2 + \sigma|q|^4)q = \frac{ih(t)}{2}\left(|q|^2q\right)_x + ip(t)(|q|^2)_x q, \\
&q(x, t) \text{ will represent a different physical quantity when using it to describe different systems. For example, when the present equation with high-order dispersion and nonlinear terms describes the pulse transmission in the femtosecond state and considers the loss in the transmission process.}
\end{align*}
\]

\[\text{(1)}\]
Figure 1: Continued.
\[ q(x, t) \] denotes the complex envelope of the electric field, \( x \) and \( t \) are the distance along the direction of propagation and time, respectively, \( f(t) \) is the dispersion coefficient, \( h(t) \) is the self-steepening coefficient, \( p(t) \) is the self-frequency shift coefficient, and \( \sigma \) is a constant. Green and Biswas [22] studied Equation (1) by using the ansatz method [23] and obtained the exact soliton solutions under some constraints on the parameters. Recently, one of the present authors has obtained analytical traveling-wave solutions to a generalized Gross-Pitaevskii (GP) equation [22] with some new time- and space-varying coefficients and external fields [24] because of their possible applications to the BECs [25–29]. Obviously, the current cubic-quintic NLSE is more complicated than the GP equation. It should be pointed out that all kinds of NLSEs with varying coefficients mentioned above [16–22] are different from Equation (1) except for reference...
Figure 2: Continued.
[17], in which the authors only studied the particular case $a_0 = 0$ using the direct but complicated integral approach.

This paper is organized as follows. In “Exact Solutions,” we first apply the trial function method to obtain its exact solutions by using a suitable transformation, and then, we illustrate the shapes of the wave amplitude of different solutions by taking appropriate parameters for those varying coefficients. Finally, in “Concluding Remarks,” we summarize the results found in this work.

2. Exact Solutions

Assume that the solution to Equation (1) is given by

$$q(x, t) = u(\xi)e^{\eta}, \xi = k(t)x + w(t), \eta = s(t)x + r(t), \quad (2)$$

where $k(t), w(t), s(t),$ and $r(t)$ are undetermined parameters related to time. Substituting them into (1) and separating the real and the imaginary parts, one finds
Figure 3: Continued.
If the solution satisfies
\[ u'(i) = F(\xi) = \sum_{i=0}^{m} a_i u^i, \]
where \( a_i, i=0,\ldots,m \) are constants and \( m \) is an integer to be determined (the value of \( m \) in Eq. (5) is determined as \( m=6 \) when using the homogeneous balance theory), then substituting (5) into (4) and setting each coefficient of \( u' \),
u^2 u' in (3), and u' (i = 1, 3, 5) in (4) to zero allow us to obtain a set of algebraic equations as follows:

\[ k'(t)x + w'(t) + 2f(t)s(t)k(t) = 0, \]
\[ [3h(t) + 2p(t)]k(t) = 0, \quad a_1 = a_3 = a_5 = 0, \]
\[ a_2 = \frac{s'(t)x + r'(t) + f(t)s^2(t)}{f(t)k'(t)}, \]
\[ a_4 = -g(t) + h(t)s(t), \]
\[ a_6 = -\frac{sg(t)}{3f(t)k'(t)}. \]  

(6)

Set

\[ k(t) = k \neq 0, s(t) = s \neq 0, r(t) = c_1 \int f(t) dt, g(t) = c_4, \]

(7)

where k, s, c_1, c_2, c_3, and c_4 are arbitrary constants, f(t) is an arbitrary function. In this work, we only consider three different cases, f(t) = 1, f(t) = t, and f(t) = 1/(1 + t), for simplicity. The other coefficients can be determined by the following forms:

\[ p(t) = \frac{3}{2} c_3 k^2 f(t), \]
\[ w(t) = -2sk \int f(t) dt, \]
\[ a_1 = a_3 = a_5 = 0, \]
\[ a_2 = c_1 + c_2^2, \]
\[ a_4 = -\frac{c_5 + sc_1}{2}, \]
\[ a_6 = -\frac{ac_2}{3}. \]

In terms of these coefficients (8), Equation (5) with m = 6 is thus simplified as

\[ (u')^2 = a_0 + a_2 u^2 + a_4 u^4 + a_6 u^6. \]  

(10)

Before studying the general case \( a_0 \neq 0 \), let us first show the soliton solutions \( u = u(\xi)e^{\eta} \) for the special case \( a_0 = 0 \) [17]:

\[ q = \left\{ \begin{array}{ll}
\left[ \frac{2a_2}{\varepsilon \sqrt{a_4^2 - 4a_2 a_6} \cos h (2\sqrt{a_4^2}) - a_4} \right]^{1/2} \cdot e^{i\eta}, & a_2 - 4a_2 a_6 > 0, \ a_2 > 0, \ \text{bright soliton solution}, \\
\left[ \frac{2a_2}{\varepsilon \sqrt{a_4 a_6 - a_4^2} \sin h (2\sqrt{a_4^2}) - a_4} \right]^{1/2} \cdot e^{i\eta}, & a_2 - 4a_2 a_6 < 0, \ a_2 > 0, \ \text{singular soliton solution}, \\
\left[ \frac{a_2}{a_4} \left( 1 + \varepsilon \tan h \left( \frac{\sqrt{a_4^2}}{2} \right) \right) \right]^{1/2} \cdot e^{i\eta}, & a_2^2 - 4a_2 a_6 = 0, \ a_2 > 0, \ \text{dark soliton solution}, \\
\left[ \frac{-a_2 a_4 \sec h^2 (\sqrt{a_4^2})}{a_4 - a_6 [1 + \varepsilon \tan h (\sqrt{a_4^2})]^2} \right]^{1/2} \cdot e^{i\eta}, & a_2 > 0, \ \varepsilon = \pm 1, \ \text{bright soliton-like solution}. \\
\end{array} \right. \]

(11)

How to find the exact solution to the general case \( a_0 \neq 0 \) becomes the main purpose of this work. Applying a transformation \( \varphi = u^{-2} \) to Equation (10) enables us to obtain

\[ \varphi' = \pm 2 \sqrt{a_0} G(\varphi), \ G(\varphi) = \varphi^3 + \frac{a_2}{a_0} \varphi^2 + \frac{a_4}{a_0} \varphi + \frac{a_6}{a_0}. \]  

(12)

It is found that Equation (12) can be divided into three different cases depending on the factor \( \Delta = B^2 - 4AC \) by the Shengin discrimination method, where the parameters A, B, and C are given by

\[ A = \frac{a_2^2 - 3a_0 a_4}{a_0^2}, \]
\[ B = \frac{a_2 a_4 - 9a_0 a_6}{a_0^2}, \]
\[ C = \frac{a_2^2 - 3a_0 a_6}{a_0^2}. \]  

(13)

Case 1. \( \Delta < 0 \)

...
Figure 4: Continued.
In this case, $G(\varphi) = 0$ has three unequal roots, $\varphi_1 < \varphi_2 < \varphi_3$: $\varphi_1 = -a_2/3a_0 - 2\sqrt{A} \cos (\theta/3)/3$ and $\varphi_{2,3} = -a_2/3a_0 + \sqrt{A}[\cos(\theta/3) \pm \sqrt{3} \sin(\theta/3)]/3$, where $\theta = \arccos((2a_2 - 3a_0B)/2a_0\sqrt{A})$. In terms of them, Equation (12) can be expressed as

$$q = \begin{cases} \frac{1}{\sqrt{a_0}(\varphi - \varphi_1)(\varphi - \varphi_2)} \left[ a_2 \varphi^2 \left( \cos(\theta/3) \pm \sqrt{3} \sin(\theta/3) \right) \right] \cos^2 \left( \sqrt{a_0}(\varphi - \varphi_1)(\varphi - \varphi_2) \right)^{1/2} \cos \theta, & a_0 > 0, \\ \sin \left[ \sqrt{\frac{a_0}{a_2}}(\varphi - \varphi_1)(\varphi - \varphi_2) \right] \cos^2 \left( \sqrt{a_0}(\varphi - \varphi_1)(\varphi - \varphi_2) \right)^{1/2} \cos \theta, & a_0 < 0, \end{cases}$$

where

$$f(t) = \begin{cases} 1, \\ t, \\ \frac{1}{1+t}, \end{cases}$$

for $a_0 > 0$. The wave propagates along the $x$-axis.

Figure 4: (a–c) The 3D wave shapes of $|q|$: (a) $f(t) = 1$, (b) $f(t) = t$, and (c) $f(t) = 1/(1+t)$. (d) The wave propagates along the $x$-axis.
Figure 5: Continued.
To understand these solutions, we illustrate the shapes of the wave amplitude for three different cases, \( f(t) = 1, f(t) = t \), and \( f(t) = 1/(1+t) \), which are all periodic waves \( (k = s = 1, a_0 = 1, a_2 = -6, a_4 = 11, a_6 = -6) \).

**Case 2.** \( \Delta > 0 \)
Figure 6: Continued.
Figure 6: (a–c) The 3D wave shapes of $|q|$: (a) $f(t) = 1$, (b) $f(t) = t$, and (c) $f(t) = 1/(1+t)$. (d) The wave propagates along the $x$-axis.
G(φ) = 0 has a single root topics are slightly different from those in Figure 1. (the parameters k = s = 1, 

Case 3. Δ = 0

A ≠ 0

G(φ) = 0 has two roots, φ1 = B/2A and φ2 = −a2/3a0 + B/3A. It is known that Equation (12) can be transformed into 𝜙′ = ±2√a0(φ − φ1)/φ − φ2, from which we are able to obtain the exact solutions to Equation (1).

\[
q = \frac{1}{\phi_1 \sec^2 \left( \sqrt{a_0} (\phi_2 - \phi_1) \xi \right) - \phi_1 \tan^2 \left( \sqrt{a_0} (\phi_2 - \phi_1) \xi \right)} \cdot e^{i\eta}, \quad a_0 \phi_1 < a_0 \phi_2, \tag{18}
\]

\[
qu = \frac{1}{\sqrt{\phi_1 \sec^2 \left( \sqrt{a_0} (\phi_2 - \phi_1) \xi \right) + \phi_1 \tan^2 \left( \sqrt{a_0} (\phi_2 - \phi_1) \xi \right)}} \cdot e^{i\eta}, \quad a_0 \phi_1 > a_0 \phi_2. \tag{19}
\]

For the case a0φ1 < a0φ2, the shapes of the periodic wave amplitude are shown in Figure 3 (k = s = 1, a0 = 1, a2 = −4, a4 = 5, and a6 = −2), which is very similar to those in Figure 1. For the case a0φ1 > a0φ2 (a0 > 0), the shapes of the wave amplitude are displayed in Figure 4 (k = s = 1, a0 = 1, a2 = −5, a4 = 8, and a6 = −4), which corresponds to a bright soliton wave. However, the case a0 < 0 generates a dark soliton wave as shown in Figure 5 (k = s = 1, a0 = −1, a2 = 4, a4 = −5, and a6 = 2). These new and interesting phenomena do not appear in other cases due to the different signs of the parameter a0.

A = 0 and B = 0, i.e., a4 = a2/3a0, a6 = a4/27a2 4/3.

G(φ) = 0 has three equal roots, φ1 = −a2/3a0. The relation 𝜙′ = ±2√a0(φ − φ1)/φ − φ2 enables us to obtain a rational function solution

\[
q = \left( \frac{3a_0 \omega^2}{3 - a_0^2 \xi^2} \right)^{1/2} \cdot e^{i\eta}, \tag{20}
\]

which represents a singular solitary wave. The corresponding shapes of the wave amplitude are illustrated in Figure 6 (k = s = 1, a0 = 1, a2 = 3, a4 = 3, and a6 = 1).

Before ending this section, we give a useful remark on these graphics. Comparing Figures 2(a)–2(c), it is found that the amplitude is the same. This implies that the variable coefficient f(t) has no effect on the amplitude. We have the same conclusion for Figures 1 and 3–5, but the amplitude in Figure 6 is changed.

3. Concluding Remarks

In this paper, we investigated a kind of nonlinear Schrödinger equation with high-order time-dependent coefficients, which describes the propagation of ultrashort optical pulses in nonlinear optical fibers. The trial function method has been used to find general solutions such as the periodic solutions (14) and (16) including the Jacobi elliptic function, the periodic wave solutions (18) involving the Jacobian elliptic function, solitary wave solutions (19), and rational function solutions (20). To describe the properties of the solutions, three different functions, f(t) = 1, f(t) = t, and f(t) = 1/(1 + t), are chosen to show the shapes of the wave amplitudes. We summarize the main results as follows:
(1) For $f(t) = 1$, Figures 1(a), 2(a), and 3(a) show the shapes of the wave amplitude of the periodic wave solutions (14)–(18), respectively. Figures 4(a) and 5(a) display the case of the solitary wave solutions (19); Figure 4(a) shows the case of a bright soliton ($a_0 > 0$), but Figure 5(a) illustrates the case of a dark soliton ($a_0 < 0$). Figure 6(a) shows the wave shapes of the rational function solutions (20). By observing Figures 1(a), 2(a), 3(a), 4(a), 5(a), and 6(a), it is found that the velocity of the pulse remains constant during propagation because $f(t)$ is a real constant.

(2) For the case $f(t) = t$, the shapes of the wave amplitude of the periodic wave solutions (14)–(18) are shown in Figures 1(b), 2(b), and 3(b), respectively. The case of the solitary wave solutions (19) is plotted in Figures 4(b) and 5(b). For example, Figure 4(b) describes a bright soliton wave ($a_0 > 0$), while Figure 5(b) corresponds to a dark soliton wave ($a_0 < 0$). The shape of the wave amplitude of the rational function solutions (20) is illustrated in Figure 6(b). It can be found that the pulse propagation velocity has a parabolic feature by observing Figures 1(b), 2(b), 3(b), 4(b), 5(b), and 6(b).

(3) For $f(t) = 1/(1 + t)$, Figures 1(c), 2(c), and 3(c) describe the periodic wave solutions (14)–(18), respectively. Figures 4(c) and 5(c) show the case of the solitary wave solutions (19). For instance, Figure 4(c) represents a bright soliton wave ($a_0 > 0$), but Figure 5(c) corresponds to a dark soliton wave ($a_0 < 0$). Figure 6(c) shows the shapes of the wave amplitude of the rational function solutions (20). It is seen from Figures 1(c), 2(c), 3(c), 4(c), 5(c), and 6(c) that the pulse propagation has the feature of a logarithmic function.

Before ending this work, we will make three useful remarks. First, it should be mentioned that the specific expressions of various exact solutions of the NLSE with varying coefficients also reflect the diversity of the solitary wave solutions. The existence of solitary wave solutions implies a perfect balance between the nonlinear effect and the dispersion effect, which usually requires peculiar conditions. Without doubt, this work will help us understand the physical phenomena described by Equation (1). Second, it is believed that the results such as optical solitons presented here will make a major impact in the area of nonlinear optics. The model which was used in this work for mathematical analysis gave the characteristics of stable solitary waves in the system. This confirms that the optical fiber described by the system can be transmitted stably for a long time. Third, the trial function method is also an effective and practical method for solving other kinds of nonlinear equations with varying coefficients.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that there is no conflict of interest regarding the publication of this paper.

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