P-MATRIX DESCRIPTION OF INTERACTION OF TWO CHARGED HADRONS AND LOW-ENERGY NUCLEAR-COULOMB SCATTERING PARAMETERS

V. A. Babenko and N. M. Petrov

Bogolyubov Institute for Theoretical Physics,
National Academy of Sciences of Ukraine, Kiev, Ukraine

The scattering of two charged strongly interacting particles is described on the basis of the $P$-matrix approach. In the $P$ matrix, it is proposed to isolate explicitly the background term corresponding to purely Coulomb interaction, whereby it becomes possible to improve convergence of the expansions used and to obtain a correct asymptotic behavior of observables at high energies. The expressions for the purely Coulomb background $P$ matrix, its poles and residues, and purely Coulomb eigenfunctions of the $P$-matrix approach are obtained. The nuclear-Coulomb low-energy scattering parameters of two charged hadrons are investigated on the basis of this approach combined with the method of isolating the background $P$ matrix. Simple explicit expressions for the nuclear-Coulomb scattering length and effective range in terms of the residual $P$ matrix are derived. These expressions give a general form of the nuclear-Coulomb low-energy scattering parameters for models of finite-range strong interaction. Specific applications of the general expressions derived in this study are exemplified by considering some exactly solvable models of strong interaction containing hard core repulsion, and, for these models, the nuclear-Coulomb low-energy scattering parameters for arbitrary values of the orbital angular momentum are found explicitly. In particular, the nuclear-Coulomb scattering length and effective range are obtained explicitly for the boundary-condition model, the model of a hard-core delta-shell potential, the Margenau model, and the model of hard-core square-well potential.

1Electronic address: pet@gluk.org
1. INTRODUCTION

The $P$-matrix approach to describing hadron-hadron interaction was first proposed by Jaffe and Low [1] and was then developed in a number of studies [2–4]. This approach is a modification of the well-known Wigner-Eisenbud $\mathcal{R}$-matrix theory [5, 6]. Within the $P$-matrix approach, the scattering amplitude is expressed in terms of the logarithmic derivative of the wave function at the surface of the strong-interaction region — that is, in terms of the $P$ matrix. Here it is assumed that the configuration space of the system is broken down into two regions, the external region, where the interaction can be described in terms of a two-particle potential, and the internal region, where strong interaction is dominant. For the $P$ matrix, a so-called dispersion formula that appears to be its pole expansion and which establishes its energy dependence can be derived on the basis of quite general assumptions. Observables can then be described in terms of a finite number of parameters.

In [3, 4, 7], a method was proposed for explicitly isolating a free background part in the $P$ matrix. This method is advantageous in that it simplifies the implementation of the $P$-matrix approach in specific applications and extends the region of its applicability. The free $P$ matrix, which corresponds to the absence of interaction, was isolated as the background part in the aforementioned studies. This is natural in dealing with the scattering of neutral particles. Here, we propose a generalization of the isolation method to the case involving charged particles, so that there is long-range Coulomb interaction in the system along with strong interaction. It is well known that, in this case, scattering theory requires a nontrivial modification. We show that the idea of explicitly isolating a background part in the $P$ matrix as put forth in [3, 4, 7] can be implemented for charged particles as well and that, for the background $P$ matrix, it is advisable in this case to take the purely Coulomb $P$ matrix — that is, the logarithmic derivative of the regular Coulomb wave function at the surface of the interaction region. It turns out that the isolation of the background Coulomb part offers the same advantages as in the absence of Coulomb interaction.

As an application of the $P$-matrix approach combined with the method for isolat-
ing the background Coulomb $P$ matrix, we study the scattering length and the effective range for low-energy nuclear-Coulomb scattering. These parameters are important physical quantities characterizing the scattering of charged hadrons and light nuclei at low energies. We obtain simple explicit expressions for the nuclear-Coulomb low-energy scattering parameters in terms of the parameters of the residual $P$ matrix; these expressions make it possible to analyze and evaluate the nuclear-Coulomb scattering length and effective range and to find them directly for finite-range strong-interaction potentials. As a matter of fact, the expressions that we obtain here determine a general form of the nuclear-Coulomb low-energy scattering parameters for models of strong interaction of finite range. In [8], expressions for the nuclear-Coulomb low-energy scattering parameters in terms of the $P$-matrix parameters were obtained without resort to the isolation method. Those expressions are more cumbersome and less convenient in applications than the present ones.

Much attention has been given to the nuclear-Coulomb low-energy scattering parameters (see, for example, [9–12]) since these physical quantities play an important role in theoretical and experimental investigations. In some studies (see [13–17]), these parameters were determined explicitly for some specific cases of separable nuclear potentials (in particular, for the Yamaguchi potential). Here, we find a general form of the nuclear-Coulomb scattering parameters for a rather broad class of local strong-interaction models — namely, for models of finite range. General expressions obtained for the low-energy parameters make it possible to determine these quantities explicitly for a number of exactly solvable strong-interaction models containing hard core repulsion. We emphasize that it is the use of the simple expressions obtained by the isolation method that made it possible to simplify significantly the relevant consideration. It should also be noted that the investigation presented here was performed for an arbitrary value of the orbital angular momentum $\ell$. 

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2. DISPERSION FORMULA FOR THE $P$ MATRIX AND NUCLEAR-COULOMB OBSERVABLES

For the elastic scattering of two charged strongly interacting particles, the radial wave function $\psi_{l k}(r)$ of relative motion in a state characterized by a specific value of the orbital angular momentum $\ell$ is regular at the origin and satisfies the radial Schrödinger equation

$$\left[ \frac{d^2}{dr^2} + E - \frac{l(l+1)}{r^2} - V(r) \right] \psi_{l k}(r) = 0 \quad (1)$$

with the potential

$$V(r) = V_s(r) + V_c(r) \quad , \quad (2)$$

where $V_s(r)$ is a short-range strong potential (by assumption, it has a finite range $R$) and $V_c(r) = 2\xi k/r$ is an ordinary Coulomb potential. Here, $\xi$ is the Coulomb parameter

$$\xi \equiv \frac{\mu e_1 e_2}{\hbar^2 k} = \frac{1}{a_B k} \quad , \quad (3)$$

where $a_B$ is the Bohr radius,

$$a_B \equiv \frac{\hbar^2}{\mu e_1 e_2} \quad , \quad (4)$$

$e_1$ and $e_2$ being the charges of the particles involved in the scattering process. We shall use the system of units where the reduced Planck constant and the doubled reduced mass are both equal to unity ($\hbar = 2\mu = 1$), so that the energy of the relative motion, $E$, is expressed in terms of the wave number $k$ as $E = k^2$.

At infinity, the radial wave function satisfies the scattering boundary condition

$$\psi_{l k}(r) \xrightarrow{r \to \infty} \bar{\psi}_{l k}(r) \equiv e^{i\delta_l(k)} [\cos \nu_l(k) \; F_l(\xi, kr) + \sin \nu_l(k) \; G_l(\xi, kr)] \quad , \quad (5)$$

where $\bar{\psi}_{l k}(r)$ is the asymptotic wave function for the continuous spectrum, while $F_l(\xi, kr)$ and $G_l(\xi, kr)$ are, respectively, the regular and the irregular Coulomb wave functions [18], whose asymptotic behavior at infinity is given by

$$F_l(\xi, kr) \xrightarrow{r \to \infty} \sin \left( kr - \xi \ln 2kr - l\pi \frac{2}{2} + \sigma_l(k) \right) \quad , \quad (6)$$

$$G_l(\xi, kr) \xrightarrow{r \to \infty} \cos \left( kr - \xi \ln 2kr - l\pi \frac{2}{2} + \sigma_l(k) \right) \quad . \quad (7)$$
Here, \( \sigma_l(k) \equiv \arg \Gamma(l + 1 + i\xi) \) is the purely Coulomb phase shift, and the total phase shift \( \delta_l(k) \) has the form

\[
\delta_l(k) = \sigma_l(k) + \nu_l(k),
\]

where \( \nu_l(k) \) is the nuclear-Coulomb phase shift.

The \( P \) matrix \( P_l(E) \) is defined in terms of the logarithmic derivative of the radial wave function at the surface of the strong-interaction region \( (r = R) \),

\[
P_l(E) \equiv \frac{R \psi'_{lk}(R)}{\psi_{lk}(R)}.
\]

In the internal region \( r \leq R \), we introduce a complete set of orthonormalized eigenfunctions \( u_{ln}(r) \) that satisfy the Schrödinger equation (1) and the homogeneous boundary conditions

\[
u_{ln}(0) = 0, \quad u_{ln}(R) = 0
\]

at the ends of the interval \([0, R]\). Nontrivial solutions that obey the conditions in (10) exist only at some energy eigenvalues \( E_{ln} \) that are determined by solving the Sturm-Liouville problem specified by Eqs. (1) and (10). The orthonormalization conditions have the form

\[
\int_0^R u_{lm} u_{ln} \, dr = \delta_{mn}.
\]

By expanding the wave function in the internal region in a series in eigenfunctions \( u_{ln}(r) \), we find that, for the \( P \) matrix, there is the dispersion formula [1,3,4]

\[
P_l(E) = P_l(0) + \sum_{n=1}^{\infty} \frac{E}{E_{ln}} \frac{\gamma_{ln}^2}{E - E_{ln}},
\]

where

\[
\gamma_{ln} \equiv \sqrt{R} u'_{ln}(R).
\]

Relation (12) involves a constant \( P_l(0) \), the \( P \) matrix at zero energy, and it is the isolation of this constant that ensures convergence of the remaining series. The dispersion formula (12), which represents a pole expansion of the \( P \) matrix, establishes the general form of its energy dependence. This dependence is completely determined by the states of the compound system which are characterized by the energy eigenvalues \( E_{ln} \) and the
residues $\gamma^2_{ln}$. These quantities in turn are controlled by the physical properties of the system in the internal region and are independent of energy $E$.

Let us now establish the relation between the $S$ matrix and the $P$ matrix. For this, we note that the wave function in the external region, $\psi_{lk}(r)$, can be represented in the general form

$$\psi_{lk}(r) = i^2 \left[ H_l^{(-)}(\xi, kr) - S_l(k) H_l^{(+)}(\xi, kr) \right], \quad r > R, \quad (14)$$

where $H_l^{(\pm)}(\xi, kr)$ are the Coulomb Jost solutions given by

$$H_l^{(\pm)}(\xi, kr) = e^{\mp i\sigma_l(k)} \left[ G_l(\xi, kr) \pm i F_l(\xi, kr) \right], \quad (15)$$

which represent the diverging and the converging waves distorted by the Coulomb potential. Accordingly, their asymptotic behavior is given by

$$H_l^{(\pm)}(\xi, kr) \xrightarrow{r\to\infty} e^{\pm i(kr - \xi \ln 2kr - \frac{l\pi}{2})}. \quad (16)$$

By using the matching conditions at the point $r = R$ and definition (9), we find that the $S$ matrix can be expressed in terms of the $P$ matrix as

$$S_l(k) = S_l^{(h)}(k) \frac{P_l^{(-)}(k) - P_l(k)}{P_l^{(+)}(k) - P_l(k)}, \quad (17)$$

where

$$S_l^{(h)}(k) \equiv \frac{H_l^{(-)}(\xi, kR)}{H_l^{(+)}(\xi, kR)} \quad (18)$$

is the $S$ matrix corresponding to the scattering on a hard core of radius $R$ in the presence of the Coulomb potential and $P_l^{(\pm)}(k)$ are the logarithmic derivatives of the diverging and converging Coulomb waves at the boundary surface,

$$P_l^{(\pm)}(k) \equiv \frac{kR H_l^{(\pm)'}(\xi, kR)}{H_l^{(\pm)}(\xi, kR)}. \quad (19)$$

Hereafter, a prime denotes differentiation with respect to the variable $\rho = kR$. The real and the imaginary part of the function $P_l^{(+)}(k)$ are usually denoted by $\triangle_l(k)$ and $s_l(k)$, respectively; since the relation $P_l^{(+)}(k) = P_l^{(-)}(k)$ obviously holds, we can write

$$P_l^{(\pm)}(k) = \triangle_l(k) \pm i s_l(k), \quad (20)$$
where the functions $s_l(k)$ and $\Delta_l(k)$ are expressed in terms of the Coulomb functions as [19]

$$s_l(k) = \frac{kR}{F^2_l(\xi, kR) + G^2_l(\xi, kR)}, \quad (21)$$

$$\Delta_l(k) = s_l(k) \left[ F_l(\xi, kR) F'_l(\xi, kR) + G_l(\xi, kR) G'_l(\xi, kR) \right]. \quad (22)$$

With the aid of Eqs. (17) and (20), it can easily be found that the nuclear-Coulomb phase shift can be represented as

$$\nu_l(k) = \zeta_l(k) + \arctan \frac{s_l(k)}{P_l(k) - \Delta_l(k)}, \quad (23)$$

where the phase shift $\zeta_l(k)$ for scattering on a hard core of radius $R$ in the presence of the Coulomb interaction is given by

$$\zeta_l(k) \equiv - \arctan \frac{F_l(\xi, kR)}{G_l(\xi, kR)}. \quad (24)$$

Expressions (17) and (23) for the observables reveal a significant drawback of the $P$-matrix approach based on the dispersion formula (12) as an approximation of the $P$ matrix: if only a finite number of terms are retained, the observables in question will have an incorrect asymptotic behavior at high energies. By way of example, we indicate that, with increasing energy, the phase shift (23) will then behave as the phase shift $\zeta_l(k)$ for scattering on a hard core; that is, it will tend to infinity,

$$\nu_l(k) \xrightarrow[k \to \infty]{} -kR + O(1). \quad (25)$$

But in fact, the phase shift must vanish at high energies, at least for regular potentials.
3. PURELY COULOMB P MATRIX

An incorrect asymptotic behavior of observables at high energies can be avoided by isolating the background part in the \( P \) matrix. In the presence of Coulomb interaction, we define the background \( P \) matrix as the purely Coulomb \( P \) matrix — that is, in terms of the logarithmic derivative of the regular Coulomb wave function,

\[
P_I^{(c)} (E) \equiv \frac{k R F_I' (\xi, k R)}{F_I (\xi, k R)}.
\]  

We recall that the regular Coulomb wave function \( F_I (\xi, \rho) \) is expressed in terms of the confluent hypergeometric function \( \Phi (a, b; z) \) as \[18\]

\[
F_I (\xi, \rho) = C_l (\xi) e^{-i \rho \rho^{l+1}} \Phi (l + 1 - i \xi, 2l + 2; 2i \rho),
\]

where

\[
C_l (\xi) \equiv \frac{2^l}{(2l + 1)!} e^{-\pi \xi/2} |\Gamma (l + 1 + i \xi)|
\]

is the Coulomb penetrability factor that is a rather complicated function of energy and which is introduced in order to ensure the required asymptotic behavior of the function \( F_I (\xi, \rho) \) at infinity [see Eq. (6)]. If, however, we consider only the internal region \( 0 \leq r \leq R \), it is more convenient to introduce a solution that does not involve the factor \( C_l (\xi) \) and which possesses simpler properties near the origin. We define such a solution \( \phi_l (\xi, \rho) \) through the relation

\[
F_I (\xi, \rho) = (2l + 1)!! C_l (\xi) \phi_l (\xi, \rho),
\]

where the factor \((2l + 1)!!\) was introduced in order that, upon switching the Coulomb interaction off, the function \( \phi_l (\xi, \rho) \) reduce to the spherical Riccati-Bessel function:

\[
\phi_l (0, \rho) = j_l (\rho).
\]

The expression for the function \( \phi_l (\xi, \rho) \) in terms of a confluent hypergeometric function can easily be found with the aid of Eqs. (27) and (29). The Coulomb \( P \) matrix as expressed in terms of the solution \( \phi_l (\xi, \rho) \) has the form

\[
P_I^{(c)} (E) = \frac{k R \phi_I' (\xi, k R)}{\phi_I (\xi, k R)},
\]
which is analogous to (26).

The positions \( E_{ln}^{(c)} \equiv k_{ln}^{(c)2} \) of the poles of the Coulomb \( P \) matrix — they depend on the Bohr radius \( a_B \) and on the interaction range \( R \) \( (E_{ln}^{(c)} = E_{ln}^{(c)}(a_B, R)) \) — are defined by the roots of the denominator of the expression on the right-hand side of (31),
\[
\phi_l \left( \frac{1}{a_B k_{ln}}, k_{ln} R \right) = 0. \tag{32}
\]

The Coulomb eigenfunctions \( u_{ln}^{(c)}(r) \), which obey the Schrödinger equation (1) with the purely Coulomb potential and the boundary conditions (10), are given by
\[
u_{ln}^{(c)}(r) = \frac{\gamma_{ln}^{(c)}}{\sqrt{R k_{ln}^{(c)}}} \phi_l (\xi_{ln}, k_{ln} R) \tag{33}\]

where \( \xi_{ln} \equiv 1/a_B k_{ln} \). And the parameters \( \gamma_{ln}^{(c)} \), which are determined from the normalization condition for the eigenfunctions, can be found if we use the Green’s theorem,
\[
u_{lk_1}(R) u_{lk_2}'(R) - u_{lk_1}'(R) u_{lk_2}(R) = (k_1^2 - k_2^2) \int_0^R u_{lk_1} u_{lk_2} dr \tag{34}\]

for two solutions to Eq. (1) that correspond to two different energy values, \( k_1^2 \) and \( k_2^2 \). By substituting (33) into (34), going over to the limit \( k_1 \to k_2 = k_{ln} \) and taking into account Eqs. (11) and (13), we find that the expression for the Coulomb residues \( \gamma_{ln}^{(c)2} \) can be recast into the form
\[
\gamma_{ln}^{(c)2} = \frac{2 \frac{E_{ln}^{(c)}}{1 - \theta_l (\xi_{ln}, k_{ln} R) / a_B R E_{ln}^{(c)} \phi_l ' (\xi_{ln}, k_{ln} R)}}{E_{ln}^{(c)} - E_{ln}^{(c)}}. \tag{35}\]

Here the function \( \theta_l (\xi, \rho) \) is the derivative of the function \( \phi_l (\xi, \rho) \) with respect to the parameter \( \xi \),
\[
\theta_l (\xi, \rho) \equiv \frac{\partial \phi_l (\xi, \rho)}{\partial \xi}, \tag{36}\]

and can be directly expressed in terms of a confluent hypergeometric function.

Thus, we have completely determined the parameters of the Coulomb \( P \) matrix (its poles and residues) and found the Coulomb eigenfunctions. The dispersion formula for the Coulomb \( P \) matrix has the form (12); that is,
\[
P_{l}^{(c)}(E) = P_{l}^{(c)}(0) + \sum_{n=1}^{\infty} \frac{E}{E_{ln}^{(c)} - E} \frac{\gamma_{ln}^{(c)2}}{E_{ln}^{(c)}}, \tag{37}\]
where the $P$ matrix at zero energy, $P_t^{(c)}(0)$, is given by expression

$$P_t^{(c)}(0) = \frac{z I_{2l}(z)}{I_{2l+1}(z)} - l, \quad (38)$$

which can be directly obtained from (31) for $k \to 0$ by using the expansion of the regular Coulomb wave function and its derivative in terms of Bessel functions [18, 20, 21]. In expression (38), $I_{\nu}(z)$ are modified Bessel functions and the dimensionless parameter $z$ is given by

$$z \equiv 2\sqrt{2R/a_B}. \quad (39)$$

So far, we have considered the case of Coulomb repulsion ($a_B > 0$). In the case of Coulomb attraction ($a_B < 0$), $P_t^{(c)}(0)$ has the form

$$P_t^{(c)}(0) = \frac{\zeta J_{2l}(\zeta)}{J_{2l+1}(\zeta)} - l, \quad (40)$$

where $J_{\nu}(\zeta)$ are Bessel functions and $\zeta \equiv 2\sqrt{2R/|a_B|}$.

4. ISOLATING THE PURELY COULOMB BACKGROUND $P$ MATRIX

In nuclear-Coulomb $P$ matrix (9), we now isolate explicitly the purely Coulomb background $P$ matrix (26), following a way that is similar to that used to isolate explicitly the free background $P$ matrix in the absence of Coulomb interaction [3, 4, 7]. We represent this transformation in the form

$$P_t(E) = P_t^{(c)}(E) + \tilde{P}_t(E). \quad (41)$$

With the aid of Eqs. (12) and (37), it can be shown that, for the residual nuclear-Coulomb $P$ matrix $\tilde{P}_t(E)$, we have the expansion

$$\tilde{P}_t(E) = \tilde{P}_t(0) + \sum_{n=1}^{\infty} \left[ \frac{E}{E_{in}E - E_{in}} \frac{\gamma_{ln}^2}{E - E_{in}^{(c)}} - \frac{E}{E_{in}^{(c)}} \frac{\gamma_{ln}^{(c)2}}{E - E_{in}^{(c)}} \right]. \quad (42)$$

By comparing the expansions (12) and (42) for the functions $P_t(E)$ and $\tilde{P}_t(E)$ we conclude that the isolation of the purely Coulomb background part in the nuclear-Coulomb $P$ matrix according to (41) amounts to a partial summation of the series in (12), where one isolates the part that corresponds to the Coulomb interaction and which
is known explicitly. This naturally improves convergence of the original series, thereby making it possible to obtain a more accurate representation of observables as functions of energy. It can be shown that the expansion in (42) converges at the same rate as a series whose general term is proportional to $1/n^4$, while the expansion in (12) converges as $1/n^2$ — that is, much more slowly.

By making transformation (41) in Eq. (17) and using the relation

\[ P_l^{(c)}(k) - P_l^{(\pm)}(k) = \frac{kRe^{\pm i\sigma_l}}{F_l(\xi, kR) H_l^{(\pm)}(\xi, kR)} \equiv \frac{1}{c_l^{(\pm)}(k)}, \]  

(43)

which can easily be verified for logarithmic derivatives, we can straightforwardly express the nuclear-Coulomb $S$ matrix $S_l^{(cs)}(E) = e^{2i\nu_l(k)}$ in terms of the residual $P$ matrix $\hat{P}_l(E)$ as

\[ S_l^{(cs)}(E) = \frac{1 + c_l^{(-)}(k) \hat{P}_l(E)}{1 + c_l^{(+)}(k) \hat{P}_l(E)}, \]  

(44)

where the functions $c_l^{(\pm)}(k)$ are determined according to (43) and obviously satisfy the relation $c_l^{(+)}(k) = c_l^{(-)}(k)$. With the aid of (44), we can easily represent the nuclear-Coulomb phase shift as

\[ \nu_l(k) = - \arctan \frac{F_l^2(\xi, kR) \hat{P}_l(E)}{kR + F_l(\xi, kR) G_l(\xi, kR) \hat{P}_l(E)}. \]  

(45)

From Eqs. (44) and (45), it is obvious that, if only a finite number of terms are retained in the pole expansion (42) for $\hat{P}_l(E)$, the $S$ matrix and the phase shift will have a correct asymptotic behavior at high energies:

\[ S_l(k) \rightarrow 1, \; \nu_l(k) \rightarrow 0. \]  

(46)

Thus, an isolation of the purely Coulomb background term in the $P$ matrix leads to a correct asymptotic behavior of the observables at high energies if the residual $P$ matrix is approximated by a finite number of pole terms. We can see that the transformation (41) has a transparent mathematical and physical substantiation and that its application provides the same advantages as in the absence of Coulomb interaction.
5. EXPRESSIONS FOR THE NUCLEAR-COULOMB LOW-ENERGY SCATTERING PARAMETERS IN TERMS OF THE RESIDUAL $P$ MATRIX PARAMETERS

A great number of studies have been devoted to the problem of generalizing and modifying effective-range theory in the presence of long-range Coulomb interaction (see, for example, [22–27]). As a result, the Coulomb-modified effective-range function $K_{cst}(E)$ was introduced, and the nuclear-Coulomb scattering length and effective range were determined for the case of $S$-wave scattering, as well as for scattering in a state characterized by an arbitrary value of the orbital angular momentum $\ell$. In [22, 25], it was shown that, in the case of an arbitrary orbital angular momentum, the effective-range expansion in the presence of Coulomb interaction has the form

$$K_l(E) \equiv (2l + 1)!! C_l^2(\xi) k^{2l+1} \left[ \cot \nu_l(k) + \frac{2\xi}{C_0^2(\xi) h(\xi)} \right] =$$

$$= -\frac{1}{a_l} + \frac{1}{2} r_l k^2 + \ldots, \quad (47)$$

where the function $h(\xi)$ is expressed in terms of the digamma function $\psi(z) \equiv \Gamma'(z)/\Gamma(z)$ as

$$h(\xi) \equiv Re \psi(1 + i\xi) - \ln |\xi|. \quad (48)$$

In the complex plane of energy $E$, the nuclear-Coulomb effective-range function $K_l(E)$ is analytic in some domain near the origin [26]; hence, it can be expanded in the Maclaurin series (47) in powers of $E$ in the vicinity of the point $E = 0$. Thus, a special role of the function $K_l(E)$ is associated with its analyticity near $E = 0$. The nuclear-Coulomb scattering length $a_l$ and effective range $r_l$ are determined in this case in terms of the coefficients in the expansion (47) of the function $K_l(E)$. We note that, in a large number of studies, the nuclear-Coulomb quantities, such as $a_l$, $r_l$, $K_l$ and others are equipped with the additional indices $c$ and $s$, which label parameters associated with, respectively, Coulomb and short-range interactions. This was done in order to distinguish these quantities from their counterparts in the absence of Coulomb interaction, which are labeled only with the index $s$. Since we do not consider here
the case where there is no Coulomb field, the indices $c$ and $s$ are suppressed on all nuclear-Coulomb quantities.

For a further analysis, it is reasonable to introduce the dimensionless inverse scattering length $\gamma_l$, and the dimensionless effective range $\rho_l$ as

$$\gamma_l \equiv \frac{l^2 a_B^{2l+1}}{4a_l},$$  \hspace{2cm} (49)$$

$$\rho_l \equiv 3 l^2 a_B^{2l-1} r_l.$$  \hspace{2cm} (50)

In the particular case of scattering by a hard core of radius $R$ ($\psi_{lk}(R) = 0$, $P_l(E) = \infty$), the nuclear-Coulomb low-energy scattering parameters $\gamma_l$ and $\rho_l$ can easily be found in the explicit form [8]

$$\gamma_l^h = \frac{K_\nu(z)}{I_\nu(z)},$$  \hspace{2cm} (51)$$

$$\rho_l^h = 1 - \mu_l \gamma_l^h + \frac{2(\lambda_l - \beta)}{I_\nu^2(z)},$$  \hspace{2cm} (52)

where the superscript $h$ denotes, as previously, a hard core and $I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions. The constants $\nu$, $\lambda_l$, and $\mu_l$ are given by

$$\nu \equiv 2l + 1,$$  \hspace{2cm} (53)$$

$$\lambda_l \equiv l(l+1),$$  \hspace{2cm} (54)$$

$$\mu_l \equiv 4\nu \lambda_l,$$  \hspace{2cm} (55)

while the dimensionless parameter $\beta$ is defined as

$$\beta \equiv \frac{R}{a_B}.$$  \hspace{2cm} (56)

As before, the parameter $z$ has the form (39). In order to render the expressions presented below less cumbersome, it is convenient to isolate explicitly, in the low-energy parameters $\gamma_l$ and $\rho_l$, the parts that correspond to scattering on a hard core. Accordingly, we set

$$\gamma_l = \gamma_l^h + \hat{\gamma}_l,$$  \hspace{2cm} (57)$$

$$\rho_l = \rho_l^h + \hat{\rho}_l,$$  \hspace{2cm} (58)
defining, in this way, the residual low-energy nuclear-Coulomb scattering parameters $\hat{\gamma}_l$ and $\hat{\rho}_l$.

Let us further express the nuclear-Coulomb scattering length and effective range in terms of the residual $P$ matrix. By substituting (45) into (47), we find that the nuclear-Coulomb effective-range function as expressed in terms of the residual $P$ matrix is given by

$$K_l(E) = (2l + 1)!!^2 C_l^2(\xi) k^{2l+1} \left[ \frac{2\xi}{C_0^2(\xi)} h(\xi) - \frac{C_l(\xi, \rho)}{F_l(\xi, \rho)} - \frac{\rho}{F_l^2(\xi, \rho)} \hat{P}_l(E) \right],$$  \hspace{1cm} (59)

where, as before, we use the notation $\rho = kR$. Let us expand the right-hand side of Eq. (59) in a Maclaurin series in powers of energy $E = k^2$. It is obvious that, as long as we are interested neither in the shape parameter nor in higher expansion coefficients, it is sufficient to retain only the terms that are linear in $E$. We further make use of the known relation for the Coulomb penetration factor [18],

$$\frac{C_l^2(\xi)}{C_0^2(\xi)} = \frac{2^{2l}}{(2l + 1)!!^2} (1^2 + \xi^2)(2^2 + \xi^2)\ldots(l^2 + \xi^2),$$  \hspace{1cm} (60)

and of the asymptotic expression for the function $h(\xi)$ at low energies [18],

$$h(\xi) \overset{\xi \to \infty}{\sim} \frac{1}{12\xi^2} + \frac{1}{120\xi^4} + \ldots.$$  \hspace{1cm} (61)

The expansions of the Coulomb wave functions in power series in energy $E$ were previously studied by many authors [20, 21, 28, 29]. To terms that are linear in energy, these expansions for the case of Coulomb repulsion can be written as

$$F_l(\xi, \rho) = \frac{(2l + 1)! C_l(\xi)}{(2\xi)^{l+1}} z^2 \left\{ I_\nu(z) - \frac{z^3}{96\xi^2} \left[ I_{\nu+1}(z) + \frac{2l}{z} I_{\nu+2}(z) \right] + \ldots \right\},$$  \hspace{1cm} (62)

$$G_l(\xi, \rho) \overset{\xi \to \infty}{\sim} \frac{(2l + 1)! C_l(\xi)}{(2\xi)^l C_0^2(\xi)} z \left\{ K_\nu(z) - \frac{z^3}{96\xi^2} \left[ K_{\nu+1}(z) - \frac{2l}{z} K_{\nu+2}(z) \right] + \ldots \right\}. $$  \hspace{1cm} (63)

The dispersion relation for the residual $P$ matrix (42) can be recast into the form

$$\hat{P}_l(E) = \sum_{n=1}^\infty \left[ \frac{\gamma_{ln}^2}{E - E_{ln}} - \frac{\gamma_{ln}^{(c)^2}}{E - E_{ln}^{(c)}} \right].$$  \hspace{1cm} (64)

This expansion contains no additional parameters and is completely determined by the quantities $E_{ln}$ and $\gamma_{ln}^2$. It can be shown that the expansion in (64) converges at the
same rate as a series whose general term is proportional to $1/n^2$. The analyticity of the residual $P$ matrix in a vicinity of the point $E = 0$ immediately follows from (64) if all energy eigenvalues differ from zero. The expansion of the residual $P$ matrix in a power series in energy $E$ can be written in the form

$$\hat{P}_l(E) = \hat{P}_l + \hat{Q}_l \rho^2 + \ldots ,$$

(65)

where

$$\hat{P}_l \equiv \hat{P}_l(0),$$

(66)

$$\hat{Q}_l \equiv \frac{1}{R^2} \hat{P}'_l(0)$$

(67)

are dimensionless expansion coefficients. We can easily express the quantities $\hat{P}_l$ and $\hat{Q}_l$ in terms of the $P$-matrix parameters as

$$\hat{P}_l = \sum_{n=1}^{\infty} \left[ \frac{\gamma_{ln}^2}{E_{ln}^{(c)}} - \frac{\gamma_{ln}^2}{E_{ln}} \right],$$

(68)

$$\hat{Q}_l = \frac{1}{R^2} \sum_{n=1}^{\infty} \left[ \frac{\gamma_{ln}^2}{E_{ln}^{(c)}} - \frac{\gamma_{ln}^2}{E_{ln}} \right].$$

(69)

We also note that, on the basis of Eq. (41), the quantities $\hat{P}_l$ and $\hat{Q}_l$ can be determined from the relations

$$P_l = P_l^{(c)} + \hat{P}_l,$$

(70)

$$Q_l = Q_l^{(c)} + \hat{Q}_l,$$

(71)

where

$$P_l \equiv P_l(0),$$

(72)

$$Q_l \equiv \frac{1}{R^2} P'_l(0)$$

(73)

are parameters in the expansion of the $P$ matrix,

$$P_l(E) = P_l + Q_l \rho^2 + \ldots .$$

(74)

The parameters $P_l^{(c)}$ and $Q_l^{(c)}$ in the expansion of the purely Coulomb background $P$ matrix can be found explicitly from Eqs. (26) and (62). The results are

$$P_l^{(c)} = l + 1 + \frac{z}{2} \frac{I_{\nu+1}(z)}{I_{\nu}(z)},$$

(75)
\[ 2\beta \left( 3Q_l^{(c)} + 1 \right) = l\nu + \left( lz - \frac{\mu_l}{z} \right) \frac{I_{\nu+1}(z)}{I_{\nu}(z)} + 2(\beta - \lambda_l) \frac{I_{\nu+1}^2(z)}{I_{\nu}^2(z)}. \] (76)

Substituting now expressions (60)-(63) and (65) into Eq. (59) and taking into account Eq. (47), we arrive at explicit expressions for the inverse scattering length \( \hat{\gamma}_l \) and the effective range \( \hat{\rho}_l \), also referred to as the dimensionless nuclear-Coulomb residual parameters. The results are given by

\[ \hat{\gamma}_l = \frac{1}{2 I_{\nu}^2 \hat{P}}, \] (77)
\[ \frac{\hat{\rho}_l}{z^2 \hat{\gamma}_l} = 3\beta \frac{\hat{Q}_l}{\hat{P}} + 4 \frac{\lambda_l - \beta}{z} \frac{I_{\nu+1}}{I_{\nu}} - l, \] (78)

where \( I_{\nu} \equiv I_{\nu}(z) \). In the particular case of interaction in the \( S \) state \( (l = 0) \), the last formulas are somewhat simplified to become

\[ \hat{\gamma} = \frac{1}{2 I_{1}^2 \hat{P}}, \] (79)
\[ \frac{\hat{\rho}}{z^2 \hat{\gamma}} = 3\beta \frac{\hat{Q}}{\hat{P}} - \frac{z}{2} \frac{I_2}{I_1}. \] (80)

In the case of Coulomb attraction \( (a_B < 0) \), all the above formulas are valid upon the substitution of conventional Bessel functions for modified ones:

\[ I_{\nu}(z) \rightarrow \frac{1}{i^\nu} J_{\nu}(\zeta), \quad K_{\nu}(z) \rightarrow \frac{\pi}{2i^\nu} Y_{\nu}(\zeta), \quad \zeta = 2\sqrt{2R/|a_B|}. \] (81)

Formulas (77) and (78) yield general explicit expressions for the nuclear-Coulomb low-energy scattering parameters in terms of the residual \( P \) matrix parameters. These expressions make it possible to obtain directly a general form of the nuclear-Coulomb scattering length and effective range for models of finite-range strong interaction.

6. NUCLEAR-COULOMB LOW-ENERGY SCATTERING
PARAMETERS FOR EXACTLY SOLVABLE MODELS CONTAINING HARD CORE REPULSION

For specific applications of the above general expressions, we will consider some exactly solvable models of finite-range strong interaction containing hard core repulsion. For these, we will find explicitly the nuclear-Coulomb low-energy scattering parameters for arbitrary values of the orbital angular momentum.
6.1. Boundary Condition Model

In the boundary condition model, the interaction in the internal region is determined by a single energy-independent parameter, the value of the logarithmic derivative of the wave function at the boundary surface — that is, the constant $P_l$. It is obvious that the parameter $Q_l$ vanishes in this case. Thus, we have

$$ P_l (E) = P_l, \quad (82) $$

$$ Q_l = 0, \quad (83) $$

whence it follows that

$$ \tilde{P}_l = P_l - P_l^{(c)}, \quad (84) $$

$$ \tilde{Q}_l = -Q_l^{(c)}. \quad (85) $$

The nuclear-Coulomb low-energy scattering parameters can then be written as

$$ \hat{\gamma}_l = \frac{1}{2 I_\nu^2 \left( P_l - P_l^{(c)} \right)}, \quad (86) $$

$$ \frac{\hat{\rho}_l}{z^2 \hat{\gamma}_l} = 3\beta \frac{Q_l^{(c)}}{P_l^{(c)} - P_l} + 4 \frac{\lambda_l - \beta}{z} \frac{I_{\nu+1}}{I_\nu} - l, \quad (87) $$

where the quantities $P_l^{(c)}$ and $Q_l^{(c)}$ are given by Eqs. (75) and (76).

6.2. Hard-Core Delta-Shell Potential

Let us consider the case where the strong interaction is described by the delta-shell potential concentrated on the sphere of radius $R$ and supplemented with a hard core of radius $R_c$ less than $R$,

$$ V_s (r) = \begin{cases} +\infty, & r < R_c \\ -\lambda \delta (r-R), & r > R_c \end{cases}. \quad (88) $$

Here, $\lambda$ is the dimensionless interaction constant. In this case, the wave function in the internal region $(r < R)$ is a linear combination of the Coulomb wave functions,

$$ \psi_{lk} (r) = A_l (k) \left[ G_l (\xi, kR_c) F_l (\xi, kr) - F_l (\xi, kR_c) G_l (\xi, kr) \right], \quad R_c < r < R \quad (89) $$
and satisfies the zero boundary condition at \( r = R_c \): \( \psi_{l}(R_c) = 0 \). At the boundary surface \( (r = R) \), the wave function is continuous, but its derivative undergoes a discontinuity,

\[
\psi'_{l}(R + 0) - \psi'_{l}(R - 0) = -\frac{\lambda}{R} \psi_{l}(R). \tag{90}\]

By using formulas (89) and (90), we find that, in the case of a hard-core delta-shell potential, the \( P \) matrix can be represented as

\[
P_{l}(E) = \frac{\rho \ G_{l}(\xi, x) \ F_{l}(\xi, \rho) - F_{l}(\xi, x) \ G_{l}(\xi, \rho)}{G_{l}(\xi, x) \ F_{l}(\xi, \rho) - F_{l}(\xi, x) \ G_{l}(\xi, \rho)} - \lambda, \tag{91}\]

where \( x \equiv kR_c \). With the aid of the definition of the background Coulomb \( P \) matrix (26) and the representation in (41), we obtain the residual \( P \) matrix for the potential (88) in the form

\[
\hat{P}_{l}(E) = \frac{\rho \ F_{l}(\xi, x) / F_{l}(\xi, \rho)}{G_{l}(\xi, x) \ F_{l}(\xi, \rho) - F_{l}(\xi, x) \ G_{l}(\xi, \rho)} - \lambda. \tag{92}\]

By means of the expansion of the Coulomb functions (62) and (63), we derive the parameters of the residual \( P \) matrix, \( \hat{P}_{l} \) and \( \hat{Q}_{l} \), according to (65). The results are

\[
\hat{P}_{l} = \frac{1}{2} \frac{I_{\nu}(y) / I_{\nu}(z)}{K_{\nu}(y) I_{\nu}(z) - I_{\nu}(y) K_{\nu}(z)} - \lambda, \tag{93}\]

\[
24\beta^{2} \hat{Q}_{l} = (lz^{2} - \mu_{l}) \left( \lambda + \hat{P}_{l} \right) + (\lambda_{l} - \beta) \times \frac{(\lambda_{l} - \alpha) / (\lambda_{l} - \beta) - 2z \ [K_{\nu}(y) I_{\nu+1}(z) + I_{\nu}(y) K_{\nu+1}(z)] I_{\nu}(y) / I_{\nu}(z) + I_{\nu}^{2}(y) / I_{\nu}^{2}(z) }{[K_{\nu}(y) I_{\nu}(z) - I_{\nu}(y) K_{\nu}(z)]^{2}}, \tag{94}\]

where \( \alpha \equiv R_{c}/a_{B} \) and \( y \equiv 2\sqrt{2\alpha} \). By substituting (93) and (94) into (77) and (78), we now find that the nuclear-Coulomb scattering length and effective range for the hard-core delta-shell potential can be represented as

\[
\hat{\gamma}_{l} = \frac{\gamma_{l}^{h}(y) - \gamma_{l}^{h}(z)}{1 - 2\lambda I_{\nu}^{2}(z) \ [\gamma_{l}^{h}(y) - \gamma_{l}^{h}(z)]}, \tag{95}\]

\[
\hat{\rho}_{l} = \frac{\rho_{l}^{h}(y) - \rho_{l}^{h}(z)}{[\gamma_{l}^{h}(y) - \gamma_{l}^{h}(z)]^{2}} + 2\lambda z I_{\nu}(z) [lz I_{\nu}(z) + 4 (\beta - \lambda_{l}) I_{\nu+1}(z)], \tag{96}\]

where \( \gamma_{l}^{h}(y) \) and \( \rho_{l}^{h}(y) \) are the low-energy nuclear-Coulomb parameters for a hard core of radius \( R_{c} \) (Eqs. (51) and (52) with substitutions \( z \rightarrow y \) and \( \beta \rightarrow \alpha \)), while \( \gamma_{l}^{h}(z) \equiv \gamma_{l}^{h} \).  

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and \( \rho_l^h (z) \equiv \rho_l^h \) are the parameters for the hard core of radius \( R \) (Eqs. (51), (52)). In the limiting case of a delta-shell potential without a core \( (R_c \to 0, \ y \to 0, \text{and} \ \gamma_l^h (y) \to \infty) \), expressions (95) and (96) reduce to the known expressions for the low-energy nuclear-Coulomb parameters for scattering on a delta potential [16, 17]

\[
\hat{\gamma}_l = -\frac{1}{2\lambda I^2_\nu (z)},
\]

(97)

\[
\hat{\rho}_l = 2\lambda z I_\nu (z) [lzI_\nu (z) + 4(\beta - \lambda_l) I_{\nu+1} (z)].
\]

(98)

6.3. Margenau Model

In the Margenau model [30], strong interaction is simulated by a square-well potential containing hard core repulsion; in addition Coulomb interaction is assumed to be absent in the internal region. The latter is justified by the fact that, in the internal region, the Coulomb interaction is much weaker than strong interaction. Thus, the total interaction in this model is described by the potential

\[
V (r) = \begin{cases} 
+\infty, & r < R_c, \\
-V_0, & R_c < r < R, \\
2\xi k/r, & r > R.
\end{cases}
\]

(99)

In this case, the wave function in the internal region has the form

\[
\psi_{lk} (r) = A_l (k) [n_l (KR_c) j_l (Kr) - j_l (KR_c) n_l (Kr)], \quad R_c < r < R,
\]

(100)

where \( K \equiv \sqrt{V_0 + E} \), and \( j_l \) and \( n_l \) are the spherical Riccati-Bessel functions. The \( P \) matrix can then be written as

\[
P_l (E) = \rho \frac{j_l (x) n_l' (\rho) - n_l (x) j_l' (\rho)}{j_l (x_0) n_l (\rho_0) - n_l (x_0) j_l (\rho_0)},
\]

(101)

where \( x \equiv KR_c \) and \( \rho \equiv KR \). From the above, we can easily determine the \( P \)-matrix parameters \( P_l \) and \( Q_l \). The results are

\[
P_l = \rho_0 \frac{j_l (x_0) n_l' (\rho_0) - n_l (x_0) j_l' (\rho_0)}{j_l (x_0) n_l (\rho_0) - n_l (x_0) j_l (\rho_0)},
\]

(102)
In this case are given by residual $P$ and taking into account Eqs. (75) and (76), we can find the parameters of the function in the internal region has the form
\[ \frac{1}{I_{2\gamma t}^2} = -2\rho_0 \frac{j_t(x_0) n_{l+1}(\rho_0) - n_l(x_0) j_{l+1}(\rho_0)}{j_t(x_0) n_l(\rho_0) - n_l(x_0) j_l(\rho_0)} - \frac{b}{I_{\nu+1} \nu} , \] (104)
where $\xi \equiv \frac{1}{a_{BR_0}}$.

6.4. Hard-Core Square-Well Potential

For the case of a strong-interaction potential in the form of a square well with a hard core,
\[ V_s(r) = \begin{cases} +\infty, & r < R_c , \\ -V_0 \theta(R-r), & r > R_c \end{cases} \] (106)
we confine ourselves to determining the nuclear-Coulomb scattering length. For the simpler case of a square-well potential without a core, the nuclear-Coulomb scattering length and the nuclear-Coulomb effective range were found in [8]. In this case, the wave function in the internal region has the form
\[ \psi_{lk}(r) = A_{lk} [G_l(\Xi, KR_c) F_l(\Xi, Kr) - F_l(\Xi, KR_c) G_l(\Xi, Kr)] , \quad R_c < r < R , \] (107)
where $\Xi \equiv \frac{1}{a_{BR}}$, and the $P$ matrix is given by
\[ P_l(E) = \rho \frac{F_l(\Xi, x) G'_l(\Xi, \rho) - G_l(\Xi, x) F'_l(\Xi, \rho)}{F_l(\Xi, x) G_l(\Xi, \rho) - G_l(\Xi, x) F_l(\Xi, \rho)} , \] (108)
where, as in the preceding subsection, $x \equiv KR_c$ and $\rho \equiv KR$. In accordance with (70) and (75), the parameter $\hat{P}_l$ of the residual $P$ matrix then assumes the form
\[ \hat{P}_l = \rho_0 \frac{F_l(\xi_0, x_0) G'_l(\xi_0, \rho_0) - G_l(\xi_0, x_0) F'_l(\xi_0, \rho_0)}{F_l(\xi_0, x_0) G_l(\xi_0, \rho_0) - G_l(\xi_0, x_0) F_l(\xi_0, \rho_0)} - \frac{z I_{\nu+1}(z)}{2 I_{\nu}(z)} - (l+1) , \] (109)
while the nuclear-Coulomb scattering length is given by

\[
\frac{1}{I^2 \gamma_l} = 2\rho_0 \frac{F_l (\xi_0, x_0) G_l' (\xi_0, \rho_0) - G_l (\xi_0, x_0) F_l' (\xi_0, \rho_0)}{F_l (\xi_0, x_0) G_l (\xi_0, \rho_0) - G_l (\xi_0, x_0) F_l (\xi_0, \rho_0)} - \frac{I_{\nu+1}}{I_\nu} - (\nu + 1). \tag{110}
\]

7. CONCLUSION

In summary, an explicit isolation of the purely Coulomb background part in the \( P \) matrix leads to a correct asymptotic behavior of physical observables at high energies when the residual \( P \) matrix is approximated by a finite number of pole terms. Concurrently, the isolation of the background \( P \) matrix makes it possible to improve convergence of the remaining expansions. The transformation in (41) has a transparent mathematical and physical substantiation, and its application provides the same advantages as in the absence of Coulomb interaction. In addition, the explicit isolation of the purely Coulomb background part in the \( P \) matrix makes it possible to obtain the simple general expressions (77) and (78) for the nuclear-Coulomb low-energy scattering parameters in terms of the residual \( P \) matrix. With the aid of these expressions, we can directly calculate the nuclear-Coulomb scattering length and effective range for finite-range strong-interaction potentials. If the Schrödinger equation for these potentials admits of an exact solution in the presence of Coulomb interaction, the nuclear-Coulomb parameters can be found explicitly. In general, the nuclear-Coulomb low-energy scattering parameters can be obtained for arbitrary short-range strong-interaction potentials at any value of the orbital angular momentum \( \ell \). On the basis of the expressions derived in the present study, we have found explicitly the nuclear-Coulomb scattering length and effective range for the boundary-condition model, for the model of a hard-core delta-shell potential, for the Margenau model, and for the hard-core square-well potential at arbitrary values of the orbital angular momentum.

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