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Limit Cycles of a Class of Polynomial Differential Systems Bifurcating from the Periodic Orbits of a Linear Center

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Abstract: In this paper, we study the number of limit cycles of a new class of polynomial differential systems, which is an extended work of two families of differential systems in systems considered earlier. We obtain the maximum number of limit cycles that bifurcate from the periodic orbits of a center using the averaging theory of first and second order.

Keywords: existence; limit cycle; averaging method; Kukles system

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1. Introduction

One of the more difficult problems in the qualitative theory of polynomial differential equations in the plane $\mathbb{R}^2$ is the study of their limit cycles. Thus a classical problem related to these polynomial differential systems is the second part of the unsolved 16th Hilbert problem [1,2], which essentially consists of finding a uniform upper bound for the maximum number of limit cycles that a planar polynomial differential system of a given degree can have.

The limit cycles problem and the center problem are concentrated on specific classes of systems. For instance, much has been written on Kolmogorov systems, Liénard systems and Kukles systems, that is, systems of the form

$$\dot{x} = -y, \quad \dot{y} = x + \lambda y + Q(x, y), \quad (1)$$

where $Q(x, y)$ is a polynomial with real coefficients of degree $n$. Bifurcation of limit cycles in Kukles systems have been tackled by several authors and by using different approaches.

In [3], Kukles gave necessary and sufficient conditions in order that (1) with $n = 3$ has a center at the origin. This cubic system without the term $y^3$ was also studied in [4] and the authors called it reduced. Christopher and Lloyd [5] presented some systems that yield at most five limit cycles bifurcating from the origin. In [6], Chavarriga et al. studied the maximum number of small amplitude limit cycles for Kukles systems which can coexist with some invariant algebraic curves. By averaging theory, bifurcation of limit cycles for a family of perturbed Kukles differential systems was studied
in [7–11]. In [8], Llibre and Mereu studied the maximum number of limit cycles of the Kukles polynomial differential systems

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - f(x) - g(x)y - h(x)y^2 - d_0 y^3,
\end{align*}
\]

where the polynomials \( f(x), g(x) \) and \( h(x) \) have degree \( n_1, n_2 \) and \( n_3 \) respectively, \( d_0^k \neq 0 \) is a real number.

Sáez and Szántó, in [12] introduced the following system

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x + \epsilon(x^2 + y^2) \left( -A + \sum_{i=1}^{n-2} q_i(x, y) - A_l \right),
\end{align*}
\]

where \( A_l > 0, q_i(0,0) \in \mathbb{R}, \) and \( \epsilon \) is a small parameter, thy proved the following result.

**Theorem 1** (See [12]). If either \( n = 2k \) or \( n = 2k - 1 \) for \( k \geq 2 \), then system (2) has at most \((k - 2)\) global limit cycles bifurcated from the unperturbed Hamiltonian center.

In [13], Rabanal computed the maximum number of limit cycles of the following differential systems

\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= x + (x^2 + y^2) \sum_{i=1}^{l} \epsilon^i (q_l(x, y) - A_l),
\end{align*}
\]

where for every \( l = 1, 2, 3, A_l > 0 \) and the polynomial \( q_l(x, y) \) has degree \( n_l - 2 \geq 1 \) with \( q_l(0,0) = 0, \) and \( \epsilon \) is a small parameter. For \( n_l = 2k_l \) or \( n_l = 2k_l - 1, k_l \geq 2, \) thy obtained the maximum number of limit cycles of the polynomial differential systems (3) bifurcating from the periodic orbits of the linear centre \( \dot{x} = y, \dot{y} = -x, \) using averaging theory

\begin{itemize}
  \item [a] of first order \( k_1 - 2, \)
  \item [b] of second order is \( \{k_2 - 2, \left\lfloor \frac{n_2 - 2}{2} \right\rfloor - 2 \}. \)
  \item [c] of third order is \( \{k_3 - 2, \left\lfloor \frac{n_3 - 2}{2} \right\rfloor - 1 \}. \)
\end{itemize}

where \( \left\lfloor . \right\rfloor \) denotes the integer part function.

Using the averaging theory, we shall study in this work the maximum number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed inside the following differential systems

\[
\begin{align*}
\dot{x} &= -y + \sum_{i=1}^{l} \epsilon^i f_n^i(x) \\
\dot{y} &= x + (x^2 + y^2) \sum_{i=1}^{l} \epsilon^i \left( -A_l + \sum_{i=1}^{m_l-2} (q_{l,i}x^i + \tilde{q}_{l,i}y^i) \right),
\end{align*}
\]

where \( A_l > 0, q_{l,i}, \tilde{q}_{l,i} \in \mathbb{R}, \) the polynomial \( f_n^i(x) \) has degree \( n_l \) and \( \epsilon \) is a small parameter. More precisely our main result is the following.

**Theorem 2.** Assume that for \( l = 1, 2 \) the constants \( A_l > 0, \) the polynomials \( f_n^i(x) \) have degree \( n_l, \) with \( n_l \geq 1. \) Suppose that \( m_l = 2k_l \) or \( m_l = 2k_l - 1 \) and \( k_l \geq 2. \) Then for \( |\epsilon| \) sufficiently small the maximum number of limit cycles of the polynomial differential systems (4) bifurcating from the periodic orbits of the linear centre \( \dot{x} = y, \dot{y} = -x, \) using averaging theory

\begin{itemize}
  \item [(a)] of first order is \( \lambda_1 = \max \left\{ \left\lfloor \frac{n_l+1}{2} \right\rfloor, k_1 - 1 \right\} \) limit cycles
(b) of second order is

\[
\lambda_2 = \max \left\{ \frac{m_1 - 2}{2}, \frac{[n_1]}{2}, \frac{[m_1 - 2]}{2} + \mu, \frac{[n_1]}{2} \right\},
\]

where

\[
\mu = \min \left\{ \frac{[n_1 - 1]}{2}, k_1 - 1 \right\}.
\]

The proof of Theorem 2 is given in Section 3. The results that we shall use from the averaging theory of first and second order for computing limit cycles are presented in Section 2.

2. The Averaging Theory of First and Second Order

Now we summarize the basic results from averaging theory that we need for proving the results of this paper.

Consider the differential system

\[
\dot{x}(t) = \varepsilon F_1(x, t) + \varepsilon^2 F_2(x, t) + \varepsilon^3 R(x, t, \varepsilon),
\]

where \( F_1, F_2 : D \times \mathbb{R} \rightarrow \mathbb{R}^n \), \( R : D \times \mathbb{R} \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n \) are continuous functions, \( T \)-periodic in the first variable, and \( D \) is an open subset of \( \mathbb{R}^n \). Assume that the following hypotheses (i) and (ii) hold.

(i) \( F_1(\cdot, t) \in C^1(D) \) for all \( t \in \mathbb{R}, F_1, F_2, R \) and \( D, F_1 \) are locally Lipschitz with respect to \( x \), and \( R \) is differentiable with respect to \( \varepsilon \), where \( D, F_1 \) indicate the Jacobian matrix of \( F_1 \) with respect to \( x \).

We define \( F_{n0} : D \rightarrow \mathbb{R} \) for \( n = 1, 2 \) as

\[
F_{10}(z) = \frac{1}{T} \int_0^T F_1(z, s)ds,
\]

\[
F_{20}(z) = \frac{1}{T} \int_0^T \left[ D_z F_1(z, s)y(z, s) + F_2(z, s) \right] ds,
\]

where

\[
y(z, s) = \int_0^s F_1(z, t)dt.
\]

(ii) For \( V \subset D \) an open and bounded set and for each \( \varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\} \), there exists \( a_\varepsilon \in V \) such that

\[
F_{10}(a_\varepsilon) + F_{20}(a_\varepsilon) = 0 \text{ and } d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0.
\]

Then, for \( |\varepsilon| > 0 \) sufficiently small there exists a \( T \)-periodic solution \( \varphi(\cdot, \varepsilon) \) of system (5) such that \( \varphi(0, \varepsilon) = a_\varepsilon \).

The expression \( d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0 \) means that the Brouwer degree of the function \( F_{10} + \varepsilon F_{20} : V \rightarrow \mathbb{R}^n \) at the fixed point \( a_\varepsilon \) is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function \( F_{10} + \varepsilon F_{20} \) at \( a_\varepsilon \) is not zero.

If \( F_{10} \) is not identically zero, then the zeros of \( F_{10} + \varepsilon F_{20} \) at mainly the zeros of \( F_{10} \) for \( \varepsilon \) sufficiently small. In this case the previous result provides the averaging theory of first order.

If \( F_{10} \) is identically zero and \( F_{20} \) is not identically zero, then the zeros of \( F_{10} + \varepsilon F_{20} \) are mainly the zeros of \( F_{20} \) for \( \varepsilon \) sufficiently small. In this case the previous result provides the averaging theory of second order. For additional information on the averaging theory see for instance [14–16].
3. Proof of Theorem 2

3.1. Proof of Statement (a) of Theorem 2

In order to apply the first order averaging method we write system (4) with \( l = 1 \), in polar coordinates \((r, \theta)\) where \( x = r \cos \theta, y = r \sin \theta, r > 0 \).

If we take \( f_{n_1}^1(x) = \sum_{i=0}^{n_1} a_i x^i \) system (4) can be written as follows

\[
\begin{align*}
\dot{r} &= \frac{\rho}{2 \pi} \int_0^{2\pi} F_1(r, \theta) d\theta, \\
\dot{\theta} &= 1 + \varepsilon \left( -\frac{1}{2} \sum_{i=0}^{n_1} a_i r^i \cos^i \theta \sin \theta + r g_1(r \cos \theta, r \sin \theta) \cos \theta \right),
\end{align*}
\]

(6)

where

\[
g_1(r \cos \theta, r \sin \theta) = -A_1 + \sum_{i=1}^{m_1-2} \left( q_{i,1} r^i \cos^i \theta + \tilde{q}_{i,1} r^i \sin^i \theta \right).
\]

Now taking \( \theta \) as the new independent variable, system (6) becomes

\[
\frac{dr}{d\theta} = \rho F_1(r, \theta) + o(r^2),
\]

where

\[
F_1(r, \theta) = \sum_{i=0}^{n_1} a_i r^i \cos^i \theta + r^2 g_1(r \cos \theta, r \sin \theta) \sin \theta.
\]

By using the notation introduced in Section 2 we have that

\[
F_{10}(r) = \frac{1}{2\pi} \int_0^{2\pi} F_1(r, \theta) d\theta,
\]

\[
F_{10}(r) = \frac{1}{2\pi} \sum_{i=0}^{n_1} a_i r^i \int_0^{2\pi} \cos^{i+1} \theta d\theta + \frac{r^2}{2\pi} \sum_{i=1}^{m_1-2} \left( q_{i,1} r^i \cos^i \theta + \tilde{q}_{i,1} r^i \sin^i \theta \right). 
\]

We know that

\[
\frac{1}{2\pi} \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta = \begin{cases} 0, & \text{if } i \text{ odd or } j \text{ odd} \\ 1, & \text{if } i \text{ and } j \text{ are even.} \end{cases}
\]

(7)

Hence

\[
F_{10}(r) = r \left( \frac{1}{2\pi} \sum_{i=0}^{n_1-1} a_{2i+1} r^{2i} \int_0^{2\pi} \cos^{2i+2} \theta d\theta + \sum_{i=0}^{k_1-2} \frac{a_{i+1}}{2^{i+1} (i+1)!} q_{2i+1} r^{2i+2} \int_0^{2\pi} \sin^{2i+2} \theta d\theta \right),
\]

for every \( m_1 \in \{2k_1, 2k_1 - 1\} \).

Now using the expressions of the integrals in Appendix A, we obtain

\[
F_{10}(r) = r \left( \sum_{i=0}^{n_1-1} \frac{a_{i+1}}{2^{i+1} (i+1)!} a_{2i+1} r^{2i} + \sum_{i=0}^{k_1-2} \frac{a_{i+1}}{2^{i+1} (i+1)!} \tilde{q}_{2i+1} r^{2i+2} \right).
\]

(8)

For \( n_1 \geq 1 \), the polynomial \( F_{10}(r) \) has at most \( \lambda_1 = \max \{ \left[ \frac{n_1-1}{2} \right], k_1 - 1 \} \) positive roots. Hence (a) of Theorem 2 is proved.
3.2. Proof of Statement (b) of Theorem 2

For proving statement (b) of Theorem 2 we shall use the second-order averaging theory. If we write

\[ f_{N_2}^2(x) = \sum_{i=0}^{n_2} b_i x^i. \]

Then system (4) with \( l = 2 \) in polar coordinates \((r, \theta), r > 0\) becomes

\[
\begin{aligned}
    r &= \varepsilon \left( \sum_{i=0}^{n_1} a_i r^i \cos^{i+1} \theta + r^2 g_1(r \cos \theta, r \sin \theta) \sin \theta \right) + \\
    &\quad + \varepsilon^2 \left( \sum_{i=0}^{n_2} b_i r^i \cos^{i+1} \theta + r^2 g_2(r \cos \theta, r \sin \theta) \sin \theta \right), \\
    \dot{\theta} &= 1 + \varepsilon \left( -\frac{1}{r} \sum_{i=0}^{n_1} a_i r^i \cos^i \theta \sin \theta + r g_1(r \cos \theta, r \sin \theta) \cos \theta \right) + \\
    &\quad + \varepsilon^2 \left( -\frac{1}{r} \sum_{i=0}^{n_2} b_i r^i \cos^i \theta \sin \theta + r g_2(r \cos \theta, r \sin \theta) \cos \theta \right),
\end{aligned}
\]

where

\[ g_2(r \cos \theta, r \sin \theta) = -A_2 + \sum_{i=1}^{m_2-2} \left( q_{1,2} r^i \cos^i \theta + \tilde{q}_{1,2} r^i \sin^i \theta \right). \]

Taking \( \theta \) as the new independent variable system, (9) can be written as

\[
\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + o(\varepsilon^3),
\]

where

\[
F_1(r, \theta) = \sum_{i=0}^{n_1} a_i r^i \cos^{i+1} \theta + r^2 \sin \theta g_1(r \cos \theta, r \sin \theta),
\]

and

\[
F_2(r, \theta) = \sum_{i=0}^{n_2} b_i r^i \cos^{i+1} \theta + r^2 \sin \theta g_2(r \cos \theta, r \sin \theta)
\]

\[
+ \cos \theta \sin \theta \left( \sum_{i=0}^{n_1} a_i r^i \cos^i \theta \right)^2
\]

\[
- r^3 \sin \theta \cos \theta (g_1(r \cos \theta, r \sin \theta))^2
\]

\[
- r (2 \cos^2 \theta - 1) g_1(r \cos \theta, r \sin \theta) \sum_{i=0}^{n_1} a_i r^i \cos^i \theta.
\]

In order to compute \( F_{20}(r) \), we need that \( F_{10}(r) \) be identically zero. Then from (8), we have

\[
\begin{aligned}
    a_{2i+1} &= -\frac{2i+2}{2i+1} \bar{q}_{2i-1,1}, & & 1 \leq i \leq \mu, \\
    a_{2i+1} &= \tilde{q}_{2i-1,1}, & & 1 \leq i \leq \lambda_1, \\
    a_1 &= 0, & & i = 0,
\end{aligned}
\]

where

\[
\mu = \min \left\{ \left\lfloor \frac{n_1-1}{2} \right\rfloor, k_1-1 \right\}, \quad \lambda_1 = \max \left\{ \left\lfloor \frac{n_1-1}{2} \right\rfloor, k_1-1 \right\}.
\]

First, using (12) and, by substituting in (10), we obtain

\[
F_1(r, \theta) = -A_1 r^2 \sin \theta + \sum_{i=0}^{\left\lfloor n_1/2 \right\rfloor} a_{2i} r^{2i} \cos^{2i+1} \theta
\]

\[
+ \sin \theta \sum_{i=1}^{\left\lfloor (n_1-2)/2 \right\rfloor} \tilde{q}_{2i,1} r^{2i+2} \cos^i \theta + \sum_{i=1}^{\left\lfloor (n_1-2)/2 \right\rfloor} \tilde{q}_{2i,1} r^{2i+2} \sin^{2i+1} \theta
\]

\[
+ \sum_{i=1}^{\mu} \tilde{q}_{2i-1,1} r^{2i+1} \left( -\frac{2i+2}{2i+1} \cos^{2i+2} \theta + \sin^{2i+2} \theta \right),
\]

\[
F_2(r, \theta) = -A_2 + \sum_{i=1}^{m_2-2} \left( q_{1,2} r^i \cos^i \theta + \tilde{q}_{1,2} r^i \sin^i \theta \right).
\]
then

\[
\frac{d}{dr} F_1(r, \theta) = -2r A_1 \sin \theta + \sum_{i=0}^{m_1} 2i q_{2i, r^{2i+1} \cos^{2i+1} \theta} + \sum_{i=1}^{m_1-2} (i+2) q_{i, r^{i+1} \cos \theta} + \sum_{i=1}^{m_1-2} (2i+2) q_{2i, r^{2i+1} \sin^{2i+1} \theta} + \sum_{i=1}^{m_1-2} (2i+1) \tilde{q}_{2i-1, 1, r^{2i-1} \left( \frac{-2i+2}{2i+1} \cos^{2i+2} \theta + \sin^{2i+1} \theta \right)}.
\]

(14)

Again, using the integrals of Appendix A, we obtain

\[
y(r, \theta) = \int_{0}^{\theta} F_1(r, \phi) d\phi
\]

\[
= \sum_{i=0}^{m_1} \frac{a_i}{2^i} \sum_{l=1}^{i} \gamma_{l, i} \sin (2l + 1) \theta + \sum_{i=1}^{m_1-2} q_{i, r^{i+1} \frac{1}{i+1} (1 - \cos^{i+1} \theta)} + \sum_{i=1}^{m_1-2} \tilde{q}_{i, r^{i+1} \frac{1}{i+1} \cos^{i+1} \theta}
\]

\[
+ \sum_{i=1}^{m_1-2} \tilde{q}_{i, r^{i+1} \frac{1}{i+1} \cos \theta} + \sum_{i=1}^{m_1-2} \tilde{q}_{i, r^{i+1} \frac{1}{i+1} \sin \theta} - A_1 \theta^2 (1 - \cos \theta).
\]

Then, taking into account that

\[
\frac{1}{2^2l} \left( \frac{2i}{i} \right) - \frac{2i+2}{2i+1} \frac{1}{2^{2i+2}} \left( \frac{2i+2}{i+1} \right) = 0,
\]

\[
y(r, \theta) = \sum_{i=0}^{m_1} \left( a_i \sum_{l=1}^{i} \gamma_{l, i} \sin (2l + 1) \theta \right)
+ \sum_{i=1}^{m_1-2} q_{i, r^{i+1} \frac{1}{i+1} (1 - \cos^{i+1} \theta)}
+ \sum_{i=1}^{m_1-2} \tilde{q}_{i, r^{i+1} \frac{1}{i+1} \cos \theta}
+ \sum_{i=1}^{m_1-2} \tilde{q}_{i, r^{i+1} \frac{1}{i+1} \sin \theta}
- A_1 \theta^2 (1 - \cos \theta),
\]

(15)

where \( \xi_{i,j}, \gamma_{k,l}, \beta_{i,j}, \rho_{i,j} \) and \( \beta'_{i,j} \) are constants.

In order to apply the second order averaging method we need to compute the corresponding function \( F_{20}(r) \) that we rewrite as

\[
F_{20}(r) = F_{20}^1(r) + F_{20}^2(r),
\]

with

\[
F_{20}^2(r) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{d}{dr} F_1(r, \theta) y(r, \theta) d\theta,
\]
and

$$F_{20}^2(r) = \frac{1}{2\pi} \int_0^{2\pi} F_2(r, \theta) d\theta.$$  

**Lemma 1.** The integral $F_{20}^1(r)$ is a polynomial in the variable $r$ given by

$$F_{20}^1(r) = \sum_{i=1}^{[\frac{m_1}{2}]} \sum_{j=0}^{[\frac{n}{2}]} \left( W_{ij} q_{2i,1} + \tilde{W}_{ij} q_{2i,1} \right) a_{2j} r^{2j+1} + \sum_{i=0}^{\mu} V_{ij} A_{ij} q_{2j+1} r^{2j+2} + \sum_{i=0}^{\mu} Z_{ij} A_{ij} q_{2j+1} r^{2j+1},$$

where

$$W_{ij} = -\frac{2j}{2j+1} l_{2j+2,0} + (2i + 2) S_{ij}, \tilde{W}_{ij} = (2i + 2) F_{ij} + 2j Q_{ij},$$

$$V_{ij} = (2i + 3) D_{ij} + \frac{1}{i+1} K_{ij}, \tilde{V}_{ij} = (2i + 2) l_{2j+2} - (2j + 1) l_{2j,0},$$

and

$$Z_i = 2 (l_{2i+2,0} - R_i),$$

where $S_{ij}, F_{ij}, Q_{ij}, D_{ij}$ and $K_{ij}$ are real constants and $l_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \cos^i \theta \sin^j \theta d\theta$.

**Proof.** From (14) and (15) we have

$$F_{20}^1(r) = h_1(r) + h_2(r) + h_3(r) + h_4(r) + h_5(r),$$

where

$$h_1(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{\mu} 2ia_{2j} r^{2i-1} \cos^{i+1} \theta y(r, \theta) d\theta,$$

$$h_2(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{\mu} (i + 2) q_{ij} r^{i+1} \cos^i \theta \sin \theta y(r, \theta) d\theta,$$

$$h_3(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{\mu} (2i + 2) \tilde{q}_{2i} r^{2i+1} \cos^{i+1} \theta y(r, \theta) d\theta,$$

$$h_4(r) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=0}^{\mu} (2i + 2) \tilde{q}_{2i} r^{2i+1} \sin^{i+1} \theta y(r, \theta) d\theta,$$

$$h_5(r) = \frac{1}{\pi} \int_0^{2\pi} -r A_{ij} \sin \theta y(r, \theta) d\theta.$$

For simplifying the expression of the polynomial $h_1(r)$, using the integrals of Appendix A, we have

$$\left(\Delta_1\right) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=0}^{[\frac{n}{2}]} 2j a_{2j} r^{2j-1} \cos^{2j+1} \theta \right) \left( \sum_{i=0}^{[\frac{m}{2}]} a_{2i} r^{2i} \sum_{i=1}^{[\frac{n}{2}]} \gamma_{ij} \sin (2i + 1) \theta \right) \theta d\theta = 0,$$
\[(\Delta_2) \frac{2\pi}{\Xi} \int_0^{2\pi} \left( \sum_{j=1}^{2n} 2ja_j r^{2j-1} \cos^{2j+1} \theta \right) \left( \sum_{s=1}^{m-1} q_{s,1} r^{s+2} \frac{1}{s+1} (1 - \cos^{s+1} \theta) \right) d\theta \]
\[= - \sum_{j=0}^{2n} \sum_{i=1}^{m-2} \frac{2j+1}{2j+1} I_{2j+2} 2ja_j q_{2j+1} r^{2j+1}, \]

\[(\Delta_3) \frac{2\pi}{\Xi} \int_0^{2\pi} \left( \sum_{j=1}^{2n} 2ja_j r^{2j-1} \cos^{2j+1} \theta \right) \left( \sum_{i=1}^{m-2} \zeta_{i,1} \cos (2i + 1) \theta \right) d\theta \]
\[= \sum_{j=0}^{2n} \sum_{i=1}^{m-2} 2jQ_{ij} a_j q_{2i+1} r^{2j+1}, \]

where

\[Q_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \left( \cos^{2j+1} \theta \right) \left( \zeta_{i,1} \cos (2i + 1) \theta \right) d\theta, \]

\[(\Delta_4) \int_0^{2\pi} \left( \sum_{j=1}^{2n} 2ja_j r^{2j-1} \cos^{2j+1} \theta \right) \left( \sum_{i=1}^{m} \zeta_{2i-1,1} r^{2i+1} \left( \sum_{l=1}^{i+1} \beta_{l,1} \sin (2l) \theta \right) \right) d\theta = 0 \]

\[(\Delta_5) \frac{2\pi}{\Xi} \int_0^{2\pi} \left( \sum_{j=1}^{2n} 2ja_j r^{2j-1} \cos^{2j+1} \theta \right) \left( -A_1 r^2 (1 - \cos \theta) \right) d\theta \]
\[= \sum_{j=0}^{2n} 2I_{2j+2} 2ja_j A_1 r^{2j+1}. \]

We have that the sum of the integrals \((\Delta_1)-(\Delta_5)\) is the polynomial \(h_1(r)\).

From the integrals of Appendix A, we have

\[(\Delta_6) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{s=1}^{m-2} (1+2)q_{s,1} r^{s+1} \cos^s \theta \sin \theta \right) \left( \sum_{j=1}^{2n} a_j r^{2j} \sum_{i=1}^j \gamma_{i,1} \sin (2i + 1) \theta \right) d\theta \]
\[= \sum_{i=1}^{m-2} \sum_{j=0}^{2n} (2i+2)S_{ij} a_j q_{2j+1} r^{2j+1}, \]

where

\[S_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \left( \cos^i \theta \sin \theta \right) \left( \sum_{i=1}^j \gamma_{i,1} \sin (2i + 1) \theta \right) d\theta. \]

\[(\Delta_7) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m-2} (j+2)q_{j,1} r^{j+2} \cos^j \theta \sin \theta \right) \left( \sum_{i=1}^{m-2} q_{i,1} r^{i+2} \frac{1}{i+1} (1 - \cos^{i+1} \theta) \right) d\theta = 0 \]

\[(\Delta_8) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m-2} (j+2)q_{j,1} r^{j+1} \cos^j \theta \sin \theta \right) \left( \sum_{i=1}^{m-2} \zeta_{i,1} \cos (2i + 1) \theta \right) d\theta = 0 \]

\[(\Delta_9) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{s=1}^{m-2} (s+2)q_{s,1} r^{s+1} \cos^s \theta \sin \theta \right) \left( \sum_{i=1}^{m-2} q_{2i-1,1} r^{2i+1} \left( \sum_{l=1}^{i+1} \beta_{l,1} \sin (2l) \theta \right) \right) d\theta \]
\[= \sum_{i=0}^{k_1-1} \sum_{j=1}^{m_1} (2i+3)D_{ij} q_{2j+1,1} \tilde{q}_{2j+1,1} r^{2j+3}, \]
where
\[
D_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \left( \cos^{2i+1} \theta \sin \theta \sum_{l=1}^{j+1} \hat{p}_{ij} \sin (2l\theta) \right) d\theta.
\]

\((\Delta_{10})\) \(\frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_1-2} (j+2)q_{j,1}r^{j+1} \cos^j \theta \sin \theta \right) \left( -A_1 r^2 (1 - \cos \theta) \right) d\theta = 0
\]

The sum of the integrals \((\Delta_6)-(\Delta_{10})\) is the polynomial \(h_2(r)\).

For finding the expression of the polynomial \(h_3(r)\), using the integrals of Appendix A, we have
\[
(\Delta_{11}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_2} (2i+2)\tilde{q}_{2j,1}r^{2i+1} \sin^{2i+1} \theta \right) \left( \sum_{j=1}^{m_2} a_{2j}r^{2j} \sum_{l=1}^{i+1} \gamma_{ij} \sin (2l+1) \theta \right) d\theta = 0
\]

where
\[
F_{ij} = \frac{1}{2\pi} \int_0^{2\pi} \left( \sin^{2i+1} \theta \sum_{l=1}^{j+1} \gamma_{ij} \sin (2l+1) \theta \right) d\theta,
\]

\[(\Delta_{12}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_2} (2j+2)\tilde{q}_{2j,1}r^{2j+1} \sin^{2j+1} \theta \right) \left( \sum_{j=1}^{m_2} \tilde{q}_{j,1}r^{j+2} \frac{1}{i+1} (1 - \cos^{i+1} \theta) \right) d\theta = 0
\]

\[(\Delta_{13}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_2} (2j+2)\tilde{q}_{2j,1}r^{2j+1} \sin^{2j+1} \theta \right) \left( \sum_{j=1}^{m_2} \tilde{q}_{2j,1}r^{2j+2} \sum_{l=1}^{i+1} \tilde{\xi}_{ij} \cos (2l+1) \theta \right) d\theta = 0
\]

\[(\Delta_{14}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_2} (2j+2)\tilde{q}_{2j,1}r^{2j+1} \sin^{2j+1} \theta \right) \left( \sum_{j=1}^{m_2} \tilde{q}_{2j-1,1}r^{2j+1} \left( \sum_{l=1}^{i+1} \tilde{\beta}_{ij} \sin (2l \theta) \right) \right) d\theta = 0
\]

\[(\Delta_{15}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{m_2} (2j+2)\tilde{q}_{2j,1}r^{2j+1} \sin^{2j+1} \theta \right) \left( -A_1 r^2 (1 - \cos \theta) \right) d\theta = 0
\]

We have that the sum of the integrals \((\Delta_{11})-(\Delta_{15})\) is the polynomial \(h_3(r)\).

From the integrals of Appendix A, we have
\[
(\Delta_{16}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{\mu} (2j+1) \tilde{q}_{2j-1,1}r^{2j} \left( -\frac{2j+2}{2j+1} \cos^{2j+2} \theta + \sin^{2j} \theta \right) \right) \left( \sum_{i=0}^{m_2} a_{2i}r^{2i} \sum_{l=1}^{i+1} \gamma_{ij} \sin (2l+1) \theta \right) d\theta = 0,
\]

\[(\Delta_{17}) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^{\mu} (2j+1) \tilde{q}_{2j-1,1}r^{2j} \left( -\frac{2j+2}{2j+1} \cos^{2j+2} \theta + \sin^{2j} \theta \right) \right) \left( \sum_{i=1}^{m_2} q_{j,1}l^{i+2} \frac{1}{i+1} (1 - \cos^{i+1} \theta) \right) d\theta
\]

\[
= \sum_{j=1}^{\mu} \sum_{i=1}^{m_2} -\frac{1}{i+1} H_{j}q_{j,1} \tilde{q}_{2j-1,1}r^{i+2} + \sum_{i=0}^{\mu} \sum_{i=1}^{m_1-2} \frac{1}{i+1} K_{ij}q_{2i+1,1} \tilde{q}_{2j-1,1}r^{2i+2} + \sum_{j=1}^{\mu} \sum_{i=1}^{k_2-2} \frac{1}{i+1} K_{ij}q_{2i+1,1} \tilde{q}_{2j-1,1}r^{2i+2} + \sum_{j=1}^{\mu} \sum_{i=1}^{k_2-2} \frac{1}{i+1} K_{ij}q_{2i+1,1} \tilde{q}_{2j-1,1}r^{2i+2} + \sum_{j=1}^{\mu} \sum_{i=1}^{k_2-2} \frac{1}{i+1} K_{ij}q_{2i+1,1} \tilde{q}_{2j-1,1}r^{2i+2} \right)\]
where
\[ H_j = (2j + 2)I_{2j+2,0} + (2j + 1)I_{0,2j}, \]
and
\[ K_{ij} = (2j + 2)I_{2j+2,0} - (2j + 1)I_{2j+2,2j}. \]

\[ (\Delta 18) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{i=1}^{m} (2j + 1) \overline{q}_{2j-1,1}r^{2j} \left( -\frac{2j + 2}{2j + 1} \cos^{2j+2} \theta + \sin^{2j} \theta \right) \right) \times \left( \sum_{i=1}^{m} \overline{q}_{2i,1}r^{2i+2} \sum_{l=1}^{i} \xi_{i,l} \cos (2l + 1) \theta \right) \, d\theta = 0, \]

\[ (\Delta 19) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{i=1}^{m} (2j + 1) \overline{q}_{2j-1,1}r^{2j} \left( -\frac{2j + 2}{2j + 1} \cos^{2j+2} \theta + \sin^{2j} \theta \right) \right) \times \left( \sum_{i=1}^{m} \overline{q}_{2i-1,1}r^{2i+2} \left( \sum_{l=1}^{i+1} \beta_{i,l} \sin (2l\theta) \right) \right) \, d\theta = 0, \]

\[ (\Delta 20) \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{i=1}^{m} (2j + 2)I_{2j+2} - (2j + 1)I_{2j,0} \right) A_{2i+1}(\theta) \, d\theta = \sum_{j=1}^{\mu} \left( 2j + 2 \right) I_{2j+2} - (2j + 1)I_{2j,0} A_{2j+1}(\theta) \, d\theta. \]

We have that the sum of the integrals \((\Delta 16)−(\Delta 20)\) is the polynomial \(h_4(r)\).

Finally, for computing the polynomial \(h_5(r)\), using the integrals of Appendix A, we have

\[ (\Delta 21) \frac{1}{2\pi} \int_0^{2\pi} \left( -2rA_1 \sin \theta \right) \left( \sum_{i=0}^{n} a_{2i}r^{2i} \sum_{l=1}^{i} \gamma_{i,l} \sin (2l + 1) \theta \right) \, d\theta = -2 \sum_{i=0}^{n} A_1 R_{i} \, d\theta, \]

where
\[ R_i = \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{l=1}^{i} \gamma_{i,l} \sin \theta \sin (2l + 1) \theta \right) \, d\theta, \]

\[ (\Delta 22) \frac{1}{2\pi} \int_0^{2\pi} \left( -2rA_1 \sin \theta \right) \left( \sum_{i=1}^{m} \overline{q}_{i,1}r^{i+2} \frac{1}{i+1} (1 - \cos^{i+1} \theta) \right) \, d\theta = 0, \]

\[ (\Delta 23) \frac{1}{2\pi} \int_0^{2\pi} \left( -2rA_1 \sin \theta \right) \left( \sum_{i=1}^{m} \overline{q}_{2i,1}r^{2i+2} \sum_{l=1}^{i} \xi_{i,l} \cos (2l + 1) \theta \right) \, d\theta = 0, \]

\[ (\Delta 24) \frac{1}{2\pi} \int_0^{2\pi} \left( -2rA_1 \sin \theta \right) \left( \sum_{i=1}^{m} \overline{q}_{2i-1,1}r^{2i+2} \left( \sum_{l=1}^{i+1} \beta_{i,l} \sin (2l\theta) \right) \right) \, d\theta = 0, \]

\[ (\Delta 25) \frac{1}{2\pi} \int_0^{2\pi} \left( -2rA_1 \sin \theta \right) \left( -A_{1}r^{2} (1 - \cos \theta) \right) \, d\theta = 0. \]

We have that the sum of the integrals \((\Delta 21)−(\Delta 25)\) is the polynomial \(h_5(r)\). Hence Lemma 1 is proved. \(\Box\)
Lemma 2. The integral $F_{20}^2(r)$ is a polynomial in the variable $r$ given by

$$F_{20}^2(r) = \sum_{s=0}^{m-1} I_{2s+2,0} b_{2s+1} r^{2s+1} + \sum_{s=0}^{k-2} I_{0,2s+2} q_{2s+1,2} r^{2s+3}$$

$$-2 \sum_{s=0}^{k-2} \sum_{j=1}^{\mu} q_{2s+1,1} \bar{q}_{2j-1,1} I_{2s+2,2j+2} r^{2i+2j+7}$$

$$+ \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} (i-j+1)(2i+2) (2i+1) I_{2i+2,2j} \bar{q}_{2i-1,1} \bar{q}_{2j-1,1} r^{2i+2j+3}.$$

Proof. Using (12) and, substituting in (11) we have

$$F_2(r, \theta) = g_1(r, \theta) + g_2(r, \theta) + g_3(r, \theta) + g_4(r, \theta),$$

where

$$g_1(r, \theta) = \sum_{i=0}^{\mu} h_i r^i \cos r^i \theta + r^2 \left( \sum_{i=1}^{\mu-2} \left( q_{i,2} r^{i \cos \theta} + \bar{q}_{i,2} r^{i \sin \theta} \right) - A_2 \right) \sin \theta,$$

$$g_2(r, \theta) = \frac{\cos \theta \sin \theta}{r} \left( \sum_{i=0}^{\mu} a_{2i} r^{2i} \cos 2i+\theta + \sum_{i=1}^{\mu} \frac{2i+2}{2i+1} \bar{q}_{2i-1,1} r^{2i+1} \cos 2i+2 \theta \right)^2,$$

$$g_3(r, \theta) = -r^3 \sin \theta \cos \theta \left( \sum_{i=1}^{\mu-2} q_{i,1} r^{i+2} \cos i \theta \sin \theta + \sum_{i=1}^{\mu} \bar{q}_{i,1} r^{2i+2} \sin 2i+1 \theta \right)$$

$$+ \sum_{i=1}^{\mu} \bar{q}_{2i-1,1} r^{2i+1} \sin 2i \theta - A_1 r^2 \sin \theta)^2,$$

and

$$g_4(r, \theta) = -r \left( 2 \cos^2 \theta - 1 \right) \left( \sum_{i=0}^{\mu} a_{2i} r^{2i} \cos 2i+1 \theta + \sum_{i=1}^{\mu} \frac{2i+2}{2i+1} \bar{q}_{2i-1,1} r^{2i+1} \cos 2i+2 \theta \right)$$

$$\times \left( \sum_{i=1}^{\mu-2} q_{i,1} r^{i+2} \cos i \theta \sin \theta + \sum_{i=1}^{\mu} \bar{q}_{i,1} r^{2i+2} \sin 2i+1 \theta \right)$$

$$+ \sum_{i=1}^{\mu} \bar{q}_{2i-1,1} r^{2i+1} \sin 2i \theta - r^2 \sin \theta A_1 \right).$$

For an explicit expression of the polynomial $F_{20}^2(r)$, using (7), we have

$$\bar{\Lambda}_1 = \frac{1}{2\pi} \int_0^{2\pi} g_1(r, \theta) d\theta$$

$$= \sum_{s=0}^{m-1} I_{2s+2,0} b_{2s+1} r^{2s+1} + \sum_{s=0}^{k-2} I_{0,2s+2} q_{2s+1,2} r^{2s+3},$$
for every \( m_2 \in \{2k_2, 2k_2 - 1\} \),

\[
\tilde{\Delta}_2 = \frac{1}{2\pi} \int_0^{2\pi} g_2(r, \theta) d\theta \\
= \frac{1}{2\pi r} \int_0^{2\pi} \cos \theta \sin \theta \left( \sum_{i=0}^{[m_2]} a_{2i} r^{2i+1} \cos^{2i+1} \theta \right)^2 d\theta \\
+ \frac{1}{2\pi r} \int_0^{2\pi} \cos \theta \sin \theta \left( \sum_{i=0}^{[\mu]} -\frac{2i + 2}{2i + 1} \tilde{q}_{2i-1, 1} r^{2i+1} \cos^{2i+2} \theta \right)^2 d\theta \\
- \frac{1}{2\pi r} \int_0^{2\pi} \cos \theta \sin \theta \left( \sum_{i=0}^{[\mu]} \frac{2j + 2}{2j + 1} q_{2i-1, 1} r^{2i+2} \cos^{2i+3} \theta \right) d\theta \\
= 0,
\]

\[
\tilde{\Delta}_3 = \frac{1}{2\pi} \int_0^{2\pi} g_3(r, \theta) d\theta = -\frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{i=1}^{[m_1]} \sum_{j=1}^{[\mu]} q_{i, 1} \tilde{q}_{2i-1, 1} r^{i+2i+6} \cos^{i+1} \theta \sin^{2i+2} \theta \right) d\theta \\
= -2 \sum_{i=0}^{k_1 - 2} \sum_{j=1}^{[\mu]} q_{2i+1, 1} \tilde{q}_{2j-1, 1} I_{2s+2, 2j+2} r^{2s+2j+7},
\]

\[
\tilde{\Delta}_4 = \frac{1}{2\pi} \int_0^{2\pi} g_4(r, \theta) d\theta \\
= 2 \sum_{i=1}^{[\mu]} \sum_{j=1}^{[\mu]} \frac{2i + 2}{2i + 1} I_{2i+4, 2j} \tilde{q}_{2i-1, 1} \tilde{q}_{2j-1, 1} r^{2i+2j+3} \\
- \sum_{i=1}^{[\mu]} \sum_{j=1}^{[\mu]} \frac{2j + 2}{2j + 1} I_{2i+2, 2j} \tilde{q}_{2i-1, 1} \tilde{q}_{2j-1, 1} r^{2i+2j+3}.
\]

From \( I_{2i+4, 2j} = \frac{2i+3}{2i+1} I_{2i+2, 2j} \), we have that

\[
\tilde{\Delta}_4 = \sum_{i=1}^{[\mu]} \sum_{j=1}^{[\mu]} \frac{(i + 1) (2i + 2)}{(i + j + 2) (2i + 1)} I_{2i+2, 2j} \tilde{q}_{2i-1, 1} \tilde{q}_{2j-1, 1} r^{2i+2j+3}.
\]

The sum of the integrals \( \tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_3 \) and \( \tilde{\Delta}_4 \) is the polynomial \( F_{20}^2(r) \). Hence Lemma 2 is proved. \( \square \)

By Lemmas 1 and 2, we have

\[
F_{20}(r) = r \left( \sum_{i=1}^{[n_2-2]} \sum_{j=0}^{[\mu]} (W_{ij} q_{2i+1} + \tilde{W}_{ij} \tilde{q}_{2i+1}) a_{2j} r^{2j+2} \right) \\
+ \sum_{i=0}^{k_1 - 2} \sum_{j=1}^{[\mu]} V_{ij} q_{2i+1, 1} \tilde{q}_{2j-1, 1} r^{2i+2j+2} \\
- \sum_{i=1}^{[\mu]} \sum_{j=1}^{[n_2-2]} \frac{1}{i+1} H_{ij} \tilde{q}_{i+1, 1} \tilde{q}_{2j-1, 1} r^{i+2j+1} + \sum_{i=0}^{[\mu]} Z_i A_1 a_{2i} r^{2i} \\
+ \sum_{j=1}^{[\mu]} \tilde{V}_{ij} A_1 \tilde{q}_{2j-1, 1} r^{2j+1} + \sum_{s=0}^{[n_2-1]} I_{2s+2, 2s} \tilde{q}_{2s+1, 1} r^{2s}.
\]
we conclude that $F_{20}$ has at most

$$\lambda_2 = \max \left\{ \left[ \frac{m_1 - 2}{2} \right], \left[ \frac{n_1}{2} \right], \left[ \frac{m_1 - 2}{2} \right] + \mu, \left[ \frac{n_1}{2} \right], \left[ \frac{n_2 - 1}{2} \right], k_2 - 1, k_1 + \mu + 1, 2\mu + 1 \right\},$$

positive roots. Hence (b) of Theorem 2 is proved.

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**Appendix A. Formulae**

In this appendix we recall some formulae that will be used during the paper, (see [17]). For $i, j, k \geq 0$ we have

\[
\int_0^\theta \cos^{2i+1} \phi d\phi = \sum_{l=1}^{i} \gamma_{i,l} \sin (2l+1) \theta, \\
\int_0^\theta \cos^i \phi \sin \phi d\phi = \frac{1}{i+1} (1 - \cos^{i+1} \theta), \\
\int_0^\theta \sin^{2i+1} \phi d\phi = \sum_{l=1}^{i} \beta_{i,l} \cos (2l+1) \theta, \\
\int_0^\theta \cos^{2i+2} \phi d\phi = \frac{1}{2^{i+2}} \left( \begin{array}{c} 2i+2 \\ i+1 \end{array} \right) \theta + \sum_{l=1}^{i+1} \beta_{i,l} \sin (2l\theta), \\
\int_0^\theta \sin^{2i} \phi d\phi = \frac{1}{2^i} \left( \begin{array}{c} 2i \\ i \end{array} \right) \theta + \sum_{l=1}^{i} \rho_{i,l} \sin (2l\theta),
\]

where $\gamma_{i,l}, \beta_{i,l}, \beta'_{i,l}$ and $\rho_{i,l}$ are non-zero constants.

\[
\int_0^{2\pi} \sin^{2i} \theta d\theta = \int_0^{2\pi} \cos^{2i} \theta d\theta = \frac{\pi \alpha_i}{2^i-1},
\]

where $\alpha_i = 3.5...(2i-1), \alpha_{i+1} = (2i+1)\alpha_i$,

\[
\int_0^{2\pi} \sin^i \theta \cos (2l+1) \theta d\theta = \int_0^{2\pi} \cos^i \theta \sin (2l+1) \theta d\theta = 0, l \geq 0,
\]
\[\int_{0}^{2\pi} \sin^l \theta \sin (2l \theta) d\theta = \int_{0}^{2\pi} \cos^l \theta \sin(2l \theta)d\theta = 0, l \geq 0,\]

\[\int_{0}^{2\pi} \sin^l \theta \sin (2l+1 \theta)\theta d\theta = \begin{cases} 0, & \text{if } i = 2k \\ \Gamma_{k,l}, & \text{if } i = 2k+1 \end{cases}, l \geq 0,\]

\[\int_{0}^{2\pi} \cos^l \theta \cos (2l+1 \theta)\theta d\theta = \begin{cases} 0, & \text{if } i = 2k \\ \Lambda_{k,l}, & \text{if } i = 2k+1 \end{cases}, l \geq 0,\]

\[\int_{0}^{2\pi} \cos^l \theta \sin \theta \sin (2l \theta)d\theta = \begin{cases} 0, & \text{if } i = 2k+1 \\ \Gamma_{k,l}, & \text{if } i = 2k \end{cases}, l \geq 1,\]

\[\int_{0}^{2\pi} \cos^l \theta \sin \theta \cos(2l+1 \theta)d\theta = 0, l \geq 0,\]

where \(\Gamma_{k,l}, \Lambda_{k,l}, \Gamma_{k,l}\) and \(\Lambda_{k,l}\) are real constants.

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