Geodesic multiplication as a tool for classical and quantum gravity*

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Abstract

Algebraic systems called the local geodesic loops and their tangent Akivis algebras are considered. Their possible role in theory of gravity is considered. Quantum conditions for the infinitesimal quantum events are proposed.

1 Introduction

Spacetime and event are two fundamental concepts of relativistic physics. According to theory of general relativity, totality of all events or the classical (pre-quantum) spacetime can be represented by a (1 + 3)-dimensional Lorentzian manifold. Evolution of the Lorentzian metric tensor is prescribed by the Einstein equations, constraining the Einstein and energy-momentum tensors of spacetime to be proportional. On the classical level, gravitation reveals itself through curvature of spacetime. In this sense the general relativity may be proclaimed to be a geometrical theory of the gravitational field.

There have been several attempts to quantize gravity. In fact, quantization of the gravitational field is one of the most intriguing problems in theoretical physics. In a sense, quantization is first of all an algebraic method for one looks for the quantum observables which must imitate the algebraic properties of the classical ones. So every algebraic aspect of the classical spacetime must be thoroughly taken into account as well.

In this paper we outline an idea of the geodesic quantization, based on construction of the local geodesic multiplication \( GM \) of spacetime points (Sec. 2). For the Minkowski spacetime of special relativity, the \( GM \) in fact coincides with the common vector addition rule. Hence, in this (Minkowskian) case the \( GM \) turns out to be a globally defined commutative and associative operation. The Abelian property of the \( GM \) is in fact due to the globally vanishing torsion and curvature of the Minkowski space.

Geodesic multiplication \([1, 2, 3]\) on manifolds with an affine connection is mathematically rather elaborated but is still not widely known for physicists. Nevertheless, it deserves attention as well.

Via the \( GM \) one can treat spacetime algebraically, as a collection of the algebraic systems called the local geodesic loops. In general relativity, the \( GM \) acquires dynamical meaning. Nonassociativity of \( GM \) is an algebraic manifestation of the curvature of spacetime (Sec. 3). Tangent spaces at spacetime points turn out to be binary-ternary algebras called the Akivis algebras. Based on this fact, the geodesic quantum conditions are proposed for the infinitesimal quantum events (Sec. 4).

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2 Local geodesic multiplication

At first let us introduce some basic algebraic notions.

A quasigroup [4, 5] is a set $G$ with a binary operation (which we denote by juxtaposition) which has the following property: in equation $gh = k$, knowledge of any two elements specifies the third one uniquely. A quasigroup with a unit element is called a loop [4, 5].

For a fixed point $e$ of spacetime $M$, choose a tangent vector $X$ from the tangent space $T_e(M)$ of $M$ at $e$. Consider a local path $t \mapsto g(t; X)$ in $M$ through point $e$ with the tangent vector $X$ at $e$:

$$ g^i(0; X) = e^i, \quad \partial_t g^i(0; X) = X^i. \quad (2.1) $$

It is well known that this path is the unique local geodesic path through $e$ in direction $X$ iff the following differential equation holds:

$$ \partial_t^2 g^i + \Gamma^i_{lm} \partial_t g^l \partial_t g^m = 0, \quad (2.2) $$

where $\Gamma^i_{lm}$ are the affine connection coefficients.

The exponential mapping $X \mapsto g := \text{Exp}_e(X) := g(1; X)$ at $e$ is known [6] to be a local diffeomorphism of a suitable neighbourhood of origin of $T_e(M)$ onto the corresponding (normal) neighbourhood of $e \in M$. Note that this property allows us treat the tangent vectors of the spacetime as infinitesimal events. We can also say that every event from the normal neighbourhood of $e$ can be generated via the exponential mapping by the corresponding tangent vector from $T_e(M)$.

The local geodesic loop at $e$ can be constructed in such a neighbourhood $M_e$ of $e$, where all required exponential mappings are well defined local diffeomorphisms. Choose in $M_e$ another local geodesic arc $h(s; Y)$ through the point $e$ with the direction $Y \in T_e(M)$. To perform the parallel transport of $X \in T_e(M)$ along this geodesic, we must solve the linear Cauchy problem

$$ \partial_s X^i + \Gamma^i_{ml} \partial_s h^l X^m = 0, \quad X'(0) = X. \quad (2.3) $$

Performing the parallel transport of $X \in T_e(M)$, we obtain at $h := \text{Exp}_e(Y)$ the tangent vector $X' := X'(1)$ in $T_h(M)$. Now, draw the local geodesic arc through $h$ in the direction $X'$, and mark point $\text{Exp}_h(X')$ on it. This point is called the product of $g$ and $h$, and it will be denoted as $gh$. Explicitly, the multiplication formula reads [1]

$$ gh = (\text{Exp}_h \circ \tau^e_h \circ \text{Exp}_e^{-1})g, \quad (2.4) $$

where $\tau^e_h : T_e(M) \rightarrow T_h(M)$ denotes the parallel transport mapping of the tangent vectors from $T_e(M)$ into $T_h(M)$ along the unique local geodesic arc joining points $e$ and $h : \tau^e_h(X) = X'$. The neighbourhood $M_e$ of $e$ with multiplication rule [2,4] is a local differentiable loop [1, 2, 3] denoted henceforth by $M_e$ as well. The unit element of $M_e$ is $e$, and the local geodesic paths through the unit element $e$ are the one–parameter subgroups of $M_e$. One can also say that the local geodesic multiplication is monoassociative:

$$ gg \cdot g = g \cdot gg, \quad \forall g \in M_e. \quad (2.5) $$

Note that the crucial part of the construction lies on Cauchy problems (2.1), (2.2) and (2.3), on existence and uniqueness of their solutions, and also on the local diffeomorphism property of the exponential mapping [6].

We can repeat the above construction and attach a local geodesic loop to all reasonable points of the spacetime. The patching conditions for the local geodesic loops attached to different points of a manifold with affine connection have been described by L.V. Sabinin [7].

One can easily check that all geodesic loops of the Minkowski spacetime are the Abelian groups. In this particular case, the geodesic multiplication can be found from the common vector addition rule. The Abelian property manifests algebraically the fact that the affine spaces are globally torsionless and flat.
3 Akivis algebras

Generally speaking, the geodesic loops need not be commutative and associative. There may exist such a triple of points \( g, h, k \) in \( M_e \) that

\[
gh \neq hg, \quad gh \cdot k \neq g \cdot hk.
\]  

(3.1)

Let us choose in \( M_e \) the local coordinates where \( e^i = 0 \) for all \( i \). Deviation of \( M_e \) from commutativity and associativity can be measured \([3]\) by the structure constants \( c^i_{lm} \) and \( A^i_{lmn} \) defined by

\[
(gh)^i - (hg)^i = C^i_{lm}g^lh^m + \cdots,
\]  

(3.2)

\[
(gh \cdot k)^i - (g \cdot hk)^i = A^i_{lmn}g^lh^mk^n + \cdots.
\]  

(3.3)

where dots mean the higher order terms. It turns out that non-commutativity and non-associativity of the local geodesic loops are intimately related to torsion and curvature of space-time. Denote the torsion and curvature tensors as \( S^i_{lm}(e) \) and \( R^i_{lmn}(e) \), respectively. The direct computations \([3]\) show that

\[
C^i_{lm} = 2S^i_{lm}(e),
\]  

(3.4)

\[
A^i_{lmn} = R^i_{lmn}(e) - \nabla_n S^i_{lm}(e),
\]  

(3.5)

where \( \nabla_n \) denotes the covariant differentiation operator.

We can now introduce \([8, 9]\) the tangent algebra \( A_e \) of \( M_e \) similarly to the tangent (Lie) algebra of a Lie group. Geometrically, the tangent algebra \( A_e \) coincides with the tangent space \( T_e(M) \) of \( M_e \) at \( e \). The product \([X, Y]\) of \( X, Y \in A_e \) is defined in \( A_e \) by

\[
[X, Y]^i := C^i_{lm}X^lY^m = -[Y, X]^i.
\]  

(3.6)

We can equip \( A_e \) with a ternary operation as well \([3, 10]\). For a triple \( X, Y, Z \in A_e \), define their triple product \((X, Y, Z)\) in \( A_e \) by

\[
(X, Y, Z)^i := A^i_{lmn}X^lY^mZ^n.
\]  

(3.7)

The tangent algebra \( A_e \) is thus a binary–ternary algebra, and it need not be a Lie algebra. In other words, there may be a triple \( X, Y, Z \in A_e \), such that the Jacobi identity fails in \( A_e \):

\[
J(X, Y, Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \neq 0.
\]  

(3.8)

Instead, for all \( X, Y, Z \) in \( A_e \), we have \([3]\) a more general identity

\[
J(X, Y, Z) = (X, Y, Z) + (Y, Z, X) + (Z, X, Y) - (X, Z, Y) - (Z, Y, X) - (Y, X, Z)
\]  

(3.9)

called the Akivis identity. The binary-ternary algebra \( A_e \) is hence called the Akivis algebra.

Comment. It is well known that the tangent algebras of the local Lie groups are the Lie algebras \([11]\). Non–Lie Akivis algebras appeared first as the tangent algebras of the local analytic loops \([10]\). The tangent algebras of the local analytic Moufang (Bol) loops turn out to be the Mal’tsev (Bol) algebras \([8] \ ([12] \ [13] \ [14]) \). These cases are quite remarkable for the following reason (generalized converse third Lie theorem): every real finite–dimensional Mal’tsev (Bol) algebra is the tangent algebra of some analytic Moufang (local Bol) loop \([15] \ [16] \ [17] \ ([12] \ [13] \ [14]) \). The converse third Lie theorem for the general Akivis algebras has been discussed in \([10] \ [18]\).
4 Quantum conditions

One can construct the Akivis algebras via non–associative algebras as well. If we denote multiplication of a non–associative algebra by juxtaposition, then its commutator–associator algebra (CA–algebra) is the one with the following commutator and associator brackets:

\[ [x, y] := xy - yx, \quad (x, y, z) := xy \cdot z - x \cdot yz. \]  

(4.1)

Anti–commutativity of the commutator bracketing is obvious, and the Akivis identity can be checked by direct calculations. The original non–associative algebra can be said to be the enveloping algebra of the corresponding CA–algebra.

Comment. It is well known that the CA–algebras of associative algebras are the Lie algebras. CA–algebras of alternative algebras are the Mal’tsev algebras \[8\]. CA–algebras of right–alternative algebras turn out to be the Bol algebras \[13, 20\]. The problem of embedding of non–Lie Akivis algebras into non–associative (enveloping) algebras (generalization of the Birkhoff–Witt theorem) is posed in \[10\] and has not been solved yet\[1\].

We can exploit the above–presented property of the CA–algebras and propose the quantum conditions for the infinitesimal quantum events, called the geodesic quantum conditions.

In a sense, every quantization is a representation of classical observables: algebraic properties of quantum observables are believed to imitate algebraic properties of the classical ones. Otherwise we are confronted with anomaly (quantum mechanical symmetry breaking) \[21, 22, 23\]. For example, in canonical quantization, the canonical algebraic structure of observables is required to be preserved. Likewise, we can try to preserve the algebraic structure of the classical infinitesimal events for the quantum ones as well.

Geodesic quantization must be a correspondence between the classical and quantum events. For an infinitesimal event \(X\) (tangent vector), let us denote the corresponding quantum (infinitesimal) event as \(Q_X\). If we believe that the infinitesimal quantum events preserve the structure of geodesic Akivis algebras, the geodesic quantum conditions read

\[ [Q_X, Q_Y] := Q_XQ_Y - Q_YQ_X = qQ_{[X,Y]}, \]  

(4.2)

and

\[ (Q_X, Q_Y, Q_Z) := Q_XQ_Y \cdot Q_Z - Q_X \cdot Q_YQ_Z = q^2Q_{(X,Y,Z)}. \]  

(4.3)

Here, the quantization constant (geodesic quantum deformation parameter) is denoted as \(q\), and the classical infinitesimal events \(X, Y, Z\) must belong to the same geodesic Akivis algebra. For \(X, Y, Z\) from \(A_e\), the brackets in (4.2), (4.3) are defined by (3.6), (3.7).

Conditions (4.2), (4.3) mean that we seek for the enveloping non–associative algebra of the geodesic Akivis algebra. In this case, the binary and ternary multiplications of the infinitesimal events are concealed, respectively, into commutator and associator of the corresponding enveloping non–associative algebra. Non–associativity of the quantum events is the price we must pay for the concealing, but we get in fact rid of the ternary structure of the Akivis algebra, which seems to be quite a beneficial and sensible compensation. Recall that in the canonical quantization, the Poisson algebra multiplication of the classical observables is concealed into associative multiplication of the enveloping associative algebra of the quantum observables.

We can also assume that the geodesic quantization rule \(X \rightarrow Q_X\) is linear. This means that for \(X = X^i\partial_i\) the corresponding quantum event \(Q_X\) reads

\[ Q_X := X^iQ_i \]  

(4.4)

and (4.2), (4.3) read

\[ [Q_l, Q_m] := Q_lQ_m - Q_mQ_l = qS_{lm}^iQ_i, \]  

(4.5)

\[ (Q_l, Q_m, Q_n) := Q_lQ_m \cdot Q_n - Q_l \cdot Q_mQ_n = q^2(R_{lmn}^i - \nabla_nS_{lm}^i)Q_i. \]  

(4.6)

This problem has been solved by I. P. Shestakov in Dokl. Akad. Nauk 368 (1999), 21–23.
Here, the torsion and the curvature tensors must be valued at the point where the tangent vectors (infinitesimal events) $\partial_i$ are taken at.

Consistency of the geodesic quantum conditions (4.2)–(4.6) with general relativity (Einstein equations), and also compatibility (patching) conditions of the quantizations at different space-time points must be inquired. The construction of observables by means of the non–associative quantum events will be a crucial problem as well.

Finally, let us note that in the early days of quantum mechanics Jordan, von Neumann and Wigner [24, 25, 26, 27] tried to describe the quantum observables in terms of commutative but non-associative (power-associative) algebras, nowadays called the commutative Jordan algebras. In [28, 30], non-associativity was suggested to be related with the elementary length. Thorough historical review and discussion about the physical meaning and evolution of non-associative structures in physics is presented in [30, 31].

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