Quantum Spin Systems on Infinite Lattices

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1 | Introduction

These lecture notes aim to introduce the mathematical tools needed to deal with quantum mechanical systems consisting of very many (for our purposes, usually infinitely many) sites or particles. There is more than one way to do this, but here we will follow the so-called operator algebraic approach. Here the observables are modelled by elements of some operator algebra, in our case this is usually a C*-algebra. This is essentially a generalisation of the usual setting in quantum mechanics, where one considers the bounded (and sometimes also unbounded) operators acting on a Hilbert space. It should become clear in due course why this generalisation is useful, and not just some inessential mathematical machinery.

A single spin-1/2 particle is one of the simplest quantum mechanical systems, especially if we forget about the spatial degrees of freedom and only think about the spin part. Nevertheless, such systems play an important part in quantum information. They are the quantum analogue, called a qubit in that context, of a bit in computer science. As such they are one of the fundamental building blocks of quantum computation. There is not much to say about a single qubit, but if we combine a number of qubits (even finitely many), there are a lot of interesting questions to be studied.

Slightly more general, simple quantum mechanical systems such as qubits can be the constituents of complex quantum systems. Recall that in statistical mechanics the goal is to describe systems made up of a large number of “simple” objects such as gas molecules. It then turns out that precise knowledge on each individual molecule (e.g. its position and momentum) is not so useful any more to say anything about the system as a whole. In quantum statistical mechanics the situation is similar. Here one considers systems made up of a large number of simple quantum systems. In typical models one can think for example of a lattice, with at each site of the lattice an atom. The atoms are considered to be pinned down at the lattice site, and hence cannot move. However, the quantum degrees of freedom the atoms can interact with nearby atoms. Such models are often encountered in condensed matter, for example.

In dealing with such systems it can be convenient to consider the so-called thermodynamic limit, which is the limit of infinite system size. This has a few advantages. First of all, one can make a clear distinction between local and global properties, without having to keep track of the system size. Moreover one does not have to worry about boundary effects and boundary conditions, which may simplify life a lot. Besides that, considering the thermodynamic limit is even necessary to consider, for example, phase transitions. Depending on your

\(^1\)An extensive discussion can be found in the introduction of [7].
interests, you may also find the fact that it leads to interesting mathematics a compelling reason to study such systems.

In these lectures we will develop the mathematical tools to deal with such systems in a mathematically rigorous way. Since the goal is to create familiarity with the tools and methods of the operator algebraic approach to quantum mechanics, some proofs are only sketched. Moreover, to simplify matters most results are often not stated in the most general way possible. Rather, the emphasis will be on quantum spin systems (rather than, say, continuous systems) on lattices.

We give a brief overview of the contents of these lectures. In the first part of the notes the necessary mathematics is introduced: we give a short introduction to operator algebras and $C^*$-algebras in particular. In addition, the relation between this abstract mathematics and quantum mechanics is explained. Then we move on to the main topic of interest: quantum systems with infinitely many sites. In the remainder, we will see how various concepts in physics, such as ground states, can be described in this setup. An other important topic that will be discussed is that of Lieb-Robinson bounds. Such bounds limit the speed with which information can propagate in the system, not unlike the speed of light in special relativity. Finally, we present a small selection of different topics related to the material here, in particular on the use of operator algebraic methods in physics. Throughout several examples of physical systems will be explained.

1.1 Prerequisites

In these notes a certain level of mathematical maturity is assumed. The reader should be familiar with the basic concepts of analysis, topology and functional analysis. In particular, convergence in and completeness of metric spaces and the notion of Hilbert space are important. An excellent book on advanced analysis is *Analysis now* by G.K. Pedersen [52]. Besides this, basic knowledge of quantum mechanics is assumed. Although not strictly necessary for the mathematical details, it provides motivation for many of the important definitions and concepts we will introduce.

1.2 Acknowledgements

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Linear maps acting on Hilbert spaces play an important role in quantum mechanics. Such a linear map is called an operator. Well-known examples are the Pauli matrix $\sigma_z$, which measures the spin in the $z$ direction, and the position and momentum operators $P$ and $Q$ of a single particle on the line. The first acts on the Hilbert space $\mathbb{C}^2$, while the latter are defined on dense subspaces of $L^2(\mathbb{R})$. An operator algebra is an algebra of such operators, usually with additional conditions such as closure in a certain topology. Here we introduce some of the basic concepts in the theory of operator algebras. The material here is standard, and by now there is a huge body of textbooks on the subject, most of which cover a substantially bigger part of the subject than these notes. Particularly recommended are the two volumes by Bratteli and Robinson [7, 8], which contain many applications to physics. Many of the topics in these lecture notes are studied there in extenso. The books by Kadison & Ringrose provide a very thorough introduction to the field, with many exercises [32, 31]. The book by Takesaki [65] (and the subsequent volumes II and III [66, 67]) is a classic, but is more technical.

Before going into the mathematical details, let us take a step back, and try to motivate the study of operator algebras. Is it really necessary, or even useful, to develop this whole mathematical machinery? What is wrong with the usual picture of a Hilbert space of wave functions, with, say, position and momentum observables? There are a few reasons one can give.

The operator algebraic approach generally helps to obtain results at the level of mathematical rigour. Especially for systems with infinitely many degrees of freedom, the algebraic approach is often more suitable. These play an important role in quantum field theory and quantum statistical mechanics. In the latter, one tries to obtain properties from systems with very many microscopic degrees of freedom. The operator algebraic approach gives a natural playground for the study of structural questions, such as the (non-)existence of translationally invariant equilibrium states in certain classes of models. Finally, the subject is of interest from a purely mathematical point of view as well, and has led to interesting collaborations between (mathematical) physicists and mathematicians.

2.1 Functional analysis: a quick review

We first recall the definition of a Hilbert space. Let $\mathcal{H}$ be a vector space over $\mathbb{C}$ $^1$. An inner product on $\mathcal{H}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that the

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$^1$One can also define Hilbert spaces over $\mathbb{R}$, but these will play no role in these lectures.
following conditions hold:

1. **Conjugate symmetry**: \( \langle \eta, \psi \rangle = \langle \psi, \eta \rangle \) for all \( \eta, \psi \in \mathcal{H} \),

2. **Linearity in the second variable**: \( \langle \psi, \lambda \eta + \xi \rangle = \lambda \langle \psi, \xi \rangle + \langle \psi, \xi \rangle \) for all \( \psi, \eta, \xi \in \mathcal{H} \) and \( \lambda \in \mathbb{C} \),

3. **Positive definiteness**: if \( \langle \psi, \psi \rangle \geq 0 \) for all \( \psi \in \mathcal{H} \), and \( \langle \psi, \psi \rangle = 0 \) if and only if \( \psi = 0 \).

If the condition that \( \langle \psi, \psi \rangle = 0 \) for \( \psi = 0 \) only is not satisfied, \( \mathcal{H} \) is called a **pre-inner product space**. One can always divide out the subspace of vectors with norm zero to obtain a proper inner product space from a pre-inner product space. Note that from the first and second condition we obtain \( \langle \lambda \psi, \eta \rangle = \lambda \langle \psi, \eta \rangle \).

With the help of an inner product a norm \( \| \cdot \| \) can be defined on \( \mathcal{H} \), by \( \| \psi \|^2 = \langle \psi, \psi \rangle \) for all \( \psi \in \mathcal{H} \). That this indeed is a norm follows from the well-known Cauchy-Schwarz inequality, which states that \( \| \langle \psi, \xi \rangle \|^2 \leq \langle \psi, \psi \rangle \langle \xi, \xi \rangle \), and the other properties of an inner product. Hence each inner product space is a normed vector space in a canonical way.

A normed space can be endowed with a **topology**, since the norm defines a distance function (or **metric**), defined by \( d(\psi, \xi) := \| \psi - \xi \| \). In other words, we can talk about convergence of sequences and of continuity. Recall that a sequence \( (\psi_n)_n \) in a metric space converges to some element \( \psi \) if and only if \( d(\psi_n, \psi) \) tends to zero as \( n \) goes to infinity. A special role is played by the **Cauchy sequences**. A sequence \( (\psi_n)_n \) in a metric space is a Cauchy sequence if for each \( \varepsilon > 0 \), there is some integer \( N \) such that \( d(\psi_n, \psi_m) < \varepsilon \) for all \( n, m \geq N \). In other words, if we look far enough in the sequence, all elements will be close to each other.

It is not so difficult to show that if a sequence converges in a metric space, it must necessarily be a Cauchy sequence. The converse is not true: there are metric spaces in which Cauchy sequences do not converge. For example, consider the space \( X = (0, 1) \) with the usual metric of \( \mathbb{R} \). Then the sequence given by \( x_n = 1/n \) is a Cauchy sequence, but it does not converge (since \( 0 \notin X \)). A metric space in which all Cauchy sequences converge is called **complete**. The most familiar example is likely the real line \( \mathbb{R} \), but note that a subspace of a complete space is not necessarily complete, as the example above shows.

In these notes we will mainly be concerned with a special class of complete spaces: the Banach spaces. A **Banach space** is a normed vector space that is complete with respect to the norm. For quantum mechanics the natural playground is the **Hilbert space**, whose definition we recall here.

**Definition 2.1.1.** A **Hilbert space** \( \mathcal{H} \) is an inner product space over \( \mathbb{C} \) that is also a Banach space. That is, it is complete with respect to the norm induced by the inner product.

**Examples of Hilbert spaces**

The easiest examples of Hilbert spaces are the finite dimensional ones, where \( \mathcal{H} = \mathbb{C}^N \) for some integer \( N \). The inner product is the usual one, that is if

\(^2\)In the mathematics literature the inner product is often linear in the **first** variable, and anti-linear in the second. We adopt the physics conventions.
ξ, η ∈ ℍ then \( \langle \xi, \eta \rangle = \sum_{i=1}^{N} \xi_i \eta_i \). It follows from basic linear algebra that this indeed is an inner product. That ℍ is complete with respect to the metric induced by the inner product essentially follows from the completeness of \( \mathbb{C}^N \). This can be seen by noting that a sequence \( \xi_n \in ℍ \) is Cauchy if and only if the sequences of coordinates \( n \mapsto (\xi_i) \), with \( i = 1, \ldots, N \) is Cauchy. Hence \( \mathbb{C}^N \) is a Hilbert space.

Next we consider infinite dimensional Hilbert spaces. Write \( ℓ^2(ℕ) \) for the set of all sequences \( x_n \), such that \( \sum_{i=1}^{∞} |x_i|^2 < ∞ \). For two such sequences \( x_n \) and \( y_n \), define

\[
\langle x_n, y_n \rangle_{ℓ^2} = \sum_{i=1}^{∞} x_i y_i. \tag{2.1.1}
\]

The first thing to check is that this definition makes sense: a priori it is not clear that the sum on the right hand side converges. Note that since \( (|x| - |y|)^2 ≥ 0 \) for all complex numbers \( x \) and \( y \), it follows that \( |x_n y_n| ≤ \frac{1}{2}(|x_n|^2 + |y_n|^2) \). Hence for \( x_n \) and \( y_n \) elements of \( ℓ^2(ℕ) \), the right hand side of equation (2.1.1) converges absolutely, and hence converges. It follows that \( ℓ^2(ℕ) \) is a vector space over \( \mathbb{C} \) and that equation (2.1.1) defines an inner product.

**Exercise 2.1.2.** Show that \( ℓ^2(ℕ) \) is complete with respect to the norm induced by the inner product (and hence a Hilbert space).

A familiar example from quantum mechanics is the Hilbert space \( L^2(ℝ) \), consisting of square integrable functions \( f : ℝ → ℂ \) with the inner product

\[
\langle f, g \rangle_{L^2} = \int f(x)\overline{g(x)} dx.
\]

The definition of such integrals is a subtle matter. It is usually implicit that the integration is with respect to the Lebesgue measure on \( ℝ \). In calculations, on the other hand, one usually has to deal with integrals of explicitly given functions and one does not have to worry about such issues. The proof that the vector space of all square integrable functions is indeed a Hilbert space requires some results of measure theory, and is outside of the scope of these notes. It can be found in most textbooks on functional analysis.

The definition of \( L^2 \) spaces can be generalised to any measure space \((X, µ)\) by using the corresponding integral in the definition of the inner product. The resulting Hilbert space is then denoted by \( L^2(X, dµ) \).

**Remark 2.1.3.** The Hilbert space \( ℓ^2(ℕ) \) can be seen as a special case of an \( L^2 \) space. To see this, note that for each sequence \( x_n \) we can find a function \( f : ℕ → ℂ \), defined by \( f(n) = x_n \). Conversely, each such function defines a sequence via the same formula. Next we define a measure \( µ \) on \( ℕ \) by \( µ(X) = \]

The appropriate setting for a mathematical theory of integration is measure theory. This will not play a role in the rest of the course. Details can be found in most graduate level texts on analysis or functional analysis.
[X], where \(|X|\) is the number of elements for any \(X \subset \mathbb{N}\). This measure is usually called the \textit{counting measure} for obvious reasons. For two functions \(f\) and \(g\) the \(L^2\) inner product then becomes

\[
\langle f, g \rangle_{L^2} = \int f(n)g(n)d\mu(n) = \sum_{i=1}^{\infty} f(n)g(n) = \langle f, g \rangle_{\ell^2},
\]

where in the last equality we identified the functions \(f\) and \(g\) with their corresponding sequences, as above. It follows that \(\ell^2(\mathbb{N}) \equiv L^2(\mathbb{N}, d\mu)\).

**Linear maps**

Once one has defined a class of mathematical objects (in our case, Hilbert spaces), the next question to answer is what kind of maps between Hilbert spaces one considers. For Hilbert spaces linear maps \(L\) are an obvious choice. That is, \(L : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) is a function such that for all \(\lambda \in \mathbb{C}\) and \(\xi, \eta \in \mathcal{H}_1\) we have

\[
L(\lambda\xi) = \lambda L(\xi) \quad \text{and} \quad L(\xi + \eta) = L(\xi) + L(\eta).
\]

We will usually omit the brackets and simply write \(L\xi\) for \(L(\xi)\).

Since Hilbert spaces carry a topology induced by the norm, it is natural to restrict to those linear maps that are continuous with respect to this topology. On the other hand, it is often convenient to work with bounded linear maps, for which there is a constant \(M > 0\) such that \(\|L\xi\| \leq M\|\xi\|\), independent of the vector \(\xi\). We should note that not all operators one encounters in quantum mechanics are bounded, for example the momentum operator of a particle on the line is not. The treatment of unbounded linear maps is a topic on its own, to which we will only return briefly when discussing Hamiltonians.

The next proposition says that bounded and continuous linear maps are actually the same. Note that this means that unbounded linear maps are not continuous. In general, they can only be defined on a \textit{subset} of a Hilbert space.

**Proposition 2.1.4.** Let \(L : V \rightarrow W\) be a linear map between two normed vector spaces. Then the following are equivalent:

1. \(L\) is continuous with respect to the norm topology,

2. \(L\) is bounded.

**Proof.** First suppose that \(L\) is bounded, so that there is a constant \(M > 0\) such that \(\|L\| \leq M\|\|\). Let \(\varepsilon > 0\) and choose \(\delta = \varepsilon/M\). Then for any \(v_1, v_2 \in V\) such that \(\|v_1 - v_2\| < \delta\), we have (with the help of linearity)

\[
\|L(v_1 - v_2)\| = \|L(v_1) - L(v_2)\| \leq M\|v_1 - v_2\| < \delta M = \varepsilon.
\]

This proves that \(L\) is continuous.

Conversely, set \(\varepsilon = 1\) and let \(\delta > 0\) be the corresponding maximal distance as in the definition of continuity. Then if \(\|v\| < \delta\) (hence the distance of \(v\) to

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4In abstract nonsense language: one has to say in which category one is working.
0 is smaller than $\delta$, it follows that $\|Lv\| \leq 1$. If $v$ is any non-zero vector, the norm of $\frac{\delta v}{2\|v\|}$ is smaller than $\delta$. By linearity it follows that
$$\|\delta Lv\| \leq 1 \iff \|Lv\| \leq \frac{2}{\delta}\|v\|.$$
This concludes the proof, since for $v = 0$ the inequality is satisfied trivially.

Hence continuous linear maps and bounded linear maps between Hilbert spaces are the same thing. This is often very convenient. For example, in many situations it is easy to construct a linear map that is bounded on some dense subset of a Hilbert space. Then this result allows us to extend the definition of the linear map to the whole Hilbert space, and obtain a bounded linear map again. We will see examples of this later.

**Definition 2.1.5.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. We write $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the set of bounded linear maps from $\mathcal{H}_1$ to $\mathcal{H}_2$. As a shorthand, $\mathfrak{B}(\mathcal{H}) := \mathfrak{B}(\mathcal{H}, \mathcal{H})$.

An element of $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ will also be called a (bounded) operator. Usually we will restrict to maps in $\mathfrak{B}(\mathcal{H})$, and call such elements simply operators on $\mathcal{H}$. Note that $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a vector space over $\mathbb{C}$. We will later study more properties of these spaces. Essentially, the theory of operator algebras amounts to studying these spaces (and subspaces thereof).

If $A$ is a bounded linear map $A : \mathcal{H}_1 \to \mathcal{H}_2$, the adjoint $A^*$ of $A$ is the unique linear map $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that
$$\langle \psi, A^* \eta \rangle_{\mathcal{H}_1} = \langle A \psi, \eta \rangle_{\mathcal{H}_2}, \quad \text{for all } \psi \in \mathcal{H}_1, \eta \in \mathcal{H}_2.$$
Of course one has to show that such a linear map $A^*$ exists and indeed is unique. To do this rigorously requires some work, but a heuristic argument goes as follows. Choose orthonormal bases $e_i$ and $f_j$ of $\mathcal{H}_1$ and $\mathcal{H}_2$. Then we can represent $A^*$ as a matrix (an “infinite” matrix if either of the Hilbert spaces is infinite dimensional) with respect to these bases. The elements $\langle e_i, A^* f_j \rangle_{\mathcal{H}_1}$ give the matrix coefficients in this basis, and hence determine the linear map $A^*$. In the physics literature the adjoint is usually called the Hermitian conjugate and denoted by a $\dagger$. We will write $^*$ for the adjoint, as is common in mathematics and mathematical physics. We will come back to the adjoint of linear maps on Hilbert spaces when we discuss $C^*$-algebras.

**Exercise 2.1.6.** Verify that the adjoint has the following properties:

1. $(A^*)^* = A$ for any linear map $A$,
2. the adjoint is conjugate linear, i.e. $(\lambda A + B)^* = \overline{\lambda} A^* + B^*$, with $\lambda \in \mathbb{C}$,
3. and $(AB)^* = B^* A^*$.

Here $AB$ stands for the composition $A \circ B$, that is, $ABv = A(B(v))$. In these notes we will always omit the composition sign of two linear maps.

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5 The proof requires the Riesz representation theorem. First note that the map $\xi \mapsto \langle A\xi, \psi \rangle$ is a bounded linear functional on $\mathcal{H}$. By the Riesz representation theorem there is a $\eta \in \mathcal{H}$ such that $\langle \eta, \xi \rangle = \langle A\xi, \psi \rangle$ for all $\xi \in \mathcal{H}$. Set $A^* \psi = \xi$. This uniquely defines a bounded operator $A^*$ with the right properties.
An operator \( A \) is called self-adjoint if \( A = A^* \) and normal if \( AA^* = A^*A \).

It is an isometry if \( V^*V = I \), where \( I \) is the identity operator. The name “isometry” comes from the fact that this implies that \( V \) is an isometric map, since

\[
\|V\psi\|^2 = \langle V\psi, V\psi \rangle = \langle \psi, V^*V\psi \rangle = \langle \psi, \psi \rangle = \|\psi\|^2.
\]

Note that an isometry is always injective. A projection \( P \) is an operator such that \( P^2 = P^* = P \). Projectors project onto closed subspaces of the Hilbert space \( \mathcal{H} \) on which they are defined. If \( V \) is an isometry, it is easy to check that \( VV^* \) is a projection.

Finally we come to the isomorphisms of Hilbert spaces. An isomorphism \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) is a bounded linear map, such that its inverse \( U^{-1} \) exists and is also a linear map. We also require that \( U \) is an isometry, \( \|U\psi\| = \|\psi\| \).

**Theorem 2.1.7.** Let \( U : \mathcal{H}_1 \to \mathcal{H}_2 \) be a bounded linear map between two Hilbert spaces. Then the following are equivalent:

1. \( U \) is an isomorphism of Hilbert spaces,
2. \( U \) is surjective and \( \langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle \) for all \( \eta, \xi \in \mathcal{H}_1 \),
3. \( U^*U = UU^* = I \), where \( I \) is the identity map. (Note that it would be more accurate to distinguish between the identity map of \( \mathcal{H}_1 \) and of \( \mathcal{H}_2 \)).

**Proof.** \( 1 \Rightarrow 2 \) Since \( U \) is invertible it must be surjective. Because \( U \) is in addition an isometry, it follows that \( \langle U\xi, U\xi \rangle = \|U\xi\|^2 = \langle \xi, \xi \rangle \) for all \( \xi \in \mathcal{H}_1 \).

Now let \( \xi, \eta \in \mathcal{H}_1 \). Then the polarisation identity, which can be easily verified, says that

\[
\langle U\xi, U\eta \rangle = \frac{1}{4} \sum_{k=0}^{4} i^k \langle U(\xi + i^k\eta), U(\xi + i^k\eta) \rangle,
\]

and the claim follows.

\( 2 \Rightarrow 3 \) Since \( U \) is an isometry it follows that \( U^*U = I \) and that \( U \) is injective. Hence \( U \) is a bijection and therefore invertible, so \( U^*UU^{-1} = U^* = U^{-1} \). It follows that \( UU^* = I \).

\( 3 \Rightarrow 1 \) The assumptions say that \( U^* \) is the inverse of \( U \). Because \( U^*U = I \) it is also clear that \( U \) is isometric.

A continuous linear map between Hilbert spaces satisfying assumption \( 3 \) of the Theorem will be called a unitary.

The theorem can be used to show that, in a sense, there are not many different Hilbert spaces. To make this precise, recall that an orthonormal basis of a Hilbert space \( \mathcal{H} \) is a set \( \{ \xi_\alpha \} \) of vectors of norm one such that \( \langle \xi_\alpha, \xi_\beta \rangle = 0 \) if \( \alpha \neq \beta \) and such that the linear span of these vectors is dense in \( \mathcal{H} \). That is, for each vector \( \xi \) and \( \varepsilon > 0 \), one can find a linear combination of finitely many of the basis vectors that is close to \( \xi \), i.e. there exist finitely many \( \lambda_\alpha \neq 0 \) such that

\[
\left\| \sum_{i=1}^{k} \lambda_\alpha \xi_\alpha - \xi \right\| < \varepsilon.
\]

The cardinality of a set of basis vectors is called the dimension of a Hilbert space. One can show that this is independent of the choice of orthonormal basis. This leads to the following result.
**CHAPTER 2. OPERATOR ALGEBRAS**

**Proposition 2.1.8.** Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if and only if they have the same dimension.

*Proof (sketch).* If $U : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary and $\{\xi_n\}$ is an orthonormal basis, then $\{U\xi_n\}$ is again orthonormal. Moreover, because $U$ is surjective, it also is a basis of $\mathcal{H}_2$.

Conversely, suppose that we have bases $\{\xi_n\}$ and $\{\eta_n\}$ of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Define a linear map $U$ by $U\xi_n = \eta_n$. This can be linearly extended to the span of $\xi_n$, and is clearly bounded. It follows that $U$ can be extended to a map of $\mathcal{H}_1$ to $\mathcal{H}_2$. One can then verify that this map must be a unitary. □

**Completion**

There is no reason why a metric space should be complete. In many constructions we will encounter in these notes this will be the case. Nevertheless, it is possible to embed a (non-complete) metric space into a space that is. A well-known example is the completion of the rational numbers $\mathbb{Q}$: this is precisely the set of real numbers $\mathbb{R}$. There is a certain “minimal” way to do this, in the sense that we only add limits of Cauchy sequences, and nothing more. Essentially this means that the original metric space is dense in the completion. Moreover, it makes sense to demand that the metric on the new space restricts to the metric on the original space (embedded in the completion).

In this section we outline this construction with an example: the construction of a Hilbert space from a (not necessarily complete) inner product space $H$. This construction is primarily of theoretical interest. It shows that a completion always exists, but the description of the completion is not very convenient in concrete calculations.

So let $H$ be a vector space over the complex numbers, and let $\langle \cdot , \cdot \rangle_H$ be an inner product on $H$. We will define a Hilbert space $\mathcal{H}$ containing $H$, and show that it is complete. First, let $\{\xi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$ be two Cauchy sequences in $H$. We say that the sequences are equivalent, and write $(\xi_n)_n \sim (\psi_n)_n$, if for each $\varepsilon > 0$ there is some $N > 0$ such that $\|\xi_n - \psi_n\| < \varepsilon$ for all $n > N$. Hence two Cauchy sequences are equivalent if they are arbitrarily close for sufficiently large $N$.

It is possible to define a scalar multiplication and addition on the set of Cauchy sequences. Let $(\psi_n)_n$ and $(\xi_n)_n$ be two Cauchy sequences. If $\lambda \in \mathbb{C}$, then $(\lambda \psi_n)_n$ is again a Cauchy sequence. Similarly, the sequence $(\psi_n + \xi_n)_n$ is Cauchy (by the triangle inequality). These operations are well-defined with respect to the equivalence relation defined above, so that $\lambda (\psi_n)_n \sim \lambda (\xi_n)_n$ if $(\psi_n)_n \sim (\xi_n)_n$ (and similarly for addition). This means that we can take the vector space $V$ of Cauchy sequences, and divide out by the equivalence relation. We will denote the corresponding vector space by $\mathcal{H}$.

There is a different way to think about $\mathcal{H}$. First of all, note that $(\psi_n)_n \sim (\xi_n)_n$ if and only if $(\psi_n - \xi_n)_n \simeq (0)_n$, where $(0)_n$ is the sequence of all zeros. The set of sequences that are equivalent to the zero sequence forms a vector space $V_0$. Then $\mathcal{H}$ is then $V/V_0$, the quotient of $V$ by the vector space of “null sequences”. Hence an element of $\mathcal{H}$ is an equivalence class $[(\psi_n)_n]$, and two

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Note: This is essentially how the real numbers are usually defined. Namely, they are limits of Cauchy sequences of rational numbers.
representatives of the same class are related by $(\psi_n)_n = (\xi_n)_n + (\eta_n)_n$, where $(\eta_n)_n \in V_0$. Sometimes one writes $(\psi_n)_n + V_0$ for this equivalence class.

At this point $H$ is a vector space over $\mathbb{C}$. The goal is to define an inner product on $H$ that is compatible, in a sense to be made precise later, with the inner product of $H$. To this end, define

$$\langle [(\psi_n)_n], [(\xi_n)_n] \rangle_H := \lim_{n \to \infty} \langle \psi_n, \xi_n \rangle_H.$$ 

There are a few things to check. First of all, the right hand side should converge for this definition to even make sense. Secondly, it should be well defined in the sense that it does not depend on the representative of the equivalence class. The first property follows because the inner product is continuous by the Cauchy-Schwartz inequality. As for the second part, consider for example $(\xi_n)_n \sim (\eta_n)_n$. Then

$$|\langle \psi_n, \xi_n \rangle - \langle \psi_n, \eta_n \rangle| = |\langle \psi_n, (\xi_n - \eta_n) \rangle| \leq \|\psi_n\| \|\xi_n - \eta_n\|.$$ 

Since $(\psi_n)_n$ is a Cauchy sequence, the sequence $\|\psi_n\|$ is uniformly bounded. The terms $\|\xi_n - \eta_n\|$ go to zero as $n$ goes to infinity (by assumption). Hence we conclude that $\{(\psi_n)_n, [(\xi_n)_n]\}_H = \{(\psi_n)_n, [(\eta_n)_n]\}_H$.

The original space $H$ can be embedded into $H$ in the following way. Let $\psi \in H$. Then $(\psi_n)_n$, where $\psi_n = \psi$ for each $n$, clearly is a Cauchy sequence. Then the embedding is given by a map $\iota : H \to H$, with $\iota(\psi) = [(\psi_n)_n]$, the equivalence class of the constant sequence with values $\psi$. Note that $\langle \eta, \psi \rangle_H = \langle \iota(\eta), \iota(\psi) \rangle_H$. In particular, the map $\iota$ is an isometry.

**Exercise 2.1.9.** Verify the details, for example show that $\iota$ is linear and that $H$ indeed is a complete metric space.

The results in this section, together with the exercise above, can be summarised as follows.

**Theorem 2.1.10.** Let $H$ be an inner product space. Then there is a Hilbert space $\mathcal{H}$ and a linear embedding $\iota : H \hookrightarrow \mathcal{H}$ such that $\iota(H)$ is dense in $\mathcal{H}$ and $\langle \psi, \xi \rangle_H = \langle \iota(\psi), \iota(\xi) \rangle_{\mathcal{H}}$.

The completion is in fact unique in the following sense. Let $\mathcal{H}'$ be another completion, that is there is an isometric linear map $\iota' : H \hookrightarrow \mathcal{H}'$ whose image is dense in $\mathcal{H}'$. Then there is a unitary $U : \mathcal{H} \to \mathcal{H}'$.

Usually $H$ is identified with its image in $\mathcal{H}$, that is, we think of $H \subset \mathcal{H}$ and every element of $\mathcal{H}$ can be approximated arbitrarily well by an element in $H$. Later we will need the completion of algebras, with respect to some norm. These can be constructed in a similar manner.

**Tensor products and direct sums**

There are few basic constructions to obtain new Hilbert spaces from (pairs of) Hilbert spaces. The simplest construction is taking subspaces. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K} \subset \mathcal{H}$ be a linear subspace. If $\mathcal{K}$ is closed in norm (that is, the limit of any Cauchy sequence in $\mathcal{K}$ is also in $\mathcal{K}$), then $\mathcal{K}$ is a Hilbert space, by restricting the inner product of $\mathcal{H}$ to $\mathcal{K}$. Note that one can define
a projection \( P : \mathcal{H} \to \mathcal{H} \) onto the subspace \( \mathcal{K} \). This projection is sometimes written as \([\mathcal{K}]\).

Next, suppose that we have two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). The \textit{direct sum} \( \mathcal{H}_1 \oplus \mathcal{H}_2 \) consists of all tuples \((\xi, \eta)\) with \( \xi \in \mathcal{H}_1 \) and \( \eta \in \mathcal{H}_2 \). This is a Hilbert space, if we define the following addition rules and inner product:

\[
\lambda_1 (\xi_1, \eta_1) + \lambda_2 (\xi_2, \eta_2) = (\lambda_1 \xi_1 + \lambda_2 \xi_2, \lambda_1 \eta_1 + \lambda_2 \eta_2),
\]

\[
\langle (\xi_1, \eta_1), (\xi_2, \eta_2) \rangle_{\mathcal{H}_1 \oplus \mathcal{H}_2} = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}_1} + \langle \eta_1, \eta_2 \rangle_{\mathcal{H}_2},
\]

where the \( \lambda_i \) are scalars. Note there is a natural isometry \( V_1 : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2 \) defined by \( V_1 \xi = (\xi, 0) \), and similarly for \( V_2 \). If \( L_i : \mathcal{H}_i \to \mathcal{K}_i \) for \( i = 1, 2 \) and \( \mathcal{K}_i \) Hilbert spaces. Then we can define a linear map \( L_1 \oplus L_2 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{K}_1 \oplus \mathcal{K}_2 \) by

\[
(L_1 \oplus L_2)(\xi, \eta) = (L_1 \xi, L_2 \eta).
\]

If \( L_1 \) and \( L_2 \) are bounded, then so is \( L_1 \oplus L_2 \). Hence in that case it is continuous, by Proposition 2.1.1. One can show that \((L_1 \oplus L_2)^* = L_1^* \oplus L_2^*\). One can easily generalise this construction to finite direct sums. It is also possible to define \textit{infinite} direct sums, but this requires a bit more care to make sure all expressions converge.

Finally we discuss the tensor product of two Hilbert spaces. This has a direct physical interpretation: if two quantum systems are described by two Hilbert spaces, the tensor product describes the combination of these two systems as a whole. Its construction is a bit more involved than that for direct sums. One way to do it is as follows. First consider the vector space \( V \) consisting of formal (finite) linear combinations of elements of the form \( \xi \otimes \eta \) for some \( \xi \in \mathcal{H}_1 \) and \( \eta \in \mathcal{H}_2 \). That is, an element of \( V \) is of the form \( \sum_{i=1}^n \lambda_i \xi_i \otimes \eta_i \). Now this is a huge space, and we want to impose some relations. That is, we will identify elements of the following form:

\[
\lambda (\xi \otimes \eta) = (\lambda \xi) \otimes \eta = \xi \otimes (\lambda \eta),
\]

\[
(\xi_1 + \xi_2) \otimes \eta = \xi_1 \otimes \eta + \xi_2 \otimes \eta, \quad (\xi_1, \eta_1) \mathcal{H}_1, (\xi_2, \eta_2) \mathcal{H}_2
\]

(2.1.2)

\[
\xi \otimes (\eta_1 + \eta_2) = \xi \otimes \eta_1 + \xi \otimes \eta_2.
\]

If we quotient \( V \) by these relations we obtain a vector space \( H \). The image of \( \xi \otimes \eta \) in this quotient will again be written as \( \xi \otimes \eta \). The vector space \( H \) can be made into an pre-Hilbert space by setting

\[
\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle_H = \langle \xi_1, \eta_1 \rangle_{\mathcal{H}_1} \langle \xi_2, \eta_2 \rangle_{\mathcal{H}_2}
\]

and extending by linearity. This induces a norm on \( H \), but in general \( H \) is \textit{not} a Hilbert space, since it will not be complete with respect to this norm. However, by the results of the previous section we can complete \( H \) to obtain a Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). Again, finite tensor products are defined analogously. We will discuss infinite tensor products in the next chapter.

Suppose again that \( L_i : \mathcal{H}_i \to \mathcal{K}_i \) are bounded linear maps. We can define a map \( L_1 \otimes L_2 : H \to K_1 \otimes K_2 \) by

\[
(L_1 \otimes L_2)(\xi \otimes \eta) = L_1 \xi \otimes L_2 \eta.
\]

This is well-defined since the \( L_i \) are linear, and hence this definition is compatible with the relations in (2.1.2). Since \( L_1 \) and \( L_2 \) are bounded, \( L_1 \otimes L_2 \)
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is bounded on the subspace $H$. To define $L_1 \otimes L_2$ on all of $H_1 \otimes H_2$ we use that $H$ is dense in the tensor product, together with Proposition 2.1.4. Since $H$ is dense in $H := H_1 \otimes H_2$, every $\xi \in H$ is the norm limit of a sequence $\xi_n$ of elements in $H$. We then define

$$(L_1 \otimes L_2)\xi := \lim_{n \to \infty} (L_1 \otimes L_2)\xi_n.$$ 

Since $L_1 \otimes L_2$ is bounded, the image of a Cauchy sequence is again Cauchy. It follows that $(L_1 \otimes L_2)\xi_n$ converges in the Hilbert space $K_1 \otimes K_2$. This defines $L_1 \otimes L_2$ on all of $H$. One can check that the definition of $L_1 \otimes L_2$ does not depend on the choice of sequence, and that it is indeed a linear map. This procedure is sometimes called extension by continuity, where we extend a bounded linear map defined on a dense subset of a complete linear space, and taking values in a complete linear space, to the whole space.

For adjoints of tensor products of linear maps we have $(L_1 \otimes L_2)^* = L_1^* \otimes L_2^*$.

2.2 Banach and $C^*$-algebras

The set $\mathfrak{B}(H)$ of bounded operators on a Hilbert space has more structure than we have discussed above. For example, it is an algebra. Recall that an algebra $\mathfrak{A}$ is a vector space on which a multiplication is defined. This multiplication should be compatible with addition, in the sense that $A(B+C) = AB + AC$ for all $A,B,C \in \mathfrak{A}$, and similarly for $(B+C)A$. Moreover, it should be compatible with multiplication by scalars in the obvious sense. We do not require our algebras to have a unit, although in most of our applications this will be the case. In this section we will try to abstract the properties of $\mathfrak{B}(H)$ to obtain the notion of a $C^*$-algebra.

A $*$-algebra is an algebra on which an (anti-linear) involution $*$ is defined. That is, there is a map $*: \mathfrak{A} \to \mathfrak{A}$ with the following properties:

1. $(A^*)^* = A$ for all $A \in \mathfrak{A}$,
2. $(AB)^* = B^*A^*$ for $A,B \in \mathfrak{A}$,
3. $(\lambda A + B)^* = \overline{\lambda}A^* + B^*$ for $A,B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$.

The first property says that the $*$-operation is involutive, and the last property describes anti-linearity. A familiar example of a $*$-algebra are the $n \times n$ matrices where the $*$-operation is given by taking the adjoint of the matrix.

So far the discussion has been purely algebraic. In our prototypical example of $\mathfrak{B}(H)$ topology enters in a natural way. As we will see later, it is for example possible to define a norm on $\mathfrak{B}(H)$. This makes it an example of a Banach algebra.

Definition 2.2.1. A Banach algebra $\mathfrak{A}$ is an algebra which is complete with respect to a norm $\| \cdot \|$. Moreover, the norm should satisfy $\|AB\| \leq \|A\| \|B\|$. A Banach $*$-algebra is a Banach algebra which is also a $*$-algebra and for which $\|A^*\| = \|A\|$ for all $A \in \mathfrak{A}$.

7We will deal almost exclusively with vector spaces over $\mathbb{C}$.
The completeness means that any Cauchy sequence $A_n$ in $A$ converges to some $A \in A$. The inequality on the norm of a product $AB$ guarantees that the multiplication in the algebra is continuous with respect to the metric topology, and similarly it follows that the $*$-operation is continuous as well. Finally, if $A$ has a unit $I$, then $\|I\| \geq 1$. A unit is an element $I$ such that $IA = AI = A$ for all $A \in A$. A unit is necessarily unique (if it exists).

**Exercise 2.2.2.** Prove these claims.

A Banach algebra already has a rich structure. However, it turns out that if we demand one extra condition on the norm, we can do much more. This extra condition is what defines a $C^*$-algebra. Such algebras will be used to model the observables in quantum mechanical systems.

**Definition 2.2.3.** A $C^*$-algebra $A$ is a Banach $*$-algebra such that the $C^*$-property holds: $\|A^*A\| = \|A\|^2$.

Note that in the condition that $\|A^*\| = \|A\|$ for a Banach algebra follows from the $C^*$-property and submultiplicativity ($\|AB\| \leq \|A\|\|B\|$) of the norm. The converse however is not true, so not any Banach $*$-algebra is a $C^*$-algebra. If a $C^*$-algebra has a unit $I$, it follows automatically that $\|I\| = 1$.

Most (if not all) of the $C^*$-algebras in these notes are unital. In cases where it makes a difference, we will only give proofs in the unital case. Nevertheless, let us briefly mention two useful techniques in dealing with non-unital algebras. First of all, let $A$ be a non-unital $C^*$-algebra. Then $A$ can always be embedded into a unital algebra $\tilde{A}$, defined as

$$\tilde{A} := \{(A, \lambda) : A \in A, \lambda \in \mathbb{C}\}.$$  

Addition and the $*$-operation act componentwise, multiplication is defined by $(A, \lambda)(B, \mu) = (\mu A + \lambda B + AB, \lambda \mu)$. A norm is defined by

$$\|(A, \lambda)\| = \sup_{B \in A, \|B\| = 1} \|A + \lambda B\|.$$

It can be shown that $\tilde{A}$ is a unital $C^*$-algebra with this norm. The unit is the element $(0, 1)$. Note that $A$ clearly is a subalgebra of $\tilde{A}$. This construction is called adjoining an identity. The other technique is that of an approximate identity. This is a net $E_\lambda$ of positive elements (i.e., elements of the form $A^*A$ for some $A$), such that $\|E_\lambda\| \leq 1$ and $\lim_\lambda E_\lambda A = \lim_\lambda A E_\lambda = A$ for all $A \in A$. An approximate identity always exists in a $C^*$-algebra.

**Example: the bounded operators on a Hilbert space**

Let $H$ be a Hilbert space. An operator $A$ on $H$ is a linear map $A : H \to H$ (or sometimes only defined on a linear subspace of $H$). It is called bounded if there is some constant $C > 0$ such that $\|A\psi\| \leq C\|\psi\|$ for all $\psi \in H$. If this is not the case, then the operator is called unbounded. Note that the bounded operators on a Hilbert space are a special case of the linear maps we studied earlier (c.f. Proposition 2.1.4). We will show that $B(H)$ is a $C^*$-algebra, where the $*$-operation is the adjoint of linear maps on Hilbert spaces defined earlier.

First, it is clear that $B(H)$ is an algebra, since we can add (or multiply with a scalar) bounded linear maps to obtain new bounded maps. Moreover,
the adjoint satisfies all the conditions of a $\ast$-operation by Exercise 2.1.6. It remains to define a suitable norm and show that it has the right properties. The norm can be defined as follows:

$$
\|A\| := \sup_{\psi \in \mathcal{H}, \psi \neq 0} \frac{\|A\psi\|}{\|\psi\|} = \sup_{\psi \in \mathcal{H}, \|\psi\| = 1} \|A\psi\|.
$$

(2.2.1)

The last equality immediately follows from linearity of $A$. That this is indeed a norm follows directly because $\| \cdot \|$ is a norm on the Hilbert space $\mathcal{H}$. Submultiplicativity follows because for any $\psi \in \mathcal{H}$,

$$
\|AB\psi\| = \|A(B\psi)\| \leq \|A\| \|B\psi\| \leq \|A\| \|B\| \|\psi\|.
$$

To show that $B(\mathcal{H})$ is complete with respect to this norm, let $A_n \in B(\mathcal{H})$ be a Cauchy sequence of operators. Let $\psi \in \mathcal{H}$. Then $n \mapsto A_n\psi$ is a Cauchy sequence of vectors in the Hilbert space, hence this converges to a vector which we will denote by $A\psi$. That is, for every $\psi \in \mathcal{H}$ we define

$$
A\psi := \lim_{n \to \infty} A_n\psi.
$$

Since each $A_n$ is linear, this defines a linear map $A$. The map $A$ is bounded and $\|A - A_n\| \to 0$ if $n \to \infty$.

**Exercise 2.2.4.** Verify this claim.

It remains to show that the $C^*$-identity for the norm holds. Note that for $A \in B(\mathcal{H})$ we have

$$
\|A\psi\|^2 = \langle A\psi, A\psi \rangle = \langle \psi, A^*A\psi \rangle \leq \|A^*A\| \|\psi\|^2
$$

for all $\psi \in \mathcal{H}$. Hence $\|A\|^2 \leq \|A^*A\| \leq \|A\| \|A^*\|$ and $\|A\| \leq \|A^*\|$. Reversing the roles of $A$ and $A^*$ shows that $\|A\| = \|A^*\|$, from which the claim follows.

**Remark 2.2.5.** The completeness proof works for any Banach space. For the adjoint, however, one needs the Hilbert space structure to define it in the first place.

**Example: commutative $C^*$-algebras**

Let $X$ be a locally compact topological space. Write for the space $C_0(X)$ of all continuous functions $f : X \to \mathbb{C}$ that vanish at infinity. That is, $f \in C_0(X)$ if and only if for each $\varepsilon > 0$, there is a compact set $K_\varepsilon \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K_\varepsilon$. To get some feeling for this condition, consider the case $X = \mathbb{R}$. Then $f \in C_0(\mathbb{R})$ if and only if $f$ is continuous and $|f(x)| \to 0$ if $x \to \pm \infty$. Examples are

$$
f(x) = \frac{1}{1 + x^2} \text{ and } f(x) = \exp(-x^2).
$$

Note that $C_0(X)$ is an algebra if we define multiplication and addition point-wise:

$$(f + g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x).$$
An involution can be defined by setting $f^*(x) = \overline{f(x)}$. Hence $C_0(X)$ is a $\ast$-algebra. We can define a norm as follows:

$$\|f\| := \sup_{x \in X} |f(x)|.$$ 

Because continuous functions on compact sets are bounded, and $f$ is small outside some compact set, it follows that $\|f\|$ is finite if $f \in C_0(X)$. Note that the $C^*$-property, $\|f^*f\| = \|f\|^2$ follows immediately from the definition.

A standard result in topology says that if $\|f_n - f\| \to \infty$ for some function $f : X \to \mathbb{C}$ and $f_n$ a sequence in $C_0(X)$, then $f$ is also continuous (and it follows that $f \in C_0(X)$). Hence $C_0(X)$ is a $C^*$-algebra.

Note that $C_0(X)$ is a commutative algebra. The unit would be the function $f(x) = 1$ for all $x$. But this is only in $C_0(X)$ if $X$ is compact, because otherwise it does not vanish at infinity. The commutative Gel’fand-Naimark theorem in $C^*$-algebras says that in fact any commutative $C^*$-algebra is of the form $C_0(X)$ for some locally compact space $X$, and $X$ is compact if and only if the $C^*$-algebra is unital.

2.3 Linear functionals and states

Recall that in quantum mechanics, a (pure) state is often represented by a wave function, representing a vector in the Hilbert space of the system. Suppose that we have such a vector $|\psi\rangle$. Then the expectation value of an observable is given by

$$A \mapsto \langle \psi | A | \psi \rangle. \quad (2.3.1)$$

This definition clearly also works for operators $A$ that are not observables in the usual sense, because they are not self-adjoint. If we allow such operators, it is clear that equation (2.3.1) is linear as a function of $A$. Moreover, it is positive: for any $A$ we have

$$\langle \psi | A^* A | \psi \rangle = \langle A\psi | A\psi \rangle = \|A\psi\|^2 \geq 0.$$ 

This can be abstracted to the setting of Banach $\ast$-algebras.

**Definition 2.3.1.** A linear functional on a Banach $\ast$-algebra $\mathfrak{A}$ is a linear map $\omega : \mathfrak{A} \to \mathbb{C}$. It is called positive if $\omega(A^* A) \geq 0$ for all $A \in \mathfrak{A}$.

Note that this is a special case of the continuous linear maps of Proposition 2.1.4. In particular a linear functional is continuous if and only if $\omega$ is bounded. We can also define the norm of a linear functional by

$$\|\omega\| := \sup_{A \in \mathfrak{A}, \|A\|=1} |\omega(A)|.$$ 

This is just a special case of the norm on bounded linear maps: the linear functionals on a Banach $\ast$-algebra $\mathfrak{A}$ can be identified with $B(\mathfrak{A}, \mathbb{C})$.

**Lemma 2.3.2 (Cauchy-Schwarz).** Let $\omega$ be a positive linear functional on a $C^*$-algebra $\mathfrak{A}$. Then for all $A, B \in \mathfrak{A}$ we have the inequality

$$|\omega(B^* A)|^2 \leq \omega(B^* B) \omega(A^* A).$$

This is a variant of the well-known Cauchy-Schwarz inequality. Moreover, $\omega(A^* B) = \omega(B^* A)$. 
Proof. Let $\lambda \in \mathbb{C}$. Since $\omega$ is positive, it follows that
\[
\omega((\lambda A + B)^* (\lambda A + B)) = |\lambda|^2 \omega(A^* A) + \overline{\lambda} \omega(A^* B) + \lambda \omega(B^* A) + \omega(B^* B) \geq 0.
\]
Because this expression has to be real for all $\lambda$, it follows that $\omega(A^* B) = \omega(B^* A)$. If we then specialise to $\lambda$ being real, we get a quadratic function of $\lambda$. Demanding that the graph of this function lies above (or on) the real line leads to the desired inequality.

Note that we have not required linear functionals to be continuous. It turns out, however, that positive linear functionals are automatically continuous, and their norm can be obtained by evaluating the linear functional on the identity of the algebra. We will not prove this result here, but it can be found in most textbooks on operator algebras (for example, [7, Prop. 2.3.11]).

**Theorem 2.3.3.** Let $\omega$ be a linear functional on a unital $C^*$-algebra $\mathfrak{A}$. Then the following are equivalent:

1. $\omega$ is a positive linear map;
2. $\omega$ is continuous and $\|\omega\| = |\omega(I)|$.

If any of these are satisfied, we have the following properties:

(a) $|\omega(A)|^2 \leq \omega(A^* A) \|\omega\|$;

(b) $|\omega(A^* BA)| \leq \omega(A^* A) \|B\|$ for all $A, B \in \mathfrak{A}$.

Again, a similar statement is true for non-unital algebras, if one replaces the unit by an approximate unit.

**The state space of a $C^*$-algebra**

Let $\mathfrak{A}$ be a $C^*$-algebra. Then a state on $\mathfrak{A}$ is a positive linear functional $\omega : \mathfrak{A} \to \mathbb{C}$ of norm one (hence $\omega(I) = 1$ if $\mathfrak{A}$ is unital). We will write $S(\mathfrak{A})$ for the set of states on $\mathfrak{A}$. It turns out that this notion of a state is the correct abstraction of states in Hilbert space quantum mechanics. We will briefly come back to this later, but for the moment note that if $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ and $\psi \in \mathcal{H}$, the map $A \mapsto \langle \psi, A\psi \rangle$ is a state (if $\psi$ is a unit vector). Recall that if $A$ is a quantum mechanical observable, this expression gives the expectation value of the measurement outcome. This is how we can think of states on $C^*$-algebras as well.

Note that $S(\mathfrak{A})$ is a metric space, where the metric is induced by the norm on linear functionals. There are also other natural notions of convergence of a sequence of states. We will need the following two later.

**Definition 2.3.4.** Let $\varphi_n$ be a sequence of states on some $C^*$-algebra $\mathfrak{A}$. We say that $\varphi_n$ converges in norm to a state $\varphi \in S(\mathfrak{A})$ if $\|\varphi_n - \varphi\| \to 0$, where the norm is the usual norm on linear functions, as defined above. We say that $\varphi_n$ converges to $\varphi$ in the weak-* topology if for each $A \in \mathfrak{A}$, one has that $|\varphi_n(A) - \varphi(A)| \to 0$. 
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Exercise 2.3.5. Let $\varphi \in \mathcal{S}(\mathfrak{A})$ and suppose that we have a sequence $\varphi_n \in \mathcal{S}(\mathfrak{A})$. Show that $\varphi_n \to \varphi$ in the norm topology implies that $\varphi_n \to \varphi$ in the weak-* topology. Is the converse also true? Prove your claim or give a counterexample.

The state space of a $C^*$-algebra $\mathfrak{A}$ in general has a very rich structure, which strongly depends on the algebra itself. Nevertheless, there are some general properties that are always true. An important property is that $\mathcal{S}(\mathfrak{A})$ is convex: if $0 \leq \lambda \leq 1$ and $\omega_1$ and $\omega_2$ are states,

$$\omega(A) := \lambda \omega_1(A) + (1 - \lambda) \omega_2(A)$$

is a state again. A pure state is a state that cannot be written as a convex combination of different states.

Definition 2.3.6. Let $\omega$ be a state on a $C^*$-algebra $\mathfrak{A}$. Then $\omega$ is called pure if $\omega = \lambda \omega_1 + (1 - \lambda) \omega_2$ for $\lambda \in (0,1)$ and some states $\omega_1, \omega_2$ implies that $\omega_1 = \omega_2 = \omega$. If $\omega$ is not pure, it is called mixed.

We would like to stress that the notion of pure and mixed states depends on the algebra $\mathfrak{A}$. If $\mathfrak{B} \subset \mathfrak{A}$ is a $C^*$-subalgebra (that is, $\mathfrak{B}$ is closed in norm in $\mathfrak{A}$ and a $C^*$-algebra) containing the unit of $\mathfrak{A}$, we can restrict a state $\omega$ on $\mathfrak{A}$ to a state $\omega|_{\mathfrak{B}}$ on $\mathfrak{B}$. Even if $\omega$ is pure, it is not necessarily so that $\omega|_{\mathfrak{B}}$ is.

Remark 2.3.7. A pure state can equivalently be defined as follows. A state $\omega$ is pure if for every positive linear functional $\varphi$ that is majorised by $\omega$, that is $0 \leq \varphi(A^*A) \leq \omega(A^*A)$ for all $A \in \mathfrak{A}$, then $\varphi = \lambda \omega$ for some $0 \leq \lambda \leq 1$. It requires some work to show that these are indeed equivalent.

Exercise 2.3.8. Let $\mathfrak{A} = M_d(\mathbb{C})$ for some positive integer $d$. Recall that a density matrix $\rho$ is a matrix such that $\rho^* = \rho$, $\rho$ is positive (i.e., all the eigenvalues are bigger than or equal to zero) and $\text{Tr}(\rho) = 1$. Show that states $\omega$ on $\mathfrak{A}$ are in one-to-one correspondence with density matrices $\rho$ such that $\omega(A) = \text{Tr}(\rho A)$ for all $A$. Also show that $\omega$ is pure if and only if the corresponding $\rho$ satisfies $\rho^2 = \rho$ and if and only if $\rho$ is a projection onto a one-dimensional subspace.

As a special case of this exercise, consider $d = 2$. Such a system is called a qubit in quantum information theory. Since the Pauli matrices (together with the identity) form a basis of the two-by-two matrices, each density matrix $\rho$ can be written as

$$\rho = \frac{1}{2} \begin{pmatrix} 1 + z & x + iy \\ x - iy & 1 - z \end{pmatrix}.$$  

Because $A \mapsto \text{Tr}(\rho A)$ should be a state, it follows that $x, y$ and $z$ must be real, and that $x^2 + y^2 + z^2 \leq 1$. Hence any state on $M_2(\mathbb{C})$ can be characterized by giving a point in the unit ball of $\mathbb{R}^3$. In this context it is called the Bloch sphere. As an exercise, one can show that the state is pure if and only if the corresponding point $(x, y, z)$ lies on the surface of the sphere, so that $x^2 + y^2 + z^2 = 1$. This example also shows that if we have a mixed state, it generally can be written in many different ways as a convex combination of pure states. Indeed, take any two antipodal points on the sphere, such that the line connecting them goes through the point $(x, y, z)$. Then the state $\rho$ can be written as a convex linear combination of the two pure states corresponding to the points on the sphere.

To finish this section, we note the following useful theorem (without proof).
Theorem 2.3.9. Let \( \mathfrak{A} \) be a unital \( C^* \)-algebra. Then the set of states \( \mathfrak{A} \) is a closed convex and compact subset (with respect to weak-* topology) of the set of all linear functionals of norm one.

This implies, for example, that if \( \varphi_n \) is a sequence of states converging to some \( \varphi \) in the weak-* topology, then \( \varphi \) is also a state. The compactness property can be used to show that any sequence of states has a converging subsequence.

Example: classical states

A nice thing about the algebraic approach to quantum mechanics is that classical mechanics can be described in the same framework. We illustrate this with an example. Consider a configuration space \( M \) of a classical system. To avoid technical details as much as possible, we assume that \( M \) is compact. A classical observable is a continuous function \( f : M \to \mathbb{R} \). Slightly more general, we may consider complex valued continuous functions. As we have seen before, these functions form a \( C^* \)-algebra \( C_0(M) \). The classical observables correspond to the self-adjoint elements of this algebra.

Now let \( \mu \) be a probability measure on the configuration space \( M \). This measure can be interpreted as our knowledge of the system: it tells us what is the state of the system with which probability. If we want to measure an observable \( f \), the expectation value of the outcome is given by

\[
\omega(f) := \int f(x) d\mu(x).
\]

Note that \( \omega(1) = 1 \) since \( \mu \) is a probability measure and that \( \omega(\overline{f}) \geq 0 \). Hence \( \omega \) is a state on the observable algebra. Conversely, one can show that any state on \( C_0(M) \) comes from a probability measure \( \mu \) (see for example [52 Thm. 6.3.4]). The pure states of \( C_0(M) \) correspond to Dirac (or point) measures. If \( \mu = \delta_x \), the Dirac measure in the point \( x \in M \), then \( \omega(f) = f(x) \) for all \( f \). This is the situation in which we know in which configuration \( x \in M \) the system is, and hence we know the outcome of each observable.

2.4 The Gel’fand-Naimark-Segal construction

In the usual Hilbert space setting of quantum mechanics, the observables of a system are represented as bounded or unbounded operators acting on some Hilbert space. It turns out that in the abstract setting we are considering here, Hilbert space is actually just around the corner. That is, given a state \( \omega \) on some \( C^* \)-algebra \( \mathfrak{A} \), one can construct a Hilbert space \( \mathcal{H}_\omega \) and a representation \( \pi \) of \( \mathfrak{A} \) into \( \mathfrak{B}(\mathcal{H}_\omega) \). This is the content of the Gel’fand-Naimark-Segal theorem. Before going into the details of this construction, we first give the basic definitions.

Definition 2.4.1. Let \( \mathfrak{A} \) be a \( C^* \)-algebra and let \( \mathcal{H} \) be a Hilbert space. A representation of \( \mathfrak{A} \) on \( \mathcal{H} \) is a *-homomorphism \( \pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}) \). That is, a linear map such that \( \pi(AB) = \pi(A)\pi(B) \) for all \( A,B \in \mathfrak{A} \). For a *-representation one has in addition that \( \pi(A^*) = \pi(A)^* \). In these notes we only consider *-representations (without explicitly mentioning so).
Thus a representation represents the elements of an abstract $C^*$-algebra as bounded operators on a Hilbert space, in such a way that the algebraic relations among the elements are preserved.

**Remark 2.4.2.** A basic result in operator theory is that for representations $\pi$ of $C^*$-algebras, one automatically has $\|\pi(A)\| \leq \|A\|$ for all $A \in A$. It follows that a representation is automatically continuous with respect to the norm topology. The proof is not difficult, but requires some definitions that we will not discuss here. See for example [31 Thm. 4.1.8] or [61 Prop. I.7.3] for a proof. Moreover, if $\pi$ is injective then it is in fact an isometry, meaning that $\|\pi(A)\| = \|A\|$.

To avoid trivialities we will always restrict to **non-degenerate** representations. A representation $\pi : A \to \mathcal{B}(\mathcal{H})$ is non-degenerate if the set $\pi(A)\mathcal{H}$ is dense in $\mathcal{H}$. If a representation is degenerate, one can always obtain a non-degenerate representation by restricting to Hilbert space to $[\pi(A)\mathcal{H}]$. A representation is called **cyclic** if there is some vector $\Omega \in \mathcal{H}$ such that $\pi(A)\Omega$ is a dense subset of $\mathcal{H}$. Such an $\Omega$ is called a **cyclic vector**. We are now in a position to state and prove one of the fundamental tools in operator algebra, the **GNS construction**.

**Theorem 2.4.3.** Let $A$ be a unital $C^*$-algebra and $\omega$ a state on $A$. Then there is a triple $(\pi_\omega, \mathcal{H}_\omega, \Omega)$, where $\mathcal{H}_\omega$ is a Hilbert space and $\pi_\omega$ a representation of $A$ on $\mathcal{H}_\omega$, such that $\Omega$ is cyclic for $\pi_\omega$ and in addition we have $\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle$, $A \in A$.

This triple is unique in the sense that if $(\pi, \mathcal{H}, \Psi)$ is another such triple, there is a unitary $U : \mathcal{H}_\omega \to \mathcal{H}$ such that $U\Omega = \Psi$ and $\pi(A) = U\pi_\omega(A)U^*$ for all $A \in A$.

**Proof.** Define the set $N_\omega = \{ A \in A : \omega(A^*A) = 0 \}$. Suppose that $A \in \mathcal{A}$ and $B \in N_\omega$. Then the Cauchy-Schwarz inequality, Lemma 2.3.2 implies that $A^*B \in N_\omega$. But this means that $N_\omega$ is a left ideal of $A$. We can then form the quotient vector space $\mathcal{H}_\omega = A/N_\omega$ and denote $[A]$ for the equivalence class of $A \in A$ in this quotient. That is, $[A] = [B]$ if and only if $A = B + N_0$ for some $N_0 \in N_\omega$. We can define an inner product on $\mathcal{H}_\omega$ by $\langle [A], [B] \rangle := \omega(A^*B)$.

This is clearly sesquilinear, and by Lemma 2.3.2 it follows that this definition is independent of the choice of representatives $A$ and $B$. Because $\omega$ is a state this linear functional is positive, and it is non-degenerate precisely because we divide out the null space of $\omega$. Indeed, if $\langle [A], [A] \rangle = 0$ it follows that $A \in N_\omega$.

By taking the completion with respect to this inner product we obtain our Hilbert space $\mathcal{H}_\omega$. Next we define the representation $\pi_\omega$ by defining the action of $\pi_\omega(A)$ on the dense subset $\mathcal{H}_\omega$ of $\mathcal{H}_\omega$. Let $[B] \in \mathcal{H}_\omega$, then we define $\pi_\omega(A)[B] := [AB]$. It is easy to check that this is well-defined. Note that for all $B \in A$ we have $\|\pi_\omega(A)[B]\|_{\mathcal{H}_\omega} = \langle [AB], [AB] \rangle = \omega(B^*A^*AB) \leq \|A\|^2\omega(B^*B) = \|A\|^2\|B\|_{\mathcal{H}_\omega}^2$. 
From the definition of $\pi$, Remark 2.4.4. The theorem is still true if one drops the condition that $\pi(A)$ is bounded on a dense subset of $\mathcal{H}_\omega$. It is also not difficult to check that $\pi(AB) = \pi(A)\pi(B)$ and $\pi(A^*) = \pi(A)^*$ is an easy check. If we define $\Omega := [I]$ it is clear that $\Omega$ is cyclic for $\pi$, by definition of $\mathcal{H}_\omega$. Moreover, for each $A \in \mathfrak{A}$,

$$\langle \Omega, \pi(A)\Omega \rangle = \langle [I], [A] \rangle = \omega(A),$$

which completes the construction of the GNS representation.

It remains to be shown that the construction is essentially unique. Suppose that $(\pi, \mathcal{H}, \Psi)$ is another such triple. Define a map $U : \mathcal{H}_\omega \to \mathcal{H}$ by setting

$$U\pi(A)\Omega = \pi(A)\Psi$$

for all $A \in \mathfrak{A}$. Note that this is a linear map of a dense subspace of $\mathcal{H}_\omega$ onto a dense subspace of $\mathcal{H}$. Moreover, for each $A, B$ we have

$$\langle U\pi(A)\Omega, U\pi(B)\Omega \rangle = \langle \pi(A)\Psi, \pi(B)\Psi \rangle = \omega(A^*B) = \langle \pi(A)\Omega, \pi(B)\Omega \rangle.$$

It follows that $U$ can be extended to a unitary operator from $\mathcal{H}_\omega$ onto $\mathcal{H}$. The property that $U\pi(A) = \pi(A)U$ can be easily verified on a dense subset of the Hilbert space.

Remark 2.4.4. The theorem is still true if one drops the condition that $\mathfrak{A}$ is unital. The proof however becomes more involved and involves so-called approximate units.

Exercise 2.4.5. Let $\mathfrak{A}$ be a $C^*$-algebra and $\omega$ a faithful state, meaning that $\omega(A^*A) > 0$ if $A \neq 0$. Show that the corresponding GNS representation is faithful (in the sense that the kernel is trivial).

Faithfulness of the state is not a necessary condition for the representation to be faithful, only a sufficient one.

A representation is a special case of a morphism between $C^*$-algebras. A $*$-morphism $\rho : \mathfrak{A} \to \mathfrak{B}$ between two $C^*$-algebras is a linear map such that $\rho(AB) = \rho(A)\rho(B)$ and $\rho(A^*) = \rho(A)^*$ for all $A, B \in \mathfrak{A}$. Hence a representation is a morphism where $\mathfrak{B} = \mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Morphisms are also always continuous: $||\rho(A)|| \leq ||A||$.

Definition 2.4.6. An automorphism $\alpha$ of a $C^*$-algebra $\mathfrak{A}$ is a morphism $\alpha : \mathfrak{A} \to \mathfrak{A}$ that is invertible, such that $\alpha^{-1}$ is a morphism as well. The set of all automorphisms of a $C^*$-algebra forms a group, which is denoted by $\text{Aut}(\mathfrak{A})$.

Exercise 2.4.7. Show that an automorphism of a $C^*$-algebra is an isometry. Also verify that if $U \in \mathfrak{U}$ is unitary, the map $\text{Ad}U$ defined by $A \mapsto UAU^*$ is an automorphism.

An automorphism that is given by conjugation with a unitary, as in the exercise, is called inner. In general, not every automorphism of a $C^*$-algebra is inner (except when $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$). Note that automorphisms preserve all the algebraic relations of the algebra. Hence they are a natural tool to model symmetries. This is the context in which we will encounter most automorphisms later in the course.
There is an important corollary that follows from the uniqueness of the GNS representation. If \( \alpha \) is an automorphism of \( \mathcal{A} \) and \( \omega \) is invariant under the action of this automorphism (for example, a ground state of a physical system is invariant under some symmetry), then \( \alpha \) is implemented by a unitary in the GNS representation. The precise statement is as follows:

**Corollary 2.4.8.** Let \( \mathcal{A} \) be a \( C^* \)-algebra and \( \alpha \) an automorphism of \( \mathcal{A} \). Suppose that \( \omega \) is a state on \( \mathcal{A} \) such that \( \omega \circ \alpha = \omega \). Then there is a cyclic representation \( (\pi, \mathcal{H}, \Omega) \) such that \( \omega(\mathcal{A}) = \langle \Omega, \pi(\mathcal{A})\Omega \rangle \) and a unitary \( U \in \mathcal{B}(\mathcal{H}) \) such that \( \pi \circ \alpha(\mathcal{A}) = U\pi(\mathcal{A})U^* \) and \( U\Omega = \Omega \).

**Proof.** Let \( (\pi, \mathcal{H}, \Omega) \) be the GNS representation for the state \( \omega \). Note that \( (\pi \circ \alpha, \mathcal{H}, \Omega) \) is another GNS triple: \( \Omega \) is again cyclic since \( \alpha \) is an automorphism, and we have \( \langle \Omega, \pi(\alpha(\mathcal{A}))\Omega \rangle = \omega(\alpha(\mathcal{A})) = \omega(\mathcal{A}) \).

By the uniqueness statement in Theorem 2.4.3 there is a unitary operator \( U \) such that \( \pi \circ \alpha(\mathcal{A}) = U\pi(\mathcal{A})U^* \). This is the unitary we were looking for.

**Remark 2.4.9.** The automorphism \( \alpha \) does not have to be inner. At first sight it may seem like a contradiction that a non-inner automorphism is implemented by a unitary operator in the GNS representation. This confusion can be resolved by noting that \( U \) does not need to be in \( \pi(\mathcal{A}) \), but only in \( \mathcal{B}(\mathcal{H}) \), the latter being bigger in general.

Essentially the same proof can be given for groups of automorphisms. For example, let \( t \mapsto \alpha_t \), \( t \in \mathbb{R} \) be a group of automorphisms, that is, we have \( \alpha_{t+s} = \alpha_t \circ \alpha_s \) and each \( \alpha_t \) is an automorphism. Suppose that \( \omega \) is an invariant state for each \( \alpha_t \). Then there is a group of unitaries \( t \mapsto U_t \) implementing these automorphisms in the GNS representation. As an example, one can think of \( \alpha_t \) being the time evolution of a quantum mechanical system, defined on the observables (the Heisenberg picture). Then \( U_t \) gives us the unitary evolution on the Hilbert space. We will study this later in more detail.

**Exercise 2.4.10.** Consider a group of automorphisms \( \alpha_t \) as above, together with an invariant state \( \omega \).

1. Show that the map \( t \mapsto U_t \) indeed defines a unitary representation (so that \( U(t+s) = U(t)U(s) \) and \( U(t)^* = U(-t) \)).

2. Suppose that \( t \mapsto \alpha_t \) is strongly continuous. This means that for each \( A \in \alpha_t \), the map \( t \mapsto \alpha_t(A) \) is continuous. Show that for each \( \psi \in \mathcal{H}_\omega \) the map \( t \mapsto U_t\psi \) is continuous (with respect to the norm topology on \( \mathcal{H}_\omega \)). Hint: first show that it is enough to show continuity as \( t \to 0 \).

These results are actually true for any topological group \( G \).

Another application is the Gel’fand-Naimark theorem, which states that in fact every \( C^* \)-algebra can be seen as some algebra of bounded operators acting on a Hilbert space. In practice this is however of limited value. The resulting representation is hard to describe explicitly, and does not give much insight. It is therefore often easier to work with the abstract \( C^* \)-algebra \( \mathcal{A} \) itself.

Before we prove this result we first return to direct sums of Hilbert spaces. At the end of Section 2.3 direct sums of a finite number of Hilbert spaces were
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defined. This can be generalised to arbitrary direct sums as follows. Let $I$ be an index set (infinite or not) and suppose that $\mathcal{H}_i$ is a Hilbert space for each $i \in I$. The direct sum $\bigoplus_{i \in I} \mathcal{H}_i$ is defined as all those $\psi = (\psi_i \in \mathcal{H}_i)_{i \in I}$ such that
\[ \sum_{i \in I} \|\psi_i\|^2_{\mathcal{H}_i} < \infty. \]

An inner product can then be defined by $\langle \psi, \eta \rangle = \sum_{i \in I} \langle \psi_i, \eta_i \rangle$, which converges precisely because of the condition on the norms. One can show that this indeed is an inner product and that it makes the direct sum into a Hilbert space. We will also write $\oplus_{i \in I} \psi_i$ for a vector $\psi$ as above.

If for each $i \in I$ we have a representation $\pi_i$ of a C*-algebra $\mathfrak{A}$ on $\mathcal{H}_i$, the direct sum representation $\pi$ can be defined by
\[ \pi(A)\psi = \bigoplus_{i \in I} \pi_i(A)\psi_i. \]

Note that this is a generalisation of the direct sums of linear maps that were discussed earlier. An easy check shows that $\pi$ is indeed a representation. We will also write $\oplus_{i \in I} \pi_i$ for this representation.

**Theorem 2.4.11 (Gel’fand-Naimark).** Every C*-algebra $\mathfrak{A}$ can be isometrically represented on some Hilbert space $\mathcal{H}$. That is, there is a faithful isometric representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$.

**Proof.** It is a fact that for a representation $\pi$ of $\mathfrak{A}$, for each $A \in \mathfrak{A}$ the inequality $\|\pi(A)\| \leq \|A\|$ holds (see Remark 2.4.2). It therefore remains to show that one can find a representation such that $\|\pi(A)\| \geq \|A\|$. To this end we will need the following result. Let $A \in \mathfrak{A}$ be positive (that is, $A = B^*B$ for some $B \in \mathfrak{A}$). Then there is a state $\omega$ on $\mathfrak{A}$ such that $\omega(A) = \|A\|$.

Now consider $B \in \mathfrak{A}$ and let $\omega$ be a state such that $\omega(B^*B) = \|B^*B\| (= \|B\|^2)$. Construct the corresponding GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega)$. Then
\[ \|\pi_\omega(B)\Omega\|^2 = \omega(B^*B) = \|B^*B\| = \|B\|^2. \]

Hence $\|\pi_\omega(B)\| \geq \|B\|$ and hence $\|\pi_\omega(B)\| = \|B\|$. To conclude the proof of the theorem, consider the representation $\pi$ defined by
\[ \pi = \bigoplus_{\omega \in \mathcal{S}(\mathfrak{A})} \pi_\omega, \]
where $\pi_\omega$ is the corresponding GNS representation. In fact it is enough to only consider the pure states (c.f. Footnote 8). It follows that $\|\pi(A)\| = \|A\|$ for all $A \in \mathfrak{A}$. \(\square\)

In the theorem we constructed a direct sum of cyclic representations. It is true in general that any non-degenerate representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ can be obtained as a direct sum of cyclic representations. To illustrate this, choose a non-zero vector $\psi \in \mathcal{H}$. Then we can consider the subspace $\mathcal{K} = [\pi(\mathfrak{A})\psi]$. The proof of this, which is not too difficult, relies on some facts that we have not discussed in the lectures. See for example [31, Thm. 4.3.4(iv)]. One can even show that $\omega$ can be chosen to be a pure state.
the closure of the set $\pi(\mathfrak{A})\psi$. Note that $\mathcal{K}$ is a Hilbert space and that we have a projection $P_{\mathcal{K}}$ that projects $\mathcal{H}$ onto this subspace. Since $\mathcal{K}$ is invariant, $\pi(A)\psi \in \mathcal{K}$ if $\psi \in \mathcal{K}$ and it follows that $P_{\mathcal{K}}\pi(A) = \pi(A)P_{\mathcal{K}}$ for all $A \in \mathfrak{A}$. The same is true for the projection on the orthogonal complement, $I - P_{\mathcal{K}}$. This allows us to write $\pi$ is a direct sum of representations,

$$\pi(A) = P_{\mathcal{K}}\pi(A)P_{\mathcal{K}} \oplus (I - P_{\mathcal{K}})\pi(A)(I - P_{\mathcal{K}})$$

It is straightforward to check this equality if we identify $\mathcal{H}$ with $\mathcal{K} \oplus \mathcal{K}^\perp$. By construction, $P_{\mathcal{K}}\pi(A)P_{\mathcal{K}}$ is a cyclic representation. We can then start over again with the representation in the second summand, and so on, to write $\pi$ as a direct sum of cyclic representations. This argument can be made fully rigorous by an application of Zorn’s Lemma.

A natural question is when it is not possible to further decompose a representation into a direct sum of representations. Such representations are called irreducible. There are in fact two natural notions of irreducibility. Let $\mathfrak{A}$ be a $C^*$-algebra acting non-degenerately on some Hilbert space $\mathcal{H}$. The non-degeneracy condition means that $\mathfrak{A}\mathcal{H}$ is dense in $\mathcal{H}$. This amounts to saying that there is no non-zero vector $\psi$ such that $A\psi = 0$ for all $A \in \mathfrak{A}$. If $\mathfrak{A}$ acts degenerately, one can always restrict $\mathfrak{A}$ to (the closure of) the subspace $\mathfrak{A}\mathcal{H}$.

The algebra $\mathfrak{A}$ is said to act topologically irreducible if the only closed subspaces left globally invariant under the action of $\mathfrak{A}$ are $\{0\}$ and $\mathcal{H}$ itself. It is said to be irreducible if $\mathfrak{A}' = \mathbb{C}I$. Here the prime denotes the commutant of $\mathfrak{A}$, that is,

$$\mathfrak{A}' = \{B \in \mathfrak{B}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathfrak{A}\}.$$ 

A perhaps surprising result is that for $C^*$-algebras both notions of irreducibility coincide.

**Proposition 2.4.12.** Let $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ be a representation of a $C^*$-algebra $\mathfrak{A}$. Then the following are equivalent:

1. $\pi$ is topologically irreducible.
2. The commutant of $\pi(\mathfrak{A})$ is equal to $\mathbb{C}I$.
3. Every non-zero vector $\psi \in \mathcal{H}$ is cyclic for $\pi$.

**Proof.** First note that if $\psi \in \mathcal{H}$ is any vector, then $\pi(\mathfrak{A})\psi$ is an invariant subspace under the action of $\pi(\mathfrak{A})$ and hence so is its closure. Hence if $\pi$ is topologically irreducible and $\psi \neq 0$ it follows that this closure must be equal to $\mathcal{H}$, so that $\psi$ is cyclic. Conversely, suppose that $\pi$ is not topologically irreducible. Consider a non-trivial closed subspace $\mathcal{K}$ and $\psi \in \mathcal{K}$ non-zero. Since $\psi$ is cyclic, the closure of $\pi(\mathfrak{A})\psi$ is equal to $\mathcal{H}$ by assumption. But by the invariance assumption, $\pi(\mathfrak{A})\psi \subset \mathcal{K}$, a contradiction. This proves the equivalence of 1 and 3.

Next suppose that the commutant is trivial. Let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$ that is invariant under $\pi(\mathfrak{A})$. Then the projection $P_{\mathcal{K}}$ onto $\mathcal{K}$ commutes with $\pi(\mathfrak{A})$, as we have seen above. Hence $P_{\mathcal{K}}\lambda I$ for some $\lambda \in \mathbb{C}$. But $P_{\mathcal{K}}^2 = P_{\mathcal{K}}$, from which it follows that $\lambda = 0$ or $\lambda = 1$ and therefore $\mathcal{K}$ is equal to $\mathcal{H}$ or the zero subspace, proving that 2 implies 1.
Finally, suppose that $\pi$ is topologically irreducible. For the sake of contradiction, let us assume that the commutant is non-trivial. It follows (using so-called spectral theory) that there must be a non-trivial projection $P$ in $\pi(\mathfrak{A})'$. But then the range of this projection is invariant under $\pi(\mathfrak{A})$, a contradiction.

**Exercise 2.4.13.** Let $\mathfrak{A} = M_2(\mathbb{C})$. For $A \in \mathfrak{A}$, write $A_{i,j}$ for the corresponding matrix elements. Define $\rho(A) = A_{1,1}$ and $\sigma(A) = \lambda A_{11} + (1 - \lambda)A_{22}$, where $0 < \lambda < 1$. Show that $\rho$ and $\sigma$ are states and find the corresponding GNS representations. Also show that in the first case the representation is irreducible, while in the second case the GNS representation can be written as the direct sum of two irreducible representations.

In Exercise 2.4.13 we obtained GNS representations for different states of $M_2(\mathbb{C})$. Note that in the notation of that exercise, $\rho$ is a pure state, while $\sigma$ is mixed. The corresponding GNS representation of $\rho$ is irreducible, while that of $\sigma$ is reducible. This is in fact a general and very useful result:

**Theorem 2.4.14.** Let $\mathfrak{A}$ be a $C^*$-algebra and $\omega$ a state on $\mathfrak{A}$. Then the corresponding GNS representation $(\pi, \mathcal{H}, \Omega)$ is irreducible if and only if $\omega$ is pure.

**Proof.** Suppose that $\omega$ is pure but $(\pi, \mathcal{H}, \Omega)$ is not irreducible. Then there is some non-zero $T \in \pi(\mathfrak{A})'$. It follows that $T^*$ also commutes with every $\pi(A)$ and hence $T + T^*$ is in the commutant. Because there is a non-trivial self-adjoint element in the commutant, it follows that there must be a non-trivial projection $P$ there as well. Consider a linear functional $\varphi$ defined by

$$\varphi(A) = \langle P\Omega, \pi_\omega(A)P\Omega \rangle.$$ 

This functional is non-zero because $P$ commutes with $\pi(\mathfrak{A})$ and $\Omega$ is cyclic for $\pi_\omega$. Moreover, it is clearly positive and we have

$$\varphi(A^*A) = \langle P\Omega, \pi_\omega(A^*A)P\Omega \rangle = \langle \Omega, \pi_\omega(A^*)P\pi_\omega(A)\Omega \rangle \leq \|P\|\omega(A^*A),$$

with Theorem 2.4.13. Hence $\varphi$ is majorised by $\omega$. On the other hand, it is easy to check that $\varphi$ is not a multiple of $\omega$. This is in contradiction with the purity of $\omega$ (see also Remark 2.3.7).

Conversely, suppose that $\pi_\omega$ is irreducible. We sketch how to show that $\omega$ must be pure. Let $\varphi$ be a positive linear functional majorised by $\omega$. We want to show that $\varphi = \lambda \omega$ for some $0 \leq \lambda \leq 1$. Consider the dense subspace $\pi_\omega(A)\Omega$ of $\mathcal{H}_\omega$. We can define a new inner product on this space by

$$\langle \pi_\omega(A)\Omega, \pi_\omega(B)\Omega \rangle_\varphi = \varphi(A^*B).$$

By assumption the new inner product is bounded by the old inner product. By the Riesz representation theorem there exists a bounded operator $T$ such that $\langle \pi_\omega(A)\Omega, \pi_\omega(B)\Omega \rangle_\varphi = \langle \pi_\omega(A)\Omega, T\pi_\omega(B)\Omega \rangle$ and $\|T\| \leq 1$ (compare with Footnote 5 on page 8). By checking on a dense subset one can show that $T$ commutes with any $\pi_\omega(A)$ and hence is of the form $\lambda I$ for some $\lambda$. It follows that $\varphi = \lambda \omega$. Since $\|T\| \leq 1$, also $\lambda \leq 1$ and the proof is complete. \(\square\)
These results can be used to study the structure of abelian $C^*$-algebras. If $\mathcal{A}$ is an abelian $C^*$-algebra, a character is defined to be a non-zero linear map $\omega: \mathcal{A} \to \mathbb{C}$ such that $\omega(AB) = \omega(A)\omega(B)$ for all $A, B \in \mathcal{A}$. One can prove that a character is automatically continuous and positive. In other words, a character is a one-dimensional representation of $\mathcal{A}$. It turns out that the characters are just the pure states of $\mathcal{A}$.

**Proposition 2.4.15.** Let $\mathcal{A}$ be an abelian $C^*$-algebra. Then $\omega$ is a character of $\mathcal{A}$ if and only if $\omega$ is a pure state.

**Proof.** We first show that a state $\omega$ on $\mathcal{A}$ is pure if and only if $\omega$ is a character. To this end, note that by the theorem above $\omega$ is pure if and only if the GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ is irreducible. But because $\mathcal{A}$ is abelian, we always have that $\pi_\omega(\mathcal{A}) \subset \pi_\omega(\mathcal{A})'$. This can only be true if $\mathcal{H}_\omega$ is one-dimensional. Hence we have

$$
\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle = \pi_\omega(A)\langle \Omega, \Omega \rangle.
$$

Since $\pi_\omega$ is a representation, it follows that $\omega(AB) = \omega(A)\omega(B)$. Conversely, since $\omega$ is a state it follows by Theorem 2.3.3 that $\omega(A^*) = \omega(A)$. Hence if $\omega(AB) = \omega(A)\omega(B)$, $\omega$ is a representation and hence by the uniqueness of the GNS construction, $\mathcal{H}_\omega$ is one-dimensional.

It remains to be shown that a character of $\mathcal{A}$ is in fact a state. We assume that $\mathcal{A}$ is unital, the non-unital case can be shown with the help of approximate units. Since $\omega$ is a character, it is positive and continuous by the comments before the proposition. Moreover, for all $A \in \mathcal{A}$ it holds that

$$
\omega(A) = \omega(AI) = \omega(A)\omega(I),
$$

hence $\omega(I) = 1$ and $\|\omega\| = 1$. 

**Remark 2.4.16.** The set of characters of an abelian $C^*$-algebra $\mathcal{A}$ is called the spectrum of the algebra. Let us write $X$ for this set. One can endow $X$ with a locally compact topology. The abelian Gel’fand-Naimark theorem mentioned earlier says that $\mathcal{A}$ is isomorphic to $C_0(X)$ for $X$ the spectrum of $\mathcal{A}$.

### 2.5 Quantum mechanics in the operator algebraic approach

Quantum mechanics can be formulated in terms of $C^*$-algebras and states. This is often dubbed the algebraic approach to quantum mechanics. It can be argued that this is indeed the right framework for quantum mechanics. One of the fathers of algebraic quantum theory was von Neumann [70], who also made many important contributions to operator algebra [71]. Indeed, he can be regarded as one of the founders of the field. Later important contributions to algebraic quantum theory were made by I.E. Segal [59], who tried to give an abstract characterization of the postulates of quantum mechanics, and Haag and Kastler [29], who apply the ideas of algebraic quantum mechanics to quantum field theory. As mentioned before, one of the driving forces behind the push for an abstract description of quantum mechanics comes from the wish to understand better systems with infinitely many degrees of freedom. Here we
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give an overview of the main points in the algebraic approach. A much more thorough analysis can be found in [3, 50] or [40, Ch. 14].

Recall that in quantum mechanics, the observables are self-adjoint operators \( A \) (which can in principle be unbounded). The eigenvalues or the spectrum of \( A \) determine the possible outcomes after measuring the observable. A (pure) state is represented by a vector (“wave function”) \( \psi \) in the Hilbert space \( \mathcal{H} \).

The expectation value of the measurement outcome is then given, according to the usual probabilistic interpretation of quantum mechanics, by the quantity \( \langle \psi, A\psi \rangle \). After the measurement outcome has been determined, the state “collapses” in accordance with the outcome.

In the abstract setting all these properties can be defined without any reference to the Hilbert space. The observables are modelled by the self-adjoint elements of a \( C^* \)-algebra \( \mathfrak{A} \). By extension we will sometimes call non-self-adjoint operators observables as well (and call \( \mathfrak{A} \) the algebra of observables), even though strictly speaking they are not. Also note that the product of two observables is not necessarily an observable again (since \( A^* B^* \neq B^* A^* \) in general). Alternatively one can work with so-called Jordan algebras where this problem does not occur. Jordan algebras are less well understood, however, and do not have many of the nice properties of \( C^* \)-algebras. The interested reader can learn about them in [7], for example.

The set of possible outcomes of a measurement is given by the spectrum of an observable. The spectrum \( \sigma(A) \) of an operator \( A \) is defined by

\[
\sigma(A) := \{ \lambda \in \mathbb{C} : (A - \lambda I) \text{ is not invertible} \}.
\]

Note that the inverse should be in the algebra \( \mathfrak{A} \) as well. One can prove that \( \sigma(A) \subset \mathbb{R} \) if \( A \) is self-adjoint. The spectrum is a generalisation of the set of eigenvalues of a matrix. Indeed, if \( A \) is an \( n \)-by-\( n \) matrix and \( \lambda \) an eigenvalue, \( A - \lambda I \) is clearly not invertible, since it maps the corresponding eigenspace to the zero space. Conversely, if \( \lambda \in \sigma(A) \), then ker(\( A \)) is non-trivial by the rank-nullity theorem. Any non-zero vector in the kernel is an eigenvector with eigenvalue \( \lambda \). For bounded operators on infinite dimensional Hilbert spaces this correspondence breaks down. Any eigenvalue is still in the spectrum, but the converse is not necessarily true. Another significant difference is that in the infinite setting, the spectrum is not necessarily discrete. Nevertheless, there is an analogue of the eigenvalue decomposition. Namely, every self-adjoint operator \( A \) can be obtained as

\[
A = \int \lambda dE(A; \lambda),
\]

10 A proper treatment would require discussion of so-called normal states and of von Neumann algebras. Since we will not need these concepts later on, we will not go into the details here.

11 This excludes unbounded operators. One can argue that this is no severe objection, since in actual experiments there will always be a bounded range in which a measurement apparatus can operate. If one considers an unbounded operator on a Hilbert space, one can show that its spectral projections are contained in a von Neumann algebra (a subclass of the \( C^* \)-algebras). These spectral projections essentially project on the subspace of states with possible measurement outcomes in a bounded range. The advantage of working with bounded operators is that they are technically much easier to handle. Nevertheless, one can incorporate unbounded operators in the framework (by passing to the GNS representation of a state, for example). We will encounter some examples of this later on.
where $E(A; \lambda)$ are the so-called spectral projections. For each interval $I$ (or more precisely, Borel subset of $\mathbb{R}$), $E(A; I)$ is a projection. These projections are in general not elements of the $C^*$-algebra. However, if one represents the algebra on some Hilbert space, the projections are elements of $B(H)$. We should note that this construction is state dependent.

Now assume that the system is in a state $\omega$ and that we want to measure some (self-adjoint) observable $S$. Let $I$ be some interval. The probability that the measurement outcome lies in this interval is given by the Born rule: it is equal to $\omega(E(S; I)^* E(S; I))$. Suppose that after measurement it is found that the value lies somewhere in an interval $I$. Let $P = E(S; I)$ be the corresponding spectral projection. Then the new state after the measurement is given by

$$\omega'(A) = \frac{\omega(PAP^*)}{\omega(P^*P)}$$

provided $\omega(P^*P) \neq 0$. The latter condition can be interpreted physically as well. Recall that projections correspond to “yes/no” experiments; they have outcome 0 or 1. If $\omega(P) = 0$, it means that the expectation value of this experiment is zero. But this means that the event occurs with probability zero. Hence the condition that $\omega(P^*P) \neq 0$ is very natural.

It is also possible to define transition probabilities \[56\]. Recall that in quantum mechanics the probability of a transition from a state $\psi$ (which we see as a vector in some Hilbert space) to a state $\xi$ is given by $|\langle \psi, \xi \rangle|^2$. We can define a similar quantity in the algebraic setting as well. If $\omega_1$ and $\omega_2$ are two pure states on some $C^*$-algebra, the transition probability is defined as

$$P(\omega_1, \omega_2) = 1 - \frac{1}{4} ||\omega_1 - \omega_2||^2. \quad (2.5.1)$$

One can show that for two vector states $\psi, \xi$ this reduces to the usual quantity in quantum mechanics \[56\].

There are some aspects that we have not mentioned so far. One of them is the dynamics of the system. In Hilbert space quantum mechanics the dynamics are induced (through the Schrödinger equation) by a (possibly unbounded) operator $H$, the Hamiltonian. In the algebraic approach a natural approach is to look at the time evolution of the observables, $t \mapsto A(t)$. This is essentially the Heisenberg picture. The time evolution induces a one-parameter group of automorphisms $t \mapsto \alpha_t$ of the observable algebra. This brings us to another advantage of the algebraic approach. Symmetries of the system can be described in a natural way by groups of automorphisms acting on the observable. Symmetries have many applications in quantum mechanics, but let us mention one in particular. The algebraic approach turns out to be a convenient framework to describe spontaneous symmetry breaking in a mathematically rigorous way \[63\]. We will come back to symmetries in more detail later.

**Inequivalent representations**

We will say that two representations $\pi$ and $\rho$ of a $C^*$-algebra are unitary equivalent (or simply equivalent) if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$ between the corresponding Hilbert spaces and in addition we have that $U \pi(A) U^* = \rho(A)$ for all $A \in \mathfrak{A}$. If this is the case, we also write $\pi \cong \rho$. In general, a
Theorem 2.5.1 was first analysed by Wick, Wightman and Wigner \[72\].

Consider a representation $\pi$ this representation. Write $\psi$ not superposable if and only if the corresponding GNS representations can consider the subspaces $H$. Projections on these subspaces will be denoted by $P$. Each irreducible representation of these relations is unitary equivalent to the representation generated by the Pauli matrices \[73\]. A similar result is true for a finite number of copies of such systems. We will see that the existence of inequivalent representations has consequences for the superposition principle.

For example, von Neumann \[69\] showed that there is only one irreducible representation of the canonical commutation relations $[P,Q] = i\hbar$ (up to unitary equivalence). The same is true for a spin-1/2 system. If we consider the algebra generated by $S_x, S_y, S_z$, satisfying $[S_i, S_j] = i\varepsilon_{ijk} S_k$ and $S_x^2 + S_y^2 + S_z^2 = \frac{3}{4} I$, each irreducible representation of these relations is unitary equivalent to the representation generated by the Pauli matrices \[73\]. A similar result is true for a finite number of copies of such systems. We will see that the existence of inequivalent representations has consequences for the superposition principle.

If $\pi : \mathfrak{A} \to \mathfrak{B}(H)$ is a representation of a $C^*$-algebra, there is an easy way to obtain different states on $\mathfrak{A}$. Take any vector $\psi \in H$ of norm one. Then the assignment $A \mapsto \langle \psi, \pi(A)\psi \rangle$ defines a state. Such states are called vector states for the representation $\pi$. Note that by the GNS construction it is clear that any state can be realised as a vector state in some representation. Consider two states $\omega_1$ and $\omega_2$ that are both vector states for the same representation $\pi$. Hence there are vectors $\psi_1$ and $\psi_2$ such that $\omega_i(A) = \langle \psi_i, \pi(A)\psi_i \rangle$ for all $A \in \mathfrak{A}$. Consider now $\psi = \alpha \psi_1 + \beta \psi_2$ with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$ and both $\alpha$ and $\beta$ are non-zero. Then $\omega(A) = \langle \psi, \pi(A)\psi \rangle$ again is a state. However, it may be the case that the resulting state is not pure (even if the $\omega_i$ are) and we have a mixture

$$\omega(A) = |\alpha|^2 \omega_1(A) + |\beta|^2 \omega_2(A). \quad (2.5.2)$$

If this is the case for any representation $\pi$ in which $\omega_1$ and $\omega_2$ are vector states, we say that the two states are not superposable or not coherent. This situation was first analysed by Wick, Wightman and Wigner \[72\].

**Theorem 2.5.1** (\[3\] Thm 6.1). Let $\omega_1$ and $\omega_2$ be pure states. Then they are not superposable if and only if the corresponding GNS representations $\pi_{\omega_1}$ and $\pi_{\omega_2}$ are inequivalent.

**Proof.** Consider a representation $\pi$ such that $\omega_1$ and $\omega_2$ are vector states in this representation. Write $\psi_i \in H$ for the corresponding vectors. Then we can consider the subspaces $H_i$ of $H$, defined as the closure of $\pi(\mathfrak{A})\psi_i$. The projections on these subspaces will be denoted by $P_i$.

Note that $\psi_i$ is, by definition, cyclic for the representation $\pi(\mathfrak{A})$ restricted to $H_i$. Let us write $\pi_i$ for these restricted representations. But since the vectors $\psi_i$ implement the state, it follows that the representation $\pi_i$ must be (unitary equivalent to) the GNS representations $\pi_{\omega_i}$. Let $U : H_1 \to H_2$ be a bounded linear map such that $U \pi_1(A) = \pi_2(A)U$ for all $A \in \mathfrak{A}$. By first taking adjoints, we then see that $U^*U \pi_1(\lambda) = \pi_2(\lambda)U^*U$. By irreducibility of $\pi_1$ it follows that $U^*U = \lambda I$ for some $\lambda \in \mathbb{C}$. In fact, $\lambda$ must be real since $U^*U$ is self-adjoint. A similar argument holds for $UU^*$. Hence by rescaling we can choose $U$ to be unitary, unless $U^*U = 0$. Hence non-zero maps $U$ intertwining the representations only exist if $\pi_1$ and $\pi_2$ are unitarily equivalent.

To get back to the original setting, let $T \in \pi(\mathfrak{A})'$. Then $P_2TP_1$ can be identified with a map $U : H_1 \to H_2$ such that $U \pi_1(A) = \pi_2(A)U$. Conversely,
any such map can be extended to an operator in $P_2 \pi(\mathfrak{A})'P_1$. Hence

$$P_2 \pi(\mathfrak{A})'P_1 = \{0\}$$

if and only if $\pi_1$ and $\pi_2$ are not unitarily equivalent.

Since $I \in \pi(\mathfrak{A})'$ it is clear that $P_1 P_2 = P_2 P_1 = 0$ if $\pi_1$ and $\pi_2$ are not equivalent. This implies that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal subspaces of $\mathcal{H}$. Hence

$$\langle \psi_2, \pi(A) \psi_1 \rangle = 0 = \langle \psi_1, \pi(A) \psi_2 \rangle.$$  

Consequently, if $\psi = \alpha \psi_1 + \beta \psi_2$ with $|\alpha|^2 + |\beta|^2 = 1$, then

$$\omega(A) := \langle \psi, \pi(A) \psi \rangle = |\alpha|^2 \omega_1(A) + |\beta|^2 \omega_2(A).$$

Hence $\omega_1$ and $\omega_2$ are not superposable.

Conversely, suppose that $\pi_{\omega_1}$ and $\pi_{\omega_2}$ are unitarily equivalent. Then there must be some unitary $U \in \pi(\mathfrak{A})'$ such that $P_2 U P_1 \neq 0$. This is only possible if there are vectors $\varphi_2 \in \mathcal{H}_2$ such that $\langle \varphi_2, U \varphi_1 \rangle \neq 0$. Since $\pi(\mathfrak{A}) \psi_1$ is dense in $\mathcal{H}_1$ (and similarly for $\mathcal{H}_2$), there must be $A_1, A_2 \in \mathfrak{A}$ such that

$$\langle \pi(A_1) \psi_2, U \pi(A_2) \psi_1 \rangle \neq 0. \quad (2.5.3)$$

Set $\varphi = U \psi_1$. Since $U$ commutes with $\pi(A)$ for every $A$, it follows that

$$\langle \varphi, \pi(A) \varphi \rangle = \omega_1(A).$$

Now consider the vector $\psi = \alpha \varphi + \beta \psi_2$. This induces a state $\omega$, and we find

$$\omega(A) - |\alpha|^2 \omega_1(A) - |\beta|^2 \omega_2(A) = \overline{\alpha} \beta \langle \psi_1, \pi(A) \psi_2 \rangle + \overline{\alpha} \omega_2(\psi_2, \pi(A) \psi_1). \quad (2.5.4)$$

Consider then $A = A_2^* A_1$, where $A_1$ and $A_2$ are as above. Then the right hand side of equation (2.5.4) becomes

$$2 \text{Re}(\overline{\alpha} \omega_2(\pi(A_1) \psi_2, U \pi(A_2) \psi_1)).$$

By choosing a suitable multiple if $A$ we can make the right hand side non-zero, because of equation (2.5.3). It follows that equation (2.5.2) does not hold.

This result shows that as soon as a $C^*$-algebra has inequivalent representations, there are states that are not coherent. That is, there are pure states $\omega_1$ and $\omega_2$ such that a superposition of those states is never pure. The proof also makes clear that if we have vector states corresponding to inequivalent representations, there can never be a transition from one state to the other, not even by applying any operation available in $\mathfrak{A}$, because $\langle \psi_1, \pi(A) \psi_2 \rangle$ is zero. Such a rule that forbids such transitions is called a superselection rule. There are many different (but strongly related) notions of a superselection rule around, see for example [18] for a discussion.

We can also compare this result with the definition of a transition probability between pure states, equation (2.5.1). Note that if $P(\omega, \sigma) \neq 0$, we must have that $\|\omega - \sigma\| < 2$. But one can show that if $\|\omega - \sigma\| < 2$, then necessarily the GNS representations $\pi_\omega$ and $\pi_\sigma$ must be unitarily equivalent [32, Cor. 10.3.8]. Hence these results are consistent: there can be no transitions between inequivalent pure states.
Exercise 2.5.2. Let $\mathfrak{A} = M_n(\mathbb{C})$. Show that there are no non-coherent states.

Remark 2.5.3. The Theorem is still true for non-pure states, only then the inequivalence condition has to be replaced with the condition that the representations are disjoint. Two representations $\pi_1$ and $\pi_2$ are disjoint if and only if no subrepresentation of $\pi_1$ is unitarily equivalent to a subrepresentation of $\pi_2$. 
3 | Infinite systems

With the preparations from the previous chapter we are now in a position to discuss the main topic of interest: quantum spin systems with infinitely many sites. We will first consider an explicit example, showing that inequivalent representations appear in a natural way. Then we discuss the appropriate mathematical framework to discuss spin systems with infinitely many sites. After that we discuss how to describe dynamics in this setting, and how to define equilibrium and ground states.

Although we only consider discrete spin systems here, most of the techniques generalise to more complicated systems. For example, instead of spin systems we can consider systems with at each site a separable Hilbert space. Or we can consider continuous systems and define, for example, creation and annihilation operators to obtain Fock space. This is the appropriate setting to discuss many-body quantum systems. Furthermore, one can consider the algebraic approach to quantum field theory [25]. This aims to give an axiomatic and mathematically fully rigorous description of quantum field theories.

3.1 (In)equivalence of representations

As mentioned before, there are fundamental differences between systems with finitely and infinitely many degrees of freedom. For example, the existence of inequivalent representations does not occur for finite systems, as was already mentioned at the end of Chapter 2. To see that this is drastically different for infinite systems, we will now consider a simple example of a system with infinitely many sites, namely that of an infinite spin chain. Later we will approach such systems more systematically, but this example already shows that going to infinitely many sites gives rise to phenomena not encountered in finite systems.

More concretely, we imagine that the sites of our system are labelled by the integers, and that at each site there is a Pauli spin system. One can think for example of the Heisenberg model. First we define the observables. At each site of the system there is a copy of the algebra generated by the usual Pauli matrices. That is, we have operators $\sigma^x_n$, $\sigma^y_n$ and $\sigma^z_n$ for each $n \in \mathbb{Z}$. These operators fulfil the following commutation relations:

$$ [\sigma^i_n, \sigma^j_m] = 2\delta_{n,m} \sum_{k=1}^3 \varepsilon_{ijk} \sigma^k_n, \quad (3.1.1) $$

This example is adapted from Chapter 2.3 of [60].
where \(i, j = x, y, z\) and \(\varepsilon_{ijk}\) is the completely anti-symmetric (Levi-Civita) symbol, together with the condition that \((\sigma^z_n)^2 = I\). Note in particular that observables acting on different sites commute. These operators and relations generate an algebra of observables. Later we will see how we can obtain a \(C^*\)-algebra from this, but for now we forget about the topological structure.

The first step in defining a representation of this algebra is to give the Hilbert space on which it acts. To this end, consider the spin in the \(z\)-direction at each site. The spin can be either up or down after a measurement in this basis. If we do this for each site, we get a sequence \(s_n\), with \(n \in \mathbb{Z}\) and \(s_n = \pm 1\). We denote \(S\) for the set of such sequences. To define the Hilbert space, a naive first attempt would be to let each such \(s \in S\) correspond to a basis vector \(|s\rangle\).

But this is a huge Hilbert space, since the set \(S\) is uncountable.\(^2\) This makes things much more complicated, and arguably isn’t very physical. For example, the algebra generated by the operators \(\sigma_n\) would act far from irreducibly on this Hilbert space. Therefore, instead we look at subsets of \(S\):

\[
S^+ := \{s_n \in S : s_n \neq 1 \text{ for finitely many } n \in \mathbb{Z}\}.
\]

That is, the set of all sequences \(s_n\) for which only finitely many spins are not pointing upwards. Note that this set is countable. The set \(S^-\) is defined analogously. The corresponding Hilbert spaces are then \(\mathcal{H}^\pm = l^2(S^\pm)\). It turns out that it is more convenient to work with an alternative description of these Hilbert spaces. Namely, consider the set of all functions \(f : S^+ \to \mathbb{C}\) such that

\[
\sum_{s_n \in S^+} |f(s_n)|^2 < \infty,
\]

together with the usual inner product. We have seen this alternative description before in Remark 2.1.6.\(^\text{\footnote{This can be seen by using a Cantor diagonal argument.}}\) We can identify the vector \(|s\rangle\) with the function \(f_s(s') = \delta_{s,s'}\), where \(\delta_{s,s'}\) is one if \(s_n = s'_n\) for all \(n\), and zero otherwise. These functions form an orthonormal basis for the Hilbert space. We can think of \(f \in \mathcal{H}^+\) as defining a vector \(|\psi\rangle = \sum_{s \in S^+} f(s) |s\rangle\).

Next we define a “spin flip” map on \(S\). Let \(n \in \mathbb{Z}\). Then \(\theta_n : S \to S\) is defined by

\[
(\theta_n(s))_k = \begin{cases} -s_k & \text{if } n = k \\ s_k & \text{otherwise.} \end{cases}
\]

Note that this flips the \(n\)-th spin, while leaving all the other spins invariant. Clearly it can be restricted to a map \(\theta_n : S^+ \to S^+\). Next we define a representation of the operators \(\sigma^z_n\) as follows. Let \(f \in \mathcal{H}^+\). Then we define

\begin{align*}
(\pi^+ (\sigma^z_n)f)(s) &= f(\theta_n(s)), \\
(\pi^+ (\sigma^y_n)f)(s) &= is_n f(\theta_n(s)), \\
(\pi^+ (\sigma^x_n)f)(s) &= s_n f(s), \\
(\pi^+ (I)f)(s) &= f(s) \tag{3.1.2}
\end{align*}

Note that these operators act as one would expect from the Pauli matrices: \(\pi^+ (\sigma^z_n)\) flips the spin at the \(n\)-th site and \(\pi^+ (\sigma^z_n)\) multiplies with the corresponding eigenvalue of the spin at the \(n\)-th site. Moreover, they satisfy the
same relations as in equation (3.1.1). So \( \pi^+ \) defines a representation of the algebra generated by the \( \sigma^k_n \).

We first show that the representation is irreducible. Note that by Proposition 2.4.12 there are a number of equivalent criteria. Here we will show that any vector is cyclic for the representation. Recall that the functions \( f_s(s') = \delta_{s,s'} \) form a basis of the Hilbert space. Since any \( s_1, s_2 \in S^+ \) differ only in a finite number of places \( n_1, \ldots, n_k \), we can transform \( f_{s_1} \) into \( f_{s_2} \) by acting with the operator

\[
\pi^+(\sigma^x_{n_1}) \cdots \pi^+(\sigma^x_{n_k})
\]

on it. Hence if we act with a finite number of operators we can transform one basis state into another. This implies that any basis vector is cyclic. This is not quite enough yet to conclude that any vector is cyclic. Consider a vector \( \psi = \sum_{k}^{i=1} \lambda_k f_{s_k} \) for certain \( s_k \in S^+ \) and \( \lambda_k \in \mathbb{C} \). Such vectors are dense in \( \mathcal{H}^+ \). Define for \( n \in \mathbb{Z} \) the following operators:

\[
P^+_n = \frac{\pi^+(I) \pm \pi^+(\sigma^z_n)}{2}.
\]

Note that \( P^+_n \) projects onto the subspace of vectors where the \( n \)-th spin is up, and similarly for \( P^-_n \). Hence if \( \psi \) is above, we can construct an operator \( P \) that projects onto one of the basis vectors in the sum. If we act with this operator on \( \psi \), and then subsequently with other operators as above, we can span a dense subspace. It follows that \( \psi \) is cyclic and therefore that \( \pi^+ \) is irreducible.

**Exercise 3.1.1.** Show that \( \pi^+ \) is irreducible by checking that only multiples of the identity commute with the representation.

In defining the representation \( \pi^+ \) we started with the subset \( S^+ \) of \( S \) where all but finitely many spins are \(+1\). We might as well have started with the set \( S^- \), defined in the obvious way: all but finitely many \( s_n \) are equal to \(-1\). Similarly as above we then define the Hilbert space \( \mathcal{H}^- \) and a representation \( \pi^- \). The representation \( \pi^- \) can be defined as in equation (3.1.2), but note that it acts on a different Hilbert space. Even though the representation is defined in essentially the same way, we will see that \( \pi^+ \) and \( \pi^- \) are not unitarily equivalent.

To see this, we will look at the polarisation of the system. For each \( N \in \mathbb{N} \), define

\[
m_N = \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma^z_n.
\]

Using the definitions we see that for any \( s, s' \in S^+ \) we have

\[
(f_{s'}, \pi^+(m_N)f_s) = \frac{1}{2N+1} \sum_{n=-N}^{N} s_n \delta_{s,s'} = \frac{\delta_{s,s'}}{2N+1} \sum_{n=-N}^{N} s_n.
\]

Since all but finitely many \( s_n \) are \(+1\), this converges to one as \( N \to \infty \) (or zero if \( s \neq s' \)). By linearity this is also true for vectors that are finite linear combinations of such base vectors. To show that it is true in general, first note that \( \| \pi^+(\sigma^z_n) \| = 1 \). By the triangle inequality it follows that \( \| \pi^+(m_N) \| \leq 1 \) for all \( N \). Because of this uniform bound it follows that

\[
\lim_{N \to \infty} \langle \psi, \pi^+(m_N)\xi \rangle = \langle \psi, \xi \rangle
\]
for all $\psi, \xi \in H^+$. We can do a similar thing for the representation $\pi^-$ acting in $H^-$. Here we find
\[ \lim_{N \to \infty} \langle \psi, \pi^-(mN)\xi \rangle = -\langle \psi, \xi \rangle, \]
hence with a minus sign (since most of the spins are pointing downwards).

Now suppose that the representations $\pi^+$ and $\pi^-$ are unitarily equivalent. Then there is a unitary map $U : H^+ \to H^-$ such that $U\pi^+(A)U^* = \pi^-(A)$ for all observables $A$. In particular, choose some $f_s \in H^+$. It follows that
\[ \langle f_s, \pi^+(mN)f_s \rangle = \langle f_s, U^*\pi^-(mN)Uf_s \rangle = \langle Uf_s, \pi^-(mN)Uf_s \rangle. \]

However, the left hand side of this expression tends to $+1$, while the right hand tends to $-1$ as $N$ goes to infinity. Hence such a unitary cannot exist.

This result can be understood intuitively as follows. To go from a vector in $H^+$ to a vector in $H^-$, we have to flip infinitely many spins around. This does not correspond to an operator in the algebra $\mathfrak{A}$ generated by the Pauli spin operators. That is, with no “physical” operation we can transform one Hilbert space to the other, where a physical operation is here understood as something that can be approximated by operations on a finite number of sites.

**Exercise 3.1.2.** Argue that there are infinitely many inequivalent representations.

### 3.2 Infinite tensor products

Suppose that we have a quantum system consisting of $n$ copies of, say, a spin-$1/2$ system. The single system is described by the Hilbert space $H_x = \mathbb{C}^2$. For the composite system we then have according to the rules of quantum mechanics $H = \bigotimes_{i=1}^n \mathbb{C}^2$, the tensor product of $n$ copies of the single site space. The observables at a single site are spanned by the Pauli matrices, and hence equal to $M_2(\mathbb{C})$. For the whole system we then have $\mathfrak{A} = \bigotimes_{i=1}^n M_2(\mathbb{C}) \cong \mathfrak{B}(\bigotimes_{i=1}^n \mathbb{C})$.

For an abstract $C^*$-algebra it is not immediately clear what the tensor product of this algebra with another $C^*$-algebra should be, and this can be a subtle issue. Other problems arise when we do not look at systems with finitely many sites any more, but infinitely many, for example in quantum spin chains. The example of inequivalent representations above already shows a fundamental difference between systems with finitely many and infinitely many sites. In this section we will introduce the proper framework to describe such systems.

**Infinite tensor product of Hilbert spaces**

One approach is to define first the tensor product of infinitely many copies of the one-site Hilbert space, say $\mathbb{C}^d$ for spin systems. Let us write $I$ for the set indexing the different sites in the system. The goal is to define $\bigotimes_{i \in I} H_i$, where in our case $H_i \cong \mathbb{C}^d$. We can try to generalise the tensor product construction of Section 2.4. That is, we first consider $\psi = (\psi_i)$ and $\xi = (\xi_i)$,
where $\psi_i, \xi_i \in \mathcal{H}_i$. The inner product between these vectors should be given by

$$\langle \psi, \xi \rangle = \prod_{i \in I} \langle \psi_i, \xi_i \rangle_{\mathcal{H}_i}.$$ 

The problem is that the product in the right hand side in general does not converge.

To solve this, choose a “reference” vector $\Omega_i \in \mathcal{H}_i$ of norm one, for every $i \in I$. Then consider only those sequences $\psi = (\psi_n)$ such that $\psi_i \neq \Omega_i$ for only finitely many $i \in I$. For such vectors the inner product clearly converges. We can then consider the linear space consisting of finite linear combinations of such vectors. The inner product can be extended to this space by linearity. Finally, we can take the completion with respect to the norm induced by this inner product to obtain a Hilbert space $\mathcal{H}$.

**Exercise 3.2.1.** Consider a one-dimensional chain of spin-$1/2$ systems. Choose as a reference vector $|\Omega_n\rangle$ at each site the spin-up vector in the $z$-basis. Define a unitary map $U : \bigotimes C^2 \rightarrow \mathcal{H}^+$, where the tensor product is the infinite tensor product as described in this section, and $\mathcal{H}^+$ is the Hilbert space defined in Section 3.1.

Once we have defined the Hilbert space we can consider the observables of the system. One could for example take $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ as the observable algebra. This has a downside, however: locality is lost in this approach. It is reasonable to assume that an experimenter can only perform measurements on a finite number of sites, and not of the whole (infinite) system at once. Hence the experimenter only has access to observables that can be approximated by strictly local observables. This is the approach we will consider later on.

Alternatively, one could at a similar construction of the observable algebra as was used for the tensor product. That is, consider observables of the form $A = \bigotimes A_i$, where $A_i \in \mathfrak{B}(\mathcal{H}_i)$ and $A_i = I$ for all but finitely many $i \in I$. Such observables act on $\bigotimes_{i \in I} \mathcal{H}_i$ in the obvious way. By taking linear combinations of such operators we obtain a $*$-algebra $\mathfrak{A}$. To get a $C^*$-algebra, we can take the double commutant $\mathfrak{M} = \mathfrak{A}''$. This is a so-called von Neumann algebra. A von Neumann algebra is in particular a $C^*$-algebra, but the converse is not true. A rather surprising result is that the resulting algebra $\mathfrak{M}$ strongly depends on the choice of reference vectors $\Omega_i$ in the construction of the tensor product [4, 53]. Hence in this approach one is faced with the decision of which sequence of vectors to take. The right choice is not at all obvious in general.

**Tensor products of $C^*$-algebras**

The (algebraic) tensor product of two $*$-algebras is defined as usual. We recall the construction here. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are $*$-algebras. Then the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$ consists of linear combinations of elements $A \otimes B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, where the following identifications are made:

$$\lambda (A) \otimes B = A \otimes (\lambda B) = \lambda (A \otimes B), \quad \lambda \in \mathbb{C}$$

$$(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B, \quad A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2.$$ 

\(a\) Note that the operator is well defined since only finitely many $A_i$ act non-trivial.
These conditions say that there is a bilinear map $L : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A} \otimes \mathfrak{B}$, defined by

$$(A_1 \otimes B_1)(A_2 \otimes B_2) := A_1 A_2 \otimes B_1 B_2,$$

$$(A \otimes B)^* = A^* \otimes B^*.$$ 

To obtain a $C^*$-algebra we have to define a norm on the algebraic tensor product, satisfying the $C^*$-property. The completion of the algebraic tensor with respect to this norm is a $C^*$-algebra, which is called the tensor product of $\mathfrak{A}$ and $\mathfrak{B}$ (with respect to the chosen norm).\footnote{Defining the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of two $C^*$-algebras is a delicate matter in general. The point is that there are, in general, different natural $C^*$-norms on the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$. These different norms lead to different completions, and hence different tensor products. There is a special class of $C^*$-algebras for which there is a unique norm on the tensor product, the nuclear algebras. If $\mathfrak{A}$ is nuclear, then there is a unique $C^*$-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ for any $C^*$-algebra $\mathfrak{B}$. Examples of nuclear algebras are $\mathcal{M}_n(\mathbb{C})$, the algebra of bounded operators on a Hilbert space, and abelian algebras. (Almost) all $C^*$-algebras we encounter will be nuclear.}

Analogous to the tensor product of bounded linear maps, defined before, it is desirable that the norm on the algebraic tensor product is a cross norm. That is, for $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, we want that

$$\|A \otimes B\| = \|A\| \|B\|,$$

where the norms on the right hand side are the norms of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. One way to obtain such a norm is to take faithful representations $\pi$ and $\rho$ of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then we define

$$\left\| \sum_{i=1}^n \pi(A_i) \otimes \rho(B_i) \right\| = \left\| \sum_{i=1}^n \pi(A_i) \otimes \rho(B_i) \right\|.$$ 

The norm on the right hand side is to be understood as the norm of the bounded linear operator on $\mathcal{H}_\pi \otimes \mathcal{H}_\rho$. This defines a norm which one can show is independent of the choice of faithful representation. The completion of $\mathfrak{A} \otimes \mathfrak{B}$ with respect to this norm is called the minimal or spatial tensor product. As mentioned in Footnote\footnote{Defining the tensor product $\mathfrak{A} \otimes \mathfrak{B}$ of two $C^*$-algebras is a delicate matter in general. The point is that there are, in general, different natural $C^*$-norms on the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$. These different norms lead to different completions, and hence different tensor products. There is a special class of $C^*$-algebras for which there is a unique norm on the tensor product, the nuclear algebras. If $\mathfrak{A}$ is nuclear, then there is a unique $C^*$-tensor product $\mathfrak{A} \otimes \mathfrak{B}$ for any $C^*$-algebra $\mathfrak{B}$. Examples of nuclear algebras are $\mathcal{M}_n(\mathbb{C})$, the algebra of bounded operators on a Hilbert space, and abelian algebras. (Almost) all $C^*$-algebras we encounter will be nuclear.} there may be other cross norms on $\mathfrak{A} \otimes \mathfrak{B}$. For the applications we consider, however, this will not be the case and we can safely write $\mathfrak{A} \otimes \mathfrak{B}$ without creating confusion about which norm to use.

### Quasi-local algebras

With the help of the tensor product defined above we can now define the observable algebra for the systems we are interested in. The observable algebra generated by the $\sigma_i^n$ defined in the first section of this chapter is an example. Here we will give a general construction of such algebras and show how we can define a norm to obtain a $C^*$-algebra. Again, we restrict to the case of quantum spin systems. The theory is developed in full generality in, e.g., [7, 8].

The starting point is a set $\Gamma$ labelling the sites of the system. We will always assume that $\Gamma$ is countable. In many cases $\Gamma$ will have some more structure, for example we can take $\Gamma = \mathbb{Z}^d$. In that case there is a natural action of the translation group on the set of sites. This will induce a group of translations
on the observable algebra, and allows us to talk about translation invariant system. An important theme will always be locality. This is concerned with operators acting only on a subset of the sites.

**Definition 3.2.2.** Let $\Gamma$ be as above. We will write $\mathcal{P}(\Gamma)$ for the set of all subsets of $\Gamma$. Similarly, $\mathcal{P}_f(\Gamma)$ is the subset of all finite subsets of $\Gamma$.

For simplicity we assume that at each site $x \in \Gamma$ there is a $d$-dimensional quantum spin system with observable algebra $A(\{x\}) := M_d(\mathbb{C})$, where $d$ is independent of the site $x$. If $\Lambda \in \mathcal{P}_f(\Gamma)$ is a finite collection of sites, the corresponding algebra of observables is given by

$$A(\Lambda) = \bigotimes_{x \in \Lambda} A(\{x\}) = \bigotimes_{x \in \Lambda} M_d(\mathbb{C}).$$

Note that this is the usual construction if one takes $n$ copies of a quantum mechanical system. The $C^*$-algebra $A(\Lambda)$ is understood as the algebra generated by all observables acting only on the sites in $\Lambda$ (and trivially elsewhere on the system).

In this way we obtain in a natural way a local net. The net structure is as follows. If $\Lambda_1 \subset \Lambda_2$ with $\Lambda_2 \in \mathcal{P}_f(\Gamma)$, then there is an inclusion of the corresponding algebras. If $A_1 \otimes \cdots \otimes A_n \in A(\Lambda_1)$ then $A_1 \otimes \cdots \otimes A_n \otimes I \cdots \otimes I$ is in $A(\Lambda_2)$, where we inserted copies of the identity operator acting on the sites of $\Lambda_2 \setminus \Lambda_1$. The assignment $\Lambda \mapsto A(\Lambda)$ is local in the following sense. If $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\Gamma)$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then we have

$$[A(\Lambda_1), A(\Lambda_2)] = \{0\},$$

where we understand $A(\Lambda_i)$ to be embedded into a sufficiently large $A(\Lambda)$ containing both $\Lambda_1$ and $\Lambda_2$, so that it makes sense to talk about the commutator.

The set of local or strictly local observables is then defined by

$$A_{loc} = \bigcup_{\Lambda \in \mathcal{P}_f(\Gamma)} A(\Lambda).$$

Note that $A_{loc}$ is a $*$-algebra. There is a $C^*$-norm on this algebra, given by the norms on each $A(\Lambda)$. The completion of $A_{loc}$ with respect to this norm is the quasi-local algebra $A$. This algebra can be interpreted as consisting of all those observables that can be approximated arbitrarily well (in norm) by observables in finite regions of space. As we have mentioned before, the physical idea is that an experimenter has only access to a bounded region of the system.

If $A \in A(\Lambda)$ for some $\Lambda \in \mathcal{P}_f(\Gamma)$ we say that $A$ is localised in $\Lambda$. The smallest such $\Lambda$ such that $A \in A(\Lambda)$ is called the support of $A$ and denoted by $\text{supp}(A)$. For convenience we will set $A(\emptyset) = \mathbb{C} I$, since multiples of the identity are contained in $A(\Lambda)$ for all $\Lambda \in \mathcal{P}_f(\Gamma)$. Analogously we can talk about observables localised in infinite regions. Let $\Lambda \subset \Gamma$, where $\Lambda$ is not necessarily finite. Then we can define

$$A(\Lambda) = \bigcup_{\Lambda_f \in \mathcal{P}_f(\Lambda)} A(\Lambda_f).$$

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5In this construction we glossed over some mathematical technicalities. Essentially, one constructs the inductive limit of a net of algebras. The inductive limit construction also works for non-trivial (but compatible with the net structure) inclusions of algebras. See e.g. [55, 64] for more details.
The bar means that we have to take the closure with respect to the norm, to obtain a $C^*$-algebra. Note that $\mathfrak{A}(\Lambda)$ can be embedded into $\mathfrak{A}$ in a natural way. We will always regard $\mathfrak{A}(\Lambda)$ as a subalgebra of $\mathfrak{A}$. Again, the interpretation of operators in $\mathfrak{A}(\Lambda)$ is that they can be approximated arbitrarily well by strictly local operators localised in $\Lambda$. Particularly important with respect to duality are complements of $\Lambda \in \mathcal{P}(\Gamma)$. We will denote the complement by $\Lambda^c$. The locality condition can be extended to the corresponding algebras, by continuity. That is, we have

$$[\mathfrak{A}(\Lambda), \mathfrak{A}(\Lambda^c)] = \{0\}$$

for any $\Lambda \in \mathcal{P}(\Gamma)$.

The construction so far is very general. In fact we see that it only depends on the dimensions of the local spin systems. Hence the algebra of observables alone does not contain much information. Rather, we need to consider additional concepts such as dynamics and states for these algebras. We will address this shortly.

**Simplicity of the quasi-local algebra**

Recall that a closed two-sided ideal $\mathcal{I}$ (or simply an ideal) of a $C^*$-algebra $\mathfrak{A}$ is a closed subspace $\mathcal{I} \subset \mathfrak{A}$ such that $AB$ and $BA$ are in $\mathcal{I}$ if $A \in \mathfrak{A}$ and $B \in \mathcal{I}$. A $C^*$-algebra is called simple if its only closed two-sided ideals are $\mathcal{I} = \{0\}$ and $\mathfrak{A} = \mathfrak{A}$.

**Exercise 3.2.3.** Let $\mathfrak{A} = M_d(\mathbb{C})$. Show that $\mathfrak{A}$ is simple. Hint: show first that if $\mathcal{I}$ is a non-trivial ideal, then it contains matrices $E_{ii}$ which are zero everywhere except on the $i$-th place on the diagonal, where it is one.

The goal is to show that the quasi-local algebra defined above is simple. To this end, we first need to introduce the quotient algebra $\mathfrak{A}/\mathcal{J}$, where $\mathcal{J}$ is a closed two-sided ideal. As a vector space, it is the quotient of $\mathfrak{A}$ by the vector space $\mathcal{J}$, hence it consists of equivalence classes $[A]$, $A \in \mathfrak{A}$ with $[A] = [B]$ if and only if $A = B + J$ for some $J \in \mathcal{J}$. This can be turned into a Banach space by defining the norm as

$$\|[A]\| = \inf_{J \in \mathcal{J}} \|A + J\|.$$ 

That this really defines a Banach space requires some work. The details can be found in, e.g. [31]. The space can be made into a $*$-algebra by setting

$$[A] \cdot [B] = [AB], \quad [A]^* = [A^*].$$

Because $\mathcal{J}$ is a closed two-sided ideal (in particular this implies that $\mathcal{J}^* = \mathcal{J}$), this is well defined. It turns out that $\mathfrak{A}/\mathcal{J}$ is a $C^*$-algebra. That is, the $C^*$-identity holds for this $*$-operation and the norm defined above.

There is a natural $*$-homomorphism $\varphi$ from $\mathfrak{A}$ into $\mathfrak{A}/\mathcal{J}$: define $\varphi(A) = [A]$. Since it is a $*$-homomorphism between $C^*$-algebras it is automatically continuous. Note that if $\mathcal{J}$ is a closed two-sided ideal, we can always find a representation of $\mathfrak{A}$ which has this ideal as its kernel. To this end, take any faithful representation $\pi$ of $\mathfrak{A}/\mathcal{J}$, which always exists by Theorem [24.11] Then $\pi \circ \varphi$ is a representation of $\mathfrak{A}$ with kernel $\mathcal{J}$.

**Proposition 3.2.4.** The quasi-local algebra $\mathfrak{A}(\mathbb{Z}^d)$ for a quantum spin system is simple.
Proof. Suppose that $\mathfrak{I}$ is an ideal of $\mathfrak{A}(\mathbb{Z}^d)$. If $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$, it is easy to check that $\mathfrak{I} \cap \mathfrak{A}(\Lambda)$ is an ideal of $\mathfrak{A}(\Lambda)$. Hence the intersection is either $\{0\}$ or $\mathfrak{A}(\Lambda)$, by Exercise 3.2.3 and because $\bigotimes_{n=1}^d M_d(\mathbb{C}) \cong M_d^d(\mathbb{C})$. In the latter case, it contains the identity and hence $\mathfrak{I} = \mathfrak{A}(\mathbb{Z}^d)$ and we are done. We may therefore assume that $\mathfrak{I} \cap \mathfrak{A}(\Lambda) = \{0\}$ for all $\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$.

Let $\pi \circ \varphi$ be a representation of $\mathfrak{A}$ with kernel $\mathfrak{I}$ as constructed above. Note that $\mathfrak{A}(\Lambda) \cap \mathfrak{I} = \{0\}$ by the argument above, for all $\Lambda \in \mathcal{P}_f(\Lambda)$. This implies that $\pi$ restricted to $\mathfrak{A}_{\text{loc}}$ is injective. Hence $\|\pi(\varphi(A))\| = \|A\|$ for all $A \in \mathfrak{A}_{\text{loc}}$. In particular, $\|\varphi(A)\| = \|A\|$. Hence, by definition of the norm on $\mathfrak{A}/\mathfrak{I}$, for each non-zero $A \in \mathfrak{A}_{\text{loc}}$ it follows that

$$\|A - J\| \geq \|A\|$$

for all $J \in \mathfrak{I}$. But if $J \in \mathfrak{I}$, there is a sequence $A_n \in \mathfrak{A}_{\text{loc}}$ such that $A_n \to J$ in norm. Since $\|A_n - J\| \geq \|A_n\|$ as above and the left-hand side converges to zero, it follows that $\|A_n\|$ must converge to zero. Hence it follows that $J$ must be zero.

Note that the kernel of a representation of a $C^*$-algebra is always a closed. We thus have the following very useful corollary, where the isometric property follows from a general result on injective $*$-homomorphisms between $C^*$-algebras.

Corollary 3.2.5. Every non-zero representation of the quasi-local algebra is faithful (and hence isometric).

This says that for any representation $\pi$ of the quasi-local algebra $\mathfrak{A}$ we take, we can identify $\mathfrak{A}$ with its image $\pi(\mathfrak{A})$. In this case it is not necessary to do the whole Gel’fand-Naimark construction as in the proof of Theorem 3.2.5: it suffices to take any state of $\mathfrak{A}$ and do the GNS representation, to identify $\mathfrak{A}$ with a closed subalgebra of $B(H)$ for some Hilbert space $H$ as the image of $\mathfrak{A}$ under this representation.

Inequivalent representations

As we have seen, there are many inequivalent representations of the observable algebra of quantum spin systems. Since representations are often obtained by applying the GNS construction to some state on an algebra, it is useful to have an criterion for unitary equivalence of the corresponding representations only in terms of the states. For states on a quasi-local algebra there is such an explicit criterion.

Proposition 3.2.6. Let $\mathfrak{A} := \mathfrak{A}(\Gamma)$ be the quasi-local observable algebra of some spin system and suppose that $\omega_1$ and $\omega_2$ are pure states on $\mathfrak{A}$. Then the following criteria are equivalent:

1. The corresponding GNS representations $\pi_1$ and $\pi_2$ are equivalent.

2. For each $\varepsilon > 0$, there is a $\Lambda_\varepsilon \in \mathcal{P}_f(\Gamma)$ such that

$$|\omega_1(A) - \omega_2(A)| < \varepsilon \|A\|,$$

for all $A \in \mathfrak{A}(\Lambda)$ with $\Lambda \in \mathcal{P}_f(\Lambda_\varepsilon)$.
Here $\mathcal{N}_\varepsilon$ is the complement of $\Lambda_\varepsilon$ in $\Gamma$.

The second condition says that for every $\varepsilon > 0$, the states $\omega_1$ and $\omega_2$ are close to each other, as long as we restrict to observables outside some (fixed!) region depending on $\varepsilon$. In other words, if we restrict to observables “far away”, the states look the same. We will omit the proof. The statement is a special case of [7, Cor. 2.6.11].

This result can be used to give an alternative proof of the inequivalence of the representations $\pi_+^\varepsilon$ and $\pi_-^\varepsilon$ defined at the beginning of this chapter. To see this, first note that the generators $\sigma^\varepsilon_n$ satisfy the relations of the Pauli matrices. Hence the operators acting on a single site $n$ generate the observable algebra $\mathfrak{A}(\{n\}) = M_2(\mathbb{C})$. We can then define the corresponding quasi-local algebra $\mathfrak{A}(\mathbb{Z})$ as above. We can then define states $\omega_+$ and $\omega_-$ on $\mathfrak{A}$ and show that the corresponding GNS representations are $\pi_+^\varepsilon$ and $\pi_-^\varepsilon$. To define these states, it is enough to first define them for strictly local observables $A \in \mathfrak{A}(\Lambda)$, and show that this can be done in a consistent way. That is, if $A \in \mathfrak{A}(\Lambda)$ with $\Lambda$ finite, then $A$ can be identified with a matrix in $A \in M_2(\mathbb{C}) \otimes \cdots \otimes \mathbb{C}$. We can then define a state on $\mathfrak{A}(\Lambda)$ by setting $\omega_\Lambda(A) = \text{Tr}(\rho_\Lambda A)$ for some density operator $A$. The consistency conditions means that if $\Lambda \subseteq \Lambda', \text{ then we should have } \text{Tr}(\rho_{\Lambda'} A) = \text{Tr}(\rho_\Lambda A \otimes I \otimes \cdots \otimes I)$. That is, if we include $\mathfrak{A}(\Lambda)$ into a bigger algebra, the value of the state should not change. Since $\mathfrak{A}_{\text{loc}}$ is dense in $\mathfrak{A}$, we can extend the state defined in this way on the local algebra to a state $\omega_+$ on $\mathfrak{A}$. Since we already know that these representations are irreducible, it follows that the states must be pure. We can then use the proposition to prove inequivalence of the representations.

Exercise 3.2.7. Find suitable states $\omega_+$ and $\omega_-$ and use the proposition to verify that the representations $\pi_+^\varepsilon$ and $\pi_-^\varepsilon$ are indeed not equivalent.

Translation symmetry

In many cases there is a natural translation symmetry acting on the system. The example that will be most relevant for us is when $\Gamma = \mathbb{Z}^d$ for some positive integer $d$. Note that $\mathbb{Z}^d$ acts on itself by addition (or translation). If $\Lambda \subseteq \Gamma$ we write $\Lambda + x$ for the same subset, translated by $x \in \mathbb{Z}^d$. This induces an action on the quasi-local algebra $\mathfrak{A}(\mathbb{Z}^d)$. Assume first for simplicity that $\mathfrak{A}(x) \in \mathfrak{A}(\{x\}) \cong M_2(\mathbb{C})$ for some $x \in \mathbb{Z}^d$. Hence the support of $A(x)$ is only the site $x$. Since by construction the observable algebra is the same at any site, there is a corresponding $A(x + y) \in \mathfrak{A}(\{x + y\})$ for every $y \in \mathbb{Z}^d$. We define $\tau_y(A(x)) = A(x + y)$. This construction works for any local operator $A \in \mathfrak{A}(\Lambda)$. Hence for every $y \in \mathbb{Z}^d$ we can define a map $\tau_y : \mathfrak{A}_{\text{loc}} \rightarrow \mathfrak{A}_{\text{loc}}$. Clearly $\tau_y$ is an automorphism of the local algebra, hence $\|\tau_y(A)\| = \|A\|$ for all $A \in \mathfrak{A}_{\text{loc}}$. We can then extend $\tau_y$ to an automorphism of $\mathfrak{A}(\mathbb{Z}^d)$ by continuity and since $\mathfrak{A}_{\text{loc}}$ is dense in $\mathfrak{A}(\mathbb{Z}^d)$. The automorphism has the property that $\tau_y(\mathfrak{A}(\Lambda)) = \mathfrak{A}(\Lambda + y)$ for all $\Lambda \subseteq \mathfrak{P}(\mathbb{Z}^d)$.

This construction can be done for any $y \in \mathbb{Z}^d$. Moreover, it is easy to check that $\tau_{x+y}(A) = \tau_x(\tau_y(A))$ and that $\tau_0(A) = A$ for all $A \in \mathfrak{A}$. Hence

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6 The statement there is more general, but for pure states, quasi-equivalence is the same as unitary equivalence. Moreover, states of the quasi-local algebra of quantum spin systems are always locally normal (a term we will not define here), because the observable algebras of finite regions are finite dimensional in that case.
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$$\tau : \mathbb{Z}^d \to \text{Aut}(\mathfrak{A}(\mathbb{Z}^d))$$ is a group of automorphisms. A translation invariant state on $$\mathfrak{A}(\mathbb{Z}^d)$$ can then be defined as a state $$\omega$$ such that $$\omega(\tau_x(A)) = \omega(A)$$ for all $$x \in \mathbb{Z}^d$$ and $$A \in \mathfrak{A}$$. As an example we can consider the infinite spin chain discussed at the beginning of this chapter. If we choose all spins in the up direction, this gives a vector in $$\mathcal{H}^+$$ and hence a state on the observable algebra. Clearly this state is translation invariant. The uniqueness of the GNS construction then implies that there is a unitary representation $$x \mapsto U(x)$$ implementing the translations in the GNS representations. That is, we have $$\pi(\alpha_x(A)) = U(x) \pi(A) U(x)^*$$ for all $$x \in \mathbb{Z}^d$$ and $$A \in \mathfrak{A}$$.

Another useful property is that if we move any local operator away far enough, it will commute with any other local operator. In fact we can say a bit more:

**Theorem 3.2.8.** Consider the quasi-local algebra $$\mathfrak{A}(\mathbb{Z}^d)$$ with $$x \mapsto \tau_x$$ the natural action of the translation group. Then for each $$A, B \in \mathfrak{A}(\mathbb{Z}^d)$$ we have

$$\lim_{|x| \to \infty} \|\tau_x(A), B\| = 0. \quad (3.2.1)$$

Here $$|x| \to \infty$$ means that $$x$$ goes to infinity in any direction.

**Proof.** We first suppose that $$A$$ and $$B$$ are strictly local. In that case, there are $$\Lambda_A, \Lambda_B \in \mathcal{P}_I(\mathbb{Z}^d)$$ such that $$A \in \mathfrak{A}(\Lambda_A)$$ and $$B \in \mathfrak{A}(\Lambda_B)$$. Note that $$\tau_x(A) \in \mathfrak{A}(\Lambda_A + x)$$. Since both $$\Lambda_A$$ and $$\Lambda_B$$ are finite, there is some integer $$N$$ such that $$(\Lambda_A + x) \cap \Lambda_B = \emptyset$$ for all $$|x| > N$$. Hence by locality we have $$[\tau_x(A), B] = 0$$ for all $$|x| > N$$ and hence equation (3.2.1) holds.

As for the general case, let $$A, B \in \mathfrak{A}$$ and let $$\varepsilon > 0$$. Since the local algebra is dense in $$\mathfrak{A}$$, there are $$A_x$$ and $$B_x$$ that are strictly local and such that $$\|A - A_x\| < \varepsilon$$ (and similarly for $$B$$). By the argument above, $$[\tau_x(A_x), B_x] = 0$$ for $$x$$ large enough. In addition we have that $$\|\tau_x(A)\| = \|A\|$$ since $$\tau_x$$ is an automorphism for each $$x$$. Hence we have

$$\lim_{|x| \to \infty} \|\tau_x(A), B\| = \lim_{|x| \to \infty} \|[\tau_x(A - A_x), B] + [\tau_x(A_x), B]\|
\leq 2\varepsilon\|B\| + \lim_{|x| \to \infty} \|[\tau_x(A_x), B - B_x] + [\tau_x(A_x), B_x]\|
\leq 2\varepsilon(\|A_x\| + \|B\|).$$

To obtain the last line we used that $$[\tau_x(A_x), B_x] = 0$$ for $$x$$ sufficiently large. The claim follows by noting that $$\|A_x\| \leq \|A\| + \varepsilon$$ by the reverse triangle inequality. \(\Box\)

The property in the proof is called asymptotic abelianness. The first applications of asymptotic abelianness were studied in [33, 58]. One useful consequence is that the set of translation invariant states is actually a simplex. This means that any translational invariant states can be obtained uniquely as a combination of extremal translational invariant states.

### 3.3 Dynamics

So far the discussion has been completely general, in that we have only treated kinematical aspects so far. For example, the quasi-local algebra essentially
depends only on the number of degrees of freedom at each site. What differentiates different systems of, say, spin-1/2 particles are their dynamics. In other words, how the observables evolve over time. One way to specify the dynamics is to specify Hamiltonian.

Recall that a Hamiltonian is a (possibly unbounded) self-adjoint operator acting on some Hilbert space $\mathcal{H}$. If $\psi \in \mathcal{H}$ is a wave function, its time evolution is governed by the Schrödinger equation (in units of $\hbar = 1$)

$$i\partial_t \psi = H\psi.$$  

The solution to this equation is given by

$$\psi(t) = \exp(-itH)\psi,$$

This converges (under suitable conditions on $H$) when acting on so-called analytic vectors $\psi$, which span a dense set.

This allows us to define a unitary $U(t) = \exp(itH)$ in $\mathcal{B}(\mathcal{H})$. Hence for the time evolved vector we have $\psi(t) = U(t)^*\psi$. From the power series expansion and the self-adjointness of $H$ it follows that $U(t)^* = U(-t)$, $U(0) = I$ and $U(t+s) = U(t)U(s)$. The latter condition says that evolving over a time $t + s$ is the same as first evolving for a time $s$ and then for a time $t$. In short, we have a one-parameter group $t \mapsto U(t)$ of unitaries. Moreover, one can show that this group is strongly continuous, in the sense that

$$\lim_{t \to 0} \|U(t)\psi - \psi\| = 0$$

for any vector $\psi \in \mathcal{H}$. Hence the Hamiltonian induces a group of unitaries that implement the time evolution on the vectors in the Hilbert space. A famous result by Stone shows that in fact any such evolution can be obtained from a “Hamiltonian”. The result can be stated as follows:

**Theorem 3.3.1 (Stone).** Let $t \mapsto U(t)$ be a strongly continuous one-parameter group of unitaries acting on a Hilbert space $\mathcal{H}$. Then there is a densely defined self-adjoint operator $H$ acting on $\mathcal{H}$ such that $U(t) = \exp(itH)$ for all $t \in \mathbb{R}$.

We will not give a proof here. It can be found, for example, in [31, Thm. 5.6.36].

Instead of looking at how the states evolve, we can also look at how the observables change (the Heisenberg picture). Let $t \mapsto U(t)$ be a one-parameter group of unitaries as above. Let $\mathfrak{A} = \mathcal{B}(\mathcal{H})$. Then we obtain a one-parameter group of automorphism $t \mapsto \alpha_t \in \text{Aut}(\mathfrak{A})$ by setting

$$\alpha_t(A) = U(t)AU(t)^*.$$  

Both viewpoints give the same physics. Indeed, for the expectation values of an observable $A$ we have at a time $t$

$$\langle A \rangle_t = \langle \psi_t, A\psi_t \rangle = \langle U(t)^*\psi, AU(t)^*\psi \rangle = \langle \psi, \alpha_t(A)\psi \rangle.$$
Hence the Schrödinger and the Heisenberg picture give the same results.

Ultimately we want to find a description of dynamics in the abstract setting of $C^*$-algebras. If $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$ this will not lead to much new. Note in particular that $U(t) \in \mathfrak{A}$. Automorphisms of this form are called inner. An automorphism of the algebra of bounded operators on a Hilbert space is always inner.

**Exercise 3.3.2.** Let $\mathcal{H}$ be a Hilbert space and consider the $C^*$-algebra $\mathfrak{B}(\mathcal{H})$. Suppose that $\psi \in \mathcal{H}$ is a unit vector. We define the projection on $\psi$, which we write as $|\psi\rangle \langle \psi|$, by

$$|\psi\rangle \langle \psi| \xi = \langle \psi | \xi \rangle \psi,$$

for $\psi \in \mathcal{H}$. Note that any one-dimensional projection is of this form.

1. Show that a projection $P \in \mathfrak{B}(\mathcal{H})$ has one-dimensional range if and only if $PAP = c_A P$ for any $A \in \mathfrak{B}(\mathcal{H})$ and $c_A \in \mathbb{C}$ may depend on $A$.

2. Let $\alpha$ be an automorphism of $\mathfrak{A}$. Show that $\alpha(|\psi\rangle \langle \psi|)$ is a one-dimensional projection.

3. Define a map $U : \mathfrak{A} \psi \rightarrow \mathcal{H}$ by $UA\psi = \alpha(A)\psi'$, where $\psi'$ is such that $\alpha(|\psi\rangle \langle \psi|) = |\psi'\rangle \langle \psi'|$. Show that $U$ is well-defined and extends to a unitary in $\mathfrak{B}(\mathcal{H})$.

4. Show that $\alpha(A) = UAU^*$ for all $A \in \mathfrak{B}(\mathcal{H})$.

Hence Stone’s theorem and this result tell us that we can always find a Hamiltonian description of a time evolution. Conversely, defining a Hamiltonian gives the time evolution on the observables as above.

If $\mathfrak{A}$ is not of the form $\mathfrak{B}(\mathcal{H})$, the matter is more complicated. For one, since we do not have a natural Hilbert space any more, it is not clear how to define the Hamiltonian at all. Of course, one can take the GNS representation of a state to obtain a Hilbert space, but the result will depend on the state. In general the Hamiltonian is not bounded, hence it cannot be an element of the $C^*$-algebra. Moreover, it is not true any more that every automorphism of $\mathfrak{A}$ is inner. That is, there may not exist unitaries that are contained in $\mathfrak{A}$ such that $\alpha(A) = UAU^*$ for all $A \in \mathfrak{A}$. Therefore we take the one-parameter group $t \mapsto \alpha_t$ as one of the fundamental objects. In the next sections we will see how we can describe the dynamics in a purely $C^*$-algebraic framework, and how this relates to the picture above.

**Derivations**

The Hamiltonian can be interpreted as the generator of time translations. It turns out that in the context of $C^*$-algebras, generators of one-parameter groups (or even semi-groups) of automorphisms can be conveniently expressed by derivations.

**Definition 3.3.3.** A (symmetric) derivation of a $C^*$-algebra $\mathfrak{A}$ is a linear map from a $*$-subalgebra $D(\delta)$ of $\mathfrak{A}$ into $\mathfrak{A}$ such that

1. $\delta(A^*) = \delta(A)^*$ for $A \in D(\delta)$,
2. $\delta(AB) = \delta(A)B + A\delta(B)$ for $A, B \in D(\delta)$. 
The algebra $D(\delta)$ is called the domain of $\delta$.

For the systems that we are interested in, one usually has that the domain of the derivation are the strictly local operators of the quasi-local algebra, i.e. $D(\delta) = \mathfrak{A}_{\text{loc}}$. Note that $\mathfrak{A}_{\text{loc}}$ is dense in $\mathfrak{A}$. This is also necessary, since in the end we want to be able to define a map that acts on all of $\mathfrak{A}$.

Derivations can be obtained from one-parameter groups of automorphisms of a $C^*$-algebra $\mathfrak{A}$. More precisely, let $t \mapsto \alpha_t$ be a one-parameter group of automorphisms of $\mathfrak{A}$. Moreover, assume that the action is strongly continuous, then the map $t \mapsto \alpha_t(A)$ is continuous in the norm topology for each $A \in \mathfrak{A}$. Define $\delta(A)$ by

$$\delta(A) = \lim_{t \to 0} \left( \frac{\alpha_t(A) - A}{t} \right)$$

(3.3.1)

if this limit exists. The set of all $A \in \mathfrak{A}$ for which this limit exists is denoted by $D(\delta)$. Note that the limit is just the derivative of $\alpha_t(A)$ at $t = 0$.

**Lemma 3.3.4.** The $\delta(A)$ defined above is a symmetric derivation. Moreover, we have that $\alpha_t(D(\delta)) \subset D(\delta)$.

**Proof.** We first show that $D(\delta)$ is a $*$-subalgebra. It is clear that it is a vector space, since the expression (3.3.1) is linear in $A$. Note that it follows that $\delta$ is a linear map. Suppose that $A \in \delta(A)$. Since $\alpha_t(A^*) = \alpha_t(A)^*$ and $\|A^*\| = \|A\|$ it follows that if $A \in D(\delta)$, then $A^* \in D(\delta)$ and $\delta(A^*) = \delta(A)^*$. To show the “Leibniz rule”, suppose that $A, B \in D(\delta)$. Note that $\alpha_t(AB) = \alpha_t(A)\alpha_t(B)$, hence we can use the product rule for derivatives. This leads to $\delta(AB) = A\delta(B) + \delta(A)B$. It follows that $D(\delta)$ is a $*$-subalgebra of $\mathfrak{A}$ and that $\delta$ is a derivation.

To show the last property, suppose that $A \in D(\delta)$ and let $s \in \mathbb{R}$. Then

$$0 = \lim_{t \to 0} \left\| \frac{\alpha_t(A) - A}{t} - \delta(A) \right\|$$

$$= \lim_{t \to 0} \left\| \alpha_s \left( \frac{\alpha_t(A) - A}{t} \right) - \alpha_s(\delta(A)) \right\|$$

$$= \lim_{t \to 0} \left\| \left( \frac{\alpha_t(sA) - sA}{t} \right) - \alpha_s(\delta(A)) \right\|.$$ 

Hence $\alpha_t(D(\delta)) \subset D(\delta)$ for any $t$. 

The next exercise shows how derivations are related to the Hamiltonian description discussed earlier.

**Exercise 3.3.5.** Suppose that $\mathfrak{A} = \mathfrak{B}(H)$ and let $\alpha_t$ be a one-parameter group of automorphisms given by $\alpha_t(A) = e^{itH}Ae^{-itH}$ for some self-adjoint $H$. Show that the corresponding derivation is

$$\delta(A) = i[H, A].$$

Verify explicitly that it is indeed a symmetric derivation. Note: If $H$ is unbounded, $\delta(A)$ will not be defined for each $A$ (since the limit will not exist). In this exercise you can ignore these convergence issues.
If we are given a derivation $\delta$, the question is if (and how) it generates a one-parameter group of automorphisms. For $A \in D(\delta)$, we can set

$$\alpha_t(A) := \exp(t\delta)(A) = A + t\delta(A) + \frac{t^2\delta^2(A)}{2} + \ldots,$$

(3.3.2)

where we set $\delta^0(A) = A$. Note that this definition makes sense only if we can interpret $\delta^2(A)$. That is, $\delta(A)$ should be in the domain of $D(\delta)$ again.

Even if this is the case, it is not at all clear if the expression even converges, and if it allows us to define an automorphism on all of $\mathfrak{A}$. Moreover, one can ask oneself in which topology this should converge, and what the continuity properties of $t \mapsto \alpha_t$ are. Such questions are studied in great detail in [7, Ch. 3] and [8, Ch. 6]. Here we will content ourselves with answering these questions for a relatively simple (but important) class of derivations, associated to spin systems with local interactions.

First, however, consider the case of Exercise 3.3.5, where $\delta(A) = i[H, A]$. We will argue, at least heuristically, why equation (3.3.2) is the right definition. To this end, we first collect the first few terms of the expansion of the exponential, to obtain (with $A \in D(\delta)$)

$$\exp(t\delta)(A) = A + it[H, A] - \frac{t^2}{2}[H,[H, A]] + \mathcal{O}(t^3).$$

On the other hand, if we expand $e^{itH} Ae^{-itH}$ up to second order in $t$, we get

$$e^{itH} Ae^{-itH} = (I + itH - \frac{t^2}{2}H^2 + \ldots)A(I - itH - \frac{t^2}{2}H^2 + \ldots) + \mathcal{O}(t^3)$$

$$= A - itAH + itHA - \frac{t^2}{2}H^2A + t^2HAH - \frac{t^2}{2}A^2H^2 + \mathcal{O}(t^3)$$

$$= A + it[H, A] - \frac{t^2}{2}[H,[H, A]] + \mathcal{O}(t^3),$$

hence we recover the automorphism from which the derivation $\delta$ was obtained.

**Finite range interactions**

The question one has to answer is which derivation is the relevant one when studying a particular system. In case of a lattice system, $\delta(A) = i[H, A]$ for some Hamiltonian $H$ is usually not really adequate. If there are infinitely many sites, $H$ will be unbounded in general and the commutator does not make any sense a priori. For example, consider a 1D infinite spin chain. Suppose that at each site the energy is given by the value of the spin in the $z$ direction. Hence in this case, $H = \sum_{n=-\infty}^{\infty} \sigma_n^z$. This is clearly not an element of the quasi-local algebra $\mathfrak{A}$. Another approach might be to first define a Hilbert space and a representation of $\mathfrak{A}$ on this Hilbert space, and then define $H$ on this Hilbert space. But this defeats the purpose of the algebraic approach.

The way out is that we can look at the interactions in a finite part of the system. From the model one wants to study it is usually clear what the interactions should be within this finite region. If the size of the region is taken to be bigger and bigger, one could hope that this converges in a suitable sense. For example, consider again the spin chain example above. Let $A$ be a local observable. Then there is some integer $N$ such that $[\sigma_n^z, A] = 0$ for all $n \geq N$, \}
by locality. Hence the following definition makes sense for all local observables $A \in \mathfrak{A}_{\text{loc}}$:

$$\delta(A) = \lim_{n \to \infty} i \sum_{k=-n}^{n} \sigma^z_k, A.$$ 

This defines a derivation with domain $D(\delta) = \mathfrak{A}_{\text{loc}}$. To consider more general examples we first need the following definition.

**Definition 3.3.6.** Consider a quantum spin system defined on a set $\Gamma$ of sites, with quasi-local algebra $\mathfrak{A}$. An interaction $\Phi$ is a map $\Phi : \mathcal{P}_f(\Gamma) \to \mathfrak{A}$ such that for each $\Lambda$ we have that $\Phi(\Lambda) \in \mathfrak{A}(\Lambda)$ and $\Phi(\Lambda)$ is self-adjoint.

As the name suggests, an interaction describes what the interactions are between the spin systems sitting at the sites of some set $\Lambda$. If $\Phi$ is an interaction, we define the local Hamiltonians by

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X),$$

where the sum is over all subsets of $\Lambda$. Note that $H_\Lambda = H^*_\Lambda$ and that $H_\Lambda \in \mathfrak{A}(\Lambda)$.

To define what it means for an interaction to be short range, we have to assume some additional structure on the set $\Gamma$ of sites. In particular, we assume that there is a metric $d(x, y)$ on $\Gamma$ such that $d(x, y) \geq 1$ if $x \neq y$ and such that for any $M > 0$ and $x \in \Gamma$, the ball of radius $M$ around $x$, that is $B_M(x) = \{ y \in \Gamma : d(x, y) \leq M \}$, contains at most $N_M$ elements, where $N_M$ is a constant independent of $x$. This condition ensures that each point $x$ has only finitely many neighbours and the maximum number of neighbours is bounded from above. The example we will usually consider is where $\Gamma = \mathbb{Z}^d$ and the metric is the taxicab or Manhattan metric, defined as

$$d(x, y) = \sum_{i=1}^{d} |x_i - y_i|.$$ 

It is easy to see that this metric fulfils all the conditions. If $\Lambda \subset \Gamma$ is any subset, we define the diameter of $\Lambda$ as $\text{diam}(\Lambda) = \sup_{x, y \in \Gamma} d(x, y)$. An interaction $\Phi$ is then set to be of finite range if there is some constant $c_\Phi > 0$ such that $\Phi(\Lambda) = 0$ whenever $\text{diam}(\Lambda) > c_\Phi$. For finite range interactions there are, as the name implies, no interactions between sites that are at least a distance $d_\Phi$ apart. We will also consider only bounded interactions, for which $\|\Phi\| := \sup_{x \in \Gamma} \sum_{\Lambda \in \mathcal{P}_f(\Gamma), x \in \Lambda} \|\Phi(\Lambda)\| < \infty$.

**Remark 3.3.7.** In the remainder we will mainly deal with bounded finite range interactions. These conditions can be relaxed in many cases to interactions whose strength decays sufficiently fast as the distance between two sites increases. Depending on the precise conditions one chooses, however, this can be done only at the expense of weaker statements on the convergence of the dynamics. For example, the dynamics may only converge for certain states. We refer to Chapter 6 of Bratteli and Robinson for more details [8].
Finally, suppose that we also have a translation symmetry of $\Gamma$. This induces a group of automorphism $\tau_x$ on $\mathfrak{A}$, as discussed above. An interaction is set to be translation invariant if $\Phi(\Lambda + x) = \tau_x(\Phi(\Lambda))$ for all $x \in \mathbb{Z}^d$ and $\Lambda \in \mathcal{P}_f(\Gamma)$, where $\mathbb{Z}^d$ is the translation group. Note that a translation invariant short range interaction is automatically bounded.

Example 3.3.8 (The quantum Heisenberg model). Consider $\Gamma = \mathbb{Z}$ with at each site of $\Gamma$ a spin-1/2 system. Define an interaction by $\Phi(n) = -h \sigma_z^n$, with $h$ some real constant, and

\[
\Phi(\{n, n+1\}) = -\frac{1}{2} (J_x \sigma_x^n \sigma_x^{n+1} + J_y \sigma_y^n \sigma_y^{n+1} + J_z \sigma_z^n \sigma_z^{n+1})
\]

for real constants $J_x, J_y$ and $J_z$. In all other cases we set $\Phi(\Lambda) = 0$. Note that $\Phi$ is a finite range interaction. Moreover, it is clear that $\Phi$ is translation invariant. For $\Lambda \subset \Gamma$ of the form $[n, m+1] \cap \mathbb{Z}$ for integers $n,m$, the corresponding local Hamiltonians are

\[
H_\Lambda = -\frac{1}{2} \sum_{k=n}^{m} (J_x \sigma_x^k \sigma_x^{k+1} + J_y \sigma_y^k \sigma_y^{k+1} + J_z \sigma_z^k \sigma_z^{k+1}) - h \sum_{k=n}^{m+1} h \sigma_z^k.
\]

The last term can be interpreted as an external field, while the terms with $J_i$ describe a coupling between nearest neighbour spins. If $J_x = J_y \neq J_z$, this model is called the Heisenberg XXZ model. An interesting fact is that the behaviour of the system strongly depends on the couplings $J_i$.

Example 3.3.9. Consider again $\Gamma = \mathbb{Z}$. A generalisation of the previous example can be obtained as follows. Choose functions $j_i : \mathbb{Z} \to \mathbb{R}$. Define an interaction $\Phi$ by

\[
\Phi(\{0\}) = h \sigma_z^0,
\]

\[
\Phi(\{0, n\}) = \sum_{k=1}^{\lambda} j_k(n) \sigma_0^k \sigma_n^k,
\]

and by requiring that the interaction is translation invariant (i.e., $\Phi(\Lambda + x) = \tau_x(\Phi(\Lambda))$ for all $\Lambda \in \mathcal{P}_f(\Gamma)$). This describes a model of spins in an external magnetic field, together with two-body interactions whose strength is determined by the functions $j_i$. Note that if the support of $j_i$ is not bounded, there are interactions between pairs of spins that are arbitrarily far away. If this is the case, the interaction is clearly not of finite range.

Now let $\Phi$ be a bounded finite range interaction of range $c_\Phi$. As before, we can define the local Hamiltonians $H_{\Lambda}$. Now suppose that $\Lambda \in \mathfrak{A}_{loc}$ is localised in some set $\Lambda_\Lambda$. Because the interaction is of finite range, there are only finitely many finite subsets $\Lambda$ of $\Gamma$ such that $[\Phi(\Lambda), A] \neq 0$. For example, consider the set of all points of $\Gamma$ with distance at most $c_\Phi$ from $\Lambda_\Lambda$. Then any finite subset $\Lambda$ of $\Gamma$ such that $\Phi(\Lambda)$ does not commute with $A$ must necessarily be contained in this set, by the finite range assumption. Hence if we consider $\sum_{\Lambda \in \mathcal{P}_f(\Gamma)} [\Phi(\Lambda), A]$, only a finite number of terms do not vanish and this sum converges to an element in $\mathfrak{A}_{loc}$. It follows that

\[
\delta(A) = \lim_{\Lambda \to \infty} i[H_{\Lambda}, A]
\]
defines a symmetric derivation with domain \( D(\delta) = \mathfrak{A}_{\text{loc}} \). Here with \( \Lambda \to \infty \) we mean the following. Consider a sequence \( \Lambda_1 \subset \Lambda_2 \subset \cdots \) of finite subsets of \( \Gamma \) such that \( \bigcup_n \Lambda_n = \Gamma \). Then with \( \lim_{\Lambda \to \infty} \) we mean that the limit \( \lim_{n \to \infty} \) converges independently of the sequence \( \Lambda_n \) that was chosen.

The goal is now to show that \( \exp(t\delta)(A) \) converges for \( A \in D(\delta) \) and \( t \) sufficiently small. Note that \( \delta(A) \in D(\delta) \) for all \( A \in D(\delta) \), hence expressions of the form \( \delta^n(A) \) are well-defined. The problem, however, is that \( \text{supp} \delta(A) \) may be bigger than the support of \( A \). A close look at the definition of \( \delta \) and the local Hamiltonians show \( H_\Lambda \) that this means that for \( \delta^n(A) \), in principle more and more interaction terms can contribute. The boundedness of \( \Phi \) will ensure that the growth of the support will not be too big, allowing us to make estimates on the norm \( \|\delta^n(A)\| \), allowing us to conclude that the exponential \( \exp(t\delta) \) converges. The ideas behind the proof date back to Robinson [57].

**Lemma 3.3.10.** Let \( \Phi \) be a bounded interaction of finite range and write \( \delta \) for the corresponding derivation. Then the algebra \( D(\delta) = \mathfrak{A}_{\text{loc}} \) is analytic for \( \delta \), in the sense that for each \( A \in D(\delta) \), \( \exp(t\delta)(A) \) is analytic as a function of \( t \), for \( t \) sufficiently small.

**Proof.** Let \( A \in \mathfrak{A}(\Lambda) \) for some finite set \( \Lambda \). By locality we have that \( [\Phi(X), A] = 0 \) whenever \( X \cap \Lambda = \emptyset \). Hence we can write

\[
\delta^n(A) = i^n \sum_{X_1 \cap S_1 \neq \emptyset} \cdots \sum_{X_n \cap S_n \neq \emptyset} [\Phi(X_n), \ldots [\Phi(X_1), A] \ldots ],
\]

where we set \( S_1 = \Lambda \) and \( S_k = S_{k-1} \cup X_{k-1} \) for \( k > 1 \). Since \( \Phi \) is a bounded interaction and the number of sites in a \( r \)-ball around any site is uniformly bounded, there is some constant \( N_\Phi \) such that \( \Phi(X) = 0 \) if \( |X| > N_\Phi \), hence it suffices to consider sets \( X_i \) such that \( |X_i| \leq N_\Phi \). This implies that \( |S_2| \leq N_\Phi - 1 + |\Lambda| \) and by induction,

\[
|S_n| \leq (n - 1)(N_\Phi - 1) + |\Lambda|.
\]

Using this observation we can make the estimate

\[
\|\delta^n(A)\| \leq 2^n \|A\| \prod_{k=1}^n ((k - 1)(N_\Phi - 1) + |\Lambda|) \|\Phi\|^n \tag{3.3.3}
\]

by successively estimating the norm of the commutators, and doing the sums.

Next we note the following inequality, which holds for all positive \( a \) and \( \lambda \): \( a^n \leq n!\lambda^{-n} \exp(a\lambda) \). This inequality can be easily verified using the expansion of the exponent. Using this inequality with \( a = (n - 1)(N_\Phi - 1) + |\Lambda| \), we obtain the estimate

\[
\|\delta^n(A)\| \leq \|A\|n!\lambda^{1-|\Lambda|} \exp \left( \frac{2\|\Phi\|e^{N_\Phi-1}}{\lambda} \right)^n. \tag{3.3.4}
\]

Using this estimate we see that \( \exp(t\delta)(A) \) converges for \( t \leq \frac{\lambda}{2\|\Phi\|e^{N_\Phi-1}} \), since \( \sum_n x^n \) converges for \( x < 1 \). Note that since the estimate is valid for all \( \lambda > 0 \), we actually get convergence for all \( t \).

\footnote{Alternatively we can consider \( \mathcal{P}_f(\Gamma) \) as the index set of a net \( \Lambda \to i[H_\Lambda, A] \), where the order relation is by inclusion. The convergence described in the text is the convergence of this net in the sense of topology.}
We define \( \alpha_t(A) = \exp(t\delta)(A) \) for \( A \in \mathfrak{A}_{\text{loc}} \). Since \( \delta \) is a derivation we have that \( \alpha_t(AB) = \alpha_t(A)\alpha_t(B) \) and that \( \alpha_t(A^*) = \alpha_t(A)^* \). However, we cannot just yet conclude that \( \alpha_t \) defines an automorphism of \( \mathfrak{A} \), since \( \alpha_t \) is only defined on a subset of \( \mathfrak{A} \). We will show that we can in fact extend \( \alpha_t \) to all of \( A \). Moreover, it can be approximated by local dynamics in the following sense. Define \( \delta \Lambda(A) = i[H, A] \) for \( \Lambda \in \mathcal{P}(\Gamma) \). Clearly, \( \lim_{\Lambda \to \infty} \delta \Lambda(A) = \delta(A) \). We will show that \( \alpha^\Lambda_t(A) := \exp(t\delta \Lambda)(A) \to \alpha_t(A) \).

**Theorem 3.3.11.** Let \( \Phi \) be as in the Lemma above. Then \( \alpha_t \) extends to a strongly continuous one-parameter group \( t \mapsto \alpha_t \) of automorphisms of \( \mathfrak{A} \) such that for each \( A \in \mathfrak{A} \) and \( t \in \mathbb{R} \) we have

\[
\lim_{\Lambda \to \infty} \| \alpha^\Lambda_t(A) - \alpha_t(A) \| = 0. \tag{3.3.5}
\]

**Proof.** As we saw before, \( \exp(t\delta \Lambda)(A) = \exp(itH_A)A\exp(-itH_A) \), hence \( \alpha^\Lambda_t \) is isometric. Now let \( A \in \mathfrak{A}(X) \) be a local operator. Since the interaction is strictly local, for each \( \Lambda \) that contains \( X \) there is an integer \( N(\Lambda) \) such that \( \delta^N \Lambda(A) = \delta^N(A) \) for all \( n \leq N(\Lambda) \). Moreover, \( N(\Lambda) \to \infty \) if \( \Lambda \to \infty \). Hence we see that

\[
\| \alpha^\Lambda_t(A) - \alpha_t(A) \| = \left\| \sum_{n=N(\Lambda)}^{\infty} \left( \frac{t^n}{n!} \delta^n \Lambda(A) - \frac{t^n}{n!} \delta^n(A) \right) \right\|.
\]

By using the estimate from equation (3.3.4) we see that this goes to zero as \( \Lambda \to \infty \) (and hence \( N(\Lambda) \to \infty \)). Hence for each strictly local observable \( A \) it follows that equation (3.3.5) holds. Since the strictly local observables form a dense subset of all the observables, and \( \alpha^\Lambda_t \) and hence \( \alpha_t \) is isometric, we can extend \( \alpha_t \) to an automorphism of all of \( \mathfrak{A} \). That is indeed an automorphism follows because \( \delta \) is a *-derivation and since \( \alpha_t(A)^{-1} = \alpha_{t^{-1}}(A) \).

It remains to show that \( t \mapsto \alpha_t \) is strongly continuous. Since \( \alpha_{t+s}(A) = \alpha_t(\alpha_s(A)) \) and \( \alpha_s \) is isometric for each \( s \), it is enough to show that \( \alpha_t(A) \to A \) as \( t \to 0 \). But this follows again by using the estimates for the norm of \( \delta^n(A) \). \( \square \)

After introducing ground states in this formalism, we will see how we can recover a Hamiltonian implementing the time translations from this formalism.

**Remark 3.3.12.** Essentially the same proof also works for two more general classes of interactions. First of all, one can consider the normed vector space of bounded finite range interactions as we have discussed here, and take the completion with respect to the norm to obtain a Banach space. Alternatively, one could consider all interactions \( \Phi \) that satisfy \( \sup_{x \in X} \sum_{\exists \lambda > 0} \| e^{\lambda X} \| \Phi(X) \| < \infty \) for some \( \lambda > 0 \). Note that bounded interactions of finite range satisfy this condition.

**Definition 3.3.13.** Let \( \mathfrak{A} \) be a \( C^* \)-algebra and \( t \mapsto \alpha_t \) a strongly continuous one-parameter group of automorphisms. Such a pair \( (\mathfrak{A}, \alpha) \) is called a \( C^* \)-dynamical system.
3.4 Ground states and thermal states

Once we have defined dynamics on a system we can try to talk about ground states and equilibrium states of such systems. Unfortunately, a rigorous treatment of most of these results would require a lot of advanced techniques from, e.g., functional analysis and complex analysis, and falls outside the scope of this course. For this reason we will omit or only sketch most of the proofs in this section, and restrict to introducing the main definitions and results. A full treatment can be found in [8].

A ground state of a Hamiltonian \( H \) acting on some Hilbert space is an eigenvector of the Hamiltonian with minimum eigenvalue. Since for physically relevant systems the energy is bounded from below, we can always normalise the Hamiltonian in such a way that \( H \Omega = 0 \) for a ground state \( \Omega \). Note that a ground state is stationary, since \( e^{-itH} \Omega = \Omega \). Hence the first condition on ground states of a \( \mathcal{C}^* \)-dynamical system \((\mathcal{A}, \alpha_t)\) is that it should be invariant with respect to each \( \alpha_t \). If \( \omega \) is a state on \( \mathcal{A} \) such that \( \omega \circ \alpha_t = \omega \) for all \( t \), it follows from Exercise [2.4.10] that in the corresponding GNS representation, \((\pi_\omega, \Omega, H_\omega)\), there is a strongly continuous one-parameter group \( t \mapsto U(t) \) of unitaries implementing the automorphisms \( \alpha_t \). By Stone’s theorem, Theorem 3.3.1, there is a self-adjoint densely defined operator \( H_\omega \) such that \( U(t) = \exp(itH_\omega) \). We require that this \( H_\omega \) is positive, denoted \( H_\omega \geq 0 \). This means that \( \langle \psi, H_\omega \psi \rangle \geq 0 \) for all \( \psi \) in the domain of \( H_\omega \). Hence a ground state will be defined as follows:

**Definition 3.4.1.** Let \( \mathcal{A} \) be a \( \mathcal{C}^* \)-algebra and \( t \mapsto \alpha_t \) a strongly continuous one-parameter group of automorphisms. A state \( \omega \) on \( \mathcal{A} \) is said to be an \( \alpha \)-ground state if \( \omega \) is invariant for each \( \alpha_t \) and if \( H_\omega \geq 0 \) for the Hamiltonian in the corresponding GNS representation.

This definition first requires to check that the ground state is invariant. Moreover, one has to construct the GNS representation and find a Hamiltonian implementing the dynamics in the representation. It is therefore desirable to have a purely \( \mathcal{C}^* \)-algebraic characterisation of ground states. A criterion on terms of the generator \( \delta \) of the time translations would be particularly useful, considering that we defined derivations that generate the dynamics in terms of interactions. Indeed, there are equivalent characterisations of ground states. For this result we first need the following lemma.

**Lemma 3.4.2.** Let \( \delta \) be a symmetric derivation generating a strongly continuous one-parameter group \( \alpha_t \) and suppose that \( \omega \) is a state on \( \mathcal{A} \) such that

\[
i \omega(A \delta(A)) \in \mathbb{R}
\]

for all \( A = A^* \in D(\delta) \). Then \( \omega(\delta(A)) = 0 \) for all \( A \in D(\delta) \) and \( \omega \circ \alpha_t = \omega \) for all \( t \).

**Proof.** Let \( A \in D(\delta) \) be self-adjoint. Then we have that

\[
i \omega(A \delta(A)) = i \overline{\omega(A \delta(A))} = -i \omega(\delta(A^*)A^*) = -i \omega(\delta(A)A).
\]

Using the Leibniz property of the derivation we obtain

\[
0 = \omega(A \delta(A)) + \omega(\delta(A)A) = \omega(\delta(A^2)).
\]
Now let $X \in D(\delta)$ be any positive operator, then we can take the square root $\sqrt{X}$ and use the formula above to conclude $\omega(\delta(X)) = 0, \text{[10]}$. Because any element of a $C^\ast$-algebra can be written as linear combination of at most four positive operators, it follows that $\omega(\delta(A)) = 0$ for all $A \in D(\delta)$.

Since $\alpha_t(A) = \exp(t\delta)(A)$ for all $A \in D(\delta)$, it follows from the expansion of the exponential and the result above that $\omega(\alpha_t(A)) = \omega(A)$ for all $A \in D(\delta)$. Since the domain is a dense subset of $\mathbb{R}$, the claim follows from continuity. \qed

Suppose that $\psi$ is a vector in $\mathcal{H}_\omega$. On the one hand, consider the equation

$$\frac{d}{dt} \bigg|_{t=0} e^{itH} \pi_\omega(A)e^{-itH} \psi = i[H_\omega, \pi_\omega(A)]\psi.$$ 

On the other hand, $e^{itH} \pi_\omega(A)e^{-itH} = \pi_\omega(\alpha_t(A))$, and hence if we take the derivative again at $t = 0$, it follows that the expression above is equal to $\pi_\omega(\delta(A))\psi$. A proper analysis requires a bit more care (in particular, since the expressions will in general not be well defined for any $\psi$ or $A$), but this can be done (see e.g. the text after [7] Prop. 3.2.28]). The result is that we have the following identity for all vectors $\psi \in D(H_\omega)$ and $A \in D(\delta)$:

$$\pi_\omega(\delta(A))\psi = i[H_\omega, \pi_\omega(A)]\psi. \quad (3.4.1)$$

In addition one can show that $\pi_\omega(D(\delta))\Omega \subset D(H_\omega)$. Another way to state the equation above is that the derivation $\delta$ is implemented as an inner derivation on the Hilbert space $\mathcal{H}_\omega$. With this observation we can now give an alternative definition of ground states.

**Theorem 3.4.3.** Let $\omega$ be a state on a $C^\ast$-algebra $\mathfrak{A}$ and suppose that $t \mapsto \alpha_t$ is a strongly continuous one-parameter group of automorphisms. Write $\delta$ for the corresponding generator. Then the following are equivalent:

1. $\omega$ is a ground state for $\alpha$,
2. $-i\omega(A^\ast \delta(A)) \geq 0$ for all $A \in \delta(\mathfrak{A})$.

*Proof.* $(\mathfrak{I} \Rightarrow \mathfrak{II})$ Let $A \in \delta(\mathfrak{A})$. Because $H_\omega$ is a positive operator it follows that $\langle \pi_\omega(\ast)\Omega, H_\omega \pi_\omega(A)\Omega \rangle \geq 0$. Hence

$$\langle \Omega \pi_\omega(A), H_\omega \pi_\omega(A)\Omega \rangle = -i\langle \pi_\omega(A)\Omega, \pi_\omega(\delta(A))\Omega \rangle = -i\omega(\ast \delta(A)) \geq 0,$$

since $\pi_\omega(\delta(A))\Omega = iH_\omega \pi_\omega(A)\Omega$ by the argument before the theorem (and since $H_\omega \Omega = 0$).

$(\mathfrak{II} \Rightarrow \mathfrak{I})$ From Lemma [3.4.2] it follows that $\omega$ is invariant with respect to $\alpha_t$, and hence we can find a Hamiltonian $H_\omega$ leaving the GNS vector $\Omega$ invariant. It remains to show that $H_\omega$ is positive. Let $\psi = \pi_\omega(A)$ for $A \in D(\delta)$. Then with the help of equation (3.4.1), we find

$$\langle \psi, H_\omega \psi \rangle = \langle \pi_\omega(A)\Omega, \pi_\omega(\delta(A))\Omega \rangle = -i\omega(\ast \delta(A)) \geq 0.$$ 

The result follows by a density argument (technically, that $\pi_\omega(D(\delta))$ is a core for $H_\omega$). \qed

---

10 A positive operator is any operator that is of the form $X = B^\ast B$ for some $B$ in the same algebra. Alternatively, its spectrum $\sigma(X)$ is contained in the positive real line. For such operators one can define a square root operation using the functional calculus. The details can be found in most books on $C^\ast$-algebras. One can show that the square root is also in the domain of $\delta$. 

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Hence we can define ground states in a purely algebraic setting by specifying the interactions of the system considering the corresponding derivation. Having a definition of ground states we can then try to answer different questions on the structure of the set of ground states. For example, it is not clear if it is non-empty. Indeed, there exist $C^*$-dynamical systems $(\mathfrak{A}, \alpha)$ without ground states. Furthermore, ground states should precisely be those states that minimize the local Hamiltonians in a suitable sense. We will come back to these points later.

**KMS states**

A basis question in statistical mechanics is how to characterise thermal states. The definition that will be given here was introduced by Hugenholtz, Haag and Winnink [28], giving a characterization of states satisfying the conditions studied by Kubo, Martin and Schwinger [36, 39]. In their honour they are called KMS states. Besides for the description of thermal states, KMS states have turned out to be extremely useful in the theory of operator algebras as well. We first give the (or rather, one of the many equivalent) definition. We first give the definition. Later we will discuss some results that will give support the this notion of thermal equilibrium states.

**Definition 3.4.4.** Let $(\mathfrak{A}, \alpha)$ be a $C^*$-dynamical system and let $\beta > 0$. Consider the strip

$$S_\beta = \{ z \in \mathbb{C} : \text{Im}(z) \in [0, \hbar\beta] \}$$

in the complex plane. A state $\omega$ on $\mathfrak{A}$ is called a KMS state if for each $A, B \in \mathfrak{A}$ there is a complex function $F_{AB}$ on $S_\beta$ such that

1. $F_{AB}$ is analytic on the interior of the strip and continuous on the boundaries,
2. $F_{AB}(t) = \omega(A\alpha_t(B))$ for all $t \in \mathbb{R}$, and
3. $F_{AB}(t + i\hbar\beta) = \omega(\alpha_t(B)A)$.

From now on we will use units where $\hbar = 1$. The parameter $\beta$ is the inverse temperature, $\beta^{-1} = k_BT$, where $T$ is the temperature and $k_B$ is Boltzmann’s constant. A ground state should be an equilibrium state at $T = 0$, and indeed one can see ground states as KMS states for $\beta = \infty$. Nevertheless, despite being similar in many aspects, there are also some fundamental differences between KMS states and ground states.

Note that the function $F_{AB}$ is given by expectation values of operators in the state $\omega$ on the boundary of the strip. This relation can be extended to values into the strip, under suitable circumstances. To discuss this, we first introduce the set of entire elements for a $C^*$-dynamical system. Note that for each $A \in \mathfrak{A}$ we have the continuous function $t \mapsto \alpha_t(A)$. The operator $A$ is said to be entire if there is an entire (i.e., complex analytic) function $f : \mathbb{C} \to \mathfrak{A}$ such that $f(t) = \alpha_t(A)$ for all $A \in \mathfrak{A}$. If $A$ is entire we will simply write $\alpha_z(A)$ for the corresponding function. Note that in the expansion of $\exp(t\delta)(A)$ for a derivation $\delta$ it is easy to plug in complex numbers instead of real numbers, to obtain $\alpha_z(A)$ for $z$ complex. A useful result is that there the set of entire elements, $\mathfrak{A}_{ent}$, is a norm dense *-subalgebra of $\mathfrak{A}$ [61 Prop. IV.4.6]. We then have the following result, whose proof can be found in [8 Prop. 5.3.7].
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Proposition 3.4.5. Let $\omega$ be a KMS state at inverse temperature $\beta$. Then for each $A \in \mathfrak{A}$ and $B \in \mathfrak{A}_\alpha$ we have that, restricted to the strip $S_\beta$,

$$F_{A,B}(z) = \omega(\mathcal{A}_\infty(B)),$$

where $F_{AB}$ is the function as in Definition 3.4.4.

Using this result we can put the claim that KMS states at $\beta = \infty$ correspond to ground states in context. Indeed, $\omega$ is a ground state if and only if the function $x \mapsto \omega(\mathcal{A}_\infty(B))$ is analytic and bounded on the upper half plane (i.e., the interior of the “strip” $S_\infty$) for all $A, B \in \mathfrak{A}_\alpha$ [Prop. 5.3.19], and continuous on the real line.

Just as for ground states there is an equivalent formulation of KMS states using only the derivation $\delta$. Again we omit the proof.

Theorem 3.4.6 (Roepstorff-Araki-Sewell). Let $(\mathfrak{A}, \alpha)$ be a C*-dynamical system and $\delta$ the corresponding derivation. Then the following are equivalent:

1. $\omega$ is a KMS state at inverse temperature $\beta$;

2. $-i\beta \omega(A^* \delta(A)) \geq \omega(A^* A) \log \left( \frac{\omega(A^* A)}{\omega(A A^*)} \right)$ for all $A \in D(\delta)$.

Here we define $x \log(x/y)$ to be zero if $x = 0$ and $+\infty$ if $x > 0$ and $y = 0$. This function is actually lower semicontinuous, which means that $\liminf_{(x,y) \to (x_0,y_0)} x \log(x/y) \geq x_0 \log(x_0/y_0)$.

We end this section on technical properties of KMS states with the following useful result, to the effect that limits of KMS states are again KMS states.

Proposition 3.4.7. Let $(\mathfrak{A}, \alpha)$ be a C*-dynamical system (where $\mathfrak{A}$ is unital) and $\delta$ the generator of $\alpha$. Suppose that for each $n \in \mathbb{N}$ we have a derivation $\delta_n$ generating a one-parameter group $t \mapsto \alpha_n^t$ of automorphisms such that $\lim_n \delta_n(A) = \delta(A)$ for all $A \in D(\delta)$. Moreover, suppose that for each $n$ we have a state $\omega_n$ which is KMS (with respect to $\alpha_n^t$) at inverse temperature $\beta_n$. Finally, suppose that $\lim \beta_n = \beta$, where we allow $\beta = +\infty$ as well. Then each weak* limit point of the sequence $\omega_n$ is a KMS state (with respect to $\alpha$) at inverse temperature $\beta$. It follows that there exists a KMS state at inverse temperature $\beta$.

Proof. Since $\mathfrak{A}$ is unital, the state space of $\mathfrak{A}$ is compact in the weak* topology, by Theorem 2.3.6. Hence weak* limit points exist. Let $\omega_{n_k}$ be a subsequence converging to a limit point $\omega$. The goal is to show that $\omega$ is a KMS state. To this end, we will distinguish the case $\beta < \infty$ and $\beta = \infty$.

Let us first assume that $\beta < \infty$. Without loss of generality, we can assume that $\beta_{n_k} < \infty$. By Theorem 3.4.6 we have for each $A \in D(\delta)$ the following inequality:

$$-i\beta_{n_k} \omega_{n_k}(A^* \delta(A)) \geq \omega_{n_k}(A^* A) \log \left( \frac{\omega_{n_k}(A^* A)}{\omega_{n_k}(A A^*)} \right)$$

This means that there is a subsequence $\omega_{n_k}$ such that $\lim_{n_k} \omega_{n_k}(A) = \omega(A)$ for the limit point $\omega$. 

\^{\dagger}\dagger
Hence \( \omega Z = \text{Tr}(\exp(A^*A)) \) is a continuous (but only lower semi-continuous), but we can still get

\[
-i\beta \omega(A^*\delta(A)) \geq \limsup_{k \to \infty} \omega_{n_k}(A^*A) \log \left( \frac{\omega_{n_k}(A^*A)}{\omega_{n_k}(AA^*)} \right)
\]

\[
\geq \liminf_{k \to \infty} \omega_{n_k}(A^*A) \log \left( \frac{\omega(A^*A)}{\omega(AA^*)} \right)
\]

\[
= \omega(A^*A) \log \left( \frac{\omega(A^*A)}{\omega(AA^*)} \right),
\]

where we used that \( \limsup \leq \liminf \) and that \( x \log(x/y) \) is lower semi-continuous.

Again by Theorem 3.4.6 it follows that \( \omega \) is a \( \beta \)-KMS state.

The case \( \beta = \infty \) remains. We will show that \( \omega \) is a ground state. First consider the case that each \( \beta_{n_k} = \infty \). Then we have

\[
-i\omega_{n_k}(A^*\delta(A)) \geq 0
\]

for all \( k \) and the result follows by taking the limit. Finally, consider the case where all \( \beta_{n_k} < \infty \). In this case we have by Theorem 3.4.6 that

\[
-i\omega(A^*\delta(A)) = \lim_{k \to \infty} -i\omega_{n_k}(A^*\delta(A))
\]

\[
\geq \limsup_{k \to \infty} \frac{1}{\beta_{n_k}} \omega_{n_k}(A^*A) \log \left( \frac{\omega_{n_k}(A^*A)}{\omega_{n_k}(AA^*)} \right)
\]

\[
\geq \limsup_{k \to \infty} \frac{1}{\beta_{n_k}} (\omega_{n_k}(A^*A) - \omega_{n_k}(AA^*)) = 0
\]

In the last line the inequality \( x \log(x/y) \geq x - y \) is used, which follows from the convexity of \( x \mapsto x \log x \).

Gibbs states

We now come to an important class of examples of KMS states, the Gibbs states. Let \( \Lambda \in \mathcal{P}_f(\Gamma) \). Then \( \mathfrak{A}(\Lambda) \) is a finite dimensional \( C^* \)-algebra and we can write any state \( \omega_\Lambda \) on \( \mathfrak{A}(\Lambda) \) in the form \( \omega_\Lambda(A) = \text{Tr}(\rho A) \) for some density matrix \( \rho \) (see Exercise 2.3.8). Consider a \( H_\Lambda = H_\Lambda^* \in \mathfrak{A}(\Lambda) \). Consider the state corresponding to the density matrix \( \rho_\Lambda = Z^{-1} \exp(-\beta H_\Lambda) \), where

\[
Z = \text{Tr}(\exp(-\beta H_\Lambda)) \]

Hence we have a state \( \omega_\Lambda \) given by

\[
\omega_\Lambda(A) = \frac{\text{Tr}(e^{-\beta H_\Lambda} A)}{\text{Tr}(e^{-\beta H_\Lambda})}.
\]

We claim that it is a KMS state for inverse temperature \( \beta \), where the time automorphism is given by \( \alpha^{\Lambda}_t(A) = e^{itH_\Lambda} A e^{-itH_\Lambda} \). Define \( F_{AB}(z) = \omega_\Lambda(A\alpha^{\Lambda}_t(B)) \) for \( z \in \mathbb{C} \). Then we have by the cyclicity of the trace, and \( t \) real,

\[
ZF_{AB}(t + i\beta) = \text{Tr} \left( e^{-\beta H_\Lambda} A e^{(-\beta + i\beta)H_\Lambda} B e^{(\beta - i\beta)H_\Lambda} \right) = \text{Tr} \left( e^{-\beta H_\Lambda} \alpha^{\Lambda}_t(B) A \right).
\]

Hence \( \omega \) satisfies the KMS conditions at inverse temperature \( \beta \) with respect to the action \( \alpha^{\Lambda}_t \). It is in fact the only state on \( \mathfrak{A}(\Lambda) \) with this property. To show

\footnote{The reader might recognize the partition function from statistical mechanics.}
this result we first need the show that each tracial state on a finite matrix algebra, that is a state for which \( \tau(AB) = \tau(BA) \), coincides with the usual trace. The proof is left as an exercise.

**Exercise 3.4.8.** Let \( A = M_n(\mathbb{C}) \). Let \( \omega \) be a linear functional on \( A \) such that \( \omega(AB) = \omega(BA) \) for all \( A, B \). Prove that \( \omega(A) = c \operatorname{Tr}(A) \) for some constant \( c \). Hint: consider matrices \( E_{ij} \) which are one on the \((i,j)\) position and zero otherwise.

Consider the algebra \( A \) as in the exercise, together with a given time evolution \( \alpha_t(A) = e^{itH}Ae^{-itH} \). It turns out that the Gibbs state at inverse temperature \( \beta \) is in fact the unique KMS state with respect to this action. This can be seen as follows (following the argument in [61]). Let \( \omega \) be a KMS state on \( A \). Note that since \( A \) is finite dimensional, it is easy to see that each element is analytical. Hence \( \omega(AB) = \omega(B\alpha_\beta(A)) \), by the KMS condition and Proposition 3.4.5. Define \( \widetilde{\omega}(A) = \omega(e^{\beta H}A) \). We then have

\[
\widetilde{\omega}(AB) = \omega(e^{\beta H}AB) = \omega(e^{\beta H}Ae^{\beta H}e^{-\beta H}B) = \omega(e^{\beta H}BA) = \widetilde{\omega}(BA),
\]

where we used the KMS condition. Hence by the exercise above, \( \widetilde{\omega}(A) = \lambda \operatorname{Tr}(A) \) for some constant \( \lambda \). The constant can be found by plugging in the identity and it follows that \( \omega \) must be the Gibbs state at inverse temperature \( \beta \).

In infinite dimensions, things are more complicated. Nevertheless, we can use the Gibbs states for finite systems to obtain KMS states. It is however no longer true in general that each KMS state can be obtained in this way.

A basic result in functional analysis, the Hahn-Banach theorem, says that we can extend \( \omega_\Lambda \) to a state \( \omega^G_\Lambda \) on all of \( \mathfrak{A} \). This extension is not unique in general, but for our purposes existence is enough. We write \( \omega^G_\Lambda \) for any such extension. Now consider an increasing sequence \( \Lambda_1 \subset \Lambda_2 \subset \ldots \) of finite subsets of \( \Gamma \) such that their union is equal to \( \Gamma \). This gives a sequence \( \omega^G_{\Lambda_n} \) of (extensions of KMS) states as above. Since the state space of \( \mathfrak{A} \) is weak-* compact, there is a subsequence \( \omega^G_{\Lambda_{n_k}} \) converging to a state \( \omega \) on \( \mathfrak{A} \). Since the convergence is in the weak-* topology, it follows that for local observables \( A \in A(\Lambda) \) we have

\[
\omega(A) = \lim_{k \to \infty} \omega_{\Lambda_{n_k}}(A),
\]

where it is understood that the limit on the right hand side is only over states corresponding to sufficiently large \( \Lambda_{n_k} \). We say that \( \omega \) is the **thermodynamic limit** of local Gibbs states.

If \( \Phi \) is a strictly local and bounded interaction, we have seen before that \( D(\delta) = A_{10c} \) for the corresponding derivation. We can then argue as in Theorem 3.3.11 and Proposition 3.4.7 that the thermodynamic limit of the local Gibbs states defined above is a \( \beta \)-KMS state for the time evolution generated by \( \Phi \). In other words, we have the following result:

**Corollary 3.4.9.** Let \( \Phi \) be a strictly local and bounded interaction. Then there is a KMS state for each \( \beta > 0 \) with respect to the dynamics generated by \( \Phi \). Moreover, the set of ground-states is non-empty.

**Remark 3.4.10.** Again, the same result can be stated for more general interactions. For example, instead of the convergence of the derivations for a
fixed $A$, as required in Proposition 3.4.7, can be loosened to demand that
\[ \lim_{\delta \to \infty} \| \alpha^\delta(A) - \alpha_0(A) \| \to 0. \]
This implies that there is a sequence $A_n \to A$ such that for the corresponding derivations we have $\delta_n(A_n) \to \delta(A)$. Note that this allows for the situation where $D(\delta_n)$ is different for each $n$.

**Von Neumann entropy**

In classical statistical mechanics the equilibrium states are characterised by minimising the free energy. We will give a quantum version of this statement, supporting the definition of KMS states. To define the free energy we first have to define the *entropy* of a state. For us it will suffice to define this only for states that are given by density matrices in finite dimensional $C^*$-algebras. The entropy can be defined for states on more general $C^*$ (or von Neumann) algebras, but that requires considerable additional machinery. The interested reader can consult [51] for the basics.

To calculate the entropy, one has to make sense of expressions of the form $\log A$ for some operator $A$. Such expressions can be made rigorous with the help of functional calculus. There are different versions of the functional calculus, assuming for example different continuity properties. We outline the procedure in the simplest case of the continuous functional calculus. Let $\mathfrak{A}$ be a unital $C^*$-algebra and let $A \in \mathfrak{A}$ be normal (i.e., $AA^* = A^*A$). Recall that the spectrum $\sigma(A)$ is defined as $\sigma(A) = \{ \lambda \in \mathbb{C} : (A - \lambda I) \text{ is not invertible} \}$. A standard result in operator theory is that the spectrum of a bounded operator is compact, and $r(A) := \sup_{\lambda \in \sigma(A)} |\lambda| = \|A\|$. The goal is to define $f(A)$ for $f \in C(\sigma(A))$. It is straightforward to define $p(A)$ for some polynomial $p(x) = \sum_{k=1}^n c_k x^k$: we set

\[ p(A) := \sum_{k=1}^n c_k A^k. \]

Note that since $A$ commutes with itself, there is no problem in defining the powers.

By the Stone-Weierstrass approximation theorem the polynomials form a dense subalgebra of $C(\sigma(A))$. In fact every $f \in C(\sigma(A))$ can be approximated uniformly by a sequence $p_n$ of polynomials. Then by a standard argument one shows that $p_n(A)$ converges (in norm) in $\mathfrak{A}$, and $f(A)$ is defined to be its limit.

It is instructive to carry out this procedure for a simple example. Consider $\mathfrak{A} = M_2(\mathbb{C})$ and let $A \in \mathfrak{A}$ be a self-adjoint matrix. Then there is a complete set of eigenvalues $\lambda_i$ with corresponding orthonormal basis $|\psi_i\rangle$. Moreover, $A = \sum_{i=1}^n \lambda_i |\psi_i\rangle\langle\psi_i|$. Since the $|\psi_i\rangle\langle\psi_i|$ are mutually commuting orthogonal projections, it is easy to calculate that for a polynomial $p$ one has $p(A) = \sum_{i=1}^n p(\lambda_i) |\psi_i\rangle\langle\psi_i|$. It follows that $f(A) = \sum_{i=1}^n f(\lambda_i) |\psi_i\rangle\langle\psi_i|$ for an arbitrary continuous function $f$ on the spectrum of $\mathfrak{A}$.

Consider for the moment a finite dimensional system with observable algebra $\mathfrak{A} = M_d(\mathbb{C})$. By Exercise 2.3.8 it follows that any state on $\mathfrak{A}$ can be uniquely written in the form $\omega(A) = \text{Tr}(\rho A)$ for some density operator $\rho$. Von Neumann defined the entropy for such a density matrix $\rho$ as $S(\rho) = -\text{Tr}(\rho \log \rho)$. Since $\rho$ is positive, there is a complete set of eigenvectors with corresponding eigenvalues $\{\lambda_i\}$. The entropy then reduces to $-\sum_i \lambda_i \log \lambda_i$, where $0 \log 0$ is defined to
be zero. This coincides with the classical entropy of a probability distribution on \(\{1, \ldots, n\}\) with \(p(i) = \lambda_i\). The von Neumann entropy plays an important role in quantum information theory, just as its classical analog, the Shannon entropy, does in classical communication theory. See [50] for more details.

Note that we have defined the entropy only for finite dimensional algebras. In the quantum spin systems we have been discussing, however, we are dealing with states on the quasi-local algebra, for which no entropy is defined. Therefore we will look at the entropy in a finite subset of the system.

**Definition 3.4.11.** Let \(\omega\) be a state on a quasi-local algebra \(\mathfrak{A}\), where the local algebras \(\mathfrak{A}(\Lambda)\) are finite dimensional matrix algebras. Let \(\Lambda\) be a finite set. Then we define the entropy \(S_\Lambda(\omega)\) by

\[
S_\Lambda(\omega) = - \text{Tr}(\rho_\Lambda \log \rho_\Lambda),
\]

where \(\rho_\Lambda\) is the unique density matrix such that \(\omega(A) = \text{Tr}(\rho_\Lambda A)\) for all \(A \in \mathfrak{A}(\Lambda)\).

Finally, we consider the relative entropy of two states. If \(\rho\) and \(\sigma\) are the density matrices of two states, their relative entropy is defined as

\[
S_{\text{rel}}(\sigma|\rho) = \text{Tr}(\rho \log \rho - \rho \log \sigma).
\]

One can show that \(S_{\text{rel}}(\rho|\sigma) \geq 0\), where we have equality if and only if \(\rho = \sigma\). Again, one can define the relative entropy of two states on the quasi-local algebra, when restricted to a finite region \(\Lambda\), in a straightforward way.

**Free energy**

Consider a finite dimensional system and a time evolution given by a Hamiltonian \(H\). Moreover, suppose that \(\rho\) is a density matrix giving a state of the system. We will also write \(\rho\) for the corresponding state. Let \(\beta > 0\). Then the free energy is given by

\[
F_\beta(\rho) = \rho(H) - \beta^{-1}S(\rho).
\]

Write \(\rho_\beta\) for the Gibbs state at inverse temperature \(\beta\). Then we have

\[
S_{\text{rel}}(\rho_\beta|\rho) = \beta(F_\beta(\rho) - \Phi(\beta)).
\]  

(3.4.2)

Here \(\Phi(\beta)\) is the value of the free energy on the Gibbs state. Since \(S_{\text{rel}} \geq 0\), it follows that the free energy is minimal when \(\rho\) is the Gibbs state at inverse temperature \(\beta\). Because \(S_{\text{rel}}(\rho_\beta|\rho) = 0\) implies that \(\rho_\beta = \rho\) it follows that the Gibbs state can be alternatively defined as the state that minimizes the free energy.

**Exercise 3.4.12.** Show that \(\Phi(\beta) = -\beta^{-1} \log \text{Tr}(\exp(-\beta H))\) and verify that equation (3.4.2) indeed holds.

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13 This amounts to requiring that \(x \mapsto x \log x\) is continuous in zero.
CHAPTER 3. INFINITE SYSTEMS

Translational invariance states

In general the entropy scales proportionally to $|\Lambda|$, so that it is not very useful to define the entropy for the whole infinite system as the limit of $S_\Lambda(\omega)$ as $\Lambda \to \infty$. One can still look at the entropy per unit of volume, at least in the case of translationally invariant systems, defined as $S(\omega) := \lim_{\Lambda \to \infty} S_\Lambda(\omega)/|\Lambda|$. Here $\Lambda \sim \infty$ is the limit in the sense of Van Hove. This is defined as follows. Assume that $\Gamma = \mathbb{Z}^d$, and choose a vector $(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$, where each $a_i > 0$. This defines a box $\Lambda_a$ as follows:

$$\Lambda_a = \{ x \in \mathbb{Z}^d : 0 \leq x_i < a_i, i = 1 \ldots d \}.$$ 

We can translate each box over a vector $na$ (with $n \in \mathbb{Z}$) to obtain a partition of $\mathbb{Z}^d$ into boxes. If $\Lambda$ is any finite subset of $\mathbb{Z}^d$, we write $n\Lambda_a(a)$ for the number of (translated) boxes as above that have non-empty intersection. Similarly, $n\Lambda_a(a)$ is the number of boxes that are contained in $\Lambda$. We then say that a sequence of sets goes to infinity in the sense of Van Hove, notation $\Lambda_n \sim \infty$ (or simply without the subscript $n$) if

$$\lim_{n \to \infty} \frac{n\Lambda_a(a)}{n\Lambda_n(a)} = 1$$

for all $a$. Note that this is a weaker form of convergence than $\Lambda \to \infty$, where we only allow a certain sequence of sets $\Lambda_1 \subset \Lambda_2 \subset \ldots$. Note that if $\Lambda_n \sim \infty$ then $\bigcup_n \Lambda_n = \mathbb{Z}^d$. This can be seen by taking a larger and larger.

Using this notion of convergence we can make sense of the entropy per unit of volume in the case of translationally invariant systems. More precisely, we have the following proposition, whose proof can be found in [8].

**Proposition 3.4.13.** For a translation invariant state $\omega$ the following limit exists

$$S(\omega) = \lim_{\Lambda \to \infty} S_\Lambda(\omega)/|\Lambda|.$$ 

It is equal to $\inf_{a \in \mathbb{Z}^d} S_{\Lambda_a}(\omega)/|\Lambda_a|$.

In a similar manner we can define the other component in the free energy defined above: the energy of a state. Again we look at translationally invariant systems. In addition we assume that we are given an interaction $\Phi$ that is also translationally invariant. The interaction should satisfy the bound

$$\|\Phi\| = \sum_{\Lambda \geq 0} \frac{||\Phi||}{|\Lambda|} < \infty,$$

which is certainly the case if $\Phi$ is of finite range. Again, for such systems we can define the mean energy per unit volume. In particular we have the following proposition.

**Proposition 3.4.14.** Let $\Phi$ be as above and suppose that $\omega$ is an invariant state. Then the following limit exists:

$$H_\Phi(\omega) = \lim_{\Lambda \to \infty} \frac{\omega(H_\Phi)}{|\Lambda|},$$
where $H^\Phi_\Lambda$ are the local Hamiltonians. The limit is equal to $\omega(E_\Phi)$, where

$$E_\Phi = \sum_{\Lambda \ni 0} \frac{\Phi(\Lambda)}{|\Lambda|}.$$ 

It follows that for translationally invariant states and ditto interactions we can again define the free energy functional by

$$F_\beta(\omega) = H_\Phi(\omega) - \beta^{-1} S(\omega).$$

Recall that in the finite dimensional case the KMS states were precisely those states that minimize the free energy functional. This is true in the present setting as well, at least if we make a slightly stronger assumption on the behaviour of the interaction. The result can be stated as follows (again we omit the proof).

**Theorem 3.4.15.** Let $\Phi$ be a translationally invariant interaction such that there is some $\lambda > 0$ such that

$$\|\Phi\|_\lambda = \sum_{\Lambda \ni 0} \|\Phi(\Lambda)\| e^{\lambda |\Lambda|} < \infty.$$ 

Write $\alpha$ for the corresponding time evolution. Then the following are equivalent for a translationally invariant state:

1. $\omega$ is a KMS state at inverse temperature $\beta$ with respect to $\alpha$;
2. $\omega$ minimizes the free energy functional $\sigma \mapsto F_\beta(\sigma)$.

For ground states there is a similar result. Under the same assumptions as in the theorem, one can show that a translationally invariant state is a ground state for the dynamics if and only if $\omega$ minimizes $H_\Phi(\omega)$.

**Remark 3.4.16.** We have restricted ourselves to translationally invariant states, but similar results are also true for general systems (under suitable assumptions on the interaction $\Phi$). In this case the mean energy and mean entropy lose their meaning. Instead one can look at arbitrary bipartitions of the system in a finite part and its complement. Then one can calculate certain entropic quantities and energies. In the calculation of the energies one has to be careful, since there are contributions coming from the interactions at the boundary of $\Lambda$, the so-called surface elements. Using these quantities again a free energy functional can be defined, now depending on the region $\Lambda$. Given a state $\omega$, one can then look at the set $C^\Lambda_\omega$ of all states that agree with $\omega$ when restricted to $\Lambda^c$. If for each region $\Lambda$ the free energy functional applied to $\omega$ is the same as the infimum of the value on all states in $C^\Lambda_\omega$, then $\omega$ is a KMS state (and vice versa).

**Thermalisation and the second law of thermodynamics**

We end the discussion of KMS states with two properties relating them to thermodynamical questions. First of all, suppose we have a large (in our case, infinite) system in equilibrium at inverse temperature $\beta$. We then consider another small (finite) system in any given state and let the systems interact.
In other words, we consider a quantum system coupled to a heat bath in some way. One expects that after some time, the small system will be driven to a thermal state at the same inverse temperature. This is called the zeroth law of thermodynamics. It turns out that this is true and in fact this condition uniquely characterizes the KMS states (under certain assumptions on the coupling between the two systems). This was first proven in [35]. An overview of the main arguments can be found in [60, pp. 114–117]. Here we will content ourselves with a brief sketch on how to describe these systems.

As for the reservoir system, we assume that it is described by a C*-dynamical system \((A_R, \alpha_R)\) with corresponding generator \(\delta_R\). The small system is described by a finite dimensional algebra \(A_f = M_d(\mathbb{C})\) with time evolution governed by a Hamiltonian \(H_f\). The system as a whole is then described by the observable algebra \(A = A_f \otimes A_R\). We can define a derivation describing the evolution when there is no coupling between the systems. It is just given by

\[
\delta(A \otimes B) = i([H, A]) \otimes I + I \otimes \delta_R(B).
\]

One can check that this indeed generates a time evolution of the form \(\alpha_t = \alpha^f_t \otimes \alpha^R_t\). To introduce a coupling between the systems, we have to add a term that acts on both sides at the same time. We assume that there is an interaction of the form \(\lambda V\), where \(\lambda\) is some parameter and \(V\) is of the form

\[
V = \sum_{j=1}^n A_j \otimes A^R_j.
\]

We assume that \(V\) is self-adjoint and in addition that the reservoir initially is in a state \(\omega\). We demand that \(\omega(A^R_j) = 0\) for all \(j\). The dynamics of the perturbed system is then generated by the derivation \(\delta(\cdot) + i\lambda [V, \cdot]\), where \(\delta\) is the derivation of the uncoupled system as above. Now suppose that the finite system is in an initial state \(\rho\), such that the combined system is in the state \(\rho \otimes \omega\). Under the time evolution the state will also undergo an evolution. At a later point \(t\) we might be interested in the state of the finite system. In general, the system as a whole is not in a product state any more, because of the interactions between the bath and the system. Nevertheless, we can still “forget” about the big system and obtain a state \(\rho_t\) of the small system. This can be done with the help of a conditional expectation. If we do this at each time \(t\), we see how the small system evolves. Because of the coupling this time evolution is in general not given by a unitary any more. Nevertheless, there is a map \(\gamma_t\) such that \(\rho_t = \gamma_t(\rho)\). Technically, \(t \mapsto \gamma_t\) is a one-parameter group of completely positive transformations on \(A_f\).

In this situation, it is shown in [35] that, roughly speaking, the following conditions are equivalent:

1. \(\omega\) is a KMS state at inverse temperature \(\beta\),

2. There is a unique state \(\rho\) on the finite system that is invariant under the dynamics induced by the coupled system. This is the Gibbs state at inverse temperature \(\beta\). Moreover, any initial state \(\rho_0\) converges to \(\rho\) as \(t \to \infty\).

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14 This is a generalisation of the partial trace operation.
Hence, as expected, the state of the small system will tend to the (unique) equilibrium state at the same temperature as the big reservoir to which it is coupled.

Finally we have a look at the second law of thermodynamics, which says that one cannot extract energy from a system at thermal equilibrium in a cyclic process (that is, by returning back to the equilibrium dynamics). Again we first investigate the question for a finite dimensional system governed by a Hamiltonian $H$. Let $\omega$ be the corresponding Gibbs state for $\beta > 0$. Note that the energy of the system is given by $\omega(H)$. We then apply a unitary operation $U$ to the system, to obtain a state $\omega_U(A) := \omega(U^*AU)$. The energy difference is then

$$\omega_U(H) - \omega(H) = \omega(U^*HU) - \omega(H) = -i\omega(U^*\delta(U)),$$

where $\delta(U) = i[H, U]$ as usual. Now represent $\omega$ by a density operator $\rho$. Then $\omega_U$ is represented by a unitary equivalent density operator. But this does not change the eigenvalues of $\rho$, and hence we conclude that $S(\omega) = S(\omega_U)$. Therefore we have

$$F_\beta(\omega) - F_\beta(\omega_U) = \omega_U(H) - \omega(H) \geq 0.$$

The inequality follows because the Gibbs state minimizes the free energy. Hence we conclude that $-i\omega(U^*\delta(U)) \geq 0$. This condition can be generalised to infinite systems in a straightforward way, leading to the condition of passivity.

**Definition 3.4.17.** Let $(\mathfrak{A}, \alpha)$ be a $C^*$-dynamical system with generator $\delta$. Suppose that $\omega$ is an invariant state. Then $\omega$ is said to be passive if

$$-i\omega(U^*\delta(U)) \geq 0$$

for all unitaries $U$ in the domain $D(\delta)$.

Note that by definition ground states are passive. In a seminal work, Pusz and Woronowicz proved that in fact any KMS state is passive [54].

**Theorem 3.4.18 (Pusz-Woronowicz).** Let $(\mathfrak{A}, \alpha)$ be a $C^*$-dynamical system. Then every $\beta$-KMS state is passive.

The condition of passivity is related to the inability to let the equilibrium system perform work, as we will outline now. Consider dynamics $\alpha_t$ generated by some derivation $\delta$. At $t = 0$ we start perturbing the dynamics. One can think of turning on a magnetic field, for example. We do this until some later time $T$, after which all perturbations are turned off again and we are back with the original dynamics. Mathematically this can be described as follows. At times $0 \leq t \leq T$, the dynamics is not generated by $\delta$ alone, but by

$$\delta^P_t(A) = \delta(A) + i[P_t, A].$$

Note that if the dynamics is generated by a Hamiltonian, this amounts to adding an extra term $P_t$ to the Hamiltonian. We will assume that $P_t : \mathbb{R} \to \mathfrak{A}$ with $P_t = P_t^*$ for all $t$. Moreover, $P_t = 0$ if $t \leq 0$ or $t \geq T$. Finally, we will assume that $t \mapsto P_t$ is differentiable.

The introduction of perturbations means that the time evolution is modified: for each $t$ we have an automorphism $\alpha^P_t$ describing the time evolution.
Note that $t \mapsto \alpha^{P}_t$ does not need to be a one-parameter group any more! Since the dynamics is now time-dependent, finding the automorphisms $\alpha^{P}_t$ is not as straightforward as in the time-independent case. Nevertheless, one can show that they exist and are related to the original dynamics.

**Theorem 3.4.19.** Let $(\mathfrak{A}, \alpha)$ be a $C^*$-dynamical system with generator $\delta$. Let $t \mapsto P_t$ be as above. Then there is a unique one-parameter family of $*$-automorphisms $\alpha^{P}_t$ solving the differential equation

$$\frac{d\alpha^{P}_t}{dt}(A) = \alpha^{P}_t(\delta(A) + i[P_t, A])$$

for all $A \in \delta(D)$ and with initial condition $\alpha^{P}_0(A) = A$. Moreover, for each $t$ there is a unitary $\Gamma^{P}_t \in \mathfrak{A}$ such that

$$\alpha^{P}_t(A) = \Gamma^{P}_t \alpha_t(A)(\Gamma^{P}_t)^*$$

for all $A \in \mathfrak{A}$.

One can give an explicit expression for $\Gamma^{P}_t$ in terms of perturbation series (see [8, Sect. 5.4.4]).

Perturbing the system means that work is performed by some external forces. The question is then, how much work is actually performed? Suppose that initially the system is in a state $\omega$ at $t = 0$. At some later time the state has evolved to $\omega_t = \omega \circ \alpha^{P}_t$. To estimate the change in energy we divide the interval $[0, T]$ up into $N$ intervals $(t_i, t_{i+1})$. If $N$ is big enough we can assume that the state $\omega_t$ doesn’t change much over the time of an interval. The difference in energy is then given by $\omega_{t_i}(P_{t_i} - P_{t_{i-1}})$. Summing over all intervals we obtain

$$L^P(\omega) = \sum_{i=1}^{N} \omega_{t_i}(P_{t_i} - P_{t_{i-1}}).$$

In the limit $N \to \infty$ this can be expressed as an integral.

Alternatively we can argue as in the finite case. The total amount of work is given by the energy of the state $\omega_T$ minus the energy of the state $\omega_0 = \omega$. Note that $\omega_T(A) = \omega(\Gamma^{P}_T \alpha_T(A)(\Gamma^{P}_T)^*)$. One then has a similar result for the energy difference of the two states as in the case of finite systems discussed above. That is, it is given by $-i\omega(\Gamma^{P}_T \delta((\Gamma^{P}_T)^*))$. In fact it is equal to $L^P(\omega)$ defined above (where strictly speaking we should use the integral formulation obtained from $N \to \infty$), that is

$$L^P(\omega) = -i\omega(\Gamma^{P}_T \delta((\Gamma^{P}_T)^*).$$

This gives us the second law of thermodynamics for KMS states. Since KMS states are passive, $-i\omega(\Gamma^{P}_T \delta((\Gamma^{P}_T)^*)) \geq 0$, hence the system has more (or equal) energy after the “cycle” of perturbing the system and going back to the original dynamics. It is therefore not possible to extract energy from the system in a cycle. This gives further evidence that the notion of a KMS state is indeed the right one to describe thermal equilibrium states.

**Remark 3.4.20.** Under some additional assumptions a converse is also true. That is, under these assumptions one can show that a passive state is a KMS
state. The extra assumptions are necessary because the set of passive states is convex. For example, a convex combination of two KMS states at different temperatures is a passive state, but it is clearly not a KMS state at a certain temperature. Therefore one needs to impose additional assumptions to somehow pick out the “pure phases”.

3.5 The toric code

We now apply the results and techniques developed in this chapter to an important example in quantum information theory, the toric code. This model was first introduced by Kitaev [34]. The reason that it is called the toric code is two-fold: the model is often considered on a torus (i.e., as a finite system with periodic boundary conditions in the $x$ and $y$ direction) and it is an example of a quantum code. Quantum codes are used to store quantum information and correct errors. We will only make some brief comments later on this aspect.

Instead of on a torus, we will consider the model on an infinite plane. That is, consider the lattice $\mathbb{Z}^2$. The set $\Gamma$ of sites is defined to be the edges between nearest-neighbour points in the lattice (see Figure 3.1). At each of these edges there is a spin-1/2 degree of freedom, with corresponding observable algebra $M_2(\mathbb{C})$. We can then define the quasi-local algebra $\mathfrak{A}(\Gamma)$ as before. Note that there is a natural action of the translation group, so that it makes sense to talk about translation invariant interactions or states.

There are two special subsets of sites that we will consider. For any vertex $v$ there are in total four edges that begin or end in that vertex (see the picture). Such a set will be called a star. Similarly, one can define a plaquette, as the edges around a vertex in the dual lattice (see the solid lines in the picture). To

Figure 3.1: Lattice on which Kitaev’s toric code is defined. The spin degrees of freedom live on the edges between the solid dots. Also indicated are a star (dashed lines) and a plaquette (solid lines). Picture from [41].
a star $s$ and a plaquette $p$ we associate the following operators:

$$A_s = \bigotimes_{j \in s} \sigma^x_j, \quad B_p = \bigotimes_{j \in p} \sigma^z_j.$$ 

An important property is that $[A_s, B_p] = 0$ for any star $s$ and plaquette $p$. This can be seen because a star and a plaquette always have an even number of edges in common. Commuting the operators at each edge give a minus sign, because of the anti-commutation of Pauli matrices. Since the number of minus signs is even the claim follows. Another property is that $A_s^2 = B_p^2 = I$.

The star and plaquette operators will be used to define the interactions of the model. Namely, for $\Lambda \in \mathcal{P}_f(\Gamma)$ we set

$$\Phi(\Lambda) = \begin{cases} -A_s & \Lambda = s \text{ for some star } s \\ -B_p & \Lambda = p \text{ for some plaquette } p \\ 0 & \text{else} \end{cases}.$$ 

Note that the interactions is of finite range and bounded. Moreover, it is translation invariant. By Theorem 3.3.11 it follows that there is a one-parameter group $t \mapsto \alpha_t$ describing the time evolution with respect to these dynamics. In the remainder of this chapter we will construct a ground state for this model.

The exposition here largely follows [1]. The toric code is actually only one example of a large class of models defined by Kitaev [34], usually called the quantum double models. For example, to each finite group $G$ one can associate such a model. The toric code corresponds to the choice $G = \mathbb{Z}_2$, the smallest non-trivial group. Using similar methods as here one can find the ground state for any group $G$ [42].

To define the ground state, the following lemma will be useful.

**Lemma 3.5.1.** Suppose that $\mathfrak{A}$ is a unital $C^*$-algebra and let $\omega$ be a state on $\mathfrak{A}$. Suppose $X \in \mathfrak{A}$ is such that $X^* = X$, the operator $X - I$ is positive, and $\omega(X) = 1$. Then for each $Y \in \mathfrak{A}$ we have that $\omega(YX) = \omega(YX^*Y)$.

**Proof.** Because $X - I$ is positive we can take the square root $\sqrt{X - I}$ (compare with footnote 10 on page 52), which is also self-adjoint. Hence we have

$$|\omega((X - I)Y)|^2 = |\omega(\sqrt{X - I}\sqrt{X - I}Y)|^2 \leq \omega(X - I)\omega(Y^*(X - I)Y) = 0,$$

where we used Lemma 2.3.2. The other equality is proved similarly. \qed

With the help of this lemma it is not so difficult to show that any state with $\omega(A_s) = \omega(B_p) = 1$ is a ground state.

**Proposition 3.5.2.** Let $\omega$ be a state such that $\omega(A_s) = \omega(B_p) = 1$ for all stars $s$ and plaquettes $p$. Then $\omega$ is a ground state of the toric code.

**Proof.** Let $A \in \mathfrak{A}(\Lambda)$ be a local operator. Write $\delta$ for the derivation corresponding to the toric code interaction. Because of locality, most terms in $\delta(A)$ vanish. We obtain

$$\delta(A) = -i \sum_{s \cap \Lambda \neq \emptyset} [A_s, A] - i \sum_{p \cap \Lambda \neq \emptyset} [B_p, A],$$

where the sum over $s$ and $p$ is over the plaquettes and stars intersecting $\Lambda$, respectively.
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where the sum as over all stars \( s \) (plaquettes \( p \)) that have non-empty intersection with the support of \( A \). Hence we have

\[
-i\omega(A^*\delta(A)) = \sum_{\omega\cap\Lambda\neq\emptyset} \omega(A^*AA_s) - \omega(A^*A\omega) + \sum_{p\cap\Lambda\neq\emptyset} \omega(A^*AB_p) - \omega(A^*B_pA).
\]

But \( I - A_s \) and \( I - B_p \) are positive operators, since they are positive multiples of projections, so that we can use Lemma 3.5.1 to write \( \omega(A^*AA_s) = \omega(A^*A) \) (and similarly for the \( B_p \) term).

We are done if we can show that \( \omega(A^*A) - \omega(A^*A_s) \) is positive, since the analogous result for \( B_p \) follows from the same argument. But we have \( \omega(A^*(I - A_s)A) \geq 0 \), because \( A^*(I - A_s)A \) is positive, since \( I - A_s \) is positive. By Theorem 3.4.3 it follows that \( \omega \) is a ground state.

Hence any state such that \( \omega(A_s) = \omega(B_p) = 1 \) is a ground state. This condition can be interpreted as the condition that the operators \( A_s \) and \( B_p \) stabilize the ground state. Indeed, if \( (\pi_0, \mathcal{H}, \Omega) \) is the corresponding GNS representation, it follows that \( \pi_0(A_s)\Omega = \pi_0(B_p)\Omega = \Omega \). One thing is not clear yet, however: it may be that there are no states that satisfy this condition. The goal is to show that such a state does exist, by considering a classical Ising model.

More concretely, consider the \( C^* \)-algebra \( \mathfrak{A}_{cl} \) that is generated by the operators \( A_s \) and \( B_p \) (for all stars \( s \) and plaquettes \( p \)). Since all generators mutually commute, it follows that \( \mathfrak{A}_{cl} \) is abelian. We want to map this algebra to a algebra of observables of a classical Ising system. Recall that this is defined as follows. Consider a set \( \Gamma_s \) of spins. In our case, we take this set to be equal to the set of stars on the lattice. A configuration assigns to every spin \( s \in \Gamma_s \) the value \(+1\) (up) or \(-1\). The set of all configurations \( S \), which can be identified with the set of functions \( f : \Gamma_s \to \{-1, 1\} \) is called the configuration space.

As discussed before, the set of observables is the set of (real-valued) functions on the configuration space that vanish at infinity. Here we will slightly extend this to all bounded functions. This makes no essential difference here. A particular example are the functions \( \sigma_s : S \to \mathbb{R} \), with \( \sigma_s(f) = f(s) \). That is, it gives the value of the spin at site \( s \). Note that \( \sigma_s^2 = 1 \) and \( \sigma_s^* = \sigma_s \). Finally, clearly all of the observables \( \sigma_s \) commute. Now we add another copy of the system, associated with the plaquettes of the Kitaev model. We write \( T \) for the corresponding configuration space and \( \tau_p \) for the corresponding observables that measure the spin at the plaquette \( p \). Hence the configuration space \( S \times T \) describes two copies of a free (non-interacting) Ising model. The idea is then to map the algebra \( \mathfrak{A}_{cl} \) to a algebra of observables of the two copies of the free Ising model. But since \( \sigma_s \) and \( \tau_p \) have the same algebraic properties as the \( A_s \) and \( B_p \), we can just map \( A_s \) to \( \sigma_s \) and \( B_p \) to \( \tau_p \). This gives an isomorphism of the corresponding algebras. In this way we can obtain a state on \( \mathfrak{A}_{cl} \) by giving a state on the Ising model. Consider the state where all spins are in the up direction (this corresponds to the Dirac measure concentrated in the corresponding element in \( S \times T \). Hence we obtain a state \( \omega_{cl} \) with \( \omega_{cl}(A_s) = \omega_{cl}(B_p) = 1 \).

We now have a state \( \omega_{cl} \) on a abelian subalgebra \( \mathfrak{A}_{cl} \) of \( \mathfrak{A} \) with the desired properties. By the Hahn-Banach extension theorem, it follows that there is a
state \( \omega \) on \( \mathfrak{A} \) that extends \( \omega_{\text{cl}} \). By the proposition above it follows that \( \omega \) is a ground state, and we will now argue that the conditions that \( \omega(A_s) = \omega(B_p) = 1 \) in fact completely determine \( \omega \) (so that there is a unique extension of \( \omega_{\text{cl}} \) to \( \mathfrak{A} \)). To see why this is true, we first consider an example. Suppose \( A = \sigma^y_j \) for some site \( j \). We want to know the value of \( \omega(\sigma_j) \). Let \( s \) be any star that contains the site \( j \). By applying Lemma 3.5.1 we find

\[
\omega(A) = \omega(A \sigma^y_j) = -\omega(\sigma^y_j A) = -\omega(\sigma_j),
\]

hence \( \omega(A) = 0 \). The second equality follows because \( \sigma^y_j \) and \( \sigma^y_j \) anti-commute.

Now consider a general local element \( A \). By linearity it suffices to show that we can calculate \( \omega(A) \) in the case that \( A \) is a monomial in the Pauli matrices, that is,

\[
A = \prod_{j \in \Lambda} \sigma^x_j,
\]

where \( \Lambda \) is the support of \( A \) and \( k_j = x, y, z \). The idea is to use the trick above repeatedly, in a systematic way. We first try to find all sites plaquettes \( p \) and stars \( s \) that have only one site in common with the support \( \Lambda \) of \( A \). For example, let \( p \) be a plaquette such that \( \Lambda \cap p = \{ j \} \) for some site \( j \). Then, by locality, we see that \( B_p \) and \( A \) commute if \( k_j = z \), and anti-commute otherwise. Similarly, for star operators we have a similar result with \( k_j = x \). Hence by the argument above, if for any of these sites we have anti-commutation, \( \omega(A) = 0 \). This forces \( k_j \) for all sites \( j \) as above. Continuing in this way it follows that \( \omega(A) \) is zero unless \( A \) is a product of star and plaquette operators. In that case, clearly we have \( \omega(A) = 1 \) by a repeated application of Lemma 3.5.1. It follows that \( \omega \) is completely determined on all local operators, and hence on all of \( \mathfrak{A} \) since the local operators are dense in \( \mathfrak{A} \) and \( \omega \) is continuous (because it is a state). Note that the state is manifestly translationally invariant.

The state \( \omega \) that we have constructed is the only translationally invariant ground state. This can be seen as follows. Suppose that \( \rho \) is any translationally invariant state. First of all, note that \( \rho(A_s) \) and \( \rho(B_p) \) are real because both operators are self-adjoint. By translational invariance, the value of \( A_s \) is the same for all stars \( s \), and similarly for \( B_p \). Now, since \( A_s^2 = I \), it follows from the Cauchy-Schwarz inequality, Lemma 2.3.2, that \( \rho(A_s) \leq 1 \). But if \( \rho(A_s) < 1 \), it is clear that \( H_\phi(\rho) > H_\phi(\omega) \), where \( H_\phi \) is the mean energy as before. By the remark made after Proposition 3.4.1 it follows that \( \rho \) cannot be a ground state. The argument for the case that \( \rho(B_p) \neq 1 \) is the same.

Remark 3.5.3. The Kitaev model is actually special in the sense that it minimizes each of the terms in the Hamiltonian individually. A system for which this is true is called completely free of frustration. This is true here, essentially, because all the terms in the Hamiltonian commute with each other. Hence one expects that they all can be diagonalised simultaneously.

With a bit more work one can in fact argue that \( \omega \) is the unique ground state, so that there are no non-translationally invariant ground states. It follows that \( \omega \) is a pure state\(^{15}\) This should be contrasted with the ground space degeneracy

\(^{15}\) This is not immediately obvious, but follows from the fact that the set of ground states is a face in the set of all states. In this case, the face consists of only one point and it follows that this point must also be an extremal point of the set of all states. Hence the state is pure.
of the model as it is usually considered. Here one takes a compact surface of genus \( g \). The genus is roughly the number of holes, so a torus has genus one. On this compact surface one can draw a (finite) graph, where the edges represent the spin degrees of freedom.\(^{16}\) Again considering the case of the torus, one can think of a square lattice, and identify the left and right edges, as well as the top and bottom edges, to get a graph that can be embedded into the torus. Once one has the graph, the edges determine the Hilbert space of the system and one can define a Hamiltonian in the model in a similar way as above. The ground state degeneracy can then be shown to be equal to \( 4^g \). Note that this is consistent with the uniqueness of the ground state on the plane, since the plane has genus zero.

We will outline why this is the case in case of the torus. Note that for the torus, there are two different non-contractible loops on the surface. If we model the torus as a square with the edges identified, these loops correspond to a horizontal and a vertical line from edge to edge. We can form such a loop by connecting edges in the graph. Let's call one such loop \( \xi_X \). To this path \( \xi_X \) we associate an operator \( F_{\xi_X} = \otimes_{j \in \xi_X} \sigma^x_j \). Note that \( F_{\xi_X} \) commutes with all star operators and all plaquette operators. Hence we cannot apply the trick above, so what \( \omega(F_{\xi_X}) \) is not determined. This essentially gives an extra parameter in the definition of ground states. If we combine this with a loop on the dual lattice (with operators \( \sigma^z_j \)), and a combination of the two, we obtain a four-dimensional ground state.

It has been the hope that this ground state space could be used to store quantum information (or qubits). The idea is that we would encode the information in a certain state in the ground space. The hope was that this would lead to a reliable storage. That is, the state should not change because of small perturbations due to the environment, such as thermal noise. This is a natural requirement if one wants to store information for a longer time, just as with a classical hard disk you expect to be able to read your files again if you turn on your computer the next day. The essential idea now is that to transform the state \( F_{\xi_X} \Omega \), where \( \Omega \) is a state in the ground space, to another (orthogonal) state in the ground space, one has to do a non-local operation, where locality is interpreted here in terms of the system size. Here, to undo the effect of the path operator, one has to apply another operator going around the loop. If we make the system big, such non-local operations are very unlikely to occur due to thermal fluctuations.\(^{14}\)

A more careful analysis unfortunately shows that that is a bit too optimistic, see for example \(^{11,12,13,37}\). For example, what can happen is that a local (pair) of excitations gets created. This is local, and won’t destroy the information in the ground state. However, there is no energy penalty for moving the excitations around. This has the effect that they may spread out very fast, to get something that is not local any more. It is now generally accepted that the toric code is not a good quantum memory, and finding one is one of the big challenges in quantum computing at the moment.

\(^{16}\) One needs to demand some additional conditions on the graph to avoid pathological cases.

\(^{17}\) This is only a small selection on the vast amount of literature on so-called self-correcting quantum memories.
4 | Lieb-Robinson bounds

One important difference between relativistic systems and non-relativistic systems (such as the ones we consider here), is that for relativistic systems one has \textit{causality}: information cannot travel faster than the speed of light. In lattice systems there is no such natural bound. Correlations can, in principle, spread arbitrarily fast. Nevertheless, under suitable conditions there is a maximum velocity in the system, which dictates how fast information can propagate through the system. This is essentially what a \textit{Lieb-Robinson bound} is.

The first of these bounds was proven by Lieb and Robinson in 1972 \cite{lieb1972}, where the authors assumed translation invariance. This allowed them to use Fourier techniques to prove the result. In the next three decades, comparatively little work was done on this type of bounds. This changed however drastically around 2004, when people realised that such bounds are very useful in studying, for example, many-body systems and quantum information. For example, Hastings \cite{hastings2004} used an improved version of the bound to prove a multi-dimensional version of the Lieb-Schultz-Mattis theorem. This theorem tells us something about the low energy excitations of gapped Hamiltonians. Around the same time, Nachtergaele and Sims \cite{Nachtergaele2004} proved a version of the Lieb-Robinson bound that does not require translational invariance. In addition, they showed that for Hamiltonians with an energy gap, one has exponential decay of correlation functions (in terms of the distance of the support of two observables). After this work, many new applications and improvements of the Lieb-Robinson bounds have been found. For more information we refer to \cite{bravyi2006, bravyi2009}.

4.1 Statement and proof

Before we can state the Lieb-Robinson theorem, we first have to make clear what kind of systems we want to consider. Here we will largely follow the work of Nachtergaele and Sims \cite{Nachtergaele2004}. The class of systems that we allow will be much wider than those considered in the previous chapter. Again, we will consider a set $\Gamma$ of sites, together with a metric $d$ on the set of sites. At each site $x \in \Gamma$ we have a finite dimensional system, as before. We allow the dimension to vary from site to site, as long as there is a (finite) upper bound $N$ to this dimension. This condition can be dropped for suitable interactions, but it will complicate the proofs.

The class of interactions that we consider will be bigger than before as well. In particular, we define the following norm of an interaction $\Phi$:

$$
\|\Phi\|_\lambda = \sup_{x \in \Gamma} \sum_{A \ni x} |A| \|\Phi(A)\| N^{|A|} e^{\lambda \text{diam}(A)},
$$

\hspace{1cm} (4.1.1)
where \( \lambda > 0 \) is some constant. We consider all such interactions for which \( \| \Phi \|_{\lambda} < \infty \). Note that this includes all bounded interactions of finite range.

We will continuously need to estimate norms of commutators. For this reason, it will be convenient to introduce the shorthand

\[
C_{A,B}(x,t) = [\alpha_t(A), B],
\]

where \( x \in \Gamma, A \in \mathfrak{A}(\{x\}) \) and \( B \in \mathfrak{A} \). Similarly we define

\[
C_B(x,t) = \sup_{A \in \mathfrak{A}(\{x\})} \frac{\|C_{A,B}(x,t)\|}{\|A\|},
\]

that is, the maximum value of \( C_{A,B}(x,t) \), taking into account irrelevant rescaling of the operator \( A \). The bound that we will prove will give an estimate on the norm of \( C_B(x,t) \). In particular, this norm will decrease exponentially as the distance between \( x \) and the support of \( B \) becomes bigger.

**Theorem 4.1.1 (Lieb-Robinson).** Let \( \lambda > 0 \). Then for all interactions \( \Phi \) with \( \| \Phi \|_{\lambda} < \infty \), sites \( x \in \Gamma \) and \( t \in \mathbb{R} \) and \( B \in \mathfrak{A} \), we have the bound

\[
C_B(x,t) \leq e^{2|t|\|\Phi\|_{\lambda}} C_B(x,0) + \sum_{y \in \Gamma, y \neq x} e^{-\lambda d(x,y)} \left(e^{2|t|\|\Phi\|_{\lambda}} - 1\right) C_B(y,0).
\]

(4.1.2)

Before we go into the proof, we discuss some specific forms of the bounds. A special and very useful case is when the operator \( B \) is strictly local, so that \( B \in \mathfrak{A}(\Lambda) \) for some \( \Lambda \in \mathcal{P}_f(\Gamma) \). If in addition \( x \notin \Lambda \), it follows by locality that \( C_B(x,0) = 0 \). Hence the first term in the bound can be dropped. The second term can be estimated as well, as is outlined below.

Another useful observation is that it is easy to extend the bounds to (local) operators \( A \) that are supported on more than one site, lets say on the finite set \( \Lambda \). Indeed, note that \( A \) can be written in the form

\[
A = \sum_{i_1,\ldots,i_k} \lambda_{i_1,\ldots,i_k} E_{i_1} \cdots E_{i_k},
\]

(4.1.3)

where the \( \lambda \) are complex coefficients and where for each \( l \), \( \{E_{i_l}\} \) is a set of matrix units of \( \mathfrak{A}(\{l\}) \) with \( l \in \Lambda \). For example, one can take the matrices that have a one in exactly one entry, and zeros elsewhere. Note that by the uniform bound on the dimension of the Hilbert spaces at each site, there are at most \( N^{2|\Lambda|} \) terms in the sum. We can then repeatedly apply the inequality

\[
\|[A_1 A_2, B]\| \leq ||A_1||\|A_2 B\| + ||A_2||\|A_1 B\|,
\]

which follows from the triangle inequality. Taking also the summations into
account, we can make the following estimates:

\[
\| [\alpha_t(A), B] \| = \left\| \sum_{i_1, \ldots, i_k} \lambda_{i_1, \ldots, i_k} [\alpha_t(E_{i_1}) \cdots \alpha_t(E_{i_k}), B] \right\|
\]

\[
\leq \| A \| \sum_{i_1, \ldots, i_k} \| [\alpha_t(E_{i_1}) \cdots \alpha_t(E_{i_k}), B] \|
\]

\[
\leq \| A \| \sum_{i_1, \ldots, i_k} \sum_{\lambda=1}^k \| \alpha_t(E_{i_\lambda}), B] \|
\]

\[
\leq \| A \| N^{2|\Lambda|} \sum_{\lambda \in \Lambda} C_B(\lambda, t).
\]

Here we used that \( \| E_{i_\lambda} \| = 1 \) in the second line, and the Lieb-Robinson theorem in the last line.

Next, if \( B \) is also strictly local, we can find a more explicit bound for \( C_B(x, t) \). In particular, we can bound the right-hand side of equation (4.1.2) as follows. First, note that by locality \( [A, B] = 0 \) if the supports of \( A \) and \( B \) are disjoint. Hence \( C_B(y, 0) = 0 \) if \( y \notin \text{supp}(B) \). If \( y \in \text{supp}(B) \), we can use the trivial bound on the commutator, and we obtain \( C_B(y, 0) \leq 2 \| B \| \chi_{\text{supp}(B)}(y) \). Here \( \chi_{\text{supp}(B)} \) is the indicator function of the set \( \text{supp}(B) \). Also note that \( \exp(-\lambda d(x, y)) \leq \exp(-\lambda d(x, \text{supp}(B))) \) if \( y \in \text{supp}(B) \). Hence we can conclude that

\[
C_B(x, t) \leq e^{2|t|\Phi(A)}C_B(x, 0) + 2|\text{supp}(B)| \| B \| e^{-\lambda d(x, \text{supp}(B))} \left( e^{2|t|\Phi(A)} - 1 \right).
\]

We might as well estimate the \( e^{2|t|\Phi(A)} - 1 \) term by dropping the \(-1\), and take all the terms together. Combining these results we obtain the following corollary.

**Corollary 4.1.2.** Consider the same setting as in Theorem 4.1.1 and suppose that \( A \in \mathfrak{A}(\Lambda_1) \) and \( B \in \mathfrak{A}(\Lambda_2) \) are local operators. Then we have the bound

\[
\| [\alpha_t(A), B] \| \leq 4 \| A \| \| B \| \| A_1 \| \| A_2 \| N^{2|\Lambda|} e^{2|t|\Phi(A)} - \lambda d(\Lambda_1, \Lambda_2).
\]

This is a very rough bound, and with some extra work (and assumptions on the metric), it is possible to improve on it. For example, if \( \Lambda_1 \cap \Lambda_2 = \emptyset \), then we can use \( e^{2|t|\Phi(A)} - 1 \) instead of \( e^{2|t|\Phi(A)} \). We will later discuss further improvements of this bound (at the expense of some mild additional assumptions), but first we will give the proof of Theorem 4.1.1. As mentioned, it largely follows [15], which in turn is based on the proof given in [8], the main difference being that no Fourier theory techniques are no longer used, to be able to dispense of the translational invariance requirement.

**Proof of Theorem 4.1.1.** Let \( x \in \Gamma \) and \( A \in \mathfrak{A}(\{x\}) \) and suppose that \( B \in \mathfrak{A} \). We first look at \( C_{A, B}(x, t) \). Note that

\[
[\alpha_t(A), B] - [A, B] = \int_0^t \frac{d}{ds} [\alpha_s(A), B] ds = \int_0^t [\alpha_s(\delta(A)), B] ds,
\]
by the fundamental theorem of calculus. We also used that the derivative at 
$t = 0$ of $\alpha_t(A)$ is equal to $\delta(A)$, and the one-parameter group property of $\alpha_t$.
Hence, writing this out we obtain

$$C_{A,B}(x,t) = C_{A,B}(x,0) + i \sum_{\Lambda \ni x} \int_0^t [\alpha_s([\Phi(\Lambda), A]), B]ds. \quad (4.1.4)$$

To get a bound for $C_B(x,t)$ we can take the norm of this expression, and 
repeatedly use the triangle inequality, to obtain

$$C_B(x,t) \leq C_B(x,0) + \sum_{\Lambda \ni x} \int_0^{|t|} \sup_{A \in \mathfrak{A}(\{x\})} \|\alpha_s([\Phi(\Lambda), A]), B]\|ds. \quad (4.1.5)$$

We want to estimate the terms under the integral sign. First note that if 
$\Lambda \in \mathcal{P}_f(\Gamma)$ and $x \in \Lambda$, then $[\Phi(\Lambda), A]$ is an element of $\mathfrak{A}(\Lambda)$ (and if $x \notin \Lambda$ it is 
zero, by locality). We can then again write it in terms of matrix units, as 
we did before in equation (4.1.3). Note that for the coefficients we must have 
$|\lambda_{i_1...i_k}| \leq 2 \|\Phi(\Lambda)\|\|A\|$. We also have that 
$\|\alpha_t(E_{i_1} \cdots E_{i_k}), B\| \leq \sum_{x \in \Lambda} C_B(x,t)$,

since $\|E_{ij}\| = 1$. Combining everything with equation (4.1.5), we arrive at the bound

$$C_B(x,t) \leq C_B(x,0) + 2 \sum_{\Lambda \ni x} \|\Phi(\Lambda)\|N^{2|\Lambda|} \int_0^{|t|} \sum_{y \in \Lambda} C_B(y,s)ds. \quad (4.1.6)$$

Note that $C_B(x,t)$ appears both on the left and the right hand side. Hence we 
can again apply the inequality to the term under the integral, and so on. We 
first introduce the following shorthand notation:

$$\varphi(x,y) := \sum_{\Lambda \ni x,y} \|\Phi(\Lambda)\|N^{2|\Lambda|}.$$

Note that the sum is over all finite sets $\Lambda$ that contain both the point $x$ and $y$.
With this notation we can rewrite the bound for $C_B(x,t)$ as

$$C_B(x,t) \leq C_B(x,0) + 2 \int_0^{|t|} \sum_{y \in \Gamma} \varphi(x,y)C_B(y,s)ds. \quad (4.1.6)$$

We can then iterate this expression, and calculate the integrals over the constants terms, to get the following series in $|t|$:

$$C_B(x,t) \leq C_B(x,0) + 2|t| \sum_{y \in \Gamma} \varphi(x,y)C_B(y,0)$$

$$+ \frac{(2|t|)^2}{2} \sum_{y, y' \in \Gamma} \varphi(y, y') \sum_{y'' \in \Gamma} \varphi(y', y'')C_B(y'', 0) +$$

$$+ \frac{(2|t|)^3}{3!} \sum_{y, y' \in \Gamma} \varphi(y, y') \sum_{y'', y''' \in \Gamma} \varphi(y', y'') \varphi(y'', y''')C_B(y''', 0) + \ldots.$$
Now set \( \varphi_\lambda(x, y) = e^{\lambda d(x, y)} \varphi(x, y) \). The idea is to first rewrite the term above in terms of \( \varphi_\lambda \). To get the idea, we will do this explicitly for the term proportional to \( |t|^2 \), the other terms are similar. We have

\[
\sum_{y \in \Gamma} \sum_{y' \in \Gamma} \varphi(x, y) \varphi(y, y') C_B(y', 0)
= \sum_{y \in \Gamma} \sum_{y' \in \Gamma} e^{-\lambda(d(x, y) + d(y, y'))} \varphi_\lambda(x, y) \varphi_\lambda(y, y') C_B(y', 0)
\leq \sum_{y \in \Gamma} \sum_{y' \in \Gamma} e^{-\lambda d(x, y')} \varphi_\lambda(x, y) \varphi_\lambda(y, y') C_B(y', 0)
= \sum_{y \in \Gamma} \sum_{y' \in \Gamma} e^{-\lambda d(x, y)} \varphi_\lambda(x, y') \varphi_\lambda(y', y) C_B(y, 0)
\]

where we used the triangle inequality to conclude that \( d(x, y') \leq d(x, y) + d(y, y') \). Doing a similar rewriting with the other terms, we arrive at the following bound

\[
C_B(x, t) \leq \sum_{y \in \Gamma} e^{-\lambda d(x, y)} f(x, y) C_B(y, 0),
\]

where \( f(x, y) \) is the function

\[
f(x, y) = \delta_{x, y} + 2|t| \varphi_\lambda(x, y) + \frac{(2|t|)^2}{2} \sum_{y' \in \Gamma} \varphi_\lambda(x, y') \varphi_\lambda(y', y) + \frac{(2|t|)^3}{3!} \sum_{y'', y' \in \Gamma} \varphi_\lambda(x, y'') \varphi_\lambda(y'', y') \varphi_\lambda(y', y) + \ldots.
\]

Note that we see that the exponential decay becomes clear from the bound now. It remains to estimate the sums over \( \varphi_\lambda \). Note that for each \( x \in \Gamma \),

\[
\sum_{y \in \Gamma} \varphi_\lambda(x, y) = \sum_{y \in \Gamma} \sum_{A \ni x, y} \|\Phi(A)\| N^2 |A| e^{\lambda d(x, y)}
\leq \sum_{A \ni x} \sum_{y \in A} \|\Phi(A)\| N^2 |A| e^{\lambda \text{diam}(A)}
= \sum_{A \ni x} |A| \|\Phi(A)\| N^2 |A| e^{\lambda \text{diam}(A)} \leq \|\Phi\|_\lambda,
\]

where we used the definition of \( \|\Phi\|_\lambda \) in the last step. Using this inequality we can estimate \( f(x, y) \) by

\[
f(x, y) \leq \delta_{x, y} + (e^{2|t|} - 1)\|\Phi\|_\lambda.
\]

Plugging this in the expression for \( C_B(x, t) \) we finally obtain

\[
C_B(x, t) \leq e^{2|t|\|\Phi\|_\lambda} C_B(x, 0) + \sum_{y \in \Gamma, y \neq x} e^{-\lambda d(x, y)} (e^{2|t|\|\Phi\|_\lambda} - 1) C_B(y, 0).
\]

This concludes the proof. \( \square \)
4.2 Locality of time evolution

We are interested in the time evolution of local observables. As the time evolves, in general the size of the support will spread out. But if Lieb-Robinson bounds apply one expects that this spread will be limited. Such considerations are important, for example, when one wants to build quantum computers based on large quantum mechanical systems. For example, if one performs some operation on a small part of the system, it is important to know how long it will take before another distant part of the system is affected by this local operation. In relativistic systems the limit is given by the speed of light. So in a sense, Lieb-Robinson bounds give an analogue of the speed of light in quantum lattice systems.

To answer this question we first consider the following question. Consider an operator \( A \) supported on some finite set \( \Lambda \). Then for any operator \( B \) whose support is disjoint from \( \Lambda \), we have that \( [A, B] = 0 \). Conversely, if \( A \) is any operator and \( [A, B] = 0 \) for any \( B \in \mathfrak{A}(\Lambda^c) \) for some finite set \( \Lambda \), then it follows that \( A \in \mathfrak{A}(\Lambda) \) (see [61, Prop. IV.1.6]). In practice, strict commutativity is a bit too much to ask, and will rule out the dynamics generated by many interesting interactions. So instead of strict commutation, we assume only that the norm of the commutator is small. The effect is that although \( A \) need not be strictly local any more, it can be approximated by an operator \( A' \) that is localised in \( \Lambda \). This is the content of the following lemma [44].

Lemma 4.2.1. Let \( \Lambda \in \mathcal{P}_f(\Gamma) \). Suppose that \( \varepsilon > 0 \) and \( A \in \mathfrak{A} \) are such that

\[
\| [A, B] \| \leq \varepsilon \| A \| \| B \| \quad \text{for all } B \in \mathfrak{A}(\Lambda^c).
\]

Then it follows that there is some \( A_\Lambda \in \mathfrak{A}(\Lambda) \) such that \( \| A_\Lambda - A \| \leq \varepsilon \| A \| \).

We will not give the proof of this statement, but instead consider as an example the simpler case of finite systems. More precisely, we assume that we are given a Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) where \( \mathcal{H}_2 \) is finite dimensional. The observable algebra is \( \mathfrak{A} = \mathfrak{B}(\mathcal{H}) = \mathfrak{B}(\mathcal{H}_1) \otimes \mathfrak{B}(\mathcal{H}_2) \). Suppose that we are given some \( A \in \mathfrak{A} \) and \( \varepsilon > 0 \) such that \( \| [A, I \otimes B] \| \leq \varepsilon \| A \| \| B \| \) for all \( B \in I \otimes \mathfrak{B}(\mathcal{H}_2) \).

The goal is to approximate \( A \) by some operator \( A' \in \mathfrak{B}(\mathcal{H}_1) \otimes I \). To achieve this, define \( A' \) by

\[
A' = \int_{U(\mathcal{H}_2)} (I \otimes U)A(I \otimes U)^* dU,
\]

where the integral is over the group of unitaries in \( \mathfrak{B}(\mathcal{H}_2) \) and the integral is with respect to the Haar measure of this group. This measure has the property that the substitutions \( U \mapsto UV \) and \( U \mapsto VU \) (with \( V \) also a unitary) leave the integral invariant. Hence if \( V \in \mathfrak{B}(\mathcal{H}_2) \), we have

\[
(I \otimes V)A' = \int_{U(\mathcal{H}_2)} (I \otimes VU)A(I \otimes U)^* dU
= \int_{U(\mathcal{H}_2)} (I \otimes U)A(I \otimes V^* U)^* dU = A'(I \otimes V),
\]

hence \( A' \in (I \otimes \mathfrak{B}(\mathcal{H}_2))' = \mathfrak{B}(\mathcal{H}_1) \otimes I \).
Finally we show that $A'$ indeed approximates $A$. Indeed, note that
\[
\|A' - A\| = \int_{\mathcal{U}(\mathcal{H}_2)} \|I \otimes U, A(I \otimes U)^* \| \leq \varepsilon \int_{\mathcal{U}(\mathcal{H}_2)} \|A\|dU = \varepsilon \|A\|,
\]
since $\|U\| = 1$ for any unitary $U$.

Now suppose that $t \in \mathbb{R}$ and $A \in \mathfrak{A}(\Lambda)$ for some $\Lambda \in \mathcal{P}_f(\Gamma)$. Moreover we suppose that the time evolution $\alpha_t$ comes from some interaction satisfying the requirements for Theorem 4.1.1. The goal is to show that $\alpha_t(A)$ can be approximated by a local observable with support on a set that is not too much bigger than $\Lambda$. Obviously, the support will generally grow bigger as $t$ becomes bigger, but we are interested in the case of fixed $t$. The idea is to use Lemma 4.2.1 so we must have a bound on commutators of that form. Corollary 4.1.2 gives such a bound, but the problem is that it depends on the support of the operator $B$, while the Lemma requires the bound to hold for all $B$, independent of the size of the support. Hence as a first step we will improve the bound in Corollary 4.1.2.

Recall that the site of the support of $B$ comes into the bound in Corollary 4.1.2 by estimating the summation in the expression $C_B(x, t)$. We proceed as before, and as a first step recall the bound
\[
C_B(x, t) \leq 2\|B\| \sum_{y \in \text{supp}(B)} e^{-\lambda(d(x,y) + 2|t|\|\Phi\|_\Lambda)}.
\]
We estimated the sum by noting that $d(x,y) \geq d(x,\text{supp}(B))$. This is of course a rough estimate, since the effect of the sites in the support of $B$ decays exponentially with the distance to $x$. Therefore we will make a more careful estimate. In particular, we can estimate the above expression as
\[
C_B(x, t) \leq 2\|B\| \sum_{k=0}^{\infty} e^{-\lambda(d(\text{supp}(B)) + k + 2|t|\|\Phi\|_\Lambda)} |\text{supp}(B) \cap (B_{k+1}(x) \setminus B_k(x))|,
\]
where $B_k(x)$ is the ball of size $k$ around $x$. That is, we break up the support of $B$ into bigger and bigger annuli of width 1. If the amount of points in such an annulus does not grow exponentially (or faster) as $k$ becomes bigger, the sum can be bounded independent of the support of $B$. This is for example the case if $\Gamma = \mathbb{Z}^d$ with the usual metric. As an example we can consider $\Gamma = \mathbb{Z}$. The more general setting is discussed in, for example, [43, 47]. Note that in the case $\Gamma = \mathbb{Z}$, if we increase the diameter of the ball $B_k(x)$ by one, we add two additional points. Hence in that case we can estimate
\[
C_B(x, t) \leq 4\|B\| e^{2|t|\|\Phi\|_\Lambda} \sum_{k=0}^{\infty} e^{-k\lambda}.
\]
Since $\lambda > 0$ the sum converges to $1/(1 - \exp(-\lambda))$. As mentioned before, a similar result holds for $\Gamma = \mathbb{Z}^d$. With this bound we find the following improved version of Corollary 4.1.2.

**Corollary 4.2.2.** Consider the same setting as in Theorem 4.1.1 with the addition that $\Gamma = \mathbb{Z}^d$. Suppose that $A \in \mathfrak{A}(\Lambda)$ and $B \in \mathfrak{A}$. Then there is a constant $C > 0$ such that
\[
\|\alpha_t(A), B\| \leq C\|A\|\|B\|\|\Lambda\| N^2 |\Lambda| e^{2|t|\|\Phi\|_\Lambda}.
\]
Using similar considerations it is also possible to get rid of the proportionality to $|\Lambda|$, but this will not be necessary for our purposes.

Now choose an $\varepsilon > 0$, $t \in \mathbb{R}$ and $A \in \mathcal{F}(\Lambda)$ for some finite set $\Lambda$. Let $C$ be as in the Corollary above. Choose $d_{\text{min}}$ such that

$$\varepsilon > C|\Lambda|N^2 |\Lambda|e^{2|t|\|\Phi\|_\lambda \gamma}.$$

Then for any $B$ whose support is at least a distance $d_{\text{min}}$ from $\Lambda$, we have the bound $\|[\alpha_t(A), B]\| < \varepsilon \|A\|\|B\|$. Hence by Lemma 4.2.1 it follows that $\alpha_t(A) \in \mathcal{F}(\Lambda')$ where $\Lambda' = \bigcup_{x \in \Lambda} B_{d_{\text{min}}}(x)$. Also note that $d_{\text{min}}$ essentially scales as $2|t|\|\Phi\|_\lambda$. Hence $2\|\Phi\|_\lambda$ gives a bound on the velocity with which the support of the time-evolved operator grows: it can be interpreted as the “group velocity” or “speed of sound” in the system.

### 4.3 Exponential decay of correlations

In relativistic quantum field theories with a mass gap, one has that correlations decay exponentially [20]. Essentially this means that if $\omega$ is the ground state, $|\omega(AB) - \omega(A)\omega(B)|$ is proportional to $\exp(-\gamma d(A, B))$, where $\gamma$ is some constant and $d(A, B)$ is the distance between the supports of $A$ and $B$. The mass gap property here means that there are only massive particles in the theory, and the lightest particle has mass $m_0$. The constant $\gamma$ in the bound depends on the speed of light, which as mentioned before, also gives rise to a Lieb-Robinson bound. A natural question is then if in the context of lattice systems, it is possible to use the Lieb-Robinson bounds we have derived here as a substitute for the speed of light, and prove an analogue of the exponential clustering theorem. This is indeed the case [45], although its proof is highly non-trivial.

To state the result we first need to give an analogue of the mass gap in the present context. This is done in terms of the spectrum of a Hamiltonian $H$. Recall that generally the Hamiltonian is a bounded operator, hence we have to be a bit more careful on how the spectrum is defined. Note that $H$ is a linear map from its domain $D(H)$ to some Hilbert space $\mathcal{H}$. The same is true for the linear map $H + \lambda I$ for some $\lambda \in \mathbb{C}$. If this map is invertible, and its inverse is a bounded linear map, then we say that $\lambda$ is in the resolvent of $H$. The spectrum is the complement (in $\mathbb{C}$) of the resolvent. We denote the spectrum by $\text{spec}(H)$. Unlike the case of bounded operators, the spectrum of an unbounded operator is generally not compact. It may also be empty, or all of $\mathbb{C}$ for example.

Note that a Hamiltonian is a positive operator. In particular this implies that its spectrum is contained in the positive real line. With the normalisation $H\Omega = 0$ we see that 0 is also in the spectrum. We then say that a Hamiltonian $H$ is gapped if there is some $\gamma > 0$ such that

$$\text{spec}(H) \cap (0, \gamma) = \emptyset.$$

This means that there are no excited states with an energy below $\gamma$. For systems with such gapped Hamiltonians one can show that exponential clustering holds. Here we state a recent version [46].

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1This property can be expressed in terms of the joint spectrum of the generators of translations in the vacuum (that is, the momentum operators). The condition is then that (expect for 0, which corresponds to the vacuum), the spectrum lies on or above the mass shell with mass $m_0$ in the forward lightcone.
Theorem 4.3.1 (Exponential clustering). Let $\Phi$ be an interaction such that $\|\Phi\|_\lambda < \infty$. Moreover, suppose that there is a unique ground state $\omega$ for the corresponding dynamics and the Hamiltonian $H$ in the ground state representation has a gap $\gamma > 0$. Then there is some constant $\mu > 0$ such that

$$|\omega(AB) - \omega(A)\omega(B)| \leq C(A, B, \gamma)e^{-\mu d(\Lambda_1, \Lambda_2)}$$

for all disjoint $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\Gamma)$ and $A \in \mathcal{A}(\Lambda_1), B \in \mathcal{A}(\Lambda_2)$, and $C(A, B, \gamma)$ is given by

$$C(A, B, \gamma) = \|A\|\|B\| \left(1 + \frac{1}{\mu d(\Lambda_1, \Lambda_2)} + c \min(|\partial \Phi(\Lambda_1)|, |\partial \Phi(\Lambda_2)|)\right),$$

for some constant $c$ (which depends on the lattice structure and the interaction).

Here $\partial \Phi(X)$ is the set of all $x \in X$, such that there is some $\Lambda \ni x$ such that $\Lambda \cap X^c \neq \emptyset$ and $\Phi(\Lambda) \neq 0$. In other words, it consists of all points such that there is some interaction term across the boundary of $X$. The uniqueness condition on the ground state can be dropped, but one has to take a bit more care in that case. We also note that one can in fact give an explicit expression for $\mu$ in terms of the gap and the interaction.
Where to go from here?

So far we have discussed the basic framework of the mathematical treatment of quantum systems with many degrees of freedom. We have only covered a small part of the literature, and certainly many important results and developments have been omitted. In this chapter we will briefly point out some of these topics. Clearly, this will be by no means a survey of the entire field. Rather, biased by the interests of the author, we pick out two specific topics: algebraic quantum field theory and the effect of local perturbations.

With respect to the topics discussed in these lecture notes, there are many important subjects left virtually untouched. With respect to the operator algebra, we have discussed only $C^*$-algebras. For physical applications, the von Neumann algebras, which form a special class of operator algebras, play an important role. Also the so-called Tomita-Takesaki theory comes into play naturally when discussing KMS states.

Another important topic is the simulation of quantum many body systems. Simulating such systems is hard in general, essentially because the dimension of the Hilbert space grows exponentially in the number of sites. Hence it takes a huge amount of resources to simulate such systems, even for relatively small system sizes. The techniques we have discussed here are not of much help there. However, one can look for certain subsets of states for which this problem is easier. One important example are the finitely correlated states or matrix product states [19]. Such states admit a compact description that could in principle be stored on a computer. This make them much more amenable to simulations that an arbitrary state on an infinite lattice. The review [68] discusses many applications, and since this review has appeared, new and improved applications have been found.

There are many more structural results on the type of models that we studied. For example, we have only briefly discussed the role of symmetries. Symmetries are often a very helpful tool in studying physics. For example, they can be used to discuss order and disorder in large systems. This is related to symmetry breaking: it is possible that even though a Hamiltonian admits a certain symmetry, this symmetry is broken in the ground state [60]. This is the mechanism that is behind the Higgs boson, but it can also appear in quantum spin systems. Symmetries can also be used to decompose invariant states into extremal invariant states, which cannot be written as a combination of distinct invariant states. This is called \textit{ergodic decomposition}. This helps in studying, for example, the structure of the set of translation invariant KMS states.
5.1 Algebraic quantum field theory

Quantum field theory (QFT) is arguably one of the most successful theories of the last century. Not withstanding the huge success of the “traditional” (mainly perturbative) methods used by physicists working in quantum field theory, these are unsatisfactory from a mathematical viewpoint, because many concepts are mathematically ill-defined. Some aspects can be made rigorous (the reader can consult, for example, the book by Glimm and Jaffe [24]), but there are still many problems. In order to study QFT in a rigorous mathematical framework, it is desirable to have an axiomatic basis for QFT as a starting point.

One such axiomatisation is given by the Wightman axioms which, in a nutshell, postulate that quantum fields are given by operator valued distributions. The classic PCT, Spin and Statistics, and All That by Streater and Wightman remains a good introduction to this framework [62]. Although this approach is a natural one coming from “ordinary” quantum field theory, it also has some drawbacks. From a mathematical point of view, one has to deal with unbounded operators. At a more conceptual level there is the criticism that the quantum fields, which in general are not observables, are like coordinates, which should not be taken as the starting point of a theory.

An alternative is provided by what is called algebraic quantum field theory (AQFT), based on the Haag-Kastler axioms. This is the framework that we will use. In essence, the fundamental objects are nets of $C^*$-algebras of observables that can be measured in some finite region of space-time. At first sight it is perhaps surprising that in this approach one considers only bounded observables, since it is well known that the position and momentum operators for a single particle are unbounded. One should keep in mind, however, that in the physical world there are always limitations on the measuring equipment, and one can always only measure a bounded set of (eigen)values.

The two approaches are in fact not as unrelated as they might appear at first sight. Under certain conditions one can move from one framework to the other (and back). See, for instance [6], and references therein. While the Wightman axioms are closer to common practice in quantum field theory, the Haag-Kastler approach is easier to work with mathematically, since one does not have to deal with unbounded operator-valued distributions.

One of the earliest works on AQFT is by Haag and Kastler [29]. By now there is a large body of work. The monograph by Haag [25] and the book by Araki [5] are particularly recommended for a review of the physical and mathematical principles underlying this (operator) algebraic approach to quantum field theory. A review can also be found in [10]. The second edition of Streater and Wightman [62] also contains a short overview.

As argued in the introduction, there are two basic principles underlying the AQFT approach. First of all, it is the algebraic structure of the observables...
that is important. The second principle is locality: in relativistic QFT it makes sense to speak about observables that describe the physical properties localised in some region of space-time (for example $T \times S$, with $T$ a time interval and $S$ a bounded region of space, say a laboratory). Moreover, by Einstein causality one can argue that observables in spacelike separated regions are compatible in that they commute. As the basic regions we consider double cones $O$, defined as the intersection of (the interior of) a forward and backward light-cone. Note that a double cone is causally complete: $O = O''$, where a prime $'$ denotes taking the causal complement. To each double cone $O$ we associate a unital $C^*$-algebra $A(O)$ of observables localised in the region $O$. Finally, note that the Poincaré group $P^+_\uparrow$ (generated by translations and Lorentz transformations) acts on double cones. We write $g \cdot O$ for the image of a double cone under a transformation $g$.

The starting point of AQFT, then, is a map $O \mapsto A(O)$. There are a few natural properties the map $O \mapsto A(O)$ should have if it is to describe (observables in) quantum field theory. For example, anything that can be localised in $O$ can be localised in a bigger region as well. This leads to the following list of axioms, now known as the Haag-Kastler axioms.

1. **Isotony:** if $O_1 \subset O_2$ then there is an inclusion $i : A(O_1) \hookrightarrow A(O_2)$. We assume the inclusions are injective unital $*$-homomorphisms. Often the algebras are realised on the same Hilbert space, and we have $A(O_1) \subset A(O_2)$.

2. **Locality:** if $O_1$ is spacelike separated from $O_2$, then the associated local observable algebras $A(O_1)$ and $A(O_2)$ commute.

3. **Poincaré covariance:** there is a strongly continuous action $x \mapsto \beta_x$ of the Poincaré group $P^+_\uparrow$ on the algebra of observables, such that

$$\beta_g(A(O)) = A(g \cdot O).$$

We will always assume that the algebras $A(O)$ are non-trivial.

**Remark 5.1.1.** Instead of Poincaré covariance one sometimes requires the weaker condition of translation covariance, see for example [9].

Note that $O \mapsto A(O)$ is a net of $C^*$-algebras, just as the local algebras of spin systems we discussed in Section 3.2. Just as in one can form the union (or more precisely, the inductive limit) $A$ of quasi-local observables. If the local algebras are all realised on the same Hilbert space, this amounts to taking $A = \bigcup_O A(O)$, where the bar denotes closure with respect to the operator norm. The algebra $A$ is called the algebra of quasi-local observables. We will usually assume that the local algebras act as bounded operators on some Hilbert space. In that case, for an arbitrary (possibly unbounded) subset $S$ of Minkowski space, we set $A(S) := \bigcup_{O \subset S} A(O)\|\|$, where the union is over all double cones contained in $S$.

In fact, in practice one usually realises the net as a net of von Neumann algebras acting on some Hilbert space. Under physically reasonable assumptions the algebras $A(O)$ are Type III factors. These are a specific type of von Neumann algebras. See [74] for a discussion of the physical significance of this.
It should be noted that in this axiomatic approach some of the constructions of "conventional" quantum field theory can be discussed. For example, field operators, particle aspects and scattering theory can be defined in this setting. This approach is particularly suited to study structural properties of quantum field theory.

Vacuum representation

The vacuum plays a special role in quantum field theory. Intuitively, it describes empty space. Alternatively, it has minimal energy. To define the notion of a vacuum state rigorously, one first defines energy decreasing operators. The precise details are not important for us (see e.g. §4.2). In essence one considers operators of the form \( Q = \int f(x)\beta_x(A) d^4x \) for some observable \( A \) and smooth function \( f \) whose Fourier transform has support disjoint from the forward light-cone \( V^+ \). The \( \beta_x \) are the translation automorphisms as in the Haag-Kastler axioms. A vacuum state then essentially is a state \( \omega_0 \) on \( A \) such that \( \omega_0(Q^*Q) = 0 \) for any such \( Q \).

One can prove that a vacuum state is translation invariant. The corresponding vacuum representation, which will be denoted by \( \pi_0 \), is then translation covariant. That is, there is a unitary representation \( x \mapsto U(x) \) such that \( \pi_0(\alpha_x(A)) = U(x)\pi_0(A)U(x)^* \) defined by \( U(x)\pi_0(A)\Omega_0 = \pi_0(\alpha_x(A))\Omega_0 \) for any \( x \) in Minkowski space and \( A \in A \). These translations are generated by unbounded operators \( P_\mu \), which have the natural interpretations of energy \( (P_0) \) and momentum \( (P_i, \text{ with } i = 1, \ldots, d - 1) \). These operators mutually commute, hence one can consider their joint spectrum. This spectrum is in fact contained in the forward light-cone, as follows from the assumptions above on \( \omega_0(Q^*Q) = 0 \) (for suitable \( Q \)). This is interpreted as "positivity of the energy". Finally, if \( \pi_0 \) is irreducible, then the vacuum vector \( \Omega_0 \) is the unique (up to a scalar) translation invariant vector. In fact, any factorial vacuum representation is automatically irreducible.

Alternatively, a vacuum representation can be characterised as a translation covariant representation such that the spectrum of the generators of these translations is contained in the forward light-cone \( V^+ \). Moreover, 0 is in the point spectrum, since the vacuum vector is invariant. A special case is a massive vacuum representation. This is a vacuum representation where 0 is an isolated point in the spectrum and there is some \( m > 0 \) such that the spectrum is contained in \( \{0\} \cup \{p : p^2 \geq m^2, p_0 > 0\} \). That is, there is a mass gap.

Particle statistics

One of the highlights of algebraic quantum field theory is the Doplicher-Roberts theorem [17]. This theorem allows us to construct a net of field operators creating certain (particle) excitations. These field net operators have the commutation properties that one would expect. For example, the field-net operators corresponding to two fermions in spacelike separated regions anti-commute, while those for bosons commute. Before we can come to this theorem, we have to explain how we describe excitations, and how one can define their statistics. The idea is to find the different superselection sectors of the theory. As before, these correspond to equivalence classes of irreducible representations of the observable algebra \( A \). In general there are many inequivalent ones, and
most are not physically relevant. For example, we would only be interested in representations which describe a finite number of excitations. Therefore one has to introduce a criterion to select the relevant representations. One such criterion is the Doplicher-Haag-Roberts one \cite{15, 16}. Let $\pi_0$ be a vacuum representation as above. Let $\mathcal{O}$ be an arbitrary double cone and write $\mathcal{O}'$ for its causal complement. This induces an algebra $\mathfrak{A}(\mathcal{O}')$ of observables that can be measured inside $\mathcal{O}'$. The DHR criterion then considers all representations $\pi$ such that for each double cone $\mathcal{O}$ it holds that

$$\pi_0 \mid \mathfrak{A}(\mathcal{O}') \cong \pi \mid \mathfrak{A}(\mathcal{O}')$$ (5.1.1)$$

where the symbol $\mid$ means that we restrict the representations to the algebra $\mathfrak{A}(\mathcal{O}')$. Intuitively speaking, it selects only those excitations that look like the vacuum when one restricts to measurements outside a double cone. It is known that this criterion rules out physically relevant theories (for example, one can always detect an electrical charge by measuring the flux through a arbitrarily large sphere, by Gauss' law), but nevertheless this class shows many of the relevant features of a theory of superselection sectors. One can also relax the criterion. For example, Buchholz and Fredenhagen use “spacelike cones” instead of double cones, and show that in massive theories one always has such localisation properties \cite{9}. In recent work, a more general approach is suggested to deal with massless excitations \cite{11}, which lead to infrared problems in the approach outlined here.

The set of representations that satisfy the criterion (5.1.1) has a rich structure. For example, one can define a product operation $\pi_1 \times \pi_2$ on such representations. This new representation can be interpreted as first creating an excitation of type $\pi_2$, then one of $\pi_1$. The ground state representation acts as the identity with respect to this product, in the sense that $\pi \times \pi_0 \cong \pi$. In this way, one can think of $n$ (particle) excitations of the class $\pi$ located in spacelike separated regions. Such configurations can be obtained as vector states in the representation $\pi^\otimes n$. Moreover, there is a canonical way to define unitary operators that interchange the order in which the excitations are created. These operators are analogous to the transposition operator in, for example, a quantum system of $n$ particles. Applying this unitary to the state vector leaves the expectation values in the state invariant, but can change the state vector itself.

This enables a study of the exchange statistics of the excitations. The result is that the interchange of excitations is described by representations of the symmetric group, when the net of observable algebras is defined on Minkowski space with at least two spatial dimensions. This representation only depends on the equivalence class of the representation. In this way we obtain the well-known classification into bosons or fermions from first principles. In a seminal work \cite{17}, Doplicher and Roberts showed how one could, starting from the superselection sectors satisfying the criterion above, construct a net of field operators. These field operators can “create” excitations. As expected, for bosonic excitations the field operators located in spacelike regions commute, whereas for fermions they anti-commute.

In lower dimensional space-times an interesting phenomenon appears: excitations are not necessarily bosonic or fermionic anymore. For example, it

\footnote{There are some additional technical conditions necessary on the representation $\pi_0$, however, the most important one being Haag duality.}
may happen that the interchange of two particles leads to a non-trivial phase. In fact, even more general (non-abelian) effects are possible. Excitations with such properties (nowadays usually called *anyons*) are expected to appear as “emergent” quasi-particles in certain condensed matter systems. They play an important role in the quest of building a quantum computer that does not suffer too much from thermal noise introduced by the environment [49]. In the algebraic quantum field theory community the possibility of such “braided” statistics has been studied by many authors, see e.g. [21, 22, 23]. The name “braided” stems from the fact that one does not obtain a representation of the symmetric group any more, but rather of the so-called braid group.

5.2 Local perturbations

The final topic that we want to discuss is that of *local perturbations*. The question is as follows. Suppose that we have some quantum systems with dynamics given by some interaction $\Phi$. Then we have a set $S_\Phi$ of ground states with respect to this dynamics. Now suppose that we perturb the dynamics by a local operator, so that the new dynamics are generated by a derivation

$$\delta(A) := \delta_\Phi(A) + i[V, A],$$

where $V$ is some local, self-adjoint operator and $\delta_\Phi$ is the derivation induced by $\Phi$. The question then is, what is the effect on the space of ground states? Is the effect local in the sense that the ground states of the perturbed system look the same as those of the unperturbed ones, when sufficiently far away from the localisation of the perturbation? Here we discuss some rigorous results by Bachmann, Michalakis, Nachtergaele and Sims on this problem [5].

The precise setting is as follows. Suppose that we have defined some dynamics $\alpha_t$. Consider a ground state for these dynamics. Then there is some Hamiltonian $H(0)$ implementing the dynamics $\alpha_t$ in the GNS representation. We now assume that the perturbed dynamics are of the form $H(s) = H(0) + \Phi(s)$, where $s \in [0, 1]$ and $\Phi(s) = \Phi(s)^\ast$. The dynamics should not change too much as a function of $s$, in the sense that there is some $M > 0$ such that $\|H'(s)\| \leq M$ for all $s \in [0, 1]$. We will also assume that there is a gap $\gamma > 0$ in the spectrum for all $s$, where $\gamma$ does not depend on $s$. In particular, we assume that

$$\text{spec } H(s) = \Sigma_1(s) + \Sigma_2(s)$$

for some sets $\Sigma_1(s)$ and $\Sigma_2(s)$ such that $\inf_{s \in [0, 1]} d(\Sigma_1(s), \Sigma_2(s)) \geq \gamma$. There is an additional (technical) smoothness condition that we need to demand. For its precise formulation, see [5, Assumption 2.1].

Now write $P(s)$ for the spectral projection on $\Sigma(s)$ \footnote{Recall that in finite dimensions this is the projection onto the corresponding eigenspaces.} Then under the conditions stated above it turns out that the projections are related by

$$P(s) = U(s)P(0)U(s)^\ast$$

(5.2.1)

for some norm-continuous path $U(s)$ of unitaries. It can be shown that these unitaries can be obtained as the unique solution of a linear differential equation [5, Prop. 2.4].
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Next we make a locality assumption on the perturbation $\Phi(s)$. Namely, assume that there is a $\Lambda \in \mathcal{P}(\Gamma)$ such that $\Phi(s) \in \mathfrak{A}(\Lambda)$ for all $s \in [0,1]$. Note that this means that $\Phi(s) \in \mathfrak{A}(\Lambda)$ up to a constant operator. Finally, we assume that there is a Lieb-Robinson bound for the perturbed dynamics. More precisely, there should exist constants $\mu > 0, C > 0$ and $v > 0$ such that for local operators $A$ and $B$ we have

$$
\|\alpha_t^{H(s)}(A), B\| \leq C\|A\|\|B\| \min(\|\text{supp}(A)\|, |\text{supp}(B)|) e^{-\mu(d(\text{supp}(A), \text{ supp}(B) - v|t|)}
$$

for all $s \in [0,1]$. Here $\alpha_t^{H(s)}(A) = e^{itH(s)}Ae^{-itH(s)}$, the dynamics generated by $H(s)$.

Under these assumptions one can show that the unitaries $U(s)$ can be approximated by local unitaries. These local unitaries are supported on sets containing $\Lambda$, and as expected, the bigger this set is, the better the approximation will be. In particular, define the following sets for $R > 0$:

$$
\Lambda_R = \{ y \in \Gamma : \exists x \in \Lambda \text{ such that } d(x, y) < R \}.
$$

With this notation we have the following result [3, Thm. 3.4]:

**Theorem 5.2.1.** There exists a subexponential function $G(R)$ and a constant $C$ such that for any $R > 0$ and for all $s \in [0,1]$, there exists a unitary $V_R(s)$ with support in $\Lambda_R$ such that

$$
\|U(s) - V_R(s)\| \leq CG \left( \frac{\gamma R}{2v} \right),
$$

where $\gamma$ and $v$ are as above.

One can show that for large enough $R$ the function $G$ is proportional to $\exp \left( -2/\sqrt{7\log(R)} \right)$.

As an example we suppose that for each $s$ we have a non-degenerate ground state, and hence it can be represented as a vector $\Omega(s)$ in Hilbert space. The ground state is separated from the rest of the spectrum by a gap of at least $\gamma$ for all $s$. Under this conditions we can show that far away from the perturbation $\Phi(s)$ the ground states look alike. In particular, consider $R > 0$ and $A \in \mathfrak{A}(\Lambda_R)$.

Then $[A, V_R(s)] = 0$ for all $s$ by locality. Moreover, note that each $P(s)$ is a one-dimensional projection because of the non-degeneracy of the ground state. This implies that $\Omega(s) = U(s)\Omega(0)$, with the help of equation [3.2.1]. Hence we have

$$
\|\langle \Omega(s) | A \Omega(s) \rangle - \langle \Omega(0) | A \Omega(0) \rangle \| = |\langle \Omega(0) | A(s) [A, U(s)] \Omega(0) \rangle |
$$

$$
= |\langle \Omega(0) | A(s) [A, V_R(s)] \Omega(0) \rangle + \langle \Omega(0) | U(s) ^* [A, U(s) - V_R(s)] \Omega(0) \rangle |
$$

$$
\leq 2\|A\|\|U(s) - V_R(s)\| \leq 2\|A\|CG \left( \frac{\gamma R}{2v} \right).
$$

Since the right hand side goes to zero as $R$ grows, we see that the states look the same far away from the perturbation.

In the same paper as in which the proof of the theorem above appeared, the authors applied similar techniques to address the question of "equivalence of phases". This is an important notion in the classification of topologically
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ordered phases. Roughly speaking, two states $\omega_0$ and $\omega_1$ are said to be in the same phase, if $\omega_0$ is the ground state of a gapped Hamiltonian $H(0)$, $\omega_1$ is the ground state of a gapped Hamiltonian $H(1)$, and there is a continuous family of gapped Hamiltonians $H(s)$ as above connecting the two Hamiltonians [12].

In the thermodynamic limit this situation can be described as follows. First consider for a finite $\Lambda \subset \Gamma$ the set $S_{\Lambda}(s)$ of all (mixtures of) states with energy in $I(s)$, where $I(s)$ is some interval containing $\Sigma_1(s)$ (and disjoint from $\Sigma_2(s)$). We can then consider an increasing sequence $\Lambda_n$ of subsets, and look at all the weak-$\star$ limit points as $n \to \infty$ of states $S_{\Lambda_n}(s)$. Note that this is the same procedure as we used before in the construction of KMS states from finite volume Gibbs states in Section 3.4. Then, for the local sets of states there are automorphisms $\alpha_{\Lambda_n}^s$ such that $S_{\Lambda_n}(s) = S_{\Lambda_n}(0) \circ \alpha_{\Lambda_n}^s$. The automorphisms $\alpha_{\Lambda_n}^s$ actually satisfy a Lieb-Robinson type of bound. This bound can be used to show that the thermodynamic of this sequence of automorphisms exists. This leads to the following theorem:

**Theorem 5.2.2.** There is an automorphism $\alpha_s$ such that $S(s) = S(0) \circ \alpha_s$. The automorphism $\alpha_s$ can be obtained from an $s$-dependent quasilocal interaction, and satisfies a Lieb-Robinson bound.
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