Analyticity of the Cauchy problem and persistence properties for a generalized Camassa-Holm equation

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Abstract

This paper is mainly concerned with the Cauchy problem for a generalized Camassa-Holm equation with analytic initial data. The analyticity of its solutions is proved in both variables, globally in space and locally in time. Then, we present a persistence property for strong solutions to the system. Finally, explicit asymptotic profiles illustrate the optimality of these results.

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1 Introduction

In this paper we consider the Cauchy problem for the following generalized Camassa-Holm equation,

\[
\begin{align*}
&\begin{cases}
    u_t - u_{txx} = \partial_x (2 + \partial_x)[(2 - \partial_x)u]^2, & t > 0, \\
    u(0, x) = u_0(x),
\end{cases} \\
\end{align*}
\tag{1.1}
\]

or equivalently

\[
\begin{align*}
&\begin{cases}
    m = u - u_{xx}, \\
    m_t = 2m^2 + (8u_x - 4u)m + (4u - 2u_x)m_x + 2(u + u_x)^2, & t > 0, \\
    m(0, x) = u(0, x) - u_{xx}(0, x) = m_0(x).
\end{cases} \\
\end{align*}
\tag{1.2}
\]

Note that \(G(x) = \frac{1}{2}e^{-|x|}\) and \(G(x) \ast f = (1 - \partial_x^2)^{-1}f\) for all \(f \in L^2(\mathbb{R})\) and \(G \ast m = u\). Then we can rewrite (1.1) as follows:

\[
\begin{align*}
&\begin{cases}
    u_t(t, x) = 4uu_x + G \ast [\partial_x(2u_x^2 + 6u^2) + \partial_x^2(u_x^2)] \\
    = 4uu_x - u_x^2 + G \ast [\partial_x(2u_x^2 + 6u^2) + u_x^2], & t > 0, \\
    u(0, x) = u_0(x).
\end{cases} \\
\end{align*}
\tag{1.3}
\]

The equation (1.1) was proposed recently by Novikov in [41]. It is integrable and belongs to the following class [41]:

\[
(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}),
\]

which has attracted much interest, particularly in the possible integrable members of (1.4).

The most celebrated integrable member of (1.4) is the well-known Camassa-Holm (CH) equation [5]:

\[
(1 - \partial_x^2)u_t = 3uu_x - 2u_xu_{xx} - uu_{xxx}. 
\tag{1.5}
\]

The CH equation can be regarded as a shallow water wave equation [5, 18]. It is completely integrable. It is completely integrable. That means that the system can be transformed into a linear flow at constant speed in suitable action-angle variables (in the sense of infinite-dimensional Hamiltonian systems), for a large class of initial data [5, 9, 19]. It also has a bi-Hamiltonian structure [8, 29], and admits exact peaked solitons of the form \(ce^{-|x-ct|}\) with \(c > 0\), which are orbitally stable [22]. It is worth mentioning that the peaked solitons present the characteristic for the travelling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [6, 11, 13, 17, 43].
The local well-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in [13, 14, 23, 42]. It was shown that there exist global strong solutions to the CH equation [10, 13, 14] and finite time blow-up strong solutions to the CH equation [10, 13, 14, 15]. The existence and uniqueness of global weak solutions to the CH equation were proved in [20, 48]. The global conservative and dissipative solutions of CH equation were investigated in [3, 4]. Finite propagation speed and persistence properties of solutions to the Camassa-Holm equation have been studied in [12, 33].

The second celebrated integrable member of (1.4) is the famous Degasperis-Procesi (DP) equation [25]:

\[(1 - \partial_x^2)u_t = 4uu_x - 3u_xu_{xx} - uu_{xxx}.\]  

The DP equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the CH shallow water equation [26]. The DP equation is integrable and has a bi-Hamiltonian structure [24]. An inverse scattering approach for the DP equation was presented in [21, 40]. Its traveling wave solutions was investigated in [36, 44].

The local well-posedness of the Cauchy problem of the DP equation in Sobolev spaces and Besov spaces was established in [30, 31, 52]. Similar to the CH equation, the DP equation has also global strong solutions [37, 53, 55] and finite time blow-up solutions [27, 28, 37, 38, 52, 53, 54, 55]. It also has global weak solutions [7, 27, 54, 55].

Although the DP equation is similar to the CH equation in several aspects, these two equations are truly different. One of the novel features of the DP different from the CH equation is that it has not only peakon solutions [24] and periodic peakon solutions [54], but also shock peakons [39] and the periodic shock waves [28].

The third celebrated integrable member of (1.4) is the known Novikov equation [41]:

\[(1 - \partial_x^2)u_t = 3uu_xu_{xx} + u^2u_{xxx} - 4u^2u_x.\]  

The most difference between the Novikov equation and the CH and DP equations is that the former one has cubic nonlinearity and the latter ones have quadratic nonlinearity.

It was showed that the Novikov equation is integrable, possesses a bi-Hamiltonian structure, and admits exact peakon solutions \(u(t, x) = \pm \sqrt{c}e^{\pm c|t - ct|}\) with \(c > 0\) [34].

The local well-posedness for the Novikov equation in Sobolev spaces and Besov spaces was studied in [46, 47, 50, 51]. The global existence of strong solutions were established in [46] under some sign conditions and the blow-up phenomena of the strong solutions were shown in [51]. The global weak solutions for the Novikov equation were studied in [45, 45].

Recently, the Cauchy problem of (1.1) in the Besov spaces \(B^{s}_{p,r}\), \(s > max\left\{\frac{1}{p}, \frac{1}{2}\right\}\) and critical Besov space \(B^{\frac{1}{2},1}_{2,1}\) has been studied in [56, 57]. To our best knowledge, analyticity and persistence properties of the Cauchy problem for (1.1) has not been studied yet. In this paper we first prove the analyticity of solutions to the system (1.1) in both variables, with \(x\) on
the line $\mathbb{R}$ and $t$ in an neighborhood of zero, provided that the initial data is analytic on $\mathbb{R}$. Furthermore, we show a persistence property of the strong solutions to (1.1). The analysis of the solutions in weighted spaces is useful to obtain information on their spatial asymptotic behavior.

The paper is organized as follows. In Section 2 we obtain the analytic solutions to the equation (1.1) on the line. In Section 3 we obtain a persistence result on solutions to (1.1) in the weight $L^p$ spaces $L^p, \phi := L^p(\mathbb{R}, \phi^p dx)$. In Section 4, we compute the spatial asymptotic profiles.

**Notations.** Since all space of functions in the following sections are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity.

## 2 Analytic solutions

Setting

$$ f(x) = x^2, \quad P_1 = \partial_x, \quad P_2 = \partial_x(1 - \partial_x^2)^{-1}, $$

we can rewrite (1.1) in the following form:

$$ \partial_t u = 2P_1(f(u)) + 2P_2(f(u_x)) + 6P_2(f(u)) + P_2P_1(f(u_x)). \tag{2.1} $$

Next, we can transform (2.1) into the following equation:

$$ \begin{align*}
\partial_t u_1 &= 2P_1(f(u_1)) + 2P_2(f(u_2)) + 6P_2(f(u_1)) + P_1P_2(f(u_2)) \\
&= P_1 \left( 2f(u_1) \right) + P_2 \left( 2f(u_2) + 6f(u_1) \right) + P_1P_2 \left( f(u_2) \right) \\
&= F_1(u_1, u_2), \\
\partial_t u_2 &= 4P_1(u_1u_2) - P_1(f(u_2)) + P_1P_2 \left( 2f(u_2) + 6f(u_1) \right) + P_2(f(u_2)) \\
&= F_2(u_1, u_2).
\end{align*} $$

Therefore we can get the following system from combining the above equation with our initial value:

$$ \begin{align*}
\partial_t u_1 &= F_1(u_1, u_2), \\
\partial_t u_2 &= F_2(u_1, u_2), \\
\quad u_1(0, x) &= u_0(x), \\
\quad u_1(0, x) &= \partial_x u_0(x).
\end{align*} \tag{2.2} $$
Define $U = (u_1, u_2)$ and $F(U) = F(u_1, u_2) = (F_1(u_1, u_2), F_2(u_1, u_2))$. Then we have

$$\begin{align*}
\frac{\partial U(t)}{\partial t} &= F(t, U(t)), \\
U(0) &= (u_0(x), \partial_x u_0(x)).
\end{align*}$$

(2.3)

Before stating our result, we need to introduce some suitable Banach spaces. For any $s > 0$, we define

$$E_s \triangleq \left\{ u \in C^\infty : \|u\|_s = \sup_{k \in \mathbb{N}} s^k \|\partial_x^k u\|_{H^1} < \infty \right\}.$$

Now we have the following analyticity result:

**Theorem 2.1.** Let $u_0 \in E_s$. There exist an $\varepsilon > 0$ and a unique solution $u$ of the Cauchy problem (1.3) that is analytic on $(-\varepsilon, \varepsilon) \times \mathbb{R}$.

We first study the properties of the space $E_s$. Let us recall some useful properties of these Banach spaces, as follows.

**Proposition 2.2.** [[1, 32, 49]]

1. $E_s$ equipped the norm $\|\cdot\|_s$ is a Banach space by the completeness of $H^1$ and the closedness of the differential operator $\partial_x$.
2. The functions in $E_s$ are real analytic on $\mathbb{R}$, namely, $E_s \subset C^\omega$.
3. For any $0 < s' < s$, $E_s \hookrightarrow E_{s'}$ with $\|u\|_{s'} \leq \|u\|_s$.
4. Let $s > 0$. There exists a constant $C > 0$, independent of $s$, such that for any $u$ and $v$ in $E_s$, we get

$$\|uv\|_s \leq C\|u\|_s \|v\|_s,$$

where $C = C(r)$ depends only on $r$. In particular, for any $s > 0$, we have

$$\|f(u) - f(v)\|_s = \|u^2 - v^2\|_s \leq C\|u + v\|_s \|u - v\|_s,$$

for any $u, v \in E_s$.
5. For any $0 < s' < s \leq 1$, we have

$$\|P_1 u\|_{s'} \leq \frac{1}{s - s'} \|u\|_s,$$  

(2.4)

$$\|P_2 u\|_s \leq \|u\|_s.$$  

(2.5)

Then Theorem 2.1 is a straightforward consequence of the following result.

**Theorem 2.3.** [[1, 32]] Let $(X_s, \|\cdot\|_s)_{0 < s \leq 1}$ be a scale of decreasing Banach spaces, such that $X_s \subset X_{s'}$ with $\|\cdot\|_{s'} \leq \|\cdot\|_s$ for any $0 < s' < s$. Consider the Cauchy problem

$$\begin{align*}
\frac{du}{dt} &= F(t, u(t)), \\
u(0) &= 0.
\end{align*}$$

(2.6)
Let \( T, \ R \) and \( C \) be positive numbers and suppose that \( F \) satisfies the following conditions:

(a) If for any \( 0 < s' < s < 1 \), the function \( t \mapsto u(t) \) is holomorphic on \( |t| < T \) and continuous on \( |t| \leq T \) with values in \( X_s \) and

\[
\sup_{|t| \leq T} ||u(t)||_s < R,
\]

then \( t \mapsto F(t, u(t)) \) is a holomorphic function on \( |t| < T \) with values in \( X_{s'} \).

(b) For any \( 0 < s' < s \leq 1 \) and any \( u, v \in B(0, R) \subset X_s \), that is, \( ||u||_s < R, \ ||v||_s < R \), we have

\[
\sup_{|t| \leq T} ||F(t, u) - F(t, v)||_{s'} \leq \frac{C}{s - s'} ||u - v||_s.
\]

(c) There exists a \( M > 0 \), such that for any \( 0 < s < 1 \),

\[
\sup_{|t| \leq T} ||F(t, 0)||_s \leq \frac{M}{1 - s}.
\]

Now we set

\[
||(u_1, u_2)||_s \triangleq \sum_{i=1}^{2} ||u_i||_s,
\]

and

\[
||F(u_1, u_2)||_s \triangleq \sum_{i=1}^{2} ||F_i(u_1, u_2)||_s.
\]

Proof. To complete the proof of Theorem 2.1 it suffices to verify the conditions (a) – (c) above.

We first focus on checking the conditions (a). To check the condition (a), we only need to derive that \( ||F(U)||_{s'} < \infty \), when \( \sup_{|t| \leq T} ||U||_s < \infty \).

For any \( u_j \in E_s, (j = 1, 2) \), we obtain

\[
||F(U)||_{s'} = ||F(u_1, u_2)||_{s'}
\]

\[
= ||F_1(u_1, u_2)||_{s'} + ||F_2(u_1, u_2)||_{s'}
\]

\[
= ||P_1(2f(u_1)) + P_2(2f(u_2) + 6f(u_1)) + P_1P_2(2f(u_2) + 6f(u_1))||_{s'}
\]

\[
+ ||4P_1(u_1u_2) - P_1(f(u_2)) + P_1P_2(2f(u_2) + 6f(u_1)) + P_2(f(u_2)))||_{s'}
\]

\[
= \frac{2}{s - s'} ||u_1^2||_s + 2||u_2^2||_{s'} + 6||u_1^2||_{s'} + \frac{1}{s - s'} ||u_2^2||_s
\]

\[
+ \frac{4}{s - s'} ||u_1u_2||_s + \frac{1}{s - s'} ||u_2^2||_s + \frac{2}{s - s'} ||u_2^2||_s + \frac{6}{s - s'} ||u_1^2||_s + ||u_2^2||_{s'}
\]

\[
\leq \frac{C}{s - s'} (||u_1||_s + ||u_2||_s)^2
\]

\[
\leq \frac{C}{s - s'} (||u_1, u_2||_s)^2,
\]
which means that system (2.3) with zero initial datum satisfies the condition (a). This completes the proof of the condition (a).

Next, we consider the condition (b).

For any \( u_j \) and \( v_j \in B(0, R) \subset E_s, \ (j = 1, 2) \), we get

\[
\| F(U) - F(V) \|_{s'} = \| F(u_1, u_2) - F(v_1, v_2) \|_{s'} \\
= \| F_1(u_1, u_2) - F_1(v_1, v_2) \|_{s'} + \| F_2(u_1, u_2) - F_2(v_1, v_2) \|_{s'} \\
\triangleq I_1 + I_2.
\]

Then, we will estimate \( I_1 \) and \( I_2 \) respectively. By Proposition [2.2] yields

\[
I_1 \leq \| 2P_1[f(u_1) - f(v_1)] \|_{s'} + \| 2P_2[f(u_2) - f(v_2)] \|_{s'} \\
+ 6\| P_2[f(u_1) - f(v_1)] \|_{s'} + \| P_1P_2[f(u_2) - f(v_2)] \|_{s'} \\
\leq \frac{2}{s - s'}\| f(u_1) - f(v_1) \|_s + 2\| f(u_2) - f(v_2) \|_s \\
+ 6\| f(u_1) - f(v_1) \|_{s'} + \frac{1}{s - s'}\| f(u_2) - f(v_2) \|_s \\
\leq \frac{2C}{s - s'}\| u_1 + v_1 \|_s\| u_1 - v_1 \|_s + 2C\| u_2 + v_2 \|_{s'}\| u_2 - v_2 \|_s' \\
+ 6C\| u_1 + v_1 \|_{s'}\| u_1 - v_1 \|_{s'} + \frac{C}{s - s'}\| u_2 + v_2 \|_s\| u_2 - v_2 \|_s \\
(2.7) \leq \frac{C(r, R)}{s - s'}\| (u_1, u_2) - (v_1, v_2) \|_s,
\]

\[
I_2 \leq \| 4P_1(u_1u_2 - v_1v_2) \|_{s'} + \| P_1[f(u_2) - f(v_2)] \|_{s'} \\
+ \| 2P_1P_2[f(u_2) - f(v_2)] \|_{s'} + \| P_1P_2[f(u_2) - f(v_2)] \|_{s'} \\
\leq \frac{4}{s - s'}\| u_1u_2 - v_1v_2 \|_s + \frac{1}{s - s'}\| f(u_2) - f(v_2) \|_s \\
+ \frac{2}{s - s'}\| f(u_2) - f(v_2) \|_s \\
+ \frac{6}{s - s'}\| f(u_1) - f(v_1) \|_s + \| f(u_2) - f(v_2) \|_{s'} \\
\leq \frac{4}{s - s'}\| u_1 - v_1 \|_s\| u_1 - v_1 \|_s + \| v_1 \|_s\| u_2 - v_2 \|_s + \frac{C}{s - s'}\| u_2 + v_2 \|_s\| u_2 - v_2 \|_s \\
+ \frac{2C}{s - s'}\| u_2 + v_2 \|_s\| u_2 - v_2 \|_s \\
+ \frac{6C}{s - s'}\| u_1 + v_1 \|_s\| u_1 - v_1 \|_s + \| u_2 + v_2 \|_{s'}\| u_2 - v_2 \|_{s'} \\
(2.8) \leq \frac{C(r, R)}{s - s'}\| (u_1, u_2) - (v_1, v_2) \|_s.
\]

This completes the proof of the condition (b).

The condition (c) can be easily obtained once our system (2.2) or (2.3) is transformed into a new system with zero initial data as in (2.6). This complete the proof of Theorem 2.1.
3 Persistence properties

In this section, we shall discuss the persistence properties for a generalized Camassa-Holm equation \( (1.1) \) in weighted \( L^p \) spaces. We can first draw some standard definitions.

**Definition 3.1.** (1) In general a weight function is simply a non-negative function.
(2) A weight function \( v: \mathbb{R} \rightarrow \mathbb{R} \) is sub-multiplicative if
\[
v(x + y) \leq v(x)v(y), \quad \forall x, y \in \mathbb{R}.
\]
(3) Given a sub-multiplicative function \( v \), by definition a positive function \( \phi \) is \( v \)-moderate if and only if
\[
\exists \ C_0 > 0 : \phi(x + y) \leq C_0 v(x)\phi(y), \quad \forall x, y \in \mathbb{R}.
\]
We say that \( \phi \) is moderate which means \( \phi \) is \( v \)-moderate for some sub-multiplicative function \( v \).

Let us recall the most standard examples of such weights:
\[
\phi(x) = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1 + |x|)^c \log(e + |x|)^d.
\]
We have (see [2]):
(1) For \( a, c, d \geq 0 \) and \( 0 \leq b \leq 1 \), such weight is sub-multiplicative.
(2) If \( a, c, d \in \mathbb{R} \) and \( 0 \leq b \leq 1 \), then \( \phi \) is moderate. More precisely, \( \phi_{a,b,c,d} \) is \( \phi_{\alpha, \beta, \gamma, \delta} \)-moderate for \( |a| \leq \alpha, \ b \leq \beta, \ |c| \leq \gamma, \ and \ |d| \leq \delta \).

Then, we give the definition for an admissible weight function:

**Definition 3.2.** An admissible weight function for \( (1.1) \) is a locally absolutely continuous function \( \phi: \mathbb{R} \rightarrow \mathbb{R} \) which satisfy

(1) \( \phi \) is \( v \)-moderate where \( v \) is some sub-multiplicative weight function such that
\[
\inf_{x \in \mathbb{R}} v(x) > 0,
\]
and
\[
\int_{\mathbb{R}} \frac{v(x)}{e^{|x|}} dx < \infty. \tag{3.1}
\]
(2) \( |\phi'(x)| \leq A|\phi(x)| \), for some \( A > 0 \) and a.e. \( x \in \mathbb{R} \).

Let us recall the following useful proposition which will be used in the proof of persistence properties.
Proposition 3.3. [2]

(1) Let $v : \mathbb{R}^n \to \mathbb{R}^+$ and $C_0 > 0$. Then following conditions are equivalent:

(i) $\forall x, y : v(x + y) \leq C_0 v(x)v(y)$.

(ii) For all $1 \leq p, q, r \leq \infty$ and for any measurable functions $f_1, f_2 : \mathbb{R}^n \to \mathbb{C}$ the weighted Young inequalities hold:

$$
\|(f_1 * f_2)v\|_r \leq C_0 \|f_1 v\|_p \|f_2 v\|_q, \quad 1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
$$

(2) Let $1 \leq p \leq \infty$ and $v$ be a sub-multiplicative weight on $\mathbb{R}^n$. The following two conditions are equivalent:

(i) $\phi$ is a $v$ – moderate weight function (with constant $C_0$).

(ii) For all measurable functions $f_1$ and $f_2$ the weighted Young estimate holds

$$
\|(f_1 * f_2)\phi\|_p \leq C_0 \|f_1 v\|_1 \|f_2 \phi\|_p.
$$

We can now state our main result on an admissible weight function.

**Theorem 3.4.** Let $T > 0$, $s > \frac{5}{2}$, and $2 \leq p \leq \infty$. Let also $u \in C([0, T], H^s)$ be a strong solution of the Cauchy problem for (1.1), such that $u|_{t=0} = u_0$ satisfies

$$
u_0 \phi \in L^p, \quad (\partial_x u_0) \phi \in L^p \quad \text{and} \quad (\partial_{xx} u_0) \phi \in L^p,$$

where $\phi$ is an admissible weight function for (1.1). Then, for all $t \in [0, T]$, we have the estimate,

$$
\|u(t)\phi\|_p + \|\partial_x u(t)\phi\|_p + \|\partial_{xx} u(t)\phi\|_p \leq \left(\|u_0 \phi\|_p + \|\partial_x u_0 \phi\|_p + \|\partial_{xx} u_0 \phi\|_p\right)e^{CMt},
$$

for some constant $C > 0$ depending only on $v$, $\phi(A, C_0, \inf_{x \in \mathbb{R}} v(x)$, and

$$
\int_{\mathbb{R}} \frac{v(x)}{|x|} dx < \infty), \quad \text{and}
$$

$$
M \equiv \sup_{t \in [0, T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty + \|\partial_{xx} u(t)\|_\infty) < \infty.
$$

The standard weights $\phi = \phi_{a,b,c,d}(x) = e^{a|x|^b}(1 + |x|)^c \log(e + |x|)^d$ is the basic example of the application of Theorem 3.3 if it satisfies the following conditions:

$$
a \geq 0, \quad c, d \in \mathbb{R}, \quad b = 1, \quad ab < 1.
$$

The restriction $ab < 1$ guarantees the validity of the condition (3.1) for a multiplicative function $v(x) \geq 0$. The restriction $b = 1$ guarantees that $|\phi'(x)| \leq A|\phi(x)|$. When $0 < b < 1$, we have $|\phi'| \to \infty$, as $x \to 0$. 

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Proof. Assume that φ is the admissible weight function. Applying the assumption \( u \in C([0,T], H^s) \), with \( s > \frac{5}{2} \), we get

\[
M \equiv \sup_{t \in [0,T]} (\|u(t)\|_\infty + \|\partial_x u(t)\|_\infty + \|\partial_{xx} u(t)\|_\infty) < \infty.
\]

The first equation in (3.3) can be rewritten as:

\[
u_t - 4uu_x - \partial_x G * [2u_x^2 + 6u^2 + \partial_x(u_x^2)] = 0,
\]

with the kernel \( G(x) = \frac{1}{\pi x} e^{-|x|} \). Let us consider the N-truncations of \( \phi_N(x) = \min\{\phi, N\} \) for any \( N \in \mathbb{Z}^+ \). Note that \( \phi_N : \mathbb{R} \to \mathbb{R} \) is a locally absolutely continuous function which satisfies

\[
\|\phi_N(x)\|_\infty \leq N, \quad |\phi'_N(x)| \leq A|\phi_N(x)| \quad a.e. on \ \mathbb{R},
\]

and

\[
\phi_N(x + y) \leq C_1 v(x) \phi_N(y), \quad \forall x, y \in \mathbb{R},
\]

where \( C_1 = \max\{C_0, \alpha^{-1}\} \), \( \alpha = \inf_{x \in \mathbb{R}} v(x) > 0 \).

As known in [2], the N-truncations \( \phi_N \) of a \( v \)-moderate weight \( \phi \) are uniformly \( v \)-moderate with respect to \( N \).

Considering the case \( 2 \leq p < \infty \). First, we give estimates on \( u\phi_N \). Multiply (3.2) by \( \phi_N|u\phi_N|^{p-2}(u\phi_N) \) and integrate to obtain

\[
\frac{1}{p} \frac{d}{dt} \left( \|u\phi_N\|_p^p \right) - 4 \int_{\mathbb{R}} |u\phi_N|^p (\partial_x u) dx
\]

\[
- \int_{\mathbb{R}} |u\phi_N|^{p-2} (u\phi_N)(\phi_N G_x * [2u_x^2 + 6u^2 + \partial_x(u_x^2)]) dx = 0.
\]

The two above integrals are clearly finite since \( \phi_N \in L^\infty(\mathbb{R}) \). And \( u(\cdot, t) \in H^s, s > \frac{5}{2} \) leads to \( \partial_x G * [2u_x^2 + 6u^2 + \partial_x(u_x^2)] \in L^1 \cap L^\infty \). Applying Hölder’s inequality, Proposition 3.3 and the condition (3.1), it follows that

\[
\frac{d}{dt} \|u\phi_N\|_p \leq 4M \|u\phi_N\|_p + \|\phi_N G_x * [2u_x^2 + 6u^2 + \partial_x(u_x^2)]\|_p
\]

\[
\leq 4M \|u\phi_N\|_p + C_1 \|G_x v\|_L^1 \|2u_x^2 + 6u^2 + \partial_x(u_x^2)\| \phi_N\|_p
\]

\[
\leq 4M \|u\phi_N\|_p + C_2 \|2u_x^2 + 6u^2 + \partial_x(u_x^2)\| \phi_N\|_p
\]

\[
\leq M(4 + 6C_2) \|u\phi_N\|_p + 6MC_2 \|u_x\phi_N\|_p,
\]

where \( C_2 \) depends only on \( v \) and \( \phi \).

Next, we will give estimates on \( u_x\phi_N \). Differentiating (3.2) with respect to \( x \)-variable, then multiplying by \( \phi_N \), yields

\[
\partial_t(u_x\phi_N) - 4u_x(u_x\phi_N) - 4u\phi_N\partial_x^2 u + 2u_x\phi_N\partial_x^2 u - \phi_N \partial_x G * [\partial_x(2u_x^2 + 6u^2) + u_x^2] = 0.
\]
3  PERSISTENCE PROPERTIES

Multiplying the above equation by \(|u_x\phi_N|^{p-2}(u_x\phi_N)|\), integrating the result equation in \(x\)-variable, and using integration by parts, we obtain

\[
\frac{1}{p} \frac{d}{dt} \left( \|u_x\phi_N\|_p^p \right) - 4 \int_\mathbb{R} |u_x\phi_N|^p \partial_x u \, dx - 4 \int_\mathbb{R} \partial_x^2 u |u_x\phi_N|^{p-2}(u_x\phi_N)(u\phi_N) \, dx
\]

(3.7) + 2 \int_\mathbb{R} |u_x\phi_N|^p \partial_x u \, dx - \int_\mathbb{R} |u_x\phi_N|^{p-2}(u_x\phi_N)(\phi_N \partial_x G [\partial_x (2u_x^2 + 6u^2) + u_x^2]) \, dx = 0.

Arguing as before, the two above integrals are clearly finite because of \(|\phi_N| \in L^\infty(\mathbb{R})\). And \(u(-t) \in H^s, s > \frac{5}{2}\) leads to \(\partial_x G [\partial_x (2u_x^2 + 6u^2) + u_x^2] \in L^1 \cap L^\infty\). Applying Hölder’s inequality, Proposition 3.3 and condition (3.1), we obtain

\[
\frac{d}{dt} \|u_x\phi_N\|_p \leq 6M(\|u_x\phi_N\|_p + \|u\phi_N\|_p) + \|\phi_N G_x [\partial_x (2u_x^2 + 6u^2) + u_x^2]\phi_N\|_p
\]

(3.8)

\[
\leq 6M(\|u_x\phi_N\|_p + \|u\phi_N\|_p) + C_1\|G_x v\|_{L^1}[\|\partial_x (2u_x^2 + 6u^2) + u_x^2]\phi_N\|_p
\]

\[
\leq 6M(\|u_x\phi_N\|_p + \|u\phi_N\|_p) + C_2[\|\partial_x (2u_x^2 + 6u^2) + u_x^2]\phi_N\|_p
\]

where \(C_2\) depends only on \(v\) and \(\phi\).

Then, we will give estimates on \(u_{xx}\phi_N\). Differentiating (3.2) twice with respect to \(x\)-variable, next multiplying by \(\phi_N\), we get

\[
\frac{d}{dt} \phi_N(u_{xx})(t) = \phi_N(4u - 2u_x)u_{xxx} - 2\phi_Nu_{xx}^2 + 8\phi_Nu_xu_{xx} - 12\phi_Nuu_x
\]

(3.9)

\[
+ \phi_N \partial_x G * [(2u_x^2 + 6u^2) + \partial_x (u_x^2)].
\]

Multiplying the above equation by \(|\phi_N u_{xx}|^{p-2}(\phi_N u_{xx})\), integrating the result equations in the \(x\)-variable, and using integration by parts and (3.3), implies

\[
\frac{d}{dt} \|u_{xx}\phi_N\|_p \leq (4A + 10)M \|\phi_N u_{xx}\|_p + 12M \|\phi_N u_x\|_p + \|\phi_N G_x [\partial_x (2u_x^2 + 6u^2) + \partial_x (u_x^2)]\|_p
\]

(3.10)

\[
\leq (4A + 10)M \|\phi_N u_{xx}\|_p + 12M \|\phi_N u_x\|_p + C_1\|G_x v\|_{L^1}[\|\partial_x (2u_x^2 + 6u^2) + u_x^2]\phi_N\|_p
\]

\[
\leq (4A + 10)M \|\phi_N u_{xx}\|_p + 12M \|\phi_N u_x\|_p + C_2[\|\partial_x (2u_x^2 + 6u^2) + u_x^2]\phi_N\|_p
\]

where we use

\[
\int_\mathbb{R} \phi_N(2u_x - 4u)u_{xx}|\phi_N u_{xx}|^{p-2}(\phi_N u_{xx}) \, dx
\]

\[
= |\int_\mathbb{R} (2u_x - 4u)[\partial_x (\phi_N u_{xx}) - \phi_N u_{xx}](\phi_N u_{xx})^{p-2}(\phi_N u_{xx}) \, dx|
\]

\[
\leq |\frac{1}{p} \int_\mathbb{R} (2u_x - 4u)\partial_x (\phi_N u_{xx}) \, dx| + |\int_\mathbb{R} (2u_x - 4u)\phi_N u_{xx}(\phi_N u_{xx})^{p-2}(\phi_N u_{xx}) \, dx|
\]

\[
\leq |\frac{1}{p} \int_\mathbb{R} (2u_x - 4u)(\phi_N u_{xx}) \, dx| + A \int_\mathbb{R} |2u_x - 4u| \phi_N u_{xx} \, dx
\]
\[
\leq \frac{1}{p} (2 \| u_{xx} \|_{L^\infty} + 4 \| u_x \|_{L^\infty}) \| \phi_N u_{xx} \|_{L^p}^p + A (2 \| u_x \|_{L^\infty} + 4 \| u \|_{L^\infty}) \| \phi_N u_{xx} \|_{L^p}^p
\]
\[ \leq (4A + 2) M \| \phi_N u_{xx} \|_{L^p}^p. \]

Now, together the inequalities (3.5), (3.8) with (3.10) and then integrating, yields
\[
\| u(t) \phi_N \|_p + \| (\partial_x u) \phi_N \|_p + \| (\partial_{xx} u) \phi_N \|_p
\leq (\| u_0 \phi_N \|_p + \| (u_0_x) \phi_N \|_p + \| (u_{0xx}) \phi_N \|_p) \exp(CMt)
\]
\[ \leq (\| u_0 \phi \|_p + \| (u_0_x) \phi \|_p + \| (u_{0xx}) \phi \|_p) \exp(CMt), \]
for all \( t \in [0, T] \).

Since \( \phi_N(x) \to \phi(x) \) as \( N \to \infty \) for a.e. \( x \in \mathbb{R} \). Recalling that \( u_0 \phi \in L^p \), \( u_{0x} \phi \in L^p \) and \( u_{0xx} \phi \in L^p \), we get
\[
\| u(t) \phi \|_p + \| (\partial_x u) \phi \|_p + \| (\partial_{xx} u) \phi \|_p
\leq (\| u_0 \phi \|_p + \| (u_0_x) \phi \|_p + \| (u_{0xx}) \phi \|_p) \exp(CMt),
\]
for all \( t \in [0, T] \). Finally, we treat the case \( p = \infty \). Noticing that \( u_0, u_{0x}, u_{0xx} \in L^2 \cap L^\infty \) and \( \phi_N \in L^\infty \), hence, we have
\[
\| u(t) \phi_N \|_q + \| (\partial_x u) \phi_N \|_q + \| (\partial_{xx} u) \phi_N \|_q
\leq (\| u_0 \phi_N \|_q + \| (u_0_x) \phi_N \|_q + \| (u_{0xx}) \phi_N \|_q) \exp(CMt)
\]
\[ \leq (\| u_0 \phi_N \|_\infty + \| (u_0_x) \phi_N \|_\infty + \| (u_{0xx}) \phi_N \|_\infty) \exp(CMt), \quad q \in [2, \infty). \]

The last term in the right-hand side is independent of \( q \). Since \( \| \phi_N \|_{L^p} \to \| \phi_N \|_{L^\infty} \) as \( p \to \infty \) for any \( \phi_N \in L^\infty \cap L^2 \), implies
\[
\| u(t) \phi_N \|_\infty + \| (\partial_x u) \phi_N \|_\infty + \| (\partial_{xx} u) \phi_N \|_\infty
\leq (\| u_0 \phi_N \|_\infty + \| (u_0_x) \phi_N \|_\infty + \| (u_{0xx}) \phi_N \|_\infty) \exp(CMt)
\]
\[ \leq (\| u_0 \phi \|_\infty + \| (u_0_x) \phi \|_\infty + \| (u_{0xx}) \phi \|_\infty) \exp(CMt). \]

The last term in the right-hand side is independent of \( N \). Now taking \( N \to \infty \) implies that the estimate (2.6) remains valid for \( p = \infty \).

Indeed, for \( a < 0 \), we have \( \phi(x) \to 0 \) as \( |x| \to \infty \): the conclusion of Theorem 3.4 remains true but we are not interested in this case. It is interesting in the following two particular cases:

**Remark 3.5.** (1) **Power weights:** Take \( \phi = \phi_{0,0,c,0} \) with \( c > 0 \). It is \( v \)-moderate, where \( v = (1 + |x|)^c \), \( c < 0 \). It can be deduced easily \( v > 0 \), \( ve^{-|x|} \in L^1 \). And sending \( p = \infty \).
From Theorem 3.4 we obtain

\[ |u(t,x)| + |\partial_x u(t,x)| + |\partial_x^2 u(t,x)| \leq C'(1 + |x|)^{-c}. \]

with the condition

\[ |u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| \leq C(1 + |x|)^{-c}. \]

Thus, we obtain the persistence properties on algebraic decay rates of strong solutions to (1.1).

(2) Exponential weights: Choose \( \phi = \phi_{a,1,0,0} \). We deduce that \( \phi(x) = e^{ax} \) if \( x \geq 0 \) and \( \phi(x) = 1 \) if \( x \leq 0 \) with \( 0 \leq a < 1 \). Such weight clearly satisfies the admissibility conditions of Definition 3.2.

From Theorem 3.4, we have

\[ |u(t,x)| + |\partial_x u(t,x)| + |\partial_x^2 u(t,x)| \leq C'e^{-ax}. \]

with the condition

\[ |u_0(x)| + |\partial_x u_0(x)| + |\partial_x^2 u_0(x)| \leq Ce^{-ax}. \]

Thus, we obtain the persistence properties on the point wise decay rates of strong solutions to (1.1).

The limit case \( a = b = 1 \) is not covered by Theorem 3.4. In other words, To obtain an application of some \( v \)-moderate weights \( \phi \) for which condition (3.1) does not hold, we need to make a modification to Theorem 3.4. Instead of assuming (3.1), we now put the weaker condition

(3.16) \( ve^{-|\cdot|} \in L^p \quad \text{where} \quad 2 \leq p \leq \infty. \)

See Theorem (3.6) below, which covers the case of such fast growing weights.

**Theorem 3.6.** Let \( 2 \leq p \leq \infty \) and \( \phi \) be a \( v \)-moderate weight function as in Definition 3.2 satisfying the condition (3.16) instead of (3.1). Let also \( u|_{t=0} = u_0 \) satisfy

\[ u_0 \phi \in L^p, \quad u_0 \phi^{\frac{1}{2}} \in L^2, \]

\[ (\partial_x u_0) \phi \in L^p, \quad (\partial_x u_0) \phi^{\frac{1}{2}} \in L^2, \]

and

\[ (\partial_x^2 u_0) \phi \in L^p, \quad (\partial_x^2 u_0) \phi^{\frac{1}{2}} \in L^2. \]

Let also \( u \in C([0,T], H^s), s > \frac{5}{2}, \) be the strong solution of the Cauchy problem for (1.1), emanating from \( u_0 \). Then,

\[ \sup_{t \in [0,T]} (\|u(t)\|_p + \|\partial_x u(t)\|_p + \|\partial_x^2 u(t)\|_p) < \infty, \]
and
\[
\sup_{t \in [0,T]} (\|u(t)\phi^{\frac{1}{2}}\|_2 + \|\partial_x u(t)\phi^{\frac{1}{2}}\|_2 + \|\partial_x^2 u(t)\phi^{\frac{1}{2}}\|_2) < \infty.
\]

Proof. It is easy to observe that \(\phi^{\frac{1}{2}}\) is a \(v^\frac{1}{2}\)-moderate weight where \(\inf_{x \in \mathbb{R}} v^\frac{1}{2}(x) > 0\), such that \(|(\phi^{\frac{1}{2}})'(x)| \leq \frac{\phi^{\frac{1}{2}}}{2}\). By condition (3.16), \(v^\frac{1}{2}e^{-|x|} \in L^2p\), hence Hölder’s inequality implies that \(v^\frac{1}{2}e^{-|x|} \in L^1\). Then an application of Theorem 3.4 with \(p = 2\) to the weight \(\phi^{\frac{1}{2}}\), yields
\[
(3.17) \quad \|u(t)\phi^{\frac{1}{2}}\|_2 + \|\partial_x u(t)\phi^{\frac{1}{2}}\|_2 + \|\partial_x^2 u(t)\phi^{\frac{1}{2}}\|_2 \leq (\|u_0\phi^{\frac{1}{2}}\|_2 + \|u_0\phi^{\frac{1}{2}}\|_2 + \|u_{0xx}\phi^{\frac{1}{2}}\|_2)e^{CMt},
\]
which along with Hölder’s inequality, implies
\[
(3.18) \quad \|[(2u^2_x + 6u^2_x) + \partial_x (u^2_x)]\phi\|_1 \leq K_0e^{2CMt},
\]
\[
(3.19) \quad \|[(\partial_x (2u^2_x + 6u^2_x) + u^2_x)]\phi\|_1 \leq K_1e^{2CMt}.
\]
The constants \(K_0\) and \(K_1\) below depend only on \(\phi\) and on the datum. By the same argument as in the proof of Theorem 3.4 (recall that \(\phi_N = \min\{\phi(x), N\}\)) for \(p < \infty\), yields
\[
(3.20) \quad \frac{d}{dt}\|u_N\|_p \leq CM\|\phi_N u\|_p + \|\phi_N \partial_x G * [(2u^2_x + 6u^2_x) + \partial_x (u^2_x)]\|_p,
\]
\[
(3.21) \quad \frac{d}{dt}\|u_x \phi_N\|_p \leq CM\|\phi_N u_x\|_{L^p} + \|\phi_N \partial_x G * [\partial_x (2u^2_x + 6u^2_x) + (u^2_x)]\|_p,
\]
and
\[
(3.22) \quad \frac{d}{dt}\|u_{xx} \phi_N\|_p \leq CM\|\phi_N u_{xx}\|_p + \|\phi_N \partial_x G * [(2u^2_x + 6u^2_x) + \partial_x (u^2_x)]\|_p.
\]

Note that \(|\partial_x G| \leq \frac{1}{2}e^{-|.|}\). Then combining Proposition 3.3, the condition (3.16) with the estimates (3.18)-(3.19), we have
\[
\|\phi_N \partial_x G * [\partial_x (2u^2_x + 6u^2_x) + (u^2_x)]\|_p \leq K_2e^{2CMt},
\]
\[
\|\phi_N \partial_x G * [(2u^2_x + 6u^2_x) + \partial_x (u^2_x)]\|_p \leq K_3e^{2CMt}.
\]
The constants \(K_2\), \(K_3\) depend only on \(\phi\) and on the datum.

Plugging the two last estimates in (3.20)-(3.22), and summing we obtain
\[
\frac{d}{dt}(\|u_N\|_p + \|u_x \phi_N\|_p + \|u_{xx} \phi_N\|_p) 
\leq CM(\|u_N\|_p + \|u_x \phi_N\|_p + \|u_{xx} \phi_N\|_p) + 2(K_2 + K_3)e^{2CMt}.
\]
Integrating and finally as before letting \(N \to \infty\), we can get the conclusion in the case \(2 \leq p < \infty\). The constants throughout the proof are independent on \(p\). Therefore, as before, the result for infinite exponents \(p = \infty\) can be established. The argument is fully similar to that of Theorem 3.4.

\(\Box\)
Remark 3.7. Choosing \( \phi(x) = \phi_{1,1,0,0}(x) = e^{|x|} \). It is \( v \)-moderate with \( v = e^{|x|} \). Let \( p = \infty \). It is easy to deduce that \( \frac{v}{e^{|x|}} \in L^\infty \). Applying Theorem 3.6 we get the strong solution such that

\[
|u(x, t)| + |\partial_x u(x, t)| + |\partial_x^2 u(x, t)| \leq Ce^{-|x|},
\]

if \( |u_0(x)|, |\partial_x u_0(x)| \) and \( |\partial_x^2 u_0(x)| \) are both bounded by \( ce^{-|x|} \).

4 Exact asymptotic profiles

In this section, we finish this paper with the proof of a asymptotic profile.

Theorem 4.1. Let \( s > \frac{5}{2} \) and \( u_0 \in H^s, u_0 \neq 0 \), such that

\[
(4.1) \quad \sup_{x \in \mathbb{R}} e^{\frac{|x|}{2}} (1 + |x|)^{\frac{3}{2}} \log(e + |x|)^d(|u_0(x)| + |(\partial_x u_0)(x)| + |(\partial_x^2 u_0)(x)|) < \infty,
\]

for some \( d > \frac{1}{2} \). Then condition (4.1) is conserved uniformly in \([0, T]\) by the strong solution \( u \in C([0, T], H^s) \) of the Camassa-Holm equation. Moreover, the following asymptotic profiles (respectively for \( x \to +\infty \) and \( x \to -\infty \)) hold:

\[
(4.2) \quad \left\{ \begin{array}{l}
\begin{aligned}
u(x, t) &= u_0(x) + e^{-x}t \left\{ \Phi(t) + C_1 (1 + x)^{-1} \log(e + x)^{-2d}(x, t) \\
&\quad + C_2 [\log(1 + x)]^{1-2d} + o(\log(1 + x)]^{1-2d}) \right\}, \\
u(x, t) &= u_0(x) - e^x t \left\{ \Psi(t) + C_1 (1 - x)^{-1} \log(e - x)^{-2d}(x, t) \\
&\quad + C_2 [\log(1 - x)]^{1-2d} + o(\log(1 - x)]^{1-2d}) \right\}.
\end{aligned}
\end{array} \right.
\]

where, for all \( t \in [0, T] \), some constants \( C_1, C_2 \) depend only on the datum and some constants \( c_1, c_2 > 0 \) independent on \( t \).

\[
(4.3) \quad c_1 \leq \Phi(t) \leq c_2, \quad c_1 \leq \Psi(t) \leq c_2.
\]

Proof. A simple application of Theorem 3.4 with \( p = \infty \) and the weight \( \phi(x) = e^{|x|} (1 + |x|)^{\frac{3}{2}} \log(e + |x|)^d \) leads to the fact that condition (4.1) is conserved uniformly in \([0, T]\) by the strong solution \( u \in C([0, T], H^s) \).

Integrating (4.3) over \([0, t]\), we obtain

\[
(4.4) \quad u(x, t) = u_0(x) + \int_0^t [4u(x, s)\partial_x u(x, s) - (\partial_x u(x, s))^2] ds
\]

where \( F(u) = 6u^2 + 2(\partial_x u)^2 \).
For $t \in [0, T]$, applying (4.1), we get
\[
| \int_0^t [4u(x, s)\partial_x u(x, s) - (\partial_x u(x, s))^2] ds | \leq \frac{C_1}{2} e^{-|x|} t (1 + |x|)^{-1} \log(e + |x|)^{-2d},
\]
\[
| \int_0^t G * (\partial_x u)^2 (x, s) ds | \leq \frac{C_1}{2} e^{-|x|} t (1 + |x|)^{-1} \log(e + |x|)^{-2d}.
\]

For $0 < t \leq T$, assume that
\[
h(x, t) = \frac{1}{t} \int_0^t F(u(x, s)) ds,
\]
and
\[
\Phi(t) = \frac{1}{2} \int_{-\infty}^{+\infty} e^y h(y, t) dy, \quad \Psi(t) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-y} h(y, t) dy.
\]
The function $(1 + |x|)^{\frac{1}{2}} \log(e + |x|)^{-d}$ belongs to $L^2$. Then we have $e^{2|x|} (|u_0(x)| + (\partial_x u_0)(x)| + (\partial^2_x u_0)(x)) \in L^2$. Applying Theorem 3.3 with $p = 2$ and the weight $\phi(x) = e^{\frac{|x|}{2}}$ yields
\[
\int_{\mathbb{R}} e^{|y|} h(y, t) dy < \infty.
\]

Because $\Phi$ and $\Psi$ is continuous at $t = 0$, we give the definition
\[
\Phi(0) = \frac{1}{2} \int_{-\infty}^{+\infty} e^y F(u_0)(y) dy \quad \text{and} \quad \Psi(0) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-y} F(u_0)(y) dy.
\]

The assumption $u_0 \neq 0$ and $u_0 \in H^s$, $s > \frac{5}{2}$, ensures the validity of estimates (4.3) with $c_1, c_2 > 0$ for all $t \in [0, T]$.

Now using $\partial_x G(x - y) = -\frac{1}{2} \text{sign}(x - y) e^{-|x-y|}$ we get
\[
- \int_0^t \partial_x G * F(u)(s) ds = e^{-x} \int_{-\infty}^{x} e^y h(y, t) dy - \frac{e^x}{2} \int_{-\infty}^{+\infty} e^{-y} h(y, t) dy - e^{-x} t [\Phi(t) - \frac{1}{2} \int_{-\infty}^{+\infty} (e^y + e^{2x-y}) h(y, t) dy].
\]

But, by the dominated convergence theorem, it follows
\[
0 \leq \int_{x}^{+\infty} (e^y + e^{(2x-y)}) h(y, t) dy \leq 2 \int_{x}^{+\infty} e^y h(y, t) dy \to 0, \quad \text{as} \quad x \to +\infty.
\]

In the same way,
\[
- \int_0^t \partial_x G * F(u)(s) ds = -e^x t [\Psi(t) - \frac{1}{2} \int_{-\infty}^{x} (e^{-y} + e^{y-2x}) h(y, t) dy],
\]
and
\[
0 \leq \int_{x}^{-\infty} (e^{-y} + e^{y-2x}) h(y, t) dy \leq 2 \int_{x}^{-\infty} 2e^{-y} h(y, t) dy \to 0, \quad \text{as} \quad x \to -\infty.
\]
By Hospital’s Rule, we have
\[
\lim_{x \to \infty} \frac{\int_{x}^{\infty} e^{\|y\|} h(t, y) dy}{\log(1 + |x|)^{1-2d}} = \frac{-e^{\|x\|} h(t, x)}{(1 - 2d) \log(1 + |x|)^{-2d} (1 + |x|)^{-1} \text{sign}(x)} \leq C_2.
\]

(4.6)

Thus, we complete the asymptotic profile of (4.2).

Applying Theorem 4.2, we immediately obtain the fact that only the zero solution can be compactly supported (or decay faster than \(e^{-|x|}\)) at two different times \(t_0\) and \(t_1\).

**Theorem 4.2.** Let \(s > \frac{5}{2}\) and \(u_0 \in H^s\), \(u_0 \neq \lambda e^{-\sqrt{x}}\), such that
\[
(4.7) \quad \sup_{x \in \mathbb{R}} e^{\frac{|x|}{2}} (1 + |x|)^{\frac{1}{2}} \log(e + |x|)^d (|u_0(x)| + |(\partial_x u_0)(x)| + |(\partial_x^2 u_0)(x)|) < \infty,
\]
for some \(d > \frac{1}{2}\). Then condition (4.7) is conserved uniformly in \([0, T]\) by the strong solution \(u \in C([0, T], H^s)\) of the Camassa-Holm equation. Moreover, the following asymptotic profiles (respectively for \(x \to +\infty\) and \(x \to -\infty\)) hold:
\[
(4.8) \quad \begin{cases}
    u(x, t) = u_0(x) + e^{-\sqrt{t}} \left\{ \Phi(t) + C_1 (1 + x)^{-1} \log(e + x)^{-2d} (x, t) \\
    \quad + C_2 \log(1 + x)^{1-2d} + o(\log(1 + x)^{1-2d}) \right\}, \\
    u(x, t) = u_0(x) - e^\sqrt{t} \left\{ \Psi(t) + C_1 (1 - x)^{-1} \log(e - x)^{-2d} (x, t) \\
    \quad + C_2 \log(1 - x)^{1-2d} + o(\log(1 - x)^{1-2d}) \right\}.
\end{cases}
\]

where, for all \(t \in [0, T]\), some constants \(c_1, c_2\) depend only on the datum and some constants \(c_1, c_2 > 0\) independent on \(t\),
\[
(4.9) \quad c_1 \leq \Phi(t) \leq c_2, \quad c_1 \leq \Psi(t) \leq c_2.
\]

**Proof.** A simple application of Theorem 3.4 with \(p = \infty\) and the weight \(\phi(x) = e^{\frac{|x|}{2}} (1 + |x|)^{\frac{1}{2}} \log(e + |x|)^d\) leads to the fact that condition (4.7) is conserved uniformly in \([0, T]\) by the strong solution \(u \in C([0, T], H^s)\).

We can rewritten (4.3) as follows:
\[
(4.10) \quad u_t(t, x) = 4uu_x - u_x^2 + \sqrt{3}u^2 - G \star [u_x^2 - \sqrt{3}u^2] + G_x \star [(\sqrt{2}u_x + \sqrt{6}u)^2].
\]

Integrating (4.10) over \([0, t]\), we obtain
\[
u(x, t) = u_0(x) + \int_0^t [4u(x, s)\partial_x u(x, s) - (\partial_x u(x, s))^2 + \sqrt{3}u^2] ds
\]

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\[
(4.11) \quad -\int_0^t G \ast [(\partial_x u)^2 - \sqrt{3}u^2](x, s)ds + \int_0^t G_x \ast [(\sqrt{2}u_x + \sqrt{6}u)](x, s)ds.
\]

For \( t \in [0, T] \), applying (4.7), we get
\[
| \int_0^t [4u(x, s)\partial_x u(x, s) - (\partial_x u(x, s))^2 + \sqrt{3}u^2]ds | \leq \frac{C_1}{2}e^{-|x|}t(1 + |x|)^{-1}\log(e + |x|)^{-2d},
\]
\[
| \int_0^t G \ast [(\partial_x u)^2 - \sqrt{3}u^2](x, s)ds | \leq \frac{C_1}{2}e^{-|x|}t(1 + |x|)^{-1}\log(e + |x|)^{-2d}.
\]

For \( 0 < t \leq T \), assume that
\[
h(x, t) = \frac{1}{t^{\frac{3}{2}}} \int_0^t (\sqrt{2}u_x + \sqrt{6}u)^2(x, s)ds, \forall x \geq 0,
\]
and
\[
\Phi(t) = \frac{1}{2} \int_{-\infty}^{+\infty} e^y h(y, t)dy, \quad \Psi(t) = \frac{1}{2} \int_{-\infty}^{+\infty} e^y h(y, t)dy.
\]

The function \((1 + |\cdot|)^{-\frac{3}{2}} \log(e + |\cdot|)^{-d} \) belongs to \( L^2 \). Then we have \( e^{|x|}(|u_0(x)| + |(\partial_x u_0)(x)| + \| (\partial^2 u_0)(x) \|) \in L^2 \). Applying Theorem 3.4 with \( p = 2 \) and the weight \( \phi(x) = e^{|x|} \) yields \( \int_{\mathbb{R}} e^{\phi(y)}h(y, t)dy < \infty \).

Because \( \Phi \) and \( \Psi \) is continuous at \( t = 0 \), we give the definition
\[
\Phi(0) = \frac{1}{2} \int_{-\infty}^{+\infty} e^y (\sqrt{2}u_{0x} + \sqrt{6}u_0)^2(y)dy, \quad \Psi(0) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-y} (\sqrt{2}u_{0x} + \sqrt{6}u_0)^2(y)dy.
\]

The assumption \( u_0 \neq \lambda e^{-\sqrt{3}x} \) and \( u_0 \in H^s, \ s > \frac{5}{2} \), ensures the validity of estimates [119] with \( c_1, c_2 > 0 \) for all \( t \in [0, T] \).

Now using \( \partial_x G(x - y) = -\frac{1}{2}\text{sign}(x - y)e^{-|x-y|} \) we get
\[
-\int_0^t \partial_x G \ast (\sqrt{2}u_{0x} + \sqrt{6}u_0)^2(s)ds = e^{-x}t \int_{-\infty}^{+\infty} e^x h^2(y, t)dy - e^{-x}t \int_{-\infty}^{+\infty} e^{-y} h^2(y, t)dy = e^{-x}t[\Phi(t) - \frac{1}{2} \int_{-\infty}^{+\infty} (e^y + e^{2x-y})h^2(y, t)dy].
\]

But, by the dominated convergence theorem, it follows
\[
(4.12) \quad 0 \leq \int_{-\infty}^{+\infty} (e^y + e^{2x-y})h^2(y, t)dy \leq 2 \int_{-\infty}^{+\infty} e^y h^2(y, t)dy \to 0, \ \text{as} \ x \to +\infty.
\]

In the same way,
\[
-\int_0^t \partial_x G \ast (\sqrt{2}u_{0x} + \sqrt{6}u_0)^2(s)ds = e^{-x}t[\Psi(t) - \frac{1}{2} \int_{-\infty}^{+\infty} (e^{-y} + e^{y-2x})h^2(y, t)dy],
\]
and
\[
0 \leq \int_{-\infty}^{+\infty} (e^{-y} + e^{y-2x})h^2(y, t)dy \leq 2 \int_{-\infty}^{+\infty} e^{-y} h^2(y, t)dy \to 0, \ \text{as} \ x \to -\infty.
\]
By Hospital’s Rule, we have

\[
\lim_{x \to \infty} \int_x^{+\infty} e^{\|y\|^2/2} dy \to \int_{x}^{+\infty} e^{\|y\|^2/2} dy
\]

\[
\frac{e^{\|x\|^2/2}}{\log(1 + \|x\|)}^{1-2d} (1 - 2d) \frac{\log(1 + \|x\|)^{-2d}(1 + \|x\|)^{-1}}{\log(1 + \|x\|)}^{-1} \text{sign}(x)
\]

(4.13)

\[
\leq C_2.
\]

Thus, we complete the asymptotic profile of (4.8).

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