CONVERGENCE AND CENTER MANIFOLDS FOR
DIFFERENTIAL EQUATIONS DRIVEN BY COLORED NOISE

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(Communicated by Rafael de la Llave)

Abstract. In this paper, we study the convergence and pathwise dynamics of random differential equations driven by colored noise. We first show that the solutions of the random differential equations driven by colored noise with a nonlinear diffusion term uniformly converge in mean square to the solutions of the corresponding Stratonovich stochastic differential equation as the correlation time of colored noise approaches zero. Then, we construct random center manifolds for such random differential equations and prove that these manifolds converge to the random center manifolds of the corresponding Stratonovich equation when the noise is linear and multiplicative as the correlation time approaches zero.

1. Introduction. This paper is concerned with the convergence and pathwise dynamics of the following random differential equation on \( \mathbb{R}^n \) driven by colored noise:

\[ \dot{u}_\delta = f(u_\delta) + \sigma(u_\delta) z_\delta(\theta_t \omega), \]

where \( f : \mathbb{R}^n \to \mathbb{R}^n \) and \( \sigma : \mathbb{R}^n \to \mathbb{R}^{n \times l} \) are nonlinear functions, and \( z_\delta(\theta_t \omega) \) is an \( l \)-dimensional colored noise with correlation time \( \delta > 0 \).

To describe the colored noise, we introduce the canonical Wiener probability space \((\Omega, \mathcal{F}, \mathbb{P})\) where

\[ \Omega = C_0(\mathbb{R}, \mathbb{R}^l) := \{ \omega \in C(\mathbb{R}, \mathbb{R}^l) : \omega(0) = 0 \} \]

with the compact-open topology, \( \mathcal{F} \) is its Borel \( \sigma \)-algebra, and \( \mathbb{P} \) is the Wiener measure. Let \( W \) be an \( l \)-dimensional Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\). Then \( W \) has the form \( W(t, \omega) = \omega(t) \) for \( \omega \in \Omega \).

2010 Mathematics Subject Classification. Primary: 60H10; Secondary: 37D10, 37H10.

Key words and phrases. Ornstein-Uhlenbeck process, colored noise, approximations, random dynamical systems, center manifolds.

This work was supported by NSFC (11501549, 11331007, 11831012), NSF (1413603), the Fundamental Research Funds for the Central Universities (YJ201646) and International Visiting Program for Excellent Young Scholars of SCU.

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Consider the Wiener shift \( \theta_t \) defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) by
\[
\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t).
\]
It is known that the probability measure \( \mathbb{P} \) is an ergodic invariant measure for \( \theta_t \).
\((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) forms a metric dynamical system, see Arnold [2].

For each \( \delta > 0 \), we consider the following stochastic differential equation on \( \mathbb{R}^l \):
\[
dz = -\frac{1}{\delta} z \, dt + \frac{1}{\delta} dW. \tag{2}
\]
This equation has a unique stationary solution given by
\[
z_\delta(t, \omega) = \frac{1}{\delta} \int_{-\infty}^{t} e^{\frac{s-t}{\delta}} \, dW(s), \tag{3}
\]
which is called an Ornstein-Uhlenbeck process or colored noise. For convenience, we denote by
\[
z_\delta(\omega) = \frac{1}{\delta} \int_{-\infty}^{0} e^{\frac{s}{\delta}} \, dW(s).
\]
Then we find that
\[
z_\delta(t, \omega) = z_\delta(\theta_t \omega) = -\int_{-\infty}^{0} \frac{1}{\delta^2} e^{\frac{s}{\delta}} \theta_t \omega(s) \, ds. \tag{4}
\]
For later purpose, we write
\[
Z_\delta(t, \omega) = \int_{0}^{t} z_\delta(s, \omega) \, ds = \int_{0}^{t} z_\delta(\theta_s \omega) \, ds. \tag{5}
\]
The colored noise has been widely used in physics and biology to study the dynamical behavior of solutions of random systems, see, e.g., [10, 20, 23, 43, 36, 45] and the references therein. Recently, the attractors and invariant manifolds of random differential equations driven by additive or linear multiplicative colored noise have been studied in [11, 12] and [18, 19], respectively.

The colored noise can be regarded as an approximation to the white noise in the sense that
\[
\lim_{\delta \to 0^+} \sup_{t \in [0, T]} \left| \int_{0}^{t} z_\delta(\theta_s \omega) \, ds - \omega(t) \right| = 0, \text{ a.s.}
\]
for each \( T > 0 \), see [11, Lemma 2.2], also [1].

A basic question is: Do the solutions and dynamics of the random equation (1) converge to those of the corresponding Stratonovich stochastic differential equation
\[
du = f(u) \, dt + \sigma(u) \circ dW? \tag{6}
\]
In this paper, we try to answer this question. We first show that the solution \( u_\delta \) of equation (1) indeed converges uniformly in mean square to a solutions of equation (6) as the correlation time \( \delta \) approaches zero. Then, we construct the random center manifolds for equation (1) and prove that the center manifolds of the equations driven by colored noise converges to the center manifolds of the corresponding Stratonovich equations when the noise is linear and multiplicative as the correlation time approaches zero.

More precisely, we state our results as follows. The first result is about the mean square convergence of the solutions of the random differential equation driven by colored noise.
Main Theorem 1. Let \( u_\delta(t, \omega, x) \) and \( u(t, \omega, x) \) be the solutions of equations (1) and (0) with initial data \( x \) at \( t = 0 \), respectively. Assume that \( f^i \in C^1_b(\mathbb{R}^n) \) and \( \sigma^i \in C^b_0(\mathbb{R}^n) \) for all \( i = 1, \cdots, n \) and \( j = 1, \cdots, l \). Then, for every \( T > 0 \) we have
\[
\lim_{\delta \to 0^+} E \left[ \sup_{t \in [0, T]} |u_\delta(t, \omega, x) - u(t, \omega, x)|^2 \right] = 0,
\]
where \( C^k_b(\mathbb{R}^n) \) is the usual space of \( C^k \) smooth functions from \( \mathbb{R}^n \) to \( \mathbb{R} \) with bounded derivatives up to order \( k \in \mathbb{N} \).

The second result is on the existence of center-manifolds at stationary solution \( u = 0 \) for the random differential equation
\[
\dot{u}_\delta = Au_\delta + f(u_\delta) + \sigma(u_\delta)z_\delta(\theta_t \omega),
\]
for each \( \delta > 0 \), where \( A \) is a partially hyperbolic \( n \times n \) matrix, \( f \) and \( \sigma \) are nonlinear terms, and \( z_\delta(\theta_t \omega) \) is \( l \)-dimensional colored noise. Assume that \( f \) and \( \sigma \) satisfy \( f(0) = 0, \sigma(0) = 0 \) and that there exist constants \( M_0, R_0 > 0 \) and \( \epsilon_0 \in (0, 1) \) such that
\[
|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \leq M_0(\|u\|^{\epsilon_0} + \|v\|^{\epsilon_0}) \|u - v\|, \ |u|, |v| \leq R_0.
\]

Main Theorem 2. Assume that \( A \) is a partially hyperbolic \( n \times n \) matrix and \( f \) and \( \sigma \) satisfy (8) with \( f(0) = 0, \sigma(0) = 0 \). Then, equation (7) has a local Lipschitz center-manifold.

Remark 1. Replacing the colored noise by a \( l \)-dimensional white noise \( W(t) \) in equation (7), we consider the corresponding Stratonovich stochastic differential equation
\[
du = (Au + f(u)) dt + \sigma(u) \circ dW.
\]
By the theory due to Kunita, etc, this equation generates a random dynamical system. However, it is not known how to establish the existence of its center manifolds.

The last result is on the convergence of the center manifolds of the random differential equations with linear multiplicative noise. More precisely, we consider the random differential equation driven by a linear multiplicative colored noise
\[
\dot{u}_\delta = Au_\delta + f(u_\delta) + u_\delta z_\delta(\theta_t \omega),
\]
and the corresponding Stratonovich stochastic differential equation
\[
du = (Au + f(u)) dt + u \circ dW.
\]
Here \( A \) is a \( n \times n \) matrix and \( f(0) = 0, z_\delta(\theta_t \omega) \) and \( W \) are one dimensional noises. Note that in this case, \( u = 0 \) is a stationary solution.

Main Theorem 3. Assume that \( A \) is a partially hyperbolic \( n \times n \) matrix and \( f \) is globally Lipschitz continuous with \( f(0) = 0 \). Then, there exists \( \epsilon_0 > 0 \) such that if the Lipschitz constant of \( f \) \( \text{Lip}(f) < \epsilon_0 \), then both equation (11) and equation (10) have global center manifolds, and the center manifold of equation (10) converges pathwise to that of equation (11).

Related to this paper, in [38], the authors investigated the center-manifolds of random equations driven by the stationary process \( \mathcal{G}(\theta_t \omega) = (\omega(t + \delta) - \omega(t))/\delta \), which is also an approximation of the white noise in the sense that
\[
\lim_{\delta \to 0^+} \sup_{t \in [0, T]} \left| \int_0^t \mathcal{G}_\delta(\theta_{s\delta} \omega) ds - \omega(t) \right| = 0.
\]
for each \( T > 0 \). Note that \( G(\theta_t\omega) \) is a local approximation of the white noise, while \( z_\delta(\theta_t\omega) \) is a non-local approximation. Because of this difference, new ideas and techniques are needed for the proof of convergence of solutions and center manifolds for the colored noise in the present paper.

Using deterministic differential equations to approximate stochastic differential equations was introduced by Wong and Zakai in their pioneer work [47, 48] in which they studied both piecewise linear approximations and piecewise smooth approximations for one-dimensional Brownian motions and proved the convergence of solutions of the approximated equations. Their work was later extended to stochastic differential equations of higher dimensions, for example, by McShane [28], Strook and Varadhan [40], Sussmann [41, 42], Ikeda, Nakao and Yamato [17], Ikeda and Watanabe [16], Shmatkov [39] and recently by Kelly and Melbourne [22]. The results of the Wong-Zakai approximations have also been generalized to stochastic differential equations driven by martingales and semimartingales, see for example, Nakao-Yamato [32], Konecny [24], Protter [35], Nakao [31], and Kurtz-Protter [25, 26].

The study of invariant manifolds dates back to Hadamard [13], Lyapunov [27] and Perron [33]. Currently, there is a large body of literature in this area, see, for example, for deterministic center-manifolds, Pliss [34], Kelley [21], Hale [14], Henry [15], Carr [4], Vanderbauwhede-Van Gils [44], Chow-Lu [6, 7] and Bates-Jones [3]. For random invariant manifolds, the reader is referred to Wanner [46], Arnold [2], Mohammed-Scheutzow [29], Schmalfuss [37], Duan-Lu-Schmalfuss [8, 9], and Mohammed-Zhang-Zhao [30].

We organize this paper as follows. In the next section, we study equations (1) and (6) and establish the convergence of the solutions as \( \delta \to 0^+ \). In section 3, we show the existence of center-manifold for random differential equations driven by colored noise with a nonlinear diffusion term. In section 4, we establish the convergence of center manifolds of equations (10).

2. Convergence of solutions of RDEs driven by colored noise. In this section, we consider the stochastic differential equation

\[
\dot{u} = f(u)dt + \sigma(u) \circ dW, \quad u(0) = x, \quad x \in \mathbb{R}^n
\]  

and its corresponding equation driven by colored noise of the form

\[
\dot{u}_\delta = f(u_\delta) + \sigma(u_\delta)z_\delta(\theta_t\omega), \quad u_\delta(0) = x, \quad x \in \mathbb{R}^n,
\]  

where \( W(t) = (W_1(t), \ldots, W_l(t)) \) is a two-sided Brownian motion and \( z_\delta(\theta_t\omega) \) is the colored noise given by (4).

Throughout this section, we assume that \( f \) and \( \sigma \) are Lipschitz continuous functions, i.e., there is a constant \( L > 0 \) such that for all \( u, v \in \mathbb{R}^n \)

\[
|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \leq L|u - v|.
\]  

For the classical Wiener space \((\Omega, \mathcal{F}, \mathbb{P})\), from the law of logarithms, it follows that there exists a \( \theta_t \)-invariant subset \( \Omega \) of \( \Omega \) of full measure with sublinear growth:

\[
\lim_{s \to \pm \infty} \frac{|\omega(s)|}{|s|} = 0.
\]

Let

\[
C_\omega = \sup_{s \in \mathbb{Q}} \frac{|\omega(s)|}{|s| + 1}.
\]
where \( Q \) is the set of rational numbers. By the pathwise continuity of the Wiener process, we find that \( C_\omega : \Omega \to \mathbb{R}^+ \) is a measurable function and
\[
|\omega(s)| \leq C_\omega(|s| + 1)
\]
for all \( s \in \mathbb{R} \). Recall that \( \theta_t \omega(s) = \omega(s + t) - \omega(t) \), it then follows that
\[
C_{\theta_t \omega} \leq 2C_\omega(|t| + 1).
\]
Thus
\[
|z_\delta(\theta_t \omega)| \leq K_\delta C_\omega(|t| + 1),
\]
where \( K_\delta := \frac{3}{2}(\delta + 1) \). This estimate plays a key role in the proof of well-posedness of equation (13).

Consider the space \((\tilde{\Omega}, \mathcal{F}, \tilde{\mathbb{P}})\) where \( \mathcal{F} = \{\Omega \cap A, \ A \in \mathcal{F}\} \) and \( \tilde{\mathbb{P}} \) is the restriction of \( \mathbb{P} \) to \( \mathcal{F} \). We will restrict our study in \((\Omega, \mathcal{F}, \mathbb{P})\), which is again denoted by \((\Omega, \mathcal{F}, \mathbb{P})\).

Let \( \mathcal{N} \) be the collection of all null sets of \((\Omega, \mathcal{F}, \mathbb{P})\). Given \( t \in \mathbb{R} \), denoted by
\[
\mathcal{F}_t := \forall s \leq t \mathcal{F}_t^s, \ \forall t \in \mathbb{R}
\]
with
\[
\mathcal{F}_t^s := \sigma(\omega(p) - \omega(q) : s \leq p \leq q \leq t) \vee \mathcal{N}, \ \forall s \leq t,
\]
where \( \sigma(\omega(p) - \omega(q) : s \leq p \leq q \leq t) \) is the smallest \( \sigma \)-algebra generated by the random variables \( \omega(p) - \omega(q) \) for \( s \leq p \leq q \leq t \). By Arnold [2, p.91], we have
\[
\mathcal{F}_t^{s+} := \cap_{u>0} \mathcal{F}_t^u = \mathcal{F}_t^{-}, \quad \mathcal{F}_t^{-} := \cap_{u<s} \mathcal{F}_t^u = \mathcal{F}_t^s
\]
and \( \theta_t^{-1} \mathcal{F}_t = \mathcal{F}_{s+t}^t \) for \( s \leq t \). Hence \((\Omega, \mathcal{F}, (\theta_t)_{t \in \mathbb{R}}, (\mathcal{F}_t^s)_{s \leq t})\) is a filtered dynamical system.

**Proposition 1.** Assume (14) holds. Then, for each \( \delta > 0 \) we have the following:

(i) equation (13) has a unique solution \( u_\delta(t, \omega, x) \) defined for all \( 0 \leq t < +\infty \);
(ii) \( u_\delta(t, \omega, x) \) is Lipschitz continuous in \( x \);
(iii) \( u_\delta(t, \omega, x) \) is \( \mathcal{F}_t \) measurable;
(iv) \( u_\delta(\cdot, \cdot, \cdot) \) is \( B((0, +\infty)) \otimes \mathcal{F} \otimes B(\mathbb{R}^n) \) measurable;
(v) \( u_\delta(t, \omega, x) \) generates a random dynamical system.

Since the proof of this proposition follows from the standard arguments, we omit it.

Let
\[
Z_\delta(t, \omega) := (Z_\delta(t, \omega_1), Z_\delta(t, \omega_2), \ldots, Z_\delta(t, \omega_l)),
\]
where for each \( 1 \leq i \leq l \),
\[
Z_\delta(t, \omega_i) := \int_0^t z_\delta(s, \omega_i) ds = \int_0^t z_\delta(\theta_s \omega_i) ds.
\]
Then random differential equation (13) can be written as
\[
\dot{u}_\delta = f(u_\delta) + \sigma(u_\delta)Z_\delta(t, \omega), \ u_\delta(0) = x, \ x \in \mathbb{R}^n.
\]
For any \( T > 0 \), in what follows, we shall show that the solutions of equation (16) converge in mean square to the solutions of equation (12) uniformly on \([0, T]\) as \( \delta \to 0^+ \).

From now on, we use \( K \) to denote a generic positive constant whose value may change from line to line, but does not depend on \( \delta \).

We first use the Burkholder-Davis-Gundy inequality to obtain the following estimates.
Lemma 2.1. Let $W_0$ be a two-sided real-valued Brownian motion and $f : \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{-\infty}^{0} |f(\tau)|^2 \, d\tau < \infty$. Then for arbitrary $p > 0$ we have

$$E \left| \int_{-\infty}^{0} f(\tau) \, dW_0(\tau) \right|^p \leq K \left( \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau \right)^{\frac{p}{2}}.$$

Particularly, $E \left| \int_{-\infty}^{0} f(\tau) \, dW_0(\tau) \right|^2 = \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau$.

Proof. Let $\tilde{W}_0(s) = W_0(-s)$ for $s \in \mathbb{R}$. Then we find that

$$\int_{-\infty}^{0} f(\tau) \, dW_0(\tau) = - \int_{0}^{+\infty} f(-s) \, d\tilde{W}(s). \quad (17)$$

Applying the Itô isometry to the right-hand side of the above identity, we get

$$E \left| \int_{0}^{\infty} f(-s) \, d\tilde{W}_0(s) \right|^2 = \int_{0}^{\infty} |f(-s)|^2 \, ds = \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau,$$

which along with (17) implies

$$E \left| \int_{-\infty}^{0} f(\tau) \, dW_0(\tau) \right|^2 = \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau.$$

On the other hand, by the Burkholder-Davis-Gundy inequality, for any $u > 0$ we have

$$E \left| \int_{0}^{u} f(-s) \, d\tilde{W}_0(s) \right|^p \leq E \sup_{t \in [0,u]} \left| \int_{0}^{t} f(-s) \, d\tilde{W}_0(s) \right|^p \leq K \left( \int_{0}^{u} |f(-s)|^2 \, ds \right)^{\frac{p}{2}}. \quad (18)$$

Therefore, we obtain

$$E \left| \int_{0}^{\infty} f(-s) \, d\tilde{W}_0(s) \right|^p \leq K \left( \int_{0}^{\infty} |f(-s)|^2 \, ds \right)^{\frac{p}{2}} = K \left( \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau \right)^{\frac{p}{2}}$$

which together with (17) yields

$$E \left| \int_{-\infty}^{0} f(\tau) \, dW_0(\tau) \right|^p \leq K \left( \int_{-\infty}^{0} |f(\tau)|^2 \, d\tau \right)^{\frac{p}{2}},$$

as desired. \hfill \square

The following lemma is a summary of basic properties of the approximations of a Brownian motion.

Lemma 2.2. For every $1 \leq i \leq l$, we have

(1) $Z_\delta(t + k\delta, \omega_i) = Z_\delta(t, \theta_{k\delta}\omega_i) + Z_\delta(k\delta, \omega_i)$ for all $k \in \mathbb{N}$ and $t \geq 0$;

(2) $Z_\delta(0, \omega_i) = 0$;

(3) $E \left( \int_{k\delta}^{(k+1)\delta} |Z_\delta(s, \omega_i)| \, ds \right)^6 \leq K\delta^3$ for all $k \in \mathbb{N}$;

(4) for any $i_1, \ldots, i_m \in \{1, \ldots, l\}$, if $p_1, p_2, \ldots, p_m \geq 1$ and $p_1 + p_2 + \cdots + p_m \leq 6$,

$$E \left[ \left( \int_{k\delta}^{(k+1)\delta} |Z_\delta(s, \omega_{i_1})| \, ds \right)^{p_1} \left( \int_{k\delta}^{(k+1)\delta} |Z_\delta(s, \omega_{i_2})| \, ds \right)^{p_2} \cdots \left( \int_{k\delta}^{(k+1)\delta} |Z_\delta(s, \omega_{i_m})| \, ds \right)^{p_m} \right] \leq K\delta^{\frac{1}{2}(p_1 + p_2 + \cdots + p_m)} \forall k \in \mathbb{N};$$
(5) for any $i_1, \ldots, i_m \in \{1, \ldots, l\}$ and $n_1, n_2, \ldots, n_m \in \mathbb{N}^+$, if $p_1, p_2, \ldots, p_m \geq 1$ and $p_1 + p_2 + \cdots + p_m \leq 6$,

$$E\left[\left(\int_0^{n_1} |Z_{\delta}(s, \omega_{i_1})| ds\right)^{p_1} \left(\int_0^{n_2} |Z_{\delta}(s, \omega_{i_2})| ds\right)^{p_2} \cdots \left(\int_0^{n_m} |Z_{\delta}(s, \omega_{i_m})| ds\right)^{p_m}\right] \leq Kn_1^{p_1} n_2^{p_2} \cdots n_m^{p_m} \delta^{\frac{3}{2}(p_1 + p_2 + \cdots + p_m)}.$$

Proof. We first notice that for any $1 \leq i \leq l$,

$$\frac{d[Z_{\delta}(t + k\delta, \omega_i) - Z_{\delta}(t, \theta_k \omega_i)]}{dt} = z_{\delta}(\theta_{t + k\delta} \omega_i) - z_{\delta}(\theta_t \omega_i) = 0.$$

This implies that $Z_{\delta}(t + k\delta, \omega_i) = Z_{\delta}(t, \theta_k \omega_i)$ is constant and hence Property (1) is true. Property (2) is obvious. Property (4) follows from the Hölder inequality and Property (3). To show Property (3), we first use Property (1) and the invariance of the probability measure $P$ under $\theta_t$ to get

$$E\left[\int_0^{(k+1)\delta} |Z_{\delta}(s, \omega_i)| ds\right]^6 = E\left[\int_0^{\delta} |Z_{\delta}(s, \omega_i)| ds\right]^6.$$

Then, by the Hölder inequality, Fubini’s theorem and Lemma 2.1, we obtain

$$E\left(\int_0^{\delta} |Z_{\delta}(s, \omega_i)| ds\right)^6 \leq \delta^5 \int_0^{\delta} E|z_{\delta}(\theta_s \omega_i)|^6 ds = \delta^5 \int_0^{\delta} E|z_{\delta}(\omega_i)|^6 ds = \delta^{-1} \int_0^{\delta} E \left| \int_0^0 e^{\frac{s}{\delta} dW_i(\tau)} \right|^6 ds \leq K\delta^{-1} \int_0^{\delta} \left[ \int_0^{\delta} e^{\frac{2\tau}{\delta}} d\tau \right]^3 ds \leq K\delta^3.$$

Similarly, for each $j = 1, \ldots, m$, we have

$$E\left(\int_0^{n_j \delta} |Z_{\delta}(s, \omega_{i_j})| ds\right)^6 \leq Kn_j^6 \delta^3,$$

which together with the Hölder inequality yields (5), and hence completes the proof.

The next lemma is on the moments of the approximations.

Lemma 2.3. For each $1 \leq i \leq l$ and $t \geq 0$, $EZ_{\delta}(t, \omega_i)^2 = t - \delta(1 - e^{-\frac{t}{\delta}})$ and $EZ_{\delta}(t, \omega_i)^4 \leq K(t + \delta)^2$.

Proof. Applying Fubini’s theorem, we first note that

$$Z_{\delta}(t, \omega_i) = \int_0^t \left[ \frac{1}{\delta} \int_{-\infty}^s e^{\frac{\tau - \delta}{\delta}} dW_i(\tau) \right] ds.$$
\begin{align*}
\frac{1}{\delta} \int_0^t e^{-\frac{t-s}{\delta}} ds \right] dW_i(\tau) + \int_0^t \left[ \frac{1}{\delta} e^{-\frac{t-s}{\delta}} \right] dW_i(\tau) \\
= \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right] dW_i(\tau) + \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right] dW_i(\tau).
\end{align*}

By the independence of $\int_0^t \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right] dW_i(\tau)$ and $\int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right] dW_i(\tau)$ and the Itô isometry, we get

$$EZ_\delta(t,\omega)^2 = E \left\{ \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right] dW_i(\tau) \right\}^2 + E \left\{ \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right] dW_i(\tau) \right\}^2$$

$$= \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right]^2 d\tau + \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right]^2 d\tau$$

$$= t - \delta (1 - e^{-\frac{t}{\delta}}).$$

Similarly, we find that

$$EZ_\delta(t,\omega)^4 = E \left\{ \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right] dW_i(\tau) + \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right] dW_i(\tau) \right\}^4$$

$$\leq KE \left\{ \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right] dW_i(\tau) \right\}^4 + KE \left\{ \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right] dW_i(\tau) \right\}^4$$

$$\leq K \left\{ \int_{-\infty}^0 \left[ e^{\frac{\tau}{\delta}} - e^{-\frac{\tau}{\delta}} \right]^2 d\tau \right\}^2 + K \left\{ \int_0^t \left[ 1 - e^{-\frac{t-s}{\delta}} \right]^2 d\tau \right\}^2$$

$$\leq K(1 - e^{-\frac{t}{\delta}})^4 \delta^2 + K(t - \frac{3\delta}{2} + 2\delta e^{-\frac{t}{\delta}} - \frac{\delta}{2} e^{-\frac{2t}{\delta}})^2$$

$$\leq K(t + \delta)^2,$$

as desired. \hfill \Box

Note that equation (16) can be written as

$$u_\delta(t,\omega) - x^i = \int_0^t f^i(u_\delta(s,\omega))ds + \sum_{j=1}^l \int_0^t \sigma^{ij}(u_\delta(s,\omega))\dot{Z}_\delta(s,\omega_j)ds.$$  \hfill (20)

Similarly, by using the Itô integral, equation (12) takes the form

$$u^t(t,\omega) - x^i = \int_0^t f^i(u(s,\omega))ds + \sum_{j=1}^l \int_0^t \sigma^{ij}(u(s,\omega))dW_j$$

$$+ \frac{1}{2} \sum_{j=1}^l \sum_{\alpha=1}^n \int_0^t (\sigma^{\alpha j} \partial_\alpha \sigma^{ij})(u(s,\omega))ds \quad \forall i = 1, \cdots, n,$$

where $\partial_\alpha \sigma^{ij} = \frac{\partial \sigma^{ij}}{\partial x^\alpha}$. Now we state our main result as follows.

**Theorem 2.4.** Assume that $f^i \in C^1_b(\mathbb{R}^n)$ and $\sigma^{ij} \in C^2_b(\mathbb{R}^n)$ for all $i = 1, \cdots, n$ and $j = 1, \cdots, l$. Then for every $T > 0$ we have

$$\lim_{\delta \to 0^+} E \left[ \sup_{t \in [0,T]} |u_\delta(t,\omega) - u(t,\omega)|^2 \right] = 0.$$
The proof of this theorem consists of four steps to derive uniform estimates on the difference of solutions of equations (20) and (21) over various time intervals. First, as in [16], we choose an integer function \( n : [0, 1] \rightarrow \mathbb{N} \) such that \( n(\delta) \uparrow +\infty \), \( n(\delta)^4 \delta \downarrow 0 \), and \( n(\delta)^4 \delta \rightarrow +\infty \) as \( \delta \downarrow 0 \). This is possible since we may take \( n(\delta) = \lfloor \delta^{-1/4.5} \rfloor \). Let \( \tilde{\delta} := n(\delta) \delta \). For any \( s \in \mathbb{R}^+ \), if \( k\tilde{\delta} \leq s < (k+1)\tilde{\delta} \), we define \( [s] = (k+1)\tilde{\delta} \) and \( |s| = k\tilde{\delta} \), respectively. Set \( m(s) = \lfloor s/\tilde{\delta} \rfloor \). We divide the interval \([0, +\infty)\) into equal subintervals of length \( \tilde{\delta} \) by using partition points

\[
0 = \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_k < \cdots ,
\]

where \( \tilde{t}_k = k\tilde{\delta} \).

For each \( i = 1, \ldots, n \), using (20) and (21), we write the difference \( u^i_\delta(t, \omega) - u^i(t, \omega) \) as a sum of four terms, i.e.,

\[
u^i_\delta(t, \omega) - u^i(t, \omega) = \Lambda(t) + \Pi(0, \tilde{\delta}) + \Pi(\tilde{\delta}, |t|\tilde{\delta}) + \Pi(|t|\tilde{\delta}, t),
\]

where

\[
\Lambda(t) := \int_0^t f^i(u(s, \omega))ds - \int_0^t f^i(u(s, \omega))ds,
\]

\[
\Pi(t_1, t_2) := \Pi_1(t_1, t_2) + \Pi_2(t_1, t_2) + \Pi_3(t_1, t_2),
\]

\[
\Pi_1(t_1, t_2) := \sum_{j=1}^l \int_{t_1}^{t_2} \sigma^{ij}(u_\delta(s, \omega))\dot{Z}_{\delta}(s, \omega_j)ds,
\]

\[
\Pi_2(t_1, t_2) := -\sum_{j=1}^l \int_{t_1}^{t_2} \sigma^{ij}(u(s, \omega))dW_j,
\]

\[
\Pi_3(t_1, t_2) := -\frac{1}{2} \sum_{j=1}^l \sum_{\alpha=1}^n \int_{t_1}^{t_2} (\sigma^{\alpha j} \partial_\alpha \sigma^{ij})(u(s, \omega))ds
\]

for \( 0 \leq t_1 < t_2 \).

First, for \( \Lambda(t) \), it is clear that we have:

**Lemma 2.5.** There is a constant \( K > 0 \) such that for each \( s_1 \in [0, T] \)

\[
E \left[ \sup_{t \in [0, s_1]} |\Lambda(t)|^2 \right] \leq K \int_0^{s_1} E|u_\delta(s, \omega) - u(s, \omega)|^2ds.
\]

Next, we estimate \( \Pi(|t|\tilde{\delta}, t) \) and have the following lemma.

**Lemma 2.6.** As \( \delta \rightarrow 0^+ \), we have

\[
E \left[ \sup_{t \in [0, T]} |\Pi(|t|\tilde{\delta}, t)|^2 \right] = o(1).
\]

**Proof.** Recall from (22) that

\[
\Pi(|t|\tilde{\delta}, t) = \Pi_1(|t|\tilde{\delta}, t) + \Pi_2(|t|\tilde{\delta}, t) + \Pi_3(|t|\tilde{\delta}, t).
\]

For \( \Pi_1(|t|\tilde{\delta}, t) \), using the Hölder inequality, we have

\[
E \left[ \sup_{t \in [0, T]} |\Pi_1(|t|\tilde{\delta}, t)|^2 \right] \leq K \sum_{j=1}^l E \sup_{t \in [0, T]} \left( \int_{|t|\tilde{\delta}}^{t} |\dot{Z}_{\delta}(s, \omega_j)|ds \right)^2
\]
\[
\leq K \sum_{j=1}^{l} \left( E \left( \sup_{t \in [0,T]} \int_{|t|\tilde{\delta}}^{t} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \right)^{\frac{1}{2}} \\
\leq K \sum_{j=1}^{l} \left[ E \left( \max_{0 \leq k \leq m(T)} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \right]^{\frac{1}{2}} \\
\leq K \sum_{j=1}^{l} \left[ \sum_{k=0}^{m(T)} E \left( \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \right]^{\frac{1}{2}}.
\]

By Lemma 2.2 (1) and (5) and the invariance of probability measure \(\mathbb{P}\), we have
\[
E \left( \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \\
= E \left( \int_{0}^{\tilde{\delta}} |\tilde{Z}_{\delta}(s+k\tilde{\delta},\omega_{j})|ds \right)^{4} \\
= E \left( \int_{0}^{\tilde{\delta}} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \\
= E \left( \int_{0}^{\tilde{\delta}} |\tilde{Z}_{\delta}(s,\omega_{j})|ds \right)^{4} \leq Kn(\delta)^{4}\delta^{2}.
\]

Thus, from (23), we have
\[
E \left[ \sup_{t \in [0,T]} |\Pi_{1}(\{t\}\tilde{\delta}, t)|^{2} \right] \\
\leq K \left[ m(T)n(\delta)^{4}\delta^{2} \right]^{\frac{1}{2}} \\
\leq K \left[ \frac{n(\delta)^{4}\delta^{2}}{n(\delta)\delta} \right]^{\frac{1}{2}} = K \left( n^{3}(\delta)\delta \right)^{\frac{1}{2}} \to 0 \text{ as } \delta \to 0^{+}.
\]

For \(\Pi_{2}(\{t\}\tilde{\delta}, t)\), we split it into two parts.
\[
\Pi_{2}(\{t\}\tilde{\delta}, t) = -\sum_{j=1}^{l} \Pi_{21}(\{t\}\tilde{\delta}, t) - \sum_{j=1}^{l} \Pi_{22}(\{t\}\tilde{\delta}, t),
\]

where
\[
\Pi_{21}(\{t\}\tilde{\delta}, t) := \sigma^{ij}(u(\{t\}\tilde{\delta}, \omega))\omega_{j}(t) - \omega_{j}(\{t\}\tilde{\delta}),
\]
\[
\Pi_{22}(\{t\}\tilde{\delta}, t) := \int_{|t|\tilde{\delta}}^{t} \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u(\{s\}\tilde{\delta}, \omega)) \right] dw_{j}(s).
\]

For \(\Pi_{21}(\{t\}\tilde{\delta}, t)\), we obtain
\[
E \left[ \sup_{t \in [0,T]} |\Pi_{21}(\{t\}\tilde{\delta}, t)|^{2} \right] \\
\leq KE \left[ \sup_{t \in [0,T]} |\omega_{j}(t) - \omega_{j}(\{t\}\tilde{\delta})|^{2} \right] \\
\leq K \left[ E \max_{0 \leq k \leq m(T)} \sup_{0 \leq t \leq \delta} |\omega_{j}(t+k\tilde{\delta}) - \omega_{j}(k\tilde{\delta})|^{4} \right]^{\frac{1}{2}}.
\]
\begin{align*}
&\leq K \left[ \sum_{k=0}^{m(T)} E|\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})|^4 \right]^{\frac{1}{2}} \\
&\leq K \left[ m(T)\tilde{\delta}^2 \right]^{\frac{1}{2}} \leq K \tilde{\delta}^2 \to 0 \text{ as } \delta \to 0^+,
\end{align*}

where the third inequality follows from martingale inequality and the fourth one follows from $\omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta})$ has the same distribution as $\omega_j(\tilde{\delta})$ and the Brownian scaling property.

For $\Pi_{22}(|t|\tilde{\delta}, t)$ we get that

\begin{align*}
E & \left[ \sup_{t \in [0,T]} |\Pi_{22}(|t|\tilde{\delta}, t)|^2 \right] \\
&\leq E \left\{ \sup_{0 \leq k \leq m(T)} \sup_{t \in [0,\delta]} \left[ \int_{k\tilde{\delta}}^{k\tilde{\delta}+t} [\sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))]dW_j(s) \right]^2 \right\} \\
&\leq \sum_{k=0}^{m(T)} E \left\{ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))] \right]^2 ds \right\} \\
&\leq K \sum_{k=0}^{m(T)} E \left\{ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))] \right]^2 ds \right\}.
\end{align*}

Here the martingale inequality is used to get the last estimate. Then, using the Itô isometry, we obtain

\begin{align*}
&\leq K \sum_{k=0}^{m(T)} \left\{ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))] \right]^2 ds \right\} \\
&= \sum_{k=0}^{m(T)} E \left\{ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))] \right]^2 ds \right\} \\
&= \sum_{k=0}^{m(T)} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} E \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s](\tilde{\delta}), \omega))] \right]^2 ds.
\end{align*}

Since $f \in C^1_b$ and $\sigma \in C^2_b$, by (21) we have that for $t > s$

\begin{align*}
E|u^i(t, \omega) - u^i(s, \omega)|^2 &\leq KE \left( \int_s^t f^i(u(r, \omega))dr \right)^2 \\
&+ KE \left( \sum_{j=1}^{l} \int_s^t \sigma^{ij}(u(r, \omega))dw_j(r) \right)^2 \\
&+ K \left( \frac{1}{2} \sum_{j=1}^{l} \sum_{\alpha=1}^{n} \int_s^t \sigma^{\alpha j} \partial_{\alpha} \sigma^{ij}(u(r, \omega))dr \right)^2 \\
&\leq K((t-s)^2 + (t-s)) \forall i = 1, \ldots, n.
\end{align*}
Hence,

\[
\sum_{k=0}^{m(T)} \int_{k\delta}^{(k+1)\delta} E \left[ \sigma^{ij}(u(s, \omega)) - \sigma^{ij}(u([s][\tilde{\delta}], \omega)) \right]^2 ds 
\leq K m(T) \tilde{\delta}^2 (\tilde{\delta} + 1) \leq K \tilde{\delta} (\tilde{\delta} + 1) \to 0 \text{ as } \delta \to 0^+.
\]

Therefore, we have

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{22}(|t|\tilde{\delta}, t)|^2 \right] = o(1) \text{ as } \delta \to 0^+.
\]

Since \( \sigma \) has bounded derivatives, we have

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{3}(|t|\tilde{\delta}, t)|^2 \right] \leq K \tilde{\delta}^2.
\]

Combining (22) and (24)-(26), we have

\[
E \left[ \sup_{t \in [0, T]} |\Pi(|t|\tilde{\delta}, t)|^2 \right] = o(1) \text{ as } \delta \to 0^+.
\]

This completes the proof of the lemma. \( \square \)

Since \( |\Pi(0, \tilde{\delta})|^2 \leq \sup_{t \in [0, T]} |\Pi(|t|\tilde{\delta}, t)|^2 \), we have

**Lemma 2.7.** The following holds

\[
E \left[ |\Pi(0, \tilde{\delta})|^2 \right] = o(1) \text{ as } \delta \to 0^+.
\]

Finally, we estimate \( \Pi(\tilde{\delta}, |t|\tilde{\delta}) \). Recall from (22) that

\[
\Pi(\tilde{\delta}, |t|\tilde{\delta}) = \Pi_1(\tilde{\delta}, |t|\tilde{\delta}) + \Pi_2(\tilde{\delta}, |t|\tilde{\delta}) + \Pi_3(\tilde{\delta}, |t|\tilde{\delta}),
\]

where

\[
\Pi_1(\tilde{\delta}, |t|\tilde{\delta}) := \sum_{j=1}^{l} \int_{0}^{\Delta_j |t|\tilde{\delta}} \sigma^{ij}(u_\delta(s, \omega)) \tilde{Z}_\delta(s, \omega_j) ds,
\]

\[
\Pi_2(\tilde{\delta}, |t|\tilde{\delta}) := -\sum_{j=1}^{l} \int_{0}^{\Delta_j |t|\tilde{\delta}} \sigma^{ij}(u(s, \omega)) dW_j,
\]

\[
\Pi_3(\tilde{\delta}, |t|\tilde{\delta}) := -\frac{1}{2} \sum_{j=1}^{l} \sum_{\alpha=1}^{n} \int_{0}^{\Delta_j |t|\tilde{\delta}} (\sigma^{\alpha j} \partial_\alpha \sigma^{ij})(u(s, \omega)) ds.
\]

We first rewrite \( \Pi_1(\tilde{\delta}, |t|\tilde{\delta}) \) and use integration by parts to have

\[
\Pi_1(\tilde{\delta}, |t|\tilde{\delta}) = -\sum_{j=1}^{l} \sum_{k=1}^{\frac{m(t)-1}{\delta}} \int_{k\delta}^{(k+1)\delta} \sigma^{ij}(u_\delta(s, \omega)) d\left(Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(s, \omega_j)\right)
\]

\[
= \sum_{j=1}^{l} \Pi_{11}(\tilde{\delta}, |t|\tilde{\delta}) + \sum_{j=1}^{l} \sum_{\alpha=1}^{n} \Pi_{12}(\tilde{\delta}, |t|\tilde{\delta}),
\]
where

\[
\Pi_{11}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta}, \omega))(Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(k\tilde{\delta}, \omega_j)),
\]

\[
\Pi_{12}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \partial_\alpha \sigma^{ij}(u_\delta(s, \omega))(f^\alpha(u_\delta(s, \omega)) \\
+ \sum_{\beta=1}^l \sigma^{\alpha \beta}(u_\delta(s, \omega)) Z_\delta(s, \omega_\beta)) \left( Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(s, \omega_j) \right) ds.
\]

From now on, we always assume that \(\delta^{0.9} \leq \tilde{\delta}\) since \(n(\delta)^{10} \delta \to +\infty\) as \(\delta \to 0^+\). Note that

\[
\int_{\tilde{\delta}}^{(t)\tilde{\delta}} \sigma^{ij}(u_\delta([s]([t](\tilde{\delta}) - \delta^{0.9}, \omega)) dW_j(s) \\
= \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \left( \omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta}) \right).
\]

We then write \(\Pi_{11}(\tilde{\delta}, [t](\tilde{\delta}))\) as a sum of three parts.

\[
\Pi_{11}(\tilde{\delta}, [t](\tilde{\delta})) = \int_{\tilde{\delta}}^{(t)\tilde{\delta}} \sigma^{ij}(u_\delta([s]([t](\tilde{\delta}) - \delta^{0.9}, \omega)) dW_j(s) + \Pi_{111}(\tilde{\delta}, [t](\tilde{\delta})) \\
+ \Pi_{1}'(\tilde{\delta}, [t](\tilde{\delta})),
\]

where

\[
\Pi_{111}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \left( \sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \right) \\
\times \left( Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(k\tilde{\delta}, \omega_j) \right),
\]

\[
\Pi_{1}'(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \left[ Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(k\tilde{\delta}, \omega_j) \\
- \left( \omega_j((k+1)\tilde{\delta}) - \omega_j(k\tilde{\delta}) \right) \right].
\]

On the one hand, by Fubini’s theorem, we find that

\[
Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(k\tilde{\delta}, \omega_j) \\
= \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \frac{1}{\delta} \int_{-\infty}^s e^{\frac{s-t}{\delta}} dW_j(\tau) \right] ds \\
= \int_{-\infty}^{k\tilde{\delta}} \left[ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \frac{1}{\delta} e^{\frac{s-t}{\delta}} ds \right] dW_j(\tau) + \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \frac{1}{\delta} e^{\frac{s-t}{\delta}} ds \right] dW_j(\tau) \\
= \int_{-\infty}^{k\tilde{\delta}} \left[ e^{\frac{s-k\tilde{\delta}}{\delta}} - e^{\frac{s-(k+1)\tilde{\delta}}{\delta}} \right] dW_j(\tau) + \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left[ 1 - e^{\frac{s-(k+1)\tilde{\delta}}{\delta}} \right] dW_j(\tau).
\]

This implies that we can split \(\Pi_{11}'(\tilde{\delta}, [t](\tilde{\delta}))\) into

\[
\Pi_{11}'(\tilde{\delta}, [t](\tilde{\delta})) = \Pi_{112}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})),
\]
Then \( \Pi \) can be rewritten as

\[
\Pi_{11}(\tilde{\delta}, [t](\tilde{\delta})) = \int_{\tilde{\delta}}^{[t](\tilde{\delta})} \sigma^{ij}(u_{\delta}([s](\tilde{\delta}) - \delta^{0.9}, \omega)) dW_j(s) + \Pi_{111}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{112}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})).
\]

Hence, we can write \( \Pi(\tilde{\delta}, [t](\tilde{\delta})) \) as

\[
\Pi(\tilde{\delta}, [t](\tilde{\delta})) = \sum_{j=1}^{l} \left( \Upsilon_{1}^{j}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{111}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{112}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) + \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})) \right)
\]

where

\[
\Upsilon_{1}^{j}(\tilde{\delta}, [t](\tilde{\delta})) := \int_{\tilde{\delta}}^{[t](\tilde{\delta})} \left( \sigma^{ij}(u_{\delta}([s](\tilde{\delta}) - \delta^{0.9}, \omega)) - \sigma^{ij}(u(s, \omega)) \right) dW_j(s),
\]

\[
\Upsilon_{2}^{\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \Pi_{12}(\tilde{\delta}, [t](\tilde{\delta})) - \frac{1}{2} \int_{\tilde{\delta}}^{[t](\tilde{\delta})} (\sigma^{\alpha j} \partial_\alpha \sigma^{ij})(u(s, \omega)) ds.
\]

Next, we estimate each term on the right hand of equation (27), respectively. We first summarize them as follows.

**Lemma 2.8.** Let \( T > 0 \) be a fixed constant. Then we have the following estimates as \( \delta \to 0^+ \):

\[
E \left[ \sup_{t \in [0, s_1]} |\Upsilon_{1}^{j}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \int_0^{s_1} E|u_{\delta}(s, \omega) - u(s, \omega)|^2 ds + o(1), \ s_1 \in [0, T],
\]

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{111}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] = o(1),
\]

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{112}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] = o(1),
\]

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{113}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] = o(1),
\]

\[
E \left[ \sup_{t \in [0, T]} |\Pi_{114}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] = o(1),
\]

\[
E \left[ \sup_{t \in [0, s_1]} |\Upsilon_{2}^{\alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \int_0^{s_1} E|u_{\delta}(s, \omega) - u(s, \omega)|^2 ds + o(1), \ s_1 \in [0, T].
\]
Proof. We prove the statements one by one.

(I) Estimate of $\gamma^i_{\{\tilde{\delta}, \{t\}^{\tilde{\delta}}\}}$

By using the martingale inequality and the Itô isometry, we obtain

$$E \left[ \sup_{t \in [0, s_1]} |\gamma^i_{\{\tilde{\delta}, \{t\}^{\tilde{\delta}}\}}|^2 \right] \leq K \int_{\tilde{\delta}}^{s_1} E |u_\delta([s]^{\tilde{\delta}}) - \delta^{0.9}, \omega) - u(s, \omega)|^2 ds$$

$$\leq K \int_{0}^{T} E |u_\delta(s, \omega) - u(s, \omega)|^2 ds$$

$$+ K \int_{\tilde{\delta}}^{s_1} E |u_\delta([s]^{\tilde{\delta}}) - \delta^{0.9}, \omega) - u_\delta(s, \omega)|^2 ds.$$ 

From (20), we have

$$|u_\delta(s, \omega) - u_\delta([s]^{\tilde{\delta}}) - \delta^{0.9}, \omega) | \leq K \left[ \tilde{\delta} + \delta^{0.9} + \sum_{d=1}^{l} \int_{[s]^{\tilde{\delta}} - \delta^{0.9}} |\tilde{Z}_\delta(r, \omega_d)| dr \right].$$

Then, changing variable $r$ to $r + [s]^{\tilde{\delta}} - \delta^{0.9}$ in the integral, and using Lemma 2.2 (1) and (5) and the $\theta_\epsilon$-invariance of $P$, we have

$$E \left[ \sup_{t \in [0, s_1]} |\gamma^i_{\{\tilde{\delta}, \{t\}^{\tilde{\delta}}\}}|^2 \right]$$

$$\leq K \int_{0}^{T} E |u_\delta(s, \omega) - u(s, \omega)|^2 ds$$

$$+ K \int_{\tilde{\delta}}^{s_1} \left[ \tilde{\delta} + \delta^{0.9} + \sum_{d=1}^{l} E \left( \int_{[s]^{\tilde{\delta}} - \delta^{0.9}} |\tilde{Z}_\delta(r, \omega_d)| dr \right)^2 \right] ds$$

$$\leq K \int_{0}^{T} E |u_\delta(s, \omega) - u(s, \omega)|^2 ds + K [\tilde{\delta} + \delta^{0.9}]^2 + (2n(\delta))^2] \delta^2$$

$$= K \int_{0}^{T} E |u_\delta(s, \omega) - u(s, \omega)|^2 ds + o(1) \text{ as } \delta \to 0^+.$$ 

This completes the proof of property (28).

(II) Estimate of $\Pi_{111}(\tilde{\delta}, \{t\}^{\tilde{\delta}})$

Recall that

$$\Pi_{111}(\tilde{\delta}, \{t\}^{\tilde{\delta}}) := \sum_{k=1}^{m(t)-1} \left( \sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \right) \times \left( Z_\delta((k + 1)\tilde{\delta}, \omega_j) - Z_\delta(k\tilde{\delta}, \omega_j) \right).$$

Using the Cauchy inequality, we have

$$E \left[ \sup_{t \in [0, T]} |\Pi_{111}(\tilde{\delta}, \{t\}^{\tilde{\delta}})|^2 \right]$$

$$\leq E \left\{ \sum_{k=1}^{m(T)-1} \left( \sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \right)^2 \right\}$$

$$\leq E \left[ \sum_{k=1}^{m(T)-1} \left( \sigma^{ij}(u_\delta(k\tilde{\delta}, \omega)) - \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \right)^2 \right].$$
Thus, by using Lemma 2.2 (1) and the $\theta_t$-invariance of $s$, we have that
\[
\left| u^j_\delta(k\delta, \omega) - u^j_\delta(k\delta - \delta^{0.9}, \omega) \right| \leq K \left[ \delta^{0.9} + \sum_{d=1}^{l} \int_{k\delta - \delta^{0.9}}^{k\delta} |\tilde{Z}_\delta(s, \omega)| ds \right].
\]

Thus, by using Lemma 2.2 (1) and the $\theta_t$-invariance of $\mathbb{P}$, we have that
\[
E \left[ \sup_{t \in [0, T]} |\Pi_{111}(\tilde{\delta}, |t|)\tilde{\delta})|^2 \right] \leq K \left\{ m(T) \left[ \sum_{k=1}^{m(T)-1} \left( \delta^{3.6} + \sum_{d=1}^{l} \int_{k\delta - \delta^{0.9}}^{k\delta} |\tilde{Z}_\delta(s, \omega)| ds \right)^4 \right] \times m(T)^2 EZ_\delta(\tilde{\delta}, \omega)^4 \right\}^{\frac{1}{2}}.
\]
Then, changing variable $s$ to $s + k\delta - \delta^{0.9}$ and using Lemma 2.2 (1) and the $\theta_t$-invariance of $\mathbb{P}$, we have that
\[
E \left[ \sup_{t \in [0, T]} |\Pi_{111}(\tilde{\delta}, |t|)\tilde{\delta})|^2 \right] \leq K [m(T)^2(\delta^{3.6} + \delta^{1.6}) \times m(T)^2 EZ_\delta(\tilde{\delta}, \omega)^4]^{\frac{1}{2}}
\]
\[
\leq K [m(T)^2(\delta^{3.6} + \delta^{1.6})m(T)^2\delta^{2}]^{\frac{1}{2}}
\]
\[
\leq K [n(\delta)^{-2}\delta^{1.6} + (n(\delta)^{5}\delta)^{-0.4}]^{\frac{1}{2}} \to 0 \text{ as } \delta \to 0^+,
\]
where in the second inequality Lemma 2.3 is used, and the fact $m(T)\tilde{\delta} \leq T$, $\tilde{\delta} = n(\delta)\delta$ and
\[
E \left( \int_0^{\delta^{0.9}} |\tilde{Z}_\delta(s, \omega)| ds \right)^4 \leq \delta^{2.7} \int_0^{\delta^{0.9}} E|z_\delta(\theta_\omega \omega)|^4 ds
\]
\[
= \delta^{2.7} \int_0^{\delta^{0.9}} E|z_\delta(\omega)|^4 ds
\]
\[
= \delta^{-1.3} \int_0^{\delta^{0.9}} E \left( \int_0^\infty e^{\frac{z_\delta}{\tau}} dW_\tau \right)^4 ds
\]
\[
\leq K \delta^{-1.3} \int_0^{\delta^{0.9}} \left[ \int_0^\infty e^{\frac{2\tau}{\tau}} d\tau \right]^2 ds \leq K \delta^{1.0}
\]
for $d = 1, \ldots, l$ are also used. This completes the proof of property (29).

(III) Estimate of $\Pi_{112}(\tilde{\delta}, |t|)\tilde{\delta})$

Note that
\[
\Pi_{112}(\tilde{\delta}, |t|)\tilde{\delta}) = - \sum_{k=1}^{m(t)-1} \sigma^{ij}(\tilde{u}_\delta(k\delta - \delta^{0.9}, \omega)) \times \int_{k\delta}^{(k+1)\delta} e^{-\frac{(k+1)\delta}{s}} dW_j(\tau).
\]
In order to estimate $\Pi_{112}(\delta, [t](\delta))$, we set

$$M_n(\omega) := -\sum_{k=1}^{n} \sigma^{ij}(u_{\delta}(k\delta - \delta^{0.9}, \omega)) \times \int_{k\delta}^{(k+1)\delta} e^{-\frac{r-(k+1)\delta}{\delta}} dW_j(\tau).$$

Put $J_n := \mathcal{F}_{(n+1)\delta}$. Since $\sigma^{ij}(u_{\delta}(k\delta - \delta^{0.9}, \omega)) \in \mathcal{F}_{k\delta-\delta^{0.9}}$ and $\int_{k\delta}^{(k+1)\delta} e^{-\frac{r-(k+1)\delta}{\delta}} dW_j(\tau) \in \mathcal{F}_{(k+1)\delta}$, $M_n(\omega)$ is a $J_n$-martingale. In fact, we have

$$E[M_{n+1}(\omega) | J_n] = M_n(\omega) - E\left[ \sigma^{ij}(u_{\delta}((n+1)\delta - \delta^{0.9}, \omega)) \int_{(n+1)\delta}^{(n+2)\delta} e^{-\frac{r-(n+2)\delta}{\delta}} dW_j(\tau) \bigg| J_n \right].$$

Hence, by the martingale inequality and Itô isometry we obtain

$$E\left[ \sup_{t \in [0,T]} |\Pi_{112}(\delta, [t](\delta))|^2 \right] \leq KE\left[ |M_{m(T)-1}(\omega)|^2 \right]$$

$$= K \left\{ \sum_{k=1}^{m(T)-1} E\left[ \left( \sigma^{ij}(u_{\delta}(k\delta - \delta^{0.9}, \omega)) \right)^2 \left( \int_{k\delta}^{(k+1)\delta} e^{-\frac{r-(k+1)\delta}{\delta}} dW_j(\tau) \right)^2 \right] \right\}$$

$$\leq K \left\{ \sum_{k=1}^{m(T)-1} E\left( \int_{k\delta}^{(k+1)\delta} e^{-\frac{r-(k+1)\delta}{\delta}} dW_j(\tau) \right)^2 \right\}$$

$$= K \left\{ \sum_{k=1}^{m(T)-1} \int_{k\delta}^{(k+1)\delta} e^{2r-(k+1)\delta} \, dr \right\}$$

$$= Km(T) \times \frac{\delta}{2} \left( 1 - e^{-\frac{2\delta}{\delta}} \right) \leq Kn(\delta)^{-1} \rightarrow 0 \text{ as } \delta \rightarrow 0^+, \quad \text{where the first equality follows from the fact for each } 1 \leq k_1 < k_2 \leq m(T) - 1,$$

$$E\left[ \sigma^{ij}(u_{\delta}(k_1\delta - \delta^{0.9}, \omega)) \int_{k_1\delta}^{(k_1+1)\delta} e^{-\frac{r-(k_1+1)\delta}{\delta}} dW_j(\tau) \right. \times \sigma^{ij}(u_{\delta}(k_2\delta - \delta^{0.9}, \omega)) \int_{k_2\delta}^{(k_2+1)\delta} e^{-\frac{r-(k_2+1)\delta}{\delta}} dW_j(\tau)$$

$$= E\left[ \sigma^{ij}(u_{\delta}(k_1\delta - \delta^{0.9}, \omega)) \int_{k_1\delta}^{(k_1+1)\delta} e^{-\frac{r-(k_1+1)\delta}{\delta}} dW_j(\tau) \times \sigma^{ij}(u_{\delta}(k_2\delta - \delta^{0.9}, \omega)) \right]$$

$$\times E\left( \int_{k_2\delta}^{(k_2+1)\delta} e^{-\frac{r-(k_2+1)\delta}{\delta}} dW_j(\tau) \right) = 0.$$
(IV) **Estimate of \( \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) \)**

For \( \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) \), using the Itô isometry, we have

\[
E \left[ \sup_{t \in [0,T]} \left| \Pi_{113}(\tilde{\delta}, [t](\tilde{\delta})) \right|^2 \right] \leq Km(T) \sum_{k=1}^{m(T)-1} E \left| \int_{-\infty}^{k\tilde{\delta} - \delta^{0.9}} e^{r-k\tilde{\delta}} dW_j(\tau) \right|^2
\]

\[
\leq Km(T)^2 \times \frac{\delta}{2} \times e^{-2\delta^{-0.1}} = K \frac{1}{n(\delta)^2} e^{2\delta^{-0.1}} \rightarrow 0,
\]

as \( \delta \rightarrow 0^+ \), where the last limit follows from \( \delta e^{2\delta^{-0.1}} \rightarrow +\infty \) as \( \delta \rightarrow 0^+ \). This completes the proof of property (31).

(V) **Estimate of \( \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})) \)**

In order to estimate \( \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})) \), we set

\[
X_n(\omega) := \sum_{k=1}^{n} \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \times \int_{k\tilde{\delta} - \delta^{0.9}}^{(k+1)\tilde{\delta} - \delta^{0.9}} e^{r-k\tilde{\delta}} dW_j(\tau).
\]

Put \( \mathcal{P}_n := \mathcal{F}_{(n+1)\tilde{\delta} - \delta^{0.9}} \). We find that \( X_n(\omega) \) is a \( \mathcal{P}_n \)-martingale. Hence, similarly to \( \Pi_{112}(\tilde{\delta}, [t](\tilde{\delta})) \), by the martingale inequality and Itô isometry we obtain

\[
E \left[ \sup_{t \in [0,T]} \left| \Pi_{114}(\tilde{\delta}, [t](\tilde{\delta})) \right|^2 \right] \leq KE \left[ X_{m(T)-1}(\omega)^2 \right]
\]

\[
= K \left\{ \sum_{k=1}^{m(T)-1} \left( \sigma^{ij}(u_\delta(k\tilde{\delta} - \delta^{0.9}, \omega)) \right)^2 \left( \int_{k\tilde{\delta} - \delta^{0.9}}^{(k+1)\tilde{\delta} - \delta^{0.9}} e^{r-k\tilde{\delta}} dW_j(\tau) \right)^2 \right\}
\]

\[
\leq K \left\{ \sum_{k=1}^{m(T)-1} \left( \int_{k\tilde{\delta} - \delta^{0.9}}^{(k+1)\tilde{\delta} - \delta^{0.9}} e^{2(r-k\tilde{\delta})} d\tau \right)^2 \right\}
\]

\[
= Km(T) \times \frac{\delta}{2} (1 - e^{-2\delta^{0.1}}) \leq Km(\delta)^{-1} \rightarrow 0 \text{ as } \delta \rightarrow 0^+,
\]

where the first equality follows from the fact for each \( 1 \leq k_1 < k_2 \leq m(T) - 1 \),

\[
E \left[ \sigma^{ij}(u_\delta(k_1\tilde{\delta} - \delta^{0.9}, \omega)) \int_{k_1\tilde{\delta} - \delta^{0.9}}^{k_1\tilde{\delta}} e^{r-k_1\tilde{\delta}} dW_j(\tau) \times \sigma^{ij}(u_\delta(k_2\tilde{\delta} - \delta^{0.9}, \omega)) \int_{k_2\tilde{\delta} - \delta^{0.9}}^{k_2\tilde{\delta}} e^{r-k_2\tilde{\delta}} dW_j(\tau) \right]
\]

\[
= E \left[ \sigma^{ij}(u_\delta(k_1\tilde{\delta} - \delta^{0.9}, \omega)) \int_{k_1\tilde{\delta} - \delta^{0.9}}^{k_1\tilde{\delta}} e^{r-k_1\tilde{\delta}} dW_j(\tau) \times \sigma^{ij}(u_\delta(k_2\tilde{\delta} - \delta^{0.9}, \omega)) \right]
\]

\[
\times E \left[ \int_{k_2\tilde{\delta} - \delta^{0.9}}^{k_2\tilde{\delta}} e^{r-k_2\tilde{\delta}} dW_j(\tau) \right] = 0.
\]
This completes the proof of property (32).

(VI) Estimate of $\Upsilon_{2}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta}))$

We first recall that

$$\Upsilon_{2}^{\alpha}(\tilde{\delta}, [t](\tilde{\delta})) = \Pi_{12}(\tilde{\delta}, [t](\tilde{\delta})) - \frac{1}{2} \int_{\tilde{\delta}}^{t(\tilde{\delta})} (\sigma^{\alpha j} \partial_{\alpha} \sigma^{ij}) (u(s, \omega)) ds,$$

where

$$\Pi_{12}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \partial_{\alpha} \sigma^{ij} (u_{\tilde{\delta}}(s, \omega)) \left( f^{\alpha}(u_{\tilde{\delta}}(s, \omega)) \right. + \left. \sum_{\beta=1}^{l} \sigma^{\alpha \beta}(u_{\tilde{\delta}}(s, \omega)) \dot{Z}_{\tilde{\delta}}(s, \omega_{\beta}) \right) (Z_{\tilde{\delta}}((k+1)\tilde{\delta}, \omega_{j}) - Z_{\tilde{\delta}}(s, \omega_{j})) ds.$$

To estimate $\Upsilon_{2}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta}))$, we rewrite it as

$$\Upsilon_{2}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) = \sum_{\beta=1}^{l} \Upsilon_{21}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{22}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{23}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})),$$

where

$$\Upsilon_{21}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \left( (\sigma^{\alpha \beta} \partial_{\alpha} \sigma^{ij}) (u_{\tilde{\delta}}(s, \omega)) - (\sigma^{\alpha \beta} \partial_{\alpha} \sigma^{ij}) (u_{\tilde{\delta}}(k\tilde{\delta}, \omega)) \right) \dot{Z}_{\tilde{\delta}}(s, \omega_{\beta}) (Z_{\tilde{\delta}}((k+1)\tilde{\delta}, \omega_{j}) - Z_{\tilde{\delta}}(s, \omega_{j})) ds,$$

$$\Upsilon_{22}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} (f^{\alpha} \partial_{\alpha} \sigma^{ij}) (u_{\tilde{\delta}}(s, \omega)) (Z_{\tilde{\delta}}((k+1)\tilde{\delta}, \omega_{j}) - Z_{\tilde{\delta}}(s, \omega_{j})) ds,$$

$$\Upsilon_{23}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{\beta=1}^{l} \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} \sigma^{\alpha \beta}(u_{\tilde{\delta}}(k\tilde{\delta}, \omega)) (Z_{\tilde{\delta}}((k+1)\tilde{\delta}, \omega_{j}) - Z_{\tilde{\delta}}(s, \omega_{j})) - \frac{1}{2} \delta_{\beta j} ds,$$

$$\Upsilon_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \frac{1}{2} \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} (\sigma^{\alpha j} \partial_{\alpha} \sigma^{ij}) (u_{\tilde{\delta}}(k\tilde{\delta}, \omega)) - (\sigma^{\alpha j} \partial_{\alpha} \sigma^{ij}) (u(s, \omega)) ds,$$

where $\delta_{\beta j}$ is Kronecker delta.
(VI-1) Estimate of $\Upsilon_{21}^{\beta, j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))$

As for $\Upsilon_{21}^{\beta, j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))$, we have

$$E \left[ \sup_{t \in [0,T]} |\Upsilon_{21}^{\beta, j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right]$$

$$\leq K \sum_{k=1}^{n} E \left[ \sum_{k=1}^{m(T)-1} \int_{k \tilde{\delta}}^{(k+1)\tilde{\delta}} |u_{\tilde{\delta}}^j(s, \omega) - u_{\tilde{\delta}}^j(k \tilde{\delta}, \omega)||\hat{Z}_{\tilde{\delta}}(s, \omega_\beta)|ds \right]$$

By (20), we have for $s \in (k \tilde{\delta}, (k+1)\tilde{\delta})$,

$$|u_{\tilde{\delta}}^j(s, \omega) - u_{\tilde{\delta}}^j(k \tilde{\delta}, \omega)| \leq K \left( \tilde{\delta} + \sum_{d=1}^{l} \int_{k \tilde{\delta}}^{(k+1)\tilde{\delta}} |\hat{Z}_{\tilde{\delta}}(r, \omega_d)|dr \right).$$

Then by (34) and Lemma 2.2(5) we get that

$$E \left[ \sup_{t \in [0,T]} |\Upsilon_{21}^{\beta, j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right]$$

$$\leq K m(T) E \left\{ \sum_{k=1}^{m(T)-1} \left[ \tilde{\delta}^2 + \sum_{d=1}^{l} \int_{k \tilde{\delta}}^{(k+1)\tilde{\delta}} |\hat{Z}_{\tilde{\delta}}(r, \omega_d)|dr \right] \right\}$$

$$(34)$$

$$\leq K m(T)^2 |\tilde{\delta}|^2 n(\delta)^4 \delta^2 + n(\delta)^6 \delta^3 |\tilde{\delta}|^2 - n(\delta)^4 \delta \to 0 \text{ as } \delta \to 0^+.$$

(61-2) Estimate of $\Upsilon_{22}^{j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))$

For $\Upsilon_{22}^{j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))$, by Lemma 2.2(5) we find

$$E \left[ \sup_{t \in [0,T]} |\Upsilon_{22}^{j, \alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \tilde{\delta}^2 E \left[ \sum_{k=1}^{m(T)-1} \int_{k \tilde{\delta}}^{(k+1)\tilde{\delta}} |\hat{Z}_{\tilde{\delta}}(s, \omega_j)|ds \right]^2$$

$$\leq K \tilde{\delta}^2 m(T) \sum_{k=1}^{m(T)-1} E \left[ \int_{k \tilde{\delta}}^{(k+1)\tilde{\delta}} |\hat{Z}_{\tilde{\delta}}(s, \omega_j)|ds \right]^2$$

$$\leq K \tilde{\delta}^2 m(T)^2 n(\delta)^2 \delta \to 0 \text{ as } \delta \to 0^+.$$
(VI-3) Estimate of \( \Upsilon_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \)

For \( \Upsilon_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \), we have that for \( s_1 \in [0, T], \)
\[
E\left[ \sup_{t \in [0, s_1]} |Y_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] 
\leq K \int_0^{s_1} E[(\sigma^j \partial_\alpha \sigma^i)(u_\delta([s](\tilde{\delta}), \omega)) - (\sigma^j \partial_\alpha \sigma^i)(u(s, \omega))|^2 ds 
\leq K \int_0^{s_1} E|u(s, \omega) - u_\delta([s](\tilde{\delta}), \omega)|^2 ds 
\leq K \int_0^{s_1} E|u(s, \omega) - u_\delta([s](\tilde{\delta}), \omega)|^2 ds + K \int_0^{s_1} E|u_\delta(s, \omega) - u_\delta([s](\tilde{\delta}), \omega)|^2 ds.
\]

Using (20), Lemma 2.2 (1) and (5), and the \( \theta_t \)-invariance of \( \mathbb{P} \), we have for each \( 1 \leq i \leq n \)
\[
E|u_i(\delta)(s, \omega) - u_i(\delta)([s](\tilde{\delta}), \omega)|^2 \leq KE\left( \tilde{\delta} + \sum_{d=1}^l \int_{[s](\tilde{\delta})} |Z_\delta(s, \omega_\delta)| ds \right)^2 
\leq K(\tilde{\delta}^2 + n(\delta)^2 \delta).
\]

Hence, we have
\[
E\left[ \sup_{t \in [0, s_1]} |Y_{24}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \int_0^{s_1} E|u(s, \omega) - u_\delta(s, \omega)|^2 ds 
+ K(\tilde{\delta}^2 + n(\delta)^2 \delta).
\]

(IVI-4) Estimate of \( \Upsilon_{23}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \)

Finally, we will prove \( E\left[ \sup_{t \in [0, T]} |Y_{23}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] = o(1) \) as \( \delta \to 0^+ \). We write
\[
Y_{23}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) = \sum_{\beta=1}^{l} Y_{231}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \sum_{\beta=1}^{l} Y_{232}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + Y_{233}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})),
\]
where
\[
Y_{231}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{k\tilde{\delta}+\delta} (\sigma^\beta \partial_\alpha \sigma^i)(u_\delta(k\tilde{\delta}, \omega)) \left\{ Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(s, \omega_j) - \frac{1}{2} \delta_{\beta j} \right\} ds,
\]
\[
Y_{232}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} (\sigma^\beta \partial_\alpha \sigma^i)(u_\delta(k\tilde{\delta}, \omega)) \left\{ Z_\delta(s, \omega_j) - \frac{1}{2} \delta_{\beta j} \right\} ds,
\]
\[
Y_{233}^{j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(t)-1} \int_{k\tilde{\delta}}^{(k+1)\tilde{\delta}} (\sigma^\beta \partial_\alpha \sigma^i)(u_\delta(k\tilde{\delta}, \omega)) \left\{ Z_\delta((k+1)\tilde{\delta}, \omega_j) - Z_\delta(s, \omega_j) + (\delta - \delta) - c(\delta) - \frac{1}{2} \delta_{\beta j} \right\} ds,
\]
\[ Y_{233}^{\alpha}(\delta, \lceil t \rceil(\delta)) := -\delta c(\delta) \times \sum_{k=1}^{m(t)-1} (\sigma^{\alpha_j} \partial_n \sigma^{i_j})(u_\delta(k\delta, \omega)) \]

and \( c(\delta) := \frac{3}{4} - e^{-\frac{\delta - 3\delta}{4}} + \frac{1}{4} e^{-\frac{2(\delta - 3\delta)}{4}}. \)

**VI-4-1) Estimate of \( Y_{231}^{\beta, j, \alpha}(\delta, \lceil t \rceil(\delta)) \)**

For \( Y_{231}^{\beta, j, \alpha}(\delta, \lceil t \rceil(\delta)) \), we find

\[
E \left[ \sup_{t \in [0, T]} |Y_{231}^{\beta, j, \alpha}(\delta, \lceil t \rceil(\delta))|^2 \right] \leq KE \left[ \sum_{k=1}^{(m(T)-1)} \int_{k\delta}^{(k+1)\delta} |\tilde{Z}_{\delta}(s, \omega_\beta)| \times |Z_{\delta}((k+1)\delta, \omega_j) - Z_{\delta}(s, \omega_j)| ds \right]^2 + Km(T)^2 \delta^2
\]

\[
\leq Km(T)^2 E \left[ \left( \int_0^{\delta} |\tilde{Z}_{\delta}(s, \omega_\beta)| |Z_{\delta}(\delta, \omega_j) - Z_{\delta}(s, \omega_j)| ds \right)^2 \right] + Km(T)^2 \delta^2
\]

\[
\leq Km(T)^2 \left\{ E \left[ \left( \int_0^{\delta} |\tilde{Z}_{\delta}(s, \omega_\beta)| ds \right)^2 \left( \int_0^{\delta} |\tilde{Z}_{\delta}(s, \omega_j)| ds \right)^2 \right] + E \left[ \left( \int_0^{\delta} |\tilde{Z}_{\delta}(s, \omega_\beta)| ds \right)^2 \left( |Z_{\delta}(\delta, \omega_j)|^2 + |Z_{\delta}(\delta, \omega_j)|^2 \right) \right] + Km(T)^2 \delta^2
\]

\[
\leq Km(T)^2 \left( 2\delta^2 + \delta \delta \right) + Km(T)^2 \delta^2 \to 0 \text{ as } \delta \to 0^+.
\]

Here in the second inequality we use the \( \theta_\tau \)-invariance of \( \mathbb{P} \) and in the last one we use Lemma 2.2 (5), the Hölder inequality, and Lemma 2.3.

**VI-4-2) Estimate of \( Y_{232}^{\beta, j, \alpha}(\delta, \lceil t \rceil(\delta)) \)**

For each \( \beta = 1, \cdots, l \), by (3) and Fubini’s theorem, we note that

\[
\tilde{Z}_{\delta}(s, \omega_\beta)[Z_{\delta}((k+1)\delta, \omega_j) - Z_{\delta}(s, \omega_j)]
\]

\[
= \frac{1}{\delta} \int_{-\infty}^{\delta} e^{-\frac{s}{\tau}} dW_\beta(\tau) \left[ \int_{-\infty}^{\delta} \left( \int_{-\infty}^{t} \frac{1}{\delta} e^{-\frac{t}{\tau}} dt \right) dW_j(\tau) + \int_{-\infty}^{\delta} \left( \int_{\tau}^{t} \frac{1}{\delta} e^{-\frac{t}{\tau}} dt \right) dW_j(\tau) \right]
\]

\[
= \frac{1}{\delta} \int_{-\infty}^{\delta} e^{-\frac{s}{\tau}} dW_\beta(\tau) \left[ \int_{-\infty}^{\delta} \left( \int_{-\infty}^{t} e^{-\frac{t}{\tau}} - e^{-\frac{(t+1)\delta}{\tau}} \right) dW_j(\tau) + \int_{-\infty}^{\delta} \left( 1 - e^{-\frac{(t+1)\delta}{\tau}} \right) dW_j(\tau) \right]
\]

\[
= \frac{1}{\delta} \left[ \int_{-\infty}^{\delta} e^{-\frac{s}{\tau}} dW_\beta(\tau) + \int_{-\infty}^{\delta} e^{-\frac{s}{\tau}} dW_\beta(\tau) \right] \times \left[ \int_{-\infty}^{\delta} \left( e^{-\frac{s}{\tau}} - e^{-\frac{(t+1)\delta}{\tau}} \right) dW_j(\tau) \right]
\]
\[
+ \int_{k\delta+\delta}^{s} \left( e^{-\frac{s}{\delta}} - e^{-\frac{\tau-(k+1)\delta}{\delta}} \right) dW_{j}(\tau) + \int_{s}^{(k+1)\delta} \left( 1 - e^{-\frac{\tau-(k+1)\delta}{\delta}} \right) dW_{j}(\tau) \]
\]
\[
= \sum_{p=1}^{4} B_{k}^{\beta,j,\delta,p}(s, \omega),
\]
for \( k\delta + \delta \leq s \leq (k+1)\delta \), where
\[
B_{k}^{\beta,j,\delta,1}(s, \omega) := \frac{1}{\delta} \times e^{-\frac{s}{\delta}}(1 - e^{-\frac{(k+1)\delta}{\delta}}) \times \int_{-\infty}^{k\delta+\delta} e^{\tau} dW_{\beta}(\tau) \times \int_{-\infty}^{k\delta+\delta} e^{\tau} dW_{j}(\tau),
\]
\[
B_{k}^{\beta,j,\delta,2}(s, \omega) := \left[ \int_{k\delta+\delta}^{s} \left( e^{-\frac{s}{\delta}} - e^{-\frac{(k+1)\delta}{\delta}} \right) dW_{j}(\tau) + \int_{s}^{(k+1)\delta} \left( 1 - e^{-\frac{(k+1)\delta}{\delta}} \right) dW_{j}(\tau) \right]
\times \frac{1}{\delta} \int_{-\infty}^{k\delta+\delta} e^{\tau} dW_{\beta}(\tau),
\]
\[
B_{k}^{\beta,j,\delta,3}(s, \omega) := \frac{1}{\delta} \int_{k\delta+\delta}^{s} e^{-\frac{s}{\delta}} dW_{\beta}(\tau) \times \int_{-\infty}^{k\delta+\delta} \left( e^{-\frac{s}{\delta}} - e^{-\frac{(k+1)\delta}{\delta}} \right) dW_{j}(\tau),
\]
\[
B_{k}^{\beta,j,\delta,4}(s, \omega) := \left[ \int_{k\delta+\delta}^{s} \left( e^{-\frac{s}{\delta}} - e^{-\frac{(k+1)\delta}{\delta}} \right) dW_{j}(\tau) + \int_{s}^{(k+1)\delta} \left( 1 - e^{-\frac{(k+1)\delta}{\delta}} \right) dW_{j}(\tau) \right]
\times \frac{1}{\delta} \int_{-\infty}^{k\delta+\delta} e^{\tau} dW_{\beta}(\tau).
\]
In order to estimate \( \Upsilon_{232}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \) for each \( \beta = 1, \ldots, l \), we rewrite it into
\[
\Upsilon_{232}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) = \Upsilon_{2321}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{2322}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{2323}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) + \Upsilon_{2324}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})),
\]
where
\[
\Upsilon_{232p}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(T)-1} \int_{k\delta+\delta}^{(k+1)\delta} \left( \sigma_{ij}^{\alpha} \partial_{u_{\beta}} \left( u_{\delta}(k\delta, \omega) \right) B_{k}^{\beta,j,\delta,p}(s, \omega) \right) ds \quad \forall p = 1, 2, 3,
\]
\[
\Upsilon_{2324}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) := \sum_{k=1}^{m(T)-1} \int_{k\delta+\delta}^{(k+1)\delta} \left( \sigma_{ij}^{\alpha} \partial_{u_{\beta}} \left( u_{\delta}(k\delta, \omega) \right) \right) B_{k}^{\beta,j,\delta,4}(s, \omega)
\left( \delta \tilde{s} - \delta \right)^{-1} c(\delta) - \frac{1}{2} \delta_{ij} \right] ds.
\]
For \( \Upsilon_{2321}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \), by the Hölder inequality and Lemma 2.1, we have that
\[
E \left[ \sup_{t \in [0, T]} \left| \Upsilon_{2321}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \right|^{2} \right]
\leq Km(T)\tilde{\delta} - \delta \sum_{k=1}^{m(T)-1} \int_{k\delta+\delta}^{(k+1)\delta} E \left[ B_{k}^{\beta,j,\delta,1}(s, \omega)^{2} \right] ds
\leq Km(T)\tilde{\delta} - \delta \sum_{k=1}^{m(T)-1} \int_{k\delta+\delta}^{(k+1)\delta} e^{4(k+1)\delta} ds
\leq Km(T)^{2}(\tilde{\delta} - \delta) \delta \leq Kn(\delta)^{-1} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0^{+}.
\]
For \( \Upsilon_{2322}^{\beta,j,\alpha}(\tilde{\delta}, [t](\tilde{\delta})) \), we consider
\[
R_{n}^{1} := \sum_{k=1}^{n} \int_{k\delta+\delta}^{(k+1)\delta} \left( \sigma^{\alpha} \partial_{u_{\beta}} \left( u_{\delta}(k\delta, \omega) \right) B_{k}^{\beta,j,\delta,2}(s, \omega) \right) ds.
\]
Note that \((\sigma^{\alpha \beta} \partial_{\alpha} \sigma^{ij}) (u_3 (k \tilde{\delta}, \omega))\) is \(\mathcal{F}^{k \tilde{\delta}}\)-measurable, \(\int_{-\infty}^{k \tilde{\delta} + \delta} e^{r - \frac{r \tau}{s}} dW_{\beta} (\tau)\) is \(\mathcal{F}_{k \tilde{\delta} + \delta}\)-measurable, and \(\int_{s}^{k \tilde{\delta} + \delta} \left( e^{r - \frac{r \tau}{s}} - e^{-\frac{r \tau}{s+(k+1)\tilde{\delta}}} \right) dW_{j} (\tau) + \int_{s}^{(k+1)\tilde{\delta}} \left( 1 - e^{-\frac{r \tau}{s+(k+1)\tilde{\delta}}} \right) dW_{j} (\tau)\) is \(\mathcal{F}_{k \tilde{\delta} + \delta}\)-measurable for any \(k \tilde{\delta} + \delta \leq s \leq (k + 1) \tilde{\delta}\). Let \(\mathcal{K}_{n} := \mathcal{F}_{(n+1)\tilde{\delta} + \delta}\). We have

\[
E(R_{n+1}^{l} | \mathcal{K}_{n}) = R_{n} + \frac{1}{\delta} \times \left( \sigma^{\alpha \beta} \partial_{\alpha} \sigma^{ij} (u_3 ((n + 1) \tilde{\delta}, \omega)) \right) \int_{-\infty}^{n \tilde{\delta} + \delta} e^{r - \frac{r \tau}{s}} dW_{\beta} (\tau) \times E \left\{ \int_{s}^{n \tilde{\delta} + \delta} \left( \left( e^{r - \frac{r \tau}{s}} - e^{-\frac{r \tau}{s+(n+1)\tilde{\delta}}} \right) dW_{j} (\tau) \right) \right\} = R_{n}^{l},
\]

where the last equality follows from the Fubini’s theorem. So \(R_{n}^{l}\) is \(\mathcal{K}_{n}\)-martingale. Without loss of generality, we always assume that \(n(\delta) \geq 1\). Then by the martingale inequality we have

\[
E \left[ \sup_{t \in [0,T]} |\Upsilon_{2322}^ {\beta,j,\alpha} (\tilde{\delta}, [t \tilde{\delta}]) | \right] \leq K E \left[ |R_{m(T)-1}^{l}| \right]
\]

\[
= K \sum_{k=1}^{m(T)-1} E \left\{ \left( \sigma^{\alpha \beta} \partial_{\alpha} \sigma^{ij} \right)^{2} (u_3 (k \tilde{\delta}, \omega)) \left[ \int_{k \tilde{\delta} + \delta}^{(k+1)\tilde{\delta}} B_{k}^{\beta,j,\alpha} (s, \omega) ds \right]^{2} \right\}
\]

\[
\leq K \frac{1}{\delta^{2}} \sum_{k=1}^{m(T)-1} \int_{k \tilde{\delta} + \delta}^{(k+1)\tilde{\delta}} \left( \int_{k \tilde{\delta} + \delta}^{s} \left( e^{r - \frac{r \tau}{s}} - e^{-\frac{r \tau}{s+(k+1)\tilde{\delta}}} \right) dW_{j} (\tau) \right)
\]
\[ + \int_s^{(k+1)\delta} \left( 1 - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) \times \int_{-\infty}^{k\delta+\delta} e^{-\frac{\tau}{m}} dW_\beta(\tau) ds \]

\[ \leq K \frac{\beta - \delta}{\delta^2} m(T-1) \sum_{k=1}^{(k+1)\delta} \int_{k\delta+\delta}^{(k+1)\delta} E \left\{ \left[ \int_s^{(k+1)\delta} \left( e^{\frac{\tau}{m}} - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) + \int_s^{(k+1)\delta} \left( 1 - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) \right] \times \int_{-\infty}^{k\delta+\delta} e^{-\frac{\tau}{m}} dW_\beta(\tau) \right\} ds \]

\[ \leq K \frac{\beta - \delta}{\delta^2} m(T-1) \sum_{k=1}^{(k+1)\delta} \int_{k\delta+\delta}^{(k+1)\delta} \left\{ E \left[ \int_s^{(k+1)\delta} \left( e^{\frac{\tau}{m}} - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) \right] \right\} \times \left( \int_{-\infty}^{k\delta+\delta} e^{-\frac{\tau}{m}} dW_\beta(\tau) \right) ds \]

\[ \leq K \frac{\beta - \delta}{\delta^2} m(T) \times (\beta^2 + \delta^3) \leq K(\beta + \delta) \to 0 \quad \text{as} \quad \delta \to 0^+ . \]

Here the first equality follows from the fact that for each \( 1 \leq k_1 < k_2 \leq m(T) - 1 , \)

\[ E \left[ (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_1 \delta, \omega)) \int_{k_1 \delta+\delta}^{(k+1)\delta} B^{\beta,j,\delta,2}_{k_1}(s, \omega) ds \right] \]

\[ \times (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_2 \delta, \omega)) \int_{k_2 \delta+\delta}^{(k+1)\delta} B^{\beta,j,\delta,2}_{k_2}(s, \omega) ds \]

\[ = E \left[ (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_1 \delta, \omega)) \int_{k_1 \delta+\delta}^{(k+1)\delta} B^{\beta,j,\delta,2}_{k_1}(s, \omega) ds \right] \]

\[ \times (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_2 \delta, \omega)) \times \frac{1}{\delta} \int_{-\infty}^{k_2 \delta+\delta} e^{\frac{\tau}{m}} dW_\beta(\tau) \]

\[ \times E \left\{ \int_{k_2 \delta+\delta}^{(k+1)\delta} e^{-\frac{\tau}{m}} \left[ \int_{k_2 \delta+\delta}^{(k+1)\delta} \left( e^{\frac{\tau}{m}} - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) + \int_s^{(k+1)\delta} \left( 1 - e^{-\frac{(k+1)\tau}{m}} \right) dW_j(\tau) \right] ds \right\} \]

\[ = E \left[ (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_1 \delta, \omega)) \int_{k_1 \delta+\delta}^{(k+1)\delta} B^{\beta,j,\delta,2}_{k_1}(s, \omega) ds \times (\sigma^{\alpha \beta} \partial_\alpha \sigma^{ij})(u_\delta(k_2 \delta, \omega)) \right] \]

\[ \times \frac{1}{\delta} \int_{-\infty}^{k_2 \delta+\delta} e^{\frac{\tau}{m}} dW_\beta(\tau) \]

\[ \times E \left\{ \int_{k_2 \delta+\delta}^{(k+1)\delta} e^{-\frac{\tau}{m}} \left( e^{\frac{\tau}{m}} - e^{-\frac{(k+1)\tau}{m}} \right) ds \right\} \]
We note that (and the last second inequality follows from the direct calculation. For each \( n \),

\[
R_{n}^{II} := \sum_{k=1}^{n} \int_{k\delta + \delta}^{(k+1)\delta} (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}(k\delta, \omega)) B_{k}^{\beta,j,\delta,3}(s, \omega) ds.
\]

Similarly to \( Y_{2322}^{\beta,j,\alpha} \), \( R_{n}^{II} \) is also \( K_{n} \)-martingale and by the martingale inequality we have

\[
E \left[ \sup_{t \in [0,T]} |Y_{2322}^{\beta,j,\alpha}(\tau, [t](\delta))|^{2} \right] \\
\leq K E \left[ |R_{m(T)}^{II}|^{2} \right] \\
= K \sum_{k=1}^{m(T)-1} E \left\{ \left( \int_{k\delta + \delta}^{(k+1)\delta} (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}(k\delta, \omega)) B_{k}^{\beta,j,\delta,3}(s, \omega) ds \right)^{2} \right\} \\
\leq K \frac{1}{\delta^{2}} \sum_{k=1}^{m(T)-1} E \left\{ \int_{k\delta + \delta}^{(k+1)\delta} e^{\frac{-s}{\delta}} ds \right\}^{2} \\
\quad \times E \left\{ \int_{k\delta + \delta}^{(k+1)\delta} e^{\frac{-s}{\delta}} ds \right\} ds \\
\leq K \frac{1}{\delta^{2}} \int_{0}^{\frac{m(T)-1}{\delta}} \left( \int_{k\delta + \delta}^{(k+1)\delta} e^{\frac{-s}{\delta}} ds \right) ds \\
\leq K \frac{1}{\delta^{2}} \int_{0}^{m(T)} \left( \int_{k\delta + \delta}^{(k+1)\delta} e^{\frac{-s}{\delta}} ds \right) ds.
\]

For \( Y_{2324}^{\beta,j,\alpha} \), we consider

\[
R_{n}^{III} := \sum_{k=1}^{n} \int_{k\delta + \delta}^{(k+1)\delta} (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}(k\delta, \omega)) \left\{ B_{k}^{\beta,j,\delta,4}(s, \omega) + \left( \delta(\delta - \delta)^{-1}c(\delta) - \frac{1}{2} \right) \delta_{\beta j} \right\} ds.
\]

For each \( n \in \mathbb{N} \), set

\[
\Gamma_{\delta}^{\beta j}(k, \omega) := \int_{k\delta + \delta}^{(k+1)\delta} \left\{ B_{k}^{\beta,j,\delta,4}(s, \omega) + \left( \delta(\delta - \delta)^{-1}c(\delta) - \frac{1}{2} \right) \delta_{\beta j} \right\} ds.
\]

We note that \( (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}(k\delta, \omega)) \) is \( \mathcal{F}_{k\delta} \)-measurable by Proposition 1 (iii) and \( B_{k}^{\beta,j,\delta,4}(s, \omega) \) is \( \mathcal{F}_{k\delta + \delta} \)-measurable for any \( k\delta + \delta \leq s \leq (k+1)\delta \) and get that

\[
E(R_{n+1}^{III} | K_{n}) = R_{n}^{III} + E \left\{ (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}((n+1)\delta), \omega)) \Gamma_{\delta}^{\beta j}(n+1, \omega) | K_{n} \right\} \\
= R_{n}^{III} + (\sigma^{\alpha\beta} \delta \sigma)^{(i)}(u_{\delta}((n+1)\delta), \omega)) E \left[ \Gamma_{\delta}^{\beta j}(n+1, \omega) \right]
\]
Next, we claim that $E \left[ \Gamma^j_\delta(n+1, \omega) \right] = 0$. Clearly, if $\beta \neq j$, $E \left[ \Gamma^\beta_j(n+1, \omega) \right] = 0$.

If $\beta = j$, recall that

$$B_{n+1}^{j,\delta}(s, \omega) = \left[ \int_s^{n+1} \left( e^{\frac{\tau}{\delta}} - e^{\frac{\tau-(n+2)\delta}{\delta}} \right) dW_j(\tau) \right. $$

$$\left. + \int_s^{n+1} \left( 1 - e^{\frac{\tau-(n+2)\delta}{\delta}} \right) dW_j(\tau) \times \frac{1}{\delta} \int_s^{n+1} e^{\frac{\tau}{\delta}} dW_j(\tau) \right].$$

Since $\int_s^{n+1} \left( 1 - e^{\frac{\tau-(n+2)\delta}{\delta}} \right) dW_j(\tau)$ is $F_{n+1}^{(n+2)\delta}$-measurable and $\int_s^{n+1} e^{\frac{\tau}{\delta}} dW_j(\tau)$ is $F_{n+1}^{(n+2)\delta}$-measurable, we find that they are independent. Thus

$$EB_{n+1}^{j,\delta}(s, \omega)$$

$$= \frac{1}{\delta} E \left[ \int_s^{n+1} \left( e^{\frac{\tau}{\delta}} - e^{\frac{\tau-(n+2)\delta}{\delta}} \right) dW_j(\tau) \times \int_s^{n+1} e^{\frac{\tau}{\delta}} dW_j(\tau) \right]$$

$$= \frac{1}{\delta} \int_s^{n+1} \left( e^{\frac{\tau}{\delta}} - e^{\frac{\tau-(n+2)\delta}{\delta}} \right) e^{\frac{\tau}{\delta}} d\tau.$$ 

By the elementary calculation, we find that

$$\int_s^{n+1} EB_{n+1}^{j,\delta}(s, \omega) ds = \delta e^{\frac{\delta}{2}} - \frac{\delta e^{2(\delta-\delta)}}{2} - \frac{5 \delta}{4} + \frac{1}{2},$$

which implies that

$$E \left[ \Gamma^j_\delta(n+1, \omega) \right] = \int_s^{n+1} \left( EB_{n+1}^{j,\delta}(s, \omega) + \left( \delta(\delta-\delta)^{-1}c(\delta) - \frac{1}{2} \right) \right) ds = 0.$$ 

Hence, we have that $E \left[ \Gamma^j_\delta(n+1, \omega) \right] = 0$ for $\beta = j$. So, $R_{n+1}^{I}$ is a $K_\omega$-martingale.

Consequently, for each $\beta = 1, \cdots, l$, by the martingale inequality we have

$$E \left[ \sup_{t \in [0,T]} |\Gamma^\beta_\delta(t, [\beta, l])|^2 \right]$$

$$\leq KE \left[ |R_{n+1}^{I}|^2 \right]$$

$$= K \sum_{k=1}^{m(T)-1} E \left\{ \left( \sigma_{\alpha,\beta} \partial_{\alpha} \sigma_{\beta} \right)^2 (u_{\delta}(k\delta, \omega)) \times \left\{ \int_{k\delta+\delta}^{(k+1)\delta} B_{k}^{j,\delta}(s, \omega) + \left( \delta(\delta-\delta)^{-1}c(\delta) - \frac{1}{2} \right) \delta \right\} ds \right\}^2$$

$$\leq K \sum_{k=1}^{m(T)-1} E \left\{ \int_{k\delta+\delta}^{(k+1)\delta} \left( \int_{k\delta+\delta}^{(k+1)\delta} e^{\frac{\tau}{\delta}} - e^{\frac{\tau-(k+1)\delta}{\delta}} dW_j(\tau) \right. \right.$$

$$\left. + \int_s^{(k+1)\delta} \left( 1 - e^{\frac{\tau-(k+1)\delta}{\delta}} \right) dW_j(\tau) \times \frac{1}{\delta} \int_s^{(k+1)\delta} e^{\frac{\tau}{\delta}} dW_j(\tau) ds \right.$$ 

$$\left. + \left( \delta c(\delta) - \frac{1}{2} (\delta - \delta) \right) \delta \right\}^2.$$
Here the first equality follows from the fact that for each $1 \leq k_1 < k_2 \leq m(T) - 1$,
\[
E \left[ (\sigma^\alpha \partial_\alpha \sigma^\beta)(u_\delta(k_1 \delta, \omega)) \Gamma^\beta_j(k_1, \omega) \times (\sigma^\alpha \partial_\alpha \sigma^\beta)(u_\delta(k_2 \delta, \omega)) \Gamma^\beta_j(k_2, \omega) \right]
\]
\[= E \left[ (\sigma^\alpha \partial_\alpha \sigma^\beta)(u_\delta(k_1 \delta, \omega)) \Gamma^\beta_j(k_1, \omega) \times (\sigma^\alpha \partial_\alpha \sigma^\beta)(u_\delta(k_2 \delta, \omega)) \right]
\]
\[\times E[\Gamma^\beta_j(k_2, \omega)] = 0.\]

Note that $c(\delta) < 1$. Hence,
\[
E \left[ \sup_{t \in [0,T]} \| \gamma_{2324}^{\beta,j,\alpha}(\delta, [t](\delta)) \|^2 \right]
\]
\[\leq K \frac{\tilde{\delta} - \delta}{\delta^2} m(T)^{-(k+1)} \sum_{k=1}^{k_\delta} \int_{k\delta + \delta}^{s} \left\{ E \left[ \int_{s}^{t} \left( e^{\frac{\tau - s}{\delta}} - e^{\frac{\tau - (k+1)\delta}{\delta}} \right) dW_j(\tau) \right]^2 \times \left[ \int_{k\delta + \delta}^{s} e^{\frac{\tau - s}{\delta}} dW_j(\tau) \right]^2 \right\} ds \]
\[+ K m(T) \delta^2 + K m(T) \delta^2
\]
\[\leq K \frac{\tilde{\delta} - \delta}{\delta^2} m(T)^{-(k+1)} \sum_{k=1}^{k_\delta} \int_{k\delta + \delta}^{s} \left\{ E \left[ \int_{s}^{t} \left( e^{\frac{\tau - s}{\delta}} - e^{\frac{\tau - (k+1)\delta}{\delta}} \right) dW_j(\tau) \right]^4 \right\}^{\frac{1}{2}} \]
\[\times \left\{ E \left[ \int_{k\delta + \delta}^{s} e^{\frac{\tau - s}{\delta}} dW_j(\tau) \right]^4 \right\}^{\frac{1}{2}} ds \]
\[+ K m(T) \delta^2 + K m(T) \delta^2.
\]

Using the Burkholder-Davis-Gundy inequality, we further have
\[
E \left[ \sup_{t \in [0,T]} \| \gamma_{2324}^{\beta,j,\alpha}(\delta, [t](\delta)) \|^2 \right]
\]
\[\leq K \frac{\tilde{\delta} - \delta}{\delta^2} m(T)^{-(k+1)} \sum_{k=1}^{k_\delta} \int_{k\delta + \delta}^{s} \left\{ E \left[ \int_{s}^{t} \left( e^{\frac{\tau - s}{\delta}} - e^{\frac{\tau - (k+1)\delta}{\delta}} \right) d\tau \times \left[ \int_{k\delta + \delta}^{s} e^{\frac{2(\tau - s)}{\delta}} d\tau \right] \right] \right\}^{\frac{1}{2}} + K m(T) \delta^2 + K m(T) \delta^2
\]
we have that as $\delta \to 0^+$.

**Proof of Theorem 2.4.** Using Lemma 2.5, Lemma 2.6, Lemma 2.7, and Lemma 2.9, as $\delta \to 0^+$.

Combining estimates (28)-(33), we have

\[
\text{(VI-4-3) Estimate of } \Upsilon_{233}^{j}\alpha(\tilde{\delta}, [t](\tilde{\delta}))
\]

For $\Upsilon_{233}^{j}\alpha(\tilde{\delta}, [t](\tilde{\delta}))$, using $c(\tilde{\delta}) < 1$ again, we obtain

\[
E \left[ \sup_{t \in [0, T]} |\Upsilon_{233}^{j}\alpha(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \delta^2 m(T)^2 \leq K n(\tilde{\delta})^{-2} \to 0 \text{ as } \delta \to 0^+.
\]

Summarizing above gives (33). \qed

**Lemma 2.9.** As $\delta \to 0^+$,

\[
E \left[ \sup_{t \in [0, s]} |\Pi(\tilde{\delta}, [t](\tilde{\delta}))|^2 \right] \leq K \int_0^{s_1} E|u(s, \omega) - u(s, \omega)|^2 ds + o(1).
\]

**Proof of Theorem 2.4.** Using Lemma 2.5, Lemma 2.6, Lemma 2.7, and Lemma 2.9, we have that as $\delta \to 0^+$,

\[
E \left[ \sup_{t \in [0, s]} |u(s, \omega) - u(s, \omega)|^2 \right] \leq K \int_0^{s_1} E|u(s, \omega) - u(s, \omega)|^2 ds + o(1).
\]

Then, Theorem 2.4 follows from Gronwall’s inequality. \qed

3. **Existence of center manifolds of RDEs driven by colored noise.** In this section, we prove the existence of a local Lipschitz center manifold of the random differential equation driven by colored noise

\[
\dot{u}_\delta = A u_\delta + f(u_\delta) + \sigma(u_\delta)z_\delta(\theta_\omega), \tag{35}
\]

for each $\delta > 0$, where $A$ is a partially hyperbolic $n \times n$ matrix, $f$ and $\sigma$ are nonlinear terms, and $z_\delta(\theta_\omega)$ is an $l$-dimensional colored noise.

For simplicity, let $F_\delta(\omega, u) := f(u) + \sigma(u)z_\delta(\omega)$. Then equation (35) can be rewritten as

\[
\dot{u}_\delta = A u_\delta + F_\delta(\theta_\omega, u_\delta). \tag{36}
\]

Assume that

\[
f(0) = 0 \text{ and } \sigma(0) = 0. \tag{37}
\]

This implies that $F_\delta(\omega, 0) = 0$. We divide the spectrum $\sigma(A)$ of matrix $A$ into

\[
\sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s,
\]

where $\sigma_u := \{ \lambda \in \sigma(A) \mid \text{Re}\lambda > 0 \}$, $\sigma_c := \{ \lambda \in \sigma(A) \mid \text{Re}\lambda = 0 \}$ and $\sigma_s := \{ \lambda \in \sigma(A) \mid \text{Re}\lambda < 0 \}$. By the assumption, $\sigma_c \neq \emptyset$. Let $E^u, E^c$ and $E^s$ denote the generalized eigenspaces corresponding to $\sigma_u$, $\sigma_c$ and $\sigma_s$, respectively. Then

\[
R^n = E^u \oplus E^c \oplus E^s
\]

with corresponding projections $P^u : \mathbb{R}^n \to E^u, P^c : \mathbb{R}^n \to E^c$ and $P^s : \mathbb{R}^n \to E^s$. Let

\[
0 < \beta < \min\{ |\text{Re}\lambda| \mid \lambda \in \sigma_u \cup \sigma_s \}.\]
It is well-known that for each $0 < \alpha < \beta$, there is a $K \geq 1$ such that
\[
|e^{At}P^c| \leq Ke^{\alpha|t|}, t \in \mathbb{R},
\]
\[
|e^{At}P^n| \leq Ke^{\beta t}, t \leq 0,
\]
\[
|e^{At}P^n| \leq Ke^{-\beta t}, t \geq 0.
\] (38)

Furthermore, we assume that there exist constants $M_0, R_0 > 0$ and $\epsilon_0 \in (0, 1)$ such that
\[
|f(u) - f(v)| + |\sigma(u) - \sigma(v)| \leq M_0(|u|^\epsilon_0 + |v|^\epsilon_0)|u - v|, \ |u|, |v| \leq R_0.
\] (39)

By (15), we observe that
\[
|F_\delta(\omega, u) - F_\delta(\omega, v)| \leq M_0(1 + K_\delta C_\omega)(|u|^\epsilon_0 + |v|^\epsilon_0)|u - v|, \ |u|, |v| \leq R_0.
\]

We note that $C_\omega$ is tempered from above and $C_{\theta \omega}$ is locally integrable in $t$.

Next, we introduce a modified equation by using a cut-off function. Let $\sigma(s)$ be a $C^\infty$ function from $\mathbb{R}$ to $[0, 1]$ with
\[
\sigma(s) = 1 \text{ for } |s| \leq 1, \ |\sigma'(s)| \leq 2.
\]

Let $\rho : \Omega \to (0, +\infty)$ be a random variable tempered from below such that $\rho(\theta \omega)$ is locally integrable in $t$. We consider a modification of $F(\omega, u)$. Let
\[
F_{\delta, \rho}(\omega, u) = \sigma \left( \frac{|u|}{\rho(\omega)} \right) F_\delta(\omega, u).
\]

An elementary calculation gives

**Lemma 3.1.** (i) $F_{\delta, \rho}(\omega, u) = F_\delta(\omega, u)$ for $|u| \leq \rho(\omega)$;

(ii) there exists a random variable $B_1(\omega) > 0$ tempered from above, $B_1(\theta \omega)$ is locally integrable in $t$, such that $|F_{\delta, \rho}(\omega, u) - F_{\delta, \rho}(\omega, v)| \leq B_1(\omega)|u|^\epsilon_0|u - v|,

\forall u, v \in \mathbb{R}^l$.

We now consider the following modification of equation (36)
\[
\dot{u}_\delta = Au_\delta + F_{\delta, \rho}(\theta \omega, u_\delta).
\] (40)

Clearly, its solutions also generate random dynamical system. Fixing $\alpha$, we define the following Banach space for $\gamma = \frac{\alpha + \beta}{2}$,
\[
C_\gamma := \{ \varphi \in C(\mathbb{R}, \mathbb{R}^n) | \sup_{t \in \mathbb{R}} e^{-\gamma|t|} |\varphi(t)| < +\infty \}
\]

with the norm
\[
|\varphi|_\gamma = \sup_{t \in \mathbb{R}} e^{-\gamma|t|} |\varphi(t)|.
\]

Then for each $\delta > 0$ we define
\[
M_{\delta}^\gamma(\omega) := \{ x_0 \in \mathbb{R}^n | u_\delta(\cdot, \omega, x_0) \in C_\gamma \}.
\]

Clearly, $M_{\delta}^\gamma(\omega)$ is not empty. In the following, we will prove that $M_{\delta}^\gamma(\omega)$ is a Lipschitz manifold given by a graph of a Lipschitz function and is invariant under the random dynamical systems generated by equation (40). The next lemma gives a description of set $M_{\delta}^\gamma(\omega)$ by using an integral equation.
Lemma 3.2. For each \( \delta > 0 \), \( x_0 \in M^c_\delta(\omega) \) if and only if there exists a function \( u_\delta(\cdot) \in C_\gamma \) with the initial value \( u_\delta(0) = x_0 \) and satisfies
\[
u_\delta(t) = e^{At} \xi + \int_0^t e^{A(t-s)} P^c F_{\delta,\rho}(\theta_{\delta}, u_\delta(s))ds + \int_t^{+\infty} e^{A(t-s)} P^u F_{\delta,\rho}(\theta_{\delta}, u_\delta(s))ds
+ \int_{-\infty}^t e^{A(t-s)} P^s F_{\delta,\rho}(\theta_{\delta}, u_\delta(s))ds,
\]
where \( \xi = P^c x_0 \).

Proof. Let \( x_0 \in M^c_\delta(\omega) \). By the variation of constants formula, for \( \tau, t \in \mathbb{R} \) we have that
\[
u_\delta(t, \omega, x_0) = e^{A(t-\tau)} u_\delta(\tau, \omega, x_0) + \int_\tau^t e^{A(t-s)} F_{\delta,\rho}(\theta_{\delta}, u_\delta(s, \omega, x_0))ds.
\]
Taking \( \tau = 0 \), we find
\[
P^c u_\delta(t, \omega, x_0) = e^{At} P^c x_0 + \int_0^t e^{A(t-s)} P^c F_{\delta,\rho}(\theta_{\delta}, u_\delta(s, \omega, x_0))ds.
\]
Note that
\[
P^u u_\delta(t, \omega, x_0) = e^{A(t-\tau)} P^u u_\delta(\tau, \omega, x_0) + \int_\tau^t e^{A(t-s)} P^u F_{\delta,\rho}(\theta_{\delta}, u_\delta(s, \omega, x_0))ds.
\]
By (38), for \( \tau > \max\{t, 0\} \) we observe that
\[
|e^{A(t-\tau)} P^u u_\delta(\tau, \omega, x_0)| \leq K e^{(\gamma - \beta)\tau} |u_\delta(\cdot, \omega, x_0)|_\gamma \to 0 \text{ as } \tau \to +\infty.
\]
Taking the limit \( \tau \to +\infty \) in (43), we obtain
\[
P^u u_\delta(t, \omega, x_0) = \int_{+\infty}^t e^{A(t-s)} P^u F_{\delta,\rho}(\theta_{\delta}, u_\delta(s, \omega, x_0))ds.
\]
Similarly, with (38) we have
\[
P^s u_\delta(t, \omega, x_0) = \int_{-\infty}^t e^{A(t-s)} P^s F_{\delta,\rho}(\theta_{\delta}, u_\delta(s, \omega, x_0))ds.
\]
Combining (42), (44) and (45), we get (41). The converse follows from a straightforward computation. This completes the proof.

The following theorem gives the existence of locally Lipschitz center manifolds for equations (35).

Theorem 3.3. Assume that (37)-(39) hold and choose the the tempered radius \( \rho(\omega) \) such that
\[
0 < \rho(\omega) < \left(\frac{12KB_1(\omega)}{\beta - \alpha}\right)^{-\frac{1}{\alpha}},
\]
then there exists a Lipschitz center manifold for the random differential equation (40) with \( \delta > 0 \) which is given by
\[
M^c_\delta(\omega) = \{\xi + h^c_\delta(\omega, \xi) | \xi \in \mathbb{E}^c\},
\]
where \( h^c_\delta(\omega, \cdot) : \mathbb{E}^c \to \mathbb{E}^u \oplus \mathbb{E}^s \) is a Lipschitz continuous mapping and satisfies \( h^c_\delta(\omega, 0) = 0 \). Consequently, it is also a local center manifold of equation (35).
Proof. We first prove that equation (41) has a unique solution \( u^\delta = u^\delta(\cdot, \omega, \xi) \) in \( C_\gamma \) which is Lipschitz continuous in \( \xi \in E^c \). To see this, we replace \( u_\delta(s) \) by \( u(s) \) on the right hand side of equation (41) and denote it by \( J^\delta_s(u, \omega, \xi) \). For each \( u \in C_\gamma, \xi \in E^c, \) and \( \omega \in \Omega \), using (38), Lemma 3.1, and (46), we have that

\[
e^{-\gamma|t|}|J^\delta_s(u, \omega, \xi)| \leq K|\xi| + \left( \frac{12}{\beta - \alpha} \right)^{-1} \left\{ \int_0^t e^{\alpha|t-s|+\gamma|\xi|} ds \right\} + \int_t^{+\infty} e^{\beta(t-s)+\gamma|\xi|} ds + \int_{-\infty}^t e^{-\beta(t-s)+\gamma|\xi|} ds |u|_\gamma \leq K|\xi| + \frac{1}{2}|u|_\gamma.
\]

This implies that the operator \( J^\delta_s(\cdot, \omega, \xi) \) maps \( C_\gamma \) into \( C_\gamma \). For each \( u, \bar{u} \in C_\gamma \), we have that

\[
|J^\delta_s(u, \omega, \xi) - J^\delta_s(\bar{u}, \omega, \xi)| \leq \int_0^t |e^{A(t-s)} \|F_{\delta, \rho}(\theta_s \omega, u(s)) - F_{\delta, \rho}(\theta_s \omega, \bar{u}(s))| ds \leq \frac{1}{2}|u - \bar{u}|_\gamma,
\]

which implies that \( J^\delta_s(\cdot, \omega, \xi) \) is a uniform contraction with respect to the parameter \( (\omega, \xi) \). Using the contraction mapping principle, we have that \( J^\delta_s(\cdot, \omega, \xi) \) has a unique fixed point \( u^\delta(\cdot, \omega, \xi) \) for each \( \xi \in E^c \) and \( \omega \in \Omega \). Clearly, \( u^\delta(\cdot, \omega, 0) = 0 \) since \( F_{\delta, \rho}(\omega, 0) \). Similarly, for all \( \xi, \xi_0 \in E^c \) we get that

\[
|u^\delta(t, \omega, \xi) - u^\delta(t, \omega, \xi_0)| e^{-\gamma|t|} \leq K|\xi - \xi_0| + \frac{1}{2}|u^\delta(\cdot, \omega, \xi) - u^\delta(\cdot, \omega, \xi_0)|_\gamma.
\]

Hence we have

\[
|u^\delta(\cdot, \omega, \xi) - u^\delta(\cdot, \omega, \xi_0)|_\gamma \leq 2K|\xi - \xi_0|.
\]

Moreover, since \( u^\delta(\cdot, \omega, \xi) \) can be an \( \omega \)-wise limit of the iteration of contraction mapping \( J^\delta_s(\cdot, \omega, \xi) \) starting at 0 and mapping a \( \mathcal{F} \)-measurable function to a measurable function, \( u^\delta(\cdot, \omega, \xi) \) is \( \mathcal{F} \)-measurable with respect to \( \omega \). On the other hand, since \( u^\delta(\cdot, \omega, \xi) \) is Lipschitz continuous in \( \xi \), by Castaing and Valadier [5, Lemma III.14], \( u^\delta(\cdot, \omega, \xi) \) is measurable with respect to \( (\omega, \xi) \). Put \( h^\delta_s(\omega, \xi) = P^u u^\delta(0, \omega, \xi) + P^s u^\delta(0, \omega, \xi) \). Then

\[
h^\delta_s(\omega, \xi) = \int_{+\infty}^0 e^{-As} P^u F_{\delta, \rho}(\theta_s \omega, u^\delta(s, \omega, \xi)) ds + \int_{-\infty}^0 e^{-As} P^s F_{\delta, \rho}(\theta_s \omega, u^\delta(s, \omega, \xi)) ds.
\]
4. Convergence of center manifolds of random equations.

$h^c_\omega(\omega,0) = 0$, and is measurable in $(\xi,\omega)$ and Lipschitz continuous in $\xi$. From Lemma 3.2 and the definition of $h^c_\omega(\omega,\xi)$, we find

$$M^c_\omega = \{ \xi + h^c_\omega(\omega,\xi) | \xi \in E^c \}.$$

Next we prove $M^c_\omega(\omega)$ is a random set, i.e., for any $x \in \mathbb{R}^n$

$$\omega \mapsto \inf_{y \in \mathbb{R}^n} |x - (P_x y + h^c_\omega(\omega, P_x y))|$$

is measurable. Note that

$$\inf_{y \in \mathbb{R}^n} |x - (P_x y + h^c_\omega(\omega, P_x y))| = \inf_{y \in \mathbb{R}^n} |x - (P_x y + h^c_\omega(\omega, P_x y))|.$$  

Then the measurability follows immediately from the continuity of $h^c_\omega(\omega, P_x y)$ in $y$ and measurability of $\omega \rightarrow h^c_\omega(\omega, P_x y)$.

In the end, we claim that $M^c_\omega(\omega)$ is invariant, i.e., for all $s \in \mathbb{R}$

$$u_{\delta}(s, \omega, M^c_\omega(\omega)) = M^c_\omega(\theta_s \omega). \tag{47}$$

We first note that for each fixed $s \in \mathbb{R}$ and $x_0 \in M^c_\omega(\omega)$, $u_{\delta}(t+s, \omega, x_0)$ is a solution of

$$\dot{u}_\delta = Au_\delta + F_{\delta,\rho}(\theta_t(\theta_s \omega), u_\delta) \quad \text{with initial condition } u_\delta(s, \omega, x_0).$$

Thus, by the uniqueness of the solution, $u_{\delta}(t, \theta_s \omega, u_{\delta}(s, \omega, x_0)) = u_{\delta}(t+s, \omega, x_0)$. Since $u_{\delta}(:,:, \omega, x_0) \in C_{\gamma, \delta}$, it then follows that $u_{\delta}(:,:, \theta_s \omega, u_{\delta}(s, \omega, x_0)) \in C_{\gamma, \delta}$. Therefore, $u_{\delta}(s, \omega, x_0) \in M^c_\omega(\theta_s \omega)$, which implies that

$$u_{\delta}(s, \omega, M^c_\omega(\omega)) \subset M^c_\omega(\theta_s \omega). \tag{48}$$

Since $u_{\delta}(s, \omega) := u_{\delta}(s, \omega, \cdot)$ is a cocycle of homeomorphisms of $\mathbb{R}^n$, by (48) we have

$$M^c_\omega(\omega) \subset u_{\delta}(s, \omega)^{-1} M^c_\omega(\theta_s \omega) = u_{\delta}(-s, \theta_s \omega) M^c_\omega(\theta_s \omega) \subset M^c_\omega(\omega).$$

Thus (47) holds. Then we complete the proof of Theorem 3.3. \hfill \Box

4. Convergence of center manifolds of random equations. In this section, we establish the convergence of the center manifolds of the random differential equation driven by colored noise

$$\dot{u}_\delta = Au_\delta + f(\delta_\omega) + u_\delta z_\delta(\theta_t \omega) \tag{49}$$

More precisely, we will show than its center manifolds converge as $\delta \rightarrow 0$ to the center manifold of the stochastic differential equation

$$du = (Au + f(u))dt + u \circ dW. \tag{50}$$

Here, we assume that $A$ is an $n \times n$ matrix with zero real parts of eigenvalues, $f$ is a globally Lipschitz continuous function with $f(0) = 0$, and $z_\delta(\theta_t \omega)$ and $W(t, \omega)$ are 1-dimensional noises.

First, we recall the following lemma from [11].

**Lemma 4.1.** For all $T_1 < T_2$ and $\omega \in \Omega$,

$$\lim_{\delta \rightarrow 0^+} \left\| \int_0^t z_\delta(\theta_s \omega)ds - \omega(t) \right\|_{C([T_1, T_2])} = 0, \tag{51}$$

where $C([T_1, T_2])$ is the space of continuous functions defined on $[T_1, T_2]$ with uniform norm.
Next, we consider a linear stochastic differential equation:

\[ dz = -z \, dt + dW. \]  \hspace{1cm} (52)

For this equation, we borrow the following results from \cite{8}:

**Lemma 4.2.** (1) For \( \omega \in \Omega \) the random variable

\[ z(\omega) = -\int_{-\infty}^{0} e^r \omega(r) \, dr \]

exists and generates a unique stationary solution of (52) given by

\[ \Omega_1 \times \mathbb{R} \ni (\omega, t) \rightarrow z(\theta t, \omega) = -\int_{0}^{t} e^r \theta \omega(r) \, dr + \omega(t). \]

The mapping \( t \rightarrow z(\theta t, \omega) \) is continuous.

(2) In particular, on \( \Omega \) we have

\[ \lim_{t \to \pm \infty} \left| z(\theta t, \omega) \right| = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z(\theta r, \omega) \, dr = 0. \]

For each \( \delta > 0 \), replacing the white noise in equation (52) by \( z(\theta t, \omega) \) we get

\[ \dot{z}_\delta = -z_\delta + z_\delta(\theta t, \omega). \]  \hspace{1cm} (53)

Let \( z(t, \omega, x) \) and \( z_\delta(t, \omega, x) \) be the solutions of equations (52) and (53) with the initial data \( x \) at \( t = 0 \), respectively. The following Lemma shows that \( z_\delta(t, \omega, x) \) is an uniform approximation of \( z(t, \omega, x) \).

**Lemma 4.3.** For any \( T_1 < T_2 \), we have

\[ \lim_{\delta \to 0^+} \left\| z_\delta(\cdot, \omega, x) - z(\cdot, \omega, x) \right\|_{C([T_1, T_2])} = 0. \]

**Proof.** We first choose a positive constant \( T \) such that \( [T_1, T_2] \subset [-T, T] \). Note that

\[ z_\delta(t, \omega, x) = x - \int_{0}^{t} z_\delta(s, \omega, x) \, ds + \int_{0}^{t} z_\delta(\theta r, \omega) \, dr, \]

\[ z(t, \omega, x) = x - \int_{0}^{t} z(s, \omega, x) \, ds + \omega(t). \]

Therefore, we get

\[ |z_\delta(t, \omega, x) - z(t, \omega, x)| \leq \int_{0}^{t} |z_\delta(\theta r, \omega) - \omega(t)| \, dr + |z_\delta(s, \omega, x) - z(s, \omega, x)| \, ds|. \]

By Gronwall’s inequality we obtain

\[ |z_\delta(t, \omega, x) - z(t, \omega, x)| \leq \sup_{t \in [0, T]} \int_{0}^{t} |z_\delta(\theta r, \omega) - \omega(t)|e^{T} \, dt \forall t \in [0, T], \]

\[ |z_\delta(t, \omega, x) - z(t, \omega, x)| \leq \sup_{t \in [-T, 0]} \int_{0}^{t} |z_\delta(\theta r, \omega) - \omega(t)|e^{T} \, dt \forall t \in [-T, 0]. \]

Then, using Lemma 4.1, we complete the proof of Lemma 4.3. \( \square \)

The next lemma is concerned with the stationary solution of equation (53).
Lemma 4.4. For all $T_1 < T_2$, the following statements hold:

1. For each $\delta > 0$, the random variable

$$z_\delta^*(\omega) = \int_{-\infty}^{0} e^r z_\delta(\theta r \omega) dr, \quad \omega \in \Omega,$$

exists and generates a stationary solution of (53) given by

$$\Omega \times \mathbb{R} \ni (\omega, t) \mapsto z_\delta^*(\theta t \omega) = \int_{-\infty}^{0} e^r z_\delta(\theta r + t \omega) dr.$$

The mapping $t \mapsto z_\delta^*(\theta t \omega)$ is continuous.

2. In particular, on $\Omega$ we have

$$\lim_{t \to \pm \infty} |z_\delta^*(\theta t \omega)| = 0, \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_{0}^{t} z_\delta^*(\theta r \omega) dr = 0,$$

uniformly with respect to $\delta \in (0, \frac{1}{2}]$.

3. In addition, for every $\omega \in \Omega$,

$$\lim_{\delta \to 0^+} \sup_{t \in [T_1, T_2]} |z_\delta^*(\theta t \omega) - z(\theta t \omega)| = 0.$$

Proof. The proof of this lemma can be found in [12]. For the reader’s convenience, we here sketch the main idea. Let $\omega \in \Omega$. We first show (1) holds. By (4), we see that for each $\delta > 0$,

$$z_\delta^*(\theta t \omega) = -\int_{-\infty}^{0} \frac{1}{\delta^2} e^r \theta r \omega(s) ds.$$

(54)

Recall that on $\Omega$,

$$\lim_{s \to \pm \infty} \frac{|\omega(s)|}{|s|} = 0.$$

(55)

We have $z_\delta^*(\omega) = \int_{-\infty}^{0} e^r z_\delta(\theta r \omega) dr$ is well-defined. Note that $z_\delta^*(\theta t \omega)$ can be written as

$$z_\delta^*(\theta r \omega) = \int_{-\infty}^{t} e^{r-t} z_\delta(\theta \omega) dr.$$

Then, $z_\delta^*(\theta t \omega)$ satisfies equation (53) and is a stationary process.

By (54), we have

$$z_\delta(\theta r + t \omega) = -\frac{1}{\delta^2} \int_{-\infty}^{0} e^r \omega(s + r + t) ds + \frac{1}{\delta} \omega(r + t).$$

Taking the integral transformation $s \to s + r + t$, we find that

$$z_\delta(\theta r + t \omega) = -\frac{1}{\delta^2} \int_{-\infty}^{r+t} e^{s-r} \omega(s) ds + \frac{1}{\delta} \omega(r + t).$$
which together with Fubini’s theorem yields that
\[
\begin{align*}
    z_3^\varepsilon(\theta_t \omega) &= \int_{-\infty}^{0} e^r \left(-\frac{1}{\delta^2} \int_{-\infty}^{r+t} e^{\frac{-\varepsilon}{r-t}} \omega(s) ds \right) dr + \frac{1}{\delta} \int_{-\infty}^{0} e^r \omega(r+t) dr \\
    &= \int_{-\infty}^{t} \frac{-1}{\delta^2} e^{\frac{-\varepsilon}{r-t}} \omega(s) \int_{s-t}^{0} e^{(1-t) \frac{r}{s}} dr ds + \frac{1}{\delta} \int_{-\infty}^{0} e^r \omega(r+t) dr \\
    &= \frac{1}{\delta(1-\delta)} \int_{-\infty}^{t} e^{\frac{-\varepsilon}{r-t}} \omega(s) ds - \frac{1}{\delta(1-\delta)} \int_{-\infty}^{t} e^{s-t} \omega(s) ds \\
    &+ \frac{1}{\delta} \int_{-\infty}^{0} e^r \omega(r+t) dr. 
\end{align*}
\] (56)

In the first integral and the second one, taking the integral transformation \( r = \frac{s-t}{\delta} \) and \( r = s-t \), separately, we have that
\[
\begin{align*}
    z_3^\varepsilon(\theta_t \omega) &= \int_{-\infty}^{0} e^r (\omega(r\delta + t) - \omega(r+t)) dr. 
\end{align*}
\] (57)

We first show that \( \frac{|z_3^\varepsilon(\theta_t \omega)|}{|t|} \rightarrow 0 \) as \( t \rightarrow -\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \). (55) implies that for any \( \varepsilon > 0 \), there exists a positive constant \( T_0 = T(\omega, \varepsilon) \) such that for all \( |s| \geq T_0 \),
\[
    |\omega(s)| \leq \varepsilon|s|. 
\] (58)

Then we have for \( t \leq -T_0 \) and \( \delta \in (0, \frac{1}{2}] \),
\[
\begin{align*}
    \frac{|z_3^\varepsilon(\theta_t \omega)|}{|t|} &\leq \frac{1}{(1-\delta)|t|} \int_{-\infty}^{0} e^r (|\omega(r\delta + t)| + |\omega(r+t)|) dr \\
    &\leq \frac{4\varepsilon}{|t|} \int_{-\infty}^{0} e^r (|r| + |t|) dr \\
    &= \frac{4\varepsilon}{|t|} \int_{-\infty}^{0} e^r |r| dr + 4\varepsilon,
\end{align*}
\]
where (58) is used in the second inequality. Clearly, there exists a positive constant \( \tilde{T}_1 = \tilde{T}_1(\omega, \varepsilon) \) such that \( |t| \geq \tilde{T}_1 \),
\[
    \frac{1}{|t|} \int_{-\infty}^{0} e^r |r| dr < 1.
\]

Choose \( \tilde{T}_2 = \max\{T_0, \tilde{T}_1\} \), as \( t \leq -\tilde{T}_2 \), we have that \( \frac{|z_3^\varepsilon(\theta_t \omega)|}{|t|} < 8\varepsilon \). Hence, \( \frac{|z_3^\varepsilon(\theta_t \omega)|}{|t|} \rightarrow 0 \) as \( t \rightarrow -\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \).

Next we show that \( \frac{|z_3^\varepsilon(\theta_t \omega)|}{|t|} \rightarrow 0 \) as \( t \rightarrow +\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \). Using (55) again, we find that there exists \( T_3 = T_3(\omega) > 0 \) such that for \( |t| \geq T_3 \),
\[
    |\omega(t)| \leq |t|.
\]

On the other hand, by the continuity of \( \omega \in \Omega \), there exists \( M(\omega) > 0 \) such that for \( |t| \leq T_3 \)
\[
    |\omega(t)| \leq M(\omega).
\]
Then we get for all \( t \in \mathbb{R} \),
\[
|\omega(t)| \leq M(\omega) + |t|.
\] (59)

By (57), we have
\[
\frac{z^*_3(\theta, \omega)}{t} = \frac{1}{(1 - \delta)t} \int_{-\infty}^{-T_*} e^r(\omega(r\delta + t) - \omega(r + t))dr
\]
\[+ \frac{1}{(1 - \delta)t} \int_{-T_*}^{0} e^r(\omega(r\delta + t) - \omega(r + t))dr,
\]
where \( T_* \) is a positive number to be specified later. For the first integral on the right hand side, we have
\[
\left| \frac{1}{(1 - \delta)t} \int_{-\infty}^{-T_*} e^r(\omega(r\delta + t) - \omega(r + t))dr \right|
\]
\[\leq \frac{4}{|t|} \int_{-\infty}^{-T_*} e^r(M(\omega) + |r| + |t|)dr
\]
\[\leq \frac{4}{|t|} \int_{-\infty}^{0} e^r(M(\omega) + |r|)dr + 4 \int_{-\infty}^{-T_*} e^r dr.
\]
We first choose a sufficiently large \( T_* > 0 \) such that \( 8 \int_{-\infty}^{-T_*} e^r dr < \varepsilon \) and note that there exists a \( T_4 = T(\omega, \varepsilon) > 0 \) such that \( |t| > T_4 \)
\[
\frac{4}{|t|} \int_{-\infty}^{0} e^r(M(\omega) + |r|)dr < \varepsilon.
\]
Fix \( T_* \). Then we find that for \( t > T_4 \),
\[
\left| \frac{1}{(1 - \delta)t} \int_{-\infty}^{-T_*} e^r(\omega(r\delta + t) - \omega(r + t))dr \right| < 2\varepsilon. \] (60)

On \( \tau \in [-T_*, 0] \), since \( |r\delta + t| > |t| - |r\delta| > |t| - |r| > |t| - T_* \) for \( \delta \in (0, \frac{1}{2}] \) and \( |r + t| > |t| - |r| > |t| - T_* \), then for \( t > T_* + T_0 \) and \( \delta \in (0, \frac{1}{2}] \),
\[
\left| \frac{1}{(1 - \delta)t} \int_{-T_*}^{0} e^r(\omega(r\delta + t) - \omega(r + t))dr \right|
\]
\[\leq \frac{4\varepsilon}{|t|} \int_{-T_*}^{0} e^r(|r| + |t|)dr
\]
\[\leq \frac{4\varepsilon}{|t|} \int_{-\infty}^{0} e^r|dr + 2\varepsilon \int_{-\infty}^{0} e^r dr. \] (61)

Combining (60) with (61), we find that there exists a \( T_5 = \max\{T_4, T_* + T_0, \tilde{T}_1\} \) such that \( t > T_5 \) and \( \delta \in (0, \frac{1}{2}] \),
\[
\left| \frac{z^*_3(\theta, \omega)}{t} \right| < 8\varepsilon.
\]

This implies that \( \frac{|z^*_3(\theta, \omega)|}{|t|} \to 0 \) as \( t \to +\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \).

We now show that \( \frac{1}{\delta} \int_{0}^{t} z^*_3(\theta, \omega) \, dr \to 0 \) as \( t \to -\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \). Recall (56). Taking the transformation \( \tau = r + t \) for the third integral in (56), we have
\[
z^*_3(\theta, \omega) = \frac{1}{\delta(1 - \delta)} \int_{-\infty}^{t} e^{\frac{-t}{\delta}} \omega(\tau) \, d\tau - \frac{1}{1 - \delta} \int_{-\infty}^{t} e^{-t} \omega(\tau) \, d\tau.
\]
Then we get that
\[
\frac{1}{t} \int_0^t \frac{1}{(1 - \delta)^t} \int_{-\infty}^t e^{\tau - \delta \tau} \omega(\tau) d\tau = \frac{1}{1 - \delta} \int_{-\infty}^t e^{\tau - \delta \tau} \omega(\tau) d\tau
\]
\[
= \frac{1}{\delta t(1 - \delta)} \int_{-\infty}^0 e^{\tau} \omega(\tau) \int_{\tau}^t e^{-\delta \tau} d\tau + \frac{1}{\delta t(1 - \delta)} \int_{-\infty}^t e^{\tau} \omega(\tau) \int_{\tau}^t e^{-\delta \tau} d\tau
\]
\[
- \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau \omega(\tau) \int_{\tau}^t e^{-\tau} d\tau - \frac{1}{(1 - \delta)} \int_{-\infty}^t e^\tau \omega(\tau) \int_{\tau}^t e^{-\tau} d\tau
\]
\[
= \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau \omega(\tau) d\tau - \frac{1}{(1 - \delta)} \int_{-\infty}^t e^\tau \omega(\tau) d\tau
\]
\[
+ \frac{1}{(1 - \delta)} \int_{-\infty}^t e^{-\tau} \omega(\tau) d\tau - \frac{1}{(1 - \delta)} \int_{-\infty}^t e^{-\tau} \omega(\tau) d\tau.
\]
Taking the integral transformations \( \tau \to \frac{\tau}{\delta} \) for the first integral, \( \tau \to \tau - t \) for the third one and \( \tau \to \frac{\tau + t}{\delta} \) for the last one, we further obtain that
\[
\frac{1}{t} \int_0^t z_3^t(\theta, \omega) d\tau = \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau (\omega(t + t) - \omega(t) + \delta \omega(t)) - \frac{1}{(1 - \delta)} \int_{-\infty}^t e^\tau (\omega(t + t) - \omega(t) + \delta \omega(t)) d\tau.
\]
As \( t \leq \min\{-T_0, -\tilde{T}_1\} \) and \( \delta \in (0, \frac{1}{2}] \), by (58) we have that
\[
\left| \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau (\omega(t + t) - \delta \omega(t)) d\tau \right| \leq \frac{4\varepsilon}{|t|} \int_{-\infty}^0 e^\tau (|\tau| + |t|) d\tau
\]
\[
= \frac{4\varepsilon}{|t|} \int_{-\infty}^0 e^|\tau| d\tau + 4\varepsilon \int_{-\infty}^0 e^\tau d\tau \leq 8\varepsilon.
\]
On the other hand, by (59) we get that for \( \delta \in (0, \frac{1}{2}] \),
\[
\left| \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau (-\omega(t) + \delta \omega(t)) d\tau \right| \leq \frac{4}{|t|} \int_{-\infty}^0 e^\tau (M(\omega) + |\tau|) d\tau.
\]
This implies that as \( t < -T_4 \) and \( \delta \in (0, \frac{1}{2}] \),
\[
\left| \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau (-\omega(t) + \delta \omega(t)) d\tau \right| \leq 8\varepsilon.
\]
Combining the above estimates, for \( t \leq \max\{-T_0, -\tilde{T}_1, -T_4\} \) and \( \delta \in (0, \frac{1}{2}] \), we find that \( \frac{1}{t} \int_0^t z_3^t(\theta, \omega) d\tau < 9\varepsilon \) and hence \( \frac{1}{t} \int_0^t z_3^t(\theta, \omega) d\tau \to 0 \) as \( t \to -\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \).

In order to estimate \( \frac{1}{t} \int_0^t z_3^t(\theta, \omega) d\tau \to 0 \) as \( t \to +\infty \), uniformly for \( \delta \in (0, \frac{1}{2}] \), we rewrite (62) as
\[
\frac{1}{t} \int_0^t z_3^t(\theta, \omega) d\tau = \frac{1}{(1 - \delta)} \int_{-\infty}^-\tau_4 e^\tau (\omega(\tau + t) - \omega(\tau) + \delta \omega(t - \delta \tau) - \delta \omega(t - \delta \tau) + t)) d\tau
\]
\[
+ \frac{1}{(1 - \delta)} \int_{-\infty}^0 e^\tau (\omega(t + t) - \omega(t) + \delta \omega(t) - \delta \omega(t)) d\tau.
\]
By (59), we find that as $\delta \in (0, \frac{1}{2}]$,

$$
\left| \frac{1}{(1-\delta)t} \int_{-\infty}^{-T_*} e^r (\omega(\tau + t) - \omega(\tau) + \delta \omega(\delta \tau) - \delta \omega(\delta \tau + t))d\tau \right|
\leq \frac{8}{|t|} \int_{-\infty}^{-T_*} e^r (M(\omega) + |\tau| + |t|)d\tau
\leq \frac{8}{|t|} \int_{-\infty}^{0} e^r (M(\omega) + |\tau|)d\tau + 8 \int_{-\infty}^{-T_*} e^r d\tau.
$$

Recall that $T_* > 0$ is chosen such that $8 \int_{-\infty}^{0} e^r d\tau < \varepsilon$ and note that $t > T_4$

$$
\frac{8}{|t|} \int_{-\infty}^{0} e^r (M(\omega) + |\tau|)d\tau < 2\varepsilon.
$$

Then we have

$$
\left| \frac{1}{(1-\delta)t} \int_{-\infty}^{-T_*} e^r (\omega(\tau + t) - \omega(\tau) + \delta \omega(\delta \tau) - \delta \omega(\delta \tau + t))d\tau \right| < 3\varepsilon.
$$

Moreover, on $\tau \in [-T_*, 0]$, since $|\tau + t| > |t| - |\tau| > T_* \text{ and } |\delta \tau + t| > |t| - |\tau| > |t| - T_*$ for $\delta \in (0, \frac{1}{2}]$, by (58) and (59) we have for $t > \max\{T_0 + T_*, T_4, T_1\}$ and $\delta \in (0, \frac{1}{2}]$,

$$
\left| \frac{1}{(1-\delta)t} \int_{-T_*}^{0} e^r (\omega(\tau + t) - \omega(\tau) + \delta \omega(\delta \tau) - \delta \omega(\delta \tau + t))d\tau \right|
\leq \frac{4\varepsilon}{t} \int_{-\infty}^{0} e^r |\tau|d\tau + 4\varepsilon \int_{-\infty}^{0} e^r d\tau + \frac{4}{t} \int_{-\infty}^{0} e^r (M(\omega) + |\tau|)d\tau < 9\varepsilon.
$$

Thus, for $t \geq \max\{T_0 + T_*, T_4, T_1\}$ and $\delta \in (0, \frac{1}{2}]$, we have $\frac{1}{t} \int_0^t z_\theta^\delta(\theta, \omega) d\tau < 12\varepsilon$.

It follows that $\frac{1}{t} \int_0^t z_\theta^\delta(\theta, \omega) d\tau \to 0$ as $t \to +\infty$, uniformly for $\delta \in (0, \frac{1}{2}]$.

(3) In (57), let $t = 0$ and we obtain

$$
z_\theta^\delta(\omega) = \frac{1}{1-\delta} \int_{-\infty}^{0} e^r (\omega(r\delta) - \omega(r))dr.
$$

Recall that $z(\omega) = -\int_{-\infty}^{0} e^r \omega(r)dr$. We have

$$
z_\theta^\delta(\omega) - z(\omega) = \frac{1}{1-\delta} \int_{-\infty}^{0} e^r \omega(r\delta)dr + (1 - \frac{1}{1-\delta}) \int_{-\infty}^{0} e^r \omega(r)dr.
$$

Since $\lim_{\delta \to 0^+} e^r \omega(r\delta) = 0$, $|e^r \omega(r\delta)| \leq e^r (M(\omega) + |r|) \text{ for } \delta \in (0, \frac{1}{2}]$ and $\int_{-\infty}^{0} e^r (|r| + M(\omega))d\tau < +\infty$. By the Dominated Convergence Theorem, we have

$$
\lim_{\delta \to 0^+} \frac{1}{1-\delta} \int_{-\infty}^{0} e^r \omega(r\delta)dr = 0.
$$

Clearly, $1 - \frac{1}{1-\delta} \to 0$ as $\delta \to 0^+$. So we get that

$$
\lim_{\delta \to 0^+} z_\theta^\delta(\omega) = z(\omega).
$$
In the end, note that $z_\delta(\theta_t \omega)$ and $z(\theta_t \omega)$ are solutions of equations (53) and (52), respectively. Then we have

$$z_\delta^*(\theta_t \omega) = z_\delta^*(\omega) - \int_0^t z_\delta^*(\theta_s \omega) \, ds + \int_0^t z_\delta(\theta_t \omega) \, dr,$$

$$z(\theta_t \omega) = z(\omega) - \int_0^t z(\theta_s \omega) \, ds + \omega(t).$$

which along with Gronwall’s inequality implies that

$$|z_\delta^*(\theta_t \omega) - z(\theta_t \omega)| \leq \left[ |z_\delta^*(\omega) - z(\omega)| + \sup_{t \in [0,T]} |\int_0^t z_\delta(\theta_t \omega) \, dr - \omega(t)| \right] e^T$$

for all $t \in [0,T]$ and

$$|z_\delta^*(\theta_t \omega) - z(\theta_t \omega)| \leq \left[ |z_\delta^*(\omega) - z(\omega)| + \sup_{t \in [-T,0]} |\int_0^t z_\delta(\theta_t \omega) \, dr - \omega(t)| \right] e^T$$

for all $t \in [-T,0]$, where $T$ is a positive constant such that $[T_1, T_2] \subset [-T, T]$. It follows from (63) and (51) that

$$\lim_{\delta \to 0^+} \| z_\delta^*(\theta \omega) - z(\theta \omega) \|_{C([T_1, T_2])} = 0.$$

This completes the proof of this Lemma.

Next, we study the center manifolds of (50) and (49). We first show that the solution operator of (50) defines a random dynamical system. To see this, we consider the random differential equation

$$\frac{dv}{dt} = Av + z(\theta_t \omega) v + F(\theta_t \omega, v), \quad (64)$$

where $F(\omega, v) = e^{-z(\omega)} f(e^{z(\omega)} v)$. Note that for each $\omega \in \Omega$, the function $F$ has the same global Lipschitz constant $L$ as $f$. Then the well-posedness of (64) follows from the standard theory of deterministic ordinary differential equations, for every fixed $\omega \in \Omega$. Hence the solution mapping

$$(t, \omega, x) \mapsto v(t, \omega, x)$$

generates a random dynamical system on $\mathbb{R}^n$, i.e., $v$ is $B(\mathbb{R}) \otimes \mathcal{F} \otimes B(\mathbb{R}^n)$ measurable and forms a cocycle:

$$v(0, \omega, x) = x, \quad \text{for all } \omega \in \Omega,$$

$$v(t + s, \omega, x) = v(t, \theta_{s \omega}, \cdot) \circ v(s, \omega, x), \quad \text{for all } t, s \in \mathbb{R}, \ \omega \in \Omega.$$

Similarly, consider the equation:

$$\frac{dv_\delta}{dt} = Av_\delta + z_\delta^*(\theta_t \omega) v_\delta + F^*_\delta(\theta_t \omega, v_\delta), \quad (65)$$

where $F^*_\delta(\omega, v) = e^{-z^*_\delta(\omega)} f(e^{z^*_\delta(\omega)} v)$. Note that the solution mapping

$$(t, \omega, x) \mapsto v_\delta(t, \omega, x)$$

of (65) also generates a random dynamical system.

For each $x \in \mathbb{R}^n$, $\omega \in \Omega$ and $\delta > 0$ we introduce the following random transformations

$$T_\delta(\omega, x) := e^{-z^*_\delta(\omega)} x \text{ and } T(\omega, x) := e^{-z(\omega)} x.$$
Clearly, for fixed $\omega \in \Omega$, their inverse transformations are
\[(T^*_\delta)^{-1}(\omega, x) = e^{z_0^*(\omega)x} \text{ and } T^{-1}(\omega, x) = e^{z(\omega)x},\]
respectively. For the sake of convenience, we denote $z^*_0(\omega) := z(\omega)$, $T^*_0(\omega, x) := T(\omega, x)$ and $v_0(t, \omega, x) := v(t, \omega, x)$. Then we have:

**Theorem 4.5.** Suppose that $v_\delta$ is the random dynamical system generated by equation (64) as $\delta = 0$ (resp. (65) as $\delta > 0$). Then
\[
(t, \omega, x) \mapsto (T^*_\delta)^{-1}(\theta t, \omega, \cdot) \circ v_\delta(t, \omega, T^*_\delta(\omega, x)) =: \hat{v}_\delta(t, \omega, x)
\]
is a random dynamical system. For any $x \in \mathbb{R}^n$ this process is a solution of equation (50) as $\delta = 0$ (resp. (49) as $\delta > 0$) and forms a random dynamical system.

The proof is similar to Proposition 3.1 in [38] and we omit it.

As in section 3, we write the spectrum $\sigma(A)$ of matrix $A$ as
\[
\sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s,
\]
where $\sigma_u := \{ \lambda \in \sigma(A) \mid \text{Re} \lambda > 0 \}$, $\sigma_c := \{ \lambda \in \sigma(A) \mid \text{Re} \lambda = 0 \}$ and $\sigma_s := \{ \lambda \in \sigma(A) \mid \text{Re} \lambda < 0 \}$. By the assumption, $\sigma_c \neq \emptyset$. Let $E_u, E_c$ and $E^s$ denote the generalized eigenspaces corresponding to $\sigma_u$, $\sigma_c$ and $\sigma_s$, respectively. Then
\[
R^n = E_u \oplus E_c \oplus E^s
\]
with corresponding projections $P^u : \mathbb{R}^n \to E_u, P_\delta : \mathbb{R}^n \to E_c$ and $P^s : \mathbb{R}^n \to E^s$.

Let
\[
0 < \beta < \min\{|\text{Re} \lambda| \mid \lambda \in \sigma_u \cup \sigma_s\}.
\]
Then (38) holds.

For the remainder of this section, we will fix such $\alpha$. For each $\gamma \in (\alpha, \beta)$, we define the Banach spaces
\[
C_{\gamma, \delta} := \{ \varphi \in C(\mathbb{R}, \mathbb{R}^n) \mid \sup_{t \in \mathbb{R}} e^{-\gamma|t| - J_{\delta}^T z^*_0(\theta t, \omega) dr} |\varphi(t)| < +\infty \}
\]
with the norm
\[
|\varphi|_{\gamma, \delta} = \sup_{t \in \mathbb{R}} e^{-\gamma|t| - J_{\delta}^T z^*_0(\theta t, \omega) dr} |\varphi(t)|,
\]
Then for all $\delta \geq 0$ we define
\[
M^c_\delta(\omega) := \{ x_0 \in \mathbb{R}^n \mid v_\delta(\cdot, \omega, x_0) \in C_{\gamma, \delta} \}.
\]
Since the stationary process $z^*_0(\theta t, \omega)$ has the exact same properties (Lemmas 4.1-4.4) as the process $\int_0^{\infty} e^{r} G_\delta(\theta t + r \omega) dr$, where $G(\omega) = \omega(\delta)/\delta$ (see Lemmas 3.1-3.4 in [38]), using the same arguments from in [38], we have that $M^c_\delta(\omega)$ is a random invariant manifold given by the graph of a Lipschitz function for all small $\delta \geq 0$. Furthermore, using the same procedure as for Theorems 3.1-3.3 in [38], we obtain the following result. For brevity we do not repeat the proof here.

**Theorem 4.6.** If
\[
KL \left( \frac{1}{\gamma - \alpha} + \frac{2}{\beta - \gamma} \right) < 1,
\]
then there exists a Lipschitz center manifold for the random differential equation (65) as $\delta > 0$ (resp. (64) as $\delta = 0$) which is given by
\[
M^c_\delta(\omega) = \{ \xi + h_\delta(\omega, \xi) | \xi \in E^c \},
\]
where $h^c_\delta(\omega, \cdot) : E^c \to E^u \oplus E^s$ is a Lipschitz continuous mapping and satisfies $h^c_\delta(\omega, 0) = 0$. Consequently,
\[
\tilde{M}^c_\delta(\omega) = (T^c_\delta)^{-1}(\omega, M^c_\delta(\omega)) = \{ \xi + e^{z^c_\delta}(\omega) h^c_\delta(\omega, e^{-z^c_\delta}(\omega) \xi) | \xi \in E^c \}
\]
is a Lipschitz center manifold of equation (49) as $\delta > 0$ (resp. (50) as $\delta = 0$). Furthermore, for any $(\omega, \xi) \in \Omega \times E^c$
\[
\lim_{\delta \to 0^+} e^{z^c_\delta}(\omega) h^c_\delta(\omega, e^{-z^c_\delta}(\omega) \xi) = e^{z}(\omega) h^c(\omega, e^{-z}(\omega) \xi),
\]
i.e., the Lipschitz center manifold of equation (49) converges pathwise to that of equation (50).

**Remark.** (1) The center manifolds obtained in Theorem 4.6 are $C^1$ smooth if $f \in C^1$. (2) Under certain conditions, similar results as Theorem 4.6 also hold for stable and unstable manifolds. (3) If $f$ is an arbitrary $C^1$ function with $f(0) = 0$ and $Df(0) = 0$, one can use the standard procedure to modify $f$ by using a smooth cut-off function such that the modified function is globally Lipschitz continuous with a desired small Lipschitz constant. Thus, applying the results obtained here, one can get the convergence of local center-manifolds of the equations driven by colored noise.

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Received for publication November 2018.

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