On several two-boundary problems for a particular class of Lévy processes

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Running head: Two-boundary problems for certain Lévy processes

Abstract

Several two-boundary problems are solved for a special Lévy process: the Poisson process with an exponential component. The jumps of this process are controlled by a homogeneous Poisson process, the positive jump size distribution is arbitrary, while the distribution of the negative jumps is exponential. Closed form expressions are obtained for the integral transforms of the joint distribution of the first exit time from an interval and the value of the overshoot through boundaries at the first exit time. Also the joint distribution of the first entry time into the interval and the value of the process at this time instant are determined in terms of integral transforms.

1 Introduction

We assume that all random variables and stochastic processes are defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), a filtered probability space, where the filtration \(\{\mathcal{F}_t\}\) satisfies the usual conditions of right-continuity and completion. A Lévy process is a \(\mathcal{F}\) -adapted stochastic process \(\{\xi(t); t \geq 0\}\) which has independent and stationary increments and whose paths are right-continuous with left limits [1]. Under the assumption that \(\xi(0) = 0\) the Laplace transform of the process \(\{\xi(t); t \geq 0\}\) has the form

\[
E[e^{-p\xi(t)}] = e^{t k(p)}, \quad \text{Re} \; p = 0,
\]

where the function \(k(p)\) is called the Laplace exponent and is given by the formula (1)

\[
k(p) = \frac{1}{t} \ln E[e^{-p\xi(t)}] = \frac{1}{2} p^2 \sigma^2 - \alpha p + \int_{-\infty}^{\infty} \left( e^{-px} - 1 + \frac{px}{1+x^2} \right) \Pi(dx).
\]

Here \(\alpha, \sigma \in \mathbb{R}\) and \(\Pi(\cdot)\) is a measure on the real line. The introduced process is a space homogeneous, strong Markov process. Note, that the distribution of the first exit time from an interval plays a crucial role in applications and its knowledge also allows to solve a number of other two-boundary problems. Let us fix \(B > 0\) and define the variable

\[
\chi(y) = \inf \{ t : y + \xi(t) \notin [0, B] \}, \quad y \in [0, B],
\]

the first exit time from the interval \([0, B]\) by the process \(y + \xi(t)\). The random variable \(\chi(y)\) is a Markov time and \(P[\chi(y) < \infty] = 1\) [6]. Exit from the interval \([0, B]\) can take place either through the upper boundary \(B\), or through the lower boundary \(0\). Introduce events: \(A^B = \{\omega : \xi(\chi(y)) > B\}\), i.e. the exit takes place through the upper boundary; \(A_0 = \{\omega : \xi(\chi(y)) < 0\}\), i.e. the exit takes place through the lower boundary. Define

\[
X(y) = (\xi(\chi(y)) - B) I_{A^B} + (-\xi(\chi(y))) I_{A_0}, \quad P[A^B + A_0] = 1,
\]

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the value of the overshoot through one of the boundaries at the first exit time, where \( I_A = I_A(\omega) \) is the indicator of the event \( A \). The first two-boundary problem for \( \text{Lévy} \) processes with the Laplace exponent of the general form (1) has been solved by Gihman and Skorokhod \((3, \text{p.306-311})\). These authors have determined the joint distribution of \( \{\xi^-(t), \xi(t), \xi^+(t)\} \), where \( \xi^+(t) = \sup_{u \leq t} \xi(u), \quad \xi^-(t) = \inf_{u \leq t} \xi(u), \quad t \geq 0 \). For a spectrally positive \( \text{Lévy} \) process the joint distribution of \( \{\chi(y), X(y)\} \) has been studied by many authors among which Emery \([5]\), Suprun and Shurenkov \([14]\). The first exit time for a spectrally one-sided \( \text{Lévy} \) process has been considered by Bertoin \([2]\), Pistorius \([15], [16]\), Kyprianou \([12]\) and others. Kadankov and Kadankova \([8]\) have suggested another approach for determining the joint distribution of \( \{\chi(y), X(y)\} \) for the \( \text{Lévy} \) process with Laplace exponent (1). Their method is based on application of one-boundary functionals \( \{\tau^x, T^x\}, \{\tau_x, T_x\}, \quad x \geq 0 \), where

\[
\tau^x = \inf \{t : \xi(t) > x\}, \quad T^x = \xi(\tau^x) - x, \quad \tau_x = \inf \{t : \xi(t) < -x\}, \quad T_x = -\xi(\tau_x) - x.
\]

Integral transforms of these joint distributions have been obtained in 60's in papers of Pecherskii and Rogozin \([17], [13]\), Borovkov \([3]\), Zolotarev \([20]\). Kadankov and Kadankova \([8]\) have used probabilistic methods (the total probability law, space homogeneity and the strong Markov property of the process) to determine the integral transforms

\[
E[e^{-sx}(y); X(y) \in du, A^B], \quad E[e^{-sx}(y); X(y) \in du, A_0]
\]

of the joint distribution of \( \{\chi(y), X(y)\} \). For a spectrally positive \( \text{Lévy} \) process several two-boundary problems have been solved in \([9]-[11]\).

In this paper we obtain the integral transforms of the distributions of a number of two-boundary functionals associated with the first exit time (Section 3) and the first entry time (Section 4) for an important particular class of \( \text{Lévy} \) processes described in Section 2. The advantage is that these are closed formulas for the transforms, with no recursions, typical for the general case.

### 2 The Poisson process with negative exponential exponent

We now give a formal definition of the process which we consider. Let \( \eta \in (0, \infty) \) be a positive random variable, and \( \gamma \) be an exponential variable with parameter \( \lambda > 0 \): \( P[\gamma > x] = e^{-\lambda x}, \quad x \geq 0 \). Introduce the random variable \( \xi \in \mathbb{R} \) by its distribution function

\[
F(x) = a e^{\lambda x} I\{x \leq 0\} + (a + (1 - a) P[\eta \leq x]) I\{x > 0\}, \quad a \in (0,1), \quad \lambda > 0.
\]

Consider a right-continuous compound Poisson process \( \xi(t) = \sum_{k=0}^{N(t)} \xi_k, \quad t \geq 0 \), where \( \{\xi_k; k \geq 1\} \) are independent random variables identically distributed as \( \xi \), \( \xi_0 = 0 \), and \( N(t) \) is a homogeneous Poisson process with intensity \( c > 0 \). Then its Laplace exponent is of the form

\[
k(p) = c \int_{-\infty}^{\infty} (e^{-xp} - 1) dF(x) = a_1 \frac{p}{\lambda - p} + a_2 (E[e^{-p\eta}] - 1), \quad c > 0, \quad \text{Re} \ p = 0,
\]

where \( a_1 = ac, \quad a_2 = (1 - a)c \). Here and in the sequel we will call such process the Poisson process with a negative exponential component. Note, that inter-arrival times of the jumps of the process \( \{\xi(t); t \geq 0\} \) are exponentially distributed with parameter \( c \). With probability \( 1-a \) there occur positive jumps of size \( \eta \), and with probability \( a \) there occur negative jumps of value \( \gamma \) that is exponentially distributed with parameter \( \lambda \). The first term of (2) is the simplest case of a rational function, while the second term is nothing but the Laplace exponent of a monotone Poisson process with positive jumps of value \( \eta \). It is well known fact \(([17])\), that in this case the equation \( k(p) - s = 0, \quad s > 0 \) has a unique root \( c(s) \in (0, \lambda) \), in the semi-plane \( \text{Re} \ p > 0 \). Denote by \( \nu_s \) an exponentially distributed random variable with parameter \( s > 0 \), independent of the process. The introduction of \( \nu_s \) allows the following short-hand notation
for the double Laplace transforms of the process, i.e. \( \int_0^\infty e^{-st} E[e^{-\nu t}] dt = E[e^{-\nu t}] \). For the double Laplace transforms of the processes \( \xi^+ (\cdot), \xi^- (\cdot) \) the following formulae hold

\[
E[e^{-\nu t}] = \frac{c(s)}{s}\frac{\lambda - p}{\lambda} c(s) - p, \quad \Re p \leq 0, \\
E[e^{\nu t}] = \frac{s\lambda}{c(s)} (p - c(s)) \Re(p, s), \quad \Re p \geq 0,
\]

where

\[
\Re(p, s) = (a_1 p + (p - \lambda)|s - a_2 (E[e^{-p\eta}] - 1)|)^{-1}, \quad \Re p \geq 0, \quad p \neq c(s).
\]

Observe that the function \( \Re(p, s) \) is analytic in the semi-plane \( \Re p > c(s) \) and \( \lim_{p \to \infty} \Re(p, s) = 0 \). Therefore, it allows a representation in the form of an absolutely convergent Laplace integral (4)

\[
\Re(p, s) = \int_0^\infty e^{-p x} R_x(s) dx, \quad \Re p > c(s).
\]

We will call the function \( R_x(s), \quad x \geq 0 \) the resolvent of the Poisson process with a negative exponential component. We assume that \( R_x(s) = 0, \) for \( x < 0 \). Note, that \( R_0(s) = \lim_{p \to \infty} p \Re(p, s) = (c + s)^{-1} \), and the equalities (3) imply

\[
P[\xi^-(\nu_x) = 0] = \frac{c(s)}{\lambda}, \quad P[\xi^+(\nu_x) = 0] = \frac{\lambda}{c(s)} \frac{s}{s + c}.
\]

The second formula of (3) yields

\[
\Re(p, s) = \frac{c(s)}{s\lambda} \frac{1}{p - c(s)} E[e^{\nu t}], \quad \Re p > c(s).
\]

The functions

\[
\frac{1}{p - c(s)} = \int_0^\infty e^{-u(p-c(s))} du, \quad \Re p > c(s),
\]

\[
E[e^{\nu t}] = \int_0^\infty e^{-u} dP[\xi^+(\nu_x) < u], \quad \Re p \geq 0,
\]

which enter the right-hand side of (6), are the Laplace transforms for \( \Re p > c(s) \). Therefore, the original functions of the left-hand side and the right-hand side of (6) coincide, and

\[
R_x(s) = \frac{c(s)}{s\lambda} \int_{x}^{\infty} e^{c(s)(x-u)} dP[\xi^+(\nu_x) < u], \quad x \geq 0,
\]

which is the resolvent representation of the Poisson process with a negative exponential component. Note, that the representation for the resolvent of the spectrally one-sided Lévy process similar to (7) was obtained by Suprun and Shurenkov [18,19]. This representation implies that \( R_x(s), \quad x \geq 0 \) is a positive, monotone, continuous, increasing function of an exponential order, i.e. there exists \( 0 < A(s) < \infty \) such that \( R_x(s) < A(s)e^{\nu c(s)} \), for all \( x \geq 0 \). Therefore,

\[
\int_0^\infty R_x(s)e^{-\alpha x} dx < \infty, \quad \alpha > c(s).
\]

Moreover, in the neighborhood of any \( x \geq 0 \) the function \( R_x(s) \) has bounded variation. Hence, the inversion formula is valid

\[
R_x(s) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\infty+i\infty} e^{sp} \Re(p, s) dp, \quad \alpha > c(s).
\]
The latter equality together with (5) determines the resolvent of the Poisson process with a negative exponential component. To derive the joint distribution of the first exit time and the value of the overshoot at the first exit time for a Poisson process with a negative exponential component, we apply a general theorem for the Lévy processes which has been proved in [8]. Before stating the theorem we mention the following results

\[
E\left[ e^{-\sigma T^x - p T^x} \right] = \left( E[e^{-\pi(p)}] \right)^{-1} E\left[ e^{-p(\pi(p) - x)}; \pi(p) > x \right], \quad \text{Re} \ p \geq 0, \\
E\left[ e^{-\sigma T^x - p T^x} \right] = \left( E[e^{p\pi(p)}] \right)^{-1} E\left[ e^{p(\pi(p) + x)}; \pi(p) > x \right], \quad \text{Re} \ p \geq 0. \tag{9}
\]

The formulae (9) have been obtained by Pecherskii and Rogozin [13]. A simple proof of these equalities is given in [8]. After some calculations it follows from (3) and (9) that the integral transforms of the joint distributions of \{\tau_x, T_x\}, \{\tau^x, T^x\} of the Poisson process with a negative exponential component satisfy the equalities

\[
E[\xi^x; T_x \in du] = (\lambda - c(s)) e^{-\lambda u} du = E[\xi^x] P[\gamma \in du], \tag{10}
\]

\[
\int_0^\infty e^{-px} E[\xi^x - z(\tau^x)] \, dx = \frac{1}{p} \left( 1 - \frac{p + z - c(s)}{z - c(s)} \frac{\mathbb{R}(p + z, s)}{\mathbb{R}(z, s)} \right), \quad \text{Re} \ p > 0, \text{ Re } z \geq 0.
\]

The first equality of (10) yields that \tau_x and \tau_x are independent. Moreover, for all \( x \geq 0 \) the value of the overshoot through the lower level \( T_x \) is exponentially distributed with parameter \( \lambda \). This feature characterizes the Poisson process with a negative exponential component. Now we state the main results on two-sided exit problems.

### 3 The first exit from an interval

We now derive the joint distribution of the first exit time and the value of the overshoot at the first exit. The following result is true for the general Lévy processes ([8]), and for convenience it is stated here as a lemma.

**Lemma 1.** Let \{\xi(t); t \geq 0\}, \xi(0) = 0 be a real-valued Lévy process whose Laplace exponent is given by (1), \( B > 0 \), \( y \in [0, B] \), \( x = B - y \). Let

\[
\chi(y) = \inf\{ t : y + \xi(t) \notin [0, B] \}, \quad X(y) = (\xi(\chi(y)) - B) I_A + (-\xi(\chi(y))) I_A
\]

be, respectively, the first exit time from the interval \([0, B]\) by the process \( y + \xi(t) \) and the value of the overshoot through the boundary at the first exit time. For \( s > 0 \), the Laplace transforms of the joint distribution of \{\chi(y), X(y)\} satisfy the following equations

\[
E[e^{-s\chi(y)}; X(y) \in du, A_B] = f^+_s(x, du) + \int_0^\infty f^+_s(x, dv) K^+_s(v, du),
\]

\[
E[e^{-s\chi(y)}; X(y) \in du, A_0] = f^-_s(y, du) + \int_0^\infty f^-_s(y, dv) K^-_s(v, du), \tag{11}
\]

where

\[
f^+_s(x, du) = E[e^{-s\tau^x} + \tau^x] \in du - \int_0^\infty E[e^{s\tau^x}; T_y \in du] E[e^{-s\tau^{y+B}}; T^{y+B} \in du],
\]

\[
f^-_s(y, du) = E[e^{-s\tau^x} + \tau^x] \in du - \int_0^\infty E[e^{s\tau^x}; T^x \in du] E[e^{-s\tau^{y+B}}; T^{y+B} \in du];
\]

and \( K^+_s(v, du) = \sum_{n=1}^\infty K^{(n)}_s(v, du, s), \quad v \geq 0 \) are the series of the successive iterations of kernels \( K_\pm(v, du, s) \). These kernels are given by

\[
K_+(v, du, s) = \int_0^\infty E[e^{-s\tau^{y+B}}; T_{y+B} \in dl] E[e^{-s\tau^{y+B}}; T^{y+B} \in du],
\]

\[
K_-(v, du, s) = \int_0^\infty E[e^{-s\tau^{y+B}}; T_{y+B} \in dl] E[e^{-s\tau^{y+B}}; T^{y+B} \in du], \tag{12}
\]
and their successive iterations \((n \in \mathbb{N} = \{1, 2, \ldots \})\) are defined by

\[
K_{\pm}^{(1)}(v, du, s) = K_{\pm}(v, du, s), \quad K_{\pm}^{(n+1)}(v, du, s) = \int_{0}^{\infty} K_{\pm}(v, dl, s) K_{\pm}(l, du, s). \quad (13)
\]

We apply now the formulae of Lemma 1 for the case when the underlying process is the Poisson process with an exponentially distributed negative component.

**Theorem 1.** Let \(\{\xi(t); t \geq 0\}, \xi(0) = 0\) be a Poisson process with a negative exponential component whose Laplace exponent is given by (2), \(B > 0, \ y \in [0, B], \ x = B - y\). Let

\[
\chi(y) = \inf\{t : y + \xi(t) \notin [0, B]\}, \quad \text{and} \quad X(y) = (\xi(\chi(y)) - B) I_{A_0} + (-\xi(\chi(y))) I_{A_0}
\]

be, respectively, the first exit time from the interval and the value of overshoot through one of the boundaries. Then for \(s > 0\),

1) the integral transforms of the joint distribution of \(\{\chi(y), X(y)\}\) satisfy the following equations

\[
E[e^{-s\chi(y)}; X(y) \in du, A_0] = e^{-\lambda u} (\lambda - c(s)) e^{-yc(s)} \left[1 - E[e^{-st\gamma - c(s)\xi(\tau^\gamma)}]\right] K(s)^{-1} du, \quad (14)
\]

\[
E[e^{-s\chi(y)}; X(y) \in du, A^B] = E[e^{-st\gamma}; T^x \in du] - E[e^{-s\chi(y)}; A_0] E[e^{-s\gamma + B}; T^\gamma + B \in du],
\]

where

\[
K(s) = 1 - E[e^{-s\gamma}] E[e^{-s\gamma + B - c(s)T^B}] = E[e^{-s\gamma + B - c(s)T^\gamma + B}] = \lambda \int_{0}^{\infty} e^{-\lambda u} E[e^{-s\gamma + B - c(s)T^\gamma + B}] du.
\]

In particular,

\[
E[e^{-s\chi(y)}; A_0] = \left(1 - \frac{c(s)}{\lambda}\right) e^{-yc(s)} \left[1 - E[e^{-st\gamma - c(s)\xi(\tau^\gamma)}]\right] K(s)^{-1}, \quad (15)
\]

\[
E[e^{-s\chi(y)}; A^B] = E[e^{-st\gamma}] - E[e^{-s\chi(y)}; A_0] E[e^{-s\gamma + B}];
\]

2) the Laplace transforms of the random variable \(\chi(y)\) satisfy the following representations

\[
E[e^{-s\chi(y)}; X(y) \in du, A_0] = e^{-\lambda (u + B)} \frac{R_x(s)}{R_B(\lambda, s)} du, \quad E[e^{-s\chi(y)}; A_0] = \frac{1}{\lambda} e^{-\lambda B} \frac{R_x(s)}{R_B(\lambda, s)},
\]

\[
E[e^{-s\chi(y)}; A^B] = 1 - \frac{R_x(s)}{R_B(\lambda, s)} \left[\frac{1}{\lambda} e^{-\lambda B} + s\lambda S_B(\lambda, s)\right] + s\lambda S_x(s),
\]

\[
\int_{0}^{\infty} e^{-st} P[\chi(y) > t] dt = \lambda \frac{R_x(s)}{R_B(\lambda, s)} \hat{S}_B(\lambda, s) - \lambda S_x(s), \quad (16)
\]

where \(R_x(s), \ x \geq 0\) is the resolvent of the process, defined by (5), (8);

\[
S_x(s) = \int_{0}^{x} R_u(s) du, \quad \hat{R}_B(\lambda, s) = \int_{B}^{\infty} e^{-\lambda u} R_u(s) du, \quad \hat{S}_B(\lambda, s) = \int_{B}^{\infty} e^{-\lambda u} S_u(s) du.
\]

**Proof.** For the Poisson process with a negative exponential component, equalities of Lemma 1 take a simplified form. Using the equalities (10) and the defining formulae (12) for the kernels \(K_{\pm}(v, du, s)\) we obtain

\[
K_{+}(v, du, s) = \left(1 - \frac{c(s)}{\lambda}\right) e^{-c(s)(v+B)} E[e^{-st\gamma + B}; T^\gamma + B \in du],
\]

\[
K_{-}(v, du, s) = e^{-\lambda u}(\lambda - c(s)) e^{-c(s)B} E[e^{-s\gamma + B - c(s)T^\gamma + B}] du,
\]
where $\gamma$ is an exponentially distributed random variable with the parameter $\lambda$, independent of the process under the consideration. Using these equalities, the method of mathematical induction and the formulae (13), we obtain the successive iterations $K_{\pm}^{(n)}(v, du, s), \ n \in \mathbb{N}$ of the kernels $K_{\pm}(v, du, s)$:

$$K_{-}^{(n)}(v, du, s) = E[e^{-\sigma T_{\gamma} + B - c(s)T_{\gamma} + B} \lambda e^{-\lambda u} \ du],$$

$$K_{+}^{(n)}(v, du, s) = e^{-v c(s)} E[e^{-s T_{\gamma}}] K(s)^{-1} E[e^{-s T_{\gamma} + B} T_{\gamma} + B \in du].$$

The series $K_{\pm}^{(n)}(v, du)$ of the successive iterations $K_{\pm}^{(n)}(v, du, s)$ are nothing but geometric series, and their sums are given by

$$K_{-}^{\pm}(v, du) = \sum_{n=1}^{\infty} K_{-}^{(n)}(v, du, s) = E[e^{-\sigma T_{\gamma} + B - c(s)T_{\gamma} + B}] K(s)^{-1} e^{-\lambda u} \ du,$$

$$K_{+}^{\pm}(v, du) = \sum_{n=1}^{\infty} K_{+}^{(n)}(v, du, s) = e^{-v c(s)} E[e^{-s T_{\gamma}}] K(s)^{-1} E[e^{-s T_{\gamma} + B} T_{\gamma} + B \in du].$$

Substituting the obtained expressions for $K_{\pm}^{(n)}(v, du)$ into (11) yields the formulae (14) of Theorem 1. Integrating the formulae (14) with respect to $u \in \mathbb{R}_{+}$ leads to formula (15) of the theorem. Now, utilizing the definition of the resolvent (5), (8) and the equalities (10) we derive the resolvent representation for the functions $E[e^{-\sigma T_{\gamma} + c(s)\xi(\tau^+)}, E[e^{-\sigma T_{\gamma}}]]$:

$$E[e^{-\sigma T_{\gamma} + c(s)\xi(\tau^+)}] = 1 - e^{-c(s)} R_{x}(s) r(c(s), s),$$

$$E[e^{-\sigma T_{\gamma}}] = 1 - \frac{s\lambda}{c(s)} R_{x}(s) + s\lambda S_{x}(s),$$

where

$$S_{x}(s) = \int_{0}^{\infty} R_{x}(s) du, \quad r(c(s), s) = \frac{d}{dp} \mathbb{R}(p, s)^{-1} \bigg|_{p=c(s)}.$$ 

Substituting these expressions into (15), we obtain the representations (16) of Theorem 1. \qed

### 4 The first entry time into an interval

The knowledge of the joint distribution of \{\chi(y), X(y)\} allows to solve another two-boundary problem, namely to determine the integral transforms of the joint distribution of the first entry time into the fixed interval by the Lévy process and the value of the process at this time. We obtain the result in Theorem 2 below for the general Lévy processes. In Theorem 3 closed expressions for the integral transforms in case of a Poisson process with a negative exponential exponent are given.

**Theorem 2.** Let \{\xi(t); t \geq 0\}, $\xi(0) = 0$ be a Lévy process whose Laplace exponent is given by (1), $B > 0$, $\chi(y) \overset{\text{def}}{=} 0$ for $y \notin [0, B]$. Let

$$\chi(y) = \inf \{t > \chi(y) : y + \xi(t) \in [0, B]\}, \quad \chi(y) = y + \xi(\chi(y)) \in [0, B], \quad y \in \mathbb{R}$$

be, respectively, the first entry time into the interval $[0, B]$ by the process $y + \xi(t)$ and the value of the process at this time. For $s > 0$, the integral transforms of the joint distribution of
\(\{\chi(y), \overline{\chi}(y)\}, \ y \in \mathbb{R}\) satisfy the following equations

\[
b^v(du, s) \overset{df}{=} E[e^{-s\chi(u+B)}; \overline{\chi}(v + B) \in du] = \int_0^\infty Q^v_+(v, dl) E[e^{-st}; B - T_l \in du] + \int_0^\infty Q^v_+(v, dl) \int_0^\infty E[e^{-st}; T_l - B \in du] E[e^{-s\tau^v}; T^v \in du], \quad v > 0,
\]

\[
b_e(du, s) \overset{df}{=} E[e^{-s\chi(u-v)}; \overline{\chi}(v - w) \in du] = \int_0^\infty Q^v_-(v, dl) E[e^{-st}; T_l \in du]
\]

\[
+ \int_0^\infty Q^v_-(v, dl) \int_0^\infty E[e^{-st}; T_l - B \in du] E[e^{-s\tau^v}; B - T^v \in du], \quad v > 0,
\]

\[
b(y, du, s) \overset{df}{=} E[e^{-s\chi(y)}; \overline{\chi}(y) \in du] = \int_0^\infty E[e^{-sx(y)}; X(y) \in dv, A^B] b^v(du, s)
\]

\[
+ \int_0^\infty E[e^{-sx(y)}; X(y) \in dv, A_0] b_e(du, s), \quad y \in [0, B],
\]

where \(\delta(x), \ x \in \mathbb{R}\) is the delta function and

\[
Q^v_+(v, du) = \delta(v - u) du + \sum_{n=1}^{\infty} Q^{(n)}_+(v, du, s), \quad v > 0.
\]

The functions \(Q^{(n)}_+(v, du, s), \ n \in \mathbb{N}\) are defined by

\[
Q^{(1)}_+(v, du, s) = Q_+(v, du, s), \quad Q^{(n+1)}_+(v, du, s) = \int_0^\infty Q^{(n)}_+(v, dl, s) Q_+(l, du, s);
\]

and they are the successive iterations of the kernels \(Q_+(v, du, s)\), which are given by

\[
Q_+(v, du, s) = \int_0^\infty E[e^{-st}; T^v - B \in dl] E[e^{-st}; T^l - B \in du],
\]

\[
Q_-(v, du, s) = \int_0^\infty E[e^{-st}; T^v - B \in dl] E[e^{-st}; T^l - B \in du].
\]

**Proof.** For the functions \(b^v(du, s)\), \(b_e(du, s)\), \(v > 0\) according to the total probability law, space homogeneity of the process and the fact that \(\tau_v\), \(\tau^v\) are Markov times, the following system of equations is valid

\[
b^v(du, s) = E[e^{-st}; B - T_v \in du] + \int_0^\infty E[e^{-st}; T_v - B \in dl] b_0(du, s),
\]

\[
b_e(du, s) = E[e^{-st}; T^v \in du] + \int_0^\infty E[e^{-st}; T^v - B \in dl] b^v(du, s).
\]

This system is similar to a system of linear equations with two variables. Substituting the expression for \(b_e(du, s)\) from the right-hand side of the second equation into the first equation yields

\[
b^v(du, s) = E[e^{-st}; B - T_v \in du] + \int_0^\infty E[e^{-st}; T_v - B \in dl] E[e^{-st}; T^l \in du]
\]

\[
+ \int_{t=0}^\infty E[e^{-st}; T_v - B \in dl] \int_{v=0}^\infty E[e^{-st}; T^l - B \in du] b^v(du, s).
\]

Changing the order of integration in the third term of the second equation implies for the function \(b^v(du, s), \ v > 0\)

\[
b^v(du, s) = \int_0^\infty Q_+(v, du, s) b^v(du, s)
\]

\[
+ E[e^{-st}; B - T_v \in du] + \int_0^\infty E[e^{-st}; T_v - B \in dl] E[e^{-st}; T^l \in du],
\]

\[7\]
which is a linear integral equation with the following kernel
\[ Q_+(v, du, s) = \int_0^\infty E[e^{-stv}; T_v - B \in du] E[e^{-stt}; T_l - B \in du], \quad v > 0. \]

We now show, that for all \( v, u > 0, \ s > s_0 > 0 \) this kernel enjoys the following property
\[ Q_+(v, du, s) < \lambda, \quad \lambda = E[e^{-stv}] E[e^{-stB}], \quad s_0 > 0. \]

Indeed, for all \( s > 0 \) it follows from
\[ E[e^{-stv}; T_v - B \in du] = E[e^{-stv+B}; T_v+B \in du] - \int_0^B E[e^{-stv}; T_v \in du] E[e^{-stt-B}; T_B-t \in du], \]
that the following chain of inequalities holds
\[ E[e^{-stv}; T_v - B \in du] \leq E[e^{-stv+B}; T_v+B \in du] \leq E[e^{-stv+B}] \leq E[e^{-stB}]. \]

Analogously we establish, that \( E[e^{-stv}; T_v - B \in du] \leq E[e^{-stv+B}; T_v+B \in du] \leq E[e^{-stv+B}] \leq E[e^{-stB}] \). These chains of the inequalities imply the following estimation for the kernel \( Q_+(v, du, s) \), for all \( v, u > 0, \ s > s_0 > 0 \)
\[ Q_+(v, du, s) = \int_0^\infty E[e^{-stv}; T_v - B \in du] E[e^{-stt}; T_l - B \in du] \leq E[e^{-stB}] E[e^{-stB}] < \lambda = E[e^{-stB}] E[e^{-s\lambda B}], \quad s_0 > 0. \]

This bound and the method of mathematical induction yield that the successive iterations \( Q_+^{(n)}(v, du, s) \) (19) of the kernels \( Q_+(v, du, s) \) for all \( v, u > 0, \ s > s_0 > 0 \) obey the inequality \( Q_+^{(n+1)}(v, du, s) < \lambda^{n+1}, n \in \mathbb{N} \). Therefore, the series of successive iterations \( \sum_{n=1}^\infty Q_+^{(n)}(v, du, s) < \lambda(1 - \lambda)^{-1} \) converges uniformly for all \( v, u > 0, \ s > s_0 > 0 \). Utilizing now the method of successive iterations ([14], p. 33) to solve the integral equation (22) yields the first equality of the theorem. The second equality of the theorem can be verified analogously. It is not difficult to establish the third equality of the theorem using the total probability law and the fact that \( \chi(y) \) is the Markov time. \( \square \)

Denote by
\[ m^s_t(du) = \int_0^\infty \lambda e^{-\lambda x} E[e^{-stx}; Tx \in du] dx, \quad P(\lambda, du) = e^{-\lambda B} (m^s_t(du) + \lambda e^{\lambda u} du). \]

**Theorem 3.** Let \( \{\xi(t); t \geq 0\}, \ \xi(0) = 0 \) be a Poisson process with a negative exponential component as specified above, \( B > 0, \ \chi(y) \) def 0, for \( y \notin [0, B] \), and let
\[ \overline{\chi}(y) = \inf \{ t > \chi(y): y + \xi(t) \in [0, B] \}, \quad \overline{\chi}(y) = y + \xi(\overline{\chi}(y)) \in [0, B], \quad y \in \mathbb{R} \]
be, respectively, the first entry time into the interval \( [0, B] \) by the process \( y + \xi(t) \) and the value of the process at this time. The integral transforms of the joint distribution of \( \{\overline{\chi}(y), \overline{\chi}(y)\} \), \( y \in \mathbb{R} \) for \( s > 0 \) satisfy the following equations
\[ b^v(y, du, s) = e^{-\tau_c(s)} \left( 1 - \frac{c(s)}{\lambda} \right) T(s)^{-1} P(\lambda, du), \quad v > 0, \]
\[ b_0(y, du, s) = m^s_t(du) + e^{Bc(s)} \left( 1 - \frac{c(s)}{\lambda} \right) T(s)^{-1} P(\lambda, du), \quad v > 0, \]
\[ b(y, du, s) = \left( 1 - \frac{c(s)}{\lambda} \right) T(s)^{-1} \left[ \frac{c(s)(B-y)}{R_B} - \frac{R_B - y}{R_B} e^{-B(\lambda - c(s))} \right] P(\lambda, du) + \frac{1}{\lambda} P(\lambda, du), \quad y \in [0, B], \]
(23)
where
\[ m_x^s(du) = E[e^{-sT^x}; T^x \in du], \quad \dot{T}_x^s(c(s)) = E[e^{-sT^x - c(s)T^x}; T^x > B], \quad x \geq 0, \]
\[ \dot{T}_x^s(c(s)) = \lambda \int_0^\infty e^{-\lambda s} \dot{T}_x^s(c(s)) \, dx, \quad T(s) = 1 - \left(1 - \frac{c(s)}{\lambda}\right) \dot{T}_x^s(c(s)) e^{-\lambda(c(s))}. \]

**Proof.** We apply now the equalities (17) of Theorem 2 to obtain the formulae (23). For this we have to calculate for the Poisson process with a negative component the kernels \( Q^\pm(v, dl, s) \), and the successive iterations \( Q^{(n)}_\pm(v, dl, s), n \in \mathbb{N} \), and the series \( Q^\pm_x(v, dl) \). Utilizing the defining formula of the kernels (21) and the formulae (10) yields
\[ Q_+(v, dl, s) = e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B E[e^{-\lambda T^\gamma}; T^\gamma - B \in dl]}, \quad v > 0, \]
\[ Q_-(v, dl, s) = \dot{T}_v^s(c(s)) e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) \lambda e^{-\lambda dl}, \quad v > 0, \]
where \( \dot{T}_v^s(c(s)) = E[e^{-sT^x - c(s)T^x}; T^x > B], \quad x \geq 0 \). Using the defining formula (19) for the successive iterations and the method of mathematical induction it follows from (24) that for \( n \in \mathbb{N} \),
\[ Q^{(n)}_+(v, dl, s) = e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B \left(\dot{T}_v^s(c(s))\right)^{n-1} E[e^{-\lambda T^\gamma}; T^\gamma - B \in dl]}, \]
\[ Q^{(n)}_-(v, dl, s) = \dot{T}_v^s(c(s)) e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) \left(\dot{T}_v^s(c(s))\right)^{n-1} \lambda e^{-\lambda dl}, \]
where
\[ \dot{T}_v^s(c(s)) = e^{-B(\lambda - c(s))} (\lambda - c(s)) \int_0^\infty e^{-\lambda x} \dot{T}_x^s(c(s)) \, dx. \]
The series \( Q^\pm_x(v, dl) \) of the successive iterations \( Q^{(n)}_\pm(v, dl, s) \) (see (18)) are just the geometric series, and their sums are given by
\[ Q^+_x(v, dl) = \delta(v - l) \, dl + e^{-vc(s)} \left(1 - \frac{c(s)}{\lambda}\right) e^{-\lambda B T(s)^{-1} E[e^{-\lambda T^\gamma}; T^\gamma - B \in dl]}, \quad v > 0, \]
\[ Q^-_x(v, dl) = \delta(v - l) \, dl + \dot{T}_v^s(c(s)) e^{-B(\lambda - c(s))} \left(1 - \frac{c(s)}{\lambda}\right) T(s)^{-1} \lambda e^{-\lambda dl}, \quad v > 0, \]
where \( T(s) = 1 - \dot{T}_v^s(c(s)) \). Substituting in the equalities (17) of Theorem 2 the expressions for the functions \( Q^\pm_x(v, dl) \), and the expressions for the functions \( E[e^{-sX(y)}; X(y) \in dv, A^B], E[e^{-sX(y)}; X(y) \in dv, A_0] \), which are given by the formulae of Theorem 1, we obtain the formulae (23) of Theorem 3. \( \square \)

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