Stable perturbative QCD predictions at moderate energies with a modified couplant

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The problem of precise evaluation of perturbative QCD predictions at moderate energies is addressed. In order to improve stability of the predictions with respect to change of the renormalization scheme it is proposed to replace the sequence of conventional renormalization group improved approximants by a sequence of modified approximants, involving a modified running coupling constant, which is free from Landau singularity and much less renormalization scheme dependent than the conventional running coupling constant. A concrete model of the modified coupling constant is proposed and it is shown, that the QCD corrections to the static interquark potential evaluated with this coupling constant are indeed much less sensitive to the scheme parameters. It is pointed out, that the modified predictions display somewhat weaker energy dependence compared to the conventional predictions, which may help accommodate in a consistent way some very low and very high energy determinations of the strong coupling constant.

As is well known, finite order QCD predictions obtained with the conventional renormalization group improved perturbation expansion depend to some extent on the choice of the renormalization scheme (RS) \[1\]. This scheme dependence is formally of higher order relative to the order of the perturbative approximant, but at moderate energies (of the order of few GeV) it may be numerically significant \[2\]. This casts some doubt on the accuracy of perturbative results obtained in this energy range. Some attempts have been made to address this problem by modifying conventional perturbative approximants \[3\]. In the following we present an alternative approach \[4\], which may offer some advantages over the previously proposed solutions.

The starting point of our analysis is the observation, that strong RS dependence of perturbative predictions is to a large extent the result of a very strong RS dependence of the running coupling parameter (couplant) itself. The N-th order couplant \(a(Q^2) = g^2(Q^2)/4\pi^2\) in the conventional approach satisfies the renormalization group (RG) equation:

\[
Q^2 \frac{da}{dQ^2} = \beta^{(N)}(a),
\]

with

\[
\beta^{(N)}(a) = -\frac{b}{2} a^2 \left[ 1 + \sum_{k=1}^{N} c_k a^k \right],
\]

where the coefficients \(c_2, c_3, \ldots\) are scheme dependent. An extreme manifestation of the strong RS-dependence of \(a(Q^2)\) is the fact, that in a large class of renormalization schemes (in particular, all the schemes with positive \(c_k\) coefficients) the couplant becomes singular at positive \(Q^2\) (Landau singularity), and the location and character of the singularity strongly depends on the scheme parameters (it is controlled mainly by the highest order term retained in the \(\beta\)-function).

However, within the perturbative approach we have some freedom in defining the effective expansion parameter. The idea is to exploit this freedom and define an alternative, less RS-dependent couplant, which could then be used in perturbative expressions instead of the conventional couplant, hopefully giving less RS-dependent predictions for physical quantities. Obviously, we would like this modified couplant to be free from the Landau singularity.

A simple way to define such a modified couplant is to replace the conventional, polynomial \(\beta\)-function \[2\] by an appropriately chosen non-
polynomial function \( \tilde{\beta}(a) \). To retain the formal consistency with the conventional approach, the expansion of the modified \( \beta \)-function in any given order should of course reproduce the required number of small-\( a \) expansion coefficients of the conventional \( \beta \)-function. Next, to ensure the absence of the Landau singularity it should behave for large \( a \) as \(-\xi a^k\), with \( \xi > 0 \) and \( k \leq 1 \). Furthermore, it would be convenient, if the modified \( \beta \)-function would be analytic in some neighbourhood of \( a = 0 \). This restriction eliminates models with exponentially small contributions, which produce \( 1/Q^n \) terms in large-\( Q^2 \) expansion for the couplant and the physical quantity itself, that interfere with terms of similar type from the operator product expansion, and sometimes are even forbidden in this expansion.

Additionally, we shall impose the condition, that \( \tilde{\beta}^{(N)}(a) \) should not contain any order-specific free parameters, i.e. it should contain only parameters, which characterize the whole sequence of the modified approximants. Indeed, if \( \tilde{\beta}^{(N)}(a) \) contains in each order new free parameters, unrelated to other parameters, this would have adverse effect on the predictive power of the modified expansion, at least if we restrict our attention to few low order approximants.

The modified couplant is obtained by integrating the renormalization group equation \( \beta \) with the modified \( \beta \)-function.

The idea of modifying the expression for the effective coupling parameter in QCD has of course a long history, and many models have been proposed in the literature \[5\]. It turns out, however, that these models are not satisfactory from the point of view described above.

A simple way to construct models of \( \tilde{\beta}^{(N)}(a) \) with the required properties is to use the so called mapping method \[6\]. One of the models analyzed by us is based on the mapping

\[
u(a) = \frac{a}{1 + \eta a}, \tag{3}
\]

where \( \eta \) is a real positive parameter. It is easy to verify, that the function of the form

\[
\tilde{\beta}^{(N)}(a) = -\frac{b}{2} [\kappa a - \kappa \nu(a) + \sum_{k=0}^{N} \tilde{c}_k \nu(a)^{k+2}]
\tag{4}
\]

where \( \kappa \) is a real positive parameter and the coefficients \( \tilde{c}_k \) have the form

\[
\tilde{c}_0 = 1 - \eta \kappa, \quad \tilde{c}_1 = c_1 + 2 \eta - \eta^2 \kappa, \\
\tilde{c}_2 = c_2 + 3c_1 \eta + 3\eta^2 - \eta^3 \kappa, \quad \text{etc.}
\]

does indeed satisfy all the criteria listed above.

The free parameters in this model function have the following interpretation: \( 1/\eta \) characterizes the value of the couplant \( a(Q^2) \), for which the nonpolynomial character of the \( \tilde{\beta}^{(N)}(a) \) becomes essential, while \( \kappa \) determines the low-\( Q^2 \) asymptotics of the modified couplant \( a(Q^2) \), which behaves in this limit as \( 1/(Q^2)^{b\kappa/2} \). The function \( \tilde{\beta}^{(N)}(a) \) is of course a viable replacement for \( \beta^{(N)}(a) \) in the \( N \)-th order of perturbation expansion for any value of these parameters. However, a clever choice of these parameters, based perhaps on some information outside of perturbation theory, may improve the quality of low order perturbative results in the modified expansion. As a first try, we choose \( \kappa = 2/b \), which ensures \( 1/Q^2 \) behavior for the modified couplant at low \( Q^2 \), as suggested by some theoretical approaches. Our procedure for fixing \( \eta \) is described further below.

The modified perturbative predictions for a physical quantity are obtained by replacing the conventional couplant in the perturbative expression by the modified couplant. To verify, whether our modified couplant does indeed improve the stability of perturbative predictions, we analysed in some detail the next-to-leading (NL) and next-next-to-leading (NNL) order predictions for several quantities, including the QCD correction \( \delta_V \) to the static interquark interaction potential \[7\], which is related to the Fourier transform of this potential in the following way:

\[
V(Q^2) = -4\pi C_F \frac{\delta_V(Q^2)}{Q^2}, \tag{5}
\]

where \( C_F = 4/3 \). The NNL order expression for \( \delta_V \) has the form:

\[
\delta_V^{(2)}(Q^2) = a \left[ 1 + r_1 a + r_2 a^2 \right], \tag{6}
\]
For three active flavors in the \( \overline{\text{MS}} \) scheme with \( \mu^2 = Q^2 \) we have \( r_1^{\overline{\text{MS}}} = 1.75 \) and \( r_2^{\overline{\text{MS}}} = 16.7998 \) (as well as \( b = 9/2, c_1 = 16/9 \) and \( c_2^{\overline{\text{MS}}} = 4.471 \)). The freedom of choice of the RS for this quantity in NL order may be parametrized by the parameter \( r_1 \), while in the NNL order we have additional free parameter, which we may choose to be the coefficient \( c_2 \) in the \( \beta \)-function. The numerical value of the couplant is then determined from the implicit equation, which results from the integration of the of the equation (7) with an appropriate boundary condition:

\[
\frac{b}{2} \ln \frac{Q^2}{\Lambda_{\overline{\text{MS}}}^2} = r_1^{\overline{\text{MS}}} - r_1 + c_1 \ln \frac{b}{a} + \frac{1}{c_1} \ln a + O(a) \tag{7}
\]

\( r_1^{\overline{\text{MS}}} \) is equal to 1.75 for three active flavors in the \( \overline{\text{MS}} \) scheme with \( \mu^2 = Q^2 \). The freedom of choice of the RS for this quantity in NL order may be parametrized by the parameter \( r_1 \), while in the NNL order we have additional free parameter, which we may choose to be the coefficient \( c_2 \) in the \( \beta \)-function. The numerical value of the couplant is then determined from the implicit equation, which results from the integration of the of the equation (7) with an appropriate boundary condition:

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In Fig. 1 we show predictions for \( \delta_G \) at \( Q^2 = 3 \text{GeV}^2 \), as a function of \( r_1 \), for several values of \( c_2 \), obtained in the conventional expansion. As we see, the differences in the predictions are quite large. In Fig. 2 we show the corresponding plot obtained in the modified expansion (with \( \eta = 4.1 \), which we justify further below). Clearly, the modified predictions are much less RS-dependent than the conventional predictions. We have verified, that similar improvement in the stability is obtained for other physical quantities, in particular for the QCD correction to the Gross-Llewellyn-Smith sum rule in deep inelastic scattering [8].

The pattern of \( r_1 \) and \( c_2 \) dependence of the modified predictions indicates, that also in the case of the modified expansion it is possible to choose the scheme according to the Principle of Minimal Sensititvity [9]. General equations defining the PMS parameters for the modified expansion are of course the same as in the case of the conventional expansion (vanishing of the partial derivatives with respect to the scheme parameters), but the resulting equations are much more complicated, because the integral of \( 1/\beta(a) \) is more involved. Let us note, that it is only in the modified expansion that the full potential of the PMS method may be realized, because only in the modified approach the predictions are finite down to \( Q^2 = 0 \) in all possible renormalization schemes.

Figure 1. \( \delta_V \) for \( n_f = 3 \), at \( Q^2 = 3 \text{GeV}^2 \), as a function of \( r_1 \), for several values of \( c_2 \), as given by the conventional perturbation expansion with \( \Lambda_{\overline{\text{MS}}}^{(3)} = 350 \text{MeV} \). Dashed line indicates the NL order prediction.

An important point in our analysis is the choice of the value of the parameter \( \eta \). To fix the value of \( \eta \) we use the phenomenological formula [10] for \( \delta_V \), which has had some success in correlating the experimental data for heavy quarkonia. We adjust \( \eta \) in such a way, that the \( Q^2 \) dependence of the NNL order modified PMS prediction for \( \delta_V \) would match the \( Q^2 \)-dependence of the phenomenological expression [10] as closely as possible. It turns out, that the best agreement is obtained for \( \eta = 4.1 \) — if the modified predictions for \( \delta_V \) are matched to coincide with the phenomenological expression at \( Q^2 = 9 \text{GeV}^2 \), then the relative difference between these two expressions is less than 1% down to \( Q^2 = 1 \text{GeV}^2 \).

Perturbative predictions obtained in the modified approach have an interesting property: they have somewhat weaker \( Q^2 \)-dependence at moderate \( Q^2 \) compared to the conventional predictions. In order to see, how significant this effect could be for phenomenology, we performed within the modified approach a fit to the experimental data for \( \delta_{\text{GLS}} \). From the experimental result given in [11] we inferred, that purely perturbative contribution to the GLS integral at 3 GeV\(^2\) is equal to
Figure 2. $\delta_V$ for $n_f = 3$, at $Q^2 = 3$ GeV$^2$, as a function of $r_1$, for several values of $c_2$, obtained using the modified perturbation expansion with $\Lambda_{\overline{MS}}^{(3)} = 350$ MeV and $\eta = 4.1$. Dashed line indicates the NL order prediction.

$\delta_{GLS}^{exp} = 0.131 \pm 0.040$, where we have added the statistical and systematic errors in quadrature. If we fit $\Lambda_{\overline{MS}}^{(3)}$ to this experimental value using the conventional NNL order expression for $\delta_{GLS}$ in the $\overline{MS}$ scheme, and then convert this parameter into the quark thresholds, we obtain $\alpha_s(M_Z^2) = 0.1126^{+0.0072}_{-0.0116}$, which is to be compared with the world average $\alpha_s(M_Z^2) = 0.1187 \pm 0.0020$.

Performing the same fit and extrapolation with the modified NNL order expression in the $\overline{MS}$ scheme (with $\eta = 4.1$) we obtain a somewhat larger value:

$\tilde{\alpha}_s(M_Z^2) = 0.1147^{+0.0084}_{-0.0127}$. The upward shift is a welcome effect. This effect is even more striking, when we perform the same fit using the PMS approximants. Using the NNL order PMS expression in the conventional expansion we obtain $\alpha_s(M_Z^2) = 0.1097^{+0.0058}_{-0.0102}$, while using the NNL order PMS approximant in the modified expansion we get $\tilde{\alpha}_s(M_Z^2) = 0.1150^{+0.0084}_{-0.0127}$. This shows, that within the the modified perturbative approach it may be easier to accommodate in a consistent way some of the very low and very high energy determinations of $\alpha_s$.

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