Nonlinear waves described by the generalized Swift-Hohenberg equation

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Abstract. We study the wave processes described by the generalized Swift–Hohenberg equation. We show that the traveling wave reduction of this equation does not pass the Kovalevskaya test. Some solitary wave solutions and kink solutions of the generalized Swift–Hohenberg equation are found. We use the pseudo–spectral algorithm to perform the numerical simulation of the wave processes described by the mixed boundary value problem for the generalized Swift–Hohenberg equation. This algorithm was tested on the obtained solutions. Some features of the nonlinear waves evolution described by the generalized Swift–Hohenberg equation are studied.

1. Introduction
The Swift-Hohenberg equation is a fundamental partial differential equation that describes pattern formation in different physical systems. For the first time this equation was found by J. Swift and P. Hohenberg for describing the hydrodynamic fluctuations at the convective instability [1] and has the form

$$u_t + 2u_{xx} + u_{xxxx} = \alpha u + \beta u^2 − \gamma u^3,$$

where $\alpha$, $\beta$ and $\gamma$ are constant parameters of equation. This equation arises in a variety fields of science. For example it describes the mechanism of shear microbands formation in nanocrystalline materials, the amplitude of optical electric field inside the cavity, the patterns inside thin vibrated granular layers [2–4] and etc. Thus, it is very important to study the wave processes described by this equation and its generalizations.

One of the interesting generalizations of equation (1), arises in literature, have the following form

$$u_t + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = \alpha u - \gamma u^{m+1},$$

where $\sigma$, $\alpha$ and $\gamma$ are the parameters. In work [5] authors define the existence condition of non stationary meromorphic solutions of equation (2), that corresponds to $\sigma \neq 0$. Using this fact they were succeed to find elliptic and simple periodic solutions. Another work, devoted to studding the equation (2) is a work [6], where the so-called snakes-and-ladders structure of equation is studied.

Here we review our results of studding the wave processes described by equation (2).
2. Exact solutions of generalized Swift–Hohenberg equation

Since equation (2) admits spatiotemporal translational symmetry, we try to find the solutions of (2) using the traveling waves variables \( u(x, y) = u(z), z = kx - \omega t \)

\[-\omega y_z + 2k^2 y_{zz} - \sigma k^3 y_{zzz} + k^4 y_{zzzz} = \alpha y - \gamma y^{m+1}.\] (3)

We show that equation (3) does not have the Painlevé property [4]. Thus the finding partial solutions of equation (3) becomes an actual task.

Equation (3) contains the term \( \gamma y^{m+1} \) with an arbitrary parameter \( m \). It prevents to use known algorithms for exact solutions searching. To overcome this difficulty we use the method presented in works [9–12] that allows us to find two types of exact solution of equation (3). First of them is a kink solution in the form

\[ y(z) = \left( \frac{-32(m + 2)(m + 4)(3m + 4)}{\gamma(11m^2 + 72m + 96)^2} \right)^{1/m} \left( \frac{1}{1 + e^{-z}} \right)^{4/m}, \] (4)

where \( m \neq 0, -2, -4, -4/3, -8/3 \). This solution exists when

\[ k^{(1,2)} = \pm \frac{2m}{\sqrt{22m^2 + 144m + 192}}, \quad \omega^{(1,2)} = \pm \frac{8m(3m + 8)(m^2 + 12m + 16)}{(11m^2 + 72m + 96)^2}, \]
\[ \sigma = \frac{2\sqrt{2}(3m + 8)}{\sqrt{11m^2 + 72m + 96}}, \quad \alpha = -\frac{32(m + 4)(3m + 4)(m + 2)}{(11m^2 + 72m + 96)^2}. \] (5)

Another exact solution is solitary wave solution which reads as

\[ y(z) = \left( \frac{32(3m^3 + 22m^2 + 48m + 32)}{\gamma(4m^4 + 60m^3 + 297m^2 + 540m + 324)} \right)^{1/m} \left( \frac{e^{-\frac{1}{4}z}}{1 + e^{-z}} \right)^{4/m}. \] (6)

This solution exists when

\[ k^{(3,4)} = \pm \frac{2m}{\sqrt{4m^2 + 30m + 36}}, \quad \omega^{(3,4)} = \pm \frac{4m^2(4m + 9)}{(m + 6)^2(2m + 3)^2}, \]
\[ \sigma = \frac{2\sqrt{2}(3m + 8)}{\sqrt{2m^2 + 15m + 18}}, \quad \alpha = 12(m + 3). \] (7)

The graphical representation of obtained solutions is given on Fig. 1, 2.

It should be noted that equation (3) admits the transformation \( k \rightarrow -k, \omega \rightarrow -\omega \) and \( z \rightarrow -z \). Thus the solutions (4) and (6) are the solutions if we make the above mentioned transformation.

3. Main results of numerical simulation

Here we consider the periodic boundary value problem for equation (2) in the following form

\[ u_t + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = \alpha u - \gamma u^{m+1}, \]
\[ u(x, 0) = u_0(x), \quad u(x, t) = u(x + H, t), \] (8)

where \( u_0(x) \) is an initial disturbance.

Numerical solution of problem (8) is constructed using the pseudo–spectral method [13–15]. The applied numerical approach was tested using the obtained solutions [4].

First of all, it should be noted that the wave behavior, described by the mixed boundary value problem (8), strongly depends on the parity of parameter \( m \). If we take in computations
Figure 1. Exact solution (4) at \( z_0 = 10, \gamma = -1, m = 3, 5, 7 \) (curves 1–3).

Figure 2. Exact solution (6) at \( z_0 = 10, \gamma = 0.1, m = 3, 5, 7 \) (curves 1–3).

Figure 3. Evolution of initial ”hat” disturbance at \( m = 2 \). (a)–(e) – \( t = 0, 1, 2.5, 6, 45 \).

an even value of parameter \( m \), initial disturbance transforms in a regular structure after a cascade of transformations. For example, if we take the disturbance of trivial solution in the form \( \sim a \exp(-\beta x^2) \), we observe three stages of transformations. The first stages corresponds to the formation of horizontal plateau on the top of disturbance. The formation of a ”snake” structure is observed on the next stage. This structure travels thought the whole computational domain. Finally, this structure becomes regular (see Fig. 3).

However, if the value of parameter \( m \) is odd, we observe the blow-up phenomenon.

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