Infrared Safe Observables in 
\( \mathcal{N} = 4 \) Super Yang-Mills Theory

L. V. Bork\(^b\), D. I. Kazakov\(^{\alpha, b}\), G. S. Vartanov\(^{\alpha, d}\) and A. V. Zhiboedov\(^{\alpha, l}\)

\(^{\alpha}\) Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna, Russia,
\(^{b}\) Institute for Theoretical and Experimental Physics, Moscow, Russia,
\(^{\alpha}\) University Center, Joint Institute for Nuclear Research, Dubna, Russia,
\(^{l}\) Moscow State University, Physics Department, Moscow, Russia.

Abstract

The infrared structure of MHV gluon amplitudes in planar limit for \( \mathcal{N} = 4 \) super Yang-Mills theory is considered in the next-to-leading order of PT. Explicit cancellation of the infrared divergencies in properly defined cross-sections is demonstrated. The remaining finite parts for some inclusive differential cross-sections in planar limit are calculated analytically. In general, contrary to the virtual corrections, they do not reveal any simple structure.

Keywords: Super Yang-Mills, Infrared safe observables, maximally helicity violating amplitudes

PACS classification codes: 11.15.-q; 11.30.Pb; 11.25.Tq

1 Introduction

In recent years remarkable progress in understanding the structure of planar \( \mathcal{N} = 4 \) SYM (supersymmetric Yang-Mills) theory has been achieved. In the planar limit this theory seems to be the first example of solvable non-trivial four-dimensional Quantum Field Theory.

The objects which were in the spotlight starting from AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence were local operators, namely the spectrum of their anomalous dimensions. They were calculated on the one hand side from the field theory approach \(^{10}\) and from the other side as the energy levels of a string in classical background \(^{2, 3}\) revealing remarkable coincidence.

Recently the quantities of interest are the so-called MHV\(^4\) scattering amplitudes. It happens that in the planar limit of \( \mathcal{N} = 4 \) SYM theory they have a truly simple structure \(^{1, 4}\). It is

---

\(^1\)Present address: Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut 14476 Golm, Germany

\(^2\)Present address: Physics Department, Princeton University, Princeton NJ 08544, USA

\(^3\)Defined as \( g \to 0; \ N_c \to \infty; \ \lambda = g^2 N_c \ fixed \)

\(^4\)MHV (maximally helicity violating) amplitudes are the amplitudes where all particles are treated as outgoing and the net helicity is equal to \( n - 4 \) where \( n \) is the number of particles. For gluon amplitudes MHV amplitudes are defined as the amplitudes in which all but two gluons have positive helicities.
useful to consider the color-ordered amplitude defined through the color decomposition

$$A_n^{(l-loop)} = g^{n-2} \left( \frac{g^2 N_c}{16\pi^2} \right)^l \sum_{\text{perm}} Tr(T^{a(1)}...T^{a(n)}) A_n^{(l)}(p_{a(1)}, ..., p_{a(n)}),$$

(1)

where $A_n$ is the physical amplitude, $A_n$ are the partial color-ordered amplitudes, and $a_i$ is the color index of $i$-th external “gluon”.

It was found that these amplitudes reveal the iterative structure which was first established in two loops [5] and then confirmed at the three loop level by Bern, Dixon and Smirnov, who formulated the ansatz [6] for the $n$-point MHV amplitudes:

$$M_n \equiv \frac{A_n}{A_n^{\text{tree}}} = 1 + \sum_{L=1}^{\infty} \left( \frac{\alpha}{4\pi} \right)^L M_n^{(L)}(\epsilon) = \exp \left[ \sum_{l=1}^{\infty} \left( \frac{\alpha}{4\pi} \right)^l (f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon)) \right],$$

(2)

where $\alpha = g^2 N_c / 4\pi$ and $E_n^{(l)}$ vanishes as $\epsilon \to 0$, $C^{(l)}$ are some finite constants, and $M_n^{(1)}(l\epsilon)$ is the $l\epsilon$-regulated one-loop $n$-point $\phi^3$ scalar amplitude.

It is not surprising that the IR divergent parts of the amplitudes factorize and exponentiate [7]. What is less obvious is that it is also true for the finite part

$$M_n(\epsilon) = \exp \left[ -\frac{1}{8} \sum_{l=1}^{\infty} \left( \frac{\alpha}{4\pi} \right)^l \left( \frac{\gamma_K^{(l)}}{(l\epsilon)^2} + \frac{2G_0^{(l)}}{l\epsilon} \right) \sum_{i=1}^{n} \left( \frac{\mu^2}{-s_{i,i+1}} \right)^{l\epsilon} + \frac{1}{4} \sum_{l=1}^{\infty} \left( \frac{\alpha}{4\pi} \right)^l \gamma_K^{(l)} F^{(1)}(0) \right],$$

(3)

where $\gamma_K$ is the so-called cusp anomalous dimension [8] and $G_0$ is the second function (dependent on the IR regularization) which defines the IR structure of the amplitude [9].

According to BDS ansatz the finite part of the amplitude is defined by the cusp anomalous dimension and a function of kinematical parameters specified at one-loop. For a four gluon amplitude one has

$$F^{(1)}_4(0) = \frac{\gamma_K}{4} \log^2 \frac{s}{t}. $$

(4)

The cusp anomalous dimension is a function of the gauge coupling, for which the four terms of the weak coupling expansion [10] and two terms of the strong coupling expansion [2, 3] are known. Integrability from both sides of the AdS/CFT correspondence leads to all-order integral equation [11] solution of which being expanded in the coupling reproduces both the series [12].

While for $n = 4, 5$ the BDS ansatz goes through all checks, namely the amplitudes were calculated up to four loops for four gluons [10] and up to two loops for five gluons [13]. However, starting from $n = 6$ it fails. The first indication of the problem was strong coupling calculation in the limit $n \to \infty$ [14] where discrepancy with the BDS formula was found. The second indication came from the amplitude/Wilson loop duality [15, 16, 17, 18] namely from the comparison of the hexagonal light-like Wilson loop and finite part to the BDS ansatz for the six-gluon amplitude. It was found that the two expressions have difference in a non-trivial function of the three (dual) conformally invariant variables [19]. The third indication appeared in the paper [20] where the analytical structure of the BDS ansatz was analyzed and starting from $n = 6$ the Regge limit factorization of the amplitude in some physical regions fails. Finally it was shown by explicit two-loop calculation [21] that the BDS ansatz is not true and it needs
to be modified by some unknown finite function, which is an open and intriguing problem. From two-loop calculation for the six-point amplitude \[21\] and hexagonal light-like Wilson loop \[22\] it was shown that the gluon amplitude/Wilson loop duality is still valid.

While all the UV divergences in $\mathcal{N} = 4$ SYM are absent in scattering amplitudes the IR ones remain and are supposed to be canceled in a properly defined quantities. Regularized expressions act like some kind of scaffolding which has to be removed to obtain eventual physical observable. It is these quantities that are the aim of our calculation. And though the Kinoshita-Lee-Nauenberg \[23\] theorem in principle tell us how to construct such quantities, explicit realization of this procedure is not simple and one can think of various possibilities. In particular, one could consider the so-called energy flow functions defined in terms of the energy-momentum tensor correlators introduced in \[24\] and considered at weak coupling regime in \[25\] and recently at strong coupling regime in \[26\]. From our side we concentrated on inclusive cross-sections in the hope that they reveal some factorization properties discovered in the regularized amplitudes. Similar questions were discussed in \[27\], where the inclusive cross-section like the IR safe observables in $\mathcal{N}=4$ SYM were constructed.

2 The Infrared Safe Observables

To perform the procedure of cancellation of the IR divergences one should have in mind that in conformal theory all the masses are zero and one has additional collinear divergences which need special care. In these work we employ the method developed in QCD parton model \[28, 29, 30, 31, 32\]. It includes two main ingredients in the cancellation of infrared divergencies coming from the loops: emission of additional soft real quanta and redefinition of the asymptotic states resulting in the splitting terms governed by the kernels of DGLAP equations. The latter ones take care of the collinear divergences.

Typical observables in QCD parton model calculations are inclusive jet cross-sections, where the total energy of scattered partons is not fixed since they are considered to be parts of the scattered hadrons. In \[29\] the algorithm of extracting divergences was developed which allows one to cancel divergences and apply numerical methods for calculation of a finite part. In our paper we choose as our observables the inclusive cross-sections with fixed initial energy and get analytical expression for the finite part of the differential cross-section. We do not assume any confinement and consider the scattering of the single parton based coherent state\[5\] being the asymptotic states of conformal field theory.

When the number of particles increases one has to specify the measurable quantity and to distinguish the particle(s) in the final state. If one wants to construct the finite quantity it is not sufficient to consider the process with the fixed number of final particles. One has to include processes with emission of additional soft and collinear massless states, i.e to consider the inclusive cross-section.

One possibility is to introduce the energy and angular resolution of the detector and to cut the phase space so that the soft quanta with total energy below the threshold as well as all the particles within the given solid angle are included. This procedure works well in QED

\[5\]squared perturbative amplitudes used in our calculation have been summed over colors, so in this sense they are colorless and there are no contradiction with statements that cancellation of IR divergences occurs only for colorless objects.
but introduces explicit dependence on the energy and angular cut off, thus violating conformal invariance.

We adopt here the other attitude and do not introduce any cut off in the phase space but rather consider the inclusive cross-section with emission of all possible particles allowed by kinematics. Then one has to specify which particle is detected. For instance, one can measure the scattering of a given particle on a given angle integrating over all the other particles. As it will be clear later in this case one still cannot avoid introducing some scale related to the definition of the asymptotic states of a theory.

3 Calculation of Inclusive Cross-sections in $\mathcal{N} = 4$ SYM theory

Our aim is to evaluate the NLO correction to the inclusive differential polarized cross section in the weak coupling limit in planar $\mathcal{N} = 4$ SYM in analytical form and to trace the cancellation of the IR divergences.

We start with the $2 \rightarrow 2$ MHV scattering amplitude with two incoming positively polarized gluons and two outgoing positively polarized gluons and consider the differential cross-section $d\sigma_{2\rightarrow2}(g^+g^+ \rightarrow g^+g^+)/d\Omega$ as a function of the scattering solid angle. The total cross-section is divergent at zero angle. Treating all the particles as outgoing this amplitude is denoted as $(-,-,++)$ MHV amplitude. At tree level the cross-section is given by

$$\frac{d\sigma_{2\rightarrow2}}{d\Omega_{13}} = \frac{1}{J} \int d\phi_2 |M_4^{(\text{tree})}|^2 S_2,$$

where $J$ is flux factor, in our case $J = s$, $s$ is standard Mandelstam variable, $d\phi_2$ is the phase volume of two-particle process (we use dimensional regularization $6$, $D = 4 - 2\epsilon$) and $S_n$ ($n = 2$ in this case) is the so-called measurement function which specifies what is really detected. $d\Omega_{13} = d\phi_{13} d\cos(\theta_{13}) \sin(\theta_{13})$, $\theta_{13}$ is the scattering angle of the particles with momenta $p_3$ with respect to $p_1$ in the center of mass frame. The matrix element is obtained from the color-ordered amplitudes via summation $[1, 4]$}

$$|M_4^{(\text{tree})(-,-,++)}|^2 = g^4 N_c^2 (N_c^2 - 1) \sum_{\sigma \in P_3} \frac{s_{12}^4}{s_{1}^{4(1)} s_{2}^{4(2)} s_{3}^{4(3)}},$$

where in all expressions we take $s_{ij} = (p_i + p_j)^2$.

Within dimensional regularization(reduction) the cross-section looks like

$$\left(\frac{d\sigma_{2\rightarrow2}}{d\Omega_{13}}\right)_{-,-,+} = \frac{\alpha^2 N_c^2}{2E^2} \left( \frac{s^2}{t^2 u^2} + \frac{s^2}{t^2} + \frac{s^2}{u^2} \right) \frac{\mu^2}{s} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \frac{4(3 + c^2)}{(1 - c^2)^2},$$

where $s, t, u$ are the Mandelstam variables, $E$ is the total energy in the center of mass frame, $c = \cos \theta_{13}$, $\mu$ and $\epsilon$ are the parameters of the dimensional regularization. The next step is to calculate the NLO corrections.

$6$ namely – FDH version of dimensionless reduction, see for details $33$.

$7$ if to be more accurate in dimensional regularization(reduction) we have $d\Omega_{13}' = d\phi_{13}' \sin(\phi_{13})^{-2\epsilon} d\cos(\theta_{13}) \sin(\theta_{13})^{-2\epsilon}$. 

4
3.1 Virtual part

To get the one-loop contribution to the differential cross section we use already known one loop contribution for the color-ordered amplitude \[34\]

\[ M_{4}^{(1-loop)}(\epsilon) = A_{4}^{(1-loop)}/A_{4}^{(tree)} = -\frac{1}{2} s t I_{4}^{(1-loop)}(s, t), \]

where \( I_{4}^{(1)}(s, t) \) is the scalar box diagram

\[ I_{4}^{(1-loop)}(s, t) = -\frac{2}{s t} \frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)} \left[ \frac{1}{\epsilon^2} \left( \frac{\mu^2}{s} \right)^{\epsilon} + \left( \frac{\mu^2}{-t} \right)^{\epsilon} \right] + \frac{1}{2} \log^2 \left( \frac{s}{-t} \right) + \frac{\pi^2}{2} + O(\epsilon). \]

Then it is straightforward to obtain the one-loop contribution to the cross-section in the planar limit

\[ \left( \frac{d\sigma_{2-2}}{d\Omega_{13}} \right)_{virt} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^{2\epsilon} 4 \left\{ \alpha \left( \frac{3}{4\pi} \right) \left[ -\frac{16}{c^2} \frac{3 + c^2}{(1 - c^2)^2} + \frac{4}{\epsilon} \left( \frac{5 + 2c + c^2}{(1 - c^2)^2} \right) \log \left( \frac{1 - c}{2} \right) \right] + \frac{5 - 2c + c^2}{(1 - c^2)^2} \log \left( \frac{1 + c}{2} \right) \right\}. \] \hspace{1cm} (8)

It should be stressed that due to conformal invariance of \( \mathcal{N} = 4 \) SYM theory at quantum level there are no UV divergences in (8) and all divergences have the IR or collinear nature. They have to cancel in the properly defined observables.

3.2 Real emission

The next step is the calculation of the amplitude with three outgoing particles. Here we have to define which is the process that we are interested in. There are several possibilities.

1. Three gluons with positive helicities: \( g^+g^+g^+ \). This is the MHV amplitude;

2. Two gluons with positive helicities and the third one with negative helicity: \( g^+g^+ \rightarrow g^+g^+g^- \). This is the anti-MHV amplitude;

3. One of three final particles is the gluon with positive helicity and the rest is the quark-antiquark pair: \( g^+g^+ \rightarrow g^+q^-\bar{q}^+ \) or \( g^+g^+ \rightarrow g^+q^+\bar{q}^- \). This is an anti-MHV amplitude;

4. One of three final particles is the gluon with positive helicity and the rest are two scalars: \( g^+g^+ \rightarrow g^+\Lambda\Lambda \). This is an anti-MHV amplitude.

If one fixes one gluon with positive helicity scattered at angle \( \theta \) and sum over all the other particles then all the processes mentioned above contribute. In case when one fixes two gluons with positive helicity and looks for the rest, only the first two options are allowed.

8There is also a \( g^+g^+ \rightarrow g^+g^-g^- \) helicity configuration, but the partial amplitudes for them are equal. We will use \((-+ +-)\) notation for both of them.

9The \( \mathcal{N} = 4 \) supermultiplet consists of a gluon \( g \), 4 fermions ("quarks") \( q^A \) and 6 real scalars \( \Lambda^{AB} \). \( A \) and \( B \) are \( SU(4)_R \) indices, \( \Lambda \) is an antisymmetric tensor. It is implied that all squared amplitudes with quarks and scalars are summed over these indices.
The cross-section of these processes can be written as

$$\frac{d\sigma_{2-3}}{d\Omega_{13}} = \frac{1}{f} \int d\phi_3 |M_3^{(\text{tree})}|^2 S_3,$$

where \(d\phi_3\) is the three-particle phase volume and \(S_3\) is the measurement function, which constrains the phase space and defines the particular observable.

We omit the details of the calculation and for the lack of space present here only the divergent parts of the calculated objects. We leave all the full answers for a separate publication and present the finite part only in the simplest case below.

When there are three identical particles in the final state one has to define which ones are detected. In case of one detectable particle one can choose the fastest one, in case of two - the two fastest. Selection of detectable particles can be achieved restricting the momentum. Thus choosing \(p_3^0 > E/3\) we fix the gluon with momentum \(p_3\) as the fastest particle. When the final particles are not identical this problem does not appear and the phase space is not restricted. In what follows we restrict the moment of the gluon by a universal value \(p_3^0 > (1 - \delta)/2E\) and keep the value of \(\delta\) arbitrary. The case of identical particles then corresponds to \(\delta = 1/3\) and the case of nonidentical particles to \(\delta = 1\). One can show that in this case the requirements of stability of observable with respect to emission of soft and collinear quanta [29] are satisfied. We show below that IR and collinear divergences cancel in observables for an arbitrary value of \(\delta\).

1. Real Emission (MHV)

$$\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{\text{Real}}^{(-+++)} = \alpha^2 N_c \left(\frac{\mu}{s}\right)^2 \frac{2\alpha}{\pi} \left\{ \frac{1}{\epsilon} \left[ 8(3+c^2) + \frac{2}{c^2(1-c^2)^2} \log(1-c^2) + \frac{2}{(1-c)^2} \log(1+c^2) \right] \right.\right.$$

$$+ \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2} + \frac{12(3+c^2)}{(1-c^2)^2} \log \frac{1-\delta}{\delta} \left.\right\} + \text{Finite part}; \tag{10}$$

Notice the singularity when \(\delta \to 1\).

2. Real Emission (anti-MHV)

$$\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{\text{Real}}^{(--++)} = \alpha^2 N_c \left(\frac{\mu}{s}\right)^2 \frac{2\alpha}{\pi} \left\{ \frac{1}{\epsilon} \left[ -12(c^2+3) \log(1-c^2) + \frac{64(12c^2+17)}{3(1-c^2)^3} \right.\right.$$

$$+ \frac{2\delta}{(1-c^2)^2} \left( \frac{2}{3}(5+3c^2)\delta^2 - (c^2+19)\delta + 2(5c^2+43) \right) + \frac{2(3c^2-24c+85)}{(1-c)(1+c^3)} \log(1-c^2) \right.\right.$$

$$\left. - \frac{8(c^2-6c+21)}{(1-c)(1+c^3)} \log(1+\delta-(1-\delta)c) - \frac{32(c^2-4c+7)}{(1+c)^3(1+c^3)(1+\delta-c(1-\delta))} \right.\right.$$

$$\left. - \frac{32(2-c)}{(1+c)^3(1+\delta-c(1-\delta))^2} + \frac{64(1-c)}{3(1+c)^3(1+\delta-c(1-\delta))^3} + (c \leftrightarrow -c) \right\} + \text{Finite part}; \tag{11}$$

3. Fermions

$$\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{\text{Real}}^{(--+\bar{q})} = \alpha^2 N_c \left(\frac{\mu}{s}\right)^2 \frac{2\alpha}{\pi} \left\{ -\frac{16}{\epsilon} \left[ \frac{(79+25c^2)}{3(1-c^2)^2} \right.\right.$$

$$+ \frac{2(3-c)^2}{(1-c)(1+c)^3} \log(1-c^2) + \frac{2(3+c)^2}{(1-c)^3(1+c)} \log(1+c^2) \right\} + \text{Finite part}; \tag{12}$$
4. Scalars

$$\left( \frac{d\sigma_{2\to 3}}{d\Omega_{13}} \right)_{\text{Real}}^{(-++\Lambda\Lambda)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{8}{\epsilon} \left[ -\frac{2(10 + 7c^2)}{(1 - c^2)^2} \right] + \text{Finite part} \right\}$$

3.3 Splitting

Taking into account emission of additional soft quanta allows one to cancel the IR divergences (double poles in $\epsilon$) but leaves the single poles originating from collinear ones. Indeed, in case of massless particles the asymptotic states (both the initial and final ones) are not well defined since a massless quanta can split into two parallel ones indistinguishable from the original. To take this into account one introduces the notion of distribution of the initial particle (gluon) with respect to the fraction of the carried momentum $z$: $g(z)$. Then the initial distribution corresponds to $g(z) = 1(1 - z)$, and the emission of a gluon leads to a splitting: the gluon carries the fraction of momentum equal $z$, while the collinear gluon - $(1 - z)$. The probability of this event is given by the so-called splitting functions $P_{gg}(z)$. In case of a gluon in a final state this corresponds to the fragmentation of the gluon into pair of gluons or pair of quarks or scalars.

Additional contributions from collinear particles in initial or final states to inclusive cross-sections have the form, respectively

$$d\sigma_{2\to 2}^{\text{spl,init}} = \frac{\alpha}{2\pi} \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \int_0^1 dz g(z) \sum_{i,j=1,2; i\neq j} d\sigma_{2\to 2}(zp_i, p_j, p_3, p_4) S_{ij}^{\text{spl,init}}(z),$$

$$d\sigma_{2\to 2}^{\text{spl,fin}} = \frac{\alpha}{2\pi} \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon d\sigma_{2\to 2}(p_1, p_2, p_3, p_4) \int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z) S_{ij}^{\text{spl,fin}}(z),$$

where the scale $Q_f^2$, sometimes called the factorization scale, belongs to the definition of the coherent asymptotic state and restricts the value of transverse momenta. The dependence of parton distribution on $Q_f^2$ is governed by the DGLAP equation. The splitting function $P_{ij}$ for each helicity configuration can be obtained as a collinear limit of the corresponding partial amplitude (see for example [35, 36] for more details).

Taking into account the splitting of initial states and the fragmentation of the final states we get the following contribution to the inclusive cross sections

1. The initial and final splitting for the MHV amplitude.

$$\left( \frac{d\sigma_{2\to 3}}{d\Omega_{13}} \right)_{\text{InSplit}}^{(-++++)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{4(c^2 + 3)}{(1 - c^2)^2} \left( \log \frac{1-c}{2} + \log \frac{1+c}{2} \right) - \frac{8(1-c)}{(1 - c^2)^2} \log \frac{1-\delta}{\delta} - \frac{16\delta(2\delta - 3)}{(1 - c^2)^2(1 - \delta)^2} \right] \right\} + \text{Finite part},$$

$$\left( \frac{d\sigma_{2\to 3}}{d\Omega_{13}} \right)_{\text{FinSplit}}^{(-++++)} = \frac{\alpha^2 N_c^2}{E^2} \left( \frac{\mu^2}{s} \right)^\epsilon \left( \frac{\mu^2}{Q_f^2} \right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{1}{\epsilon} \left[ -\frac{4(c^2 + 3)}{(1 - c^2)^2}\log \frac{1-\delta}{\delta} \right] \right\};$$
2. The initial and final splitting for the anti-MHV amplitude

\[
\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{InSplit}}}^{(-++-)} = \frac{\alpha'^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{1}{\epsilon} \left(8(c^2 + 3)\right) \log \delta - \frac{64(12c^2 + 17)}{3(1-c^2)^3} \right. \\
- \frac{4\delta}{(1-c^2)^2} \left(\frac{2}{3}(1 + c^2)\delta + (c^2 - 5)\delta + 2(c^2 + 17)\right) + \left(\frac{4(c^3 - 15c^2 + 51c - 45)}{(1-c^2)(1+c)^3}\right) \log \frac{1-c}{2} \\
+ \frac{8(c^2 - 6c + 21)}{(1-c)(1+c)^3} \log \frac{1+\delta - c(1-\delta)}{2} + \frac{32(c^2 - 4c + 7)}{(1+c)^3(1-\delta)(1+\delta - c(1-\delta))} \\
- \frac{32(2-c)}{(1+c)^3(1-\delta)^2} + \frac{64(1-c)}{3(1+c)^3(1-\delta)(1+\delta - c(1-\delta))} + (c \leftrightarrow -c) \right\} + \text{Finite part},
\]

(18)

\[
\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{FnSplit}}}^{(-++\bar{q}d)} = \frac{\alpha'^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{14(c^2 + 3)}{\epsilon (1-c^2)^2} \right. \\
- \frac{2(3-c)^2}{(1-c)(1+c)^3} \log \frac{1-c}{2} + \frac{2(3+c)^2}{(1-c)^3(1+c)} \log \frac{1+c}{2} \right\} + \text{Finite part};
\]

(20)

3. The initial splitting for the quark final states (\(\delta = 1\))

\[
\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{InSplit}}}^{(--\bar{q}d)} = \frac{\alpha'^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{16(c^2 + 3)}{\epsilon (1-c^2)^2} \right. \\
- \frac{2(3-c)^2}{(1-c)(1+c)^3} \log \frac{1-c}{2} + \frac{2(3+c)^2}{(1-c)^3(1+c)} \log \frac{1+c}{2} \right\} + \text{Finite part};
\]

(21)

4. The initial splitting for the scalar final states (\(\delta = 1\))

\[
\left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{InSplit}}}^{(-++\bar{A}A)} = \frac{\alpha'^2 N_c^2}{E^2} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{\mu^2}{Q_f^2}\right)^\epsilon \frac{\alpha}{\pi} \left\{ \frac{8}{\epsilon} \left[ \frac{2(10 + 7c^2)}{(1-c)^2} \right] \\
- \frac{3(5-c)}{(1+c)^3} \log \frac{1-c}{2} - \frac{3(5+c)}{(1-c)^3} \log \frac{1+c}{2} \right\} + \text{Finite part}. 
\]

4 IR safe observables in \(\mathcal{N} = 4\) SYM

In the NLO there are two sets of amplitudes, namely the MHV and anti-MHV amplitudes, which contribute to the observables. The leading order 4-gluon amplitude is both MHV and anti-MHV and we split it into two parts. Then one can construct three types of infrared-safe quantities in the NLO of perturbation theory, namely

- pure gluonic MHV amplitude

\[
A^{MHV} = \frac{1}{2} \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{Virt}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{Real}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{InSplit}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{FnSplit}}}^{(--++)};
\]

(22)

- pure gluonic anti-MHV amplitude

\[
B^{anti-MHV} = \frac{1}{2} \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{Virt}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{Real}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{InSplit}}}^{(--++)} + \left(\frac{d\sigma_{2-3}}{d\Omega_{13}}\right)_{_{\text{FnSplit}}}^{(--++)}; 
\]

(23)
• anti-MHV amplitude with fermions or scalars from the full $\mathcal{N} = 4$ supermultiplet

$$C_{\text{Matter}} = \left( \frac{d\sigma_{2\rightarrow 3}}{d\Omega_{13}} \right)_{\text{Real}} + \left( \frac{d\sigma_{2\rightarrow 3}}{d\Omega_{13}} \right)_{\text{InSplit}}. \quad (24)$$

We would like to stress once more that in each expression $\text{(22, 23, 24)}$ all IR divergences cancel for arbitrary $\delta$ and only the finite part is left.

Defining now the physical condition for the observation we get several infrared-safe inclusive cross-sections

• Registration of two fastest gluons of positive helicity

$$A^{\text{MHV}}|_{\delta=1/3} + B^{\text{anti-MHV}}|_{\delta=1}; \quad (25)$$

• Registration of one fastest gluon of positive helicity

$$A^{\text{MHV}}|_{\delta=1/3} + B^{\text{anti-MHV}}|_{\delta=1/3} + C_{\text{Matter}}|_{\delta=1}; \quad (26)$$

• Anti-MHV cross-section

$$B^{\text{anti-MHV}}|_{\delta=1} + C_{\text{Matter}}|_{\delta=1}. \quad (27)$$

Relative simplicity of the virtual contribution $\text{(8)}$ which does not contain any special functions but logs suggests similar structure of the real part. However this is not the case. While the singular terms are simple enough and cancel completely the finite parts are usually cumbersome and contain polylogarithms. The only expression where they cancel corresponds to $\delta = 1$ case which is possible only for the last set of observables, namely for the anti-MHV cross-section $\text{(27)}$. Choosing the factorization scale to be $Q_f = E$ we get

$$\left( \frac{d\sigma}{d\Omega_{13}} \right)_{\text{AntiMHV}} = \frac{4\alpha^2 N_c^2}{E^2} \left\{ \frac{3 + c^2}{(1 - c^2)^2} \right. \left. - \frac{\alpha}{4\pi} \left[ 2 \left( \frac{c^4 + 2c^3 + 4c^2 + 6c + 19}{(1 - c)^2(1 + c)^4} \log^2 \left( \frac{1 - c}{2} \right) + \frac{6\alpha^2(3c^2 + 13)}{9(1 - c^2)^2} \right) \right] \right\} \quad (28)$$

One can see that even this expression does not repeat the Born amplitude and does not have any simple structure. Note, that the finite answer depends on the factorization scale. This dependence comes from the asymptotic states which violate conformal invariance of the Lagrangian. This dependence seems to be unavoidable and reflects the act of measurement. Construction of observables which do not contain any external scale remains an open question.
5 Discussion

To solve the model might have different meaning. Calculation of divergences and understanding of their structure is very useful but surely not enough. The knowledge of the S-matrix would be the final goal though the definition of the S-matrix in conformal theory is a problem. Even in the absence of the UV divergences there are severe IR problems and matrix elements do not exist after removal of regularization. The experience of QCD, which is very similar to $\mathcal{N}=4$ SYM theory from the point of view of the IR problems, tells us that in inclusive cross-sections the IR divergences cancel and one has finite physical observables. However, one either has to redefine the asymptotic states or consider the scattering of "hadrons". In both the cases one has to introduce some parton distributions which are the functions of a fraction of momenta and, in higher orders, of momenta transfer. This leads to appearance of a factorization scale which breaks conformal invariance. This means that we do not have meaningful observables in a pure conformal theory. The finite observables of the type considered here are the inclusive cross-sections

$$d\sigma_{\text{incl}}^{\text{obs}} = \sum_{n=2}^{\infty} \int dz_1 q_1(z_1, \frac{Q^2_{1}}{\mu^2}) \int dz_2 q_2(z_2, \frac{Q^2_{2}}{\mu^2}) \prod_{i=1}^{n} \int dx_i q_i(x_i, \frac{Q^2_{i}}{\mu^2}) \times$$

$$\times d\sigma^{2-n}(z_1 p_1, z_2 p_2, \ldots) S_n(\{z\}, \{x\}) = g^4 \sum_{L=0}^{\infty} \left( \frac{g^2}{16\pi^2} \right)^L d\sigma_L^{\text{finite}}(s, t, u, Q^2_f),$$

which besides the kinematical variables contain the dependence on the factorization scale.

Remarkable factorization properties of the MHV amplitudes accumulated in the BDS ansatz (with the so far unknown modification) suggest the way of receiving "exact" results for the amplitudes on shell. However, as we have already mentioned, it is the finite part that we are really for. Unfortunately, our calculation has demonstrated that the simple structure of the amplitudes governed by the cusp anomalous dimension has been totally washed out by complexity of the real emission matrix elements integrated over the phase space. This means that either $\mathcal{N}=4$ SYM theory does not allow such a simple factorizable solution or that we considered the inappropriate observables. Alternatively one may rely on simplification in particular kinematic regime like the Regge limit where the exponentiation is expected.

There is an interesting duality between the MHV amplitudes and the Wilson loop, between the weak and the strong coupling regime \cite{15,16,17,18}. Probably it would be possible using the AdS/CFT correspondence to construct the IR safe observables at the strong coupling limit (similarly to what we did here) and to shed some light on the "true" calculable objects in conformal theories.

Acknowledgements

We would like to thank A.Gorsky, A.Kotikov, L. Lipatov, R. Roiban, A. Slavnov for valuable discussions. In particular, we thank G.Korchemsky for useful comments and the list of references. Financial support from RFBR grant # 08-02-00856 and grant of the Ministry of Education and Science of the Russian Federation # 1027.2008.2 is kindly acknowledged. Two of us LB and GV are partially supported by the Dynasty foundation.
References

[1] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, Phys. Rev. Lett. 56, 2459 (1986);
F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with n Gluons*, Nucl. Phys. B 306, 759 (1988).

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, Nucl. Phys. B 636 (2002) 99 [arXiv:hep-th/0204051].

[3] S. Frolov and A. A. Tseytlin, *Semiclassical quantization of rotating superstring in AdS(5) X S(5)*, JHEP 0206 (2002) 007 [arXiv:hep-th/0204226].

[4] M. L. Mangano and S. J. Parke, *Multiparton amplitudes in gauge theories*, Phys. Rept. 200, 301 (1991) [arXiv:hep-th/0509223].

[5] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, Phys. Rev. Lett. 91 (2003) 251602 [arXiv:hep-th/0309040].

[6] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, Phys. Rev. D 72 (2005) 085001 [arXiv:hep-th/0505205].

[7] A. H. Mueller, *On The Asymptotic Behavior Of The Sudakov Form-Factor*, Phys. Rev. D 20 (1979) 2037;
J. C. Collins, *Algorithm To Compute Corrections To The Sudakov Form-Factor*, Phys. Rev. D 22 (1980) 1478;
A. Sen, *Asymptotic Behavior Of The Sudakov Form-Factor In QCD*, Phys. Rev. D 24 (1981) 3281;
G. P. Korchemsky, *Double logarithmic asymptotics in QCD*, Phys. Lett. B 217 (1989) 330;
L. Magnea and G. Sterman, *Analytic continuation of the Sudakov form-factor in QCD*, Phys. Rev. D 42 (1990) 4222;
G. P. Korchemsky, *Sudakov Form-Factor In QCD*, Phys. Lett. B 220 (1989) 629.

[8] S. V. Ivanov, G. P. Korchemsky and A. V. Radyushkin, *Infrared Asymptotics Of Perturbative QCD: Contour Gauges*, Yad. Fiz. 44 (1986) 230 [Sov. J. Nucl. Phys. 44 (1986) 145].
G. P. Korchemsky and A. V. Radyushkin, *Loop Space Formalism And Renormalization Group For The Infrared Asymptotics Of QCD*, Phys. Lett. B 171 (1986) 459.

[9] L. J. Dixon, L. Magnea and G. Sterman, *Universal structure of subleading infrared poles in gauge theory amplitudes*, [arXiv:0805.3515 [hep-ph]];
L. F. Alday, *Universal structure of subleading infrared poles at strong coupling*, [arXiv:0904.3983 [hep-th]].

[10] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, *The Four-Loop Planar Amplitude and Cusp Anomalous Dimension in Maximally Supersymmetric Yang-Mills Theory*, Phys. Rev. D 75 (2007) 085010 [arXiv:hep-th/0610248];
F. Cachazo, M. Spradlin and A. Volovich, *Four-Loop Cusp Anomalous Dimension From Obstructions*, Phys. Rev. D 75 (2007) 105011 [arXiv:hep-th/0612309].

[11] B. Eden and M. Staudacher, *Integrability and transcendentality*, J. Stat. Mech. 0611 (2006) P014 [arXiv:hep-th/0603157].
N. Beisert, B. Eden and M. Staudacher, Transcendentality and crossing, J. Stat. Mech. 0701 (2007) P021 [arXiv:hep-th/0610251].

[12] B. Basso, G. P. Korchemsky, J. Kotanski, Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling, Phys. Rev. Lett. 100:091601, (2008).

[13] F. Cachazo, M. Spradlin and A. Volovich, Iterative structure within the five-particle two-loop amplitude, Phys. Rev. D 74, 045020 (2006) [arXiv:hep-th/0602228];
Z. Bern, M. Czakon, D. A. Kosower, R. Roiban and V. A. Smirnov, Two-Loop Iteration of Five-Point \( N=4 \) Super-Yang-Mills Amplitudes, Phys. Rev. Lett. 97, 181601 (2006) [arXiv:hep-th/0604074].

[14] L. F. Alday and J. Maldacena, Comments on gluon scattering amplitudes via AdS/CFT, JHEP 0711 (2007) 068 [arXiv:0710.1060 [hep-th]].

[15] L. F. Alday and J. M. Maldacena, Gluon scattering amplitudes at strong coupling, JHEP 0706 (2007) 064 [arXiv:0705.0303 [hep-th]].

[16] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, Conformal properties of four-gluon planar amplitudes and Wilson loops, Nucl. Phys. B 795 (2008) 385 [arXiv:0707.0243 [hep-th]].

[17] A. Brandhuber, P. Heslop and G. Travaglini, MHV Amplitudes in \( N=4 \) Super Yang-Mills and Wilson Loops, Nucl. Phys. B 794 (2008) 231 [arXiv:0707.1153 [hep-th]].

[18] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, On planar gluon amplitudes/Wilson loops duality, Nucl. Phys. B 795 (2008) 52 [arXiv:0709.2368].

[19] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, The hexagon Wilson loop and the BDS ansatz for the six-gluon amplitude, Phys. Lett. B 662 (2008) 456 [arXiv:0712.4138 [hep-th]].

[20] J. Bartels, L. N. Lipatov, A. S. Vera, BFKL Pomeron, Reggeized gluons and Bern-Dixon-Smirnov amplitudes, [arXiv:0802.2065 [hep-th]].

[21] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory, [arXiv:0803.1465 [hep-th]].

[22] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Hexagon Wilson loop = six-gluon MHV amplitude, Nucl. Phys. B 815 (2009) 142 [arXiv:0803.1466 [hep-th]].

[23] T. Kinoshita, Mass singularities of Feynman amplitudes, J. Math. Phys. 3 (1962) 650;
T. D. Lee, M. Nauenberg, Degenerate Systems and Mass Singularities, Phys. Rev. 133 (1964) B1549.

[24] N. A. Sveshnikov and F. V. Tkachov, Jets and quantum field theory, Phys. Lett. B 382 (1996) 403 [arXiv:hep-ph/9512370].
M. Testa, Exploring the light-cone through semi-inclusive hadronic distributions, JHEP 9809 (1998) 006 [arXiv:hep-ph/9807204].

[25] G. P. Korchemsky, G. Oderda and G. Sterman, Power corrections and nonlocal operators, [arXiv:hep-ph/9708346].
G. P. Korchemsky and G. Sterman, Power corrections to event shapes and factorization, Nucl. Phys. B 555, 335 (1999)
[26] D. M. Hofman and J. Maldacena, *Conformal collider physics: Energy and charge correlations*, JHEP **0805**, 012 (2008) [arXiv:0803.1467 [hep-th]].

[27] W. L. van Neerven, *Infrared Behavior Of On-Shell Form-Factors In A N=4 Supersymmetric Yang-Mills Field Theory*, Z. Phys. C **30**, 595 (1986).

[28] S. D. Ellis, Z. Kunszt, D. E. Soper, *The One Jet Inclusive Cross-Section at Order $\alpha_s^3$. 1. Gluons Only*, Phys. Rev. D **40**:2188,1989.

[29] S. D. Ellis, Z. Kunszt, D. E. Soper, *The One Jet Inclusive Cross-Section at Order $\alpha_s^3$. Quarks and gluons*, Phys. Rev. Lett. **64** (1990) 2121.

[30] Z. Kunszt, D. E. Soper, *Calculation of jet cross-sections in hadron collisions at order $\alpha_s^3$*, Phys. Rev. D **46** (1992) 192.

[31] S. Catani, M. H. Seymour, *A General algorithm for calculating jet cross-sections in NLO QCD*, Nucl. Phys. B**485** (1997) 291 [hep-ph/9605323].

[32] B. A. Kniehl, A. V. Kotikov, Z. V. Merebashvili, O. L. Veretin, *Heavy-quark pair production in polarized photon-photon collisions at next-to-leading order: Fully integrated total cross sections*, Phys. Rev. D **79**:114032,2009 [arXiv:0905.1649 [hep-ph]].

[33] Z. Bern, D. A. Kosower, *The Computation of loop amplitudes in gauge theories*, Nucl. Phys. B **379** (1992) 451.

[34] M. B. Green, J. H. Schwarz and L. Brink, *N=4 Yang-Mills And N=8 Supergravity As Limits Of String Theories* Nucl. Phys. B **198**, 474 (1982).

[35] S. Frixione, Z. Kunszt, A. Signer, *Three-jet cross sections to next-to-leading order*, Nucl. Phys. B **467** (1996) 399. [hep-ph/9703305].

[36] L. J. Dixon, *Calculating scattering amplitudes efficiently*, [arXiv:hep-ph/9601359].