DOUBLE BIFURCATION DIAGRAMS AND FOUR POSITIVE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS VIA TIME MAPS

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(Communicated by Rafael Ortega)

Abstract. In this paper, we consider the existence and exactness of multiple positive solutions for the nonlinear boundary value problem

\[
\begin{align*}
-\frac{u''(x)}{\lambda} &= f(u), \quad 0 < x < 1, \\
\frac{u(0)}{l} &= 0, \\
\frac{1}{u(1)} + u'(1) + [1 - u(1)]u(1) &= 0,
\end{align*}
\]

where \( \lambda > 0 \) is a bifurcation parameter, \( f(u) > 0 \) for \( u > 0 \). We give complete descriptions of the structure of bifurcation curves and determine the existence and multiplicity of positive solutions of the above problem for \( f(u) = e^u, f(u) = a^u(a > 0), f(u) = u^p(p > 0), f(u) = e^u - 1, f(u) = a^u - 1(a > 1) \) and \( f(u) = (1 + u)^p(p > 0) \). Our methods are based on a detailed analysis of time maps.

1. Introduction. In this paper, we study the bifurcation diagrams and multiple positive solutions of the nonlinear boundary value problem

\[
\begin{align*}
-\frac{u''(x)}{\lambda} &= f(u), \quad 0 < x < 1, \\
\frac{u(0)}{l} &= 0, \\
\frac{1}{u(1)} + u'(1) + [1 - u(1)]u(1) &= 0,
\end{align*}
\]

where \( \lambda > 0 \) is a bifurcation parameter, \( f(u) > 0 \) for \( u > 0 \).

Our technique is a careful analysis of the so-called time-map \( G(\lambda, \rho) \) (or \( H(\lambda, \rho) \)), a function defined by an elliptic integral, which measures the time an orbit takes to get from one boundary line to another. In fact, the time map method was widely used in Laetsch [14], Crandall and Rabinowitz [5], Smoller and Wasserman [22], Castro and Shivaji [3], Liu and Zhang [18], Wang and Yeh [23], Cheng [4], Addou and Wang [1], Brubaker and Pelesko [2], Pan and Xing [19, 20], Zhang and Feng [24] and Huang, Cheng, Wang and Chuang [7].

2000 Mathematics Subject Classification. 34B18, 74G35.
Key words and phrases. Double bifurcation diagrams, existence and multiplicity of positive solutions, nonlinear boundary conditions, time map.
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The motivation for this study comes from Goddard et al. in [6] and Hung et al. in [10]. In [6], J. Goddard II, E.K. Lee and R. Shivaji have considered the reaction-diffusion model with nonlinear boundary condition given by

\[
\begin{align*}
&u_t = d\Delta u + \lambda f(u), \quad x \in \Omega, \\
&d\alpha(u) \frac{\partial u}{\partial \eta} + (1 - \alpha(u))u = 0, \quad x \in \partial\Omega,
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( n \geq 1 \), \( \Delta \) is the Laplace operator, \( \lambda \) is a positive parameter, \( d \) is the diffusion coefficient, \( \alpha(u) \) is the outward normal derivative, \( f: [0, \infty) \to [0, \infty) \) is a smooth function, and \( \alpha(u) \in [0, 1] \) is a nondecreasing smooth function. They discussed the existence of at least two positive radial solutions for \( \lambda \gg 1 \) when \( \Omega \) is an annulus in \( \mathbb{R}^n \). Further, they discussed the existence of a double S-shaped bifurcation curve and the existence of six positive solutions for a certain range of positive \( \lambda \) when \( n = 1, \Omega = (0, 1) \), and \( f(u) = e^au + \) with \( a \gg 1 \). However, their results are all based on numerical computations. K. Hung, S. Wang and C. Yu [10] made further investigation on this problem and proved the results of [6] theoretically.

As Goddard et al. in [6] and Hung et al. in [10] pointed out that studying problem (1.1) is equivalent to analyzing the following two boundary problems

\[
\begin{align*}
&-u''(x) = \lambda f(u), \quad 0 < x < 1, \\
&u(0) = u(1) = 0, \quad (1.2)
\end{align*}
\]

and

\[
\begin{align*}
&-u''(x) = \lambda f(u), \quad 0 < x < 1, \\
&u(0) = 0, \quad u'(1) = -c. \quad (1.3)
\end{align*}
\]

More precisely, the positive solutions of (1.2) and (1.3) are the positive solutions of (1.1).

When \( f(u) = e^u \), (1.2) is known as one-dimensional Liouville-Bratu-Gelfand problem. The classical Liouville-Bratu-Gelfand problem is concerned with positive solutions of the equation

\[
\begin{align*}
&-\Delta u = \lambda e^u, \quad x \in \Omega, \\
&u = 0, \quad x \in \partial\Omega, \quad (1.4)
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \). The equation arises from a model of combustible gas dynamics. Nontrivial solutions of problem (1.4) are steady states for the thermal reaction process. Here \( \lambda > 0 \) is known as the Frank-Kamenetskii parameter. In [15, 16], Liang and Wang studied the classification and evolution of bifurcation curves of positive solutions for the one-dimensional perturbed Gelfand equation with Dirichlet-Neumann boundary conditions given by

\[
\begin{align*}
&-u''(x) = \lambda e^{au} \bigg(\frac{au}{a + u}\bigg), \quad 0 < x < 1, \\
&u(0) = 0, \quad u'(1) = -c, \quad (1.5)
\end{align*}
\]

where \( 4 \leq a < \infty \). They proved that the shape of the bifurcation curve depended on the value of \( a, c \). The proofs given in [15, 16] are more subtle and complicated. The results substantially improve and generalize those given in [10].

The earlier study of problem (1.2) is in [17, 11]. Since then, problem (1.2) has been exuberantly studied in the past decades. There are two main tools to study the exact multiplicity of positive solutions and the shape of the bifurcation curve of problem (1.2). One way is bifurcation theory, see [12, 13, 21] and the references cited therein. In particular, we would like to mention some excellent results of
Shi and Shivaji [21]. Using bifurcation theory, in [21], Shi and Shivaji obtained the exact multiplicity of positive solutions of problem (1.2) with $f(0) < 0$, which is called semipositone problem, and proved that the bifurcation diagram of (1.2) looks exactly like one of the following two graphs:

Fig.1. Reversed S-shaped curve.  Fig.2. Broken reversed S-shaped curve.

Another way is time map method. In [8], Hung and Wang studied the bifurcation curve and exact multiplicity of positive solutions of problem (1.2) by using time map analysis, and gave an application to the perturbed Gelfand problem

\[
\begin{aligned}
&u''(x) + \lambda \exp\left(\frac{a u}{a + u}\right) = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{aligned}
\]

(1.6)

where $a > 0$ is the activation energy parameter. See Fig.3. The authors proved that, if $a \geq a^* \approx 4.166$ for some constant $a^*$ defined in [8, Theorem 2.2(i)], the bifurcation curve $S$ of (1.6) is exactly S-shaped on the $(\lambda, \|u\|_\infty)$-plane. More precisely, there exist two positive numbers $\lambda_* < \lambda^*$ such that (1.6) has exactly three positive solutions for $\lambda_* < \lambda < \lambda^*$, exactly two positive solutions for $\lambda = \lambda_*$ and $\lambda = \lambda^*$, and exactly one positive solution for $0 < \lambda < \lambda_*$ and $\lambda > \lambda^*$. Recently, Huang and Wang [9] proved that there exists a critical bifurcation value $a_0 \approx 4.069$ such that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve is exactly S-shaped for $a > a_0$ and is monotone increasing for $0 < a \leq a_0$. That is, they proved the long-standing conjecture for the one-dimensional perturbed Gelfand problem.

Fig.3. Exactly S-shaped bifurcation curve $S$ of (1.6).
We define the bifurcation diagrams of positive solutions of (1.1), (1.2) and (1.3), respectively as follows:

\[ C = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \} , \]
\[ S = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.2)} \} , \]
\[ \tilde{S} = \{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.3)} \} . \]

Clearly, \( C = S \cup \tilde{S} \). In this paper, on the \((\lambda, \|u\|_\infty)\)-plane, we will give bifurcation curve \( S \) (resp. \( \tilde{S} \)) for six cases: see Fig. 6, Fig. 8, Fig. 10, Fig. 12, Fig. 14 and Fig. 16.

For problem (1.3), Goddard, Lee and Shivaji [6] discussed the existence of a \( S \)-shaped bifurcation curve by using numerical computations. In [10], Hung, Wang and Yu gave rigorous theoretical proofs of some computational results of [6]. In [15, 16], Liang and Wang gave more perfect results on (1.5). Notice that these results are all on exponential nonlinearity. In this paper, we not only give some new results on exponential nonlinearity by establishing a new time map, but also study problem (1.1) with other nonlinearities of \( f(u) = a^u (a > 0) \), \( f(u) = u^p (p > 0) \), \( f(u) = e^{u^a} - 1 \), \( f(u) = a^u - 1(a > 1) \) and \( f(u) = (1 + u)^p(p > 0) \). In our main results, we obtain the bifurcation curve \( C = S \cup \tilde{S} \) of problem (1.1) on the \((\lambda, \|u\|_\infty)\)-plane, and prove that problem (1.1) has multiple positive solutions for a certain range of positive \( \lambda \). However, the exact shape of bifurcation curve \( \tilde{S} \) and the exact number of problem (1.3) need further investigation.

The rest of the paper is organized as follows: In Section 2, we introduce a new time map and analyze the time map which plays a key role in the paper. In Section 3, the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases of \( f(u) = e^u \) and \( f(u) = a^u (a > 0) \) will be stated. In Section 4, we give the bifurcation diagrams and multiple positive solutions of problem (1.1) for the case \( f(u) = u^p (p > 0) \). In Section 5, we establish the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases \( f(u) = e^u - 1 \) and \( f(u) = a^u - 1(a > 1) \), and we determine the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases of \( f(u) = (1 + u)^p(p > 0) \) in Section 6.

2. Time map. In this paper, to prove our main results, we define a new time map, which is different from that of Smoller and Wasserman [22], Goddard, Lee and Shivaji [6] and Huang, Wang and Yu [10].

As is well known, (1.2) is equivalent to
\[
\sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda},
\]
where \( \rho = \|u\|_\infty, \ F(u) = \int_0^u f(s)ds \). We define a new time map as follows:
\[
G(\lambda, \rho) = \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}}, \quad (\lambda, \rho) \in \Sigma, \tag{2.1}
\]
where
\[
\Sigma = \{ (\lambda, \rho) | 0 < \lambda < +\infty, 0 < \rho < +\infty \}.
\]
Then positive solutions of (1.2) correspond to
\[
\|u\|_\infty = \rho \text{ and } G(\lambda, \rho) = 1. \tag{2.2}
\]
Thus, studying of the number of positive solutions of (1.2) is equivalent to studying the shape of the time map $G(\lambda, \rho)$ on $\Sigma$. By calculation, we can get that
\[
\frac{\partial}{\partial \rho} G(\lambda, \rho) = \frac{\sqrt{2}}{2} \int_0^\rho \frac{\theta'(s) - \theta(s)}{\rho \sqrt{\lambda (F(\rho) - F(s))}} ds,
\] (2.3)
where $\theta(s) = 2F(s) - sf(s)$.

On the other hand, by [7], problem (1.3) is equivalent to
\[
\sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{2}} \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda},
\] where $0 \leq q \leq \rho$ satisfies $2\lambda(F(\rho) - F(q)) = 1$.

We define
\[
H(\lambda, \rho) = \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{\lambda (F(\rho) - F(s))}} - \frac{1}{\sqrt{2}} \int_0^{F^{-1}(F(\rho) - \frac{\lambda}{4\pi})} \frac{ds}{\sqrt{\lambda (F(\rho) - F(s))}}, \quad (\lambda, \rho) \in \Gamma
\] (2.4)
as the time map, where
\[
\Gamma = \{(\lambda, \rho) | 0 < \lambda < +\infty, \rho_0(\lambda) \leq \rho < +\infty\}, \quad \rho_0(\lambda) = F^{-1}(\frac{1}{2\lambda}).
\]
Then positive solutions of (1.3) correspond to
\[
\|u\|_\infty = \rho \quad \text{and} \quad H(\lambda, \rho) = 1.
\] (2.5)

We give some properties of $G(\lambda, \rho)$ in the following Lemma 2.1. The proof of Lemma 2.1 is similar to Theorem 1.1 in [23], so we omit it.

**Lemma 2.1.** (i) For fixed $\lambda > 0$,
\[
\lim_{\rho \to 0^+} G(\lambda, \rho) = \frac{\pi}{\sqrt{\lambda m_0}}, \quad \lim_{\rho \to +\infty} G(\lambda, \rho) = \frac{\pi}{\sqrt{\lambda m_\infty}},
\]
where
\[
m_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad m_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.
\]
(ii) For fixed $\rho > 0$, $G(\lambda, \rho)$ is decreasing on $\lambda$, and
\[
\lim_{\lambda \to 0^+} G(\lambda, \rho) = +\infty, \quad \lim_{\lambda \to +\infty} G(\lambda, \rho) = 0.
\]

To study the number of positive solutions of (1.3), we need to analyze the shape of $H(\lambda, \rho)$. We investigate several important properties of $H(\lambda, \rho)$ in the following Lemmas.

By (2.1),(2.4) and Lemma 2.1 we have

**Lemma 2.2.** (i) For $(\lambda, \rho) \in \Gamma$, $H(\lambda, \rho) \leq G(\lambda, \rho)$.
(ii) For fixed $\lambda > 0$, $H(\lambda, \rho_0(\lambda)) = G(\lambda, \rho_0(\lambda))$.
(iii) For fixed $\rho > 0$, $H(\lambda, \rho)$ is decreasing on $\lambda$, and $\lim_{\lambda \to +\infty} H(\lambda, \rho) = 0$. 
Proof. We need only to prove (iii). By (2.4) we have

\[ \frac{\partial}{\partial \lambda} H(\lambda, \rho) = \frac{\partial}{\partial \lambda} G(\lambda, \rho) + \frac{1}{2\sqrt{2} \lambda^2} \int_0^{F^{-1}(F(\rho) - \frac{1}{\lambda})} \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{2\lambda^2} f(q) \]

\[ = - \frac{1}{2\lambda} \left[ \sqrt{2} \int_0^{\rho} \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} - \frac{1}{\sqrt{2}} \int_0^{F^{-1}(F(\rho) - \frac{1}{\lambda})} \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} + \frac{1}{\lambda f(q)} \right], \]

then we have \( \frac{\partial}{\partial \lambda} H(\lambda, \rho) < 0 \). It follows that \( H(\lambda, \rho) \) is decreasing on \( \lambda \). Thus Lemma 2.2 is proved.

Lemma 2.3. If \( f(u) > 0 \) for \( u > 0 \) and \( f'(u)F(u) < f^2(u) \), then \( \eta(\lambda) = H(\lambda, \rho_0(\lambda)) \) is decreasing on \( \lambda \), and \( \lim_{\lambda \to 0^+} \eta(\lambda) = +\infty, \lim_{\lambda \to +\infty} \eta(\lambda) = 0 \).

Proof.

\[ \eta(\lambda) = \sqrt{2} \int_0^{\rho_0(\lambda)} \frac{ds}{\sqrt{\lambda(F(\rho_0(\lambda)) - F(s))}} = \sqrt{2} \int_0^{1} \frac{\rho_0(\lambda)}{\sqrt{\lambda(F(\rho_0(\lambda)) - F(\rho_0(\lambda)t))}} dt \]

\[ = \sqrt{2} \int_0^{1} \frac{\rho_0(\lambda)}{\sqrt{\lambda(\frac{1}{2\lambda} - F(\rho_0(\lambda)t))}} dt. \] (2.6)

Let \( y = \lambda(\frac{1}{2\lambda} - F(\rho_0(\lambda)t)) \), we have

\[ \eta(\lambda) = \sqrt{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{y}f(F^{-1}(\frac{1}{2} - y))} dy = \sqrt{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{1}{2} - y}} \frac{1}{\sqrt{f(F^{-1}((\frac{1}{2} - y)\frac{1}{\lambda}))}} dy. \]

The map \( \lambda \mapsto \frac{1}{\sqrt{f(F^{-1}((\frac{1}{2} - y)\frac{1}{\lambda}))}} \) is a composition of \( t \mapsto \frac{F(t)}{f(t)} \) and \( t = F^{-1}((\frac{1}{2} - y)\frac{1}{\lambda}) \). Therefore

\[ \eta'(\lambda) = \sqrt{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{1}{2} - y}} \frac{f^2(t) - f'(t)F(t)}{f^2(t) - f(F^{-1}((\frac{1}{2} - y)\frac{1}{\lambda}))} \frac{1}{f(F^{-1}((\frac{1}{2} - y)\frac{1}{\lambda}))} (-\frac{1}{2} - y) \frac{1}{\lambda^2}) dy < 0. \]

Then \( \eta(\lambda) \) is decreasing on \( \lambda \).

By the third formular of (2.6), we can easily obtain that

\[ \lim_{\lambda \to 0^+} \eta(\lambda) = +\infty, \lim_{\lambda \to +\infty} \eta(\lambda) = 0. \]

Remark 1. It is clear that \( f(u) = e^u, f(u) = u^a (a > 0), f(u) = u^p(p > 0), f(u) = e^u - 1, f(u) = (1 + u)^p(p > 0) \) and \( f(u) = a^u - 1(a > 1) \) satisfy assumptions in Lemma 2.3.
Fig. 4. Graph of $G(\lambda, \rho)$ for fixed $\lambda$ in the case $f(u) = e^u$.

Fig. 5. Graphs of $G(\lambda, \rho)$ and $H(\lambda, \rho)$ for $\lambda_0, \lambda_1, \lambda_2$ in the case $f(u) = e^u$. 
Fig. 6. Bifurcation diagram of $C = S \cup \tilde{S}$ of (1.1) in the case $f(u) = e^u$.

3. $f(u) = e^u$. In this section, the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases of $f(u) = e^u$ and $f(u) = a^u (a > 0)$ will be stated. The proofs of Lemma 3.1 and Lemma 3.2 are similar to that of Theorem 1.1 in [23], so we omit it.

**Lemma 3.1** (See Fig. 4). For fixed $\lambda > 0$, \[ \lim_{\rho \to 0^+} G(\lambda, \rho) = 0, \lim_{\rho \to +\infty} G(\lambda, \rho) = 0 \] and $G(\lambda, \rho)$ has exactly one critical point $(\lambda, \rho^*)$, a maximum.

Thus we can get that

**Lemma 3.2.** There exists $\bar{\lambda} > 0$ such that:
(i) If $0 < \lambda < \bar{\lambda}$, (1.2) has exactly two positive solutions.
(ii) If $\lambda = \bar{\lambda}$, (1.2) has exactly one positive solution.
(iii) If $\lambda > \bar{\lambda}$, (1.2) has no positive solution.

**Lemma 3.3.** There exists $\bar{\lambda} > 0$ such that:
(i) If $\lambda > \bar{\lambda}$, then $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) > 0$.
(ii) If $\lambda = \bar{\lambda}$, then $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = 0$.
(iii) If $0 < \lambda < \bar{\lambda}$, then $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) < 0$.

**Proof.** Noticing that $f(u) = e^u$ and $F(u) = \int_0^u f(s)ds$, we have
\[ F^{-1}\left(F(\rho) - \frac{1}{2\lambda}\right) = \ln\left(e^\rho - \frac{1}{2\lambda}\right). \]

This and (2.4) show that
\[ \frac{\partial}{\partial \rho} H(\lambda, \rho) = \frac{\partial}{\partial \rho} G(\lambda, \rho) + \frac{1}{2} \int_0^{\ln\left(e^\rho - \frac{1}{2\lambda}\right)} \frac{\lambda f(\rho)}{\left(\lambda(F(\rho) - F(s))\right)^{\frac{1}{2}}} ds - \frac{e^\rho}{e^\rho - \frac{1}{2\lambda}}. \]
Then
\[ \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = \frac{\partial}{\partial \rho} G(\lambda, \rho_0(\lambda)) - \left(1 + \frac{1}{2\lambda}\right). \]  
(3.1)

By (2.1) and direct computation, we have
\[ G(\lambda, \rho) = \frac{2\sqrt{2}}{\sqrt{\lambda}} e^{-\frac{\rho}{2}} \ln \left(\sqrt{e^\rho} + \sqrt{e^\rho - 1}\right). \]  
(3.2)

Then
\[ \frac{\partial}{\partial \rho} G(\lambda, \rho) = -\frac{\sqrt{2}}{\sqrt{\lambda}} e^{-\frac{\rho}{2}} \left[\ln(\sqrt{e^\rho} + \sqrt{e^\rho - 1}) - \frac{e^\rho}{\sqrt{e^\rho} \sqrt{e^\rho - 1}}\right]. \]

It follows that
\[ \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = -2 \frac{\ln(\sqrt{1 + \frac{1}{2\lambda} + \sqrt{\frac{1}{2\lambda}}})}{\sqrt{1 + 2\lambda}} + 1 - \frac{1}{2\lambda}. \]  
(3.3)

Then
\[ \frac{d}{d\lambda} \left(\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda))\right) = \frac{\sqrt{1 + 2\lambda} + 2\lambda \ln \left(\sqrt{1 + \frac{1}{2\lambda} + \sqrt{\frac{1}{2\lambda}}}\right)}{\lambda(1 + 2\lambda)^{\frac{1}{2}}} + \frac{1}{2\lambda^2} > 0. \]

On the other hand, by Lemma 2.1 (i) we can see there exists \( \lambda^* \) such that \( \rho^* = \rho_0(\lambda^*) \), and \( (\lambda^*, \rho^*) \) is the maximum. Then \( \frac{\partial}{\partial \rho} G(\lambda^*, \rho^*) = 0 \). By (3.1) we have
\[ \frac{\partial}{\partial \rho} H(\lambda^*, \rho_0(\lambda^*)) < 0. \]
Let \( \lambda \to +\infty \) in (3.1), we can see for large \( \lambda \), \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) > 0 \). Then we obtain the results of Lemma 3.3.

For the notational convenience, we denote
\[ h(\lambda) = \sup \left\{ H(\lambda, \rho) : \rho \in (\rho_0(\lambda), +\infty) \right\}. \]

**Lemma 3.4.** If \( h(\lambda) \neq +\infty \), then \( h(\lambda) \) is decreasing.

**Proof.** Assume \( \lambda_1 < \lambda_2 \). Then there exists \( \bar{\rho} \geq \rho_0(\lambda_2) \) such that \( h(\lambda_2) = H(\lambda_2, \bar{\rho}) \).
If \( \bar{\rho} > \rho_0(\lambda_1) \), then by Lemma 2.2 (iii) we have
\[ h(\lambda_1) \geq H(\lambda_1, \bar{\rho}) > H(\lambda_2, \bar{\rho}) = h(\lambda_2). \]
If \( \bar{\rho} \leq \rho_0(\lambda_1) \), then by Lemma 2.3 there exists \( \lambda_1 < \lambda_3 < \lambda_2 \) such that
\[ h(\lambda_1) \geq H(\lambda_1, \rho_0(\lambda_1)) > H(\lambda_3, \rho_0(\lambda_3)) \geq H(\lambda_2, \bar{\rho}) = h(\lambda_2). \]
Then \( h(\lambda) \) is decreasing. This finishes the proof of Lemma 3.4.

**Theorem 3.5** (See Fig.6). Consider problem (1.1) with \( f(u) = e^u \). Then there exist positive numbers \( \lambda_0 < \lambda_1 < \lambda_2 \equiv 3.512 \) such that:
(i) If \( \lambda_0 < \lambda < \lambda_1 \), (1.1) has at least 4 positive solutions.
(ii) If \( 0 < \lambda \leq \lambda_0 \) or \( \lambda = \lambda_1 \), (1.1) has at least 3 positive solutions.
(iii) If \( \lambda_1 < \lambda < \lambda_2 \), (1.1) has at least 2 positive solutions.
(iv) If \( \lambda = \lambda_2 \), (1.1) has a unique positive solution.
(v) If \( \lambda > \lambda_2 \), (1.1) has no positive solution.

**Proof.** By (2.2), Lemma 2.1 and Lemma 3.4, there exists \( \lambda_2 > 0 \) such that \( G(\lambda_2, \rho_*) = 1, G(\lambda, \rho) < 1 \) for \( \lambda > \lambda_2, \rho > 0 \) (see Fig.5.). Then (1.1) has no positive solution for \( \lambda > \lambda_2 \), has exact one positive solution for \( \lambda = \lambda_2 \). (By a similar computation to [9, (8)], we have \( \lambda_2 \approx 3.512 \).)
By Lemma 2.2 and Lemma 3.1, we have $\lim_{\rho \to +\infty} H(\lambda, \rho) = 0$.

By Lemma 2.2 (ii), we can see $H(\lambda_2, \rho_0(\lambda_2)) = G(\lambda_2, \rho_0(\lambda_2)) < 1$. Combining this with Lemma 2.3 there exists $\lambda_0 < \lambda_2$ such that $H(\lambda_0, \rho_0(\lambda_0)) = 1$. By Lemma 2.2 and (3.2), we have

$$H(\lambda_0, \rho_0(\lambda_0)) = \frac{4}{\sqrt{1 + 2\lambda_0}} \ln \left( \sqrt{1 + \frac{1}{2\lambda_0}} + \sqrt{\frac{1}{2\lambda_0}} \right) = 1. \quad (3.4)$$

Then we have

$$\ln \left( \sqrt{1 + \frac{1}{2\lambda_0}} + \sqrt{\frac{1}{2\lambda_0}} \right) = \frac{\sqrt{1 + 2\lambda_0}}{4}. \quad (3.5)$$

Substituting (3.5) into (3.3) we have

$$\frac{\partial}{\partial \rho} H(\lambda_0, \rho_0(\lambda_0)) = \frac{1}{2} - \frac{1}{2\lambda_0}.$$ 

Solving (3.5) by mathematica (or matlab) we have $\lambda_0 \approx 1.678 > 1$, then $H_\rho(\lambda_0, \rho_0(\lambda_0)) > 0$. By Lemma 2.1 and Lemma 2.2, we have that (1.1) has at least 3 positive solutions for $0 < \lambda \leq \lambda_0$.

By Lemma 2.2 (iii) there exists $\lambda_1 \in (\lambda_0, \lambda_2)$, $\bar{\rho} > 0$ such that $H(\lambda_1, \bar{\rho}) = \max_{\rho} H(\lambda_1, \rho) = 1$. Then (1.1) has at least 4 positive solutions for $\lambda_0 < \lambda < \lambda_1$ and has at least 3 positive solutions for $\lambda = \lambda_1$. Thus, we complete the proof of Theorem 3.1.

**Remark 2.** We can extend the above results to the case $f(u) = a^u$. 

**Corollary 1** (See Fig7). Let $f(u) = a^u (a > 1)$ in (1.1). Then for fixed $\lambda > 0$, $\lim_{\rho \to 0^+} G(\lambda, \rho) = 0$, $\lim_{\rho \to +\infty} G(\lambda, \rho) = 0$ and $G(\lambda, \rho)$ has exactly one critical point ($\lambda, \rho^*$), a maximum.

**Corollary 2.** Let $f(u) = a^u (a > 1)$ in (1.1). Then there exists $\bar{\lambda} > 0$ such that:

(i) If $0 < \lambda < \bar{\lambda}$, (1.2) has exactly two positive solutions.
(ii) If $\lambda = \bar{\lambda}$, (1.2) has exactly one positive solution.
(iii) If $\lambda > \bar{\lambda}$, (1.2) has no positive solution.
Corollary 3 (See Fig.8). Let \( f(u) = a^u (a > 1) \) in (1.1). Then we have same results as Theorem 3.1.

Proof. The proof is similar to Theorem 3.1, so we omit it.

The case of \( 0 < a < 1 \) is completely different from the case \( a > 1 \).

Corollary 4 (See Fig.7). Let \( f(u) = a^u (0 < a < 1) \) in (1.1). Then for fixed \( \lambda > 0 \),

\[
\lim_{\rho \to +\infty} G(\lambda, \rho) = +\infty \quad \text{and} \quad G(\lambda, \rho) \text{ is increasing on } \rho.
\]

Corollary 5 (See Fig.8). Let \( f(u) = a^u (0 < a < 1) \) in (1.1). Then problem (1.2) has exactly one positive solution for all \( \lambda > 0 \).

Corollary 6 (See Fig.8). Let \( f(u) = a^u (0 < a < 1) \) in (1.1). Then there exists \( \lambda^* \) such that:

(i) If \( 0 < \lambda \leq \lambda^* \), (1.1) has exactly one positive solution.

(ii) If \( \lambda > \lambda^* \), (1.1) has at least two positive solutions.

Proof. By computation we have

\[
\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = \frac{\partial}{\partial \rho} G(\lambda, \rho_0(\lambda)) - \left( 1 + \frac{\ln a}{2\lambda} \right).
\]  

(3.6)

Then if \( 0 < a < 1 \), we have \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) > 0 \).

In fact, by (2.3) and (3.6) we can see \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) \) is decreasing on \( \lambda \). By computation we have

\[
\frac{\partial}{\partial \rho} G(\lambda, \rho) = \frac{\sqrt{2}\sqrt{-\ln a}}{\sqrt{\lambda}} \left[ \arctan \left( \frac{\ln a}{2\lambda} \right) + \frac{\sqrt{1 - a^\rho}}{\sqrt{1 + a^\rho}} \right].
\]

Then if \( a < 1 \), we have \( \frac{\partial}{\partial \rho} G(\lambda, \rho_0(\lambda)) > 0 \).

It follows that \( \lim_{\lambda \to +\infty} \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = 1 > 0 \). Then \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) \) is increasing on \( \lambda \). By (2.4) we have

\[
H(\lambda, \rho) = \frac{1}{\sqrt{2}} \left( \int_0^\rho \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} + \int_{F^{-1}(F(\rho) - \frac{1}{\sqrt{2}})}^{\rho} \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} \right).
\]

Then by Corollary 3.1 we have \( \lim_{\rho \to +\infty} H(\lambda, \rho) = +\infty \). Thus we give the proof of Corollary 3.6.
For the special case of \( a = 1 \), we obtain the exact numbers of positive solutions and determine the exactness of bifurcation curves for problem (1.1).

**Corollary 7** (See Fig.7). Let \( f(u) = a^u \) in (1.1) with \( a = 1 \), i.e. \( f(u) = 1 \). Then for fixed \( \lambda > 0 \), \( \lim_{\rho \to 0^+} G(\lambda, \rho) = 0 \), \( \lim_{\rho \to +\infty} G(\lambda, \rho) = +\infty \) and \( G(\lambda, \rho) \) is increasing on \( \rho \).

**Proof.** Noticing that \( f(u) = 1 \), then \( F(u) = \int_0^u f(s) \, ds = u \), \( \theta(s) = 2F(s) - sf(s) = s \). This, (2.3) and Lemma 2.1 show that Corollary 3.7 holds. \( \square \)

**Corollary 8** (See Fig.8). Let \( f(u) = a^u \) in (1.1) with \( a = 1 \), i.e. \( f(u) = 1 \). Then problem (1.2) has exactly one positive solution for all \( \lambda > 0 \).

**Corollary 9** (See Fig.8). Let \( f(u) = a^u \) in (1.1) with \( a = 1 \), i.e. \( f(u) = 1 \). Then problem (1.1) has exactly one positive solution for \( 0 < \lambda \leq 2 \), has exactly two positive solutions for \( \lambda > 2 \).

**Proof.** Noticing that \( F(u) = u \), then by (2.2), we have

\[
G(\lambda, \rho) = \frac{2\sqrt{2}}{\sqrt{\lambda}}\sqrt{\rho}.
\]  

(3.7)

It follows from (3.7) and (2.2) that \( \rho = \frac{1}{8} \).

Similarly, by (2.4), we obtain that

\[
H(\lambda, \rho) = \frac{\sqrt{2}}{\sqrt{\lambda}}\sqrt{\rho} + \frac{1}{\lambda}.
\]  

(3.8)

From (3.8) and (2.5), we have that

\[
\rho = \frac{1}{2} \left( \lambda + \frac{1}{\lambda} - 2 \right).
\]  

(3.9)

Noticing \( \rho > \frac{1}{4\lambda} \) (see (2.4)), we have \( \lambda > 2 \). Moreover, from (3.9), it is easy to see that \( \rho \) is increasing on \( \lambda \). This gives the proof of Corollary 3.9. \( \square \)

Fig.9. Graphs of \( G(\lambda, \rho) \) for fixed \( \lambda \) in the case \( f(u) = u^p \).
(iii) If $0 < p < 1$, then $G(\lambda, \rho)$ is increasing on $\rho$, and $\lim_{\rho \to 0^+} G(\lambda, \rho) = 0$.

$$\lim_{\rho \to +\infty} G(\lambda, \rho) = +\infty.$$  

Proof. By computation we have

$$G(\lambda, \rho) = \frac{\sqrt{2(p+1)}}{\sqrt{\lambda}} \rho^{\frac{1}{p+1}} \int_0^1 \frac{dt}{\sqrt{1 - t^{p+1}}}.$$  

Then we easily obtain the results of Lemma 4.1. □

**Lemma 4.2.** If $p \neq 1$, then for all $\lambda > 0$, (1.2) has exact one positive solution; if $p = 1$, then (1.2) has no positive solution for $\lambda \neq \pi^2$, has infinitely positive solutions for $\lambda = \pi^2$.

**Lemma 4.3.** For fixed $\lambda > 0$, $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) < 0$ and

$$\lim_{\rho \to +\infty} H(\lambda, \rho) = \begin{cases} 0, & \ p > 1, \\ \frac{\pi}{2\sqrt{\lambda}}, & \ p = 1, \\ +\infty, & \ p < 1. \end{cases}$$  

(4.1)

Proof. By (2.4) we have

$$\frac{\partial}{\partial \rho} H(\lambda, \rho) = \frac{\partial}{\partial \rho} G(\lambda, \rho) + \frac{1}{2\sqrt{2}} \int_0^{F^{-1}(F(\rho) - \frac{1}{2\lambda})} \frac{f(\rho)}{[\lambda(F(\rho) - F(s))]^{\frac{3}{2}}} \, ds - \frac{f(\rho)}{f(F^{-1}(F(\rho) - \frac{1}{2\lambda}))}.$$  

(4.2)

Since $\lim_{\rho \to \rho_0(\lambda)} F^{-1}(F(\rho) - \frac{1}{2\lambda}) = 0$, we compute and obtain that

$$\lim_{\rho \to \rho_0(\lambda)} \frac{\partial}{\partial \rho} H(\lambda, \rho) = -\infty.$$  

By Lemma 2.2 (i) and Lemma 4.1 (i) we have $\lim_{\rho \to +\infty} H(\lambda, \rho) = 0$ for $p > 1$.

If $p = 1$, then combining the fact that $\lim_{\rho \to +\infty} F^{-1}(F(\rho) - \frac{1}{2\lambda}) = +\infty$ with (2.4) and Lemma 4.1 (ii) we have $\lim_{\rho \to +\infty} H(\lambda, \rho) = \frac{\pi}{2\sqrt{\lambda}}$.

If $p < 1$, by (2.4) we have

$$H(\lambda, \rho) = \frac{1}{\sqrt{2}} \left( \int_0^\rho \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} + \int_{F^{-1}(F(\rho) - \frac{1}{2\lambda})}^\rho \frac{ds}{\sqrt{\lambda(F(\rho) - F(s))}} \right).$$

Then by Lemma 4.1 (iii) we have $\lim_{\rho \to +\infty} H(\lambda, \rho) = +\infty$. Lemma 4.3 is proved. □

For convenience we introduce the following notation

$$g(\lambda) = \inf \left\{ H(\lambda, \rho) | \rho \in (\rho_0(\lambda), +\infty) \right\}.$$  

**Lemma 4.4.** If $g(\lambda) \neq 0$, then $g(\lambda)$ is decreasing.

Proof. Let $\lambda_1 < \lambda_2$. Then there exist $\rho_1$ and $\rho_2$ such that

$$g(\lambda_1) = H(\lambda_1, \rho_1), \quad g(\lambda_2) = H(\lambda_2, \rho_2).$$

By Lemma 2.2 (iii), we have

$$g(\lambda_1) = H(\lambda_1, \rho_1) > H(\lambda_2, \rho_1) > H(\lambda_2, \rho_2) = g(\lambda_2).$$

Then $g(\lambda)$ is decreasing. □
Theorem 4.5 (See Fig.10). Consider (1.1) with $f(u) = u^p$ ($p > 0$). Then we have

(i) If $p > 1$, then there exists $\lambda_* > 0$ such that (1.1) has exactly one positive solution for $\lambda \geq \lambda_*$, and has at least two positive solutions for $0 < \lambda < \lambda_*$. 

(ii) If $p = 1$, then there exists $\lambda_1 = \frac{\pi^2}{4}, \lambda_2 = \pi^2$ such that (1.1) has at least one positive solution for $\frac{\pi^2}{4} < \lambda \leq \pi^2$, and has no positive solution for $0 < \lambda \leq \frac{\pi^2}{4}$ and $\lambda > \pi^2$. 

(iii) If $p < 1$, then there exist $0 < \lambda_* < \lambda^*$ such that (1.1) has exactly one positive solution for $\lambda < \lambda_*$, has at least three positive solutions for $\lambda_* < \lambda < \lambda^*$, and has at least two positive solutions for $\lambda = \lambda_*$ and $\lambda \geq \lambda^*$. 

Proof. By Lemma 2.1-Lemma 2.3 and Lemma 4.1-Lemma 4.4, it is easy to obtain the results of Theorem 4.1, so we omit it. 

Fig.10. Bifurcation diagrams of $C = S \cup \tilde{S}$ of (1.1) in the case $f(u) = u^p$.

Fig.11. Graph of $G(\lambda, \rho)$ for fixed $\lambda$ in the case $f(u) = e^u - 1$. 
Fig. 12. Bifurcation diagram of \( C = S \cup \overline{S} \) of (1.1) in the case \( f(u) = e^u - 1 \).

5. \( f(u) = e^u - 1 \). In this section, we establish the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases \( f(u) = e^u - 1 \) and \( f(u) = a^u - 1 (a > 1) \).

**Lemma 5.1** (See Fig. 11). If \( f(u) = e^u - 1 \), then for fixed \( \lambda > 0 \), \( G(\lambda, \rho) \) is decreasing on \( \rho \), and

\[
\lim_{\rho \to 0^+} G(\lambda, \rho) = \frac{\pi}{\sqrt{\lambda}}, \quad \lim_{\rho \to +\infty} G(\lambda, \rho) = 0.
\]

*Proof.* It is easy obtained by Lemma 2.1 and (2.3). So we omit it.

**Lemma 5.2** (See Fig. 12). If \( f(u) = e^u - 1 \), then there exists \( \lambda^* = \pi^2 \) such that:

(i) If \( 0 < \lambda \leq \lambda^* \), (1.2) has exactly one positive solution.

(ii) If \( \lambda > \lambda^* \), (1.2) has no positive solution.

**Lemma 5.3.** If \( f(u) = e^u - 1 \), then for fixed \( \lambda > 0 \), \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) < 0 \) and

\[
\lim_{\rho \to +\infty} H(\lambda, \rho) = 0.
\]

*Proof.* It is similar to that of Lemma 4.3. So we omit it.

**Theorem 5.4** (See Fig. 12). Consider (1.1) with \( f(u) = e^u - 1 \). Then there exist \( 0 < \lambda_* < \lambda^* = \pi^2 \) such that:

(i) If \( 0 < \lambda < \lambda_* \), (1.1) has at least two positive solution.

(ii) If \( \lambda_* \leq \lambda \leq \lambda^* \), (1.1) has exactly one positive solution.

(iii) If \( \lambda > \lambda^* \), (1.1) has no positive solution.

We can extend the above results to the case \( f(u) = a^u - 1 (a > 1) \).

**Corollary 10.** Consider (1.1) with \( f(u) = a^u - 1 (a > 1) \). Then there exist \( 0 < \lambda_* < \lambda^* = \frac{\pi^2}{\ln a} \) such that:

(i) If \( 0 < \lambda < \lambda_* \), (1.1) has at least two positive solution.

(ii) If \( \lambda_* \leq \lambda \leq \lambda^* \), (1.1) has exactly one positive solution.

(ii) If \( \lambda > \lambda^* \), (1.1) has no positive solution.

**Remark 3.** If \( a \leq 1 \), then \( f(u) = a^u - 1 \) does not satisfy \( f(u) > 0 \) for \( u > 0 \). So, we only consider the case \( a > 1 \) for \( f(u) = a^u - 1 \).
6. $f(u) = (1 + u)^p(p > 0)$. In this section, we study the bifurcation diagrams and multiple positive solutions of problem (1.1) for the cases of $f(u) = (1 + u)^p(p > 0)$.

Lemma 6.1 (See Fig.13). Assume $f(u) = (1 + u)^p(p > 0)$. Then for fixed $\lambda > 0$, we have:

(i) If $p > 1$, then $\lim_{\rho \to 0^+} G(\lambda, \rho) = 0$, $\lim_{\rho \to +\infty} G(\lambda, \rho) = 0$ and $G(\lambda, \rho)$ has exactly one critical point $(\lambda, \rho^*)$, a maximum.

(ii) If $p = 1$, then $G(\lambda, \rho)$ is increasing on $\rho$ and $\lim_{\rho \to 0^+} G(\lambda, \rho) = 0$, $\lim_{\rho \to +\infty} G(\lambda, \rho) = \frac{\pi}{\sqrt{\lambda}}$.

(iii) If $0 < p < 1$, then $G(\lambda, \rho)$ is increasing on $\rho$, and $\lim_{\rho \to 0^+} G(\lambda, \rho) = 0$,

$$\lim_{\rho \to +\infty} G(\lambda, \rho) = +\infty.$$  

Proof. If $p > 1$, by Theorem 1.1 of [23] we obtain the result of (i). If $0 < p \leq 1$, then by (2.3) we can easily obtain that $\theta'(u) > 0$. It follows that $\frac{\partial}{\partial \rho} G(\lambda, \rho) > 0$, i.e. $G(\lambda, \rho)$ is increasing on $\rho$. Several limits can be obtained by calculation. So we omit it.

Lemma 6.2. Assume $f(u) = (1 + u)^p(p > 0)$.

If $p > 1$, then there exists $\lambda > 0$ such that:

(i) If $0 < \lambda < \hat{\lambda}$, (1.2) has exactly two positive solutions.

(ii) If $\lambda = \hat{\lambda}$, (1.2) has exactly one positive solution.

(iii) If $\lambda > \hat{\lambda}$, (1.2) has no positive solution.

If $p = 1$, then there exists $\lambda_* = \pi^2$ such that:

(i) If $0 < \lambda \leq \lambda_*$, (1.2) has exactly one positive solution.

(ii) If $\lambda > \lambda_*$, (1.2) has no positive solution.

If $0 < p < 1$, then for all $\lambda > 0$, (1.2) has exactly one positive solution.

Lemma 6.3. For fixed $\lambda > 0$,

$$\lim_{\rho \to +\infty} H(\lambda, \rho) = \begin{cases} 0, & p > 1, \\ \frac{\pi}{2\sqrt{\lambda}}, & p = 1, \\ +\infty, & p < 1. \end{cases} \quad (6.1)$$

Proof. It is similar to that of Lemma 4.3. So we omit it.

Lemma 6.4. $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda))$ is increasing on $\lambda$ and $\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) < 0$ for any $\lambda > 0$. 

Fig.13. Graphs of $G(\lambda, \rho)$ for fixed $\lambda$ in the case $f(u) = (1 + u)^p(p > 0)$.
Proof. Let \( v = \frac{1+s}{1+p} \). Then, by (2.1) we have

\[
G(\lambda, \rho) = \frac{\sqrt{2(p+1)}}{\sqrt{\lambda}} \int_0^\rho \frac{ds}{\sqrt{(1+\rho)^{p+1} - (1+s)^{p+1}}}
\]

\[
= \frac{\sqrt{2(p+1)}}{\sqrt{\lambda}}(1+\rho)^{\frac{1-p}{2}} \int_{\frac{1}{1+p}}^1 \frac{dv}{\sqrt{1-v^{p+1}}}
\]

Then

\[
\frac{\partial}{\partial \rho} G(\lambda, \rho) = \frac{\sqrt{2(p+1)}}{\sqrt{\lambda}} \left[ \frac{1-p}{2} (1+\rho)^{-\frac{1+p}{2}} \int_{\frac{1}{1+p}}^1 \frac{dv}{\sqrt{1-v^{p+1}}} + (1+\rho)^{-1} \frac{1}{\sqrt{(1+\rho)^{p+1} - 1}} \right],
\]

and

\[
\frac{\partial}{\partial \rho} G(\lambda, \rho_0(\lambda))
\]

\[
= \sqrt{2(p+1)} \left[ \frac{1-p}{2} \left( \frac{1}{\sqrt{\lambda + \frac{p+1}{2}}} \int_{\frac{1}{1+p}}^1 \frac{dv}{\sqrt{1-v^{p+1}}} \right) + \left( 1 + \frac{p+1}{2\lambda} \right)^{-\frac{1+p}{2}} \right].
\]

It follows that

\[
\frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = \sqrt{2(p+1)} \left[ \frac{1-p}{2} \left( \frac{1}{\sqrt{\lambda + \frac{p+1}{2}}} \int_{\frac{1}{1+p}}^1 \frac{dv}{\sqrt{1-v^{p+1}}} \right) + \left( 1 + \frac{p+1}{2\lambda} \right)^{-\frac{1+p}{2}} \right]
\]

\[
+ \left( 1 + \frac{p+1}{2\lambda} \right)^{-\frac{1+p}{2}} - \left( 1 + \frac{p+1}{2\lambda} \right)^{\frac{1+p}{2}}.
\]

By (6.2) it is easy to see that \( \lim_{\lambda \to +\infty} \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) = 0 \) and \( \frac{\partial}{\partial \rho} H(\lambda, \rho_0(\lambda)) \) is increasing on \( \lambda \) if \( p \geq 1 \). This gives the proof of Lemma 6.4.

**Theorem 6.5** (See Fig. 14). Assume \( f(u) = (1+u)^p \) \((p > 1)\). Then there exist

0 < \( \lambda_* \) < \( \lambda^* \) such that:

(i) If \( 0 < \lambda < \lambda_* \), (1.1) has at least three positive solutions.

(ii) If \( \lambda_* \leq \lambda < \lambda^* \), (1.1) has at least two positive solutions.

(iii) If \( \lambda = \lambda^* \), (1.1) has exactly one positive solution.

(iv) If \( \lambda > \lambda^* \), (1.1) has no positive solution.

**Proof.** By Lemma 6.1-Lemma 6.4 we easily obtain the results of Theorem 6.1. So we omit it.
If $0 < p \leq 1$, we use the method of [6] and [10] to prove the existence of positive solution. For convenience, we give some definitions and lemmas similar to [6] and [10].

Considering problem (1.2), we define

$$G(\rho) = \sqrt{2} \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \quad \text{for} \quad 0 < \rho < \infty.$$ 

Then the positive solutions of (1.2) correspond to

$$\|u\|_\infty = \rho \quad \text{and} \quad G(\rho) = \sqrt{\lambda}.$$
Considering problem (1.3), for $0 \leq \eta < \rho < \infty$, we define
\[
\tilde{H}(\rho, \eta) = 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^\eta \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{F(\rho) - F(\eta)}}.
\] (6.3)

We investigate several important properties of $\tilde{H}(\rho, \eta)$ through Lemma 6.5.

**Lemma 6.6** (See Fig. 15). Consider $\tilde{H}(\rho, \eta)$ with $0 \leq \eta < \rho < \infty$. Then the following assertions (i)-(v) hold:

(i) For any fixed $\rho > 0$, $\tilde{H}(\rho, \eta)$ is a monotone decreasing function of $\eta$ on $[0, \rho)$. Moreover,
\[
\lim_{\eta \to \rho^-} \tilde{H}(\rho, \eta) = -\infty.
\]

(ii) \[
\lim_{\rho \to 0^+} \tilde{H}(\rho, 0) = -\infty,
\]
\[
\lim_{\rho \to +\infty} \tilde{H}(\rho, 0) = \begin{cases}
\sqrt{2\pi}, & p = 1, \\
+\infty, & p < 1.
\end{cases}
\]

(iii) For fixed $0 < p \leq 1$, there exists a unique $\rho_0 < 0.28$ such that
\[
\tilde{H}(\rho, 0) = \begin{cases}
< 0, & \rho \in (0, \rho_0), \\
= 0, & \rho = \rho_0, \\
> 0, & \rho \in (\rho_0, \infty).
\end{cases}
\] (6.4)

(iv) For fixed $0 < p \leq 1$, there exists a unique function $\eta(\rho) \in C[\rho_0, \infty) \cap C^1(\rho_0, \infty)$ such that
\[
\eta(\rho) = \begin{cases}
0, & \rho = \rho_0, \\
> 0, & \rho \in (\rho_0, \infty),
\end{cases}
\]
and $\tilde{H}(\rho, \eta(\rho)) = 0$ for $\rho \geq \rho_0$.

(v) For $\rho > \rho_0$, the function $\eta(\rho)$ satisfies
\[
\eta'(\rho) = \frac{[F(\rho) - F(\eta(\rho))]^2 \left[2\sqrt{2\pi}G'(\rho) + \int_0^{\eta(\rho)} \frac{f(s)}{[F(\rho) - F(\eta(\rho))]^{3/2}} ds\right] + f(\rho)}{2[F(\rho) - F(\eta(\rho))] + f(\eta(\rho))}.
\] (6.5)

**Proof.** The proof of Lemma 6.5(i) is similar to that of Lemma B in [6]. The proof of Lemma 6.5(v) is similar to that of Lemma 3.1 in [10]. So we omit it.

We prove Lemma 6.5(ii). By (6.3), we have
\[
\tilde{H}(\rho, 0) = 2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{F(\rho)}}.
\] (6.6)

By (2.1) and Lemma 6.1 (ii) (iii) it is easy to see that \[
\lim_{\rho \to 0^+} \tilde{H}(\rho, 0) = -\infty,
\]
\[
\lim_{\rho \to +\infty} \tilde{H}(\rho, 0) = \begin{cases}
\sqrt{2\pi}, & p = 1, \\
+\infty, & p < 1.
\end{cases}
\]

Now we prove Lemma 6.5(iii). By (6.6) we have
\[
\tilde{H}'(\rho, 0) = 2 \frac{d}{d\rho} \left( \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} + f'(\rho) \right) \left( \frac{1}{2[F(\rho)]^{3/2}} \right).
\]
By Lemma 6.1 (ii) (iii) we can see $H'(\rho, 0) > 0$, combining this with (ii) we have there exists a unique $\rho_0$ such that
\[
H(\rho, 0) \begin{cases} < 0, & \rho \in (0, \rho_0), \\ = 0, & \rho = \rho_0, \\ > 0, & \rho \in (\rho_0, \infty). \end{cases}
\]

Now we prove that $\rho_0 < 0.28$.
By (6.6) and the fact $f(s) = (1+s)^p(p \leq 1)$ we have
\[
2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} > 2 \int_0^\rho \frac{ds}{\sqrt{f(\rho)(\rho - s)}} > 2 \frac{1}{\sqrt{1+\rho}} \int_0^\rho \frac{ds}{\sqrt{\rho - s}} = \frac{4\sqrt{p}}{\sqrt{1+\rho}},
\]
and $\frac{1}{\sqrt{F(\rho)}} < \frac{1}{\sqrt{p}}$. Then we have
\[
H(\rho, 0) > \frac{4\sqrt{p}}{\sqrt{1+\rho}} - \frac{1}{\sqrt{p}}.
\]
Letting $\frac{4\sqrt{p}}{\sqrt{1+\rho}} - \frac{1}{\sqrt{p}} = 0$, we have $\rho \approx 0.279$. It follows that $\rho_0 < 0.28$.
(iv) follows by (i)-(iii). The proof of Lemma 6.5 is now complete. 

Thus, for each $\rho \in [\rho_0, +\infty)$, there is a unique $\eta(\rho) \in [0, \rho)$ such that $H(\rho, \eta(\rho)) = 0$. Then we can define the time map of (1.3) as follows:
\[
H_1(\rho, \eta(\rho)) = \frac{1}{2[F(\rho) - F(\eta(\rho))]} \forall \rho \in [\rho_0, \infty), \ \eta(\rho) \in [0, \rho). \quad (6.7)
\]

By [6, Theorem 3.3] and [10, Lemma 3.1], positive solution $u$ of (1.3) correspond to
\[
\|u\|_\infty = \rho \text{ and } H_1(\rho, \eta(\rho)) = \lambda. \quad (6.8)
\]

Thus, studying of the number of positive solutions of (1.3) is equivalent to studying the shape of time map $H_1(\rho, \eta(\rho))$ on $[\rho_0, \infty)$. (see [6]).

**Lemma 6.7.** Consider $H_1(\rho, \eta(\rho))$ and $G(\rho)$ for $\rho \in [\rho_0, \infty)$. Then the following assertions hold:
(i) $H_1(\rho_0, \eta(\rho_0)) = [G(\rho_0)]^2$ and $H_1(\rho, \eta(\rho)) < [G(\rho)]^2$ for $\rho > \rho_0$.
(ii) \[
\lim_{\rho \to +\infty} H_1(\rho, \eta(\rho)) = \begin{cases} \pi^2, & p = 1, \\ \frac{4}{p}, & +\infty, \ p < 1. \end{cases}
\]
(iii) For $0 < p \leq 1$, $\lim_{\rho \to \rho_0^+} \frac{d}{d\rho} H_1(\rho, \eta(\rho)) > 0$.

**Proof.** The proof of Lemma 6.6(i) is similar to that of Lemma 3.2 in [10].
Now we prove Lemma 6.6(ii). By (6.3),(6.7) and (6.8) we have
\[
2 \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^{\eta(\rho)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \frac{1}{\sqrt{F(\rho) - F(\eta(\rho))}},
\]
then
\[
\sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{2}} \int_0^{\eta(\rho)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \frac{1}{\sqrt{2}\sqrt{F(\rho) - F(\eta(\rho)))}}.
\]
Similar to Lemma 4.3, we can prove
\[
\lim_{\rho \to +\infty} H_1(\rho, \eta(\rho)) = \begin{cases} 
\frac{\pi^2}{4}, & p = 1, \\
+\infty, & p < 1.
\end{cases}
\]

(iii) The proof is similar to that of Theorem 2.1 in [10]. By (6.7) and Lemma 6.5 (v) we have
\[
\lim_{\rho \to \rho_0^+} \frac{d}{d\rho} H_1(\rho, \eta(\rho)) = -\frac{f(\rho) - f(\eta(\rho))\eta'(\rho)}{2[F'(\rho) - F(\eta(\rho))]}.
\]

where
\[
\lim_{\rho \to \rho_0^+} [f(\rho) - f(\rho_0)\eta'(\rho)] = -\frac{2[F(\rho_0)]^2}{2F(\rho_0) + 1} \left[ \sqrt{2G'(\rho_0)} - \frac{f(\rho_0)}{\sqrt{F(\rho_0)}} \right],
\]
in which
\[
\sqrt{2G'(\rho_0)} - \frac{f(\rho_0)}{\sqrt{F(\rho_0)}} = \int_0^{\rho_0} \frac{h(s)}{\rho_0[F(\rho_0) - F(s)]^{3/2}} ds,
\]
where
\[
h(s) = 2F(\rho_0) - \rho_0f(\rho_0) - 2F(s) + sf(s) - 2\rho_0f(\rho_0)[F(\rho_0) - F(s)].
\]
It is easy to see that \(h(\rho_0) = 0\). We compute that
\[
h'(s) = f(s) \left[ -1 + \frac{ps}{1 + s} + 2\rho_0f(\rho_0) \right] < 0
\]
because of \(\rho_0 < 0.28\). Then \(h(s) > 0\) and
\[
\lim_{\rho \to \rho_0^+} \frac{d}{d\rho} H_1(\rho, \eta(\rho)) > 0.
\]

This finishes the proof of Lemma 6.6.

By Lemma 6.5 and Lemma 6.6 we can directly obtain the following Theorem 6.2.

**Theorem 6.8** (See Fig. 16.). (I) Assume \(f(u) = (1 + u)^p\) \((p = 1)\). Then there exist
\(0 < \lambda_0 < \frac{\pi^2}{4}\) such that:
(i) If \(0 < \lambda \leq \lambda_0\) or \(\frac{\pi^2}{4} \leq \lambda < \pi^2\), (1.1) has exactly one positive solution.
(ii) If \(\lambda_0 < \lambda < \frac{\pi^2}{4}\), (1.1) has at least two positive solutions.
(iii) If \(\lambda \geq \pi^2\), (1.1) has no positive solution.

(II) Assume \(f(u) = (1 + u)^p\) \((0 < p < 1)\). Then there exist \(0 < \lambda_0 < \frac{\pi^2}{4}\) such that:
(i) If \(0 < \lambda \leq \lambda_0\), (1.1) has exactly one positive solution.
(ii) If \(\lambda > \lambda_0\), (1.1) has at least two positive solutions.
Fig. 16. Bifurcation diagrams of $C = S \cup \overline{S}$ of (1.1) in the case $f(u) = (1 + u)^p (p \leq 1)$.

Acknowledgements. This work is sponsored by the National Natural Science Foundation of China (11301178, 11371117), the Beijing Natural Science Foundation of China (1163007) and the Scientific Research Project of Construction for Scientific and Technological Innovation Service Capacity (KM201611232019). The authors are grateful to anonymous referees for their constructive comments and suggestions, which has greatly improved this paper.

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Received March 2017; revised January 2018.

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