Effective Field Theories as Asymptotic Series: From QCD to Cosmology.

Ariel R. Zhitnitsky

Physics Department, University of British Columbia, 6224 Agricultural Road, Vancouver, BC V6T 1Z1, Canada

Abstract

We present some generic arguments demonstrating that an effective Lagrangian $L_{\text{eff}}$ which, by definition, contains operators $O^n$ of arbitrary dimensionality in general is not convergent, but rather an asymptotic series. It means that the behavior of the far distant terms has a specific factorial dependence $L_{\text{eff}} \sim \sum_n c_n O^n$, $c_n \sim n!$, $n \gg 1$.

We discuss a few apparently different problems, which however have something in common— the aforementioned $n!$— behavior:

1. Effective long-distance theory describing the collective fields in QCD;
2. Effective Berry phase potential which is obtained by integrating over the fast degrees of freedom. As is known, the Berry potential is associated with induced local gauge symmetry and might be relevant for the compactification problem at the Planck scale.
3. Nonlocal Lagrangians introduced by Georgi for appropriate treatment of the effective field theories without power expanding.
4. The so-called improved action in lattice field theory where the new, higher dimensional operators have been introduced into the theory in order to reduce the lattice artifacts.
5. Cosmological constant problem and vacuum expectation values in gravity.

We discuss some applications of this, seemingly pure academic phenomenon, to various physical problems with typical energies from 1GeV to the Plank scale.

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1 e-mail address: arz@physics.ubc.ca
1 Introduction

Today it is widely believed that all of our present realistic field theories are actually not fundamental, but effective theories. The standard model is presumably what we get when we integrate out modes of very high energy from some unknown theory, and like any other effective field theory, its lagrangian density contains terms of arbitrary dimensionality, though the terms in the Lagrangian density with dimensionality greater than four are suppressed by negative powers of a very large mass $M$. Even in QCD, for the calculation of processes at a few GeV we would use an effective field theory with heavier quarks integrated out, and such an effective theory necessarily involves terms in the Lagrangian of unlimited dimensionality.

The basic idea behind effective field theories is that a physical process at energy $E \ll M$ can be described in terms of an expansion in $E/M$, see recent reviews [2], [3], [4]. In this case we can limit ourselves by considering only a few first leading terms and neglect the rest. In this paper we discuss not this standard formulation of the problems, but rather, we are interested in the behavior of the coefficients of the very high dimensional operators in the expansion. We shall demonstrate that these coefficients $c_n$ grow as fast as a factorial $n!$ for sufficiently large $n$. Thus, the series under discussion is not a convergent, but an asymptotic one. Such a behavior raises problems both of fundamental nature, concerning the status of the expansion and of practical importance, as to whether divergences can be associated with new physical phenomena. It means, first of all, that in order to make sense, such a theory should be defined by some specific prescription, for example, by Borel transformation.

Let us note, that our remark about the factorial dependence of the series for large $n \gg 1$ is an absolutely irrelevant issue for the analysis of standard problems when we are interested in the low energy limit only. We have nothing new to say about these issues.

However, sometimes we need to know the behavior of whole series when the distant terms in the series might be important. In this case the analysis of the large order terms in the expansion has some physical meaning.

Such a situation may occur in a variety of different problems as will be discussed in a more detail later in the text. Now let us mention that in general it occurs when the energy scale $E$ is close to $M$ or/and when two or more intermediate, not well separated scales, come into the game [1].

This letter is organized in the following way. In the next section we argue, by analyzing a couple of examples, that the factorial behavior of the coefficients in front of the high dimensional operators, is a very general property of effective field theories [3].

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2The generality of this phenomenon can be compared with the well known property of the large order behavior in perturbative series [3]. As is known a variety of different field theories (gauge theories, in particular) exhibits a factorial growth of the coefficients in the
In the last section we discuss some possible applications of the obtained results to different field theories with very different scales (from QCD problems to the cosmological constant problem).

2 Basic Examples.

2.1 Main Idea.

We begin our analysis with the following remark. An effective field theory can be considered as an particular case of the more general idea of the Wilson operator product expansion (OPE). It has been demonstrated recently \[3\], that the OPE for some specific correlation functions (heavy-light quark system \(\bar{Q}q\)) in QCD is an asymptotic, and not a convergent series. The general arguments of the paper \[3\] have been explicitly tested in QCD \(2\) (where the vacuum structure as well as the spectrum of the theory is known) with the same conclusion concerning the asymptotic nature of OPE \[4\]. In both cases the arguments were based on the dispersion relations and the general properties of the spectrum of the theory. However, the experience with large order behavior in perturbative series \[5\] teaches us that the factorial growth of the coefficients is of very general nature and it is not specific property of some Green functions.

Thus, we expect that the asymptotic nature of the OPE has a much more general origin and it is not related to the specific correlation functions, for which it was found for the first time \[3\].

To be more specific and in order to explain what is going on with the effective theory when we integrate out the heavy degrees of freedom, let us consider QED with one heavy electron of mass \(M\). The effective field theory for photons can be obtained by integrating out the fermion degrees of freedom. The most general solution of this problem is not known, however in the case of a specific (constant) external electric field \(E\) the corresponding expression for \(L_{\text{eff}}\) is known (see e.g. the textbook \[8\]). In order to find the OPE coefficients for the high dimensional operators \(E^n\) one can expand \(L_{\text{eff}}\) in power of \(E\):

\[
L_{\text{eff}} = M^4 \sum_n c_n \left(\frac{E}{M^2}\right)^n. \tag{1}
\]

Of course, the eq.\((1)\) is not the most general form, because it does not contain all possible operators, in particular those operators which would contain some terms with derivatives \(\sim \partial_\mu E\). Our goal now is to demonstrate perturbative expansion with respect to coupling constant. This growth in perturbative expansion is very different from the phenomenon we are discussing, where the factorial behavior is related to high dimensional operators, and not to coupling constant expansion. However, in spite of the apparent difference of these phenomena, actually they have some common general origin. We shall discuss this connection later.
that we do have a factorial behavior already in this simple case where we select only some specific class of operators, namely those $\sim E_n$.

Our next step is as follows. First of all we shall find an exact formula for the $n$–dependence of the coefficients $c_n$; secondly, we give a qualitative explanation of why such a factorial behavior takes place. Our argumentation will be so general in form that it will be perfectly clear that this phenomenon is very universal in nature.

The effective Lagrangian for the problem can be written in the following way:\cite{8}:

$$L_{\text{eff}} = \frac{1}{8\pi^2} \int_0^\infty \frac{ds}{s^2} [E \text{cth}(Es) - \frac{1}{s}] e^{-isM^2},$$  \hspace{1cm} (2)

where we denote the external field $E$ together with its coupling constant $e$.

We expand this expression in $E$ using the formula

$$\frac{1}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$  \hspace{1cm} (3)

where $B_k$ are Bernoulli numbers. For large $k$ these numbers as is known exhibit factorial growth:

$$B_{2n} = 2(-1)^{n+1}(2n)! \sum_{r=1}^{\infty} \frac{1}{(2\pi r)^{2n}} \sim 2(-1)^{n+1}(2n)! \frac{1}{(2\pi)^{2n}}, \hspace{1cm} n \gg 1. \hspace{1cm} (4)$$

Thus, the coefficients $c_n$ in the OPE (1) are factorially divergent for large $n$:

$$c_{2n} = \frac{1}{8\pi^2} 2^{2n} B_{2n} \frac{(2n-3)!}{(2n)!} \sim (2n)!. \hspace{1cm} (5)$$

In particular, for $n = 2$ this formula reproduces the well-known Euler-Heisenberg Effective lagrangian $L_{EH}$, which is nothing but the first nontrivial term in the series (1):

$$L_{EH} = \frac{2}{45M^4} \left(\frac{e^2}{4\pi}\right)^2 E^4, \hspace{1cm} (6)$$

We have redefined the coupling constant $e$ in this expression to present the formula in a standard way.

Now, how one can understand this factorial behavior (3) in simple terms? We suggest the following almost trivial explanation which however is a very universal in nature.

Let us look at the function $L_{\text{eff}}(z)$ (1) as an analytical function of the complex variable $z = E/M^2$ for which the standard dispersion relations hold. The factorial growth of the coefficients in the real part of $L_{\text{eff}}(z)$ implies that the corresponding imaginary part has a very specific behavior $\text{Im} L_{\text{eff}}(z) \sim e^{-1/z}$ which follows from the dispersion relations:

$$f(z) \sim \sum_n f_n z^n \hspace{0.5cm} f_n \sim (a)^n n! \sim \int \frac{dz'}{(z')^{n+2}} \text{Im} f(z') \longleftrightarrow \text{Im} f(z') \sim e^{-\frac{a}{z}} \hspace{1cm} (7)$$
Here we have introduced an arbitrary analytical function $f(z)$ to be more general.

At the same time, an imaginary part of the amplitude, as is known, is related to a real physical process: the pair-creation in the strong external field. We have fairly good physical intuition of what kind of dependence on the field one could expect for such a physical process. Namely, as we shall discuss later, this process can be thought of as a penetration through a potential barrier in the quasi-classical approximation. So, from a physical point of view we would expect that the $E^-$ dependence should have the following form $ImL_{eff}(E) \sim e^{-1/E}$. As we shall see this is exactly the case for our QED example (7) and in a full agreement with what the dispersion relations (7) tell us.

Now, we would like to present the explicit formula for the probability of pair creation in the constant electric field $E$. It is given by (see e.g. [8]):
\begin{equation}
    w = -\frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^2} [E\text{cth}(Es) - \frac{1}{s}] \text{Im}(e^{-isM^2}).
\end{equation}

The “only” difference with the formula (2) is the replacement $Re(e^{-isM^2}) \Rightarrow Im(e^{-isM^2})$. However, this replacement completely modifies the analytical structure. Indeed, the explicit calculation of the coefficients in the power expansion for imaginary part in the formula (8) leads to the following integrals which are zero $\int dz \sin(z)z^{2n-3} \sim \sin[(n-1)\pi] = 0$. Thus, the imaginary part is not expandable at $E = 0$ in agreement with our arguments about a singular behavior at this point $\sim e^{-1/E}$.

Fortunately, a direct calculation can be performed with the following final result, explicitly demonstrating the $e^{-1/z}$ structure (see e.g. [8]):
\begin{equation}
    w = \frac{E^2}{4\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-\frac{nM^2\pi}{E}).
\end{equation}

A few comments are in order. First, the behavior $w(z) \sim e^{-1/Z}$ is exactly what we expected. It can be interpreted as penetration through a potential barrier in the quasi-classical approximation. Indeed, the standard formula for the ionization of a state with bound energy $-V \sim 2M$ and external field $E$ is proportional to
\begin{equation}
    \sim \exp(-2\int dx \sqrt{2M(V - Ex)} \sim \exp(-\frac{\text{const}.M^2}{E})
\end{equation}
which qualitatively explains the exact result (9).

We are not pretending here to have derived new result in QED. All these classical formulae have been well known for a many years. Rather, we wanted

\footnote{This integral can be reduced, in according to Cauchy theorem, to the calculation of the contributions from the poles of the $cthz$ function.}
to explain, by analyzing this QED example, the main source of the $n!$ dependence in the Effective lagrangian. The Effective lagrangian, by definition, is a series of operators of arbitrary dimensions constructed from the light fields $E$. This is presumably obtained from some underlying field theory by integrating out the heavy fields of mass $M$. It is perfectly clear that the probability of the physical creation of the heavy particles with mass $M$ in external field $E$ is strongly suppressed $\sim \exp\left(-\frac{1}{E}\right)$. The dispersion relations thus unambiguously imply that the coefficients in the real part of the effective lagrangian are factorialy large.

We believe that this simple explanation is so universal in form that it can be applied to almost arbitrary nontrivial effective field theories leading to the same conclusion about factorial behavior. We shall consider another explanation of the same phenomenon later in the text, but now we would like to note that the relation between imaginary and real parts of the amplitudes of course is well known, and heavily used in particle physics. In particular, the recent analysis of the $n!$ behavior in the perturbative $\alpha_s^n$ expansion shows [9], that the physical multiparticle cross section (the imaginary part) is exponentially small. This important result is a simple consequence of dispersion relations similar to eq.(7).

We would like to come back to formula (5) to explain this factorial behavior in the OPE one more time from an absolutely independent point of view. Again, we use QED as an example to demonstrate an idea, however, as we shall see, the arguments which follow are much more general and universal in nature.

As is known, almost all nontrivial field theories exhibit factorial growth of coefficients in the perturbative expansion with respect to coupling constant $\alpha_s$. This factorial dependence can be understood as the rapid growth of the number of Feynman graphs $\alpha_s^4$.

Now, how one can understand the nature of the Wilson OPE in terms of the Feynman graphs? As is known the computational recipe of the coefficients in the OPE is simple: it is necessary to separate large and small distance physics. Large distance physics is presented by operators of light fields; the small distance contribution is explicitly calculated from the underlying field theory. Technically, in order to carry out this program, we cut the perturbative graphs in all possible ways over the photon lines (in general case, a photon field will be replaced by some light degrees of freedom). These lines present the external light fields. They are combined together in the specific way to organize all possible operators. The coefficients in front of these operators can be explicitly calculated and they are determined by

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4 Do not confuse this perturbative expansion with OPE and Effective lagrangian we are dealing with. These series are very different in nature, but they both exhibit an factorial growth.

5 Here we do not discuss the so-called renormalons which give the same factorial dependence, but have a very different origin.
the small distance physics.

From this technical explanation of the calculation of the coefficients in the Wilson OPE it should be clear, that if the underlying theory possesses factorial growth in the perturbative expansion, the Effective lagrangian constructed from this theory exhibits the same factorial behavior for the high dimensional operators. The moral of this argument is very simple: the factorial growth of the perturbative expansion in the underlying theory can not disappear without trace. It will show up in the coefficients of the high dimensional operators in the Effective lagrangian obtained from the underlying theory.

Having demonstrated the main result on factorial growth of the coefficients (in an Effective lagrangian) as universal phenomenon, we would like to discuss a few more examples.

2.2 Nonlocal Lagrangian.

The main goal of this section is the demonstration of the fact that in general the so-called nonlocal lagrangians [10], [1] exhibit the same feature we have been discussing in the previous section. Namely, irrespective of the “smearing” prescription of the nonlocal part of interaction, the corresponding Effective action, obtained in the standard way, will exhibit the factorial growing coefficients for the high dimensional operators.

Before going into details, let us recall a few general results concerning nonlocal lagrangians. First of all, we refer to the old review paper [10] on this subject regarding the motivations. The recent interest on this subject was renewed in ref. [1] where it was advocated that such a lagrangian is the useful tool to deal with a physical situation in which the scales are not well separated. Anyhow, our main interest at the moment is not a physical application, but rather, the demonstration of some universal property for such kind of system. The next relevant remark concerning a nonlocal lagrangian is as follows: the nonlocal, lowest dimensional coupling constant (let us say, quartic) in general case can induce some changes in the coupling terms with larger number of fields (let say, six eight, ...). Thus, we are forced to consider an effective Lagrangian with operators with an arbitrary number of low energy fields. To be more specific, we shall consider the following effective lagrangian for the scalar field $\phi$ discussed in ref. [1]:

$$L_{int} = \sum_{r=1}^{\infty} G_{2r} \phi^{2r}$$  \hspace{1cm} (10)

Here, $G_{2r}$ are some nonlocal functions which are analytical in the region of definition, depend as a consequence of momentum conservation on $2r - 1$ linearly independent momenta, and may have dimensions proportional to some power of an implicit scale of nonlocality $\Lambda$. We assume in what follows that the nonlocal couplings in the bare action [10] are of order of one (we
mean by this that there is no strong dependence on \( r \), like \( r! \) or so); we shall demonstrate in this case that the interactions will give the factorial growing coefficients for the high dimensional operators in the corresponding Effective lagrangian obtained from (10).

We start our analysis from the well understood \( \phi^4 \) interaction. The issue of whether the interaction is local or nonlocal is not relevant for the analysis of the large order behavior in Effective theory. As we have discussed in the previous section, in order to calculate the coefficients for the high dimensional operators, we have to: a). calculate the number of graphs for the given order \( n \), b). cut the internal lines to organize the operators of the maximal dimensions. For \( \phi^4 \) theory, it is well known\([11],[5]\), that the large order behavior of perturbative series is \( n! \). When we cut lines in order to produce the external operators, each cut gives two external \( \phi^2 \) fields. Thus, we get operator \( \phi^{2n} \) in the corresponding Effective lagrangian with coefficient \( n! \) in front of it (or, what is the same, we expect the following behavior for the \( n \)-th term \( L_{\text{eff}}^{(n)} \) in the effective Lagrangian \( L_{\text{eff}}^{(n)} \sim (\frac{n}{2})! \phi^n \).

We would like to generalize this result for the bare action with arbitrary dimensions (10). In the course of these calculations we shall reproduce the \( (\frac{n}{2})! \) behavior mentioned above. We shall demonstrate also that the essential result will not be changed with the increasing of dimensions of the vertices \( r \) (10), provided that \( n \gg r \). The last condition is required for the method to be applicable.

Let us remind that the Lipatov’s idea \([11],[3]\) of the calculation of large order behavior in a field theory is to present the coefficients \( Z_k \) in the perturbative expansion \( Z(g) = \sum Z_k g^k \) through a contour integral in the complex \( g \)-plane:

\[
Z_k \sim \int D\phi \oint \frac{dg}{g^{k+1}} e^{-S(\phi)}, \tag{11}
\]

where \( S(\phi) \) is the action of the scalar field theory \( \frac{1}{4} \phi^4 \) and \( D\phi \) is the standard measure for the functional integral which defines the theory (We discuss here the perturbative expansion for the Grand Partition Function \( Z(g) \). An arbitrary correlation function can be considered in an analogous way.). If the theory possesses the classical instanton solution, then the calculation of the integral over \( g \) can be done through steepest descent method. This method is justified only for small \( g \). But for the large order \( k \), the integral over \( g \) is dominated by the small \( g \) contribution. Indeed, in our specific case of \( \phi^4 \) field theory the classical instanton solution has the following property \( \phi_{\text{cl}} \sim 1/\sqrt{g}, \ [11] \). This can be seen from the saddle point equations for \( g_0(k) \) and \( \phi_{\text{cl}}(k) \) (the actual equations are differential equations, of course, but we are keeping the track only on external parameter \( k \), disregarding all complications related to the coordinate \( x \) dependence):

\[
\frac{k}{g_0} + \frac{\phi_{\text{cl}}^4}{4} = 0, \quad -\Delta \phi_{\text{cl}} + g_0 \phi_{\text{cl}}^3 = 0, \quad \implies \tag{12}
\]
\[ \phi_{cl} \sim \frac{1}{\sqrt{g_0}} \sim \sqrt{k}, \quad g_0 \sim \frac{-1}{k}, \quad S_{cl} \sim \phi_{cl}^2 \sim \frac{1}{g_0} \sim k. \]

From these equations it is clearly seen that the classical action \( S_{cl} \sim 1/g_0 \sim k \) is parametrically large for the large external parameter \( k \). Thus, the semiclassical approximation is completely justified.

The generalization of these formulae for the more complicated action \( \sim \phi^{2r} \) is straightforward: Instead of (12) we have the following behavior:

\[ \phi_{cl} \sim \sqrt{kr}, \quad g_0 \sim \frac{-1}{k^{r-1}}, \quad S_{cl} \sim \phi_{cl}^2 \sim k. \] (13)

Thus, the method is applicable for the large \( k \) and for any finite number \( r \), where the classical action is large and the coupling constant is small. From these formulae one can calculate the the large order behavior in perturbative series with bare action \( \sim g\phi^{2r} \). The result is \( g^k(rk - k)! \). This growth is much faster than we found previously for \( \phi^4 \) theory with \( r = 2 \). However, the coefficients in the Effective lagrangian for the operator \( \phi^n \) grow in the same way as before \( \sim \left( \frac{n}{2} \right)! \). The technical explanation for that is simple: when we cut the lines in order to produce an external operator, the dimension of the obtained operator \( \sim \phi^{2k(r-1)} \) would be higher than for \( \phi^4 \) theory. Thus, for the operator \( \phi^n \) the coefficients in Effective lagrangian have the same growth \( \sim \left( \frac{n}{2} \right)! \) as we already mentioned.

It would be interesting to understand this result in somewhat different way. Essentially, what we need to calculate is the number of graphs which contribute to the \( n \) point correlation function \( Z^{(n)} \sim \langle \phi(x_1)\phi(x_2)...\phi(x_n) \rangle \). Such a calculation can be done within the same Lipatov’s technique. The only technical difference in comparison with the calculation of the large order behavior for the Grand Partition Function \( Z(g) \) itself is following: We have to substitute in the first approximation the classical solution \( \phi_{cl} \) in place of the external \( \phi \) fields. More precisely,

\[ Z^{(n)} \sim \int D\phi \int \frac{dg}{g} \phi(x_1)\phi(x_2)...\phi(x_n)e^{-S(\phi)} \sim \] (14)

\[ \int D\phi \int \frac{dg}{g} e^{-S(\phi_{cl})} \phi_{cl}(x_1)\phi_{cl}(x_2)...\phi_{cl}(x_n) \sim \left( \sqrt{n} \right)^n \sim \left( \frac{n}{2} \right)! \]

In this formula we took into account that the classical field depends on \( n \) as \( \phi_{cl} \sim \sqrt{n}, \) (13) and the total number of external fields in the correlation function is equal \( n \). The semiclassical approximation we have used in the derivation (14) is justified as far as number \( (\frac{n}{2}) \gg 1 \). Only in this case the integral over \( g \) is dominated by the small \( g \) contribution and instanton calculus can be applied.

The factorial dependence (14) can be interpreted as the rapid growth of the number of Feynman graphs. As we see the dependence on \( n \) remains the
same irrespectively to the form of the bare vortices provided that \( n \gg r \). This is in agreement with what we discussed before and related to the fact that the essential part of classical solution \( \phi_{cl} \sim \sqrt{n} \) remains the same for arbitrary \( r \). Such a behavior suggests that all terms from the bare action give more or less the same contributions to the coefficients in the Effective theory. To obtain the total number of graphs which contribute to the operator \( \phi^n \) we should sum up all terms coming from all possible vertices \( r \ll n \). It gives essentially the same \((n/2)!\) behavior because
\[
\sum_{r=2}^{r=n} c_r r^{n/2} (n/2)! \sim (n/2)! \quad n \gg 1.
\]
We do not expect any special cancellations between different terms which may kill this growth. The contribution from the higher order operators \( r \simeq n \) can not be estimated in the same way, but one could expect that the growth of the coefficients could be even more severe in this case.

The moral is: We certainly have a divergent series for Effective lagrangian induced by some unknown full theory no matter what the starting point is. We shall discuss some applications of this result in the conclusion.

2.3 A few more examples

In this subsection we are going to discuss a few more examples from very different fields of physics:

a). Collective fields in QCD;

b). Berry phase as a dynamical field in compactification problem;

c). Lattice field theories.

d) Gravity at Plank scale.

We shall demonstrate that the phenomenon of the asymptotic nature of an Effective lagrangian is a very universal one. This universality is the common feature which characterizes these so different fields of physics we mentioned above. a). We start from the QCD, as underlying theory. The problem in this case can be formulated in the following way (see recent paper [12] on this subject and references therein). How one can integrate over small distance physics in order to extract the long-distance dynamics? An appropriate way to implement this program is: a). introduce the collective degrees of freedom, colorless mesons, as the external sources into the underlying lagrangian; b) integrate over the quarks and gluons with high frequencies by introducing the normalization point \( \mu \). The obtained Effective lagrangian is the \( 1/\mu \) expansion where operators are expressed in terms of the external fields as well as low-energetic quarks and gluons. Our remark is: the coefficients in this expansion grow factorially with the dimension of the operators. We postpone the discussion of the physical meaning of this result to the Conclusion. Let us note, that the procedure of obtaining the Effective lagrangian in this case is not much different from the case we discussed previously. The only new element is the introduction of the collective fields which were not present in our original lagrangian. However, this does not effect the general arguments on the \( n! \) behavior.
Indeed, one can consider the quark-antiquark external lines (instead of the collective meson fields) for the calculation of the coefficients in the OPE, as discussed in the previous section. In this case, all arguments on $n!$ behavior can be applied in a straightforward way. Thus, we expect a factorial behavior of the coefficients for the Effective QCD lagrangian, as well as for the chiral lagrangian, as its particular case. An exact formula for the coefficients depends on the operator under consideration. This is because the different fields (gluons, quarks, mesons), which are constituents of the operator are not equally weighted. However the precise expression for the coefficients in terms of constituents of these operators is not a relevant issue at the moment. We shall discuss consequences of this result in the Conclusion.

b). We continue our short review of different models by analyzing the so-called Berry phase as a dynamical gauge field[13]. There are a few applications of this idea. We consider only one of them. As is known, the standard philosophy of compactification at the Plank scale is the assumption of a very high gauge invariance at this scale which will be broken at lower scales. It is quite possible that some of gauge symmetries are dynamically induced rather than a required principle. We refer to the recent paper [14] on this subject for details and references. Here we would like to demonstrate that the Effective lagrangian for the induced dynamical Berry field is not a convergent, but an asymptotic series. As usual, the Effective lagrangian is obtained by integrating over the fast degrees of freedom; the Berry field itself is considered as a slow variable. The Effective lagrangian is understood as a theory describing the dynamics of these slow fields.

To be more specific, if one integrates over the compactified space coordinates, than one obtains an Effective lagrangian which depends on Berry’s potential $A_\mu = -iu^\dagger \partial_\mu u$. Here $u$ is an original fermion field considered as a fast variable. Now the situation clearly resembles QCD where the underlying lagrangian does not contain meson fields. They will appear and become dynamical variables after integrating over the fast quark fields. The same situation takes place in the case under consideration where the Berry potential can be thought of as a composite of $u$ and $u^\dagger$ original fields.

Now all previous QCD- arguments regarding the $n!$ growth of coefficients in the Effective lagrangian can be applied to the present case. We end up with the same conclusion that the Effective lagrangian $L_{\text{eff}}(A_\mu)$ as a function of the Berry potential is an asymptotic series. Of course, there is a huge difference in scales between QCD and the theory under consideration: in former case the parameter of expansion is $1/\mu$ with $\mu \simeq 1 GeV$; in later one the scale is the Plank scale $M_P \simeq 10^{19} GeV$. However, there is no fundamental difference between these two models in the way of obtaining the corresponding Effective lagrangians: in both cases the slow fields can

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6The very different approach[13] leads essentially to the similar conclusion about the asymptotic nature of the adiabatic expansion
be considered as the composite of the original fields. A symmetry prevents them from getting a mass: for the $\pi$ meson it is the chiral symmetry; for the Berry field $A_\mu$ it is a gauge symmetry. Thus, both fields can be considered as soft variables and the philosophy of Effective lagrangian can be applied. The integration over the fast variables, as we argued earlier, leads to the $n!$ growth of the coefficients in the Effective lagrangians in both cases.

c). Our next example is the lattice QCD. As is known, the main idea in lattice QCD is to replace continuous spacetime variables by a discrete lattice. Then the path integral defining the QCD can be evaluated numerically. If we denote $a$ as the lattice spacing, then the standard discretization of the QCD action has errors of $O(a^2)$ that are large when the lattice spacing is not small enough. This was the reason to suggest the so-called improved action for lattice QCD\cite{16} (for recent development see \cite{17}). The improved discretization has been designed in such a way that finite $a$-errors are systematically removed by introducing new (nonrenormalizable) interactions into the lattice action. All coefficients of the new interactions are determined by demanding that the discretized action reproduces continuum physics to a given accuracy. In particular, the Wilson action contains all terms proportional to $a^2, a^4, ...$ beyond the desired gluon kinetic term\cite{16}:

$$1 - \frac{1}{3} Re \, Tr U_{pl} = r_0 \sum_{\mu\nu} Tr(F_{\mu\nu}F_{\mu\nu}) + a^2 \sum_i r_i R^i +$$

$$0(a^4) + ... + \sum_{n,i} a^{2n} r_{i,2n} Q^{i,2n},$$

where $U_{pl}$ is the product of link matrices on a plaquette $P$; $R^i$ is the set of operators of dimension six; the $r_i$ are coefficients in the OPE of the plaquette. For higher dimensional operators we introduced the corresponding notations $Q^{i,2n}$ and $r_{i,2n}$ with the index $n$ labeling the dimension of the operator, and the index $i$ classifying different operators with given dimension.

Our remark is: The coefficients in the expansion (15) are factorially growing with the dimension of the operators. We shall discuss the physical consequences of this statement in the Conclusion. Now, we would like to explain this $n!$ growth in the following way: The lattice action is defined in terms of the link operator

$$U_{x,\mu} = P \exp[-ig \int_{x}^{x+aq} A \cdot dy]$$

with the simplest choice of path for the integral as a straight line joining $x$ and $x + a\mu$. A single plaquette contribution can be thought of as a Wilson loop surrounding this point $x$ with radius $a$. As is known, the Wilson loop can be interpreted as the creation of a heavy quark-antiquark pair which propagates for a time $a$ and finally annihilates. It can be interpreted as a
forward and backward propagating of one heavy quark as well. Anyhow, one can interpret the action \( (15) \) as the effective action which is obtained after integrating out the heavy quarks with mass \( a^{-1} \). As usual, to give some sense to the Effective lagrangian which presumably describes the dynamics of light degrees of freedom, the mass of the auxiliary heavy quarks should be much larger than the characteristic scale in QCD: \( a^{-1} \gg 1 \text{GeV} \). Once this interpretation in terms of the heavy quark has been made, we have reduced our problem to the previously discussed case \( (1) \).

**d**). Our last, but not least example is the effective field theory of gravity. We refer to the recent review \[18\] on this subject for a general introduction and references. The only remark we would like to make here is the following. Nowadays it is generally accepted that the Einstein lagrangian

\[
S_{\text{grav}} = \int d^4x \sqrt{g} \frac{2}{\kappa^2} R
\]

is only the first local term of the expansion of a more complicated theory (string?). Thus, general relativity should be considered as an effective field theory with infinitely many terms allowed by general coordinate invariance. As usual, in the effective theory description, only the first term in the expansion plays a role at low energy \( E \ll M_{\text{Plank}} \). If we were not interested in quantum effects at the Plank scale with \( E \simeq M_{\text{Plank}} \), eq. (17) would be the end of the story. However we intend to discuss physics at the Plank scale, thus we would like to write down the Effective lagrangian in the most general form:

\[
S_{\text{eff}} = \int d^4x \sqrt{g} [\Lambda + \frac{2}{\kappa^2} R + c_1 R^2 + c_2 R_{\mu\nu} R^{\mu\nu} + \sum_n c_n Q^n + L_{\text{matter}} + L_{\text{dilaton}} + L_{\text{inflaton}} + \ldots],
\]

where the operators \( Q^n \) are high dimensional operators constructed from the relevant fields (\( R_{\mu\nu} \), dilaton, inflaton \( \phi \), gauge fields \( F_{\mu\nu} \), etc). Our remark here is that **the coefficients in the Effective lagrangian describing even the pure gravity theory, exhibit factorial growth.** The arguments which support this statement are the same as before: if the underlying theory (in our case it is given by lagrangian (17) possesses factorial growth in the perturbative expansion, the Effective lagrangian constructed from this theory exhibits the same factorial behavior for the high dimensional operators.

As we already mentioned, the factorial behavior of coefficients in the perturbative expansion can be understood as the fast increase in the number of Feynman diagrams. In pure Yang Mills theory we know well that such a growth does take place\[5\]. We can interpret this growth as a manifestation of the three- and four- gluon vertices which lead to the factorially divergent

\[7\] Any extra fields may only increase this growth.
number of the diagrams. In the case of gravity (17) we expect the same factorial behavior because of the nonlinear nature of the interaction (17) similar to a gauge theory. Of course, there is a big difference between those two, related to the fact that gravity is not a renormalizable field theory. However the only relevant point for our purposes is that the coefficients are factorially growing the dimension of the operator increases. The possible physical consequences of this phenomenon will be discussed in the last section.

3 Instead of conclusion

3.1 General summary

In this letter we have presented two independent sets of arguments which support the idea that almost any nontrivial Effective lagrangian obtained by integrating out some heavy fields and/or fast degrees of freedom, is non-convergent, but an asymptotic series.

The first set of arguments is based on the idea that the imaginary part of the amplitude related to the probability of the physical creation of a heavy particle, is exponentially small $\sim \exp(-\frac{1}{E})$. The dispersion relations in this case unambiguously imply that the coefficients of the expansion in the real part of the corresponding amplitude exhibit an factorial dependence. Once these coefficients are found to be factorially large, we can forget about the way the result was derived, we can forget about the external auxiliary field $E$ which we heavily used in our arguments. Coefficients in the OPE do not depend on the applied field $E$, no matter how small it is.

The second line of reasoning is based on the analysis of the large order behavior of the perturbative series. As we have argued, if the underlying theory possesses factorial growth of the coefficients of the perturbative series, than the corresponding Effective lagrangian constructed from this theory will exhibit the same factorial behavior for the high dimensional operators.

We believe that both of these lines of arguments are so general in form that almost all nontrivial Effective lagrangian will demonstrate $n!$ behavior. We believe that this phenomenon is universal in nature.

Now we would like to discuss some physical consequences which might result from this phenomenon. As we mentioned in Introduction, we have nothing new to say in the case of analysis of low energy phenomena for which the small expansion parameter is $\lambda \equiv E/M \ll 1$. In such a case, the exact formula is approximated perfectly well by the first term of the asymptotic expansion and we can safely forget about all the rest. However, very often the situation is not so fortunate and the expansion parameter $\lambda \sim 1$, (let say $1/3$ or $1/2$). In this event people try to improve the situation by considering the next to -leading terms or even next to next to -leading order. If the series were convergent, these efforts would be worthwhile. However, as we argued in this letter, an Effective lagrangian , in general, is represented
by an asymptotic, not a convergent series. Thus, one may ask the following general questions:

a). How many terms one should keep in the Effective lagrangian for the best approximation of an exact formula for the given parameter $\lambda$?

b). What is the fundamental uncertainty (related to our lack of knowledge of the higher dimensional operators) one should expect for an Effective lagrangian represented as an asymptotic series?

Let us recall that the standard perturbative expansion in QCD is also asymptotic series. For this case the answers on the questions a). and b). are well known\footnote{Our thanks to A. I. Karazhyan for useful discussions.}. In particular, as is known, the pole mass of a heavy quark suffers from an intrinsic uncertainty of order $\Lambda_{QCD}$. Another example is the fundamental uncertainty of perturbative calculations of the correlation function for the light quarks\footnote{To discuss this question we take the simplest asymptotic expansion of QCD (15) to illustrate a concrete example of our discussion.}

We believe that the asymptotic nature of the OPE and Effective lagrangian, in particular, will lead to a similar fundamental uncertainty for some physically interesting characteristics. In particular, as we argued in \cite{7}, any hopes to improve the standard QCD sum rules (like the idea advocated in \cite{19}) by summing up a certain subset of the power corrections and ignoring all the rest, is fundamentally an erroneous idea because of the asymptotic nature of the OPE. A similar example which has been discussed recently is the OPE for $\tau$ decay\footnote{To discuss this question we take the simplest asymptotic expansion of QCD (15) to illustrate a concrete example of our discussion.}. It was argued that the tail of the condensate series may be quite noticeable in the nonperturbative analysis of the hadronic $\tau$ decay.

Therefore, the moral is: if the parameter of the asymptotic expansion is not small enough, the two questions formulated above might have some phenomenological relevance. The effective description of QCD which has been discussed in the previous section is one example. We believe that the lattice calculations (also discussed in the previous section) is another example of the same kind. Indeed, as we argued in the previous section the expansion (15) is an asymptotic series. Thus, we can formulate the following question: How many terms in the asymptotic expansion should be kept for the given lattice size $a$ in order to get the best possible accuracy? The same question can be reformulated in somewhat different way: What is the fundamental uncertainty of the lattice calculations which are associated with the tail of the high dimensional operators in the Effective lagrangian (15)?

### 3.2 Cosmological constant problem

We wish to discuss some consequences of the factorial behavior in the Effective lagrangian (18) for gravity separately. Let us recall that the natural scale of the cosmological term $\Lambda$ is the Plank scale. Indeed, the most popular cosmology today, the inflationary scenario (for a review see \cite{20} and \cite{21})
assumes that our universe passed through an era in which the cosmological
term dominated, and it is a total mystery why we should be left in a universe
with an almost vanishing vacuum energy. Of course we do not know the an-
twer to this question, but we would like to suggest the following scenario
which is based on the asymptotic nature of the effective lagrangian (18).

Let us assume that at the very early epoch the gravity field as well as
other relevant for inflation fields (scalar,...) exhibit some nonzero vacuum
expectation values (VEV), which we shall call the condensates. We believe
that this is very likely to happen in gravity at the Plank scale in analogy with
the phenomenon of gluon condensation in QCD at 1GeV scale. We introduce
the notation $\langle \Phi \rangle$ for the condensate of any relevant field: a scalar field which
people usually introduce to describe inflation (inflaton), dilaton or a gravity
field itself. The natural scale for such a condensate is, of course, the Plank
scale. For the higher dimensional operators $Q^n$ from eq.(18) we assume that
there is a factorization rule which allows us to estimate the higher order
condensates in the following way $\langle Q^n \rangle \sim \langle \Phi^n \rangle \sim \langle \Phi \rangle^n$. We note that this
assumption is not crucial for our purposes, but, rather, is a simplification
which allows us to demonstrate the main idea in a very simple way. The
similar assumption in QCD is justified in the limit in which the number of
colors $N_c \to \infty$. Given that these assumptions have been made, we can use
the Borel representation formula for the asymptotic series (18):

$$\langle Leff \rangle \sim \sum_{n=0}^{n=\infty} n!(-1)^n \langle \Phi \rangle^n = \int_0^\infty \frac{dt}{t(t + \langle \Phi \rangle)} \exp(-\frac{1}{t}) \tag{19}$$

Now we would like to briefly discuss the vacuum structure of de Sit-
ter Space. In different words, we would like to discuss the parameter $\langle \Phi \rangle$
from eq. (19). We refer to the recent papers [23]-[24] on this subject (see
references to previous papers therein). The main result of these investiga-
tions is the observation that the higher order quantum gravity corrections
to the different physical values in general are infrared divergent. In particu-
lar, the divergence is observed in the vacuum correlator $\langle \phi^2 \rangle$. Probably, this
divergence has power-like behavior in time rather than exponential one, as
previously thought. It may force us to take some nonperturbative dynam-

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8 We assumed in this formula that the series is Borel summable. This is may or may
not be the case; however we believe that the Borel non-summability of an expansion does
not signal an inconsistency or ambiguity of the theory. The Borel prescription is just
one of many summation methods and need not be applicable everywhere. For Borel-non-
summable cases, one could expect the sign ($-$) in the denominator of eq.(19). Thus, some
prescription, based on the physics consideration, should be given in order to evaluate an
integral like that. Some new physics usually accompanies such a phenomenon, but we do
not go into details here. Rather, we would like to mention the non-Borel summable example
of the principal chiral field theory at large $N$ [22]. In this case, the explicit solution is
known. The coefficients grow factorially with the order and the series is non-Borel summable.
Nevertheless, the physical observables are perfectly exist, the exact result can be recovered
by special prescription which uses a non-trivial procedure of analytical continuation.
ics into account, which we do not know. Instead we introduce some small phenomenological parameter $\epsilon(\tau)$ into the VEV

$$\langle \Phi \rangle \to \frac{\langle \Phi \rangle}{\epsilon(\tau)}, \quad \epsilon \to 0$$

in order to account for this new physics responsible for the infrared divergences mentioned above.

One can see in this case that the integral which describes the vacuum energy

$$\langle H_{eff} \rangle \sim \int_0^{\infty} \exp\left(\frac{-1}{t} \right) \frac{dt}{t(t + \langle \Phi \rangle \epsilon(\tau))} \sim \epsilon \to 0$$

(20)

goes to zero at small $\epsilon$. As we mentioned above, the effect (20) does not crucially depend on our assumptions about the factorization properties for the condensates $\langle \Phi^n \rangle \sim \langle \Phi \rangle^n$ as neither on our assumption of exact factorial dependence of the coefficients $c_n = n!$. Both of these effects presumably lead (apart from $n!$) to some mild $n-$dependence which can be easily implemented into the formula (20) by introducing some smooth function $f(t)$ whose moments $\int f(t) t^{-n-2} \exp\left(\frac{-1}{t} \right) dt$ exactly reproduce a $n-$dependence of the coefficients as well as of the condensates. If this function is mild enough, it will not destroy the relation (20), but might change some numerical coefficients. Besides that, a condensate might have, along with singular part proportional to $\langle \Phi \rangle \epsilon(\tau)$, a regular part as well $\langle \Phi \rangle \epsilon(\tau) + \text{const}$. As can be seen from the representation (20) this does not destroy the eq.(20).

Few remarks are in order. The vanishing of the vacuum energy is the consequence of the asymptotic nature of the effective lagrangian and the infrared properties of the VEVs. All others simplified assumptions which have been made for technical reasons do not affect the phenomenon. Vanishing of the vacuum energy (20) can be interpreted (after inflation, when all relevant condensates presumably go to zero) as the vanishing of the cosmological constant, the only relevant operator in the Effective lagrangian (all other terms are marginal or irrelevant operators).

As our last remark, we would like to note that the strong infrared dependence of the vacuum condensate $\langle \Phi \rangle$ is not a unique property of de Sitter gravity. Two-dimensional QCD with a large number of colors also exhibits a strong infrared dependence. In particular, the so-called mixed vacuum condensates can be exactly calculated in this theory in the chiral limit ($m_q \to 0$) and exhibit the following dependence on the infrared parameter $m_q$ [24]:

$$\frac{1}{2n} \langle \bar{q} (ig\epsilon_{\lambda\sigma} G_{\lambda\sigma} \gamma_5)^n q \rangle = \left( -\frac{g^2 \langle \bar{q} q \rangle}{2m_q} \right)^n \langle \bar{q} q \rangle,$$

(21)

\footnote{We could consider lagrangian instead of hamiltonian with the same result.}
where $q$ is a quark field and $G_{\lambda\sigma}$ is a gluon field of $QCD_2(N = \infty)$. The chiral condensate $\langle \bar{q}q \rangle$ in this theory can be calculated exactly \[26\]. It does not vanish without contradicting the Coleman theorem. The very important feature of this formula: it diverges in the chiral limit $m_q \to 0$, where the parameter $m_q$ plays the role of the infrared regulator of the theory. Now, if we considered the asymptotic series constructed from these condensates

$$\sum_{n=0}^{n=\infty} (-1)^n n! a_n \langle \bar{q}(i g \epsilon_{\lambda\sigma} G_{\lambda\sigma} \gamma_5) q \rangle^n \sim \int_0^\infty \exp\left(-\frac{1}{t}\right) \frac{dt f(t)}{t(t + \frac{1}{m_q})} \sim m_q \to 0, \tag{22}$$

we would get result of zero for this series, in spite of the fact that each term on the left hand side diverges in the chiral limit and irrespective of the precise behavior of the coefficients $a_n$! Of course this is only a toy example which however can give us a hint of what might happen in real Nature.

We close this section by noting that the vanishing of the vacuum energy in this scenario does not require any fine tuning of parameters. Rather, it is a very natural consequence of the asymptotic origin of the Effective lagrangian and of the infrared behavior of the VEVs. The problem of naturality within an Effective lagrangian approach has been discussed more than once. In the given context the cosmological constant problem has been discussed recently in \[4\] with the following main conclusion: If a relevant operator appears in the Effective field theory with a coefficient much less than a typical scale without a symmetry reason, it should be taken as a warning for effective field theory dogma.

We hope to have suggested here a natural scenario for the vanishing of the coefficient for a relevant operator which is not based on symmetry considerations. We close this section with the following remark. If this scenario works (as we hope), it means first of all, that all related problems should be explained at the same time within the same approach. In particular, we expect \[27\] that an inflationary scenario, which is the most popular cosmology today, can be understood in terms of the same physical variables within the same philosophy.

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References
[1] V.Bhansli and H.Georgi, Running Nonlocal Lagrangians, hep-ph/9205242, May 1992.

[2] H.Georgi, Annu.Rev.Nuc.Part.Sci., 43, (1994), 355.

[3] A.Manohar, Effective Field Theories, hep-ph/9508245.

[4] D. Kaplan, Effective Field Theories, hep-ph/9506035.

[5] Current Physics-Sources and Comments, vol.7, "Large Order Behavior of Perturbative Theory", eds J.C.Gillou and J.Zinn-Justin, 1990.

[6] M.A.Shifman, Theory of Pre-Asymptotic Effects in weak Inclusive Decays, TPI-MINN-94/17-T, Talk at the Workshop, Minneapolis, hep-ph/9405246;
Talk at PASCOS, Baltimore, March 1995, TPI-MINN-95/15-T, hep-ph/9505283;
Talk at XVIII Kazimier Meeting, TPI-MINN-95/32-T, Ames, May, 1995, hep-ph/9511469.

[7] A.Zhitnitsky, Lessons from QCD\(_2\) (N → ∞): Vacuum structure, Asymptotic Series, Instantons and all that. hep-ph/9510366.

[8] C.Itzykson and J.Zuber, Quantum Field Theory, McGraw-Hill Book Company, 1980.

[9] V.I.Zakharov, Nucl.Phys. B377, (1992), 501; Nucl.Phys. B385, (1992), 452.

[10] D.A.Kirzhnits, Sov.Phys.Usp. 9, (1967), 692.

[11] L.N.Lipatov, Sov.Phys. JETP 45, (1977), 216.

[12] P.H.Damgaard, H.B.Nielsen and R.Sollacher, Nucl.Phys. B414, (1994), 541.

[13] A.Shapere and F.Wilczek, Geometrical phases in physics, World scientific, Singapore, 1989.

[14] K.Kikkawa and H.Tamura, Int.J.Mod.Phys. A10, (1995), 1597.

[15] M.Berry, Proc.R.Soc.Lond. A 414, (1987), 31, see in [13].

[16] K.Symanzik, Nucl.Phys. B226, (1983), 187, 205.

[17] G.P.Lepage et al., Phys.Rev. D46, (1992), 4052; G.Lepage and P.Mackenzie, Phys.Rev. D48, (1993), 2250.

[18] J.F.Donoghue, gr-qc/9512024, hep-ph/9512287.
[19] S.Mikhailov and A.Radyushkin, JETP Lett. 43, (1986), 712; Phys. Rev, D45, (1992), 1754.

[20] A.Linde, Particle Physics and Inflationary Cosmology, harwood Academic Publishers, 1990.

[21] R.H.Brandenberger, Invited lectures at TASI-94, Boulder. Proceedings,ed J.Donoghue, World Scientific,1995, Singapore.

[22] V.Fateev, V.Kazakov and P.Wiegmann, Nucl.Phys. B424, (1994), 505.

[23] A.D.Dolgov, M.B.Einhorn, V.I.Zakharov, Phys.Rev. D52, (1995), 717; Acta Phys. Polon 26, (1995), 65.

[24] N.C.Tsamis and R.P.Woodard, Phys.Lett.B301,(1993),351; Ann.Phys. 238,(1995), 1.

[25] B.Chibisov and A.Zhitnitsky, Phys.Lett. B362, (1995), 105. [hep-ph/9502258]

[26] A.Zhitnitsky, Phys.Lett.B165,(1985),405; Sov.J.Nucl.Phys.43, (1986), 999; Sov.J.Nucl.Phys.44,(1986),139.

[27] R.Brandenberger and A.Zhitnitsky, in preparation, January,1996.