Properties of Rank Metric Codes

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Abstract

This paper investigates general properties of codes with the rank metric. First, we investigate asymptotic packing properties of rank metric codes. Then, we study sphere covering properties of rank metric codes, derive bounds on their parameters, and investigate their asymptotic covering properties. Finally, we establish several identities that relate the rank weight distribution of a linear code to that of its dual code. One of our identities is the counterpart of the MacWilliams identity for the Hamming metric, and it has a different form from the identity by Delsarte.

I. Introduction

Although the rank has long been known to be a metric implicitly and explicitly (see, for example, [1]), the rank metric was first considered for error control codes (ECCs) by Delsarte [2]. The potential applications of rank metric codes to wireless communications [3], [4], public-key cryptosystems [5], and storage equipments [6], [7] have motivated a steady stream of works [6]–[18], described below, that focus on their properties.

The majority of previous works focus on rank distance properties, code construction, and efficient decoding of rank metric codes, and the seminal works in [2], [7], [8] have made significant contribution to these topics. Independently in [2], [7], [8], a Singleton bound (up to some variations) on the minimum rank distance of codes was established, and a class of codes achieving the bound with equality was constructed. We refer to this class of codes as Gabidulin codes henceforth. In [2], [8], analytical expressions to compute the weight distribution of linear codes achieving the Singleton bound with equality were also derived. In [6], it was shown that Gabidulin codes are optimal for correcting crisscross errors (referred to as lattice-pattern errors in [6]). In [7], it was shown that Gabidulin codes are also optimal in the sense of a Singleton bound in crisscross weight, a metric considered in [7], [11], [19] for crisscross errors. Decoding algorithms that parallel the extended Euclidean algorithm and the Peterson-Gornstein-Zierler
algorithm were introduced for Gabidulin codes in [8] and [7], respectively. In [2], the counterpart of the MacWilliams identity, which relates the rank distance enumerator of a code to that of its dual code, was established using association schemes. Following the works in [2], [7], [8], the construction in [8] was extended in [13] and the properties of subspace subcodes and subfield subcodes were considered in [14], [20]; the counterparts of the Welch-Berlekamp algorithm and the Berlekamp-Massey algorithm were considered in [21] and [22] respectively for Gabidulin codes; the error performance of Gabidulin codes was investigated in [11], [16], [17].

Some previous works focus on the packing and covering properties of rank metric codes. Both packing and covering properties are significant for ECCs, and packing and covering radii are basic geometric parameters of a code, important in several respects [23]. For instance, the covering radius can be viewed as a measure of performance: if the code is used for error correction, then the covering radius is the maximum weight of a correctable error vector [24]; if the code is used for data compression, then the covering radius is a measure of the maximum distortion [24]. The Hamming packing and covering radii of ECCs have been extensively studied (see, for example, [25]–[27]), whereas the rank packing and covering radii have received relatively little attention. It was shown that nontrivial perfect rank metric codes do not exist in [9], [10], [15]. In [12], a sphere covering bound for rank metric codes was introduced. Generalizing the concept of rank covering radius, the multi-covering radii of codes with the rank metric were defined in [28]. Bounds on the volume of balls with rank radii were also derived [18].

In this paper, we investigate packing, covering, and rank distance properties of rank metric codes. The main contributions of this paper are:

- In Section III, we establish further properties of elementary linear subspaces (ELS’s) [17], and investigate properties of balls with rank radii. In particular, we derive both upper and lower bounds on the volume of balls with given rank radii that are tighter than their respective counterpart in [18]. These technical results are used later in our investigation of the sphere covering properties of rank metric codes.
- In Section IV, we study the packing properties of rank metric codes, and also derive the asymptotic maximum code rate for a code with given relative minimum rank distance.
- In Section V, we study the covering properties of rank metric codes, and derive both upper and lower bounds on the minimal cardinality of a code with given length and rank covering radius. Our new bounds are tighter than the bounds introduced in [12]. Using the sphere covering bound, we also establish additional sphere covering properties for linear rank metric codes, and prove that some classes of rank metric codes have maximal covering radius. Finally, we establish the asymptotic
minimum code rate for a code with given relative covering radius.

- In Section [VI], we study the rank weight properties of linear codes. We show that, similar to the MacWilliams identity for the Hamming metric, the rank weight distribution of any linear code can be expressed as an analytical expression of that of its dual code. It is also remarkable that our MacWilliams identity for the rank metric has a similar form to that for the Hamming metric. Despite the similarity, our new identity is proved using a different approach based on linear spaces. The intermediate results obtained using our approach offer interesting insight. We also derive identities that relate moments of the rank weight distribution of a linear code to those of its dual code.

We provide the following remarks on our results:

1) The concept of elementary linear subspace was introduced in our previous work [17]. It has similar properties to those of a set of coordinates, and as such has served as a useful tool in our derivation of properties of the rank metric (see Section [III]), covering properties of rank metric codes in general (see Section [V]), and Gabidulin codes (see [17]). Although our results may be derived without the concept of ELS, we have adopted it in this paper since it enables readers to easily relate our approach and results to their counterparts for Hamming metric codes.

2) Both the matrix form [2], [7] and the vector form [8] for rank metric codes have been considered in the literature. Following [8], in this paper the vector form over $\text{GF}(q^m)$ is used for rank metric codes although their rank weight is defined by their corresponding $m \times n$ code matrices over $\text{GF}(q)$ [8]. The vector form is chosen in this paper since our results and their derivations for rank metric codes can be related to their counterparts for Hamming metric codes.

3) In [2], the MacWilliams identity is given between the rank distance enumerator sequences of two dual codes using the generalized Krawtchouk polynomials. Based on a different proof, we establish the same identity for linear rank metric codes, although our identity is expressed using different parameters which are shown to be the generalized Krawtchouk polynomials as well. We also present this identity in weight enumerator polynomial form (cf. Theorem [2]). In their polynomial forms, the MacWilliams identities for both the rank and the Hamming metrics are similar to each other. Furthermore, the polynomial form allows us to derive further identities (cf. Propositions [18] and [19]) between the rank weight distribution of linear dual codes.

The rest of the paper is organized as follows. Section [II] gives a brief review of necessary background to keep this paper self-contained. In Section [III] we derive some further properties of ELS’s and balls of rank radii. In Sections [IV] and [V] we investigate the packing and covering properties respectively of
II. Preliminaries

A. Rank metric and elementary linear subspaces

Consider an $n$-dimensional vector $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in \text{GF}(q^m)^n$. The field $\text{GF}(q^m)$ may be viewed as an $m$-dimensional vector space over $\text{GF}(q)$. The rank weight of $\mathbf{x}$, denoted as $\text{rk}(\mathbf{x})$, is defined to be the maximum number of coordinates in $\mathbf{x}$ that are linearly independent over $\text{GF}(q)$ [8]. Note that all ranks are with respect to $\text{GF}(q)$ unless otherwise specified in this paper. The coordinates of $\mathbf{x}$ thus span a linear subspace of $\text{GF}(q^m)^n$, denoted as $S(\mathbf{x})$, with dimension equal to $\text{rk}(\mathbf{x})$. For all $\mathbf{x}, \mathbf{y} \in \text{GF}(q^m)^n$, it is easily verified that $d_R(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} \text{rk}(\mathbf{x} - \mathbf{y})$ is a metric over $\text{GF}(q^m)^n$, referred to as the rank metric henceforth [8]. The minimum rank distance of a code $C$, denoted as $d_R(C)$, is simply the minimum rank distance over all possible pairs of distinct codewords. When there is no ambiguity about $C$, we denote the minimum rank distance as $d_R$.

In [17], we introduced the concept of elementary linear subspace (ELS). If there exists a basis set $B$ of vectors in $\text{GF}(q)^n$ for a linear subspace $V \subseteq \text{GF}(q^m)^n$, we say $V$ is an elementary linear subspace and $B$ is an elementary basis of $V$. We denote the set of all ELS’s of $\text{GF}(q^m)^n$ with dimension $v$ as $E_v(q^m,n)$. An ELS has properties similar to those for a set of coordinates [17], and they are summarized as follows. A vector has rank $\leq r$ if and only if it belongs to some ELS with dimension $r$. For any $V \in E_v(q^m,n)$, there exists $\tilde{V} \in E_{n-v}(q^m,n)$ such that $V \oplus \tilde{V} = \text{GF}(q^m)^n$, where $\oplus$ denotes the direct sum of two subspaces. For any vector $\mathbf{x} \in \text{GF}(q^m)^n$, we denote the projection of $\mathbf{x}$ on $V$ along $\tilde{V}$ as $\mathbf{x}_V$, and we remark that $\mathbf{x} = \mathbf{x}_V + \mathbf{x}_{\tilde{V}}$.

B. The Singleton bounds

It can be shown that $d_R \leq d_H$ [8], where $d_H$ is the minimum Hamming distance of the same code. Due to the Singleton bound for block codes, the minimum rank distance of an $(n,k)$ block code over $\text{GF}(q^m)$ thus satisfies

$$d_R \leq n - k + 1. \tag{1}$$

An alternative bound on the minimum rank distance is also given in [29]:

$$d_R \leq \frac{m}{n} (n - k) + 1. \tag{2}$$

For $n \leq m$, the bound in (1) is tighter than that in (2). When $n > m$ the bound in (2) is tighter.
When \( n \leq m \), a class of codes satisfying (1) with equality was first proposed in [8] and then generalized in [13]. Let \( g = (g_0, g_1, \ldots, g_{n-1}) \) be linearly independent elements of \( \text{GF}(q^m) \), then the code defined by the generator matrix

\[
G = \begin{pmatrix}
g_0 & g_1 & \cdots & g_{n-1} \\
[g_0] & [g_1] & \cdots & [g_{n-1}] \\
\vdots & \vdots & \ddots & \vdots \\
[g_{k-1}] & [g_{k-1}] & \cdots & [g_{n-1}]
\end{pmatrix},
\]

where \( [i] = q^{ai} \) with \( a \) being an integer prime to \( m \), is called a generalized Gabidulin code generated by \( g = (g_0, g_1, \ldots, g_{n-1}) \); it has dimension \( k \) and minimum rank distance \( d_R = n - k + 1 \) [13].

A class of codes satisfying (2) with equality was proposed in [29]. It consists of cartesian products of a generalized Gabidulin code with length \( n = m \). Let \( \mathcal{G} \) be an \((m, k, d_R = m - k + 1)\) generalized Gabidulin code over \( \text{GF}(q^m) \), and let \( \mathcal{G}^l \overset{\text{def}}{=} \mathcal{G} \times \ldots \times \mathcal{G} \) be the code obtained by \( l \) cartesian products of \( \mathcal{G} \). Thus \( \mathcal{G}^l \) is a code over \( \text{GF}(q^m) \) with length \( ml \), dimension \( kl \), and minimum rank distance \( d_R = m - k + 1 \) [29].

C. Covering radius and excess

The covering radius \( \rho \) of a code \( C \) with length \( n \) over \( \text{GF}(q^m) \) is defined to be the smallest integer \( \rho \) such that all vectors in the space \( \text{GF}(q^m)^n \) are within distance \( \rho \) of some codeword of \( C \) [27]. It is the maximal distance from any vector in \( \text{GF}(q^m)^n \) to the code \( C \). That is, \( \rho = \max_{x \in \text{GF}(q^m)^n} \{d(x, C)\} \).

Also, if \( C \subset C' \), then the covering radius of \( C \) is no more than the minimum distance of \( C' \). Finally, a code \( C \) with length \( n \) and minimum distance \( d \) is called a maximal code if there does not exist any code \( C' \) with same length and minimum rank distance such that \( C \subset C' \). A maximal code has covering radius \( \rho \leq d - 1 \).

Van Wee [30], [31] derived several bounds on codes with Hamming covering radii based on the excess of a code, which is determined by the number of codewords covering the same vectors. Below are some key definitions and results in [30], [31]. For all \( V \subseteq \text{GF}(q^m)^n \) and a code \( C \) with covering radius \( \rho \), the excess on \( V \) by \( C \) is defined to be

\[
E_C(V) \overset{\text{def}}{=} \sum_{c \in C} |B^H_{\rho}(c) \cap V| - |V|,
\]

where \( B^H_{\rho}(c) \) denotes a ball centered at \( c \) with Hamming radius \( \rho \). The excess on \( \text{GF}(q^m)^n \) by \( C \) is given by \( E_C(\text{GF}(q^m)^n) = |C| \cdot V^H_{\rho}(q^m, n) - q^{mn} \), where \( V^H_{\rho}(q^m, n) \) denotes the volume of a ball with Hamming radius \( \rho \). Also, if \( \{W_i\} \) is a family of disjoint subsets of \( \text{GF}(q^m)^n \), then \( E_C(\bigcup_i W_i) = \sum_i E_C(W_i) \).

Suppose \( Z \overset{\text{def}}{=} \{z \in \text{GF}(q^m)^n | E_C(\{z\}) > 0\} \) [30], i.e., \( Z \) is the set of vectors covered by at least two
codewords in $C$. Note that $z \in Z$ if and only if $|B^m_{\rho}(z) \cap C| \geq 2$. It can be shown that $|Z| \leq E_C(Z) = E_C(GF(q^m)^n) = |C| \cdot V^m_n(q^m, n) - q^{mn}$.

Although the above definitions and properties were developed for the Hamming metric, they are in fact independent of the underlying metric and thus are applicable to the rank metric as well.

D. Notations

In order to simplify notations, we shall occasionally denote the vector space $GF(q^m)^n$ as $F$. We denote the number of vectors of rank $u$ ($0 \leq u \leq \min\{m, n\}$) in $GF(q^m)^n$ as $N_u(q^m, n)$. It can be shown that $N_u(q^m, n) = [n]_u^m \alpha(m, u)$ [8], where $\alpha(m, 0) \text{ def } = 1$ and $\alpha(m, u) \text{ def } = \prod_{i=0}^{u-1} (q^m - q^i)$ for $u \geq 1$. The $[n]_u$ term is often referred to as a Gaussian polynomial [32], defined as $[n]_u \text{ def } = \alpha(n, u)/\alpha(u, u)$. Note that $[n]_u$ is the number of $u$-dimensional linear subspaces of $GF(q)^n$. We refer to all vectors in $GF(q^m)^n$ within rank distance $r$ of $x \in GF(q^m)^n$ as a ball of rank radius $r$ centered at $x$, and denote it as $B_r(x)$. Its volume, which does not depend on $x$, is denoted as $V_r(q^m, n) = \sum_{u=0}^{r} N_u(q^m, n)$. We also define $\beta(m, 0) \text{ def } = 1$ and $\beta(m, u) \text{ def } = \prod_{i=0}^{u-1} [m-1]_i$ for $u \geq 1$, which are used in Section [VI]. These terms are closely related to Gaussian polynomials: $\beta(m, u) = [n]_u \beta(u, u)$ and $\beta(m + u, m + u) = [n+u]_u \beta(m, m) \beta(u, u)$.

III. TECHNICAL RESULTS

A. Further properties of ELS’s

Lemma 1: Any vector $x \in GF(q^m)^n$ with rank $r$ belongs to a unique ELS $V \in E_r(q^m, n)$.

Proof: The existence of $V \in E_r(q^m, n)$ has been proved in [17]. Thus we only prove the uniqueness of $V$, with elementary basis $\{v_i\}_{i=0}^{r-1}$. Suppose $x$ also belongs to $W$, where $W \in E_r(q^m, n)$ has an elementary basis $\{w_j\}_{j=0}^{r-1}$. Therefore, $x = \sum_{i=0}^{r-1} a_i v_i = \sum_{j=0}^{r-1} b_j w_j$, where $a_i, b_j \in GF(q^m)$ for $0 \leq i, j \leq r - 1$. By definition, we have $\mathcal{G}(x) = \mathcal{G}(a_0, \ldots, a_{r-1}) = \mathcal{G}(b_0, \ldots, b_{r-1})$, therefore $b_j$’s can be expressed as linear combinations of $a_i$’s, i.e., $b_j = \sum_{i=0}^{r-1} c_{j,i} a_i$ where $c_{j,i} \in GF(q)$. Hence

$$x = \sum_{j=0}^{r-1} b_j w_j = \sum_{j=0}^{r-1} \sum_{i=0}^{r-1} c_{j,i} a_i w_j = \sum_{i=0}^{r-1} a_i u_i,$$

where $u_i = \sum_{j=0}^{r-1} c_{j,i} w_j \in GF(q)^n$. Now consider $X$, the matrix obtained by expanding the coordinates of $x$ with respect to the basis $\{a_i\}_{i=0}^{m-1}$. For $0 \leq i \leq r - 1$, the $i$-th row of $X$ is given by the vector $v_i$ by definition and by the vector $u_i$ from Eq. (5). Therefore $v_i = u_i \in W$, and hence $V \subseteq W$. However, dim($V$) = dim($W$), and thus $V = W$. ■

Lemma 1 shows that an ELS is analogous to a subset of coordinates since a vector $x$ with Hamming weight $r$ belongs to a unique subset of $r$ coordinates, often referred to as the support of $x$. 

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In [17], it was shown that an ELS always has a complementary elementary linear subspace. The following lemma enumerates such complementary ELS’s.

**Lemma 2:** Suppose $V \in E_v(q^m, n)$ and $A \subseteq V$ is an ELS with dimension $a$, then there are $q^{a(v-a)}$ ELS’s $B$ such that $A \oplus B = V$. Furthermore, there are $q^{a(v-a)} \binom{v}{a}$ such ordered pairs $(A, B)$.

**Proof:** First, remark that $\dim(B) = v - a$. The total number of sets of $v - a$ linearly independent vectors over $GF(q)$ in $V \setminus A$ is given by $N = (q^v - q^a)(q^v - q^{a+1}) \cdots (q^v - q^{v-1}) = q^{a(v-a)}\alpha(v-a, v-a)$. Note that each set of linearly independent vectors over $GF(q)$ constitutes an elementary basis set. Thus, the number of possible $B$ is given by $N$ divided by $\alpha(v-a, v-a)$, the number of elementary basis sets for each $B$. Therefore, once $A$ is fixed, there are $q^{a(v-a)}$ choices for $B$. Since the number of $a$-dimensional subspaces $A$ in $V$ is $\binom{v}{a}$, the total number of ordered pairs $(A, B)$ is hence $q^{a(v-a)} \binom{v}{a}$. \hfill $\blacksquare$

Puncturing a vector with full Hamming weight results in another vector with full Hamming weight. Lemma 3 below shows that the situation for vectors with full rank is similar.

**Lemma 3:** Suppose $V \in E_v(q^m, n)$ and $u \in V$ has rank $v$, then $\text{rk}(u_A) = a$ and $\text{rk}(u_B) = v - a$ for any $A \in E_a(q^m, n)$ and $B \in E_{v-a}(q^m, n)$ such that $A \oplus B = V$.

**Proof:** First, $u_A \in A$ and hence $\text{rk}(u_A) \leq a$ by [17, Proposition 2]; similarly, $\text{rk}(u_B) \leq v - a$. Now suppose $\text{rk}(u_A) < a$ or $\text{rk}(u_B) < v - a$, then $v = \text{rk}(u) \leq \text{rk}(u_A) + \text{rk}(u_B) < a + v - a = v$. \hfill $\blacksquare$

It was shown in [17] that the projection $u_A$ of a vector $u$ on an ELS $A$ depends on both $A$ and its complement $B$. The following lemma further clarifies the relationship: changing $B$ always modifies $u_A$, provided that $u$ has full rank.

**Lemma 4:** Suppose $V \in E_v(q^m, n)$ and $u \in V$ has rank $v$. For any $A \in E_a(q^m, n)$ and $B \in E_{v-a}(q^m, n)$ such that $A \oplus B = V$, define the functions $f_u(A, B) = u_A$ and $g_u(A, B) = u_B$. Then both $f_u$ and $g_u$ are injective.

**Proof:** Consider another pair $(A', B')$ with dimensions $a$ and $v - a$ respectively. Suppose $A' \neq A$, then $u_{A'} \neq u_A$. Otherwise $u_A$ belongs to two distinct ELS’s with dimension $a$, which contradicts Lemma 1. Hence $u_{A'} \neq u_A$ and $u_{B'} = u - u_{A'} \neq u - u_A = u_B$. The argument is similar if $B' \neq B$. \hfill $\blacksquare$

### B. Properties of balls with rank radii

**Lemma 5:** For $0 \leq r \leq \min\{n, m\}$,

$$
q^{r(m+n-r)} \leq V_r(q^m, n) < q^{r(m+n-r)+\sigma(q)},
$$

where $\sigma(q) = \frac{1}{\ln(q)} \sum_{k=1}^{\infty} \frac{1}{k(q^k-1)}$ is a decreasing function of $q$ satisfying $\sigma(q) < 2$ for $q \geq 2$ [17].

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we associate one subspace \( r_k(\mathbb{F}(q^m)) \) with diameter. Show below that balls with rank radii do not necessarily maximize the cardinality of a set with a given diameter. Without loss of generality, we assume that the center of the ball is 0. For any \( x \in \mathbb{F}(q^m)^n \), we associate one subspace \( \mathfrak{S} \) of \( \mathbb{F}(q^m) \) such that \( \dim(\mathfrak{S}) = r \) and \( \mathfrak{S}(x) \subseteq \mathfrak{S} \). We consider the vectors \( y \in \mathbb{F}(q^m)^{n-r} \) such that \( \mathfrak{S}(y) \subseteq \mathfrak{S} \). There are \( q^{rn} \) choices for \( x \) and, for a given \( x \), \( q^{r(n-r)} \) choices for \( y \). Thus the total number of vectors \( z = (x, y) \in \mathbb{F}(q^m)^n \) is \( q^{r(m+n-r)} \). Since \( \mathfrak{S}(z) \subseteq \mathfrak{S} \), we have \( \text{rk}(z) \leq r \). Thus, \( V_r(q^m, n) \geq q^{r(m+n-r)} \).

We remark that both bounds in (6) are tighter than their respective counterparts in [18, Proposition 1]. More importantly, the two bounds in (6) differ only by a factor of \( q^{\sigma(q)} \), and thus they not only provide a good approximation of \( V_r(q^m, n) \), but also accurately describe the asymptotic behavior of \( V_r(q^m, n) \).

The diameter of a set is defined to be the maximum distance between any pair of elements in the set [25, p. 172]. For a binary vector space \( \mathbb{F}(2)^n \) and a given diameter \( 2r < n \), Kleitman [33] proved that balls with Hamming radius \( r \) maximize the cardinality of a set with a given diameter. However, when the underlying field for the vector space is not \( \mathbb{F}(2) \), the result is not necessarily valid [27, p. 40]. We show below that balls with rank radii do not necessarily maximize the cardinality of a set with a given diameter.

**Proposition 1:** For \( 3 \leq n \leq m \) and \( 2 \leq 2r < n \), there exists \( S \subseteq \mathbb{F}(q^m)^n \) with diameter 2r such that \( |S| > V_r(q^m, n) \).

**Proof:** The set \( S \triangleq \{(x_0, \ldots, x_{n-1}) \in \mathbb{F}(q^m)^n | x_{2r} = \cdots = x_{n-1} = 0 \} \) has diameter 2r and cardinality \( q^{2mr} \). For \( r = 1 \), we have \( V_1(q^m, n) = 1 + \frac{(q^n-1)(q^{m-1})}{(q-1)} < q^{2m} \). For \( r \geq 2 \), we have \( V_r(q^m, n) < q^{r(n+m)-(r^2-\sigma(q))} \) by Lemma 5. Since \( r^2 > 2 > \sigma(q) \), we obtain \( V_r(q^m, n) < q^{(n+m)} \leq |S| \).

The intersection of balls with Hamming radii has been studied in [27, Chapter 2], and below we investigate the intersection of balls with rank radii.

**Lemma 6:** If \( 0 \leq r, s \leq n \) and \( c_1, c_2 \in \mathbb{F}(q^m)^n \), then \( |B_r(c_1) \cap B_s(c_2)| \) depends on \( c_1 \) and \( c_2 \) only through \( d_e(c_1, c_2) \).

**Proof:** First, without loss of generality, we assume \( c_1 = 0 \), and we denote \( \text{rk}(c_2) = e \). We can express \( c_2 \) as \( uB \), where \( u = (u_0, \ldots, u_{e-1}, 0, \ldots, 0) \in \mathbb{F}(q^m)^n \) has rank \( e \) and \( B \in \mathbb{F}(q)^{n \times n} \) has full rank. For any \( x \in B_r(0) \cap B_s(u) \) we have \( \text{rk}(xB) = \text{rk}(x) \leq r \) and \( \text{rk}(xB - c_2) = \text{rk}(x - u) \leq s \). Thus there is a bijection between \( B_r(0) \cap B_s(uB) \) and \( B_r(0) \cap B_s(u) \). Hence \( |B_r(0) \cap B_s(uB)| = |B_r(0) \cap B_s(u)| \), that is, \( |B_r(0) \cap B_s(uB)| \) does not depend on \( B \).

Since \( |B_r(0) \cap B_s(uB)| \) is independent of \( B \), we assume \( B = I_{n \times n} \) without loss of generality henceforth. The nonzero coordinates of \( u \) all belong to a basis set \( \{u_i\}_{i=0}^{m-1} \) of \( \mathbb{F}(q^m) \). Let \( x =
\((x_0, \ldots, x_{n-1}) \in B_r(0) \cap B_s(u)\), then we can express \(x_j\) as \(x_j = \sum_{i=0}^{m-1} a_{i,j}u_i\) with \(a_{i,j} \in \text{GF}(q)\) for \(0 \leq j \leq n - 1\). Suppose \(v = (v_0, \ldots, v_{e-1}, 0, \ldots, 0) \in \text{GF}(q^m)^n\) also has rank \(e\), then the nonzero coordinates of \(v\) all belong to a basis set \(\{v_i\}_{i=0}^{m-1}\) of \(\text{GF}(q^m)^n\). We define \(\bar{x} = (\bar{x}_0, \ldots, \bar{x}_{n-1}) \in \text{GF}(q^m)^n\) such that \(\bar{x}_j = \sum_{i=0}^{m-1} a_{i,j}v_i\) for \(0 \leq j \leq n - 1\). We remark that \(\text{rk}(\bar{x}) = \text{rk}(x) \leq r\) and \(\text{rk}(\bar{x} - v) = \text{rk}(x - u) \leq s\). Thus there is a bijection between \(B_r(0) \cap B_s(v)\) and \(B_r(0) \cap B_s(u)\). Hence \(|B_r(0) \cap B_s(u)|\) depends on the vector \(u\) only through its rank \(e\).

**Proposition 2:** If \(0 \leq r, s \leq n, c_1, c_2, c_1', c_2' \in \text{GF}(q^m)^n\) and \(d_s(c_1, c_2) > d_s(c_1', c_2')\), then

\[
|B_r(c_1) \cap B_s(c_2)| \leq |B_r(c_1') \cap B_s(c_2')|.
\]

**Proof:** It suffices to prove (7) when \(d_s(c_1, c_2) = d_s(c_1', c_2') + 1 = e + 1\). By Lemma 6, we can assume without loss of generality that \(c_1 = c_1' = 0\), \(c_2' = (0, c_1, \ldots, c_e, 0, \ldots, 0)\) and \(c_2 = (c_0, c_1, \ldots, c_e, 0, \ldots, 0)\), where \(c_0, \ldots, c_e \in \text{GF}(q^m)\) are linearly independent.

We will show that an injective mapping \(\phi\) from \(B_r(c_1) \cap B_s(c_2)\) to \(B_r(c_1') \cap B_s(c_2')\) can be constructed. We consider vectors \(z = (z_0, z_1, \ldots, z_{n-1}) \in B_r(c_1) \cap B_s(c_2)\). We thus have \(\text{rk}(z) \leq r\) and \(\text{rk}(u) \leq s\), where \(u = (u_0, u_1, \ldots, u_{n-1}) = z - c_2 = (z_0 - c_0, z_1 - c_1, \ldots, z_{n-1})\). We also define \(\tilde{z} = (z_1, \ldots, z_{n-1})\) and \(\tilde{u} = (u_1, \ldots, u_{n-1})\). We consider three cases for the mapping \(\phi\), depending on \(\bar{z}\) and \(\bar{u}\).

- **Case I:** \(\text{rk}(\bar{u}) \leq s - 1\). In this case, \(\phi(z) \overset{\text{def}}{=} z\). We remark that \(\text{rk}(z - c_2') \leq \text{rk}(\bar{u}) + 1 \leq s\) and hence \(\phi(z) \in B_r(c_1') \cap B_s(c_2')\).

- **Case II:** \(\text{rk}(\bar{u}) = s\) and \(\text{rk}(\bar{z}) \leq r - 1\). In this case, \(\phi(z) \overset{\text{def}}{=} (z_0 - c_0, z_1, \ldots, z_{n-1})\). We have \(\text{rk}(\phi(z)) \leq \text{rk}(\bar{z}) + 1 \leq r\) and \(\text{rk}(\phi(z) - c_2') = \text{rk}(z - c_2) \leq s\), and hence \(\phi(z) \in B_r(c_1') \cap B_s(c_2')\).

- **Case III:** \(\text{rk}(\bar{u}) = s\) and \(\text{rk}(\bar{z}) = r\). Since \(\text{rk}(u) = s\), we have \(z_0 - c_0 \in \mathcal{S}(\bar{u})\). Similarly, since \(\text{rk}(z) = r\), we have \(z_0 \in \mathcal{S}(\bar{z})\). Denote \(\dim(\mathcal{S}(\bar{u}, \bar{z})) = d\) \((d \geq s)\). For \(d > s\), let \(c_0, \ldots, c_{d-1}\) be a basis of \(\mathcal{S}(\bar{u}, \bar{z})\) such that \(c_0, \ldots, c_{s-1} \in \mathcal{S}(\bar{u})\) and \(c_s, \ldots, c_{d-1} \in \mathcal{S}(\bar{z})\). Note that \(c_0 \in \mathcal{S}(\bar{u}, \bar{z})\), and may therefore be uniquely expressed as \(c_0 = c_u + c_z\), where \(c_u \in \mathcal{S}(c_0, \ldots, c_{s-1}) \subseteq \mathcal{S}(\bar{u})\) and \(c_z \in \mathcal{S}(c_s, \ldots, c_{d-1}) \subseteq \mathcal{S}(\bar{z})\). If \(d = s\), then \(c_z = 0 \in \mathcal{S}(\bar{z})\). In this case, \(\phi(z) \overset{\text{def}}{=} (z_0 - c_2, z_1, \ldots, z_{n-1})\). Remark that \(z_0 - c_z \in \mathcal{S}(\bar{z})\) and hence \(\text{rk}(\phi(z)) = r\). Also, \(z_0 - c_z = z_0 - c_0 + c_u \in \mathcal{S}(\bar{u})\) and hence \(\text{rk}(\phi(z) - c_2') = s\). Therefore \(\phi(z) \in B_r(c_1') \cap B_s(c_2')\).

We now verify that the mapping \(\phi\) is injective. Suppose there exists \(z'\) such that \(\phi(z') = \phi(z)\). Since \(\phi(z)\) only modifies the first coordinate of \(z\), the last \(n - 1\) coordinates of \(z\) and \(z'\) are equal and so are the last \(n - 1\) coordinates of \(z - c_2\) and \(z' - c_2\). Hence \(z\) and \(z'\) belong to the same case. It can be easily verified that for each case above, \(\phi\) is injective. Hence \(\phi(z') = \phi(z)\) implies that \(z' = z\). Therefore \(\phi\) is injective, and \(|B_r(c_1) \cap B_s(c_2)| \leq |B_r(c_1') \cap B_s(c_2')|\).
Corollary 1: If $0 \leq r, s \leq n$, $c_1, c_2, c_1', c_2' \in \text{GF}(q^m)^n$ and $d_k(c_1, c_2) \geq d_k(c_1', c_2')$, then

$$|B_r(c_1) \cup B_s(c_2)| \geq |B_r(c_1') \cup B_s(c_2')|.$$ (8)

Proof: The result follows from

$$|B_r(c_1) \cup B_s(c_2)| = V_r(q^m, n) + V_s(q^m, n) - |B_r(c_1) \cap B_s(c_2)|.$$ □

We now quantify the volume of the intersection of two balls with rank radii for some special cases, which will be used in Section \( \text{V-B} \).

Proposition 3: If $c_1, c_2 \in \text{GF}(q^m)^n$ and $d_k(c_1, c_2) = r$, then $|B_r(c_1) \cap B_1(c_2)| = 1 + (q^m - q^r)[11] + (q^r - 1)[1]$. Proof: The claim holds for $r = m$ trivially, and we assume $r < m$ henceforth. By Lemma 6, we can assume $c_2 = 0$ and hence $rk(c_1) = r$ without loss of generality. By Lemma 1, the vector $c_1$ belongs to a unique ELS $V \in E_r(q^m, n)$. First of all, it is easy to check that $y = 0 \in B_r(c_1) \cap B_1(0)$. We consider a nonzero vector $y \in B_1(0)$ with rank 1. Firstly, if $y \in V$, then $c_1 - y \in V$. We hence have $rk(c_1 - y) \leq r$ and $y \in B_r(c_1)$. Note that there are $(q^m - 1)[1]$ such vectors. Secondly, if $y \notin V$ and $\mathcal{G}(y) \subseteq \mathcal{G}(c_1)$, then $\mathcal{G}(c_1 - y) \subseteq \mathcal{G}(c_1)$. We hence have $rk(c_1 - y) \leq r$ and $y \in B_r(c_1)$. Note that there are $(q^r - 1)([11] - [1])$ such vectors. Finally, suppose $y \notin V$ and $\mathcal{G}(y) \notin \mathcal{G}(c_1)$. Denote the linearly independent coordinates of $c_1$ as $\alpha_0, \ldots, \alpha_{r-1}$ and a nonzero coordinate of $y$ as $\alpha_r \notin \mathcal{G}(c_1)$, where $\{\alpha_i\}_{i=0}^{m-1}$ is a basis set of $\text{GF}(q^m)$. Then the matrix $C_1 - Y$ obtained by expanding the coordinates of $c_1 - y$ according to the basis $\{\alpha_i\}$ has row rank $r + 1$. Therefore $rk(c_1 - y) = r + 1$, and $y \notin B_r(c_1)$. □

Proposition 4: If $c_1, c_2 \in \text{GF}(q^m)^n$ and $d_k(c_1, c_2) = r$, then $|B_s(c_1) \cap B_{r-s}(c_2)| = q^s(q^{r-s})[rs]$ for $0 \leq s \leq r$.

Proof: By Lemma 6, we can assume that $c_1 = 0$, and hence $rk(c_2) = r$. By Lemma 1, $c_2$ belongs to a unique ELS $V \in E_r(q^m, n)$. We first prove that all vectors $y \in B_s(0) \cap B_{r-s}(c_2)$ are in $V$. Let $y = y_V + y_W$, where $W \in E_{n-r}(q^m, n)$ such that $V \oplus W = \text{GF}(q^m)^n$. We have $y_V + (c_2 - y)_V = c_2$, with $rk(y_V) \leq rk(y) \leq s$ and $rk((c_2 - y)_V) \leq rk(c_2 - y) \leq r - s$. Therefore, $rk(y_V) = rk(y) = s$, $rk((c_2 - y)_V) = rk(c_2 - y) = r - s$, and $\mathcal{G}(y_V) \cap \mathcal{G}((c_2 - y)_V) = \{0\}$. Since $rk(y_V) = rk(y)$, we have $\mathcal{G}(y_W) \subseteq \mathcal{G}(y_V)$; and similarly $\mathcal{G}((c_2 - y)_V) \subseteq \mathcal{G}((c_2 - y)_V)$. Altogether, we obtain $\mathcal{G}(y_W) \cap \mathcal{G}((c_2 - y)_W) = \{0\}$. However, $y_W + (c_2 - y)_W = 0$, and hence $y_V = (c_2 - y)_W = 0$. Therefore, $y \in V$.

We now prove that $y$ is necessarily the projection of $c_2$ onto some ELS $A$ of $V$. If $y \in V$ satisfies $rk(y) = s$ and $rk(c_2 - y) = r - s$, then $y$ belongs to some ELS $A$ and $c_2 - y \in B$ such that $A \oplus B = V$. We hence have $y = c_{2,A}$ and $c_2 - y = c_{2,B}.$

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On the other hand, for any \( A \in E_s(q^m, n) \) and \( B \in E_{r-s}(q^m, n) \) such that \( A \oplus B = V \), \( c_{2,A} \) is a vector of rank \( s \) with distance \( r - s \) from \( c_2 \) by Lemma 3. By Lemma 4 all the \( c_{2,A} \) vectors are distinct. There are thus as many vectors \( y \) as ordered pairs \( (A, B) \). By Lemma 2 there are \( q^s(r-s) \) such pairs, and hence \( q^{s(r-s)} \) vectors \( y \).

As shown in Figure 1 only the outmost layers of two balls of radii \( s \) and \( r - s \) intersect when the distance between the two centers is \( r \). Proposition 4 quantifies the volume of the intersection in Figure 1.

The problem of the intersection of three balls with rank radii is more complicated since the volume of the intersection of three balls with rank radii is not completely determined by the pairwise distances between the centers. We give a simple example to illustrate this point: consider \( GF(2^2)^3 \) and the vectors \( c_1 = c_1' = (0, 0, 0) \), \( c_2 = c_2' = (1, \alpha, 0) \), \( c_3 = (\alpha, 0, 1) \), and \( c_3' = (\alpha, \alpha + 1, 0) \), where \( \alpha \) is a primitive element of the field. It can be verified that \( d_R(c_1, c_2) = d_R(c_2, c_3) = d_R(c_3, c_1) = 2 \) and \( d_R(c_1', c_2') = d_R(c_2', c_3') = d_R(c_3', c_1') = 2 \). However, \( B_1(c_1) \cap B_1(c_2) \cap B_1(c_3) = \{ (\alpha + 1, 0, 0) \} \), whereas \( B_1(c_1') \cap B_1(c_2') \cap B_1(c_3') = \{ (1, 0, 0), (0, \alpha + 1, 0), (\alpha, \alpha, 0) \} \). We remark that this is similar to the problem of the intersection of three balls with Hamming radii discussed in [27, p. 58], provided that the underlying field \( GF(q^m) \) is not \( GF(2) \).

**IV. PACKING PROPERTIES OF RANK METRIC CODES**

Combining (1) and (2) and generalizing slightly to account for nonlinear codes, we can show that the cardinality \( K \) of a code \( C \) over \( GF(q^m) \) with length \( n \) and minimum rank distance \( d_R \) satisfies

\[
K \leq \min \left\{ q^{m(d_R+1)}, q^{n(m-d_R+1)} \right\}. \tag{9}
\]

In this paper, we call the bound in (9) the Singleton bound\(^1\) for codes with the rank metric, and refer to codes that attain the Singleton bound as maximum rank distance (MRD) codes.

\(^1\)The Singleton bound in [7] has a different form since array codes are defined over base fields.
For any given parameter set \( n, m, \) and \( d_R \), explicit construction for MRD codes exists. For \( n \leq m \) and \( d_R \leq n \), generalized Gabidulin codes can be constructed. For \( n > m \) and \( d_R \leq m \), an MRD code can be constructed by transposing a generalized Gabidulin code of length \( m \) and minimum rank distance \( d_R \) over \( \text{GF}(q^m) \), and this code is not necessarily linear over \( \text{GF}(q^m) \). Although maximum distance separable (MDS) codes, which attain the Singleton bound for the Hamming metric, exist only for limited block length over any given field, MRD codes can be constructed for any block length \( n \) and minimum rank distance \( d_R \) over arbitrary fields \( \text{GF}(q^m) \). This has significant impact on the packing properties of rank metric codes as explained below.

The sphere packing problem we consider is as follows: given a finite field \( \text{GF}(q^m) \), length \( n \), and radius \( r \), what is the maximum number of non-intersecting balls with radius \( r \) that can be packed into \( \text{GF}(q^m)^n \)? The sphere packing problem is equivalent to finding the maximum cardinality \( A(q^m, n, d) \) of a code over \( \text{GF}(q^m) \) with length \( n \) and minimum distance \( d \geq 2r + 1 \): the spheres of radius \( r \) centered at the codewords of such a code do not intersect one another. Furthermore, when these non-intersecting spheres centered at all codewords cover the whole space, the code is called a perfect code.

For the Hamming metric, although nontrivial perfect codes do exist, the optimal solution to the sphere packing problem is not known for all the parameter sets [25]. In contrast, for rank metric codes, although nontrivial perfect rank metric codes do not exist [9], [10], we show that MRD codes provide an optimal solution to the sphere packing problem for any set of parameters. For given \( n, m, \) and \( r \), let us denote the maximum cardinality among rank metric codes over \( \text{GF}(q^m) \) with length \( n \) and minimum distance \( d_R \geq 2r + 1 \) as \( A_R(q^m, n, d_R) \). For \( d_R > \min\{n, m\} \), \( A_R(q^m, n, d_R) = 1 \). For \( d_R \leq \min\{n, m\} \), \( A_R(q^m, n, d_R) = \min \{q^m(n-d_R+1), q^n(m-d_R+1)\} \). Note that the maximal cardinality is achieved by MRD codes for all parameter sets. Hence, MRD codes admit the optimal solutions to the sphere packing problem for rank metric codes.

The performance of Hamming metric codes of large block length can be studied in terms of asymptotic bounds on the relative minimum distance in the limit of infinite block length. In this section, we aim to derive the asymptotic form of \( A_R(q^m, n, d_R) \) in the case where both block length and minimum rank distance go to infinity. However, this cannot be achieved for finite \( m \) since the minimum rank distance is no greater than \( m \). Thus, we consider the case where \( \lim_{n \to \infty} \frac{m}{n} = b \), where \( b \) is a constant.

Define \( \delta \doteq \lim_{n \to \infty} \frac{d_R}{n} \) and \( a(\delta) \doteq \lim_{n \to \infty} \sup \left[ \log_{q^m} A_R(q^m, n, \lfloor bn \rfloor) \right] \), where \( a(\delta) \) represents the maximum possible code rate of a code which has relative minimum distance \( \delta \) as its length goes to infinity. We can thus determine the maximum possible code rate \( a(\delta) \) of a code.

**Proposition 5:** For \( 0 \leq \delta \leq \min\{1, b^{-1}\} \), the existence of MRD codes for all parameter sets implies
\[ a(\delta) = \min \{1 - \delta, 1 - b\delta\}. \quad (10) \]

V. COVERING PROPERTIES OF RANK METRIC CODES

A. The sphere covering problem

In this section, we are interested in the sphere covering problem for the rank metric. This problem can be stated as follows: given an extension field \( \text{GF}(q^m) \), length \( n \), and radius \( \rho \), we want to determine the minimum number of balls of rank radius \( \rho \) which cover \( \text{GF}(q^m)^n \) entirely. The sphere covering problem is equivalent to finding the minimum cardinality \( K_R(q^m, n, \rho) \) of a code over \( \text{GF}(q^m) \) with length \( n \) and rank covering radius \( \rho \). We remark that if \( C \) is a code over \( \text{GF}(q^m) \) with length \( n \) and covering radius \( \rho \), then its transpose code \( C^T \) is a code over \( \text{GF}(q^n) \) with length \( m \) and the same covering radius. Therefore, \( K_R(q^m, n, \rho) = K_R(q^n, m, \rho) \), and without loss of generality we shall assume \( n \leq m \) henceforth in this section.

We remark that \( K_R(q^m, n, 0) = q^{mn} \) and \( K_R(q^m, n, n) = 1 \) for all \( m \) and \( n \). Hence we assume \( 0 < \rho < n \) throughout this section. Two bounds on \( K_R(q^m, n, \rho) \) can be easily derived.

**Proposition 6:** For a code over \( \text{GF}(q^m) \) with length \( n \) and covering radius \( 0 < \rho < n \), we have

\[
\left\lfloor \frac{q^{mn}}{V_\rho(q^m, n)} \right\rfloor + 1 \leq K_R(q^m, n, \rho) \leq q^{m(n-\rho)}.
\]

**Proof:** The lower bound is a straightforward generalization of the bound given in [12]. Note that the only codes with cardinality \( \frac{q^{mn}}{V_\rho(q^m, n)} \) are perfect codes. However, there are no nontrivial perfect codes for the rank metric [9]. Therefore, \( K_R(q^m, n, \rho) > \frac{q^{mn}}{V_\rho(q^m, n)} \). The upper bound follows from \( \rho \leq n - k \) for any \((n, k)\) linear code [27], and hence any linear code with covering radius \( \rho \) has cardinality \( \leq q^{m(n-\rho)} \).

We refer to the lower bound in (11) as the sphere covering bound.

For a code over \( \text{GF}(q^m) \) with length \( n \) and covering radius \( 0 < \rho < n \), we have \( K_R(q^m, n, \rho) \leq K_H(q^m, n, \rho) \), where \( K_H(q^m, n, \rho) \) is the minimum cardinality of a (linear or nonlinear) code over \( \text{GF}(q^m) \) with length \( n \) and Hamming covering radius \( \rho \). This is because any code with Hamming covering radius \( \rho \) has rank covering radius \( \leq \rho \). Since \( K_H(q^m, n, \rho) \leq q^{m(n-\rho)} \), this provides a tighter bound than the one given in Proposition 6.

**Lemma 7:** For all \( m > 0 \) and nonnegative \( n, n', \rho, \) and \( \rho' \), we have

\[
K_R(q^m, n + n', \rho + \rho') \leq K_R(q^m, n, \rho) K_R(q^m, n', \rho').
\]

(12)
In particular, we have

\[ K_r(q^m, n + 1, \rho + 1) \leq K_r(q^m, n, \rho), \tag{13} \]

\[ K_r(q^m, n + 1, \rho) \leq q^m K_r(q^m, n, \rho). \tag{14} \]

**Proof:** First, (12) follows directly from [12, Proposition 4]. In particular, when \((n', \rho') = (1, 1)\) and \((n', \rho') = (1, 0)\), we obtain \((13)\) and \((14)\) respectively. \[\blacksquare\]

**B. Lower bounds for the sphere covering problem**

We will derive two nontrivial lower bounds on \(K_r(q^m, n, \rho)\). First, we adapt the bound given in [34, Theorem 1].

**Proposition 7:** For all \(q^m, n, \) and \(0 < \rho < n\), we have

\[ K_r(q^m, n, \rho) \geq \left\lfloor \frac{q^{mn} - A_k(q^m, n, 2\rho + 1)q^{\rho^2}\left[\frac{2\rho}{\rho}\right]}{V_{\rho}(q^m, n) - q^{\rho^2}\left[\frac{2\rho}{\rho}\right]} \right\rfloor, \tag{15} \]

provided that the denominator on the right hand side (RHS) is positive.

**Proof:** Suppose \(C\) is a code over \(GF(q^m)\) with length \(n\) and rank covering radius \(\rho\), and let \(C_0\) be a maximal subcode of \(C\) with minimum rank distance \(d' \geq 2\rho + 1\). If \(d' > n\), we choose \(C_0\) to be a single codeword. \(C_0\) thus covers \(|C_0|V_{\rho}(q^m, n)\) vectors. Define \(C_1 = C \setminus C_0\) (\(C_1\) is not empty, otherwise \(C\) would be a nontrivial perfect code) and for any \(c_1 \in C_1\), let \(f(c_1)\) denote the number of vectors covered by \(c_1\) which are not covered by \(C_0\). Since \(C_0\) is maximal, there exists at least one codeword \(c_0 \in C_0\) such that \(d_r(c_0, c_1) \leq 2\rho\). We have \(f(c_1) \leq V_{\rho}(q^m, n) - q^{\rho^2}\left[\frac{2\rho}{\rho}\right]\), where the equality corresponds to when there is only one such \(c_0\) and \(d_r(c_0, c_1) = 2\rho\) by Proposition [2]. In that case, Proposition [4] implies that there are \(q^{\rho^2}\left[\frac{2\rho}{\rho}\right]\) vectors covered by both \(c_0\) and \(c_1\). Thus, we have

\[ q^{mn} \leq |C_0|V_{\rho}(q^m, n) + \sum_{c_1 \in C_1} f(c_1) \leq |C_0|V_{\rho}(q^m, n) + (|C| - |C_0|) \left( V_{\rho}(q^m, n) - q^{\rho^2}\left[\frac{2\rho}{\rho}\right]\right) = |C| \left( V_{\rho}(q^m, n) - q^{\rho^2}\left[\frac{2\rho}{\rho}\right]\right) + |C_0|q^{\rho^2}\left[\frac{2\rho}{\rho}\right]. \]

We have \(|C_0| \leq A_k(q^m, n, d') \leq A_k(q^m, n, 2\rho + 1)\), and the result follows. \[\blacksquare\]

We remark that \(A_k(q^m, n, 2\rho + 1) = q^{m(n-2\rho)}\) if \(2\rho + 1 \leq n\), or 1 otherwise. Next, we obtain both sufficient and necessary conditions under which the bound is nontrivial, i.e., when the denominator on the RHS of (15) is positive.

**Lemma 8:** The denominator on the RHS of (15) is positive if \(\rho(m + n - 3\rho) \geq \sigma(q)\). Also, the denominator in (15) is positive only if \(m + n \geq 3\rho\).
Proof: We first prove the sufficient condition. We need to show \( V_\rho(q^m, n) > q^{\rho^2 \lceil \frac{2}{\rho} \rceil} \). By Lemma 5, \( V_\rho(q^m, n) \geq q^{\rho(m+n-\rho)} \). By [17, Lemma 1], we have \( \lceil \frac{2}{\rho} \rceil < q^{\rho^2 + \sigma(q)} \). Therefore, the denominator in (15) is positive if \( \rho(m + n - \rho) \geq 2\rho^2 + \sigma(q) \).

We now prove the necessary condition. Note that \( \alpha(n, \rho) \leq q^{n \rho} \) and \( \alpha(2\rho, \rho) \geq q^{2\rho^2 - \tau(q)} \), where \( \tau(q) = \log_q \left( \frac{q^2}{q-1} \right) \) [17, Lemma 2]. Now suppose \( \rho(m + n - 3\rho) < -\tau(q) \), then \( q^{\rho(m+n-3\rho)+\tau(q)} < 1 \). This implies \( \frac{\alpha(n, \rho)}{\alpha(2\rho, \rho)} q^{\rho(m-\rho)} < 1 \), and hence \( \lceil n \rceil q^{m \rho} < \lceil \frac{2}{\rho} \rceil q^{\rho^2} \). By [17, Lemma 13], we obtain \( V_\rho(q^m, n) \leq \lceil n \rceil q^{m \rho} < q^{\rho^2 \lceil \frac{2}{\rho} \rceil} \). Therefore, \( V_\rho(q^m, n) - q^{\rho^2 \lceil \frac{2}{\rho} \rceil} > 0 \) only if \( \rho(m + n - 3\rho) \geq -\tau(q) \).

Finally, \( 0 < \tau(q) < 1 \) for \( q \geq 2 \) and hence \( \rho(m + n - 3\rho) \geq 0 \).

Before deriving the second nontrivial lower bound, we need the following adaptation of [31, Lemma 8]. Let \( C \) be a code with length \( n \) and rank covering radius \( \rho \) over \( GF(q^m) \). We define \( A \equiv \{ x \in GF(q^m)^n | d_k(x, C) = \rho \} \).

Lemma 9: For \( x \in A \setminus Z \) and \( 0 < \rho < n \), we have

\[
E_C(B_1(x)) \geq \epsilon,
\]

where \( \epsilon \equiv \left\lfloor \frac{(q^m - q^\rho)(\lceil n \rceil - \lceil \rho \rceil)}{q^\rho \lceil \rho + 1 \rceil} \right\rfloor + (q^m - q^\rho) \left( \lceil \rho \rceil - \lceil n \rceil \right) \).

Proof: Since \( x \notin Z \), there is a unique \( c_0 \in C \) such that \( d_k(x, c_0) = \rho \). By Proposition 3, we have \( |B_\rho(c_0) \cap B_1(x)| = 1 + (q^m - q^\rho) \left( \lceil \rho \rceil \right) + (q^\rho - 1) \left( \lceil n \rceil \right) \). For any codeword \( c_1 \in C \) satisfying \( d_k(x, c_1) = \rho + 1 \), by Proposition 4, we have \( |B_\rho(c_1) \cap B_1(x)| = q^\rho \left( \lceil \rho + 1 \rceil \right) \). Finally, for all other codewords \( c_2 \in C \) at distance \( > \rho + 1 \) from \( x \), we have \( |B_\rho(c_2) \cap B_1(x)| = 0 \). Denoting \( N \equiv \{ c_1 \in C | d_k(x, c_1) = \rho + 1 \} \), we obtain

\[
E_C(B_1(x)) = \sum_{c \in C} |B_\rho(c) \cap B_1(x) - |B_1(x)|
\]

\[
= (q^m - q^\rho) \left( \lceil \rho \rceil \right) + N q^\rho \left( \lceil \rho + 1 \rceil \right) - \left( \lceil n \rceil \right) (q^m - q^\rho)
\]

\[
\equiv (q^m - q^\rho) \left( \lceil \rho \rceil - \lceil n \rceil \right) \mod \left( q^\rho \left( \lceil \rho + 1 \rceil \right) \right).
\]

The proof is completed by realizing that \( (q^m - q^\rho) \left( \lceil n \rceil - \lceil \rho \rceil \right) < 0 \), while \( E_C(B_1(x)) \) is a non-negative integer.

Proposition 8: If \( \epsilon > 0 \), then

\[
K_R(q^m, n, \rho) \geq \left\lfloor \frac{q^{mn}}{V_\rho(q^m, n) - \frac{1}{2} N_\rho(q^m, n)} \right\rfloor,
\]

where \( \delta \equiv V_1(q^m, n) - q^{\rho - 1} \lceil \rho \rceil - 1 + 2\epsilon \).
The proof of Proposition 8 provided in Appendix A uses the approach in the proof of [31, Theorem 6] and is based on the concept of excess reviewed in Section II-C. We remark that, unlike the bound given in Proposition 7, the bound in Proposition 8 is always applicable. The lower bounds in (15) and (17), when applicable, are at least as tight as the sphere covering bound given in (11).

C. Upper bounds for the sphere covering problem

From the perspective of covering, the following lemma gives a characterization of MRD codes in terms of ELS’s.

Lemma 10: Let \( C \) be an \((n, k)\) linear code over \( \text{GF}(q^m) \) \((n \leq m)\). \( C \) is an MRD code if and only if \( C \oplus V = \text{GF}(q^m)^n \) for all \( V \in E_{n-k}(q^m, n) \).

Proof: Suppose \( C \) is an \((n, k, n-k+1)\) MRD code. It is clear that \( C \cap V = \{0\} \) and hence \( C \oplus V = \text{GF}(q^m)^n \) for all \( V \in E_{n-k}(q^m, n) \).

Conversely, suppose \( C \oplus V = \text{GF}(q^m)^n \) for all \( V \in E_{n-k}(q^m, n) \). Then \( C \) does not contain any nonzero codeword of weight \( \leq n-k \), and hence its minimum distance is \( n-k+1 \).

Let \( \alpha_0, \alpha_1, \ldots, \alpha_{m+\rho-1} \in \text{GF}(q^{m+\rho}) \) be a basis set of \( \text{GF}(q^{m+\rho}) \) over \( \text{GF}(q) \), and let \( \beta_0, \beta_1, \ldots, \beta_{m-1} \) be a basis of \( \text{GF}(q^n) \) over \( \text{GF}(q) \). We define the linear mapping \( f \) between two vector spaces \( \text{GF}(q^m) \) and \( \mathcal{S}_m \) by \( f(\beta_i) = \alpha_i \) for \( 0 \leq i \leq m-1 \). This can be generalized to \( n \)-dimensional vectors, by applying \( f \) componentwise. We thus define \( \bar{f} : \text{GF}(q^m)^n \rightarrow \text{GF}(q^{m+\rho})^n \) such that for any \( v = (v_0, \ldots, v_{n-1}) \), \( \bar{f}(v) = (f(v_0), \ldots, f(v_{m-1})) \). This function \( \bar{f} \) is a linear bijection from \( \text{GF}(q^m)^n \) to its image \( \mathcal{S}_m^n \).

Lemma 11: For all \( r \) and any \( V \in E_r(q^m, n) \), \( \bar{f}(V) \subset W \), where \( W \in E_r(q^{m+\rho}, n) \).

Proof: We first show that \( \bar{f} \) preserves the rank. Suppose \( u \in \text{GF}(q^m)^n \). Let us denote the matrix formed after extending the coordinates of \( u \) with respect to the basis \( \{\beta_i\} \) as \( U \). The extension of \( \bar{f}(u) \) with respect to the basis \( \{\alpha_i\} \) is given by \( \bar{U} = \begin{pmatrix} U \\ 0 \end{pmatrix} \). We thus have \( \text{rk}(\bar{U}) = \text{rk}(U) \), and \( \text{rk}(\bar{f}(u)) = \text{rk}(u) \).

Let \( B = \{b_i\} \) be a basis of \( V \in E_r(q^m, n) \) with vectors of rank one. Then for all \( i \), \( \bar{f}(b_i) \) has rank one and \( \{\bar{f}(b_i)\} \) form a basis, and hence \( \bar{f}(V) \subset W \), where \( W \in E_r(q^{m+\rho}, n) \) with \( \{\bar{f}(b_i)\} \) as a basis.

Proposition 9: Let \( C \) be an \((n, n-\rho, \rho+1)\) MRD code with covering radius \( \rho \). Then the code \( \bar{f}(C) \) is a code of length \( n \) over \( \text{GF}(q^{m+\rho}) \) with cardinality \( q^n(n-\rho) \) and covering radius \( \rho \).

Proof: The other parameters for the code are obvious, and it suffices to establish the covering radius. Let \( \mathcal{I}_\rho \) be a subspace of \( \text{GF}(q^{m+\rho}) \) with dimension \( \rho \) such that \( \mathcal{S}_m \oplus \mathcal{I}_\rho = \text{GF}(q^{m+\rho}) \). Any
\( u \in GF(q^{m+\rho})^n \) can be expressed as \( u = v + w \), where \( v \in \mathbb{F}_q^n \) and \( w \in \mathbb{F}_q^n \). Hence \( \text{rk}(w) \leq \rho \), and \( w \in W \) for some \( W \in E_\rho(q^{m+\rho}, n) \). By Lemma 10 we can express \( v \) as \( v = \bar{\bar{f}}(c + e) = \bar{f}(c) + \bar{\bar{f}}(e) \), where \( c \in C \) and \( e \in V \), such that \( \bar{f}(V) \subset W \). Eventually, we have \( u = \bar{f}(c) + \bar{\bar{f}}(e) + w \), where \( \bar{f}(e) + w \in W \), and thus \( d(u, \bar{f}(c)) \leq \rho \). Thus \( \bar{f}(C) \) has covering radius \( \leq \rho \). Finally, it is easy to verify that the covering radius of \( \bar{f}(C) \) is exactly \( \rho \).

**Corollary 2:** We have

\[
K_r(q^m, n, \rho) \leq q^{\max\{m-\rho, n\}(n-\rho)}.
\] (18)

**Proof:** We can construct an \((n, n - \rho)\) MRD code \( C \) over \( GF(q^m) \) with covering radius \( \rho \), where \( \mu = \max\{m - \rho, n\} \). By Proposition 9 \( \bar{f}(C) \), where \( \bar{f} \) maps \( GF(q^m)^n \) into a subset of \( GF(q^m)^n \), has covering radius \( \rho \). Note that \( |\bar{f}(C)| = |C| = q^\mu(n-\rho) \).

We can use the properties of \( K_r(q^m, n, \rho) \) in Lemma 7 in order to obtain two tighter bounds when \( \rho \geq m - n \).

**Proposition 10:** Given fixed \( m \), \( n \), and \( \rho \), for any \( n \geq l > 0 \) and \((n_i, \rho_i)\) for \( 0 \leq i \leq l - 1 \) so that

\[
0 < n_i \leq n, 0 \leq \rho_i \leq n_i, \text{ and } n_i + \rho_i \leq m \text{ for all } i, \text{ and } \sum_{i=0}^{l-1} n_i = n \text{ and } \sum_{i=0}^{l-1} \rho_i = \rho,
\]

we have

\[
K_r(q^m, n, \rho) \leq \min_{\{n_i, \rho_i\}_0} \left\{ q^{m(n-\rho)-\sum \rho_i(n_i-\rho_i)} \right\}.
\] (19)

**Proof:** By Lemma 7, we have \( K_r(q^m, n, \rho) \leq \prod_i K_r(q^m, n_i, \rho_i) \) for all possible sequences \( \{\rho_i\} \) and \( \{n_i\} \). For all \( i \), we have \( K_r(q^m, n_i, \rho_i) \leq q^{(m-\rho_i)(n-\rho_i)} \) by Corollary 2 and hence \( K_r(q^m, n, \rho) \leq q^{\sum (m-\rho_i)(n_i-\rho_i)} = q^{m(n-\rho)-\sum \rho_i(n_i-\rho_i)} \).

It is clear that the upper bound in (19) is tighter than the upper bound in (18). It can also be shown that it is tighter than the bound in (13).

The following upper bound is an adaptation of [27, Theorem 12.1.2].

**Proposition 11:** For any \( m, n \leq m \), and \( \rho < n \), there exists a code over \( GF(q^m) \) of length \( n \) and covering radius \( \rho \) with cardinality

\[
K_r(q^m, n, \rho) \leq \left\lfloor \frac{1}{1 - \log_{q^m} (q^{mn} - V_\rho(q^m, n))} \right\rfloor + 1.
\] (20)

Our proof, given in Appendix B, adopts the approach used to prove [27, Theorem 12.1.2].

**Proposition 12:** For all \( m, n \leq m \), \( \rho < n \), we have

\[
K_r(q^m, n, \rho) \leq \frac{q^{mn}}{V_\rho(q^m, n)} [1 + \ln(V_\rho(q^m, n))].
\] (21)

**Proof:** Consider the square 0-1 matrix \( A \) of order \( q^{mn} \), where each row and each column represents a vector in \( GF(q^n)^n \). Set \( a_{i,j} = 1 \) if and only if the sphere with rank radius \( \rho \) centered at vector \( i \) covers the vector \( j \). There are thus exactly \( V_\rho(q^m, n) \) ones in each row and each column of \( A \). Note that any \( q^{mn} \times K \) submatrix \( C \) of \( A \) with no all-zeros rows represents a code with cardinality \( K \) and covering
radius $\rho$. Applying the Johnston-Stein-Lovász theorem [27, Theorem 12.2.1] to $A$, we can find such a submatrix with $K \leq \frac{1}{V_\rho(q^m,n)} [q^{mn} + q^{mn} \ln(V_\rho(q^m,n))]$.

The tightest bounds on $K_\rho(q^m,n,\rho)$ known so far are given in Table II for $q = 2, 2 \leq m \leq 7, 2 \leq n \leq m$, and $1 \leq \rho \leq 6$.

D. Covering properties of linear rank metric codes

For a linear code with given covering radius, the sphere covering bound also implies a lower bound on its dimension.

**Proposition 13:** An $(n,k)$ linear code over $GF(q^m)$ with rank covering radius $\rho$ satisfies

$$n - \rho - \frac{\rho(n - \rho) + \sigma(q)}{m} + 1 \leq k \leq n - \rho. \quad (22)$$

**Proof:** The upper bound directly follows the upper bound in (11). We now prove the lower bound. By the sphere covering bound, we have $q^{mk} > \frac{q^{mn}}{V_\rho(q^m,n)}$. However, by Lemma 5 we have $V_\rho(q^m,n) < q^{\rho(m+n-\rho)+\sigma(q)}$ and hence $q^{mk} > q^{mn-\rho(m+n-\rho)-\sigma(q)}$.

We do not adapt the bounds in (15) and (17) as their advantage over the lower bound in (22) is not significant. Table II lists the values of the bounds in Proposition 13 on the dimension of a linear code with given covering radius for $4 \leq m \leq 8$ and $4 \leq n \leq m$. Note that only $1 < \rho < n - 1$ are considered in Table II since, as shown below, the dimension of a linear code with given covering radius can be completely determined when $\rho \in \{0,1,n-1,n\}$.

**Proposition 14:** Let $C$ be an $(n,k)$ linear code over $GF(q^m)$ ($n \leq m$) with rank covering radius $\rho$. Then $k = n - \rho$ if $\rho \in \{0,1,n-1,n\}$ or $\rho(n-\rho) \leq m - \sigma(q)$, or if $C$ is a generalized Gabidulin code or an ELS.

**Proof:** The cases $\rho \in \{0,1,n-1,n\}$ are straightforward. In all other cases, since $k \leq n - \rho$ by Proposition 13 it suffices to prove that $k \geq n - \rho$. First, suppose $\rho = 1$, then $k$ satisfies $q^{mk} > \frac{q^{mn}}{V_\rho(q^m,n)}$ by the sphere covering bound. However, $V_1(q^m,n) < q^{m+n} \leq q^{2m}$, and hence $k > n - 2$. Second, if $\rho(n-\rho) \leq m - \sigma(q)$, then $0 < \frac{1}{m} (\rho(n-\rho) + \sigma(q)) \leq 1$ and $k \geq n - \rho$ by Proposition 13. Third, if $C$ is an $(n,k,n-k+1)$ generalized Gabidulin code with $k < n$, then there exists an $(n,k+1,n-k)$ generalized Gabidulin code $C'$ such that $C \subset C'$. We have $\rho \geq d_k(C') = n - k$, as noted in Section II-C and hence $k \geq n - \rho$. The case $k = n$ is straightforward. Finally, if $C$ is an ELS of dimension $k$, then for all $x$ with rank $n$ and for any $e \in C$, $d_k(x,e) \geq r_k(x) - r_k(e) \geq n - k$.

A similar argument can be used to bound the covering radius of the cartesian products of generalized Gabidulin codes.
Corollary 3: Let $G$ be an $(n, k, d_k)$ generalized Gabidulin code $(n \leq m)$, and let $G^l$ be the code obtained by $l$ cartesian products of $G$ for $l \geq 1$. Then the rank covering radius of $G^l$ satisfies $\rho(G^l) \geq d_k - 1$.

Note that when $n = m$, $G^l$ is a maximal code, and hence Corollary 3 can be further strengthened.

Corollary 4: Let $G$ be an $(m, k, d_k)$ generalized Gabidulin code over $GF(q^m)$, and let $G^l$ be the code obtained by $l$ cartesian products of $G$. Then $\rho(G^l) = d_k - 1$.

The tightest bounds for the dimension of linear codes with given covering radius known so far are given in Table II for $q = 2$, $4 \leq m \leq 8$, $4 \leq n \leq m$, and $2 \leq \rho \leq 6$.

E. Asymptotic covering properties

Table II provides solutions to the sphere covering problem for only small values of $m$, $n$, and $\rho$. Next, we study the asymptotic covering properties when both blocklength and minimum rank distance go to infinity. As in Section IV, we consider the case where $\lim_{n \to \infty} \frac{n}{m} = b$, where $b$ is a constant. In other words, these asymptotic covering properties provide insights on the covering properties of long rank metric codes over large fields.

The asymptotic form of the bounds in (6) are given in the lemma below.

Lemma 12: For $0 \leq \delta \leq \min\{1, b^{-1}\}$, $\nu(\delta) \triangleq \lim_{n \to \infty} \left[ \frac{\log_q V_{[\delta m]}(q^m, n)}{n} \right] = \delta(1 + b - b\delta)$.

Proof: By Lemma 5 we have $q^{d_k(m+n-d_k)} \leq V_{d_k}(q^m, n) < q^{d_k(m+n-d_k)+\sigma(q)}$. Taking the logarithm and dividing by $n$, this becomes $\delta(1 + b - b\delta) \leq \log_q V_{[\delta m]}(q^m, n)/n < \delta(1 + b - b\delta) + \frac{\sigma(q)}{mn}$. The proof is concluded by taking the limit when $n$ tends to infinity.

Define $r \triangleq \frac{a}{n}$ and $k(r) = \lim_{n \to \infty} \inf \left[ \frac{\log_q K_{[\rho]}(q^m, n, \rho)}{n} \right]$. The bounds in (11) and (21) together solve the asymptotic sphere covering problem.

Theorem 1: For all $b$ and $r$, we have

$$k(r) = (1 - r)(1 - br). \quad (23)$$

Proof: By Lemma 12 the sphere covering bound in (11) asymptotically becomes $k(r) \geq (1 - r)(1 - br)$. Also, from the bound in (21), we have

$$K_{[\rho]}(q^m, n, \rho) \leq \frac{q^{mn}}{V_{[\rho]}(q^m, n)} \left[ 1 + \ln(V_{[\rho]}(q^m, n)) \right]$$

$$\leq \frac{q^{mn}}{V_{[\rho]}(q^m, n)} \left[ 1 + mn \ln(q) \right]$$

$$\log_{q^{mn}} K_{[\rho]}(q^m, n, \rho) \leq \log_{q^{mn}} \frac{q^{mn}}{V_{[\rho]}(q^m, n)} + O((mn)^{-1} \ln(mn)).$$

By Lemma 12 this asymptotically becomes $k(r) \leq (1 - r)(1 - br)$. Note that although we assume $n \leq m$ above for convenience, both bounds in (11) and (21) hold for any values of $m$ and $n$. \n
VI. MACWilliams identity

For all $v \in \text{GF}(q^n)^n$ with rank weight $r$, the rank weight function of $v$ is defined as $f_r(v) = y^rx^{n-r}$. Let $C$ be a code of length $n$ over $\text{GF}(q^n)$. Suppose there are $A_i$ codewords in $C$ with rank weight $i$ ($0 \leq i \leq n$), then the rank weight enumerator of $C$, denoted as $W^n_C(x, y)$, is defined to be

$$W^n_C(x, y) = \sum_{v \in C} f_r(v) = \sum_{i=0}^{n} A_i y^i x^{n-i}. $$

A. $q$-product of homogeneous polynomials

Let $a(x, y; m) = \sum_{i=0}^{r} a_i(m) y^i x^{r-i}$ and $b(x, y; m) = \sum_{j=0}^{s} b_j(m) y^j x^{s-j}$ be two homogeneous polynomials in $x$ and $y$ of degrees $r$ and $s$ respectively with coefficients $a_i(m)$ and $b_j(m)$ respectively. $a_i(m)$ and $b_j(m)$ for $i, j \geq 0$ in turn are real functions of $m$, and are assumed to be zero unless otherwise specified.

**Definition 1 ($q$-product):** The $q$-product of $a(x, y; m)$ and $b(x, y; m)$ is defined to be the homogeneous polynomial of degree $(r+s)$ $c(x, y; m) = a(x, y; m) \ast b(x, y; m) = \sum_{u=0}^{r+s} c_u(m) y^u x^{r+s-u}$, with

$$c_u(m) = \sum_{i=0}^{u} q^i s a_i(m) b_{u-i}(m-i). $$

(24)

We shall denote the $q$-product by $\ast$ henceforth. For $n \geq 0$ the $n$-th $q$-power of $a(x, y; m)$ is defined recursively: $a(x, y; m)^{[0]} = 1$ and $a(x, y; m)^{[n]} = a(x, y; m)^{[n-1]} \ast a(x, y; m)$ for $n \geq 1$.

We provide some examples to illustrate the concept. It is easy to verify that $x \ast y = xy$, $y \ast x = qyx$, $yx \ast x = qyx^2$, and $yx \ast (q^m - 1)y = (q^m - q)y^2x$. Note that $x \ast y \neq y \ast x$. It is easy to verify that the $q$-product is neither commutative nor distributive in general. However, it is commutative and distributive in some special cases as described below.

**Lemma 13:** Suppose $a(x, y; m) = a$ is a constant independent from $m$, then $a(x, y; m) \ast b(x, y; m) = b(x, y; m) \ast a(x, y; m) = ab(x, y; m)$. Also, if $\deg[c(x, y; m)] = \deg[a(x, y; m)]$, then $[a(x, y; m) + c(x, y; m)] \ast b(x, y; m) = a(x, y; m) \ast b(x, y; m) + c(x, y; m) \ast b(x, y; m)$, and $b(x, y; m) \ast [a(x, y; m) + c(x, y; m)] = b(x, y; m) \ast a(x, y; m) + b(x, y; m) \ast c(x, y; m)$.

The homogeneous polynomials $a_i(x, y; m) = [x + (q^m - 1)y]^{[i]}$ and $b_i(x, y; m) = (x - y)^{[i]}$ are very important to our derivations below. The following lemma provides the analytical expressions of $a_i(x, y; m)$ and $b_i(x, y; m)$.
Lemma 14: For \( i \geq 0 \), \( \sigma_i \overset{\text{def}}{=} \frac{i(i-1)}{2} \). For \( l \geq 0 \), we have \( y[l] = q^\sigma_l y^l \) and \( x[l] = x^l \). Furthermore,

\[
a_l(x, y; m) = \sum_{u=0}^{l} \binom{l}{u} \alpha(m, u)y^ux^{l-u},
\]

\[
b_l(x, y; m) = \sum_{u=0}^{l} \binom{l}{u} (-1)^u q^{\sigma_u} y^ux^{l-u}.
\]

Note that \( a_l(x, y; m) \) is the rank weight enumerator of \( GF(q^m) \). The proof of Lemma 14 is given in Appendix [C].

Definition 2 (q-transform): We define the q-transform of \( a(x, y; m) = \sum_{i=0}^{r} a_i(m) y^i x^{r-i} \) as the homogeneous polynomial \( \bar{a}(x, y; m) = \sum_{i=0}^{r} a_i(m) y[i]^\ast x[r-i] \).

Definition 3 (q-derivative [35]): For \( q \geq 2 \), the q-derivative at \( x \neq 0 \) of a real-valued function \( f(x) \) is defined as

\[
f(1)(x) \overset{\text{def}}{=} \frac{f(qx) - f(x)}{(q-1)x}.
\]

For any real number \( a \), \( f(x + ag(x))^{(1)} = f(1)(x) + ag(1)(x) \) for \( x \neq 0 \). For \( \nu \geq 0 \), we shall denote the \( \nu \)-th q-derivative (with respect to \( x \)) of \( f(x, y) \) as \( f^{(\nu)}(x, y) \). The 0-th q-derivative of \( f(x, y) \) is defined to be \( f(x, y) \) itself.

Lemma 15: For \( 0 \leq \nu \leq l \), \( (x^l)^{(\nu)} = \beta(l, \nu)x^{l-\nu} \). The \( \nu \)-th q-derivative of \( f(x, y) = \sum_{i=0}^{r} f_i y^i x^{r-i} \) is given by \( f^{(\nu)}(x, y) = \sum_{i=0}^{r-l-\nu} f_i \beta(i, \nu) y^i x^{r-i\nu} \). Also,

\[
a^{(\nu)}_l(x, y; m) = \beta(l, \nu) a_{l-\nu}(x, y; m)
\]

\[
b^{(\nu)}_l(x, y; m) = \beta(l, \nu) b_{l-\nu}(x, y; m).
\]

The proof of Lemma 15 is given in Appendix [D].

Lemma 16 (Leibniz rule for the q-derivative): For two homogeneous polynomials \( f(x, y) = \sum_{i=0}^{r} f_i y^i x^{r-i} \) and \( g(x, y) = \sum_{j=0}^{s} g_j y^j x^{s-j} \) with degrees \( r \) and \( s \) respectively, the \( \nu \)-th \( (\nu \geq 0) \) q-derivative of their q-product is given by

\[
[f(x, y) * g(x, y)]^{(\nu)} = \sum_{l=0}^{\nu} \binom{\nu}{l} q^{(\nu-l)(r-l)} f^{(l)}(x, y) * g^{(\nu-l)}(x, y).
\]

The proof of Lemma 16 is given in Appendix [E]. Lemma 16 gives the \( \nu \)-th q-derivative of q-products of homogeneous polynomials.

The \( q^{-1} \)-derivative is similar to the q-derivative.

Definition 4 (q^{-1}-derivative): For \( q \geq 2 \), the q-derivative at \( y \neq 0 \) of a real-valued function \( g(y) \) is defined as

\[
g^{(1)}(y) = \frac{g(q^{-1}y) - g(y)}{(q^{-1} - 1)y}.
\]
For \( \nu \geq 0 \), we shall denote the \( \nu \)-th \( q^{-1} \)-derivative (with respect to \( y \)) of \( g(x, y) \) as \( g^{(\nu)}(x, y) \). The 0-th \( q^{-1} \)-derivative of \( g(x, y) \) is defined to be \( g(x, y) \) itself.

**Lemma 17:** For \( 0 \leq \nu \leq l \), the \( \nu \)-th \( q^{-1} \)-derivative of \( y^l \) is \( (y^l)^{(\nu)} = q^{\nu(1-n)+\sigma_{\nu}} \beta(l, \nu) y^{l-\nu} \). Also,
\[
\begin{align*}
a_l^{(\nu)}(x, y; m) &= \beta(l, \nu) q^{-\sigma_{\nu}} \alpha(m, \nu) a_{l-\nu}(x, y; m - \nu) \tag{30} \\
b_l^{(\nu)}(x, y; m) &= (-1)^{\nu} \beta(l, \nu) b_{l-\nu}(x, y; m). \tag{31}
\end{align*}
\]

The proof of Lemma 17 is given in Appendix F.

**Lemma 18 (Leibniz rule for the \( q^{-1} \)-derivative):** For two homogeneous polynomials \( f(x, y; m) = \sum_{i=0}^{r} f_i y^r x^{r-i} \) and \( g(x, y; m) = \sum_{j=0}^{s} g_j y^s x^{s-j} \) with degrees \( r \) and \( s \) respectively, the \( \nu \)-th (\( \nu \geq 0 \)) \( q^{-1} \)-derivative of their \( q \)-product is given by
\[
[f(x, y; m) \ast g(x, y; m)]^{(\nu)} = \sum_{l=0}^{\nu} \begin{pmatrix} \nu \\ l \end{pmatrix} q^{l(s-n)+l} f^{(l)}(x, y; m) \ast g^{(\nu-l)}(x, y; m - l). \tag{32}
\]

The proof of Lemma 18 can be found in Appendix G.

**B. The dual of a vector**

For all \( u, v \in GF(q^m)^n \), let \( u \cdot v \) denote the standard inner product of \( u \) and \( v \). For any linear subspace \( L \subseteq GF(q^m)^n \), \( L^\perp \) is defined as \( \{ u \in GF(q^m)^n | u \cdot v = 0 \ \forall v \in L \} \) and referred to as the dual of \( L \).

As an important step toward our main result, we derive the rank weight enumerator of \( \langle v \rangle^\perp \), where \( v \in GF(q^m)^n \) is an arbitrary vector and \( \langle v \rangle \) is defined as \( \{ av : a \in GF(q^m) \} \). Note that \( \langle v \rangle \) can be viewed as an \( (n, 1) \) linear code over \( GF(q^m) \) with a generator matrix \( v \). It is remarkable that the rank weight enumerator of \( \langle v \rangle^\perp \) depends only on the rank of \( v \).

Berger [36] has determined that the linear isometries for the rank distance are given by the scalar multiplication by a non-zero element of \( GF(q^m) \), and multiplication on the right by a nonsingular matrix \( B \in GF(q)^{n \times n} \). We say that two codes \( C \) and \( C' \) are rank-equivalent if there exists a linear isometry \( f \) for the rank distance such that \( f(C) = C' \).

**Lemma 19:** Suppose \( v \) has rank \( r \geq 1 \). Then \( L = \langle v \rangle^\perp \) is rank-equivalent to \( C \times GF(q^m)^{n-r} \), where \( C \) is an \( (r, r-1, 2) \) MRD code and \( \times \) denotes cartesian product.

**Proof:** We can express \( v \) as \( v = \tilde{v}B \), where \( \tilde{v} = (v_0, \ldots, v_{r-1}, 0 \ldots, 0) \) has rank \( r \), and \( B \in GF(q)^{n \times n} \) has full rank. Remark that \( \tilde{v} \) is the parity-check of the code \( C \times GF(q^m)^{n-r} \), where \( C = \langle (v_0, \ldots, v_{r-1}) \rangle^\perp \) is an \( (r, r-1, 2) \) MRD code. It can be easily checked that \( u \in L \) if and only if \( \tilde{u} \equiv uB^T \in \langle \tilde{v} \rangle \). Therefore, \( \langle \tilde{v} \rangle^\perp = LB^T \), and hence \( L \) is rank-equivalent to \( \langle v \rangle^\perp = C \times GF(q^m)^{n-r} \).
We hence derive the rank weight enumerator of an \((r, r-1, 2)\) MRD code. Note that the rank weight distribution of linear Class-I MRD codes has been derived in [2], [8]. However, we will use our results to give an alternative derivation of the rank weight distribution of linear Class-I MRD codes later, and thus we shall not use the result in [2], [8] here.

**Proposition 15:** Suppose \(v_r \in GF(q^m)^r\) has rank \(r (0 \leq r \leq m)\). The rank weight enumerator of \(L_r = \langle v \rangle^\perp\) depends on only \(r\) and is given by

\[
W^r_{L_r}(x, y) = q^{-rm} \left\{ [x + (q^m-1)y]^r + (q^m-1)(x-y)^r \right\}.
\] (33)

**Proof:** We first prove that the number of vectors with rank \(r\) in \(L_r\), denoted as \(A_{r,r}\), depends only on \(r\) and is given by

\[
A_{r,r} = q^{-m} (\alpha(m, r) + (q^m-1)(-1)^r q^{qr})
\] (34)

by induction on \(r (r \geq 1)\). Eq. (34) clearly holds for \(r = 1\). Suppose Eq. (34) holds for \(r = \bar{r} - 1\).

We consider all the vectors \(u = (u_0, \ldots, u_{\bar{r}-1}) \in L_{\bar{r}}\) such that the first \(\bar{r} - 1\) coordinates of \(u\) are linearly independent. Remark that \(u_{\bar{r}-1} = -v_{\bar{r}-1} - v_{\bar{r}-2} = v_{\bar{r}-1} - v_{\bar{r}-2} = v_{\bar{r}-1} - v_{\bar{r}-2}\).

Thus there are \(N_{\bar{r}-1}(q^m, \bar{r} - 1) = \alpha(m, \bar{r} - 1)\) such vectors \(u\). Among these vectors, we will enumerate the vectors \(t\) whose last coordinate is a linear combination of the first \(\bar{r} - 1\) coordinates, i.e., \(t = (t_0, \ldots, t_{\bar{r}-2}, \sum_{i=0}^{\bar{r}-2} a_i t_i) \in L_{\bar{r}}\) where \(a_i \in GF(q)\) for \(0 \leq i \leq \bar{r} - 2\).

Remark that \(t \in L_{\bar{r}}\) if and only if \(t_0, \ldots, t_{\bar{r}-2}, (v_0 + a_0 v_{\bar{r}-1}, \ldots, v_{\bar{r}-2} + a_{\bar{r}-2} v_{\bar{r}-1}) = 0\). It is easy to check that \(v(a) = (v_0 + a_0 v_{\bar{r}-1}, \ldots, v_{\bar{r}-2} + a_{\bar{r}-2} v_{\bar{r}-1})\) has rank \(\bar{r} - 1\). Therefore, if \(a_0, \ldots, a_{\bar{r}-2}\) are fixed, then there are \(A_{\bar{r}-1,\bar{r}-1}\) such vectors \(t\). Also, suppose \(\sum_{i=0}^{\bar{r}-2} t_i v_i + v_{\bar{r}-1} \sum_{i=0}^{\bar{r}-2} b_i t_i = 0\). Hence \(\sum_{i=0}^{\bar{r}-2} (a_i - b_i) t_i = 0\), which implies \(a = b\) since \(t_i\)'s are linearly independent. That is, \(\langle v(a) \rangle \cap \langle v(b) \rangle = \{0\}\) if \(a \neq b\). We conclude that there are \(q^{\bar{r}-1} A_{\bar{r}-1,\bar{r}-1}\) vectors \(t\). Therefore, \(A_{\bar{r},\bar{r}} = \alpha(m, \bar{r} - 1) - q^{\bar{r}-1} A_{\bar{r}-1,\bar{r}-1} = q^{-m} \alpha(m, \bar{r}) + (q^m-1)(-1)^{\bar{r}} q^{qr}\).

Denote the number of vectors with rank \(p\) in \(L_r\) as \(A_{r,p}\). We have \(A_{r,p} = \left[\frac{r}{p}\right] A_{r,p}\) [8], and hence \(A_{r,p} = \left[\frac{r}{p}\right] q^{-m} [\alpha(m, p) + (q^m-1)(-1)p q^{sp}]\). Thus, \(W^r_{L_r}(x, y) = \sum_{p=0}^{r} A_{r,p} x^{r-p} y^p = q^{-m} \left\{ [x + (q^m-1)y]^r + (q^m-1)(x-y)^r \right\}\).}

We comment that Proposition [15] in fact provides the rank weight distribution of any \((r, r-1, 2)\) MRD code.

**Lemma 20:** Let \(C_0 \subseteq GF(q^m)^r\) be a linear code with rank weight enumerator \(W^r_{C_0}(x, y)\), and for \(s \geq 0\), let \(W^r_{C_s}(x, y)\) be the rank weight enumerator of \(C_s \triangleq C_0 \times GF(q^m)^s\). Then \(W^r_{C_s}(x, y)\) only depends on \(s\) and is given by

\[
W^r_{C_s}(x, y) = W^r_{C_0}(x, y) \ast [x + (q^m-1)y]^s,
\] (35)
Proof: For $s \geq 0$, denote $W^r_C(x, y) = \sum_{u=0}^{r+s} B_{s,u} y^u x^{r+s-u}$. We will prove that
\[
B_{s,u} = \sum_{i=0}^{u} q^i B_{0,i} \binom{s}{u-i} \alpha(m-i, u-i)
\]
(36)
by induction on $s$. Eq. (36) clearly holds for $s = 0$. Now assume (36) holds for $s = \bar{s} - 1$. For any $x_{\bar{s}} = (x_0, \ldots, x_{r+\bar{s}-1}) \in \mathcal{C}_{\bar{s}}$, we define $x_{\bar{s}-1} = (x_0, \ldots, x_{r+\bar{s}-2}) \in \mathcal{C}_{\bar{s}-1}$. Then $\text{rk}(x_{\bar{s}}) = u$ if and only if either $\text{rk}(x_{\bar{s}-1}) = u$ and $x_{r+\bar{s}-1} \in \mathfrak{S}(x_{\bar{s}-1})$ or $\text{rk}(x_{\bar{s}-1}) = u - 1$ and $x_{r+\bar{s}-1} \notin \mathfrak{S}(x_{\bar{s}-1})$. This implies $B_{s,u} = q^u B_{\bar{s}-1,u} + (q^m - q^{u-1}) B_{\bar{s}-1,u-1} = \sum_{i=0}^{u} q^i B_{0,i} \binom{s}{u-i} \alpha(m-i, u-i)$. \hfill \blacksquare

Combining Lemma 19, Proposition 15, and Lemma 20, the rank weight enumerator of $\langle \mathbf{v} \rangle^\perp$ can be determined at last.

Proposition 16: For $\mathbf{v} \in \text{GF}(q^m)^n$ with rank $r \geq 0$, the rank weight enumerator of $\mathcal{C} = \langle \mathbf{v} \rangle^\perp$ depends on only $r$, and is given by
\[
W^r_C(x, y) = q^{-m} \left\{ [x + (q^m - 1)y]^n + (q^m - 1)(x - y)^r \right\} \ast [x + (q^m - 1)y]^{n-r} \right\}.
\]
(37)

C. MacWilliams identity for the rank metric

Using the results in Section 6B, we now derive the MacWilliams identity for rank metric codes. But first, we give two definitions from [25] that are needed in our derivation.

Definition 5: Let $\mathbb{C}$ be the field of complex numbers. Let $a \in \text{GF}(q^m)$ and let $\{1, \alpha_1, \ldots, \alpha_{m-1}\}$ be a basis set of $\text{GF}(q^m)$. We thus have $a = a_0 + a_1 \alpha_1 + \ldots + a_{m-1} \alpha_{m-1}$, where $a_i \in \text{GF}(q)$ for $0 \leq i \leq m-1$. Finally, let $\zeta \in \mathbb{C}$ be a primitive $q$-th root of unity, $\chi(a) \overset{\text{def}}{=} \zeta^{a_0}$ provides a mapping from $\text{GF}(q^m)$ to $\mathbb{C}$.

Definition 6 (Hadamard transform): For a mapping $f$ from $\text{GF}(q^m)^n$ to $\mathbb{C}$, the Hadamard transform of $f$, denoted as $\hat{f}$, is defined to be
\[
\hat{f}(\mathbf{v}) \overset{\text{def}}{=} \sum_{\mathbf{u} \in \text{GF}(q^m)^n} \chi(\mathbf{u} \cdot \mathbf{v}) f(\mathbf{u}),
\]
(38)
where $\mathbf{u} \cdot \mathbf{v}$ denotes the inner product of $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathcal{C}$ be an $(n, k)$ linear code over $\text{GF}(q^m)$, and let $W^r_C(x, y) = \sum_{i=0}^{n} A_i y^i x^{n-i}$ be its rank weight enumerator and $W^r_{\mathcal{C}^\perp}(x, y) = \sum_{j=0}^{n} B_j y^j x^{n-j}$ be the rank weight enumerator of its dual code $\mathcal{C}^\perp$.

Theorem 2: For any $(n, k)$ linear code $\mathcal{C}$ and its dual code $\mathcal{C}^\perp$ over $\text{GF}(q^m)$,
\[
W^r_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W^r_{\mathcal{C}}(x + (q^m - 1)y, x - y),
\]
(39)
where $\tilde{W}^r_{\mathcal{C}}$ is the $q$-transform of $W^r_{\mathcal{C}}$. Equivalently,
\[
\sum_{j=0}^{n} B_j y^j x^{n-j} = q^{-mk} \sum_{i=0}^{n} A_i (x - y)^i \ast [x + (q^m - 1)y]^{n-i}.
\]
(40)
Proof: We have \( \text{rk}(\lambda \mathbf{u}) = \text{rk}(\mathbf{u}) \) for all \( \lambda \in \text{GF}(q^m)^* \) and all \( \mathbf{u} \in \text{GF}(q^m)^n \). We want to determine \( \hat{f}_k(\mathbf{v}) \) for all \( \mathbf{v} \in \text{GF}(q^m)^n \). By Definition 6, we can split the summation in Eq. (38) into two parts:

\[
\hat{f}_k(\mathbf{v}) = \sum_{\mathbf{u} \in \mathcal{L}} \chi(\mathbf{u} \cdot \mathbf{v}) f_k(\mathbf{u}) + \sum_{\mathbf{u} \in F \setminus \mathcal{L}} \chi(\mathbf{u} \cdot \mathbf{v}) f_k(\mathbf{u}),
\]

where \( \mathcal{L} = \langle \mathbf{v} \rangle^\perp \). If \( \mathbf{u} \in \mathcal{L} \), then \( \chi(\mathbf{u} \cdot \mathbf{v}) = 1 \) by Definition 5 and the first summation is equal to \( W^R_k(x, y) \). For the second summation, we gather vectors into groups of the form \( \{\lambda \mathbf{u}_1\} \), where \( \lambda \in \text{GF}(q^m)^* \) and \( \mathbf{u}_1 \cdot \mathbf{v} = 1 \). We remark that for \( \mathbf{u} \in F \setminus \mathcal{L} \) (see [25, Chapter 5, Lemma 9])

\[
\sum_{\lambda \in \text{GF}(q^m)^*} \chi(\lambda \mathbf{u}_1 \cdot \mathbf{v}) f_k(\lambda \mathbf{u}_1) = f_k(\mathbf{u}_1) \sum_{\lambda \in \text{GF}(q^m)^*} \chi(\lambda) = -f_k(\mathbf{u}_1).
\]

Hence the second summation is equal to \( -\frac{1}{q^m-1} W^R_k(x, y) \). This leads to \( \hat{f}_k(\mathbf{v}) = \frac{1}{q^m-1} [q^m W^R_k(x, y) - W^R_k(x, y)] \). Using \( W^R_k(x, y) = [x + (q^m - 1)y]^{[n]} \) and Proposition 16 we obtain \( \hat{f}_k(\mathbf{v}) = (x - y)^r \ast [x + (q^m - 1)y]^{n-r} \).

By [25, Chapter 5, Lemma 11], any mapping \( f \) from \( F \) to \( \mathbb{C} \) satisfies \( \sum_{\mathbf{v} \in \mathcal{L}} f(\mathbf{v}) = \frac{1}{|\mathcal{L}|} \sum_{\mathbf{v} \in \mathcal{L}} \hat{f}(\mathbf{v}) \). Applying this result to \( f_k(\mathbf{v}) \), we obtain (39) and (40).

Also, \( B_j \)'s can be explicitly expressed in terms of \( A_i \)'s.

Corollary 5: We have

\[
B_j = \frac{1}{|\mathcal{C}|} \sum_{i=0}^{n} A_i P_j(i; m, n),
\]

where

\[
P_j(i; m, n) \overset{\text{def}}{=} \sum_{l=0}^{j} \begin{pmatrix} i \\ l \end{pmatrix} \begin{pmatrix} n - i \\ j - l \end{pmatrix} (-1)^{j-l} q^{\alpha l} q^{l(m-i)} \alpha (m - l, j - l). \tag{42}
\]

Proof: We have \( (x - y)^{[i]} \ast (x + (q^m - 1)y)^{[n-i]} = \sum_{j=0}^{n} P_j(i; m, n) y^j x^{n-j} \). The result follows from Theorem 2.

Note that although (41) is the same as that in [2, (3.14)], \( P_j(i; m, n) \) in (42) are different from \( P_j(i) \) in [2, (A10)] and their alternative forms in [37]. We can show that

Proposition 17: \( P_j(x; m, n) \) in (42) are the generalized Krawtchouk polynomials.

The proof is given in Appendix H. Also, it was pointed out in [37] that \( \frac{P_j(x; m, n)}{P_j(0; m, n)} \) is actually a basic hypergeometric function.

Proposition 17 shows that \( P_j(x; m, n) \) in (42) are an alternative form for \( P_j(i) \) in [2, (A10)], and hence our results in Corollary 5 are equivalent to those in [2, Theorem 3.3].
D. Moments of the rank distribution

Next, we investigate the relationship between moments of the rank distribution of a linear code and those of its dual code. Our results parallel those in [25, p. 131].

**Proposition 18:** For \(0 \leq \nu \leq n\),
\[
\sum_{i=0}^{n-\nu} \binom{n-i}{\nu} A_i = q^{m(k-\nu)} \sum_{j=0}^{\nu} \binom{n-j}{n-\nu} B_j. \tag{43}
\]

**Proof:** First, apply Eq. (41) to \(C^\perp\). We obtain \(A_i = q^{m(k-n)} \sum_{j=0}^{\nu} B_j P_i(j; m, n)\), and hence
\[
\sum_{i=0}^{n-\nu} \binom{n-i}{\nu} A_i = q^{m(k-n)} \sum_{j=0}^{n} B_j \sum_{i=0}^{n} \binom{n-i}{\nu} P_i(j; m, n).
\]
We have \(\sum_{i=0}^{n-\nu} \binom{n-i}{\nu} P_i(j; m, n) = q^{m(n-\nu)} \binom{n-j}{n-\nu} [38, (29)]\), and the result follows.

**Proof:** First, applying Theorem 2 to \(C^\perp\), we obtain
\[
\sum_{i=0}^{n} A_i y^i x^{n-i} = q^{m(k-n)} \sum_{j=0}^{n} B_j b_j(x, y; m) * a_{n-j}(x, y; m). \tag{44}
\]
Next, we apply the \(q\)-derivative with respect to \(x\) to Eq. (44) \(\nu\) times. By Lemma 15 the left hand side (LHS) becomes \(\sum_{i=0}^{n-\nu} \beta(n-i, \nu) A_i y^i x^{n-i-\nu}\), while by Lemma 16 the right hand side (RHS) reduces to \(q^{m(k-n)} \sum_{j=0}^{n} B_j \psi_j(x, y)\), where
\[
\psi_j(x, y) \overset{\text{def}}{=} [b_j(x, y; m) * a_{n-j}(x, y; m)]^{(\nu)} = \sum_{l=0}^{\nu} \binom{\nu}{l} q^{(\nu-l)(j-l)} b_j^{(l)}(x, y) * a_{n-j}^{(\nu-l)}(x, y; m).
\]
By Lemma 15, \(b_j^{(l)}(x, y; m) = \beta(j, l)(x-y)^{j-l}\) and \(a_{n-j}^{(\nu-l)}(x, y; m) = \beta(n-j, \nu-l) a_{n-j-\nu+l}(x, y; m)\). It can be verified that for any homogeneous polynomial \(b(x, y; m)\) and for any \(s \geq 0\), \((b * a_s)(1, 1; m) = q^{ms}b(1, 1; m)\). Also, for \(x = y = 1\), \(b_j^{(l)}(1, 1; m) = \beta(j, l)\delta_{j,l}\). We hence have \(\psi_j(1, 1) = 0\) for \(j > \nu\), and \(\psi_j(1, 1) = \binom{\nu}{j} \beta(j, j) \beta(n-j, \nu-j) q^{m(n-\nu)}\) for \(j \leq \nu\). Since \(\beta(n-j, \nu-j) = \binom{n-j}{\nu-j} \beta(\nu-j, \nu-j)\) and \(\beta(\nu,j) = \binom{\nu}{j} \beta(j, j) \beta(n-j, \nu-j) - \beta(j, \nu-j)\), \(\psi_j(1, 1) = \binom{\nu-j}{\nu} \beta(\nu, \nu) q^{m(n-\nu)}\). Applying \(x = y = 1\) to the LHS and rearranging both sides using \(\beta(n-i, \nu) = \binom{n-i}{\nu} \beta(\nu, \nu)\), we obtain (43).

**Corollary 6:** Let \(d'_k\) be the minimum rank distance of \(C^\perp\). If \(0 \leq \nu < d'_k\), then
\[
\sum_{i=0}^{n-\nu} \binom{n-i}{\nu} A_i = q^{m(k-\nu)} \binom{n}{\nu}. \tag{45}
\]

**Proof:** We have \(B_0 = 1\) and \(B_1 = \ldots = B_\nu = 0\).

Using the \(q^{-1}\)-derivative, we obtain another relationship.

**Proposition 19:** For \(0 \leq \nu \leq n\),
\[
\sum_{i=\nu}^{n} \binom{i}{\nu} q^{\nu(n-i)} A_i = q^{m(k-\nu)} \sum_{j=0}^{\nu} \binom{n-j}{n-\nu} (-1)^j q^\sigma \alpha(m-j, \nu-j) q^{j(\nu-j)} B_j. \tag{46}
\]
The proof of Proposition 19 is similar to that of Proposition 18 and is given in Appendix I. Similarly, when \( \nu \) is less than the minimum distance of the dual code, Proposition 19 can be simplified.

Corollary 7: If \( 0 \leq \nu < d^d_{r_k} \), then

\[
\sum_{i=0}^{n-1} [i]_{\nu} q^{\nu(n-i)} a_i = q^{m(k-\nu)} [n]_{\nu} a(m, \nu). \tag{47}
\]

Proof: We have \( B_0 = 1 \) and \( B_1 = \cdots = B_\nu = 0 \).]

Following [25], we refer to the LHS of Eq. (43) and (46) as moments of the rank distribution of \( \mathcal{C} \).

E. Relation to Delsarte’s results

Delsarte [2] also derived the MacWilliams identity for rank metric codes, and below we briefly relate our results to those by Delsarte.

Delsarte [2] considered array codes with the rank metric. In [2], the inner product between two \( m \times n \) matrices \( A \) and \( B \) over \( GF(q) \) is defined as \( A \cdot B \triangleq Tr(AB^T) \). Two matrices \( A \) and \( B \) are orthogonal if \( \chi(A \cdot B) = 1 \), where \( \chi \) is a nontrivial character of the additive group \( GF(q) \), and dual codes are defined using this orthogonality. Delsarte then established an analytical expression (cf. [2, Theorem 3.3]) between the rank distance enumerator of an array code and that of its dual.

Clearly the definitions of dual codes are different in our work and [2]. However, we show below the two definitions collide when dual bases are used. With a slight abuse of notation, the inner products between two vectors and two matrices are both denoted by \( \cdot \) and dual codes of both vector and array codes are denoted by \( \perp \). For all vectors \( \mathbf{x} \in GF(q^m)^n \), we expand \( \mathbf{x} \) into a matrix with respect to the basis \( B = \{ \beta_i \}_{i=0}^{m-1} \) of \( GF(q^m) \) over \( GF(q) \) and refer to the matrix \( \{ x_{i,j} \}_{i,j=0}^{m-1,n-1} \) as \( \mathbf{x}_B \). That is, \( x_j = \sum_{i=0}^{m-1} x_{i,j} \beta_i \) for \( 0 \leq j < n \). For a code \( \mathcal{C} \) of length \( n \) over \( GF(q^m) \), we denote \( \mathcal{C}_B \triangleq \{ \mathbf{x}_B \in GF(q^{mn}) | \mathbf{x} \in \mathcal{C} \} \). Clearly, if \( \mathcal{C} \) is an \( (n,k) \) linear code over \( GF(q^m) \), then \( \mathcal{C}_B \) is an \( (mn,mk) \) linear array code over \( GF(q) \).

Lemma 21: Let \( E = \{ \epsilon_i \}_{i=0}^{m-1} \) and \( P = \{ \phi_j \}_{j=0}^{m-1} \) be dual bases of \( GF(q^m) \) over \( GF(q) \). Then for any \( \mathbf{a}, \mathbf{b} \in GF(q^m)^n \), \( \text{Trace}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a}_E \cdot \mathbf{b}_P \).

Proof: We have \( \mathbf{a} \cdot \mathbf{b} = \sum_j a_j b_j = \sum_j \sum_i \sum_k a_{i,j} \epsilon_i b_{k,j} \phi_k \). Applying the trace function on both sides, we obtain

\[
\text{Trace}(\mathbf{a} \cdot \mathbf{b}) = \sum_j \sum_i \sum_k a_{i,j} b_{k,j} \text{Trace}(\epsilon_i \phi_k) = \sum_j \sum_i a_{i,j} b_{i,j} = \text{Tr}(\mathbf{a}_E \mathbf{b}_P^T). \tag{48}
\]

Proposition 20: For an \( (n,k) \) code \( \mathcal{C} \) over \( GF(q^m) \) and dual bases \( E \) and \( P \) of \( GF(q^m) \) over \( GF(q) \), \( (\mathcal{C}^\perp)_E = (\mathcal{C}_P)^\perp \).
Proof: Let $v \in C^\perp$, then for any $u \in C$, $v \cdot u = 0$. Hence $v_E \cdot u_P = 0$ by Lemma [21] and $\chi(v_E \cdot u_P) = 1$ for all $u_P \in C_P$. Therefore $v_E \in (C_P)^\perp$ and $(C^\perp)_E \subseteq (C_P)^\perp$. Since $|(C^\perp)_E| = |(C_P)^\perp|$, we conclude that $(C^\perp)_E = (C_P)^\perp$.

Proposition [20] implies that our identity in Corollary [5] can be derived from Delsarte’s identity in [2, Theorem 3.3]. Suppose $C$ is an $(n,k)$ linear code over $GF(q^m)$ with rank weight distribution $A_i$ and its dual code $C^\perp$ has rank distribution $B_j$. Let $E$ and $P$ are dual bases of $GF(q^m)$ over $GF(q)$. Note that $C_P$ and $(C^\perp)_E$ have the same rank distribution as $C$ and $C^\perp$, respectively. Also by Proposition [20] $C_P$ and $(C^\perp)_E$ are dual array codes. Thus applying Delsarte’s identity to $C_P$ results in Corollary [5]. Note that the rank distance enumerator and rank weight enumerator are the same for linear codes.

Our results in this section are different from Delsarte’s results in several aspects. First, $P_j(i; m, n)$ in [42] are different from $P_j(i)$ in [2, (A10)] and their alternative forms in [37]. In Proposition [17] we show that $P_j(x; m, n)$ are actually the generalized Krawtchouk polynomials, and hence $P_j(i; m, n)$ in [42] are equivalent to $P_j(i)$ in [2, (A10)]. Second, our approach to proving the MacWilliams identity is different, and intermediate results of our proof offer interesting insights (see Lemma [19] and Proposition [16]). Third, our MacWilliams identity is also expressed in a polynomial form (Theorem [2]) similar to that in [25], and the polynomial form allows us to derive other identities (see, for example, Propositions [18] and [19]) that relate the rank distribution of a linear code to that of its dual.

VII. Conclusions

In this paper, we investigate the packing, covering, and weight properties of rank metric codes. We show that MRD codes not only are optimal in the sense of the Singleton bound, but also provide the optimal solution to the sphere packing problem. We also derive bounds for the sphere covering problem and establish the asymptotic minimum code rate for a code with given relative covering radius. Finally, we establish identities that relate the rank weight distribution of a linear code to that of its dual code.
APPENDIX

The proofs in this section use some well-known properties of Gaussian polynomials [32]:

\[
\begin{align*}
\binom{n}{k} &= \binom{n}{n-k} \quad (48) \\
\binom{n}{k} &= \binom{n-1}{k} + q^{n-k-1} \binom{n-1}{k-1} \quad (49) \\
&= q^k \binom{n-1}{k} + \binom{n-1}{k} \quad (50) \\
&= \frac{q^{n-k-1}}{q^{n-k-1}} \binom{n}{k} \quad (51) \\
&= \frac{q^{n-k+1} - 1}{q - 1} \binom{n}{k} \quad (52) \\
\binom{n}{k} \binom{k}{l} &= \binom{n}{n-k} \binom{k}{n-k} \quad (53)
\end{align*}
\]

A. Proof of Proposition 8

We first establish a key lemma.

Lemma 22: If \( z \in Z \) and \( 0 < \rho < n \), then

\[
|A \cap B_1(z)| \leq V_1(q^m, n) - q^{\rho-1} \binom{\rho}{1}. \quad (54)
\]

Proof: By definition of \( \rho \), there exists \( c \in C \) such that \( d_R(z, c) \leq \rho \). By Proposition 2, \( |B_1(z) \cap B_{\rho-1}(c)| \) gets its minimal value for \( d_R(z, c) = \rho \), which is \( q^{\rho-1} \binom{\rho}{1} \) by Proposition 4. A vector at distance \( \leq \rho - 1 \) from any codeword does not belong to \( A \). Therefore, \( B_1(z) \cap B_{\rho-1}(c) \subseteq B_1(z) \setminus A \), and hence

\[
|A \cap B_1(z)| = |B_1(z)| - |B_1(z) \setminus A| \leq V_1(q^m, n) - |B_1(z) \cap B_{\rho-1}(c)|. \quad \blacksquare
\]

For a code \( C \) with covering radius \( \rho \) and \( \epsilon \geq 1 \),

\[
\gamma \overset{\text{def}}{=} \epsilon \binom{q^m - |C|V_{\rho-1}(q^m)}{n} - (\epsilon - 1) \binom{|C|V_{\rho}(q^m, n) - q^m}{n} \quad (55)
\]

\[
\leq \epsilon |A| - (\epsilon - 1)|Z| \quad (56)
\]

\[
\leq \epsilon |A| - (\epsilon - 1)|A \cap Z|
\]

\[
= \epsilon |A \setminus Z| + |A \cap Z|,
\]

where (56) follows from \( |Z| \leq |C|V_{\rho}(q^m, n) - q^m \), given in Section II-C.

\[
\gamma \leq \sum_{a \in A \setminus Z} EC(B_1(a)) + \sum_{a \in A \cap Z} EC(B_1(a)) \quad (57)
\]

\[
= \sum_{a \in A} EC(B_1(a)),
\]

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where (57) follows from Lemma 9 and $|A \cap Z| \leq E_C(A \cap Z)$.

\[
\begin{align*}
\gamma &\leq \sum_{a \in A} \sum_{x \in B_1(a) \cap Z} E_C(\{x\}) \\
&= \sum_{x \in Z} \sum_{a \in B_1(x) \cap A} E_C(\{x\}) \\
&= \sum_{x \in Z} |A \cap B_1(x)| E_C(\{x\}),
\end{align*}
\]

where (58) follows from the fact the second summation is over disjoint sets \(\{x\}\). Using Lemma 22, we obtain

\[
\begin{align*}
\gamma &\leq \sum_{x \in Z} \left( V_1(q^m, n) - q^{\rho-1} \binom{\rho}{1} \right) E_C(\{x\}) \\
&= \left( V_1(q^m, n) - q^{\rho-1} \binom{\rho}{1} \right) E_C(Z) \\
&= \left( V_1(q^m, n) - q^{\rho-1} \binom{\rho}{1} \right) (|C| V_\rho(q^m, n) - q^{mn}).
\end{align*}
\]

Combining (59) and (55), we obtain (17).

\[\text{(59)}\]

B. Proof of Proposition 11

\[\text{(58)}\]

Proof: Given a radius \(\rho\) and a code \(C\), denote by \(P_\rho(C)\) the set of vectors in \(\text{GF}(q^m)^n\) that are at distance \(> \rho\) from \(C\). To simplify notations, \(Q \overset{\text{def}}{=} q^{mn}\) and \(p_\rho(C) \overset{\text{def}}{=} Q^{-1}|P_\rho(C)|\). Let us denote the set of all codes over \(\text{GF}(q^m)\) of length \(n\) and cardinality \(K\) as \(S_K\). Clearly \(|S_K| = (Q^K)\). Let us calculate the average value of \(p_\rho(C)\) for all codes \(C \in S_K\):

\[
\begin{align*}
\frac{1}{|S_K|} \sum_{C \in S_K} p_\rho(C) &= \frac{1}{|S_K|} Q^{-1} \sum_{C \in S_K} |P_\rho(C)| = \frac{1}{|S_K|} Q^{-1} \sum_{C \in S_K} \sum_{x \in F|d_x(x, C) > \rho} 1 \\
&= \frac{1}{|S_K|} Q^{-1} \sum_{x \in F} \sum_{C \in S_K} |d_x(x, C) > \rho| 1 \\
&= \frac{1}{|S_K|} Q^{-1} \sum_{x \in F} \left( Q - V_\rho(q^m, n) \right) K \\
&= \frac{Q - V_\rho(q^m, n)}{K} \left/ \left( Q^K \right) \right. \\
&= \frac{Q - V_\rho(q^m, n)}{K} \left/ \left( Q^K \right) \right. \\
\end{align*}
\]

(60)
Eq. (60) comes from the fact that there are \((Q - V_{\rho}(q^m, n))\) codes with cardinality \(K\) that do not cover \(x\). For all \(K\), there exists a code \(C' \in S_K\) for which \(p_{\rho}(C')\) is no more than the average, that is:

\[
p_{\rho}(C') \leq \left( Q \right)^{-1} \left( Q - V_{\rho}(q^m, n) \right) K \\
\leq (1 - Q^{-1} V_{\rho}(q^m, n))^K.
\]

Let us choose \(K = \left[ -\log_Q \left( 1 - Q^{-1} V_{\rho}(q^m, n) \right) \right] + 1\) so that \(K \log_Q \left( 1 - Q^{-1} V_{\rho}(q^m, n) \right) < -1\) and hence \(p_{\rho}(C') = (1 - Q^{-1} V_{\rho}(q^m, n))^K < Q^{-1}\). It follows that \(|p_{\rho}(C')| < 1\), and \(C'\) has covering radius at most \(\rho\).

\[\square\]

C. Proof of Lemma 14

The proof is by induction on \(l\). Clearly all the claims hold for \(l = 0\). Suppose \(y^{[l-1]} = q^{\sigma_{l-1}} y^{\tilde{l}-1}\) for \(\tilde{l} \geq 1\), then \(y^{[l]} = y^{[l-1]} * y = q^{\tilde{l}-1} q^{\sigma_{l-1}} y^{\tilde{l}} = q^{\sigma_{l}} y^{\tilde{l}}\). The proof for \(x^{[l]}\) is similar.

Suppose Eq. (25) holds for \(l = \tilde{l} - 1\). We have \(a_l(x, y; m) = a_{l-1}(x, y; m) * (x + (q^m - 1)y) = \sum_{u=0}^{l} a_{l,u} y^u x^{\tilde{l}-u}\). By (24), we have

\[
a_{\tilde{l},u} = q^u a_{\tilde{l}-1,u} + q^{u-1} a_{\tilde{l}-1,u-1} (q^{m-u+1} - 1)
\]

\[
= q^u \left[ \tilde{l} - 1 \right] \left( m, u \right) + q^{u-1} \left[ \tilde{l} - 1 \right] \left( m, u-1 \right) (q^{m-u+1} - 1)
\]

\[
= \left( q^u \left[ \tilde{l} - 1 \right] \left( m, u \right) + q^{\tilde{l} - 1} \left( m, u-1 \right) \right) \alpha(m, u)
\]

\[
= \left[ \tilde{l} \right] \left( m, u \right) \alpha(m, u), \tag{62}
\]

where Eq. (62) follows Eq. (49).

Suppose Eq. (26) holds for \(l = \tilde{l} - 1\). We have \(b_l(x, y; m) = b_{l-1}(x, y; m) * (x - y) = \sum_{u=0}^{l} b_{l,u} y^u x^{\tilde{l}-u}\). By (24), we have

\[
b_{\tilde{l},u} = q^u b_{\tilde{l}-1,u} - q^{u-1} b_{\tilde{l}-1,u-1}
\]

\[
= q^u \left[ \tilde{l} - 1 \right] \left( -1 \right) q^{\sigma_u} + q^{u-1} \left[ \tilde{l} - 1 \right] \left( -1 \right)^u q^{\sigma_{u-1}}
\]

\[
= \left( q^u \left[ \tilde{l} - 1 \right] \left( -1 \right) q^{\sigma_u} + q^{\tilde{l} - 1} \left( -1 \right)^u q^{\sigma_u} \right)
\]

\[
= \left[ \tilde{l} \right] \left( -1 \right) q^{\sigma_u}, \tag{63}
\]

where Eq. (63) also follows Eq. (49).
D. Proof of Lemma 15

The proof is by induction on $\nu$. Clearly all the claims hold for $\nu = 0$. The $\nu$-th $q$-derivative of $x^{l}$ follows Definition 3. Suppose Eq. (27) holds for $\nu = \bar{\nu} - 1$, then

$$a^{(\nu)}_{l}(x, y; m) = \left[ \beta(l, \bar{\nu} - 1) a_{l-\bar{\nu}+1}(x, y; m) \right]^{(1)}$$

$$= \beta(l, \bar{\nu} - 1) \sum_{i=1}^{l-\bar{\nu}+1} \left[ l - \bar{\nu} + 1 \atop i \right] \alpha(m, l - \bar{\nu} + l - i) y^{l-\bar{\nu}+l-1-i} x^{i-1}$$

$$= \beta(l, \bar{\nu} - 1) \left[ l - \bar{\nu} + 1 \atop 1 \right] \sum_{i=0}^{l-\bar{\nu}} \left[ l - \nu \atop i \right] \alpha(m, l - \bar{\nu} + i) y^{l-\bar{\nu}-i} x^{i}$$

(64)

where Eq. (64) follows Eq. (53).

Similarly, suppose Eq. (28) holds for $\nu = \bar{\nu} - 1$, hence

$$b^{(\nu)}_{l}(x, y; m) = \left[ \beta(l, \bar{\nu} - 1) b_{l-\bar{\nu}+1}(x, y; m) \right]^{(1)}$$

$$= \beta(l, \bar{\nu} - 1) \sum_{i=1}^{l-\bar{\nu}+1} \left[ l - \bar{\nu} + 1 \atop i \right] (-1)^{l-\bar{\nu}+l-i} q^{i-\bar{\nu}+1} y^{l-\bar{\nu}+l-1-i} x^{i-1}$$

$$= \beta(l, \bar{\nu}) \sum_{i=0}^{l-\bar{\nu}} \left[ l - \nu \atop i \right] (-1)^{l-\bar{\nu}-i} q^{i-\bar{\nu}} y^{l-\bar{\nu}-i} x^{i}$$

$$= \beta(l, \bar{\nu}) b_{l-\bar{\nu}}(x, y; m).$$

E. Proof of Lemma 16

We consider homogeneous polynomials $f(x, y; m) = \sum_{i=0}^{\varphi} f_{i} y^{i} x^{r-i}$ and $u(x, y; m) = \sum_{i=0}^{\varphi} u_{i} y^{i} x^{r-i}$ of degree $r$ as well as $g(x, y; m) = \sum_{j=0}^{\sigma} g_{j} y^{j} x^{s-j}$ and $v(x, y; m) = \sum_{j=0}^{\sigma} v_{j} y^{j} x^{s-j}$ of degree $s$. First, we need a technical lemma.

Lemma 23: If $u_{r} = 0$, then

$$\frac{1}{x} (u(x, y; m) * v(x, y; m)) = \frac{u(x, y; m)}{x} * v(x, y; m).$$

(65)

If $v_{s} = 0$, then

$$\frac{1}{x} (u(x, y; m) * v(x, y; m)) = u(x, qy; m) * \frac{v(x, y; m)}{x}.$$

(66)

Proof: Suppose $u_{r} = 0$, then $\frac{u(x,y; m)}{x} = \sum_{i=0}^{r-1} u_{i} y^{i} x^{r-1-i}$. Hence

$$\frac{u(x,y; m)}{x} * v(x, y; m) = \sum_{k=0}^{r+s-1} \left( \sum_{l=0}^{k} q^{l} u_{l}(m) v_{k-l}(m - l) \right) y^{k} x^{r+s-1-k}$$

$$= \frac{1}{x} (u(x, y; m) * v(x, y; m)).$$
Suppose \( v_\nu = 0 \), then
\[
\frac{v(x,y,m)}{x} = \sum_{j=0}^{s-1} v_j y^j x^{s-1-j}.
\]
Hence
\[
u(x,y;m) = \frac{v(x,y;m)}{x} = \frac{1}{x}(u(x,y;m) \ast v(x,y;m)).
\]

We now give a proof of Lemma 16.

**Proof:** In order to simplify notations, we omit the dependence of the polynomials \( f \) and \( g \) on the parameter \( m \). The proof is by induction on \( \nu \). For \( \nu = 1 \), we have
\[
[f(x,y) \ast g(x,y)]^{(1)} = \frac{1}{(q-1)x} \left[ f(qx,y) \ast g(qx,y) - f(qx,y) \ast g(x,y) \right]
+ f(qx,y) \ast g(x,y) - f(x,y) \ast g(x,y)
\]
\[
= \frac{1}{(q-1)x} \left[ f(qx,y) \ast (g(qx,y) - g(x,y)) + (f(qx,y) - f(x,y)) \ast g(x,y) \right]
= f(qx,y) \ast \frac{g(qx,y) - g(x,y)}{(q-1)x} + f(qx,y) - f(x,y) \ast g(x,y)
\]
\[
= q^\nu f(x,y) \ast g^{(1)}(x,y) + f^{(1)}(x,y) \ast g(x,y),
\]
where Eq. (67) follows Lemma 23.

Now suppose Eq. (29) is true for \( \nu = \bar{\nu} \). In order to further simplify notations, we omit the dependence of the various polynomials in \( x \) and \( y \). We have
\[
(f \ast g)^{(\bar{\nu}+1)} = \sum_{l=0}^{\bar{\nu}} \left[ \bar{\nu} \atop l \right] q^{(\bar{\nu}-l)(r-l)} f^{(l)} \ast g^{(\bar{\nu}-l+1)}
\]
\[
= \sum_{l=0}^{\bar{\nu}} \left[ \bar{\nu} \atop l \right] q^{(\bar{\nu}-l)(r-l)} f^{(l)} \ast g^{(\bar{\nu}-l+1)} + \bar{\nu} + 1 \sum_{l=1}^{\bar{\nu}+1} \left[ \bar{\nu} \atop l - 1 \right] q^{(\bar{\nu}+1-l)(r-l+1)} f^{(l)} \ast g^{(\bar{\nu}-l+1)}
\]
\[
= \sum_{l=1}^{\bar{\nu}+1} \left[ \bar{\nu} + 1 \atop l \right] q^{(\bar{\nu}+1-l)(r-l)} f^{(l)} \ast g^{(\bar{\nu}-l+1)},
\]
where Eq. (68) follows Eq. (63), and Eq. (70) follows Eq. (49).
F. Proof of Lemma 17

Proof: The proof is by induction on \( \nu \). Clearly all the claims hold for \( \nu = 0 \). The \( \nu \)-th \( q^{-1} \)-derivative of \( y^l \) follows from Definition 4. Suppose Eq. (30) holds for \( \nu = \bar{\nu} - 1 \), then

\[
a_l^{(\nu)}(x, y; m) = \beta(l, \bar{\nu} - 1)q^{\sigma_l - 1}\alpha(m, \bar{\nu} - 1)a_{l-\bar{\nu}+1}(x, y; m)
\]

\[
= \beta(l, \bar{\nu} - 1)q^{\sigma_l - 1}\alpha(m, \bar{\nu} - 1)\sum_{i=0}^{l-\bar{\nu}+1} \binom{l - \bar{\nu} + 1}{i} (m - \bar{\nu} + 1, i) x^{l-\bar{\nu}+1-i} q^{l-1-i} y^{i-1}
\]

\[
= \beta(l, \bar{\nu})q^{\sigma_l}\alpha(m, \bar{\nu})a_{l-\bar{\nu}}(x, y; m, \bar{\nu}).
\]

Suppose Eq. (31) holds for \( \nu = \bar{\nu} - 1 \), then

\[
b_l^{(\nu)}(x, y; m) = (-1)^{\bar{\nu}-1}\beta(l, \bar{\nu} - 1)b_{l-\bar{\nu}+1}(x, y; m)
\]

\[
= (-1)^{\bar{\nu}-1}\beta(l, \bar{\nu})\sum_{i=1}^{l-\bar{\nu}+1} \binom{l - \bar{\nu} + 1}{i-1} (q^{\sigma_l - 1} - 1)(m - \bar{\nu}, i - 1)x^{l-\bar{\nu}+1-i} q^{l-1-i} y^{i-1}
\]

\[
= (-1)^{\bar{\nu}}\beta(l, \bar{\nu})b_{l-\bar{\nu}}(x, y; m).
\]

G. Proof of Lemma 18

We consider homogeneous polynomials \( f(x, y; m) = \sum_{i=0}^{r} f_i y^i x^{r-i} \) and \( u(x, y; m) = \sum_{i=0}^{r} u_i y^i x^{r-i} \) of degree \( r \) as well as \( g(x, y; m) = \sum_{j=0}^{s} g_j y^j x^{s-j} \) and \( v(x, y; m) = \sum_{j=0}^{s} v_j y^j x^{s-j} \) of degree \( s \). First, we need a technical lemma.

Lemma 24: If \( u_0 = 0 \), then

\[
\frac{1}{y}(u(x, y; m)) \ast v(x, y; m)) = q^s \frac{u(x, y; m)}{y} * v(x, y; m - 1).
\]

If \( v_0 = 0 \), then

\[
\frac{1}{y}(u(x, y; m)) \ast v(x, y; m)) = u(x, qy; m) * \frac{v(x, y; m)}{y}.
\]
Proof: Suppose \( u_0 = 0 \), then \( \frac{u(x,y;m)}{y} = \sum_{i=0}^{r-1} u_{i+1} x^{r-1-i} y^i \). Hence
\[
q^s \frac{u(x,y;m)}{y} * v(x,y;m-1) = q^s \sum_{k=0}^{r+s-1} \left( \sum_{l=0}^{k} q^{l} u_{l+1} v_{k-l}(m-1-l) \right) x^{r+s-1-k} y^k
\]
\[
= q^s \sum_{k=1}^{r+s} \left( \sum_{l=1}^{k} q^{l-1} u_{l} v_{k-l}(m-l) \right) x^{r+s-k} y^{k-1}
\]
\[
= \frac{1}{y} (u(x,y;m) * v(x,y;m)).
\]

Suppose \( v_0 = 0 \), then \( \frac{v(x,y;m)}{y} = \sum_{j=0}^{s-1} v_{j+1} x^{s-1-j} y^j \). Hence
\[
u(x,y;m) * \frac{v(x,y;m)}{y} = \sum_{k=0}^{r+s-1} \left( \sum_{l=0}^{k} q^{l(s-1)} q^{l} u_{l} v_{k-l}(m-l) \right) x^{r+s-1-k} y^k
\]
\[
= \sum_{k=1}^{r+s} \left( \sum_{l=0}^{k-1} q^{l} u_{l} v_{k-l}(m-l) \right) x^{r+s-k} y^{k-1}
\]
\[
= \frac{1}{y} (u(x,y;m) * v(x,y;m)).
\]

We now give a proof of Lemma 18.

Proof: The proof is by induction on \( \nu \). For \( \nu = 0 \), the result is trivial. For \( \nu = 1 \), we have
\[
[f(x,y;m) * g(x,y;m)]^{(1)} = \frac{1}{(q^{-1} - 1)y} [f(x,q^{-1}y;m) * g(x,q^{-1}y;m) - f(x,y;m) * g(x,y;m)]
\]
\[
+ f(x,q^{-1}y;m) * g(x,y;m) - f(x,y;m) * g(x,y;m)]
\]
\[
= \frac{1}{(q^{-1} - 1)y} [f(x,q^{-1}y;m) * (g(x,q^{-1}y;m) - g(x,y;m))]
\]
\[
+ (f(x,q^{-1}y;m) - f(x,y;m)) * g(x,y;m)]
\]
\[
= f(x,y;m) * \frac{g(x,q^{-1}y;m) - g(x,y;m)}{(q^{-1} - 1)y}
\]
\[
+ q^s \frac{f(x,q^{-1}y;m) - f(x,y;m)}{(q^{-1} - 1)y} * g(x,y;m - 1)
\]
\[
= f(x,y;m) * g^{(1)}(x,y;m) + q^s f^{(1)}(x,y;m) * g(x,y;m - 1),
\]
where (73) comes from Lemma 24.
Now suppose Equation (32) is true for $\bar{\nu}$. In order to further simplify notations, we omit the dependence of the various polynomials in $x$ and $y$. We have

$$
[f(m) + g(m)]^{(\bar{\nu}+1)} = \sum_{l=0}^{\bar{\nu}} \left[ \frac{\bar{\nu}}{l} \right] q^{l(s-\bar{\nu}+l)} \left[ f^{(l)}(m) * g^{(\bar{\nu}-l)}(m-l) \right]^{(1)}
$$

$$
= \sum_{l=0}^{\bar{\nu}} \left[ \frac{\bar{\nu}}{l} \right] q^{l(s-\bar{\nu}+l)} \left( f^{(l)}(m) * g^{(\bar{\nu}-l+1)}(m-l) \right)
+ q^{s-\bar{\nu}+l} f^{(l+1)}(m) * g^{(\bar{\nu}-l)}(m-l-1)
$$

$$
= \sum_{l=0}^{\bar{\nu}} \left[ \frac{\bar{\nu}}{l} \right] q^{l(s-\bar{\nu}+l)} f^{(l)}(m) * g^{(\bar{\nu}-l+1)}(m-l)
+ \sum_{l=1}^{\bar{\nu}+1} \left[ \frac{\bar{\nu}}{l-1} \right] q^{l(s-\bar{\nu}+l-1)} f^{(l)}(m) * g^{(\bar{\nu}-l+1)}(m-l)
$$

$$
= \sum_{l=1}^{\bar{\nu}+1} \left( q^{l} \left[ \frac{\bar{\nu}}{l} \right] + \left[ \frac{\bar{\nu}}{l-1} \right] \right) q^{l(s-\bar{\nu}+l-1)} f^{(l)}(m) * g^{(\bar{\nu}-l+1)}(m-l)
+ f(m) * g^{(\bar{\nu}+1)}(m) + q^{s(\bar{\nu}+1)} f^{(\bar{\nu}+1)}(m) * g(m-\bar{\nu}+1)
$$

$$
= \sum_{l=0}^{\bar{\nu}+1} \left[ \frac{\bar{\nu}+1}{l} \right] q^{l(s-\bar{\nu}+l+1)} f^{(l)}(m) * g^{(\bar{\nu}-l+1)}(m-l)
$$

where (75) comes from (74) and (76) comes from (50).  

H. Proof of Proposition 17

Proof: It was shown in [37] that the generalized Krawtchouk polynomials are the only solutions to the recurrence

$$
P_{j+1}(i+1; m+1, n+1) = q^{j+1} P_{j+1}(i+1; m, n) + q^{j} P_{j}(i; m, n)
$$

(77)
with initial conditions $P_j(0; m, n) = \left[ \begin{array}{c} n \\ j \end{array} \right] \alpha(m, j)$. Clearly, our polynomials satisfy these initial conditions.

We hence show that $P_j(i; m, n)$ satisfy the recurrence in Eq. (77). We have

$$P_{j+1}(i+1; m+1, n+1) = \sum_{l=0}^{i+1} \left[ \begin{array}{c} i+1 \\ l \end{array} \right] \left[ \begin{array}{c} n-l \\ j+1-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (m+1-l, j+1-l)$$

$$= \sum_{l=0}^{i+1} \left[ \begin{array}{c} i+1 \\ l \end{array} \right] \left[ \begin{array}{c} m+1-l \\ j+1-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l)$$

$$+ \sum_{l=0}^{i+1} \left[ \begin{array}{c} i+1 \\ l \end{array} \right] \sum_{l=0}^{i+1} \left[ \begin{array}{c} i+1 \\ l \end{array} \right] \{ q^j + [j-1] \} \{ q^{i+1-l} \left[ \begin{array}{c} m-l \\ j+1-l \end{array} \right] + \left[ m-l \right] \} \cdot \ldots$$

$$\cdots (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l)$$

$$= \sum_{l=0}^{i} \left[ \begin{array}{c} i \\ l \end{array} \right] q^{j+1-l} \left[ \begin{array}{c} m-l \\ j+1-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l)$$

$$+ \sum_{l=0}^{i} q^i \left[ \begin{array}{c} i \\ l \end{array} \right] q^{j+1-l} \left[ \begin{array}{c} m-l \\ j-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l)$$

$$+ \sum_{l=1}^{i+1} \left[ \begin{array}{c} i \\ l-1 \end{array} \right] \left[ \begin{array}{c} m-l \\ j+1-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l)$$

$$+ \sum_{l=1}^{i+1} \left[ \begin{array}{c} i \\ l-1 \end{array} \right] \left[ \begin{array}{c} m-l \\ j-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j+1-l),$$

where (78) follows from (50). Let us denote the four summations in the right hand side of Eq. (79) as $A$, $B$, $C$, and $D$ respectively. We have $A = q^{j+1} P_{j+1}(i; m, n)$, and

$$B = \sum_{l=0}^{i} \left[ \begin{array}{c} i \\ l \end{array} \right] \left[ \begin{array}{c} m-l \\ j-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j-l) (q^{n-i+l} - q^j),$$

$$C = \sum_{l=0}^{i} \left[ \begin{array}{c} i \\ l \end{array} \right] q^i \left[ \begin{array}{c} i \\ l \end{array} \right] \left[ \begin{array}{c} m-l-1 \\ j-l \end{array} \right] (-1)^l+1 q^{\sigma_{i+1}} q^{(l+1)(n-i)\alpha} (n-i, j-l)$$

$$= -q^{j+n-i} \sum_{l=0}^{i} \left[ \begin{array}{c} i \\ l \end{array} \right] \left[ \begin{array}{c} m-l \\ j-l \end{array} \right] (-1)^l q^{\sigma_l} q^{(n-i)\alpha} (n-i, j-l) q^{m-j-1} q^{m-l-1} q^{m-j-1} q^{m-l-1},$$

$$D = -q^{n-i} \sum_{l=0}^{i} \left[ \begin{array}{c} i \\ l \end{array} \right] \left[ \begin{array}{c} m-l \\ j-l \end{array} \right] (-1)^l q^{\sigma_{i+1}} q^{(n-i)\alpha} (n-i, j-l) q^{j+l} q^{m-j-1} q^{m-l-1},$$
where (81) follows from (51) and (82) follows from both (82) and (51). Combining (80), (81), and (82), we obtain

\[ B + C + D = \sum_{l=0}^{i} \left[ \binom{i}{l} \binom{m - l}{j - l} \right] (1)^l q^{\sigma_i + i} q^{l(n-i)} \alpha(n - i, j - l) \ldots \]

\[ \ldots \left\{ q^{n-i+l} - q^l \right\} q^{m-1} \right\} \]

\[ = -q^l P_j(i; m, n). \]

\[ \square \]

1. Proof of Proposition [19]

Before proving Proposition [19], we need two technical lemmas.

Lemma 25: For all \( m, \nu, \) and \( l, \) we have

\[ \delta(m, \nu, j) \triangleq \sum_{i=0}^{j} \left[ \binom{j}{i} \right] (1)^i q^{\sigma_i} \alpha(m - i, \nu) = \alpha(\nu, j) \alpha(m - j, \nu - j) q^{j(m-j)}. \] (83)

Proof: The proof is by induction on \( j. \) When \( j = 0, \) the claim trivially holds. Let us suppose it holds for \( j. \) We have

\[ \delta(m, \nu, j + 1) = \sum_{i=0}^{j+1} \left[ \binom{j+1}{i} \right] (1)^i q^{\sigma_i} \alpha(m - i, \nu) \]

\[ = \sum_{i=0}^{j+1} \left( q^{i} \left[ \binom{j+1}{i} \right] + \left[ \binom{j+1}{i} - \binom{j}{i-1} \right] \right) (1)^i q^{\sigma_i} \alpha(m - i, \nu) \] (84)

\[ = \sum_{i=0}^{j} q^{i} \left[ \binom{j}{i} \right] (1)^i \delta(m - i, \nu) + \sum_{i=0}^{j+1} \left[ \binom{j+1}{i} - \binom{j}{i-1} \right] q^{i} \delta(m - i, \nu) \]

\[ = \sum_{i=0}^{j} q^{i} \left[ \binom{j}{i} \right] (1)^i \delta(m - i, \nu) - \sum_{i=0}^{j} \left[ \binom{j}{i} \right] (1)^i \delta(m - i, \nu) \]

\[ = \alpha(\nu, j + 1) \alpha(m - j - 1, \nu - j - 1) q^{(j+1)(m-j-1)}, \]

where (84) follows from (50).

Lemma 26: For all \( n, \nu, \) and \( j, \) we have

\[ \theta(n, \nu, j) \triangleq \sum_{l=0}^{j} \left[ \binom{j}{l} \right] q^{l(n-\nu)} (-1)^l q^{\sigma_l} \alpha(\nu - l, j - l) = (-1)^j q^{\sigma_j} \left[ \frac{n-j}{n-\nu} \right]. \] (85)
Proof: The proof goes by induction on \( j \). When \( j = 0 \), the claim trivially holds. Let us suppose it holds for \( \tilde{j} \). We have

\[
\theta(n, \nu, \tilde{j} + 1) = \sum_{l=0}^{\tilde{j}+1} \binom{\tilde{j} + 1}{l} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} + 1 - l) = \sum_{l=0}^{\tilde{j}+1} \binom{\tilde{j}}{l} + q^{\tilde{j}+1-l} \left( \binom{\tilde{j}}{l-1} \right) \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} + 1 - l) \tag{86}
\]

\[
= \sum_{l=0}^{\tilde{j}} \binom{\tilde{j}}{l} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} - l) (q^{\nu - l} - q^{\tilde{j}-l}) + \sum_{l=1}^{\tilde{j}+1} q^{\tilde{j}-l+1} \binom{\tilde{j}}{l-1} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} - l + 1), \tag{87}
\]

where (86) follows from (49). Let us denote the first and second summations in the right hand side of (87) as \( A \) and \( B \), respectively. We have

\[
A = (q^{\nu} - q^{\tilde{j}}) \sum_{l=0}^{\tilde{j}} \binom{\tilde{j}}{l} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} - l) = (q^{\nu} - q^{\tilde{j}}) \theta(n - 1, \nu, \tilde{j}) = (q^{\nu} - q^{\tilde{j}})(-1)^j q^{\nu j} \left( \binom{n - \tilde{j}}{n - \nu} \right), \tag{88}
\]

and

\[
B = \sum_{l=1}^{\tilde{j}} q^{\tilde{j}-l} \binom{\tilde{j}}{l-1} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (l+1)(n-\nu) (-1)^{l+1} q^{\nu l+1} \alpha(\nu - l, \tilde{j} - l) = -q^{\tilde{j}+n-\nu} \sum_{l=0}^{\tilde{j}} \binom{\tilde{j}}{l} \left( \binom{n - \tilde{j}}{\nu - l} \right) q^l (n-\nu) (-1)^l q^{\nu l} \alpha(\nu - l, \tilde{j} - l) = -q^{\tilde{j}+n-\nu} \theta(n - 1, \nu - 1, \tilde{j}) = -q^{\tilde{j}+n-\nu} (-1)^j q^{\nu j} \left( \binom{n - \tilde{j}}{n - \nu} \right). \tag{89}
\]

Combining (86), (88), and (89), we obtain

\[
\theta(n, \nu, \tilde{j} + 1) = (-1)^{\tilde{j}} q^{\nu j} \left( (q^{\nu} - q^{\tilde{j}}) \left( \binom{n - \tilde{j}}{n - \nu} \right) - q^{\tilde{j}+n-\nu} \left( \binom{n - \tilde{j}}{n - \nu} \right) \right) = (-1)^{\tilde{j}+1} q^{\nu j+1} \left( \binom{n - \tilde{j}}{n - \nu} \right) \left( (q^{\nu-j} - 1) \frac{q^{n-\nu} - 1}{q^{\nu-j} - 1} + q^{n-\nu} \right) \tag{90}
\]

\[
= (-1)^{\tilde{j}+1} q^{\nu j+1} \left( \binom{n - \tilde{j}}{n - \nu} \right), \tag{91}
\]

where (90) follows from (52).
We now give a proof of Proposition 19.

Proof: We apply the $q^{-1}$-derivative with respect to $y$ to Eq. (44) $\nu$ times, and we apply $x = y = 1$.

By Lemma 17 the left hand side (LHS) becomes

$$
\sum_{i=\nu}^{n} q^{\nu(1-i)+\sigma_{\nu}} \beta(i, \nu) A_i = q^{\nu(1-n)+\sigma_{\nu}} \beta(\nu, \nu) \sum_{i=\nu}^{n} \left[ \nu \right] q^{\nu(n-i)} A_i.
$$

(92)

The right hand side (RHS) becomes $q^{m(k-n)} \sum_{j=0}^{n} B_j \psi_j(1, 1)$, where

$$
\psi_j(x, y) \overset{\text{def}}{=} [b_j(x, y; m) \ast a_{n-j}(x, y; m)]^{[\nu]}
$$

(93)

$$
= \sum_{l=0}^{\nu} \left[ \nu \right] q^{(n-j-\nu+l)} b_j^{(l)}(x, y; m) \ast a_{n-j}^{(\nu-l)}(x, y; m-l)
$$

(94)

$$
= \sum_{l=0}^{\nu} \left[ \nu \right] q^{(n-j-\nu+l)} (-1)^{j} \beta(j, l) \beta(n-j, \nu-l) q^{-\sigma_{\nu}} \ldots
$$

(95)

$$
\ldots \quad \beta(\nu, \nu) q^{-\sigma_{\nu}} \sum_{l=0}^{\nu} \left[ j \right] \left[ n-j \right] \nu-l \right] q^{(n-j)} (-1)^{j} q^{\sigma_{\nu}} \ldots
$$

(96)

where (93) and (94) follow from Lemmas 18 and 17 respectively.

We have

$$
[b_{j-l} \ast \alpha(m-l, \nu-l) a_{n-j-\nu+l}](1, 1; m-\nu) \ldots
$$

$$
= \sum_{u=0}^{n-\nu} \sum_{i=0}^{u} q^{(n-j-\nu+l)} \left[ j-l \right] \left[ n-j-l \right] \nu-l \right] \left[ u-i \right] \nu-i \right] \alpha(m-\nu-i, u-i)
$$

$$
= q^{m-n-\nu-j+l} \sum_{i=0}^{j-l} \left[ j-l \right] \left[ n-j-l \right] \nu-l \right] \alpha(m-l, \nu-l)
$$

(95)

$$
= q^{m-n-\nu-j+l} \alpha(\nu-l, j-l) \alpha(m-j, \nu-j) q^{j-l}(m-j),
$$

(96)

where (95) follows from Lemma 25.

Hence

$$
\psi_j(1, 1) = \beta(\nu, \nu) q^{m-n-\nu-j+1-n} + \sigma_{\nu} \alpha(m-j, \nu-j) q^{j-\nu-j} \ldots
$$

$$
\ldots \sum_{l=0}^{j} \left[ \nu \right] \left[ n-j \right] \nu-l \right] \alpha(\nu-l, j-l)
$$

$$
= \beta(\nu, \nu) q^{m-n-\nu-j+1-n} + \sigma_{\nu} \alpha(m-j, \nu-j) q^{j-\nu-j} (-1)^{j} q^{\sigma_{\nu}} \left[ n-j \right],
$$

(96)

where (96) follows from Lemma 26. Incorporating this expression for $\psi_j(1, 1)$ in the definition of the RHS and rearranging both sides, we obtain the result.
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| $m$ | $n$ | $\rho = 1$ | $\rho = 2$ | $\rho = 3$ | $\rho = 4$ | $\rho = 5$ | $\rho = 6$ |
|-----|-----|-----------|-----------|-----------|-----------|-----------|-----------|
| 2   | 2   | b 3-4 A   | 1         |           |           |           |           |
| 3   | 2   | b 4 B     | 1         |           |           |           |           |
|     | 3   | b 11-32 C | a 2-4 C   | 1         |           |           |           |
| 4   | 2   | b 7-8 B   | 1         |           |           |           |           |
|     | 3   | b 40-64 B | b 4-8 C   | 1         |           |           |           |
|     | 4   | c 293-1024 C | b 10-64 C | a 2-8 C   | 1         |           |           |
| 5   | 2   | b 12-16 B | 1         |           |           |           |           |
|     | 3   | b 154-256 B | b 6-8 B   | 1         |           |           |           |
|     | 4   | b 2267-4096 B | b 33-256 C | a 3-8 C   | 1         |           |           |
|     | 5   | b 34894-2^{17} C | b 233-2979 E | b 10-128 C | a 2-8 C   | 1         |           |
| 6   | 2   | b 23-32 B | 1         |           |           |           |           |
|     | 3   | b 601-1024 B | a 10-16 B | 1         |           |           |           |
|     | 4   | b 17822-2^{15} B | b 123-256 B | b 6-16 C   | 1         |           |           |
|     | 5   | b 550395-2^{20} B | b 1770-2^{14} C | c 31-256 C | a 3-16 C   | 1         |           |
|     | 6   | c 17318410-2^{26} C | c 27065-424990 E | c 214-4299 E | c 9-181 D | a 2-16 C   | 1         |
| 7   | 2   | b 44-64 B | 1         |           |           |           |           |
|     | 3   | b 2372-4096 B | a 19-32 B | 1         |           |           |           |
|     | 4   | b 141231-2^{18} B | c 484-1024 B | b 10-16 B  | 1         |           |           |
|     | 5   | b 8735289-2^{24} B | b 13835-2^{15} B | b 112-1024 C | a 5-16 C  | 1         |           |
|     | 6   | b 549829402-2^{30} B | c 42229-2^{22} C | b 1584-2^{15} C | b 31-746 E | a 3-16 C  | 1         |
|     | 7   | b 34901004402-2^{37} C | c 13205450-244855533 E | b 23978-596534 E | c 203-5890 E | a 8-242 D | a 2-16 C  | 1         |

**TABLE I**

**Bounds on $K_R(q^m, n, \rho)$, for $2 \leq m \leq 7$, $2 \leq n \leq m$, and $1 \leq \rho \leq 6$.** For each set of parameters, the tightest lower and upper bounds on $K_R(q^m, n, \rho)$ are given, and letters associated with the numbers are used to indicate the tightest bound. The lower case letters a–c correspond to the lower bounds in (11), (15), and (17) respectively. The upper case letters A–E denote the upper bounds in (11), (18), (19), (20), and (21) respectively.
| $m$ | $n$ | $\rho = 2$ | $\rho = 3$ | $\rho = 4$ | $\rho = 5$ | $\rho = 6$ |
|-----|-----|-----------|-----------|-----------|-----------|-----------|
| 4   | 4   | 1-2       | 1         | 0         |           |           |
| 5   | 4   | 1-2       | 1         | 0         |           |           |
|     | 5   | 2-3       | 1-2       | 1         | 0         |           |
| 6   | 4   | 2         | 1         | 0         |           |           |
|     | 5   | 2-3       | 1-2       | 1         | 0         |           |
|     | 6   | 3-4       | 2-3       | 1-2       | 1         | 0         |
| 7   | 4   | 2         | 1         | 0         |           |           |
|     | 5   | 2-3       | 1-2       | 1         | 0         |           |
|     | 6   | 3-4       | 2-3       | 1-2       | 1         | 0         |
|     | 7   | 4-5       | 3-4       | 2-3       | 1-2       | 1         |
| 8   | 4   | 2         | 1         | 0         |           |           |
|     | 5   | 3         | 2         | 1         | 0         |           |
|     | 6   | 3-4       | 2-3       | 1-2       | 1         | 0         |
|     | 7   | 4-5       | 3-4       | 2-3       | 1-2       | 1         |
|     | 8   | 5-6       | 3-5       | 2-4       | 1-3       | 1-2       |

**TABLE II**

**Bounds on $k$ for $q = 2$, $4 \leq m \leq 8$, $4 \leq n \leq m$, and $2 \leq \rho \leq 6$.**