THE STRONG LEFSCHETZ PROPERTY IN CODIMENSION TWO

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Abstract. Every artinian quotient of $K[x,y]$ has the strong Lefschetz property if $K$ is a field of characteristic zero or is an infinite field whose characteristic is greater than the regularity of the quotient. We improve this bound in the case of monomial ideals. Using this we classify when both bounds are sharp. Moreover, we prove that the artinian quotient of a monomial ideal in $K[x,y]$ always has the strong Lefschetz property, regardless of the characteristic of the field, exactly when the ideal is lexsegment. As a consequence we describe a family of non-monomial complete intersections that always have the strong Lefschetz property.

1. Introduction

Let $K$ be an infinite field of arbitrary characteristic, and let $I$ be a homogeneous artinian ideal in $S = K[x_1, \ldots, x_n]$. The quotient $S/I$ is said to have the strong Lefschetz property if there exists a linear form $\ell \in [S/I]_1$ such that for all integers $d \geq 0$ and $t \geq 1$ the map $\times \ell^t : [S/I]_d \to [S/I]_{d+t}$ has maximal rank. In this case, $\ell$ is called a strong Lefschetz element of $S/I$. If the maps have maximal rank for $t = 1$, then $S/I$ is said to have the weak Lefschetz property, and $\ell$ is called a weak Lefschetz element of $S/I$.

The Lefschetz properties have been studied extensively; see the recent survey by Migliore and Nagel [17] and the references contained therein. The interest in these properties largely stems from constraints on the Hilbert functions of quotients that have the weak or strong Lefschetz property (see, e.g., [2, 9, 18]).

Until recently, most results have focused on characteristic zero or on at least three variables. For artinian quotients of $K[x, y]$, this is not without reason: the weak Lefschetz property always holds, regardless of characteristic. This was explicitly proven for characteristic zero by Harima, Migliore, Nagel, and Watanabe in [9, Proposition 4.4] (see the note following the next theorem for more on the characteristic zero case). It was proven for arbitrary characteristic by Migliore and Zanello in [18, Corollary 7], though it was not specifically stated therein as noted by Li and Zanello in [12, Remark 2.6] (see also [7]).

Theorem 1.1. [18, Corollary 7] Every artinian ideal in $K[x,y]$ has the weak Lefschetz property, regardless of the characteristic of $K$.

Further still, the strong Lefschetz property is known to hold when the characteristic is zero or greater than the regularity of the quotient.

Theorem 1.2. Let $I$ be a homogeneous artinian ideal in $R = K[x,y]$, where $K$ is a field of characteristic $p \geq 0$. Then $R/I$ has the strong Lefschetz property if $p = 0$ or $p > \text{reg } R/I$.

This result has a varied history. The characteristic zero part was first explicitly given by Harima, Migliore, Nagel, and Watanabe [9, Proposition 4.4]. Their proof relies on the
generic initial ideal being strongly-stable. Recall that the generic initial ideal is strongly-stable in characteristic zero but also in characteristics larger than the largest exponent of a minimal generator of the ideal (see, e.g., [10] Proposition 4.2.4(b))). Hence the proof of [9] Proposition 4.4] extends to the positive characteristic restriction given above. Using a different approach, Basili and Iarrobino proved a much stronger result [11, Theorem 2.16] which reduces to the theorem as stated above. Further still, Iarrobino has pointed out to us that the characteristic zero part follows from a much earlier result of Briançon [3] and the positive characteristic part follows from an earlier result of his own [11, Theorem 2.9].

In this paper we consider the presence of the strong Lefschetz property for homogeneous artinian quotients of $R = K[x, y]$, where the characteristic of $K$ is positive. In Section 2 we recall some needed definitions and introduce the width function of a monomial ideal. The possible width functions are classified in Proposition 2.1 which is analogous to Macaulay’s Theorem for Hilbert functions. In Section 3 we derive conditions to determine when the multiplication map $\times t : [R/I]_d \to [R/I]_{d+t}$ has maximal rank for monomial quotients of $R$.

Section 4 contains the main results of this paper. In particular, Theorem 4.2 bounds the characteristics in which the strong Lefschetz property can be absent from monomial quotients by means of the width function. From this we recover Theorem 1.2 using different techniques than used in [9] and [11]. Furthermore, we classify when the bounds in Theorem 4.2 and Theorem 1.2 are sharp in Corollary 4.7 and Corollary 4.8 respectively. In Theorem 4.11 we show that a monomial quotient always has the strong Lefschetz property if and only if it is an artinian quotient of a lexsegment ideal.

We close with some observations in Section 5. We use Proposition 5.2 to show that there exist non-monomial complete intersections that always have the strong Lefschetz property, and thus the presence of the strong Lefschetz property for complete intersections is not determined by the ci-type. In Sections 5.2 and 5.3 we briefly describe connections to enumerative combinatorics and the weak Lefschetz property in codimension three, respectively.

Throughout the remainder of this paper $R = K[x, y]$, where $K$ is an infinite field of characteristic $p \geq 0$.

2. The width function

Let $I$ be a homogeneous ideal of $S = K[x_1, \ldots, x_n]$. Recall that each component $[S/I]_d$ is a finite dimensional $K$-vector space, and the Hilbert function of $S/I$ is the function $h_{S/I} : \mathbb{N}_0 \to \mathbb{N}$, where $h(d) := h_{S/I}(d) := \dim_K [S/I]_d$. If there is an integer $r$ such that $h(i) > 0$ if and only if $0 \leq i \leq r$, then $S/I$ is said to be artinian; in this case, $r$ is the regularity of $S/I$ and is denoted reg $S/I$. If $S/I$ is artinian and $r = \text{reg } S/I$, then we call the finite sequence $(h(0), \ldots, h(r))$, where $h = h_{S/I}$, the $h$-vector of $S/I$. Further still, the initial degree of $I$ is the smallest degree of a minimal generator of $I$ and is denoted $\text{indeg } I$. Thus $[S/I]_i \cong [S]_i$ for $0 \leq i < \text{indeg } I$.

2.1. Lexsegment ideals & Macaulay’s Theorem.

Suppose $x_1 > \cdots > x_n$ in $S$. The lexicographic order on the monomials in $S$ is given by $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if either $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the leftmost nonzero component of the vector $(b_1, \ldots, b_n) - (a_1, \ldots, a_n)$ is negative. On the other hand, the reverse lexicographic order on the monomials in $S$ is given by $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if either $\sum_{i=1}^n a_i > \sum_{i=1}^n b_i$ or $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and the rightmost nonzero component of the
vector \((b_1, \ldots, b_n) - (a_1, \ldots, a_n)\) is positive. Notice that in the case of two variables \(n = 2\), these two orders are the same.

A monomial ideal \(I\) in \(S\) is **lexsegment in degree** \(d\) if for any two monomials \(u, v \in [I]_d\) and any monomial \(m \in [S]_d\) such that \(u \leq m \leq v\) in the lexicographic order, then \(m \in I\). If \(I\) is lexsegment in every degree, then \(I\) is said to be a (completely) lexsegment ideal. Further, a lexsegment ideal \(I\) is called an **initial lexsegment ideal** if \(x^d \in I\) for every degree \(d\) such that \([I]_d \neq \emptyset\).

In order to state Macaulay’s Theorem (see, e.g., [10, Theorem 6.3.8]), we must first define some notation. Let \(A = (a_1, \ldots, a_s)\) be a nonnegative integer \(s\)-tuple. This allows us to immediately classify the Hilbert functions of ideals in two variables.

**Theorem 2.1** (Macaulay’s Theorem). Let \(h : \mathbb{N}_0 \to \mathbb{N}_0\) be a function. The following statements are equivalent:

(i) \(h\) is the Hilbert function of a standard graded \(K\)-algebra,

(ii) \(h\) is the Hilbert function of an initial lexsegment quotient in \(h(1)\) variables, and

(iii) \(h(0) = 1\) and \(h(d + 1) \leq h(d)(d)\) for all \(d \geq 1\).

This allows us to immediately classify the Hilbert functions of ideals in two variables.

**Proposition 2.2.** Let \(h : \mathbb{N}_0 \to \mathbb{N}_0\) be a function. Then \(h\) is the Hilbert function of some (proper) homogeneous quotient in \(R = K[x, y]\) if and only if there exists a nonnegative integer \(d\) so that \(h(j) = j + 1\) for \(0 \leq j \leq d\) and \(h(j) \geq h(j + 1) \geq 0\) for all \(j \geq d\).

Moreover, we classify the Hilbert functions that force a monomial ideal to be lexsegment.

**Lemma 2.3.** Suppose \(h : \mathbb{N}_0 \to \mathbb{N}_0\) is the Hilbert function of some quotient of \(R = K[x, y]\). Then every monomial quotient \(R/I\) with \(h_{R/I} = h\) is a lexsegment quotient if and only if for every nonnegative integer \(d\) such that \(h(d) > h(d + 1)\) then \(h(d + 1) = h(d + 2)\).

**Proof.** We prove the negation of the desired statement in two parts. Moreover, all comparisons of monomials in \(R = K[x, y]\) are in the lexicographic order with \(x > y\), and all ordered sets are presented in ascending order.

Suppose that there exists a nonnegative integer \(d\) such that \(h(d) > h(d + 1)\) and \(h(d + 1) \neq h(d+2)\). By Proposition 2.2, once a Hilbert function is weakly decreasing, then it must remain so; hence \(h(d + 1) > h(d+2)\). Let \(I\) be the initial lexsegment ideal with Hilbert function \(h\), as guaranteed by Macaulay’s Theorem (see Theorem 2.1). By construction, \([I]_d\) is spanned by \(a = d + 1 - h(d)\) monomials of degree \(d\), in particular, by the set \(A = \{x^{d-a+1}y^{a-1}, \ldots, x^d\}\). Similarly, \([I]_{d+1}\) and \([I]_{d+2}\) are spanned by \(b = d + 2 - h(d+1)\) monomials of degree \(d+1\) and \(c = d + 3 - h(d+2)\) monomials of degree \(d+2\), respectively; let \(B = \{x^{d-b+2}y^{b-1}, \ldots, x^{d+1}\}\) and \(C = \{x^{d-c+3}y^{c-1}, \ldots, x^{d+2}\}\) be those monomials.

Let \(B’ = \{x^{d-b+1}y^b, x^{d-b+3}y^{b-2}, \ldots, x^{d+1}\}\). Since \(h(d) > h(d + 1)\), we have that \(b - a \geq 2\) and so every product of a member of \(A\) with either \(x\) or \(y\) is in \(B’\); in particular, \(x^{d-a+1}y^{a-1}y\) is in \(B’\) since \(a \leq b - 2\). Further, since \(h(d + 1) > h(d + 2)\), we have that \(c - b \geq 2\). Hence every product of a member of \(B’\) with \(x\) or \(y\) is in \(C\); in particular, \(x^{d-b+1}y^{b}y\) is in \(C\) since \(b + 1 \leq c - 1\). Let \(J\) be given by \([J]_i = [I]_i\) for \(i \neq d + 1\) and suppose \([J]_{d+1}\) spanned by \(B’\). Then \(J\) is an ideal with Hilbert function \(h_{R/J} = h\) but is not lexsegment in degree \(d + 1\).

Now suppose that there exists a monomial quotient \(R/I\) with \(h_{R/I} = h\) that is not lexsegment. That is, there is a degree, say, \(d + 1\), such that \([I]_{d+1}\) is spanned by \(d + 2 - h(d+1)\)
monomials such that there are at most \( d - h(d + 1) \) pairs of consecutive monomials. For every monomial \( m \) in \([I]_d\), \( \{ym, xm\} \) forms a consecutive pair in \([I]_{d+1}\), so there can be at most \( d - h(d + 1) \) monomials spanning \([I]_d\). Since exactly \( d + 1 - h(d) \) monomials span \([I]_d\), then \( h(d) + 1 \leq h(d) \). Moreover, for every monomial \( m' \) in \([I]_{d+1}\), the monomials \( xm' \) and \( ym' \) are in \([I]_{d+2}\). However, consecutive monomials overlap in exactly one multiple. Hence there are at least \( 2(d + 2 - h(d + 1)) - (d - h(d + 1)) = d + 4 - h(d + 1) \) monomials in \([I]_{d+2}\). Since exactly \( d + 3 - h(d + 2) \) monomials span \([I]_{d+2}\), then \( h(d + 2) + 1 \leq h(d + 1) \). Thus we have that \( h(d) > h(d + 1) > h(d + 2) \). \[\square\]

2.2. The width function of a monomial ideal.

Throughout this section, all comparisons of monomials in \( R = K[x, y] \) are in the lexicographic order with \( x > y \).

Let \( I \) be a (not necessarily artinian or lexsegment) monomial ideal of \( R \). The \textit{width function} of \( R/I \) is the function \( w_{R/I} : \mathbb{N}_0 \to \mathbb{N}_0 \) defined as follows. If \( 0 \leq d < \text{indeg} I \), then \( w_{R/I}(d) = 0 \). Suppose \( d \geq \text{indeg} I \). Let \( b \) be the smallest integer so that \( x^b y^{d-b} \in I \), and let \( c \) be the largest integer so that \( x^c y^{d-c} \in I \). Then \( w_{R/I}(d) = c - b + 1 \) is the “width” of the monomials in \([I]_d\). If \( R/I \) is artinian and \( r = \text{reg} R/I \), then we call the finite sequence \((w(0), \ldots, w(r))\), where \( w = w_{R/I} \), the \textit{w-vector} of \( R/I \).

Example 2.4. Let \( I = (x^6, x^3y, xy^5, y^5) \). Then the h-vector of \( R/I \) is \((1, 2, 3, 4, 4, 2)\) and the w-vector of \( R/I \) is \((0, 0, 0, 0, 1, 5)\). Notice that in degree 4, the only monomial in \([I]_4\) is \( x^3y \), hence \( w_{R/I}(4) = 1 \). However, in degree 5, the monomials in \([I]_5\) are \( y^5, xy^4, x^2y^2 \), and \( x^4y \), so \( w_{R/I}(5) = 5 \).

We now classify the possible width functions of monomial ideals.

Proposition 2.5. Let \( w : \mathbb{N}_0 \to \mathbb{N}_0 \) be a function, and let \( R = K[x, y] \). The following statements are equivalent:

(i) \( w = w_{R/I} \), where \( I \) is a monomial ideal,
(ii) \( w = w_{R/I} \), where \( I \) is an initial lexsegment ideal,
(iii) \( w(d) = d + 1 - h_{R/I}(d) \), where \( I \) is an initial lexsegment ideal, and
(iv) either \( w(d) = d + 1 \) for all \( d \geq 0 \) or there exists an integer \( m > 0 \) so that \( w(d) = 0 \) for \( d < m \) and \( 1 \leq w(d) < w(d + 1) \leq d + 2 \), for \( d \geq m \).

Proof. Clearly (ii) implies (i). We proceed by showing (i) \(\Rightarrow\) (iv) \(\Rightarrow\) (iii) \(\Rightarrow\) (ii).

(i) \(\Rightarrow\) (iv): Let \( I \) be a monomial ideal with width function \( w = w_{R/I} \). If \( R = I \), then \( w(d) = d + 1 \) for all \( d \). Suppose \( I \neq R \), then \( \text{indeg} I \geq 1 \). By construction, \( w(d) = 0 \) for \( d < \text{indeg} I \). Let \( d \geq \text{indeg} I \), and set \( b = \min\{i \in \{x^i y^{d-i} \in I \} \} \) and \( c = \max\{i \in \{x^i y^{d-i} \in I \} \} \), i.e., \( w(d) = c - b + 1 \). Clearly \( 0 \leq b \leq c \leq d \), so \( 1 \leq w(d) \leq d + 1 \). Moreover, since \( x^b y^{d-b} \) and \( x^c y^{d-c} \) are both in \( I \), then \( x^b y^{d+1-b} \) and \( x^{c+1} y^{d-c} \) are both in \( I \). Thus \( b \geq \min\{i \in \{x^i y^{d+1-i} \} \} \) and \( c + 1 \leq \max\{i \in \{x^i y^{d+1-i} \in I \} \} \), and so \( w(d + 1) \geq c + 1 - b + 1 > w(d) \).

(iv) \(\Rightarrow\) (iii): If \( w(d) = d + 1 \) for all \( d \geq 0 \), then \( w(d) = d + 1 - h_{R/I}(d) \), where \( I = R \), which is clearly an initial lexsegment ideal. Suppose now \( w : \mathbb{N}_0 \to \mathbb{N}_0 \) is a function such that there exists an integer \( m > 0 \) so that \( w(d) = 0 \) for \( d < m \) and \( 1 \leq w(d) < w(d + 1) \leq d + 2 \), for \( d \geq m \). Define \( h : \mathbb{N}_0 \to \mathbb{N}_0 \) by \( h(d) = d + 1 - w(d) \) for \( d \geq 0 \). Thus \( h(0) = 1 \), \( h(1) \leq 2 \), and \( h(d) = d + 1 \) for \( 0 \leq d < m \). Moreover, since \( w(d) < w(d + 1) \leq d + 2 \) for \( d \geq m \), then \( h(d) = d + 1 - w(d) \geq d + 2 - w(d + 1) = h(d + 1) \geq 0 \). Hence by Proposition 2.2, \( h \) is a Hilbert function of some proper homogeneous quotient of \( R \). Thus by Macaulay’s Theorem (see Theorem 2.1), there exists an initial lexsegment ideal \( I \) such that \( h = h_{R/I} \).
(iii) ⇒ (ii): Let \( w(d) = d + 1 - h_{R/I}(d) \), where \( I \) is an initial lexsegment ideal. As \( I \) is lexsegment in degree \( d \), for all \( d \), then there are \( d + 1 - h_{R/I}(d) = w(d) \) monomials of degree \( d \) in \( I \). Moreover, these \( w(d) \) monomials are consecutive in the lexicographic order and so \( w_{R/I}(d) = w(d) \).

Thus the four statements (i)–(iv) are indeed equivalent. \( \square \)

Further, we classify the width functions that force a monomial ideal to be lexsegment.

**Lemma 2.6.** Suppose \( w: \mathbb{N}_0 \to \mathbb{N}_0 \) is the width function of some monomial quotient of \( R = K[x, y] \). Then every monomial quotient \( R/I \) with \( w_{R/I} = w \) is lexsegment if and only if for every nonnegative integer \( d \) we have \( 0 \leq w(d + 1) - w(d) \leq 2 \).

**Proof.** We prove the negation of the desired statement in two parts. Moreover, all comparisons of monomials in \( R = K[x, y] \) are in the lexicographic order with \( x > y \), and all ordered sets are presented in ascending order.

Suppose that there exists a nonnegative integer \( d \) so that \( w(d + 1) - w(d) > 2 \). Let \( I \) be the initial lexsegment ideal with width function \( w_{R/I} = w \), as guaranteed by Proposition 2.3. Hence \( [I]_{d+1} \) is spanned by the \( w(d + 1) \) monomials \( B = \{x^{d-w(d+1)} y^{w(d+1)-1}, \ldots, x^{d+1}\} \).

Notice that the next to last monomial of \( B \), namely \( x^{d-w(d+1)+2} y^{w(d+1)-1}, \ldots, x^{d+1} \), is not a multiple of a monomial in \( [I]_d \), since \( x^{d-w(d+1)+1} y^{w(d)-1} \) is the smallest monomial in \( [I]_d \) and \( w(d) \leq w(d+1) - 3 \).

Let \( B' \) be \( B \setminus \{x^{d-w(d+1)+1} y^{w(d)-1}\} \). Hence if \( J \) given by \( [J]_i = [I]_i \), for \( i \neq d + 1 \) and \( [J]_{d+1} \) spanned by \( B' \), then \( J \) is an ideal with width function \( w_{R/J} = w \), and \( J \) is not lexsegment in degree \( d + 1 \).

Now suppose that there exists a monomial quotient \( R/I \) with \( w_{R/I} = w \) that is lexsegment in every degree \( i \leq d \) but not lexsegment in degree \( d + 1 \). Thus there are \( w(i) \) consecutive monomials in \( I \) of degree \( i \) for \( i \leq d \), and there are less than \( w(d + 1) \) monomials in \( I \) of degree \( d + 1 \). By Proposition 2.3 we have \( w(d) < w(d + 1) \). Since \( I \) is not lexsegment in degree \( d + 1 \), then there must be at least one monomial that is not consecutive to one of the \( w(d) + 1 \) consecutive multiples of the \( w(d) \) consecutive monomials in degree \( d \). Thus the width in degree \( d + 1 \) must be at least two larger than \( w(d) + 1 \) (one for the absent monomial and one for the guaranteed non-consecutive monomial). That is, \( w(d+1) \geq w(d) + 3 \). \( \square \)

For a monomial ideal \( I \) of \( R \), the number of degree \( d \) monomials not in \( I \) between the smallest and largest degree \( d \) monomials in \( I \) can be derived easily from the width function and the Hilbert function. This number can be thought of as the “lexsegment defect” of \( I \) in degree \( d \).

**Lemma 2.7.** Let \( I \) be a monomial ideal in \( R \), where the monomials of \( R \) are ordered lexicographically. The number of degree \( d \) monomials not in \( I \) between the smallest and largest degree \( d \) monomials in \( I \) is \( w_{R/I}(d) + h_{R/I}(d) - (d + 1) \), which is zero if and only if \( I \) is lexsegment in degree \( d \).

**Proof.** For \( d \geq 0 \), there are \( d + 1 - h_{R/I}(d) \) degree \( d \) monomials not in \( I \), thus there are \( w_{R/I}(d) + 1 - h_{R/I}(d) \) degree \( d \) monomials not in \( I \) between the smallest and largest degree \( d \) monomials in \( I \). Furthermore, \( I \) is lexsegment in degree \( d \) if and only if there are no missing monomials between the smallest and largest degree \( d \) monomials in \( I \). \( \square \)

3. Maximal rank multiplication maps
Throughout this section, all comparisons of monomials in $R = K[x,y]$ are in the lexicographic order with $x > y$, and all ordered sets are presented in ascending order.

Recall that a monomial algebra has the weak (strong) Lefschetz property exactly when the sum of the variables is a weak (strong) Lefschetz element.

**Proposition 3.1.** ([5] Proposition 2.2) Let $I$ be a monomial artinian ideal in the ring $S = K[x_1, \ldots, x_n]$. Then $S/I$ has the weak (strong) Lefschetz property if and only if $x_1 + \cdots + x_n$ is a weak (strong) Lefschetz element of $S/I$.

In particular, Theorem 1.1 immediately implies that we need only look at the maps between equi-dimensional components.

**Lemma 3.2.** Let $I$ be a monomial artinian ideal in $R$. Then $R/I$ has the strong Lefschetz property if and only if the map $\times (x + y)^t : [R/I]_d \to [R/I]_{d+t}$ is a bijection for all integers $d \geq 0$ and $t \geq 1$ where $h_{R/I}(d) = h_{R/I}(d+t) = d+1$.

**Proof.** The presence of the strong Lefschetz property clearly implies the second condition.

Suppose now that the second condition holds. Let $\varphi_{d,d+t}$ be the map $\times (x + y)^t : [R/I]_d \to [R/I]_{d+t}$, and so $\varphi_{d,a+b+c} = \varphi_{a+b,a+b+c} \circ \varphi_{a,a+b}$.

Theorem 1.1 implies that $\varphi_{d,d+1}$ always has maximal rank. In particular, $\varphi_{d,d+1}$ is injective for $0 \leq d < \text{indeg } I$ and surjective for $\text{indeg } I \leq d$. Since $\varphi_{d,d+t} = \varphi_{d+t-1,d+t} \circ \cdots \circ \varphi_{d,d+1}$, if $d + t \leq \text{indeg } I$ (respectively, $d > \text{indeg } I$), then each term in the composition is injective (respectively, surjective) and so $\varphi_{d,d+t}$ is injective (respectively, surjective).

Now suppose that $d < \text{indeg } I < d + t$. Since $d + t > \text{indeg } I$, then $h(d+t) - 1 \leq \text{indeg } I$ and so $h(d+t) - 1 = h(d + t)$ by Proposition 2.2. Thus, by assumption, $\varphi_{h(d+t)-1,d+t}$ is a bijection. If $d < h(d+t) - 1$, then $\varphi_{d,d+t} = \varphi_{h(d+t)-1,d+t} \circ \varphi_{d,h(d+t)-1}$. Since $h(d+t) - 1 \leq \text{indeg } I$, $\varphi_{d,h(d+t)-1}$ is injective. Hence $\varphi_{d,d+t}$ is injective as the composition of injective functions is injective. On the other hand, if $d > h(d+t) - 1$, then $\varphi_{h(d+t)-1,d+t} = \varphi_{d,d+t} \circ \varphi_{h(d+t)-1,d}$. Hence $\varphi_{d,d+t}$ is surjective as $\varphi_{h(d+t)-1,d+t}$ is, by assumption, a bijection. \[\square\]

Let $I$ be a monomial ideal in $R$, and let $d \geq 0$ and $t \geq 1$ be any integers such that $h(d) = h(d + t) = d + 1$. Suppose the ordered monomials $\{x^{b_0}y^{d+t-b_0}, \ldots, x^{b_d}y^{d+t-b_d}\}$ span $[R/I]_{d+t}$. Note that the ordering implies that $0 \leq b_0 < \cdots < b_d \leq d + t$. Then the map $\times (x + y)^t : [R/I]_d \to [R/I]_{d+t}$ is given by the $(d+1) \times (d+1)$ matrix $N_{R/I}(d,d+t)$, where the entry $(i,j)$ is the coefficient on $x^{b_i}y^{d+t-b_j}$ in $x^iy^{d-i}(x + y)^t$, i.e., $(b_j^t - i)$.

**Example 3.3.** Let $I = (x^{10}, y^7)$. Then $h_{R/I}(5) = h_{R/I}(10) = 6$. Thus $N_{R/I}(5,5)$ is the $6 \times 6$ matrix

$$
\begin{pmatrix}
5 & 10 & 10 & 5 & 1 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 \\
0 & 1 & 10 & 10 & 10 & 5 \\
0 & 0 & 1 & 10 & 10 & 10 \\
0 & 0 & 0 & 1 & 10 & 10 \\
0 & 0 & 0 & 0 & 1 & 5
\end{pmatrix}.
$$

The determinant of this matrix is $210 = 2 \cdot 3 \cdot 5 \cdot 7$, and so the map $\times (x + y)^5 : [R/I]_5 \to [R/I]_{10}$ has maximal rank if and only if char $K = 0$ or char $K > 7$.

The determinant of $N_{-}(d,d+t)$ can be given by a closed form.
Lemma 3.4. Let $I$ be a monomial ideal in $R$, and let $d \geq 0$ and $t \geq 1$ be integers so that $h_{R/I}(d) = h_{R/I}(d + t) = d + 1$. Then $\left| \det N_{R/I}(d, d + t) \right|$ is

$$\prod_{0 \leq i < j \leq s-r} (b_{r+j}-b_{r+i}) \sum_{i=0}^{s-r} \frac{(t+i)!}{(t+s-b_{r+i})!(b_{r+i}-r)!},$$

where the ordered monomials $\{x^{b_o}y^{d+t-b_o}, \ldots, x^{b_l}y^{d+t-b_l}\}$ span $[R/I]_{d+t}$, $r := \max(\{0\} \cup \{k+1 \mid b_k = t+k\})$.

If $r = s + 1$, then the determinant is one; otherwise, the largest term of the above closed form is $w_{R/I}(d + t) - 1$, and it appears exactly once.

Proof. For $0 \leq i < r$, we have $b_i = i$, and so the $(i, j)$ entry of $N_{R/I}(d, d + t)$ is $\binom{t}{j-i}$, i.e., $1$ if $i = j$ and $0$ if $i > j$. Similarly, for $s < j \leq d$, we have that $b_j = t + j$ and so the $(i, j)$ entry of $N_{R/I}(d, d + t)$ is $\binom{t}{t+j-i}$, i.e., is $1$ if $i = j$ and $0$ if $i < j$.

Partitioning $N_{R/I}(d, d + t)$ into a block matrix with square diagonal matrices with sizes $r, s - r + 1$, and $d - s$, respectively, yields

$$N_{R/I}(d, d + t) = \begin{pmatrix} U & A & 0 \\ 0 & \hat{N} & 0 \\ 0 & B & L \end{pmatrix},$$

where $U$ is a square upper-triangular matrix with ones on the diagonal, $L$ is a square lower-triangular matrix with ones on the diagonal, and $\hat{N}$ is an $(s-r+1) \times (s-r+1)$ square matrix with entry $(i, j)$ given by $\binom{t}{b_{r+i}-(r+i)}$. Using the block matrix formula for the determinant (twice), we have

$$\det N_{R/I}(d, d + t) = \det U \cdot \det \hat{N} \cdot \det L = \det \hat{N}.$$

Set $A := t$, $n := s - r + 1$, and $L_j := b_{r+j-1} - r + 1$. Then the determinant evaluation of $\hat{N}$, hence of $N_{R/I}(d, d + t)$, follows immediately from Equation (12.5). Re-indexing to start from $0$ instead of $1$ yields the stated result.

If $r = s + 1$, then the products are all empty, hence the determinant is one. Suppose $r \leq s$. Notice that $b_s < t + s$ and $b_r > r$. Hence the largest term in the first product $b_s - b_r$ is less than $t + s - r$, and the largest term in the numerator of the second product is $t + s - r$. The largest term in the denominator of the second product is the maximum of $t + s - b_r$ and $b_s - r$, both of which are less than $t + s - r$. Hence the largest term of the products in the formula is $t + s - r$.

Notice that by the definitions of $r$ and $s$, $x^r y^{d+t-r}$ and $x^{t+s} y^{d-s}$ are the lexicographically smallest and largest monomials in $[I]_{d+t}$, respectively. Hence $w_{R/I}(d + t) = t + s - r + 1$ and so the largest term of the products in the formula is $w_{R/I}(d + t) - 1$.

The matrix $\hat{N}$ in the preceding proof is the matrix $N_{R/J}(s - r, s - r + t)$, where $J$ is the ideal $\langle x^{b_r-r} y^{l+s-b_s}, \ldots, x^{b_s-r} y^{l+s-b_s} + (x, y)^{t+s-r+1} \rangle$.
Example 3.5. Let \( I = (x^{15}, x^{10} y^2, x^2 y^9, y^{15}) \). Then \( h_{R/I}(9) = 10 = h_{R/I}(14) \), so \( N_{R/I}(9, 14) \) is the 10 \( \times \) 10 matrix

\[
\begin{pmatrix}
1 & 5 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 10 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 10 & 5 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 5 & 10 & 5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 5 & 10 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 & 0 \\
\end{pmatrix}
\]

Notice that the upper-left 2 \( \times \) 2 matrix is upper-triangular, the lower-left 2 \( \times \) 2 matrix is lower-triangular, and the central 6 \( \times \) 6 matrix has determinant 210, and so the map \( \times (x + y)^5 : [R/I]_9 \to [R/I]_{14} \) has maximal rank if and only if \( \text{char } K = 7 \).

Indeed, the central 6 \( \times \) 6 matrix is the same as the matrix in Example 3.3 and is the matrix \( \hat{N} \) in the proof of Lemma 3.4. Furthermore, following the preceding remark, we note that \( d = 9, t = 5, r = 2, \) and \( s = 7 \) for \( R/I \) in degree 14. Hence \( \hat{N} \) is the matrix \( N_{R/J}(s - r, s - r + t) = N_{R/J}(5, 10) \), where \( J = (x^{10}, x^3 y^7, x^2 y^8, x y^9, y^{10}) \). Notice that \( R/J \) and the quotient considered in Example 3.3 are the same in degrees 5 and 10.

Analysing the formula in Lemma 3.4, we determine exactly when \( |\det N| = 1 \).

Corollary 3.6. Let \( I \) be a monomial ideal in \( R \), and let \( d \geq 0 \) and \( t \geq 1 \) be integers so that \( h_{R/I}(d) = h_{R/I}(d + t) = d + 1 \). Then the following statements are equivalent:

(i) \( |\det N_{R/I}(d, d + t)| = 1 \),
(ii) \( w_{R/I}(d + t) = t, \) and
(iii) \( I \) is lexsegment in degree \( d + t \).

Specifically, if \( t = 1 \), then \( |\det N_{R/I}(d, d + t)| = 1 \).

Proof. By Lemma 2.7, \( w_{R/I}(d + t) = d + t - 1 - h_{R/I}(d + t) = t \) if and only if \( I \) is lexsegment in degree \( d \), hence (ii) is equivalent to (iii).

Suppose \([R/I]_{d+t}\) is spanned by the ordered monomials \( \{x^{b_0} y^{d+t-b_0}, \ldots, x^{b_d} y^{d+t-b_d}\} \). Set \( r := \max(\{0\} \cup \{k + 1 \mid b_k = k\}) \) and \( s := \min(\{d\} \cup \{k - 1 \mid b_k = t + k\}) \). Then \( r < b_r < \cdots < b_s < s + t \). By Lemma 3.4, \( |\det N_{R/I}(d, d + t)| \) is

\[
\prod_{0 \leq i < j \leq s - r} (b_{r+j} - b_{r+i}) \prod_{i=0}^{s-r} \frac{(t+i)!}{(t+s-b_{r+i})!(b_{r+i}-r)!}.
\]

Further, notice that \( w_{R/I}(d + t) = s - r + t + 1 \).

Now we prove (i) is equivalent to (ii). Let \( r, s, \) and \( t \) be integers such that \( 0 \leq r \leq s + 1 \) and \( t \geq 2 \). Let \( b_r, \ldots, b_s \) be integers such that \( r < b_r < \cdots < b_s < s + t \). Define \( D(r, s, t, \{b_r, \ldots, b_s\}) \) to be

\[
\prod_{0 \leq i < j \leq s - r} (b_{r+j} - b_{r+i}) \prod_{i=0}^{s-r} \frac{(t+i)!}{(t+s-b_{r+i})!(b_{r+i}-r)!}.
\]
Clearly \( D(r, s, t, \{b_r, \ldots, b_s\}) \geq 1 \) for all valid arguments.

Notice that \( D(r, s, t, \{b_r, \ldots, b_s\}) = 1 \) if \( r = s + 1 \), as the products are all empty. Furthermore, if \( t = 1 \) and \( r \leq s \), then \( r < b_r < \cdots < b_s < s + 1 \) implies that there are at least \( s - r + 1 \) distinct integers exclusively between \( r \) and \( s + 1 \); however, there are only \( s - r \) such integers. Hence if \( t = 1 \), then \( r = s + 1 \), and we can define \( D(s + 1, s + 1, 1, \emptyset) = 1 \) for every \( s \). (We also note that the case when \( t = 1 \) follows immediately from Theorem [4].)

**Step 1: Base case.** Note that \( D(s + 1, s + 1, t, \emptyset) = 1 \) for all \( s \) and \( t \). Further, if \( r = s \) and \( t \geq 2 \), then \( D(r, r, t, \{b_r, \ldots, b_s\}) = \binom{s}{r} \geq t \), as \( 1 \leq b_r - r \leq t - 1 \).

**Step 2: Increasing \( t \).** Assume \( r < s \) and \( t \geq 1 \). Clearly since \( b_s < s + t \), then \( b_s < s + t + 1 \). Thus \( (r, s, t + 1, \{b_r, \ldots, b_s\}) \) forms a valid argument for \( D(\cdot) \). Indeed, we can rewrite \( D(r, s, t + 1, \{b_r, \ldots, b_s\}) \) in terms of \( D(r, s, t, \{b_r, \ldots, b_s\}) \) as follows:

\[
D(r, s, t + 1, \{b_r, \ldots, b_s\}) = \prod_{0 \leq i < j \leq s - r} (b_{r+i} - b_{r+i}) \prod_{i=0}^{s-r} \frac{(t+1)!}{(t+1) + s - b_{r+i}!(b_{r+i} - r)!} D(r, s, t, \{b_r, \ldots, b_s\}).
\]

For \( 0 \leq i \leq s - r \), we have \( b_{r+i} > r+i \) and so \( (t+1) + s - b_{r+i} \leq t + s + 1 - (r+i+1) = t + s - r - i \). Further still, \( t + 1 + i = t + s - r - j + 1 \), where \( j = s - r - i \), i.e., \( 0 \leq j \leq s - r \). Thus we have

\[
\prod_{i=0}^{s-r} (t+1) = \prod_{j=0}^{s-r} (t + s - r - j + 1) \geq \prod_{j=0}^{s-r} (t + s - r - j) \geq \prod_{i=0}^{s-r} (t + 1 + s - b_{r+i}).
\]

Hence \( \prod_{i=0}^{s-r} \frac{(t+1)!}{(t+1) + s - b_{r+i}!} > 1 \) and \( D(r, s, t + 1, \{b_r, \ldots, b_s\}) > D(r, s, t, \{b_r, \ldots, b_s\}) \geq 1 \).

Thus, if \( t \geq 2 \) and \( b_s < s + t - 1 \), then \( D(r, s, t, \{b_r, \ldots, b_s\}) > 1 \).

**Step 3: When \( b_s = s + t - 1 \).** Suppose \( r < s \), \( t \geq 2 \), and \( b_s = s + t - 1 \). Since \( r \leq s \), then \( r \leq (s - 1) + 1 \) and so \( (r, s - 1, t, \{b_r, \ldots, b_{s-1}\}) \) forms a valid argument for \( D(\cdot) \). Indeed, we can rewrite \( D(r, s, t, \{b_r, \ldots, b_s\}) \) in terms of \( D(r, s - 1, t, \{b_r, \ldots, b_{s-1}\}) \) as follows:

\[
D(r, s, t, \{b_r, \ldots, b_s\}) = \prod_{0 \leq i < j \leq s - r} (b_{r+j} - b_{r+i}) \prod_{i=0}^{s-r} \frac{(t+1)!}{(t+1) + s - b_{r+i}!(b_{r+i} - r)!} D(r, s - 1, t, \{b_r, \ldots, b_{s-1}\})
\]

\[
= \left( \frac{(t + s - r)!}{(t + s - b_s)!(b_s - r)!} \prod_{0 \leq t < s - r} \frac{b_s - b_{r+i}}{t + s - b_{r+i}} \right) D(r, s - 1, t, \{b_r, \ldots, b_{s-1}\})
\]

\[
= \left( \frac{(t + s - r)(t + s - 1 - b_{r+i})}{t + s - b_{s-1}} \right) D(r, s - 1, t, \{b_r, \ldots, b_{s-1}\}).
\]

Notice though, since \( r < b_{s-1} \), we have \( t + s - r > t + s - b_{s-1} \) and so \( \frac{t + s - r}{t + s - b_{s-1}} > 1 \). Thus, \( D(r, s, t, \{b_r, \ldots, b_s\}) > D(r, s - 1, t, \{b_r, \ldots, b_{s-1}\}) \geq 1 \).

**Conclusion.** We thus see that \( D(r, s, t, \{b_r, \ldots, b_s\}) = 1 \) if and only if \( r = s + 1 \). This is equivalent to \( w_{R/I}(d+t) = t \), as \( w_{R/I}(d+t) = s - r + t + 1 \). \( \square \)
4. The strong Lefschetz property

Let $I$ be a homogeneous ideal of $S = K[x_1, \ldots, x_n]$, and fix a term order $<$ on the monomials of $S$ (e.g., the lexicographic order). The initial ideal $\text{in}_<(g \cdot I)$ that is fixed on some Zariski open subset of $GL_n(K)$ is called the \textit{generic initial ideal} of $I$ with respect to $<$. By Example 4.2.8, for every prime $p > 0$, $\text{gin}(x^p, y^p) = (x^p, y^p)$ if the characteristic of $K$ is $p$.

Remark 4.1. The generic initial ideal is known to be strongly stable if the characteristic of $K$ is zero (see, e.g., [10, Proposition 4.2.6]). For ideals in $K[x, y]$, being strongly stable is equivalent to being lexsegment. However, the generic initial ideal is not so well-behaved in positive characteristic. Indeed, Example 4.2.8 shows that, for every prime $p > 0$, $\text{gin}(x^p, y^p) = (x^p, y^p)$ if the characteristic of $K$ is $p$.

Recall that an artinian ideal has the weak (strong) Lefschetz property exactly when its generic initial ideal with respect to the reverse lexicographic order has the weak (strong) Lefschetz property.

Proposition 4.2. [19, Proposition 2.8] Let $I$ be a homogeneous artinian ideal in $S = K[x_1, \ldots, x_n]$, and let $J$ be the generic initial ideal of $I$ with respect to the reverse lexicographic order. Then $S/I$ has the weak (strong) Lefschetz property if and only if $S/J$ has the weak (strong) Lefschetz property.

Further, for $R = K[x, y]$ the lexicographic and reverse lexicographic orders are identical. In some cases, we can use the above result to lift statements about monomial ideals to statements about homogeneous ideals.

4.1. Bounds on the absence of the strong Lefschetz property.

We first use the width function to bound the characteristics in which the strong Lefschetz property or the strong Lefschetz property can be absent. See Corollary 4.7 for a classification of when this bound is sharp.

Theorem 4.3. Let $I$ be a monomial artinian ideal in $R = K[x, y]$, where $K$ is a field of characteristic $p > 0$. Then $R/I$ has the strong Lefschetz property if $p \geq w_{R/I}(\text{reg } R/I)$.

Proof. Let $d \geq 0$ and $t \geq 1$ be integers such that $h(d) = h(d + t) = d + 1$. Then the largest factor of the determinant of $N_{R/I}(d, d + t)$ is $w_{R/I}(d + t) - 1$ by Lemma 3.4. Hence $N_{R/I}(d, d + t)$ has maximal rank if $p \geq w_{R/I}(d + t)$.

Further, we have that $w_{R/I}(\text{reg } R/I) \geq w_{R/I}(i)$ for all $0 \leq i \leq \text{reg } R/I$ by Proposition 2.5. This implies that every matrix $N_{R/I}(d, d + t)$ has maximal rank if $p \geq w_{R/I}(\text{reg } R/I)$. Therefore, $R/I$ has the strong Lefschetz property by Lemma 3.2 if $p \geq w_{R/I}(\text{reg } R/I)$.

We now recover Theorem 1.2 with different techniques than those used in [9] and [11]. (Nota bene: We have not assumed Theorem 1.2 up to this point; doing so offers no benefit as our approach and desired results are different.) In particular, by weakening the bound in the preceding proof we can generalise from monomial ideals to homogeneous ideals. See Corollary 4.8 for a classification of when this bound is sharp.

Theorem 4.4. Let $I$ be a homogeneous artinian ideal in $R = K[x, y]$, where $K$ is a field of characteristic $p > 0$. Then $R/I$ has the strong Lefschetz property if $p > \text{reg } R/I$.

Proof. Let $J = \text{gin } I$ under the (reverse) lexicographic order. Then by Theorem 4.3, $R/J$ has the strong Lefschetz property if $p \geq w_{R/J}(\text{reg } R/J)$. By Proposition 2.5, we have that $w_{R/J}(\text{reg } R/J) \leq \text{reg } R/J + 1$, and since the generic initial ideal preserves the Hilbert function
we have \( \text{reg} R/J = \text{reg} R/I \) as \( J \) is artinian. Hence \( R/I \) has the strong Lefschetz property if \( p > \text{reg} R/I \) by Proposition 1.2.

**Remark 4.5.** The bound in the preceding theorem sometimes holds in higher codimension. Let \( I \) be a monomial complete intersection in \( S = K[x_1, \ldots, x_n] \), i.e., \( I = (x_1^{d_1}, \ldots, x_n^{d_n}) \). Then we have that \( S/I \) has the strong Lefschetz property if \( p > \text{reg} S/I \) by [6, Theorem 3.6(ii)]. Moreover, this bound is sharp in some cases; in particular, if \( p = \text{reg} S/I \), then \( S/I \) fails to have the strong Lefschetz property by [6, Theorem 3.6(i)]. See Corollary 4.8 for a similar statement about the sharpness of the bound in the preceding corollary.

On the other hand, the bound is not true in general, even as a bound for the failure of the weak Lefschetz property. Let \( I = (x^{20}, y^{20}, z^{20}, x^3y^8z^{13}) \) be an ideal of \( S = K[x, y, z] \). In this case, \( \text{reg} S/I = 50 \), and \( S/I \) has the weak Lefschetz property if and only if the characteristic of \( K \) is not one of the following primes: 2, 3, 5, 7, 11, 17, 19, 23, or 20554657. This example comes from [7], wherein the presence and absence of the weak Lefschetz property for monomial ideals in codimension three is considered.

We now consider for which characteristics the strong Lefschetz property is absent.

**Lemma 4.6.** Let \( I \) be a monomial artinian ideal in \( R = K[x, y] \). Suppose there exists a \( j \) such that \( \text{indeg} I \leq j \leq \text{reg} R/I \), \( w_{R/I}(j) \neq j + 1 - h_{R/I}(j) \), and \( w_{R/I}(j) - 1 = \text{char} K \) is prime. Then \( R/I \) fails to have the strong Lefschetz property.

**Proof.** Let \( d = h_{R/I}(j) - 1 \) and \( t = j - d \), then \( h_{R/I}(d) = d + 1 = h_{R/I}(d + t) \). By Lemma 3.4, since \( w_{R/I}(d + t) > d + 1 - h_{R/I}(d + t) \), we have that the largest factor of the formula giving the determinant of \( N_{R/I}(d + t) \) is \( w_{R/I}(d + t) - 1 = \text{char} K \). This implies that the map \( \times(x + y) : [R/I]_d \rightarrow [R/I]_{d+t} \) fails to have maximal rank, i.e., \( R/I \) fails to have the strong Lefschetz property.

Using this we classify exactly when the bound in Theorem 4.3 is sharp.

**Corollary 4.7.** Let \( I \) be a monomial artinian ideal in \( R = K[x, y] \), and suppose \( p = w_{R/I}(\text{reg} R/I) - 1 \) is prime. Then \( R/I \) fails to have the strong Lefschetz property in characteristic \( p \) if and only if \( x^{\text{reg} R/I}, y^{\text{reg} R/I} \in I \).

**Proof.** Suppose \( p = w_{R/I}(\text{reg} R/I) - 1 \) is prime. If one of \( x^{\text{reg} R/I} \) and \( y^{\text{reg} R/I} \) is not in \( I \), then \( R/I \) can only fail to have the strong Lefschetz property in characteristics smaller than \( p = w_{R/I}(\text{reg} R/I) - 1 \) by Theorem 4.3.

On the other hand, suppose \( x^{\text{reg} R/I}, y^{\text{reg} R/I} \in I \). Then \( w_{R/I}(\text{reg} R/I) \neq \text{reg} R/I + 1 - h_{R/I}(\text{reg} R/I) \), and hence \( R/I \) fails to have the strong Lefschetz property by Lemma 4.6.

As with Theorems 4.3 and 1.2, the above pair of results can be extended to homogeneous ideals, if we strengthen the restrictions on the ideals. This in turn classifies exactly when the bound in Theorem 1.2 is sharp.

**Corollary 4.8.** Let \( I \) be a homogeneous artinian ideal in \( R = K[x, y] \), where \( K \) is a field of characteristic \( p > 0 \). If \( p \leq \text{reg} R/I \) and \( x^p, y^p \in \text{gin} I \) (under the lexicographic order), then \( R/I \) fails to have the strong Lefschetz property in particular, if \( \text{reg} R/I \) is prime and \( p = \text{reg} R/I \), then \( R/I \) fails to have the strong Lefschetz property if and only if \( x^p, y^p \in \text{gin} I \) (under the lexicographic order).

**Proof.** Let \( J = \text{gin} I \) under the (reverse) lexicographic order. If \( x^p, y^p \in J \), then \( w_{R/J}(p) = p + 1 \). Since \( p \leq \text{reg} R/I = \text{reg} R/J \), then \( w_{R/J}(p) \neq p + 1 - h_{R/J}(p) \). Hence, by Lemma 4.6
$R/J$ fails to have the strong Lefschetz property, and so $R/I$ fails to have the strong Lefschetz property by Proposition 4.2.

However, the bounds can be far from sharp in some cases.

**Example 4.9.** Ignoring lexsegment ideals (which always have the strong Lefschetz property by Theorem 4.11), the bounds in Theorems 4.3 and 1.2 are far from sharp in some cases. In particular, consider the ideal $I_n = (x^{2n}, y^2)$, where $n \geq 1$. By [6, Lemma 4.2(i)], we have that $I_n$ has the strong Lefschetz property if and only if the characteristic of $K$ divides $2^n$, i.e., char $K = 2$. However, reg $R/I_n = 2^n$, so the regularity bound is sharp if and only if $n = 1$. Moreover, $w_{R/I_n}(\text{reg } R/I_n) = 2^n + 1$, so the width bound is never sharp for this family.

**Remark 4.10.** Corollary 4.8 further implies that the maximal degree of a minimal generator is not a good bound. For example, let $I_p = (x^{(p+1)/2}, y^{(p+3)/2})$, where $p$ is an odd prime. Then reg $R/I_p = p$ and $x^p, y^p \in I_p$, so by Corollary 4.8 $R/I_p$ fails to have the strong Lefschetz property in characteristic $p$. Notice that the maximal generating degree of a minimal generator of $I_p$ is $\frac{p+3}{2}$, which is less than $p$ for $p > 3$.

### 4.2. Forcing the presence of the strong Lefschetz property.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring, where $K$ is a field of characteristic zero. Then Migliore and Zanello [18, Theorem 5] classified the Hilbert functions of artinian quotients of $S$ that force the weak Lefschetz property to hold. Similarly, Zanello and Zylinski [20, Corollary 3.3] proved all artinian quotients of $S$ with a given Hilbert function have the strong Lefschetz property if and only if the initial lexsegment ideal with the given Hilbert function has the strong Lefschetz property.

In some cases, the strong Lefschetz property always holds, regardless of characteristic. The following theorem classifies the monomial ideals with this property. (See Proposition 5.2 for an infinite family of non-monomial ideals with this property.)

**Theorem 4.11.** Let $I$ be a monomial artinian ideal in $R = K[x, y]$. Then $R/I$ always has the strong Lefschetz property, regardless of the characteristic of $K$, if and only if $I$ is a lexsegment ideal.

**Proof.** Suppose $I$ is not a lexsegment ideal. Then there is a degree $j \geq \text{indeg } I$ such that $I$ is not lexsegment in degree $j$. Let $d = h_{R/I}(j) - 1$ and $t = j - d$, then $h_{R/I}(d) = d + 1 = h_{R/I}(d + t)$. By Corollary 3.6 we see that $N_{R/I}(d, d + t)$ does not always have maximal rank, hence $R/I$ does not always have the strong Lefschetz property.

Suppose now that $I$ is a lexsegment ideal. Then by Corollary 3.6 the matrices $N_{R/I}(d, d + t)$, where $h(d) = h(d + t) = d + 1$, always have maximal rank. Therefore, $R/I$ always has the strong Lefschetz property, regardless of the characteristic of $K$, by Lemma 3.2.

This implies that no Hilbert function (see Proposition 2.2) or width function (see Proposition 2.5) can force the strong Lefschetz property to be absent in some characteristic for all ideals with the given Hilbert or width function. On the other hand, we can describe a large class of Hilbert functions and width functions that force the strong Lefschetz property to be present.

Using Lemma 2.3 we classify the Hilbert functions that force monomial ideals to always have the strong Lefschetz property, regardless of characteristic.

**Proposition 4.12.** Suppose $h : \mathbb{N}_0 \to \mathbb{N}_0$ is the Hilbert function of some homogeneous artinian quotient of $R = K[x, y]$. Then every monomial quotient of $R$ such that $h_{R/I} = h$
has the strong Lefschetz property, regardless of the characteristic of $K$, if and only if for every nonnegative integer $d$ such that $h(d) > h(d+1)$ then $h(d+1) = h(d+2)$.

**Proof.** Combine Lemma 2.3 and Theorem 4.11.

Hence, we can force homogeneous ideals with these Hilbert functions to have the strong Lefschetz property.

**Corollary 4.13.** Suppose $h : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is the Hilbert function of some homogeneous artinian quotient of $R = K[x, y]$. If for every nonnegative integer $d$ such that $h(d) > h(d+1)$ then $h(d+1) = h(d+2)$, then every homogeneous artinian quotient $R/I$ with $h_{R/I} = h$ has the strong Lefschetz property.

**Proof.** Let $R/I$ be some homogeneous artinian quotient with $h_{R/I} = h$. Since $\text{gin} I$ preserves the Hilbert function, we have that $h_{R/\text{gin} I} = h$. By Proposition 4.12, $R/\text{gin} I$ has the strong Lefschetz property, and so $R/I$ has the strong Lefschetz property by Proposition 4.2.

**Remark 4.14.** We must be careful with the ring in which we consider an ideal to be. For example, the ideal $I = (x^2 + y^2, x^3 + y^3)$ is in $R = K[x, y]$, regardless of the field $K$. However, $I$ is not artinian in a field of characteristic two. Indeed, if char $K = 2$, then $(x, y) = (1, 1)$ is a non-trivial common zero of the generators of $I$. However, in all other characteristics $R/I$ is artinian and has $h$-vector $(1, 2, 2, 1)$.

Suppose char $K \neq 2$. The reduced Gröbner basis for $I$ is $(x^2 + y^2, xy^2 - y^3, y^4)$, and so the initial ideal of $I$ is in $I = (x^2, xy^2, y^4)$. Notice that in $I$ is lexsegment, and so always has the strong Lefschetz property by Theorem 4.11. Using [19, Proposition 2.9], the latter implies that $R/I$ also has the strong Lefschetz property, if char $K \neq 2$.

Moreover, we classify the width functions that force monomial ideals to always have the strong Lefschetz property, regardless of characteristic, using Lemma 2.6.

**Proposition 4.15.** Suppose $w : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is the width function of some monomial artinian quotient of $R = K[x, y]$. Then every monomial artinian quotient $R/I$ such that $w_{R/I} = w$ has the strong Lefschetz property, regardless of the characteristic of $K$, if and only if for every nonnegative integer $d$ we have $0 \leq w(d+1) - w(d) \leq 2$.

**Proof.** Combine Lemma 2.6 and Theorem 4.11.

Hence, we can force homogeneous ideals with these width functions to have the strong Lefschetz property.

**Corollary 4.16.** Suppose $w : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is the width function of some homogeneous artinian quotient of $R = K[x, y]$. If for every nonnegative integer $d$ we have $0 \leq w(d+1) - w(d) \leq 2$, then every homogeneous artinian quotient $R/I$ with $w_{R/\text{gin} I} = w$ has the strong Lefschetz property.

**Proof.** Let $R/I$ be some homogeneous artinian quotient with $w_{R/\text{gin} I} = w$. By Proposition 4.12, $R/\text{gin} I$ has the strong Lefschetz property, and so $R/I$ has the strong Lefschetz property by Proposition 4.2.

5. Observations

We close with some observations and connections.
5.1. Complete intersections.

Let \( I = (f_1, \ldots, f_n) \) be a homogeneous ideal in \( S = K[x_1, \ldots, x_n] \). Then \( I \) is said to be a complete intersection of type \((\deg f_1, \ldots, \deg f_n)\) if the generators of \( I \) form a regular sequence in \( S \). This is equivalent to \( S/I \) being artinian. We considered the presence of the strong Lefschetz property in positive characteristic for monomial complete intersections in \([6]\). Here we show that the two of the results therein do not hold for non-monomial ideals.

The following lemma classifies the presence of the strong Lefschetz property in positive characteristic for monomial complete intersections of type \((a, b)\) in general for non-monomial complete intersections of type \((2,2)\) and \((3,3)\). Indeed, the following result shows that there exist non-monomial complete intersections of type \((2,2)\) and \((3,3)\) that \( R \) does not have the strong Lefschetz property for any \( p \). This is equivalent to \( S/\mathfrak{m} \) being artinian.

**Lemma 5.1.** \([6, Lemma 4.2]\) Let \( R = K[x, y] \) and \( p \) be the characteristic of \( K \). Then the following statements both hold.

(i) \( R/(x^a, y^b) \), for \( a \geq 2 \), has the strong Lefschetz property if and only if \( p \) does not divide \( a \).

(ii) \( R/(x^a, y^b) \), for \( a \geq 3 \), has the strong Lefschetz property if and only if \( p = 2 \) and \( a = 2 \pmod{4} \) or \( p \neq 2 \) and \( a \) is not equivalent to \(-1, 0, 1 \pmod{p} \).

This implies that every monomial complete intersection of type \((a, b)\), where \( 2 \leq a \leq 3 \) and \( a \leq b \), fails to have the strong Lefschetz property for some positive characteristic. (Furthermore, by Theorem \([11]\) we see that the only monomial complete intersections in \( R \) that always have the strong Lefschetz property are of type \((a, 1)\), where \( a \geq 1 \).) This is not true for arbitrary complete intersections of the same types. Indeed, the following result shows that there exist non-monomial complete intersections of type \((a, b)\), where \( 2 \leq a \leq 3 \) and \( a \leq b \), that always have the strong Lefschetz property.

**Proposition 5.2.** Let \( R = K[x, y] \). Then the following non-monomial homogeneous artinian ideals always have the strong Lefschetz property, regardless of the characteristic of \( K \).

(i) \((x^a, xy^{b-1} + y^b)\), where \( b \geq 2 \), and

(ii) \((x^a, x^2y^{b-2} + y^b)\), where \( b \geq 3 \).

**Proof.** Suppose that \( I = (x^2, xy^{b-1} + y^b) \), where \( b \geq 2 \). Let \( G = \{x^2, xy^{b-1} + y^b, y^{b+1}\} \). Clearly \( G \) is a reduced Gröbner basis for \( I \), regardless of the characteristic of \( K \). So in \( I = (x^2, xy^{b-1}, y^{b+1}) \).

On the other hand, suppose \( I = (x^3, x^2y^{b-2} + y^b) \), where \( b \geq 3 \). Let \( G = \{x^3, x^2y^{b-2} + y^b, xy^b, y^{b+2}\} \). Clearly \( G \) is a reduced Gröbner basis for \( I \), regardless of the characteristic of \( K \). So in \( I = (x^3, x^2y^{b-2}, xy^b, y^{b+2}) \).

In either case, \( I \) is an artinian lexsegment ideal, and thus \( R/I \) always has the strong Lefschetz property by Theorem \([11]\). Thus using \([19, Proposition 2.9]\), we see that \( R/I \) always has the strong Lefschetz property. \( \square \)

The following theorem characterises the presence of the strong Lefschetz property in positive characteristic for monomial complete intersections of type \((d, d)\).

**Theorem 5.3.** \([6, Theorem 4.9]\) Let \( R = K[x, y] \), where \( p \) is the characteristic of \( K \), and \( I_d = (x^d, y^d) \), where \( d \geq 2 \). Then \( R/I_d \) has the strong Lefschetz property if and only if \( p = 0 \) or \( 2d - 2 < p^s \), where \( s \) is the largest integer such that \( p^{s-1} \) divides \((2d - 1)(2d + 1)\).

This result implies that \( I = (x^p, y^p) \), where \( p \) is prime, fails to have the strong Lefschetz property in characteristic \( p \). However, Proposition \([5, Proposition 5.2]\) implies that this result does not hold in general for non-monomial complete intersections of type \((2,2)\) and \((3,3)\). We further checked this for non-monomial complete intersections of type \((p, p)\), where \( 5 \leq p \leq 41 \) is prime.
Conjecture 5.4. Let $p$ be an odd prime. Suppose $I = (x^p, x^{(p+1)/2}y^{(p-1)/2} + y^p)$ is an ideal of $R = K[x, y]$, where $K$ is a field of characteristic $p$. Then $R/I$ has the strong Lefschetz property.

Remark 5.5. Clearly the set $G = \{ x^p, x^{(p+1)/2}y^{(p-1)/2} + y^p, x^{(p-1)/2}y^p, y^{(3p+1)/2} \}$ is a reduced Gröbner basis for $I$ as in the preceding conjecture, regardless of the characteristic of $K$. Hence in $I = (x^p, x^{(p+1)/2}y^{(p-1)/2}, x^{(p-1)/2}y^p, y^{(3p+1)/2})$, and so in $I$ is lexsegment if and only if $p = 3$.

From the examples in Proposition 5.2 and Conjecture 5.4 if true, we see that the absence of the strong Lefschetz property for monomial complete intersections of a fixed type does not imply the same for non-monomial complete intersections of the same type.

5.2. Families of non-intersecting lattice paths.

Let $I$ be a monomial artinian ideal in $R = K[x, y]$, and let $d$ and $t$ be integers such that $0 \leq d < \text{indeg } I$ and $\text{indeg } I \leq d + t < \text{reg } R/I$. The matrix $N_{R/I}(d, d + t)$ can be identified with a matrix involving lattice paths in a particular sub-lattice of $\mathbb{Z}^2$. Moreover, if $N_{R/I}(d, d + t)$ is square, then its determinant provides a meaningful interpretation. We first must make several definitions.

A lattice path in a sub-lattice $L \subset \mathbb{Z}^2$ is a finite sequence of vertices of $L$ so that all single steps move either to the right (horizontal) or up (vertical). Given any two vertices $A = (u, v), E = (x, y) \in \mathbb{Z}^2$, the number of lattice paths in $\mathbb{Z}^2$ from $A$ to $E$ is the binomial coefficient \( \binom{x-u+y-v}{x-u} \), as each lattice path has precisely $x - u + y - v$ steps and $x - u$ of these must be horizontal steps. (Note that if $E$ is left of or below $A$, then the number of lattice paths is zero.)

Let $L_{R/I}$ be the finite sub-lattice of $\mathbb{Z}^2$ consisting of the point $(i, j)$, that is, $x^i y^j$ is non-zero in $R/I$. We denote by $L_{R/I}(d, d + t)$ the sub-lattice $L_{R/I}$ with two sets of distinguished vertices:

(i) the vertices of $L_{R/I}$ along the line $x + y = d$ are labeled $A_i = (a_i, d - a_i)$, where $0 \leq i \leq m$ and $a_0 < \cdots < a_m$, and

(ii) the vertices of $L_{R/I}$ along the line $x + y = d + t$ are labeled $E_j = (b_j, d + t - b_j)$, where $0 \leq j \leq n$ and $b_0 < \cdots < b_n$.

By careful observation we notice that the distinguished vertices correspond precisely to the monomials spanning $[R/I]_d$ and $[R/I]_{d+t}$, respectively.

The lattice path matrix of $L_{R/I}(d, d + t)$ is the $(m+1) \times (n+1)$ matrix $N = N(L_{R/I}(d, d + t))$ with entries $N(i,j)$ defined to be the number of lattice paths in $\mathbb{Z}^2$ from $A_i$ to $E_j$, i.e., $\binom{b_j-a_i+(d+t-b_j)-(d-a_i)}{b_j-a_i}$. Clearly then, $N(L_{R/I}(d, d + t)) = N_{R/I}(d, d + t)$.

A family of non-intersecting lattice paths is a finite collection of lattice paths such that no two lattice paths in the family have common points. Moreover, when $m = n$ the matrix $N(L_{R/I}(d, d + t)) = N_{R/I}(d, d + t)$ is square, and its determinant is given in Lemma 3.4. We use a theorem first given by Lindström in [13] Lemma 1] and stated independently in [8] Theorem 1] by Gessel and Viennot to interpret this determinant.

Theorem 5.6. [13] Lemma 1] & [8] Theorem 1] Let $A_1, \ldots, A_m$ and $E_1, \ldots, E_m$ be distinguished vertices of $\mathbb{Z}^2$. Define the $m \times m$ matrix $N$ to have entry $N(i,j)$ defined to be the number of lattice paths in $\mathbb{Z}^2$ from $A_i$ to $E_j$. Then

$$\det N = \sum_{\lambda \in \Gamma_m} \text{sgn} (\lambda) P_\lambda^+(A \to E)$$
where, for each permutation $\lambda \in \mathfrak{S}_m$, $P_\lambda^+(A \to E)$ is the number of families of non-intersecting lattice paths going from $A_i$ to $E_{\lambda(i)}$.

**Example 5.7.** Recall that in Example 3.3, the matrix $N_{R/I}(d, d + t)$ was given, where $I = (x^{10}, y^7)$ and $d = t = 5$. Figure 5.1 shows the lattice $L_{R/I}(d, d + t)$ with the distinguished vertices marked and a family of non-intersecting lattice paths in grey. By Example 3.3, the determinant of $N_{R/I}(d, d + t)$ is 210, so there are 210 families of non-intersecting lattice paths from $A_i$ to $E_i$ in $L_{R/I}(d, d + t)$.

![Figure 5.1](image-url)

**Figure 5.1.** The lattice $L_{R/(x^{10}, y^7)}(5, 10)$ with distinguished vertices marked and a family of non-intersecting lattice paths shown in grey.

Notice that when $m = n$, i.e., the number of $A_i$ vertices is the same as the number of $E_j$ vertices, then the path starting at $A_i$ in a family of non-intersecting lattice paths must end at $E_i$, for $0 \leq i \leq m$. Hence by Theorem 5.6, $\det N(L_{R/I}(d, d + t))$ is the number of non-intersecting lattice paths in $L_{R/I}(d, d + t)$.

5.3. **Connections to the weak Lefschetz property in codimension three.**

The connection between the strong Lefschetz property in codimension two and families of non-intersecting lattice paths described in Section 5.2 is similar to the connection of the latter to the weak Lefschetz property in codimension three, as described in [7]. Thus we see that the strong Lefschetz property in codimension two is intimately related to the weak Lefschetz property in codimension three.

In particular, let $J$ be a monomial artinian ideal in $R = K[x, y]$. We see that the multiplication map $\times (x + y)^t : [R/I]_d \to [R/I]_{d+t}$ has maximal rank exactly when the multiplication map $\times (x + y + z) : [S/J]_{d+t-1} \to [S/J]_{d+t}$ has maximal rank, where $J = I + (z^t)$ is an ideal of $S = K[x, y, z]$ (where we abuse notation to see $I$ as a monomial ideal of $S$). This follows as their cokernels, $[R/(I, (x + y)^t)]_{d+t}$ and $[S/(J, x + y + z)]_{d+t}$, are isomorphic. This connection has been used more generally; see, e.g., [4], [6], [14], and [16].

Using the above connection and Theorem 1.2 we can bound the failure of the weak Lefschetz property for certain codimension three monomial ideals.

**Proposition 5.8.** Let $J$ be a monomial artinian ideal in $S = K[x, y, z]$, where exactly one generator of $J$ is divisible by $z$, up to a change of variables. Then $S/J$ has the weak Lefschetz property if the characteristic of $K$ is at least $\text{reg } S/J$. 


Proof. Since \( J \) is artinian and has exactly one generator divisible by \( z \), then that generator must be \( z^t \) for some positive \( t \). Moreover, the remaining generators of \( J \) must be monomials in \( x \) and \( y \) only; let \( I \) be the ideal generated by these monomials. Then \( I \) can be seen as a monomial artinian ideal in \( R = K[x, y] \).

By Theorem 1.2, \( R/I \) has the strong Lefschetz property if \( \text{char} \ K \geq \text{reg} \ R/I \). Moreover, \( \text{reg} \ S/J \geq \text{reg} \ R/I \), so \( R/I \) has the strong Lefschetz property if \( \text{char} \ K \geq \text{reg} \ S/J \). Thus we see that \( S/J \) has the weak Lefschetz property if the characteristic of \( K \) is at least \( \text{reg} \ S/J \). \( \square \)

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