MODIFIED SCATTERING FOR THE KLEIN-GORDON EQUATION WITH THE CRITICAL NONLINEARITY IN THREE DIMENSIONS

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

SATOSHI MASAKI*

Department systems innovation
Graduate school of Engineering Science, Osaka University
1-3, Machikaneyamacho, Toyonaka, Osaka 560-8531, Japan

JUN-ICHI SEGATA

Mathematical Institute, Tohoku University
6-3, Aoba, Aramaki, Aoba-ku, Sendai 980-8578, Japan

Abstract. In this paper, we consider the final state problem for the nonlinear Klein-Gordon equation (NLKG) with a critical nonlinearity in three space dimensions: \( (\Box + 1)u = \lambda |u|^{2/3}u \), \( t \in \mathbb{R}, x \in \mathbb{R}^3 \), where \( \Box = \partial_t^2 - \Delta \) is d’Alembertian. We prove that for a given asymptotic profile \( u_{ap} \), there exists a solution \( u \) to (NLKG) which converges to \( u_{ap} \) as \( t \to \infty \). Here the asymptotic profile \( u_{ap} \) is given by the leading term of the solution to the linear Klein-Gordon equation with a logarithmic phase correction. Construction of a suitable approximate solution is based on the combination of Fourier series expansion for the nonlinearity used in our previous paper [23] and smooth modification of phase correction by Ginibre and Ozawa [6].

1. Introduction. This paper is devoted to the study of the final state problem for the nonlinear Klein-Gordon equation with a critical nonlinearity in three space dimensions:

\[
\begin{aligned}
(\Box + 1)u &= \lambda |u|^{2/3}u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3, \\
-u - u_{ap} &\to 0 \quad \text{in } L^2 \quad \text{as } t \to +\infty,
\end{aligned}
\]

where \( \Box = \partial_t^2 - \Delta \) is d’Alembertian, \( u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \) is an unknown function, \( u_{ap} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \) is a given function, and \( \lambda \) is a non-zero real constant. The aim of this paper is to find a proper choice of the function \( u_{ap} \) so that the equation (1.1) admits a nontrivial solution. In other words, we want to determine a “right” asymptotic behavior which actually takes place. This is a continuation of our previous study of the two dimensional case in [24].

Let us briefly review known results on the global existence and long time behavior of solutions to the more general nonlinear Klein-Gordon equation:

\[
(\Box + 1)u = \lambda |u|^{p-1}u, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^d,
\]

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* Corresponding author.
where \( p > 1 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \). The linear scattering theory indicates that the power \( p = 1 + 2/d \) will be a borderline between the short and the long range scattering theories, since point-wise decay of a solution to the linear Klein-Gordon equation is \( O(t^{-d/2}) \) as \( t \to \infty \). This formal observation was firstly justified by Glassey [7], Matsumura [25] and Georgiev and Yordanov [5] for \( p \leq 1 + 2/d \). More precisely, they proved that solutions to (1.2) do not scatter to a solution to the linear Klein-Gordon equation if \( 1 < p \leq 1 + 2/d \). Later, Georgiev and Lecente [4] obtained a point-wise decay estimate for small solutions to (1.2) for \( p > 1 + 2/d \) with \( d = 1, 2, 3 \). Furthermore, Hayashi and Naumkin [11] proved that a small solution to (1.2) scatters to a solution to the linear Klein-Gordon equation if \( p > 1 + 2/d \) and \( d = 1, 2 \). Notice that it is still an open problem for the asymptotic behavior of small solution to (1.2) when \( p \) is close to \( 1 + 2/d \) and \( d \geq 3 \). See [9, 17, 29, 30, 31, 33] for the small data scattering when \( d \geq 3 \) and \( p \) is large.

For the critical case \( p = 1 + 2/d \) and \( d = 1 \), Georgiev and Yordanov [5] studied point-wise decay of a solution to the initial value problem. Delort [1] obtained an asymptotic profile of a global solution to the equation. His proof is based on hyperbolic coordinates and the compactness of the support of the initial data was assumed. See also Lindblad and Soffer [19] for an alternative proof of [1]. The compact support assumption in [1] was later removed by Hayashi and Naumkin in [8] by using the vector field approach.

Recently, the authors [24] consider (1.2) with \( p = 1 + 2/d \) and \( d = 2 \) and specify an asymptotic profile \( u_{ap} \) that allows a unique solution \( u \) which converges to \( u_{ap} \) as \( t \to \infty \). The asymptotic profile \( u_{ap} \) has the same form as in the \( d = 1 \) case. Namely, it is given by the leading term of a solution to the linear Klein-Gordon equation with a logarithmic phase correction. The key ingredient is to extract a resonant term, which determines the shape of the phase correction, by means of a Fourier series expansion of the nonlinearity. In this paper, we consider (1.1), which is the final value problem for (1.2) with \( p = 1 + 2/d \) and \( d = 3 \). Because the power becomes a fractional number, the argument in the two dimensional case [24] is not directly applicable. To deal with the nonlinearity, we use the argument used in Ginibre and Ozawa [6].

Let us introduce the asymptotic profile \( u_{ap} \) which we work with. To this end, we first recall that the leading term of a solution to the linear Klein-Gordon equation

$$
\begin{aligned}
&\{(\square + 1)v = 0 \quad t \in \mathbb{R}, \ x \in \mathbb{R}^3,
&v(0, x) = \phi_0(x), \quad \partial_t v(0, x) = \phi_1(x) \quad x \in \mathbb{R}^3
\end{aligned}
$$

is given by

$$
t^{-\frac{2}{3}} \mathbf{1}_{\{|x|<1\}}(t, x) \langle \mu \rangle \overset{\rho(\mu)}{\hat{\mathcal{S}}}(\mu, t) = t^{-\frac{2}{3}} \mathbf{1}_{\{|x|<1\}}(t, x) \langle \mu \rangle \overset{\rho(\mu)}{\hat{\mathcal{S}}}(\mu, t) = t^{-\frac{2}{3}} \mathbf{1}_{\{|x|<1\}}(t, x)
$$

under suitable smoothness and decay assumptions on the initial data, where \( \mu = \mu(t, x) := x/\sqrt{t^2 - |x|^2}, \mathbf{1}_\Omega(t, x) \) is the characteristic function supported on \( \Omega \subset \mathbb{R}^{1+3} \), and \( \rho \geq 0 \) and \( \beta \in [0, 2\pi) \) are given by the relation

$$
\rho(\mu)e^{i\beta(\mu)} = e^{-i\frac{\mu}{2}} (\langle \mu \rangle \hat{\phi}_0(\mu) - i\hat{\phi}_1(\mu)),
$$

see [12] for instance. Remark that the leading term vanishes outside the light cone. The fact that such function is a leading term implies that the solution itself is small in the region.
For a given final state \((\phi_0, \phi_1)\), we define the asymptotic profile \(u_{\text{ap}}\) by
\[
u_{\text{ap}}(t, x) := t^{-\frac{5}{2}} \mathbf{1}_{\{|x| < t\}}(t, x) (\mu)^{\frac{5}{2}} \rho(\mu) \Re e^{i(\mu) t + \Psi(\mu) \log t + \beta(\mu))},
\] (1.3)
where the phase correction term is given by
\[
\Psi(\mu) = -\frac{\lambda \Gamma\left(\frac{11}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{7}{2}\right)} \rho(\mu)^{\frac{5}{2}}.
\] (1.4)
Remark that the coefficient comes from the first Fourier-cosine coefficient of a \(2\pi\)-periodic function \(\cos \theta|^{2/3} \cos \theta\). The final state \((\phi_0, \phi_1)\) is taken from the function space \(Y\) defined by
\[
Y := \{(\phi_0, \phi_1) \in S'(\mathbb{R}^3) \times S'(\mathbb{R}^3) : \| (\phi_0, \phi_1) \|_Y < \infty\},
\]
\[
\|(\phi_0, \phi_1)\|_Y := \|\phi_0\|_{H_x^2} + \| x\phi_0\|_{H_x^3} + \| x^2 \phi_0\|_{H_x^4}
+ \| \phi_1\|_{H_x^3} + \| x\phi_1\|_{H_x^2} + \| x^2 \phi_1\|_{H_x^3}.
\]
The main result in this paper is as follows.

**Theorem 1.1.** Let \((\phi_0, \phi_1) \in Y\). For \(3/4 < \gamma < 5/6\), there exists \(\varepsilon = \varepsilon(\gamma) > 0\) such that if \(\| (\gamma)^{3/2} \rho \|_{L_{\text{loc}}^\infty} < \varepsilon\) then there exist \(T \geq 3\) and a unique solution \(u(t)\) for the equation (1.1) satisfying
\[
\sup_{t \geq T} \left(\| u - u_{\text{ap}} \|_{L^\infty((t, \infty) ; H_x^3)} + \| u - u_{\text{ap}} \|_{L^\infty((t, \infty) ; H_x^{1/2})} \right) < \infty,
\] (1.5)
where the asymptotic profile \(u_{\text{ap}}\) is defined from \((\phi_0, \phi_1)\) via (1.3).

**Remark 1.1.** Remark that if \(\rho\) is non-trivial then \(\| u_{\text{ap}}(t) \|_{H_x^{1/2}}\) does not decay in time, and so (1.5) implies that we successfully extract the leading term of the solution. Since \(u_{\text{ap}}\) is supported on the light cone \(\{|x| \leq t\}\), (1.5) also shows that the contribution of the solution from the outside of the light cone is much smaller than that from the inside.

**Remark 1.2.** The same result holds true for equations with a general critical nonlinearity \(F(u) : \mathbb{R} \to \mathbb{R}\) satisfying \(F(\lambda u) = \lambda^{5/3} F(u)\) for all \(\lambda > 0\) and \(u \in \mathbb{R}\). See Remark 2.2 below for the detail. We also note that the final state problem for the one dimensional cubic Klein-Gordon equation is studied by Hayashi and Naumkin [10].

**Remark 1.3.** Concerning the scattering results for the Klein-Gordon equation with the critical quasilinear nonlinearity, the readers can consult Moriyama [26], Katayama [13], Sunagawa [34] for one dimensional cubic case and Ozawa, Tsutaya and Tsutsumi [28], Delort, Fang and Xue [2], Kawahara and Sunagawa [15], Katayama, Ozawa and Sunagawa [14] for the two dimensional quadratic case.

The rest of the paper is organized as follows. In Section 2, we exhibit the outline of the proof of Theorem 1.1. We construct a solution with the desired property by applying the contraction principle to the integral equation of Yang-Feldman type associated with (1.1) around a suitable approximate solution. The crucial points of the proof are summarized as Propositions 2.1 and 2.3. Then, we prove Proposition 2.1 in Section 3 and Proposition 2.3 in Section 4.
2. Outline of the proof of Theorem 1.1. In this section, we give an outline of the proof of Theorem 1.1.

2.1. On the solvability of the final state problem. We first remark that solvability of the final value problem (1.1) and validity of the desired asymptotics are both reduced to the appropriateness of the choice of the asymptotic behavior of $u_{ap}$.

Let $A(t, x)$ be a given asymptotic profile of a solution to (1.1). We show that if $A(t, x)$ is well-chosen then we obtain a solution which asymptotically behaves like $A(t, x)$. Let $N(u) = \lambda |u|^{2/3} u$.

**Proposition 2.1.** For any $\gamma > 3/4$, there exists $\eta = \eta(\gamma) > 0$ such that if $A(t, x)$ satisfies

\begin{equation}
\sup_{t \geq T_0} t^{\frac{\gamma}{2}} \|A(t)\|_{L^\infty} \leq \eta, \tag{2.1}
\end{equation}

\begin{equation}
\sup_{t \geq T_0} t^{1 + \gamma} \|(\Box + 1)A(t) - N(A(t))\|_{L^1_\rho} < \infty \tag{2.2}
\end{equation}

for some $T_0 \geq 3$, then there exist $T \geq T_0$ and a unique solution $u \in C([T, \infty); H^1_{\rho})$ for the equation (1.1) satisfying

\begin{equation}
\sup_{T \geq T} t^{\gamma} \left( \|u - A\|_{L^\infty((t, \infty); H^\frac{1}{2}_\rho)} + \|u - A\|_{L^{\frac{10}{3}}((t, \infty); L^{\frac{10}{3}}_\rho)} \right) < \infty \tag{2.3}
\end{equation}

for the same $\gamma$.

The proposition will be proven in Section 3. It is worth mentioning that there is no information available on the shape of the support of a solution $u$ even though the function $A$ is supported on the light cone. However, in such a case, the above asymptotics (2.3) shows that at least the solution $u$ decays in the outside of the support of $A$.

2.2. Choice of an appropriate asymptotic profile. An easy choice is $A = u_{ap}$. However, it does not work well. Hence, we choose a suitable $A$ that satisfies the assumptions (2.1) and (2.2) and

\begin{equation}
\sup_{T \geq T} t^{\gamma} \left( \|u_{ap} - A\|_{L^\infty((t, \infty); H^\frac{1}{2}_\rho)} + \|u_{ap} - A\|_{L^{\frac{10}{3}}((t, \infty); L^{\frac{10}{3}}_\rho)} \right) < \infty, \tag{2.4}
\end{equation}

Then, the solution obtained by Proposition 2.1 from such profile $A$ possesses the desired asymptotics.

The obstacle in three-dimensional case lies in the fact that the phase correction term $\Psi$, given in (1.4), has the fractional power term $\rho^{2/3}$. The power comes from the nonlinearity. Notice that because of the fractional power, we may not estimate $(\Box + 1) u_{ap}$, in general. To overcome the difficulty, we employ the argument in Ginibre and Ozawa [6]. We introduce a modified phase corrector

$$\tilde{\Psi}(s, \mu) := - \frac{\lambda \Gamma(\frac{11}{6})}{\sqrt{\pi} \Gamma\left(\frac{4}{3}\right)} \tilde{\rho}(s, \mu)^{\frac{2}{3}}, \quad \tilde{\rho}(s, \mu) = \sqrt{\rho(\mu)^2 + s^{-1}(\mu)^{-\frac{3}{4}}},$$

and an auxiliary approximate solution

$$\tilde{u}_{ap}(t, x) = t^{-\frac{3}{2}} 1_{\{|x| < t\}}(t, x) \langle \mu \rangle^{\frac{2}{3}} \rho(\mu) \text{Re} e^{it(\alpha(t, \mu) + \beta(\mu))}, \tag{2.5}$$

where $\mu = \mu(t, x) := x/\sqrt{t^2 - |x|^2}$ and

$$\alpha(t, \mu) := (\mu)^{-1} t + \tilde{\Psi}(t, \mu) \log t.$$


we have to find the main part of \(N\). In [24], a Fourier series expansion is introduced for this purpose. Here, we split \(A\) we recall the construction of the two dimensional case [24]. This is the idea of the proof. For readers convenience, in (2.2) holds for any \(\eta > 0\) and \(\gamma < 5/6\), there exists \(\varepsilon\) such that if \(||(\cdot)^{3/2}\|_{L^\infty}\leq \varepsilon||\) then \(A\) satisfies \((2.1)\) for some \(T_0 \geq 3\).

Together with Proposition 2.1, this proposition implies Theorem 1.1. Section 4 is devoted to the proof of the above proposition.

\textbf{Remark 2.2.} Let us consider a generalization of Theorem 1.1 to any real-valued nonlinearity satisfying \(F(\lambda u) = \lambda^{5/3}F(u)\) for any \(\lambda > 0\) and \(u \in \mathbb{R}\). Notice that this class of nonlinearity is written as \(F(u) = \lambda_1|u|^{\frac{5}{3}}u + \lambda_2|u|^{\frac{5}{3}}\). Theorem 1.1
corresponds to the case where \( \lambda_2 = 0 \). By means of the following lemma, we see that the nonlinearity \( \lambda_2 |u|^{5/3} \) does not contain a resonant part, and so that we can treat the above general nonlinearity by the same argument.

**Lemma 2.4.** Let \( \bar{c}_n := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos \theta |^\frac{3}{2} \cos n \theta d\theta \) for \( n \geq 0 \). Then, \( \bar{c}_n = 0 \) for odd \( n \) and

\[
\bar{c}_n = \frac{2(-1)^{\frac{n}{2}} \Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{3n-5}{6}\right)}{\sqrt{\pi} \Gamma\left(-\frac{5}{6}\right) \Gamma\left(\frac{3n+11}{6}\right)}
\]

for even \( n \). In particular, \( \bar{c}_n = O(n^{-8/3}) \) as \( n \to \infty \).

**Proof.** We follow the argument in [22]. It is easy to see that \( \bar{c}_n = 0 \) for odd \( n \) and

\[
\bar{c}_n = \frac{4}{\pi} \int_{0}^{\pi/2} \cos \theta \frac{3}{2} \cos n \theta d\theta
\]

for even \( n \). Let \( b_m = \frac{4}{\pi} \int_{0}^{\pi/2} \cos \theta \frac{3}{2} \cos 2m \theta d\theta \). Then, one sees from integration by parts that

\[
b_{m+1} - b_m = -\frac{3(2m+1)}{8}(b_{m+1} + b_m),
\]

which shows \( b_{m+1} = -\frac{m-5/6}{m+11/6} b_m \). Solving the relation, we obtain

\[
b_m = (-1)^m \frac{\Gamma(m - \frac{5}{6})}{\Gamma(m + \frac{11}{6})} \beta
\]

with some constant \( \beta \). Since \( b_0 = \frac{4}{\pi} \int_{0}^{\pi/2} (\cos \theta)^{5/3} d\theta = \frac{2\Gamma(4/3)}{\sqrt{\pi} \Gamma(11/6)} \), we have

\[
\beta = \frac{2\Gamma(4/3)}{\sqrt{\pi} \Gamma(-5/6)}.
\]

This completes the proof. \( \square \)

3. **Proof of Proposition 2.1.** In this section, we prove Proposition 2.1. The proof is essentially the same as in the two dimensional case [24]. The following inhomogeneous Strichartz estimates associated with the Klein-Gordon equation is crucial for the proof. Let

\[
G[g](t) := \int_{t}^{\infty} \sin((t-\tau)\sqrt{1-\Delta})(1-\Delta)^{-1/2} g(\tau) d\tau.
\]

**Lemma 3.1.** Let \( 2 \leq q \leq 6 \) and \( 2/p + 3/q = 3/2 \). Then we have

\[
\|G[g]\|_{L^p_t((T,\infty),L^q_x)} \leq C \|(1-\Delta)^{\frac{1}{2p} - \frac{5}{3q}} g\|_{L^p_t((T,\infty),L^q_x)},
\]

\[
\|G[g]\|_{L^p_t((T,\infty),L^\infty_x)} \leq C \|(1-\Delta)^{\frac{1}{2p} - \frac{5}{3q}} g\|_{L^p_t((T,\infty),L^\infty_x)},
\]

\[
\|G[g]\|_{L^p_t((T,\infty),L^2_x)} \leq C \|(1-\Delta)^{\frac{1}{2p} - \frac{5}{3q}} g\|_{L^p_t((T,\infty),L^2_x)}.
\]

**Proof.** The above inequalities follow from combination of the \( L^p-L^q \) estimate for the solution to the Klein-Gordon equation by [20] with the duality argument by [36] for the non-endpoint case \( q \neq 6 \) and the argument by [16] for the endpoint case \( q = 6 \). Since the proof is now standard, we omit the detail. \( \square \)
Proof of Proposition 2.1. We introduce

$$X_T = \{ w \in C([T, \infty) ; L^2_T) ; \| w \|_{X_T} < \infty \}$$

for $T \geq 3$, where

$$\| w \|_{X_T} = \sup_{t \geq T} t^{3} \left( \| w \|_{L^\infty((t, \infty) ; H^\frac{1}{2})} + \| w \|_{L^{\infty, u}((t, \infty) ; L^\frac{u}{2})} \right).$$

For $R > 0$ and $T > 0$, we define

$$\tilde{X}_T(R) = \{ w \in C([T, \infty) ; L^2_T) ; \| w \|_{X_T} \leq R \}.$$ 

The function space $X_T$ is a Banach space with the norm $\| \cdot \|_{X_T}$ and $\tilde{X}_T(R)$ is a complete metric space with the $\| \cdot \|_{X_T}$-metric.

We put $v = u - A$. Then the equation (1.1) is equivalent to

$$(\Box + 1) v = N(v + A) - N(A) - F,$$  \hspace{1cm} (3.2)

where

$$F := (\Box + 1) A - N(A)$$

The associate integral equation to the equation (3.2) is

$$v = G[\{ N(v + A) - N(A) \} - F],$$  \hspace{1cm} (3.3)

where $G$ is given by (3.1). It suffices to show the existence of a unique solution $v$ to the equation (3.3) in $X_T$ for suitable $\eta > 0$ and $T \geq T_0$. We prove this assertion by the contraction argument. Define the nonlinear operator $\Phi$ by

$$\Phi v := G[\{ N(v + A) - N(A) \} - F]$$

for $v \in \tilde{X}_T(R)$. We show that $\Phi$ is a contraction map on $\tilde{X}_T(R)$ if $R > 0$, $T \geq T_0$, and $\eta > 0$ are suitably chosen. Let $v \in \tilde{X}_T(R)$ and $t \geq T$. Since

$$|N(v + A) - N(A)| \leq \int_0^1 |N'(A + \theta v) - N'(A)| d\theta |v| + |N'(A)||v|$$

$$\leq C(|v|^{\frac{5}{7}} + |A|^{\frac{5}{7}})|v|,$$

we see from the assumptions on $v$ and Lemma 3.1 that

$$\| \Phi v \|_{L^\infty((t, \infty) ; H^\frac{1}{2})} + \| \Phi v \|_{L^{\infty, u}((t, \infty) ; L^\frac{u}{2})}$$

$$\leq C\left( \| v \|_{L^\infty((t, \infty) ; H^\frac{1}{2})} + \| N' \|_{L^{\infty, u}((t, \infty) ; L^\frac{u}{2})} \right)$$

$$+ \| (1 - \Delta)^{-\frac{1}{4}} F \|_{L^1((t, \infty) ; L^2_T)}$$

$$\leq C\left( \frac{\| v \|_{L^\infty((t, \infty) ; H^\frac{1}{2})}}{\| v \|_{L^\infty((t, \infty) ; H^\frac{1}{2})}} \left( \int_t^\infty \| v(\tau) \|_{L^2_T} d\tau \right)^{\frac{1}{2}} + \int_t^\infty \| A(\tau) \|_{L^2_T} d\tau \right)^{\frac{1}{2}}$$

$$+ \int_t^\infty \| F(\tau) \|_{L^2_T} d\tau \right)$$

$$\leq C \left( R^\frac{5}{2} t^{-\frac{3}{2}} + \int_t^\infty R^\frac{5}{2} \eta R^{-1-\gamma} d\tau + \int_t^\infty M R^{-1-\gamma} d\tau \right)$$

$$\leq C t^{-\gamma} (R^\frac{5}{2} t^{-\frac{3}{2} \gamma + \frac{1}{2}} + R\eta^\frac{5}{2} + M),$$

where $M$ is an upper bound on the right hand side of (2.2). Therefore we obtain

$$\| \Phi v \|_{X_T} \leq C_1 \left( R^\frac{5}{2} T^{-\frac{3}{2} \gamma + \frac{1}{2}} + R\eta^\frac{5}{2} + M \right),$$  \hspace{1cm} (3.4)
In the same way as above, for \( v_1, v_2 \in \tilde{X}_T(R) \), we can show

\[
\| \Phi v_1 - \Phi v_2 \|_{X_T} \leq C_2((\| v_1 \|_{X_T}^T + \| v_2 \|_{X_T}^T)^{-\frac{\gamma}{2} + \frac{1}{2}} + \eta^\frac{3}{4})\| v_1 - v_2 \|_{X_T}.
\] (3.5)

We first fix \( R \) so that \( C_1 M \leq R/2 \). Then, using the fact that \( \gamma > 3/4 \), we are able to choose a sufficiently large \( T > 0 \) and a sufficiently small \( \eta > 0 \) such that

\[
C_1(R^{\frac{3}{2}}T^{-\frac{3}{2} + \frac{1}{2} + \frac{1}{2}} + R\eta^\frac{3}{4} + M) \leq R,
\]

\[
C_2(R^{\frac{3}{2}}T^{-\frac{3}{2} + \frac{1}{2} + \frac{1}{2}} + \eta^\frac{3}{4}) \leq \frac{1}{2},
\]

For such \( R, T, \eta \), there exists a unique solution to the integral equation (3.3) in \( \tilde{X}_T(R) \). The uniqueness of solutions to the equation (3.3) in \( X_T \) follows from the first inequality of the estimate (3.5) for solutions \( v_1 \in X_T \) and \( v_2 \in X_T \). Hence the equation (3.3) has a unique solution in \( X_T \). This completes the proof of Proposition 2.1.

\( \square \)

4. Proof of Proposition 2.3. In this section, we prove Proposition 2.3. Since (2.1) is trivial, we prove (2.4) and (2.2) in Sections 4.2 and 4.3, respectively, after preparing preliminary estimates in Section 4.1. Hereafter we always restrict our attention to the region \( |x| < t \) and \( t \geq 3 \).

We introduce new variables \((s, \mu)\) by \( s = t \) and \( \mu = x/\sqrt{t^2 - |x|^2} \). Then, we have

\[
\partial_t = \partial_s - s^{-1}\langle \mu \rangle^2 \mu \cdot \nabla \mu,
\]

\[
\partial_x \mu = s^{-1}\langle \mu \rangle \partial_{\mu_s} + s^{-1}\langle \mu \rangle \mu \cdot \nabla \mu
\]

and

\[
\Box = \partial_t^2 - 2s^{-1}\langle \mu \rangle^2 \mu \cdot \nabla \mu - s^{-2}(\mu)^2 \Delta \mu
\]

\[
- 3s^{-2}(\mu)^2 \mu \cdot \nabla \mu - s^{-2}(\mu)^2 \sum_{1 \leq i, j \leq 3} \mu_i \mu_j \partial_i \partial_j.
\]

Also remark that

\[
\| f(t, \mu(t, x)) \|_{L^p_v(|x| < t)} = s^{\frac{3}{2}} \| \langle \mu \rangle^{-\frac{3}{2}} f(s, \mu) \|_{L^p_v(\mathbb{R}^3)}
\]

for any \( p \in (0, \infty) \).

4.1. Preliminaries. We collect preliminary estimates.

Lemma 4.1. For \( n, m \in \mathbb{R} \),

\[
(\Box_{t, x} + 1)(s^{-m}e^{in(\mu)^{-1}s}) = -(n^2 - 1)s^{-m}e^{in(\mu)^{-1}s}
\]

\[
- in(2m - d)s^{-m-1}(\mu)e^{in(\mu)^{-1}s} + m(m + 1)s^{-m-2}e^{in(\mu)^{-1}s},
\]

where \( d = 3 \) is the spatial dimension.

Proof. It follows by a direct calculation. \( \square \)

Recall that \( \tilde{\rho}(s, \mu) = \sqrt{\rho(\mu)^2 + s^{-4}(\mu)^{-3}} \). An elementary inequality

\[
\max(\rho(\mu), s^{-\frac{1}{2}}(\mu)^{-\frac{1}{2}}) \leq \tilde{\rho}(s, \mu) \leq \rho(\mu) + s^{-\frac{1}{2}}(\mu)^{-\frac{1}{2}}
\]

will be useful. The following will be used to estimate the error that comes from the phase modification.
Lemma 4.2. For $s \geq 3$, we have the following inequality:
\[ \rho(\mu)(\tilde{\rho}(s, \mu) - \rho(\mu)) \lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}}, \]
where the implicit constant is independent of $\rho$.

Proof. By a direct calculation, we obtain
\begin{align*}
\rho(\mu)(\tilde{\rho}(s, \mu) - \rho(\mu)) &= \frac{1}{3} \rho(\mu) \int_{0}^{1} (\rho(\mu)^{2} + \theta s^{-1} \langle \mu \rangle^{-3})^{-\frac{3}{2}} s^{-1} \langle \mu \rangle^{-3} d\theta \\
&\lesssim \frac{1}{3} \rho(\mu) \int_{0}^{1} (\rho(\mu)^{2})^{-\frac{3}{2}} (\theta s^{-1} \langle \mu \rangle^{-3})^{-\frac{3}{2}} s^{-1} \langle \mu \rangle^{-3} d\theta = C s^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}}
\end{align*}
for every $\mu \in \mathbb{R}^{3}$. Thus we obtain the desired inequality. \qed

Now, we turn to the estimate of the derivatives of the modified phase part $\tilde{\Psi}(s, \mu) = -(\lambda c_{1}/2)\tilde{\rho}(s, \mu)^{2/3}$.

Lemma 4.3. For $s \geq 3$, we have the following inequalities:
\begin{align*}
|\partial_{\mu_{j}} \tilde{\Psi}(s, \mu)| &\lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-1}, \quad (4.6) \\
|\partial_{\mu_{j}} \tilde{\Psi}(s, \mu)| &\lesssim s^{-1-\frac{5}{2}} \langle \mu \rangle^{-\frac{3}{2}}, \quad (4.7) \\
|\rho(\mu) \partial_{\mu_{j}} \tilde{\Psi}(s, \mu)| &\lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}}, \quad (4.8) \\
|\rho(\mu)^{2} \tilde{\Psi}(s, \mu) \partial_{\mu_{j}} \tilde{\Psi}(s, \mu)| &\lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}}, \quad (4.9) \\
|\rho(\mu) \partial_{\mu_{j}}^{2} \tilde{\Psi}(s, \mu)| &\lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}} \langle \mu \rangle^{-\frac{3}{2}}, \quad (4.10)
\end{align*}
where the implicit constants are independent of $\rho$.

Proof. The first four inequalities (4.6)-(4.9) follow from
\[ \partial_{\mu_{j}} \tilde{\Psi}(s, \mu) = C s^{-2} \langle \mu \rangle^{-3} \tilde{\rho}(s, \mu)^{\frac{2}{3}}, \]
and (4.5). Similarly,
\[ \partial_{\mu_{j}}^{2} \tilde{\Psi}(s, \mu) = C_{1} s^{-3} \langle \mu \rangle^{-3} \tilde{\rho}(s, \mu)^{-\frac{5}{3}} + C_{2} s^{-4 \frac{5}{3}} \langle \mu \rangle^{-6} \tilde{\rho}(s, \mu)^{-\frac{7}{3}}, \]
yields the last inequality (4.10). \qed

Lemma 4.4. For $s \geq 3$, we have the following inequalities:
\begin{align*}
|\partial_{\mu_{j}} \tilde{\Psi}(s, \mu)| &\lesssim s^{\frac{3}{2}} \langle \mu \rangle^{\frac{3}{2}} |\partial_{\mu_{j}} \rho(\mu)| + s^{-\frac{3}{2}} \langle \mu \rangle^{-2}, \quad (4.11) \\
|\rho(\mu)^{2} \partial_{\mu_{j}} \tilde{\Psi}(s, \mu) \partial_{\mu_{k}} \tilde{\Psi}(s, \mu)| &\lesssim |\nabla_{\mu \rho(\mu)}|^{2} + s^{-1} \langle \mu \rangle^{-5}, \quad (4.12) \\
|\rho(\mu) \partial_{\mu_{j}} \partial_{\mu_{k}} \tilde{\Psi}(s, \mu)| &\lesssim \rho(\mu)^{\frac{3}{2}} |\nabla_{\mu \rho(\mu)}|^{2} + s^{\frac{3}{2}} \langle \mu \rangle^{\frac{3}{2}} \langle \nabla_{\mu \rho(\mu)} \rangle^{2} + s^{-\frac{3}{2}} \langle \mu \rangle^{-3} \rho(\mu), \quad (4.13) \\
|\rho(\mu) \partial_{\mu_{j}} \partial_{\mu_{k}} \tilde{\Psi}(s, \mu)| &\lesssim s^{-\frac{3}{2}} \langle \mu \rangle^{-1} |\partial_{\mu_{j}} \rho(\mu)| + s^{-\frac{3}{2}} \langle \mu \rangle^{-2} \rho(\mu), \quad (4.14)
\end{align*}
where the implicit constants are independent of $\rho$. 

Proof. The first two inequalities (4.11) and (4.12) follow from
\[ \partial_{\mu_j} \Psi = C_1 \rho(s, \mu) \frac{d}{ds} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j) \]
and (4.5). To obtain the inequality (4.13), we use
\[ \partial_{\mu_j} \partial_{\mu_k} \Psi = C_2 \rho(s, \mu) \frac{d}{ds} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j) \]
\[ \times (\rho(\mu) \partial_{\mu_k} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_k) \]
\[ + C_1 \rho(s, \mu) \frac{d}{ds} (\rho(\mu) \partial_{\mu_j} \rho(\mu) + \rho(\mu) \partial_{\mu_k} \rho(\mu)) \]
\[ + C_1 \rho(s, \mu) \frac{d}{ds} (15s^{-1} \langle \mu \rangle^{-7} \mu_j \mu_k - 3s^{-1} \langle \mu \rangle^{-5} \delta_{jk}), \]
where \( \delta_{jk} \) is the Kronecker delta. The inequality (4.14) is a consequence of
\[ \partial_{\mu_j} \partial_{\mu_k} \Psi = C_3 s^{-2} \langle \mu \rangle^{-3} \rho(s, \mu) \frac{d}{ds} (\rho(\mu) \partial_{\mu_j} \rho(\mu) - 3s^{-1} \langle \mu \rangle^{-5} \mu_j) \]
\[ + 3C_1 s^{-2} \langle \rho(s, \mu) \rangle^{-\frac{3}{2}} \langle \mu \rangle^{-5} \mu_j. \]
This completes the proof of Lemma 4.4. \( \square \)

4.2. Proof of (2.4). We now show that \( A \) satisfies (2.4) for \( \gamma < 5/6 \).

Proof. Since
\[ \left| e^{\frac{ic}{c^2} \rho(s, \mu) \frac{d}{ds}} - e^{\frac{ic}{c^2} \rho(\mu) \frac{d}{ds}} \right| \lesssim C(\rho(s, \mu)^{\frac{3}{2}} - \rho(\mu)^{\frac{3}{2}}) \log t \]
for \( t \geq 3 \), we deduce from (4.4) and Lemma 4.2 that
\[ \| \bar{u}_{ap} - u_{ap} \|_{L^2_{\mu}} \lesssim (\log t) \| \langle \mu \rangle^{-1} \rho(\mu) \frac{d}{ds} (\rho(s, \mu)^{\frac{3}{2}} - \rho(\mu)^{\frac{3}{2}}) \|_{L^2_{\mu}(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}} \log t \]
and
\[ \| \bar{u}_{ap} - u_{ap} \|_{L^2_{\mu}} \lesssim t^{-\frac{3}{2}} - \frac{3}{2} \log t. \]
Further, we see from (4.5) that
\[ \| \bar{v}_{ap} \|_{L^2_{\mu}(|x| < t)} \lesssim t^{-1} \| \langle \rho \rangle^{\frac{3}{2}} + t^{-\frac{3}{2}} \langle \mu \rangle^{-1} \rho \|_{L^2_{\mu}} \lesssim t^{-1} \| \rho \|_{L^2_{\mu}}^{\frac{3}{2}} + t^{-\frac{3}{2}} \| \rho \|_{L^2_{\mu}} \] (4.15)
and
\[ \| \bar{v}_{ap} \|_{L^2_{\mu}(\mathbb{R}^3)} \lesssim t^{-\frac{3}{2}} (t^{-1} \| \langle \mu \rangle^{\frac{3}{2}} \rho \|_{L^2_{\mu}}^{\frac{3}{2}} + t^{-\frac{3}{2}} \| \rho \|_{L^2_{\mu}}) \].
Similarly, we have
\[ \| \nabla_x \bar{u}_{ap} - \nabla_x u_{ap} \|_{L^2_{\mu}} + \| \nabla_x \bar{v}_{ap} \|_{L^2_{\mu}} \lesssim t^{-\frac{3}{2}} (\log t) \| \langle \phi_0, \phi_1 \rangle \|_{Y}^{\frac{3}{2}}. \]
Indeed, in view of (4.2), the leading term with respect to \( t \) appears only when the derivative \( \nabla_x \) hits \( e^{in(\mu)^{-1} t} \). Furthermore, in that case, \( \nabla_x e^{in(\mu)^{-1} t} = -in\mu e^{in(\mu)^{-1} t} \) and so the estimate is essentially the same.

Combining these estimates, we conclude that \( A \) satisfies (2.4) as long as \( \gamma < 5/6 \). \( \square \)
4.3. **Proof of (2.2).** To complete the proof of Proposition 2.3, we prove \( A \) satisfies the condition (2.2) for \( \gamma < 5/6 \). Note that

\[
\Box + 1)A - N(A) = ((\Box + 1)\tilde{u}_{ap} - N_r) + ((\Box + 1)\tilde{u}_{ap} - N_{tr}) + (N(\tilde{u}_{ap}) - N(A)).
\]

The third term of the right hand side is estimated easily. Indeed, we see that

\[
\|N(A) - N(\tilde{u}_{ap})\|_{L^2_t} \lesssim (\|\tilde{u}_{ap}\|_{L^\infty_t} + \|\tilde{v}_{ap}\|_{L^\infty_t})^2 \|\tilde{v}_{ap}\|_{L^2_t}.
\]

By (4.4) and the embedding \( H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \), we have

\[
\|\tilde{u}_{ap}\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho e^{i\beta}\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho e^{i\beta}\|_{H^2_t}.
\]

On the other hand, by means of (4.4), (4.5), and \( \sum_{n \geq 2} \frac{|c_n|}{n-1} < \infty \), we have

\[
\|\tilde{v}_{ap}\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho e^{i\beta}\|_{L^\infty_t} \lesssim t^{-\frac{3}{2}} \|\rho e^{i\beta}\|_{H^2_t} \lesssim 2 \|\rho e^{i\beta}\|_{H^2_t}.
\]

for \( t \geq 3 \). Similarly, noting that \( H^2(\mathbb{R}^3) \hookrightarrow H^{3/5}(\mathbb{R}^3) \hookrightarrow L^{10/3}(\mathbb{R}^3) \), one sees from (4.15) that

\[
\|\tilde{v}_{ap}\|_{L^2_t} \lesssim t^{-1} \left( \|\rho e^{i\beta}\|_{H^2_t} \right)^{\frac{3}{2}}
\]

for \( t \geq 3 \). Combining the above inequalities, we obtain

\[
\|N(A) - N(\tilde{u}_{ap})\|_{L^2_t} \lesssim t^{-2} \left( \|\rho e^{i\beta}\|_{H^2_t} \right)^{\frac{3}{2}}.
\]

Thus, this term is acceptable.

Hence, we estimate the first and the second terms, in what follows.

**Proposition 4.5.** For \( t \geq 3 \),

\[
\|((\Box_{t,x} + 1)\tilde{u}_{ap} - N_r)\|_{L^2_t(\{|x| < t\})} \lesssim t^{-1} \left( \log t \right) \|((\phi_0, \psi_1))\|_{Y}^2
\]

holds.

**Proof.** To show the inequality (4.17), we begin with the computation of the linear part:

\[
((\Box + 1)\tilde{u}_{ap} = \text{Re} \left[ ((\Box + 1)(s^{-\frac{3}{2}}(\mu)^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i(\mu)^{-1}s} + i\tilde{\Psi}(s, \mu) \log s) \right].
\]

We split

\[
((\Box + 1)(s^{-\frac{3}{2}}(\mu)^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i(\mu)^{-1}s} + i\tilde{\Psi}(s, \mu) \log s)
= (((\Box + 1)s^{-\frac{3}{2}}e^{i(\mu)^{-1}s}\mu)^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i(\mu)^{-1}s} + i\tilde{\Psi}(s, \mu) \log s)
+ s^{-\frac{3}{2}}e^{i(\mu)^{-1}s}((\mu)^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s, \mu) \log s})
+ 2\partial_t((s^{-\frac{3}{2}}e^{i(\mu)^{-1}t})\tilde{\Psi}(s, \mu) \log s)
+ 2\nabla_x((s^{-\frac{3}{2}}e^{i(\mu)^{-1}s} \cdot \nabla_x((\mu)^{\frac{3}{2}}\rho(\mu)e^{i\beta}e^{i\tilde{\Psi}(s, \mu) \log s})
= I_1 + I_2 + I_3 + I_4.
\]

In light of Lemma 4.1, we have

\[
I_1 = \frac{15}{4} s^{-\frac{3}{2}}(\mu)^{\frac{3}{2}}\rho(\mu)e^{i(\alpha + \beta)}.
\]

(4.19)
By (4.3), one sees that
\[ I_2 = s^{-\frac{7}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle \frac{3}{2} \rho(\mu) e^{i\beta} \partial_s^2 e^{i\tilde{\Psi}(s,\mu) \log s} \]
\[ - 2s^{-\frac{3}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^2 \nabla \partial_s (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s} \]
\[ - s^{-\frac{7}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^2 \Delta (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s} \]
\[ - 3s^{-\frac{3}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^2 \nabla \partial_s (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s} \]
\[ - s^{-\frac{7}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^2 \sum_{1 \leq j,k \leq 3} \mu_j \mu_k \partial_j \partial_k (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s} \]
\[ =: J_1 + J_2 + J_3 + J_0 + J_4. \]
Moreover, since
\[ \partial_t (s^{-\frac{7}{2}} e^{i(\mu^{-1})s}) = i \langle \mu \rangle s^{-\frac{7}{2}} e^{i(\mu^{-1})s} - \frac{3}{2} s^{-\frac{3}{2}} e^{i(\mu^{-1})s}, \]
\[ \nabla_x e^{i(\mu^{-1})s} = -i \mu e^{i(\mu^{-1})s}, \]
we have
\[ I_4 = 2is^{-\frac{7}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^3 (\mu \cdot \nabla \partial_s (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s}) \]
and
\[ I_5 = -2s^{-\frac{7}{2}} \tilde{\Psi}(s,\mu) (\langle \mu \rangle^2 \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \]
\[ - 2s^{-\frac{3}{2}} (\log s) \partial_s \tilde{\Psi}(s,\mu) (\langle \mu \rangle^2 \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \]
\[ - 2is^{-\frac{3}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^3 (\mu \cdot \nabla \partial_s (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s})) \]
\[ - 3is^{-\frac{7}{2}} \tilde{\Psi}(s,\mu) (\langle \mu \rangle^2 \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \]
\[ - 3is^{-\frac{7}{2}} (\log s) \partial_s \tilde{\Psi}(s,\mu) (\langle \mu \rangle^2 \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \]
\[ + 3s^{-\frac{7}{2}} e^{i(\mu^{-1})s} \langle \mu \rangle^2 (\mu \cdot \nabla \partial_s (\langle \mu \rangle^2) \frac{3}{2} \rho(\mu) e^{i\beta} e^{i\tilde{\Psi}(s,\mu) \log s})) \]
\[ =: -2s^{-\frac{7}{2}} \tilde{\Psi}(s,\mu) (\langle \mu \rangle^2 \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) + J_5 - I_4 + J_6 + J_7 - J_0. \]
From (4.18), (4.19), (4.20), (4.21) and (4.22) we reach to
\[ (\Box + 1) u_{ap} - N_t = \lambda c_1 s^{-\frac{7}{2}} (\langle \mu \rangle^2 \frac{3}{2} - \rho(\mu) \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \]
\[ + \Re I_1 + \sum_{k=1}^7 \Re J_k. \]
To estimate the right hand side of (4.23), we use lemmas in Section 4.1.

We will estimate the right hand side of (4.23) in $L^2_t(|x| < t)$. Thanks to (4.4) and Lemma 4.2, we have
\[ \| \lambda c_1 s^{-\frac{7}{2}} (\langle \mu \rangle^2 \frac{3}{2} - \rho(\mu) \frac{3}{2} \rho(\mu) e^{i(\alpha + \beta)}) \|_{L^2_t(|x| < t)} \lesssim t^{-1 - \frac{5}{8}}. \]
Furthermore, it follows from Lemma 4.3 that
\[ \| \Re (J_5 + J_7) \|_{L^2_t(|x| < t)} \lesssim t^{-1 - \frac{5}{8}} \log t. \]
By (4.4) and (4.19), we obtain
\[ \| \Re I_1 \|_{L^2_t(|x| < t)} \lesssim t^{-2} \| \rho \|_{L^2_t(\mathbb{R}^3)} \lesssim t^{-2} \| \rho \|_{L^2_t}. \]
Similarly,
\[
\|\text{Re} \, J_0\|_{L^2(|x|<t)} \lesssim t^{-2}(\|\langle \cdot \rangle^{-\frac{3}{4}} \rho\|_{L^\infty_t(L^\infty_x(\mathbb{R}^3))}^2 + \|\langle \cdot \rangle^{-2} \rho\|_{L^\infty_t(L^2_x(\mathbb{R}^3))}) \leq t^{-2} \|\rho\|_{H^1}(\|\rho\|_{H^1})^{\frac{3}{2}}.
\]

To estimate \( J_1 \), we note that
\[
\partial^2_s e^{i\tilde{\Psi}(s,\mu)} \log s
= i\partial^2_s \tilde{\Psi}(s,\mu) \log s e^{i\tilde{\Psi}(s,\mu)} + 2is^{-1}\partial_s \tilde{\Psi}(s,\mu) e^{i\tilde{\Psi}(s,\mu)} \log s
\]
\[- (\partial_s \tilde{\Psi}(s,\mu) \log s)^2 e^{i\tilde{\Psi}(s,\mu)} \log s - 2s^{-1}\partial_s \tilde{\Psi}(s,\mu) \log s e^{i\tilde{\Psi}(s,\mu)} \log s
\]
\[- is^{-2}\tilde{\Psi}(s,\mu) e^{i\tilde{\Psi}(s,\mu)} \log s - s^{-2}\tilde{\Psi}(s,\mu)^2 e^{i\tilde{\Psi}(s,\mu)} \log s.
\]

Then, one sees from Lemma 4.3 that
\[
\|J_1\|_{L^2(|x|<t)} \lesssim t^{-2-\frac{3}{8}} \log t + t^{-2-\frac{3}{8}} + t^{-2-\frac{3}{8}} (\log t)^2
\]
\[+ t^{-3}(\log t)\|\langle \cdot \rangle^{-4} \rho \|_{L^\infty_t(L^\infty_x(\mathbb{R}^3))} + t^{-2}\|\langle \cdot \rangle^{-1} \rho(t)\|_{L^\infty_t(L^2_x(\mathbb{R}^3))}^2
\]
\[+ t^{-2}\|\langle \cdot \rangle^{-1} \rho(t)\|_{L^\infty_t(L^2_x(\mathbb{R}^3)).}
\]

Using (4.5) and the Hölder and the Sobolev inequalities, we obtain
\[
\|J_1\|_{L^2(|x|<t)} \lesssim t^{-2}(\|\rho\|_{H^1})^{\frac{3}{2}}.
\]

We next estimate \( J_3 \) and \( J_4 \). To this end, we remark that
\[
\partial_j \partial_k (\mu)^{\frac{3}{2}} \rho(\mu) e^{i\tilde{\Psi}(s,\mu)} \log s
= (\partial_j \partial_k (\mu)^{\frac{3}{2}} \rho(\mu) e^{i\tilde{\Psi}(s,\mu)}) \log s
\]
\[+ i(\partial_j (\mu)^{\frac{3}{2}} \rho(\mu) e^{i\tilde{\Psi}(s,\mu)}) \partial_k \tilde{\Psi}(s,\mu) \log s e^{i\tilde{\Psi}(s,\mu)} \log s
\]
\[+ i(\partial_k (\mu)^{\frac{3}{2}} \rho(\mu) e^{i\tilde{\Psi}(s,\mu)}) \partial_j \tilde{\Psi}(s,\mu) \log s e^{i\tilde{\Psi}(s,\mu)} \log s
\]
\[- (\mu)^{\frac{3}{2}} \rho(\mu) e^{i\tilde{\Psi}(s,\mu)} \log s e^{i\tilde{\Psi}(s,\mu)} \log s.
\]

Hence, it follows from Lemma 4.4 that
\[
\|J_3 + J_4\|_{L^2(|x|<t)}
\lesssim t^{-2} \left( \|\langle \cdot \rangle^\frac{3}{2} \nabla^2 (\langle \cdot \rangle^\frac{3}{2} \rho e^{i\tilde{\beta}})\|_{L^\infty_x} + (\log t) \|\langle \cdot \rangle^3 \rho \|_{L^\infty_x} \right)
\]
\[+ t^{-3}(\log t) \left( \|\langle \cdot \rangle^\frac{3}{2} \nabla (\langle \cdot \rangle^\frac{3}{2} \rho e^{i\tilde{\beta}})\|_{L^\infty_x} + \|\langle \cdot \rangle^3 \rho \|_{L^2_x} \right)
\]
\[+ t^{-2}(\log t) \left( \|\langle \cdot \rangle^{-\frac{3}{2}} \nabla (\langle \cdot \rangle^\frac{3}{2} \rho e^{i\tilde{\beta}})\|_{L^\infty_x} + \|\rho\|_{L^2_x} \right)
\]
\[+ t^{-2}(\log t)^2 \|\langle \cdot \rangle^3 \rho \|_{L^2_x} + t^{-3}(\log t)^2 \|\langle \cdot \rangle^{-2} \rho \|_{L^\infty_x}.
\]

By an argument similar to that in (4.16), we have
\[
\|\text{Re}(J_3 + J_4)\|_{L^2(|x|<t)} \lesssim t^{-1-\frac{3}{8}} (\log t) \|\langle \cdot \rangle^3 \rho e^{i\tilde{\beta}}\|_{H^2} + \|\langle \cdot \rangle^3 \rho\|_{H^2}^2.
\]
Finally, we estimate $J_2$. Since
\[
\partial_{\mu_j} \partial_s (\langle \mu \rangle \frac{2}{\rho(\mu)} e^{i\beta} e^{i\bar{\Psi}(s,\mu) \log s})
= (\partial_{\mu_j} (\langle \mu \rangle \frac{2}{\rho(\mu)} e^{i\beta})) (i \partial_s \bar{\Psi}(s,\mu) \log s + is^{-1} \bar{\Psi}(s,\mu)) e^{i\bar{\Psi}(s,\mu) \log s}
+ \langle \mu \rangle \frac{2}{\rho(\mu)} e^{i\beta} (i \partial_{\mu_j} \partial_s \bar{\Psi}(s,\mu) \log s + is^{-1} \partial_{\mu_j} \bar{\Psi}(s,\mu)) e^{i\bar{\Psi}(s,\mu) \log s}
- \langle \mu \rangle \frac{2}{\rho(\mu)} e^{i\beta} (\partial_{\mu_j} \bar{\Psi}(s,\mu) \log s + s^{-1} \bar{\Psi}(s,\mu)) \partial_{\mu_j} \bar{\Psi}(s,\mu) (\log s) e^{i\bar{\Psi}(s,\mu) \log s},
\]
develop from Lemmas 4.3 and 4.4 that
\[
\|J_2\|_{L^2_x(I|\tau|<t)} \lesssim t^{1 - \frac{n}{2}} \|\langle \cdot \rangle \frac{2}{\rho(\mu)} \|_{L^2} + \|\langle \cdot \rangle \frac{2}{\rho(\mu)} \|_{H^2} + \|\langle \cdot \rangle \frac{2}{\rho(\mu)} \|_{H^1}.
\]
Substituting (4.24), (4.25), (4.26), (4.27), (4.28), (4.29), (4.30) into (4.23), we obtain (4.16) because \(\|\langle \cdot \rangle \frac{2}{\rho(\mu)} \|_{H^2} + \|\langle \cdot \rangle \frac{2}{\rho(\mu)} \|_{H^1} \lesssim \|\langle \mu \rangle \|_{Y}\). This completes the proof of Proposition 4.5.

Next we give an estimate for difference between \((\Box + 1)\bar{u}_{ap}\) and the non-resonant part \(N_n\).

**Proposition 4.6.** For all \(n \geq 2\) and \(t \geq 3\), we have
\[
\|(\Box + 1) v_n - N_n\|_{L^2_x(I|\tau|<t)} \lesssim |c_n| t^{-2} \langle \|\langle \mu \rangle \|_{Y} \rangle^{\frac{5}{2}}, \tag{4.31}
\]
where \(N_n\) is given by
\[
N_n = \lambda c_n t^{\frac{n}{2}} \mathbf{1}_{\{\tau|<t\}} \langle \mu \rangle \frac{2}{\rho(\mu)} \text{Re} e^{i\mu}.
\]
In particular,
\[
\|(\Box + 1) u_{ap} - N_{ap}\|_{L^2_x(I|\tau|<t)} \lesssim t^{-2} \langle \|\langle \mu \rangle \|_{Y} \rangle^{\frac{5}{2}}.
\]
**Proof.** Denoting \(d_n = -\lambda c_n/(n^2 - 1)\), we have
\[
((\Box + 1) v_n = d_n \text{Re} ((\Box + 1) (s^{-\frac{2}{n}} \langle \mu \rangle \frac{2}{\rho(\mu)} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
\]
As in the previous case, we split
\[
d_n((\Box + 1)(s^{-\frac{2}{n}} \langle \mu \rangle \frac{2}{\rho(s,\mu)} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
= d_n((\Box + 1)s^{-\frac{2}{n}} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
+ d_n(s^{-\frac{2}{n}} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s}) \partial_s ((\mu \frac{2}{\rho(s,\mu)} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
+ 2d_n \partial_s (s^{-\frac{2}{n}} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s}) \partial_s ((\mu \frac{2}{\rho(s,\mu)} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
- 2d_n \nabla_x (s^{-\frac{2}{n}} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s}) \cdot \nabla_x ((\mu \frac{2}{\rho(s,\mu)} e^{i\mu} e^{i\bar{\Psi}(s,\mu) \log s})
= I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}.
\]
By means of Lemma 4.1,

\[ I_{1,n} = \lambda c_n s^{-\frac{2}{p}} \langle \mu \rangle^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\alpha + \beta)} - 2i n d_n s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\alpha + \beta)} \]

\[ + \frac{3d_n}{4} s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\alpha + \beta)} =: \lambda c_n s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\alpha + \beta)} + K_{1,n} + K_{2,n}. \]

(4.33)

In a similar way, by using (4.3) we have

\[ I_{2,n} = d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} \partial_e^2 (\rho(s, \mu)^{\frac{2}{p}} e^{in(s, \mu) \log s}) \]

\[ - 2d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - 3d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ =: J_{1,n} + J_{2,n} + J_{3,n} + \frac{3}{2} K_{3,n} + J_{4,n}. \]

(4.34)

By using the identities

\[ \partial_i (s^{-\frac{2}{p}} e^{in(\mu)^{-1}s}) = in(\mu)s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} - \frac{5}{2} s^{-\frac{2}{p}} e^{in(\mu)^{-1}s}, \]

\[ \nabla_{\mu} e^{in(\mu)^{-1}s} = -in\mu e^{in(\mu)^{-1}s}, \]

we obtain

\[ I_{4,n} = 2i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} \]

\[ \times (\mu \cdot \{ s^{-1}(\mu) \nabla_{\mu} + s^{-1}(\mu)(\mu \cdot \nabla_{\mu}) \}) \{ (\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \} \]

\[ = 2i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{3}{p}} (\mu \cdot \nabla_{\mu} ((\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s})) \]

(4.35)

and

\[ I_{3,n} = -2n^2 d_n s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} e^{in(\mu)^{-1}s} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - 2n^2 d_n s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} (\log s)(\mu)^{\frac{2}{p}} e^{in(\mu)^{-1}s} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ + 2i n d_n s^{-\frac{2}{p}} (\mu)^{\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - 2i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{3}{p}} (\mu \cdot \nabla_{\mu} ((\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s})) \]

\[ - 5i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ - 5i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ + 5i n d_n s^{-\frac{2}{p}} e^{in(\mu)^{-1}s} (\mu)^{\frac{2}{p}} \tilde{\Psi}(s, \mu)(\mu)^{\frac{2}{p}} \rho(s, \mu)^{\frac{2}{p}} \rho(\mu)e^{in(\beta)} e^{in(s, \mu) \log s} \]

\[ =: K_{4,n} + K_{5,n} - I_{4,n} + J_{5,n} + J_{7,n} + K_{6,n} + \frac{5}{2} K_{3,n}. \]

(4.36)
Substituting (4.33), (4.34), (4.35) and (4.36) into (4.32), we conclude that
\[\begin{align*}
(\Box + 1)v_n - N_n &= \lambda c_n s^{-\frac{7}{4}} (\rho(s, \mu)^{\frac{3}{2}} - \rho(\mu)^{\frac{3}{2}}) \rho(\mu) \Re(e^{i\alpha + \beta}) \\
&+ \sum_{k=1}^{7} \Re J_{k,n} + \sum_{k=1}^{6} \Re K_{k,n}.
\end{align*}\]  

The first term of the right hand side is \(O(|c_n|^{|s-1/6|})\) in \(L^2_t(|x| < t)\) with the help of (4.4) and Lemma 4.2. The estimates for \(J_{k,n}\) are similar to those for the corresponding \(J_k\). The differences are that the additional decay effect of order \(O(t^{-1})\) makes them all higher order terms, that each term is multiplied by \(\langle \mu \rangle \tilde{\rho}(s, \mu)^{2/3}\), and that the order in \(n\) is at most \(O(n^2|d_n|) = O(|c_n|) = O(n^{-8/3})\) as \(n \to \infty\) because the phase parts are differentiated at most twice. The terms \(K_{k,n}\) are new but the estimates for \(K_{k,n}\) are done in a similar way. This completes the proof of Proposition 4.6.

\[\square\]

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E-mail address: masaki@sigmath.es.osaka-u.ac.jp
E-mail address: segata@m.tohoku.ac.jp