A QUESTION OF FROHARDT ON 2-GROUPS, AND SKEW TRANSLATION QUADRANGLES OF EVEN ORDER

KOEN THAS

Abstract. We solve a fundamental question posed in Frohardt’s 1988 paper [7] on finite 2-groups with Kantor families, by showing that finite groups with a Kantor family \((\mathcal{F}, \mathcal{F}^*)\) having distinct members \(A, B \in \mathcal{F}\) such that \(A^* \cap B^*\) is a central subgroup of \(H\) and the quotient \(H/(A^* \cap B^*)\) is abelian cannot exist if the center of \(H\) has exponent 4 and the members of \(\mathcal{F}\) are elementary abelian. In a similar way, we solve another old problem dating back to the 1970s by showing that finite skew translation quadrangles of even order \((t, t)\) are always translation generalized quadrangles.

Contents

1. Introduction 1
2. Synopsis of definitions 3
3. Notation 5
4. Preliminary properties 5
5. A new subquadrangle lemma 6
6. Proof of the first main result 8
7. STGQs of order \((t, t)\) with \(t\) even are TGQs 9
References 10

1. Introduction

In a stunning and seminal paper of 1988 [7], Frohardt solved a big part of a famous question of Kantor which states that a finite group admitting a Kantor family is a \(p\)-group. Such groups are very important since they produce generalized quadrangles, the central Tits buildings of rank 2. In fact, groups with Kantor families yield the most important tool to construct generalized quadrangles, and a large literature exists on these objects — see for instance [14] chapters 8–10 and [16] chapters 10 and 11. Generalized quadrangles constructed from Kantor families — by a procedure which is explained in section 2.4 — carry an interesting automorphism group which locally fixes some point line-wise, and are called “elation generalized quadrangles” (EGQs). The theory of EGQs was initially devised through the 1970s and 1980s (see [14] chapters 8–10 and [20]) in order to study automorphisms, and especially Moufang conditions, of generalized quadrangles from a local point of view, much like translation planes were studied in the theory of projective planes. Together with a deep combinatorial theory, this enabled an almost entirely synthetic-geometric proof of the celebrated results of Fong and Seitz [5, 6] on split BN-pairs of rank 2, in the case of type \(B_2\) BN-pairs — see [14] chapters 8 and 9, [16] chapters 10 and 11, and also the related recent note [22].

In [7], Frohardt also studied groups \(H\) with Kantor families \((\mathcal{F}, \mathcal{F}^*)\) which have distinct members \(A, B \in \mathcal{F}\) such that \(A^* \cap B^* \leq Z(H)\) (where \(Z(H)\) denotes the center of \(H\)) and \(H/(A^* \cap B^*)\) is abelian. Such Kantor families are fundamental, since almost all examples of Kantor families have this natural property — see, e.g., the monograph [20] or Payne’s famous paper [12]. We also refer to section 2.4 for more background information on this matter. Only a very few classes of finite

2000 Mathematics Subject Classification. 05B25, 20B10, 20B25, 20E42, 51E12.
groups are known which have Kantor families; the most used ones are elementary abelian groups, and Heisenberg groups defined over finite fields in odd characteristic, of dimension 3 or 5 (see e.g. the monograph [20]). They all satisfy Frohardt’s condition.

**Theorem 1.1** (D. Frohardt [7], 1988). Let $H$ be a finite group with a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ having distinct members $A, B \in \mathcal{F}$ such that $A^* \cap B^* \leq Z(H)$ and $H/(A^* \cap B^*)$ is abelian. Then one of the following three cases occurs:

1. $Z(H)$ and the elements of $\mathcal{F}$ are elementary abelian.
2. $Z(H)$ is an elementary abelian 2-group and the elements of $\mathcal{F}$ have exponent 4.
3. $Z(H)$ has exponent 4 and the elements of $\mathcal{F}$ are elementary abelian 2-groups.

In loc. cit., Frohardt asked whether cases (2) and (3) actually occur.

In 2006, Rostermundt [15] and, independently Thas [18], constructed an infinite class of examples (related to Hermitian varieties in projective dimension 3), which fitted in class (2), so class (3) became the final challenge. Frohardt’s problem resisted many years of tries (see for instance the work of Hachenberger [8]), and the problem also occurs in the large literature of translation nets. Besides, the existence of elements in class (3) presents one of the main obstacles in classifying so-called “skew translation generalized quadrangles” (STGQs) — see [22, 24], and also [25, 1]. We also refer to section 2.5 for some milestone results in that classification theory.

In this paper, we resolve the question by showing that class (3) is empty. The proof consists of a mixture of synthetic and geometric reasoning. Most of the proof was obtained about 4 years ago, but the key lemma, Lemma 5.1, which constructs certain subquadrangles from local data and can be seen as a group-theoretical generalization of a result in [17], was only found very recently, in January 2018. It uses the finer geometry of the projective representation (through the André-Bruck-Bose construction) of the local translation plane at the regular elation point, once we have shown that in case (3), the parameters of hypothetical examples necessarily are of type $(t, t)$. The details can be found in section 5.

As a bonus of this new technique, we are able to solve another long-standing open problem on which virtually no progress has been made since Payne’s first paper on skew translation generalized quadrangles [11] from 1975, marking the birth of the theory. In its long history, no examples were found of STGQs of even order $(t, t)$ for which the elation group was not abelian — when $t$ is odd, the theory is entirely different, since then it is easy to prove that such a group cannot be abelian at all. Since one expects that such STGQs would have to meet the requirements of Frohardt’s theorem, and since for STGQs of order $(t, t)$ the elements of $\mathcal{F}$ are always elementary abelian (see [23] for those details), such examples would conjecturally live in class (3) of Theorem 1.1. And indeed they do, if they would exist.

The best result up to now was the same result when $(t, t) = (8, 8)$ [1], with a very long and computer-aided technical proof. Our proof of the general solution is very short (see sections 5 and 7): it states the any finite STGQ of even order $(t, t)$ necessarily is a translation generalized quadrangle — that is, the associated elation group is an elementary abelian 2-group, and hence the quadrangle has a projective representation in some projective space. This essentially means that there is no “proper” theory of STGQs in this case.

On the combinatorial level, there is also an other interesting aspect to this result. If every line incident with some point in a finite generalized quadrangle of even order $(t, t)$ is regular (see section 2.2 for a formal definition of this very important notion), then it can be shown that the point itself is also regular ([14, section 1.5.2]). Our second main result provides a group-theoretical converse: if a finite EGG of even order $(t, t)$ has a regular elation point (this is another way of defining STGQs in this specific case), then it is a translation quadrangle, so all the lines incident with $x$ are regular.

**Organization.** In sections 2 and 8 we introduce the basic notions that we will need. In section 4 we make a number of synthetic observations; we essentially show that in case (3) of Frohardt’s theorem, it is sufficient to work with Kantor families of type $(t, t)$. This is done in the geometrical
language of generalized quadrangles. In section 5, we obtain the new subquadrangle lemma. Then, in section 6, we show that Frohardt’s class (3) is empty. And finally, in the last section, we show that STGQs of even order \((t, t)\) are always translation quadrangles.

2. Synopsis of definitions

Let \(\Gamma\) be a thick generalized quadrangle (GQ). It is a rank 2 geometry \(\Gamma = (P, B, I)\) (where we call the elements of \(P\) “points” and those of \(L\) “lines”) such that the following axioms are satisfied:

(a) there are no ordinary digons and triangles contained in \(\Gamma\);
(b) each two elements of \(P \cup L\) are contained in an ordinary quadrangle;
(c) there exists an ordinary pentagon.

It can be shown that there are exist constants \(s\) and \(t\) such that each point is incident with \(t + 1\) lines and each line is incident with \(s + 1\) points. We say that \((s, t)\) is the order of \(\Gamma\). Note that an ordinary quadrangle is just a (necessarily thin) GQ of order \((1, 1)\) — we call such a subgeometry also “apartment” (of \(\Gamma\)).

2.1. Subquadrangles. A subquadrangle (subGQ) \(\Gamma' = (P', B', I')\) of a generalized quadrangle \(\Gamma = (P, B, I)\), is a GQ for which \(P' \subseteq P\), \(B' \subseteq B\), and \(I'\) is the incidence relation which is induced by \(I\) on \((P' \times B') \cup (B' \times P')\).

2.2. Regularity. Let \(\Gamma\) be a thick GQ of order \((s, t)\). If \(X\) and \(Y\) are (not necessarily different) points, or lines, \(X \sim Y\) denotes the fact that there is a line incident with both, or a point incident with both. Then \(X^\perp := \{Y \mid Y \sim X\}\), and if \(S\) is a point set, or a line set, \(S^\perp := \cap_{s \in S} s^\perp\) and \(S^\perp\perp := (S^\perp)^\perp\).

A particularly important example is the case where \(S = \{X, Y\}\) is a set of distinct noncollinear points (or nonconcurrent lines, but this is merely the dual situation, which we leave to the reader); then each line incident with \(X\) is incident with precisely one point of \(\{X, Y\}^\perp\) (so if \(\Gamma\) is finite, \(|\{X, Y\}^\perp| = t + 1\)). The set \(\{X, Y\}^{\perp\perp}\) consists of all points which are collinear with every point of \(\{X, Y\}^\perp\), so

\[
(1) \quad \{X, Y\} \subseteq \{X, Y\}^{\perp\perp}\]

Let \(Z\) be any point of \(\{X, Y\}^{\perp}\); if each line incident with \(Z\) is incident with exactly one point of \(\{X, Y\}^{\perp\perp}\), then this property is independent of the choice of \(Z\), and we say that \(\{X, Y\}\) is regular. In the finite case, we could equivalently have asked that \(|\{X, Y\}^{\perp\perp}| = t + 1\).

We call a point/line \(X\) regular if \(\{X, Y\}\) is regular for all \(Y \not\sim X\).

2.3. Symmetry. Isomorphisms and automorphisms of generalized quadrangles are defined in the usual manner. See chapter 1 of [14]. The automorphism group of a GQ \(\Gamma\) will be denoted by \(\operatorname{Aut}(\Gamma)\).

Let \(X\) be a point or a line in a thick GQ \(\Gamma\). A symmetry with center \(X\) (in the case of a point) or axis \(X\) (in the case of a line) is an element of \(\operatorname{Aut}(\Gamma)\) that fixes each element of \(X^\perp\). We say that \(X\) is a center of symmetry (point case) or an axis of symmetry (line case) if there exist \(Y\) and \(Z\) in \(X^\perp\) such that \(Y \not\sim Z\), for which the group of all symmetries \(S(X)\) with center/axis \(X\) acts transitively on \(\{Y, Z\}^{\perp\perp} \setminus \{X\}\). This definition does not depend on the choice of \((Y, Z)\), and one easily shows that “transitive” implies “sharply transitive.” In the finite, we could also have asked that

\[
(2) \quad |S(X)| = t
\]

if the order of \(\Gamma\) be \((s, t)\).

Note that a center/axis of symmetry is necessarily regular.
2.4. Kantor families. In this section, we will recall the very important Kantor family construction. Groups with Kantor families produce generalized quadrangles, and vice versa, a certain type of generalized quadrangle gives rise to Kantor families. We will only define finite Kantor families, as that is the case we will need in due course.

So suppose \( K \) is a finite group of order \( u^v v \) for positive integers \( u \) and \( v \), both at least 2. A Kantor family of type \((u, v)\) in \( K \) is a pair \((F, F^*)\) of sets of subgroups of \( K \) for which the properties below are satisfied:

(a) \(|F| = |F^*| = v + 1\), and there is a bijection \(* : F \mapsto F^*\) which maps each \( A \in F \) to \( A^* \in F^*\) such that \( A \leq A^*\);

(b) for each \( A \in F \), \(|A| = u\) and \(|A^*| = uv\);

(c) if \( A, B, C \) are different elements in \( F \), then \( AB \cap C = \{\text{id}\}\);

(d) if \( A \) and \( B \) are different elements in \( F \), then \( A \cap B^* = \{\text{id}\}\).

From the data \( (K, (F, F^*)) \) one constructs a thick GQ \( \Gamma = \Gamma(K, (F, F^*)) \) of order \((u, v)\) as follows. Its points are a symbol \((\infty, x)\), left cosets of type \( gA^* \) \((A \in F \text{ and } g \in K)\), and the elements of \( K \). Its lines are symbols \([A] \in F\) and left cosets of type \( gA \) \((A \in F \text{ and } g \in K)\). The point \((\infty, x)\) is incident with all lines of the first type, and no other lines. A line \([A] \in F\) is also incident with all points \( gA^* \). All other incidences are (reversed) containment. The group \( K \) acts naturally as an automorphism group on \( \Gamma \), by left multiplication on the cosets, while fixing the symbolic point and symbolic lines. Then \( K \) fixes all the lines incident with \((\infty, x)\), and acts sharply transitivity on the points of \( \Gamma \) which are not collinear with \((\infty, x)\).

Conversely, let \( \Omega \) be a thick GQ of order \((u, v)\), with an automorphism group \( L \) which fixes some point \( x \) linewise while acting sharply transitivity on the points not collinear with \( x \). Then \(|L| = u^2 v \). Now take one arbitrary point \( y \) not collinear with \( x \) (due to the transitivity of \( L \) on these points, this choice is not important). For each line \( L_{\text{id}}y \), let \( u \) be the point which is incident with \( U \) and collinear with \( x \); let \( L_{\text{id}}y := L_{\text{id}}x \cdot \text{then with } F := \{ L_{\text{id}} \mid U_{\text{id}}y \} \text{ and } F^* := \{ L_{\text{id}} \mid U_{\text{id}}y \}, \quad (F, F^*) \text{ is a Kantor family of type } (u, v) \text{ in } L \). Also, we have a natural isomorphism

\[
\Gamma(L; (F, F^*)) \mapsto \Omega
\]

which maps \( L \) to itself and \((\infty, x)\) to \( x \).

2.5. Skew translation quadrangles. If \( (\Gamma, K) \) is an EGQ with elation point \( x \), and \( x \) is a center of symmetry such that the corresponding symmetry group \( S \) is a subgroup of \( K \), then we call \((\Gamma, K)\) a skew translation generalized quadrangle (STGQ). In terms of Kantor families (using the notation of the previous subsection), a Kantor family \((F, F^*)\) gives rise to an STGQ if and only if there is a normal subgroup \( C \) of \( K \) such that for each \( A \in F \), we have \( A^* = AC \) (see [11] or [14] chapter 10)); \( C \) then corresponds to the group of symmetries with center the elation point. Note that for different \( A, B, C \) \in F, we have \( A^* \cap B^* = C \).

This type of GQ is very general: each known finite generalized quadrangle which is not isomorphic to the Hermitian quadrangle \( H(4, q^2) \) for some finite field \( F_q \), is, modulo point-line duality, either an STGQ, or the Payne-derived quadrangle of an STGQ. More details on this statement can be found in [24]; for a more or less recent but rather detailed census on the known GQs, we also refer to [13].

We recall some basic (“decisive”) classification results in the large theory of STGQs. Much more can be found in [24].

The first result was first obtained by the author in [24] (see [23] for chronological details in its conception). It was also obtained in [11] with virtually the same proof.

Theorem 2.1. An STGQ of odd order \((t, t)\) is isomorphic to the symplectic quadrangle \( W(t) \).

The symplectic quadrangle \( W(t) \) arises as the \( \mathbb{F}_t \)-rational points in a projective space \( \mathbb{P}^3(\mathbb{F}_t) \), together with the absolute lines with respect to a symplectic polarity of \( \mathbb{P}^3(\mathbb{F}_t) \) (see [14] chapter 3); it is one of the natural geometric modules of the group \( \text{PSp}_4(t) \).

The most investigated type of STGQ is arguably the class of "flock quadrangles." All the known flock quadrangles arise through Kantor families in one and the same type of elation group: finite
Heisenberg groups of dimension 5 (over a finite field). The next result proves the converse, and is taken from [21].

**Theorem 2.2.** If $(\Gamma, K)$ is an EGQ of order $(s, t)$ and $K$ is isomorphic to a Heisenberg group of dimension 5 over $\mathbb{F}_t$, then $\Gamma$ is a flock quadrangle.

The following result is one of the only known results on STGQs of even order $(t, t)$; its proof takes up more than half of the paper [1], and is computer-aided.

**Theorem 2.3.** An STGQ of order $(8, 8)$ is a translation generalized quadrangle.

In the last section of the present paper, we will handle all STGQs of even order.

3. Notation

Let $K$ a finite group with a Kantor family $(\mathcal{F}, \mathcal{F}^*)$ of type $(s, t)$. For certain distinct elements $A, B \in \mathcal{F}$, we suppose that $K/(A^* \cap B^*)$ is abelian and that $A^* \cap B^* \leq Z(K)$, the center of $K$. Put $A^* \cap B^* =: \mathcal{S}$. We want to work solely in case (3) of Frohardt’s theorem in sections 4 and 5 so that the elements of $\mathcal{F}$ are elementary abelian, but $Z(K) =: Z$ has exponent 4.

The associated generalized quadrangle with parameters $(s, t)$ is denoted $\Gamma^x$, or $\Gamma$. The Kantor family is defined relative to the point $z \neq x$.

We say that a group $G \leq K$ satisfies (*) if for any element $\alpha$ of $G$, we have that if $\alpha$ fixes some point $y \sim x \neq y$, $\alpha$ also fixes $xy$ pointwise. We say that $G \leq K$ satisfies (*) at $U \parallel x$ if this property is locally fulfilled at $U$.

4. Preliminary properties

In this section we observe some initial properties which narrow down the possibilities for elements in Theorem 1.1(3). In the final part of the proof of the main theorem, we will not need all these properties, but it is interesting to see how far one can go “synthetically.”

First note that for any line $U \parallel x$, $K$ acts transitively on $U \setminus \{x\}$. As $\mathcal{S} \leq Z(K)$, it hence follows that $\mathcal{S}$ fixes $[A]$ and $[B]$ pointwise (so $\mathcal{S}$ is in the kernel of the action of $K$ on the point set of $[A]$, and $[B]$). As $K/\mathcal{S}$ is abelian and transitive on both $[A] \setminus \{x\}$ and $[B] \setminus \{x\}$, we have that $K$ has (*) at $[A]$ and $[B]$.

Let $z \in Z^x$; if $z$ would fix an affine line, $z$ is in some conjugate of an element of $\mathcal{F}$, implying that $z^2 = 1$. So we may assume that $z$ does not have this property. Now suppose there is an affine line $U$ such that $U^z$ does not intersect $U$. Take any element $V \in \{U, U^z\}^\perp$ which is not incident with $x$; choose the element $\beta$ in $K_U$ which sends $V^z$ to $V$; then $z\beta$ fixes $U$, and $[z, \beta] = 1$, so $id = (z\beta)^2 = z^2\beta^2 = z^2$. So we may suppose that $z$ fixes each point of $x^\perp$, i.e., $z$ is a symmetry with center $x$. In its turn, this implies that $z \in \mathcal{S}$.

4.1. Reduction of parameters. Now suppose that $z$ is a (nontrivial) symmetry with center $x$, and suppose that $U \sim [A]$ is an affine line. Let $\alpha \in A^*$ be an element which sends $U^z$ to $U$, such that $z\alpha \neq id$ (and note that such $\alpha$ exist, of course). Then $z\alpha$ fixes $U$. As $[z, \alpha] = id$, $z$ is an involution if and only if $\alpha$ is an involution. As $(\mathcal{F}, \mathcal{F}^*)$ is a Kantor family, we can write $A^* = A\mathcal{S}$ (with $A \cap \mathcal{S} = \{id\}$). We can write $\alpha = z^{-1}a$, with $a \in K_U$, so that the fixed points structure of $\alpha$ is that of $a$ — precisely the points incident with $[A]$. Suppose $\alpha$ does not fix affine lines. Applying a theorem of Benson, we conclude

\[(t + 1)(s + 1) + st \equiv st + 1 \mod s + t,
\]

which gives a contradiction unless $s = t$ is even. In the other cases, $\alpha$ must fix affine lines, so that it is an involution, and hence $z$ also is. So if $s \neq t$, each element of $Z$ is an involution, showing that case (3) of Frohardt’s theorem cannot occur under these assumptions.
4.2. The centralizers $C_B(\alpha)$. Now consider the case where $s = t$, and $t$ is even. Note that $S$ only consists of symmetries and involutions. Suppose $\beta \in S^\times$ is not an involution (it is then a symmetry); compose $\beta$ with a hypothetical involution $i$ in $Z$ which is not a symmetry. Then $\beta i \in Z$ is not a symmetry, so it is an involution, and $id = (\beta i)^2 = \beta^2 i^2 = \beta^2$, contradiction. So all involutions in $S$ are symmetries, and whence all elements of $S$ are symmetries. It follows easily that $x$ is a regular point.

If $K$ is abelian, the theory of translation generalized quadrangles gives us that $K$ is elementary abelian (see [16, chapter 3]), so we may assume that $K$ be not abelian. By Proposition 3.1 of Hachenberger [8], it follows that $Z(K) = S$.

Let $C \neq D$ be in $T$; then $[H,H] = [C,D]$ (as $H = CD(S) = ((cd)^2)|c \in C, d \in D$). As this group is a subgroup of $S$, it follows that $[H,H]$ is elementary abelian. As $\Phi = H^2[H,H]$, it easily follows that $\Phi$ is the elementary abelian subgroup of $S$ of all squares. Then on the other hand, if $s = \gamma^2 \in S$ is a square, there is an $e \in E \in T$ and $c \in S$ such that $\gamma = ec$ (as $H = \cup_{A \in T} A^*$, and $s = ecec = c^2$). So $\Phi = S^2$. Also, for $A \in T \setminus \{E\}$, there is a unique $B \in T$ such that $\gamma \in AE$ — write $\gamma ab$ with $a \in A$ and $b \in B$. Then $(ab)^2 = \gamma^2 = s$, so that $s \in [H,H]$. We have obtained that $[H,H] = S^2 = \Phi$.

Let $A \neq B$ be arbitrary in $T$. Now let $\alpha \in A^\times$ be arbitrary, and let $\beta \in B^\times$ be such that $[\alpha, \beta] = id$. Put $T \setminus \{A,B\} = \{C_1, \ldots, C_{t-1}\}$. Then for each $i \in \{1, \ldots, t-1\}$ there is precisely one triple $(c_i, \beta_i, s_i) \in C_i \times B \times S$ such that $\alpha \beta = c_i \beta_i s_i$. Note that the maps $\mu : \{1, \ldots, t-1\} \rightarrow S^\times: j \rightarrow s_j$ and $\mu' : \{1, \ldots, t-1\} \rightarrow B \setminus \{\beta\}: j \rightarrow \beta_j$ are surjective. Suppose that $s_j^2 = id$; then as $(\alpha \beta)^2 = id$, $[\beta_c, \beta_d] = id$, and so $[\beta_c, c_i \beta_d s_i] = id = [\beta_c, \alpha \beta]$, implying that $[\alpha, \beta_c] = id$. The converse is also true. So

$$|\text{involutions in } S|(\text{including id}) = |C_B(\alpha)| =: \ell.$$ (5)

On the other hand we have $t = |\alpha B| \times |C_B(\alpha)|$, so for any such $\alpha$, the size of $\alpha^K$ (which is the same as $\alpha^B$) is a constant ($\frac{t}{\ell}$) independent of the choice of $\alpha$. As $\alpha^K = \alpha^B$, it also follows that $\alpha$ fixes precisely $|C_B(\alpha)| = \ell$ points of $[A] \setminus \{x\}$ linewise.

Note at this point that if we prove that $S$ is an elementary abelian 2-group, then each $A^*$ is also elementary abelian, and as $K$ is covered by the $A^*$s, it follows that $K$ itself is elementary abelian, contradiction with the assumption that $K$ has exponent 4.

In the rest of this paper, we will denote the subgroup of $S$ that consists of all its involutions, by $J$.

4.3. Extra info on $J$, and $\ell$. Keeping the same notation as before, we recall that $[H,H] = [C,D] = ((cd)^2)|c \in C, d \in D|$. Each $(cd)^2 = cdc = [c,d]$ is an element in $S$ which obviously is an involution if it is nontrivial, so $[H,H] \leq J$. It follows that

$$\ell = |J| \geq |[H,H]| = \frac{t}{\ell},$$ (6)

so that $\ell \geq \sqrt{t}$.

5. A new subquadrangle lemma

In this section, we will prove an extremely useful lemma which produces subquadrangles from subplanes in a derived substructure. We use essentially the same notation as before, but the setting is more general. In [17] such results were already obtained in less general circumstances, but without the presence of elation groups. Let me first define the basic object of our interest. Let $\Gamma$ be a thick finite GQ of order $(t,t)$. Suppose $\Gamma$ has some regular point $x$. Then an affine plane $\pi(x)$ of order $t$ can be constructed as follows. Its points are sets $\{u,v\}^{1,1}$, where $u$ and $v$ are noncollinear points in $x^\perp$. The lines are the points in $x^\perp \setminus \{x\}$. Incidence is natural. The parallel classes correspond to the lines incident with $x$ (so we can see $x$ as the line at infinity of $\pi(x)$).

In [17], the same structure is considered in a quadrangle of order $(s,t)$ (one then gets a net instead of an affine plane), and subplanes of order $t$ are used to construct subquadrangles of order $(t,t)$. Such results are obviously highly applicable, but they are of no use in the present setting. On
the other hand, if $\Gamma$ is as in the previous section, then $\pi(x)$ is a translation plane with translation group $K/S$, and translation planes always have proper subplanes, if the order is not a prime. Those can be small (for instance, defined over the kernel in the projective representation of the translation plane). So in stark contrast with the setting of [17], such planes in general have a much smaller number of parallel classes than the translation plane itself. In terms of group theory, we want to consider the group extension

$$1 \mapsto S \mapsto K \mapsto T \mapsto 1,$$

and a subplane $\pi'$ in $\pi(x)$ with translation group $T'\leq T$; the objective then is to define a subGQ $\Gamma$ of $\Gamma$ which also has $x$ as a regular point, and which induces the translation plane $\pi'$. Of course, we know that $K$ contains a subgroup $\tilde{K}$ containing $S$ such that $\tilde{K}/S = T'$, but that group will not do in general (as such groups need not produce subquadrangles), as the reader will see below.

**Lemma 5.1 (Subquadrangle Lemma).** Let $(\Gamma, K)$ be a thick finite elation generalized quadrangle of order $(t, t)$ with elation group $K$ and regular elation point $x$. If $\pi'$ is an affine subplane of the affine plane $\pi(x)$, which is a translation plane of order $r$ with translation group induced by the translation group of $\Pi(x)$, then $\Gamma$ contains subGQs of order $(r, r)$ which are also elation quadrangles with as elation group a subgroup of $K$, and with regular elation point $x$.

**Proof.** First of all, note that as $x$ is a regular point, $x$ is a center of symmetry (see [24] as a general reference on the structure of STGQs); denote the corresponding group of symmetries by $S$ (and note that $S$ is a subgroup of $K$). Then $K/S = T$ naturally is a translation group for $\pi(x)$. Suppose $\pi'$ as is in the statement, and let the corresponding translation group be $T' \leq T$. Let $z \neq x$ be arbitrary but fixed, and construct the Kantor family $(\mathcal{F}, \mathcal{F}^*)$ with respect to this point. We denote the lines on $z$ by $U_i$ (indexed by a set $S$ of order $t + 1$), and put $K_{U_i} := A_i$. Put $\text{proj}_{A_i} z = a_i$. We use the notation $[A_i]$ both for the line of $\Gamma$ incident with $a_i$, and also for the corresponding point at infinity of $\pi(x)$. To $z$ corresponds a point $\{x, z\}^{1+1}$ of $\pi(x)$. Without loss of generality, we suppose that this is also a point of $\pi'$.

With respect to $\{x, z\}^{1+1}$, the translation plane $\pi(x)$ is defined by the congruence partition (see [9] chapter VII, section 3) $\{T_i := A_i/S \mid i \in S\}$, where $A_i/S \cong A_i S / S$ for each $i$. One notes that $A_i \cap S = \{1\}$. Now let $S' \subseteq S$ be such that the indexes in $S'$ correspond to the points at infinity of $\pi'$; then the congruence partition of $\pi'$ is given by

$$\{T'_i := T_i \cap T' \mid i \in S'\}.$$

For each $j \in S'$, define $A_j'$ as the subgroup of $A_j$ defined by

$$A_j'/S = T_j' \cong A_j' S/S.$$

Now consider the subgroup

$$K' := \left\langle A_j' \mid j \in S' \right\rangle$$

of $K$, and let $S'$ be the kernel of the action of $K'$ on the plane $\pi'$; then obviously $S'$ is a subgroup of $S$. For each $j \in S'$, define $A_j'^*$ as $A_j' S'$. Define $\mathcal{F}' := \{A_j' \mid j \in S'\}$ and $\mathcal{F}^{**} := \{A_j'^* \mid j \in S'\}$.

Claim: $(\mathcal{F}', \mathcal{F}^{**})$ is a Kantor family with parameters $(r, |S'|)$. Put $|S'| = \sigma$. First note that $|K'| = r^2 \sigma$ by its mere definition. Also, note that $\sigma \neq 1$, as otherwise $\mathcal{F}'$ defines a congruence partition in $K'$, which would yield triangles in $\Gamma$. By definition of the elements in $\mathcal{F}'$ and $\mathcal{F}^{**}$, we have, for each $j \in S'$, that $|A_j'| = r$ and $|A_j'^*| = r \sigma$. All the other required properties follow from the fact that for each such $j$, $A_j' \leq A_j$ and $A_j'^* \leq A_j^*$. The claim is proved.

It follows that the Kantor family $(\mathcal{F}', \mathcal{F}^{**})$ defines a subquadrangle $\Gamma'$ of order $(r, \sigma)$ of $\Gamma$ that contains $x$ and $z$, and that $K' \leq K$ is an elation group for the elation point $x$. Also, since for all $m \neq n$ in $S'$ we have that

$$A_m'^* \cap A_n'^* = S',$$

the Kantor family $(\mathcal{F}', \mathcal{F}^{**})$ of $\Gamma$ contains subquadrangles of order $(r, \sigma)$.
it easily follows that $x$ is a regular point of $\Gamma'$ with group of symmetries $S'$ (cf. section 2.4). ■

Note that in general, $S$ is not a subgroup of $K'$ (cf. the discussion before the statement of the subGQ lemma).

Since $\pi(x)$ is a translation plane of order $t$, or since $\Gamma$ is an EGQ of order $(t, t)$, we know that $t$ is the power of a prime $p$, cf. [19]. The next corollary uses this fact.

**Corollary 5.2** (Existence of classical subquadrangles). Use the notation of Lemma 5.1. Put $t = p^h$ with $p$ a prime. Then $\Gamma$ contains subquadrangles of order $(p, p)$ isomorphic the symplectic quadrangle $W(p)$, which enjoy all the properties of Lemma 5.1.

**Proof.** Any translation plane of order $t = p^h$ contains sub translation planes of order $p$ (which is easily seen in the André-Bruck-Bose representation of the plane in a projective space over $\mathbb{F}_p$ — see [10] Theorem 1.5 and consequent remark). The corresponding subGQs of order $(p, p)$ which arise from the construction of Lemma 5.1 are EGQs with a regular point $x$. Since $p$ is prime, we have that such subGQs are classical by Bloemen, Thas and Van Maldeghem [2], and since $p$ is regular, we have that such subGQs are indeed isomorphic to $W(p)$ by loc. cit. ■

**Remark 5.3.** Lemma 5.1 can to some extent be generalized to elation generalized quadrangles of general order $(s, t)$ with a regular point, and can also be adapted to the infinite case. We will do this in the forthcoming paper [25]. It is of no concern now.

6. PROOF OF THE FIRST MAIN RESULT

We keep using the notation of section 3 and section 1. Let $a \in A \in \mathcal{I}$, and suppose $a \neq id$. Then $a$ fixes precisely $\ell + 1$ points incident with $[\infty]$ pointwise. Let $v$ be a point incident with $[A]$ which is not such a point.

Consider the translation plane $\pi(x)$. Applying the André-Bruck-Bose construction (see [10] Theorem 1.5 and consequent remark) over $\mathbb{F}_2$, we represent $\pi(x)$ in a projective space $\mathbb{P}^{2h}(\mathbb{F}_2)$, where $t = 2^h$. So we have a set $S$ of $t + 1$ mutually disjoint $(h - 1)$-spaces in some hyperplane $\kappa$ of $\mathbb{P}^{2h}(\mathbb{F}_2)$, and the GQ-points in $x^\perp \setminus \{x\}$, which are lines of $\pi(x)$, correspond to the $h$-spaces in $\mathbb{P}^{2h}(\mathbb{F}_2)$ which intersect $\kappa$ in an element of $S$. The GQ-sets of type $\{u, v\}^\perp$ with $u$ and $v$ noncollinear GQ-points in $x^\perp$, which are the points of $\pi(x)$ not incident with the translation line at infinity, correspond to the $\mathbb{F}_2$-points of $\mathbb{P}^{2h}(\mathbb{F}_2) \setminus \kappa$. The GQ-point $x$, which is the translation line of $\pi(x)$, corresponds to the set $S$, and the GQ-lines incident with $x$, which are the points of the translation line of $\pi(x)$, correspond to the elements of $S$.

The translation group $T = K/S$ corresponds to the group of translations of $\mathbb{P}^{2h}(\mathbb{F}_2)$ with axis $\kappa$; it is an elementary abelian 2-group of size $t^2 = 2^{2h}$.

Suppose $U \mathbf{1}z$ is the line of $\Gamma$ for which $K_U = A$. Let $U \cap [A] = u$. Let $\delta \in S$ be the element corresponding to $[A]$. Let $\alpha$ be the $h$-space in $\mathbb{P}^{2h}(\mathbb{F}_2)$ corresponding with $u$, and let $\beta$ be the $h$-space corresponding with $v$; we have that $\alpha \cap \beta = \delta$. Consider the translation of $\mathbb{P}^{2h}(\mathbb{F}_2)$ which $a$ induces on $\mathbb{P}^{2h}(\mathbb{F}_2)$; it is an involution with center a point $c \in \delta$ (we also denote it by $a$). Let $L$ be any line (a $\mathbb{P}^1(\mathbb{F}_2)$) in $\alpha$ which is not contained in $\kappa$, and which is incident with $c$; then $L^\perp = a$. Now consider a line $L'$ in $\beta$ but not contained in $\kappa$, which contains $c$ as well, and such the the projective plane $\langle L, L' \rangle$ meets $\kappa$ in a line which is not contained in $\delta$. Note that such a line $L'$ exists: just let it vary in the $h$-space $\beta$: then

$$f : L' \mapsto \langle L, L' \rangle \cap \kappa$$

is an injection, and as $\dim(\delta) = \dim(\beta) - 1$, the existence follows. Note that $a$ fixes $\langle L, L' \rangle$.

Applying Lemma 5.1 and Corollary 5.2 we conclude that $\Gamma$ has a subquadrangle $\mathbb{1}z$ isomorphic to $W(2)$ which contains $x, u$ and $v$ (a GQ of order $(2, 2)$ is isomorphic to $W(2)$; see [13] chapter 6), and for which $a$ is an element in its elation group. Now $a$ fixes these points, and also the $W(2)$-lines on $x$, as well as the $W(2)$-lines on $u$ (since $a$ fixes $u$ linewise). Let $VU \mathbf{1}v$ in $\mathbb{1}z, V \neq [A]$. Let $U'\mathbf{1}u$
also be in $\Gamma$ with $U' \neq \{A\}$; then $\{V, U'\}$ is regular in $\Gamma$ (by [14] §3.2.1 and §3.3). Let $W$ and $X$ be the lines of $\Gamma$ in $\{V, U'\} - \{\{A\}\}$; it is clear that $W^a = X$ and $X^a = W$, as $a$ fixes the lines in $\Gamma$ on $u$ and $x$, and as $a$ fixes $\langle L, L' \rangle$. Since $V$ meets both $W$ and $X$, and since $v^a = v$, we conclude that $V^a = V$, contradiction, as $a$ does not fix any line on $v$ different from $\{A\}$, by assumption.

It follows that $\ell = t$, that is, $a$ is a symmetry with axis $\{A\}$. Since the choice of $A$ and $a \in A^x$ was arbitrary, each line incident with $x$ is an axis of symmetry, and hence $\Gamma$ is indeed a TGQ with translation group $K$. But then $K$ is an elementary abelian 2-group by [10] Theorem 3.4.2, contradicting the assumption that $\Gamma$ was a member of class (3) in Theorem 7.1. (Other way: since $a$ is a symmetry with axis $\{A\}$, one observes that $\{a, K\} = \{\text{id}\}$, so that $a \in Z(K) \setminus S$, and this contradicts the fact that $Z(K) = S$ — see [12])

This concludes the proof of the main result. ■

**Remark 6.1.** We could also have directly proved, relying on the fact that $s = t$ and using Lemma 5.1 that each $a \in A$, with $A \in T$ arbitrary, is a symmetry with axis $\{A\}$. (On the other hand, once the work in section 4 is done, the “indirect way” is very fast.) We will illustrate this idea in detail in the next section.

## 7. STGQS OF ORDER $(t, t)$ WITH $t$ EVEN ARE TGQS

We now turn to the case of STGQs of order $(t, t)$ with $t$ even. The main difference with the setting of the Frohardt’s question, is that we know a priori now that the parameters of the quadrangle are of the form $(t, t)$ (with $t$ even), but we do not know that $S$ is a subgroup of the center of $K$. So a lot of information is lost in comparison with our analysis in Frohardt’s setting.

The main theorem of this final section tells us that there is no “proper” theory of STGQs if the order is $(t, t)$, $t$ even.

**Theorem 7.1.** Let $(\Gamma, K)$ be a skew translation quadrangle with elation point $x$, of order $(t, t)$, where $t$ is even. Then $(\Gamma, K)$ is a translation generalized quadrangle. In particular, $K$ is an elementary abelian 2-group.

**Proof.** Since $(\Gamma, K)$ is an STGQ, we know that $x$ is regular, and that $\pi(x)$ is a translation plane with translation group $T := K/S$, where $S$ is the group of symmetries with center $x$.

We use the exact same notation as in the previous section. So $A = KU, U$ is a line meeting $\{A\}$ in a point $u$ different from $x$. Now $v$ is any point incident with $\{A\}$ and different from $x$ and $u$, and $a \in A^x$; also, $c$ denotes the center of $a$ as before. Etc.

The $(h+1)$-space $\langle \alpha, \beta \rangle$ meets $\kappa$ in an $h$-space which contains $\delta$, so it meets any other element of $S$ in precisely one point. For $W \in S \setminus \{\delta\}$, we will denote this point by $w$. We fix the line $L$ in $\alpha$, and consider the plane $\langle cw, L \rangle$; this plane is contained in $\langle \alpha, \beta \rangle$ and meets $\beta$ in a line $\omega(W)$ which is not contained in $\delta$. Applying Lemma 5.1 and Corollary 5.2 we conclude that $\Gamma$ has a subquadrangle $\Gamma$ isomorphic to $W(2)$ which contains $x$, $u$ and $v$, and also the GQ-line $\bar{W}$ on $x$ that corresponds to $W$. Also, the elation group $K'$ as defined in Lemma 5.1 contains the element $a$ by construction, and the line $\bar{U}$ is also a line of $\Gamma$. Let $V$ be arbitrary, but different from $\{A\}$. Since the line $\bar{W}Lx$ is arbitrary, it follows that we can find such a subGQ with the same properties as before, and containing $\bar{V}$. Since $a$ fixes $U$, we now conclude as in the previous section that $\bar{V}$ is also fixed by $a$. So $a$ fixes every line in $\{A\}$ which is not incident with $u$. Now we can take $U$ to be some line in this set to conclude that $a$ is a symmetry with axis $\{A\}$.

So as in the previous section, we finally conclude that $(\Gamma, K)$ is a TGQ, and $K$ is an elementary abelian 2-group. ■

The following corollary is a culmination of Theorem 7.1 and Theorem 2.1.

**Corollary 7.2** (Classification of STGQs of order $(t, t)$). A finite STGQ of order $(t, t)$ is either isomorphic to the symplectic quadrangle $W(t)$, or is a translation generalized quadrangle. ■
References

[1] J. Bamberg, S. P. Glasby and E. Swartz. AS-configurations and skew-translation generalised quadrangles, J. Algebra 421 (2015), 311–330.
[2] I. Bloemen, J. A. Thas and H. Van Maldeghem. Elation generalized quadrangles of order $(p, t)$, $p$ prime, are classical, Special issue on orthogonal arrays and affine designs, Part I, J. Statist. Plann. Inference 56 (1996), 49–55.
[3] F. Buekenhout. (ed.) Handbook of Incidence Geometry. Buildings and foundations. North-Holland, Amsterdam, 1995, xii+1420 pp.
[4] I. Cardinali and S. E. Payne. $q$-Clan Geometries in Characteristic 2, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.
[5] P. Fong and G. M. Seitz. Groups with a $(B, N)$-pair of rank 2. I, Invent. Math. 21 (1973), 1–57.
[6] P. Fong and G. M. Seitz. Groups with a $(B, N)$-pair of rank 2. II, Invent. Math. 24 (1974), 191–239.
[7] D. Frohardt. Groups which produce generalized quadrangles, J. Combin. Theory Ser. A 48 (1988), 139–145.
[8] D. Hachenberger. Groups admitting a Kantor family and a factorized normal subgroup, Des. Codes Cryptogr. 8 (1996), 135–143.
[9] N. Knarr. Translation Planes. Foundations and construction principles, Lecture Notes in Math. 1611, Springer-Verlag, Berlin, 1995.
[10] E. Payne. Flock generalized quadrangles and related structures: an update, in Generalized Polygons, Proceedings of the Academy Contact Forum “Generalized Polygons” (Brussels, October 20, 2000), pp. 61–98, Brussels, Belgium, 2000.
[11] S. E. Payne and J. A. Thas. Finite Generalized Quadrangles. Second edition, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2009.
[12] R. L. Rostermundt. Elation groups of the Hermitian surface $H(3, q^2)$ over a finite field of characteristic 2, Innov. Incidence Geom. 5 (2007), 117–128.
[13] J. A. Thas, K. Thas and H. Van Maldeghem. Translation Generalized Quadrangles, Series in Pure Mathematics 26, World Scientific, Singapore, 2006.
[14] K. Thas. A theorem concerning nets arising from generalized quadrangles with a regular point, Des. Codes Cryptogr. 25 (2002), 247–253.
[15] K. Thas. Some basic questions and open problems in the theory of elation generalized quadrangles, and their solutions, Bulletin Belgian Math. Soc. — Simon Steven 12 (2006), 909–918.
[16] K. Thas. Order in building theory, in Surveys in Combinatorics 2011, London Math. Soc. Lecture Note Ser. 392, Cambridge University Press, 2011, pp. 235–331.
[17] K. Thas. A Course on Elation Quadangles, European Math. Soc., Zürich, 2012.
[18] K. Thas. Isomorphisms of groups related to flocks, J. Algebraic Combin. 36 (2012), 111–121.
[19] K. Thas. Local half Moufang quadrangles, Des. Codes Cryptogr. 68 (2013), 319–324.
[20] K. Thas. Classification of skew translation generalized quadrangles, I, Discrete Math. Theor. Comput. Sci. 17 (2015), 89–96.
[21] K. Thas. Central aspects of skew translation quadrangles, I, J. Algebraic Combin., 52 pp., to appear.
[22] J. Tits. Buildings of Spherical Type and Finite BN-Pairs, Lecture Notes in Math. 386, Springer, Berlin, 1974.
[23] J. Tits and R. M. Weeds. Moufang Polygons, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

Ghent University, Department of Mathematics, Krijgslaan 281, S25, B-9000 Ghent, Belgium
Email address: koen.thas@gmail.com