The symmetry of the Kuramoto system and the essence of the cluster synchronization

Guilhua Tian\textsuperscript{1,2}, Songhua Hu\textsuperscript{1,2,3}\textsuperscript{*}

\textsuperscript{1}School of Science, Beijing University of Posts And Telecommunications. Beijing 100876, China.
\textsuperscript{2}State Key Laboratory of Information Photonics and Optical Communications, Beijing University of Posts And Telecommunications. Beijing 100876, China. and
\textsuperscript{3}School of Electronic and Information Engineering, North China Institute of Science and Technology, Yanjiao 065201, China.

The cluster synchronization (CS) is a very important characteristic for the higher harmonic coupling Kuramoto system. A novel method from the symmetry transformation is provided, and it gives CS a profoundly mathematical explanation and clear physical annotation. Detailed numerical studies for the order parameters in various conditions confirm the theoretical predictions from this new view of the symmetry transformation. The work is very beneficial to the further study on CS in various systems.

PACS numbers: 05.45.Xt, 05.45.-a

As the simplest and the most celebrated one, the Kuramoto model captures the main property of the collective synchronization, and is applied in many physical, biological and social systems, including electrochemical oscillators, Josephson junction arrays, cardiac pacemaker cells, circadian rhythms in mammals, network structure and neural network\textsuperscript{[1]-[5]}. Many generalizations of the Kuramoto model have been investigated. Including the large inertia in the generalized Kuramoto model, the transition from the incoherent state to the collective synchronization became of the first order\textsuperscript{[3, 6]}. Noise also can push the incoherent stationary state to become stable\textsuperscript{[4, 5]}. When the universal coupling strength $K$ becomes oscillator-dependent and correlated with the frequency, the explosive synchronization (ES) appears\textsuperscript{[5]-[13]}. ES is also an abrupt, of the first-like phase transformation. The identical synchronization (ES) appears\textsuperscript{[9]-[13]}. ES is also an independent and correlated with the frequency, the explosive synchronization (ES) appears\textsuperscript{[9]-[13]}. ES is also an useful and effective way to understand the CS thanks to unavailability in obtaining the analytic form for the intricate asymmetry density function for the second harmonic coupling case\textsuperscript{[28], [29]}. The symmetric viewpoint is applied and CS of the two groups of $m$ and $N - m$ oscillators is connected with their symmetry groups of the dynamics $S_m \times S_{N-m}$\textsuperscript{[28], [29]}. The symmetry group $S_N$ is only suited for the identical oscillators in the Kuramoto model. However, it is still very difficult to obtain clear analytical results by the self-consistent approach and detailed understanding of the stability of the asynchronous states is still missing\textsuperscript{[20, 32]}. As the simplest and the most celebrated one, the Kuramoto model captures the main property of the collective synchronization, and is applied in many physical, biological and social systems, including electrochemical oscillators, Josephson junction arrays, cardiac pacemaker cells, circadian rhythms in mammals, network structure and neural network\textsuperscript{[1]-[5]}. Many generalizations of the Kuramoto model have been investigated. Including the large inertia in the generalized Kuramoto model, the transition from the incoherent state to the collective synchronization became of the first order\textsuperscript{[3, 6]}. Noise also can push the incoherent stationary state to become stable\textsuperscript{[4, 5]}. When the universal coupling strength $K$ becomes oscillator-dependent and correlated with the frequency, the explosive synchronization (ES) appears\textsuperscript{[5]-[13]}. ES is also an abrupt, of the first-like phase transformation. The identical oscillators with the nonlocal coupling strength will give rise to the new chimera state, which is the combination of the coherent state and the incoherent state for the identical oscillators\textsuperscript{[14]-[18]}. All the above examples are of the first harmonic coupling as $H(\theta_j - \theta_i) = K_{ij} \sin(\theta_j - \theta_i)$. Whenever the higher globally coupling harmonic term is introduced, interesting phenomena appear, like the cluster synchronization (CS) or multi-entrainment, and switching of the oscillators between different clusters with the external force\textsuperscript{[19]-[20]}. The higher harmonic coupling is dominating in $\phi$-Josephson junction\textsuperscript{[21], [22]}, in the electrochemical oscillators in higher voltage\textsuperscript{[19], [23], [24]}, in neuronal networks with learning and network adaption\textsuperscript{[25]-[30]}. CS is the most outstanding feature of the Kuramoto model with higher order harmonic coupling.

CS or multi-entrainment has been investigated by the method of self-consistent approach in Refs.\textsuperscript{[19], [20], [22]-[32]}. Neural network actually studied the combination of the first and second harmonic couplings in the generalized Kuramoto model\textsuperscript{[25]-[32]}, which is also treated in Ref.\textsuperscript{[20], [33]}. In the $N$ identical oscillators’ case, the symmetry viewpoint is applied and CS of the two groups of $m$ and $N - m$ oscillators is connected with their symmetry groups of the dynamics $S_m \times S_{N-m}$\textsuperscript{[28], [29]}. The symmetry group $S_N$ is only suited for the identical oscillators in the Kuramoto model. However, it is still very difficult to obtain clear analytical results by the self-consistent approach and detailed understanding of the stability of the asynchronous states is still missing\textsuperscript{[20, 32]}. The higher order coupling strength $K > 0$ is assumed. The higher

\begin{equation}
\dot{\theta}_n = \omega_n + \frac{1}{N} \sum_{j=1}^{N} K \sin(\theta_j - \theta_n),
\end{equation}
harmonic coupling of the generalized Kuramoto model is

$$\dot{\theta}_n = \bar{\omega}_n + \frac{1}{N} \sum_{j=1}^{N} \bar{K}_m \sin(m(\theta_j - \theta_n)).$$  \hspace{1cm} (2)

The order parameter is defined as

$$re^{i\Psi} = \frac{1}{N} \sum_{j=1}^{N} \cos{\theta}_j.$$  \hspace{1cm} (3)

Ref.[19] shows that the critical parameter $\bar{K}_m$, relating with the corresponding $m$-th order parameter is the same $\bar{K}_m = 2\Delta$ for all integers $m$, where $2\Delta$ is the width of the original Lorentz frequency distribution in Ref.[19]. In the case of small strength $\bar{K}_m < \bar{K}_m$, the term $\omega_n$ dominates the change of the phase $\theta_n$ and the whole phase system is in the incoherent state. Whenever $\bar{K}_m$ exceeds $\bar{K}_m$, the second terms in Eq.(2) predominate and CS emerges[19].

Why the critical coupling $\bar{K}_m$ is the same for all different integers $m$? Is it only a coincidence or is there some underlying reason? The study in the letter shows that the symmetry that the Kuramoto system keeps is responsible for.

By introduction of the transformation

$$\phi = m\theta, \omega_n = m\bar{\omega}_n, K = m\bar{K}_m.$$  \hspace{1cm} (4)

Eq.(2) takes the form

$$\dot{\phi}_n = \omega_n + \frac{1}{N} \sum_{j=1}^{N} K \sin(\phi_j - \phi_n),$$  \hspace{1cm} (5)

which is the same as Eq. (1) of the standard Kuramoto model[13].

The generalized order parameters is defined as $R_m = \frac{1}{N} \sum_{j=1}^{N} e^{i\phi_j} = |R_1|e^{i\phi_1}$, and Eq.(3) becomes $\dot{\phi}_n = \bar{\omega}_n + \frac{K}{N} (R_1 e^{-i\phi_n} - R_1^* e^{i\phi_n})$. In the limit $N \rightarrow \infty$, the density function $f(\phi, \omega, t)$ is introduced to describe the distribution of the phases at a given frequency and satisfies the continuous equations

$$\partial_t f + \partial_\phi \left[ \left( \omega + \frac{K}{2N} \left( R_1 e^{-i\phi} - R_1^* e^{i\phi} \right) \right) f \right] = 0.$$  \hspace{1cm} (6)

The solutions to Eq.(2) could be obtained from the ones to Eq.(5), where OA mechanism could be utilized when the distribution of the natural frequency is the Lorentz’s one[34].

The density distribution function $f(\phi, \omega, t)$ is a periodic function satisfying $f(\phi + 2\pi, \omega, t) = f(\phi, \omega, t)$, and in the case $m = 2$ it corresponds to the symmetric one in Ref.[19]. It is easy to see that in the stationary state $f(\phi, \omega, t) = f(\phi, \omega, t)|_{t=\infty}$ the system is partially synchronous whenever $K > 2\Delta$ for the Lorentz distribution of the natural frequency of the phases with $\Delta$ its width[19, 34, 40]. Hence the same critical parameter $\bar{K}_m$ is realized for all $m$-th order parameters. Combination of the symmetry transformation [41], $f(\phi, \omega, t)$ can completely determine the evolution of the dynamics and this is the key point in the latter to study CS.

We apply both the transformation [41] and the distribution function $f(\phi, \omega, t)$ with its periodic property to investigate the order parameters for the oscillators in several special cases, and make the corresponding predictions on the order parameters. Then we integrate Eq.(4) directly for these cases and obtain the corresponding numerical order parameters directly from the numerical integration. The later numerical results confirm the former prediction. The details are divided into five groups in the following.

(a)If the initial oscillators’ phases are uniformly distributed in $(0, 2\pi)$, together with the transformations [41], the order parameter $r$ in Eq.(3) in the large $N$ limit turns out as

$$re^{i\Psi} = \frac{1}{m} \int_{-\infty}^{+\infty} \int_{0}^{2m\pi} f(\phi, \omega, t)e^{\frac{2\pi i}{m} \omega} d\omega d\phi = 0,$$  \hspace{1cm} (7)

where the upper integral number is $2m\pi$ due to the transformations [41]. The symmetry property of the distribution function $f(\phi + 2\pi, \omega, t) = f(\phi, \omega, t)$ makes the order parameter $r = 0$, no matter what a great coupling strength is applied. This is actually the manifestation of the cluster property of the higher harmonic coupling. The $m$-term harmonic coupling will give rise to the corresponding $m$ clusters, and the phase oscillators in one cluster behave completely the same way as those in another cluster. The order parameter $r$ can only take the zero value, which is a typical manifestations for the cluster phenomena of the higher harmonic coupling in the system.

The numerical studies in Fig.1 confirm the above ideas of $r \equiv 0$ with the uniform initial distribution in $(0, 2\pi)$ of the phases and different coupling strengths and different higher-term couplings. When $K < K_c$, all parameters $r$ in the three cases in Fig.1 is similar, and the oscillators are incoherent. Of course, the symmetry properties in Eq.(4) will force parameters $r$ in $m \geq 2$ will be more smaller than that in $m = 1$ case.
strengths $K$ surpass $K_c$, $r \approx 1$ is obtained for usual Kuramoto model ($m=1$), and $r \approx 0$ stands out in $m \geq 2$ in Fig.1 which indicates the formation of the clustering synchronization. Fig.2 further gives clustering synchronization for the case of $K = 5$, $m = 2$, $m = 3$ by the phases’ position in the circles in the middle and the left panels.

**FIG. 2:** The cluster synchronization of the phases in the circles corresponding the parameters in the second panel in Fig.1 with the same parameters $K = 5$ but different $m = 1, 2, 3$. There are 1000 oscillators and their positions are indicated by the small blue circles in the big circles.

Furthermore, the more higher harmonic coupling is applied, the more clusters appear and the smaller section of the whole range $2\pi$ every cluster occupies. Therefore, very higher harmonic coupling of the oscillators will produce pseudo-synchronization where the large number of the clusters will globally behave the same way with the oscillators within each very small cluster being any state.

**FIG. 3:** Schematic diagrams $r(t)$ of initial uniform phases distribution in $(0, A)$ with the same parameters $K = 5$, $m = 2$ but different $A$.

(b) When the initial phases distribution $(0, A)$ is narrowed and much smaller than $\frac{2\pi}{m}$, and the coupling strength is stronger than the critical $K_c$, and with every term in the second part $\frac{1}{N} \sum_{j=1}^{N} K_m \sin m(\theta_j - \theta_n)$ in Eq.(2) has the same effect as the corresponding term in ordinary Kuramoto (1), and they together dominate over the first part and attract all oscillators to the synchronic state. So, the initial synchronized state in one cluster will remain synchronized all the time, just like in the ordinary Kuramoto model and the order parameter

$$r = | \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\phi, \omega, t)e^{i\phi} d\omega d\phi| \approx r_1^m \approx 1$$

In this case, almost all oscillators are synchronized in the only one cluster, and no other cluster synchronization exists. See the phases in the circles in the second ($A = 0.75\pi$, $m = 2$) and third ($A = 0.8\pi$, $m = 2$) panels in Fig.3 for details. Note there are several phases are left opposite to the synchronized one in the two panels.

(c) When $A$ of the initial distribution $(0, A)$ goes beyond $\frac{2(m-1)\pi}{m}$ but less than $2\pi$, there approximately emerges $m$ cluster synchronization. As the order parameter $r \approx 0$ is achieved, which is shown in Fig.3 with $A = 1.2\pi$, $1.25\pi$ in the case of $m = 2$, $K = 5$ for the Gauss frequency distribution with $K_c < 2$. In this case, the initial range of $\phi = m\theta$ exceeds $2(m-1)\pi$, and guarantee the validation of the transformation (3) and Eq.(7). So CS is again connected with the symmetry of the Kuramoto model (2). The fourth ($A = 1.25\pi$, $m = 2$) and fifth ($A = 1.2\pi$, $m = 2$) panels in Fig.3 agree with this analysis.

(d) When $A$ of the initial distribution $(0, A)$ lies in $\frac{2(m-1)\pi}{m}$ to $\frac{2\pi}{m}$ with $n < m$ and $m$ greater than 2, the partial CS appears indicated by the order parameter $r$ approaches neither 0 nor 1. The order parameter takes the form

$$r(A) = | \int_{-\infty}^{+\infty} \int_{0}^{mA} f(\phi, \omega, t)e^{i\phi} d\omega d\phi|,$$

Generally, in the stationary state, the upper bound $mA$ could be replaced by $2n\pi$ in most cases. Roughly, $r(A) \approx | \frac{1}{\pi}(\sum_{j=0}^{n} e^{2\pi i j})r_1^m |$, where $r_1^m$ is the first clus-
ter’s order parameter, and is near 1 in the case \( K > K_c \). The numerical studies in Fig.4 crudely illustrate the feature for \( r(A) \). Define
\[
   r(n) = \left| \int_{-\infty}^{+\infty} f(\phi, \omega, t)e^{i\Omega t}d\phi \right|, 
\]
but different much large than \( K_r \). The order parameters is defined as
\[
   \theta \equiv \frac{1}{n} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} f(\phi, \omega, t)e^{i\Omega t}d\phi d\theta, 
\]
for \( m_0 < r(A) < r(n - 1) \). For example, \( r(n) \) takes the following numerical values \( r(1) = 1, r(2) \approx 0.8666, r(3) \approx 0.6777, r(4) \approx 0.4333, r(5) = 0.2, r(6) = 0 \) for the case \( m = 6 \) and \( K > K_c \). Fig.4 shows the data for \( r(A) \) meets the constrain as above mentioned. This again tells one that the distribution function \( f(\phi, \omega, t) \) can be used to obtain the parameter \( r(A) \) through the symmetry transformation (4).

![FIG. 5: Schematic diagrams \( r(t) \) of initial uniform phases distribution in \((0, \pi)\) with the same parameters \( K = 5, m = 2 \). The only difference for the two cases is their initial values for the phases.](image)

When \( A \) of the initial distribution \((0, A)\) is \( \frac{2n\pi}{m} \), the dynamics of the model is very sensitive to the initial conditions. The order parameters is defined as
\[
   r(A) = \frac{1}{n} \left| \left( \int_{-\infty}^{+\infty} \int_{0}^{2\pi} f(\phi, \omega, t)e^{i\Omega t}d\phi d\theta \right) + A \right|. 
\]

The oscillators with initial phase very near \( \frac{2n\pi}{m} \) can either lag into the \( n \)-th cluster or drift forward to the \((n + 1)\)-th cluster. \( A \) is the related order parameter for the entering the \((n + 1)\)-th cluster, and the fraction of the oscillators for this kind is very sensitive to the system’s initial conditions. So do the parameters \( A \) and \( r(A) \). \( \bar{n} \) in Eq.(9) is in \((n, n + 1)\) depending on parameter \( A \). All are shown in Fig.7.

![FIG. 6: Schematic demonstration of the translation properties for \( r(t) \) of initial uniform distribution of the phases in \((B, B + A)\) with the same coupling strength \( K = 5, K > K_c \), the same \( m = 6 \) but different \( B \).](image)

We now conclude the symmetric viewpoint. The transformations (4) make it possible to relate the results of Eq.(3) with that of Eq.(2). Furthermore, the periods for both \( m \theta \) in Eq.(2) and \( \phi \) in Eq.(3) are identical, that is, \( 2\pi \). The case \( K_r > K_c \) is the same case \( K > K_c \) for the transformed model (3). So the initial phases distribution in \((0, A)\) in Eq.(2) is equivalent to the initial distribution \((0, mA)\) in Eq.(3). So when \( A \in \left( \frac{2n\pi}{m}, \frac{2(n + 1)\pi}{m} \right) \), the initial phases distribution for variable \( \phi \) is \((0, 2(n + 1)\pi)\). Every \( 2\pi \) distribution in Eq.(3) will be a Kuramoto model and will be synchronized when \( K > K_c \). Therefore the distribution \((0, 2(n + 1)\pi)\) for \( \phi \) will be equivalent to \( n + 1 \) Kuramoto models, that is, the \( n + 1 \) cluster synchronization. This is the root of the formula (8). Mathematically, the above results means that the phases lines

In the following, physical explanations are given on the basis of attraction and repulsion interaction among the different oscillators.

The coupling strength \( K > 0 \) in Eq.(11) is attracting to synchronization and is repulsive to synchronization as \( K < 0 \). The incoherent state will stable in the case \( K < 0 \). Concerning the attractive and repulsive properties of the coupling parameter, Hong and Strogatz have studied the identical oscillators with some couples others negatively (the contrarians) and some of positive coupling (the conformist). The contrarians like to be anti-phase with the mean field and the conformist is easy to in-phase. Also other interesting phenomena like traveling wave occurs \([38]\). There also are references investigating the phenomena \([39]-[44]\). In Ref.\([39]\), the \( N \) identical
phases with the non-linear coupling is studied and the positiveness and negativeness of the coupling parameter is controlled by the non-linearity coupling. The dynamics of the system is

$$\dot{\theta}_i = (1 - eZ^2_i) \sum_{j=1}^{N_i} K_1 \sin(\theta_j - \theta_i), \quad (10)$$

where the local order parameter $Z_i = |\frac{1}{N_i} \sum_{j=1}^{N_i} e^{im\theta_j}|$. The repulsive coupling is realized if $1 - eZ^2_i < 0$. The nonlinear coupling will result in phase-locked states, while the large nonlinear coupling will give rise to multistable, periodic and chaotic states. In the neural network, the fast studying model is attributed as stable, periodic and chaotic states. In the neural network, the fast studying model is attributed as stable, periodic and chaotic states [39].

While the large nonlinear coupling will result in phase-locked states, the second Harmonic Kuramoto model. In this way, the attracting and repulsive coupling parameter is achieved depending the difference of the two phases of the two oscillators [20]-[23]. Similarly, the attracting and repulsive properties of the coupling strength are the key to explain the cluster phenomena in the higher odder harmonic coupling Kuramoto.

For the parameter $\bar{K}_m > K_c > 0$, the coupling strength can be either attracting or repulsive depending the difference of the two phases. If all $(\theta_j - \theta_i)$ are less than $\frac{2\pi}{m}$, then $m(\theta_j - \theta_i) < 2\pi$ and they can be collectively synchronous and form a cluster and most oscillators synchronous in the cluster. Further increase of the range of the phases over $\frac{2\pi}{m}$, the oscillators with phases greater than $\frac{2\pi}{m}$ will be repulsed by the oscillators with phases less than $\frac{2\pi}{m}$, so large phase difference will form for these two kinds of oscillators. Because the most oscillators cluster synchronically, the repulsion to the oscillators with phases larger than $\frac{2\pi}{m}$ is dominant, hence, these oscillators could only oscillators with phases large than $\frac{2\pi}{m}$ and the second cluster emerges. All along this way, more clusters will appear as the range of the phases becomes larger and larger.

Conclusion and discussion: in generalized Kuramoto with the higher order harmonic coupling, the view from the symmetry transformation gives the explanation to CS both profound mathematical insight and clear physical understanding. Detailed numerical studies confirm the symmetric analysis. The similar analysis could extend to the forced Kuramoto model $\dot{\theta}_n = \omega_n + \frac{1}{N} \sum_{j=1}^{N} K_m \sin(\theta_j - \theta_n) + F_n(t)$, with the force taking the form of $F_n(t) = F \sin(\Omega t)$ in the neural learning network. Whenever the force is correlated with the oscillators, like $F(t) = F \sin(\Omega t - \theta_n)$, there is no symmetry group transformation like Eq.(4). Neither can the Kuramoto model with mixed higher harmonic orders coupling have symmetry group transformation like Eq.(4). So new ideas are needed to be explored in the two cases.

Acknowledgments

The work was partly supported by the National Natural Science of China (No. 10875018) and the Major State Basic Research Development Program of China (973 Program: No.2010CB923202).

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[45] After completing our work, we notice Ref.[30] has a similar transformation for $m = 2$ case in the fast study model.
[46] The critical coupling strength for $K_m$ is the same $2\Delta$ for the Lorentz frequency distribution, and one should not use the relation $K = mK_m$ to deduce the critical strength for Eq.(2). The correct deduction is from the fact that the critical strength both for oscillators in $\theta$ or in $\phi$ is obtained from the density function $f(\phi, \omega, t)$. The conclusion is the critical strengths for $K$ and $K_m$ are the same for both Lorentz frequency distribution and other frequency distributions.