RIPless compressed sensing from anisotropic measurements

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Abstract

Compressed sensing is the art of reconstructing a sparse vector from its inner products with respect to a small set of randomly chosen measurement vectors. It is usually assumed that the ensemble of measurement vectors is in isotropic position in the sense that the associated covariance matrix is proportional to the identity matrix. In this paper, we establish bounds on the number of required measurements in the anisotropic case, where the ensemble of measurement vectors possesses a non-trivial covariance matrix. Essentially, we find that the required sampling rate grows proportionally to the condition number of the covariance matrix. In contrast to other recent contributions to this problem, our arguments do not rely on any restricted isometry properties (RIP’s), but rather on ideas from convex geometry which have been systematically studied in the theory of low-rank matrix recovery. This allows for a simple argument and slightly improved bounds, but may lead to a worse dependency on noise (which we do not consider in the present paper).

Keywords: Compressed sensing, $\ell_1$ minimization, the LASSO, the Dantzig selector, restricted isometries, anisotropic ensembles, sparse regression, operator Bernstein inequalities, non-commutative large deviation estimates, the golfing scheme. Subject Classification: (94A12, 60D05, 90C25).

1. Introduction and Results

Compressed sensing is a highly active research field in statistics and signal analysis \cite{1, 2, 3, 4}. It can be thought of as being concerned with establishing Nyquist-type sampling theorems for signals which are sparse, rather than band-limited.

More precisely, let $x \in \mathbb{C}^n$ be a vector with no more than $s$ non-zero entries (i.e. $x$ is $s$-sparse). Suppose we have no information about $x$ apart from its sparsity and the inner products $\langle a_i, x \rangle$, $i = 1, \ldots, m$ between $x$ and $m \ll n$ vectors $a_i$. The central question is: under what conditions on $m$ and the $a_i$’s is it possible to uniquely and computationally efficiently recover $x$? Early celebrated results \cite{1, 2, 3} established e.g. that if the measurement vectors $\{a_i\}_{i=1}^m$ are randomly chosen discrete Fourier vectors.

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and $m = O(s \log n)$, then, with high probability, the unknown vector $x$ is the unique minimizer of the $\ell_1$-norm in the affine space defined by the known inner products.

The precise statement of our results in this introductory section will follow very closely the exhibition in [5]. The reason for this approach, and the relation of the present paper with other work (in particular [6]), is stated in Section 2.

We make the following definitions: Let $F$ be a distribution of random vectors on $\mathbb{C}^m$. Let $a_1, \ldots, a_m$ be a sequence of i.i.d. random vectors drawn from $F$. Define the sampling matrix

$$A := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} e_i a_i^*,$$

where $e_1, \ldots, e_m$ denote the canonical basis vectors of $\mathbb{C}^m$. Once more, let $x$ be an $s$-sparse vector. We aim to prove that with high probability the solution $x^*$ to the convex optimization problem

$$\min_{\bar{x} \in \mathbb{C}^n} \|\bar{x}\|_1 \quad \text{subject to} \quad A\bar{x} = Ax,$$

(1)

is unique and equal to $x$ given that the number of measurements $m$ is large enough.

It turns out that the required size of $m$ depends only on two simple properties of the ensemble $F$. These are identified below:

**Completeness** We require that the ensemble $F$ is complete in the sense that the covariance matrix $\Sigma = \mathbb{E}[aa^*] / 2$ is invertible. The condition number of $\Sigma$ will be denoted by $\kappa$.

Most of the previous work has focused on the case where the covariance matrix is proportional to the identity matrix $\Sigma \propto I$ (however, see Section 2). We refer to this case as the isotropic one.

In order to describe the second relevant property of the ensemble, we have to fix a scale. Indeed, note that the minimizer of the convex problem (1) is invariant under re-scaling of the ensemble (i.e. substituting $a_i$ by $\nu a_i$ for a number $\nu \neq 0$). The same is true for the condition number $\kappa$. Thus, we are free to pick an advantageous scale, without affecting the notions introduced so far. In the isotropic case, a natural normalization convention [5] consists in requiring that $\mathbb{E}[aa^*] = I$. This option is not available in the more general, anisotropic case, we are interested in here. Instead, we will implicitly demand from now that

$$\lambda_{\max}(\mathbb{E}[aa^*]) = \lambda_{\min}(\mathbb{E}[aa^*])^{-1},$$

(2)

where $\lambda_{\max}, \lambda_{\min}$ denote the maximal and the minimal eigenvalue respectively. In the isotropic case, this reduces to the normalization $\mathbb{E}[aa^*] = I$ used in [5].

The fact that (2) can always be achieved (and further properties that follow from it) will be established in Lemma 8 below. With this convention, we define:

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2 Recall that the condition number of a matrix is the ratio between its largest and its smallest singular value.
Incoherence  The \textit{incoherence parameter} is the smallest number $\mu$ such that
\[
\max_{1 \leq i \leq n} |\langle a, e_i \rangle|^2 \leq \mu, \quad \max_{1 \leq i \leq n} |\langle a, \mathbb{E}[aa^*]^{-1} e_i \rangle|^2 \leq \mu
\] holds almost surely.

The previously known isotropic result we aim to generalize is:

\textbf{Theorem 1} ([5]). Let $x$ be an $s$-sparse vector in $\mathbb{R}^n$. If we demand isotropy ($\mathbb{E}[aa^*] = 1$) and if the number of measurements fulfills
\[
m \geq C_\omega \mu s \log n,
\]
then the solution $x^*$ of the convex program (1) is unique and equal to $x$ with probability at least $1 - \frac{5}{n} - e^{-\omega}$.

In the statement above, $C_\omega$ may be chosen as $C_0 (1 + \omega)$ for some positive numerical constant $C_0$.

Our main theorem reads:

\textbf{Theorem 2} (Main Theorem). Let $x \in \mathbb{C}^n$ be an $s$-sparse vector, let $\omega \geq 1$. If the number of measurements fulfills
\[
m \geq C_\kappa \mu^2 s \log n,
\]
then the solution $x^*$ of the convex program (1) is unique and equal to $x$ with probability at least $1 - e^{-\omega}$.

In the statement above, $C$ is a constant less than 18044. For $n, s$ sufficiently large, the value may be improved to $C \leq 228$. We have made no attempts to optimize these constants.

Comparing these two theorems, we see that the effect of dropping the isotropy constraint on the ensemble can essentially be captured in a single, simple quantity: the condition number $\kappa$ of the covariance matrix. All other minor differences between Theorem 1 and Theorem 2 result from slightly different proof techniques.

1.1. Improvements

A first way of improving the result is based on a definition borrowed from [6, Def. 1.2].

\textbf{Definition 3.} The largest and smallest $s$-sparse eigenvalue of a matrix $X$ are given by
\[
\lambda_{\text{max}}(s, X) := \max_{v, \|v\|_2 \leq s} \frac{\|Xv\|_2}{\|v\|_2}, \quad \lambda_{\text{min}}(s, X) := \min_{v, \|v\|_2 \leq s} \frac{\|Xv\|_2}{\|v\|_2},
\]
\footnote{In fact, our definition differs very slightly from [6]: their $\rho_{\text{max}}(s, X)$ is the square of our $\lambda_{\text{max}}(s, X)$. We opted for this change because the notions defined here reduce to the ordinary eigenvalues in the case of $s = n$.}
where \( \| v \|_0 = |\text{supp}(v)| \) denotes the cardinality of the support (i.e. the sparsity) of \( v \). The \( s \)-sparse condition number \(^4\) of \( X \) is

\[
\text{cond}(s, X) := \frac{\lambda_{\max}(s, X)}{\lambda_{\min}(s, X)}.
\]

Based on this notion, one can state a strictly stronger version of the Main Theorem (which is the form we will prove in Section 3):

**Theorem 4.** With

\[
\kappa_s := \max \{ \text{cond}(s, \Sigma), \text{cond}(s, \Sigma^{-1}) \},
\]

the conclusion of the main Theorem \(^2\) continues to hold if the lower bound on \( m \) is weakened to

\[
m \geq C \mu \kappa_s \omega^2 s \log n,
\]

for the same constant \( C \).

We further suspect that the second incoherence condition in \(^3\) can be relaxed. Two alternative bounds not relying on that condition are stated in Proposition \(^5\) below. (The modifications of our proof necessary to arrive at these improved estimates will be sketched after Lemma \(^9\).)

**Proposition 5.** Let \( K \) be a constant such that

\[
2 \| [aa^*, \mathbb{E}[aa^*]^{-1}] \|_\infty \leq K
\]

holds almost surely, where \( [\cdot, \cdot] \) denotes the commutator \( ([A, B] = AB - BA) \) and \( \| \cdot \|_\infty \) is the operator norm.

If the requirement \(^3\) is not necessarily fulfilled, the conclusions of Theorem 2 remain valid if the sampling rate is bounded below by either

\[
m \geq C \kappa \mu s \omega^2 s \log n
\]

or

\[
m \geq C (\kappa \mu s + K) \omega^2 \log n.
\]

The commutator bound \(^5\) is particularly relevant for ensembles corresponding to non-uniform samples from an orthogonal basis. In that case, \( \mathbb{E}[aa^*] \) and \( aa^* \) commute with probability one, so that \( K \) may be chosen to be zero.

There is another degree of freedom which we have not yet systematically explored: Note that the minimizer of the convex optimization \(^1\) does not change if we re-scale

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\(^4\) Estimating \( \text{cond}(s, X) \) is equivalent to computing the RIP constants of \( X \) (c.f. e.g. \(^3\)). There are currently no tractable methods known for computing these numbers for any concrete set of matrices. We want to emphasize that while the mathematical concept of “RIP constants” appears in our sharpened result, its use here is completely different from the way it would be employed in RIP-based approaches to compressed sensing. To wit, we apply the concept to the expected sensing matrix (and its inverse), but not to any actual instances.
individual vectors $a_i \mapsto \nu_i a_i$ for some set of non-zero numbers $\nu_i$. While we have chosen a global scale for the covariance matrix (c.f. Lemma 8), the individual weights remain free parameters that may be used to optimize the sampling rate. Pursuing this problem further seems likely to be fruitful.

We remark that the incoherence conditions can be relaxed to hold only with high probability. This opens up our results to, for example, the case of Gaussian measurement vectors. The details can be developed in complete analogy to Ref. [5, Appendix B].

Lastly, all statements remain true if the measurement vectors are drawn “without replacement” instead of independently – c.f. [8] for details.

2. Relation with previous work and history

Most results on sparse vector recovery have relied on certain conditions that quantify how much a given sampling matrix $A$ distorts the geometry of the set of all sparse vectors. By far the most prominent example in that regard is the restricted isometry property (RIP) [3, 6] which measures the extent to which $A$ deviates from preserving Euclidean distances between sparse vectors. Conceptually close variations of the RIP include the restricted eigenvalue condition introduced in [9], or the restricted correlation assumption [10]. Another example is the width property advanced in [11]—a Banach space-theoretic condition that seems to be weaker than the RIP.

From roughly 2008 on, the conceptually strongly related problem of recovering a low-rank matrix from few expansion coefficients with respect to a fixed matrix basis has come more and more into focus [12, 13]. There seems to be no easy way to directly translate the geometric approaches mentioned above to the general low-rank matrix recovery problem. Instead, the pioneering publications on the matrix problem used fairly elaborate methods from convex duality theory [12, 13]. (However, c.f. [14, 15, 16] for interesting special cases where RIP-based techniques are applicable to low-rank matrix recovery problems; and [17] for a related “restricted strong convexity” property with consequences for matrix recovery).

In [18, 19] the second author and his collaborators introduced a simplified approach to the low-rank matrix recovery problem. While these works still build on the convex framework of [12, 13], they incorporate several new ideas. These include the use of non-commutative large deviation theorems originating from quantum information theory [20, 21], randomized constructions based on i.i.d. samples of the measurement vectors, and a certain iterative “golfing scheme” for the construction of inexact dual certificates. These techniques were later modified and adapted to the original sparse-vector setting in [5]. This showed that the conceptual closeness of the matrix and the vector theory may be used to devise very similar proofs.

This “RIPless” approach to compressed sensing leads arguably to simpler proofs and gives tighter bounds at least for the noise-free recovery problem. As far as we know, RIP-based arguments still perform superior in the important noisy regime.

The work [5] did not include a systematic study of non-isotropic ensembles (however, “small” deviations from isotropy were discussed in Appendix B). In fact, E. Candès [5] suggested to us the problem of finding a generalization of the golfing
scheme that could cope with anisotropic ensembles. This has been achieved by the first author of this paper during a research project under the supervision of the second author \[22\]. This explains the close relation between \[5\] and the present work.

An analysis of anisotropic compressed sensing within the original RIP framework has been carried out by other authors, most notably in \[6\]. Since their paper does not directly address the noise-free case, a direct comparison of statements is difficult. The closest result to ours seems to appear in Section 1.3, where a bound of

\[ m \geq O(sM^2 \log n \log^3 (s \log n)) \]

for the sampling rate is given. The quantity \( M \) is an upper bound on the largest coefficient for the measurement vectors \( a_i \), related to our parameter \( \mu \). The big-O notation hides a constant proportional to \( \kappa (\rho^{-1} \) in the language of \[6\]). Thus, the basic structure of the solutions is very similar. However, some important differences are these:

- We do not incur the \( \log^{3} \)-term, which is a major advantage of our method. Up to a constant factor, our required sampling rate corresponds to the theoretical lower limit.

- The result in \[6\] holds uniformly in the sense that with their probability of success, one obtains a sampling matrix which works simultaneously for all sparse vectors. This is not the case for us.

- We have proved no results on noise-resilience. While, following \[5\], it should be straightforward to do so, the results may be worse than the RIP-based ones in \[6\].

- The proof methods are completely different.

3. Proof

The proof is conceptually close to \[5\], which in turn closely resembles \[19\]. Here we give a largely self-contained presentation.

3.1. Notation

Throughout this paper, we will use the following conventions: If a statement holds almost surely, we will abbreviate this by a.s. In the case of vectors, \( \| \cdot \|_p \) denotes the \( \ell_p \)-norm, whereas in the operator case \( \| \cdot \|_p \) refers to the Schatten-\( p \) norm (i.e. the \( \ell_p \)-norm of the singular values). The letter \( z \) will always denote a vector in \( \mathbb{C}^n \) supported on a set \( T \) of cardinality at most \( s \) (i.e. \( z \) is \( s \)-sparse). \( T^c \) shall denote the complement of \( T \), and \( P_T \) (\( P_{T^c} \)) refers to the orthogonal projector onto the set of all vectors supported on \( T \) (\( T^c \)). Finally we will use the following technical definitions:

\[ X = (\mathbb{E}[aa^*])^{-1} = \Sigma^{-2}, \quad X_T = P_T XP_T. \]
3.2. Large deviation bounds

A central role in the argument is played by certain large deviation bounds for sums of matrix-valued random variables. These have been introduced in [20] in the context of quantum information theory. The first application to matrix completion and compressed sensing problems, as well as the first “Bernstein version” taking variance information into account, appeared in [18, 19]. The version we will be making use of derives from Theorem 1.6 in [21].

Proposition 6 (Matrix Bernstein inequality [21]). Consider a finite sequence \( \{M_k\} \in \mathbb{C}^{d \times d} \) of independent, random matrices. Assume that each random matrix satisfies \( \mathbb{E} [M_k] = 0 \) and \( \|M_k\|_\infty \leq B \) a.s. and define

\[
\sigma^2 := \max \left\{ \| \sum_k \mathbb{E} (M_k M_k^*) \|_\infty, \| \sum_k \mathbb{E} (M_k^* M_k) \|_\infty \right\}.
\]

Then for all \( t \geq 0 \),

\[
\text{Pr} \left( \| \sum_k M_k \|_\infty \geq t \right) \leq 2d \exp \left( -\frac{t^2/2}{\sigma^2 + Bt/3} \right). \tag{6}
\]

We will also require a vector-valued deviation estimate. While one could in principle obtain such a statement by applying Proposition 6 to diagonal matrices, a direct argument does away with the dimension factor \( d \) on the r.h.s. of (6). This will save a logarithmic factor in the sampling rate of the Main Theorem. The particular vector-valued Bernstein inequality below is based on the exposition in [23] (Chapter 6.3, equation (6.12)), with a direct proof appearing in [19].

Proposition 7 (Vector Bernstein inequality). Let \( \{g_k\} \in \mathbb{C}^d \) be a finite sequence of independent random vectors. Suppose that \( \mathbb{E} [g_k] = 0 \) and \( \|g_k\|_2 \leq B \) a.s. and put \( \sigma^2 \geq \sum_k \mathbb{E} \left[ \|g_k\|_2^2 \right] \). Then for all \( 0 \leq t \leq \sigma^2 / B \):

\[
\text{Pr} \left( \left\| \sum_k g_k \right\|_2 \geq t \right) \leq \exp \left( -\frac{t^2}{8\sigma^2} + \frac{1}{4} \right).
\]

3.3. Fundamental estimates

We adopt the structure and nomenclature of this section from [5]. The following elementary bounds will be used repeatedly:

\[
|\langle a_k, z \rangle|^2 \leq s\mu \|z\|_2^2, \quad |\langle a_k, Xz \rangle|^2 \leq s\mu \|z\|_2^2, \tag{7}
\]

\[
\|P_T a_k\|_2^2 \leq \mu s, \quad \|P_T X a_k\|_2^2 \leq \mu s. \tag{8}
\]

Also, we will always assume that \( m \geq s \).
Lemma 8 (Scaling). Let \( \tilde{a} \) be a random vector such that \( \mathbb{E}[\tilde{a}\tilde{a}^*] \) is invertible. There is a number \( \nu \) such that, with \( a := \nu \tilde{a} \), it holds that
\[
\kappa_s = \lambda_{\max}(s, \mathbb{E}[aa^*]) = \lambda_{\min}(s, \mathbb{E}[aa^*])^{-1}
\]
for all \( 1 \leq s \leq n \). This rescaled ensemble fulfills:
\[
\kappa_s \mu \geq 1. \tag{9}
\]

Proof. The first assertion follows immediately for \( \nu = (\lambda_{\max}(s, \mathbb{E}[\tilde{a}\tilde{a}^*])\lambda_{\min}(s, \mathbb{E}[\tilde{a}\tilde{a}^*]))^{-\frac{1}{4}} \).

For the second claim: By definition \( \mu \geq \max_i |\langle a, e_i \rangle|^2 \) holds almost surely, so that in particular
\[
\mu \geq \mathbb{E} \left[ \max_i |\langle a, e_i \rangle|^2 \right].
\]
For every \( i \), the function \( a \mapsto |\langle a, e_i \rangle|^2 \) is convex, which implies that
\[
\mu \geq \mathbb{E} \left[ \max_i |\langle a, e_i \rangle|^2 \right] = \max_i e_i^* (aa^*) e_i
\]
is convex (as the pointwise maximum of convex functions). Hence, by Jensen’s inequality,
\[
\mathbb{E} \left[ \max_i |\langle a, e_i \rangle|^2 \right] \geq \max_i e_i^* \mathbb{E} [aa^*] e_i = \max_i (e_i, \mathbb{E} [aa^*]) \geq \lambda_{\min}(1, \mathbb{E} [aa^*]) \geq \lambda_{\min}(s, \mathbb{E} [aa^*]).
\]
Therefore \( \mu \geq \lambda_{\min}(s, \mathbb{E} [aa^*]) \). Together with \( \kappa_s = \lambda_{\min}^{-1}(s, \mathbb{E} [aa^*]) \), this implies \( \mu \kappa_s \geq 1 \).

The estimates in this proof are tight in the sense that there are ensembles for which each inequality above turns into an equality. A straightforward example for such an ensemble is given by picking super-normalized Fourier basis vectors \( f_k \) (with coefficients \( (f_k)_l = e^{\frac{2\pi i kl}{n}} \)) according to the uniform probability distribution.

Lemma 9 (Local isometry). Let \( T \) and \( P_T \) be as in the notation section. Then for each \( 0 \leq \tau \leq \frac{1}{2} \):
\[
\Pr (\| P_T (XA^* A - \mathbb{1}) P_T \|_\infty \geq \tau) \leq 2s \exp \left( -\frac{m}{s\mu \kappa_s} \frac{\tau^2}{2 (1 + 2\tau/3)} \right)
\]

Proof. Let us decompose the relevant expression:
\[
P_T (XA^* A - \mathbb{1}) P_T = \frac{1}{m} \sum_{i=1}^m M_k,
\]
Thus:

\[ \|M_k\|_\infty \leq \|P_T X a_k a_k^* P_T\|_2 + 1 \]
\[ = \|P_T X a_k\|_2 \|a_k^* P_T\|_2 + 1 \]
\[ \leq \mu s + 1 \leq 2\mu s \kappa_s =: B. \]

Furthermore:

\[
\|E[M_k^* M_k]\|_\infty = \|E[(P_T (X a_k a_k^* - I) P_T) (P_T (a_k a_k^* X - I) P_T)]\|_\infty \\
= \|E[P_T X a_k a_k^* P_T a_k a_k^* X P_T] - E[P_T X a_k a_k^* P_T] + P_T]\|_\infty \\
= \|E[P_T (X a_k \langle a_k, P_T a_k \rangle a_k^* X - I) P_T]\|_\infty \\
\leq \max (\|\mu s E[P_T X a_k a_k^* X P_T]\|_\infty , 1) \\
\leq \max (\mu s \|X_T\|_\infty , 1) \leq \max (\mu s \kappa_s, 1) = \mu s \kappa_s.
\]

Similarly,

\[
\|E[M_k^* M_k]\|_\infty = \|E[P_T (a_k \langle a_k, X P_T X a_k \rangle a_k^* - I) P_T]\|_\infty \\
\leq \max (\|s \mu E[P_T a_k a_k^* P_T]\|_\infty , 1) \leq \max (s \mu \|P_T X^{-1} P_T\|_\infty , 1) \leq \mu s \kappa_s.
\]

Thus:

\[
\max \left\{ \|\sum_{k=1}^m E(M_k^* M_k)\|_\infty , \|\sum_{k=1}^m E(M_k^* M_k)\|_\infty \right\} \leq ms \mu s \kappa_s =: \sigma^2.
\]

Applying the Matrix Bernstein inequality for \(s\)-dimensional matrices \((P_T (X A^* A - I) P_T)\) has rank at most \(s\) with \(t = mt\) yields the desired result.

The estimate (10) is the only place in the proof where the second incoherence property in (3) is essentially used. A careful analysis shows that in all other cases, one can do without it, possibly at the price of replacing \(\kappa_s\) by \(\kappa\) (which is the reason why we have not spelled it out). In order to obtain the results of Proposition 5 the bound (10) has to be modified. To arrive at (4), use

\[
\|E[M_k^* M_k]\|_\infty \leq E[\|M_k^* M_k\|_\infty] \\
\leq E[\|P_T a_k a_k^* P_T \langle a_k, X P_T X a_k \rangle\|_\infty] \\
\leq s \mu E[\|a_k, X P_T X a_k\|] = s \mu E[\|a_k a_k^* X P_T X\|] \\
= s \mu \tr (X^{-1} X P_T X) = s \mu \tr (P_T X) \leq s^2 \mu s \kappa_s.
\]

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And for (5):
\[
\|\mathbb{E} [P_T a_k a_k^* X P_T X a_k a_k^* P_T]\|_{\infty} = \|\mathbb{E} [P_T X a_k a_k^* X P_T a_k a_k^* P_T]\|_{\infty} + 2 \|\mathbb{E} [P_T a_k a_k^* X P_T X a_k a_k^* P_T]\|_{\infty} \\
\leq \mu s \kappa_s + K \|P_T X X^{-1} P_T\|_{\infty} = \mu s \kappa_s + K.
\]

**Lemma 10 (Low-distortion).** Let $z, T, P_T$ be as in the notation section. For each $0 \leq \tau \leq 1$ it holds that
\[
\Pr (\|P_T (1 - A^* A) z\|_2 \geq \tau \|z\|_2) \leq \exp \left( -\frac{m \tau^2}{16 s \mu \kappa_s} + \frac{1}{4} \right).
\]

**Proof.** The structure of the proof closely follows the one of Lemma 9. Set $g_k := P_T (1 - a_k a_k^* X) z$.

We bound
\[
\|g_k\|_2 = \|P_T (1 - a_k a_k^* X) z\|_2 \\
\leq \|z\|_2 + \|P_T a_k \langle a_k, X z \rangle\|_2 \\
\leq \|z\|_2 + s \mu \|z\|_2 \leq 2 s \mu \kappa_s \|z\|_2 =: B
\]

and
\[
\mathbb{E} [\|g_k\|^2] \leq \mathbb{E} [\|P_T a_k \langle a_k, X z \rangle\|^2] + \|z\|^2 \\
= \mathbb{E} [\|P_T a_k \|^2 \langle a_k, X z \rangle^2] + \|z\|^2 \\
\leq s \mu \mathbb{E} [\langle X z, a_k \rangle \langle a_k, X z \rangle] + \|z\|^2 \\
= s \mu \langle X z, \mathbb{E} [a_k a_k^* X z] \rangle + \|z\|^2 \\
= s \mu \langle X z, z \rangle + \|z\|^2 \leq 2 s \mu \kappa_s \|z\|^2
\]

so that
\[
\sum_{k=1}^m \mathbb{E} [\|g_k\|^2] \leq 2 m s \mu \kappa_s \|z\|^2 =: a^2
\]

and thus $\frac{a^2}{\tau^2} = m \|z\|_2^2$. The advertised statement follows by applying the vector Bernstein inequality for $t = m \tau$.

**Lemma 11 (Off-support incoherence).** Let $z, P_T$ again be as in the notation section. Then for each $\tau \geq 0$:
\[
\Pr (\|P_T A^* A X z\|_\infty \geq \tau \|z\|_2) \leq 2n \exp \left( -\frac{3 m \tau^2}{2 \mu \kappa_s (3 + \sqrt{s} \tau)} \right)
\]
Proof. Fix \(i \in T^c\) and use the following decomposition:

\[
\langle e_i, A^* X z \rangle = \frac{1}{m} \sum_{k=1}^{m} M_k,
\]

where \(M_k := \langle e_i, a_k a_k^* X z \rangle = \langle e_i, a_k \rangle \langle a_k, X z \rangle\). Note that we have:

\[
\mathbb{E}[M_k] = \langle e_i, \mathbb{E}[a_k a_k^*] X z \rangle = \langle e_i, z \rangle = 0,
\]

because \(e_i \in T^c\). Bound

\[
|M_k| = |\langle e_i, a_k \rangle \langle a_k, X z \rangle| \leq \sqrt{s \mu \kappa_s} ||z||_2 =: B,
\]

and

\[
\mathbb{E}[M_k M_k^*] = \mathbb{E}[M_k^* M_k] = \mathbb{E}[|\langle a_k, e_i \rangle|^2 |\langle a_k, X z \rangle|^2] \\
\leq \mu \mathbb{E}[|X z, a_k a_k^* X z|] = \mu \langle X z, z \rangle \\
\leq \mu \|X_T\|_\infty \|z\|_2 \leq \mu \kappa_s \|z\|_2^2.
\]

Therefore we can set \(\sigma^2 := m \mu \kappa_s \|z\|_2^2\). Applying the Matrix Bernstein inequality for \(d = 1\) and the union bound over all \(i \in T^c\) yields the claim. \(\square\)

Lemma 12 (Uniform off-support incoherence). Let \(T^c, P_T\) be as in the notation section. For \(0 \leq \tau \leq 1\) we have

\[
\Pr \left( \max_{i \in T^c} \|P_T X A^* e_i\|_2 \geq \tau \right) \leq n \exp \left( -\frac{m \tau^2}{8 s \mu \kappa_s} + \frac{1}{4} \right)
\]

Proof. Fix \(i \in T^c\) and decompose:

\[
P_T X A^* e_i = \frac{1}{m} \sum_{k=1}^{m} g_k,
\]

where \(g_k := \langle a_k, e_i \rangle P_T X a_k\). It holds that \(\mathbb{E}[g_k] = 0\). Next, bound

\[
\|g_k\|_2 = |\langle a_k, e_i \rangle| \|P_T X a_k\|_2 \leq s \mu =: B.
\]

Furthermore:

\[
\mathbb{E}[\|g_k\|_2^2] \leq \sum_{i \in T} \mu \mathbb{E}[\langle e_i, X a_k a_k^* X e_i \rangle] \leq \sum_{i \in T} \mu \|X_T\|_\infty \leq s \mu \kappa_s.
\]

We can therefore set \(\sigma^2 := m s \mu \kappa_s\) and apply the Vector Bernstein inequality for \(t = m \tau\). Noting that \(\sigma^2 / B = m \kappa_s \geq m\) finishes the proof. \(\square\)
3.4. Convex geometry

Our aim is to prove that the solution \( x^\star \) to the optimization problem (11) equals the unknown vector \( x \). One way of assuring this is by exhibiting a dual certificate [24]. This method was first introduced in [2] and is now standard. We will use a relaxed version of this first introduced in [19] and later adapted from matrices to vectors in [5]. Our version further adapts the statement to the anisotropic setting.

**Lemma 13** (Inexact duality). Let \( x \in \mathbb{C}^n \) be a \( s \)-sparse vector, let \( T = \text{supp} (x) \). Assume that

\[
\| (P_T X A^* A P_T)^{-1} \|_\infty \leq 2, \tag{11}
\]
\[
\max_{i \in T^c} \| P_T X A e_i \|_2 \leq 1 \tag{12}
\]

and that there is a vector \( v \) in the row space of \( A \) obeying

\[
\| v_T - \text{sgn} (x) \|_2 \leq \frac{1}{4} \tag{13}
\]
\[
\| v_{T^c} \|_\infty \leq \frac{1}{4} \tag{14}
\]

Then the solution \( x^\star \) of the convex program (11) is unique and equal to \( x \).

**Proof.** Let \( \hat{x} = x + h \) be a solution of the minimization procedure. We note that feasibility requires \( Ah = 0 \). To prove the claim it suffices to show \( h = 0 \). Observe:

\[
\| \hat{x} \|_1 = \| x + h_T \|_1 + \| h_{T^c} \|_1
\]
\[
= \langle \text{sgn} (x + h_T), x + h_T \rangle + \| h_{T^c} \|_1
\]
\[
\geq \langle \text{sgn} (x), x \rangle + \langle \text{sgn} (x), h_T \rangle + \| h_{T^c} \|_1
\]
\[
\geq \| x \|_1 - |\langle \text{sgn} (x), h_T \rangle| + \| h_{T^c} \|_1.
\]

Feasibility requires \( \langle v, h \rangle = 0 \) (since \( v \) is in the row space of \( A \)) and therefore:

\[
|\langle \text{sgn} (x), h_T \rangle| = |\langle \text{sgn} (x) - v_T, h_T \rangle + \langle v_T, h_T \rangle|
\]
\[
= |\langle \text{sgn} (x) - v_T, h_T \rangle - \langle v_{T^c}, h_{T^c} \rangle|
\]
\[
\leq |\langle \text{sgn} (x) - v_T, h_T \rangle| + |\langle v_{T^c}, h_{T^c} \rangle|
\]
\[
\leq \frac{1}{4} \| h_T \|_2 \| h_{T^c} \|_2 + |\langle v_{T^c}, h_{T^c} \rangle|
\]
\[
\leq \frac{1}{4} \| h_T \|_2 + |\langle v_{T^c}, h_{T^c} \rangle|,
\]

where we have used (13). Together with:

\[
|\langle v_{T^c}, h_{T^c} \rangle| \leq \| v_{T^c} \|_\infty \| h_{T^c} \|_1 \leq \frac{1}{4} \| h_{T^c} \|_1,
\]

this implies:

\[
|\langle \text{sgn} (x), h_T \rangle| \leq \frac{1}{4} \left( \| h_T \|_2 + \| h_{T^c} \|_1 \right).
\]
Furthermore due to (11) and (12) it holds that
\[
\| h_T \|_2 = \| (P_T X A^* A P_T)^{-1} (P_T X A^* A) h_T \|_2 \\
= \| (P_T X A^* A P_T)^{-1} (P_T X A^* A) (h - h_{T^*}) \|_2 \\
= \| - (P_T X A^* A P_T)^{-1} (P_T X A^* A) h_{T^*} \|_2 \\
\leq 2 \| P_T X A^* A P_T h \|_2 \\
\leq 2 \max_{i \in T^*} \| P_T X A^* A e_i \|_2 \| h_{T^*} \|_1 \\
\leq 2 \| h_{T^*} \|_1,
\]
All this together implies:
\[
\| \hat{x} \|_1 \geq \| x \|_1 - \frac{1}{4} \| h_T \|_2 + \frac{3}{4} \| h_{T^*} \|_1 \\
\geq \| x \|_1 + \frac{1}{4} \| h_{T^*} \|_1.
\]
Consequently \( \| \hat{x} \|_1 = \| x \|_1 \) demands \( \| h_{T^*} \|_1 = 0 \), which in turn implies \( \| h_T \|_2 = 0 \), because \( \| h_T \|_2 \leq 2 \| h_{T^*} \|_1 \). Therefore \( h = 0 \), which corresponds to a unique minimizer \( \hat{x} = x \).

3.5. Construction of the certificate

It remains to show that a dual certificate \( v \) as described in Lemma 13 can indeed be constructed. We will prove:

**Lemma 14.** Let \( x \in \mathbb{C}^n \) be an \( s \)-sparse vector, let \( \omega \geq 1 \). If the number of measurements fulfills
\[
m \geq 18044 \kappa_s \mu \omega^2 s \log n,
\]
then with probability at least \( 1 - e^{-\omega} \), the constraints (11) (12) will hold and a vector \( v \) with the properties required for Lemma 13 exists.

This lemma immediately implies the Main Theorem.

The proof employs a recursive procedure (dubbed the “golfing scheme”) to construct a sequence \( v_i \) of vectors converging to a dual certificate with high probability. The technique has been developed in [18, 19] in the context of low-rank matrix recovery problems and has later been refined for compressed sensing in [5]. Here, we further modify the construction to handle anisotropic ensembles.

**Proof.** The recursive scheme consists of \( l \) iterations. The \( i \)-th iteration depends on three parameters: \( m_i \in \mathbb{N}; c_i, t_i \in \mathbb{R} \) which will be chosen in the course of the later analysis. To initialize, set
\[
v_0 = 0
\]
(the \( v_i \) for \( 1 \leq i \leq l \) will be defined iteratively below). We will use the notation
\[
q_i = \text{sgn}(x) - P_T v_i.
\]
The $i$-th step of the scheme proceeds according to the following protocol: We sample $m_i$ vectors from the ensemble $F$. Let $\tilde{A}$ be the $m_i \times n$-matrix whose rows consists of these vectors. We check whether the following two conditions are met:

\[
\left\| P_T \left( \frac{1}{m_i} \tilde{A}^* \tilde{A} X \right) P_T q_{i-1} \right\|_2 \leq c_i \| q_{i-1} \|_2, \tag{15}
\]
\[
\left\| \frac{m_i}{m} P_T \tilde{A}^* \tilde{A} X P_T q_{i-1} \right\|_\infty \leq t_i \| q_{i-1} \|_2. \tag{16}
\]

If so, set

\[
A_i = \tilde{A}, \quad v_i = \frac{m_i}{m} A_i^* A_i X P_T (\text{sgn}(x) - v_{i-1}) + v_{i-1}
\]

and proceed to step $i + 1$. If either of (15), (16) fails to hold, repeat the $i$-th step with a fresh batch of $m_i$ vectors drawn from $F$. Denote the number of repetitions of the $i$-th step by $r_i$.

We now analyze the properties of the above recursive construction. The following identities are easily verified by repeating the given transformations inductively:

\[
v := v_i = \frac{m_i}{m} A_i^* A_i X P_T (\text{sgn}(x) - v_{i-1}) + v_{i-1}
\]

\[
= \frac{m_i}{m} A_i^* A_i X P_T q_{i-1} + v_{i-1}
\]

\[
= \ldots = \sum_{i=1}^l \frac{m_i}{m} A_i^* A_i X P_T q_{i-1}, \tag{17}
\]

\[
q_i = \text{sgn}(x) - P_T v_i
\]

\[
= \text{sgn}(x) - P_T \left( \frac{m_i}{m} A_i^* A_i X P_T (\text{sgn}(x) - v_{i-1}) + v_{i+1} \right)
\]

\[
= (\text{sgn}(x) - P_T v_{i-1}) - \frac{m_i}{m} A_i^* A_i X P_T (\text{sgn}(x) - v_{i-1})
\]

\[
= P_T \left( \frac{1}{m_i} A_i^* A_i X \right) q_{i-1}
\]

\[
= \ldots = \prod_{j=1}^i P_T \left( \frac{1}{m_i} A_j^* A_j X \right) P_T \text{sgn}(x). \tag{18}
\]
Together with (15) and (16), one obtains

\[ \|q_l\|_2 \leq \prod_{i=1}^l c_i \|q_{i-1}\|_2 \leq \prod_{i=1}^l c_i \|q_0\|_2 = \sqrt{s} \prod_{i=1}^l c_i, \]

\[ \|v_T\|_\infty = \left\| P_T \left( \sum_{i=1}^l \frac{m_i}{m} A_i^* A_i X P_T q_{i-1} \right) \right\|_\infty \leq \sum_{i=1}^l \left\| \frac{m_i}{m} P_T A_i^* A_i X P_T q_{i-1} \right\|_2 \leq \sum_{i=1}^l t_i \|q_{i-1}\|_2 \leq \sqrt{s} \left( t_1 + \sum_{i=2}^l t_i \prod_{j=1}^{i-1} c_j \right). \]

Following [19], we choose the parameters \( l, c_i, t_i \) as

\[ l = \left\lceil \frac{1}{2} \log_2 s \right\rceil + 2, \quad c_1 = c_2 = \frac{1}{2 \sqrt{\log n}}, \quad t_1 = t_2 = \frac{1}{8 \sqrt{s}}, \]

and for \( i \geq 3 \)

\[ t_i = \frac{\log n}{8 \sqrt{s}}, \quad c_i = \frac{1}{2}. \]

A short calculation then yields

\[ \|v_T\|_\infty \leq \frac{1}{4}, \quad \|v - \text{sgn}(x_T)\|_2 = \|q_l\|_2 \leq \frac{1}{4}, \]

which are conditions (13) and (14).

Next, we need to establish that the total number

\[ \sum_{i=1}^l m_i r_i \]

of sampled vectors remains small with high probability. More precisely, we will bound the probability

\[ p_3 := \Pr \left( (r_1 > 1) \text{ or } (r_2 > 1) \text{ or } \sum_{i=1}^l r_i \geq l' \right) \]

for some \( l' \) to be chosen later.

To that end, denote by \( p_1(i) \) the probability that (15) fails to hold in any given batch of the \( i \)-th step. Analogously, let \( p_2(i) \) be the probability of failure for (16). Lemmas [10] and [11] give the estimates

\[ p_1(i) \leq \exp \left( -\frac{m_i c_i^2}{16s \mu_\kappa_s} + \frac{1}{4} \right), \quad p_2(i) \leq 2n \exp \left( -\frac{3m_i t_i^2}{2 \mu_\kappa_s (3 + \sqrt{s} t_i)} \right). \]
We choose
\[ l' = 4(\omega + \log 12 + \frac{2}{3}l), \quad m_1 = m_2 = 694\kappa_s\mu \omega s \log n, \]
and for \( i \geq 3 \)
\[ m_i = 694\kappa_s\mu \omega s. \]
Such a choice can be guaranteed by a total sampling rate \( m \geq 18044\kappa_s\mu \omega^2 s \log n \) and ensures
\[ p_1(i) + p_2(i) \leq \frac{1}{6}e^{-\omega} \leq \frac{1}{12} \]
for all \( i \). (It is easily seen that for for \( n \gg 1 \), a bound of \( m \geq 228\kappa_s\mu \omega^2 s \log n \) is sufficient. The constants appearing here are highly unlikely to be optimal.) Note that
\[ \sum_{i=1}^{l} r_i \geq l' \]
only if fewer than \( l \) of the first \( l' \) batches of vectors satisfied both (15) and (16). This implies that
\[ \Pr \left( \sum_{i=1}^{l} r_i \geq l' \right) \leq \Pr(N \leq l-1)_{\text{Bin}(l', \frac{11}{12})}, \]
where the r.h.s. is the probability of obtaining fewer than \( l \) outcomes in a binomial process with \( l' \) repetitions and individual success probability \( 11/12 \). We bound this quantity using a standard concentration bound from [25] (C. McDiarmid’s section "Concentration"):
\[ \Pr \left( |\text{Bin}(n, p) - np| > \tau \right) \leq 2 \exp \left( -\frac{\tau^2}{3np} \right). \]
This yields \( \Pr \left( \sum_{i=1}^{l} r_i \geq l' \right) \leq \frac{1}{6}e^{-\omega} \) for our choice of \( l' \). Putting things together, we have
\[ p_3 \leq 3 \frac{1}{6}e^{-\omega} = \frac{1}{2}e^{-\omega}, \]
according to the union bound. In addition, we have to take into account that properties (11) and (12) can fail as well. We denote these probabilities of failure by \( p_4 \) and \( p_5 \). Lemmas 9 and 12 give:
\[ p_4 \leq 2\exp \left( -\frac{6m}{7s\mu \kappa_s} \right), \quad p_5 \leq n\exp \left( -\frac{m}{8s\mu \kappa_s} + \frac{1}{4} \right). \]
Our sampling rate \( m \) guarantees \( p_4 \leq \frac{1}{4}e^{-\omega} \) as well as \( p_5 \leq \frac{1}{4}e^{-\omega} \). Applying the union bound now yields our desired overall error bound \( (p_3 + p_4 + p_5 \leq e^{-\omega}). \)

\[ \square \]
4. Conclusion and Outlook

In this paper, we have shown that proof techniques based on duality theory and the “golfing scheme” are versatile enough to handle the situation where the ensemble of measurement vectors is not isotropic.

An obvious future line of research would be to translate these results to the low-rank matrix recovery problem. Given the high degree of similarity between [19] and [5], this should be a conceptually straight-forward task. This would further generalize the scope of this proof method, beyond ortho-normal operator bases [19] and tight frames [20].

Also, Proposition 5 suggests that the second incoherence property 3 can be relaxed or maybe even disposed of. We leave this as an open problem.

5. Acknowledgments

We thank E. Candès for suggesting the problem treated here, and him, Y. Plan, P. Jung, and P. Walk for insightful discussions. Financial support from the Excellence Initiative of the German Federal and State Governments (grant ZUK 43), the German Science Foundation (DFG grants CH 843/1-1 and CH 843/2-1), the Swiss National Science Foundation, and the Swiss National Center of Competence in Research “Quantum Science and Technology” is gratefully acknowledged.

References

[1] E. Candès, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Transactions on Information Theory 52 (2006) 489 – 509.

[2] E. Candes, T. Tao, Near-optimal signal recovery from random projections: Universal encoding strategies?, IEEE Transactions on Information Theory 52 (2006) 5406 – 5425.

[3] D. Donoho, Compressed sensing, IEEE Transactions on Information Theory 52 (2006) 1289 – 1306.

[4] M. Wainwright, Sharp thresholds for high-dimensional and noisy sparsity recovery using $\ell_1$-constrained quadratic programming (Lasso), IEEE Transactions on Information Theory 55 (2009) 2183 –2202.

[5] E. J. Candes, Y. Plan, A probabilistic and RIPless theory of compressed sensing, IEEE Transactions on Information Theory 57 (2011) 7235–7254.

[6] M. Rudelson, S. Zhou, Reconstruction from anisotropic random measurements, preprint: arXiv:1106.1151 (2011).

[7] A. Juditsky, A. Nemirovski, On verifiable sufficient conditions for sparse signal recovery via $\ell_1$ minimization, Mathematical Programming 127 (2011) 57–88.
[8] D. Gross, V. Nesme, Note on sampling without replacing from a finite collection of matrices, preprint: arXiv:1001.2738 (2010).

[9] P. J. Bickel, Y. Ritov, A. B. Tsybakov, Simultaneous analysis of Lasso and Dantzig selector, The Annals of Statistics 37 (2009) 1705–1732.

[10] P. J. Bickel, Discussion: The Dantzig selector: Statistical estimation when \( p \) is much larger than \( n \), The Annals of Statistics (2007) 2352–2357.

[11] B. S. Kashin, V. N. Temlyakov, A remark on compressed sensing, Mathematical notes 82 (2007) 748–755.

[12] E. Candes, B. Recht, Exact matrix completion via convex optimization, Foundations of Computational Mathematics 9 (2009) 717–772.

[13] E. Candes, T. Tao, The power of convex relaxation: Near-Optimal matrix completion, IEEE Transactions on Information Theory 56 (2010) 2053–2080.

[14] Y. Liu, Universal low-rank matrix recovery from Pauli measurements, Adv. in Neural Information Processing Systems 24 (2011) 1638–1646.

[15] E. J. Candes, Y. Plan, Tight oracle inequalities for low-rank matrix recovery from a minimal number of noisy random measurements, IEEE Transactions on Information Theory 57 (2011) 2342–2359.

[16] S. T. Flammia, D. Gross, Y.-K. Liu, J. Eisert, Quantum tomography via compressed sensing: error bounds, sample complexity and efficient estimators, New Journal of Physics 14 (2012) 095022.

[17] S. Negahban, M. J. Wainwright, Restricted strong convexity and weighted matrix completion: Optimal bounds with noise, The Journal of Machine Learning Research 13 (2012) 1665–1697.

[18] D. Gross, Y. Liu, S. T. Flammia, S. Becker, J. Eisert, Quantum state tomography via compressed sensing, Physical Review Letters 105 (2010) 150401.

[19] D. Gross, Recovering low-rank matrices from few coefficients in any basis, IEEE Transactions on Information Theory 57 (2011) 1548–1566.

[20] R. Ahlswede, A. Winter, Strong converse for identification via quantum channels, IEEE Transactions on Information Theory 48 (2002) 569–579.

[21] J. Tropp, User-Friendly tail bounds for sums of random matrices, Foundations of Computational Mathematics (2011) 1–46.

[22] R. Kueng, Efficient recovery of sparse vectors using anisotropic ensembles, research project, ETH Zürich, June 2011.

[23] M. Ledoux, M. Talagrand, Probability in Banach Spaces: Isoperimetry and Processes, Springer, Berlin, 1991.
[24] D. P. Bertsekas, A. Nedi, A. E. Ozdaglar, Convex analysis and optimization, Athena Scientific, Belmont, MA, 2003.

[25] M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, B. Reed, Probabilistic methods for algorithmic discrete mathematics, Vol. 16, Springer, 1998.

[26] M. Ohliger, V. Nesme, D. Gross, Y. Liu, J. Eisert, Continuous-variable quantum compressed sensing, preprint arXiv:1111.0853 (2011).