Probabilistic model predictive safety certification for learning-based control
Kim P. Wabersich, Lukas Hewing, Andrea Carron, Melanie N. Zeilinger

Abstract—Reinforcement learning (RL) methods have demonstrated their efficiency in simulation environments. However, many applications for which RL offers great potential, such as autonomous driving, are also safety critical and require a certified closed-loop behavior in order to meet safety specifications in the presence of physical constraints. This paper introduces a concept, called probabilistic model predictive safety certification (PMPSC), which can be combined with any RL algorithm and provides provable safety certificates in terms of state and input chance constraints for potentially large-scale systems. The certificate is realized through a stochastic tube that safely connects the current system state with a terminal set of states, that is known to be safe. A novel formulation in terms of a convex receding horizon problem allows a recursively feasible real-time computation of such probabilistic tubes, despite the presence of possibly unbounded disturbances. A design procedure for MPSC relying on bayesian inference and recent advances in probabilistic set invariance is presented. Using a numerical car simulation, the method and its design procedure are illustrated by enhancing a simple RL algorithm with safety certificates.

Index Terms—Reinforcement learning (RL), Stochastic systems, Predictive control, Safety

I. INTRODUCTION

WHILE the field of reinforcement learning demonstrated various classes of learning-based control methods in research-driven applications [1], [2], hardly any results have been successfully transferred to industrial applications that are safety critical, i.e. applications that are subject to physical and safety constraints. In industrial applications, successful control methods are often of simple structure, such as the Proportional–Integral–Derivative (PID) controller [3] or linear state feedback controller [4], which require an expert to cautiously tune them manually. Manual tuning is generally time consuming and therefore expensive, especially in the presence of safety specifications. Modern control methods, such as model predictive control (MPC), tackle this problem by providing safety guarantees w.r.t. adequate system and disturbance models by design, reducing manual tuning requirements. The various successful applications of MPC to safety critical systems, see e.g. [5], [6] for an overview, reflect these capabilities.

While provable safety of control methods facilitates the overall design procedure, tuning of various parameters, such as the cost function in order to achieve a desired closed-loop behavior, still needs to be done manually and requires a reasonable amount of experience. In contrast, RL methods using trial-and-error procedures are often more intuitive to design and are capable of autonomously finding a better policy. The downside of many RL algorithms, however, is that explicit consideration of physical system limitations and safety requirements at each time step cannot be addressed, often due to the complicated inner workings, which limits their applicability in many industrial applications [7].

This paper aims at addressing this problem by introducing a probabilistic model predictive safety certification (PMPSC) scheme for learning-based controllers, which can equip any controller with probabilistic constraint satisfaction guarantees. The scheme is motivated by the following observation. Often, an MPC controller with a short prediction horizon is sufficient in order to provide safety of a system during closed-loop operation, even though the same horizon would not be enough to achieve a desired performance. For example, in case of autonomous driving, checking if it is possible to transition the car into a safe set of states (e.g. brake down to low velocity) can be done efficiently by solving an open loop optimal control problem with relatively small planning horizon (e.g. using maximum deceleration). At the same time, a much longer planning horizon for an MPC controller, or even another class of control policies would be required in order to provide a comfortable and foresightful driving experience.

This motivates the combination of ideas from MPC with RL methods in order to achieve safe and high performance closed-loop system operation with a possibly small amount of manual tuning required. More precisely, a learning-based input action is certified as safe, if it leads to a safe state, i.e., a state for which a potentially low-performance but online computable and safe backup controller exists for all future times. By repeatedly computing such a backup controller for the state predicted one step ahead after application of the learning input, it gets either certified as safe and is applied, or it is overwritten by the previous safe backup controller. The resulting concept can be seen as a safety filter that only filters proposed learning signals, for which we cannot guarantee constraint satisfaction in the future.

Contributions: We provide a safety certification framework which allows for enhancing arbitrary learning-based control methods with safety guarantees[1] and which is suitable for possibly large-scale systems with continuous and chance

1Inputs provided by a human can be similarly enhanced by the safety certification scheme, which relates e.g. to the concept of electronic stabilization control from automotive engineering.
constrained input and state spaces. In order to enable efficient implementation and scalability, we provide an online algorithm together with a data-driven synthesis method to compute backup solutions that can be realized by real-time capable and established model predictive control (MPC) solvers, e.g. [8], [9], [10]. Compared to previously presented safety frameworks for learning-based control, e.g. [11], the set of safe state and action pairs is implicitly represented through an online optimization problem, enabling us to circumvent its explicit offline computation, which generally suffers from the curse of dimensionality.

Different from related concepts, like those presented in [12], [13], [14], [15], we consider possibly nonlinear stochastic systems that can be represented as linear systems with bounded model uncertainties and possibly unbounded additive noise. For this class of systems, we present an automated, parametrization free and data-driven design procedure that is tailored to the context of learning the system dynamics. Using the example of safely learning to track a trajectory with a car, we show how to construct a safe reinforcement learning algorithm using our framework in combination with a basic policy search algorithm.

II. RELATED WORK

Driven by the rapid progress in reinforcement learning there is also a growing awareness regarding safety aspects of machine learning systems [17], see e.g. [16] for a comprehensive overview. As opposed to most methods developed in the context of safe RL, the approach presented in this paper keeps the system safe at all times, including exploration, and considers continuous state and action spaces. This is possible through the use of models and corresponding uncertainty estimates of the system, which can be sequentially improved by, e.g., a RL algorithm to allow greater exploration.

In model-free safe reinforcement learning methods, policy search algorithms have been proposed, e.g. [17], which provide safety guarantees in expectation by solving a constrained policy optimization using a modified trust-region policy gradient method [18]. Efficient policy tuning with respect to best worst-case performance (also worst-case stability under physical constraints) can be achieved using Bayesian min-max optimization, see e.g. [19], or by safety constrained Bayesian optimization as e.g. in [20], [21]. These techniques share the limitation that they need to be tailored to a task-specific class of policies. Furthermore, most techniques require to repeatedly execute experiments, which prohibits fully autonomous safe learning in ‘closed-loop’.

In [22], a method was developed that allows to analyze a given closed-loop system (under an arbitrary RL policy) with respect to safety, based on a probabilistic system model. An extension of this method is presented in [23], where the problem of updating the policy is investigated and practical implementation techniques are provided. The techniques require an a-priori known Lyapunov function and Lipschitz continuity of the closed-loop learning system. In the context of model-based safe reinforcement learning, several learning-based model predictive control approaches are available. The method proposed in [24] conceptually provides deterministic guarantees on robustness, while statistical identification tools are used to identify the system in order to improve performance. In [25], the mentioned scheme has been tested and validated onboard using a quadcopter. In [26], a robust constrained learning-based model predictive control algorithm for path-tracking in off-road terrain is studied. The experimental evaluation shows, that the scheme is safe and conservative during initial trials, when model uncertainty is high, and very performant, once the model uncertainty is reduced. Regarding safety, [27] presents a learning model predictive control method, that provides theoretical guarantees in case of Gaussian process model estimates. For iterative tasks, [28] proposes a learning model predictive control scheme, that can be applied to linear system models with bounded disturbances. Instead of using model predictive control techniques, PILCO [29] allows to calculate analytic policy gradients and achieves good data efficiency, based on non-parametric Gaussian process regression.

The previously discussed literature provides specific reinforcement learning algorithms, that are tied to a specific, mostly model predictive control based, policy. In contrast, the proposed concept uses MPC-based ideas in order to establish safety, independent of a specific reinforcement learning policy. This offers the opportunity to apply RL for learning more complex tasks than, e.g., steady-state stabilization, which is usually considered in model predictive control. Many reinforcement learning algorithms are able to maximize rewards from a black-box function, i.e. rewards that are only available from measurements, which would not be possible using a model predictive controller, where the cost enters the corresponding online optimization problem explicitly.

Closely related to the approach proposed in this paper, the concept of a safety framework for learning-based control emerged from robust reachability analysis, robust invariance, as well as classical Lyapunov-based methods [30], [31], [32]. The concept consists of a safe set in the state space and a safety controller as originally proposed in [33] for the case of perfectly known system dynamics in the context of safety barrier functions. While the system state is contained in the safe set, any feasible input (including learning-based controllers) can be applied to the system. However, if such an input would cause the system to leave the safe set, the safety controller interferes. Since this strategy is compatible with any learning-based control algorithm, it serves as a universal safety certification concept. Previously proposed concepts are limited to a robust treatment of the uncertainty, in order to provide rigorous safety guarantees. This potentially results in a conservative system behavior, or even ill-posedness of the overall safety requirement e.g. in case of frequently considered Gaussian distributed additive system noise which has unbounded support.

Compared to previous research using similar model predictive control based safety mechanisms such as [12], [13], [14], [15], we introduce a probabilistic formulation of the safe set and consider safety in probability for all future times, allowing one to prescribe a desired degree of conservatism. The proposed method only requires an implicit description of the safe set as opposed to an explicit representation, which enables scalability with respect to the state dimension, while being independent of a particular RL algorithm.
III. Preliminaries and Problem Statement

A. Notation

The set of symmetric matrices of dimension $n$ is denoted by $S^n$, the set of positive (semi-) definite matrices by $(S^n)^+$, the set of integers in the interval $[a, b] \subset \mathbb{R}$ by $\mathcal{I}_{[a,b]}$, and the set of integers in the interval $[a, \infty) \subset \mathbb{R}$ by $\mathcal{I}_{\geq a}$. The Minkowski sum of two sets $A_1, A_2 \subset \mathbb{R}$ is denoted by $A_1 \oplus A_2 := \{ a_1 + a_2 | a_1 \in A_1, a_2 \in A_2 \}$ and the Pontryagin set difference by $A_1 \ominus A_2 := \{ a_1 \in A_1 | a_1 + a_2 \in A_2, \forall a_2 \in A_2 \}$.

The i-th row and i-th column of a matrix $A \in \mathbb{R}^{n \times m}$ is denoted by row$_i(A)$ and col$_i(A)$, respectively. The expression $x \sim Q_x$ means that a random variable $x$ is distributed according to the distribution $Q_x$. And the set of symmetric matrices of dimension $n$ is denoted by $\mathcal{S}(n)$, the set of positive (semi-) definite matrices by $(\mathcal{S}(n))^+$.

For a random variable $\mu$, the means that a random variable $\mu$ has expectation equals $\mu$. For a random variable $x$, $\mathbb{E}(x)$ and $\text{var}(x)$ denote the expected value and the variance.

B. Problem statement

We consider a-priori unknown nonlinear, time-invariant discrete-time dynamical systems of the form

$$x(k+1) = f(x(k), u(k)) + w_s(k), \quad \forall k \in \mathcal{I}_{\geq 0}$$

subject to polytopic state and input constraints $x(k) \in \mathcal{X}$, $u(k) \in \mathcal{U}$, and i.i.d. stochastic disturbances $w_s(k) \sim Q_w$. For controller design, we consider an approximate model description of the following form

$$x(k+1) = A x(k) + B u(k) + w_m(x(k), u(k)) + w_s(k),$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are typically obtained from linear system identification techniques, see e.g. [34], and $w_m(x(k), u(k), w_s(k))$ accounts for model errors.

In order to provide safety certificates, we require that the model error $w_m(x(k), u(k))$ is contained with a certain probability in a model error set $\mathcal{W}_m \subset \mathbb{R}^n$, where $\mathcal{W}_m$ is chosen based on available data $D = \{(x_i, u_i), f(x_i, u_i) + w_s, i=1\}$.

Assumption III.1 (Bounded model error). The deviation between the true system (1) and the corresponding model (2) is bounded by

$$\Pr\left( \forall k \in \mathcal{I}_{\geq 0}, (x(k) \in \mathcal{X}, u(k) \in \mathbb{R}^n, w_m(x(k), u(k), w_s) \in \mathcal{W}_m) \right) \geq p_m$$

where $p_m > 0$ denotes the probability level.

A principled way to infer a system model of the form (2) from available data is discussed in Section (V).

In this paper, system safety is defined as a required degree of constraint satisfaction in the form of probabilistic state and input constraints, i.e., as chance-constraints of the form

$$\Pr(x(k) \in \mathcal{X}) \geq p_x, \quad \Pr(u(k) \in \mathcal{U}) \geq p_u,$$

for all $k \in \mathcal{I}_{\geq 0}$ with probabilities $p_x, p_u \geq 0$.

The overall goal is to certify safety of arbitrary control signals $u_c(k) \in \mathbb{R}^m$, e.g., provided by an RL algorithm. This is achieved by means of a safety policy, which is computed in real time based on the current system state $x(k)$ and the proposed input $u_c(k)$. A safety policy consists of a safe input $u_S(k)$ at time $k$ and a safe backup trajectory that guarantees safety with respect to the constraints (4) when applied in future time instances. The safety policy is updated at every time step, such that the first input equals $u_c(k)$ if that is safe and otherwise implements a minimal safe modification. More formally:

Definition III.2. An input $u_c(k)$ is certified as safe for system (1) at time step $k$ and state $x(k)$ w.r.t. a safety policy $\pi_S : \mathcal{I}_{\geq k} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, if $\pi_S(k) = u_c(k)$ and $u(k) = \pi_S(k, x(k), u_c(k))$ for $k \geq k$ keeps the system safe, i.e. satisfies (4) for all $k \geq k$.

By assuming that a safety policy can be found for the initial system state, Definition III.2 implies the following safety algorithm. At every time step, safety of a proposed input $u_c(k)$ is verified using the safety policy according to Definition III.2. If safety cannot be verified, the proposed input is modified and $u(k) = \pi_S(k, x(k), u_c(k))$ is applied to the system instead, ensuring safety until the next step and learning input pair can be certified as safe again. The set of initial states for which $\pi_S$ ensures safety can thus be interpreted as a safe set of system states and represents a probabilistic variant of the safe set definition in [12].

In the following, we present a method to compute a safety policy $\pi_S$ for uncertain models of the form (2) making use of model predictive control (MPC) techniques, which provide real-time feasibility and scalability of the approach while aiming at a large safe set implicitly defined by the safety policy.

IV. PROBABILISTIC MODEL PREDICTIVE SAFETY CERTIFICATION

The fundamental idea of model predictive safety certification, which was introduced for linear deterministic systems in [12], is the on-the-fly computation of a safety policy $\pi_S$, that ensures constraint satisfaction at all times in the future. The safety policy is specified using MPC methods, i.e., an input sequence is computed that safely steers the system to a terminal safe set $\mathcal{X}_f$, which can be done efficiently in real-time. The first input is selected as the learning input if possible, in which case it is certified as safe, or selected as ‘close’ as possible to the learning input otherwise. A specific choice of the terminal safe set $\mathcal{X}_f$ allows to show that a previous solution at time $k-1$ implies the existence of a feasible solution at time $k$, ensuring safety for all future times. Such terminal sets $\mathcal{X}_f$ can, e.g., be a neighborhood of a locally stable steady-state of the system (1), or a possibly conservative set of states for which a safe controller is known.

A. Nominal model predictive safety certification scheme

In order to introduce the basic idea of the presented approach, we introduce a nominal model predictive safety certification (NMPC) scheme under the simplifying assumption that the system dynamics (1) are perfectly known, time-independent, and without noise, i.e. $x(k+1) = f(x(k), u(k)) \forall k \in \mathcal{I}_{\geq 0}$. The mechanism to construct the safety policy for certifying a
Assumption IV.1 (Nominal invariant terminal set). There exists a nominal terminal invariant set $\mathcal{X}_f \subseteq \mathcal{X}$ and a corresponding control law $\kappa_f : \mathcal{X}_f \to \mathcal{U}$, such that for all $x \in \mathcal{X}_f$ it holds $\kappa_f(x) \in \mathcal{U}$ and $f(x, \kappa_f(x)) \in \mathcal{X}_f$.

Assumption IV.1 provides recursive feasibility of optimization problem (5) and therefore infinite-time constraint satisfaction, i.e., if there exists a feasible solution at time $k$, there exists one at $k+1$ and therefore at all future times, see e.g. [35].

The safety certification scheme then works as follows. Consider a measured system state $x(k-1)$, for which (5) is feasible and the input trajectory $\{u^*_i\}_{i=0}^{N-1}$ is computed. After applying the first input to the system $u(k-1) = u^*_0$, the resulting state $x(k)$ is measured again. Because it holds in the nominal case that $x(k) = x^*_1$, a valid input sequence $\{u^*_1, \ldots, u^*_N\}$ is known from the previous time step, which satisfies constraints and steers the state to the safe terminal set $\mathcal{X}_f$ as indicated by the brown trajectory in Figure 1 (left). Safety of a proposed learning input $u_{\mathcal{L}}$ is certified by solving optimization problem (5), which, if feasible for $u_{0:k} = u_{\mathcal{L}}$, provides the green trajectory in Figure 1 (left) such that $u_{\mathcal{L}}$ can be safely applied to the system. Should problem (5) not be feasible for $u_{0:k} = u_{\mathcal{L}}$, it returns an alternative input sequence that safely guides the system towards the safe set $\mathcal{X}_f$. The first element of this sequence $u^*_0$ is chosen as close as possible to $u_{\mathcal{L}}$ and is applied to system (1) instead of $u_{\mathcal{L}}$. Due to recursive feasibility, i.e., knowledge of the brown trajectory in Figure 1 (left), such a solution always exists, ensuring safety.

In the context of learning-based control, the true system dynamics are rarely known accurately. In order to derive a probabilistic version of the NMPS scheme that accounts for uncertainty in the system model (2), in the following, we leverage advances in probabilistic stochastic model predictive control (NSPC) [36], [37], [38], [39].
control \cite{36}, based on so-called probabilistic reachable sets.

**B. Probabilistic model predictive safety certification scheme**

In the case of uncertain system dynamics, the safety policy consists of two components following a tube-based MPC concept \cite{35}. The first component considers a nominal state of the system $z(k)$ driven by linear dynamics, and computes a nominal safe trajectory $\{z_{m,k},v_{m,k}\}$, which is similar to the case of perfectly known dynamics introduced in the previous section. The second component consists of an auxiliary controller, which acts on the deviation $e(k)$ of the true system state from the nominal one and ensures that the true state $x(k)$ remains close to the nominal trajectory. Specifically, it guarantees that $e(k)$ stays within a set $R$, often called the ‘tube’, with probability at least $p_e$. Together, the resulting safety policy is able to steer the system state $z(k)$ within the probabilistic tube along the nominal trajectory towards the safe terminal set.

We first define the main components and assumptions, in order to then introduce the probabilistic model predictive safety certification (PMPSC) problem together with the proposed controller. Define with $z(k) \in \mathbb{R}^n$ and $v(k) \in \mathbb{R}^n$ the nominal system states and inputs, as well as the nominal dynamics according to model \cite{2} as

$$ z(k+1) = Az(k) + Bv(k), \quad k \in \mathcal{I}_{\geq 0} \tag{7} $$

with initial condition $z(0) = x(0)$. For example, one might choose matrices $(A, B)$ in the context of learning time-invariant linear systems based on the maximum likelihood estimate of the true system dynamics. Denote $e(k) := x(k) - z(k)$ as the error (deviation) between the true system state, evolving according to \cite{1}, and the nominal system state following \cite{7}. The controller is then defined by augmenting the nominal input with an auxiliary feedback on the error, in the case of a linear system \cite{7} a linear state feedback controller $K_R$

$$ u(k) = v(k) + K_R(x(k) - z(k)), \tag{8} $$

which keeps the real system state $x(k)$ close to the nominal system state $z(k)$, i.e. keeps the error $e(k)$ small, if $K_R \in \mathbb{R}^{m \times n}$ is chosen such that it stabilizes system \cite{7}. By Assumption \textbf{III.1} the model error $w_m(x(k),u(k))$ is contained in $\mathbb{W}_m$ for all time steps with probability $p_m$. Therefore, we drop the state and input dependencies in the following and simply refer to $w_m(k)$ as model mismatch at time $k$, such that the error dynamics can be expressed as

$$ e(k+1) = e(k+1) - z(k+1) $$

$$ = f(x(k),u(k)) + w_s(k) - Az(k) - Bv(k) $$

$$ = f(x(k),u(k)) - Ax(k) - Bu(k) $$

$$ + Az(k) + Bu(k) + w_s(k) - Az(k) - Bv(k) $$

$$ = (A + BK_R)e(k) + w_m(k) + w_s(k). \tag{9} $$

By setting the initial nominal state to the real state, i.e., $z(0) = x(0) \Rightarrow e(0) = 0$, the goal is to keep the evolving error $e(k)$, i.e. the deviation to the nominal reference trajectory, small in probability with levels $p_e$ and $p_u$ for state and input constraints \cite{4}, respectively. This requirement can be formalized using the concept of probabilistic reachable sets introduced in \cite{37,38,36}.

**Definition IV.2.** A set $R$ is a probabilistic reachable set (PRS) at probability level $p$ for system \cite{9} if

$$ e(0) = 0 \Rightarrow \Pr(e(k) \in R) \geq p, \quad \text{for all } k \in \mathcal{I}_{\geq 0}. \tag{10} $$

In Section \textbf{IV.A} it is shown how to compute PRS sets $R_x, R_u$, corresponding to state and input chance constraints \cite{4}, in order to fulfill the following Assumption.

**Assumption IV.3 (Probabilistic tube).** There exists a linear state feedback matrix $K_R^* \in \mathbb{R}^{m \times n}$, that stabilizes system \cite{1}. The corresponding PRS sets for the error dynamics \cite{9} with probability levels $p_e$ and $p_u$ are denoted by $R_{x,e}, R_{u,e} \subseteq \mathbb{R}^n$.

Based on Assumption \textbf{IV.3} it is possible to define deterministic constraints on the nominal system \cite{7}, that capture the chance constraints \cite{4}, by choosing $\mathbb{X} \otimes R_x$ and $\mathbb{U} \otimes K_R^* R_u$ as depicted in Figure \textbf{1}(right), in which the grey circles illustrate the PRS centered around the predicted nominal state $z$, such that they contain the true state $x$ with probability $p_x$. Through calculation of the nominal safety policy towards a safe terminal set within the tightened constraints, and by application of \cite{8}, finite-time chance-constraint satisfaction upon the planning horizon $k+N$ follows directly by Definition \textbf{IV.2} and Assumption \textbf{IV.3}. In order to provide ‘infinite’ horizon safety through recursive feasibility of \cite{8}, we require a terminal invariant set for the nominal system state $Z_f$ similar to Assumption \textbf{IV.1} which is contained in the tightened constraints.

**Assumption IV.4 (Nominal terminal set).** There exists a terminal invariant set $Z_f \subseteq \mathbb{X} \otimes R_x$ and a corresponding control law $\kappa_f : Z_f \rightarrow \mathbb{U} \otimes K_R^* R_u$ such that for all $z \in Z_f$ it holds $\kappa_f(z) \in \mathbb{U} \otimes K_R^* R_u$ and $Az + Bu \in Z_f.$

In classical tube-based and related stochastic MPC methods, the nominal system \cite{7} is reset at each time step in order to minimize the nominal objective. While a re-initialization of the nominal-, and therefore also of the error system, works in a robust setting, it prohibits a direct probabilistic analysis using PRS according to Definition \textbf{IV.2} that only provide statements about the autonomous error system, starting from time $k = 0$ and evolving linearly for all future times. Consequently and in contrast to classical formulations, we virtually simulate \cite{7} via \cite{36}, which leads to the error dynamics \cite{9} despite online replanning of the nominal trajectory at each time step, compare also with Figure \textbf{2} accompanying the proof of Theorem \textbf{IV.5}.

Building on the tube-based controller structure, an input is certified as safe if it can be represented in the form of \cite{8} by selecting $v_0^*$ accordingly. Otherwise an alternative input is provided ensuring that $\Pr(e(k) \in R_x) \geq p_x, \Pr(e(k) \in R_u) \geq p_u$ for all $k \in \mathcal{I}_{\geq 0}$. Combining this mechanism with the assumptions from above yield the main result of the paper.

**Theorem IV.5.** Let Assumptions \textbf{IV.3} and \textbf{IV.4} hold. If \cite{5} is feasible for $z_{0,k}^* = x(0)$, then system \cite{1} under the control law \cite{8} with $v(k) = v_{0,k}^*$ resulting from the PMPSC
problem \((\ref{eq:3})\) is safe for all \(u \in \mathcal{U}(k)\) and for all times, i.e., the chance constraints \((\ref{eq:5})\) are satisfied for all \(k \geq 0\).

**Proof.** We begin by investigating the error dynamics under \((\ref{eq:3})\). By \((\ref{eq:3})\) it follows that \(e(k)\) evolves for all \(k \in \mathbb{Z}_{\geq 0}\), despite re-optimizing \(v_{1|k}, z_{1|k}\), based on \(u \in \mathcal{U}(k)\) according to \((\ref{eq:3})\) at every time step, see also Figure 2. Therefore \(\Pr(e(k) \in \mathcal{R}_x) \geq p_x\) and \(\Pr(e(k) \in \mathcal{R}_u) \geq p_u\) for all \(k \in \mathbb{Z}_{\geq 0}\) by Assumption \((\ref{eq:5})\).

Next, Assumption \((\ref{eq:4})\) provides recursive feasibility of optimization problem \((\ref{eq:3})\), i.e. if there exists a feasible solution at time \(k\), there will always exist one at \(k+1\), specifically \(\{v_{1|k}, \ldots, v_{N-1|k}, f(z_{N|k})\}\) is a feasible solution, which implies feasibility of \((\ref{eq:3})\) for all \(k \geq 0\) by induction.

Finally, by recursive feasibility it follows that \(z(k) \in \mathbb{X} \cap \mathcal{R}_x\) and \(v(k) \in \mathbb{U} \cap K \mathcal{R}_u\) for all \(k \in \mathbb{Z}_{\geq 0}\), implying in combination with \(\Pr(e(k) \in \mathcal{R}_x) \geq p_x\) and \(\Pr(e(k) \in \mathcal{R}_u) \geq p_u\) for all \(k \in \mathbb{Z}_{\geq 0}\) that \(\Pr\{x(k) = z(k) + e(k) \in \mathbb{X}\} \geq p_x\) and \(\Pr\{v(k) = v(k) + K \mathcal{R}e(k) \in \mathbb{U}\} \geq p_u\) for all \(k \in \mathbb{Z}_{\geq 0}\).

We therefore proved, that if \((\ref{eq:3})\) is feasible for \(z_{0|0} = x(0)\), \((\ref{eq:3})\) will always provide a control input such that constraints \((\ref{eq:5})\) are satisfied.

**Remark IV.6** (Recursive feasibility despite unbounded disturbances). Various recent stochastic model predictive control approaches, which consider chance-constraints in the presence of unbounded additive noise, are also based on constraint tightening (see \((\ref{eq:5}), (\ref{eq:4}), (\ref{eq:6}), (\ref{eq:7})\)). The techniques employ a recovery mechanism in order to account for infeasibility of the online MPC problem. In contrast, the proposed formulation offers inherent recursive feasibility, even for unbounded disturbance realizations \(w_s(k)\), by simulating the nominal system and therefore not optimizing over \(z_0\). While optimization over \(z_0\) usually enables state feedback in tube-based stochastic model predictive control methods, we incorporate state feedback through the cost function of \((\ref{eq:3})\). A similar strategy can also be used for recursively feasible stochastic MPC schemes as presented in \((\ref{eq:42})\).

**V. Data based design**

In order to employ the PMPSC scheme the following components must be provided: The model description \((\ref{eq:1})\) of the true system \((\ref{eq:1})\) (Section \((\ref{sec:mod})\), the probabilistic error tubes \(\mathcal{R}_x, \mathcal{R}_u\) based on the model \((\ref{eq:1})\) according to Assumption \((\ref{eq:5})\) (Section \((\ref{sec:mod})\), and the nominal terminal set \(\mathbb{Z}_f\), which provides recursive feasibility according to Assumption \((\ref{eq:4})\) (Section \((\ref{sec:mod})\)). In this section, we present efficient techniques for computing those components that are tailored to the learning-based control setup by solely relying on available data collected from the system.

**A. Model design**

For simplicity, we focus on systems that can be approximated by linear Bayesian regression \((\ref{eq:43}), (\ref{eq:44})\) of the form

\[
x(k+1) = \theta^T(z) + w_s(k)
\]

with unknown parameter matrix \(\theta \in \mathbb{R}^{n \times n+m}\), which is inferred from noisy measurements \(\mathcal{D} = \{(x_i, u_i), y_k\}_{k=1}^{N} = 1\) with \(y_k = f(x(k), u(k)) + w_s(k), w_s(k) \sim \mathcal{Q}_{w_s}\), using a prior distribution \(\mathcal{Q}_\theta\) on the parameters \(\theta\). Note that distribution pairs \(\mathcal{Q}_{w_s}\) and \(\mathcal{Q}_\theta\) that allow for efficient posterior computation, e.g. Gaussian distributions, usually exhibit infinite support, i.e. \(w_s(k) \in \mathbb{R}^n\), which can generally not be treated robustly using, e.g. the related method presented in \((\ref{eq:12})\). In the following, we present one way of obtaining the required model error set \(\mathcal{W}_m\) using confidence sets of the posterior distribution \(\mathcal{Q}_\theta|\mathcal{D}\).
We start by describing the set of all realizations of (11), which contain the true system with probability \( p_m \). To this end, let the confidence region at probability level \( p_m \) of the random vector \( \theta \sim Q_{\theta|D} \), denoted by \( E_{p_m}(Q_{\theta|D}) \), be defined such that
\[
\Pr(\theta \in E_{p_m}(Q_{\theta|D})) \geq p_m
\]
and compare the corresponding set of system dynamics against the expected system dynamics, which is in the considered case given by
\[
E(\theta)^\top \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} = 1[A,B].
\]
(13)

Note that the model error between (11) and (13) is unbounded by definition, if we consider an unbounded domain as required by (3) since \( \lim_{\|x,u\| \to \infty} \|((A,B) - \tilde{\theta}^\top)(x,u)\| = \infty \) for any \( \theta \in E_{p_m}(Q_{\theta|D}) \) such that \( \tilde{\theta} \neq \theta \). We therefore make the practical assumption, that the model error is bounded outside a sufficiently large ‘outer’ state and input space \( X_o \times U_o \supseteq X \times U \), as illustrated in Figure 3 which relates to Assumption III.1 as follows.

**Assumption V.1 (Bounded model error).** The set
\[
\mathcal{W}_m := \{ w_m \in \mathbb{R}^n | \forall (x,u,\theta) \in X_o \times U_o \times E_{p_m}(Q_{\theta|D}) \}
\]
\[
Ax + Bu + w_m = \tilde{\theta}^\top(z)
\]
is an overbound of \( \mathcal{W}_m \) according to Assumption III.1, i.e.
\[
\mathcal{W}_m \subseteq \mathcal{W}_m.
\]

A simple but efficient computation scheme for overapproximating \( \mathcal{W}_m \) using \( E_{p_m}(Q_{\theta|D}) \) can be developed for the special case of a Gaussian prior distribution \( \text{col}_i(\tilde{\theta}) \sim N(0,\Sigma^0) \) and Gaussian distributed process noise \( w_{s}(k) \sim N(0,\Sigma^2) \). We begin with the posterior distribution \( Q_{\theta|D} \) of \( \theta \) conditioned on data \( D \) given by
\[
p(\text{col}_i(\theta)|D) = N(\sigma_s^{-1}C_i^{-1}X\text{col}_i(y),C_i^{-1})
\]
(14)

where \( \text{row}_i(X) = \phi(x_i,u_i)^\top \), \( \text{row}_i(y) = y_i^\top \), and \( C_i = \sigma_s^{-1}XX^\top + (\Sigma^0)^{-1} \), see e.g. [45, 43].

Using the posterior distribution \( Q_{\theta|D} \) according to (14) we compute a polytopic overapproximation of \( E_{p_m}(Q_{\theta|D}) \) in a second step, which can be used in order to finally obtain an approximation of \( \mathcal{W}_m \) and therefore \( \mathcal{W}_m \) since \( \mathcal{W} \subseteq \mathcal{W}_m \) by Assumption V.1. To this end, consider the vectorized model parameters \( \text{vec}(\theta) \) and their confidence set \( E_{p_m}(Q_{\text{vec}(\theta)}) = \{ \text{vec}(\theta) \in \mathbb{R}^{n^2+mn} | \text{vec}(\theta)^\top C_{\text{vec}(\theta)} \leq \chi^2_{n^2+mn}(p_m) \} \), where \( \chi^2_{n^2+mn}(p_m) \) is the chi-squared distribution of degree \( n^2 + mn \) and
\[
C := \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_n \end{pmatrix}
\]
is the posterior covariance according to (14). A computationally cheap outer approximation of \( E_{p_m}(Q_{\text{vec}(\theta)}) \) can be obtained by picking its major axes \( \{ \tilde{\theta}_i \}_{i=1}^{n^2+mn} \) using singular value decomposition of \( C \), which provide the vertices of an inner polytopic approximation \( \text{co}(\{ \tilde{\theta}_i \}_{i=1}^{n^2+mn}) \) of \( E_{p_m}(Q_{\text{vec}(\theta)}) \). Scaling this inner approximation by \( \sqrt{n^2+mn} \) yields vertices of an outer polytopic approximation of \( E_{p_m}(Q_{\text{vec}(\theta)}) \) given by the convex hull \( \text{co}(\{ \tilde{\theta}_i \}_{i=1}^{n^2+mn}) \) with \( \tilde{\theta}_i = \sqrt{n^2+mn}\theta_i \).

Based on this outer approximation of \( E_{p_m}(Q_{\text{vec}(\theta)}) \), it is possible to compute a corresponding outer approximation of \( \mathcal{W}_m \) as follows. Due to convexity of \( \text{co}(\{ \tilde{\theta}_i \}_{i=1}^{n^2+mn}) \), it is sufficient to impose
\[
\theta_i^\top(x,u) - (Ax + Bu) \in \mathcal{W}_m
\]
(15)

for all \( i \in I_{1,n^2+mn} \), \( x \in X_o \), \( u \in U_o \), since by definition of \( E_{p_m}(Q_{\text{vec}(\theta)}) \) and Assumption V.1 we have with probability at least \( p_m \) that there exist \( \lambda_i(x,u) \geq 0 \) with \( \sum_{i=1}^{n^2+mn} \lambda_i(x,u) = 1 \) such that
\[
f(x,u) = \tilde{\theta}_i^\top(z)
\]
\[
= \sum_{i=1}^{n^2+mn} \lambda_i(x,u) \tilde{\theta}_i^\top(z)
\]
\[
\leq \sum_{i=1}^{n^2+mn} \lambda_i(x,u) (Ax + Bu + \tilde{\mathcal{W}}_m)
\]
\[
\in \{ Ax + Bu \} \oplus \tilde{\mathcal{W}}_m.
\]

Therefore, (15) can be used in order to construct an outer approximation \( \mathcal{W}_m = \{ w \in \mathbb{R}^n | \|w\|_2 \leq w_{\max} \} \), where
\[
w_{\max} := \max_{i \in I_{1,n^2+mn}} \left( \max_{x \in X_o,u \in U_o} \left\| (\theta_i^\top - (A B)) (x,u) \right\|_2 \right)
\]

with \( (A B) := E(\theta)^\top, \{ \tilde{\theta}_i \}_{i=1}^{n^2+mn} = \sqrt{2} \{ \tilde{\theta}_i \}_{i=1}^{n^2+mn} \) and \( \tilde{\theta}_i \), the major axes of \( E_{p_m}(Q_{\text{vec}(\theta)}) \).

**Remark V.2 (Gaussian processes).** Instead of linear Bayesian regression, one could also use Gaussian process regression by assuming that the system dynamics (11) has bounded norm in a reproducing kernel Hilbert space. The model error set \( \mathcal{W}_m \) can then be derived using the bound presented in [47, Theorem 2]. By using, e.g., the nominal linear model (2) as prior mean function and a stationary kernel function, the uncertainty tends to a constant value around the prior mean function (2) for values of \( x \) outside of \( X_o \), meaning that \( X_o \) would have to be chosen according to the length-scale.
B. Calculation of \( \mathcal{R} \) for uncertain linear dynamics and unbounded disturbances

In this subsection, we provide a method to compute a PRS set \( \mathcal{R} \) with pre-specified probability level \( p \) according to Assumption [V.3] that can be used for obtaining both, a PRS \( \mathcal{R}_x \) at probability level \( p_x \) corresponding to the state constraints \( \mathcal{E}_x \), and a PRS \( \mathcal{R}_u \) at probability level \( p_u \) for input constraints, respectively. As commonly done in the context of related MPC methods, the computations are based on choosing a stabilizing tube controller \( K_{\mathcal{R}} \) in \( \mathcal{E}_x \) first, e.g. using LQR design, in order to efficiently compute the PRS \( \mathcal{R}_x \) and \( \mathcal{R}_u \) afterwards.

The proposed PRS computation distinguishes between an error resulting from model mismatch between (1) and (2), and an error caused by possibly unbounded process noise due to \( w_s(k) \sim Q_{w_s} \). The error system \( \mathcal{E}_x \) admits a decomposition \( e(k) = e_m(k) + e_s(k) \) with \( e(0) = e_m(0) = e_s(0) = 0 \) and
\[

e_m(k) = (A + BK_{\mathcal{R}})e_m(k) + w_m(k), \quad e_s(k) = (A + BK_{\mathcal{R}})e_s(k) + w_s(k).
\]

While the fact that \( Q_{w_s} \) is known allows us to compute a PRS with respect to \( e_s(k) \) as described in [V-B b), the model error \( e_m(k) \) is state and input dependent with unknown distribution. Therefore, we bound \( e_m \) robustly in probability using the concept of robust invariance, i.e., a robustly positive invariant set accounts deterministically for all possible model mismatches \( w_m(k) \in W_m \) at probability level \( p_m \) according to the following definition.

**Definition V.3.** A set \( \mathcal{E}_s \) is said to be a robustly positive invariant set (RIS) for system (17) if \( e_m(0) \in \mathcal{E} \) implies that \( e_m(k) \in \mathcal{E} \) for all \( k \in I_{\geq 0} \). It is called a RIS at probability level \( p_m \) if \( \Pr(\mathcal{E} \text{ is RIS}) \geq p_m \).

This enables us to state the following lemma for the cumulated error \( e(k) \) according to (17).

**Lemma V.4.** If \( \mathcal{E} \) is a RIS at probability level \( p_m \) for the model error system (17) and \( \mathcal{R}_s \) is a PRS for the disturbance error system (18) at probability level \( p_u \), then \( \mathcal{R} = \mathcal{E} \oplus \mathcal{R}_s \) is a PRS for the cumulated error system (19) at probability level \( p_m p_s \).

**Proof.** By the definition of the Minkowski sum, \( e_m(k) \in \mathcal{E} \) and \( e_s(k) \in \mathcal{R}_s \) implies \( e(k) \in \mathcal{R} \) for any \( k \in I_{\geq 0} \). Choosing \( e_s(0) = e_m(0) = 0 \) yields for all \( k \in I_{\geq 0} \)
\[
\Pr(e(k) \in \mathcal{R}) \geq \Pr(e_m(k) \in \mathcal{E} \wedge e_s(k) \in \mathcal{R}_s) = \Pr(e_m(k) \in \mathcal{E}) \Pr(e_s(k) \in \mathcal{R}_s) \geq p_m p_s,
\]
which proves the result.

**Proof.**
\[
\Pr(e(k) \in \mathcal{R}) \geq \Pr(e_m(k) \in \mathcal{E} \wedge e_s(k) \in \mathcal{R}_s) = \Pr(e_m(k) \in \mathcal{E}) \Pr(e_s(k) \in \mathcal{R}_s) \geq p_m p_s,
\]

which proves the result.

Lemma V.4 allows for computing the PRS \( \mathcal{R}_s \), that accounts for the stochastic disturbances (18) independently of the RIS \( \mathcal{E} \), dealing with the model uncertainty \( W_m \). In the following we present one option for determining \( \mathcal{R}_s \) and refer to [30] for further computation methods. Then, based on the model obtained from Section V-A, an optimization problem for the synthesis of \( \mathcal{E} \) is given.

a) PRS \( \mathcal{R}_s \) for stochastic errors: Using the variance \( \text{var}(Q_{w_s}) \) and the Chebyshev bound, a popular way in order to compute \( \mathcal{R}_s \) is given by solving the Lyapunov equation
\[
A_d^T \Sigma_\infty A_d - \Sigma_\infty = -\text{var}(Q_{w_s}) \quad \text{for} \quad \Sigma_\infty,
\]
yielding \( \mathcal{R}_s = \{ e_s \in \Re^n | e_s^T \Sigma_\infty e_s \leq \tilde{p} \} \) with probability level
\[
p = 1 - n_x/\tilde{p}, \quad \text{see e.g. [30]}
\]
Furthermore, if \( Q_{w_s} \) is a normal distribution, \( \mathcal{R}_s \) with \( \tilde{p} = \chi^2_0(p) \) is a PRS of probability level \( p \), where \( \chi^2_0(p) \) is the quantile function of the \( n \)-dimensional chi-squared distribution.

b) RIS \( \mathcal{E} \) for ellipsoidal model errors: Given a bound on the model error according to Assumption III.1 of the form
\[
W_m = \{ w \in \Re^n | w^T Q^{-1} w \leq 1 \} \quad \text{with} \quad Q^{-1} := I_n w_{\max}^2 \quad \text{using} \quad [19],
\]
we can make use of methods from robust control [48] in order to construct a possibly small set \( \mathcal{E}_s = \{ e | e^T P e \leq \alpha \} \) at probability level \( p_m \) by solving
\[
\max_{\alpha^{-1}} \quad \alpha^{-1}
\]
s.t.:
\[
\begin{pmatrix}
A_d^T P A_d - \tau_0 P & A_d^T P - \tau_1 Q^{-1} \\
P A_d & P - \tau_1 Q^{-1}
\end{pmatrix} \preceq 0
\]
\[
1 - \tau_0 - \tilde{p} \tau_1 \alpha^{-1} \geq 0,
\]
\[
\tau_0, \tau_1 > 0,
\]
where \( P \in S_{++}^n \) has to be pre-selected using, e.g., the infinite horizon LQR cost \( x(k)^T P x(k) \), corresponding to the LQR feedback \( u(k) = K_{\mathcal{R}} x(k) \). As pointed out in [49], optimization problem [19] has a monotonicity property in the bilinearity \( \tau_1 \alpha^{-1} \) such that it can be efficiently solved using bisection on the variable \( \alpha^{-1} \). A more advanced design procedure, yielding less conservative robust invariant sets, can be found, e.g., in [50].

In summary, based on the uncertainty of the linear model with respect to the true model (1) inferred from data and by solving (17), we obtain a RIS \( \mathcal{E}_o \) at probability level \( p_m \), which contains the model error (17). Together with the PRS \( \mathcal{R}_s \) from Section V-B a), Lemma V.4 provides the overall PRS for the error system (9), which is given by \( \mathcal{R} = \mathcal{E}_o \oplus \mathcal{R}_s \) at probability level \( p_m p_s \). Note that the ratio between \( p_m \) and \( p_s \) can be freely chosen in order to obtain overall tubes \( \mathcal{R}_x, \mathcal{R}_u \) at probability levels \( p_x = p_s^{\mathcal{R}_x} p_m^s \) and \( p_u = p_s^{\mathcal{R}_u} p_m^u \) according to the chance constraints (4).

C. Iterative construction of the terminal safe set \( \mathcal{Z}_f \)

While the terminal constraint (6d) in combination with Assumption [V.4] is key in order to provide a safe backup control policy \( \pi_2 \) for all future times, it can restrict the feasible set of (6). The goal is therefore to provide a large terminal set \( \mathcal{Z}_f \) yielding potentially less conservative modifications of the proposed learning-based control input \( u_c \) according to (6).

This can be iteratively achieved by recycling previously calculated solutions to (6), starting from a potentially conservative initial terminal set \( \mathcal{Z}_f \) according to Assumption [V.4]. Such an initialization can be computed using standard invariant set methods for linear systems, see e.g. [51] and references therein. Note, that the underlying idea of iteratively enlarging the terminal set is related to the concepts presented, e.g., in [52, 53].
Let the set of nominal initial states obtained from successfully solved instances of (5) be denoted by \( z^*(k) = \{ z^*_n(x(i)), i \in \mathcal{I}_{[1, k]} \} \).

**Proposition V.5.** If Assumption IV.4 holds for \( \mathbb{Z}_f \) and (6) is convex, then the set \( \mathbb{Z}_f^0 := co(z^*(k)) \cup \mathbb{Z}_f \) satisfies Assumption IV.4.

**Proof.** We proceed similar to the proof of [12] Theorem IV.2. Let \( z \in \mathbb{Z}_f^0 \) and note, that if (6) convex, then the feasible set is a convex set, see e.g. [45], and therefore \( co(z^*(k)) \) is a subset of the feasible set. From here, it follows together with the fact that the system dynamics is linear, that there exist multipliers \( \lambda_i \geq 0 \), \( \Sigma_{i=1}^N \lambda_i = 1 \) such that we have \( z = \Sigma_{i=1}^N \lambda_i z^*_n(x(i)) \) as well as corresponding state and input trajectories \( \{ \Sigma_{i=1}^N \lambda_i z^*_n(x(i)) \} \) and \( \{ \Sigma_{i=1}^N \lambda_i v^*_n(x(i)) \} \), \( \Sigma_{i=1}^N \lambda_i \delta^*_n(x(i)) \) that satisfy state, input, and terminal constraints due to the convexity of these sets. We can therefore explicitly state

\[
\kappa_f(z(k), z(k)) = \begin{cases} \Sigma_{i=1}^N \lambda_i v^*_n(x(i)), & \text{if } k \in \mathcal{I}_{[0, N-1]}, \\
\kappa_f(z), & \text{else}
\end{cases}
\]

as the required nominal terminal control law according to Assumption IV.4. Noting that \( co(z^*(k)) \subseteq X \oplus \mathcal{R}_x \) and \( v^*_n \subseteq \mathbb{U} \oplus K_R \mathcal{R}_u \) by (60), (66) shows that for all \( z \in \mathbb{Z}_f^0 \) there exists a control law \( \kappa_f \) according to Assumption IV.4 which completes the proof.

**D. Overall MPSC design procedure**

Given the methods presented in this section, the MPSC problem synthesis from data can be summarized as follows.

- **Step 1:** Construct the model representation (2) based on measurements as described in Section V.A
- **Step 2:** Compute the corresponding PRS \( \mathcal{R}_{x}, \mathcal{R}_{u} \) according to Section V.B
- **Step 3:** Initialize \( \mathbb{Z}_f = \{0\} \) (principled ways in order to calculate less restrictive \( \mathbb{Z}_f \) can be found for example in [51]) and collect the nominal trajectories during closed-loop operation to iteratively enlarge \( \mathbb{Z}_f \) according to (21).
- **Step 4:** Collected state measurements can in addition be used to reduce model uncertainty, allowing tighter bounds on \( w_m \) and recomputation of \( \mathcal{R}_{x}, \mathcal{R}_{u} \), which enables greater exploration of the system in the future.

**VI. NUMERICAL EXAMPLE: SAFELY LEARNING TO CONTROL A CAR**

In this section, we apply the proposed PMPSC scheme in order to safely learn how to drive a simulated autonomous car along a desired trajectory without leaving a narrow road. For the car simulation we consider the dynamics

\[
\begin{align*}
\dot{x} &= v \cos(\theta) \\
\dot{y} &= v \sin(\theta) \\
\dot{\theta} &= (v/L) \tan(\delta) \frac{1}{1+(u/v_{\text{CH}})} \\
\dot{\delta} &= (1/T_s)(u_\delta - \delta) \\
\dot{v} &= a \\
\dot{a} &= (1/T_a)(u_a - a),
\end{align*}
\]

with position \( (x, y) \) in world coordinates, orientation \( \theta \), velocity \( v \), acceleration \( a \), and steering angle \( \delta \), where the acceleration rate is modeled by a first-order lag with respect to the desired acceleration (system input) \( u_a \), and the angular velocity of the steering angle is also modeled by a first-order lag w.r.t. the desired steering angle (system input) \( u_\delta \). The system is subject to the state and input constraints \( |\delta| \leq 0.7 \, \text{[rad]} \), \( |v| \leq 19.8 \, \text{[m s}^{-1}] \), \( -6 \leq a \leq 2 \, \text{[m s}^{-2}] \), \( |u_\delta| \leq 0.7 \, \text{[rad]} \), and \( -6 \leq u_a \leq 2 \, \text{[m s}^{-2}] \), for which the true car dynamics can be approximately represented by (21), e.g. [54], with parameters \( T_\delta = 0.08 \, [\text{s}] \), \( T_a = 0.3 \, [\text{s}] \), \( L = 2.9 \, [\text{m}] \), and \( v_{\text{CH}} = 20 \, [\text{m s}^{-2}] \). The system is discretized with a sampling time of 0.1 [s].

The learning task is to find a control law, that tracks a periodic reference trajectory on a narrow road, which translates in an additional safety constraint \( |y| \leq 1 \). The terminal set according to Assumption IV.4 is defined as the road center with angles \( \theta = \delta = 0 \) and acceleration \( a = 0 \), which is a safe set for (21) with \( \kappa_f = 0 \). The planning horizon is selected to \( N = 30 \) and the model (2) as well as the PRS set \( \mathcal{R}_x = \mathcal{R}_u \) is updated using a probabilistic model predictive certification framework with a probability level 96% is computed based on a 40 second state and input trajectory according to Section V. D.

**VII. CONCLUSION**

This paper has introduced a methodology to enhance arbitrary RL algorithms with safety guarantees during the process of learning. The scheme is based on a data-driven, linear belief approximation of the system dynamics, that is used in order to compute safety policies for the learning-based controller 'on-the-fly'. By proving existence of a safety policy policy at
APPENDIX

A. Details of numerical example

The model is computed according to Section V-A based on measurements of system (21) as depicted in Figure 6 sensor noise $\sigma_s = 0.01$ and prior distribution $\Sigma_x^0 = 10I_n$. The state feedback

$$K_R = \begin{pmatrix} -1.25 & -0.05 & -2.34 & -0.75 & -0.19 & 0.02 \\ 0.02 & -0.69 & -0.03 & 0.02 & -5.04 & -2.85 \end{pmatrix}$$

according to Assumption IV.3 is computed according to the mean dynamics of (13) using LQR design.

Applying the procedure described in Section V-B yields the PRS set $R = \{ x \in \mathbb{R}^6 | x^\top P x \leq 1 \}$ with

$$P = \begin{pmatrix} 11.62 & -0.06 & 12.34 & 4.42 & 0.14 & 0.10 \\ -0.06 & 10.54 & 0.29 & 0.03 & 30.176 & 6.74 \\ 12.34 & 0.29 & 51.44 & 12.20 & 2.29 & 0.67 \\ 4.42 & 0.03 & 12.20 & 15.60 & 0.24 & 0.08 \\ 0.14 & 30.17 & 2.29 & 0.24 & 188.51 & 46.48 \\ 0.10 & 6.74 & 0.67 & 0.08 & 46.48 & 27.69 \end{pmatrix}$$

and tightened input and state constraints $||u_d|| \leq 0.75$ [rad], $-5.4 \leq u_a \leq 1.3$ [m s$^{-2}$], $||y|| \leq 0.58$ [m], $||f|| \leq 0.45$ [rad], $||v|| \leq 19.8$ [m s$^{-1}$], $-4.7 \leq a \leq 0.21$ [m s$^{-2}$], computed using the MPT-Toolbox [56].

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