ESSENTIAL MANIFOLDS WITH EXTRA STRUCTURES

SERGII KUTSAK

Abstract. We consider classes of algebraic manifolds \( \mathcal{A} \), of symplectic manifolds \( \mathcal{S} \), of symplectic manifolds with the hard Lefschetz property \( \mathcal{HS} \) and the class of cohomologically symplectic manifolds \( \mathcal{CS} \). For every class of manifolds \( \mathcal{C} \) we denote by \( \mathcal{EC}(\pi, n) \) a subclass of \( n \)-dimensional rationally essential manifolds with fundamental group \( \pi \). In this paper we prove that for all the above classes with symplectically aspherical form the condition \( \mathcal{EC}(\pi, 2n) \neq \emptyset \) implies that \( \mathcal{EC}(\pi, 2n-2) \neq \emptyset \) for every \( n > 2 \). Also we prove that all the inclusions \( \mathcal{EA} \subset \mathcal{EH}\mathcal{S} \subset \mathcal{ES} \subset \mathcal{EC}\mathcal{S} \) are proper.

1. Introduction

Let \( M \) be a closed, connected, orientable manifold of dimension \( n \) and let \( \pi \) be the fundamental group of \( M \). Recall that a map \( f : M \to K(\pi, 1) \) is called a classifying map for \( M \) if \( f \) induces an isomorphism \( f_* : \pi_1(M, x_0) \to \pi_1(K(\pi, 1), f(x_0)) \) for all \( x_0 \in M \). It is well-known that a classifying map exists and is unique up to homotopy. Gromov called a manifold \( M \) inessential if there exists a classifying map \( f : M \to K(\pi, 1) \) to the \((n-1)\)-skeleton of \( K(\pi, 1) \). Otherwise \( M \) is called essential [Gr1]. Gromov noticed that manifolds with positive scalar curvature tend to be inessential. He introduced several classes of essential manifolds (hyperspherical, hypereuclidean, enlargeable, [Gr2]) for which he jointly with Lawson proved that manifolds of those classes cannot carry a metric with positive scalar curvature [GL]. The following is found in reference [DR, Lemma 2.4].

1.1. Proposition. An orientable \( n \)-manifold \( M \) is essential if and only if the homomorphism \( f_* : H_n(M) \to H_n(K(\pi, 1)) \) induced by the classifying map \( f \) is nontrivial. Equivalently, if the image of the fundamental class \([M] \in H_n(M)\) under \( f_* \) is nontrivial in the \( n \)th integral homology group \( H_n(K(\pi, 1)) \) of the Eilenberg-MacLane space \( K(\pi, 1) \).
For example, the real projective space $\mathbb{R}P^{2n+1}$ is an essential manifold. Every aspherical manifold (for example, the torus $T^n$, a compact orientable surface $M_g$ of genus $g$) is essential. There are no simply connected essential manifolds of positive dimension.

1.2. Definition. Let $M$ be a closed, connected, orientable manifold of dimension $n$ and let $\pi$ be the fundamental group of $M$. We say that manifold $M$ is rationally essential if a classifying map $f : M \to K(\pi, 1)$ induces nontrivial homomorphism $f_* : H_n(M; \mathbb{Q}) \to H_n(K(\pi, 1); \mathbb{Q})$.

Clearly, every rationally essential manifold is essential but not vice versa: $\mathbb{R}P^{2n+1}$ is not rationally essential.

Clearly, if $H_n(K(\pi, 1)) = 0$ then there are no essential (and hence rationally essential) $n$-manifolds with the fundamental group $\pi$. The converse also holds: if $\pi$ is a finitely presented group and $H_n(\pi; \mathbb{Q}) \neq 0$ then there exists a rationally essential $n$-manifold with the fundamental group $\pi$, see Theorem 3.1 below.

Brunnbauer and Hanke gave a characterization of Gromov type classes of rationally essential manifolds with given fundamental group in terms of group homology [BH]. In this paper we consider similar problem for some symplectic type classes.

Given a class of manifolds $C$ we denote by $\mathcal{E}C$ the subclass that consists of rationally essential manifolds. Here we consider the following classes:

$$\mathcal{A} \subset \mathcal{HS} \subset \mathcal{S} \subset \mathcal{CS}$$

where $\mathcal{A}$ is the class of algebraic manifolds, $\mathcal{S}$ is the class of symplectic manifolds, $\mathcal{HS}$ is the class of symplectic manifolds with the hard Lefschetz property, and $\mathcal{CS}$ is the class of cohomologically symplectic manifolds (see sections 3 and 4 below). It is known that all the above inclusions of classes are proper [C, TO, G1, DGMS]. We will show that the inclusions of the essential counterparts are also proper.

For every class of manifolds $C$ we denote by $\mathcal{C}(\pi, n)$ a subclass of $n$-dimensional manifolds with fundamental group $\pi$. This paper is an attempt to study the following question.

**MAIN PROBLEM.** For which values $\pi$ and $n$, is $\mathcal{EC}(\pi, n)$ non-empty?

In particular, in the paper we address the following conjecture proposed by Dranishnikov and Rudyak:
CONJECTURE. For the first three above classes for $n > 2$ the condition $\mathcal{E}C(\pi, 2n) \neq \emptyset$ implies that $\mathcal{E}C(\pi, 2n - 2) \neq \emptyset$.

We prove for all the above classes a weaker version of the conjecture that deals with symplectically aspherical manifolds, see Section 3 for the definition.

Note that every complex projective algebraic manifold $V$ is symplectic: the corresponding symplectic form is given by the Kähler form $[G_{\text{H}}, \text{page 109}]$. In particular, we are able to speak about symplectically aspherical algebraic manifolds.

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3. Preliminaries

The following fact is known (see for example $[BH]$, $[Dr]$). Since there is no detailed proof of it in print, we give a complete proof here.

3.1. Theorem. For every finitely presented group $\pi$ and every integer $n$ if $H_n(\pi; \mathbb{Q}) \neq 0$ then for every nontrivial element $\alpha \in H_n(\pi; \mathbb{Q})$ there exists a closed, connected, orientable $n$-manifold $M$, an integer $k \neq 0$ and a map $f : M \to K(\pi, 1)$ such that $f_*([M]) = k\alpha$ and $f_* : \pi_1(M) \to \pi_1(K(\pi, 1))$ is a group isomorphism.

Proof. Let $\pi$ be a finitely presented group and let $n$ be an integer such that $H_n(\pi; \mathbb{Q}) \neq 0$. Take any nontrivial element $\alpha$ in $H_n(\pi; \mathbb{Q})$. Because of a theorem of Thom, there exist a closed $n$-manifold $N$, an integer $k \neq 0$ and a map $g : N \to K(\pi, 1)$ such that $g_*([N]) = k\alpha$, see e.g. $[R$, Theorem IV.7.36$]. Suppose that $g_* : \pi_1(N) \to \pi_1(K(\pi, 1))$ is not surjective. Let $\alpha : [0, 1] \to K(\pi, 1)$ be a loop such that $[\alpha] \in \pi_1(K(\pi, 1)) \setminus \text{Im}(g_*)$ and $\alpha(0) = \alpha(1) = y_0$. Without loss of generality we can assume that $y_0 \in \text{Im}(g)$ since the fundamental groups of $K(\pi, 1)$ based at different points are isomorphic because $K(\pi, 1)$ is path connected. Take $x_0 \in N$ such that $f(x_0) = y_0$. Consider chart $(\bar{U}, \varphi)$ on $N$ such that $\varphi(U) = \mathbb{R}^n$ and $\varphi(x_0) = 0$. Now define function $h : \mathbb{R}^n \to \mathbb{R}^n$ in generalized spherical coordinates as follows

$$h(r, \theta_1, ..., \theta_{n-1}) = \begin{cases} 0, & \text{if } 0 \leq r \leq 1, \\ (r - 1, \theta_1, ..., \theta_{n-1}), & \text{if } r > 1. \end{cases}$$

To perform a surgery on a manifold $N$ we shall define a new function $\tilde{g}$ by: $\tilde{g}(x) = g(x)$ if $x \notin U$, $\tilde{g}(x) = g(\varphi^{-1}h\varphi(x))$ if $x \in U$. Then $\tilde{g}$ is
homotopic to $g$ because $h$ is homotopic to the identity map on $\mathbb{R}^n$. Let $D$ be the preimage under $\varphi$ of the unit ball in $\mathbb{R}^n$ centered at 0. Now we perform a surgery on the manifold $N$. There exists an embedding $i : S^0 \times D^n \to N$ such that $i(S^0 \times D^n) \subseteq D$ and $x_0 \notin i(S^0 \times D^n)$. Form a new manifold from the union of $N \times I$ and $D^1 \times D^n$ by attaching $S^0 \times D^n$ to its image under $i \times 1$. We can extend map $\tilde{g} \times 1$ by defining $\tilde{g}$ on $D^1 \times D^n$ as follows

$$\tilde{g}(t, x) = \alpha(t) \text{ for all } (t, x) \in D^1 \times D^{n-1}.$$ 

Connect point $x_0$ with points $(0, c), (1, c)$ in $D^1 \times D^{n-1}$ for some $c \in D^n$ by paths $\gamma_1(t), \gamma_2(t)$ respectively. Let $\beta(t) = (t, c) \in D^1 \times D^n$ for all $t \in [0, 1]$. Then $(\tilde{g} \times 1)_*(\gamma_1 \beta \gamma_2^{-1}) = \alpha$. So we can construct a manifold $\tilde{N}$ and a map $\tilde{g} : \tilde{N} \to K(\pi, 1)$ such that $\tilde{g}_*([\tilde{N}]) = k\alpha$ and $\tilde{g}_*$ induces an epimorphism on fundamental groups. Now we want to perform surgeries that annihilate the elements that generate the kernel of $\tilde{g}_*$. Note that since $\tilde{N}$ is orientable then every loop $\gamma$ in $\tilde{N}$ can be homotoped to a loop $\tilde{\gamma}$ that has trivial normal bundle in $\tilde{N}$. Clearly, if a loop $\tilde{\gamma}$ is trivial then the loop $\alpha \tilde{\gamma} \alpha^{-1}$ is also trivial for every path $\alpha : [0, 1] \to \tilde{N}$ such that $\alpha(1) = \tilde{\gamma}(0)$. Since $\text{Ker}(\tilde{g}_*)$ is normally finitely generated [MW] then we can perform surgery on $\tilde{N}$ finitely many times to construct a manifold $M$ and a map $f : M \to K(\pi, 1)$ that induces isomorphism $f_* : \pi_1(M) \to \pi_1(K(\pi, 1))$ and such that $f_*([M]) = k\alpha$.

Note that every oriented manifold of dimension $\leq 2$ is essential, an oriented 3-manifold $M$ is essential iff the group $\pi_1(M)$ is not free, [CG, RO].

3.2. Definition. We define a cohomology class $v \in H^m(X; G)$ to be aspherical if $v = f^*(a)$ for a classifying map $f : X \to K(\pi_1(X), 1)$ and some $a \in H^m(K(\pi_1(X), 1); G)$.

Note that if a class $v$ is aspherical and $v^k \neq 0$ then $v^k$ is aspherical.

3.3. Proposition. Let $M$ be a closed, orientable manifold of dimension $km$, and let $u \in H^m(M; \mathbb{Q})$ be an aspherical class. If $u^k \neq 0$, then $M$ is rationally essential.

3.4. Definition. A symplectic structure on a smooth manifold $M$ is a non-degenerate skew-symmetric closed 2-form $\omega \in \Omega^2(M)$. A symplectic manifold is a pair $(M, \omega)$ where $M$ is a smooth manifold and $\omega$ is a symplectic structure on $M$.

The non-degeneracy condition means that for all $p \in M$ we have the property that if $\omega(v, w) = 0$ for all $w \in T_p M$ then $v = 0$. The
skew-symmetry condition means that for all \( p \in M \) we have \( \omega(v, w) = -\omega(w, v) \) for all \( v, w \in T_p M \). The closed condition means that the exterior derivative \( d \omega \) of \( \omega \) is identically zero. Since each odd-dimensional skew-symmetric matrix is singular, we see that \( M \) has even dimension. Every symplectic \( 2n \)-dimensional manifold \((M, \omega)\) is orientable since the \( n \)-fold wedge product \( \omega \wedge \ldots \wedge \omega \) never vanishes.

3.5. **Definition.** A symplectic manifold \((M, \omega)\) is *symplectically aspherical* if

\[
\int_{S^2} f^* \omega = 0
\]

for every smooth map \( f : S^2 \to M \).

Clearly, if \( \pi_2(M) = 0 \) then a symplectic manifold \((M, \omega)\) is symplectically aspherical. However there are symplectically aspherical manifolds with nontrivial \( \pi_2 \), [G2] [IKRT].

3.6. **Remark.** The cohomology class \([\omega]\) in a symplectically aspherical manifold \((M, \omega)\) is aspherical. It follows from classical results of Hopf, [H] (see also [CLOT, Theorem 8.17], [B, Theorem 5.2]).

In view of this remark and Proposition 3.3 we have the following corollary

3.7. **Corollary.** Every closed symplectically aspherical manifold is rationally essential.

To proceed, we need the following theorems, see e.g [M, page 41].

3.8. **Theorem** (Lefschetz Hyperplane Theorem). Let \( V \) be a complex projective algebraic variety of complex dimension \( k \) which lies in the complex projective space \( \mathbb{C}P^n \), and let \( P \) be a hyperplane in \( \mathbb{C}P^n \) which contains the singular points (if any) of \( V \). Then the relative homotopy groups \( \pi_r(V, V \cap P) \) are equal to zero for all \( r < k \).

Note that \( V \cap P \) is a manifold (i.e. non-singular variety) if \( V \) is.

3.9. **Theorem** (Donaldson [D]). Let \( L \to V \) be a complex line bundle over a compact symplectic manifold \((V, \omega)\) with compatible almost-complex structure, and with the first Chern class \( c_1(L) = \left[ \frac{\omega}{2\pi} \right] \). Then there is a constant \( C \) such that for all large \( k \) there is a section \( s \) of \( L^\otimes k \) with

\[
|\bar{\partial} s| < \frac{C}{\sqrt{k}} |\partial s|
\]

on the zero set of \( s \).
3.10. **Theorem** (Donaldson \[D\]). Let \( W_k \) be the zero-set of a section \( s \) of \( L^{\otimes k} \to V \) satisfying the conditions of Theorem 3.9. When \( k \) is sufficiently large the inclusion \( i : W_k \to V \) induces an isomorphism on homotopy groups \( \pi_p \) for \( p \leq n - 2 \) and a surjection on \( \pi_{n-1} \).

In view of Theorem 3.9 and Theorem 3.10 we obtain the following corollary

3.11. **Corollary.** Let \((M, \omega)\) be a closed symplectic manifold of dimension \(2n\) such that the cohomology class \([\omega]\) is integral. Then there exists a symplectic submanifold \(V\) of \(M\) of codimension 2 such that inclusion \(i : V \to M\) induces an isomorphism on homotopy groups \(\pi_p\) for \(p \leq n - 2\) and a surjection on \(\pi_{n-1}\). Furthermore, the homology class \([V]\) in \(M\) is the Poincaré dual to a class \(r[\omega]\) for some integer \(r\).

**Proof.** The proof follows from Theorem 3.9 and Theorem 3.10 with \(\omega\) normalized such that \(c_1(L) = [\omega]\). Let \(V\) be the zero-set of a section \(s\) of \(L^{\otimes k} \to M\) as in Theorem 3.9. Then inequality (3.1) guarantees the existence of symplectic structure on \(V\). So \(V\) is a symplectic submanifold of \(M\) of codimension 2. The homology class of \(V\) is Poincaré dual to the first Chern class of \(L^{\otimes k}\) up to a multiplicative constant \(r\). Finally, according to Theorem 3.10 the inclusion \(i : V \to M\) induces an isomorphism on homotopy groups \(\pi_p\) for \(p \leq n - 2\) and a surjection on \(\pi_{n-1}\). \(\square\)

4. **Classes of essential manifolds**

4.1. **Theorem.** Assume that \(M\) is a complex projective algebraic manifold of (real) dimension \(2k\) which lies in the complex projective space \(\mathbb{C}P^N\). Suppose also that \(M\) is symplectically aspherical. Then for every integer \(m\) with \(2 \leq m \leq k\) there exists a rationally essential algebraic manifold \(V\) of dimension \(2m\) with fundamental group isomorphic to \(\pi_1(M)\).

**Proof.** The case \(m = k\) is the Corollary 3.7. By induction, it suffices to prove the theorem for \(m = k - 1\). Indeed, assume that \(\dim M = 2k > 4\) and let \(V = M \cap \mathbb{C}P^{N-1}\). If we prove that \(V\) is a rationally essential complex algebraic manifold with \(\dim V = 2k - 2 > 4\) and the fundamental group \(\pi = \pi_1(M)\), we apply the previous argument for \(V\) instead of \(M\). Because of the Theorem 3.8 \(\pi_r(M, V) = 0\) for \(r < k - 1\). From the exactness of the homotopy sequence

\[
\pi_2(M, V) \to \pi_4(V) \to \pi_1(M) \to \pi_1(M, V)
\]

it follows that

\[
\pi_1(M) \simeq \pi_1(V) \simeq \pi\] since \(\pi_2(M, V) \simeq \pi_1(M, V) \simeq 0\).
Hence $V$ is a complex algebraic manifold with fundamental group isomorphic to $\pi$, and $\dim V = \dim M - 2$. It remains to prove that $V$ is rationally essential. But this follows from Corollary 3.7 because the induced Kähler form on $V$ is aspherical. \hfill $\Box$

4.2. Theorem. Let $(M, \omega)$ be a closed symplectically aspherical manifold of dimension $2n > 2$ with fundamental group $\pi$. Then for every $k$ such that $2 \leq k \leq n$ there exists a symplectically aspherical manifold $V$ of dimension $2k$ with fundamental group isomorphic to $\pi$.

Proof. We prove the theorem by induction. Similarly to the proof of Theorem 4.1, it suffices to prove the case $k = n - 1$. Without loss of generality, we can assume that the cohomology class $[\omega]$ is integral (see [IKRT, Prop. 1.5]). Let $M$ be a manifold as in Corollary 3.11. Then, for $n > 2$, the inclusion $i : V \to M$ induces an isomorphism on the fundamental groups $\pi_1(V) \to \pi_1(M)$. Now, $V$ is a symplectic manifold with symplectic structure $i^\ast \omega$ induced from $M$. It is clear that

$$\int_{S^2} g^\ast i^\ast \omega = 0$$

for every map $g : S^2 \to V$. Thus $(V, i^\ast \omega)$ is a symplectically aspherical manifold of dimension $2n - 2$ with $\pi_1(V) = \pi$. \hfill $\Box$

4.3. Definition. A symplectic manifold $(M^{2n}, \omega)$ has the hard Lefschetz property (HLP) if the map

$$L^k_{[\omega]} : H^{n-k}_{DR}(M^{2n}) \to H^{n+k}_{DR}(M^{2n}), \quad L^k_{[\omega]}([x]) = [\omega^k \wedge x]$$

is an isomorphism for all $k = 0, \ldots, n$.

For example, the Hard Lefschetz Theorem says that every Kähler manifold has HLP, see [GH, page 122].

4.4. Theorem. Let $(M, \omega)$ be a symplectically aspherical manifold of dimension $2n > 2$ with fundamental group $\pi$ and having HLP. Then for every $m$ such that $2 \leq m \leq n$ there exists a symplectically aspherical manifold $(V, \eta)$ of dimension $2m$ with fundamental group isomorphic to $\pi$ and having HLP.

Proof. We follow the proof of Theorem 4.2 and must prove that the manifold $V$ as in Theorem 4.2 has HLP. First, we need to show that $L^k_{[\omega]} : H^{n-1-k}(V) \to H^{n-1+k}(V)$ is an isomorphism for all $k = 0, \ldots, n - 1$ where $\omega^\ast$ is the pullback of $\omega$ under inclusion $i : V \to M$. We need to consider separately the case when $k = 0$. So fix any $k$ such that $0 < k \leq n - 1$. Since $H^{n-1-k}(V)$ and
$H^{n-1+k}(V)$ have the same dimension, it suffices to show that $L^k_k$ is a monomorphism. Consider the following commutative diagram

$$
\begin{array}{ccc}
H^{n-1-k}(M) & \xrightarrow{L^k_k} & H^{n-1+k}(M) \\
\downarrow i^*_1 & & \downarrow i^*_2 \\
H^{n-1-k}(V) & \xrightarrow{L^k_k} & H^{n-1+k}(V)
\end{array}
$$

where $L^k_k$ is a monomorphism because $L^k_{k+1}$ is an isomorphism. It follows from Corollary 3.11 and Whitehead theorem (see [S, page 399]) that $i^*_1$ is an isomorphism. Hence it suffices to show that $i^*_2$ is a monomorphism on the $\text{Im}(L^k_k)$. Assume that $\alpha \in H^{n-1-k}(M)$ is nontrivial and $i^*_2(\alpha \smile \omega^k) = 0$. Then

$$
0 \neq r([M] \smile (\alpha \smile \omega^{k+1})) = r([M] \smile (\alpha \smile \omega^k)) \smile \omega \\
= r([M] \smile \omega) \smile (\alpha \smile \omega^k) = i^*_s([V]) \smile (\alpha \smile \omega^k) \\
i^*_s([V] \smile i^*_2(\alpha \smile \omega^k)) = 0.
$$

This is a contradiction. So $L^k_{[\omega^*]}$ is an isomorphism for all $k = 1, \ldots, n-1$.

If $k = 0$ then it is obvious that $L^0_{[\omega^*]} : H^{n-1}(V) \to H^{n-1}(V)$ is an isomorphism. Thus $V$ is a symplectically aspherical manifold of dimension $2n - 2$ with fundamental group $\pi$ having the HLP.

Now we can apply the above procedure to $V$, and the result follows by induction. \qed

4.5. **Definition** (Lupton-Oprea [LO]). A manifold $M$ of dimension $2n$ is **cohomologically symplectic** (or, briefly, c-symplectic) if there exists a closed differential 2-form $\omega$ on $M$ such that $[\omega]^n \neq 0$.

Clearly, not all c-symplectic manifolds are symplectic. For example, $\mathbb{C}P^2 \# \mathbb{C}P^2$ is c-symplectic but is not symplectic [GT].

4.6. **Theorem.** Let $(M, \omega)$ be a c-symplectic manifold of dimension $2n > 2$ with fundamental group $\pi$ and with aspherical c-symplectic form. Then for every $m$ such that $2 \leq m \leq n$ there exists a c-symplectic manifold $(V, \eta)$ of dimension $2m$ with fundamental group isomorphic to $\pi$ and with aspherical c-symplectic form.

**Proof.** Let $f : M \to K(\pi, 1)$ be a classifying map for $M$. Then $\omega = f^* a$ for some $a \in H^2(K(\pi, 1))$. There exists a $(2n - 2)$-dimensional submanifold $N$ of $M$ such that $[N] = r\eta$ for some $r \in \mathbb{Z}$, where $\eta =$
Let \( i : N \to M \) be the inclusion of \( N \) into \( M \).
We want to show that \((i^*\omega)^{n-1} \neq 0\). Suppose that \((i^*\omega)^{n-1} = 0\). Then
\[
0 \neq r([M] \wedge \omega^n) = r([M] \wedge \omega^{n-1}) = i_*([N]) \wedge \omega^{n-1} = i_*([N] \wedge (i^*\omega)^{n-1}) = 0.
\]
This is a contradiction. Hence \((i^*\omega)^{n-1} \neq 0\). By using surgery we can construct a manifold \( N' \) and a map \( i' : N' \to M \) that induces an isomorphism on the fundamental groups. Moreover, there exist a manifold \( W \) with \( \partial W = N \sqcup N' \) and a map \( g : W \to M \) that extends \( i \) and \( i' \). In other words, the singular manifolds \( i : N \to M \) and \( i' : N' \to M \) are bordant:
\[
\begin{array}{ccc}
N & \xrightarrow{j} & W & \xleftarrow{j'} & N' \\
\searrow i & & \downarrow g & \swarrow i' & \\
& & M & & 
\end{array}
\]
where \( j \) and \( j' \) are the inclusions. Thus \( i'_*([N']) = i_*([N]) \). Now
\[
\langle (i^*\omega)^{n-1}, [N'] \rangle = \langle \omega^{n-1}, i'_*([N']) \rangle = \\
= \langle \omega^{n-1}, i_*([N]) \rangle = \langle (i^*\omega)^{n-1}, [N] \rangle \neq 0,
\]
so \((i^*\omega)^{n-1} \neq 0\). Thus \((N', i^*\omega)\) is a c-symplectic manifold of dimension \( 2n - 2 \) with fundamental group isomorphic to \( \pi \). Clearly, \( i^*\omega \) is an aspherical form because \( i^*\omega = (f \circ i')^*a \). The result follows by induction.

**4.7. Proposition.** There is an example of a rationally essential 4-dimensional c-symplectic manifold \( M \) which is not symplectic.

**Proof.** Let \( \Sigma \) be an aspherical 4-dimensional homology sphere (see [RaT]). We consider the connected sum \( M = \mathbb{C}P^2 \# \mathbb{C}P^2 \# \Sigma \) and show that it does not admit an almost complex structure. According to the result of Ehresmann and Wu, a compact 4-manifold \( M \) has an almost complex structure with first Chern class \( c_1 \in H^2(M, \mathbb{Z}) \) if and only if \( c_1 \) reduces modulo 2 to the second Stiefel-Whitney class \( w_2 \) and
\[
c_1^2([M]) = 3\tau + 2\chi,
\]
where \( \chi \) is the Euler characteristic of \( M \) and \( \tau \) is its signature ([MS, page 119]). A routine computation shows that \( \chi = 4, \tau = 2 \) and \( c_1^2([M]) \) is the sum of squares of two integers. But 14 cannot be represented in such form. Hence \( M \) does not admit an almost complex structure and therefore is not a symplectic manifold because every symplectic
manifold admits a compatible almost complex structure. Furthermore, $\Sigma = K(\pi_1(\Sigma), 1)$, and the collapsing map $f : M \to \Sigma$ has degree 1. Thus $M$ is a rationally essential manifold since the homomorphism induced by $f$ on the 4th homology groups $f_* : H_4(M; \mathbb{Q}) \to H_4(\Sigma; \mathbb{Q})$ is nontrivial.

Since $\Sigma$ is a homology sphere, the collapsing map $i : M \to \mathbb{C}P^2 \# \mathbb{C}P^2$ induces the isomorphism

$$i^* : H^2(\mathbb{C}P^2; \mathbb{R}) \oplus H^2(\mathbb{C}P^2; \mathbb{R}) \to H^2(M; \mathbb{R}).$$

Let $\{[\omega_1], [\omega_2]\}$ be a basis of $H^2(\mathbb{C}P^2; \mathbb{R}) \oplus H^2(\mathbb{C}P^2; \mathbb{R})$. Then $i^*([\omega_1] + [\omega_2])^2 \neq 0$ in $H^4(M; \mathbb{R})$. Hence $M$ is a c-symplectic manifold. □

4.8. Remark. Note that the Dranishnikov-Rudyak conjecture is not true for c-symplectic manifolds. Consider a rationally essential c-symplectic manifold $M = \mathbb{C}P^4 \# \mathbb{C}P^4 \# (\Sigma \times \Sigma)$ with fundamental group $\pi_1(M) \cong \pi_1(\Sigma) \times \pi_1(\Sigma)$. Since $\Sigma \times \Sigma$ is the Eilenberg-MacLane space $K(\pi_1(\Sigma), 1)$ and $H_6(S^1 \times \Sigma; \mathbb{Q})$ is trivial then there does not exist a rationally essential 6-manifold with fundamental group isomorphic to $\pi_1(M)$.

4.9. Theorem. All the inclusions of classes

$$\mathcal{E}A \subset \mathcal{E}HS \subset \mathcal{E}S \subset \mathcal{E}CS$$

are proper.

Proof. First we prove that the inclusion $\mathcal{E}A \subset \mathcal{E}HS$ is proper. Let $\mathbb{H}$ be the Heisenberg manifold. Then the blow-up $M$ of $\mathbb{H} \times \mathbb{H}$ along a torus is a symplectic manifold that satisfies the hard Lefschetz property and has nontrivial triple Massey product [C]. Since $\mathbb{H}$ is an aspherical manifold then $\mathbb{H} \times \mathbb{H}$ is the Eilenberg-MacLane space. So $M$ is a rationally essential manifold because there exists a degree 1 (classifying) map $f : M \to \mathbb{H} \times \mathbb{H}$. Note that $M$ is not algebraic since it has non-trivial Massey product, while all Kähler (and therefore algebraic) manifolds are formal spaces, [DGMS], and hence all their Massey products are trivial.

Now we prove that the inclusion $\mathcal{E}HS \subset \mathcal{E}S$ is proper. Consider the Kodaira-Thurston manifold $KT$ obtained by taking the product of the Heisenberg manifold $\mathbb{H}$ and the circle $S^1$. It is well-known that $KT$ is a symplectic manifold. The Kodaira-Thurston manifold is rationally essential because it is a nilmanifold and it can not have the hard Lefschetz property because a symplectic nilmanifold of Lefschetz type is diffeomorphic to a torus [BG].
We have already shown that the inclusion $\mathcal{ES} \subset \mathcal{ECS}$ is proper, see Proposition 4.7 above.

The Dranishnikov-Rudyak conjecture cannot be reduced to the aspherical case in view of the following

4.10. **Proposition.** The blow up of a 4-torus at a single point $M = T^4 \# \mathbb{CP}^2$ is an algebraic manifold which does not admit an aspherical symplectic form.

**Proof.** Let $\omega$ be a symplectic form on $M$. Then $\int_M \omega^2 \neq 0$. We can obtain a form $\omega'$ on $\mathbb{CP}^2$ that extends the restriction of $\omega$ on $\mathbb{CP}^2 \setminus D$ such that $\int_{\mathbb{CP}^2} \omega'^2 \neq 0$ where $D$ is a small enough disk. Then there exists a map $f : S^2 \to \mathbb{CP}^2 \setminus D$ with $\int_{S^2} f^* \omega' \neq 0$ because if we assume that $\int_{S^2} f^* \omega' = 0$ for all maps $f : S^2 \to \mathbb{CP}^2 \setminus D$ then $[\omega'] = 0$ in $H^2(\mathbb{CP}^2; \mathbb{R})$. Therefore $[\omega']^2 = 0$ and $\int_{\mathbb{CP}^2} \omega^2 = 0$ which contradicts to the choice of $\omega'$. Consider $f : S^2 \to \mathbb{CP}^2 \setminus D$ such that $\int_{S^2} f^* \omega' \neq 0$. Since $\omega$ and $\omega'$ coincide on $\mathbb{CP}^2 \setminus D$ then $\int_{S^2} f^* \omega \neq 0$. Thus $\omega$ is not an aspherical symplectic form.

It is natural to consider the class of Kähler manifolds $\mathcal{K}$ and ask whether the inclusions $\mathcal{EA} \subset \mathcal{EK} \subset \mathcal{ES}$ are proper. It is known that inclusions $\mathcal{A} \subset \mathcal{K} \subset \mathcal{S}$ are proper $\textbf{[V],[C]}$ and manifold $M$ in Theorem 4.9 shows that inclusion $\mathcal{EK} \subset \mathcal{ES}$ is also proper. Note that $M$ is not Kähler because it is not formal.

4.11. **Question.** Does there exist an essential Kähler manifold that is not algebraic?

4.12. **Question.** In view of the theorems proved above we may ask whether the Dranishnikov-Rudyak conjecture holds true for the class of Kähler manifolds with aspherical Kähler form.

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Sergii Kutsak, Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, FL 32601, USA

E-mail address: sergiikutsak@ufl.edu