Abstract

We consider a deterministic discrete-time model of fire spread introduced by Hartnell [1995] and the problem of minimizing the number of burnt vertices when deploying a limited number of firefighters per timestep. While only two firefighters per timestep are needed in the two dimensional lattice to contain any outbreak, we prove a conjecture of Wang and Moeller [2002] that $2d - 1$ firefighters per timestep are needed to contain a fire outbreak starting at a single vertex in the $d$-dimensional square lattice for $d \geq 3$; we also prove that in the $d$-dimensional lattice, $d \geq 3$, for each positive integer $f$ there is some outbreak of fire such that $f$ firefighters per timestep are insufficient to contain the outbreak. We prove another conjecture of Wang and Moeller that the proportion of elements in the three-dimensional grid $P_n \times P_n \times P_n$ which can be saved with one firefighter per time step when an outbreak starts at one vertex goes to 0 as $n$ gets large. Finally, we use integer programming to prove results about the minimum number of timesteps needed and minimum number of burnt vertices when containing a fire outbreak in the two dimensional square lattice with two firefighters per timestep.

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1 Introduction

Hartnell [5] introduced a deterministic discrete-time model of fire spread on a graph $G$ and considered how firefighters can act to stop a fire outbreak. In this model, an outbreak of...
fire starts at a set of root vertices of \( G \) at time \( t = 0 \). In response, firefighters are placed at the vertices \( a_{1,1}, a_{1,2}, \ldots, a_{1,c_1} \) at time \( t = 1 \), where the firefighters defend or protect each vertex from the spreading fire. The fire then spreads from burning vertices to non-defended neighbors. Firefighters are again deployed to defend the vertices \( a_{2,1}, \ldots, a_{2,c_2} \) at time \( t = 2 \) (the vertices \( a_{1,1}, a_{1,2}, \ldots, a_{1,c_1} \) remain defended), and the fire spreads again. The process continues until the fire can no longer spread. We say that the fire outbreak is contained after \( t \) time steps if there is some finite time \( t \) such that after the disease spreads during time \( t \), only a finite number of vertices are burnt and the disease can no longer spread. The motivating question is to find an optimal sequence of defended vertices that minimizes the total number of burnt vertices.

The fire spread model is also relevant in epidemiology. Traditionally, epidemiological models assume that the population being studied is well-mixed in the sense that any pair of individuals are just as likely to come in contact and transmit a disease as any other. However, recently epidemiologists have attempted to incorporate spatial information into their models [7, 1, 2]. The model of fire spread presented above can be considered as modeling a perfectly contagious disease with no cure, where vertices adjacent to infected vertices become infected at every discrete time step and, once infected, remain infected from then on. The response allowed is only a limited number of vaccinations of non-infected vertices per time step. The limited number of vaccinations is particularly relevant to real-world situations because of limited availability of the vaccine or limited numbers of health personnel to administer the vaccine. The main question is of course to minimize the total number of infected vertices.

In this work we study fire containment on square grids. Grids are a natural class of graphs on which to consider both fire and disease spread since they are often used to represent geographic areas. Both Wang and Moeller [8] and MacGillivray and Wang [6] studied grids to find algorithms for containment. Wang and Moeller showed that two firefighters per time step is sufficient to contain a fire outbreak in a two dimensional square grid, and conjectured that \( 2d - 1 \) firefighters are necessary to contain a fire outbreak in a \( d \) dimensional square grid. We prove this conjecture in section 2. Fogarty [3] showed that two firefighters suffice in the two dimensional square lattice to contain any finite outbreak of fire where an arbitrarily large but finite number of vertices are initially on fire. However, we prove that for any fixed number \( f \) of firefighters, there is a finite outbreak of fire in which \( f \) firefighters per time step are insufficient to contain the outbreak. We also prove the conjecture of Wang and Moeller that the proportion of elements in the three-dimensional grid \( P_n \times P_n \times P_n \) which can be saved by using one firefighter per time step when an outbreak at one vertex occurs goes to 0 as \( n \) gets large.

In section 3 we provide an alternate proof using integer programming of Wang and Moeller’s result that the minimum number of time steps needed to contain a fire outbreak in a two dimensional square grid when using two firefighters per time step is 8. We also use this technique to prove that the minimum number of burned vertices in such an outbreak is 18.

We use the following terminology to describe the fire spread and firefighter response. During the \( t \)th time step for \( t > 0 \), firefighters are deployed and then the fire spreads. If we describe the state of vertices at the beginning of the \( t \)th time step, we mean before the firefighters are deployed during the \( t \)th time step. If we describe the state of vertices at the end of the \( t \)th time step, or equivalently, at the end of \( t \) time steps, we mean after the
fire has spread during the $t^{th}$ time step. A firefighter may defend neither a burnt vertex nor a previously defended vertex. Once fire has spread to a vertex $v$, we say that $v$ is a burnt vertex. After being burnt or defended, a vertex remains in that state until the process ends. In addition to the burnt and defended vertices, we say that a vertex $v$ is saved at the end of the $t^{th}$ time step if there is no path from $v$ to the root consisting only of burnt and non-defended vertices at the end of the $t^{th}$ time step.

We consider the infinite $d$-dimensional square grids $\mathbb{L}^d$. The vertices of $\mathbb{L}^d$ are the points of $\mathbb{R}^d$ with integer coordinates, and $x$ is adjacent to $y$ if and only if $x$ is distance 1 from $y$ in the usual Euclidean $\ell_2$ metric.

## 2 Three and Higher Dimensional Square Grids

Wang and Moeller proved in [8] that an outbreak starting at a single point in a regular graph of degree $r$ can be contained if $r - 1$ firefighters can be deployed per time step. Specifically, for the $d$-dimensional square grid $\mathbb{L}^d$, $2d - 1$ firefighters suffice to contain an outbreak starting at a single point. They conjectured that this bound is tight, and we present a proof of this conjecture here.

Wang and Moeller observed that at least two firefighters per time step are needed for containment in $\mathbb{L}^3$, and Fogarty showed in [3] that at least three firefighters per time step are needed to contain the outbreak. Her main theorem involves a “Hall-type condition” for the graph, which provides a lower bound for how fast the fire can spread. The theorem considers the front of the fire, which is the set of burnt vertices farthest from the root. The theorem states that if this front grows quickly (i.e., at least $f$) regardless of its precise shape, then it cannot be contained by deploying $f$ firefighters per time step. Theorem 2 strengthens Fogarty’s theorem by considering initial growth of the fire that is faster than $f$ so that the fire reaches a “critical mass” and can sustain growth of the front by at least $f$ from that point onward.

First we state some definitions.

**Definition 1.** Let $D_k$ denote the set of vertices in a rooted graph $G$ that are distance $k$ from the root vertex $r$. Let $r_k$ denote the number of firefighters in $D_{k+1}, D_{k+2}, \ldots$ at the end of the $k^{th}$ time step. These firefighters can be thought of as “reserve” firefighters since they are not adjacent to the fire when deployed. We define $r_0$ to be 0. Let $B_k \subseteq D_k$ denote the number of burned vertices in $D_k$ at the end of the $k^{th}$ time step.

**Theorem 2.** Let $G$ be a rooted graph, $h$ a positive integer, and $a_0, a_1, \ldots, a_h$ positive integers each at least $f$ such that the following holds:

1. Every $A \subseteq D_0, A \neq \emptyset$, satisfies $|N(A) \cap D_1| \geq |A| + a_0$.
2. For $1 \leq k \leq h$, every $A \subseteq D_k$ where $|A| \geq 1 + \sum_{i=0}^{k-1} (a_i - f)$ satisfies $|N(A) \cap D_{k+1}| \geq |A| + a_k$.
3. For $k > h$, every $A \subseteq D_k$ such that $|A| \geq 1 + \sum_{i=0}^{h} (a_i - f)$ satisfies $|N(A) \cap D_{k+1}| \geq |A| + f$. 

3
Suppose that at most \( f \) firefighters per time step are deployed. Then

\[
|B_n| \geq \begin{cases} 
1 & \text{if } n = 0, \\
1 + r_n + \sum_{i=0}^{n-1} (a_i - f) & \text{if } 1 \leq n \leq h + 1, \\
1 + r_n + \sum_{i=0}^{h} (a_i - f) & \text{if } n > h + 1,
\end{cases} \tag{1}
\]

regardless of the sequence of firefighter placements. Specifically, \( f \) firefighters per time step are insufficient to contain an outbreak that starts at the root vertex.

Proof. Let \( p_{n+1} \) denote the number of firefighters placed in \( D_{n+1} \) at time \( n + 1 \), and let \( p_{\leq n} \) denote the number of reserve firefighters placed in \( D_{n+1} \) during time steps \( 1, \ldots, n \). Note that

\[
r_{n+1} \leq (r_n - p_{\leq n}) + (f - p_{n+1}) = r_n + f - p_{n+1} - p_{\leq n}. \tag{2}
\]

This follows since \( r_n - p_{\leq n} \) is the number of firefighters placed in \( D_{n+2}, D_{n+3}, \ldots \) for times \( 1, \ldots, n \), and at most \( f - p_{n+1} \) firefighters are available to be placed in \( D_{n+2}, D_{n+3}, \ldots \) at time \( n + 1 \). Strict inequality occurs if a firefighter is placed in \( D_k \) for \( k < n + 1 \) at time \( n + 1 \).

We prove (1) by induction on \( n \). For \( n = 0 \), \( |B_0| = 1 \) holds trivially. We assume the result holds for \( n, 0 \leq n \leq h \), and prove the result for \( n + 1 \). By the inductive hypothesis,

\[
|B_n| \geq \begin{cases} 
1 & \text{if } n = 0, \\
1 + r_n + \sum_{i=0}^{n-1} (a_i - f) & \text{if } 1 \leq n \leq h,
\end{cases} \tag{3}
\]

and so by hypotheses (1) and (2)

\[
|N(B_n) \cap D_{n+1}| \geq |B_n| + a_n. \tag{4}
\]

Thus,

\[
|B_{n+1}| = |N(B_n) \cap D_{n+1}| - p_{n+1} - p_{\leq n} \\
\geq |B_n| + a_n - p_{n+1} - p_{\leq n}, \quad \text{by (4)},
\]

\[
\geq 1 + r_n + \sum_{i=0}^{n-1} (a_i - f) + a_n - p_{n+1} - p_{\leq n}, \quad \text{by (3)},
\]

\[
= 1 + (r_n + f - p_{n+1} - p_{\leq n}) + \sum_{i=0}^{n-1} (a_i - f) + (a_n - f)
\]

\[
\geq 1 + r_{n+1} + \sum_{i=0}^{n} (a_i - f), \quad \text{by (2)}.
\]

This proves (1) for \( 0 \leq n \leq h + 1 \).

We now prove (1) for \( n \geq h + 1 \) using induction on \( n \). Note that (1) holds for \( n = h + 1 \) from above. We thus assume (1) holds for \( n \geq h + 1 \), and we prove the result for \( n + 1 \). By inductive hypothesis,

\[
|B_n| \geq 1 + r_n + \sum_{i=0}^{h} (a_i - f), \tag{5}
\]
and so by hypothesis \(3\) holds for \(n > h\). Thus,

\[
|B_{n+1}| = |N(B_n) \cap D_{n+1}| - p_{n+1} - p_{\leq n}
\geq |B_n| + f - p_{n+1} - p_{\leq n}, \text{ by } (1),
\]

\[
\geq 1 + r_n + \sum_{i=0}^{h} (a_i - f) + f - p_{n+1} - p_{\leq n}, \text{ by } (5),
\]

\[
= 1 + (r_n + f - p_{n+1} - p_{\leq n}) + \sum_{i=0}^{h} (a_i - f)
\geq 1 + r_{n+1} + \sum_{i=0}^{h} (a_i - f), \text{ by } (2). \tag*{\blacksquare}
\]

This completes the proof of Theorem 2.

We now turn our attention to square lattices of dimension three and higher. It will prove convenient to partition these lattices into identical subgraphs.

**Definition 3.** The orthants of \(\mathbb{R}^d\) are the \(2^d\) regions defined by the hyperplanes \(x_i = -1/2\) in \(\mathbb{R}^d, i = 1, \ldots, d\). Let the orthants in \(\mathbb{L}^d\) be the subsets of vertices that lie in each orthant of \(\mathbb{R}^d\). Thus, the \(j\)th coordinates of all the vectors in a given orthant of \(\mathbb{R}^d\) are all non-negative or are all negative, for \(j = 1, \ldots, d\). Let \(D_k^+\) denote the vertices of \(D_k \subset \mathbb{L}^d\) in the orthant whose elements are all non-negative.

Let \(v = (v_1, v_2, \ldots, v_d)\) be an element of \(D_k \subset \mathbb{L}^d\). Let \(c_i(v)\) denote \(v_i\), and for a set \(A \subset D_k\) define \(A_i \equiv \{v \in A : c_i(v) = r\}\). Let \(v_{\rightarrow i}\) denote \((v_1, v_2, \ldots, v_i', v_{i+1}, \ldots, v_d) \in D_{k+1}\), where \(v_i' = v_i + 1\) if \(v_i \geq 0\) or \(v_i' = v_i - 1\) if \(v_i < 0\). Thus, \(v_{\rightarrow i}\) is in the same orthant as \(v\).

**Lemma 4.** In \(\mathbb{L}^d\) for \(d \geq 3\), if \(A \subset D_k\) where \(|A| \geq 2d - 2\), then \(|N(A) \cap D_{k+1}| \geq |A| + 2d - 2\).

**Proof.** Given any nonempty set \(A \subset D_k \subset \mathbb{L}^d\) completely contained in one orthant, we will show that

\[
|N(A) \cap D_{k+1}| \geq |A| + d - 1, \text{ for any } d. \tag{6}
\]

We form a set \(B \subset N(A) \cap D_{k+1}\) in the following way:

1. For each \(v \in A\), add \(v_{\rightarrow 1}\) to \(B\).
2. For each \(2 \leq j \leq d\), let \(r_j\) be the value of the \(j\)th coordinate of elements of \(A\) that is greatest in absolute value. For each \(v \in A_i^{j+1}\), add \(v_{\rightarrow j}\) to \(B\).

Each vector added to \(B\) in step 1 is unique, and each vector added to \(B\) in step 2 is also unique since the \(j\)th coordinate was chosen to be maximum. Thus, \(|N(A) \cap D_{k+1}| \geq |B| \geq |A| + d - 1\).

Let \(A \subset D_k \subset \mathbb{L}^d\). If \(A\) is not completely contained in one orthant, then let \(A\) be partitioned as

\[A = A_1 \cup A_2 \cup \cdots \cup A_q,\]
where each $A_\ell$ is in a different orthant $O_\ell$. By [1], $|N(A_\ell) \cap D_{k+1}| \geq |A_\ell| + d - 1$. Note also that the corresponding sets $B_\ell$ in the proof above for $A_\ell$ do not overlap since they are in different orthants. Hence,

$$|N(A) \cap D_{k+1}| \geq \sum_{\ell=1}^{q} |N(A_\ell) \cap O_\ell \cap D_{k+1}|$$

$$\geq \sum_{\ell=1}^{q} [|A_\ell| + d - 1]$$

$$\geq |A| + 2d - 2.$$  

Thus, we may assume that $A$ is completely contained in one orthant, and, without loss of generality, we assume that all coordinates of elements of $A$ are non-negative.

We now proceed to prove the lemma by induction on $d$. Let $A \subseteq D^+_k \subseteq \mathbb{L}^d$, where $|A| \geq 2d - 2$. Suppose that $d = 3$. Let $n_i$ denote the number of nonempty $A_i$, or, equivalently, the number of distinct $i$th coordinates of elements of $A$. Let $i'$ be a coordinate where $n_i$ is maximized. We claim that $n_{i'} \geq 3$. If $n_{i'} = 1$, then $A$ contains only one element, which is a contradiction since $|A| \geq 2d - 2 = 4$. If $n_{i'}$ is 2, then each coordinate has only two different values it can assume. However, the sum of the coordinates must remain $k$. It is straightforward to verify that the maximum number of elements in $A$ is 3, which contradicts the fact that $|A| \geq 4$. Thus, $n_{i'} \geq 3$.

For each $r$ where $A_r$ is nonempty, form a set $\hat{A}_r \subseteq D^d_{k-r} \subseteq \mathbb{L}^d$ by eliminating the $i'$ coordinate of each element in $A_r$; thus, the $\hat{A}_r$’s are the parts of $A$ contained in the slices of $\mathbb{L}^d$ taken in direction $i'$. By [1], $N(\hat{A}_r) \cap D^d_{k-r+1} \geq |\hat{A}_r| + d - 2$. For each $v$ in $N(\hat{A}_r) \cap D^d_{k-r+1}$, form an element $\tilde{v}$ in $N(A_r') \cap D^d_{k+1}$ by inserting $r$ as the $i'$ coordinate. Notice that these elements are distinct when the $i'$ coordinates are distinct. Let $m$ be the maximum $r$ such that $A_r'$ is nonempty, or equivalently, the largest $i'$ coordinate. For each $v \in A_m'$, we also have $v_{-i'} = N(A) \cap D_{k+1}$, and these vectors are distinct from any formed above because the $i'$ coordinate is larger. Thus,

$$|N(A) \cap D_{k+1}| \geq \sum_{r:A_r' \neq \emptyset} \left( |A_r| + d - 2 \right) + |A_m'|$$

$$\geq |A| + n_{i'}(d - 2) + |A_m'|.$$  

(7)

Since $|A_m'| \geq 1$, [1] implies that

$$|N(A) \cap D_{k+1}| \geq |A| + 3d - 5,$$  

(8)

and when $d = 3$,

$$|N(A) \cap D_{k+1}| \geq |A| + 4 = |A| + 2d - 2.$$

Now suppose that $d > 3$. Again let $n_i$ denote the number of nonempty $A_i$, and let $i'$ be a coordinate where $n_i$ is maximized. If $n_{i'} \geq 3$, then using the same construction as in the $d = 3$ case, we have [1], and since $d > 3$, $|N(A) \cap D_{k+1}| \geq |A| + 2d - 2$. If $n_{i'} = 1$, then $A$
contains only one element, which is a contradiction since \(|A| \geq 2d - 2 \geq 4\). We are thus left with the case \(n_i = 2\). Let \(m\) be the maximum \(r\) such that \(A_{r'}^{i'}\) is nonempty, or equivalently, the largest \(i'\) coordinate of elements of \(A\), and let \(r' \neq m\) be the minimum value of \(r\) where \(A_{r'}^{i'}\) is nonempty. If \(|A_m'| \geq 2\), then using the same construction as in the \(n_i \geq 3\) case, we have by (7)

\[
|N(A) \cap D_{k+1}| \geq |A| + n_i'(d - 2) + |A_m'|
\]

\[
\geq |A| + (2d - 4) + 2, \text{ since } |A_m'| \geq 2,
\]

\[
\geq |A| + 2d - 2.
\]

If \(|A_m'| = 1\), then we again use the construction from the \(n_i \geq 3\) case. However, \(|A_{r'}^{i'}| \geq 2d - 3\), so by induction,

\[
|N(A_{r'}^{i'}) \cap D_{d-k-r'+1}^{d-1}| \geq |A_{r'}^{i'}| + 2d - 4.
\]

Here, the notation \(D_{d-k-r'+1}^{d-1}\) means the set \(D_z \subseteq \mathbb{L}_{d-1}\), emphasizing the dimension of \(\mathbb{L}_{d-1}\). For each \(v \in N(A_{r'}^{i'}) \cap D_{d-k-r'+1}^{d-1}\), form an element \(\tilde{v}\) in \(N(A_{r'}^{i'}) \cap D_{d-k+1}^{d-1}\) by inserting \(r'\) as the \(i'\) coordinate. Additionally, for the single vector \(v \in A_m^{i'}\) and \(1 \leq j \leq d\), \(v_{-j} \in N(A) \cap D_{k+1}\), and these vectors are distinct from those formed above because the \(i'\) coordinate is larger. Thus,

\[
|N(A) \cap D_{k+1}| \geq \left(|A_{r'}^{i'}| + 2d - 4\right) + d
\]

\[
= |A| + 3d - 3, \text{ since } |A_{r'}^{i'}| = |A| + 1, \]

\[
\geq |A| + 2d - 2, \text{ since } d > 3.
\]

Lemma 4 provides the long-term growth of the front, Condition 3 needed for Theorem 2.

The next lemma gives the complementary requirements.

**Lemma 5.** In \(\mathbb{L}_d\) for \(d \geq 3\), if \(A \subseteq D_1\) where \(|A| \geq 2\), then \(|N(A) \cap D_2| \geq |A| + 4d - 6\).

**Proof.** Let \(A \subseteq D_1 \subseteq \mathbb{L}_d\) where \(|A| \geq 2\). Every vector \(v \in A\) is of the form \((0, \ldots, x_i, \ldots, 0)\), where \(x_i = \pm 1\). Each vector \(v \in A\) has \(2(d - 1)\) neighbors in \(D_2\) formed by replacing each of the zero coordinates in \(v\) with \(\pm 1\), and one neighbor formed by replacing 1 in the \(i\)th coordinate with 2 or replacing \(-1\) with \(-2\). If \(v\) and \(v'\) are vectors of \(A\) with nonzero entries in different coordinates, then \(v\) and \(v'\) share exactly one neighbor in \(D_2\). If \(v\) and \(v'\) have nonzero entries in the same coordinate, then \(v\) and \(v'\) share no neighbors in \(D_2\). Thus,

\[
|N(A) \cap D_2| \geq \left|A\right|(2(d - 1) + 1) - \left(|A|\right)
\]

\[
= \left|A\right|(2d - \frac{|A|}{2} - \frac{1}{2})
\]

\[
\geq \left|A\right| + \left|A\right| \left(2d - \frac{|A|}{2} - \frac{3}{2}\right).
\]

It is straightforward to use calculus to verify that

\[
\left|A\right| \left(2d - \frac{|A|}{2} - \frac{3}{2}\right) \geq 4d - 6,
\]

7
where \(d \geq 3\) and \(2 \leq |A| \leq 2d\), and so
\[
|N(A) \cap D_2| \geq |A| + 4d - 6.
\]

**Theorem 6.** In \(\mathbb{L}^d\), \(2d - 1\) firefighters are needed to contain an outbreak of fire starting at a single vertex.

**Proof.** Since \(\mathbb{L}^d\) is vertex transitive, we may assume that the root vertex where the fire outbreak starts is the origin. We use Theorem 2 with \(f = 2d - 2\), \(h = 1\), \(a_0 = 2d - 1\), and \(a_1 = 4d - 6\). The one element set \(D_0\) has \(2d\) neighbors in \(D_1\) so hypothesis 1 of Theorem 2 holds, Lemma 3 shows hypothesis 2 of Theorem 2 holds for \(k = 1\), and Lemma 4 shows hypothesis 3 holds for \(k > 1\). By Theorem 2, \(2d - 2\) firefighters are insufficient to contain an outbreak starting at the origin.

Fogarty also showed in 3 that two firefighters suffice in \(\mathbb{L}^2\) to contain any finite outbreak of fire where an arbitrarily large but finite number of vertices are initially on fire. However, we prove for \(\mathbb{L}^d\) where \(d \geq 3\) that for any fixed number \(f\) of firefighters, there is a finite outbreak of fire in which \(f\) firefighters per time step are insufficient to contain the outbreak.

First we establish the following lemma. Essentially, the lemma says that if we have a “front” of \(x\) elements, then it will grow outwards by at least \(\Omega(\sqrt{x})\) in the next time step.

**Lemma 7.** Let \(f\) be any positive integer. If \(A \subseteq D^+_k \subseteq \mathbb{L}^3\) where \(|A| > \frac{1}{2}(f - 1)(f - 2)\), then \(|N(A) \cap D^+_{k+1}| \geq |A| + f\).

**Proof.** Let \(A \subseteq D^+_k \subseteq \mathbb{L}^3\) be a set where \(|A| > \frac{1}{2}(f - 1)(f - 2)\). The elements of \(B := \{v_{\rightarrow 1} : v \in A\}\) are distinct vertices in \(N(A) \cap D^+_{k+1}\), and the set \(B\) has cardinality equal to \(|A|\). Therefore, it suffices to show that if \(|A| > \frac{1}{2}(f - 1)(f - 2)\), then there are at least \(f\) distinct elements of the form \(v_{\rightarrow j}\) which are not elements of \(B\), where \(v \in A\) and \(j \in \{2, 3\}\).

Let \(m\) be the largest first coordinate of elements of \(A\), and let \(t\) be the smallest first coordinate of elements of \(A\). Recall that the sets \(A^+_1\), \(r = t, t+1, \ldots, m\), partition \(A\). Let \(\sigma_r\) equal \(|A^+_r|\), so that \(\sum_{r=t}^m \sigma_r = |A|\). Note that \(\sigma_t, \sigma_m > 0\).

Suppose some \(\sigma_r\) is equal to zero, where \(t < r < m\). Then \(A\) is partitioned into the sets \(A_1\) consisting of all elements of \(A\) with first coordinate greater than \(r\) and \(A_2\) consisting of all elements of \(A\) with first coordinate less than \(r\). Clearly \(N(A_1) \cap N(A_2) \cap D^+_{k+1} = \emptyset\). Define \(A'_1 := \{v_{\rightarrow 2} : v \in A_1\}\) and \(A'_2 := \{v_{\rightarrow 1} : v \in A_2\}\), so that \(A'_1\) and \(A'_2\) are subsets of \(D^+_{k+1}\). Since \(A'_1\) is simply a translate of \(A_1\) by 1 in the first coordinate, \(N(A'_1) \cap D^+_{k+1}\) is a translate of \(N(A_1) \cap D^+_{k+1}\) by 1 in the first coordinate. Similarly, \(N(A'_2) \cap D^+_{k+2}\) is a translate of \(N(A_2) \cap D^+_{k+1}\) by 1 in the second coordinate. Thus, we have that
\[
|N(A'_1 \cup A'_2) \cap D^+_{k+2}| \leq |N(A'_1) \cap D^+_{k+2}| + |N(A'_2) \cap D^+_{k+2}|
\]
\[
= |N(A_1) \cap D^+_{k+1}| + |N(A_2) \cap D^+_{k+1}|
\]
\[
= |N(A) \cap D^+_{k+1}|,
\]
where the last equality follows since \(N(A_1) \cap D^+_{k+1}\) and \(N(A_2) \cap D^+_{k+1}\) do not intersect. However, \(A'_1 \cup A'_2\) has the same size as \(A\), but the separation between the largest first coordinate of elements of \(A'_1 \cup A'_2\) and the smallest first coordinate of \(A'_1 \cup A'_2\) is less than
Therefore, by induction on \( m - t \) we reduce to the case where no \( \sigma_r \) is equal to zero, i.e., there is an element of \( A \) with first coordinate \( r \) for every \( t \leq r \leq m \).

Consider the sets \( S_r = \{ v_{t,j} : v \in A^1_r, j \in \{2,3\} \} \subseteq N(A) \cap D^+_{k+1} \). Observe that the cardinality of \( S_r \) is at least \( \sigma_r + 1 \). Clearly all \( S_r \) are disjoint, since all elements of \( S_r \) have first coordinate \( r \). The elements of \( S_r \) have \( t \) as their first coordinate, while all elements of \( B \) have first coordinates at least \( t + 1 \), so no elements of \( S_t \) are in \( B \). Furthermore, for all \( r > t \), if an element of \( S_r \) is in \( B \), then by considering its first coordinate, the element must be in the set \( \{ v_{t+1,j} : v \in A^1_{t+1} \} \). In particular, this set has size \( \sigma_{r-1} \). If \( \sigma_r + 1 > \sigma_{r-1} \), then there are at least \( \sigma_r + 1 - \sigma_{r-1} \) elements in \( S_r \) not in \( B \). Therefore, the number of elements in \( N(A) \cap D^+_{k+1} \) that are not in \( B \) is bounded below by

\[
g(\sigma) := \sum_{r=t}^{m} \max(0, \sigma_r + 1 - \sigma_{r-1}),
\]

with the convention that \( \sigma_{t-1} = 0 \).

Now take any nonzero sequence \( \sigma_t, \sigma_{t+1}, \ldots, \sigma_m \). We claim that if \( g(\sigma) < f \), then \( \sum_{r=t}^{m} \sigma_r \leq \frac{1}{2}(f - 1)(f - 2) \), which would complete the proof of the theorem. Suppose we have some sequence \( \sigma_t, \sigma_{t+1}, \ldots, \sigma_m \) with \( g(\sigma) < f \). First, suppose that there exists some \( r > t \) where \( \sigma_r \geq \sigma_{r-1} \). Then adding 1 to \( \sigma_{r-1} \) decreases the \( r \)-th term of \( [4] \) by 1, possibly adds 1 to the \((r-1)\)-st term, and leaves all other terms unchanged; in particular, it does not increase the value of \( g(\sigma) \) and increases \( \sum \sigma_r \). Therefore, we can reduce to the case where \( \sigma \) is strictly decreasing.

Next, suppose we have \( \sigma_r < \sigma_{r-1} - 1 \) for some \( t < r \leq m \). Then adding 1 to \( \sigma_r \) leaves all terms of \( [2] \) unchanged. Similar to before, this operation does not change \( g(\sigma) \), while increasing \( \sum \sigma_r \). Doing this repeatedly, we reduce to the case where

\[
\sigma_{r-1} = \sigma_r + 1
\]

for all \( t < r \leq m \). However, this case is easy to evaluate; each term in \( [2] \) is zero except the \( r = t \) term, which is equal to \( \sigma_t + 1 \). Since \( g(\sigma) = \sigma_t + 1 < f \), \( \sigma_t < f - 1 \). Since \( \sigma_m > 0 \), \( \sum_{r=t}^{m} \sigma_r \) is at most the sum of the first \( f - 2 \) positive integers. Thus,

\[
\sum_{r=t}^{m} \sigma_r \leq \frac{1}{2}(f - 1)(f - 2).
\]

This allows us to prove the following theorem.

Theorem 8. For any dimension \( d \geq 3 \) and any fixed positive integer \( f \), \( f \) firefighters per time step are not sufficient to contain all finite outbreaks in \( \mathbb{L}^d \).

Proof. Since \( \mathbb{L}^3 \) is contained in \( \mathbb{L}^d \) for \( d \geq 3 \), it suffices to prove the statement for \( d = 3 \). We consider an initial outbreak consisting of all of \( D^+_{k} \) for \( k \) large enough so that \( |D^+_{k}| > \frac{1}{2}(f - 1)(f - 2) \). To show that \( f \) firefighters are insufficient to contain this outbreak, we will construct a related graph that captures the essential disease dynamics and then invoke Theorem 2. Let \( G \) be the subgraph of \( \mathbb{L}^3 \) induced by vertices with non-negative coordinates that are distance at least \( k \) from the origin. Let \( G' \) be the graph formed from \( G \) by identifying
all of the vertices in $D_k^+$ as a single vertex $r$. An edge exists between vertices $x$ and $y$ in $G'$ if $xy$ is an edge in $G$ or if $x = r$ and $y \in N_G(D_k^+)$. Let $D'_t$ denote the set of vertices in $G'$ that are distance $i$ from the root $r$. By Lemma 7,

$$|N(D_k^+) \cap D_{k+1}^+| \geq |D_k^+| + f > \frac{1}{2}(f-1)(f-2) + f,$$

and so

$$|N(r) \cap D'_1| > (|D'_0| - 1) + \frac{1}{2}(f-1)(f-2) + f.$$ 

If $A' \subseteq D'_i$, where $i > 0$ and $|A'| > \frac{1}{2}(f-1)(f-2)$, then $A'$ corresponds to a set $A \subseteq D_{k+i}^+$ and by Lemma 7

$$|N(A) \cap D_{k+i+1}^+| \geq |A| + f,$$

and hence

$$|N(A') \cap D_{i+1}^+| \geq |A'| + f.$$ 

By Theorem 2 with $h = 0$, and $a_0 = \frac{1}{2}(f-1)(f-2) + f$, $f$ firefighters are insufficient to contain an outbreak starting at $r$ in $G'$, and hence $f$ firefighters are insufficient to contain an outbreak consisting of all of $D_k^+$ in $\mathbb{L}^3$.

The essential problem here is that for $d \geq 3$, the boundary of an outbreak grows faster than the constant number of firefighters deployed at a given time step. Indeed, in dimension $d$, the boundary grows as a polynomial of degree $d-2$. This motivates the following ambitious conjecture.

**Conjecture 9.** Suppose that $f(t)$ is a function on $\mathbb{N}$ with the property that $\frac{f(t)}{t^{d-2}}$ goes to 0 as $t$ gets large. Then there exists some outbreak on $\mathbb{L}^d$ which cannot be contained by deploying $f(t)$ firefighters at time $t$.

A weaker conjecture would require $f(t)$ to be a polynomial.

Lemma 7 also allows us to resolve another conjecture of Wang and Moeller in [8]. They conjectured that as $n$ gets large, the proportion of elements in the three-dimensional grid $P_n \times P_n \times P_n$ which can be saved by using one firefighter per time step when an outbreak at one vertex occurs goes to 0. We prove this conjecture in the following

**Theorem 10.** Let $v$ be any vertex of $P_n \times P_n \times P_n$, for $n \geq 1$. Then the maximum number of vertices which can be saved by deploying one firefighter per time step with an initial outbreak at $v$ grows at most as $O(n^2)$. In particular, the proportion of vertices which can be saved goes to 0 as $n$ gets large.

**Proof.** We prove the theorem in the case $v = (0, 0, 0)$. The general statement easily follows by splitting $P_n \times P_n \times P_n$ into orthants with apex $v$. We actually prove a stronger statement. Consider the graph $G$ induced from the lattice $\mathbb{L}^3$ by vertices with non-negative coordinates and distance at most $3n$ from the origin $v$. We prove the theorem for the graph $G$. Note that $G$ contains $P_n \times P_n \times P_n$ as an induced subgraph.

We claim that $|B_t| - r_t \geq \frac{t^2 + 2}{2}$ for all $t$ regardless of what firefighter placements are made. Since there are $\binom{t+2}{2} = \frac{t^2 + 3t + 2}{2}$ vertices in $D_t^+$, this statement is saying that at the end of the $t$th time step the number of reserve firefighters together with the unburned vertices
(including defended vertices) in $D_t^+$ cannot exceed $t$. By considering time up to $t = 3n$, when all vertices have had a chance to be burned, at most $1 + 2 + \ldots + 3n = O(n^2)$ vertices are unburned. This implies the same conclusion for $P_n \times P_n \times P_n$.

The proof of the claim is by induction. At the end of the 0th time step, there are no reserve firefighters, and one vertex in $D_1$ is burned; the difference is $1 - 0 = 1 \geq 1 = \frac{0^2+0+2}{2}$ as desired.

Suppose $t \geq 0$, and suppose that the statement is true for $t$. Then

$$|B_t| \geq \frac{t^2 + t + 2}{2} > \frac{1}{2} t(t + 1). \quad (11)$$

Let $f = t + 2$. By Lemma 7

$$|N(B_t) \cap D_{t+1}^+| \geq |B_t| + f. \quad (12)$$

As in the proof of Theorem 2 let $p_{t+1}$ denote the number of firefighters placed in $D_{t+1}^+$ at time $t + 1$, and let $p_{\leq t}$ denote the number of reserve firefighters placed in $D_{t+1}^+$ during time steps $1, \ldots, t$. Thus,

$$|B_{t+1}| - r_{t+1} = \left[|N(B_t) \cap D_{t+1}^+| - p_{t+1} - p_{\leq t}\right] - r_{t+1}
\geq |N(B_t) \cap D_{t+1}^+| - r_t - 1, \quad \text{by } (2),
\geq |B_t| + f - r_t - 1, \quad \text{by } (12),
\geq \frac{t^2 + t + 2}{2} + (t + 2) - 1, \quad \text{by } (11),
\geq \frac{(t + 1)^2 + (t + 1) + 2}{2}.

Hence the claim follows.

In practice, one can ensure when an outbreak starts at $(0, 0, 0)$ that $t$ vertices in $D_t^+$ are unburned at time $t$. However, because the fire doubles back on itself, it is unclear that one can actually save a quadratic number of vertices. Wang and Moeller exhibit the construction of building a “fire wall” by defending all of the vertices at distance $k$ from $(n, n, n)$. In order for this to be effective, we must be able to cover all $\frac{(k+1)(k+2)}{2}$ such vertices in the $3n - k$ time steps it takes the fire to reach this hyperplane. This yields $k = O(\sqrt{n})$. The number of vertices saved is the number of vertices at distance $k$ or less from $(n, n, n)$, which is $\frac{(k+1)(k+2)(k+3)}{6}$. This is $O(k^3) = O(n^{3/2})$. Therefore, the optimal number of vertices saved given an initial outbreak at $(0, 0, 0)$ in the grid graph $P_n \times P_n \times P_n$ when deploying one firefighter per time step is between $O(n^{3/2})$ and $O(n^2)$.

### 3 Two Dimensional Square Grid

According to Wang and Moeller in [8], Hartnell, Finbow, and Schmeisser first proved that an outbreak of fire in $L^2$ starting at a single vertex can be contained using two firefighters per time step. Their sequence of firefighter placements contained the outbreak at the end of 11 time steps. Wang and Moeller showed that the disease cannot be contained at the
end of 7 time steps when using two firefighters per time step and presented a sequence of firefighter placements that attains this minimum. Their sequence allows 18 vertices to be burned. Surprisingly, Wang and Moeller do not comment on whether their solution attains the minimum number of burned vertices. In fact, 18 is the minimum number of burned vertices, and we prove this using integer programming. The same technique also gives a computer proof of Wang and Moeller’s result that at least 8 time steps are needed. Their proof relies heavily on case analysis.

The tightness in the following theorem is due to Wang and Moeller [8].

**Theorem 11.** In $L^2$, if an outbreak of fire starts on a single vertex, then when using two firefighters per time step at least 18 vertices are burned. This bound is tight.

**Proof.** We formulate an integer program using the boolean variables $b_{x,t}$ and $d_{x,t}$. The variable $b_{x,t}$ is 1 if and only if vertex $x$ is burned at or before time $t$, and $d_{x,t}$ is 1 if and only if $x$ is defended at or before time $t$. We wish to minimize the total number of vertices that become burned. For the integer program to be implementable with a finite number of variables and constraints, we restrict the graph to $L = \{(x, y) \in L^2 : |x| \leq \ell$ and $|y| \leq \ell\}$ and $0 \leq t \leq T$, where $\ell$ and $T$ are chosen to be sufficiently large that the fire never reaches the boundary and is completely contained by time $T$. In the actual computations performed, $\ell = 6$ and $T = 9$ proved sufficient. We choose $T > 8$ to ensure that the fire is actually contained and does not grow in the last time step.

The integer program is

$$\text{minimize } \sum_{x \in L} b_{x,T}$$

subject to:

- $b_{x,t} + d_{x,t} - b_{y,t-1} \geq 0$, for all $x \in L$, $y \in N(x)$, and $1 \leq t \leq T$, \hspace{1cm} (13)
- $b_{x,t} + d_{x,t} \leq 1$, for all $x \in L$ and $1 \leq t \leq T$, \hspace{1cm} (14)
- $b_{x,t} - b_{x,t-1} \geq 0$, for all $x \in L$ and $1 \leq t \leq T$, \hspace{1cm} (15)
- $d_{x,t} - d_{x,t-1} \geq 0$, for all $x \in L$ and $1 \leq t \leq T$, \hspace{1cm} (16)
- $\sum_{x \in L} (d_{x,t} - d_{x,t-1}) \leq 2$, for $1 \leq t \leq T$, \hspace{1cm} (17)
- $b_{x,0} = \begin{cases} 1 & \text{if } x \text{ is the origin,} \\ 0 & \text{otherwise,} \end{cases}$ for all $x \in L$, \hspace{1cm} (18)
- $d_{x,0} = 0$, for all $x \in L$, \hspace{1cm} (19)
- $b_{x,t}, d_{x,t} \in \{0, 1\}$, for all $x \in L$ and $0 \leq t \leq T$. \hspace{1cm} (20)

Condition (13) enforces the spread of the fire while respecting vertices defended by a firefighter. Note that vertices can spontaneously combust, catching fire, but the minimization of the objective function ensures that this does not happen in the optimal solution. Condition (14) prevents a firefighter from defending a burnt vertex, while conditions (15) and (16) ensure that once a vertex is burnt or defended, it stays in that state. Condition (17) only allows two firefighters per time step. Conditions (18) and (19) give the initial conditions at time $t = 0$, and condition (20) makes the program a binary integer program.
Figure 1: Optimal solution of the integer program used in the proof of Theorem 11. The fire outbreak starts at time 0 at the root, and then spreads to the black vertices at the times written next to the vertices. The square firefighters $a_i$ are placed at time $i$. This placement of two firefighters per time step in $L^2$ completely contains the outbreak in 8 time steps, allowing only the minimum number of 18 burned vertices.

The integer program was solved in about 1.83 hours using the GNU Linear Programming Kit [4] running on a Pentium IV 2.6GHz processor, and 18 was the minimum number of burnt vertices at time $t = 9$. Figure 1 shows the minimum solution. The fire was completely contained and thus did not reach the sides of $L$. Also note that the solution presented by Wang and Moeller in [8] also allows only 18 burnt vertices but is slightly different from the solution presented here. □

Lemma 12. If an outbreak of fire in $L^2$ is contained by 14 defended vertices and $(x, y)$ is a burnt vertex, then $|x| \leq 5$ and $|y| \leq 5$.

Proof. Suppose that $(x, y)$ is a burnt vertex, and, without loss of generality, that $x > 5$. Since $(x, y)$ is burnt, there is a path $v_0 = (x, y), v_1, v_2, \ldots, v_t = (0, 0)$ from $(x, y)$ to the origin consisting of burnt vertices. For each $0 \leq a \leq 6$, there is a vertex $v_{\rho(a)}$ such that the first coordinate of $v_{\rho(a)}$ is $a$. Since the fire is contained, there must be a defended vertex above and below each of these seven vertices, and there must be at least one defended vertex with first coordinate less than 0 and one with first coordinate greater than $x$. But this requires 16 defended vertices, resulting in a contradiction. □

Theorem 13 (Wang and Moeller). In $L^2$, if an outbreak of fire starts at a single vertex, then the fire cannot be contained at the end of 7 time steps when using two firefighters per time step. Thus, at least 8 time steps are needed to contain the fire, and this bound is tight.

Proof. We use a similar integer program to the one used in the proof of Theorem 11. By Lemma 12, if the outbreak can be contained after 7 time steps, then no burnt vertex will
have either coordinate equaling 6 in absolute value. We thus use the finite grid $L$ where $\ell = 6$, and we use the objective function

$$\text{minimize } \sum_{x=(a,b) \in L, |a|=6 \text{ or } |b|=6} b_{x,T}.$$  

If the disease can be contained after 7 time steps, then the optimal value of the objective function will be 0. All of the conditions from the previous integer program are included except condition (17) is changed to

$$\sum_{x \in L} (d_{x,t} - d_{x,t-1}) \leq \begin{cases} 2 & \text{for } 1 \leq t \leq 7, \\ 0 & \text{for } 8 \leq t \leq T. \end{cases}$$  \tag{21}$$

This prevents firefighters from being used after 7 time steps.

The integer program with $T = 9$ was solved in about 40 minutes using the GNU Linear Programming Kit running on a Pentium M 900MHz processor. The minimum value was 1, meaning that in every feasible solution, the fire burned a vertex with one coordinate equaling 6 in absolute value. This contradicts Lemma 12, and so at least 8 time steps are needed to contain an outbreak in $\mathbb{L}^2$ when using two firefighters per time step. \qed

4 Future Work

There are many avenues for future work in models of responses to fire and disease spread. For infinite graphs, we can ask the same question as for the infinite square grids: What is the minimum number of firefighters needed per time step so that only a finite number of vertices are burned? Percolation is a related topic whose methods may also apply here.

From the viewpoint of an arsonist or bioterrorist, one would like to find the most vulnerable vertices in a graph $G$. A vertex $v$ is most vulnerable if a fire outbreak starting at $v$ burns the most vertices of $G$ given an optimal firefighter response. Can the most vulnerable vertices in a graph be determined without knowing the optimal firefighter response? Perhaps they could then be preemptively defended. From the viewpoint of a network architect, we would like to design graphs that are resistant to such attacks. Similar questions can also be asked if there are $k$ initial outbreaks of fire.

Finally, MacGillivray and Wang observed that the firefighter problem can be viewed as a one-player game. Suppose that the fire has a choice, too: the fire can only spread to $d$ neighbors each time step. This forms a two-player game. What strategy should the firefighters use to minimize the number of burned vertices?

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