Maximum hands-off feedback control for finite-time stabilization

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Abstract—This work proposes a way to time-sparsify pre-designed stabilizing feedback controls. For this purpose, a finite-time stabilizing sliding-mode feedback control law is considered. For the finite-time stabilizing controller, an on-off switching signal is designed to optimally time-sparsify the feedback control signals. Due to the non-Lipschitz nature of the sliding-mode controller, this requires the use of a non-smooth Pontryagin Maximum Principle. Numerical experiments are presented to verify the theoretical results and illustrate the advantages of the proposed ideas.

I. INTRODUCTION

Optimality in time is a fundamental notion in optimal control. A classical and important problem is minimum-time control (or time-optimal control)\(^{[1]}\),\(^{[2]}\), where the total time duration of control is minimized from a given initial state to another terminal state under a magnitude constraint on the control (commonly, inputs are constrained as per \(|u(t)| \leq 1|\)). It is well known that the minimum-time control is of bang-bang type: the optimal control signal is binary, taking only the extremum values \(\pm 1|\).

In the context of sparsity, a maximum hands-off control has been recently proposed in \(^{[3]}\), where the time duration is minimized on which the control takes non-zero values, under the same setting as the minimum-time control mentioned above. Maximum hands-off control brings advantages in some networked control setups and also provides an alternative when environmental factors are to be taken care of. From an optimization perspective, maximum hands-off control is also called an \(L_0\)-optimal control or sparse control, since it minimizes the \(L_0\) norm of the control signal to obtain the sparsest (and feasible) control. Maximum hands-off control (or \(L_0\)-optimal control) was shown to be of bang-off-bang type: the control signal is ternary taking only values \(\pm 1|\) or 0 \(^{[4]}\). This property is important when considering an \(L_1\) relaxation \(^{[3]}\), \(^{[5]}\) for numerical computations. If the \(L_1\)-optimal control, also fuel-optimal control \(^{[6]}\), is bang-off-bang, then the \(L_1\) optimal control coincides with the \(L_0\) optimal control. Also, in \(^{[7]}\) the tradeoff between sparsity and time optimality is studied using mixed \(L_0\) and time optimization.

To date, maximum hands-off control has only been studied using open-loop finite-horizon optimisations. Closed-loop feedback has merely been incorporated as an afterthought, using a combination of model predictive control \(^{[8]}\) and self-triggered control ideas \(^{[9]}\), \(^{[10]}\). Unfortunately, the use of such feedback methods for maximum hands-off control relies upon heavy numerical computations, see \(^{[3]}\). This raises the necessity to describe maximum hands-off control as an explicit function of the state, that is, formulating hands-off state feedback policies. To the best of our knowledge, there is no work on the construction of such explicit state (or output) feedback control policies with optimal time-sparsity.

In this paper, we propose a novel hands-off control formulation with explicit state feedback. Our main idea consists in designing an (optimal) binary signal, which switches a given non-sparse controller on and off. Under suitable assumptions, our method leads to sparse feedback control policies with guaranteed closed loop stability properties. To present our ideas, we focus on the double integrator, for which we adapt the finite-time stabilizing feedback controller proposed in \(^{[11]}\). The obtained control falls under the broad category of sliding mode control. Sliding mode control offers efficient means of constructing finite-time stabilizing feedback that is robust to matched disturbances. Conventional sliding mode control is non-smooth, but higher order sliding modes have also been proposed to add smoothness, necessary for our approach. A detailed review can be found in \(^{[12]}\), \(^{[13]}\), \(^{[14]}\), \(^{[15]}\) and references therein. Classical sliding mode control involves a finite-time reaching phase to a surface (a level set) in the state-space on which the closed-loop trajectories ‘slide’ to asymptotically converge to the origin. To achieve finite-time convergence to the origin, a more modern approach involves nonlinear sliding surfaces called ‘terminal sliding mode’ (see e.g. \(^{[16]}\)). The control design presented in \(^{[11]}\) contains a terminal sliding mode and hence the trajectories converge to the origin in finite time. Asymptotic stability of linear systems with sparse control has also been studied in the context of intermittent feedback in our previous works \(^{[17]}\), \(^{[18]}\), \(^{[19]}\). However, optimality of the time-sparsity was not addressed in these articles. In comparison with previous contributions on sparse control, we here propose a means for optimally time-sparsifying nonlinear finite-time feedback. This retains the robustness and stability properties of feedback control, while adding the resource efficient optimal time-sparsity feature to the control.
An outline of the remainder of this manuscript is as follows: Mathematical preliminaries are revisited in the next section. Section III formulates the new sparse control problem and derives necessary optimality conditions using a non-smooth Pontryagin maximum principle [20, Theorem 22.26]. Section IV presents numerical results to illustrate the effectiveness of the proposed method. Section V draws conclusions.

Notation

For any positive integer $d$, the vector space $\mathbb{R}^d$ is equipped with the standard Euclidean norm. The Euclidean open unit ball of radius $r$ centered at $y \in \mathbb{R}^d$ is denoted by $B(y, r)$. If $S \subset \mathbb{R}^d$ and $r > 0$, then $S^{(r)} := \bigcup_{y \in S} B(y, r)$ is the $r$-dilation of $S$. The Lebesgue measure on $\mathbb{R}$ is denoted by Leb. The indicator function over a measurable set $S$ is denoted $1_S(u)$ and defined as $1_S(u) = 1$ for $u \in S$ and 0 otherwise.

If $m$ is a positive integer, a map $\varphi: \mathbb{R}^d \to \mathbb{R}^m$ is locally Lipschitz continuous if there exists $L > 0$ such that the inequality $\|\varphi(y_1) - \varphi(y_2)\| \leq L\|y_1 - y_2\|$ holds for all $y_1, y_2 \in S \subset \mathbb{R}^d$; here $L$ is a rank of $\varphi$. The symbol $\text{sgn}(s)$ denotes the signum function and takes values $+1$, 0 and $-1$ for $s > 0$, $s = 0$ and $s < 0$ respectively.

II. PRELIMINARIES

In this section, we introduce mathematical definitions and preliminary results that will be critical in understanding and proving the optimal control results presented in subsequent sections.

**Definition II.1.** [11, p. 679] Consider the dynamical system,

$$\dot{z} = f(z)$$

where $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous on an open neighborhood of the origin and $f(0) = 0$. Denote the solution of (1) starting at $z(0) = \pi$ by $\rho_t(\pi)$ for $t \in [0, \bar{t}]$ with some $\bar{t} > 0$. The origin is said to be a finite-time-stable equilibrium of (1) if there exists an open neighborhood $N \subset \mathcal{D}$ of the origin and a function $T: N \setminus \{0\} \to [0, +\infty[$, called the settling time, such that the following statements hold:

- Finite-time convergence: For every $\pi \in N \setminus \{0\}$, $\rho_t(\pi)$ is defined for $t \in [0, T(\pi)]$, $\rho_t(\pi) \in N \setminus \{0\}$ for $t \in [0, T(\pi)]$ and $\lim_{t \to T(\pi)^+} \rho_t(\pi) = 0$.

- Lyapunov Stability: For any open set $V_\epsilon$, such that $0 \in V_\epsilon \subset N$, there exists an open set $V_0$ such that $0 \in V_0 \subset N$ and such that for every $\pi \in V_0 \setminus \{0\}$, $\rho_t(\pi) \in V_\epsilon$ for $t \in [0, T(\pi)]$.

The origin is said to be a globally finite-time-stable equilibrium if it is a finite-time-stable equilibrium and $\mathcal{D} = N = \mathbb{R}^n$.

We need the following adaptation of [20, Theorem 22.26]:

**Theorem II.2.** Let $\bar{t} \in [0, +\infty[$ and let $U \subset \mathbb{R}^m$ be a non-empty Borel measurable set. Let a lower semicontinuous instantaneous cost function

$$\Lambda : \mathbb{R}^d \times U \to \mathbb{R},$$

and a continuously differentiable terminal cost function

$$\ell : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

be given. Consider the optimal control problem

$$\min_u \quad \ell(x(0), x(\bar{t})) + \int_0^{\bar{t}} \Lambda(x(t), u(t)) \, dt$$

subject to

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) & \text{for a.e. } t \in [0, \bar{t}], \\ u(t) \in U & \text{for a.e. } t \in [0, \bar{t}], \\ x(0), x(\bar{t}) \in E \subset \mathbb{R}^d \times \mathbb{R}^d, \end{cases}$$

where $\bar{t} > 0$ is the terminal time, $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is Borel measurable in $(x, u)$, and $E$ is a closed set.

For an $\eta \in [0, 1]$, we define the Hamiltonian $H^\eta$ by

$$H^\eta(p, x, u) := (p, f(x, u)) - \eta \Lambda(x, u),$$

where

$$(p, x, u) \in \mathbb{R}^d \times \mathbb{R}^d \times U.$$
\[ \dot{p}(t) = \partial_C H^\eta(p(t), x, u_\ast(t)) |_{x=x_\ast(t)} \]  
for a.e. \( t \in [0, \hat{t}] \),  
(6)

where \( \partial_C \) denotes the generalized gradient (with respect to the second argument), the Hamiltonian maximum condition  
\[ H^\eta(p(t), x_\ast(t), u_\ast(t)) = \sup_{v \in \mathbb{U}} H^\eta(p(t), x_\ast(t), v) \]  
for a.e. \( t \in [0, \hat{t}] \),  
(7)

as well as the constancy of the Hamiltonian  
\[ H^\eta(p(t), x_\ast(t), u_\ast(t)) = h \]  
for a.e. \( t \in [0, \hat{t}] \), some \( h \in \mathbb{R} \).  
(8)

In the above result, the quadruple \( (\eta, p(t), x_\ast(t), u_\ast(t)) \) is known as the extremal lift of the optimal state-action trajectory  
\[ [0, \hat{t}] \ni t \mapsto (x_\ast(t), u_\ast(t)) \].

The number \( \eta \) is called the abnormal multiplier. The abnormal case — when \( \eta = 0 \) — may arise, e.g., when the constraints of the optimal control problem are so tight that the cost function plays no role in determining the solution. For instance, we have an abnormal case when the optimal solution \( t \mapsto (x_\ast(t), u_\ast(t)) \) is “isolated” in the sense that there is no other solution satisfying the end-point constraints in the vicinity — as measured by the supremum norm — of the optimal solution.

III. Maximum Hands-off Control with Feedback

Whilst our approach is in principle applicable to a variety of systems, we here focus our attention to the double integrator with state variables \( x(t), y(t) \in \mathbb{R} \) and control \( \nu \in \mathbb{R} \), described as per:

\[
\begin{align*}
\dot{x}(t) &= y(t) \\
\dot{y}(t) &= \nu(t), \quad x(0) = \bar{x}, \quad y(0) = \bar{y}
\end{align*}
\]

Seminal results in [11] propose a finite-time stabilizing controller for the double integrator dynamics (10). We recall the following special case of [11, Proposition 1] on a finite-time feedback law.

**Proposition III.1.** The origin is a globally finite-time-stable equilibrium of (10) (as in Definition II.1) under the control law,  
\[ \nu(t) := \psi(x(t), y(t)) = -y(t)^{1/3} - (x(t) + \frac{3}{5} y(t)^{5/3})^{1/5}. \]

(11)

**Remark III.2.** The closed-loop vector field obtained by substituting (11) in (10) is locally Lipschitz everywhere except on the \( x \)-axis and the set  
\[ S_{1/3} = \{(x, y) \in \mathbb{R}^2 : (x + \frac{3}{5} y^{5/3}) = 0\} \]  
(see [11, Remark 2]). The closed-loop vector field is transversal to the \( x \)-axis everywhere but the origin and therefore Lemma 2 of [21] can be used to claim existence of a unique forward solutions It is also fairly straightforward to conclude that \( S_{1/3} \) is a positively invariant set and therefore initial conditions in the set \( S_{1/3} \) will ‘slide’ along \( S_{1/3} \) to the origin in finite-time.

We now define the following optimal control problem in order to time-sparsify over \([0, \hat{t}]\), with the finite-time control law (11).

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} (x(\hat{t})^2 + y(\hat{t})^2) + \mu \int_0^\hat{t} h(u(t)) \, dt \\
\text{subject to} & \quad \begin{cases}
\dot{x}(t) = y(t) \\
\dot{y}(t) = -u(t) \left[y(t)^{1/3} + (x(t) + \frac{3}{5} y(t)^{5/3})^{1/5}\right], \\
u(t) \in [0, 1] \quad \text{for a.e. } t \in [0, \hat{t}], \quad u \text{ Lebesgue measurable}, \\
(x(0), y(0)) = (\bar{x}, \bar{y}), \\
(x(\hat{t}), y(\hat{t})) \in S_{1/3} \subset \mathbb{R}^2,
\end{cases}
\end{align*}
\]

(12)

with \( S_{1/3} \) being a closed set as defined earlier and some \( \mu > 0 \). For a real number \( \eta \), we define the Hamiltonian \( H^\eta \) by

\[
H^\eta(p_1(t), p_2(t), x(t), y(t), u(t)) := p_1(t)y(t) - p_2(t)u(t) \\
\left[y(t)^{1/3} + (x(t) + \frac{3}{5} y(t)^{5/3})^{1/5}\right] - \eta \mu h(u(t)), \\
\left[(p_1(t), p_2(t)), (x(t), y(t)), u(t)\right] \in \mathbb{R}^2 \times \mathbb{R}^2 \times [0, 1].
\]

In (12) the stage-cost \( h(u) \) is a sparsity cost on the control encoded via the \( L_1 \) norm or the \( L_0 \) semi-norm. This will be elaborated upon in further results. We also make the following feasibility assumption.

**Assumption III.3.** The transfer time \( \hat{t} \) is greater than \( T(\bar{x}, \bar{y}) \) which is the finite-time required for the following autonomous dynamics to converge to the set \( S_{1/3} \):

\[
\begin{align*}
\dot{x}(t) &= y(t) \\
\dot{y}(t) &= -\left[y(t)^{1/3} + (x(t) + \frac{3}{5} y(t)^{5/3})^{1/5}\right], \quad x(0) = \bar{x}, \quad y(0) = \bar{y}
\end{align*}
\]

(13)

It can be observed that such \( T(\bar{x}, \bar{y}) \) exist for any \( (\bar{x}, \bar{y}) \in \mathbb{R}^2 \) by the finite-time stability claim in Proposition III.1 and then subsequently referring to Definition II.1. \( \square \)

We are now ready to state our main result in the following theorem. We define a placeholder

\[
\Sigma(x(t), y(t)) := \left[y(t)^{1/3} + (x(t) + \frac{3}{5} y(t)^{5/3})^{1/5}\right]
\]

to aid a compact presentation below.

**Theorem III.4.** Consider the optimal control problem in (12) and refer to the notations introduced in this section. Then the problem stated in (12) is feasible under Assumption III.3. **Further**, if \([0, \hat{t}] \ni t \mapsto (x_\ast(t), y_\ast(t), u_\ast(t)) \in \mathbb{R} \times \mathbb{R} \times [0, 1] \) is
a local minimizer of (12), then there exists a scalar, \( \eta = 1 \) or 0 and an absolutely continuous map \((p_1, p_2) := p : [0, \hat{t}] \rightarrow \mathbb{R}^2\) satisfying the following conditions for a.e. \( t \in [0, \hat{t}]\):

- If \( \eta = 1 \) and \( h(u(t)) = |u(t)| \) (\( L_1 \) optimal),

\[
(p(0), -p(\hat{t})) \in \mathbb{R}^2 \times \{(x(\hat{t}), y(\hat{t}))\} + \text{span}\{(1, y(\hat{t})^{2/3}) \mid y \in \mathbb{R}\}
\]  

(Transversality)

\[
-\dot{p}(t) = \begin{pmatrix}
M_{11}(x(t), y(t), u(t)) & M_{12}(x(t), y(t), u(t)) \\
M_{21}(x(t), y(t), u(t)) & M_{22}(x(t), y(t), u(t))
\end{pmatrix}^\top p(t) \quad \text{(Adjoint equation)}
\]

with,

\[
M_{11}(x(t), y(t), u(t)) := 0; \quad M_{12}(x(t), y(t), u(t)) := 1;
\]

\[
M_{21}(x(t), y(t), u(t)) := -\frac{u(t)}{5} \left( x(t) + \frac{3}{5}y(t)^{\frac{2}{3}} \right) - \frac{4}{9} y(t)^{\frac{2}{3}};
\]

\[
M_{22}(x(t), y(t), u(t)) := -u(t) \left[ \frac{1}{3} y(t)^{-\frac{2}{3}} + \frac{1}{5} x(t) + \frac{3}{5} y(t)^{\frac{2}{3}} \right] \quad \text{(15)}
\]

\[
u_*(t) \in \begin{cases}
\{0\} & p_2(t) \Sigma(x_*(t), y_*(t)) > -\mu \\
\{1\} & p_2(t) \Sigma(x_*(t), y_*(t)) < -\mu \\
[0, 1] & p_2(t) \Sigma(x_*(t), y_*(t)) = -\mu
\end{cases}
\]  

(Hamiltonian maximization)

\[
H^n(p_1(t), p_2(t), x_*(t), y_*(t), u_*(t)) = h,
\]

for a.e. \( t \in [0, \hat{t}] \) and some \( h \in \mathbb{R} \). \hspace{1cm} (Hamiltonian constancy)

- If \( \eta = 0 \),

\[
(p(0), -p(\hat{t})) \in \mathbb{R}^2 \times (x(\hat{t}), y(\hat{t}))+ \text{span}\{(1, y(\hat{t})^{2/3}) \mid y \in \mathbb{R}\}
\]  

(Transversality)

\[
-\dot{p}(t) = \begin{pmatrix}
M_{11}(x(t), y(t), u(t)) & M_{12}(x(t), y(t), u(t)) \\
M_{21}(x(t), y(t), u(t)) & M_{22}(x(t), y(t), u(t))
\end{pmatrix}^\top p(t) \quad \text{(Adjoint equation)}
\]

with \( M_{ij}(x(t), y(t), u(t)) \) for \( i, j \in \{1, 2\} \) is as defined in (15).

\[
u_*(t) \in \begin{cases}
\{0\} & p_2(t) \Sigma(x_*(t), y_*(t)) > 0 \\
\{1\} & p_2(t) \Sigma(x_*(t), y_*(t)) < 0 \\
[0, 1] & p_2(t) \Sigma(x_*(t), y_*(t)) = 0
\end{cases}
\]  

(Hamiltonian maximization)

\[
H^n(p_1(t), p_2(t), x_*(t), y_*(t), u_*(t)) = h,
\]

for a.e. \( t \in [0, \hat{t}] \) and some \( h \in \mathbb{R} \). \hspace{1cm} (Hamiltonian constancy)

Proof. We first prove the feasibility claim of the theorem. It is evident that choosing \( u(t) = 1 \) for all \( t \in [0, \hat{t}] \) results in the finite-time-stable closed-loop dynamics (13). Further, the trajectories of the aforementioned closed-loop dynamics reach the set \( \delta_{1/3} \) in some finite time \( T(\overline{x}, \overline{y}) < \hat{t} \) by Assumption III.3. Therefore, the control \( u = 1 \) satisfies all the constraints of the optimal control problem (12) and results in convergence to \( \delta_{1/3} \) at time \( \hat{t} \) (once the closed-loop trajectories reach \( \delta_{1/3} \) they remain on it due to the positive invariance of \( \delta_{1/3} \)); hence \( u(t) = 1 \) is a feasible action.

The necessary conditions in Theorem III.4 are obtained by applying Theorem II.2 to the optimal control problem (12). In order to ensure that Theorem II.2 can in fact be applied to (12), we first verify the assumptions. Lower semi-continuity of \( h(u(t)) = |u(t)| \) or \( -1 \), and the Borel measurability of the closed-loop vector field is immediately evident; so is the continuous differentiability of the terminal cost \( \frac{1}{2}(x^2 + y^2) \). \( \delta_{1/3} \) is closed since it is the inverse of a closed set under a continuous function. The key aspect of the assumptions that needs attention is the Lipschitz-like condition in (3). Since the stage-cost, \( h(u(t)) \) is independent of the states, the condition (3) reduces to finding a map \( k : [0, 1] \rightarrow \mathbb{R} \) such that the vector field \( f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2 \) defined below satisfies a Lipschitz condition for all \( x, y \) in a neighborhood of the optimal \((x_*(t), y_*(t))\), \( u(t) \in U \) at a.e. \( t \in [0, \hat{t}] \) with \( k(u) \) being its rank.

\[
f(x, y, u) = \begin{pmatrix}
\gamma y^{1/3} + (x + \frac{3}{5} y^{2/3})^{1/5}
\end{pmatrix},
\]

From Remark III.2, it is evident that there exists a positive constant \( \gamma \) such that choosing \( k(u) = \gamma u \) satisfies (3).
However, the condition fails on the $x$-axis and the set $S_{1/3}$ as stated in the remark. The vector field $f(x, y, u)$ can be shown to be transversal to the $x$-axis everywhere but the origin [11]. Further, our final condition is to ‘reach’ the set $S_{1/3}$ and therefore we avoid both these contentious regimes if $u_\ast(t) = 1$ on the $x$-axis. Under these conditions, Theorem II.2 is applicable to (12). The necessary conditions can be obtained by directly evaluating (4)-(9) for the optimal control problem (12) with $\eta = 0$ or 1 and $h(u(t)) = |u(t)|$ or $-1(u(t))$.

Remark III.5. To the best of our knowledge, Theorem III.4 is the first result on maximum hands-off, explicit state-feedback control that guarantees closed-loop finite-time stability. Several mechanical systems under linearization possess a double integrator structure and therefore can benefit from the sparse control design suggested by Theorem III.4. Sparse control also reduces data to be transmitted over control channels for networks of double integrators.

Remark III.6. In Theorem III.4, requiring the final states to converge to the origin is not possible because this involves ‘sliding’ on $S_{1/3}$. The vector field $f(x, y, u)$ does not satisfy the Lipschitz criteria on $S_{1/3}$ and Theorem II.2 cannot be applied to obtain necessary conditions. We therefore specify final conditions to lie on $S_{1/3}$. However, $S_{1/3}$ is a positively invariant set and therefore any terminal state on $S_{1/3}$ is transferred in finite-time to the origin, thus justifying our relaxation.

Remark III.7. The $L_1$ optimal control is known to be identical to the $L_0$ optimal control whenever the $L_1$ optimal control signal turns out to be ‘bang-off-bang’ [3, Theorem 8]. Therefore as per Theorem III.4, the $L_1$ and $L_0$ optimal control signals are identical almost everywhere if $p_2(t)\Sigma(x_\ast(t), y_\ast(t)) \neq -\mu$ for almost any $t \in [0, \bar{t}]$.

IV. NUMERICAL EXPERIMENTS

In this section we illustrate the optimal control problem (12) with $h(u(t)) = |u(t)|$ ($L_0$ optimal) and $\mu = 1$. The initial condition is $(\overline{x}, \overline{y}) = (-4.5, 0.5)$, and the key time instants are $\bar{t} = 5.9s$, $t_f = 7.4s$. The settling time for this initial data is, $T(\overline{x}, \overline{y}) \approx 5.45s$. Since, $\bar{t} > T(\overline{x}, \overline{y}) > \overline{T}(\overline{x}, \overline{y})$ feasibility is guaranteed. The simulations are terminated once $\|(x(t), y(t))\| \leq 5 \times (10)^{-4}$. The final conditions at $t = \bar{t}$ are set to lie in the dilated set $S_{1/3}^{(\epsilon)}$ with $\epsilon = 0.1$. This is because the necessary conditions result in a very stiff differential equation close to $S_{1/3}$. For $t \in [\bar{t}, \bar{t}]$ we set $u(t) = 1$.

The optimal control described by Theorem III.4 is applied to the simulation and corresponding results are depicted in Figures 1 to 3. Figure 1 shows the phase portrait of the system, the curve from $(\bar{x}, \bar{y})$ to $(\hat{x}, \hat{y}) := (x(\hat{t}), y(\hat{t}))$ is the optimal part and corresponds to $t \in [0, \hat{t}]$ while the curve from $(\hat{x}, \hat{y})$ to the origin corresponds to $u(t) = 1$ with $t \in [\hat{t}, t_f]$. Figure 2 plots the optimal control trajectory $u_\ast$. It is evident from Figure 2 that the controller remains non zero only for a part of their activation times from point $(\bar{x}, \bar{y})$ to point $(\hat{x}, \hat{y})$ fulfilling the sparsity requirement, while at point $(\hat{x}, \hat{y})$ it is switched to $u(t) = 1$ which ensures the trajectories reach $S_{1/3}$ and follow it to the origin at time $t_f$. It is also evident from Figure 1 that given sufficient time $t_f$ the control can be $u(t) = 0$ too for $t \in [\hat{t}, t_f]$. In that case the phase portrait in Figure 1 will continue on the horizontal path from $[\bar{t}, t_f]$ before reaching $S_{1/3}$. It is interesting to observe from Figure 2 that the total control activation time is lesser than the settling time $T(\overline{x}, \overline{y})$. Finally, Figure 3 shows the evolution of the states with time and indicates convergence to $S_{1/3}^{(\epsilon)}$ at $\bar{t} = 5.9s$ beyond which the trajectories move towards the sliding surface, eventually sliding on $S_{1/3}$ to converge to the origin at $t_f = 7.4s$. In order to illustrate the fact that the robustness properties of the feedback are preserved upon time-sparsification, we insert a sinusoidal disturbance of magnitude 0.05 units to the $y$-dynamics with the control action remaining exactly as before (Figure 2). The corresponding phase-portrait is shown in Figure 4. We notice that the earlier horizontal segments of the phase portrait in Figure 1 are no longer so,
observed that the final conditions hit the dilation due to the introduction of the disturbance. However, it was justifying sparsification. Future work will be geared towards

is less than the settling time for the nominal dynamics, thus from the simulations is that the total control activation time the optimal control and state trajectories. A key observation for optimality which is then solved numerically to arrive at

smooth cost is used to arrive at the necessary conditions finite-time stability property is preserved with minimum an optimal on-off signal is designed to guarantee that the time stabilizing controller from the literature was used and stabilizing state feedback control laws. A classical finite-

tackling more complex and nonlinear dynamical systems in this framework to evolve a general means of computing timesparse feedback laws.

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