On the Global Existence and Blowup Phenomena of Schrödinger Equations with Multiple Nonlinearities

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Abstract
In this paper, we consider the global existence and blowup phenomena of the following Cauchy problem

\[
\begin{cases}
-i u_t = \Delta u - V(x)u + f(x, |u|^2)u + (W \ast |u|^2)u, & x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\]

where \(V(x)\) and \(W(x)\) are real-valued potentials with \(V(x) \geq 0\) and \(W\) is even, \(f(x, |u|^2)\) is measurable in \(x\) and continuous in \(|u|^2\), and \(u_0(x)\) is a complex-valued function of \(x\). We obtain some sufficient conditions and establish two sharp thresholds for the blowup and global existence of the solution to the problem. These results can be looked as the supplement to Chapter 6 of [3]. In addition, our results extend those of [17] and improve some of [15].

Keywords: Nonlinear Schrödinger equation; Global existence; Blow up in finite time; Sharp threshold.

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1 Introduction
In this paper, we are interested in the global existence and blowup phenomena of the following Cauchy problem

\[
\begin{cases}
-i u_t = \Delta u - V(x)u + f(x, |u|^2)u + (W \ast |u|^2)u, & x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) = u_0(x) \in \Sigma, & x \in \mathbb{R}^N,
\end{cases}
\]

where \(V(x)\) and \(W(x)\) are real-valued potentials with \(V(x) \geq 0\) and \(W\) is even, \(f(x, |u|^2)\) is measurable in \(x\) and continuous in \(|u|^2\), and \(u_0(x)\) is a complex-valued function of \(x\),

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and Σ is a natural Hilbert space:

\[ \Sigma = \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) |u|^2 dx < +\infty \} \]  

(1.2)

with the inner product

\[ \langle \varphi, \psi \rangle = \int_{\mathbb{R}^N} [\varphi \bar{\psi} + \nabla \varphi \cdot \nabla \bar{\psi} + V(x) \varphi \bar{\psi}] dx \]  

(1.3)

and the norm

\[ \| u \|_{\Sigma}^2 = \int_{\mathbb{R}^N} [|u|^2 + |\nabla u|^2 + V(x)|u|^2] dx. \]  

(1.4)

The model (1.1) appears in the theory of Bose-Einstein condensation, nonlinear optics and theory of water waves (see [3, 5, 6, 8, 9, 13]).

In convenience, we will give some assumptions on \( V, f \) and \( W \) as follows.

(V1) \( V(x) \geq 0 \) and \( V \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) for \( r \geq 1, r > \frac{N}{2} \) or

(V2) \( V(x) \in S_1^r \), \( V(x) \geq 0 \) and \( |D^\alpha V| \) is bounded for all \( |\alpha| \geq 2 \). Here \( S_1^r \) is the complementary set of \( S_1 = \{ V(x) \text{ satifies (V1)} \} \).

(f1) \( f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \) is measurable in \( x \) and continuous in \( |u|^2 \) with \( f(x, 0) = 0 \). Assume that for every \( k > 0 \) there exists \( L(k) < +\infty \) such that \( |f(x, s_1) - f(x, s_2)| \leq L(k)|s_1 - s_2| \) for all \( 0 \leq s_1 < s_2 < k \). Here

\[ L(k) \in C([0, \infty)), \quad \text{if } N = 1 \]
\[ L(k) \leq C(1 + k^\alpha) \quad \text{with } 0 \leq \alpha < \frac{2}{N-2}, \quad \text{if } N \geq 2. \]  

(1.5)

(W1) \( W \) is even and \( W \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N) \) for some \( q \geq 1, q > \frac{N}{2} \).

Denote \( \frac{1}{(N-2)^+} = +\infty \) when \( N = 1, 2 \) and \( (N-2)^+ = N - 2 \) when \( N \geq 3 \).

First, we consider the local well-posedness of (1.1). We have a proposition as follows.

**Proposition 1.1.** (Local Existence Result) Assume that (f1) and (W1) are true, \( V(x) \) satisfies (V1) or (V2), \( u_0 \in \Sigma \). Then there exists a unique solution \( u \) of (1.1) on a maximal time interval \([0, T_{\max})\) such that \( u \in C(\Sigma; [0, T_{\max})) \) and either \( T_{\max} = +\infty \) or else

\[ T_{\max} < +\infty, \quad \lim_{t \to T_{\max}} \| u(\cdot, t) \|_{\Sigma} = +\infty. \]

**Definition 1.1.** If \( u \in C(\Sigma; [0, T]) \) with \( T = \infty \), we say that the solution \( u \) of (1.1) exists globally. If \( u \in C(\Sigma; [0, T)) \) with \( T < +\infty \) and \( \lim_{t \to T} \| u(\cdot, t) \|_{\Sigma} \rightarrow +\infty \), we say that the solution \( u \) of (1.1) blows up in finite time.

Our main topic is the global existence and blowup phenomena of the solution to (1.1), which is directly motivated by [3]. Since Cazevave established some results on blowup and global existence of the solutions to (1.1) with (V1), (f1) and (W1) in [3], we are interested in the parallel problems such as: What are the results about the blowup
and global existence of the solutions to (1.1) with (V2), (f1) and (W1)? How can we establish the sharp threshold for global existence and blowup of the solution to (1.1)?

About the topic of global existence and blowup in finite time, there are many results on the special cases of (1.1). However, we only cite some very related references which only gave some sufficient conditions on global existence and blowup of the solution to the special case of (1.1). We will show how all the cited results give coherence and connection to our paper below. A special case of (1.1) is

\[
\begin{cases}
-i u_t = \Delta u + f(|u|^2)u, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}
\]

In [7], Glassey established some blowup results for (1.6). In [1], Berestycki and Cazenave established the sharp threshold for blowup of (1.6) with supercritical nonlinearity by considering a constrained variational problem. In [16], Weinsten presented a relationship between the sharp criterion for the global solution of (1.6) and the best constant in the Gagliardo-Nirenberg’s inequality. In [4], Cazenave and Weisseler established the local existence and uniqueness of the solution to (1.6) with

\[f(|u|^2)u = |u|^p u,\]

Very recently, Tao et al. in [15] studied the Cauchy problem (1.6) with

\[f(|u|^2)u = \mu|u|^{p_1} u + \nu|u|^{p_2} u,\]

where \(\mu\) and \(\nu\) are real numbers, \(0 < p_1 < p_2 < \frac{4}{N-2}\) with \(N \geq 3\). This type of nonlinearity brings the failure of the equation in (1.6) to be scale invariant and it cannot satisfy the conditions of the blowup theorem in [7] in some cases. Tao et al. established the results on local and global well-posedness, asymptotic behavior (scattering) and finite time blowup under some assumptions. These papers above have given some sufficient conditions on global existence and blowup of the solution or established the sharp threshold for the special case of (1.6). Naturally, we want to establish a new sharp threshold for global existence and blowup of the solution to (1.6) in this paper, which will generalize or even improve these results above.

The following Cauchy problem

\[
\begin{cases}
-i u_t = \frac{1}{2} \Delta u - V(x)u + |u|^p u, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]

is also a special case of (1.1). If \(p < \frac{4}{N}\), in [13], Oh obtained the local well-posedness and global existence results of (1.7) under some conditions on \(V(x)\). If \(\frac{4}{N} \leq p < \frac{4}{(N-2)^+}\), in [17], Zhang established a sharp threshold for the global existence and blowup of the solutions to (1.7) with \(V(x) = |x|^2\). Another special case of (1.1) is the following Cauchy problem of Schrödinger-Hartree equation:

\[
\begin{cases}
-i u_t = \Delta u + (W \ast |u|^2)u, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]

Using a contraction mapping argument and energy estimates, Hitoshi obtained the local and global existence results on (1.8) in [8]. More recently, Miao et al. studied the global
well-posedness and scattering for the mass-critical Hartree equation with radial data in [11] and global well-posedness, scattering and blowup for the energy-critical, focusing Hartree equation with the radial case in [12]. And in [10], Li et al. also dealt with the focusing energy-critical Hartree equation, they prove that the maximal-lifespan $I = \mathbb{R}$, moreover, the solution scatters in both time directions. However, there are few results on the sharp threshold for global existence and blowup of the solution to (1.8). Therefore, we want to establish a sharp threshold for global existence and blowup of the solution to (1.8) under some conditions.

Now we will introduce some notations. Denote

$$F(x, |u|^2) = \int_0^{|u|^2} f(x, s) ds, \quad G(|u|^2) = \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx$$ \hspace{1cm} (1.9)

$$h(u) = -V(x)u + f(x, |u|^2)u + (W * |u|^2)u,$$ \hspace{1cm} (1.10)

$$H(u) = -\frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2)dx + \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx.$$ \hspace{1cm} (1.11)

Mass ($L^2$ norm)

$$M(u) := \left( \int_{\mathbb{R}^N} |u(x, t)|^2 dx \right)^{\frac{1}{2}}.$$ \hspace{1cm} (1.12)

Energy

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2)dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 dx.$$ \hspace{1cm} (1.13)

In [3], Cazenave obtained some sufficient conditions on blowup and global existence of the solution to (1.1) with (V1), (f1) and (W1). The following two theorems can be looked as the parallel results to Corollary 6.1.2 and Theorem 6.5.4 of [3] respectively.

**Theorem 1.** (Global Existence) Assume that $u_0 \in \Sigma$, (V2) and (f1) are true, and

$$W^+ \in L^q(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$ \hspace{1cm} (1.14)

for some $q \geq 1$, $q \geq \frac{N}{2}$ (and $q > 1$ if $N = 2$). Here $W^+ = \max(W, 0)$. Suppose further that there exist constants $c_1$ and $c_2$ such that $F(x, |u|^2) \leq c_1|u|^2 + c_2|u|^{2p+2}$ with $0 < p < \frac{2}{N}$. Then the solution of (1.L) exists globally. That is, $\|u(\cdot, t)\|_{\Sigma} < +\infty$ for all $0 < t < +\infty$.

**Theorem 2.** (Blowup in Finite Time) Assume that $u_0 \in \Sigma$, $|x|u_0 \in L^2(\mathbb{R}^N)$, (V2), (f1) and (W1) are true. Suppose further that

$$(N + 2)F(x, |u|^2) - N|u|^2 f(x, |u|^2) \leq 0,$$ \hspace{1cm} (1.15)

$$2V(x) + (x \cdot \nabla V) \geq 0 \quad a.e.,$$ \hspace{1cm} (1.16)

$$2W(x) + (x \cdot \nabla W) \leq 0 \quad a.e.$$ \hspace{1cm} (1.17)
If \( (1) \ E(u_0) < 0 \) or \( (2) \ E(u_0) = 0 \) and \( \exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 dx < 0 \),

then the solution of \((1.1)\) will blow up in finite time. That is, there exists \( T_{\max} < \infty \) such that

\[
\lim_{t \to T_{\max}} \|u(\cdot, t)\|_{\Sigma} = \infty.
\]

Denote

\[
Q(u) := 2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} (x \cdot \nabla) |u|^2 dx
\]

\[
+ N \int_{\mathbb{R}^N} [F(x, |u|^2) - |u|^2 f(x, |u|^2)] dx + \frac{1}{2} \int_{\mathbb{R}^N} ((x \cdot \nabla W) \ast |u|^2) |u|^2 dx. \tag{1.18}
\]

We will establish the first type of sharp threshold as follows.

**Theorem 3.** (Sharp Threshold I) Assume that \( V(x) \equiv 0 \) and \( W \in L^q(\mathbb{R}^N) \) with \( \frac{N}{q} < q < \frac{N}{2} \). Suppose further that \( f(x, 0) = 0 \) and there exist constants \( c_1, c_2, c_3 > 0 \) and \( \frac{2}{q} < p_1, p_2, l < \frac{2}{(N-2)^+} \) such that

\[
IF(x, |u|^2) \leq |u|^2 f(x, |u|^2) - F(x, |u|^2) \leq c_1 |u|^{2p_1 + 2} + c_2 |u|^{2p_2 + 2}, \tag{1.19}
\]

\[
NIW(x) + (x \cdot \nabla W) \leq 0 \leq c_3 W(x) + (x \cdot \nabla W). \tag{1.20}
\]

Let \( \omega \) be a positive constant satisfying

\[
d_1 := \inf_{\{u \in \Sigma \setminus \{0\}, Q(u) = 0\}} (\omega \|u\|_2^2 + E(u)) > 0, \tag{1.21}
\]

where \( Q(u) \) is defined by \((1.18)\). Suppose that \( u_0 \in H^1(\mathbb{R}^N) \) satisfies

\[
\omega \|u_0\|_2^2 + E(u_0) < d_1.
\]

Then

\( (1) \) If \( Q(u_0) > 0 \), the solution of \((1.1)\) exists globally;

\( (2) \) If \( Q(u_0) < 0 \), \( |x|u_0 \in L^2(\mathbb{R}^N) \) and \( \exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 dx < 0 \), the solution of \((1.1)\) blows up in finite time.

**Remark 1.1.** Theorem 3 is only suitable for \((1.1)\) with \( V(x) \equiv 0 \). To establish the sharp threshold for \((1.1)\) with \( V(x) \neq 0 \), we will construct a type of cross constrained variational problem and establish some cross-invariant manifolds. First, we introduce some functionals as follows:

\[
I_\omega(u) = \omega \|u\|_2^2 + E(u), \tag{1.22}
\]

\[
S_\omega(u) = 2\omega \|u\|_2^2 + \int_{\mathbb{R}^N} \left\{|\nabla u|^2 + V(x)|u|^2 - f(x, |u|^2)|u|^2 - (W \ast |u|^2)|u|^2\right\} dx. \tag{1.23}
\]

Denote the Nehari manifold

\[
\mathcal{N} := \{ u \in \Sigma \setminus \{0\}, \ S_\omega(u) = 0 \}, \tag{1.24}
\]
and cross-manifold

\[ \mathcal{CM} := \{ u \in \Sigma \setminus \{0\}, \ S_\omega(u) < 0, \ Q(u) = 0 \}. \]  

(1.25)

And define

\[ d_N := \inf_N I_\omega(u), \]  

(1.26)

\[ d_M := \inf_{\mathcal{CM}} I_\omega(u), \]  

(1.27)

\[ d_{II} := \min(d_N, d_M). \]  

(1.28)

In Section 5, we will prove that \( d_{II} \) is always positive. Therefore, it is reasonable to define the following cross-manifold

\[ \mathcal{K} := \{ u \in \Sigma \setminus \{0\} : I_\omega(u) < d_{II}, \ S_\omega(u) < 0, \ Q(u) < 0 \}. \]  

(1.29)

We give the second type of sharp threshold as follows

**Theorem 4.** (Sharp Threshold II) Assume that (f1), (W1) and (1.19). Suppose that

\[ W(x) \geq 0, \quad NIW(x) + (x \cdot \nabla W) \leq 0 \]  

(1.30)

and there exists a positive constant \( c \) such that

\[ NIW(x) + (x \cdot \nabla V) \geq cV(x) \geq 0 \]  

(1.31)

with the same \( l \) in (1.19). Assume further that the function \( f(x, |u|^2) \) satisfies \( f(x, 0) = 0 \) and

\[ f(x, |u|^2) \leq f(x, k^2|u|^2), \quad f_s'(x, k^2|u|^2) \leq f_s'(x, |u|^2), \]  

(1.32)

\[ F(x, k^2|u|^2) - k^2|u|^2 f(x, k^2|u|^2) \leq k^2[F(x, |u|^2) - |u|^2 f(x, |u|^2)] \]  

(1.33)

for \( k > 1 \). Here \( f_s'(x, z) \) is the value of the partial derivative of \( f(x, s) \) with respect to \( s \) at the point \( (x, z) \). If \( u_0 \in \Sigma \) satisfies \( |x|u_0 \in L^2(\mathbb{R}^N) \) and \( I_\omega(u_0) = \omega\|u_0\|^2 + E(u_0) < d_{II}, \) then the solution of (1.1) blows up in finite time if and only if \( u_0 \in \mathcal{K} \).

**Remark 1.2.** (1) \( f(x, |u|^2) \leq f(x, k^2|u|^2) \) implies that \( k^2F(x, |u|^2) \leq F(x, k^2|u|^2) \) for \( k > 1 \).

(2) The blowup of solution to (1.1) will benefit from the condition \( V(x) \geq 0 \). In some cases, the blowup of the solution to (1.1) can be delayed or prevented by the introduction of potential(see [2] and the references therein).

This paper is organized as follows: In Section 2, we will prove Proposition 1.1, recall some results of [3] and give some other properties. In Section 3, we will prove Theorem 1 and 2. In Section 4, we establish the sharp threshold for (1.1) with \( V(x) \equiv 0 \). In Section 5, we will prove Theorem 4.
2 Preliminaries

In the sequel, we use $C$ and $c$ to denote various finite constants, their exact values may vary from line to line.

First, we will give the proof of Proposition 1.1.

**The proof of Proposition 1.1:** If (V1) is true, then there exist $V_1(x) \in L^r(\mathbb{R}^N)$ with $r \geq 1$, $r > \frac{N}{2}$, and $V_2(x) \in L^\infty(\mathbb{R}^N)$ such that

$$V(x) = V_1(x) + V_2(x).$$

Noticing that $0 < \frac{2r}{r-1} < \frac{2N}{N-2}$, using Hölder’s and Soblev’s inequalities, we have

$$\int_{\mathbb{R}^N} V(x)|u|^2 \, dx = \int_{\mathbb{R}^N} V_1(x)|u|^2 \, dx + \int_{\mathbb{R}^N} V_2(x)|u|^2 \, dx \leq \left( \int_{\mathbb{R}^N} |V(x)|^r \, dx \right)^{\frac{1}{r}} \left( \int_{\mathbb{R}^N} |u|^{2r} \, dx \right)^{\frac{1}{2r}} + C \int_{\mathbb{R}^N} |u|^2 \, dx$$

for any $u \in H^1(\mathbb{R}^N)$. Consequently, we have

$$\|u\|_{H^1} \leq \|u\|_{\Sigma} \leq C\|u\|_{H^1},$$

which means that $\Sigma = H^1(\mathbb{R}^N)$ if $V \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for $r \geq 1$, $r > \frac{N}{2}$. By the results of Theorem 3.3.1 in [3], we have the local well-posedness result of (1.1) in $H^1(\mathbb{R}^N)$.

If (V2), (f1) and (W1) are true, similar to the proof of Theorem 3.5 in [13], we can establish the local well-posedness result of (1.1) in $\Sigma$. Roughly, we only need to replace $|u|^{p+1}u$ by $f(x, |u|^2)u + (W*|u|^2)u$ in the proof, and we can obtain the similar results under the assumptions of (V2), (f1) and (W1). We omit the detail here. □

Noticing that $3h(u)\bar{u} = 0$ and $h(u) = H'(u)$, following the method of [7] and the discussion in Chapter 3 of [3], one can obtain the conservation of mass and energy. We give the following proposition without proof.

**Proposition 2.1.** Assume that $u(x, t)$ is a solution of (1.1). Then

$$M(u) = \left( \int_{\mathbb{R}^N} |u(x, t)|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^N} |u_0(x)|^2 \, dx \right)^{\frac{1}{2}} = M(u_0),$$

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} \{ |\nabla u|^2 + V(x)|u|^2 - F(x, |u|^2) \} \, dx - G(|u|^2) = E(u_0)$$

for any $0 \leq t < T_{\text{max}}$.

We will recall some results on blowup and global existence of the solution to (1.1) with (V1), (f1) and (W1).
**Theorem A** (Corollary 6.1.2 of [3]) Assume that (V1), (f1) and (1.14). Suppose that there exist $A \geq 0$ and $0 \leq p < \frac{2}{N}$ such that

$$F(|u|^2) \leq A|u|^2(1 + |u|^{2p}).$$

Then the maximal strong $H^1$-solution of (1.1) is global and $\sup\{\|u\|_{H^1} : t \in \mathbb{R}\} < \infty$ for every $u_0 \in H^1(\mathbb{R}^N)$.

**Theorem B** (Theorem 6.5.4 of [3]) Assume that (V1), (f1), (W1) and (1.13)–(1.17). If $u_0 \in H^1(\mathbb{R}^N)$, $|x|u_0 \in L^2(\mathbb{R}^N)$ and $E(u_0) < 0$, then the $H^1$-solution of (1.1) will blow up in finite time.

Let $J(t) = \int_{\mathbb{R}^N} |x|^2|u|^2 \, dx$. After some elementary computations, we obtain

$$J'(t) = 4\Im \int_{\mathbb{R}^N} (x \cdot \nabla u) \bar{u} \, dx, \quad J''(t) = 4Q(u).$$

We have the following proposition

**Proposition 2.2.** Assume that $u(x, t)$ is a solution of (1.1) with $u_0 \in \Sigma$ and $|x|u_0 \in L^2(\mathbb{R}^N)$. Then the solution to (1.1) will blow up in finite time if either

1. there exists a constant $c < 0$ such that $J''(t) = 4Q(u) \leq c < 0$ or
2. $J''(t) = 4Q(u) \leq 0$ and $J'(0) = \Im \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0$.

**Proof:** Since $u_0 \in \Sigma$ and $|x|u_0 \in L^2(\mathbb{R}^N)$, we have

$$|J'(0)| < 4 \int_{\mathbb{R}^N} |(x \bar{u}_0|\nabla u_0)| \, dx \leq 8 \int_{\mathbb{R}^N} (|\nabla u_{10}|^2 + |xu_{10}|^2) \, dx < +\infty.$$

(1) If $J''(t) \leq c < 0$, integrating it from 0 to $t$, we get $J'(t) < ct + J'(0)$. Since $c < 0$, we know that there exists a $t_0 = \max(0, \frac{J'(0)}{c})$ such that $J'(t) < J'(t_0) < 0$ for $t > t_0$. On the other hand, we have

$$0 \leq J(t) = J(t_0) + \int_{t_0}^{t} J'(s) \, ds < J(t_0) + J'(t_0)(t - t_0),$$

which implies that there exists a $T_{\text{max}} < +\infty$ satisfying

$$\lim_{t \to T_{\text{max}}} J(t) = 0.$$  \hspace{1cm} (2.4)

Using the inequality

$$\|g\|_2^2 \leq \frac{2}{N} \|\nabla g\|_2 \|xg\|_2 \quad \text{if} \quad g \in H^1(\mathbb{R}^N), \ xg \in L^2(\mathbb{R}^N)$$

and noticing that $\|u(\cdot, t)\|_2 = \|u_0\|_2$, we have

$$\lim_{t \to T_{\text{max}}} \|u\|_{\Sigma} = +\infty.$$  \hspace{1cm} (2.5)

(2) Similar to (2.3), we can get

$$0 \leq J(t) \leq J(0) + J'(0)t,$$

which implies that the solution will blow up in a finite time $T_{\text{max}} \leq \frac{J(0)}{J'(0)}$. \hfill \Box
3 The sufficient conditions on global existence and blowup in finite time

In this section, we will prove Theorem 1 and 2, which give some sufficient conditions on global existence and blowup of the solution to (1.1).

The proof of Theorem 1: Letting $W^+ = W_1 + W_2$, where $W_1 \in L^\infty$ and $W_2 \in L^q$ with $q > \frac{N}{2}$, using Hölder’s and Young’s inequalities, we obtain

$$
\int_{\mathbb{R}^N} (W_2 \ast (uv)) w z dx \leq \|W_2\|_{L^4} \|u\|_{L^r} \|v\|_{L^r} \|w\|_{L^r} \|z\|_{L^r}
$$

with $r = \frac{4q}{2q - 1}$. Especially, we have

$$
\int_{\mathbb{R}^N} (W_2 \ast |u|^2)|u|^2 dx \leq \|W_2\|_{L^4} \|u\|_{L^r}^4. \quad (3.1)
$$

Using (3.1) and Gagliardo-Nirenberg’s inequality, we get

$$
\frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx \leq \|W_1\|_{L^\infty} \|u\|_{L^2}^4 + \|W_2\|_{L^q} \|u\|_{L^r}^4 \frac{4q}{2q - 1}
$$

$$
\leq \|W_1\|_{L^\infty} \|u\|_{L^2}^4 + C \|W_2\|_{L^q} \|\nabla u\|_{L^2} \|u\|_{L^2}^{\frac{8q - 2N}{L^2}}. \quad (3.2)
$$

Using Young’s inequality, from (3.2), we have

$$
C \|W_2\|_{L^q} \|\nabla u\|_{L^2} \|u\|_{L^2}^{\frac{4q - N}{L^2}} \leq \varepsilon \|\nabla u\|_{L^2}^2 + C(\varepsilon, \|W_2\|_{L^q}) \|u\|_{L^2}^{\frac{8q - 2N}{L^2}} \quad (3.3)
$$

for some $\varepsilon > 0$. Noticing that $F(x, |u|^2) \leq c_1 |u|^2 + c_2 |u|^{2p - 2}$, using Gagliardo-Nirenberg’s inequality and (3.3) with $\varepsilon = \frac{1}{4}$, we get

$$
E(u_0) = \frac{1}{2} \left( \int_{\mathbb{R}^N} \left( |\nabla u_0|^2 + V(x)|u_0|^2 - F(x, |u_0|^2) \right) dx \right) - \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u_0|^2)|u_0|^2 dx
$$

$$
= \frac{1}{2} \left( \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V(x)|u|^2 - F(x, |u|^2) \right) dx \right) - \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx
$$

$$
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V(x)|u|^2 - c_1 |u|^2 - c_2 |u|^{2p + 2} \right) dx \right)
$$

$$
- \|W_1\|_{L^\infty} \|u\|_{L^2}^4 - C \|W_2\|_{L^q} \|\nabla u\|_{L^2} \|u\|_{L^2}^{\frac{4q - N}{L^2}}
$$

$$
\geq \frac{1}{2} \left( \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V(x)|u|^2 - c_1 |u|^2 \right) dx \right)
$$

$$
- c_2 C_N \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{pN}{2}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{2 + p(2 - N)}{2}}
$$

$$
- \|W_1\|_{L^\infty} \|u\|_{L^2}^4 - \frac{1}{4} \|\nabla u\|_{L^2}^2 - C \|u\|_{L^2}^{\frac{8q - 2N}{L^2}}. \quad (3.4)
$$
Since \( \|u\|_2 = \|u_0\|_2 \), from (3.4), we can obtain
\[
4E(u_0) + C\|u_0\|_{L^2}^2 + C\|u_0\|_{L^2}^4 + C\|u_0\|_{L^2}^{8-2N/2} 
\geq \int_{\mathbb{R}^N} V(x)|u|^2dx + \int_{\mathbb{R}^N} |\nabla u|^2dx \left( 1 - c \left( \int_{\mathbb{R}^N} |\nabla u|^2dx \right)^{\frac{8N}{2} - 1} \right). \tag{3.5}
\]

Since \( p < \frac{2}{N} \) means that \( \frac{8N}{2} - 1 < 0 \), (3.5) implies that \( \|u\|_{L^2}^2 \) is always controlled by
\[4E(u_0) + C\|u_0\|_{L^2}^2 + C\|u_0\|_{L^2}^4 + C\|u_0\|_{L^2}^{8-2N/2}.\]
That is, the solution of (1.1) exists globally. \( \square \)

**Remark 3.1.** We will give some examples of \( V(x) \), \( f(x, |u|^2) \) and \( W(x) \). It is easy to verify that they satisfy the conditions of Theorem 1.

Example 1. \( V(x) = |x|^2 \), \( W(x) = e^{-\pi |x|^2} \) and \( f(x, |u|^2) = b|u|^{2p} \) with \( b \) is a real constant and \( 0 < p < \frac{2}{N} \).

Example 2. \( V(x) = |x|^2 \), \( W(x) = \frac{|x|^2}{1+|x|^2} \) and \( f(x, |u|^2) = b|u|^{2p} \ln(1 + |u|^2) \) with \( b \) is a real constant and \( 0 < p < \frac{2}{N} \).

**The proof of Theorem 2:** Set
\[y(t) = J'(t) = 4\overline{\alpha} \int_{\mathbb{R}^N} (x \cdot \nabla u)\bar{u}dx. \tag{3.6}\]
Using (1.15)-(1.17), we have
\[
y'(t) = 8 \int_{\mathbb{R}^N} |\nabla u|^2dx - 4 \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2dx 
+ 4N \int_{\mathbb{R}^N} [F(x, |u|^2) - |u|^2f(x, |u|^2)]dx 
+ 2 \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |u|^2\} |u|^2dx 
= 16E(u) + 4 \int_{\mathbb{R}^N} [-2V(x) - (x \cdot \nabla V)]|u|^2 + [(N+2)F(x, |u|^2) - N|u|^2f(x, |u|^2)]dx 
+ 2 \int_{\mathbb{R}^N} \{(2W + (x \cdot \nabla W)) \ast |u|^2\}|u|^2dx \leq 16E(u) = 16E(u_0) < 0. \tag{3.7}\]

From (3.6) and (3.7), we obtain
\[
||xu(x, t)||_{L^2}^2 \leq \|xu_0\|_{L^2}^2 + 4t\overline{\alpha} \int_{\mathbb{R}^N} \bar{u}_0(x \cdot \nabla u_0)dx + 8t^2E(u_0). \tag{3.8}\]

Since \( ||xu(x, t)||_{L^2}^2 \geq 0 \), whether (1) or (2), (3.8) will be absurd for \( t > 0 \) large enough. Therefore, the solution of (1.1) will blow up in finite time. \( \square \)

**Remark 3.2.** We will give some examples of \( V(x) \), \( W(x) \) and \( f(x, |u|^2) \). It is easy to verify that they satisfy the conditions of Theorem 2.

Example 1. \( V(x) = |x|^2 \), \( W(x) = |x|^{-2} \) and \( f(x, |u|^2) = b|u|^{2p} \) with \( b > 0 \) and \( p > \frac{2}{N} \) with \( N \geq 3 \).

Example 2. \( V(x) = |x|^2 \), \( W(x) = |x|^{-2} \) and \( f(x, |u|^2) = b|u|^{2p} \ln(1 + |u|^2) \) with \( b > 0 \) and \( p \geq \frac{2}{N} \) with \( N \geq 3 \).
4 The sharp threshold for global existence and blowup of the solution to (1.1) with $V(x) \equiv 0$ and $W \in L^q(\mathbb{R}^N)$ with $\frac{N}{4} < q < \frac{N}{2}$

In this section, we will establish the sharp threshold for global existence and blowup of the solution to (1.1) with $V(x) \equiv 0$ and $W \in L^q(\mathbb{R}^N)$ with $\frac{N}{4} < q < \frac{N}{2}$.

The proof of Theorem 3. We will proceed in four steps.

Step 1. We will prove $d_I > 0$. $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $Q(u) = 0$ mean that

$$2 \int_{\mathbb{R}^N} |\nabla u|^2 dx = N \int_{\mathbb{R}^N} |u|^2 f(x, |u|^2) - F(x, |u|^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |u|^2\} |u|^2 dx$$

$$\leq \frac{N(l + 1)}{l} \int_{\mathbb{R}^N} [c_1 |u|^{2p_1 + 2} + c_2 |u|^{2p_2 + 2}] dx + C \int_{\mathbb{R}^N} (W \ast |u|^2) |u|^2 dx$$

$$\leq C \|u\|_{2p_1 + 2}^2 + C \|u\|_{2p_2 + 2}^2 + C \|W\|_{L^q} \|u\|_{\frac{4q}{2q - r}}^4.$$

Using Gagliardo-Nirenberg’s and Hölder’s inequalities, we can get

$$2 \leq C(\|\nabla u\|_2^\frac{Np_q}{2} (\|u\|_2^{2p_1 + 1} - \frac{Np_q}{2}) + C(\|\nabla u\|_2^\frac{Np_q}{2} (\|u\|_2^{2p_1 + 1} - \frac{Np_q}{2}))$$

$$+ C(\|\nabla u\|_2^\frac{N}{2} (\|u\|_2^{2p_2 + 1} - \frac{N}{2}))$$

$$\leq C \left\{ (\|\nabla u\|_2^2 + \|u\|_2^{2p_1 + 1}) + (\|\nabla u\|_2^2 + \|u\|_2^{2p_1 + 1}) + (\|\nabla u\|_2^2 + \|u\|_2^{2p_2 + 1}) \right\}.$$

That is,

$$\|\nabla u\|_2^2 + \|u\|_2^2 \geq C > 0 \quad (4.1)$$

if $Q(u) = 0$ and $u \in H^1(\mathbb{R}^N) \setminus \{0\}$.

On the other hand, if $Q(u) = 0$, we have

$$2 \int_{\mathbb{R}^N} |\nabla u|^2 dx = N \int_{\mathbb{R}^N} |u|^2 f(x, |u|^2) - F(x, |u|^2) dx - \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |u|^2\} |u|^2 dx$$

$$\geq Nl \int_{\mathbb{R}^N} F(x, |u|^2) dx + \frac{Nl}{2} \int_{\mathbb{R}^N} \{W \ast |u|^2\} |u|^2 dx,$$

that is,

$$- \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2) dx - \frac{1}{4} \int_{\mathbb{R}^N} \{W \ast |u|^2\} |u|^2 dx \geq - \frac{1}{Nl} \int_{\mathbb{R}^N} |\nabla u|^2 dx. \quad (4.2)$$

Using (4.2), we can obtain

$$\omega \|u\|_2^2 + E(u) = \omega \|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2) dx - \frac{1}{4} \int_{\mathbb{R}^N} \{W \ast |u|^2\} |u|^2 dx$$

$$\geq \omega \|u\|_2^2 + \left( \frac{1}{2} - \frac{1}{Nl} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

$$\geq \min\{\omega, \left( \frac{1}{2} - \frac{1}{Nl} \right) \} (\|\nabla u\|_2^2 + \|u\|_2^2) \geq C > 0$$
from \((1.1)\). Hence
\[ d_I > 0. \]

Step 2. Denote
\[ K_+ = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ Q(u) > 0, \ \omega \|u\|_2^2 + E(u) < d_I \} \]
and
\[ K_- = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \ Q(u) < 0, \ \omega \|u\|_2^2 + E(u) < d_I \}. \]

We will prove that \(K_+\) and \(K_-\) are invariant sets of \((1.1)\) with \(V(x) \equiv 0\) and \(W \in L^q(\mathbb{R}^N)\) with \(\frac{N}{2} < q < \frac{N}{2}\). That is, we need to show that \(u(\cdot, t) \in K\) for all \(t \in (0, T_{\max})\) if \(u_0 \in K_+\). Since \(\|u\|_2\) and \(E(u)\) are conservation quantities for \((1.1)\), we have
\[ u(\cdot, t) \in H^1(\mathbb{R}^N) \setminus \{0\}, \ \omega \|u(\cdot, t)\|_2^2 + E(u(\cdot, t)) < d_I \quad (4.3) \]
for all \(t \in (0, T_{\max})\) if \(u_0 \in K_+\). We need to prove that \(Q(u(\cdot, t)) \geq 0\). Otherwise, assume that there exists a \(t_1 \in (0, T_{\max})\) satisfying \(Q(u(\cdot, t_1)) = 0\) by the continuity. Note that \((4.3)\) implies
\[ \omega \|u(\cdot, t_1)\|_2^2 + E(u(\cdot, t_1)) < d_I. \]

However, the inequality above and \(Q(u(\cdot, t_1)) = 0\) are contradictions to the definition of \(d_I\). Therefore, \(Q(u(\cdot, t)) > 0\). Consequently, \((4.3)\) and \(Q(u(\cdot, t)) > 0\) imply that \(u(\cdot, t) \in K_+\). That is, \(K_+\) is an invariant set of \((1.1)\) with \(V(x) \equiv 0\) and \(W \in L^q(\mathbb{R}^N)\) with \(\frac{N}{2} < q < \frac{N}{2}\). Similarly, we can prove that \(K_-\) is also an invariant set of \((1.1)\) with \(V(x) \equiv 0\) and \(W \in L^q(\mathbb{R}^N)\) with \(\frac{N}{2} < q < \frac{N}{2}\).

Step 3. Assume that \(Q(u_0) > 0\) and \(\omega \|u_0\|_2^2 + E(u_0) < d_I\). By the results of Step 2, we have \(Q(u(\cdot, t)) > 0\) and \(\omega \|u(\cdot, t)\|_2^2 + E(u(\cdot, t)) < d_I\). That is,
\[
-2\|\nabla u(\cdot, t)\|_2^2 \leq -N \int_{\mathbb{R}^N} |u|^2 f(x, |u|^2) - F(x, |u|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |u|^2\} |u|^2 \, dx
\]
\[
< -NI \int_{\mathbb{R}^N} F(x, |u|^2) \, dx - \frac{NI}{2} \int_{\mathbb{R}^N} \{W \ast |u|^2\} |u|^2 \, dx,
\]
and
\[
d_I > \omega \|u(\cdot, t)\|_2^2 + \frac{1}{2}\|\nabla u(\cdot, t)\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2) \, dx - \frac{1}{4} \int_{\mathbb{R}^N} \{W \ast |u|^2\} |u|^2 \, dx.
\]

The two inequalities imply that
\[
\omega \|u(\cdot, t)\|_2^2 + (\frac{1}{2} - \frac{1}{NI})\|\nabla u(\cdot, t)\|_2^2 < d_I.
\]

which means that
\[ \|u(\cdot, t)\|_{H^1(\mathbb{R}^N)} < \infty, \]
i.e., the solution exists globally.
Step 4. Assume that $Q(u_0) < 0$ and $\omega\|u_0\|_2^2 + E(u_0) < d_I$. By the results of Step 2, we obtain $Q(u(\cdot, t)) < 0$ and $\omega\|u(\cdot, t)\|_2^2 + E(u(\cdot, t)) < d_I$. Hence we get

$$J''(t) = 4Q(u) < 0, \quad J'(0) = 4\int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0.$$ 

By the results of Proposition 2.2, the solution will blow up in finite time. $\square$

As a corollary of Theorem 3, we obtain the sharp threshold for global existence and blowup of the solution of (1.6) as follows.

**Corollary 4.1.** Assume that $f(x, 0) = 0$ and (1.14). Let $\omega$ be a positive constant satisfying

$$d_I' := \inf_{\{u \in \Sigma \setminus \{0\}, Q_1(u) = 0\}} (\omega\|u\|_2^2 + E(u)) > 0. \quad (4.4)$$

Here

$$Q_1(u) := 2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} [F(x, |u|^2) - |u|^2 f(x, |u|^2)] \, dx. \quad (4.5)$$

Suppose that $u_0 \in H^1(\mathbb{R}^N)$ satisfies

$$\omega\|u_0\|_2^2 + E(u_0) < d_I.'$$

Then

1. If $Q_1(u_0) > 0$, the solution of (1.6) exists globally;
2. If $Q_1(u_0) < 0$, $|x|u_0 \in L^2(\mathbb{R}^N)$ and $\exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0$, the solution of (1.6) blows up in finite time.

**Remark 4.1.** In Theorem 1.5 of [15], Tao et al. proved that:

Assume that $u(x, t)$ is a solution of (1.6) with $f(x, |u|^2)u = \mu|u|^{p_1}u + \nu|u|^{p_2}u$, where $\mu > 0$, $\nu > 0$, $p_1 = 1 < p_2 = \frac{4}{N-2}$ with $N \geq 3$, $\exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0$, $|x|u_0 \in L^2(\mathbb{R}^N)$ and $E(u_0) < 0$. Then blowup occurs.

Corollary 4.1 improve the result above. In fact, if $f(x, |u|^2)u = \mu|u|^{p_1}u + \nu|u|^{p_2}u$, then

$$Q_1(u) = 4E(u) - \frac{(Np_1 - 4)\mu}{(p_1 + 2)} \|u\|_{p_1+2}^{p_1+2} - \frac{(Np_2 - 4)\nu}{(p_2 + 2)} \|u\|_{p_2+2}^{p_2+2} \leq E(u),$$

hence $E(u_0) < 0$ implies that $Q_1(u_0) < 0$. That is, our blowup condition is weaker than theirs. On the other hand, our conclusion is still true if $0 < E(u_0) < d_I' - \omega\|u_0\|_2^2$ with $Q_1(u_0) < 0$, $\exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0$ and $|x|u_0 \in L^2(\mathbb{R}^N)$. In other words, our result is stronger than theirs if $\omega\|u_0\|_2^2 + E(u_0) < d_I'$ with $Q_1(u_0) < 0$, $\exists \int_{\mathbb{R}^N} (x \cdot \nabla u_0) \bar{u}_0 \, dx < 0$ and $|x|u_0 \in L^2(\mathbb{R}^N)$.

**Remark 4.2.** We will give some examples of $f(x, |u|^2)$ and $W(x)$. It is easy to verify that they satisfy the conditions of Theorem 3.

Example 4.1. $W(x) \equiv 0$, $f(x, |u|^2) = c|u|^{2q_1} + d|u|^{2q_2}$ with $c < 0$, $d > 0$ and $q_2 > \frac{4}{N}$, $q_2 > q_1 > 0$. 

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Example 4.2. \( W(x) \equiv 0, f(x, |u|^2) = b|u|^{2b} \ln(1 + |u|^2) \) with \( b > 0 \) and \( p > \frac{2}{N} \).

Example 4.3. Let \( f(x, |u|^2) \) be one of those in Examples 4.1 and 4.2. And Let

\[
W(x) = \begin{cases} 
\frac{1}{|x|^N}, & |x| \leq 1, \\
\varphi(x), & 1 \leq |x| \leq 2, \\
\frac{1}{|x|^N}, & |x| \geq 2, 
\end{cases}
\]

where \( 2 < Nl < \frac{N}{q} < K \), and \( \varphi(x) \) satisfies

\[
Nl\varphi(x) + (x \cdot \nabla \varphi) \leq 0 \leq c_3\varphi(x) + (x \cdot \nabla \varphi)
\]

when \( 1 \leq |x| \leq 2 \) and makes \( W(x) \) be smooth. Obviously, \( W \in L^q(\mathbb{R}^N) \).

5 Sharp threshold for the blowup and global existence of the solution to (1.1)

Theorem 4 extend the results of [17] to more general case. Moreover, we need subtle estimates and more sophisticated analysis in the proof.

5.1 Some invariant manifolds

In this subsection, we will prove that \( d_N, d_M, d_{II} > 0 \), and construct some invariant manifolds.

Proposition 5.1.1. Assume that the conditions of Theorem 4 hold. Then \( d_N > 0 \).

Proof: Assume that \( u \in \Sigma \setminus \{0\} \) satisfying \( S_\omega(u) = 0 \). Using Gagliardo-Nirenberg’s and Young’s inequalities, we have

\[
2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} [\|\nabla u\|^2 + V(x)|u|^2]dx
\]

\[
= \int_{\mathbb{R}^N} |u|^2f(x, |u|^2)dx + \int_{\mathbb{R}^N} (W * |u|^2)|u|^2dx
\]

\[
\leq \frac{l+1}{l} \int_{\mathbb{R}^N} \left[ c_1|u|^{2p_l+2} + c_2|u|^{2p_2+2} \right] dx + \|W_1\|_{L^\infty}\|u\|_2^2 + \|W_2\|_{L^\infty}\|u\|_4^4
\]

\[
\leq C(\|\nabla u\|_2^{\frac{Np_1}{2}} (\|u\|_2^{2})^{p_1+1 - \frac{Np_1}{2}} + C(\|\nabla u\|_2^{\frac{Np_2}{2}} (\|u\|_2^{2})^{p_2+1 - \frac{Np_2}{2}}
\]

\[
+ \|W_1\|_{L^\infty}\|u\|_2^2 + C\|W_2\|_{L^\infty}\|\nabla u\|_2^\frac{N}{2} \|u\|_2^\frac{N}{2}
\]

\[
\leq C(\|\nabla u\|_2^{\frac{Np_1}{2}} (\|u\|_2^{2})^{p_1+1 - \frac{Np_1}{2}} + C(\|\nabla u\|_2^{\frac{Np_2}{2}} (\|u\|_2^{2})^{p_2+1 - \frac{Np_2}{2}}
\]

\[
+ C\|u\|_2^4 + \|\nabla u\|_2^4 + C(\|W_2\|_{L^\infty})\|u\|_4^4. \quad (5.1)
\]
Using Hölder’s inequality, from (5.1), we can obtain
\[
2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx
\leq C \left(2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\right)^{p_1+1}
+ C \left(2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\right)^{p_2+1}
+ C \left(2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\right)^2.
\] (5.2)

(5.2) implies that
\[
2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx \geq C > 0
\] (5.3)
for some positive constant \(C\).

On the other hand, if \(S_\omega(u) = 0\), we get
\[
\omega\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx
= \frac{1}{2} \int_{\mathbb{R}^N} f(x, |u|^2)|u|^2dx + \frac{1}{2} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2dx
\geq \min(l + 1, 2) \left(\frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2)dx + \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2dx\right).
\] (5.4)

From (5.4), we obtain
\[
I_\omega(u) = \omega\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2 - F(x, |u|^2))dx - G(|u|^2)
\geq \min \left(\frac{l}{2(l+1)}, \frac{1}{4}\right) \left(2\omega\|u\|_2^2 + \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\right)
\geq C > 0.
\] (5.5)

Consequently,
\[
d_N = \inf_{\mathcal{N}} I_\omega(u) > C > 0.
\]

Now, we will give some properties of \(I_\omega(u), S_\omega(u)\) and \(Q(u)\). We have a proposition as follows.

**Proposition 5.1.2.** Assume that \(Q(u)\) and \(S_\omega(u)\) are defined by (1.18) and (1.23). Then we have

(i) There at least exists a \(w^* \in \Sigma \setminus \{0\}\) such that
\[
S_\omega(w^*) = 0, \quad Q(w^*) = 0.
\] (5.6)
(ii) There at least exists a \( w^* \in \Sigma \setminus \{0\} \) such that

\[
S_\omega(w^*) < 0, \quad Q(w^*) = 0.
\] (5.7)

**Proof:** (i) Noticing the assumptions on \( V(x), W(x) \) and \( f(x, |u|^2) \), similar to the proof of Theorem 1.7 in [14], it is easy to prove that there exists a \( w^* \in \Sigma \setminus \{0\} \) satisfying

\[
2\omega w^* + V(x)w^* - \Delta w^* = f(x, |w^*|^2)w^* + (W \ast |w^*|^2)w^* \quad \text{in} \quad \mathbb{R}^N.
\] (5.8)

Multiplying \((5.8)\) by \( w^* \) and integrating over \( \mathbb{R}^N \) by part, we get \( S_\omega(w^*) = 0 \).

Multiplying \((5.8)\) by \( (x \cdot \nabla w^*) \) and integrating over \( \mathbb{R}^N \) by part, we obtain the Pohozaev’s identity:

\[
N\omega||w^*||^2 + \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla w^*|^2 dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x)|w^*|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (x \cdot \nabla V)|w^*|^2 dx
= \frac{N}{2} \int_{\mathbb{R}^N} F(x, |w^*|^2) dx + \frac{N}{2} \int_{\mathbb{R}^N} (W \ast |w^*|^2)|w^*|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} ((x \cdot \nabla W) \ast |w^*|^2)|w^*|^2 dx.
\] (5.9)

From \( S_\omega(w^*) = 0 \) and \((5.9)\), we can get \( Q(w^*) = 0 \).

(ii) Letting \( v_{k,\lambda}(x) = kw^*(\lambda x) \) for \( k > 0 \) and \( \lambda > 0 \), we can obtain

\[
S_\omega(v_{k,\lambda}) = 2\omega k^2 \int_{\mathbb{R}^N} |w^*(\lambda x)|^2 dx + k^2 \int_{\mathbb{R}^N} |\nabla w^*(\lambda x)|^2 dx + k^2 \int_{\mathbb{R}^N} V(x)|w^*(\lambda x)|^2 dx
- k^2 \int_{\mathbb{R}^N} |w^*(\lambda x)|^2 f(x, k^2|w^*(\lambda x)|^2) dx - k^4 \int_{\mathbb{R}^N} (W \ast |w^*(\lambda x)|^2)|w^*(\lambda x)|^2 dx,
\] (5.10)

\[
Q(v_{k,\lambda}) = 2k^2 \int_{\mathbb{R}^N} |\nabla w^*(\lambda x)|^2 dx - k^2 \int_{\mathbb{R}^N} (x \cdot \nabla V)|w^*(\lambda x)|^2 dx
- N \int_{\mathbb{R}^N} [k^2|w^*(\lambda x)|^2 f(x, k^2|w^*(\lambda x)|^2) - F(x, k^2|w^*(\lambda x)|^2) dx
+ \frac{k^4}{2} \int_{\mathbb{R}^N} ((x \cdot \nabla W) \ast |w^*(\lambda x)|^2)|w^*(\lambda x)|^2 dx.
\] (5.11)

Looking \( S_\omega(v_{k,\lambda}) \) and \( Q(v_{k,\lambda}) \) as the functions of \((k, \lambda)\), setting \( g(k, \lambda) = S_\omega(v_{k,\lambda}) \) and \( \eta(k, \lambda) = Q(v_{k,\lambda}) \), we get that \( g(1,1) = 0 \) and \( \eta(1,1) = 0 \). And we want to prove that there exists a pair of \((k, \lambda)\) such that \( g(k, \lambda) = S_\omega(v_{k,\lambda}) < 0 \) and \( \eta(k, \lambda) = Q(v_{k,\lambda}) = 0 \). Since \( \eta(1,1) = 0 \), we know that the image of \( \eta(k, \lambda) \) and the plane \( \eta = 0 \) intersect in the space of \((k, \lambda, \eta)\) and form a curve \( \eta(k, \lambda) = 0 \). Hence there exist many positive real number pairs \((k, \lambda)\) relying on \( w^* \) such that \( Q(v_{k,\lambda}) = 0 \) near \((1,1)\) with \( k > 1 \). On the other hand, under the assumptions of \( V(x) \) and \( W(x) \), it is easy to see that \( g(k, 1) < 0 \) for any \( k > 1 \). By the continuity, we can choose a pair of \((k, \lambda)\) near \((1,1)\) with \( k > 1 \) satisfies both \( Q(v_{k,\lambda}) = 0 \) and \( S_\omega(v_{k,\lambda}) < 0 \). Letting \( u^* = v_{k,\lambda} \) for this \((k, \lambda)\), we get that \( S_\omega(u^*) < 0 \) and \( Q(u^*) = 0 \). □
Proposition 5.1.2 means that $\mathcal{CM}$ is not empty and $d_M$ is well defined. Moreover, we have

**Proposition 5.1.3.** Assume that the conditions of Theorem 4 hold. Then $d_M > 0$.

**Proof:** $u \in \Sigma \setminus \{0\}$ and $S_\omega(u) < 0$ imply that

$$2 \omega \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2| \, dx$$

$$< \int_{\mathbb{R}^N} |u|^2 f(x, |u|^2) \, dx + \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx$$

$$\leq \frac{l + 1}{l} \int_{\mathbb{R}^N} |c_1|u|^{2p_1 + 2} + c_2|u|^{2p_2 + 2} \, dx$$

$$+ \|W_1\|_{L^\infty} |u|^4_{L^2} + C\|W_2\|_{L^0} \|\nabla u\|_{L^2}^p \|u\|_{L^2}^{\frac{4q - N}{q}}. \tag{5.12}$$

Similar to (5.1) and (5.2), from (5.12), we have

$$2 \omega \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2| \, dx \geq C > 0. \tag{5.13}$$

On the other hand, if $Q(u) = 0$, we have

$$2 \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 \, dx$$

$$= N \int_{\mathbb{R}^N} [|u|^2 f(x, |u|^2) - F(x, |u|^2)] \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) * |u|^2\} |u|^2 \, dx$$

$$\geq Nl \int_{\mathbb{R}^N} F(x, |u|^2) \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) * |u|^2\} |u|^2 \, dx,$$

that is,

$$- \frac{1}{2} \int_{\mathbb{R}^N} F(x, |u|^2) \, dx + \frac{1}{4Nl} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) * |u|^2\} |u|^2 \, dx$$

$$\geq - \frac{1}{Nl} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2Nl} \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 \, dx. \tag{5.14}$$

Using (1.19), (1.30), (1.31), (5.13) and (5.14), we can get

$$I_\omega(u) = \omega \int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 - F(x, |u|^2)|u|^2 \, dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |u|^2)|u|^2 \, dx$$

$$\geq \omega \int_{\mathbb{R}^N} |u|^2 dx + \frac{Nl - 2}{2Nl} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2Nl} \int_{\mathbb{R}^N} [NIV(x) + (x \cdot \nabla W)|u|^2 \, dx$$

$$- \frac{1}{4Nl} \int_{\mathbb{R}^N} \{[NIV + (x \cdot \nabla W)] * |u|^2\} |u|^2 \, dx$$

$$\geq C \left(2 \omega \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx\right)$$

$$\geq C > 0. \tag{5.15}$$
Consequently,

\[ d_M = \inf_{CM} I_\omega(u) > C > 0. \]

By the conclusions of Proposition 5.1.1 and Proposition 5.1.3, we have

\[ d_{II} = \min\{d_M, d_N\} > 0. \quad (5.16) \]

Now we define the following manifolds:

\[ K := \{ u \in \Sigma \setminus \{0\} : I_\omega(u) < d_{II}, \ S_\omega(u) < 0, \ Q(u) < 0 \}, \]

\[ K_+ := \{ u \in \Sigma \setminus \{0\} : I_\omega(u) < d_{II}, \ S_\omega(u) < 0, \ Q(u) > 0 \}, \]

\[ R_+ := \{ u \in \Sigma \setminus \{0\} : I_\omega(u) < d_{II}, \ S_\omega(u) > 0 \}. \]

The following proposition will show some properties of \( K, K_+ \) and \( R_+ \):\(^\square\)

Proposition 5.1.4 Assume that the conditions of Theorem 4 hold. Then

(i) \( K, K_+ \) and \( R_+ \) are not empty.

(ii) \( K, K_+ \) and \( R_+ \) are invariant manifolds of \( f \).

Proof: (i) In order to prove \( K \) is not empty, we only need to find that there are at least exists a \( w \in K \). For \( w^* \in \Sigma \setminus \{0\} \) satisfies \( S_\omega(w^*) = 0 \) and \( Q(w^*) = 0 \), letting \( w_\rho = \rho w^* \) for \( \rho > 0 \), we have

\[ S_\omega(w_\rho) = \rho^2 \int_{\mathbb{R}^N} \left\{ 2\omega|w^*|^2 + |\nabla w^*|^2 + V(x)|w^*|^2 \right\} dx \]

\[ - \int_{\mathbb{R}^N} \rho^2 |w^*|^2 f(x, \rho^2 |w^*|^2) dx - \rho^4 \int_{\mathbb{R}^N} (W*|w^*|^2)|w^*|^2 dx, \]

\[ Q(w_\rho) = \rho^2 \int_{\mathbb{R}^N} \left( 2|\nabla w^*|^2 - (x \cdot \nabla V)|w^*|^2 \right) dx \]

\[ + N \int_{\mathbb{R}^N} [F(x, \rho^2 |w^*|^2) - \rho^2 |w^*|^2 f(x, \rho^2 |w^*|^2)] dx \]

\[ + \frac{1}{2} \rho^4 \int_{\mathbb{R}^N} (x \cdot \nabla W)*|w^*|^2 |w^*|^2 dx, \]

\[ I_\omega(w_\rho) = \frac{1}{2} \rho^2 \int_{\mathbb{R}^N} \left\{ 2\omega|w^*|^2 + |\nabla w^*|^2 + V(x)|w^*|^2 \right\} dx \]

\[ - \frac{1}{2} \int_{\mathbb{R}^N} F(x, \rho^2 |w^*|^2) dx - \frac{1}{4} \rho^4 \int_{\mathbb{R}^N} (W*|w^*|^2)|w^*|^2 dx. \]

Since \( f(x, |w^*|^2) < f(x, \rho^2 |w^*|^2) \) and \( \rho^2 F(x, |w^*|^2) < F(x, \rho^2 |w^*|^2) \) for \( \rho > 1 \) and from (1.33), we can obtain

\[ S_\omega(w_\rho) < \rho^2 S_\omega(w^*) = 0, \quad Q(w_\rho) < \rho^2 Q(w^*) = 0 \quad (5.20) \]

for any \( \rho > 1 \). Noticing \( d_{II} > 0 \), we also can choose \( \rho > 1 \) closing to 1 enough such that

\[ I_\omega(w_\rho) < \rho^2 I_\omega(w^*) < d_{II}. \quad (5.21) \]

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for any $0 < \rho < 1$. Noticing $d_{II} > 0$, we also can choose $0 < \rho < 1$ closing to 1 enough such that $I_\omega(w_\rho) < d_{II}$ by continuity, which implies that $w_\rho \in \mathcal{R}_+$. That is, $\mathcal{R}_+$ is not empty.

For $w^* \in \Sigma$ satisfies $S_\omega(w^*) < 0$ and $Q(w^*) = 0$, letting $w_\sigma = \sigma w^*$ for $\sigma > 0$, we have

$$Q(w_\sigma) = \sigma^2 \int_{\mathbb{R}^N} (2|\nabla w^*|^2 - (x \cdot \nabla V)|w^*|^2)dx$$

$$- \int_{\mathbb{R}^N} N[\sigma^2|w^*|^2 f(x, \sigma^2|w^*|^2) - F(x, \sigma^2|w^*|^2)]dx$$

$$+ \frac{1}{2} \sigma^4 \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} |w^*|^2dx,$$

$$S_\omega(w_\sigma) = \sigma^2 \int_{\mathbb{R}^N} \{2\omega|w^*|^2 + |\nabla w^*|^2 + V(x)|w^*|^2\} dx$$

$$- \int_{\mathbb{R}^N} \sigma^2|w^*|^2 f(x, \sigma^2|w^*|^2)dx - \sigma^4 \int_{\mathbb{R}^N} (W \ast |w^*|^2)|w^*|^2dx,$$

$$I_\omega(w_\sigma) = \frac{1}{2} \sigma^2 \int_{\mathbb{R}^N} \{2\omega|w^*|^2 + |\nabla w^*|^2 + V(x)|w^*|^2\} dx$$

$$- \frac{1}{2} \int_{\mathbb{R}^N} F(x, \sigma^2|w^*|^2)dx - \frac{1}{4} \sigma^4 \int_{\mathbb{R}^N} (W \ast |w^*|^2)|w^*|^2dx.$$

Since $\phi(\sigma) = Q(w_\sigma)$ is a smooth function of $\sigma$ and $Q(w^*) = 0$, we have $\phi(1) = 0$. If $\phi'(1) \neq 0$, then there exists a $\sigma_0 > 0$ such that $Q(u_\sigma) = \phi(\sigma) > 0$ for $\sigma \in (1, \sigma_0)$ if $\sigma_0 > 1$ (or $\sigma \in (\sigma_0, 1)$ if $\sigma_0 < 1$). By continuity, we can choose such $\sigma_0$ closing to 1 enough such that $S_\omega(w_\sigma) < 0$ and $I_\omega(w_\sigma) < d_{II}$ for $\sigma \in (1, \sigma_0)$ if $\sigma_0 > 1$ (or $\sigma \in (\sigma_0, 1)$ if $\sigma_0 < 1$). That is, $w_\sigma \in \mathcal{K}_+$ and $\mathcal{K}_+$ is not empty.

If $\phi'(1) = 0$, from $\phi(1) = 0$ and $\phi'(1) = 0$, we can respectively obtain

$$- N \int_{\mathbb{R}^N} [|w^*|^2 f(x, |w^*|^2) - F(x, |w^*|^2)]dx$$

$$= - N \int_{\mathbb{R}^N} |w^*|^4 f'_s(x, |w^*|^2)dx + \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} |w^*|^2dx$$

and

$$Q(w^*) = \int_{\mathbb{R}^N} (2|\nabla w^*|^2 - (x \cdot \nabla V)|w^*|^2 - N|w^*|^4 f'_s(x, |w^*|^2)) dx$$

$$+ \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} |w^*|^2dx.$$
Letting \( w_\sigma = \sigma w^* \), we have
\[
Q(w_\sigma) = \sigma^2 \int_{\mathbb{R}^N} (2|\nabla w^*|^2 - (x \cdot \nabla V)|w^*|^2 - N|w^*|^4 f_\sigma(x, \sigma^2|w^*|^2)) \, dx \\
+ \sigma^4 \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} \, |w^*|^2 \, dx \\
> \sigma^2 \int_{\mathbb{R}^N} (2|\nabla w^*|^2 - (x \cdot \nabla V)|w^*|^2 - N|w^*|^4 f_\sigma(x, |w^*|^2)) \, dx \\
+ \sigma^4 \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} \, |w^*|^2 \, dx \\
= \sigma^2 Q(w^*) + (\sigma^4 - \sigma^2) \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |w^*|^2\} \, |w^*|^2 \, dx > 0 \quad (5.23)
\]
for \( 0 < \sigma < 1 \). By continuity, we can choose such \( \sigma \) closing to 1 enough such that \( S_\omega(w_\sigma) < 0 \) and \( I_\omega(w_\sigma) < d_{II} \). That is to say, \( w_\sigma \in \mathcal{K}_+ + \mathcal{K}_+ \) is not empty.

(ii) In order to prove that \( \mathcal{K} \) is the invariant manifold of (1.1), we need to show that: If \( u_0 \in \mathcal{K} \), then solution \( u(x, t) \) of (1.1) satisfies \( u(x, t) \in \mathcal{K} \) for any \( t \in [0, T) \).

Assume that \( u(x, t) \) is a solution of (1.1) with \( u_0 \in \mathcal{K} \). Then we can obtain
\[
I_\omega(u(\cdot, t)) = E(u(\cdot, t)) + \omega \|u(\cdot, t)\|^2 = E(u_0) + \omega \|u_0\|^2 = I_\omega(u_0) < d_{II} \quad (5.24)
\]
for \( t \in [0, T) \). Next we prove that \( S_\omega(u(\cdot, t)) < 0 \) for \( t \in [0, T) \). Otherwise, by continuity, there exists a \( t_0 \in (0, T) \) such that \( S_\omega(u(\cdot, t_0)) = 0 \) because of \( S_\omega(u_0) < 0 \).

Since \( \|u(\cdot, t)\|^2 = \|u_0\|^2 \) and \( u_0 \in \Sigma \setminus \{0\} \), it is easy to see that \( u(\cdot, t_0) \in \Sigma \setminus \{0\} \). By the definitions of \( d_{II} \) and \( d_{II} \), we know that \( I_\omega(u(\cdot, t_0)) \geq d_{II} \), which is a contradiction to \( I_\omega(u(\cdot, t)) < d_{II} \) for \( t \in [0, T) \). Hence \( S_\omega(u(\cdot, t)) < 0 \) for all \( t \in [0, T) \).

Now we only need to prove that \( Q(u(\cdot, t)) < 0 \) for \( t \in [0, T) \). Otherwise, since \( Q(u_0) < 0 \), there exists a \( t_1 \in (0, T) \) such that \( Q(u(\cdot, t_1)) = 0 \) by continuity. And \( S_\omega(u(\cdot, t_1)) < 0 \) means that \( u(\cdot, t_1) \in CM \). By the definitions of \( d_{II} \) and \( d_{II} \), we obtain \( I_\omega(u(\cdot, t_1)) \geq d_{II} \), which is a contradiction to \( I_\omega(u(\cdot, t)) < d_{II} \) for \( t \in [0, T) \). Hence \( Q(u(\cdot, t)) < 0 \) for all \( t \in [0, T) \).

By the discussions above, we know that: \( u(x, t) \in \mathcal{K} \) for any \( t \in [0, T) \) if \( u_0 \in \mathcal{K} \), which means that \( \mathcal{K} \) is the invariant manifold of (1.1).

Similarly, we can prove that \( \mathcal{K}_+ \) and \( \mathcal{R}_+ \) are also invariant manifolds of (1.1). \( \square \)

**Remark 5.1.1.** By the definitions of \( d_{II}, d_{II}, d_{II}, \mathcal{K}, \mathcal{K}_+ \) and \( \mathcal{R}_+ \), it is easy to see that
\[
\{u \in \Sigma \setminus \{0\} : I_\omega(u) < d_{II}\} = \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{R}_+.
\]

### 5.2 The proof of Theorem 4

The proof of Theorem 4 depends on the following two lemmas.

**Lemma 5.2.1.** Assume that the conditions of Theorem 4 hold. Then the solutions of (1.1) with \( u_0 \in \mathcal{K} \) will blow up in finite time.
Proof: Since \( u_0 \in \mathcal{K} \) and \( \mathcal{K} \) is the invariant manifold of (1.1), we have \( Q(u(x,t)) < 0 \), \( S_\omega(u(x,t)) < 0 \) and \( I_\omega(u(x,t)) < d_{II} \).

Under the conditions of Theorem 4, we have \( J''(t) = 4Q(u) < 0 \) and \( J'(0) < 0 \). By the results of Proposition 2.2, the solution \( u(x,t) \) will blow up in finite time. The conclusion of this lemma is true. \( \square \)

On the other hand, we have a parallel result on global existence.

**Lemma 5.2.2.** Assume that the conditions of Theorem 4 hold. If \( u_0 \in \mathcal{K}_+ \) or \( u_0 \in \mathcal{R}_+ \), then the solutions of (1.1) exists globally.

**Proof:** Case 1: Assume that \( u(x,t) \) is a solution of (1.1) with \( u_0 \in \mathcal{K}_+ \). Since \( \mathcal{K}_+ \) is an invariant manifold of (1.1), we know that \( u(\cdot,t) \in \mathcal{K}_+ \), which means that \( I_\omega(u(\cdot,t)) < d_{II} \) and \( Q(u(\cdot,t)) > 0 \). \( Q(u(\cdot,t)) > 0 \) and (1.1) imply that

\[
2 \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} (x \cdot \nabla V)|u|^2 dx \\
\geq NL \int_{\mathbb{R}^N} F(x,|u|^2)dx - \frac{1}{2} \int_{\mathbb{R}^N} \{(x \cdot \nabla W) \ast |u|^2\} |u|^2 dx. \tag{5.25}
\]

By the definition of \( I_\omega(u) \) and using (5.25), we have

\[
d_{II} > I_{\omega}(u(\cdot,t)) = \omega \int_{\mathbb{R}^N} |u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \\
- \frac{1}{2} \int_{\mathbb{R}^N} F(x,|u|^2)dx - \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx \\
\geq \omega \int_{\mathbb{R}^N} |u|^2 dx + \frac{NL - 2}{2NL} \int_{\mathbb{R}^N} |\nabla u|^2 dx \\
+ \int_{\mathbb{R}^N} \frac{NLV(x) + (x \cdot \nabla V)}{2NL} |u|^2 dx \\
- \frac{1}{4NL} \int_{\mathbb{R}^N} \{ [NLW + (x \cdot \nabla W)] \ast |u|^2 \} |u|^2 dx \\
\geq C \left( \int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x)|u|^2 dx \right). \tag{5.26}
\]

(5.26) means that \( u(x,t) \) exists globally.

Case 2: Assume that \( u(x,t) \) is a solution of (1.1) with \( u_0 \in \mathcal{R}_+ \). Since \( \mathcal{R}_+ \) is also an invariant manifold of (1.1), we know that \( u(x,t), \in \mathcal{R}_+ \), which means that \( I_\omega(u(\cdot,t)) < d_{II} \) and \( S_\omega(u(\cdot,t)) > 0 \). Since \( S_\omega(u) > 0 \), we can get

\[
\omega \|\nabla u\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2) dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^N} f(x,|u|^2)|u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx \\
\geq \min(l + 1, 2) \left( \frac{1}{2} \int_{\mathbb{R}^N} F(x,|u|^2)dx + \frac{1}{4} \int_{\mathbb{R}^N} (W \ast |u|^2)|u|^2 dx \right). \tag{5.27}
\]
From (5.27), we can obtain
\[
I_\omega(u) = \omega\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2 - F(x,|u|^2))dx - G(|u|^2)
\]
\[
\geq \min \left( \frac{l}{(l+1)^2}, \frac{1}{2} \right) \left( \omega\|u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx \right).
\] (5.28)

(5.28) implies that the solution \( u(x, t) \) exists globally. \( \square \)

**The proof of Theorem 4:** By the results of Lemma 5.2.1, Lemma 5.2.2, we know that Theorem 4 is right. \( \square \)

As a corollary of Theorem 4, we obtain a sharp threshold for the blowup in finite time and global existence of the solution of (1.8) as follows

**Corollary 5.1.** Assume that \( f(x,|u|^2) \equiv 0, V(x) \equiv 0, W(x) > 0 \) for all \( x \in \mathbb{R}^N, W(0) > 0 \) and \( \omega \in I_\omega \). Then the solution of (1.8) blows up in finite time if and only if

\[
u \equiv 0, W(x) + (x \cdot \nabla W) \leq 0.
\]

If \( u_0 \in H^1(\mathbb{R}^N), |x|u_0 \in L^2(\mathbb{R}^N) \) and \( I_\omega(u_0) = \omega\|u_0\|_2^2 + \|E(u_0)\|_2 > d_{11} \), then the solution of (1.8) blows up in finite time if and only if \( u_0 \notin \mathcal{K} \).

**Remark 5.2.1.** A typical example of (1.8) is

\[
\begin{cases}
-\Delta u = (|x|^{-k} \ast |u|^2)u, & x \in \mathbb{R}^N, \ t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\] (5.29)

which is also a special case of (1.1) with \( V(x) \equiv 0, f(x, |u|^2) \equiv 0 \) and \( W(x) = |x|^{-K} \) with \( 2 < Nl < K < \frac{N}{q} < 4 \). Letting \( W = W_1 + W_2 \) with

\[
W_1(x) = \begin{cases} 0, & |x| \leq \gamma_1, \\
0, & |x| > \gamma_1,
\end{cases} \quad \text{and} \quad W_2(x) = \begin{cases} |x|^{-K}, & |x| \leq \gamma_1, \\
0, & |x| > \gamma_1,
\end{cases}
\]

we can see that \( W_1 \in L^\infty(\mathbb{R}^N) \) and \( W_2 \in L^q(\mathbb{R}^N) \) with some \( \frac{N}{q} < q \). Corollary 5.1 gives the sharp threshold for blowup and global existence of the solution to (5.29).

We will give some examples of \( V(x), f(x, |u|^2) \) and \( W(x) \). It is easy to verify that they satisfy the conditions of Theorem 4.

**Example 1.** \( V(x) = |x|^2, W(x) = a|x|^{-K} \) with \( 2 < Nl < K < \frac{N}{q} < 4 \) for \( x \in \mathbb{R}^N \) and \( f(x, |u|^2) = b|u|^{2p_1} + c|u|^{2p_2} \) with \( a > 0, b > 0, c > 0 \) and \( p_2 > p_1 > \frac{2}{N} \).

**Example 2.** \( V(x) = |x|^2, W(x) = a|x|^{-K} \) with \( 2 < Nl < K < \frac{N}{q} < 4 \) for \( x \in \mathbb{R}^N \) and \( f(x, |u|^2) = c|u|^{2q_1} + d|u|^{2q_2} \) with \( a > 0, c \) is a real number, \( d > 0 \) and \( q_2 > \frac{2}{N}, q_2 > q_1 > 0 \).

**Example 3.** \( V(x) = \frac{|x|^2}{1 + |x|^2}, W(x) = a|x|^{-K} \) with \( 2 < Nl < K < \frac{N}{q} < 4 \) for \( x \in \mathbb{R}^N \) and \( f(x, |u|^2) = b|u|^{2p} \ln(1 + |u|^2) \) with \( a > 0, b > 0 \) and \( p > \frac{2}{N} \).
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