Quantum anomaly and anomalous Josephson effect in inversion asymmetric Weyl semimetals

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We study a Josephson junction involving an inversion-asymmetric Weyl semimetal in presence of time reversal symmetric (TRS) or time reversal symmetry broken tilt in the Weyl spectra. We reveal that both types of tilts in the Weyl nodes lead to a Josephson 0-π transition and a zero bias supercurrent. Strikingly, the TRS tilt gives rise to a pure valley Josephson current (VJC) and TRS broken tilt induces a pure chirality Josephson current (CJC) in this system. The VJC and CJC are the manifestation of valley symmetry broken and \( Z_2 \) symmetry broken by the respective tilt. We obtain reversal of a pure VJC and pure CJC even in the zero bias condition controllable by the junction length. Our analysis of controllability of valley and chirality dependent transport in inversion asymmetric Weyl semimetal junction could allow applications in valleytronics and chiralitytronics, respectively. The tilt induced Josephson effect provides an alternative route for supercurrent 0-π transition, different from the conventional ferromagnetism Josephson junctions where the spin polarization is essential. In the long junction and zero temperature limit, VJC and CJC are associated with an anomaly of Goldstone bosons at \( \pi \) superconducting phase difference.

**Introduction:**- Recent discovery of Weyl fermions in a number of materials draws intensive attention owing to their novel physics associated with Weyl nodes\(^{1-9}\). Weyl semimetals (WSMs) are gapless topological materials whose low energy excitations are Weyl fermions, particles that play significant roles in quantum field theory. The energy spectra show a pair of strongly degenerate Weyl cones having opposite chirality separated in momentum space. The stability of Weyl nodes requires at least time reversal (TR) or inversion (IR) symmetry is broken in the system. The minimal model of TR broken WSM contains single pair of Weyl nodes, whereas, an IR broken WSM involves four Weyl nodes with total zero chirality\(^{10,11}\). Chirality is an intrinsic property of Weyl fermions and can be understood as topological charge of a Weyl node. The possibility to probe and manipulate the chirality and valley of a WSM, are thus remains an important issue in this context. Several studies have been devoted to such chirality and valley-dependent physics of WSMs, recently\(^{12-15}\).

The topological nontrivial nature provides promising transport properties of WSMs including anomalous hall effect\(^{16-20}\), Andreev reflection\(^{21,22}\), magnetotransport\(^{23,24}\). Josephson junction present another complementary route to investigate the anomalous transport properties of topological superconductors. Recently, Josephson junction based on Dirac/Weyl semimetals have been fabricated and investigated\(^{25-29}\). To the date, most of experimentally discovered WSMs break inversion symmetry including TaAs family\(^{10,11}\). The chirality of Cooper pairs remains a well defined property in Josephson junction of inversion-asymmetric WSM. A chirality dependent Josephson transport has been predicted in those materials in response to a Zeeman field\(^{14}\). Spin polarization plays an essential role to realize this phenomena.

The linear energy dispersions near Weyl nodes generally tilted along certain momentum direction. The Lorentz symmetry is spontaneously broken in WSMs by the tilted dispersions\(^{29,30}\). Weyl fermions are categorized into two types, type-I and type-II, depending whether the tilts exceed the Fermi velocity of electrons or not. Although, the tilts of Weyl cone does not change the topology of the energy bands, but it largely influence the quantum transport\(^{31-34}\). One of the intriguing phenomena occurs in a Josephson junction of TR broken WSM, where Cooper pairs acquire an extra momentum due to tilt\(^{35}\). This brings an unusual oscillations in the Josephson current including Josephson 0-π transition.

Thus it is natural to ask whether tilt can leads to anomalous effects in the Josephson junction involving inversion-asymmetric WSMs. Particulariy, whether tilts can probe the chirality and valley physics in WSMs.

In this article, we study the Josephson effect of an inversion broken WSM Josephson junction with proximity induced s-wave superconductor. The interplay between the tilts, valley and chirality of Weyl nodes, and s-wave superconductor resulting in unusual behaviors of supercurrent through the junction. We find that tilt introduces an extra phase in the current-phase relations (CPRs). The tilt induced phase causes supercurrent reversal and Josephson \( \phi \) junction in each valley and chirality sectors. The phase shift is controllable by the junction length and doping. The current phase relation is dominated by second harmonics in the supercurrent 0-π transition. The tilt induced phase shifts the current phase relations (CPRs) of opposite valleys and chirality differently. The unequal phase initiates many exotic phenomena. Remarkably, a finite VJC and CJC develops, respectively, in presence of TRS and TRS broken tilts. A pure VJC and CJC reversal are possible by tuning the junction length, even in zero bias condition. In the long junction and zero temperature limit, a singularity of VJC and CJC when the tilt induced phase disappears, signifies a quantum anomaly of Goldstone bosons at \( \pi \) superconducting phase difference.
Model Hamiltonian and setup: We start by considering the general low-energy Hamiltonian describing a single Weyl node,

\[ H_W(q) = \hbar v_F q \sigma_0 + \hbar v_F \hat{n}_{ij} q \sigma_j \]  

where \( v_F \) is the Fermi velocity without tilt. \( \sigma_0 \) and \( \sigma_j = \sigma_1, \sigma_2, \sigma_3 \) are the \( 2 \times 2 \) identity matrix and Pauli matrices, respectively. \( \hat{n}_{ij} \) is the anisotropy in the spectrum and \( \chi = \text{Det}(\hat{n}_{ij}) \) is determined by the chirality of the given node. The energy dispersion of Eq. (1) is given by

\[ E_{\pm}(q) = \hbar v_F q \pm \hbar v_F \sqrt{\sum_j q_j (\hat{n}^T)_{ij} q_j} = T(q) \pm U(q), \]

where \( T(q) \) and \( U(q) \) regarded as the kinetic and potential parts. The ratio \( v_t/v_F \) measures the tilt of the Weyl cone. The Weyl cones in type-I if \( T(q) < U(q) \) (i.e., \( v_t < v_F \)) and in type-II if \( T(q) > U(q) \) (i.e., \( v_t > v_F \)). We restrict our study in type-I case.

An inversion broken but TR preserved WSM has minimum four Weyl nodes. The pair of Weyl nodes have same chirality and if one of them appears at \( q \) then another must be at \(-q\). Since the total chirality is zero, which garantees that, there must be another TR pair of Weyl nodes with opposite chirality. The Hamiltonians of TR pair are related with each other by, \( q_i \leftrightarrow -q_i \) and \( \sigma_i \leftrightarrow -\sigma_i \) with \( T \) being the time reversal operator. Thus two TR-related nodes must have opposite tilt. The tilts of two opposite chiral nodes can have any arbitrary value since they are not symmetry related. However, mirror symmetry is very common in large numbers of inversion asymmetric WSMs, specifically in TaAs, TaP, NbAs, and NbP families. The mirror symmetry relates the opposite chiral Weyl nodes by flipping the sign of momentum along the mirror axis. Consequently, the opposite chiral Weyl nodes have opposite tilt. At low energy the minimal model can be written as a sum of four effective Weyl Hamiltonians,

\[ H_W = \sum_k \sum_{\gamma} \sum_{\nu} \hat{\Psi}^\dagger_{\gamma,k} H_{\gamma,k} \hat{\Psi}_{\gamma,k} \]

where \( \gamma = 1, 2, 3 \) and \( \nu = 1, 2, 4 \) in the four Weyl nodes. The nodes 1 and 3 carry positive chirality while nodes 2 and 4 carry negative chirality. Pair of opposite chiral nodes i.e., \( \gamma = 1, 3 \) and \( \gamma = 2, 4 \) are related to each other by a mirror plane perpendicular to the \( z \)-axis. The linearized Hamiltonian at each Weyl nodes is given by

\[ H_{1(3)}(k) = C_{1(3)} k_x \sigma_0 + (k_x \sigma_x + k_y \sigma_y + k_z \sigma_z) \]  

\[ H_{2(4)}(k) = C_{2(4)} k_x \sigma_0 + (k_x \sigma_x + k_y \sigma_y - k_z \sigma_z) \]

where the spinor \( \Psi_{1(3)}(k) = (c_{1(3),k}^{(B)}, c_{1(3),k}^{(A)}) \) and \( \Psi_{2(4)}(k) = (c_{2(4),k}^{(A)}, c_{2(4),k}^{(B)}) \). \( c_{\gamma,k}^{(A)}, c_{\gamma,k}^{(B)} \) are annihilation operators with orbital indices \( A \) and \( B \). The tilts of two opposite chiral nodes can be related to each other by \( C_1 = -C_3 \) and \( C_2 = -C_4 \). On the other hand, TR is broken if the tilt coefficients in each chirality sectors are equal i.e., \( C_1 = C_3 \) and \( C_2 = C_4 \). Additionally, the mirror symmetry: \( R_z H(k_x, k_y, -k_z) R_z = H(k_x, k_y, k_z) \) relates the tilt coefficients of opposite chiral Weyl nodes by \( C_2 = -C_1 \) and \( C_4 = -C_3 \), in both cases. The Hamiltonians in each chirality sectors are invariant under TR operator \( T \), i.e., \( \{ H_{\pm}(q), T \} = 0 \), in presence of TRS tilt. The TR-operator reads \( T = -i \gamma_5 K \), where \( K \) denotes the complex conjugation. Here, the positive and negative chiral Hamiltonians are defined as, \( H_+ = \text{diag}(H_1, H_3) \) and \( H_- = \text{diag}(H_2, H_4) \). For an inversion symmetry broken WSM, there exists an emergent symmetry operation \( U \) which connect the two chirality sectors by \( U H U^{-1} = H_\pm \). The tilted operator \( U \) call the \( Z_2 \) exchange symmetry and the operator reads \( U = e^{i\pi \sigma_x \gamma_5 R_k / 2} \), where \( R_k \) is the reflection operator about \( yz \) plane. Then it is easy to check that the TRS tilt preserves the \( Z_2 \) symmetry whereas, the TRS broken tilt breaks this symmetry. This is due to fact that the valley degeneracy is broken by TRS tilts. In presence of TRS broken tilit, the position of the Weyl nodes of positive and negative chirality are shifted oppositely by \( \Delta k_z \), which breaks the \( Z_2 \) exchange symmetry.

We consider an inversion asymmetric WSM sandwich between two s-wave superconducting WSM. The superconductivity in WSM regions can be proximity induced by superconducting electrode. The BdG Hamiltonian for positive chirality sector is given by,

\[ H_{\text{BdG}}^+(\phi) = \begin{pmatrix} H_{\text{BdG}}^{(1)}(\phi) & 0 \\ 0 & H_{\text{BdG}}^{(2)}(\phi) \end{pmatrix} \]

in the Nambu basis \( (\Psi_{1,\phi}, \Psi_{1,\downarrow}, \Psi_{3,\phi}, -\Psi_{3,\downarrow}, \Psi_{1,\uparrow}, \Psi_{3,\uparrow}, \Psi_{1,\downarrow}, -\Psi_{1,\uparrow}) \). The \( \theta \) in the above equation is \( 4 \times 4 \) null matrix. The BdG Hamiltonian in Eq. (4) decouples into two blocks. The diagonal Hamiltonian part \( H_{\text{BdG}}^{(1)}(\phi) \) is given by,

\[ H_{\text{BdG}}^{(1)}(\phi) = -iC \partial_r \sigma_0 \nu_x - \mu(\sigma_0 \nu_z - i\nu_y \sigma_0 - i\nu_x \sigma_y) \]

The \( 2 \times 2 \) Pauli matrices \( \nu_i \) \( i = x, y, z \) acting on particle-hole space. Here, \( \Delta_r(\phi) = \Delta(\theta |z| - L/2) \) is the pairing potential, \( \phi \) is the phase difference and \( L \) is the junction length; \( \Theta(z) \) the Heaviside function and \( \text{sgn}(z) \) is the sign function. The chemical potential is taken as: \( \mu(r) = \mu_S \text{sgn}(\Omega(z) - L/2) \). The Hamiltonian \( H_{\text{BdG}}^{(2)}(\phi) \) can be obtained by replacing \( C \) by \( -C \) in Eq. (4), in case of TRS tilt. Thus the two blocks in Eq. (4) become different i.e., \( H_{\text{BdG}}^{(1)(\phi)} \neq H_{\text{BdG}}^{(2)(\phi)} \). These two Hamiltonians become identical in case of TRS broken tilt. Similarly, the BdG Hamiltonian of negative chirality sector \( H_{\text{BdG}}^{-}(\phi) \) is obtained by using the Hamiltonian in Eqs. (4). The BdG Hamiltonians of two chirality sectors are identical in case of TRS tilt because it preserves the \( Z_2 \) exchange symmetry. However, they become different in presence of TRS broken tilt.
The energy dispersions relation in superconducting region are calculated by diagonalizing the Hamiltonian in Eq. (5), and given by,

$$E_s = C q_z \pm \sqrt{\Delta_0^2 + (\mu_s \pm q)^2}$$

(6)

In the normal region, the energy dispersions are obtained by diagonalizing the Hamiltonian of electron and hole separately. The dispersion relations are given by,

$$E_{e(h)} = \pm \sqrt{q_{p}^2 + q_{p}^2 + C q_z} + (\pm)\mu$$

(7)

where the subscripts e(h) denote the electrons (holes) excitation spectra. The propagating wavectors of quasiparticles with incident energy $E$ and transverse momentum $k_p$ are obtained from Eq. (7) and are given by,

$$q_{1(2)} = \frac{-C(\mu_N + E) \pm \sqrt{(\mu_N + E)^2 - (1 - C^2)q_p^2}}{1 - C^2}$$

(8)

$$q_{3(4)} = \frac{C(\mu_N - E) \pm \sqrt{(\mu_N - E)^2 - (1 - C^2)q_p^2}}{1 - C^2}$$

(9)

with $q_p = \sqrt{q_{1(2)}^2 + q_{3(4)}^2}$. The electron and hole wavefunctions in the normal region read,

$$\Psi_{in(out)}^{e} = e^{i\Omega_{1(2)}z} (1 \ Q_{1(2)} \ 0 \ 0)$$

(10)

$$\Psi_{in(out)}^{h} = e^{i\Omega_{3(4)}z} (0 \ 0 \ 1 \ Q_{3(4)})$$

(11)

where,

$$\Omega_{1(2)} = \frac{q_p e^{i\theta}}{q_{1(2)} + q_{1(2)}}; \ Q_{3(4)} = \frac{q_p e^{i\theta}}{q_{1(2)} + q_{3(4)}}$$

(12)

with $q_{1(2)} = \sqrt{q_{p}^2 + q_{1(2)}^2}$ and $q_{3(4)} = \sqrt{q_{p}^2 + q_{3(4)}^2}$. To calculate the Josephson current in the junction to first obtain the energy spectrum for the Andreev bound states in the normal region. This is done by matching the wavefunctions at the two SN interfaces ($z = 0$ and $z = L$) in the junction and solving the allowed energy values. The boundary conditions lead to an $8 \times 8$ matrix $M$ for the above eight scattering coefficients. The condition $\det(M) = 0$ initiate the non-trivial relation between the Andreev bound state $E_b$ with the superconducting phase difference $\phi$. The ABSs in case of TRS tilt is found as follows,

$$E_b^{(1)} = \Delta \sqrt{1 - C^2} \frac{F_1}{F_3} + \frac{F_2}{F_3} \sin^2 \phi_B^{(1)}$$

(13)

in which the expressions of $F_1$, $F_2$ and $F_3$ are written explicitly as,

$$F_1 = (Q_1 Q_4 + Q_2 Q_3) - (Q_1 Q_3 + Q_2 Q_4) \cos(\Delta q L/2) + \frac{(Q_1 Q_4 + \alpha^2 Q_1 Q_2)}{\alpha^2} \sin^2(\Delta q L/2)$$

$$F_2 = (Q_1 - Q_2)(Q_3 - Q_4)$$

$$F_3 = (Q_2 Q_3 + Q_1 Q_4) - (Q_2 Q_4 + Q_1 Q_3) \cos(\Delta q L)$$

(14)

with $\Delta q = (q_1 - q_2)$. The expression of phase $\phi_B^{(1)}$ is given by,

$$\phi_B^{(1)} = -\frac{(q_1 + q_2) L}{2} + \frac{\phi}{2} + \frac{\phi}{2}$$

(15)

with $\phi = 2\mu_N CL/(1 - C^2)$. The ABSs of BdG Hamiltonian $H_{BdG}^{(2)}$ in case of TRS tilt, is obtained by replacing $Q_1 \leftrightarrow Q_3$, $Q_2 \leftrightarrow Q_4$, $\alpha \rightarrow 1/\alpha$, $\phi \rightarrow -\phi$, and is given by,

$$E_b^{(2)} = \Delta \sqrt{1 - C^2} \frac{F_1}{F_3} + \frac{F_2}{F_3} \sin^2 \phi_B^{(2)}$$

(16)

where the phase $\phi_B^{(2)} = (\phi - \phi)/2$. Here, the ABSs of the two chirality sectors are equal because of the $Z_2$ symmetry. We define the total Josephson current as,

$$J_{tot}(\phi) = J^{(1)}(\phi) + J^{(2)}(\phi)$$

(17)

and valley Joseph current as,

$$J_{valley}(\phi) = J^{(1)}(\phi) - J^{(2)}(\phi)$$

(18)

where, $J^{(1)}(\phi)$ and $J^{(2)}(\phi)$ are the Josephson currents derived respectively from Eq. (13) and Eq. (10), contrarily, in case of TRS broken tilt, the BdG Hamiltonians $H_{BdG}^{(1)}$ and $H_{BdG}^{(2)}$ are equal. Thus, the ABSs, $E_b^{(1)}(\phi)$ and $E_b^{(2)}(\phi)$ are degenerate. However, in this situation the ABSs of positive ($E_b^{(1)}(\phi)$) and negative ($E_b^{(2)}(\phi)$) chirality sectors become different as a consequence of $Z_2$ symmetry breaking. By straightforward algebra we get, $E_b^{(1)}(\phi) = E_b^{(1)}(\phi)$ and $E_b^{(2)}(\phi) = E_b^{(2)}(\phi)$. So, the total Josephson current here $J_{tot}(\phi) = J^{+}(\phi) + J^{-}(\phi)$, has same expression as in Eq. (17). The chirality Joseph current is defined: $J_{ch1}(\phi) = J^{+}(\phi) - J^{-}(\phi)$ and again has same expression as in Eq. (18). The Josephson currents $J_{valley}(\phi)$ and $J_{ch1}(\phi)$ vanishes in absence of TRS tilt and TRS breaking tilt, respectively.
Total Josephson current:- We consider the BdG Hamiltonian of the positive chirality sector in Eq.(4) in case of TRS tilt. The TRS tilt induced phase $\phi_t$ breaks the valley degeneracy and it appears with an opposite sign in the currents $J^{(1)}(\phi)$ and $J^{(2)}(\phi)$. Thereby these two currents become different i.e., $J^{(1)}(\phi) \neq J^{(2)}(\phi)$. However, they are equal in absence of TRS tilt. The presence of phase shift $\phi_t$ leads to an anomalous Josephson current at zero- bias (i.e., $\phi = 0$) $J^{(1)}(\phi = 0) \neq 0$, $J^{(2)}(\phi = 0) \neq 0$. Thus the system indeed realizes the Josephson $\phi$ junction which is controllable by the parameter $\phi_t$, i.e., by tuning the Josephson length and doping for any arbitrary value of tilt. We define an operator $T_{BdG} = -i\sigma_y K \tau_0$, which relate the two BdG Hamiltonians of positive chirality sector as follows: $T_{BdG} \mathcal{H}_{BdG}^{(1)}(\phi) T_{BdG}^{-1} = \mathcal{H}_{BdG}^{(2)}(-\phi)$. As a result, the Andreev levels have the following symmetry: $\mathcal{E}^{(1)}(\phi) = \mathcal{E}^{(2)}(-\phi)$ and subsequently, the current satisfies $J^{(1)}(\phi) = -J^{(2)}(-\phi)$. Thus the total current $J_{tot}(\phi)$ is an odd function of $\phi$ ($J_{tot}(\phi) = -J_{tot}(-\phi)$). On the other hand, the TRS broken tilt induced phase $\phi_t$ breaks $\mathcal{Z}_2$ symmetry and consequently it appears with an opposite sign in the currents $J^+(\phi)$ and $J^-(\phi)$. These two currents are equal in absence of TRS broken tilt. An anomalous current also occurs at zero-bias i.e., $J^+(\phi = 0) \neq 0$, $J^-(\phi = 0) \neq 0$. The BdG Hamiltonians of two chirality sectors are related to each other by, $S_{BdG} \mathcal{H}_{BdG}^{(1)}(\phi) S_{BdG}^{-1} = \mathcal{H}_{BdG}^{(2)}(-\phi)$, where the operator $S_{BdG}$ is the product of $T_{BdG}$ and $U_{BdG}$. Consequently, the Andreev levels have the following symmetry: $\mathcal{E}^+(\phi) = \mathcal{E}^-(\phi)$ and the current satisfies $J^+(\phi) = -J^-(\phi)$. Thus, the total current $J_{tot}(\phi) = J^+(\phi) + J^-(\phi)$ is also an odd function of $\phi$. We have shown the current phase relation (CPR) in Fig.1. The left panel of Fig.1 displays the CPR of individual Josephson current components. The non-zero value of zero-bias supercurrents (when $\phi_t \neq 0$) indicates that the system is in Josephson $\phi$-junction. The zero-bias supercurrent vanishes if and only if the tilt induced phase $\phi_t = 0$. This only occurs in absence of tilt. Thus, the realization of anomalous Josephson currents are different from the earlier studies. The $\phi$ junction discussed mainly in time-reversal broken system like in presence of Zeeman field and spin-orbit coupling or in a junction of SC-Ferromagnet-SC. The right panel of Fig.1 displays the CPR of total Josephson current ($J_{tot}(\phi)$). Since the current $J_{tot}(\phi)$ is an odd function, it always vanishes at $\phi = n\pi$. The junction is in 0-state for these values of $\phi_t$. Thus the Josephson current 0 to $\pi$ transition can be realized by tuning the phase parameter $\phi_t$ both in TRS tilt and TRS broken tilt. The red and blue solid lines in Fig.1 represent the Josephson 0 and $\pi$-junction, respectively. The dotted curves representing the CPR other than 0 or $\pi$ Josephson junction. The CPR can generally be expanded in Fourier series with all the harmonics as follows: $J_{tot}(\phi) = \sum_n [J_n \sin n\phi + I_n \cos n\phi]$. Since the current $J_{tot}(\phi)$ in each chirality sector has time reversal symmetry, all the coefficients $I_n$ of cosine terms are zero. The leading term of $J_{tot}(\phi)$ is $\sin \phi$ and the presence of higher harmonic terms causes an extra dip or peak in the dotted curves for $\phi \in [0, \pi]$. At $\phi_t = \pi/2$, the current has $\pi$ periodicity and is dominated by the second harmonic term $\sin(2\phi)$. This is shown by black dotted line.

Valley/chirality Josephson current:- The breaking of valley degeneracy (by TRS tilts) and $\mathcal{Z}_2$ symmetry (by TRS broken tilts) causes the striking phenomenon of finite VJC ($J_{valley}$) and finite CJC ($J_{chiral}$), respectively. However, these two currents share the same expression. The valley/chirality current is an even function of $\phi$, i.e., $J_{valley/chi}(\phi = \phi_{valley/chi}(-\phi)$ and has $2\pi$ periodicity in $\phi$. This signifies that the $J_{valley/chi}(\phi)$ exists at $\phi = n\pi$ where, the $J_{tot}(\phi)$ is identically zero. Thus a pure VJC/CJC can be found at $\phi = n\pi$. This is one of the astonishing results here. It is possible to transfer a pure valley/chiral supercurrent across the junction. This could have important implications in superconductor base valleytronics/chiraltronics. An anomalous VJC/CJC also persists at zero bias condition. Fig.2 displays the total and valley/chirality Josephson current for two different values of $\phi_t$. The maxima of $|J_{valley/chi}(\phi)|$ and zero of $J_{tot}(\phi)$ occur respectively at $\phi = n\pi$, indicated by arrows. A pure VJC/CJC exists at these points of $\phi$. The
zero-bias VJC/CJC is an oscillatory function of $\phi_t$, given as: $J_{\text{valley/chi}}(\phi = 0) \sim \sin \phi_t$. Thus, a reversible pure VJC/CJC at zero bias is feasible by appropriately choosing the phase $\phi_t$. However, these currents reversible are possible to occur at any value $\phi$. In Fig.(2), the red solid lines get reversed into the blue solid line by tuning the phase $\phi_t$.

**Josephson current in long junction and Quantum anomaly:-** The VJC and CJC are closely linked respectively with the valley symmetry breaking and $Z_2$ symmetry breaking in this system. We now explain how the currents VJC and CJC are associated with quantum anomaly of Goldstone bosons. In the long junction ($L \gg \xi$) and zero temperature limit, the Andreev bound states deep in the superconducting gap ($\epsilon \ll \Delta$) have only significant contributions to the Josephson currents\cite{19}. The low energy excitations around a Fermi surface are described by the helical model with Hamiltonian,

$$H = v_F (q_z + \frac{C_{\mu N}}{1 - C^2}) s_z$$  \hspace{0.5cm} (19)

with $v_F = (\mu^2_N - (1 - C^2)k_F^2)^{1/2}/\mu_N$. The Pauli matrix $s_z$ acting on two valleys in case of TRS tilted whereas it acting on two chirality sectors in case of TRS breaking tilt. The ABS spectrum is given by,

$$\epsilon_{\pm} = \pm \frac{\pi v_F}{L} \left( n + \frac{1}{2} + \frac{\phi \pm \phi_t}{2\pi} \right)$$  \hspace{0.5cm} (20)

Compare to the short junction limit, the energy levels here are linear in $\phi \pm \phi_t$ and cross at $\phi \pm \phi_t = \pi$. The Josephson current are obtained by the method of Bosonization\cite{20,24}. For a given transverse momentum $k_p$ the current expression is given by\cite{18},

$$j_{\eta} = \frac{2\pi v_F}{L} \left( \frac{\phi \pm \eta \phi_t}{\pi} - sgn(\phi + \eta \phi_t - \pi) - 1 \right)$$  \hspace{0.5cm} (21)

where $\eta = \pm 1$ representing two valleys or two chirality sectors. By summing over $k_p$, the total Josephson current at $\phi = \pi$ is given by,

$$J_{\eta} = \frac{2\mu_N^2}{3L} \left( \eta \frac{\phi_t}{\pi} - sgn(\eta \phi_t) \right)$$  \hspace{0.5cm} (22)

Here, we neglect the term $C^2$. Thus the valley/chirality Josephson current for small value of $\phi_t$ is given by,

$$J_{\text{valley/chi}} = \frac{2\mu_N^2}{3L} \left( 2\phi_t - 2sgn(\phi_t) \right)$$  \hspace{0.5cm} (23)

The current in Eq.(23) shows a discontinuous jump at $\phi_t = 0$ i.e, $C = 0$. The discontinuity in the current signifies that the valley or chiral Josephson current is finite even if the system preserves the valley symmetry or $Z_2$ symmetry. Thus this discontinuity of currents associated with the quantum anomaly of Cooper pairs.

**Conclusions:-** We find that tilt induced phase breaks the valley degeneracy and $Z_2$ symmetry, respectively, in time reversal preserved and time reversal breaking system. The tilting can lead to a Josephson current $0-\pi$ transition and Josephson $\phi$ junction in an inversion-asymmetric WSM Josephson junction. In particular, the TRS tilts give rise to a pure VJC whereas, TRS breaking tilts produce a finite pure CJC which can exist even at zero bias condition. Consequently, in the long junction and at zero temperature, VJC and CJC are closely related to a quantum anomaly of Goldstone bosons. A pure VJC and CJC reversible can be realized by tuning the phase $\phi_t$, which means the reversible can occur by tuning the length $L$ or doping $\mu_N$ for a fixed value of tilt $C$ of Weyl nodes. The mechanism of phase shift in the Josephson current here is completely different from the conventional ferromagnetic Josephson junctions. The spin polarization is essential in the later case.

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Appendix S1: Analogy with Ferromagnetic Josephson Junction

The Hamiltonian of a single Weyl node in presence of tilt is given by,
\[ H(k) = Ck_z \sigma_0 + (k_x \sigma_x + k_y \sigma_y + k_z \sigma_z) \] (S1.1)

The equienergy surface equations of the above Hamiltonian are given by,
\[ \left[ k_z - \frac{CE}{(1-C^2)} \right]^2 + \frac{k_x^2}{(1-C^2)} + \frac{k_y^2}{(1-C^2)} = \frac{E^2}{(1-C^2)^2} \] (S1.2)

So, the Fermi surfaces are ellipsoids centered at \((\pm \frac{C \pm E}{(1-C^2)}, 0, 0)\). The following transformation,
\[ k'_z = k_z - \frac{CE}{(1-C^2)}, k'_x = \frac{k_x}{\sqrt{1-C^2}}, k'_y = \frac{k_y}{\sqrt{1-C^2}}, \mathcal{E}' = \frac{E}{(1-C^2)} \] (S1.3)

transform the surface to a sphere in the new coordinations,
\[ \mathcal{E}'^2 = k'_z^2 + k'_x^2 + k'_y^2 \] (S1.4)

From Eq.(S1.3), it is clear that the tilting term shifted the momentum \(k_z\) of an electron. For a TRS tilt, the momentum of time reversal pair Weyl nodes are shifted in the opposite direction. Thus, a pair of electrons at the Fermi surface acquire a centre of mass momentum \(\Delta k = 2\mathcal{E}/(1-C^2)\), at normal incidence. This is akin to the ferromagnetic Josephson junction where the centre of mass momentum originated via the spin splitting. Similarly, the shifting of momentum occurs in the negative chirality sector with an equal magnitude. In contrast, the momentum of opposite chiral electrons are shifted equally in the opposite direction in case of TRS broken tilt. We have shown the momentum shift of Weyl nodes both TRS and TRS broken tilt in Fig.(S3). Blue and red circles are representing positive and negative Weyl nodes. The positive chiral nodes 1 and 3 are shifted oppositely by \(\pm \Delta k\) in momentum space in presence of TRS tilt. Similarly, the negative chiral nodes 2 and 4 are shifted oppositely by \(\pm \Delta k\) in momentum space. As a result, the net chiral momentum shift is zero. On the other hand, the positive and negative chiral nodes are shifted oppositely by \(\pm \Delta k\) in momentum space in case of TRS broken tilt. The net chiral momentum shift gives rise to chirality Josephson current in a Josephson junction involving inversion asymmetric WSM.

Appendix S2: Scattering Wavefunctions, Andreev Bound states and Josephson current

1. TRS tilt

We consider the BdG Hamiltonian of positive chirality sector in presence of TRS tilt. The matrix form of the Hamiltonian is given,
\[ \mathcal{H}^{+}_{BdG}(\phi) = \begin{pmatrix} \mathcal{H}^{(1)}_{BdG}(\phi) & 0 \\ 0 & \mathcal{H}^{(2)}_{BdG}(\phi) \end{pmatrix} \] (S2.1)

The matrix form of \(\mathcal{H}^{(1)}_{BdG}\) is written as,
\[ \mathcal{H}^{(1)}_{BdG}(\phi) = \begin{pmatrix} H_1(k) - \mu & \Delta(\phi) \\ \Delta^*(\phi) & \mu - TH_1(k)T^{-1} \end{pmatrix} \] (S2.2)

where \(\Delta(r) = |\Delta| e^{i\text{sgn}(z)\phi/2}\.\) Here, we consider that doping \(\mu_s\) in superconducting region to be large. The diagonalization of Eq.(S2.2) yields the eigenvalues,
\[ \mathcal{E}_s^{(1)} = Ck_z \pm \sqrt{\Delta^2 + (\mu_s \pm k_z)^2} \] (S2.3)
FIG. S3. Momentum shift due to tilt of an inversion asymmetric Weyl nodes with mirror symmetry. Blue and red circles indicates Weyl nodes of positive and negative chiralities, respectively. Right (left) panel shows the momentum shift of Weyl nodes in case of TRS (TRS broken) tilt.

where first $+(-)$ sign denotes the electronlike (holelike) excitations while second $+(-)$ sign denotes the conduction (valance) band. In the two superconducting regions, the basis function are,

$$
\phi_{qe}(z) = \left( \alpha e^{i\beta_1} \ 0 \ e^{-i\phi_t} \right)^T e^{ik_{se}^+ z} \\
\phi_{qe}^-(z) = \left( 0 \ \frac{z_{se}^+}{\alpha} \ 0 \ e^{-i\phi_t} \right)^T e^{ik_{se}^- z} \\
\phi_{qh}(z) = \left( 0 \ e^{i\phi_t} \ \alpha e^{i\beta_2} \right)^T e^{ik_{sh}^+ z} \\
\phi_{qh}^-(z) = \left( e^{i\phi_t} \ 0 \ \frac{z_{sh}^+}{\alpha} \ 0 \right)^T e^{ik_{sh}^- z}
$$

where $t \in \{R, L\}$ labels the superconducting pairing phase on the left and right hand sides $\phi_L = -\phi/2$ and $\phi_R = \phi/2$, respectively. Here,

$$
\beta_{1(2)} = \arccos \left[ \frac{-E \mp \mathcal{C}\mu_s}{\Delta \sqrt{(1-\mathcal{C}^2)}} \right] \quad \text{(S2.5)}
$$

and

$$
\alpha = \sqrt{\frac{1-\mathcal{C}}{1+\mathcal{C}}} \quad \text{(S2.6)}
$$

The wavevectors $k_{se(h)}^\pm$ are obtained from Eq. (S2.22) and given below,

$$
k_{se}^\pm = \pm \mu_s - \mathcal{C} \mathcal{E} \pm \sqrt{(\mathcal{E} \mp \mathcal{C}\mu_s)^2 - (1-\mathcal{C}^2)\Delta^2} \quad \text{(S2.7)}
$$

$$
k_{sh}^\pm = \pm \mu_s - \mathcal{C} \mathcal{E} \mp \sqrt{(\mathcal{E} \mp \mathcal{C}\mu_s)^2 - (1-\mathcal{C}^2)\Delta^2} \quad \text{(S2.8)}
$$
The above expressions in Eq. (S2.8) of quasiparticles wave-vectors takes the form \( k_{+} = \pm (\mu_{s} + \Omega) \) and \( k_{-} = \pm (\mu_{s} - \Omega) \), respectively for \( C = \pm i \). Here, \( \Omega = i\sqrt{\Delta^{2} - E^{2}} \) for subgap energies \( E \leq \Delta \), while \( \Omega = \text{sgn}(E)\sqrt{E^{2} - \Delta^{2}} \) for \( E > \Delta \). To obtain Andreev bound states, We impose the boundary conditions along \( z \),

\[
\Psi_{L}^{S}(z = 0) = \Psi_{N}(z = 0), \quad \Psi_{N}(z = L) = \Psi_{R}^{S}(z = L)
\]

(S2.9)

where \( \Psi_{L}^{S}(z) \), \( \Psi_{R}^{S}(z) \) and \( \Psi_{N}(z) \) are respectively the wavefunctions in the left superconductor, right superconductor and normal reion. The wavefunctions in the three different regions are written explicitly,

\[
\Psi_{L}^{S} = t_{1} \phi_{q_{c}-} + t_{2} \phi_{q_{h}-}
\]

\[
\Psi_{N} = a_{1} \Psi_{in}^{e} + a_{2} \Psi_{out}^{e} + a_{3} \Psi_{in}^{h} + a_{4} \Psi_{out}^{h}
\]

\[
\Psi_{R}^{S} = t_{3} \phi_{q_{c}+} + t_{4} \phi_{q_{h}+}
\]

(S2.10)

Here, \( t_{i} \) and \( a_{i} \) (\( i = 1, 2, 3, 4 \)) are the scattering amplitudes of quaiiparticles (electron and hole) in three different regions. The form of different components \( \Psi_{in(out)}^{e} \) and \( \Psi_{in(out)}^{h} \) are given,

\[
\Psi_{in(out)}^{e} = e^{i\beta_{i}(z)} (1 \quad Q_{i(2)} \quad 0 \quad 0)
\]

(S2.11)

\[
\Psi_{in(out)}^{h} = e^{i\beta_{i}(z)} (0 \quad 0 \quad 1 \quad Q_{i(4)})
\]

(S2.12)

where,

\[
Q_{1(2)} = \frac{k_{p}e^{i\theta}}{\pm k_{2} - k_{1(2)}}; \quad Q_{3(4)} = \frac{k_{p}e^{i\theta}}{\pm k_{2} - k_{3(4)}}
\]

(S2.13)

The wavevectors \( k_{i} \) are given,

\[
k_{1(2)} = -C(\mu_{N} + E) \pm \sqrt{(\mu_{N} + E)^{2} - (1 - C^{2})k_{p}^{2}}; \quad k_{3(4)} = C(\mu_{N} - E) \pm \sqrt{(\mu_{N} - E)^{2} - (1 - C^{2})k_{p}^{2}}
\]

(S2.14)

where \( k_{p} = \sqrt{k_{p}^{2} + k_{2}^{2}} \), \( k_{+} = \sqrt{k_{p}^{2} + k_{1(2)}^{2}} \), \( k_{-} = \sqrt{k_{p}^{2} + k_{3(4)}^{2}} \) and \( \theta = \tan^{-1}(k_{p}/k_{2}) \). The boundary conditions in Eq. (S2.9) leads to eight linear equations in the matrix form \( \mathcal{M} \mathcal{X} = 0 \) where \( \mathcal{M} \) is a \( 8 \times 8 \) matrix and \( \mathcal{X} \) is the column vector containing the eight scattering coefficients. The Andreev bound states are obtained by demanding the nozero solutions of these equations or equivalently from the condition det(\( \mathcal{M} \)) = 0. Here we take \( \mu_{N} \), \( \mu_{S} \) \( \gg \Delta \) and concentrate on the short junction limit \( L \ll \xi = h\nu/\Delta \). This allows us to neglect the contributions from the Andreev states \( \mathcal{E}_{b} > \Delta \). In the short junction limit, the quasiparticles wavectors are related to each other by \( q_{3(4)} = -q_{1(2)} \). The condition det(\( \mathcal{M} \)) = 0 lead to the following equation,

\[
\mathcal{F}_{3} \cos \beta = \mathcal{F}_{1} + \mathcal{F}_{2} \sin^{2} \phi_{B}^{(1)}
\]

(S2.15)

To obtain the above form, we take the assumption \( \beta_{1} \simeq \beta_{2} = \beta = \arccos(\frac{\mathcal{E}}{\Delta \sqrt{(1 - C^{2})}}) \). We obtain the ABSs from Eq. (S2.15) and the analytical form is given below.

\[
\mathcal{E}_{b}^{(1)} = \Delta \sqrt{1 - C^{2}} \sqrt{\frac{\mathcal{F}_{1}}{\mathcal{F}_{3}}} + \frac{\mathcal{F}_{2}}{\mathcal{F}_{3}} \sin^{2} \phi_{B}^{(1)}
\]

(S2.16)

The analytical expressions of \( \mathcal{F}_{1}, \mathcal{F}_{2} \) and \( \mathcal{F}_{3} \) are given,

\[
\mathcal{F}_{1} = (Q_{1}Q_{4} + Q_{2}Q_{3}) - (Q_{1}Q_{3} + Q_{2}Q_{4}) \cos^{2}(\Delta qL/2)
\]

\[
+ \frac{(Q_{3}Q_{4} + \alpha^{2}Q_{1}Q_{2})}{\alpha^{2}} \sin^{2}(\Delta qL/2)
\]

\[
\mathcal{F}_{2} = (Q_{1} - Q_{2})(Q_{3} - Q_{4})
\]

\[
\mathcal{F}_{3} = (Q_{2}Q_{3} + Q_{1}Q_{4}) - (Q_{2}Q_{4} + Q_{1}Q_{3}) \cos(\Delta qL)
\]

(S2.17)

with \( \Delta q = (q_{1} - q_{2}) \). The expression of phase \( \phi_{B}^{(1)} \) is given by,

\[
\phi_{B}^{(1)} = \frac{(k_{1} + k_{2})L}{2} + \phi = \phi + \phi_{B}^{(1)}
\]

(S2.18)
with \( \phi_i = 2\mu_N CL/(1 - C^2) \). The Josephson current at low temperature \( T \ll \Delta/k_B \), \( k_B \) is the Boltzmann constant) is determined from the ABSs by,

\[
I(\phi) = -\frac{2e}{\hbar} \sum_b \frac{\partial \mathcal{E}_b}{\partial \phi} f(\mathcal{E}_b)
\]

where \( f(\mathcal{E}_b) \) is the Fermi-Dirac distribution function. The Josephson current density now obtain as,

\[
J(\phi) = \frac{W^2}{(2\pi)^2} \int I(\phi)dq_x dq_y \tag{S2.20}
\]

where, \( W \) is the dimension in both \( x \) and \( y \)-direction. We now consider the BdG Hamiltonian \( \mathcal{H}^{(2)}_{BdG} \) in Eq.(S2.1). The matrix form of \( \mathcal{H}^{(2)}_{BdG} \) is given by,

\[
\mathcal{H}^{(2)}_{BdG}(\phi) = \begin{pmatrix}
H_3(k) - \mu & \Delta(\phi) \\
\Delta^*(\phi) & \mu - TH_3(k)T^{-1}
\end{pmatrix} \tag{S2.21}
\]

The Hamiltonian \( \mathcal{H}^{(2)}_{BdG} \) is thus obtained by replacing \( C \) by \(-C\) in Eq.(S2.2). The eigenvalues are given by,

\[
\mathcal{E}_s^{(2)} = -Ck_z \pm \sqrt{\Delta^2 + (\mu_x \pm k_z)^2} \tag{S2.22}
\]

We similarly construct wavefunctions in three different regions by replacing \( C \) by \(-C\). This replacement effectively leads to the followings substitution: \( Q_1 \leftrightarrow Q_3 \), \( Q_2 \leftrightarrow Q_1 \), \( \alpha \rightarrow 1/\alpha \), \( \phi_i \rightarrow -\phi_i \). Thus the analytical expressions of \( F_1 \), \( F_2 \) and \( F_3 \) are remained invariant. The ABSs are given,

\[
E_b^{(2)} = \sqrt{1 - C^2} \sqrt{F_1 + F_2 \sin^2 \phi_b^{(2)}} \tag{S2.23}
\]

with \( \phi_b^{(2)} = (\phi - \phi_i)/2 \). The ABSs of negative and positive chirality sectors are equal due to \( \mathbb{Z}_2 \) symmetry (see Eq.(S3.15)).

2. TRS breaking tilt

We now consider the BdG Hamiltonian of positive chirality sector in presence of TRS breaking tilt. In this case the diagonal Hamiltonian in Eq.(S2.1) are same i.e., \( \mathcal{H}_{BdG}^{(1)}(\phi) = \mathcal{H}_{BdG}^{(2)}(\phi) \). However, due to \( \mathbb{Z}_2 \) breaking the ABSs of negative and positive chirality sectors are not equal.

Appendix S3: Symmetry Analysis of Josephson current

At low energy and considering the mirror symmetry, the Hamiltonians of positive and negative chirality in presence of a TRS tilt, are given by,

\[
H_+(r) = -iC\partial_z \sigma_0 \tau_z - i\tau_0(\partial_x \sigma_x + \partial_y \sigma_y + \partial_z \sigma_z)
\]

\[
H_-(r) = -iC\partial_z \sigma_0 \tau_z + i\tau_0(\partial_x \sigma_x + \partial_y \sigma_y + \partial_z \sigma_z)
\]

whereas for a TRS broken tilt the above Hamiltonians are written as

\[
H_+(r) = -iC\partial_z \sigma_0 \tau_0 - i\tau_0(\partial_x \sigma_x + \partial_y \sigma_y + \partial_z \sigma_z)
\]

\[
H_-(r) = iC\partial_z \sigma_0 \tau_0 + i\tau_0(\partial_x \sigma_x + \partial_y \sigma_y + \partial_z \sigma_z)
\]

The time reversal operator \( T = -i\tau_x \sigma_y K \), commutes with the Hamiltonians in Eq.(S3.1) i.e., \([H_{\pm}(r), T] = 0\) whereas it does not commute with the Hamiltonians in Eq.(S3.2). In addition there exists an symmetry operation

\[
U = i\tau_y \sigma_y R_x
\]

where \( R_y \) is the reflection operator about the \( yz \)-plane. In absence of tilt, the opposite chiral sector follow the symmetry: \( UH_+(r)U^{-1} = H_-(r) \). We call \( U \) as the \( \mathbb{Z}_2 \) (exchange) operator. The TRS tilt preserve the \( \mathbb{Z}_2 \) symmetry whereas TRS broken tilt breaks this symmetry. Thus the TRS broken tilt breaks both the \( \mathbb{Z}_2 \) symmetry and \( T \).
simultaneously. However, the TRS broken tilted system preserve the combined symmetry defined by the product of $\mathcal{T}$ and $\mathcal{U}$ i.e.,

$$\mathcal{T} \mathcal{U}_+ (r)(\mathcal{T} \mathcal{U})^{-1} = H_- (r) \quad \text{(S3.4)}$$

The BdG Hamiltonian for the positive chirality sector is given by,

$$\mathcal{H}^{+}_{\text{BdG}}(\phi) = \begin{pmatrix} \mathcal{H}^{(1)}_{\text{BdG}}(\phi) & 0 \\ 0 & \mathcal{H}^{(2)}_{\text{BdG}}(\phi) \end{pmatrix} \quad \text{(S3.5)}$$
in which the Hamiltonians $\mathcal{H}^{(1)}_{\text{BdG}}(\phi)$ is given by,

$$\mathcal{H}^{(1)}_{\text{BdG}}(\phi) = \begin{pmatrix} -i \Delta e^{-i \text{sgn}(z) \phi/2} & i \Delta e^{i \text{sgn}(z) \phi/2} \\ -i \Delta e^{-i \text{sgn}(z) \phi/2} & \mu_0 - H_1^1(r) \end{pmatrix} \quad \text{(S3.6)}$$

and $\mathcal{H}^{(2)}_{\text{BdG}}(\phi)$ is given by,

$$\mathcal{H}^{(2)}_{\text{BdG}}(\phi) = \begin{pmatrix} -i \Delta e^{-i \text{sgn}(z) \phi/2} & i \Delta e^{i \text{sgn}(z) \phi/2} \\ -i \Delta e^{-i \text{sgn}(z) \phi/2} & \mu_0 - H_3^1(r) \end{pmatrix} \quad \text{(S3.7)}$$

We define extended time reversal operator $\mathcal{T}_{\text{BdG}}$ as,

$$\mathcal{T}_{\text{BdG}} = \begin{pmatrix} -i \sigma_y \mathcal{K} & 0 \\ 0 & -i \sigma_y \mathcal{K} \end{pmatrix} \quad \text{(S3.8)}$$

By using $\mathcal{T}_{\text{BdG}}$, we find for a TRS tilt,

$$\mathcal{T}_{\text{BdG}} \mathcal{H}^{(1)}_{\text{BdG}}(\phi) \mathcal{T}_{\text{BdG}}^{-1} = \mathcal{H}^{(2)}_{\text{BdG}}(-\phi) \quad \text{(S3.9)}$$

The BdG equation for the Hamiltonian $\mathcal{H}^{(1)}_{\text{BdG}}(\phi)$ is described by,

$$\mathcal{H}^{(1)}_{\text{BdG}}(\phi) \psi_n = E^{(1)}_n(\phi) \psi_n \quad \text{(S3.10)}$$

where $E_n$ and $\psi_n$ are eigenstate and eigenfunction labeled by an index $n$. By using Eq. (S3.9), the BdG equation Eq. (S3.10) can be transformed to

$$\mathcal{H}^{(2)}_{\text{BdG}}(-\phi) \mathcal{T}_{\text{BdG}} \psi_n = E^{(1)}_n(\phi) \mathcal{T}_{\text{BdG}} \psi_n \quad \text{(S3.11)}$$

From Eqs. (S3.10) and (S3.11), it is clear that $\mathcal{H}^{(1)}_{\text{BdG}}(\phi)$ and $\mathcal{H}^{(2)}_{\text{BdG}}(-\phi)$ have same eigenvalues which leads to the following symmetry,

$$E^{(1)}_n(\phi) = E^{(2)}_n(-\phi) \quad \text{(S3.12)}$$

Using the formula for the Josephson current,

$$J(\phi) = -\frac{2e}{\hbar} \sum_n \frac{\partial E_n(\phi)}{\partial \phi} f(E_n) \quad \text{(S3.13)}$$

one can find the following relation for the Josephson current.

$$J^{(1)}(\phi) = -J^{(2)}(-\phi) \quad \text{(S3.14)}$$

However, for a TRS broken tilt the Hamiltonians $\mathcal{H}^{(1)}_{\text{BdG}}(\phi)$ and $\mathcal{H}^{(2)}_{\text{BdG}}(\phi)$ are identical i.e., $\mathcal{H}^{(1)}_{\text{BdG}}(\phi) = \mathcal{H}^{(2)}_{\text{BdG}}(\phi)$. Consequently, the Josephson currents are equal i.e., $J^{(1)}(\phi) = J^{(2)}(\phi)$. We now construct BdG Hamiltonian $H^\pm_{\text{BdG}}(\phi)$ in a similar way. It can be easily shown that,

$$\mathcal{U}_{\text{BdG}} \mathcal{H}^{\pm}_{\text{BdG}}(\phi) \mathcal{U}_{\text{BdG}}^{-1} = H^\mp_{\text{BdG}}(\phi) \quad \text{(S3.15)}$$

holds for a TRS tilt with $\mathcal{U}_{\text{BdG}} = \text{diag} \{ \mathcal{U}, \mathcal{U} \}$. The symmetry of Eq. (S3.15) leads to the symmetry in ABS as $E^+_n(\phi) = E^-_n(\phi)$ and consequently in the Josephson current as $J^+(\phi) = J^-(\phi)$. Thus the chirality Josephson current vanishes in this model system. In contrast, for a TRS broken tilt, the symmetry of BdG Hamiltonian of opposite chirality sectors is given by,

$$(\mathcal{T} \mathcal{U})_{\text{BdG}} \mathcal{H}^+_\text{BdG}(\phi)(\mathcal{T} \mathcal{U})_{\text{BdG}}^{-1} = H^-_{\text{BdG}}(-\phi) \quad \text{(S3.16)}$$

The above symmetry in Eq. (S3.16) leads to the symmetry in ABS as $E^+_n(\phi) = E^-_n(-\phi)$ and consequently in the Josephson current as $J^+(\phi) = -J^-(\phi)$. Thus, $J^+(\phi) \neq J^-(\phi)$ which produces a finite chirality Josephson current in this model system.