Abstract

The Heisenberg inequality $\Delta x \Delta p \geq \hbar/2$ can be replaced by an exact equality, for suitably chosen measures of position and momentum uncertainty, which is valid for all wavefunctions. The significance of this “exact” uncertainty relation is discussed, and results are generalised to angular momentum and phase, photon number and phase, time and frequency, and to states described by density operators. Connections to optimal estimation of an observable from the measurement of a second observable, Wigner functions, energy bounds and entanglement are also given.
I  INTRODUCTION

One of the remarkable features of quantum mechanics is the property that certain observables cannot simultaneously be assigned arbitrarily precise values. This property does not compromise claims of completeness for the theory, since it may consistently be asserted that such observables cannot simultaneously be measured to an arbitrary accuracy \[1\]. Thus the Heisenberg inequality

\[ \Delta X \Delta P \geq \hbar/2 \] (1)

is generally taken to reflect an essential incompleteness in the applicability of classical concepts of position and momentum to physical reality.

It was recently noted that this fundamental inequality can be greatly strengthened. In particular, one may define a measure of position uncertainty \(\delta X\) (which arises naturally in classical statistical estimation theory), and a measure of nonclassical momentum uncertainty \(\Delta P_{nc}\) (which arises from a natural decomposition of the momentum operator), such that \[2\]

\[ \delta X \Delta P_{nc} = \hbar/2 \] (2)

for all wavefunctions. Such an equality may be regarded as an exact uncertainty relation, and may be shown to imply the usual Heisenberg inequality Eq. (1). Hence, perhaps paradoxically, the uncertainty principle of quantum mechanics may be given a quantitatively precise form.

In Ref. \[2\] the above exact uncertainty relation was merely noted in passing, with the emphasis being on other properties of \(\delta X\) and \(\Delta P_{nc}\). Similarly, while the very existence of an exact form of the uncertainty principle was recently shown to provide a sufficient basis for moving from the classical equations of motion to the Schrödinger equation \[3\], the corresponding exact uncertainty relation Eq. (2) was only briefly mentioned. The purpose of this paper, therefore, is to study the physical significance of Eq. (2) in some detail, including its extensions to other pairs of conjugate observables and to general states described by density operators.

In the following section it is shown that quantum observables such as momentum, position, and photon number have a natural decomposition, into the sum of a classical and a nonclassical component. The classical component corresponds to the best possible measurement of the observable, on a given state, which is compatible with measurement of the conjugate observable. Complementarity implies that the classical component cannot be equivalent to the observable itself, i.e., there is in general an nontrivial nonclassical component. It is this nonclassical component which reflects the
mutual incompatibility of pairs of conjugate observables, and the magnitude of which appears in the exact uncertainty relations to be derived [e.g., $\Delta P_{nc}$ in Eq. (2)]. The decomposition into classical and nonclassical components is also related in a natural manner to quantum continuity equations and to quasiclassical properties of the Wigner function.

In Sec. III a measure of uncertainty is defined for continuous random variables, which plays a fundamental role in classical estimation theory, and which also provides a direct measure of the robustness of the variable with respect to Gaussian diffusion processes. This measure, the “Fisher length” of the variable, may of course be calculated for quantum observables as well, and appears as $\delta X$ in the exact uncertainty relation in Eq. (2).

The ingredients of classical/nonclassical decompositions and Fisher lengths are combined in Sec. IV to obtain a number of exact uncertainty relations, such as Eq. (3) and the equality

$$\delta \Phi \Delta N_{nc} = 1/2$$

for phase and photon number, valid for all pure states. These relations generalise to inequalities for states described by density operators, and are far stronger than the corresponding Heisenberg-type inequalities. It is shown that a bound on Fisher length leads to an entropic lower bound for the groundstate energies of quantum systems, and results are generalised to an exact uncertainty relation for time and frequency, and to higher dimensions.

In Sec. V it is shown that the decomposition of an observable of a given quantum system into classical and nonclassical components is essentially nonlocal in nature, being dependent in general on manipulations performed on a second system with which the first is entangled. The significance of the relevant exact uncertainty relations is discussed, with particular reference to EPR-type states.

A formal generalisation of exact uncertainty relations, to arbitrary pairs of quantum observables, is noted in Sec. VI. Moreover, it is shown that a result of Ivanovic [4], for complete sets of mutually complementary observables on finite Hilbert spaces (such as the Pauli spin matrices), may be reinterpreted as an exact uncertainty relation for the “collision lengths” of the observables.

Conclusions are given in Sec. VII.

II CLASSICAL AND NONCLASSICAL COMPONENTS OF QUANTUM OBSERVABLES
A Momentum

The nonclassical momentum uncertainty $\Delta P_{nc}$ appearing in Eq. (3) is defined via a natural decomposition of the momentum observable $P$ into “classical” and “nonclassical” components,

$$P = P_{cl} + P_{nc}. \quad (3)$$

This decomposition is state-dependent, and will be defined explicitly further below. In particular, it will be shown that the classical component, $P_{cl}$, corresponds to the best estimate of momentum for a given quantum state compatible with a position measurement. Moreover, the average error of this best estimate will be shown to correspond to the variance $(\Delta P_{nc})^2$ of the nonclassical component. In Secs. II.B and II.C it will further be shown that $P_{cl}$ is related to the momentum flow in a classical continuity equation following from the Schrödinger equation, and to an average momentum arising naturally from quasiclassical properties of the Wigner function. However, it is the “best estimate” interpretation above that provides the most general basis for generalisation to other observables.

As a starting point, recall that in classical mechanics one can simultaneously obtain precise values for position and momentum, whereas in quantum mechanics one must choose to accurately measure either one or the other. It is therefore reasonable to ask the following question: If I measure one of these observables precisely, on a known quantum state, then what is the best estimate I can make for the value of the other observable? Such an estimate of momentum from the measurement of position will be called a classical estimate of $P$, since it assigns simultaneous values to $X$ and $P$.

It will be shown that the best classical estimate of $P$, given the measurement result $X = x$ on a quantum system described by wavefunction $\psi(x)$, is given by

$$P_{cl}(x) = \frac{\hbar}{2i} \left( \frac{\psi'(x)}{\psi(x)} - \frac{\psi^*(x)}{\psi^*(x)} \right) = \hbar [\arg \psi(x)]'. \quad (4)$$

More generally, for a quantum system described by density operator $\rho$, one has

$$P_{cl}(x) := \frac{\langle x|P\rho + \rho P|x\rangle/2}{\langle x|\rho|x\rangle} \quad (5)$$

(which reduces to the first expression for $\rho = |\psi\rangle\langle \psi|$). Note that this estimate is equivalent to measurement of the Hermitian operator

$$P_{cl} = \int dx \, P_{cl}(x)|x\rangle\langle x| \quad (6)$$
on state $\rho$, which by construction commutes with $X$. The experimentalist’s procedure is thus to (i) prepare the system in state $\rho$; (ii) measure the position $X$; and (iii) for result $X = x$ calculate $P_{cl}(x)$. As stated above, this procedure yields the best possible estimate of the momentum of the system that is compatible with simultaneous knowledge of the position of the system.

It is important to note that $P_{cl}(x)$ and $P_{cl}$ should, strictly speaking, explicitly indicate their dependence on a given state $\rho$, e.g., via the notation $P_{cl}(x|\rho)$ and $P_{cl}^\rho$ respectively. This would in particular be necessary if one wished to evaluate the expectation value $\text{tr}[\sigma P_{cl}^\rho]$ for some density operator $\sigma$ other than $\rho$. However, since in fact expectation values will only be evaluated for the corresponding state $\rho$ throughout this paper, explicit notational dependence on the state may be conveniently dispensed with, without leading to ambiguity. Similar remarks apply to the nonclassical momentum component $P_{nc}$ in Eq. (3).

To prove that $P_{cl}(x)$ above provides the best classical estimate of $P$, consider some general classical estimate $\tilde{P}(x)$ for momentum associated with measurement result $X = x$ for state $\rho$. This estimate is then equivalent to measurement of the operator $\tilde{P} = \int dx \tilde{P}(x)|x\rangle\langle x|$, and hence the average error of the estimate may be quantified by

$$E_P := \langle (P - \tilde{P})^2 \rangle = \langle P^2 \rangle + \langle \tilde{P}^2 \rangle - 2 \langle P \tilde{P} \rangle,$$

(7)

where $\langle A \rangle$ denotes $\text{tr}[\rho A]$. But, using the cyclic property of the trace operation and evaluating the trace in the position representation,

$$\langle \tilde{P}P + P\tilde{P} \rangle = \int dx \langle x|\tilde{P}P + P\tilde{P}|x\rangle$$

$$= \int dx \tilde{P}(x)|x\rangle P\rho + \rho P|x\rangle$$

$$= 2 \int dx \langle x|\rho\rangle \tilde{P}(x)P_{cl}(x) = 2\langle \tilde{P}P_{cl} \rangle,$$

and hence

$$E_P = \langle P^2 \rangle + \langle \tilde{P}^2 \rangle - 2\langle \tilde{P}P_{cl} \rangle$$

$$= \langle P^2 \rangle - \langle P_{cl}^2 \rangle + \langle (\tilde{P} - P_{cl})^2 \rangle.$$

(8)

Since the last term is positive, the average error is therefore minimised by the choice $\tilde{P} = P_{cl}$ as claimed.

The nonclassical momentum component $P_{nc}$ is implicitly defined via Eqs. (3) and (4). From Eq. (3) one finds that the expectation values of the
observables $P$ and $P_{cl}$ are always equal (for the corresponding state $\rho$), i.e.,

$$
\langle P \rangle = \langle P_{cl} \rangle, \quad \langle P_{nc} \rangle = 0.
$$

(9)

Hence the quantum momentum $P$ in Eq. (3) can also be interpreted as the sum of an average momentum, $P_{cl}$, and a nonclassical momentum fluctuation, $P_{nc}$. Moreover, the magnitude of this fluctuation is simply related to the minimum average error: choosing $\tilde{P} = P_{cl}$ implies from Eqs. (3), (7) and (8) that

$$
E_{P}^{\text{min}} = \langle (P - P_{cl})^2 \rangle = \langle P_{nc}^2 \rangle = \langle P^2 \rangle - \langle P_{cl}^2 \rangle.
$$

(10)

It will be seen that, as a consequence of the exact uncertainty relation Eq. (2), this error does not vanish for any state (although it may be arbitrarily small). Note from Eqs. (4) and (11) that the nonclassical fluctuation strength $\Delta P_{nc}$ in Eq. (2) is a fully operational quantity, as it may be determined from the measured distributions of $P$ and $P_{cl}$.

Several formal properties further support the physical significance of the decomposition in Eq. (3). First, the classical and nonclassical components are linearly uncorrelated, i.e.,

$$
\text{Var} P = \text{Var} P_{cl} + \text{Var} P_{nc},
$$

(11)

as follows immediately from Eqs. (9) and (10). This implies a degree of statistical, and hence physical, independence for $P_{cl}$ and $P_{nc}$. Second, the classical momentum component commutes with the conjugate observable $X$ while the nonclassical component does not, i.e.,

$$
[X, P_{cl}] = 0, \quad [X, P_{nc}] = i\hbar.
$$

(12)

Hence it is the nonclassical component of $P$ which generates the fundamental quantum property $[X, P] = i\hbar$. Finally, when the decomposition is generalised to more than one dimension (see Sec. IV.E), one finds that the commutativity property $[P_j^i, P_k^j] = 0$ for the vector components of momentum is preserved by the decomposition, i.e.,

$$
[P_{cl}^j, P_{cl}^k] = 0 = [P_{nc}^j, P_{nc}^k].
$$

(13)

The decomposition in Eq. (3) attempts to demarcate classical and nonclassical momentum properties. It is therefore reasonable to hope that the nonclassical component $P_{nc}$ in particular might play a fundamental role in describing the essence of what is “quantum” about quantum mechanics. This is indeed the case. A derivation of the Schrödinger equation as
a consequence of adding a nonclassical momentum fluctuation to a classical ensemble (with strength inversely proportional to the uncertainty in position), has recently been given \[3\]. In this paper it will be shown that the nonclassical components of quantum observables, such as position, momentum and angular momentum, satisfy exact uncertainty relations such as Eq. (3). It will further be shown that the decomposition of observables into classical and nonclassical components helps to distinguish between local and nonlocal features of quantum entanglement.

B Angular momentum

Angular momentum takes quantized values in quantum mechanics, but continuous values in classical mechanics. Hence it is not immediately clear whether a decomposition into classical and nonclassical contributions can exist, analogous to Eq. (3). A similar remark may be made for photon number. However, it will be seen that discreteness *per se* imposes no impediment (see also Sec. VI).

For simplicity, consider a rigid rotator confined to the \( xy \)-plane, with angular momentum

\[
J = J_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi},
\]

moment of inertia \( I \), and phase angle \( \phi \). If a phase-dependent potential \( V(\phi) \) acts on the rotator (eg, \( V(\phi) = mg \cos \phi \) for a pendulum), then the corresponding Hamiltonian is

\[
H = J^2/(2I) + V(\phi).
\]

A pure state of the rotator has corresponding angular momentum and phase representations

\[
|\psi\rangle = \sum_j \psi_j |j\rangle = \int_0^{2\pi} d\phi f(\phi) |\phi\rangle,
\]

where \( |j\rangle \) is the eigenstate of angular momentum \( \hbar j \), \( |\phi\rangle \) is the phase eigenset \((2\pi)^{-1/2} \sum_j e^{-ij\phi} |j\rangle\), and the phase wavefunction \( f(\phi) \) is related to the amplitudes \( \psi_j \) by

\[
f(\phi) = \langle \phi | \psi \rangle = (2\pi)^{-1/2} \sum_j \psi_j e^{ij\phi}.
\]

By analogy with Eq. (3), the angular momentum can be decomposed into classical and nonclassical components,

\[
J = J_{cl} + J_{nc},
\]

(14)
with

$$J_{cl} = \int d\phi J_{cl}(\phi) |\phi\rangle \langle \phi|$$  \hspace{1cm} (15)

$$J_{cl}(\phi) = \langle \phi | J_{cl} | \phi \rangle = \frac{\langle \phi | J \rho \phi + \rho J | \phi \rangle}{\langle \phi | \rho | \phi \rangle}$$  \hspace{1cm} (16)

in analogy to Eqs. (8) and (10) respectively. One may show that $J_{cl}(\phi)$ is the best estimate of angular momentum compatible with a measurement of phase for state $\rho$, and that

$$\langle J \rangle = \langle J_{cl} \rangle, \quad \langle J_{nc} \rangle = 0,$$  \hspace{1cm} (17)

$$\text{Var} J = \text{Var} J_{cl} + \text{Var} J_{nc}$$  \hspace{1cm} (18)

in analogy to Eqs. (9) and (11). Note from these properties that one also has

$$\langle J^2 \rangle = \langle J_{cl}^2 \rangle + \langle J_{nc}^2 \rangle,$$

and hence the kinetic energy $\langle J^2 \rangle/(2I)$ splits into a classical contribution and a nonclassical contribution. An exact uncertainty relation for $J_{nc}$ and phase angle will be derived in Sec. IV.

It is of interest to point out an alternative approach to the decomposition in Eq. (14), based on the continuity equation for the phase probability density. In particular, restricting to a pure state $\rho = |\psi\rangle \langle \psi|$ for convenience, multiplying the Schrödinger equation for the phase wavefunction $f(\phi)$ by $f^*(\phi)$ and taking the imaginary part yields the continuity equation

$$\partial |f|^2 / \partial t + (\partial / \partial \phi) [ |f|^2 I^{-1} J_{cl}(\phi) ] = 0,$$  \hspace{1cm} (19)

with $J_{cl}(\phi)$ defined as above. Thus $I^{-1} J_{cl}$ is the angular stream velocity associated with members of a classical ensemble of rotators described by phase density $|f|^2$, and hence $J_{cl}$ is the corresponding angular momentum.

A similar “dynamical” approach, based on the continuity equation

$$\partial |\psi|^2 / \partial t + (\partial / \partial x) [ |\psi|^2 m^{-1} P_{cl}(x) ] = 0$$

for the position probability density, was given in Ref. 2 as the basis for defining the momentum decomposition of Eq. (3). However, such approaches are in general only applicable for systems with Hamiltonians quadratic in the observable of interest.
C Wigner function approach

In this subsection another approach to the decomposition of position and momentum observables is noted, based on an analogy between classical phase space distributions and the Wigner function. In this approach \( P_{cl} \) appears as the natural quantum analogue of a classical average momentum.

The Wigner function \( W(x,p) \) corresponding to density operator \( \rho \) is defined by

\[
W(x,p) := \frac{1}{(2\pi \hbar)^{1/2}} \int d\xi e^{-i\frac{p}{\hbar}\xi} \langle x - \xi/2 | \rho | x + \xi/2 \rangle.,
\]

(20)

and behaves like a joint probability density for position and momentum to the extent that

\[
\langle x | \rho | x \rangle = \int dp W(x,p)
\]

\[
\langle p | \rho | p \rangle = \int dx W(x,p).
\]

However, \( W(x,p) \) can typically take negative values, and is hence fundamentally nonclassical in nature.

Now, if \( \rho(x,p) \) is a classical joint probability density on phase space, then one can define the average momentum associated with position \( x \) by \( p_{av}(x) = \int dp \rho \text{prob}(p|x) \) where \( \text{prob}(p|x) \) denotes the conditional probability that the momentum is equal to \( p \) at position \( x \), i.e., \( \text{prob}(p|x) = \rho(x,p)/\int dp \rho(x,p) \). The classical average momentum at position \( x \) is thus

\[
p_{av}(x) = \int dp \rho \text{prob}(x,p) / \int dp \rho(x,p).
\]

This immediately suggests defining an analogous quantum average momentum associated with position \( x \) by

\[
P_{av}(x) := \frac{\int dp p W(x,p)}{\int dp W(x,p)},
\]

(21)

yielding a natural decomposition of the momentum observable \( P \) into an average component and a fluctuation component:

\[
P = P_{av} + P_{fluc},
\]

(22)

where \( P_{av} = \int dx P_{av}(x) |x\rangle\langle x| \).
Remarkably, this is equivalent to the decomposition in Eq. (3). In particular, one has the identities

\[ P_{av} \equiv P_{cl}, \quad P_{fluc} \equiv P_{nc}. \]  

This follows by first substituting Eq. (20) into Eq. (21) and using integration by parts, to give

\[
\langle x|\rho|x \rangle P_{av}(x) = (2\pi\bar{\hbar})^{-1} \int\int dpd\xi \left[i\hbar \frac{d}{d\xi} e^{-ip\xi/\hbar} \right] \langle x - \xi/2|\rho|x + \xi/2 \rangle
\]

\[
= (\hbar/i) \int d\xi \delta(\xi) (d/d\xi) \langle x - \xi/2|\rho|x + \xi/2 \rangle
\]

\[
= (\hbar/i) \frac{d}{d\xi} \langle x - \xi/2|\rho|x + \xi/2 \rangle \bigg|_{\xi=0}.
\]

Expanding in momentum eigenkets then yields

\[
\langle x|\rho|x \rangle P_{av}(x) = 2^{-1} \int\int dpdp' \langle p + p'|p|\rho|p' \rangle e^{ix(p-p')/\hbar}
\]

\[
= 2^{-1} \int\int dpdp' \langle p|P\rho + \rho P|p' \rangle e^{ix(p-p')/\hbar}
\]

\[
= \langle x|P\rho + \rho P|x \rangle / 2
\]

as required.

The Wigner function thus enables an alternative approach to the decomposition in Eq. (3), which moreover reinforces the interpretation of Eq. (9), that the momentum of a quantum particle comprises a nonclassical fluctuation about a classical average. As an immediate application, note that in obvious analogy to Eqs. (21)-(23) one may define the corresponding decomposition of the position observable \( X \) into classical and nonclassical components via

\[
X = X_{cl} + X_{nc},
\]

\[
X_{cl}(p) = \int dx xW(x,p) / \int dx W(x,p),
\]

where \( X_{cl} = \int dp X_{cl}(p)|p\rangle\langle p| \). This agrees with the analogous definition based on Eq. (9), corresponding to a “best estimate” approach, and also with the definition given in Ref. [2] based on a semiclassical continuity equation.
D  Photon number

Determining a classical component of the photon number \( N \) is reasonably straightforward. However, because the observable conjugate to \( N \) is not represented by a Hermitian operator, the notion of a decomposition \( N = N_{cl} + N_{nc} \) has to be generalised. The reader not interested in the technical details of this generalisation may wish merely to note Eqs. (28), (29) and (31) below, which are analogous to Eqs. (16), (17) and (18) respectively.

The most general description of an observable \( A \), consistent with standard quantum theory, is via a probability operator measure (POM), i.e., via a set of positive operators \( \{A_j\} \) which sum to the identity operator \([8]\). The probability of result \( A = a_j \) for a measurement of \( A \) on state \( \rho \) is given by \( \text{tr}[\rho A_j] \). For the special case of an observable with a Hermitian operator representation, \( A_j \) is just the projection onto the eigenspace corresponding to \( a_j \).

The phase observable \( \Phi \) conjugate to the photon number observable \( N \) is described by the continuous POM \( \{\phi \langle \phi \rangle \} \) with \([8, 9, 10]\)

\[
|\phi\rangle := (2\pi)^{-1/2} \sum_n e^{in\phi} |n\rangle,
\]  

(25)

where \( |n\rangle \) denotes the eigenstate corresponding to \( n \) photons. The probability density for obtaining phase value \( \phi \) for a measurement of \( \Phi \) on state \( \rho \) is therefore

\[
p(\phi|\rho) = \text{tr}[\rho|\phi\rangle\langle\phi|] = \langle\phi|\rho|\phi\rangle.
\]  

(26)

The phase kets \( |\phi\rangle \) may be recognised as eigenkets of the (non-Hermitian) Susskind-Glogower phase operator \([11]\), and are not mutually orthogonal.

As per Sec. II.A, one may consider a classical estimate \( \tilde{N}(\phi) \) of photon number based on measurement result \( \Phi = \phi \) for state \( \rho \). Note that such an estimate corresponds to a POM observable \( \tilde{N} \), with measurement outcome determined by a measurement of \( \Phi \). Thus \( \tilde{N} \) and \( \Phi \) are compatible observables, being jointly measurable.

To determine the best classical estimate of \( N \), one has to choose an appropriate measure of error. Here a difficulty arises: one cannot in general add or subtract POM observables as they do not have algebraic representations as operators. Hence the expression \( \langle (N - \tilde{N})^2 \rangle \) analogous to Eq. (7) is not well defined. However, evaluating Eq. (5) in the position representation yields the equivalent expression

\[
\mathcal{E}_P = \int dx \langle x|\tilde{P}(x)|P(x)|\rho|P - \tilde{P}(x)|x\rangle,
\]  

and hence one may analogously define

\[
\mathcal{E}_N = \int d\phi \langle \phi|(N - \tilde{N}(\phi))|N - \tilde{N}(\phi)|\phi\rangle
\]  

(27)
for the average error of a classical estimate of \( N \). It follows, precisely as per the minimisation of \( \mathcal{E}_P \) in Sec. II.A, that the best classical estimate of photon number is given by

\[
N_{cl}(\phi) := \frac{\langle \phi | N \rho + \rho N | \phi \rangle / 2}{\langle \phi | \rho | \phi \rangle}.
\]

(28)

The classical photon number observable, \( N_{cl} \), thus shares formal similarities with \( P_{cl} \) and \( J_{cl} \) in Eqs. (5) and (16) respectively. Moreover, it is straightforward to show that

\[
\langle N \rangle = \langle N_{cl} \rangle
\]

(29)
in analogy to Eqs. (9) and (17). However, the algebraic difficulty mentioned above again arises in the definition of a corresponding nonclassical photon number observable \( N_{nc} \). In particular, the formal expression

\[
N = N_{cl} + N_{nc}
\]

is not well defined, since \( N_{cl} \) does not have a Hermitian operator representation in general. This difficulty does not in fact pose a problem for obtaining an exact uncertainty relation for phase and photon number (as \( \text{Var} N - \text{Var} N_{cl} \) can be substituted for \( (\Delta N_{nc})^2 \)), but for completeness will be resolved further below.

Note first that, irrespective of the existence of a formal decomposition into classical and nonclassical observables, one can define a decomposition of the average energy \( \langle H \rangle = \hbar \omega \langle N + 1/2 \rangle \) into classical and quantum components by

\[
\langle H \rangle = E_{cl} + E_{nc},
\]

(30)

where \( E_{cl} := \hbar \omega \langle N_{cl} \rangle \). For the particular case of an eigenstate of \( n \) photons it follows via Eq. (28) that

\[
E_{cl} = n \hbar \omega, \quad E_{nc} = \frac{1}{2} \hbar \omega.
\]

Thus the nonclassical energy is precisely the vacuum energy for such states.

Finally, to define a POM observable \( N_{nc} \) which can be regarded as representing the nonclassical component of photon number, it is simplest to exploit formal similarities between photon number and angular momentum. In particular, extend the Hilbert space to include a set of “negative photon number” states \( \{|n\} : n = -1, -2, -3, \ldots, \} \), and define the extended photon operator \( \mathcal{N} \)

\[
\mathcal{N} = \sum_{n=-\infty}^{\infty} n\{|n\}\langle n|.
\]


and the (mutually orthogonal) extended phase states

\[ |\phi^*\rangle = (2\pi)^{-1/2} \sum_{n=-\infty}^{\infty} e^{in\phi} |n\rangle. \]

This is formally analogous to the case of angular momentum considered in Sec. II.B, and in particular one may define the operator decomposition

\[ N^* = N^*_c + N^*_nc \]

analogous to Eq. (14).

Consider now the projection operator

\[ E = \sum_{n=0}^{\infty} |n\rangle\langle n|, \]

which projects onto the original Hilbert space. For any “physical” state, i.e., any state with no negative photon number components, one has \( \rho = E\rho E \). Thus, substituting \( E\rho E \) for \( \rho \) in the expression for \( N^*_c(\phi) \) analogous to Eq. (28), and noting the identities \( E|\phi^*\rangle = |\phi\rangle \), \( EN^* = N = NE \) and their conjugates, one finds that \( N^*_c(\phi) = N_c(\phi) \) for such states. Moreover, any given POM observable \( \{A^*_j\} \) on the extended Hilbert space is mapped by \( E \) to the POM observable \( \{A_j \equiv EA^*_j E\} \) on the original Hilbert space. It is straightforward to check that \( E \) maps \( N^* \) and \( N^*_c \) to \( N \) and \( N_c \) respectively (for states with no negative energy components). Hence one may define the observable \( N_{nc} \) as the POM mapped to by \( E \) from \( N^*_nc \) (i.e., as the POM obtained by applying \( E \) to the projections onto the eigenspaces of \( N^*_nc \)).

Under the above definition of \( N_{nc} \) one has the statistical independence property

\[ \text{Var}N = \text{Var}N_c + \text{Var}N_{nc} \]

in analogy to Eqs. (11) and (18). Indeed, this result is a trivial consequence of the corresponding relation for \( N^*, N^*_c \) and \( N^*_nc \). More generally, for any state \( \rho \) with no negative photon number components one has

\[ \text{tr}[\rho A_j^*] = \text{tr}[E\rho E A_j^*] = \text{tr}[\rho E A_j^* E] = \text{tr}[\rho A_j] \]

for any POM observable \( A^* \). The statistical properties of \( A^* \) and \( A \) are therefore identical for any such state. Thus \( N_c \) and \( N_{nc} \) inherit all statistical properties of \( N^*_c \) and \( N^*_nc \) respectively, including Eq. (31).

### III FISHER LENGTH

#### A Position

The uncertainty measure \( \Delta P_{nc} \) in Eq. (2) is now well defined - it is the rms uncertainty of the nonclassical momentum component \( P_{nc} \). However, it still
remains to define the measure of position uncertainty $\delta X$ in Eq. (2). This is done below for the general case of continuous observables taking values over the entire set of real numbers, such as position and momentum, while the case of periodic observables such as phase is treated in Sec. III.B. Note that $\delta X$ is a purely classical measure of uncertainty, requiring no reference to quantum theory whatsoever.

For a random variable $X$ which takes values over the whole range of real numbers, there are of course many possible ways to quantify the spread of the corresponding distribution $p(x)$. Thus, for example, one may choose the rms uncertainty $\Delta X$, the collision length $1/\int dx \, p(x)^2$, or the ensemble length $\exp[-\int dx \, p(x) \ln p(x)]$. All of these examples have the desirable properties of having the same units as $X$, scaling with $X$, and vanishing in the limit as $p(x)$ approaches a delta function.

A further uncertainty measure satisfying the above properties is

$$\delta X := \left[ \int_{-\infty}^{\infty} dx \, p(x) \left( \frac{d \ln p(x)}{dx} \right)^2 \right]^{-1/2}.$$  (32)

While this measure may appear unfamiliar to physicists, it is in fact closely related to the well known Cramer-Rao inequality that lies at the heart of statistical estimation theory:

$$\Delta X \geq \delta X.$$  (33)

Thus $\delta X$ provides a lower bound for $\Delta X$. Indeed, more generally, $\delta X$ provides the fundamental lower bound for the rms uncertainty of any unbiased estimator for $X$. The bound in Eq. (33) is tight, being saturated if and only if $p(x)$ is a Gaussian distribution.

Eq. (33) is more usually written in the form $\text{Var} X \geq 1/F_X$, where $F_X = (\delta X)^{-2}$ is the “Fisher information” associated with translations of $X$. It is hence appropriate to refer to $\delta X$ as the Fisher length. From Eq. (32) it is seen that the Fisher length may be regarded as a measure of the length scale over which $p(x)$ (or, more precisely, $\ln p(x)$) varies rapidly.

Basic properties of the Fisher length are: (i) $\delta Y = \lambda \delta X$ for $Y = \lambda X$; (ii) $\delta X \to 0$ as $p(x)$ approaches a delta function; (iii) $\delta X \leq \Delta X$ with equality only for Gaussian distributions; and (iv) $\delta X$ is finite for all distributions. This last property follows since the integral in Eq. (32) can vanish only if $p(x)$ is constant everywhere, which is inconsistent with $\int dx \, p(x) = 1$.

The Fisher length has the unusual feature that it depends on the derivative of the distribution. Moreover, for this reason it vanishes for distributions which are discontinuous - to be expected from the above interpretation of
\( \delta X \), since such distributions vary infinitely rapidly over a zero length scale \((\delta X = 0)\) may be shown by replacing such a discontinuity at point \( x_0 \) by a linear interpolation over an interval \([x_0 - \epsilon, x_0 + \epsilon]\) and taking the limit \( \epsilon \to 0 \). The Fisher length also vanishes for a distribution that is zero over some interval (since \( \ln p(x) \) in Eq. (32) changes from \(-\infty\) to a finite value over any neighbourhood containing an endpoint of the interval). While these features imply that \( \delta X \) is not a particularly useful uncertainty measure for such distributions (similarly, \( \Delta X \) is not a particularly useful measure for the Cauchy-Lorentz distribution \((a/\pi)(a^2 + x^2)^{-1}\)), they are precisely the features that lead to a simple proof that the momentum uncertainty is infinite for any quantum system with a position distribution that is discontinuous or vanishes over some interval (as will be shown in Sec. IV).

One further property of Fisher length worthy of note is its alternative interpretation as a “robustness length”. In particular, suppose that a variable described by \( p(x) \) is subjected to a Gaussian diffusion process, i.e., \( \dot{p} = \gamma p'' + \sigma p' \) for diffusion constant \( \gamma \) and drift velocity \( \sigma \). It then follows from Eq. (32) and de Bruijn’s identity [17] that the rate of entropy increase is given by
\[
\dot{S} = \frac{\gamma}{(\delta X)^2}.
\]
(34)
Since a high rate of entropy increase corresponds to a rapid spreading of the distribution, and hence nonrobustness to diffusion, this inverse-square law implies that the Fisher length \( \delta X \) is a direct measure of robustness. Hence \( \delta X \) may also be referred to as a robustness length. This characterisation of robustness is explored for quantum systems in Ref. [2].

Finally, note that Fisher length is not restricted to position observables, but may be calculated as per Eq. (32), for any observable which takes values over the entire set of real numbers, such as momentum.

B Phase

For a periodic random variable the corresponding Fisher length is defined in a slightly modified manner, and satisfies a correspondingly modified Cramer-Rao inequality. In particular, for a phase variable \( \Phi \) with associated period \( 2\pi \) and periodic phase distribution \( p(\phi) \) one defines
\[
\delta \Phi := \left[ \int_0^{2\pi} d\phi p(\phi) \left( \frac{d\ln p(\phi)}{d\phi} \right)^2 \right]^{-1/2}.
\]
(35)
This quantity satisfies many of the same properties as \( \delta X \) above, and again may be interpreted as a robustness length. However, \( \delta \Phi \) is distinguished from \( \delta X \) in two important respects.
First, due to the compact support of \( p(\phi) \), it is possible for \( p(\phi) \) to be a uniform distribution, with \( \delta \Phi = \infty \). Thus \( \delta \Phi \) perhaps somewhat overestimates the spread of a uniform distribution! (just as \( \Delta X \) overestimates the spread of a Cauchy-Lorentz distribution). Note this property implies that a uniform phase distribution is infinitely robust to diffusion - it simply cannot spread any further. This property is also precisely what is needed for the existence of an exact uncertainty relation between phase and photon number, as will be seen in Sec. IV.

Second, and more importantly, \( \delta \Phi \) satisfies a modified form of the Cramer-Rao inequality in Eq. (33). In particular, for a periodic phase distribution \( p(\phi) \), define the “variance” about an arbitrary angle \( \theta \) by

\[
\text{Var}_\theta \Phi := \int_{-\pi}^{\theta+\pi} d\phi (\phi - \theta)^2 p(\phi),
\]

with corresponding rms uncertainty \( \Delta_\theta \Phi \). One may then derive the Cramer-Rao type inequality

\[
\Delta_\theta \Phi \geq |1 - 2\pi p(\theta + \pi)| \delta \Phi.
\]

Note that for a distribution highly peaked about a mean value \( \theta \) one will typically have \( p(\theta + \pi) \ll 1 \), and hence this inequality reduces to \( \Delta_\theta \Phi \geq \delta \Phi \) in analogy to Eq. (33).

To obtain Eq. (37), note that integration by parts and the periodicity of \( p(\phi) \) gives

\[
\int_{-\pi}^{\theta+\pi} d\phi p'(\phi)(\phi - \theta) = 2\pi p(\theta + \pi) - 1.
\]

But from the Schwarz inequality one has

\[
\left[ \int_{-\pi}^{\theta+\pi} d\phi p'(\phi)(\phi - \theta) \right]^2 \leq \int_{-\pi}^{\theta+\pi} d\phi p'(\phi)^2 / p(\phi) \int_{-\pi}^{\theta+\pi} d\phi p(\phi)(\phi - \theta)^2.
\]

Eq. (37) then follows via the definitions in Eqs. (35) and (36). Note that equality holds only in the case that the Schwarz inequality is saturated, i.e., when the two terms in square brackets in the first equality above are proportional. This occurs when \( p(\phi) \) is a (truncated) Gaussian or inverted Gaussian, centred on \( \theta \).
IV  EXACT UNCERTAINTY RELATIONS

A  Position and momentum

In the previous two sections the quantities $\Delta P_{nc}$ and $\delta X$ have been motivated and discussed on completely independent grounds. One is a measure of uncertainty for the nonclassical component of momentum, while the other is a measure of uncertainty for position that appears naturally in the contexts of classical statistical estimation theory and Gaussian diffusion processes.

It is a remarkable fact that for all pure states these two quantities are related by the simple equality in Eq. (2), repeated here for convenience:

$$\delta X \Delta P_{nc} = \hbar / 2.$$  (38)

Thus the Fisher length of position is inversely proportional to the strength of the nonclassical momentum fluctuation. Note from Eqs. (11) and (33) that $\Delta P \geq \Delta P_{nc}$ and $\Delta X \geq \delta X$ respectively. Hence the Heisenberg uncertainty relation Eq. (1) is an immediate consequence of this exact quantum uncertainty relation.

A simple proof of Eq. (38) was given in Ref. [2]; a more general result, valid for density operators, is proved below. Before proceeding to the proof, however, several simple consequences of the exact uncertainty relation in Eq. (38) are noted.

First, recalling that $\delta X$ vanishes for position distributions that are discontinuous or are zero over some interval (see Sec. III.A), it follows immediately from Eq. (38) that $\Delta P_{nc}$ is infinite in such cases. From Eq. (11) $\Delta P$ is then also infinite. Note that this conclusion cannot be derived from the Heisenberg inequality Eq. (11), nor from the entropic uncertainty relation for position and momentum [20]. The exact uncertainty relation Eq. (38) is thus significantly stronger than the latter inequalities.

A second related consequence worth mentioning is a simple proof that any well-localized state, i.e., one for which the position distribution vanishes outside some finite interval, has an infinite energy (at least for any potential energy that is bounded below at infinity). This is immediately implied by the property

$$E = (8m)^{-1} \hbar^2 (\delta X)^{-2} + \langle P^2 \rangle / (2m) + \langle V(x) \rangle.$$  (39)

(following from Eqs. (10) and (38)), noting that $\delta X = 0$ for such states. Note that this “paradox” of standard quantum mechanics (that there are no states which are both well-localised and have finite energy) is a consequence
of the simple external potential model, rather than of some deep incompleteness of the theory. Note also that this property is purely quantum in nature, since the divergent term vanishes in the limit $\hbar \to 0$.

Third, the property $\delta X < \infty$ (see Sec. III.A) immediately implies from the exact uncertainty relation Eq. (38) that $\Delta P_{nc}$ can never vanish, i.e.,

$$\Delta P_{nc} > 0.$$  \hspace{1cm} (40)

Thus all quantum states necessarily have a nonzero degree of nonclassicality associated with them [21]. This may be regarded as further support for the physical significance of the decomposition into classical and nonclassical components.

Eq. (38) for pure states will now be proved as a special case of the more general inequality

$$\delta X \Delta P_{nc} \geq \hbar / 2,$$  \hspace{1cm} (41)

holding for states described by density operators. While not an exact uncertainty relation, this inequality is still much stronger than the corresponding Heisenberg inequality in Eq. (1). Not only is it saturated for all pure states (not just the “minimum uncertainty” states), but it implies that properties such as Eq. (40) hold for any quantum state.

Inequality (41) is an immediate consequence of Eq. (10) and the relations

$$\frac{\hbar^2}{4(\delta X)^2} + \langle P_{cl}^2 \rangle = \int dx \frac{|\langle x | P \rho | x \rangle|^2}{\langle x | \rho | x \rangle} \leq \langle P^2 \rangle,$$  \hspace{1cm} (42)

which hold for all density operators $\rho$. The equality in Eq. (42) is obtained by substituting Eqs. (5) and (6) for the classical momentum component $P_{cl}$, and the representation

$$\langle x | P \rho | x \rangle = \mu \langle \mu | \rho | \mu \rangle \langle \nu | \rho | \nu \rangle \leq \langle x | P \rho P | x \rangle \langle x | \rho | x \rangle.$$  \hspace{1cm} \(\frac{1}{\hbar^2} \int dx \frac{\langle x | P \rho P | x \rangle}{\langle x | \rho | x \rangle},\)

for the Fisher length, following from the definition of $\delta X$ in Eq. (12) and the identity $(d/dx) \langle x | A | x \rangle = (i/\hbar) \langle x | [P, A] | x \rangle$ (derived by expanding in momentum eigenkets). The inequality in Eq. (12) is obtained by defining the states $|\mu\rangle = \rho^{1/2} |P \rangle$, $|\nu\rangle = \rho^{1/2} |x \rangle$, and using the Schwarz inequality

$$|\langle x | P \rho | x \rangle|^2 = |\langle \mu | \nu \rangle|^2 \leq \langle \mu | \mu \rangle \langle \nu | \nu \rangle = \langle x | P \rho P | x \rangle \langle x | \rho | x \rangle.$$  \hspace{1cm} (43)

Remarkably, for the special case of a pure state, direct substitution of $\rho = |\psi\rangle \langle \psi|$ into the integral in Eq. (12) yields equality on the righthand side, and hence the exact uncertainty relation Eq. (38).
Finally, note that a similar derivation may be given for the *conjugate* uncertainty relation
\[
\Delta X_{nc}\delta P \geq \hbar/2,
\] (44)
again saturated by pure states. This relation similarly implies the Heisenberg inequality; requires the variance in position to be infinite for states with momentum distributions that are discontinuous or which vanish over a continuous range of momentum values; and implies that the variance of the nonclassical component of position is strictly positive.

**B Energy bounds**

Eqs. (10) and (41) immediately yield the lower bound
\[
E \geq (8m)^{-1}h^2(\delta X)^{-2} + \langle V \rangle
\] (45)
for the average energy \(E\) of any state. Moreover, from Eqs. (4) and (38), this bound is saturated for all *real wavefunctions*, such as energy eigenstates. It follows that bounds for energy may be obtained via corresponding bounds on the Fisher length \(\delta X\).

For example, consider the case of the one-dimensional Coulomb potential \(V(x) = -Zq^2/|x|\). From Eqs. (1) and (9) of Ref. [22] one has the bound
\[
(\delta X)^{-2} \geq 4\langle |x|^{-1}\rangle^2,
\]
and hence from Eq. (15) the lower bound
\[
E \geq (2m)^{-1}h^2\langle |x|^{-1}\rangle^2 - Zq^2\langle |x|^{-1}\rangle
\]
for energy. Minimising with respect to \(\langle |x|^{-1}\rangle\) then yields the lower bound
\[
E_0 \geq -Z^2q^4m/(2\hbar^2)
\]
for the groundstate energy. The righthand side is, fortuitously, the correct groundstate energy, and this result may be generalized to the three-dimensional case via the formalism in Sec. IV.E below.

A number of upper and lower bounds for the Fisher length are given by Romera and Dehesa [22], and by Dembo et al. [23], which yield corresponding bounds on energy. Eq. (34) of the latter reference provides an interesting connection between groundstate energy estimation and the entropy of the position observable. In particular, the “isoperimetric inequality” [23]
\[
\delta X \leq (2\pi e)^{-1/2}e^S,
\]
where \(S = -\int dx p(x) \ln p(x)\) is the position entropy, implies via Eq. (15) the general entropic lower bound
\[
E \geq (4m)^{-1}\pi e^2h^2e^{-2S} + \langle V \rangle.
\] (46)
Eq. (46) may be exploited to estimate groundstate energies by maximising the position entropy for a given value of $\langle V \rangle$. Note this gives a lower bound on $E_0$, in contrast to the usual upper bounds provided by variational methods. For example, for a harmonic oscillator with $V(x) = m\omega^2 x^2 / 2$, the entropy is well known to be maximised for a given value of $\langle x^2 \rangle$ by a Gaussian distribution. Substituting such a distribution into Eq. (46) and minimising with respect to $\langle x^2 \rangle$ then yields the estimate $E_0 \geq \hbar \omega / 2$, where the right-hand side is in fact the correct groundstate energy (because the groundstate probability distribution is indeed Gaussian).

As a further example of Eq. (46), consider a particle bouncing in a uniform gravitational field, with $V(x) = mgx$ for $x \geq 0$. For a fixed value $\langle x \rangle = \lambda$ one finds that the entropy is maximised by the exponential distribution $p(x) = \lambda^{-1} \exp(-x/\lambda)$ ($x \geq 0$), yielding the lower bound

$$E \geq \pi \hbar^2 (4me^2\lambda^2)^{-1} + mg\lambda.$$ Minimizing with respect to $\lambda$ then gives the estimate

$$E_0 \geq (3/2)[\pi/(2e)]^{1/3} (mg^2\hbar^2)^{1/3} \approx 1.249 (mg^2\hbar^2)^{1/3},$$

which is comparable to the exact value of $(mg^2\hbar^2)^{1/3} a_0 \approx 1.856 (mg^2\hbar^2)^{1/3}$ obtained by solving the Schrödinger equation [24], where $a_0$ denotes the first Airy function zero.

C Phase, angular momentum and photon number

The exact uncertainty relations

$$\delta \Phi \Delta J_{nc} = \hbar / 2,$$

$$\delta \Phi \Delta N_{nc} = 1/2,$$

for phase and angular momentum and for phase and photon number respectively, may be proved exactly as per Eq. (38) above, and are valid for all pure states. For more general states described by density operators the right-hand sides become lower bounds.

It follows, for example, that the variance of angular momentum is infinite for states with phase distributions which are discontinuous or vanish over some interval. Similarly, the photon number variance is infinite for states with a discontinuous phase distribution [25]. Conversely, consider the case of a photon number eigenstate. From Eq. (31) it follows that both the classical and nonclassical fluctuations in photon number vanish, and hence from the
exact uncertainty relation in Eq. (48) that the Fisher length $\delta\Phi$ is infinite, i.e., such states have a uniform phase distribution (see Sec. III.B). Thus the exact uncertainty relation in Eq. (48) is sufficiently strong to exhibit the complementary nature of phase and photon number. Similar remarks may be of course be made for the case of angular momentum.

The exact uncertainty relations may be used to derive the Heisenberg-type inequalities

$$\Delta\phi \Delta J \geq |1 - 2\pi p(\theta + \pi)|\hbar/2,$$

$$\Delta\phi \Delta N \geq |1 - 2\pi p(\theta + \pi)|/2.$$  (49)

These follow directly from Eqs. (47) and (48), using the modified Cramer-Rao inequality Eq. (37) and the additivity of variances in Eqs. (18) and (31). Similar inequalities have been previously given by Pegg and Barnett [26] and by Shapiro [9]. Note that these inequalities are not of sufficient strength to draw the conclusions obtained above from the exact uncertainty relations. Note further that for continuous phase distributions one can always choose the reference angle $\theta$ such that the righthand sides trivially vanish.

D  Time and frequency

In classical signal processing theory, a signal is represented by a normalized time-varying amplitude $a(t)$. Since such signals typically obey linear propagation laws, their analysis usually relies heavily on the frequency representation $A(f)$ of $a(t)$, given by the Fourier transform

$$A(f) = \int dt a(t)e^{2\pi if t}.$$  (51)

This relation is formally similar to the connection between position and momentum amplitudes in quantum mechanics, and in particular one has the well known time-frequency uncertainty relation

$$\Delta f \Delta t \geq (4\pi)^{-1}$$  (52)

in analogy to the Heisenberg inequality Eq. (1).

The “instantaneous frequency” of the signal at time $t$ is defined as

$$f_{\text{inst}}(t) := (2\pi)^{-1}(d/dt)[\arg a(t)],$$  (53)

which from Eq. (3) is seen to be analogous to the classical component of momentum. Thus there is a corresponding decomposition of frequency,

$$f = f_{\text{inst}} + f_{\text{fluc}},$$  (54)
into an instantaneous frequency component and a fluctuating frequency component, analogous to Eq. (3). As per Sec. II.A, the instantaneous frequency may be interpreted as the best possible estimate of the frequency of the signal at a given time.

The purpose of this subsection is to point out the exact uncertainty relation

$$\Delta f_{\text{fluc}} \delta t = \left[ \text{Var} f - \text{Var} f_{\text{inst}} \right]^{1/2} \delta t = (4\pi)^{-1}$$

for frequency and time. This is formally equivalent to the relation for position and momentum in Eq. (38), and may be proved in precisely the same manner.

The exact uncertainty relation implies that the instantaneous frequency $f_{\text{inst}}$ is a good estimate of frequency precisely when the “Fisher time” $\delta t$ is large. Moreover, causal signals, defined to be those for which $a(t)$ vanishes for all times less than some initial time [27], must have $\delta t = 0$ (see Sec. III.A), and hence it follows that $\Delta f = \infty$ for such signals. The same conclusion holds for any signal for which $a(t)$ is discontinuous or vanishes over some interval. Note that these conclusions cannot be derived from the weaker inequality Eq. (52) (which itself follows as a consequence of the exact uncertainty relation and the Cramer-Rao inequality in Eq. (33)).

### E Higher dimensions

Exact uncertainty relations for vector observables are of interest not only because the world is not one-dimensional, but because some physical properties, such as entanglement, require more than one dimension for their discussion. It will therefore be indicated here how Eq. (2) may be generalised to the case of $n$-vectors $\mathbf{X}$ and $\mathbf{P}$. This case has also been briefly considered in Ref. [2]. For simplicity only pure states will be considered.

First, one has the vector decomposition

$$\mathbf{P} = \mathbf{P}_{\text{cl}} + \mathbf{P}_{\text{nc}}$$

into classical and nonclassical components, where $\mathbf{P}_{\text{cl}}$ commutes with $\mathbf{X}$, and

$$\mathbf{P}_{\text{cl}}(\mathbf{x}) = \langle \mathbf{x} | \mathbf{P}_{\text{cl}} | \mathbf{x} \rangle = \frac{\hbar}{2i} \left( \nabla_{\psi} - \nabla_{\psi^*} \right) = \hbar \nabla [\text{arg } \psi]$$

is the best estimate of $\mathbf{P}$ from measurement value $\mathbf{X} = \mathbf{x}$ for state $\psi$ (one may also derive $\mathbf{P}_{\text{cl}}(\mathbf{x})$ from continuity equations or a Wigner function as per Secs. II.B and II.C). Note that since the vector components of $\mathbf{P}$ commute,
as do the vector components of \( P_{cl} \), then

\[
[P^j_{nc}, P^k_{nc}] = [P^j - P^j_{cl}, P^k - P^k_{cl}] = (\hbar^2 / i)(\partial_j \partial_k - \partial_k \partial_j) \arg[\psi] = 0,
\]
as claimed in Eq. (13). In analogy to Eqs. (9) and (11) one may derive \( \langle P \rangle = \langle P_{cl} \rangle \) and the generalized linear independence property

\[
\text{Cov}(P) = \text{Cov}(P_{cl}) + \text{Cov}(P_{nc}),
\]
where the \( n \times n \) covariance matrix of \( n \)-vector \( A \) is defined by the matrix coefficients

\[
[Cov(A)]_{jk} = \langle A_j A_k \rangle - \langle A_j \rangle \langle A_k \rangle.
\]
Second, the notion of Fisher length for one dimension is generalized to

\[
\text{FCov}(X) := \left\{ \int d^n x \rho(x) [\nabla \ln p(x)] [\nabla \ln p(x)]^T \right\}^{-1},
\]
with equality for Gaussian distributions.

One may show by direct calculation of \( \text{Cov}(P_{cl}) \) that the generalized exact uncertainty relation

\[
\text{FCov}(X) \text{Cov}(P_{nc}) = (\hbar / 2)^2 I_n
\]
holds for all pure states, where \( I_n \) denotes the \( n \times n \) unit matrix. The corresponding Heisenberg matrix inequality follows immediately from Eqs. (58), (61) and (62) as

\[
\text{Cov}(X) \text{Cov}(P) \geq (\hbar / 2)^2 I_n.
\]
obtained by multiplying Eq. (62) on the left by the inverse of FCov(X).
This yields a generalization of the one-dimensional exact uncertainty relation Eq. (62) for each individual vector component of X and P. A further choice is to take the square root of the determinant of both sides of Eq. (62), to give the corresponding “volume” equality

\[ \delta V_X \Delta V_{P_{nc}} = (\hbar/2)^n, \] (64)

where the Fisher volume \( \delta V \) and the covariance volume \( \Delta V \) are defined as the square roots of the determinants of the respective covariance matrices. For \( n = 1 \) this relation reduces to Eq. (2).

V ENTANGLEMENT AND CORRELATION

Consider now the case of two one-dimensional particles, with respective position and momentum observables \((X^{(1)}, P^{(1)})\) and \((X^{(2)}, P^{(2)})\). Such a system corresponds to \( n = 2 \) in Sec. IV.E, and the corresponding nonclassical momentum components associated with wavefunction \( \psi \) follow from Eqs. (56) and (57) as

\[ P^{(1)}_{nc} = P^{(1)} - \hbar \frac{\partial \arg \psi(x_1,x_2)}{\partial x_1}, \quad P^{(2)}_{nc} = P^{(2)} - \hbar \frac{\partial \arg \psi(x_1,x_2)}{\partial x_2}. \] (65)

For entangled states (e.g., a superposition of two product states), it follows that the nonclassical momentum of particle 1 will typically depend on the position observable of particle 2, and vice versa. Hence if some unitary transformation (e.g., a position displacement) is performed on the second particle, then the nonclassical momentum of the first particle is typically changed.

The decomposition into classical and nonclassical components is therefore nonlocal: the decomposition of a single-particle observable typically depends upon actions performed on another particle with which the first is entangled. Conversely, all such decompositions are invariant under actions performed on a second unentangled particle. The nonlocality inherent in quantum entanglement is thus reflected to some degree by the nonlocality of classical/nonclassical decompositions.

The exact uncertainty relation corresponding to the decomposition of momentum in Eq. (65) is given by the matrix equality of Eq. (62), with \( n = 2 \). This leads to three independent inequalities, as discussed in Sec. IV.E, two of which may be chosen as generalizations of the exact uncertainty relation in Eq. (2) for each individual particle. The third independent
inequality could, for example, be chosen as the volume inequality in Eq. (64). However, a different choice provides an interesting connection with the Pearson correlation coefficient of classical statistics. In particular, this coefficient is defined for two compatible observables \(A\) and \(B\), in terms of the coefficients \(C_{jk}\) of the corresponding covariance matrix \(\operatorname{Cov}(A,B)\), by

\[
\rho_P(A,B) := C_{12}/(C_{11}C_{22})^{1/2}, \tag{66}
\]

and provides a measure of the degree to which \(A\) and \(B\) are linearly correlated. It ranges between -1 (a high degree of linear correlation with negative slope) and +1 (a high degree of linear correlation with positive slope). One may analogously define the “Fisher” correlation coefficient in terms of the coefficients \(C_{jk}^F\) of the corresponding Fisher covariance matrix \(\operatorname{FCov}(A,B)\), with

\[
\rho_F(A,B) := C_{12}^F/(C_{11}^FC_{22}^F)^{1/2}. \tag{67}
\]

This again provides a measure of correlation ranging between -1 and +1, and is equal to the Pearson correlation coefficient for all Gaussian distributions.

The third equality may now be chosen as the simple correlation relation

\[
\rho_P(P_{(1)nc},P_{(2)nc}) + \rho_F(X^{(1)},X^{(2)}) = 0, \tag{68}
\]

as may be verified by direct calculation from Eq. (62). Thus, for example, if the nonclassical momentum components of particles 1 and 2 are positively correlated then the position observables are negatively correlated, and vice versa. More generally, the degree of nonclassical momentum correlation is seen to be precisely determined by the degree of position correlation. The exact uncertainty relation in Eq. (62) thus constrains both uncertainty and correlation.

A nice example is provided by the approximate EPR state

\[
\psi(x_1,x_2) = Ke^{-(x_1-x_2-a)^2/4\sigma^2}e^{-(x_1+x_2)^2/4\tau^2}e^{ip_0(x_1+x_2)/(2\hbar)},
\]

where \(K\) is a normalisation constant and \(\sigma \ll 1 \ll \tau\) in suitable units. One may then calculate

\[
\langle X^{(1)} - X^{(2)} \rangle = a, \quad \text{Var}(X^{(1)} - X^{(2)}) = \sigma^2 \ll 1,
\]

\[
\langle P^{(1)} + P^{(2)} \rangle = p_0, \quad \text{Var}(P^{(1)} + P^{(2)}) = \hbar^2/\tau^2 \ll 1,
\]

and hence \(\psi\) is an approximate eigenstate of the relative position and the total momentum, i.e., one may write

\[
X^{(1)} - X^{(2)} \approx a, \quad P^{(1)} + P^{(2)} \approx p_0. \tag{69}
\]
This state is thus an approximate version of the (nonnormalizable) ket considered by Einstein, Podolsky and Rosen in connection with the completeness of the quantum theory \[28\].

For state $\psi$ one finds from Eq. (57) that the classical components of momentum are constant, each being equal to $p_0/2$. Hence one has $\text{Cov} P_{nc} = \text{Cov} P$ from Eq. (58). Then, since equality holds in Eq. (61) for Gaussian distributions, the exact uncertainty relation corresponding to $\psi$ follows from Eq. (62) as

$$\text{Cov}(X)\text{Cov}(P) = (\hbar/2)^2 I_n.$$  \(70\)

Eq. (68) reduces to (recalling that $r_P$ and $r_F$ are equivalent for Gaussian distributions) the correlation relation

$$r_P(X) + r_P(P) = 0.$$  

This latter result is consistent with Eq. (69), which implies that $X^{(1)}$ and $X^{(2)}$ are highly positively correlated for state $\psi$ [$r_P(X) \approx 1$], while $P^{(1)}$ and $P^{(2)}$ are highly negatively correlated [$r_P(P) \approx -1$].

Finally, it is of interest to consider the effect of measurements on the approximate EPR state $\psi$. First, for a position measurement on particle 2, with result $X^{(2)} = x$, the state of particle 1 collapses to the wavefunction obtained by substituting $x_2 = x$ and renormalising. It follows that the classical momentum component $P^{(1)}_{cl}$ remains equal to $p_0/2$. Hence the momentum decomposition of particle 1 is not altered by knowledge of $X^{(2)}$.

Conversely, for a momentum measurement on particle 2 with result $P^{(2)} = p$, one finds via straightforward calculation of the appropriate Gaussian integrals that the state of particle 1 collapses to the wavefunction

$$\psi(x_1|P^{(2)} = p) = K' e^{-\frac{(x_1 + a/2)^2}{(\sigma^2 + \tau^2)/4}} e^{i\tilde{p}x_1}/\hbar,$$

where $K'$ is a normalisation constant and

$$\tilde{p} = \frac{\sigma^2 p + \tau^2 (p_0 - p)}{\sigma^2 + \tau^2}.$$

It follows that the classical momentum component $P^{(1)}_{cl}$ is not invariant under a measurement of $P^{(2)}$, changing from $p_0/2$ to $\tilde{p}$. Hence there is a “nonlocal” effect on the classical/nonclassical decomposition of momentum for particle 1, brought about by a measurement of $P^{(2)}$. This effect is a reflection of the strong correlation between $P^{(1)}$ and $P^{(2)}$ for state $\psi$. In particular, note that since $\sigma << 1 << \tau$, one has $\tilde{p} \approx p_0 - p$, as might well be expected from Eq. (69).
VI NON-CONJUGATE AND DISCRETE OBSERVABLES

Exact uncertainty relations can be formally extended in a very general way to arbitrary pairs of Hermitian observables. Unfortunately, the physical significance of such an extension is not entirely clear, as will be seen below. However, for the case of a complete set of mutually complementary observables on a finite Hilbert space it will be shown that results in the literature provide a very satisfactory form of exact uncertainty relation.

First, consider the case of any two observables $A$ and $B$ represented by Hermitian operators, and for state $\rho$ define

$$B^A_{cl} := \sum_a |a\rangle \langle a| \frac{\langle a|B\rho + \rho B|a\rangle}{\langle a|\rho|a\rangle}/2.$$  \hfill (71)

Here $|a\rangle$ denotes the eigenket of $A$ with eigenvalue $a$, and the summation is replaced by integration for continuous ranges of eigenvalues.

Clearly the above expression generalises Eqs. (5) and (6), and indeed $B^A_{cl}$ may be interpreted as providing the best estimate of $B$ compatible with measurement of $A$ on state $\rho$. Note that $A^A_{cl} = A$, i.e., $A$ is its own best estimate. One may further define $B^A_{nc}$ via the decomposition

$$B = B^A_{cl} + B^A_{nc},$$

and obtain the relations

$$\langle B \rangle = \langle B^A_{cl} \rangle, \quad \text{Var}B = \text{Var}B^A_{cl} + \text{Var}B^A_{nc}$$

for state $\rho$, in analogy to Eqs. (3) and (11).

If one is then prepared to define the quantity $\delta^B A$ by

$$(\delta^B A)^{-2} = \sum_a \frac{\langle a|(|i/h)|B|\rho||a\rangle^2}{\langle a|\rho|a\rangle},$$

in analogy to Eq. (33), then precisely as per the derivation of Eq. (41) one may show that

$$(\delta^B A) \Delta B^A_{nc} \geq \hbar/2,$$ \hfill (72)

with equality for all pure states.

Thus there is a very straightforward generalisation of Eq. (2) to arbitrary pairs of observables. A difficulty is, however, to provide a meaningful statistical interpretation of $\delta^B A$. Note in particular that, unlike the Fisher length
δX, this quantity is not a functional of the probability distribution ⟨a|ρ|a⟩ in general. Possibly, noting the commutator which appears in the definition of δBA, one can interpret this quantity as a measure of the degree to which a measurement of A can distinguish between B-generated translations of state ρ, i.e., between unitary transformations of the form $e^{ixB/\hbar}\rho e^{-ixB/\hbar}$ [8]. Here such an attempt will not be made.

Finally, it is pointed out that a rather different type of exact uncertainty relation exists for a set of $n+1$ mutually complementary observables $A_1, A_2, \ldots, A_{n+1}$ on an n-dimensional Hilbert space. Such sets are defined by the property that the distribution of any member is uniform for an eigenstate of any other member, and are known to exist when n is a power of a prime number [29]. As an example one may choose $n = 2$, and take $A_1, A_2$ and $A_3$ to be the Pauli spin matrices.

Let $L$ denote the collision length of probability distribution $\{p_1, p_2, \ldots p_n\}$, defined by

$$L := 1/\sum_j (p_j)^2.$$  

Note that $L$ is equal to 1 for a distribution concentrated on a single outcome, and is equal to $n$ for a distribution spread uniformly over all n possible outcomes. It hence provides a direct measure of the spread of the distribution over the space of outcomes [13].

One may show that

$$\sum_i 1/L_i = 1 + \text{tr}[\rho^2] \leq 2,$$  \hspace{1cm} (73)

where $L_i$ denotes the collision length of observable $A_i$ for state $\rho$. This reduces to a strict equality for all pure states, and thus provides an exact uncertainty relation for the collision lengths of any set of $n+1$ mutually complementary observables. For example, if $L_j = 1$ for some observable $A_j$ (minimal uncertainty), then $L_i = n$ for all $i \neq j$ (maximal uncertainty).

Ivanovic has shown that Eq. (73) can be used to derive an entropic uncertainty relation for the $A_i$ [4], while Brukner and Zeilinger have interpreted Eq. (73) as an additivity property of a particular “information” measure [30].

**VII CONCLUSIONS**

It has been shown that the uncertainty principle has in fact an element of certainty: the lack of knowledge about an observable is, for any wavefunction, precisely determined by the lack of knowledge about the conjugate
observable. The measures of lack of knowledge must of course be chosen appropriately (as the nonclassical fluctuation strength and the Fisher length). What is remarkable is that such measures can be chosen at all.

The exact uncertainty relations in Eqs. (2), (47), (48) and (62) are formal consequences of the Fourier transformations which connect the representations of conjugate quantum observables. Hence they may be extended to any domain in which such transformations have physical significance. This includes, for example, the time-frequency domain considered in Sec. IV.D, as well as Fourier optics and image processing.

It would be of interest to determine whether exact uncertainty relations exist for relativistic systems. One is hampered in direct attempts by difficulties associated with one-particle interpretations of the Klein-Gordon and Dirac equations. It would perhaps therefore be more fruitful to first consider extensions to general field theories.

Finally, note that the definition of the Fisher covariance matrix in Eq. (60) suggests an analogous definition of a “Wigner” covariance matrix WCov, defined via the coefficients of its matrix inverse

\[ [\text{WCov}^{-1}]_{jk} := \int d^2n_z W^{-1} \frac{\partial W}{\partial z_j} \frac{\partial W}{\partial z_k}. \]

Here \( W \) denotes the Wigner function of the state, and \( z \) denotes the phase space vector \((x, p)\). It would be of interest to determine to what degree this matrix is well-defined, and to what extent its properties characterise nonclassical features of quantum states.

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