SUBSOLUTIONS OF TIME-PERIODIC HAMILTON-JACOBI EQUATIONS

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ABSTRACT. We prove the existence of $C^1$ critical subsolutions of the Hamilton-Jacobi equation for a time-periodic Hamiltonian system. We draw a consequence for the Minimal Action functional of the system.

1. Introduction

The purpose of this note is to generalize to time-periodic Hamiltonian systems some results that are known for autonomous (time-free) systems, namely Theorems 1.3 and 1.6 of [FS04], and Theorem 1 of [Mt03].

We call time-periodic Hamiltonian a $C^2$ function $H : T^* M \times \mathbb{T} \to \mathbb{R}$, where $M$ is a closed, connected manifold, and $\mathbb{T}$ is the unit circle, such that the restriction of $H$ to any subset $T^*_x M \times \{t\}$, for $(x,t) \in M \times \mathbb{T}$, is strictly convex and superlinear (see [Mr91] which originated this line of research). We make the additional assumption that the Hamiltonian flow of $H$ is complete. The $\mathbb{T}$ factor is understood as a periodic dependance on time, whence the name. The Hamilton-Jacobi equation (HJc) is

$$\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}, t) = c$$

where the unknown $u$ is a $C^1$ function $M \times \mathbb{T} \to \mathbb{R}$, and $c \in \mathbb{R}$ is a constant. In general there may be no solution at all. One possible way around this fact is to look for solutions in a weak sense, say, viscosity solutions (see [F], [BeR04]). Another is to look for subsolutions, i.e. $C^1$ functions $u$ such that

$$\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}, t) \leq c.$$ 

The two approaches turn out to be connected, as shown by [FS04].

Since $M \times \mathbb{T}$ is compact, any function is a subsolution for a sufficiently large $c$, so the set $I$ of $c \in \mathbb{R}$ such that (HJc) has a subsolution is not empty. A subsolution of (HJc) is a subsolution of (HJc'), for any $c' \geq c$, so $I$ is an interval, unbounded to the right. Due to the convexity and superlinearity of $H$, and to the compactness of $M \times \mathbb{T}$, $H$ is bounded below, so $I$ must be bounded to the left. Its infimum is called the critical value of $H$, and denoted $\alpha(H)$.

It is natural to ask whether $I$ is closed, i.e. whether $\alpha(H) \in I$. A subsolution of (HJo(H)), if it exists, is called critical. When $H$ is autonomous the answer to the latter question is provided by Theorem 1.2 of [FS04]:

**Theorem 1** (Fathi-Siconolfi). There exists a $C^1$ critical subsolution.

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We extend this theorem to the time-periodic case in Section 3. The idea of (the first step of) the proof is borrowed from [BBa], and uses the estimates of Section 2.2, which were proved in [Mt03] for the autonomous case.

The Hamiltonian $H$ being convex and superlinear, we may take advantage of the Lagrangian formulation of Classical Mechanics. Define

$$L : TM \times \mathbb{T} \to \mathbb{R}$$

$$(x, v, t) \mapsto \sup_{p \in T^*_x M} \left\{ \langle p, v \rangle - H(x, p, t) \right\}$$

then $L$ is $C^2$, fiberwise strictly convex and superlinear. It defines, via the Euler-Lagrange equation, a flow $\Phi_t$ on $TM \times \mathbb{T}$ which is complete since it is the conjugate, under Legendre Transform, of the Hamiltonian flow of $H$.

Define $\mathcal{M}_{\text{inv}}$ to be the set of $\Phi_t$-invariant, compactly supported, Borel probability measures on $TM \times \mathbb{T}$. Mather showed that the function (called action of the Lagrangian on measures)

$$\mathcal{M}_{\text{inv}} \to \mathbb{R}$$

$$\mu \mapsto \int_{TM \times \mathbb{T}} L d\mu$$

is well defined and has a minimum. It turns out that this minimum is $-\alpha(H)$. For this reason $\alpha(H)$ is also denoted $\alpha(L)$. A measure achieving the minimum is called $L$-minimizing.

One drawback of this characterization of the critical value is that when you want to test the minimality of a measure, you first need to check invariance. With this in mind, an important corollary of Theorem 1.3 of [FS04] is Theorem 1.6 of [FS04], which is itself an elaboration on a theorem proved by Mañé in [Mn96], and was proved by Bangert ([Ba99]) in the special case when the Lagrangian is a Riemannian metric.

**Definition 2.** A probability measure $\mu$ on $TM$ is called closed if

$$\int_{TM} \|v\| d\mu(x, v) < +\infty,$$

and for every smooth function $f$ on $M$, we have

$$\int_{TM \times \mathbb{T}} df(x).v d\mu(x, v) = 0.$$

Mather proved in [Mr91] that every invariant measure is closed.

**Theorem 3** (Fathi-Siconolfi). We have

$$-\alpha(0) = \min \left\{ \int_{TM} L d\mu : \mu \text{ is closed} \right\}.$$

Moreover, every closed measure that achieves the minimum above is invariant under the Euler-Lagrange flow of $L$, and is thus a minimizing measure.

The strength of this theorem is that it allows to work with measures without having to verify a priori that they are invariant. We give an appropriate definition of a closed measure for a time-periodic Hamiltonian system in Section 3.1 and indicate how the proof of Theorem 3 carries over to that case.

The critical value is thus a useful tool for selecting interesting invariant subsets; for instance the supports of minimizing measures (Mather set), or the Aubry set (see below). The following classical trick gives us more milk.
from the same cow. If \( \omega \) is a closed one-form on \( M \), then \( L - \omega \) is again a convex and superlinear Lagrangian, and it has the same Euler-lagrange flow as \( L \). Besides, by Mather’s Lemma (invariant measures are closed) if \( \mu \in \mathcal{M}_{inv} \), the integral \( \int_{TM \times T} \omega d\mu \) only depends on the cohomology class of \( \omega \). Then the minimum over \( \mathcal{M}_{inv} \) of \( \int (L - \omega) d\mu \) is actually a function of the cohomology class of \( \omega \), the opposite of which is called the \( \alpha \)-function of the system. An \( (L - \omega) \)-minimizing measure is also called \( (L, \omega) \)-minimizing or \( (L, c) \)-minimizing if \( c \) is the cohomology of \( \omega \). To sum up

\[
\alpha_L: H^1(M, \mathbb{R}) \to \mathbb{R}
\]

\[
c \mapsto - \min \left\{ \int_{TM \times T} (L - \omega) d\mu : \mu \in \mathcal{M}_{inv} \ [\omega] = c \right\}.
\]

In particular \( \alpha(L) = \alpha_L(0) \). We shall omit the subscript \( L \) when no ambiguity is possible. Mather proved that \( \alpha \) is convex and superlinear. The analogy with the Lagrangian goes no further; in general \( \alpha \) is neither strictly convex, nor \( C^1 \) (see [Mt97]). The regions where \( \alpha \) is not strictly convex (being convex, it must then be affine) are called faces of \( \alpha \). By Proposition 6 of [Mt03] (see [Be02] for the time-periodic case) changing the cohomology class within a given face does not select any new dynamics. The presence of faces is often correlated with some rationality properties of homology classes (see [Mt03], Corollary 3). Understanding this phenomenon is the motivation for Theorem 1 of [Mt03], which we extend to the time-periodic case in the last section. The proof uses both the estimates of Section 2, and the existence of a \( C^1 \) subsolution, instead of Whitney’s Extension Theorem as in [Mt03].

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2. Preliminaries

2.1. Some Weak KAM theory. In this section we briefly recall a few definition, referring the reader to the bibliography ([P], and [CIS] for the time-periodic case) for more information. Define, for all \( n \in \mathbb{N} \),

\[
h_n: (M \times T) \times (M \times T) \to \mathbb{R}
\]

\[
((x, t), (y, s)) \mapsto \min \int_t^{s+n} L(\gamma, \dot{\gamma}, t) dt + n\alpha(0)
\]

where the minimum is taken over all absolutely continuous curves \( \gamma: [t, s + n] \to M \) such that \( \gamma(t) = x \) and \( \gamma(s + n) = y \). Note that we abuse notation, denoting by the same \( t \) an element of \( T = \mathbb{R}/\mathbb{Z} \) or the corresponding point in \( [0, 1[ \). The Peierls barrier is then defined as

\[
h: (M \times T) \times (M \times T) \to \mathbb{R}
\]

\[
((x, t), (y, s)) \mapsto \lim \inf_{n \to \infty} h_n ((x, t), (y, s)).
\]

The Aubry set is

\[
\mathcal{A}_0 := \{(x, t) \in M \times T : h((x, t), (x, t)) = 0\}.
\]

We say a function \( f: M \times T \to \mathbb{R} \) is \( (L, \alpha(0)) \)-dominated if for every absolutely continuous curve \( \gamma: [a, b] \to M \) with \( b \geq a \) we have

\[
\int_a^b (L(\gamma, \dot{\gamma}, t) + \alpha(0)) dt \geq f(\gamma(b), b) - f(\gamma(a), a).
\]
Such functions exist and are Lipschitz ([F], Lemma 4.2.2), hence almost everywhere differentiable by Rademacher’s theorem; wherever the derivative exists, they are subsolutions of the Hamilton-Jacobi equation (see [FS04]), that is
\[ \frac{\partial f}{\partial t} + H(x, \frac{\partial f}{\partial x}, t) \leq \alpha(0). \]

A forward (resp. backward) weak KAM solution is a function \( u \) which is \((L, \alpha(0))\)-dominated and, for every \((x, t) \in M \times \mathbb{T}\), there exists an absolutely continuous curve \( \gamma: [t, +\infty] \to M \) (resp. \( \gamma: [-\infty, t] \to M \)) such that \( \gamma(t) = x \) and, for every \( s \in [t, +\infty] \) (resp. \( s \in [-\infty, t] \)), we have
\[ \int_t^s (L(\gamma(t), \dot{\gamma}(t), t) + \alpha(0)) \, dt = u(\gamma(s), \dot{\gamma}(s), s) - u(\gamma(t), \dot{\gamma}(t), t). \]

For every forward weak KAM solution \( u_+ \) there exists a unique backward weak KAM solution \( u_- \) such that \( u_+ \leq u_- \), \( u_+ = u_- \) in \( A_0 \) ([F], Theorem 5.12). The pair \((u_+, u_-)\) is then called a weak KAM conjugate pair. It is a remarkable fact that for all \((x, t), (y, s) \in M \times \mathbb{T}\)
\[ h((x, t), (y, s)) = \sup \{ u_-(y, s) - u_+(x, t) \} \]
where the supremum is taken over all weak KAM conjugate pairs \((u_+, u_-)\) ([F], Corollary 5.37).

### 2.2. An estimate.
To clear up the notation, we assume \( \alpha(0) = 0 \) by replacing \( L \) with \( L - \alpha(0) \). Take \( \epsilon > 0 \). Let \( N(\epsilon) \in \mathbb{N}^* \) be the smallest integer such that
\[ \forall n \geq N(\epsilon), \forall (x, \tau), (y, \sigma) \in M \times \mathbb{T}, h_n((x, \tau), (y, \sigma)) \geq h((x, \tau), (y, \sigma)) - \epsilon. \]
Let \((u_-, u_+)\) be a weak KAM conjugate pair such that \( (u_- - u_+)^{-1}(0) = A_0 \). Define \( A_\epsilon := (u_- - u_+)^{-1}([2\epsilon, +\infty]) \). Let \( a, b \) be elements of \( \mathbb{R} \cup \pm \infty \) and let \( \gamma: [a, b] \to M \) be an absolutely continuous curve. Denote by \( \text{Leb} \) the normalized Lebesgue measure on \( \mathbb{R} \), by \( \text{Int} \) the integer part, and set \( \mu_\gamma([a, b]) = \text{Leb}(\gamma^{-1}(A_\epsilon)) \). Then

**Lemma 4.** We have:
\[ \int_a^b L(\gamma(t), \dot{\gamma}(t), t) \, dt \geq u_+(\gamma(b), b) - u_+(\gamma(a), a) + \epsilon \text{Int} \left( \frac{\mu_\gamma([a, b])}{N(\epsilon)} \right). \]

**Proof.** Define inductively a sequence in \( \mathbb{R} \cup \pm \infty \) by \( t_0 := a \) and
\[ t_{i+1} := \max \{ t_i \leq t \leq b: t - t_i \geq N(\epsilon), \mu_\gamma([a, b]) \leq N(\epsilon) \} \]
Set \( n_i := \text{Int}(t_{i+1} - t_i) \); we have \( n_i \geq N(\epsilon) \). Note that \( \forall i \geq 1, (\gamma(t_i), t_i) \in A_\epsilon \); this is the reason why we need a max in the above formula. Also, denoting \( n = \max \{ i: t_i \leq b \} \), we have
\[ n = \text{Int} \left( \frac{\mu_\gamma([a, b])}{N(\epsilon)} \right) \]
since \( \mu_\gamma([t_i, t_{i+1}]) = N(\epsilon) \).
Now, we have
\[
\int_a^b L(\gamma(t), \dot{\gamma}(t), t) dt = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} L(\gamma(t), \dot{\gamma}(t), t) dt + \int_{t_n}^b L(\gamma(t), \dot{\gamma}(t), t) dt \\
\geq \sum_{i=0}^{n-1} h_{n_i}((\gamma(t_{i+1}), t_{i+1}), (\gamma(t_i), t_i)) \\
+ u_+(\gamma(b), b) - u_+(\gamma(t_n), t_n).
\]
Since \( n_i \geq N(\epsilon) \), we have
\[
h_{n_i}((\gamma(t_{i+1}), t_{i+1}), (\gamma(t_i), t_i)) \geq h((\gamma(t_{i+1}), t_{i+1}), (\gamma(t_i), t_i)) - \epsilon \\
\geq u_-(\gamma(t_{i+1}), t_{i+1}) - u_+(\gamma(t_i), t_i) - \epsilon
\]
whence
\[
\int_a^b L(\gamma(t), \dot{\gamma}(t), t) dt \geq \sum_{i=0}^{n-1} [u_-(\gamma(t_{i+1}), t_{i+1}) - u_+(\gamma(t_i), t_i) - \epsilon] \\
+ u_+(\gamma(b), b) - u_+(\gamma(t_n), t_n) \\
= \sum_{i=1}^{n-1} [u_-(\gamma(t_{i+1}), t_{i+1}) - u_+(\gamma(t_i), t_i) - \epsilon] \\
+ u_+(\gamma(b), b) - u_+(\gamma(a), a)
\]
and, because \((\gamma(t_{i+1}), t_{i+1}) \in A_\epsilon\),
\[
\int_a^b L(\gamma(t), \dot{\gamma}(t), t) dt \geq u_+(\gamma(b), b) - u_+(\gamma(a), a) + n\epsilon
\]
which proves the Lemma. \(\square\)

2.3. Consequence of the estimate.

Lemma 5. There exists a \(C^2\) non-negative function \(W : M \times \mathbb{T} \rightarrow \mathbb{R}\) which is positive outside \(A_0\) and zero inside \(A_0\), such that \(\alpha(L - W) = \alpha(L)\) and \(A_0(L - W) = A_0(L)\).

Proof. First we point out that, denoting \(\chi_\epsilon\) the characteristic function of \(A_\epsilon\), Lemma 4 may be rewritten

(1)
\[
\int_a^b L(\gamma(t), \dot{\gamma}(t), t) dt \geq u_+(\gamma(b), b) - u_+(\gamma(a), a) + \frac{\epsilon}{N(\epsilon)} \int_a^b \chi_\epsilon(\gamma(t), t) dt - \epsilon
\]
since for each \(i\) we have
\[
\int_{t_i}^{t_{i+1}} \chi_\epsilon(\gamma(t), t) dt = N(\epsilon)
\]
and
\[
\int_{t_n}^b \chi_\epsilon(\gamma(t), t) dt \leq N(\epsilon).
\]
The map
\[
\chi := \sup_{n \in \mathbb{N}} \frac{2^{-n}}{N(2^{-n})} \chi_{2^{-n}}
\]
is integrable by Lebesgue’s Monotone Convergence Theorem. So, taking the supremum over \( n \in \mathbb{N} \) in Equation (1) we get

\[
\int_{a}^{b} L(\gamma(t), \dot{\gamma}(t), t) dt \geq u_+(\gamma(b), b) - u_+(\gamma(a), a) + \int_{a}^{b} \chi(\gamma(t), t) dt - 1.
\]

Now pick a \( C^2 \) function \( W : M \times \mathbb{T} \rightarrow \mathbb{R} \) which is positive outside of \( A_0 \), and such that

\[
\forall (x, t) \in M \times \mathbb{T}, \ 0 \leq W(x, t) \leq \chi(x, t).
\]

First let us verify that such a function exists. For every \( n \) in \( \mathbb{N} \) we can find a \( C^2 \) map \( W_n : M \times \mathbb{T} \rightarrow \mathbb{R} \) with \( C^2 \)-norm \( \leq 1 \) and such that

\[
\forall (x, t) \in M \times \mathbb{T}, \ 0 \leq W_n(x, t) \leq \frac{2^{-n-1}}{N(2^n)} \chi_{2^n}(x, t).
\]

Now consider \( W := \sum_{n \geq 0} W_n \), then \( W \) is \( C^2 \), non-negative, and

\[
\forall (x, t) \in A_{n+1} \cap (M \times \mathbb{T} \setminus A_n), \ W(x, t) \leq \sum_{k \geq n+1} \frac{2^{-k-1}}{N(2^k)} \leq \frac{2^n}{N(2^n)} \leq \chi(x, t)
\]

the latter inequality being true because \((x, t) \notin A_n\).

It remains to be seen that \( \alpha(L - W) = \alpha(L) \).

First, note that since \( W \) is non-negative, for any real number \( c \), a subsolution of \( (HJc) \) for \( H + W \) is also a subsolution of \( (HJc) \) for \( H \) so \( 0 = \alpha(H) \leq \alpha(H + W) \).

Conversely, let \( \mu \) be an ergodic \((L - W)\)-minimizing measure and let \( \gamma : \mathbb{R} \rightarrow M \) be a curve such that \((\gamma, \dot{\gamma}, t)\) is a \( \mu \)-generic orbit. We have, for all \( s, t \) in \( \mathbb{R} \):

\[
\int_{a}^{b} \{ L(\gamma(t), \dot{\gamma}(t), t) - W(\gamma(t), t) \} dt \geq \int_{a}^{b} \chi(\gamma(t), t) dt - 1
\]

thus by Birkhoff’s Ergodic Theorem

\[
\int (L - W) d\mu \geq 0.
\]

This proves that \( 0 = \alpha(L) \geq \alpha(L - W) \) so \( \alpha(L) = \alpha(L - W) \).

Let us pause for a moment to prove

**Proposition 6.** There exists a critical subsolution which is strict at every point of \( M \times \mathbb{T}^1 \setminus A_0 \).

**Remark 7.** The autonomous case of this Proposition ([FS04], Proposition 6.1) is the first step of the proof of Theorem 1.3 of [FS04]. The idea of the proof that follows is borrowed from [BBa].
Proof. Take a weak KAM solution $u$ for $L - W$, where $W$ is given by Lemma 5. Recall that the Hamiltonian corresponding to $L - W$ under Legendre transform is $H + W$. At every point of differentiability of $u$ we have
\[
\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}, t) + W(x, t) \leq \alpha_{L - W}(0) = \alpha_L(0)
\]
that is,
\[
\frac{\partial u}{\partial t} + H(x, \frac{\partial u}{\partial x}, t) \leq \alpha_L(0) - W(x, t)
\]
so $u$ is a subsolution for $L$, strict outside of $A_0$. □

Observe that, since we know from [CIS] that any critical subsolution is actually a solution of $(HJ\alpha(H))$ in $A_0$, the latter Proposition implies the following characterization of the Aubry set:

**Proposition 8.** A point $(x, t) \in M \times \mathbb{T}$ is in $A_0$ if and only if no critical subsolution of $(HJ)$ is strict at $(x, t)$.

Now let us come back to the proof of Lemma 5. We still have to find $W$ such that $A_0(L - W) = A_0(L)$. First note that since $W$ is non-negative, and $0 = \alpha(H) \leq \alpha(H + W)$, any critical subsolution of $(HJ)$ for $H + W$ is also a critical subsolution of $(HJ)$ for $H$. Besides, $W$ being positive outside $A_0$, such a subsolution is strict (for $H$) outside $A_0$. By Proposition 5 this implies $A_0(L - W) \supset A_0(L)$.

For the converse inclusion we may need to modify $W$. Assume there exists a $W_1$ such that $0 \leq W \leq W_1$, all inequalities being strict outside $A_0$, and $\alpha(H + W_1) = \alpha(H + W)$. This can be achieved by replacing $W$ with $W/2$ and taking $W$ as $W_1$. Then a critical subsolution for $H + W_1$ is also a critical subsolution for $H + W$, and it is strict for $H + W$ outside $A_0$, which proves $A_0(L - W) \subset A_0(L)$.

□

3. Subsolutions

Now we extend to the time-periodic case Theorem 1.3 of [FS04]:

**Theorem 9.** There exists a $C^1$ critical subsolution which is strict at every point of $M \times \mathbb{T} \setminus A_0$.

At this point we assume the reader has Theorem 9.2 of [FS04] before his eyes and explain how it applies. Take

- $N := M \times \mathbb{T}$
- $f := u$ given by Proposition 5
- $A := A_0(L) = A_0(L - W)$
- $B := \text{the domain of } du$; $B$ has full measure and $du$ is defined in $B$ and continuous in $A$
- since we do not require the $C^1$ subsolution to approximate the strict subsolution, we do not need to specify $\epsilon$

\[
F := \{(x, p, t, \tau) \in T(M \times \mathbb{T}) \setminus A_0: \tau + H(x, p, t) \leq \alpha_L(0) - W(x, t)\}
\]
\[ O := \left\{ (x, p, t, \tau) \in T(M \times T^1) \setminus A_0 : \tau + H(x, p, t) < \alpha_L(0) - \frac{1}{2} W(x, t) \right\}. \]

Then Theorem 9.2 of [FS04] yields a function \( g \) that is the required \( C^1 \) critical subsolution, strict at every point of \( M \times T^1 \setminus A_0 \). \( \square \)

### 3.1. Closed measures

If we are going to extend Theorem 9 to time-periodic systems we have to integrate functions on \( T(M \times T) \) with respect to measures that are only defined on \( TM \times T \). The crucial point in the proof of Mather’s lemma is that invariant measures are supported on curves in \( TM \) of type \((\gamma(t), \dot{\gamma}(t))\). In the time-dependant setting we are considering curves in \( M \times T \) of type \((\gamma(t), t)\) so their velocities are \((\gamma(t), t, \dot{\gamma}(t), 1)\). So the measures on \( T(M \times T) \) that we shall use are concentrated on the hypersurface \( \{(x, t, v, 1) : (x, v, t) \in TM \times T\} \) in \( T(M \times T) \). This leads to the following

**Definition 10.** A probability measure \( \mu \) on \( TM \times T \) is called closed if

\[
\int_{TM \times T} \|v\| d\mu(x, v, t) < +\infty,
\]

and for every smooth function \( f \) on \( M \times T \), we have

\[
\int_{TM \times T} df(x, t).v.(1) d\mu(x, v, t) = 0.
\]

Then Mather’s lemma and its proof carry over without modification.

Let us sketch briefly how the proof of Theorem 1.6 of [FS04] applies to the time-periodic case. The first part of the proof consists of showing that a closed measure that realizes the minimum is supported inside \( A_0 \). To make it work in the time-periodic case it suffices to replace every occurrence of \( H(x, d_x u) \) by \( \partial_t u + H(x, \partial_x u, t) \). Then apply Proposition 10.3 of [FS04] with \( N = M \times T \) instead of \( M \), and you’re done.

### 4. Minimal Action

#### 4.1. Preliminaries

Since the \( \alpha \)-function of \( L \) is convex, at every point its graph has a supporting hyperplane. We call face of \( \alpha \) the intersection of the graph of \( \alpha \) with one of its supporting hyperplane. By Fenchel (a.k.a. convex) duality it is equivalent to study the differentiability of \( \beta \) or to study the faces of \( \alpha \). If \( c \) is a cohomology class, we call \( F_c \) the largest face of \( \alpha \) containing \( c \) in its relative interior, and \( \text{Vect} F_c \) the underlying vector space of the affine space it generates in \( H^1(M, \mathbb{R}) \). We call \( \bar{V}_c \) the underlying vector space of the affine space generated by pairs \((c', \alpha(c') - \alpha(c))\) where \( c' \in F_c \). Replacing, if necessary, \( L \) by \( L - \omega \) where \( [\omega] = c \), we only need consider the case when \( c = 0 \). Likewise, replacing \( L \) with \( L - \alpha(0) \) we may assume \( \alpha(0) = 0 \).

**Definition 11.** Let \( \tilde{E}_0 \) be the set of \((c, \tau) \in H^1(M \times T, \mathbb{R}) = H^1(M, \mathbb{R}) \times H^1(T, \mathbb{R}) \) such that there exists a smooth closed one-form \( \omega \) on \( M \times T \) with \([\omega] = (c, \tau)\) and \( \text{supp}(\omega) \cap A_0 = \emptyset \). Let \( E_0 \) be the canonical projection of \( \tilde{E}_0 \) to \( H^1(M, \mathbb{R}) \).
Definition 12. Let \( \tilde{G}_0 \) be the set of \((c, \tau) \in H^1(\mathbb{M} \times \mathbb{T}, \mathbb{R}) = H^1(\mathbb{M}, \mathbb{R}) \times H^1(\mathbb{T}, \mathbb{R}) \) such that there exists a continuous closed one-form \( \omega \) on \( \mathbb{M} \times \mathbb{T} \) with \([\omega] = (c, \tau)\) and
\[
\omega(x, t, v, \tau) = 0 \forall (x, t) \in \mathcal{A}_0 \subset \mathbb{M} \times \mathbb{T}, \forall (v, \tau) \in T(x, t)\mathbb{M} \times \mathbb{T}.
\]

Let \( G_0 \) be the canonical projection of \( \tilde{G}_0 \) to \( H^1(\mathbb{M}, \mathbb{R}) \).

Now we can state the main result of this section

Theorem 13. The following inclusions hold true:
\[
E_0 \subset \text{Vect}F_0 \subset G_0.
\]

In view of the above definitions we shall need to integrate one forms on \( \mathbb{M} \times \mathbb{T} \) with respect to invariant measures. We denote by \( \int \omega d\mu \) the expression
\[
\int_{\mathbb{M} \times \mathbb{T}} \omega(x, t) \cdot (v, 1) d\mu(x, v, t).
\]

The following lemma is useful.

Lemma 14. If \( \omega \) is a closed one form on \( \mathbb{M} \times \mathbb{T} \), with \([\omega] = (c, \tau) \in H^1(\mathbb{M}, \mathbb{R}) \times H^1(\mathbb{T}, \mathbb{R}) \), and \( \mu \) is an \((L, c)\)-minimizing measure, then
\[
\int (L - \omega) d\mu = -\alpha(c) - \tau.
\]

Proof. Consider a closed one-form \( \omega_1 \) on \( \mathbb{M} \) such that \([\omega_1] = c\). Denote \( \tilde{\tau} \) the constant one-form \( \tau dt \) on \( \mathbb{T} \). Then \( \omega_1 \oplus \tilde{\tau} \) is a one-form on \( \mathbb{M} \times \mathbb{T} \), cohomologous to \( \omega \). Let \( f \) be a smooth function on \( \mathbb{M} \times \mathbb{T} \) such that \((\omega_1, \tilde{\tau}) = \omega + df\). Then by Mather’s lemma (invariant measures are closed)
\[
\int (L - \omega) d\mu = \int (L - (\omega_1 \oplus \tilde{\tau})) d\mu.
\]

On one hand \( \int (L - \omega_1) d\mu = -\alpha(c) \) since \( \mu \) is \((L, c)\)-minimizing. On the other hand, since \( \mu \) is a probability measure, we have \( \int \tilde{\tau} d\mu = \int \tau d\mu = \tau \). The lemma is proved.

4.2. Proof of \( E_0 \subset \text{Vect}F_0 \). Pick \( c \in E_0 \). Let \( \tau \in H^1(\mathbb{T}, \mathbb{R}) \) and \( \omega \) a closed one-form on \( \mathbb{M} \times \mathbb{T}^1 \) be such that \( \text{supp}(\omega) \cap \mathcal{A}_0 = \emptyset \) and \([\omega] = (c, \tau)\). Since \( \text{supp}(\omega) \) is compact there exists \( \epsilon > 0 \) such that
\[
\mu(x, t) - u_+(x, t) \geq 2\epsilon \forall (x, t) \in \text{supp}(\omega).
\]

By \textit{a priori} compactness there exists a compact subset \( K \) in \( TM \times \mathbb{T}^1 \) such that for all \( \theta \in [-1, 1] \), for all \( L + \theta \omega \)-minimizing measure \( \mu \), the support of \( \mu \) is contained in \( K \). Let \( \delta \) be such that
\[
\forall (x, v, t) \in K, |\delta \omega(x, t)(v, 1)| \leq \frac{\epsilon}{N(\epsilon)}.
\]

Let \( \mu \) be an ergodic \((L + \delta \omega)\)-minimizing measure and let \( \gamma : \mathbb{R} \rightarrow \mathbb{M} \) be a \( \mu \)-generic orbit. We have, for all \( s \leq t \):
\[
\left| \int_t^s \delta \omega(\gamma, \dot{\gamma}, t) dt \right| \leq (t - s) \frac{\epsilon}{N(\epsilon)} \leq \epsilon \text{Int} \left( \frac{t - s}{N(\epsilon)} \right) + \epsilon
\]
whence
\[ \int_t^s (L + \delta \omega)(\gamma, \dot{\gamma}, t)dt \geq u_+(\gamma(t), t) - u_+(\gamma(s), s) - \epsilon \]

thus by Birkhoff’s Ergodic Theorem \( \int (L + \delta \omega)d\mu \geq 0 \). Now by Lemma 4.
\[ \int (L + \delta \omega)d\mu = -\alpha(\delta c) - \delta \tau \text{ so } \alpha(\delta c) \leq -\delta \tau. \]

Likewise, \( \alpha(-\delta c) \leq \delta \tau \) thus \( \alpha(\delta c) + \alpha(-\delta c) \leq 0 \). On the other hand by convexity of \( \alpha \) the reverse inequality is true : \( \alpha(\delta c) + \alpha(-\delta c) \geq 0 \) so the inequalities \( \alpha(\delta c) \leq -\delta \tau \) and \( \alpha(-\delta c) \leq \delta \tau \) are actually equalities. This means that \( \alpha \) restricted to the line segment \([-\delta c, \delta c]\), is affine with slope \(-\tau\), which proves that \(-\tau = \alpha(c)\) and \(\delta c \in F_0\) whence \(c \in \text{Vect}F_0\). \(\square\)

4.3. **Proof ofVect** \(F_0 \subset G_0\). Pick \(c\) in the interior of \(F_0\). Note that by Proposition 6 of \(\text{Mf03}\) we have \(A_c = A_0\). Take \(\omega\) a smooth closed one-form on \(M\) such that \(\omega|c = c\). Let \(u_0\) (resp. \(u_1\)) be a \(C^1\) subsolution for \(L\) (resp. \(L - \omega\)). Then for all \((x, v, t) \in \tilde{A}_0\), we have
\[
\begin{align*}
\frac{\partial u_0}{\partial x}(x, t) &= \frac{\partial L}{\partial v}(x, v, t) \\
\frac{\partial u_1}{\partial x}(x, t) &= \frac{\partial L}{\partial v}(x, v, t) - \omega_x(v)
\end{align*}
\]

Observe that the Hamiltonian paired by Legendre transform with \(L - \omega\) is \((x, p, t) \mapsto H(x, p + \omega_x, t) := H_\omega(x, p, t)\). Thus
\[ \forall (x, t) \in A_0 \ H_\omega(x, \frac{\partial u_1}{\partial x}(x, t), t) = H(x, \frac{\partial u_0}{\partial x}(x, t), t). \]

On the other hand in \(A_0\) \(u_0\) and \(u_1\) are solutions of the Hamilton-Jacobi equation:
\[
\begin{align*}
\frac{\partial u_0}{\partial t}(x, t) + H(x, \frac{\partial u_0}{\partial x}(x, t), t) &= \alpha(0) \\
\frac{\partial u_1}{\partial t}(x, t) + H_\omega(x, \frac{\partial u_1}{\partial x}(x, t), t) &\quad = \alpha(c)
\end{align*}
\]

whence
\[ \frac{\partial(u_1 - u_0)}{\partial t}(x, t) = \alpha(c) - \alpha(0) \quad \forall (x, v, t) \in \tilde{A}_0. \]

Consider the closed one-form \(\tilde{\omega}\) on \(M \times \mathbb{T}\) defined by
\[ \tilde{\omega}(x, v, \tau) := \omega_x(v) + (\alpha(0) - \alpha(c))\tau. \]

The cohomology class of \(\tilde{\omega}\) is \((c, \alpha(0) - \alpha(c))\) and \(\tilde{\omega} = d(u_0 - u_1)\) in \(A_0\) so replacing \(\tilde{\omega}\) by the continuous one-form \(\omega - d(u_0 - u_1)\) we see that \(c \in G_0\). \(\square\)

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